

Lagrangian $Sp(3)$ BRST symmetry for irreducible gauge theories

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Abstract

The Lagrangian $Sp(3)$ BRST symmetry for irreducible gauge theories is constructed in the framework of homological perturbation theory. The canonical generator of this extended symmetry is shown to exist. A gauge-fixing procedure specific to the standard antibracket-antifield formalism, that leads to an effective action, which is invariant under all the three differentials of the $Sp(3)$ algebra, is given.

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1 Introduction

The crucial feature of the BRST method [1]–[13] is the recursive pattern of homological perturbation theory [14]–[20], which allows one to prove the existence of the BRST symmetry itself. The machinery of homological perturbation theory has been adapted to cover the BRST-anti-BRST transformation in both Lagrangian and Hamiltonian formulations [21]–[38]. In the context of extended BRST symmetries, the Hamiltonian version of the $Sp(3)$ BRST symmetry has recently been developed in [39]–[41].

In this paper we show that the methods of homological perturbation theory can be extended in order to cover the construction of the Lagrangian

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$Sp(3)$ BRST symmetry in the irreducible case. This can be done by triplicating each differential appearing in the antibracket-antifield BRST formalism. We begin with the triplication of the gauge transformations of a given irreducible theory, which allows us to determine the correct ghost spectrum. Consequently, we develop a proper construction of the exterior longitudinal tricomplex to ensure that the cohomologies associated with each of the three exterior longitudinal derivatives are isomorphic to the cohomology of the standard exterior longitudinal derivative along the gauge orbits from the BRST description of the initial irreducible theory. The most difficult part is the construction of the Koszul-Tate tricomplex. Due to the fact that the complete description of the gauge orbits obtained by triplicating the gauge symmetries is not accompanied by the triplication of the equations of motion, each of the three Koszul-Tate differentials contains, besides the standard canonical part, also a noncanonical component, which acts on some supplementary antifields (bar and tilde variables). The Koszul-Tate tricomplex is constructed to furnish a triresolution of the algebra of smooth functions defined on the stationary surface of field equations. The problem of constructing the $Sp(3)$ BRST algebra is further transferred at the level of its canonical generator, that is solution to the so-called classical master equation of the $Sp(3)$ formalism. By means of properly extending the homological perturbation theory, the canonical generator is shown to exist. The final step of our treatment consists in giving a gauge-fixing procedure that ensures the invariance of the effective action under all three BRST symmetries that compose the Lagrangian $Sp(3)$ algebra.

The paper is organized as follows. Section 2 reviews the basic aspects of the standard antibracket-antifield formalism. In Section 3 we present the main ideas of the Lagrangian BRST $Sp(3)$ theory. Section 4 focuses on the construction of the exterior longitudinal tricomplex, while Section 5 approaches the construction of the Koszul-Tate triresolution. In Section 6 we prove the existence of the canonical generator of the $Sp(3)$ theory as solution to the extended classical master equation. The gauge-fixing procedure is accomplished in Section 7. Section 8 exemplifies the general procedure for abelian gauge fields. In Section 9 we present the main conclusions of the paper.

2 Brief review of standard antibracket-antifield formalism

We begin with a theory described by the Lagrangian action

$$S_0[\Phi^{\alpha_0}] = \int d^D x \mathcal{L}(\Phi^{\alpha_0}, \partial_{\mu_1} \Phi^{\alpha_0}, \dots, \partial_{\mu_1 \dots \mu_k} \Phi^{\alpha_0}), \quad (1)$$

invariant under the (infinitesimal) gauge transformations

$$\delta_\varepsilon \Phi^{\alpha_0} = Z_{\alpha_1}^{\alpha_0} \varepsilon^{\alpha_1}, \quad (2)$$

written in De Witt's condensed notations. The fields and gauge parameters are assumed to respectively display the Grassmann parities $\epsilon(\Phi^{\alpha_0}) \equiv \epsilon_{\alpha_0}$, $\epsilon(\varepsilon^{\alpha_1}) \equiv \epsilon_{\alpha_1}$, while those of the gauge generators are of course $\epsilon(Z_{\alpha_1}^{\alpha_0}) = \epsilon_{\alpha_0} + \epsilon_{\alpha_1}$. The action $S_0[\Phi^{\alpha_0}]$ is supposed to be a local functional, so it may depend on the fields and their space-time derivatives up to a finite order, say, k . In addition, the gauge transformations (2) are taken to be irreducible, i.e., the gauge generators are independent¹. We work in the general case of an open gauge algebra

$$\frac{\delta^R Z_{\alpha_1}^{\alpha_0}}{\delta \Phi^{\beta_0}} Z_{\beta_1}^{\beta_0} - (-)^{\epsilon_{\alpha_1} \epsilon_{\beta_1}} \frac{\delta^R Z_{\beta_1}^{\alpha_0}}{\delta \Phi^{\beta_0}} Z_{\alpha_1}^{\beta_0} = Z_{\gamma_1}^{\alpha_0} C_{\alpha_1 \beta_1}^{\gamma_1} - \frac{\delta^R S_0}{\delta \Phi^{\beta_0}} M_{\alpha_1 \beta_1}^{\beta_0 \alpha_0}, \quad (3)$$

where the structure functions $C_{\alpha_1 \beta_1}^{\gamma_1}$ and $M_{\alpha_1 \beta_1}^{\beta_0 \alpha_0}$ may involve the fields, possess the Grassmann parities $\epsilon(C_{\alpha_1 \beta_1}^{\gamma_1}) = \epsilon_{\alpha_1} + \epsilon_{\beta_1} + \epsilon_{\gamma_1}$, $\epsilon(M_{\alpha_1 \beta_1}^{\beta_0 \alpha_0}) = \epsilon_{\alpha_0} + \epsilon_{\beta_0} + \epsilon_{\alpha_1} + \epsilon_{\beta_1}$, and display the symmetry properties

$$C_{\alpha_1 \beta_1}^{\gamma_1} = -(-)^{\epsilon_{\alpha_1} \epsilon_{\beta_1}} C_{\beta_1 \alpha_1}^{\gamma_1}, \quad (4)$$

$$M_{\alpha_1 \beta_1}^{\beta_0 \alpha_0} = -(-)^{\epsilon_{\alpha_1} \epsilon_{\beta_1}} M_{\beta_1 \alpha_1}^{\beta_0 \alpha_0} = -(-)^{\epsilon_{\alpha_0} \epsilon_{\beta_0}} M_{\alpha_1 \beta_1}^{\alpha_0 \beta_0}. \quad (5)$$

The upper index R (L) signifies the right (left) derivative. Taking into account the restrictions imposed by the Jacobi identity, we are led to some new structure functions, $D_{\alpha_1 \beta_1 \gamma_1}^{\alpha_0 \delta_1}$, defined by

$$\begin{aligned} & \frac{\delta^R C_{\alpha_1 \beta_1}^{\delta_1}}{\delta \Phi^{\alpha_0}} Z_{\gamma_1}^{\alpha_0} - C_{\alpha_1 \tau_1}^{\delta_1} C_{\beta_1 \gamma_1}^{\tau_1} + \\ & (-)^{\epsilon_{\alpha_1}(\epsilon_{\beta_1} + \epsilon_{\gamma_1})} \left(\frac{\delta^R C_{\beta_1 \gamma_1}^{\delta_1}}{\delta \Phi^{\alpha_0}} Z_{\alpha_1}^{\alpha_0} - C_{\beta_1 \tau_1}^{\delta_1} C_{\gamma_1 \alpha_1}^{\tau_1} \right) + \end{aligned}$$

¹This means that the solutions to the equation $Z_{\alpha_1}^{\alpha_0} f^{\alpha_1} \approx 0$ are trivial, $f^{\alpha_1} = \frac{\delta^R S_0}{\delta \Phi^{\alpha_0}} f^{\alpha_0 \alpha_1}$, where the weak equality ' \approx ' refers to the stationary surface of field equations, $\Sigma : \frac{\delta^R S_0}{\delta \Phi^{\alpha_0}} = 0$.

$$\begin{aligned}
& (-)^{\epsilon_{\gamma_1}(\epsilon_{\alpha_1} + \epsilon_{\beta_1})} \left(\frac{\delta^R C_{\gamma_1 \alpha_1}^{\delta_1}}{\delta \Phi^{\alpha_0}} Z_{\beta_1}^{\alpha_0} - C_{\gamma_1 \tau_1}^{\delta_1} C_{\alpha_1 \beta_1}^{\tau_1} \right) = \\
& 3 \frac{\delta^R S_0}{\delta \Phi^{\alpha_0}} D_{\alpha_1 \beta_1 \gamma_1}^{\alpha_0 \delta_1}.
\end{aligned} \tag{6}$$

The procedure can be continued step by step in order to reveal the entire higher-order tensor structure provided by the gauge theory under discussion.

At the level of the BRST formalism, the entire gauge structure of a theory is completely captured by the BRST differential, s . The key point of the antibracket-antifield formalism is represented by the construction of the BRST differential along the general line of homological perturbation theory. The BRST operator starts like $s = \delta + D + \dots$, where δ is the Koszul-Tate differential, and D is the exterior longitudinal derivative along the gauge orbits. The Koszul-Tate operator provides an homological resolution of $C^\infty(\Sigma)$ (smooth functions on the stationary surface), while D takes into account the gauge invariances on Σ . The main feature of s is its nilpotency, $s^2 = 0$. Denoting by $(,)$ the antibracket, and by S the canonical generator of the Lagrangian BRST symmetry, $s\bullet = (\bullet, S)$, the nilpotency of s is equivalent to the classical master equation

$$(S, S) = 0, \tag{7}$$

with the boundary condition

$$S = S_0 + \Phi_{\alpha_0}^* Z_{\alpha_1}^{\alpha_0} \eta^{\alpha_1} + \dots, \tag{8}$$

where $\Phi_{\alpha_0}^*$ represent the antifields associated with the original fields, and η^{α_1} are the ghosts corresponding to the gauge parameters. If we make the notation Φ^α for the fields (original fields and ghosts) and Φ_α^* for their antifields, such that $(\Phi^\alpha, \Phi_\beta^*) = \delta_\beta^\alpha$, we have that the Koszul-Tate differential and the exterior longitudinal derivative along the gauge orbits can be canonically generated through

$$\delta \Phi_\alpha^* = (\Phi_\alpha^*, S)|_{\text{ghosts}=0}, \tag{9}$$

$$D \Phi^\alpha = (\Phi^\alpha, S)|_{\text{antifields}=0}. \tag{10}$$

The classical master equation is equivalent to the family of equations $\delta \begin{smallmatrix} [k+1] \\ S \end{smallmatrix} = D \begin{smallmatrix} [k] \\ S \end{smallmatrix} \left[\begin{smallmatrix} [0] \\ S, \dots, S \end{smallmatrix} \right]$, where $S = \sum_{k=0}^{\infty} \begin{smallmatrix} [k] \\ S \end{smallmatrix}$, $\text{res} \begin{pmatrix} [k] \\ S \end{pmatrix} = k$ and $\text{gh} \begin{pmatrix} [k] \\ S \end{pmatrix} = 0$. The degree denoted by res controls the grading of the Koszul-Tate complex and is named resolution degree, $\text{res}(\delta) = -1$. The existence of the solution to the above family is guaranteed by the acyclicity of the Koszul-Tate differential

at positive resolution degrees. With the help of the solution to the master equation one constructs the gauge-fixed action

$$S_\psi = S \left[\Phi^\alpha, \Phi_\alpha^* = \frac{\delta^L \psi}{\delta \Phi^\alpha} \right], \quad (11)$$

where $\psi[\Phi^\alpha]$ is the gauge-fixing fermion, which is traditionally chosen to depend only on the fields.

3 General ideas of the Lagrangian $Sp(3)$ BRST symmetry

Let us investigate what happens if we consider a complete redundant description of the gauge orbits obtained by triplicating the gauge transformations (2)

$$\delta_\varepsilon \Phi^{\alpha_0} = Z_{\alpha_1}^{\alpha_0} (\varepsilon_1^{\alpha_1} + \varepsilon_2^{\alpha_1} + \varepsilon_3^{\alpha_1}). \quad (12)$$

Alternatively, we can write down (12) under a matrix-like form

$$\delta_\varepsilon \Phi^{\alpha_0} = Z_{A_1}^{\alpha_0} \varepsilon^{A_1}, \quad (13)$$

where

$$Z_{A_1}^{\alpha_0} = (Z_{\alpha_1}^{\alpha_0}, Z_{\alpha_1}^{\alpha_0}, Z_{\alpha_1}^{\alpha_0}), \quad \varepsilon^{A_1} = \begin{pmatrix} \varepsilon_1^{\alpha_1} \\ \varepsilon_2^{\alpha_1} \\ \varepsilon_3^{\alpha_1} \end{pmatrix}, \quad A_1 = (\alpha_1, \alpha_1, \alpha_1). \quad (14)$$

Consequently, the gauge generators $Z_{A_1}^{\alpha_0}$ will be no longer independent, but two-stage reducible, with the reducibility relations and reducibility matrices respectively expressed by

$$Z_{A_1}^{\alpha_0} Z_{B_1}^{A_1} = 0, \quad Z_{B_1}^{A_1} Z_{\gamma_1}^{B_1} = 0, \quad (15)$$

$$Z_{B_1}^{A_1} = \begin{pmatrix} \mathbf{0} & \delta_{\beta_1}^{\alpha_1} & -\delta_{\beta_1}^{\alpha_1} \\ -\delta_{\beta_1}^{\alpha_1} & \mathbf{0} & \delta_{\beta_1}^{\alpha_1} \\ \delta_{\beta_1}^{\alpha_1} & -\delta_{\beta_1}^{\alpha_1} & \mathbf{0} \end{pmatrix}, \quad Z_{\gamma_1}^{B_1} = \begin{pmatrix} -\delta_{\gamma_1}^{\beta_1} \\ -\delta_{\gamma_1}^{\beta_1} \\ -\delta_{\gamma_1}^{\beta_1} \end{pmatrix}. \quad (16)$$

According to the ideas of the usual antifield-BRST theory [13], we can construct a standard BRST symmetry for the system described by the action (1), subject to the second-stage reducible gauge transformations (12).

Our main concern is to go deeper and see if there is possible to generate an extended Lagrangian BRST symmetry of the type $Sp(3)$ for our gauge theory, i.e., if we can algebraically construct three anticommuting differentials $(s_a)_{a=1,2,3}$

$$s_a s_b + s_b s_a = 0, \quad a, b = 1, 2, 3. \quad (17)$$

In other words, we investigate the construction of a differential tricomplex $(s_1, s_2, s_3, \mathcal{A})$, trigraded in terms of the ghost tridegree $\text{trigh} = (\text{gh}_1, \text{gh}_2, \text{gh}_3)$, where

$$\text{trigh}(s_1) = (1, 0, 0), \text{trigh}(s_2) = (0, 1, 0), \text{trigh}(s_3) = (0, 0, 1), \quad (18)$$

such that each of three differentials decomposes like

$$s_a = \delta_a + D_a + \dots \quad (19)$$

We will refer to this differential tricomplex as the $Sp(3)$ *BRST tricomplex*. The above decomposition is made according to the following ideas:

1. The three operators $(\delta_a)_{a=1,2,3}$ should define a differential tricomplex $(\delta_1, \delta_2, \delta_3, \mathcal{A}')$, trigraded by the resolution tridegree, denoted by $\text{trires} = (\text{res}_1, \text{res}_2, \text{res}_3)$, with $\text{trires}(\delta_1) = (-1, 0, 0)$, $\text{trires}(\delta_2) = (0, -1, 0)$, $\text{trires}(\delta_3) = (0, 0, -1)$. Moreover, this triple complex is required to furnish a triresolution of $C^\infty(\Sigma)$, where Σ is the stationary surface of field equations, $\Sigma : \delta^R S_0 / \delta \Phi^{\alpha_0} = 0$. We will call it the *Koszul-Tate triresolution*. (For a more detailed approach to triresolutions, see Appendix A.)
2. The three operators $(D_a)_{a=1,2,3}$ will act on a certain trigraded algebra \mathcal{A}'' such that $(D_1, D_2, D_3, \mathcal{A}'')$ is a differential tricomplex (the *exterior longitudinal tricomplex*), the corresponding tridegree being denoted by $\text{trideg} = (\text{deg}_1, \text{deg}_2, \text{deg}_3)$, with $\text{trideg}(D_1) = (1, 0, 0)$, $\text{trideg}(D_2) = (0, 1, 0)$, $\text{trideg}(D_3) = (0, 0, 1)$. In addition, we ask that the cohomologies associated with each $(D_a)_{a=1,2,3}$ are isomorphic to the cohomology of the standard exterior longitudinal derivative along the gauge orbits from the BRST description of the initial irreducible theory.

The relationship between these three types of gradings is that if a function(al) F has $\text{trires}(F) = (r_1, r_2, r_3)$ and $\text{trideg}(F) = (d_1, d_2, d_3)$, then its ghost tridegree is $\text{trigh}(F) = (g_1, g_2, g_3)$, where $g_a = d_a - r_a$, $a = 1, 2, 3$. In order to implement the three symmetries $(s_a)_{a=1,2,3}$ at the Lagrangian level, we will construct three different antibrackets, consider three extended classical master equations, and show that there exists a common solution S to these equations, that generates the three BRST-like symmetries. In consequence, we have to associate three antifields with each field (original fields and ghosts). Just like in the case of the antibracket-antifield BRST-anti-BRST symmetry, the triplication of gauge symmetries is not accompanied by a triplication of the equations that define the stationary surface. This is

why we will need some supplementary variables (bar and tilde variables) in the antifield sector to kill some nontrivial co-cycles in the homologies of the Koszul-Tate operators. As a result, each of the Koszul-Tate differentials will decompose as a sum between a canonical and a noncanonical part, where the noncanonical operators will act only on these supplementary antifields. Regarding the exterior longitudinal tricomplex, we will see that a good starting point is the triplication of the gauge symmetries and the introduction of the corresponding reducibility relations, which will allow us to determine the correct ghost spectrum.

It is interesting to notice another aspect induced by the triplication of the gauge symmetries. We can always define a simple differential complex associated with a given tricomplex. For example, (D, \mathcal{A}'') , with

$$D = D_1 + D_2 + D_3, \quad (20)$$

can be regarded as the simple differential complex associated with the tri-complex $(D_1, D_2, D_3, \mathcal{A}'')$ if we grade the algebra \mathcal{A}'' according to $\deg = \deg_1 + \deg_2 + \deg_3$, hence $\deg(D) = 1$. The (weak) nilpotency and anticommutativity of $(D_a)_{a=1,2,3}$ on \mathcal{A}'' , $D_a D_b + D_b D_a \approx 0$, $a, b = 1, 2, 3$, together with this simple graduation, imply that

$$(D_a D_b + D_b D_a \approx 0, \ a, b = 1, 2, 3) \Leftrightarrow D^2 \approx 0, \quad (21)$$

where the weak equality refers to the stationary surface. Moreover, D is nothing but the exterior longitudinal derivative associated with this new reducible and complete description of the gauge orbits (*extended exterior longitudinal derivative*), and the cohomology of D is isomorphic to those of $(D_a)_{a=1,2,3}$, and therefore isomorphic to that of the standard exterior longitudinal derivative. Along the same line, (δ, \mathcal{A}') , with

$$\delta = \delta_1 + \delta_2 + \delta_3, \quad (22)$$

graded by the *total resolution degree* $\text{res} = \text{res}_1 + \text{res}_2 + \text{res}_3$ ($\text{res}(\delta) = -1$) is precisely a simple differential complex corresponding to $(\delta_1, \delta_2, \delta_3, \mathcal{A}')$, and we have that

$$(\delta_a \delta_b + \delta_b \delta_a = 0, \ a, b = 1, 2, 3) \Leftrightarrow \delta^2 \approx 0. \quad (23)$$

The operator δ (*extended Koszul-Tate differential*) clearly furnishes a resolution of the algebra $C^\infty(\Sigma)$, the relationship between the triresolution and resolution properties being

$$(H_{0,0,0}(\delta_a) = C^\infty(\Sigma)) \Leftrightarrow H_0(\delta) = C^\infty(\Sigma), \quad (24)$$

$$(H_{i,j,k}(\delta_a) = 0, \ i, j, k \geq 0, \ i + j + k > 0) \Leftrightarrow H_l(\delta) = 0, \ l > 0, \quad (25)$$

where $H_{i,j,k}(\delta_a)$ signifies the space of elements F with $\text{tri}(\text{res}(F)) = (i, j, k)$, that are δ_a -closed modulo δ_a -exact, and $H_l(\delta)$ means the cohomological space spanned by the objects G with $\text{res}(G) = l$, that are δ -closed modulo δ -exact.

Finally, from the $Sp(3)$ differential tricomplex $(s_1, s_2, s_3, \mathcal{A})$ we can construct the simple differential complex (s, \mathcal{A}) graded according to the *total ghost number* $\text{tgh} = \text{gh}_1 + \text{gh}_2 + \text{gh}_3$, where

$$s = s_1 + s_2 + s_3, \quad (26)$$

and the *extended BRST differential* s has $\text{tgh}(s) = 1$. It is clear that it splits as

$$s = \delta + D + \dots, \quad (27)$$

where δ and D are exactly the extended Koszul-Tate differential, respectively, the extended exterior longitudinal derivative introduced before. The relationship between the three simple degrees is expressed by $\text{tgh} = \text{deg} - \text{res}$. The rest of the terms, denoted by ' \dots ', are required in order to ensure the nilpotency of s , i.e.,

$$s^2 = 0, \quad (28)$$

which is linked to the $Sp(3)$ algebra defining relations (17) through

$$(s_a s_b + s_b s_a = 0, \ a, b = 1, 2, 3) \Leftrightarrow s^2 = 0. \quad (29)$$

These observations stay at the basis of the procedure we are going to develop in order to investigate the existence of the Lagrangian BRST symmetry of the type $Sp(3)$: instead of explicitly proving the existence of $(s_a)_{a=1,2,3}$ satisfying the properties mentioned in the above, we will prove the existence of the associated extended BRST symmetry and show that it splits in exactly three pieces matching the trigraduation.

4 Exterior longitudinal tricomplex. Tricanonical structure

We begin with the triplication of the gauge generators and gauge parameters like in (14), as well as with the corresponding reducibility functions (16). In this way, our departure point is a second-stage reducible complete description of the gauge orbits associated with the initial irreducible one. Following the lines of the standard BRST formalism related to the construction of the exterior longitudinal complex in the reducible case, we introduce the ghosts η^{A_1} for the gauge generators $Z_{A_1}^{\alpha_0}$, the ghosts of ghosts π^{B_1} for the first-stage reducibility functions $Z_{B_1}^{A_1}$, and the ghosts of ghosts of ghosts λ^{γ_1} for the

second-stage reducibility functions $Z_{\gamma_1}^{B_1}$, such that the entire ghost spectrum is given by

$$\eta^{A_1} \equiv (\eta_1^{\alpha_1}, \eta_2^{\alpha_1}, \eta_3^{\alpha_1}), \quad \pi^{B_1} \equiv (\pi_1^{\beta_1}, \pi_2^{\beta_1}, \pi_3^{\beta_1}), \quad \lambda^{\gamma_1}, \quad (30)$$

where the Grassmann parities of the ghosts are valued like

$$\epsilon(\eta_a^{\alpha_1}) = \epsilon_{\alpha_1} + 1, \quad \epsilon(\pi_a^{\beta_1}) = \epsilon_{\beta_1}, \quad \epsilon(\lambda^{\gamma_1}) = \epsilon_{\gamma_1} + 1, \quad (31)$$

where $a = 1, 2, 3$. The algebra \mathcal{A}'' will then be the polynomial algebra generated by η^{A_1} , π^{B_1} and λ^{γ_1} , with coefficients from $C^\infty(\Sigma)$. In order to properly grade this algebra, we set the obvious trigraduation

$$\text{trideg}(\eta_1^{\alpha_1}) = (1, 0, 0), \quad \text{trideg}(\eta_2^{\alpha_1}) = (0, 1, 0), \quad \text{trideg}(\eta_3^{\alpha_1}) = (0, 0, 1), \quad (32)$$

$$\text{trideg}(\pi_1^{\beta_1}) = (0, 1, 1), \quad \text{trideg}(\pi_2^{\beta_1}) = (1, 0, 1), \quad \text{trideg}(\pi_3^{\beta_1}) = (1, 1, 0), \quad (33)$$

$$\text{trideg}(\lambda^{\gamma_1}) = (1, 1, 1), \quad \text{trideg}(\Phi^{\alpha_0}) = (0, 0, 0). \quad (34)$$

By applying the rules of the standard Lagrangian BRST formalism, we can construct the extended exterior longitudinal derivative D associated with this new reducible description of the gauge orbits. The action of D on the original fields, as well as on the generators of the polynomial algebra \mathcal{A}'' , is defined by

$$D\Phi^{\alpha_0} = Z_{A_1}^{\alpha_0} \eta^{A_1}, \quad D\eta^{A_1} = Z_{B_1}^{A_1} \pi^{B_1} + \dots, \quad (35)$$

$$D\pi^{B_1} = Z_{\gamma_1}^{B_1} \lambda^{\gamma_1} + \dots, \quad D\lambda^{\gamma_1} = \dots, \quad (36)$$

where the terms generically denoted by ‘ \dots ’ are such as to ensure $D^2 \approx 0$, and they can be explicitly determined by using the formulas (3–6) related to the gauge structure of the investigated theory. In terms of the accompanying simple graduation, we have that $\deg(\Phi^{\alpha_0}) = 0$, $\deg(\eta_a^{\alpha_1}) = 1$, $\deg(\pi_a^{\alpha_1}) = 2$, $a = 1, 2, 3$, $\deg(\lambda^{\alpha_1}) = 3$. On behalf of the relations (20) and (35–36), we can write down the complete definitions of the three exterior longitudinal derivatives on the generators from the exterior longitudinal tricomplex in the context of the trigraduation governed by trideg , under the form

$$D_a \Phi^{\alpha_0} = Z_{\alpha_1}^{\alpha_0} \eta_a^{\alpha_1}, \quad (37)$$

$$D_a \eta_b^{\alpha_1} = \varepsilon_{abc} \pi_c^{\alpha_1} + \frac{1}{2} (-)^{\epsilon_{\beta_1}} C_{\beta_1 \gamma_1}^{\alpha_1} \eta_a^{\gamma_1} \eta_b^{\beta_1}, \quad (38)$$

$$\begin{aligned} D_a \pi_b^{\alpha_1} &= -\delta_{ab} \lambda^{\alpha_1} + \frac{1}{2} (-)^{\epsilon_{\beta_1}+1} C_{\beta_1 \gamma_1}^{\alpha_1} \eta_a^{\gamma_1} \pi_b^{\beta_1} + \\ &\frac{1}{12} (-)^{\epsilon_{\delta_1}} \varepsilon_{bcd} C_{\beta_1 \gamma_1}^{\alpha_1} C_{\delta_1 \varepsilon_1}^{\gamma_1} \eta_a^{\varepsilon_1} \eta_c^{\delta_1} \eta_d^{\beta_1}, \end{aligned} \quad (39)$$

$$D_a \lambda^{\alpha_1} = \frac{1}{2} (-)^{\epsilon_{\beta_1}} C_{\beta_1 \gamma_1}^{\alpha_1} \eta_a^{\gamma_1} \lambda^{\beta_1} + \frac{1}{12} (-)^{\epsilon_{\delta_1}+1} \left(C_{\beta_1 \gamma_1}^{\alpha_1} C_{\delta_1 \epsilon_1}^{\gamma_1} - (-)^{\epsilon_{\beta_1}(\epsilon_{\delta_1}+\epsilon_{\epsilon_1})} C_{\delta_1 \gamma_1}^{\alpha_1} C_{\epsilon_1 \beta_1}^{\gamma_1} \right) \eta_a^{\epsilon_1} \eta_b^{\delta_1} \pi_b^{\beta_1} \quad (40)$$

where ε_{abc} are completely antisymmetric and constant, with $\varepsilon_{123} = +1$. It can be checked by direct computation that $(D_1, D_2, D_3, \mathcal{A}'')$ in the presence of the definitions (37–40) indeed determines a differential tricomplex, and, in addition, that the cohomologies associated with each $(D_a)_{a=1,2,3}$ are isomorphic to the cohomology of the standard exterior longitudinal derivative.

As we have previously noticed, we intend to define a generator that is common to all three BRST-like symmetries $(s_a)_{a=1,2,3}$. In view of this, we define three antibrackets, to be denoted by $(,)_a$, with $a = 1, 2, 3$, which further requires the introduction of three antifields, conjugated to each field/ghost in one of the antibrackets. In order to make this tricanonical structure compatible with the trigraduation, and preserve the symmetry between its components, we set the ghost tridegrees of the antibrackets as $\text{trigh}((,)_1) = (1, 0, 0)$, $\text{trigh}((,)_2) = (0, 1, 0)$, $\text{trigh}((,)_3) = (0, 0, 1)$. It is understood that the antibrackets satisfy all the basic properties of the antibracket in the standard antibracket-antifield formalism [8], [13], such as odd Grassmann behaviour, etc. Moreover, we ask that the trigrading governing the exterior longitudinal tricomplex does not interfere with that characteristic to the Koszul-Tate triresolution, so we impose that $\text{trideg}(\delta_a) = (0, 0, 0)$, $\text{trires}(D_a) = (0, 0, 0)$, $\text{trires}(\Phi^A) = (0, 0, 0)$, which remains valid with respect to the corresponding simple gradings, where Φ^A is a collective notation for all fields and ghosts

$$\Phi^A \equiv (\Phi^{\alpha_0}, \eta_a^{\alpha_1}, \pi_a^{\alpha_1}, \lambda^{\alpha_1}). \quad (41)$$

Thus, we are led to the following antifield spectrum

$$\Phi_A^{*(a)} \equiv (\Phi_{\alpha_0}^{*(a)}, \eta_{b\alpha_1}^{*(a)}, \pi_{b\alpha_1}^{*(a)}, \lambda_{\alpha_1}^{*(a)}), \quad a, b = 1, 2, 3. \quad (42)$$

The main features of the antifields are

$$\epsilon(\Phi_A^{*(a)}) = \epsilon(\Phi^A) + 1, \quad (43)$$

$$\text{trires}(\Phi_A^{*(1)}) = (gh_1(\Phi^A) + 1, gh_2(\Phi^A), gh_3(\Phi^A)) = -\text{trigh}(\Phi_A^{*(1)}), \quad (44)$$

$$\text{trires}(\Phi_A^{*(2)}) = (gh_1(\Phi^A), gh_2(\Phi^A) + 1, gh_3(\Phi^A)) = -\text{trigh}(\Phi_A^{*(2)}), \quad (45)$$

$$\text{trires}(\Phi_A^{*(3)}) = (gh_1(\Phi^A), gh_2(\Phi^A), gh_3(\Phi^A) + 1) = -\text{trigh}(\Phi_A^{*(3)}), \quad (46)$$

and they result from the basic properties of the antibrackets plus the correlation among the various trigradings. The upper index (a) indicates in which antibracket is an antifield $\Phi_A^{*(a)}$ conjugated to a field Φ^A , such that the fundamental antibrackets read as

$$\left(\Phi^A, \Phi_B^{*(a)}\right)_b = \delta_B^A \delta_b^a. \quad (47)$$

For notational simplicity, we will make the convention to represent a function(al) F of resolution tridegree (r_1, r_2, r_3) by $\overset{[r_1, r_2, r_3]}{F}$ and one of ghost tridegree (g_1, g_2, g_3) by $\overset{(g_1, g_2, g_3)}{F}$. Similarly, we will use the notations $\overset{[r]}{F}$ and $\overset{(g)}{F}$ in connection with the total resolution degree, respectively, total ghost number. Thus, for a function(al) depending only on the antifields, $\overset{[r_1, r_2, r_3]}{F}$, we can write $\overset{(-r_1, -r_2, -r_3)}{F}$, and the same for res and tgh. Concretely, we have that the antifields (42) have the following resolution tridegrees

$$\overset{[1,0,0]^*(1)}{\Phi}_{\alpha_0}, \overset{[0,1,0]^*(2)}{\Phi}_{\alpha_0}, \overset{[0,0,1]^*(3)}{\Phi}_{\alpha_0}, \overset{[2,0,0]^*(1)}{\eta}_{1\alpha_1}, \overset{[1,1,0]^*(2)}{\eta}_{1\alpha_1}, \overset{[1,0,1]^*(3)}{\eta}_{1\alpha_1}, \quad (48)$$

$$\overset{[1,1,0]^*(1)}{\eta}_{2\alpha_1}, \overset{[0,2,0]^*(2)}{\eta}_{2\alpha_1}, \overset{[0,1,1]^*(3)}{\eta}_{2\alpha_1}, \overset{[1,0,1]^*(1)}{\eta}_{3\alpha_1}, \overset{[0,1,1]^*(2)}{\eta}_{3\alpha_1}, \overset{[0,0,2]^*(3)}{\eta}_{3\alpha_1}, \quad (49)$$

$$\overset{[1,1,1]^*(1)}{\pi}_{1\alpha_1}, \overset{[0,2,1]^*(2)}{\pi}_{1\alpha_1}, \overset{[0,1,2]^*(3)}{\pi}_{1\alpha_1}, \overset{[2,0,1]^*(1)}{\pi}_{2\alpha_1}, \overset{[1,1,1]^*(2)}{\pi}_{2\alpha_1}, \overset{[1,0,2]^*(3)}{\pi}_{2\alpha_1}, \quad (50)$$

$$\overset{[2,1,0]^*(1)}{\pi}_{3\alpha_1}, \overset{[1,2,0]^*(2)}{\pi}_{3\alpha_1}, \overset{[1,1,1]^*(3)}{\pi}_{3\alpha_1}, \overset{[2,1,1]^*(1)}{\lambda}_{\alpha_1}, \overset{[1,2,1]^*(2)}{\lambda}_{\alpha_1}, \overset{[1,1,2]^*(3)}{\lambda}_{\alpha_1}, \quad (51)$$

and hence the total resolution degrees

$$\overset{[1]^*(a)}{\Phi}_{\alpha_0}, \overset{[2]^*(a)}{\eta}_{b\alpha_1}, \overset{[3]^*(a)}{\pi}_{b\alpha_1}, \overset{[4]^*(a)}{\lambda}_{\alpha_1}, \quad a, b = 1, 2, 3. \quad (52)$$

Employing the usual ideas of the BRST formalism, *we ask that the extended exterior longitudinal derivative is inferred by means of the total antibracket $(,) = (,)_1 + (,)_2 + (,)_3$ with a generator S of total ghost number equal to zero.* Actually, we will ask more, and take S to be of ghost tridegree $(0, 0, 0)$. In this manner we are certain that $(D_a)_{a=1,2,3}$ will be recovered via the antibrackets $((,)_a)_{a=1,2,3}$ with one and the same generator

$$D_a \Phi^A = \left(\Phi^A, S\right)_a \Big|_{\Phi_B^{*(b)}=0}. \quad (53)$$

Then, from (53) and (37–39), it is easy to see that the generator S begins like

$$\begin{aligned} S &= S_0 [\Phi^{\alpha_0}] + \Phi_{\alpha_0}^{*(a)} Z_{\alpha_1}^{\alpha_0} \eta_a^{\alpha_1} + \varepsilon_{abc} \eta_{b\alpha_1}^{*(a)} \pi_c^{\alpha_1} - \pi_{a\alpha_1}^{*(a)} \lambda^{\alpha_1} + \dots = \\ &\overset{[0]}{S} + \overset{[1]}{S} + \overset{[2]}{S} + \overset{[3]}{S} + \dots \end{aligned} \quad (54)$$

The first term, $S_0[\Phi^{\alpha_0}] = S^{[0]}$ is not necessary in order to get the relations (53), but the reason for considering it will be clear at the level of the Koszul-Tate triresolution. The notations $S^{[n]}$ are motivated by the standard approach, and refer to a decomposition according to the total resolution degree. All the terms involved so far with the functional S satisfy the condition to have the ghost tridegree equal to $(0, 0, 0)$.

5 Koszul-Tate triresolution

We have reached now the construction of the Koszul-Tate tricomplex. Actually, as we want to reproduce the action of the Koszul-Tate operators on the antifields through the antibrackets between the antifields and the generator S at all ghosts equal to zero, the form of the Koszul-Tate operators is somehow fixed by the form of S . Unfortunately, this property cannot be ensured in the framework of the Lagrangian $Sp(3)$ BRST formalism as we require in the meantime that the Koszul-Tate tricomplex generates a triresolution of $C^\infty(\Sigma)$. The incompatibility between these two requirements is induced by the fact that the complete description of the gauge orbits obtained by triplicating the gauge symmetries is not accompanied by the triplication of the equations of motion. Then, as we cannot weaken the triresolution property, we admit in change that the Koszul-Tate operators are not entirely reproduced by means of the antibrackets, or, in other words, we expect that these differentials decompose in a canonical and a noncanonical part.

The boundary conditions (54) allow us to write $\delta = \delta_1 + \delta_2 + \delta_3$ on the polynomial algebra \mathcal{A}' generated by the antifields $(\Phi_A^{*(a)})_{a=1,2,3}$, with coefficients from $C^\infty(I)$, where I is the space of all field histories. Initially, $(\delta_a)_{a=1,2,3}$ are defined on the generators of \mathcal{A}' through

$$\delta_a \Phi_{\alpha_0}^{*(b)} = \left(\Phi_{\alpha_0}^{*(b)}, S \right)_a \Big|_{\text{ghosts}=0} = -\delta_{ab} \frac{\delta^L S_0}{\delta \Phi^{\alpha_0}}, \quad (55)$$

$$\delta_a \eta_{c\alpha_1}^{*(b)} = \left(\eta_{c\alpha_1}^{*(b)}, S \right)_a \Big|_{\text{ghosts}=0} = (-)^{\epsilon_{\alpha_1}} \delta_{ab} \Phi_{\alpha_0}^{*(c)} Z_{\alpha_1}^{\alpha_0}, \quad (56)$$

$$\delta_a \pi_{c\alpha_1}^{*(b)} = \left(\pi_{c\alpha_1}^{*(b)}, S \right)_a \Big|_{\text{ghosts}=0} = (-)^{\epsilon_{\alpha_1}+1} \delta_{ab} \varepsilon_{cde} \eta_{e\alpha_1}^{*(d)}, \quad (57)$$

$$\delta_a \lambda_{\alpha_1}^{*(b)} = \left(\lambda_{\alpha_1}^{*(b)}, S \right)_a \Big|_{\text{ghosts}=0} = (-)^{\epsilon_{\alpha_1}+1} \delta_{ab} \pi_{d\alpha_1}^{*(d)}, \quad (58)$$

and they obviously respect the grading properties: $\text{trires}(\delta_1) = (-1, 0, 0)$, $\text{trires}(\delta_2) = (0, -1, 0)$, $\text{trires}(\delta_3) = (0, 0, -1)$, $\text{res}(\delta) = \text{res}(\delta_a) = -1$. However, there appear two major problems linked to the above definitions. First

of all, δ is not a differential as δ^2 fails to vanish on the antifields $\pi_{c\alpha_1}^{*(b)}$ and $\lambda_{\alpha_1}^{*(b)}$. Moreover, δ cannot realize a resolution of $C^\infty(\Sigma)$, and $(\delta_a)_{a=1,2,3}$ also cannot determine a triresolution of the same algebra, as long as there appear nontrivial co-cycles at positive resolution degrees. Indeed, $\Phi_A^{*(2)}$ and $\Phi_A^{*(3)}$ are co-cycles for δ_1 , $\Phi_A^{*(1)}$ and $\Phi_A^{*(3)}$ for δ_2 , $\Phi_A^{*(1)}$ and $\Phi_A^{*(2)}$ for δ_3 , while all $\Phi_A^{*(1)} - \Phi_A^{*(2)}$, $\Phi_A^{*(2)} - \Phi_A^{*(3)}$ and $\Phi_A^{*(3)} - \Phi_A^{*(1)}$ are nontrivial co-cycles for δ .

An appropriate way to remedy both inconveniences is to add some new variables in the antifield sector, such that all the above co-cycles become trivial, and then modify the actions of the Koszul-Tate operators on $\pi_{c\alpha_1}^{*(b)}$ and $\lambda_{\alpha_1}^{*(b)}$ in such a way that they become nilpotent. Analysing the structure and the resolution tridegrees of the nontrivial co-cycles mentioned before, it looks that we need to introduce three such new variables (bar variables) for each triplet of antifields

$$(\Phi_A^{*(1)}, \Phi_A^{*(2)}, \Phi_A^{*(3)}) \rightarrow (\bar{\Phi}_A^{(1)}, \bar{\Phi}_A^{(2)}, \bar{\Phi}_A^{(3)}), \quad (59)$$

with the properties

$$\epsilon(\bar{\Phi}_A^{(a)}) = \epsilon(\Phi_A^{*(a)}) + 1 = \epsilon(\Phi^A), \quad a = 1, 2, 3, \quad (60)$$

$$\text{trires}(\bar{\Phi}_A^{(1)}) = (gh_1(\Phi^A), gh_2(\Phi^A) + 1, gh_3(\Phi^A) + 1), \quad (61)$$

$$\text{trires}(\bar{\Phi}_A^{(2)}) = (gh_1(\Phi^A) + 1, gh_2(\Phi^A), gh_3(\Phi^A) + 1), \quad (62)$$

$$\text{trires}(\bar{\Phi}_A^{(3)}) = (gh_1(\Phi^A) + 1, gh_2(\Phi^A) + 1, gh_3(\Phi^A)), \quad (63)$$

(which further induce that $\text{trigh}(\bar{\Phi}_A^{(a)}) = -\text{trires}(\bar{\Phi}_A^{(a)})$, and $\text{res}(\bar{\Phi}_A^{(a)}) = \text{tgh}(\Phi^A) + 2 = -\text{tgh}(\bar{\Phi}_A^{(a)})$), and on which $(\delta_a)_{a=1,2,3}$ act like

$$\delta_a \bar{\Phi}_A^{(b)} = (-)^{\epsilon(\Phi^A)} \varepsilon_{abc} \Phi_A^{*(c)}, \quad (64)$$

where the phase-factor in (64) was chosen for convenience. More precisely, the bar variables and their properties are expressed by

$$\begin{matrix} [0,1,1]^{(1)} & [1,0,1]^{(2)} & [1,1,0]^{(3)} & [1,1,1]^{(1)} & [2,0,1]^{(2)} & [2,1,0]^{(3)} \\ \bar{\Phi}_{\alpha_0} & \bar{\Phi}_{\alpha_0} & \bar{\Phi}_{\alpha_0} & \bar{\eta}_{1\alpha_1} & \bar{\eta}_{1\alpha_1} & \bar{\eta}_{1\alpha_1} \end{matrix}, \quad (65)$$

$$\begin{matrix} [0,2,1]^{(1)} & [1,1,1]^{(2)} & [1,2,0]^{(3)} & [0,1,2]^{(1)} & [1,0,2]^{(2)} & [1,1,1]^{(3)} \\ \bar{\eta}_{2\alpha_1} & \bar{\eta}_{2\alpha_1} & \bar{\eta}_{2\alpha_1} & \bar{\eta}_{3\alpha_1} & \bar{\eta}_{3\alpha_1} & \bar{\eta}_{3\alpha_1} \end{matrix}, \quad (66)$$

$$\begin{matrix} [0,2,2]^{(1)} & [1,1,2]^{(2)} & [1,2,1]^{(3)} & [1,1,2]^{(1)} & [2,0,2]^{(2)} & [2,1,1]^{(3)} \\ \bar{\pi}_{1\alpha_1} & \bar{\pi}_{1\alpha_1} & \bar{\pi}_{1\alpha_1} & \bar{\pi}_{2\alpha_1} & \bar{\pi}_{2\alpha_1} & \bar{\pi}_{2\alpha_1} \end{matrix}, \quad (67)$$

$$\begin{matrix} [1,2,1]^{(1)} & [2,1,1]^{(2)} & [2,2,0]^{(3)} & [1,2,2]^{(1)} & [2,1,2]^{(2)} & [2,2,1]^{(3)} \\ \bar{\pi}_{3\alpha_1} & \bar{\pi}_{3\alpha_1} & \bar{\pi}_{3\alpha_1} & \bar{\lambda}_{\alpha_1} & \bar{\lambda}_{\alpha_1} & \bar{\lambda}_{\alpha_1} \end{matrix}, \quad (68)$$

while the definitions of δ_a acting on them read as

$$\delta_a \bar{\Phi}_{\alpha_0}^{(b)} = (-)^{\epsilon_{\alpha_0}} \varepsilon_{abc} \Phi_{\alpha_0}^{*(c)}, \quad \delta_a \bar{\eta}_{d\alpha_1}^{(b)} = (-)^{\epsilon_{\alpha_1}+1} \varepsilon_{abc} \eta_{d\alpha_1}^{*(c)}, \quad (69)$$

$$\delta_a \bar{\pi}_{d\alpha_1}^{(b)} = (-)^{\epsilon_{\alpha_1}} \varepsilon_{abc} \pi_{d\alpha_1}^{*(c)}, \quad \delta_a \bar{\lambda}_{\alpha_1}^{(b)} = (-)^{\epsilon_{\alpha_1}+1} \varepsilon_{abc} \lambda_{\alpha_1}^{*(c)}. \quad (70)$$

On account of (69–70), we change the definitions (57–58) like

$$\delta_a \pi_{c\alpha_1}^{*(b)} = (-)^{\epsilon_{\alpha_1}+1} \delta_{ab} \left(\varepsilon_{cde} \eta_{e\alpha_1}^{*(d)} + \bar{\Phi}_{\alpha_0}^{(c)} Z_{\alpha_1}^{\alpha_0} \right), \quad (71)$$

$$\delta_a \lambda_{\alpha_1}^{*(b)} = (-)^{\epsilon_{\alpha_1}+1} \delta_{ab} \left(\pi_{d\alpha_1}^{*(d)} + \bar{\eta}_{d\alpha_1}^{(d)} \right). \quad (72)$$

At this stage, one can see that we have removed all the initial nontrivial co-cycles from the homologies of δ_a and δ , and implemented the nilpotency on the antifields $\pi_{c\alpha_1}^{*(b)}$. In the meantime, we have also created new nontrivial co-cycles ($\bar{\Phi}_A^{(a)}$ respectively in the homologies of δ_a , and $\bar{\Phi}_A^{(1)} + \bar{\Phi}_A^{(2)} + \bar{\Phi}_A^{(3)}$ in that of δ), and have not yet restored the nilpotency on the generators $\lambda_{\alpha_1}^{*(b)}$. In view of this, the next step will be to still enlarge the antifield spectrum by some supplementary variables (tilde variables), one for each triplet of bar variables

$$(\bar{\Phi}_A^{(1)}, \bar{\Phi}_A^{(2)}, \bar{\Phi}_A^{(3)}) \rightarrow \tilde{\Phi}_A, \quad (73)$$

with the properties

$$\epsilon(\tilde{\Phi}_A) = \epsilon(\bar{\Phi}_A^{(a)}) + 1 = \epsilon(\Phi^A) + 1, \quad (74)$$

$$\text{trires}(\tilde{\Phi}_A) = (gh_1(\Phi^A) + 1, gh_2(\Phi^A) + 1, gh_3(\Phi^A) + 1), \quad (75)$$

or, equivalently,

$$\begin{matrix} [1,1,1] & [2,1,1] & [1,2,1] & [1,1,2] \\ \tilde{\Phi}_{\alpha_0}, & \tilde{\eta}_{1\alpha_1}, & \tilde{\eta}_{2\alpha_1}, & \tilde{\eta}_{3\alpha_1}, \end{matrix} \quad (76)$$

$$\begin{matrix} [1,2,2] & [2,1,2] & [2,2,1] & [2,2,2] \\ \tilde{\pi}_{1\alpha_1}, & \tilde{\pi}_{2\alpha_1}, & \tilde{\pi}_{3\alpha_1}, & \tilde{\lambda}_{\alpha_1}. \end{matrix} \quad (77)$$

We define the actions of δ_a on the tilde variables through

$$\delta_a \tilde{\Phi}_A = (-)^{\epsilon(\Phi^A)+1} \bar{\Phi}_A^{(a)}, \quad (78)$$

hence

$$\delta_a \tilde{\Phi}_{\alpha_0} = (-)^{\epsilon_{\alpha_0}+1} \bar{\Phi}_{\alpha_0}^{(a)}, \quad \delta_a \tilde{\eta}_{b\alpha_1} = (-)^{\epsilon_{\alpha_1}} \bar{\eta}_{b\alpha_1}^{(a)}, \quad (79)$$

$$\delta_a \tilde{\pi}_{b\alpha_1} = (-)^{\epsilon_{\alpha_1}+1} \bar{\pi}_{b\alpha_1}^{(a)}, \quad \delta_a \tilde{\lambda}_{\alpha_1} = (-)^{\epsilon_{\alpha_1}} \bar{\lambda}_{\alpha_1}^{(a)}, \quad (80)$$

and adjust the definitions (72) as

$$\delta_a \lambda_{\alpha_1}^{*(b)} = (-)^{\epsilon_{\alpha_1}+1} \delta_{ab} \left(\pi_{d\alpha_1}^{*(d)} + \bar{\eta}_{d\alpha_1}^{(d)} - \tilde{\Phi}_{\alpha_0} Z_{\alpha_1}^{\alpha_0} \right). \quad (81)$$

In this manner, we removed all nontrivial co-cycles at positive triresolution and resolution degrees from the homologies of δ_a and δ , and, meanwhile, recovered the nilpotency of δ .

In consequence, the total Koszul-Tate operator splits as

$$\delta = \delta_{\text{can}} + V, \quad (82)$$

with $\text{res}(\delta) = \text{res}(\delta_{\text{can}}) = \text{res}(V) = -1$, where δ_{can} gives the part from δ inferred via a canonical action, and the operator V acts only on the bar and tilde variables

$$V = \bar{V} + \tilde{V}. \quad (83)$$

Taking into account the relation $\delta = \delta_1 + \delta_2 + \delta_3$ and the trigraduation, we have that

$$\delta_a = \delta_{\text{cana}} + V_a, \quad a = 1, 2, 3, \quad (84)$$

$$V_a = \bar{V}_a + \tilde{V}_a, \quad a = 1, 2, 3, \quad (85)$$

$$\delta_{\text{cana}\bullet} = (\bullet, S)_a|_{\text{ghosts}=0}, \quad (86)$$

$$\bar{V}_a\bullet = (-)^{\epsilon(\Phi^A)} \varepsilon_{abc} \Phi_A^{*(c)} \frac{\delta^{R\bullet}}{\delta \bar{\Phi}_A^{(b)}}, \quad \tilde{V}_a\bullet = (-)^{\epsilon(\Phi^A)+1} \bar{\Phi}_A^{(a)} \frac{\delta^{R\bullet}}{\delta \tilde{\Phi}_A}, \quad (87)$$

hence

$$\text{trires}(V_1) = \text{trires}(\bar{V}_1) = \text{trires}(\tilde{V}_1) = (-1, 0, 0) = -\text{trigh}(V_1), \quad (88)$$

$$\text{trires}(V_2) = \text{trires}(\bar{V}_2) = \text{trires}(\tilde{V}_2) = (0, -1, 0) = -\text{trigh}(V_2), \quad (89)$$

$$\text{trires}(V_3) = \text{trires}(\bar{V}_3) = \text{trires}(\tilde{V}_3) = (0, 0, -1) = -\text{trigh}(V_3). \quad (90)$$

Under these circumstances, it can be shown that $(\delta_1, \delta_2, \delta_3, \mathcal{A}')$ indeed realizes a triresolution of $C^\infty(\Sigma)$, where the correct \mathcal{A}' is given by the polynomial algebra in the generators $(\Phi_A^{*(a)}, \bar{\Phi}_A^{(a)}, \tilde{\Phi}_A)$, with coefficients from $C^\infty(I)$, and the actions of the three Koszul-Tate differentials are defined with the help of the relations (55–56), (69–71) and (79–81). Consequently, the simple complex (δ, \mathcal{A}') graded in terms of total resolution degree, res , furnishes a resolution of $C^\infty(\Sigma)$. In conclusion, we had to give up the canonical action of the Koszul-Tate differentials in favour of constructing a true triresolution of the algebra $C^\infty(\Sigma)$. Some of the most important properties of the triresolutions, which will be used in the sequel at the proof of the existence of S , are given in Appendix A.

The modification of the canonical part of δ_a like in (71) and (81) attracts a change in the boundary conditions for S

$$\begin{aligned} S = & S_0[\Phi^{\alpha_0}] + \Phi_{\alpha_0}^{*(a)} Z_{\alpha_1}^{\alpha_0} \eta_a^{\alpha_1} + \left(\varepsilon_{abc} \eta_{b\alpha_1}^{*(a)} + \bar{\Phi}_{\alpha_0}^{(c)} Z_{\alpha_1}^{\alpha_0} \right) \pi_c^{\alpha_1} - \\ & \left(\pi_{a\alpha_1}^{*(a)} + \bar{\eta}_{a\alpha_1}^{(a)} - \tilde{\Phi}_{\alpha_0} Z_{\alpha_1}^{\alpha_0} \right) \lambda^{\alpha_1} + \dots \end{aligned} \quad (91)$$

6 Extended classical master equation

Once we have determined the extended exterior longitudinal tricomplex and realized a triresolution of $C^\infty(\Sigma)$, we are able to identify the algebra \mathcal{A} with the polynomial algebra

$$\mathcal{A} = \mathbf{C} \left[\Phi_A^{*(a)}, \bar{\Phi}_A^{(a)}, \tilde{\Phi}_A \right] \otimes C^\infty(I) \otimes \mathbf{C} \left[\Phi^A \right], \quad (92)$$

trigraded now in terms of the ghost tridegree $\text{trigh} = (\text{gh}_1, \text{gh}_2, \text{gh}_3)$. We extend the actions of δ_a on the fields and ghosts by requiring that

$$\delta_a \Phi^A = 0, \quad a = 1, 2, 3, \quad (93)$$

which ensures that $\text{trigh}(\delta_1) = (1, 0, 0)$, $\text{trigh}(\delta_2) = (0, 1, 0)$, $\text{trigh}(\delta_3) = (0, 0, 1)$ and $\text{tgh}(\delta_a) = \text{tgh}(\delta) = 1$. Now, we have to show that there exist three nilpotent operators $s_a : \mathcal{A} \rightarrow \mathcal{A}$, such that $(s_1, s_2, s_3, \mathcal{A})$ is a triple complex trigraded by trigh , and the extended BRST differential $s = s_1 + s_2 + s_3$ starts like $\delta + D + \dots$, where the supplementary terms are chosen such that $s^2 = 0$. Combining these results with those of the standard BRST formalism, we are led to the construction of the generator S with $\text{tgh}(S) = 0$, such that

$$s\bullet = (\bullet, S) + V\bullet, \quad (94)$$

where $(\bullet) = (\bullet)_1 + (\bullet)_2 + (\bullet)_3$. Demanding the nilpotency of s , we are led to solving the classical master equation of the Lagrangian $Sp(3)$ formalism

$$\frac{1}{2} (S, S) + VS = 0, \quad (95)$$

on account of the Jacobi identity for the antibracket and of the fact that V behaves like a derivation with respect to the total antibracket. Following the usual rules of homological perturbation theory, the equation (95) is equivalent to the tower of equations

$$\delta \begin{matrix} [k+1] \\ S \end{matrix} = D \begin{matrix} [k] \\ S \end{matrix} \left[\begin{matrix} [0] \\ S \end{matrix}, \dots, \begin{matrix} [k] \\ S \end{matrix} \right], \quad k \geq 0, \quad (96)$$

where

$$S = \sum_{p=0}^{\infty} \begin{matrix} [p] \\ S \end{matrix}, \quad \text{res} \left(\begin{matrix} [p] \\ S \end{matrix} \right) = p, \quad \text{tgh} \left(\begin{matrix} [p] \\ S \end{matrix} \right) = 0, \quad (97)$$

while the boundary conditions on S are (see (91))

$$\begin{matrix} [0] \\ S \end{matrix} = S_0 [\Phi^{\alpha_0}], \quad \begin{matrix} [1] \\ S \end{matrix} = \Phi_{\alpha_0}^{*(a)} Z_{\alpha_1}^{\alpha_0} \eta_a^{\alpha_1}, \quad (98)$$

$$S^{[2]} = \left(\varepsilon_{abc} \eta_{b\alpha_1}^{*(a)} + \bar{\Phi}_{\alpha_0}^{(c)} Z_{\alpha_1}^{\alpha_0} \right) \pi_c^{\alpha_1} + \dots, \quad (99)$$

$$S^{[3]} = - \left(\pi_{a\alpha_1}^{*(a)} + \bar{\eta}_{a\alpha_1}^{(a)} - \tilde{\Phi}_{\alpha_0} Z_{\alpha_1}^{\alpha_0} \right) \lambda^{\alpha_1} + \dots. \quad (100)$$

Equations (96) are nothing but the equations of the standard BRST theory, so the proof of the existence of their solutions is well-known. Indeed, it is enough to prove that $D^{[k]}$ is δ -closed for $k \geq 1$ in order to prove that the equations (96) possess solutions ($S^{[0]}$ and $S^{[1]}$ are purely boundary terms, so they are completely expressed by (98)). On the other hand, the δ -closure of $D^{[k]}$ results from the Jacobi identity for the total antibracket, so equations (96) possess solutions. Thus, the only matter to be dealt with remains the proof of the fact that s splits in precisely three pieces, $s = s_1 + s_2 + s_3$, of ghost tridegrees $(1, 0, 0)$, $(0, 1, 0)$, respectively, $(0, 0, 1)$, or, equivalently, that S is of ghost tridegree $(0, 0, 0)$.

Definition 1 *Let $F \in \mathcal{A}$. If F satisfies $\text{tgh}(F) = l > 0$ (respectively, $\text{tgh}(F) = l \geq 0$), then F is said to be of positive ghost tridegree (respectively, nonnegative ghost tridegree) if it can be decomposed as*

$$F = \sum_{i+j+k=l} F^{(i,j,k)}, \quad (101)$$

where $i \geq 0$, $j \geq 0$ and $k \geq 0$. (We understand that by virtue of the above notation one has $\text{trigh} F^{(i,j,k)} = (i, j, k)$.) Then, it is easy to see that the subset of \mathcal{A} provided by the polynomials in the ghosts and antifields with coefficients from $C^\infty(I)$ of positive ghost tridegree (respectively, nonnegative ghost tridegree) is a subalgebra, to be denoted by \mathcal{A}_{++} (respectively, \mathcal{A}_+). In particular, we have that $\mathcal{A}_{++} \subset \mathcal{A}_+$.

Next, we prove a positivity theorem, that will be useful in the sequel.

Theorem 1 (positivity theorem) *Let $F \in \mathcal{A}_{++}$ be such that $\text{res}(F) = m > 0$ and $\delta F = 0$. Then, there exists $P \in \mathcal{A}_+$ such that $\delta P = F$.*

Proof. We consider the component $F^{\{i,j,k\};[p,q,r]}$ of F with the properties $\text{trideg} \left(F^{\{i,j,k\};[p,q,r]} \right) = (i, j, k)$ and $\text{tri-res} \left(F^{\{i,j,k\};[p,q,r]} \right) = (p, q, r)$. The assumption $\text{res}(F) = m$ allows us to write that

$$F = \sum_{i,j,k} \left(\sum_{p+q+r=m} F^{\{i,j,k\};[p,q,r]} \right). \quad (102)$$

The condition $\delta F = 0$ implies that

$$\delta \left(\sum_{p+q+r=m} \frac{\{i,j,k\};[p,q,r]}{F} \right) = 0, \quad (103)$$

for every fixed triplet (i, j, k) , while the condition $F \in \mathcal{A}_{++}$ ensures that in the last sum appear only terms with $p \leq i$, $q \leq j$ and $r \leq k$, where $i + j + k > m$. Then, it follows that we are in the conditions of Theorem 4 (see the Appendix A), which enables us to state that there exists an element $\frac{\{i,j,k\}}{P}$ with $\text{trideg} \left(\frac{\{i,j,k\}}{P} \right) = (i, j, k)$ and $\text{res} \frac{\{i,j,k\}}{P} = m + 1$, such that

$$\sum_{p+q+r=m} \frac{\{i,j,k\};[p,q,r]}{F} = \delta \left(\frac{\{i,j,k\}}{P} \right), \quad (104)$$

which can be represented like

$$\frac{\{i,j,k\}}{P} = \sum_{\bar{p}+\bar{q}+\bar{r}=m+1} \frac{\{i,j,k\};[\bar{p},\bar{q},\bar{r}]}{P}, \quad (105)$$

where in the last sum are involved only terms with $\bar{p} \leq i$, $\bar{q} \leq j$ and $\bar{r} \leq k$, such that the total ghost degree of every subcomponent is $i + j + k - m - 1 \geq 0$ as $i + j + k > m$, which shows that $\frac{\{i,j,k\}}{P} \in \mathcal{A}_+$. Consequently, (102) and (104) show that we have

$$F = \delta P, \quad (106)$$

with

$$P = \sum_{i,j,k} \frac{\{i,j,k\}}{P}, \quad (107)$$

where $P \in \mathcal{A}_+$. This proves the theorem.

In terms of the notions introduced in the above, we can equivalently reformulate the property that S is of ghost tridegree $(0, 0, 0)$ like $S \in \mathcal{A}_+$. Indeed, if $\frac{(l,m,n)}{S}$ denotes the component of S with $\text{trigh} \left(\frac{(l,m,n)}{S} \right) = (l, m, n)$, then the condition $\text{tgh}(S) = 0$ implies that $l + m + n = 0$, while $S \in \mathcal{A}_+$ gives that $l \geq 0$, $m \geq 0$, $n \geq 0$, which ensures that indeed $l = 0$, $m = 0$, $n = 0$, hence $\text{trigh}(S) = (0, 0, 0)$. This equivalent property is proved by the next theorem.

Theorem 2 *Let S be a solution of the classical master equation (95). Then, S may be chosen of nonnegative ghost tridegree, $S \in \mathcal{A}_+$.*

Proof. We develop S according to the total resolution degree like in (97). Suppose that $\overset{[p]}{S}$ is of nonnegative ghost tridegree for $p \leq k$. Then, we have to prove that $\overset{[k+1]}{S}$ may be chosen to be of nonnegative ghost tridegree. As we have seen, the equation (95) is equivalent to the family (96). (We are only interested in the equations with $k > 0$, as $\overset{[0]}{S}$ and $\overset{[1]}{S}$ are purely boundary terms, completely given by (98), and they obviously display nonnegative ghost tridegrees.) The explicit form of $\overset{[k]}{D}$ is given by

$$\begin{aligned} \overset{[k]}{D} \left[\overset{[0]}{S}, \dots, \overset{[k]}{S} \right] = & -\frac{1}{2} \left(\sum_{m=1}^k \left(\overset{[m]}{S}, \overset{[k-m+1]}{S} \right)_{(\Phi, \Phi^*)} + \sum_{m=2}^k \left(\overset{[m]}{S}, \overset{[k-m+2]}{S} \right)_{(\eta, \eta^*)} + \right. \\ & \left. \sum_{m=3}^k \left(\overset{[m]}{S}, \overset{[k-m+3]}{S} \right)_{(\pi, \pi^*)} + \sum_{m=4}^k \left(\overset{[m]}{S}, \overset{[k-m+4]}{S} \right)_{(\lambda, \lambda^*)} \right), \end{aligned} \quad (108)$$

where $(\cdot)_{(\Phi, \Phi^*)}$ represents the total antibracket constructed with respect to the fields Φ^{α_0} and their antifields, $(\cdot)_{(\eta, \eta^*)}$ is the total antibracket involving only the ghosts $\eta_a^{\alpha_1}$ and their antifields, and so on. On account of the properties of the total antibracket, it is easy to see that in the right hand-side of (108) appear only components from \mathcal{A}_{++} , hence $\overset{[k]}{D} \in \mathcal{A}_{++}$. Indeed, let us consider a more general term of the type $\left(\overset{[p]}{S}, \overset{[q]}{S} \right)$, with $p \leq k$, $q \leq k$, where (\cdot) signifies the total antibracket with respect to all pairs field(ghost)/antifield. Then, as by assumption $\overset{[p]}{S}$ and $\overset{[q]}{S}$ are of nonnegative ghost tridegree, while the possible ghost tridegrees of the total antibracket are $(1, 0, 0)$, $(0, 1, 0)$ or $(0, 0, 1)$, it follows that $\left(\overset{[p]}{S}, \overset{[q]}{S} \right)$ is of positive ghost tridegree. Thus, the right hand-side of the equation (96) is of positive ghost tridegree. On the other hand, $\delta \overset{[k]}{D} = 0$ ($k > 0$), such that we are in the conditions of the positivity theorem. So, there exists $\overset{[k+1]}{S}$ such that $\delta \overset{[k+1]}{S} = \overset{[k]}{D}$, and, moreover, $\overset{[k+1]}{S}$ is of nonnegative ghost tridegree, $\overset{[k+1]}{S} \in \mathcal{A}_+$. This ends the proof.

As a consequence of the above theorem, we have that indeed $\text{trigh}(S) = (0, 0, 0)$, which shows that s splits precisely into three pieces $(s_a)_{a=1,2,3}$, of ghost tridegrees $(1, 0, 0)$, $(0, 1, 0)$, respectively, $(0, 0, 1)$. Moreover, we have that the classical master equation of the $Sp(3)$ formalism (95) is equivalent to three equations corresponding to the three different antibrackets

$$\frac{1}{2} (S, S)_a + V_a S = 0, \quad a = 1, 2, 3. \quad (109)$$

In this way, we have shown that there exists a generator of the Lagrangian BRST $Sp(3)$ symmetry, and hence this symmetry can be constructed for any (irreducible) gauge theory. However, on the one hand, the generator S is neither s_1 , nor s_2 , nor s_3 invariant. On the other hand, we need to infer a proper gauge-fixed action that can be used in the path integral, which is s_a -invariant. In this light, we expose in the sequel a gauge-fixing procedure that implements the s_a -invariances of the gauge-fixed action and presents in addition the desirable feature of ensuring a direct equivalence with the standard antibracket-antifield BRST formalism.

7 Gauge-fixing procedure

We begin by restoring an anticanonical structure for all the variables (and, in particular, for the bar and tilde ones) for bringing the classical master equation of the BRST $Sp(3)$ formalism to a more familiar form. In view of this, we focus for the moment on a single antibracket, for example on the first one, and forget about the other two, $(\cdot)_2$ and $(\cdot)_3$. In order to preserve the tricanonical structure, we cannot declare the existing variables, excepting Φ^A and $\Phi_A^{*(1)}$, conjugated in the first antibracket. This is why we need to extend the algebra of the BRST $Sp(3)$ tricomplex \mathcal{A} by adding the variables

$$(\rho_2^A, \rho_3^A, \kappa_1^A, \mu_2^A, \mu_3^A, \nu_1^A), \quad (110)$$

with the properties

$$\epsilon(\rho_2^A) = \epsilon(\rho_3^A) = \epsilon(\nu_1^A) = \epsilon(\Phi^A), \quad (111)$$

$$\epsilon(\kappa_1^A) = \epsilon(\mu_2^A) = \epsilon(\mu_3^A) = \epsilon(\Phi^A) + 1, \quad (112)$$

$$\text{trigh}(\rho_2^A) = (\text{gh}_1\Phi^A - 1, \text{gh}_2\Phi^A, \text{gh}_3\Phi^A + 1), \quad (113)$$

$$\text{trigh}(\rho_3^A) = (\text{gh}_1\Phi^A - 1, \text{gh}_2\Phi^A + 1, \text{gh}_3\Phi^A), \quad (114)$$

$$\text{trigh}(\nu_1^A) = (\text{gh}_1\Phi^A, \text{gh}_2\Phi^A + 1, \text{gh}_3\Phi^A + 1), \quad (115)$$

$$\text{trigh}(\kappa_1^A) = (\text{gh}_1\Phi^A - 1, \text{gh}_2\Phi^A + 1, \text{gh}_3\Phi^A + 1), \quad (116)$$

$$\text{trigh}(\mu_2^A) = (\text{gh}_1\Phi^A, \text{gh}_2\Phi^A + 1, \text{gh}_3\Phi^A), \quad (117)$$

$$\text{trigh}(\mu_3^A) = (\text{gh}_1\Phi^A, \text{gh}_2\Phi^A, \text{gh}_3\Phi^A + 1), \quad (118)$$

$$(\Phi_A^{*(3)}, \rho_2^B)_1 = \delta_A^B, \quad (\rho_3^A, \Phi_B^{*(2)})_1 = \delta_B^A, \quad (\bar{\Phi}_A^{(1)}, \kappa_1^B)_1 = \delta_A^B, \quad (119)$$

$$\left(\mu_3^A, \bar{\Phi}_B^{(2)}\right)_1 = \delta_B^A, \quad \left(\bar{\Phi}_A^{(3)}, \mu_2^B\right)_1 = \delta_A^B, \quad \left(\nu_1^A, \tilde{\Phi}_B\right)_1 = \delta_B^A. \quad (120)$$

So, if S is solution to the equation (95), then

$$S_1 = S + \Phi_A^{*(2)} \mu_2^A + \Phi_A^{*(3)} \mu_3^A + \bar{\Phi}_A^{(1)} \nu_1^A, \quad (121)$$

satisfies the equation

$$(S_1, S_1)_1 = 0, \quad (122)$$

which is nothing but the standard classical master equation in the first antibracket. As a consequence, we can apply the gauge-fixing procedure from the standard antibracket-antifield approach. In view of this, we have to choose a certain fermionic functional ψ_1 that depends on half of the variables from the enlarged BRST tricomplex, and eliminate the other half with its help. We notice that, according to the fundamental antibrackets (119–120) and $\left(\Phi^A, \Phi_B^{*(1)}\right)_1 = \delta_B^A$, the variables Φ^A , $\Phi_A^{*(3)}$, ρ_3^A , $\bar{\Phi}_A^{(1)}$, μ_3^A , $\bar{\Phi}_A^{(3)}$ and ν_1^A are regarded as ‘fields’, while $\Phi_A^{*(1)}$, ρ_2^A , $\Phi_A^{*(2)}$, κ_1^A , $\bar{\Phi}_A^{(2)}$, μ_2^A and $\tilde{\Phi}_A$ are viewed like their corresponding ‘antifields’. We choose to eliminate the variables $\Phi_A^{*(1)}$, ρ_2^A , ρ_3^A , κ_1^A , $\bar{\Phi}_A^{(2)}$, $\bar{\Phi}_A^{(3)}$ and $\tilde{\Phi}_A$, and, moreover, to take the fermion ψ_1 such as to implement the gauge-fixing conditions

$$\rho_2^A = \rho_3^A = \kappa_1^A = 0. \quad (123)$$

Then, we have to take ψ_1 not to depend on the variables conjugated to those appearing in (123), namely, $\Phi_A^{*(3)}$, $\Phi_A^{*(2)}$ and $\bar{\Phi}_A^{(1)}$, but only on the remaining ones, i.e., Φ^A , μ_2^A , μ_3^A and ν_1^A . (The variables are eliminated from the theory with the aid of the well-known relations antifield = $\frac{\delta^L \psi_1}{\delta(\text{field})}$, field = $-\frac{\delta^L \psi_1}{\delta(\text{antifield})}$, where it is understood that the ‘field’ is conjugated to the corresponding ‘antifield’ in the first antibracket.) In view of this, we take ψ_1 of the form

$$\psi_1 = (-)^{1+\epsilon(\Phi^B)(\epsilon(\Phi^A)+1)} \frac{\delta^{2L} \psi [\Phi^A]}{\delta \Phi^A \delta \Phi^B} \mu_3^A \mu_2^B + \frac{\delta^L \psi [\Phi^A]}{\delta \Phi^A} \nu_1^A, \quad (124)$$

where $\psi [\Phi^A]$ is an arbitrary fermionic functional involving only the fields and ghosts. From (124) we find the gauge-fixing conditions (123), as well as

$$\Phi_A^{*(1)} = (-)^{1+\epsilon(\Phi^C)(\epsilon(\Phi^B)+1)} \frac{\delta^{3L} \psi}{\delta \Phi^A \delta \Phi^B \delta \Phi^C} \mu_3^B \mu_2^C + \frac{\delta^{2L} \psi}{\delta \Phi^A \delta \Phi^B} \nu_1^B \equiv f_A(\Phi, \mu, \nu), \quad (125)$$

$$\bar{\Phi}_A^{(2)} = (-)^{\epsilon(\Phi^A)} \frac{\delta^{2L} \psi}{\delta \Phi^A \delta \Phi^B} \mu_2^B \equiv g_{2A}(\Phi, \mu), \quad (126)$$

$$\bar{\Phi}_A^{(3)} = (-)^{\epsilon(\Phi^A)} \frac{\delta^{2L} \psi}{\delta \Phi^A \delta \Phi^B} \mu_3^B \equiv g_{3A}(\Phi, \mu), \quad (127)$$

$$\tilde{\Phi}_A = \frac{\delta^L \psi}{\delta \Phi^A} \equiv h_A(\Phi), \quad (128)$$

such that the gauge-fixed action will be given by

$$\begin{aligned} S_{1\psi} [\Phi^A, \Phi_A^{*(2)}, \Phi_A^{*(3)}, \bar{\Phi}_A^{(1)}, \mu_2^A, \mu_3^A, \nu_1^A] = \\ S [\Phi^A, \Phi_A^{*(1)} = f_A, \Phi_A^{*(2)}, \Phi_A^{*(3)}, \bar{\Phi}_A^{(1)}, \bar{\Phi}_A^{(2)} = g_{2A}, \bar{\Phi}_A^{(3)} = g_{3A}, \tilde{\Phi}_A = h_A] + \\ \Phi_A^{*(2)} \mu_2^A + \Phi_A^{*(3)} \mu_3^A + \bar{\Phi}_A^{(1)} \nu_1^A. \end{aligned} \quad (129)$$

In consequence, after applying the gauge-fixing procedure, we have reached the gauge-fixed action (129). Moreover, it can be simply shown that it leads to an effective action which is s_a -invariant, and that can be further used in the path integral. Although correct, the gauge-fixing procedure developed so far is not completely satisfactory in a sense that will be made clear below. Let us consider the simple case of an abelian gauge algebra with gauge generators $Z_{\alpha_1}^{\alpha_0}$ independent of the fields, such that $\epsilon_{\alpha_0} = \epsilon_{\alpha_1}$. In this case, the solution to the master equation (122) is completely given by (121), where S is entirely expressed by the boundary conditions (91). We take a gauge-fixing fermion ψ that is quadratic in the fields Φ^A . From (125–128) it follows that the functions f_A , g_{2A} and g_{3A} will not involve the Φ^A 's, while h_A are linear in Φ^A . Then, h_A induces in the gauge-fixed action the term $h_{\alpha_1}(\Phi^A) \lambda^{\alpha_1}$. Because $\epsilon(h_{\alpha_1}(\Phi^A)) = \epsilon_{\alpha_1} + 1 = \epsilon_{\alpha_0} + 1$, and h_{α_1} are linear in Φ^A , it results that h_{α_1} cannot depend on the original fields, Φ^{α_0} (as $\epsilon(\Phi^{\alpha_0}) = \epsilon_{\alpha_0}$). This means that at the level of the gauge-fixed action (129) the only dependence of the original fields appears in the original Lagrangian action itself, hence the gauge-fixed action is degenerate (due to the gauge invariances (2)). Thus, in order to infer a correct gauge-fixed action we must take ψ at least cubic in the Φ^A 's. Now, we assume that ψ is cubic in the Φ^A 's. In this situation, from (125–128) we observe that the functions f_A , g_{2A} , g_{3A} and h_A will depend in general on Φ^{α_0} , being quadratic in the variables they involve, such that they will induce within the gauge-fixed action (129) a dependence of Φ^{α_0} that is manifested through interaction terms that are cubic in the various variables. In conclusion, the gauge-fixed action cannot contain a gauge-fixing term of the type $b^{\alpha_1} f_{\alpha_1}(\Phi^{\alpha_0})$, where $f_{\alpha_1}(\Phi^{\alpha_0}) = 0$ signify the gauge conditions on the original fields. This fact represents an inconvenient because the presence of such a term is necessary for a proper relationship with the standard antibracket-antifield formalism.

In order to surpass this inconvenient, we proceed as follows. We introduce the purely gauge fields φ^{α_1} (that do not enter the original action) with the gauge invariances $\delta_\xi \varphi^{\alpha_1} = \xi^{\alpha_1}$, where the gauge parameters ξ^{α_1} have statistics

opposite to that of the gauge conditions $f_{\alpha_1}(\Phi^{\alpha_0})$, hence we have

$$\epsilon(\varphi^{\alpha_1}) = \epsilon(\xi^{\alpha_1}) = \epsilon(f_{\alpha_1}) + 1. \quad (130)$$

According to the general theory exposed in section 4, we introduce the additional ghost sector

$$\left(\begin{matrix} (1,0,0)^{\alpha_1} & (0,1,0)^{\alpha_1} & (0,0,1)^{\alpha_1} & (0,1,1)^{\alpha_1} & (1,0,1)^{\alpha_1} & (1,1,0)^{\alpha_1} & (1,1,1)^{\alpha_1} \\ C_1 & C_2 & C_3 & p_1 & p_2 & p_3 & l \end{matrix} \right), \quad (131)$$

displaying the Grassmann parities

$$\epsilon(C_a^{\alpha_1}) = \epsilon(l^{\alpha_1}) = \epsilon(f_{\alpha_1}), \quad \epsilon(p_a^{\alpha_1}) = \epsilon(f_{\alpha_1}) + 1, \quad a = 1, 2, 3. \quad (132)$$

For notational simplicity, we make the collective notation

$$\varphi^I = (\varphi^{\alpha_1}, C_a^{\alpha_1}, p_a^{\alpha_1}, l^{\alpha_1}). \quad (133)$$

Thus, as explained in sections 4 and 5, the antifield spectrum will contain the variables

$$(\varphi_I^{*(a)}, \bar{\varphi}_I^{(a)}, \tilde{\varphi}_I), \quad a = 1, 2, 3, \quad (134)$$

whose properties result from the general formulas (43–46), (60–63) and (74–75) adapted to the additional field/ghost spectrum. As the sector corresponding to the new fields does not interfere in any point with the original one, the solution to the master equation of the BRST $Sp(3)$ formalism associated with the overall gauge theory will be

$$\bar{S} = S + \varphi_{\alpha_1}^{*(a)} C_a^{\alpha_1} + (\varepsilon_{abc} C_{b\alpha_1}^{*(a)} + \bar{\varphi}_{\alpha_1}^{(c)}) p_c^{\alpha_1} - (p_{a\alpha_1}^{*(a)} + \bar{C}_{a\alpha_1}^{(a)} - \tilde{\varphi}_{\alpha_1}) l^{\alpha_1}. \quad (135)$$

Now, we reprise the gauge-fixing procedure exposed in the above, but with respect to the larger gauge theory. We give up the second and third antibrackets, and introduce the variables $(r_2^I, r_3^I, k_1^I, m_2^I, m_3^I, n_1^I)$ with the properties respectively of the type (111–120). Subsequently, we pass to the functional

$$\bar{S}_1 = \bar{S} + \Phi_A^{*(2)} \mu_2^A + \Phi_A^{*(3)} \mu_3^A + \bar{\Phi}_A^{(1)} \nu_1^A + \varphi_I^{*(2)} m_2^I + \varphi_I^{*(3)} m_3^I + \bar{\varphi}_I^{(1)} n_1^I, \quad (136)$$

which is of course solution to the standard classical master equation in the first antibracket, $(\bar{S}_1, \bar{S}_1)_1 = 0$. Now, we have to choose a fermionic functional $\bar{\psi}_1$, with the help of which we eliminate half of the variables in favour of the other half. We act like before, and eliminate, besides $\Phi_A^{*(1)}, \rho_2^A, \rho_3^A, \kappa_1^A, \bar{\Phi}_A^{(2)}, \bar{\Phi}_A^{(3)}$ and $\tilde{\Phi}_A$, also the variables $\varphi_I^{*(1)}, r_2^I, r_3^I, k_1^I, \bar{\varphi}_I^{(2)}, \bar{\varphi}_I^{(3)}$ and $\tilde{\varphi}_I$ via the gauge-fixing conditions (123) together with

$$r_2^I = r_3^I = k_1^I = 0. \quad (137)$$

This can be done through

$$\bar{\psi}_1 = (-)^{1+\epsilon(\Phi^A)(\epsilon(\Phi^\Delta)+1)} \frac{\delta^{2L}\bar{\psi}'[\Phi^\Delta]}{\delta\Phi^\Delta\delta\Phi^A} \mu_3^\Delta \mu_2^\Delta + \frac{\delta^L\bar{\psi}'[\Phi^\Delta]}{\delta\Phi^\Delta} \nu_1^\Delta, \quad (138)$$

where the fermionic functional $\bar{\psi}'$ depends only on $\Phi^\Delta = (\Phi^A, \varphi^I)$. Also, we employed the notations $\mu_2^\Delta = (\mu_2^A, m_2^I)$, $\mu_3^\Delta = (\mu_3^A, m_3^I)$, $\nu_1^\Delta = (\nu_1^A, n_1^I)$. In order to implement some linear gauge conditions, we take the functional $\bar{\psi}'$ of the special form

$$\bar{\psi}' = \bar{\psi}[\Phi^\Delta] + \varphi^{\alpha_1} f_{\alpha_1}(\Phi^{\alpha_0}), \quad (139)$$

where $f_{\alpha_1}(\Phi^{\alpha_0})$ is linear in the fields Φ^{α_0} . The elimination process gives, at the level of the larger gauge theory, some relations of the type (125–128), with A, B, C replaced by $\Delta, \Delta', \Delta''$, and ψ by $\bar{\psi}'$. We remark that the elimination of the variables connected with the fields Φ^{α_0} and φ^{α_1} produces the terms

$$\Phi_{\alpha_0}^{*(1)} = N_{\alpha_1\alpha_0} n_1^{\alpha_1} + \dots, \quad \varphi_{\alpha_1}^{*(1)} = (-)^{\epsilon_{\alpha_0}(\epsilon(f_{\alpha_1})+1)} N_{\alpha_1\alpha_0} \nu_1^{\alpha_0} + \dots, \quad (140)$$

$$\bar{\Phi}_{\alpha_0}^{(2)} = (-)^{\epsilon_{\alpha_0}} N_{\alpha_1\alpha_0} m_2^{\alpha_1} + \dots, \quad \bar{\varphi}_{\alpha_1}^{(2)} = (-)^{(\epsilon_{\alpha_0}+1)(\epsilon(f_{\alpha_1})+1)} N_{\alpha_1\alpha_0} \mu_2^{\alpha_0} + \dots, \quad (141)$$

$$\bar{\Phi}_{\alpha_0}^{(3)} = (-)^{\epsilon_{\alpha_0}} N_{\alpha_1\alpha_0} m_3^{\alpha_1} + \dots, \quad \bar{\varphi}_{\alpha_1}^{(3)} = (-)^{(\epsilon_{\alpha_0}+1)(\epsilon(f_{\alpha_1})+1)} N_{\alpha_1\alpha_0} \mu_3^{\alpha_0} + \dots, \quad (142)$$

$$\tilde{\Phi}_{\alpha_0} = N_{\alpha_1\alpha_0} \varphi^{\alpha_1} + \dots, \quad \tilde{\varphi}_{\alpha_1} = f_{\alpha_1}(\Phi^{\alpha_0}) + \dots, \quad (143)$$

where $N_{\alpha_1\alpha_0} = \frac{\delta^L f_{\alpha_1}}{\delta\Phi^{\alpha_0}}$.

The gauge-fixed action $\bar{S}_{1\bar{\psi}'}[\Phi^\Delta, \Phi_\Delta^{*(2)}, \Phi_\Delta^{*(3)}, \bar{\Phi}_\Delta^{(1)}, \mu_2^\Delta, \mu_3^\Delta, \nu_1^\Delta]$ will be constructed from (136) where we eliminate some of the ‘fields’ and ‘antifields’ with the help of the expressions resulting from $\bar{\psi}'$, such that the path integral can be written as

$$Z_{\bar{\psi}'} = \int \mathcal{D}\Phi^\Delta \mathcal{D}\Phi_\Delta^{*(2)} \mathcal{D}\Phi_\Delta^{*(3)} \mathcal{D}\bar{\Phi}_\Delta^{(1)} \mathcal{D}\mu_2^\Delta \mathcal{D}\mu_3^\Delta \mathcal{D}\nu_1^\Delta \exp(i\bar{S}_{1\bar{\psi}'}), \quad (144)$$

being understood that $\Phi_\Delta^{*(a)} = (\Phi_A^{*(a)}, \varphi_I^{*(a)})$, $\bar{\Phi}_\Delta^{(a)} = (\bar{\Phi}_A^{(a)}, \bar{\varphi}_I^{(a)})$. If we take $\bar{\psi}$ not to depend on φ^{α_1} , $C_a^{\alpha_1}$ and $p_a^{\alpha_1}$, the latter relations in (143) yield the term $f_{\alpha_1}(\Phi^{\alpha_0})l^{\alpha_1}$ in the gauge-fixed action, which represents the standard gauge-fixing term. The gauge-fixing procedure developed in relation with the larger theory is more flexible. Excepting the fact that we can obtain a standard gauge-fixing term, it is no longer necessary that $\bar{\psi}$ is cubic in the Φ^Δ ’s as the term that fixes the gauge invariances of the original fields comes

now from the second piece in (139), so $\bar{\psi}$ may be taken to be quadratic (see also the example).

Now, we define the effective action. This is obtained by enforcing the gauge-fixing conditions directly in the solution \bar{S}_1 of the classical master equation in the first antibracket with the help of some Lagrange multipliers

$$S_{\text{eff}} = \bar{S}_1 + \left(\Phi_{\Delta}^{*(1)} - \frac{\delta^L \bar{\psi}_1}{\delta \Phi_{\Delta}} \right) \mu_1^{\Delta} + \left(\bar{\Phi}_{\Delta}^{(2)} - \frac{\delta^L \bar{\psi}_1}{\delta \mu_3^{\Delta}} \right) \nu_2^{\Delta} + \left(\bar{\Phi}_{\Delta}^{(3)} + \frac{\delta^L \bar{\psi}_1}{\delta \mu_2^{\Delta}} \right) \nu_3^{\Delta} + \left(\tilde{\Phi}_{\Delta} - \frac{\delta^L \bar{\psi}_1}{\delta \nu_1^{\Delta}} \right) \omega^{\Delta}, \quad (145)$$

where the Lagrange multipliers have the properties

$$\mu_1^{\Delta} = (\mu_1^A, m_1^I), \quad \nu_2^{\Delta} = (\nu_2^A, n_2^I), \quad \nu_3^{\Delta} = (\nu_3^A, n_3^I), \quad \omega^{\Delta} = (\omega^A, o^I), \quad (146)$$

$$\epsilon(\mu_1^{\Delta}) = \epsilon(\omega^{\Delta}) = \epsilon(\Phi^{\Delta}) + 1, \quad \epsilon(\nu_2^{\Delta}) = \epsilon(\nu_3^{\Delta}) = \epsilon(\Phi^{\Delta}). \quad (147)$$

Taking into account the formulas (138), (125–128) with A, B, C replaced by $\Delta, \Delta', \Delta''$, and ψ by $\bar{\psi}'$, we find that the effective action reads as

$$S_{\text{eff}} = \bar{S} + \sum_{a=1}^3 \left(\Phi_{\Delta}^{*(a)} \mu_a^{\Delta} + \bar{\Phi}_{\Delta}^{(a)} \nu_a^{\Delta} \right) + \left(\tilde{\Phi}_{\Delta} - \frac{\delta^L \bar{\psi}'}{\delta \Phi_{\Delta}} \right) \omega^{\Delta} + (-)^{\epsilon(\Phi^{\Delta'})} \frac{\delta^{3L} \bar{\psi}'}{\delta \Phi_{\Delta} \delta \Phi_{\Delta'} \delta \Phi_{\Delta''}} \mu_3^{\Delta''} \mu_2^{\Delta'} \mu_1^{\Delta} - \frac{\delta^{2L} \bar{\psi}'}{\delta \Phi_{\Delta} \delta \Phi_{\Delta'}} \sum_{a=1}^3 (\nu_a^{\Delta'} \mu_a^{\Delta}), \quad (148)$$

so the corresponding path integral is of the type

$$Z_{\text{eff}} = \int \mathcal{D}\Phi^{\Delta} \mathcal{D}\Phi_{\Delta}^{*(a)} \mathcal{D}\bar{\Phi}_{\Delta}^{(a)} \mathcal{D}\tilde{\Phi}_{\Delta} \mathcal{D}\mu_a^{\Delta} \mathcal{D}\nu_a^{\Delta} \mathcal{D}\omega^{\Delta} \exp(iS_{\text{eff}}). \quad (149)$$

If one integrates in Z_{eff} over the auxiliary variables $\mu_1^{\Delta}, \nu_2^{\Delta}, \nu_3^{\Delta}, \omega^{\Delta}, \Phi_{\Delta}^{*(1)}, \bar{\Phi}_{\Delta}^{(2)}, \bar{\Phi}_{\Delta}^{(3)}$ and $\tilde{\Phi}_{\Delta}$, one reobtains the path integral (144).

Let us work in the sequel with the second antibracket and forget about the other two. In this case we need to introduce the variables $(\mu_1^{\Delta}, \mu_3^{\Delta}, \nu_2^{\Delta})$ respectively conjugated in the second antibracket to $(\bar{\Phi}_{\Delta}^{(3)}, \bar{\Phi}_{\Delta}^{(1)}, \tilde{\Phi}_{\Delta})$, and work with the gauge-fixing fermion

$$\bar{\psi}_2 = (-)^{1+\epsilon(\Phi^{\Delta})(\epsilon(\Phi^{\Delta})+1)} \frac{\delta^{2L} \bar{\psi}'}{\delta \Phi_{\Delta} \delta \Phi^{\Delta}} [\Phi^{\Delta}] \mu_1^{\Delta} \mu_3^{\Delta} + \frac{\delta^L \bar{\psi}'}{\delta \Phi_{\Delta}} [\Phi^{\Delta}] \nu_2^{\Delta}, \quad (150)$$

while the gauge-fixing conditions resulting from $\bar{\psi}_2$ are obtained with the help of the same $\bar{\psi}'$ like in (139), and will be enforced within the effective

action through the Lagrange multipliers $(\mu_2^\Delta, \nu_1^\Delta, \nu_3^\Delta, \omega^\Delta)$. Similarly, if we focus on the third antibracket, then we add $(\mu_1^\Delta, \mu_2^\Delta, \nu_3^\Delta)$, which are taken to be conjugated in the third antibracket to $(\bar{\Phi}_\Delta^{(2)}, \bar{\Phi}_\Delta^{(1)}, \tilde{\Phi}_\Delta)$, and consider the gauge-fixing fermion

$$\bar{\psi}_3 = (-)^{1+\epsilon(\Phi^\Delta)(\epsilon(\Phi^\Delta)+1)} \frac{\delta^{2L}\bar{\psi}'[\Phi^\Delta]}{\delta\Phi^\Delta\delta\Phi^\Delta} \mu_2^\Delta \mu_1^\Delta + \frac{\delta^L\bar{\psi}'[\Phi^\Delta]}{\delta\Phi^\Delta} \nu_3^\Delta. \quad (151)$$

The consequent gauge-fixing conditions are implemented within the effective action via the Lagrange multipliers $(\mu_3^\Delta, \nu_1^\Delta, \nu_2^\Delta, \omega^\Delta)$. Thus, we can derive the path integral (149) working with any of the antibrackets, so we conclude that (149) is s_a -invariant. This follows from the general properties of the formalism developed until now. One can also check by direct computation that the effective action (148) is invariant under the ‘gauge-fixed’ BRST $Sp(3)$ transformations

$$s_a \Phi^\Delta = (-)^{\epsilon(\Phi^\Delta)} \mu_a^\Delta, \quad (152)$$

$$s_a \Phi_\Delta^{*(b)} = \delta_{ab} \frac{\delta^R \bar{S}}{\delta \Phi^\Delta}, \quad (153)$$

$$s_a \bar{\Phi}_\Delta^{(b)} = \varepsilon_{abc} \Phi_\Delta^{*(c)}, \quad (154)$$

$$s_a \tilde{\Phi}_\Delta = \bar{\Phi}_\Delta^{(a)}, \quad (155)$$

$$s_a \mu_b^\Delta = (-)^{\epsilon(\Phi^\Delta)} \varepsilon_{abc} \nu_c^\Delta, \quad (156)$$

$$s_a \nu_b^\Delta = (-)^{\epsilon(\Phi^\Delta)} \delta_{ab} \omega^\Delta, \quad (157)$$

$$s_a \omega^\Delta = 0, \quad (158)$$

up to some ‘skew-symmetric’ combinations of equations of motion. In this way, the announced aim, of obtaining an effective action that is s_a -invariant, has been accomplished. This completes our formalism.

8 Example

Let us exemplify the general theory developed so far. In order to emphasize the key features of our method, we consider the simple case of abelian gauge fields. We begin with the Lagrangian action

$$S_0[A^\alpha] = -\frac{1}{4} \int d^4x F_{\alpha\beta} F^{\alpha\beta}, \quad (159)$$

invariant under the abelian and irreducible gauge transformations $\delta_\varepsilon A^\alpha = \partial^\alpha \varepsilon$, where $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. The analogies between the general formalism

and this example are: $\alpha_0 \rightarrow \alpha$, $\Phi^{\alpha_0} \rightarrow A^\alpha$, $\alpha_1 \rightarrow 1$, $\varepsilon^{\alpha_1} \rightarrow \varepsilon$, $Z_{\alpha_1}^{\alpha_0} \rightarrow \partial^\alpha$. According to the general ideas exposed in Section 3, we triplicate the gauge parameters, and consider the second-stage reducible gauge transformations $\delta_\varepsilon A^\alpha = \partial^\alpha \varepsilon_1 + \partial^\alpha \varepsilon_2 + \partial^\alpha \varepsilon_3$. Then, the ghost spectrum reads as

$$\begin{matrix} (1,0,0) & (0,1,0) & (0,0,1) & (0,1,1) & (1,0,1) & (1,1,0) & (1,1,1) \\ \eta_1, & \eta_2, & \eta_3, & \pi_1, & \pi_2, & \pi_3, & \lambda \end{matrix}, \quad (160)$$

where

$$\epsilon(\eta_a) = \epsilon(\lambda) = 1, \quad \epsilon(\pi_a) = 0, \quad a = 1, 2, 3. \quad (161)$$

The antifield spectrum and its properties result from the general line exposed in sections 4 and 5. The solution to the classical master equation (95) of the $Sp(3)$ formalism reads as

$$\begin{aligned} S = \int d^4x \left(-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + A_\alpha^{*(a)} \partial^\alpha \eta_a + \varepsilon_{abc} \eta_b^{*(a)} \pi_c + \right. \\ \left. \bar{A}_\alpha^{(c)} \partial^\alpha \pi_c - \left(\pi_a^{*(a)} + \bar{\eta}_a^{(a)} \right) \lambda + \tilde{A}_\alpha \partial^\alpha \lambda \right), \end{aligned} \quad (162)$$

and it actually reduces to the boundary conditions (98–100).

Initially, we will fix the gauge without introducing the supplementary fields φ^{α_1} . In view of this, we pass to the first antibracket and discard the other two. Consequently, we need the additional variables (110) in order to obtain that every variable has its conjugated in the first antibracket. Thus, the solution S_1 to the standard master equation in the first antibracket (122) reads as

$$\begin{aligned} S_1 = S + \int d^4x \left(A_\alpha^{*(2)} \mu_2^{(A)\alpha} + \sum_{a=1}^3 \left(\eta_a^{*(2)} \mu_2^{(\eta_a)} + \pi_a^{*(2)} \mu_2^{(\pi_a)} \right) + \lambda^{*(2)} \mu_2^{(\lambda)} + \right. \\ A_\alpha^{*(3)} \mu_3^{(A)\alpha} + \sum_{a=1}^3 \left(\eta_a^{*(3)} \mu_3^{(\eta_a)} + \pi_a^{*(3)} \mu_3^{(\pi_a)} \right) + \lambda^{*(3)} \mu_3^{(\lambda)} + \bar{A}_\alpha^{(1)} \nu_1^{(A)\alpha} + \\ \left. \sum_{a=1}^3 \left(\bar{\eta}_a^{(1)} \nu_1^{(\eta_a)} + \bar{\pi}_a^{(1)} \nu_1^{(\pi_a)} \right) + \bar{\lambda}^{(1)} \nu_1^{(\lambda)} \right), \end{aligned} \quad (163)$$

where we put an extra upper index for distinguishing between the various types of μ 's and ν 's associated with different fields that carry the same index A . We employ a gauge-fixing fermion of the type (124) for performing the gauge-fixing procedure. As we need to choose a definite form for the fermionic functional ψ , let us examine a little bit the form of S_1 . The ‘gauge-fixing’ term for the electromagnetic field is yielded via the elimination of \tilde{A}_α from the term $\tilde{A}_\alpha \partial^\alpha \lambda$ present into the solution S . Then, we have to put in ψ a term that is at least quadratic in A^α , such that $\tilde{A}_\alpha = \delta^L \psi / \delta A^\alpha$ effectively

depends on Maxwell's field. For definiteness, we choose this term to be precisely quadratically, $\psi \sim A^\alpha A_\alpha$. As ψ is fermionic, the term $A^\alpha A_\alpha$ should be multiplied by a scalar fermionic ghost. There are actually four ghosts that satisfy this requirement, namely, $(\eta_a)_{a=1,2,3}$ and λ . As the ghost of ghost of ghost λ is already implied in $\tilde{A}_\alpha \partial^\alpha \lambda$, we are tempted to take $\psi \sim A^\alpha A_\alpha \lambda$ for forcing the couplings in the gauge-fixed action to involve less fields/ghosts. In the meantime, we avoid other terms in ψ for not complicating unnecessarily the gauge-fixed action. In conclusion, we try ψ under the form

$$\psi = \int d^4x \left(\frac{1}{2} A^\alpha A_\alpha \lambda \right), \quad (164)$$

with the help of which we infer (see formulas (125–128))

$$A_\alpha^{*(1)} = \mu_{3\alpha}^{(A)} \mu_2^{(\lambda)} - \mu_{2\alpha}^{(A)} \mu_3^{(\lambda)} + \lambda \nu_1^{(A)\alpha} + A_\alpha \nu_1^{(\lambda)}, \quad (165)$$

$$\eta_1^{*(1)} = \eta_2^{*(1)} = \eta_3^{*(1)} = \pi_1^{*(1)} = \pi_2^{*(1)} = \pi_3^{*(1)} = 0, \quad (166)$$

$$\lambda^{*(1)} = -\mu_{3\alpha}^{(A)} \mu_2^{(A)\alpha} + A_\alpha \nu_1^{(A)\alpha}, \quad \bar{A}_\alpha^{(2)} = \lambda \mu_{2\alpha}^{(A)} + A_\alpha \mu_2^{(\lambda)}, \quad (167)$$

$$\bar{\eta}_1^{(2)} = \bar{\eta}_2^{(2)} = \bar{\eta}_3^{(2)} = \bar{\pi}_1^{(2)} = \bar{\pi}_2^{(2)} = \bar{\pi}_3^{(2)} = 0, \quad (168)$$

$$\bar{\lambda}^{(2)} = -A_\alpha \mu_2^{(A)\alpha}, \quad \bar{A}_\alpha^{(3)} = \lambda \mu_{3\alpha}^{(A)} + A_\alpha \mu_3^{(\lambda)}, \quad (169)$$

$$\bar{\eta}_1^{(3)} = \bar{\eta}_2^{(3)} = \bar{\eta}_3^{(3)} = \bar{\pi}_1^{(3)} = \bar{\pi}_2^{(3)} = \bar{\pi}_3^{(3)} = 0, \quad (170)$$

$$\bar{\lambda}^{(3)} = -A_\alpha \mu_3^{(A)\alpha}, \quad \tilde{A}_\alpha = A_\alpha \lambda, \quad (171)$$

$$\tilde{\eta}_1 = \tilde{\eta}_2 = \tilde{\eta}_3 = \tilde{\pi}_1 = \tilde{\pi}_2 = \tilde{\pi}_3 = 0, \quad \tilde{\lambda} = \frac{1}{2} A^\alpha A_\alpha. \quad (172)$$

Substituting the above relations in (163), we obtain a gauge-fixed action of the type (129). Integrating in the resulting gauge-fixed path integral over some auxiliary fields, we arrive at

$$Z_{1\psi} = \int \mathcal{D}A^\alpha \mathcal{D}\eta_a \mathcal{D}\pi_a \mathcal{D}\lambda \exp \left(i S'_{1\psi} \right), \quad (173)$$

where

$$S'_{1\psi} = \int d^4x \left(-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} - \sum_{a=1}^3 \lambda (\partial_\alpha \pi_a) (\partial^\alpha \eta_a) + \lambda A_\alpha \partial^\alpha \lambda \right). \quad (174)$$

All the terms in the gauge-fixed action excluding those appearing in the original Lagrangian action describe couplings of order three among fields and ghosts. Now, it appears clearly that we cannot reach the familiar form of the gauge-fixed action for Maxwell's theory involving a term $b \partial_\alpha A^\alpha$ that

enforces the Lorentz gauge condition $\partial_\alpha A^\alpha = 0$, or a Gaussian term of the type $b \left(\partial_\alpha A^\alpha + \frac{1}{2}b \right)$, where b is an auxiliary bosonic field.

In the sequel, we will fix the gauge following the more elaborated line exposed in the final part of Section 7. We intend to implement the gauge-fixing conditions $f_{\alpha_1}(\Phi^{\alpha_0}) = 0 \rightarrow \partial_\alpha A^\alpha = 0$. In view of this, we add a fermionic scalar field φ subject to the gauge transformation $\delta_\xi \varphi = \xi$. The solution to the classical master equation of the $Sp(3)$ BRST formalism for the overall gauge theory has the form

$$\bar{S} = S + \int d^4x \left(\varphi^{*(a)} C_a + \left(\varepsilon_{abc} C_b^{*(a)} + \bar{\varphi}^{(c)} \right) p_c - \left(p_a^{*(a)} + \bar{C}_a^{(a)} - \tilde{\varphi} \right) l \right), \quad (175)$$

(see (135) where the index α_1 has been suppressed). In order to pass to the solution of the master equation in the first antibracket, we enlarge the field and antifield spectra with the fields of the type r , k , m and n , such that we arrive at (see (136))

$$\begin{aligned} \bar{S}_1 = \bar{S} + \int d^4x & \left(A_\alpha^{*(2)} \mu_2^{(A)\alpha} + \sum_{a=1}^3 \left(\eta_a^{*(2)} \mu_2^{(\eta_a)} + \pi_a^{*(2)} \mu_2^{(\pi_a)} \right) + \lambda^{*(2)} \mu_2^{(\lambda)} + \right. \\ & A_\alpha^{*(3)} \mu_3^{(A)\alpha} + \sum_{a=1}^3 \left(\eta_a^{*(3)} \mu_3^{(\eta_a)} + \pi_a^{*(3)} \mu_3^{(\pi_a)} \right) + \lambda^{*(3)} \mu_3^{(\lambda)} + \bar{A}_\alpha^{(1)} \nu_1^{(A)\alpha} + \\ & \sum_{a=1}^3 \left(\bar{\eta}_a^{(1)} \nu_1^{(\eta_a)} + \bar{\pi}_a^{(1)} \nu_1^{(\pi_a)} \right) + \bar{\lambda}^{(1)} \nu_1^{(\lambda)} + \varphi^{*(2)} m_2^{(\varphi)} + l^{*(2)} m_2^{(l)} + \varphi^{*(3)} m_3^{(\varphi)} + \\ & \sum_{a=1}^3 \left(C_a^{*(2)} m_2^{(C_a)} + p_a^{*(2)} m_2^{(p_a)} \right) + \sum_{a=1}^3 \left(C_a^{*(3)} m_3^{(C_a)} + p_a^{*(3)} m_3^{(p_a)} \right) + \\ & \left. l^{*(3)} m_3^{(l)} + \bar{\varphi}^{(1)} n_1^{(\varphi)} + \sum_{a=1}^3 \left(\bar{C}_a^{(1)} n_1^{(C_a)} + \bar{p}_a^{(1)} n_1^{(p_a)} \right) + \bar{l}^{(1)} n_1^{(l)} \right). \end{aligned} \quad (176)$$

The gauge-fixing process relies on the choice of a certain fermionic functional of the type (138), with the fermion $\bar{\psi}'$ like in (139). We take $\bar{\psi}[\Phi^\Delta] \rightarrow \int d^4x \left(A^\alpha \partial_\alpha \lambda + \frac{1}{2} \varphi l \right)$ and $f_{\alpha_1}(\Phi^{\alpha_0}) \rightarrow \partial_\alpha A^\alpha$, such that

$$\bar{\psi}' = \int d^4x \left(A^\alpha \partial_\alpha \lambda + \varphi \left(\frac{1}{2} l + \partial_\alpha A^\alpha \right) \right). \quad (177)$$

Eliminating some ‘fields’ and ‘antifields’ from (176) with the help of (177), as explained in Section 7, and further integrating in the resulting path integral over some auxiliary variables, we finally deduce

$$Z_{1\bar{\psi}'} = \int \mathcal{D}A^\alpha \mathcal{D}\varphi \mathcal{D}\eta_a \mathcal{D}C_a \mathcal{D}\pi_a \mathcal{D}p_a \mathcal{D}\lambda \mathcal{D}l \exp \left(i \bar{S}'_{1\bar{\psi}'} \right), \quad (178)$$

where

$$\begin{aligned} \bar{S}'_{1\bar{\psi}'} = \int d^4x \left(-\frac{1}{4} F_{\alpha\beta} F^{\alpha\beta} + \left(\frac{1}{2} l + \partial_\alpha A^\alpha \right) l - (\partial_\alpha \varphi) (\partial^\alpha \lambda) + \right. \\ \left. \sum_{a=1}^3 ((\partial_\alpha p_a) (\partial^\alpha \eta_a) + (\partial^\alpha C_a) (\partial_\alpha \pi_a)) \right). \end{aligned} \quad (179)$$

We observe that we were able to enforce a Gaussian term in the gauge-fixed action, of the type $\left(\frac{1}{2}l + \partial_\alpha A^\alpha\right)l$, as the ghost of ghost of ghost l corresponding to the supplementary scalar fermionic field φ is bosonic. If we want to implement the Lorentz gauge-fixing condition $\partial_\alpha A^\alpha = 0$, it is enough to discard the term $\frac{1}{2}\varphi l$ from (177), obtaining thus $\bar{S}'_{1\bar{\psi}'}$ without the term $\frac{1}{2}(l)^2$. In the meantime, we notice that $\bar{\psi}'$ is quadratic in the fields, which was not possible for a fermion of the type ψ (which must be at least cubic, see (164)). Finally, we remark that the gauge-fixed action (179) bear the trace of the triplication. Indeed, the terms $(\partial_\alpha p_a) (\partial^\alpha \eta_a)$ correspond to the triplication of the gauge transformations, $(\partial_\alpha \pi_a) (\partial^\alpha C_a)$ are associated with the induced first-stage reducibility relations, while $(\partial_\alpha \varphi) (\partial^\alpha \lambda)$ are correlated with the second-stage reducibility relations. In this context, p_a , C_a and φ play the role of Lagrangian antighosts respectively corresponding to η_a , π_a and λ . This completes the analysis of the model under study.

9 Conclusion

To conclude with, in this paper we have shown that the $Sp(3)$ BRST symmetry for irreducible theories can be developed in the antibracket-antifield formulation by adapting the methods of homological perturbation theory. The key point of our approach is the construction of a Koszul-Tate triresolution of the algebra of smooth functions defined on the stationary surface of field equations, that allows us to apply a positivity-like theorem for triresolutions. With the canonical generator of the $Sp(3)$ BRST symmetry at hand, we give a gauge-fixing procedure specific to the standard antibracket-antifield formalism, that leads to an effective action which is s_a -invariant. The general procedure is finally applied on abelian gauge fields.

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A Appendix: basic properties of triresolutions

Definition 2 Let \mathcal{A}_0 be an algebra and \mathcal{A}' be a trigraded algebra with the tridegree called resolution tridegree, $\text{trires} = (\text{res}_1, \text{res}_2, \text{res}_3)$, where all res_a ($a = 1, 2, 3$) are assumed to be nonnegative integers, $\text{res}_a \geq 0$, $a = 1, 2, 3$. We define the total resolution degree as $\text{res} = \text{res}_1 + \text{res}_2 + \text{res}_3$. Let $\delta : \mathcal{A}' \rightarrow \mathcal{A}'$ be a differential of total resolution degree minus one, $\delta^2 = 0$, $\text{res}(\delta) = -1$, such that $\text{res}(\delta a) = \text{res}(a) - 1$ when $\text{res}(a) \geq 1$, or $\text{res}(\delta a) = 0$ when $\text{res}(a) = 0$, in which case $\delta a = 0$. One says that the simple differential complex (δ, \mathcal{A}') induces a triresolution $((\delta_a)_{a=1,2,3}, \mathcal{A}')$ of the algebra \mathcal{A}_0 if and only if:

1. The differential δ splits as the sum among three derivations only, $\delta = \delta_1 + \delta_2 + \delta_3$, with $\text{trires}(\delta_1) = (-1, 0, 0)$, $\text{trires}(\delta_2) = (0, -1, 0)$, and $\text{trires}(\delta_3) = (0, 0, -1)$ (no extra piece, say, of resolution tridegree $(-2, 1, 0)$ occurs). The nilpotency of δ induces that $(\delta_a)_{a=1,2,3}$ are three anticommuting differentials, $\delta_a \delta_b + \delta_b \delta_a = 0$, $a, b = 1, 2, 3$.

2. We have that

$$H_{0,0,0}(\delta_a) = \mathcal{A}_0, \quad H_{i,j,k}(\delta_a) = 0, \quad a = 1, 2, 3, \quad i, j, k \geq 0, \quad i + j + k > 0, \quad (180)$$

$$H_0(\delta) = \mathcal{A}_0, \quad H_l(\delta) = 0, \quad l > 0, \quad (181)$$

where $H_{i,j,k}(\delta_a)$ signifies the space of elements with $\text{trires} = (i, j, k)$, that are δ_a -closed modulo δ_a -exact, and $H_l(\delta)$ means the cohomological space spanned by the objects with $\text{res} = l$, that are δ -closed modulo δ -exact.

Theorem 3 Let $((\delta_a)_{a=1,2,3}, \mathcal{A}')$ be a triresolution and $\overset{[i,j,k]}{F} \in \mathcal{A}'$ (assuming $i + j + k > 0$), be such that

$$\left(\delta_a \overset{[i,j,k]}{F} = 0, \quad a = 1, 2, 3 \right) \Leftrightarrow \delta \overset{[i,j,k]}{F} = 0. \quad (182)$$

Then, $\overset{[i,j,k]}{F}$ can be represented as

$$\overset{[i,j,k]}{F} = \delta_3 \delta_2 \delta_1 \overset{[i+1,j+1,k+1]}{F}. \quad (183)$$

Proof. From (182) it follows that $\delta_3 \overset{[i,j,k]}{F} = 0$, which further yields that there exists an element $\overset{[i,j,k+1]}{R}$ such that

$$\overset{[i,j,k]}{F} = \delta_3 \overset{[i,j,k+1]}{R}, \quad (184)$$

since $H_{i,j,k}(\delta_3) = 0$ for $i + j + k > 0$. But one also has $\delta_2 \overset{[i,j,k]}{F} = 0$, hence $\delta_2 \delta_3 \overset{[i,j,k+1]}{R} = 0$, which is equivalent to $\delta_3 \left(\delta_2 \overset{[i,j,k+1]}{R} \right) = 0$ on account of the anticommutativity between δ_2 and δ_3 . This means that there exists an object $\overset{[i,j-1,k+2]}{R'}$ such that

$$\delta_2 \overset{[i,j,k+1]}{R} = \delta_3 \overset{[i,j-1,k+2]}{R'}. \quad (185)$$

Similarly, we have that $\delta_1 \overset{[i,j,k]}{F} = 0$, which induces that $\delta_3 \left(\delta_1 \overset{[i,j,k+1]}{R} \right) = 0$ by means of the anticommutativity between δ_1 and δ_3 . Therefore, there exists a certain $\overset{[i-1,j,k+2]}{R''}$ with the property

$$\delta_1 \overset{[i,j,k+1]}{R} = \delta_3 \overset{[i-1,j,k+2]}{R''}. \quad (186)$$

If we apply: (i) δ_2 on (185); (ii) δ_1 on (186); (iii) δ_1 on (185), δ_2 on (186), and add the results, then we find the equations

$$\delta_2 \overset{[i,j-1,k+2]}{R'} = \delta_3 \overset{[i,j-2,k+3]}{R'}, \quad (187)$$

$$\delta_1 \overset{[i-1,j,k+2]}{R''} = \delta_3 \overset{[i-2,j,k+3]}{R''}, \quad (188)$$

$$\delta_1 \overset{[i,j-1,k+2]}{R'} + \delta_2 \overset{[i-1,j,k+2]}{R''} = \delta_3 \overset{[i-1,j-1,k+3]}{Q}. \quad (189)$$

Next, we perform the following steps: we apply (i) δ_2 on (187); (ii) δ_1 on (188); (iii) δ_1 on (187), δ_2 on (189), and add the resulting equations; (iv) δ_2 on (188), δ_1 on (189), and add the resulting equations, then we get

$$\delta_2 \overset{[i,j-2,k+3]}{R'} = \delta_3 \overset{[i,j-3,k+4]}{R'}, \quad (190)$$

$$\delta_1 \overset{[i-2,j,k+3]}{R''} = \delta_3 \overset{[i-3,j,k+4]}{R''}, \quad (191)$$

$$\delta_1 \overset{[i,j-2,k+3]}{R'} + \delta_2 \overset{[i-1,j-1,k+3]}{Q} = \delta_3 \overset{[i-1,j-2,k+4]}{Q'}, \quad (192)$$

$$\delta_2 \overset{[i-2,j,k+3]}{R''} + \delta_1 \overset{[i-1,j-1,k+3]}{Q} = \delta_3 \overset{[i-2,j-1,k+4]}{Q''}. \quad (193)$$

From now on, we obviously derive a tower of descent equations of the form

$$\delta_2 \overset{[i,j-m,k+m+1]}{R'} = \delta_3 \overset{[i,j-m-1,k+m+2]}{R'}, \quad (194)$$

$$\delta_1 \begin{matrix} [i-m,j,k+m+1] \\ R'' \end{matrix} = \delta_3 \begin{matrix} [i-m-1,j,k+m+2] \\ R'' \end{matrix}, \quad (195)$$

$$\delta_1 \begin{matrix} [i,j-m,k+m+1] \\ R' \end{matrix} + \delta_2 \begin{matrix} [i-1,j-m+1,k+m+1] \\ Q' \end{matrix} = \delta_3 \begin{matrix} [i-1,j-m,k+m+2] \\ Q' \end{matrix}, \quad (196)$$

$$\delta_2 \begin{matrix} [i-m,j,k+m+1] \\ R'' \end{matrix} + \delta_1 \begin{matrix} [i-m+1,j-1,k+m+1] \\ Q'' \end{matrix} = \delta_3 \begin{matrix} [i-m,j-1,k+m+2] \\ Q'' \end{matrix}, \quad (197)$$

for $m \geq 3$. The last equations in (196–197) read as

$$\delta_2 \begin{matrix} [i-1,0,k+j+2] \\ Q' \end{matrix} = 0, \quad m = j + 1, \quad (198)$$

$$\delta_1 \begin{matrix} [0,j-1,k+i+2] \\ Q'' \end{matrix} = 0, \quad m = i + 1, \quad (199)$$

while the last equations in (194–195) are expressed by

$$\delta_2 \begin{matrix} [i,0,k+j+1] \\ R' \end{matrix} = 0, \quad m = j, \quad (200)$$

$$\delta_1 \begin{matrix} [0,j,k+i+1] \\ R'' \end{matrix} = 0, \quad m = i. \quad (201)$$

Their solutions are obviously given by

$$\begin{matrix} [i-1,0,k+j+2] \\ Q' \end{matrix} = \delta_2 \begin{matrix} [i-1,1,k+j+2] \\ M' \end{matrix}, \quad (202)$$

$$\begin{matrix} [0,j-1,k+i+2] \\ Q'' \end{matrix} = \delta_1 \begin{matrix} [1,j-1,k+i+2] \\ M'' \end{matrix}, \quad (203)$$

$$\begin{matrix} [i,0,k+j+1] \\ R' \end{matrix} = \delta_2 \begin{matrix} [i,1,k+j+1] \\ N' \end{matrix}, \quad (204)$$

$$\begin{matrix} [0,j,k+i+1] \\ R'' \end{matrix} = \delta_1 \begin{matrix} [1,j,k+i+1] \\ N'' \end{matrix}. \quad (205)$$

Introducing the solutions (202) and (204) in the equation (196) corresponding to $m = j$, respectively, the solutions (203) and (205) in the equation (197) associated with $m = i$, we further infer the solutions to these two equations like

$$\begin{matrix} [i-1,1,k+j+1] \\ Q' \end{matrix} = \delta_1 \begin{matrix} [i,1,k+j+1] \\ N' \end{matrix} + \delta_2 \begin{matrix} [i-1,2,k+j+1] \\ M' \end{matrix} - \delta_3 \begin{matrix} [i-1,1,k+j+2] \\ M' \end{matrix}, \quad (206)$$

$$\begin{matrix} [1,j-1,k+i+1] \\ Q'' \end{matrix} = \delta_2 \begin{matrix} [1,j,k+i+1] \\ N'' \end{matrix} + \delta_1 \begin{matrix} [2,j-1,k+i+1] \\ M'' \end{matrix} - \delta_3 \begin{matrix} [1,j-1,k+i+2] \\ M'' \end{matrix}. \quad (207)$$

In order to solve the equations (194) and (195) for $m = j - 1$, respectively, $m = i - 1$, we employ one more time the solutions (204), respectively, (205), which allows us to determine

$$\begin{matrix} [i,1,k+j] \\ R' \end{matrix} = \delta_2 \begin{matrix} [i,2,k+j] \\ N' \end{matrix} - \delta_3 \begin{matrix} [i,1,k+j+1] \\ N' \end{matrix}, \quad (208)$$

$$\begin{matrix} [1,j,k+i] \\ R'' \end{matrix} = \delta_1 \begin{matrix} [2,j,k+i] \\ N'' \end{matrix} - \delta_3 \begin{matrix} [1,j,k+i+1] \\ N'' \end{matrix}. \quad (209)$$

Using the results (206) and (208) in the equation (196) for $m = j - 1$, respectively, (207) and (209) in the equation (197) for $m = i - 1$, we reach their solutions in the form

$$\begin{matrix} [i-1,2,k+j] \\ Q' \end{matrix} = \delta_1 \begin{matrix} [i,2,k+j] \\ N' \end{matrix} + \delta_2 \begin{matrix} [i-1,3,k+j] \\ M' \end{matrix} - \delta_3 \begin{matrix} [i-1,2,k+j+1] \\ M' \end{matrix}, \quad (210)$$

$$\begin{matrix} [2,j-1,k+i] \\ Q'' \end{matrix} = \delta_2 \begin{matrix} [2,j,k+i] \\ N'' \end{matrix} + \delta_1 \begin{matrix} [3,j-1,k+i] \\ M'' \end{matrix} - \delta_3 \begin{matrix} [2,j-1,k+i+1] \\ M'' \end{matrix}. \quad (211)$$

Reprising the same operations, namely, inserting step by step the solutions into the equations (194–197) with decreasing m , we find the following solutions to the equations (190–193)

$$\begin{matrix} [i,j-2,k+3] \\ R' \end{matrix} = \delta_2 \begin{matrix} [i,j-1,k+3] \\ N' \end{matrix} - \delta_3 \begin{matrix} [i,j-2,k+4] \\ N' \end{matrix}, \quad (212)$$

$$\begin{matrix} [i-2,j,k+3] \\ R'' \end{matrix} = \delta_1 \begin{matrix} [i-1,j,k+3] \\ N'' \end{matrix} - \delta_3 \begin{matrix} [i-2,j,k+4] \\ N'' \end{matrix}, \quad (213)$$

$$\begin{matrix} [i-1,j-1,k+3] \\ Q \end{matrix} = \delta_1 \begin{matrix} [i,j-1,k+3] \\ N' \end{matrix} + \delta_2 \begin{matrix} [i-1,j,k+3] \\ M' \end{matrix} - \delta_3 \begin{matrix} [i-1,j-1,k+4] \\ M' \end{matrix}, \quad (214)$$

$$\begin{matrix} [i-1,j-1,k+3] \\ Q \end{matrix} = \delta_2 \begin{matrix} [i-1,j,k+3] \\ N'' \end{matrix} + \delta_1 \begin{matrix} [i,j-1,k+3] \\ M'' \end{matrix} - \delta_3 \begin{matrix} [i-1,j-1,k+4] \\ M'' \end{matrix}. \quad (215)$$

At this stage, we notice that there appears a first restriction due to (214–215), namely,

$$\begin{aligned} \delta_1 \begin{matrix} [i,j-1,k+3] \\ N' \end{matrix} + \delta_2 \begin{matrix} [i-1,j,k+3] \\ M' \end{matrix} - \delta_3 \begin{matrix} [i-1,j-1,k+4] \\ M' \end{matrix} = \\ \delta_2 \begin{matrix} [i-1,j,k+3] \\ N'' \end{matrix} + \delta_1 \begin{matrix} [i,j-1,k+3] \\ M'' \end{matrix} - \delta_3 \begin{matrix} [i-1,j-1,k+4] \\ M'' \end{matrix}. \end{aligned} \quad (216)$$

In the meantime, the solution to the equations (187–188) is expressed by

$$\begin{matrix} [i,j-1,k+2] \\ R' \end{matrix} = \delta_2 \begin{matrix} [i,j,k+2] \\ N' \end{matrix} - \delta_3 \begin{matrix} [i,j-1,k+3] \\ N' \end{matrix}, \quad (217)$$

$$\begin{matrix} [i-1,j,k+2] \\ R'' \end{matrix} = \delta_1 \begin{matrix} [i,j,k+2] \\ N'' \end{matrix} - \delta_3 \begin{matrix} [i-1,j,k+3] \\ N'' \end{matrix}. \quad (218)$$

Looking now at the equation (189), it only gives some new restrictions as it should be satisfied for every of the solutions (214) and (215). Thus if we use the results given by (217–218) in the equation (189), where we replace

$[i-1, j-1, k+3]$

alternatively with the solutions (214) and (215), we arrive at the new restrictions

$$\begin{aligned} \delta_1 \delta_2 \begin{matrix} [i, j, k+2] \\ N' \end{matrix} - \delta_1 \delta_3 \begin{matrix} [i, j-1, k+3] \\ N' \end{matrix} + \delta_2 \delta_1 \begin{matrix} [i, j, k+2] \\ N'' \end{matrix} - \delta_2 \delta_3 \begin{matrix} [i-1, j, k+3] \\ N'' \end{matrix} = \\ \delta_3 \delta_1 \begin{matrix} [i, j-1, k+3] \\ N' \end{matrix} + \delta_3 \delta_2 \begin{matrix} [i-1, j, k+3] \\ M' \end{matrix}, \end{aligned} \quad (219)$$

$$\begin{aligned} \delta_1 \delta_2 \begin{matrix} [i, j, k+2] \\ N' \end{matrix} - \delta_1 \delta_3 \begin{matrix} [i, j-1, k+3] \\ N' \end{matrix} + \delta_2 \delta_1 \begin{matrix} [i, j, k+2] \\ N'' \end{matrix} - \delta_2 \delta_3 \begin{matrix} [i-1, j, k+3] \\ N'' \end{matrix} = \\ \delta_3 \delta_2 \begin{matrix} [i-1, j, k+3] \\ N'' \end{matrix} + \delta_3 \delta_1 \begin{matrix} [i, j-1, k+3] \\ M'' \end{matrix}. \end{aligned} \quad (220)$$

The restrictions (216) and (219–220) are simultaneously satisfied if and only if there exist some elements X' , X'' , Y' , Y'' , Z' , Z'' and W such that

$$\begin{matrix} [i, j, k+2] \\ N' \end{matrix} = \begin{matrix} [i, j, k+2] \\ N'' \end{matrix} + \delta_1 \begin{matrix} [i+1, j, k+2] \\ X' \end{matrix} + \delta_2 \begin{matrix} [i, j+1, k+2] \\ X'' \end{matrix} + \delta_3 \begin{matrix} [i, j, k+3] \\ W \end{matrix}, \quad (221)$$

$$\begin{matrix} [i-1, j, k+3] \\ N'' \end{matrix} = \begin{matrix} [i-1, j, k+3] \\ M' \end{matrix} + \delta_2 \begin{matrix} [i-1, j+1, k+3] \\ Y' \end{matrix} + \delta_3 \begin{matrix} [i-1, j, k+4] \\ Y'' \end{matrix} + \delta_1 \begin{matrix} [i, j, k+3] \\ W \end{matrix}, \quad (222)$$

$$\begin{matrix} [i, j-1, k+3] \\ N' \end{matrix} = \begin{matrix} [i, j-1, k+3] \\ M'' \end{matrix} + \delta_1 \begin{matrix} [i+1, j-1, k+3] \\ Z' \end{matrix} + \delta_3 \begin{matrix} [i, j-1, k+4] \\ Z'' \end{matrix} - \delta_2 \begin{matrix} [i, j, k+3] \\ W \end{matrix}, \quad (223)$$

where, in addition, there must exist an object T that correlates Y'' and Z'' with $\begin{matrix} [i-1, j-1, k+4] \\ M' \end{matrix}$ and $\begin{matrix} [i-1, j-1, k+4] \\ M'' \end{matrix}$ involved with the right hand-sides of the solutions (214–215) via the relations

$$-\delta_1 \begin{matrix} [i, j-1, k+4] \\ Z'' \end{matrix} + \delta_2 \begin{matrix} [i-1, j, k+4] \\ Y'' \end{matrix} = \begin{matrix} [i-1, j-1, k+4] \\ M' \end{matrix} - \begin{matrix} [i-1, j-1, k+4] \\ M'' \end{matrix} + \delta_3 \begin{matrix} [i-1, j-1, k+5] \\ T \end{matrix}. \quad (224)$$

Thus, we have completely elucidated the solutions to the equations (187–189). We are now left only with the equations (185–186), which are obviously solved by

$$\begin{matrix} [i, j, k+1] \\ R \end{matrix} = \delta_2 \begin{matrix} [i, j+1, k+1] \\ N' \end{matrix} - \delta_3 \begin{matrix} [i, j, k+2] \\ N' \end{matrix}, \quad (225)$$

$$\begin{matrix} [i, j, k+1] \\ R \end{matrix} = \delta_1 \begin{matrix} [i+1, j, k+1] \\ N'' \end{matrix} - \delta_3 \begin{matrix} [i, j, k+2] \\ N'' \end{matrix}, \quad (226)$$

where $\begin{matrix} [i, j, k+2] \\ N' \end{matrix}$ and $\begin{matrix} [i, j, k+2] \\ N'' \end{matrix}$ are linked through the relation (221). On the other hand, the last two formulas imply a new restriction, namely

$$\delta_2 \begin{matrix} [i, j+1, k+1] \\ N' \end{matrix} - \delta_3 \begin{matrix} [i, j, k+2] \\ N' \end{matrix} = \delta_1 \begin{matrix} [i+1, j, k+1] \\ N'' \end{matrix} - \delta_3 \begin{matrix} [i, j, k+2] \\ N'' \end{matrix}. \quad (227)$$

Using now (221) in (227), it follows that the last restriction is equivalent to

$$\delta_1 \begin{pmatrix} [i+1,j,k+1] \\ N'' \end{pmatrix} - \delta_3 \begin{pmatrix} [i+1,j,k+2] \\ X' \end{pmatrix} = \delta_2 \begin{pmatrix} [i,j+1,k+1] \\ N' \end{pmatrix} + \delta_3 \begin{pmatrix} [i,j+1,k+2] \\ X'' \end{pmatrix}, \quad (228)$$

which is satisfied if and only if there exist some elements F , G and H such that

$$\begin{pmatrix} [i+1,j,k+1] \\ N'' \end{pmatrix} - \delta_3 \begin{pmatrix} [i+1,j,k+2] \\ X' \end{pmatrix} = -\delta_2 \begin{pmatrix} [i+1,j+1,k+1] \\ F \end{pmatrix} + \delta_1 \begin{pmatrix} [i+2,j,k+1] \\ G \end{pmatrix}, \quad (229)$$

$$\begin{pmatrix} [i,j+1,k+1] \\ N' \end{pmatrix} + \delta_3 \begin{pmatrix} [i,j+1,k+2] \\ X'' \end{pmatrix} = \delta_1 \begin{pmatrix} [i+1,j+1,k+1] \\ F \end{pmatrix} + \delta_2 \begin{pmatrix} [i,j+2,k+1] \\ H \end{pmatrix}. \quad (230)$$

Finally, substituting any of the solutions (225–226) (which now coincide) into the equation (184) and taking into account (229–230), we deduce that its solution is provided by

$$\begin{pmatrix} [i,j,k] \\ F \end{pmatrix} = \delta_3 \delta_2 \delta_1 \begin{pmatrix} [i+1,j+1,k+1] \\ F \end{pmatrix}, \quad (231)$$

which proves the theorem.

Theorem 4 Let $\begin{pmatrix} [m] \\ F \end{pmatrix} \in \mathcal{A}'$, with $\text{res} \left(\begin{pmatrix} [m] \\ F \end{pmatrix} \right) = m > 0$, be such that

$$\begin{pmatrix} [m] \\ F \end{pmatrix} = \sum_{p+q+r=m} \begin{pmatrix} [p,q,r] \\ F \end{pmatrix}. \quad (232)$$

Assume that: 1. $\delta \begin{pmatrix} [m] \\ F \end{pmatrix} = 0$, and 2. in the sum (232) only terms with $p \leq i$, $q \leq j$, $r \leq k$ occur, for some i, j, k , such that $i+j+k > m$ (strictly). Then, we can represent $\begin{pmatrix} [m] \\ F \end{pmatrix}$ under the form

$$\begin{pmatrix} [m] \\ F \end{pmatrix} = \delta \begin{pmatrix} [m+1] \\ P \end{pmatrix}, \quad (233)$$

where

$$\begin{pmatrix} [m+1] \\ P \end{pmatrix} = \sum_{\bar{p}+\bar{q}+\bar{r}=m+1} \begin{pmatrix} [\bar{p},\bar{q},\bar{r}] \\ P \end{pmatrix}, \quad (234)$$

involves only terms with $\bar{p} \leq i$, $\bar{q} \leq j$ and $\bar{r} \leq k$.

Proof. According to the assumption 2, it follows that

$$\begin{aligned} \begin{pmatrix} [m] \\ F \end{pmatrix} &= \begin{pmatrix} [i,j,m-i-j] \\ F \end{pmatrix} + \begin{pmatrix} [i-1,j,m-i-j+1] \\ F \end{pmatrix} + \begin{pmatrix} [i,j-1,m-i-j+1] \\ F \end{pmatrix} + \\ &\dots + \begin{pmatrix} [i,m-i-k,k] \\ F \end{pmatrix} + \dots + \begin{pmatrix} [m-j-k,j,k] \\ F \end{pmatrix}, \end{aligned} \quad (235)$$

being understood that $\overset{[a,b,c]}{F} = 0$ if $a < 0$, or $b < 0$, or $c < 0$. It is clear that $\overset{[i,j,m-i-j]}{F}$ is the term in F with the lowest third component of the resolution tridegree (if $m - i - j < 0$, then this piece vanishes). Then, from $\delta \overset{[m]}{F} = 0$, it follows that $\delta_3 \overset{[i,j,m-i-j]}{F} = 0$, such that, using $H_{i',j',k'}(\delta_3) = 0$, for $i' + j' + k' > 0$, we deduce that

$$\overset{[i,j,m-i-j]}{F} = \delta_3 \overset{[i,j,m-i-j+1]}{P'}, \quad (236)$$

where $\overset{[i,j,m-i-j+1]}{P'} \equiv 0$ if $m - i - j < 0$. In the meantime, we have that $m - i - j + 1 \leq k$ because $m < i + j + k$. If we subtract $\delta \overset{[i,j,m-i-j+1]}{P'}$ from $\overset{[m]}{F}$, we infer that

$$\begin{aligned} \overset{[m]}{F} - \delta \overset{[i,j,m-i-j+1]}{P'} &= \overset{[i-1,j,m-i-j+1]}{F'} + \overset{[i,j-1,m-i-j+1]}{F'} + \dots \\ &+ \overset{[i,m-i-k,k]}{F} + \dots + \overset{[m-j-k,j,k]}{F}. \end{aligned} \quad (237)$$

Next, we act similarly and remove the terms of lowest third component of the resolution tridegree in the right hand-side of (237), namely, $\overset{[i-1,j,m-i-j+1]}{F'}$ + $\overset{[i,j-1,m-i-j+1]}{F'}$. If we proceed in the same manner, we reach the last step

$$\overset{[m]}{F} - \delta \tilde{P} = \overset{[m-j-k,j,k]}{F'}, \quad (238)$$

such that $\delta \overset{[m]}{F} = 0$ becomes equivalent to

$$\begin{aligned} \left(\delta \overset{[m-j-k,j,k]}{F'} = 0 \right) &\Leftrightarrow \\ \left(\delta_1 \overset{[m-j-k,j,k]}{F'} = 0, \delta_2 \overset{[m-j-k,j,k]}{F'} = 0, \delta_3 \overset{[m-j-k,j,k]}{F'} = 0 \right). \end{aligned} \quad (239)$$

On the other hand, the previous theorem ensures that

$$\begin{aligned} \overset{[m-j-k,j,k]}{F'} &= \delta_1 \delta_2 \delta_3 \overset{[m-j-k+1,j+1,k+1]}{Q} = \\ &(\delta_1 + \delta_2 + \delta_3) \delta_2 \delta_3 \overset{[m-j-k+1,j+1,k+1]}{Q}, \end{aligned} \quad (240)$$

due to the nilpotency and anticommutativity of δ_2 and δ_3 , such that we can further write

$$\overset{[m-j-k,j,k]}{F'} = \delta \overset{[m-j-k+1,j,k]}{P'}, \quad (241)$$

with $\frac{[m-j-k+1,j,k]}{P'} = \delta_2 \delta_3 \frac{[m-j-k+1,j+1,k+1]}{Q}$. Then, we find that

$$\frac{[m]}{F} = \delta \left(\frac{[m+1]}{P} \right), \quad (242)$$

where

$$\frac{[m+1]}{P} = \frac{[i,j,m-i-j+1]}{P'} + \dots + \frac{[i,m-i-k+1,k]}{P'} + \dots + \frac{[m-j-k+1,j,k]}{P'}. \quad (243)$$

It is now easy to see that $m - i - j + 1 \leq k$, $m - i - k + 1 \leq j$ and $m - j - k + 1 \leq i$, as we supposed that $m < i + j + k$, hence $\frac{[m+1]}{P}$ includes only terms of resolution tridegrees $(\bar{p}, \bar{q}, \bar{r})$, with $\bar{p} \leq i$, $\bar{q} \leq j$ and $\bar{r} \leq k$, such that $\bar{p} + \bar{q} + \bar{r} = m + 1$. This ends the proof.

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