

# QCD/String holographic mapping and high energy scattering amplitudes

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We find a one to one mapping between low energy string dilaton states in AdS bulk and high energy glueball states on the corresponding boundary. This holographic mapping leads to a relation between bulk and boundary scattering amplitudes. From this relation and the dilaton action we find the appropriate momentum scaling for high energy QCD amplitudes at fixed angles.

The idea that strong interactions should have a description in terms of strings is not new [1] (see [2] for a discussion). However, understanding the non trivial relation between string theory and QCD is still a defying task. A remarkable step in this direction was given by Maldacena [3], conjecturing the equivalence of large  $N$  limit of  $SU(N)$  superconformal field theories and supergravity/string theory in a higher dimensional anti-de Sitter spacetime (AdS/CFT correspondence) [4–6]. In this correspondence, bulk fields act as classical sources for boundary correlation functions[4–8]. One of the striking features of this correspondence, pointed out soon by Witten [5], is that it is a realization of the holographic principle: “The degrees of freedom of a quantum theory with gravity can be mapped on the corresponding boundary”[9–13]. From general arguments of the AdS/CFT correspondence one finds that there is an isomorphism between the Hilbert spaces of bulk and boundary theories [14–17].

A long standing puzzle for the string description of strong interactions, is the high energy scattering at fixed angles. In string theory such a process is soft (the amplitudes decay exponentially with energy) while both experimental data and QCD theoretical predictions[18, 19] indicate a hard behavior (amplitudes decaying with a power of energy). A solution for this puzzle was proposed recently by Polchinski and Strassler[20] based on the AdS/CFT scenario. They introduced an energy scale (associated with the lightest glueball mass) by cutting the AdS space and taking a slice analogous to that of the Randall-Sundrum model [21, 22]. This AdS slice is considered as an approximation for the space dual to a confining gauge theory which can be associated with QCD. Then they found the correct high energy QCD amplitude for glueball scattering at fixed angles starting from the string amplitude in 10 dimensions and integrating over the warped AdS extra dimension, weighted by the dilaton wave function.

A general relation between string and QCD states would certainly be non trivial. However, for the particular process of glueball high energy scattering at fixed angles the result of Polchinski and Strassler seems to suggest that the states of strings and QCD could be related

explicitly. At low energies string theory can be approximated by a supergravity action involving the dilaton field. In this regime one can look for a mapping between states of the dilaton and of boundary fields.

In this letter we consider the same kind of AdS slice of Polchinski and Strassler [20] and propose a one to one mapping between bulk low energy string dilaton states and boundary composite operators associated with QCD glueballs. This mapping is inspired in a simpler version found in ref.[23] between Fock spaces of scalar field theories in AdS bulk and boundary. Here we consider glueball operators and map their scattering amplitude into the bulk dilaton amplitude. Note that bulk and boundary operators should be on-shell in order to represent in and out asymptotic states of scattering processes. Then using the low energy string action and restricting to fixed angle scattering (in both theories) we find that the glueball amplitudes scale as

$$\mathcal{M} \sim s^{(4-\Delta)/2} \quad (1)$$

where  $\sqrt{s}$  is the energy and  $\Delta$  is the sum of the scaling dimensions of glueball operators. This is the expected behavior for high energy QCD[18, 19] and is in agreement with Polchinski and Strassler [20].

Let us consider a type IIB string theory. At energies much lower than the string scale  $1/\sqrt{\alpha'}$  this theory can be described by the supergravity action[4, 24]

$$S = \frac{1}{2\kappa^2} \int d^{10}x \sqrt{G} e^{-2\Phi} \left[ \mathcal{R} + G^{MN} \partial_M \Phi \partial_N \Phi + \dots \right] \quad (2)$$

where  $G^{MN}$  is the ten dimensional metric,  $\mathcal{R}$  is the Ricci scalar curvature,  $\Phi$  is the dilaton field and  $\kappa \sim g(\alpha')^2$ . We identify this ten dimensional space as  $AdS_5 \times S^5$  with radius  $R$  and measure

$$ds^2 = \frac{R^2}{z^2} \left( dz^2 + (d\vec{x})^2 - dt^2 \right) + R^2 d\Omega_5^2, \quad (3)$$

where the coordinates  $\Omega_5$  describe a five dimensional sphere  $S^5$ .

Following Polchinski and Strassler [20] we cut the AdS space in order to obtain a boundary theory with an energy scale (mass gap). We consider an AdS slice with “size”  $z_{max}$  representing an infrared cut off associated with the mass of the lightest glueball. We also consider the dilaton to be in the  $s$  – wave state, so we will not take into account variations with respect to  $S^5$  coordinates. Thus, the action becomes

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$$S = \frac{\pi^3 R^8}{4\kappa^2} \int d^4x \int_0^{z_{max}} \frac{dz}{z^3} e^{-2\Phi} \times \left[ \mathcal{R} + (\partial_z \Phi)^2 + \eta^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi + \dots \right] \quad (4)$$

where  $\eta^{\mu\nu}$  is the four dimensional Minkowski metric.

The solution for the free dilaton field, that will be used to build up the bulk asymptotic states in the scattering process can be cast into the form [25]

$$\Phi(z, \vec{x}, t) = \sum_{p=1}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{z^2 J_2(u_p z)}{z_{max} w_p(\vec{k}) J_3(u_p z_{max})} \times \{ \mathbf{a}_p(\vec{k}) e^{-i w_p(\vec{k}) t + i \vec{k} \cdot \vec{x}} + h.c. \}, \quad (5)$$

with  $0 \leq z \leq z_{max}$  and  $w_p(\vec{k}) = \sqrt{u_p^2 + \vec{k}^2}$ ,  $h.c.$  means hermitean conjugate and  $u_p$  are defined by

$$u_p z_{max} = \chi_{2,p} \quad (6)$$

such that the Bessel function satisfies  $J_2(\chi_{2,p}) = 0$ . If we had considered bulk fields with mass  $M$  the order of this Bessel function would be  $\nu = \sqrt{4 + M^2 R^2}$ . The operators  $\mathbf{a}_p$ ,  $\mathbf{a}_p^\dagger$  satisfy the commutation relations

$$[\mathbf{a}_p(\vec{k}), \mathbf{a}_{p'}^\dagger(\vec{k}')] = 2(2\pi)^3 w_p(\vec{k}) \delta_{pp'} \delta^3(\vec{k} - \vec{k}'). \quad (7)$$

In the AdS/CFT correspondence bulk scalars couple to composite conformal boundary operators of dimension  $d = 2 + \sqrt{4 + M^2 R^2}$ .

On the boundary ( $z = 0$ ) of the AdS slice we consider massive composite operators  $\Theta(\vec{x}, t)$  that will represent glueballs. The corresponding creation-annihilation operators are assumed to satisfy the algebra (for asymptotic states)

$$[\mathbf{b}(\vec{K}), \mathbf{b}^\dagger(\vec{K}')] = 2(2\pi)^3 w(\vec{K}) \delta^3(\vec{K} - \vec{K}'), \quad (8)$$

where  $w(\vec{K}) = \sqrt{\vec{K}^2 + \mu^2}$ . Note that this commutation relation holds only for asymptotic states. In general, the creation-annihilation operators associated with a composite field like  $\Theta(\vec{x}, t)$  cannot satisfy such a free field relation.

A mapping between theories that live in different dimensions is not trivial. In the standard AdS/CFT framework there is a correspondence between (on shell) string theory in the AdS bulk and (off shell) conformal field theory on the boundary. Here we are considering a different and simpler situation of string theory in a low energy regime where it can be approximated by a supergravity action. In particular, we are interested in the dilaton field which is the bulk dual to the boundary glueball operator. So we are looking for a mapping between two field theories defined in different dimensions. Further, we want to relate asymptotic states of bulk and boundary theories so that we will consider (on shell) quantized

free fields. If both field theories had continuous momenta it would be impossible to find a one to one mapping between quantum states. However, as we consider just a slice of AdS, the spectrum of the momentum associated with the axial direction is discrete. Actually even when one considers not just a slice but the whole AdS space one must include a boundary at infinity, compactifying the space and finding again a discrete spectrum in the axial direction, as discussed in [25]. Then the continuous part of bulk and boundary momenta  $\vec{k}$  and  $\vec{K}$  have the same dimensionality.

This discretization makes it possible to establish a one to one mapping between bulk and boundary momenta ( $\vec{k}, u_p$ ) and  $\vec{K}$ . We assume a trivial mapping between their angular parts and then look for a relation between their moduli  $K \equiv |\vec{K}|$ ,  $k \equiv |\vec{k}|$ . The idea is that one can relate the points of an enumerable set of  $p$  lines with those of a single line by dividing the latter into intervals and mapping each line  $p$  into a corresponding interval. So we introduce a sequence of energy scales  $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3 \dots$  in order to map each interval of the boundary momentum modulus  $\mathcal{E}_{p-1} < K \leq \mathcal{E}_p$  with  $p = 1, 2, \dots$  into the entire range of the transverse bulk momentum modulus  $k$ , corresponding to some fixed axial momentum  $u_p$ . This mapping between bulk and boundary momenta allow us to map the corresponding creation-annihilation operators (7,8). For the first energy interval, corresponding to  $u_1$ , defined as  $0 \leq K \leq \mathcal{E}_1$ , we can write[23]

$$\begin{aligned} k \mathbf{a}_1(\vec{k}) &= K \mathbf{b}(\vec{K}) \\ k \mathbf{a}_1^\dagger(\vec{k}) &= K \mathbf{b}^\dagger(\vec{K}). \end{aligned} \quad (9)$$

This mapping must preserve the physical consistency of both theories. In particular Poincare invariance should not be broken neither for the boundary theory nor for the bulk theory at a fixed  $z$ . This is obtained imposing that the canonical commutation relations (7,8) are preserved by the mapping (9). Then substituting eq. (9) into relation (7) and using eq. (8) we find an equation in terms of the moduli of bulk and boundary momenta which solution can be written as[23]

$$k = \frac{u_1}{2} \left[ \frac{\mathcal{E}_1 + \sqrt{\mathcal{E}_1^2 + \mu^2}}{K + \sqrt{K^2 + \mu^2}} - \frac{K + \sqrt{K^2 + \mu^2}}{\mathcal{E}_1 + \sqrt{\mathcal{E}_1^2 + \mu^2}} \right]. \quad (10)$$

Similar relations can be obtained for the other intervals  $\mathcal{E}_{p-1} < K \leq \mathcal{E}_p$  with  $p = 2, 3, \dots$ . One might wonder if the trivial mapping  $\vec{k} = \vec{K}$  would also be a solution. However this would not provide a one to one mapping between the entire bulk momenta ( $\vec{k}, u_p$ ) and the boundary momenta  $\vec{K}$  for all different values of  $p$  as long as there is only one boundary field with mass  $\mu$ .

The momentum operators in the bulk and boundary theories are respectively:

$$(\vec{P}, u) = \sum_p \int \frac{d^3k}{2(2\pi)^3} \frac{\mathbf{a}_p^\dagger(\vec{k}) \mathbf{a}_p(\vec{k})}{\sqrt{k^2 + u_p^2}} (\vec{k}, u_p) \quad (11)$$

$$\vec{\Pi} = \int \frac{d^3 K}{2(2\pi)^3} \frac{\mathbf{b}^\dagger(\vec{K})\mathbf{b}(\vec{K})}{\sqrt{K^2 + \mu^2}} \vec{K} \quad (12)$$

Note that Poincare invariance in the  $\vec{x}$  directions holds both in the boundary and bulk theories since the canonical commutation relations (7) and (8) are preserved by the mapping.

We are considering a high energy glueball scattering on the four dimensional AdS boundary. This will be mapped into a scattering process of dilaton states in the effective low energy string theory. The equation (10) is understood as the relation between bulk and boundary momenta for the particles involved in the scatterings. Identifying  $\mu$  with the mass of the lightest glueball and choosing the AdS size as

$$z_{max} \sim \frac{1}{\mu} \quad (13)$$

we find that  $u_1 \sim \mu$ , once  $z_{max} \sim 1/u_1$  according to eq. (6). Note that the size  $z_{max}$  can then be interpreted as an infrared cutoff for the boundary theory.

Further, we can take  $\mathcal{E}_1$  large enough so that the momenta associated with the high energy glueball scattering can fit into the region  $\mu \ll K \ll \mathcal{E}_1$ . Then we can approximate relation (10) as

$$k \approx \frac{\mathcal{E} \mu}{2K}, \quad (14)$$

where we defined  $\mathcal{E}_1 \equiv \mathcal{E}$  and disregarded the other energy intervals associated with higher axial momenta

$u_p, p \geq 2$ . Note that this mapping together with the conditions  $\mu \ll K \ll \mathcal{E}$  imply that  $\mu \ll k \ll \mathcal{E}$ .

Choosing the string scale to be of the same order of the high energy cut off of the boundary theory, i.e.,  $\mathcal{E} \sim 1/\sqrt{\alpha'}$ , we find that the momenta  $k$  associated with string theory correspond to a low energy regime well described by the supergravity approximation (4).

The equations (9) and (14) represent a one to one holographic mapping between bulk dilaton and boundary glueball states. Now we are going to use these equations to relate the corresponding scattering amplitudes.

Let us consider, in the bulk string theory, a scattering of 2 particles in the initial state and  $m$  particles in the final state, with all particles having axial momentum  $u_1$ . The  $S$  matrix reads

$$\begin{aligned} S_{Bulk} &= \langle \vec{k}_3, u_1; \dots; \vec{k}_{m+2}, u_1; out | \vec{k}_1, u_1; \vec{k}_2, u_1; in \rangle \\ &= \langle 0 | \mathbf{a}_{out}(\vec{k}_3) \dots \mathbf{a}_{out}(\vec{k}_{m+2}) \mathbf{a}_{in}^+(\vec{k}_1) \mathbf{a}_{in}^+(\vec{k}_2) | 0 \rangle, \end{aligned} \quad (15)$$

where  $\mathbf{a} \equiv \mathbf{a}_1$  and the *in* and *out* states are defined as  $|\vec{k}, u_1\rangle = \mathbf{a}^+(\vec{k})|0\rangle$ .

Now using the mapping between creation-annihilation operators (9) one can rewrite the above  $S$  matrix in terms of boundary operators. Considering fixed angle scattering, we take the bulk momenta to be of the form  $k_i = \gamma_i k$  and the boundary momenta  $K_i = \Gamma_i K$ , where  $\gamma_i$  and  $\Gamma_i$  are constants with  $i = 1, 2, \dots, m+2$ . Then

$$\begin{aligned} S_{Bulk} &\sim \langle 0 | \mathbf{b}_{out}(\vec{K}_3) \dots \mathbf{b}_{out}(\vec{K}_{m+2}) \mathbf{b}_{in}^+(\vec{K}_1) \mathbf{b}_{in}^+(\vec{K}_2) | 0 \rangle \left(\frac{K}{k}\right)^{m+2} \\ &\sim \langle \vec{K}_3, \dots, \vec{K}_{m+2}, out | \vec{K}_1, \vec{K}_2, in \rangle \left(\frac{K}{k}\right)^{m+2} K^{(m+2)(d-1)}, \end{aligned} \quad (16)$$

where the composite operators on the boundary have scaling dimension  $d$  and then their *in* and *out* states are  $|\vec{K}\rangle \cong K^{1-d} \mathbf{b}^+(\vec{K})|0\rangle$ , within the regime  $K \gg \mu$ .

Using the relation (14) between bulk and boundary momenta we get

$$S_{Bulk} \sim S_{Bound.} \left(\frac{\sqrt{\alpha'}}{\mu}\right)^{m+2} K^{(m+2)(1+d)} \quad (17)$$

As the scattering amplitudes  $\mathcal{M}$  are related to the corresponding  $S$  matrices (for non equal *in* and *out* states) by

$$\begin{aligned} S_{Bulk} &= \mathcal{M}_{Bulk} \delta^4(k_1^\rho + k_2^\rho - k_3^\rho - \dots - k_{m+2}^\rho) \\ S_{Bound.} &= \mathcal{M}_{Bound.} \delta^4(K_1^\rho + K_2^\rho - K_3^\rho - \dots - K_{m+2}^\rho) \end{aligned} \quad (18)$$

we find a relation between bulk and boundary scattering amplitudes

$$\begin{aligned} \mathcal{M}_{Bound.} &\sim \mathcal{M}_{Bulk} S_{Bound.} (S_{Bulk})^{-1} \left(\frac{K}{k}\right)^4 \\ &\sim \mathcal{M}_{Bulk} K^{8-(m+2)(d+1)} \left(\frac{\sqrt{\alpha'}}{\mu}\right)^{2-m} \end{aligned} \quad (19)$$

Now we must evaluate the bulk amplitude from the string low energy effective action (4). The momentum dependence of this amplitude can be determined using dimensional arguments. Note that the global constant  $R^8/\kappa^2$  associated with this action is dimensionless and the only dimensionfull parameters are  $z_{max} \sim 1/\mu$  and the Ricci scalar  $\mathcal{R} \sim 1/R^2$ . As  $\mu \ll k$  the relevant contribution to the bulk amplitude will not involve  $z_{max}$ . Further, choosing the condition  $1/R \ll k$  we can dis-

regard the term involving the Ricci scalar. This condition does not fix completely the AdS radius  $R$  and we additionally impose that  $\mu \ll 1/R$ . This implies that  $z_{max} \gg R$ . Then, if one regularizes the divergence ( $z = 0$ ) of the bulk action by cutting the axial coordinate  $z$  at  $R$  as in [4], one still has a large portion of the original AdS space:  $R \leq z \leq z_{max}$ . This guarantees that we keep the interesting AdS region which is an approximation for the near horizon geometry of  $N$  coincident D3-branes, as in the Maldacena duality.

Taking into account the normalization of the states  $|k, u_1\rangle$  one sees that  $\mathcal{M}_{Bulk}$  is dimensionally  $[\text{Energy}]^{4-n}$ , where  $n$  is the total number of scattered particles. As  $k$  is the only dimensionfull quantity that is relevant at leading order for the bulk scattering in the regime considered we find:

$$\mathcal{M}_{Bulk} \sim k^{2-m}. \quad (20)$$

Using again the relation between bulk and boundary momenta (14) and inserting this result in the boundary amplitude (19) we get

$$\mathcal{M}_{Boundary} \sim K^{4-\Delta}, \quad (21)$$

where  $\Delta = (m+2)d$  is the total scaling dimension of the scattering particles associated with glueballs on the four dimensional boundary. Considering  $K \sim \sqrt{s}$  we find the expected QCD scaling behavior[18, 19]

$$\mathcal{M}_{Boundary} \sim s^{2-\Delta/2}. \quad (22)$$

This shows that the bulk/boundary one to one mapping (9), (14) can be used to obtain the hard scattering behavior of high energy glueballs at fixed angles, from a low energy approximation of string theory.

It is interesting to relate the different energy scales used in the above derivation of the scattering amplitudes and check their consistency. The scales we discussed are

$$\mu \ll \frac{1}{R} \ll \frac{1}{\sqrt{\alpha'}}. \quad (23)$$

Note that the relation between the AdS radius  $R$ , the number of coincident branes  $N$ , the string coupling constant  $g$  and scale  $\alpha'$  is  $R^4 \sim gN(\alpha')^2$ . Then the above condition between  $R$  and  $\alpha'$  corresponds to the 't Hooft limit[1]. Assuming that the dimensionless quantity  $\mu R$  is the parameter that relates the energy scales we find  $\sqrt{\alpha'} = \mu R^2$  so that the lightest glueball mass is

$$\mu^2 = \frac{1}{gN\alpha'}. \quad (24)$$

This result is in agreement with [26]. Further, the above relation between the energy scales together with the condition that  $k \gg 1/R$  and the mapping between bulk and boundary momenta (14) imply that

$$\mu \ll K \ll \frac{1}{R} \ll k \ll \frac{1}{\sqrt{\alpha'}}, \quad (25)$$

so that the absolute values of  $k$  are greater than those of  $K$ , although the boundary scattering is a high energy process (with respect to  $\mu$ ) while the bulk scattering is a low energy process (with respect to  $1/\sqrt{\alpha'}$ ). Furthermore  $K$  and  $k$  in this regime are inversely proportional showing a kind of infrared-ultraviolet duality as expected from holography. We hope that the kind of mapping discussed in this paper might be extended to other physical processes or energy regimes. In particular we have recently applied this mapping to obtain the scalar glueball mass spectrum[27].

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