Areal Theory*

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Abstract

New features are described for models with multi-particle area-dependent potentials, in any number of dimensions. The corresponding many-body field theories are investigated for classical configurations. Some explicit solutions are given, and some conjectures are made about chaos in such field theories.

Area-dependent potentials, $V\left(A = \mathbf{r}_1 \wedge \mathbf{r}_2\right)$, or their mathematical equivalents, appear in several physical problems of contemporary interest. Notable among these problems are Yang-Mills theory (especially for spatially homogeneous configurations, with only time dependence) [3], extended supersymmetric field theories with (pseudo)scalar self-interactions [18, 13, 20], and more recently, membrane models [11, 16]. In the last century, Feynman [14] even assigned to students an exercise involving such potentials: Show the energy spectrum for a quantized areal potential model is discrete and quantum particles cannot escape from such a potential, even though the spectrum for the classical model is continuous and classical particles can escape along special trajectories for which $A = 0$. Many of us pondered this problem over the intervening years, especially in the field theory context [4]. Meanwhile, Barry Simon worked out five or six solutions to Feynman's exercise and published them [25].

Classically, such potential models are interesting insofar as they may provide simple examples of chaotic systems [24, 2]. While there are special trajectories for which the motion is quite regular, including those for which particles can escape (such as straight line, free particle motion), for most trajectories this is not the case. The current consensus is such models are not integrable in the Liouville sense, and do not admit the construction of a Lax pair. Is this really so?

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Quantum mechanically, there are obviously more interesting questions to ask about such potential models than simply whether the energy spectrum is discrete. In particular, just what *is* that discrete spectrum? Is the quantized model, or some simple variant of it, completely integrable [19], even if the classical is not? Do quantum effects sufficiently ameliorate any chaotic classical trajectories [17] to permit closed-form solutions for the wave functions, or exotic but useful forms [5] for the propagator?

With these questions in mind, I will describe in this talk[1] some new features of models with area-dependent potentials. As non-relativistic many-body field theories, I believe that area-dependent models are very interesting. I will argue this beginning with the classical field versions. I explicitly solve these for some special configurations, and then investigate general situations. The classical field theory problem always reduces to solving the linear Schrödinger equation in a self-consistent effective potential. In general, the effective potential for the $V = A$ model is anisotropic and linear, while for the $V = A^2$ model it is anisotropic and quadratic. In both cases the effective potential is determined from the initial data by a closed set of coupled time-dependent equations. I will analyze these equations, especially for the case of spherically symmetric particle density, and I will solve them in that special situation. However, I have not completely solved the area-dependent models for general, anisotropic initial data. Much more work is needed for the $V = A^2$ model. Additional work is also needed to construct the quantum versions of these field theories. These particular models seem to be ripe for exploration using deformation quantization [1, 7, 10] and ideas from non-commutative geometry [15].

To begin, we construct a many-body field theory on the plane with a three-body potential that is just the *signed* area of the three-body triangle (*not* the absolute value of the area). We take a collection of *three types of non-identical particles*, i.e. three particle "species", each represented by its own local field ψ_a , $a = 1, 2, 3$. In this case, the Hamiltonian has no lower bound for configurations with large but negative A, so the model is reminiscent of a linear potential single-particle QM. (Actually there is more than reminiscing going on here as we shall soon see.)

[1] In the actual talk, I began with a simple model of N distinguishable point particles on the plane, with $V = A$. This is based on work in collaboration with Alexios Polychronakos and Cosmas Zachos [9]. We have completely understood this model's classical and quantum properties. Since the potential is quadratic, this is not very difficult to do, in principle, but nevertheless it was necessary to develop an efficient formalism to handle an arbitrary number of particles. We have done this using so-called cyclotomic coordinates that have previously been used to analyze toroidal membrane dynamics [12]. I view the analysis of this $V = A$ model as a precursor to point particle models with $V = A^2$. These latter models are much more interesting, but are not so easy to solve. While I have made a modicum of progress in an unfinished attempt to construct a Lax pair for certain variants of $V = A^2$ models, in the point particle approach [23], I will not discuss this here. Instead, I will shift from point particle models to field theories.

Explicitly, we define a multi-particle configuration by the field theory Lagrangian

$$L = \sum_{a=1,2,3} i \int (d\mathbf{r})\, \psi_a^* (\mathbf{r}) \frac{\partial}{\partial t} \psi_a (\mathbf{r}) - H \ .$$

It should be understood that all the fields also depend on a common time, t, but usually we will not explicitly indicate this. We then use the obvious symmetrical Hamiltonian

$$\begin{aligned} H = & \int (d\mathbf{r}) \sum_{a=1,2,3} \psi_a^* (\mathbf{r}) \left[\tfrac{-\nabla^2}{2m} \right] \psi_a (\mathbf{r}) \\ & + k \iiint (d\mathbf{r}_1) (d\mathbf{r}_2) (d\mathbf{r}_3) \ A (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \ \left| \psi_1 (\mathbf{r}_1) \psi_2 (\mathbf{r}_2) \psi_3 (\mathbf{r}_3) \right|^2 \end{aligned}$$

involving A, (twice) the area of the triangle formed by the three particles.

$$A (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) = \mathbf{r}_1 \wedge \mathbf{r}_2 + \mathbf{r}_2 \wedge \mathbf{r}_3 + \mathbf{r}_3 \wedge \mathbf{r}_1 \ .$$

This may be rewritten (exactly) as

$$H = \sum_{a=1,2,3} \int (d\mathbf{r}) \, \psi_a^* (\mathbf{r}) \left[\tfrac{-\nabla^2}{2m} \right] \psi_a (\mathbf{r}) + \tfrac{1}{3} \sum_{a=1,2,3} \int (d\mathbf{r}) \left(S_a [\psi] + \mathbf{V}_a [\psi] \cdot \mathbf{r} \right) |\psi_a (\mathbf{r})|^2$$

where we have introduced the combinations $S_a [\psi] + \mathbf{V}_a [\psi] \cdot \mathbf{r}$ for $a = 1, 2, 3$ to serve as *effective potentials* for the three fields. These effective potentials are defined in terms of the fields as

$$\begin{aligned} V_a^i [\psi] &= \tfrac{1}{2} k \varepsilon^{ij} \sum_{b,c=1,2,3} \varepsilon^{abc} \iint (d\mathbf{r}_2) (d\mathbf{r}_3) \ (\mathbf{r}_2 - \mathbf{r}_3)^j \ \left| \psi_b (\mathbf{r}_2) \psi_c (\mathbf{r}_3) \right|^2 \\ S_a [\psi] &= \tfrac{1}{2} k \varepsilon^{abc} \iint (d\mathbf{r}_2) (d\mathbf{r}_3) \ (\mathbf{r}_2 \wedge \mathbf{r}_3) \ \left| \psi_b (\mathbf{r}_2) \psi_c (\mathbf{r}_3) \right|^2 \end{aligned}$$

Note that S_a and $\mathbf{V}_a$ do *not* depend on ψ_a but *do* depend on the other two field configurations, $\psi_{b \neq a}$. Further observe that $S_a [\psi] = 0$ if *either* of the two ψ's involved are of definite parity (either even or odd functions of $\mathbf{r}$), and $V_a^i [\psi] = 0$ if *both* of the two ψ's involved are of definite parity. Thus the special configurations where all three species have definite parity reduce to free fields! Again, this is a model on the plane, so each species coordinate $\mathbf{r}_a$ is a two-vector. To construct a potential linear in the signed area in higher dimensions, we would need to take an inner product of $A (\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ with some constant 2-form.

Like the point-particle theory in [9], this classical field theory is always *solvable*, even in the general situation when all the fields do *not* have definite parity. Unlike [9], we work in real coordinates here. The field equations in terms of the effective potentials are then:

$$i \frac{\partial}{\partial t} \psi_a (\mathbf{r}, t) + \frac{1}{2m} \nabla^2 \psi_a (\mathbf{r}, t) = \left(S_a [\psi] + \mathbf{V}_a [\psi] \cdot \mathbf{r} \right) \psi_a (\mathbf{r}, t) \qquad \text{no sum } a.$$

For each species we have just Schrödinger's equation with a linear (in r^i) potential, albeit a non-isotropic linear potential, in general, with non-constant (in t) configuration-dependent coefficients. From the above definitions we obtain

$$S_a\left[\psi\right] = \tfrac{1}{2}k \sum_{\substack{b,c=1,2,3 \\ i,j=1,2}} \varepsilon^{ij}\varepsilon^{abc} N_b R_b^i N_c R_c^j \ , \quad V_a^i\left[\psi\right] = k \sum_{\substack{b,c=1,2,3 \\ j=1,2}} \varepsilon^{ij}\varepsilon^{abc} N_b R_b^j N_c$$

$$N_a R_a^j \equiv \int (d\mathbf{r})\ r^j\ \left|\psi_a\left(\mathbf{r},t\right)\right|^2 \ , \quad N_a P_a^j \equiv -i \int (d\mathbf{r})\ \psi_a^*\left(\mathbf{r},t\right) \overleftrightarrow{\nabla}^j \psi_a\left(\mathbf{r},t\right) \ ,$$
$$N_a \equiv \int (d\mathbf{r})\ \left|\psi_a\left(\mathbf{r},t\right)\right|^2 ,$$

with no sum over a in any of these. While these are indeed configuration-dependent coefficients, the implicit field dependence of the coefficients is completely tractable. From the field equations, the coefficients in the effective potential obey simple first order time-derivative equations:

$$\frac{d}{dt}N_a = 0 \ , \quad \frac{d}{dt}R_a^k = \frac{1}{2m}\, P_a^k \ ,$$
$$\frac{d}{dt}P_a^j = -2V_a^j = -2k\varepsilon^{ij} \sum_{b,c=1,2,3} \varepsilon^{abc} N_b N_c R_b^j \ , \quad \frac{d}{dt}\left(\sum_{a=1,2,3} N_a P_a^j\right) = 0 \ .$$

The last equation represents conservation of the system's total momentum, while the next to last equation shows that the individual particle species momenta are not separately conserved, in general.

These first order equations combine to yield linear second order equations with *constant* coefficients for any given initial data (hence easily solved).

$$m\frac{d^2}{dt^2}R_a^i = k \sum_{j,b} M_{(ia)(jb)} R_b^j \ , \qquad M_{(ia)(jb)} = \varepsilon^{ij} \sum_{c=1,2,3} \varepsilon^{acb} N_c N_b \qquad (\text{no sum } b) \ .$$

The time-*in*dependent eigenvalues of M are 0 and $\pm\sqrt{N_1 N_2 N_3}\sqrt{N_1+N_2+N_3}$, with each of these three possibilities occurring twice. (Note how the three point-particle mechanics eigenvalues in [9] are obtained by setting all the N's equal to one.) Therefore the R's, and hence all the terms in the effective potentials, can be solved for in terms of the initial data. The three independent classes of solutions for the R's, corresponding to the three eigenvalues of M, are functions linear in t, real exponentials, and oscillations. The time dependencies of the terms in the effective potential then follow from their expression in terms of the R's. Finally, to complete the solution, the linear Schrödinger equations for the individual fields must be solved using the solutions for the effective potentials.

I believe, and assert without proof, that this last step can always be carried out, although there is some possibility of unusual behavior for the fields because

the potential terms in the Schrödinger equation are combinations of polynomials and complex exponentials in t. Technically, I cannot rigorously rule out some sort of chaotic behavior in the phases of the fields, yet, but since the behavior of all the field bilinears is so deterministic, including those involving spatial derivatives, my current opinion is that chaotic behavior in the fields, if any, must be limited to only time-dependent phases and is completely innocuous. (I would devote more time to this issue, but once again, I view this model as just a warm-up exercise for the A^2 model presented below.) I leave it to the reader to compare the above system of equations to those of the standard isotropic oscillator [21, 22].

For consistency with the treatment of the point-particle case in [9], and to warm the hearts of conformal field theorists, the above analysis should perhaps be re-done using complex coordinates. Also, a generalization to a polygon area potential, involving N types of particles and $A\left(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N\right) = \mathbf{r}_1 \wedge \mathbf{r}_2 + \mathbf{r}_2 \wedge \mathbf{r}_3 + \cdots + \mathbf{r}_N \wedge \mathbf{r}_1$, is straightforward. An elegant touch would be to introduce a tensor $E^{a_1 \cdots a_N}$ on the particle type such that the combination $A\left(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_N\right) E^{a_1 \cdots a_N}$ is symmetric under any pair interchange $\mathbf{r}_i, a_i \leftrightarrow \mathbf{r}_j, a_j$. Again the most streamlined way to do this on the plane is to work with complex coordinates, as was the case in the point-particle situation.

We next construct an area-squared field theory, where the potential is A^2, a quartic function of field coordinates. This can be incorporated into a many-body field theory with only a single complex field ψ in any number of spatial dimensions n. The Hamiltonian for the model is

$$\begin{aligned} H[\psi] = & \frac{1}{2m} \int d^n s \, \partial_i \bar{\psi} \partial_i \psi \\ & + \tfrac{1}{3} k \int d^n s_1 \int d^n s_2 \int d^n s_3 \, |\psi(s_1)|^2 |\psi(s_2)|^2 |\psi(s_3)|^2 \left[s_{12}^2 s_{23}^2 - (s_{12} \cdot s_{23})^2 \right] \end{aligned}$$

where $\mathbf{s}_{ab} \equiv \mathbf{s}_a - \mathbf{s}_b$ for each ab pair of the identical particles' coordinate vectors. The equations of motion from $L[\psi] = \int d^n s \, i \bar{\psi} \partial_t \psi - H[\psi]$ are

$$i \partial_t \psi(r) + \frac{1}{2m} \nabla^2 \psi(r) = V[\psi](r) \, \psi(r)$$

In this case the effective potential corresponds to a non-isotropic oscillator

$$\begin{aligned} V[\psi](r) &= k \int d^n s_2 \int d^n s_3 \, |\psi(s_2)|^2 |\psi(s_3)|^2 \left[s_{12}^2 s_{23}^2 - (s_{12} \cdot s_{23})^2 \right] \big|_{s_1 = r} \\ &= K_{ij} r^i r^j + B_i r^i + Z = K_{ij} \left(r^i r^j - 2 R^i r^j \right) + Z \\ &= K_{ij} \left(r^i - R^i \right) \left(r^j - R^j \right) + Z - K_{ij} R^i R^j \end{aligned}$$

with the definitions and relations

$$\begin{aligned} K_{ij} &\equiv k \int d^n s_2 \int d^n s_3 \, |\psi(s_2)|^2 |\psi(s_3)|^2 \left(\delta^{ij} s_{23}^2 - s_{23}^i s_{23}^j \right) \\ &= 2kN^2 \left(\left(S - R^2 \right) \delta_{ij} - \left(S_{ij} - R_i R_j \right) \right) \end{aligned}$$

$$B_i \equiv -2k \int d^n s_2 \int d^n s_3 \; |\psi(s_2)|^2 \, |\psi(s_3)|^2 \left(s_2^i \, s_{23}^2 - s_{23}^i \, s_2 \cdot s_{23} \right)$$
$$= 4kN^2 \left(S_{ij} R_j - S R_i \right) = -2K_{ij} R_j$$

$$Z \equiv k \int d^n s_2 \int d^n s_3 \; |\psi(s_2)|^2 \, |\psi(s_3)|^2 \left(s_2^2 s_3^2 - (s_2 \cdot s_3)^2 \right) = kN^2 \left(S^2 - S_{ij} S_{ji} \right)$$

$$N \equiv \int d^n s \; |\psi(s)|^2 \;\; , \quad R_i \equiv \frac{1}{N} \int d^n s \, s_i \, |\psi(s)|^2 \;\; , \quad P_i[\psi] \equiv \frac{-i}{N} \int d^n s \left(\bar{\psi} \overleftrightarrow{\partial_i} \psi \right) \;\; ,$$
$$S_{ij} \equiv \frac{1}{N} \int d^n s \, s_i s_j \, |\psi(s)|^2 \;\; , \quad S = \sum_{j=1}^{n} S_{jj} \;\; , \quad K_{ij} = 2kN^2 \left(S\delta_{ij} - S_{ij} \right) \;\; .$$

Note that everything in $V[\psi]$ factorizes into integrals of bilinear densities. Actually, this is not too surprising given the polynomial character and factorized form of the terms of the three-body potential. So the effective potential for the area^2 model becomes

$$V[\psi](r) = 2kN^2 \left[S(r-R)^2 - S_{ij} \left(r^i - R^i \right) \left(r^j - R^j \right) \right] + 2kN^2 \left(S_{ij} R^i R^j - S R^2 \right)$$
$$+ kN^2 \left(S^2 - S_{ij} S_{ij} \right) - 2kN^2 \left(R^2 r^2 - (r \cdot R)^2 \right)$$

As already noted, this is a non-isotropic quadratic potential. Note the presence of the last term $R^2 r^2 - (r \cdot R)^2 = (\mathbf{r} \wedge \mathbf{R})^2$ which is the squared area of the parallelogram formed from r and R. The negative coefficient of this term and the potential instability it would produce for large $r \perp R$ are completely offset by the additional presence of $2kN^2 S r^2$ in the effective potential, along with the standard inequality $S \geq R^2$. Also note that V is non-zero only through the field spreading out around the center of mass location. Were $|\psi(s)|^2$ a delta-function at R, all coefficients would vanish in V.

Also note the absence of *nontrivial* "zero modes" for the quadratic part of V (i.e. those zero-modes that would persist even in the CM frame, and not those associated with the free translation of the CM). This is straightforward to see, even though for the field coordinates alone $\det(s_i s_j) = 0 = \det(s^2 \delta_{ij} - s_i s_j)$, and so these coordinate matrices have zero eigenvalues. However, $\langle \det S \rangle \neq \det \langle S \rangle$ just as $\langle x^2 \rangle \neq \langle x \rangle^2$. In fact $\det(S\delta_{ij} - S_{ij}) = \frac{1}{N^n} \det \int d^n s \; (s^2 \delta_{ij} - s_i s_j) \; |\psi(s)|^2 \geq 0$. Moreover, in the CM $\det K > 0$ *and all the eigenvalues of* K_{ij} *are greater than zero.* Or equivalently, all the eigenvalues of $S\delta_{ij} - S_{ij}$ are positive. Proof: For any real vector v, we have $v_i (S\delta_{ij} - S_{ij}) v_j = \frac{1}{N} \int d^n s \; \left(s^2 v^2 - (s \cdot v)^2 \right) |\psi(s)|^2$. But the integrand is non-negative by the Schwarz inequality $s^2 v^2 - (s \cdot v)^2 \geq 0$. Thus any eigenvalue λ of K_{ij} is non-negative, $\lambda \geq 0$. Now to rule out the case where any eigenvalue vanishes in the CM, we note the Schwarz inequality is only an equality $s^2 v^2 - (s \cdot v)^2 = 0$ if $s_i \propto v_i$. Therefore the only way to have a zero eigenvalue is to have the full integrand $\left(s^2 v^2 - (s \cdot v)^2 \right) |\psi(s)|^2$ restricted to a line where $s_i \propto v_i$. Forget that! There is always some transverse spread in $|\psi(s)|^2$. Any reasonable

wave function (or classical field) in $n \geq 2$ dimensions does not have support just on a line. So $V[\psi](r)$ always has a quadratic r^2 part and therefore the field is always localized around the CM. Or, perhaps more physically, all the particles are bound by the effective potential to the CM.

For solutions the energy is

$$H[\psi] = \frac{1}{8m} NT + kN^3 \left(\mathcal{S}^2 - \mathcal{S}_{ij}\mathcal{S}_{ij} \right)$$

using $\mathcal{S}_{ij} = S_{ij} - R_i R_j$, $\mathcal{S} = \mathcal{S}_{jj}$, $T = T_{jj}$, with the stress-tensor defined as

$$T_{ij}[\psi] \equiv \frac{-1}{N} \int d^n s \, \left(\bar{\psi} \overleftrightarrow{\partial_i} \overleftrightarrow{\partial_j} \psi \right) = \frac{-4}{N} \int d^n s \, \left(\bar{\psi} \partial_i \partial_j \psi \right)$$

Now what is the time-dependence of the three effective potential coefficients K_{ij}, B_i, and Z? Or rather, what is the time-dependence of the quantities that compose them? Obviously, N, R_i, and P_i obey the equations

$$\frac{d}{dt} N = 0 \, , \qquad \frac{d}{dt} P_i = 0 \, , \qquad \frac{d}{dt} R_i = \frac{1}{2m} P_i$$

with the solutions

$$N(t) = N_0 \equiv N \, , \qquad P_i(t) = P_{0i} \equiv P_i \, , \qquad R_i(t) = R_{0i} + \frac{1}{2m} P_i t$$

But what about S_{ij}? With the definition (note that $Q_{ij} \neq Q_{ji}$)

$$Q_{ij}[\psi] \equiv \frac{-i}{N} \int d^n s \, s_i \, \left(\bar{\psi} \overleftrightarrow{\partial_j} \psi \right) = \frac{-2i}{N} \int d^n s \, s_i \, \left(\bar{\psi} \partial_j \psi \right) - i\delta_{ij}$$

we find

$$\frac{d}{dt} S_{ij} = \frac{1}{2m} \left(Q_{ij} + Q_{ji} \right)$$

So we must include Q_{ij} into the mix of differential equations. It's time derivative is

$$\frac{d}{dt} Q_{ij} = \frac{1}{2m} T_{ij} - 4 S_{ik} K_{kj} - 2 R_i B_j$$

where $T_{ij}[\psi]$ is the previous stress-tensor. We are one step away from closing the system. We just need the final time derivative

$$\frac{d}{dt} T_{ij} = -4 \left(K_{ik} Q_{kj} + K_{jk} Q_{ki} \right) - 2 P_i B_j - 2 P_j B_i$$

The system of equations is now closed. The problem then is to solve these equations to obtain S_{ij} and hence the effective potential in terms of the initial field configuration . Note that only the symmetric tensor part of Q_{ij} varies with time.

Let's break down the tensors into somewhat more traditional combinations. The symmetric and antisymmetric parts of Q_{ij} are (almost) familiar from rotations and (at least when traced) coordinate rescaling.

$$J_{ij} = Q_{ij} - Q_{ji} \ , \qquad D_{ij} = Q_{ij} + Q_{ji}$$

These nicely occur above with their values for (or about) the CM removed. Thus define

$$\mathcal{D}_{ij} = D_{ij} - (R_i P_j + R_j P_i) \ , \qquad \mathcal{J}_{ij} = J_{ij} - (R_i P_j - R_j P_i)$$

Similarly remove from the quadrupole and stress tensors the CM coordinate and momentum dyads.

$$\mathcal{S}_{ij} = S_{ij} - R_i R_j \ , \qquad \mathcal{T}_{ij} = T_{ij} - P_i P_j$$

Incorporating momentum and angular momentum conservation, i.e. $B_i = -2K_{ij}R^j$ and $K_{ik}S_{kj} - R_j K_{ik} R^k = K_{jk}S_{ki} - R_i K_{jk} R^k$ (just a matrix commutator, $[\mathcal{S}, K]_{ij} = 0$), the time evolution equations are then

$$\begin{aligned}
&\frac{d}{dt}\mathcal{S}_{ij} = \frac{1}{2m}\,\mathcal{D}_{ij} \ , \qquad \frac{d}{dt}N = 0 \ , \qquad \frac{d}{dt}\mathcal{J}_{ij} = 0 \\
&\frac{d}{dt}\mathcal{D}_{ij} = \frac{1}{m}\,\mathcal{T}_{ij} - 4\mathcal{S}_{ik}K_{kj} - 4\mathcal{S}_{jk}K_{ki} = \frac{1}{m}\,\mathcal{T}_{ij} - 4\left\{\mathcal{S}, K\right\}_{ij} \\
&\frac{d}{dt}\mathcal{T}_{ij} = -2K_{ik}\left(\mathcal{D}_{kj} + \mathcal{J}_{kj}\right) - 2K_{jk}\left(\mathcal{D}_{ki} + \mathcal{J}_{ki}\right) = -2\left\{K, \mathcal{D}\right\}_{ij} - 2\left[K, \mathcal{J}\right]_{ij}
\end{aligned} \tag{1}$$

where now $K_{ij} = 2kN^2\left(\mathcal{S}\delta_{ij} - \mathcal{S}_{ij}\right)$. Anyone skilled in the black arts of nuclear physics should feel comfortable with equations of this sort.

An invariant arises from taking traces. Thus

$$\frac{d}{dt}\left(\operatorname{tr}\mathcal{T}_{ij} + 8mkN^2\left(\mathcal{S}^2 - \operatorname{tr}\mathcal{S}_{ij}^2\right)\right) = 0 \ .$$

Now, this is essentially just the energy in the center-of-mass.

$$P^2 + \operatorname{tr}\mathcal{T}_{ij} + 8mkN^2\left(\mathcal{S}^2 - \operatorname{tr}\mathcal{S}_{ij}^2\right) = \operatorname{tr}T_{ij} + 8mkN^2\left(\mathcal{S}^2 - \operatorname{tr}\mathcal{S}_{ij}^2\right) = 8mH[\psi]/N$$

where

$$H[\psi] = \frac{1}{8m}NT + kN^3\left(\mathcal{S}^2 - \mathcal{S}_{ij}\mathcal{S}_{ij}\right) \ .$$

As a special situation, but really without loss of generality, consider co-moving with the system in the CM, where $R_i = 0 = P_i$. Then the effective potential coefficients reduce to

$$K_{ij} = 2kN^2\left(\mathcal{S}\delta_{ij} - \mathcal{S}_{ij}\right) \ , \qquad B_i = 0 \ , \qquad Z = kN^2\left(\mathcal{S}^2 - \mathcal{S}_{ij}\mathcal{S}_{ij}\right)$$

with the $\mathcal{S}_{ij}$, $\mathcal{D}_{ij}$, and $\mathcal{T}_{ij}$ time derivatives unchanged in form. (Note in the CM $S_{ij} = \mathcal{S}_{ij}$, $D_{ij} = \mathcal{D}_{ij}$, and $T_{ij} = \mathcal{T}_{ij}$.)

For now, I cannot solve the non-isotropic equations (1). Perhaps closed-form solutions can be obtained in various limiting situations, but most probably the system is chaotic for general initial data[2]. Nevertheless, I can solve exactly a further specialization, which *is* with loss of generality. Suppose the various densities are actually isotropic about the CM. Then we have

$$\mathcal{D}_{ij} = \frac{1}{n}\mathcal{D}\delta_{ij} \ , \quad \mathcal{S}_{ij} = \frac{1}{n}\mathcal{S}\delta_{ij} \ , \quad \mathcal{T}_{ij} = \frac{1}{n}\mathcal{T}\delta_{ij} \ , \quad K_{ij} = \frac{1}{n}K\delta_{ij} = 2kN^2\left(\frac{n-1}{n}\right)\mathcal{S}\delta_{ij}$$

$$K = 2kN^2\left(n-1\right)\mathcal{S} \ , \quad Z = kN^2\left(\frac{n-1}{n}\right)\mathcal{S}^2$$

and the time derivatives are $\frac{d}{dt}K_{ij} = 2kN^2\left(\frac{n-1}{n}\right)\frac{1}{2m}\mathcal{D}\delta_{ij}$, etc. These may all be traced now w.l.o.g. to yield

$$\frac{d}{dt}K = \frac{1}{m}\,kN^2\left(n-1\right)\mathcal{D} \ , \qquad \frac{d}{dt}Z = \frac{1}{m}\,kN^2\left(\frac{n-1}{n}\right)\mathcal{S}\mathcal{D} \ , \qquad \frac{d}{dt}\mathcal{S} = \frac{1}{2m}\,\mathcal{D} \ ,$$

and the remaining equations

$$\frac{d}{dt}\mathcal{D} = \frac{1}{m}\,\mathcal{T} - 16kN^2\left(\frac{n-1}{n}\right)\mathcal{S}^2 \ , \quad \frac{d}{dt}\mathcal{T} = -8kN^2\left(\frac{n-1}{n}\right)\mathcal{S}\mathcal{D}$$

Using $\frac{d}{dt}\mathcal{S} = \frac{1}{2m}\,\mathcal{D}$ the $\mathcal{T}$ equation becomes $\frac{d}{dt}\mathcal{T} = -8mkN^2\left(\frac{n-1}{n}\right)\frac{d}{dt}\mathcal{S}^2$ which integrates immediately to

$$\mathcal{T}\left(t\right) = \mathcal{T}_0 + 8mkN^2\left(\frac{n-1}{n}\right)\left(\mathcal{S}_0^2 - \mathcal{S}\left(t\right)^2\right)$$

Inserting this into the $\mathcal{D}$ equation and using $\frac{d}{dt}\mathcal{S} = \frac{1}{2m}\,\mathcal{D}$ once again, we have a second-order equation for $\mathcal{S}$.

$$2m\,\frac{d^2}{dt^2}\mathcal{S} = \frac{1}{m}\,\mathcal{T}_0 + 8kN^2\left(\frac{n-1}{n}\right)\mathcal{S}_0^2 - 24kN^2\left(\frac{n-1}{n}\right)\mathcal{S}^2$$

Thus $\mathcal{S}$ alone evolves as a nonlinear oscillator subject to a cubic self-interaction. That is, the effective potential governing the time evolution of $\mathcal{S}$ is $\mathcal{V}\left(\mathcal{S}\right) = a\mathcal{S}^3 - b\mathcal{S}$, where the constant, initial-data-dependent coefficients are $a = 8kN^2\left(\frac{n-1}{n}\right)$, $b = \frac{1}{m}\,\mathcal{T}_0 + 8kN^2\left(\frac{n-1}{n}\right)\mathcal{S}_0^2$, and are both positive. So the potential has a local minimum/maximum at $3a\mathcal{S}^2 = b$ or $\mathcal{S}_{\min\,/\,\max} = \pm\sqrt{\frac{b}{3a}}$, hence

$$\mathcal{S}_{\min\,/\,\max} = \pm\sqrt{\frac{1}{3}\mathcal{S}_0^2 + \frac{1}{24mkN^2}\left(\frac{n}{n-1}\right)\mathcal{T}_0}$$

[2]Since giving this talk, extensive numerical investigations have shown that the model is indeed chaotic for general initial data [8].

Only the local minimum here is physical. Recall that we are in the CM, so that $D = \mathcal{D}$, $T = \mathcal{T}$, and most importantly the inequality $\mathcal{S} = S \equiv \frac{1}{N}\int d^n s\, s^2\, |\psi(s)|^2 \geq 0$,so we see that physically $\mathcal{S}$ is constrained to be positive (and $\mathcal{S} = 0$ only for $|\psi(s)|^2$ ultra-localized at the origin, like a delta-function $\cdots$ a collapse/condensation of the matter field).

Is it possible that we encounter singularities here, in a finite time? By it's definition we are constrained to have $S \geq 0$, but for "special" initial data, perhaps the equation evolves S to zero, the boundary of the unphysical $S < 0$ region, hence the matter field would collapse to the origin. After all, the equation for S is that of a nonlinear "cubic" oscillator. Do we have here a baby version of black-hole physics?

The answer is no, $S = 0$ is never achieved, as we now explain. The effective potential for $\mathcal{S}$ will require $\mathcal{S} > 0$ to be true *for all times* if and only if $\mathcal{S}_0 > 0$ and $\mathcal{E}(\mathcal{S}) < 0$, where $\mathcal{E}$ is the conserved *effective energy* for $\mathcal{S}$.

$$\mathcal{E}(\mathcal{S}) \equiv m\left(\frac{d}{dt}\mathcal{S}\right)^2 + \mathcal{V}(\mathcal{S}) = \mathcal{E}_0$$

$$\mathcal{E}_0 = \frac{1}{4m}\mathcal{D}_0^2 + 8kN^2\left(\tfrac{n-1}{n}\right)\mathcal{S}_0^3 - \left(\frac{1}{m}\mathcal{T}_0 + 8kN^2\left(\tfrac{n-1}{n}\right)\mathcal{S}_0^2\right)\mathcal{S}_0 = \frac{1}{4m}\left(\mathcal{D}_0^2 - 4\,\mathcal{T}_0\mathcal{S}_0\right)$$

Consider the two terms on the RHS of this last expression. The standard QM uncertainty relation (basically the Schwarz inequality) for any two hermitean operators $\mathbf{a}$ and $\mathbf{b}$, assuming $\langle\mathbf{a}\rangle = 0 = \langle\mathbf{b}\rangle$, is $\langle\mathbf{a}^2\rangle\langle\mathbf{b}^2\rangle \geq \left\langle\frac{-i}{2}(\mathbf{ab}-\mathbf{ba})\right\rangle^2 + \left\langle\frac{1}{2}(\mathbf{ab}+\mathbf{ba})\right\rangle^2$. Each of the two terms on the RHS of the inequality are positive. Usually we discard the second of these RHS terms and keep the first, especially in the case for canonically conjugate variables with $\mathbf{xp}-\mathbf{px} = i\hbar$. But we could equally well discard the first RHS term to obtain$\langle\mathbf{a}^2\rangle\langle\mathbf{b}^2\rangle - \left\langle\frac{1}{2}(\mathbf{ab}+\mathbf{ba})\right\rangle^2 \geq 0$. This is a strict inequality for conjugate variables. Applying this to the situation at hand we conclude $4\mathcal{TS} - \mathcal{D}^2 > 0$. Hence the cubic oscillator system above is always bound to the stable minimum, with $\mathcal{E} < 0$. This is what we wanted to show.

We will discuss static solutions in more detail below. For now, we note that these are possible if we are at the local minimum of $\mathcal{V}(\mathcal{S})$ with $\mathcal{D} = 0$, where we require

$$\mathcal{S}_{\min} = \mathcal{S}_0 = \sqrt{\frac{1}{3}\mathcal{S}_0^2 + \frac{1}{24mkN^2}\left(\frac{n}{n-1}\right)\mathcal{T}_0}\,, \quad \text{or} \quad \mathcal{T}_0 = 16mkN^2\left(\frac{n-1}{n}\right)\mathcal{S}_0^2\,,$$

as well as $\mathcal{D}_0 = 0$, of course. The fields themselves may be worked out explicitly in this case and are nothing but Gaussians.

Even in the non-static case the conserved energy for $\mathcal{S}$ reduces it's determination to quadrature and allows us to solve for the $\mathcal{S}$ motion in textbook fashion.

$$\frac{d}{dt}\mathcal{S} = \pm\sqrt{\left(\mathcal{E}_0 - \mathcal{V}(\mathcal{S})\right)/m}$$

$$\pm\frac{t}{\sqrt{m}}=\int_{\mathcal{S}_0}^{\mathcal{S}(t)}\frac{d\mathcal{S}}{\sqrt{\mathcal{E}_0-\mathcal{V}(\mathcal{S})}}=\frac{1}{\sqrt{a}}\int_{\mathcal{S}_0}^{\mathcal{S}(t)}\frac{d\mathcal{S}}{\sqrt{(\mathcal{S}_{hi}-\mathcal{S})(\mathcal{S}-\mathcal{S}_{mid})(\mathcal{S}-\mathcal{S}_{lo})}}$$

where for $\mathcal{E}_0<0$, $\{\mathcal{S}_{lo},\mathcal{S}_{mid},\mathcal{S}_{hi}\}$ are the three distinct zeroes of the cubic $\mathcal{E}_0+b\mathcal{S}-a\mathcal{S}^3$ with $\mathcal{S}_{lo}<0<\mathcal{S}_{mid}\leq\mathcal{S}(t)\leq\mathcal{S}_{hi}$. The RHS here is an elliptic integral.

$$\int_{\mathcal{S}_{mid}}^{\mathcal{S}(t)}\frac{\sqrt{\mathcal{S}_{hi}-\mathcal{S}_{lo}}\ dS}{\sqrt{4(\mathcal{S}_{hi}-S)(S-\mathcal{S}_{mid})(S-\mathcal{S}_{lo})}}=K\left(\sqrt{\tfrac{\mathcal{S}_{hi}-\mathcal{S}_{mid}}{\mathcal{S}_{hi}-\mathcal{S}_{lo}}}\right)-F\left(\sqrt{\tfrac{\mathcal{S}_{hi}-\mathcal{S}(t)}{\mathcal{S}_{hi}-\mathcal{S}_{mid}}},\sqrt{\tfrac{\mathcal{S}_{hi}-\mathcal{S}_{mid}}{\mathcal{S}_{hi}-\mathcal{S}_{lo}}}\right)$$

K and F are the standard elliptic integrals, with $F(1,z)=K(z)$, and $F(0,z)=0$. Thus $\mathcal{S}(t)$ is an elliptic function. The solution oscillates between the distinct turning points $\mathcal{S}_{mid}\leq\mathcal{S}(t)\leq\mathcal{S}_{hi}$ with period

$$\tfrac{1}{2}t_{period}=\sqrt{\tfrac{m}{a}}\int_{\mathcal{S}_{mid}}^{\mathcal{S}_{hi}}\frac{d\mathcal{S}}{\sqrt{(\mathcal{S}_{hi}-\mathcal{S})(\mathcal{S}-\mathcal{S}_{mid})(\mathcal{S}-\mathcal{S}_{lo})}}=\sqrt{\tfrac{4m}{a(\mathcal{S}_{hi}-\mathcal{S}_{lo})}}K\left(\sqrt{\tfrac{\mathcal{S}_{hi}-\mathcal{S}_{mid}}{\mathcal{S}_{hi}-\mathcal{S}_{lo}}}\right)$$

We may extend the analysis to higher dimensions for models with potentials depending on higher multi-particle coordinate forms. Assume a $(d+1)$-body potential U in $n\geq d$ dimensions which is a (d-form)^2. That is

$$F(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\cdots,\mathbf{r}_d,\mathbf{r}_{d+1})=(\mathbf{r}_1-\mathbf{r}_2)\wedge(\mathbf{r}_2-\mathbf{r}_3)\wedge\cdots\wedge(\mathbf{r}_d-\mathbf{r}_{d+1})$$

$$\begin{aligned}U(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_{d+1})&=kF^2(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_{d+1})\\&=k\ \delta^{i_1i_2\cdots i_d}_{j_1j_2\cdots j_d}\ \mathbf{r}^{i_1}_{12}\mathbf{r}^{j_1}_{12}\mathbf{r}^{i_2}_{23}\mathbf{r}^{j_2}_{23}\mathbf{r}^{i_3}_{34}\mathbf{r}^{j_3}_{34}\cdots\mathbf{r}^{i_d}_{dd+1}\mathbf{r}^{j_d}_{dd+1}\end{aligned}$$

$$\begin{aligned}H=&\int(d\mathbf{r})\ \psi^*(\mathbf{r})\left[-\frac{1}{2m}\nabla^2\right]\psi(\mathbf{r})\\&+\frac{1}{d+1}\int\cdots\int(d\mathbf{r}_1)\cdots(d\mathbf{r}_{d+1})\ U(\mathbf{r}_1,\mathbf{r}_2,\cdots,\mathbf{r}_{d+1})\ \left|\psi(\mathbf{r}_{d+1})\cdots\psi(\mathbf{r}_2)\,\psi(\mathbf{r}_1)\right|^2\end{aligned}$$

with $\int\cdots\int(d\mathbf{r}_2)\cdots(d\mathbf{r}_{d+1})\ \ U(\mathbf{r},\mathbf{r}_2,\cdots,\mathbf{r}_{d+1})\ \ \left|\psi(\mathbf{r}_{d+1})\cdots\psi(\mathbf{r}_2)\right|^2\ \ \psi(\mathbf{r})$ in the field equation.

If we assume spherically symmetric fields we always obtain from this isotropic harmonic effective potentials.

$$\begin{aligned}&\int\cdots\int(d\mathbf{r}_2)\cdots(d\mathbf{r}_{d+1})\ U(\mathbf{r},\mathbf{r}_2,\cdots,\mathbf{r}_{d+1})\ \left|\psi(\mathbf{r}_{d+1})\cdots\psi(\mathbf{r}_2)\right|^2\\&=k(n-1)(n-2)\cdots(n-d+1)\left(dM_0r^2+M_2\right)\left(\frac{M_2}{n}\right)^{d-1}\end{aligned}$$

where $M_k\equiv\int(d\mathbf{r})\,(\mathbf{r}^2)^{k/2}\,|\psi(r)|^2=\Omega_n\int_0^\infty|\psi|^2\,r^{k+n-1}dr$ and $\Omega_n=2\pi^{n/2}/\Gamma(n/2)$. As usual the interpretation (at least in the quantum theory) is that $N\equiv M_0$ is the

total number of particles. Thus the field equation for such spherically symmetric solutions becomes

$$i\frac{\partial}{\partial t}\psi\left(r\right)+\frac{1}{2m}\left(\frac{d^2}{dr^2}\psi\left(r\right)+\frac{n-1}{r}\frac{d}{dr}\psi\left(r\right)\right)=\frac{M_2^{d-1}\left(n-1\right)!}{n^{d-1}\left(n-d\right)!}\left(dM_0r^2+M_2\right)k\psi\left(r\right)\ .$$

For a Gaussian ansatz, $\psi\left(\mathbf{r},t\right)=C\exp\left(-\frac{1}{2}c\mathbf{r}^2\right)\exp\left(-iEt\right)$, this field equation reduces to

$$E+\frac{1}{2m}\left(c^2r^2-nc\right)=k\frac{\left(n-1\right)!}{\left(n-d\right)!}\left(\frac{N}{2c}\right)^{d-1}\left(dNr^2+\frac{nN}{2c}\right)$$

and so we have a solution provided

$$\frac{1}{2m}c^{d+1}=k\frac{\left(n-1\right)!}{\left(n-d\right)!}N^d\frac{d}{2^{d-1}}\ ,\qquad E=\frac{nc}{2m}\left(1+\frac{1}{2d}\right)$$

The E above is almost, but not quite, the energy per particle. That is, the value of H, the total energy, is almost but not quite EN for the spherically symmetric solution. For the Gaussian ansatz

$$H=\frac{ncN}{4m}\left(1+\frac{1}{d}\right)$$

The above suggests what to do also in the situation where ψ is not necessarily spherically symmetric. For any $U\left(\mathbf{r},\mathbf{r}_2,\mathbf{r}_3\cdots,\mathbf{r}_d,\mathbf{r}_{d+1}\right)$ which is a quadratic form in the components of $\mathbf{r}$ we may write quite generally

$$Z+\mathbf{r}^iB_i+\mathbf{r}^i\mathbf{r}^jK_{ij}=\int\cdots\int\left(d\mathbf{r}_2\right)\cdots\left(d\mathbf{r}_{d+1}\right)\ U\left(\mathbf{r},\mathbf{r}_2,\cdots,\mathbf{r}_{d+1}\right)\ \left|\psi\left(\mathbf{r}_{d+1}\right)\cdots\psi\left(\mathbf{r}_2\right)\right|^2\ .$$

This is true for the case at hand. $U\left(\mathbf{r},\mathbf{r}_2,\mathbf{r}_3\cdots,\mathbf{r}_d,\mathbf{r}_{d+1}\right)$ is a quadratic form in the components of $\mathbf{r}$ (as well as quadratic in all the other individual $\mathbf{r}_a$) even though it is of order $2d$ altogether. Hence

$$K_{ij}=k\delta^{ii_2\cdots i_d}_{jj_2\cdots j_d}\int\cdots\int\left(d\mathbf{r}_2\right)\cdots\left(d\mathbf{r}_{d+1}\right)\ \mathbf{r}_{23}^{i_2}\mathbf{r}_{23}^{j_2}\cdots\mathbf{r}_{dd+1}^{i_d}\mathbf{r}_{dd+1}^{j_d}\ \left|\psi\left(\mathbf{r}_{d+1}\right)\cdots\psi\left(\mathbf{r}_2\right)\right|^2$$

$$B_i=-2k\delta^{ii_2\cdots i_d}_{j_1j_2\cdots j_d}\int\cdots\int\left(d\mathbf{r}_2\right)\cdots\left(d\mathbf{r}_{d+1}\right)\ \mathbf{r}_2^{j_1}\ \mathbf{r}_{23}^{i_2}\mathbf{r}_{23}^{j_2}\cdots\mathbf{r}_{dd+1}^{i_d}\mathbf{r}_{dd+1}^{j_d}\ \left|\psi\left(\mathbf{r}_{d+1}\right)\cdots\psi\left(\mathbf{r}_2\right)\right|^2$$

$$Z=k\delta^{i_1i_2\cdots i_d}_{j_1j_2\cdots j_d}\int\cdots\int\left(d\mathbf{r}_2\right)\cdots\left(d\mathbf{r}_{d+1}\right)\ \mathbf{r}_2^{i_1}\mathbf{r}_2^{j_1}\ \mathbf{r}_{23}^{i_2}\mathbf{r}_{23}^{j_2}\cdots\mathbf{r}_{dd+1}^{i_d}\mathbf{r}_{dd+1}^{j_d}\ \left|\psi\left(\mathbf{r}_{d+1}\right)\cdots\psi\left(\mathbf{r}_2\right)\right|^2$$

It is now straightforward to use the field equations

$$i\frac{\partial}{\partial t}\psi\left(\mathbf{r}\right)+\frac{1}{2m}\nabla^2\psi\left(\mathbf{r}\right)=\left(Z+\mathbf{r}^iB_i+\mathbf{r}^i\mathbf{r}^jK_{ij}\right)\psi\left(\mathbf{r}\right)$$

to determine a closed set of time-derivative equations obeyed by the coefficients in the effective potential. Once these are solved, either in special situations or

perhaps more generally, then the field equation itself is to be solved using the time-dependence determined for the coefficients, in a self-consistent way. Sounds easy, even if it is not in practice, but perhaps the resulting non-isotropic equations can always be solved in closed form in some limit, such as large n.

There was neither enough time in the talk nor enough space in this written version to discuss either the supersymmetric extensions of these models or their quantization using deformation methods [9, 6]. These subjects will be treated elsewhere.

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