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Noncommutative Chiral Anomaly and The Dirac-Ginsparg-Wilson Operator

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Abstract

It is shown that local axial anomaly in $2-$dimensions emerges naturally in the *gauge-invariant quantized* Ginsparg-Wilson relation if one postulates an underlying noncommutative structure of spacetime . Indeed if one first regularizes the $2-$d plane with a fuzzy sphere, i.e with a $(2l+1)\times(2l+1)$ matrix model , then one immediately finds that the gauge-invariant Dirac operator D_{GF} is different from the chiral-invariant Dirac operator D_{CF} . The fact that gauge states are different from chiral states stems from the free noncommutative Dirac operator D_F which is found to be inconsistent with chiral symmetry at high frequencies . Furthermore it splits such that $\frac{1}{D_F}=\theta\Gamma^R+\frac{1}{D_\Lambda}$ where $\theta=\frac{1}{2l+1}$ is the noncommutativity parameter , Γ^R is the noncommutative chirality and D_Λ is identified as the correct Dirac operator on this matrix model . D_Λ has essentially the same IR spectrum as D_F , both operators share the same continuum limit yet D_Λ is consistent with chiral symmetry on the UV modes . The second step of this fuzzy regularization consists simply of approximating the exact Dirac operator D_F with D_Λ . This regularization is shown to be gauge-invariant and yields to an anomaly which in terms of Dirac operators depends only on the difference $\frac{1}{D_F}-\frac{1}{D_\Lambda}$. Although the difference $\theta\Gamma^R$ drops from the spectrum in the limit , its contribution to the variation of the measure is shown to be not zero , it gives exactly the canonical anomaly.

1 Introduction

The two key concepts in the discussion of anomalies are always , gauge invariance and regularization. Indeed Fujikawa [17] in his original derivation of the anomaly showed that quantum chiral symmetry becomes anomalous because the fermion measure is not actually invariant under chiral transformations , and the anomaly is obtained from regularizing large eigenvalues of the underlying Dirac operator. The regularization prescription which is thus to be adopted must always be gauge invariant. A simple reason is a physical one which has to do with the fact that the anomaly has direct observable consequences (for example the abelian $U(1)$ axial anomaly provides precisely the rate of the neutral electromagnetic pion decay $\pi\longrightarrow 2\gamma$) .

This quantum anomalous behaviour of chiral symmetry does not necessarily mean that this symmetry is broken , it only means that one can not regulate in a gauge-invariant , chiral-invariant manner . In particular , the famous Nielsen-Ninomiya theorem [20] states that if one maintains chiral symmetry at all stages, then one cannot avoid the doubling of fermions in the usual lattice formulation and hence the anomaly is zero [see [21] and references therein]. The anomaly here vanishes only because the effect of the different doublers cancel among themselves and not because the chiral symmetry is being maintained explicitly throughout.

Fuzzy physics [[1, 2] , see also [3] and references therein] , like lattice gauge theories, is aiming for a nonperturbative regularization of chiral gauge theories. Discretization in fuzzy physics is achieved by treating the underlying spacetimes as phase spaces then quantizing them in a canonical fashion which means in particular that we are effectively replacing the underlying spacetimes by noncommutative matrix models or fuzzy manifolds [3, 4] . As a consequence , this regularization will preserve all symmetries and topological features of the problem . Indeed a fuzzy space is by construction a discrete lattice-like structure which serves to regularize , it allows for an exact chiral invariance to be be formulated , but still the fermion-doubling problem is completely avoided [10] . The same thing was also shown to hold for fermions on fuzzy $\mathbf{CP}^2$ despite the fact that these latter do not carry spin quantum numbers but instead carry spin$_c$ quantum numbers , in other words the left-handed fermions and the right-handed fermions on $\mathbf{CP}^2$ transform differently under the spin group $SU(2)\times U(1)$ of $\mathbf{CP}^2$ [[22] and the last reference of [2]] . All this seems to suggest that fuzzy methods do not suffer from any of the limitations put by the above Nielsen-Ninomiya theorem on lattice schemes .

Global chiral anomalies on these models were , along with other topological non-trivial field configurations , formulated in [11, 12, 23] , while local anomaly on fuzzy $\mathbf{S}^2_F$ was treated first in [8] then in [26, 25] . The relevance of Ginsparg-Wilson relations in noncommutative matrix models was noted first in [10] and [27] then in [24] . In this article we will show that despite the fact that the concept of point is lacking on fuzzy $\mathbf{S}^2$, we can go beyond global considerations and define a "fuzzy" axial anomaly associated with a "fuzzy" $U(1)$ global chiral symmetry. This "fuzzy" axial anomaly is however shown to be identically zero for each finite approximation and only by requiring physical states in the continuum limit to be gauge- and chiral-symmetric in the sense we explain that we reproduce the canonical theta term .

The plan of the paper is as follows . Section 2 contains a brief description of fuzzy $\mathbf{S}^2$. Noncommutative $U(1)$ gauge theory and fermion action on $\mathbf{S}^2_F$ are introduced in section 3 . In particular we show in this section that the gauge-invariant Dirac operator D_{GF} is different from the chiral-invariant Dirac operator D_{CF} . Section 4 contains the main results of the paper , we show that a) The measure is completely symmetric under chiral transformations for all finite matrix models , b) The quantized gauge-invariant Ginsparg-Wilson relation contains already the anomaly and c) The free Dirac Operator D_F provides a gauge-invariant regularization consistent with chiral symmetry which is shown to reproduce the anomaly in the continuum limit. In section 5 we derive the divergence of the current using the star product and the Dixmier trace and show that the two methods are equivalent. We conclude in section 6 with a summary.

2 The Noncommutative Fuzzy Sphere $\mathbf{S}_F^2$

One practical way of thinking about the fuzzy sphere is as a regulator . In other words , instead of replacing Euclidean space-time $\mathbf{R}^2$ by a lattice , we implement the regularization prescription given by the substitution $\mathbf{R}^2 \longrightarrow \mathbf{S}_F^2$ where $\mathbf{S}_F^2$ is the fuzzy sphere [see [1, 2] and the extensive list of publications in [3]]. We now briefly review it .

The fuzzy sphere is a noncommutative space and as such it must be described by a spectral triple $(\mathbf{A}, \mathbf{H}, D_F)$ [4] . Beside the Dirac operator D_F which will be described shortly , the most important other ingredient in the description of the fuzzy sphere , in terms of Connes' noncommutative geometry [4] , is the algebra $\mathbf{A}$ of $(2l+1)\times(2l+1)$ matrices Mat_{2l+1} . This algebra can be obtained as follows .

As embedded in $\mathbf{R}^3$ the commutative unit sphere is described by three commuting coordinates $n_a = x_a$, $a = 1, 2, 3$ satisfying the constraint $\sum_a n_a^2 = 1$. In terms of these commutative coordinates any function on $\mathbf{S}^2$ can be expanded as follows

$$f(\vec{n}) = \sum_{i_1,...,i_k} f_{i_1,...,i_k} n_{i_1}...n_{i_k} \equiv \sum_{km} f_{km} Y_{km}(\theta, \phi), \tag{1}$$

where the Y_{km}'s are the usual spherical harmonics and the sum over k goes from 0 to ∞ . The claim is that the geometry of $\mathbf{S}^2$ can be completely reconstructed in terms of the algebra $\mathcal{A}$ of all functions f on $\mathbf{S}^2$.

To obtain the noncommutative fuzzy $\mathbf{S}^2$ we simply need to replace consistently the algebra $\mathcal{A}$ by a suitable algebra of operators the elements of which will fail to commute in general . In the case of $\mathbf{S}^2$ which is a co-adjoint orbit, i.e $\mathbf{S}^2 = SU(2)/U(1)$ this process is not arbitrary as one can simply quantize the underlying symplectic structure to get the algebra of operators $\mathbf{A}$. One finds that any operator $\hat{f}$ of $\mathbf{A}$ has now the expansion

$$\hat{f}(\vec{n}^F) = \sum_{i_1,...,i_k} f_{i_1,...,i_k} n_{i_1}^F...n_{i_k}^F, \tag{2}$$

where n_i^F's are defined by

$$n_i^F = \frac{L_i}{\sqrt{l(l+1)}}. \tag{3}$$

L_i's satisfy $[L_i, L_j] = i\epsilon_{ijk}L_k$ and $\sum_{i=1}^3 L_i^2 = l(l+1)$ respectively , in other words L_i's are the generators of the IRR l of $SU(2)$, and therefore all the sums in (2) terminate [1, 2, 3] . This fact can be better seen by rewriting (2) in terms of Polarization tensors , namely

$$\hat{f} = \sum_{k=0}^{2l} \sum_{m=-k}^{k} f_{km} T_{km}(l). \tag{4}$$

[For an extensive analysis of the properties of $T_{km}(l)$'s see [5]] . Remark that n_i^F's clearly satisfy the requirements

$$(n_1^F)^2 + (n_2^F)^2 + (n_3^F)^2 = 1, \tag{5}$$

and

$$[n_i^F, n_j^F] = \frac{i}{\sqrt{l(l+1)}} \epsilon_{ijk} n_k^F. \tag{6}$$

(2),(3),(5) and (6) define what we call fuzzy $\mathbf{S}^2$ or $\mathbf{S}^2_F$. Loosely speaking , $\mathbf{S}^2_F$ is the algebra $\mathbf{A}$ above which is nothing else but the algebra Mat_{2l+1} of $(2l+1)\times(2l+1)$ matrices . $\mathbf{H}$ in the spectral triple $(\mathbf{A},\mathbf{H},D_F)$ is simply the Hilbert space on which the above IRR l of $SU(2)$ is acting .

Fuzzy Derivations are by definition the generators of the adjoint action of $SU(2)$, in other words the derivative of the fuzzy function $\hat{f}$ in the space-time direction a is $adL_a\hat{f}$ defined by

$$adL_a(\hat{f})=[L_a,\hat{f}]. \tag{7}$$

This is motivated by the fact that continuum derivations $\mathcal{L}_a=-i\epsilon_{abc}n_b\partial_c$ generate rotations on $\mathbf{S}^2$, which also provide the adjoint IRR of $SU(2)$, and hence it is only natural to stress this property once we go to the fuzzy .

As one can already notice all these definitions are in a very close analogy with the case of continuum $\mathbf{S}^2$ where the algebra of functions $\mathcal{A}$ plays the same role played here by $\mathbf{A}$. In fact the continuum limit is defined by $l\longrightarrow\infty$ where the fuzzy coordinates n_i^F's approach the ordinary coordinates n_i's , i.e

$$n_i^F\longrightarrow n_i \quad \text{with } \vec{n}^2=1 \quad \text{and} \quad [n_i,n_j]=0, \tag{8}$$

and where the algebra $\mathbf{A}$ tends to the algebra $\mathcal{A}$ in the sense that

$$\hat{f}\longrightarrow f(\vec{n})=\sum_{i_1,...,i_k} f_{i_1,...,i_k}n_{i_1}...n_{i_k}, \tag{9}$$

and

$$adL_i(\hat{f})\longrightarrow\mathcal{L}_i(f)(\vec{n}). \tag{10}$$

Now the sums in (9) are not cut-off and one can have in the expansion polynomials of arbitrary degrees in n_i's . Formally one writes $\mathcal{A}=Mat_\infty$ and think of the fuzzy sphere as having a finite number of points equal to $2l+1$ which will diverge in the continuum limit $l\longrightarrow\infty$.

3 Noncommutative Fuzzy Actions

3.1 Noncommutative Fuzzy YM Action

A scalar field on the noncommutative space $\mathbf{S}^2_F$ is a general matrix $\hat{f}$ of the algebra Mat_{2l+1} which admits the momentum-space expansion (4) . Classical actions for these scalar fields and their quantum theories were put forward in [6] . A vector field A_a^F can be similarly defined by an expansion of the form

$$A_a^F=\sum_{k=0}^{2l}\sum_{m=-k}^{k}A_a(km)T_{km}(l)\ ,\ A_a^{F+}=A_a^F, \tag{11}$$

i.e each component A_a^F is a $(2l+1)\times(2l+1)$ matrix . The modes $A_a(km)$ are complex numbers satisfying $A_a(km)^*=(-1)^mA_a(k-m)$ where for each momentum (km) the corresponding triple $(A_1^F(km),A_2^F(km),A_3^F(km))$ obviously transforms as an $SO(3)-$vector . Writing a gauge principle for this matrix vector field is not difficult , indeed the action takes the usual form

$$S_{YMF} \quad = \quad \frac{1}{4e^2}\frac{1}{2l+1}Tr_l F^F_{ab}F^F_{ab}, \tag{12}$$

where the trace Tr_l is defined on the Hilbert space $\mathbf{H}$ whereas the curvature F^F_{ab} is given by

$$\begin{aligned} F^F_{ab} &\equiv [D^F_a, D^F_b] - i\epsilon_{abc}D^F_c \ , \ D^F_a = L_a + A^F_a \\ &= [L_a, A^F_b] - [L_b, A^F_a] + [A^F_a, A^F_b] - i\epsilon_{abc}A^F_c. \end{aligned} \tag{13}$$

Gauge transformations are implemented by unitary transformations acting on the $(2l+1)-$dimensional Hilbert space of the irreducible representation l of $SU(2)$. These transformations are $D^F_a \longrightarrow D^{F'}_a = U^L D^F_a U^{L+}$, $A^F_a \longrightarrow A^{F'}_a = U^L A^F_a U^{L+} + U^L[L_a, U^{L+}]$ and $F^F_{ab} \longrightarrow F^{F'}_{ab} = U^L F^F_{ab} U^{L+}$ where $U^L = e^{i\Lambda^L}$ and $\Lambda^L = \Lambda^{L+}$ is an element of the algebra $\mathbf{A}$ of $(2l+1)\times(2l+1)$ matrices acting on the left . U^L's define then fuzzy $U(1)$ gauge theory .

A final remark is to note that the gauge field $\vec{A}^F$ has three components and hence an extra condition is needed in order to project this gauge field onto two dimensions . One adopts here the prescription of [14] , i.e we impose on the gauge filed $\vec{A}^F$ the gauge-invariant condition

$$D^F_a D^F_a = l(l+1). \tag{14}$$

The full quantization of this model and its continuum planar limit will be reported elsewhere [7] .

3.2 Gauge-Invariant Dirac Operator and Chiral Fermion in NCG

The Dirac operator is a key ingredient in Connes noncommutative geometry (NCG) especially in connection with metric spaces and spin structures [4] . In this section we will show explicitly that the properties of the Dirac operator at very short distances are indeed at the origin of the axial anomaly . More precisely we will show that the gauge-invariant Dirac operator D_{GF} on the matrix model (6) is incompatible with chiral fermion in the sense that chiral symmetry in the presence of gauge fields is governed by a completely different Dirac operator D_{CF} . The consequence of this fact will on the other hand be analyzed in great detail in the next section .

First we find the gauge-invariant Dirac operator D_{GF} on fuzzy $\mathbf{S}^2$, i.e a finite dimensional gauge-invariant matrix action for fermion to be added to (12). In close analogy with the free fermion action on continuum $\mathbf{S}^2$ given by [8, 9]

$$S = \int_{\mathbf{S}^2}\frac{d\Omega}{4\pi}\bar{\chi}\mathcal{D}\chi \ , \ \mathcal{D} = \sigma_a\mathcal{L}_a + 1 \ , \ \mathcal{L}_a = -i\epsilon_{abc}x_b\partial_c, \tag{15}$$

the free fermion action on fuzzy $\mathbf{S}^2$ is defined by

$$S_F \quad = \quad \frac{1}{2l+1}Tr_l\bar{\psi}_F D_F\psi_F \ , \ D_F = \sigma_a[L_a, ...] + 1. \tag{16}$$

$\mathcal{D}$ and D_F are precisely the Dirac operators on continuum $\mathbf{S}^2$ and fuzzy $\mathbf{S}^2$ respectively [10, 3, 11, 12] . The continuum spinor χ is an element of $\mathcal{A}\otimes\mathbf{C}^2$ where $\mathcal{A}$ is the algebra of function on continuum $\mathbf{S}^2$, while the fuzzy spinor ψ_F is an element of $\mathbf{A}\otimes\mathbf{C}^2\equiv Mat_{2l+1}\otimes\mathbf{C}^2$. Both spinors are of dimension $(\text{mass})^{\frac{1}{2}}$, $\bar{\chi}=\chi^+$, $\bar{\psi}_F=\psi_F^+$ and σ_i's are Pauli matrices .

It is a trivial fact that the above continuum Dirac operator $\mathcal{D}$ admits a chirality structure , i.e we can define a chirality operator γ such that $\gamma^2=1$, $\{\gamma,\mathcal{D}\}=0$, i.e $\gamma=\sigma_a n_a$. The Grosse-Klimčík-Prešnajder Dirac operator D_F on fuzzy $\mathbf{S}^2$ admits also a chirality operator which can be seen as follows , first we rewrite D_F in the form [10, 3, 11, 12]

$$\frac{1}{2l+1}D_F = \frac{1}{2}(\Gamma^R+\Gamma^L), \tag{17}$$

where $L_a^L\equiv L_a$'s and $-L_a^R$'s are the generators of the IRR l of $SU(2)$ which act on the right of the algebra $\mathbf{A}$. Γ^R and Γ^L , on the other hand , are the operators

$$\Gamma^L = \frac{1}{l+\frac{1}{2}}[\vec{\sigma}.\vec{L}^L+\frac{1}{2}] \ , \ \Gamma^R=\frac{1}{l+\frac{1}{2}}[-\vec{\sigma}.\vec{L}^R+\frac{1}{2}]. \tag{18}$$

Γ^R is the chirality operator as was shown originally in [13] , this choice is also motivated by the fact that $\Gamma^R\longrightarrow-\gamma$, when $l\longrightarrow\infty$ (see latter for the minus sign), $(\Gamma^R)^2=\Gamma^R$, $(\Gamma^R)^+=\Gamma^R$, and $[\Gamma^R,\hat{f}]=0$ for any $\hat{f}\in\mathbf{A}$. However this Γ^R does not exactly anticommute with the Dirac operator since

$$\Gamma^R D_F+D_F\Gamma^R = \frac{1}{l+\frac{1}{2}}D_F^2. \tag{19}$$

Despite this problem , one can show that (D_F,Γ^R) defines a chiral structure on fuzzy $\mathbf{S}^2$ which satisfies $a)$ the Ginsparg-Wilson relation , $b)$ is without fermion doubling and $c)$ has the correct continuum limit [10, 3]. The absence of fermion doubling for example can be easily seen by comparing the spectrum of D_F given by $D_F(j)=\{\pm(j+\frac{1}{2}),j=\frac{1}{2},\frac{3}{2},...,2l-\frac{1}{2}\}\cup\{j+\frac{1}{2} \quad , \quad j=2l+\frac{1}{2}\}$ with the spectrum of $\mathcal{D}$ given by $\mathcal{D}(j)=\{\pm(j+\frac{1}{2}),j=\frac{1}{2},\frac{3}{2},...,\infty\}$ [12] . As one can immediately see there is no fermion doubling and the spectrum of D_F is simply cut-off at the top eigenvalue $j=2l+\frac{1}{2}$ if compared with the continuum spectrum [10, 3] .

However the situation when one includes gauge field is quite different. Gauging the continuum Dirac operator $\mathcal{D}$ means invoking the minimal replacement $\mathcal{L}_a\longrightarrow\mathcal{L}_a+A_a$, and therefore we end up with the Dirac operator $\mathcal{D}_G=\mathcal{D}+\sigma_a A_a$. Correspondingly the gauge-invariant fermion action in the continuum is given by

$$S_G = \int\frac{d\Omega}{4\pi}\Big[\bar{\chi}\sigma_a\mathcal{L}_a(\chi)+\bar{\chi}\chi+\bar{\chi}\sigma_a A_a\chi\Big]. \tag{20}$$

As usual , gauge transformations act on χ , $\bar{\chi}$ as follows $\chi\longrightarrow\chi^{'}=U\chi$, $\bar{\chi}\longrightarrow\bar{\chi}^{'}=\bar{\chi}U^+$. Similarly , the fuzzy gauged Dirac operator is defined by

$$D_{GF}=D_F+\sigma_a A_a^F, \tag{21}$$

while the fuzzy analogue of the action (20) is given by

$$S_{GF} = \frac{1}{2l+1}Tr_l\Big[\bar{\psi}_F\sigma_a[L_a,\psi_F]+\bar{\psi}_F\psi_F+\bar{\psi}_F\sigma_a A_a^F\psi_F\Big], \tag{22}$$

This gauge-invariant fermion fuzzy action is invariant under gauge transformations provided the spinor ψ_F is simultaneously transformed as $\psi_F \longrightarrow \psi_F^{'} = U^L\psi_F$, $\bar{\psi}_F \longrightarrow \bar{\psi}_F^{'} = \bar{\psi}_F U^{L+}$.

The quantum theory of the noncommutative Schwinger model given by the action (12)+(22) will be considered in great detail elsewhere [7] , in here we will only touch on the corresponding quantum chiral structure and its anomalous behaviour . The claim is that (22) is not chiral-invariant , nevertheless chiral symmetry can be exactly defined for all finite values of l and yet an anomalous behaviour remains in the continuum limit . It is obtainable as we will show from a combination of fuzzy path integrals and noncommutative Ginsparg-Wilson relations and the requirement of gauge invariance .

3.3 The Chiral-invariant Dirac Operator D_C and its Complex Action

On continuum $\mathbf{S}^2$ exact chiral invariance of the classical action is expressed by the anticommutation relation $\gamma\mathcal{D} + \mathcal{D}\gamma = 0$ which is in fact the limit of the Ginsparg-Wilson relation (19). We now show that even with (19) exact chiral invariance can be constructed consistently on fuzzy $\mathbf{S}^2$. The fermion action (20) was already shown to be gauge invariant , but under the canonical continuum chiral transformations $\chi \longrightarrow \chi^{'} = \chi + \lambda\gamma\chi$, $\bar{\chi} \longrightarrow \bar{\chi}^{'} = \bar{\chi} + \lambda\bar{\chi}\gamma$, one can show that it is invariant only if the gauge field is constrained to satisfy $x_aA_a = 0$ which is a consequence of the identity $\mathcal{D}_G\gamma + \gamma\mathcal{D}_G = 2\vec{A}.\vec{n} = 2\phi$. From the continuum limit of (14) it is obvious that this constraint is satisfied and hence chiral symmetry is maintained. However one wants also to formulate chiral symmetry without the need to use any constraint on the gauge field , indeed the action

$$\int \frac{d\Omega}{4\pi}\Big[\bar{\chi}\mathcal{D}\chi + \bar{\chi}\hat{\sigma}_a A_a\chi\Big]. \tag{23}$$

is strictly chiral invariant for arbitrary gauge configurations . $\hat{\sigma}_a$ is the Clifford algebra projected onto the sphere , i.e $\hat{\sigma}_a = \mathcal{P}_{ab}\sigma_b$, $\mathcal{P}_{ab} = \delta_{ab} - n_an_b$. The action (23) is however still gauge invariant because of the identity $n_a\mathcal{L}_a = 0$. Action (23) can be rewritten as follows

$$S_C = \int \frac{d\Omega}{4\pi}\Big[\bar{\chi}\mathcal{D}\chi + \epsilon_{abc}\bar{\chi}Z_bn_cA_a\chi\Big] \ , \ Z_a = \frac{i}{2}[\gamma,\sigma_a] \tag{24}$$

from which we can define the new chiral-invariant Dirac operator $\mathcal{D}_C = \mathcal{D} + \epsilon_{abc}Z_bn_cA_a$. The difference between the continuum gauge-invariant Dirac operator $\mathcal{D}_G$ and the continuum chiral-invariant Dirac operator $\mathcal{D}_C$ is proportional to the normal component ϕ of the gauge field , i.e $\mathcal{D}_C = \mathcal{D}_G - \gamma\phi$, and hence they are essentially identical (by virtue of the constraint $n_aA_a = 0$) and the spectrum of the theory seems to be both gauge invariant as well as chiral invariant . The Fuzzy analogue of (24) is the action

$$S_{CF} = \frac{1}{2l+1}Tr_l\Big[\bar{\psi_F}D_F\psi_F + \epsilon_{abc}\bar{\psi_F}Z_b^Fn_c^FA_a^F\psi_F\Big] \ , \ Z_a^F = \frac{i}{2}[\Gamma^L\sigma_a + \sigma_a\Gamma^R], \tag{25}$$

which is invariant under the fuzzy chiral transformations

$$\begin{aligned}\psi_F \longrightarrow \psi_F^{'} &= \psi_F + \Gamma^R \psi_F \lambda^L + O(\lambda^L) \\ \bar{\psi}_F \longrightarrow \bar{\psi}_F^{'} &= \bar{\psi}_F - \lambda^L \bar{\psi}_F \Gamma^L + O(\lambda^L).\end{aligned} \tag{26}$$

Remark that the chiral parameter λ^L in (26) is a general $(2l+1)\times(2l+1)$ matrix which is small in the sense of coherent states [15, 16] . Furthermore in order for this parameter to correspond to a global transformation it must only be a function of the Casimir $(\vec{L}^L)^2$. The structure of the above chiral transformations (26) is of course uniquely dictated by the Ginsparg-Wilson relation (19) which can also be put in the form $\Gamma^L D_F - D_F \Gamma^R = 0$ [18]. Remark also that the spinor $\bar{\psi}$ can not now be identified with ψ^+ although the theory is still classical and despite the fact that in gauge theory $\bar{\psi}$ is $\equiv \psi^+$. This result can be interpreted as the statement that on the finite dimensional matrix model (6) chiral and gauge symmetries are not commuting symmetries , i.e chiral states are different from gauge states at very short distances (corresponding to the UV part of the spectrum) . Indeed we can define from (25) a new Dirac operator

$$D_{CF} = D_F + \epsilon_{abc} Z_b^F x n_c^F A_a^F \tag{27}$$

which is very different from D_{GF} and hence the two operators do not commute. Now , the operator D_{CF} tends in the large l limit to $\mathcal{D}_C$ so that both operators D_{CF} and D_{GF} have the same continuum limit [Recall that in this limit we have $\Gamma^R \longrightarrow -\gamma$, $\Gamma^L \longrightarrow \gamma$ which are consequences of the requirements $\Gamma^L D_F - D_F \Gamma^R = 0 \longrightarrow \gamma\mathcal{D} + \mathcal{D}\gamma = 0$, $Z_k^F \longrightarrow Z_k$ and hence $D_{CF} \longrightarrow \mathcal{D}_C$] .

Finally let us say that the action (22) is not chiral invariant while the action (25) is complex and not gauge invariant (in fact the above D_{CF} is not self-adjoint but as we have just said it is the correct fuzzy analogue of $\mathcal{D}_C$ from the point of view of chiral invariance and if we insist that chiral transformations take the canonical form (26)) . The kinetic term in both actions (22) and (25) is however the same and hence we do not loose the nice feature of having in the noncommutative case a free spectrum which is identical to the continuum spectrum with a natural cut-off .

4 Chiral Anomaly

4.1 Quantum measure and The Path Integral

The quantum theory of interest is defined through the following path integral

$$\int \mathcal{D}A_i^F \int \mathcal{D}\psi_F \mathcal{D}\bar{\psi}_F e^{-S_{GF}}. \tag{28}$$

Despite the fact that S_{GF} is not chiral-invariant , it is the correct starting point as one wants to maintain gauge-invariance throughout . The anomaly anyway arises from the non-invariance of the measure under chiral transformations [17] and hence we will focus on this measure and show explicitly that for all finite approximations of the noncommutative Schwinger model this measure is in fact invariant unless the properties of the Dirac operator D_{GF} are also taken into account in the evaluation of the trace .

In a matrix model such as (22) manipulations on the quantum measure have a precise meaning . Indeed and by following [17] we first expand the fuzzy spinors ψ_F , $\bar{\psi}_F$ in terms of the eigentensors $\phi(\mu, A)$ of the Dirac operator D_{GF}, write $\psi_F = \sum_\mu \theta_\mu \phi(\mu, A)$, $\bar{\psi}_F = \sum_\mu \bar{\theta}_\mu \phi^+(\mu, A)$ where θ_μ's , $\bar{\theta}_\mu$'s are independent sets of Grassmanian variables, and $\phi(\mu, A)$'s are defined by $D_{GF}\phi(\mu, A) = \lambda_\mu(A)\phi(\mu, A)$, and normalized such that $\frac{1}{2l+1}Tr_l\phi^+(\mu, A)\phi(\nu, A) = \delta_{\mu\nu}$. For weak fuzzy gauge fields μ stands for j , k and m which are the eigenvalues of $\vec{J}^2 = (\vec{K}+\frac{\vec{\sigma}}{2})^2$, $\vec{K}^2 = (\vec{L}^L - \vec{L}^R)^2$ and J_3 respectively . As we will show, and similarly to [17] , what really matters in the calculation is the asymptotic behaviour when $A_i^F \longrightarrow 0$ of $\phi(\mu, A)$'s and $\lambda_\mu(A)$'s given by $\lambda_\mu(A) \longrightarrow j(j+1) - k(k+1) + \frac{1}{4}$ and $\phi(\mu, A) \longrightarrow \sqrt{2l+1}\sum_{k_3,\sigma} C^{jm}_{kk_3\frac{1}{2}\sigma} T_{kk_3}(l)\chi_{\frac{1}{2}\sigma}$ [3, 12, 5] . The quantum measure is therefore well defined and it is given by

$$\mathcal{D}\psi_F\mathcal{D}\bar{\psi}_F = \prod_\mu d\theta_\mu d\bar{\theta}_\mu \longrightarrow \prod_{k=0}^{2l}\prod_{j=k-\frac{1}{2}}^{k+\frac{1}{2}}\prod_{m=-j}^{j} d\theta_{kjm}d\bar{\theta}_{kjm} \tag{29}$$

A canonical calculation shows that the above quantum measure changes under the fuzzy chiral transformations (26) as follows

$$\int \mathcal{D}A_i^F \int \mathcal{D}\psi_F{}' \mathcal{D}\bar{\psi}_F' e^{-S_{GF}'} = \int \mathcal{D}A_i^F \int \mathcal{D}\psi_F\mathcal{D}\bar{\psi}_F e^{S_{\theta F}} e^{-S_{GF}-\Delta S_{GF}}, \tag{30}$$

with

$$\Delta S_{GF} = -\frac{1}{2l+1}Tr_l\lambda^L[L_a, \bar{\psi}_F\sigma_a\Gamma^R\psi_F] + \frac{1}{2l+1}Tr_l\lambda^L\bar{\psi}\big[\sigma_a\Gamma^R - \Gamma^L\sigma_a\big]A_a^F\psi_F, \tag{31}$$

where we can see explicitly in the second term in ΔS_{GF} why (22) is not chiral-invariant . This extra piece is zero for zero gauge fields and is generically of the order of $1/l$. It is due to edge effects introduced by fuzzification , more precisely it is due to the fact that in the discrete the chirality operator Γ^R does not exactly anti-commute with the tangent Clifford algebra $\sigma_a P_{ab}$ in the sense that we have $[\sigma_a\Gamma^R - \Gamma^L\sigma_a]P_{ab} \neq 0$. In the continuum this is an identity , since the normal component of $\vec{A}^F$ is identically zero by virtue of (14) , and therefore $\Delta\Gamma^L$ is not needed , i.e $[\sigma_a\gamma + \gamma\sigma_a]\mathcal{P}_{ab} = 0$. P_{ab} and $\mathcal{P}_{ab}$ are simply the projectors (in the fuzzy and in the continuum respectively) on the sphere [26, 28].

The theta term is on the other hand given by

$$S_{\theta F} \quad = \quad -\frac{1}{2l+1}\sum_\mu Tr_l\lambda^L\phi^+(\mu, A)(\Gamma^R - \Gamma^L)\phi(\mu, A). \tag{32}$$

The problem with (31) and (32) is in fact not edge effects but rather gauge-invariance (for example $\phi(\mu, A)$ transforms as $U^L\phi(\mu, A)$ for the Dirac equation to be gauge-invariant). In here we adopt a different , more economical , route in defining chiral invariance which is consistent with gauge symmetry . We start by simply changing the chiral transformations (26) to

$$\begin{aligned}\psi_F \longrightarrow \psi_F' &= \psi_F + \Gamma^R\psi_F\lambda^L \\ \bar{\psi}_F \longrightarrow \bar{\psi}_F' &= \bar{\psi}_F - \lambda^L\bar{\psi}_F\hat{\Gamma}^L,\end{aligned} \tag{33}$$

where

$$\hat{\Gamma}^L = \frac{1}{l+\frac{1}{2}}(\sigma_a(L_a + A_a^F) + \frac{1}{2}) = \Gamma^L + \frac{1}{l+\frac{1}{2}}\vec{\sigma}.\vec{A}^F. \tag{34}$$

(33) reduces in the limit to the usual transformations yet it gurantees gauge invariance in the noncommutative fuzzy setting since $\hat{\Gamma}^L$ transforms as $U^L\hat{\Gamma}^L U^{L+}$ under gauge transformations. Indeed the change of the action under these transformations is now given by

$$\begin{aligned}\Delta S_{GF} &= -\frac{1}{2l+1}Tr_l\lambda^L[L_a, \bar{\psi_F}\sigma_a\Gamma^R\psi_F] + \frac{1}{2l+1}Tr_l\lambda^L\bar{\psi}_F\Bigg[(\sigma_a\Gamma^R - \Gamma^L\sigma_a)A_a^F - \frac{1}{l+\frac{1}{2}}\vec{\sigma}.\vec{A}^F D_{GF}\Bigg]\psi_F \\ &= -\frac{1}{2l+1}Tr_l\lambda^L[L_a, \bar{\psi_F}\sigma_a\Gamma^R\psi_F] - \frac{i}{(2l+1)^2}\epsilon_{abc}Tr_l\lambda^L\bar{\psi}_F\sigma_c F_{ab}^F\psi_F,\end{aligned} \tag{35}$$

where in particular we have used equation (14) . We still have an edge effect, i.e the extra piece above vanishes in the continuum limit but now it is exactly gauge-invariant . The theta term is now also gauge-invariant and is given by

$$S_{\theta F} = -\frac{1}{2l+1}\sum_{\mu} Tr_l\lambda^L\phi^+(\mu, A)(\Gamma^R - \hat{\Gamma}^L)\phi(\mu, A). \tag{36}$$

Next because the Dirac operator D_{GF} is self-adjoint on $Mat_{2l+1}\otimes\mathbf{C}^2$, the states $\phi(\mu, A)$'s form a complete set and hence one must have the identity

$$\frac{1}{2l+1}\sum_{\mu}\phi^{AB\alpha}(\mu, A)\phi^{+CD\beta}(\mu, A) = \delta^{\alpha\beta}\delta^{AD}\delta^{BC}. \tag{37}$$

Due to the finiteness of the matrix model , it is an identity easy to check that the theta term $S_{\theta F}$ is zero , indeed we have

$$S_{\theta F} = -\Bigg(Tr_l\lambda^L\Bigg)\Bigg(Tr_l tr_2(\Gamma^R - \hat{\Gamma}^L)\Bigg)\equiv 0. \tag{38}$$

tr_2 is the $2-$dimensional spin trace , i.e $tr_2\mathbf{1} = 2$, $tr_2\sigma_a = 0$, etc . In fact it is this trace which actually vanishes , i.e $tr_2(\Gamma^R - \hat{\Gamma}^L) = 0$, and hence the correct axial anomaly requires more than the naive evaluation of the trace .

4.2 Noncommutative Ginsparg-Wilson Relation

We will now undertake the task of carefully analyzing the spectrum of the theory and consequently derive the continuum limit of the axial anomaly . We first start with the free theory and rewrite the Ginsparg-Wison relation (19) as $(\Gamma^R - \Gamma^L)D_F + D_F(\Gamma^R - \Gamma^L) = 0$ which means that in the absence of gauge field we must have $tr[\Gamma^R - \Gamma^L] = 0$ where the trace is taken in the space of spinors . Chiral invariance of D_{CF} can be expressed by the Ginsparg-Wilson relation $\Gamma^L D_{CF} - D_{CF}\Gamma^R = 0$ and hence $tr[\Gamma^R - \Gamma^L]$ is also $= 0$ if the effect of the gauge field is taken into account through D_{CF} . However if we include the gauge field through the gauge-invariant Dirac operator D_{GF} we compute instead

$$\{\Gamma^R - \Gamma^L, D_{GF}\} = \frac{2}{2l+1}\Bigg[N_a A_a^F - \frac{i}{2}\epsilon_{abc}\sigma_c F_{ab} + \sigma_a A_a^F + [L_a, A_a^F]\Bigg], \tag{39}$$

where $N_a = -2(L_a^R + L_a)$ and where F_{ij} is the *would-be* continuum curvature $F_{ab}^F = F_{ab} + [A_a^F, A_b^F]$. The naive limit of this equation is $\{\gamma, \mathcal{D}_G\} = 2\phi = 0$. In other words , in the continuum interacting theory one might be tempted to conclude that $Tr\gamma = 0$ which we know is wrong in the presence of gauge fields. Noncommutative geometry , as it is already obvious from equation (39), already gives us the structure of the chiral anomaly , indeed (39) can furthermore be rewritten in the form

$$\left[\Gamma^R - \Gamma^L - \frac{2}{2l+1}D_{GF}\right]^2 - 4 = \frac{2i}{(2l+1)^2}\epsilon_{abc}\sigma_c(F_{ab} + F_{ab}^F) - \frac{4}{(2l+1)^2}(A_a^F)^2,$$

or equivalently in the gauge-covariant form

$$\{\Gamma^R - \hat{\Gamma}^L, D_{GF}\} = -\frac{2i}{2l+1}\epsilon_{abc}\sigma_c F_{ab}^F, \tag{40}$$

which we find remarkable . We have used above the identity $\{D_{GF}, \sigma_a A_a^F\} = \frac{1}{2}\{N_a, A_a^F\} + i\epsilon_{abc}\sigma_c F_{ab}^F + \sigma_a A_a^F - \frac{i}{2}\epsilon_{abc}\sigma_c F_{ab}$. We have also used extensively the identities $(\Gamma^R - \Gamma^L)^2 - 4 = -\frac{4D_F^2}{(2l+1)^2}$, $D_{GF}^2 = D_{GF} + (L_a - L_a^R + A_a^F)^2 + \frac{i}{2}\epsilon_{abc}\sigma_c F_{ab}^F$ as well as the constraint (14) . Now if we trace both sides of the above equation we obtain the exact answer

$$-tr(\Gamma^R - \Gamma^L) = \frac{i}{2l+1}\epsilon_{abc} tr\big(\frac{1}{D_{GF}}\sigma_c F_{ab}^F\big) + \frac{1}{2l+1} tr\big(\frac{1}{D_{GF}} J\big)$$

where

$$\begin{aligned} J &= -A_a^F N_a - i\epsilon_{abc}\sigma_c[A_a^F, A_b^F] - 4i\epsilon_{abc}\sigma_c L_a^R L_b - 2\sigma_a A_a^F - i\epsilon_{abc}\sigma_c N_b A_a^F \\ &= -2D_{GF}\sigma_a A_a^F + 4i\sqrt{l(l+1)}D_{Gw} \ , \end{aligned} \tag{41}$$

where D_{Gw} is now obviously gauge-covariant , in fact it is exactly the gauged Watamuras Dirac operator [13] , i.e

$$D_{Gw} = \epsilon_{abc}\sigma_a \frac{D_b^F}{\sqrt{l(l+1)}} L_c^R. \tag{42}$$

This operator has the continuum limit $\mathcal{D}_{Gw} = i\gamma\mathcal{D}_G - i\phi = i\gamma\mathcal{D}_G$ where the normal component ϕ vanishes by virtue of the constraint (14) . Recall that $\mathcal{D}_G = \mathcal{D} + \sigma_a A_a$ is the continuum limit of D_{GF} (more on this below). Hence we obtain

$$-tr(\Gamma^R - \hat{\Gamma}^L) = \frac{i}{2l+1}\epsilon_{abc} tr\big(\frac{1}{D_{GF}}\sigma_c F_{ab}^F\big) + \frac{4i\sqrt{l(l+1)}}{2l+1} tr\big(\frac{1}{D_{GF}} D_{Gw}\big). \tag{43}$$

The symbol tr denotes the trace in the space of fuzzy spinors , i.e $tr(X) = \sum_\mu \phi^+(\mu, A) X \phi(\mu, A)$. In particular the cyclic property of the trace does not hold because of the non-commutativity of the different ingredients , in other words the result of this trace is still an element in the algebra $\mathbf{A} = Mat_{2l+1}$. The full gauge-invariant chiral anomaly (36) takes now the form

$$S_{\theta F} = \frac{i}{(2l+1)^2}\epsilon_{abc} Tr_l \lambda^L tr\big(\frac{1}{D_{GF}}\sigma_c F_{ab}^F\big) + \frac{4i\sqrt{l(l+1)}}{(2l+1)^2} Tr_l \lambda^L tr\big(\frac{1}{D_{GF}} D_{Gw}\big),$$

or by using (37)

$$S_{\theta F} = \left[\frac{1}{2l+1} Tr_l \lambda^L\right]\left[i\epsilon_{abc} Tr_l\left(tr_2(\frac{1}{D_{GF}}\sigma_c) F_{ab}^F\right) + 4i\sqrt{l(l+1)} Tr_l tr_2(\frac{1}{D_{GF}} D_{Gw})\right]. \tag{44}$$

4.3 Noncommutative Geometry As A Regulator

It is not difficult to observe that the naive continuum limit $l \longrightarrow \infty$ of the above equation is

$$(8l^2)\Big(\int\frac{d\Omega}{4\pi}\lambda\Big)\Big(\int\frac{d\Omega}{4\pi}tr_2(\gamma)\Big)=\Big[\int\frac{d\Omega}{4\pi}\lambda\Big]\Big[(2il)\epsilon_{abc}\int\frac{d\Omega}{4\pi}\Big(tr_2(\frac{1}{\mathcal{D}_G}\sigma_c)F_{ab}\Big)+(8l^2)\int\frac{d\Omega}{4\pi}tr_2(\gamma)\Big].$$

By dividing across by l^2 this becomes an identity $0=0$. In other words the fact that $S_{\theta F}=0$ in the fuzzy does not necessarily mean that the anomaly vanishes in the continuum . The claim (which we will explain shortly in detail) is that the Dirac operator D_F is inconsistent *in the continuum limit with chiral symmetry on top modes* which is exactly the source of the anomaly. Hence a gauge-invariant regularization of this behaviour is required to recover the anomaly in the continuum . We adopt the prescription

$$\begin{aligned}
S_{\theta\Lambda} &\equiv \Big[\frac{1}{2l+1}Tr_l\lambda^L\Big]\Big[i\epsilon_{abc}Tr_l\Big(tr_2(\frac{1}{D_{G\Lambda}}\sigma_c)F^F_{ab}\Big)+4i\sqrt{l(l+1)}Tr_l tr_2(\frac{1}{D_{G\Lambda}}D_{w\Lambda})\Big]\\
&= \Big[\frac{1}{2l+1}Tr_l\lambda^L\Big]\Big[i\epsilon_{abc}Tr_l\Big(tr_2(\frac{1}{D_{G\Lambda}}-\frac{1}{D_{GF}})\sigma_c F^F_{ab}\Big)\Big]\\
&+ \Big[\frac{1}{2l+1}Tr_l\lambda^L\Big]\Big[4i\sqrt{l(l+1)}Tr_l\Big(tr_2\Big(\frac{1}{D_{G\Lambda}}D_{w\Lambda}-\frac{1}{D_{GF}}D_{Gw}\Big)\Big)\Big],
\end{aligned}$$

where we have simply made the replacement $D^{-1}_{GF}\longrightarrow D^{-1}_{G\Lambda}$, $D_{Gw}\longrightarrow D_{w\Lambda}$ in the first line above, then used again $S_{\theta F}=0$. The operator $\frac{1}{D_{GF}}D_{Gw}$ as we saw tends in the limit to $-i\gamma$, i.e it behaves as a chirality , whereas the regulator Λ is meant only to regularize the free Dirac operator D_F and not alter chiralities. Therefore a natural choice for $D_{w\Lambda}$ which does not alter chiralities is such that

$$\frac{1}{D_{G\Lambda}}D_{w\Lambda}=\frac{1}{D_{GF}}D_{Gw}.$$

In perturbation theory the Dirac operator D_{GF} admits the expansion

$$\begin{aligned}
\frac{1}{D_{GF}} &= \frac{1}{D_F}-\frac{1}{D_F}\sigma_a A^F_a\frac{1}{D_F}+\frac{1}{D_F}\sigma_a A^F_a\frac{1}{D_F}\sigma_a A^F_a\frac{1}{D_F}-...\\
&= \frac{1}{D_F}+\sum_{n=1}(-1)^n\frac{1}{D_F}\sigma_a A^F_a\frac{1}{D_F}....\sigma_a A^F_a\frac{1}{D_F},
\end{aligned}$$

and hence one deduce immediately that to linear terms in the gauge field [24], the anomaly is given by

$$S_{\theta\Lambda} = \Big[\frac{1}{2l+1}Tr_l\lambda^L\Big]\Big[i\epsilon_{abc}Tr_l\Big(tr_2(\frac{1}{D_\Lambda}-\frac{1}{D_F})\sigma_c F^F_{ab}\Big)\Big]. \qquad (45)$$

The regulator Λ will now be defined precisely . We first show that the regulator Λ is essentially provided by the chiral properties of the free Dirac operator D_F in the UV domain and hence most contributions to the chiral anomaly (45) are coming from high frequency modes of the spectrum. Next recall the continuum limits $\vec{x}^R\longrightarrow\vec{x}$, $\vec{x}^F\longrightarrow\vec{x}$, $\Gamma^R\longrightarrow-\gamma$, $\Gamma^L\longrightarrow\gamma$, and in particular remark that because $\vec{x}^R$ tends to $\vec{x}$ in the limit $l\longrightarrow\infty$, the extra minus sign in the commutation relations of these x_i^R's becomes completely unobserved in the limit which is indeed expected . But from the free fuzzy eigenvalues equation

$$\frac{D_F}{2l+1}\phi(\mu,0)=\frac{\Gamma^R+\Gamma^L}{2}\phi(\mu,0)=\frac{\lambda_\mu}{2l+1}\phi(\mu,o), \tag{46}$$

i.e equation (17) (recall that μ stands for $k=0,...,2l$, $j=k-\frac{1}{2},k+\frac{1}{2}$ and $m=-j,...,+j$ and $\lambda_\mu=j(j+1)-k(k+1)+\frac{1}{4}$) we can easily see that for all the infrared modes $\mu<<2l$, the limits $\Gamma^R{\longrightarrow}-\gamma$, $\Gamma^L{\longrightarrow}\gamma$ are indeed satisfied since both left and right hand sides of (46) vanish in the continuum limit , i.e

$$Lim_{l\longrightarrow\infty}\big[\frac{D_F}{2l+1}\phi(\mu,0)\big]=Lim_{l\longrightarrow\infty}\big[\frac{\Gamma^R+\Gamma^L}{2}\phi(\mu,0)\big]=0. \tag{47}$$

But for ultraviolet modes $\mu{\sim}2l$ we have instead

$$Lim_{l\longrightarrow\infty}\big[\frac{D_F}{2l+1}\phi(\mu,0)\big]=\pm1\ ,\ \text{whereas}\ Lim_{l\longrightarrow\infty}\big[\frac{\Gamma^R+\Gamma^L}{2}\phi(\mu,0)\big]=0. \tag{48}$$

The sign is $+1$ if λ_μ is a positive energy eigenvalue and -1 if λ_μ is a negative energy eigenvalue . Using the identity $\frac{2l+1}{D_F}=\frac{(2l+1)^2}{D_F^2}\frac{\Gamma^R+\Gamma^L}{2}$ first , then taking the limit $l{\longrightarrow}\infty$ one can rewrite (48) in the form

$$Lim_{l\longrightarrow\infty}\big[\frac{2l+1}{D_F}\phi(\mu,0)\big]=\pm1\ ,\ \text{whereas}\ Lim_{l\longrightarrow\infty}\big[\frac{2}{\Gamma^R+\Gamma^L}\phi(\mu,0)\big]=0. \tag{49}$$

All this means in particular that either 1) the limits $\Gamma^R{\longrightarrow}-\gamma$ and $\Gamma^L{\longrightarrow}\gamma$ are not valid in the UV domain or that 2) the free Dirac operator D_F is not appropriate in describing chiral symmetry in the UV . Since one wants to maintain chiral symmetry explicitly throughout, the limits $\Gamma^R{\longrightarrow}-\gamma$, $\Gamma^L{\longrightarrow}\gamma$ should always hold true and one must instead replace D_F^{-1} (when restricted to the UV modes) with a new Dirac operator D_Λ^{-1} with the key property of having vanishing eigenvalues for $\mu{\sim}2l$. Such a Dirac operator D_w already exists in the literature , it was found originally by Watamuras [13] and used extensively in [10, 11] to construct , among other topological quantities , the global anomaly on the noncommutative matrix model (6) . In here and as it turns out this Dirac operator is also very useful in extracting the local anomaly on the continuum limit of (6).

In order to define D_Λ^{-1} more precisely , we proceed simply by writing identities . We have

$$\begin{aligned}\frac{1}{D_F}&=\frac{1}{D_F^2}\frac{2l+1}{2}(\Gamma^R+\Gamma^L)\\&=\frac{1}{2l+1}\frac{1}{D_F^2}\Gamma^R\Big(D_F^2+2i\sqrt{l(l+1)}D_w\Big),\end{aligned}$$

or equivalently

$$\frac{1}{D_F}=\frac{\Gamma^R}{2l+1}+2i\frac{\sqrt{l(l+1)}}{2l+1}\frac{1}{D_F^2}\Gamma^RD_w, \tag{50}$$

where we have used the results $\Gamma^R\Gamma^L=-1+\frac{1}{2(l+\frac{1}{2})^2}\big(D_F^2+2i\sqrt{l(l+1)}D_w\big)$, $\Gamma^RD_F-D_F\Gamma^L=0$, and D_w is exactly the Watamura Dirac operator given by

[13, 3]

$$D_w=\epsilon_{abc}\sigma_a\frac{L_b^L}{\sqrt{l(l+1)}}L_c^R. \tag{51}$$

We now recall few facts important to us here about this Dirac operator D_w. In the continuum $\mathcal{D}_w$ is related to $\mathcal{D}$ by $\mathcal{D}_w=i\gamma\mathcal{D}$ and hence both operators $\mathcal{D}_w$ and $\mathcal{D}$ have the same spectrum (that does not mean that they commute in fact $\{\mathcal{D}_w,\mathcal{D}\}=0$) . In the fuzzy , the spectrum of D_F (the fuzzy analogue of $\mathcal{D}$) is simply cut-off at the top modes $j=2l+\frac{1}{2}$ while the spectrum of D_w (the fuzzy analogue of $\mathcal{D}_w$) is quite deformed , in particular the eigenvalues of D_w when $j=2l+\frac{1}{2}$ are now exactly zero while for other large j's these eigenvalues are very small which is precisely the correct behaviour we want at the top modes [13, 3]. In fact as a consequence of this behaviour we have the exact anticommutation relation $\{\Gamma^R,D_w\}=0$.

The remarkable identity (50) gives essentially the prescription we want . One can immediately see that the behaviour of the operator $\frac{2l+1}{D_F}$ as a sign operator in the UV (see equation (49)) comes entirely from the first term Γ^R in (50) since the second term clearly vanishes. Remark also that the operator $2i\frac{\sqrt{l(l+1)}}{2l+1}\frac{1}{D_F^2}\Gamma^R D_w$ has the limit

$$Lim_{l\longrightarrow\infty}\left[2i\frac{\sqrt{l(l+1)}}{2l+1}\frac{1}{D_F^2}\Gamma^R D_w\right]=\frac{1}{\mathcal{D}}, \tag{52}$$

i.e it has the same limit as $\frac{1}{D_F}$ and hence it has also the same continuum spectrum . In fact even in the noncommutative fuzzy setting the IR part of the spectrum of the operator $2i\frac{\sqrt{l(l+1)}}{2l+1}\frac{1}{D_F^2}\Gamma^R D_w$ differs from that of $\frac{1}{D_F}$ only by correction of the order of $1/l$. The Dirac operator D_Λ^{-1} can therefore be identified with

$$\frac{1}{D_\Lambda}\equiv 2i\frac{\sqrt{l(l+1)}}{2l+1}\frac{1}{D_F^2}\Gamma^R D_w, \tag{53}$$

or equivalently

$$\frac{1}{D_\Lambda}-\frac{1}{D_F}=-\frac{\Gamma^R}{2l+1}. \tag{54}$$

This equation is viewed properly as a *noncommutative regularization prescription* dictated by the requirement that the free Dirac operator must be consistently chiral-invariant while taking the continuum limit . The operator $\frac{1}{D_\Lambda}$ satisfies all the desired requirement a) its spectrum in the IR is essentially identical to the spectrum of $\frac{1}{D_F}$, b) It vanishes in the UV as required by chiral symmetry , see equation (49) . c) Removing the regulator here is therefore the same as taking the continuum limit in which case the two operators $\frac{1}{D_F}$ and $\frac{1}{D_\Lambda}$ become identical . The contribution of the difference (which is here $-\frac{\Gamma^R}{2l+1}$) drops in the continuum limit from the spectrum yet its contribution to the variation of the quantum measure (45) does not vanish , this is exactly the anomaly.

To summarize , *the regulator* Λ *consists simply of approximating in a gauge-invariant manner the exact Dirac operator* $\frac{1}{D_F}$ *with the operator* $2i\frac{\sqrt{l(l+1)}}{2l+1}\frac{1}{D_F^2}\Gamma^R D_w$ and one obtains as a result the following anomaly on $\mathbf{S}^2_F$

$$\begin{aligned} S_{\theta\Lambda} &= \left[\frac{1}{2l+1}Tr_l\lambda^L\right]\left[i\epsilon_{abc}Tr_l\left(tr_2(\frac{1}{D_\Lambda}-\frac{1}{D_F})\sigma_c F^F_{ab}\right)\right] \\ &= -\left[\frac{1}{2l+1}Tr_l\lambda^L\right]\left[i\epsilon_{abc}\frac{1}{2l+1}Tr_l\left(tr_2\left(\Gamma^R\sigma_c\right)F^F_{ab}\right)\right] \\ &= \left[\frac{1}{2l+1}Tr_l\lambda^L\right]\left[\frac{2\sqrt{l(l+1)}}{l+\frac{1}{2}}i\epsilon_{abc}\frac{1}{2l+1}Tr_l\left(n^R_c F^F_{ab}\right)\right], \end{aligned} \tag{55}$$

where we have used the identity $tr_2\Gamma^R\sigma_c = -\frac{2\sqrt{l(l+1)}}{l+\frac{1}{2}}n^R_c$. Using the $SU(2)$ coherent states $|\vec{n},l>$ or equivalently the corresponding star product on $\mathbf{S}^2$ given in [15, 16] , then taking the continuum limit $l\longrightarrow\infty$ one recovers the anomaly on commutative $\mathbf{S}^2$ to be

$$S_\theta = \left[\int\frac{d\Omega}{4\pi}\lambda(\vec{n})\right]\left[i\frac{e}{2\pi}\epsilon_{abc}\int\frac{d\Omega}{4\pi}n_c F_{ab}\right], \tag{56}$$

where for convenience we have scaled the electric charge from the field strength as follows $<\vec{n},l|F^F_{ab}|\vec{n},l>=\frac{e}{4\pi}F_{ab}(\vec{n})$.

5 Dixmier Trace As An Alternative Continuum Limit

The claim of [9] is that the correct approximation of the continuum action (15) is not the action (16) but rather the action

$$S^{'}_F = \frac{1}{2l+1}Tr_l\frac{1}{|D_F|^2}\bar{\chi}_F D_F\chi_F\ . \tag{57}$$

This can be motivated as follows . The continuum action (15) can be rewritten in terms of the Dixmier tarce in the form[4]

$$\begin{aligned} S &= Tr_\omega\frac{1}{|\mathcal{D}|^2}\bar{\chi}\mathcal{D}\chi \\ &= Tr_\omega\bar{\psi}\mathcal{D}\psi, \end{aligned} \tag{58}$$

where $\psi=\frac{1}{|\mathcal{D}|}\chi$. The correct spinor is of course χ and not ψ . Tr_ω is the Dixmier trace which is defined "roughly speaking" as the coefficient of the logarithmic divergence of the ordinary trace , which turns out to be , for continuum manifolds , equal to the ordinary integral. In the continuum, its definition can be given by Connes trace theorem [4]

$$Tr_\omega|D|^{-d}f=\int_M dx^1\wedge dx^2\wedge...\wedge dx^d\sqrt{detg(x)}f(x),$$

where d and D are the dimension and the Dirac operator of the Riemannian spin manifold M , and $g_{\mu\nu}(x)$ is the metric on M.

The free fermionic action (16) on fuzzy $\mathbf{S}^2$ is essentially the fuzzy analogue of the second line of (58) , namely

$$S_F = Tr_{\omega,F}\bar{\psi}_F D_F \psi_F \ , \ Tr_{\omega,F} = \frac{1}{2l+1}Tr_l. \tag{59}$$

Now the fuzzy analogue of the first line of (58) (which is the action (57)) is not equal to (59) because the fuzzy Dixmier trace $Tr_{\omega,F}$ does not satisfy any of the properties satisfied by Tr_ω and which were needed in proving (58). In particular the crucial property of Tr_ω which is that the Dixmier trace of operators of order higher than one vanishes identically , is not satisfied by $Tr_{\omega,F}$. The correct noncommutative fuzzy spinor when approaching the limit is therefore not ψ_F but rather χ_F defined by $\psi_F = \frac{1}{|D_F|}\chi_F$.

For completeness we now show that even with this assumption both actions (57) and (59) yields the same physics at the limit . In particular we show that the divergence of the current given by equation (35) has (upto an overall logarithmic divergence) the same continuum limit for both (57) and (59) , i.e the star product used in deriving the final result (56) is upto an overall logarithmic divergence the same as the Dixmier trace . Indeed it is a trivial statement that (35) tends in the continuum limit (in the sense of the star product) to the canonical result

$$\Delta S_{GF} \quad = \quad \int \frac{d\Omega}{4\pi}\lambda(\vec{n})\mathcal{L}_a\big(\bar{\psi}\sigma_a\gamma\psi\big)(\vec{n}). \tag{60}$$

To show this result in the case of the Dixmier trace is however a more involved exercise .

The first step towards this end is to use the fact that ψ_F is not the correct spinor and that we need to substitute it with $\frac{1}{|D_F|}\chi_F$ in ΔS_{GF} . The second step is to write explicitly the trace in the Hilbert space $\mathbf{H}$ in terms of its $SU(2)-$coherent states's basis $|\vec{n},l>$. We then obtain in the large l limit

$$\Delta S_{GF}\simeq g_l\int\frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}\mathcal{L}_a(\lambda)(\vec{n}_1)\bar{\chi}(\vec{n}_1)[\frac{1}{|D_F|}]_{n_1n_2l}\sigma_a\gamma(\vec{n}_2)[\frac{1}{|D_F|}]_{n_2n_1l}\chi(\vec{n}_1),$$

The notation is $O_{n_1n_2l} =< \vec{n}_1,l|O|\vec{n}_2,l>$ and where g_l is given by $g_l = -\frac{1}{(2l+1)(4\pi)^4}$. We have also used above the fact that in the continuum limit $l\longrightarrow\infty$ the coherent state $|\vec{n},l>$ becomes localized at $\vec{n}$. The next step is crucial and involves the use of heat kernel techniques to extract the continuum answer from the above action [9, 19]. These considerations, which we will explain next, were first put forward in [9] and were used to study the continuum limit of the free scalar as well as the free fermionic fuzzy actions on $\mathbf{S}^2_F$. The assumption here is that these considerations are still valid in the presence of gauge fields if these latter can be treated as weak perturbations in the large l limit . The spectrum of the fuzzy Dirac operator D_F was shown to be precisely the spectrum of the continuum Dirac operator $\mathcal{D}$ only cut-off at the top eigenvalue $j = 2l + \frac{1}{2}$, and therefore in the limit $l\longrightarrow\infty$, it is reasonable to assume that the operator $|D_F|^{-1}$ will converge to its continuum counterpart $|\mathcal{D}_c|^{-1}$ whose eigenvalues smaller than the cut-off eigenvalue $\Lambda(l)^{-1} = |D_F(j = 2l + \frac{1}{2})|^{-1} = |2l + 1|^{-1}$ are simply set to zero . The above action takes then the form

$$\Delta S_{GF}\simeq g_l\int\frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}\mathcal{L}_a(\lambda)(\vec{n}_1)\bar{\chi}(\vec{n}_1)[\frac{1}{|\mathcal{D}_c|}]_{n_1n_2}\sigma_a\gamma(\vec{n}_2)[\frac{1}{|\mathcal{D}_c|}]_{n_2n_1}\chi(\vec{n}_1). \tag{61}$$

At the large l limit , the truncated operator $|\mathcal{D}_c|^{-1}$ converges weakly to the usual continuum operator $|\mathcal{D}|^{-1}$ because of its boundedness , and therefore the high frequency behaviour of D_F is irrelevant . However in order to obtain the expected logarithmic divergence at the limit $l \longrightarrow \infty$, it is not enough , as one can check , to simply replace in (61) all the operators $|\mathcal{D}_c|^{-1}$ by their weak limit $|\mathcal{D}|^{-1}$. To recover this crucial logarithmic divergence , the high frequency behaviour , which is irrelevant for individual operators , is important to take into account in the whole expression . As was shown in [9], the proper way to take the continuum limit is to leave in the action (61) only any one truncated operator and substitute the other with its weak limit. We have then

$$\Delta S_{GF} \simeq g_l \int \frac{d\Omega_1}{4\pi}\frac{d\Omega_2}{4\pi}\mathcal{L}_a(\lambda)(\vec{n}_1)\bar{\chi}(\vec{n}_1)[\frac{1}{|\mathcal{D}_c|}]_{n_1n_2}\sigma_a\gamma(\vec{n}_2)[\frac{1}{|\mathcal{D}|}]_{n_2n_1}\chi(\vec{n}_1). \quad (62)$$

We need now to evaluate the heat kernels $[\frac{1}{|\mathcal{D}_c|}]_{n_1n_2}$ and $[\frac{1}{|\mathcal{D}|}]_{n_2n_1}$. To this end we first remark that a reliable approximation for $|\mathcal{D}_c|^{-1}$ is $|\mathcal{D}_l|^{-1}$ whose modes with eigenvalues smaller than $|\Lambda(l)|^{-1}$ are exponentially suppressed , i.e [9]

$$\frac{1}{|\mathcal{D}_l|} \equiv \frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty \frac{dt}{t}t^{\frac{1}{2}}e^{-\mathcal{D}_l^2 t} \simeq \frac{1}{\Gamma(\frac{1}{2})}\int_{T(l)}^\infty \frac{dt}{t}t^{\frac{1}{2}}e^{-\mathcal{D}^2 t}, \quad (63)$$

where we have substituted $\mathcal{D}$ for $\mathcal{D}_l$ but restricted the integration to $t \geq T(l) = \frac{1}{\Lambda(l)^2}$, the eigenvalues $\mathcal{D}^2 >> \Lambda^2(l)$ will then all be suppressed and that is precisely $\mathcal{D}_l$. At $l \longrightarrow \infty$, the logarithmically divergent piece in (62) will come from the limit when $\vec{n}_1 \longrightarrow \vec{n}_2$, i.e from contact terms . This can be understood from the fact that the heat kernel $G_l(\vec{n}_1,\vec{n}_2) = [\frac{1}{|\mathcal{D}_l|}]_{n_1n_2}$ has a singularity as $\vec{n}_1 \longrightarrow \vec{n}_2$, which is coming from its short time behavior [19] , and therefore this heat kernel is effectively given by the integral

$$G_l(\vec{n}_1,\vec{n}_2) = \frac{1}{\Gamma(\frac{1}{2})}\int_{T(l)}^{T_0} \frac{dt}{t}t^{\frac{1}{2}} < \vec{n}_1|e^{-\mathcal{D}^2 t}|\vec{n}_2 >, \quad (64)$$

where T_0 is any small number larger than $T(l)$. Putting it differently , the contribution to the heat kernel $G_l(\vec{n}_1,\vec{n}_2)$ coming from the integration over $t \geq T_0$ is a finite quantity which vanishes at the large l limit . In the expression for the heat kernel above , we are now allowed to use the asymptotic formula at short times $< \vec{n}_1|e^{-\mathcal{D}^2 t}|\vec{n}_2 > \simeq \frac{1}{4\pi t}e^{-\frac{|\vec{n}_1-\vec{n}_2|^2}{4t}}$ and hence [9, 19]

$$G_l(\vec{n}_1,\vec{n}_2) = \frac{1}{\Gamma(\frac{1}{2})}\int_{T(l)}^{T_0} \frac{dt}{t}t^{\frac{1}{2}}\frac{1}{4\pi t}e^{-\frac{|\vec{n}_1-\vec{n}_2|^2}{4t}}. \quad (65)$$

If we let l goes to infinity in the above equation , the left hand side will give the heat kernel $G(\vec{n}_1,\vec{n}_2) = [\frac{1}{|\mathcal{D}|}]_{n_1n_2}$, whereas in the right hand side $T(l) \longrightarrow 0$. We then obtain a formula for the other heat kernel, namely

$$G(\vec{n}_1,\vec{n}_2) = \frac{1}{\Gamma(\frac{1}{2})}\int_0^\infty \frac{dt}{t}t^{\frac{1}{2}}\frac{1}{4\pi t}e^{-\frac{|\vec{n}_1-\vec{n}_2|^2}{4t}} = \frac{1}{2\pi}\frac{1}{|\vec{n}_1-\vec{n}_2|}, \quad (66)$$

where we have also let T_0 goes to infinity as this will only add regular terms when $\vec{n}_1 \longrightarrow \vec{n}_2$ but allows us to evaluate the integral exactly .

The relevant limit , $l \longrightarrow \infty$, $\vec{n}_1 \longrightarrow \vec{n}_2$, suggests the following change of variables , $2\vec{n} = \vec{n}_1 + \vec{n}_2$ and $\vec{\xi} = \vec{n}_1 - \vec{n}_2$ in (62) . Clearly $\vec{n}_1$ and $\vec{n}_2$ are normalized such that $\vec{n}_1^2 = \vec{n}_2^2 = 1$ and therefore $\vec{n}^2 = 1 - \frac{1}{4}\vec{\xi}^2$ which means that $\vec{n}$ will generate a sphere in the limit of interest $\vec{n}_1 \longrightarrow \vec{n}_2$ or $\vec{\xi} \longrightarrow 0$, and hence one can take $d\Omega_1 d\Omega_2 = d^2\vec{\xi} d\Omega = \xi d\xi d\Omega_\xi d\Omega$. The statement here is that the continuum limit is the same if we perform the above change of coordinates and just integrate $\vec{n}$ over a sphere whereas integrate $\vec{\xi}$ over a plane, since the short distance behaviour in the $\vec{\xi}-$variables is really what matters after all . Using the expansions $f(\vec{n}_1) == f(\vec{n}) + \frac{\xi^i}{2}\partial_i f(\vec{n}) + O(\xi^2)$ and $f(\vec{n}_2) = f(\vec{n}) - \frac{\xi^i}{2}\partial_i f(\vec{n}) + O(\xi^2)$ in (62) we obtain

$$\Delta S_{GF} \simeq \frac{g_l}{(4\pi)^2} \int_{\mathbf{S}^2} d^2\vec{\xi} d\Omega \Big[\mathcal{L}_a(\lambda)(\vec{n})\bar{\chi}(\vec{n}) G_l(\xi)\sigma_a \gamma(\vec{n}) G(\xi)\chi(\vec{n}) + O(\xi^2)\Big], \quad (67)$$

where linear terms in $\vec{\xi}$ vanish by rotational invariance . Now by using the identity

$$\int d^2\vec{\xi} G_l(\xi) G(\xi) = \int d\xi G_l(\xi) = \frac{1}{2\pi} lnl + \text{ finite terms}, \quad (68)$$

we obtain the final answer , which is the ordinary total divergence of the axial current , namely

$$\Delta S_{GF} \longrightarrow \Delta S_G = -\frac{2g_l lnl}{(4\pi)^2} \int_{\mathbf{S}^2} \frac{d\Omega}{4\pi} \lambda(\vec{n}) \mathcal{L}_a \big(\bar{\chi}\sigma_a \gamma\chi\big)(\vec{n}). \quad (69)$$

The overall normalization $\frac{1}{\mathcal{N}} = -\frac{2g_l lnl}{(4\pi)^2} = \frac{1}{(4\pi)^6}\frac{lnl}{l}$ clearly vanishes in the limit of large l and can be accounted for by simply redefining the Dirac operator D_Λ in (54) as follows

$$\frac{1}{D_\Lambda} - \frac{1}{D_F} = -\frac{1}{\mathcal{N}} \frac{\Gamma^R}{2l+1}. \quad (70)$$

This redefinition does not alter any of the properties of the Dirac operator D_Λ as $\mathcal{N} \longrightarrow \infty$ when $l \longrightarrow \infty$. Putting together this last result (69) with the result (56) for global parameters λ , we obtain the local chiral anomaly equation

$$\mathcal{L}_a(\bar{\chi}\sigma_a\gamma\chi)(\vec{n}) = i\frac{e}{2\pi}\epsilon_{abc} n_c F_{ab}. \quad (71)$$

6 Conclusion

We showed that on the noncommutative matrix model $\mathbf{S}_F^2$ given by the commutation relations (6) the gauge-invariant Dirac operator D_{GF} (equation (21)) is different from the chiral-invariant Dirac operator D_{CF} (equation (27)) . The kinetic term in the corresponding actions (22) and (25) is however the same and hence in both cases we have the nice feature of having a free spectrum which is identical to the continuum spectrum with a natural cut-off . In particular as we go deep in the UV domain the gauge-invariant modes do not coincide with the chiral-invariant modes and the net effect is essentially the source of the anomaly in two dimensions . D_{GF} and D_{CF} become identical only at large distances where modes can be both chiral- and gauge-symmetric simultaneously.

The fact that gauge states are different from chiral states stems already from the free noncommutative Dirac operator D_F which was found to be inconsistent with chiral symmetry at high energies (see equation (49)) . Furthermore it was found to split such that

$$\frac{1}{D_F} = \theta\Gamma^R + \frac{1}{D_\Lambda}\ ,\ \theta = \frac{1}{2l+1}$$

where θ is the noncommutativity parameter or in some sense the lattice spacing, Γ^R is the noncommutative chirality and D_Λ is identified as the correct Dirac operator on this matrix model . D_Λ given by (54) has essentially the same IR spectrum as D_F , both operators share the same noncommutative continuum limit yet D_Λ is consistent with chiral symmetry on the UV modes in the sense of equation (49).

The second step of the fuzzy regularization implemented in this paper consisted simply of approximating the exact Dirac operator D_F with D_Λ . This regularization was shown to yield in the gauge-invariant quantized Ginsparg-Wilson relation (45), i.e

$$S_{\theta\Lambda} = \left[\frac{1}{2l+1}Tr_l\lambda^L\right]\left[i\epsilon_{abc}Tr_l\left(tr_2(\frac{1}{D_\Lambda} - \frac{1}{D_F})\sigma_c F^F_{ab}\right)\right] \tag{72}$$

to an anomaly which in terms of Dirac operators depends only on the difference $\frac{1}{D_F} - \frac{1}{D_\Lambda}$. The difference $\theta\Gamma^R$ clearly drops from the spectrum in the limit yet its contribution to the variation of the measure under chiral transformations is not zero , it gives exactly the anomaly (56).

The divergence of the axial current was also derived using two methods , the star product and the Dixmier trace , and the theta term found.

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