Ellipsoidal, Cylindrical, Bipolar and Toroidal Wormholes in 5D Gravity

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Abstract

In this paper we construct and analyze new classes of wormhole and flux tube-like solutions for the 5D vacuum Einstein equations. These 5D solutions possess generic local anisotropy which gives rise to a gravitational running or scaling of the Kaluza-Klein "electric" and "magnetic" charges of these solutions. It is also shown that it is possible to self–consistently construct these anisotropic solutions with various rotational 3D hypersurface geometries (*i.e.* ellipsoidal, cylindrical, bipolar and toroidal). The local anisotropy of these solutions is handled using the technique of anholonomic frames with their associated nonlinear connection structures [1]. Through the use of the anholonomic frames the metrics are diagonalized, in contrast to holonomic coordinate frames where the metrics would have off–diagonal components. In the local isotropic limit these solutions are shown to be equivalent to spherically symmetric 5D wormhole and flux tube solutions.

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I. INTRODUCTION

The first solutions describing black holes and wormholes in 4D and higher dimensional gravity were spherical symmetric solutions with diagonal metrics [2]. Later Salam, Strathee and Perracci [3] showed that including off–diagonal components in higher dimensional metrics is equivalent to including gauge fields. They concluded that geometrical gauge fields could act as sources of exotic matter necessary for the construction of a wormhole. Refs. [4, 5] examined locally isotropic solutions with off–diagonal metric components for 5D vacuum Einstein equations. These solutions were similar to spherically symmetric 4D wormhole or flux tube metrics with "electric" and/or "magnetic" fields running along the throat of the wormhole. These "electromagnetic" fields arose as a consequence of the off–diagonal elements of the metric. By varying certain free parameters of the off–diagonal elements of the 5D metrics it was possible to change the relative strengths of the fields in the wormhole's throat, and to change the longitudinal and transverse size of the wormhole's throat. In [6] the we constructed *anisotropic* wormhole and flux tube solutions, which reduced to the solutions of [4, 5] in the isotropic limit. The anisotropy of these metrics was handled using the method of anholonomic frames with associated nonlinear connections, which has been developed by one of the authors (SV) in Refs. [1]. It was shown that these anisotropic solutions exhibited a variation or running of the "electromagnetic" parameters as a result of the angular anisotropies and/or through variations of the extra spatial dimension.

In this paper we extend the investigation of [6] by applying the anholonomic frames method to construct anisotropic wormhole and flux tube solutions to 5D Kaluza–Klein theory which possess a range of different symmetries (elliptic, cylindrical, bipolar, toroidal). We will discuss the physical consequences of these solutions, in particular the variation of the "electromagnetic" parameters (*e.g.* the "electric" and "magnetic" charges associated with the solutions). This variation of the "electromagnetic" charges, which here occurs in the context of a higher dimensional gravity theory, can be likened to the variation or running of electric charge that occurs when a real electric charge is placed into some dielectric medium or in a quantum vacuum where quantum fluctuations produce a scale dependent electric charge. We will sometimes loosely refer to this gravitational variation of the "electromagnetic" parameters of the solutions as the gravitational running, scaling or renormalization of the charges of the solutions.

II. ANHOLONOMIC FRAMES AND 5D VACUUM EINSTEIN EQUATIONS

In this section we outline the basic formulas for 5D Einstein gravity, and introduce the method of anholonomic frames. We construct locally anisotropic metrics which are generalizations of those considered in Ref. [6]. These 5D metrics have a mixture of holonomic and anholonomic variables, and are most naturally dealt with using anholonomic frames. Finally we analyze the physical and mathematical properties of these 5D, locally anisotropic vacuum solutions.

A. Metric ansatz

Let us consider a 5D pseudo–Riemannian spacetime of signature $(+,-,-,-,-)$ and denote the local coordinates $u^\alpha = (x^i, y^a) = (x^1, x^2, x^3, y^4 = s, y^5 = p),$ – or more compactly $u = (x, y)$ – where the Greek indices are split into two subsets x^i (holonomic coordinates) and y^a (anholonomic coordinates) labeled respectively by Latin indices $i, j, k, ... = 1, 2, 3$ and $a, b, ... = 4, 5$. The local coordinate bases, $\partial_\alpha = (\partial_i, \partial_a),$ and their duals, $d^\alpha = (d^i, d^a)\,,$ are written respectively as

$$\partial_\alpha \equiv \frac{\partial}{du^\alpha} = (\partial_i = \frac{\partial}{dx^i}, \partial_a = \frac{\partial}{dy^a}) \tag{1}$$

and

$$d^\alpha \equiv du^\alpha = (d^i = dx^i, d^a = dy^a). \tag{2}$$

We can treat an arbitrary coordinate, x^i or $y^a,$ as space-like (x, y, z), time-like (t) or as the 5^{th} spatial coordinate (χ). The aim is then to study anisotropies and anholonomic constraints for various coordinates.

With respect to the coordinate frame base (2) the 5D pseudo-Riemannian metric

$$dS^2 = g_{\alpha\beta} du^\alpha du^\beta \tag{3}$$

with its metric coefficients $g_{\alpha\beta}$ parameterized as

$$\begin{bmatrix} g_1 + {w_1}^2 h_4 + {n_1}^2 h_5 & w_1 w_2 h_4 + n_1 n_2 h_5 & w_1 w_3 h_4 + n_1 n_3 h_5 & w_1 h_4 & n_1 h_5 \\ w_1 w_2 h_4 + n_1 n_2 h_5 & g_2 + {w_2}^2 h_4 + {n_2}^2 h_5 & w_2 w_3 h_4 + n_2 n_3 h_5 & w_2 h_4 & n_2 h_5 \\ w_1 w_3 h_4 + n_1 n_3 h_5 & w_3 w_2 h_4 + n_2 n_3 h_5 & g_3 + {w_3}^2 h_4 + {n_3}^2 h_5 & w_3 h_4 & n_3 h_5 \\ w_1 h_4 & w_2 h_4 & w_3 h_4 & h_4 & 0 \\ n_1 h_5 & n_2 h_5 & n_3 h_5 & 0 & h_5 \end{bmatrix}, \tag{4}$$

The ansatz functions of this metric are smooth function of the form:

$$
\begin{aligned}
g_1 &= 1, \qquad g_{2,3} = g_{2,3}(x^2, x^3) = \epsilon_{2,3} \exp\left[2b_{2,3}(x^2, x^3)\right], && (5)\\
h_{4,5} &= h_{4,5}(x^2, x^3, s) = \exp[2f_{4,5}(x^2, x^3, s)], && (6)\\
w_1 &= w_1(x^2), \qquad w_{2,3} = w_{2,3}(x^2, x^3, s), \\
n_1 &= n_1(x^2), \qquad n_{2,3} = n_{2,3}(x^2, x^3, s);
\end{aligned}
$$

The ansatz functions of the metric are taken to depend on two isotropic variables (x^2, x^3) and on one anisotropic variable, $y^4 = s$.

Metric (3) can be greatly simplified into the form

$$\delta S^2 = g_{ij}\,(x)\,dx^i dx^i + h_{ab}\,(x,s)\,\delta y^a \delta y^b, \qquad (7)$$

with diagonal coefficients

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & g_2 & 0 \\ 0 & 0 & g_3 \end{bmatrix} \text{ and } h_{ab} = \begin{bmatrix} h_4 & 0 \\ 0 & h_5 \end{bmatrix} \qquad (8)$$

if instead of coordinate bases (1) and (2) one used anholonomic frames (anisotropic bases)

$$\delta_\alpha \equiv \frac{\delta}{du^\alpha} = (\delta_i = \partial_i - N_i^b(u)\,\partial_b, \partial_a = \frac{\partial}{dy^a}) \qquad (9)$$

and

$$\delta^\alpha \equiv \delta u^\alpha = (\delta^i = dx^i, \delta^a = dy^a + N_k^a(u)\,dx^k) \qquad (10)$$

where the N–coefficients are parametrized as

$$N_1^4 = w_1, \qquad N_{2,3}^4 = w_{2,3} \text{ and } N_1^5 = n_1, \qquad N_{2,3}^5 = n_{2,3}$$

They define an associated nonlinear connection (N–connection) structure. (see Refs [1, 7]). Here, we shall not emphasize the N–connection formalism. The anisotropic frames (9) and (10) are anholonomic because, in general, they satisfy some anholonomic relations,

$$\delta_\alpha \delta_\beta - \delta_\beta \delta_\alpha = W_{\alpha\beta}^\gamma \delta_\gamma,$$

with nontrivial anholonomy coefficients

$$
\begin{aligned}
W_{ij}^k &= 0, \qquad W_{ai}^k = 0, \qquad W_{ab}^k = W_{ab}^c = 0, && (11)\\
W_{ij}^a &= -\Omega_{ij}^a, \qquad W_{bj}^a = -\partial_b N_j^a, \qquad W_{ia}^b = \partial_a N_j^b,
\end{aligned}
$$

where

$$\Omega^a_{ij} = \delta_j N^a_i - \delta_i N^a_j.$$

Conventionally, the N–coefficients decompose spacetime objects (*e.g.* tensors, spinors and connections) into objects with mixed holonomic–anholonomic characteristics. The holonomic parts of an object are indicated with indices of type $i, j, k, ...$, while the anholonomic parts have indices of type $a, b, c, ...$. Tensors, metrics and linear connections with coefficients defined with respect to anholonomic frames (9) and (10) are distinguished (d) by N–coefficients into holonomic and anholonomic subsets and are called d–tensors, d–metrics and d–connections.

B. Einstein equations in holonomic–anholonomic variables

The main "trick" of the anholonomic frames method for integrating Einstein's equations in general relativity and various (super)string and higher / lower dimension gravitational theories, is to find the coefficients N^a_j such that the block matrices g_{ij} and h_{ab} are diagonalized [1, 7]. This greatly simplifies computations. With respect to such anholonomic frames the partial derivatives are N–elongated (locally anisotropic).

Metric (3) with coefficients (4) (or equivalently, the d–metric (7) with coefficients (8)) is assumed to solve the 5D Einstein equations

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \kappa \Upsilon_{\alpha\beta}, \tag{12}$$

where κ and $\Upsilon_{\alpha\beta}$ are respectively the coupling constant and the energy–momentum tensor. For most of the paper we will consider vacuum solutions, $\Upsilon_{\alpha\beta} = 0$. The nontrivial components of the Ricci tensor (details of the computations are given in Refs. [6, 7]), for the ansatz, are

$$R^2_2 = R^3_3 = -\frac{1}{2g_2 g_3}\left[g_3^{\bullet\bullet} - \frac{g_2^{\bullet} g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2^{''} - \frac{g_2^{'} g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2}\right], \tag{13}$$

$$R^4_4 = R^5_5 = -\frac{\beta}{2h_4 h_5}, \tag{14}$$

$$R_{42} = -w_2 \frac{\beta}{2h_5} - \frac{\alpha_2}{2h_5}, \qquad R_{43} = -w_3 \frac{\beta}{2h_5} - \frac{\alpha_3}{2h_5}, \tag{15}$$

$$R_{52} = -\frac{h_5}{2h_4}\left[n_2^{**} + \gamma n_2^{*}\right], \qquad R_{53} = -\frac{h_5}{2h_4}\left[n_3^{**} + \gamma n_3^{*}\right], \tag{16}$$

where

$$\alpha_2 = h_5^{*\bullet} - \frac{h_5^*}{2}\left(\frac{h_4^\bullet}{h_4} + \frac{h_5^\bullet}{h_5}\right) = h_5^*\left(\ln|f_5^*| + f_5 - f_4\right)^\bullet, \qquad h_5^* \neq 0; \tag{17}$$

$$\alpha_3 = h_5^{*\prime} - \frac{h_5^*}{2}\left(\frac{h_4'}{h_4} + \frac{h_5'}{h_5}\right) = h_5^*\left(\ln|f_5^*| + f_5 - f_4\right)', \qquad h_5^* \neq 0; \tag{18}$$

$$\beta = h_5^{**} - \frac{h_5^*}{2}\left(\frac{h_5^*}{h_5} + \frac{h_4^*}{h_4}\right) = h_5^*\left(\ln|f_5^*| + f_5 - f_4\right)^*, \qquad h_5^* \neq 0 \tag{19}$$

$$\gamma = \frac{3}{2}\frac{{h_5}^*}{h_5} - \frac{{h_4}^*}{h_4} = \left[3f_5 - 2f_4\right]^*. \tag{20}$$

The partial derivatives are denoted as $h^\bullet = \partial h/\partial x^2, f' = \partial f/\partial x^3$ and $f^* = \partial f/\partial s.$ We have given the formulas both in terms of $h_{4,5}$ and $f_{4,5}$ since we will need this later.

Formulas (13)–(16) were obtained with respect to anholonomic frames for a fixed linear connection adapted to the N–connection structure, called the canonical distinguished connection [8]. (Miron and Anastasiei introduced this connection on vector bundles, but it can be used in a similar fashion on (pseudo) Riemannian spaces if the N–connection is considered). The coefficients of a distinguished connection $\Gamma^\alpha{}_{\beta\gamma} = \left(L^i{}_{jk}, L^a{}_{bk}, C^i{}_{jc}, C^a{}_{bc}\right),$ are computed from the formulas

$$\begin{aligned} L^i{}_{jk} &= \frac{1}{2}g^{in}\left(\delta_k g_{nj} + \delta_j g_{nk} - \delta_n g_{jk}\right), \\ L^a{}_{bk} &= \partial_b N_k^a + \frac{1}{2}h^{ac}\left(\delta_k h_{bc} - h_{dc}\partial_b N_k^d - h_{db}\partial_c N_k^d\right), \\ C^i{}_{jc} &= \frac{1}{2}g^{ik}\partial_c g_{jk},\ C^a{}_{bc} = \frac{1}{2}h^{ad}\left(\partial_c h_{db} + \partial_b h_{dc} - \partial_d h_{bc}\right). \end{aligned} \tag{21}$$

The coefficients in (21) reduce to the Christoffel symbols if the metric components g_{ij} depend only on x–variables, the h_{ab} depend only on y–variables, and the N–connection vanishes. We emphasize that if the anholonomic frames are introduced into consideration, there is a certain class of linear connections which satisfy the metricity condition for a given metric, or inversely, there is a certain class of metrics which satisfy the metricity conditions for a given linear connection (this result was originally obtained by A. Kawaguchi [9] in 1937. Details can be found in [8]; see Theorems 5.4 and 5.5 in Chapter III). So, we need to state explicitly what type of linear connection is used for the definition of the curvature and Ricci tensor if the space–time is provided with an anholonomic frame structure. In this work and in Refs. [6, 7] the linear connection is considered to be of the form (21). The off–diagonal metrics studied in this paper will be compatible with the canonical linear connection, but may not have a trivial limit to a diagonal holonomic metric.

The scalar curvature is

$$R = 2\left(R_2^2 + R_4^4\right).$$

using this along with the components of the Ricci tensor in Eqs. (13)-(16) one can show that for the metric ansatz (4) the coefficients of the energy–momentum d–tensor satisfy

$$\Upsilon_1^1 = \Upsilon_2^2 + \Upsilon_4^4, \qquad \Upsilon_2^2 = \Upsilon_3^3 = \Upsilon_2, \qquad \Upsilon_4^4 = \Upsilon_5^5 = \Upsilon_4,$$

with respect to anholonomic bases (9) and (10). Thus the Einstein equations can be written as

$$R_2^2 = -\kappa\Upsilon_4, \qquad R_4^4 = -\kappa\Upsilon_2, \qquad R_{4\widehat{i}} = \kappa\Upsilon_{4\widehat{i}}, \qquad R_{5\widehat{i}} = \kappa\Upsilon_{5\widehat{i}}, \tag{22}$$

where $\widehat{i} = 2, 3$.

With this setup it is possible to construct very general classes of solutions to these equations [6, 7] which describe locally anisotropic solitons, black holes, black tori and wormhole solutions.

C. General properties of the anisotropic vacuum solutions

In the vacuum case Eqs. (22) reduce to:

$$g_3^{\bullet\bullet} - \frac{g_2^{\bullet}g_3^{\bullet}}{2g_2} - \frac{(g_3^{\bullet})^2}{2g_3} + g_2^{''} - \frac{g_2^{'}g_3^{'}}{2g_3} - \frac{(g_2^{'})^2}{2g_2} = 0, \tag{23}$$

$$h_5^{**} - \frac{h_5^*}{2}\left(\frac{h_5^*}{h_5} + \frac{h_4^*}{h_4}\right) = 0, \tag{24}$$

$$\beta w_{2,3} + \alpha_{2,3} = 0, \tag{25}$$

$$n_{2,3}^{**} + \gamma n_{2,3}^{*} = 0. \tag{26}$$

We now discuss general features for the d–metric coefficients, $(g_2, g_3), (h_4, h_5)$, and the N–connection coefficients $w_{2,3}$ and $n_{2,3}$ which solve this system of equations:

1. Eq. (23) relates two functions $g_2(x^2, x^3)$ and $g_3(x^2, x^3)$ and their partial derivatives in the isotropic coordinates x^2 and x^3. If one of the functions is fixed, by some symmetry and boundary conditions the second function is found by solving a second order partial differential equation. For example by redefinition of the coordinates or a conformal transformation one can transform g_3, (or conversely, g_2) into a constant. Using this technique one of the authors (SV) was able to construct various 2D soliton – dilaton and black hole-like configurations [7]

2. Eq. (24) contains partial derivatives of only the anisotropic coordinate s, and relates the two functions $h_4(x^2, x^3, s)$ and $h_5(x^2, x^3, s)$. By fixing one of these functions the second one is found by solving a second or first order differential equation in s (the x–variables being treated as parameters). These equations reduce to the Bernoulli equations [10], and are satisfied by two arbitrary functions $h_{4,5}(x^2, x^3)$ for which $h^*_{4,5} = 0$. Thus there are three classes of solutions:

   - Class A, for which $h^*_4 = 0$, $h^*_5 \neq 0$;
   - Class B, for which $h^*_5 = 0$, $h^*_4 \neq 0$;
   - Class C, both $h^*_{4,5} \neq 0$.

   If the condition $h^*_5 \neq 0$ is satisfied, we can write (24), in f–variables (see (19)), as

   $$(\ln |f^*_5| + f_5 - f_4)^* = 0,$$

   which is solved by arbitrary functions $f_5(x^2, x^3, s)$ and

   $$f_4(x^1, x^2, s) = f_{4[0]}(x^1, x^2) + \ln |f^*_5| + f_5, \tag{27}$$

   Bracketed subscripts indicate "constants" of integration with respect to the s variable. The general solution of (24) expressing h_5 via h_4 is

   $$\begin{aligned} h_5(x^2, x^3, s) &= \left[h_{5[1]}(x^2, x^3) + h_{5[2]}(x^2, x^3) \int \sqrt{h_4(x^2, x^3, s)} ds, \right]^2 \\ &= h_{5[0]}(x^2, x^3)[1 + \varpi(x^2, x^3)s]^2, \qquad h^*_4 = 0, \end{aligned} \tag{28}$$

   the integration "constants", $f_{5[0,1,2]}(x^2, x^3)$ and $\varpi(x^2, x^3)$, are determined by boundary conditions and locally anisotropic limits as well as from the requirement that the Eqs. (25) and (26) are compatible. Conversely, for a given h_5, the general solution of (24) is (27) which can be rewritten with respect to variables $h_{4,5}$, as

   $$h_4(x^2, x^3, s) = h_{4[0]}(x^2, x^3) \left[\left(\sqrt{|h_5(x^2, x^3, s)|} \right)^* \right]^2 \tag{29}$$

3. If the functions $h_4(x^2, x^3, s)$ and $h_5(x^2, x^3, s)$ are known, then Eqs. (25) become linearly independent algebraic equations for $w_{2,3}$

   $$w_{2,3}\beta + \alpha_{2,3} = 0,$$

If in the case of vacuum Einstein equations $h_5^* = 0$, we have $\alpha_{\widehat{i}} = \beta = 0$ and as a consequence, Eq. (25) becomes trivial, allowing arbitrary values of the functions $w_{2,3}(x^2, x^3, s)$. For $h_5^* \neq 0$ we must impose the condition $\alpha_{2,3} = 0$, or identify these values with the corresponding non–diagonal components of the energy–momentum tensor. We also note that ansatz (4) admits an arbitrary function $w_1(x^2)$ which is not contained in the vacuum Einstein equations. This function can be fixed by requiring that it be compatible with some locally isotropic solutions.

4. Eqs. (26) can be solved in general form if the functions $h_4(x^2, x^3, s)$ and $h_5(x^2, x^3, s)$ (and therefore the coefficient γ from (20)) are known,

$$
\begin{aligned}
n_{2,3}(x^2, x^3, s) &= n_{2,3[0]}(x^2, x^3) + n_{2,3[1]}(x^2, x^3) \int \frac{h_4(x^2, x^3, s)}{h_5^{3/2}(x^2, x^3, s)} ds, \qquad \gamma \neq 0; \\
n_{2,3}(x^2, x^3, s) &= n_{2,3[0]}(x^2, x^3) + n_{2,3[1]}(x^2, x^3) s, \qquad \gamma = 0,
\end{aligned}
\tag{30}
$$

where the functions $n_{2,3[0]}(x^2, x^3)$ and $n_{2,3[1]}(x^2, x^3)$ are defined from some boundary conditions. Again the ansatz (4) admits another arbitrary function $n_1(x^2)$ which is not contained in the vacuum Einstein equations. This function can be fixed by requiring compatibility with some locally isotropic solutions.

If the metric coefficients h_4 and h_5 are solutions to Eq. (24) then one can define two new $[\widehat{h}_4 = \eta_4 h_4,\ \widehat{h}_5 = \eta_5 h_5]$ solutions. We call the functions $\eta_{4,5} = \eta_{4,5}(x^2, x^3, s)$ gravitational polarizations since they modify the behavior of the metric coefficients h_4 and h_5 in a manner similar to how a material modifies the behavior of electric and magnetic fields in media.

The "renormalization" of $h_{4,5}$ into $\widehat{h}_{4,5}$ results in the "renormalization" of $n_{2,3}$: $n_{2,3} \to \widehat{n}_{2,3}$ from formula (30) with $h_{4,5} \to \widehat{h}_{4,5}$ and $\gamma \to \widehat{\gamma}$.

III. LOCALLY ISOTROPIC WORMHOLES, FLUX TUBES, AND ANISOTROPIC RUNNING OF CONSTANTS

We give a brief review of the locally isotropic wormhole and flux tube solutions (DS-solutions) constructed in Refs. [5, 11], and their anisotropic generalization proposed in Ref. [6]. The isotropic DS-solutions represent 5D gravitational field configurations which carry "electric" and/or "magnetic" charges. Various authors have studied related 5D solutions: Liu and Wesson investigated 5D solitonic solutions [12]; they also considered 5D charged black holes [13]; 5D wormhole configurations with electromagnetic charges were studied in Ref. [14]; Ref. [15] looks at 5D solutions with magnetic charge; a general reference for 5D Kaluza-Klein theory and solutions is [16]. The anisotropic constructions considered in this paper are slightly different from Ref. [6]. Here we use a fixed conformal factor, in order to construct anisotropic solutions with background geometries more general than spherical. The study of these more general background geometries will be carried out in section **V**.

A. 5D Locally isotropic wormholes and flux tubes

Ref. [11] considered the following spherically symmetric 5D metric, with off–diagonal terms

$$
\begin{aligned}
ds^2_{(DS)} &= e^{2\nu(r)}dt^2 - dr^2 - a(r)(d\theta^2 + \sin^2\theta d\varphi^2) \\
&- r_0^2 e^{2\psi(r)-2\nu(r)}\left[d\chi_{(DS)} + \omega(r)dt + n\cos\theta d\varphi\right]^2,
\end{aligned} \tag{31}
$$

$\chi_{(DS)}$ is the 5^{th} coordinate; r, θ, φ are $3D$ spherical coordinates; n is an integer; $r \in \{-R_0, +R_0\}$ ($R_0 \leq \infty$) and r_0 is a constant. All functions $\nu(r), \psi(r)$ and $a(r)$ were considered to be even functions of r satisfying $\nu'(0) = \psi'(0) = a'(0) = 0$. Here we shall study a particular class of this metric, with $\nu(r) = 0$. We also introduce a new 5^{th} coordinate

$$
\chi = \chi_{(DS)} - \mu(\theta,\varphi)^{-1}\int d\xi(\theta,\varphi)
$$

for which

$$
d\chi_{(DS)} + n\cos\theta d\varphi = d\chi + n\cos\theta d\theta
$$

and

$$
\frac{\partial\xi}{\partial\varphi} = \mu n\cos\theta, \qquad \frac{\partial\xi}{\partial\theta} = -\mu n\cos\theta,
$$

if the factor $\mu(\theta,\varphi)$ is taken, for instance,

$$
\mu(\theta,\varphi) = \exp(\theta - \varphi)\ |\cos\theta|^{-1}.
$$

This redefinition of the 5^{th} coordinate $\chi_{(DS)} \to \chi$, with $d\chi$ elongated by N–coefficients proportional to t, r, θ (isotropic coordinates) allows us to consider anisotropies on coordinates (φ, χ). The metric (31), in coordinates $(t, r, \theta, \varphi, \chi)$, and for $\nu(r) = 0$ is equivalently rewritten as

$$
ds^2_{(DS)} = dt^2 - dr^2 - a(r)(d\theta^2 + \sin^2\theta d\varphi^2) - r_0^2 e^{2\psi(r)}\left[d\chi + \omega(r)dt + n\cos\theta d\theta\right]^2 \tag{32}
$$

This form of the metric will be used to find new, anisotropic solutions of Einstein's equations. The coefficient $\omega(r)$ in (32) is treated as the t –component of the electromagnetic potential and $n \cos\theta$ as the θ-component. These electromagnetic potentials lead to the metric having radial Kaluza-Klein "electrical" and "magnetic" fields. The 5D Kaluza-Klein "electric" field is

$$E_{KK} = r_0 \omega' e^{3\psi} = q_0 / a(r) \tag{33}$$

the "electric" charge $q_0 = r_0 \omega'(0)$ can be parametrized as

$$q_0 = 2\sqrt{a(0)} \sin \alpha_0.$$

The corresponding dual, "magnetic" field is

$$H_{KK} = Q_0 / a(r) \tag{34}$$

with "magnetic" charge $Q_0 = n r_0$ parametrized as

$$Q_0 = 2\sqrt{a(0)} \cos \alpha_0,$$

The following "circle" relation

$$\frac{(q_0^2 + Q_0^2)}{4a(0)} = 1 \tag{35}$$

relates the "electric" and "magnetic" charges. As the free parameters of the metric are varied there are five classes of solutions with the properties:

1. $Q_0 = 0$ or $H_{KK} = 0$, a wormhole–like "electric" object;
2. $q_0 = 0$ or $E_{KK} = 0$, a finite "magnetic" flux tube;
3. $q_0 = Q_0$ or $H_{KK} = E_{KK}$, an infinite "electromagnetic" flux tube;
4. $Q_0 < q_0$ or $H_{KK} < E_{KK}$, a wormhole–like "electromagnetic" object;
5. $Q_0 > q_0$ or $H_{KK} > E_{KK}$, a finite, "magnetic–electric" flux tube.

Metric (32) is a particular example of a d–metric of type (7), with the ansatz functions given by (4), or equivalently (8). For the coordinates $x^1 = t, x^2 = r, x^3 = \theta, y^4 = s = \chi, y^5 = p = \varphi$ the set of ansatz functions

$$\begin{aligned} g_1 &= 1, \qquad g_2 = -1, \qquad g_3 = -a(r), \\ h_4 &= -a(r) \sin^2\theta, \qquad h_5 = -r_0^2 e^{2\psi(r)}, \\ w_i &= 0, \qquad n_1 = \omega(r), \qquad n_2 = 0, \qquad n_3 = n \cos\theta \end{aligned} \tag{36}$$

defines a trivial, locally isotropic solution of the vacuum Einstein equations (23)–(26) which satisfies the conditions $h^*_{4,5} = 0.$ We next deal with anisotropic deformations of this solution.

B. Anisotropic generalizations of DS–solution

The simplest way to obtain anisotropic wormhole / flux tube solutions [6] is to take r_0^2 from (32) or (36)) not as a constant, but as "renormalized" via $r_0^2 \to \widehat{r}_0^2 = \widehat{r}_0^2(r, \theta, s).$

1. *DS-solutions with anisotropy via $s = \chi$*

From the isotropic solution (36) we generate an anisotropic solution of Class A by taking

$$\widehat{h}_4(r, \theta) = h_4(r, \theta) = -a(r) \sin^2 \theta,$$

with $\eta_4 = 1$ so that $\widehat{h}^*_4 = h^*_4 = 0,$ but $\widehat{h}^*_5 = \eta^*_5(r, \theta, \chi) h_5(r) \neq 0.$ Using Eq. (28) we parametrize

$$\widehat{r}_0^2(\chi) \simeq r^2_{0(0)}[1 + \varpi(r, \theta)\chi]^2 \tag{37}$$

so that

$$\begin{aligned} h_5(r, \theta, \chi) &= \eta_5(r, \theta, \chi) h_5(r) \\ h_{5[0]}(r, \theta) &= h_5(r) = -r_0^2 e^{2\psi(r)}, \qquad \eta_5(r, \theta, \chi) = [1 + \varpi(r, \theta)\chi]^2. \end{aligned}$$

Under the conditions in this subsection β and $\alpha_{2,3}$ from Eqs. (17) (18) (19) (and therefore $w_{2,3}$) can be arbitrary functions. Here we will require $w_{2,3} \to 0$ in the locally isotropic limit, $\varpi\chi \to 0$. From Eq. (30) $n_{2,3}$ depends on the anisotropic variable $s = \chi$ in the following way

$$n_3(r, \theta, \chi) = n_{3[0]}(r, \theta) + n_{3[1]}(r, \theta)[1 + \varpi_0 \chi]^{-2}$$

with $\varpi(r, \theta) = \varpi_0 = const$. We obtain the locally isotropic limit of (36), for $\varpi\chi \to 0$ if we fix the boundary conditions with $n_{2[0,1]} = 0, n_{3[0]} = 0$, $n_{3[1]}(r, \theta) = n \cos \theta$ and $n_1 = \omega(r).$

The 5D gravitational vacuum polarization induced by variation of "constant" $\widehat{r}_0(\chi)$ renormalizes the electromagnetic charge as $q(\chi) = \widehat{r}_0(\chi)\omega'(r = 0)$. In terms of the angular parametrization the "electric" charge becomes

$$q(\chi) = 2\sqrt{a(0)} \sin \alpha(\chi),$$

The "electric" field from (33) becomes

$$E_{KK} = \frac{q(\chi)}{a(r)}.$$

The renormalization of the magnetic charge, $Q_0 \to Q(\chi)$, can be obtained using the renormalized "electric" charge in relationship (35) and solving for $Q(\chi)$. The form of (35) implies that the running of $Q(\chi)$ will be the opposite that of $q(\chi)$. For example, if $q(\chi)$ increases with χ then $Q(\chi)$ will decrease. The locally anisotropic polarizations $\alpha(\chi)$ are either defined from experimental data or computed from a quantum model of 5D gravity. With the coordinates taken as $x^1 = t, x^2 = r, x^3 = \theta, y^4 = s = \chi, y^5 = p = \varphi$, one can construct a locally anisotropic solution of the vacuum Einstein equations (23)–(26) by making the following identifications for the ansatz functions from (32)

$$\begin{aligned} g_1 &= 1, \qquad g_2 = -1, \qquad g_3 = -a(r), \\ \widehat{h}_4 &= h_4 = -a(r)\sin^2\theta, \qquad \eta_4 = 1 \\ \widehat{h}_5 &= \eta_5 h_5, \qquad h_5(r) = -r_0^2 e^{2\psi(r)}, \qquad \eta_5 = [1+\varpi_0\chi]^2 \\ w_i &= 0, \qquad n_1 = \omega(r), \qquad n_2 = 0, \qquad n_3 = n\cos\theta[1+\varpi_0\chi]^{-2}, \end{aligned} \tag{38}$$

This generalizes the DS–solution (32) by allowing the Kaluza–Klein electric and magnetic charges to be dependent on (*i.e.* scale with) the 5^{th} coordinate $s=\chi$. We will call these the χ –solutions).

2. *DS-solutions with anisotropy via φ*

In a similar fashion we can consider anisotropic dependencies with respect to $s=\varphi$. These will be called φ–solutions. The simplest option is to take $h_5^* = 0$ but $h_4^* \neq 0$, *i.e.* to define a solution with

$$\begin{aligned} \widehat{h}_4(r,\theta,\varphi) &= \eta_4 h_4(r), \qquad h_4(r) = -r_0^2 e^{2\psi(r)} \\ \widehat{h}_5(r,\theta) &= h_5(r,\theta) = -a(r)\sin^2\theta, \\ \eta_4 &= \exp[\varpi(r,\theta,\varphi)], \qquad \eta_5 = 1, \end{aligned}$$

this allows $w_{2,3}$ to take arbitrary values since β and $\alpha_{2,3}$ from Eqs. (25) vanish. For small polarizations we can approximate

$$\widehat{h}_4(r,\theta,\varphi) = h_{4[0]}(r,\theta)[1+\varpi\varphi].$$

The general solution of (26) for $\widehat{\gamma} = -(\ln|\widehat{h}_4|)^*$ is

$$n_{2,3}(r,\theta,\varphi) = n_{2,3[0]}(r,\theta) + n_{2,3[1]}(r,\theta)\int \exp\varpi(r,\theta,\varphi)d\varphi;$$

we take $w_1 = \omega(r), w_2 = 0, w_3 = n\cos\theta$, $n_{2[0]}(r,\theta) = 0$ and $n_{3[0]}(r,\theta) = n\cos\theta$, $n_{,3[1]}(r,\theta) = 1$ which are compatible with the local isotropic limit (*i.e.* $\varpi(r,\theta,\varphi) \to 0$ and $\int \varpi(r,\theta,\varphi)d\varphi \to 0$). Taking the coordinates as $x^1 = t, x^2 = r, x^3 = \theta, y^4 = s = \varphi, y^5 = p = \chi$ the following form for the ansatz functions

$$\begin{aligned}
g_1 &= 1, \qquad g_2 = -1, \qquad g_3 = -a(r), \\
\widehat{h}_4 &= \eta_4 h_4, \qquad h_4 = -r_0^2 e^{2\psi(r)}, \qquad \eta_4 = \exp[\varpi(r,\theta,\varphi)], \\
\widehat{h}_5 &= h_5, \qquad h_5 = -a(r)\sin^2\theta, \qquad \eta_5 = 1, \\
w_1 &= \omega(r), \qquad w_2 = 0, \qquad w_3 = n\cos\theta, \\
n_1 &= 0, \qquad n_{2,3} = n_{2,3[1]}(r,\theta)\int \exp\varpi(r,\theta,\varphi)d\varphi,
\end{aligned} \tag{39}$$

gives a locally anisotropic generalization of the DS–metric (32) for anisotropic dependencies on the angle φ.

We have constructed two classes of locally anisotropic generalizations of the DS–solution : $s = \varphi$ (*i.e.* anisotropic angular polarizations) or $s = \chi$ (*i.e.* dependence of the Kaluza–Klein charges on the 5^{th} coordinate). If the metric (32), describing these two classes of solutions, were given with respect to a coordinate frame (1) non–diagonal terms would occur, and the study of these solutions would be more difficult.

IV. GRAVITATIONAL θ–POLARIZATION OF KALUZA-KLEIN CHARGES

We can further generalize the forms (38) and (39) to generate new solutions of the 5D vacuum Einstein equations with deformations of the constants r_0^2 and n with respect to the θ variable. These θ deformations take the form of the equation for an ellipsoid in polar coordinates. This again leads to varying electric, q, and magnetic, Q, charges.

A. Gravitational renormalization of Kaluza-Klein charges via variable r_0

In this subsection we give a solution for which the Kaluza-Klein charges are gravitationally renormalized by the radius becoming dependent on θ (*i.e.* in Eq. (35) $a(0) \to a(\theta)$.

1. θ–renormalization of charges for χ–solutions

The easiest way to obtain such θ–polarizations for the χ–solutions of (28) and (37) is to consider the coordinates as $x^1 = t, x^2 = r, x^3 = \theta, y^4 = s = \chi, y^5 = p = \varphi$, and let the ansatz functions take the form

$$
\begin{aligned}
g_1 &= 1, \qquad g_2 = -1, \qquad g_3 = -a(r), \\
\widehat{h}_4 &= h_4 = -a(r)\sin^2\theta, \qquad \eta_4 = 1, \\
\widehat{h}_5 &= \eta_5 h_5(r), \qquad h_5(r) = -r_0^2 e^{2\psi(r)}, \qquad \eta_5 = [1+\varepsilon_r\cos\theta]^{-2}[1+\varpi_0\chi]^2, \\
w_i &= 0, \qquad n_1 = \omega(r), \qquad n_2 = 0, \qquad n_3 = n\cos\theta\,[1+\varpi_0\chi]^{-2},
\end{aligned}
\tag{40}
$$

where η_5 is the polarizaton and

$$
\widehat{r}_0^2(r,\theta,\chi) \simeq r_{0(0)}^2\,[1+\varepsilon_r\cos\theta]^{-2}\,[1+\varpi_0\chi]^2,
$$

where ε_r is the eccentricity. The "constant", $\widehat{r}_0$, has both an elliptic variation in θ (*i.e.* $r_{0(0)}\,[1+\varepsilon_r\cos\theta]^{-1}$) and a linear variation on the 5^{th} coordinate (*i.e.* $[1+\varpi(r,\theta)\chi]$).

For these χ–solutions with elliptic variations as given above the 5D Kaluza-Klein charges get renormalized through the elliptic variation of $\widehat{r}_0(r,\theta,\varepsilon_r,\chi)$. This renormalizes the "electric" charge as

$$
q(\theta,\chi) = r_0\sqrt{\eta_5(\theta,\varepsilon_r,\chi)}\,\omega'(0) = \sqrt{\eta_5(\theta,\varepsilon_r,\chi)}\,q_0
$$

or in terms of the angular parametrization

$$
q(\theta,\chi) = 2\sqrt{a(0)}\sqrt{\eta_5(\theta,\varepsilon_r,\chi)}\sin\alpha_0,
$$

The "electric" field (33) transforms into

$$
E_{KK} = \frac{q(\theta,\chi)}{a(r)} = \frac{q_0}{(\sqrt{\eta_5})^{-1}a(r)}
$$

$(\sqrt{\eta_5})^{-1}$ can be treated as an anisotropic, gravitationally induced permittivity. The renormalization of the magnetic charge, $Q_0 \to Q(\theta,\chi)$, can be obtained from Eq. (35) using $q(\theta,\chi)$ from above. In this case the corresponding dual, "magnetic" field is $H_{KK} = Q(\theta,\chi)/a(r)$ with the "magnetic" charge $Q_0 = nr_0$ given by

$$
Q = 2\sqrt{a(0)}\sqrt{\eta_5(\theta,\varepsilon_r,\chi)}\cos\alpha_0,
$$

These gravitationally polarized charges satisfy the circumference equation (35) with variable radius $2\sqrt{a(0)}\sqrt{\eta_5(\theta,\varepsilon_r,\chi)}$

$$\frac{(q_0^2+Q_0^2)}{4a(0)\eta_5(\theta,\varepsilon_r,\chi)}=1. \tag{41}$$

2. *Elliptic renormalization of charges for φ–solutions*

The φ–solutions can also be modified to have an elliptic variation with respect to θ. As in the case of the χ–solutions this gives an effective gravitational renormalization of the charges. With the coordinates defined as $x^1=t, x^2=r, x^3=\theta, y^4=s=\varphi, y^5=p=\chi$ the form of this variation of the φ – solutions is

$$\begin{aligned}
g_1 &= 1, \qquad g_2=-1, \qquad g_3=-a(r), \\
\widehat{h}_4 &= \eta_4 h_4, \qquad h_4=-r_0^2 e^{2\psi(r)}, \qquad \eta_4=[1+\varepsilon_r\cos\theta]^{-2}\exp\left[\varpi(r,\theta,\varphi)\right], \\
\widehat{h}_5 &= h_5=-a(r)\sin^2\theta, \qquad \eta_5=1, \qquad \widehat{h}_5^*=h_5^*=0, \\
w_1 &= \omega(r), \qquad w_2=0, \qquad w_3=n\cos\theta, \\
n_1 &= 0, \qquad n_{2,3}=n_{2,3[1]}(r,\theta)\int \exp\varpi(r,\theta,\varphi)d\varphi,
\end{aligned} \tag{42}$$

The renormalized charges arise as in the previous example via $r_0^2\rightarrow \widehat{r}_0^2(r,\theta,\varphi)$. The "electric" charge becomes

$$q(\theta,\varphi)=r_0\sqrt{\eta_4(\theta,\varepsilon_r,\varphi)}\omega'(0)=\sqrt{\eta_4(\theta,\varepsilon_r,\varphi)}q_0$$

in terms of the angular parametrization this becomes

$$q(\theta,\varphi)=2\sqrt{a(0)}\sqrt{\eta_4(\theta,\varepsilon_r,\varphi)}\sin\alpha,$$

The "electric" field (33) transforms into

$$E_{KK}=\frac{q(\theta,\varphi)}{a(r)}=\frac{q_0}{(\sqrt{\eta_4})^{-1}a(r)}$$

$(\sqrt{\eta_4})^{-1}$ can be treated as an anisotropic gravitationally induced permittivity depending on the angular variables. The renormalized magnetic charge, $Q_0\rightarrow Q(\theta,\varphi)$, can be determined using Eq. (35) and $q(\theta,\varphi)$ giving

$$Q(\theta,\varphi)=2\sqrt{a(0)}\sqrt{\eta_4(\theta,\varepsilon_r,\varphi)}\cos\alpha_0,$$

The "magnetic" field is then $H_{KK} = Q(\theta,\varphi)/a(r)$. The gravitationally polarized charges satisfies Eq. (35) with a variable radius of $2\sqrt{a(0)}\sqrt{\eta_4(\theta,\varepsilon_r,4)}$.

$$\frac{(q_0^2+Q_0^2)}{4a(0)\eta_4(\theta,\varepsilon_r,\varphi)} = 1. \tag{43}$$

There is again an elliptical variation in θ, and also an anisotropic dependence in φ.

Comparing formulas (41) and (43) we find that there are two type of anisotropic gravitational polarizations of the charges: in the first case the running with respect to the 5^{th} coordinate is emphasized; in the second case the anisotropy comes just from the angular variables. In both cases there is an elliptical dependence on θ.

B. Gravitational renormalization of Kaluza-Klein charges via r_0 and n

A different class of solutions from those given in Eqs. (38), (39) and (40), (42) can be constructed if, in addition to r_0, we allow the n in the $n\cos\theta$ term in Eq. (32) to vary. For the χ–solutions this variable n will affect n_3, while for the φ–solutions it will affect w_3. The variability of r_0 and n is parameterized using the gravitational vacuum polarizations $\kappa_r(r,\theta,s)$ and $\kappa_n(r,\theta,s)$ as

$$r_0 \rightarrow \widehat{r}_0 = r_0/\kappa_r(r,\theta,s) \ \text{ and } n \rightarrow \widehat{n} = n/\kappa_n(r,\theta,s)$$

where $\kappa_r(r,\theta,s) = [\sqrt{\eta_4(r,\theta,s)}]^{-1}$ or $= [\sqrt{\eta_5(r,\theta,s)}]^{-1}$. The polarized charges are

$$q = q_0/\kappa_r = 2\sqrt{a(0)}\sin\alpha_0/\kappa_r$$

and

$$Q = Q_0/\kappa_n = 2\sqrt{a(0)}\cos\alpha_0/\kappa_n,$$

Using these charges in Eq. (35) gives the formula for an ellipse in the charge space coordinates (q_0,Q_0),

$$\frac{q_0^2}{4a(0)\kappa_r^2} + \frac{Q_0^2}{4a(0)\kappa_n^2} = 1, \tag{44}$$

the axes of the ellipse are $2\sqrt{a(0)}\kappa_r$ and $2\sqrt{a(0)}\kappa_n$. Formula (44) contains formulas (41) and (43) as special cases.

The form of the χ and φ – solutions from the previous two subsections gets modified by this "elliptic" renormalization of the Kaluza-Klein charges.

- With the coordinates defined as $x^1 = t, x^2 = r, x^3 = \theta, y^4 = s = \chi, y^5 = p = \varphi$ the χ–solutions with $\kappa_r = [\sqrt{\eta_5\,(r,\theta,\chi)}]^{-1}$ take the form

$$
\begin{aligned}
g_1 &= 1, \qquad g_2 = -1, \qquad g_3 = -a(r), \\
\widehat{h}_4 &= h_4 = -a(r)\sin^2\theta, \qquad \widehat{h}_4^* = 0, \qquad h_4^* = 0, \qquad \eta_4 = 1, \\
\widehat{h}_5 &= \eta_5 h_5, \qquad h_5 = -r_0^2 e^{2\psi(r)}, \qquad \eta_5 = 1/\kappa_r^2(r,\theta,\chi), \\
w_i &= 0, \qquad n_1 = \omega(r), \qquad n_2 = 0, \qquad n_3 = n\cos\theta/\kappa_n(r,\theta,\chi).
\end{aligned} \tag{45}
$$

- With the coordinates defined as $x^1 = t, x^2 = r, x^3 = \theta, y^4 = s = \varphi, y^5 = p = \chi$ the φ–solutions with $\kappa_r = [\sqrt{\eta_4\,(r,\theta,\chi)}]^{-1}$ take the form

$$
\begin{aligned}
g_1 &= 1, \qquad g_2 = -1, \qquad g_3 = -a(r), \\
\widehat{h}_4 &= \eta_4 h_4, \qquad h_4 = -r_0^2 e^{2\psi(r)}, \qquad \eta_4 = 1/\kappa_r^2(r,\theta,\varphi), \\
\widehat{h}_5 &= h_5 = -a(r)\sin^2\theta, \qquad \eta_5 = 1, \qquad h_5^* = 0 \\
w_1 &= \omega(r), \qquad w_2 = 0, \qquad w_3 = n\cos\theta/\kappa_n\,(r,\theta,\varphi)\,, \\
n_1 &= 0, \qquad n_{2,3} = n_{2,3[1]}(r,\theta)\int \ln|\kappa_r(r,\theta,\varphi)|d\varphi,
\end{aligned} \tag{46}
$$

V. WORMHOLES IN ELLIPSOIDAL, CYLINDRICAL, BIPOLAR AND TOROIDAL BACKGROUNDS

The locally anisotropic wormhole / flux tube solutions presented in the previous sections are anisotropic deformations from a spherical 3D hypersurface background. These solutions can be generalized to other rotational hypersurface geometry backgrounds. In this section we will give the explicit forms for these generalized solutions and analyze their basic properties. The notations and metric relations for the 3D Euclidean rotational hypersurfaces that we use will be those of Ref. [17].

A. Elongated rotation ellipsoid hypersurfaces

An elongated rotation ellipsoid hypersurface (a 3D e–ellipsoid) is given by the formula

$$
\frac{x^2+y^2}{\sigma^2-1} + \frac{z^2}{\sigma^2} = \tilde{a}^2(r), \tag{47}
$$

where $\sigma \geq 1$, and x, y, z here are the usual Cartesian coordinates. $\tilde{a}(r)$ is similar to the radius in the spherical symmetric case. The 3D, ellipsoidal coordinate system is defined

$$x = \tilde{a} \sinh u \sin v \cos s, \qquad y = \tilde{a} \sinh u \sin v \sin s, \qquad z = \tilde{a} \; \cosh u \cos v, \tag{48}$$

where $\sigma = \cosh u$ and $0 \leq u < \infty, \; 0 \leq v \leq \pi, \; 0 \leq s < 2\pi$. The hypersurface metric is

$$g_{uu} = g_{vv} = \tilde{a}^2 \left(\sinh^2 u + \sin^2 v \right), \qquad g_{ss} = \tilde{a}^2 \sinh^2 u \sin^2 v. \tag{49}$$

It will be more useful to consider a conformally transformed metric, where the components in Eq.(49) are multiplied by the conformal factor $\tilde{a}^{-2} \left(\sinh^2 u + \sin^2 v \right)^{-1}$, giving

$$\begin{aligned} ds^2_{(3e)} &= du^2 + dv^2 + g_{ss}(u,v) ds^2 \\ g_{ss}(u,v) &= \sinh^2 u \sin^2 v / (\sinh^2 u + \sin^2 v). \end{aligned} \tag{50}$$

B. Flattened rotation ellipsoid hypersurfaces

In a similar fashion we consider the hypersurface equation for a flattened rotation ellipsoid (a 3D f–ellipsoid),

$$\frac{x^2 + y^2}{1 + \sigma^2} + \frac{z^2}{\sigma^2} = \tilde{a}^2(r), \tag{51}$$

here $\sigma \geq 0$ and $\sigma = \sinh u$. In this case the 3D coordinate system is defined as

$$x = \tilde{a} \cosh u \sin v \cos s, \qquad y = \tilde{a} \cosh u \sin v \sin s, \qquad z = \tilde{a} \sinh u \cos v, \tag{52}$$

where $0 \leq u < \infty, \; 0 \leq v \leq \pi, \; 0 \leq s < 2\pi$. The hypersurface metric is

$$g_{uu} = g_{vv} = \tilde{a}^2 \left(\sinh^2 u + \cos^2 v \right) \qquad g_{\varphi\varphi} = \tilde{a}^2 \sinh^2 u \cos^2 v, \tag{53}$$

Again for later convenience we consider a conformally transformed version of this metric

$$\begin{aligned} ds^3_{(3f)} &= du^2 + dv^2 + g_{ss}(u,v) ds^2, \\ g_{ss}(u,v) &= \sinh^2 u \cos^2 v / (\sinh^2 u + \cos^2 v). \end{aligned} \tag{54}$$

C. Ellipsoidal cylindrical hypersurfaces

The formula for an ellipsoidal cylindrical hypersurface is

$$\frac{x^2}{\sigma^2} + \frac{y^2}{\sigma^2 - 1} = \rho^2, \;\; z = s, \tag{55}$$

where $\sigma \geq 1$. The 3D radial coordinate is given as $\tilde{a}^2 = \rho^2 + s^2$. The 3D coordinate system is defined

$$x = \rho \cosh u \cos v, \qquad y = \rho \sinh u \sin v, \qquad z = s,$$

where $\sigma = \cosh u$ and $0 \leq u < \infty,\ 0 \leq v \leq \pi$. Using the expressions for x, y and Eq. (55) we can make the change $\rho(x, y) \to \rho(u, v)$. The hypersurface metric is

$$g_{uu} = g_{vv} = \rho^2(u, v)\ \left(\sinh^2 u + \sin^2 v\right), \qquad g_{ss} = 1;$$

we will again consider a conformally transformed version of this metric

$$\begin{aligned} ds^2_{(3c)} &= du^2 + dv^2 + g_{ss}(u, v, \rho(u, v))ds^2, \\ g_{ss}(u, v) &= 1/\rho^2(u, v)\ (\sinh^2 u + \sin^2 v). \end{aligned} \tag{56}$$

D. Bipolar coordinates

Now we consider a bipolar hypersurface given by the formula

$$\left(\sqrt{x^2 + y^2} - \frac{\tilde{a}(r)}{\tan\xi}\right)^2 + z^2 = \frac{\tilde{a}^2(r)}{\sin^2\xi}, \tag{57}$$

which describes a hypersurface obtained by rotating the circles

$$\left(y - \frac{\tilde{a}(r)}{\tan\xi}\right)^2 + z^2 = \frac{\tilde{a}^2(r)}{\sin^2\xi}$$

around the z axis; because $|\tan\xi|^{-1} < |\sin\xi|^{-1}$, the circles intersect the z axis. The relationship between the Cartesian coordinates and the bipolar coordinates is

$$x = \frac{\tilde{a}(r)\sin\xi\cos s}{\cosh\tau - \cos\xi}, \qquad y = \frac{\tilde{a}(r)\sin\xi\sin s}{\cosh\tau - \cos\xi}, \qquad z = \frac{\tilde{a}(r)\sinh\tau}{\cosh\tau - \cos\xi},$$

where $-\infty < \tau < \infty, 0 \leq \xi < \pi, 0 \leq s < 2\pi$. The hypersurface metric is

$$g_{\tau\tau} = g_{\xi\xi} = \frac{\tilde{a}^2(r)}{(\cosh\tau - \cos\xi)^2}, \qquad g_{ss} = \frac{\tilde{a}^2(r)\sin^2\xi}{(\cosh\tau - \cos\xi)^2},$$

which, after multiplication by the conformal factor $(\cosh\tau - \cos\sigma)^2/\rho^2$ becomes

$$ds^2_{(3b)} = d\tau^2 + d\xi^2 + g_{ss}(\xi)ds^2, \qquad g_{ss}(\xi) = \sin^2\xi. \tag{58}$$

E. Toroidal coordinates

Now we consider a toroidal hypersurface with nontrivial topology given by the formula

$$\left(\sqrt{x^2+y^2}-\widetilde{a}(r)\ (\coth\xi)\right)^2+z^2=\frac{\widetilde{a}^2(r)}{\sinh^2\xi}, \tag{59}$$

the relationship to the Cartesian coordinates is given by

$$x=\frac{\widetilde{a}(r)\sinh\tau\cos s}{\cosh\tau-\cos\xi},\qquad y=\frac{\widetilde{a}(r)\sin\xi\sin s}{\cosh\tau-\cos\xi},\qquad z=\frac{\widetilde{a}(r)\sinh\xi}{\cosh\tau-\cos\xi},$$

where $-\pi<\xi<\pi, 0\leq\tau<\infty, 0\leq s<2\pi$. The hypersurface metric is

$$g_{\sigma\sigma}=g_{\tau\tau}=\frac{\widetilde{a}^2(r)}{(\cosh\tau-\cos\xi)^2},\qquad g_{ss}=\frac{\widetilde{a}^2(r)\sin^2\xi}{(\cosh\tau-\cos\xi)^2}$$

After multiplication by the conformal factor $(\cosh\tau-\cos\sigma)^2/\widetilde{a}^2(r)$ this takes the same form as (58)

$$ds^3_{(3t)}=d\tau^2+d\xi^2+g_{ss}(\xi)ds^2,\qquad g_{ss}(\xi)=\sin^2\xi, \tag{60}$$

Although this looks identical to the metric in (58) the coordinates (τ,ξ,s) have different meanings in each case. This can be seen by the different ranges for the two cases.

F. Anisotropic wormholes in rotation deformed hypersurface backgrounds

In order to construct wormholes which exhibit the various 3D geometries cataloged above, we will associate one of the ansatz functions of the wormhole solutions with $g_{ss}(x^2,x^3)$. For the χ–solutions this is accomplished by letting $h_4=g_{ss}(x^2,x^3)$; for the φ–solutions this is accomplished letting by $h_5=g_{ss}(x^2,x^3)$.

The construction of such solutions is based on the assumption that $g_{ss}(x^2,x^3)$ for the five non-spherical geometries listed above is to be taken as $h_4=g_{ss}(x^2,x^3)$ (for χ–solutions), or as $h_5=g_{ss}(x^2,x^3)$ (for $\varphi(z)$–solutions). In each case $h_{4,5}$ is multiplied by corresponding gravitational polarizations, $\eta_{4,5}$, so as to give wormhole/flux tube configurations of the form (38) , (40) , (45) (for χ–solutions) or configurations of the form (39), (42), (46) (for $\varphi(z)$–solutions).

1. The χ–solutions

For the five 3D geometries given above, the d–metrics (8) for the χ–solutions from Eqs. (38), (40), (45), have the coordinates defined as

$$x^k = \begin{cases} (t,u,v), 0 \leq u < \infty,\ 0 \leq v \leq \pi, \cosh u \geq 1,\ \text{ellipsoid (47)};\\ (t,u,v), 0 \leq u < \infty,\ 0 \leq v \leq \pi, \sinh u \geq 0,\ \text{ellipsoid (51)};\\ (t,u,v), 0 \leq u < \infty,\ 0 \leq v \leq \pi, \cosh u \geq 1, \text{cylinder (55)};\\ (t,\tau,\xi), -\infty < \tau < \infty, 0 \leq \xi < \pi, \text{bipolar (57)};\\ (t,\tau,\xi), 0 \leq \tau < \infty, -\pi < \xi < \pi, \text{torus (59)}; \end{cases}$$

$$y^4 = s = \chi, \qquad y^5 = p = \begin{cases} \varphi \in [0,2\pi),\ \text{ellipsoids ; bipolar ; torus};\\ z \in (-\infty,\infty), \text{cylinder}; \end{cases}$$

and the ansatz functions given as

$$g_1 = 1, \qquad g_2 = -1, \qquad g_3 = -1, \tag{61}$$

$$\widehat{h}_4 = \eta_4 h_4, \qquad h_4 = g_{ss}(x^2,x^3) = \begin{cases} \frac{\sinh^2 u \sin^2 v}{\sinh^2 u + \sin^2 v},\ \text{ellipsoid (50)};\\ \frac{\sinh^2 u \cos^2 v}{\sinh^2 u + \cos^2 v}, \text{ellipsoid (54)};\\ \frac{\rho^{-2}(u,v)}{\sinh^2 u + \sin^2 v}, \text{cylinder (55)};\\ \sin^2\xi, \text{bipolar (57)}; \text{torus (59)}; \end{cases}$$

$$\eta_4 = \left[\left(\sqrt{\widehat{h}_5(x^2,x^3,\chi)}\right)^*\right]^2, \text{see (27)}, \tag{62}$$

$$\eta_5 = \begin{cases} [1+\varpi_0\chi]^2,\ \text{see (38)};\\ \left[1+\varepsilon_r \cos x^3\right]^{-2}[1+\varpi_0\chi]^2,\ \text{see (40)};\\ 1/\kappa_r^2(x^2,x^3,\chi),\ \text{see (45)}; \end{cases}$$

$$\widehat{h}_5 = \eta_5 h_5, \qquad h_5(x^2,x^3,\chi) = -r_0^2 \exp\{2\psi[r(x^2,x^3,\chi)]\}; \qquad w_i = 0;$$

$$r = \widetilde{a}^{(invers)}\left(x^2,x^3,\chi\right) \ \text{from} \quad (47);\ (51);\ (55);\ \ (57); (59);$$

$$n_1 = \omega(x^2), \qquad n_2 = 0, \qquad n_3 = n\cos x^2 \times \begin{cases} [1+\varpi_0\chi]^{-2},\ \text{see (38)};\\ [1+\varpi_0\chi]^{-2},\ \text{see (40)};\\ 1/\kappa_n(x^2,x^3,\chi),\ \text{see (45)}. \end{cases}$$

The formulas (61) describe wormhole / flux tube configurations which are defined self–consistently in the various rotational hypersurface backgrounds listed above. As in the case of the spherical background these solutions have an anisotropic deformation with respect to the given hypersurface backgrounds.

2. The φ–solutions

Now we give the form of the d–metric (8) for the φ–solutions embedded in the five 3D rotational hypersurfaces. The coordinates are taken as

$$
x^k = \left\{
\begin{array}{l}
(t,u,v), 0 \leq u < \infty,\ \ 0 \leq v \leq \pi, \cosh u \geq 1,\ \text{ellipsoid (47)};\\
(t,u,v), 0 \leq u < \infty,\ \ 0 \leq v \leq \pi, \sinh u \geq 0,\ \text{ellipsoid (51)};\\
(t,u,v), 0 \leq u < \infty,\ \ 0 \leq v \leq \pi, \cosh u \geq 1, \text{cylinder (55)};\\
(t,\tau,\xi), -\infty < \tau < \infty, 0 \leq \xi < \pi, \text{bipolar (57)};\\
(t,\tau,\xi), 0 \leq \tau < \infty, -\pi < \xi < \pi, \text{torus (59)};
\end{array}
\right.
$$

$$
y^4 = s = \left\{
\begin{array}{l}
\varphi \in [0,2\pi),\ \text{ellipsoid ; bipolar ; torus;}\\
z \in (-\infty,\infty), \text{cylinder;}
\end{array}
\right. \qquad y^5 = p = \chi,
$$

and the a [illegible]

$$
g_1 = 1, \qquad g_2 = -1, \qquad g_3 = -1, \tag{63}
$$

$$
\widehat{h}_4 = \eta_4 h_4, \qquad h_4 = g_{ss}(x^2,x^3) = \left\{
\begin{array}{l}
\frac{\sinh^2 u \sin^2 v}{\sinh^2 u + \sin^2 v},\ \text{ellipsoid (50)};\\
\frac{\sinh^2 u \cos^2 v}{\sinh^2 u + \cos^2 v}, \text{ellipsoid (54)};\\
\frac{\rho^{-2}(u,v)}{\sinh^2 u + \sin^2 v}, \text{cylinder (55)};\\
\sin^2 \xi, \text{bipolar (57); torus (59)};
\end{array}
\right.
$$

$$
\eta_4 = \left\{
\begin{array}{l}
\exp[\varpi(x^2,x^3,s],\ \text{see (39)};\\
\left[1 + \varepsilon_r \cos x^3\right]^{-2} \exp[\varpi(x^2,x^3,s)],\ \text{see (42)};\\
1/\kappa_r^2(x^2,x^3,s),\ \text{see (46)};
\end{array}
\right. \tag{64}
$$

$$
\eta_5 = \left\{
\begin{array}{l}
\eta_{5[0]}\left(x^2,x^3\right) + \eta_{5[1]}\left(x^2,x^3\right) \int \eta_4\left(x^2,x^3,s\right) ds,\ \text{for } \widehat{h}_4^* \neq 0,\\
\eta_{5[0]}\left(x^2,x^3\right) + \eta_{5[1]}\left(x^2,x^3\right) s;\ \text{for } \widehat{h}_4^* = 0;
\end{array}
\right.
$$

$$
\widehat{h}_5 = \eta_5 h_5, \qquad h_5(x^2,x^3,s) = -r_0^2 \exp\{2\psi[r(x^2,x^3,s)]\};
$$

$$
r = \widetilde{a}^{(invers)}\left(x^2,x^3,s\right)\ \text{from}\quad (47);\ (51);\ (55);\quad (57); (59);
$$

$$
w_1 = \omega(x^2), \qquad w_2 = 0, \qquad w_3 = n \cos x^2 \times \left\{
\begin{array}{l}
1,\ \text{see (39)};\\
1,\ \text{see (42)};\\
1/\kappa_n(x^2,x^3,s),\ \text{see (46)}.
\end{array}
\right.
$$

$$
n_1 = 0, \qquad n_{2,3} = n_{2,3[0]}\left(x^2,x^3\right) \times \left\{
\begin{array}{l}
\int \exp \varpi_0\left(x^2,x^3,s\right) ds,\ \text{see (39)};\\
\int \exp \varpi_0\left(x^2,x^3,s\right) ds,\ \text{see (42)};\\
\int |\ln \kappa_n(x^2,x^3,s)| ds,\ \text{see (46)}.
\end{array}
\right.
$$

Formulas (63) describe a large class of wormhole / flux tube configurations which are defined self–consistently in the various 3D rotation hypersurface backgrounds. The deformations in this case come from the angular coordinate $s = \varphi$ (for the ellipsoid, bipolar and toroidal cases) or from the axial coordinate $s = z$ (for the cylindrical case). These "deformation" or anisotropic coordinates, φ or z, are also third coordinates about which the rotation of the hypersurfaces occurs.

VI. CONCLUSIONS

The construction of wormhole and/or flux tube solutions in modern string theory, extra dimensional gravity and quantum chromodynamics is of fundamental importance in understanding these theories (especially their non-perturbative aspects). Such solutions are difficult to find, and the solutions which are known usually have a high degree of symmetry. In this paper we have applied the method of anholonomic frames to construct the general form of wormholes and flux tubes in 5D Kaluza-Klein theory. These solutions have local anisotropy which would make their study using holonomic frames difficult. This helps to demonstrate the usefulness of the anholonomic frames method in studying anisotropic solutions. Most physical situations do not possess a high degree of symmetry, and so the anholonomic frames method provides a useful mathematical framework for studying these less symmetric configurations.

The key result of this paper is the demonstration that off–diagonal metrics in 5D Kaluza–Klein theory can be parametrized into forms that define new, interesting classes of solutions of Einstein's vacuum equations. These solutions represent wormhole and flux tube configurations which are locally anisotropic. These anisotropic solutions reduce to previously known spherically symmetric wormhole metrics [4, 5, 11] in the local isotropic limit. These anisotropic solutions also extend the idea of Salam, Strathee and Perracci [3] that including off–diagonal components in higher dimensional metrics gives rise to gauge fields and charges. Not only do we find "electric" and"magnetic" charges for our solutions, but the anisotropies in the 5^{th} coordinate (χ) and/or in the angular coordinate (φ) give a gravitational scaling or running of these Kaluza-Klein charges. Such a gravitational scaling of charges could provide an experimental signature for the presence of extra dimensions (*i.e.* if some charge were observed to exhibit a running which was not in agreement with that given by 4D quantum field theory this could be evidence for a gravitational running of the charge).

In the first part of this paper these anisotropic solutions were constructed as deformations from a spherical background. In the final section of this paper we showed that it is possible to construct a large variety of such anisotropic solutions as deformations from various background geometries: elliptic (elongated and flattened), cylindrical, toroidal an bipolar.

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