

# SPECTRAL GEOMETRY OF SPACETIME<sup>1</sup>

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ABSTRACT. Spacetime, understood as a globally hyperbolic manifold, may be characterized by spectral data using a 3+1 splitting into space and time, a description of space by spectral triples and by employing causal relationships, as proposed earlier. Here, it is proposed to use the Hadamard condition of quantum field theory as a smoothness principle.

## 1. INTRODUCTION

Classical spacetime is to appear from a quantum theory. Though it is not clear at present how this is to come about, there is some possibility that the spacetime manifold will be offered by nature not in the form of a definition used by some textbook but rather in the form of spectral data which appear naturally in quantum theory. Here, such a spectral description of spacetime is discussed.

In A. Connes' noncommutative geometry [1–3], a spin manifold is described using a spectral triple. However, due to the indefinite metric of spacetime, this scheme is not directly applicable. The problem can be circumvented by a Hamiltonian description in which spacetime is foliated by spacelike hypersurfaces, separately describable by spectral triples and related to each other in a certain way [4]. Such spectral data can be considerably compressed if causal relationships are exploited [5]. This is reviewed in Section 2.

In Section 3, the idea that the spectral data originate from a quantum theory is taken seriously. The Hadamard condition of quantum field theory in curved spacetime [6–17] is reviewed and proposed as a possible principle to ensure smoothness.

The conclusion summarizes the presented view and contains some speculations on how the spectral data as a whole may be generated.

## 2. SPACETIME SPECTRAL DATA

A spin manifold with positive definite metric can be described [1–3] by a certain spectral triple  $(A, \mathcal{H}, D, J, \gamma)$ . Here,  $A$  is a commutative pre- $C^*$ -algebra represented (faithfully) on a Hilbert space  $\mathcal{H}$ ,  $D$  is an unbounded selfadjoint operator on  $\mathcal{H}$ ,  $J$  is an antiunitary conjugation and  $\gamma$  is a grading operator on  $\mathcal{H}$ . These structures satisfy a well known set of conditions given in [3]

*Note 1.* The above spectral description is chosen in such a way so as to make sense in rather general situations, also in the case when the algebra  $A$  is not commutative. In this work which is limited to classical spacetime with a very simple particle content this generalization will not be used directly but only as an indication that the used framework is mathematically a natural one.

In the special setting considered here the algebra  $A$  is to be the algebra of smooth functions  $C^\infty(M)$  on the spin manifold  $M$ ,  $\mathcal{H}$  is the Hilbert space  $L^2(M, S)$  of

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sections of the spin bundle,  $D$  is the Dirac operator  $\mathcal{D}$ ,  $J$  is the charge conjugation and  $\gamma$  is the volume element.

From the spectral triple, it is possible to construct the full geometry of the spin manifold with positive definite metric. In particular, the distance between points  $p$  and  $q$  on the manifold can be given as the maximal value difference in points  $p, q$  for functions with derivative at most 1:

$$(1) \quad d(p, q) = \sup \{ |f(p) - f(q)| ; \| [\mathcal{D}, f] \| \leq 1 \}$$

The assumption of the definiteness of the metric in the above description is essential. A simple indication of that is visible in the breakdown of Equation (1): The distance between two arbitrary points on a manifold with Lorentzian signature cannot be given by a general length-extremum principle.

While the whole geometry of spacetime with Lorentzian signature cannot be dealt with by simply taking over the results from Euclidean signature, it is still possible to describe spacelike hypersurfaces by spectral triples. The whole spacetime can then be foliated by such spacelike hypersurfaces  $\Sigma_t$  and by supplying a time evolution, a lapse function  $N$  and shift vector field  $N^i$ , as in the ADM (Arnowitt-Deser-Misner) formalism [18], the whole manifold can be described in spectral geometry [4].

In order to avoid inessential technical problems, it will be assumed that the spacetime manifold  $M$  is globally hyperbolic, i.e., topologically of the form  $\Sigma \times \mathbb{R}$  with  $\Sigma$  a compact (spin structure allowing) space manifold and with the imbeddings  $\iota_t : \Sigma \times \{t\} \rightarrow \Sigma_t \subset M$  providing a foliation by spacelike Cauchy hypersurfaces.

Thus, at each time, there is a spectral triple  $(A_t, \mathcal{H}_t, D_t, J_t, \gamma_t)$  describing  $\Sigma_t$  which together with the lapse  $N$ , the shift  $N^i$  and algebra isomorphisms  $\iota_{t_2} \iota_{t_1}^{-1} : A_{t_1} \rightarrow A_{t_2}$  identifying hypersurfaces at different times.

The Hilbert spaces  $\mathcal{H}_t$  can be identified if they are understood as initial data of a spinor field  $\psi$  with mass  $m$  obeying the Dirac equation

$$(2) \quad (\mathcal{D} - m)\psi = 0$$

and equipped with the invariant inner product

$$(3) \quad (\psi_1, \psi_2) = \int_{\Sigma_t} \bar{\psi}_1 \gamma^\mu \psi_2 d\Sigma_\mu$$

with  $d\Sigma_\mu$  being the volume element on the hypersurface and with  $\gamma^\mu$  being the Clifford generators. Thus, spinor fields given on different spacelike hypersurfaces are identified, if they are the restrictions of the same solution of the Dirac equation and the given inner product does not depend on the hypersurface on which it is calculated.

This Hilbert space has then the meaning of the classical phase space of the spinor field  $\psi$ . On this Hilbert space, the mutually isomorphic algebras  $A_t$  of smooth functions at different times  $t$  are represented differently, allowing to compare spinor fields localized at different times and thus to examine causal contacts between hypersurfaces at different times. It is the examination of the causal contacts between hypersurfaces that allows drastically to reduce the required spectral data [5].

The knowledge of causal contacts allows in particular to examine the intersection of the light cone originating at a point  $x \in \Sigma_{t_1}$  with a hypersurface  $\Sigma_{t_2}$  at a later time. The light cone intersects the hypersurface  $\Sigma_{t_2}$  in a sphere. To first order in time difference, the size of the sphere gives information on the distance between the hypersurfaces  $\Sigma_{t_1}$ ,  $\Sigma_{t_2}$  and thus provides the lapse function. The centre of the sphere lies in the normal direction from the tip of the light cone and allows thus to identify the shift. Since the sphere has to first order in time difference a constant radius, independent of direction, also the metric on the hypersurface can be

recovered from the causal relationships, up to a conformal factor. The conformal factor can, however, be obtained from the knowledge of the volume element  $\gamma_t$ . At that point one can dispose of the metric information in the Dirac operators  $D_t$  and keep only its sign  $F_t = D_t |D_t|^{-1}$  which contains information on spatial smoothness.

The spacetime interval between two events  $p_t, q_{t+dt}$  occurring at times  $t, t + dt$  can be given as

$$(4) \quad ds^2 = dl^2(p_t, q_{t+dt}) - dr^2(p_t, dt)$$

where

$$(5) \quad dl^2(p_t, q_{t+dt}) = \left( \frac{3}{4\pi} \times \frac{\text{volume of ball with centre above } p_t \text{ and } q_{t+dt} \text{ on its boundary}}{\text{volume of ball with centre above } p_t \text{ and } q_{t+dt} \text{ on its boundary}} \right)^{\frac{2}{3}}$$

$$(6) \quad dr^2(p_t, dt) = \left( \frac{3}{4\pi} \times \frac{\text{volume of ball given at time } t + dt \text{ by light cone originating in } p_t}{\text{volume of ball given at time } t + dt \text{ by light cone originating in } p_t} \right)^{\frac{2}{3}}$$

Thus a spectral description of a spacetime manifold may be given by a family of algebras  $A_t$  represented on a Hilbert space  $\mathcal{H}$  together with families of sign operators  $F_t$ , volume elements  $\gamma_t$  and antiunitary involutions  $J_t$ , as given above<sup>1</sup>.

However, at this point, it is not possible to spell out a full set of requirements on the spectral data sufficient to reconstruct a spacetime manifold *without* previous assurance that the data really give a manifold. In particular, there is no insurance that the foliated spacetime to be obtained from the data will be smooth in the time direction. It is the content of the next section to propose such a global smoothness principle.

### 3. THE HADAMARD CONDITION

The spectral data in the previous section were lacking a smoothness principle. However, if the idea is taken seriously that the phase space of the spinor field  $\psi$  which plays the role of the representation space  $\mathcal{H}$  of the spectral data corresponds to a physical field then such a principle is naturally present.

In that case, the classical phase space  $\mathcal{H}$  generates via the canonical anticommutation relations a quantum field algebra and the field is then described by its state on the algebra. However, not all states on the field algebra are believed to be physical, since only for very special ones one can define a sensible stress-energy tensor, as required by R.Wald [20] and only for those states the semi-classical Einstein equations are meaningful. A sufficient condition for states to be physical in this sense is the Hadamard condition [6–17] which, after a reformulation due to Radzikowski [12, 13], is understood to be a positive energy condition on the two-point function of the state. The technical definition based on micro-local analysis can be rephrased in the following way:

**Definition 1.** The singular structure of the two-point function (or of any other distribution) is characterized by its wave front set. It is given by all directions in cotangent space along which the symbol of any pseudodifferential operator, acting on the two-point function and producing a smooth result, has to vanish.

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<sup>1</sup>The here given exposition is based on things like measuring the volume of spheres. This is in principle possible but is to be understood to be rather of conceptual value than of practical use since it is not easy to do calculations in this way. Explicit expressions for the spacetime geometry, i.e., for the spatial metric, the lapse and the shift can be obtained from considering commutators of functions in  $A_t$  at different times among themselves and with the Hamiltonian generating the family  $A_t$ , as given in [19]. These results can then be used for practical calculations.

A two-point function is said to satisfy the Hadamard condition if its wave front set contains only positive frequencies propagating forward in time and negative frequencies backward in time.

States satisfying the Hadamard condition, so called Hadamard states have the remarkable property that if they fulfill the Hadamard condition on one Cauchy hypersurface they do so on the full globally hyperbolic spacetime as a consequence of its smoothness. It is therefore possible to turn this argument around and to state the following smoothness principle:

**Smoothness Principle.** The time evolution given by the spectral data  $(A_t, \mathcal{H}, F_t, J_t, \gamma_t)$  has to preserve Hadamard states.

*Remark 1.* The smoothness principle is a necessary condition. It is at this point not clear to what degree it provides full spacetime smoothness (see also 1) of the Conclusion), though, in examples, it can be shown to rule out non-smoothness in the time direction.

*Remark 2.* The two-point function is fully determined by operators on the classical phase space. This can be done in a particularly simple way if the two-point function stems from a quasi-free pure state  $\omega$  in which case one only needs a complex structure  $J_\omega$ . It is thus sufficient to work with the classical phase space of the spinor field to verify the Hadamard condition, a shadow of the full quantum theory. Since the Heisenberg picture is used, the Hadamard state  $\omega$  (and thus  $J_\omega$ ) is chosen once for all times and is in particular independent of the foliation of the considered spacetime. In this sense, the smoothness principle is a global one.

*Remark 3.* There is a wider class of states, the adiabatic states [14–17, 21–23] which satisfy a weaker, generalized Hadamard condition. They reside in the same folio as the Hadamard states [24]<sup>2</sup> and are easier to construct and to work with. Unfortunately, it is not known yet whether they allow for a regularized stress-energy tensor but they are a potential alternative to the Hadamard states, providing thus an alternative to the smoothness principle.

#### 4. CONCLUSION

Spacetime, a globally hyperbolic manifold can be described by spectral data  $(A_t, \mathcal{H}, F_t, \gamma_t, J_t)$  of Section 2 and supplied with the smoothness principle of Section 3.

However, the spectral data are not to be understood to be in their final form. A number of open problems and speculations remain and are to be sorted out in further investigations:

- (1) The smoothness principle may render the data  $F_t$  describing spatial smoothness superfluous but this is not clear at the moment.
- (2) The spectral data contain not only information on spacetime but also pre-select a foliation of spacetime. To avoid this, one may attempt to pack the algebras  $A_t$  into one greater algebra.
- (3) The algebras  $A_t$  are extremely important in cutting the classical spacetime out of the operator algebra on  $\mathcal{H}$  but were not given a physical interpretation in the same way as was done for the spinor field. They are the structures that provide the classical meaning of position. One possibility is to interpret them as the shadow of another quantum field interacting with the spinors field. In particular, if some particles of that other quantum field would have zero rest mass, it would be understandable that they would be a primary decoherence inducing environment [26, 27], producing

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<sup>2</sup>see also [25]

thus a pointer basis, the eigenbasis of  $A_t$ . A minimal model would take as the field in question the  $U(1)$  gauge field inherently associated with a spinor field. The physical content of this theory would then be quantum electrodynamics on a curved background with the electrons localized by a bath of photons.

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