UNIVERSITÀ DEGLI STUDI DI LECCE

FACOLTÀ DI SCIENZE
Dipartimento di Fisica

# Ring Division Algebras, Self Duality and Supersymmetry

Tesi di dottorato di ricerca in Fisica
presentata da
*Khaled Safwat Abdel-Khalek Mostafa*

Relatori
Professor *Pietro Rotelli*
The Exam Commission
Professor *Federico Cesaroni*
Professor *Boris Konopelchenko*
Professor *Emilia D'Anna*
External Member
Professor *Antoine Van Proeyen*

<ins>Ciclo XII</ins>

Anno Accademico 1999-2000

# *Acknowledgements*

*First and formost my gratitude goes to Allah my god for his continuous help, for helping me in choosing the topic of the thesis, the supervisors and the place. Prof. M. Shalaby (Ain Shames University/Cairo) was responsible for introducing me to theoretical physics and for this I owe him much. I feel really grateful to Prof. P. Rotelli for giving me the opportunity to work with and learn from him many beautiful ideas and I would like to thank him for many enjoyable discussions during the last four years. I feel indebted to Prof. C. Imbimbo for many constructive criticisims of this thesis and for helping me to have a better understanding of my work. Over the last four years, I have had several fruitful discussions with Prof. G. Thompson to whom I am thankful. I would like to acknowledge Prof. A. Zichichi and the third world laboratory for financial support at a critical period of my thesis. Many thanks also to Prof. S. Marchiafava and F. Englert for encouragements. My gratitude to the members of the Physics department at the* Università di Lecce *for the very stimulating environment I have found here. Last but not least, this thesis is dedicated to my family.*

# Contents

# Chapter 1

# INTRODUCTION

## 1.1 Mathematics

Imaginary numbers appeared in mathematics a long time ago. For example, Nicolas Chuquet (1445–1500) wrote “Triparty en la science des nombres” where he introduced an exponential notation, allowing positive, negative and zero powers. He showed that some equations lead to imaginary solutions but rejected them “tel nombre est ineperible”. Geronimo Cardano (1501–1576) wrote “Ars magna”, found solutions to polynomials which lead to square roots of negative quantities but also rejected them “as subtle as it is useless”. The first to consider imaginary numbers was Rafael Bombelli (1530–1590) who published “Algebra” and proposes the “wild idea” that one can use these square roots of negative numbers to get to the real solutions by using conjugation. Albert Girard (1595–1632) publishes “Invention nouvelle en l'algebra” retaining all imaginary roots because they show the general principles in the formation of an equation from its roots. Rene Descartes (1596–1650) coins the term “imaginary” for terms involving square roots of negative numbers but takes their existence as a sign that the problem is insoluble. Reviving some speculation, Gottfried Leibniz (1646–1716) says that imaginary numbers are halfway between existence and nonexistence. Sustaining algebra by geometry, John Wallis (1616–1703) was the first to represent complex numbers geometrically in his book “Algebra” published in 1673. Roger Cotes (1682–1716) deduces that $\exp\left(\sqrt{-1}\, a\right) = \cos(a) + \sqrt{-1} \sin(a)$ but his result was largely ignored. But then a new era begins, it was Leonhard Euler who brought complex numbers from the shadow to the daylight, he invents the symbol $i$ for $\sqrt{-1}$ and works extensively with imaginary numbers, for example, he shows that a complex number to the power of a complex number is also a complex number. Jean d'Alembert's (1717–1783) constructs functions of complex variables, obtaining what later is called the Cauchy–Riemann equation. Caspar Wessel (1745–1818) discovers that complex numbers can be represented graphically on a two dimensional plane, what we now call the "Argand" or "Guassian" representation of complex numbers. Modern complex analysis may be dated to the book of Augustin Cauchy (1789–1857) "Memoire sur les integrales definies, prises entre des limites imaginaires" which contains his integral theorems on residues. Then the work of Augustus de Morgan (1806–1871) and Carl Gauss (1777–1855) opens the way to what later becomes complex numbers analysis. So finally, what was rejected as useless quantities become the heart of mathematics.

While the discovery and acceptance of complex numbers took a long time, the histroy of quaternions and octonions is much shorter. Quaternions were discovered by a single man [1], William Hamilton (1805–1865). Trying to generalize his "Theory of Algebraic Couples", where he constructs a rigorous algebra of complex numbers as number pairs for the first time, he identifies $x + iy$ with its $\mathbb{R}^2$ coordinates $(x, y)$. After many years of trial and error, Hamilton discovers quaternions on Monday 16 October 1843 and defines a vector subspace $ai + bj + ck$ by elements which may be interpreted as an $R^3$ coordinate system $(a, b, c)$ but $i, j, k$ are not commutative. As early as 1845, Octonions were introduced by Arthur Cayley and John Graves independently [2][3].

Quaternions and octonions may be presented as a linear algebra over the field of real numbers $\mathbb{R}$ with a general element of the form

$$Y = y_0 e_0 + y_i e_i, \qquad y_0, y_i \in \mathbb{R} \tag{1.1}$$

where $i = 1, 2, 3$ for quaternions $\mathbb{H}$ and $i = 1..7$ for octonions $\mathbb{O}$. We always use Einstein's summation convention. The $e_i$ are imaginary units, for quaternions

$$\begin{aligned} e_i e_j &= -\delta_{ij} + \epsilon_{ijk} e_k, && (1.2)\\ e_i e_0 &= e_0 e_i = e_i, && (1.3)\\ e_0 e_0 &= e_0, && (1.4) \end{aligned}$$

where $\delta_{ij}$ is the Kronecker delta and $\epsilon_{ijk}$ is the three dimensional Levi–Cevita tensor, as $e_0 = 1$ when there is no confusion we omit it. Octonions have the same structure, only we must replace $\epsilon_{ijk}$ by the octonionic structure constant $f_{ijk}$ which is completely antisymmetric and equal to one for any of the following three cycles

$$123,\ 145,\ 176,\ 246,\ 257,\ 347,\ 365. \tag{1.5}$$

The important feature of real, complex, quaternions and octonions is the existence of an inverse for any non-zero element. For the generic quaternionic or octonionic element given in (1.1), we define the conjugate $Y^*$ as an involution $(Y^*)^* = Y$, such that

$$Y^* = y_0 e_0 - y_i e_i, \tag{1.6}$$

introducing the norm as $N(Y) \equiv \|Y\| = YY^* = Y^*Y$ then the inverse is

$$Y^{-1} = \frac{Y^*}{\|Y\|}. \tag{1.7}$$

The Norm is nondegenerate and positively definite. We have the decomposition property

$$\|XY\| = \|X\| \quad \|Y\| \tag{1.8}$$

$N(xy)$ being nondegenerate and positive definite obeys the axioms of the scalar product and our algebra is called a normed algebra. The uniqueness and beauty of real, complex, quaternionic and octonionic numbers stem from Hurwitz' theorem [4]:

> Each normed composition algebra with a unit element is isomorphic to one of the following algebras: to the algebra of real numbers, to the algebra of complex numbers, to the quaternion algebra or to the octonion algebra.

Another important mathematical property of the ring algebra is the following: For the set $\mathcal{S}$ defined by

$$\mathcal{S} \quad = \{X \mid \|X\| = 1\} \tag{1.9}$$

where $X$ is a ring division element then from the decomposition property (1.8), we have a closure structure

$$X, Y \in \mathcal{S} \quad \to Z = XY \in \mathcal{S} \tag{1.10}$$

even for octonions which do not admit a group structure (group is defined for associative algebra). This beautiful closure can be extended to a generic ring division element by scaling. For any two generic ring division elements $W, V$ we construct

$$\widetilde{W} = \frac{W}{\|W\|}, \qquad \widetilde{V} = \frac{V}{\|V\|} \tag{1.11}$$

hence

$$\left\|\widetilde{W}\right\| = \left\|\widetilde{V}\right\| = \left\|\widetilde{W}\widetilde{V}\right\| = 1. \tag{1.12}$$

A geometric meaning of this closure is the parallelizability of ring division spheres. For $\|X\| = 1$

$$\begin{array}{lll} \mathbb{C} & x_0^2 + x_1^2 & = 1 \quad defines \quad a\ unit \quad S^1 \\ \mathbb{H} & x_0^1 + x_1^2 + x_2^2 + x_3^2 & = 1 \quad defines \quad a\ unit \quad S^3 \\ \mathbb{O} & x_0^1 + x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 & = 1 \quad defines \quad a\ unit \quad S^7 \end{array} \tag{1.13}$$

the parallelizability means that there is such an $X$ that defines *globally* $1, 3, 7$ vector fields for $S^1, S^3, S^7$ respectively [5][6].

Another importance of ring division algebras is their relevance to the classification of real Clifford algebras $Cliff\,(m, n)$. A task that had been achieved by Atiyah, Bott and Shapiro [7]. Their appearance is clear through the Bott periodicity. We also wish to mention the work of Milnor[1], where for the first time in history a diffeomorphic non homeomorphic structure was found. There are 28 of such structures over $S^7$, the fake $S^7$ [8]. As a matter of fact, these fake $S^7$ are the higher bundles of the four dimensional $SO\,(4)$ instanton solutions[9]. Donaldson received in 1986 the Field Medal for his work about the infinite diffeomorphic non-homeomorphic $\mathbb{R}^4$. An idea that he got by carefully studying the space of solutions of the four dimensional $n = 1$ "quaternionic" instanton. Finally, ring division algebras have a connection with homotopies, Hopf fibrarion and many other interesting topics.

It is clear that the history of these other elements of the ring division algebra, quaternions and octonions, are much shorter than the complex one. Maybe, it has not been yet fully written. With the hindsight of complex numbers, these new hypercomplex numbers were immediately accepted, perhaps only the question of their utility in physics is still to be discovered and may yet involve much discussion and take a long time.

## 1.2 Physics

Non associative algebra appeared for the first time in physics when Jordan, van Neuman and Wigner introduced commutative but non-associative opera-

[1]According to legend, Milnor presented his first acheivements as an assignment. On one occasion, he was late for the class of Fox (the Father of the american Knot theory). During that lesson, Fox explained his way of doing research. Usually he writes the most difficult 10 questions and tries to solve any one of them. As an example, he wrote for his students the 10 questions that kept him busy at that time. Fortunatley, John Milnor came late that day, he saw the quetions, he thought they were homework. After the class he worked hard untill the next morning. Then before the class of the next day, he approached Fox expressing his desire to change his field of research because he only managed to solve one of the 10 questions. Fox was totally surprised. But then later Milnor continued to surprise the world especially by his Field Medal's diffeomorphic non-homeomorphic structure.

tors – Jordan algebras – for the construction of a new quantum mechanics[10]. More recently ,after the proposed eightfold way by Gell-Mann and Ne'eman, there were some octonionic rivals for $SU(3)$ such as $G_2$, $SO(7)$, $SO(8)$ and others[11]. Possible octonionic internal symmetries were considered by different people such as Souriau and Kastler [12], Pais [13], Tiomno [14], Gamba [15], and Penny [16]. The inclusion $SU(3) \subset G_2 \subset SO(7) \subset SO(8)$ lead many authors to consider the relationship between the nuclear strong field and octonions. There are a lot of papers dealing with the $SO(8)$ symmetry [17]. The potential significance of non-associative algebras for the generalization of classical dynamics was pointed out by Nambu [18]. Generalizing the Liouville theorem, about the conservation of the phase space volume, Nambu introduced a new generalization of Poisson brackets which may be interpreted as an associator. Nambu in his paper made some remarks about non-associative algebras, octonions and Jordan algebras. As another application of octonions, allowing parastatistics and the paraquark model, Freund showed in [19] that color gauging is only possible in the octonionic realization of the paraquark model. Octonions have even been applied to gravitation. Vollendorf [20] constructed a bilocal field theory with the group $SO(4,4)$ instead of the Lorentz group and with a system of 24 coupled differential equations to explain the five conservation laws of charge, hypercharge, baryon number and the two lepton number then known.

From the early seventies and up to the present time, octonions have been applied with some success to different important problems such as quark confinement, grand unified models (GUT). A serious step was taken by Günaydin and Gürsey [21][22] when they present in their work a systematic study of the octonionic algebraic structure. If we follow the theory of observable states developed by Birkhoff and van Neumann [23], we can only have observable states in Hilbert spaces over associative normed algebras. The standard quantum mechanics explores the Hilbert space over complex numbers. The quaternion case with a quaternionic scalar product was developed by Finkelstein, Jauch and Emch [24]. The general hope was to introduce isospin degrees of freedom by enlarging the quantum Hilbert space. But Jauch proved that quaternionic representations of the poincaré group did not generate any new states. The reader is referred to the book of Adler for a modern formulation of quaternionic field theory [25]. Another interesting idea is the use of hypercomplex Hilbert space but with a complex scalar product, Goldstein, Horwitz and Biedenharn [26], considered octonionic Hilbert space with a complex scalar product as early as 1962. The abandon of associativity means the existence of nonobservable states. Günaydin and Gürsey used these nonobservable states to explain the quark confinement phenomena.

Later on octonions entered the Grand Unified Theories era with different applications. In [27], Gürsey suggested that the exceptional group $F_4$ might be used to describe the internal charge space of particles. Even the other exceptional groups $E_6, E_7, E_8$ have been utilized to provide larger GUT models [28]

$$SU(3)_c \times SU(2)_L \times U(1)_Y \subset SU(5) \subset SO(10) \subset E_6 \subset E_8\,. \tag{1.14}$$

As a matter of fact, nowadays, $E_6 \times U\,(1)$ is the most promising scheme from the superstrings phenomenological point of view. Günaydin constructed an exceptional realization of the Lorentz group in [29]. The exceptional $SL\,(2, \mathbb{C})$ multiplets generate a non-associative algebra. Exceptional supergroups were also introduced and investigated in [29].

Starting from the eighties, new applications of ring division algebras in physics were found. The instanton problem, supersymmetry, supergravity, superstrings and recently branes technology. We give references in the appropriate chapters of this thesis to the history of the first two topics. Application of quaternions and octonions to supergravity spontaneous compactification was a very important and active field of research during the mid eighties. Especially compactification of $d = 11$ supergravity over $S^7$ to 4 dimensions. It is an impossible task to list all the relevant papers, so we direct the interested reader to the physics report [30] written by Duff, Nilsson and Pope where a lot of references are given. We just mention that the first indication of the octonionic nature of this problem appeared in the Englert solution of $d = 11$ supergravity compactification over $S^7$[31] and a systematic study along this line has been carried out in [32][33][34][35]. The relations between superstrings (p-branes) and octonions had been considered from many different points of view, the reader may consult the references given in [36] for details.

## 1.3 Outline of The Thesis

Superstrings promise the possibility of unifying all the fundamental forces of nature. Abandoning the idea of point particle seems necessary to incorporate general relativity with renormalizable field theory. One of the puzzles of this string program is its double facet [37][38]. On the one hand, we can work over the two dimensional string sheet where we can use the powerful methods of conformal field theory. On the other hand working with the ten dimensional space-time Green–Shwarz (GS) supersymmetric action, non–perturbative effects may be seen. Understanding the relations between these two different formalisms of string theory is important but many features of the GS formalism still have to be elucidated. For example, the the algebra is given on–shell and the action exists only in certain space–time dimensions, $d = 3, 4, 6, 10$. Only for ten dimensions, the quantum anomalies of the model cancel i.e. $d = 10$ is a very special case of an already a very restrictive class. The relationship between gamma matrices needed for the existence of the GS picture is the same as that used to prove the existence of simple supersymmetric Yang-Mills models (containing only a gauge field and a spinor) in the same dimensions $d = 3, 4, 6, 10$ [41]. A complete comprehension of this fact is important.

At the quantum level, we know how to proceed in a perturbative fashion using Feynman diagrams. Starting from quantum chromodynamics QCD, and its quark confinement problem, theoretician searched for non–perturbative phenomena. Self–dual Yang–Mills solutions, *instantons* [39][40], can never be evaluated using perturbation theory. It is widely believed that in superstrings, non–perturbative solutions are of great significance and they may give interesting phenomenological applications that can be tested experimentally.

In short, off–shell formalism of SSYM and self–duality are very critical topics which may improve our theoretical knowledge and can be used as toy models for testing new approaches.

The object of this thesis is to investigate in a systematic way the relations between ring division algebras, off–shell SSYM and higher dimensional self–duality. The starting point is to understand how ring division algebras are specific representations of Clifford algebras. We present this analysis in chapter II. For complex numbers the discussion is simple. Even for quaternionic numbers, once the non-commutativity is taken into account, the formulation can be fully analyzed and understood easily. As octonions are non-associative numbers, we need to work harder to clarify many different subtleties. In chapter II we concentrate upon the Clifford structure that can be extracted from octonions. To make the picture clearer, we show what will happen if we go beyond the ring division algebras limit. The Clifford structure is no longer faithful.

In chapter III, we continue our study of octonions, we show that they are endowed with additional useful characteristics. Fixing the direction of action, octonions exhibit soft Lie algebra properties which we call a soft seven sphere [42][43]. Soft Lie algebras are elements that close under the action of the commutator with structure functions of coordinates system that parameterize a hidden space (the gauge manifold). We study this scheme in full details, we compute the structure functions explicitly with different degrees of complication.

In Chapter IV, we start to investigate the physical applicability of the soft seven sphere. The self duality conditions play a fundamental role for any non–perturbative effects in point particle or string field theory [44]. The higher dimensional self dual constraints have properties distinct from the four dimensional one. As a first exercise in the use of the soft seven sphere, we show how to reformulate a quartic eight dimensional self duality condition into a quadratic form. Thus, we put the Grossman–Kephart–Stasheff condition (GKS) [45] into a form much similar to the four dimensional equation.

In Chapter V, we will discuss supersymmetry. In particular, the off-shell simple supersymmetric Yang–Mills models SSYM. We shall show that important characteristics can be seen clearly only by using the ring division approach. The ten dimensional case will be very special. For example, we recover Berkovits formulation for the $d = 10$ off–shell SSYM in a very transparent way.

# Chapter 2

# Hypercomplex Structures

The starting point of this chapter is to know how to translate some real $n \times n$ matrices $\mathbb{R}(n)$ to their corresponding complex, quaternionic and octonionic representations ( shaeffer bimodule representation for octonions [46]). It is well known from a mathematical point of view that any $\mathbb{R}^{2n}$ is trivially a $\mathbb{C}^n$ complex manifold and any $\mathbb{R}^{4n}$ is also a trivial quaternionic manifold $\mathbb{H}^n$. Furthermore, any $\mathbb{R}^{8n}$ is a trivial $\mathbb{O}^n$ octonionic manifold, in the sense that the seven sphere can always be embedded in $\mathbb{R}^8$. As any $\mathbb{R}^n \times \mathbb{R}^n$ is isomorphic as a vector space to the space of $n \times n$ matrices $\mathbb{R}(n)$ [47], we would expect

$$\begin{aligned} \mathbb{C}(n) \times \mathbb{C}(n) &\rightarrow \mathbb{R}(2n); && (2.1)\\ \mathbb{H}(n) \times \mathbb{H}(n) &\rightarrow \mathbb{R}(4n); && (2.2)\\ \mathbb{O}(n) \times \mathbb{O}(n) &\rightarrow \mathbb{R}(8n). && (2.3) \end{aligned}$$

Even if in this thesis we only work with matrices, there is a hidden geometric and topological underlying structure behind this algebraic construction.

Any hypercomplex manifold has a well defined *local* hypercomplex structure that can be put into the matrix form that we shall develop in this chapter. Lifting this local hypercomplex structure to a global one is not always possible. It amounts to dividing the manifold into local patches where the almost structure is well defined and gluing together these different patches insuring the existence of a (differentiable) structure function that transfers the local hypercomplex structure from one patch to another. If this can be achieved over all the manifold then our space admits a global hypercomplex structure. From the geometric point of view, one should prove the vanishing of the "Nehijinus tensors" [48][49]. From the topological point of view, one should overcome global obstructions. The story is very similar to the existence of spinorial manifolds. Actually, our almost hypercomplex structures, when represented as matrices, close as Clifford algebra over certain Euclidean spaces.

## 2.1 Complex Structure

For complex variables, one can represent any complex number z as an element of $\mathbb{R}^2$

$$z = z_0 e_0 + z_1 e_1 \equiv Z = \begin{pmatrix} z_0 \\ z_1 \end{pmatrix}.$$

The action of $e_0 = 1$ and $e_1$ induce the following matrix transformations on Z ,

$$e_0 z = z e_0 = z \equiv \mathbb{E}_0 Z = \mathbf{1}_2 Z = Z,$$

where $\mathbf{1}_n$ will always mean the $n \times n$ identity matrix, while

$$\begin{aligned} e_1 z &= z e_1 = z_0 e_1 - z_1 && (2.4) \\ &\equiv \mathbb{E}_1 Z && (2.5) \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} -z_1 \\ z_0 \end{pmatrix}. && (2.6) \end{aligned}$$

Of course, we have

$$(\mathbb{E}_0)^2 = \mathbf{1}_2, \qquad (\mathbb{E}_1)^2 = -\mathbf{1}_2. \tag{2.7}$$

Now, there is a problem, these two matrices $\mathbb{E}_0$ and $\mathbb{E}_1$ alone are not sufficient to form a basis for $R(2)$. The solution of our dilemma is straightforward. If we also take into account that

$$z^* = z_0 - z_1 e_1 \equiv Z^* = \begin{pmatrix} z_0 \\ -z_1 \end{pmatrix}$$

we find

$$\begin{aligned} Z^* &= \tilde{\mathbb{E}}_0 Z && (2.8) \\ &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} z_0 \\ -z_1 \end{pmatrix}, && (2.9) \end{aligned}$$

and

$$\begin{aligned} e_1 z^* &= z^* e_1 = z_0 e_1 + z_1 = e_1^* z && (2.10) \\ &\equiv \mathbb{E}_1 Z^* = \tilde{\mathbb{E}}_1 Z && (2.11) \\ &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_0 \\ -z_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} z_0 \\ z_1 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_0 \end{pmatrix}. && (2.12) \end{aligned}$$

Obviously with these four matrices $\{\mathbb{E}_0, \tilde{\mathbb{E}}_0, \mathbb{E}_1, \tilde{\mathbb{E}}_1\}$, $\mathbb{C}(n) \to \mathbb{R}(2n)$ is proved.

## 2.2 Sp(1) Structure

The quaternionic algebra is given by $e_i.e_j = -\delta_{ij} + \epsilon_{ijk}e_k \Longleftrightarrow [e_i, e_j] = 2\epsilon_{ijk}e_k$, where $\epsilon_{ijk}$ is the three dimensional Levi-Civita tensor ($\epsilon_{123} = 1$) and $i, j, k = 1, 2, 3$. Being non commutative, one must distinguish between right and left multiplication. In 1989, writing a quaternionic Dirac equation [50], Rotelli introduced a "barred" momentum operator with right action of $e_1$

$$-\partial_\mu \mid e_1 \tag{2.13}$$

such that

$$[\ (-\partial_\mu \mid e_1)\psi \equiv -\partial_\mu \psi e_1\ ] \quad . \tag{2.14}$$

In recent papers [51], partially barred quaternions

$$q + p \mid e_1 \qquad [\ q,\ p \in \mathbb{H}\ ] \quad , \tag{2.15}$$

have been used to formulate a quaternionic quantum mechanics and quaternionic field theory. From the viewpoint of group structure, these barred operators are very similar to complexified quaternions [52]

$$q + \mathcal{I}p \tag{2.16}$$

where the imaginary unit $\mathcal{I}$ commutes with the quaternionic imaginary units ($e_1$, $e_2$, $e_3$), but in physical problems, like eigenvalue calculations, tensor products, relativistic equations solutions, they give different results. A complete generalization for quaternionic multiplication is represented by the following barred operators

$$\mathbb{H}|\mathbb{H} = q_1 + q_2 \mid e_1 + q_3 \mid e_2 + q_4 \mid e_3 \qquad [\ q_{1,\ldots,4} \in \mathbb{H}\ ] \quad , \tag{2.17}$$

and was developed a long time ago. As early as 1912, they had been used by Conway and Silberstein [53][54] to reformulate special relativity and electromagnetism in a pure quaternionic language. Look to Synge [55] for a review. The set of $\mathbb{H}|\mathbb{H}$ numbers with its 16 linearly independent elements, form a basis of $GL(4, \mathbb{R})$. They have been revived recently to write down a one-component Dirac equation [56].

Let us now show how to represent these quaternionic operators as real $4 \times 4$ matrices. Like in the complex case, we represent *any quaternionic number (as distinct from an "operator")* as a column vector

$$q = q_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 \equiv Q = \begin{pmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{pmatrix} \quad ,$$

then

$$e_1.q = q_0 e_1 - q_1 + q_2 e_3 - q_3 e_2 \equiv \mathbb{E}_1 Q \tag{2.18}$$

and so on forth. The canonical left quaternionic structures [48][57] over $\mathbb{R}^4$ are

$$\mathbb{E}_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} ; \mathbb{E}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} ;$$

$$\mathbb{E}_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} ; \tag{2.19}$$

such that

$$\mathbb{E}_i \mathbb{E}_j = (-\delta_{ij} + \epsilon_{ijk} \mathbb{E}_k) \quad , \quad (\mathbb{E}_i)^2 = -\mathbf{1}_4, \tag{2.20}$$

Using Rotelli's notation, right action is given by

$$(1|e_1)q = qe_1 = q_0 e_1 - q_1 - q_2 e_3 + q_3 e_2 \tag{2.21}$$

$$\equiv 1|\mathbb{E}_1 \; Q \tag{2.22}$$

and so on for $1|e_2, 1|e_3$. Our canonical right quaternionic structures are

$$1|\mathbb{E}_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} ; 1|\mathbb{E}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} ;$$

$$1|\mathbb{E}_3 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} , \tag{2.23}$$

and

$$1|\mathbb{E}_i \; 1|\mathbb{E}_j = (-\delta_{ij} - \epsilon_{ijk} 1|\mathbb{E}_k) \quad , \quad (1|\mathbb{E}_i)^2 = -\mathbf{1}_4 \quad . \tag{2.24}$$

We can write these left/right quaternionic structures compactly as

$$(\mathbb{E}_i)_{\mu\nu} = (\delta_{0\mu}\delta_{i\nu} - \delta_{0\nu}\delta_{i\mu} - \epsilon_{i\mu\nu}) \tag{2.25}$$

$$(1|\mathbb{E}_i)_{\mu\nu} = (\delta_{0\mu}\delta_{i\nu} - \delta_{0\nu}\delta_{i\mu} + \epsilon_{i\mu\nu}) \tag{2.26}$$

where $\mu, \nu$ run from 0 to 3 or explicitly

$$\begin{aligned}\mathbb{E}_{i\mu\nu} &= -\epsilon_{i\mu\nu} \qquad \text{if } \mu,\nu = 1,2,3. &(2.27)\\ \mathbb{E}_{i0\nu} &= -\delta_{i\nu} \qquad , \qquad \mathbb{E}_{i\mu 0} = \delta_{i\mu} \qquad , \qquad \mathbb{E}_{i00} = 0. &(2.28)\end{aligned}$$

and

$$\begin{aligned}1|\mathbb{E}_{i\mu\nu} &= \epsilon_{i\mu\nu} \qquad \text{if } \mu,\nu = 1,2,3. &(2.29)\\ 1|\mathbb{E}_{i0\nu} &= -\delta_{i\nu} \qquad , \qquad 1|\mathbb{E}_{i\mu 0} = \delta_{i\mu} \qquad , \qquad 1|\mathbb{E}_{i00} = 0. &(2.30)\end{aligned}$$

These mathematical quaternionic structures are the 't Hooft eta symbols [40] well known in physics and we can check that

$$1|\mathbb{E}_{i\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\ 1|\mathbb{E}_{i\alpha\beta} \qquad \mathit{and} \qquad \mathbb{E}_{i\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\ \mathbb{E}_{i\alpha\beta}. \tag{2.31}$$

Having $\mathbb{E}_i, 1|\mathbb{E}_i$ enables us to find any generic operator $\mathbb{E}_i|\mathbb{E}_j$ corresponding to $e_i|e_j$

$$(e_i|e_j)\ \ q = e_i\ \ (1|e_j)\ \ \ q = e_i q e_j \equiv (\mathbb{E}_i|\mathbb{E}_j)\ \ \ Q = \mathbb{E}_i\ \ (1|\mathbb{E}_j)\ \ \ \ Q, \tag{2.32}$$

then we have the 16 base elements of the operator in $\mathbb{H}|\mathbb{H}$

$$\left\{\begin{array}{c} 1, e_1, e_2, e_3, \\ 1|e_1, e_1|e_1, e_2|e_1, e_3|e_1, \\ 1|e_2, e_1|e_2, e_2|e_2, e_3|e_2, \\ 1|e_3, e_1|e_3, e_2|e_3, e_3|e_3 \end{array}\right\}. \tag{2.33}$$

And their corresponding matrix representations

$$\left\{\begin{array}{c} \mathbf{1}_4, \mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3, \\ 1|\mathbb{E}_1, \mathbb{E}_1|\mathbb{E}_1, \mathbb{E}_2|\mathbb{E}_1, \mathbb{E}_3|\mathbb{E}_1, \\ 1|\mathbb{E}_2, \mathbb{E}_1|\mathbb{E}_2, \mathbb{E}_2|\mathbb{E}_2, \mathbb{E}_3|\mathbb{E}_2, \\ 1|\mathbb{E}_3, \mathbb{E}_1|\mathbb{E}_3, \mathbb{E}_2|\mathbb{E}_3, \mathbb{E}_3|\mathbb{E}_3 \end{array}\right\}. \tag{2.34}$$

We can thus deduce the following group structure for our quaternionic operators

- $su(2)_{Left}$

$$\begin{aligned} e_i e_j &= -\delta_{ij} + \epsilon_{ijk} e_k, &(2.35)\\ su(2)_{Left} &\sim \{e_1, e_2, e_3\} \sim \{\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3\}\,. &(2.36)\end{aligned}$$

- $su(2)_{Right}$

$$\begin{aligned} 1|e_j 1|e_i &= 1|(e_i e_j) = -\delta_{ij} - \epsilon_{ijk} 1|e_k, & (2.37)\\ su(2)_{Right} &\sim \{a_1 = 1|e_2, a_2 = 1|e_1, a_3 = 1|e_3\} \\ &\sim \{a_1 = 1|\mathbb{E}_2, a_2 = 1|\mathbb{E}_1, a_3 = 1|\mathbb{E}_3\} & (2.38) \end{aligned}$$

such that [1]

$$a_i a_j = -\delta_{ij} + \epsilon_{ijk} a_k. \qquad (2.39)$$

- $so(4) \sim su(2)_{Left} \times su(2)_{Right}$, as

$$e_i.1|e_j = 1|e_j.e_i = e_i|e_j, \quad i.e. \quad [e_i,\ 1|e_j] = 0, \qquad (2.40)$$

thus

$$so(4) \sim \ \{e_1, e_2, e_3, 1|e_1, 1|e_2, 1|e_3\}. \qquad (2.41)$$

- $spin(2,3)$ - and its subgroups - can be realized by a Clifford algebra construction, e.g.

$$\gamma_1 = e_3, \quad \gamma_2 = e_2, \quad \gamma_3 = e_1|e_1, \quad \gamma_4 = e_1|e_2, \quad \gamma_5 = e_1|e_3, \qquad (2.42)$$

$$\{\gamma_\alpha, \gamma_\beta\} = 2diag(-,-,+,+,+). \qquad (2.43)$$

By explicit calculation, one finds (in the basis given above)

$$\begin{aligned} spin(2,3) &\sim \ the\ set\ of\ [\gamma_\alpha, \gamma_\beta] \qquad \alpha, \beta = 1..5 \quad , \\ &\sim \ \{e_1, 1|e_1, 1|e_2, 1|e_3, e_2|e_1, e_3|e_1, e_2|e_2, e_3|e_2, e_2|e_3, e_3|e_3\}. \end{aligned} \qquad (2.44)$$

The reason that eqn.(2.42) can lead to eqn.(2.43) is that

$$e_i e_j \ (1|e_k) + e_j e_i \ (1|e_k) = 0.$$

This construction was first introduced by Synge [55] to give a quaternionic formulation of special relativity ($so(1,3)$).

- Also at the matrix level the full set $\mathbb{H}|\mathbb{H}$ closes as an algebra, indeed using the above equations we find

$$\begin{aligned} 1|e_i \ \ e_j|e_k &= \epsilon_{kil} \ \ e_j|e_l, & (2.45)\\ e_i \ \ e_j|e_k &= \epsilon_{ijl} \ \ e_l|e_k, & (2.46)\\ e_i|e_j \ \ e_m|e_n &= \epsilon_{iml} \ \ \epsilon_{njp} \ \ e_l|e_p. & (2.47) \end{aligned}$$

[1] We rename $a_1 = 1|e_2$ and $a_2 = 1|e_1$ so that equation (2.39) comes with the standard sign.

We have used Maple [58] to prove that the 16 matrices of $\{\mathbb{H}|\mathbb{H}\}$ are linearly independent so that they can form a basis for any $\mathbb{R}(4)$ as we claimed in (2.2). Omitting the identity, the set of 15 elements $\{\mathbb{H}|\mathbb{H}\} \setminus 1$ closes an $sl(4, \mathbb{R})$ algebra.

## 2.3 Octonionic Structure

We now summarize our notation for the octonionic algebra. There is a number of equivalent ways to represent the octonions multiplication table. Fortunately, it is always possible to choose an orthonormal basis $(e_0, \ldots, e_7)$ such that

$$\varphi = \varphi_0 + \varphi_m e_m \qquad ( \varphi_{0,\ldots,7} \quad \in \mathbb{R} ), \tag{2.48}$$

where $e_m$ are elements obeying the noncommutative and nonassociative algebra

$$e_m e_n = -\delta_{mn} + f_{mnp} e_p( \quad m, n, p = 1..7 ), \tag{2.49}$$

with $f_{mnp}$ totally antisymmetric and equal to unity for the seven different three cycles

123, 145, 176, 246, 257, 347, 365

(each cycle represents a quaternionic subalgebra). We can define an associator as follows, for any three octonionic numbers $a, b$ and $c$,

$$\{a,\ b,\ c\} \equiv (ab)c - a(bc), \tag{2.50}$$

where in each term on the right-hand we must, first of all, perform the multiplication in brackets. Note that for real, complex and quaternionic numbers the associator is trivially null. For octonionic imaginary units however we have

$$\{e_m,\ e_n,\ e_p\} \equiv (e_m e_n)e_p - e_m(e_n e_p) = 2C_{mnps} e_s, \tag{2.51}$$

with $C_{mnps}$ totally antisymmetric and equal to unity for the seven combinations

1247, 1265, 2345, 2376, 3146, 3157, 4567.

Working with octonionic numbers the associator (2.51) is non-vanishing for any three elements which are not in the same three cycles, however, the “alternative condition” is always fulfilled

$$\{a,\ b,\ c\} + \{c,\ b,\ a\} = 0. \tag{2.52}$$

Due to the non-associativity, representing any form of octonions by matrices seems impossible. Nevertheless, we overcome these problems by introducing left/right-octonionic operators and fixing the direction of action. We discuss in the next subsection their relation to $GL(8, \mathbb{R})$ where we present our translation idea and give some explicit examples which allow us to establish the isomorphism between our left/right octonionic operators and $GL(8, \mathbb{R})$.

Let us summarize the main points of the translation idea leaving the details to the next subsection. Exactly as in the quaternionic case, it seems natural to define and investigate the existence of barred operators

$$\mathbb{O}_0 \; + \; \mathbb{O}_m|e_m \qquad [\; \mathbb{O}_{0,\dots,7} \;\; \text{octonions} \;] \quad . \tag{2.53}$$

We first observe that an octonionic barred operator, $a|b$, which acts on octonionic functions, $\varphi$,

$$a|b \;\; \varphi \equiv a\varphi b \quad ,$$

is not a well defined object. For $a \neq b$ the triple product $a\varphi b$ could be either $(a\varphi)b$ or $a(\varphi b)$. So, in order to avoid this ambiguity (due to the nonassociativity of the octonionic numbers) we need to introduce left/right-barred operators. We will define left-barred operators by $a)b$, with $a$ and $b$ which represent octonionic numbers [46][59]. They act on octonionic functions $\varphi$ as follows

$$a)b \;\; \varphi = (a\varphi)b \quad . \tag{2.54}$$

In a similar way we can introduce right-barred operators $a(b$, defined by

$$a(b \;\; \varphi = a(\varphi b) \quad . \tag{2.55}$$

Obviously, there are barred-operators which are associative like

$$1)a = 1(a \equiv 1|a \quad .$$

Furthermore, because of the alternativity condition (2.52) ,

$$a)a = a(a \equiv a|a \quad .$$

At first glance it seems that we must consider the following 106 barred-operators:

$$\begin{array}{lll}
1, \; e_m, \; 1|e_m & & (15 \text{ elements}) \; , \\
e_m|e_m & & (7) \; , \\
e_m)e_n & (m \neq n) & (42) \; , \\
e_m(e_n & (m \neq n) & (42) \; , \\
(m, \; n = 1, ..., 7) \quad . & &
\end{array}$$

Nevertheless, *it is possible to prove that each right-barred operator can be expressed by a suitable combination of left-barred operators.* For example, from eq. (2.52), by posing $a = e_m$ and $c = e_n$, we quickly obtain

$$e_m(e_n + e_n(e_m \;\; \equiv \;\; e_m)e_n + e_n)e_m \quad . \tag{2.56}$$

So we can represent the most general octonionic operator by only left-barred objects

$$\mathbb{O}_0 + \sum_{m=1}^{7} \mathbb{O}_m) e_m \qquad [\ \mathbb{O}_{0,\dots,7} \ \ \text{octonions}\ ] \quad , \tag{2.57}$$

reducing to 64 independent elements the previous 106. This number of 64 suggests a correspondence between our barred octonions (2.57) and $GL(8,\mathbb{R})$.

In subsection (2.3.2), we focus our attention on the group $GL(4,\mathbb{C}) \subset GL(8,\mathbb{R})$. In doing so, we will find that only particular combinations of octonionic barred operators give us suitable candidates for the $GL(4,\mathbb{C})$ translation.

### 2.3.1 Octonionic Operators and $8 \times 8$ Real Matrices

In order to explain the idea of *translation*, let us look explicitly at the action of the operators $1 \mid e_1$ and $e_2$, on a generic octonionic function $\varphi$

$$\varphi = \varphi_0 + e_1\varphi_1 + e_2\varphi_2 + e_3\varphi_3 + e_4\varphi_4 + e_5\varphi_5 + e_6\varphi_6 + e_7\varphi_7 \quad [\ \varphi_{0,\dots,7} \in \mathbb{R}\ ]. \tag{2.58}$$

We have

$$1|e_1 \quad \varphi \ \equiv \ \varphi e_1 = e_1\varphi_0 - \varphi_1 - e_3\varphi_2 + e_2\varphi_3 - e_5\varphi_4 + e_4\varphi_5 + e_7\varphi_6 - e_6\varphi_7, \tag{2.59}$$

$$e_2\varphi \ = \ e_2\varphi_0 - e_3\varphi_1 - \varphi_2 + e_1\varphi_3 + e_6\varphi_4 + e_7\varphi_5 - e_4\varphi_6 - e_5\varphi_7. \tag{2.60}$$

If we represent our octonionic ("state") function $\varphi$ by the following real column vector

$$\Phi \ \leftrightarrow \ \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} \quad , \tag{2.61}$$

we can rewrite eqs. (2.59–2.60) in matrix form,

$$
1|\mathbb{E}_1\Phi = \begin{pmatrix} -\varphi_1 \\ \varphi_0 \\ \varphi_3 \\ -\varphi_2 \\ \varphi_5 \\ -\varphi_4 \\ -\varphi_7 \\ \varphi_6 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} \tag{2.62}
$$

$$
\mathbb{E}_2\Phi = \begin{pmatrix} -\varphi_2 \\ \varphi_3 \\ \varphi_0 \\ -\varphi_1 \\ -\varphi_6 \\ -\varphi_7 \\ \varphi_4 \\ \varphi_5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} \tag{2.63}
$$

In this way we can immediately obtain a real matrix representation for the octonionic barred operators $1 \mid e_1$ and $e_2$. Following this procedure we can construct the complete set of translation rules for the imaginary unit operators $e_m$ and the barred operators $1 \mid e_m$ (appendix A).

At first glance it seems that our translation doesn't work. If we extract the matrices corresponding to $e_1$, $e_2$ and $e_3$, namely,

$$
\mathbb{E}_1 = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},
$$

$$\mathbb{E}_2 = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\mathbb{E}_3 = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix},$$

we find

$$\mathbb{E}_1\mathbb{E}_2 \neq \mathbb{E}_3 \quad . \tag{2.64}$$

In obvious contrast with the octonionic relation

$$e_1 e_2 = e_3 \quad . \tag{2.65}$$

This paradox is easily understood. In deducing our translation rules, we understand octonions as operators, and so they must be applied to a certain octonionic function, or state, $\varphi$, and `not` upon another "operator". So the octonionic relation

$$e_3\varphi \quad = (e_1e_2)\varphi \tag{2.66}$$

is indeed translated by

$$\mathbb{E}_3\varphi \quad , \tag{2.67}$$

whereas,

$$e_1(e_2\varphi) \quad [\ \neq e_3\varphi\ ] \tag{2.68}$$

becomes

$$\mathbb{E}_1\mathbb{E}_2\varphi \quad [\ \neq \mathbb{E}_3\varphi\ ] \quad . \tag{2.69}$$

For $e_m$ and $1|e_n$, we have simple multiplication rules. In fact, utilizing the associator properties we find

$$\begin{aligned} e_m \ \ [e_n\varphi] &= (e_me_n)\ \varphi + (e_m\varphi)\ e_n - e_m\ (\varphi e_n)\ , & (2.70)\\ [\varphi e_m]\ \ e_n &= \varphi\ (e_me_n) - (e_m\varphi)\ e_n + e_m\ (\varphi e_n)\ . & (2.71) \end{aligned}$$

Thus[2],

$$
\begin{aligned}
e_m \;.\; e_n &\equiv -\delta_{mn} + f_{mnp}e_p + e_m)e_n - e_m(e_n \;, & (2.72)\\
1|e_n \;.\; 1|e_m &\equiv -\delta_{mn} + f_{mnp}e_p - e_m)e_n + e_m(e_n \;. & (2.73)
\end{aligned}
$$

The previous relation can be immediately rewritten in matrix form as follows [46]

$$
\begin{aligned}
\mathbb{E}_m\mathbb{E}_n &\equiv -\delta_{mn} + f_{mnp}\mathbb{E}_p + [1|\mathbb{E}_n,\ \mathbb{E}_m] \quad , & (2.74)\\
1|\mathbb{E}_n 1|\mathbb{E}_m &\equiv -\delta_{mn} + f_{mnp}1|\mathbb{E}_p + [\mathbb{E}_m,\ 1|\mathbb{E}_n] \quad . & (2.75)
\end{aligned}
$$

Introducing a new matrix multiplication, " $\circ$ ", related to the standard matrix multiplication (row by column) by

$$\mathbb{E}_m \circ \mathbb{E}_n \equiv \mathbb{E}_m\mathbb{E}_n - [1|\mathbb{E}_n,\ \mathbb{E}_m] \quad , \tag{2.76}$$

we can quickly reformulate the nonassociative octonionic algebra (in terms of matrices this time) by

$$\mathbb{E}_m \circ \mathbb{E}_n = -\delta_{mn} + f_{mnp}\mathbb{E}_p \quad . \tag{2.77}$$

Working with left/right barred operators we now show how the nonassociativity is realized with our matrix translation. Such operators enable us to reproduce the octonions nonassociativity by the matrix algebra. Consider for example

$$e_3)e_1 \quad \varphi \equiv (e_3\varphi) \;\; e_1 = e_2\varphi_0 - e_3\varphi_1 + \varphi_2 - e_1\varphi_3 - e_6\varphi_4 - e_7\varphi_5 + e_4\varphi_6 + e_5\varphi_7 \quad . \tag{2.78}$$

This equation will be translated into

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0\\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0\\ \varphi_1\\ \varphi_2\\ \varphi_3\\ \varphi_4\\ \varphi_5\\ \varphi_6\\ \varphi_7 \end{pmatrix} = \begin{pmatrix} \varphi_2\\ -\varphi_3\\ \varphi_0\\ -\varphi_1\\ \varphi_6\\ \varphi_7\\ -\varphi_4\\ -\varphi_5 \end{pmatrix} \tag{2.79}$$

[2]We have used square brackets on the L.H.S. in the previous two equations in order to avoid confusion between the L.H.S. and the last term on the R.H.S. in the following two equations which might occur if we had employed $e_m\,(e_n\varphi)$ for the L.H.S. of (2.70) etc.

Whereas,

$$e_3(e_1 \quad \varphi \;\equiv\; e_3(\varphi e_1) \;=\; e_2\varphi_0 - e_3\varphi_1 + \varphi_2 - e_1\varphi_3 + e_6\varphi_4 + e_7\varphi_5 - e_4\varphi_6 - e_5\varphi_7 \quad , \tag{2.80}$$

will become

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \varphi_0 \\ \varphi_1 \\ \varphi_2 \\ \varphi_3 \\ \varphi_4 \\ \varphi_5 \\ \varphi_6 \\ \varphi_7 \end{pmatrix} = \begin{pmatrix} \varphi_2 \\ -\varphi_3 \\ \varphi_0 \\ -\varphi_1 \\ -\varphi_6 \\ -\varphi_7 \\ \varphi_4 \\ \varphi_5 \end{pmatrix} \tag{2.81}$$

The nonassociativity is then reproduced since left and right barred operators, like

$$e_3)e_1 \quad \text{and} \quad e_3(e_1$$

are represented by different matrices. The complete set of translation rules for left/right-barred operators is given in appendix A.

The full matrix representation for left/right barred operators can be quickly obtained by suitable multiplications of the matrices $\mathbb{E}_m$ and $1|\mathbb{E}_n$. By direct calculations we can extract the matrices which correspond to the operators

$$e_m)e_n \qquad \textit{and} \qquad e_m(e_n \qquad ,$$

which we call, respectively,

$$\mathbb{E}_m)\mathbb{E}_n \qquad \textit{and} \qquad \mathbb{E}_m(\mathbb{E}_n \quad .$$

Since our left/right barred operators can be represented by an ordered action of the operators $e_m$ and $1 \mid e_n$, we can relate the matrices $\mathbb{E}_m)\mathbb{E}_n$ and $\mathbb{E}_m(\mathbb{E}_n$ to the matrices $\mathbb{E}_m$ and $1|\mathbb{E}_m$:

$$\mathbb{E}_m)\mathbb{E}_n \quad \equiv \quad 1|\mathbb{E}_n \quad \mathbb{E}_m \quad , \tag{2.82}$$

$$\mathbb{E}_m(\mathbb{E}_n \quad \equiv \quad \mathbb{E}_m \quad 1|\mathbb{E}_n \quad . \tag{2.83}$$

The previous discussions concerning the octonions nonassociativity and the isomorphism between $GL(8,\mathbb{R})$ and barred octonions, can now be elegantly summarized as follows.

`1 - Matrix representation for octonions nonassociativity acting on certain octonionic number.`

$$\mathbb{E}_m)\mathbb{E}_n \neq \mathbb{E}_m(\mathbb{E}_n \qquad [\, 1|\mathbb{E}_n\mathbb{E}_m \neq \mathbb{E}_m 1|\mathbb{E}_n \quad \text{for } m \neq n] \quad .$$

`2 - Isomorphism between` $GL(8,\mathbb{R})$ `and barred octonions.`

If we rewrite our 106 barred operators by real matrices:

$$\begin{array}{lr}
1,\ \mathbb{E}_m,\ 1|\mathbb{E}_m & (15 \text{ matrices}) \ , \\
\mathbb{E}_m|\mathbb{E}_m \equiv \mathbb{E}_m \quad 1|\mathbb{E}_m = 1|\mathbb{E}_m \quad \mathbb{E}_m & (7) \ , \\
\mathbb{E}_m)\mathbb{E}_n \equiv 1|\mathbb{E}_n \quad \mathbb{E}_m \qquad (m \neq n) & (42) \ , \\
\mathbb{E}_m(\mathbb{E}_n \equiv \mathbb{E}_n \quad 1|\mathbb{E}_m \qquad (m \neq n) & (42) \ , \\
(m,\ n = 1, ..., 7) \quad ; &
\end{array}$$

we have two different basis for $GL(8,\mathbb{R})$:

$$(1) \qquad 1\ ,\ \mathbb{E}_m\ ,\ 1|\mathbb{E}_m\ ,\ 1|\mathbb{E}_n\ \ \mathbb{E}_m,\ \ 1|\mathbb{E}_m\ \ \mathbb{E}_m \quad ,$$

$$(2) \qquad 1\ ,\ \mathbb{E}_m\ ,\ 1|\mathbb{E}_m\ ,\ \mathbb{E}_m\ \ 1|\mathbb{E}_n,\quad 1|\mathbb{E}_m\ \ \mathbb{E}_m.$$

We now note some difficulties due to the nonassociativity of octonions. When we translate from barred octonions to $8 \times 8$ real matrices there is no problem. For example, in the octonionic equation

$$e_4\{[(e_6\varphi)e_1]e_5\} \quad , \tag{2.84}$$

we quickly recognize the following left/right octonionic operators,

$$e_6)e_1 \quad \textit{followed by} \quad e_4(e_5 \qquad .$$

Hence we can translate eq. (2.84) into

$$[\mathbb{E}_4(\mathbb{E}_5] \quad [\mathbb{E}_6)\mathbb{E}_1] \ \ \varphi \quad . \tag{2.85}$$

But in going from $8 \times 8$ real matrices to octonions we must be careful of the ordering. For example, with A, B matrices

$$AB\ \varphi \tag{2.86}$$

can be understood for translation purposes as

$$(AB)\varphi \quad , \tag{2.87}$$

or

$$A(B\varphi) \quad . \tag{2.88}$$

In order to avoid confusion we translate eq. (2.86) by eq. (2.88). In general when brackets are absent we shall choose the convention that

$$ABC \ldots Z\varphi \equiv A(B(C \ldots (Z\varphi) \ldots)) \quad . \tag{2.89}$$

### 2.3.2 Octonionic Operators and $4 \times 4$ Complex Matrices

Some complex groups play a critical role in physics. No one can deny the importance of $U(1,\mathbb{C})$ and $SU(2,\mathbb{C})$. In relativistic quantum mechanics, $GL(4,\mathbb{C})$ is implicit in writing the Dirac equation. Starting from our $GL(8,\mathbb{R})$, we should be able to extract its subgroup $GL(4,\mathbb{C})$. Whence we should be able to translate the famous Dirac-gamma matrices and write down a four dimensional one-component octonionic Dirac equation [60].

Let us show how we can extract our 32 basis of $GL(4,\mathbb{C})$: Working with the symplectic decomposition of octonionic "states"

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \leftrightarrow \psi_1 + e_2\psi_2 + e_4\psi_3 + e_6\psi_4 \qquad [\ \psi_{1,\dots,4} \in \mathbb{C}(1,\ e_1)\ ] \quad . \tag{2.90}$$

we analyze the action of left-barred operators on our octonionic wave functions $\psi$. For example, we find

$$\begin{aligned} 1|e_1 \quad \psi \equiv \quad & \psi e_1 && = e_1\psi_1 + e_2(e_1\psi_2) + e_4(e_1\psi_3) + e_6(e_1\psi_4) && (2.91)\\ & e_2\psi && = -\psi_2 + e_2\psi_1 - e_4\psi_4^* + e_6\psi_3^* \quad , && (2.92)\\ e_3)e_1 \quad \psi \equiv \quad & (e_3\psi)e_1 && = \psi_2 + e_2\psi_1 + e_4\psi_4^* - e_6\psi_3^* \quad . && (2.93) \end{aligned}$$

Following the same methodology of the previous section, we can immediately note a correspondence between the complex matrix $i\mathbf{1}_{4\times4}$and the octonionic operator $1 \mid e_1$

$$\begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \leftrightarrow 1|e_1 \quad . \tag{2.94}$$

This translation does not work for all barred operators. Let us show this, explicitly. For example, we cannot find a $4 \times 4$ complex matrix which, acting upon

$$\begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \quad ,$$

gives the column vector

$$e_2\psi = \begin{pmatrix} -\psi_2 \\ \psi_1 \\ -\psi_4^* \\ \psi_3^* \end{pmatrix} \quad or \quad e_3)e_1\psi = \begin{pmatrix} \psi_2 \\ \psi_1 \\ \psi_4^* \\ -\psi_3^* \end{pmatrix}$$

and so we have not the possibility to relate

$$e_2 \qquad or \qquad e_3)e_1$$

with a complex matrix. Nevertheless, a combined action of these two operators gives us

$$e_2\psi + (e_3\psi)\ \ e_1 = 2e_2\psi_1 \quad ,$$

and allows us to represent the octonionic barred sum

$$e_2\ +\ e_3)e_1 \quad , \tag{2.95}$$

by the $4 \times 4$ complex matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad . \tag{2.96}$$

Following this procedure we can represent our generic $4 \times 4$ complex matrix by octonionic barred operators (but not necessarily the contrary). The explicit correspondence rules are given in appendix B.

We conclude our discussion upon the relationship between barred operators and $4 \times 4$ complex matrices, by noting that the 32 basis elements of $GL(4, \mathbb{C})$ can be extracted in a different way from the 64 generators of $GL(8, \mathbb{R})$. It is well known that any complex matrix can be rewritten as a real matrix by the following isomorphism ($\sigma$ are the standard Pauli matrices),

$$1 \ \leftrightarrow\ \mathbf{1}_{2\times 2} \qquad \text{and} \qquad i \ \leftrightarrow\ -i\sigma_2 \quad .$$

The situation at the lowest order is

$$GL(2, \mathbb{R}) \qquad \text{generators :} \qquad \mathbf{1}_{2\times 2}\ ,\ \sigma_1\ ,\ \ -i\sigma_2\ ,\ \sigma_3 \quad ; \tag{2.97}$$

$$GL(1, \mathbb{C}) \qquad \text{isomorphic :} \qquad \mathbf{1}_{2\times 2}\ ,\ \ -i\sigma_2 \quad . \tag{2.98}$$

In a similar way (choosing appropriate combinations of left-barred octonionic operators, in which only $\pm\mathbf{1}_{2\times 2}$and $\pm i\sigma_2$appear) we can extract from $GL(8, \mathbb{R})$ the 32 basis elements of $GL(4, \mathbb{C})$. For further details see appendix B.

## 2.4 Beyond Octonions and Clifford Algebras

Going to higher dimensions, we define "hexagonions" ($\mathbb{X}$) by introducing a new element $e_8$ such that

$$\begin{aligned} \mathbb{X} &= \mathbb{O}_1 + \mathbb{O}_2 e_8 \\ &= x_0 e_0 + \ldots + x_{16} e_{16}. \quad x_\mu \in \mathbb{R} \end{aligned} \tag{2.99}$$

and

$$e_i e_j = -\delta_{ij} + C_{ijk} e_k. \tag{2.100}$$

Now, we have to find a suitable form of $C_{ijk}$. Recalling how the structure constant is written for octonions

$$\begin{aligned} \mathbb{O} &= \mathbb{Q}_1 + \mathbb{Q}_2 e_4 \\ &= x_0 e_0 + \ldots + x_7 e_7, \end{aligned} \tag{2.101}$$

where $\mathbb{Q}$ are quaternions, we have already chosen the convention $e_1 e_2 = e_3$ which is extendable to (2.101), we set $e_1 e_4 = e_5$, $e_2 e_4 = e_6$and $e_3 e_4 = e_7$, but we still lack the relationships between the remaining possible triplets, $\{e_1, e_6, e_7\}$; $\{e_2, e_5, e_7\}$; $\{e_3, e_5, e_6\}$ which can be fixed by using

$$\begin{aligned} e_1 e_6 = e_1(e_2 e_4) = -(e_1 e_2) e_4 = -e_3 e_4 = -e_7, \\ e_2 e_5 = e_2(e_1 e_4) = -(e_2 e_1) e_4 = +e_3 e_4 = +e_7, \\ e_3 e_5 = e_3(e_1 e_4) = -(e_3 e_1) e_4 = -e_2 e_4 = -e_6. \end{aligned}$$

These define all the structure constants for octonions. Returning to $\mathbb{X}$, we have the seven octonionic conditions, and the decomposition (2.99). We set $e_1 e_8 = e_9$, $e_2 e_8 = e_A$, $e_3 e_8 = e_B$, $e_4 e_8 = e_C$, $e_5 e_8 = e_D$, $e_6 e_8 = e_E$, $e_7 e_8 = e_F$ where $A = 10$, $B = 11$, $C = 12$, $D = 13$, $E = 14$ and $F = 15$. The other elements of the multiplication table may be chosen in analogy with (2.101). Explicitly, the 35 hexagonionic triplets are

$$\begin{array}{ccccccc} (123), & (145), & (246), & (347), & (257), & (176), & (365), \\ (189), & (28A), & (38B), & (48C), & (58D), & (68E), & (78F), \\ (1BA), & (1DC), & (1EF), & (29B), & (2EC), & (2FD), & (3A9), \\ (49D), & (4AE), & (4BF), & (3FC), & (3DE), & (5C9), & (5AF), \\ (5EB), & (6FD), & (6CA), & (6BD), & (79E), & (7DA), & (7CB). \end{array}$$

This can be extended for any generic higher dimensional field $\mathbb{F}^n$.

It can be shown by using some combinatorics that the number of such triplets $N$ for a general $\mathbb{F}^n$field is $(n > 1)$

$$N = \frac{(2^n - 1)!}{(2^n - 3)! \quad 3!}, \tag{2.102}$$

giving

| $\mathbb{F}^n$ | $n$ | $dim$ | $N$ |
|---|---|---|---|
| $\mathbb{Q}$ | 2 | 4 | 1 |
| $\mathbb{O}$ | 3 | 8 | 7 |
| $\mathbb{X}$ | 4 | 16 | 35 |

*and so on.*

One may notice that for any non-ring division algebra ($\mathbb{F}$, $n > 3$), $N > dim(\mathbb{F}^n)$ except when dim $= \infty$, i.e. a functional Hilbert space with a Cliff(0,$\infty$) structure. Does this inequality have any connection with the ring division structure of the $(S^1, S^3, S^7)$ spheres ? Yes, that is what we are going to show now.

Following, the same translation idea projecting our algebra $\mathbb{X}$ over $\mathbb{R}^{16}$, any $\mathbb{E}_i$ is given by a relation similar to that for $\mathbb{Q}$

$$(\mathbb{E}_i)_{\alpha\beta} = \delta_{i\alpha}\delta_{\beta 0} - \delta_{i\beta}\delta_{\alpha 0} + C_{i\alpha\beta}. \tag{2.103}$$

But contrary to the quaternions and octonions, the Clifford algebra closes only for a subset of these $E_i$'s, namely

$$\{\mathbb{E}_i, \mathbb{E}_j\} = -2\delta_{ij} \quad \text{for} \quad i,j,k = 1\ldots 8 \ \textit{not}\ 1...15. \tag{2.104}$$

Because we have lost the ring division structure. By careful investigation, we find that another ninth $\mathbb{E}_i$ can be constructed, in agreement with the Clifford algebra classification [7]. There is no standard[3] 16 dimensional representation for $Cliff\,(15)$. Following this procedure, we can give a simple way to write real Clifford algebras over any arbitrary dimensions.

Sometimes, a specific multiplication table may be favored. For example in soliton theory, the existence of a symplectic structure related to the bi-hamiltonian formulation of integrable models is welcome. It is known from the Darboux theorem, that locally a symplectic structure is given up to a minus sign by

$$\mathcal{J}_{dim\times dim} = \begin{pmatrix} 0 & -\mathbf{1}_{\frac{dim}{2}} \\ \mathbf{1}_{\frac{dim}{2}} & 0 \end{pmatrix}, \tag{2.105}$$

this fixes the following structure constants

$$C_{(\frac{dim}{2})1(\frac{dim}{2}+1)} = -1, \tag{2.106}$$

$$C_{(\frac{dim}{2})2(\frac{dim}{2}+2)} = -1, \tag{2.107}$$

$$\vdots \tag{2.108}$$

$$C_{(\frac{dim}{2})(\frac{dim}{2}-1)(dim-1)} = -1, \tag{2.109}$$

[3]Look to [57] for a non standard representation.

which is the decomposition that we have chosen in (2.101) for octonions

$$C_{415} = C_{426} = C_{437} = -1. \tag{2.110}$$

Generally our symplectic structure is

$$\left(1|\mathbb{E}_{\left(\frac{dim}{2}\right)}\right)_{\alpha\beta} = \delta_{0\alpha}\delta_{\beta\left(\frac{dim}{2}\right)} - \delta_{0\beta}\delta_{\alpha\left(\frac{dim}{2}\right)} - \epsilon_{\alpha\beta\left(\frac{dim}{2}\right)}. \tag{2.111}$$

Moreover some other choices may exhibit a relation with number theory and Galois fields [61]. It is highly non-trivial how Clifford algebraic language can be used to unify many distinct mathematical notions such as Grassmanian [62], complex, quaternionic and symplectic structures.

The main result of this section, the non-existence of 16 dimensional representation of $Cliff\,(0,15)$ is in agreement with the Atiyah–Bott–Shapiro classification of real Clifford algebras [7]. In this context, the importance of ring division algebras can also be deduced from the Bott periodicity [63]. Another interesting observation, if we interpret the complex, quaternions, octonions eigenfunctions as real spinors, we find that for

$$\begin{array}{lrcll} complex & Z^t E_1 Z & = & 0, & \quad (2.112) \\ quaternions & Q^t \mathbb{E}_i Q & = & 0 \quad i = 1..3, & \quad (2.113) \\ octonions & \Phi^t \mathbb{E}_i \Phi & = & 0 \quad i = 1..7. & \quad (2.114) \end{array}$$

These states ($Z$, $Q$, $\Phi$) are called pure spinors as first coined by Cartan. These pure spinors play an important role for minimal surfaces, integrable models, twistor calculus, and string theory [64].

# Chapter 3

# The Soft Seven Sphere

Ring division algebras play fundamental roles in mathematics from algebra to geometry and topology with many different applications. The applicability of real and complex numbers in physics is not in question. Quaternions which may be represented as Pauli matrices are also important. the use of octonions in physics is the problem. In the first section of this chapter, we introduce what we mean by the word "soft" algebra, generally we follow closely the presentation of Sohnius[43]. Sohnius used soft algebras with structure functions that vary over space-time as well as over an internal gauge manifold. In this thesis we use only soft algebras with structure functions that vary over the internal gauge manifold (the fiber) not the base space-time manifold. In the second section, we introduce the seven sphere as a soft algebra [42], we calculate its structure functions explicitly and also discuss some relevant points such as the validity of the Jacobi identity. Furthermore, we emphasis some important features such as the pointwise reduction, closure, and some other consistency checks. In the last section, we reformulate some standard Lie group results putting them in a form suitable to subsequent applications.

## 3.1 Sohnuis' Idea

Trying to find a suitable framework for supersymmetric theories, Sohnius introduced the notion of soft gauge algebras i.e. algebras where the structure constants become structure functions of space-time and gauge-dependent fields. He proceeded as follows: For any Lie group, we know that a generic element can be written as $exp(i\varepsilon_i L_i)$ where $\varepsilon$ are finite numbers of parameters and $L_i$ are our Lie algebras elements. A field $A$, that lives in a certain representation of our algebras, transforms infinitesimally as

$$A \longrightarrow A + \delta_{\varepsilon_1} A \quad \text{with} \quad \delta_{\varepsilon_1} A = -i[A, \varepsilon_1^i L_i] \qquad . \tag{3.1}$$

Sohnuis considered the special case when the commutator of two successive transformations leads to

$$[\delta_{\varepsilon_1}, \delta_{\varepsilon_2}]\Phi = -i[\Phi, \varepsilon_1^i \varepsilon_2^j f_{ij}{}^k(\varphi) L_k] = -i\left[\Phi, \varepsilon_3^k L_k\right] \qquad , \tag{3.2}$$

where $\varphi$ is a coordinate system for the internal gauge manifold and $f_{ij}{}^k(\varphi)$ are "the structure functions" of our soft algebra defined by

$$[\delta_{\varepsilon_1^i}, \delta_{\varepsilon_2^j}] = \delta_{f^{ij}{}_k(\varphi)\varepsilon_3^k}. \tag{3.3}$$

In standard local gauge theory, we have $\varepsilon(x)$ and we need a gauge field that transforms inhomogeneously:

$$\delta_\varepsilon A_\mu^i = \partial_\mu \varepsilon(x) + A_\mu^j \varepsilon(x)^i f_{ij}{}^k(\varphi) \qquad . \tag{3.4}$$

In this thesis, we will always assume

$$\partial_\mu f_{ijk}(\varphi) = 0. \tag{3.5}$$

By the $\varphi$ dependence of $f_{ijk}(\varphi)$, we mean a dependence on the internal gauged space which is another manifold distinct from our space-time $x$. To develop a representation, we use

$$\delta_\varepsilon \Phi = \varepsilon(x)\delta(L_i)\Phi, \tag{3.6}$$

and

$$[\delta(L_i), \delta(L_j)] = a f_{ij}{}^k(\varphi)\delta(L_k) \tag{3.7}$$

where $a \in \mathbb{R}$. In this thesis we set $a = 2$. Let's see how octonions may be treated as a soft algebra exactly in the sense of eq. (3.7).

## 3.2 Octonions and the Soft Seven Sphere

We start by recalling the non-associative octonion algebra. A generic octonion number is

$$\varphi = \varphi_0 e_0 + \varphi_i e_i = \varphi_\mu e_\mu. \qquad [i = 1..7, \mu = 0..7, \varphi_\mu \in \mathbb{R}] \tag{3.8}$$

and its associator

$$[e_i, e_j, e_k] = (e_i e_j)e_k - e_i(e_j e_k) \qquad , \tag{3.9}$$

is non-zero for any three elements that are not in the same three cycles and is completely antisymmetric The following formula or any of its generalization is thus ambiguous

$$e_1e_5e_7 = \begin{cases} (e_1e_5)e_7 = -e_3 \\ \text{or} \\ e_1(e_5e_7) = e_3 \end{cases} \tag{3.10}$$

so the best way is to define the action of the imaginary units in a certain direction. In the spirit of Englert, Sevrin, Troost, Van Proeyen and Spindel[42] (also look at [46]) we define the left action of octonionic operators $\delta_i$ by

$$\delta_i\varphi = (e_i\varphi) \tag{3.11}$$

implying that the following equation is well defined

$$\delta_i\delta_j\varphi = \delta_i(\delta_j\varphi) = \delta_i(e_j\varphi) = e_i(e_j\varphi), \tag{3.12}$$

then eq. (3.10) reads unambiguously as

$$\delta_1\delta_5e_7 = (e_1(e_5e_7)) = e_3. \tag{3.13}$$

Since octonions are also non-commutative, we must also differentiate between left and right action. Using the barred notation [50], we introduce right action as

$$1|\delta_i\ \varphi = (\varphi e_i) \qquad , \tag{3.14}$$

for example

$$(1|\delta_i)(1|\delta_j)\varphi = (1|\delta_i)(1|\delta_j\varphi) = 1|\delta_i(\varphi e_j) = ((\varphi e_j)e_i) \qquad . \tag{3.15}$$

As we shall see shortly, we can express the associator in terms of left and right operators. The imaginary octonionic units generate the seven sphere $S^7$which has many properties similar to Lie algebras and/or Lie groups. $S^7$and Lie groups are the only non-flat compact parallelizable manifolds [65][66][67].

The important point for evaluating any Lie algebra is the commutator, so let's examine

$$\begin{aligned} [\delta_i,\delta_j]\varphi &= e_i(e_j\varphi) - e_j(e_i\varphi) && (3.16)\\ &= 2f_{ijk}e_k\varphi - 2[e_i,e_j,\varphi] && (3.17)\\ &= 2f_{ijk}e_k\varphi + 2[e_i,\varphi,e_j]. && (3.18)\end{aligned}$$

now the last term can be written as

$$\begin{aligned} [e_i,\varphi,e_j] &= (e_i\varphi)e_j - e_i(\varphi e_j) && (3.19)\\ &= -\delta_i(1|\delta_j)\varphi + (1|\delta_j)\delta_i\varphi && (3.20)\\ &= -[\delta_i, 1|\delta_j]\varphi, && (3.21)\end{aligned}$$

thus our commutator can be rewritten as

$$
\begin{aligned}
[\delta_i, \delta_j]\varphi &= 2f_{ijk}e_k\varphi + 2[e_i, \varphi, e_j] && (3.22)\\
&= 2f_{ijk}\delta_k\varphi - 2[\delta_i, 1|\delta_j]\varphi. && (3.23)
\end{aligned}
$$

Note that right operators are necessary because the last term the associator can never be written in terms of left operators alone.

After simple calculations, one concludes that the octonionic imaginary units are determined completely by (3.23) and the following equations

$$
\begin{aligned}
[1|\delta_i, 1|\delta_j]\,\varphi &= -2f_{ijk}1|\delta_k\varphi - 2[\delta_i, 1|\delta_j]\varphi && (3.24)\\
\{\delta_i, \delta_j\}\varphi &= -2\delta_{ij}\varphi && (3.25)\\
\{1|\delta_i, 1|\delta_j\}\varphi &= -2\delta_{ij}\varphi \qquad , && (3.26)
\end{aligned}
$$

where the $\delta_{ij}$ in (3.25) and (3.26) are the standard Kronecker delta tensor.

It has been proved in [42][68], using three different ways, that the $\delta_i$ algebra is associative. Thus a representation theory in terms of matrices should be, in principle, possible. Indeed, in the last chapter, we have derived an algebra completely isomorphic to (3.23,3.24,3.25,3.26) by exploiting the idea that octonions can be used as a basis for any $8 \times 8$ real matrix. we have two sets of matrices, essentially,

$$
\begin{array}{rcrcl}
\delta_i & \Longleftrightarrow & (\mathbb{E}_i)_{\mu\nu} & = & \delta_{0\mu}\delta_{i\nu} - \delta_{0\nu}\delta_{i\mu} - f_{i\mu\nu}, \\
1|\delta_i & \Longleftrightarrow & (1|\mathbb{E}_i)_{\mu\nu} & = & \delta_{0\mu}\delta_{i\nu} - \delta_{0\nu}\delta_{i\mu} + f_{i\mu\nu}.
\end{array} \qquad (3.27)
$$

The set of matrices $\mathbb{E}_i$ and $1|\mathbb{E}_i$ have appeared in different octonionic works e.g. [21][32][35][69][70][71]. Furthermore, they correspond, as we have already said, to the 't Hooft eta symbols (2.25-2.29). We suggest that their most appropriate names should be *the canonical left and right octonionic structure at the north/south pole of the seven sphere*. By explicit calculation, one finds that

$$
\begin{aligned}
[\mathbb{E}_i, \mathbb{E}_j]\varphi &= 2f_{ijk}\mathbb{E}_k\varphi - 2[\mathbb{E}_i, 1|\mathbb{E}_j]\,\varphi && (3.28)\\
[1|\mathbb{E}_i, 1|\mathbb{E}_j]\,\varphi &= -2f_{ijk}1|\mathbb{E}_k\,\varphi - 2[\mathbb{E}_i, 1|\mathbb{E}_j]\,\varphi && (3.29)\\
\{\mathbb{E}_i, \mathbb{E}_j\}\varphi &= -2\delta_{ij}\varphi && (3.30)\\
\{1|\mathbb{E}_i, 1|\mathbb{E}_j\}\varphi &= -2\delta_{ij}\varphi && (3.31)
\end{aligned}
$$

where $\varphi$ is represented by a column matrix

$$
\varphi^t = \begin{pmatrix} \varphi_0 & \varphi_1 & \varphi_2 & \varphi_3 & \varphi_4 & \varphi_5 & \varphi_6 & \varphi_7 \end{pmatrix}
$$

The word "isomorphic" above is justified since $\{\delta_i\}$ is associative [42][68] and the same holds obviously for our $\{\mathbb{E}_i\}$ as they are written in terms of matrices. Our Jacobian identities are

$$[\delta_i,[\delta_j,\delta_k]]\varphi+[\delta_j,[\delta_k,\delta_i]]\varphi+[\delta_k,[\delta_i,\delta_j]]\varphi=0 \qquad , \tag{3.32}$$

or

$$[\mathbb{E}_i,[\mathbb{E}_j,\mathbb{E}_k]]\varphi+[\mathbb{E}_j,[\mathbb{E}_k,\mathbb{E}_i]]\varphi+[\mathbb{E}_k,[\mathbb{E}_i,\mathbb{E}_j]]\varphi=0 \qquad . \tag{3.33}$$

We shall return to these identities again at the end of this section.

Fixing the direction of the application for any imaginary octonionic units extracts a part of the algebra that respects associativity, but a certain price has to be paid. The presence of the $\varphi$ is essential and either of $\{\mathbb{E}_i\}$ or $\{1|\mathbb{E}_i\}$ is an open algebra, they don't close upon the action of the commutator. Here comes the second step, the soft Lie algebra idea. It is clear that the right hand side of (3.22 or 3.28) has a complicated $\varphi$ dependence. Knowing that the seven sphere has a torsion that varies from one point to another [32][65][66] and mimicking the Lie group case where the structure constants are proportional to the fixed group torsion, it is natural to propose that (3.22) may be redefined [42] as

$$[\delta_i,\delta_j]\varphi=2f^{(+)}_{ijk}(\varphi)\delta_k\varphi \qquad , \tag{3.34}$$

where $f^{(+)}_{ijk}(\varphi)$ are structure functions that vary over the whole $S^7$manifold. It is clear that our $\delta_i$ play the same role of the $\delta(L_i)$ defined in the previous subsection eq.(3.7). These structure functions $f^{(+)}_{ijk}(\varphi)$ were computed previously using different properties of the associator and some other octonionic identities in [32][42][65][66][68]. Here we use our matrix representation to give another alternative way to calculate $f^{(+)}_{ijk}(\varphi)$

$$[\mathbb{E}_i,\mathbb{E}_j]\ \varphi=2f^{(+)}_{ijk}(\varphi)\mathbb{E}_k\ \varphi \qquad . \tag{3.35}$$

Let's do it for the following example

$$[\mathbb{E}_1,\mathbb{E}_2]\ \varphi=2f^{(+)}_{12k}(\varphi)\mathbb{E}_k\ \varphi \qquad , \tag{3.36}$$

which is equivalent to the following eight equations ,

$$\begin{aligned}
\varphi_3 &= \varphi_1 f^{(+)}_{121}(\varphi)+\varphi_2 f^{(+)}_{122}(\varphi)+\varphi_3 f^{(+)}_{123}(\varphi)\\
&\quad +\varphi_4 f^{(+)}_{124}(\varphi)+\varphi_5 f^{(+)}_{125}(\varphi)+\varphi_6 f^{(+)}_{126}(\varphi)+\varphi_7 f^{(+)}_{127}(\varphi),\\
\varphi_2 &= \varphi_2 f^{(+)}_{123}(\varphi)-\varphi_0 f^{(+)}_{121}(\varphi)-\varphi_3 f^{(+)}_{122}(\varphi)\\
&\quad -\varphi_5 f^{(+)}_{124}(\varphi)+\varphi_4 f^{(+)}_{125}(\varphi)+\varphi_7 f^{(+)}_{126}(\varphi)-\varphi_6 f^{(+)}_{127}(\varphi),
\end{aligned}$$

$$
\begin{aligned}
\varphi_1 &= \varphi_0 f^{(+)}_{122}(\varphi) - \varphi_3 f^{(+)}_{121}(\varphi) + \varphi_1 f^{(+)}_{123}(\varphi) \\
&\quad + \varphi_6 f^{(+)}_{124}(\varphi) + \varphi_7 f^{(+)}_{125}(\varphi) - \varphi_4 f^{(+)}_{126}(\varphi) - \varphi_5 f^{(+)}_{127}(\varphi), \\
\varphi_0 &= \varphi_2 f^{(+)}_{121}(\varphi) - \varphi_1 f^{(+)}_{122}(\varphi) + \varphi_0 f^{(+)}_{123}(\varphi) \\
&\quad + \varphi_7 f^{(+)}_{124}(\varphi) - \varphi_6 f^{(+)}_{125}(\varphi) + \varphi_5 f^{(+)}_{126}(\varphi) - \varphi_4 f^{(+)}_{127}(\varphi), \\
\varphi_7 &= \varphi_0 f^{(+)}_{124}(\varphi) - \varphi_5 f^{(+)}_{121}(\varphi) - \varphi_6 f^{(+)}_{122}(\varphi) \\
&\quad - \varphi_7 f^{(+)}_{123}(\varphi) + \varphi_1 f^{(+)}_{125}(\varphi) + \varphi_2 f^{(+)}_{126}(\varphi) + \varphi_3 f^{(+)}_{127}(\varphi), \\
\varphi_6 &= \varphi_7 f^{(+)}_{122}(\varphi) - \varphi_4 f^{(+)}_{121}(\varphi) - \varphi_6 f^{(+)}_{123}(\varphi) \\
&\quad + \varphi_1 f^{(+)}_{124}(\varphi) - \varphi_0 f^{(+)}_{125}(\varphi) + \varphi_3 f^{(+)}_{126}(\varphi) - \varphi_2 f^{(+)}_{127}(\varphi), \\
\varphi_5 &= \varphi_7 f^{(+)}_{121}(\varphi) + \varphi_4 f^{(+)}_{122}(\varphi) - \varphi_5 f^{(+)}_{123}(\varphi) \\
&\quad - \varphi_2 f^{(+)}_{124}(\varphi) + \varphi_3 f^{(+)}_{125}(\varphi) + \varphi_0 f^{(+)}_{126}(\varphi) - \varphi_1 f^{(+)}_{127}(\varphi), \\
\varphi_4 &= \varphi_6 f^{(+)}_{121}(\varphi) - \varphi_5 f^{(+)}_{122}(\varphi) - \varphi_4 f^{(+)}_{123}(\varphi) \\
&\quad + \varphi_3 f^{(+)}_{124}(\varphi) + \varphi_2 f^{(+)}_{125}(\varphi) - \varphi_1 f^{(+)}_{126}(\varphi) - \varphi_0 f^{(+)}_{127}(\varphi).
\end{aligned}
\tag{3.37}
$$

We now solve these equations for the seven unknown $f^{(+)}_{12i}(\varphi)$. We find

$$f^{(+)}_{121}(\varphi) = f^{(+)}_{122}(\varphi) = 0, \tag{3.38}$$

$$f^{(+)}_{123}(\varphi) = \frac{{\varphi_0}^2 - {\varphi_6}^2 - {\varphi_5}^2 + {\varphi_2}^2 - {\varphi_4}^2 + {\varphi_1}^2 + {\varphi_3}^2 - {\varphi_7}^2}{r^2}$$

and

$$f^{(+)}_{124}(\varphi) = +2\frac{\varphi_0\varphi_7 - \varphi_5\varphi_2 + \varphi_6\varphi_1 + \varphi_3\varphi_4}{r^2}, \tag{3.39}$$

$$f^{(+)}_{125}(\varphi) = -2\frac{\varphi_0\varphi_6 - \varphi_3\varphi_5 - \varphi_1\varphi_7 - \varphi_2\varphi_4}{r^2}, \tag{3.40}$$

$$f^{(+)}_{126}(\varphi) = +2\frac{\varphi_0\varphi_5 - \varphi_1\varphi_4 + \varphi_7\varphi_2 + \varphi_3\varphi_6}{r^2}, \tag{3.41}$$

$$f^{(+)}_{127}(\varphi) = -2\frac{\varphi_0\varphi_4 + \varphi_6\varphi_2 + \varphi_1\varphi_5 - \varphi_3\varphi_7}{r^2}, \tag{3.42}$$

where

$$r^2 = (\varphi_0^2 + \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2 + \varphi_5^2 + \varphi_6^2 + \varphi_7^2). \tag{3.43}$$

Along the same lines we can calculate all the structure functions, we give all of them in Appendix C. What we have just calculated is commonly called the (+) torsion[32], we can find the (–) torsion by replacing the left by right multiplication in (3.35)

$$[1|\mathbb{E}_i, 1|\mathbb{E}_j]\ \varphi = 2 f^{(-)}_{ijk}(\varphi) 1|\mathbb{E}_k\ \varphi. \tag{3.44}$$

we find

$$f^{(-)}_{121}(\varphi) = f^{(-)}_{122}(\varphi) = 0, \tag{3.45}$$

$$f^{(-)}_{123}(\varphi) = -\frac{{\varphi_0}^2 - {\varphi_6}^2 - {\varphi_4}^2 + {\varphi_2}^2 + {\varphi_1}^2 - {\varphi_7}^2 - {\varphi_5}^2 + {\varphi_3}^2}{r^2},$$

and

$$f^{(-)}_{124}(\varphi) = +2\frac{\varphi_0\varphi_7 + \varphi_5\varphi_2 - \varphi_6\varphi_1 - \varphi_3\ ,\varphi_4}{r^2}, \tag{3.46}$$

$$f^{(-)}_{125}(\varphi) = -2\frac{\varphi_0\varphi_6 + \varphi_3\varphi_5 + \varphi_1\varphi_7 + \varphi_2\varphi_4}{r^2}, \tag{3.47}$$

$$f^{(-)}_{126}(\varphi) = +2\frac{\varphi_0\varphi_5 + \varphi_1\varphi_4 - \varphi_7\varphi_2 - \varphi_3\ ,\varphi_6}{r^2}, \tag{3.48}$$

$$f^{(-)}_{127}(\varphi) = -2\frac{\varphi_0\varphi_4 - \varphi_6\varphi_2 - \varphi_1\varphi_5 + \varphi_3\varphi_7}{r^2}, \tag{3.49}$$

the remaining $f^{(-)}_{ijk}(\varphi)$ are listed in appendix C.

Let's pause for a moment and note some of the evident features of these $f^{(\pm)}_{ijk}(\varphi)$,

- One notices immediately that at $\varphi^t$=(1,0,0,0,0,0,0,0) / (-1,0,0,0,0,0,0,0), the north / south pole (NP/SP), we recover the octonionic structure constants: $f^{(+)}_{ijk}(NP/SP) = -f^{(-)}_{ijk}(NP/SP) = f_{ijk}$ and any non-standard cycles vanishes e.g. $f^{(\pm)}_{567}(NP/SP) = 0$.

- Our construction started from a given multiplication table and as there are different choices [72][73], we can have different families.

- Restricting ourselves to $S^3$, we have the quaternionic structure constants i.e. for $\varphi_0^2 + \varphi_1^2 + \varphi_2^2 + \varphi_3^2 = 1$, and $\varphi_4 = \varphi_5 = \varphi_6 = \varphi_7 = 0$ we have $f_{123}(\varphi) = \epsilon_{123}$ given in eq. (2.20) and all other $f$'s vanish.

- Our matrix representation $\{\mathbb{E}_0, \mathbb{E}_i\}$ is 8 dimensional and isomorphic to $\{\delta_0, \delta_i\}$. Of course constraining ourselves to a seven sphere of radius $r$, $(\varphi^\mu\varphi_\mu = r^2)$, $\mathbb{E}_i$ or $1|\mathbb{E}_i$ $(i = 1..7)$ are completely legitimate representation of the seven imaginary octonionic units as our $\mathbb{E}_i$ or $1|\mathbb{E}_i$ $(i = 1, 2, 3)$ defined in the introduction are a representation of imaginary quaternion units.

- Over $\mathrm{S}^7$, $\partial_{\mathrm{S}^7} f^{(\pm)}_{ijk}(\varphi) \neq 0$. This is a very important characteristic of the seven sphere.

To manifest the $\varphi$ dependence, let's give some examples. To simplify the notations, we use here $(i, j, k)^{(\pm)}$ for $f^{(\pm)}_{ijk}(\varphi)$

- As we said before at $(\varphi_0 = 1, \varphi_i = 0)$, the nonvanishing cocycles are

$$(1,2,3)^{(+)} = (1,4,5)^{(+)} = (1,7,6)^{(+)} = (2,4,6)^{(+)} = (2,5,7)^{(+)} = 1$$
$$(3,4,7)^{(+)} = (3,6,5)^{(+)} = 1, \tag{3.50}$$
$$(1,2,3)^{(-)} = (1,4,5)^{(-)} = (1,7,6)^{(-)} = (2,4,6)^{(-)} = (2,5,7)^{(-)} = -1$$
$$(3,4,7)^{(-)} = (3,6,5)^{(-)} = -1, \tag{3.51}$$

and zero otherwise. Another non-trivial example is at $\left(\varphi_\mu = \frac{\mu+1}{\sqrt{204}}\right)$, we find

$$\begin{array}{lll}
(1,2,3)^{(+)} = -12/17, & (2,5,7)^{(+)} = 4/51, & (1,5,6)^{(+)} = 1/51, \\
(1,4,5)^{(+)} = -6/17, & (1,7,6)^{(+)} = 8/51, & (3,6,5)^{(+)} = 0, \\
(4,3,7)^{(+)} = -2/51, & (4,2,6)^{(+)} = 3/17, & \\
(1,2,4)^{(+)} = 4/17, & (1,5,2)^{(+)} = -8/17, & (3,5,4)^{(+)} = -44/51, \\
(5,6,7)^{(+)} = -10/17, & (1,3,4)^{(+)} = -5/17, & (4,1,6)^{(+)} = -14/17, \\
(1,5,7)^{(+)} = -40/51, & (3,5,7)^{(+)} = -2/17, & (3,1,6)^{(+)} = -14/51, \\
(2,3,5)^{(+)} = -23/51, & (1,7,4)^{(+)} = -4/17, & (4,5,6)^{(+)} = -16/51, \\
(2,6,5)^{(+)} = -38/51, & (1,6,2)^{(+)} = -8/17, & (6,3,2)^{(+)} = -22/51, \\
(1,3,5)^{(+)} = -10/51, & (2,4,3)^{(+)} = -2/17, & (4,3,6)^{(+)} = -20/51, \\
(3,7,6)^{(+)} = -13/17, & (1,2,7)^{(+)} = -1/17, & (4,2,7)^{(+)} = -16/17, \\
(2,7,3)^{(+)} = -16/17, & (2,6,7)^{(+)} = -4/51, & (7,1,3)^{(+)} = -28/51, \\
(2,4,5)^{(+)} = 2/17, & (4,7,5)^{(+)} = -7/51, & (4,6,7)^{(+)} = -10/51,
\end{array} \tag{3.52}$$

and

$$\begin{array}{lll}
(1,2,3)^{(-)} = 12/17, & (1,4,5)^{(-)} = 6/17, & (1,7,6)^{(-)} = -8/51, \\
(2,4,6)^{(-)} = 3/17, & (3,5,7)^{(-)} = 8/51, & (3,4,7)^{(-)} = -2/51, \\
(4,6,7)^{(-)} = 4/51, & (1,7,3)^{(-)} = -2/3, & (1,4,3)^{(-)} = -8/51, \\
(1,2,4)^{(-)} = -4/51, & (1,5,6)^{(-)} = -4/51, & (1,2,5)^{(-)} = -31/51, \\
(1,2,7)^{(-)} = -2/51, & (3,1,5)^{(-)} = -2/51, & (1,4,6)^{(-)} = -46/51, \\
(1,4,7)^{(-)} = -3/17, & (2,5,7)^{(-)} = -4/51, & (2,4,7)^{(-)} = -50/51, \\
(2,5,3)^{(-)} = -6/17, & (4,6,5)^{(-)} = -8/51, & (1,7,5)^{(-)} = -12/17, \\
(3,6,1)^{(-)} = -3/17, & & (3,6,2)^{(-)} = -10/17, \\
(2,4,5)^{(-)} = -2/51, & (2,4,3)^{(-)} = 0, & (3,4,5)^{(-)} = -47/51 \\
(3,4,6)^{(-)} = -6/17, & (3,6,5)^{(-)} = 0, & (2,5,6)^{(-)} = -12/17, \\
(2,3,7)^{(-)} = -3/17, & (2,6,7)^{(-)} = 0, & (3,6,7)^{(-)} = -12/17, \\
(1,2,6)^{(-)} = -6/17, & (4,5,7)^{(-)} = 0, & (6,5,7)^{(-)} = -35/51.
\end{array} \tag{3.53}$$

We have some kind of dynamical Lie algebra of seven generators with structure "constants" that change their values from one point to another. Let us emphasis the difference between considering $\mathbb{E}_i$ as an open algebra or as elements of a soft seven sphere, observe that

$$[\mathbb{E}_1, \mathbb{E}_2] \ = 2\mathbb{E}_3 - 2\left[\mathbb{E}_1, 1|\mathbb{E}_2\right], \tag{3.54}$$

but

$$\begin{aligned}[\mathbb{E}_1, \mathbb{E}_2]\,\Phi \ &= 2f^{(+)}_{123}(\varphi)\,\mathbb{E}_3\Phi \quad +2f^{(+)}_{124}(\varphi)\,\mathbb{E}_4\Phi \\ &+2f^{(+)}_{125}(\varphi)\,\mathbb{E}_5\Phi \quad +2f^{(+)}_{126}(\varphi)\,\mathbb{E}_6\Phi + 2f^{(+)}_{127}(\varphi)\,\mathbb{E}_7\Phi.\end{aligned} \tag{3.55}$$

At the NP

$$\Phi^t_{NP} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{3.56}$$

we still have

$$[\mathbb{E}_1, \mathbb{E}_2] = 2\mathbb{E}_3 - 2\left[\mathbb{E}_1, 1|\mathbb{E}_2\right] \tag{3.57}$$

whereas

$$\begin{aligned}[\mathbb{E}_1, \mathbb{E}_2]\,\Phi_{NP} \ &= \ 2f^{(+)}_{12k}(\varphi_{NP})\mathbb{E}_k\Phi_{NP} \\ &= \ 2\mathbb{E}_3\Phi_{NP}\end{aligned}$$

$$\begin{pmatrix} 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

At $\Phi_W \equiv \left(\varphi_\mu = \frac{\mu+1}{\sqrt{204}}\right)$, we still have

$$[\mathbb{E}_1, \mathbb{E}_2] = 2\mathbb{E}_3 - 2\left[\mathbb{E}_1, 1|\mathbb{E}_2\right] \tag{3.58}$$

we find that

$$\begin{aligned}[\mathbb{E}_1, \mathbb{E}_2]\,\Phi_W &= 2f^{(+)}_{12k}(\varphi_W)\mathbb{E}_k\Phi_W \\ &= \left(-\frac{24}{17}\mathbb{E}_3 + \frac{8}{17}\mathbb{E}_4 + \frac{16}{17}\mathbb{E}_5 + \frac{16}{17}\mathbb{E}_6 - \frac{2}{17}\mathbb{E}_7\right)\Phi_W\end{aligned}$$

$$\begin{pmatrix} 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{204}} \\ \frac{2}{\sqrt{204}} \\ \frac{3}{\sqrt{204}} \\ \frac{4}{\sqrt{204}} \\ \frac{5}{\sqrt{204}} \\ \frac{6}{\sqrt{204}} \\ \frac{7}{\sqrt{204}} \\ \frac{8}{\sqrt{204}} \end{pmatrix} = \begin{pmatrix} \frac{-4}{51}\sqrt{51} \\ \frac{-1}{17}\sqrt{51} \\ \frac{2}{51}\sqrt{51} \\ \frac{1}{51}\sqrt{51} \\ \frac{8}{51}\sqrt{51} \\ \frac{-7}{51}\sqrt{51} \\ \frac{2}{17}\sqrt{51} \\ \frac{-5}{51}\sqrt{51} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & \frac{24}{17} & \frac{-8}{17} & \frac{-16}{17} & \frac{-16}{17} & \frac{2}{17} \\ 0 & 0 & \frac{24}{17} & 0 & \frac{-16}{17} & \frac{8}{17} & \frac{-2}{17} & \frac{-16}{17} \\ 0 & \frac{-24}{17} & 0 & 0 & \frac{-16}{17} & \frac{2}{17} & \frac{8}{17} & \frac{16}{17} \\ \frac{-24}{17} & 0 & 0 & 0 & \frac{2}{17} & \frac{16}{17} & \frac{-16}{17} & \frac{8}{17} \\ \frac{8}{17} & \frac{16}{17} & \frac{16}{17} & \frac{-2}{17} & 0 & 0 & 0 & \frac{24}{17} \\ \frac{16}{17} & \frac{-8}{17} & \frac{-2}{17} & \frac{-16}{17} & 0 & 0 & \frac{-24}{17} & 0 \\ \frac{16}{17} & \frac{2}{17} & \frac{-8}{17} & \frac{16}{17} & 0 & \frac{24}{17} & 0 & 0 \\ \frac{-2}{17} & \frac{16}{17} & \frac{-16}{17} & \frac{-8}{17} & \frac{-24}{17} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{204}} \\ \frac{2}{\sqrt{204}} \\ \frac{3}{\sqrt{204}} \\ \frac{4}{\sqrt{204}} \\ \frac{5}{\sqrt{204}} \\ \frac{6}{\sqrt{204}} \\ \frac{7}{\sqrt{204}} \\ \frac{8}{\sqrt{204}} \end{pmatrix}.$$

We are not making a projection but a reformulation of the algebra. This fact should always be kept in mind. The same happens in a non-trivial way for the Jacobi identity, i.e.

$$\left(f_{ijm}(\varphi)f_{mkt}(\varphi) + f_{jkm}(\varphi)f_{mit}(\varphi) + f_{kim}(\varphi)f_{mjt}(\varphi)\right)\mathbb{E}_t\varphi = 0, \tag{3.59}$$

but, in general,

$$\left(f_{ijm}(\varphi)f_{mkt}(\varphi) + f_{jkm}(\varphi)f_{mit}(\varphi) + f_{kim}(\varphi)f_{mjt}(\varphi)\right) \neq 0. \tag{3.60}$$

Another important feature is

$$[\mathbb{E}_i, 1|\mathbb{E}_j] = -2f_{ijk}\left(\varphi\right)\mathbb{E}_k + [\mathbb{E}_i, \mathbb{E}_j] = -2f_{ijk}\left(\varphi\right)1|\mathbb{E}_k + [1|\mathbb{E}_i, 1|\mathbb{E}_j] \tag{3.61}$$

which is equal to zero iff $i = j$ but for the soft seven sphere, $[\mathbb{E}_i, 1|\mathbb{E}_j]\,\varphi = 0$ not only for $i = j$ but also at the NP/SP for any i,j.

Lastly over any group manifold the left torsion equals minus the right torsion, but for $S^7$ this is not in general true.

## 3.3 More General Solutions

In the previous section , we have used brute force to calculate $f^{(+)}_{ijk}(\varphi)$. There is another way, smarter and easier. We have the following situation

$$\mathbb{E}_i\mathbb{E}_j\,\varphi = \left(-\delta_{ij} + f^{(+)}_{ijk}(\varphi)\,\mathbb{E}_k\right)\varphi \tag{3.62}$$

but our $\mathbb{E}_i$ defines what is called a pure spinor as we mentioned at the end of chapter II,

$$\varphi^t\mathbb{E}_i\varphi = 0 \tag{3.63}$$

thus

$$\varphi^t\left(\mathbb{E}_i\mathbb{E}_j\right)\varphi = \varphi^t\left(-\delta_{ij}\right)\varphi, \tag{3.64}$$

using

$$\left(\mathbb{E}_k\right)^{-1} = -\mathbb{E}_k \tag{3.65}$$

we find

$$\varphi^t\left(-\mathbb{E}_k\mathbb{E}_i\mathbb{E}_j\right)\varphi = \varphi^t\left(f^{(+)}_{ijk}(\varphi)\right)\varphi \tag{3.66}$$

but

$$\varphi^t\varphi = r^2 \tag{3.67}$$

which gives us

$$f^{(+)}_{ijk}(\varphi) = \frac{\varphi^t\left(-\mathbb{E}_k\mathbb{E}_i\mathbb{E}_j\right)\varphi}{r^2}. \tag{3.68}$$

and

$$f^{(-)}_{ijk}(\varphi) = \frac{\varphi^t\left(-1|\mathbb{E}_k\;\;1|\mathbb{E}_i\;\;1|\mathbb{E}_j\right)\varphi}{r^2}. \tag{3.69}$$

There is another interesting property to note

$$\varphi^t\left[\mathbb{E}_i, 1|\mathbb{E}_j\right]\varphi = 0 \tag{3.70}$$

which may be the generalization of the standard Lie algebra relation, left and right action commute everywhere over the group manifold.

The left and right torsions that we have constructed are not the only parallelizable torsions of $\mathrm{S}^7$. Our $\mathbb{E}_i$ and $1|\mathbb{E}_i$ are given in terms of the octonionic structure constants (3.27) i.e. the torsion at NP/SP. Considering two new points, we may define new sets of $\mathbb{E}_i$ and $1|\mathbb{E}_i$. As $\mathrm{S}^7$ contains an infinity of points, practically, we have an infinity of parallelizable torsion. If our method is self contained and sufficient, we should be able to construct these infinity of pointwise structures. Indeed, $\mathbb{E}_i(\varphi)$ and $1|\mathbb{E}_i(\varphi)$ are in general

$$\begin{array}{rcrcl} \delta_i & \Longleftrightarrow & (\mathbb{E}_i(\varphi))_{\mu\nu} & = & \delta_{0\mu}\delta_{i\nu} - \delta_{0\nu}\delta_{i\mu} - f^{(+)}_{i\mu\nu}(\varphi), \\ 1|\delta_i & \Longleftrightarrow & (1|\mathbb{E}_i(\varphi))_{\mu\nu} & = & \delta_{0\mu}\delta_{i\nu} - \delta_{0\nu}\delta_{i\mu} + f^{(-)}_{i\mu\nu}(\varphi), \end{array} \tag{3.71}$$

in complete analogy with (3.23,3.24,3.25,3.26). Of course the soft Algebra idea should hold here as well as for the special $(\mathbb{E}_i, 1|\mathbb{E}_i)$ constructed in terms of the north pole torsion. Repeating the calculation in terms of $(\mathbb{E}_i(\varphi), 1|\mathbb{E}_i(\varphi))$. Let us introduce a new vector field $\lambda$,

$$\lambda^t = \begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 & \lambda_7 \end{pmatrix}. \tag{3.72}$$

We define two new generalized structure functions

$$[\mathbb{E}_i(\varphi), \mathbb{E}_j(\varphi)]\,\lambda = 2f_{ijk}^{(++)}(\varphi,\lambda)\mathbb{E}_k(\varphi)\lambda \tag{3.73}$$

$$[1|\mathbb{E}_i(\varphi), 1|\mathbb{E}_j(\varphi)]\,\lambda = 2f_{ijk}^{(--)}(\varphi,\lambda)1|\mathbb{E}_k(\varphi)\lambda \tag{3.74}$$

where $f_{ijk}^{(\pm\pm)}(\varphi,\lambda)$ have a very complicated structure,

$$f_{ijk}^{(++)}(\varphi,\lambda) = \frac{\lambda^t\left(-\mathbb{E}_k(\varphi)\,\mathbb{E}_i(\varphi)\,\mathbb{E}_j(\varphi)\right)\lambda}{r^2}, \tag{3.75}$$

$$f_{ijk}^{(--)}(\varphi,\lambda) = \frac{\lambda^t\left(-1|\mathbb{E}_k(\varphi)\quad 1|\mathbb{E}_i(\varphi)\quad 1|\mathbb{E}_j(\varphi)\right)\lambda}{r^2} \tag{3.76}$$

as examples, we list four of them in Appendix D. We will use them later when we study some applications.

## 3.4 Some Group Theory

An arbitrary octonion can be associated to $\mathbb{R}^8 = \mathbb{R}\oplus\mathbb{R}^7$ [69] where $\mathbb{R}$ denotes the subspace spanned by the identity $e_0 = 1$. Octonions with unit length define the octonionic unit sphere $S^7$. The isometries of octonions is described by $O(8)$ which may be decomposed as

$$O(8): \qquad H \oplus K \oplus E \tag{3.77}$$

where $H$ is the 14 parameters $G_2$ algebra of the automorphism group of *octonions*, K is the torsionful seven sphere $SO(7)/G_2$ and our E is the round seven sphere $SO(8)/SO(7)$. In fact the different three non-equivalent representation of O(8) - the vectorial so(8) and the two different spinorial $spin^L(8)$ and $spin^R(8)$, which are related by triality, can be realized by suitable left and right octonionic multiplication. The reduction of O(8) to O(7) induces $so(8) \longrightarrow so(7)\oplus 1$, $spin^R(8) \longrightarrow spin(7)$ and $spin^L(8) \longrightarrow spin(7)$.

We would like to show how to generate these different Lie algebras entirely from our canonical left/right octonionic structures. We start from the $8\times 8$ gamma matrices $\gamma^i_{\mu\nu}$ in seven dimensions, using $\delta_{ij}$ as our flat Euclidean metric,

$$\{\gamma^i, \gamma^j\} = 2\delta^{ij}\mathbf{1}_8 \qquad , \tag{3.78}$$

where $i, j, \ldots = 1, 2, \ldots 7$ and $\mu, \nu, \ldots = 0, 1, 2, \ldots 7$. We can use either of the following choices

$$\gamma^j_+ = i\mathbb{E}_j \quad or \quad \gamma^j_- = i1|\mathbb{E}_j \qquad , \tag{3.79}$$

of course the $i$ in the right hand sides is the imaginary complex unit. This relates our antisymmetric, Hermitian and hence purely imaginary gamma matrices to the canonical octonionic left/right structures. The antisymmetric product of two gamma matrices will be denoted by

$$\gamma^{ij} = \gamma^{[i}\gamma^{j]} \qquad , \tag{3.80}$$

and we have [1]

$$\gamma^i\gamma^j\gamma^k = \frac{1}{4!}\epsilon^{ijklmnp}\gamma^l\gamma^m\gamma^n\gamma^p \qquad . \tag{3.81}$$

The matrices $\gamma^{ij}$ span the 21 generators $J^{ij}$ of spin(7) in its eight-dimensional spinor representation. The spinorial representation of spin(7) can be enlarged to the left/right handed spinor representation of spin(8) by different ways. The easiest one is to include either of $\pm\mathbb{E}_i$ or $\pm 1|\mathbb{E}_i$ [21][70][71] defining $J^i = J^{i0}$, so(8) can be written as

$$\begin{aligned} [J^i, J^i] &= 2J^{ij} && (3.82)\\ [J^i, J^{mn}] &= 2\delta^{im}J^n - 2\delta^{in}J^m && (3.83)\\ [J^{ij}, J^{kl}] &= 2\delta^{jk}J^{il} + 2\delta^{il}J^{jk} - 2\delta^{ik}J^{jl} - 2\delta^{jl}J^{ik}. && (3.84)\end{aligned}$$

The automorphism group of octonions is $G_2 \subset SO(7) \subset SO(8)$. A suitable basis for $G_2$ is [21][69][70][71]

$$H_{ij} = f_{ijk}\left(\mathbb{E}_k - 1|\mathbb{E}_k\right) - \frac{3}{2}\left[\mathbb{E}_i, 1|\mathbb{E}_j\right] \qquad , \tag{3.85}$$

which implies the linear relations

$$f_{ijk}H_{jk} = 0 \qquad , \tag{3.86}$$

These constraints enforce $H_{ij}$ to generate the 14 dimensional vector space of $G_2$. There are different ways to represent the remaining seven generators, denoted here by K,

$$\frac{so(7)}{G_2} \;:\; K_v^{\pm i} = \pm\frac{1}{2}\left(\mathbb{E}_i - 1|\mathbb{E}_i\right) \qquad , \tag{3.87}$$

$$\frac{spin(7)}{G_2} \;:\; K_s^{\pm i} = \pm\left(\frac{1}{2}\mathbb{E}_i + 1|\mathbb{E}_i\right) \qquad , \tag{3.88}$$

$$\frac{\overline{spin(7)}}{G_2} \;:\; \overline{K}_s^{\pm i} = \mp\left(\mathbb{E}_i + \frac{1}{2}1|\mathbb{E}_i\right) \qquad , \tag{3.89}$$

[1] It is interesting to note that this equation may be used as an alternative definition for the octonionic multiplication table.

Defining the conjugate representation[2] by

$$\overline{\mathbb{E}} = -1|\mathbb{E} \qquad \text{and} \qquad \overline{1|\mathbb{E}} = -\mathbb{E} \qquad , \tag{3.90}$$

(3.87) is self-conjugate while (3.88) is octonionic-conjugate to (3.89). The vector representation so(7) generated by $H_{ij} \oplus K_v^{\pm i}$ is seven dimensional because $K_v^{\pm i} e_0 = 0$ whereas the spin(7) representation generated by $H_{ij} \oplus K_s^{\pm i}$ is eight dimensional.

To make apparent the role of the automorphism group $G_2$, the different commutators of $\mathbb{E}$ and $1|\mathbb{E}$ may be written as

$$\begin{aligned}
[\mathbb{E}_i, \mathbb{E}_j] &= \frac{1}{3}\left(4H_{ij} + 2f_{ijk}\mathbb{E}_k + 4f_{ijk}1|\mathbb{E}_k\right) \qquad , & (3.91)\\
[1|\mathbb{E}_i, 1|\mathbb{E}_j] &= \frac{1}{3}\left(4H_{ij} - 4f_{ijk}\mathbb{E}_k - 2f_{ijk}1|\mathbb{E}_k\right) \qquad , & (3.92)\\
[\mathbb{E}_i, 1|\mathbb{E}_j] &= \frac{1}{3}\left(-2H_{ij} + 2f_{ijk}\mathbb{E}_k - 2f_{ijk}1|\mathbb{E}_k\right) \qquad . & (3.93)
\end{aligned}$$

or $G_2$ given by

$$H_{ij} = \frac{1}{2}\left([\mathbb{E}_i, \mathbb{E}_j] + [1|\mathbb{E}_i, 1|\mathbb{E}_j] + [\mathbb{E}_i, 1|\mathbb{E}_j]\right) \qquad . \tag{3.94}$$

Thus as we promised, the $\mathbb{E}$ and $1|\mathbb{E}$ are the necessary and the sufficient building blocks for expressing the different Lie algebras and coset representations related to the seven sphere. Note that all the constructions given in this section start from the Clifford algebra relation (3.79), and the formulation holds equally for $\mathbb{E}\,(\varphi)$ and $1|\mathbb{E}\,(\varphi)$.

[2] n.b. this definition is not matrix conjugation.

# Chapter 4

# Soft Seven Sphere Self Duality

The word instanton has been coined for solutions of elliptic non linear field equations in Euclidean space time, with boundary conditions at infinity in such a way that stable topological properties emerge [39]. The study of Euclidean Yang-Mills fields involves many mathematical items falling under the headings : differential geometry (fiber bundles, connections), differential topology (characteristic classes, index theory), and algebraic geometry (twistors, holomorphic bundles) which makes it a rich and unique subject. We review the standard $d = 4$ dimensional instanton solution in the first section.

In higher dimensions $d > 4$, there is no unique way to define self-duality. It is a matter of prejudice. One tries to conserve as much as possible of the four dimensions duality characteristics. Particularly in 8 dimensions, there have been a lot of proposals [45],[74]—[83]. There are some relationships between these apparently distinct constructions. Each author of these proposals concentrated on a certain aspect of the 4 dimensional self-duality which they considered a good starting point for the generalization to 8 dimensions.

In the second section of this chapter, we list the main features of the GKS instanton, then in the last section we present the soft seven sphere instanton. We will show how the soft seven sphere can be used to reformulate the GKS in a way very similar to the four dimensional case.

## 4.1 The 4 Dimensional Instanton

Consider an $SU(2)$ classical gauge field over four dimensional Euclidean space $\mathbb{R}^4$. Let the gauge potential be $A_\mu = A_\mu^a t^a$, while the field strength is defined as the commutator of the covariant derivative $(D_\mu \equiv \partial_\mu + A_\mu)$,

$$F_{\mu\nu} \equiv [D_\mu, D_\nu] = F_{\mu\nu}^a t^a = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \,, \tag{4.1}$$

where the generators of the Lie algebra are $t^a = -i\frac{\sigma^a}{2}$, and $\left[t^a, t^b\right] = i\epsilon^{abc}t^c$. We want to find the $A^a_\mu$ which minimizes the Euclidean action

$$S_E = \int d^4x \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} = -\frac{1}{2}\int d^4x \, tr\left(F_{\mu\nu}F^{\mu\nu}\right). \tag{4.2}$$

We rewrite it as

$$S_E = \frac{1}{8}\int d^4x \left\{\left[F^a_{\mu\nu} \pm {}^*F^a_{\mu\nu}\right]^2 \mp 2F^a_{\mu\nu}\,{}^*F^{a\mu\nu}\right\} \tag{4.3}$$

where the dual field strength ${}^*F$ is defined by (the four dimensional Levi–Cevita tensor $\epsilon_{0123} = 1$)

$${}^*F^a_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{a\alpha\beta}. \tag{4.4}$$

Hence the lower bound for the action is

$$S_E \geq \mp\frac{1}{4}\int d^4x F^a_{\mu\nu}\,{}^*F^{a\mu\nu} \tag{4.5}$$

The equality sign holds iff

$$F^a_{\mu\nu} \pm {}^*F^a_{\mu\nu} = 0 \tag{4.6}$$

which is the self/antiself duality condition (SD/ASD). The self/antiself duality condition when combined with the Bianchi identities

$$D_\mu F_{\nu\rho} + D_\rho F_{\mu\nu} + D_\nu F_{\rho\mu} = 0 \Leftrightarrow \epsilon^{\omega\mu\nu\rho} D_\mu F_{\nu\rho} = 0, \tag{4.7}$$

yields the equations of motion

$$\epsilon^{\omega\mu\nu\rho} D_\mu F_{\nu\rho} = D_\mu\,{}^*F^{\mu\omega} = \pm D_\mu F^{\mu\omega} = 0. \tag{4.8}$$

Since the self/antiself duality equation is only first order in derivatives, it is much easier to solve than the second order field equations.

In order for $S_E$ to remain finite, we require

$$F_{\mu\nu} \overset{|x|\to\infty}{\longrightarrow} 0 \tag{4.9}$$

which is automatically valid if

$$A_\mu \overset{|x|\to\infty}{\longrightarrow} g^{-1}(x)\,\partial_\mu g(x) \tag{4.10}$$

where $g(x) \in SU(2)$ is a gauge transformation. Thus we have the following situation: In general $F_{\mu\nu} \neq 0$ inside a volume $\mathbb{R}^4$, but vanishes at the infinite boundary $\partial E^4 = S^3_\infty$, a three dimensional sphere, where the gauge potential $A_\mu$ approaches a pure gauge. At the boundary $x \in S^3_\infty$, our gauge transformation $g(x) \in SU(2)$, are mappings

$$g : S^3_\infty \longrightarrow SU(2) \simeq S^3. \tag{4.11}$$

These mappings are classified according to the homotopy classes determined by the topological winding number $w$

$$\pi_3(SU(2)) \simeq \pi_3\left(S^3\right) \simeq \{w\} = \mathbb{Z}. \tag{4.12}$$

The topological number $w$ is related to the Chern number

$$Ch_2 = \frac{1}{8\pi^2}\int d^4x\, \epsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\alpha\beta}. \tag{4.13}$$

Defining the topological charge

$$N = \int d^4x\, n(x) \qquad\qquad n(x) = \frac{e^2}{32\pi^2} F^a_{\mu\nu}\, {}^*F^{a\mu\nu} \tag{4.14}$$

this is then an integer, since it is the topological number corresponding to the mapping of the three-dimensional sphere into the gauge group $SU(2)$. One can then get the following expression for the euclidean action if (4.6) is satisfied

$$S_E = \mp\frac{8\pi^2}{e^2}N. \tag{4.15}$$

The simplest solution to the ASD condition ($F_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$), corresponding to a value of the topological charge $N = 1$, [39][40] is given by:

$$A^a_\mu = \frac{x^2}{x^2+\lambda^2}\left(g^{-1}(x)\,\partial_\mu g(x)\right)^a = -\frac{2\left(\mathbb{E}^a\right)_{\mu\nu} x^\nu}{x^2+\lambda^2} \tag{4.16}$$

where

$$g(x) = \frac{x_0\mathbf{1}_4 + ix.\sigma}{\sqrt{x^2}} \tag{4.17}$$

and $\mathbb{E}^a$ are the canonical left quaternionic structures given in eq.(2.25). Leading to

$$F^a_{\mu\nu} = \frac{4\lambda^2\left(\mathbb{E}^a\right)_{\mu\nu}}{\left(x^2+\lambda^2\right)^2}. \tag{4.18}$$

Such a solution has a natural quaternionic formulation [84][85]. Consider

$$g(x) = \frac{x^\mu e_\mu}{|x|} \tag{4.19}$$

where $e_\mu$ are our four quaternionic units, introducing the self dual $SO(4)$ basis

$$\vartheta_{\mu\nu} = \frac{1}{2}\left(\bar{e}_\mu e_\nu - \bar{e}_\nu e_\mu\right) \tag{4.20}$$

such that $(\bar{e}_0 = e_0, \bar{e}_i = -e_i)$, we find

$$\vartheta_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\vartheta^{\alpha\beta} \tag{4.21}$$

and

$$\left[\vartheta_{\mu\nu}, \vartheta_{\alpha\beta}\right] = 2\left(\delta_{\alpha\mu}\vartheta_{\beta\nu} - \delta_{\alpha\nu}\vartheta_{\beta\mu} - \delta_{\beta\mu}\vartheta_{\alpha\nu} + \delta_{\beta\nu}\vartheta_{\alpha\mu}\right). \tag{4.22}$$

For $A_\mu$ given by

$$A_\mu = \frac{x^2}{x^2+\lambda^2}\left(g^{-1}(x)\,\partial_\mu g(x)\right) = -\frac{2\vartheta_{\mu\nu}x^\nu}{x^2+\lambda^2} \tag{4.23}$$

we have

$$F_{\mu\nu} = \frac{4\lambda^2\vartheta_{\mu\nu}}{\left(x^2+\lambda^2\right)^2} \Longrightarrow F_{\mu\nu} = -\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}. \tag{4.24}$$

The self-dual solution $F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$ can be found simply by replacing $e_\mu \to 1|e_\mu$ leading to

$$g(x) = \frac{x^\mu 1|e_\mu}{|x|}. \tag{4.25}$$

The corresponding $SO(4)$ antiself–dual basis are

$$\bar{\vartheta}_{\mu\nu} = \frac{1}{2}\left(1|\bar{e}_\mu 1|e_\nu - 1|\bar{e}_\nu 1|e_\mu\right) \Longrightarrow \bar{\vartheta}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}\bar{\vartheta}^{\alpha\beta}. \tag{4.26}$$

Then with the choice

$$A_\mu = \frac{x^2}{x^2+\lambda^2}\left(g^{-1}(x)\,\partial_\mu g(x)\right) = -\frac{2\bar{\vartheta}_{\mu\nu}x^\nu}{x^2+\lambda^2}, \tag{4.27}$$

we have

$$F_{\mu\nu} = \frac{4\lambda^2\bar{\vartheta}_{\mu\nu}}{\left(x^2+\lambda^2\right)^2} \Longrightarrow F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} \tag{4.28}$$

The higher $n > 1$ instantons, Atiyah, Drinfield, Hitchin and Manin (ADHM) solutions, are given naturally in terms of quaternions [85].

Lastly, we would like to show how to get static monopole solutions by field redefinition of the instanton problem. We consider only static monopoles, purely magnetic and which are solutions of an equation called the Bogomolny condition[84]. The model is defined over $\mathbb{R}^3$. Such monopoles are called BPS states, they correspond to an $su\,(2)$ valued pair of a gauge field $A_i$ $(i = 1, 2, 3)$ and a scalar field $\phi$. The action of the monopole system is

$$\mathcal{L} = \frac{1}{2}\int d^3x \left[B^i B_i + (D_i\phi)\left(D^i\phi\right) - \lambda\left(\phi^2 - a^2\right)^2\right], \tag{4.29}$$

$B_i$ and $D_i\phi$ are defined by

$$B_i = \epsilon_{ijk}\left(\partial^j A^k + A^j A^k\right), \qquad D_i\phi = \partial_i\phi + [A_i, \phi]. \tag{4.30}$$

The field equations for this static system are

$$\epsilon_{ijk} D^j B^k = [\phi, D_i\phi], \tag{4.31}$$
$$D^i D_i\phi = 2\lambda\phi^3 - 2\lambda a^2\phi \tag{4.32}$$

whereas the Bianchi identiy is

$$D_i B_i = 0, \tag{4.33}$$

Being a second order system of partial differential equations (4.31–4.33), they are hard to solve. However, if we take the limit $\phi^2 = a^2$, we impose a first order equation

$$B_i = D_i\phi \tag{4.34}$$

which is the Bogomolny equation. We can relate the self duality condition (4.6) to the Bogomolny condition if we redefine $\phi$ as a fourth component $A_0$ of the gauge field $A_i$ i.e. $A_\mu \equiv (A_0 = \phi, A_i)$. We then have

$$D_i\phi = F_{i0}, \qquad B_i = \frac{1}{2}\epsilon_{ijk}F^{jk} \tag{4.35}$$

by substitution in (4.34), we recover the self duality condition.

## 4.2 The Grossman–Kephart–Stasheff Instanton

The four dimensional self–duality notion proved to be a very powerful tool both of physics and mathematics, so it is natural to investigate the occurrence of a similar condition in higher dimensions. As we have already said, there is no standard way to express the self duality equation in $d > 4$ dimensions. In a generic $d$ dimensions, a $p$ form $v$ is

$$v = \frac{1}{p!} v_{\alpha_1\cdots\alpha_p} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p}, \qquad \alpha_1\ ,\ldots,\alpha_p = 1\ldots d \tag{4.36}$$

and the dual form is

$$ {}^*v = \frac{1}{d!}\epsilon^{\alpha_1\cdots\alpha_d} v_{\alpha_1\cdots\alpha_p} dx^{\alpha_{p+1}} \wedge \cdots \wedge dx^{\alpha_d}. \tag{4.37} $$

which means, in $d$ dimensions the dual of a $p$ form is a $d-p$ form. Knowing that the Yang-Mills field strength can be written as a two form

$$ F = \frac{1}{2} F_{\alpha_1\alpha_2} dx^{\alpha_1} \wedge dx^{\alpha_2}, \tag{4.38} $$

then the dual of a 2 form is another 2 form iff $d = 4$. Constraining ourselves to Yang-Mills models, we have at disposal just the one form gauge field $A = A_i dx^i$ or the 2 form field strength $F$.

In eight dimensions, it is not obvious how we should proceed. There have been different suggestions. For example:

- The Fubini–Nicolai [75] or the Corrigan-Devchand-Fairlie-Nuyts [76] instanton: the authors insist on the existence of "squaring" i.e. the action can be written as the square of self-dual fields.

- There exist also some promising generic higher dimensional self-duality conditions that are not just restricted to 8 dimensions but also go beyond that limit. For example, the Ivanon-Popov [81] proposal where a Clifford-algebraic structure is used. Another example is the geometric Bais-Batenburg [80] self-duality which is based upon hypercomplex structures over appropriate manifolds.

Here, we would like to concentrate on another proposal. Grossman–Kephart-Stasheff [45] suggested a condition, for eight dimensions, that has deep topological roots: The last Hopf map $S^{15} \xrightarrow{S^7} S^8$, conformal invariance and spin structure over $S^8$[83].

Working over the 8 dimensional euclidean space $\mathbb{R}^8$. The (GKS) self duality condition is (the eight dimensional Levi–Cevita tensor $\epsilon_{01234567} = 1$)

$$ F^a_{\alpha_1\alpha_2} F^a_{\alpha_3\alpha_4} = \frac{1}{4!}\epsilon_{\alpha_1\ldots\alpha_8} F^{a\alpha_5\alpha_6} F^{a\alpha_7\alpha_8}. \tag{4.39} $$

where there is summation over the Lie algebra indices $a$. Grossman, Kephart and Stasheff insisted upon the conformal invariance of the Yang-Mills action in 4 dimensions. In eight dimensions the Yang-Mills action is not conformally invariant hence they considered the functional

$$ \mathcal{A} = \int d^8x \ \left( F^a_{\mu_1\mu_2} F^a_{\mu_3\mu_4} F^{b\mu_1\mu_2} F^{b\mu_3\mu_4} \right) \tag{4.40} $$

which upon the use of the self duality condition (4.39), takes a form similar to the fourth Chern class

$$\int_{S^8} \epsilon_{\alpha_1 \dots \alpha_8} F^{a\alpha_1\alpha_2} F^{a\alpha_3\alpha_4} F^{b\alpha_5\alpha_6} F^{b\alpha_7\alpha_8}. \tag{4.41}$$

In the search for solutions of the GKS duality condition and requiring that $F_{\mu_1\mu_2} \xrightarrow{S^7_\infty \sim \{|x| \to \infty\}} 0$, so that $A_{\mu_1}$ must be a pure gauge at infinity

$$A_\mu \xrightarrow{S^7_\infty} g^{-1}(x)\, \partial_\mu g(x). \tag{4.42}$$

where $g(x)$ is a gauge transformation. GKS assumed the following form for $g(x)$

$$g(x) = \frac{x^\mu \mathbb{E}_\mu}{\sqrt{x^2}} \tag{4.43}$$

where $\mathbb{E}_\mu$ are given by (3.27), whence,

$$g^\dagger(x)\, g(x) = 1 \qquad and \qquad Det\,(g(x)) = 1. \tag{4.44}$$

Note that, for $\hat{x}^\mu = \frac{x^\mu}{\sqrt{x^2}}$, we have $\hat{x}^\mu \hat{x}_\mu = 1$ i.e. $g(x)$ parameterize a unit $S^7$. For the boundary condition given above, we have the following situation:

$$g(x) : S^7_\infty \longrightarrow S^7 \tag{4.45}$$

Such a class of maps are classified according to the seventh homotopy group of the seven sphere

$$\pi_7\left(S^7\right) = n \in \mathbb{Z} \tag{4.46}$$

i.e. there is no map between solutions of different n. They lie in different classes. In particular, there can be no map between the trivial $n = 0$ $(A_\mu = 0)$ field configuration and $n > 0$ $(A_\mu \neq 0)$. Any $n \neq 0$ is stable and will never decay. GKS proposed the following ansatz for $A_\mu$ $(n = 1)$ solution

$$A_\mu = \frac{x^2}{x^2 + \lambda^2}\left(g^{-1}(x)\, \partial_\mu g(x)\right). \tag{4.47}$$

which solves (4.39). Now, let's mention the difference between this GKS duality and the standard four dimensional duality considered in the previous section.

- It is not derived from an action. $\mathcal{A}$ has no quadratic term in the derivatives i.e. there is no kinetic term.
- The GKS duality is valid only for a specific representation 8 of a specific group $SO(8)$ in contrast to the self–duality condition which is well defined for any representation of any simple Lie group.
- The GKS solution is not a solution of the $\mathbb{R}^8$ Yang–Mills field equations.

## 4.3 Eight Dimensional Soft Self Duality

Now we would like to reformulate in a quadratic form the GKS self duality condition. We work over $\mathbb{R}^8$. Contrary to the standard Yang-Mills gauge field, the soft gauge field strength carries a dependence upon the internal manifold. It is important to know where we stand over the $S^7_{gauge}$, as the structure functions vary from one point to another. For a soft gauge field $A_\mu(x) \equiv A^i_\mu \mathbb{E}_i$, the insertion of the $\varphi$ is essential for the closure of the commutator

$$\begin{aligned} [A_\mu(x), A_\nu(x)]\,\varphi &\equiv A^i_\mu(x) A^j_\nu(x)\,[\mathbb{E}_i, \mathbb{E}_j]\,\varphi && (4.48)\\ &= 2 f^{(+)}_{ijk}\,(\varphi)\, A^i_\mu(x) A^j_\nu(x) \mathbb{E}_k \varphi. && (4.49)\end{aligned}$$

thus the field strength is given by

$$\begin{aligned} F_{\mu\nu}(x,\varphi) &= F_{\mu\nu}(x)\ \varphi \\ &= F^i_{\mu\nu}\,(x)\,\mathbb{E}_i \varphi \\ &= \left(\frac{\partial A_\nu(x)}{\partial x^\mu} - \frac{\partial A_\mu(x)}{\partial x^\nu} + [A_\mu(x), A_\nu(x)]\right)\varphi. \qquad (4.50)\end{aligned}$$

The critical point for the self-duality condition is the existence of a fourth rank tensor. Adding a zero index to extend $f^{(\pm)}_{ijk}(\varphi)$ from $\mathbb{R}^7$ to $\mathbb{R}^8$, we define a fourth rank tensor $\eta_{\alpha\beta\mu\nu}(\varphi)$ which is equal to

$$\eta^{(\pm)}_{0ijk}(\varphi) = f^{(\pm)}_{ijk}(\varphi) \qquad , \qquad (4.51)$$

and zero elsewhere. The proposed generalization of the four dimensional self duality is the following *soft self duality condition*

$$F\ (x,\varphi) = \star F\ (x,\varphi), \qquad (4.52)$$

or in terms of components

$$F_{0i}\ (x,\varphi) = \frac{1}{2}\eta^{(\pm)}_{0ijk}(\varphi) F^{jk}(x,\varphi) \qquad , \qquad (4.53)$$

note that $\eta_{\alpha\beta\mu\nu}(\varphi)$ varies over the seven sphere. To proceed, we require the vanishing of $F_{\mu\nu}$ at the infinite $S^7_\infty$, thus $A_\mu$ (at $S^7_\infty$) must be a pure gauge $A_\mu = g^{-1}(x)\,\partial_\mu g(x)$, where our gauge transformation $g(x)$ is a map from the spatial $S^7_\infty$ to the gauge space $S^7_{gauge}$ [1]. Consider an $S^7$ element

$$g(x) = \frac{\mathbb{E}_\mu x^\mu}{\sqrt{x^2}}, \qquad (4.54)$$

[1]Soft seven sphere gauge transformations reduce to the standard Yang-Mills theory at any single point over the seven sphere. The soft seven sphere gauge field $A_\mu$ transforms as $A\mu\,(x)\,\varphi \to g^{-1}\,(A_\mu\,(x) + \partial_\mu)\,g\varphi$ and the Field strength $F_{\mu\nu}\,(x)$ transforms as $F_{\mu\nu}\,(x,\varphi) \to g^{-1}\,(F_{\mu\nu}\,(x))\,g\varphi$. The presence of the $\varphi$ is essential.

the self-dual gauge solution of the self dual condition is exactly the GKS ansatz

$$A_{\mu}^{(+)}(x)=\frac{x^{2}}{x^{2}+\lambda^{2}}g^{-1}(x)\partial_{\mu}g(x)=-\frac{\Xi_{\mu\nu}^{(+)}x^{\nu}}{x^{2}+\lambda^{2}} \tag{4.55}$$

where the $\Xi_{\mu\nu}^{(+)}$ is given by

$$\Xi_{\mu\nu}^{(+)}=\frac{1}{2}\left(\mathbb{E}_{\mu}^{t}\mathbb{E}_{\nu}-\mathbb{E}_{\nu}^{t}\mathbb{E}_{\mu}\right). \tag{4.56}$$

We call $\Xi_{\mu\nu}^{(+)}$ the self dual tensor, because

$$\Xi_{oi}^{(+)}\varphi=\frac{1}{2}\eta_{0ijk}^{(+)}(\varphi)\Xi^{(+)jk}\varphi. \tag{4.57}$$

After substituting $A_{\mu}^{(+)}(x)$ into (4.50), we find

$$F_{\mu\nu}^{(+)}(x,\varphi)=2\frac{(\Xi_{\mu\nu}^{(+)})\lambda^{2}}{(\lambda^{2}+x^{2})^{2}}\varphi \tag{4.58}$$

which is obviously soft self dual (4.53).

Another problem of the GKS instanton that can be overcame in the soft seven sphere framework is the compatibility of the equation of motion and the self duality condition. We find by explicit calculation that our solution (4.55) satisfies the Yang-Mills equation of motion for a soft gauge field

$$D_{\mu}F_{\mu\nu}\left(x\right)\varphi=\partial_{\mu}F_{\mu\nu}\left(x\right)\varphi+[A_{\mu}\left(x\right),F_{\mu\nu}\left(x\right)]\varphi=0. \tag{4.59}$$

Of course the four dimensional case is more powerful because the self duality is related directly to the Bianchi identity which does not hold in higher dimensions. However, in our case the soft self duality is compatible with the equation of motion whereas for Grossman-Kephart-Stasheff instanton, one must work over curved space–time with certain condition for the metric in order to satisfy both the self duality and the equation of motion. To construct a static seven dimensional monopole, we proceed by static dimensional reduction from $\mathbb{R}^{8}$ to $\mathbb{R}^{7}$. Identifying

$$\begin{aligned} A_{0} &= \phi, && \qquad (4.60)\\ F_{ij} &= f_{ijk}\left(\varphi\right)B^{k} \qquad i,j,k=1..7 && \qquad (4.61)\end{aligned}$$

then the self duality will be reduced to

$$\left(B_{i}\right)\varphi=\left(D_{i}\phi\right)\varphi. \tag{4.62}$$

Using the soft seven sphere, we can easily generate new solutions of GKS dualities. Working with $\mathbb{E}(\varphi)$, replacing and $g(x)$ by

$$g(x,\varphi) = \frac{\mathbb{E}_\mu(\varphi)\, x^\mu}{\sqrt{x^2}} \tag{4.63}$$

the resulting gauge field given in terms

$$\Xi^{(++)}_{\mu\nu}(\varphi) = \frac{1}{2}\left(\mathbb{E}^t_\mu(\varphi)\,\mathbb{E}_\nu(\varphi) - \mathbb{E}^t_\nu(\varphi)\,\mathbb{E}_\mu(\varphi)\right) \tag{4.64}$$

leads to

$$A^{(++)}_\mu(x) = \frac{x^2}{x^2+\lambda^2} g^{-1}(x,\varphi)\partial_\mu g(x,\varphi) = -\frac{\Xi^{(++)}_{\mu\nu}(\varphi)\, x^\nu}{x^2+\lambda^2} \tag{4.65}$$

$$F^{(++)}_{\mu\nu}(x,\varphi) = 2\frac{(\Xi^{(++)}_{\mu\nu}(\varphi))_{\mu\nu}\,\lambda^2}{(\lambda^2+x^2)^2} \tag{4.66}$$

which also satisfy (4.39) and (4.53).

# Chapter 5

# Hypercomplex SSYM Models

Day after day, supersymmetry consolidates its position in theoretical physics. Even if it was introduced more than 25 years ago, there are still problems with the geometric basis of extended ($N > 1$) supersymmetry. The situation of the extended superspace is far less satisfactory than the original N=1 superspace. At the level of the algebra the on-shell formalism closes up to modulo of the classical equations of motion. This fact seems odd at the quantum level since the equations of motion receive loop corrections[1]. The superspace introduces an elegant supermanifold with different enlarged superconnections, where some are truly integrable in the sense of having zero supercurvature. In principle, the extended superspace should be a very powerful tool for quantum calculations.

Before starting, we feel obliged to mention something about the history of the following conjecture: Ring Division Algebras $\mathbb{K} \equiv \{$ real $\mathbb{R}$, complex $\mathbb{C}$, quaternions $\mathbb{H}$, octonions $\mathbb{O}$ $\}$ are relevant to simple supersymmetric Yang-Mills. The first hint, as mentioned by Schwarz [86] comes from the number of propagating Bose and Fermi degrees of freedom which is one for $d = 3$, two for $d = 4$, four for $d = 6$ and eight for $d = 10$ suggesting a correspondence with real $\mathbb{R}$, complex $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$. Kugo and Townsend [87] investigated in detail the relationship between $\mathbb{K}$ and the irreducible spinorial representation of the Lorentz group in $d = 3, 4, 6, 10$, building upon the following chain of isomorphisms

$$\begin{aligned} so\,(2,1) &\iff sl(2,\mathbb{R}) \\ so(3,1) &\iff sl(2,\mathbb{C}) \\ so(5,1) &\iff sl(2,\mathbb{H}). \end{aligned}$$

[1] Also, the supersymmetry transformations receive corrections and one should test the closure of the algebra order by order in perturbation theory.

They conjectured that $so(9,1) \Longleftrightarrow sl(2,\mathbb{O})$, the correct relation turned out to be

$$so(9,1) \Longleftrightarrow sl(2,\mathbb{O}) \oplus G_2$$

as has been shown by Chung and Sudbery [88], i.e. the dimension of $Sl\,(2,\mathbb{O})$ is 31. Also in [87], a quaternionic treatment of the $d=6$ case is presented. Later, Evans made a systematic investigation of the relationship between SSYM and ring division algebra in a couple of papers. In the first [89], he simplified the construction of SSYM by proving a very important identity between gamma matrices by using the intrinsic triality of ring division algebra instead of the "tour de force" used originally by Brink, Scherck and Schwarz [41] via Fierz identities generalized to $d>4$ dimensions. Then, in the second paper [90], Evans made the connection even clearer by showing how the auxiliary fields are really related to ring division algebras. For $d=3,4,6,10$ we need $k=0,1,3,7$ auxiliary fields respectively. An alternative approach for the octonionic case was introduced by Berkovits [91] who invented a larger supersymmetric transformation called generalized supersymmetry in [92]. There has also been a twistor attempt by Bengtsson and Cederwall [93]. For more references about the octonionic case and ten dimensional physics one may consult references in [94] and its extension to p-branes by Belecowe and Duff [95]. The early work of Nilsson may be relevant [96][97] too.

As a first step towards an extended superspace, we address the point of the algebraic auxiliary fields for simple N=1 supersymmetric Yang-Mills (SSYM) definable only in $d=3,4,6$ and 10 dimensions [41]. The important point is: *While the physical fields couple to ring division left action the auxiliary ones couple to right action (or vice versa).* To admit a closed off-shell supersymmetric algebra, left and right action must commute i.e. we should have a parallelizable associative algebra. For d = 6, quaternions work fine but for d = 10, the only associative seven dimensional algebra that is known is the soft seven sphere. We shall show below how this works. In this chapter, we use the same symbols (left action $\equiv \mathbb{E}_j$, right action $\equiv 1|\mathbb{E}_j$) for either complex, quaternionic or octonionic numbers and each case should be distinguished by the range of the indices $j$ which run from 1 to $(1,3,7)$ for complex, quaternions and octonions respectively.

## 5.1 On–Shell SSYM in $d=3,4,6,10$

In this section we follow a notation which is a mixture between that of Evans [89][90] and Supersolutions [98]. The Minkowskian metric has signature $\eta_{MN} \equiv (-,+,\ldots,+)$ and spinorial indices have range $n$. Our generalized gamma matrices $\Gamma$ and $\tilde{\Gamma}$ are *real symmetric* and we will never raise or lower the spinors indices. We define $(\Gamma_M)_{ab}$ of lower spinorial indices, whereas the upper ones are defined by $\left(\tilde{\Gamma}_0\right)^{ab} = -(\Gamma_0)_{ab}$ and $\left(\tilde{\Gamma}\right)^{ab} = (\Gamma)_{ab}$, whence

$$\Gamma^M \tilde{\Gamma}^N + \Gamma^N \tilde{\Gamma}^M = 2\eta^{MN}$$

or in terms of components

$$\left(\Gamma^M\right)_{ab} \left(\tilde{\Gamma}^N\right)^{bc} + \left(\Gamma^N\right)_{ab} \left(\tilde{\Gamma}^M\right)^{bc} = 2\eta^{MN}\delta_a^c. \tag{5.1}$$

with $a, b$ of range $n$.

Simple supersymmetric Yang-Mills models are composed of gauge fields $A_M$, spinor fields $\Psi^a$ and the Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F_{MN}F^{MN} + \frac{1}{2}\Psi^t\Gamma^M\nabla_M\Psi, \tag{5.2}$$

where $\nabla_M \equiv \partial_M + A_M$; $F_{MN} \equiv [\nabla_M, \nabla_N]$ and we have suppressed here the Lie algebra indices. We first ask in which dimensions $d$ and for what type of spinorial field $\Psi$ the action (5.2) is supersymmetric? Assume the spinorial field has $n$ components. Upon quantization, the gauge field has $d-2$ *physical degrees of freedom* while the spinor field has $n/2$ physical degrees of freedom. A supersymmetric model must have equal number of bosonic and fermionic degrees of freedom, hence $d = 2 + \frac{n}{2}$ and since for spinors $n = 2, 4, 8, 16$ this lead to $d = 3, 4, 6, 10$. We work with *real bases for spinors and vectors* [90]. Introducing an n components Grassmann variable $\xi$, we postulate the supersymmetry transformation $\delta_\xi$ to be

$$\begin{aligned} \delta_\xi A_M &= \xi^a (\Gamma_M)_{ab} \Psi^b, \\ \delta_\xi \Psi^a &= \frac{1}{2}\xi^b \left(\tilde{\Gamma}^M\right)^{ac} \left(\Gamma^N\right)_{cb} F_{MN}. \end{aligned} \tag{5.3}$$

We have to check the invariance of the Lagrangian. The variation of $F_{MN}$ is

$$\delta_\xi F_{MN} = \xi^a (\Gamma_N)_{ab} \nabla_M \Psi^b - \xi^a (\Gamma_M)_{ab} \nabla_N \Psi^b, \tag{5.4}$$

the variation of the first term of our $\mathcal{L}$ gives

$$\delta_\xi \left(-\frac{1}{4}F_{MN}F^{MN}\right) = -\xi^a (\Gamma_M)_{ab} F^{MN} \nabla_N \Psi^b. \tag{5.5}$$

Now, taking into account $\nabla_M = \partial_M + A_M$, making the variation of the second term of the Lagrangian (5.2) and using as a basis of our Lie algebra $\Psi^z t^z$ with

$[t^x, t^y] = c^{xyz} t^z$,

$$\begin{aligned} \delta_\xi \left( \frac{1}{2} \left( \Gamma^M \right)_{ab} \Psi^a \nabla_M \Psi^b \right) &= \xi^a \left( \Gamma^N \right)_{ab} F_{MN} \nabla^M \Psi^b \\ & -\frac{1}{2} \xi^c \left( \Gamma_M \right)_{ab} \left( \Gamma^M \right)_{cd} \left( \Psi^{az} c_{xyz} \Psi^{dx} \Psi^{by} \right) \\ & + \textit{total derivative.} \end{aligned} \tag{5.6}$$

We have used the Bianchi identity in the above derivation. The SSYM action (5.2) is invariant under (5.3) if

$$\xi^c \left( \Gamma_M \right)_{ab} \left( \Gamma^M \right)_{cd} \left( \Psi^{az} c_{xyz} \Psi^{dx} \Psi^{by} \right) = 0 \tag{5.7}$$

$$\Rightarrow Q_{abcd} = \left( \Gamma_M \right)_{ab} \left( \Gamma^M \right)_{cd} + \left( \Gamma_M \right)_{bd} \left( \Gamma^M \right)_{ca} + \left( \Gamma_M \right)_{da} \left( \Gamma^M \right)_{cb} = 0. \tag{5.8}$$

We conclude from this calculation that $\mathcal{L}$ is invariant under (5.3) in any dimension for abelian algebras with any spin representation because (5.7) is trivially zero for $c_{xyz} = 0$. Now, we want to find the solution of (5.8). As our complex numbers, quaternions, octonions form a Clifford algebra of signature $Cliff(0,1), Cliff\,(0,3)\,, Cliff\,(0,7)$ respectively, we have

$$\begin{aligned} (\mathbb{E}_k)_{\mu\nu} (\mathbb{E}_j)_{\lambda\nu} + (\mathbb{E}_j)_{\mu\nu} (\mathbb{E}_k)_{\lambda\nu} &= 2\delta_{kj} \delta_{\mu\lambda}, \\ (\mathbb{E}_k)_{\mu\nu} (\mathbb{E}_j)_{\mu\lambda} + (\mathbb{E}_j)_{\mu\nu} (\mathbb{E}_k)_{\mu\lambda} &= 2\delta_{kj} \delta_{\nu\lambda}, \\ (\mathbb{E}_k)_{\mu\nu} (\mathbb{E}_k)_{\lambda\zeta} + (\mathbb{E}_k)_{\lambda\nu} (\mathbb{E}_k)_{\mu\zeta} &= 2\delta_{\mu\lambda} \delta_{\nu\zeta}, \end{aligned} \tag{5.9}$$

and the same holds equally well for $(1|\mathbb{E}_j)$. As had been noticed by Evans, these are direct consequences of the ring division triality[89]. We can construct immediately two sets of gamma matrices as follows

$$\begin{aligned} (\Gamma_0) &= \begin{pmatrix} -\mathbf{1}_{\frac{n}{2}} & 0 \\ 0 & -\mathbf{1}_{\frac{n}{2}} \end{pmatrix}; \\ (\Gamma_j) = \begin{pmatrix} 0 & \mathbb{E}_j \\ -\mathbb{E}_j & 0 \end{pmatrix}&; \quad (1|\Gamma_j) = \begin{pmatrix} 0 & 1|\mathbb{E}_j \\ -1|\mathbb{E}_j & 0 \end{pmatrix}, \\ (\Gamma_{d-2}) = \begin{pmatrix} 0 & \mathbf{1}_{\frac{n}{2}} \\ \mathbf{1}_{\frac{n}{2}} & 0 \end{pmatrix}&; \quad (\Gamma_{d-1}) = \begin{pmatrix} \mathbf{1}_{\frac{n}{2}} & 0 \\ 0 & -\mathbf{1}_{\frac{n}{2}} \end{pmatrix}, \end{aligned} \tag{5.10}$$

where $j = 1..d-3$, thus

$$\Gamma^M \tilde{\Gamma}^N + \Gamma^N \tilde{\Gamma}^M = 1|\Gamma^M 1|\tilde{\Gamma}^N + 1|\Gamma^N 1|\tilde{\Gamma}^M = 2\eta^{MN}$$

or in terms of components

$$\begin{aligned}\left(\Gamma^M\right)_{ab}\left(\tilde{\Gamma}^N\right)^{bc}+\left(\Gamma^N\right)_{ab}\left(\tilde{\Gamma}^M\right)^{bc} &= \left(1|\Gamma^M\right)_{ab}\left(1|\tilde{\Gamma}^N\right)^{bc}+\left(1|\Gamma^N\right)_{ab}\left(1|\tilde{\Gamma}^M\right)^{bc} \\ &= 2\eta^{MN}\delta^c_a. \end{aligned} \tag{5.11}$$

automatically, our Γ's satisfy the identity (5.8) for both left and right actions. Indeed by direct calculation using (5.9), we see that

$$\Gamma_{Ma(b}\Gamma^M_{cd)} = 1|\Gamma_{Ma(b}1|\Gamma^M_{cd)} = 0. \tag{5.12}$$

Consequently, the spin representation decomposes into

$$SPIN\,(1,2) \iff SL\,(2,R) \tag{5.13}$$
$$SPIN\,(1,3) \iff SL\,(2,C) \tag{5.14}$$
$$SPIN\,(1,5) \iff SL\,(2,H) \tag{5.15}$$

and using the soft sphere, it seems that $soft\ SPIN\,(1,9) \iff SL\,(2, soft\ S^7)$.

We still have to show that the commutators of (5.3) close to a supersymmetric algebra. For any arbitrary field $V$ in our Lagrangian, we need to check that

$$\boxed{[\delta_\xi,\delta_\chi]\,V = 2\xi^a\chi^b\left(\Gamma^M\right)_{ab}\partial_M V\ .} \tag{5.16}$$

As we are working on–shell, the algebra should closes modulo the fermionic equation of motion

$$\left(\Gamma^M\right)_{ab}\nabla_M\Psi^a = 0, \qquad \forall b \tag{5.17}$$

and gauge transformation. Using (5.3), we can easily check the closure for the gauge field $A_M$

$$\begin{aligned}[\delta_\xi,\delta_\chi]\,A_M &= -\frac{1}{2}\xi^a\chi^b\left(\left(\tilde{\Gamma}^P\right)^{cd}\left(\Gamma_M\right)_{bc}\left(\Gamma^N\right)_{ad}+\left(\tilde{\Gamma}^P\right)^{cd}\left(\Gamma_M\right)_{ac}\left(\Gamma^N\right)_{bd}\right)F_{PN} \\ &= 2\xi^a\chi^b\left(\Gamma^N\right)_{ab}F_{NM}\end{aligned} \tag{5.18}$$

where we have used $F_{PN} = -F_{NP}$. To check the close for the fermionic field is a little bit lengthy, but straightforward

$$\begin{aligned}[\delta_\xi,\delta_\chi]\,\Psi^c &= \underline{\xi^a\chi^b\left(\tilde{\Gamma}^M\right)^{cd}\left(\Gamma^N\right)_{de}\left(\Gamma_N\right)_{ab}\nabla_M\Psi^e} - \xi^a\chi^b\left(\tilde{\Gamma}^M\right)^{cd}Q_{abde}\nabla_M\Psi^e \\ &= 2\xi^a\chi^b\left(\Gamma^M\right)_{ab}\nabla_M\Psi^c - \xi^a\chi^b\left(\tilde{\Gamma}^M\right)^{cd}Q_{abde}\nabla_M\Psi^e,\end{aligned} \tag{5.19}$$

where we have used the fermionic equation of motion to simplify the underlined term. Thus the supersymmetry closes iff $Q_{abde} = 0$. This is true both for the abelian and nonabelian cases. In summary to *close the algebra and to have an invariant Lagrangian* $Q_{abde}$ *must vanishes* for both the abelian and the non-abelian case.

## 5.2 Representation of the Supersymmetry Algebra

For a theory to be supersymmetric, it is necessary that its particle content form a representation of the supersymmetry algebra. Using the gamma matrices representation given in the previous section, we show how to describe the representations of our supersymmetry algebra in $d = 3, 4, 6, 10$. From (5.16), we deduce our supersymmetry algebra as

$$\boxed{\{Q_a, Q_b\} = 2\,(\Gamma)_{ab}\,\partial_\mu \equiv -2\,(\Gamma)_{ab}\,P_\mu} \tag{5.20}$$

where $Q_a$ are the supersymmetry generators and transform as spin-half operators under the angular momentum algebra. Moreover, the supersymmetry generators commute with the momentum operator $P_\mu$ and hence, with $P^2$. Therefore, all states in a given representation of the algebra have the same mass. For our case we will be concerned with the massless representation only. For massless states, we can always go to a frame where $P^\mu = M(1, .., 1)$. Then the supersymmetry algebra becomes

$$\boxed{\{Q_a, Q_b\} = \begin{pmatrix} 0 & 0 \\ 0 & 4M \end{pmatrix} = -2M\,(\Gamma_+)_{ab}}\,.$$

where

$$(\Gamma_+) = (\Gamma_0) + (\Gamma_{d-1}) = -2\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_{\frac{n}{2}} \end{pmatrix}. \tag{5.21}$$

It is convenient to rescale our generators as

$$a_\mu = \frac{1}{\sqrt{2M}} Q_\mu\,,$$

for $\mu = 0..(\frac{n}{2} - 1)$, where $(\frac{n}{2} - 1) = 0, 1, 3, 7$ for $d = 3, 4, 6, 10$ respectively. Then, the supersymmetry algebra takes the form

$$\{a_\mu, a_\nu\} = -\delta_{\mu\nu}\,,$$

This is a Clifford algebra with $\frac{n}{2}$ generators. We can now proceed in two different ways:

1- To retrieve the standard complex representation of our supersymmetry algebra, we have to pair our generators

$$\begin{array}{lll} d=3 & a_0 & \\ d=4 & b=a_0+ia_1 & b^*=a_0-ia_1 \\ d=6 & \left\{\begin{array}{l} b_1=a_0+ia_1 \\ b_1^*=a_0-ia_1 \end{array}\right. & \begin{array}{l} b_2=a_3+ia_4 \\ b_2^*=a_3-ia_4 \end{array} \\ d=10 & \left\{\begin{array}{l} b_1=a_0+ia_1 \\ b_1^*=a_0-ia_1 \\ b_2=a_3+ia_4 \\ b_2^*=a_3-ia_4 \end{array}\right. & \begin{array}{l} b_3=a_5+ia_6 \\ b_3^*=a_5-ia_6 \\ b_4=a_7+ia_8 \\ b_4^*=a_7-ia_8 \end{array} \end{array} \tag{5.22}$$

leading to *case I*

$$\begin{array}{llllll} d=3 & a_0 & & & & \\ d=4 & \{b,b\} & = \{b^*,b^*\} & =0, & \{b,b^*\} & =-2 \\ d=6 & \{b_i,b_j\} & = \left\{b_i^*,b_j^*\right\} & =0, & \left\{b_i,b_j^*\right\} & =-2\delta_{ij} \quad i,j=1,2 \\ d=10 & \{b_i,b_j\} & = \left\{b_i^*,b_j^*\right\} & =0, & \left\{b_i,b_j^*\right\} & =-2\delta_{ij} \quad i,j=1..4 \end{array} \tag{5.23}$$

2- We can work with hypercomplex numbers, then we have *case II*

$$\begin{array}{lll} d=3 & a_0 & \\ d=4 & \left\{\begin{array}{l} b=a_0+e_1a_1 \\ b^*=a_0-e_1a_1 \end{array}\right. & e_1 \textit{ is the imaginary complex unit} \\ d=6 & \left\{\begin{array}{l} b_1=a_0+e_ia_i \\ b_1^*=a_0-e_ia_i \end{array}\right. & \begin{array}{l} e_i \textit{ are imaginary quaternion units} \\ i=1\cdots 3 \end{array} \\ d=10 & \left\{\begin{array}{l} b_1=a_0+e_ia_i \\ b_1^*=a_0-e_ia_i \end{array}\right. & \begin{array}{l} e_i \textit{ are imaginary octonionic units} \\ i=1\cdots 7 \end{array} \end{array} \tag{5.24}$$

## 5.3 The SSYM Auxiliary Fields Problem

One may ask oneself: Why are auxiliary fields important? There are many convincing reasons. Let us mention jut five of them.

1- Only in the presence of auxiliary fields is the supersymmetry manifest. Indeed, when we use superspace, we can write our supersymmetric models in a form clearly invariant under Lorentz transformation as well as supersymmetry. The clearest example is the superspace formulation of $N = 1$ supersymmetric theories.

2- As we saw in the first section of this chapter, the closure of the supersymmetric algebra is achieved only by using the field equations of motion. At the quantum level, the equations of motion get corrected and consequently the supersymmetric algebra will be realized in a highly non-trivial fashion, Look to [99].

3- The use of the Lagrangian formulation of field theory is usually advocated on the basis of symmetries arguments. Hence making the symmetry manifest is a priority.

4- Feynman diagrams with superfields explains naturally many of the "miracle" cancellations in supersymmetric models.

5- Supersymmetry is only constructed systematically when we use superspace. In principle, the superspace formulation should provide us with all the details, the supersymmetry transformations, the full interaction Lagrangian, even the constraints must be derived in agreement with the super Bianchi identities.

Unfortunately, we don't have a complete superspace treatment in $d > 4$. The number of auxiliary fields can be counted easily.

For SSYM, we have the following off–shell degrees of freedom (ofdf)

$$\begin{array}{lccccc} d\backslash ofdf & \Psi & & A_\mu & & \\ 3 & 2 & - & 2 & = & 0 \\ 4 & 4 & - & 3 & = & 1 \\ 6 & 8 & - & 5 & = & 3 \\ 10 & 16 & - & 9 & = & 7 \end{array} \tag{5.25}$$

The $d = 3$ case is trivial in the sense that it contains no auxiliary fields. For $d = 4$, a superspace formalism based on $SL\,(2, \mathbb{C})$ is needed to formulate our supersymmetric YM model in a manifestly invariant way. Such a superspace treatment provides us automatically with the needed single auxiliary field. In $d = 6, 10$, it is conjectured that an $SL\,(2, \mathbb{H})$ and $SL\,(2, \mathbb{O})$ are needed. Here, we try to support this conjecture by a different argument and we hope that the tools presented may help in the future to find the full superspace formulation.

Using Evans ansatz [89], SSYM are composed of: Gauge fields $A_M$, spinors $\Psi^a$, $j\,(= 1..d - 3)$ algebraic auxiliary fields $K^j$. The gauge group indices will be suppressed in the following. The Lagrangian density is

$$\mathcal{L} = -\frac{1}{4}F_{MN}F^{MN} + \frac{i}{2}\Psi^t\Gamma^M\nabla_M\Psi + \frac{1}{2}K^2, \tag{5.26}$$

where $\nabla_M \equiv \partial_M + A_M$; $F_{MN} \equiv [\nabla_M, \nabla_N]$ and the $\Gamma$ are given in (5.10). The Lagrangian is invariant up to a total derivative iff (5.8) holds. Our supersymmetry transformations are[2]

$$\begin{aligned}
\delta_\eta A_M &= i\eta\Gamma_M\Psi, \\
\delta_\eta \Psi^\alpha &= \frac{1}{2}F_{MN}\left(\Gamma_{MN}\ \eta\right)^\alpha + K^j\left(\Lambda_j\right)^\alpha_\beta \eta^\beta, \\
\delta_\eta K_j &= i\left(\Gamma^M\nabla_M\Psi\right)_\alpha\left(\Lambda_j\right)^\alpha_\beta\eta^\beta,
\end{aligned} \tag{5.27}$$

where $\Lambda_j$ are some real antisymmetric matrices $\left(\Lambda_j\right)^t = -\left(\Lambda_j\right)$ [3] and Lorentz transformations are generated by $\Gamma_{MN} \equiv \tilde{\Gamma}_{[M}\Gamma_{N]}$. Imposing the closure of the supersymmetry infinitesimal transformations

$$[\delta_\epsilon, \delta_\eta] = 2i\epsilon^t\Gamma^M\eta\partial_M\ . \tag{5.28}$$

The closure on $A_M$ yields

$$\Gamma_M\Lambda_j - \Lambda_j\Gamma_M = 0. \tag{5.29}$$

In addition to this condition the closure on $K^j$ also requires

$$\Lambda_j\Lambda_h + \Lambda_h\Lambda_j = -2\delta_{jh}. \tag{5.30}$$

While closure on the fermionic field $\Psi^\alpha$ holds iff

$$\left(\Gamma^M\right)_{\alpha\beta}\left(\tilde{\Gamma}_M\right)^{\gamma\delta} = 2\delta^\gamma_{(\alpha}\delta^\delta_{\beta)} + 2\left(\Lambda_j\right)^\gamma_{(\alpha}\left(\Lambda_j\right)^\delta_{\beta)}\,.$$

Now, we continue in a different way to Evans. To construct $\Lambda_j$, we first notice from (5.30) that the $\Lambda_j$ form a real Clifford algebra, and from (5.29) that they commute with our space-time $\Gamma_M$ Clifford algebra. The solution of the auxiliary field problem for $d = 3, 4, 6$ dimensions is then simply

$$\Lambda_j = \begin{pmatrix} 1|\mathbb{E}_j & 0 \\ 0 & 1|\mathbb{E}_j \end{pmatrix}, \tag{5.31}$$

because

$$\{1|\mathbb{E}_j, 1|\mathbb{E}_h\} = -2\delta_{jh}, \tag{5.32}$$

and

$$[\mathbb{E}_j, 1|\mathbb{E}_h] = 0. \tag{5.33}$$

[2]Contrary to [90], we set $\Lambda_P = \tilde{\Lambda}^P$ from the strart.

[3]We will mention shortly how to relax this condition.

Of course this solution is not unique. For example, if someone had started with $1|\Gamma_M$, he would have found $\Lambda_j = \begin{pmatrix} \mathbb{E}_j & 0 \\ 0 & \mathbb{E}_j \end{pmatrix}$. In general, we can relax the conditions of antisymmetricity of $\Lambda$ and the symmetricity of $\Gamma$. One writes any $\Gamma$ and expand it in terms left/right action $(\mathbb{E}_{i,} 1|\mathbb{E}_j, \mathbb{E}_m|\mathbb{E}_n)$ then the $\Lambda$ will be given in terms of $(1|\mathbb{E}_{i,}\mathbb{E}_j, \mathbb{E}_n|\mathbb{E}_m)$.

For $d = 10$, working with octonions the situation is different. From chapter 3, we know that octonionic left and right action commutes only when applied to $\varphi$,

$$\varphi^t \left[\mathbb{E}_j, 1|\mathbb{E}_h\right] \varphi = 0, \tag{5.34}$$

and $\varphi$ is just an 8 dimensional column matrix. Up to now, we have not restricted $\varphi$ by any other conditions. With two different $\varphi$, $\left(\varphi^{(1)}, \varphi^{(2)}\right)$, we impose now the conditions that $\varphi^{(i)}$ be fermionic fields. We express our 16 dimensional Grassmanian variables $\epsilon, \eta$ of eqn.(5.28) in terms of $\varphi$,

$$\begin{aligned} \epsilon &= \eta^t \\ &\Downarrow \\ \epsilon &= \begin{pmatrix} \varphi^{(1)} & \varphi^{(2)} \end{pmatrix}; \qquad \eta = \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} \end{aligned} \tag{5.35}$$

We now rederive (5.28) for the octonions. The closure conditions of our algebra, *without omitting the Grassmanian variables* are

$$\begin{aligned} \eta^t \left(\Gamma_M \Lambda_j - \Lambda_j \Gamma_M\right) \eta &= 0, \\ \eta^t \left(\Lambda_j \Lambda_h + \Lambda_h \Lambda_j\right) \eta &= \eta^t \left(-2\delta_{jh}\right) \eta, \\ \eta^t \left(\left(\Gamma^M\right)_{\alpha\beta} \left(\tilde{\Gamma}_M\right)^{\gamma\delta}\right) \eta &= \eta^t \left(2\delta^\gamma_{(\alpha}\delta^\delta_{\beta)} + 2\left(\Lambda_j\right)^\gamma_{(\alpha} \left(\Lambda_j\right)^\delta_{\beta)}\right) \eta, \end{aligned} \tag{5.36}$$

which are satisfied for the octonionic representation

$$\left(\Gamma_j\right)_{ab} = \begin{pmatrix} 0 & \mathbb{E}_j \\ -\mathbb{E}_j & 0 \end{pmatrix}, \qquad \Lambda_j = \begin{pmatrix} 1|\mathbb{E}_j & 0 \\ 0 & 1|\mathbb{E}_j \end{pmatrix}. \tag{5.37}$$

By interchanging left/right action, we have different solutions as in the quaternionic case. In summary, while the fermionic fields couple to left/right action through the gamma matrices, the auxiliary fields couple to right/left action through the $\Lambda$. For the octonionic case *the presence of the Grassmanian variables is essential.* Contrary to the standard supersymmetry transformation, our Grassman variables are the same, which is identical to the result obtained by Berkovits in [91]. According to Evans [92], the attractive feature of this scheme is that the Lagrangian (5.26) and the transformation

(5.27) are manifestly invariant under the generalized Lorentz group $SO(1,9)$. In our formulation, we can show some additional characteristic. In some cases, the (5.35) condition may be relaxed, for equal $j$ or $h$ (no summation)

$$\left.\begin{array}{c} \varphi^t\ \mathbb{E}_j\ \left[\mathbb{E}_j, 1|\mathbb{E}_h\right]\varphi \\ \varphi^t\ \ 1|\mathbb{E}_i\ \ \left[\mathbb{E}_j, 1|\mathbb{E}_h\right]\varphi \\ \varphi^t\ E_h\ \left[\mathbb{E}_j, 1|\mathbb{E}_h\right]\varphi \\ \varphi^t\ \ 1|E_h\ \ \left[\mathbb{E}_j, 1|\mathbb{E}_h\right]\varphi \end{array}\right\} = 0. \tag{5.38}$$

i.e. relating $\epsilon$ and $\eta$ by an $S^7$ is also allowed.

Now, Let us show what will happen to $spin\,(1,9)$ when we transform it to $soft\ spin\,(1,9)$

$$\begin{aligned} soft\ spin\,(1,9) &\sim \left[\Gamma_i, \Gamma_j\right]\eta \\ &= \left[\begin{pmatrix} 0 & \mathbb{E}_i \\ -\mathbb{E}_i & 0 \end{pmatrix}, \begin{pmatrix} 0 & \mathbb{E}_j \\ -\mathbb{E}_j & 0 \end{pmatrix}\right]\begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} \\ &= -\begin{pmatrix} 0 & \left[\mathbb{E}_i, \mathbb{E}_j\right] \\ \left[\mathbb{E}_i, \mathbb{E}_j\right] & 0 \end{pmatrix}\begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix} \\ &= -\begin{pmatrix} 0 & f^{(+)}_{ijk}\left(\varphi^{(2)}\right)\mathbb{E}_k \\ f^{(+)}_{ijk}\left(\varphi^{(1)}\right)\mathbb{E}_k & 0 \end{pmatrix}\begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \end{pmatrix}. \end{aligned} \tag{5.39}$$

Lastly, let us make some comments about a possible superspace. It seems that the best way to find the $d = 6, 10$ superspace for SSYM is by defining some quaternionic and octonionic Grassmann variables that decompose the corresponding spinors into an $SL\,(2, H)$ and an $SL\,(2, soft\ S^7)$ respectively

$$\left\{\theta_\alpha, \theta_\beta\right\} = \left\{\bar{\theta}_{\dot{\alpha}}, \bar{\theta}_{\dot{\beta}}\right\} = \left\{\theta_\alpha, \bar{\theta}_{\dot{\beta}}\right\} = 0, \tag{5.40}$$

where $\alpha = 1, 2$ over quaternions or octonions. We know that the supersymmetry generators $Q_\alpha$ are derived from right multiplication

$$Q_\alpha = \left(\partial_\alpha - 1|\Gamma^\mu_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}P_\mu\right) \tag{5.41}$$

$$Q^\alpha = \left(-\partial^\alpha + \bar{\theta}_{\dot{\beta}}1|\tilde{\Gamma}^{\mu\dot{\beta}\alpha}P_\mu\right) \tag{5.42}$$

also

$$\bar{Q}^{\dot{\alpha}} = \left(\partial^{\dot{\alpha}} - 1|\tilde{\Gamma}^{\mu\dot{\alpha}\alpha}\theta_\alpha P_\mu\right) \tag{5.43}$$

$$\bar{Q}_{\dot{\alpha}} = \left(-\partial_{\dot{\alpha}} + \theta^\alpha 1|\Gamma_{\alpha\dot{\alpha}}P_\mu\right) \tag{5.44}$$

whereas the covariant derivative $D_\alpha$ are obtained by left action

$$D_\alpha = \left(\partial_\alpha + \Gamma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}P_\mu\right) \tag{5.45}$$

$$D^\alpha = \left(-\partial^\alpha - \bar\theta_{\dot\beta}\tilde\Gamma^{\mu\dot\beta\alpha}P_\mu\right) \tag{5.46}$$

also

$$\bar D^{\dot\alpha} = \left(\partial^{\dot\alpha} + \tilde\Gamma^{\mu\dot\alpha\alpha}\theta_\alpha P_\mu\right) \tag{5.47}$$

$$\bar D_{\dot\alpha} = \left(-\partial_{\dot\alpha} - \theta^\alpha\Gamma_{\alpha\dot\alpha}P_\mu\right) \tag{5.48}$$

Leading to a result acceptable but different from the standard $N = 1$, $d = 4$ superspace,

$$\begin{aligned}
\{Q_\alpha, \bar Q_{\dot\alpha}\} &= -2\,(1|\Gamma^\mu_{\alpha\dot\alpha})\,P_\mu\ ,\\
\{Q_\alpha, Q_\beta\} &= \{\bar Q_{\dot\alpha}, \bar Q_{\dot\beta}\}\ =\ 0\ ,\\
\\
\{D_\alpha, \bar D_{\dot\alpha}\} &= 2\Gamma^\mu_{\alpha\dot\alpha}P_\mu\ ,\\
\{D_\alpha, D_\beta\} &= \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\}\ =\ 0\ ,
\end{aligned}$$

and iff left and right action commute, we restore

$$\begin{aligned}
\{Q_\alpha, \bar D_{\dot\alpha}\} &= \{D_\alpha, \bar Q_{\dot\alpha}\} = 0\ ,\\
\{Q_\alpha, D_\beta\} &= \{\bar D_{\dot\alpha}, \bar Q_{\dot\beta}\}\ =\ 0\ .
\end{aligned}$$

On the other hand for octonions we would have the weaker conditions,

$$\begin{aligned}
\begin{pmatrix}\varphi^{(1)} & \varphi^{(2)}\end{pmatrix}\{Q_\alpha, \bar D_{\dot\alpha}\}\begin{pmatrix}\varphi^{(1)}\\ \varphi^{(2)}\end{pmatrix} &= \begin{pmatrix}\varphi^{(1)} & \varphi^{(2)}\end{pmatrix}\{D_\alpha, \bar Q_{\dot\alpha}\}\begin{pmatrix}\varphi^{(1)}\\ \varphi^{(2)}\end{pmatrix} = 0\ ,\\
\begin{pmatrix}\varphi^{(1)} & \varphi^{(2)}\end{pmatrix}\{Q_\alpha, D_\beta\}\begin{pmatrix}\varphi^{(1)}\\ \varphi^{(2)}\end{pmatrix} &= \begin{pmatrix}\varphi^{(1)} & \varphi^{(2)}\end{pmatrix}\{\bar D_{\dot\alpha}, \bar Q_{\dot\beta}\}\begin{pmatrix}\varphi^{(1)}\\ \varphi^{(2)}\end{pmatrix} =\ 0\ .
\end{aligned}$$

The commutation of left and right actions is not just needed for associativity but for the invariance under supersymmetry transformation

$$\delta_\xi \equiv \xi Q + \bar\xi\bar Q \tag{5.49}$$

because only the associativity ensures

$$\begin{pmatrix}\varphi^{(1)} & \varphi^{(2)}\end{pmatrix}[\delta_\xi, D_\alpha]\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = \begin{pmatrix}\varphi^{(1)} & \varphi^{(2)}\end{pmatrix}\left[\delta_\xi, \bar D_{\dot\alpha}\right]\begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix} = 0, \tag{5.50}$$

since $\delta_\xi$ is left action and $D_\alpha$ is right action which is a very important relation in the standard $N = 1$ superspace for the invariance of the Lagrangian under supersymmetry transformation. We hope to return to this point in a future work.

# Chapter 6

# Conclusions

In this thesis, we have presented a systematic study of the hidden faithful Clifford algebraic structure in the different types of ring division algebras. This relationship had been elaborated by going beyond octonions to hexagonions. We have then dedicated a complete chapter to octonions. They are not as useless as often believed.. They may be safely employed once the non-associativity has been bypassed. The necessary ingredients are:

- *Fixing the direction of action by introducing the $\delta$ operator.*
- *Closing the $\delta$ algebra by using structure functions $f_{ijk}^{(+)}(\varphi)$.*
- *Matrix representation of the $\delta$ algebra. The $\mathbb{E}$ or $\mathbb{E}(\varphi)$ can be found and their structure functions can be computed easily.*

During this analysis, we have introduced and discussed the soft seven sphere. There maybe different applications of the soft seven sphere in physics [100] [101] [102] [103]. We have given two such cases where the ring division algebras occupies a special position. Self-duality and SSYM are two promising places where the soft seven sphere proves to be useful and indeed essential. In our formulation, *we find a new eight dimensional feature, that had never appeared before, the existence of an infinite family of dualities. By moving over the gauged seven sphere, we define new conditions and we have new solutions. We have parameterized all these conditions and solutions in terms of the coordinate system over the gauged seven sphere.*

By defining the soft–duality condition, we have tried to retain as much as possible of the four dimensional case. We then showed how new solutions of the GKS condition can be easily found .

*For SSYM, the new and old off–shell formulations can be rederived in a systematic and uniform fashion.* We believe that the interplay between left and right ring division operators is not a coincidence but an intrinsic property of supersymmetry that needs further study. By interchanging left and right action, we have many different solutions. Again the octonionic ten dimensional case is very special. It will be interesting to apply the ideas presented here into the GS context.

We hope that this work constitute a first step in the correct direction for further application of the soft seven sphere algebra.

# Appendix A

# The First Appendix

In this appendix we give the translation rules between octonionic left-right barred operators and $8\times8$ real matrices. In order to simplify our presentation we introduce the following notation:

$$\{\, a,\ b,\ c,\ d\,\}_{(1)} \ \equiv\ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} ,\ \ \{\, a,\ b,\ c,\ d\,\}_{(2)} \equiv \begin{pmatrix} 0 & a & 0 & 0 \\ b & 0 & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & d & 0 \end{pmatrix} , \tag{A.1}$$

$$\{\, a,\ b,\ c,\ d\,\}_{(3)} \ \equiv\ \begin{pmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ c & 0 & 0 & 0 \\ 0 & d & 0 & 0 \end{pmatrix} ,\ \ \{\, a,\ b,\ c,\ d\,\}_{(4)} \equiv \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix} , \tag{A.2}$$

where $a,\ b,\ c,\ d$ and 0 represent $2\times2$ real matrices. As elsewhere by $\sigma_1$, $\sigma_2$, $\sigma_3$ we mean the standard Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad . \tag{A.3}$$

The only necessary translation rules that we need to know explicitly are the following

$$
\begin{array}{llllllll}
e_1 & \leftrightarrow & \{ & -i\sigma_2, & -i\sigma_2, & -i\sigma_2, & i\sigma_2 & \}_{(1)} \\
e_2 & \leftrightarrow & \{ & -\sigma_3, & \sigma_3, & -\mathbf{1}, & \mathbf{1} & \}_{(2)} \\
e_3 & \leftrightarrow & \{ & -\sigma_1, & \sigma_1, & -i\sigma_2, & -i\sigma_2 & \}_{(2)} \\
e_4 & \leftrightarrow & \{ & -\sigma_3, & \mathbf{1}, & \sigma_3, & -\mathbf{1} & \}_{(3)} \\
e_5 & \leftrightarrow & \{ & -\sigma_1, & i\sigma_2, & \sigma_1, & i\sigma_2 & \}_{(3)} \\
e_6 & \leftrightarrow & \{ & -\mathbf{1}, & -\sigma_3, & \sigma_3, & \mathbf{1} & \}_{(4)} \\
e_7 & \leftrightarrow & \{ & -i\sigma_2, & -\sigma_1, & \sigma_1, & -i\sigma_2 & \}_{(4)} \\
1 \mid e_1 & \leftrightarrow & \{ & -i\sigma_2, & i\sigma_2, & i\sigma_2, & -i\sigma_2 & \}_{(1)} \\
1 \mid e_2 & \leftrightarrow & \{ & -\mathbf{1}, & \mathbf{1}, & \mathbf{1}, & -\mathbf{1} & \}_{(2)} \\
1 \mid e_3 & \leftrightarrow & \{ & -i\sigma_2, & -i\sigma_2, & i\sigma_2, & i\sigma_2 & \}_{(2)} \\
1 \mid e_4 & \leftrightarrow & \{ & -\mathbf{1}, & -\mathbf{1}, & \mathbf{1}, & \mathbf{1} & \}_{(3)} \\
1 \mid e_5 & \leftrightarrow & \{ & -i\sigma_2, & -i\sigma_2, & -i\sigma_2, & -i\sigma_2 & \}_{(3)} \\
1 \mid e_6 & \leftrightarrow & \{ & -\sigma_3, & \sigma_3, & -\sigma_3, & \sigma_3 & \}_{(4)} \\
1 \mid e_7 & \leftrightarrow & \{ & -\sigma_1, & \sigma_1, & -\sigma_1, & \sigma_1 & \}_{(4)}
\end{array}
$$

The remaining rules can be easily constructed remembering that

$$
\begin{array}{lll}
e_m \mid e_m & \leftrightarrow 1|E_m \;\; E_m & , \\
 & \leftrightarrow \;\; E_m \;\; 1|E_m & , \\
e_m \;)\; e_n & \leftrightarrow 1|E_n \;\; E_m & , \\
e_m \;(\; e_n & \leftrightarrow \;\; E_m \;\; 1|E_n & .
\end{array}
$$

For example,

$$
e_1 \mid e_1 \;\; \leftrightarrow \;\; \begin{pmatrix} -i\sigma_2 & 0 & 0 & 0 \\ 0 & -i\sigma_2 & 0 & 0 \\ 0 & 0 & -i\sigma_2 & 0 \\ 0 & 0 & 0 & i\sigma_2 \end{pmatrix} \begin{pmatrix} -i\sigma_2 & 0 & 0 & 0 \\ 0 & i\sigma_2 & 0 & 0 \\ 0 & 0 & i\sigma_2 & 0 \\ 0 & 0 & 0 & -i\sigma_2 \end{pmatrix}
$$
$$
= \{ -\mathbf{1},\; \mathbf{1},\; \mathbf{1},\; \mathbf{1} \}_{(1)} \quad ,
$$

$$
e_3 \;)\; e_1 \;\; \leftrightarrow \;\; \begin{pmatrix} -i\sigma_2 & 0 & 0 & 0 \\ 0 & i\sigma_2 & 0 & 0 \\ 0 & 0 & i\sigma_2 & 0 \\ 0 & 0 & 0 & -i\sigma_2 \end{pmatrix} \begin{pmatrix} 0 & -\sigma_1 & 0 & 0 \\ \sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sigma_2 \\ 0 & 0 & -i\sigma_2 & 0 \end{pmatrix}
$$
$$
= \{ \sigma_3,\; \sigma_3,\; \mathbf{1},\; -\mathbf{1} \}_{(2)} \quad ,
$$

and

$$
e_3 \;(\; e_1 \;\; \leftrightarrow \;\; \begin{pmatrix} 0 & -\sigma_1 & 0 & 0 \\ \sigma_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i\sigma_2 \\ 0 & 0 & -i\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} -i\sigma_2 & 0^0 & 0 & \\ 0 & i\sigma_2 & 0 & 0 \\ 0 & 0 & i\sigma_2 & 0 \\ 0 & 0 & 0 & -i\sigma_2 \end{pmatrix}
$$
$$
= \{ \sigma_3,\; \sigma_3,\; -\mathbf{1},\; \mathbf{1} \}_{(2)} \quad .
$$

Following this procedure any matrix representation of right/left barred operators can be obtained. Using Mathematica [58], we have proved the linear independence of the 64 elements which represent the most general octonionic operator

$$\mathbb{O}_0 + \sum_{m=1}^{7} \mathbb{O}_m \ ) \ e_m \quad .$$

So our barred operators form a complete basis for any $8 \times 8$ real matrix and this establishes the isomorphism between $GL(8, \mathbb{R})$ and barred octonions.

# Appendix B

# The Second Appendix

We have given the action of barred operators on the octonionic functions (states)

$$\psi = \psi_1 + e_2\psi_2 + e_4\psi_3 + e_6\psi_4 \qquad [\ \psi_{1,\dots,4} \in C(1,\ e_1)\ ] \quad .$$

In the following we will use the notation

$$e_2 \ \rightarrow \ \{-\psi_2,\ \psi_1,\ -\psi_4^*,\ \psi_3^*\} \quad ,$$

to indicate

$$e_2\psi \ = \ -\psi_2 + e_2\psi_1 - e_4\psi_4^* + e_6\psi_3^* \quad .$$

As occurred in the previous appendix we need to know only the action of the barred operators $e_m$ and $1 \mid e_m$

$$\begin{array}{llllll}
e_1 & \rightarrow \{ & e_1\psi_1, & -e_1\psi_2, & -e_1\psi_3, & -e_1\psi_4 \ \} \\
e_2 & \rightarrow \{ & -\psi_2, & \psi_1, & -\psi_4^*, & \psi_3^* \ \} \\
e_3 & \rightarrow \{ & -e_1\psi_2, & -e_1\psi_1, & -e_1\psi_4^*, & e_1\psi_3^* \ \} \\
e_4 & \rightarrow \{ & -\psi_3, & \psi_4^*, & \psi_1, & -\psi_2^* \ \} \\
e_5 & \rightarrow \{ & -e_1\psi_3, & e_1\psi_4^*, & -e_1\psi_1, & -e_1\psi_2^* \ \} \\
e_6 & \rightarrow \{ & -\psi_4, & -\psi_3^*, & \psi_2^*, & \psi_1 \ \} \\
e_7 & \rightarrow \{ & e_1\psi_4, & e_1\psi_3^*, & -e_1\psi_2^*, & e_1\psi_1 \ \} \\
1 \mid e_1 & \rightarrow \{ & e_1\psi_1, & e_1\psi_2, & e_1\psi_3, & e_1\psi_4 \ \} \\
1 \mid e_2 & \rightarrow \{ & -\psi_2^*, & \psi_1^*, & \psi_4^*, & -\psi_3^* \ \} \\
1 \mid e_3 & \rightarrow \{ & e_1\psi_2^*, & -e_1\psi_1^*, & e_1\psi_4^*, & -e_1\psi_3^* \ \} \\
1 \mid e_4 & \rightarrow \{ & -\psi_3^*, & -\psi_4^*, & \psi_1^*, & \psi_2^* \ \} \\
1 \mid e_5 & \rightarrow \{ & e_1\psi_3^*, & -e_1\psi_4^*, & -e_1\psi_1^*, & e_1\psi_2^* \ \} \\
1 \mid e_6 & \rightarrow \{ & -\psi_4^*, & \psi_3^*, & -\psi_2^*, & \psi_1^* \ \} \\
1 \mid e_7 & \rightarrow \{ & -e_1\psi_4^*, & -e_1\psi_3^*, & e_1\psi_2^*, & e_1\psi_1^* \ \}
\end{array}$$

From the previous correspondence rules we immediately obtain the others barred operators. We give, as an example, the construction of the operator $e_4 \mid e_7$. We know that

$$e_4 \;\rightarrow\; \{-\psi_3,\ \psi_4^*,\ \psi_1,\ -\psi_2^*\}$$

and

$$1 \mid e_7 \;\rightarrow\; \{-e_1\psi_4^*,\ -e_1\psi_3^*,\ e_1\psi_2^*,\ e_1\psi_1^*\} \quad . \tag{B.1}$$

Combining these operators we find

$$\{-e_1(-\psi_2^*)^*,\ -e_1\psi_1^*,\ e_1(\psi_4^*)^*,\ e_1(-\psi_3)^*\} \quad ,$$

and so

$$e_4 \mid e_7 \;\rightarrow\; \{e_1\psi_2,\ -e_1\psi_1^*,\ e_1\psi_4,\ -e_1\psi_3^*\} \quad .$$

As remarked at the end of subsection IV-b, we can extract the 32 basis elements of $GL(4,\mathbb{C})$ directly by suitable combinations of the 64 basis elements of $GL(8,\mathbb{R})$. We must choose the combination which have only $\mathbf{1}_{2\times 2}$ and $-i\sigma_2$ as matrix elements. Nevertheless we must take care in manipulating our octonionic barred operators. If we wish to extract from $GL(8,\mathbb{R})$ the 32 elements which characterize $GL(4,\mathbb{C})$ we need to change the octonionic basis of $GL(8,\mathbb{R})$. In fact, the natural choice for the symplectic octonionic representation

$$\psi \;=\; (\varphi_0 + e_1\varphi_1) + e_2(\varphi_2 + e_1\varphi_3) + e_4(\varphi_4 + e_1\varphi_5) + e_6(\varphi_6 + e_1\varphi_7) \quad ,$$

requires the following real counterpart

$$\tilde{\varphi} \;=\; \varphi_0 + e_1\varphi_1 + e_2\varphi_2 - e_3\varphi_3 + e_4\varphi_4 - e_5\varphi_5 + e_6\varphi_6 + e_7\varphi_7 \quad .$$

whereas we used in subsection IV-a the following basis

$$\varphi \;=\; \varphi_0 + e_1\varphi_1 + e_2\varphi_2 + e_3\varphi_3 + e_4\varphi_4 + e_5\varphi_5 + e_6\varphi_6 + e_7\varphi_7 \quad .$$

The changes in the signs of $e_3\varphi_3$ and $e_5\varphi_5$ implies a modification in the generators of $GL(8,\mathbb{R})$. For example, $e_2$ and $e_3 \mid e_1$ now read

$$e_2 \;\equiv\; \{-\mathbf{1},\ \mathbf{1},\ -\sigma_3,\ \sigma_3\}_{(2)} \qquad \text{and} \qquad e_3 \mid e_1 \;\equiv\; \{\mathbf{1},\ \mathbf{1},\ \sigma_3,\ -\sigma_3\}_{(2)} \quad .$$

i.e. the change of basis induces the following modifications

$$\mathbf{1} \rightleftharpoons \sigma_3 \quad .$$

Their appropriate combination gives

$$\frac{e_2 + e_3 \mid e_1}{2} \;\equiv\; \{0,\ \mathbf{1},\ 0,\ 0\}_{(2)} \quad \overset{complexifing}{\longrightarrow} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ,$$

as required by eq. (2.96).

# Appendix C

# The Third Appendix

The seven standard cycles are given by

$$
\begin{aligned}
f^{(+)}_{123}(\varphi) &= \frac{{\varphi_0}^2 - {\varphi_6}^2 - {\varphi_5}^2 + {\varphi_2}^2 - {\varphi_4}^2 + {\varphi_1}^2 + {\varphi_3}^2 - {\varphi_7}^2}{r^2}, \\
f^{(+)}_{145}(\varphi) &= \frac{{\varphi_0}^2 - {\varphi_6}^2 + {\varphi_4}^2 - {\varphi_2}^2 + {\varphi_1}^2 - {\varphi_7}^2 + {\varphi_5}^2 - {\varphi_3}^2}{r^2}, \\
f^{(+)}_{176}(\varphi) &= \frac{{\varphi_0}^2 + {\varphi_6}^2 - {\varphi_4}^2 - {\varphi_2}^2 + {\varphi_1}^2 + {\varphi_7}^2 - {\varphi_5}^2 - {\varphi_3}^2}{r^2}, \\
f^{(+)}_{246}(\varphi) &= \frac{{\varphi_0}^2 + {\varphi_6}^2 + {\varphi_4}^2 + {\varphi_2}^2 - {\varphi_1}^2 - {\varphi_7}^2 - {\varphi_5}^2 - {\varphi_3}^2}{r^2}, \\
f^{(+)}_{257}(\varphi) &= \frac{{\varphi_0}^2 - {\varphi_6}^2 - {\varphi_4}^2 + {\varphi_2}^2 - {\varphi_1}^2 + {\varphi_7}^2 + {\varphi_5}^2 - {\varphi_3}^2}{r^2}, \\
f^{(+)}_{347}(\varphi) &= \frac{{\varphi_0}^2 - {\varphi_6}^2 + {\varphi_4}^2 - {\varphi_2}^2 - {\varphi_1}^2 + {\varphi_7}^2 - {\varphi_5}^2 + {\varphi_3}^2}{r^2}, \\
f^{(+)}_{365}(\varphi) &= \frac{{\varphi_0}^2 + {\varphi_6}^2 - {\varphi_4}^2 - {\varphi_2}^2 - {\varphi_1}^2 - {\varphi_7}^2 + {\varphi_5}^2 + {\varphi_3}^2}{r^2},
\end{aligned}
\tag{C.1}
$$

where

$$r^2 = (\varphi_0^2 + \varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2 + \varphi_5^2 + \varphi_6^2 + \varphi_7^2) \tag{C.2}$$

and the non-standard subset

$$
\begin{aligned}
f^{(+)}_{124}(\varphi) &= +2\frac{\varphi_0\varphi_7 - \varphi_5\varphi_2 + \varphi_6\varphi_1 + \varphi_3\varphi_4}{r^2}, \\
f^{(+)}_{125}(\varphi) &= -2\frac{\varphi_0\varphi_6 - \varphi_3\varphi_5 - \varphi_1\varphi_7 - \varphi_2\varphi_4}{r^2}, \\
f^{(+)}_{126}(\varphi) &= +2\frac{\varphi_0\varphi_5 - \varphi_1\varphi_4 + \varphi_7\varphi_2 + \varphi_3\varphi_6}{r^2},
\end{aligned}
$$

$$
\begin{aligned}
f^{(+)}_{127}(\varphi) &= -2\frac{\varphi_0\varphi_4+\varphi_6\varphi_2+\varphi_1\varphi_5-\varphi_3\varphi_7}{r^2},\\
f^{(+)}_{143}(\varphi) &= +2\frac{\varphi_0\varphi_6+\varphi_3\varphi_5+\varphi_2\varphi_4-\varphi_1\varphi_7}{r^2},\\
f^{(+)}_{146}(\varphi) &= -2\frac{\varphi_3\varphi_0-\varphi_4\varphi_7-\varphi_1\varphi_2-\varphi_5\varphi_6}{r^2},\\
f^{(+)}_{175}(\varphi) &= +2\frac{\varphi_3\varphi_0-\varphi_1\varphi_2+\varphi_5\varphi_6+\varphi_4\varphi_7}{r^2},\\
f^{(+)}_{247}(\varphi) &= -2\frac{\varphi_0\varphi_1-\varphi_7\varphi_6-\varphi_4\varphi_5-\varphi_3\varphi_2}{r^2},\\
f^{(+)}_{147}(\varphi) &= +2\frac{\varphi_2\varphi_0-\varphi_4\varphi_6+\varphi_5\varphi_7+\varphi_1\varphi_3}{r^2},\\
f^{(+)}_{243}(\varphi) &= -2\frac{\varphi_0\varphi_5+\varphi_1\varphi_4+\varphi_7\varphi_2-\varphi_3\varphi_6}{r^2},\\
f^{(+)}_{253}(\varphi) &= +2\frac{\varphi_0\varphi_4-\varphi_1\varphi_5+\varphi_6\varphi_2+\varphi_3\varphi_7}{r^2},\\
f^{(+)}_{173}(\varphi) &= -2\frac{\varphi_0\varphi_5-\varphi_7\varphi_2-\varphi_3\varphi_6-\varphi_1\varphi_4}{r^2},\\
f^{(+)}_{245}(\varphi) &= +2\frac{\varphi_3\varphi_0+\varphi_5\varphi_6-\varphi_4\varphi_7+\varphi_1\varphi_2}{r^2},\\
f^{(+)}_{256}(\varphi) &= +2\frac{\varphi_0\varphi_1-\varphi_3\varphi_2+\varphi_7\varphi_6+\varphi_4\varphi_5}{r^2},\\
f^{(+)}_{361}(\varphi) &= +2\frac{\varphi_0\varphi_4+\varphi_3\varphi_7+\varphi_1\varphi_5-\varphi_6\varphi_2}{r^2},\\
f^{(+)}_{362}(\varphi) &= -2\frac{\varphi_0\varphi_7-\varphi_3\varphi_4-\varphi_5\varphi_2-\varphi_6\varphi_1}{r^2},\\
f^{(+)}_{345}(\varphi) &= -2\frac{\varphi_2\varphi_0-\varphi_5\varphi_7-\varphi_1\varphi_3-\varphi_4\varphi_6}{r^2},\\
f^{(+)}_{346}(\varphi) &= +2\frac{\varphi_0\varphi_1+\varphi_3\varphi_2+\varphi_7\varphi_6-\varphi_4\varphi_5}{r^2},\\
f^{(+)}_{367}(\varphi) &= +2\frac{\varphi_2\varphi_0+\varphi_4\varphi_6+\varphi_5\varphi_7-\varphi_1\varphi_3}{r^2},\\
f^{(+)}_{135}(\varphi) &= -2\frac{\varphi_0\varphi_7-\varphi_3\varphi_4+\varphi_6\varphi_1+\varphi_5\varphi_2}{r^2},\\
f^{(+)}_{156}(\varphi) &= -2\frac{\varphi_2\varphi_0+\varphi_1\varphi_3+\varphi_4\varphi_6-\varphi_5\varphi_7}{r^2},\\
f^{(+)}_{237}(\varphi) &= +2\frac{\varphi_0\varphi_6-\varphi_2\varphi_4+\varphi_3\varphi_5+\varphi_1\varphi_7}{r^2},\\
f^{(+)}_{267}(\varphi) &= -2\frac{\varphi_3\varphi_0-\varphi_5\varphi_6+\varphi_4\varphi_7+\varphi_1\varphi_2}{r^2},\\
f^{(+)}_{357}(\varphi) &= +2\frac{\varphi_0\varphi_1+\varphi_4\varphi_5-\varphi_7\varphi_6+\varphi_3\varphi_2}{r^2},\\
f^{(+)}_{456}(\varphi) &= -2\frac{\varphi_0\varphi_7+\varphi_5\varphi_2+\varphi_3\varphi_4-\varphi_6\varphi_1}{r^2},
\end{aligned}
$$

$$\begin{aligned}
f^{(+)}_{457}(\varphi) &= +2\frac{\varphi_0\varphi_6+\varphi_2\varphi_4+\varphi_1\varphi_7-\varphi_3\varphi_5}{r^2},\\
f^{(+)}_{467}(\varphi) &= -2\frac{\varphi_0\varphi_5+\varphi_1\varphi_4+\varphi_3\varphi_6-\varphi_7\varphi_2}{r^2},\\
f^{(+)}_{567}(\varphi) &= +2\frac{\varphi_0\varphi_4-\varphi_6\varphi_2-\varphi_1\varphi_5-\varphi_3\varphi_7}{r^2}.
\end{aligned} \tag{C.3}$$

For right actions, the standard cocycles are

$$\begin{aligned}
f^{(-)}_{123}(\varphi) &= -\frac{{\varphi_0}^2-{\varphi_6}^2-{\varphi_4}^2+{\varphi_2}^2+{\varphi_1}^2-{\varphi_7}^2-{\varphi_5}^2+{\varphi_3}^2}{r^2},\\
f^{(-)}_{145}(\varphi) &= -\frac{{\varphi_0}^2-{\varphi_6}^2+{\varphi_4}^2-{\varphi_2}^2+{\varphi_1}^2-{\varphi_7}^2+{\varphi_5}^2-{\varphi_3}^2}{r^2},\\
f^{(-)}_{176}(\varphi) &= -\frac{{\varphi_0}^2+{\varphi_6}^2-{\varphi_4}^2-{\varphi_2}^2+{\varphi_1}^2+{\varphi_7}^2-{\varphi_5}^2-{\varphi_3}^2}{r^2},\\
f^{(-)}_{246}(\varphi) &= -\frac{{\varphi_0}^2+{\varphi_6}^2+{\varphi_4}^2+{\varphi_2}^2-{\varphi_1}^2-{\varphi_7}^2-{\varphi_5}^2-{\varphi_3}^2}{r^2},\\
f^{(-)}_{257}(\varphi) &= -\frac{{\varphi_0}^2-{\varphi_6}^2-{\varphi_4}^2+{\varphi_2}^2-{\varphi_1}^2+{\varphi_7}^2+{\varphi_5}^2-{\varphi_3}^2}{r^2},\\
f^{(-)}_{347}(\varphi) &= -\frac{{\varphi_0}^2-{\varphi_6}^2+{\varphi_4}^2-{\varphi_2}^2-{\varphi_1}^2+{\varphi_7}^2-{\varphi_5}^2+{\varphi_3}^2}{r^2},\\
f^{(-)}_{365}(\varphi) &= -\frac{{\varphi_0}^2+{\varphi_6}^2-{\varphi_4}^2-{\varphi_2}^2-{\varphi_1}^2-{\varphi_7}^2+{\varphi_5}^2+{\varphi_3}^2}{r^2},
\end{aligned} \tag{C.4}$$

while the non-standard cocycles are

$$\begin{aligned}
f^{(-)}_{124}(\varphi) &= +2\frac{\varphi_0\varphi_7+\varphi_5\varphi_2-\varphi_6\varphi_1-\varphi_3\varphi_4}{r^2},\\
f^{(-)}_{125}(\varphi) &= -2\frac{\varphi_0\varphi_6+\varphi_3\varphi_5+\varphi_1\varphi_7+\varphi_2\varphi_4}{r^2},\\
f^{(-)}_{126}(\varphi) &= +2\frac{\varphi_0\varphi_5+\varphi_1\varphi_4-\varphi_7\varphi_2-\varphi_3\varphi_6}{r^2},\\
f^{(-)}_{127}(\varphi) &= -2\frac{\varphi_0\varphi_4-\varphi_6\varphi_2-\varphi_1\varphi_5+\varphi_3\varphi_7}{r^2},\\
f^{(-)}_{143}(\varphi) &= +2\frac{\varphi_0\varphi_6-\varphi_3\varphi_5-\varphi_2\varphi_4+\varphi_1\varphi_7}{r^2},\\
f^{(-)}_{146}(\varphi) &= -2\frac{\varphi_3\varphi_0+\varphi_4\varphi_7+\varphi_1\varphi_2+\varphi_5\varphi_6}{r^2},\\
f^{(-)}_{175}(\varphi) &= +2\frac{\varphi_3\varphi_0+\varphi_1\varphi_2-\varphi_5\varphi_6-\varphi_4\varphi_7}{r^2},\\
f^{(-)}_{247}(\varphi) &= -2\frac{\varphi_0\varphi_1+\varphi_7\varphi_6+\varphi_4\varphi_5+\varphi_3\varphi_2}{r^2},
\end{aligned}$$

$$
\begin{aligned}
f^{(-)}_{147}(\varphi) &= +2\frac{\varphi_2\varphi_0+\varphi_4\varphi_6-\varphi_5\varphi_7-\varphi_1\varphi_3}{r^2},\\
f^{(-)}_{243}(\varphi) &= -2\frac{\varphi_0\varphi_5-\varphi_1\varphi_4-\varphi_7\varphi_2+\varphi_3\varphi_6}{r^2},\\
f^{(-)}_{253}(\varphi) &= +2\frac{\varphi_0\varphi_4+\varphi_1\varphi_5-\varphi_6\varphi_2-\varphi_3\varphi_7}{r^2},\\
f^{(-)}_{173}(\varphi) &= -2\frac{\varphi_0\varphi_5+\varphi_7\varphi_2+\varphi_3\varphi_6+\varphi_1\varphi_4}{r^2},\\
f^{(-)}_{245}(\varphi) &= +2\frac{\varphi_3\varphi_0-\varphi_5\varphi_6+\varphi_4\varphi_7-\varphi_1\varphi_2}{r^2},\\
f^{(-)}_{256}(\varphi) &= +2\frac{\varphi_0\varphi_1+\varphi_3\varphi_2-\varphi_7\varphi_6-\varphi_4\varphi_5}{r^2},\\
f^{(-)}_{361}(\varphi) &= +2\frac{\varphi_0\varphi_4-\varphi_3\varphi_7-\varphi_1\varphi_5+\varphi_6\varphi_2}{r^2},\\
f^{(-)}_{362}(\varphi) &= -2\frac{\varphi_0\varphi_7+\varphi_3\varphi_4+\varphi_5\varphi_2+\varphi_6\varphi_1}{r^2},\\
f^{(-)}_{345}(\varphi) &= -2\frac{\varphi_2\varphi_0+\varphi_5\varphi_7+\varphi_1\varphi_3+\varphi_4\varphi_6}{r^2},\\
f^{(-)}_{346}(\varphi) &= +2\frac{\varphi_0\varphi_1-\varphi_3\varphi_2-\varphi_7\varphi_6+\varphi_4\varphi_5}{r^2},\\
f^{(-)}_{367}(\varphi) &= +2\frac{\varphi_2\varphi_0-\varphi_4\varphi_6-\varphi_5\varphi_7+\varphi_1\varphi_3}{r^2},\\
f^{(-)}_{135}(\varphi) &= -2\frac{\varphi_0\varphi_7+\varphi_3\varphi_4-\varphi_6\varphi_1-\varphi_5\varphi_2}{r^2},\\
f^{(-)}_{156}(\varphi) &= -2\frac{\varphi_2\varphi_0-\varphi_1\varphi_3-\varphi_4\varphi_6+\varphi_5\varphi_7}{r^2},\\
f^{(-)}_{237}(\varphi) &= +2\frac{\varphi_0\varphi_6+\varphi_2\varphi_4-\varphi_3\varphi_5-\varphi_1\varphi_7}{r^2},\\
f^{(-)}_{267}(\varphi) &= -2\frac{\varphi_3\varphi_0+\varphi_5\varphi_6-\varphi_4\varphi_7-\varphi_1\varphi_2}{r^2},\\
f^{(-)}_{357}(\varphi) &= +2\frac{\varphi_0\varphi_1-\varphi_4\varphi_5+\varphi_7\varphi_6-\varphi_3\varphi_2}{r^2},\\
f^{(-)}_{456}(\varphi) &= -2\frac{\varphi_0\varphi_7-\varphi_5\varphi_2-\varphi_3\varphi_4+\varphi_6\varphi_1}{r^2},\\
f^{(-)}_{457}(\varphi) &= +2\frac{\varphi_0\varphi_6-\varphi_2\varphi_4-\varphi_1\varphi_7+\varphi_3\varphi_5}{r^2},\\
f^{(-)}_{467}(\varphi) &= -2\frac{\varphi_0\varphi_5-\varphi_1\varphi_4-\varphi_3\varphi_6+\varphi_7\varphi_2}{r^2},\\
f^{(-)}_{567}(\varphi) &= +2\frac{\varphi_0\varphi_4+\varphi_6\varphi_2+\varphi_1\varphi_5+\varphi_3\varphi_7}{r^2}.
\end{aligned}
\tag{C.5}
$$

# Appendix D

# The Fourth Appendix

We give below some examples of the torsionful structure functions,

$$
\begin{aligned}
f_{123}^{(++)}(\varphi,\lambda) =&(y_0{}^2\,x_3{}^2 + y_0{}^2\,x_2{}^2 - y_0{}^2\,x_5{}^2 - y_0{}^2\,x_4{}^2 + y_0{}^2\,x_6{}^2 - y_0{}^2\,x_7{}^2 - \\
&+ y_0{}^2\,x_1{}^2 + y_7{}^2\,x_4{}^2 + y_7{}^2\,x_6{}^2 - 4\,y_0\,y_6\,x_1\,x_7 \\
&- y_3{}^2\,x_4{}^2 - y_0{}^2\,x_4{}^2 - y_0{}^2\,x_6{}^2 - y_0{}^2\,x_7{}^2 \\
&- y_3{}^2\,x_6{}^2 + 4\,y_0\,y_4\,x_3\,x_7 - 4\,y_0\,y_7\,x_5\,x_2 - 4\,y_0\,y_7\,x_3\,x_4 \\
&+ 4\,y_0\,y_7\,x_6\,x_1 + y_1{}^2\,x_1{}^2 - y_1{}^2\,x_7{}^2 - y_1{}^2\,x_5{}^2 + y_1{}^2\,x_2{}^2 \\
&- y_1{}^2\,x_4{}^2 - y_1{}^2\,x_6{}^2 + y_1{}^2\,x_3{}^2 - 4\,y_0\,y_5\,x_1\,x_4 \\
&- 4\,y_0\,y_5\,x_3\,x_6 + 4\,y_0\,y_4\,x_2\,x_6 + 4\,y_0\,y_6\,x_3\,x_5 \\
&- 4\,y_0\,y_6\,x_2\,x_4 + 4\,y_1\,y_4\,x_7\,x_2 + 4\,y_0\,y_4\,x_1\,x_5 \\
&+ 4\,y_1\,y_5\,x_1\,x_5 + 4\,y_1\,y_4\,x_1\,x_4 - y_2{}^2\,x_4{}^2 - 4\,y_1\,y_5\,x_3\,x_7 \\
&+ y_2{}^2\,x_2{}^2 + 4\,y_1\,y_6\,x_3\,x_4 + 4\,y_1\,y_6\,x_5\,x_2 + y_2{}^2\,x_1{}^2 \\
&+ 4\,y_1\,y_6\,x_6\,x_1 - y_2{}^2\,x_6{}^2 + 4\,y_3\,y_5\,x_2\,x_4 - y_7{}^2\,x_1{}^2 \\
&- y_2{}^2\,x_5{}^2 + 4\,y_3\,y_4\,x_3\,x_4 - 4\,y_3\,y_7\,x_6\,x_2 + 4\,y_3\,y_7\,x_3\,x_7 \\
&+ 4\,y_1\,y_7\,x_1\,x_7 - 4\,y_3\,y_4\,x_5\,x_2 - y_6{}^2\,x_1{}^2 + y_6{}^2\,x_5{}^2 \\
&- y_4{}^2\,x_3{}^2 + 4\,y_3\,y_4\,x_6\,x_1 - y_6{}^2\,x_2{}^2 + 4\,y_3\,y_5\,x_3\,x_5 \\
&+ 4\,y_3\,y_6\,x_7\,x_2 + y_6{}^2\,x_4{}^2 + 4\,y_2\,y_7\,x_1\,x_4 + y_6{}^2\,x_6{}^2 \\
&+ y_6{}^2\,x_7{}^2 - 4\,y_3\,y_6\,x_1\,x_4 + 4\,y_3\,y_6\,x_3\,x_6 - y_6{}^2\,x_3{}^2 \\
&+ y_5{}^2\,x_4{}^2 + y_5{}^2\,x_6{}^2 + y_5{}^2\,x_5{}^2 + 4\,y_3\,y_5\,x_1\,x_7 + y_5{}^2\,x_7{}^2 \\
&- 4\,y_1\,y_4\,x_3\,x_6 + y_2{}^2\,x_3{}^2 - y_3{}^2\,x_7{}^2 - 4\,y_1\,y_7\,x_2\,x_4 \\
&+ 4\,y_1\,y_7\,x_3\,x_5 - y_2{}^2\,x_7{}^2 - 4\,y_1\,y_5\,x_6\,x_2 - 4\,y_3\,y_7\,x_1\,x_5 \\
&- y_5{}^2\,x_2{}^2 - y_5{}^2\,x_3{}^2 + 4\,y_0\,y_5\,x_2\,x_7 - 4\,y_2\,y_6\,x_5\,x_1 \\
&+ 4\,y_2\,y_6\,x_6\,x_2 + y_3{}^2\,x_1{}^2 - y_3{}^2\,x_5{}^2 + 4\,y_2\,y_5\,x_5\,x_2 \\
&+ y_7{}^2\,x_7{}^2 - y_7{}^2\,x_2{}^2 + y_4{}^2\,x_5{}^2 - y_5{}^2\,x_1{}^2 + y_7{}^2\,x_5{}^2
\end{aligned}
$$

$$
\begin{aligned}
&+ 4\, y_2\, y_7\, x_3\, x_6 - {y_7}^2\, {x_3}^2 + 4\, y_2\, y_7\, x_7\, x_2 + 4\, y_2\, y_5\, x_6\, x_1 \\
&- 4\, y_2\, y_5\, x_3\, x_4 - 4\, y_2\, y_6\, x_3\, x_7 - 4\, y_2\, y_4\, x_7\, x_1 + {y_4}^2\, {x_7}^2 \\
&- 4\, x_0\, y_1\, y_6\, x_7 - 4\, x_0\, y_3\, y_7\, x_4 + 4\, x_0\, y_2\, y_5\, x_7 \\
&- 4\, x_0\, y_3\, y_5\, x_6 + {x_0}^2\, {y_0}^2 + {x_0}^2\, {y_3}^2 + {x_0}^2\, {y_1}^2 - {x_0}^2\, {y_6}^2 \\
&+ {x_0}^2\, {y_2}^2 - {x_0}^2\, {y_4}^2 - {x_0}^2\, {y_7}^2 - {x_0}^2\, {y_5}^2 + {y_4}^2\, {x_6}^2 \\
&+ {y_4}^2\, {x_4}^2 - {y_4}^2\, {x_2}^2 + 4\, y_2\, y_4\, x_3\, x_5 + 4\, y_2\, y_4\, x_2\, x_4 \\
&- {y_4}^2\, {x_1}^2 + {y_3}^2\, {x_2}^2 + {y_3}^2\, {x_3}^2 - 4\, x_0\, y_2\, y_7\, x_5 \\
&+ 4\, x_0\, y_2\, y_4\, x_6 - 4\, x_0\, y_2\, y_6\, x_4 + 4\, x_0\, y_3\, y_6\, x_5 \\
&- 4\, x_0\, y_0\, y_7\, x_7 - 4\, x_0\, y_1\, y_5\, x_4 - 4\, x_0\, y_0\, y_5\, x_5 \\
&+ 4\, x_0\, y_3\, y_4\, x_7 - 4\, x_0\, y_0\, y_4\, x_4 + 4\, x_0\, y_1\, y_7\, x_6 \\
&- 4\, x_0\, y_0\, y_6\, x_6 + 4\, x_0\, y_1\, y_4\, x_5)\ / \\
&(({y_0}^2 + {y_1}^2 + {y_2}^2 + {y_3}^2 + {y_4}^2 + {y_5}^2 + {y_6}^2 + {y_7}^2) \\
&({x_0}^2 + {x_5}^2 + {x_3}^2 + {x_1}^2 + {x_2}^2 + {x_7}^2 + {x_6}^2 + {x_4}^2))
\end{aligned}
$$

$$
\begin{aligned}
f^{(++)}_{127}\,(\varphi,\lambda) =& -2(y_0\, y_4\, {x_5}^2 - 2y_0\, y_6 x_1 x_3 - 2y_0\, y_5 x_4 x_5 \\
&+ 2\, y_0\, y_5\, x_6\, x_7 + 2\, y_0\, y_5\, x_2\, x_3 + {y_1}^2\, x_6\, x_2 + {y_1}^2\, x_5\, x_{11}\, x_3 - 2y_0\, y_5\, x_4\, x_5 \\
&- 2\, y_1\, y_6\, x_1\, x_2 - y_1\, y_5\, {x_3}^2 - y_0\, y_4\, {x_1}^2 + y_0\, y_4\, {x_3}^2 \\
&- 2\, y_1\, y_6\, x_4\, x_7 + 2\, y_1\, y_6\, x_5\, x_6 + 2\, y_1\, y_4\, x_4\, x_5 \\
&+ 2\, y_1\, y_3\, x_1\, x_7 + 2\, y_1\, y_4\, x_6\, x_7 + 2\, y_1\, y_4\, x_2\, x_3 \\
&- {y_1}^2\, x_3\, x_7 + 2\, y_2\, y_5\, x_5\, x_6 - {y_3}^2\, x_6\, x_2 - {y_3}^2\, x_1\, x_5 \\
&+ 2\, y_1\, y_3\, x_3\, x_5 - 2\, y_1\, y_3\, x_2\, x_4 - y_1\, y_5\, {x_1}^2 + y_1\, y_5\, {x_5}^2 \\
&+ y_1\, y_5\, {x_7}^2 + y_2\, y_6\, {x_6}^2 - y_2\, y_6\, {x_4}^2 - y_1\, y_5\, {x_6}^2 \\
&- y_1\, y_5\, {x_4}^2 + y_1\, y_5\, {x_2}^2 - {y_0}^2\, x_3\, x_7 + {y_0}^2\, x_5\, x_1 \\
&+ {y_0}^2\, x_6\, x_2 - 2\, y_0\, y_6\, x_4\, x_6 - 2\, y_0\, y_6\, x_5\, x_7 - y_0\, y_4\, {x_2}^2 \\
&+ {y_2}^2\, x_6\, x_2 + y_2\, y_6\, {x_7}^2 - y_2\, y_6\, {x_2}^2 - y_2\, y_6\, {x_5}^2 \\
&- y_2\, y_6\, {x_3}^2 - y_0\, y_4\, {x_7}^2 + y_0\, y_4\, {x_6}^2 - y_0\, y_4\, {x_4}^2 \\
&+ {y_2}^2\, x_5\, x_1 - {y_2}^2\, x_3\, x_7 + 2\, y_0\, y_3\, x_6\, x_1 + 2\, y_2\, y_3\, x_1\, x_4 \\
&- 2\, y_0\, y_3\, x_3\, x_4 - 2\, y_0\, y_3\, x_5\, x_2 - 2\, y_2\, y_4\, x_5\, x_7 \\
&+ 2\, y_2\, y_4\, x_4\, x_6 - 2\, y_2\, y_4\, x_1\, x_3 - 2\, y_4\, y_7\, x_6\, x_1 \\
&+ 2\, y_4\, y_7\, x_2\, x_5 - 2\, y_5\, y_7\, x_7\, x_1 - 2\, y_5\, y_7\, x_3\, x_5 \\
&- 2\, y_5\, y_7\, x_2\, x_4 - 2\, y_6\, y_7\, x_7\, x_2 + 2\, y_6\, y_7\, x_1\, x_4 \\
&+ {x_0}^2\, y_0\, y_4 + 2\, y_2\, y_3\, x_7\, x_2 + 2\, y_2\, y_3\, x_3\, x_6 + x_0\, {y_7}^2\, x_4 \\
&- x_0\, {y_3}^2\, x_4 - x_0\, {y_4}^2\, x_4 + x_0\, {y_2}^2\, x_4 + x_0\, {y_0}^2\, x_4 \\
&- x_0\, {y_5}^2\, x_4 - 2\, y_2\, y_5\, x_1\, x_2 + 2\, y_2\, y_5\, x_4\, x_7
\end{aligned}
$$

$$
\begin{aligned}
&- 2\, y_4\, y_7\, x_3\, x_4 + y_2\, y_6\, {x_1}^2 + {y_3}^2\, x_3\, x_7 + y_3\, y_7\, {x_4}^2 \\
&+ y_3\, y_7\, {x_6}^2 - y_3\, y_7\, {x_2}^2 + y_3\, y_7\, {x_5}^2 - y_3\, y_7\, {x_3}^2 \\
&- y_3\, y_7\, {x_1}^2 + y_3\, y_7\, {x_7}^2 - {y_4}^2\, x_6\, x_2 + {y_4}^2\, x_7\, x_3 \\
&- {y_4}^2\, x_1\, x_5 - {y_5}^2\, x_6\, x_2 - {y_5}^2\, x_5\, x_1 + {y_5}^2\, x_3\, x_7 \\
&+ {y_7}^2\, x_6\, x_2 + {y_7}^2\, x_5\, x_1 - {y_6}^2\, x_5\, x_1 - {y_6}^2\, x_2\, x_6 \\
&+ {y_6}^2\, x_3\, x_7 - {y_7}^2\, x_3\, x_7 + {x_0}^2\, y_2\, y_6 - 2\, y_6\, y_7\, x_3\, x_6 \\
&- {x_0}^2\, y_3\, y_7 + {x_0}^2\, y_1\, y_5 - x_0\, {y_6}^2\, x_4 + x_0\, {y_1}^2\, x_4 \\
&- 2\, x_0\, y_0\, y_3\, x_7 - 2\, x_0\, y_4\, y_7\, x_7 - 2\, x_0\, y_1\, y_6\, x_3 \\
&- 2\, x_0\, y_2\, y_3\, x_5 + 2\, x_0\, y_5\, y_7\, x_6 + 2\, x_0\, y_0\, y_5\, x_1 \\
&+ 2\, x_0\, y_0\, y_6\, x_2 - 2\, x_0\, y_2\, y_4\, x_2 + 2\, x_0\, y_1\, y_3\, x_6 \\
&+ 2\, x_0\, y_2\, y_5\, x_3 - 2\, x_0\, y_6\, y_7\, x_5 - 2\, x_0\, y_1\, y_4\, x_1)\ / \\
&(({y_0}^2 + {y_1}^2 + {y_2}^2 + {y_3}^2 + {y_4}^2 + {y_5}^2 + {y_6}^2 + {y_7}^2) \\
&({x_0}^2 + {x_5}^2 + {x_3}^2 + {x_1}^2 + {x_2}^2 + {x_7}^2 + {x_6}^2 + {x_4}^2))
\end{aligned}
$$

# Bibliography


[1] W. R. Hamilton, *Elements of Quaternions* (Chelsea Publishing Co., New York, 1969).

[2] J. T. Graves, *Mathematical Papers* (1843).

[3] A. Cayley, Phil. Mag. (London) **26**, 210 (1845). A. Cayley, *Papers Collected Mathematical Papers* (Cambridge 1889).

[4] A. Hurwitz, Nachr. Gesell. Wiss. Göttingeg, Math. Phys. Kl., 309.
A. Hurwitz, *Mathematische Werke Band II, Zahlentheorie, Algebra und Geometrie*, pag. 565 (Birkhaüser, Basel, 1933).

[5] J. F. Adams, Ann. Math. 75 (1962) 603.

[6] D. Husemoller, Fibre Bundles, Mc-Graw-Hill Book Company, 1966.

[7] M. F. Atiyah, R. Bott and A. Shapiro, Topology **3** (Suppl. 1) (1964) 3.

[8] J. Milnor, Ann. Math. 64 (1956) 399.

[9] Yamagishi, Phys. Lett. 134B (1984) 47.

[10] P. Jordan, J Von Neumann and E. Wigner, Ann. of Math. 35 (1934) 29.

[11] R. E. Behrends, J. Dreitlein, C. Fronsdal and B.W. Lee, Rev. Mod. Phys. 34 (1962) 1.
D.R. Speiser and J. Tarski J. Math. Phys. 4 (1963) 588.

[12] J. Souriau and D. Kastler, C. R. Acad. Sci. Paris 250 (1960) 2807.

[13] A. Pais, Phys. Rev. Lett. 7 (1961) 291; J. Math. Phys. 3 (1962) 1135.

[14] J. Tiomno, in "Theoretical Physics", (Tireste Simanr 1962), Vienna 1963, pp 251–264.

[15] A. Gamba, J. Math. Phys. 8 (1967) 775.

[16] R. Penny, Nouvo Cimento B3 (1971) 95.

[17] B. d'Espagnet, Group Theoretical Methods Proc. 1962 Int. Conf. on High Energy Physics, CERN, Geneva, 1962, pp 917.
Y. Ne'eman, Phys. Lett. 4 (1963) 81, Phys. Rev. Lett. 13 (1964) 769.
Y. Ne'eman and I. Ozwath, Pys. Rev. B138 (1965) 1474.
F. Gursey, Ann. of Phys. 12 (1961) 91.
D. Horn and Y.Ne'eman, Nouvo Cimento 31 (1964) 879.
M. Gourdin, Nouvo Cimento 30 (1963) 587.
G. Cocho and B. El mesen, Rev. mixic. fis. 13 (1964) 7.
G. Cocho, Phys. Rev. B137 (1965) 1255.
P. Gueret, J. P. Vigier and W. Tait, Nouvo Cimento A17 (1973) 663.

[18] Y. Nambu, Phys. Rev. D7 (1973) 2405.

[19] P. G. O. Freund, Phys. Rev. D13 (1976) 2322.

[20] F. Vollendorf, Z. Naturforsch. 30 (1975) 642; Z. Naturforsch. 30 (1975) 891.

[21] M. Gunaydin and F. Gursey, J. Math. Phys. 14 (1973) 1651.

[22] M. Gunaydin and F. Gursey, Phys. Rev. D9 (1974) 3387.

[23] Birkhoff and Van Neuman, Ann. of Math. 77 (1936) 823.

[24] D. Finkelstein, J. M. Jauch, S. Schiminovitch and D. Speiser, J. Math. Phys. 3 (1962) 207, Helv. Phys. Acta 35 (1962) 328.
D. Finkelstein, J. M. Jauch, D. Speiser, J. Math. Phys. 4 (1963) 136.
G. G. Emch, Helv. Phys. Acta. 36 (1963) 739, 770.

[25] S. Adler, Quaternionic quantum mechanics and quantum fields, New York, NY., Oxford University Press, 1995.

[26] H. H. Goldstein and L. P. Horwitz, Math. Ann. 154 (1964) 1; 164 (1966) 316.
L. P. Horwitz and L. C. Biedenharn, Helv. Phys. Acta 38 (1965) 385.

[27] F. Gursey,"Exceptional Groups And Elementary Particles. (Talk)," *In Nijmegen 1975, Proceedings, Group Theoretical Methods In Physics, Berlin 1976, 225-233.*

[28] F. Gursey, P. Ramond and P. Sikivie, Phys. Lett. B60 (1976) 177. P. Sikivie and F. Gursey, Phys. Rev. D16 (1977) 816.

[29] M. Gunaydin, Nuovo Cimento 29A (1975) 467.

[30] M. J. Duff, B. E. Nilsson and C. N. Pope, Phys. Rept. 130 (1986) 1.

[31] F. Englert, Phys. Lett. Phys. Lett. B119 (1982) 339.

[32] M. Rooman, Nucl. Phys. **B236** (1984) 501.

[33] F. Gursey and C. Tze, Phys. Lett. B127 (1983) 191.

[34] T. Dereli, M. Panahimoghaddam, A. Sudbery and R. W. Tucker, Phys. Lett. B126 (1983) 33.

[35] B. de Wit and H. Nicolai, Nucl. Phys. B231 (1984) 506.

[36] M. J. Duff, R. R. Khuri and J. X. Lu, Phys. Rep. **259** (1995) 213, J. A. Harvey and A. Strominger, Phys. Rev. Lett. **66** (1991) 549, M. J. Duff, J. M. Evans, R. R. Khuri, J. X. Lu, R. Minasian, "The Octonionic Membrane", hep-th/9706124, B. S. Acharya, J. M. Figueroa-O'Farrill, M. O'Loughlin, and B. Spence, "Euclidean D-branes and higher dimensional gauge theory", hep-th/9707118, B. S. Acharya, M. O'Loughlin, and B. Spence, Nucl. Phys. **503B** (1997), 657, L. Baulieu, H. Kanno, and I. M. Singer, "Cohomological Yang-Mills theory in eight dimensions", hep-th/9705127, L. Baulieu, H. Kanno, and I. M. Singer, "Special quantum field theories in eight and other dimensions", hep-th/9704167, J. M. Figueroa-O'Farrill, C. Köhl, and B. Spence, "Supersymmetric Yang-Mills, octonionic instantons and triholomorphic curves", hep-th/9710082, G. Gibbons, G. Papadopoulos, and K. Stelle, "HKT and OKT geometries on soliton black hole moduli spaces", hep-th/9706207, C. M. Hull, "Higher dimensional Yang-Mills theories and topological terms", hep-th/9710165, J. M. Figueroa-O'Farrill, "Gauge theory and the Division Algebras", hep-th/9710168.

[37] M.B. Green, J.H. Schwarz and E. Witten, Superstring theory, Cambridge, University Press, 1987.

[38] J. Polchinski, String theory, Cambridge, University Press, 1998.

[39] A. A. Belavin, A. M. Polyakov, A. S. Schwarz and Y. Tyupkin, Phys. Lett. **59B** (1975) 85.

[40] G. 't Hooft, Phys. Rev. **14D** (1976) 3432.

[41] L. Brink, J. Scherk and J. H. Schwarz, Nucl. Phys. **121B** (1977) 77.

[42] F. Englert, A. Servin, W. Troost, A. Van Proeyen and Ph. Spindel, J. Math. Phys. **29** (1988) 281.

[43] M. Sohnius, Z. Phys. **C18** (1983) 229.

[44] M. J. Duff, R. R. Khuri and J. X. Lu, Phys. Rep. **259** (1995) 213.

[45] B. Grossman, T.W. Kephart and J.D. Stasheff, Commun. Math. Phys. **96** (1984) 431.
B. Grossman, T.W. Kephart and J.D. Stasheff, Phys. Lett. **B220** (1989) 431

[46] R. D. Schafer, An introduction to non-associative algebrae (Academic Press, New York, 1966).

[47] R. Gilmore, Lie groups, Lie algebras, and some of their applications, New York, Wiley, 1974.

[48] K. Yano and M. Kon, Structures on Manifolds, Series in Pure Mathematics - volume 3, (World Scientific 1984).

[49] D. V. Alekseevsky and S. Marchiafava, Annali di Matematica Pura ed Applicata (IV), CLXXI (1996) 205.

[50] P. Rotelli, Mod. Phys. Lett. A., **4** (1989) 933.

[51] S. De Leo and P. Rotelli, Prog. Theor. Phys. 92 (1994) 917, J.Phys. G22 (1996) 1137.

[52] K. Morita, Prog.Theor.Phys. 67 (1982) 1860, Prog.Theor.Phys. 68 (1982) 2159.

[53] A. W. Conway, Proc. Irish Academy, vol xxix, Section A, No. 1, (read feb. 1911).

[54] L. Sillberstein, Phil. Mag. S. 6, Vol. 23 No 137 (May 1912), 790.

[55] J. L. Synge, "Quaternions, Lorentz Transformations and the Conway–Dirac–Eddington Matrices", (Dublin, Dublin Institute for Advanced Studies, 1972).

[56] S. De Leo, Int. J. Mod. Phys. A11 (1996) 3973.

[57] K. Abdel-Khalek, Int. J. Mod. Phys. A13 (1998) 569.

[58] Maple, Maple is a registered trademark of waterloo Maple, Inc.

[59] S. De Leo and K. Abdel-Khalek, J. Math. Phys., 38 (1997) 582.

[60] S. De Leo and K. Abdel-Khalek, Prog. Theor. Phys. 96 (1996) 833.

[61] G. Dixon, BRX TH-372, hep-th/9503053.

[62] K. Abdel-Khalek, Int. J. Mod. Phys. A13 (1998) 223.

[63] K. B.Marathe and G. Martucci, The mathematical foundations of gauge theories, Amsterdam, North-Holland, 1992.

[64] P. Budinich, Commun. Math. Phys. 107 (1986) 455.

[65] E. Cartan and J. A. Schouten, Proc. Kon. Akad. Wet. Amesterdam 29 (1926) 803, 933.

[66] E. Cartan, J. Math. Pures et Appl. 6 (1927) 1.

[67] J. A. Wolf, J. Diff. Geom. **6** (1972) 317.

[68] M. Cederwall and C. R.Preitschopf, Comm. Math. Phys. **167** (1995) 373.

[69] J. Lukierski and P. Minnaert, Phys. Lett. **129B** (1983) 392.

[70] M. Gunaydin and H. Nicolai, Phys. Lett. **351B** (1995) 169.

[71] M. Gunaydin and S.V. Ketov, Nucl. Phys. **B467** (1996) 215.

[72] J. Schray and C.A. Manogue, hep-th/9407179.

[73] C.A. Manogue and J. Schray, J. Math. Phys. **34** (1993) 3746 hep-th/9302044.

[74] R.S. Ward, Nucl. Phys. **B236** (1984) 381.

[75] S. Fubini and H. Nicolai, Phys. Lett. **155B** (1985) 369.

[76] E. Corrigan, C. Devchand, D.B. Fairlie and J. Nuyts, Nucl. Phys. **B214** (1983) 452.

[77] E. Corrigan, C. Devchand, D. B. Fairlie and J. Nuyts, Nucl. Phys. **214B** (1983) 452.

[78] D. B. Fairlie and J. Nuyts, J. Phys. **A17** (1984) 2867.

[79] D.B. Fairlie and J. Nuyts, J. Math. Phys. **25** (1984) 2025.

[80] F.A. Bais and P. Batenburg, Nucl. Phys. **B269** (1986) 363.

[81] T. A. Ivanova and A. D. Popov, Lett. Math. Phys. **24** (1992) 85.

[82] T. A. Ivanova, Phys.Lett. **315B** (1993) 277.

[83] G. Landi, Lett. Math. Phys. **11** (1986) 171.

[84] C. Nash and S. Sen, Topology and geometry for physicists, London, Academic Press 1983.

[85] M. .F. Atiyah, V. G. Drinfield, N. J. Hitchin, and Yu. I. Manin, Phys. Lett. **65A** (1978) 185.

[86] J. H. Schwarz, "Introduction To Supersymmetry", Presented at 28th Scottish Universities Summer School in Physics, Edinburgh, Scotland, Jul 28 - Aug 17, 1985.

[87] T. Kugo and P. Townsend, Nucl. Phys. **B221** (1983) 357.

[88] K. W. Chung and A. Sudbery, Phys. Lett. **B198** (1987) 161.

[89] J. M. Evans, Nucl. Phys. **B298** (1988) 92.

[90] J. M. Evans, Nucl. Phys. **B310** (1988) 44 .

[91] N. Berkovits, Phys. Lett. **B318** (1993) 104.

[92] J. M. Evans, Phys. Lett. **B334** (1994) 105. J. M. Evans, STRINGS AND SYMMETRIES, Edited by Gulen Aktas et al, Springer-Verlag (Lecture Notes in Physics, 447) 1995.

[93] I. Bengtsson, M. Cederwall, Nucl. Phys. **B302**, (1988) 81.

[94] H. Tachibana, K. Imaeda, Nuovo Cimento **B104**, (1989) 91. D. B. Fairlie and C.A. Manogue, Phys. Rev. **D36** (1987) 475, C.A. Manogue, A. Sudbery, Phys. Rev. **D40** (1989) 4073. I. Oda, T. Kimura, A. Nakamura, Prog. Theor. Phys. **80** (1988) 367. M. Cederwall, Phys. Lett. **B210** (1988) 169, E. Corrigan, T.J. Hollowood, Commun. Math. Phys. **122** (1989) 393, E. Corrigan, T.J. Hollowood, Phys. Lett. **B203**, (1988) 47, R. Foot, G.C. Joshi, Int. J. Theor. Phys. **28** (1989) 1449, R. Foot, G.C. Joshi, Phys. Lett. **B199**, (1987) 203.

[95] M.P. Blencowe and M.J. Duff, Nucl. Phys. **B310** (1988) 387.

[96] B. E. Nilsson, "Off-Shell Fields For The Ten-Dimensional Supersymmetric Yang-Mills Theory," GOTEBORG-81-6.

[97] B. E. Nilsson, Class. Quant. Grav. **3** (1986) L41.

[98] P. Deligne, D. S. Freed, Supersolutions, hep-th/9901094.

[99] S. J. Gates, M. T.Grisaru, M. Rocek and W. Siegel, Superspace: or one thousand and one lessons in supersymmetry, London, Benjamin/Cummings, 1983.

[100] N. Berkovits, Phys. Lett. **241B** (1990) 497.

[101] N. Berkovits, Nucl. Phys. **358B** (1991) 169.

[102] L. Brink, M. Cederwall and C. R. Preitschopf, Phys. Lett. **311B** (1993) 76.

[103] J. A. H. Samtleben, Nucl. Phys. **453B** (95) 429.