SEARCHING FOR A CONNECTION BETWEEN MATROID THEORY AND STRING THEORY

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Abstract

We make a number of observations about matter-ghost string phase, which may eventually lead to a formal connection between matroid theory and string theory. In particular, in order to take advantage of the already established connection between matroid theory and Chern-Simons theory, we propose a generalization of string theory in terms of some kind of Kahler metric. We show that this generalization is closely related to the Kahler-Chern-Simons action due to Nair and Schiff. We also add new information about the relationship between matroid theory, $D = 11$ supergravity and Chern-Simons formalism.

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I. INTRODUCTION

Although the key principle in M-theory[1–3] and string theory[4] is unknown there is accumulating evidence for the existence of some kind of duality principle. In fact, duality is the key physical concept that relates the five known superstring theories in 9+1 dimensions (i.e., nine space and one time), Type I, Type IIA, Type IIB, Heterotic SO(32) and Heterotic $E_8 \times E_8$, which may now be understood as different manifestations of M-theory. Thus, anticipating the possibility that duality is the basic principle in M-theory, one may be interested in the mathematical structure necessary to make sense of such a duality principle. The idea is similar to the role played by tensor analysis which gives a mathematical sense to the postulate of relativity "the laws of physics are the same for every observer". In two previous works we proposed the possibility that such a mathematical structure could be realized through the so called matroid theory.[5] Matroid theory, which can be understood as a generalization of graph theory and matrix theory, has the duality symmetry among its key basic concepts. In fact, in contrast to graphs in which duality can only be considered in connection with planar graphs, matroid theory has the remarkable property that every matroid has a unique dual matroid. As an example of the importance of the duality property in matroid theory let us just mention a theorem due to Whitney[5]: if $M_1, .., M_p$ and $M_1^{'}, .., M_p^{'}$ are the components (blocks) of the matroids M and M' respectively, and if $M_i^{'}$ is the dual of M_i $(i = 1, ..., p)$ then M' is dual of M and conversely, if M and M' are dual matroids then $M_i^{'}$ is dual of M_i. Moreover, in a general context, we have the remarkable proposition that if a statement μ in the theory of matroids has been proved true, then also its dual μ^* is true.

Of course, the question is how to achieve such a relationship between matroid theory and M-theory. Especially if we do not even know the formal partition function associated to M-theory. As a first step in this direction, one may attempt to see if matroid theory is linked somehow to $D = 11$ supergravity which is one of the manifestations of M-theory. In fact, it has been shown[6] that the Fano matroid and its dual are closely related to Englert's compactification[7] of $D = 11$ supergravity. This result is physically interesting because it allows a connection between the fundamental Fano matroid or its dual[8] and octonions which, at the same time, are one of the alternative division algebras.[9] In reference 10, we made further progress on this program, incorporating matroid theory on quantum Yang-Mills theory in the context of Chern-Simons action. Our mechanism was based on a theorem due to Thistlethwaite[11] which connects the Jones polynomial for alternating knots with the Tutte polynomial for graphs. Since Witten[12] showed that

Jones polynomial can be understood in three dimensional terms through a Chern-Simons formalism, it became evident that we achieved a bridge between matroid theory and Chern-Simons formalism.

In this article, we further pursue the idea of relating matroid theory with M-theory. Since the five fundamental strings are different vacuum limits of M-theory, it seems natural to try to find first a link between matroid theory and string theory. In this context there are a number of observations that indicate that this idea makes sense. First, since Chern-Simons formalism is closely linked to conformal field theory and Matrix theory, which in turn are related to string theory, one should expect a connection of the form: matroid-theory→Chern-Simons-theory→string-theory. Second, since strings are closely related to knots, which in turn are related in one to one correspondence to signed graphs, one should expect a link of the form: matroids→signed graphs→knots→strings. Finally, we can in effect combine the two previous observations in the form: matroids→signed graphs→knots→Chern-Simons-formalism→strings.

In order to achieve our goal, we study the possibility that, in the string phase of matter-ghost coupling, the world sheet metric and the target space-time metric become unified in just one metric. We show, in some detail, that such a unified metric may be a certain kind of Kahler metric. This observation lead us to consider the Kahler-Chern-Simons action as the key bridge to connect matroid theory and string theory.

The plan of this work is as follows. In section 2, we briefly review matroid theory. In section 3, we closely follow the Ref. 6 adding new information about the connection between matroid theory and $D = 11$ supergravity . In section 4, we briefly review Ref. 10 and propose a possible extension of the relation between matroid theory and Witten's partition function for knots. In section 5, we propose a generalized Polyakov string action with the property of unifying the world-sheet metric and the target space-time metric. Finally, in section 6, we make some final comments.

II. A BRIEF REVIEW OF MATROID THEORY

At present matroid theory, also called combinatorial geometry or pregeometry, can be understood as the combinatorial analogue of K-theory. In fact, the axioms of K-theory are very similar to the properties achieved with the Tutte-Gotendiek invariants for matroids. This interpretation emerged from a great number of contributions from a several mathematicians since 1935 with the pioneer work of Whitney[5] on "Abstract properties of linear dependence". In the same year, Birkhoff[13] established the connection between simple matroids and geometric lattices. In 1936, MacLane[14] gave an interpretation of matroids in terms of projective geometry. And an important progress to the subject was given in 1958 by Tutte[8] who introduced the concept of homotopy for matroids. The fascination of this subject among the combinatorial mathematicians can be appreciated from the large body of information about matroid theory. In fact, there is a large number of books about matroid theory. For background information on this subject the reader should consult Oxley[15] and Welsh.[16] We also recommend the books of Wilson,[17] Kung[18] and Ribnikov.[19]

It is known that in graph theory only planar graphs have an associated dual graph. For instance, the Kuratowski theorem assures that the complete graph K_5 and the bipartita graph $K_{3,3}$ which are not planar do not have an associated dual graph. In a sense, matroid theory arose as an attempt to solve this lack of duality symmetry. The attractive feature is that in matroid theory every matroid has an associated unique dual matroid. In particular the matroid associated to K_5, let us say $M(K_5)$, has a dual $M^*(K_5)$. The important aspect is that $M^*(K_5)$ is not graphic, that is, it can not be represented by a graph. This is, of course, an indication that matroid theory is a generalization of graph theory. Therefore, the great advantage of matroid theory is that it provides us with a mathematical structure in which the concepts of duality of planar graphs is extended to graphs that are not planar.

Another interesting aspect that motivates the subject is that linear dependence in algebra can be understood as a particular case of matroid theory. In fact, matroid theory leads to matroids that are not even representable by a vector space or by matrices, extending the concept of orthogonality in vector spaces. Summarizing, we can say that by extending the concept of duality in vector spaces and planar graphs, matroid theory accomplishes a generalization of both graph theory and matrix theory.

Mathematically, a matroid is defined as follows: a matroid M is a pair (E, I), where E, called the ground set, is a non-empty finite set, and I is a non-empty collection of subsets of E satisfying the following two properties:

$(I\ i)$ any subset of an independent set is independent.

$(I\ ii)$ if K and J are independent sets with $K \subseteq J$, then there is an element e contained in J but not in K, such that $K \cup \{e\}$ is independent.

Members of I are called independent sets of M; other sets are called dependent. Therefore, the definition itself of a matroid divides all possible subsets of E in two types: independent and dependent subsets. Thus, we see that, even from the beginning, matroids have the dual structure independent-dependent. From this point of view, it is not a surprise to find eventually that every matroid has an associated dual matroid.

A base is defined to be any maximal independent set. Similarly, the minimal dependent set is called a circuit. By repeatedly using the property $(I\,ii)$ it is straightforward to show that any two bases have the same number of elements.

An alternative definition of a matroid in terms of bases is as follows:

A matroid M is a pair $(E, \mathcal{B})$, where E is a non-empty finite set and $\mathcal{B}$ is a non-empty collection of subsets of E (called bases) satisfying the following properties:

$(\mathcal{B}\ i)$ no base properly contains another base;

$(\mathcal{B}\ ii)$ if B_1 and B_2 are bases and if b is any element of B_1, then there is an element g of B_2 with the property that $(B_1 - \{b\}) \cup \{g\}$ is also a base.

A matroid can also be defined in terms of circuits:

A matroid M is a pair $(E,\ C)$, where E is a non-empty finite set, and C is a non-empty collection of subsets of E (called circuits) satisfying the following properties.

$(C\ i)$ no circuit properly contains another circuit;

$(C\ ii)$ if C_1 and C_2 are two distinct circuits each containing an element c, then there exists a circuit in $C_1 \cup C_2$ which does not contain c.

If we start with any of the three definitions then one finds that the other two follow as theorems. For example, it is possible to prove that (I) implies $(\mathcal{B})$ and (C). In other words, these three definitions are equivalent. There are other definitions also equivalent to these three, but for the purpose of this work it is not necessary to consider all of them.

As we noticed previously, even from the initial structure of a matroid theory we find relations such as independent-dependent structure which suggests duality. The dual of M, denoted by M^*, is defined as a pair $(E, \mathcal{B}^*)$, where $\mathcal{B}^*$ is a non-empty collection of subsets of E formed with the complements of the bases of M. An immediate consequence of this definition is that every matroid has a dual and this dual is unique. It also follows that the double-dual M^{**} is equal to M. Moreover, if S is a subset of E, then the size of the largest independent set contained in S is called the rank of S and is denoted by $\rho(S)$. If $M = M_1 + M_2$ and $\rho(M) = \rho(M_1) + \rho(M_2)$ we shall say that M is separable. Any maximal non-separable part of M is a block of M. An important theorem due to Whitney[5] is that if $M_1, .., M_p$ and $M_1^{'}, .., M_p^{'}$ are the blocks of the matroids M and M' respectively, and if $M_i^{'}$ is the dual of M_i $(i = 1, ..., p)$. Then M' is dual of M. Conversely, let M and M' be dual matroids, and let $M_1, .., M_p$ be blocks of M. Let $M_1^{'}, .., M_p^{'}$ be the corresponding submatroids of M'. Then $M_1^{'}, .., M_p^{'}$ are the blocks of M', and $M_i^{'}$ is dual of M_i.

Over the last years matroid theory has been growing very rapidly. There are already well established formalisms for oriented matroids[20] and bias matroids.[21] The former can be understood as a generalization of oriented graphs and the latter as an extension of signed graphs. In each one of these branches of matroid theory there are very interesting theorems and results, some of which we shall mention in the next sections.

III. MATROID THEORY AND SUPERGRAVITY

Here, we briefly review the main results of Ref. 6 and add some new observations. In Ref. 6 we showed that the Fano matroid F_7may be connected with octonions which, in turn, are related to the Englert's compactification of $D = 11$ supergravity.

A Fano matroid F_7 is the matroid defined on the ground set

$$E = \{1, 2, 3, 4, 5, 6, 7\}$$

whose bases are all those subsets of E with three elements except $f_1 = \{1, 2, 3\}$, $f_2 = \{5, 1, 6\}$, $f_3 = \{6, 4, 2\}$, $f_4 = \{4, 3, 5\}$, $f_5 = \{4, 7, 1\}$, $f_6 = \{6, 7, 3\}$ and $f_7 = \{5, 7, 2\}$. The circuits of the Fano matroid are precisely these subsets and its complements. It follows that these circuits define the dual F_7^* of the Fano matroid.

Let us write the set E in the form E=$\{e_1, e_2, e_{3,}e_{4,}e_5, e_6, e_7\}$. Thus, the subsets used to define the Fano matroid now become $f_1 = \{e_1, e_2, e_3\}$, $f_2 = \{e_5, e_1, e_6\}$, $f_3 = \{e_6, e_4, e_2\}$, $f_4 = \{e_4, e_3, e_5\}$, $f_5 = \{e_4, e_7, e_1\}$, $f_6 = \{e_6, e_7, e_3\}$ and $f_7 = \{e_5, e_7, e_2\}$. The key idea in Ref. 6 was to identify the quantities e_i, where $i = 1, ...,7$, with the octonionic imaginary units. Specifically, we write an octonion q in the form $q = q_0e_0 + q_1e_1 + q_2e_2 + q_3e_3 + q_4e_4 + q_5e_5 + q_6e_6 + q_7e_7$, where q_0 and q_i are real numbers. Here, e_0 denotes the identity. The product of two octonions can be obtained from the formula:

$$e_ie_j = -\delta_{ij} + \psi_{ij}^k e_k, \tag{1}$$

where δ_{ij} is the Kronecker delta and $\psi_{ijk} = \psi_{ij}^l \delta_{lk}$ is the fully antisymmetric structure constants, with $i, j, k = 1, ..., 7$. By taking the ψ_{ijk} equals 1 or -1 for each one of the seven combinations f_i we may derive all the values of ψ_{ijk}.

The octonion (Cayley) algebra is not associative, but alternative. This means that the basic associator of any three imaginary units is

$$\langle e_i, e_j, e_k\rangle = (e_ie_j)e_k - e_i(e_je_k) = \varphi_{ijkm}e_m, \tag{2}$$

where φ_{ijkl} is a fully antisymmetric four index tensor. It turns out that φ_{ijkl} and ψ_{ijk} are related by the expression

$$\varphi_{ijkl} = (1/3!)\epsilon_{ijklmnr}\psi_{mnr}, \tag{3}$$

where $\epsilon_{ijklmnr}$ is the completely antisymmetric Levi-Civita tensor, with $\epsilon_{12...7} = 1$. It is interesting to observe that given the numerical values f_i for the indices of ψ_{mnr} and using (3) we get the other seven subsets of E with four elements of the dual Fano matroid F_7^*. For instance, if we take f_1 then we have ψ_{123} and (3) gives φ_{4567} which leads to the circuit subset $\{4, 5, 6, 7\}$ of F_7^*.

Therefore, this shows that the Fano matroid and its dual are closely related to octonions which at the same time are an essential part of the Englert's solution of absolute parallelism on S^7 of $D = 11$ supergravity. It is important to mention that the Fano matroid is the only minimal binary irregular matroid. Just as octonions are central mathematical objects in division algebras, this property makes the Fano matroid a central mathematical object in matroid theory. $D = 11$ supergravity is on the other hand an important physical structure in M-theory. Therefore we have here a link between three apparently unrelated important objects in its own field: Fano matroid (matroid theory) $\leftrightarrow$octonions (algebra) $\leftrightarrow$ $D = 11$ supergravity (unify fundamental physics).

We would like to make some further observations between the Fano matroid and octonions. Consider the subsets $h_1 = \{v_1, v_2, v_3\}$, $h_2 = \{v_5, v_1, v_6\}$, $h_3 = \{v_6, v_2, v_4\}$, $h_4 = \{v_4, v_3, v_5\}$, $h_5 = \{v_4, v_7, v_1\}$, $h_6 = \{v_6, v_7, v_3\}$ and $h_7 = \{v_5, v_7, v_2\}$. If we identify v_i, where $i = 1, ...,7$, of these subsets with the columns of the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix}, \tag{4}$$

we notice that the matrix A provides a representation of the Fano matroid F_7. Now, suppose that the Fano matroid is extended to a structure in which the sets $\{v_i, v_j, v_k\}$ corresponding to the h_i are replaced by the completely antisymmetric object

$$(v_i, v_j, v_k) = -(v_i, v_k, v_j) = -(v_j, v_i, v_k). \tag{5}$$

For instance, we may replace $h_1 = \{v_1, v_2, v_3\}$ by $\hat{h}_1 = (v_1, v_2, v_3)$. Specifically, we define the extended Fano matroid $\hat{F}_7$ as the pair $(E, \hat{\mathcal{B}})$ in which $\hat{\mathcal{B}}$ is the set of three elements (v_i, v_j, v_k) except the completely antisymmetric quantities $\hat{h}_1 = (v_1, v_2, v_3)$, $\hat{h}_2 = (v_5, v_1, v_6)$, $\hat{h}_3 = (v_6, v_2, v_4)$, $\hat{h}_4 = (v_4, v_3, v_5)$, $\hat{h}_5 = (v_4, v_7, v_1)$, $\hat{h}_6 = (v_6, v_7, v_3)$ and $\hat{h}_7 = (v_5, v_7, v_2)$. The generalization from F_7 to $\hat{F}_7$ is very similar to the transition from graphs to digraphs (or oriented graphs) in which the edges of the original graph, let us say $\{a, b\}$, are changed to an ordering set, $(a, b) = -(b, a)$. The important point is that if there exists such a transition between F_7 and $\hat{F}_7$ then we discover that $\hat{F}_7$ almost determine completely the octonion algebra, essentially because (v_i, v_j, v_k) for the different h_i become closely related to the structure constants ψ_{mnr} associated to octonions. In fact, there is an extension of matroid theory which seems to be what these observations suggest for the Fano matroid, namely oriented matroids.[20]

In order to define oriented matroids it is necessary to define first what signed circuits are. A signed circuit X is a circuit with the partition (X^+, X^-) into two sets: X^+ the set of positive elements of X, and X^- its set of negative elements.

An oriented matroid $\mathcal{M}$ is a pair $(E, \mathcal{C})$, where E is a non-empty finite set, and $\mathcal{C}$ is a non-empty collection of subsets of E (called signed circuits) satisfying the following properties.

$(\mathcal{C}\ i)$ no circuit properly contains another circuit.

$(\mathcal{C}\ ii)$ if $\mathcal{C}_1$ and $\mathcal{C}_2$ are two distinct signed circuits, $\mathcal{C}_1 \neq$-$\mathcal{C}_2$, and $c \in \mathcal{C}_1^+ \cap \mathcal{C}_2^-$ then there exists a third circuit $\mathcal{C}_3 \in \mathcal{C}$ with $\mathcal{C}_3^+ \subseteq (\mathcal{C}_1^+ \cap \mathcal{C}_2^+)/\{c\}$ and $\mathcal{C}_3^- \subseteq (\mathcal{C}_1^- \cap \mathcal{C}_2^-)/\{c\}$.

It is not difficult to see that by forgetting signs, this definition of oriented matroids reduces to the definition of ordinary (non-oriented) matroids.

An alternative but equivalent way to define an oriented matroid is as follows: An oriented matroid $\mathcal{M}$ is a pair (E, χ), where E is a non-empty finite set and χ (called chirotope) is a mapping $E^r \to \{-1, 0, 1\}$, with r the rank on E, satisfying the following properties.

(χi) χ is not identically zero

$(\chi ii)\chi$ is alternating

(χiii) for all $x_1, x_2, ..., x_r$ and $y_1, y_2, ..., y_r$ such that

$$\chi(x_1, x_2, ..., x_r)\chi(y_1, y_2, ..., y_r) \neq 0$$

there exists an $i \in \{1, 2, 3, 4, 5, 6, 7\}$ such that

$$\chi(y_i, x_2, ..., x_r)\chi(y_1, y_2, ..., y_{i-1}, x_1, y_{i+1}, ..., y_r) = \chi(x_1, x_2, ..., x_r)\chi(y_1, y_2, ..., y_r).$$

For a vector configuration χ can be identified as

$$\chi(i_1, ..., i_r) \equiv sign \det(v_{i_1}, ..., v_{i_r}) \in \{-1, 0, 1\}$$

and (χiii) turns out to be related to the Grassmann-Plucker relation.

Returning to the case of the Fano matroid it is tempting to identify h_i with the chirotope $\chi(i_1, i_2, i_3) = sign \det(v_{i_1}, v_{i_2}, v_{i_3})$. But, in Ref. 45 it is noted that the Fano matroid is not orientable. Specifically, one can verify that the Fano matroid does not satisfy the property (χiii). Nevertheless, it is interesting to observe that one may write the formula[46]

$$\psi_{i_1 i_2 i_3} + \chi(i_1, i_2, i_3) = C_{i_1 i_2 i_3},$$

where $C_{i_1 i_2 i_3} \in \{-1, 1\}$ may be identified with the uniform matroid $U_{3,7}$ which is an excluded minor for $GF(5)$-representability, where $GF(q)$ denotes a finite field of order q. In this sense the Fano matroid and the octonions look as complementary concepts of the oriented uniform matroid $M(U_{3,7})$ structure.

It is worth remarking that the structure of $\hat{F}_7$ not necessarily corresponds, in a straightforward way, to oriented matroids for the following observations. An important problem in matroid theory is to see which matroids can be mapped into a set of vectors in a vector space over a given field. When such a map exists we are speaking about a coordinatization (or representation) of the matroid over the field. A matroid which has a coordinatization over $GF(2)$ is called binary. Furthermore, a matroid which has a coordinatization over every field is called regular. It turns out that regular matroids are of fundamental importance in matroid theory, among other things, because they play a similar role as planar graphs in graph theory.[17] It is known that a graph is planar if and only if it contains no subgraph homeomorphic to K_5 or $K_{3,3}$. The analogue of this theorem for matroids was proved by Tutte.[8] In fact, Tutte proved that a matroid is regular if and only if it is binary and has no minor isomorphic to the Fano matroid or the dual of this.

The important point is that an algebra, like the octonion algebra, is a vector space with an additional multiplicative operation. If it could be possible to identify this additional product with a kind of rule for the bases of $\mathcal{B}$ in a given matroid then we could speak about a representation of a matroid (with this additional product) in terms of an algebra instead of just the corresponding vector space. At present, we have not been able to find in the literature this kind of structure for matroids. But it seems to us that our identification of $\hat{F}_7$ with octonions may provide an example of matroids associated with an algebra rather than with just the corresponding vector space.

IV. MATROID THEORY AND CHERN-SIMONS THEORY

Here, we shall briefly review the main results of Ref. 10 about the connection between matroid theory and Chern-Simons theory. For this purpose let us introduce the Witten's partition function

$$Z(L, k) = \int DA \exp(S_{cs}) \prod_{r=1}^{n} W(L_r, \rho_r), \tag{6}$$

where S_{CS} is the Chern-Simons action

$$S_{CS} = \frac{k}{2\pi} \int_{M^3} Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A), \tag{7}$$

and $W(C_i, \rho_i)$ is the Wilson line

$$W(L_r, \rho_r) = Tr_{\rho_r} P \exp(\int_{L_r} A_i^a T_a dx^i). \tag{8}$$

Here, $A = A_i^a T_a dx^i$, with T_a the generators of the Lie algebra of G and the symbol P means the path-ordering along the knots L_r. If we choose $M^3 = S^3$, $G = SU(2)$ and $\rho_r = C^2$ for all the link components then the Witten's partition function (6) reproduces the Jones polynomial

$$Z(L, k) = V_L(t), \tag{9}$$

where

$$t = e^{\frac{2\pi i}{k}} \tag{10}$$

and $V_L(t)$ denotes the Jones polynomial satisfying the skein relation;

$$t^{-1} V_{L_+} - t V_{L_-} = (\sqrt{t} - \frac{1}{\sqrt{t}}) V_{L_0}, \tag{11}$$

where L_+, L_- and L_0 are the standard notation for overcrossing, undercrossing and zero crossing.

On the other hand, Thistlethwaite[11] showed that if L is an alternating link and $G(L)$ the corresponding planar graph, then the Jones polynomial $V_L(t)$ is equal to the Tutte polynomial $T_G(-t, -t^{-1})$ up to a sign and a factor power of t. Specifically, we have

$$V_L(t) = (-t^{\frac{3}{4}})^{w(L)} t^{\frac{-(\rho - n)}{4}} T_G(-t, -t^{-1}), \tag{12}$$

where $w(L)$ is the writhe and ρ and n are the rank and the nullity of G, respectively. Here, $V_L(t)$ is the Jones polynomial of alternating link L. The Tutte polynomial associated to each graph G is a polynomial $T_G(x, x^{-1})$ with the property that if G is composed solely of isthmus and loops then $T_G(x, x^{-1}) = x^I x^{-l}$, where I is the number of isthmuses and l is the number of loops. The polynomial T_G satisfies the skein relation

$$T_G = T_{G'} + T_{G''}, \tag{13}$$

where G' and G'' are obtained by delating and contracting respectively an edge that is neither a loop nor an isthmus of G.

On the other hand, a theorem due to Tutte allows to compute $T_G(-t, -t^{-1})$ from the maximal trees of G. In fact, Tutte proved that if $\mathcal{B}$ denotes the set of maximal trees in a graph G, $i(\mathcal{B})$ denotes the number of internally active edges in G, and $e(B)$ refers to the number of the externally active edges in G (with respect to a given maximal tree $B \in \mathcal{B}$) then the Tutte polynomial is given by the formula

$$T_G(-t, -t^{-1}) = \sum_{B \subseteq \mathcal{B}} x^{i(B)} x^{-e(B)}, \tag{14}$$

where the sum is over all elements of $\mathcal{B}$.

The important point is that the Tutte polynomial $T_G(-t, -t^{-1})$ computed according to (14) uses the concept of a graphic matroid $M(G)$ defined as the pair $(E, \mathcal{B})$, where E is the set of edges of G. This remarkable connection between the Tutte polynomial and a matroid allows in fact a relation between the partition function $Z(L, k)$ given in (6) and matroid theory. This is because according to (12) the Tutte polynomial $T_G(-t, -t^{-1})$ is related to the Jones polynomial $V_L(t)$ which in turn according to (9) is connected to the partition function $Z(L, k)$. Specifically, for $M^3 = S^3$, $G = SU(2)$, $\rho_r = C^2$ for all alternating link components of L, we have the relation

$$Z(L, k) = V_L(t) = (-t^{\frac{3}{4}})^{w(L)} t^{\frac{-(\rho - n)}{4}} T_G(-t, -t^{-1}). \tag{15}$$

Thus, the matroid $(E, \mathcal{B})$ used to compute $T_G(-t, -t^{-1})$ can be associated not only to $V_L(t)$, but also to $Z(L, k)$. Therefore, we have found a bridge which links the matroid formalism $(E, \mathcal{B})$ and the partition function $Z(L, k)$. This may allow to bring many concepts of matroid theory to fundamental physics and conversely, different results in fundamental physics may be used as an inspiration to further develop matroid theory. As an example of the former remark let us just mention how the duality concept in matroid theory can be used as a symmetry of $Z(L, k)$.

First of all, it is known that in matroid theory the concept of duality is of fundamental importance. For example, there is a remarkable theorem that assures that every matroid has a dual. So, the question arises about what are the implications of this theorem in Chern-Simons formalism. In order to address this question let us first make a change of notation $T_G(-t, -t^{-1}) \to T_{M(G)}(t)$ and $Z(L, k) \to Z_{M(G)}(k)$. The idea of this notation is to emphasize the connection between matroid theory, Tutte polynomial and Chern- Simons partition function. Consider the planar dual graph G^* of G. In matroid theory we have $M(G^*)$ $=M^*(G)$. Therefore, the duality property of the Tutte polynomial

$$T_G(-t,-t^{-1}) = T_{G^*}(-t^{-1},-t) \tag{16}$$

can be expressed as

$$T_{M(G)}(t) = T_{M^*(G)}(t^{-1}) \tag{17}$$

and consequently from (15) we discover that for the partition function $Z_{M(G)}(k)$ we should have the dual property

$$Z_{M(G)}(k) = Z_{M^*(G)}(-k). \tag{18}$$

As a second example let us first mention another theorem due to Withney:[5] If $M_1,..,M_p$ and $M_1^{'},..,M_p^{'}$ are the components (or blocks) of the matroids M and M' respectively, and if $M_i^{'}$ is the dual of M_i $(i=1,...,p)$. Then M' is dual of M. Conversely, let M and M' be dual matroids, and let $M_1,..,M_p$ be components of M. Let $M_1^{'},..,M_p^{'}$ be the corresponding submatroids of M'. Then $M_1^{'},..,M_p^{'}$ are the components of M', and $M_i^{'}$ is dual of M_i. Thus, according to (18) we find that

$$Z_{M_i(G_i)}(k) = Z_{M_i^{\prime}(G_i)}(-k) \tag{19}$$

if and only if

$$Z_{M(G)}(k) = Z_{M'(G)}(-k), \tag{20}$$

where G_i are the components or blocks of G.

Our discussion has been, so far, based on alternating links L. This kind of links is an important, but relatively small subclass of links. In fact, there is a one to one correspondence between links and signed graphs and a link is alternating if the signed graph representation has all edges with the same sign. Therefore, in order to generalize the procedure it turns necessary to have a generalized Thustlethwaite's [11] theorem for any signed graph not just for those of the same sign. Fortunately, Thustlethwaite himself,[11] and later Kauffman,[22] precisely generalized the original Thustlethwaite's theorem for planar unsigned graphs.

Theorem: Let G be a planar signed graph. Let $K(G)$ be the knot/link diagram corresponding to G. Then $\langle K(G)\rangle = T_G(A,B,x,y)$. The bracket polynomial for knots and links is a specialization of the generalized Tutte polynomial for signed graphs.

Here, A, B,and d are commuting variables associated to the link. A and B correspond to A-channel, B-channel respectively, while the parameter d is used as a factor of normalization in order to make $T_G(A,B,d)$ invariant under the Reidemeister moves II and III.

Furthermore, Kauffman showed that $T_G = T_G(A, B, x, y)$ has a spanning tree expansion of the form

$$T_G = \sum_{H \subseteq \mathcal{B}} \Lambda(H), \tag{21}$$

where $\Lambda(H)$ denotes the product of the contribution of the edges of G relative activities of the maximal trees H in G.

In principle, since up to a normalized factor, measuring the orientability of the link, $\langle K(G) \rangle \leftrightarrow CS$, in order to find a generalization of our procedure we need to relate T_G with matroid theory. A $T_G \longleftrightarrow$ *matroids* connection is given by (21) in the sense that the sum is over all maximal trees H in G. Notice, however, that the maximal trees H are associated to the underlying graph (without signs) of the signed graph and not to the signed graph itself.

It is known that matroids associated to signed graphs are called bias matroids.[21] It turns out that bias matroids are interesting by themselves, but unfortunately the subject about this kind of matroids has not been developed for our purpose and it appears that many of the interesting properties of ordinary matroids are lost. Nevertheless, the idea of writing T_G as a sum over bias matroids seems interesting and deserves further study.

It may help to mention in this direction that Crapo[23] proposed an alternative possibility to write T_G as a sum over all spanning subsets of E, rather than over maximal trees. This idea is motivated from the observation that in this case the rank function ρ becomes an important concept and can be used to generalize T_G to matroid theory. A generalization for signed graphs of the Crapo's polynomial has been proposed by Murasugi[24] and by Shwarzler and Welsh.[25] Let us briefly mention these two polynomials.

Murasugi introduced the following polynomial. Let $\Gamma(r, s)$ denote the set of all spanning subgraph S of G. Then $T_G(x, y, z)$ is defined by

$$T_G(x, y, z) = \sum_{k,\rho} \left\{ \sum_{S \in \Gamma(r,s)} x^{P(S)-N(S)} \right\} y^{k(S)-1} z^{|S|-\rho(S)}, \tag{22}$$

where $P(S)$ and $N(S)$ denote the number of positive and negative edges in S respectively. It is interesting to note that $\beta_0 = k = r + 1$ and $\beta_1 = n = \mid S \mid - \rho(S)$, where n is the nullity and β_i denotes the i Betti number of S as a $1-$complex. Although this polynomial uses the rank and the nullity concepts, the fact that the sum is over all spanning subgraphs Γ means that $T_G(x, y, z)$ is also applied only to the underlying unsigned graph G associated to the signed graph. Furthermore, the Murasugi polynomial does not have a direct relation with the Kauffmann polynomial.

On the other hand, Shwarzler and Welsh[25] proved that the Kauffmann polynomial associated to a link L can be expressed in terms of the associated signed graph $G(L)$ as follows

$$T_G(A,B) = A^{|E^-|-|E^+|}(-A^2 - A^{-2})^{\rho(G)} \sum_{S \subseteq E} A^{4(\rho(S)-|S^-|)} B^{\rho(G)+|S|-2\rho(S)}, \quad (23)$$

where $B = -A^4 - 1$ and for any subset $S \subseteq E(G)$, S^+ and S^- denote the positive and negative signed part respectively. It is important to remark that Shwarzler and Welsh showed that (23) is a specialization of a more general polynomial for signed matroids. In fact, Shwarzler and Welsh proposed an eight variables polynomial which contains as specialization not only the Kauffman bracket polynomial but also the Tutte polynomial of a matroid, the partition function of the anisotropic Ising model and the Kauffman-Murasugi polynomial of signed graphs (For further details see Ref. 25).

V. MATROID THEORY AND STRING THEORY

In the literature,[26–28] several attempts have been done to connect Chern-Simons formalism with string theory. One of the most interesting[26] comes from the idea that at some level the decoupling of ghost and matter does not hold. In this case, matter fields and ghosts become mixed and the standard string theory should be replaced by some kind of topological string theory.[29] It has been shown[27] that some topological string theories perturbatively coincide with Chern-Simons theory. So, in this sense Chern-Simons theory is equivalent to topological string theory. However, the problem arises when it is attempted to relate Chern-Simons theory with fundamental strings. In fact it has been shown[30] that in the pure Chern-Simons formalism there are not enough degrees of freedom to reproduce not only the induced gravity but the toroidal compactification of heterotic string.

These observations are, of course, important in order to find a matroid theory and string theory connection and eventually M-theory connection. In the previous section we explained a matroid theory-Chern-Simons theory relation via Tutte and Jones polynomials. It is clear then that what we should look for is some kind of generalization of fundamental strings which may provide the bridge between fundamental strings and topological strings.

The generalized fundamental strings could be the topological membrane[31] itself , but this is likely to be reduced to the topological strings rather than to fundamental strings. Another possibility is the membrane theory or any other p-brane[32], but it has been shown[33] that through double dimensional reduction these are reduced to fundamental strings rather than to topological strings. So, although there is the hope that at some level 3D topological field theory may lead to fundamental strings, the correct formulation of such a theory is at present unknown.

In this section, we propose an alternative generalization of fundamental strings which seems closer to our purpose than the already known alternative of topological membranes or p-branes.

The idea comes from the observation that in the Polyakov type action the world sheet metric and the target space-time metrics are decoupled. But it seems natural to think that at a more fundamental level when ghost and matter fields are mixed the decoupling between such two metrics is no longer true. Therefore the desired generalization of string theory must be based on a unified metric of the world sheet and target space-time metrics.

Let us clarify these observations. For this purpose, let us first consider the Polyakov action

$$S = \frac{1}{2}\int d^2\xi\sqrt{-g}g^{ab}(\xi)\partial_a x^\mu \partial_b x^\nu G_{\mu\nu}(x), \tag{24}$$

where $g_{ab}(\xi)$ and $G_{\mu\nu}(x)$, with $\mu,\nu = 1,...,D$, are the world sheet metric and the target space-time metric, respectively. We observe from (24) that the two metrics $g_{ab}(\xi)$ and $G_{\mu\nu}(x)$ play very different roles; $g_{ab}(\xi)$ determines the world sheet metric swept out by the string in its dynamical evolution, while $G_{\mu\nu}(x)$ determines the background metric where the string is moving. Therefore, classically $g_{ab}(\xi)$ and $G_{\mu\nu}(x)$ are unrelated. However, this is no longer true at the quantum level. For instance, it is well known that in a consistent quantum string theory $g_{ab}(\xi)$ plays an essential role to fix the size of the matrix $G_{\mu\nu}(x)$: $D = 26$ in the bosonic case. This kind of relation between $g_{ab}(\xi)$ and $G_{\mu\nu}(x)$ is, however, in a certain sense superficial because in the critical dimension 26 matter fields decouple from the corresponding ghost with associated central charge $c = -26$. The important observation is that, as it was mentioned in the introduction, at a deeper level the decoupling between matter fields and ghost must be no longer true and therefore one should expect that in such a case there must be a unified framework for the two metrics $g_{ab}(\xi)$ and $G_{\mu\nu}(x)$.

Consider the line element

$$ds^2 = G_{(\hat{\mu}\hat{\nu})}(x^{\hat{\alpha}})dx^{\hat{\mu}} \otimes dx^{\hat{\nu}}, \tag{25}$$

where the indices $\hat{\mu}$, $\hat{\nu}$ run from 1 to $2D$, the symbol $\otimes$ means tensor product and $G_{(\hat{\mu}\hat{\nu})} = G_{(\hat{\nu}\hat{\mu})}$. Suppose that (25) can be written as

$$ds^2 = G_{(\mu\nu)}(x^{\hat{\alpha}})dx^\mu \otimes dx^\nu + G_{(\mu\nu)}(x^{\hat{\alpha}})dy^\mu \otimes dy^\nu. \tag{26}$$

Here, we assume that $G_{(\mu A)} = G_{(A\mu)} = 0$, with $A = D+1, ..., 2D$ and we identify $x^A \to y^\mu$ and $G_{(AB)} \to G_{(\mu\nu)}$. It is not difficult to see that (26) can be rewritten as

$$ds^2 = G^{ab}_{(\mu\nu)}(x^{\hat{\alpha}})dx^\mu_a \otimes dx^\nu_b, \tag{27}$$

where $x^\mu_1 \equiv x^\mu$ and $x^\mu_2 = y^\mu$ and we assumed that $G^{11}_{(\mu\nu)} = G^{22}_{(\mu\nu)}$ and $G^{12}_{(\mu\nu)} = G^{21}_{(\mu\nu)} = 0$.

On the other hand, if we use the definition

$$z^\mu = x^\mu + iy^\mu, \tag{28}$$

we find that (26) can be written in the alternative way

$$ds^2 = G_{(\mu\nu)}(x^{\hat{\alpha}})dz^\mu \otimes d\bar{z}^\nu. \tag{29}$$

In this scenario, since $dx^{\hat{\mu}} \otimes dx^{\hat{\nu}}$ is a second-rank symmetric tensor the same results follow if we consider the most general hermitian metric

$$G_{\hat{\mu}\hat{\nu}}(x^{\hat{\alpha}}) = G_{(\hat{\mu}\hat{\nu})}(x^{\hat{\alpha}}) + iG_{[\hat{\mu}\hat{\nu}]}(x^{\hat{\alpha}}). \tag{30}$$

Here, $G_{[\hat{\mu}\hat{\nu}]}$ denotes an antisymmetric tensor metric. Of course, $G_{\hat{\mu}\hat{\nu}}$ satisfies the hermitian condition $G_{\hat{\mu}\hat{\nu}} = G^\dagger_{\hat{\mu}\hat{\nu}}$.

Now, consider the metric $G_{\hat{\mu}\hat{\nu}}(x^{\hat{\alpha}})$ given in (30), in connection with the exterior product

$$\Omega = \frac{1}{2}G_{\hat{\mu}\hat{\nu}}(x^{\hat{\alpha}})dx^{\hat{\mu}} \wedge dx^{\hat{\nu}}. \tag{31}$$

Using the exterior product property $dx^{\hat{\mu}} \wedge dx^{\hat{\nu}} = -dx^{\hat{\nu}} \wedge dx^{\hat{\mu}}$, we see that (31) leads to

$$\Omega = \frac{i}{2}G_{[\hat{\mu}\hat{\nu}]}(x^{\hat{\alpha}})dx^{\hat{\mu}} \wedge dx^{\hat{\nu}}. \tag{32}$$

Assuming that $G_{[\mu A]} = G_{[A\mu]} = 0$ we find that (32) becomes

$$\Omega = \frac{i}{2}G_{[\mu\nu]}(x^{\hat{\alpha}})dx^\mu \wedge dx^\nu + \frac{i}{2}G_{[AB]}(x^{\hat{\alpha}})dy^A \wedge dy^B. \tag{33}$$

We again make the identification $x^A \to y^\mu$ and $G_{[AB]} \to G_{[\mu\nu]}$. The formula (33) can be rewritten as

$$\Omega = \frac{i}{2}G_{[\mu\nu]}(x^{\hat{\alpha}})(dx^\mu \wedge dx^\nu + dy^\mu \wedge dy^\nu). \tag{34}$$

Introducing $x^\mu_1 \equiv x^\mu$ and $x^\mu_2 = y^\mu$ and assuming that $G^{12}_{[\mu\nu]} = G^{21}_{[\mu\nu]} = 0$ and $G^{11}_{[\mu\nu]} = G^{22}_{[\mu\nu]} \neq 0$ we find that (34) leads to

$$\Omega = \frac{i}{2} G^{ab}_{[\mu\nu]}(x^{\hat{\alpha}}) dx^\mu_a \wedge dx^\nu_b. \tag{35}$$

On the other hand, using the definition (28) for z^μ we find that (34) can also be written as

$$\Omega = \frac{i}{2} G_{[\mu\nu]}(x^{\hat{\alpha}}) dz^\mu \wedge d\bar{z}^\nu. \tag{36}$$

Summarizing we have shown that if $G^{11}_{(\mu\nu)} = G^{22}_{(\mu\nu)} \neq 0$ and $G^{12}_{(\mu\nu)} = G^{21}_{(\mu\nu)} = 0$, and $G^{11}_{[\mu\nu]} = G^{22}_{[\mu\nu]} \neq 0$ and $G^{12}_{[\mu\nu]} = G^{21}_{[\mu\nu]} = 0$ then

$$ds^2 = G^{ab}_{\mu\nu}(x^\alpha_c) dx^\mu_a \otimes dx^\nu_b \tag{37}$$

is equivalent to

$$ds^2 = G_{\mu\nu}(z^\alpha, \bar{z}^\beta) dz^\mu \otimes d\bar{z}^\nu, \tag{38}$$

and that

$$\Omega = \frac{i}{2} G^{ab}_{\mu\nu}(x^\alpha_c) dx^\mu_a \wedge dx^\nu_b \tag{39}$$

is equivalent to

$$\Omega = \frac{i}{2} G_{\mu\nu}(z^\alpha, \bar{z}^\beta) dz^\mu \wedge d\bar{z}^\nu. \tag{40}$$

We recognize in (38) and (40) the formulae used to define the Kahler metric which in addition satisfies the condition $d\Omega = 0$. Therefore we have shown that under certain anzats the metric $G^{ab}_{\mu\nu}(x^\alpha_c)$ can be identified with the Kahler metric. This shows that it makes mathematical sense to consider a metric of the form $G^{ab}_{\mu\nu}(x^\alpha_c)$.

Our goal is now to use the metric $G^{ab}_{\mu\nu}(x^\alpha_c)$ in connection with string theory. We find that there are at least two different ways to achieve this. In fact, in the first case we have the action

$$S_1 = \frac{1}{2} \int d^2\xi \sqrt{-g} g^{ab}(\xi) \partial_a x^\mu_c \partial_b x^\nu_d G^{cd}_{\mu\nu}(x), \tag{41}$$

while in the second case we have[34]

$$S_2 = \frac{1}{2} \int d^2\xi G^{ab}_{\mu\nu}(\xi, x) \partial_a x^\mu \partial_b x^\nu. \tag{42}$$

For our purpose to relate matroid theory with string theory both possibilities look attractive. The action S_1 may be useful to understand T-duality or S-duality because of its property of being symmetric under the interchange of coordinates $x \leftrightarrow y$. However, S_2 is closer to our idea of unified worldsheet-target spacetime metrics when matter and ghost mix.

In fact, in the particular case

$$G_{\mu\nu}^{ab} = \sqrt{g} g^{ab} G_{\mu\nu}, \tag{43}$$

one sees that S_2 is reduced to the Polyakov action (24). This shows that ordinary bosonic string theory is contained in a theory associated to (42). Another particular case for $G_{\mu\nu}^{ab}$ is

$$G_{\mu\nu}^{ab} = \sqrt{g} g^{ab} G_{\mu\nu} + i\varepsilon^{ab} B_{\mu\nu}, \tag{44}$$

where $B_{\mu\nu} = -B_{\nu\mu}$ is a two form and ε^{ab} is the completely antisymmetric tensor with $\varepsilon^{12} = 1$. The choice (44) leads to a generalized bosonic string theory, the so called nonlinear sigmal model in two dimensions, in which the string propagates in a background determined not only by gravity but by the antisymmetric two form gauge field B with associated field strength $dH = B$. Finally, the third example is provided precisely for what we have already discussed when $G_{\mu\nu}^{ab}$ is identified with a Kahler metric.

Here, we are not particularly interested in developing the full theory implied by S_2, but to point out how S_2 can be related to matroid theory via Chern-Simons theory. For this purpose it seems to be convenient to start by recalling briefly how the Kahler structure is related to Chern-Simons theory.

There are a number of restrictions which a background field must satisfy in order to have a consistent string theory. Perhaps one of the most important is the anomaly cancellation fixed by the constraint[4]

$$dH = trR \wedge R - trF \wedge F, \tag{45}$$

where R is the curvature associated to $G_{\mu\nu}$. The formula (45) is an important restriction for the possible compactifications. One of the most attractive solutions of (45) is when the ten dimensional space-time vacuum state is given by $\mathrm{T}^4 \times K$, where T^4 is a maximally symmetric four dimensional spacetime and K is a six dimensional Kahler manifold.

Now, it is known that a Kahler metric determines a Kahler manifold, so Kahler metric is related to string theory through (45). In turn (45) contains the second Chern class $\int trF \wedge F$ which reduces to the Chern-Simons form. Therefore, Kahler metric is closely related to Chern-Simons formalism in string theory. Consequently, the action S_2 with the choice of $G_{\mu\nu}^{ab}$ as a Kahler metric establishes a connection between matroid theory and string theory via Chern-Simons formalism.

VI. COMMENTS

In the present work we have shown that the Kahler metric may provide an important bridge to connect matroid theory and string theory. Specifically using a generalized string theory we established the following identifications: $string - theory \leftrightarrow Kahler - structure$; $Chern - Simons - theory \leftrightarrow Kahler - structure$ and $matroid - theory \leftrightarrow Chern - Simons - theory$. Moreover, it is natural to expect that this kind of relations suggest a direct link between the generalized string theory described by (42), which by convenience we shall call Kahler-string action, and the pure Chern-Simons theory. But at first sight it is improbable that this more direct relation exists. The reason is that pure Chern-Simons theory does not provide us enough degrees of freedom to describe the dynamics even for the heterotic string theory. Therefore, these observations suggest that there must exist a generalized Chern-Simons action which is reduced to the Kahler-string action (42). Of course, the Kahler structure must play an important role in this generalized Chern-Simons theory. Happily, such a theory along this idea has already been proposed. In fact, some years ago Nair and Schiff[35] proposed what they called Kahler-Chern-Simons theory. The action proposed by Nair and Schiff has the form

$$S_{KCS} = \frac{k}{2\pi} \int_{M^3 \times R} Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)\Omega, + Tr(F\Phi + \Phi F), \qquad (46)$$

where F is the field strength in $\mathrm{M}^4 \times R$, Φ is a Lie algebra value $(2,0)$ form and $M^4$is a Kahler manifold of real dimension four. It turns out that S_{KCS} provides an action description of antiself-dual gauge fields (instantons) on a four dimensional Kahler manifolds. Here, we are interested in seeing if under quantization S_{KCS} is reduced to the Kahler-string action (42). It can be shown, however, that up to WSW term, S_{KCS} leads to the action

$$S_3 = \frac{1}{2} \int d^{2+2}\xi \sqrt{-g} g^{ab}_{ij}(\xi) \frac{\partial x^\mu}{\partial \xi^a_i} \frac{\partial x^\nu}{\partial \xi^a_i} G_{\mu\nu}(x), \qquad (47)$$

rather than (44). Here, g^{ab}_{ij} can be identified with a Kahler metric on M^4, while $G_{\mu\nu}(x)$ is given by $G_{\mu\nu}(x) = \partial_\mu U \partial_\nu U^{-1}$, where U is a locally defined G-valued function related to the gauge field A by $A = U^{-1}dU$. Therefore, we conjecture that there must be a slightly different action from (46), with the property of reducing to (42).

Nevertheless, the action S_{KCS} may be of special interest to relate matroid theory not only to string theory but to M-theory itself. In fact, it is known that S_{KCS} leads to a theory in terms of fields in a target space of $N = 2$ strings.[36] In turn, $N = 2$ strings is one of the main proposals of M-theory.[37] Moreover, it has been pointed out in Ref. 38 that a number of similarities exist between the other main proposal of M-theory, namely Matrix theory.[39] In principle, for S_{KCS} one may repeat the formalism of section 3. In fact, consider the partition function

$$Z_{KCS}(L, k) = \int DA \exp(S_{KCS}) \prod_{r=1}^{n} W(L_r, \rho_r), \tag{48}$$

where $W(L_i, \rho_i)$ are the Wilson lines defined in (8). It is tempting to speculate that $Z_{KCS}(L, k)$ must be related to some knot invariant in a similar way that $Z_{CS}(L, k)$ is related to the Jones polynomials. From this idea, since knots are in one to one correspondence with signed graphs one should expect to find the desired relation between matroid theory and $Z_{KCS}(L, k)$, which may lead eventually to a matroid theory and M-theory connection.

Another possible route to connect matroid theory with M-theory comes from the work of Gopakumar and Vafa[40−41] who have proved that topological strings are closely related to M- theory. Since Chern-Simons formalism is linked to topological strings,[27] it seems that we are closer to make the matroid theory the underlying mathematical structure of M-theory (see Ref. 42 for interesting observations about M- theory).

Finally, besides the possible connection between matroid theory and string theory the present formalism may be of special interest for quantum gravity based on the Ashtekar formalism[43] (see also Ref. 44 and references therein). The most interesting solutions of the Ashtekar constraints correspond to the Witten's partition function. Consequently, the duality symmetries (18) may also play an important role in such solutions. It is known that the Vasiliev invariants become an important tool in the loop solutions of quantum canonical gravity in the Ashtekar formalism. Since the Vasiliev invariants can be understood as a generalization of the Jones polynomials, it may be interesting for further research to investigate whether matroid theory can be connected to such invariants.

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