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Analytic Study of Nonperturbative Solutions in Open String Field Theory

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Abstract

We propose an analytic framework to study the nonperturbative solutions of Witten's open string field theory. The method is based on the Moyal star formulation where the kinetic term can be split into two parts. The first one describes the spectrum of two identical half strings which are independent from each other. The second one, which we call midpoint correction, shifts the half string spectrum to that of the standard open string. We show that the nonlinear equation of motion of string field theory is exactly solvable at zeroth order in the midpoint correction. An infinite number of solutions are classified in terms of projection operators. Among them, there exists only one stable solution which is identical to the standard butterfly state. We include the effect of the midpoint correction around each exact zeroth order solution as a perturbation expansion which can be formally summed to the complete exact solution.

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I. INTRODUCTION

Starting with the original work of Witten [1] string field theory has been strongly tied with noncommutative geometry. This is in the spirit of treating the open string field as an infinite dimensional matrix, while the definition of the star product is formally identical to matrix multiplication. This viewpoint has been pursued in more detail by explicitly splitting the left and right degrees of freedom in the split string formalism [2].

Recently, this formal correspondence played a major rôle in importing basic ideas of noncommutative geometry to string field theory. One of the stimulating ideas is the vacuum string field theory (VSFT) proposal [3]. With an assumption on a simplified kinetic term to describe the tachyon vacuum, the classical solutions that would describe the D-brane are given by the noncommutative soliton [4] (=projector). It is well known that projectors are the fundamental geometrical objects in noncommutative geometry since they represent the K-homology group. The proof of the VSFT conjecture on the kinetic term was, however, difficult and there remained many open questions.

The Moyal formulation [5][6][7][8][9] (MSFT) is an explicit representation of Witten's string field theory in terms of the Moyal product, which is the main language in noncommutative geometry. Unlike previous proposals of the split string field framework, particular attention was paid to solve the ambiguity at the midpoint. In the context of MSFT, the subtlety is reflected in the form of the associativity anomaly [6] of the infinite dimensional matrices T, R, v, w which provide the change of variables from the open string coordinates to the canonical pairs of the Moyal product. The anomaly is resolved by deforming the non-associative algebra among T, R, v, w to an associative one, by introducing a regulated version of these matrices. All the elements of string field theory such as, perturbative spectrum, Neumann matrices and Feynman rules, were explicitly written in terms of the regularized framework, and their equivalence to other frameworks, when the regulator is removed, was demonstrated in [7][8]. Other proposals of the Moyal formalism [10][11] are equivalent to the original one [9].

In this paper, we take a step toward the classification of nonperturbative solutions of string field theory. The method is based on the splitting of the kinetic term into two parts. The first one gives a description of the open string where the left and right half strings completely decouple. The second term gives the correction to the split string description to recover the correct spectrum of the open string. We call it the "midpoint correction" since it carries the information that two half strings are indeed connected at the midpoint.

We will show that there exists a basis which diagonalizes the first part of the kinetic term and the nonlinear interaction term at the same time. The combined nonlinear system (namely Witten's action without the mid-point correction of the kinetic term) becomes a completely solvable matrix model. We can obtain all the exact solutions that are invariant under translations by using the projection operators under the star product. These provide a basis for classifying all nonperturbative solutions of open string field theory on the D25 brane. We derive the spectrum at zeroth order in the midpoint correction and then compute the effect of the midpoint as a perturbation.

The midpoint correction has been neglected in the literature because the coefficients in front of it seems to disappear in the naive limit of the regularization. It has, however, a finite contribution to the spectrum and cannot be discarded. In this paper, we propose to treat it as a correction to any of the zeroth order exact solutions, and provide a formal expression that sums up the perturbation series to the full exact solutions.

The situation turns out to be similar to the discussions of noncommutative scalar field theory [4]. There, in large θ (noncommutativity parameter) limit, the potential term gives the dominant contribution, and the equation of motion is solved by projectors. For finite θ, the kinetic term is incorporated as a perturbation [4][12][13] which gives corrections to the noncommutative soliton. In string field theory, usually there is no such parameter θ which can be adjusted to justify the validity of the perturbation expansion. However, in the Moyal formulation with a regulator, we have some freedom to choose the string oscillator frequencies (denoted as κ_n), while keeping the basic algebraic structure of string field theory, including important relations such as the nonlinear relations of Neumann coefficients derived by Gross and Jevicki [14]. While these parameters get fixed to the usual ones $\kappa_n = n$ at the end, we may use the behavior of the theory as a function of this degree of freedom to define the perturbation theory in the intermediate steps. In section IV, we give a construction of the exact solution of the full theory as the expansion of the midpoint correction along this idea. The perturbative expansion can be determined uniquely at each order with a condition (relating to the stability) on the spectrum of the open string solution. The perturbation series can then be summed up to a formal expression that represents the full exact solution.

II. THE SETUP

A. Moyal star formulation

The starting point of our discussion is Witten's action in the Siegel gauge written in the Moyal star formulation of String Field Theory (MSFT),

$$S(A) = -\int d^d\bar{x}\, Tr\left(\frac{1}{2\alpha'}A\star(L_0-1)A+\frac{g}{3}A\star A\star A\right). \tag{2.1}$$

The kinetic term is given by the Virasoro operator which is a second order differential operator acting on string fields $A\left(\bar{x},\xi,\xi^{gh}\right)$. We specify this operator later since it will be the main focus of this section. The string field is a function of the midpoint coordinate $\bar{x}$ and matter and fermionic bc ghosts coordinates, $\xi \equiv (x_e^\mu, p_e^\mu)$, $(\mu = 0, 1, \cdots, d-1, (d = 26))$ and $\xi^{gh} \equiv \left(x_e^b, x_e^c, p_e^b, p_e^c\right)$. The even index e specifies the mode and takes its value in $\{e = 2, 4, \cdots, 2N\}$ where N is the large integer which is introduced for the regularization. ξ (resp. ξ^{gh}) are Grassmann even (resp. odd) coordinates in noncommutative Moyal space with the star product specified by [5, 8, 16, 17][1]

$$[x_e^\mu, x_{e'}^\nu]_\star = i\theta\delta_{ee'}\eta^{\mu\nu}\,, \quad \left\{x_e^b, p_{e'}^b\right\}_\star = \{x_e^c, p_{e'}^c\}_\star = \theta'\delta_{ee'}\,. \tag{2.2}$$

In Appendix A we summarize MSFT notation that is used in this paper, including precise definitions of the Bogoliubov transformation between conventional open string operators and Moyal coordinates.

We note the additional free parameters in regulated MSFT. Namely, the spectrum parameters κ_e and κ_o that correspond to string oscillator frequencies can be chosen freely (see Appendix A for the notation) as the definition of the regularized theory. While they will be fixed at the end to $\kappa_n = n$ to reproduce the correct open string defined in Eq.(A2), the algebraic framework of MSFT is well defined for any κ_n as a function of n.

The equivalence of MSFT, with its regulator, to the conventional operator formulation has already been established in the following sense [8],

1. The spectrum of the kinetic term is identical to the conventional open string at large N. That is, the propagator in perturbation theory is identical in both theories.
2. The Neumann coefficients of the oscillator formalism were computed directly from the Moyal product [7][16], and these were shown to satisfy the Gross-Jevicki nonlinear relations [14] for any frequencies κ_n. Their simple expression explains as well agrees with spectroscopy [15], and agree numerically with other computations at large N.

These facts are sufficient to guarantee the equivalence of two formulations of string field theory in the computation of any perturbative string amplitudes [8]. Our formulation is well adapted for the discussion of nonperturbative string physics which will be the main topic of this paper.

We note that the conformal symmetry is lost in the regularized theory since we truncate the number of oscillators at level $2N$. Such truncation is, however, indispensable for the

[1] Compared to [8], we redefined the Grassmann variables with odd label to even ones $x_e^b, x_e^c, p_e^b, p_e^c$ as

$$x_e^b = \kappa_e^{-1} S x_o^{gh}\,, \quad p_e^b = \kappa_e S p_o^{gh}\,, \quad x_e^c = T y_o^{gh}\,, \quad p_e^c = \bar{R} q_o^{gh}\,, \quad (S := \kappa_e T \kappa_o^{-1})\,.$$

The form of the Moyal $\star$ product (Eq.(70) in [8]) is invariant under this transformation.

correct treatment of the associativity anomaly in the fundamental matrices [6] and also the Neumann coefficients. We note that every explicit computation of string field theory so far is based on some cut-off (level-truncation and so on). The additional freedom in the spectrum is actually directly related to the loss of conformal symmetry. We have to fix it at the end of the computation by taking the large N limit.

One of the main goals of string field theory (and also the main goal of this paper) is to solve the nonlinear equation of motion,

$$(L_0 - 1)A + \alpha' g A \star A = 0 \ . \tag{2.3}$$

The solutions, except for the trivial one ($A = 0$), describe nonperturbative backgrounds of open string theory which should be related to D-branes. It is widely believed that there exists a unique solution A_0 which describes the tachyon vacuum where D-branes are annihilated. There exist a large amount of numerical evidence [18] that confirms the existence and the desirable properties of such solution (D-brane tension, the absence of open string propagation and so on).

While the achievement of the numerical study is very impressive, it is still indispensable to develop an analytic framework to study the tachyon vacuum and other nonperturbative phenomena in string theory. In the operator formulation a difficulty originates from the fact that the basis which diagonalizes the kinetic term (conventional Fock space representation, which is its main tool) gives a complicated expression for the star product including the Neumann coefficients.

In the Moyal star formulation, on the other hand, the star product that produces (2.2) is trivial, while the kinetic term becomes off-diagonal. The complication of the kinetic term is, however, manageable and has not been a hindrance in developing the formalism for practical computations, such as Feynman graphs [8], and as we will see in the following, for nonperturbative solutions.

B. Splitting of kinetic term

We first translate the Virasoro operator from the conventional operator language to the Moyal star formulation by following the path in references [7, 8, 16].

The Virasoro operator L_0 written in terms of the standard oscillator notation is

$$L_0^{osc} = \frac{1}{2}\alpha_0^2 + \sum_{n=1}^{2N} \eta_{\mu\nu}\alpha_{-n}^{\mu}\alpha_n^{\nu} + \sum_{n=1}^{2N} \kappa_n (b_{-n}c_n + c_{-n}b_n) \tag{2.4}$$

where we explicitly truncate the number of oscillators and rewrite the frequency from n to κ_n. The commutation relation among oscillators with generic frequency is given in Eq.(A1). After translating the oscillators into the Moyal space as in Appendix A, we arrive at the expression of L_0 as a second order differential operator acting on the Moyal field A in the Siegel gauge (θ, θ' are arbitrary parameters which define the noncommutativity, and $l_s = \sqrt{2\alpha'}$),

$$
\begin{aligned}
L_0 &= L_0^{matter} + L_0^{ghost} , \qquad (2.5)\\
L_0^{matter} &= \sum_{e>0} \left(-\frac{l_s^2}{2}\frac{\partial^2}{\partial x_e^2} - \frac{\theta^2}{8 l_s^2}\kappa_e^2 \frac{\partial^2}{\partial p_e^2} + \frac{1}{2 l_s^2}\kappa_e^2 x_e^2 + \frac{2 l_s^2}{\theta^2} p_e^2 \right) + \frac{1}{2}\left(1+\bar{w}w\right)\beta_0^2 \\
&\quad + \frac{i l_s}{2}\beta_0 \sum_{e>0} w_e \frac{\partial}{\partial x_e} - \frac{1}{1+\bar{w}w}\frac{2 l_s^2}{\theta^2}\left(\sum_{e>0} w_e p_e\right)^2 - \frac{d}{2}\sum_{n=1}^{2N}\kappa_n , \qquad (2.6)\\
L_0^{ghost} &= i \sum_{e>0} \left(\frac{\partial}{\partial x_e^b}\frac{\partial}{\partial x_e^c} + \frac{\theta'^2}{4}\kappa_e^2 \frac{\partial}{\partial p_e^b}\frac{\partial}{\partial p_e^c} + \kappa_e^2 x_e^b x_e^c + \frac{4}{\theta'^2} p_e^b p_e^c \right) \\
&\quad - \frac{i}{1+\bar{w}w}\left(\sum_e w_e \frac{\partial}{\partial x_e^b}\right)\left(\sum_{e'} w_{e'} \frac{\partial}{\partial x_{e'}^c}\right) + \sum_{n=1}^{2N}\kappa_n . \qquad (2.7)
\end{aligned}
$$

$\beta_0 = -il_s \frac{\partial}{\partial \bar{x}} = -il_s \frac{\partial}{\partial x_0}$ represents the center of mass momentum, which can be written as a derivative of the midpoint coordinate $\bar{x}$. We note that the expression consists of mostly diagonal combination of the Moyal variables. The off-diagonal pieces with coefficient $(1 + \bar{w}w)^{-1}$ appear because of the Bogoliubov transformation from the odd modes with spectrum κ_o, to the even modes with spectrum κ_e. This complication of the kinetic term is the cost of the simplification of the star product in the Moyal product formulation.

At this stage, the equation of motion appears as horribly complicated – it is nonlinear, it contains an infinite order of derivatives through the star product, and is off-diagonal through the terms proportional to $(1 + \bar{w}w)^{-1}$. However, there exists a critical simplification of L_0 which saves us from most of these difficulties. The trick is to rewrite the diagonal pieces of L_0 by using the star product in the following form. It gives the splitting of L_0 into two parts containing the symbols $\mathcal{L}_0$,γ [7, 8, 16],

$$
(L_0 - 1)A = (\mathcal{L}_0(\beta_0) \star A + A \star \mathcal{L}_0(-\beta_0)) + \gamma A, \qquad (2.8)
$$

with [2]

$$
\begin{aligned}
\mathcal{L}_0(\beta_0) &= \mathcal{L}_0^{matter}(\beta_0) + \mathcal{L}_0^{ghost} - \frac{1}{2} , \qquad \gamma = \gamma^{matter} + \gamma^{ghost} , \qquad (2.9)\\
\mathcal{L}_0^{matter}(\beta_0) &= \sum_{e>0}\left(\frac{l_s^2}{\theta^2}p_e^2 + \frac{\kappa_e^2}{4 l_s^2}x_e^2 - \frac{l_s}{\theta} w_e p_e \beta_0 \right) + \frac{1}{4}(1+\bar{w}w)\beta_0^2 - \frac{d}{4}\sum_{n>0}\kappa_n , \\
\mathcal{L}_0{}^{ghost} &= i \sum_{e>0}\left(\frac{2}{\theta'^2} p_e^b p_e^c + \frac{\kappa_e^2}{2} x_e^b x_e^c \right) + \frac{1}{2}\sum_{n>0}\kappa_n , \qquad (2.10)
\end{aligned}
$$

[2] The expression $A \star \mathcal{L}_0(-\beta_0)$ is to be understood that β_0 is a derivative applied on $A(\bar{x})$ on its left, even though it is written on the right for convenience of notation.

$$\gamma^{matter} = -\frac{1}{1+\bar{w}w}\frac{2l_s^2}{\theta^2}\left(\sum_{e>0} w_e p_e\right)^2 ,$$

$$\gamma^{ghost} = -\frac{i}{1+\bar{w}w}\left(\sum_{e>0} w_e \frac{\partial}{\partial x_e^b}\right)\left(\sum_{e'>0} w_{e'}\frac{\partial}{\partial x_{e'}^c}\right) . \tag{2.11}$$

The star products with the *field* $\mathcal{L}_0$ reproduces the diagonal part of the *differential operator* $L_0 - 1$, while the ordinary product with γ gives the off-diagonal part of $L_0 - 1$.

The action of γ can also be written by using the star product while it cannot be split as left and right multiplications alone,

$$\gamma^{matter} A = -\frac{l_s^2}{2\theta^2(1+\bar{w}w)}\sum_{e,e'} w_e w_{e'}\eta_{\mu\nu}\left\{p_e^\mu, \{p_{e'}^\nu, A\}_\star\right\}_\star , \tag{2.12}$$

$$\gamma^{ghost} A = -\frac{i}{\theta'^2(1+\bar{w}w)}\sum_{e,e'} w_e w_{e'}\left\{p_e^b, [p_{e'}^c, A]_\star\right\}_\star . \tag{2.13}$$

These formulae will be useful in the concrete computation in section IV.

We emphasize that γ depends only on the rank one quantity $\hat{p}^{\mu,b,c} = (1+\bar{w}w)^{-1/2}\sum_e w_e p_e^{\mu,b,c}$, which is basically a single string momentum mode. This mode was first pointed out as the source of associativity anomalies of the star product in string field theory [6]. The strange mode that was later discovered in the continuous Moyal formalism [10] is given by $\bar{p}^{\mu,b,c} = (1+\bar{w}w)^{-1/2}\hat{p}^{\mu,b,c}$ up to a numerical constant as shown in [7][9]. As we will see, it is due to this mid-point complication that string field theory is not trivially solvable. We describe our strategy to solve the equation of motion with this splitting in the next subsection.

We explain the basic role of the two parts of L_0. If the γ term were absent, the spectrum of the free string would depend only on the frequencies κ_e since $\mathcal{L}_0$ (aside from the last constant term) depends only on κ_e. The string spectrum coming only from the two terms with $\mathcal{L}_0$ looks like the string spectrum of both even and odd oscillators, but with the frequency of the odd oscillator κ_o adjusted to be equal to the frequency of the neighboring even oscillator $\kappa_o \to \kappa_{o+1} = \kappa_e$. On further reflection, one can see that the $\mathcal{L}_0$ term by itself essentially describes the kinetic term of the two half strings which the original open string is composed of. However, the open string has a different spectrum, namely the even modes have frequencies κ_e and the odd modes have frequencies κ_o. The second term γ precisely fixes the discrepancy of the spectrum from the half string description. It carries the information of the midpoint where two half strings are connected, as shown in [7][8]. In this sense, we will refer to γ as the "midpoint correction".

The factor $(1+\bar{w}w)^{-1}$ appears to vanish naively in the open string limit Eq.(A2). However, this is misleading because in computations one obtains factors of $\bar{w}w$ in the numerator that produce finite contributions by the γ term. This is the mechanism of anomalies [6]. It is because of this subtlety that this term has been largely missed in the split string literature.

We note that the splitting of L_0 into $\mathcal{L}_0$ part and γ term is not unique. Namely one may obtain the same L_0 by the shift,

$$\mathcal{L}_0 \to \mathcal{L}_0 - f(\xi, \xi^{gh}), \quad \gamma A \to \gamma A + \left\{ f(\xi, \xi^{gh}), A \right\}_\star . \tag{2.14}$$

The framework of our analysis explained in the next subsection will not be basically affected by such a change as long as $f(\xi)$ is quadratic with respect to ξ, ξ^{gh}. It modifies, however, the half string spectrum and the Fock space structure discussed in the following sections. A fact mentioned above, namely that γ depends only on the rank one quantity $\hat{p}$, will be generally broken in such an arbitrary shift. Such a complicated choice of the splitting between $\mathcal{L}_0$ and γ may be useful to define a sensible perturbation expansion in the open string limit of Eq.(A2). We will come back to this issue later.

C. Strategy

Due to the separation of the kinetic term, we rewrite the action in the following form,

$$S = -\int d^d\bar{x}\, Tr \left(\frac{1}{2\alpha'} A \star (L_0 - 1) A + \frac{g}{3} A \star A \star A \right) = -\left(S_1 + S_2 + S_3 \right) . \tag{2.15}$$

$$S_1 = \frac{1}{\alpha'} \int d^d\bar{x}\, Tr A \star \mathcal{L}_0(\beta_0) \star A , \tag{2.16}$$

$$S_2 = \frac{1}{2\alpha'} \int d^d\bar{x}\, Tr A \star (\gamma A) , \tag{2.17}$$

$$S_3 = \frac{g}{3} \int d^d\bar{x}\, Tr A \star A \star A . \tag{2.18}$$

Before pursuing the nonperturbative analysis, let us emphasize that the conventional perturbation expansion of open string field theory is successfully reproduced in our MSFT formalism. In the conventional perturbative case, the classical equation of motion that comes from the quadratic part $S_1 + S_2$, namely $(L_0 - 1)\, A = 0$, gives the spectrum of string states associated with the oscillator frequencies (κ_e, κ_o) [7][8]. The interaction among these perturbative states is given by S_3 where only the use of the Moyal product is sufficient to compute interactions. Pursuing this in our formalism gives results that are in agreement with the conventional oscillator approach to the open string field theory [14]. In particular, our MSFT formalism including ghosts, replaces the complicated Neumann coefficients of the oscillator approach to express the interaction part explicitly [7][16]. Our theory includes a consistent regulator, and with it numerical estimates of certain quantities have been compared successfully to numerical results obtained in other formalisms [16]. Thus, we are confident that we have the correct theory to explore nonperturbative phenomena.

By dividing the kinetic term into S_1 and S_2, we can pursue the alternative splitting of the action. Namely we first solve the system with $S_1 + S_3$. As we discuss in the following sections, there is a basis which diagonalizes these two terms ($\mathcal{L}_0$ and $\star$-product) at the same time[3]. It gives a major simplification of the equation of motion and we can solve the nonlinear equation analytically at the classical level. The solutions are given in terms of the projection operators and should be regarded as defining nonperturbative vacua of open string field theory in the limit $S_2 \to 0$. This is in some sense similar to the VSFT proposal [3] although we expand the system from the different vacuum. The midpoint correction S_2 will be introduced as the "perturbation" to the exact solutions of $S_1 + S_3$.

At the level of the equation of motion, this strategy is equivalent to writing (2.3) as[4],

$$\mathcal{L}_0 \star A + A \star \mathcal{L}_0 + \alpha' g A \star A = -\epsilon\gamma A \ . \tag{2.19}$$

and treat the right hand side as a source term. We introduced formally an expansion parameter ϵ which will be used to describe the order of perturbation. We must put $\epsilon = 1$ at the end. We will first solve the equation in the absence of the source term, and later include the source for the complete solution.

At first glance, even without the source term, we are still left with a nonlinear differential equation of infinite order and the analytic study of such an equation of motion seems impossible. However, here the methods of noncommutative geometry come in handy for any operator $\mathcal{L}_0$. Thus, at the formal level, one may find solutions $A = A_P$ labeled by projectors P in the following form,

$$A_P = -\frac{2}{\alpha' g}\mathcal{L}_0 \star P. \tag{2.20}$$

Here P is any projector which satisfies the following properties,

$$P \star P = P \ , \qquad [P, \mathcal{L}_0] = 0 \ . \tag{2.21}$$

Once we have a solution A_P of the homogeneous equation, the corrections to it due to γ are taken into account by the following integral equation which is equivalent to an exact formal solution[5] of Eq.(2.19)

$$A = A_P - \int_0^\infty d\tau \ e_\star^{-\tau(\mathcal{L}_0 + \alpha' g A_P)} \star \left(\alpha' g \left(A - A_P\right)_\star^2 + \epsilon\gamma A\right) \star e_\star^{-\tau(\mathcal{L}_0 + \alpha' g A_P)}. \tag{2.24}$$

[3] There is another basis which diagonalize S_2 and S_3 at the same time and we may carry out the program which parallels our discussion in the following (see appendix B). However, there is no basis which diagonalized all three terms S_i ($i = 1, 2, 3$). It gives the essential difficulty to obtain the tachyon vacuum in the analytic form.

[4] We omit the $\bar{x}$ derivative (or β_0) since our main focus in this paper is the study of the translational invariant solutions.

[5] To verify this, consider the star anticommutator of both sides of the integral equation with the quantity $(\mathcal{L}_0 + \alpha' g A_P)$. The left side is $\{(\mathcal{L}_0 + \alpha' g A_P), A\}_\star$ while the right side, in addition to

Since A appears on both sides, one approach to obtaining an explicit solution for A is by recursion. As will be discussed below, this amounts to a perturbative expansion of A in powers of ϵ,

$$A = A^{(0)} + \epsilon A^{(1)} + \epsilon^2 A^{(2)} + \cdots \ , \tag{2.25}$$

with the lowest order being $A^{(0)} = A_P$. The full perturbative series is given later in section IV. This analysis could in fact be pursued for any $\mathcal{L}_0$.

A natural question is (1) does such projector P indeed exist, (2) can all such projectors be written explicitly for given $\mathcal{L}_0$, (3) does this exhaust all the solutions of Eq.(2.19) with $\epsilon = 0$? The answer is formally yes to all three questions, as follows.

As in the situation in the noncommutative soliton [4], the oscillator representation of the Moyal product gives an equivalent but more transparent means to analyze such a problem. In this language, we take $\mathcal{L}_0$ as a hamiltonian. The rank one projectors which commute with $\mathcal{L}_0$ can be constructed schematically as outer products $P_\lambda = |\lambda\rangle\langle\lambda|$ of the normalized eigenstates of the hamiltonian $\mathcal{L}_0|\lambda\rangle = \lambda|\lambda\rangle$, with $\langle\lambda|\lambda'\rangle = \delta_{\lambda\lambda'}$. Finding solutions of the form (2.20) reduces to finding eigenstates of $\mathcal{L}_0$. But this is an easy task for our $\mathcal{L}_0$ since it is the hamiltonian of a collection of harmonic oscillators[6]. A careful treatment along this scenario is given in the next section.

After we find an analytic form for $A^{(0)}$, we use Eq.(2.19) recursively to determine the expansion of the analytic solution of full equation of motion,

$$\left\{\mathcal{L}_0{}', A^{(k)}\right\}_\star \ = \ -\gamma A^{(k-1)} - \alpha' g \sum_{i=1}^{k-1} A^{(i)} \star A^{(k-i)} \ , \tag{2.26}$$

$$\mathcal{L}_0{}' \ \equiv \ \mathcal{L}_0 + \alpha' g A^{(0)} \ . \tag{2.27}$$

This is, of course, equivalent to the iterative solution of the integral equation in Eq.(2.24),

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$\left\{\left(\mathcal{L}_0 + \alpha' g A_P\right), A_P\right\}_\star$, produces a total τ derivative under the integral sign

$$\int_0^\infty d\tau \, \frac{\partial}{\partial\tau}\left[e_\star^{-\tau\left(\mathcal{L}_0+\alpha' g A_P\right)} \star \left(\alpha' g \left(A - A_P\right)_\star^2 + \epsilon\gamma A\right) \star e_\star^{-\tau\left(\mathcal{L}_0+\alpha' g A_P\right)}\right] . \tag{2.22}$$

Assuming a positive spectrum for the hamiltonian $\left(\mathcal{L}_0 + \alpha' g A_P\right)$, the integral contributes only at the boundary $\tau = 0$. We will return later to discuss the issue of the spectrum of $\left(\mathcal{L}_0 + \alpha' g A_P\right)$, for now we proceed formally. Inserting the result of the integral, the left and right sides of the equation yield

$$\left\{\left(\mathcal{L}_0 + \alpha' g A_P\right), A\right\}_\star = \left\{\left(\mathcal{L}_0 + \alpha' g A_P\right), A_P\right\}_\star - \left(\alpha' g \left(A - A_P\right)_\star^2 + \epsilon\gamma A\right) . \tag{2.23}$$

Rearranging this equation we obtain back our full equation of motion in Eq.(2.19) after using the fact that A_P is a solution of the homogeneous equation.

[6] In the Moyal language, the projectors we want as functions of the noncommutative space $P_\lambda(\xi)$ are known as the Wigner distributions for all the quantum states of the harmonic oscillator. These are well known in the literature in the case of a single harmonic oscillator [19].

but we will use a matrix formalism that is convenient in the case of oscillators. We will show that there is no obstruction to solving Eq.(2.26) order by order and one can determine $A^{(k)}$ uniquely for any starting point $A^{(0)}$. The explicit form of the first order correction and formal solution for any $A^{(k)}$ is given in section IV.

III. PHYSICS AT $\gamma = 0$

A. Splitting limit

The nature of the system at zeroth order in the γ term may be understood as follows. Let us begin by examining L_0 in the absence of the gamma term. The remaining part of L_0 has no information about the odd frequencies κ_o and the resulting spectrum of the modified L_0 corresponds to string oscillators that are both even and odd, but with the odd frequencies κ_o, instead of being arbitrary, replaced by the neighboring even frequency κ_e

$$\kappa_o \to \kappa_{o+1} = \kappa_e \ , \text{ for } \ o = 1, 3, 5, \cdots , \tag{3.1}$$

while κ_e is still arbitrary as is usual in regulated MSFT. This exactly characterizes the spectrum produced by the $\mathcal{L}_0$ part of L_0 in Eq.(2.19) in the absence of γ. Indeed, an inspection of $\mathcal{L}_0$ shows that it contains only the even frequencies κ_e. Given the fact that the star product is independent of the frequencies, the system $S_1 + S_3$ has no information on how to correct the frequencies of the odd oscillators from κ_{o+1} to κ_o.

Conversely if we insert Eq.(3.1) in the regulated MSFT system, a major simplification occurs in the defining matrices and vectors U, v, w. Namely U becomes the identity matrix and v, w vanishes. The vanishing of w_e immediately implies $S_2 = 0$, and the open string splits into independent half strings. We recall that the meaning of U, v, w is to give Bogoliubov transformation from variables with frequency κ_o to those with κ_e. If there is no difference between the sets of frequencies, there is no need to perform Bogoliubov transformation and the matrix which define it becomes trivial.

The MSFT formalism gives us the ability to consider this limit as well as the corrections. Intuitively one may think that a small change in the frequencies may not change the physics of the system drastically. Indeed, Eq.(3.1) is a small change in κ_o when o is sufficiently large. Hence a more interesting situation is to consider MSFT in the limit (3.1) for o larger than some number, $o > 2N$ while leaving both κ_e, κ_o arbitrary for $e, o \leq 2N$. In that case the trivialization of U, v, w applies only to the modes above $2N$, and γ gets contributions only from the modes up to $2N$. As long as $\gamma \neq 0$ the string does not split into two independent halves. So, it seems worth studying such limits, at least for the higher modes, since the formalism simplifies drastically while the physics (depending on the specific question) may be about the same. Furthermore, since we have complete control of the corrections, one can test both analytically and numerically the size of the correction. Such more complicated approximation scenarios are under study. But in the present paper we do not take any limits on the κ_e, κ_o; we simply study the system in powers of γ but for arbitrary κ_e, κ_o.

B. Oscillator representation

The simplification achieved in the splitting limit is elegantly rewritten in the oscillator notation[7]. We diagonalize the action of $\mathcal{L}_0$ and the star product at the same time in the matrix notation.

We introduce the creation and annihilation operators of the matter sector [7] and the ghost sector [8] [16] (in even mode variable)

$$\beta^\mu_e = \frac{1}{\sqrt{\kappa_e}}\left(-i\epsilon(e)\frac{\kappa_e}{2l_s}x^\mu_e + \frac{l_s}{\theta}p^\mu_e\right), \tag{3.2}$$

$$\beta^b_e = \left(-i\epsilon(e)\frac{\kappa_e}{2}x^b_e + \frac{1}{\theta'}p^c_e\right), \quad \beta^c_e = \left(\frac{1}{2}x^c_e - \epsilon(e)\frac{i}{\theta'\kappa_e}p^b_e\right) \tag{3.3}$$

which satisfy the canonical commutation relations with respect to $\star$ product,

$$[\beta^\mu_e, \beta^\nu_{e'}]_\star = \eta^{\mu\nu}\epsilon(e)\delta_{e+e'}, \qquad \left\{\beta^b_e, \beta^c_{e'}\right\}_\star = \delta_{e+e'}. \tag{3.4}$$

In terms of these oscillators, one can rewrite $\mathcal{L}_0$ as

$$\mathcal{L}_0 = \sum_{e>0}\kappa_e\left(\beta^\mu_{-e}\star\beta^\nu_e\eta_{\mu\nu} + \beta^b_{-e}\star\beta^c_e + \beta^c_{-e}\star\beta^b_e\right) - \nu, \quad \nu = \frac{1}{2} - \frac{d-2}{4}\left(\sum_{e>0}\kappa_e - \sum_{o>0}\kappa_o\right), \tag{3.5}$$

We note that both the star product (3.4) and the kinetic term (3.5) are diagonal with respect to the basis (3.2–3.3). It makes the splitting limit completely solvable.

We introduce the (nonperturbative) vacuum state through the relations,

$$\beta^\mu_e\star A_0 = \beta^b_e\star A_0 = \beta^c_e\star A_0 = A_0\star\beta^\mu_{-e} = A_0\star\beta^b_{-e} = A_0\star\beta^c_{-e} = 0\ . \tag{3.6}$$

The state that solves them becomes

$$A_0 \sim \exp\left(-\sum_e\left(\frac{\kappa_e}{2l_s^2}(x_e)^2 + \frac{2l_s^2}{\theta^2\kappa_e}(p_e)^2 + i\kappa_e x^b_e x^c_e + \frac{4i}{\theta'^2\kappa_e}p^b_e p^c_e\right)\right). \tag{3.7}$$

[7] We note that a somewhat similar representation was considered in [20] in the expansion around the sliver solution by neglecting the midpoint correction.

This is called "butterfly state" [8] in the literature [3] and satisfies the projector condition,

$$A_0 \star A_0 = A_0 \,, \quad A_0^* = A_0 \ . \tag{3.8}$$

This is the critical simplification by neglecting the mixing term. As we see in the following, the states generated by multiplying the creation operators from the left or the annihilation operators from the right diagonalize both $\mathcal{L}_0$ and the star product.

To illustrate our idea we use a simplified situation with only one pair oscillators a and $a^\dagger$ satisfying $[a, a^\dagger] = 1$. The orthonormal basis is given explicitly as,

$$\phi_{nm}(x,p) = \frac{1}{\sqrt{n!m!}}(a^\dagger)^n_\star \star A_0 \star (a)^n_\star \,, \quad \phi_{nm} \star \phi_{rs} = \delta_{mr}\phi_{ns} \ , \tag{3.9}$$

$$\hat{N} \star \phi_{nm} = n\phi_{nm} \,, \quad \phi_{nm} \star \hat{N} = m\phi_{nm} \,, \quad \hat{N} = a^\dagger a \ . \tag{3.10}$$

The orthogonality with respect to $\star$ product is essential in the following. In the above case, it can be proved by the commutation relation and the condition for A_0 (3.7). ϕ_{nn} becomes mutually orthogonal projectors,

$$\phi_{nn} \star \phi_{mm} = \delta_{nm}\phi_{mm} \ . \tag{3.11}$$

The multi-oscillator extension of above basis is given simply by direct product. We introduce the multi-index symbol

$$\mathbf{n} = \left\{ n_e^i | n_e^i \geq 0, e = 2, 4, \cdots, 2N, \ i = 0, \cdots, d; \ n_e^i = 0, 1 \text{ for } \ i = b, c \right\} \ , \tag{3.12}$$

and introduce the states $\phi_{\mathbf{nm}} = \prod_e \prod_i \phi^{\{i,e\}}_{n_e^i, m_e^i}$. We denote the set of multi-indices $\{\mathbf{n}\} = \mathbf{Z}^{\otimes Nd} \otimes \mathbf{Z}_2^{\otimes 2N}$ as $\mathcal{B}$. The basis satisfies

$$\mathcal{L}_0 \star \phi_{\mathbf{nm}} = \lambda_{\mathbf{n}}\phi_{\mathbf{nm}} \,, \quad \phi_{\mathbf{nm}} \star \mathcal{L}_0 = \lambda_{\mathbf{m}}\phi_{\mathbf{nm}} \,, \quad \lambda_{\mathbf{n}} \equiv \left(\sum_i \sum_{e>0} \kappa_e n_e^i \right) - \nu \ , \tag{3.13}$$

$$\phi_{\mathbf{nm}} \star \phi_{\mathbf{rs}} = \delta_{\mathbf{mr}}\phi_{\mathbf{ns}} \ . \tag{3.14}$$

We expand $A = \sum_{\mathbf{nm}} a_{\mathbf{nm}}(\bar{x})\phi_{\mathbf{mn}}$ and put it in the Lagrangian, we obtain $S_1 + S_3 = S^{matrix} + \delta S$ with

$$S^{matrix} = \int d^d\bar{x} \,\mathrm{Tr} \left(\frac{1}{2}\partial_{\bar{x}} a(\bar{x}) \cdot \partial_{\bar{x}} a(\bar{x}) + \frac{1}{\alpha'}\Lambda \cdot a \cdot a + \frac{g}{3} a \cdot a \cdot a \right) \ , \tag{3.15}$$

$$\delta S = \frac{\bar{w}w}{2} \int d^d\bar{x} \,\mathrm{Tr}\, \partial_{\bar{x}} a \cdot \partial_{\bar{x}} a + \frac{2i}{\theta} \int d^d\bar{x} \,\mathrm{Tr}\, a \cdot (\sum_e w_e P_e) \cdot \partial_{\bar{x}} a \ . \tag{3.16}$$

[8] In fact, we can show that this A_0 corresponds to (twisted) butterfly state: $e^{-\frac{1}{2}(L^m_{-2} + L'^g_{-2})} c_1|0\rangle$ in the limit $\kappa_e = e, \kappa_o = o, N = \infty$ [16].

where $\Lambda_{\mathbf{nm}} = \lambda_{\mathbf{n}}\delta_{\mathbf{nm}}$, the trace Tr is over the $\mathbf{n}$ indices and $\cdot$ is the matrix product. P_e is the matrix that corresponds to the star multiplication of $p_e = \frac{\sqrt{\kappa_e}\theta}{2l_s}(\beta_e + \beta_e^\dagger)$. δS becomes off-diagonal but does not affect the equation of motion with the translational invariance. In the splitting limit (3.1), δS vanishes because $w = 0$. S^{matrix} has a structure which is very similar to $c = d$ matrix model except that the kinetic term contains a mass term Λ which is not proportional to the identity matrix. It describes a characteristic feature of string theory that the color index $\mathbf{n}$ has a certain mass $\lambda_{\mathbf{n}}$.

C. Translational invariant solutions

In this section, we construct the translational invariant (=independent of $\bar{x}$) solutions in the splitting limit. It is quite interesting that the equation of motion becomes completely solvable and we can give an explicit form of the arbitrary solutions. Each solution describes a nonperturbative vacuum of string field theory while it may be stable or unstable. We also derive the open string spectrum around each vacuum explicitly while it becomes rather trivial.

The equation of motion obtained from the action (3.15,3.16) is

$$(\lambda_{\mathbf{n}} + \lambda_{\mathbf{m}})a_{\mathbf{nm}} + \alpha' g \sum_{\mathbf{k}} a_{\mathbf{nk}} a_{\mathbf{km}} = 0 \ . \tag{3.17}$$

This equation has the following significant property which we call "reducibility". Namely for any (finite) subset of $\mathcal{B}' = \{\mathbf{k}_1, \cdots, \mathbf{k}_n\} \in \mathcal{B}$, one can consistently restrict the equation of motion by replacing a to its rank n sub-matrix $a_{\mathbf{kl}}$ with $\mathbf{k}, \mathbf{l} \in \mathcal{B}'$. In other words, one can consistently put $A_{\mathbf{nm}} = 0$ if $\mathbf{n}$ or $\mathbf{m}$ do not belong to $\mathcal{B}'$ without any conflict with the equation of motion. In short, the equation of motion can be truncated to the diagonal finite dimensional sub-matrix of A.

For the simplest case $n = 1$, the equation of motion reduces to a scalar relation $2\lambda_{\mathbf{n}} a_{\mathbf{nn}} + \alpha' g (a_{\mathbf{nn}})^2 = 0$ for $\mathbf{n} \in \mathcal{B}'$. A nonvanishing solution is given by $A = -\frac{2}{\alpha' g}\lambda_{\mathbf{n}}\phi_{\mathbf{nn}}$ by using rank one projector $\phi_{\mathbf{nn}}$. More general diagonal solutions can be written by superposing mutually orthogonal projectors in the subset $\mathcal{B}'$,

$$A_{\mathcal{B}'} = -\frac{2}{\alpha' g} \sum_{\mathbf{n} \in \mathcal{B}'} \lambda_{\mathbf{n}} \phi_{\mathbf{nn}} = -\frac{2}{\alpha' g} \mathcal{L}_0 \star P_{\mathcal{B}'} \ , \tag{3.18}$$

$$P_{\mathcal{B}'} = \sum_{\mathbf{n} \in \mathcal{B}'} \phi_{\mathbf{nn}} \ , \quad P_{\mathcal{B}'} \star P_{\mathcal{B}'} = P_{\mathcal{B}'} \ . \tag{3.19}$$

We note that there are infinite number of exact and analytic solutions for the different choices of the subset $\mathcal{B}'$. Solution of the form (3.18) will be referred to the diagonal solutions.

Actually the solution is not restricted to the diagonal ones. To see it, one write the equation of motion in the matrix form,

$$\Lambda \cdot a + a \cdot \Lambda + g\alpha' a \cdot a = 0\,. \tag{3.20}$$

By shifting $a = a' - \Lambda/(\alpha' g)$, the equation of motion becomes

$$(a')^2 = \Lambda^2/(\alpha' g)^2\;. \tag{3.21}$$

We take the subset $\mathcal{B}'$ in such a way that for every $\mathbf{n} \in \mathcal{B}'$, $\lambda_{\mathbf{n}}^2$ is the same, namely the degenerate eigenspace in the right hand side of the above equation. The equation of motion restricted to the subset $\mathcal{B}'$ becomes $(a')^2 = (\lambda^2/(\alpha' g)^2)\, I$ where I is the identity matrix and λ is the degenerate eigenvalue. This equation obviously has off-diagonal solutions. As an example, we pick $n = 2$. A family of matrices which satisfy this relation is $A = \sum_{i=1,2,3} q_i \sigma_i$ with $\sum_i q_i^2 = \lambda^2/(\alpha' g)^2$ (σ_i are the Pauli matrices).

The solvability of the classical equation of motion (3.15) implies there exist a similar solvability even at the quantum level if we ignore the $\bar{x}$ dependence. We give a short comment on the analogy with two matrix model in Appendix C.

We would like to interpret each solution of string field theory as a new (unstable) D-brane which is related to the original D25 brane[9]. In order to make this statement more explicit, we expand the action around the solution,

$$S[A_{\mathcal{B}'} + A'] = \frac{4V}{3\alpha'^3 g^2} \mathrm{Tr}\left(\mathcal{L}_0{}^3_\star \star P_{\mathcal{B}'}\right) \tag{3.22}$$

$$+ \int d^d\bar{x}\, \mathrm{Tr}\left(\frac{1}{2}\partial_{\bar{x}} A' \star \partial_{\bar{x}} A' + \frac{1}{\alpha'}\mathcal{L}_0 \star (1 - 2P_{\mathcal{B}'}) \star A' \star A' + \frac{g}{3} A' \star A' \star A'\right) \tag{3.23}$$

The first term (where V is the volume of space-time) gives the tension of the (un)stable D-brane

$$T_{\mathcal{B}'} = \frac{4}{3\alpha'^3 g^2} \sum_{\mathbf{n}\in\mathcal{B}'} \lambda_{\mathbf{n}}^3\;. \tag{3.24}$$

The second term shows $\mathcal{L}_0$ is replaced by a new $\mathcal{L}_0{}'$ on the (un)stable D-brane,

$$\mathcal{L}_0 = \sum_{\mathbf{n}\in\mathcal{B}} \lambda_{\mathbf{n}} \phi_{\mathbf{nn}} \to \mathcal{L}_0{}' \equiv \mathcal{L}_0 \star (1 - 2P_{\mathcal{B}'}) = \sum_{\mathbf{n}\in\mathcal{B}-\mathcal{B}'} \lambda_{\mathbf{n}} \phi_{\mathbf{nn}} - \sum_{\mathbf{n}\in\mathcal{B}'} \lambda_{\mathbf{n}} \phi_{\mathbf{nn}}\,. \tag{3.25}$$

We note that the mass squared of the matrix component $A_{\mathbf{nm}}$ is given by the sum of the contribution from the half strings $\lambda_{\mathbf{n}} + \lambda_{\mathbf{m}}$. The above argument shows that the contribution changes its sign when the label $\mathbf{n}$ is included in the set $\mathcal{B}'$.

[9] It is not obvious if the open string at zeroth order in γ (splitting limit) is related to D-branes. However, we use this terminology in a generalized "background" where the open string has the frequencies (κ_e, κ_o), which become (κ_e, κ_e) when γ is neglected.

D. Tachyon vacuum

Suppose we start from the theory

$$\lambda_{\mathbf{n}} < 0 \quad \text{if and only if} \quad \mathbf{n} \in \mathcal{B}_0 \tag{3.26}$$

for some subset $\mathcal{B}_0 \subset \mathcal{B}$. In such theory, (at least) the matrix components $A_{\mathbf{nn}}$ ($\mathbf{n} \in \mathcal{B}_0$) become the tachyonic modes. Our arguments in this section clearly show that if we use the solution of motion $A_{\mathcal{B}_0}$ and re-expand around that solution, all the negative contributions from the half string changes sign and the tachyonic modes disappear. This is precisely the definition of the tachyonic vacuum.

In the splitting limit, for a specific parameter choice $\kappa_e = e$ we have $\nu = 1/2$. There is only one tachyon $\mathcal{B}_0 = \{\mathbf{0}\}$ and the tachyon vacuum becomes,

$$A = \frac{1}{\alpha' g}\phi_{\mathbf{00}} \ . \tag{3.27}$$

This is the butterfly state in our notation. We note that we obtain the butterfly state as the approximate solution (namely by neglecting S_2) and this is not the exact solution for the full system $S_1 + S_2 + S_3$.

The action around this vacuum takes the following form (after the shift of the vacuum energy),

$$S[A] = \mathrm{Tr}\left(\frac{1}{2}\partial_{\bar{x}} A \star \partial_{\bar{x}} A + \frac{1}{\alpha'}\mathcal{L}_{vac} \star A \star A + \frac{g}{3} A \star A \star A\right) , \quad \mathcal{L}_{vac} = \sum_{\mathbf{n}} |\lambda_{\mathbf{n}}| \phi_{\mathbf{nn}} , \tag{3.28}$$

with all eigenvalues $|\lambda_{\mathbf{n}}|$ positive. This is the action for the "vacuum string field theory" in the splitting limit.

Our description of the solutions at zeroth order in γ shares many properties in common with the conventional VSFT proposal. One of the most outstanding characterizations is the rôle of the projector for describing the exact solutions of the classical equation of motion. On the other hand, there are a few points which are different from the VSFT proposal.

The first point is the form of the solutions. They contain the action of Virasoro operator $-2\mathcal{L}_0 \star P$ instead of the simple projector itself as in VSFT proposal. In a sense, our solution is closer to the solution $\Psi = Q_L I$ proposed in the purely cubic theory [21] (after the replacement of the identity by the projector). It is due to the fact that the kinetic term always remains in the expansion around any exact solution.

A second point is the nature of the tachyon vacuum. As we have seen it is characterized only by the absence of tachyonic modes in the spectrum and the open string propagation seems to survive. Namely, the cohomology defined by the quadratic term at the tachyonic vacuum does not appear to be trivial. In the usual proposal, the tachyon vacuum is where there is no open string propagation since it is the point where D-branes annihilate.

It is, of course, not very clear to which extent we should take such "discrepancies" seriously. In the Siegel gauge there are an infinite number of subsidiary conditions that must be applied on our solutions. We have not implemented yet these conditions. It is likely that the ground state of the potential energy already satisfies these conditions, but only a subset or none of the remaining extremal states would.

A better approach to investigate this issue may be to construct the full BRST operator in the Moyal formalism. This appears possible at $N = \infty$, but with an infinite number of modes the issue of the midpoint is plagued with anomalies and it is difficult to be confident that we have complete control of the anomalies by working directly at $N = \infty$. On the other hand, at finite N we have not figured out a substitute for the Virasoro algebra that would be needed to construct the BRST operator. At this point it appears quite likely that, like L_0, the full BRST operator Q_B (a differential operator) also has a representation similar to Eq.(2.19), namely

$$Q_B A = \mathcal{Q} \star A + A \star \mathcal{Q} + qA. \tag{3.29}$$

We hope to report on this aspect in a future publication. Armed with such a star product representation of Q_B we can give a similar analysis to what we have presented in this paper, and then we can answer the issues of the cohomology at the tachyonic vacuum.

It is interesting to point out the following observation in relation to closed strings. The spectrum at zeroth order in γ (split string with $\kappa_o = \kappa_e$) conceptually is close to the *closed string* spectrum, especially if we consider that each half string imitates the independent modes from the left and right movers on a closed string[10]. This begins to give a clue on how the graviton can be described as part of open string field theory.

IV. INCLUSION OF MIDPOINT CORRECTION

In Witten's string field theory, the solution which describes the tachyonic vacuum is one of the most important goal. In our language, it corresponds to solving the equation of motion (2.3) without assuming $\gamma = 0$. Since we have already solved the equation of motion analytically in $\gamma = 0$ limit, it is sensible to introduce the effect of γ as perturbation. For this purpose, we replace γ by $\epsilon\gamma$ with an expansion parameter ϵ. We expand A as

$$A = A^{(0)} + \epsilon A^{(1)} + \epsilon^2 A^{(2)} + \cdots , \tag{4.1}$$

[10] A related remark was made in [22].

and use $A^{(0)}$ as the solution at $\epsilon = 0$. When the spectral asymmetry between κ_e and κ_o is very small, we would obtain the converging series which describe the exact solution.

We note that there is a formal analogy between our case and the analysis of the noncommutative soliton in the scalar field theory where the equation of motion becomes

$$[a, [a^\dagger, \phi]] + \theta V(\phi)_\star = 0 \ . \tag{4.2}$$

In the usual scenario, we first solve the second part by assuming the parameter θ, which is a measure of noncommutativity, is very large. The solution is given by the noncommutative soliton as $\phi = t\phi_0$ where $\phi_0 \star \phi_0 = \phi_0$ and $V'(t) = 0$. The first term is later included as the perturbation to such solutions. Then one finds that while some solutions are stable an instability emerges for some of the solutions [13]. Here we try to investigate our system from a similar point of view. In our case, a similar rôle is played by the spectral parameters κ_n. We can make the γ term very small by choosing them very close to the splitting limit.

We start from the rank 1 solution characterized by some harmonic oscillator state labeled by $\mathbf{n}_0$

$$A^{(0)} = -\frac{2}{\alpha' g}\lambda_{\mathbf{n}_0}\phi_{\mathbf{n}_0\mathbf{n}_0} \ . \tag{4.3}$$

We put Eq.(4.1) into the equation of motion and pick up $O(\epsilon^k)$ coefficients. In the first order $(k = 1)$ we obtain,

$$\mathcal{L}_0{}' \star A^{(1)} + A^{(1)} \star \mathcal{L}_0{}' = -\gamma A^{(1)} \equiv B^{(1)} \ , \tag{4.4}$$

$$\mathcal{L}_0{}' \equiv \mathcal{L}_0 + \alpha' g A^{(0)} \equiv \sum_{\mathbf{k}} \lambda'_{\mathbf{k}}\phi_{\mathbf{kk}} = \sum_{\mathbf{k}} \lambda_{\mathbf{k}}(1 - 2\delta_{\mathbf{n}_0\mathbf{k}})\phi_{\mathbf{kk}} \ . \tag{4.5}$$

The eigenvalues of the shifted $\mathcal{L}_0{}'$ are exactly the same as the modified spectrum of the half string on the unstable D-brane which corresponds to $A^{(0)}$ as in the previous sections. If we expand $A^{(1)} = \sum_{\mathbf{nm}} a^{(1)}_{\mathbf{nm}}\phi_{\mathbf{mn}}$ and $-\gamma A^{(0)} = \sum_{\mathbf{nm}} b^{(1)}_{\mathbf{nm}}\phi_{\mathbf{mn}}$, the solution to the perturbation expansion becomes,

$$(\lambda'_{\mathbf{n}} + \lambda'_{\mathbf{m}})a^{(1)}_{\mathbf{nm}} = b^{(1)}_{\mathbf{nm}} \ . \tag{4.6}$$

This equation has a unique solution as long as $(\lambda'_{\mathbf{n}} + \lambda'_{\mathbf{m}}) \neq 0$. We note that $(\lambda'_{\mathbf{n}} + \lambda'_{\mathbf{m}})$ gives the mass squared of the open string on the (unstable) D-brane. The recursion relation breaks if there exist massless excitations. Such modes exist if (i)there exists $\mathbf{m}(\neq \mathbf{n}_0)$ such that $\lambda_{\mathbf{m}} = \lambda_{\mathbf{n}_0}$ (we suppose $\lambda_{\mathbf{n}_0} \neq 0$). or (ii) $\lambda'_{\mathbf{n}} = 0$ for some $\mathbf{n}$. In such situations, we need to impose $b^{(1)}_{\mathbf{nm}} = b^{(1)}_{\mathbf{mn}} = 0$ in order to have a perturbation expansion with a nontrivial solution. However, this type of constraint becomes rather nontrivial when we need to solve the higher order equation.

For the situation (i), we believe that the rank 1 solutions become singular in the perturbation series. One resolution for the degenerate case is to consider the higher rank projector and only begin with the solution of the form $A^{(0)} = -\frac{2\lambda}{\alpha' g}\sum_{i\in\Lambda}\phi_{ii}$ where Λ is the set of indices with $\lambda_i = \lambda$. Starting from this solution, it is not possible to have $\lambda'_{\mathbf{n}} + \lambda'_{\mathbf{m}} = 0$, and the recursion formula becomes consistent and have a unique solution. For the situation (ii), there does not seem to exist such a cure. One possibility is, however, to shift the splitting of L_0 into $\mathcal{L}_0$ and γ slightly, $\mathcal{L}_0 \to \mathcal{L}_0 + b$, $\gamma \to \gamma - 2b$. This shifts the eigenvalues of $\mathcal{L}_0$ by a constant and escape the singularity mentioned above[11].

As an explicit example, we present the first order correction if we take the butterfly state as the zeroth term $A_0 = \frac{2}{\alpha' g}\nu\phi_{\mathbf{00}}$. We use the oscillator representation of the action of gamma terms to the string fields,

$$
\begin{aligned}
\gamma^{matter} A &= \frac{-1}{8(1+\bar{w}w)}\sum_{e,e'>0}\sqrt{\kappa_e}w_e\sqrt{\kappa_{e'}}w_{e'}\left\{\beta^\mu_e + \beta^{\mu\dagger}_e, \left\{\beta^\nu_{e'} + \beta^{\nu\dagger}_{e'}, A\right\}_\star\right\}_\star \eta_{\mu\nu}\,, && (4.9)\\
\gamma^{gh} A &= \frac{1}{4(1+\bar{w}w)}\sum_{e,e'>0} w_e\kappa_e w_{e'}\left\{\beta^c_e - \beta^{c\dagger}_e, \left[\beta^b_{e'} + \beta^{b\dagger}_{e'}, A\right]_\star\right\}_\star\,. && (4.10)
\end{aligned}
$$

The first order correction is given as,

$$
\begin{aligned}
\alpha' g A_1 &= \frac{(d-2)\delta}{4}\phi_{\mathbf{00}} + \frac{\nu}{4(1+\bar{w}w)}\sum_{e,e'>0}\left(\frac{w_e w_{e'}}{\kappa_e+\kappa_{e'}}D^1_{ee'} + \frac{2w_e w_{e'}}{\kappa_e+\kappa_{e'}-2\nu}D^2_{ee'}\right)\,, && (4.11)\\
\delta &\equiv \frac{\sum_{e>0}\kappa_e w_e^2}{1+\bar{w}w}\,, && (4.12)\\
D^1_{ee'} &= \eta_{\mu\nu}\sqrt{\kappa_e\kappa_{e'}}\left(\beta^{\mu\dagger}_e\star\beta^{\nu\dagger}_{e'}\star\phi_{\mathbf{00}} + \phi_{\mathbf{00}}\star\beta^\nu_{e'}\star\beta^\mu_e\right)
\end{aligned}
$$

[11] We have to mention that such a dangerous situation seems to appear at least naively. In the open string limit (A2), the vacuum energy ν defined in (3.5) becomes divergent. If we use the zeta function regularization to obtain a finite value for ν, we need to use,

$$\sum_{e>0} e - \sum_{o>0} o = 2(\zeta(-1) - \zeta(-1,1/2)) = 2\left(-\frac{1}{24}-\frac{1}{12}\right) = -\frac{1}{4}\,. \qquad (4.7)$$

For the critical dimension $d = 26$, it makes $\nu = 2$. With such a choice for ν, there exists a "graviton-like" excitation $\beta^{\mu\dagger}_2\star\phi_{\mathbf{00}}\star\beta^\nu_2$ which becomes exactly massless. Furthermore one can easily check the right-hand side of (4.4) is also nonvanishing (dilaton-like excitation),

$$B_1 \propto \frac{1}{1+\bar{w}w}(w_2)^2\eta_{\nu\mu}\beta^{\nu\dagger}_2\star\phi_{\mathbf{00}}\star\beta^\mu_2 + \cdots\,. \qquad (4.8)$$

The situation is, however, very delicate. If we take the naive open string limit $\bar{w}w \to \infty$, this term also vanishes. This is the usual problem of taking the naive limit. The proposal in MSFT [6][7][8] is to use the finite N regularization in all the intermediate computation and take the large N limit only at the end of the calculation. The divergence which we encounter is caused by the use of the zeta-function regularization (4.7) at the intermediate step of the computation which becomes quite dangerous. The correct prescription will be to take ν unfixed and solve the recursion and only take the limit (4.7) after we sum over all the perturbation expansion.

$$+2\kappa_e \left(\beta_e^{c\dagger} \star \beta_{e'}^{b\dagger} \star \phi_{\mathbf{00}} + \phi_{\mathbf{00}} \star \beta_{e'}^{b} \star \beta_e^{c}\right) \ , \tag{4.13}$$

$$D_{ee'}^2 \ = \ \eta_{\mu\nu}\sqrt{\kappa_e\kappa_{e'}}\beta_e^{\mu\dagger} \star \phi_{\mathbf{00}} \star \beta_{e'}^{\nu} - \kappa_e \left(\beta_e^{c\dagger} \star \phi_{\mathbf{00}} \star \beta_{e'}^{b} + \beta_{e'}^{b\dagger} \star \phi_{\mathbf{00}} \star \beta_e^{c}\right) \ . \tag{4.14}$$

We note that the first order correction is small compared with the zeroth order term if $\bar{w}w << 1$, namely in the vicinity of the splitting limit (3.1). On the other hand, in the open string limit (A2), while the combination $w_e/\sqrt{1+\bar{w}w}$ becomes very small (which are the coefficients of $D_{1,2}$), $\bar{w}w$, δ and ν are naively divergent. In this sense, the applicability of the perturbation series in the open string limit seems to be quite subtle.

One possibility to overcome this difficulty is to use the ambiguity of the splitting of $\mathcal{L}_0$ and γ which is mentioned in section II B. With some careful choice, for example, it seems that one can remove the divergence in ν and δ which appear at the first order. The problem of higher corrections, however, is very delicate and we would like to postpone the careful treatment of these problems to a future publication.

Due to the correction to the tachyon vacuum, the formula for the brane tension should also be modified. We expand the action in the following form,

$$S[A^{(0)} + \epsilon A^{(1)} + \epsilon^2 A^{(2)} + \cdots] = S^{(0)} + \epsilon S^{(1)} + \epsilon^2 S^{(2)} + \cdots \ . \tag{4.15}$$

If we start the perturbation series around the solution $A^{(0)} = -\frac{2}{\alpha' g}\mathcal{L}_0 \star P$ (with $\mathcal{L}_0 \star P = P \star \mathcal{L}_0$, $P \star P = P$), the zeroth term is given in Eq.(3.22). The first order correction is given by

$$S^{(1)} = -\frac{2}{\alpha'^2 g} V \mathrm{Tr}\, \left(\mathcal{L}_0{}_\star^2 \star P \star A^{(1)}\right) \ . \tag{4.16}$$

In the perturbation around the butterfly state, we evaluate the tension as,[12]

$$T = \frac{1}{\alpha'^3 g^2}\left(-\frac{4}{3}\nu^3 - \epsilon\frac{\nu^2\delta}{2}(d-2)\right) + O(\epsilon^2) \ . \tag{4.17}$$

We can continue the perturbation expansion for higher k. The recursion formula is already given in (2.26). With the above redefinition of A_0 for the degenerate case, we can solve this equation term by term for any spectrum uniquely. The second order perturbation is, for example, given as,

$$A^{(2)} \ = \ (-L_0')^{-1}\left(\gamma A^{(1)}\right) + \alpha' g (-L_0')^{-1}(A^{(1)} \star A^{(1)}) \ , \tag{4.18}$$

$$\text{with} \quad A^{(1)} = (-L_0')^{-1}\left(\gamma A^{(0)}\right) \ . \tag{4.19}$$

where L_0' applied on any field is defined by $L_0' A \equiv \mathcal{L}_0{}' \star A + A \star \mathcal{L}_0{}'$, while $(L_0')^{-1}$ on any field is given by $(L_0')^{-1} A = \int_0^\infty d\tau \ e_\star^{-\tau\mathcal{L}_0{}'} \star A \star e_\star^{-\tau\mathcal{L}_0{}'}$, as explained in Eq.(2.24) and footnote (5).

[12] This will be compared to the ordinary D25-brane tension after a self consistent normalization of the action in MSFT[16].

More directly, both L_0' and $(L_0')^{-1}$ are simple algebraic expressions in the matrix notation of Eqs.(4.4-4.6).

Actually one may obtain a formal expression for A for the entire perturbation sum. For that purpose, we introduce the "dressed propagator",

$$(-L_0')^{-1} + (-L_0')^{-1}\epsilon\gamma(-L_0')^{-1} + (-L_0')^{-1}\epsilon\gamma(-L_0')^{-1}\epsilon\gamma(-L_0')^{-1}\cdots = (-L_0' - \epsilon\gamma)^{-1}, \quad (4.20)$$

and "dressed version" of $A^{(1)}$ and star product,

$$\tilde{A}^{(1)} \equiv (-L_0' - \epsilon\gamma)^{-1}\gamma A^{(0)}, \quad A \bullet B \equiv \alpha' g(-L_0' - \epsilon\gamma)^{-1}(A \star B) . \quad (4.21)$$

We note that the "bullet product $\bullet$" is not associative product. One may then claim that full wave function A can be expressed as,

$$\begin{aligned} A &= A^{(0)} + \epsilon\tilde{A}^{(1)} + \epsilon^2\tilde{A}^{(1)} \bullet \tilde{A}^{(1)} + \epsilon^3\left((\tilde{A}^{(1)} \bullet \tilde{A}^{(1)}) \bullet \tilde{A}^{(1)} + \tilde{A}^{(1)} \bullet (\tilde{A}^{(1)} \bullet \tilde{A}^{(1)})\right) + \cdots \\ &= A^{(0)} + \sum_{n=1}^{\infty}\epsilon^n\left(\text{(all possible associations of) } \underbrace{\tilde{A}^{(1)} \bullet \cdots \bullet \tilde{A}^{(1)}}_{n}\right) . \end{aligned} \quad (4.22)$$

For example, ϵ^4 term is given as,

$$\begin{aligned} &\tilde{A}^{(1)} \bullet (\tilde{A}^{(1)} \bullet (\tilde{A}^{(1)} \bullet \tilde{A}^{(1)})) + (\tilde{A}^{(1)} \bullet \tilde{A}^{(1)}) \bullet (\tilde{A}^{(1)} \bullet \tilde{A}^{(1)}) + \tilde{A}^{(1)} \bullet ((\tilde{A}^{(1)} \bullet \tilde{A}^{(1)}) \bullet \tilde{A}^{(1)}) \\ &+(\tilde{A}^{(1)} \bullet (\tilde{A}^{(1)} \bullet \tilde{A}^{(1)})) \bullet \tilde{A}^{(1)} + ((\tilde{A}^{(1)} \bullet \tilde{A}^{(1)}) \bullet \tilde{A}^{(1)}) \bullet \tilde{A}^{(1)}. \end{aligned}$$

A proof of the formula (4.22) is given by the use of the recursion formula (2.26). An easier proof is to write down the equation of motion for the deviation $A' \equiv A - A^{(0)} = \epsilon A^{(1)} + \epsilon^2 A^{(2)} + \cdots$,

$$-(L_0' + \epsilon\gamma)A' = \epsilon\gamma A^{(0)} + \alpha' g A' \star A' . \quad (4.23)$$

From above definitions, one may rewrite it as, $A' = \epsilon\tilde{A}^{(1)} + A' \bullet A'$. We use this relation recursively to obtain (4.22), for example,

$$A' = \epsilon\tilde{A}^{(1)} + (\epsilon\tilde{A}^{(1)} + A' \bullet A') \bullet (\epsilon\tilde{A}^{(1)} + A' \bullet A') = \cdots . \quad (4.24)$$

With this explicit formula, we claim that there exists a unique solution (4.22) to the full string field equation for each solution in the splitting limit as long as the perturbation expansion is convergent. We hope that the careful analysis in the vicinity of the splitting limit will reveal some nature of the dynamics in the open string limit.

We note that the solutions becomes the projector with respect to Witten's star product $\star$ only at the splitting limit and the perturbation breaks such simplicity.

V. CONCLUSION

We have seen in this paper that the splitting limit gives a system where the translational invariant solutions are solved analytically in terms of the projection operators. We argued that this is an analog of the large θ limit in the noncommutative scalar field theory. We can introduce the midpoint effect as a perturbation series which is analogous to the finite θ case. In the development of open string field theory, this gives the first example where the rôle of noncommutative solitons is explicitly demonstrated with a careful treatment of the midpoint correction. We believe that it gives a firm ground upon which the relation between noncommutative geometry and open string field theory will be discussed in the future.

There are, of course, many topics which should be clarified in a future study. One of the most interesting directions is to find the analytic solution for $\gamma \neq 0$ in a closed form. While this appears difficult because we cannot find a basis which diagonalizes S_i $(i = 1, 2, 3)$ simultaneously, there may be a possibility that a few of the exact solutions, in particular the true vacuum, could be derived with some insight.

In our description, the tachyon vacuum still seems to have an open string spectrum (modulo the extra gauge invariance conditions in the Siegel gauge). Elimination of these modes would be possible only when the exact solution is found in a closed form, and the remaining gauge invariance conditions are imposed.

A related issue is the BRST symmetry. So far with arbitrary choice of the spectrum κ_n as a function of n, we cannot define the nilpotent BRST operator. The merit of our approach is that one can handle the midpoint correction at finite N. While the BRST charge exists in the open string limit at $\kappa_n = n$, it is very challenging to see how the BRST operator will be affected in that limit by the midpoint correction we have emphasized.

Another essential question is whether the splitting limit itself can be interpreted as a "real string" defined by some kind of (B)CFT. As an example, we take $\kappa_e = e$ and $N = \infty$. The spectrum of the model is generated by two set of oscillators $a_e^\dagger$ (and a_e) action on the vacuum state from left (and right). In a sense each of two half string behaves exactly like the original open string. This is somewhat similar to the closed string excitation where two sets of oscillators are left and right moving modes. Since there is no $L_0 - \bar{L}_0 = 0$ constraint in the splitting limit, it is certainly different from the closed string. However this analogy may have some implication of the nature of the tachyon vacuum where we are supposed to have only the closed string excitations.

While it is more speculative, we may comment on the relation with D-branes. Usually in BCFT, D-branes are described by boundary states in the closed string Hilbert space. We note that the open string Hilbert space in the splitting limit has a similar structure as the closed string. The similarity may imply that the deformation of the open string Hilbert space is needed to describe the D-brane as a projector in the open string Hilbert space. This kind of comment might be helpful if we want to perform a similar analysis in a generic closed string background.

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Appendix A: Definitions in MSFT

We review some basic definitions in the Moyal star formulation of string field theory [5–8].

To provide a regulator in MSFT, we use an explicit truncation of the number of oscillators ($1 \leq |n| \leq 2N$) and introduce the $2N$ arbitrary "frequency" parameters κ_n ($n = 1, \cdots, 2N$). These appear in the commutation relations among the oscillators, such as

$$[\alpha_n^\mu, \alpha_{n'}^\nu] = \kappa'_n \delta_{n+n'} \eta^{\mu\nu} \ , \quad \kappa'_n \equiv \epsilon(n)\kappa_{|n|} \ . \tag{A1}$$

In the following we need to distinguish the frequency for even and odd labels, and write them as κ'_e, κ'_o, where e (resp. o) runs over even (resp. odd) numbers in the range of n.

In the definition of the canonical variables in Moyal space, there is a Bogoliubov transformation $U_{-e,o}$ from the oscillators labeled with odd numbers o to those labeled with even numbers e. In [6–8], U is related to a set of special matrices and vectors T, R, S, v', w'. These are all functions of the frequency parameters κ_e, κ_o. In the limit (which we will refer to as "the open string limit")

$$N \to \infty \ , \quad \kappa_o \to o \ , \quad \kappa_e \to e \ , \tag{A2}$$

the basic matrix U and vectors w'_e, v'_e (for both positive and negative integers e, o) are

$$U_{-e,o} \to \frac{2}{\pi} \frac{i^{o-e-1}}{o-e} \ , \quad w'_e \to i^{-e+2} \ , \quad v'_o \to \frac{2}{\pi} \frac{i^{o-1}}{o} \quad \text{and} \ \ \bar{w}' w' \to \infty. \tag{A3}$$

Compared to the notation w_e, v_o that we also use for positive integers, w' and v' are defined as

$$w'_e = w_{|e|}/\sqrt{2}, \ \ v'_o = v_{|o|}/\sqrt{2}. \tag{A4}$$

In the regulated version of MSFT with finite N these matrices are deformed as functions of arbitrary κ_e, κ_o as follows

$$U_{-e,o} = \frac{w'_e v'_o \kappa'_o}{\kappa'_e - \kappa'_o} \ , \quad U^{-1}_{-o,e} = \frac{w'_e v'_o \kappa'_e}{\kappa'_e - \kappa'_o} \ , \quad U U^{-1} = U^{-1} U = 1 \ , \tag{A5}$$

$$w_e = i^{2-e}\frac{\prod_{o'>0}|\kappa_e^2/\kappa_{o'}^2-1|^{\frac{1}{2}}}{\prod_{e'(\neq e)>0}|\kappa_e^2/\kappa_{e'}^2-1|^{\frac{1}{2}}}, \quad v_o = i^{o-1}\frac{\prod_{e'>0}|1-\kappa_o^2/\kappa_{e'}^2|^{\frac{1}{2}}}{\prod_{o'(\neq o)>0}|1-\kappa_o^2/\kappa_{o'}^2|^{\frac{1}{2}}}\ , \tag{A6}$$

$$T_{eo} = U_{-e,o} + U_{e,o}, \quad R_{oe} = U^{-1}_{-o,e} + U^{-1}_{o,e}, \quad S_{eo} = U_{-e,o} - U_{e,o} = U^{-1}_{-o,e} - U^{-1}_{o,e}\ , \tag{A7}$$

where a bar on a matrix means its transpose. Of course, these expressions reduce to their limiting values in Eq.(A3) in the large N limit. They satisfy the following relations for arbitrary κ_n (including the limit of Eq.(A2))

$$U^{-1} = {\kappa'_o}^{-1}\bar{U}\kappa'_e = \bar{U} + v'\bar{w}'\ , \quad U\bar{U} = 1 - \frac{w'\bar{w}'}{1+\bar{w}'w}\ , \quad v' = \bar{U}w' \tag{A8}$$

$$TR = RT = 1\ , \quad S\bar{S} = \bar{S}S = 1\ , \quad T = \kappa_e^{-1}S\kappa_o,\ R = \kappa_o^{-1}\bar{S}\kappa_e. \tag{A9}$$

In Eq.(A8) the first formula implies that U changes the spectrum from κ'_o to κ'_e, whereas the second one gives the origin of the midpoint correction. The v'_o and w'_e vectors are related to each other through the third relation. This is only a partial list of relations; for the complete set of relations see [7].

In most explicit computations it is much more efficient to use the relations among the matrices rather than their explicit messy expressions. We emphasize that these relations hold for any values of κ_n, including those of the limiting case in Eq.(A2). Therefore, performing analytic computations at finite N is often not any more difficult than performing them at infinite N.

In the Moyal language, the action of the oscillators α on the string field is represented by the action of β oscillators defined in Eq.(3.2). Suppose the oscillator state $|\psi\rangle$ corresponds to Moyal field A (in Siegel gauge),

$$\alpha_e^\mu|\psi\rangle \ \longleftrightarrow\ \hat{\beta}_e^\mu A \equiv \sqrt{\frac{\kappa_e}{2}}(\beta_e^\mu \star A - A\star\beta_{-e}^\mu) - w'_e\beta_0^\mu A\ , \tag{A10}$$

$$\alpha_o^\mu|\psi\rangle \ \longleftrightarrow\ \hat{\beta}_o^\mu A \equiv \sum_{e\neq 0}(\bar{\beta}_e^\mu A)U_{-e,o} = \sqrt{\frac{\kappa_e}{2}}(\beta_o^\mu\star A + A\star\beta_{-o}^\mu)\,, \tag{A11}$$

$$b_e|\psi\rangle \ \longleftrightarrow\ \hat{\beta}_e^b A \equiv \frac{1}{\sqrt{2}}\left(\beta_e^b\star A + (-1)^{|A|}\star\beta_{-e}^b\right)\ , \tag{A12}$$

$$b_o|\psi\rangle \ \longleftrightarrow\ \hat{\beta}_o^b A \equiv \sum_{e\neq 0}(\bar{\beta}_e^b A)U_{-e,o} \equiv \frac{1}{\sqrt{2}}\left(\beta_o^b\star A - (-1)^{|A|}A\star\beta_{-o}^b\right)\ , \tag{A13}$$

$$c_e|\psi\rangle \ \longleftrightarrow\ \hat{\beta}_e^c A \equiv \frac{1}{\sqrt{2}}\left(\beta_e^c\star A - (-1)^{|A|}A\star\beta_{-e}^c\right)\ , \tag{A14}$$

$$c_o|\psi\rangle \ \longleftrightarrow\ \hat{\beta}_o^c A \equiv \sum_{e\neq 0}(U^{-1}_{o,-e}\bar{\beta}_e^c)A \equiv \frac{1}{\sqrt{2}}\left(\beta_o^c\star A + (-1)^{|A|}\star\beta_{-o}^c\right) \tag{A15}$$

where β_o is Bogoliubov transformation of β_e, namely

$$\beta_o^\mu = \sum_{e\neq 0}\beta_e^\mu U_{-e,o}\ , \quad \beta_o^b = \sum_{e\neq 0}\beta_e^b U_{-e,o}\ , \quad \beta_o^c = \sum_{e\neq 0}U^{-1}_{o,-e}\beta_e^c\ . \tag{A16}$$

Note that $\hat{\beta}^{\mu,b,c}_{e,o}$, $\bar{\beta}^{\mu,b,c}_{e,o}$ are differential operators, but $\beta_{e,o}$ are fields multiplied with the Moyal star. We can prove that the even differential operators $\hat{\beta}_e$, $\bar{\beta}_e$ satisfy the standard (anti-) commutation relation for the even mode oscillators,

$$\left[\hat{\beta}^{\mu}_{e}, \hat{\beta}^{\nu}_{e'}\right] = \left[\bar{\beta}^{\mu}_{e}, \bar{\beta}^{\nu}_{e'}\right] = \eta^{\mu\nu}\kappa'_{e}\delta_{e+e'}\ , \quad \left\{\hat{\beta}^{b}_{e}, \hat{\beta}^{c}_{e'}\right\} = \left\{\bar{\beta}^{b}_{e}, \bar{\beta}^{c}_{e'}\right\} = \delta_{e+e'}\ . \tag{A17}$$

On the other hand, after Bogoliubov transformation, $\hat{\beta}^{p}_{o}$ satisfies the odd mode commutation relation,

$$\left[\hat{\beta}^{\mu}_{o}, \hat{\beta}^{\nu}_{o'}\right] = \kappa'_{o}\delta_{o+o'}\ , \quad \left\{\hat{\beta}^{b}_{o}, \hat{\beta}^{c}_{o'}\right\} = \delta_{o+o'}\ . \tag{A18}$$

For arbitrary frequencies, one can define the perturbative string states. In particular, the perturbative vacuum in the oscillator language is mapped to the gaussian function [7, 16],

$$A_0 \sim \exp\left(-\bar{\xi}^{\mu}M_0\xi_{\mu} - 2\bar{\xi}^{b}M_0^{gh}\xi^{c}\right), \quad \bar{\xi}^{b} = (\bar{x}^{b}_{e}\ \bar{p}^{b}_{e}), \quad \bar{\xi}^{c}_{2} = (\bar{x}^{c}_{e}\ \bar{p}^{c}_{e})\ , \tag{A19}$$

$$M_0 = \begin{pmatrix} \frac{\kappa_e}{2l_s^2} & 0 \\ 0 & \frac{2l_s^2}{\theta^2}T\kappa_o^{-1}\bar{T} \end{pmatrix}, \quad M_0^{gh} = \begin{pmatrix} \frac{i}{2}\bar{R}\kappa_o R & 0 \\ 0 & \frac{2i}{\theta'^2}\kappa_e^{-1} \end{pmatrix}\ . \tag{A20}$$

Similarly one can construct the Moyal map of coherent states which correspond to adding a linear term in the exponent of the gaussian above. With this setup, the Moyal star is used to compute generalizations of Neumann coefficients. It is shown [7] that they satisfy the basic nonlinear relations given by Gross and Jevicki even for arbitrary frequencies. The generalized Neumann matrices $\left(V^{[rs]}_{n}\right)_{kl}$, $\left(V^{[rs]}_{n}\right)_{k0}$, $\left(V^{[rs]}_{n}\right)_{00}$ for any n-string vertex are shown to be simple explicit functions of the single matrix $t_{eo} = \kappa_e^{1/2}T_{eo}\kappa_o^{-1/2}$. Diagonalizing this single matrix diagonalizes simultaneously all Neumann matrices. This explains and justifies the notion of Neumann spectroscopy for arbitrary oscillator frequencies. These results were initially obtained in the matter sector (or with bosonized ghosts) but by now they have been generalized to include also the fermionic ghost sector [16]. In the open string limit we fix the frequencies to Eq.(A2). In this limit our generalized Neumann matrices agree with other computations of these coefficients.

Furthermore, for arbitrary frequencies κ_n, using the Moyal star, one can also compute open string amplitudes [8] including the ghost sector [16].

The regulator is removed by taking the limit in Eq.(A2) at the end of computations. As emphasized in [8], taking such a limit at the Lagrangian level is wrong because of anomalies and leads to inconsistent results. We note that we break the conformal symmetry explicitly when we work at finite N or arbitrary values of κ_n. This is the cost to pay to resolve the associativity anomaly among the basic relations. We expect that the conformal symmetry is re-established in the limit of Eq.(A2).

It has been shown that this regularization scheme gives the correct results in explicit computations, including the spectrum of L_0, perturbative states, Neumann coefficients, string

Feynman graphs, and numerical estimates of certain quantities computed with other methods in the literature.

Appendix B: Solvability of $S_2 + S_3$

In this appendix, we show that the combination $S_2 + S_3$ is also solvable as in the combination $S_1 + S_3$ considered in the text. For simplicity we consider only the matter sector and keep just one space-time component. We change variable (symplectic transformation) from x_e, p_e to y_e, q_e such that $y_2 = \frac{1}{\sqrt{1+\bar{w}w}} \sum_e w_e p_e$. The e.o.m. becomes $q_2^2 A + \alpha' g A \star A = 0$ but since $q_4, q_6, \cdots$ and $y_4, y_6, \cdots$ are irrelevant, we wrote it simply as, (neglecting subscript 2),

$$q^2 A(y,q) + \alpha' g (A \star A)(y,q) = 0 \ . \tag{B1}$$

While the first term does not split as before, one may always write it as,

$$\frac{1}{4} \{q, \{q, A\}_\star\}_\star + \alpha' g A \star A = 0 \ . \tag{B2}$$

While the kinetic term does not split, we may use the same trick as before to write down one family of solutions. For that purpose, we prepare the wave function which is diagonal with respect to q,

$$q \star \phi(k,l) = k\, \phi(k,l), \qquad \phi(k,l) \star q = l\, \phi(k,l) \qquad k, l \in \mathbf{R}\,, \tag{B3}$$

$$\phi(k,l) \star \phi(k',l') = \delta(l-k')\, \phi(k,l') \ . \tag{B4}$$

The solution to this definition is given as,

$$\phi(k,l) = \delta\left(q - (k+l)/2\right) e^{i(k-l)y} \ . \tag{B5}$$

If we expand $A = \int dkdl A(k,l)\phi(l,k)$, e.o.m. can be rewritten in terms of $A(k,l)$ as

$$\left(\frac{k+l}{2}\right)^2 A(k,l) + \alpha' g \int dr A(k,r) A(r,l) = 0 \ . \tag{B6}$$

A family of solutions which is similar to those given in the previous section can be written as,

$$A_\theta(k,l) = -\frac{k^2}{\alpha' g} \theta_\Sigma(k) \delta(k-l) \tag{B7}$$

with

$$\theta_\Sigma(k) = \begin{cases} 1 & k \in \Sigma \\ 0 & \text{otherwise} \end{cases} \tag{B8}$$

where Σ is a certain range in $\mathbf{R}$. We note that the projector $\phi(k,k) = \delta(q-k)$ does not depend on the coordinate y.

The tension is computed similarly as

$$S[A_\Sigma] = \frac{V}{6\alpha'^3 g^2} \int_\Sigma dk\, k^6 \ . \tag{B9}$$

The volume factor V appears here because A does not depend on the coordinate. Since the projector is defined over the continuum variable, we expect any nontrivial solution obtained here will be unstable except for the trivial one $A = 0$.

Appendix C: Integrability of the matrix model

The fact that we can solve the translational invariant solutions of Eq.(3.15) implies that it is also integrable even at the quantum level as long as we neglect the $\bar{x}$ dependence (namely zero dimensional model). With this simplification, we argue that it reduces to the two matrix model and is indeed solvable.

We consider the partition function with the source term,

$$Z[J] = \int [da] \exp\left(-\mathrm{Tr}(a\Lambda a) - \frac{1}{3}\mathrm{Tr}(a^3) - \mathrm{Tr}(Ja)\right) \ . \tag{C1}$$

The following change of variable,

$$a = -\Lambda + a' \tag{C2}$$

kills the quadratic term and the partition function becomes,

$$Z[J] = e^{-\frac{2}{3}\mathrm{Tr}(\Lambda^3)+\mathrm{Tr}(J\Lambda)} \int [da'] \exp\left(-\frac{1}{3}\mathrm{Tr}(a'^3) - \mathrm{Tr}((J-\Lambda^2)a')\right) \ . \tag{C3}$$

This is the partition function of the purely cubic theory with the modified source term $J \to J' = J - (\Lambda)^2$.

If we ignore the prefactor, the problem is now reduced to solve the partition function,

$$Z[J'] \propto \int [da'] \exp\left(-\frac{1}{3}\mathrm{Tr}(a'^3) - \mathrm{Tr}(J'a')\right) \ . \tag{C4}$$

One interesting aspect of this integration is that the off-diagonal part of the matrix integration can be exactly performed. The measure of the integration of Hermite matrix a can be replaced by

$$[da'] = d\vec{a}\,[dU](\Delta(a))^2 \tag{C5}$$

where we use the decomposition $a' = UaU^\dagger$ by using unitary matrix U and eigenvalues $\vec{a}$ of A' ($a = \mathrm{diag}(\vec{a})$). $\Delta(a) = \prod_{i<j}(a_i - a_j)$ is van der Monde determinant. After this decomposition, Eq.(C4) becomes,

$$\int d\vec{a}\,(\Delta(a))^2 e^{-\frac{1}{3}\sum_i a_i^3} \int [dU] \exp(-\mathrm{Tr}J'UaU^{-1}) \tag{C6}$$

The integration over unitary matrix can be performed by using the famous formula proved by Itzykson, Zuber and Brezin [23],

$$\int [dU] \exp\left(\frac{1}{t}\mathrm{Tr}(AUBU^{-1})\right) = c(\Delta(a)\Delta(b))^{-1} \det\left[\exp(\frac{1}{t}a_j b_k)\right] \ , \tag{C7}$$

where a, b are the eigenvalues of matrices A, B, $c = t^{N(N-1)/2}\prod_{j=1}^{n} j!$. Eq.(C4) becomes finally,

$$Z[J'] \propto \int d\vec{a}\, \frac{\Delta(a)}{\Delta(\varphi)} \exp\left(-\frac{1}{3}\sum_i a_i^3 - \sum_i \varphi_i a_i\right) \tag{C8}$$

where φ's are the eigenvalues of J'.

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