Sewing string tree vertices with ghosts using canonical forms

Leonidas Sandoval Junior*
Department of Mathematics
Centro de Ciências Tecnológicas
UDESC - Universidade do Estado de Santa Catarina - Brazil

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Abstract

We effectively sew two vertices with ghosts in order to obtain a third, composite vertex in the most general case of cycling transformations. In order to do this, we separate the vertices into two parts: a bosonic oscillator part and a ghost oscillator part and write them as canonical forms.

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1 Introduction

Sewing of vertices in order to build larger ones is a technique that has been used since the early days of String Theory. In [1], it has been shown how to use the Group Theoretic approach to String Theory [2] in order to perform the sewing of two string vertices taking into account their ghost structure. Here we shall perform the sewing of two vertices with ghosts using another procedure, writing the two original vertices in terms of canonical forms (see the appendix) and then performing the sewing explicitly. This method has been used in [3] and [4] in order to build multiloop string amplitudes for the bosonic string. This is done in order to check the result obtained in [1] and also to present an example of the use of canonical forms in the sewing of vertices.

The sewing procedure consists on taking two distinct vertices U_1 and $U_2^\dagger$, the first with N_1 strings and the second with N_2 strings, and choosing one string from each vertex to be sewn together. We shall be choosing leg E from vertex U_1 and leg F from vertex U_2. The sewn legs give rise to what we call propagator $\mathcal{P}$. This procedure is well explained in [5] and [1].

In order to perform the sewing, we shall divide the so called physical vertex [6] of an open bosonic string into a bosonic and a fermionic part. This is the vertex that has the correct number of ghosts. It can be written as

$$\begin{aligned} U &= \left(\prod_{\substack{i=1\\ i\neq E}}^{N} {}_i\langle 0|\right) \exp\left[-\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N_1}\sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1}V_j\right) a_{m\mu}^j\right] \exp\left[\sum_{\substack{i,j=1\\ i\neq j}}^{N}\sum_{n=2}^{\infty}\sum_{m=-1}^{\infty} c_n^i E_{nm}\left(\Gamma V_i^{-1}V_j\right) b_m^j\right] \\ &\quad \times \prod_{r=-1}^{1}\sum_{j=1}^{N}\sum_{s=-1}^{1E_{rs}(V_j)b_s^j} \times \prod_{\substack{i=1\\ i\neq a,b,c}}^{N}\sum_{j=1}^{N}\sum_{n=-1}^{\infty} e_n^{ij} b_n^j \ , \end{aligned} \tag{1}$$

where $a_{n\mu}^i$ are bosonic oscillators, and b_n^i and c_n^i are ghost oscillators. The matrices $D_{nm}(\gamma)$ and $E_{nm}(\gamma)$ depend on the cyclings and are given by [3] [4]

$$D_{n0}(\gamma) = \frac{1}{\sqrt{n}}\left[\gamma(0)\right]^n \ , \tag{2}$$

*E-mail address: dma2lsj@dcc.fej.udesc.br

$$
\begin{aligned}
D_{nm}(\gamma) &= \sqrt{\frac{m}{n}}\,\frac{1}{m!}\frac{\partial^m}{\partial z^m}\left[\gamma(z)\right]^n\Big|_{z=0} \ , && (3)\\
D_{00}(\gamma) &= \frac{1}{2}\ln\left[\frac{d}{dz}\gamma(z)\right]\Big|_{z=0} \ , && (4)\\
E_{nm}(\gamma) &= \frac{1}{(m+1)!}\frac{\partial^{m+1}}{\partial z^{m+1}}\left[(\gamma z)^{n+1}\left(\frac{\partial}{\partial z}\gamma z\right)^{-1}\right]\Big|_{z=0} . && (5)
\end{aligned}
$$

These matrices have the following multiplication properties:

$$
\begin{aligned}
&\sum_{p=1}^{\infty} D_{np}(\gamma_1)D_{pm}(\gamma_2) + D_{n0}(\gamma_1)\delta_{0m} + \delta_{0n}D_{0m}(\gamma_2) = D_{nm}(\gamma_1\gamma_2) \ ; && (6)\\
&\sum_{t=-1}^{1E_{rt}(}\gamma_1)E_{ts}(\gamma_2) = E_{rs}(\gamma_1\gamma_2) \quad , \quad r,s,t = -1,0,1 \ ; && (7)\\
&E_{rn}(\gamma) = 0 \quad , \quad r = -1,0,1 \quad , \quad n \geq 2 \ , && (8)\\
&\sum_{p=-1}^{\infty} E_{np}(\gamma_1)E_{pm}(\gamma_2) = E_{nm}(\gamma_1\gamma_2) \quad , \quad n,m \geq -1 \ ; && (9)\\
&\sum_{p=2}^{\infty} E_{np}(\gamma_1)E_{pm}(\gamma_2) = E_{nm}(\gamma_1\gamma_2) - \sum_{r,s=-1}^{1E_{nr}(}\gamma_1)E_{rs}(\gamma_2)\delta_{sm} \quad , \quad n,m \geq 2 \ . && (10)
\end{aligned}
$$

The last term of equation (1) is the product of $N-3$ ghost oscillators b_n^i (N legs minus arbitrary legs a, b and c) and is necessary for the vertex to have the correct ghost number. The coefficients e_n^{ij} are given by [7]

$$
e_n^{ij} = \sum_{m=-1}^{\infty} k_m^{ij} E_{mn}(\gamma_m^j) \ , \tag{11}
$$

where

$$
\begin{aligned}
&\gamma_{-1}^j = V_j \ , && (12)\\
&\gamma_0^j = \exp\left(-\mathcal{L}_0^j \ln a_0^j\right)\exp\left(-\sum_{n=1}^{\infty}\bar{a}_n^j\mathcal{L}_n^j\right) \ , && (13)\\
&\gamma_p^j = \exp\left(-\sum_{n=p+1}^{\infty}\bar{a}_n^j\mathcal{L}_n^j\right) \ , \ p \geq 1 \ , && (14)
\end{aligned}
$$

where $\mathcal{L}_n^j$ are the generators of the bosonic and ghost parts of the conformal algebra.

2 Bosonic oscillator part

Now we shall perform the sewing for the bosonic part of the vertex. The bosonic part of vertex U_1 is given by

$$
V_1^{\rm osc} = \left(\prod_{\substack{i=1\\ i\neq E}}^{N_1} {}_i\langle 0|\right)\exp\left[-\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{N_1}\sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1}V_j\right) a_{m\mu}^j\right] \ . \tag{15}
$$

Isolating all terms related to leg E, we have the following formula for the modified vertex $V_1^{\rm osc}$:

$$
V_1^{\rm osc} = \left(\prod_{\substack{i=1\\ i\neq E}}^{N_1} {}_i\langle 0|\right)\exp\left[-\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq E}}^{N_1}\sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1}V_j\right) a_{m\mu}^j\right. \tag{16}
$$

$$\times\ {}_E\langle 0|\exp\left[-\sum_{\substack{i=1\\i\neq E}}^{N_1}\sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1}V_E\right) a_{m\mu}^E\right] . \tag{17}$$

This may be written in terms of a canonical form (see the Appendix) [3]:

$$V_1^{\rm osc} = \left(\prod_{\substack{i=1\\i\neq E}}^{N_1} {}_i\langle 0|\right) {}_E\langle 0|\exp\left(-\sum_{n=1}^{\infty} B_n^{1\mu} a_{n\mu}^E\right)\exp(-\phi_1) \tag{18}$$

where

$$B_n^{1\mu} = \sum_{\substack{i=1\\i\neq E}}^{N_1}\sum_{n=0}^{\infty} a_m^{\mu i} D_{mn}\left(\Gamma V_i^{-1}V_E\right) , \tag{19}$$

$$\phi_1 = \frac{1}{2}\sum_{\substack{i,j=1\\i\neq j\\i,j\neq E}}^{N_1}\sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1}V_j\right) a_{m\mu}^j + \sum_{\substack{i=1\\i\neq E}}^{N_1}\sum_{n=0}^{\infty} a_n^{\mu i} D_{n0}\left(\Gamma V_i^{-1}V_E\right) a_{0\mu}^E . \tag{20}$$

Similarly, considering vertex $V_2^{\rm osc\dagger}$, we must change

$$a_m^{\mu F} \to a_m^{\mu F\dagger} = -a_{-m}^{\mu F} \tag{21}$$

so that[1]

$$V_2^{\rm osc\dagger} = \left(\prod_{\substack{i=1\\i\neq F}}^{N_2} {}_i\langle 0|\right)\exp\left[-\frac{1}{2}\sum_{\substack{i,j=1\\i\neq j\\i,j\neq F}}^{N_2}\sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1}V_j\right) a_{m\mu}^j\right]$$
$$\times \exp\left[\sum_{\substack{i=1\\i\neq F}}^{N_2}\sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1}V_E\right) a_{-m\mu}^F\right] |0\rangle_F , \tag{23}$$

which can be written

$$V_2^{\rm osc\dagger} = \left(\prod_{\substack{i=1\\i\neq F}}^{N_2} {}_i\langle 0|\right)\exp\left(-\sum_{n=1}^{\infty} a_{-n}^{\mu F} A_{n\mu}^2\right)\exp(-\phi_2)|0\rangle_F \tag{24}$$

where

$$A_{n\mu}^2 = -\sum_{\substack{i=1\\i\neq F}}^{N_2}\sum_{n=0}^{\infty} D_{nm}\left(\Gamma V_F^{-1}V_i\right) a_{m\mu}^i , \tag{25}$$

$$\phi_2 = \frac{1}{2}\sum_{\substack{i,j=1\\i\neq j\\i,j\neq F}}^{N_2}\sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1}V_j\right) a_{m\mu}^j + \sum_{\substack{i=1\\i\neq F}}^{N_2}\sum_{n=0}^{\infty} a_n^{\mu i} D_{n0}\left(\Gamma V_i^{-1}V_F\right) a_{0\mu}^F . \tag{26}$$

[1] The minus sign in the expression for the Hermitian conjugate is because of the minus sign in the commutation relation of the $a_n^{\mu i}$ oscillators:

$$[a_n^{\mu i}, a_m^{\nu j}] = \frac{n}{|n|}\delta_{n,-m}\eta^{\nu\mu}\delta^{ij} . \tag{22}$$

The oscillator part of the propagator is purely a conformal transformation [5]:

$$\mathcal{P}_{\rm osc} = V_E^{-1} V_F \Gamma \ . \tag{27}$$

Using the properties of coherent states (see the Appendix), we may represent this as [4]

$$\mathcal{P}_{\rm osc} =: \exp\left[\sum_{n,m=0}^{\infty} a_{-n}^{\mu E} D_{nm}\left(V_E^{-1} V_F \Gamma\right) a_{m\mu}^{E} - \sum_{n=1}^{\infty} a_{-n}^{\mu E} a_{n\mu}^{E}\right] : , \tag{28}$$

or, in terms of a canonical form,

$$\begin{aligned} \mathcal{P}_{\rm osc} \quad &= \quad \exp\left(-\sum_{n=1}^{\infty} a_{-n}^{\mu E} A_{n\mu}^{3}\right) : \exp\left\{-\sum_{n,m=1}^{\infty} a_{-n}^{\mu E}\left[C_{nm}^{3} - \delta_{nm}\right] a_{m\mu}^{E}\right\} : \\ &\quad \times \exp\left(-\sum_{n=1}^{\infty} B_n^{3\mu} a_{n\mu}^{E}\right) \exp(-\phi_3) \end{aligned} \tag{29}$$

with

$$A_{n\mu}^{3} = -D_{n0}\left(V_F^{-1} V_E \Gamma\right) a_{0\mu}^{E} \ , \tag{30}$$

$$B_n^{3\mu} = a_0^{\mu E} D_{0n}\left(V_E^{-1} V_F \Gamma\right) \ , \tag{31}$$

$$C_{nm}^{3} = -D_{nm}\left(V_E^{-1} V_F \Gamma\right) \ , \tag{32}$$

$$\phi_3 = a_0^{\mu E} D_{00}\left(V_E^{-1} V_F \Gamma\right) a_{0\mu}^{E} \ . \tag{33}$$

Using now the multiplication rule for two canonical forms (as in the Appendix), we obtain

$$\begin{aligned} V_1^{\rm osc}\mathcal{P}_{\rm osc} \quad &= \quad \left(\prod_{\substack{i=1 \\ i\neq E}}^{N_1} {}_i\langle 0|\right) {}_E\langle 0| \exp\left(-\sum_{n=1}^{\infty} a_{-n}^{E\mu} A_{n\mu}^{4}\right) \times : \exp\left\{-\sum_{n,m=1}^{\infty} a_{-n}^{\mu E}\left[C_{nm}^{4} - \delta_{nm}\right] a_{m\mu}^{E}\right\} : \\ &\quad \times \exp\left(-\sum_{n=1}^{\infty} B_n^{4\mu} a_{n\mu}^{E}\right) \exp(-\phi_4) \end{aligned} \tag{34}$$

with

$$A_{n\mu}^{4} = -D_{n0}\left(V_F^{-1} V_E \Gamma\right) a_{0\mu}^{E} \ , \tag{35}$$

$$B_n^{4\mu} = a_0^{\mu E} D_{0n}\left(V_E^{-1} V_F \Gamma\right) + \sum_{\substack{i=1 \\ i\neq E}}^{N_1} \sum_{m=0}^{\infty} \sum_{p=1}^{\infty} a_m^{\mu i} D_{mp}\left(\Gamma V_i^{-1} V_E\right) D_{pn}\left(V_E^{-1} V_F \Gamma\right) \ , \tag{36}$$

$$C_{nm}^{4} = -D_{nm}\left(V_E^{-1} V_F \Gamma\right) \ , \tag{37}$$

$$\begin{aligned} \phi_4 \quad &= \quad \frac{1}{2} \sum_{\substack{i,j=1 \\ i\neq j \\ i,j\neq E}}^{N_1} \sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1} V_j\right) a_{m\mu}^{j} \\ &\quad + \sum_{\substack{i=1 \\ i\neq E}}^{N_1} \sum_{n=0}^{\infty} a_n^{\mu i} D_{n0}\left(\Gamma V_i^{-1} V_E\right) a_{0\mu}^{E} + a_0^{\mu E} D_{00}\left(V_E^{-1} V_F \Gamma\right) a_{0\mu}^{E} \\ &\quad + \sum_{\substack{i=1 \\ i\neq E}}^{N_1} \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} a_m^{\mu i} D_{nm}\left(\Gamma V_i^{-1} V_E\right) D_{n0}\left(V_E^{-1} V_F \Gamma\right) a_{0\mu}^{E} \ . \end{aligned} \tag{38}$$

Because of the vacuum ${}_E\langle 0|$, the first two exponentials do not give any contribution, and so we end up with

$$V_1^{\rm osc}\mathcal{P}_{\rm osc} = \left(\prod_{\substack{i=1\\ i\neq E}}^{N_1} {}_i\langle 0|\right) {}_E\langle 0| \exp\left(-\sum_{n=1}^{\infty} B_n^{4\mu} a_{n\mu}^E\right) \exp(-\phi_4) \ . \tag{39}$$

Now we make use of property (6) of the $D_{nm}(\gamma)$ matrices in order to simplify the expressions for the coefficients of $V_1^{\rm osc}\mathcal{P}_{\rm osc}$. We then have

$$\begin{aligned}\sum_{n=1}^{\infty} B_n^{4\mu} a_{n\mu}^E \quad &= \quad \sum_{\substack{i=1\\ i\neq E}}^{N_1} \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} a_m^{\mu i} D_{mn}\left(\Gamma V_i^{-1} V_F \Gamma\right) a_{n\mu}^E \\ &\quad + \sum_{n=1}^{\infty} \left(a_0^{\mu E} + \sum_{\substack{i=1\\ i\neq E}}^{N_1} a_0^{\mu i}\right) D_{0n}\left(V_E^{-1} V_F \Gamma\right) a_{n\mu}^E \ ,\end{aligned} \tag{40}$$

$$\begin{aligned}\phi_4 \quad &= \quad \frac{1}{2} \sum_{\substack{i=1\\ i\neq j\\ i,j\neq E}}^{N_1} \sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1} V_j\right) a_{m\mu}^j + \sum_{\substack{i=1\\ i\neq E}}^{N_1} \sum_{n=0}^{\infty} a_n^{\mu i} D_{n0}\left(\Gamma V_i^{-1} V_F \Gamma\right) a_{0\mu}^E \\ &\quad + \left(a_0^{\mu E} + \sum_{\substack{i=1\\ i\neq E}}^{N_1} a_0^{\mu i}\right) D_{00}\left(V_E^{-1} V_F \Gamma\right) a_{0\mu}^E \ .\end{aligned} \tag{41}$$

Since

$$a_0^{\mu E} + \sum_{\substack{i=1\\ i\neq E}}^{N_1} a_0^{\mu i} = \sum_{i=1}^{N} a_0^{\mu i} = 0 \ , \tag{42}$$

which is implied by momentum conservation on vertex $V_1^{\rm osc}$, we have the following coefficients for $V_1^{\rm osc}\mathcal{P}_{\rm osc}$:

$$B_n^{4\mu} \quad = \quad \sum_{\substack{i=1\\ i\neq E}}^{N_1} \sum_{m=0}^{\infty} a_m^{\mu i} D_{mn}\left(\Gamma V_i^{-1} V_F \Gamma\right) \ , \tag{43}$$

$$\phi_4 \quad = \quad \frac{1}{2} \sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq E}}^{N_1} \sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1} V_j\right) a_{m\mu}^j + \sum_{\substack{i=1\\ i\neq E}}^{N_1} \sum_{n=0}^{\infty} a_n^{\mu i} D_{n0}\left(\Gamma V_i^{-1} V_F \Gamma\right) a_{0\mu}^E \ . \tag{44}$$

So, the effect of multiplying $V_1^{\rm osc}$ by $\mathcal{P}_{\rm osc}$ is just to change the cycling transformations in vertex $V_1^{\rm osc}$ in the following way:

$$V_E \longrightarrow V_F \Gamma \ . \tag{45}$$

Now we multiply $V_1^{\rm osc}\mathcal{P}_{\rm osc}$ by $V_2^{\rm osc\dagger}$, obtaining the composite vertex

$$V_c^{\rm gh} = V_1^{\rm osc}\mathcal{P}_{\rm osc} V_2^{\rm osc\dagger} = \left(\prod_{\substack{i=1\\ i\neq E,F}}^{N_1+N_2} {}_i\langle 0|\right) {}_E\langle 0| \left(-\sum_{n=1}^{\infty} a_{-n}^{\mu E} A_{n\mu}\right) \left(-\sum_{n=1}^{\infty} B_n^{\mu} a_{n\mu}^E\right) \exp(-\phi)|0\rangle_F \tag{46}$$

where

$$A_{n\mu} \quad = \quad -\sum_{\substack{i=1\\ i\neq F}}^{N_2} \sum_{n=0}^{\infty} D_{nm}\left(\Gamma V_F^{-1} V_i\right) a_{m\mu}^i \ , \tag{47}$$

$$B_n^\mu = \sum_{\substack{i=1\\i\neq E}}^{N_1} \sum_{m=0}^{\infty} a_m^{\mu i} D_{mn}\left(\Gamma V_i^{-1} V_F \Gamma\right) , \tag{48}$$

$$\begin{aligned}
\phi = & \frac{1}{2} \sum_{\substack{i,j=1\\i\neq j\\i,j\neq E}}^{N_1} \sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1} V_j\right) a_{m\mu}^j + \frac{1}{2} \sum_{\substack{i,j=1\\i\neq j\\i,j\neq F}}^{N_2} \sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1} V_j\right) a_{m\mu}^j \\
& + \sum_{\substack{i=1\\i\neq E}}^{N_1} \sum_{n=0}^{\infty} a_n^{\mu i} D_{n0}\left(\Gamma V_i^{-1} V_F \Gamma\right) a_{0\mu}^E + \sum_{\substack{i=1\\i\neq F}}^{N_2} \sum_{n=0}^{\infty} a_n^{\mu i} D_{n0}\left(\Gamma V_i^{-1} V_F\right) a_{0\mu}^F \\
& + \sum_{\substack{i=1\\i\neq E}}^{N_1} \sum_{\substack{i=1\\i\neq F}}^{N_2} \sum_{n,m=0}^{\infty} \sum_{p=1}^{\infty} a_n^{\mu i} D_{np}\left(\Gamma V_i^{-1} V_F \Gamma\right) D_{pm}\left(\Gamma V_F^{-1} V_j\right) a_{m\mu}^j .
\end{aligned} \tag{49}$$

Using the multiplication properties (6) of the $D_{nm}(\gamma)$ matrices, we get

$$\begin{aligned}
\phi = & \frac{1}{2} \sum_{\substack{i,j=1\\i\neq j\\i,j\neq E}}^{N_1} \sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1} V_j\right) a_{m\mu}^j + \frac{1}{2} \sum_{\substack{i,j=1\\i\neq j\\i,j\neq F}}^{N_2} \sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1} V_j\right) a_{m\mu}^j \\
& + \sum_{\substack{i=1\\i\neq E}}^{N_1} \sum_{\substack{i=1\\i\neq F}}^{N_2} \sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1} V_j\right) a_{m\mu}^j + \sum_{\substack{i=1\\i\neq E}}^{N_1} \sum_{n=0}^{\infty} a_n^{\mu i} D_{n0}\left(\Gamma V_i^{-1} V_F \Gamma\right) \left(a_{0\mu}^E + \sum_{\substack{j=1\\j\neq F}}^{N_2} a_{0\mu}^j \right) \\
& + \sum_{\substack{i=1\\i\neq F}}^{N_2} \sum_{n=0}^{\infty} a_n^{\mu i} D_{n0}\left(\Gamma V_i^{-1} V_F\right) \left(a_{0\mu}^F + \sum_{\substack{j=1\\j\neq E}}^{N_1} a_{0\mu}^j \right) .
\end{aligned} \tag{50}$$

Considering now the identification of legs E and F and the momentum conservation for vertices V_1^{osc} and $V_2^{\text{osc}\dagger}$, this becomes simply

$$\phi = \frac{1}{2} \sum_{\substack{i,j=1\\i\neq j\\i,j\neq E}}^{N_1+N_2} \sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1} V_j\right) a_{m\mu}^j . \tag{51}$$

Because of the two vacua ${}_E\langle 0|$ and $|0\rangle_F$, the terms $A_{n\mu}$ and B_n^μ will not give any contribution and we will end up just with the contribution of the phase ϕ. So we have that

$$V_c^{\text{osc}} = \left(\prod_{\substack{i=1\\i\neq E,F}}^{N_1+N_2} {}_i\langle 0| \right) \exp\left[-\frac{1}{2} \sum_{\substack{i,j=1\\i\neq j\\i,j\neq E}}^{N_1+N_2} \sum_{n,m=0}^{\infty} a_n^{\mu i} D_{nm}\left(\Gamma V_i^{-1} V_j\right) a_{m\mu}^j \right] \tag{52}$$

which is the right expression for the composite vertex V_c^{osc}.

2.1 Ghost part

We will now consider the ghost part of the vertex. The ghost part of a physical vertex U with N legs is given by

$$U^{\text{ghost}} = \left(\prod_{\substack{i=1\\i\neq E}}^{N} {}_i\langle 0| \right) \exp\left[\sum_{\substack{i,j=1\\i\neq j}}^{N} \sum_{n=2}^{\infty} \sum_{m=-1}^{\infty} c_n^i E_{nm}\left(\Gamma V_i^{-1} V_j\right) b_m^j \right.$$

$$\times \prod_{r=-1}^{1} \sum_{j=1}^{N} \sum_{s=-1}^{1} E_{rs}(V_j) b_s^j \times \prod_{\substack{i=1 \\ i \neq a,b,c}}^{N} \sum_{j=1}^{N} \sum_{n=-1}^{\infty} e_n^{ij} b_n^j \;, \tag{53}$$

Making use of the integral of some anticommuting variables β_r $(r = -1, 0, 1)$ and η^i $(i = 1, \ldots, N;\ i \neq a, b, c)$, the ghost part of this vertex can be written [4]

$$\begin{aligned} U^{\text{gh}} &= \left(\prod_{i=1}^{N} {}_i\langle 0| \right) \int \prod_{r=-1}^{1} d\beta_r \int \prod_{\substack{i=1 \\ i \neq a,b,c}}^{N} d\eta^i \exp \left[\sum_{\substack{i,j=1 \\ i \neq j}}^{N} \sum_{n=2}^{\infty} \sum_{m=-1}^{\infty} c_n^i E_{nm} \left(\Gamma V_i^{-1} V_j \right) b_m^j \right] \\ &\times \exp \left[\beta_r \sum_{j=1}^{N} \sum_{s=-1}^{1} E_{rs}(V_j) b_s^j \right] \exp \left[\eta^i \sum_{j=1}^{N} \sum_{n=-1}^{\infty} e_n^{ij} b_n^j \right] \;. \end{aligned} \tag{54}$$

The ghost part of vertex U_1 can be written in a similar way. Isolating all terms related with leg E, we obtain

$$\begin{aligned} U_1^{\text{gh}} &= \left(\prod_{\substack{i=1 \\ i \neq E}}^{N_1} {}_i\langle 0| \right) {}_E\langle 0| \int \prod_{r=-1}^{1} d\beta_r \int \prod_{\substack{i=1 \\ i \neq b,c}}^{N_1} d\eta^i \exp \left[\sum_{\substack{i,j=1 \\ i \neq j \\ i,j \neq E}}^{N_1} \sum_{n=2}^{\infty} \sum_{m=-1}^{\infty} c_n^i E_{nm} \left(\Gamma V_i^{-1} V_j \right) b_m^j \right] \\ &\times \exp \left[\sum_{\substack{i=1 \\ i \neq E}}^{N_1} \sum_{n=2}^{\infty} \sum_{m=-1}^{\infty} c_n^i E_{nm} \left(\Gamma V_i^{-1} V_E \right) b_m^E \right] \exp \left[\sum_{\substack{i=1 \\ i \neq E}}^{N_1} \sum_{n=2}^{\infty} \sum_{m=-1}^{\infty} c_n^E E_{nm} \left(\Gamma V_E^{-1} V_i \right) b_m^i \right] \\ &\times \exp \left[\beta_r \sum_{\substack{j=1 \\ j \neq E}}^{N_1} \sum_{s=-1}^{1} E_{rs}(V_j) b_s^j + \beta_r \sum_{s=-1}^{1} E_{rs}(V_E) b_s^E \right] \\ &\times \exp \left[\eta^i \sum_{\substack{j=1 \\ j \neq E}}^{N_1} \sum_{n,m=-1}^{\infty} e_n^{ij} E_{nm}(V_{0j}^{-1} V_j) b_m^j + \eta^i \sum_{n,m=-1}^{\infty} e_n^{iE} E_{nm}(V_{0E}^{-1} V_E) b_m^E \right] \;. \end{aligned} \tag{55}$$

This may be written in terms of a canonical form[2] [4]:

$$U_1^{\text{gh}} = \left(\prod_{\substack{i=1 \\ i \neq E}}^{N_1} {}_i\langle 0| \right) {}_E\langle 0| \int \prod_{r=-1}^{1} d\beta_r \int \prod_{\substack{i=1 \\ i \neq b,c}}^{N_1} d\eta^i \exp \phi_1 \exp \left(\sum_{n=-1}^{\infty} C_n^1 b_n^E \right) \exp \left(\sum_{n=2}^{\infty} c_n^E D_n^1 \right) \tag{56}$$

where

$$\begin{aligned} \phi_1 &= \sum_{\substack{i,j=1 \\ i \neq j \\ i,j \neq E}}^{N_1} \sum_{n=2}^{\infty} \sum_{m=-1}^{\infty} c_n^i E_{nm} \left(\Gamma V_i^{-1} V_j \right) b_m^j + \beta_r \sum_{\substack{j=1 \\ j \neq E}}^{N_1} \sum_{s=-1}^{1} E_{rs}(V_j) b_s^j \\ &+ \eta^i \sum_{\substack{j=1 \\ j \neq E}}^{N_1} \sum_{n,m=-1}^{\infty} e_n^{ij} E_{nm}(V_{0j}^{-1} V_j) b_m^j \;, \end{aligned} \tag{57}$$

$$C_n^1 = \sum_{\substack{i=1 \\ i \neq E}}^{N_1} \sum_{m=2}^{\infty} c_m^i E_{mn} \left(\Gamma V_i^{-1} V_E \right) + \beta_r \sum_{s=-1}^{1} E_{rs}(V_E) \delta_{sn} + \sum_{m=-1}^{\infty} \eta^i e_m^{iE} E_{mn}(V_{0E}^{-1} V_E) \;, \tag{58}$$

[2]See the Appendix for the definition of canonical forms for ghosts.

$$D_n^1 = \sum_{\substack{i=1\\i\neq E}}^{N_1} \sum_{m=-1}^{\infty} E_{nm}\left(\Gamma V_E^{-1} V_i\right) b_m^i \ . \tag{59}$$

Similarly, the physical ghost vertex $U_2^{\rm gh\dagger} \equiv C_F U_2^{0\rm gh\dagger}$ can be obtained by making the transformations

$$b_n^i \to b_{-n}^i \quad , \quad c_n^i \to -c_{-n}^i \tag{60}$$

and by using the Hermitian conjugate $|0\rangle_F$ of vacuum $_F\langle 0|$:

$$\begin{aligned} U_2^{\rm gh\dagger} &= \left(\prod_{\substack{i=1\\i\neq F}}^{N_2} {}_i\langle 0|\right) \int \prod_{r=-1}^{1} d\beta_r \int \prod_{\substack{i=1\\i\neq d,g,h}}^{N_2} d\mathcal{M}^i \exp\left[\sum_{\substack{i,j=1\\i\neq j\\i,j\neq F}}^{N_2} \sum_{n=2}^{\infty}\sum_{m=-1}^{\infty} c_n^i E_{nm}\left(\Gamma V_i^{-1} V_j\right) b_m^j\right] \\ &\times \exp\left[\sum_{\substack{i=1\\i\neq F}}^{N_2} \sum_{n=2}^{\infty}\sum_{m=2}^{\infty} b_{-m}^F c_n^i E_{nm}\left(\Gamma V_i^{-1} V_F\right)\right] \\ &\times \exp\left[-\sum_{\substack{i=1\\i\neq F}}^{N_2} \sum_{n=2}^{\infty}\sum_{m=-1}^{\infty} E_{nm}\left(\Gamma V_F^{-1} V_i\right) b_m^i c_{-n}^F\right] \exp\left[\beta_r \sum_{\substack{j=1\\j\neq F}}^{N_2} \sum_{s=-1}^{1 E_{rs}(V_j) b_s^j}\right] \\ &\times \exp\left[\mathcal{M}^i \sum_{\substack{j=1\\j\neq E}}^{N_2} \sum_{n,m=-1}^{\infty} e_n^{ij} E_{nm}(V_{0j}^{-1} V_j) b_m^j + \sum_{n=2}^{\infty}\sum_{m=-1}^{\infty} b_{-m}^F \mathcal{M}_i e_n^{iF} E_{nm}(V_{0F}^{-1} V_F)\right] |0\rangle_F \end{aligned} \tag{61}$$

where some terms obtained by following the procedure described above have been annihilated by the Hermitian conjugate vacuum $|0\rangle_F$.

This vertex can be written as

$$U_2^{\rm gh\dagger} = \left(\prod_{\substack{i=1\\i\neq F}}^{N_2} {}_i\langle 0|\right) \int \prod_{r=-1}^{1} d\beta_r \int \prod_{\substack{i=1\\i\neq d,g,h}}^{N_2} d\mathcal{M}^i \exp\phi_2 \exp\left(\sum_{n=2}^{\infty} A_n^2 b_{-n}^F\right) \exp\left(\sum_{n=-1}^{\infty} c_{-n}^F B_n^2\right) |0\rangle_F \tag{62}$$

where

$$\begin{aligned} \phi_2 &= \sum_{\substack{i,j=1\\i\neq j\\i,j\neq F}}^{N_2} \sum_{n=2}^{\infty}\sum_{m=-1}^{\infty} c_n^i E_{nm}\left(\Gamma V_i^{-1} V_j\right) b_m^j + \beta_r \sum_{\substack{j=1\\j\neq F}}^{N_2} \sum_{s=-1}^{1 E_{rs}(V_j) b_s^j} \\ &+ \mathcal{M}^i \sum_{\substack{j=1\\j\neq F}}^{N_2} \sum_{n,m=-1}^{\infty} e_n^{ij} E_{nm}(V_{0j}^{-1} V_j) b_m^j \ , \end{aligned} \tag{63}$$

$$A_n^2 = -\sum_{\substack{i=1\\i\neq F}}^{N_2} \sum_{m=2}^{\infty} c_m^i E_{mn}\left(\Gamma V_i^{-1} V_F\right) - \sum_{m=-1}^{\infty} \mathcal{M}^i e_m^{iF} E_{mn}(V_{0F}^{-1} V_F) \ , \tag{64}$$

$$B_n^2 = \sum_{\substack{i=1\\i\neq F}}^{N_2} \sum_{p=2}^{\infty}\sum_{m=-1}^{\infty} \delta_{np} E_{pm}\left(\Gamma V_F^{-1} V_i\right) b_m^i \ . \tag{65}$$

The ghost part of the propagator is given by

$$\mathcal{P} = V_E^{-1} V_F \Gamma \ . \tag{66}$$

Being a conformal transformation, can be written in terms of the following canonical form [4]:

$$\mathcal{P}_{\mathrm{gh}} = :\exp\left[\sum_{n,m=-1}^{\infty} c_{-n}^{E}\left(E_{nm}^{3}-\delta_{nm}\right)b_{m}^{E}\right]\exp\left(\sum_{n=2}^{\infty}\sum_{r=-1}^{1c_n^E G_{nr}^3 b_r^E}\right)$$
$$\times\exp\left[\sum_{n,m=2}^{\infty} b_{-n}^{E}\left(F_{nm}^{3}-\delta_{nm}\right)c_{m}^{E}\right]: \tag{67}$$

with

$$E_{nm}^{3} = E_{nm}\left(V_E^{-1}V_F\Gamma\right) \ , \tag{68}$$
$$G_{nr}^{3} = -\sum_{s=-1}^{1} E_{ns}\left(V_F^{-1}V_E\Gamma\right)E_{sr}\left(\Gamma V_E^{-1}V_F\Gamma\right) \ , \tag{69}$$
$$F_{nm}^{3} = E_{nm}\left(V_E^{-1}V_F\Gamma\right) \ . \tag{70}$$

Multiplying now U_1^{gh} by $\mathcal{P}_{\mathrm{gh}}$, we obtain

$$U_1^{\mathrm{gh}}\mathcal{P}_{\mathrm{gh}} = \left(\prod_{\substack{i=1\\i\neq E}}^{N_1} {}_i\langle 0|\right){}_E\langle 0|\int\prod_{r=-1}^{1} d\beta_r\int\prod_{\substack{i=1\\i\neq b,c}}^{N_1} d\eta^i\exp\phi_4 :\exp\left(\sum_{n=2}^{\infty}\sum_{r=-1}^{1c_n^E G_{nr}^4 b_r^E}\right):$$
$$\times\exp\left(\sum_{n=-1}^{\infty} C_n^4 b_n^E\right)\exp\left(\sum_{n=2}^{\infty} c_n^E D_n^4\right) \tag{71}$$

where

$$\phi_4 = \phi_1 \ , \tag{72}$$
$$G_{nr}^{4} = G_{nr}^{3} \ , \tag{73}$$
$$C_n^4 = \sum_{\substack{i=1\\i\neq E}}^{N_1}\sum_{m=2}^{\infty} c_m^i E_{mn}\left(\Gamma V_i^{-1}V_F\Gamma\right) + \beta_r\sum_{s=-1}^{1} E_{rs}\left(V_F\Gamma\right)\delta sn\eta_i\sum_{m=-1}^{\infty} e_m^{iE}E_{mn}\left(V_{0E}^{-1}V_F\Gamma\right) \ , \tag{74}$$
$$D_n^4 = \sum_{\substack{i=1\\i\neq E}}^{N_1}\sum_{m=-1}^{\infty} E_{nm}\left(V_F^{-1}V_i\right)b_m^i - \sum_{\substack{i=1\\i\neq E}}^{N_1}\sum_{r,s=-1}^{1} E_{nr}\left(V_F^{-1}V_E\Gamma\right)E_{rs}\left(\Gamma V_E^{-1}V_i\right)b_s^i \tag{75}$$

and some of the terms have been annihilated by the vacuum ${}_E\langle 0|$. In the expressions above, we have used multiplication properties (7-10) of matrices $E_{nm}(\gamma)$.

The combination of the second and fourth terms in the expression above yields

$$:\exp\left(\sum_{n=2}^{\infty}\sum_{r=-1}^{1c_n^E G_{nr}^4 b_r^E}\right):\exp\left(\sum_{n=2}^{\infty} c_n^E D_n^4\right) =$$
$$-\sum_{n=2}^{\infty}\sum_{r,s=-1}^{1c_n^E E_{ns}}\left(V_F^{-1}V_E\Gamma\right)\left[E_{sr}\left(\Gamma V_E^{-1}V_F\Gamma\right)b_r^E + \sum_{\substack{i=1\\i\neq E}}^{N_1} E_{sr}\left(\Gamma V_E^{-1}V_i\right)b_r^i\right] \ . \tag{76}$$

Using multiplication property (7) we can write this as

$$-\sum_{n=2}^{\infty}\sum_{r,s,t=-1}^{1} c_n^E E_{ns}\left(V_F^{-1}V_E\Gamma\right)E_{st}\left(\Gamma V_E^{-1}\right)\left[E_{tr}\left(V_F\Gamma\right)b_r^E + \sum_{\substack{i=1\\i\neq E}}^{N_1} E_{tr}\left(V_i\right)b_r^i\right] \tag{77}$$

what is zero due to the overlap identities of $U_1^{\rm gh}\mathcal{P}_{\rm gh}$.

We can now rewrite the combination $U_1^{\rm gh}\mathcal{P}_{\rm gh}$ as [4]

$$U_1^{\rm gh}\mathcal{P}_{\rm gh} = \left(\prod_{\substack{i=1\\ i\neq E}}^{N_1} {}_i\langle 0|\right) {}_E\langle 0| \int \prod_{r=-1}^{1} d\beta_r \int \prod_{\substack{i=1\\ i\neq b,c}}^{N_1} d\eta^i \exp\phi_5 \exp\left(\sum_{n=-1}^{\infty} C_n^5 b_n^E\right) \exp\left(\sum_{n=2}^{\infty} c_n^E D_n^5\right) \tag{78}$$

where now

$$\begin{aligned}
\phi_5 &= \sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq E}}^{N_1} \sum_{n=2}^{\infty}\sum_{m=-1}^{\infty} c_n^i E_{nm}\left(\Gamma V_i^{-1}V_j\right) b_m^j + \beta_r \sum_{\substack{j=1\\ j\neq E}}^{N_1}\sum_{s=-1}^{1} E_{rs}(V_j) b_s^j \\
&\quad + \eta^i \sum_{\substack{j=1\\ j\neq E}}^{N_1}\sum_{n,m=-1}^{\infty} e_n^{ij} E_{nm}(V_{0j}^{-1}V_j) b_m^j\ , \qquad (79)\\
C_n^5 &= \sum_{\substack{i=1\\ i\neq E}}^{N_1}\sum_{m=2}^{\infty} c_m^i E_{mn}\left(\Gamma V_i^{-1} V_F\Gamma\right) + \beta_r \sum_{s=-1}^{1} E_{rs}\left(V_F\Gamma\right)\delta_{sn} + \eta^i \sum_{m=-1}^{\infty} e_m^{iE} E_{mn}\left(V_{0E}^{-1}V_F\Gamma\right)\ , \qquad (80)\\
D_n^5 &= \sum_{\substack{i=1\\ i\neq E}}^{N_1}\sum_{m=-1}^{\infty} E_{nm}\left(V_F^{-1}V_i\right) b_m^i\ . \qquad (81)
\end{aligned}$$

So the effect of inserting the propagator into $U_1 G$ is to make the following change:

$$V_E \to V_E\mathcal{P} = V_F\Gamma \tag{82}$$

what was expected considering the bosonic oscillator case.

Now we multiply $U_1^{\rm gh}\mathcal{P}_{\rm gh}$ by $U_2^{\rm gh\dagger}$ in order to obtain the composite vertex $U_c^{\rm gh}$. The only remaining term which is not annihilated by the two vacua associated to legs E and F is the resulting phase ϕ, so that the result obtained is[3]

$$U_c^{\rm gh} = \left(\prod_{\substack{i=1\\ i\neq E,F}}^{N_1+N_2} {}_i\langle 0|\right) \int \prod_{r=-1}^{1} d\beta_r \int \prod_{\substack{i=1\\ i\neq b,c}}^{N_1} d\eta^i \int \prod_{\substack{i=1\\ i\neq d,g,h}}^{N_2} d\mathcal{M}^i \exp\phi \tag{83}$$

where

$$\begin{aligned}
\phi &= \sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq E}}^{N_1}\sum_{n=2}^{\infty}\sum_{m=-1}^{\infty} c_n^i E_{nm}\left(\Gamma V_i^{-1}V_j\right) b_m^j + \sum_{\substack{i,j=1\\ i\neq j\\ i,j\neq F}}^{N_2}\sum_{n=2}^{\infty}\sum_{m=-1}^{\infty} c_n^i E_{nm}\left(\Gamma V_i^{-1}V_j\right) b_m^j \\
&\quad + \sum_{\substack{i=1\\ i\neq E}}^{N_1}\sum_{\substack{j=1\\ j\neq F}}^{N_2}\sum_{n=2}^{\infty}\sum_{m=-1}^{\infty} c_n^i E_{nm}\left(\Gamma V_i^{-1}V_j\right) b_m^j + \sum_{\substack{j=1\\ j\neq E}}^{N_1}\sum_{\substack{i=1\\ i\neq F}}^{N_2}\sum_{n=2}^{\infty}\sum_{m=-1}^{\infty} c_n^i E_{nm}\left(\Gamma V_i^{-1}V_j\right) b_m^j \\
&\quad + \beta_r \sum_{\substack{j=1\\ j\neq E}}^{N_1}\sum_{s=-1}^{1} E_{rs}(V_j) b_s^j + \beta_r \sum_{\substack{i=1\\ i\neq E}}^{N_2}\sum_{s=-1}^{1} E_{rs}(V_i) b_s^i \\
&\quad + \eta^i \sum_{\substack{j=1\\ j\neq E}}^{N_1}\sum_{n=-1}^{\infty} e_n^{ij} E_{nm}(V_{0j}^{-1}V_j) b_m^j + \eta^i \sum_{\substack{j=1\\ j\neq F}}^{N_2}\sum_{n,m=-1}^{\infty} e_m^{iE} E_{mn}\left(V_{0E}^{-1}V_j\right) b_n^j \\
&\quad + \mathcal{M}^i \sum_{\substack{j=1\\ j\neq F}}^{N_2}\sum_{n=-1}^{\infty} e_n^{ij} E_{nm}(V_{0j}^{-1}V_j) b_m^j + \mathcal{M}^i \sum_{\substack{j=1\\ j\neq E}}^{N_1}\sum_{n,m=-1}^{\infty} e_m^{iF} E_{mn}\left(V_{0F}^{-1}V_j\right) b_n^j \qquad (84)
\end{aligned}$$

[3]We write $U_c^{\rm gh} \equiv U_1^{\rm gh}\mathcal{P}_{\rm gh}U_2^{\rm gh\dagger}$.

where we have set some terms to zero using property (8) and the overlap identities for vertices U_1^{gh} and $U_2^{\mathrm{gh}\dagger}$.

Performing the integrations of the anticommuting variables introduced before, we then obtain

$$\begin{aligned} U_c^{\mathrm{gh}} = & \left(\prod_{\substack{i=1 \\ i\neq E,F}}^{N_1+N_2} {}_i\langle 0| \right) \exp \left[\sum_{\substack{i,j=1 \\ i\neq j \\ i,j\neq E,F}}^{N_1+N_2} \sum_{n=2}^{\infty} \sum_{m=-1}^{\infty} c_n^i E_{nm} \left(\Gamma V_i^{-1} V_j \right) b_m^j \right] \times \prod_{r=-1}^{1} \sum_{\substack{i=1 \\ i\neq E,F}}^{N_1+N_2} \sum_{s=-1}^{1 E_{rs}(V_i) b_s^i} \\ & \times \prod_{\substack{i=1 \\ i\neq b,c}}^{N_1} \left[\sum_{\substack{j=1 \\ j\neq E}}^{N_1} \sum_{n,m=-1}^{\infty} e_n^{ij} E_{nm}(V_{0j}^{-1} V_j) b_m^j + \sum_{\substack{j=1 \\ j\neq F}}^{N_2} \sum_{n,m=-1}^{\infty} e_m^{iE} E_{mn} \left(V_{0E}^{-1} V_j \right) b_n^j \right] \\ & \times \prod_{\substack{i=1 \\ i\neq d,g,h}}^{N_2} \left[\sum_{\substack{j=1 \\ j\neq E}}^{N_1} \sum_{n,m=-1}^{\infty} e_m^{iF} E_{mn} \left(V_{0F}^{-1} V_j \right) b_n^j + \sum_{\substack{j=1 \\ j\neq F}}^{N_2} \sum_{n,m=-1}^{\infty} e_n^{ij} E_{nm}(V_{0j}^{-1} V_j) b_m^j \right] . \end{aligned} \tag{85}$$

This expression for the composite vertex matches exactly the one obtained using the overlap identities [1].

3 Conclusions

Using explicit sewing techniques, we have sewn two vertices together in order to become a composite vertex. The calculations have been done with the correct ghost numbers for each vertex and the result has both BRST invariance and the correct ghost counting. It verifies the results obtained using overlap identities and the Group Theoretic method for String Theory.

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A Coherent states and canonical forms

In this appendix we present the concepts of coherent states and canonical forms for bosonic and ghost oscillators [3] [4].

A.1 Coherent states

Given operators a_n with commutation relations

$$[a_n, a_m^\dagger] = n\delta_{nm} \ , \tag{86}$$

a *coherent state* $|f\rangle$ is defined by[4]

$$|f\rangle = \mathrm{e}^{fa^\dagger}|0\rangle \ . \tag{87}$$

Some of the properties of coherent states are:

1) $a|f\rangle = f|f\rangle$ so that $\mathrm{e}^{ga}|f\rangle = \mathrm{e}^{gf}|f\rangle$;

2) $\mathrm{e}^{ga^\dagger} = |f+g\rangle$;

[4] We will not be writing the index n of a_n until it becomes necessary.

3) $\langle f|g\rangle = \mathrm{e}^{f^\dagger g}$;

4) $x^{na^\dagger a}|f\rangle = |x^n f\rangle$;

5) $1 = \frac{1}{\pi}\int d(\mathrm{Re}f)d(\mathrm{Im}f)\mathrm{e}^{-|f|^2}|f\rangle\langle f|$;

6) Defining the tensor product of coherent states as

$$|f_1, f_2, \dots, f_n, \dots\rangle = \prod_{n=1}^{\infty} \mathrm{e}^{f_n a_n^\dagger}|0\rangle \ , \tag{88}$$

we have

$$: \exp\left[\sum_{n,m=1}^{\infty} a_n^\dagger (C_{nm} - \delta_{nm}) a_m\right] : |f_1, f_2, \dots, f_n, \dots\rangle = |\sum_{n=1}^{\infty} C_{1m}f_m, C_{2m}f_m, \dots, C_{nm}f_m, \dots\rangle \ . \tag{89}$$

Symbolically, we can write this property as

$$: \mathrm{e}^{a^\dagger (C-1) a} : |f\rangle = |Cf\rangle \ . \tag{90}$$

Application: since $x^{a^\dagger a}|f\rangle = |xf\rangle$ and $: \mathrm{e}^{a^\dagger (x-1)a} : |f\rangle = |xf\rangle$, we have

$$x^{a^\dagger a} =: \mathrm{e}^{a^\dagger (x-1) a} : \tag{91}$$

when applied to coherent states.

A.2 Canonical forms

There will be two definitions here for canonical forms. In terms of bosonic oscillators [3] α_n, $[\alpha_n, \alpha_m] = \delta_{n,-m}$, a canonical form $\mathcal{O}$ is any operator which can be written as

$$\mathcal{O} = \exp\left(-\sum_{n=1}^{\infty} \alpha_{-n} A_n\right) : \exp\left[\sum_{n,m=1}^{\infty} \alpha_{-n}\left(C_{nm} - \delta_{nm}\right)\alpha_m\right] : \exp\left(-\sum_{n=1}^{\infty} B_n \alpha_n\right) \mathrm{e}^{-\phi} \tag{92}$$

where ϕ, A_n and B_n do not depend on α_n or α_{-n}. The product of two canonical forms $\mathcal{O}_1$ and $\mathcal{O}_2$ is again a canonical form $\mathcal{O}$ with

$$A_n = A_n^1 + \sum_{m=1}^{\infty} C_{nm}^1 A_m^2 \ , \tag{93}$$

$$B_n = B_n^2 + \sum_{m=1}^{\infty} B_m^1 C_{mn}^2 \ , \tag{94}$$

$$C_{nm} = \sum_{p=1}^{\infty} C_{np}^1 C_{pm}^2 \ , \tag{95}$$

$$\phi = \phi_1 + \phi_2 + \sum_{n=1}^{\infty} B_n^1 A_n^2 \ . \tag{96}$$

In terms of the ghost oscillators [4], a canonical form $\mathcal{C}$ can be defined as any operator that can be written as

$$\begin{aligned} \mathcal{C} \ &= \ \mathrm{e}^{\phi} \exp\left(\sum_{n=2}^{\infty} A_n b_{-n}\right) \exp\left(\sum_{n=-1}^{\infty} B_n c_{-n}\right) \\ &\times : \exp\left[\sum_{n,m=-1}^{\infty} c_{-n}\left(E_{nm} - \delta_{nm}\right) b_m\right] \exp\left(\sum_{n=2}^{\infty} \sum_{r=-1}^{1 c_n G_{nr} b_r}\right) \\ &\times \exp\left[\sum_{n,m=2}^{\infty} b_{-n}\left(F_{nm} - \delta_{nm}\right) c_m\right] : \exp\left(\sum_{n=-1}^{\infty} C_n b_n\right) \exp\left(\sum_{n=2}^{\infty} D_n c_n\right) \ . \end{aligned} \tag{97}$$

Again, the product of two canonical forms $\mathcal{C}_1$ and $\mathcal{C}_2$ will be a canonical form $\mathcal{C}$ with coefficients

$$\phi = \phi_1 + \phi_2 - \sum_{n=-1}^{\infty} C_n^1 B_n^2 - \sum_{n=2}^{\infty} D_n^1 A_n^2 + \sum_{n=2}^{\infty} \sum_{r=-1}^{1 A_n^2 G_{nr}^1 B_r^2} , \tag{98}$$

$$A_n = A_n^1 + \sum_{n=2}^{\infty} A_m^2 F_{mn}^1{}^T , \tag{99}$$

$$B_n = B_n^1 + \sum_{n=-1}^{\infty} B_m^2 E_{mn}^1{}^T , \tag{100}$$

$$E_{nm} = \sum_{p=-1}^{\infty} E_{np}^1 E_{pm}^2 , \tag{101}$$

$$F_{nm} = \sum_{p=2}^{\infty} F_{np}^1 F_{pm}^2 , \tag{102}$$

$$G_{nr} = G_{nr}^2 + \sum_{m=2}^{\infty} \sum_{s=-1}^{1 F_{nm}^2 G_{ms}^1 E_{sr}^2} , \tag{103}$$

$$C_n = C_n^2 - \sum_{m=2}^{\infty} \sum_{r=-1}^{1 A_m^2 G_{mr}^1 E_{rn}^2 +} \sum_{m=-1}^{\infty} C_m^1 E_{mn}^2 , \tag{104}$$

$$D_n = D_n^2 + \sum_{m=2}^{\infty} \sum_{r=-1}^{1 F_{nm}^2 G_{mr}^1 B_r^2 +} \sum_{m=2}^{\infty} D_m^1 F_{mn}^2 . \tag{105}$$

where A^T means the transpose matrix of A.

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