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# Options for Gauge Groups in Five-Dimensional Supergravity [1]

**John Ellis**[†] [2], **Murat Günaydin**[†‡] [3] and **Marco Zagermann**[*] [4]

[†] *Theory Division, CERN, 1211 Geneva 23, Switzerland*

[‡] *Physics Department, Pennsylvania State University, University Park, PA 16802, USA*

[*] *Fachbereich Physik, Martin-Luther-Universität Halle-Wittenberg, Friedemann-Bach-Platz 6, D-06099 Halle, Germany*

**Abstract**

Motivated by the possibility that physics may be effectively five-dimensional over some range of distance scales, we study the possible gaugings of five-dimensional $\mathcal{N}=2$ supergravity. Using a constructive approach, we derive the conditions that must be satisfied by the scalar fields in the vector, tensor and hypermultiplets if a given global symmetry is to be gaugeable. We classify all those theories that admit the gauging of a compact group that is either Abelian or semi-simple, or a direct product of a semi-simple and an Abelian group. In the absence of tensor multiplets, either the gauge group must be semi-simple or the Abelian part has to be $U(1)_R$ and/or an Abelian isometry of the hyperscalar manifold. On the other hand, in the presence of tensor multiplets the gauge group cannot be semi-simple. As an illustrative exercise, we show how the Standard Model $SU(3)\times SU(2)\times U(1)$ group may be gauged in five-dimensional $\mathcal{N}=2$ supergravity. We also show how previous special results may be recovered within our general formalism.

[1] Work supported in part by the National Science Foundation under Grant Number PHY-9802510.
[2] John.Ellis@cern.ch
[3] murat@phys.psu.edu
[4] zagermann@physik.uni-halle.de

# 1 Introduction

There is currently much interest in the possibility that extra dimensions may appear at distance scales that are large relative to the inverse of the Planck length $1/M_P \sim 10^{-33}$ cm or the Grand Unification scale $1/M_{GUT} \sim 10^{-30}$ cm, and possibly at scales accessible to experiments. It is therefore important to understand what gauge groups and what matter representations are possible in various dimensions and what restrictions on the underlying 'Theory of Everything' may be provided by some variant of eleven-dimensional M theory.

One particular scenario for extra dimensions is the original proposal that eleven-dimensional M theory might be compactified on some Calabi-Yau manifold down to five dimensions [1]. The fifth dimension would then be just a few orders of magnitude larger than the Planck length or the GUT scale, and five-dimensional supergravity would be the appropriate effective low-energy field theory over this range of scales. In this scenario, the $SU(3) \times SU(2) \times U(1)$ gauge fields of the Standard Model would be restricted to a brane at one end of the fifth dimension, and there would be another 'hidden' gauge group restricted to another brane at its other end. Subsequently, elaborations with other gauge groups appearing on intermediate branes have also been studied [2].

In all this class of scenarios, a good characterization of the options available in the effective intermediate five-dimensional theory [3, 4] that governs the dynamics in the bulk between the branes is essential. For example, this effective theory frequently plays an essential rôle in mediating supersymmetry breaking between the brane on which it originates and the brane where the Standard Model is localized [5].

Analyses of this class of scenarios have been in the context of five-dimensional supergravity with only *Abelian* gaugings [3]. This assumption was motivated by the fact that the Hořava-Witten scenario [1] yields a gauging of an Abelian isometry of the universal hypermoduli space, which originates from the non-vanishing $\boldsymbol{G}$ flux in the underlying eleven-dimensional theory [4, 6]. Supplementary motivation came from the more general expectation that the Standard Model gauge group would be localized on one brane.

Calabi-Yau manifolds generically do not possess continuous non-Abelian global symmetries that are candidates for gauging the five-dimensional supergravity theory. On the other hand, such symmetries may appear at singular points in the moduli space of Calabi-Yau manifolds, leading to the possible appearance of enhanced gauge symmetries [7]. Moreover, non-perturbative M-theory dynamics may favour some alternatives to Calabi-Yau compactification possessing global symmetries that might be gauged.

One should also remain open to the possibility that the $SU(3) \times SU(2) \times U(1)$ gauge group of the Standard Model might *not* be restricted to a four-dimensional brane in this higher-dimensional space. A strong argument against the latter possibility seems to be provided by the excellent agreement of the values of the gauge couplings measured at low energies with the predictions of supersymmetric gauge theories in four dimensions [8]. However, it has been observed that gauge-coupling unification is also possible [9], in some approximation, even if the Standard Model gauge group extends into a fifth dimension. Therefore the possibility of such an extension cannot, perhaps, be rejected absolutely.

For all these reasons, we think it important to characterize what gauge groups may be possible in five-dimensional supergravity, and at what price in terms of restrictions on the scalar manifold associated (presumably) with the compactification from higher dimensions, in particular its global symmetries.

Previous analyses have focussed on five-dimensional supergravity theories with scalar manifolds in particular symmetry classes. In this paper, we attempt a systematic classification of all the options for the five-dimensional gauge group, noting in each case the appropriate conditions on the corresponding scalar manifolds. As a special case, we mention how the $SU(3) \times SU(2) \times U(1)$ gauge group of the Standard Model may be obtained in a suitable five-dimensional supergravity theory, not that we recommend it for any particular phenomenological reasons, but simply as an interesting exercise illustrating our general results.

The outline of this paper is as follows: In Section 2, we recall the relevant properties of ungauged $\mathcal{N} = 2$ supergravity theories in five dimensions. Our emphasis is on the global symmetry groups, $G$, of these theories and their 'gaugeable' subgroups $K \subset G$. As shown, the least trivial part of a classification of admissible gauge groups lies in the classification of the gaugeable isometries of the vector multiplet moduli space. In Section 3, which constitutes the main part of this paper, we give such a classification. To be precise, we classify all those theories that admit the gauging of a compact group $K$ that is either Abelian or semi-simple or a direct product of a semi-simple and an Abelian group. We illustrate our results with the example of $SU(3) \times SU(2) \times U(1)$ in Section 4, and summarize and draw some conclusions from our results in Section 5. Finally, Appendix A contains a few explicit examples illustrating our general discussion.

## 2 Ungauged Five-Dimensional $\mathcal{N} = 2$ Supergravity and its Possible Gaugings

Gauged supergravity theories are supergravity theories in which some vector fields $A_\mu^I$ are coupled to matter fields $\Phi^A$ via gauge covariant derivatives of the form

$$D_\mu \Phi^A \equiv \nabla_\mu \Phi^A + g A_\mu^I (T_I)^A{}_B \Phi^B \tag{2.1}$$

Here, $\nabla_\mu$ denotes the ordinary space-time-covariant derivative, $g$ is some coupling constant, and the $(T_I)^A{}_B$ are the representation matrices for the matter fields $\Phi^A$. If the gauge group is non-Abelian, there are, in addition, self-couplings among the vector fields $A_\mu^I$. A supergravity theory without such 'gauge' couplings is generally termed 'ungauged' [5].

Typically, the local gauge symmetry of a gauged supergravity theory reduces to a global, i.e., rigid, symmetry of an underlying ungauged supergravity theory when the gauge coupling $g$ is turned off. In these cases, one can iteratively construct the gauged supergravity

[5]The terms 'gauged' and 'ungauged' supergravity are only used for theories in which the supergravity sector and the gauge sector show a certain degree of entanglement. This typically happens when the supergravity multiplet contains vector fields that are candidates for gauge fields. A prominent example for which this is *not* the case is four-dimensional $\mathcal{N} = 1$ supergravity coupled to $\mathcal{N} = 1$ super-Yang-Mills theory with or without chiral matter multiplets, as in the minimally supersymmetric extension of the Standard Model. In four and five dimensions, one needs at least eight supercharges for the supergravity multiplet to contain at least one vector field, so that the term 'gauged supergravity' is commonly used in these dimensions only for theories with $\mathcal{N} \geq 2$ supersymmetry.

theories from their ungauged relatives via the Noether procedure. To this end, one first selects a 'gaugeable' subgroup, $K$, of the total global symmetry group, $G$, of the underlying ungauged Lagrangian. One then covariantizes the relevant derivatives *à la* (2.1), so as to turn the former *global* symmetry group $K$ into a *local* gauge symmetry. This typically breaks supersymmetry, but, if the gauge group $K$ was appropriately chosen, supersymmetry can be restored by adding a few additional terms to the Lagrangian and the transformation laws.

In this Section we recall the appropriate criteria for a group $K \subset G$ to be gaugeable in the context of five-dimensional $\mathcal{N} = 2$ supergravity theories. In the remainder of this paper we then look for solutions to these constraints.

### 2.1 General Formalism

The minimal amount of supersymmetry in five space-time dimensions corresponds to eight real supercharges, and is generally referred to as $\mathcal{N} = 2$ supersymmetry. The $R$-symmetry group of the underlying Poincaré superalgebra is $USp(2)_R \cong SU(2)_R$. The five-dimensional $\mathcal{N} = 2$ supergravity multiplet can be coupled to vector multiplets, self-dual tensor multiplets and hypermultiplets. The field contents of these multiplets are as follows [6].

- The supergravity multiplet
$$(e_\mu^m, \psi_\mu^i, A_\mu)$$
contains the fünfbein $e_\mu^m$, an $SU(2)_R$ doublet of gravitini $\psi_\mu^i$: $i = 1, 2$ and a vector field $A_\mu$.

- A vector multiplet
$$(A_\mu, \lambda^i, \varphi)$$
consists of a vector field $A_\mu$, an $SU(2)_R$ doublet of spin-1/2 gaugino fermions $\lambda^i$: $i = 1, 2$ and one real scalar field $\varphi$.

- A tensor multiplet has the same field content as a vector multiplet, but with the vector field $A_\mu$ replaced by a two-form field $B_{\mu\nu}$ satisfying odd-dimensional duality as explained below.

- A hypermultiplet
$$(\zeta^A, q^X)$$
comprises two spin-1/2 fermions (hyperini) $\zeta^A$: $A = 1, 2$, and four real scalar fields $q^X$: $X = 1, \ldots, 4$. The hyperini are inert under $SU(2)_R$, which is why we have not used the $SU(2)_R$ doublet index $i$ for these fields.

[6] Our space-time conventions coincide with those of [10, 11, 12, 13], i.e., all fermions are symplectic Majorana spinors, the metric signature is $(-, +, +, +, +)$, and $\mu, \nu \ldots$ and $m, n, \ldots$ denote curved and flat space-time indices, respectively.

When the theory is *ungauged*, vector and tensor fields can always be dualized into each other and are physically equivalent, so one does not have to distinguish between vector and tensor multiplets at the level of the ungauged theory. However, this equivalence between vector and tensor multiplets does not hold for certain *gauged* theories, as we discuss in more detail below.

The ungauged coupling of $n$ vector and $m$ hypermultiplets to supergravity was worked out in [10, 14]. The bosonic sector of such a theory consists of

- the fünfbein $e_\mu^m$,
- $(n+1)$ vector fields $A_\mu^{\tilde{I}}$: $\tilde{I}, \tilde{J} \ldots = 0, 1, \ldots, n$, where we have combined the graviphoton with the $n$ vector fields from the $n$ vector multiplets to form a single $(n+1)$-plet of vector fields,
- $n$ scalar fields $\varphi^x$: $x, y, \ldots = 1, \ldots, n$ from the $n$ vector multiplets,
- $4m$ scalar fields $q^X$: $X, Y, \ldots = 1, \ldots, 4m$ from the $m$ hypermultiplets,

The $(n+4m)$ scalar fields $\{\varphi^x, q^X\}$ parametrize a Riemannian manifold $\mathcal{M}$ of (real) dimension $(n+4m)$, which was found to factorize [14]:

$$\mathcal{M} = \mathcal{M}_{VS} \times \mathcal{M}_Q, \tag{2.2}$$

where $\mathcal{M}_{VS}$ is an $n$-dimensional real manifold [10], which is 'very special' in a sense defined below and parametrized by the scalar fields $\varphi^x$, and $\mathcal{M}_Q$ denotes a quaternionic manifold of real dimension $4m$ parametrized by the hyperscalars $q^X$ [15].

Introducing the Maxwell-type field strengths $F_{\mu\nu}^{\tilde{I}} \equiv 2\partial_{[\mu} A_{\nu]}^{\tilde{I}}$, the bosonic part of the Lagrangian reads [10, 14]

$$\begin{aligned} e^{-1}\mathcal{L}_{bosonic} &= -\frac{1}{2}R - \frac{1}{4}\overset{\circ}{a}_{\tilde{I}\tilde{J}} F_{\mu\nu}^{\tilde{I}} F^{\mu\nu\tilde{J}} - \frac{1}{2} g_{\tilde{x}\tilde{y}} (\partial_\mu \varphi^{\tilde{x}})(\partial^\mu \varphi^{\tilde{y}}) \\ &\quad - \frac{1}{2} h_{XY} (\partial_\mu q^X)(\partial^\mu q^Y) + \frac{e^{-1}}{6\sqrt{6}} C_{\tilde{I}\tilde{J}\tilde{K}} \varepsilon^{\mu\nu\rho\sigma\lambda} F_{\mu\nu}^{\tilde{I}} F_{\rho\sigma}^{\tilde{J}} A_\lambda^{\tilde{K}}. \end{aligned} \tag{2.3}$$

Here, $e \equiv \det(e_\mu^m)$, whereas $g_{xy}(\varphi)$ and $h_{XY}(q)$ denote, respectively, the metrics on the scalar manifolds $\mathcal{M}_{VS}$ and $\mathcal{M}_Q$. The quantity $\overset{\circ}{a}_{\tilde{I}\tilde{J}}(\varphi)$ is symmetric in its indices and depends on the scalar fields $\varphi^x$. The completely symmetric tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$, by contrast, is *constant*, i.e., it does *not* depend on any of the scalar fields. Because of this, the Lagrangian is invariant under the Maxwell-type transformations

$$A_\mu^{\tilde{I}} \longrightarrow A_\mu^{\tilde{I}} + \partial_\mu \Lambda^{\tilde{I}} \tag{2.4}$$

even though $A_\mu^{\tilde{I}}$ appears explicitly in the $F \wedge F \wedge A$ term in (2.3). Despite this invariance, the above theories are still referred to as 'ungauged', as we discussed at the beginning of this Section.

The tensor $C_{\tilde{I}\tilde{J}\tilde{K}}$ turns out to determine completely the part of the Lagrangian that is due to the supergravity and the vector multiplets [10]. In particular, it completely determines the metric of the 'very special' manifold $\mathcal{M}_{VS}$. To be more explicit, the $C_{\tilde{I}\tilde{J}\tilde{K}}$ define a cubic polynomial

$$N(h) := C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} \tag{2.5}$$

in $(n+1)$ real variables $h^{\tilde{I}}$: $\tilde{I} = 0, \ldots, n$, which endows $\mathbb{R}^{(n+1)}$ with the metric

$$a_{\tilde{I}\tilde{J}}(h) := -\frac{1}{3} \frac{\partial}{\partial h^{\tilde{I}}} \frac{\partial}{\partial h^{\tilde{J}}} \ln N(h). \tag{2.6}$$

The $n$-dimensional 'very special' manifold $\mathcal{M}_{VS}$ can then be represented as the hypersurface [10]

$$N(h) = C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}} = 1 \tag{2.7}$$

with the metric $g_{xy}$ on $\mathcal{M}_{VS}$ being the induced metric of this hypersurface in the "ambient" space with the metric (2.6), and furthermore we have $\overset{\circ}{a}_{\tilde{I}\tilde{J}}(\varphi) = a_{\tilde{I}\tilde{J}}|_{N=1}$.

## 2.2 The Global Symmetries and their Possible Gaugings

In this subsection we give a general overview of the different types of global symmetries of the ungauged Lagrangian (2.3), and give a pre-classification of the possible types of gaugings.

### 2.2.1 Case I: No Hypermultiplets

We first consider theories without hypermultiplets, which we also describe as 'Maxwell-Einstein supergravity theories' (MESGTs). In these cases, the $C_{\tilde{I}\tilde{J}\tilde{K}}$ determine the entire theory, and any (infinitesimal) linear transformation

$$h^{\tilde{I}} \longrightarrow M^{\tilde{I}}_{\tilde{J}} h^{\tilde{J}} \tag{2.8}$$

$$A^{\tilde{I}}_{\mu} \longrightarrow M^{\tilde{I}}_{\tilde{J}} A^{\tilde{J}}_{\mu} \tag{2.9}$$

that leaves the $C_{\tilde{I}\tilde{J}\tilde{K}}$ invariant:

$$M^{\tilde{I}'}_{(\tilde{I}} C_{\tilde{J}\tilde{K})\tilde{I}'} = 0, \tag{2.10}$$

extends to a *global* symmetry of the entire Lagrangian. We call $G_{VS}$ the group generated by all these symmetry transformations, i.e., the invariance group of the cubic polynomial $N(h)$. The group $G_{VS}$ has to be a subgroup of the isometry group, $Iso(\mathcal{M}_{VS})$, of the scalar manifold $\mathcal{M}_{VS}$, which becomes manifest if one rewrites the kinetic term of the scalar fields as [18, 10]

$$-\frac{1}{2} g_{xy} (\partial_\mu \varphi^x)(\partial^\mu \varphi^y) = \frac{3}{2} C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} \partial_\mu h^{\tilde{J}} \partial^\mu h^{\tilde{K}}|_{N=1}.$$

In most cases, $G_{VS}$ and $Iso(\mathcal{M}_{VS})$ are the same, but there are some counterexamples [18, 19] in which some isometries of $\mathcal{M}_{VS}$ do not extend to global symmetries of the full Lagrangian, i.e., to symmetries of the $C_{\tilde{I}\tilde{J}\tilde{K}}$. In such cases, it is then necessary to distinguish between the invariance group of the pure scalar sector, $Iso(\mathcal{M}_{VS})$, and the symmetry group of the entire Lagrangian, $G_{VS}$, because only the latter can be gauged.

Regardless of the possible existence of this geometric symmetry group $G_{VS}$ (for generic $C_{\tilde{I}\tilde{J}\tilde{K}}$, $G_{VS}$ might very well be trivial), every MESGT is in any case invariant under global transformations of the $R$-symmetry group $SU(2)_R$. As mentioned at the beginning of this Section, $SU(2)_R$ acts only on the indices $i$ of the fermions, not on the 'geometric' indices $(\tilde{I}, x)$. As a consequence, the total *global* symmetry group of a MESGT factorizes:

$$\text{Global invariance group of a MESGT} = G_{VS} \times SU(2)_R.$$

On quite general grounds, one thus obtains the following list of conceivable types of gauge groups [11, 12, 17]:

- $U(1)_R \subset SU(2)_R$,
- $K \subset G_{VS}$,
- $U(1)_R \times K$,
- $SU(2)_R \times K$ with $K \supset SU(2)$.

Here, $K$ denotes some 'gaugeable' subgroup of $G_{VS}$ (see below). The gauging of $U(1)_R$ turns out to be a necessary prerequisite for obtaining Anti-de Sitter ground states [11, 12, 16]. On the other hand, the gauging of $U(1)_R$ does not interfere with the gauging of a subgroup $K$ of $G_{VS}$ [12] [7]. This is no longer true if one wants to gauge the *entire* $R$-symmetry group $SU(2)_R$, which *requires* the simultaneous gauging of a subgroup $K \subset G_{VS}$ that itself contains an $SU(2)$ subgroup $SU(2) \subset K$ [17]. From this it follows that the non-trivial part of a more explicit gauge group classification lies in the classification of the possible gauge groups $K \subset G_{VS}$.

What are the constraints on such gauge groups $K$? According to (2.9), the $(n+1)$ vector fields $A_\mu^{\tilde{I}}$ transform in a (not necessarily irreducible) $(n+1)$-dimensional representation of the global invariance group $G_{VS}$. The minimal consistency requirement for a subgroup $K \subset G_{VS}$ to be gaugeable is therefore that this $(n+1)$-dimensional representation contains the adjoint of $K$ as a subrepresentation. In the most general case, one therefore has the decomposition [8]:

$$(n+1)_{G_{VS}} \longrightarrow \text{adj}(K) \oplus \text{singlets}(K) \oplus \text{non-singlets}(K). \tag{2.11}$$

Two cases have to be distinguished:

(i) When the above decomposition contains no non-singlets of $K$ beyond the adjoint, it was shown in [11] that the gauging can always be performed and that the resulting theory has no scalar potential, unless one also gauges $U(1)_R$ [12] or $SU(2)_R$ [17] in addition to $K$.

(ii) If, on the other hand, non-singlets beyond the adjoint do occur, the corresponding non-singlet vector fields have to be converted to self-dual tensor fields $B_{\mu\nu}$ in order for the

[7] We should point out one subtle point in this regard. The gauge field of $U(1)_R$ must be a linear combination of those vector fields that are singlets of $K$.

[8] For $K$ Abelian the adjoint of $K$ and the $K$-singlets should be identified.

gauging to be compatible with supersymmetry [12]. At the linearized level, these tensor fields fulfill a first-order field equation of the form [20]

$$dB = im * B, \tag{2.12}$$

where $*$ denotes the Hodge dual, $m$ is a massive parameter proportional to the gauge coupling $g$, and all internal indices have been suppressed for simplicity. Because of this equation, the two-form fields $B_{\mu\nu}$ are no longer equivalent to vector fields when the gauge coupling is non-zero.

For later reference, we split the index $\tilde{I}$ according to

$$\tilde{I} = (I, M), \tag{2.13}$$

where $I, J, K, \ldots = 1, \ldots n_V$ collectively denote the vector fields in the adjoint as well as the $K$-singlets, and the $M, N, P, \ldots = 1, \ldots, n_T$ refer to the non-singlets of $K$, i.e., the tensor fields.

The presence of self-dual tensor fields introduces two important new features into the theory:

- Consistency with supersymmetry now requires the existence of a non-vanishing scalar potential, $P^{(T)}$, which can be written in the form [12]

$$P^{(T)} = \frac{3}{4} g_{xy} K_I^x K_J^y h^I h^J, \tag{2.14}$$

  where the $K_J^x$ denote the Killing vectors on $\mathcal{M}_{VS}$ corresponding to the subgroup $K \subset G_{VS} \subset Iso(\mathcal{M}_{VS})$ of its isometry group [9]. This potential is manifestly positive definite and hence can not lead to AdS ground states, unless one also gauges $U(1)_R$ [16].

- The presence of the tensor fields implies several new restrictions on the $C_{\tilde{I}\tilde{J}\tilde{K}}$ and the admissible gauge groups $K \subset G_{VS}$ [12]. Supersymmetry now demands that the coefficients of the type $C_{MNP}$ and $C_{IJM}$ have to vanish:

$$C_{MNP} = C_{IJM} = 0. \tag{2.15}$$

  Furthermore, the transformation matrices $\Lambda_{IN}^M$ of the non-singlets have to be

$$\Lambda_{IM}^N = \frac{2}{\sqrt{6}} \Omega^{NP} C_{MPI} \Longleftrightarrow \Omega_{NP} \Lambda_{IM}^P = \frac{2}{\sqrt{6}} C_{MNI}, \tag{2.16}$$

  where $\Omega_{MN}$ and $\Omega^{MN}$ are antisymmetric and inverse to each other:

$$\Omega_{PN} \Omega^{NM} = \delta_P^M.$$

[9] As mentioned earlier, and contrary to what happens in four dimensions [21, 22], this potential vanishes when no tensor fields are present. This can be seen directly from (2.14), taking into account the fact that the very special geomety of $\mathcal{M}_{VS}$ implies [11] that $K_{\tilde{I}}^x h^{\tilde{I}} = 0$ when the summation goes over the *full* set of indices $\tilde{I}$.

For the inverse $\Omega^{MN}$ to exist, $n_T$ obviously has to be even. The symmetry of the $C_{IMN}$ and equation (2.16) further imply

$$\Lambda_{IN}^{P}\Omega_{PM} + \Omega_{NP}\Lambda_{IM}^{P} = 0 \quad \text{or} \quad \Lambda_I^T \cdot \Omega + \Omega \cdot \Lambda_I = 0, \tag{2.17}$$

i.e., the non-singlets have to transform in a *symplectic* representation of the gauge group $K$ [12].

In Section 3, we exploit these restrictions and classify those $C_{\tilde{I}\tilde{J}\tilde{K}}$ that meet all these requirements. Having physical applications in mind, however, we only consider *compact* gauge groups $K$ that are either
(i) Abelian or
(ii) semi-simple or
(iii) a direct product of an Abelian and a semi-simple group.

#### 2.2.2 Case II: The General Case with Hypermultiplets

When hypermultiplets are present [13, 14], there is an additional global symmetry group, $Iso(\mathcal{M}_Q)$, the isometry group of the quaternionic target space $\mathcal{M}_Q$ of the hyperscalars [15]. However, as the hypermultiplets do not contain any vector fields themselves, any gauging of the quaternionic isometries has to be 'external', i.e., it has to be done with the vector fields $A_\mu^I$ of the supergravity and/or vector multiplets.

Two cases should be distinguished (see also [13, 6, 22]).

(i) If one wants to gauge an Abelian subgroup $K \subset Iso(\mathcal{M}_Q)$, one needs at least $\dim(K)$ vector fields, i.e., $n_V = (\dim(K) - 1)$ vector multiplets. No other restriction has to be satisfied in the vector multiplet sector.

(ii) If $K \subset Iso(\mathcal{M}_Q)$ is non-Abelian, one needs at least $n_V = \dim(K)$ vector multiplets, but now one also needs the gauge fields to transform in the adjoint of $K$. This means that, just as in the case without hypermultiplets, $K$ now also has to be a gaugeable subgroup of $G_{VS}$.

To summarize, the gauging of a given non-trivial group of quaternionic isometries imposes the same constraints on the gaugeable subgroups of the very special geometry as in the case without the hypermultiplets. We therefore focus on a classification of the gaugeable isometries of the very special geometry. Having solved that problem, the classification of the gaugeable quaternionic isometries is then equivalent to a classification of all isometry groups of all possible quaternionic manifolds [10]. A deeper understanding of this problem would also provide information on the possible matter representations in five-dimensional gauged supergravities, which is also important for the reasons mentioned in the Introduction. However, this lies beyond the scope of this paper: for some recent results, see [24].

[10] The *homogeneous* quaternionic manifolds were classified in [23].

# 3 Very Special Manifolds with Gaugeable Compact Isometries

Our goal is to classify the cubic polynomials

$$N(h) = C_{\tilde{I}\tilde{J}\tilde{K}} h^{\tilde{I}} h^{\tilde{J}} h^{\tilde{K}}$$

that have a non-trivial invariance group, $G_{VS}$, with a gaugeable compact subgroup $K \subset G_{VS}$.

Our classification is constructive, in that we write down the possible building blocks of such polynomials, i.e., of the underlying coefficients $C_{\tilde{I}\tilde{J}\tilde{K}}$. Besides the restrictions imposed by the gauging, these building blocks have to satisfy one additional constraint, which is already present in the ungauged theory. This constraint has to do with the fact that a given set of $C_{\tilde{I}\tilde{J}\tilde{K}}$ uniquely determines the tensor $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$ in the kinetic term of the vector fields as well as the metric $g_{xy}$ of the very special manifold $\mathcal{M}_{VS}$. Both $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$ and $g_{xy}$ have to be positive definite in order to be physically meaningful.

In general, it appears difficult to see when this is the case, because of the complicated expressions one usually gets when evaluating (2.6) on the hypersurface $N(h) = 1$. Fortunately, however, there is a basis of the ambient space $\mathbb{R}^{(n+1)} \supset \mathcal{M}_{VS}$, the 'canonical basis' [10], in which these positivity properties become manifest. In this canonical basis, the $C_{\tilde{I}\tilde{J}\tilde{K}}$ take the form

$$\begin{aligned} C_{000} &= 1 \\ C_{00i} &= 0 \\ C_{0ij} &= -\frac{1}{2}\delta_{ij} \\ C_{ijk} &= \text{arbitrary} \end{aligned} \tag{3.1}$$

with $i, j, k, \ldots = 1, \ldots, n$. As indicated, the coefficients of the type $C_{ijk}$ may be chosen at will, i.e., they parametrize the remaining freedom one has in deforming the manifold $\mathcal{M}_{VS}$ without spoiling the positivity properties of $g_{xy}$ and $\overset{\circ}{a}_{\tilde{I}\tilde{J}}$.

In the above basis, the invariance condition (2.10)

$$M^{\tilde{I}'}{}_{(\tilde{I}} C_{\tilde{J}\tilde{K})\tilde{I}'} = 0 \tag{3.2}$$

restricts the transformation matrices $M^{\tilde{I}'}{}_{\tilde{I}}$ to be of the form (see also [23]):

$$\begin{aligned} M^0{}_0 &= 0 \\ M^i{}_0 &= M^0{}_i \\ M^i{}_j &= S_{ij} + A_{ij}, \end{aligned} \tag{3.3}$$

where $S_{ij}$ is symmetric in $i$ and $j$, and $A_{ij}$ is antisymmetric. The matrix $S_{ij}$ is given by

$$S_{ij} = M^k{}_0 C_{kij}, \tag{3.4}$$

whereas $A_{ij}$ is subject to the constraint

$$C_{l(ij}A_{k)l} = M^m_{\ \ 0}\left[C_{lm(i}C_{jk)l} - \frac{1}{2}\delta_{m(i}\delta_{jk)}\right]. \tag{3.5}$$

We are only interested in *compact* symmetries of the $C_{\tilde{I}\tilde{J}\tilde{K}}$. These are generated by the antisymmetric part of $M^{\tilde{I}}_{\ \tilde{J}}$, i.e., we have to set $M^i_{\ 0} = M^0_{\ i} = 0$ and are left with

$$M^{\tilde{I}}_{\ \tilde{J}} = \begin{pmatrix} 0 & 0 \\ 0 & A_{ij} \end{pmatrix} \tag{3.6}$$

with

$$A_{ij} = -A_{ji} \Longleftrightarrow A_{ij} \in \mathfrak{so}(n) \tag{3.7}$$

$$C_{l(ij}A_{k)l} = 0. \tag{3.8}$$

Hence, a compact symmetry group of the cubic polynomial $N(h)$ is given by the subgroup of the $SO(n)$ rotations of the $h^i$ that also leave the coefficients $C_{ijk}$ invariant [11]. All we have to do then is to classify the possible $C_{ijk}$ that preserve *gaugeable* subgroups $K$ of this $SO(n)$.

### 3.1 The Most Symmetric Case: $C_{ijk} = 0$

We start this classification with the simplest case

$$C_{ijk} = 0 \tag{3.9}$$

for all $i, j, k, \ldots = 1, \ldots, n$. In this most symmetric case, the polynomial $N(h)$ is obviously invariant under the full $SO(n)$. In fact, it is easy to see that (3.9) automatically implies $M^0_{\ i} = M^i_{\ 0} = 0$ via the constraint (3.5), i.e., there are no non-compact symmetries, and $SO(n)$ is the full symmetry group of $N(h)$. It is interesting to note that the manifolds based on (3.9) are in general not homogeneous, i.e., they are not contained in the classification of homogeneous very special manifolds given in [23]. Their peculiar geometry can best be seen by introducing the following 'radial coordinate' for the scalar manifold

$$r^2 = \frac{3}{2}\sum_{i=1}^{n} h^i h^i.$$

The hypersurface condition then takes the form

$$N = h^0[(h^0)^2 - r^2] = 1,$$

[11]This also implies that the action of a compact gauge group $K \subset G_{VS} \subset Iso(\mathcal{M}_{VS})$ has always at least one fixed point on $\mathcal{M}_{VS}$, namely the 'base point' [10] $h^I_c = (1, 0, \ldots, 0) \in \mathcal{M} \subset \mathbb{R}^{n+1}$, which is left invariant under the action of $SO(n) \supset K$. This in turn guarantees the existence of at least one critical point of the potential $P^{(T)}$ related to the tensor fields, because $K^x_I = 0$ at this point - see (2.14). Obviously, this critical point corresponds to a Minkowski ground state of the theory (unless $U(1)_R$ is also gauged [16]), and it can be shown that this ground state is $\mathcal{N} = 2$ supersymmetric.

which can be rewritten in terms of the 'lightcone' coordinates $r_\pm = \frac{1}{2}(h^0 \pm r)$ as

$$r_+ r_- (r_+ + r_-) = 4.$$

This hypersurface has two disconnected components . The topology of each connected component of the full hypersurface is of the form

$$\mathcal{M}_{VS} = \aleph \times S^{n-1},$$

where $\aleph$ is the surface in the $(h^0, r)$ plane given by $N = 1$.

We now turn to the gaugeable subgroups of $G_{VS} = SO(n)$. The components $h^i$ transform in the $\mathbf{n}$ of $SO(n)$. Any gaugeable compact subgroup $K \subset G_{VS}$ must therefore be a subgroup of $SO(n)$ such that the adjoint representation of $K$ is contained in the $\mathbf{n}$ of $SO(n)$. However, the adjoint of *any* compact group $K$ is always embeddable in the defining representation of any $SO(n)$ with $n \geq \dim(K)$, because the positive-definite Cartan-Killing form $\kappa_{ab}$ provides an invariant metric for the adjoint of $K$. Hence, *any* compact group $K$ with $\dim(K) \leq n$ can be gauged if (3.9) holds. If $n - \dim(K) =: r > 0$, one has $(r+1)$ spectator vector fields, one of them being $A^0_\mu$, which can be identified with the graviphoton. By construction, the other $\dim(K)$ vector fields transform in the adjoint of $K$ and act as $K$-gauge fields. The spectator vector fields can in principle be used to gauge also $U(1)_R$ and/or Abelian isometries of the hyperscalar manifold $\mathcal{M}_Q$, if they exist.

Note that the gaugings described above do not introduce any tensor fields. The only way to obtain a theory with tensor fields in the above model is by gauging an $SO(2)$ subgroup of $SO(n)$: $n \geq 2$, with $A^0_\mu$ being the $SO(2)$ gauge field. This follows because the transformation matrices $\Lambda^M_{IN}$ of such tensor fields would have to be related to some $C_{IMN}$ via (2.16). In the case at hand, i.e., with $C_{ijk} = 0$, such coefficients could only come from the $C_{0ij}$ with $I = 0$ - see (3.1). Thus $A^0_\mu$ would be the only vector field that could couple to such tensor fields, and the latter can only be charged with respect to a single $SO(2)$ subgroup of $SO(n)$.

We discuss such Abelian gaugings with tensor fields in a slightly more general context in Section 3.3.

We now consider cubic polynomials $N(h)$ with *non*-trivial $C_{ijk}$. These polynomials can be viewed as deformations of the simplest case (3.9). Since there are no completely symmetric invariant tensors of rank three in the $\mathbf{n}$ of $SO(n)$, such deformations will in general break $SO(n)$ to a subgroup. We are only interested in the case where this surviving symmetry group (or a subgroup thereof) can be gauged. As usual, we refer to this gaugeable subgroup of $SO(n)$ as $K$. Note also that, whereas the case $C_{ijk} = 0$ does not in general lead to homogeneous spaces, some of the deformations with $C_{ijk} \neq 0$ do.

### 3.2 Nontrivial $C_{ijk}$ without Tensor Fields

We first consider the case where the gauging of $K$ does not involve tensor fields. In this case, the $\mathbf{n}$ of $SO(n)$ decomposes according to

$$\mathbf{n} = \text{adjoint}(K) \oplus \text{singlets}(K).$$

Assuming the above decomposition, an Abelian factor of $K$ could not act non-trivially on anything. Thus, when no tensor fields are present, a compact gauge group $K \subset G_{VS}$ has to be semi-simple [12].

We split the indices $i = 1, \ldots, n$ as follows:

$$i = (a, \alpha), \tag{3.10}$$

where $a, b, \ldots = 1, \ldots, p \equiv \dim(K)$ correspond to the adjoint of $K$, and $\alpha, \beta, \ldots = 1, \ldots, r$ label the $r$ singlets, where $p + r = n$.

Before we proceed, we note that the term of the form

$$C_{0ij} h^0 h^i h^j = -\frac{1}{2} h^0 \delta_{ij} h^i h^j$$

appearing in the canonical basis (3.1) now reads

$$C_{0ij} h^0 h^i h^j = -\frac{1}{2} h^0 (\delta_{ab} h^a h^b + \delta_{\alpha\beta} h^\alpha h^\beta). \tag{3.11}$$

Our goal is to find all possible deformations of the relation $C_{ijk} = 0$ (3.9) that are consistent with the invariance under $K$. Clearly, coefficients of the form $C_{a\alpha\beta}$ transform in the adjoint of $K$ and can therefore never be invariant under $K$ transformations when $K$ is semi-simple. Indeed, any such non-trivial $C_{a\alpha\beta}$ would correspond to an Abelian ideal of $K$, in contradiction to the assumption of semi-simplicity. Hence, we have

$$C_{a\alpha\beta} = 0. \tag{3.12}$$

It remains to discuss the coefficients of the following forms.

(i) $C_{\alpha\beta\gamma}$:
Since the $h^\alpha$ are $K$-singlets, any $C_{\alpha\beta\gamma}$ are consistent with $K$ invariance.

(ii) $C_{\alpha ab}$:
In order to be invariant under $K$, $C_{\alpha ab}$ has to be an invariant symmetric tensor of rank 2 of the adjoint representation of $K$. The only such object is the Cartan-Killing form $\kappa_{ab}$ of $K$. However, in order for the $\delta_{ab}$ term in (3.11) to be invariant under $K$, one has to work in a basis where $\kappa_{ab} = \delta_{ab}$, so that any term $C_{\alpha ab}$ must be of the form

$$C_{\alpha ab} = c_\alpha \delta_{ab}$$

with some arbitrary constants $c_\alpha$.

(iii) $C_{abc}$:
In order for this term to be invariant under the action of $K$, it has to be equal to a completely symmetric invariant tensor of rank 3 of the adjoint representation of $K$.

[12] Of course, one could still gauge $U(1)_R$ and/or an Abelian subgroup of $Iso(\mathcal{M}_Q)$ in addition to $K \subset G_{VS}$.

As was already emphasized in [12], such tensors exist only for the groups $SU(N)$ with $N \geq 3$ (or products thereof), where they are given by the Gell-Mann $d$ symbols:

$$d_{abc} = \mathrm{Tr}(T_a\{T_b, T_c\})$$

with the $T_a$ being the generators of $SU(N)$. Hence, if $K = SU(N)$: $N \geq 3$, or if $K$ is a product of such $SU(N)$ factors, a term $C_{abc} = d_{abc}$ can be introduced without spoiling the $K$ invariance of the cubic polynomial $N(h)$. As an interesting side remark, we note that an $SU(N)$ gauging with $C_{abc} = d_{abc}$ leads to a quantization condition for the gauge coupling constant of $K$ [25], whereas an $SU(N)$ gauging with $C_{abc} = 0$ does not. The reason for this difference is the non-triviality of the Chern-Simons term in the case $C_{abc} = d_{abc}$: see [25] for further details.

### 3.3 Non-Trivial $C_{ijk}$ with Tensor Fields

Before we start with the classification of the possible $C_{ijk}$, we first prove the following

**Observation:** If tensor fields are present, a compact gauge group $K$ has to have at least one Abelian factor.

*Proof:* We first recall that a compact group $K \subset G_{VS}$ can act non-trivially only on the $h^i$: $i = 1, \ldots, n$, i.e., $h^0$ has to be inert under $K$. Hence, all the tensor field indices $M, N, \ldots = 1, \ldots, 2m \equiv n_T$ must be among the $i, j, \ldots = 1, \ldots, n$. We therefore split the index $i$ as follows

$$i = (I', M),$$

where the indices $I', J', \ldots = 1, \ldots, (n-2m)$ label the vector fields $A_\mu^i$ that are *not* dualized to tensor fields. The total set of vector fields that survive the tensor field dualization $A_\mu^M \to B_{\mu\nu}^M$ is thus given by

$$A_\mu^I = (A_\mu^0, A_\mu^{I'}).$$

We recall that the $h^M$ transform as follows (cf. (2.8)) under $K$:

$$h^M \longmapsto \Lambda_{IN}^M h^N,$$

with

$$\Lambda_{IM}^N = \frac{2}{\sqrt{6}} \Omega^{NP} C_{IMP}. \tag{3.13}$$

being the $K$ transformation matrices of the tensor fields $B_{\mu\nu}^M$. Furthermore, we note that the term

$$C_{0ij} h^0 h^i h^j = -\frac{1}{2} h^0 \delta_{ij} h^i h^j$$

appearing in the canonical basis (3.1) contains the term

$$C_{0MN} h^0 h^M h^N = -\frac{1}{2} h^0 \delta_{MN} h^M h^N. \tag{3.14}$$

The presence of this term has two important consequences:

(i) There is always a non-vanishing matrix $\Lambda_{0M}^{N}$ given by (3.13), which, in the case at hand, becomes

$$\Lambda_0 = -\frac{1}{\sqrt{6}}\Omega^{-1}. \tag{3.15}$$

(ii) Since $h^0$ is inert under $K$, the $K$ invariance of the term (3.14) requires the matrices $\Lambda_{IM}^{N}$ to be orthogonal:

$$\Lambda_I^T + \Lambda_I = 0. \tag{3.16}$$

Recalling that the $\Lambda_{IM}^{N}$ also have to be symplectic (2.17):

$$\Lambda_I^T \cdot \Omega + \Omega \cdot \Lambda_I = 0, \tag{3.17}$$

we have

$$\begin{aligned} \Omega \cdot [\Lambda_0, \Lambda_I] \cdot \Omega \quad &\overset{(3.15)}{=} \quad -\frac{1}{\sqrt{6}}[\Lambda_I \cdot \Omega - \Omega \cdot \Lambda_I] \\ &\overset{(3.16)}{=} \quad \frac{1}{\sqrt{6}}[\Lambda_I^T \cdot \Omega + \Omega \cdot \Lambda_I] \\ &\overset{(3.17)}{=} \quad 0, \end{aligned}$$

i.e., the (non-trivial) matrix $\Lambda_0$ commutes with all the $\Lambda_I$, and $K$ has to have at least one Abelian factor, which acts nontrivially on the tensor fields via $\Lambda_{0N}^{M}$. *Q.E.D.*

As a corollary of (3.16) and (3.17), we note that, choosing $\Omega = i\sigma_2 \otimes \mathbf{1}_m$, each matrix $\Lambda_I$ has to be of the form

$$\Lambda = \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \text{ with } \begin{cases} X = -X^T \\ Y = Y^T, \end{cases} \tag{3.18}$$

where $X$ and $Y$ are real $(m \times m)$-matrices. Obviously, $X + iY$ is anti-Hermitian, i.e., an element of $\mathfrak{u}(m)$ (the above is nothing but the standard embedding of $\mathfrak{u}(m)$ into $\mathfrak{sp}(2m, \mathbb{R})$ or $\mathfrak{so}(2m)$). This already shows that the allowed representation matrices $\Lambda_{IM}^{N}$, and hence the allowed coefficients $C_{IMN}$, are in one-to-one correspondence with *unitary* $m$-dimensional representations of the compact gauge group $K$.

We now return to our classification of the possible coefficients $C_{ijk}$ in the presence of tensor fields. Due to the above Observation, $K$ has to have at least one Abelian factor. We first cover the case when $K = K' \times U(1)$ with $K'$ semi-simple, and then the case when $K = U(1)^l$ is purely Abelian. The most general case is then obtained by rather obvious combinations.

### 3.3.1 $K = K' \times U(1)$

We first assume $K = K' \times U(1)$ with $K'$ semi-simple and with both factors acting non-trivially on the same set of tensor fields. The $\mathbf{n}$ of $SO(n)$ then decomposes with respect to $K'$ as

$$\mathbf{n} = \text{adjoint}(K') \oplus \text{singlets}(K') \oplus \text{non-singlets}(K'),$$

where, by assumption, the $U(1)$ factor acts non-trivially only on the non-singlets of $K'$. Consequently, we split the index $i = 1, \ldots, n$ into three subsets of indices:

$$i = (a, \alpha, M), \tag{3.19}$$

where, $a, b, \ldots = 1, \ldots, p \equiv \dim(K')$ correspond to the adjoint of $K'$; $\alpha, \beta, \ldots = 1, \ldots, r$ label the $r$ singlets; and $M, N, \ldots = 1, \ldots, 2m$ refer to the $2m$ non-singlets: $p + r + 2m = n$.

As explained in Section 2.2.1, the presence of the non-singlets $h^M$ requires the conversion of the corresponding vector fields $A_\mu^M$ to antisymmetric tensor fields $B_{\mu\nu}^M$. For consistency, the coefficients of the form $C_{IJM}$ and $C_{MNP}$ then have to vanish (see (2.15)). Recalling that, in our current notation, the index $I$ comprises the indices $(0, a, \alpha)$, the set of possibly non-vanishing coefficients $C_{ijk}$ therefore shrinks to

$$C_{\alpha\beta\gamma}, \quad C_{\alpha ab}, \quad C_{\alpha\beta a}, \quad C_{abc}, \quad C_{aMN}, \quad C_{\alpha MN}.$$

The allowed $C_{ijk}$ are constrained by the requirement that they be invariant under $K$. The coefficients of the type $C_{\alpha\beta a}$ are $U(1)$ singlets, but they transform in the adjoint of $K'$ and can therefore never contain any singlets of $K'$ when $K'$ is semi-simple (see above). Hence,

$$C_{\alpha\beta a} = 0,$$

and we are left with the following.

(i) $C_{\alpha\beta\gamma}$:
Any coefficient of this type would automatically be inert under $K$, and can therefore have any arbitrary value.

(ii) $C_{\alpha ab}$:
This term is a $U(1)$ singlet. As explained in our discussion of the corresponding term for the case without tensor fields, the only possible form of this term consistent with invariance under $K'$ is

$$C_{\alpha ab} = c_\alpha \delta_{ab},$$

with arbitrary constants $c_\alpha$.

(iii) $C_{abc}$
As explained earlier, this term can be either zero or equal to the $d$ symbols of $SU(N)$, if $K' = SU(N)$: $N \geq 3$, or if $K'$ is a product of such $SU(N)$ factors.

(iv) $C_{aMN}$:
We first note that, in general, any term of the form $C_{IMN}$ with $I \in \{0, a, \alpha\}$ is automatically invariant under $K$. In fact, under a $K$ transformation, it transforms as

$$C_{IMN} \longmapsto f_{JI}^K C_{KMN} + \Lambda_{JM}^P C_{IPN} + \Lambda_{JN}^P C_{IMP},$$

which vanishes automatically because of relation (3.13) and the fact that the $\Lambda_{IM}^N$ generate a representation of $K$:

$$[\Lambda_I, \Lambda_J] = \Lambda_K f_{IJ}^K. \tag{3.20}$$

The $C_{aMN}$ are uniquely determined by the $\Lambda^N_{aM}$ via (3.13). All we have to do then is to classify the possible $K'$ representation matrices $\Lambda^N_{aM}$. From our discussion around (3.18), however, it follows that the possible $\Lambda^N_{aM}$ are in one-to-one correspondence with $m$-dimensional unitary representations of $K'$. Since $K'$ is compact, *any* representation of $K'$ can be chosen to be unitary, and any such unitary representation can be embedded into $(2m \times 2m)$ matrices of the form (3.18) to form a possible set of $\Lambda^N_{aM}$ or, equivalently, a possible set of $C_{aMN}$.

(v) $C_{\alpha MN}$:
The $C_{\alpha MN}$ also give rise to transformation matrices $\Lambda^N_{\alpha M}$ via (3.13). Since, by assumption, our gauge group is $K = K' \times U(1)$, and the $\Lambda^N_{aM}$ already generate $K'$, the $\Lambda^N_{\alpha M}$ are either zero or they correspond to the $U(1)$ factor. However, we already know that the (non-vanishing) matrix $\Lambda^N_{0M}$ generates this $U(1)$ factor - see the proof at the beginning of this subsection. Since we assumed only one $U(1)$ factor, the $\Lambda^N_{\alpha M}$ can be at most proportional to $\Lambda^N_{0M}$, otherwise they would give rise to another, independent, Abelian factor in the gauge group $K$. For the $C_{\alpha MN}$ this means that they can be at most (remember that $C_{0MN} = -(1/2)\delta_{MN}$)

$$C_{\alpha MN} = d_\alpha \delta_{MN}$$

for some constants $d_\alpha$. In that case, the $U(1)$ gauge field would be the linear combination

$$A_\mu[U(1)] = \left[-\frac{1}{2}A^0_\mu + d_\alpha A^\alpha_\mu\right].$$

### 3.3.2 $K = U(1)^l$

We now come to the case when $K = U(1)^l$ is purely Abelian. We assume for simplicity that all the $U(1)$ factors act on the same set of tensor fields. If there were Abelian groups acting on mutually disjoint sets of tensor fields, the cubic polynomial would simply decompose into several subpieces of the type to be described below.

Assuming now the above gauge group structure, the $\mathbf{n}$ of $SO(n)$ decomposes as follows:

$$\mathbf{n} = \text{singlets}(K) \oplus \text{non-singlets}(K).$$

We denote the singlets of $K$ by $\alpha, \beta, \ldots = 1, \ldots, r$ and the non-singlets by $M, N, \ldots = 1, \ldots, 2m$, i.e., we split

$$i = (\alpha, M).$$

The possible non-vanishing $C_{ijk}$ are now the following.

(i) $C_{\alpha\beta\gamma}$:
These coefficients are automatically singlets of $K$, and can therefore be chosen arbitrarily.

(ii) $C_{\alpha MN}$:
Via (3.13), these coefficients are related to the $K$-transformation matrices $\Lambda^N_{\alpha M}$,

which are again of the form (3.18). The maximal Abelian subgroup of $U(m)$ is $m$-dimensional, so that $K$ can be at most $U(1)^m$. In the special case $K = U(1)$, the same arguments that were used in the case $K = K' \times U(1)$ apply, and the $C_{\alpha MN}$ could be at most

$$C_{\alpha MN} = d_\alpha \delta_{MN}$$

for some constants $d_\alpha$. In this case, the $U(1)$ gauge field would again be the linear combination

$$A_\mu[U(1)] = \left[ -\frac{1}{2} A^0_\mu + d_\alpha A^\alpha_\mu \right].$$

It is now rather straightforward to construct more general cubic polynomials by various combinations of the above basic building blocks.

We close this subsection with a comment on the nature of the tensor fields. As we have seen, a compact gauge group $K \subset G_{VS}$ has to be semi-simple when *no* tensor fields are introduced. Conversely, when tensor fields *are* present, a compact gauge group $K \subset G_{VS}$ can *never* be semi-simple; it has to contain at least one Abelian factor. This suggests the following interpretation.

If a compact group $K \subset G_{VS}$ is gauged, and tensor fields have to be introduced, one has at least one $\mathcal{N} = 2$ supersymmetric Minkowski ground state of the potential $P^{(T)}$ (see the footnote on page 11). The tensor multiplets should therefore admit an interpretation as $\mathcal{N} = 2$ Poincaré supermultiplets, at least for compact $K$. Since the tensor fields satisfy a massive field equation, such a multiplet would necessarily have to be massive. This is consistent with the form of the scalar potential $P^{(T)}$ in (2.14), which can be easily shown to be quadratic in the $h^M$. Due to their $K$ transformation properties, the $h^M$ have a natural interpretation as parametrizing the scalar fields in the tensor multiplets. Thus, $P^{(T)}$ can be interpreted as providing the mass terms for the massive scalars in the (massive) tensor multiplets. Such a massive tensor multiplet would have to be a centrally-charged BPS multiplet in order to have the same number of degrees of freedom as the massless vector multiplet from which it emerged. Indeed, the five-dimensional $\mathcal{N} = 2$ Poincaré superalgebra with central charges has precisely one such BPS multiplet with exactly the right field content (see, e.g., [26, 20]). It is then tempting to identify the $U(1)$ factor in the (compact) gauge group $K$ with the (necessarily gauged) central charge of the corresponding Poincaré superalgebra.

Note that the whole situation changes when one gauges $U(1)_R$ as well as $K$. As shown in [16], this kind of gauging typically leads to a $\mathcal{N} = 2$ supersymmetric AdS ground state, and the tensor multiplets would then have a natural interpretation as the self-dual tensor multiplets of the $\mathcal{N} = 2$ AdS superalgebra described in [27].

# 4 An Illustrative Exercise: The Standard Model Gauge Group

As an illustration of the general analysis of Section 3, we now demonstrate how to obtain the Standard Model gauge group $K_{SM} = SU(3) \times SU(2) \times U(1)$ within five-dimensional supergravity.

Since the dimension of the Standard Model gauge group is $\dim(K_{SM}) = 12$, we need at least twelve vector fields, i.e., at least eleven vector multiplets in addition to the supergravity multiplet. In addition to this minimal field content, there might be additional vector multiplets and/or some tensor multiplets. We first discuss the case without any tensor multiplets.

### 4.1 Case 1: No Tensor Multiplets

When there are no tensor multiplets, all the vector fields have to transform in the adjoint representation of $K_{SM}$, or they must be singlets under the gauge group, as discussed in Section 2.2.1. Since the adjoint of the $U(1)$ factor of $K_{SM}$ is trivial, this $U(1)$ factor has to act trivially on all the vector fields. In order to obtain fields charged under the $U(1)$ factor without introducing tensor fields, one would therefore have to gauge a $U(1)_R$ subgroup of the $R$-symmetry group and/or an Abelian isometry of the hypermultiplet scalar manifold $\mathcal{M}_Q$ (provided such an isometry exists). Neither of these Abelian gaugings interferes with the classification of the admissible very special manifolds $\mathcal{M}_{VS}$. We can thus, as in Section 3.2, restrict our attention to the semi-simple part of $K_{SM}$.

Working in the canonical basis, the $(n+1)$ vector fields $A_\mu^{\tilde{I}}$ are split into

$$A_\mu^{\tilde{I}} = (A_\mu^0, A_\mu^i)$$

with $i = 1, \ldots, n$ $(n \geq 11)$, and the $C_{\tilde{I}\tilde{J}\tilde{K}}$ are of the form

$$\begin{aligned} C_{000} &= 1 \\ C_{00i} &= 0 \\ C_{0ij} &= -\frac{1}{2}\delta_{ij} \\ C_{ijk} &= \text{not yet fixed} \end{aligned} \tag{4.1}$$

A compact symmetry group acts trivially on $A_\mu^0$, so that the adjoint vector fields of $SU(2)$ and $SU(3)$ have to be recruited from the $A_\mu^i$, which we therefore split into

$$A_\mu^i = (A_\mu^{\hat{a}}, A_\mu^{\bar{a}}, A_\mu^{\alpha}),$$

where $A_\mu^{\hat{a}}$ and $A_\mu^{\bar{a}}$ denote the adjoint vector fields of $SU(2)$ and $SU(3)$, respectively, whereas the $A_\mu^{\alpha}$ stand for additional $K_{SM}$ singlets (which may or may not be present).

As described in Section 3.2, the coefficients $C_{ijk}$ are now restricted by their $SU(2) \times SU(3)$ invariance to take the following forms:

$$\begin{aligned} C_{\alpha\beta\gamma} &= \text{arbitrary} \\ C_{\alpha\beta\hat{a}} &= 0 \\ C_{\alpha\beta\bar{a}} &= 0 \\ C_{\alpha\hat{a}\hat{b}} &= c_\alpha \delta_{\hat{a}\hat{b}} \end{aligned}$$

$$
\begin{aligned}
C_{\alpha\bar{a}\bar{b}} &= d_\alpha \delta_{\bar{a}\bar{b}} \\
C_{\alpha\hat{a}\bar{b}} &= 0 \\
C_{\hat{a}\hat{b}\hat{c}} &= 0 \\
C_{\hat{a}\hat{b}\bar{c}} &= 0 \\
C_{\hat{a}\bar{b}\bar{c}} &= 0 \\
C_{\bar{a}\bar{b}\bar{c}} &= b d_{\bar{a}\bar{b}\bar{c}},
\end{aligned}
\tag{4.2}
$$

where $C_{\alpha\beta\gamma}$, $c_\alpha$, $d_\alpha$ and $b$ denote some arbitrary coefficients, and the $d_{\bar{a}\bar{b}\bar{c}}$ are the $d$ symbols of $SU(3)$. As mentioned earlier, there is no such term for the $SU(2)$ factor - see (4.2). A number of remarks are now relevant.

**Remark 1:** A linear combination of the $SU(2) \times SU(3)$ singlets $A_\mu^0$ and $A_\mu^\alpha$ could always be used to gauge $U(1)_R$ and/or an Abelian isometry of the hyperscalar manifold $\mathcal{M}_{VS}$. Similarly, the $SU(2)$ and the $SU(3)$ gauge fields $A_\mu^{\hat{a}}$ and $A_\mu^{\bar{a}}$ could always be used to gauge $SU(2)$ and $SU(3)$ subgroups of $Iso(\mathcal{M}_Q)$, provided such subgroups exist. Depending on the particular quaternionic manifold one considers, one would then get hypermultiplets transforming in certain representations of $K_{SM}$ (if this is what wants to have).

**Remark 2:** As mentioned in Section 3.3, a non-zero value for $b$ would lead to a quantization condition for the $SU(3)$ coupling constant in the sense described in [25].

**Remark 3:** If $n$ satisfies its lower bound $n = 11$, i.e., if there are no $A_\mu^\alpha$, and $A_\mu^0$ is the only $K_{SM}$ singlet, one has two options:

(i) $b = 0$:
corresponding to the simple case $C_{ijk} = 0$ described in Section 3.1,

(ii) $b \neq 0$:
leading to a quantization condition for the $SU(3)$ coupling constant - see Remark 2 above.

Thus, the minimal case $n = 11$ is fairly restrictive and allows only for a one-parameter family of scalar manifolds $\mathcal{M}_{VS}$. The price one has to pay for this rigidity is that the $U(1)$ factor of the Standard Model gauge group would have to be gauged with the only remaining vector field $A_\mu^0$, so that *all* the vector fields would have to participate in the gauging, including the *graviphoton*. If, for some reason, one does not want the graviphoton to be part of the Standard Model gauge fields, one would need at least $n = 12$, which then introduces more arbitrariness into the theory via the new undetermined coefficients $C_{\alpha\beta\gamma}$, $c_\alpha$, $d_\alpha$.

**Remark 4:** None of the 'minimal' cases with $n = 11$, described in Remark 3, corresponds to a symmetric space $\mathcal{M}_{VS}$. In order to implement the Standard Model gauge group in a model based on a *symmetric* space $\mathcal{M}_{VS}$, one needs $n \geq 12$, i.e., at least one additional singlet $A_\mu^\alpha$. The corresponding values for $C_{\alpha\beta\gamma}$, $c_\alpha$, $d_\alpha$ and $b$ can be read off from equations (A.1) and (A.2) in the Appendix.

**Remark 5:** If there are at least three $A_\mu^\alpha$, and if the $C_{\alpha\beta\gamma}$, $c_\alpha$, $d_\alpha$ are chosen appropriately, i.e., as described in Section 3.2, one could introduce further non-Abelian gauge factors. Similarly – if this is desired – one could consider embedding $K_{SM}$ into larger gauge groups like $SU(5)$, $SO(10)$ etc. and write out the resulting restrictions on the $C_{\tilde{I}\tilde{J}\tilde{K}}$. We leave these extensions as exercises.

### 4.2 Case 2: The Presence of Tensor Fields

We now consider the case with tensor fields. Self-dual tensor fields always have to be charged under some gauge group [12]. In our case, this group could simply be $K_{SM}$ itself, or some part of it. On the other hand, the tensor fields could also be charged under some other gauge group factor which does not belong to the Standard Model gauge group $K_{SM}$. In order to keep the degree of complexity at a minimum, we only consider the case when $K_{SM}$ is indeed the full gauge group, and the tensor fields are charged under $K_{SM} = SU(3) \times SU(2) \times U(1)$. This is then exactly the case we considered in Section 3.3.1, and we can simply quote the results of that Section. As the tensor fields always come in pairs, we now need $n \geq 11 + 2 = 13$.

We again work in the canonical basis, but now split the index $i$ as follows

$$i = (\hat{a}, \bar{a}, \alpha, M), \tag{4.3}$$

where $\hat{a}$ and $\bar{a}$ correspond to the adjoint of $SU(2)$ and $SU(3)$, respectively, whereas $\alpha$ refers to the singlets and $M$ to the non-singlets (i.e., the tensor fields) of $K_{SM}$.

The admissible $C_{ijk}$ are now given by (see Section 3.3.1):

$$\begin{aligned}
C_{\alpha\beta\gamma} &= \text{arbitrary} \\
C_{\alpha\beta\hat{a}} &= 0 \\
C_{\alpha\beta\bar{a}} &= 0 \\
C_{\alpha\hat{a}\hat{b}} &= c_\alpha \delta_{\hat{a}\hat{b}} \\
C_{\alpha\bar{a}\bar{b}} &= d_\alpha \delta_{\bar{a}\bar{b}} \\
C_{\alpha\hat{a}\bar{b}} &= 0 \\
C_{\hat{a}\hat{b}\hat{c}} &= 0 \\
C_{\hat{a}\hat{b}\bar{c}} &= 0 \\
C_{\hat{a}\bar{b}\bar{c}} &= 0 \\
C_{\bar{a}\bar{b}\bar{c}} &= b d_{\bar{a}\bar{b}\bar{c}} \\
C_{M\bar{a}\bar{b}} &= 0 = C_{M\hat{a}\hat{b}} = C_{M\hat{a}\bar{b}} = C_{M\bar{a}\alpha} = C_{M\hat{a}\alpha} = C_{M\alpha\beta} \\
C_{\hat{a}MN} &= \frac{\sqrt{6}}{2} \Omega_{MP} \Lambda^P_{\hat{a}N} \\
C_{\bar{a}MN} &= \frac{\sqrt{6}}{2} \Omega_{MP} \Lambda^P_{\bar{a}N} \\
C_{\alpha MN} &= e_\alpha \delta_{MN} \\
C_{MNP} &= 0.
\end{aligned} \tag{4.4}$$

Here, $C_{\alpha\beta\gamma}$, $c_\alpha$, $d_\alpha$, $e_\alpha$ and $b$ are again arbitrary coefficients, which might or might not be zero, and $d_{\bar{a}\bar{b}\bar{c}}$ again stand for the $SU(3)$ $d$ symbols. The matrices $\Lambda^P_{\hat{a}N}$ and $\Lambda^P_{\bar{a}N}$ are, respectively, the $SU(2)$ and $SU(3)$ transformation matrices of the tensor fields. They can be related to $(\frac{n_T}{2})$-dimensional unitary representations of $SU(2)$ and $SU(3)$ via (3.18), where $n_T$ denotes the (even) number of tensor fields. As for the $U(1)$ factor, the tensor fields would transform via the representation matrix $\Lambda \sim \Omega^{-1}$ as in (3.15), with the corresponding $U(1)$ gauge field being the linear combination

$$A_\mu[U(1)] = \left[-\frac{1}{2}A_\mu^0 + e_\alpha A_\mu^\alpha\right].$$

(see the last item in Section 3.3.1).

Once again, one finds that the minimal case $n = 13$ leads to a very small number of choices for $\mathcal{M}_{VS}$, and requires the graviphoton to be one of the Standard Model gauge fields. To be more precise, the coefficients $C_{\alpha\beta\gamma}$, $c_\alpha$, $d_\alpha$, $e_\alpha$ have to vanish, because there is no $A_\mu^\alpha$, and the $SU(2)$ and $SU(3)$ transformation matrices $\Lambda^P_{\tilde{a}N}$ and $\Lambda^P_{\tilde{\alpha}N}$ would have to vanish because there is no non-trivial representation of these groups of the form (3.18) for the minimal case $n_T = 2$: any such representation would be related to *one*-dimensional (and hence trivial) unitary representations of $SU(2)$ and $SU(3)$ via (3.18). Thus, in the minimal embedding of the Standard Model gauge group with two tensor fields, the tensor fields form an $U(1) \cong SO(2)$ doublet and are inert under $SU(2) \times SU(3)$, and the only free parameter is the coefficient $b$.

Departure from the minimal value $n = 13$ then again introduces more arbitrariness into the theory because of the new unconstrained coefficients $C_{\alpha\beta\gamma}$, $c_\alpha$, $d_\alpha$, $e_\alpha$, which, in the absence of any further selection principle, can have any value.

# 5 Summary and Conclusions

We gave in the Introduction various motivations for considering the possible gaugings of five-dimensional $\mathcal{N} = 2$ supergravity. Whereas globally supersymmetric $\mathcal{N} = 2$ Yang-Mills theories in five dimensions can be studied for any compact gauge group without very stringent restrictions on the field content [28], it is not *a priori* clear what new restrictions are imposed by the non-linear structures introduced by a coupling to supergravity. Since gravity plays an important rôle in the current interest in five-dimensional theories, it is therefore important to analyze the constraints local supersymmetry imposes on the gauge sector.

In general, this is a difficult geometrical problem, which helps explain why most studies in the past focussed on theories with very peculiar classes of scalar manifolds. In fact, almost all the known concrete examples involved symmetric [10, 11, 19, 18, 12] or at least homogeneous spaces [23, 12]. However, thanks to the very special geometry of the five-dimensional vector multiplet moduli space encoded in the coefficients $C_{\tilde{I}\tilde{J}\tilde{K}}$, this geometrical problem can be reduced to a purely algebraic one. The entire analysis boils down to a classification of the possible $C_{\tilde{I}\tilde{J}\tilde{K}}$ that are consistent with invariance under the gauge group $K$.

We have solved this algebraic classification problem for all compact gauge groups that are semi-simple, or Abelian, or a direct product of a semi-simple and an Abelian group.

Our algebraic approach allowed us to go beyond the limitations set by the restriction to homogeneous or symmetric spaces. In fact, from the viewpoint of possible gauge symmetries, symmetric and homogeneous spaces are just particular examples of much larger classes of possible scalar manifolds.

Our main results can be summarized as follows.

(i) $K$ semi-simple:

*Any* compact semi-simple group $K$ can be gauged provided one respects certain constraints on the field content and on the couplings encoded in the $C_{\tilde{I}\tilde{J}\tilde{K}}$. These constraints can be found in Section 3.1 and 3.2. The key results are

- One always needs *at least* $n = \dim(K)$ vector multiplets, i.e., there is always at least one spectator vector field which can be identified with the graviphoton. Note that this no longer holds true for non-compact gauge groups. There, one can construct examples in which *all* the vector fields, including the graviphoton, act as the gauge fields of $K$ [11].
- In the minimal case $n = \dim(K)$, the scalar manifold $\mathcal{M}_{VS}$ is *fixed* whenever $K$ does not contain an $SU(N)$ factor with $N \geq 3$. If, on the other hand, $K$ does contain $SU(N)$ factors with $N \geq 3$, each such $SU(N)$ factor gives rise to one undetermined parameter in the $C_{\tilde{I}\tilde{J}\tilde{K}}$ and hence in the resulting scalar manifold $\mathcal{M}_{VS}$, as is illustrated by the Standard Model example discussed in Section 4. The minimal case $n = \dim(K)$ does not in general lead to symmetric spaces.
- If $K$ is purely semi-simple and compact, tensor fields are ruled out, because they would need at least one $U(1)$ factor in the gauge group. Again this result no longer holds true for non-compact gauge groups, where one could also have tensors for purely semi-simple $K$ [11, 12].
- As a by-product of the previous item, we found a natural interpretation of the tensor multiplets in terms of massive BPS multiplets of the centrally-extended Poincaré superalgebra, and also as self-dual tensor multiplets of the corresponding AdS superalgebra. Which of these two interpretations applies depends whether one also gauges $U(1)_R$ or not, as we discuss at the end of Section 3.

(ii) $K$ Abelian:

There are essentially two ways to implement an Abelian gauge group $K$. If the Abelian gauge group is $U(1)_R$ and/or an Abelian isometry of the hypermultiplet moduli space $\mathcal{M}_Q$, no restriction on the very special geometry of the vector multiplet sector is imposed: the very special geometry is blind towards such gaugings.

The other possibility, which is the one we focused on in this paper, is when the Abelian gauge group acts non-trivially on the very special manifold $\mathcal{M}_{VS}$, i.e., when one gauges an Abelian isometry of $\mathcal{M}_{VS}$. This case *always* requires tensor fields charged under $K$.

(iii) $K = K_{semi-simple} \times K_{Abelian}$:

This is essentially a combination of (i) and (ii), so, again, if the Abelian factor acts non-trivially on $\mathcal{M}_{VS}$, one must have some tensor fields charged under this Abelian factor.

The only new feature is now that the tensor fields can also be charged with respect to the semi-simple part of the gauge group. This is an interesting difference from the analogous $\mathcal{N} = 4$ theories [29], where the tensor fields can only be charged with respect to a one-dimensional Abelian group. As for the possible $K$ representations of the tensor fields, we found that they are in one-to-one corresponence with unitary representations of $K$.

In this paper, we have provided five-dimensional model-builders with a necessary toolkit, enabling them to construct the most general theory with any given gauge group. As an example of such a construction, we considered the Standard Model gauge group as a toy model in Section 4.

The matter content allowable in a general five-dimensional $\mathcal{N} = 2$ supergravity theory requires a further discussion of the hypermultiplet sector, which goes beyond the scope of this paper. Another worthwhile extension of the present work would be to consider the analogous classification problem for gaugings of six-dimensional supergravity. There is increasing interest in six-dimensional models of particle physics: see [30] and references therein. So far, phenomenological constructions have not incorporated explicitly the constraints that would be imposed by local supersymmetry in six dimensions [31], which are even stronger than those in five dimensions.

We foresee a fruitful continuation of the dialogue between model-building and explorations of the structures of higher-dimensional supergravity theories.

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# Appendix

## A Gauge Theories in Families of Symmetric Spaces

As an illustration of the more abstract discussion in Section 3, we show in this Appendix how to recover some well-known examples in the language used in that Section. These examples correspond to the scalar manifolds

- $\mathcal{M}_{VS} = SO(1,1) \times SO(n-1,1)/SO(n-1)$:
  (the 'generic Jordan family' [10])
- $\mathcal{M}_{VS} = SO(n,1)/SO(n)$
  (the 'generic non-Jordan family' [19])
- $\mathcal{M}_{VS} = SL(3,\mathbb{C})/SU(3)$,

which, apart from three additional cousins of the last one, exhaust all the very special manifolds that are symmetric spaces [19, 18].

### A.1 $\mathcal{M}_{VS} = SO(1,1) \times SO(n-1,1)/SO(n-1)$

In the canonical basis, the corresponding cubic polynomial is given by

$$N(h) = \left[(h^0)^3 - \frac{3}{2}h^0\delta_{ij}h^ih^j - \frac{1}{\sqrt{2}}(h^1)^3 + \frac{3}{\sqrt{2}}h^1[(h^2)^2 + \ldots + (h^n)^2]\right]. \tag{A.1}$$

In terms of the framework in Section 3, this polynomial can be interpreted in different ways, depending on which group $K$ one chooses as the gauge group. Using indices

$$\begin{aligned} \alpha &= 1 \\ a &= 2, \ldots, n, \end{aligned}$$

for example, it could correspond to one of the theories where a semi-simple group $K \subset SO(n-1) \subset SO(n)$ with adjoint$(K) \subset (\mathbf{n-1}) \subset \mathbf{n}$ can be gauged without the introduction of tensor fields, as in Section 3.2.

However, one can also interpret the indices $\{2, \ldots n\}$ (or a subset thereof) as tensor field indices $M, N \ldots$. This would then correspond to an $SO(2) \subset SO(n)$ gauging with tensor fields, with the $SO(2)$ gauge field being proportional to the linear combination $[A_\mu^0 - \sqrt{2}A_\mu^1]$, as in Section 3.3.2.

Other interpretations involving combinations of the above are of course also possible. This illustrates that, in general, for one and the same manifold $\mathcal{M}_{VS}$, various different types of gaugings are possible, and, conversely, that the $C_{\tilde{I}\tilde{J}\tilde{K}}$ we constructed in Sections 3.2 and 3.3 might sometimes describe the same manifold $\mathcal{M}_{VS}$.

We note finally that the transformation $h^{\tilde{I}} \mapsto \tilde{h}^{\tilde{I}}$ with

$$\begin{aligned} \tilde{h}^0 &= \frac{1}{\sqrt{3}}[h^0 - \sqrt{2}h^1] \\ \tilde{h}^1 &= \frac{1}{\sqrt{3}}[\sqrt{2}h^0 + h^1] \\ \tilde{h}^{\tilde{I}} &= h^{\tilde{I}} \text{ for } \tilde{I} = 2, \ldots n \end{aligned}$$

leads to the following simple form

$$N(\tilde{h}) = \left(\frac{3}{2}\right)^{\frac{3}{2}} \left(\sqrt{2}\tilde{h}^0[(\tilde{h}^1)^2 - (\tilde{h}^2)^2 - \ldots - (\tilde{h}^n)^2]\right),$$

which is no longer in the canonical basis, but makes the full non-compact symmetry $Iso(\mathcal{M}_{VS}) = G_{VS} = SO(1,1) \times SO(n-1,1)$ manifest.

### A.2 $\mathcal{M}_{VS} = SO(n,1)/SO(n)$

In the canonical basis, the corresponding cubic polynomial reads

$$N(h) = \left[(h^0)^3 - \frac{3}{2}h^0\delta_{ij}h^ih^j + \frac{1}{\sqrt{2}}(h^1)^3 + \frac{3}{2\sqrt{2}}h^1[(h^2)^2 + \ldots + (h^n)^2]\right]. \tag{A.2}$$

This is, apart from two (important) prefactors, of the same form as the polynomials of the generic Jordan family. Therefore, the discussion of the possible compact gauge groups $K$ is very similar and is not repeated here. Giving up the canonical basis, the above polynomial can also be simplified by a coordinate transformation similar to that described for the generic Jordan family. The definition

$$\begin{aligned}
\tilde{h}^0 &= \frac{1}{\sqrt{3}}[h^0 + \sqrt{2}h^1] \\
\tilde{h}^1 &= \frac{1}{\sqrt{3}}[\sqrt{2}h^0 - h^1] \\
\tilde{h}^{\tilde{I}} &= h^{\tilde{I}} \text{ for } \tilde{I} = 2, \ldots n
\end{aligned}$$

leads to

$$N(\tilde{h}) = \left(\frac{3}{2}\right)^{\frac{3}{2}} \left(\sqrt{2}\tilde{h}^0(\tilde{h}^1)^2 - \tilde{h}^1[(\tilde{h}^2)^2 + \ldots + (\tilde{h}^n)^2]\right).$$

We note that not all isometries of the scalar manifolds $\mathcal{M}_{VS}$ in this family are symmetries of the full $\mathcal{N} = 2$ supergravity [18]. As stressed earlier, only the subgroup of the isometry group that leaves $N(\tilde{h})$ invariant gets extended to a symmetry group of the full supergravity. In this case it turns out to be the $(n-1)$-dimensional Euclidean subgroup of $SO(n,1)$.

### A.3 $\mathcal{M} = SL(3,\mathbb{C})/SU(3)$

In this model, which corresponds to the Jordan algebra, $J_3^{\mathbb{C}}$, of complex Hermitian $(3 \times 3)$ matrices [10], the index $i$ runs from $1$ to $8$. We first decompose this index according to $i = (a, 4, M)$ with

$$\begin{aligned}
a, b, \ldots &= 1, \ldots, 3 \\
M, N, \ldots &= 5, \ldots, 8.
\end{aligned}$$

In the canonical basis, the underlying cubic polynomial can then be written as

$$\begin{aligned}
N(h) = \Bigg[ &(h^0)^3 - \frac{3}{2}h^0\delta_{ij}h^ih^j + \frac{3}{\sqrt{2}}h^4[\delta_{ab}h^ah^b - \frac{1}{2}\delta_{MN}h^Mh^N] \\
&- \frac{1}{\sqrt{2}}(h^4)^3 + \left(\frac{3}{2}\right)^{3/2}\gamma_{aMN}h^ah^Mh^N \Bigg],
\end{aligned} \tag{A.3}$$

where

$$\begin{aligned}
\gamma_1 &= \mathbf{1}_2 \otimes \sigma_1 \\
\gamma_2 &= -\sigma_2 \otimes \sigma_2 \\
\gamma_3 &= \mathbf{1}_2 \otimes \sigma_3.
\end{aligned}$$

This form makes it easy to verify that one can gauge an $SU(2) \times U(1)$ group acting non-trivially on a set of four tensor fields $B_{\mu\nu}^M$, as in Section 3.3.1.

The $SU(2)$ vector fields are $A^a_\mu$, and the $U(1)$ gauge field is proportional to the linear combination $[\sqrt{2}A^0_\mu + A^4_\mu]$. This kind of gauging was examined in detail in [17].

On the other hand, the above polynomial can also be understood differently. After some relabelling, one finds that the above polynomial is just

$$N(h) = \left[(h^0)^3 - \frac{3}{2}h^0\delta_{ij}h^i h^j + d_{ijk}h^i h^j h^k\right],$$

where $i, j, \ldots = 1, \ldots, 8$, with the $d_{ijk}$ being the $d$ symbols of $SU(3)$. In this form, it becomes obvious that one can also gauge $SU(3)$ without introducing any tensor fields, as in [11] and our discussion in Section 3.2.

Finally, a somewhat more concise form of (A.3) is obtained via a transformation to the new coordinates

$$\begin{aligned}\tilde{h}^0 &= \frac{1}{\sqrt{3}}(\sqrt{2}h^0 + h^4)\\ \tilde{h}^4 &= \frac{1}{\sqrt{3}}(h^0 - \sqrt{2}h^4)\\ \tilde{h}^{\tilde{I}} &= h^{\tilde{I}} \text{ for } \tilde{I} \neq 0, 4,\end{aligned}$$

which no longer correspond to the canonical basis. In terms of these,

$$N(\tilde{h}) = \left(\frac{3}{2}\right)^{\frac{3}{2}}\left(\sqrt{2}\tilde{h}^4\eta_{\alpha\beta}\tilde{h}^\alpha\tilde{h}^\beta + \gamma_{\alpha MN}\tilde{h}^\alpha\tilde{h}^M\tilde{h}^N\right), \tag{A.4}$$

where

$$\begin{aligned}\alpha, \beta, \ldots &= 0, 1, 2, 3\\ \eta_{\alpha\beta} &= \mathrm{diag}(+, -, -, -)\\ \gamma_0 &= -\mathbf{1}_4.\end{aligned}$$

This is the parametrization used in [10]. Indeed, it is now easy to verify that (A.4) is nothing but the determinant of

$$\tilde{\mathbf{h}} = \left(\frac{3}{2}\right)^{\frac{1}{2}}\begin{pmatrix}\sqrt{2}\tilde{h}^4 & \tilde{h}^5 - i\tilde{h}^7 & \tilde{h}^6 - i\tilde{h}^8\\ \tilde{h}^5 + i\tilde{h}^7 & \tilde{h}^0 + \tilde{h}^3 & \tilde{h}^1 - i\tilde{h}^2\\ \tilde{h}^6 + i\tilde{h}^8 & \tilde{h}^1 + i\tilde{h}^2 & \tilde{h}^0 - \tilde{h}^3\end{pmatrix},$$

i.e., the determinant of an element $\tilde{\mathbf{h}}$ of the Jordan algebra $J_3^{\mathbb{C}}$ of complex Hermitian $(3\times3)$-matrices [10].

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