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Bulk-Boundary Propagator in Liouville Theory on a Disc

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Abstract

We study Liouville theory on worldsheets with boundary using the solutions of Knizhnik-Zamolodchikov equation involving a degenerate representation of the Virasoro algebra. The expression for bulk-boundary propagator on a disc is proposed.

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Liouville theory is an interacting two-dimensional conformal field theory that has many applications in string theory. It was originally studied as a theory of two-dimensional gravity, and recently it has also applied to the study of strings propagating near the singularities of manifolds. It plays a key role in understanding the holography in string theory, since most of the known backgrounds realizing the holography have a radial direction which is described by a linear-dilaton like theory. To study the property of D-branes on such backgrounds from the CFT viewpoint, it is therefore necessary to understand the Liouville theory on open worldsheets.

Liouville theory on open worldsheets was analyzed in [1, 2] where some basic correlators were derived. They were based on bootstraps and free field techniques which were first developed in [3, 4, 5] for the case without boundary. See also [6, 7, 8]. Our analysis in this letter is along these path.

Liouville theory on a worldsheet with boundary is defined by the action

$$\begin{aligned} I &= \frac{1}{8\pi}\int_\Sigma d^2\sigma\sqrt{g}\left[g^{mn}\partial_m\phi\partial_n\phi+\sqrt{2}QR\phi+8\pi\mu e^{\sqrt{2}b\phi}\right] \\ &\quad +\frac{1}{4\pi}\int_{\partial\Sigma}d\xi g^{1/4}\left[\sqrt{2}QK\phi+4\pi\mu_B e^{b\phi/\sqrt{2}}\right], \end{aligned} \tag{1}$$

where K is the curvature of the boundary and μ_B is called the boundary cosmological constant. μ_B can take different values for each connected component of the boundary and, what is more, μ_B can be different for each boundary segment bounded by two boundary vertex operators.

When μ and μ_B vanish, the theory reduces to a free theory with the stress tensor

$$T=-\frac{1}{2}(\partial\phi\partial\phi-\sqrt{2}Q\partial^2\phi) \tag{2}$$

and the center $c=1+6Q^2$.

We have two kinds of primary fields in Liouville theory with boundary. Bulk primaries $V_\alpha(z)=e^{\sqrt{2}\alpha\phi(z)}$ correspond to closed string modes while boundary primaries $B_\alpha(x)=e^{\alpha\phi(x)/\sqrt{2}}$ correspond to open string modes. They are of conformal weight $h_\alpha=\alpha(Q-\alpha)$. We mainly use coordinates $z,w,\dots$ for positions of bulk fields and $x,y,\cdots$ for boundary fields.

As was studied in [1, 2], it is natural to regard μ_B as labeling D-branes or Cardy states in Liouville theory. The classification of Cardy states has been done in [2] using the modular property of Virasoro characters. Consistent Cardy states are expressed as linear combination of Ishibashi states, and the coefficients are proportional to the one-point function of a bulk field $\langle V_\alpha(z)\rangle$ on a disc. This was obtained in [1]:

$$\begin{aligned} \langle V_\alpha(z)\rangle_{\mu_B} &= U(\alpha;s)|z-\bar{z}|^{-2h_\alpha}, \\ U(\alpha;s) &= 2\cosh\left[2\pi s(2\alpha-Q)\right]\left\{\mu\pi\gamma(b^2)\right\}^{(Q-2\alpha)/2b}(2\alpha-Q)\Gamma[(2\alpha-Q)b]\Gamma[(2\alpha-Q)/b], \end{aligned} \tag{3}$$

where $\gamma(x)=\Gamma(x)/\Gamma(1-x)$ and we introduced a new parameter s for labeling the Cardy states $|s\rangle_c$. It is related to μ_B via

$$\mu_B=\cosh(2\pi sb)\left(\frac{\mu}{\sin\pi b^2}\right)^{1/2}. \tag{4}$$

Our goal is to obtain the expression for bulk-boundary propagator $\langle V_\alpha(z)B_\beta(x)\rangle_s$ on a disc. What we have to determine is the structure constant $R(\alpha,\beta;s)$ defined by

$$\langle V_\alpha(z)B_\beta(x)\rangle_s = |z-\bar{z}|^{h_\beta-2h_\alpha}|z-x|^{-2h_\beta}R(\alpha,\beta;s). \tag{5}$$

To obtain this we use the $(1+2)$-point function with an auxiliary insertion of a degenerate operator $B_{-b/2}(y)$. Conformal invariance restricts the correlator to take the form

$$\begin{aligned}
&\left\langle V_\alpha(z)B_\beta(x)B_{-b/2}(y)\right\rangle^{\text{(upper half plane)}}_{s_L,s_R} \\
&= (x-z)^{-h_\beta-h_{-b/2}}(x-\bar{z})^{-h_\beta+h_{-b/2}}|z-\bar{z}|^{h_\beta-2h_\alpha+h_{-b/2}}(y-\bar{z})^{-2h_{-b/2}} \\
&\quad\times\left\langle V_\alpha(0)B_\beta(1)B_{-b/2}(\eta)\right\rangle^{\text{(disc)}}_{s_L,s_R}, \qquad \eta=\tfrac{(y-z)(x-\bar{z})}{(y-\bar{z})(x-z)}
\end{aligned} \tag{6}$$

This correlator has two boundary segments since there are two boundary operator insertions. The suffices L,R mean the left or right of $B_{-b/2}$ on the real axis. The two boundary cosmological constants must satisfy $s_L\pm s_R=\pm' ib/2$ so as to be consistent with the fusion rule[2]. Otherwise the null vector actually does not decouple.

The $(1+2)$-point function can be expressed in a number of ways:

$$\begin{aligned}
&\left\langle V_\alpha(0)B_\beta(1)B_{-b/2}(\eta)\right\rangle^{\text{(disc)}}_{s_L,s_R} \\
&= c_+(\beta;s_L,s_R)R(\alpha,\beta-\tfrac{b}{2};s_R)G_{\alpha,\beta}(\eta)e^{i\pi b\beta/2} \\
&\quad + c_-(\beta;s_L,s_R)R(\alpha,\beta+\tfrac{b}{2};s_R)G_{\alpha,Q-\beta}(\eta)e^{i\pi b(Q-\beta)/2} \\
&= c_+(\beta;s_R,s_L)R(\alpha,\beta-\tfrac{b}{2};s_L)G_{\alpha,\beta}(\eta e^{-2\pi i})e^{-i\pi b\beta/2} \\
&\quad + c_-(\beta;s_R,s_L)R(\alpha,\beta+\tfrac{b}{2};s_L)G_{\alpha,Q-\beta}(\eta e^{-2\pi i})e^{-i\pi b(Q-\beta)/2},
\end{aligned} \tag{7}$$

where the equality should hold at least up to overall factors. Here $G_\beta(\eta)$ is the following solution of KZ equation:

$$G_{\alpha,\beta}(\eta)=\eta^{b\alpha}(1-\eta)^{b\beta}F\left(b(2\alpha+\beta-Q-\tfrac{b}{2}),b(\beta-\tfrac{b}{2}),b(2\beta-b);1-\eta\right). \tag{8}$$

We assume that the correlator is analytic on the complex η-plane except on the positive half of the real axis. The expression $\eta e^{-2\pi i}$ indicates going clockwise around $\eta=0$ once. $c_\pm(\beta;s_L,s_R)$ are the OPE coefficients:

$$B_\beta(x)B_{-b/2}(y)_{s_L,s_R}\overset{x<y}{\sim}|y-x|^{b\beta}c_+(\beta;s_L,s_R)B_{\beta-b/2}(x)+|y-x|^{b(Q-\beta)}c_-(\beta;s_L,s_R)B_{\beta+b/2}(x) \tag{9}$$

They are calculated as the free field correlators with an appropriate insertion of boundary screening operator $S_B=\int_{\partial\Sigma}B_b$:

$$\begin{aligned}
c_+(\beta;s_L,s_R) &= \left\langle B_\beta B_{-b/2}B_{Q-\beta+b/2}\right\rangle_{\text{free}} = 1, \\
c_-(\beta;s_L,s_R) &= \left\langle(-\mu_B S_B)B_\beta B_{-b/2}B_{Q-\beta+b/2}\right\rangle_{\text{free}} \\
&= \frac{2b^2}{\pi}\Gamma(1-2b\beta)\Gamma(2b\beta-bQ)\cos\pi(b\beta-\tfrac{bQ}{2})(\mu\pi\gamma(b^2))^{1/2}\cos\pi(b\beta-\tfrac{bQ}{2}\mp 2ibs_R) \\
&\qquad (s_R=s_L\pm\tfrac{ib}{2}).
\end{aligned} \tag{10}$$

We can derive some recursion relations between $R(\alpha, \beta; s)$ from the transformation property and monodromy of hypergeometric functions. We shall do this by rewriting the $(1+2)$-point function as a linear sum of the solutions $H_{\alpha,\beta}(\eta)$ and $H_{Q-\alpha,\beta}(\eta)$ that diagonalize the monodromy around $\eta = 0$. They are defined as

$$H_{\alpha,\beta}(\eta) = \eta^{b\alpha}(1-\eta)^{b\beta} F\left(b(2\alpha+\beta-Q-\tfrac{b}{2}), b(\beta-\tfrac{b}{2}), b(2\alpha-b); \eta\right) \tag{11}$$

and are related to $G_{\alpha,\beta}$ via

$$\begin{aligned} (G_{\alpha,\beta}, G_{\alpha,Q-\beta}) &= (H_{\alpha,\beta}, H_{Q-\alpha,\beta}) \begin{bmatrix} x_{\beta,\alpha} & x_{Q-\beta,\alpha} \\ x_{\beta,Q-\alpha} & x_{Q-\beta,Q-\alpha} \end{bmatrix}, \\ x_{\beta,\alpha} &= \frac{\Gamma(2b\beta-b^2)\Gamma(1+b^2-2b\alpha)}{\Gamma(1+\frac{b^2}{2}-2b\alpha+b\beta)\Gamma(b\beta-\frac{b^2}{2})}. \end{aligned} \tag{12}$$

There are subtle phase factors arising in deriving the recursion relation, which should be handled carefully so that the resultant relation is symmetric under the exchange of s_L and s_R. We must also pay attention to the relation

$$\left\langle V_\alpha(0) B_\beta(1) B_{-b/2}(\eta)\right\rangle^{(\mathrm{disc})}_{s_L,s_R} = \eta^{-2h_{-b/2}} \left\langle V_\alpha(0) B_\beta(1) B_{-b/2}(\eta^{-1})\right\rangle^{(\mathrm{disc})}_{s_R,s_L}, \tag{13}$$

which merely corresponds to flipping the real axis of the upper half-plane. We finally find the following relation:

$$iX^+ \begin{bmatrix} c_+(\beta; s_R, s_L) R(\alpha, \beta-\frac{b}{2}; s_L) \\ c_-(\beta; s_R, s_L) R(\alpha, \beta+\frac{b}{2}; s_L) \end{bmatrix} = X^- \begin{bmatrix} c_+(\beta; s_L, s_R) R(\alpha, \beta-\frac{b}{2}; s_R) \\ c_-(\beta; s_L, s_R) R(\alpha, \beta+\frac{b}{2}; s_R) \end{bmatrix} \tag{14}$$

where the components of $X^\pm$ are given by

$$x^\pm_{\beta,\alpha} = e^{\pm i\pi b(\alpha+\frac{\beta}{2}-\frac{3Q}{4})} x_{\beta,\alpha}. \tag{15}$$

The phases are fixed so that we can find a matrix Y satisfying $YX^\mp = \pm iX^\pm$, ensuring that the relation is symmetric under the exchange of s_L and s_R.

Let us pose a while and see the consistency of the relation (14) with the reflection symmetry of the vertices. The structure constant $R(\alpha, \beta; s)$ should be invariant under the reflection of the bulk vertex operator $V_\alpha \Leftrightarrow V_{Q-\alpha}$:

$$V_\alpha = D(\alpha) V_{Q-\alpha}, \quad D(\alpha) = -[\mu\pi\gamma(b^2)]^{(Q-2\alpha)/b} \frac{\Gamma((2\alpha-Q)b)\Gamma((2\alpha-Q)/b)}{\Gamma(-(2\alpha-Q)b)\Gamma(-(2\alpha-Q)/b)}. \tag{16}$$

and the reflection of the boundary operator $B_\beta \Leftrightarrow B_{Q-\beta}$:

$$\begin{aligned} B_\beta &= d(\beta; s, s) B_{Q-\beta}, \\ d(\beta; s_L, s_R) &= \frac{\mathbf{G}(Q-2\beta)\mathbf{G}(2\beta-Q)^{-1}\{\mu\pi\gamma(b^2)b^{2-2b^2}\}^{(Q-2\beta)/2b}}{\mathbf{S}(\beta+is_L+is_R)\mathbf{S}(\beta-is_L-is_R)\mathbf{S}(\beta+is_L-is_R)\mathbf{S}(\beta-is_L+is_R)}. \end{aligned} \tag{17}$$

Here we used some of the functions Υ, $\mathbf{S}$, $\mathbf{G}$ introduced in [3, 4, 1]. They are characterized by the shift relations

$$\Upsilon(x+b) = b^{1-2bx}\gamma(bx)\Upsilon(x), \qquad \mathbf{S}(x+b) = 2\sin(\pi bx)\mathbf{S}(x), \qquad \mathbf{G}(x+b) = \frac{b^{\frac{1}{2}-bx}}{\sqrt{2\pi}}\Gamma(bx)\mathbf{G}(x), \tag{18}$$

and those with b replaced with $1/b$. We can check that if we can find a solution $R(\alpha,\beta;s)$ of the recursion relation, then its reflections

$$D(\alpha)R(Q-\alpha,\beta;s), \quad d(\beta;s,s)R(\alpha,Q-\beta;s) \tag{19}$$

are also solutions of the same recursion relation.

By inserting $s_R = s_L + \frac{ib}{2}$ into (14) we obtain partial difference equations for $R(\alpha,\beta;s)$. To write down the solution, it seems that the most efficient way is to take the Fourier transform:

$$\tilde{R}(\alpha,\beta;p) = \tfrac{1}{2}\int_{-\infty}^{\infty} ds e^{4\pi sp} R(\alpha,\beta;s), \qquad R(\alpha,\beta,s) = -i\int_{-i\infty}^{i\infty} dp e^{-4\pi sp}\tilde{R}(\alpha,\beta,p), \tag{20}$$

in terms of which the difference equation becomes simply

$$\begin{aligned} & \tilde{R}(\alpha,\beta;p+b)\sin\pi b(p+\alpha-\tfrac{\beta}{2}+\tfrac{Q}{2})\sin\pi b(p-\alpha-\tfrac{\beta}{2}+\tfrac{3Q}{2}) \\ = \ & \tilde{R}(\alpha,\beta;p)\sin\pi b(p+\alpha+\tfrac{\beta}{2}-\tfrac{Q}{2})\sin\pi b(p-\alpha+\tfrac{\beta}{2}+\tfrac{Q}{2}) \\ = \ & \tilde{R}(\alpha,\beta+b;p+\tfrac{b}{2})\frac{\pi^2 b^2(\mu\pi\gamma(b^2))^{1/2}\Gamma(1-2b\beta)\Gamma(1-b^2-2b\beta)}{\Gamma(b\beta)\Gamma(1-b\beta)^3\Gamma(1-b\beta-2b\alpha+bQ)\Gamma(1-b\beta+2b\alpha-bQ)}. \end{aligned} \tag{21}$$

A solution is given by

$$\begin{aligned} \tilde{R}(\alpha,\beta;p) \ = \ & 2\pi(\mu\pi\gamma(b^2)b^{2-2b^2})^{(Q-2\alpha-\beta)/2b}\frac{\mathbf{G}(Q)\mathbf{G}(\beta)\mathbf{G}(Q-2\beta)\Upsilon(2\alpha)}{\mathbf{G}(Q-\beta)^3\mathbf{G}(2Q-2\alpha-\beta)\mathbf{G}(2\alpha-\beta)} \\ & \times\mathbf{S}(p+\tfrac{\beta+2\alpha-Q}{2})\mathbf{S}(p+\tfrac{\beta-2\alpha+Q}{2})\mathbf{S}(-p+\tfrac{\beta+2\alpha-Q}{2})\mathbf{S}(-p+\tfrac{\beta-2\alpha+Q}{2}). \end{aligned} \tag{22}$$

The normalization was fixed so that we have $R(\alpha,0;s) = U(\alpha;s)$.

We have to analyze the transformation property of the above solution under the reflection (17). To do this we check whether or not the value of $R(\alpha,\beta;s)$ at $\beta = Q$ is as required from the reflection symmetry:

$$d(\beta;s,s)\times(-i)\int dp e^{-4\pi sp}\tilde{R}(\alpha,Q-\beta;p) \overset{\beta\to 0}{\to} U(\alpha;s). \tag{23}$$

In doing this we have to be careful for the fact that when a factor like $e^{4\pi sp_0}$ is multiplied onto the inverse Fourier transform (p-integration), it shifts the contour of p-integration and possibly picks up some poles. A calculation shows that our structure constant satisfies the above equation, ensuring the reflection symmetry at $\beta = 0$.

As we have seen above, the solution was obtained by solving the recursion relation (14) and imposing the reflection symmetry. The normalization was fixed by matching with $U(\alpha;s)$ in the limit $\beta\to 0$. Any solution satisfying these conditions should be equal to the above $R(\alpha,\beta;s)$ for all $\beta\in\{b\mathbf{Z}+b^{-1}\mathbf{Z}\}$ due to recursion relation (14) and that associated with the degenerate field $B_{-1/2b}$. In this sense the solution is unique.

It is known from the perturbative analysis based on path-integral formalism that the correlators diverge when the non-conservation of Liouville momentum can be screened by a non-negative integer number of screening operators. Our $R(\alpha,\beta;s)$ indeed has the corresponding poles at $2\alpha+\beta=Q-nb$. For the first few of them we can also check that the residue is precisely equal to the free-field correlators as expected.

If a bulk operator V_α approaches the boundary, it can be expanded as a sum over boundary fields B_β. One naively expects that the expansion obeys the following formula

$$\begin{aligned} V_\alpha(z) &\overset{z\to x}{\sim} \tfrac{1}{2}\int d\beta|z-\bar{z}|^{h_\beta-2h_\alpha}R(\alpha,\beta;s)B_{Q-\beta}(x) \\ &= \tfrac{1}{2i}\int d\beta dp|z-\bar{z}|^{h_\beta-2h_\alpha}e^{-4\pi sp}\tilde{R}(\alpha,\beta;p)B_{Q-\beta}(x). \end{aligned} \tag{24}$$

where the β-integration should be done over $\beta\in\frac{Q}{2}+i\mathbf{R}$ for $\alpha\in\frac{Q}{2}+i\mathbf{R}$, and suitably deformed for generic α in order to ensure the analyticity. Similar deformation of contour is also assumed for p-integration. As a special case, take as α those values corresponding to degenerate representation of the Virasoro algebra:

$$2\alpha_{k,l}=Q-kb-lb^{-1},\quad (k,l\in\mathbf{Z}_{\geq 1}). \tag{25}$$

We then expect that V_α is expanded into boundary degenerate operators $B_{\beta_{m,n}}$ with

$$2\beta_{m,n}=Q-mb-nb^{-1},\quad (m=1,3,\cdots,2k-1,\quad n=1,3,\cdots,2l-1). \tag{26}$$

Our $R(\alpha,\beta;s)$ agrees with this expectation. To see this, note first that the integral over β and p has only contribution from poles due to the vanishing factor $\Upsilon(2\alpha)$ in the integrand. In order to cancel this factor, we actually need the degeneration of two poles that pinch the integration contour. By analyzing the location and the degeneracy of the poles of

$$\begin{aligned} &\tilde{R}(\alpha,\beta;s)d(Q-\beta;s,s) \\ &\sim \int dp\frac{\mathbf{G}(2\beta-Q)\Upsilon(2\alpha)\mathbf{S}(p+\frac{\beta+2\alpha-Q}{2})\mathbf{S}(p+\frac{\beta-2\alpha+Q}{2})\mathbf{S}(-p+\frac{\beta+2\alpha-Q}{2})\mathbf{S}(-p+\frac{\beta-2\alpha+Q}{2})}{\mathbf{G}(\beta)\mathbf{G}(Q-\beta)\mathbf{G}(2Q-2\alpha-\beta)\mathbf{G}(2\alpha-\beta)} \end{aligned} \tag{27}$$

we find that the degenerate bulk field $V_{\alpha_{k,l}}$ is expanded into degenerate boundary fields $B_{\beta_{m,n}}$ with (m,n) precisely given in (26). Note that the reflection coefficient $d(\beta;s,s)$ becomes singular when β belongs to a degenerate representation. Therefore, only $B_{\beta_{m,n}}$ (and not their reflection transform) can appear with finite coefficient in the expansion of $V_{\alpha_{k,l}}$.

The Fourier transformed structure constant $\tilde{R}(\alpha,\beta;p)$ may well be thought of as the fusion coefficient of an Ishibashi state $|p\rangle\!\rangle$ with a boundary primary B_β. It seems therefore reasonable that the bulk-boundary structure constant becomes simpler if we take Ishibashi states rather than Cardy states. In [2] the Cardy states $|m,n\rangle_{\mathrm{c}}$ corresponding to degenerate representations of Virasoro algebra have also been constructed and analyzed in some detail. From the comparison of the wave functions for Cardy states $|s\rangle_{\rm c}$ and $|m,n\rangle_{\rm c}$, it is expected that the bulk-boundary structure constant for degenerate Cardy states might be given by

$$R(\alpha,\beta)_{m,n} = 2i\int_{-i\infty}^{i\infty} dp\,\sin(2\pi mpb)\sin(2\pi np/b)\tilde{R}(\alpha,\beta;p), \tag{28}$$

with $\tilde{R}(\alpha,\beta;p)$ given in (22). On the other hand, it follows from the fusion rule that only degenerate boundary operators $B_{\beta_{r,s}}$ with

$$r = 1,3,\cdots,2m-1, \qquad s = 1,3,\cdots,2n-1$$

can appear on a degenerate Cardy state $|m,n\rangle_{\rm c}$. To prove the consistency of all these we need further study of boundary degenerate operators and degenerate Cardy states.

Of all the basic structure constants in Liouville theory on a disc, it remains to calculate the three-point function of boundary operators. The boundary degenerate operator $B_{-b/2}$ and the associated KZ equation will be useful also there.

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