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Hybrid Superstrings in NS-NS Plane Waves

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Abstract

By using the hybrid formalism, superstrings in four-dimensional NS-NS plane waves are studied in a manifest supersymmetric manner. This description of the superstring is obtained by a field redefinition of the RNS worldsheet fields and defined as a topological $N = 4$ string theory. The Hilbert space consists of two types of representations describing short and long strings. We study the physical spectrum to find boson-fermion asymmetry in the massless spectrum of the short string. Some massive spectrum of the short string and the massless spectrum of the long string are also studied.

§1. Introduction

Plane-wave backgrounds are exact string vacua and give examples of string theories in non-compact curved space-times. These backgrounds are obtained by Penrose limit of AdS spaces and have attracted much attention to study the AdS/CFT duality beyond the supergravity approximation.[1)] Despite much progress in this subject, analyses do not have preserved whole supersymmetry of these backgrounds so far. The superstring in R-R plane waves has been quantized in the light-cone gauge[2)] or using a non-covariant formalism.[3)] Only a part of supersymmetry are linearly realized in these formulations. For superstrings in NS-NS plane waves, on the other hand, we can use the Ramond-Neveu-Schwarz (RNS) formalism for covariant quantization in which, however, the space-time supersymmetry is not manifest. This is not always necessary but desirable to make transparent any cancellations coming from the supersymmetry.

In this paper, we study the four-dimensional superstring in NS-NS plane waves in terms of the hybrid formalism which has been developed in Refs. 6) and 7), and applied to several compactified string theories.[8)] This description of the superstring is obtained by a field redefinition of the RNS worldsheet fields and manifestly preserve all isometries of the background including supersymmetry.

Strings in NS-NS plane waves are described by the Nappi-Witten model,[9)] which is the WZW model on the group manifold H_4 generated by the four-dimensional Heisenberg algebra:

$$\begin{aligned} &[\mathcal{J},\mathcal{P}] = \mathcal{P}, \qquad [\mathcal{J},\mathcal{P}^*] = -\mathcal{P}^*, \\ &[\mathcal{P},\mathcal{P}^*] = \mathcal{F}. \end{aligned} \tag{1·1}$$

The Hilbert space of the NW model consists of two distinct representations, discrete (type II and III) and continuous (type I). The model has a spectral flow symmetry and all flowed representations must be also included.[10),11)] The spectrally flowed discrete representations are viewed as describing short strings localized in the transverse space to the plane wave. The flowed continuous representations define long string states propagating in the whole four-dimensional space. These structure is similar to the spectrum of the string in AdS_3,[12)] which in fact obtained by its Penrose limit.[5)]

The RNS superstring in this background is expressed by superconformal field theories of the type $H_4 \times \mathcal{M}$, where H_4 denotes the super NW model and $\mathcal{M}$ is represented by an arbitrary $N = 2$ unitary superconformal field theory with $c = 9$.[5)] This $\mathcal{M}$ sector represents a Calabi-Yau compactification if we project into the integral sector of the $U(1)_R$ charge $I_{\mathcal{M},0}$. However, it was found in 5) that the string vacua have enhanced supersymmetry if we take a generalized GSO projection which restricts the *total* $U(1)_R$ charge to be integer, while fractional $I_{\mathcal{M},0}$ is allowed. We adopt this weak GSO projection throughout.

The hybrid superstrings in NS-NS plane waves are related to the RNS suprestrings by a field redefinition of worldsheet fields. We show how to perform a redefinition from RNS fields to hybrid fields making all the space-time supersymmetry manifest. The model can be formulated as a $N=4$ topological string theory.[7)]

Then we examine the physical spectrum at several lower mass levels. We find that the massless spectrum of the short string has boson-fermion asymmetry, which is allowed without breaking supersymmetry. There are two massless bosons without fermionic partners, which can be clarified due to the manifest supersymmetry. Some massive spectrum of the short string and the massless spectrum of the long string are also studied in a manifest supersymmetric manner.

This paper is organized as follows. In section 2 we begin with a brief review of the super-NW model in the RNS formalism. The space-time supercharges are given in the form satisfying the conventional supersymmetry algebra. Hybrid worldsheet fields are introduced in section 3 by a redefinition of RNS worldsheet fields. The model is reformulated as a topological $N=4$ string theory. In section 4, we construct the Hilbert space of the hybrid superstring by using hybrid fields, which consists of two sectors representing short and long strings respectively. The physical spectrum is studied in section 5. It is found that the massless spectrum of the short string has boson-fermion asymmetry. The results are summarized with some discussions in section 6.

§2. RNS superstrings in the NS-NS plane waves

The RNS superstrings propagating in the four-dimensional NS-NS plane waves is provided by superconformal field theories of the type $H_4 \times \mathcal{M}$.[4),5)] Here H_4 denotes the super-NW model described by the super WZW model on the four-dimensional Heisenberg group. The Hilbert space of this model is constructed by representations of the H_4 super current algebra[*)]

$$
\begin{aligned}
J(z)P(w) &\sim \frac{P(w)}{z-w}, & J(z)P^*(w) &\sim -\frac{P^*(w)}{z-w}, \\
P(z)P^*(w) &\sim \frac{1}{(z-w)^2} + \frac{F(w)}{z-w}, & J(z)F(w) &\sim \frac{1}{(z-w)^2}, \\
\psi_P(z)\psi_{P^*}(w) &\sim \frac{1}{z-w}, & \psi_J(z)\psi_F(w) &\sim \frac{1}{z-w},
\end{aligned}
$$

*) We refer only to the holomorphic sector in this paper. It can be easily combined with the anti-holomorphic sector if necessary.[5),10)–12)]

$$J(z)\psi_P(w) \sim \psi_J(z)P(w) \sim \frac{\psi_P(w)}{z-w}, \qquad J(z)\psi_{P^*}(w) \sim \psi_J(z)P^*(w) \sim -\frac{\psi_{P^*}(w)}{z-w},$$
$$P(z)\psi_{P^*}(w) \sim \psi_P(z)P^*(w) \sim \frac{\psi_F(w)}{z-w}. \tag{2·1}$$

Representations of this algebra are easily obtained by using a free-field realization[10)]

$$\begin{aligned} J &= i\partial X^-, & F &= i\partial X^+, \\ P &= e^{iX^+}(i\partial Z + \psi^+\psi), & P^* &= e^{-iX^+}(i\partial Z^* - \psi^+\psi^*), \\ \psi_F = \psi^+ \qquad \psi_J &= \psi^-, & \psi_P = e^{iX^+}\psi, \qquad \psi_{P^*} &= e^{-iX^+}\psi^*, \end{aligned} \tag{2·2}$$

where operator products of free fields are defined by

$$\begin{aligned} X^+(z)X^-(w) &\sim Z(z)Z^*(w) \sim -\log(z-w), \\ \psi^+(z)\psi^-(w) &\sim \psi(z)\psi^*(w) \sim \frac{1}{z-w}. \end{aligned} \tag{2·3}$$

The zero modes of bosonic currents provide generators of the space-time symmetry (1·1):

$$\begin{aligned} \mathcal{J} &= \oint \frac{dz}{2\pi i} i\partial X^-, & \mathcal{F} &= \oint \frac{dz}{2\pi i} i\partial X^+, \\ \mathcal{P} &= \oint \frac{dz}{2\pi i} e^{iX^+}(i\partial Z + \psi^+\psi), & \mathcal{P}^* &= \oint \frac{dz}{2\pi i} e^{-iX^+}(i\partial Z^* - \psi^+\psi^*). \end{aligned} \tag{2·4}$$

The $N=1$ worldsheet superconformal symmetry is actually enhanced to $N=2$ generated by

$$\begin{aligned} T_{H_4} &= -\partial X^+\partial X^- - \partial Z\partial Z^* - \frac{1}{2}\psi^+\partial\psi^- - \frac{1}{2}\psi^-\partial\psi^+ - \frac{1}{2}\psi\partial\psi^* - \frac{1}{2}\psi^*\partial\psi, \\ G^+_{H_4} &= \psi^+ i\partial X^- + \psi i\partial Z^*, \qquad G^-_{H_4} = \psi^- i\partial X^+ + \psi^* i\partial Z, \\ I_{H_4} &= \psi^+\psi^- + \psi\psi^*. \end{aligned} \tag{2·5}$$

The model has the central charge $c=6$, which is the same with the superstring in the flat four-dimensional space-time.

The $\mathcal{M}$ sector is represented by an arbitrary unitary representation of $N=2$ rational superconformal field theory with $c=9$. We denote generators of this $N=2$ superconformal symmetry by $(T_{\mathcal{M}}, G^{\pm}_{\mathcal{M}}, I_{\mathcal{M}})$.

In order to covariantly quantize the RNS superstring, fermionic ghosts (b,c) and bosonic ghosts (β,γ) must be introduced. These superconformal ghosts satisfy

$$c(z)b(z) \sim \gamma(z)\beta(w) \sim \frac{1}{z-w}, \tag{2·6}$$

and have $N=1$ superconformal invariance generated by

$$
\begin{aligned}
T_{gh} &= -2b\partial c - \partial bc - \frac{3}{2}\beta\partial\gamma - \frac{1}{2}\partial\beta\gamma, \\
G_{gh} &= \frac{3}{2}\beta\partial c + \partial\beta c - 2b\gamma.
\end{aligned} \tag{2·7}
$$

The physical Hilbert space is defined by the cohomology $\mathcal{H}_{phys} = \mathrm{Ker}Q_{BRST}/\mathrm{Im}Q_{BRST}$ of the BRST charge

$$
Q_{BRST} = \oint \frac{dz}{2\pi i}\left(c\left(T_m + \frac{1}{2}T_{gh}\right) + \gamma\left(G_m + \frac{1}{2}G_{gh}\right)\right), \tag{2·8}
$$

where

$$
T_m = T_{H_4} + T_{\mathcal{M}}, \qquad G_m \;=\; G^+_{H_4} + G^-_{H_4} + G^+_{\mathcal{M}} + G^-_{\mathcal{M}}. \tag{2·9}
$$

Then we bosonize the worldsheet fermions and the $U(1)$ current in the $\mathcal{M}$ sector as

$$
\psi^+\psi^- = i\partial H_0, \qquad \psi\psi^* = i\partial H_1, \tag{2·10a}
$$

$$
I_{\mathcal{M}} = -\sqrt{3}i\partial H_2, \tag{2·10b}
$$

where bosons $H_I(z)$ $(I=0,1,2)$ satisfy the standard OPE's

$$
H_I(z)H_J(w) \sim -\delta_{IJ}\log(z-w). \tag{2·11}
$$

The superconformal ghosts are also bosonized by[13)]

$$
\begin{aligned}
&c = e^{\sigma}, \qquad b = e^{-\sigma}, \\
&\gamma = \eta e^{\phi} = e^{\phi-\chi}, \\
&\beta = e^{-\phi}\partial\xi = \partial\chi e^{-\phi+\chi},
\end{aligned} \tag{2·12}
$$

with

$$
\begin{aligned}
&\phi(z)\phi(w) \sim -\log(z-w), \\
&\sigma(z)\sigma(w) \sim \chi(z)\chi(w) \sim +\log(z-w).
\end{aligned} \tag{2·13}
$$

Here it is important that since the zero-mode ξ_0 is not included in these formulas, the Hilbert space of the original bosonic ghosts (β,γ) is different from the one of the bosonized fields (ϕ,ξ,η) (or (ϕ,χ)). The former (latter) is called small (large) Hilbert space $\mathcal{H}_{small}$ ($\mathcal{H}_{large}$). This extension of the Hilbert space is essential to realize the supersymmetry.

In order to obtain supersymmetric spectrum in the RNS formalism, we must impose GSO projection which guarantees the mutual locality of space-time supercharges. If we take a weak GSO condition

$$I_0 = I_{H_4,0} + I_{\mathcal{M},0} \in \mathbb{Z}, \tag{2·14}$$

the model has the enhanced supersymmetry generated by four supercharges[5)]

$$\begin{aligned}
\mathcal{Q}^{\pm\pm}_{-\frac{1}{2}} &= \oint \frac{dz}{2\pi i} e^{-\frac{\phi}{2}} e^{\pm iX^+} e^{\frac{i}{2}(H_0 \pm (H_1 + \sqrt{3}H_2))}, \\
\mathcal{Q}^{\pm\mp}_{-\frac{1}{2}} &= \oint \frac{dz}{2\pi i} e^{-\frac{\phi}{2}} e^{\frac{i}{2}(-H_0 \pm (H_1 - \sqrt{3}H_2))}.
\end{aligned} \tag{2·15}$$

The subscript $-\frac{1}{2}$ indicates that these operators are given in the $-\frac{1}{2}$ picture, which is generally read from the eigenvalue of the operator

$$\mathcal{R} = \oint \frac{dz}{2\pi i} (\xi - \partial\phi). \tag{2·16}$$

We note that these supercharges obey a peculiar algebra due to the infinite degeneracy of pictures. This algebra is equivalent to the supersymmetry only in the on-shell physical amplitudes. We change the picture of the half of the supercharges $(\mathcal{Q}^{--}_{-\frac{1}{2}}, \mathcal{Q}^{+-}_{-\frac{1}{2}})$ to $+\frac{1}{2}$ so that th [illegible]

$$\begin{aligned}
\mathcal{Q}^{--}_{\frac{1}{2}} &= \oint \frac{dz}{2\pi i} \left\{ Q_{BRST}, \xi e^{-\frac{\phi}{2}} e^{-iX^+} e^{\frac{i}{2}(H_0 - (H_1 + \sqrt{3}H_2))} \right\}, \\
&= \oint \frac{dz}{2\pi i} e^{-iX^+} \Bigg(\eta b e^{\frac{3}{2}\phi + \frac{i}{2}(H_0 - (H_1 + \sqrt{3}H_2))} \\
&\qquad + i\partial X^+ e^{\frac{\phi}{2} - \frac{i}{2}(H_0 + (H_1 + \sqrt{3}H_2))} + i\partial Z^* e^{\frac{\phi}{2} + \frac{i}{2}(-H_0 - (H_1 - \sqrt{3}H_2))} \\
&\qquad - \psi^+ e^{\frac{\phi}{2} + \frac{i}{2}(H_0 - (H_1 + \sqrt{3}H_2))} - G^-_{\mathcal{M}} e^{\frac{\phi}{2} + \frac{i}{2}(H_0 - (H_1 + \sqrt{3}H_2))} \Bigg), \\
\mathcal{Q}^{+-}_{\frac{1}{2}} &= \oint \frac{dz}{2\pi i} \left\{ Q_{BRST}, \xi e^{-\frac{\phi}{2}} e^{\frac{i}{2}(-H_0 + (H_1 - \sqrt{3}H_2))} \right\}, \\
&= \oint \frac{dz}{2\pi i} \Bigg(\eta b e^{\frac{3}{2}\phi + \frac{i}{2}(-H_0 + (H_1 - \sqrt{3}H_2))} - i\partial X^- e^{\frac{\phi}{2} - \frac{i}{2}(-H_0 - (H_1 - \sqrt{3}H_2))} \\
&\qquad + i\partial Z e^{\frac{\phi}{2} - \frac{i}{2}(H_0 + (H_1 + \sqrt{3}H_2))} - G^-_{\mathcal{M}} e^{\frac{\phi}{2} + \frac{i}{2}(-H_0 + (H_1 + \sqrt{3}H_2))} \Bigg).
\end{aligned} \tag{2·17}$$

---

*) Rigorously speaking, the relative signs between terms in the explicit forms (2·17) are not fixed without specifying the cocycle factors usually omitted. This fact makes practical calculations difficult although it is often fixed by Lorentz covariance of the result. This complexity disappears in the hybrid formalism, which is actually one of the important advantages of the hybrid formalism.

Since we will not refer to supercharges in the other pictures, we simply denote $(\mathcal{Q}^{\pm+}_{-\frac{1}{2}}, \mathcal{Q}^{\pm-}_{\frac{1}{2}}) = (\mathcal{Q}^{\pm+}, \mathcal{Q}^{\pm-})$ for the rest of this paper. These picture changed supercharges together with (2·4) generate the supersymmetry algebra in the NS-NS plane waves[5)]

$$\begin{aligned}
[\mathcal{J}, \mathcal{P}] &= \mathcal{P}, & [\mathcal{J}, \mathcal{P}^*] &= -\mathcal{P}^*, \\
[\mathcal{P}, \mathcal{P}^*] &= \mathcal{F}, & & \\
\left[\mathcal{J}, \mathcal{Q}^{\pm\pm}\right] &= \pm\mathcal{Q}^{\pm\pm}, & \left[\mathcal{J}, \mathcal{Q}^{\pm\mp}\right] &= 0, \\
\left[\mathcal{Q}^{-+}, \mathcal{P}\right] &= -\mathcal{Q}^{++}, & \left[\mathcal{Q}^{+-}, \mathcal{P}^*\right] &= \mathcal{Q}^{--}, \\
\left\{\mathcal{Q}^{++}, \mathcal{Q}^{--}\right\} &= \mathcal{F}, & \left\{\mathcal{Q}^{-+}, \mathcal{Q}^{+-}\right\} &= -\mathcal{J}, \\
\left\{\mathcal{Q}^{++}, \mathcal{Q}^{+-}\right\} &= \mathcal{P}, & \left\{\mathcal{Q}^{-+}, \mathcal{Q}^{--}\right\} &= \mathcal{P}^*.
\end{aligned} \tag{2·18}$$

Before closing this section, it is useful to reconsider physical state conditions in the RNS formalism. Although the physical states is defined by the BRST cohomology in $\mathcal{H}_{small}$, we must extend it to $\mathcal{H}_{large}$ to perform a field redefinition to hybrid fields since, as mentioned above, $\mathcal{H}_{small}$ is not enough to realize the space-time supersymmetry. Therefore, we must generalize the physical state conditions to

$$\begin{aligned}
&Q_{BRST}|\psi\rangle = 0, \\
&\qquad |\psi\rangle \sim |\psi\rangle + \delta|\psi\rangle, \qquad \delta|\psi\rangle \ = \ Q_{BRST}|\Lambda\rangle, \\
&\quad \eta_0|\psi\rangle = \eta_0|\Lambda\rangle \ = \ 0,
\end{aligned} \tag{2·19a}$$

where $|\psi\rangle, |\Lambda\rangle \in \mathcal{H}_{large}$. In addition to these cohomology conditions, we require that the physical states have ghost number one

$$Q_{gh}|\psi\rangle = |\psi\rangle, \tag{2·19b}$$

counted by the charge*)

$$Q_{gh} = \oint \frac{dz}{2\pi i} \left(cb - \xi\eta\right). \tag{2·20}$$

These conditions (2·19) have following natural interpretation as a topological $N=4$ string theory.

Let us note that there is the hidden twisted $N=4$ superconformal symmetry generated by

$$\begin{aligned}
T &= T_m + T_{gh}, \\
G^+ &= J_{BRST},
\end{aligned}$$

*) This definition of the ghost number is related to the familiar one $N_c = \oint \frac{dz}{2\pi i}(cb-\gamma\beta)$ by $Q_{gh} = N_c - \mathcal{R}$. The difference is a constant in a fixed picture.

$$
\begin{aligned}
&= c\,(T_m + T_{\beta\gamma}) + \gamma G_m - \gamma^{2b} + c\partial cb + \partial(c\xi\eta) + \partial^{2c}, \\
G^- = b, \qquad \widetilde{G}^+ \; &= \; \eta, \qquad \widetilde{G}^- \; = \; \xi T - b\,\{Q_{BRST}, \xi\} + \partial^2\xi, \\
I^{++} = \eta c, \qquad I^{--} \; &= \; b\xi, \qquad I \; = \; cb - \xi\eta,
\end{aligned} \tag{2·21}
$$

in $\mathcal{H}_{large}$. Physical state conditions (2·19a) and ghost number condition (2·19b) can be written in terms of these $N=4$ generators as

$$
G_0^+|\psi\rangle = 0, \qquad \delta|\psi\rangle = G_0^+|\Lambda\rangle, \tag{2·22a}
$$

$$
I_0|\psi\rangle = |\psi\rangle, \tag{2·22b}
$$

$$
\widetilde{G}_0^+|\psi\rangle = \widetilde{G}_0^+|\Lambda\rangle = 0, \tag{2·22c}
$$

which can be naturally interpreted as physical state conditions in the topological $N=4$ string theory.[7] Since η_0-cohomology is trivial we can always solve Eqs. (2·22c) by $|\psi\rangle = \widetilde{G}_0^+|V\rangle$ and $|\Lambda\rangle = \widetilde{G}_0^+|\Lambda^-\rangle$ to rewrite (2·22) in the more symmetric forms:

$$
G_0^+\widetilde{G}_0^+|V\rangle = 0, \tag{2·23a}
$$

$$
\delta|V\rangle = G_0^+|\Lambda^-\rangle + \widetilde{G}_0^+|\widetilde{\Lambda}^-\rangle, \tag{2·23b}
$$

$$
I_0|V\rangle = 0. \tag{2·23c}
$$

In this paper, we call the first condition (2·23a) the equation of motion and the second (2·23b) the gauge transformation following terminologies in string field theory. These conditions will be used to study physical spectrum in section 5.

§3. Hybrid superstrings in the NS-NS plane waves

In this section, we develop the hybrid formalism for superstrings in the NS-NS plane waves. We first introduce hybrid fields by finding a field redefinition from RNS fields, which allows whole space-time supersymmetry to be manifest. We rewrite worldsheet superconformal generators using these new fields to formulate the model as a topological $N=4$ string theory.

As explained in the previous section, the basic fields of the super NW model are free fields $(X^\pm, Z, Z^*, \psi^\pm, \psi, \psi^*)$ and superconformal ghosts (b, c, β, γ). We also add them the boson H_2 come from the $U(1)$ current (2·10b) in the $\mathcal{M}$ sector, which we need to define supercharges (2·15). Although bosonic fields $(X^\pm, Z, Z^*)$ are common to the hybrid formalism,[*)] remaining fields must be rearranged to obtain basic fields in the hybrid formalism. Describing these

[*)] These bosons are not exactly the same in two formalisms but related by the similarity transformation (3·6).

fields in terms of six free bosons $(H_0, H_1, H_2, \phi, \chi, \sigma)$ with the help of bosonization formulas (2·10) and (2·12), we perform a linear transformation

$$
\begin{aligned}
\phi_{--} &= -\frac{i}{2}H_0 - \frac{i}{2}H_1 - \frac{i}{2}\sqrt{3}H_2 + \frac{1}{2}\phi, \\
\phi_{+-} &= \frac{i}{2}H_0 + \frac{i}{2}H_1 - \frac{i}{2}\sqrt{3}H_2 + \frac{1}{2}\phi, \\
\phi_{++} &= -\frac{i}{2}H_0 + \frac{i}{2}H_1 + \frac{i}{2}\sqrt{3}H_2 - \frac{3}{2}\phi + \chi + \sigma, \\
\phi_{-+} &= \frac{i}{2}H_0 - \frac{i}{2}H_1 + \frac{i}{2}\sqrt{3}H_2 - \frac{3}{2}\phi + \chi + \sigma, \\
\rho &= \sqrt{3}H_2 + 3i\phi - 2i\chi - i\sigma, \\
\widehat{H}_2 &= H_2 + \sqrt{3}i\phi - \sqrt{3}i\chi,
\end{aligned}
\tag{3·1}
$$

and then define fermionic fields as

$$
\theta^{\alpha\alpha'} = e^{\phi_{\alpha\alpha'}}, \qquad p_{\alpha\alpha'} = e^{-\phi_{\alpha\alpha'}}, \qquad (\alpha, \alpha' = \pm) \tag{3·2}
$$

which satisfy

$$
\theta^{\alpha\alpha'}(z) p_{\beta\beta'}(w) \sim \frac{\delta^\alpha_\beta \delta^{\alpha'}_{\beta'}}{z - w}. \tag{3·3}
$$

The basic fields of the hybrid superstrings are finally defined by Green-Schwarz-like fields with an additional boson $(X^\pm, Z, Z^*, \theta^{\alpha\alpha'}, p_{\alpha\alpha'}, \rho)$. The $U(1)$ boson in the $\mathcal{M}$ sector is also modified to $\widehat{H}_2$, which requires modifications of superconformal generators to $(\widehat{T}_\mathcal{M}, \widehat{G}^\pm_\mathcal{M}, \widehat{I}_\mathcal{M})$ uniquely determined by change of the $U(1)$ current

$$
\widehat{I}_\mathcal{M} = -\sqrt{3}i\partial\widehat{H}_2. \tag{3·4}
$$

We note that these new generators completely (anti-)commute with the hybrid fields.

The space-time supercharges (2·15) are written by using these hybrid fields:

$$
\begin{aligned}
\mathcal{Q}^{++} &= \oint \frac{dz}{2\pi i} e^{iX^+} p_{--}, \\
\mathcal{Q}^{--} &= \oint \frac{dz}{2\pi i} e^{-iX^+} \Big(p_{++} + i\partial X^+ \theta^{--} + (i\partial Z^* + \theta^{-+} p_{++})\theta^{+-} + e^{-i\rho}\theta^{-+}\widehat{G}^-_\mathcal{M} \Big), \\
\mathcal{Q}^{-+} &= \oint \frac{dz}{2\pi i} p_{+-}, \\
\mathcal{Q}^{+-} &= \oint \frac{dz}{2\pi i} \Big(p_{-+} - i\partial X^- \theta^{+-} + i\partial Z \theta^{--} - e^{-i\rho}\theta^{++}\widehat{G}^-_\mathcal{M} \Big).
\end{aligned}
\tag{3·5}
$$

However, these supercharges are not symmetric, which leads a complicated hermiticity property to hybrid fields.[14)] These fields are chiral coordinates in the sense that a half of supercharges $\mathcal{Q}^{++}$ and $\mathcal{Q}^{-+}$ are simple superderivatives p_{--} and p_{+-} (except for factors $e^{\pm iX^+}$).

In order to obtain symmetric supercharges and hybrid fields with proper hermiticity, we must further perform a similarity transformation generated by

$$U = \oint \frac{dz}{2\pi i}\bigg(- e^{-i\rho}\theta^{++}\theta^{-+}\widehat{G}_M^- + \frac{1}{2}i\partial X^+\theta^{++}\theta^{--} + \frac{1}{2}i\partial Z^*\theta^{++}\theta^{+-} + \frac{1}{2}i\partial X^-\theta^{+-}\theta^{-+} + \frac{1}{2}i\partial Z\theta^{-+}\theta^{--} + \frac{1}{4}\theta^{-+}\theta^{++}\partial(\theta^{--}\theta^{+-})\bigg). \tag{3·6}$$

In fact, the space-time supercharges (3·5) have symmetric forms

$$\begin{aligned}
\mathcal{Q}^{++} &= \oint \frac{dz}{2\pi i}e^{iX^+}\left(p_{--} + \frac{1}{2}i\partial X^+\theta^{++} + \frac{1}{2}(i\partial Z - \theta^{+-}p_{--})\theta^{-+} + \frac{1}{8}\partial(\theta^{-+}\theta^{++})\theta^{+-}\right),\\
\mathcal{Q}^{--} &= \oint \frac{dz}{2\pi i}e^{-iX^+}\left(p_{++} + \frac{1}{2}i\partial X^+\theta^{--} + \frac{1}{2}(i\partial Z^* + \theta^{-+}p_{++})\theta^{+-} - \frac{1}{8}\partial(\theta^{--}\theta^{+-})\theta^{-+}\right),\\
\mathcal{Q}^{-+} &= \oint \frac{dz}{2\pi i}\left(p_{+-} - \frac{1}{2}i\partial X^-\theta^{-+} + \frac{1}{2}i\partial Z^*\theta^{++} - \frac{1}{8}\partial(\theta^{-+}\theta^{++})\theta^{--}\right),\\
\mathcal{Q}^{+-} &= \oint \frac{dz}{2\pi i}\left(p_{-+} - \frac{1}{2}i\partial X^-\theta^{+-} + \frac{1}{2}i\partial Z\theta^{--} + \frac{1}{8}\partial(\theta^{--}\theta^{+-})\theta^{++}\right),
\end{aligned} \tag{3·7}$$

after the similarity transformation.

We can also provide the topological $N = 4$ superconformal generators (2·21) using the hybrid fields. The $N = 2$ subalgebra is first given by

$$\begin{aligned}
T &= -\partial X^+\partial X^- - \partial Z\partial Z^* - p_{\alpha\alpha'}\partial\theta^{\alpha\alpha'} + \frac{1}{2}\partial\rho\partial\rho + \frac{1}{2}i\partial^2\rho + \widehat{T}_{\mathcal{M}} + \frac{1}{2}\partial\widehat{I}_{\mathcal{M}},\\
G^+ &= e^{-i\rho}\left(d_{--}d_{+-} + \frac{1}{8}\partial^2\theta^{-+}\theta^{++} + \frac{1}{8}\theta^{-+}\partial^2\theta^{++} - \frac{1}{4}\partial^2(\theta^{-+}\theta^{++})\right) + \widehat{G}_M^+,\\
G^- &= e^{i\rho}\left(d_{-+}d_{++} + \frac{1}{8}\partial^2\theta^{--}\theta^{+-} + \frac{1}{8}\theta^{--}\partial^2\theta^{+-} - \frac{1}{4}\partial^2(\theta^{--}\theta^{+-})\right) + \widehat{G}_M^-,\\
I &= i\partial\rho - \sqrt{3}i\partial\widehat{H}_2,
\end{aligned} \tag{3·8}$$

where

$$\begin{aligned}
d_{--} &= p_{--} - \frac{1}{2}i\partial X^+\theta^{++} - \frac{1}{2}i\partial Z\theta^{-+} + \frac{1}{4}\theta^{-+}\theta^{++}\partial\theta^{+-} - \frac{1}{8}\partial(\theta^{-+}\theta^{++})\theta^{+-},\\
d_{+-} &= p_{+-} + \frac{1}{2}i\partial X^-\theta^{-+} - \frac{1}{2}i\partial Z^*\theta^{++} - \frac{1}{4}\theta^{-+}\theta^{++}\partial\theta^{--} + \frac{1}{8}\partial(\theta^{-+}\theta^{++})\theta^{--},\\
d_{++} &= p_{++} - \frac{1}{2}i\partial X^+\theta^{--} - \frac{1}{2}i\partial Z^*\theta^{+-} - \frac{1}{4}\theta^{--}\theta^{+-}\partial\theta^{-+} + \frac{1}{8}\partial(\theta^{--}\theta^{+-})\theta^{-+},\\
d_{-+} &= p_{-+} + \frac{1}{2}i\partial X^-\theta^{+-} - \frac{1}{2}i\partial Z\theta^{--} + \frac{1}{4}\theta^{--}\theta^{+-}\partial\theta^{++} - \frac{1}{8}\partial(\theta^{--}\theta^{+-})\theta^{++},
\end{aligned} \tag{3·9}$$

are local currents of supercovariant derivatives. It is also useful to introduce bosonic supercovariant derivatives as

$$\Pi^+ = i\partial X^+ + \frac{1}{2}\theta^{+-}\partial\theta^{-+} - \frac{1}{2}\partial\theta^{+-}\theta^{-+},$$

$$\begin{aligned}
\mathit{\Pi}^- &= i\partial X^- - \frac{1}{2}\theta^{++}\partial\theta^{--} + \frac{1}{2}\partial\theta^{++}\theta^{--}, \\
\mathit{\Pi}_Z &= i\partial Z - \frac{1}{2}\theta^{++}\partial\theta^{+-} + \frac{1}{2}\partial\theta^{++}\theta^{+-}, \\
\mathit{\Pi}_Z^* &= i\partial Z^* - \frac{1}{2}\theta^{-+}\partial\theta^{--} + \frac{1}{2}\partial\theta^{-+}\theta^{--}.
\end{aligned} \tag{3·10}$$

These supercovariant derivatives and $\partial\theta^{\alpha\alpha'}$ form a closed superalgebra

$$\begin{aligned}
d_{--}(z)d_{++}(w) &\sim -\frac{\mathit{\Pi}^+(w)}{z-w}, & d_{--}(z)d_{-+}(w) &\sim -\frac{\mathit{\Pi}_Z(w)}{z-w}, \\
d_{++}(z)d_{+-}(w) &\sim -\frac{\mathit{\Pi}_Z^*(w)}{z-w}, & d_{+-}(z)d_{-+}(w) &\sim \frac{\mathit{\Pi}^-(w)}{z-w}, \\
\mathit{\Pi}^+(z)\mathit{\Pi}^-(w) &\sim \frac{1}{(z-w)^2}, & \mathit{\Pi}_Z(z)\mathit{\Pi}_Z^*(w) &\sim \frac{1}{(z-w)^2}, \\
d_{--}(z)\mathit{\Pi}^-(w) &\sim -\frac{\partial\theta^{++}(w)}{z-w}, & d_{--}(z)\mathit{\Pi}_Z^*(w) &\sim -\frac{\partial\theta^{-+}(w)}{z-w}, \\
d_{++}(z)\mathit{\Pi}^-(w) &\sim -\frac{\partial\theta^{--}(w)}{z-w}, & d_{++}(z)\mathit{\Pi}_Z(w) &\sim -\frac{\partial\theta^{+-}(w)}{z-w}, \\
d_{+-}(z)\mathit{\Pi}^+(w) &\sim \frac{\partial\theta^{-+}(w)}{z-w}, & d_{+-}(z)\mathit{\Pi}_Z(w) &\sim -\frac{\partial\theta^{++}(w)}{z-w}, \\
d_{-+}(z)\mathit{\Pi}^+(w) &\sim \frac{\partial\theta^{+-}(w)}{z-w}, & d_{-+}(z)\mathit{\Pi}_Z^*(w) &\sim -\frac{\partial\theta^{--}(w)}{z-w}.
\end{aligned} \tag{3·11}$$

We note here that these supercovariant derivatives are *supercovariant* only in the sense that they (anti-)commute with a half of supercharges $Q^{\pm\mp}$.

One can show that complicated forms of $G^{\pm}$ in (3·8) are rewritten as

$$\begin{aligned}
G^+ &= e^{-i\rho} {}^{\times}_{\times} d_{--}d_{+-} {}^{\times}_{\times} + \widehat{G}_M^+, \\
G^- &= e^{i\rho} {}^{\times}_{\times} d_{-+}d_{++} {}^{\times}_{\times} + \widehat{G}_M^-,
\end{aligned} \tag{3·12}$$

by introducing the new normal ordering ${}^{\times}_{\times}$ with respect to the currents $d_{\alpha\alpha'}$. Whole generators (2·21) of the topological $N=4$ superconformal symmetry are then provided by

$$\begin{aligned}
T &= -\partial X^+\partial X^- - \partial Z\partial Z^* - p_{\alpha\alpha'}\partial\theta^{\alpha\alpha'} + \frac{1}{2}\partial\rho\partial\rho + \frac{1}{2}i\partial^2\rho + \widehat{T}_M + \frac{1}{2}\partial\widehat{I}_M, \\
G^+ &= e^{-i\rho} {}^{\times}_{\times} d_{--}d_{+-} {}^{\times}_{\times} + \widehat{G}_M^+, \\
G^- &= e^{i\rho} {}^{\times}_{\times} d_{-+}d_{++} {}^{\times}_{\times} + \widehat{G}_M^-, \\
\widetilde{G}^+ &= e^{2i\rho-\sqrt{3}i\widehat{H}_2} {}^{\times}_{\times} d_{-+}d_{++} {}^{\times}_{\times} + e^{i\rho-\sqrt{3}i\widehat{H}_2}\widehat{G}_M^-, \\
\widetilde{G}^- &= e^{-2i\rho+\sqrt{3}i\widehat{H}_2} {}^{\times}_{\times} d_{--}d_{+-} {}^{\times}_{\times} + e^{-i\rho+\sqrt{3}i\widehat{H}_2}\widehat{G}_M^+, \\
I^{++} &= e^{i\rho-\sqrt{3}i\widehat{H}_2}, \qquad I^{--} = e^{-i\rho+\sqrt{3}i\widehat{H}_2}, \\
I &= i\partial\rho - \sqrt{3}i\partial\widehat{H}_2.
\end{aligned} \tag{3·13}$$

Physical states are defined by conditions (2·23) using zero modes G_0^+, $\widetilde{G}_0^+$ and I_0 of these generators. The supercharges (3·7) (anti-)commute with them, which guarantees the physical spectrum to be supersymmetric.

Finally we rewrite the picture counting operator (2·16) interpreted in the hybrid formalism as the R-charge operator:

$$\mathcal{R} = \oint \frac{dz}{2\pi i} \left(i\partial\rho - \frac{1}{2}(\theta^{++}p_{++} + \theta^{-+}p_{-+} - \theta^{--}p_{--} - \theta^{+-}p_{+-}) \right). \tag{3·14}$$

This is useful to identify whether each component of superfields is space-time boson or fermion. The field having (half-)integral R-charge is space-time boson (fermion) since it comes from the NS-(R-)sector in the RNS formalism.

§4. Spectral flow and the Hilbert space of the hybrid superstring

Now we study the Hilbert space of the hybrid superstring. Using the hybrid fields, the H_4 currents are realized as

$$\begin{aligned} J &= i\partial X^-, & F &= i\partial X^+, \\ P &= e^{iX^+}(i\partial Z - \theta^{+-}p_{--}), & P^* &= e^{-iX^+}(i\partial Z^* + \theta^{-+}p_{++}). \end{aligned} \tag{4·1a}$$

We can extend this H_4 current algebra to a superalgebra, which is an analog of the super current algebra (2·1), by introducing space-time supercoordinates (and their conjugates)

$$\begin{aligned} \Theta^{\pm\mp} &= \theta^{\pm\mp}, & \mathcal{P}_{\pm\mp} &= p_{\pm\mp}, \\ \Theta^{\pm\pm} &= e^{\pm iX^+}\theta^{\pm\pm}, & \mathcal{P}_{\pm\pm} &= e^{\mp iX^+}p_{\pm\pm}, \end{aligned} \tag{4·1b}$$

together with an extra $U(1)$ current $i\partial\rho$. The Hilbert space of the hybrid superstring is constructed by representations of this current superalgebra. We can expand these currents as

$$\begin{aligned} J(z) &= \sum_n J_n z^{-n-1}, & F(z) &= \sum_n F_n z^{-n-1}, \\ P(z) &= \sum_n P_n z^{-n-1}, & P^*(z) &= \sum_n P^*_n z^{-n-1}, \\ \Theta^{\pm\mp}(z) &= \sum_n \Theta^{\pm\mp}_n z^{-n}, & \mathcal{P}_{\pm\mp}(z) &= \sum_n (\mathcal{P}_{\pm\mp})_n z^{-n-1}, \\ \Theta^{\pm\pm}(z) &= \sum_n \Theta^{\pm\pm}_n z^{-n}, & \mathcal{P}_{\pm\pm}(z) &= \sum_n (\mathcal{P}_{\pm\pm})_n z^{-n-1}, \\ i\partial\rho(z) &= \sum_n \rho_n z^{-n-1}, \end{aligned} \tag{4·2}$$

where mode operators satisfy the superalgebra

$$
\begin{aligned}
[J_n, P_m] &= P_{n+m}, & [J_n, P^*_m] &= -P_{n+m}, \\
[J_n, F_m] &= n\delta_{n+m,0}, & [P_n, P^*_m] &= F_{n+m} + n\delta_{n+m,0}, \\
\left[J_n, \Theta^{\pm\pm}_m\right] &= \pm\Theta^{\pm\pm}_{n+m}, & [J_n, (\mathcal{P}_{\pm\pm})_m] &= \mp(\mathcal{P}_{\pm\pm})_{n+m}, \\
\left[P_n, \Theta^{--}_m\right] &= -\Theta^{+-}_{n+m}, & [P_n, (\mathcal{P}_{+-})_m] &= (\mathcal{P}_{--})_{n+m}, \\
\left[P^*_n, \Theta^{++}_m\right] &= \Theta^{-+}_{n+m}, & [P^*_n, (\mathcal{P}_{-+})_m] &= -(\mathcal{P}_{++})_{n+m}, \\
\left\{\Theta^{\pm\pm}_n, (\mathcal{P}_{\pm\pm})_m\right\} &= \delta_{n+m,0}, & \left\{\Theta^{\pm\mp}_n, (\mathcal{P}_{\pm\mp})_m\right\} &= \delta_{n+m,0}, \\
[\rho_n, \rho_m] &= -n\delta_{n+m,0}.
\end{aligned}
\tag{4·3}
$$

Since the hybrid fields already provide a free field realization (4·1), we can easily obtain representations of this superalgebra (4·3). As in the H_4 (super) current algebra,[10),11)5)] only the non-trivial point is the existence of the spectral flow symmetry, *i.e.* the superalgebra (4·3) is preserved by replacement

$$
\begin{aligned}
J_n &\longrightarrow J_n, & F_n &\longrightarrow F_n + p\delta_{n,0}, \\
P_n &\longrightarrow P_{n+p}, & P^*_n &\longrightarrow P^*_{n-p}, \\
\Theta^{\pm\mp}_n &\longrightarrow \Theta^{\pm\mp}_n, & (\mathcal{P}_{\pm\mp})_n &\longrightarrow (\mathcal{P}_{\pm\mp})_n, \\
\Theta^{\pm\pm}_n &\longrightarrow \Theta^{\pm\pm}_{n\pm p}, & (\mathcal{P}_{\pm\pm})_n &\longrightarrow (\mathcal{P}_{\pm\pm})_{n\mp p}, \\
\rho_n &\longrightarrow \rho_n,
\end{aligned}
\tag{4·4}
$$

for any integer $p \in \mathbf{Z}$. The Hilbert space contains all spectrally flowed representations classified into two types describing short and long strings.[11)]

4.1. *The Hilbert space of short strings*

The Hilbert space of short strings in the hybrid formalism include all spectrally flowed type II representations[10)] ($0 < \eta < 1$) defined by

$$
\begin{aligned}
J_0|j,\eta,p,l\rangle &= j|j,\eta,p,l\rangle, & F_0|j,\eta,p,l\rangle &= (\eta+p)|j,\eta,p,l\rangle, \\
J_n|j,\eta,p,l\rangle &= 0, \quad (n>0), & F_n|j,\eta,p,l\rangle &= 0, \quad (n>0), \\
P_n|j,\eta,p,l\rangle &= 0, \quad (n\geq -p), & P^*_n|j,\eta,p,l\rangle &= 0, \quad (n>p), \\
(\mathcal{P}_{\pm\mp})_n|j,\eta,p,l\rangle &= 0, \quad (n\geq 0), & \Theta^{\pm\mp}_n|j,\eta,p,l\rangle &= 0, \quad (n>0), \\
(\mathcal{P}_{++})_n|j,\eta,p,l\rangle &= 0, \quad (n\geq p+1), & \Theta^{++}_n|j,\eta,p,l\rangle &= 0, \quad (n>-p-1), \\
(\mathcal{P}_{--})_n|j,\eta,p,l\rangle &= 0, \quad (n\geq -p), & \Theta^{--}_n|j,\eta,p,l\rangle &= 0, \quad (n>p), \\
\rho_0|j,\eta,p,l\rangle &= (l-\eta)|j,\eta,p,l\rangle, & \rho_n|j,\eta,p,l\rangle &= 0, \quad (n>0),
\end{aligned}
\tag{4·5}
$$

where $l = 0, \pm 1$. The ρ_0 eigenvalue is fixed so that it make the supercurrents $G^{\pm}$ in (3·13) periodic and select a unique representative from infinitely degenerated states due to the pictures.

The explicit representations are easily constructed in terms of the hybrid fields by noting that the transverse fields $(Z, Z^*, \theta^{\pm\pm}, p_{\pm\pm})$ obey the twisted boundary condition:

$$
\begin{aligned}
i\partial Z(e^{2\pi i}z) &= e^{-2\pi i\eta} i\partial Z(z), & i\partial Z^*(e^{2\pi i}z) &= e^{2\pi i\eta} i\partial Z^*(z), \\
\theta^{\pm\pm}(e^{2\pi i}z) &= e^{\mp 2\pi i\eta}\theta^{\pm\pm}(z), & p_{\pm\pm}(e^{2\pi i}z) &= e^{\pm 2\pi i\eta} p_{\pm\pm}(z).
\end{aligned}
\tag{4·6}
$$

Then, the hybrid fields can be expanded as

$$
\begin{aligned}
i\partial X^{\pm}(z) &= \sum_n \alpha_n^{\pm} z^{-n-1}, \\
i\partial Z(z) &= \sum_n Z_{n+\eta} z^{-n-\eta-1}, & i\partial Z^*(z) &= \sum_n Z^*_{n-\eta} z^{-n+\eta-1}, \\
\theta^{\pm\mp}(z) &= \sum_n \theta_n^{\pm\mp} z^{-n}, & p_{\pm\mp}(z) &= \sum_n (p_{\pm\mp})_n z^{-n-1}, \\
\theta^{\pm\pm}(z) &= \sum_n \theta^{\pm\pm}_{n\pm\eta} z^{-n\mp\eta}, & p_{\pm\pm}(z) &= \sum_n (p_{\pm\pm})_{n\mp\eta} z^{-n\pm\eta-1},
\end{aligned}
\tag{4·7}
$$

where the oscillators satisfy the canonical (anti-)commutation relations

$$
\begin{aligned}
\left[\alpha_n^+, \alpha_m^-\right] &= n\delta_{n+m,0}, & \left[Z_{n+\eta}, Z^*_{m-\eta}\right] &= (n+\eta)\delta_{n+m,0}, \\
\left\{\theta_n^{\pm\mp}, (p_{\pm\mp})_m\right\} &= \delta_{n+m,0}, & \left\{\theta^{\pm\pm}_{n\mp\eta}, (p_{\pm\pm})_{m\pm\eta}\right\} &= \delta_{n+m,0},
\end{aligned}
\tag{4·8}
$$

The flowed type II representations are simply realized as Fock states of these oscillators (and ρ_n) on the ground state

$$
\begin{aligned}
\alpha_0^-|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= j|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle, & \alpha_0^+|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= (\eta + p)|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle, \\
\alpha_n^{\pm}|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= 0, \quad (n > 0), \\
Z_{n+\eta}|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= 0, \quad (n \geq 0), & Z^*_{n-\eta}|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= 0, \quad (n > 0), \\
(p_{+-})_0|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= |\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle \frac{\partial}{\partial\theta}, & \theta_0^{+-}|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= |\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle\theta, \\
(p_{-+})_0|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= |\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle \frac{\partial}{\partial\bar\theta}, & \theta_0^{-+}|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= |\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle\bar\theta, \\
(p_{\pm\mp})_n|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= 0, \quad (n > 0), & \theta_n^{\pm\mp}|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= 0, \quad (n > 0), \\
(p_{++})_{n-\eta}|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= 0, \quad (n > 0), & \theta^{++}_{n+\eta}|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= 0, \quad (n \geq 0), \\
(p_{--})_{n+\eta}|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= 0, \quad (n \geq 0), & \theta^{--}_{n-\eta}|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= 0, \quad (n > 0), \\
\rho_0|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= (l-\eta)|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle, & \rho_n|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle &= 0, \quad (n \geq 0),
\end{aligned}
\tag{4·9}
$$

where we have diagonalized zero modes $\alpha_0^\pm (= p^\pm)$ and $\theta_0^{\pm\pm}$, and denoted their eigenvalues by $\boldsymbol{p} = (j, p)$ and $\boldsymbol{\theta} = (\theta, \bar{\theta})$. The short string states are obtained by multiplying the Fock states by a superfield $\Psi(\boldsymbol{p}, \boldsymbol{\theta})$ on which $\frac{\partial}{\partial\theta}$ and $\frac{\partial}{\partial\bar{\theta}}$ act. Since Z and Z^* do not have zero modes, the short string is localized and cannot reach infinity in the transverse space.

The total Hilbert space is obtained by the tensor product of this Fock space and unitary representations describing the $\mathcal{M}$ sector. Since an arbitrary unitary representation of the $N = 2$ superconformal field theory is characterized by dimension Δ and $U(1)$ charge Q, we can formally define a unitary representation by

$$\begin{aligned} \widehat{L}_{0,\mathcal{M}}|\Delta, Q\rangle &= \Delta|\Delta, Q\rangle, \\ \widehat{I}_{0,\mathcal{M}}|\Delta, Q\rangle &= Q|\Delta, Q\rangle. \end{aligned} \tag{4·10}$$

The short string is represented by Fock states on the ground state

$$|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l; \Delta, Q\rangle = |\eta, \boldsymbol{p}, \boldsymbol{\theta}, l\rangle \otimes |\Delta, Q\rangle. \tag{4·11}$$

For later use, we note that this ground state have eigenvalues

$$\begin{aligned} L_0|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l; \Delta, Q\rangle = \Big((\eta + p)j - \frac{1}{2}(l - \eta)(l - \eta + 1) \\ + \Delta - \frac{1}{2}Q - \frac{1}{2}\eta(1 - \eta)\Big)|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l; \Delta, Q\rangle, \end{aligned} \tag{4·12}$$

$$I_0|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l; \Delta, Q\rangle = (l - \eta + Q)|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l; \Delta, Q\rangle, \tag{4·13}$$

$$\mathcal{R}|\eta, \boldsymbol{p}, \boldsymbol{\theta}, l; \Delta, Q\rangle = |\eta, \boldsymbol{p}, \boldsymbol{\theta}, l; \Delta, Q\rangle \left((l - \frac{1}{2}) + \frac{1}{2}(\theta\frac{\partial}{\partial\theta} - \bar{\theta}\frac{\partial}{\partial\bar{\theta}})\right), \tag{4·14}$$

where constant terms in (4·12) and (4·14) are easily derived by using the bosonization (3·2). The $U(1)$ charge condition (2·23c) together with the I_0 eigenvalue (4·13) imposes that the charge Q of the short string must be fractional.

4.2. *The Hilbert space of long strings*

The long-string Hilbert space is given by spectrally flowed type I representations $(\eta = 0)$.[10] Mode expansions are easily obtained by setting $\eta = 0$ in the previous expressions except for the transverse coordinates $(Z, Z^*, \theta^{\pm\pm}, p_{\pm\pm})$ having additional zero-modes. The Fock vacuum is defined by

$$\begin{aligned} &\alpha_0^-|\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, l\rangle = j|\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, l\rangle, &&\alpha_0^+|\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, l\rangle = p|\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, l\rangle, \\ &\alpha_n^\pm|\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, l\rangle = 0, \quad (n > 0), && \\ &Z_0|\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, l\rangle = q|\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, l\rangle, &&Z_0^*|\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, l\rangle = q^*|\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, l\rangle, \\ &Z_n|\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, l\rangle = 0, \quad (n > 0), &&Z_n^*|\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, l\rangle = 0, \quad (n > 0), \end{aligned}$$

$$
\begin{aligned}
(p_{+-})_0|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= |\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle\frac{\partial}{\partial\theta}, & \theta_0^{+-}|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= |\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle\theta,\\
(p_{-+})_0|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= |\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle\frac{\partial}{\partial\bar{\theta}}, & \theta_0^{-+}|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= |\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle\bar{\theta},\\
(p_{\pm\mp})_n|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= 0,\quad (n>0), & \theta_n^{\pm\mp}|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= 0,\quad (n>0),\\
(p_{--})_0|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= |\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle\frac{\partial}{\partial\tilde{\theta}}, & \theta_0^{--}|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= |\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle\tilde{\theta},\\
(p_{++})_0|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= |\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle\frac{\partial}{\partial\bar{\tilde{\theta}}}, & \theta_0^{++}|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= |\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle\bar{\tilde{\theta}},\\
(p_{--})_n|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= 0,\quad (n>0), & \theta_n^{--}|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= 0,\quad (n>0),\\
(p_{++})_n|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= 0,\quad (n>0), & \theta_n^{++}|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= 0,\quad (n>0),\\
\rho_0|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= l|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle, & \rho_n|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle &= 0,\quad (n>0),
\end{aligned}
\tag{4·15}
$$

where $\boldsymbol{q}=(q,q^*)$ and $\tilde{\boldsymbol{\theta}}=(\tilde{\theta},\bar{\tilde{\theta}})$ are the additional zero modes. The coefficient superfield in this sector is a function of zero modes $(\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}})$. The long strings can freely propagate in the four-dimensional space $(X^{\pm},Z,Z^*)$.

The long string is represented by Fock states on the ground state

$$
|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l;\mathit{\Delta},Q\rangle = |\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l\rangle\otimes|\mathit{\Delta},Q\rangle, \tag{4·16}
$$

having eigenvalues

$$
L_0|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l;\mathit{\Delta},Q\rangle = (pj+qq^*-\frac{1}{2}l(l+1)+\mathit{\Delta}-\frac{1}{2}Q)|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l;\mathit{\Delta},Q\rangle, \tag{4·17}
$$

$$
I_0|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l;\mathit{\Delta},Q\rangle = (l+Q)|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l;\mathit{\Delta},Q\rangle, \tag{4·18}
$$

$$
\mathcal{R}|\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l;\mathit{\Delta},Q\rangle = |\boldsymbol{p},\boldsymbol{q},\boldsymbol{\theta},\tilde{\boldsymbol{\theta}},l;\mathit{\Delta},Q\rangle\left(l+\frac{1}{2}\left(\theta\frac{\partial}{\partial\theta}+\tilde{\theta}\frac{\partial}{\partial\tilde{\theta}}-\bar{\theta}\frac{\partial}{\partial\bar{\theta}}-\bar{\tilde{\theta}}\frac{\partial}{\partial\bar{\tilde{\theta}}}\right)\right). \tag{4·19}
$$

The $U(1)$ charge condition (2·23c) together with the I_0 eigenvalue (4·18) leads that the long string must have integral Q.

§5. Physical spectrum

In this section we study the physical spectrum at lower mass levels explicitly. We concentrate on the states whose $\mathcal{M}$ sector is provided by (anti-)chiral primary states characterized by $\mathit{\Delta}=\frac{|Q|}{2}$, then solve the physical state conditions (2·23).

5.1. *physical states in the short string sector*

We first examine physical states at mass levels $N=0,\eta,1-\eta$ in the short string sector. One can easily show that it is enough to study the $l=1,0$ cases since there is no physical state having p_0-momentum $l=-1$ at these levels. The $U(1)$ charge condition (2·23c) and the chirality condition $\Delta = \frac{|Q|}{2}$ lead $\Delta = -\frac{Q}{2} = \frac{1}{2}(1-\eta)$ for the $l=1$ case and $\Delta = \frac{Q}{2} = \frac{\eta}{2}$ for the $l=0$ case.

Let us start to consider the oscillator ground state $N=0$ with $l=1$ provided by

$$|V\rangle = |1\rangle \Psi^{(\frac{1}{2})}(\boldsymbol{p}, \boldsymbol{\theta}). \tag{5·1}$$

We denote here the state (4·11) with $l=1$ by $|1\rangle$ and use this abbreviation in this subsection, for simplicity. A half of supersymmetry is realized on the coefficient superfield $\Psi^{(\frac{1}{2})}$ by*)

$$Q^{-+} = \frac{\partial}{\partial\theta} - \frac{1}{2}j\bar{\theta}, \qquad Q^{+-} = \frac{\partial}{\partial\bar{\theta}} - \frac{1}{2}j\theta. \tag{5·2}$$

Superscript $(\frac{1}{2})$ of the coefficient superfield indicates that its first component has R-charge $\frac{1}{2}$, which can be read from Eq. (4·14). The physical state conditions (2·23) lead manifestly supersymmetric conditions on the superfield:

$$D\bar{D}\Psi^{(\frac{1}{2})} = 0, \tag{5·3a}$$
$$\delta\Psi^{(\frac{1}{2})} = \bar{D}\Lambda^{(1)}, \tag{5·3b}$$

where $\Lambda^{(1)}$ is an arbitrary gauge parameter superfield and supercovariant derivatives are defined by

$$D = \frac{\partial}{\partial\theta} + \frac{1}{2}j\bar{\theta}, \qquad \bar{D} = \frac{\partial}{\partial\bar{\theta}} + \frac{1}{2}j\theta. \tag{5·4}$$

These conditions (5·3) can be easily solved by taking an appropriate gauge as

$$\Psi^{(\frac{1}{2})} = \bar{\theta}\bar{\phi}^{(0)}(p, j=0). \tag{5·5}$$

The physical component $\bar{\phi}^{(0)}$ is a space-time boson and identified with the *tachyon like* state obtained in Ref. 5). The solution (5·5) also shows that there is no fermionic massless physical state, *i.e.* the physical spectrum has boson-fermion asymmetry. This is only possible for the massless $(j=0)$ state on which the supercharges (5·2) anti-commute.

For the oscillator ground state with $l=0$

$$|V\rangle = |0\rangle \Psi^{(-\frac{1}{2})}(\boldsymbol{p}, \boldsymbol{\theta}), \tag{5·6}$$

physical state conditions are provided by

$$\bar{D}D\Psi^{(-\frac{1}{2})} = 0,$$
$$\delta\Psi^{(-\frac{1}{2})} = D\Lambda^{(-1)}, \tag{5·7}$$

*) Another half relates different mass states as a part of the DDF operators discussed in section 6.

and the solution has a similar form to (5·5) as

$$\Psi^{(-\frac{1}{2})} = \theta\phi^{(0)}(p, j=0). \tag{5·8}$$

The massless boson $\phi^{(0)}$ has no fermionic partner and is identified with *graviton like* state in Ref. 5).

Next we consider two massive cases $N = \eta, 1-\eta$. General states at the level $N = \eta$ are expanded by three Fock states as

$$|V\rangle = (\Pi_Z^*)_{-\eta}|l\rangle\Psi^{(l-\frac{1}{2})}(\boldsymbol{p},\boldsymbol{\theta}) + (d_{++})_{-\eta}|l\rangle\Phi^{(l)}(\boldsymbol{p},\boldsymbol{\theta}) + \theta^{--}_{-\eta}|l\rangle\Xi^{(l)}(\boldsymbol{p},\boldsymbol{\theta}). \tag{5·9}$$

Since we take a supercovariant basis created by $((\Pi_Z^*)_{-\eta}, (d_{++})_{-\eta}, \theta^{--}_{-\eta})$, the coefficient fields are superfields, *i.e.* their supersymmetry transformations are generated by supercharges (5·2). The equations of motion for the $l=1$ case can be written as

$$\begin{aligned} D\left(\bar{D}\Xi^{(1)} + \eta\Psi^{(\frac{1}{2})}\right) &= 0, \\ \bar{D}\left(\Xi^{(1)} - (p+\eta)D\Psi^{(\frac{1}{2})}\right) + \left((p+\eta)j+\eta\right)\Psi^{(\frac{1}{2})} &= 0, \end{aligned} \tag{5·10}$$

with the gauge transformations

$$\begin{aligned} \delta\Psi^{(\frac{1}{2})} &= \bar{D}\Lambda^{(1)}, \\ \delta\Phi^{(1)} &= \Sigma^{(1)}, \\ \delta\Xi^{(1)} &= -\eta\Lambda^{(1)}. \end{aligned} \tag{5·11}$$

Taking $\Phi^{(1)} = \Xi^{(1)} = 0$ gauge, the physical state is described by an anti-chiral superfield obeying $D\Psi^{(\frac{1}{2})} = 0$ and the on-shell condition $((p+\eta)j+\eta)\Psi^{(\frac{1}{2})} = 0$. The anti-chiral superfield has the explicit form

$$\Psi^{(\frac{1}{2})} = \psi^{(\frac{1}{2})} + \bar{\theta}\bar{\phi}^{(0)} - \frac{1}{2}\theta\bar{\theta}j\psi^{(\frac{1}{2})}, \tag{5·12}$$

containing one boson $\bar{\phi}^{(0)}$ and one fermion $\psi^{(\frac{1}{2})}$.

For the case of $l=0$, the equations of motion

$$\begin{aligned} \bar{D}\left(\Xi^{(0)} - (p+\eta)D\Psi^{(-\frac{1}{2})}\right) &= 0, \\ D\left(\Xi^{(0)} - (p+\eta)\Phi^{(0)}\right) &= 0, \\ (p+\eta)\bar{D}D\Phi^{(0)} + \eta D\Psi^{(-\frac{1}{2})} + \left[D,\bar{D}\right]\Xi^{(0)} &= \left((p+\eta)j+\eta\right)\Phi^{(0)}, \\ (p+\eta)D\left(\bar{D}\Xi^{(0)} + \eta\Psi^{(-\frac{1}{2})}\right) &= \left((p+\eta)j+\eta\right)\Xi^{(0)}, \end{aligned} \tag{5·13}$$

and the gauge transformations

$$
\begin{aligned}
\delta\Psi^{(-\frac{1}{2})} &= \Sigma^{(-\frac{1}{2})} + D\Lambda^{(-1)}, \\
\delta\Phi^{(0)} &= D\Sigma^{(-\frac{1}{2})}, \\
\delta\Xi^{(0)} &= (p+\eta)D\Sigma^{(-\frac{1}{2})},
\end{aligned}
\tag{5·14}
$$

can be solved by taking $\Psi^{(-\frac{1}{2})} = 0$ gauge as

$$
\Xi^{(0)} = (p+\eta)\frac{1}{j}\bar{D}D\Phi^{(0)}. \tag{5·15}
$$

The physical state is identified with an unconstrained superfield $\Phi^{(0)}$ obeying the on-shell condition $((p+\eta)j+\eta)\,\Phi^{(0)} = 0$.

We can easily repeat the analysis to study physical spectrum at level $N = 1-\eta$. The states at this level are generally

$$
|V\rangle = (\Pi_Z)_{-1+\eta}|l\rangle\Psi^{(l-\frac{1}{2})}(\boldsymbol{p},\boldsymbol{\theta}) + (d_{--})_{-1+\eta}|l\rangle\Phi^{(l-1)}(\boldsymbol{p},\boldsymbol{\theta}) + \theta^{++}_{-1+\eta}|l\rangle\Xi^{(l-1)}(\boldsymbol{p},\boldsymbol{\theta}). \tag{5·16}
$$

For $l = 1$, the equations of motion and the gauge transformations are given by

$$
\begin{aligned}
D\left(\Xi^{(0)} - (p+\eta)\bar{D}\Psi^{(\frac{1}{2})}\right) &= 0, \\
(p+\eta)D\bar{D}\Phi^{(0)} + (1-\eta)\bar{D}\Psi^{(\frac{1}{2})} - \left[D,\bar{D}\right]\Xi^{(0)} &= ((p+\eta)j+1-\eta)\,\Phi^{(0)}, \\
(p+\eta)\bar{D}\left(D\Xi^{(0)} + (1-\eta)\Psi^{(\frac{1}{2})}\right) &= ((p+\eta)j+1-\eta)\,\Xi^{(0)}, \\
\bar{D}\left(\Xi^{(0)} - (p+\eta)\Phi^{(0)}\right) &= 0,
\end{aligned}
\tag{5·17}
$$

and

$$
\begin{aligned}
\delta\Psi^{(\frac{1}{2})} &= \Sigma^{(\frac{1}{2})} + \bar{D}\Lambda^{(1)}, \\
\delta\Phi^{(0)} &= \bar{D}\Sigma^{(\frac{1}{2})}, \\
\delta\Xi^{(0)} &= (p+\eta)\bar{D}\Sigma^{(\frac{1}{2})},
\end{aligned}
\tag{5·18}
$$

respectively. The physical state is given by an unconstrained superfield $\Phi^{(0)}$ obeying $((p+\eta)j+1-\eta)\,\Phi^{(0)} = 0$. The superfield $\Psi^{(\frac{1}{2})}$ can be gauged away and $\Xi^{(0)}$ is solved by $\Phi^{(0)}$.

We can also solve the physical state conditions for $l = 0$

$$
\begin{aligned}
D\left(\Xi^{(-1)} - (p+\eta)\bar{D}\Psi^{(-\frac{1}{2})}\right) + ((p+\eta)j+1-\eta)\,\Psi^{(-\frac{1}{2})} &= 0, \\
\bar{D}\left(D\Xi^{(-1)} + (1-\eta)\Psi^{(-\frac{1}{2})}\right) &= 0,
\end{aligned}
\tag{5·19}
$$

and

$$\delta\Psi^{(-\frac{1}{2})} = D\Lambda^{(-1)},$$
$$\delta\Phi^{(-1)} = \Sigma^{(-1)},$$
$$\delta\Xi^{(-1)} = -(1-\eta)\Lambda^{(-1)}, \tag{5·20}$$

by a chiral superfield $\Psi^{(-\frac{1}{2})}$ obeying

$$\bar{D}\Psi^{(-\frac{1}{2})} = 0,$$
$$((p+\eta)j + 1 - \eta)\,\Psi^{(-\frac{1}{2})} = 0. \tag{5·21}$$

The explicit form of the chiral superfield is

$$\Psi^{(-\frac{1}{2})} = \psi^{(-\frac{1}{2})} + \theta\phi^{(0)} + \frac{1}{2}\theta\bar{\theta}j\psi^{(-\frac{1}{2})}. \tag{5·22}$$

In short, the physical spectrum at these massive levels contains two types of multiplets, (anti-)chiral and unconstrained. The latter is reducible and decomposes into two (chiral and anti-chiral) multiplets.

5.2. *physical states in the long string sector*

In the long string sector, we examine only the ground state with $\Delta = Q = 0$. The $U(1)$ charge condition (2·23c) leads $l = 0$ and then the state is given by

$$|V\rangle = |\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}, 0; 0, 0\rangle V^{(0)}(\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{\theta}, \tilde{\boldsymbol{\theta}}). \tag{5·23}$$

The physical state conditions become

$$\left(D\bar{D}\bar{\tilde{D}}\tilde{D} - \tilde{D}\bar{D}\bar{\tilde{D}}D\right)V^{(0)} = 0, \tag{5·24a}$$
$$\delta V^{(0)} = \tilde{D}D\Lambda^{(-1)} + \bar{D}\bar{\tilde{D}}\bar{\Lambda}^{(1)}, \tag{5·24b}$$

where the supercovariant derivatives in this sector have the form

$$D = \frac{\partial}{\partial\theta} + \frac{1}{2}j\bar{\theta} - \frac{1}{2}q^{*}\bar{\tilde{\theta}}, \qquad \bar{D} = \frac{\partial}{\partial\bar{\theta}} + \frac{1}{2}j\theta - \frac{1}{2}q\tilde{\theta},$$
$$\tilde{D} = \frac{\partial}{\partial\tilde{\theta}} - \frac{1}{2}p\bar{\tilde{\theta}} - \frac{1}{2}q\bar{\theta}, \qquad \bar{\tilde{D}} = \frac{\partial}{\partial\bar{\tilde{\theta}}} - \frac{1}{2}p\tilde{\theta} - \frac{1}{2}q^{*}\theta. \tag{5·25}$$

These conditions (5·24) are essentially the same with those for the four-dimensional vector multiplet. If we take the WZ gauge

$$V = \frac{1}{2}\tilde{\theta}\bar{\tilde{\theta}}A_{+} - \frac{1}{2}\theta\bar{\theta}A_{-} + \frac{1}{2}\tilde{\theta}\bar{\theta}A + \frac{1}{2}\theta\bar{\tilde{\theta}}A^{*}$$
$$+ \theta\bar{\theta}\bar{\tilde{\theta}}\tilde{\lambda} - \tilde{\theta}\bar{\theta}\bar{\tilde{\theta}}\lambda + \theta\tilde{\theta}\bar{\tilde{\theta}}\bar{\lambda} - \theta\tilde{\theta}\bar{\tilde{\theta}}\bar{\tilde{\lambda}} + \theta\bar{\theta}\tilde{\theta}\bar{\tilde{\theta}}\mathcal{D}, \tag{5·26}$$

the equations of motion (5·24a) leads the Maxwell equations

$$
\begin{aligned}
&\frac{1}{4}j(pA_- + jA_+ + qA^* + q^*A) - \frac{1}{2}(pj + qq^*)A_- = 0,\\
&\frac{1}{4}p(pA_- + jA_+ + qA^* + q^*A) - \frac{1}{2}(pj + qq^*)A_+ = 0,\\
&\frac{1}{4}q^*(pA_- + jA_+ + qA^* + q^*A) - \frac{1}{2}(pj + qq^*)A^* = 0,\\
&\frac{1}{4}q(pA_- + jA_+ + qA^* + q^*A) - \frac{1}{2}(pj + qq^*)A = 0,
\end{aligned}
\tag{5·27}
$$

the massless Dirac equations

$$
\begin{aligned}
&(q^*\bar{\tilde{\lambda}} - j\bar{\lambda}) = 0, \qquad (p\bar{\tilde{\lambda}} + q\bar{\lambda}) = 0,\\
&(q\tilde{\lambda} - j\lambda) = 0, \qquad (p\tilde{\lambda} + q^*\lambda) = 0,
\end{aligned}
\tag{5·28}
$$

and $\mathcal{D} = 0$ for auxiliary field. The massless spectrum of the long string is thus the vector multiplet in the four-dimensional *free-field space* $(X^\pm, Z, Z^*)$.

§6. Summary and Discussions

In this paper, we have studied four-dimensional superstrings in the NS-NS plane-wave backgrounds by using the hybrid formalism. This description of the superstring has been obtained by a field redefinition of the worldsheet fields in the super-NW model.[5)] Since we have adopted a weak GSO projection restricting only the total $U(1)_R$ charge to be integer, the model has enhanced supersymmetry which is manifest in the hybrid formalism. The Hilbert space consists of two sectors describing the short and the long strings, and including all the spectrally flowed representations of type II and I, respectively.[10),11)] Then we have studied physical states to find boson-fermion asymmetry in the massless spectrum of the short string. There are two massless bosons, called tachyon like and graviton like in Ref. 5), but no fermionic partners. We have also identified massive physical states at level $N = \eta, 1-\eta$ in the short string sector and massless physical states in the long string sector. The massless physical spectrum of the long string is the vector multiplet freely propagating in the four-dimensional space $(X^\pm, Z, Z^*)$.

The massive physical states obtained by solving physical state conditions are also created by acting the DDF operators

$$
\begin{aligned}
\mathcal{P}_n &= \oint \frac{dz}{2\pi i} e^{i\left(\frac{n+\eta}{p+\eta}\right)X^+} \left(i\partial Z - \left(\frac{n+\eta}{p+\eta}\right)\theta^{+-}p_{--} \right),\\
\mathcal{P}^*_n &= \oint \frac{dz}{2\pi i} e^{i\left(\frac{n-\eta}{p+\eta}\right)X^+} \left(i\partial Z^* - \left(\frac{n-\eta}{p+\eta}\right)\theta^{-+}p_{++} \right),
\end{aligned}
$$

$$\mathcal{Q}_n^{++} = \oint \frac{dz}{2\pi i} e^{i\left(\frac{n+\eta}{p+\eta}\right)X^+}\Bigg(p_{--} + \frac{1}{2} i\partial X^+ \theta^{++}$$
$$+ \frac{1}{2}\left(i\partial Z - \left(\frac{n+\eta}{p+\eta}\right)\theta^{+-}p_{--}\right)\theta^{-+} + \frac{1}{8}\partial(\theta^{-+}\theta^{++})\theta^{+-}\Bigg),$$
$$\mathcal{Q}_n^{--} = \oint \frac{dz}{2\pi i} e^{i\left(\frac{n-\eta}{p+\eta}\right)X^+}\Bigg(p_{++} + \frac{1}{2} i\partial X^+ \theta^{--}$$
$$+ \frac{1}{2}\left(i\partial Z^* - \left(\frac{n-\eta}{p+\eta}\right)\theta^{-+}p_{++}\right)\theta^{+-} - \frac{1}{8}\partial(\theta^{--}\theta^{+-})\theta^{-+}\Bigg), \qquad (6\cdot1)$$

on the massless physical states.[5)] They contain $\mathcal{P} = \mathcal{P}_p,\ \mathcal{P}^* = \mathcal{P}^*_{-p},\ \mathcal{Q}^{\pm\pm} = \mathcal{Q}^{\pm\pm}_{\pm p}$ and generate an affine extension of the supersymmetry algebra (2·18):

$$\begin{aligned}
[\mathcal{J}, \mathcal{P}_n] &= \left(\frac{n+\eta}{p+\eta}\right)\mathcal{P}_n, & [\mathcal{J}, \mathcal{P}^*_n] &= \left(\frac{n-\eta}{p+\eta}\right)\mathcal{P}^*_n,\\
[\mathcal{P}_n, \mathcal{P}^*_m] &= \left(\frac{n+\eta}{p+\eta}\right)\mathcal{F}\delta_{n+m,0}, & [\mathcal{J}, \mathcal{Q}^{\pm\mp}] &= 0,\\
\left[\mathcal{J}, \mathcal{Q}_n^{++}\right] &= \left(\frac{n+\eta}{p+\eta}\right)\mathcal{Q}_n^{++}, & \left[\mathcal{J}, \mathcal{Q}_n^{--}\right] &= \left(\frac{n-\eta}{p+\eta}\right)\mathcal{Q}_n^{--},\\
\left[\mathcal{Q}^{-+}, \mathcal{P}_n\right] &= -\left(\frac{n+\eta}{p+\eta}\right)\mathcal{Q}_n^{++}, & \left[\mathcal{Q}^{+-}, \mathcal{P}^*_n\right] &= -\left(\frac{n-\eta}{p+\eta}\right)\mathcal{Q}_n^{--},\\
\left\{\mathcal{Q}_n^{++}, \mathcal{Q}_m^{--}\right\} &= \mathcal{F}\delta_{n+m,0}, & \left\{\mathcal{Q}^{+-}, \mathcal{Q}^{-+}\right\} &= \mathcal{J},\\
\left\{\mathcal{Q}^{+-}, \mathcal{Q}_n^{++}\right\} &= \mathcal{P}_n, & \left\{\mathcal{Q}^{-+}, \mathcal{Q}_n^{--}\right\} &= \mathcal{P}^*_n,
\end{aligned} \qquad (6\cdot2)$$

which provides the enhanced space-time symmetry. This symmetry is obtained by taking the Penrose limit of $\mathcal{N}=2$ superconformal symmetry being the isometry of the $AdS_3 \times S^1$ background.[5)] It is interesting to trace this limit by studying hybrid superstrings in the $AdS_3 \times S^1$, which is under investigation and will be reported elsewhere.[15)]

It is also interesting to understand general aspects of the holographic duality in the plane wave backgrounds,[5),11)] which requires further consideration. We hope that the manifest supersymmetry in the hybrid formalism shed new light on future development.

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