Casimir Energies in Light of Quantum Field Theory

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We study the Casimir problem as the limit of a conventional quantum field theory coupled to a smooth background. The Casimir energy diverges in the limit that the background forces the field to vanish on a surface. We show that this divergence cannot be absorbed into a renormalization of the parameters of the theory. As a result, the Casimir energy of the surface and other quantities like the surface tension, which are obtained by deforming the surface, cannot be defined independently of the details of the coupling between the field and the matter on the surface. In contrast, the energy density away from the surface and the force between rigid surfaces are finite and independent of these complications.

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The vacuum energy of fluctuating quantum fields that are subject to boundary conditions has been studied intensely over the half-century since Casimir predicted a force between grounded metal plates[1–3]. The plates change the zero-point energies of fluctuating fields and thereby give rise to forces between the rigid bodies or stresses on isolated surfaces. The Casimir force between grounded metal plates has now been measured quite accurately and agrees with his prediction [4–6].

Casimir forces arise from interactions between the fluctuating fields and matter. Nevertheless, it is traditional to study idealized "Casimir problems" where the physical interactions are replaced *ab initio* by boundary conditions. In this Letter we study under what circumstances this replacement is justified. A real material cannot constrain modes of the field with wavelengths much smaller than the typical length scale of its interactions. In contrast, a boundary condition constrains all modes. The sum over zero point energies is highly divergent in the ultraviolet and these divergences depend on the boundary conditions. Subtraction of the vacuum energy in the absence of boundaries only removes the worst divergence (quartic in three space dimensions).

The fact that the energy of a fluctuating field diverges when a boundary condition is imposed has been known for many years[7, 8]. Schemes have been proposed to cancel these divergences by introducing new, *ad hoc* surface dependent counterterms[9] or quantum boundary functions[10]. We are not interested in such a formal solution to the problem. The method of renormalization in continuum quantum field theory without boundaries (QFT) provides the only physical way to regulate, discuss, and eventually remove divergences. Therefore we propose to embed the Casimir calculation in QFT and study its renormalization. After renormalization, if any quantity is still infinite in the presence of the boundary condition, it will depend in detail on the properties of the material that provides the physical ultraviolet cutoff and will not exist in the idealized Casimir problem. Similar subtleties were addressed in the context of dispersive media in Ref. [11].

It is straightforward to write down a QFT describing the interaction of the fluctuating field ϕ with a static background field $\sigma(\mathbf{x})$ and to choose a limit involving the shape of $\sigma(\mathbf{x})$ and the coupling strength between ϕ and σ that produces the desired boundary conditions on specified surfaces. We have developed the formalism required to compute the resulting vacuum energy in Ref. [12]. Here we focus on Dirichlet boundary conditions on a scalar field. Our methods can be generalized to the physically interesting case of conducting boundary conditions on a gauge field.

Ideally, we seek a Casimir energy that reflects only the effects of the boundary conditions and not on any other features of $\sigma(\mathbf{x})$. Therefore we do not specify any action for σ except for the standard counterterms induced by the ϕ-σ interaction. The coefficients of the counterterms are fixed by renormalization conditions applied to perturbative Green's functions. Once the renormalization conditions have been fixed by the definitions of the physical parameters of the theory, there is no ambiguity and no further freedom to make subtractions. Moreover, the renormalization conditions are independent of the particular choice of background $\sigma(\mathbf{x})$, so it makes sense to compare results for different choices of $\sigma(\mathbf{x})$, *ie.* different geometries. Having been fixed in perturbation theory, the counterterms are fixed once and for all and must serve to remove the divergences that arise for any physically sensible $\sigma(\mathbf{x})$. The Yukawa theory with coupling g in three space dimensions gives a textbook example: The $g\bar{\psi}\sigma\psi$ coupling generates divergences in low order Feynman diagrams proportional to σ^2, σ^4 and $(\partial\sigma)^2$ and therefore requires one to introduce a mass, a quartic self-coupling, and a kinetic term for σ. This is the only context in which one can study the fluctuations of a fermion coupled to a scalar background in three dimensions.

In this Letter we study the vacuum fluctuations of a real scalar field ϕ coupled to a scalar background $\sigma(\mathbf{x})$ with coupling λ, $\mathcal{L}_{\rm int}(\phi,\sigma) = \frac{1}{2}\lambda\,\sigma(\mathbf{x})\,\phi^2(\mathbf{x},t)$. In the limit where $\sigma(\mathbf{x})$ becomes a δ-function on some surface $\mathcal{S}$ and where $\lambda\to\infty$, it is easy to verify that all modes of ϕ must vanish on $\mathcal{S}$. We call this the *Dirichlet limit.* It consists of the *sharp limit*, where $\sigma(\mathbf{x})$ gets concentrated on $\mathcal{S}$, followed by the *strong coupling limit*, $\lambda\to\infty$. In general, we find that the divergence of the vacuum energy in the Dirichlet limit cannot be renormalized. Generally, even the sharp limit does not lead to a finite Casimir energy except in one dimension, where the sharp limit exists but the Casimir energy diverges as $\lambda\ln\lambda$ in the strong coupling limit.

This divergence indicates that *the Casimir energy of a scalar field forced to vanish on a surface in any dimension is infinite.* However, all is not lost. The unrenormalizable divergences are localized on $\mathcal{S}$, so quantities that do not probe $\mathcal{S}$ are well defined. For example, it is straightforward to show that the vacuum *energy density* away from $\mathcal{S}$ is well defined in the Dirichlet limit, even though the energy density on $\mathcal{S}$ diverges[13]. We expect that this is true in general. The forces between rigid bodies are also finite in the Dirichlet limit. But any quantity whose definition requires a deformation or change in area of $\mathcal{S}$ will pick up an infinite contribution from the surface energy density and therefore diverge. For example, we will see explicitly that the vacuum contribution to the *stress* on a (generalized) Dirichlet sphere in two or more dimensions is infinite.

The remainder of this Letter is organized as follows: First we briefly review our computational method and discuss the structure of the counterterms required by renormalization. Then we present two examples, leaving the details to Ref. [12]. We begin with the simplest Casimir problem: two Dirichlet points on a line, where we can compare our results with standard calculations that assume boundary conditions from the start [2]. We find that the renormalized Casimir energy is infinite but the Casimir force is finite in the Dirichlet limit. We show how the QFT approach resolves inconsistencies in the standard calculation. Next we study the Dirichlet circle in two dimensions. We demonstrate explicitly that the renormalized Casimir stress on the circle diverges in the sharp limit. We show that the divergence is associated with a simple Feynman diagram and will persist in three dimensions (the "Dirichlet sphere") and beyond.

We define the bare Casimir energy to be the vacuum energy of a quantum field ϕ coupled to a background field σ by $\mathcal{L}_{\rm int}(\phi,\sigma)$, minus the vacuum energy in the absence of σ. This quantity can be written as the sum over the shift in the zero-point energies of all the modes of ϕ relative to the trivial background $\sigma=0$, $E_{\rm bare}[\sigma] = \frac{\hbar}{2}\sum_n(\omega_n[\sigma]-\omega_n^{(0)})$. Equivalently, using the effective action formalism of QFT, $E_{\rm bare}[\sigma]$ is given by the sum of all 1-loop Feynman diagrams with at least one external σ field. The entire Lagrangian is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{2}\phi^2\sigma(\mathbf{x}) + \mathcal{L}_{\rm CT}[\sigma]\,, \qquad (1)$$

where $\mathcal{L}_{\rm CT}[\sigma]$ is the counterterm Lagrangian required by renormalization. Combining its contribution to the energy with $E_{\rm bare}[\sigma]$ yields the renormalized energy $E_{\rm cas}[\sigma]$. We have taken the dynamics of the background field $\sigma(\mathbf{x})$ to include only the ϕ-σ coupling and the counterterms required by renormalization theory. The frequencies $\{\omega[\sigma]\}$ are determined by $-\nabla^2\phi(\mathbf{x})+(m^2+\lambda\sigma(\mathbf{x}))\phi(\mathbf{x}) = \omega^2[\sigma]\phi(\mathbf{x})$. This is a renormalizable quantum field theory, so $E_{\rm cas}[\sigma]$ will be finite for any smooth σ and finite λ. We use the method developed in Ref. [12] to compute the Casimir energy of the background configuration *exactly* while still performing all the necessary renormalizations in the perturbative sector. The interested reader should consult Ref. [14] for an introduction to the method and Ref. [15] for applications. We assume that the background field $\sigma(\mathbf{x})$ is sufficiently symmetric to allow the scattering amplitude to be expanded in partial waves, which we label by ℓ. We express the *renormalized* Casimir energy as a sum over bound states $\omega_{\ell j}$ plus an integral over continuum modes with $\omega=\sqrt{k^2+m^2}$,

$$E_{\rm cas}[\sigma] = \sum_\ell N_\ell\left[\sum_j \frac{\omega_{\ell j}}{2} + \int_0^\infty \frac{dk}{2\pi}\omega(k)\frac{d}{dk}\left[\delta_\ell(k)\right]_N\right] + E_{\rm FD}^N + E_{\rm CT} \qquad (2)$$

where N_ℓ denotes the multiplicity, δ_ℓ the scattering phase shift and $\frac{1}{\pi}\frac{d\delta_\ell}{dk}$ the continuum density of states in the $\ell^{\rm th}$ partial wave. The subscript N on δ_ℓ indicates that the first N terms in the Born expansion of δ_ℓ have been subtracted. These subtractions are compensated exactly by the contribution of the first N Feynman diagrams, $E_{\rm FD}^N = \sum_{i=1}^N E_{\rm FD}^{(i)}$. In eq. (2) $E_{\rm CT}$ is the contribution of the counterterm Lagrangian, $\mathcal{L}_{\rm CT}$. Both $E_{\rm FD}^N$ and $E_{\rm CT}$ depend on the ultraviolet cutoff $1/\epsilon$, but $E_{\rm FD}^N + E_{\rm CT}$ remains finite as $\epsilon \to 0$. One can think of ϵ as the standard regulator of dimensional regularization, although our methods are not wedded to any particular regularization scheme. After subtraction, the k-integration in eq. (2) converges and can be performed numerically for any choice of $\sigma(\mathbf{x})$. It is convenient for computations to rotate the integration contour to the imaginary k-axis giving

$$E_{\rm cas}[\sigma] = \sum_\ell N_\ell \int_m^\infty \frac{dt}{2\pi}\, \frac{t}{\sqrt{t^2-m^2}}\, \left[\beta_\ell(t)\right]_N + E_{\rm FD}^N + E_{\rm CT}\,, \tag{3}$$

where $t = -ik$. The real function $\beta_\ell(t)$ is the logarithm of the *Jost function* for imaginary momenta, $\beta_\ell(t) \equiv \ln F_\ell(it)$. Efficient methods to compute $\beta_\ell(t)$ and its Born series can be found in Ref. [12]. The renormalized Casimir energy density for finite λ, $\epsilon_{\rm cas}(\mathbf{x})$, can also be written as a Born subtracted integral along the imaginary k-axis plus contributions from counterterms and low order Feynman diagrams [12].

In less than three dimensions only the lowest order Feynman diagram diverges, so only a counterterm linear in σ is necessary, $\mathcal{L}_{\rm CT} = c_1 \lambda \sigma(\mathbf{x})$. Since the tadpole graph is also local, we can fix the coefficient c_1 by requiring a complete cancellation, $E_{\rm FD}^{(1)} + E_{\rm CT} = 0$. In three dimensions it is necessary to subtract two terms in the Born expansion of $\beta_\ell(t)$ and add back the two lowest order Feynman graphs explicitly. The counterterm Lagrangian must be expanded to include a term proportional to σ^2, $\mathcal{L}_{\rm CT} = c_1\lambda\sigma(\mathbf{x}) + c_2\frac{\lambda^2}{2}\,\sigma^2(\mathbf{x})$. The new term cancels the divergence generated by the vacuum polarization diagram $E_{\rm FD}^{(2)}$, but it does not completely cancel $E_{\rm FD}^{(2)}$, because $E_{\rm FD}^{(2)}$ it is not simply proportional to $\int d^3x \sigma^2(\mathbf{x})$. To fix c_2 we can only demand that it cancels $E_{\rm FD}^{(2)}$ at a specified momentum scale $p^2 = M^2$. Different choices of M correspond to different models for the self-interactions of σ and give rise to finite changes in the Casimir energy.

We use eq. (3) and the analogous expression for the energy density with backgrounds that are strongly localized about $\mathcal{S}$, but not singular, to see how the Dirichlet limit is approached. It is straightforward to relate the Casimir energy density at the point $\mathbf{x}$ to the Green's function at $\mathbf{x}$ in the background σ, and then to show that it is finite as long as $\sigma(\mathbf{x}) = 0$. Thus we find that the Casimir *energy density* at any point away from $\mathcal{S}$ goes to a finite limit as $\sigma \to \delta_{\mathcal{S}}(\mathbf{x})$ and $\lambda \to \infty$ and that the result coincides with that found in boundary condition calculations. We also find a finite and unambiguous expression for the renormalized Casimir energy density where $\sigma(\mathbf{x})$ is nonzero, as long as it is nonsingular and the coupling strength is finite. But as we approach the sharp limit, the renormalized energy density on $\mathcal{S}$ diverges, and this divergence cannot be renormalized.

By analyzing the Feynman diagrams that contribute to the effective energy we can deduce some general results about possible divergences in the Casimir energy and energy density in the sharp limit. In particular, the divergences that occur in the Casimir energy in the sharp limit come from low-order Feynman diagrams. Specifically, using dimensional analysis it is possible to show that in n space dimensions the Feynman diagram with m external insertions of σ is finite in the sharp limit if $m > n$.

Although less sophisticated methods can be used to obtain the energy density at points away from $\mathcal{S}$ where renormalization is unnecessary, as far as we know only our method can be used to define and study the Casimir energy density where $\sigma(\mathbf{x})$ is nonzero and therefore on $\mathcal{S}$ in the Dirichlet limit.

Consider, as a pedagogical example, a real, massive scalar field $\phi(t,x)$ in one dimension, constrained to vanish at $x = -a$ and a. The standard approach, in which the boundary conditions are imposed *a priori*, gives an energy [2]

$$\widetilde{E}_2(a) = -\frac{m}{2} - \frac{2a}{\pi}\int_m^\infty dt\, \frac{\sqrt{t^2-m^2}}{e^{4at}-1}\,, \tag{4}$$

where the tilde denotes the imposition of the Dirichlet boundary condition at the outset. From this expression one obtains an attractive force between the two Dirichlet points, given by

$$\widetilde{F}(a) = -\frac{d\widetilde{E}_2}{d(2a)} = -\int_m^\infty \frac{dt}{\pi}\, \frac{t^2}{\sqrt{t^2-m^2}(e^{4at}-1)}\,. \tag{5}$$

In the massless limit, we have $\widetilde{E}_2(a) = -\pi/48a$ and $\widetilde{F}(a) = -\pi/96a^2$.

These results are not internally consistent, suggesting that the calculation has been oversimplified: As $a \to \infty$, $\widetilde{E}_2(a) \to -m/2$, indicating that the energy of an isolated "Dirichlet point" is $-m/4$. As $a \to 0$ we also have a single Dirichlet point, but $\widetilde{E}_2(a) \to \infty$ as $a \to 0$. Also note that $\widetilde{E}_2(a)$ is well defined as $m \to 0$, but we know on general grounds that scalar field theory becomes infrared divergent in one dimension when $m \to 0$.

We study this problem by coupling $\phi(t,x)$ to the static background field $\sigma(x) = \delta(x+a) + \delta(x-a)$ with coupling strength λ as in eq. (1). An elementrary calculation gives the renormalized Casimir energy for finite λ,

$$E_2(a,\lambda) = \int_m^\infty \frac{dt}{2\pi}\frac{1}{\sqrt{t^2-m^2}}\left\{t\ln\left[1+\frac{\lambda}{t}+\frac{\lambda^2}{4t^2}(1-e^{-4at})\right]-\lambda\right\} \tag{6}$$

The same method can be applied to an isolated point giving,

$$E_1(\lambda) = \int_m^\infty \frac{dt}{2\pi}\frac{t\ln\left[1+\frac{\lambda}{2t}\right]-\frac{\lambda}{2}}{\sqrt{t^2-m^2}} \tag{7}$$

For any finite coupling λ, the inconsistencies noted in $\widetilde{E}_2(a)$ do not afflict $E_2(a,\lambda)$: As $a\to\infty$, $E_2(a,\lambda)\to 2E_1(\lambda)$, and as $a\to 0$, $E_2(a,\lambda)\to E_1(2\lambda)$. Also $E_2(a,\lambda)$ diverges logarithmically in the limit $m\to 0$ as it should. The force, obtained by differentiating eq. (6) with respect to $2a$, agrees with eq. (5) in the limit $\lambda\to\infty$.

Note, however, that $E_2(a,\lambda)$ *diverges* like $\lambda\log\lambda$ as $\lambda\to\infty$. Thus the renormalized Casimir energy in a sharp background diverges as the Dirichlet boundary condition is imposed, a physical effect which is missed if the boundary condition is applied at the outset.

The Casimir *energy density* for $x\neq\pm a$ can be calculated assuming Dirichlet boundary conditions from the start simply by subtracting the density in the absence of boundaries without encountering any further divergences [2],

$$\begin{aligned}\epsilon_2(x,a) &= -\frac{m}{8a}-\int_m^\infty\frac{dt}{\pi}\frac{\sqrt{t^2-m^2}}{e^{4at}-1}-\frac{m^2}{4a}\sum_{n=1}^\infty\frac{\cos\left[\frac{n\pi}{a}(x-a)\right]}{\sqrt{(\frac{n\pi}{2a})^2+m^2}}\ \text{for } |x|<a\\ \epsilon_2(x,a) &= -\frac{m^2}{2\pi}K_0(2m|x-a|)\ \text{for } |x|>a\,.\end{aligned} \tag{8}$$

The Casimir energy density for finite λ was computed in Ref. [12]. In the limit $\lambda\to\infty$ it agrees with eq. (8) except at $x=\pm a$ where it contains an extra, singular contribution. If one integrates eq. (8) over all x, ignoring the singularities at $x=\pm a$, one obtains eq. (4). Including the contributions at $\pm a$ gives eq. (6).

This simple example illustrates our principal results: In the Dirichlet limit the renormalized Casimir energy diverges because the energy density on the "surface," $x=\pm a$ diverges. However the Casimir force and the Casimir energy density for all $x\neq\pm a$ remain finite and equal to the results obtained by imposing the boundary conditions *a priori*, eqs. (5) and (8).

A scalar field in two dimensions constrained to vanish on a circle of radius a presents a more complex problem. We decompose the energy density in a shell of width dr at a radius r into a sum over angular momenta, $\epsilon(r)=\sum_{\ell=0}^\infty\epsilon_\ell(r)$, where $\epsilon_\ell(r)$ can be written as an integral over imaginary momentum $t=-ik$ of the partial wave Green's function at coincident points $G_\ell(r,r;it)$ and its radial derivatives. First suppose we fix $\sigma(\mathbf{x})=\delta(r-a)$ and consider $r\neq a$. It is easy to see that the difference $[G_\ell(r,r,it)]_0$ between the full Green's function $G_\ell(r,r,it)$ and the free Green's function $G_\ell^{(0)}(r,r,it)$ vanishes *exponentially* as $t\to\infty$. For finite λ, both the t-integral and the ℓ-sum are uniformly convergent so $\lambda\to\infty$ can be taken under the sum and integral. The resulting energy density, $\tilde{\epsilon}(r)$, agrees with that obtained when the Dirichlet boundary condition, $\phi(a)=0$, is assumed from the start. As in one dimension, nothing can be said about the total energy because $\tilde{\epsilon}(r)$ is not defined at $r=a$, but unlike that case the integral of $\tilde{\epsilon}(r)$ now diverges even in the sharp limit for finite λ.

To understand the situation better, we take $\sigma(\mathbf{x})$ to be a narrow Gaußian of width w centered at $r=a$ and explore the sharp limit where $w\to 0$ and $\sigma(\mathbf{x})\to\delta(r-a)$. For $w\neq 0$, σ does not vanish at any value of r, so $[G_\ell(r,r,it)]_0$ no longer falls exponentially at large t, and subtraction of the first Born approximation to $G_\ell(r,r,it)$ is necessary. As in one dimension, the compensating tadpole graph can be canceled against the counterterm, $c_1\lambda\sigma(\mathbf{x})$. The result is a renormalized Casimir energy density, $\epsilon(r,w,\lambda)$, and Casimir energy, $E(w,\lambda)=\int_0^\infty dr\epsilon(r,w,\lambda)$, both of which are finite. However as $w\to 0$ both $\epsilon(a,w,\lambda)$ and $E(w,\lambda)$ diverge, indicating that the renormalized Casimir energy of the Dirichlet circle is infinite.

The divergence originates in the order λ^2 Feynman diagram. We study this diagram by subtracting the *second* Born approximation to $G_\ell(r,r,it)$ and adding back the equivalent diagram explicitly. Then the ℓ-sum and t-integral no longer diverge in the sharp limit. In the limit $w\to 0$, the diagram contributes

$$\lim_{w\to 0}E_{\rm FD}^{(2)} = -\frac{\lambda^2\,a^2}{8}\int_0^\Lambda dp\,J_0^2(ap)\,\arctan\frac{p}{2m} \tag{9}$$

which diverges like $\ln \Lambda$. The divergence originates in the high momentum components in the Fourier transform of $\sigma(r) = \delta(r-a)$, not in the high energy behavior of a loop integral, and therefore cannot be renormalized. Taking $\lambda \to \infty$ only makes the divergence worse. Because it varies with the radius of the circle, this divergence gives an infinite contribution to the surface tension. This divergence only gets worse in higher dimensions (in contrast to the claim of Ref. [16]). For example, for $\sigma(r) = \delta(r-a)$ in three dimensions the *renormalized* two point function is proportional to an integral over p of a function proportional to $\lambda^2 a^4 p^2 j_0^2(pa) \ln p$ at large p. The integral diverges like $\Lambda \ln \Lambda$. Such divergences cancel when we compute the force between rigid bodies, but not in the case of stresses on isolated surfaces.

In summary, by implementing a boundary condition as the limit of a less singular background, we are able to study the divergences that arise when a quantum field is forced to vanish on a prescribed surface. For all cases we have studied, the renormalized Casimir energy, defined in the usual sense of a continuum quantum field theory, diverges in the Dirichlet limit. Physical cutoffs (like the plasma frequency in a conductor) regulate these divergences, which are localized on the surface. On the other hand the energy density away from the surfaces or quantities like the force between rigid bodies, for which the surfaces can be held fixed, are finite and independent of the cutoffs. Observables that require a deformation or change in area of $\mathcal{S}$ cannot be defined independently of the other material stresses that characterize the system. Similar studies are underway for fluctuating fermion and gauge fields, leading to Neumann and mixed boundary conditions with the same types of divergences.

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[1] H. B. G. Casimir, Kon. Ned. Akad. Wetensch. Proc. **51**, 793 (1948).

[2] V.M. Mostepanenko and N.N. Trunov, *The Casimir Effect and its Application*, Clarendon Press, Oxford (1997), K. A. Milton, *The Casimir Effect: Physical Manifestations Of Zero-Point Energy*, River Edge, USA: World Scientific (2001).

[3] M. Bordag, U. Mohideen and V. M. Mostepanenko, Phys. Rept. **353**, 1 (2001) [arXiv:quant-ph/0106045].

[4] S. K. Lamoreaux, Phys. Rev. Lett. **78**, 5 (1997).

[5] U. Mohideen and A. Roy, Phys. Rev. Lett. **81**, 4549 (1998) [arXiv:physics/9805038].

[6] G. Bressi, G. Carugno, R. Onofrio, and G. Ruoso, Phys. Rev. Lett. **88**, 041804 (2002)[arXiv:quant-ph/0203002]

[7] P. Candelas and D. Deutsch Phys. Rev. **D20** (1979) 3063.

[8] P. Candelas, Ann. Phys. **143** (1982) 3063.

[9] K. Symanzik, Nucl. Phys. **B190** (1981) 1.

[10] A. Actor, Ann. Phys. **230** (1994) 303, Fort. Phys. **43** (1995) 141.

[11] G. Barton, J. Phys. A **34**, 4083 (2001).

[12] N. Graham, R. L. Jaffe, V. Khemani, M. Quandt, M. Scandurra and H. Weigel, Nucl. Phys. B **645**, 49 (2002) [arXiv:hep-th/0207120].

[13] The energy density away from $\mathcal{S}$ is interesting in its own right; since it can be negative, it may induce interesting effects in general relativity: F. J. Tipler, Phys. Rev. Lett. **37**, 879 (1976), S. W. Hawking, Phys. Rev. D **46**, 603 (1992), L. H. Ford and T. A. Roman, Phys. Rev. D **51**, 4277 (1995), K. D. Olum, Phys. Rev. Lett. **81**, 3567 (1998), [arXiv:gr-qc/9805003]. K. D. Olum and N. Graham, [arXiv:gr-qc/0205134].

[14] For a comprehensive review see: N. Graham, R. L. Jaffe and H. Weigel [arXiv:hep-th/0201148] in M. Bordag, ed., *Proceedings of the Fifth Workshop on Quantum Field Theory Under the Influence of External Conditions*, Intl. J. Mod. Phys. A **17** (2002) No. 6 & 7.

[15] E. Farhi, N. Graham, P. Haagensen and R. L. Jaffe, Phys. Lett. B **427**, 334 (1998) [arXiv:hep-th/9802015], E. Farhi, N. Graham, R.L. Jaffe, and H. Weigel, Phys. Lett. B **475**, 335 (2000) [arXiv:hep-th/9912283], Nucl. Phys. B **585**, 443 (2000) [arXiv:hep-th/0003144], Nucl. Phys. B **630**, 241 (2002) [arXiv:hep-th/0112217].

[16] K. A. Milton, arXiv:hep-th/0210081.