Harrison Cohomology and Abelian Deformation Quantization on Algebraic Varieties

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Abstract

Abelian deformations of ordinary algebras of functions are studied. The role of Harrison cohomology in classifying such deformations is illustrated in the context of simple examples chosen for their relevance to physics. It is well known that Harrison cohomology is trivial on smooth manifolds and that, consequently, abelian $*$-products on such manifolds are trivial to first order in the deformation parameter. The subject is nevertheless interesting; first because varieties with singularities appear in the physical context and secondly, because deformations that are trivial to first order are not always (indeed not usually) trivial as exact deformations. We investigate cones, to illustrate the situation on algebraic varieties, and we point out that the coordinate algebra on (anti-) de Sitter space is a nontrivial deformation of the coordinate algebra on Minkowski space – although both spaces are smooth manifolds.

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1 Introduction

Deformation quantization has become a dynamic subject in mathematics as well as in physics. The general setting is a (commutative and associative) algebra of functions, and the most important case is the algebra of differentiable functions on a symplectic space, the observables of classical mechanics. The problem is to construct an associative algebra, generally non commutative, on the same space, as a formal or exact deformation of this algebra.

The Poisson bracket of classical mechanics plays an important role. A deformation 'in the direction of the Poisson bracket' is a formal associative product on the algebra of formal power series,

$$f * g = fg + \sum_{n \geq 0} \hbar^n C_n(f,g),$$

such that $C_1(f,g) = \frac{1}{2}\{f,g\}$. The ordinary product fg and the Poisson bracket $\{f,g\}$ are extended to formal power series in $\hbar$. The emphasis on this type

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deformation can be explained, not only by the physical applications, but by the fact that, on a smooth manifold, every associative deformation is equivalent to one with C_1 antisymmetric. If C_1 is linear in df and in dg, and antisymmetric, then by the associativity of the $*$-product it follows that $f, g \mapsto C_1(f, g)$ is a Poisson bracket.

The existence of a Poisson bracket turns the manifold into a Poisson manifold but not, in general, into a symplectic space. The existence and the construction of $*$-products on an arbitrary Poisson manifold has recently been investigated, with great success [K, T].

Abelian deformations represent another type of generalization. Such deformations are not trivial. In the first place, though it is true that on a smooth manifold every first order, abelian deformation is trivial, the same is not true of exact deformations. In the second place, it is interesting to go beyond the context of smooth manifolds; and in particular, to attempt the construction of $*$-products on algebraic varieties with singularities.

Abelian $*$-products have turned up in at least one physical context: in attempts to quantize Nambu mechanics[D], where methods inspired by second quantization had to be used. First order abelian deformations are classified by Harrison cohomology[B, H], which is trivial on smooth manifolds. In order to overcome this difficulty attempts have been made [N, P] to generalize algebraically the usual notion [G] of (Gerstenhaber) deformations, taking a deformation parameter which acts by automorphisms (to the right and to the left) on the original algebra.

Dealing with algebras of functions (in particular, polynomials) on manifolds or varieties, it is natural to ask what types of singularities would be required for the existence of a non trivial, commutative $*$-product. This paper was planned as a study of simple examples of algebraic varieties, notably cones. However, in the course of this study, it became clear that interesting, non trivial deformations exist even on smooth manifolds (including $\mathbb{R}^n$). These are exact deformations: the deformed product is exact to all orders of λ, but the real surprise was the realization that there are exact, abelian, associative $*$-products, on smooth manifolds, of the form

$$f * g = fg + \lambda C(f, g)$$

(with no terms of higher order in λ), that are non trivial as exact deformations.

Physical applications, some that are immediate and some that are more remote, include the following:

(a) The conformal anomaly, briefly discussed in Section 2.6.

(b) A cohomological classification of operator product expansions in quantum field theories.

(c) A view of classical (anti-) de Sitter field theory as an abelian deformation of Minkowski field theory; see Section 3. We hope that this may lead to a better understanding of some difficult aspects of anti-de Sitter physics.

(d) Within the program of geometric or $*$-product quantization on co-adjoint orbits, there are unsolved difficulties in the case of conic orbits. Most interesting is the quantization of Keplerian systems.

(e) One may look into the possibility of finding central extensions of Virasoro algebras on algebraic varieties, for example, on cones.

Section 2 initiates our study by detailed calculations on very simple examples. This paper considers only algebraic varieties, $\mathbb{R}^n$or subvarieties of $\mathbb{R}^n$ defined by the vanishing of a finite set of polynomials. For physical applications it may be that the most interesting algebraic varieties are the cones,

$$\mathbb{R}^n/g, \quad g = \sum g^{ij} x_i x_j, \quad g^{ij} \in \mathbb{R}, \quad i,j = 1, ...n.$$

The simplest case, $n = 2$ and $g = x^2 - y^2 = uv$ is investigated in Section 2.3.

Section 3 deals in great detail with a deformed, abelian algebra of functions on Minkowski space that is isomorphic to an algebra of functions on (anti-) de Sitter space.

The remaining sections study Hochschild (including Harrison) homology and cohomology, on cones of any dimension. The main purpose is to develop an understanding of the relationship between singularities and cohomology and, in particular, the role of singularities within a program of deformation quantization.

2 First Examples

We shall investigate the Harrison cohomology for a commutative algebra, in the simplest context.

2.1. Harrison cohomology and abelian $*$-products.
Harrison's complex for a commutative and associative algebra $\mathcal{A}$ is a subcomplex of the Hochschild complex. The cochains are valued in the unital augmentation of the algebra itself and familiar formulas for the differential apply. Thus if E is a one-form, then

$$dE(f,g) = fE(g) - E(fg) + E(f)g,$$

which vanishes for derivations, and if C is a two-form, then

$$dC(f,g,h) = fC(g,h) - C(fg,h) + C(f,gh) - C(f,g)h.$$

A first order, abelian deformation of (the product of) the algebra is a new product on the same space, a $*$-product,

$$f * g = fg + \lambda C(f,g), \quad C(f,g) = C(g,f), \quad f,g \in \mathcal{A},$$

such that, to first order in λ, $(f*g)*h = f*(g*h)$. Here λ is a formal parameter and it is implicit that the original algebra must be extended to the commutative algebra of polynomials in λ. The condition of associativity is equivalent, to first order in λ, to the condition that C be closed, $dC = 0$.

We continue to discuss first order deformations, until further notice. If there is a one-form E, such that $C = dE$, then the deformation is said to be trivial. The reason for this is as follows. For $f \in \mathcal{A}$, let $f_\lambda := f + \lambda E(f)$. Then to first order in λ,

$$f_\lambda g_\lambda = (fg)_\lambda + \lambda C(f_\lambda, g_\lambda),$$

so that the first order mapping $f \mapsto f + \lambda E(f)$ is a first order isomorphism from $\mathcal{A}$ to the deformed algebra.

2.2. A trivial $*$-product.

Let $\mathcal{A}$ be the coordinate algebra of $\mathbb{R}^n$,

$$A = \mathbb{C}[x_1, ..., x_n],$$

and $\mathcal{A}_\lambda$ the same with a deformed product. Notice that this algebra does not have a unit. To prove that every deformation is trivial to all orders (we are here talking about formal deformations by infinite series that may or not converge for any $\lambda \in \mathbb{C}$), let

$$\Phi : \mathcal{A} \to \mathcal{A}, \quad \Phi(f*) = f,$$

where $f*$ is obtained from f by replacing $x_i x_j \ldots$ by $x_i * x_j * \ldots$. This map takes a Poincaré-Birkhoff-Witt basis for $\mathcal{A}_\lambda$ to a Poincaré-Birkhoff-Witt basis for $\mathcal{A}$ and is an algebra isomorphism. To relate this to the foregoing discussion of first order deformations, consider the one-form

$$\lambda E(f) = (f*) - f, \quad f* = \Phi^{-1}(f).$$

We have $((fg)*) = (f*) * (g*)$ and to first order in λ,

$$\begin{aligned}\lambda dE(f,g) &= [(f*) * (g*) - fg] - f[(g*) - g] - [(f*) - f]g \\ &= f * g - fg \;=\; \lambda C(f,g).\end{aligned}$$

For a more concrete example take $n = 1, C(f,g) = f_2 g_2$, where $f = f_1 + x f_2$ is the unique decomposition of f in terms of two even polynomials f_1, f_2. In this case $C = dE$ with

$$E(f) = -\frac{1}{2x}(\partial f_1 + x \partial f_2) = -\frac{1}{2x}\partial f + \frac{1}{2x} f_2.$$

This is a polynomial, though each term separately is not. But the first term is formally closed (a derivation), and

$$dE(f,g) = -\frac{1}{2x}\big(g_2 f_1 + g_2 f_1 - f g_2 - g f_2\big) = f_2 g_2 = C(f,g).$$

To find a nontrivial deformation we must go beyond this example, and one way to go is to introduce relations.

2.3. Algebras defined by relations.

Let M be the algebraic variety

$$M = \mathbb{R}^n / R,$$

where R is a set of polynomial relations. Let $\mathcal{A}$ be the coordinate algebra of M, namely

$$\mathcal{A} = \mathbb{C}[x_1, ..., x_n]/R.$$

Let $\mathcal{A}_\lambda$ be a deformed algebra and let $\Phi : \mathcal{A} \to \mathcal{A}$ be the map introduced above, that takes $f* \mapsto f$. Let R_* be the relations of $\mathcal{A}_\lambda$ and $R_\lambda = \Phi R_*$; then the two algebras are isomorphic if there is an invertible mapping on $\mathbb{R}^n$ that takes R_λ to R.

2.4. First non trivial example.

Let $n = 1$, N a natural number, $r \in \mathbb{R}$, and $R = x^N - r$. The Poincaré-Birkhoff-Witt basis of $\mathcal{A} = \mathbb{C}[x]/R$ is $\{1, x, x^2, \ldots, x^{N-1}\}$. Define a deformed product by setting

$$x^k * x^l = \begin{cases} x^{k+l} & , \quad k + l < N, \\ x^{k+l} + \lambda x^{k+l-N} & , \quad k + l \geq N. \end{cases}$$

Then $R_* = (x*)^N - (r + \lambda),\ R_\lambda = x^N - (r + \lambda)$ and a trivializing map is given by

$$x \mapsto (1 + \lambda/r)^{1/N} x.$$

Two things can go wrong. (1) This map does not exist if $r = 0$, and in that case the deformation is not trivial. (2) If $r \neq 0$, then the trivializing map is an infinite power series in λ.

We have $f * g = fg + \lambda C(f, g)$, with

$$C(x^k, x^l) = \begin{cases} 0 \quad\quad\ , & k + l < N, \\ x^{k+l-N}, & k + l \geq N. \end{cases}$$

There is no need to add higher order corrections in $f * g$; the product as it stands is associative, to all orders in λ. We have $C = dE$, with $E(x^k) = (k/Nr)x^k, \quad k = 1, ..., N - 1$.

Conclusion: Although C is first order, exact, and the $*$-product is associative to all orders, nevertheless an infinite power series was needed to trivialize it. In this particular case the appearance of infinite series is more or less innocuous, but that is not always the case, as we shall see.

2.5. Conic sections.

Let $\mathcal{A} = \mathbb{C}[x, y]/R$, with $R = y^2 - x^2 - r^2$. There is a unique decomposition

$$f = f_1 + y f_2,$$

where f_1, f_2 are polynomials in x. Take $C(f, g) = f_2 g_2$. This two-form is closed. The deformed relation is $y^2 - x^2 - (r^2 + \lambda)$. These relations define equivalent algebraic varieties if $r^2 \neq 0$ (when $r \neq 0$ there is a neighbourhood of $\lambda = 0$ in which the two varieties are equivalent), so in this case the deformation is trivial. But if $r = 0$, when the initial variety is a cone and has a singularity at $x = y = 0$, the deformation is not trivial. We shall return to cones in Section 4.

The polynomial one form

$$E(f) = \frac{1}{2r^2}[x\partial f_1 + y f_2 + xy\partial f_2] = \frac{x}{2r^2}\partial f + \frac{1}{2y} f_2$$

(in the last expression $\partial y = x/y$) is formally cohomologous to $E'(f) := \frac{1}{2y} f_2$, and

$$dE(f, g) = dE'(f, g) = -\frac{1}{2y}[f_1 g_2 + f_2 g_1 - f g_2 - f_2 g] = f_2 g_2.$$

This shows that the first order deformation is trivial when $r^2 \neq 0$. The derivation included in the definition of E, to eliminate the non polynomiality of E', does not exist when $r^2 = 0$. We notice that the obstruction is the constant term in f_2; in other words the value of $C(f, g)$ at the origin.

2.6. Application, conformal anomaly ?

We may consider the algebra of polynomials on Dirac's cone. Of interest is the projective cone, one studies functions with a fixed degree of homogeneity, normally a negative integer. In scalar field theory the degree is -1. The projective cone can be covered by two charts. If y is a set of coordinates for $\mathbb{R}^6$ then a pair of charts is given by $x_\mu^\pm = y_\mu/(y_5 \pm y_6),\ \mu = 1, 2, 3, 4$, each mapping most of the projective cone onto Minkowski space. To define a function on compactified Minkowski space we may take a pair of functions on Minkowski space, with the relation $f_1(x) = f_2(x/x^2)$. The ordinary product would be $fg = (f_1, f_2)(g_1, g_2) = (f_1g_1, f_2g_2)$. It can probably be deformed, and such deformations may perhaps relate to the conformal anomaly, but that will be left for another time.

3 Field theory on anti-de Sitter space

Let $\mathcal{A} = \mathbb{C}[x^0, ..., x^3]$, the coordinate algebra of polynomial functions on Minkowski space, without constant term. The metric is $g_{ij}dx^i dx^j = (dx^0)^2 - \sum_{i=1}^3 (dx^i)^2$. For every $a \in \mathcal{A}$ let $a = a_+ + a_-$ be the decomposition into an even and an odd part. We regard the algebra as the space of pairs,

$$A = \{(a_+, a_-)\},$$

with the product

$$AB = (a_+b_+ + a_-b_-, a_+b_- + a_-b_+),$$

or equivalently as the even subalgebra of $\mathbb{C}[x^0, \ldots, x^3, y]/(y^2 - 1)$, with elements

$$a = a_+ + ya_-.$$

We deform the product, setting

$$A * B = AB - \rho x^2 a_- b_-, \quad x^2 := g_{ij}x^i x^j, \quad \rho > 0.$$

The deformed algebra is the even subalgebra of

$$\mathbb{C}[x^0, \ldots, x^3, y]/(y^2 - 1 + \rho x^2),$$

which is a coordinate algebra of polynomial functions on anti-de Sitter space.

Referring back to Section 2.3, we notice a great deal of similarity. The above $*$-product is associative to all orders in ρ, no higher order terms are needed. Furthermore, $C(a, b) = \rho x^2 a_- b_-$ defines an exact two-form, $C = dE$ with $E(a) = (\rho/2)x^2 a_-$. But the "trivializing" map takes $a \mapsto a_+ + y\sqrt{1 + \rho x^2}a_-$, and here we find an infinite power series in the generators of the algebra, not just in the parameter. And this series is not entire but has singularities, for any ρ different from zero. There is no sensible point of view that would allow us to regard the two algebras as equivalent.

Conclusion. The algebra of coordinate functions on anti-de Sitter space is a nontrivial deformation of that of Minkowski space; this in spite of the fact that both spaces are smooth manifolds. The deformations are not classified by Harrison cohomology, trivial in this case. It is true that Harrison cohomology does classify first order deformations, but it must be understood that, even if $a * b = ab + \lambda C(a, b)$, with no higher order terms, and this is associative to all orders, trivialization is not localized at finite order. Vanishing of Harr^2 merely tells us that the term of first order in λ can be removed, to be replaced by terms of higher order.

Abelian $*$-products thus appear in two different ways. In the cases when $\mathrm{Harr}^2 = 0$ the obstructions are singularities of the "trivializing" map. But this does not diminish our interest in trying to understand those abelian $*$-products that owe their non-triviality to the existence of Harrison cohomology, and that is the subject of the rest of this paper.

4 Hochshild homology of the cone uv = 0

This section and the next take up the study of the cone algebra introduced in 2.5, with a more convenient choice of coordinates.

Let $M = \{u, v \in \mathbb{R}^2, uv = 0\}$, and let $\mathcal{A}$ be the commutative algebra

$$\mathcal{A} = \mathbb{C}[u, v]/uv.$$

We study the complex

$$0 \leftarrow C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} \cdots$$

where $C_n \in \mathcal{A}^{\otimes n}$, and

$$\partial(a_1 \otimes a_2 \otimes \cdots \otimes a_n) = a_1 a_2 \otimes a_3 \otimes \cdots - a_1 \otimes a_2 a_3 \otimes a_4 \otimes \cdots + \cdots + (-1)^n a_1 \otimes \cdots \otimes a_{n-1} a_n.$$

Let Z_n, B_n denote the subspaces of closed, respectively exact subspaces of C_n:

$$Z_n = \{a \in C_n, \partial a = 0\}, \quad B_n = \partial C_{n+1}.$$

Theorem 4.1 *The Hochschild homology $H_n = Z_n/B_n$ of this complex is of dimension 2 for $n \geq 1$, and is generated as a vector space by $u \otimes v \otimes u \otimes \cdots$ and $v \otimes u \otimes v \otimes \cdots$.*

Proof. For every n-chain a we have

$$\partial(u \otimes a) = ua_1 \otimes a_2 \otimes \cdots - u \otimes da,$$

therefore every n-chain with initial factor ua_1 is cohomologous with one with initial factor u, and every n-chain is cohomologous with $u \otimes a + v \otimes b$. Now suppose that $u \otimes a$ is closed. Then

$$u \otimes a = u \otimes vb_1 \otimes b_2 \otimes \cdots \approx u \otimes v \otimes db,$$

and so on. Since no n-chain a with each factor a_i linear in u, v is exact, the theorem is proved. $\square$

5 Hochschild cohomology of the cone uv = 0

The cochains are valued in the unital augmentation $\mathcal{A}_1$ of $\mathcal{A}$, and

$$dC(a_1, \ldots, a_n) = a_1 C(a_2, ..., a_n) - C(da) + (-1)^n C(a_1, \ldots, a_{n-1}) a_n.$$

For any n-cochain C, let C' denote the restriction of C to *closed*, *linear* chains. Thus C' is defined by the values $C'(a) = C(a)$, with $a = u \otimes v \otimes u \otimes \cdots$ and $a = v \otimes u \otimes v \otimes \cdots$. The set of all restricted cochains form a complex with differential

$$d'C'(a) = a_1 C'(a_2, ...a_n) + (-1)^n C'(a_1, ..., a_{n-1}) a_n$$

A restricted $2n$-cochain is closed if

$$C'_{2n}(u, v, ..., v) = C'_{2n}(v, u, ..., u),$$

and exact if it is valued in $\mathcal{A}$. A restricted $2n+1$-form is closed if

$$vC'_{2n+1}(u,v,...,u) + uC'_{2n+1}(v,u,...,v) = 0,$$

which implies that $C(u,...,u) = u\alpha(u), C(v,...,v) = v\beta(v)$, with $\alpha(u),\beta(v)$ in the unital augmentation of $\mathcal{A}$, and it is exact if $n>1$ and $\alpha+\beta\in\mathcal{A}$. The cohomology class of $2n$-cochain is the value of $C(u,v,u,\cdots)$ at the origin.

Proposition 5.1. *For every closed n-cochain $\tilde{C}$ of the restricted complex there is a closed Hochschild cochain C such that the restriction C' of C is equal to $\tilde{C}$.*

Proof. To find such C we begin by setting $C' = \tilde{C}$ for linear, closed arguments, then attempt to define C for nonlinear arguments by induction in the polynomial degree, using

$$dC(a) = a_1C(a_2,...) - C(da) + (-1)^nC(...,a_{n-1})a_n = 0,$$

with the observation that the argument of the middle term is of higher total polynomial degree than the others. The obstruction is $da = 0$ with a not linear; thus $a = db$. For an inductive proof, we must show that, for $a = db$, the last relation holds by virtue of similar relations involving arguments of lower order. In fact,

$$\begin{aligned} dC(db) &= b_1b_2C(b_3,...,b_{n+1}) - b_1C(db^+) \\ &\quad + (-1)^n\big(C(db^-)b_{n+1} + (-1)^{n+1}C(b_1,...,b_{n-1})b_nb_{n+1}\big), \end{aligned}$$

where $b^+ = b_2\otimes\cdots\otimes b_{n+1}$ and $b^- = b_1\otimes\cdots b_n$. By the induction hypothesis the second and third term are

$$\begin{aligned} &- b_1\Big(b_2C(b_3,\cdots,b_{n+1}) + (-1)^nC(b_2,\cdots,b_n)b_{n+1}\Big) \\ &+ (-1)^n\Big(b_1C(b_2,\cdots,b_n) + (-1)^nC(b_1,\cdots,b_n)\Big)b_{n+1}, \end{aligned}$$

and substitution gives the desired result, zero. $\square$

Proposition 5.1. *If C is any closed cochain, and its restriction is exact, then C is exact.*

Proof. We must show that there is E such that

$$C(a) = a_1E(a_2,...) - E(da) + (-1)^nE(...,a_{n-1})a_n.$$

By hypothesis this is true if a is linear and closed, and that establishes a basis for induction in the total polynomial degree of a. The obstruction is again $a = db$, and it is necessary to show that $C(db)$ is the same as

$$\begin{aligned} &b_1b_2E(b_3,\cdots,b_{n+1}) - b_1E(db^+) \\ &+ (-1)^n\Big(E(db^-)b_{n+1} + (-1)^nE(b_1,\cdots,b_{n-1})b_nb_{n+1}\Big). \end{aligned}$$

The induction hypothesis gives us $C(b^+)$ and $C(b^-)$; we use that to eliminate $E(db+)$ and $E(db^-)$ to get

$$\begin{aligned} &b_1b_2E(b_3,\cdots E(b_3,\cdots,b_{n+1}) + E(b_1,\cdots,b_{n-1})b_nb_{n+1} \\ &- b_1\Big(b_2E(b_3,\cdots,b_{n+1}) + (-1)^{n+1}E(b_2,\cdots,b_n)b_{n+1} - C(b^+)\Big) \\ &+ (-1)^n\Big(b_1E(b_2,\cdots,b_n) + (-1)^{n+1}E(b_1,\cdots,b_{n-1})b_n - C(b^-)\Big)b_{n+1}, \end{aligned}$$

which is precisely $C(db)$. $\square$

Together, the two propositions give us the following result:

Theorem 5.3 *The cohomology of the Hochshild complex is equivalent to the cohomology of the restricted complex.*

6 Geometric cochains of the cone $uv = 0$

We intend to discover which, if any, geometric properties of the variety are reflected in the Hochschild complex.

6.1. Points. Consider a collection of "geometric n-cochains" $q_1 \otimes \cdots \otimes q_n$, where q_i is a point in M, and the pairing

$$\langle q_1 \otimes \cdots \otimes q_n | a_1 \otimes \cdots \otimes a_n \rangle = \delta_{nm} a_1(q_1) \cdots a_n(q_n),$$

of geometric cochains with Hochschild chains. Here $a_1, \cdots, a_m$ are the same m-chains as before, with $a_i \in \mathcal{A}$, the algebra of coordinate functions on $M = \mathbb{C}[u, v]/uv$.

The differential is $dq_1 = q_1 \otimes q_1$ and

$$d(q_1 \otimes q_2 \otimes \cdots) = (q_1 \otimes q_1) \otimes q_3 \otimes \cdots - q_1 \otimes (q_2 \otimes q_2) \otimes q_3 \otimes \cdots + \cdots \;;$$

this ensures the relation of duality,

$$\langle dq|a \rangle = \langle p|da \rangle.$$

In this complex of points there is no cohomology. Notice that these cochains are distributions (delta-functions) valued in $\mathbb{C}$.

The intuitive meaning of this is that the points do not contain any information about those geometric properties of the variety that are related to the existence of deformations. Instead, the tangent vectors do contain such information. The tangent space is two-dimensional at the singular point, one-dimensional elsewhere.

6.2. Points and tangent vectors. Let p_i denote a tangent vector at the point q_i, and consider chains made up of both q's and p's. The pairing is

$$\langle q_1 \otimes p_2 \otimes \cdots | a_1 \otimes a_2 \otimes \cdots \rangle = a_1(q_1)(pa_2)(q_2) \cdots.$$

To preserve duality we define

$$\begin{aligned} &d(p_1 \otimes q_2 \otimes \cdots) = dp_1 \otimes q_2 \otimes \cdots - p_1 \otimes (dq_2) \otimes \cdots + \cdots, \\ &dq_1 = q_1 \otimes q_1, \quad dp_2 = p_2 \otimes q_2 + q_2 \otimes p_2. \end{aligned}$$

Let subscript zero refer to the origin, then $q_0 = 0$ and $p_0 \otimes p_0$ is closed. The 2-cohomology is carried by $p_0 \otimes p_0'$. In particular,

$$\langle \partial_u \otimes \partial_v | u \otimes v \rangle = 1.$$

Conclusion. The existence of the antisymmetric part of H^2 on the 2-cone is clearly a reflection of the fact that the dimension of tangent space is discontinuous at the singularity, but I cannot say that the existence of the symmetric part has been clarified in any deep sense. I hope to improve this type of analysis and to generalize it for algebraic manifolds in general.

7 Homology of cones

Consider the algebra $\mathbb{C}[x_1, \cdots, x_n]/g$, where $g = g(x)$ is a second order, homogeneous polynomial,

$$g = g^{ij}x_ix_j.$$

The proof of the following is similar to that of Theorem 4.1; what is used is the fact that the relations are second order, homogeneous.

Theorem 7.1. *Every chain is cohomologous to a linear chain, and no linear chain is exact, so H_k can be identified with the space of closed, linear chains.*

Proposition 7.2. *Let L_k denote the space of alternating, linear k-chains, then $H_1 = L_1$ and for $k \geq 2, H_k = \oplus_{p \leq k/2} L_{k-2p}$.*

Examples. For small k the coefficients of representatives of the homology are

$$\begin{aligned}
&k = 2 : \alpha^{ij} + \alpha g^{ij},\\
&k = 3 : \alpha^{ijk} + \alpha^i g^{jk} - \alpha^j g^{ik} + \alpha^k g^{ij},\\
&k = 4 : \alpha^{ijkl} + \alpha^{kl} g^{ij} + \alpha^{il} g^{jk} - \alpha^{jl} g^{ik} + \alpha^{ij} g^{kl} - \alpha^{kj} g^{li} + \alpha^{ki} g^{lj}\\
&\qquad\qquad + \alpha(g^{il} g^{jk} - g^{ik} g^{jl} + g^{ij} g^{kl}),
\end{aligned}$$

where the α's are complex, alternating.

8 Cohomology of cones

Exactly as in the case of the 2-cone, one proves that $H^*(\mathrm{Hom}(\mathcal{A}^{\otimes n}, \mathcal{A}))$ is the same as for the restricted complex based on linear, closed chains. For example:

Restricted one-forms are closed (never exact) if

$$dC(x_i \wedge x_j) = 0 = g^{ij} dC(x_i \otimes x_j),$$

which reduces to $g^{ij} x_i C(x_j) = 0$.

Restricted 2-forms are defined for arguments $x_i \wedge x_j$ and $g^{ij} x_i \otimes x_j$. A calculation shows that a 2-form is closed if

$$g^{ij} x_i C(x_j \wedge x_k) = 0,$$

and exact if it is symmetric and valued in $\mathcal{A}$.

Restricted 3-forms are defined on $x_i \wedge x_j \wedge x_k$ and on

$$\bar{\alpha} := (\alpha^i g^{jk} - \alpha^j g^{ik} + \alpha^k g^{ij})(x_i \otimes x_j \otimes x_k).$$

They are closed if

$$g^{ij} x_i C(x_j \wedge x_k \wedge x_l) = 0$$

and

$$(g^{il} g^{jk} - g^{jl} g^{ik} + g^{kl} g^{ij}) C(x_j \otimes x_k \otimes x_l) = 0,$$

and exact if they vanish on $x_i \wedge x_j \wedge x_k$ and, for some $Q^i \in \mathcal{A}$, $C(\bar{\alpha}) = \sum_i x_i Q^i$.

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