Natural TeV-Scale Gravity and coupling constant unification, in Heterotic M-Theory, with the usual hidden and visible sectors swapped

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Abstract

I consider a class of Grand Unified models, in which E8 is broken to $\text{SU}(3) \times \text{SU}(2) \times \text{SU}(2) \times \text{SU}(2) \times \text{U}(1)_Y$, then to $\text{SU}(3) \times \text{SU}(2)_{diag} \times \text{U}(1)_Y$. The breaking of $(\text{SU}(2))^3$ to $\text{SU}(2)_{diag}$ reduces the SU(2) coupling constant, at unification, by a factor of $1/\sqrt{3}$, so that the ratio of the SU(3) and $\text{SU}(2)_{diag}$ coupling constants, at unification, is equal to the ratio observed at about 1 TeV. By choosing a suitable alignment of $\text{U}(1)_Y$, and introducing a generalization of the CKM matrix, the $\text{U}(1)_Y$ coupling constants of the observed fermions, at unification, can also be arranged to have the ratios, to the SU(3) coupling constant, that are observed at about 1 TeV. This suggests a model of Heterotic M-Theory, with a standard embedding of the spin connection in one of the E8s, but with the visible sector now having the E8 that is unbroken at unification. The universe is pictured as a thick pipe, where the long direction of the pipe represents the four extended dimensions, the circumference represents a compact six-manifold, and the radial direction represents the eleventh dimension. The inner radius of the pipe is about 10^{-19} metres, and the outer radius of the pipe is about 10^{-14} metres. We live on the inner surface of the pipe. The low-energy generation structure and the high mass of the top quark follow from the breaking pattern, and gravity and the Yang-Mills interactions are unified at about 1 TeV. Parity breaking must occur spontaneously in four dimensions, rather than being inherited from ten dimensions. The stability of the proton might be correlated with the entries in the CKM matrix.

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1   Introduction

It has been observed by Arkani-Hamed, Dimopoulos, and Dvali, (ADD), [1, 2] that in models with n extra compact dimensions, if the non-gravitational fields are confined to a zone, in the compact dimensions, whose size is small compared to the zone in which the gravitational fields propagate, the fundamental scale of gravity can be reduced from the Planck mass to the TeV scale, because the effective four-dimensional Planck mass, M_{Pl}, at distances larger than the size, R_{ADD}, of the gravity-only compact dimensions, is related to the fundamental gravity scale, M_{Gr}, by the relation:

$$M_{Pl} = (R_{ADD} M_{Gr})^{\frac{n}{2}} M_{Gr} \tag{1}$$

Perhaps the most natural such model is Heterotic M-Theory [3, 4]. In this model the universe is eleven-dimensional, and pictured as the Cartesian product of the four extended dimensions, a six-dimensional compact space, and a one-dimensional interval. The fields of eleven-dimensional supergravity propagate in the eleven-dimensional bulk, and couple to a ten-dimensional supersymmetric Yang-Mills theory, with an E8 gauge group, at each of the two ten-dimensional "ends" of the universe. If there is N = 1 supersymmetry in the four extended dimensions, then, as originally shown by Candelas, Horowitz, Strominger, and Witten, (CHSW), [5], the six-dimensional compact space has SU(3) holonomy, and the spin connection, on the six-dimensional compact space, is naturally embedded in the E8 gauge group at one of the two "ends" of the universe, leaving the E8 gauge group at the other "end" of the universe, unbroken. The different "instanton numbers" of the E8 gauge fields, at the two ten-dimensional "ends" of the universe, then force the six-dimensional compact space to have a different volume at each of the two "ends" of the universe, as demonstrated by Witten, [6]. The volume of the six-dimensional compact space is larger at the "end" of the universe that has the spin connection embedded in its E8 gauge field, and this is usually assumed to be the visible "end" of the universe, because the embedding of the spin connection breaks the E8 gauge group to E6, and enables the Weyl condition on the ten-dimensional spinor to be transmitted to a four-dimensional spinor in the **27** of E6, thus giving a natural explanation for the observed parity violation of the couplings of the fermions to the SU(2) gauge field.

It is possible for the volume of the six-dimensional compact space, at the "end" of the universe where it is larger, to be much larger than at the "end" of the universe where it is smaller, and Witten, [6], considered such a limit, in order to derive a lower

bound on Newton's constant, assuming that the visible "end" of the universe is the "end" where the volume of the six-dimensional compact space is larger, and showed that this bound is approximately saturated, in the framework of conventional Grand Unification. Witten effectively worked to first order in the dimensionless parameter $\frac{1}{m_{11}^3 \sqrt{V}}$, where m_{11} is the eleven-dimensional Planck mass, and V is the proper volume of the six-dimensional compact space, at the "end" of the universe where it is largest. This parameter has to be small compared to 1, for the eleven-dimensional description to be valid. Witten showed that, within his linearized approximation, the proper volume of the six-dimensional compact space decreased linearly with proper distance, along the eleventh dimension, from the "end" where the volume is largest, with the coefficient being a constant factor of order 1, times

$$\frac{1}{m_{11}^3} \left| \int_X \omega \wedge \frac{\mathrm{tr} F \wedge F - \frac{1}{2}\mathrm{tr} R \wedge R}{8\pi^2} \right| \tag{2}$$

Here ω is the Kahler form of the six-dimensional compact space X, and F is the field strength of either of the E8's. The integral in (2) is equal to $V^{\frac{1}{3}}$, times a constant factor of order 1, that is the same for all models, times a number M, that depends on integer-valued topological invariants, and has a fixed value for a particular model. Thus, up to a constant factor of order 1, the proper length, ρ, of the universe, in the eleventh dimension, is limited by

$$\rho < \frac{m_{11}^3 V^{\frac{2}{3}}}{|M|} \tag{3}$$

since the volume of the six-dimensional compact space vanishes, at the end of the universe where it is smaller, when this bound is saturated.

M has usually been of order 1, in specific models. Here I want to consider, instead, models in which $|M|$ has a fixed *large* value. It is hard to find a formula for M in the literature, and it is by no means clear that models, with $|M|$ large, actually exist. Comparing two papers [7, 8] by Choi et al., suggests that M might have the form of a sum of integer-valued quantities, over the independent periods ω_I of the integer (1,1) cohomology, divided by the cube root of a triple sum of other integer-valued quantities, over the ω_I. I am not certain that this is the correct way to compare the two papers, but if it is, then since the number of independent ω_I has the finite value $h = \dim H^{1,1}(X)$, we would expect that in general M is of order $h/(h^3)^{\frac{1}{3}}$, and thus of order 1. However, since the integer-valued quantities in the numerator and the denominator have different

structures, it is reasonable to conjecture that models with large $|M|$ can nevertheless exist. Here I shall simply assume that models with large $|M|$ exist, and study their properties.

We now have two small parameters, $\frac{1}{m_{11}^3\sqrt{V}}$ and $\frac{1}{|M|}$. However, $|M|$ cannot be large compared to $m_{11}^3\sqrt{V}$, since, with (3), that would imply $\rho \ll V^{\frac{1}{6}}$ even when the inequality in (3) is saturated, which is a geometric impossibility, since the smallest possible proper volume of the six-dimensional compact space, at a proper distance ρ from the end of the universe where it is largest, is about $(V^{\frac{1}{6}} - \rho)^6$, which for $\rho \ll V^{\frac{1}{6}}$ is not much smaller than V, whereas we already know that when (3) is saturated, the volume of the six-dimensional compact space vanishes, at the end of the universe where it is smaller. The largest possible value of $|M|$ is about $m_{11}^3\sqrt{V}$, so that for fixed M, the smallest possible value of V is about $\frac{M^2}{m_{11}^6}$. If V takes its smallest possible value, for a given fixed value of M, then when the inequality in (3) is saturated, we have $\rho \simeq V^{\frac{1}{6}}$, so that the seven compact dimensions, which, in a pictorial two-dimensional representation, looked like a cylinder to start with, have changed shape, first into a truncated cone, and finally into a disk with a hole in the middle. When V takes this minimum value, for the given fixed value of M, the upper bound on ρ given by (3), and the lower bound on ρ given by the geometrical constraint, precisely coincide. Such a configuration has a smaller range of allowed small deformations than a more general configuration, so it is reasonable to conjecture that such a configuration might be stable. This gives a natural realization of the ADD proposal, if we assume that the visible "end" of the universe is actually the one where the volume of the six-dimensional compact space is smaller, which is the one whose E8 gauge group is *not* broken by embedding the spin connection into it. If the six-dimensional compact space is roughly isotropic, we can use equation (1), with $n = 7$. For example, if M_{Gr} is about 1 TeV, $R_{ADD}M_{Gr}$ is about 40,000, and R_{ADD} is about 10^{-14} metres. Identifying M_{Gr} with M_{11}, we see that the fixed instanton number, M, needed to force R_{ADD} up to this value, is about 10^{14}. This number determines the four-dimensional Planck mass, but is more in the nature of an "accidental" constant of nature, like the cosmological baryon to entropy ratio, frozen at its present value in the first moments of the universe. In a pictorial three-dimensional representation, the universe looks like a thick pipe, where the long direction of the pipe represents the four extended dimensions, the circumferential direction represents the compact six-manifold, and the radial direction represents the eleventh dimension. If unification occurs at about a TeV, the inner radius of the pipe is about 10^{-19} metres,

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and the outer radius of the pipe is about 10^{-14} metres. We live on the inner surface of the pipe. It is important to bear in mind that if models with large $|M|$ do exist, their compact six-dimensional spaces might be extremely complicated, a fact which this pictorial representation omits.

If unification occurs at around a TeV, the original motivation for the CHSW construction [5], which was to have $N = 1$ supersymmetry at low energy, in order to stabilize the gauge hierarchy of conventional Grand Unification, is no longer relevant. However many of the results of CHSW are still very useful. For example, they find that if the four extended dimensions are maximally symmetric, then they must be flat, so that the four-dimensional cosmological constant vanishes. Furthermore Witten's M-Theory calculation [6], which I used above, is based on finding a solution of the eleven-dimensional field equations, that preserves supersymmetry. Therefore I shall assume that all the assumptions made by CHSW and Witten still apply, with the single exception that the assumed hidden and visible sectors are swapped.

Choosing the visible end of the universe to be the one without the spin connection embedded in its gauge group, means that the CHSW mechanism, for generating a chiral spinor in four dimensions, from the Weyl condition on the spinor in ten dimensions, no longer operates. The four-dimensional action, obtained by dimensional reduction, will be invariant under parity, so that parity invariance must be spontaneously broken in four dimensions. On dimensional reduction, the ten-dimensional Yang-Mills spinor field reduces to a sum of terms, each of which has the form of a Cartesian product of a four-dimensional spinor field, depending on position in the four extended dimensions, and a six-dimensional spinor field, depending on position in the six-dimensional compact space. There is one term in the sum for each independent normal mode of the spinor field on the six-dimensional compact space, and each term in the sum corresponds to an independent four-dimensional spinor field.

CHSW find that when the holonomy group of the six-dimensional compact space is precisely SU(3), which is one of the conditions to have precisely $N = 1$ supersymmetry in four dimensions, there are precisely two real covariantly constant spinors, η and $i\gamma\eta$, on the six-dimensional compact space. It is natural to assume that $\frac{1}{2}(\eta + i\gamma\eta)$ and $\frac{1}{2}(\eta - i\gamma\eta)$ correspond, respectively, to a single massless left-handed spinor field, and a single massless right-handed spinor field, in four dimensions, each in the **248** of E8, which is both the adjoint and the fundamental. We shall find that the left-handed states, of all the observed fermions and antifermions, can be fitted into a single left-

handed E8 fundamental, except for the left-handed components of the top antiquark. It is therefore natural to suppose that there are no other massless fermion modes on the six-dimensional compact space, and that the left-handed components of the top antiquark come from the lightest massive fermion mode on the internal compact space, which is why the top quark is so heavy.

The general prospects for obtaining dynamical symmetry breaking in E8 Grand Unification, of the kind required to push up the masses of the unobserved chiral partners of the observed fermions, have been discussed recently by Adler [9], who also gives a useful history of E8 Grand Unification, and many references to relevant recent work. Here I shall concentrate on reconciling E8 Grand Unification with the ADD proposal.

The possibility of realizing the ADD proposal in Heterotic M-Theory, in the context of non-standard embeddings of the spin connection, was noted by Benakli [10], and by Cerdeño and Muñoz [11].

2   Gauge coupling unification

If the ADD proposal is to be combined with Grand Unification, then the gauge couplings have to unify at the TeV scale, rather than at 10^{16} GeV. One way this could work is if the running of the coupling constants somehow accelerates, so that the couplings run to their conventional unification values at the TeV scale, rather than at 10^{16} GeV. This possibility was studied by Dienes, Dudas, and Gherghetta [12, 13], and by Arkani-Hamed, Cohen, and Georgi [14].

An alternative possibility is to embed $SU(3) \times SU(2) \times U(1)$ into the Grand Unification group in an unusual way, so that the values of the coupling constants, at unification, are equal to their observed values, as evolved conventionally to the TeV scale. Usually the coupling constant of a simple non-Abelian subgroup of a Grand Unification group, at unification, is equal to the coupling constant of the Grand Unification group, irrespective of how the subgroup is embedded in the Grand Unification group. An exception occurs [10, 14] if the initial breaking of the Grand Unification group produces N copies of the of the simple subgroup, and the N copies of the simple subgroup then break into their "diagonal" subgroup. In this case, after the second stage of the breaking, the coupling constant of the "diagonal" subgroup is equal to $\frac{1}{\sqrt{N}}$ times the coupling constant of the Grand Unification group. Effectively, the gauge field, in each of the N copies of the simple non-Abelian subgroup, becomes equal to $\frac{1}{\sqrt{N}}$

times the "diagonal" gauge field, plus massive vector terms that can be ignored at low energies. The sum of the N copies of the Yang-Mills action, of the simple non-Abelian subgroup, then becomes equal to the Yang-Mills action of the "diagonal" subgroup, whose coupling constant is $\frac{1}{\sqrt{N}}$ times the coupling constant of the Grand Unification group.

Looking at the observed values of the reciprocals of the $SU(3) \times SU(2) \times U(1)$ fine structure constants, at M_Z, normalized so as to meet at unification in $SU(5)$ Grand Unification, [15], (Mohapatra [16], page 22):

$$
\begin{aligned}
\alpha_3^{-1}(M_Z) &= 8.47 \pm .22 \\
\alpha_2^{-1}(M_Z) &= 29.61 \pm .05 \\
\alpha_1^{-1}(M_Z) &= 58.97 \pm .05
\end{aligned}
\tag{4}
$$

we see that they are quite close to being in the ratios 1, 3, 6.

If we evolve them in the MSSM, [16], then α_3^{-1} and α_2^{-1} reach an exact ratio of 1, 3, at 1.32 TeV, at which point α_3^{-1} is equal to 9.73. At this point, α_1^{-1} is equal to 56.20, which is 4% off being 6 times α_3^{-1}. Alternatively, if we evolve them in the SM, [17], then α_3^{-1} and α_2^{-1} reach an exact ratio of 1, 3, at 413 GeV, at which point α_3^{-1} is equal to 10.12. At this point, α_1^{-1} is equal to 58.00, which is 4% off being 6 times α_3^{-1}.

Thus it is natural to consider the breaking of E8 to $SU(3) \times SU(2) \times SU(2) \times SU(2) \times U(1)_Y$, followed by the breaking of $(SU(2))^3$ to $SU(2)_{diag}$, and seek an embedding of $U(1)_Y$ that gives the correct hypercharges at unification. I have summarized the required left-handed fermions of the first generation, together with their hypercharges, Y, [17], the coefficients of their $U(1)_Y$ couplings in $SU(5)$ Grand Unification, and the required coefficients of their $U(1)_Y$ couplings, in Table 1. Here I have assumed that α_3^{-1} and α_1^{-1} are in the ratio 1, 6, at unification, but it would be useful to study models that achieve this within a few percent, since the correct form of running to unification is not yet known. Since the running of the coupling constants is always by small amounts, the additional states in these models, not yet observed experimentally, will not alter the unification mass, or the value of the $SU(3)$ coupling constant at unification, which is equal to the E8 coupling constant at unification, by a large amount. Thus this class of models generically predicts that the unification mass is about a TeV, and the E8 fine structure constant, at unification, is about $\frac{1}{10}$.

Witten [6] finds that the Grand Unification fine structure constant, at unification, is equal to $\frac{(4\pi)^{\frac{2}{3}}}{2m_{11}^6 V_v} = \frac{2.7}{m_{11}^6 V_v}$, where m_{11} is the eleven-dimensional Planck mass, and V_v

First generation LH states				
Multiplet	Y	$SU(3) \times SU(2)$ content	$SU(5)$ coefficient	required coefficient
$\begin{pmatrix} u_R & u_G & u_B \\ d_R & d_G & d_B \end{pmatrix}$	$\frac{1}{3}$	(3,2)	$\frac{1}{\sqrt{60}}$	$\frac{1}{\sqrt{360}}$
$\begin{pmatrix} \bar{u}_R & \bar{u}_G & \bar{u}_B \end{pmatrix}$	$-\frac{4}{3}$	$(\bar{3},1)$	$\frac{-4}{\sqrt{60}}$	$\frac{-4}{\sqrt{360}}$
$\begin{pmatrix} \bar{d}_R & \bar{d}_G & \bar{d}_B \end{pmatrix}$	$\frac{2}{3}$	$(\bar{3},1)$	$\frac{2}{\sqrt{60}}$	$\frac{2}{\sqrt{360}}$
$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}$	-1	(1,2)	$\frac{-3}{\sqrt{60}}$	$\frac{-3}{\sqrt{360}}$
$\begin{pmatrix} e^+ \end{pmatrix}$	2	(1,1)	$\frac{6}{\sqrt{60}}$	$\frac{6}{\sqrt{360}}$
$\begin{pmatrix} \bar{\nu}_e \end{pmatrix}$	0	(1,1)	absent	0

Table 1: Weak hypercharge, $SU(3) \times SU(2)$ assignments, coefficient of the coupling to the $U(1)_Y$ vector boson in $SU(5)$, and the required coefficient of the coupling to the $U(1)_Y$ vector boson, for the left-handed fermions of the first generation.

is the volume of the six-dimensional compact space at the visible end of the universe. Hence $m_{11}^6 V_v = 27$, so if R_6 denotes the diameter of the inner surface of the pipe, then $m_{11} R_6$ is about 1.7. Thus $\frac{1}{R_6}$, the Grand Unification mass, and the eleven-dimensional Planck mass can all be close to 1 TeV. Witten's long wavelength expansion [6] is a good approximation throughout most of the volume of the pipe, but breaks down near the inner surface of the pipe. The Yang-Mills coupling constants of the fields on the outer surface of the pipe are extremely small, and they are probably effectively free fields, even at cosmological distances. However they will interact gravitationally, with the same value of Newton's constant as we observe, and are candidates to form part of the dark matter of the universe.

It is convenient to use an $SU(9)$ basis for E8. On breaking E8 to $SU(9)$, the **248** of E8 splits to the **80**, **84**, and **$\overline{84}$** of $SU(9)$. Here the **80** is the adjoint of $SU(9)$, the **84** has three totally antisymmetrized $SU(9)$ fundamental subscripts, and the **$\overline{84}$** has three totally antisymmetrized $SU(9)$ antifundamental subscripts. We can proceed in close analogy to the $SU(5)$ model. The fundamental representation generators $(t_\alpha)_{ij}$ are normalized to satisfy [17]

$$\text{tr}\,(t_\alpha t_\beta) = \frac{\delta_{\alpha\beta}}{2} \tag{5}$$

The generators of the required representations are as follows:

$$\textit{Antifundamental} \qquad\qquad (T_\alpha)_{ij} = - (t_\alpha)_{ji} \tag{6}$$

$$\textit{Adjoint} \qquad\qquad (T_\alpha)_{ij,km} = (t_\alpha)_{ik}\, \delta_{mj} - \delta_{ik}\, (t_\alpha)_{mj} \tag{7}$$

$$\mathbf{84} \qquad\qquad (T_\alpha)_{ijk,mpq} = \frac{1}{6} \left((t_\alpha)_{im}\, \delta_{jp} \delta_{kq} \pm \text{ seventeen terms} \right) \tag{8}$$

$$\overline{\mathbf{84}} \qquad\qquad (T_\alpha)_{ijk,mpq} = \frac{1}{6} \left(- (t_\alpha)_{mi}\, \delta_{pj} \delta_{qk} \pm \text{ seventeen terms} \right) \tag{9}$$

where the additional terms in (8) and (9) antisymmetrize with respect to permutations of (i,j,k), and with respect to permutations of (m,p,q). We can check directly that these generators satisfy the same commutation relations as $(t_\alpha)_{ij}$, with the same structure constants.

The breaking of E8 to $SU(3) \times SU(2) \times SU(2) \times SU(2) \times U(1)_Y$ can be studied by analyzing the breaking of $SU(9)$ to $SU(3) \times SU(2) \times SU(2) \times SU(2) \times U(1)_Y$. It is convenient to use block matrix notation. Each $SU(9)$ fundamental index is replaced by a pair of indexes, an upper-case letter and a lower-case letter. The upper-case letter runs from 1 to 4, and indicates which subgroup in the sequence $SU(3) \times SU(2) \times SU(2) \times SU(2)$ the block belongs to. Thus $B = 1$ denotes the $SU(3)$, $B = 2$ denotes the first $SU(2)$, $B = 3$ denotes the second $SU(2)$, and $B = 4$ denotes the third $SU(2)$. The lower-case index is a fundamental index for the subgroup identified by the upper-case index it belongs to. It is important to note that the range of a lower-case index depends on the value of the upper-case index it belongs to, so we have to keep track of which lower-case indexes belong to which upper-case indexes. Each $SU(9)$ antifundamental index is treated in the same way, except that the lower-case index is now an antifundamental index for the appropriate subgroup. The summation convention is applied to both upper-case letters and lower-case letters that derive from an $SU(9)$ fundamental or antifundamental index, but we have to remember that lower-case indexes are to be summed over first, because their ranges of summation depend on the values of the upper-case indexes they belong to. Each $SU(9)$ adjoint representation index, which in the notation above, is a lower-case Greek letter, is replaced by a pair of indexes, an upper-case letter and a lower-case letter, where the upper-case letter runs from 1 to 5, and identifies which subgroup in the sequence $SU(3) \times SU(2) \times SU(2) \times SU(2) \times U(1)_Y$ a generator belongs to, and the lower-case letter runs over all the generators of the subgroup identified by the upper-case letter it belongs to. When an upper-case adjoint

representation index takes the value 5, the associated lower-case index takes a single value, 1. The summation convention is applied to a lower-case letter that derives from an SU(9) adjoint representation index, but *not* to an upper-case letter that derives from an SU(9) adjoint representation index.

We can now list all the blocks in the **80**, the **84**, and the $\overline{\mathbf{84}}$, and display their SU(3) × SU(2) content. This is displayed in Table 2 for the **80**, and in Table 3 for the $\overline{\mathbf{84}}$, with all the lower-case indexes suppressed.

The SU(9) generators, in the SU(3) × SU(2) × SU(2) × SU(2) × U(1)$_Y$ subgroup, may be taken as follows, in the block matrix notation.

$$\left(t^{(9)}_{Aa} \right)_{BiCj} = \delta_{AB}\delta_{AC} \left(t_{Aa} \right)_{ij} \qquad (1 \le A \le 4) \tag{10}$$

$$\begin{aligned}
\left(t^{(9)}_{51} \right)_{BiCj} &= \tfrac{1}{\theta} \left(\sigma_1\delta_{1B}\delta_{1C}\delta_{ij} + \sigma_2\delta_{2B}\delta_{2C}\delta_{ij} + \sigma_3\delta_{3B}\delta_{3C}\delta_{ij} + \sigma_4\delta_{4B}\delta_{4C}\delta_{ij} \right) \\
&= \tfrac{1}{\theta} \left(\sum_{A=1}^{4} \sigma_A\delta_{AB}\delta_{AC}\delta_{ij} \right)
\end{aligned} \tag{11}$$

Here $(t_{Aa})_{ij}$ denotes the fundamental representation generator number a, of non-Abelian subgroup number A, in the list above. Thus for $A = 1$, the subgroup is SU(3), a runs from 1 to 8, and i and j each run from 1 to 3, while for $A = 2$, 3, or 4, the subgroup is SU(2), a runs from 1 to 3, and i and j each run from 1 to 2.

σ_1, σ_2, σ_3, and σ_4 are real numbers parametrizing the embedding of the U(1)$_Y$ subgroup in SU(9), and thus in E8, and θ is a normalization factor.

In using this notation, we have to take sensible precautions, such as grouping within brackets, to keep track of which lower-case indexes belong to which upper-case indexes. In equation (11), it would be wrong to "factor out" the δ_{ij}, because it represents a 3 by 3 matrix in one term, and a 2 by 2 matrix in the other three terms.

The tracelessness of $\left(t^{(9)}_{51} \right)_{BiCj}$ implies:

$$0 = 3\sigma_1 + 2 \left(\sigma_2 + \sigma_3 + \sigma_4 \right) \tag{12}$$

and the normalization condition (5) implies:

$$\theta^2 = 6\sigma_1^2 + 4 \left(\sigma_2^2 + \sigma_3^2 + \sigma_4^2 \right) \tag{13}$$

As an example, I consider the states in the left-handed $\overline{\mathbf{84}}$. The covariant derivative is [17]

$$\mathcal{D}_\mu = \partial_\mu - igA_{\mu\alpha}T_\alpha \tag{14}$$

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States in the **80**				
Blocks	Number of distinct blocks	SU(3) × SU(2) content	Number of states	coefficient of coupling to U(1)
ψ_{11}	1	(8,1)	8	0
ψ_{22}	1	(1,3)	3	0
ψ_{33}	1	(1,3)	3	0
ψ_{44}	1	(1,3)	3	0
ψ_{diag} ψ_{diag} ψ_{diag}	not applicable	$(1,1) + (1,1)+$ $+(1,1)$	3	0
ψ_{12}	1	(3,2)	6	$\frac{\sigma_1 - \sigma_2}{\theta}$
ψ_{13}	1	(3,2)	6	$\frac{\sigma_1 - \sigma_3}{\theta}$
ψ_{14}	1	(3,2)	6	$\frac{\sigma_1 - \sigma_4}{\theta}$
ψ_{21}	1	$(\bar{3},2)$	6	$\frac{-\sigma_1 + \sigma_2}{\theta}$
ψ_{31}	1	$(\bar{3},2)$	6	$\frac{-\sigma_1 + \sigma_3}{\theta}$
ψ_{41}	1	$(\bar{3},2)$	6	$\frac{-\sigma_1 + \sigma_4}{\theta}$
ψ_{23}	1	$(1,3) + (1,1)$	4	$\frac{\sigma_2 - \sigma_3}{\theta}$
ψ_{24}	1	$(1,3) + (1,1)$	4	$\frac{\sigma_2 - \sigma_4}{\theta}$
ψ_{34}	1	$(1,3) + (1,1)$	4	$\frac{\sigma_3 - \sigma_4}{\theta}$
ψ_{32}	1	$(1,3) + (1,1)$	4	$\frac{-\sigma_2 + \sigma_3}{\theta}$
ψ_{42}	1	$(1,3) + (1,1)$	4	$\frac{-\sigma_2 + \sigma_4}{\theta}$
ψ_{43}	1	$(1,3) + (1,1)$	4	$\frac{-\sigma_3 + \sigma_4}{\theta}$

Table 2: The states in the **80**, organized by their SU(3) × SU(2) × SU(2) × SU(2) assignments, showing their SU(3) × SU(2)$_{diag}$ content, and the coefficients of their couplings to a U(1) gauge field, parametrized as in equation (11).

States in the $\overline{\mathbf{84}}$					
Blocks			Number of distinct blocks	SU(3) × SU(2) content	Number of states	coefficient of coupling to U(1)
ψ_{111}			1	$(1,1)$	1	$\frac{-3\sigma_1}{\theta}$
ψ_{211}	ψ_{121}	ψ_{112}	1	$(3,2)$	6	$\frac{-2\sigma_1-\sigma_2}{\theta}$
ψ_{311}	ψ_{131}	ψ_{113}	1	$(3,2)$	6	$\frac{-2\sigma_1-\sigma_3}{\theta}$
ψ_{411}	ψ_{141}	ψ_{114}	1	$(3,2)$	6	$\frac{-2\sigma_1-\sigma_4}{\theta}$
ψ_{221}	ψ_{212}	ψ_{122}	1	$(\bar{3},1)$	3	$\frac{-\sigma_1-2\sigma_2}{\theta}$
ψ_{331}	ψ_{313}	ψ_{133}	1	$(\bar{3},1)$	3	$\frac{-\sigma_1-2\sigma_3}{\theta}$
ψ_{441}	ψ_{414}	ψ_{144}	1	$(\bar{3},1)$	3	$\frac{-\sigma_1-2\sigma_4}{\theta}$
ψ_{123} ψ_{213} ψ_{231} ψ_{132} ψ_{312} ψ_{321}			1	$(\bar{3},3)+(\bar{3},1)$	12	$\frac{-\sigma_1-\sigma_2-\sigma_3}{\theta}$
ψ_{124} ψ_{214} ψ_{241} ψ_{142} ψ_{412} ψ_{421}			1	$(\bar{3},3)+(\bar{3},1)$	12	$\frac{-\sigma_1-\sigma_2-\sigma_4}{\theta}$
ψ_{134} ψ_{314} ψ_{341} ψ_{143} ψ_{413} ψ_{431}			1	$(\bar{3},3)+(\bar{3},1)$	12	$\frac{-\sigma_1-\sigma_3-\sigma_4}{\theta}$
ψ_{222}	ψ_{333}	ψ_{444}	these three blocks are empty			
ψ_{223}	ψ_{232}	ψ_{322}	1	$(1,2)$	2	$\frac{-2\sigma_2-\sigma_3}{\theta}$
ψ_{224}	ψ_{242}	ψ_{422}	1	$(1,2)$	2	$\frac{-2\sigma_2-\sigma_4}{\theta}$
ψ_{332}	ψ_{323}	ψ_{233}	1	$(1,2)$	2	$\frac{-\sigma_2-2\sigma_3}{\theta}$
ψ_{334}	ψ_{343}	ψ_{433}	1	$(1,2)$	2	$\frac{-2\sigma_3-\sigma_4}{\theta}$
ψ_{442}	ψ_{424}	ψ_{244}	1	$(1,2)$	2	$\frac{-\sigma_2-2\sigma_4}{\theta}$
ψ_{443}	ψ_{434}	ψ_{344}	1	$(1,2)$	2	$\frac{-\sigma_3-2\sigma_4}{\theta}$
ψ_{234} ψ_{324} ψ_{342} ψ_{432} ψ_{423} ψ_{243}			1	$(1,4)+(1,2)+$ $+(1,2)$	8	$\frac{-\sigma_2-\sigma_3-\sigma_4}{\theta}$

Table 3: The states in the $\overline{\mathbf{84}}$, organized by their SU(3) × SU(2) × SU(2) × SU(2) assignments, showing their SU(3) × SU(2)$_{diag}$ content, and the coefficients of their couplings to a U(1) gauge field, parametrized as in equation (11).

so, for unbroken SU(9), the massless Dirac action in this case is [17]:

$$\bar{\psi}i\gamma^\mu\mathcal{D}_\mu\psi \;\;=\;\; \bar{\psi}i\gamma^\mu\left(\partial_\mu - igA_{\mu\alpha}T_\alpha\right)\psi \;\;=$$

$$=\;\; \bar{\psi}_{ijk}i\gamma^\mu\left(\partial_\mu\frac{1}{6}\left(\delta_{mi}\delta_{pj}\delta_{qk}\pm\text{ five terms}\right)\right.$$

$$\left. -igA_{\mu\alpha}\frac{1}{6}\left(-\left(t_\alpha\right)_{mi}\delta_{pj}\delta_{qk}\pm\text{ seventeen terms}\right)\right)\psi_{mpq} =$$

$$=\;\; \bar{\psi}_{ijk}i\gamma^\mu\partial_\mu\psi_{ijk} - 3gA_{\mu\alpha}\bar{\psi}_{ijk}\gamma^\mu\left(t_\alpha\right)_{mi}\psi_{mjk} \qquad (15)$$

where I used (9), the antisymmetry of $\bar{\psi}_{ijk}$ and ψ_{mpq} in their indexes, and the relabelling of dummy indexes. $\bar{\psi}_{ijk}$ are the right-handed **84** states, and ψ_{mpq} are the left-handed $\overline{\mathbf{84}}$ states.

Breaking SU(9) to SU(3) × SU(2) × SU(2) × SU(2) × U(1)$_Y$, and using the block matrix notation, this becomes:

$$\bar{\psi}_{BiCjDk}i\gamma^\mu\partial_\mu\psi_{BiCjDk} - 3g\sum_{A=1}^{5}A_{\mu Aa}\bar{\psi}_{BiCjDk}\gamma^\mu\left(t_{Aa}^{(9)}\right)_{EmBi}\psi_{EmCjDk} \;\;=$$

$$=\;\; \left(\bar{\psi}_{BiCjDk}i\gamma^\mu\partial_\mu\psi_{BiCjDk} - 3g\sum_{A=1}^{4}A_{\mu Aa}\bar{\psi}_{AiCjDk}\gamma^\mu\left(t_{Aa}\right)_{mi}\psi_{AmCjDk}\right.$$

$$\left. -3gA_{\mu 51}\frac{1}{\theta}\sum_{A=1}^{4}\sigma_A\bar{\psi}_{AiCjDk}\gamma^\mu\psi_{AiCjDk}\right) \qquad (16)$$

where I used (10) and (11). We can now extract the covariant derivative Dirac action terms for the various entries in Table 3, and thus determine the coefficients of their couplings to $A_{\mu 51}$. For example, a block in ψ_{BiCjDk}, where two upper-case indexes take the value 1, and the remaining upper-case index takes the value 2, 3, or 4, is a candidate to be a (3,2) quark multiplet. The sum of all terms in (16), where two upper-case indexes take the value 1, and the remaining upper-case index takes the value 2, is:

$$\left(3\bar{\psi}_{1i1j2k}i\gamma^\mu\partial_\mu\psi_{1i1j2k} - 6gA_{\mu 1a}\bar{\psi}_{1i1j2k}\gamma^\mu(t_{1a})_{mi}\psi_{1m1j2k} - 3gA_{\mu 2a}\bar{\psi}_{2i1j1k}\gamma^\mu(t_{2a})_{mi}\psi_{2m1j1k}\right.$$

$$\left. -6gA_{\mu 51}\frac{1}{\theta}\sigma_1\bar{\psi}_{1i1j2k}\gamma^\mu\psi_{1i1j2k} - 3gA_{\mu 51}\frac{1}{\theta}\sigma_2\bar{\psi}_{2i1j1k}\gamma^\mu\psi_{2i1j1k}\right) \qquad (17)$$

Now ψ_{1i1j2k} is antisymmetric under swapping the two SU(3) antifundamental indexes i and j, so that we may write:

$$\psi_{1i1j2k} = \varepsilon_{ijm}\phi_{mk} \qquad (18)$$

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and analogously:

$$\bar{\psi}_{1i1j2k} = \varepsilon_{ijm}\bar{\phi}_{mk} \qquad (19)$$

We can then use relations such as

$$\bar{\psi}_{1i1j2k}\gamma^{\mu}\psi_{1i1j2k} = \varepsilon_{ijp}\bar{\phi}_{pk}\gamma^{\mu}\varepsilon_{ijm}\phi_{mk} = 2\delta_{pm}\bar{\phi}_{pk}\gamma^{\mu}\phi_{mk} = 2\bar{\phi}_{mk}\gamma^{\mu}\phi_{mk} \qquad (20)$$

and

$$\bar{\psi}_{1i1j2k}\gamma^{\mu}\left(t_{1a}\right)_{mi}\psi_{1m1j2k} = \varepsilon_{ijp}\bar{\phi}_{pk}\gamma^{\mu}\left(t_{1a}\right)_{mi}\varepsilon_{mjq}\phi_{qk} =$$

$$= \left(\delta_{im}\delta_{pq} - \delta_{iq}\delta_{pm}\right)\bar{\phi}_{pk}\gamma^{\mu}\left(t_{1a}\right)_{mi}\phi_{qk} = \bar{\phi}_{pk}\gamma^{\mu}\left(t_{1a}\right)_{ii}\phi_{pk} - \bar{\phi}_{mk}\gamma^{\mu}\left(t_{1a}\right)_{mi}\phi_{ik} =$$

$$= -\bar{\phi}_{mk}\gamma^{\mu}\left(t_{1a}\right)_{mi}\phi_{ik} \qquad (21)$$

to express (17) as:

$$6\Big(\bar{\phi}_{ij}i\gamma^{\mu}\partial_{\mu}\phi_{ij} + gA_{\mu 1a}\bar{\phi}_{ij}\gamma^{\mu}\left(t_{1a}\right)_{ik}\phi_{kj} - gA_{\mu 2a}\bar{\phi}_{ij}\gamma^{\mu}(t_{2a})_{mj}\phi_{im}$$

$$+ gA_{\mu 51}\frac{1}{\theta}\left(-2\sigma_1 - \sigma_2\right)\bar{\phi}_{ij}\gamma^{\mu}\phi_{ij}\Big) \qquad (22)$$

Thus we see that the index i of ϕ_{ij} is an SU(3) fundamental index. The SU(2) antifundamental is equivalent to the fundamental, the relation being given by matrix multiplication by ε_{jk}, and we could, if we wished, make a further transformation to replace the SU(2) antifundamental index j of ϕ_{ij}, by an index that is manifestly in the SU(2) fundamental. When $(SU(2))^3$ is broken to $SU(2)_{diag}$, the $A_{\mu 2a}$, in the third term in (22), will be replaced, at low energy, by $\frac{1}{\sqrt{3}}B_{\mu}$, where B_{μ} is the gauge field of $SU(2)_{diag}$. The overall factor of 6 can be absorbed into the normalizations of ϕ_{ij} and $\bar{\phi}_{ij}$, so from the fourth term in (22), we can read off what the coefficient of $gA_{\mu 51}\bar{\phi}_{ij}\gamma^{\mu}\phi_{ij}$ would be, if the $\bar{\phi}_{ij}i\gamma^{\mu}\partial_{\mu}\phi_{ij}$ term had standard normalization, and thus complete the entries in the second row of Table 3.

The entries in the third column of Tables 2 and 3 can be completed by similar methods. The entries in the fifth column of Table 3 can be completed by a simple mnemonic: for each upper-case index, of the untransformed ψ_{BiCjDk}, that takes the value A, $1 \le A \le 4$, include a term $-\frac{1}{\theta}\sigma_A$. For Table 2, the mnemonic is that when the index B of ψ_{BiCj} takes the value A, $1 \le A \le 4$, so that i is in the fundamental of non-Abelian subgroup number A, include a term $\frac{1}{\theta}\sigma_A$, and when the index C of ψ_{BiCj} takes the value A, $1 \le A \le 4$, so that j is in the antifundamental of non-Abelian subgroup number A, include a term $-\frac{1}{\theta}\sigma_A$.

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Indeed, suppose we extract all terms from (16) such that ψ_{BiCjDk} has n_A upper-case indexes with the value A, $1 \leq A \leq 4$, so that $n_1 + n_2 + n_3 + n_4 = 3$. We get $\frac{3!}{n_1!n_2!n_3!n_4!}$ contributions from the $\bar{\psi}_{BiCjDk} i\gamma^\mu \partial_\mu \psi_{BiCjDk}$ term. The number of times we get σ_A, from the third term in (16), is $\frac{2!}{\tilde{n}_1!\tilde{n}_2!\tilde{n}_3!\tilde{n}_4!}$, where $\tilde{n}_B = n_B$ if $B \neq A$, and $\tilde{n}_A = n_A - 1$. But this is equal to $\frac{2!n_A}{n_1!n_2!n_3!n_4!}$. The factor $\frac{2!}{n_1!n_2!n_3!n_4!}$ combines with the explicit factor of 3, in the third term in (16), to produce the same overall factor of $\frac{3!}{n_1!n_2!n_3!n_4!}$ as found for the first term, so the coefficient of the contributions from the third term, if the contributions from the first term had standard normalization, would be $-\frac{1}{\theta} \sum_{A=1}^{4} n_A \sigma_A$. The mnemonic for Table 2 can be justified in a similar manner.

We know that we have to find couplings of the observed fermions, to the $U(1)_Y$ gauge field, that are smaller than those found in the SU(5) model [15, 16], by an overall factor that is within a few percent of $\frac{1}{\sqrt{6}}$, so it is useful to apply the same techniques to calculate the corresponding coefficients in the SU(5) model. In this case, the relations (5) and (12) completely determine the U(1) generator, up to sign, and we find the entries in the fourth column of Table 1. The entries in the fifth column have been filled in, assuming the overall factor is exactly $\frac{1}{\sqrt{6}}$.

Comparing Tables 1, 2, and 3, we see that for most of the entries in Table 1, the left-handed **80** and **$\overline{84}$** can accomodate three left-handed generations, with states to spare. The three ψ_{diag} states, in Table 2, are natural candidates for the left-handed antineutrinos, so a certain linear combination of the left-handed antineutrinos, observed indirectly in oscillation experiments, together with the corresponding linear combination of the right-handed neutrinos, is the superpartner of the $U(1)_Y$ gauge field, or in other words, of $\cos\theta_W A_\mu - \sin\theta_W Z_\mu$, [17], where A_μ denotes the photon, and Z_μ denotes the neutral weak vector boson.

For the $(\bar{3}, 1)$ left-handed antiquark states, we see that we have exactly the right number of states to accomodate three generations, but there is a snag. For both the ψ_{221} family and the ψ_{123} family, the sum of the three entries, in the fifth column of Table 3, is equal to

$$-\frac{1}{\theta}\left(3\sigma_1 + 2\sigma_2 + 2\sigma_3 + 2\sigma_4\right) \tag{23}$$

which by (12), is equal to zero. Thus we cannot accomodate three families of antiquarks whose weak hypercharges, Y, do not add up to zero. The best we can do is accomodate two up-type antiquarks, and four down-type antiquarks. Thus the left-handed top antiquark cannot be accomodated, and must come, like the right-handed top quark,

from a non-zero Kaluza-Klein mode of the six-dimensional compact space that forms the inner surface of the cross-section of the pipe. Thus the top quark is the first observed state with a substantial admixture, specifically 50%, from a non-zero Kaluza-Klein mode.

The multiplets for which we have to find a Y-coefficient of the smallest non-zero magnitude, specifically within a few percent of $\frac{1}{\sqrt{360}}$, are the (3,2) quark multiplets, for which we have six multiplets, (three in the **80**, and three in the **$\overline{84}$**), to accomodate the three observed multiplets.

Let us first try to find a one-to-one correspondence between three of the six available multiplets, and the three observed multiplets. There are then several cases to consider, depending on which three of the six multiplets are assumed to be the observed ones. In each case, requiring that the coefficient of the coupling to the $U(1)_Y$ gauge field be equal for the three observed multiplets, together with the tracelessness condition (12) and the normalization condition (13), fixes the coefficient uniquely, up to sign, and we find that the possible values of the coefficient are $\pm\frac{1}{2}$, $\pm\frac{1}{\sqrt{24}}$, and $\pm\frac{1}{\sqrt{132}}$. None of these is within a few percent of $\frac{1}{\sqrt{360}}$, so we have exhausted the freedom to vary the embedding of $U(1)_Y$, without finding a solution.

There is another possibility, however. We know that the masses of the observed fermions break $SU(2) \times U(1)_Y$ invariance, conserving only the electromagnetic $U(1)_Q$ subgroup, whose gauge field is the photon. Furthermore, we expect, from the existence of the CKM matrix, [18, 19, 17], that the low-energy mass matrix, between the E8 states of definite $SU(3) \times SU(2) \times U(1)_Y$ quantum numbers, will be non-diagonal, and can include cross terms that violate both $SU(2)$ conservation and $U(1)_Y$ conservation, provided that $SU(3) \times U(1)_Q$ is conserved. In particular, the possibility that the low-energy mass matrix includes cross terms between states of different Y, means that for the (3,2) states, for example, the three low-energy mass eigenstates can be three mutually orthogonal linear combinations of the six available (3,2) states, and that each of these linear combinations can include states with different Y coefficients.

We can think of the sum of the couplings of the six (3,2) states, to the $U(1)_Y$ gauge field, as a bilinear form between the six left-handed (3,2) states, and the six corresponding right-handed $(\bar{3}, 2)$ states, which happens to be diagonal, in the sense that there are no cross terms between the different multiplets. We can represent this bilinear form schematically as

$$\bar{\phi}\, Y\, \phi \tag{24}$$

where Y is a diagonal six by six matrix, whose diagonal matrix elements are the Y-coefficients of the six (3,2) states, read from the fifth columns of Tables 2 and 3. Now suppose that $\phi = U\chi$, and correspondingly, $\bar{\phi} = \bar{\chi}\, U^\dagger$, where the components of χ and $\bar{\chi}$ correspond to the low-energy mass eigenstates, and U is unitary. Then (24) is equal to

$$\bar{\chi}\, U^\dagger Y U \chi \tag{25}$$

In general, $\left(U^\dagger Y U\right)_{ij}$ will be non-diagonal. However, so far only the matrix elements between the lightest three of the six mass eigenstates, which I shall take to be the states 1, 2, and 3, have been observed experimentally. This submatrix is a multiple of the three by three unit matrix, with the multiple being $\frac{1}{\sqrt{360}}$, within a few percent.

The masses of the three heavy mass eigenstates, not yet observed, break SU(2) $\times$ U(1)$_Y$ invariance, and thus can not be more than a few hundred GeV, [9], so these states should be produced copiously at the LHC. In addition, the off-diagonal matrix elements, in (25), between the three light mass eigenstates, and the three heavy mass eigenstates, will result in new point-like contributions to processes such as $\bar{u}u \to \gamma\gamma$ and $\bar{u}u \to Z^0 Z^0$, resulting from the exchange of one of the heavy fermions.

Using the unitarity of U, the low-energy mass eigenstates, χ_i and $\bar{\chi}_i$, are expressed in terms of ϕ_i and $\bar{\phi}_i$ by:

$$\chi_i = \left(U_{ki}\right)^* \phi_k \tag{26}$$

$$\bar{\chi}_i = \bar{\phi}_k U_{ki} \tag{27}$$

Now, temporarily dropping the summation convention, if we write $Y_{km} = y_k \delta_{km}$, where y_k are the Y-coefficients of the six (3,2) states, read from the fifth column of Tables 2 and 3, we can write:

$$\left(U^\dagger Y U\right)_{ij} = \sum_{k=1}^{6} y_k \left(U_{ki}\right)^* \left(U_{kj}\right) \tag{28}$$

Finally, restoring the summation convention, the unitarity of U can be expressed as:

$$\left(U_{ki}\right)^* U_{kj} = \delta_{ij} \tag{29}$$

Looking at (26), (27), (28), and (29), we see that all the information about the three observed mass eigenstates, and the matrix elements of weak hypercharge between them, is contained in the six Y-coefficients, y_k, read from the fifth column of Tables 2 and 3, and the matrix elements U_{ki}, $1 \le k \le 6$, $1 \le i \le 3$, and their complex conjugates. We

State	Y coefficient	Y coefficient when $\sigma_3 = \sigma_4$
ψ_{122}	$\frac{-\sigma_1-2\sigma_2}{\theta}$	$-\frac{\sigma_1+2\sigma_2}{\theta}$
ψ_{133}	$\frac{-\sigma_1-2\sigma_3}{\theta}$	$\frac{\sigma_1+2\sigma_2}{2\theta}$
ψ_{144}	$\frac{-\sigma_1-2\sigma_4}{\theta}$	$\frac{\sigma_1+2\sigma_2}{2\theta}$
ψ_{123}	$\frac{-\sigma_1-\sigma_2-\sigma_3}{\theta}$	$-\frac{\sigma_1+2\sigma_2}{4\theta}$
ψ_{124}	$\frac{-\sigma_1-\sigma_2-\sigma_4}{\theta}$	$-\frac{\sigma_1+2\sigma_2}{4\theta}$
ψ_{134}	$\frac{-\sigma_1-\sigma_3-\sigma_4}{\theta}$	$\frac{\sigma_1+2\sigma_2}{2\theta}$

Table 4: Y coefficients of the left-handed $(\bar{3},1)$ antiquark multiplets

do not need to know the U_{ki} for $4 \leq i \leq 6$. The matrix elements $(U_{ki})^*$, $1 \leq k \leq 6$, $1 \leq i \leq 3$, can moreover be displayed conveniently, by writing out the relations (26) explicitly, for $1 \leq i \leq 3$.

This approach can be adapted directly to the other $\mathrm{SU}(3) \times \mathrm{SU}(2)$ multiplets, by making the appropriate replacements for the number of available multiplets, (here 6), and the number of observed multiplets, (here 3).

The most stringent constraints are likely to come from the left-handed $(\bar{3},1)$ antiquark multiplets, since the number of available multiplets is 6, and the number of observed multiplets is 5, (since the left-handed top antiquark multiplet must come from a non-zero Kaluza Klein state). Let us try for a solution with $\sigma_3 = \sigma_4$. We then find, from (12), that $\sigma_3 = \sigma_4 = -\left(\frac{3\sigma_1+2\sigma_2}{4}\right)$, so that the Y-coefficients of the six left-handed $(\bar{3},1)$ antiquark multiplets are as in Table 4.

We can identify ψ_{133} and ψ_{144} as up-type antiquarks, with Y coefficient $\frac{\sigma_1+2\sigma_2}{2\theta}$, and, using equation (28), ψ_{123}, ψ_{124}, and $\frac{1}{\sqrt{2}}(\psi_{122}+\psi_{134})$, as down-type antiquarks, with Y coefficient $-\frac{\sigma_1+2\sigma_2}{4\theta}$. Thus, from Table 1, we require

$$\frac{\sigma_1+2\sigma_2}{2\theta} = \frac{-4}{\sqrt{360}} \tag{30}$$

With (13), this implies

$$44\sigma_2^2 + 44\sigma_1\sigma_2 - 13\sigma_1^2 = 0 \tag{31}$$

The two solutions may be chosen as in Table 5.

Choosing one or the other of these two solutions for the $\mathrm{U}(1)_Y$ generator, we can now seek solutions for three generations of each of the other fermion multiplets in Table 1.

	σ_1	σ_2	σ_3	σ_4	θ
Solution A	2	$-1-2\sqrt{\frac{6}{11}}$ -2.4770979	$-1+\sqrt{\frac{6}{11}}$ -0.2614511	$-1+\sqrt{\frac{6}{11}}$ -0.2614511	$6\sqrt{\frac{15}{11}}$ 7.0064905
Solution B	-2	$1-2\sqrt{\frac{6}{11}}$ -0.4770979	$1+\sqrt{\frac{6}{11}}$ 1.7385489	$1+\sqrt{\frac{6}{11}}$ 1.7385489	$6\sqrt{\frac{15}{11}}$ 7.0064905

Table 5: The two solutions for the $\mathrm{U}(1)_Y$ generator found for the $(\bar{3}, 1)$ LH antiquark multiplets, assuming $\sigma_3 = \sigma_4$.

We have already noted that the three ψ_{diag} states, in the **80**, are natural candidates for the three left-handed antineutrinos. For each of the other four multiplets in Table 1, there are at least twice as many available multiplets, in the **80** and the $\overline{\mathbf{84}}$, as we require for the three observed generations. Therefore we can simplify the search for linear combinations of the available multiplets, that have the correct Y-coefficients, and no cross-terms between the observed fermions, by assuming that each observed fermion state is a linear combination of at most two of the available multiplets, and that each available multiplet contributes to at most one observed fermion state. There are then automatically no cross terms between distinct observed fermions i and j in equation (28), and if, for example, the two available multiplets contributing to the observed fermion state i, are a and b, then equation (28) expresses the Y coefficient of the observed state, y_i, in terms of the Y coefficients of the available multiplets a and b, as

$$y_i = y_a |U_{ai}|^2 + y_b |U_{bi}|^2 \tag{32}$$

Equation (29) reduces to:

$$|U_{ai}|^2 + |U_{bi}|^2 = 1 \tag{33}$$

The problem of finding the observed fermion state i thus reduces to solving (32) and (33), as simultaneous linear equations for $|U_{ai}|^2$ and $|U_{bi}|^2$, with y_i being the required Y coefficient, read from the fifth column of Table 1, and y_a and y_b being the Y coefficients of the available multiplets a and b, read from the fifth column of Table 2 or Table 3, after substituting in the parameters of the chosen solution A or B for the $\mathrm{U}(1)_Y$ generator, from Table 5. The solution of (32) and (33) is

$$|U_{ai}|^2 = \frac{y_i - y_b}{y_a - y_b} \ , \qquad |U_{bi}|^2 = \frac{y_a - y_i}{y_a - y_b} \tag{34}$$

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so we find a fermion state i, with $|U_{ai}|^2 \geq 0$ and $|U_{bi}|^2 \geq 0$, if, and only if, y_i lies between y_a and y_b. Thus we can obtain a solution of this type, provided we can assign the observed fermion states to mutually disjoint pairs of available multiplets, such that the Y coefficient of each observed fermion state lies between the Y coefficients of the pair of available multiplets it is assigned to.

Let us try for a solution using solution A from Table 5. Then for the (3,2) quark multiplets, we find that the Y coefficients of ψ_{211}, ψ_{311}, and ψ_{411} are negative, and the Y coefficients of ψ_{12}, ψ_{13}, and ψ_{14} are positive, and greater than the required value, $\frac{1}{\sqrt{360}}$, so a possible solution is (ψ_{211},ψ_{12}), (ψ_{311},ψ_{13}), (ψ_{411},ψ_{14}). For the (1,2) lepton doublets, we find that all eight (1,2) multiplets in the $\overline{\mathbf{84}}$ have positive Y coefficients, whereas the required Y coefficient, $\frac{-3}{\sqrt{360}}$, is negative. However, since the SU(2) fundamental is equivalent to the antifundamental, the relation being given by matrix multiplication by ε_{ij}, we can also use the (1,2) multiplets in the left-handed $\mathbf{84}$, which have negative Y coefficients. Six of the (1,2) multiplets in the left-handed $\mathbf{84}$ have negative Y coefficients of larger magnitude than the required value, and two have negative Y coefficients of smaller magnitude than the required value, so we can obtain a solution, for example, with two pairs of (1,2) multiplets coming from the $\mathbf{84}$, and one pair of (1,2) multiplets with one member from the $\mathbf{84}$, and one member from the $\overline{\mathbf{84}}$. For the e^+, μ^+, and τ^+ (1,1) states, we find that ψ_{32} and ψ_{42} have exactly the required Y coefficient, $\frac{6}{\sqrt{360}}$, and thus may be identified with two of these states. The third state may be obtained as a linear combination of ψ_{111} in the $\mathbf{84}$, and, for example, ψ_{23} or ψ_{24}. Finally, we note that ψ_{34} and ψ_{43} have zero Y coefficient, and can mix with the three ψ_{diag} antineutrino states.

Once the observed fermion states with a given SU(3)$\times$SU(2) content, and a given Y coefficient, have been obtained, they can be mixed among themselves without producing cross-terms. A further mixing possibility, without producing cross-terms, is to mix E8 states that have the same SU(3) $\times$ SU(2) content and Y coefficients, among themselves, before assigning them to mutually disjoint pairs associated with the observed fermions.

The stability of the proton, in this class of models, must be a dynamical property of the vacuum, which is not yet understood. Thus the natural way to attempt to fit the observed stability of the proton, would be to attempt to adjust the substantial number of mixing angles, relating the observed fermion states to the E8 states, so as to cancel the most dangerous proton decay modes. Some of these mixing angles will affect the

CKM matrix, so this suggests that the entries in the CKM matrix might be correlated with the stability of the proton.

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