Bosonic vacuum wave functions from the BCS–type wave function of the ground state of the massless Thirring model

M. Faber* and A. N. Ivanov†‡

April 25, 2020

Atominstitut der Österreichischen Universitäten, Arbeitsbereich Kernphysik und Nukleare Astrophysik, Technische Universität Wien, Wiedner Hauptstr. 8-10, A-1040 Wien, Österreich

Abstract

A BCS–type wave function describes the ground state of the massless Thirring model in the chirally broken phase. The massless Thirring model with fermion fields quantized in the chirally broken phase bosonizes to the quantum field theory of the free massless (pseudo)scalar field (Eur. Phys. J. C **20**, 723 (2001)). The wave functions of the ground state of the free massless (pseudo)scalar field are obtained from the BCS wave function by averaging over quantum fluctuations of the Thirring fermion fields. We show that we obtain wave functions, orthogonal, normalized and non–invariant under shifts of the massless (pseudo)scalar field. This testifies the spontaneous breaking of the field–shift symmetry in the quantum field theory of a free massless (pseudo)scalar field.

*E–mail: faber@kph.tuwien.ac.at, Tel.: +43–1–58801–14261, Fax: +43–1–5864203

†E–mail: ivanov@kph.tuwien.ac.at, Tel.: +43–1–58801–14261, Fax: +43–1–5864203

‡Permanent Address: State Polytechnic University, Department of Nuclear Physics, 195251 St. Petersburg, Russian Federation

A recent analysis of the massless Thirring model [1] has shown that the wave function $|\Omega\rangle$ of the ground state of the massless Thirring model is of BCS form

$$|\Omega\rangle = \prod_{k^1}[u_{k^1} + v_{k^1}\, a^{\dagger}(k^1)b^{\dagger}(-k^1)]|0\rangle, \tag{1}$$

where $|0\rangle$ is the perturbative, chiral symmetric vacuum, $a^{\dagger}(k^1)$ and $b^{\dagger}(-k^1)$ are creation operators of fermions and anti–fermions with momentum k^1 and $-k^1$, respectively, u_{k^1} and v_{k^1} are coefficient functions [1]

$$u_{k^1} = \sqrt{\frac{1}{2}\left(1 + \frac{|k^1|}{\sqrt{(k^1)^2 + M^2}}\right)}\quad,\quad v_{k^1} = \varepsilon(k^1)\sqrt{\frac{1}{2}\left(1 - \frac{|k^1|}{\sqrt{(k^1)^2 + M^2}}\right)}, \tag{2}$$

where $\varepsilon(k^1)$ is the sign function.

The wave function (1) describes the fermions in a finite volume L. According to Yoshida [2] in the limit $L \to \infty$ the wave function can be transcribed into the form

$$|\Omega\rangle = \exp\Big\{\int^{\infty}_{-\infty} dk^1\, \tilde{\Phi}(k^1)\, [a^{\dagger}(k^1)b^{\dagger}(-k^1) - b(-k^1)a(k^1)]\Big\}\, |0\rangle, \tag{3}$$

where the phase $\Phi(k^1)$ is defined by

$$\tilde{\Phi}(k^1) = \frac{1}{2}\,\arctan\Big(\frac{M}{k^1}\Big). \tag{4}$$

Under chiral rotations

$$\begin{aligned}\psi(x) &\to \psi\,'(x) = e^{i\gamma^5\alpha_{\rm A}}\psi(x),\\ \bar{\psi}(x) &\to \bar{\psi}\,'(x) = \bar{\psi}(x)\, e^{i\gamma^5\alpha_{\rm A}},\end{aligned} \tag{5}$$

where $\psi(x)$ and $\bar{\psi}(x)$ are operators of the massless Thirring fermion fields, the wave function (3) transforms as follows [1]

$$\begin{aligned}&|\Omega\rangle \to |\Omega;\alpha_{\rm A}\rangle = \\ &= \exp\Big\{\int^{\infty}_{-\infty} dk^1\tilde{\Phi}(k^1)[a^{\dagger}(k^1)b^{\dagger}(-k^1)e^{-2i\alpha_{\rm A}\varepsilon(k^1)} - b(-k^1)a(k^1)e^{+2i\alpha_{\rm A}\varepsilon(k^1)}]\Big\}|0\rangle.\end{aligned} \tag{6}$$

As has been shown in [1] the wave functions $|\Omega;\alpha_{\rm A}\rangle$ and $|\Omega;\alpha\,'_{\rm A}\rangle$ (5) are orthogonal for $\alpha\,'_{\rm A} \neq \alpha_{\rm A}\,({\rm mod}\,2\pi)$, i.e. $\langle\alpha\,'_{\rm A};\Omega|\Omega;\alpha_{\rm A}\rangle = \delta_{\alpha'_{\rm A}\alpha_{\rm A}}$.

In the following we prove that the bosonized version of the wave functions (6) are also orthogonal. We express (6) in terms of the fermion field operators $\psi(x)$ and $\bar{\psi}(x)$ using

$$\begin{aligned}a(k^1) &= \frac{1}{\sqrt{2\pi}}\int^{\infty}_{-\infty} dx^1\,\frac{u^{\dagger}(k^1)}{\sqrt{2k^0}}\,\psi(0,x^1)\,e^{\,-\,ik^1x^1},\\ b(-k^1) &= \frac{1}{\sqrt{2\pi}}\int^{\infty}_{-\infty} dy^1\,\psi^{\dagger}(0,y^1)\,\frac{v(-k^1)}{\sqrt{2k^0}}\,e^{\,+\,ik^1y^1}.\end{aligned} \tag{7}$$

The wave functions $u(k^1)$ and $v(k^1)$ are defined by [1]

$$u(k^1) = \sqrt{2k^0}\begin{pmatrix}\theta(+k^1)\\ \theta(-k^1)\end{pmatrix}\quad,\quad v(k^1) = \sqrt{2k^0}\begin{pmatrix}+\theta(+k^1)\\ -\,\theta(-k^1)\end{pmatrix}. \tag{8}$$

They are normalized to $u^\dagger(k^1)u(k^1) = v^\dagger(k^1)v(k^1) = 2k^0$ and $\theta(\pm k^1)$ are Heaviside functions. Substituting (7) in (6) we get

$$|\Omega;\alpha_{\rm A}\rangle = \exp\Big\{\int^{\infty}_{-\infty}dx^1\int^{\infty}_{-\infty}dy^1\,\Phi(x^1-y^1)\,\bar{\psi}(0,x^1)\gamma^5 e^{+2i\gamma^5\alpha_{\rm A}}\psi(0,y^1)\Big\}|0\rangle, \tag{9}$$

where $\Phi(x^1-y^1)$ is the original of $\tilde{\Phi}(k^1)$

$$\begin{aligned}\Phi(x^1-y^1) &= \int^{\infty}_{-\infty}\frac{dk^1}{4\pi}\arctan\Big(\frac{M}{k^1}\Big)e^{\,+\,ik^1(x^1-y^1)} = \\ &= \int^{\infty}_{-\infty}\frac{dk^1}{4\pi}\Big[\frac{\pi}{2}-\arctan\Big(\frac{k^1}{M}\Big)\Big]\,e^{+ik^1(x^1-y^1)} = \\ &= \frac{\pi}{4}\,\delta(x^1-y^1)+\int^{\infty}_{0}\frac{dk^1}{2\pi i}\arctan\Big(\frac{k^1}{M}\Big)\,\sin(k^1(x^1-y^1)) = \\ &= \frac{\pi}{4}\,\delta(x^1-y^1)+\frac{M}{x^1-y^1}\int^{\infty}_{0}\frac{dk^1}{2\pi i}\,\frac{\cos(k^1(x^1-y^1))}{(k^1)^2+M^2} = \\ &= \frac{\pi}{4}\,\delta(x^1-y^1)+\frac{1}{4i}\,\frac{e^{-M|x^1-y^1|}}{x^1-y^1} = \frac{1}{4i}\,\frac{e^{-M|x^1-y^1|}}{x^1-y^1-i\,0}.\end{aligned} \tag{10}$$

The function $\Phi(x^1-y^1)$ has the property $\Phi^*(x^1-y^1) = \Phi(y^1-x^1)$, which is important in order to provide a normalization of the wave function (9), $\langle\alpha_{\rm A};\Omega|\Omega;\alpha_{\rm A}\rangle = 1$.

Integrating over y^1 [1] we obtain

$$|\Omega;\alpha_{\rm A}\rangle = \exp\Big\{\frac{\pi}{2}\int^{\infty}_{-\infty}dx^1\bar{\psi}(0,x^1)\gamma^5 e^{\,+\,2i\gamma^5\alpha_{\rm A}}\psi(0,x^1)\Big\}|0\rangle, \tag{11}$$

According to the Nambu–Jona–Lasinio condition [3] the operators of the massless fermion fields $\psi(0,x^1)$ at time zero should be equal to the operators of the massive fermion fields $\Psi(0,x^1)$ with dynamical mass M, i.e. $\psi(0,x^1) = \Psi(0,x^1)$. This yields

$$\begin{aligned}|\{\Psi\};\Omega;\alpha_{\rm A}\rangle &= \exp\Big\{\frac{\pi}{2}\int^{\infty}_{-\infty}dx^1\bar{\Psi}(0,x^1)\gamma^5 e^{+2i\gamma^5\alpha_{\rm A}}\Psi(0,x^1)\Big\}|0\rangle = \\ &= \exp\{i\,F[\Psi,\bar{\Psi};\alpha_{\rm A}]\}|0\rangle,\end{aligned} \tag{12}$$

where the operator $\exp\{i\,F[\Psi,\bar{\Psi};\alpha_{\rm A}]\}$ is defined by

$$\exp\{i\,F[\Psi,\bar{\Psi};\alpha_{\rm A}]\} = \exp\Big\{\frac{\pi}{2}\int^{\infty}_{-\infty}dx^1\bar{\Psi}(0,x^1)\gamma^5 e^{+2i\gamma^5\alpha_{\rm A}}\Psi(0,x^1)\Big\}. \tag{13}$$

For convenience we have replaced $|\Omega;\alpha_{\rm A}\rangle$ by $|\{\Psi\};\Omega;\alpha_{\rm A}\rangle$ in order to underscore that the wave function of the ground state of the massless Thirring model in the chirally broken phase is a functional of the dynamical fermion fields.

As has been shown in [1] the massless Thirring model bosonizes to the quantum field theory of the free massless (pseudo)scalar field $\vartheta(x)$. In order to find the wave function of the ground state of the massless (pseudo)scalar field $\vartheta(x)$ we have to average the operator (13) over the dynamical fermion degrees of freedom. This can be carried out within the path–integral approach. We would like to accentuate that the integration over fermion degrees of freedom of the operator (13) should be understood as the bosonization of the operator $\exp\{\,i\,F[\Psi,\bar{\Psi};\alpha_{\rm A}]\}$ that means the replacement of fermion degrees of freedom by boson ones, $\exp\{\,i\,F[\Psi,\bar{\Psi};\alpha_{\rm A}]\}\to\exp\{\,i\,B[\vartheta;\alpha_{\rm A}]\}$. The integration over fermion degrees of freedom does not touch the wave function $|0\rangle$, which can be taken away in the functional integral.

In the massless Thirring model the generating functional of Green functions we define by [1]

$$\begin{aligned}&Z_{\rm Th}[J,\bar{J}]=\int\mathcal{D}\vartheta\,Z_{\rm Th}[\vartheta;J,\bar{J}]=\\&=\int\mathcal{D}\vartheta\,\mathcal{D}\Psi\,\mathcal{D}\bar{\Psi}\,\exp\Big\{i\int d^2x\,[\bar{\Psi}(x)(i\hat{\partial}-M\,e^{+i\gamma^5\vartheta(x)})\Psi(x)+\bar{J}(x)\Psi(x)+\bar{\Psi}(x)J(x)]\Big\},\end{aligned}\tag{14}$$

where $J(x)$ and $\bar{J}(x)$ are external sources of dynamical fermions.

The bosonized version of the operator $\exp\{\,i\,F[\Psi,\bar{\Psi};\alpha_{\rm A}]\}$ is defined by

$$\begin{aligned}&\exp\{\,i\,B[\vartheta;\alpha_{\rm A}]\}=\frac{1}{Z_{\rm Th}[\vartheta;0,0]}\int\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\,\exp\{\,i\,F[\Psi,\bar{\Psi};\alpha_{\rm A}]\}\\&\times\exp\Big\{i\int d^2x\,[\bar{\Psi}(x)(i\hat{\partial}-M\,e^{\,+\,i\gamma^5\vartheta(x)})\Psi(x)+\bar{J}(x)\Psi(x)+\bar{\Psi}(x)J(x)\Big]\Big\}\Big|_{J=\bar{J}=0}=\\&=\frac{1}{Z_{\rm Th}[\vartheta;0,0]}\int\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\,\exp\{i\int d^2x\,[\bar{\Psi}(x)\Big(i\hat{\partial}-M\,e^{\,+\,i\gamma^5\vartheta(x)}\Big)\Psi(x)\\&+\frac{\pi}{2}\,\delta(x^0)\,\bar{\Psi}(x)\gamma^5\,e^{\,+\,2i\gamma^5\alpha_{\rm A}}\,\Psi(x)]\Big\},\end{aligned}\tag{15}$$

By a chiral rotation $\Psi(x)\to e^{-i\gamma^5\vartheta(x)/2}\Psi(x)$ we obtain

$$\begin{aligned}&\exp\{\,i\,B[\vartheta;\alpha_{\rm A}]\}=\\&=\frac{1}{Z_{\rm Th}[\vartheta;0,0]}\int\mathcal{D}\Psi\mathcal{D}\bar{\Psi}\,\exp\Big\{i\int d^2x\,[\bar{\Psi}(x)\Big(i\hat{\partial}+\frac{1}{2}\,\gamma^{\mu}\varepsilon_{\mu\nu}\partial^{\nu}\vartheta(x)-M\Big)\Psi(x)\\&+\frac{\pi}{2}\,\delta(x^0)\bar{\Psi}(x)\gamma^5\,e^{\,+\,i\gamma^5(2\alpha_{\rm A}-\vartheta(x))}\,\Psi(x)]\Big\}.\end{aligned}\tag{16}$$

A possible chiral Jacobian, induced by this chiral rotation [1,4], is canceled by the contribution of $Z_{\rm Th}[\vartheta;0,0]$ in the denominator.

Below we are going to show that integrating over fermion fields and keeping leading terms in the large M expansion [1] we get the following expression for the operator $\exp\{\,i\,B[\vartheta;\alpha_{\rm A}]\}$

$$\exp\{\,i\,B[\vartheta;\alpha_{\rm A}]\}=\exp\Big\{i\,\frac{\pi}{2}\,\frac{M}{g}\int^{\infty}_{-\infty}dx^1\,\sin\Big(\beta\vartheta(0,x^1)-2\alpha_{\rm A}\Big)\Big\},\tag{17}$$

where we have also used the gap–equation for the dynamical mass M (see Eq.(1.14) of Ref.[1]). The coupling constant β is related to the coupling constant g of the massless Thirring model [1,5]

$$\frac{8\pi}{\beta^2}=1-e^{-2\pi/g}.\tag{18}$$

Expression (17) has been obtained as follows. Integration over the fermion fields gives the expression [1]

$$\exp\{\,i\,B[\vartheta;\alpha_{\rm A}]\} = \exp\Big\{i\int d^2x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{1}{4\pi}\int\prod_{\ell}^{n-1}\frac{d^2x_\ell d^2k_\ell}{(2\pi)^2}\,e^{\,-\,ik_\ell\cdot(x_\ell - x)}$$
$$\times\int\frac{d^2k}{\pi i}\,{\rm tr}\Big\{\frac{1}{M-\hat{k}}\Phi(x_1)\frac{1}{M-\hat{k}-\hat{k}_1}\Phi(x_2)\ldots\Phi(x_{n-1})\frac{1}{M-\hat{k}-\hat{k}_1-\ldots-\hat{k}_{n-1}}\Phi(x)\Big\}, \tag{19}$$

where we have denoted

$$\Phi(x_k) = -\frac{\pi}{2}\,\delta(x^0_k)i\gamma^5\,e^{i\gamma^5(2\alpha_{\rm A}\,-\,\vartheta(x_k))} \tag{20}$$

with $x_0 = x$ and $k = 0,1,2,\ldots,n-1$. For the subsequent calculation of the momentum and space–time integrals we suggest to use some kind of Pauli–Villars regularization. We smear the δ–functions $\delta(x^0_k)$ with the scales M_j

$$\Phi(x_k;M_j) = -\frac{\pi}{2}\,f(M_jx^0_k)i\gamma^5\,e^{i\gamma^5(2\alpha_{\rm A}\,-\,\vartheta(x_k))}. \tag{21}$$

We require that in the limit $M_j\to\infty$ the function $f(M_jx^0_k)$ converges to the δ–function $\delta(x^0_k)$. Introducing then the coefficients C_j, which satisfy the constraints

$$\sum^N_{j=1}C_j = 1,\quad \sum^N_{j=1}C_jM^n_j = 0\,,\;n = 1,2,\ldots\,, \tag{22}$$

we rewrite (19) as follows

$$\exp\{\,i\,B[\vartheta;\alpha_{\rm A}]\} = \exp\Big\{i\sum^N_{j=1}C_j\int d^2x\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{1}{4\pi}\int\prod_{\ell}^{n-1}\frac{d^2x_\ell d^2k_\ell}{(2\pi)^2}\,e^{\,-\,ik_\ell\cdot(x_\ell - x)}$$
$$\times\int\frac{d^2k}{\pi i}\,{\rm tr}\Big\{\frac{1}{M-\hat{k}}\Phi(x_1;M_j)\frac{1}{M-\hat{k}-\hat{k}_1}\Phi(x_2;M_j)\ldots\Phi(x_{n-1};M_j)$$
$$\times\frac{1}{M-\hat{k}-\hat{k}_1-\ldots-\hat{k}_{n-1}}\Phi(x;M_j)\Big\}. \tag{23}$$

Taking, first, the large M expansion for the calculation of momentum integrals [1] and using the constraints (22) we get (17).

Expression (17) can be obtained directly from (13) by using our bosonization rules (see Eq.(3.24) of Ref.[1])

$$\begin{aligned}\bar{\Psi}(0,x^1)\gamma^5\Psi(0,x^1) &= +\,i\,\frac{M}{g}\,\sin(\beta\,\vartheta(0,x^1)),\\ \bar{\Psi}(0,x^1)\Psi(0,x^1) &= -\,\frac{M}{g}\,\cos(\beta\,\vartheta(0,x^1)).\end{aligned} \tag{24}$$

Substituting (24) in (13) we arrive at (17).

Using the operator (17) we define the bosonic wave function

$$\begin{aligned}|\{\vartheta\};\Omega;\alpha_{\rm A}\rangle &= \exp\{\,i\,B[\vartheta;\alpha_{\rm A}]\}|0\rangle = \\ &= \exp\Big\{i\,\frac{\pi}{2}\,\frac{M}{g}\int^{\infty}_{-\infty}dx^1\,\sin\Big(\beta\vartheta(0,x^1) - 2\alpha_{\rm A}\Big)\Big\}|0\rangle,\end{aligned} \tag{25}$$

which is normalized to unity. Now we have to show that such wave functions are orthogonal for $\alpha\,'_{\rm A} \neq \alpha_{\rm A}\,({\rm mod}\,2\pi)$

$$\begin{aligned}&\langle\alpha\,'_{\rm A};\Omega;\{\vartheta\}|\{\vartheta\};\Omega;\alpha_{\rm A}\rangle = \Big\langle 0\Big|\exp\Big\{+\,i\,\pi\,\frac{M}{g}\,\sin(\alpha\,'_{\rm A} - \alpha_{\rm A})\\ &\times\int^{\infty}_{-\infty}dx^1\cos(\beta\,\vartheta(0,x^1) - \alpha\,'_{\rm A} - \alpha_{\rm A})\Big\}\Big|0\Big\rangle = \lim_{L\to\infty}\exp\Big\{+\,i\,\pi\,\frac{LM}{g}\,\sin(\alpha\,'_{\rm A} - \alpha_{\rm A})\Big\}\\ &\times\Big\langle 0\Big|\exp\Big\{-\,2\pi i\,\frac{M}{g}\,\sin(\alpha\,'_{\rm A} - \alpha_{\rm A})\int^{\infty}_{-\infty}dx^1\sin^2\Big(\frac{\beta}{2}\,\vartheta(0,x^1) - \frac{\alpha\,'_{\rm A} + \alpha_{\rm A}}{2}\Big)\Big\}\Big|0\Big\rangle = \\ &= \delta_{\alpha\,'_{\rm A}\alpha_{\rm A}}\Big\langle 0\Big|\exp\Big\{-\,2\pi i\,\frac{M}{g}\,\sin(\alpha\,'_{\rm A} - \alpha_{\rm A})\int^{\infty}_{-\infty}dx^1\sin^2\Big(\frac{\beta}{2}\,\vartheta(0,x^1) - \frac{\alpha\,'_{\rm A} + \alpha_{\rm A}}{2}\Big)\Big\}\Big|0\Big\rangle = \\ &= \delta_{\alpha\,'_{\rm A}\alpha_{\rm A}}.\end{aligned} \tag{26}$$

As has been shown in [1] the massless Thirring model with fermion fields quantized in the chirally broken phase bosonizes to the free massless (pseudo)scalar field theory with the Lagrangian $\mathcal{L}(x) = \frac{1}{2}\,\partial_{\mu}\vartheta(x)\partial^{\mu}\vartheta(x)$, which is invariant under shifts of the field $\vartheta(x) \to \vartheta\,'(x) = \vartheta(x) + \alpha$ with $\alpha \in \mathbb{R}^1$ [6].

The wave function (25) describes the ground state of the free massless (pseudo)scalar field $\vartheta(x)$. Since this wave function is not invariant under the field–shifts, the symmetry is spontaneously broken. The quantitative characteristic of the spontaneously broken phase in the quantum field theory of the free massless (pseudo)scalar field $\vartheta(x)$ is a non–vanishing spontaneous magnetization $\mathcal{M} = 1$ [6]. This confirms fully the existence of the chirally broken phase in the massless Thirring model obtained in [1,4–6].

References

[1] M. Faber and A. N. Ivanov, Eur. Phys. J. C **20**, 723 (2001), hep-th/0105057.

[2] K. Yoshida, Nucl. Phys. B **187**, 103 (1981).

[3] Y. Nambu and G. Jona–Lasinio, Phys. Rev. **122**, 345 (1961).

[4] M. Faber and A. N. Ivanov, hep–th/0112183.

[5] M. Faber and A. N. Ivanov, hep–th/0205249.

[6] M. Faber and A. N. Ivanov, Eur. Phys. J. C **24**, 653 (2002), hep–th/0112184; hep–th/0204237; hep–th/0206244; hep–th/0206034.