On $SU(2)$ Yang-Mills Theory and the Maximal Abelian Gauge

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Abstract

The problem of identifying the dynamical degrees of freedom for $SU(2)$ gauge theory is discussed. After studying $SU(N)$ theories, it is shown that classical pure $SU(2)$ gauge theory is equivalent to an abelian theory. Finally, we prove that in the Euclidean formulation, the Maximal Abelian Gauge correctly identifies the abelian degrees of freedom of the $SU(2)$ theory.

Gauge field theories play a major role in High Energy Physics. The dynamics of gauge theories has been investigated mainly with perturbative methods. However, not all physical phenomena can be explained within perturbation theory. For example, the understanding of quark confinement in hadrons requires non-perturbative techniques applied to Quantum Chromodynamics (QCD). To explore the non-perturbative regimes one would like to identify the relevant dynamical degrees of freedom and to solve the field equations. For QCD the gluon degrees of freedom are, typically, described as copies of the QED photon field. This language seems to be useful to understand the high energy limit of QCD, but not for its low energy regime.

Recently, a number of authors [1, 2, 3] addressed the problem of the relevant degrees of freedom for $SU(2)$ Yang-Mills theory. A parameterization of the gauge fields was proposed and effective actions in terms of scalar fields derived. Furthermore, a number of results have been extended to $SU(N)$ theories [4, 5, 6].

For $SU(2)$, in [1] Faddeev and Niemi proposed a Skyrme like effective action to describe the low energy limit, written in terms of what they assumed to be the relevant degrees of freedom. Their work was inspired on a suggestion from 'tHooft and Polyakov [7, 8], that tried to describe color confinement as a dual Meissner effect, and on a generalization of the Wu-Yang ansatz [9]. Along the same lines, in [4] they parameterized the gluon fields[1] for an $SU(N)$ gauge theory and wrote a generalized Skyrme effective action for $SU(N)$ Yang-Mills theory.

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[1] See also [5].

In [2, 10] Cho wrote a parameterization of the gauge fields for $SU(2)$ in terms of a covariant constant scalar field in the adjoint representation and a photon-like field. Then, he proved abelian dominance for Wilson loops. Meanwhile, and within the philosophy of the work of Faddeev and Niemi, in [11] another generalized Skyrme-Faddeev action was derived as an effective action for the low energy limit of $SU(2)$ gauge theory.

For $SU(N)$ theories, Li *et al.* [6] used the technique suggested in [2] to write a general parameterization for the gluon fields from which they built an effective action.

So far the main work has been centered on building effective actions describing the low energy limit of $SU(N)$ gauge theories. Presently, a number of slightly different effective actions has been derived. It would be interesting to investigate how they are related to each other. In this paper we do not seek to discuss effective actions. Instead we look for possible ways of identifying the dynamical degrees of freedom via gauge fixing of the original theory.

The paper is organized as follows. We use the procedure of [2] to write the non-abelian field A_μ^a for $SU(N)$ theories. In particular, for $SU(2)$ we built a general parameterization of the gauge field. It follows that the dynamics of classical pure $SU(2)$ maps the dynamics of an abelian theory. Finally, we prove that, in the Euclidean formulation, the Maximal Abelian Gauge reduces the gluon field A_μ^a to the photon field of the hidden abelian theory and comment on possible implications for lattice simulations.

1 Gluon Fields for $SU(N)$ Gauge Theories

The lagrangian for $SU(N)$ gauge theories reads

$$\mathcal{L} = -\frac{1}{4}\, F_{\mu\nu}^a\, F^{a\,\mu\nu} \tag{1}$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f_{abc} A_\mu^b A_\nu^c\, ; \tag{2}$$

A_μ^a are the gluon fields.

Let n^a be a covariant constant real scalar field in the adjoint representation. From the definition it follows that

$$D_\mu n^a = \partial_\mu n^a + ig\left(F^b\right)_{ac} A_\mu^b n^c = 0\,; \tag{3}$$

the generators of the adjoint representation are defined in the usual way $\left(F^b\right)_{ac} = -i\, f_{bac}$. Given a gluon field it is always possible to solve the above equations for the scalar field n^a. In this way it is possible to define a map from the gluon field to n. Let us reverse the argument. Multiplying (3) by $\left(F^d\right)_{ea} n^d$ and solving the equations for the gauge fields we get, after some algebra,

$$
\begin{aligned}
&\left(F^d\right)_{ea} n^d\, \partial_\mu n^a + ig\left(n \cdot A_\mu\right) n^e - ig\left(n \cdot n\right) A_\mu^e + \\
&\left[\left(D^b D^e\right)_{dc} n^d n^c - \left(D^d D^c\right)_{ed} n^d n^c\right] A_\mu^b = 0\,,
\end{aligned}
\tag{4}
$$

where

$$\left(D^a\right)_{bc} = d_{abc} = \frac{1}{4}\mathrm{Tr}\left(\lambda^a \left\{\lambda^b, \lambda^c\right\}\right) \tag{5}$$

and λ^a are the Gell-Mann matrices. Writing the gauge fields as

$$A_\mu^a = \hat{A}_\mu n^a + X_\mu^a, \tag{6}$$

equations (4) become

$$\left(F^d\right)_{ea} n^d\, \partial_\mu n^a + ig\,(n \cdot X_\mu)\, n^e - ig\,(n \cdot n) X_\mu^e +$$
$$(d_{dba}\, d_{ace} - d_{dca}\, d_{aeb})\, n^d\, n^c X_\mu^b = 0. \tag{7}$$

By a convenient definition of $\hat{A}_\mu$ it follows, without loss of generality, that

$$X_\mu^a = \frac{1}{ig}\left(F^d\right)_{ea} \frac{n^d\, \partial_\mu n^a}{n \cdot n} + Y_\mu^a \tag{8}$$

$$n \cdot Y_\mu = 0. \tag{9}$$

In terms of $\hat{A}_\mu$, n^a and Y_μ^a, the gauge fields are given by

$$A_\mu^a = \hat{A}_\mu n^a + \frac{1}{ig}\left(F^c\right)_{ab} \frac{n^c\, \partial_\mu n^b}{n \cdot n} + Y_\mu^a\,; \tag{10}$$

with n and Y verifying the constraints

$$n \cdot Y_\mu = 0\,, \tag{11}$$

$$D_\mu n^a = 0. \tag{12}$$

If (10) together with (11) and (12) provide a complete parameterization of the gluon fields, the total number of independent fields on both sides of (10) should be the same. On the l.h.s., the number of degrees of freedom is $2(N^2 - 1)$. For the r.h.s., depending on how you perform the counting[2] it can be made as large as $2(N^2 - 1)$. Let us consider $SU(2)$ for simplicity. For the r.h.s., the number of d.o.f being 3 from n, 2 (3) for a massless (massive) $\hat{A}_\mu$ and possible additional d.o.f. coming from Y_μ^a. Despite the constraints, easily one arrives at a number of 6 independent fields.

Now, let us look at the gauge transformation properties of n, Y and $\hat{A}$. Since n is covariant constant, it follows that $-i\left(F^c\right)_{ab} n^c\, \partial_\mu n^b /(n \cdot n)$ belongs to the adjoint representation of the gauge group. Demanding that Y is also in the adjoint representation, then

$$\hat{A}_\mu \longrightarrow \hat{A}_\mu + \frac{1}{g} \frac{n \cdot \partial_\mu \omega}{n \cdot n}\,. \tag{13}$$

Constraints (11) and (12) are scalars under the gauge group and the parameterization (10) provides a complete gauge invariant decomposition of the gluon field A_μ^a.

For $SU(N)$ gauge theories it follows from (3) and (10) that

$$n\partial_\mu n = \frac{1}{2}\, \partial_\mu n^2 = 0, \tag{14}$$

and one can always choose $n^2 = 1$.

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[2]At this point one should alert the reader that the fields n, $\hat{A}_\mu$ and Y_μ^a should not be regarded as free fields and it is not clear that the usual naive counting provides the correct answer.

2 $SU(2)$ Gauge Theory

In this section we are going to study the implications of (10), (11) and (12) for $SU(2)$ gauge theory. The scalar field n can be parametrized by the functions θ_1 and θ_2 as

$$n = \begin{pmatrix} \sin\theta_1 \, \cos\theta_2 \\ \sin\theta_1 \, \sin\theta_2 \\ \cos\theta_1 \end{pmatrix}. \tag{15}$$

The field Y is orthogonal to n and can be parametrized in terms of scalar fields belonging to the adjoint representation of the gauge group as (see the appendix for definitions)

$$Y_\mu^a = B_\mu m^a + C_\mu p^a, \tag{16}$$

where B_μ and C_μ are gauge invariant vector fields. The constraint (3) requires

$$B_\mu = C_\mu = 0 \, ; \tag{17}$$

note that the second term in (10) generates components along p and m directions. Then, the gluon field becomes

$$
\begin{aligned}
A_\mu^1 &= \hat{A}_\mu \, \sin\theta_1 \, \cos\theta_2 - \frac{1}{g} \sin\theta_1 \, \cos\theta_1 \, \cos\theta_2 \, \partial_\mu\theta_2 - \frac{1}{g} \sin\theta_2 \, \partial_\mu\theta_1 \,, \\
A_\mu^2 &= \hat{A}_\mu \, \sin\theta_1 \, \sin\theta_2 - \frac{1}{g} \sin\theta_1 \, \cos\theta_1 \, \sin\theta_2 \, \partial_\mu\theta_2 + \frac{1}{g} \cos\theta_2 \, \partial_\mu\theta_1 \,, \\
A_\mu^3 &= \hat{A}_\mu \, \cos\theta_1 + \frac{1}{g} \sin^2\theta_1 \, \partial_\mu\theta_2 \,,
\end{aligned}
\tag{18}
$$

the gluon field tensor

$$F_{\mu\nu}^a = n^a \, \mathcal{F}_{\mu\nu} \tag{19}$$

with

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu \tag{20}$$

and

$$\mathcal{A}_\mu = \hat{A}_\mu - \frac{1}{g} \cos\theta_1 \, (\partial_\mu\theta_2) \,. \tag{21}$$

The action is

$$S = -\frac{1}{4} \int d^4x \, \mathcal{F}^2 \tag{22}$$

and the classical equations of motion

$$n^a \, \partial_\nu \mathcal{F}^{\mu\nu} = 0 \,. \tag{23}$$

Ignoring the trivial solution, (23) shows that classical pure $SU(2)$ is equivalent to an abelian theory, with the abelian field given by (21). For the quantum theory, due to the nonlinear relation between the original gauge fields and $\mathcal{A}$,

$$
\begin{aligned}
A_\mu^1 &= \mathcal{A}_\mu \, \sin\theta_1 \, \cos\theta_2 - \frac{1}{g} \sin\theta_2 \, \partial_\mu\theta_1 \,, \\
A_\mu^2 &= \mathcal{A}_\mu \, \sin\theta_1 \, \sin\theta_2 + \frac{1}{g} \cos\theta_2 \, \partial_\mu\theta_1 \,, \\
A_\mu^3 &= \mathcal{A}_\mu \, \cos\theta_1 + \frac{1}{g} \partial_\mu\theta_2 \,,
\end{aligned}
\tag{24}
$$

the relation is not so simple.

If classical $SU(2)$ Yang-Mills theory is equivalent to an abelian theory, is it possible to keep only the abelian field by choosing a convenient gauge? The question can be answered positively in the Euclidean formulation.

From now on we will assume to be in the Euclidean formulation. In the maximal abelian gauge (MAG) we look for fields which maximize

$$
\begin{aligned}
R[A] &= -\int d^4x \left[\left(A_\mu^1\right)^2 + \left(A_\mu^2\right)^2 \right] \\
&= -\int d^4x \left[\left(\mathcal{A}_\mu\right)^2 \sin^2\theta_1 \sin^2\theta_2 + \frac{1}{g}\left(\partial\theta_1\right)^2 \right]
\end{aligned} \tag{25}
$$

along each gauge orbit. The absolute maximum of (25) is realized by two classes of fields : i) $\mathcal{A} = \partial^2\theta_1 = 0$ (type I) and ii) $\sin^2\theta_1 = 0$ (type II), both having $A_\mu^1 = A_\mu^2 = 0$. The solutions of type I are vacuum solutions ($F = 0$) and can be gauged to the null solution. For type II solutions the gluon field is

$$
A_\mu^3 = \pm\mathcal{A}_\mu + \frac{1}{g}\partial_\mu\theta_2 \tag{26}
$$

and by a gauge transformation [12] can be reduced (up to a sign) to the abelian field $\mathcal{A}$. Note that the type II solution is essentially the $\hat{A}_\mu$ field,

$$
A_\mu^3 = \begin{cases} \hat{A}_\mu + \frac{2}{g}\partial_\mu\theta_2\,, \\ \pm\hat{A}_\mu\,, \\ -\hat{A}_\mu + \frac{2}{g}\partial_\mu\theta_2\,. \end{cases} \tag{27}
$$

For what concerns the quantum theory, the jacobian relating A_μ^a and $\mathcal{A}_\mu$ is now unitary, i.e., in the Maximal Abelian Gauge, pure $SU(2)$ Yang-Mills theory is equivalent to an abelian theory.

The inclusion of fermions changes the relation between the two theories. In terms of the abelian field $\mathcal{A}$ the fermionic covariant derivative in the MAG is

$$
\left(D_\mu^a\right) = \begin{pmatrix} \partial_\mu \\ \partial_\mu \\ \partial_\mu + \frac{ig}{2}\sigma^3\mathcal{A} \end{pmatrix}. \tag{28}
$$

$SU(2)$ gauge theory is formally identical to strong coupled QED with two opposite charges per quark flavor. In principle, the quest for solutions of the classical theory looks much simpler in the MAG compared to the original formulation. Therefore, MAG opens a window to investigate the relevance of the different classical configurations to the characteristics of the quark dynamics [14].

The above results for the MAG can be combined with lattice simulations to test abelian dominance and, hopefully, understand the role of Gribov copies on different observables. Our analysis shows that condition (25) does not remove completely the gauge ambiguity. We have used the remaining freedom to gauge A_μ^3 to the abelian field. However, even after doing that we are left with two solutions $A_\mu^3 = \pm\mathcal{A}_\mu$ per space-time point, meaning that the number of Gribov copies is infinite.

On the lattice the situation is similar. If we are close to the continuum, one can write the link variables as

$$
\mathcal{U}_\mu(x) = \exp\left\{\frac{i}{2}ga\mathcal{A}_\mu\sigma^3\right\} = \begin{pmatrix} e^{\pm i\mathcal{A}_\mu} & 0 \\ 0 & e^{\mp i\mathcal{A}_\mu} \end{pmatrix}, \tag{29}
$$

i.e. the number of Gribov grows with lattice volume. However, knowing that the link should look like (29), one can test a given gauge fixing algorithm to check if it identifies correctly the absolute maximum of (25) and in this way study the influence of Gribov copies on the Green's functions of our theory [13].

3 Results and Conclusions

In this paper we argued that it is possible to write the gluon fields for $SU(N)$ gauge theories in terms of scalar and vector fields. Carefully chosen, these fields can simplify the classical equations of motion.

For $SU(2)$ Yang-Mills theory our choice of the parameterization shows that the classical theory is equivalent to an abelian theory. Furthermore, it was proved that in the MAG (Euclidean formulation) the quantum theory is equivalent to an abelian theory. This result shows abelian dominance for the $SU(2)$ theory. In what concerns Gribov copies, we found that in the MAG the number of copies is infinite. This is true also for lattice simulations. For lattice calculations we suggest a test to check if gauge fixing algorithms correctly identify the absolute maximum of the lattice version of (25).

Despite additional technical complications, our discussion for $SU(2)$ can be extended to $SU(3)$ [15] and other higher rank special unitary groups. From our study, naively, one expects to observe richer structure as the rank of the group is increased. Currently, we are involved in extending this work to QCD.

Appendix

In three dimensions one can define the following orthogonal unitary vectors

$$ n = \left(\begin{array}{c} \sin\theta_1 \, \cos\theta_2 \\ \sin\theta_1 \, \sin\theta_2 \\ \cos\theta_1 \end{array} \right) \, , \; m = \left(\begin{array}{c} \cos\theta_1 \, \cos\theta_2 \\ \cos\theta_1 \, \sin\theta_2 \\ -\sin\theta_1 \end{array} \right) \, , \; p = \left(\begin{array}{c} \sin\theta_2 \\ -\cos\theta_2 \\ 0 \end{array} \right) \, . \quad (30) $$

The unitary vectors n, m and p verify the relations

$$ \partial_\mu n = m \, (\partial_\mu \theta_1) - p \sin\theta_1 \, (\partial_\mu \theta_2) \, , \quad (31) $$
$$ \partial_\mu m = -n \, (\partial_\mu \theta_1) - p \cos\theta_1 \, (\partial_\mu \theta_2) \, , \quad (32) $$
$$ \partial_\mu p = (n \sin\theta_1 + m \cos\theta_1) \, (\partial_\mu \theta_2) \quad (33) $$

and

$$ \epsilon_{abc} \, n^b \, p^c = m^a \quad \text{and circular permutations.} \quad (34) $$

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