Integrable XYZ Model with Staggered Anisotropy Parameter

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Abstract

We apply to the XYZ model the technique of construction of integrable models with staggered parameters, presented recently for the XXZ case in [1]. The solution of modified Yang–Baxter equations is found and the corresponding integrable zig-zag ladder Hamiltonian is calculated. The result is coinciding with the XXZ case in the appropriate limit.

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1 Introduction

Recently a technique was proposed [1] which allows to construct a zig-zag ladder type of integrable models on the basis of known integrable chain models. It was successfully realised on models with basic $sl(n)$ symmetries; $sl(2)$ case (XXZ model) in [1], $sl(3)$ case (anisotropic $t-J$ model) in [2] and general $sl(n)$ case in the [3]. The main element of this construction is the possibility to stagger the parameters of known integrable R-matrix along chain and time directions. Together with alternation of the anisotropy parameter Δ [1, 2] the staggered shift of the spectral parameter u by some additional model parameter θ was also introduced, which caused the appearance of the next to nearest neighbour interaction terms in the resulting local Hamiltonian.

The motivation for considering this type of integrable models is based on the observation that the phenomenological Chalker-Coddington model [4] for edge excitations in the Hall effect after its reformulation as a lattice field theory [5] and calculation of disorder over random phases becomes a Hubbard type model with staggered disposition of R-matrices [6]. Therefore it is meaningful to start the investigation of staggered models from simple cases.

In this letter we generalise the construction of the staggered XXZ model and construct a staggered integrable XYZ model. The corresponding generalised Yang–Baxter Equations (YBE), which are the condition of integrability [7, 8], have a solution. Therefore, one can define a model on a two rung ladder, the Hamiltonian of which is presented here.

2 Basic Definitions and Staggered YBE's

We start by writing down the modification of basic ingredients of the Algebraic Bethe Ansatz (ABA) technique [7, 8] appropriate for our purposes.

Let us consider $\mathbb{Z}_2$ graded quantum $V_{j,\rho}(v)$ (with $j = 1, ...N$ as a chain index) and auxiliary $V_{a,\sigma}(u)$ spaces, where $\rho, \sigma = 0, 1$ are the grading indices. Consider R-matrices, which act on the direct product of spaces $V_{a,\sigma}(u)$ and $V_{j,\rho}(v)$, $(\sigma, \rho = 0, 1)$, mapping them on the intertwined direct product of $V_{a,\bar{\sigma}}(u)$ and $V_{j,\bar{\rho}(v)}$ with the complementary $\bar{\sigma} = (1 - \sigma)$, $\bar{\rho} = (1 - \rho)$ indices

$$R_{aj,\sigma\rho}(u, v) : \quad V_{a,\sigma}(u) \otimes V_{j,\rho}(v) \to V_{j,\bar{\rho}}(v) \otimes V_{a,\bar{\sigma}}(u). \tag{2.1}$$

It is convenient to introduce two transmutation operations ι_1 and ι_2 with the property $\iota_1^2 = \iota_2^2 = id$ for the quantum and auxiliary spaces correspondingly, and to define the operators $R_{aj,\sigma\rho}$ as follows

$$\begin{aligned} R_{aj,00} &\equiv R_{aj}, \qquad R_{aj,01} \equiv R_{aj}^{\iota_1}, \\ R_{aj,10} &\equiv R_{aj}^{\iota_2}, \qquad R_{aj,11} \equiv R_{aj}^{\iota_1\iota_2}. \end{aligned} \tag{2.2}$$

The introduction of the $\mathbb{Z}_2$ grading of quantum spaces in time direction means that we have now two monodromy operators $T_\rho, \rho = 0, 1$, which act on the space $V_\rho(u) = \prod_{j=1}^N V_{j,\rho}(u)$ by mapping it on $V_{\bar{\rho}}(u) = \prod_{j=1}^N V_{j,\bar{\rho}}(u)$

$$T_\rho(v, u) \qquad : V_\rho(u) \to V_{\bar{\rho}}(u), \qquad \rho = 0, 1. \tag{2.3}$$

It is clear now, that the monodromy operator of the model, which is defined by translational invariance by two steps in the time direction and which determines the partition function, is the product of two monodromy operators

$$T(v, u) = T_0(v, u)T_1(v, u). \tag{2.4}$$

The $\mathbb{Z}_2$ grading of auxiliary spaces along the chain direction means that the $T_0(u, v)$ and $T_1(u, v)$ monodromy matrices are defined according to the following staggered product of the $R_{aj}(v, u)$ and $\bar{R}_{aj}^{\iota_2}(v, u)$

matrices:

$$
\begin{aligned}
T_1(v,u) &= \prod_{j=1}^{N} R_{a,2j-1}(v,u)\bar{R}^{\iota_2}_{a,2j}(v,u) \\
T_0(v,u) &= \prod_{j=1}^{N} \bar{R}^{\iota_1}_{a,2j-1}(v,u)R^{\iota_1\iota_2}_{a,2j}(v,u),
\end{aligned} \tag{2.5}
$$

where the notation $\bar{R}$ denotes a different parametrisation of the $R(v,u)$-matrix via spectral parameters v and u and can be considered as an operation over R with property $\bar{\bar{R}} = R$. For the integrable models where the intertwiner matrix $R(v-u)$ simply depends on the difference of the spectral parameters v and u this operation is a shift of its argument u:

$$
\bar{R}(u) = R(\bar{u}), \qquad \bar{u} = \theta - u, \tag{2.6}
$$

where θ is an additional model parameter. We will consider this case in this paper.

As it is well known in Bethe Ansatz technique [7, 8], the sufficient condition for the commutativity of transfer matrices $\tau(u) = TrT(u)$ with different spectral parameters is the YBE. For our case we have a set of two equations [1]

$$
R_{12}(u,v)\bar{R}^{\iota_1}_{13}(u)R_{23}(v) = R^{\iota_1}_{23}(v)\bar{R}_{13}(u)\tilde{R}_{12}(u,v) \tag{2.7}
$$

and

$$
\tilde{R}_{12}(u,v)R^{\iota_1\iota_2}_{13}(u)\bar{R}^{\iota_2}_{23}(v) = \bar{R}^{\iota_1\iota_2}_{23}(v)R^{\iota_2}_{13}(u)R_{12}(u,v)\ . \tag{2.8}
$$

3 Staggered XYZ Heisenberg chain

The R-matrix of the ordinary XYZ Heisenberg model can be represented as

$$
R(u) = \begin{pmatrix} a(u) & & & d(u) \\ & b(u) & c(u) & \\ & c(u) & b(u) & \\ d(u) & & & a(u) \end{pmatrix} \tag{3.1}
$$

Inputting this expression of $R(u)$ into the YBE (2.7) we obtain the following three sets of equations on $a(u), b(u), c(u), d(u)$. The first set is:

$$
\begin{aligned}
a(u,v)a^{\iota_1}(\bar{u})a(v) + d(u,v)c^{\iota_1}(\bar{u})d(v) &= a^{\iota_1}(v)a(\bar{u})\tilde{a}(u,v) + d^{\iota_1}(v)c(\bar{u})\tilde{d}(u,v), \\
a(u,v)b^{\iota_1}(\bar{u})b(v) + d(u,v)d^{\iota_1}(\bar{u})c(v) &= b^{\iota_1}(v)b(\bar{u})\tilde{a}(u,v) + c^{\iota_1}(v)d(\bar{u})\tilde{d}(u,v), \\
b(u,v)b^{\iota_1}(\bar{u})a(v) + c(u,v)d^{\iota_1}(\bar{u})d(v) &= a^{\iota_1}(v)b(\bar{u})\tilde{b}(u,v) + d^{\iota_1}(v)d(\bar{u})\tilde{c}(u,v), \\
b(u,v)a^{\iota_1}(\bar{u})b(v) + c(u,v)c^{\iota_1}(\bar{u})c(v) &= b^{\iota_1}(v)a(\bar{u})\tilde{b}(u,v) + c^{\iota_1}(v)c(\bar{u})\tilde{c}(u,v),
\end{aligned} \tag{3.2}
$$

which are satisfied automatically in case of trivial ι_1 operation, as it happened in ordinary XYZ model.

The second set

$$
\begin{aligned}
a(u,v)b^{\iota_1}(\bar{u})c(v) + d(u,v)d^{\iota_1}(\bar{u})b(v) &= c^{\iota_1}(v)a(\bar{u})\tilde{b}(u,v) + b^{\iota_1}(v)c(\bar{u})\tilde{c}(u,v), \\
b(u,v)a^{\iota_1}(\bar{u})c(v) + c(u,v)c^{\iota_1}(\bar{u})b(v) &= c^{\iota_1}(v)b(\bar{u})\tilde{a}(u,v) + b^{\iota_1}(v)d(\bar{u})\tilde{d}(u,v),
\end{aligned}
$$

$$\begin{aligned}
a(u,v)c^{\iota_1}(\bar u)a(v)+d(u,v)a^{\iota_1}(\bar u)d(v)&=c^{\iota_1}(v)a(\bar u)\tilde c(u,v)+b^{\iota_1}(v)c(\bar u)\tilde b(u,v),\\
c(u,v)a^{\iota_1}(\bar u)c(v)+b(u,v)c^{\iota_1}(\bar u)b(v)&=a^{\iota_1}(v)c(\bar u)\tilde a(u,v)+d^{\iota_1}(v)a(\bar u)\tilde d(u,v),\\
b(u,v)c^{\iota_1}(\bar u)c(v)+c(u,v)a^{\iota_1}(\bar u)b(v)&=a^{\iota_1}(v)b(\bar u)\tilde c(u,v)+d^{\iota_1}(v)d(\bar u)\tilde b(u,v),\\
b(u,v)d^{\iota_1}(\bar u)d(v)+c(u,v)b^{\iota_1}(\bar u)a(v)&=b^{\iota_1}(v)a(\bar u)\tilde c(u,v)+c^{\iota_1}(v)c(\bar u)\tilde b(u,v),
\end{aligned}\tag{3.3}$$

is reducing to staggered XXZ models equations [1] in case of $d(u)=0$. Finally we have a third set of equations

$$\begin{aligned}
a(u,v)a^{\iota_1}(\bar u)d(v)+d(u,v)c^{\iota_1}(\bar u)a(v)&=a^{\iota_1}(v)d(\bar u)\tilde c(u,v)+d^{\iota_1}(v)b(\bar u)\tilde b(u,v),\\
b(u,v)b^{\iota_1}(\bar u)d(v)+c(u,v)d^{\iota_1}(\bar u)a(v)&=a^{\iota_1}(v)c(\bar u)\tilde d(u,v)+d^{\iota_1}(v)a(\bar u)\tilde a(u,v),\\
b(u,v)d^{\iota_1}(\bar u)a(v)+c(u,v)b^{\iota_1}(\bar u)d(v)&=b^{\iota_1}(v)d(\bar u)\tilde a(u,v)+c^{\iota_1}(v)b(\bar u)\tilde d(u,v),\\
a(u,v)d^{\iota_1}(\bar u)b(v)+d(u,v)b^{\iota_1}(\bar u)c(v)&=a^{\iota_1}(v)d(\bar u)\tilde b(u,v)+d^{\iota_1}(v)b(\bar u)\tilde c(u,v),\\
a(u,v)c^{\iota_1}(\bar u)d(v)+d(u,v)a^{\iota_1}(\bar u)a(v)&=b^{\iota_1}(v)b(\bar u)\tilde d(u,v)+c^{\iota_1}(v)d(\bar u)\tilde a(u,v),\\
a(u,v)d^{\iota_1}(\bar u)c(v)+d(u,v)b^{\iota_1}(\bar u)b(v)&=a^{\iota_1}(v)a(\bar u)\tilde d(u,v)+d^{\iota_1}(v)c(\bar u)\tilde a(u,v),
\end{aligned}\tag{3.4}$$

which represents the non-trivial part of XYZ model. In the XXZ case, when $d(u)=0$, this set of equations disappears.

Our aim is now to find a solution of equations (3.2), (3.3), (3.4) with non-trivial ι_1 operation. It can be seen easily that by defining

$$\begin{aligned}
a^{\iota_1}(u)&=a(u), &\quad \tilde a(u,v)&=a(u,v),\\
b^{\iota_1}(u)&=-b(u), &\quad \tilde b(u,v)&=-b(u,v)\\
c^{\iota_1}(u)&=c(u), &\quad \tilde c(u,v)&=c(u,v)\\
d^{\iota_1}(u)&=-d(u), &\quad \tilde d(u,v)&=-d(u,v)
\end{aligned}\tag{3.5}$$

the eqs. (3.2) obeyed identically, while the eqs. (3.3) and (3.4) are reducing to the corresponding equations of the staggered case. Therefore the ordinary choice of parametrisation of $a(u), b(u), c(u)$ and $d(u)$ via the Jacobi elliptic functions

$$\begin{aligned}
a(u)&=\mathrm{sn}(u+\eta)=a^{\iota_1}(u), &\quad a(u,v)&=\mathrm{sn}(\bar u-v+\eta)=\tilde a(u,v),\\
b(u)&=\mathrm{sn}u=-b^{\iota_1}(u), &\quad b(u,v)&=\mathrm{sn}(\bar u-v)=-\tilde b(u,v),\\
c(u)&=\mathrm{sn}\eta=c^{\iota_1}(u), &\quad c(u,v)&=\mathrm{sn}\eta=\tilde c(u,v),\\
&\; d(u)=k\mathrm{sn}\eta\mathrm{sn}u\mathrm{sn}(u+\eta)=-d^{\iota_1}(u),\\
&\; d(u,v)=k\mathrm{sn}\eta\mathrm{sn}(\bar u-v)\mathrm{sn}(\bar u-v+\eta)=-\tilde d(u,v),
\end{aligned}\tag{3.6}$$

fulfils the remaining equations (3.3) and (3.4).

Let us analyse now the second set of YBE's (2.8). First we conclude immediately from the solution (3.5) of the previous set of YBE's that the operation $\tilde{}$ is simply coinciding with the operation ι_1. Then it is easy to see, that if we define ι_2 operation in (2.8) as

$$R^{\iota_2}(u)=R^{\iota_1}(-u)\ , \tag{3.7}$$

then the second set (2.8) of YBE's coincides with the first set (2.7) after additional action on it by ι_1. This means that the relation (3.7) is ensuring the fulfilment of the second set (2.8) of YBE's. Therefore we have

$$a^{\iota_2}(u)=a(-u),\quad b^{\iota_2}(u)=-b(-u),\quad c^{\iota_2}(u)=c(-u),\quad d^{\iota_2}(u)=-d(-u). \tag{3.8}$$

It is now necessary to emphasise that, as in ordinary case, the parameters Δ and k defined as

$$\Delta=\frac{a^2+b^2-c^2-d^2}{2ab}, \qquad k\mathrm{sn}^2\eta=\frac{cd}{ab} \tag{3.9}$$

are constants (as well as Δ^{ι_1} and k^{ι_1}). They are defining the anisotropy of the model in the z and y directions. As one can see, now, due to solution (3.6), the anisotropy parameter Δ is staggered along the chain and time directions.

Hence, we have found the solution of staggered YBE's (2.7), (2.8) and can calculate now the Hamiltonian of corresponding integrable model.

4 The Transfer matrix and the Hamiltonian

Having the solution of graded YBE's one can start the calculation of the monodromy matrix and the Hamiltonian. According to formula (2.4) the monodromy matrix of the model is

$$\begin{aligned}
T^{ab,\ i_1...i_{2N}}_{cd,\ k_1...k_{2N}}(u,\theta) &\equiv T_0{}^{a,\ i_1...i_{2N}}_{c,\ j_1...j_{2N}}(u,\theta)\ T_1{}^{b,\ j_1...j_{2N}}_{d,\ k_1...k_{2N}}(u,\theta)\\
&= \left(R^{\iota_1}_{0,1}{}^{a\ i_1}_{a_1 j_1}(\bar u)\ R^{\iota_1\iota_2}_{0,2}{}^{a_1 i_2}_{a_2 j_2}(u)\ R^{\iota_1}_{0,3}{}^{a_2 i_3}_{a_3 j_3}(\bar u)...\ R^{\iota_1\iota_2}_{0,2N}{}^{a_{2N-1} i_{2N}}_{c\quad j_{2N}}(u)\right)\\
&\quad \left(R_{0'1}{}^{b\ j_1}_{b_1 k_1}(u)\ R^{\iota_2}_{0',2}{}^{b_1 j_2}_{b_2 k_2}(\bar u)\ R_{0',3}{}^{b_2 j_3}_{b_3 k_3}(u)...\ R^{\iota_2}_{0',2N}{}^{b_{2N-1} j_{2N}}_{d\quad k_{2N}}(\bar u)\right) \qquad (4.1)\\
&= \left(R^{\iota_1}_{0,1}{}^{a\ i_1}_{a_1 j_1}(\bar u)\ R_{0',1}{}^{b\ j_1}_{b_1 k_1}(u)\ R^{\iota_1\iota_2}_{0,2}{}^{a_1 i_2}_{a_2 j_2}(u)\ R^{\iota_2}_{0',2}{}^{b_1 j_2}_{b_2 k_2}(\bar u)\right)...\\
&\quad \left(R^{\iota_1}_{0,2N-1}{}^{a_{2N-2} i_{2N-1}}_{a_{2N-1} j_{2N-1}}(\bar u)\ R_{0',2N-1}{}^{b_{2N-2} j_{2N-1}}_{b_{2N-1} k_{2N-1}}(u)\ R^{\iota_1\iota_2}_{0,2N}{}^{a_{2N-1} i_{2N}}_{c\quad j_{2N}}(u)\ R^{\iota_2}_{0',2N}{}^{b_{2N-1} j_{2N}}_{d\quad k_{2N}}(\bar u)\right). \qquad (4.2)
\end{aligned}$$

Lets write now an explicit formula for the R-matrix in terms of Pauli matrices

$$\begin{aligned}
R_{0,r}{}^{ai}_{cj}(u) = a(u)\left(\sigma_0{}^a_c\sigma_r{}^i_j+\bar\sigma_0{}^a_c\bar\sigma_r{}^i_j\right)+b(u)\left(\sigma_0{}^a_c\bar\sigma_r{}^i_j+\bar\sigma_0{}^a_c\sigma_r{}^i_j\right)+\\
c(u)\left(\sigma_0^+{}^a_c\sigma_r^-{}^i_j+\sigma_0^-{}^a_c\sigma_r^+{}^i_j\right)+d(u)\left(\sigma_0^+{}^a_c\sigma_r^+{}^i_j+\sigma_0^-{}^a_c\sigma_r^-{}^i_j\right),
\end{aligned} \tag{4.3}$$

where 0 and r refer to auxiliary and quantum spaces respectively and

$$\sigma\equiv\frac12\left(\sigma^4+\sigma^z\right),\quad \bar\sigma\equiv\frac12\left(\sigma^4-\sigma^z\right),\quad \sigma^+\equiv\frac12\left(\sigma^x+i\sigma^y\right),\quad \sigma^-\equiv\frac12\left(\sigma^x-i\sigma^y\right).$$

Then transfer matrix is defined as trace of monodromy matrix (4.1) over auxiliary spaces and at the zero value of the spectral parameter u is equal to

$$\begin{aligned}
\tau^{i_1...i_{2n}}_{k_1...k_{2n}}(0,\theta) &\equiv T^{ab,i_1...i_{2N}}_{ab,k_1...k_{2N}}(0,\theta) &(4.4)\\
&= tr_0 tr_{0'}\prod_{r=1}^{N}\left(R^{\iota_1}{}^{a_{2r-1} i_{2r-1}}_{a_{2r}\ j_{2r-1}}(\theta)R^{b_{2r-1} j_{2r-1}}_{b_{2r}\ k_{2r}}(0)R^{a_{2r}\ i_{2r}}_{a_{2r+1} j_{2r}}(0)R^{\iota_1}{}^{b_{2r}\ j_{2r}}_{b_{2r+1} k_{2r}}(-\theta)\right) &(4.5)\\
&= tr_0 tr_{0'}\prod_{r=1}^{N}\left(\mathrm{sn}^2\eta(\mathrm{sn}^2\eta-\mathrm{sn}^2\theta)P_{0,2r}P_{0',2r-1}\right) &(4.6)\\
&= \left(\mathrm{sn}^2\eta(\mathrm{sn}^2\eta-\mathrm{sn}^2\theta)\right)^N\delta^{i_1}_{k_3}\delta^{i_2}_{k_4}...\delta^{i_{2N-2}}_{k_{2N}}\delta^{i_{2n-1}}_{k_1}\delta^{i_{2N}}_{k_2}, &(4.7)
\end{aligned}$$

i.e. up to overall multiplier it is becoming an operator that shifts along the chain by two units, i.e. a translation operator. Here we have used

$$R_{0,r}{}^{ai}_{cj}(0)=\mathrm{sn}\eta\cdot\delta^i_c\delta^a_j\equiv\mathrm{sn}\eta\cdot P_{0,r}$$

where $P_{0,r}$ is an operator permuting auxiliary (0-th) and quantum (r-th) spaces. We also used the relation (unitarity property of R)

$$R^{\iota_1}{}^{a_{2r-1}i_{2r-1}}_{a_{2r}\ \ b_{2r}}(\theta)R^{\iota_1}{}^{b_{2r}a_{2r}}_{b_{2r+1}k_{2r}}(-\theta) = \left(\mathrm{sn}^2\eta - \mathrm{sn}^2\theta\right)\delta^{a_{2r-1}}_{k_{2r}}\delta^{i_{2r-1}}_{b_{2r+1}},$$

which holds due to following identities for elliptic functions:

$$a(u)a(-u) + d(u)d(-u) = \mathrm{sn}^2\eta - \mathrm{sn}^2 u = b(u)b(-u) + c(u)c(-u)$$

$$a(u)d(-u) + d(u)a(-u) = 0 = b(u)c(-u) + c(u)b(-u).$$

The functions $a(u)$, $b(u)$, $c(u)$ and $d(u)$ are given by (3.6) Now we can turn to calculation of the Hamiltonian

$$\frac{d}{du}\log\tau\Big|_{u=0} = \mathrm{sn}^2\eta(\mathrm{sn}^2\eta - \mathrm{sn}^2 u)\ \ \mathcal{H}^{j_1\ldots j_{2N}}_{k_1\ldots k_{2N}}\ P^{i_3\ldots i_{2N}\,i_1\,i_2}_{j_1\ldots j_{2N-1}j_{2N}} \tag{4.8}$$

One can see, that the Hamiltonian of the model is reduces to the sum of contributions from the derivatives of quartic products of neighbour R-matrices inside of brackets in (4.5) after the permutation of indices. It appears that the differentiation of two terms in the bracket again contributes to the identity operator and provides a constant contribution to Hamiltonian. Therefore the non-trivial contribution looks like

$$\left(R^{\iota_1}{}^{i_2 i_3}_{a_2 a_3}(\theta)R'{}^{i_1 a_3}_{a_1 k_1}(0)R^{\iota_1}{}^{a_1 a_2}_{k_3 k_2}(-\theta)\delta^{i_4}_{k_4}\ldots\delta^{i_{2N}}_{k_{2N}} - R^{\iota_1}{}^{i_2 i_3}_{a_2 a_3}(\theta)R'{}^{a_2 i_4}_{k_4 a_4}(0)R^{\iota_1}{}^{a_3 a_4}_{k_3 k_2}(-\theta)\delta^{i_5}_{k_5}\ldots\delta^{i_1}_{k_1}\right) + \ldots \tag{4.9}$$

After some calculations by use of properties of elliptic functions and (3.9) one can find the following expression for the Hamiltonian (without the constant part)

$$\mathcal{H} \ = \ \sum_{r=1}^{N}\mathcal{H}_{2r-1,2r,2r+1,2r+2}, \tag{4.10}$$

$$\begin{aligned}
&\mathrm{sn}^2\eta(\mathrm{sn}^2\eta - \mathrm{sn}^2 u)\mathcal{H}_{2r-1,2r,2r+1,2r+2} = \qquad\qquad (4.11)\\
&\quad \frac{1}{2}\left(a(u)a(-u) - c(u)c(-u)\right)a'(0)\left[\sigma^z_{2r-1}\sigma^z_{2r+1} - \sigma^z_{2r}\sigma^z_{2r+2}\right]\\
&\quad +\frac{b(u)}{2}\left(a(u) - a(-u)\right)\left(b'(0) - kc^2(u)d'(0)\right)\left[\sigma^+_{2r-1}\sigma^-_{2r+1} + \sigma^-_{2r-1}\sigma^+_{2r+1} - \sigma^+_{2r}\sigma^-_{2r+2} - \sigma^-_{2r}\sigma^+_{2r+2}\right]\\
&\quad +\frac{b(u)}{2}\left(a(u) - a(-u)\right)\left(d'(0) - kc^2(u)b'(0)\right)\left[\sigma^+_{2r-1}\sigma^+_{2r+1} + \sigma^-_{2r-1}\sigma^-_{2r+1} - \sigma^+_{2r}\sigma^+_{2r+2} - \sigma^-_{2r}\sigma^-_{2r+2}\right]\\
&\quad -b(u)c(u)a'(0)\left[\sigma^z_{2r}(\sigma^+_{2r+1}\sigma^-_{2r+2} - \sigma^-_{2r+1}\sigma^+_{2r+2}) + (\sigma^+_{2r-1}\sigma^-_{2r} - \sigma^-_{2r-1}\sigma^+_{2r})\sigma^z_{2r+1}\right]\\
&\quad -\frac{b(u)}{2}\left(a(u) + a(-u)\right)\left(b'(0) - kc^2(u)d'(0)\right)\left[\sigma^z_{2r+1}(\sigma^+_{2r}\sigma^-_{2r+2} - \sigma^-_{2r}\sigma^+_{2r+2}) + (\sigma^+_{2r-1}\sigma^-_{2r+1} - \sigma^-_{2r-1}\sigma^+_{2r+1})\sigma^z_{2r}\right]\\
&\quad +\frac{c(u)}{2}\left(a(u) - a(-u)\right)\left(b'(0) - kb^2(u)d'(0)\right)\left[\sigma^z_{2r-1}(\sigma^+_{2r}\sigma^-_{2r+1} - \sigma^-_{2r}\sigma^+_{2r+1}) + (\sigma^+_{2r}\sigma^-_{2r+1} - \sigma^-_{2r}\sigma^+_{2r+1})\sigma^z_{2r+2}\right]\\
&\quad +a(u)d(u)a'(0)\left[\sigma^z_{2r}(\sigma^+_{2r+1}\sigma^+_{2r+2} - \sigma^-_{2r+1}\sigma^-_{2r+2}) - (\sigma^+_{2r-1}\sigma^+_{2r} - \sigma^-_{2r-1}\sigma^-_{2r})\sigma^z_{2r+1}\right]\\
&\quad +\frac{b(u)}{2}\left(a(u) + a(-u)\right)\left(d'(0) - kc^2(u)b'(0)\right)\left[\sigma^z_{2r+1}(\sigma^+_{2r}\sigma^+_{2r+2} - \sigma^-_{2r}\sigma^-_{2r+2}) - (\sigma^+_{2r-1}\sigma^+_{2r+1} - \sigma^-_{2r-1}\sigma^-_{2r+1})\sigma^z_{2r}\right]\\
&\quad +\frac{c(u)}{2}\left(a(u) - a(-u)\right)\left(d'(0) - kb^2(u)b'(0)\right)\left[\sigma^z_{2r-1}(\sigma^+_{2r}\sigma^+_{2r+1} - \sigma^-_{2r}\sigma^-_{2r+1}) - (\sigma^+_{2r}\sigma^+_{2r+1} - \sigma^-_{2r}\sigma^-_{2r+1})\sigma^z_{2r+2}\right]
\end{aligned}$$

This expression for the Hamiltonian due to next to nearest neighbour interactions can be easier understood as one, written on a zig-zag chain.

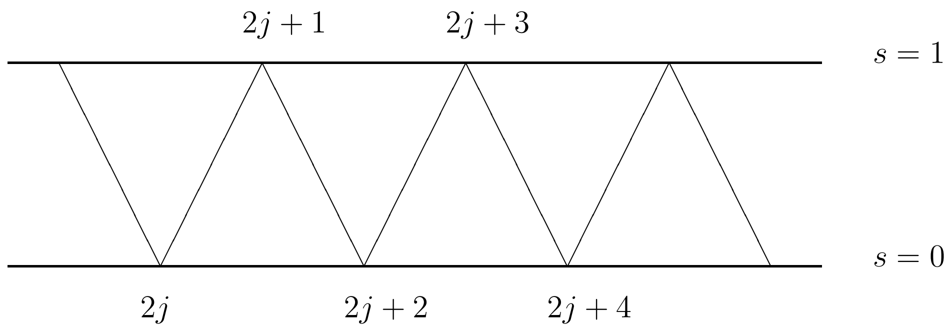

Figure 1: Zig-zag ladder chain

Let us introduce two chains consisting of the even and odd sites of the original chain and label them by $s=0$ and 1 correspondingly. Make now zig-zag rungs as it is shown in the Fig.1 and introduce the following labelling of Pauli matrices

$$\vec{\sigma}_{j,s} = \vec{\sigma}_{2j+s}, \qquad s = 0,1 \tag{4.12}$$

Then, substituting the expression (3.6) for $a(u), b(u), c(u), d(u)$ to (4.10) and after some interesting cancellations of some terms due to identities on elliptic functions, one can write the following zig-zag ladder Hamiltonian

$$\begin{aligned}
&(1-k^2\mathrm{sn}^2\eta\,\mathrm{sn}^2\theta)(\mathrm{sn}^2\eta-\mathrm{sn}^2\theta)\mathcal{H}_{j,s} = \\
&= \frac{(-1)^{s+1}}{2}\mathrm{sn}^2\theta\,\mathrm{cn}\eta\,\mathrm{dn}\eta(1-k^2\mathrm{sn}^4\eta)\left[\sigma^1_{j,s}\sigma^1_{j+1,s}+\sigma^2_{j,s}\sigma^2_{j+1,s}-\sigma^3_{j,s}\sigma^3_{j+1,s}\right] \\
&+ \hat{\epsilon}^{abc}_s\sigma^a_{j,s}\sigma^b_{j,s+1}\sigma^c_{j+1,s}+\hat{\tau}^{abc}_s\sigma^a_{j,s}\sigma^b_{j,s+1}\sigma^c_{j+1,s}, \qquad s=0,1
\end{aligned} \tag{4.13}$$

where the anisotropic antisymmetric tensors $\hat{\epsilon}^{abc}_s$ and $\hat{\tau}^{abc}_s$ are defined as follows:

$$\begin{aligned}
&\hat{\epsilon}^{3+-}_0 = -\hat{\epsilon}^{3-+}_0 = -\hat{\epsilon}^{+-3}_0 = \hat{\epsilon}^{-+3}_0 = -\hat{\epsilon}^{3+-}_1 = \hat{\epsilon}^{3-+}_1 = \hat{\epsilon}^{+-3}_1 = -\hat{\epsilon}^{-+3}_1 = -\mathrm{sn}\theta\,\mathrm{sn}\eta\,\mathrm{cn}\eta\,\mathrm{dn}\eta(1-k^2\mathrm{sn}^2\eta\,\mathrm{sn}^2\theta), \\
&\hat{\epsilon}^{+3-}_0 = -\hat{\epsilon}^{-3+}_0 = \hat{\epsilon}^{+3-}_1 = -\hat{\epsilon}^{-3+}_1 = \mathrm{sn}\eta\,\mathrm{sn}\theta\,\mathrm{cn}\theta\,\mathrm{dn}\theta(1-\mathrm{sn}^4\eta)
\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}
&\hat{\tau}^{3++}_0 = -\hat{\tau}^{3--}_0 = \hat{\tau}^{++3}_0 = -\hat{\tau}^{--3}_0 = -\hat{\tau}^{3++}_1 = \hat{\tau}^{3--}_1 = -\hat{\tau}^{++3}_1 = \hat{\tau}^{--3}_1 = -k\,\mathrm{sn}\eta\,\mathrm{sn}\theta\,\mathrm{cn}\eta\,\mathrm{dn}\eta(\mathrm{sn}^2\eta-\mathrm{sn}^2\theta), \\
&\hat{\tau}^{+3+}_0 = \hat{\tau}^{-3-}_0 = \hat{\tau}^{+3+}_1 = \hat{\tau}^{-3-}_1 = 0\,.
\end{aligned} \tag{4.15}$$

This tensors define the interaction terms of topological character. The XXZ limit (k=0) of the Hamiltonian (4.13) coincides with the corresponding expression found in [1].

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