

An Invariant action for Noncommutative Gravity in Four-Dimensions

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Abstract

Two main problems face the construction of noncommutative actions for gravity with star products: the complex metric and finding an invariant measure. The only gauge groups that could be used with star products are the unitary groups. I propose an invariant gravitational action in $D=4$ dimensions based on the constrained gauge group $U(2,2)$ broken to $U(1,1) \times U(1,1)$. No metric is used, thus giving a naturally invariant measure. This action is generalized to the noncommutative case by replacing ordinary products with star products. The four dimensional noncommutative action is studied and the deformed action to first order in deformation parameter is computed.

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In noncommutative field theory based on the Moyal star product [1], [2] the only gauge theories that can be used are based on unitary algebras. The presence of a constant background B-field for open or closed strings with D-branes lead to the noncommutativity of space-time coordinates. The Einstein-Hilbert action can be constructed either by insuring diffeomorphism invariance or local Lorentz invariance [3],[4]. This program faces difficulties when ordinary products are replaced with star products. In this case, it is not an easy matter to define a generalization of Riemannian geometry. Noncommutative Riemannian geometry has been developed for noncommutative spaces based on the spectral triple [5],[6]. The difficult part in applying this formalism is to determine the deformed spectral triple. In particular, the deformed Dirac operator is needed in order to apply this formalism to noncommutative spaces where the algebra is deformed with the star product. One must also find an invariant measure. There is, however, some recent progress on such formulation [7]. Recently, the effective action for gravity on noncommutative branes in presence of constant background B-field was derived and found to be non-covariant [8]. This conforms to the expectation that in this case space-time coordinates do not commute.

The approach based on gauging the Lorentz algebra also have problems, mainly that the metric becomes complex, and the antisymmetric part of the metric may have non-physical propagating modes [9]. Finding an invariant measure is also problematic in this approach. One way to avoid the problem of finding an invariant measure is to require the action to be an invariant D-form in a D-dimensional space [10],[11]. Experience with building gauge invariant actions which are also D-forms in a D-dimensional space tells us that these actions are usually topological, and therefore cannot describe gravity in dimensions of four or higher [12]. This is usually avoided by imposing constraints on some components of the gauge field strengths which, in some cases, is equivalent to a torsion free metric theory [13]. Constraints insure that the action, although metric independent, is not topological. The metric is then identified with some components of the gauge fields. Such constraints usually break the gauge group into a subgroup. In the noncommutative field theoretic approach to gravity this works after the constraints are imposed, provided that both the gauge group and the remaining subgroup are of the unitary type. There is a formulation of noncommutative gauge theories where the gauge group could also be of the orthogonal or symplectic type, but it turned out that there are problems associated with this formulation [14],[15],[16]. There is an alternative interpretation in the case where the constraints could be solved for some of the gauge fields in terms of the others. In this case one can insist on preserving gauge invariance in a non-linear fashion, while changing the gauge transformations of those gauge fields that are now dependent in such a way as to preserve the constraints [13]. In this paper we give an invariant four-dimensional gravitational action and then generalize it to the noncommutative case. The action is based on gauging the group $U(2,2)$ broken by constraints to $U(1,1) \times U(1,1)$. One obtains, depending on the constraints, topological gravity, Einstein gravity or conformal gravity. This construction can be extended to the noncommutative case by replacing ordinary products with star products. We derive the deformed curvatures, the deformed action and compute corrections to first order in the deformation parameter θ using the Seiberg-Witten map. We show that in this approach it is only possible to deform Gauss-Bonnet topological gravity, or conformal gravity but not Einstein gravity.

The noncommutative gravitational action was derived in dimensions two and three [17],[18],[19]. In four-dimensions the smallest unitary group that contains both the spin-connection and the vierbein which spans the group $SO(1,4)$ or $SO(2,3)$ is $U(2,2)$ or $U(1,3)$. For definiteness we will consider the group $U(2,2)$. The constraints should keep the $SO(1,3)$ subgroup invariant. The appropriate subgroup is $U(1,1) \times U(1,1)$. To be precise we define the $U(2,2)$ algebra as the set of 4×4 matrices M satisfying [20]

$$g^\dagger \Gamma_4 g = \Gamma_4,$$

where the 4×4 gamma matrices Γ_a , $a = 1, 2, 3, 4$ are the basis of a Clifford algebra

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab},$$

and where we have adopted the notation $\Gamma_4 = i\Gamma_0$ and $x^4 = ix^0$. The gauge fields A_μ satisfy

$$A_\mu^\dagger = -\Gamma_4 A_\mu \Gamma_4$$

and transform according to

$$A_\mu^g = g^{-1} A_\mu g + g^{-1} \partial_\mu g.$$

We can write

$$A = \left(ia_\mu + b_\mu \Gamma_5 + e_\mu^a \Gamma_a + f_\mu^a \Gamma_a \Gamma_5 + \frac{1}{4} \omega_\mu^{ab} \Gamma_{ab} \right) dx^\mu,$$

where

$$\Gamma_5 = \Gamma_1 \Gamma_2 \Gamma_3 \Gamma_4, \quad \Gamma_{ab} = \frac{1}{2} (\Gamma_a \Gamma_b - \Gamma_b \Gamma_a).$$

Let

$$\begin{aligned} D &= d + A, \\ D^2 &= F = (dA + A^2), \end{aligned}$$

so that F transforms covariantly $F^g = g^{-1} F g$. Decomposing the field strength in terms of the Clifford algebra generators

$$F_{\mu\nu} = iF_{\mu\nu}^1 + F_{\mu\nu}^5 \Gamma_5 + F_{\mu\nu}^a \Gamma_a + F_{\mu\nu}^{a5} \Gamma_a \Gamma_5 + \frac{1}{4} F_{\mu\nu}^{ab} \Gamma_{ab},$$

where $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$, then the components are given by

$$\begin{aligned} F_{\mu\nu}^1 &= \partial_\mu a_\nu - \partial_\nu a_\mu, \\ F_{\mu\nu}^5 &= \partial_\mu b_\nu - \partial_\nu b_\mu + 2e_\mu^a f_{\nu a} - 2e_\nu^a f_{\mu a}, \\ F_{\mu\nu}^a &= \partial_\mu e_\nu^a - \partial_\nu e_\mu^a + \omega_\mu^{ab} e_{\nu b} - \omega_\nu^{ab} e_{\mu b} + 2f_\mu^a b_\nu - 2f_\nu^a b_\mu, \\ F_{\mu\nu}^{a5} &= \partial_\mu f_\nu^a - \partial_\nu f_\mu^a + \omega_\mu^{ab} f_{\nu b} - \omega_\nu^{ab} f_{\mu b} + 2e_\mu^a b_\nu - 2e_\nu^a b_\mu, \\ F_{\mu\nu}^{ab} &= \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + 4(e_\mu^a e_\nu^b - f_\mu^a f_\nu^b) - \mu \longleftrightarrow \nu, \end{aligned}$$

We can impose the constraints

$$F_{\mu\nu}^a + F_{\mu\nu}^{a5} = 0, \quad \text{or} \quad F_{\mu\nu}^a - F_{\mu\nu}^{a5} = 0,$$

which break the gauge group $U(2, 2)$ to $U(1, 1) \times U(1, 1)$ with generators

$$(1 \pm \Gamma_5) \{1, \Gamma_{ab}\}$$

One can solve the above constraints to determine ω_μ^{ab} in terms of $e_\mu^{a\pm} = e_\mu^a \pm f_\mu^a$ and b_μ . We can rewrite the constraints in the form

$$\partial_\mu e_\nu^{a+} - \partial_\nu e_\mu^{a+} + \omega_\mu^a e_\nu^{b+} - \omega_\nu^a e_\mu^{b+} + 2e_\mu^{a+} b_\nu - 2e_\nu^{a+} b_\mu = 0,$$

or

$$\partial_\mu e_\nu^{a-} - \partial_\nu e_\mu^{a-} + \omega_\mu^a e_\nu^{b-} - \omega_\nu^a e_\mu^{b-} - 2e_\mu^{a-} b_\nu + 2e_\nu^{a-} b_\mu = 0,$$

which imply that $\omega_\mu^{ab} = \omega_\mu^{ab}(e_\mu^{a+}, b_\mu)$ or $\omega_\mu^{ab} = \omega_\mu^{ab}(e_\mu^{a-}, -b_\mu)$. The solutions which recover the Einstein action are obtained by imposing both sets of constraints simultaneously as these imply

$$f_\mu^a = \alpha e_\mu^a, \quad b_\mu = 0,$$

where α is an arbitrary parameter.

The action which is invariant under the remaining $U(1, 1) \times U(1, 1)$ group is given by [21],[22],

$$I = i \int_M \text{Tr} (\Gamma_5 F \wedge F)$$

where $F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu$. Notice that Γ_5 commutes with the generators $\{1, \Gamma_5, \Gamma_{ab}\}$ of $U(1, 1) \times U(1, 1)$ thus insuring the invariance of the action. This action is metric independent, and one expects the space-time metric to be generated from the gauge fields e_μ^a and f_μ^a . To see this we write the action when both sets of constraints are imposed simultaneously and the only independent field is e_μ^a . The action reduces to

$$I = \frac{i}{4} \int_M d^4 x \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} (R_{\mu\nu}^{ab} + 8(1 - \alpha^2) e_\mu^a e_\nu^b) (R_{\rho\sigma}^{cd} + 8(1 - \alpha^2) e_\rho^c e_\sigma^d)$$

There are three possibilities $|\alpha| < 1$, $|\alpha| = 1$ and $|\alpha| > 1$. The case $|\alpha| = 1$ gives only the Gauss-Bonnet term and is topological. The cases with $|\alpha| < 1$ and $|\alpha| > 1$ give also the scalar curvature and cosmological constants with opposite signs. The abelian gauge field a_μ decouples. This theory is different from the usual gauge formulations in that it has more vacua, and it allows for solutions with arbitrary cosmological constant. We could have restricted ourselves to $SU(2, 2)$ instead of $U(2, 2)$ as the gauge field a_μ decouples, but we did not do so because such a choice is not allowed in the noncommutative case. When only one of the constraints is imposed, then the form of the action does not change, where e_μ^{a+} is taken to be the independent field, we should solve for e_μ^{a-} from its equation of motion. It is known that the action in this case gives conformal supergravity [20].

We are now ready to deal with formulating an action for gravity which is invariant under the star product. One of the main difficulties we mentioned in previous work is that the metric defined by $g_{\mu\nu} = e_\mu^a * e_{\nu a}$ is complex [9] and one has to obtain the correct action for the non-symmetric part (or the complex part) of the metric [23],[24]. The other problem is related to finding an invariant

measure with respect to the star product [25]. Both of these problems could be solved by adopting the formalism given above. We shall show that the deformed vierbein \tilde{e}_μ^a remains real. Gauge invariance with constraints eliminates some of the superfluous degrees of freedom. The constraints also make it possible to have non-topological actions with the advantage of not introducing a metric. The vierbeins are gauge fields corresponding to the broken generators. The action being a 4 form in $D=4$ dimensions is automatically invariant under the star product. The gauge fields transform according to

$$\tilde{A}^g = \tilde{g}_*^{-1} * \tilde{A} * \tilde{g} + \tilde{g}_*^{-1} * d\tilde{g},$$

where \tilde{g} satisfies

$$\tilde{g}_*^{-1} * \tilde{g} = 1, \quad \tilde{g}^\dagger * \Gamma_4 * \tilde{g} = \Gamma_4,$$

and the gauge field strength is

$$\tilde{F} = (d\tilde{A} + \tilde{A} * \tilde{A}),$$

where

$$\tilde{A} = \tilde{A}_\mu dx^\mu, \quad \tilde{F} = \frac{1}{2} \tilde{F}_{\mu\nu} dx^\mu \wedge dx^\nu,$$

and the coordinates x^μ satisfy

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \quad [\partial_\mu, \partial_\nu] = 0, \quad dx^\mu \wedge dx^\nu = -dx^\nu \wedge dx^\mu,$$

which insures that $d^2 = 0$. We use the property

$$\begin{aligned} \tilde{A} * \tilde{A} &= \tilde{A}_\mu^I * \tilde{A}_\nu^J T_I T_J dx^\mu \wedge dx^\nu \\ &= \frac{1}{2} \left(\tilde{A}_\mu^I * \tilde{A}_\nu^J [T_I, T_J] + \tilde{A}_\mu^I * \tilde{A}_\nu^J \{T_I, T_J\} \right) dx^\mu \wedge dx^\nu, \end{aligned}$$

where we have defined both the symmetric and antisymmetric star products by

$$\begin{aligned} f *_s g &\equiv \frac{1}{2} (f * g + g * f) = fg + \left(\frac{i}{2}\right)^2 \theta^{\mu\nu} \theta^{\kappa\lambda} \partial_\mu \partial_\kappa f \partial_\nu \partial_\lambda g + O(\theta^4). \\ f *_a g &\equiv \frac{1}{2} (f * g - g * f) = \left(\frac{i}{2}\right) \theta^{\mu\nu} \partial_\mu f \partial_\nu g + \left(\frac{i}{2}\right)^3 \theta^{\mu\nu} \theta^{\kappa\lambda} \theta^{\alpha\beta} \partial_\mu \partial_\kappa \partial_\alpha f \partial_\nu \partial_\lambda \partial_\beta g + O(\theta^5). \end{aligned}$$

and T_I are the Lie algebra generators. Notice that both commutators and anti-commutators appear in the products, making it necessary to consider only the unitary groups. The advantage in using the Dirac matrix representation is that all the generators corresponding to an even number of gamma matrices form the subgroup $U(1,1) \times U(1,1)$ of $U(2,2)$ while the generators corresponding to an odd number of gamma matrices belong to the coset space $\frac{U(2,2)}{U(1,1) \times U(1,1)}$. Therefore one can constrain some of the field strengths corresponding to the generators with an odd number of gamma matrices to zero thus breaking the symmetry. It is more difficult to solve the constraints in the noncommutative case. We shall make use of the Seiberg-Witten map to do this. The S-W map is defined by the relation [2]

$$\tilde{g}_*^{-1} * \tilde{A}(A) * \tilde{g} + \tilde{g}_*^{-1} * d\tilde{g} = \tilde{A}(g^{-1} A g + g^{-1} dg),$$

and whose solution is equivalent to [2]

$$\begin{aligned}\delta\tilde{A}_\mu(\theta) &= -\frac{i}{4}\delta\theta^{\nu\rho}\left\{\tilde{A}_\nu,\left(\partial_\rho\tilde{A}_\mu+\tilde{F}_{\rho\mu}\right)\right\}_*, \\ \delta\tilde{\lambda}(\theta) &= \frac{i}{4}\delta\theta^{\nu\rho}\left\{\partial_\nu\lambda,A_\rho\right\}_*,\end{aligned}$$

where we have defined $\tilde{g}=e^{\tilde{\lambda}}$ and $\tilde{g}=e^{\tilde{\lambda}}$. These transformations do not preserve the constraints. To make these transformations compatible with the constraints one can follow the same procedure as in the commutative case. This is done by first solving the constraints and determining the dependent fields in terms of the independent ones and then modifying the transformations of these dependent fields in such a way as to preserve the constraints.

The constraints are given by

$$\tilde{F}_{\mu\nu}^a + \tilde{F}_{\mu\nu}^{a5} = 0, \quad \text{or} \quad \tilde{F}_{\mu\nu}^a - \tilde{F}_{\mu\nu}^{a5} = 0,$$

and the action invariant under $U(1,1) \times U(1,1)$ is

$$I = i \int_M Tr \left(\Gamma_{D+1} \tilde{F} * \tilde{F} \right).$$

Notice that we can write $\tilde{F} = \frac{1}{2}\tilde{F}_{\mu\nu}dx^\mu \wedge dx^\nu$ and $\tilde{F} * \tilde{F} = \frac{1}{2^2}\tilde{F}_{\mu_1\mu_2} * \tilde{F}_{\mu_3\mu_4}dx^{\mu_1} \wedge dx^{\mu_2} \wedge dx^{\mu_3} \wedge dx^{\mu_4}$. The gauge fields \tilde{A}_μ are decomposed as in the commutative case. The field strengths are given by

$$\begin{aligned}\tilde{F}_{\mu\nu}(1) &= i \left(\partial_\mu \tilde{a}_\nu - \partial_\nu \tilde{a}_\mu \right) \\ &\quad + 2 \left(-\tilde{a}_\mu *_a \tilde{a}_\nu + \tilde{b}_\mu *_a \tilde{b}_\nu + \tilde{e}_\mu^a *_a \tilde{e}_{\nu a} - \tilde{f}_\mu^a *_a \tilde{f}_{\nu a} - \frac{1}{4}\tilde{\omega}_\mu^{ab} *_a \tilde{\omega}_{\nu ab} \right), \\ \tilde{F}_{\mu\nu}(\Gamma_5) &= \partial_\mu \tilde{b}_\nu - \partial_\nu \tilde{b}_\mu + 2 \left(\tilde{e}_\mu^a *_s \tilde{f}_{\nu a} - \tilde{f}_\mu^a *_s \tilde{e}_{\nu a} \right) \\ &\quad + 2 \left(\tilde{b}_\mu *_a \tilde{a}_\nu + \tilde{a}_\mu *_a \tilde{b}_\nu \right) + \frac{1}{8}\epsilon_{abcd}\tilde{\omega}_\mu^{ab} *_a \tilde{\omega}_\nu^{cd}, \\ \tilde{F}_{\mu\nu}(\Gamma_{ab}) &= \frac{1}{4} \left(\partial_\mu \tilde{\omega}_\nu^{ab} - \partial_\nu \tilde{\omega}_\mu^{ab} + \tilde{\omega}_\mu^{ac} *_s \tilde{\omega}_{\nu c}^b - \tilde{\omega}_\mu^{bc} *_s \tilde{\omega}_{\nu c}^a \right) \\ &\quad + \frac{i}{2} \left(\tilde{a}_\mu *_a \tilde{\omega}_\nu^{ab} + \tilde{\omega}_\mu^{ab} *_a \tilde{a}_\nu \right) - \frac{1}{4}\epsilon^{ab}{}_{cd} \left(\tilde{b}_\mu *_a \tilde{\omega}_\nu^{cd} + \tilde{\omega}_\mu^{cd} *_a \tilde{b}_\nu \right) \\ &\quad - 4\epsilon^{ab}{}_{cd} \left(\tilde{e}_\mu^c *_a \tilde{f}_\nu^d + \tilde{f}_\mu^d *_a \tilde{e}_\nu^c \right) + \left(\tilde{e}_\mu^a *_s \tilde{e}_\nu^b - \tilde{e}_\nu^a *_s \tilde{e}_\mu^b - \tilde{f}_\mu^a *_s \tilde{f}_\nu^b + \tilde{f}_\nu^a *_s \tilde{f}_\mu^b \right),\end{aligned}$$

for the generators with an even number of gamma matrices, and by

$$\begin{aligned}\tilde{F}_{\mu\nu}(\Gamma_a) &= \partial_\mu \tilde{e}_\nu^a - \partial_\nu \tilde{e}_\mu^a + \tilde{\omega}_\mu^{ac} *_s \tilde{e}_{\nu c} + \tilde{e}_\mu^c *_s \tilde{\omega}_{\nu c}^a \\ &\quad - 2 \left(\tilde{b}_\mu *_s \tilde{f}_\nu^a - \tilde{f}_\mu^a *_s \tilde{b}_\nu \right) + 2i \left(\tilde{a}_\mu *_a \tilde{e}_\nu^a + \tilde{e}_\mu^a *_a \tilde{a}_\nu \right) \\ &\quad + \frac{1}{2}\epsilon^a{}_{bcd} \left(\tilde{f}_\mu^b *_a \tilde{\omega}_\nu^{cd} + \tilde{\omega}_\mu^{cd} *_a \tilde{f}_\nu^b \right), \\ \tilde{F}_{\mu\nu}(\Gamma_a\Gamma_5) &= \partial_\mu \tilde{f}_\nu^a - \partial_\nu \tilde{f}_\mu^a + \tilde{\omega}_\mu^{ac} *_s \tilde{f}_{\nu c} + \tilde{f}_\mu^c *_s \tilde{\omega}_{\nu c}^a \\ &\quad - 2 \left(\tilde{b}_\mu *_s \tilde{e}_\nu^a - \tilde{e}_\mu^a *_s \tilde{b}_\nu \right) + 2i \left(\tilde{a}_\mu *_a \tilde{f}_\nu^a + \tilde{f}_\mu^a *_a \tilde{a}_\nu \right) \\ &\quad + \frac{1}{2}\epsilon^a{}_{bcd} \left(\tilde{e}_\mu^b *_a \tilde{\omega}_\nu^{cd} + \tilde{\omega}_\mu^{cd} *_a \tilde{e}_\nu^b \right),\end{aligned}$$

for the generators with an odd number of gamma matrices. In four dimensions, the action is

$$\begin{aligned}
I &= i \int_M \text{Tr} \left(\Gamma_5 \tilde{F} * F \right) \\
&= i \int_M d^4 x \epsilon^{\mu\nu\rho\sigma} \text{Tr} \left(\Gamma_5 \tilde{F}_{\mu\nu} * \tilde{F}_{\rho\sigma} \right) \\
&= i \int_M d^4 x \epsilon^{\mu\nu\rho\sigma} \left(2 \tilde{F}_{\mu\nu}^1 *_s \tilde{F}_{\rho\sigma}^5 + \epsilon_{abcd} \tilde{F}_{\mu\nu}^{ab} *_s \tilde{F}_{\rho\sigma}^{cd} \right).
\end{aligned}$$

Notice that although only the symmetric star product appears there are linear corrections in θ to the commutative action. As in the commutative case, the constraints have to be solved for $\tilde{\omega}_\mu^{ab}$ in terms of \tilde{e}_μ^{a+} or \tilde{e}_μ^{a-} , \tilde{b}_μ and \tilde{a}_μ . However, unlike the commutative case, it is not possible to impose both constraints simultaneously after setting $\tilde{b}_\mu = 0$ because of the presence of the $\mp e^\pm \omega$ term in $\tilde{F}_{\mu\nu}^a \pm \tilde{F}_{\mu\nu}^{a5}$. These two constraints become incompatible except in the special case where $\tilde{e}_\mu^{a-} = 0$, which corresponds to deforming the Gauss-Bonnet action. If only one constraint is imposed and $\tilde{\omega}_\mu^{ab}$ is determined from the constraint, the independent fields are \tilde{e}_μ^{a+} , \tilde{e}_μ^{a-} , \tilde{b}_μ and \tilde{a}_μ resulting in deformed conformal supergravity. It is not possible to obtain a deformation of Einstein gravity as the constraints could not be imposed simultaneously.

One can expand this action perturbatively in powers of θ . This can be done by using the Seiberg-Witten map for \tilde{e}_μ^{a+} , \tilde{e}_μ^{a-} , \tilde{b}_μ and \tilde{a}_μ . These expressions are then used in the above constraint to determine $\tilde{\omega}_\mu^{ab}$. It is instructive to carry this procedure to first order in θ . Applying the Seiberg-Witten map, one gets

$$\begin{aligned}
\tilde{e}_\mu^{a\pm} &= e_\mu^{a\pm} + \frac{1}{2} \theta^{\kappa\rho} \left(a_\kappa \partial_\rho e_\mu^{a\pm} + e_\kappa^{a\pm} (2 \partial_\rho a_\mu - \partial_\mu a_\rho) \right. \\
&\quad \left. \mp \frac{i}{4} \epsilon_{abcd} (e_\kappa^{b\pm} (\partial_\rho \omega_\mu^{cd} + F_{\rho\mu}^{cd}) + \omega_\kappa^{cd} \partial_\rho e_\mu^{b\pm}) \right) + O(\theta^2) \\
&\equiv e_\mu^{a\pm} + \frac{1}{2} \theta^{\kappa\rho} e_{\mu\kappa\rho}^{a\pm} + O(\theta^2)
\end{aligned}$$

$$\begin{aligned}
\tilde{a}_\mu &= a_\mu + \frac{1}{2} \theta^{\kappa\rho} \left(a_\kappa (2 \partial_\rho a_\mu - \partial_\mu a_\rho) - b_\kappa (\partial_\rho b_\mu + F_{\rho\mu}^5) \right. \\
&\quad - e_\kappa^a (\partial_\rho e_\mu^a + F_{\rho\mu}^a) + f_\kappa^a (\partial_\rho f_\mu^a + F_{\rho\mu}^{a5}) \\
&\quad \left. + \frac{1}{8} \omega_\kappa^{ab} (\partial_\rho \omega_\mu^{ab} + F_{\rho\mu}^{ab}) \right) + O(\theta^2) \\
&\equiv a_\mu + \frac{1}{2} \theta^{\kappa\rho} a_{\mu\kappa\rho} + O(\theta^2)
\end{aligned}$$

$$\begin{aligned}
\tilde{b}_\mu &= b_\mu + \frac{1}{2} \theta^{\kappa\rho} \left(b_\kappa (2 \partial_\rho a_\mu - \partial_\mu a_\rho) + a_\kappa (\partial_\rho b_\mu + F_{\rho\mu}^5) \right. \\
&\quad \left. - \frac{i}{8} \epsilon_{abcd} \omega_\kappa^{ab} (\partial_\rho \omega_\mu^{cd} + F_{\rho\mu}^{cd}) \right) + O(\theta^2) \\
&\equiv b_\mu + \frac{1}{2} \theta^{\kappa\rho} b_{\mu\kappa\rho} + O(\theta^2)
\end{aligned}$$

We do not take $\tilde{\omega}_\mu^{ab}$ as given by the S-W map, but instead substitute the above expressions in the constraint equation to determine its value. First we write

$$\tilde{\omega}_\mu^{ab} = \omega_\mu^{ab} + \frac{1}{2}\theta^{\kappa\rho}\omega_{\mu\kappa\rho}^{ab} + O(\theta^2)$$

then the constraint becomes

$$\begin{aligned} \tilde{F}_{\mu\nu}^{a+} = F_{\mu\nu}^{a+} + \frac{1}{2}\theta^{\kappa\rho} & (\partial_\mu e_{\nu\kappa\rho}^{a+} - \partial_\nu e_{\mu\kappa\rho}^{a+} + \omega_\mu^{ac}e_{\nu\kappa\rho}^{c+} - \omega_\nu^{ac}e_{\mu\kappa\rho}^{c+} \\ & + \omega_{\mu\kappa\rho}^{ac}e_\nu^{c+} - \omega_{\nu\kappa\rho}^{ac}e_\mu^{c+} \mp 2(b_{\mu\kappa\rho}e_\nu^{a+} - b_{\nu\kappa\rho}e_\mu^{a+}) \\ & - 2(\partial_\kappa a_\mu \partial_\rho e_\nu^{a+} - \partial_\kappa a_\nu \partial_\rho e_\mu^{a+})) + O(\theta^2) \end{aligned}$$

Substituting $\tilde{F}_{\mu\nu}^{a+} = 0$ and $F_{\mu\nu}^{a+} = 0$, we can solve for $\omega_{\mu\kappa\rho}^{ab}$ to obtain:

$$\omega_{\mu\kappa\rho}^{ab} = \frac{1}{2}(e^{\nu b+}C_{\mu\nu\kappa\rho}^a - e^{\nu a+}C_{\mu\nu\kappa\rho}^b + e^{\sigma a+}e^{\nu b+}e_{\mu\sigma}^+C_{\sigma\nu\kappa\rho})$$

where

$$\begin{aligned} C_{\mu\nu\kappa\rho}^a = - & (\partial_\mu e_{\nu\kappa\rho}^{a+} - \partial_\nu e_{\mu\kappa\rho}^{a+} + \omega_\mu^{ac}e_{\nu\kappa\rho}^{c+} - \omega_\nu^{ac}e_{\mu\kappa\rho}^{c+} \\ & - 2(\partial_\kappa a_\mu \partial_\rho e_\nu^{a+} - \partial_\kappa a_\nu \partial_\rho e_\mu^{a+})) \end{aligned}$$

To find the deformed action we first calculate

$$\begin{aligned} \tilde{F}_{\mu\nu}^1 &= F_{\mu\nu}^1 + \frac{1}{2}\theta^{\kappa\rho}(\partial_\mu a_{\nu\kappa\rho} - \partial_\nu a_{\mu\kappa\rho} - \partial_\kappa a_\mu \partial_\rho a_\nu + \partial_\kappa b_\mu \partial_\rho b_\nu \\ &+ \frac{1}{2}(\partial_\kappa e_\mu^{a+} \partial_\rho e_\nu^{a-} - \partial_\kappa e_\nu^{a+} \partial_\rho e_\mu^{a-}) - \frac{1}{4}\partial_\kappa \omega_\mu^{ab} \partial_\rho \omega_\nu^{ab}) + O(\theta^2) \\ &\equiv F_{\mu\nu}^1 + \frac{1}{2}\theta^{\kappa\rho}F_{\mu\nu\kappa\rho}^1 + O(\theta^2) \end{aligned}$$

$$\begin{aligned} \tilde{F}_{\mu\nu}^{ab} &= F_{\mu\nu}^{ab} + \frac{1}{2}\theta^{\kappa\rho}(\partial_\mu \omega_{\nu\kappa\rho}^{ab} - \partial_\nu \omega_{\mu\kappa\rho}^{ab} + \omega_\mu^{ac}\omega_{\nu\kappa\rho}^{cb} - \omega_\nu^{ac}\omega_{\mu\kappa\rho}^{cb} - \omega_\mu^{bc}\omega_{\nu\kappa\rho}^{ca} + \omega_\nu^{bc}\omega_{\mu\kappa\rho}^{ca} \\ &+ 4(e_\mu^{a+}e_{\nu\kappa\rho}^{a-} - e_\nu^{a+}e_{\mu\kappa\rho}^{a-} - e_\mu^{a-}e_{\nu\kappa\rho}^{a+} + e_\nu^{a-}e_{\mu\kappa\rho}^{a+}) \\ &- 8i\epsilon_{abcd}(\partial_\kappa e_\mu^{c+}\partial_\rho e_\nu^{d-} - \partial_\kappa e_\nu^{c+}\partial_\rho e_\mu^{d-}) - 2(\partial_\kappa a_\mu \partial_\rho \omega_\nu^{ab} - \partial_\kappa a_\nu \partial_\rho \omega_\mu^{ab}) \\ &- i\epsilon_{abcd}(\partial_\kappa b_\mu \partial_\rho \omega_\nu^{cd} - \partial_\kappa b_\nu \partial_\rho \omega_\mu^{cd})) + O(\theta^2) \\ &\equiv F_{\mu\nu}^{ab} + \frac{1}{2}\theta^{\kappa\rho}F_{\mu\nu\kappa\rho}^{ab} + O(\theta^2) \end{aligned}$$

Notice that all the above expressions are real. The appearance of $i\epsilon_{abcd}$ is due to the convention $x^4 = ix^0$ so that $i\epsilon_{1234} = \epsilon_{1230} = 1$. Therefore the conformal gravity action to first order in θ is given by

$$I = i \int d^4x \epsilon^{\mu\nu\lambda\sigma} (\epsilon_{abcd}F_{\mu\nu}^{ab}F_{\lambda\sigma}^{cd} + \theta^{\kappa\rho}(2e_\mu^{a+}e_\nu^{a-}F_{\lambda\sigma\kappa\rho}^1 + \epsilon_{abcd}F_{\mu\nu}^{ab}F_{\lambda\sigma\kappa\rho}^{cd})) + O(\theta^2)$$

where we have dropped total derivative terms. The deformation to the Gauss-Bonnet action is obtained from the above expression by setting $e_\mu^{a-} = 0$. It would be instructive to compare this action with the one obtained from the Born-Infeld effective action in String theory where the field $B_{\mu\nu}$ has a constant background

[8]. One can also compare these results by following the results of Jackiw-Pi [26] by defining covariant coordinate transformations on noncommutative spaces. More importantly is to compare this result with the spectral action for a deformed spectral triple $(\tilde{\mathcal{A}}, \tilde{H}, \tilde{D})$ where $\tilde{\mathcal{A}} = l(\mathcal{A})$, \mathbb{I} is the left twist operator [27]. The difficult part is to obtain the deformed operator \tilde{D} and it is hoped that the above formulation will give some hints on how to find the appropriate Dirac operator.

To summarize, we have proposed a four-dimensional gravitational action valid for both commutative and noncommutative field theories. This action differs from the familiar gravitational action in that it allows for other vacua besides those of the metric theory. The noncommutativity is obtained by replacing ordinary products with star products. The action is gauge invariant and do not involve explicit use of the metric. Only conformal gravity or Gauss-Bonnet topological gravity could be generalized to the noncommutative case as the constraints imposed on the gauge field strengths should be self-consistent. For some of the vacuum solutions, one of the gauge fields is identified with the vierbein, and the theory becomes metric. It will be interesting to study how to generalize this proposal to higher dimensions. There are no fundamental obstacles to this approach in even dimensions. In odd dimensions, however, it is not possible to impose constraints in such a way as to preserve a smaller unitary group including the spin-connection generators of $SO(2n+1)$. It appears that in odd dimensions the only gravitational actions which are generalizable to the noncommutative case are of the Chern-Simons type [28],[29], and therefore must be topological. Finally, one can study the supersymmetric version of the four-dimensional gravitational action by considering the graded Lie-algebra $U(2,2|1)$.

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References

- [1] A. Connes, M. R. Douglas, A. Schwartz, *JHEP* **9802** (1998) 003.
- [2] N. Seiberg, E. Witten, *JHEP* **9909** (1999) 032.
- [3] R. Utiyama, *Phys. Rev.* **101** (1956) 1597.
- [4] T. W. B. Kibble, *J. Math. Phys.* **2** (1961) 212.
- [5] A. H. Chamseddine, G. Felder, J. Fröhlich, *Comm. Math. Phys.* **155** (1993) 109; A. H. Chamseddine, J. Fröhlich, O. Grandjean, *J. Math. Phys.* **36** (1995) 6255.
- [6] A. Connes *J. Math. Phys.* **36** (1995) 6194.
- [7] A. Connes and M. Dubois-Violette, *math.QA/0107070*.
- [8] F. Ardalan, H. Arfai, M. R. Garousi and A. Ghodsi, *hep-th/0204117*.

- [9] A. H. Chamseddine, *Comm. Math. Phys.* **218** (2001) 283.
- [10] Y. M. Cho, *Phys. Rev.* **D14** (1976) 3341.
- [11] A. H. Chamseddine, P. C. West, *Nucl. Phys.* **B129** (1977) 39.
- [12] D. Lovelock, *J. Math. Phys.* **12** (1971) 498.
- [13] P. van Nieuwenhuizen, M. Kaku, P. Townsend, *Phys. Rev.* **D17** (1978) 3179.
- [14] L. Bonora, M. Schnabl, M. Sheikh-Jabbari, A. Tomasiello, *Nucl. Phys.* **B589** (2000) 461.
- [15] B. Jurco, S. Schraml, P. Schupp, J. Wess, *Eur. Phys. J.* **C17** (2000) 521.
- [16] A. H. Chamseddine, *Phys. Lett.* **B504** (2001) 33.
- [17] S. Cacciatori, A. H. Chamseddine, D. Klemm, L. Martucci, W. A. Sabra, D. Zanon, *hep-th/0203038*.
- [18] V. P. Nair, *hep-th/0112114*.
- [19] S. Cacciatori, D. Klemm, L. Martucci, D. Zanon, *hep-th/0201103*.
- [20] P. van Nieuwenhuizen, *Phys. Rep.* **68** (1981) 189.
- [21] F. Mansouri, *Phys. Rev.* **D16** (1977) 2456.
- [22] A. H. Chamseddine, *Ann. Phys.* **113** (1978) 219.
- [23] A. Einstein, E. Strauss, *Ann. Math.* **47** (1946) 47, 731.
- [24] E. Schrödinger, *Space-Time Structure*, Cambridge University Press, 1985.
- [25] H. Nishino, S. Rajpott, *hep-th/0108123*.
- [26] R. Jackiw and S.-Y. Pi, *Phys. Rev. Lett.* **88** (2002) 111603.
- [27] Alain Connes and Giovanni Landi, *Comm. Math. Phys.* **221** (2001) 141.
- [28] A. H. Chamseddine, J. Fröhlich, *J. Math. Phys.* **35** (1994) 5195.
- [29] A. P. Polychronakos, *JHEP* **0011** (2000) 008.