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Rings of short $\mathcal{N}=3$ superfields in three dimensions and M-theory on $AdS_4 \times N^{0,1,0}$. [†]

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Abstract

In this paper we investigate three–dimensional superconformal gauge theories with $\mathcal{N}=3$ supersymmetry. Independently from specific models, we derive the shortening conditions for unitary representations of the $\mathrm{Osp}(3|4)$ superalgebra and we express them in terms of differential constraints on three dimensional $\mathcal{N}=3$ superfields. We find a ring structure underlying these short representations, which is just the direct generalization of the chiral ring structure of $\mathcal{N}=2$ theories. When the superconformal field theory is realized on the world–volume of an M2–brane such superfield ring is the counterpart of the ring defined by the algebraic geometry of the 8–dimensional cone transverse to the brane. This and other arguments identify the $\mathcal{N}=3$ superconformal field theory dual to M-theory compactified on $\mathrm{AdS}_4 \times N^{0,1,0}$. It is an $\mathcal{N}=3$ gauge theory with $\mathrm{SU(N)} \times \mathrm{SU(N)}$ gauge group coupled to a suitable set of hypermultiplets, with an additional Chern Simons interaction. The AdS/CFT correspondence can be directly verified using the recently worked out Kaluza Klein spectrum of $N^{0,1,0}$ and we find a perfect match. We also note that besides the usual set of BPS conformal operators dual to the lightest KK states, we find that the composite operators corresponding to certain massive KK modes are organized into a massive spin $\frac{3}{2}$ $\mathcal{N}=3$ multiplet that might be identified with the super-Higgs multiplet of a spontaneously broken $\mathcal{N}=4$ theory. We investigate this intriguing and inspiring feature in a separate paper.

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1 Introduction

There is evidence that all M-theory or Type II string backgrounds of the form $AdS_{p+2} \times X^{d-p-2}$ in d-dimensions, where X^{d-p-2} is an Einstein manifold, are dual to CFT's in $p+1$ dimensions, living on the world-volume of p-branes [1]. Many supergravity solutions associated with coset spaces X^{d-p-2} are known and have been studied in the eighties. It is therefore interesting to identify the associated CFT's and compare the KK spectrum with the spectrum of conformal operators. The dual CFT is realized on the world-volume of p-branes living at the singularities of $C(X^{d-p-2})$, the cone over X^{d-p-2} [2, 3, 4]. Unfortunately, there is no general method for determining the world-volume theory for branes in curved space-time, so one can only rely on geometrical intuition. Consider first the AdS$_5$ case. There are only two supersymmetric five-dimensional coset spaces, S^5 and $T^{1,1}$. S^5 is at the origin of the original Maldacena conjecture. The dual four-dimensional CFT dual to AdS$_5 \times T^{1,1}$ have been identified in [2]. Many checks of this identification can be found in the literature [2, 5, 6, 7]. In the AdS$_4$ case, there is a richer zoo of seven-dimensional coset spaces, corresponding to supersymmetric backgrounds of M-theory [8]. In [9], we proposed candidate dual CFT's for the two $\mathcal{N} = 2$ solutions, $Q^{1,1,1}$ [10] and $M^{1,1,1}$ [11], using intuition from toric geometry. The KK spectrum and the properties of wrapped M5-branes associated to baryons nicely fit with the CFT expectations. A candidate dual for the $\mathcal{N} = 2$ solution $V^{5,2}$, which does not admit toric description, has been proposed in [12]. The purpose of this paper, which is the natural continuation of [9], is to discuss the $\mathcal{N} = 3$ solution AdS$_4 \times N^{0,1,0}$.

$N^{0,1,0}$ can be written as $\mathrm{SU}(3)/\mathrm{U}(1)$ [13]. It has isometry $\mathrm{SU}(3) \times \mathrm{SU}(2)$ and preserves $\mathcal{N} = 3$ supersymmetry. Using geometrical arguments similar to those in [2, 9], one is led to consider an $\mathcal{N} = 4$ theory $\mathrm{SU(N)} \times \mathrm{SU(N)}$ with three hypermultiplets in the bi-fundamental representation of the two gauge groups. It was proposed in [14] that the $\mathcal{N} = 3$ CFT can be just obtained by adding an $\mathcal{N} = 3$ preserving Chern-Simons term. We shall give evidence for this proposal by carefully comparing the observables in the CFT and the excitations of the supergravity background. The complete KK spectrum of M-theory on AdS$_4 \times N^{0,1,0}$ has been recently computed [15, 16]. Both KK and conformal field theory composite operators fall in representations of the superalgebra $\mathrm{Osp}(3|4)$ and can be conveniently described in terms of three-dimensional $\mathcal{N} = 3$ superfields. In this paper, we first derive a general formalism for studying $\mathcal{N} = 3$ superfields and the $\mathrm{Osp}(3|4)$ shortening conditions and we then apply it to the comparison between KK states and CFT composite operators. We shall exhibit the CFT supermultiplets of composite operators associated to all the short multiplets belonging to the KK spectrum [1]. Indeed the analysis of the $\mathcal{N} = 3$ solution reveals that all the general features which were common to the $T^{1,1}$, $Q^{1,1,1}$, $M^{1,1,1}$ and $V^{5,2}$ compactifications [5, 7, 9, 12] still hold true also for $N^{0,1,0}$. In particular, there are long multiplets with protected rational dimensions. We show that (in analogy with the other compactifications) many of them can be identified with CFT multiplets obtained by tensoring massless and short multiplets, as suggested in [7]. We focus, in particular, on a very special long multiplet, which contains the volume of the internal manifold as one of the scalar components, and it is therefore *universal* for

[1]One could also make an independent check of the dimension of supersingletons in the CFT by looking at the baryonic operators [6, 9], which can be realized as wrapped M5-branes. Since such a calculation would simply be a repetition of known calculations that reveals no new feature we skip such additional check, which should be straightforward.

all compactification. In $\mathcal{N}=2$ compactification, the volume multiplet is a long vector multiplet. In $\mathcal{N}=3$ it becomes a long gravitino multiplet. In $N^{0,1,0}$, it has the right quantum numbers to be generated in a superHiggs mechanism, suggesting that the theory is a spontaneously broken phase of an $\mathcal{N}=4$ theory. This intriguing phenomenon will be investigated in a forthcoming publication [17].

The plan of this work is as follows. In section 2 we introduce the $\mathcal{N}=3$ superspace formalism and derive the shortening conditions of the $\mathrm{Osp}(3|4)$ irreducible representations in terms of differential constraints on primary conformal superfields. In section 3 we discuss the general structure of $\mathcal{N}=3$ three dimensional gauge theories using the component formalism and emphasizing the role of the Chern Simons interaction. In section 4 we identify the $\mathcal{N}=3$ gauge theory whose conformal fixed point realizes the AdS/CFT correspondence with the $N^{0,1,0}$ compactification of M-theory, while section 5 is devoted to test this correspondence. Finally section 6 briefly discusses, from a CFT point of view the long rational spin $\frac{3}{2}$ supermultiplet that suggests an interpretation in terms of superHiggs mechanism and that will be the focus of a forthcoming paper.

2 Three dimensional $\mathcal{N}=3$ superspace

In order to simplify the study of unitary irreducible representations of the $\mathrm{Osp}(3|4)$ superconformal algebra (see eq. (A.2) of [18]), we introduce a three dimensional $\mathcal{N}=3$ superspace formalism. This allows us to identify the short representations as particular constrained superfields. To this effect we introduce six Grassmann coordinates, $\theta^i{}_\alpha$, transforming as three Majorana bispinors and as a triplet of the $\mathrm{SO}(3)_R$ R-symmetry subalgebra:

$$\begin{aligned} [T^{ij},T^{kl}] &= -i\,(\delta^{jk}\,T^{il}-\delta^{ik}\,T^{jl}-\delta^{jl}\,T^{ik}+\delta^{il}\,T^{jk})\,, \\ T^{ij} &= \theta^i_\alpha\frac{\partial}{\partial\theta^j_\alpha}-\theta^j_\alpha\frac{\partial}{\partial\theta^i_\alpha}\,. \end{aligned} \tag{2.1}$$

The other relevant generators have the following representation:

$$\begin{aligned} P_\mu &= -i\partial_\mu\,, \\ q^{\alpha i} &= \frac{\partial}{\partial\theta^i_\alpha}+\frac{1}{2}\not\partial^\alpha{}_\beta\theta^{\beta i}\,, \\ \{q^{\alpha i},q^{\beta j}\} &= -i\,\delta^{ij}\,\not P^{\alpha\beta}\,, \\ [T^{ij},q^{\alpha k}] &= -\,i\,(\delta^{jk}\,q^{\alpha i}-\delta^{ik}\,q^{\alpha j})\,. \end{aligned} \tag{2.2}$$

We furthermore introduce the supercovariant derivatives:

$$\begin{aligned} \mathcal{D}^{\alpha i} &= \frac{\partial}{\partial\theta^i_\alpha}-\frac{1}{2}\not\partial^\alpha{}_\beta\theta^{\beta i}\,, \\ \{\mathcal{D}^{\alpha i},\mathcal{D}^{\beta j}\} &= i\,\delta^{ij}\,\not P^{\alpha\beta}\,, \qquad \{q^{\alpha i},\mathcal{D}^{\beta j}\}=0\,, \end{aligned} \tag{2.3}$$

in terms of which the shortening conditions can be expressed.

It is convenient to use the spherical irreducible basis of R-symmetry representations rather than the cartesian one, so that the Grassmann coordinates are renamed as in the following example:

$$\begin{cases} \theta^+ &= -\frac{1}{\sqrt{2}}(\theta^1+i\theta^2) \\ \theta^0 &= \theta^3 \\ \theta^- &= \frac{1}{\sqrt{2}}(\theta^1-i\theta^2) \end{cases} \tag{2.4}$$

An $\mathcal{N}=3$ superfield $\Theta = \Theta(x,\theta)$ is a function of the bosonic x^μ and Grassmann coordinates θ^i, whose expansion in powers of θ^0 gives us the decomposition of the corresponding $\mathrm{Osp}(3|4)$ representation in $\mathcal{N}=2$ supermultiplets.

We are mainly interested in conformal primary superfields (see [19, 20]), defined by:

$$\Theta(x,\theta) = \exp\left[\mathrm{i}\, x_\mu P^\mu + \theta^i q^i\right] \Theta(0)\,, \tag{2.5}$$

where $\Theta(0)$ is a primary field: $[s^i_\alpha, \Theta(0)] = [K_\mu, \Theta(0)] = 0$. An irreducible representation of the superfield $\Theta(x,\theta)$ is characterized by the Cartan labels of its highest weight state $\Theta(0)$, namely its scaling dimension, its SO(1,2) Lorentz and $\mathrm{SO}(3)_{\boldsymbol{R}}$ R-symmetry quantum numbers. We denote the R-symmetry isospin with the suffix $\boldsymbol{J}$ and the Lorentz character with a set of spinorial indices spanning the SO(1,2) irreducible representation.

2.1 Short $\mathrm{Osp}(3|4)$ representations as constrained superfields

In this section we analyze the differential constraints on the superfields that force their components to transform into short $\boldsymbol{BPS}$ representations of the $\mathrm{Osp}(3|4)$ superalgebra. [2] As in the best known case of $\mathcal{N}=2$ superfields, we find the existence of two kinds of constraints. The first one is a first order differential constraint given by:

$$\mathcal{D}_{\alpha_1} \otimes_{h.w.} \Theta^{J\ (\alpha_1\cdots\alpha_n)}(x,\theta) = 0\,, \tag{2.6}$$

or

$$\mathcal{D}_\alpha \otimes_{h.w.} \Theta^J(x,\theta) = 0 \tag{2.7}$$

for scalar superfields (without Lorentz indices). Here the tensor product refers to the $\mathrm{SO}(3)_{\boldsymbol{R}}$ isospin and “h.w” stands for highest weight. This means that only the $\mathrm{SO}(3)_{\boldsymbol{R}}$ highest weight part of the tensor product (2.6), between the isospin triplet $\mathcal{D}$ and the superfield Θ^J, is put to zero.

In complete analogy with the $\mathcal{N}=2$ case, we have another kind of constraint. It is a second order differential constraint, and it is allowed only for scalar (from the Lorentz viewpoint) superfields (of any isospin):

$$\mathcal{D}_\alpha \otimes_{h.w.} \mathcal{D}^\alpha \otimes_{h.w.} \Theta^J = 0\,. \tag{2.8}$$

The superconformal covariance of equations (2.6) and (2.8) poses some constraints on the conformal dimensions of the superfields.

2.1.1 The $\mathcal{N}=3$ analogue of the chiral ring

Let us now analyze in more details the constraint (2.7) for the lowest Lorentz and isospin representations. The most interesting case to analyze is that of a scalar (from the Lorentz viewpoint) superfield. In this case the Lorentz character allows the existence of a **ring structure** which generalizes the **chiral ring** of $\mathcal{N}=2$ theories and seems to be a common feature shared by all the three dimensional superconformal field theories. The ring

[2] When this paper was nearly finished we learned of the recent work by Ferrara and Sokatchev [21] that analyzes the differential constraints to be imposed, in harmonic superspace, on $\mathcal{N}=8$ superfields in order to describe short representations of the algebra $\mathrm{Osp}(8|4)$. Quite likely our results for $\mathrm{Osp}(3|4)$ can be described in that general formalism. A comparison is postponed to future investigations

multiplicative operation is given by extracting the highest weight irreducible part from the ordinary (tensor) product of two short superfields of isospin J and J' respectively:

$$\Theta^J \times \widetilde{\Theta}^{J'} = \left(\Theta \otimes_{h.w.} \widetilde{\Theta}\right)^{J+J'} . \tag{2.9}$$

Indeed one can show that:

$$\left.\begin{array}{l} \mathcal{D}^{J=1} \otimes_{h.w.} \Theta^J = 0 \\ \mathcal{D}^{J=1} \otimes_{h.w.} \widetilde{\Theta}^{J'} = 0 \end{array}\right\} \Longrightarrow \mathcal{D}^{J=1} \otimes_{h.w.} (\Theta \otimes_{h.w.} \widetilde{\Theta})^{J+J'} = 0 . \tag{2.10}$$

The simplest case of short scalar superfield, apart from the trivial constant, is that of isospin $J=1/2$. In this case the shortening condition (2.7) reads:

$$\left\{\begin{array}{c} \mathcal{D}^+\Theta^+ = 0 , \\ \sqrt{2}\mathcal{D}^0\Theta^+ + \mathcal{D}^+\Theta^- = 0 , \\ \mathcal{D}^-\Theta^+ + \sqrt{2}\mathcal{D}^0\Theta^- = 0 , \\ \mathcal{D}^-\Theta^- = 0 . \end{array}\right. \tag{2.11}$$

To make contact with $\mathcal{N}=2$ superspace formalism of [18] it is useful to expand the most general form of $\Theta^{J=1/2}$ in powers of θ^0. So we have:

$$\begin{pmatrix} \Theta^+ \\ \Theta^- \end{pmatrix} = \begin{pmatrix} \Phi_s \\ \Psi_s^\dagger \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{D}^+\Psi_s^\dagger \\ \mathcal{D}^-\Phi_s \end{pmatrix} \theta^0 , \tag{2.12}$$

where Φ_s and Ψ_s are two $\mathcal{N}=2$ **supersingletons**, namely they are two functions of x^μ and $\theta^\pm$ fulfilling the constraints $\mathcal{D}^+\Phi_s = \mathcal{D}^-\mathcal{D}^-\Phi_s = 0$. Hence we see that the direct generalization of the $\mathcal{N}=2$ supersingleton is the $\mathcal{N}=3$ short scalar superfield of minimum isospin. Let us now look at the case $J=1$, whose most general form is:

$$\begin{pmatrix} \Theta^+ \\ \Theta^0 \\ \Theta^- \end{pmatrix} = \begin{pmatrix} \Phi \\ \Sigma \\ \Psi^\dagger \end{pmatrix} - \begin{pmatrix} \mathcal{D}^+\Sigma \\ \frac{1}{2}(\mathcal{D}^-\Phi + \mathcal{D}^+\Psi^\dagger) \\ \mathcal{D}^-\Sigma \end{pmatrix} \theta^0 - \frac{1}{8} \begin{pmatrix} \mathcal{D}^+\mathcal{D}^+\Psi^\dagger \\ 2\mathcal{D}^+\mathcal{D}^-\Sigma \\ \mathcal{D}^-\mathcal{D}^-\Phi \end{pmatrix} (\theta^0\theta^0) , \tag{2.13}$$

where Φ and Ψ are $\mathcal{N}=2$ chiral superfields and Σ is a linear superfield ($\mathcal{D}^+\mathcal{D}^+\Sigma = \mathcal{D}^-\mathcal{D}^-\Sigma = 0$), which is a conserved (massless) vector current. Hence the superfield (2.13) represents the direct generalization of the $\mathcal{N}=2$ massless vector.

The $\mathcal{N}=3$ short scalar superfields of higher isospin can be obtained by multiplying smaller ones following the ring operation, i.e. by tensoring and taking the maximum isospin irreducible part. It is interesting to analyze the $\mathcal{N}=2$ field content, i.e. the single independent θ^0 components of such superfields. This gives an analytical version of the algebraic $\mathcal{N}=3 \to \mathcal{N}=2$ decomposition of the short multiplets (see tables (5.1) and (5.2)). The first thing to note is that the shortening constraint (2.6) implies that the only independent components are the $\theta^0=0$ restrictions of the $\mathcal{N}=3$ superfields. In the case of integer isospin we always obtain the same pattern:

$$\begin{pmatrix} \Phi^1\Phi^2\ldots\Phi^k \\ \Sigma^1\Phi^2\ldots\Phi^k + \ldots + \Phi^1\ldots\Phi^{(k-1)}\Sigma^k \\ \ldots \\ \Sigma^1\Psi^{2\dagger}\ldots\Psi^{k\dagger} + \ldots + \Psi^{1\dagger}\ldots\Psi^{(k-1)\dagger}\Sigma^k \\ \Psi^{1\dagger}\Psi^{2\dagger}\ldots\Psi^{k\dagger} \end{pmatrix}^{J=k}_{\theta^0=0} \begin{array}{cc} \leftarrow & \text{chiral} \\ \leftarrow & \text{short vector} \\ \leftarrow & 2k-1 \text{ long vectors} \\ \leftarrow & \text{short vector} \\ \leftarrow & \text{chiral} \end{array} \tag{2.14}$$

The half-integer isospin chiral superfields have a completely analogous structure. The only difference is that each field contains an odd number of $\mathcal{N}=2$ supersingletons. Thus the corresponding states are not observed in the Kaluza Klein spectrum of supergravity compactifications.

2.1.2 The $\mathcal{N}=3$ short gravitinos

Let us now analyze the second order constraint (2.8), which yields the $\mathcal{N}=3$ short gravitinos. The lowest isospin case ($J=0$) corresponds to the massless gravitino superfield:

$$\Theta = \Sigma + G_\alpha \theta^{0\,\alpha} + \tfrac{1}{4}(\mathcal{D}^+\mathcal{D}^-)\Sigma(\theta^0\theta^0)\,, \tag{2.15}$$

where G^α is an $\mathcal{N}=2$ massless gravitino ($\mathcal{D}^\pm_\alpha G^\alpha = 0$) and Σ a linear superfield, namely a massless vector.

Analogously we can derive the form of the most general $J=1$ short spinor superfield:

$$\begin{pmatrix} \Theta^+ \\ \Theta^0 \\ \Theta^- \end{pmatrix} = \begin{pmatrix} \Sigma^+ \\ \Sigma^0 \\ \Sigma^- \end{pmatrix} + \begin{pmatrix} G^+\theta^0 \\ G^0\theta^0 \\ G^-\theta^0 \end{pmatrix} + \text{derivative terms}\,, \tag{2.16}$$

which has six $\mathcal{N}=2$ independent components:

- two short gravitinos ($\mathcal{D}^+G^+ = \mathcal{D}^-G^- = 0$);
- one long gravitino, G^0;
- two short vectors: $(\mathcal{D}^+\mathcal{D}^+)\Sigma^+ = (\mathcal{D}^-\mathcal{D}^-)\Sigma^- = 0$;
- one long vector, Σ^0.

This $\mathcal{N}=2$ superfield content perfectly fits the algebraic decomposition of table (5.2). Short gravitinos of higher isospin can be obtained by composing the $J=0$ short gravitino with chiral superfields of any J. Obviously, even in this case, half-integer isospin gravitinos are not observed in the Kaluza Klein spectra, due to the presence of an odd number of supersingletons.

2.1.3 The $\mathcal{N}=3$ short gravitons

The $\mathcal{N}=3$ short graviton multiplets are realized by spinor superfields fulfilling the first order constraint (2.6). Again, the massless case corresponds to the lowest ($J=0$) isospin superfield:

$$\Theta^\alpha = G^\alpha + T^{(\alpha\beta)}\theta^0_\beta - \tfrac{1}{4}\partial^\alpha{}_\beta G^\beta(\theta^0\theta^0)\,, \tag{2.17}$$

where G^α is an $\mathcal{N}=2$ massless gravitino ($\mathcal{D}^\pm_\alpha G^\alpha = 0$) and $T^{(\alpha\beta)}$ is a massless graviton ($\mathcal{D}^\pm_\alpha T^{(\alpha\beta)} = 0$).

In an analogous way we can derive the form of the most general $J=1$ short spinor superfield:

$$\begin{pmatrix} \Theta^{+\,\alpha} \\ \Theta^{0\,\alpha} \\ \Theta^{-\,\alpha} \end{pmatrix} = \begin{pmatrix} G^{+\,\alpha} \\ G^{0\,\alpha} \\ G^{-\,\alpha} \end{pmatrix} + \begin{pmatrix} T^{+(\alpha\beta)} \\ T^{0\,(\alpha\beta)} \\ T^{-(\alpha\beta)} \end{pmatrix} \theta^0_\beta + \text{derivative terms}\,, \tag{2.18}$$

which has six $\mathcal{N}=2$ independent components (see table (5.2)):

- two short gravitons ($\mathcal{D}^+_\alpha T^{+(\alpha\beta)} = \mathcal{D}^-_\alpha T^{-(\alpha\beta)} = 0$) ;
- one long graviton, $T^{0\,(\alpha\beta)}$;
- two short gravitinos ($\mathcal{D}^+ G^+ = \mathcal{D}^- G^- = 0$) ;
- one long gravitino, G^0 .

Short gravitons of higher isospin can be obtained by composing the $J = 0$ massless graviton with chiral superfields of any J. Again, half-integer isospin gravitons, containing an odd number of supersingletons, are not observed in the Kaluza Klein spectra.

3 $\mathcal{N} = 3$ gauge theory in three dimensions

In this section we discuss the structure of a three dimensional gauge theory with $\mathcal{N} = 3$ supersymmetry. In paper [18] we have already given the general form of an $\mathcal{N} = 2$ three–dimensional gauge theory and the $\mathcal{N} = 3$ case is just a particular case in that class since a theory with $\mathcal{N} = 3$ SUSY, must *a fortiori* be an $\mathcal{N} = 2$ theory. In [18] we have also considered, within the $\mathcal{N} = 2$ class, the case of $\mathcal{N} = 4$ theories. These are obtained through dimensional reduction of an $\mathcal{N}_4 = 2$ theory in four–dimensions. Indeed since each $D = 4$ Majorana spinor splits, under dimensional reduction on a circle $\mathbb{S}^1$, into two $D = 3$ Majorana spinors, the number of three–dimensional supercharges is just twice the number of $D = 4$ supercharges:

$$\mathcal{N}_3 = 2 \times \mathcal{N}_4 \tag{3.1}$$

The $\mathcal{N}_3 = 3$ case corresponds to an intermediate situation. It is an $\mathcal{N}_3 = 2$ theory with the field content of an $\mathcal{N}_3 = 4$ one, but with additional $\mathcal{N}_3 = 2$ interactions that respect three out of the four supercharges obtained through dimensional reduction. Using an $\mathcal{N} = 2$ superfield formalism and the notion of twisted chiral multiplets it was shown in [22] that for abelian gauge theories these additional $\mathcal{N}_3 = 3$ interactions are

1. A Chern Simons term, with coefficient α
2. A mass-term with coefficient $\mu = \alpha$ for the chiral field Y^I in the adjoint of the color gauge group. By this latter we denote the complex field belonging, in four dimensions, to the $\mathcal{N}_4 = 2$ gauge vector multiplet.

In this section we want to retrieve the same result in the component formalism which is better suited to discus the relation between the world–volume gauge theory and the geometry of the transverse cone $\mathcal{C}(X^7)$. Then we dismiss superfields and turning to components we discuss the general form of a non abelian $\mathcal{N} = 3$ gauge theory in three dimensions.

3.1 The field content and the interactions

Our strategy is that of writing the $\mathcal{N} = 3$ gauge theory as aspecial case of an $\mathcal{N} = 2$ theory, whose general form was derived in [18]. For this latter the field content is given by:

$$\begin{array}{|c|c|c|c|}\hline \text{multipl. type } / SO(1,2) \text{ spin} & 1 & \frac{1}{2} & 0 \\ \hline \text{vector multipl.} & \underbrace{A^I_\mu}_{\text{gauge field}} & \underbrace{\left(\lambda^{+I},\lambda^{-I}\right)}_{\text{gauginos}} & \underbrace{M^I}_{\text{real scalar}} \\ \hline \text{chiral multip.} & & \underbrace{\left(\chi^{+i},\chi^{-i^*}\right)}_{\text{chiralinos}} & \underbrace{z^i,\ \overline{z}^{i^*}}_{\text{complex scalars}} \\ \hline \end{array} \tag{3.2}$$

and without Fayet Iliopoulos terms, which do not exist in non abelian gauge theories with no U(1) factors, the complete Lagrangian has the following form:

$$\mathcal{L}^{\mathcal{N}=2} = \mathcal{L}^{kinetic} + \mathcal{L}^{fermion\ mass} + \mathcal{L}^{potential}\,, \tag{3.3}$$

where

$$\begin{aligned} \mathcal{L}^{kinetic} &= \Big\{\eta_{ij^*}\nabla_\mu z^i \nabla^\mu \overline{z}^{j^*} - \tfrac{1}{2}\eta_{ij^*}\left(\chi^{-j^*}\nabla\!\!\!\!/\,\chi^{+i} + \chi^{+i}\nabla\!\!\!\!/\,\chi^{-j^*}\right) \\ &\quad - g_{IJ}F^I_{\mu\nu}F^{J\,\mu\nu} - \alpha\left(g_{IJ}F^I_{\mu\nu}A^J_\rho + f_{IJK}A^I_\mu A^J_\nu A^K_\rho\right)\epsilon^{\mu\nu\rho} \\ &\quad + \tfrac{1}{2}g_{IJ}\nabla_\mu M^I\nabla^\mu M^J - \tfrac{1}{4}g_{IJ}\left(\lambda^{-I}\nabla\!\!\!\!/\,\lambda^{+J} + \lambda^{+I}\nabla\!\!\!\!/\,\lambda^{-J}\right)\Big\}\,d^3x \end{aligned} \tag{3.4}$$

$$\begin{aligned} \mathcal{L}^{fermion\ mass} &= \Big\{\tfrac{i}{2}\left(\chi^{+i}\partial_i\partial_j W(z)\chi^{+j} - \chi^{-i^*}\partial_{i^*}\partial_{j^*}\overline{W}(\overline{z})\chi^{-j^*}\right) \\ &\quad - \tfrac{i}{2}f_{IJK}M^I\lambda^{-J}\lambda^{+K} - i\chi^{-j^*}M^I(T_I)_{ij^*}\chi^{+i} \\ &\quad - \left(\chi^{-i^*}\lambda^{+I}(T_I)_{i^*j}z^j - \chi^{+i}\lambda^{-I}(T_I)_{ij^*}\overline{z}^{j^*}\right) \\ &\quad - \tfrac{1}{2}\alpha g_{IJ}\lambda^{-I}\lambda^{+J}\Big\}\,d^3x \end{aligned} \tag{3.5}$$

$$\mathcal{L}^{potential} = -U(z,\overline{z})d^3x\,, \tag{3.6}$$

the scalar potential admitting the following general expression

$$\begin{aligned} U(z,\overline{z},M) &= \partial_i W(z)\eta^{ij^*}\partial_{j^*}\overline{W}(\overline{z}) \\ &\quad + \tfrac{1}{2}g^{IJ}\left(\overline{z}^{i^*}(T_I)_{i^*j}z^j\right)\left(\overline{z}^{k^*}(T_J)_{k^*l}z^l\right) \\ &\quad + \overline{z}^{i^*}M^I(T_I)_{i^*j}\eta^{jk^*}M^J(T_J)_{k^*l}z^l \\ &\quad 2\alpha^2 g_{IJ}M^IM^J - 2\alpha M^I\left(\overline{z}^{i^*}(T_I)_{i^*j}z^j\right) \end{aligned} \tag{3.7}$$

and the *superpotential* $W(z)$ being an arbitrary holomorphic function of the chiral scalars z^i. Our index notations and conventions are given in the appendices.

The $\mathcal{N}=3$ case is obtained when the following conditions are fulfilled:

- The spectrum of chiral multiplets and their representation assignments under the gauge and flavor groups are as follows

$$z^i = \left\{\begin{array}{lllll} \sqrt{2}\,Y^I & \mathbf{adj}\ [\mathcal{G}_{gauge}] & \times & \mathbf{id}\ [\mathcal{G}_{flavor}] \\ g\,u^a & \mathbf{R}_g\ [\mathcal{G}_{gauge}] & \times & \mathbf{R}_f\ [\mathcal{G}_{flavor}] \\ g\,v_a & \overline{\mathbf{R}}_g^{-1}\ [\mathcal{G}_{gauge}] & \times & \overline{\mathbf{R}}_f^{-1}\ [\mathcal{G}_{flavor}] \end{array}\right. \Rightarrow \eta^{ik^*}T^I_{k^*j} = \left\{\begin{array}{l} i\,f^I_{\ JK} \\ (T^I)_a^{\ b} \\ -(\overline{T}^I)_a^{\ b} \end{array}\right. \tag{3.8}$$

$\mathbf{R}_g$, and $\mathbf{R}_f$ being two complex representations of G_{gauge} and G_{flavor}, respectively.

- The superpotential $W(z)$ has the following form:

$$W(Y,u,v) = g_{IJ}\left(2\,g\,Y^I\,v_a\,T^{J|a}{}_b\,u^b + 2\,\alpha\,Y^I\,Y^J\right) \tag{3.9}$$

The reason why these two choices make the theory $\mathcal{N}_3 = 3$ invariant is simple: the first choice corresponds to assuming the field content of an $\mathcal{N}_3 = 4$ theory which is necessary since $\mathcal{N}_3 = 3$ and $\mathcal{N}_3 = 4$ supermultiplets are identical. The second choice introduces an interaction that preserves $\mathcal{N}_3 = 3$ supersymmetry but breaks (when $\alpha \neq 0$) $\mathcal{N}_3 = 4$ supersymmetry. We can appreciate the last statement if we rewrite the Lagrangian in such a way that its invariance against the $\mathrm{so}(3)_{\mathrm{R}}$ R-symmetry becomes manifest. To this effect we begin by recalling that viewed from an $\mathcal{N}_3 = 3$ or $\mathcal{N}_3 = 4$ vantage point the chiral fields u^a, v_a are the bosonic elements of a hypermultiplet and can be organized into a quaternion, according to the rule:

$$Q^a = \begin{pmatrix} u^a & i\,\overline{v}^a \\ i\,v_a & \overline{u}_a \end{pmatrix} \equiv q^{a|0}\mathbb{1} + iq^{A|x}\sigma_x \tag{3.10}$$

In this way the transformation of the hypermultiplet u^a, v_a under gauge or flavor generators can be rewritten as follows:

$$\begin{aligned} \delta^I\mathbf{Q} &= i\hat{T}^I\mathbf{Q} \\ \delta^I\begin{pmatrix} u^a & i\overline{v}^a \\ iv_a & \overline{u}_a \end{pmatrix} &= i\begin{pmatrix} T^{I|a}{}_b & \\ & -\overline{T}^I{}_a{}^b \end{pmatrix}\begin{pmatrix} u^b & i\overline{v}^b \\ iv_b & \overline{u}_b \end{pmatrix} \end{aligned} \tag{3.11}$$

where the $T^{I|a}{}_b$ realize a representation of $\mathcal{G}$ in terms of $n \times n$ hermitian matrices. We define $\overline{T}^I{}_a{}^b \equiv \left(T^{I|a}{}_b\right)^*$.

Under the $\mathrm{SO}(3)_{\boldsymbol{R}}$ R–symmetry the hypermultiplets transform as an SU(2) doublet, in the sense that for each $\mathcal{U} \in \mathrm{SU}(2)_{\boldsymbol{R}} \sim \mathrm{SO}(3)_{\boldsymbol{R}}$ the quaternion varies as follows:

$$\delta_R\,Q^a = Q^a\,\mathcal{U} \tag{3.12}$$

On the other hand the auxiliary fields that appear in the gaugino's supersymmetry transformation rules vary, under R–symmetry in the triplet representation of SO(3). Their on–shell values constitute the so called triholomorphic momentum map. This is a unimodular quaternion bilinear constructed by means of the gauge group generators. Explicitly one sets:

$$\mathcal{P}^I = \tfrac{1}{2}\,i\left(\bar{\mathbf{Q}}\,\hat{T}^I\,\mathbf{Q}\right) = \begin{pmatrix} \mathcal{P}^I_3 & \mathcal{P}^I_+ \\ \mathcal{P}^I_- & -\mathcal{P}^I_3 \end{pmatrix} \tag{3.13}$$

where:

$$\begin{aligned} \mathcal{P}^I_3 &= -\left(\bar{u}_a\,T^{I|a}{}_b\,u^b - \bar{v}^a\,\bar{T}^{I|}{}_a{}^b\,v_b\right) \\ \mathcal{P}^I_- &= 2\delta_{ac}\bar{v}^c\,T^{I|a}{}_b\,u^b = 2v_a\,T^{I|a}{}_b\,u^b \\ \mathcal{P}^I_+ &= -\left(\mathcal{P}^I_-\right)^* = -\,2\bar{v}^a\,\bar{T}^{I|}{}_a{}^b\,\bar{u}_b \end{aligned} \tag{3.14}$$

The first form of $\mathcal{P}^I_-$ explicitly exhibits the SU(2) covariance in the sense that $(u^a, \bar{v}^a)$ is a doublet. The second expression will be interpreted later. Out of the triholomorphic momentum map we extract the three components of a real $\mathrm{SO}(3)_{\mathbf{R}}$ trivector. Explicitly we set:

$$\begin{aligned} \mathcal{P}^I_\ell &\equiv \left\{\mathcal{P}^I_1, \mathcal{P}^I_2, \mathcal{P}^I_3\right\} \\ \mathcal{P}^I_- &= -i\left(\mathcal{P}^I_1 + \mathcal{P}^I_2\right) \\ \mathcal{P}^I_+ &= -i\left(\mathcal{P}^I_1 - \mathcal{P}^I_2\right) \end{aligned} \tag{3.15}$$

There is another $\mathrm{SO}(3)_{\mathbf{R}}$ real trivector in the theory which is composed by the complex scalar field Y^I in the adjoint representation of the gauge group together with the real scalar M^I belonging to the $\mathcal{N}=2$ vector multiplet. Explicitly we set:

$$\phi^I_\ell = \left(\begin{array}{c} -\mathrm{Im}Y^I \\ \mathrm{Re}Y^I \\ \frac{1}{2}\,M^I \end{array}\right) \tag{3.16}$$

Inserting eq.(3.9) into the general $\mathcal{N}_3 = 2$ formula (3.7) and using the notations of eq.s(3.15,3.16) we can rewrite the final form of the $\mathcal{N}=3$ scalar potential in a way that exhibits manifest invariance under $\mathrm{so}(3)$ R-symmetry and is a sum of squares:

$$\begin{aligned} U &= g_{IJ}\,\delta^{\ell m}\left[2\sqrt{2}\,\alpha\,\phi^I_\ell + \tfrac{1}{\sqrt{2}}\,g\,\mathcal{P}^I_\ell + \mathcal{Q}^I_\ell\right]\left[2\sqrt{2}\,\alpha\,\phi^J_m + \tfrac{1}{\sqrt{2}}\,g\,\mathcal{P}^J_m + \mathcal{Q}^J_m\right] \\ &\quad + 4\,g^2\,g_{IJ}\,\delta^{\ell m}\,\phi^I_\ell\,\phi^J_m\,\left[\bar{u}_a\,(T^I\,T^J)^a{}_b\,u^b + \bar{v}^a(\bar{T}^I\bar{T}^J)_a{}^b v_b\right] \end{aligned} \tag{3.17}$$

where:

$$\mathcal{Q}^I_\ell = \sqrt{2}\,\epsilon_{\ell mn}\,\phi^P_m\,\phi^Q_n\,f^I{}_{PQ} \tag{3.18}$$

The classical vacua of the $\mathcal{N}=3$ theory are immediately determined from eq.(3.17). One has:

$$\phi^I_\ell = 0 \tag{3.19}$$

$$\mathcal{P}^J_\ell(u,v) = 0 \tag{3.20}$$

Eq. (3.19) lifts the Coulomb branch of the theory setting to zero the vev.s of the scalar fields in the adjoint representation of the gauge group. Eq. (3.20), instead identifies the manifold of classical vacua with the HyperKähler quotient of the flat HyperKähler manifold spanned by the hypermultiplets $u^a, \bar{v}^a$ with respect to the triholomorphic action of the gauge group. The locus defined by (3.20) is the zero level set of the triholomorphic momentum map and it has to be further modded out by the action of U(1). When the generator $T^{I|a}_b = \mathrm{i}\delta^a_b$ is a U(1)–generator, eq.s (3.14) just reproduce the definition of the flag variety $\mathbb{F}(1,2;3) \simeq \mathrm{SU}(3)/\mathrm{U}(1)\times\mathrm{U}(1)$ which is the base manifold of $N^{0,1,0}$ seen as a circle bundle (see [9]). This is what we explain in more details in the next section.

4 The $\mathcal{N}=3$ gauge theory corresponding to the $N^{0,1,0}$ compactification

Having clarified the structure of a generic $\mathcal{N}=3$ gauge theory let us consider the specific one associated with the $N^{0,1,0}$ seven–manifold. As explained in [9] (see eq.(B.1) of that paper) the manifold $N^{0,1,0}$ is the circle bundle inside $\mathcal{O}(1,1)$ over the flag manifold $\mathbb{F}(1,2;3)$. In other words we have

$$N^{0,1,0} \xrightarrow{\pi} \mathbb{F}(1,2;3) \tag{4.1}$$

where, by definition,

$$\mathbb{F}(1,2;3) \equiv \frac{\mathrm{SU}(3)}{H_1 \times H_2} \tag{4.2}$$

is the homogeneous space obtained by modding $\mathrm{SU}(3)$ with respect to its maximal torus:

$$H_1 = \exp\left[i\theta_1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right] \quad ; \quad H_2 = \exp\left[i\theta_2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\right] \tag{4.3}$$

Furthermore as also explained in [9] (see eq.(B.2)), the base manifold $\mathbb{F}(1,2;3)$ can be algebraically described as the following quadric

$$\sum_{i=1}^{3} u^i\, v_i = 0 \tag{4.4}$$

in $\mathbb{P}^2 \times \mathbb{P}^{2*}$, where u^i and v_i are the homogeneous coordinates of $\mathbb{P}^2$ and $\mathbb{P}^{2*}$, respectively.

Hence a complete description of the metric cone $\mathcal{C}\left(N^{0,1,0}\right)$ can be given by writing the following equations in $\mathbb{C}^3 \times \mathbb{C}^{3*}$:

$$\mathcal{C}\left(N^{0,1,0}\right) = \left\{ \begin{array}{rcll} |u^i|^2 - |v_i|^2 & = & 0 & \text{fixes equal the radii of } \mathbb{P}^2 \text{ and } \mathbb{P}^{2*} \\ 2\, u^i\, v_i & = & 0 & \text{cuts out the quadric locus} \\ \left(u^i\, e^{i\theta}, v_i\, e^{-i\theta}\right) & \simeq & (u^i, v_i) & \text{identifies points of } \mathrm{U}(1) \text{ orbits} \end{array} \right. \tag{4.5}$$

Eq.s (4.5) can be easily interpreted as the statement that the cone $\mathcal{C}\left(N^{0,1,0}\right)$ is the HyperKähler quotient of a flat three-dimensional quaternionic space with respect to the triholomorphic action of a $\mathrm{U}(1)$ group. Indeed the first two equations in (4.5) can be rewritten as the vanishing of the triholomorphic momentum map of a $\mathrm{U}(1)$ group. It suffices to identify:

$$\begin{array}{rcl} \mathcal{P}_3 & = & -\left(|u^i|^2 - |v_i|^2\right) \\ \mathcal{P}_- & = & 2 v_i u^i \end{array} \tag{4.6}$$

Comparing with eq.s (3.19) we see that the cone $\mathcal{C}(N^{0,1,0})$ can be correctly interpreted as the space of classical vacua in an abelian $\mathcal{N}=3$ gauge theory with $\mathbf{3}$ hypermultiplets in the fundamental representation of a flavor group $\mathrm{SU}(3)$.

This suggests that the $\mathcal{N}=3$ non–abelian gauge theory whose infrared conformal point is dual to supergravity compactified on $\mathrm{AdS}_4 \times N^{0,1,0}$ has the following structure:

$$\begin{array}{lccc} \text{gauge group} & \mathcal{G}_{gauge} & = & \mathrm{SU(N)}_1 \times \mathrm{SU(N)}_2 \\ \text{flavor group} & \mathcal{G}_{flavor} & = & \mathrm{SU}(3) \\ \text{color representations of hypermultiplets} & \begin{bmatrix} u \\ v \end{bmatrix} & \Rightarrow & \begin{bmatrix} (\mathbf{N}_1, \bar{\mathbf{N}}_2) \\ (\bar{\mathbf{N}}_1, \mathbf{N}_2) \end{bmatrix} \\ \text{flavor representations of hypermultiplets} & \begin{bmatrix} u \\ v \end{bmatrix} & \Rightarrow & \begin{bmatrix} (\mathbf{3}, \bar{\mathbf{3}}) \\ (\bar{\mathbf{3}}, \mathbf{3}) \end{bmatrix} \end{array} \tag{4.7}$$

More explicitly and using an $\mathcal{N}=2$ notation we can say that the field content of our theory is given by the following chiral fields, that are all written as $N \times N$ matrices:

$$\begin{array}{rclll} Y_1 & = & (Y_1)^{\Lambda_1}{}_{\Sigma_1} & \text{adjoint of } \mathrm{SU(N)}_1 \\ Y_2 & = & (Y_2)^{\Lambda_2}{}_{\Sigma_2} & \text{adjoint of } \mathrm{SU(N)}_2 \\ u^i & = & (u^i)^{\Lambda_1}{}_{\Sigma_2} & \text{in the } (\mathbf{3}, \mathbf{N}_1, \bar{\mathbf{N}}_2) \\ v_i & = & (v_i)_{\Sigma_1}{}^{\Lambda_2} & \text{in the } (\mathbf{3}, \bar{\mathbf{N}}_1, \mathbf{N}_2) \end{array} \tag{4.8}$$

and the superpotential can be written as follows:

$$W = 2\,\left[g_1 \mathrm{Tr}\left(Y_1\, u^i\, v_i\right) + g_2 \mathrm{Tr}\left(Y_2\, v_i\, u_i\right) + \alpha_1 \mathrm{Tr}\left(Y_1\, Y_1\right) + \alpha_2\, \mathrm{Tr}\left(Y_2\, Y_2\right)\right] \tag{4.9}$$

where $g_{1,2}, \alpha_{1,2}$ are the gauge coupling constants and Chern Simons coefficients associated with the $\mathrm{SU(N)}_{1,2}$ simple gauge groups, respectively. Setting:

$$\begin{array}{rcl} g_1 & = & g_2 = g \\ \alpha_1 & = & \pm\alpha_2 = \alpha \end{array} \tag{4.10}$$

and integrating out the two fields $Y_{1,2}$ that have received a mass by the Chern Simons mechanism we obtain the effective quartic superpotential:

$$W^{eff} = -\tfrac{1}{2}\,\tfrac{g^2}{\alpha}\,\left[\mathrm{Tr}\left(v_i\, u^i\, v_j\, u^j\right) \pm \mathrm{Tr}\left(u^i\, v_i\, u^j\, u_j\right)\right] \tag{4.11}$$

The vanishing relation one obtains from the above superpotential are the following ones:

$$u^i\, v_j\, u^j = \pm u^j\, v_j\, u^i \quad ; \quad v_i\, u^j\, v_j = \pm v_j\, u^j\, v_i \tag{4.12}$$

Consider now the chiral conformal superfields one can write in this theory:

$$\Phi^{i_1\, i_2\, \dots\, i_k}_{j_1\, j_2\, \dots\, j_k} \equiv \mathrm{Tr}\left(u^{(i_1}\, v_{(j_1}\, u^{i_2}\, v_{j_2}\, \dots\, u^{i)_k}\, v_{j_k)}\right) \tag{4.13}$$

where the round brackets denote symmetrization on the indices. The above operators have k indices in the fundamental representation of $\mathrm{SU(3)}$ and k indices in the antifundamental one, but they are not yet assigned to the irreducible representation:

$$M_1 = M_2 = k \tag{4.14}$$

as it is predicted both by general geometric arguments and by the explicit evaluation of the Kaluza Klein spectrum of hypermultiplets [16]. To be irreducible the operators (4.13) have to be traceless. This is what is implied by the vanishing relation (4.12) if we choose the minus sign in eq.(4.10).

Notice that for $N^{0,1,0}$ the form of the superpotential, which is dictated by the Chern-Simons term, is strongly reminiscent of the superpotential considered in [2]. The CFT theory associated with $N^{0,1,0}$ has indeed many analogies with the simpler cousin $T^{1,1}$. There is however also a crucial difference. We recognize a general phenomenon that we already discussed in the $M^{1,1,1}$ and $Q^{1,1,1}$ compactifications [9]. The moduli space of vacua of the abelian theory is isomorphic to the cone $\mathcal{C}\left(N^{0,1,0}\right)$. When the theory is promoted to an non-abelian one, there are naively conformal operators whose existence is in contradiction with geometric expectations and with the KK spectrum, in this case the hypermultiplets that do not satisfy relation (4.14). Differently from what happens for $T^{1,1}$ [2], the superpotential which can be added to the theory is not sufficient for eliminating these redundant non-abelian operators.

5 Tests of correspondence

In this section we present the basic checks of the correspondence between the $\mathcal{N}=3$ superconformal gauge theory just discussed and the $N^{0,1,0}$ compactification of M-theory. Here we identify the whole set of BPS composite operators dual to the short supermultiplets of the KK spectrum. In the next section we analyze the non-BPS composite operators dual to certain massive KK modes, which seem to be organized into the Higgs supermultiplet of a spontaneously broken $\mathcal{N}=4$ supergravity.

5.1 Comparison with the KK spectrum

Let us briefly summarize the KK spectrum of the AdS$_4 \times N^{0,1,0}$ compactification of $11D$ supergravity, organized into $\mathcal{N}=3$ supermultiplets [15, 16]. These are listed in table 5.1 and 5.2, where we give their decomposition in $\mathcal{N}=2$ supermultiplets and their flavor quantum numbers. The ultrashort multiplets are:

$\mathcal{N}$=3 multiplet	$\begin{cases} \text{R}-\text{charge} \\ \text{SU(3)}-\text{irrep} \end{cases}$	$\mathcal{N}$=2 multiplets
massless graviton	$\begin{cases} J=0 \\ M_1=M_2=0 \end{cases}$	$\begin{cases} 1 \text{ massless graviton} \\ 1 \text{ massless gravitino} \end{cases}$
massless vector	$\begin{cases} J=1 \\ M_1=M_2=0 \end{cases}$	$\begin{cases} 1 \text{ massless vector} \\ 2 \text{ chiral mult.} \end{cases}$
massless vector	$\begin{cases} J=1 \\ M_1=M_2=1 \end{cases}$	$\begin{cases} 1 \text{ massless vector} \\ 2 \text{ chiral mult.} \end{cases}$

(5.1)

The short multiplets are:

$\mathcal{N}$=3 multiplet	$\begin{cases} \text{R}-\text{charge} \\ \text{SU(3)}-\text{irrep} \end{cases}$	$\mathcal{N}$=2 multiplets
short graviton	$\begin{cases} J=k\geq 1 \\ M_1=M_2=k \end{cases}$	$\begin{cases} 2 & \text{short gravitons} \\ 2k-1 & \text{long gravitons} \\ 2 & \text{short gravitinos} \\ 2k-1 & \text{long gravitinos} \end{cases}$
short gravitino	$\begin{cases} J=k+1, \quad k\geq 0 \\ M_1=k, \;\; M_2=k+3 \end{cases}$	$\begin{cases} 2 & \text{short gravitinos} \\ 2k+1 & \text{long gravitinos} \\ 2 & \text{short vectors} \\ 2k+1 & \text{long vectors} \end{cases}$
short vector	$\begin{cases} J=k, \quad k\geq 2 \\ M_1=M_2=k \end{cases}$	$\begin{cases} 2 & \text{chiral mult.} \\ 2 & \text{short vectors} \\ 2k-1 & \text{long vectors} \end{cases}$

(5.2)

5.1.1 The fundamental supersingletons

In complete analogy with the $\mathcal{N}=2$ CFT's analyzed in [9], the building blocks of all the superconformal primary fields are the supersingletons. In this case we have at our disposal the isospin doublet in the fundamental representation of the flavor group SU(3):

$$\Theta^{i\,J=1/2} = \begin{pmatrix} \Theta^{+i} \\ \Theta^{-i} \end{pmatrix}^{\mathcal{N}=3} = \begin{pmatrix} U^i \\ i\overline{V}^i \end{pmatrix}^{\mathcal{N}=2} - \frac{\sqrt{2}}{2}\,\theta^0 \begin{pmatrix} i\mathcal{D}^+\overline{V}^i \\ \mathcal{D}^- U^i \end{pmatrix}^{\mathcal{N}=2}, \tag{5.3}$$

and the conjugate doublet

$$\Theta_j^{\;J=1/2} = \begin{pmatrix} \Theta_j^+ \\ \Theta_j^- \end{pmatrix}^{\mathcal{N}=3} = \begin{pmatrix} -iV_j \\ \overline{U}_j \end{pmatrix}^{\mathcal{N}=2} - \frac{\sqrt{2}}{2}\theta^0 \begin{pmatrix} \mathcal{D}^+\overline{U}_j \\ -i\mathcal{D}^- V_j \end{pmatrix}^{\mathcal{N}=2} . \tag{5.4}$$

The $\mathcal{N}=2$ superfield components U^i and V_i are supersingletons:

$$\begin{aligned} U^i(x,\theta^\pm) &= u^i(x) + \theta^+ \chi_u^{-\,i}(x) + \tfrac{1}{2}\theta^+ \partial\!\!\!/ u^i(x)\theta^- , \\ V^i(x,\theta^\pm) &= v_i(x) + \theta^- \chi_{v\,i}^+(x) - \tfrac{1}{2}\theta^+ \partial\!\!\!/ v_i(x)\theta^- . \end{aligned} \tag{5.5}$$

The lowest components, the so-called *Di*'s, are the scalar fields u^i and v_i discussed in the previous section and realizing the homogeneous coordinates of $\mathbb{P}^2 \times \mathbb{P}^{2*}$. Their color representations are given in (4.7) and is the same for the *Rac*'s χ_u and χ_v.

5.1.2 Field theory realization of the chiral ring

The generator of the chiral ring of our CFT is the highest weight part of the tensor product of (5.3) times its conjugate doublet, i.e. the $\mathrm{SO(3)_R}$ triplet

$$\Theta^i_j{}^{\,J=1} = Tr \begin{pmatrix} -iU^i V_j \\ \frac{\sqrt{2}}{2}\left(U^i\overline{U}_j + \overline{V}^i V_j\right) \\ i\overline{V}^i\overline{U}_j \end{pmatrix}^{\mathcal{N}=2} + \mathcal{O}(\theta^0) , \tag{5.6}$$

where the trace refers to the color indices. From the flavor viewpoint, we can extract the two irreducible pieces belonging to the symmetric tensor product of the $\mathbf{3}$ and $\bar{\mathbf{3}}$ of $\mathrm{SU(3)}$. They contain the two massless vectors in the Kaluza Klein spectrum (see table (5.1)):

- the adjoint,
$$\Sigma^i{}_j \equiv \frac{\sqrt{2}}{2} Tr \left(U^i\overline{U}_j + \overline{V}^i V_j\right) - \text{flavor trace} \tag{5.7}$$
corresponding to the conserved current of the global $\mathrm{SU(3)}$ flavor;
- the singlet,
$$\Sigma \equiv \frac{\sqrt{2}}{2} Tr \left(U^i\overline{U}_i + \overline{V}^i V_i\right) \tag{5.8}$$
corresponding to the baryonic $\mathrm{U(1)}$ global symmetry.

By composing several massless vectors we obtain the whole chiral ring of superfields, containing $\mathcal{N}=2$ chiral fields and short vectors with the right flavor quantum numbers, as listed in table (5.1).

5.1.3 Field theory realization of the short gravitinos

Let us come to the short gravitinos. Remember that we basically have at our disposal the $\mathcal{N}=3$ supersingleton of (5.3), which we will simply call Θ^i, and its conjugate Θ_j. Let us consider the following composite operator:

$$\Theta^{(ijk)} = f^{lm(i} Tr \left[\Theta^j \Theta^{k)} \Theta_l \Theta_m\right] , \tag{5.9}$$

where f^{ijk} are the SU(3) structure constants and the round brackets mean symmetrization. From the isospin viewpoint, (5.9) is a triplet, while it transforms in the three time symmetric tensor product of the $\mathbf{3}$ of SU(3), in agreement with the $J=1$ short gravitino of the $N^{0,1,0}$ Kaluza Klein spectrum (see table 5.2). By construction, the operator (5.9) is a short gravitino, namely it satisfies the second order differential constraint of eq. (2.8). The $\mathcal{N}=2$ superfield content (see eq. 2.16) is given by:

$$\Sigma^{+(ijk)} = f^{lm(i}\, U^j U^{k)} \left(V_l \overline{U}_m - V_m \overline{U}_l\right); \tag{5.10}$$

$$\Sigma^{0\,(ijk)} = \sqrt{2} i f^{lm(i}\, U^j \overline{V}^{k)} \left(V_l \overline{U}_m - V_m \overline{U}_l\right); \tag{5.11}$$

$$\Sigma^{-\,(ijk)} = -f^{lm(i}\, \overline{V}^j \overline{V}^{k)} \left(V_l \overline{U}_m - V_m \overline{U}_l\right); \tag{5.12}$$

$$G_\alpha^{+(ijk)} = f^{lm(i}\, U^j U^{k)} \left(\overline{U}_l \mathcal{D}^+_\alpha \overline{U}_m - \overline{U}_m \mathcal{D}^+_\alpha \overline{U}_l\right); \tag{5.13}$$

$$G_\alpha^{-(ijk)} = -f^{lm(i}\, \overline{V}^j \overline{V}^{k)} \left(V_l \mathcal{D}^-_\alpha V_m - V_m \mathcal{D}^-_\alpha V_l\right). \tag{5.14}$$

The $\mathcal{N}=3$ short gravitinos of higher isospin are obtained by extracting the highest weight part from the product of operators in the chiral ring with (5.9):

$$\Theta^{(i_1\cdots i_{k-1}klm)\ J=k}_{\ (j_1\cdots j_{k-1})} = Tr\left[\underbrace{\Theta^{i_1}_{\ j_1} \otimes_{h.w.} \cdots \otimes_{h.w.} \Theta^{i_{k-1}}_{\ j_{k-1}}}_{k-1 \text{ objects}} \otimes_{h.w.} \Theta^{(klm)}\right]. \tag{5.15}$$

5.1.4 Field theory realization of the short gravitons

Let us now consider the composite superfield:

$$\Theta_\alpha^{J=0} = Tr\left[\Theta^i \otimes \mathcal{D}_\alpha \otimes \Theta_i - \Theta_i \otimes \mathcal{D}_\alpha \otimes \Theta^i\right], \tag{5.16}$$

where the scalar part is extracted from the isospin tensor product. It is straightforward to show that this superfield is a short graviton (2.17) by construction. From the $\mathcal{N}=2$ viewpoint, it is composed by:

- the massless graviton supermultiplet of the $\mathcal{N}=2$ subalgebra:

$$T_{(\alpha\beta)} = Tr\Big[V \partial\!\!\!/_{(\alpha\beta)} \overline{V} - \overline{V} \partial\!\!\!/_{(\alpha\beta)} V - U \partial\!\!\!/_{(\alpha\beta)} \overline{U} + \overline{U} \partial\!\!\!/_{(\alpha\beta)} U + 2\mathcal{D}^-_{(\alpha} U\, \mathcal{D}^+_{\beta)} \overline{U} + 2\mathcal{D}^+_{(\alpha} \overline{V}\, \mathcal{D}^-_{\beta)} V\Big]; \tag{5.17}$$

- the conserved current relative to the third supersymmetry charge, completing the $\mathcal{N}=3$ supersymmetry algebra:

$$G_\alpha = i Tr\left[U \mathcal{D}^-_\alpha V - V \mathcal{D}^+_\alpha U + \overline{V} \mathcal{D}^+_\alpha \overline{U} - \overline{U} \mathcal{D}^+_\alpha \overline{V}\right]. \tag{5.18}$$

All together, $T_{\alpha\beta}$ and G_α constitute the supermultiplet containing the energy-momentum tensor, the $\mathcal{N}=3$ supersymmetry charges and the $\mathcal{N}=3$ R-symmetry currents.

Once again, the short gravitons of the CFT are realized by composing (5.16) with the chiral ring operators and taking the highest weight part of isospin and flavor quantum numbers:

$$\Theta^{(i_1\cdots i_k)\ J=k}_{\alpha\ (j_1\cdots j_k)} = Tr\left[\underbrace{\Theta^{i_1}_{\ j_1} \otimes_{h.w.} \cdots \otimes_{h.w.} \Theta^{i_k}_{\ j_k}}_{k \text{ objects}} \otimes_{h.w.} \Theta_\alpha^{\ J=0}\right]. \tag{5.19}$$

It is interesting to note that some of the $\mathcal{N}=2$ components of the short gravitons (5.19) of $J\geq 1$ (precisely, the second highest helicity states) are long gravitons with the following particular structure:

$$\Phi \sim \text{conserved vector current} \times \text{chiral operator} \times \text{stress} - \text{energy tensor}\,. \tag{5.20}$$

These long $\mathcal{N}=2$ multiplets have nonetheless rational conformal dimensions, belonging to a short $\mathcal{N}=3$ graviton multiplet. Furthermore, they have the same structure of some long multiplets of rational conformal dimension identified in type IIB [7] as well as in $\mathcal{N}=2$ (see eq. (6.64) of [9]) M-theory compactifications. This suggests that the existence of such *rational multiplets* in non-maximally supersymmetric AdS compactifications could be explained by the presence of a residual form of higher supersymmetry, possibly spontaneously broken. This explanation is confirmed by a second feature common to all the $\mathcal{N}=3$ AdS_4 compactifications of $11D$ supergravity: the presence of a superHiggs multiplet, that we discuss in the next section.

6 The Universal SuperHiggs multiplet

Finally we consider the CFT realization of a long gravitino multiplet that has integer conformal dimension:

$$E_0 = 3 \tag{6.1}$$

and it is neutral with respect to the flavor group SU(3). It was found in the spectrum of the $AdS_4\times N^{0,1,0}$ compactification [16] but, as we shall argue in a forthcoming paper [17], it has a universal character, since it would appear with the same quantum numbers and the same conformal dimension (6.1) in any other Freund Rubin compactification of $D=11$ supergravity with $\mathcal{N}=3$ residual supersymmetry. In [17] we shall discuss its interpretation as superHiggs multiplet in a partial supersymmetry breaking $\mathcal{N}=4$ to $\mathcal{N}=3$. Here we want to stress its universality also from the CFT point of view.

Consider the following scalar composite superfield:

$$\mathcal{SH} = \mathrm{Tr}[\underbrace{\Theta_\Sigma \otimes \Theta_\Sigma \otimes \Theta_\Sigma}_{J=0}] = \mathrm{Tr}\left[\Theta^+_\Sigma \Theta^0_\Sigma \Theta^-_\Sigma\right]\,, \tag{6.2}$$

where Θ_Σ is the field strength superfield, i.e. a real $J=1$ short superfield (see eq. 2.13) generalizing the linear multiplet of $\mathcal{N}=2$ gauge theories:

$$\Theta_\Sigma = \begin{pmatrix} Y \\ \Sigma \\ -Y^\dagger \end{pmatrix} + \mathcal{O}(\theta^0) = \tag{6.3}$$

$$\begin{pmatrix} Y + (\theta^+\chi^-) + (\theta^+\theta^+)H + \frac{1}{2}(\lambda^+\theta^0) + \frac{1}{2}(\theta^0\theta^+)P \\ -\frac{1}{2}(\theta^0\theta^0)H^\dagger - \frac{i}{2}(\theta^0\gamma^{\mu\nu}\theta^+)F_{\mu\nu} \\ \\ -M + \frac{1}{2}(\lambda^+\theta^-) + \frac{1}{2}(\lambda^-\theta^+) + \frac{1}{2}(\theta^+\theta^-)P - \frac{i}{2}(\theta^-\gamma^{\mu\nu}\theta^+)F_{\mu\nu} \\ +\frac{1}{2}(\theta^0\chi^-) - \frac{1}{2}(\theta^0\chi^+) + (\theta^0\theta^+)H - (\theta^0\theta^-)H^\dagger + \frac{1}{4}(\theta^0\theta^0)P \\ \\ -Y^\dagger - (\theta^-\chi^+) - (\theta^-\theta^-)H^\dagger + \frac{1}{2}(\lambda^-\theta^0) + \frac{1}{2}(\theta^0\theta^-)P \\ +\frac{1}{2}(\theta^0\theta^0)H - \frac{i}{2}(\theta^-\gamma^{\mu\nu}\theta^0)F_{\mu\nu} \end{pmatrix} + \begin{array}{c}\text{derivative} \\ \text{terms.}\end{array}$$

From eq. (6.2) it is possible to identify all the field components of the superHiggs multiplet, which turn out to be related, through the AdS/CFT correspondence, to certain Kaluza Klein modes of the $N^{0,1,0}$ compactification. Of particular interest, for its clear geometrical interpretation, is the scalar component of zero isospin and conformal dimension 6. The corresponding supergravity state is given by the *breathing mode*, responsible for a uniform dilatation of the internal manifold X^7. It can be extracted by integrating the superfield $\mathcal{SH}$ with respect to the Grassmann measure $d^6\theta$:

$$\int d^2\theta^+ d^2\theta^- d^2\theta^0 \, \mathcal{SH} = Tr\left[3iHH^\dagger P + \frac{1}{4}\epsilon^{\lambda\mu\nu}\epsilon^{\rho\sigma\tau}F_{\lambda\mu}F_{\nu\rho}F_{\sigma\tau}\right] + \text{derivatives}. \tag{6.4}$$

The supergravity interpretation of this field as the volume mode of X^7 is a clear sign of the universality of the whole multiplet (it does not depend on any specific characteristic of the internal manifold). As we will show in [17] this is true for all the components of the multiplet.

From the CFT viewpoint, the composite operator (6.4) is the $\mathcal{N}=3$ supersymmetrization of the following third power of the gauge field strength:

$$\epsilon^{\lambda\mu\nu}\epsilon^{\rho\sigma\tau}F_{\lambda\mu}F_{\nu\rho}F_{\sigma\tau}\,,$$

whose dimension appears to be protected by some, so far unknown, non-renormalization theorem. Indeed a closely similar situation appears in type IIB AdS$_5$ compactifications, where the volume mode of the internal manifold is dual to the CFT operator F^4, of dimension 8, which is known to satisfy some non-renormalization theorem. This consideration suggests that the operator (6.4) could originate by the low energy expansion of an analogue of the Dirac Born Infeld for the $M2$-brane, as well as the operator F^4 comes from the α' expansion of the DBI Lagrangian of the $D3$-brane. F^4 is indeed the operator that coupled to the background breathing mode on the D3-brane world-volume.

In this perspective, the universality of the (properly supersymmetrized) third power of F could be understood: it should be traced back to the existence of a universal Lagrangian term for the $M2$-brane.

The explicit presence of $F_{\mu\nu}$ in the previous formulae deserves some comments. In three dimensions, the vector multiplet is not conformal and it does not make sense to consider it an elementary degree of freedom at the conformal point. The only singletons in three dimensions are hypermultiplets. Only hypermultiplets indeed appeared in the matching of the KK spectrum with the short multiplets of conformal operators that we discussed in the previous sections. The vector multiplet fields in the previous equations should be regarded as expressed in terms of the singletons at the conformal point, using the equations of motion, for example. Alternatively, we may consider the previous equations as operators in the three dimensional gauge theory that has the CFT as the IR limit. The previous discussion suggests that these operators become conformal operators at the fixed point.

7 Conclusions and perspectives

The identification and the study of conformal field theories dual to AdS supergravity compactifications is not a mere exercise of classification nor a simple test of AdS/CFT correspondence. As it is the case for the $N^{0,1,0}$ solution considered in this paper, a careful analysis of the properties of the theory, both on the CFT and on the supergravity side, may lead to surprising discoveries.

The most interesting lesson we have learned about non-maximally compactifications of $\mathcal{M}$-theory regards the existence of some universal features which do not depend on the geometrical details of the compactification manifold, but only on the degree of supersymmetry of the solution.

From the supergravity viewpoint, we find that all the massless multiplets, related to symmetries of the theory, are always coupled to long *shadow* multiplets. Some of these can be interpreted as the massive (super)Higgs multiplets of some spontaneously broken (super)symmetry. This phenomenon is particularly interesting for the most general symmetries, such as the group of AdS space-time isometries and/or its supersymmetries. In the $\mathcal{N}=3$ case, for instance, the *shadow* multiplet of the massless graviton, related to the $\mathrm{Osp}(3|4)$ supergroup, is a massive gravitino multiplet with same quantum numbers of the first: it is a superHiggs multiplet. Hence every $\mathcal{N}=3$ solution of the form $\mathrm{AdS}_4 \times X^7$, independently from X^7, turns out to be the broken phase of some not better specified $\mathcal{N}=4$ supergravity. The deepest implications of this fact are analyzed in [17].

Here we want to briefly discuss the consequences of the field theory counterpart of this phenomenon. The AdS/CFT prescriptions imply that the composite operators dual to the supergravity *shadow* multiplets have protected conformal dimensions. This fact is quit surprising because they are not organized in short multiplets, suggesting the existence of some non-trivial non-renormalization theorem, whose investigation is left to future speculations.

Another possible development is given by the $\mathcal{M}$-brane interpretation of the CFT dual of the most universal *shadow* multiplet: the *shadow* of the stress-energy tensor, i.e. the *breathing mode* of the internal manifold. Its existence is independent even from the degree of supersymmetry of the theory, hence it must come from a universal term of the $\mathcal{M}2$-brane action, not directly related to the background. The identification of such a term could shed new light on the microscopic structure of the $\mathcal{M}$-theory.

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A Conventions on spinors

In accordance with [18], spinor indices ($\alpha, \beta, \gamma \ldots$) are contracted *from eight to two* and are raised and lowered with $\epsilon_{\alpha\beta}$:

$$\begin{aligned} \psi^\alpha \equiv \epsilon^{\alpha\beta}\psi_\beta \qquad & \epsilon_{12} = \epsilon^{21} = 1 \\ \psi_\alpha \equiv \epsilon_{\alpha\beta}\psi^\beta \qquad & \epsilon_{\alpha\gamma}\epsilon^{\gamma\beta} = \delta_\alpha^\beta \\ (\psi\phi) \equiv \psi_\alpha\phi^\alpha = -\psi^\alpha\phi_\alpha = & \phi_\alpha\psi^\alpha \equiv (\phi\psi)\,. \end{aligned} \tag{A.1}$$

We choose the following representation of the $SO(1,2)$ Clifford algebra:

$$\left\{ \begin{array}{rcl} \gamma^0 & = & -i\sigma^2 \\ \gamma^1 & = & \sigma^3 \\ \gamma^2 & = & \sigma^1 \end{array} \right. \qquad \begin{array}{r} \eta_{\mu\nu} = \mathrm{diag}(-++) \\ \gamma^\mu \equiv \gamma^{\mu|\alpha}{}_\beta \\ [\gamma^\mu, \gamma^\nu] = 2\epsilon^{\mu\nu\rho}\gamma_\rho\,, \end{array} \tag{A.2}$$

hence the symmetry properties of the gamma matrices are:

$$\left\{ \begin{array}{lclcl} \gamma^{\mu|\alpha\beta} & \equiv & \epsilon^{\beta\gamma}\gamma^{\mu|\alpha}{}_{\gamma} & = & \gamma^{\mu|\beta\alpha} \\ \gamma^{\mu}{}_{\alpha\beta} & \equiv & \epsilon_{\alpha\gamma}\gamma^{\mu|\gamma}{}_{\beta} & = & \gamma^{\mu}{}_{\beta\alpha} \end{array} \right. \tag{A.3}$$

so that

$$(\psi\gamma^\mu\phi) = -(\phi\gamma^\mu\psi)\,. \tag{A.4}$$

Complex conjugation acts as:

$$(\psi^\alpha)^\dagger \equiv \bar{\psi}_\alpha\,, \tag{A.5}$$

so that

$$(\psi\phi)^\dagger = \bar{\phi}\bar{\psi} = (\bar{\psi}\bar{\phi}), \tag{A.6}$$

and, with our choice of gamma matrices,

$$(\psi\gamma^\mu\phi)^\dagger = -(\bar{\phi}\gamma^\mu\bar{\psi}) = (\bar{\psi}\gamma^\mu\bar{\phi})\,. \tag{A.7}$$

The spinorial derivatives act in the following way:

$$\frac{\partial}{\partial\theta^i_\alpha}\theta^j_\beta = -\frac{\partial}{\partial\theta^{i\beta}}\theta^{j\alpha} = \delta^j_i\delta^\alpha_\beta \tag{A.8}$$

and the supercovariant derivatives are:

$$\begin{aligned} \mathcal{D}^+ &\equiv -\left(\frac{\partial}{\partial\theta^-} + \tfrac{1}{2}\partial\!\!\!/\theta^+\right), \\ \mathcal{D}^- &\equiv -\left(\frac{\partial}{\partial\theta^+} + \tfrac{1}{2}\partial\!\!\!/\theta^-\right), \\ \mathcal{D}^0 &\equiv \left(\frac{\partial}{\partial\theta^0} - \tfrac{1}{2}\partial\!\!\!/\theta^0\right). \end{aligned} \tag{A.9}$$

B Notes on the $\mathcal{N}=2$ superfields

Here we briefly review the differential constraint defining the $\mathcal{N}=2$ short superfield and their field decomposition, to fix the notations adopted in the paper.

- **The chiral superfield**
  Identified by the constraint:

$$\mathcal{D}^+\Phi(x,\theta^\pm) = 0\,. \tag{B.1}$$

  In components is given by

$$\begin{aligned} \Phi(x,\theta^\pm) &= z(x) + \theta^+\chi^-(x) + (\theta^+\theta^+)H(x) \\ &+ \tfrac{1}{2}\theta^+\gamma^\mu\theta^-\partial_\mu z(x) + \tfrac{1}{4}(\theta^+\theta^+)\theta^-\gamma^\mu\partial_\mu\chi^-(x) \\ &+ \tfrac{1}{16}(\theta^+\theta^+)(\theta^-\theta^-)\Box z(x)\,. \end{aligned} \tag{B.2}$$

- **The supersingleton**
  The $\mathcal{N}=2$ supersingleton is defined by

$$\left\{ \begin{array}{rcl} \mathcal{D}^+\Phi_s(x,\theta^\pm) & = & 0 \\ (\mathcal{D}^-\mathcal{D}^-)\Phi_s(x,\theta^\pm) & = & 0\,. \end{array} \right. \tag{B.3}$$

In components is given by

$$\Phi_s(x,\theta^\pm) = z(x) + \theta^+\chi^-(x) + \tfrac{1}{2}\theta^+\gamma^\mu\theta^-\partial_\mu z(x)\,, \tag{B.4}$$

where z and χ^- are on-shell massless fields:

$$\begin{cases} \Box z = 0\,, \\ \partial\!\!\!/\chi^- = 0\,. \end{cases} \tag{B.5}$$

- **The short gravitino**
  The short gravitino, defined by:

$$\mathcal{D}^+_\alpha G^{+\alpha}(x,\theta^\pm) = 0\,, \tag{B.6}$$

is given by

$$\begin{aligned} G^{+\alpha}(x,\theta^\pm) &= \lambda_L + A\!\!\!/^+\theta^- + A\!\!\!/^-\theta^+ + \phi^-\theta^+ \\ &+ \lambda_T^{+-}(\theta^+\theta^-) + \tfrac{1}{2}(\theta^+\lambda_T^{+-})\theta^- + (\theta^+\theta^+)\lambda_T^{--} \\ &+ (\theta^+\gamma^\mu\theta^-)\psi_\mu + (\theta^+\theta^+)Z\!\!\!/\theta^- + \text{derivative terms}\,. \end{aligned} \tag{B.7}$$

- **The massless gravitino**
  The massless gravitino, defined by:

$$\mathcal{D}^+_\alpha G^\alpha(x,\theta^\pm) = \mathcal{D}^-_\alpha G^\alpha(x,\theta^\pm) = 0\,, \tag{B.8}$$

is given by

$$\Phi^\alpha(x,\theta^\pm) = \lambda_L + A\!\!\!/^+\theta^- + A\!\!\!/^-\theta^+ + (\theta^+\gamma^\mu\theta^-)\psi_\mu + \text{derivative terms}\,, \tag{B.9}$$

where the spinor λ_L and the gravitino ψ_m are massless:

$$\partial\!\!\!/\lambda_L = \epsilon^{\mu\nu\rho}\gamma_\mu\partial_\nu\psi_\rho = 0\,, \tag{B.10}$$

while the two vectors are in Lorentz gauge:

$$\partial\cdot A^+ = \partial\cdot A^- = 0\,. \tag{B.11}$$

- **The gauge potential superfield**
  Identified by the reality constraint

$$V^\dagger = V\,, \tag{B.12}$$

can be parametrized as:

$$\begin{aligned} V(x,\theta^+,\theta^-) &= C(x) + \theta^+\psi^-(x) + \theta^-\psi^+(x) \\ &+ (\theta^+\theta^+)B(x) + (\theta^-\theta^-)B^\dagger(x) \\ &- \tfrac{i}{2}\theta^+\gamma^\mu\theta^- A_\mu(x) + \tfrac{1}{2}(\theta^+\theta^-)M(x) \\ &+ \tfrac{1}{4}(\theta^+\theta^+)\theta^-\left[\lambda^-(x) + \gamma^\mu\partial_\mu\psi^-(x)\right] \\ &+ \tfrac{1}{4}(\theta^-\theta^-)\theta^+\left[\lambda^+(x) + \gamma^\mu\partial_\mu\psi^+(x)\right] \\ &+ \tfrac{1}{8}(\theta^+\theta^+)(\theta^-\theta^-)\left[P(x) + \tfrac{1}{2}\Box C(x)\right]\,. \end{aligned} \tag{B.13}$$

The gauge transformation

$$V \;\to\; V+\Phi+\Phi^\dagger\,, \tag{B.14}$$

corresponds to

$$\left\{\begin{array}{lllllll} C & \to & C+z+\bar z & \qquad\qquad & P & \to & P \\ \psi^\pm & \to & \psi^\pm+\chi^\pm & & \lambda^\pm & \to & \lambda^\pm \\ B & \to & B+H & & M & \to & M \\ A_\mu & \to & A_\mu+i(\partial_\mu z-\partial_\mu\bar z) & & & & \end{array}\right. \tag{B.15}$$

In Wess Zumino gauge, V reduces to:

$$\begin{aligned} V(x,\theta^+,\theta^-) = &-\tfrac{i}{2}\theta^+\gamma^\mu\theta^- A_\mu(x)+\tfrac{1}{2}(\theta^+\theta^-)M(x)\\ &+\tfrac{1}{4}(\theta^+\theta^+)\theta^-\lambda^-(x)+\tfrac{1}{4}(\theta^-\theta^-)\theta^+\lambda^+(x)+\tfrac{1}{8}(\theta^+\theta^+)(\theta^-\theta^-)P(x)\,. \end{aligned} \tag{B.16}$$

- **The field strength**
The gauge invariant super field strength is a real **linear superfield**:

$$\mathcal{D}^+\mathcal{D}^+\Sigma=\mathcal{D}^-\mathcal{D}^-\Sigma=0\,. \tag{B.17}$$

It is derived by the potential superfield V by:

$$\begin{aligned} \Sigma \equiv \mathcal{D}^+_\alpha\mathcal{D}^{-\alpha}V = \mathcal{D}^-_\alpha\mathcal{D}^{+\alpha}V = &\\ -M+\tfrac{1}{2}(\lambda^+\theta^-)+\tfrac{1}{2}(\lambda^-\theta^+)+\tfrac{1}{2}(\theta^-\theta^+)P-\tfrac{i}{2}(\theta^-\gamma^{\mu\nu}\theta^+)F_{\mu\nu}&\\ -\tfrac{1}{8}(\theta^-\theta^-)\theta^+\partial\!\!\!/\lambda^+-\tfrac{1}{8}(\theta^+\theta^+)\theta^-\partial\!\!\!/\lambda^-+\tfrac{1}{16}(\theta^+\theta^+)(\theta^-\theta^-)\Box M\,.& \end{aligned} \tag{B.18}$$

where

$$F_{\mu\nu}\equiv\tfrac{1}{2}\left(\partial_\mu A_\nu-\partial_\nu A_\mu\right)\,. \tag{B.19}$$

- **The SYM and CS action**
The abelian SYM action is:

$$\begin{aligned} -4\int d^3x\,\Sigma^2|_{(\theta^+\theta^+)(\theta^-\theta^-)} = -4\int d^3x\,d^2\theta^+\,d^2\theta^-\,\Sigma^2&\\ =\int d^3x\,\left\{-\tfrac{1}{4}\left(\lambda^+\partial\!\!\!/\lambda^-+\lambda^-\partial\!\!\!/\lambda^+\right)+\tfrac{1}{2}P^2-\tfrac{1}{2}\partial^\mu M\partial_\mu M-F_{\mu\nu}F^{\mu\nu}\right\}\,.& \end{aligned} \tag{B.20}$$

The supersymmetric generalization of the Chern Simons term is:

$$\begin{aligned} 4\int d^3x\,\Sigma V|_{(\theta^+\theta^+)(\theta^-\theta^-)} &= 4\int d^3x\,d^2\theta^+\,d^2\theta^-\,\Sigma V\\ &=\int d^3x\,\left\{-\epsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho-\tfrac{1}{2}\lambda^+\lambda^--PM\right\}\,. \end{aligned} \tag{B.21}$$

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