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Super Liouville Theory with Boundary

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Abstract

We study $N = 1$ super Liouville theory on worldsheets with and without boundary. Some basic correlation functions on a sphere or a disc are obtained using the properties of degenerate representations of superconformal algebra. Boundary states are classified by using the modular transformation property of annulus partition functions, but there are some of those whose wave functions cannot be obtained from the analysis of modular property. There are two ways of putting boundary condition on supercurrent, and it turns out that the two choices lead to different boundary states in quality. Some properties of boundary vertex operators are also presented. The boundary degenerate operators are shown to connect two boundary states in a way slightly complicated than the bosonic case.

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1. Introduction

Conformal field theories on worldsheets with boundary play an important role in understanding some aspects of string theory, since it gives a worldsheet description of D-branes. Conformal field theories in general have large symmetry which includes Virasoro symmetry, and they are generated by holomorphic currents. On worldsheets without boundary there are two copies of the same symmetry algebra corresponding to the left- and the right-moving sectors, and on the boundary of worldsheets the two are related to each other by certain boundary condition which preserves one copy of the symmetry algebra. From the representation theoretical point of view the classification of *boundary states* reduces to that of possible boundary conditions and their solutions.

On the other hand, some conformal field theories are endowed with a Lagrangian, and it is also available even for worldsheets with boundary if suitable boundary terms are incorporated. From this viewpoint, the classification of boundary states corresponds to that of possible boundary terms along with the boundary conditions on the fields.

These two viewpoints have been shown to be consistent in [1, 2] for Liouville theory. The boundary states are classified in a complete way through the analysis of modular property of annulus partition functions, and some boundary states correspond to the addition of a boundary interaction term with certain values of coupling constant. It was also the first case where the classification of Cardy states was done for non-compact CFTs having continuous spectrum of representations. Based on this idea the Liouville theory with boundary has been analyzed in [3, 4], and the analysis of boundary states has also been made recently for more involved CFTs such as CFT on Euclidean AdS_3[5, 6, 7, 8].

In this paper we consider the $N = 1$ supersymmetric extension of Liouville theory in the presence of boundary, using the techniques developed in [1, 2]. This theory has been analyzed for decades and some old references include [9, 10, 11, 12, 13, 14, 15, 16]. One will face some complexity due to the presence of NS and R sectors, and a careful analysis reveals what kind of new features arises as a result of supersymmetrization. For the case without boundary, the exact results for basic correlators has been obtained in [17, 18]. There has also been some recent works on the case with boundary[19]. Since super Liouville theory is one of the simplest CFTs with $N = 1$ worldsheet supersymmetry, our result should contain many of the properties which all the $N = 1$ supersymmetric CFTs have in common.

This paper is organized as follows. In section 2 we analyze the $N = 1$ super Liouville theory without boundary, especially on a sphere. The calculation of basic correlation functions which has been done in [17, 18] are reviewed. We first summarize the spectrum of degenerate representations in $N = 1$ superconformal algebra. Then we calculate various structure constants using the most fundamental degenerate operators which will be denoted as $\Theta^{\epsilon\bar{\epsilon}}_{-b/2}$, under the reasonable assumption that the product of them with any operators are expanded into two discrete terms. The consistency of this assumption is investigated by solving the differential equation for four-point functions containing $\Theta^{\epsilon\bar{\epsilon}}_{-b/2}$. In section 3 we analyze the theory on worldsheets with boundary, especially on a disc. The modular property of annulus partition functions are investigated, from which we obtain the wave functions for some Cardy states. There are some others whose wave functions cannot be determined from the analysis of modular property, and we determine them through the analysis of disc one-point functions. The two-point functions of boundary operators are also obtained. The results for reflection coefficients are consistent with the argument of density of open string states, but it turns out that the reflection coefficients differ for each operator in a single supermultiplet. The last section gives a brief summary of our results and some discussions.

2. $\mathcal{N}=1$ Super Liouville Theory

The supersymmetric extension of Liouville theory was found in [20]. It is described by a boson ϕ and its superpartner ψ, and the action on flat worldsheet reads

$$I = \frac{1}{2\pi}\int d^2z d\bar{\theta}d\theta D\Phi\bar{D}\Phi + 2i\mu\int d^2z d\bar{\theta}d\theta e^{b\Phi}, \tag{2.1}$$

where we employed the superfield formalism

$$\Phi = \phi + i\theta\psi + i\bar{\theta}\bar{\psi} + i\theta\bar{\theta}F, \quad D = \partial_\theta + \theta\partial_z, \quad \bar{D} = \partial_{\bar{\theta}} + \bar{\theta}\partial_{\bar{z}}. \tag{2.2}$$

The reader should note that there is a linear dilaton coupling hidden in the action. In [21] it was analyzed as a two-dimensional theory of supergravity with superconformal symmetry. Superfield expression for linear dilaton coupling can be found in [22].

We regard this theory as a free CFT of ϕ and ψ with a linear dilaton coupling,

$$I = \frac{1}{2\pi}\int d^2z\left[\partial\phi\bar{\partial}\phi + \tfrac{QR\phi}{4} + \psi\bar{\partial}\psi + \bar{\psi}\partial\bar{\psi} - F^2\right], \quad Q = b + b^{-1} \tag{2.3}$$

perturbed by the following interaction

$$2i\mu\int d^2z\left[ibFe^{b\phi} + b^2\psi\bar{\psi}e^{b\phi}\right]. \tag{2.4}$$

In what follows we shall neglect the auxiliary field F which yields a contact interaction, assuming the analyticity of correlators or OPEs that allow us to calculate any quantity by the continuation from the region where contact interactions can be neglected. See [23, 24, 14] for more detailed argument on this point. Thus we shall treat the super Liouville theory as the free CFT of ϕ and ψ perturbed by

$$S_{\rm int} \equiv 2i\mu b^2\int d^2z\psi\bar{\psi}e^{b\phi}. \tag{2.5}$$

The stress tensor T and the supercurrent T_F of the free theory are given by the Feigin-Fuchs representation

$$\begin{aligned} T &= -\tfrac{1}{2}(\partial\phi\partial\phi - Q\partial^2\phi + \psi\partial\psi), \\ T_F &= i(\psi\partial\phi - Q\partial\psi). \end{aligned} \tag{2.6}$$

They satisfy the super Virasoro algebra with $c=\frac{3\hat{c}}{2}=\frac{3}{2}(1+2Q^2)$

$$\begin{aligned} T(z)T(0) &\sim \frac{3\hat{c}}{4z^4}+\frac{2T(0)}{z^2}+\frac{\partial T(0)}{z}, \\ T(z)T_F(0) &\sim \frac{3T_F(0)}{2z^2}+\frac{\partial T_F(0)}{z}, \\ T_F(z)T_F(0) &\sim \frac{\hat{c}}{z^3}+\frac{2T(0)}{z}. \end{aligned} \tag{2.7}$$

We shall concentrate on the left-moving sector for the time being, in order to clarify the symmetry structure of the theory. We work with the primaries $V_\alpha=e^{\alpha\phi}$ and their superpartners $\Lambda_\alpha=-i\alpha\psi e^{\alpha\phi}$, which satisfy

$$\begin{aligned} T(z)V_\alpha(0) &\sim \frac{h_\alpha V_\alpha(0)}{z^2}+\frac{\partial V_\alpha(0)}{z}, \\ T(z)\Lambda_\alpha(0) &\sim \frac{(h_\alpha+\frac{1}{2})\Lambda_\alpha(0)}{z^2}+\frac{\partial\Lambda_\alpha(0)}{z}, \\ T_F(z)V_\alpha(0) &\sim \frac{\Lambda_\alpha(0)}{z}, \\ T_F(z)\Lambda_\alpha(0) &\sim \frac{2h_\alpha V_\alpha(0)}{z^2}+\frac{\partial V_\alpha(0)}{z}, \end{aligned} \tag{2.8}$$

with $h_\alpha=\alpha(Q-\alpha)/2$. They are NS vertices and correspond to space-time bosons. We also consider the R vertices corresponding to space-time fermions, which are given by spin fields $\Theta^\pm_\alpha=\sigma^\pm e^{\alpha\phi}$. They obey the following transformation property

$$\begin{aligned} T(z)\Theta^\pm_\alpha(0) &\sim \frac{(h_\alpha+\frac{1}{16})\Theta^\pm_\alpha(0)}{z^2}+\frac{\partial\Theta^\pm_\alpha(0)}{z}, \\ T_F(z)\Theta^\pm_\alpha(0) &\sim \frac{p_\alpha\Theta^\mp_\alpha(0)}{\sqrt{2}z^{3/2}}+\cdots, \qquad p_\alpha=\frac{i(Q-2\alpha)}{2}. \end{aligned} \tag{2.9}$$

The most important property of spin fields is that the supercurrent T_F becomes double-valued around them. The spin field $\sigma^\pm$ are defined to satisfy

$$\psi(z)\sigma^\pm(0)\sim\frac{\sigma^\mp(0)}{\sqrt{2}z^{1/2}}. \tag{2.10}$$

We can analyze the theory perturbatively, by expanding any quantity as a power series in the cosmological constant. However, due to the momentum conservation in Linear dilaton theory, any correlators of operators of definite Liouville momentum have only one contribution from a specific order of μ. This can easily be seen by employing the path integration approach and perform the integration over the zero-mode of ϕ as discussed in [25]. Then we find, for example,

$$\left\langle \textstyle\prod_i V_{\alpha_i}(z_i)\right\rangle = b^{-1}\Gamma(-N)\left\langle \textstyle\prod_i V_{\alpha_i}(z_i)S_{\rm int}^N\right\rangle_{\rm Wick} \tag{2.11}$$

where the suffix "Wick" represents the ordinary Wick contraction with respect to free fields and N is defined by

$$bN=Q(1-g)-\sum_i\alpha_i \tag{2.12}$$

for worldsheets with g handles. Although the expression (2.11) can be used to evaluate correlators by first assuming N to be a non-negative integer and then extending the result to generic N, we do not use it this way. We would rather read off from it one important property that the correlator diverges when a non-negative integer insertions of $S_{\rm int}$ can screen the non-conserving Liouville momentum, and the residue of the divergence is given by the free field correlator with an appropriate number of $S_{\rm int}$ inserted.

2.1. Two-point functions on a sphere

Here we re-derive the basic correlation functions on a sphere which were obtained by [17, 18] as an introduction to our method to analyze the theory on a disc. The most important among them are the two-point functions. We first consider the following one:

$$\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\rangle = |z_{12}|^{-4h_{\alpha_1}}2\pi\left\{\delta(p_1+p_2)+\delta(p_1-p_2)D(\alpha_1)\right\} \qquad (\alpha_i=\tfrac{Q}{2}+ip_i). \tag{2.13}$$

The global superconformal symmetry yields some relations between two-point correlators:

$$\begin{aligned}\alpha_1\alpha_2\left\langle\psi V_{\alpha_1}(z_1)\psi V_{\alpha_2}(z_2)\right\rangle &= 2h_{\alpha_1}z_{12}^{-1}\left\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\right\rangle,\\ \alpha_1\alpha_2\left\langle\bar\psi V_{\alpha_1}(z_1)\bar\psi V_{\alpha_2}(z_2)\right\rangle &= 2h_{\alpha_1}\bar z_{12}^{-1}\left\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\right\rangle,\\ \alpha_1^2\alpha_2^2\left\langle\psi\bar\psi V_{\alpha_1}(z_1)\psi\bar\psi V_{\alpha_2}(z_2)\right\rangle &= -4h_{\alpha_1}^2|z_{12}|^{-2}\left\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\right\rangle,\end{aligned} \tag{2.14}$$

so that all the two-point functions of NSNS sector vertices are described by a single structure constant, $D(\alpha)$. To study the two-point functions of RR-sector vertices, we adopt the following convention. We first define the spin fields $\bar\sigma^{\pm}$ in the right-moving sector by the equations

$$\bar\sigma^{\pm}(0)\bar\psi(z)\sim\frac{i\bar\sigma^{\mp}(0)}{\sqrt2\,\bar z^{1/2}}. \tag{2.15}$$

Then the spin fields,

$$\Theta_\alpha^{\epsilon\bar\epsilon}(z,\bar z)\equiv\sigma^{\epsilon\bar\epsilon}e^{\alpha\phi}(z,\bar z)\equiv\sigma^{\epsilon}\bar\sigma^{\bar\epsilon}e^{\alpha\phi}(z,\bar z), \quad (\epsilon,\bar\epsilon=\pm) \tag{2.16}$$

are shown to satisfy the OPE relations

$$\begin{aligned}T_F(z)\Theta_\alpha^{\epsilon,\bar\epsilon}(0) &\sim \frac{p_\alpha\Theta_\alpha^{-\epsilon,\bar\epsilon}(0)}{\sqrt2 z^{3/2}},\\ -i\Theta_\alpha^{\epsilon\bar\epsilon}(0)\bar T_F(\bar z) &\sim \frac{p_\alpha\Theta_\alpha^{\epsilon,-\bar\epsilon}(0)}{\sqrt2\bar z^{3/2}}.\end{aligned} \tag{2.17}$$

We assume that $\sigma^+,\bar\sigma^+$ commute and $\sigma^-,\bar\sigma^-$ anti-commute with fermions. Thus $\Theta_\alpha^{\pm\pm}$ commute with fermions while $\Theta_\alpha^{\pm\mp}$ anti-commute. Using this, the superconformal Ward identity becomes

$$\begin{aligned}&\oint\frac{dw}{2\pi i}(w-z)^{\frac12}(w-z')^{\frac12}T_F(w)\Theta_\alpha^{\epsilon\bar\epsilon}(z)\Theta_{\alpha'}^{\epsilon'\bar\epsilon'}(z')\{\cdots\}\\ &\quad= \frac{1}{\sqrt2}(z-z')^{\frac12}\left[p_\alpha\Theta_\alpha^{-\epsilon,\bar\epsilon}(z)\Theta_{\alpha'}^{\epsilon'\bar\epsilon'}(z')\{\cdots\}+i\epsilon\bar\epsilon p_{\alpha'}\Theta_\alpha^{\epsilon\bar\epsilon}(z)\Theta_{\alpha'}^{-\epsilon',\bar\epsilon'}(z')\{\cdots\}\right]\end{aligned}$$

$$
\begin{aligned}
&\quad + \epsilon\bar{\epsilon}\epsilon'\bar{\epsilon}'\Theta_\alpha^{\epsilon\bar{\epsilon}}(z)\Theta_{\alpha'}^{\epsilon'\bar{\epsilon}'}(z') \oint \frac{dw}{2\pi i}(w-z)^{\frac{1}{2}}(w-z')^{\frac{1}{2}} T_F(w) \{\cdots\}, \\
&\oint \frac{d\bar{w}}{2\pi i}(\bar{w}-\bar{z})^{\frac{1}{2}}(\bar{w}-\bar{z}')^{\frac{1}{2}} \{\cdots\} \Theta_\alpha^{\epsilon\bar{\epsilon}}(z)\Theta_{\alpha'}^{\epsilon'\bar{\epsilon}'}(z')\bar{T}_F(\bar{w}) \\
&= \frac{1}{\sqrt{2}}(\bar{z}-\bar{z}')^{\frac{1}{2}} \left[\{\cdots\} p_{\alpha'}\Theta_\alpha^{\epsilon\bar{\epsilon}}(z)\Theta_{\alpha'}^{\epsilon',-\bar{\epsilon}'}(z') + \{\cdots\} i\epsilon'\bar{\epsilon}' p_\alpha \Theta_\alpha^{\epsilon,-\bar{\epsilon}}(z)\Theta_{\alpha'}^{\epsilon',\bar{\epsilon}'}(z')\right] \\
&\quad + \epsilon\bar{\epsilon}\epsilon'\bar{\epsilon}' \oint \frac{dw}{2\pi i}(w-z)^{\frac{1}{2}}(w-z')^{\frac{1}{2}} \{\cdots\} \bar{T}_F(\bar{w}) \cdot \Theta_\alpha^{\epsilon\bar{\epsilon}}(z)\Theta_{\alpha'}^{\epsilon'\bar{\epsilon}'}(z').
\end{aligned} \tag{2.18}
$$

The normalization of spin fields is given by the following correlators:

$$
\left\langle \sigma^{\pm\pm}(z)\sigma^{\pm\pm}(0)\right\rangle_{\text{free}} = |z|^{-\frac{1}{4}}, \quad \left\langle \sigma^{\pm\mp}(z)\sigma^{\pm\mp}(0)\right\rangle_{\text{free}} = i|z|^{-\frac{1}{4}}, \tag{2.19}
$$

where one should be careful for that only Grassmann-even combinations can have non-vanishing correlators. Note also that all these are related via superconformal transformations. From this, we put the following ansatz for the two-point functions of RR vertices:

$$
\begin{aligned}
-i\left\langle \Theta_{\alpha_1}^{\pm\mp}(z_1)\Theta_{\alpha_2}^{\pm\mp}(z_2)\right\rangle = \left\langle \Theta_{\alpha_1}^{\pm\pm}(z_1)\Theta_{\alpha_2}^{\pm\pm}(z_2)\right\rangle &= |z_{12}|^{-4h_{\alpha_1}-\frac{1}{4}} \cdot 2\pi\delta(p_1+p_2), \\
i\left\langle \Theta_{\alpha_1}^{\pm\mp}(z_1)\Theta_{\alpha_2}^{\mp\pm}(z_2)\right\rangle = \left\langle \Theta_{\alpha_1}^{\pm\pm}(z_1)\Theta_{\alpha_2}^{\mp\mp}(z_2)\right\rangle &= |z_{12}|^{-4h_{\alpha_1}-\frac{1}{4}} \cdot 2\pi\delta(p_1-p_2)\tilde{D}(\alpha_1).
\end{aligned} \tag{2.20}
$$

These two-point functions give relations between operators carrying Liouville momentum α and $Q-\alpha$:

$$
\begin{aligned}
D(\alpha) &= \frac{V_\alpha}{V_{Q-\alpha}} = \frac{\alpha\psi V_\alpha}{(Q-\alpha)\psi V_{Q-\alpha}} = \frac{\alpha\bar{\psi}V_\alpha}{(Q-\alpha)\bar{\psi}V_{Q-\alpha}} = \frac{\alpha^2\psi\bar{\psi}V_\alpha}{(Q-\alpha)^2\psi\bar{\psi}V_{Q-\alpha}}, \\
\tilde{D}(\alpha) &= \frac{\epsilon\bar{\epsilon}\Theta_\alpha^{\epsilon\bar{\epsilon}}}{\Theta_{Q-\alpha}^{-\epsilon,-\bar{\epsilon}}}.
\end{aligned} \tag{2.21}
$$

They are referred to as the reflection relation in what follows. $D(\alpha), \tilde{D}(\alpha)$ are called the reflection coefficients.

The ansatz (2.20) for the two-point functions of spin fields might seem peculiar at first sight, because it leads to the reflection relation which flips the indices $\epsilon, \bar{\epsilon}$ as well as the momentum. In the following we obtain the reflection coefficients using some properties of degenerate primary fields, and there we will convince ourselves that the above ansatz is the only one which is consistent with the OPEs involving degenerate operators.

Degenerate fields and their OPEs

Let us summarize here some basic properties of operators belonging to degenerate representations of superconformal algebra. As was found in [26, 27, 28], they are given by the following Liouville momentum $\alpha_{r,s}$

$$
\alpha_{r,s} \equiv \frac{1}{2}(Q - rb - sb^{-1}), \tag{2.22}
$$

in close analogy with the case of bosonic Liouville theory. The difference is that the degenerate representations with odd $r+s$ sit in the R sector, while those with even $r+s$ are in the NS sector. They are known to have null states at level $rs/2$. The corresponding vertex operators are $V_{\alpha_{r,s}}$ ($r+s$ even) or $\Theta^{\pm}_{\alpha_{r,s}}$ ($r+s$ odd).

We will frequently use the most fundamental degenerate operators $\Theta^{\pm}_{-b/2}$ (and $\Theta^{\pm}_{-1/2b}$) in the following analysis. The property of the corresponding degenerate primary *states* $|2,1\rangle_{\pm}$ are summarized as follows:

$$G_0\,|2,1\rangle_{\pm} = \tfrac{i(2b^2+1)}{2\sqrt{2}b}\,|2,1\rangle_{\mp}\,, \qquad G_{-1}\,|2,1\rangle_{\pm} + \tfrac{i\sqrt{2}}{b}L_{-1}\,|2,1\rangle_{\mp} = 0. \tag{2.23}$$

We first discuss the OPEs involving these degenerate operators to find out the expressions for reflection coefficients. Then in later sections we will give a detailed analysis of the four-point functions involving them by solving the associated differential equations.

Here we just assume that the OPEs of $\Theta^{\epsilon\bar{\epsilon}}_{-b/2}$ with arbitrary primary fields or spin fields yield only two discrete terms. This assumption will be justified in later sections by analyzing the four-point functions. We begin with one particular example:

$$\Theta^{++}_{-b/2}(z_1)V_\alpha(z_2) \sim |z_{12}|^{b\alpha}C_+(\alpha)\Theta^{++}_{\alpha-b/2}(z_2) + |z_{12}|^{b(Q-\alpha)}C_-(\alpha)\Theta^{--}_{\alpha+b/2}(z_2), \tag{2.24}$$

where the coefficients $C_\pm(\alpha)$ are calculable using the techniques of free CFT as proposed in [29] and utilized in the analysis of bosonic Liouville theory in [1]:

$$\begin{aligned} C_+(\alpha) &= \lim_{z_1\to z_2} \frac{\left\langle \Theta^{++}_{Q-\alpha+b/2}(w)\Theta^{++}_{-b/2}(z_1)V_\alpha(z_2)\right\rangle_{\text{free}}}{|z_{12}|^{b\alpha}\left\langle \Theta^{++}_{Q-\alpha+b/2}(w)\Theta^{++}_{\alpha-b/2}(z_2)\right\rangle_{\text{free}}} = 1, \\ C_-(\alpha) &= \lim_{z_1\to z_2} \frac{\left\langle \Theta^{--}_{Q-\alpha-b/2}(w)\Theta^{++}_{-b/2}(z_1)V_\alpha(z_2)(-S_{\text{int}})\right\rangle_{\text{free}}}{|z_{12}|^{b\alpha}\left\langle \Theta^{--}_{Q-\alpha-b/2}(w)\Theta^{--}_{\alpha+b/2}(z_2)\right\rangle_{\text{free}}} \\ &= \mu\pi b^2\gamma(\tfrac{bQ}{2})\gamma(1-b\alpha)\gamma(b\alpha-\tfrac{bQ}{2}). \end{aligned} \tag{2.25}$$

Here we introduced the notation $\gamma(x)\equiv\Gamma(x)/\Gamma(1-x)$. The free field correlator in the numerator can be evaluated by using

$$\left\langle \sigma^{\pm\pm}(z_1)\sigma^{\mp\mp}(z_2)\psi\bar{\psi}(z_3)\right\rangle = i\left\langle \sigma^{\pm\mp}(z_1)\sigma^{\mp\pm}(z_2)\psi\bar{\psi}(z_3)\right\rangle = \frac{i}{2}|z_{12}|^{3/4}|z_{13}z_{23}|^{-1}, \tag{2.26}$$

which follows from (2.10), (2.15) and (2.19). Summarizing similar OPE relations we obtain

$$\Theta^{\epsilon\bar{\epsilon}}_{-b/2}(z_1)V_\alpha(z_2) \quad\sim\quad |z_{12}|^{b\alpha}\Theta^{\epsilon\bar{\epsilon}}_{\alpha-b/2}(z_2) + \epsilon\bar{\epsilon}|z_{12}|^{b(Q-\alpha)}C_-(\alpha)\Theta^{-\epsilon,-\bar{\epsilon}}_{\alpha+b/2}(z_2), \tag{2.27}$$

Taking the superconformal transformation of both sides we obtain

$$\begin{aligned} &\sqrt{2}z_{21}^{1/2}\Theta^{\epsilon\bar{\epsilon}}_{-b/2}(z_1)\alpha\psi V_\alpha(z_2) \\ &\quad\sim\quad \epsilon\bar{\epsilon}|z_{12}|^{b\alpha}\alpha\Theta^{-\epsilon,\bar{\epsilon}}_{\alpha-b/2}(z_2) - |z_{12}|^{b(Q-\alpha)}(Q-\alpha)C_-(\alpha)\Theta^{\epsilon,-\bar{\epsilon}}_{\alpha+b/2}(z_2), \\ &-i\sqrt{2}\bar{z}_{21}^{1/2}\Theta^{\epsilon\bar{\epsilon}}_{-b/2}(z_1)\alpha\bar{\psi}V_\alpha(z_2) \\ &\quad\sim\quad |z_{12}|^{b\alpha}\alpha\Theta^{\epsilon,-\bar{\epsilon}}_{\alpha-b/2}(z_2) - \epsilon\bar{\epsilon}|z_{12}|^{b(Q-\alpha)}(Q-\alpha)C_-(\alpha)\Theta^{-\epsilon,\bar{\epsilon}}_{\alpha+b/2}(z_2), \\ &-2i|z_{21}|\Theta^{\epsilon\bar{\epsilon}}_{-b/2}(z_1)\alpha^2\psi\bar{\psi}V_\alpha(z_2) \\ &\quad\sim\quad \epsilon\bar{\epsilon}|z_{12}|^{b\alpha}\alpha^2\Theta^{-\epsilon,-\bar{\epsilon}}_{\alpha-b/2}(z_2) + |z_{12}|^{b(Q-\alpha)}(Q-\alpha)^2C_-(\alpha)\Theta^{\epsilon\bar{\epsilon}}_{\alpha+b/2}(z_2). \end{aligned} \tag{2.28}$$

Combining them with the reflection relations (2.21), we obtain the following recursion relations between structure constants:

$$D(\alpha) = C_-(\alpha)\tilde{D}(\alpha + b/2), \quad \tilde{D}(\alpha - b/2) = C_-(Q - \alpha)D(\alpha). \tag{2.29}$$

To find the OPE of $\Theta^{\epsilon\bar{\epsilon}}_{-b/2}$ with generic spin fields, let us start with

$$-2\Theta^{+-}_{-b/2}(z_1)\Theta^{-+}_{\alpha}(z_2) \sim \tilde{C}_+(\alpha)|z_{12}|^{b\alpha+\frac{3}{4}}\psi\bar{\psi}V_{\alpha-b/2}(z_2) + \tilde{C}_-(\alpha)|z_{12}|^{b(Q-\alpha)-\frac{1}{4}}V_{\alpha+b/2}(z_2). \tag{2.30}$$

The free field integrals with screenings give

$$\tilde{C}_+(\alpha) = 1, \quad \tilde{C}_-(\alpha) = 2i\mu\pi b^2\gamma(\tfrac{bQ}{2})\gamma(\tfrac{1}{2} - b\alpha)\gamma(b\alpha - \tfrac{b^2}{2}) = 2iC_-(Q - \alpha - b/2). \tag{2.31}$$

Collecting similar OPE relations we have

$$\begin{aligned} -2\Theta^{\pm\mp}_{-b/2}(z_1)\Theta^{\mp\pm}_{\alpha}(z_2) &\sim 2i\Theta^{\pm\pm}_{-b/2}(z_1)\Theta^{\mp\mp}_{\alpha}(z_2) \\ &\sim |z_{12}|^{b\alpha+\frac{3}{4}}\psi\bar{\psi}V_{\alpha-b/2}(z_2) + \tilde{C}_-(\alpha)|z_{12}|^{b(Q-\alpha)-\frac{1}{4}}V_{\alpha+b/2}(z_2). \end{aligned} \tag{2.32}$$

Taking its superconformal transformations we obtain

$$\begin{aligned} -\sqrt{2}\Theta^{\pm\pm}_{-b/2}(z_1)\Theta^{\pm\mp}_{\alpha}(z_2) &\sim i\sqrt{2}\Theta^{\pm\mp}_{-b/2}(z_1)\Theta^{\pm\pm}_{\alpha}(z_2) \\ &\sim \bar{z}_{12}^{\frac{1}{2}}|z_{12}|^{b\alpha-\frac{1}{4}}\bar{\psi}V_{\alpha-b/2}(z_2) + \frac{(2\alpha+b)\tilde{C}_-(\alpha)}{2(2Q-2\alpha-b)}z_{12}^{\frac{1}{2}}|z_{12}|^{b(Q-\alpha)-\frac{1}{4}}\psi V_{\alpha+b/2}(z_2), \\ -\sqrt{2}\Theta^{\mp\pm}_{-b/2}(z_1)\Theta^{\pm\pm}_{\alpha}(z_2) &\sim i\sqrt{2}\Theta^{\pm\pm}_{-b/2}(z_1)\Theta^{\mp\pm}_{\alpha}(z_2) \\ &\sim -z_{12}^{\frac{1}{2}}|z_{12}|^{b\alpha-\frac{1}{4}}\psi V_{\alpha-b/2}(z_2) + \frac{(2\alpha+b)\tilde{C}_-(\alpha)}{2(2Q-2\alpha-b)}\bar{z}_{12}^{\frac{1}{2}}|z_{12}|^{b(Q-\alpha)-\frac{1}{4}}\bar{\psi}V_{\alpha+b/2}(z_2), \\ -\Theta^{\pm\pm}_{-b/2}(z_1)\Theta^{\pm\pm}_{\alpha}(z_2) &\sim i\Theta^{\pm\mp}_{-b/2}(z_1)\Theta^{\pm\mp}_{\alpha}(z_2) \\ &\sim -|z_{12}|^{b\alpha-\frac{1}{4}}V_{\alpha-b/2}(z_2) + \frac{(2\alpha+b)^2\tilde{C}_-(\alpha)}{4(2Q-2\alpha-b)^2}|z_{12}|^{b(Q-\alpha)+\frac{3}{4}}\psi\bar{\psi}V_{\alpha+b/2}(z_2). \end{aligned} \tag{2.33}$$

Combining them with the reflection relations(2.21), we obtain the same set of recursion relations as (2.29). Here one can also see that the reflection of the Liouville momentum of spin fields should be accompanied by the flip of signs $\epsilon, \bar{\epsilon}$, because in the right hand side of the above OPEs there are always two NS operators of opposite fermion number (when counted for either one of the left/right sectors separately).

A simple solution of (2.29) satisfying the unitarity

$$D(\alpha)D(Q - \alpha) = \tilde{D}(\alpha)\tilde{D}(Q - \alpha) = 1 \tag{2.34}$$

and exhibiting the $b \leftrightarrow 1/b$ duality can be found rather easily:

$$D(\alpha) = -\left(\mu\pi\gamma(bQ/2)\right)^{\frac{Q-2\alpha}{b}}\frac{\Gamma(b(\alpha-\frac{Q}{2}))\Gamma(\frac{1}{b}(\alpha-\frac{Q}{2}))}{\Gamma(-b(\alpha-\frac{Q}{2}))\Gamma(-\frac{1}{b}(\alpha-\frac{Q}{2}))}, \tag{2.35}$$

$$\tilde{D}(\alpha) = \left(\mu\pi\gamma(bQ/2)\right)^{\frac{Q-2\alpha}{b}}\frac{\Gamma(\frac{1}{2}+b(\alpha-\frac{Q}{2}))\Gamma(\frac{1}{2}+\frac{1}{b}(\alpha-\frac{Q}{2}))}{\Gamma(\frac{1}{2}-b(\alpha-\frac{Q}{2}))\Gamma(\frac{1}{2}-\frac{1}{b}(\alpha-\frac{Q}{2}))} \tag{2.36}$$

Before moving on to the analysis of three-point functions, we note that the correlators involving NS-R vertices are difficult to analyze since the screening operator becomes double-valued around them. Therefore, they are not well-defined vertex operators from the viewpoint of a perturbed free CFT.

We also note that the following relations for products of two spin fields hold:

$$\Theta^{\epsilon\bar{\epsilon}}_{\alpha_1}(z_1)\Theta^{\tau\bar{\tau}}_{\alpha_2}(z_2) = -i\bar{\epsilon}\tau\Theta^{-\epsilon,\bar{\epsilon}}_{\alpha_1}(z_1)\Theta^{-\tau,\bar{\tau}}_{\alpha_2}(z_2) = -i\bar{\epsilon}\tau\Theta^{\epsilon,-\bar{\epsilon}}_{\alpha_1}(z_1)\Theta^{\tau,-\bar{\tau}}_{\alpha_2}(z_2), \tag{2.37}$$

in all the expressions for two-point functions and OPEs obtained so far. We assume this to hold in arbitrary correlation functions containing two spin fields. As an evidence, the analysis of four-point functions becomes much simpler without any contradiction if we employ this relation. However, one should not expect this relation to hold in correlators containing more than two spin fields.

2.2. Three-point functions on a sphere

We first put the following ansatz for them:

$$\begin{array}{rclcrcl} \langle V_{\alpha_1}V_{\alpha_2}V_{\alpha_3}\rangle &=& C_1(\alpha_1,\alpha_2,\alpha_3), && \left\langle V_{\alpha_1}\Theta^{\pm\pm}_{\alpha_2}\Theta^{\mp\mp}_{\alpha_3}\right\rangle &=& \tilde{C}_1(\alpha_1;\alpha_2,\alpha_3), \\ -i\alpha_1\langle \psi V_{\alpha_1}V_{\alpha_2}V_{\alpha_3}\rangle &=& C_2(\alpha_1,\alpha_2,\alpha_3), && \left\langle V_{\alpha_1}\Theta^{\pm\pm}_{\alpha_2}\Theta^{\pm\mp}_{\alpha_3}\right\rangle &=& \tilde{C}_2(\alpha_1;\alpha_2,\alpha_3), \\ -\alpha_1^2\left\langle \psi\bar{\psi} V_{\alpha_1}V_{\alpha_2}V_{\alpha_3}\right\rangle &=& C_3(\alpha_1,\alpha_2,\alpha_3), && \left\langle V_{\alpha_1}\Theta^{\pm\pm}_{\alpha_2}\Theta^{\pm\pm}_{\alpha_3}\right\rangle &=& \tilde{C}_3(\alpha_1;\alpha_2,\alpha_3), \end{array} \tag{2.38}$$

where we omit the coordinate dependence which can easily be restored knowing the conformal weights of operators:

$$\langle\mathcal{O}_1(z_1)\mathcal{O}_2(z_2)\mathcal{O}_3(z_3)\rangle \sim z_{12}^{h_3-h_1-h_2}z_{23}^{h_1-h_2-h_3}z_{13}^{h_2-h_3-h_1}\bar{z}_{12}^{\bar{h}_3-\bar{h}_1-\bar{h}_2}\bar{z}_{23}^{\bar{h}_1-\bar{h}_2-\bar{h}_3}\bar{z}_{13}^{\bar{h}_2-\bar{h}_3-\bar{h}_1} \tag{2.39}$$

Other three-point functions are related to the above expressions via left-right symmetry, or are obtained by taking the superconformal transformations of them. Using the superconformal symmetry we can also find that $C_{1,2,3}(\alpha_i)$ are symmetric in the three arguments.

To obtain them we again use the degenerate field $\Theta^{\epsilon\bar{\epsilon}}_{-b/2}$. We focus on the left-moving sector of the theory and consider the four-point functions in chiral CFT:

$$\begin{array}{rcl} \left\langle V_{\alpha_3}(z_3)V_{\alpha_2}(z_2)\Theta^{\epsilon}_{-b/2}(z_0)\Theta^{\tau}_{\alpha_1}(z_1)\right\rangle &=& f^{\epsilon\tau}_{00}(z_i), \\ -i\alpha_2\left\langle V_{\alpha_3}(z_3)\psi V_{\alpha_2}(z_2)\Theta^{\epsilon}_{-b/2}(z_0)\Theta^{\tau}_{\alpha_1}(z_1)\right\rangle &=& f^{\epsilon\tau}_{01}(z_i), \\ -i\alpha_3\left\langle \psi V_{\alpha_3}(z_3)V_{\alpha_2}(z_2)\Theta^{\epsilon}_{-b/2}(z_0)\Theta^{\tau}_{\alpha_1}(z_1)\right\rangle &=& f^{\epsilon\tau}_{10}(z_i), \\ -\alpha_2\alpha_3\left\langle \psi V_{\alpha_3}(z_3)\psi V_{\alpha_2}(z_2)\Theta^{\epsilon}_{-b/2}(z_0)\Theta^{\tau}_{\alpha_1}(z_1)\right\rangle &=& f^{\epsilon\tau}_{11}(z_i). \end{array} \tag{2.40}$$

As was shown in [17] and reviewed in the following, a set of differential equations among them can be derived from the superconformal symmetry and the degeneracy of $\Theta_{-b/2}$ which has two independent solutions. This justifies the assumption in the previous paragraph that the

OPEs involving $\Theta^{\epsilon\epsilon}_{-b/2}$ yield only two discrete terms. However, in order to obtain the three-point structure constants it is sufficient to know that the first four of the above sixteen, $f^{\epsilon\tau}_{00}$, satisfies an ordinary hypergeometric differential equation. If we simply *assume* this we can skip the lengthy analysis of the differential equation and easily write down the solution, since the behavior when z_0 approach either of $z_{1,2,3}$ is known from the OPE formula.

Differential equation for correlators $\langle VV\Theta\Theta\rangle$

Firstly, the above sixteen functions are related via global superconformal transformation. By multiplying $\oint \frac{dw}{2\pi i}(w-z_0)^{1/2}(w-z_1)^{1/2}T_F(w)$ onto the product of operators in the bracket we find

$$\begin{aligned}
z_{30}^{1/2}z_{31}^{1/2}f^{\epsilon\tau}_{10}+z_{20}^{1/2}z_{21}^{1/2}f^{\epsilon\tau}_{01}+\frac{z_{01}^{1/2}}{\sqrt{2}}(p_{-b/2}-\epsilon\tau p_{\alpha_1})f^{-\epsilon,\tau}_{00} &= 0,\\
z_{30}^{1/2}z_{31}^{1/2}f^{\epsilon\tau}_{11}+z_{20}^{1/2}z_{21}^{1/2}(\partial_2+\frac{h_2}{z_{20}}+\frac{h_2}{z_{21}})f^{\epsilon\tau}_{00}-\frac{z_{01}^{1/2}}{\sqrt{2}}(p_{-b/2}-\epsilon\tau p_{\alpha_1})f^{-\epsilon,\tau}_{01} &= 0,\\
z_{30}^{1/2}z_{31}^{1/2}(\partial_3+\frac{h_3}{z_{30}}+\frac{h_3}{z_{31}})f^{\epsilon\tau}_{00}-z_{20}^{1/2}z_{21}^{1/2}f^{\epsilon\tau}_{11}-\frac{z_{01}^{1/2}}{\sqrt{2}}(p_{-b/2}-\epsilon\tau p_{\alpha_1})f^{-\epsilon,\tau}_{10} &= 0,\\
z_{30}^{1/2}z_{31}^{1/2}(\partial_3+\frac{h_3}{z_{30}}+\frac{h_3}{z_{31}})f^{\epsilon\tau}_{01}-z_{20}^{1/2}z_{21}^{1/2}(\partial_2+\frac{h_2}{z_{20}}+\frac{h_2}{z_{21}})f^{\epsilon\tau}_{10}&\\
+\frac{z_{01}^{1/2}}{\sqrt{2}}(p_{-b/2}-\epsilon\tau p_{\alpha_1})f^{-\epsilon,\tau}_{11} &= 0.
\end{aligned} \tag{2.41}$$

Here the relation (2.37) was assumed. By introducing a new set of functions $F^{\epsilon\tau}_{ij}(\eta)$ of the cross ratio $\eta=\frac{z_{01}z_{23}}{z_{03}z_{21}}$:

$$\begin{aligned}
f^{\epsilon\tau}_{00} &= \textstyle\prod_{i>j}z_{ij}^{\mu_{ij}}F^{\epsilon\tau}_{00}, & \textstyle\sum_{j\neq 0}\mu_{0j} &= -2h_0-1/8,\\
f^{\epsilon\tau}_{01} &= \left(\tfrac{z_{30}^2z_{21}}{z_{32}^2z_{01}z_{20}}\right)^{1/2}\textstyle\prod_{i>j}z_{ij}^{\mu_{ij}}F^{\epsilon\tau}_{01}, & \textstyle\sum_{j\neq 1}\mu_{1j} &= -2h_1-1/8,\\
f^{\epsilon\tau}_{10} &= \left(\tfrac{z_{30}z_{21}^2}{z_{01}z_{31}z_{32}^2}\right)^{1/2}\textstyle\prod_{i>j}z_{ij}^{\mu_{ij}}F^{\epsilon\tau}_{10}, & \textstyle\sum_{j\neq 2}\mu_{2j} &= -2h_2,\\
f^{\epsilon\tau}_{11} &= \left(\tfrac{z_{30}z_{21}}{z_{20}z_{31}z_{32}^2}\right)^{1/2}\textstyle\prod_{i>j}z_{ij}^{\mu_{ij}}F^{\epsilon\tau}_{11}, & \textstyle\sum_{j\neq 3}\mu_{3j} &= -2h_3,
\end{aligned} \tag{2.42}$$

The above relations can be rewritten in a simpler way:

$$\begin{aligned}
\eta(1-\eta)\partial_\eta &\equiv D\\
F^{\epsilon\tau}_{10}+F^{\epsilon\tau}_{01}+\tfrac{1}{\sqrt{2}}(p_{-b/2}-\epsilon\tau p_{\alpha_1})\eta F^{-\epsilon,\tau}_{00} &= 0,\\
F^{\epsilon\tau}_{11}-\{D+\mu_{23}+\eta(h_2+\mu_{21})\}F^{\epsilon\tau}_{00}-\tfrac{1}{\sqrt{2}}(p_{-b/2}-\epsilon\tau p_{\alpha_1})F^{-\epsilon,\tau}_{01} &= 0,\\
\{D+\mu_{23}+\eta(h_3+\mu_{03})\}F^{\epsilon\tau}_{00}-F^{\epsilon\tau}_{11}-\tfrac{1}{\sqrt{2}}(p_{-b/2}-\epsilon\tau p_{\alpha_1})F^{-\epsilon,\tau}_{10} &= 0,\\
\{D+\mu_{23}-1+\eta(h_3+\mu_{03}+1)\}F^{\epsilon\tau}_{01}&\\
+\{D+\mu_{23}-1+\eta(h_2+\mu_{12}+1)\}F^{\epsilon\tau}_{10}+\tfrac{1}{\sqrt{2}}(p_{-b/2}-\epsilon\tau p_{\alpha_1})\eta F^{-\epsilon,\tau}_{11} &= 0.
\end{aligned} \tag{2.43}$$

Secondly, we derive another set of differential equations by translating the equation for null states

$$(L_{-1}G_0+\frac{2b^2+1}{4}G_{-1})\,|2,1\rangle=(L_{-1}G_0+\frac{bp_{-b/2}}{2i}G_{-1})\,|2,1\rangle=0 \tag{2.44}$$

into a differential equation for correlators involving the corresponding degenerate operator. Here we deal with correlators involving only two spin fields one of which is degenerate, and use the following translation law:

$$\begin{aligned}
(\mathcal{L}_n\Theta^{\epsilon}_{-b/2})(z_0)\Theta^{\tau}_{\alpha_1}(z_1) &= \oint_{z_0}\frac{dz_2}{2\pi i}z_{01}^{-1}z_{20}^{n+1}z_{21}^{-n+1}T(z_2)\Theta^{\epsilon}_{-b/2}(z_0)\Theta^{\tau}_{\alpha_1}(z_1),\\
(\mathcal{G}_n\Theta^{\epsilon}_{-b/2})(z_0)\Theta^{\tau}_{\alpha_1}(z_1) &= \oint_{z_0}\frac{dz_2}{2\pi i}z_{01}^{-1/2}z_{20}^{n+1/2}z_{21}^{-n+1/2}T_F(z_2)\Theta^{\epsilon}_{-b/2}(z_0)\Theta^{\tau}_{\alpha_1}(z_1),
\end{aligned} \tag{2.45}$$

and write down the differential equation in the following way:

$$\left\langle\cdots(\sqrt{2}ib^{-1}\mathcal{L}_{-1}\Theta^{-\epsilon}_{-b/2}+\mathcal{G}_{-1}\Theta^{\epsilon}_{-b/2})(z_0)\cdot\Theta^{\tau}_{\alpha_1}(z_1)\right\rangle=0. \tag{2.46}$$

This yields a set of differential equations for the functions $F^{\epsilon\tau}_{ij}$:

$$\begin{aligned}
0 &= -i\sqrt{2}b^{-1}(D-\mu_{02}-(1-\eta)\mu_{03})F^{-\epsilon\tau}_{00}+\frac{1-\eta}{\eta}F^{\epsilon\tau}_{01}+\frac{1}{\eta}F^{\epsilon\tau}_{10},\\
0 &= -i\sqrt{2}b^{-1}(D-\mu_{02}+\frac{1}{2}-(1-\eta)(\mu_{03}+1))F^{-\epsilon\tau}_{01}\\
&\quad-(1-\eta)F^{\epsilon\tau}_{11}+(D+\mu_{23}+\eta(\mu_{12}+3h_2))F^{\epsilon\tau}_{00},\\
0 &= -i\sqrt{2}b^{-1}(D-\mu_{02}-(1-\eta)(\mu_{03}+\frac{1}{2}))F^{-\epsilon\tau}_{10}\\
&\quad-(1-\eta)(D+\mu_{23}+\eta(\mu_{03}-h_3))F^{\epsilon\tau}_{00}+F^{\epsilon\tau}_{11},\\
0 &= -i\sqrt{2}b^{-1}(D-\mu_{02}+\frac{1}{2}-(1-\eta)(\mu_{03}+\frac{1}{2}))F^{-\epsilon\tau}_{11}\\
&\quad+\frac{1-\eta}{\eta}(D+\mu_{23}-1+\eta(\mu_{03}+1-h_3))F^{\epsilon\tau}_{01}\\
&\quad+\frac{1}{\eta}(D+\mu_{23}-1+\eta(\mu_{12}+1+3h_2))F^{\epsilon\tau}_{10}.
\end{aligned} \tag{2.47}$$

Obviously, (2.43) and (2.47) are redundant. Indeed, (2.43) yields only two relations between $\{F^{\epsilon\tau}_{00},F^{-\epsilon\tau}_{01},F^{-\epsilon\tau}_{10},F^{\epsilon\tau}_{11}\}$, so that we are left with two independent functions out of four after imposing the superconformal symmetry. (2.47) also contains only two independent equations, giving a second-ordered differential equation for one of the remaining two independent functions. Thus the solution for $f^{\epsilon\tau}_{00}$ is written in terms of the hypergeometric function,

$$\begin{aligned}
f^{\epsilon\tau}_{00}(z_{0,1,2,3}) &= z_{12}^{h_3-h_1-h_2-h_0-1/8}z_{23}^{h_1-h_2-h_3+h_0+1/8}z_{13}^{h_2-h_3-h_1+h_0}z_{03}^{-2h_0-1/8}\\
&\quad\times\eta^{\mu_{01}}(1-\eta)^{\mu_{02}}F(A,B;C;\eta),\\
A &= \mu_{01}+\mu_{02}+\frac{b\alpha_3}{2}+2h_0+\frac{1}{8},\\
B &= \mu_{01}+\mu_{02}+\frac{b(Q-\alpha_3)}{2}+2h_0+\frac{1}{8},\\
C &= bQ+2\mu_{01}+4h_0+\frac{1}{4},\\
\mu_{01} &= \frac{bQ+\epsilon\tau b(2\alpha_1-Q)}{4}-\frac{1}{8}\quad\text{or}\quad\frac{bQ-\epsilon\tau b(2\alpha_1-Q)}{4}+\frac{3}{8},\\
\mu_{02} &= \frac{b\alpha_2}{2}\quad\text{or}\quad\frac{b(Q-\alpha_2)}{2}.
\end{aligned} \tag{2.48}$$

The two choices for μ_{01} give two independent solutions, while two choices of μ_{02} lead to the same solution owing to the formula of hypergeometric functions.

By taking its square in a crossing symmetric way we can construct the four-point function on a sphere involving a degenerate field,

$$\begin{aligned}\left\langle V_{\alpha_3}(z_3)V_{\alpha_2}(z_2)\Theta^{++}_{-b/2}(z_0)\Theta^{--}_{\alpha_1}(z_1)\right\rangle \\ = \ |z_{12}|^{2(h_3-h_1-h_2-h_0-1/8)}|z_{23}|^{2(h_1-h_2-h_3+h_0+1/8)}|z_{13}|^{2(h_2-h_3-h_1+h_0)}|z_{03}|^{-4h_0-1/4} \\ \times\{P_1G_1(\alpha_1,\alpha_2,\alpha_3;\eta)G_1(\alpha_1,\alpha_2,\alpha_3;\bar\eta) \\ +P_2G_2(Q-\alpha_1,\alpha_2,\alpha_3;\eta)G_2(Q-\alpha_1,\alpha_2,\alpha_3;\bar\eta)\}\,,\end{aligned} \tag{2.49}$$

where $G_{1,2}$ are given by (we use the notation like $p_{-1-2+3}\equiv -p_1-p_2+p_3$)

$$\begin{aligned}G_1(\alpha_i;\eta) &= \eta^{\frac{b\alpha_1}{2}+\frac{3}{8}}(1-\eta)^{\frac{b\alpha_2}{2}}F(\tfrac{ib}{2}p_{+1+2+3}+\tfrac{3}{4},\ \tfrac{ib}{2}p_{+1+2-3}+\tfrac{3}{4};\ ibp_1+\tfrac{3}{2};\ \eta)\\ G_2(\alpha_i;\eta) &= \eta^{\frac{b\alpha_1}{2}-\frac{1}{8}}(1-\eta)^{\frac{b\alpha_2}{2}}F(\tfrac{ib}{2}p_{+1+2+3}+\tfrac{1}{4},\ \tfrac{ib}{2}p_{+1+2-3}+\tfrac{1}{4};\ ibp_1+\tfrac{1}{2}\,;\eta)\end{aligned} \tag{2.50}$$

and the coefficients $P_{1,2}$ can be fixed from the crossing symmetry up to an overall normalization:

$$\begin{aligned}\frac{P_1}{P_2} &= -(\tfrac{1}{2}+ibp_1)^{-2}\gamma(\tfrac{1}{2}-ibp_1)^2\\ &\quad\times\gamma(\tfrac{3}{4}+\tfrac{ib}{2}p_{+1+2+3})\gamma(\tfrac{3}{4}+\tfrac{ib}{2}p_{+1+2-3})\gamma(\tfrac{3}{4}+\tfrac{ib}{2}p_{+1-2+3})\gamma(\tfrac{3}{4}+\tfrac{ib}{2}p_{+1-2-3}).\end{aligned} \tag{2.51}$$

Considering in the same way, we find that there is no crossing symmetric solution for four point functions like $\left\langle V_{\alpha_3}V_{\alpha_2}\Theta^{++}_{-b/2}\Theta^{+-}_{\alpha_1}\right\rangle$, so that they must vanish. The correlator $\left\langle V_{\alpha_3}V_{\alpha_2}\Theta^{++}_{-b/2}\Theta^{++}_{\alpha_1}\right\rangle$ are obtained simply by replacing α_1 with $Q-\alpha_1$ in the above.

From the above solutions we can derive a recursion relation for three-point structure constants $C_i(\alpha_{1,2,3})$ and $\tilde C_i(\alpha_{1,2,3})$. From the fact that some four-point functions vanish it follows that

$$C_2(\alpha_1,\alpha_2,\alpha_3)=\tilde C_2(\alpha_1,\alpha_2,\alpha_3)=0. \tag{2.52}$$

By comparing the limit $z_0\to z_1$ of (2.49) with the OPE formula (2.32) we find

$$\begin{aligned}\frac{P_1}{P_2}(p_1,p_2,p_3) &= -\frac{1}{(\alpha_1-\frac{b}{2})^2}\frac{\tilde C_+(\alpha_1)C_3(\alpha_1-\frac{b}{2},\alpha_2,\alpha_3)}{\tilde C_-(\alpha_1)C_1(\alpha_1+\frac{b}{2},\alpha_2,\alpha_3)},\\ \frac{P_1}{P_2}(-p_1,p_2,p_3) &= \frac{1}{(2Q-2\alpha_1-b)^2}\frac{\tilde C_-(\alpha_1)C_3(\alpha_1+\frac{b}{2},\alpha_2,\alpha_3)}{\tilde C_+(\alpha_1)C_1(\alpha_1-\frac{b}{2},\alpha_2,\alpha_3)}.\end{aligned} \tag{2.53}$$

These give a set of recursion relations for $C_1(\alpha_1,\alpha_2,\alpha_3)$ and $C_3(\alpha_1,\alpha_2,\alpha_3)$, whose solution can be expressed in terms of Υ function introduced in [30, 31]. It is defined by

$$\ln\Upsilon(x) = \int_0^\infty\frac{dt}{t}\left[e^{-2t}(\tfrac{Q}{2}-x)^2-\frac{\sinh^2[(Q/2-x)t]}{\sinh[bt]\sinh[t/b]}\right], \tag{2.54}$$

and satisfies the following relations

$$\Upsilon(x+b)=\Upsilon(x)b^{1-2bx}\gamma(bx),\quad \Upsilon(x+\tfrac{1}{b})=\Upsilon(x)b^{2x/b-1}\gamma(x/b),\quad \Upsilon(x)=\Upsilon(Q-x). \tag{2.55}$$

It has zeroes at

$$\Upsilon(x)=0 \ \text{ at } \ x=-mb-nb^{-1},\ x=Q+mb+nb^{-1} \ \ (m,n\in\mathbf{Z}_{\geq 0}). \tag{2.56}$$

If we define

$$\Upsilon_{\rm NS}(x)=\Upsilon(\tfrac{x}{2})\Upsilon(\tfrac{x+Q}{2}),\quad \Upsilon_{\rm R}(x)=\Upsilon(\tfrac{x+b}{2})\Upsilon(\tfrac{x+b^{-1}}{2}), \tag{2.57}$$

the solution for the recursion relation can be expressed as

$$\begin{aligned} C_1(\alpha_i) &= \left\{\mu\pi\gamma(\tfrac{bQ}{2})b^{1-b^2}\right\}^{\frac{Q-\Sigma\alpha_i}{b}}\frac{\Upsilon'_{\rm NS}(0)}{\Upsilon_{\rm NS}(\alpha_{1+2+3}-Q)}\frac{\Upsilon_{\rm NS}(2\alpha_1)\Upsilon_{\rm NS}(2\alpha_2)\Upsilon_{\rm NS}(2\alpha_3)}{\Upsilon_{\rm NS}(\alpha_{1+2-3})\Upsilon_{\rm NS}(\alpha_{2+3-1})\Upsilon_{\rm NS}(\alpha_{3+1-2})}, \\ C_3(\alpha_i) &= i\left\{\mu\pi\gamma(\tfrac{bQ}{2})b^{1-b^2}\right\}^{\frac{Q-2A}{b}}\frac{2\Upsilon'_{\rm NS}(0)}{\Upsilon_{\rm R}(\alpha_{1+2+3}-Q)}\frac{\Upsilon_{\rm NS}(2\alpha_1)\Upsilon_{\rm NS}(2\alpha_2)\Upsilon_{\rm NS}(2\alpha_3)}{\Upsilon_{\rm R}(\alpha_{1+2-3})\Upsilon_{\rm R}(\alpha_{2+3-1})\Upsilon_{\rm R}(\alpha_{3+1-2})}, \end{aligned} \tag{2.58}$$

where we used the notations like $\alpha_{1+2-3}\equiv\alpha_1+\alpha_2-\alpha_3$. The functions $\Upsilon_{\rm NS},\Upsilon_{\rm R}$ are also useful to construct reflection-symmetric quantities because of the relations

$$\begin{aligned} D(\alpha) &= (\mu\pi\gamma(\tfrac{bQ}{2})b^{1-b^2})^{\frac{Q-2\alpha}{b}}\frac{\Upsilon_{\rm NS}(2\alpha)}{\Upsilon_{\rm NS}(2Q-2\alpha)}, \\ \tilde{D}(\alpha) &= (\mu\pi\gamma(\tfrac{bQ}{2})b^{1-b^2})^{\frac{Q-2\alpha}{b}}\frac{\Upsilon_{\rm R}(2\alpha)}{\Upsilon_{\rm R}(2Q-2\alpha)}. \end{aligned} \tag{2.59}$$

The three-point structure constants containing spin fields can be obtained in a similar way. This time we take the limit $z_0\to z_2$ of the solution (2.49). Using the formulae for hypergeometric functions we find

$$\begin{aligned} &\left\langle V_{\alpha_3}(z_3)V_{\alpha_2}(z_2)\Theta^{++}_{-b/2}(z_0)\Theta^{--}_{\alpha_1}(z_1)\right\rangle \\ &\quad= |z_{12}|^{2(h_3-h_1-h_2-h_0-1/8)}|z_{23}|^{2(h_1-h_2-h_3+h_0+1/8)}|z_{13}|^{2(h_2-h_3-h_1+h_0)}|z_{03}|^{-4h_0-1/4} \\ &\quad\times\{Q_1H(\alpha_1,\alpha_2,\alpha_3;1-\eta)H(\alpha_1,\alpha_2,\alpha_3;1-\bar\eta) \\ &\qquad\quad+Q_2H(\alpha_1,Q-\alpha_2,\alpha_3;1-\eta)H(\alpha_1,Q-\alpha_2,\alpha_3;1-\bar\eta)\}, \end{aligned} \tag{2.60}$$

where H is given by

$$H(\alpha_i;1-\eta)=\eta^{\frac{b\alpha_1}{2}+\frac{3}{8}}(1-\eta)^{\frac{b\alpha_2}{2}}F(\tfrac{ib}{2}p_{+1+2+3}+\tfrac{3}{4},\,\tfrac{ib}{2}p_{+1+2-3}+\tfrac{3}{4};\,ibp_2+1;\,1-\eta), \tag{2.61}$$

and the ratio Q_1/Q_2 reads

$$\frac{Q_1}{Q_2}=b^2p_2^2\gamma(-ibp_2)^2\gamma(\tfrac{3}{4}+\tfrac{ib}{2}p_{1+2+3})\gamma(\tfrac{3}{4}+\tfrac{ib}{2}p_{1+2-3})\gamma(\tfrac{1}{4}+\tfrac{ib}{2}p_{-1+2+3})\gamma(\tfrac{1}{4}+\tfrac{ib}{2}p_{-1+2-3}). \tag{2.62}$$

By making a comparison with the OPE formula (2.27) we find recursion relations

$$\begin{aligned} \frac{Q_1}{Q_2}(p_1,p_2,p_3) &= \frac{C_+(\alpha_2)\tilde{C}_1(\alpha_3;\alpha_2-\frac{b}{2},\alpha_1)}{C_-(\alpha_2)\tilde{C}_3(\alpha_3;\alpha_2+\frac{b}{2},\alpha_1)}, \\ \frac{Q_1}{Q_2}(-p_1,p_2,p_3) &= \frac{C_+(\alpha_2)\tilde{C}_3(\alpha_3;\alpha_2-\frac{b}{2},\alpha_1)}{C_-(\alpha_2)\tilde{C}_1(\alpha_3;\alpha_2+\frac{b}{2},\alpha_1)}. \end{aligned} \tag{2.63}$$

This can be straightforwardly solved and we obtain

$$\tilde{C}_1(\alpha_3;\alpha_2,\alpha_1) = \left\{\mu\pi\gamma(\tfrac{bQ}{2})b^{1-b^2}\right\}^{\frac{Q-2A}{b}} \frac{\Upsilon'_{\rm NS}(0)\Upsilon_{\rm NS}(2\alpha_3)\Upsilon_{\rm R}(2\alpha_2)\Upsilon_{\rm R}(2\alpha_1)}{\Upsilon_{\rm R}(\alpha_{1+2+3}-Q)\Upsilon_{\rm R}(\alpha_{1+2-3})\Upsilon_{\rm NS}(\alpha_{2+3-1})\Upsilon_{\rm NS}(\alpha_{3+1-2})},$$
$$\tilde{C}_3(\alpha_3;\alpha_2,\alpha_1) = \left\{\mu\pi\gamma(\tfrac{bQ}{2})b^{1-b^2}\right\}^{\frac{Q-2A}{b}} \frac{\Upsilon'_{\rm NS}(0)\Upsilon_{\rm NS}(2\alpha_3)\Upsilon_{\rm R}(2\alpha_2)\Upsilon_{\rm R}(2\alpha_1)}{\Upsilon_{\rm NS}(\alpha_{1+2+3}-Q)\Upsilon_{\rm NS}(\alpha_{1+2-3})\Upsilon_{\rm R}(\alpha_{2+3-1})\Upsilon_{\rm R}(\alpha_{3+1-2})}. \quad (2.64)$$

We can also check that these three-point structure constants are consistent with the reflection symmetry.

Although these three-point structure constants (2.58) and (2.64) have complicated form, each factor has a clear physical meaning. Note first that the zeroes of $\Upsilon_{\rm NS}, \Upsilon_{\rm R}$ are at

$$\begin{aligned} \Upsilon_{\rm NS}(x) = 0 \qquad & \text{at} \ \ x = -mb - nb^{-1}, \ x = Q + mb + nb^{-1} \ \ (m+n \ \ \text{even}), \\ \Upsilon_{\rm R}(x) = 0 \qquad & \text{at} \ \ x = -mb - nb^{-1}, \ x = Q + mb + nb^{-1} \ \ (m+n \ \ \text{odd}). \end{aligned} \quad (2.65)$$

Each three-point structure constant therefore has eight sequences of poles. This agree with our naive expectation, since by combining the perturbative consideration, $b \leftrightarrow b^{-1}$ and the reflection symmetry we can easily guess that, for example, $C_1(\alpha_i)$ diverges at

$$(\alpha_1 \text{ or } Q-\alpha_1) + (\alpha_2 \text{ or } Q-\alpha_2) + (\alpha_3 \text{ or } Q-\alpha_3) = Q - mb - nb^{-1} \quad (m+n \ \ \text{even}). \quad (2.66)$$

The last condition is because of the fermionic nature of the screening operators. The pole structure of other structure constants can also be understood in the same way if we take into account that the reflection of spin fields also flips their Grassmann parity.

Differential equation for correlators $\langle\Theta\Theta\Theta\Theta\rangle$

Let us analyze here the correlation functions of four spin fields, one of which is degenerate. The reason for this is that we will use the solution to obtain the one-point structure constant on a disc in the next section. The solution can also be used to cross-check the three-point structure constants which were obtained previously.

We first consider the holomorphic sector as in the previous case, and begin by introducing some notations:

$$\left\langle \Theta^{\epsilon_0}_{-b/2}(z_0)\Theta^{\epsilon_1}_{\alpha_1}(z_1)\Theta^{\epsilon_2}_{\alpha_2}(z_2)\Theta^{\epsilon_3}_{\alpha_3}(z_3)\right\rangle \equiv f^{\epsilon_0\epsilon_1\epsilon_2\epsilon_3}(z_i). \quad (2.67)$$

Then we translate the expression for the null vector (2.44) into differential equation for correlators. In doing this, note first that the Ramond algebra is generated by L_1 and G_{-1}. So it suffices to give the rule of translation for these two:

$$\begin{aligned} \left\langle \mathcal{L}_1\Theta^{\epsilon_0}_{-b/2}(z_0)\textstyle\prod_i\Theta^{\epsilon_i}_{\alpha_i}(z_i)\right\rangle &= \oint_{z_0}\frac{dz_4}{2\pi i}z_{40}^2\left\langle T(z_4)\Theta^{\epsilon_0}_{-b/2}(z_0)\textstyle\prod_i\Theta^{\epsilon_i}_{\alpha_i}(z_i)\right\rangle, \\ \left\langle \mathcal{G}_{-1}\Theta^{\epsilon_0}_{-b/2}(z_0)\textstyle\prod_i\Theta^{\epsilon_i}_{\alpha_i}(z_i)\right\rangle &= \oint_{z_0}\frac{dz_4}{2\pi i}\left(\frac{z_{41}z_{42}z_{43}}{z_{40}z_{01}z_{02}z_{03}}\right)^{\frac{1}{2}}\left\langle T_F(z_4)\Theta^{\epsilon_0}_{-b/2}(z_0)\textstyle\prod_i\Theta^{\epsilon_i}_{\alpha_i}(z_i)\right\rangle. \end{aligned} \quad (2.68)$$

Using them, the rule for other generators can be obtained easily:

$$\left\langle \mathcal{G}_0\Theta^{\epsilon_0}_{-b/2}(z_0)\cdot\textstyle\prod_i\Theta^{\epsilon_i}_{\alpha_i}(z_i)\right\rangle$$

$$
\begin{aligned}
&= \frac{1}{3}\oint_{z_0}\frac{dz_4}{2\pi i}\left(\frac{z_{40}z_{41}z_{42}z_{43}}{z_{01}z_{02}z_{03}}\right)^{1/2}\left(\frac{z_{01}}{z_{41}}+\frac{z_{02}}{z_{42}}+\frac{z_{03}}{z_{43}}\right)\left\langle T_F(z_4)\Theta^{\epsilon_0}_{-b/2}(z_0)\Pi_i\Theta^{\epsilon_i}_{\alpha_i}(z_i)\right\rangle,\\
&\left\langle \mathcal{L}_{-1}\Theta^{\epsilon_0}_{-b/2}(z_0)\cdot\Pi_i\Theta^{\epsilon_i}_{\alpha_i}(z_i)\right\rangle\\
&= \frac{1}{3}\oint_{z_0}\frac{dz_4}{2\pi i}\frac{z_{41}z_{42}z_{43}}{z_{01}z_{02}z_{03}}\left(\frac{z_{01}}{z_{41}}+\frac{z_{02}}{z_{42}}+\frac{z_{03}}{z_{43}}\right)\left\langle T(z_4)\Theta^{\epsilon_0}_{-b/2}(z_0)\Pi_i\Theta^{\epsilon_i}_{\alpha_i}(z_i)\right\rangle. \qquad (2.69)
\end{aligned}
$$

Then the equation for the null vector (where we use the notation $\underline{\epsilon}\equiv-\epsilon$),

$$
\sqrt{2}ib^{-1}\mathcal{L}_{-1}\left\langle\Theta^{\underline{\epsilon}_0}_{-b/2}(z_0)\cdot\Pi_i\Theta^{\epsilon_i}_{\alpha_i}(z_i)\right\rangle+\mathcal{G}_{-1}\left\langle\Theta^{\epsilon_0}_{-b/2}(z_0)\cdot\Pi_i\Theta^{\epsilon_i}_{\alpha_i}(z_i)\right\rangle=0, \qquad (2.70)
$$

can be recast into the form of a differential equation,

$$
\begin{aligned}
&\epsilon_0\left(\frac{z_{12}z_{13}}{z_{10}}\right)^{1/2}p_1 f^{\epsilon_0\underline{\epsilon}_1\epsilon_2\epsilon_3}+\epsilon_0\epsilon_1\left(\frac{z_{21}z_{23}}{z_{20}}\right)^{1/2}p_2 f^{\epsilon_0\epsilon_1\underline{\epsilon}_2\epsilon_3}+\epsilon_0\epsilon_1\epsilon_2\left(\frac{z_{31}z_{32}}{z_{30}}\right)^{1/2}p_3 f^{\epsilon_0\epsilon_1\epsilon_2\underline{\epsilon}_3}\\
&= \frac{2i}{b}(z_{01}z_{02}z_{03})^{1/2}\left\{\partial_0+\frac{2h_0+\frac{1}{8}}{3}(z_{01}^{-1}+z_{02}^{-1}+z_{03}^{-1})\right\}f^{\underline{\epsilon}_0\epsilon_1\epsilon_2\epsilon_3}. \qquad (2.71)
\end{aligned}
$$

As in the previous case we rescale the functions $f^{\epsilon_0\epsilon_1\epsilon_2\epsilon_3}$ in the following way:

$$
f^{\epsilon_0\epsilon_1\epsilon_2\epsilon_3}=\prod_{i,j}z_{ij}^{\mu_{ij}}F^{\epsilon_0\epsilon_1\epsilon_2\epsilon_3}(\eta),\quad \eta=\frac{z_{01}z_{23}}{z_{03}z_{21}},\quad \sum_{i(\neq j)}\mu_{ij}=-2h_j-\frac{1}{8}. \qquad (2.72)
$$

Then the above equation can be rewritten into the form

$$
\begin{aligned}
&2ib^{-1}\epsilon_0\mathcal{D}F^{\underline{\epsilon}_0\epsilon_1\epsilon_2\epsilon_3}\\
&\quad = p_1(\eta-1)^{1/2}F^{\epsilon_0\underline{\epsilon}_1\epsilon_2\epsilon_3}+\epsilon_1 p_2(-\eta)^{1/2}F^{\epsilon_0\epsilon_1\underline{\epsilon}_2\epsilon_3}+\epsilon_1\epsilon_2 p_3\eta^{1/2}(1-\eta)^{1/2}F^{\epsilon_0\epsilon_1\epsilon_2\underline{\epsilon}_3},\\
&\mathcal{D} = \eta(1-\eta)\partial_\eta+(1-\eta)(\mu_{01}+\tfrac{1-2bQ}{8})-\eta(\mu_{02}+\tfrac{1-2bQ}{8}). \qquad (2.73)
\end{aligned}
$$

According to the signs of ϵ_i, there are sixteen components of $F^{\epsilon_0\epsilon_1\epsilon_2\epsilon_3}$. The above equations separate them into two groups, each containing eight components with even(odd) number of minus signs in ϵ_i. We try to reduce the number of independent components further by putting the assumption

$$
F^{\epsilon_0\epsilon_1\epsilon_2\epsilon_3}=c(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)F^{\underline{\epsilon}_0\underline{\epsilon}_1\underline{\epsilon}_2\underline{\epsilon}_3}. \qquad (2.74)
$$

The consistency with (2.73) yields

$$
\begin{aligned}
&c(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)=c(\underline{\epsilon}_0,\underline{\epsilon}_1,\underline{\epsilon}_2,\underline{\epsilon}_3)=\pm1,\\
&c(\epsilon_0,\epsilon_1,\epsilon_2,\epsilon_3)=-c(\underline{\epsilon}_0,\underline{\epsilon}_1,\epsilon_2,\epsilon_3)=c(\underline{\epsilon}_0,\epsilon_1,\underline{\epsilon}_2,\epsilon_3)=-c(\underline{\epsilon}_0,\epsilon_1,\epsilon_2,\underline{\epsilon}_3). \qquad (2.75)
\end{aligned}
$$

Denoting $c(+,+,+,+)=\xi$ and $F_{0,1,2,3}=(F^{++++},F^{--++},F^{-+-+},F^{-++-})$ we have

$$
\begin{aligned}
2ib^{-1}\mathcal{D}F_0 &= -\ p_1(\eta-1)^{1/2}F_1\ -\ p_2(-\eta)^{1/2}F_2\ -\ p_3\eta^{1/2}(1-\eta)^{1/2}F_3,\\
2ib^{-1}\mathcal{D}F_1 &= +\ p_1(\eta-1)^{1/2}F_0\ +\xi p_2(-\eta)^{1/2}F_3\ -\xi p_3\eta^{1/2}(1-\eta)^{1/2}F_2,\\
2ib^{-1}\mathcal{D}F_2 &= -\xi p_1(\eta-1)^{1/2}F_3\ +\ p_2(-\eta)^{1/2}F_0\ +\xi p_3\eta^{1/2}(1-\eta)^{1/2}F_1,\\
2ib^{-1}\mathcal{D}F_3 &= +\xi p_1(\eta-1)^{1/2}F_2\ -\xi p_2(-\eta)^{1/2}F_1\ +\ p_3\eta^{1/2}(1-\eta)^{1/2}F_0.
\end{aligned} \qquad (2.76)
$$

One can see that the above system of differential equations exhibits a symmetry in three generic spin fields. It is also consistent with the reflection symmetry: for example, F_1 obeys the same equation as that for F_0 with the signs of p_2 and p_3 flipped. Here we will not go into any further detail to determine ξ or choose explicitly one appropriate branch for each square root, since different choices lead to different correlators and we would like to study their mutual relation later in detail.

Let us step aside for a while and try another way to construct the correlators (2.67). Recall that the correlators are characterized by the analyticity and the asymptotic behavior around $z_0 \sim z_{1,2,3}$ dictated by the OPE formula:

$$f^{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}(z_i) \overset{z_0\to z_i}{\sim} z_{0i}^{\frac{b\alpha_i}{2}-\frac{1}{8}},\quad z_{0i}^{\frac{b\alpha_i}{2}+\frac{3}{8}},\quad z_{0i}^{\frac{b(Q-\alpha_i)}{2}-\frac{1}{8}},\quad z_{0i}^{\frac{b(Q-\alpha_i)}{2}+\frac{3}{8}}. \tag{2.77}$$

In the previous analysis of reflection coefficients it was assumed that the OPE of $\Theta^{\epsilon\bar{\epsilon}}_{-b/2}$ with generic spin fields (of definite chirality) yield only two discrete terms. However, since the differential equation is of the fourth order, there should be four independent solutions. Therefore, in solving the differential equation we should not adhere to the idea that each spin field in any correlator has a definite chirality. Thus we assume that the leading order behavior of the four independent solutions should be given by (2.77).

Let us then put an assumption that the correlator can be expressed as a double contour integral of the following form:

$$f^{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}(z_i) = \prod_{i<j} z_{ij}^{\rho_{ij}} \int dw dw' \prod_i (z_i - w)^{\nu_i}(z_i - w')^{\nu'_i}(w - w')^{\lambda} \tag{2.78}$$

as one can guess by analogy with simpler cases where we encounter with hypergeometric functions. The global conformal invariance yields the conditions on the exponents ν_i, ν'_i, λ and ρ_{ij}:

$$\sum_i \nu_i = \sum_i \nu'_i = -\lambda - 2, \qquad \sum_{j(\neq i)} \rho_{ij} + \nu_i + \nu'_i = -2h_i - \frac{1}{8}. \tag{2.79}$$

Analyzing the behavior at, say, $z_0 \sim z_1$ we find that the double integral approximately breaks into several terms with different asymptotic behavior. In doing this, note first that the limit $z_0 \to z_1$ can also be viewed as the limit $z_2 \to z_3$. Then there are two possibilities for w to be either near $z_{0,1}$ or near $z_{2,3}$. Similarly, there are also two possibilities for w', so that they altogether give four terms in the limit $z_0 \sim z_1$:

$$f^{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}(z_i) \sim z_{01}^{\rho_{01}},\quad z_{01}^{\rho_{01}+1+\nu'_0+\nu'_1},\quad z_{01}^{\rho_{01}+1+\nu_0+\nu_1},\quad z_{01}^{\rho_{01}+\lambda+2+\nu_0+\nu_1+\nu'_0+\nu'_1}. \tag{2.80}$$

Comparing this with the OPE formula, we can easily find that λ must be 0 or ± 1. Among these three, $\lambda = 1$ is the only solution consistent with $h_0 + \frac{1}{16} = \frac{3}{16}(1 - 2bQ)$. Going back to the analysis of the equation (2.76), we are thus lead to conjecture that after setting $\mu_{01} = \mu_{02} = 0$ in (2.73), the function $F_0(\eta)$ should be given by

$$F_0(p_1, p_2, p_3; \eta) \quad = \quad \eta^{\frac{b\alpha_1}{2}+\frac{3}{8}}(1-\eta)^{\frac{b\alpha_2}{2}+\frac{3}{8}} \int dw dw' \left[w'(w'-1)(w'-\eta)\right]^{-\frac{3}{4}}$$

$$\times\, w^{-\frac{3}{4}+\frac{ib}{2}p_{-1+2+3}}(w-1)^{-\frac{3}{4}+\frac{ib}{2}p_{1-2+3}}(w-\eta)^{-\frac{3}{4}-\frac{ib}{2}p_{1+2+3}}(w-w')$$
$$\equiv\ \eta^{\frac{b\alpha_1}{2}+\frac{3}{8}}(1-\eta)^{\frac{b\alpha_2}{2}+\frac{3}{8}}\int dwdw' f(w,w';\eta)(w-w'), \tag{2.81}$$

with certain integration contours for w and w'. Other three functions should be obtained from the symmetry of the equation (2.76): by flipping the signs of $p_{1,2,3}$ and make a suitable change of coordinates we find a set of contour integrals

$$\begin{aligned}
\mathcal{F}_0(\eta) &= \eta^{\frac{b\alpha_1}{2}+\frac{3}{8}}(1-\eta)^{\frac{b\alpha_2}{2}+\frac{3}{8}}\int dwdw' f(w,w';\eta)(w-w'),\\
\mathcal{F}_1(\eta) &= \eta^{\frac{b\alpha_1}{2}+\frac{3}{8}}(1-\eta)^{\frac{b\alpha_2}{2}-\frac{1}{8}}\int dwdw' f(w,w';\eta)(ww'-w-w'+\eta),\\
\mathcal{F}_2(\eta) &= \eta^{\frac{b\alpha_1}{2}-\frac{1}{8}}(1-\eta)^{\frac{b\alpha_2}{2}+\frac{3}{8}}\int dwdw' f(w,w';\eta)(ww'-\eta),\\
\mathcal{F}_3(\eta) &= \eta^{\frac{b\alpha_1}{2}-\frac{1}{8}}(1-\eta)^{\frac{b\alpha_2}{2}-\frac{1}{8}}\int dwdw' f(w,w';\eta)(ww'-w\eta-w'\eta+\eta),
\end{aligned} \tag{2.82}$$

satisfying the following differential equations

$$\begin{aligned}
2ib^{-1}\mathcal{D}\mathcal{F}_0(\eta) &= p_1(1-\eta)^{1/2}\mathcal{F}_1(\eta)+p_2\eta^{1/2}\mathcal{F}_2(\eta)-p_3\eta^{1/2}(1-\eta)^{1/2}\mathcal{F}_3(\eta),\\
2ib^{-1}\mathcal{D}\mathcal{F}_1(\eta) &= p_1(1-\eta)^{1/2}\mathcal{F}_0(\eta)+p_2\eta^{1/2}\mathcal{F}_3(\eta)-p_3\eta^{1/2}(1-\eta)^{1/2}\mathcal{F}_2(\eta),\\
2ib^{-1}\mathcal{D}\mathcal{F}_2(\eta) &= -p_1(1-\eta)^{1/2}\mathcal{F}_3(\eta)+p_2\eta^{1/2}\mathcal{F}_0(\eta)+p_3\eta^{1/2}(1-\eta)^{1/2}\mathcal{F}_1(\eta),\\
2ib^{-1}\mathcal{D}\mathcal{F}_3(\eta) &= -p_1(1-\eta)^{1/2}\mathcal{F}_2(\eta)+p_2\eta^{1/2}\mathcal{F}_1(\eta)+p_3\eta^{1/2}(1-\eta)^{1/2}\mathcal{F}_0(\eta),\\
\mathcal{D} &\equiv \eta(1-\eta)\left[\partial_\eta+\frac{1-2bQ}{8\eta}+\frac{1-2bQ}{8(\eta-1)}\right].
\end{aligned} \tag{2.83}$$

Hence they can be used to express the solutions of (2.76). They are not yet functions because the contours are not specified. However, a notable property is that, as far as the form of the integrand is concerned, the four transform into one another under the change of integration variables. Some typical ones are given below:

$$\begin{aligned}
w\to 1-w,\quad w'\to 1-w' &: (\mathcal{F}_0,\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)(\eta)\to(-\mathcal{F}_0,\mathcal{F}_2,\mathcal{F}_1,\mathcal{F}_3)(1-\eta)_{(p_1,p_2)\to(p_2,p_1)},\\
w\to\eta w^{-1} &: (\mathcal{F}_0,\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)(\eta)\to(\mathcal{F}_2,\mathcal{F}_3,\mathcal{F}_0,\mathcal{F}_1)(\eta)_{(p_1,p_3)\to-(p_1,p_3)},\\
w'\to\eta w'^{-1} &: (\mathcal{F}_0,\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)(\eta)\to(\mathcal{F}_2,-\mathcal{F}_3,\mathcal{F}_0,-\mathcal{F}_1)(\eta).
\end{aligned} \tag{2.84}$$

Now that we have found a way to express the solutions of the differential equation in an integral form, we have to investigate the property of them under the monodromy and find some formulae for the change of basis like those of hypergeometric functions. This can be done straightforwardly because the functions $\mathcal{F}_i(\eta)$ in the above are all expressible as sums of products of two hypergeometric functions.

By fixing the integration contours and making a suitable rescaling, we define the function $\mathcal{F}_0$ in the following way:

$$\begin{aligned}
\mathcal{F}_0(p_1,p_2,p_3;\eta) \equiv\ & \frac{8\Gamma(\frac{1}{2}+ibp_1)\Gamma(\frac{3}{2})}{\Gamma(\frac{1}{4}+\frac{ib}{2}p_{1+2-3})\Gamma(\frac{1}{4}+\frac{ib}{2}p_{1-2+3})\Gamma(\frac{1}{4})^2}\\
&\times\eta^{\frac{b\alpha_1}{2}+\frac{3}{8}}(1-\eta)^{\frac{b\alpha_2}{2}+\frac{3}{8}}\int_0^1 dwdw'[w'(1-w')(1-w'\eta)]^{-\frac{3}{4}}\\
&\times w^{-\frac{3}{4}+\frac{ib}{2}p_{1+2-3}}(1-w)^{-\frac{3}{4}+\frac{ib}{2}p_{1-2+3}}(1-w\eta)^{-\frac{3}{4}-\frac{ib}{2}p_{1+2+3}}(w'-w).
\end{aligned} \tag{2.85}$$

Using the same normalization and contour to define all the functions $\mathcal{F}_i$, we find $\mathcal{F}_0 = \mathcal{F}_1$ and $\mathcal{F}_2 = \mathcal{F}_3$. Thus we introduce the following functions

$$\begin{aligned}
-4\mathcal{G}_0(p_1,p_2,p_3;\eta) &\equiv \mathcal{F}_0(p_1,p_2,p_3;\eta) = \mathcal{F}_1(p_1,p_2,p_3;\eta) \\
&= \left(\frac{1+2ibp_{1-2+3}}{1+2ibp_1}\right)\eta^{\frac{b\alpha_1}{2}+\frac{3}{8}}(1-\eta)^{\frac{b\alpha_2}{2}+\frac{3}{8}}F(\tfrac{3}{4},\,\tfrac{5}{4},\,\tfrac{3}{2};\,\eta) \\
&\quad \times F(\tfrac{3}{4}+\tfrac{ib}{2}p_{1+2+3},\,\tfrac{1}{4}+\tfrac{ib}{2}p_{1+2-3},\,\tfrac{3}{2}+ibp_1;\,\eta) - (p_{2,3}\to -p_{2,3}), \\
2\mathcal{G}_1(p_1,p_2,p_3;\eta) &\equiv \mathcal{F}_2(p_1,p_2,p_3;\eta) = \mathcal{F}_3(p_1,p_2,p_3;\eta) \\
&= \left(\frac{1+2ibp_{1-2+3}}{1+2ibp_1}\right)\eta^{\frac{b\alpha_1}{2}-\frac{1}{8}}(1-\eta)^{\frac{b\alpha_2}{2}-\frac{1}{8}}F(-\tfrac{1}{4},\,\tfrac{1}{4},\,\tfrac{1}{2};\,\eta) \\
&\quad \times F(\tfrac{3}{4}+\tfrac{ib}{2}p_{1+2+3},\,\tfrac{1}{4}+\tfrac{ib}{2}p_{1+2-3},\,\tfrac{3}{2}+ibp_1;\,\eta) + (p_{2,3}\to -p_{2,3})
\end{aligned} \tag{2.86}$$

We furthermore introduce the notations

$$\mathcal{G}_2(p_1,p_2,p_3;\eta) = \mathcal{G}_0(-p_1,p_2,-p_3;\eta), \quad \mathcal{G}_3(p_1,p_2,p_3;\eta) = \mathcal{G}_1(-p_1,p_2,-p_3;\eta), \tag{2.87}$$

so that the solution of the differential equations for $f^{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}$ of (2.67) be linear combinations of the functions $\mathcal{G}_i(p_1,p_2,p_3;\eta)$ or of $\mathcal{G}_i(p_1,p_2,-p_3;\eta)$. Note that the hypergeometric functions in the above which do not depend on p_i can also be written as

$$\begin{aligned}
F(\tfrac{3}{4},\tfrac{5}{4},\tfrac{3}{2};\eta) &= (1-\eta)^{-1/2}\left(\frac{2}{1+\sqrt{1-\eta}}\right)^{1/2}, \\
F(-\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{2};\eta) &= \left(\frac{1+\sqrt{1-\eta}}{2}\right)^{1/2},
\end{aligned} \tag{2.88}$$

so that they are related to the four-point functions of spin operators in Ising model[32]. The above functions $\mathcal{G}_)$ are not single valued on the entire $\mathbf{CP}^1$, and the basis of solutions are chosen so as to diagonalize the monodromy around $\eta = 0$. Our next task is to find the transition coefficients giving a relation between different bases. Using the formula for hypergeometric functions they are given by

$$\begin{pmatrix} \mathcal{G}_0(p_1,p_2,p_3;\eta) \\ \mathcal{G}_1(p_1,p_2,p_3;\eta) \\ \mathcal{G}_2(p_1,p_2,p_3;\eta) \\ \mathcal{G}_3(p_1,p_2,p_3;\eta) \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -x_{+-} & -x_{+-} & x_{++} & x_{++} \\ -x_{+-} & x_{+-} & -x_{++} & x_{++} \\ x_{--} & -x_{--} & -x_{-+} & x_{-+} \\ x_{--} & x_{--} & x_{-+} & x_{-+} \end{pmatrix}\begin{pmatrix} \mathcal{G}_0(p_2,p_1,p_3;1-\eta) \\ \mathcal{G}_1(p_2,p_1,p_3;1-\eta) \\ \mathcal{G}_2(p_2,p_1,p_3;1-\eta) \\ \mathcal{G}_3(p_2,p_1,p_3;1-\eta) \end{pmatrix} \tag{2.89}$$

$$\begin{aligned}
x_{++} &= \frac{\Gamma(\frac{1}{2}+ibp_1)\Gamma(\frac{1}{2}+ibp_2)}{\Gamma(\frac{3}{4}+\frac{ib}{2}p_{1+2+3})\Gamma(\frac{1}{4}+\frac{ib}{2}p_{1+2-3})}, \\
x_{+-} &= \frac{\Gamma(\frac{1}{2}+ibp_1)\Gamma(\frac{1}{2}-ibp_2)}{\Gamma(\frac{3}{4}+\frac{ib}{2}p_{1-2-3})\Gamma(\frac{1}{4}+\frac{ib}{2}p_{1-2+3})}, \\
x_{-+} &= \frac{\Gamma(\frac{1}{2}-ibp_1)\Gamma(\frac{1}{2}+ibp_2)}{\Gamma(\frac{3}{4}+\frac{ib}{2}p_{-1+2-3})\Gamma(\frac{1}{4}+\frac{ib}{2}p_{-1+2+3})}, \\
x_{--} &= \frac{\Gamma(\frac{1}{2}-ibp_1)\Gamma(\frac{1}{2}-ibp_2)}{\Gamma(\frac{3}{4}+\frac{ib}{2}p_{-1-2+3})\Gamma(\frac{1}{4}+\frac{ib}{2}p_{-1-2-3})}.
\end{aligned} \tag{2.90}$$

Using the above formulae we shall then try to find monodromy invariant combinations of the left and right sectors. First, there is a "diagonal" product of the following form:

$$\begin{aligned}&\Big[x_{--}x_{-+}\Big(\mathcal{G}_0\bar{\mathcal{G}}_0+\mathcal{G}_1\bar{\mathcal{G}}_1\Big)+x_{+-}x_{++}\Big(\mathcal{G}_2\bar{\mathcal{G}}_2+\mathcal{G}_3\bar{\mathcal{G}}_3\Big)\Big](p_1,p_2,p_3;\eta)\\ &=\ \Big[x_{--}x_{+-}\Big(\mathcal{G}_0\bar{\mathcal{G}}_0+\mathcal{G}_1\bar{\mathcal{G}}_1\Big)+x_{-+}x_{++}\Big(\mathcal{G}_2\bar{\mathcal{G}}_2+\mathcal{G}_3\bar{\mathcal{G}}_3\Big)\Big](p_2,p_1,p_3;1-\eta)\end{aligned}\tag{2.91}$$

where we denoted $\bar{\mathcal{G}}_i(p_i;\eta)\equiv\mathcal{G}_i(p_i;\bar{\eta})$. Up to an overall constant, it has the following asymptotic behavior at $\eta\sim 0$:

$$\begin{aligned}&\left(\frac{ibp_{2-3}}{1+2ibp_1}\right)^2|\eta|^{b\alpha_1+\frac{3}{4}}+|\eta|^{b(Q-\alpha_1)-\frac{1}{4}}\gamma(\tfrac{1}{2}+ibp_1)^2\gamma(\tfrac{1}{4}+\tfrac{ib}{2}p_{-1+2+3})\gamma(\tfrac{1}{4}+\tfrac{ib}{2}p_{-1-2-3})\\ &\qquad\times\gamma(\tfrac{3}{4}+\tfrac{ib}{2}p_{-1+2-3})\gamma(\tfrac{3}{4}+\tfrac{ib}{2}p_{-1-2+3})\\ &+|\eta|^{b\alpha_1-\frac{1}{4}}+|\eta|^{b(Q-\alpha_1)+\frac{3}{4}}\left(\frac{ibp_{2+3}}{1-2ibp_1}\right)^2\gamma(\tfrac{1}{2}+ibp_1)^2\gamma(\tfrac{1}{4}+\tfrac{ib}{2}p_{-1+2+3})\gamma(\tfrac{1}{4}+\tfrac{ib}{2}p_{-1-2-3})\\ &\qquad\times\gamma(\tfrac{3}{4}+\tfrac{ib}{2}p_{-1+2-3})\gamma(\tfrac{3}{4}+\tfrac{ib}{2}p_{-1-2+3})\end{aligned}\tag{2.92}$$

Let us compare the asymptotic behavior of the above solution with the previous OPE analysis based on the assumption that $\Theta^{\epsilon\bar{\epsilon}}_{-b/2}\Theta^{\tau\bar{\tau}}_{\alpha}$ is expanded into two discrete terms. Apart from the coordinate dependences which are irrelevant, the four-point functions should obey the following asymptotic behavior:

$$\begin{aligned}&\Big\langle\Theta^{--}_{-b/2}(z_0)\Theta^{++}_{\alpha_1}(z_1)\Theta^{++}_{\alpha_2}(z_2)\Theta^{++}_{\alpha_3}(z_3)\Big\rangle\\ &\quad\sim\ |z_{01}|^{b\alpha_1+\frac{3}{4}}\left(\frac{ibp_{2-3}}{1+2ibp_1}\right)^2\tilde{C}_1(\alpha_1-\tfrac{b}{2};\alpha_2,\alpha_3)+|z_{01}|^{b(Q-\alpha_1)-\frac{1}{4}}\frac{\tilde{C}_-(\alpha_1)\tilde{C}_3(\alpha_1+\frac{b}{2};\alpha_2,\alpha_3)}{2i}\\ &\Big\langle\Theta^{++}_{-b/2}(z_0)\Theta^{++}_{\alpha_1}(z_1)\Theta^{++}_{\alpha_2}(z_2)\Theta^{++}_{\alpha_3}(z_3)\Big\rangle\\ &\quad\sim\ |z_{01}|^{b\alpha_1-\frac{1}{4}}\tilde{C}_3(\alpha_1-\tfrac{b}{2};\alpha_2,\alpha_3)+|z_{01}|^{b(Q-\alpha_1)+\frac{3}{4}}\left(\frac{ibp_{2-3}}{1-2ibp_1}\right)^2\frac{\tilde{C}_-(\alpha_1)\tilde{C}_1(\alpha_1+\frac{b}{2};\alpha_2,\alpha_3)}{2i}\end{aligned}\tag{2.93}$$

Comparing them with (2.92) we find that the crossing symmetric solution (2.91) of the differential equation corresponds to the "sum" of four-point functions

$$\begin{aligned}&x_{--}x_{-+}(\mathcal{G}_0\bar{\mathcal{G}}_0+\mathcal{G}_1\bar{\mathcal{G}}_1)+x_{+-}x_{++}(\mathcal{G}_2\bar{\mathcal{G}}_2+\mathcal{G}_3\bar{\mathcal{G}}_3)\\ &\quad\sim\ \Big\langle\Theta^{--}_{-b/2}\Theta^{++}_{\alpha_1}\Theta^{++}_{\alpha_2}\Theta^{++}_{\alpha_3}\Big\rangle+\Big\langle\Theta^{++}_{-b/2}\Theta^{--}_{\alpha_1}\Theta^{++}_{\alpha_2}\Theta^{++}_{\alpha_3}\Big\rangle\\ &\qquad+\Big\langle\Theta^{++}_{-b/2}\Theta^{++}_{\alpha_1}\Theta^{--}_{\alpha_2}\Theta^{++}_{\alpha_3}\Big\rangle+\Big\langle\Theta^{++}_{-b/2}\Theta^{++}_{\alpha_1}\Theta^{++}_{\alpha_2}\Theta^{--}_{\alpha_3}\Big\rangle.\end{aligned}\tag{2.94}$$

By flipping the momentum of one of the three generic spin fields we obtain another diagonal type solution of the differential equation, and these two are the only diagonal solutions we could find. One might think that a more careful analysis would lead to another solution and thereby enable us to write down the expression for each of the summand in the above. However, there should not be any more solutions once we admit that all the solutions are expressed in terms of $\mathcal{G}_i(p_i;\eta)$. It is also difficult to argue that the solutions be duplicated due to the ambiguity in choosing the square-root branches. Thus we conclude that there are only two independent solutions of diagonal type.

The above result seems to indicate that in correlation functions involving spin fields, all one can fix by hand is the total chirality of the product of spin fields and not the chirality of each spin field. Nevertheless, our previous analysis of two- and three-point structure constants still remains valid since they involve no more than two spin fields.

Finally, there are also solutions of the off-diagonal type:

$$\begin{aligned}&\left[x_{--}x_{-+}(\mathcal{G}_0\bar{\mathcal{G}}_1-\mathcal{G}_1\bar{\mathcal{G}}_0)-x_{+-}x_{++}(\mathcal{G}_2\bar{\mathcal{G}}_3-\mathcal{G}_3\bar{\mathcal{G}}_2)\right](p_1,p_2,p_3;\eta)\\ &\quad= \quad -\left[x_{--}x_{+-}(\mathcal{G}_0\bar{\mathcal{G}}_1-\mathcal{G}_1\bar{\mathcal{G}}_0)-x_{-+}x_{++}(\mathcal{G}_2\bar{\mathcal{G}}_3-\mathcal{G}_3\bar{\mathcal{G}}_2)\right](p_2,p_1,p_3;1-\eta)\\ &\quad\sim \quad \left\langle\Theta^{--}_{-b/2}\Theta^{+-}_{\alpha_1}\Theta^{+-}_{\alpha_2}\Theta^{++}_{\alpha_3}\right\rangle+\left\langle\Theta^{++}_{-b/2}\Theta^{-+}_{\alpha_1}\Theta^{+-}_{\alpha_2}\Theta^{++}_{\alpha_3}\right\rangle\\ &\qquad+\left\langle\Theta^{++}_{-b/2}\Theta^{+-}_{\alpha_1}\Theta^{-+}_{\alpha_2}\Theta^{++}_{\alpha_3}\right\rangle+\left\langle\Theta^{++}_{-b/2}\Theta^{+-}_{\alpha_1}\Theta^{+-}_{\alpha_2}\Theta^{--}_{\alpha_3}\right\rangle. \end{aligned} \tag{2.95}$$

3. Super Liouville Theory with Boundary

If the worldsheet has a boundary, the action has boundary terms:

$$\begin{aligned} I &= \frac{1}{2\pi}\int_\Sigma d^2z\left[\partial\phi\bar\partial\phi+\tfrac{QR\phi}{4}+\psi\bar\partial\psi+\bar\psi\partial\bar\psi\right]+2i\mu b^2\int_\Sigma d^2z\psi\bar\psi e^{b\phi}\\ &\quad+\oint_{\partial\Sigma}dx\left[\tfrac{QK\phi}{4\pi}+a\mu_B b\psi e^{b\phi/2}\right], \end{aligned} \tag{3.1}$$

where K is the curvature of the boundary which is defined by the Euler number formula

$$\frac{1}{4\pi}\int_\Sigma\sqrt{g}R+\frac{1}{2\pi}\int_{\partial\Sigma}g^{1/4}K=\chi=2-2g-h \tag{3.2}$$

for worldsheets Σ with g handles and h holes. The coupling μ_B will be referred to as the boundary cosmological constant as in the bosonic case. It can take different values for different connected components of the boundary, and it may also jump at points where the boundary primary operators are inserted. The boundary interaction term contains a Grassmann odd constant a satisfying $a^2=1$, in order to avoid the Lagrangian becoming Grassmann odd[33, 34, 19].

On worldsheets with boundary we can insert operators on the boundary. One of the most fundamental boundary operator is

$$B_\beta(x)=e^{\beta\phi/2}(x)=e^{\beta\phi_L}(x), \tag{3.3}$$

and of course there are some other operators like

$$\psi B_\beta=\psi e^{\beta\phi_L},\quad \Theta^\pm_\beta=\sigma^\pm e^{\beta\phi_L}. \tag{3.4}$$

In free-field scheme the correlators of bulk- and boundary fields on the upper half-plane can be calculated as usual using mirror image techniques. One can also perform the integration over the zero-mode of ϕ and derive that any correlator diverges when

$$Q(2-2g-h)-2\sum_{i\ (\text{bulk})}\alpha_i\ -\ \sum_{j\ (\text{boundary})}\beta_j\ =\ b(2n+n_B) \tag{3.5}$$

for non-negative integers n and n_B, and the residue is given by a sum of free field correlators with n bulk and n_B boundary screening operators.

3.1. Classification of boundary states

The classification of boundary states can be done by studying the modular property of annulus partition functions. We follow the same path as the reference [2] which analyzed the bosonic Liouville theory. We first introduce the Ishibashi states, and then classify all the possible Cardy states by expressing them as superpositions of Ishibashi states. Note that in super Liouville theory there is a freedom in choosing the spin structure on the worldsheet, so that we have to consider the characters with $(-)^F$ inserted as well as the ordinary ones.

Boundary states in generic superconformal field theory are defined as the solutions of the boundary condition on currents on the real axis:

$$T(z) = \bar{T}(\bar{z}), \quad T_F(z) = \zeta \bar{T}_F(\bar{z}). \tag{3.6}$$

Mapping the upper half-plane onto a unit disc the above condition can be rewritten into the following form

$$L_n = \bar{L}_{-n}, \quad G_r = -i\zeta \bar{G}_{-r}. \tag{3.7}$$

The boundary states $|B;\zeta\rangle$ and $\langle B;\zeta|$ are therefore solutions of the equations

$$\begin{aligned} \langle B;\zeta|\,(L_n - \bar{L}_{-n}) &= \langle B;\zeta|\,(G_r + i\zeta\bar{G}_{-r}) &= 0, \\ (L_n - \bar{L}_{-n})\,|B;\zeta\rangle &= (G_r - i\zeta\bar{G}_{-r})\,|B;\zeta\rangle &= 0. \end{aligned} \tag{3.8}$$

The NS (R) boundary states satisfy the above condition with $r \in \mathbf{Z} + \frac{1}{2}$ ($r \in \mathbf{Z}$).

Given a highest weight state $|h;\mathrm{NS}\rangle$ of the superconformal algebra, one can construct the Ishibashi state $|h;+,\zeta\rangle\!\rangle_{\mathrm{I}}$ in the following way:

$$|h;+,\zeta\rangle\!\rangle_{\mathrm{I}} = |h;\mathrm{NS}\rangle_L\,|h;\mathrm{NS}\rangle_R + (\text{descendants}) \tag{3.9}$$

In the same way one can construct the Ishibashi state $|h;-,\zeta\rangle\!\rangle_{\mathrm{I}}$ from a highest weight state $|h;\mathrm{R}^{\pm}\rangle$ in the R sector:

$$\begin{aligned} |h;-,\zeta\rangle\!\rangle_{\mathrm{I}} &= |h;\mathrm{R}^+\rangle_L\,|h;\mathrm{R}^+\rangle_R - i\zeta\,|h;\mathrm{R}^-\rangle_L\,|h;\mathrm{R}^-\rangle_R + (\text{descendants}), \\ {}_{\mathrm{I}}\langle\!\langle h;-,\zeta\,| &= \langle h;\mathrm{R}^+|_R\,\langle h;\mathrm{R}^+|_L + i\zeta\,\langle h;\mathrm{R}^-|_R\,\langle h;\mathrm{R}^-|_L + (\text{descendants}), \end{aligned} \tag{3.10}$$

where we assumed $\langle h;\mathrm{R}^+|h;\mathrm{R}^+\rangle = \langle h;\mathrm{R}^-|h;\mathrm{R}^-\rangle$ to be nonzero. Note that we will only consider the combinations of even total chirality in what follows. From these definitions it follows that the annulus partition function bounded by two Ishibashi states is given by the character:

$$\begin{aligned} {}_{\mathrm{I}}\langle\!\langle h;+,\zeta\,|\,e^{i\pi\tau_c(L_0+\bar{L}_0-\frac{c}{12})}\,|k;+,\zeta'\rangle\!\rangle_{\mathrm{I}} &= \delta_{h,k}\mathrm{Tr}_{h(\mathrm{NS})}[e^{2i\pi\tau_c(L_0-\frac{c}{24})}(\zeta\zeta')^F], \\ {}_{\mathrm{I}}\langle\!\langle h;-,\zeta\,|\,e^{i\pi\tau_c(L_0+\bar{L}_0-\frac{c}{12})}\,|k;-,\zeta'\rangle\!\rangle_{\mathrm{I}} &= \delta_{h,k}\mathrm{Tr}_{h(\mathrm{R})}[e^{2i\pi\tau_c(L_0-\frac{c}{24})}(\zeta\zeta')^F]. \end{aligned} \tag{3.11}$$

Here the expression is symbolic in the sense that the delta symbol $\delta_{h,k}$ merely represents the Ishibashi states diagonalize the annulus partition function as seen from the closed string channel. R Ishibashi states and NS Ishibashi states are orthogonal to each other.

Cardy states are defined by the property that the multiplicity of open string modes between two of them is given by the fusion coefficient $\mathcal{N}_{h,k}^{\ \ l}$. In superconformal field theories they are given by the sum of NS and R pieces:

$$
{}_{\mathrm{C}}\langle\langle h;\zeta\,| = {}_{\mathrm{C}}\langle\langle h;+,\zeta\,| + {}_{\mathrm{C}}\langle\langle h;-,\zeta\,| , \qquad |\,h;\zeta\rangle\rangle_{\mathrm{C}} = |\,h;+,\zeta\rangle\rangle_{\mathrm{C}} + |\,h;-,\zeta\rangle\rangle_{\mathrm{C}} \,. \tag{3.12}
$$

NS (R) piece of any Cardy state is itself a solution of boundary condition on currents, and should be expressed as a superposition of NS (R) Ishibashi states. The partition function on an annulus bounded by two of them as seen from the open string channel is expressed as the sum of characters with coefficients $\mathcal{N}_{h,k}^{\ \ l}$

$$
Z_{(h,\pm,\zeta),(k,\pm,\zeta')}(\tau_o) \ = \ \left\{ \begin{array}{ll} \mathcal{N}_{h,k}^{\ \ l}\mathrm{Tr}_{l(\mathrm{NS})}[e^{2i\pi\tau_o(L_0-\frac{c}{24})}(\pm 1)^F] & (\zeta=\zeta'), \\ \mathcal{N}_{h,k}^{\ \ l}\mathrm{Tr}_{l(\mathrm{R})}[e^{2i\pi\tau_o(L_0-\frac{c}{24})}(\pm 1)^F] & (\zeta=-\zeta'). \end{array}\right. \tag{3.13}
$$

Let us then consider the boundary states in super Liouville theory. First of all, as the labels h of representations of the superconformal algebra, we use the Liouville momentum p for non-degenerate representations or the index (r,s) for degenerate ones. The characters for non-degenerate representations are given by ($q\equiv e^{2\pi i\tau}$)

$$
\begin{aligned}
\mathrm{Tr}_{p(\mathrm{NS})}q^{L_0-\frac{c}{24}} &= \chi^+_{p(\mathrm{NS})}(\tau) = q^{\frac{p^2}{2}-\frac{1}{16}}\textstyle\prod_{n=1}(1-q^n)^{-1}(1+q^{n-\frac{1}{2}}),\\
\mathrm{Tr}_{p(\mathrm{NS})}(-)^F q^{L_0-\frac{c}{24}} &= \chi^-_{p(\mathrm{NS})}(\tau) = q^{\frac{p^2}{2}-\frac{1}{16}}\textstyle\prod_{n=1}(1-q^n)^{-1}(1-q^{n-\frac{1}{2}}),\\
\mathrm{Tr}_{p(\mathrm{R})}q^{L_0-\frac{c}{24}} &= \chi^+_{p(\mathrm{R})}(\tau) = 2q^{\frac{p^2}{2}}\textstyle\prod_{n=1}(1-q^n)^{-1}(1+q^n).
\end{aligned} \tag{3.14}
$$

Note that $\mathrm{Tr}_{\mathrm{R}}(-)^F q^{L_0-\frac{c}{24}}$ vanishes for any representations except for the R vacuum. They obey the following modular transformation property

$$
\begin{aligned}
\chi^+_{u(\mathrm{NS})}(\tau) &= \int_{-\infty}^{\infty} dp e^{2i\pi pu}\chi^+_{p(\mathrm{NS})}(-1/\tau),\\
2^{1/2}\chi^-_{u(\mathrm{NS})}(\tau) &= \int_{-\infty}^{\infty} dp e^{2i\pi pu}\chi^+_{p(\mathrm{R})}(-1/\tau),\\
2^{-1/2}\chi^+_{u(\mathrm{R})}(\tau) &= \int_{-\infty}^{\infty} dp e^{2i\pi pu}\chi^-_{p(\mathrm{NS})}(-1/\tau).
\end{aligned} \tag{3.15}
$$

For degenerate representations, the characters are given by those of corresponding Verma modules subtracted by those of null submodules:

$$
\begin{aligned}
\chi^+_{r,s(\mathrm{NS})} &= \chi^+_{\frac{i}{2}(rb+sb^{-1})\ (\mathrm{NS})} - \chi^+_{\frac{i}{2}(rb-sb^{-1})\ (\mathrm{NS})},\\
\chi^-_{r,s(\mathrm{NS})} &= \chi^-_{\frac{i}{2}(rb+sb^{-1})\ (\mathrm{NS})} - (-)^{rs}\chi^-_{\frac{i}{2}(rb-sb^{-1})\ (\mathrm{NS})},\\
\chi^+_{r,s(\mathrm{R})} &= \chi^-_{\frac{i}{2}(rb+sb^{-1})\ (\mathrm{R})} - \chi^-_{\frac{i}{2}(rb-sb^{-1})\ (\mathrm{R})}.
\end{aligned} \tag{3.16}
$$

We would like to find the expression for the wave functions $\Psi(p;h_\zeta)$ which express the NS and R Cardy states as superpositions of Ishibashi states belonging to normalizable representations:

$$
\begin{aligned}
{}_{\rm C}\langle\langle h;\pm,\zeta| &= 2\int_0^\infty \frac{dp}{2\pi}\Psi_\pm(p;h_\zeta)\,{}_{\rm I}\langle\langle p;\pm,\zeta|\,, \\
|h;\pm,\zeta\rangle\rangle_{\rm C} &= 2\int_0^\infty \frac{dp}{2\pi}\,|p;\pm,\zeta\rangle\rangle_{\rm I}\,\Psi^\dagger_\pm(p;h_\zeta).
\end{aligned}
\tag{3.17}
$$

Here we have taken care for the equivalence of representations with momentum p and $-p$. The Ishibashi states are normalized to satisfy

$$
\begin{aligned}
{}_{\rm I}\langle\langle p,+,\zeta|\,e^{i\pi\tau_c(L_0+\bar{L}_0-\frac{c}{12})}\,|p',+,\zeta'\rangle\rangle_{\rm I} &= 2\pi\delta(p-p')\chi^{\zeta\zeta'}_{p({\rm NS})}(\tau_c), \\
{}_{\rm I}\langle\langle p,-,\zeta|\,e^{i\pi\tau_c(L_0+\bar{L}_0-\frac{c}{12})}\,|p',-,\zeta'\rangle\rangle_{\rm I} &= \sqrt{2}\pi\delta(p-p')\chi^{\zeta\zeta'}_{p({\rm R})}(\tau_c).
\end{aligned}
\tag{3.18}
$$

We also assume that $\Psi^\dagger_\pm(p;h_\zeta)=\Psi_\pm(-p;h_\zeta)$.

There is one important notice regarding the equivalence of R Ishibashi states under the reflection $p\to -p$. Recall that the highest weight states in the R sector can be created by multiplying a spin operator onto the vacuum. Therefore, if we write the R Ishibashi states as in (3.10), we obtain

$$
\begin{aligned}
{}_{\rm I}\langle\langle p;-,\zeta| &= \left\langle\Theta^{++}_{Q/2+ip}\right| - \zeta\left\langle\Theta^{--}_{Q/2+ip}\right| + (\text{descendants}), \\
|p;-,\zeta\rangle\rangle_{\rm I} &= \left|\Theta^{++}_{Q/2-ip}\right\rangle - \zeta\left|\Theta^{--}_{Q/2-ip}\right\rangle + (\text{descendants}).
\end{aligned}
\tag{3.19}
$$

As a consequence, if $\zeta=+1$ there arises a minus sign in flipping the sign of p. Hence the wave functions for R Cardy states should depend also on how the supercurrents in the left- and the right sectors are glued: they must be odd functions of p for $\zeta=1$ and even functions for $\zeta=-1$. On the other hand, there is no such subtleties for NS Ishibashi states so that one may well expect that the NS wave functions do not depend on ζ.

The open/closed duality for annulus partition functions together with the obvious fusion relation $\mathcal{N}_{(1,1),h}^{\;\;k}=\delta_h^k$ yields

$$
\begin{aligned}
\Psi_+(p;u_\zeta)\Psi_+(-p;(1,1)_\zeta) &= \pi\cos(2\pi pu), \\
\Psi_+(p;(r,s)_\zeta)\Psi_+(-p;(1,1)_\zeta) &= 2\pi\sinh(\pi prb)\sinh(\pi ps/b), \\
\Psi_-(p;u_\zeta)\Psi_-(-p;(1,1)_\zeta) &= \pi\cos(2\pi pu), \\
\Psi_-(p;(r,s)_\zeta)\Psi_-(-p;(1,1)_\zeta) &= 2\pi\sinh(\pi prb+\tfrac{i\pi rs}{2})\sinh(\tfrac{\pi ps}{b}-\tfrac{i\pi rs}{2})
\end{aligned}
\tag{3.20}
$$

Another equation comes from the fact that the wave functions $\Psi_\pm(p;h_\zeta)$ are proportional to the disc one-point functions of $V_{\frac{Q}{2}+ip}$ or $\Theta_{\frac{Q}{2}+ip}$. Therefore they must be consistent with the reflection relations:

$$
\Psi_+(p;h_\zeta)=D(\tfrac{Q}{2}+ip)\Psi_+(-p;h_\zeta),\qquad \Psi_-(p;h_\zeta)=-\zeta\tilde{D}(\tfrac{Q}{2}+ip)\Psi_-(-p;h_\zeta).
\tag{3.21}
$$

These equations determine the form of almost all the wave functions:

$$
\Psi_+(p;(1,1)_\zeta) \;=\; 2^{\frac12}\pi^{\frac32}(\mu\pi\gamma(\tfrac{bQ}{2}))^{-ip/b}\left\{ip\Gamma(-ipb)\Gamma(-\tfrac{ip}{b})\right\}^{-1},
$$

$$\begin{aligned}
\Psi_+(p;(r,s)_\zeta) &= -2^{\frac{1}{2}}\pi^{-\frac{1}{2}}(\mu\pi\gamma(\tfrac{bQ}{2}))^{-ip/b}\cdot ip\Gamma(ipb)\Gamma(\tfrac{ip}{b})\sinh(\pi prb)\sinh(\pi ps/b),\\
\Psi_+(p;u_\zeta) &= -2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}(\mu\pi\gamma(\tfrac{bQ}{2}))^{-ip/b}\cdot ip\Gamma(ipb)\Gamma(\tfrac{ip}{b})\cos(2\pi pu),\\
\Psi_-(p;(1,1)_-) &= 2^{\frac{1}{2}}\pi^{\frac{3}{2}}(\mu\pi\gamma(\tfrac{bQ}{2}))^{-ip/b}\left\{\Gamma(\tfrac{1}{2}-ipb)\Gamma(\tfrac{1}{2}-\tfrac{ip}{b})\right\}^{-1},\\
\Psi_-(p;(r,s)_-) &= 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}(\mu\pi\gamma(\tfrac{bQ}{2}))^{-ip/b}\Gamma(\tfrac{1}{2}+ipb)\Gamma(\tfrac{1}{2}+\tfrac{ip}{b})\\
&\qquad\times\sinh(\pi prb+\tfrac{i\pi rs}{2})\sinh(\tfrac{\pi ps}{b}-\tfrac{i\pi rs}{2}),\\
\Psi_-(p;u_-) &= 2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}(\mu\pi\gamma(\tfrac{bQ}{2}))^{-ip/b}\Gamma(\tfrac{1}{2}+ipb)\Gamma(\tfrac{1}{2}+\tfrac{ip}{b})\cos(2\pi pu).
\end{aligned}\tag{3.22}$$

All we are left with is the R wave functions $\Psi_-(p;h_+)$. However, due to the requirement that they must be a product of Gamma functions multiplied by an *odd* function of p, one can actually find no analytic expressions for them. Consequently, one should conclude that the R wave functions cannot be found by analyzing the modular property. They will be proposed later by an analysis of one-point function on a disc and, moreover, we will find that the degenerate Cardy states $(r,s)_\zeta$ must satisfy $r+s=\text{even(odd)}$ for $\zeta=-1(+1)$. If we accept this, the absence of $(1,1)_+$ state explains why we could not find the wave functions the Cardy states with $\zeta=1$ from the modular property.

It might seem strange that one can find the R Cardy states with $\zeta=-1$ only, because naively one tends to think that the two choices for the boundary conditions on supercurrent should be equivalent. However, it turned out that the two choices are actually inequivalent and affects the parity of the wave functions under the sign-change of the momentum p.

3.2. One-point functions of bulk operators

Let us try to reproduce these wave functions from a different approach, by calculating the one-point functions on a disc. We define various one-point structure constants by the equations

$$\begin{aligned}
\langle V_\alpha(z)\rangle_{u_\zeta} &= U_+(\alpha;u_\zeta)|z-\bar{z}|^{-2h_\alpha},\\
\langle\Theta_\alpha^{\epsilon\epsilon}(z)\rangle_{u_\zeta} &= U_-(\alpha,\epsilon;u_\zeta)|z-\bar{z}|^{-2h_\alpha-\frac{1}{8}}.
\end{aligned}\tag{3.23}$$

The one-point functions of spin fields $\langle\Theta_\alpha^{\epsilon,-\epsilon}\rangle$ always vanish because we restricted the R boundary states to have even total chirality in (3.10). The periodicity of supercurrents when we go around the boundary of the disc is unambiguously determined by how many spin fields are inserted on the disc. All the other one-point functions are zero or obtained by superconformal transformations from the above ones. For example, the one-point function of descendants in the NS sector is given by

$$\left\langle\psi\bar{\psi}V_\alpha(z)\right\rangle_{u_\zeta} = i\zeta\cdot(Q-\alpha)\alpha^{-1}U_+(\alpha;u_\zeta)|z-\bar{z}|^{-2h_\alpha-1},\tag{3.24}$$

and the one-point functions of spin fields depend on the index ϵ in the following way:

$$\langle\Theta_\alpha^{\epsilon\epsilon}(z)\rangle_{u_\zeta} = -\zeta\left\langle\Theta_\alpha^{-\epsilon,-\epsilon}(z)\right\rangle_{u_\zeta},\tag{3.25}$$

in consistency with the previous analysis of modular property.

To obtain the one-point structure constants, we derive a set of recursion relations for them from the solution of differential equation for two-point functions with one degenerate operator. Let us first consider

$$\begin{aligned}\left\langle V_\alpha(z)\Theta^{\epsilon\epsilon}_{-b/2}(w)\right\rangle_{u_\zeta} &= |w-\bar{z}|^{-4h_{-b/2}-\frac{1}{4}}|z-\bar{z}|^{2h_{-b/2}-2h_\alpha-\frac{1}{8}}\times \\ &\times\Big\{C_+(\alpha)U_-(\alpha-\tfrac{b}{2},\epsilon;u_\zeta)\eta^{\frac{b\alpha}{2}}(1-\eta)^{-\frac{b^2}{4}+\frac{3}{8}}F(\tfrac{1-b^2}{2}+ibp_\alpha,\,\tfrac{1-b^2}{2},\,1+ibp_\alpha;\eta) \\ &+C_-(\alpha)U_-(\alpha+\tfrac{b}{2},-\epsilon;u_\zeta)\eta^{\frac{b(Q-\alpha)}{2}}(1-\eta)^{-\frac{b^2}{4}+\frac{3}{8}}F(\tfrac{1-b^2}{2}-ibp_\alpha,\,\tfrac{1-b^2}{2},\,1-ibp_\alpha;\eta)\Big\},\end{aligned} \tag{3.26}$$

where $\eta=\frac{|z-w|^2}{|z-\bar{w}|^2}$ and $C_\pm(\alpha)$ are the OPE coefficients defined in (2.25). The coefficients are chosen so that the behavior when the two bulk operators approach each other agrees with the OPE analysis. On the other hand, when $\Theta^{\epsilon\epsilon}_{-b/2}$ approaches the boundary, it should be expanded as a discrete sum of boundary degenerate operators:

$$\Theta^{\epsilon\epsilon}_{-b/2}(w)\to|w-\bar{w}|^{-2h_{-b/2}+h_{-b}+\frac{3}{8}}r_+\psi B_{-b}(w)+|w-\bar{w}|^{-2h_{-b/2}-\frac{1}{8}}r_-B_0(w) \tag{3.27}$$

with certain coefficients $r_\pm$. Comparing this with the behavior of the solution around $\eta\sim 1$ we obtain a recursion relation:

$$\begin{aligned}&C_+(\alpha)U_-(\alpha-\tfrac{b}{2},\epsilon;u_\zeta)\frac{\Gamma(b\alpha+\frac{1-b^2}{2})\Gamma(-b^2)}{\Gamma(b\alpha-b^2)\Gamma(\frac{1-b^2}{2})} \\ &+C_-(\alpha)U_-(\alpha+\tfrac{b}{2},-\epsilon;u_\zeta)\frac{\Gamma(\frac{3+b^2}{2}-b\alpha)\Gamma(-b^2)}{\Gamma(1-b\alpha)\Gamma(\frac{1-b^2}{2})} = r_-(\epsilon,\zeta)U_+(\alpha;u_\zeta)\end{aligned} \tag{3.28}$$

The coefficient r_- can be calculated using free fields

$$r_-=-ab\mu_B|w-\bar{w}|^{-\frac{3b^2}{4}-\frac{3}{8}}\int dx\left\langle\Theta^{\epsilon\epsilon}_{-b/2}(w)\psi B_b(x)B_Q(y)\right\rangle=-\sqrt{2}\pi\hat{r}b\mu_B\Gamma(-b^2)\Gamma(\tfrac{1-b^2}{2})^{-2}, \tag{3.29}$$

where $\hat{r}$ is related to the free field correlator on a disc:

$$\langle\psi(x)\sigma^{\epsilon\epsilon}(w)\rangle_\zeta=2^{-1/2}a\hat{r}(\epsilon,\zeta)|x-w|^{-1}|w-\bar{w}|^{\frac{3}{8}}. \tag{3.30}$$

For this correlator to be non-vanishing, we have to identify σ^ϵ with $\bar{\sigma}^{-\epsilon}$ on the real axis up to some constants.

Another recursion relation can be obtained from the analysis of the correlation functions of two spin fields on a disc:

$$\begin{aligned}2i\left\langle\Theta^{-\epsilon,-\epsilon}_{-b/2}(z)\Theta^{\epsilon\epsilon}_\alpha(w)\right\rangle_{u_\zeta} &= |z-\bar{w}|^{-4h_{-b/2}-\frac{1}{4}}|w-\bar{w}|^{-2h_\alpha+2h_{-b/2}}\times \\ &\Big\{-2i\zeta\tilde{C}_+(\alpha)U_+(\alpha-\tfrac{b}{2};u_\zeta)\mathcal{G}_0(p_\alpha,p_{-\frac{b}{2}},-p_\alpha;\eta) \\ &+\tilde{C}_-(\alpha)U_+(\alpha+\tfrac{b}{2};u_\zeta)\mathcal{G}_3(p_\alpha,p_{-\frac{b}{2}},-p_\alpha;\eta)\Big\},\end{aligned} \tag{3.31}$$

where $\eta=\left|\frac{z-w}{z-\bar{w}}\right|^2$ and the coefficients are determined from the consistency with the OPE of two spin fields, as before. Note that there seems to be another possibility of writing down the solution using $\mathcal{G}_i(p_\alpha,p_{-b/2},p_\alpha)$ instead of $\mathcal{G}_i(p_\alpha,p_{-b/2},-p_\alpha)$. However, it will not lead to a recursion relation consistent with the analysis of modular property. As was discussed in the previous section, it seems that we must use appropriate solutions of differential equation according to the total chirality.

Around $\eta \sim 1$ the above solution can be expressed as a certain linear combination of $\mathcal{G}_i(p_{-b/2}, p_\alpha, -p_\alpha; 1-\eta)$. Note that all the four of them show up, as opposed to what one naively expects as a limit of $\Theta^{\epsilon\epsilon}_{-b/2}$ approaching the boundary. According to our observation in the previous section, this is due to the fact that we can only fix the total chirality, so that the correlator $\left\langle \Theta^{\epsilon\epsilon}_{-b/2}\Theta^{\epsilon\epsilon}_{\alpha}\right\rangle$ is actually mixed with $\left\langle \Theta^{\epsilon,-\epsilon}_{-b/2}\Theta^{\epsilon,-\epsilon}_{\alpha}\right\rangle$ with equal weights.

The terms proportional to $\mathcal{G}_3(p_{-b/2}, p_\alpha, -p_\alpha; 1-\eta)$ can be identified with the contribution from the case where $\Theta^{\epsilon\epsilon}_{-b/2}$ approaches the boundary and turns into the boundary identity operator. Thus we obtain a recursion relation

$$\begin{aligned}\sqrt{2}\lambda^{-2} i r_-(-\epsilon,\zeta) U_-(\alpha,\epsilon;u_\zeta) \quad = \quad & -2i\zeta \tilde{C}_+(\alpha) U_+(\alpha - \tfrac{b}{2}; u_\zeta) \frac{\Gamma(b\alpha - \frac{b^2}{2})\Gamma(-b^2)}{\sqrt{2}\Gamma(b\alpha - b^2 - \frac{1}{2})\Gamma(\frac{1-b^2}{2})} \\ & + \tilde{C}_-(\alpha) U_+(\alpha + \tfrac{b}{2}; u_\zeta) \frac{\Gamma(1 + \frac{b^2}{2} - b\alpha)\Gamma(-b^2)}{\sqrt{2}\Gamma(\frac{1}{2} - b\alpha)\Gamma(\frac{1-b^2}{2})}. \qquad (3.32)\end{aligned}$$

Here a factor λ^{-2} was inserted, because the solution of the differential equation is actually a mixture of correlators as mentioned above and it is not known how they are mixed in generic solutions.

Let us solve the system of two recursion relations. We first put the following ansatz:

$$\begin{aligned}U_+ &= -2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}(\mu\pi\gamma(\tfrac{bQ}{2}))^{\frac{2\alpha-Q}{2b}}(\alpha - \tfrac{Q}{2})\Gamma(b(\alpha-\tfrac{Q}{2}))\Gamma(\tfrac{1}{b}(\alpha-\tfrac{Q}{2}))\hat{U}_+, \\ U_- &= \lambda 2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}(\mu\pi\gamma(\tfrac{bQ}{2}))^{\frac{2\alpha-Q}{2b}}\Gamma(\tfrac{1}{2} + b(\alpha-\tfrac{Q}{2}))\Gamma(\tfrac{1}{2}+\tfrac{1}{b}(\alpha-\tfrac{Q}{2}))\hat{U}_-. \qquad (3.33)\end{aligned}$$

to simplify the recursion relation into the following form

$$\begin{aligned}\hat{U}_-(\alpha - \tfrac{b}{2}, -\epsilon; u_\zeta) + \hat{U}_-(\alpha + \tfrac{b}{2}, \epsilon; u_\zeta) &= 2\lambda^{-1}\hat{r}(-\epsilon,\zeta)\mu_B \hat{U}_+(\alpha;u_\zeta)\left(\tfrac{1}{2\mu}\cos\tfrac{\pi b^2}{2}\right)^{\frac{1}{2}}, \\ -\zeta\hat{U}_+(\alpha - \tfrac{b}{2}; u_\zeta) + \hat{U}_+(\alpha + \tfrac{b}{2}; u_\zeta) &= 2\lambda^{-1}\hat{r}(-\epsilon,\zeta)\mu_B \hat{U}_-(\alpha,\epsilon;u_\zeta)\left(\tfrac{1}{2\mu}\cos\tfrac{\pi b^2}{2}\right)^{\frac{1}{2}}. \qquad (3.34)\end{aligned}$$

The solution for $\zeta = -1$ is given by

$$\begin{aligned}\hat{U}_+(\alpha;u_-) &= \cosh(\pi(2\alpha - Q)u), \\ \hat{U}_-(\alpha,\epsilon;u_-) &= \cosh(\pi(2\alpha - Q)u), \\ \mu_B &= \lambda\left(\tfrac{2\mu}{\cos(\pi b^2/2)}\right)^{1/2}\cosh(\pi u b). \qquad (3.35)\end{aligned}$$

together with $\hat{r}(\epsilon,-) = 1$. The solution for $\zeta = +1$ becomes

$$\begin{aligned}\hat{U}_+(\alpha;u_+) &= \cosh(\pi(2\alpha - Q)u), \\ \hat{U}_-(\alpha,\epsilon;u_+) &= \epsilon\sinh(\pi(2\alpha - Q)u), \\ \mu_B &= \lambda\left(\tfrac{2\mu}{\cos(\pi b^2/2)}\right)^{1/2}\sinh(\pi u b). \qquad (3.36)\end{aligned}$$

with the condition that $\hat{r}(\epsilon,+) = -\epsilon$. The above conditions on $\hat{r}$ are met when we assume $\psi = -\zeta\bar{\psi}$ on the boundary (real axis). The results for $\zeta = -1$ agree with those obtained in [19].

Our result gives the relation between the label u of Cardy states and the boundary cosmological constant μ_B. Remarkably, the relation is different according to the choice of boundary condition on supercurrent. The one-point structure constants for spin fields also differ according to ζ. Although the corresponding wave function for R Cardy states with $\zeta = 1$ could not be obtained from the modular property, it should be possible to account for this quantity using the modular property.

Due to the subtlety in the correspondence between the solutions of differential equation and the correlators, we are left with an undetermined constant λ. In later subsection we will see that λ should be unity. However, we simply set $\lambda = 1$ for the time being until we check it using the $(3,1)$ degenerate boundary operator.

One-point functions for degenerate boundary states

In the same way we can analyze the one-point structure constants for boundary states belonging to degenerate representations. The main difference as compared to the previous analysis is that they have no interpretation in terms of boundary interaction term. For bosonic Liouville theory, it was found that the geometry of the open worldsheet becomes a pseudosphere[2], so that the boundary is infinitely far from generic points in the bulk. In this case, disc two-point functions are expected to factorize to products of one-point functions in the limit where the two operators approach the boundary.

The recursion relations for degenerate boundary states are obtained by a simple modification of (3.28) and (3.32):

$$\begin{aligned}
U_-(-\tfrac{b}{2},\epsilon)U_+(\alpha) &= C_+(\alpha)U_-(\alpha-\tfrac{b}{2},\epsilon)\frac{\Gamma(b\alpha+\frac{1-b^2}{2})\Gamma(-b^2)}{\Gamma(b\alpha-b^2)\Gamma(\frac{1-b^2}{2})} \\
&\quad + C_-(\alpha)U_-(\alpha+\tfrac{b}{2},-\epsilon)\frac{\Gamma(\frac{3+b^2}{2}-b\alpha)\Gamma(-b^2)}{\Gamma(1-b\alpha)\Gamma(\frac{1-b^2}{2})}, \\
U_-(-\tfrac{b}{2},-\epsilon)U_-(\alpha,\epsilon) &= -\zeta\tilde{C}_+(\alpha)U_+(\alpha-\tfrac{b}{2})\frac{\Gamma(b\alpha-\frac{b^2}{2})\Gamma(-b^2)}{\Gamma(b\alpha-b^2-\frac{1}{2})\Gamma(\frac{1-b^2}{2})} \\
&\quad + \tilde{C}_-(\alpha)U_+(\alpha+\tfrac{b}{2})\frac{\Gamma(1+\frac{b^2}{2}-b\alpha)\Gamma(-b^2)}{2i\Gamma(\frac{1}{2}-b\alpha)\Gamma(\frac{1-b^2}{2})}
\end{aligned} \tag{3.37}$$

Assuming

$$\begin{aligned}
U_+(\alpha) &= \hat{U}_+(\alpha)\left\{\mu\pi\gamma(\tfrac{bQ}{2})\right\}^{-\alpha/b}\frac{(\alpha-\frac{Q}{2})\Gamma(b(\alpha-\frac{Q}{2}))\Gamma(\frac{1}{b}(\alpha-\frac{Q}{2}))}{(-\frac{Q}{2})\Gamma(-\frac{bQ}{2})\Gamma(-\frac{Q}{2b})}, \\
U_-(\alpha,+) &= \hat{U}_-(\alpha)\left\{\mu\pi\gamma(\tfrac{bQ}{2})\right\}^{-\alpha/b}\frac{\Gamma(\frac{1}{2}+b(\alpha-\frac{Q}{2}))\Gamma(\frac{1}{2}+\frac{1}{b}(\alpha-\frac{Q}{2}))}{(-\frac{Q}{2})\Gamma(-\frac{bQ}{2})\Gamma(-\frac{Q}{2b})}
\end{aligned} \tag{3.38}$$

together with (3.25), they are simplified to the following form:

$$\begin{aligned}
\hat{U}_-(-\tfrac{b}{2})\hat{U}_+(\alpha) &= \hat{U}_-(\alpha-\tfrac{b}{2})-\zeta\hat{U}_-(\alpha+\tfrac{b}{2}), \\
\hat{U}_-(-\tfrac{b}{2})\hat{U}_-(\alpha) &= \hat{U}_+(\alpha-\tfrac{b}{2})-\zeta\hat{U}_+(\alpha+\tfrac{b}{2}).
\end{aligned} \tag{3.39}$$

Solving them together with those obtained by $b \leftrightarrow \frac{1}{b}$, we find the following solutions:

$$\begin{aligned}\hat{U}_+(\alpha) &= \frac{\sin[\pi br(\alpha-\frac{Q}{2})]\sin[\frac{\pi s}{b}(\alpha-\frac{Q}{2})]}{\sin\frac{\pi brQ}{2}\sin\frac{\pi sQ}{2b}},\\ \hat{U}_-(\alpha) &= -i^{r+s}\frac{\sin\pi r[\frac{1}{2}+b(\alpha-\frac{Q}{2})]\sin\pi s[\frac{1}{2}+\frac{1}{b}(\alpha-\frac{Q}{2})]}{\sin\frac{\pi brQ}{2}\sin\frac{\pi sQ}{2b}},\end{aligned} \tag{3.40}$$

where r, s are integers whose sum must be even for $\zeta = -1$ and odd for $\zeta = +1$. There exists an ambiguity of $\pm$ sign in front of $\hat{U}_-$ as is obvious from the structure of the recursion relation. The minus sign was chosen from the consistency with the analysis of modular property. The results for $\zeta = -1$ again agree with [19]. By making a comparison with the wave functions obtained in the previous subsection, we find that the solutions of (3.37) can all be expressed in terms of them:

$$\begin{aligned}U_+(\alpha;(r,s)_\zeta) &= \frac{\Psi_+(-i(\alpha-\frac{Q}{2});(r,s)_\zeta)}{\Psi_+(\frac{iQ}{2};(r,s)_\zeta)},\\ U_-(\alpha,+;(r,s)_\zeta) &= \frac{\Psi_-(-i(\alpha-\frac{Q}{2});(r,s)_\zeta)}{\Psi_+(\frac{iQ}{2};(r,s)_\zeta)},\end{aligned} \tag{3.41}$$

if we define the R wave functions for degenerate representations as follows:

$$\begin{aligned}\Psi_-(p;(r,s)_\zeta) &= -i^{r+s}2^{\frac{1}{2}}\pi^{-\frac{1}{2}}(\mu\pi\gamma(\tfrac{bQ}{2}))^{-ip/b}\Gamma(\tfrac{1}{2}+ipb)\Gamma(\tfrac{1}{2}+\tfrac{ip}{b})\\ &\quad\times\sin[\pi r(\tfrac{1}{2}+ipb)]\sin[\pi s(\tfrac{1}{2}+\tfrac{ip}{b})].\end{aligned} \tag{3.42}$$

The appearance of $\Psi_+(\frac{iQ}{2};(r,s)_\zeta)$ expresses that the one-point functions are normalized by the zero-point function with the same boundary condition.

Our result for one-point structure constants for degenerate representations shows that we should associate the representations of NS(R) superalgebras to the boundary states with $\zeta = -1(+1)$. This property can also be observed by a detailed analysis of the boundary two-point functions.

3.3. Two-point functions of boundary operators

We would like then to find the expression for two-point functions of boundary operators on a disc. Equivalently, we shall find the boundary reflection coefficients $d(\beta|u,u')$ and $\tilde{d}(\beta|u,u')$ defined by

$$\begin{aligned}(B_\beta)_{u'_\zeta,u_\zeta} &= d(\beta|u_\zeta,u'_\zeta)(B_{Q-\beta})_{u'_\zeta,u_\zeta},\\ (\Theta^\epsilon_\beta)_{u'_{-\zeta},u_\zeta} &= a\tilde{d}(\beta,\epsilon|u_\zeta,u'_{-\zeta})(\Theta^{-\epsilon}_{Q-\beta})_{u'_{-\zeta},u_\zeta}.\end{aligned} \tag{3.43}$$

Here the boundary states on both sides of the boundary operators are expressed by two pairs of (u,ζ). To obtain these coefficients, all we have to do is to study the OPE of generic boundary operator with boundary degenerate operator $\Theta^\epsilon_{-b/2}$. Note that, as is the case with bosonic Liouville theory with boundary, the consistency with fusion algebra requires that the two boundary states appearing in the two sides of boundary degenerate operators should be related to each other. It is expected that the two boundary states (u,ζ) and (u',ζ') connected by $\Theta^{\epsilon}_{-b/2}$ are related via

$$\pm\ u \pm' u' = \frac{ib}{2}, \quad \zeta' = -\zeta. \tag{3.44}$$

The condition is actually more stringent as we will see below by a detailed analysis.

Let us consider the OPE formula

$$\begin{aligned}
&\Theta^{\epsilon}_{-b/2}(x)_{u''_{-\zeta},u'_{\zeta}} \times B_{\beta}(y)_{u'_{\zeta},u_{\zeta}} \\
&\quad \to \ \left\{ |x-y|^{\frac{b\beta}{2}} c_{+} \cdot \Theta^{\epsilon}_{\beta-b/2}(x) + |x-y|^{\frac{b(Q-\beta)}{2}} a c_{-} \cdot \Theta^{-\epsilon}_{\beta+b/2}(x) \right\}_{u''_{-\zeta},u_{\zeta}}
\end{aligned} \tag{3.45}$$

and calculate the coefficients $c_{\pm}(\beta,\epsilon|u,u',u'';\zeta)$ using free fields. One finds $c_{+}=1$ as usual, and that c_{-} is given in terms of free field correlators with one boundary screening operator inserted. The latter consists of three terms corresponding to three boundary segments, and we have to take the sum of these three carefully. The result reads

$$\begin{aligned}
&c_{-}(\beta,\epsilon|u,u',u'',\zeta) \\
&= \frac{-b}{\sqrt{2} r_{\sigma\sigma}(-\epsilon,\zeta)} \left\{ r_{\sigma\sigma\psi}(-\epsilon,\zeta)\mu_B \frac{\Gamma(1-b\beta)\Gamma(b\beta-\frac{bQ}{2})}{} \Gamma(1-\frac{bQ}{2}) \right. \\
&\qquad + r_{\sigma\sigma\psi}(-\epsilon,\zeta)\mu'_B \frac{\Gamma(1-b\beta)\Gamma(\frac{bQ}{2})}{} \Gamma(1+\frac{bQ}{2}-b\beta) + r_{\sigma\psi\sigma}(-\epsilon,\zeta)\mu''_B \frac{\Gamma(b\beta-\frac{bQ}{2})\Gamma(\frac{bQ}{2})}{} \Gamma(b \\
&= \frac{-b r_{\sigma\psi\sigma}(-\epsilon,\zeta)}{\sqrt{2}\pi r_{\sigma\sigma}(-\epsilon,\zeta)} \Gamma(\tfrac{bQ}{2})\Gamma(1-b\beta)\Gamma(b\beta-\tfrac{bQ}{2}) \\
&\quad \times \left\{ -i\epsilon\mu_B \cos\tfrac{\pi b^2}{2} + i\epsilon\mu'_B \cos\pi(b\beta-\tfrac{b^2}{2}) + \mu''_B \sin\pi b\beta \right\}
\end{aligned} \tag{3}$$

where the coefficients such as $r_{\sigma\sigma}$ represent the prefactors arising in front of free field correlators:

$$\begin{aligned}
\langle\sigma^{\epsilon}(x)\sigma^{\epsilon}(y)\rangle_{\zeta,-\zeta,\zeta} &= r_{\sigma\sigma}(\epsilon,\zeta)|x-y|^{-1/8}, \\
\langle\sigma^{\epsilon}(x)\sigma^{-\epsilon}(y)\psi(z)\rangle_{\zeta,-\zeta,\zeta} &= \tfrac{1}{\sqrt{2}} r_{\sigma\sigma\psi}(\epsilon,\zeta)|x-y|^{3/8}|y-z|^{-1/2}|x-z|^{-1/2}, \\
\langle\sigma^{\epsilon}(x)\psi(y)\sigma^{-\epsilon}(z)\rangle_{\zeta,-\zeta,\zeta} &= \tfrac{1}{\sqrt{2}} r_{\sigma\psi\sigma}(\epsilon,\zeta)|x-z|^{3/8}|z-y|^{-1/2}|y-z|^{-1/2}, \\
\langle\psi(x)\sigma^{\epsilon}(y)\sigma^{-\epsilon}(z)\rangle_{\zeta,-\zeta,\zeta} &= \tfrac{1}{\sqrt{2}} r_{\psi\sigma\sigma}(\epsilon,\zeta)|y-z|^{3/8}|x-y|^{-1/2}|x-z|^{-1/2}.
\end{aligned} \tag{3.47}$$

There are some relations among them due to the requirement of the consistency with cyclic permutations and the analyticity in the upper half-plane with respect to the coordinate of ψ. Fixing them in the following way:

$$\frac{r_{\sigma\psi\sigma}(\epsilon,-1)}{r_{\sigma\sigma}(\epsilon,-1)} = 1, \qquad \frac{r_{\sigma\psi\sigma}(\epsilon,1)}{r_{\sigma\sigma}(\epsilon,1)} = -i\epsilon, \tag{3.48}$$

and calculating further we obtain

$$\begin{aligned}
\zeta=-1 \Rightarrow \quad c_{-}(\beta,\epsilon|u,u',\zeta) &= \frac{2i\epsilon b}{\pi}(\mu\pi\gamma(\tfrac{bQ}{2}))^{\frac{1}{2}}\Gamma(1-b\beta)\Gamma(b\beta-\tfrac{bQ}{2}) \\
&\quad \times \sin\tfrac{b\pi}{2}(iu'+iu+\epsilon\beta)\sin\tfrac{b\pi}{2}(iu'-iu+\epsilon\beta), \\
\zeta=+1 \Rightarrow \quad c_{-}(\beta,\epsilon|u,u',\zeta) &= \frac{2ib}{\pi}(\mu\pi\gamma(\tfrac{bQ}{2}))^{\frac{1}{2}}\Gamma(1-b\beta)\Gamma(b\beta-\tfrac{bQ}{2}) \\
&\quad \times \cos\tfrac{b\pi}{2}(iu'+iu+\epsilon\beta)\sin\tfrac{b\pi}{2}(iu'-iu+\epsilon\beta)
\end{aligned} \tag{3.49}$$

under the assumption $u''=u'-\frac{i\epsilon b}{2}$. On the other hand, when $u''=u'+\frac{i\epsilon b}{2}$ the coefficient c_- does not reduce to such a simple form. This reflects the fact that the boundary states (u,ζ) and $(-u,\zeta)$ are not strictly equivalent when $\zeta=1$. Hence it is expected that

$$\left[\Theta^{+}_{-b/2}\right]_{(u-\frac{ib}{2})_{-\zeta},u_\zeta}, \qquad \left[\Theta^{-}_{-b/2}\right]_{(u+\frac{ib}{2})_{-\zeta},u_\zeta} \tag{3.50}$$

are the only boundary $(2,1)$ degenerate operators that are indeed degenerate. Combining the reflection equivalence with the OPE formula as in the spherical case we obtain the following recursion relations

$$\begin{aligned} d(\beta|u_\zeta,u'_\zeta)c_-(Q-\beta,\epsilon|u,u',\zeta) &= \tilde{d}(\beta-\tfrac{b}{2},\epsilon|u_\zeta,(u'-\tfrac{ib\epsilon}{2})_{-\zeta}), \\ \tilde{d}(\beta+\tfrac{b}{2},-\epsilon|u_\zeta,(u'-\tfrac{ib\epsilon}{2})_{-\zeta})c_-(\beta,\epsilon|u,u',\zeta) &= d(\beta|u_\zeta,u'_\zeta). \end{aligned} \tag{3.51}$$

In the same way, let us consider the OPE

$$\begin{aligned} &\Theta^{\epsilon}_{-b/2}(x)_{u''_{-\zeta},u'_\zeta}\times\psi B_\beta(y)_{u'_\zeta,u_\zeta} \\ &\quad\to \left\{|x-y|^{\frac{b\beta}{2}}c'_+\cdot\Theta^{-\epsilon}_{\beta-b/2}(x)+|x-y|^{\frac{b(Q-\beta)}{2}}ac'_-\cdot\Theta^{\epsilon}_{\beta+b/2}(x)\right\}_{u''_{-\zeta},u_\zeta} \end{aligned} \tag{3.52}$$

and calculate the coefficients using free fields. One finds

$$\begin{aligned} c'_+ &= \frac{r_{\sigma\sigma\psi}(-\epsilon,\zeta)}{\sqrt{2}r_{\sigma\sigma}(-\epsilon,\zeta)} = \left\{\frac{-i\epsilon}{\sqrt{2}}\ (\zeta=-1)\ \ ;\ \ \frac{1}{\sqrt{2}}\ (\zeta=1)\right\} \\ c'_- &= \frac{-b}{2\pi r_{\sigma\sigma}(\epsilon,\zeta)}\frac{Q-\beta}{\beta}\Gamma(1-b\beta)\Gamma(\tfrac{bQ}{2})\Gamma(b\beta-\tfrac{bQ}{2}) \\ &\quad\times\left\{r_{\sigma\sigma\psi\psi}(\epsilon,\zeta)\mu_B\cos\tfrac{\pi b^2}{2}+r_{\sigma\sigma\psi\psi}(\epsilon,\zeta)\mu'_B\cos\pi(b\beta-\tfrac{b^2}{2})-r_{\sigma\psi\sigma\psi}(\epsilon,\zeta)\mu''_B\sin\pi b\beta\right\} \\ &= \frac{-b(Q-\beta)}{2\pi\beta}\Gamma(1-b\beta)\Gamma(\tfrac{bQ}{2})\Gamma(b\beta-\tfrac{bQ}{2}) \\ &\quad\times\left\{\mu_B\cos\tfrac{\pi b^2}{2}+\mu'_B\cos\pi(b\beta-\tfrac{b^2}{2})-i\epsilon\mu''_B\sin\pi b\beta\right\}. \end{aligned} \tag{3.53}$$

Here we used the free field correlators

$$\begin{aligned} &\langle\sigma^\epsilon(x_1)\sigma^\epsilon(x_2)\psi(y_1)\psi(y_2)\rangle_{\zeta,-\zeta,\zeta} \\ &\quad= \frac{r_{\sigma\sigma\psi\psi}(\epsilon,\zeta)}{2}\frac{(x_1-y_1)(x_2-y_2)+(x_1-y_2)(x_2-y_1)}{|x_1-y_1|^{1/2}|x_1-y_2|^{1/2}|x_2-y_1|^{1/2}|x_2-y_2|^{1/2}|x_1-x_2|^{1/8}|y_1-y_2|}, \\ &\langle\sigma^\epsilon(x_1)\psi(y_1)\sigma^\epsilon(x_2)\psi(y_2)\rangle_{\zeta,-\zeta,\zeta} \\ &\quad= \frac{r_{\sigma\psi\sigma\psi}(\epsilon,\zeta)}{2}\frac{(x_1-y_1)(x_2-y_2)-(x_1-y_2)(y_1-x_2)}{|x_1-y_1|^{1/2}|x_1-y_2|^{1/2}|x_2-y_1|^{1/2}|x_2-y_2|^{1/2}|x_1-x_2|^{1/8}|y_1-y_2|}. \end{aligned} \tag{3.54}$$

Assuming again $u''=u'-\frac{i\epsilon b}{2}$, the coefficients can be rewritten further:

$$\begin{aligned} \zeta=-1\Rightarrow\quad c'_+(\beta,\epsilon|u,u',\zeta) &= -i\epsilon2^{-\frac{1}{2}}, \\ c'_-(\beta,\epsilon|u,u',\zeta) &= -\frac{2^{\frac{1}{2}}b}{\pi}\frac{Q-\beta}{\beta}(\mu\pi\gamma(\tfrac{bQ}{2}))^{\frac{1}{2}}\Gamma(1-b\beta)\Gamma(b\beta-\tfrac{bQ}{2}) \\ &\quad\times\cos\tfrac{b\pi}{2}(iu'+iu+\epsilon\beta)\cos\tfrac{b\pi}{2}(iu'-iu+\epsilon\beta), \end{aligned}$$

$$\zeta = +1 \Rightarrow \quad c'_+(\beta,\epsilon|u,u',\zeta) = 2^{-\frac{1}{2}},$$
$$c'_-(\beta,\epsilon|u,u',\zeta) = \frac{2^{\frac{1}{2}}ib}{\pi}\frac{Q-\beta}{\beta}(\mu\pi\gamma(\tfrac{bQ}{2}))^{\frac{1}{2}}\Gamma(1-b\beta)\Gamma(b\beta-\tfrac{bQ}{2}) \times \sin\tfrac{b\pi}{2}(iu'+iu+\epsilon\beta)\cos\tfrac{b\pi}{2}(iu'-iu+\epsilon\beta). \qquad (3.55)$$

If we require that the reflection relation holds also for descendants ψB_β, possibly with the coefficient different from that of the primaries

$$\beta(\psi B_\beta)_{u'_\zeta,u_\zeta} = d'(\beta|u_\zeta,u'_\zeta)(Q-\beta)(\psi B_{Q-\beta})_{u'_\zeta,u_\zeta}, \qquad (3.56)$$

we obtain another set of recursion relations

$$\begin{aligned}
\tfrac{Q-\beta}{\beta}d'(\beta|u_\zeta,u'_\zeta)c'_-(Q-\beta,\epsilon|u,u',\zeta) &= c'_+(\beta,\epsilon|u,u',\zeta)\tilde{d}(\beta-\tfrac{b}{2},-\epsilon|u_\zeta,(u'-\tfrac{ib\epsilon}{2})_{-\zeta}),\\
\tfrac{Q-\beta}{\beta}d'(\beta|u_\zeta,u'_\zeta)c'_+(Q-\beta,\epsilon|u,u',\zeta) &= c'_-(\beta,\epsilon|u,u',\zeta)\tilde{d}(\beta+\tfrac{b}{2},\epsilon|u_\zeta,(u'-\tfrac{ib\epsilon}{2})_{-\zeta}).
\end{aligned} \qquad (3.57)$$

It is straightforward to write down the solutions of the recursion relations (3.51) and (3.57) in terms of the functions $\mathbf{G}$ and $\mathbf{S}$ introduced in [1]. They are defined in the following way:

$$\log\mathbf{G}(x) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-Qt/2}-e^{-xt}}{(1-e^{-bt})(1-e^{-t/b})}+\frac{e^{-t}}{2}(\tfrac{Q}{2}-x)^2+\tfrac{1}{t}(\tfrac{Q}{2}-x)\right] \qquad (3.58)$$

$$\log\mathbf{S}(x) = \int_0^\infty \frac{dt}{t}\left[\frac{2x-Q}{t}-\frac{\sinh[(x-Q/2)t]}{2\sinh[bt/2]\sinh[t/2b]}\right]. \qquad (3.59)$$

The function $\mathbf{G}(x)$ has zeroes at $x=-mb-nb^{-1}$ $(m,n\in\mathbf{Z})$ and no poles. The functions $\mathbf{S}$ and Υ are expressed in terms of $\mathbf{G}$:

$$\mathbf{S}(x) = \mathbf{G}(Q-x)/\mathbf{G}(x), \quad \Upsilon(x) = \mathbf{G}(Q-x)\mathbf{G}(x). \qquad (3.60)$$

The shift relations for $\mathbf{G}$ and $\mathbf{S}$

$$\begin{aligned}
\mathbf{G}(x+b) &= \mathbf{G}(x)(2\pi)^{-\frac{1}{2}}b^{\frac{1}{2}-bx}\Gamma(bx), & \mathbf{G}(x+\tfrac{1}{b}) &= \mathbf{G}(x)(2\pi)^{-\frac{1}{2}}b^{\frac{x}{b}-\frac{1}{2}}\Gamma(x/b),\\
\mathbf{S}(x+b) &= \mathbf{S}(x)2\sin(\pi bx), & \mathbf{S}(x+\tfrac{1}{b}) &= \mathbf{S}(x)2\sin(\pi x/b)
\end{aligned} \qquad (3.61)$$

can be used to write down the solutions of the recursion relations. If we define the functions $\mathbf{G}_{\rm NS},\mathbf{G}_{\rm R}$ and $\mathbf{S}_{\rm NS},\mathbf{S}_{\rm R}$ by

$$\begin{aligned}
\mathbf{G}_{\rm NS}(x) &= \mathbf{G}(\tfrac{x}{2})\mathbf{G}(\tfrac{x+Q}{2}), & \mathbf{G}_{\rm R}(x) &= \mathbf{G}(\tfrac{x+b}{2})\mathbf{G}(\tfrac{x+b^{-1}}{2}),\\
\mathbf{S}_{\rm NS}(x) &= \mathbf{S}(\tfrac{x}{2})\mathbf{S}(\tfrac{x+Q}{2}), & \mathbf{S}_{\rm R}(x) &= \mathbf{S}(\tfrac{x+b}{2})\mathbf{S}(\tfrac{x+b^{-1}}{2}),
\end{aligned} \qquad (3.62)$$

the solution becomes

$$\begin{aligned}
d(\beta|u_-,u'_-) &= \frac{(\mu\pi\gamma(\frac{bQ}{2})b^{1-b^2})^{\frac{Q-2\beta}{2b}}\mathbf{G}_{\rm NS}(Q-2\beta)\mathbf{G}_{\rm NS}(2\beta-Q)^{-1}}{\mathbf{S}_{\rm NS}(\beta+iu+iu')\mathbf{S}_{\rm NS}(\beta-iu+iu')\mathbf{S}_{\rm NS}(\beta+iu-iu')\mathbf{S}_{\rm NS}(\beta-iu-iu')},\\
d'(\beta|u_-,u'_-) &= \frac{(\mu\pi\gamma(\frac{bQ}{2})b^{1-b^2})^{\frac{Q-2\beta}{2b}}\mathbf{G}_{\rm NS}(Q-2\beta)\mathbf{G}_{\rm NS}(2\beta-Q)^{-1}}{\mathbf{S}_{\rm R}(\beta+iu+iu')\mathbf{S}_{\rm R}(\beta-iu+iu')\mathbf{S}_{\rm R}(\beta+iu-iu')\mathbf{S}_{\rm R}(\beta-iu-iu')},
\end{aligned}$$

$$
\begin{aligned}
\tilde{d}(\beta,\epsilon|u_-,u'_+) &= \frac{i\epsilon\left(\mu\pi\gamma(\frac{bQ}{2})b^{1-b^2}\right)^{\frac{Q-2\beta}{2b}}\mathbf{G}_{\mathrm{R}}(Q-2\beta)\mathbf{G}_{\mathrm{R}}(2\beta-Q)^{-1}}{\mathbf{S}_{\mathrm{NS}}(\beta+iu+i\epsilon u')\mathbf{S}_{\mathrm{NS}}(\beta-iu+i\epsilon u')\mathbf{S}_{\mathrm{R}}(\beta+iu-i\epsilon u')\mathbf{S}_{\mathrm{R}}(\beta-iu-i\epsilon u')},\\
d(\beta|u_+,u'_+) &= \frac{\left(\mu\pi\gamma(\frac{bQ}{2})b^{1-b^2}\right)^{\frac{Q-2\beta}{2b}}\mathbf{G}_{\mathrm{NS}}(Q-2\beta)\mathbf{G}_{\mathrm{NS}}(2\beta-Q)^{-1}}{\mathbf{S}_{\mathrm{R}}(\beta+iu+iu')\mathbf{S}_{\mathrm{NS}}(\beta-iu+iu')\mathbf{S}_{\mathrm{NS}}(\beta+iu-iu')\mathbf{S}_{\mathrm{R}}(\beta-iu-iu')},\\
d'(\beta|u_+,u'_+) &= \frac{-\left(\mu\pi\gamma(\frac{bQ}{2})b^{1-b^2}\right)^{\frac{Q-2\beta}{2b}}\mathbf{G}_{\mathrm{NS}}(Q-2\beta)\mathbf{G}_{\mathrm{NS}}(2\beta-Q)^{-1}}{\mathbf{S}_{\mathrm{NS}}(\beta+iu+iu')\mathbf{S}_{\mathrm{R}}(\beta-iu+iu')\mathbf{S}_{\mathrm{R}}(\beta+iu-iu')\mathbf{S}_{\mathrm{NS}}(\beta-iu-iu')},\\
\tilde{d}(\beta,\epsilon|u_+,u'_-) &= \frac{i\epsilon\left(\mu\pi\gamma(\frac{bQ}{2})b^{1-b^2}\right)^{\frac{Q-2\beta}{2b}}\mathbf{G}_{\mathrm{R}}(Q-2\beta)\mathbf{G}_{\mathrm{R}}(2\beta-Q)^{-1}}{\mathbf{S}_{\mathrm{NS}}(\beta+iu'-i\epsilon u)\mathbf{S}_{\mathrm{NS}}(\beta-iu'-i\epsilon u)\mathbf{S}_{\mathrm{R}}(\beta+iu'+i\epsilon u)\mathbf{S}_{\mathrm{R}}(\beta-iu'+i\epsilon u)}.
\end{aligned}
\tag{3.63}
$$

They satisfy the unitarity condition and are consistent with the equivalence of boundary states u and $-u$ when $\zeta=-1$. Note also that the structure of poles of these quantities implies that we should identify $\zeta=-1(+1)$ boundary states with the representations for NS(R) superalgebras.

From the above result one can easily read off that d and d' differ, and $\tilde{d}$ does depend on ϵ. This means that the reflection coefficients of boundary operators differ for each operators in a single supermultiplet at least if the transformation law is defined in a naive way. As a consequence, it follows that the supersymmetry transformation and the reflection do not commute. However, from the representation theory involving degenerate representations it is natural since, if we assume (3.50), it follows that different operators in a single (degenerate) supermultiplet connect two boundary states in a different way. As a simple example, let us consider how the two boundary states on the two sides of B_{-b} or ψB_{-b} are related to each other. If these operators are regarded as created by multiplying two degenerate operators $\Theta^{\epsilon}_{-b/2}$, the only possibilities are

$$
[B_{-b}]_{(u\pm ib)_\zeta,u_\zeta}\,,\qquad [\psi B_{-b}]_{u_\zeta,u_\zeta}\,. \tag{3.64}
$$

This is reasonable if we make comparison with the OPE relations

$$
\begin{aligned}
[V_{-b}]\times[V_\alpha] &\to [V_{\alpha-b}]+\left[\psi\bar\psi V_\alpha\right]+[V_{\alpha+b}]\,,\\
\left[\psi\bar\psi V_{-b}\right]\times[V_\alpha] &\to \left[\psi\bar\psi V_{\alpha-b}\right]+[V_\alpha]+\left[\psi\bar\psi V_{\alpha+b}\right].
\end{aligned}
\tag{3.65}
$$

The generalization of this argument to higher degenerate representation is straightforward. Let us denote boundary degenerate operators as $B_{-kb-hb^{-1}},\psi B_{-kb-hb^{-1}}$ or $\Theta^{\epsilon}_{-kb-hb^{-1}}$ using two non-negative half-integers k,h. Then the two boundary states with labels u and u' are related via

$$
u'=u+i(r-k)b+i(s-h)b^{-1},\quad (0\le r\le 2k,\quad 0\le s\le 2h) \tag{3.66}
$$

where $r+s$ must be even for $B_{-kb-hb^{-1}},\ \Theta^{+}_{-kb-hb^{-1}}$ and odd for $\psi B_{-kb-hb^{-1}},\ \Theta^{-}_{-kb-hb^{-1}}$. Of course this is merely a conjecture, and the consistency should be proven by a more detailed analysis of this model.

Density of open string states

The reflection coefficients can be regarded as phase shifts, so are related to the density of certain open string states through the formulae of the following form:

$$\begin{aligned}\frac{1}{2\pi i}\frac{d}{ds}\log d(\tfrac{Q}{2}+is|u_\zeta,u'_\zeta) &= \rho(s|u_\zeta,u'_\zeta),\\ \frac{1}{2\pi i}\frac{d}{ds}\log d'(\tfrac{Q}{2}+is|u_\zeta,u'_\zeta) &= \rho'(s|u_\zeta,u'_\zeta),\\ \frac{1}{2\pi i}\frac{d}{ds}\log \tilde{d}(\tfrac{Q}{2}+is,\epsilon|u_\zeta,u'_\zeta) &= \tilde{\rho}(s,\epsilon|u_\zeta,u'_\zeta).\end{aligned} \tag{3.67}$$

Using the integral expression for $\mathbf{S}$ and discarding terms independent of the labels u and u', some of them are given by

$$\begin{aligned}\rho(s|u_-,u'_-) &\sim 2\int_{-\infty}^{\infty}dp\frac{e^{2\pi ips}\cosh(\pi Qp)\cos(2\pi up)\cos(2\pi u'p)}{\sinh(2\pi bp)\sinh(2\pi p/b)}\\ &= \int_{-\infty}^{\infty}\frac{dp}{\pi}e^{2\pi ips}\left[\Psi_+(p;u_-)\Psi_+^\dagger(p;u'_-)+\Psi_-(p;u_-)\Psi_-^\dagger(p;u_-)\right],\\ \rho'(s|u_-,u'_-) &\sim 2\int_{-\infty}^{\infty}dp\frac{e^{2\pi ips}\cosh\pi(b-b^{-1})p\cos(2\pi up)\cos(2\pi u'p)}{\sinh(2\pi bp)\sinh(2\pi p/b)}\\ &= \int_{-\infty}^{\infty}\frac{dp}{\pi}e^{2\pi ips}\left[\Psi_+(p;u_-)\Psi_+^\dagger(p;u'_-)-\Psi_-(p;u_-)\Psi_-^\dagger(p;u_-)\right].\end{aligned} \tag{3.68}$$

This agrees with the analysis of modular property since the annulus partition function bounded by two Cardy states u_- and u'_- is given by

$$\begin{aligned}Z_{u_-,u'_-} &= \int_{-\infty}^{\infty}\frac{dp}{2\pi}\left[2\Psi_+(p;u_-)\Psi_+^\dagger(p;u'_-)\chi^+_{p(\mathrm{NS})}(\tau_c)+\sqrt{2}\Psi_-(p;u_-)\Psi_-^\dagger(p;u'_-)\chi^+_{p(\mathrm{R})}(\tau_c)\right]\\ &= \frac{1}{2}\int_{-\infty}^{\infty}ds\left[\rho(s|u_-,u'_-)\left\{\chi^+_{s(\mathrm{NS})}+\chi^-_{s(\mathrm{NS})}\right\}(\tau_o)\right.\\ &\qquad\left.+\rho'(s|u_-,u'_-)\left\{\chi^+_{s(\mathrm{NS})}-\chi^-_{s(\mathrm{NS})}\right\}(\tau_o)\right].\end{aligned} \tag{3.69}$$

The other quantities can be written as

$$\begin{aligned}\rho(s|u_+,u'_+) &= \int_{-\infty}^{\infty}\frac{dp}{\pi}e^{2\pi ips}\left[\Psi_+(p;u_+)\Psi_+^\dagger(p;u'_+)+\Psi_-(p;u_+)\Psi_-^\dagger(p;u'_+)\right],\\ \rho'(s|u_+,u'_+) &= \int_{-\infty}^{\infty}\frac{dp}{\pi}e^{2\pi ips}\left[\Psi_+(p;u_+)\Psi_+^\dagger(p;u'_+)-\Psi_-(p;u_+)\Psi_-^\dagger(p;u'_+)\right],\\ \tilde{\rho}(s,\epsilon|u_-,u'_+) &= \int_{-\infty}^{\infty}\frac{dp}{\pi}e^{2\pi ips}\left[\Psi_+(p;u_-)\Psi_+^\dagger(p;u'_+)+\epsilon\Psi_-(p;u_-)\Psi_-^\dagger(p;u'_+)\right],\\ \tilde{\rho}(s,\epsilon|u_+,u'_-) &= \int_{-\infty}^{\infty}\frac{dp}{\pi}e^{2\pi ips}\left[\Psi_+(p;u_+)\Psi_+^\dagger(p;u'_-)+\epsilon\Psi_-(p;u_+)\Psi_-^\dagger(p;u'_-)\right].\end{aligned} \tag{3.70}$$

if we assume that the R wave function for $\zeta=1$ is given by

$$\Psi_-(p;u_+) = i2^{-\frac{1}{2}}\pi^{-\frac{1}{2}}(\mu\pi\gamma(\tfrac{Q}{2}))^{-ip/b}\Gamma(\tfrac{1}{2}+ipb)\Gamma(\tfrac{1}{2}+\tfrac{ip}{b})\sin(2\pi pu). \tag{3.71}$$

Hence, in all the cases the phase shifts give the density of open string states with definite worldsheet fermion number. This also suggests that the wave function for R Cardy states with u_+ is given by (3.71), in consistency with the analysis of one-point structure constants.

As a further check, let us consider the OPE of boundary operators involving $(3,1)$ degenerate representation. Above all, the OPE coefficients involving them depend on both of μ and μ_B. The consistency with the previous arguments fixes the constant λ which was left undetermined.

Let us consider the OPE relation

$$\begin{aligned} &[B_{-b}]_{u''_\zeta,u'_\zeta}\,[B_\beta]_{u'_\zeta,u_\zeta} \\ &\quad\to\ \hat{c}_+\,[B_{\beta-b}]_{u''_\zeta,u_\zeta} + \hat{c}_0\,[\psi B_\beta]_{u''_\zeta,u_\zeta} + \hat{c}_-\,[B_{\beta+b}]_{u''_\zeta,u_\zeta} \end{aligned} \tag{3.72}$$

with $u'' = u' + ib$, and use $\hat{c}_\pm$ to derive another recursion relation for boundary reflection coefficient. Here again $\hat{c}_+ = 1$, so we concentrate on the calculation of $\hat{c}_-$. There are largely two contributions to $\hat{c}_-$, which are proportional to μ and μ_B^2, respectively. Calculating first the contribution proportional μ using free fields, one finds

$$2\mu b^2\zeta\int d^2z|z|^{-2b\beta}|1-z|^{2b^2}|z-\bar{z}|^{-b^2-1} = -2\mu b^2\zeta I_0\sin(\pi b^2)\sin^2(\pi b\beta) \tag{3.73}$$

where I_0 is given by

$$I_0 = -\frac{\gamma(\frac{bQ}{2})}{2\pi\sin\pi b^2}\Gamma(1-b\beta)\Gamma(1-\tfrac{bQ}{2}-b\beta)\Gamma(b\beta)\Gamma(b\beta-\tfrac{bQ}{2}). \tag{3.74}$$

There are six contributions proportional to μ_B^2, since there are six ways of inserting two boundary screening operators onto the boundary divided into three segments. Restoring λ, they are summarized into the following form

$$\begin{aligned} &b^2 I_0\Big\{-\mu_B^2\sin(\pi b^2)\cos\tfrac{\pi b^2}{2} - {\mu'_B}^2\sin(\pi b\beta)\cos\pi(b\beta-\tfrac{b^2}{2}) + {\mu''_B}^2\sin(\pi b\beta)\cos\pi(b\beta+\tfrac{b^2}{2})\Big\} \\ &+2b^2 I_0\sin\tfrac{\pi b^2}{2}\Big\{\mu_B\mu'_B\cos\tfrac{\pi b^2}{2}\cos\pi(b\beta-\tfrac{b^2}{2}) - \mu_B\mu''_B\cos\tfrac{\pi b^2}{2}\cos\pi(b\beta+\tfrac{b^2}{2}) \\ &\qquad + \mu'_B\mu''_B\cos\pi(b\beta-\tfrac{b^2}{2})\cos\pi(b\beta+\tfrac{b^2}{2})\Big\} \\ &(\zeta=-1) \\ =\ & -2\mu\lambda^2 b^2 I_0\sin(\pi b^2)\Big\{\sin^2(\pi b\beta) + 4\sin\tfrac{b\pi}{2}(iu'+iu+\beta)\sin\tfrac{b\pi}{2}(iu'-iu+\beta) \\ &\qquad\times\cos\tfrac{b\pi}{2}(iu'+iu+\beta+b)\cos\tfrac{b\pi}{2}(iu'-iu+\beta+b)\Big\} \\ &(\zeta=1) \\ =\ & +2\mu\lambda^2 b^2 I_0\sin(\pi b^2)\Big\{\sin^2(\pi b\beta) - 4\cos\tfrac{b\pi}{2}(iu'+iu+\beta)\sin\tfrac{b\pi}{2}(iu'-iu+\beta) \\ &\qquad\times\sin\tfrac{b\pi}{2}(iu'+iu+\beta+b)\cos\tfrac{b\pi}{2}(iu'-iu+\beta+b)\Big\}. \end{aligned} \tag{3.75}$$

One can see that when $\lambda = 1$ there are some cancellation between the contributions from the bulk and the boundary, and the result for $\hat{c}_-$ yields a recursion relation for $d(\beta|u_\zeta, u'_\zeta)$ consistent with the analysis using $\Theta^\epsilon_{-b/2}$.

4. Concluding remarks

In this paper the $N = 1$ super Liouville theory was analyzed on a sphere and on a disc. The analysis was based on the approach developed in [29, 31, 1, 2] for bosonic Liouville theory. Various quantities were obtained in a form very similar to the bosonic case. However, there are some new features in $N = 1$ theory largely due to the fermionic nature of screening operators. As one of the consequences, the reflection of spin fields are always accompanied by the flip of the chirality. We also presented all the solutions of differential equations for four-point functions containing one degenerate spin operator $\Theta^{\epsilon}_{-b/2}$. For four-point functions of four spin fields, the differential equation become of the fourth order and there are therefore four independent solutions in apparent contradiction with the assumption that the product of $\Theta^{\epsilon}_{-b/2}$ with any operators is decomposed into two discrete terms. However, our solutions obey a special transformation property under the change of basis so that the crossing symmetric combination of the left and the right sectors can be identified with a particular sum of four-point functions of spin fields.

Contrarily to the bosonic case, the analysis of modular property of annulus partition functions was not sufficient to obtain all the wave functions that define Cardy states. It was also found that the two ways of putting boundary condition on supercurrent leads to two boundary states which differ in quite a non-trivial way. Indeed, we were unable to find the R Cardy states with $\zeta = 1$ from the modular analysis. We found the wave functions for them through the analysis of disc one-point functions and found the correspondence between $\zeta = -1(+1)$ boundary states and the representations of NS(R) superalgebras.

The two-point functions of boundary operators were also obtained and the density of open string states which can be read off from them were shown to be consistent with the analysis of modular property. Remarkably, the reflection coefficients for boundary operators depend on the label of boundary states in such a way that they are different for different components in the same supermultiplet. To understand this we need a more detailed analysis of the property of boundary operators.

Our analysis have shown that the non-compact superconformal theory with boundary can be analyzed using the techniques developed in the analysis of bosonic theory, if an appropriate care is taken. It would then be interesting to analyze similar superconformal theories or those with higher worldsheet supersymmetry in the same way.

Note added

For disc one-point functions, after the submission of this paper we were informed of the preceding analysis [19] which covers the $\zeta = -1$ case of our result.

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