

A NEW MAXIMALLY SUPERSYMMETRIC BACKGROUND OF IIB SUPERSTRING THEORY

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ABSTRACT. We present a maximally supersymmetric IIB string background. The geometry is that of a conformally flat lorentzian symmetric space G/K with solvable G , with a homogeneous five-form flux. We give the explicit supergravity solution, compute the isometries, the 32 Killing spinors, and the symmetry superalgebra, and then discuss T-duality and the relation to M-theory.

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1. INTRODUCTION

There are precisely four types of maximally supersymmetric solutions of eleven-dimensional supergravity [16, 5, 7]. The first three types are the familiar cases of flat eleven-dimensional space (Minkowski space and its toroidal compactifications), $\text{AdS}_4 \times S^7$ and $\text{AdS}_7 \times S^4$. The fourth solution with 32 Killing spinors is less well-known and was discovered by Kowalski-Glikman [16]. In what follows we shall refer to this as the KG space or solution. Our purpose here is to show that IIB supergravity [18, 17, 10] has, in addition to flat space and $\text{AdS}_5 \times S^5$, another maximally supersymmetric solution which is analogous to the KG space.

Eleven-dimensional supergravity, with bosonic fields the metric ds^2 and the four-form field strength F_4 , has pp-wave solutions [11] of the

form

$$\begin{aligned} ds^2 &= 2dx^+ dx^- + H(x^i, x^-)(dx^-)^2 + \sum_{i=1}^9 (dx^i)^2 \\ F_4 &= dx^- \wedge \varphi \end{aligned} \quad (1)$$

where $H(x^i, x^-)$ obeys

$$\triangle H = \frac{1}{12}|\varphi|^2. \quad (2)$$

Here \triangle is the laplacian in the transverse euclidean space \mathbb{E}^9 with coordinates x^i and $\varphi(x^i, x^-)$ is (for each x^-) a closed and coclosed 3-form in \mathbb{E}^9 . Observe that $\partial/\partial x^+$ is a covariantly constant null vector. This solution preserves at least 16 supersymmetries.

An interesting subclass of these metrics are those in which

$$H(x^i, x^-) = \sum_{i,j} A_{ij} x^i x^j \quad (3)$$

where $A_{ij} = A_{ji}$ is a constant symmetric matrix. Remarkably, these metrics describe lorentzian symmetric spaces G/K where $K = \mathbb{R}^9$ and G is a solvable group depending on A [1, 8]. Moreover, they are indecomposable (that is, they are not locally isometric to a product) when the matrix A_{ij} is non-degenerate [1]. As H grows quadratically with $|x^i|$, these are not conventional gravitational wave solutions. Such solutions with covariantly constant F_4 are spacetimes with a null homogeneous flux, referred to as Hpp-waves in [8]. The solutions are parameterised (up to overall scale and permutations) by the eigenvalues of A_{ij} and all have at least 16 Killing spinors. Surprisingly, there is precisely one non-trivial choice of A_{ij} (up to diffeomorphisms and overall scale transformations), for which the solution has 32 Killing spinors. This is the KG solution, with

$$\begin{aligned} A_{ij} &= \begin{cases} -\frac{1}{9}\mu^2 \delta_{ij} & i, j = 1, 2, 3 \\ -\frac{1}{36}\mu^2 \delta_{ij} & i, j = 4, 5, \dots, 9 \end{cases} \\ \varphi &= \mu dx^1 \wedge dx^2 \wedge dx^3, \end{aligned} \quad (4)$$

where μ is a parameter which can be set to any nonzero value by a change of coordinates.

The IIB supergravity also has pp-wave solutions with metric of the form (1) but now the transverse space is \mathbb{E}^8 and so $i = 1, \dots, 8$. In particular, we shall show that if the only non-vanishing flux is that of the self-dual five-form field strength F_5 and the dilaton is constant, then there is a pp-wave solution with metric similar to that of (1). The self-dual five-form F_5 is null and associated with a closed self-dual 4-form φ on the transverse \mathbb{E}^8 provided that H satisfies an equation similar to (2). Again these solutions generically preserve 16 supersymmetries. If H is of the form (3), then the metric again describes a lorentzian symmetric space G/\mathbb{R}^8 ; the solvable group G will be discussed in detail in Section 6.

This solution describes an Hpp-wave with null homogeneous five-form flux. Furthermore, we will show that there is precisely one choice of A_{ij} (up to diffeomorphisms and overall scale transformations) for which the solution has 32 Killing spinors. In particular, we shall find that

$$\begin{aligned} ds^2 &= 2dx^- dx^+ - 4\lambda^2 \sum_{i=1}^8 (x^i)^2 (dx^-)^2 + \sum_{i=1}^8 (dx^i)^2 \\ F_5 &= \lambda dx^- \wedge (dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8) \end{aligned} \quad (5)$$

preserves all supersymmetry of IIB supergravity, where λ is a parameter which can be set to any desired nonzero value by a coordinate transformation. For this solution

$$\begin{aligned} A_{ij} &= -4\lambda^2 \delta_{ij} \\ \varphi &= \lambda(\omega + \star\omega) , \end{aligned} \quad (6)$$

where

$$\omega = dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \quad (7)$$

and \star denotes the Hodge dual in the transverse \mathbb{R}^8 .

The isometry groups of the 11-dimensional solutions $\text{AdS}_4 \times S^7$ and $\text{AdS}_7 \times S^4$ are $\text{SO}(3, 2) \times \text{SO}(8)$ and $\text{SO}(6, 2) \times \text{SO}(5)$, respectively, both 38-dimensional. The isometry group of the KG solution [2, 8] is again 38-dimensional but it is non-semisimple. In fact, it is isomorphic to the extension, by a one-dimensional group of outer automorphisms, of the semidirect product $H(9) \ltimes (\text{SO}(3) \times \text{SO}(6))$, with $H(9)$ a 9-dimensional Heisenberg group. This semidirect product group is the quotient by a central element of $\text{CSO}(3, 2) \times \text{CSO}(6, 2)$, a certain group contraction¹ of both $\text{SO}(3, 2) \times \text{SO}(8)$ and of $\text{SO}(6, 2) \times \text{SO}(5)$. For the IIB theory, the isometry group of $\text{AdS}_5 \times S^5$ is the 30-dimensional group $\text{SO}(4, 2) \times \text{SO}(6)$, while the isometry group of our new maximally supersymmetric space is again 30-dimensional and, as we will show, is related to the $\text{CSO}(4, 2) \times \text{CSO}(4, 2)$ contraction of $\text{SO}(4, 2) \times \text{SO}(6)$.

The physical significance of these solutions in string and M-theory is puzzling, particularly for the Hpp-wave solutions that preserve all supersymmetry which are in the same universality class of solutions as Minkowski and $\text{AdS} \times S$ spaces. Nevertheless, unlike the M-theory Hpp-wave, the IIB Hpp-wave that we have found in this paper may be more amenable to investigation at least in string perturbation theory. This is because the dilaton in this background is constant and so the string coupling constant can be kept small everywhere. Arguments similar to those in [15] which utilise the form of the geometry and the 32 supersymmetries should imply that the new solution is an exact solution of IIB string theory and that the KG solution should

¹The group $\text{CSO}(p, q)$ was defined in [12] as the group contraction of $\text{SO}(p+q)$ or $\text{SO}(p, q)$ preserving a metric with q positive eigenvalues and p zero ones, so that, for example, $\text{CSO}(p, 1)$ is the Euclidean group $\text{ISO}(p)$.

be an exact solution of M-theory. However, such arguments are rather formal in the absence of an explicit quantisation of string theory in RR backgrounds or of M-theory.

2. IIB FIELD EQUATIONS AND KILLING SPINORS

We shall consider IIB supergravity backgrounds specified by a triple (M, g, F_5) where M is a ten-dimensional oriented spin manifold, g is a lorentzian metric and F_5 is a self-dual five-form. The dilaton is constant and all other fields are set to zero. The relevant field equations are the Einstein equations and the self-duality constraint on F_5 . A Killing spinor for such a background need only satisfy the equation associated with the vanishing of the gravitino supersymmetry transformation, as all other conditions are satisfied automatically for our ansatz. For the maximally supersymmetric solution, the Killing spinor equations imply the field equations.

In our conventions, the metric is mostly plus with signature $(9, 1)$. The Clifford algebra $\text{Cl}(9, 1)$, into which the spin group $\text{Spin}(9, 1)$ embeds, admits a basis spanned by real gamma-matrices Γ^a satisfying

$$\{\Gamma^a, \Gamma^b\} = +2\eta^{ab} \mathbb{1}$$

such that Γ^0 is skewsymmetric and hence skewhermitian, whereas the Γ^i , for $i = 1, \dots, 9$ are symmetric and hence hermitian. In this basis, the charge conjugation matrix C is Γ^0 . The group $\text{Spin}(9, 1)$ has two real irreducible representations of dimension 16: S_+ and S_- , the spinors of positive and negative chirality, respectively. Let S_\pm denote the corresponding spin bundles over M . The two spinor parameters in the supersymmetry transformations have positive chirality, so each is a section of S_+ . It is convenient to combine the two spinors into a single complex-valued 16-component chiral spinor field, which is a section of the complexified Weyl spinor bundle $S := S_+ \otimes \mathbb{C}$.

The Einstein equation for our ansatz is (in the Einstein frame)

$$R_{MN} - \frac{1}{2}g_{MN}R = \frac{1}{6}F_{L_1 \dots L_4 M} F^{L_1 \dots L_4}{}_N,$$

where $M, N = 0, 1, \dots, 9$. In the string frame, the background (M, g, F) is invariant under an infinitesimal supersymmetry transformation with spinor parameter ε if and only if ε satisfies the Killing spinor equation

$$\mathcal{D}_M \varepsilon = 0$$

where the supercovariant derivative is

$$\mathcal{D}_M \varepsilon = \nabla_M \varepsilon + \frac{i}{192} e^\phi F_{ML_1 \dots L_4} \Gamma^{L_1 \dots L_4} \varepsilon,$$

and ϕ is the (constant) dilaton, ∇ is the spin connection and products $\Gamma_{M \dots N}$ of Γ matrices are skewsymmetrised with strength one. It is

convenient to rescale the 5-form to absorb the prefactor, bringing the connection to the simpler form

$$\mathcal{D}_M \varepsilon = \nabla_M \varepsilon + \frac{i}{24} F_{ML_1 \dots L_4} \Gamma^{L_1 \dots L_4} \varepsilon . \quad (8)$$

In what follows we will present a background (M, g, F) for which this connection is flat: (M, g) will be an indecomposable lorentzian symmetric space G/K with solvable G (referred to as a Cahen–Wallach (CW) space in [8]) and F a parallel null form. This background is the IIB analogue of a maximally supersymmetric solution of M-theory originally discovered in [16]. It is satisfying to see that in this way it is not just the $\text{AdS} \times S$ vacua that M-theory and IIB string theory have in common, but also these more exotic ones.

3. A FAMILY OF LORENTZIAN SYMMETRIC SPACES

The classification of riemannian symmetric spaces is a classic piece of mathematics dating back to the work of Élie Cartan. In contrast, the classification of pseudo-riemannian spaces is a much harder problem, because there is no decomposition theorem which reduces the problem to studying symmetric splits of a Lie algebra $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ such that the natural action of \mathfrak{k} on \mathfrak{p} is irreducible. In fact, one is unavoidably led to consider those cases where \mathfrak{k} acts reducibly but indecomposably. The canonical example is a space which possesses a parallel null vector, since the existence of such a vector reduces the holonomy, but does not necessarily decompose the space, as the metric is degenerate when restricted to the null direction. Despite this difficulty, Cahen and Wallach [1] classified the lorentzian symmetric spaces; although so far the general pseudo-riemannian case remains unclassified. Cahen and Wallach proved that in dimensions $d \geq 3$ an indecomposable lorentzian symmetric space is locally isometric either to (anti-) de Sitter space or to a metric in a $(d-3)$ -dimensional family of symmetric spaces having a solvable transvection group.² It is precisely this family of spaces which plays a role in the construction described in this paper. A similar construction was discussed recently in [8] in the context of eleven-dimensional supergravity. The same construction applies here, and we refer the reader to that paper for more details.

Consider the metric

$$ds^2 = 2dx^+ dx^- + \sum_{i,j=1}^8 A_{ij} x^i x^j (dx^-)^2 + \sum_{i=1}^8 dx^i dx^i , \quad (9)$$

where $A_{ij} = A_{ji}$ is a symmetric matrix and $i = 1, \dots, 8$. This metric is indecomposable if and only if A_{ij} is nondegenerate. If $A_{ij} \neq 0$, we can rotate the coordinates x^i and rescale $x^\pm \rightarrow c^{\pm 1} x^\pm$ to bring A_{ij} to a diagonal form with eigenvalues lying on the unit sphere in \mathbb{R}^8 .

²If a symmetric space is of the form G/K , then G is called the transvection group.

The moduli space of the indecomposable CW metrics (9) is therefore seven-dimensional: the seven-sphere minus the singular locus consisting of degenerate A_{ij} 's and modded out by the natural action of the symmetric group \mathfrak{S}_8 .

We introduce a coframe $\theta^a = (\theta^{\hat{i}}, \theta^{\hat{+}}, \theta^{\hat{-}})$, given by

$$\theta^{\hat{i}} = dx^i \quad \theta^{\hat{-}} = dx^- \quad \theta^{\hat{+}} = dx^+ + \frac{1}{2} \sum_{i,j} A_{ij} x^i x^j dx^-, \quad (10)$$

so that the metric is

$$ds^2 = 2\theta^{\hat{+}}\theta^{\hat{-}} + \sum_{\hat{i}=1}^8 \theta^{\hat{i}}\theta^{\hat{i}}.$$

Frame indices are raised and lowered using the flat metric η_{ab} . Note that we do not need to distinguish between the upper indices $\hat{i}, \hat{-}$ and $i, -$ or between the lower indices $\hat{i}, \hat{+}$ and $i, +$, and in what follows we will omit the hats except for upper flat indices $\hat{+}$ and lower ones $\hat{-}$.

The first structure equation

$$d\theta^a + \omega^a_b \wedge \theta^b = 0,$$

gives the only nonzero components of the spin-connection ω^{ab} as

$$\omega^{\hat{+}i} = -\omega^{i\hat{+}} = \sum_j A_{ij} x^j dx^-. \quad (11)$$

The covariant derivative on spinors is given by

$$\nabla_M = \partial_M + \frac{1}{2} \omega_M^{ab} \Sigma_{ab},$$

where $\Sigma_{ab} = \frac{1}{2} \Gamma_{ab}$ are the spin generators. In our case we see that $\nabla_+ = \partial_+$ and $\nabla_i = \partial_i$, whereas

$$\nabla_- = \partial_- + \frac{1}{2} \sum_{i,j} A_{ij} x^j \Gamma_+ \Gamma_i.$$

The components of the Riemann curvature tensor of (M, g) are given by

$$R_{-i-j} = -A_{ij},$$

with all other components not related to this by symmetry vanishing. The Ricci curvature has nonzero component $R_{--} = -\text{tr } A$, and the scalar curvature vanishes. In particular, (M, g) is Ricci-flat if and only if A_{ij} is trace-free. It also follows that (M, g) is conformally flat if and only if A_{ij} is a scalar matrix; that is, $A_{ij} \propto \delta_{ij}$.

The holonomy of the metric (9) is $\mathbb{R}^8 \subset \text{SO}(9, 1)$. In terms of Lorentz transformations, the holonomy group acts by null rotations. The parallel forms are the constant functions, together with $dx^- \wedge \varphi$ where φ is any constant-coefficient form

$$\varphi = \sum_{1 \leq i_1 < i_2 < \dots < i_p \leq 8} c_{i_1 i_2 \dots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_p}.$$

Notice, in particular, that all such parallel forms are null.

4. A MAXIMALLY SUPERSYMMETRIC SOLUTION AND KILLING SPINORS

Motivated by the KG solution, we consider the ansatz

$$\begin{aligned} ds^2 &= 2dx^+dx^- + \sum_{i,j=1}^8 A_{ij}x^i x^j (dx^-)^2 + \sum_{i=1}^8 dx^i dx^i \\ F_5 &= \lambda dx^- \wedge (dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 + dx^5 \wedge dx^6 \wedge dx^7 \wedge dx^8), \end{aligned} \quad (12)$$

where λ is a real constant. The metric is clearly that of a CW space and F_5 is chosen to be null and self-dual. It was seen in the previous section that F_5 is parallel.

In order to write the supersymmetric covariant derivative (2) in the above ansatz, let us decompose it as $\mathcal{D}_M = \nabla_M + \Omega_M$, where

$$\Omega_M := \begin{cases} 0 & M = + \\ \lambda(I + J) & M = - \\ -\lambda\Gamma_+ \Gamma_M I & M = 1, 2, 3, 4 \\ -\lambda\Gamma_+ \Gamma_M J & M = 5, 6, 7, 8 \end{cases} \quad (13)$$

and we have introduced the notation $I := \Gamma_{1234}$ and $J := \Gamma_{5678}$. Notice that $I^2 = J^2 = \mathbb{1}$ and $IJ = JI$. Finally notice that since $\Gamma_+^2 = 0$,

$$\Omega_i \Omega_j = 0 \quad \text{for all } i, j = 1, 2, \dots, 8. \quad (14)$$

We now determine the precise form of the metric (i.e., the matrix A_{ij}) for which the supersymmetric covariant derivative (8) is flat, this being a necessary (and locally sufficient) condition for maximal supersymmetry.

Let ε be a Killing spinor; that is, a section of \mathbb{S} obeying $\mathcal{D}\varepsilon = 0$. Since $\nabla_+ = \partial_+$ and $\Omega_+ = 0$, we see that ε is independent of x^+ . Similarly from

$$\partial_j \varepsilon = -i\Omega_j \varepsilon \quad (15)$$

and equation (14), we see that $\partial_i \partial_j \varepsilon = 0$, whence ε is at most linear in x^i . Let us write it as

$$\varepsilon = \chi + \sum_j x^j \varepsilon_j,$$

where the spinors χ and ε_j are only functions of x^- . From equation (15) we see that $\varepsilon_j = -i\Omega_j \chi$, so that any Killing spinor ε takes the form

$$\varepsilon = \left(\mathbb{1} - i \sum_j x^j \Omega_j \right) \chi, \quad (16)$$

where the spinor χ only depends on x^- . The dependence on x^+ is fixed from the one remaining equation

$$\partial_- \varepsilon + \frac{1}{2} \sum_{i,j} A_{ij} x^j \Gamma_+ \Gamma_i \varepsilon + i\lambda(I+J)\varepsilon = 0 ,$$

which will also imply an integrability condition fixing the matrix A_{ij} .

Inserting the above expression (16) for ε into this equation and after a little bit of algebra (using repeatedly that $\Gamma_+^2 = 0$), we find

$$\begin{aligned} & \chi' + i\lambda(I+J)\chi \\ & + \sum_i x^i \left(\frac{1}{2} \sum_j A_{ij} \Gamma_+ \Gamma_j + \lambda(I+J)\Omega_i - \lambda\Omega_i(I+J) \right) \chi = 0 , \end{aligned}$$

where \prime denotes derivative with respect to x^- . Using

$$(I+J)\Omega_i - \Omega_i(I+J) = 2\lambda\Gamma_+ \Gamma_i \quad \text{for all } i ,$$

this can be rewritten as

$$\chi' + i\lambda(I+J)\chi = \sum_i x^i \left(2\lambda^2 \Gamma_i + \frac{1}{2} \sum_j A_{ij} \Gamma_j \right) \Gamma_+ \chi ,$$

The left-hand side is independent of x^+ as χ is, while the right-hand side has explicit x^+ dependence, and so both sides must vanish separately. The vanishing of the left-hand side is a first-order linear ordinary differential equation with constant coefficients, which has a unique solution for each initial value. The number of supersymmetries is then the dimension of the space of such initial values for which the right-hand side vanishes. The right-hand side will vanish for any initial value spinor that is annihilated by Γ_+ , and so there are always (at least) 16 real (or 8 complex) Killing spinors, and the solution is half-BPS for arbitrary A_{ij} . However, if A_{ij} is chosen so that the bracket on the right-hand side vanishes, then the right-hand side vanishes for arbitrary spinors χ , and the solution is maximally supersymmetric.

The right-hand side will vanish for all χ if and only if

$$A_{ij} = -4\lambda^2 \delta_{ij} ,$$

which, with ansatz (12), becomes the solution (5) presented in the introduction. The parameter λ , which hides the dependence on the (constant) dilaton, can be set to any desired (nonzero) value by rescaling the coordinates $x^\pm \mapsto c^{\pm 1} x^\pm$. We remark that for this choice of A_{ij} , the CW metric (9) is conformally flat.

The Killing spinors are easy to find, as they obey a first order ordinary differential equation

$$\chi' = -i\lambda(I+J)\chi .$$

The solution is given by the matrix exponential

$$\chi(x^-) = \exp(-\lambda x^- i(I+J)) \psi ,$$

where $\psi = \chi(0)$ is an arbitrary constant spinor. Since I and J commute, we can write the solution as

$$\chi(x^-) = \exp(-i\lambda x^- I) \exp(-i\lambda x^- J) \psi,$$

and since $I^2 = J^2 = \mathbb{1}$, we can write this as

$$\chi(x^-) = (\cos(\lambda x^-) \mathbb{1} - i \sin(\lambda x^-) I) (\cos(\lambda x^-) \mathbb{1} - i \sin(\lambda x^-) J) \psi,$$

from where the Killing spinors ψ can be read off using (16). Indeed we find that

$$\varepsilon(\psi) = \left(1 - i \sum_{j=1}^8 x^j \Omega_j \right) (\cos(\lambda x^-) \mathbb{1} - i \sin(\lambda x^-) I) \times (\cos(\lambda x^-) \mathbb{1} - i \sin(\lambda x^-) J) \psi. \quad (17)$$

The Killing spinor depends non-trivially on all the coordinates of spacetime apart from x^+ . It is periodic in x^- with period $2\pi/\lambda$.

5. A IIB PP-WAVE IN A HOMOGENEOUS FLUX

IIB supergravity admits a more general solution than those that have been discussed so far. In particular,

$$\begin{aligned} ds^2 &= 2dx^+ dx^- + H(x^i, x^-) (dx^-)^2 + ds^2(\mathbb{E}^8) \\ F_5 &= dx^- \wedge (\omega + \star\omega) \end{aligned} \quad (18)$$

is a solution provided that for fixed x^- , $H : \mathbb{E}^8 \rightarrow \mathbb{R}$ satisfies the Poisson equation

$$\triangle H = -32|\omega|^2 \quad (19)$$

where $\omega(x^i, x^-)$ is, for each, x^- , a closed and co-closed 4-form in the transverse space \mathbb{E}^8 . For example if $\omega = \lambda dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$, one can set

$$H = \frac{q}{|x|^6} + A_{ij} x^i x^j, \quad (20)$$

where

$$\text{tr } A = -32\lambda^2. \quad (21)$$

Such a solution has the interpretation of a IIB pp-wave in the presence of null homogeneous F_5 flux. Generic such solutions have 16 Killing spinors. These are

$$\varepsilon(\psi) = (\cos(\lambda x^-) \mathbb{1} - i \sin(\lambda x^-) I) (\cos(\lambda x^-) \mathbb{1} - i \sin(\lambda x^-) J) \psi,$$

where now the constant spinor ψ satisfies, $\Gamma_+ \psi = 0$. Again the Killing spinors are independent of x^+ and periodic in the null coordinate x^- with period $2\pi/\lambda$, but they are now also independent of the x^i .

6. THE SYMMETRY SUPERALGEBRA

The maximally supersymmetric solution (5) is invariant under the Lie algebra \mathfrak{S}_0 generated by those Killing vectors which also preserve F_5 , and under the supersymmetries generated by the 32 Killing spinors. Together these symmetries form a Lie superalgebra \mathfrak{S} whose even part is \mathfrak{S}_0 and whose odd part \mathfrak{S}_1 is 32-dimensional. The structure of \mathfrak{S} can be found by calculating the symmetry algebra [6, 19], as was done for the KG solution in [8].

The isometries of the CW space (M, g) were discussed in detail in [8]. The isometry algebra of the metric ds^2 consists of two parts. The first part is the 18-dimensional Lie algebra \mathfrak{g} of the transvection group G , with generators we denote $\{e_i^*, e_i, e_+, e_-\}$; whereas the second part is the subalgebra of $\mathfrak{so}(8)$ leaving A_{ij} invariant. In this case, since A_{ij} is diagonal, this is all of $\mathfrak{so}(8)$, with generators M_{ij} and corresponding Killing vectors $\xi_{M_{ij}} = x^i \partial_j - x^j \partial_i$. However not all these isometries are symmetries of the full solution because F_5 is not invariant under all of $\mathfrak{so}(8)$, but only under a subalgebra $\mathfrak{so}(4) \oplus \mathfrak{so}(4) \subset \mathfrak{so}(8)$. In summary, the complete set of Killing vectors preserving the supergravity solution (5) is the following:

$$\begin{aligned} \xi_{e_+} &= -\partial_+, & \xi_{e_-} &= -\partial_-, \\ \xi_{e_i} &= -\cos(2\lambda x^-) \partial_i - 2\lambda \sin(2\lambda x^-) x^i \partial_+ & i = 1, \dots, 8, \\ \xi_{e_i^*} &= -2\lambda \sin(2\lambda x^-) \partial_i + 4\lambda^2 \cos(2\lambda x^-) x^i \partial_+ & i = 1, \dots, 8, \\ \xi_{M_{ij}} &= x^i \partial_j - x^j \partial_i & i, j = 1, \dots, 4 \text{ and } i, j = 5, \dots, 8, \end{aligned}$$

where we denote the Killing vector field associated with $X \in \mathfrak{S}_0$ by ξ_X .

As F_5 is parallel, it is homogeneous; that is, invariant under the transvection group. It is straightforward to calculate the Lie algebra, finding $\mathfrak{S}_0 = \mathfrak{g} \ltimes (\mathfrak{so}(4) \oplus \mathfrak{so}(4))$. If $\{M_{ij}\}$, for $i, j = 1, \dots, 4$ and $i, j = 5, \dots, 8$ are the generators of $\mathfrak{so}(4) \oplus \mathfrak{so}(4)$, then the Lie brackets of \mathfrak{S}_0 are

$$\begin{aligned} [e_-, e_i] &= e_i^* & [e_-, e_i^*] &= -4\lambda^2 e_i & [e_i^*, e_j] &= -4\lambda^2 \delta_{ij} e_+ \\ [M_{ij}, e_k] &= -\delta_{ik} e_j + \delta_{jk} e_i & [M_{ij}, e_k^*] &= -\delta_{ik} e_j^* + \delta_{jk} e_i^*, \end{aligned} \quad (22)$$

where, in the second line, $1 \leq i, j, k \leq 4$ and $5 \leq i, j, k \leq 8$. The symmetry algebra \mathfrak{S}_0 of the solution (5) is 30-dimensional and non-semisimple. It is interesting to note that the dimension of this algebra coincides with that of the $\text{AdS}_5 \times S^5$ maximally supersymmetric solution. A similar fact has previously been observed for eleven-dimensional supergravity and we believe this ‘‘coincidence’’ deserves further investigation.

The spacetime is a coset space G/K where K is the \mathbb{R}^8 subgroup of G whose Lie algebra is generated by $\{e_i^*\}$. Equivalently, we may think of the spacetime as a connected component of the coset $\mathbb{S}/(\text{ISO}(4) \times$

$\text{ISO}(4)$], where \mathbf{S} the isometry group with Lie algebra \mathfrak{S}_0 and where $\text{ISO}(4)$ is the Euclidean group $\mathbb{R}^4 \rtimes \text{SO}(4)$. The Lie algebra of $\text{ISO}(4) \times \text{ISO}(4)$ is generated by $\{e_i^*, M_{ij}\}$.

To explore the structure of the symmetry algebra $\mathfrak{S}_0 = \mathfrak{g} \times (\mathfrak{so}(4) \oplus \mathfrak{so}(4))$ further, it will be useful to decompose the $\mathfrak{so}(8)$ vectors v^i into $\mathfrak{so}(4) \oplus \mathfrak{so}(4)$ representations, $v^a, v^{a'}$, where $a, b = 1, \dots, 4$ are vector indices for the first $\mathfrak{so}(4)$, and $a', b' = 1, \dots, 4$ are vector indices for the second $\mathfrak{so}(4)$. Then $\{M_{ab}, e_a\}$ and $\{M_{ab}, e_a^*\}$ both generate $\mathfrak{iso}(4)$ subalgebras. The subalgebra of \mathfrak{S}_0 with generators $\{M_{ab}, e_a, e_a^*, e_+\}$ is isomorphic to the Lie algebra $\mathfrak{cso}(4, 1, 1)$, as defined in [12]. This arises as a contraction of $\mathfrak{so}(5, 1)$ as follows. Decomposing the 15 generators into $\mathfrak{so}(4) \oplus \mathfrak{so}(1, 1)$ representations, one obtains the $\mathfrak{so}(4)$ generators $\{M_{ab}\}$, the $\mathfrak{so}(1, 1)$ generator U and the generators $Y_{a,i}$ in the $(\underline{4}, \underline{2})$ representation. Rescaling the Y by a factor ξ and U by a factor ξ^2 and then taking the limit $\xi \rightarrow 0$ gives the contracted Lie algebra of $\mathfrak{cso}(4, 1, 1)$, with the algebra given as above with $U = e_+$ and $(Y_{ai}) = (e_a, e_a^*)$.

Similarly, there is a $\mathfrak{cso}(4, 1, 1)$ generated by $\{M_{a'b'}, e_{a'}, e_{a'}^*, e_+\}$. The algebra generated by $\{M_{ij}, e_i, e_i^*, e_+\}$ is closely related to $\mathfrak{cso}(4, 2) \oplus \mathfrak{cso}(4, 2)$. Indeed if we let $\{M_{ab}, e_a, e_a^*, U\}$ denote the generators of the first $\mathfrak{cso}(4, 1, 1)$ and we let $\{M_{a'b'}, e_{a'}, e_{a'}^*, U'\}$ be those of the second, then the algebra generated by $\{M_{ij}, e_i, e_i^*, e_-\}$ is obtained from $\mathfrak{cso}(4, 1, 1) \oplus \mathfrak{cso}(4, 1, 1)$ by setting $U = U' = e_+$, so that the Lie algebra is the quotient $(\mathfrak{cso}(4, 1, 1) \oplus \mathfrak{cso}(4, 1, 1))/\mathbb{R}$ by the central subalgebra generated by $U - U'$. Finally, we extend by the outer automorphism generated by e_- , and we see that \mathfrak{S}_0 is an extension of $(\mathfrak{cso}(4, 1, 1) \oplus \mathfrak{cso}(4, 1, 1))/\mathbb{R}$ by the single generator e_- .

If one were to periodically identify the x^+ coordinate, then the non-compact generator e_+ becomes a compact generator of translations in this circular direction, and $\mathfrak{cso}(4, 1, 1)$ in the above becomes $\mathfrak{cso}(4, 2)$, which is the contraction of $\mathfrak{so}(6)$ or $\mathfrak{so}(4, 2)$ obtained by decomposing into $\mathfrak{so}(4) \oplus \mathfrak{so}(2)$ representations and scaling as above. Then \mathfrak{S}_0 becomes an extension of $(\mathfrak{cso}(4, 2) \oplus \mathfrak{cso}(4, 2))/\mathfrak{so}(2)$ by the single generator e_- , which will be a compact generator if x^- is periodically identified, and non-compact otherwise. Similarly, the symmetry algebra for the KG solution with periodic x^+ is an extension of $(\mathfrak{cso}(3, 2) \oplus \mathfrak{cso}(6, 2))/\mathfrak{so}(2)$ by a single generator, while for non-periodic x^+ it is an extension of $(\mathfrak{cso}(3, 1, 1) \oplus \mathfrak{cso}(6, 1, 1))/\mathbb{R}$. Other real forms arise if one compactifies a coordinate $z = ax^+ + bx^-$.

To compute the commutator of the even and odd generators of the symmetry superalgebra, we also need to find the spinorial Lie derivative of the Killing spinors along the Killing vector directions. The spinorial Lie derivative L_ξ along a Killing vector direction ξ is defined as

$$L_\xi = \nabla_\xi + \frac{1}{4} \nabla_M \xi_N \Gamma^{MN}. \quad (23)$$

Properties of the spinorial Lie derivative have been given in [6, 8]. For our solution (5), the non-vanishing spinorial Lie derivatives of the Killing spinors are

$$\begin{aligned}
L_{e_-}\varepsilon(\psi) &= \varepsilon(i\lambda(I+J)\psi) \\
L_{e_i}\varepsilon(\psi) &= \varepsilon(-i\lambda I\Gamma^i\Gamma_+\psi) \quad i = 1, \dots, 4 \\
L_{e_i}\varepsilon(\psi) &= \varepsilon(-i\lambda J\Gamma^i\Gamma_+\psi) \quad i = 5, \dots, 8 \\
L_{e_i^*}\varepsilon(\psi) &= \varepsilon(-2\lambda^2 I\Gamma^i\Gamma_+\psi) \quad i = 1, \dots, 4 \\
L_{e_i^*}\varepsilon(\psi) &= \varepsilon(-2\lambda^2 J\Gamma^i\Gamma_+\psi) \quad i = 5, \dots, 8 \\
L_{M_{ij}}\varepsilon(\psi) &= \varepsilon(\tfrac{1}{2}\Gamma^{ij}\psi) \quad i, j = 1, \dots, 4 \text{ and } i, j = 5, \dots, 8,
\end{aligned}$$

where L_X for $X \in \mathfrak{g}$ stands for L_{ξ_X} . Using these expressions, the commutators of the even and odd generators of the symmetry superalgebra are

$$\begin{aligned}
[e_+, Q] &= 0 \quad [e_-, Q] = i\lambda(I+J)Q \\
[e_i, Q] &= -i\lambda I\Gamma_i\Gamma_+Q \quad i = 1, \dots, 4 \\
[e_i, Q] &= -i\lambda J\Gamma_i\Gamma_+Q \quad i = 5, \dots, 8 \\
[e_i^*, Q] &= -2\lambda^2 I\Gamma_i\Gamma_+Q \quad i = 1, \dots, 4 \\
[e_i^*, Q] &= -2\lambda^2 J\Gamma_i\Gamma_+Q \quad i = 5, \dots, 8 \\
[M_{ij}, Q] &= \tfrac{1}{2}\Gamma_{ij}Q \quad i, j = 1, \dots, 4 \text{ and } i, j = 5, \dots, 8,
\end{aligned}$$

where Q are the odd generators which are complex Weyl spinors.

To find the anticommutators between two odd generators of the symmetry superalgebra, one has to compute the expression $V = \bar{\varepsilon}_1 \Gamma^M \varepsilon_2 \partial_M$, where $\varepsilon_1 = \varepsilon_1(\psi_1)$ and $\varepsilon_2 = \varepsilon_1(\psi_2)$ are Killing spinors. Of course, V is a Killing vector for all (ψ_1, ψ_2) . In particular for the solution (5), we find

$$\begin{aligned}
V &= -\bar{\psi}_1 \Gamma^- \psi_2 \xi_{e_-} - \bar{\psi}_1 \Gamma^+ \psi_2 \xi_{e_+} - \sum_{i=1}^8 \bar{\psi}_1 \Gamma^i \psi_2 \xi_{e_i} \\
&\quad + \frac{i}{2\lambda} \sum_{i=1}^4 \bar{\psi}_1 \Gamma^i I \psi_2 \xi_{e_i^*} + \frac{i}{2\lambda} \sum_{i=5}^8 \bar{\psi}_1 \Gamma^i J \psi_2 \xi_{e_i^*} \\
&\quad + \bar{\psi}_1 \sum_{i,j=1}^4 (i\lambda \Gamma^- I) \psi_2 M_{ij} + \bar{\psi}_1 \sum_{i,j=5}^8 (i\lambda \Gamma^- J) \psi_2 M_{ij}.
\end{aligned}$$

From this expression one can read off the anticommutators

$$\begin{aligned} \{Q, Q\} = & -\Gamma^- C^{-1} e_- - \Gamma^+ C^{-1} e_+ - \sum_{i=1}^8 \Gamma^i C^{-1} e_i \\ & + \frac{i}{2\lambda} \sum_{i=1}^4 \Gamma^i I C^{-1} e_i^* + \frac{i}{2\lambda} \sum_{i=5}^8 \Gamma^i J C^{-1} e_i^* \\ & + i\lambda \sum_{i,j=1}^4 \Gamma^- \Gamma^{ij} I C^{-1} M_{ij} + i\lambda \sum_{i,j=5}^8 \Gamma^- \Gamma^{ij} J C^{-1} M_{ij} . \end{aligned}$$

This concludes the computation of the structure constants of the symmetry superalgebra \mathfrak{G} .

The solutions of the type (18) are also invariant under the action of a superalgebra but now the action of the associated supergroup is not transitive acting on superspace. The bosonic part of the superalgebra is the semidirect product of the transvection Lie algebra \mathfrak{g} and the subalgebra of $\mathfrak{so}(8)$ which preserves both the quadratic form A and F_5 . This has been explained in a similar setting in [8]. A similar computation to that presented above reveals that the non-vanishing commutators of even and odd generators of the superalgebra are

$$\begin{aligned} [e_-, Q] &= i\lambda(I + J)Q \\ [M_{ij}, Q] &= \frac{1}{2}\Gamma_{ij}Q, \end{aligned}$$

where $\Gamma_+ Q = 0$ and M_{ij} are those generators of $\mathfrak{so}(8)$ that leave both the metric and F_5 invariant. Then the anticommutator of the odd generators is

$$\begin{aligned} \{Q, Q\} = & -\Gamma^+ C^{-1} e_+ - \sum_{i=1}^8 \Gamma^i C^{-1} e_i \\ & + \frac{i}{2\lambda} \sum_{i=1}^4 \Gamma^i I C^{-1} e_i^* + \frac{i}{2\lambda} \sum_{i=5}^8 \Gamma^i J C^{-1} e_i^* . \end{aligned}$$

7. T-DUALITY, COMPACTIFICATION AND M-THEORY

The metric and five-form field strength of the maximally supersymmetric IIB solution that we have presented has isometries generated by both ∂_+ and ∂_- and so is invariant under translations in the space coordinate $x = x^+ + x^-$ and in the time coordinate $t = x^+ - x^-$. However, the Killing spinors depend non-trivially on all the coordinates of space-time apart from x^+ , and are periodic in x^- with period $2\pi/\lambda$. They are then periodic in both u and t , both with the same period $2\pi/\lambda$. As ∂_x is a Killing vector, we can periodically identify the u coordinate, $x \sim x + 2\pi R$ for any R to obtain a IIB supergravity solution. However, for general radii for which $R \neq n/2\lambda$ for any integer n , the periodicity

in \mathbf{n} will be inconsistent with the periodicity of the Killing spinors and the compactified solution will admit no Killing spinors and preserve no supersymmetries. Remarkably, there is supersymmetry enhancement for special values of the radius. If $R = n/\lambda$ for some integer \mathbf{n} , then the solution is maximally supersymmetric with 32 Killing spinors with periodic boundary conditions on the circle. If $R = (2n + 1)/2\lambda$ for some integer \mathbf{n} , then the solution again has 32 Killing spinors, but now with anti-periodic boundary conditions on the circle.

For the KG solution (1),(3),(4) with $x = x^+ + x^-$ compactified on a spacelike circle of radius R , the situation is a little more complicated, and at some special values of the radius there are 16 Killing spinors, and at others there are 32. From [8], for generic R , all supersymmetry is broken. If $R = 4n/\mu$ for some integer \mathbf{n} , then there are at least 16 periodic Killing spinors and if $R = 2(2n + 1)/\mu$, there are at least 16 anti-periodic Killing spinors. If $R = 12n/\mu$ for some integer \mathbf{n} , then there are 32 periodic Killing spinors and if $R = 6(2n + 1)/\mu$, there are 32 anti-periodic Killing spinors.

Dimensionally reducing the IIB solution on the circle gives a solution of 9-dimensional $\mathbf{N} = 2$ supergravity, which is an H0-brane in the terminology of [8]. As the Lie derivatives of all the Killing spinors with respect to ∂_x are non-zero, no supersymmetries are preserved by the dimensional reduction and this H0-brane solution of the 9-dimensional supergravity theory has no Killing spinors. Equivalently, the Lie algebra generator associated with the Killing vector ∂_x does not commute with the \mathbf{Q} generators, so restricting to the \mathbf{n} -independent sector breaks all supersymmetry; for further discussion on this point see for example [9, 8]. Similarly, reducing the KG solution on a circle gives a IIA solution with no Killing spinors [8].

The IIB solution on a circle can be T-dualised to obtain a solution of the IIA theory on a dual circle of radius $\tilde{R} = 1/R$. This is a fundamental string solution wrapping the circle with both electric and magnetic flux corresponding to the RR 3-form gauge field \mathbf{C}_3 . This can then be lifted to a solution of 11-dimensional supergravity which is a membrane in a background with both electric and magnetic flux corresponding to the 11-dimensional 3-form gauge field \mathbf{C}_3 .

This string or H1-brane solution of the IIA supergravity theory has no Killing spinors that are independent of the dual coordinate \tilde{x} , as otherwise the dimensionally reduced 9-dimensional solution would be supersymmetric. We now turn to the question of whether this IIA solution could have Killing spinors with non-trivial dependence on the coordinate \tilde{x} , which is identified $\tilde{x} \sim \tilde{x} + 2\pi\tilde{R}$; these might arise only for special values of \tilde{R} . Each such Killing spinor would then lift to a Killing spinor of the 11-dimensional solution. The maximally supersymmetric solutions of 11-dimensional supergravity have been classified [16, 5, 7]

and this membrane or H2-brane is not one of them, and so we conclude that the IIA solution cannot have 32 Killing spinors, hence the IIA supergravity solution has strictly less supersymmetry than the IIB supergravity solution.

However, the situation is different at the level of string theory, as the IIB string is T-dual to the IIA string: when on a circle, the IIA and IIB strings are the same theory, but written in terms of different variables. The two T-dual solutions must both preserve the same amount of supersymmetry at the level of string theory, even though they do not at the level of the supergravity theories. This is an example of what has been called “supersymmetry without supersymmetry” [3]. In [4], the $\text{AdS}_5 \times S^5$ solution of IIB supergravity was T-dualised along the Hopf fibres of S^5 . The S^5 is a circle bundle over \mathbb{CP}^2 and dualising on the fibres gives a IIA solution $\text{AdS}_5 \times \mathbb{CP}^2 \times S^1$, which does not have a spin structure. The missing fermions arise from winding modes, and the maximal supersymmetry of the IIB solution now becomes a IIA fermionic symmetry that mixes ordinary field theory modes with winding modes, and so is a non-perturbative symmetry from the world-sheet point of view [4]. The situation here is similar. The 32 supersymmetries of the IIB string solution are present in the T-dual IIA string solution, but they are symmetries relating ordinary field theory modes to winding modes, and these stringy supersymmetries cannot be seen at the level of the supergravity theory. Similarly, the supersymmetries of the IIA solution (if any) give rise to extra stringy supersymmetries of the IIB theory. In the M-theory solution, the 32 IIB supersymmetries become symmetries of M-theory that involve membrane winding modes.

Similarly, if the time coordinate t is identified with period $2\pi T$, there will be no Killing spinors unless $T = n/2\lambda$ for integer n , while at these values there will be 32 Killing spinors which will be periodic if n is even and anti-periodic if n is odd. A timelike T-duality will give a solution of the IIA theory [14] which can then be lifted to the M theory of [13].

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