Particle Weights and their Disintegration II

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November 6, 2002

Abstract

The first article in this series presented a thorough discussion of particle weights and their characteristic properties. In this part a disintegration theory for particle weights is developed which yields pure components linked to irreducible representations and exhibiting features of improper energy-momentum eigenstates. This spatial disintegration relies on the separability of the Hilbert space as well as of the C^*-algebra. Neither is present in the GNS-representation of a generic particle weight so that we use a restricted version of this concept on the basis of separable constructs. This procedure does not entail any loss of essential information insofar as under physically reasonable assumptions on the structure of phase space the resulting representations of the separable algebra are locally normal and can thus be continuously extended to the original quasi-local C^*-algebra.

1 Introduction

As announced in the first part of this series of articles, the present paper is concerned with a disintegration theory for the highly reducible representations associated with particle weights. This endeavour is suggested by the expectation that elementary physical systems are connected with pure particle weights, giving rise to irreducible representations of the quasi-local C^*-algebra $\mathfrak{A}$. Accordingly, the sesquilinear forms on the left ideal $\mathfrak{L}$ of localizing operators, constructed from physical states of bounded energy by passing to the limit at asymptotic times, ought to be decomposable in the form

$$\sigma(L_1{}^* A L_2) = \sum_{i,j}{}' \int d\mu_{i,j}(\boldsymbol{p}) \, \langle L_1;\boldsymbol{p}\, j | A | L_2;\boldsymbol{p}\, i\rangle, \quad L_1, L_2 \in \mathfrak{L}, A \in \mathfrak{A}, \tag{1.1}$$

as motivated by a corresponding result of Araki and Haag [2, Theorem 4] for *massive* theories. Here the kets $|L_1;\boldsymbol{p}\, j\rangle$ and $|L_2;\boldsymbol{p}\, i\rangle$ denote normalizable vectors resulting from the localization of the improper energy-momentum eigenkets $|\boldsymbol{p}\, j\rangle$ and $|\boldsymbol{p}\, i\rangle$ with L_1 and L_2, respectively.

The approach to this problem in the present article is the decomposition of the GNS-representation pertaining to a generic particle weight into a direct integral of irreducible representations (spatial disintegration):

$$(\pi_w, \mathscr{H}_w) \simeq \int_X^\oplus d\nu(\xi) \, (\pi_\xi, \mathscr{H}_\xi). \tag{1.2}$$

The standard disintegration theory as expounded in the literature on C^*-algebras (e. g., cf. [13]) depends on the separability of the representation Hilbert space and even on the separability of the C^*-algebra. Before being able to make use of this theory, one therefore first has to give a separable reformulation of the concepts of local quantum physics and the notion of localizing operators derived from it. The smoothness of the latter with respect to Poincaré transformations turns out to be essential in order that the concept of particle weight be stable in the course of this kind of disintegration. According to an argument due to Buchholz, the resulting pure particle weights can be classified with regard to their mass and spin even in the case of charged systems (cf. [10] and [17, Section VI.2.2]). The necessity of passing to separable constructs in the disintegration raises the question as to the uniqueness of the result (1.2). An answer can be given by use of a compactness criterion due to Fredenhagen and Hertel, imposing restrictions on the phase space of quantum field theory. In theories complying with this assumption, the particle weight representations turn out to be locally normal. This information can then be used to show that no essential information about the physical systems gets lost by the aforementioned technical restrictions.

The first part of Section 2 presents the separable reformulation of concepts necessary to apply the standard theory of spatial disintegration to particle weight representations. This reformulation depends on a technical result, given in Appendix A, concerning the existence of norm-separable C^*-subalgebras lying strongly dense in a given one. The second part defines the concept of *restricted particle weights* arising from the standard notion in the separable context. Finally, the third part of Section 2 is devoted to the precise formulation of the disintegration theorem. In Section 3, the compactness criterion due to Fredenhagen and Hertel is applied to regain representations of the intact quasi-local algebra $\mathfrak{A}$ by use of local normality. Proofs of the results of Sections 2 and 3 have been collected in Sections 4 and 5, respectively. The Conclusions give an outlook on questions arising from the results presented and comment on an alternative (Choquet) approach to disintegration theory.

2 Disintegration of Particle Weights

2.1 Separable Reformulation of Local Quantum Physics and its Associated Algebra of Detectors

The theory of spatial disintegration of representations $(\overline{\pi}, \overline{\mathscr{H}})$ of a C^*-algebra $\overline{\mathfrak{A}}$ is a common theme of the pertinent textbooks [12, 13, 28, 23, 6], an indispensable presupposition being that of separability of the Hilbert space $\overline{\mathscr{H}}$ and even of the algebra $\overline{\mathfrak{A}}$ in their respective uniform topologies. Note that in this respect the statements of [6, Section 4.4] are incorrect (cf. also [7, Corrigenda]). While being concerned with a separable Hilbert space is common from a physicist's point of view, the corresponding requirement on the C^*-algebra $\overline{\mathfrak{A}}$ is too restrictive to be encountered in physically reasonable theories from the outset. So first of all *countable* respectively *separable* versions of the fundamental assumptions of local quantum field theory in terms of the net $\mathscr{O} \mapsto \mathfrak{A}(\mathscr{O})$ and of the Poincaré symmetry group $\mathsf{P}_+^\uparrow$ have to be formulated, before one can benefit from the extensive theory at hand.

2.1.1 Countable Collections P^c of Poincaré Transformations and $\mathscr{R}^c$ of Spacetime Regions

We start with denumerable dense subgroups $\mathsf{L}^c \in \mathsf{L}_+^\uparrow$ and $\mathsf{T}^c \in \mathbb{R}^{s+1}$ of Lorentz transformations and spacetime translations, respectively, and get a countable dense subgroup of $\mathsf{P}_+^\uparrow$ via the semi-direct product: $\mathsf{P}^c = \mathsf{L}^c \ltimes \mathsf{T}^c$. Subjecting the standard diamonds with *rational* radii, centred around the origin, to elements of P^c yields a countable family $\mathscr{R}^c$ of open bounded regions. It is invariant under P^c, covers all of $\mathbb{R}^{s+1}$ and contains arbitrarily small regions in the sense that any region in Minkowski space contains an element of this denumerable collection as a subset.

2.1.2 Net $\mathscr{O}_k \mapsto \mathfrak{A}^\bullet(\mathscr{O}_k)$ of Separable C^*-Algebras on Selected Regions

As shown in Appendix A, any unital C^*-algebra of operators on a separable Hilbert space $\mathscr{H}$ contains a norm-separable unital C^*-subalgebra which lies dense in it with respect to the strong operator topology. Applying this observation to the local C^*-algebras $\mathfrak{A}(\mathscr{O})$ of the defining positive-energy representation, we can associate with any algebra $\mathfrak{A}(\mathscr{O}_k)$, $\mathscr{O}_k \in \mathscr{R}^c$, a countable unital *-subalgebra $\mathfrak{A}^c(\mathscr{O}_k)$ over the field $\mathbb{Q}+i\mathbb{Q}$ that is strongly dense in $\mathfrak{A}(\mathscr{O}_k)$. Defining $\mathfrak{A}^\bullet(\mathscr{O}_k)$ as the C^*-algebra (over $\mathbb{C}$) which is generated by the union of all $\alpha_{(\Lambda,x)}(\mathfrak{A}^c(\mathscr{O}_i))$, where $(\Lambda,x) \in \mathsf{P}^c$ and $\mathscr{O}_i \in \mathscr{R}^c$ run through all combinations for which $\Lambda\mathscr{O}_i + x \subseteq \mathscr{O}_k$, we get a norm-separable algebra with $\mathfrak{A}^c(\mathscr{O}_k) \subseteq \mathfrak{A}^\bullet(\mathscr{O}_k) \subseteq \mathfrak{A}(\mathscr{O}_k)$, so that $\mathfrak{A}^\bullet(\mathscr{O}_k)$ turns out to be strongly dense in $\mathfrak{A}(\mathscr{O}_k)$. By construction, the resulting net $\mathscr{O}_k \mapsto \mathfrak{A}^\bullet(\mathscr{O}_k)$ fulfills the conditions of isotony, locality and covariance (imposed on the defining net) with respect to $\mathscr{R}^c$ and P^c. The countable *-algebra $\mathfrak{A}^c$ over $\mathbb{Q}+i\mathbb{Q}$, generated by the union of all the algebras $\mathfrak{A}^c(\mathscr{O}_k)$, $\mathscr{O}_k \in \mathscr{R}^c$, and thus invariant under transformations from P^c, lies uniformly dense in the C^*-inductive limit $\mathfrak{A}^\bullet$ of the net $\mathscr{O}_k \mapsto \mathfrak{A}^\bullet(\mathscr{O}_k)$ and even strongly dense in the quasi-local algebra $\mathfrak{A}$ itself: $\mathfrak{A}^c \subseteq \mathfrak{A}^\bullet \subseteq \mathfrak{A}$.

2.1.3 Countable Space $\mathfrak{L}_0^c$ of Almost Local Vacuum Annihilation Operators

Into the restricted setting of local quantum physics defined above, we now introduce the denumerable counterpart of the vector space $\mathfrak{L}_0$ of almost local vacuum annihilation operators [25, Definition 2.3]. First of all, note that it is possible to select a *countable* subspace over $\mathbb{Q}+i\mathbb{Q}$ in $\mathfrak{L}_0$, which consists of almost local vacuum annihilation operators with energy-momentum transfer in arbitrarily small regions, in the following way: $\complement\overline{V}_+$ admits a countable cover $\{\Gamma_n\}_{n\in\mathbb{N}}$ consisting of compact and convex subsets with the additional property that any bounded region in $\complement\overline{V}_+$ contains one of these. Likewise, the Lorentz group $\mathsf{L}_+^\uparrow$, being locally compact, can be covered by a countable family of arbitrarily small compact sets $\{\Theta_m\}_{m\in\mathbb{N}}$ as well. Selecting dense sequences of functions from the corresponding L^p-spaces with compact support, we get a countable family of operators in $\mathfrak{L}_0$ by regularizing the elements of $\mathfrak{A}^c$ with tensor products of these functions. Supplement this selection by all orders of partial derivatives with respect to the canonical coordinates around $(\mathbf{1},0)$, and apply all transformations from P^c to these constructs. As a result, one gets a sequence of vacuum annihilation operators, comprising elements with energy-momentum transfer in arbitrarily small regions, which generates a countable subspace $\mathfrak{L}_0^c$ over the field $\mathbb{Q}+i\mathbb{Q}$ in $\mathfrak{L}_0$ that is invariant under transformations from P^c and under taking partial derivatives of any order. When this construct is to be used in connection with a given particle weight $\langle\,.\,|\,.\,\rangle$ that is non-negative by definition, it does not cause any problems to supplement the selection in such a way that the imminent restriction of $\langle\,.\,|\,.\,\rangle$ to a subset of $\mathfrak{L}$ can be protected from getting trivial.

The operators in $\mathfrak{L}_0^c$ do not yet meet the requirements for formulating the disintegration theory. It turns out to be necessary to have precise control over the behaviour of their derivatives. To this end, we further regularize elements of $\mathfrak{L}_0^c$ by use of a *countable* set of test functions F on $\mathsf{P}_+^\uparrow$ with compact support containing the unit $(\mathbf{1},0)$. The resulting Bochner integrals

$$\alpha_F(L_0) = \int_{\mathsf{S}_F} d\mu(\Lambda,x)\, F(\Lambda,x)\, \alpha_{(\Lambda,x)}(L_0), \quad L_0 \in \mathfrak{L}_0^c, \tag{2.1}$$

are elements of the C^*-algebra $\mathfrak{A}^\bullet$ as well as of $\mathfrak{L}_0$ according to [25, Lemma 4.6], their energy-momentum transfer being contained in $\bigcup_{(\Lambda,x)\in\operatorname{supp}F}\Lambda\Gamma$ if that of $L_0 \in \mathfrak{L}_0^c$ belongs to Γ. The specific property of operators of type (2.1) in contrast to those from $\mathfrak{L}_0^c$ is that their differentiability with respect to the Poincaré group can be expressed in terms of derivatives of the infinitely differentiable test function F, thus implementing the desired governance over the properties of these derivatives. By choosing the support of the functions F small enough, one can impose an energy-momentum transfer in arbitrarily small regions on the operators $\alpha_F(L_0)$ as was the case for the elements of $\mathfrak{L}_0^c$ itself. Furthermore, a particle weight that did not vanish on the set $\mathfrak{L}_0^c$ is also non-zero when restricted to all of the operators $\alpha_F(L_0)$ constructed in (2.1). This fact is a consequence of the commutability of $|\,.\,\rangle$ and the integral defining $\alpha_F(L_0)$ [25, Lemma 5.4] in connection with the continuity of the particle weight under Poincaré transformations. The denumerable set of these special vacuum annihilation operators together with all their partial derivatives of arbitrary order (that share this specific style of construction) will be denoted $\overline{\mathfrak{L}_0^c}$ in the sequel. It might happen that two elements of $\overline{\mathfrak{L}_0^c}$ are connected by a Poincaré transformation not yet included in P^c. For technical reasons, which are motivated by the exigencies for the proof of the central Theorem 2.4 of this Section, we supplement P^c by all of the (countably many) transformations arising in this way and consider henceforth the countable subgroup $\overline{\mathsf{P}^c} \subseteq \mathsf{P}_+^\uparrow$ generated by them. $\overline{\mathfrak{L}_0^c}$ is then invariant under the operation of taking derivatives as well as under all transformations in $\overline{\mathsf{P}^c}$. Its image under *all* Poincaré transformations is denoted $\overline{\mathfrak{L}_0}$.

2.1.4 Countable Versions $\mathfrak{A}^c$ of the Quasi-Local Algebra, $\mathfrak{L}^c$ of the Left Ideal of Localizing Operators and $\mathfrak{C}^c$ of the Algebra of Detectors

Finally, we give the definitions for the counterparts of localizing operators and of detectors in the present setting [25, Definitions 2.4 and 2.5]. $\overline{\mathfrak{A}^c} \subseteq \mathfrak{A}^\bullet$ denotes the denumerable, unital *-algebra over $\mathbb{Q}+i\mathbb{Q}$ which is generated by $\mathfrak{A}^c \cup \overline{\mathfrak{L}_0^c}$. It is stable with respect to P^c and uniformly dense in $\mathfrak{A}^\bullet$. The countable counterpart $\mathfrak{L}^c$ of the left ideal $\mathfrak{L}$ in $\mathfrak{A}$ is defined as the linear span with respect to the field $\mathbb{Q}+i\mathbb{Q}$ of operators of the form $L = A\,L_0$ with $A \in \overline{\mathfrak{A}^c}$ and $L_0 \in \overline{\mathfrak{L}_0^c}$:

$$\mathfrak{L}^c \doteq \overline{\mathfrak{A}^c}\ \overline{\mathfrak{L}_0^c} = \operatorname{span}_{\mathbb{Q}+i\mathbb{Q}}\big\{A\,L_0 : A \in \overline{\mathfrak{A}^c}, L_0 \in \overline{\mathfrak{L}_0^c}\big\}. \tag{2.2}$$

It is a left ideal of the algebra $\overline{\mathfrak{A}^c}$, invariant under transformations from P^c. A countable *-subalgebra $\mathfrak{C}^c \subseteq \mathfrak{C}$ is introduced via

$$\mathfrak{C}^c \doteq \mathfrak{L}^{c*}\,\mathfrak{L}^c = \operatorname{span}_{\mathbb{Q}+i\mathbb{Q}}\big\{L_1{}^* L_2 : L_1, L_2 \in \mathfrak{L}^c\big\}. \tag{2.3}$$

2.2 Restricted Particle Weights

We shall now make use of the above constructs and define and investigate the restriction of a given particle weight in their terms. In doing so, one has to ensure that those properties established in Section 3 of [25] for generic particle weights and critical in their physical interpretation are still valid for the restricted version. The following theorem collects the list of relevant properties which are distinguished by the fact that they survive in the process of spatial disintegration. All the statements are readily checked on the grounds of [25, Theorem 3.12 and Proposition 3.13].

Theorem 2.1. *Let $(\pi_w, \mathscr{H}_w)$ be the GNS-representation associated with a given particle weight $\langle\,.\,|\,.\,\rangle$ and consider the restriction $|\,.\,\rangle^\bullet \doteq |\,.\,\rangle \restriction \mathfrak{L}^c$. The closure of its range is a separable Hilbert subspace $\mathscr{H}^\bullet$ of $\mathscr{H}_w$ that carries a non-zero, non-degenerate representation $\pi^\bullet$ of the C^*-algebra $\mathfrak{A}^\bullet$ defined by the restriction $\pi^\bullet \doteq \pi_w \restriction \mathfrak{A}^\bullet$, the representatives having their limited domain as well as range on $\mathscr{H}^\bullet$. Let furthermore $\{\alpha^\bullet_{(\Lambda,x)} \doteq \alpha_{(\Lambda,x)} \restriction \mathfrak{A}^\bullet : (\Lambda,x) \in \mathsf{P}_+^\uparrow\}$ denote the restriction of the initial automorphism group to $\mathfrak{A}^\bullet$. Then:*
(i) $|\,.\,\rangle^\bullet$ is a $(\mathbb{Q}+i\mathbb{Q})$-linear map from $\mathfrak{L}^c$ onto a dense subspace of $\mathscr{H}^\bullet$ such that the representation $\pi^\bullet$ acts on this space according to

$$\pi^\bullet(A)|L\rangle^\bullet = |AL\rangle^\bullet, \quad A \in \overline{\mathfrak{A}^c}, \quad L \in \mathfrak{L}^c. \tag{2.4}$$

(ii) $|\,.\,\rangle^\bullet$ allows for an extension to any operator L in $\overline{\mathfrak{L}_0}$ such that

$$\mathsf{P}_+^\uparrow \ni (\Lambda,x) \mapsto \big|\alpha^\bullet_{(\Lambda,x)}(L)\big\rangle^\bullet \in \mathscr{H}^\bullet \tag{2.5}$$

is a continuous mapping.

(iii) The definition $U^\bullet(x) \doteq U_w(x) \upharpoonright \mathscr{H}^\bullet$, $x \in \mathbb{R}^{s+1}$, yields a strongly continuous unitary representation of spacetime translations with a corresponding spectral measure $E^\bullet(\Delta) \doteq E_w(\Delta) \upharpoonright \mathscr{H}^\bullet$, Δ any Borel set, that is supported by a displaced forward light cone $\overline{V}_+ - q$, $q \in \overline{V}_+$. In the representation $(\pi^\bullet, \mathscr{H}^\bullet)$ these unitaries implement the spacetime translations via

$$U^\bullet(x)\pi^\bullet(A)U^\bullet(x)^* = \pi^\bullet(\alpha_x^\bullet(A)), \quad A \in \mathfrak{A}^\bullet, \quad x \in \mathbb{R}^{s+1}. \tag{2.6a}$$

On the subset $\{|L\rangle^\bullet : L \in \overline{\mathfrak{L}_0}\}$ of $\mathscr{H}^\bullet$ they act according to

$$U^\bullet(x)|L\rangle^\bullet = \big|\alpha_x^\bullet(L)\big\rangle^\bullet, \quad L \in \overline{\mathfrak{L}_0}, \tag{2.6b}$$

and for $L \in \mathfrak{L}^c \cup \overline{\mathfrak{L}_0}$ with energy-momentum transfer in the Borel set $\Delta \subseteq \mathbb{R}^{s+1}$ there holds the relation

$$E^\bullet(\Delta)|L\rangle^\bullet = |L\rangle^\bullet. \tag{2.6c}$$

Definition 2.2. Any system that complies with the complete list of properties given in Theorem 2.1 will be called a *restricted particle weight* henceforth.

The spectral property (2.6c) constitutes the basis for the proof of the Cluster Property as formulated in [25, Proposition 3.14]. The arguments presented there can be adopted literally, on condition that the obvious substitutions are observed, to implement it in the present setting.

Proposition 2.3 (Cluster Property for Restricted Particle Weights)**.** *Let L_i and L_i' be elements of $\mathfrak{L}_0^c$ with energy-momentum transfer in Γ_i respectively Γ_i', and let $A_i \in \mathfrak{A}^c$, $i = 1, 2$, be almost local. Then, for a restricted particle weight,*

$$\mathbb{R}^s \ni \boldsymbol{x} \mapsto {}^\bullet\big\langle L_1{}^* A_1 L_1' \big| \alpha_{\boldsymbol{x}}^\bullet(L_2{}^* A_2 L_2') \big\rangle^\bullet = {}^\bullet\big\langle L_1{}^* A_1 L_1' \big| U^\bullet(\boldsymbol{x}) \big| L_2{}^* A_2 L_2' \big\rangle^\bullet \in \mathbb{C}$$

is a function in $L^1(\mathbb{R}^s, d^s x)$.

2.3 Spatial Disintegration of Restricted Particle Weights

In this Subsection we shall establish the spatial disintegration of a (restricted) particle weight in terms of pure ones. In Theorem 2.1 the particle weight $\langle\,.\,|\,.\,\rangle$ defined in the framework of the full theory was associated with the representation $(\pi^\bullet, \mathscr{H}^\bullet)$ of the norm-separable C^*-algebra $\mathfrak{A}^\bullet$ on the separable Hilbert space $\mathscr{H}^\bullet$. This construction makes available the method of spatial disintegration expounded in the relevant literature. In order to express $\pi^\bullet$ in terms of an integral of irreducible representations, a last preparatory step has to be taken: a *maximal abelian von Neumann algebra* $\mathfrak{M}$ in the commutant of $\pi^\bullet(\mathfrak{A}^\bullet)$ has to be selected [13, Theorem 8.5.2]. Our choice of such an algebra is determined by the objective to get to a disintegration in terms of restricted particle weights, i. e., one has to provide for the possibility to establish the relations (2.6).

The unitary representation $x \mapsto U^\bullet(x)$ of spacetime translations has generators with joint spectrum in a displaced forward light cone. Through multiplication by suitably chosen exponential factors $\exp(iqx)$ with fixed $q \in \overline{V}_+$, we can pass to another representation which likewise implements the spacetime translations but has spectrum contained in $\overline{V}_+$. Then [4, Theorem IV.5] implies that one can find a third strongly continuous unitary representation of this kind with elements belonging to $\pi^\bullet(\mathfrak{A}^\bullet)''$, the weak closure of $\pi^\bullet(\mathfrak{A}^\bullet)$ [6, Corollary 2.4.15]. This result can again be tightened up by use of [5, Theorem 3.3] in the sense that among all the representations complying with the above features there exists exactly one which is characterized by the further requirement that the lower boundary of the joint spectrum of its generators be Lorentz invariant. It is denoted

$$\mathbb{R}^{s+1} \ni x \mapsto U_{can}^\bullet(x) \in \pi^\bullet(\mathfrak{A}^\bullet)''. \tag{2.7a}$$

At this point it turns out to be significant that the C^*-algebra $\mathfrak{A}^\bullet$ has been constructed in Section 2.1 by using local operators so that the reasoning given in [5] applies to the present situation. Another unitary representation can be defined through

$$x \mapsto V^\bullet(x) \doteq U^\bullet_{can}(x) U^\bullet(x)^{-1}. \tag{2.7b}$$

By their very construction, all the operators $V^\bullet(x)$, $x \in \mathbb{R}^{s+1}$, are elements of $\pi^\bullet(\mathfrak{A}^\bullet)'$. The maximal commutative von Neumann algebra $\mathfrak{M}$ that we are going to work with in the sequel is now selected in compliance with the condition

$$\left\{V^\bullet(x) : x \in \mathbb{R}^{s+1}\right\}'' \subseteq \mathfrak{M} \subseteq \Big(\pi^\bullet(\mathfrak{A}^\bullet) \cup \left\{U^\bullet(x) : x \in \mathbb{R}^{s+1}\right\}\Big)'. \tag{2.8}$$

Theorem 2.4. *Let $\langle\, .\, |\, .\, \rangle$ be a generic particle weight with representation $(\pi_w, \mathscr{H}_w)$ inducing, by Theorem 2.1, a restricted particle weight with representation $(\pi^\bullet, \mathscr{H}^\bullet)$ of the separable C^*-algebra $\mathfrak{A}^\bullet$ on the separable Hilbert space $\mathscr{H}^\bullet$. Select a maximal abelian von Neumann algebra $\mathfrak{M}$ such that* (2.8) *is fulfilled. Then there exist a standard Borel space X, a bounded positive measure ν on X, and a field of restricted particle weights indexed by $\xi \in \mathrm{X}$ such that the following assertions hold true:*

(i) The field $\xi \mapsto (\pi_\xi, \mathscr{H}_\xi)$ is a ν-measurable field of irreducible representations of $\mathfrak{A}^\bullet$.

(ii) The non-zero representation $(\pi^\bullet, \mathscr{H}^\bullet)$ is unitarily equivalent to their direct integral

$$(\pi^\bullet, \mathscr{H}^\bullet) \simeq \int_{\mathrm{X}}^{\oplus} d\nu(\xi)\, (\pi_\xi, \mathscr{H}_\xi), \tag{2.9a}$$

and, with W denoting the corresponding unitary operator, the vectors in both spaces are linked up by the relation

$$W\,|L\rangle^\bullet = \left\{|L\rangle_\xi : \xi \in \mathrm{X}\right\} \doteq \int_{\mathrm{X}}^{\oplus} d\nu(\xi)\, |L\rangle_\xi, \quad L \in \mathfrak{L}^c \cup \overline{\mathfrak{L}_0}. \tag{2.9b}$$

Here, in an obvious fashion, $|\, .\, \rangle_\xi$ denotes the linear mapping characteristic for the restricted ξ-particle weight (cf. Theorem 2.1).

(iii) The von Neumann algebra $\mathfrak{M}$ coincides with the algebra of operators that are diagonalisable with respect to (2.9a)*: any operator $T \in \mathfrak{M}$ corresponds to an essentially bounded measurable complex-valued function g_T according to*

$$W\,T\,W^* = \int_{\mathrm{X}}^{\oplus} d\nu(\xi)\, g_T(\xi)\, \mathbf{1}_\xi, \tag{2.9c}$$

where $\mathbf{1}_\xi$, $\xi \in \mathrm{X}$, are the unit operators of the algebras $\mathscr{B}(\mathscr{H}_\xi)$, respectively.

(iv) Let $x \mapsto U_\xi(x)$ denote the unitary representation which implements the spacetime translations in the restricted ξ-particle weight according to (2.6a)*, and let the operator $E_\xi(\Delta) \in \mathscr{B}(\mathscr{H}_\xi)$ designate the corresponding spectral projection associated with the Borel set $\Delta \subseteq \mathbb{R}^{s+1}$. Then the fields of operators*

$$\xi \mapsto U_\xi(x) \qquad \textit{and} \qquad \xi \mapsto E_\xi(\Delta)$$

are measurable and satisfy, for any x and any Borel set Δ, the following equations:

$$W\,U^\bullet(x)\,W^* = \int_{\mathrm{X}}^{\oplus} d\nu(\xi)\, U_\xi(x), \tag{2.9d}$$

$$W\,E^\bullet(\Delta)\,W^* = \int_{\mathrm{X}}^{\oplus} d\nu(\xi)\, E_\xi(\Delta). \tag{2.9e}$$

(v) In each Hilbert space $\mathscr{H}_\xi$ there exists a canonical choice of a strongly continuous unitary representation $x \mapsto U_\xi^{can}(x)$ of spacetime translations in terms of operators from $\pi_\xi(\mathfrak{A}^\bullet)'' = \mathscr{B}(\mathscr{H}_\xi)$. It is distinguished by the fact that it implements the spacetime translations in the representation $(\pi_\xi, \mathscr{H}_\xi)$ and that the joint spectrum of its generators P_ξ^c lies in the closed forward light cone $\overline{V}_+$. Moreover, for given x, the field of unitaries $\xi \mapsto U_\xi^{can}(x)$ is measurable. This representation is defined by

$$U_\xi^{can}(x) \doteq \exp(i\, p_\xi x)\, U_\xi(x), \quad x \in \mathbb{R}^{s+1}, \tag{2.9f}$$

where p_ξ is the unique vector in $\mathbb{R}^{s+1}$ that is to be interpreted as the sharp energy-momentum corresponding to the respective particle weight.

The range of energy-momenta p_ξ arising in the above disintegration is not under control as yet; in particular its connection with the geometric momentum as encoded in the support of the velocity function h that appears in the construction of particle weights [25, Section 3] is an open question. Moreover, the spatial disintegration presented above is subject to arbitrariness in two respects. There exist different constructions of the type expounded in Subsection 2.1 and therefore, according to Theorem 2.1, one has to deal with a number of different restricted particle weights derived from the GNS-representation $(\pi_w, \mathscr{H}_w)$. As a result, the object to be disintegrated according to Theorem 2.4 is by no means uniquely fixed. Upon selection of a particular one complying with the requirements of this theorem, there still remains an ambiguity as to the choice of maximal abelian von Neumann algebra with respect to which the disintegration is to be performed. Nevertheless, these interesting open questions arise on the basis that a disintegration of general particle weights into pure ones, representing elementary systems, has successfully been constructed.

3 Phase Space Restrictions and Local Normality

A number of criteria have been introduced into the analysis of generic quantum field theories in order to implement the quantum mechanical fact based on the uncertainty principle that only a finite number of linearly independent states can be fitted into a bounded region of phase space; the final aim being a selection criterion which singles out quantum field theoretic models with a complete particle interpretation. These attempts can be traced back to the year 1965 when Haag and Swieca [18] proposed a compactness condition, imposing an effective restriction on phase space. They argued that in theories with a particle interpretation the set of bounded local excitations of the vacuum with restricted energy ought to be *compact*. Buchholz and Wichmann [11] formulated a strengthened version of this criterion in 1986 on the basis of thermodynamic considerations, requiring that the set considered by Haag and Swieca should be *nuclear*. This determines a maximal value for the number of local degrees of freedom for physical states of bounded energy as the relevant set lies in an infinite-dimensional parallelepiped with summable edge lengths. Another approach to phase space restrictions is dual to the preceding concepts in reversing the order of localization and energy restriction. Here physical states of bounded energy are considered with their domain confined to local algebras. Fredenhagen and Hertel [15] proposed in 1979 that the resulting subsets of $\mathfrak{A}(\mathscr{O})^*$ are to be *compact*. Finally, a *nuclear* version of this criterion has been formulated in [26] that implies all the others. The interrelationship between these various concepts is treated in [9]. There still is room for different formulations of phase space restrictions as, e. g., investigated by Buchholz, D'Antoni and Longo in [8] and by Guido and Longo in [16].

In the present context, we want to make use of the Compactness Condition proposed by Fredenhagen and Hertel to show that, under this physically motivated presupposition, the arbitrariness in the choice of a separable C^*-subalgebra $\mathfrak{A}^\bullet$ of the quasi-local algebra $\mathfrak{A}$ in Section 2 can be removed.

Compactness Condition 3.1 (Fredenhagen-Hertel)**.**
A local quantum field theory satisfies the Fredenhagen-Hertel Compactness Condition if for each pair of a bounded Borel set $\Delta' \subseteq \mathbb{R}^{s+1}$ and of a bounded region $\mathscr{O}$ in Minkowski space the mapping

$$T^{\mathscr{O}}_{\Delta'} : \mathfrak{A}(\mathscr{O}) \to \mathscr{B}(\mathscr{H}) \qquad A \mapsto T^{\mathscr{O}}_{\Delta'}(A) \doteq E(\Delta')AE(\Delta')$$

has the property that the images of bounded subsets of $\mathfrak{A}(\mathscr{O})$ are precompact subsets of $\mathscr{B}(\mathscr{H})$ with respect to its uniform topology. In the present situation precompactness (= total boundedness) is equivalent to relative compactness [22, Chapter One, § 4, 5.].

To demonstrate the main result of this Section, Theorem 3.5, we have to make use of the concept of Δ-bounded particle weights as introduced in [25, Definition 3.15].

Definition 3.2. A particle weight is said to be Δ-*bounded*, if to any bounded Borel subset Δ' of $\mathbb{R}^{s+1}$ there exists another such set $\overline{\Delta} \supseteq \Delta + \Delta'$, so that the GNS-representation $(\pi_w, \mathscr{H}_w)$ of the particle weight and the defining representation are connected by the following inequality, valid for any $A \in \mathfrak{A}$,

$$\|E_w(\Delta')\pi_w(A)E_w(\Delta')\| \leqslant c \cdot \|E(\overline{\Delta})AE(\overline{\Delta})\| \tag{3.1}$$

with a suitable positive constant c that is independent of the Borel sets involved. Obviously, Δ ought to be a bounded Borel set as well.

This restriction can be motivated on physical grounds as opposed to mere technical needs, since, according to [25, Lemma 3.16], the asymptotic functionals constructed by use of physical states of bounded energy give rise to particle weights of this special kind. The corresponding GNS-representations $(\pi_w, \mathscr{H}_w)$ then meet the Fredenhagen-Hertel Compactness Condition if the underlying local quantum field theory does, and the same holds true for the corresponding restricted particle weights.

Proposition 3.3. *Suppose that the given local quantum field theory complies with the Compactness Condition of Fredenhagen and Hertel.*
(i) If $\langle\ .\ |\ .\ \rangle$ *is a* Δ*-bounded particle weight on* $\mathfrak{L} \times \mathfrak{L}$*, then the associated GNS-representation* $(\pi_w, \mathscr{H}_w)$ *of the quasi-local algebra* $\mathfrak{A}$ *is subject to the compactness condition as well.*

(ii) The restricted particle weight associated with the above GNS-representation by virtue of Theorem 2.1 likewise inherits the compactness property.

Under the presupposition of the Compactness Condition, a similar result holds for the irreducible representations $(\pi_\xi, \mathscr{H}_\xi)$ arising in the spatial disintegration of the restricted particle weight by virtue of Theorem 2.4 if the domain of ξ is further astricted.

Proposition 3.4. *Let* $(\pi_w, \mathscr{H}_w)$ *be the GNS-representation of the quasi-local algebra* $\mathfrak{A}$ *corresponding to the* Δ*-bounded particle weight* $\langle\ .\ |\ .\ \rangle$*, and let* $(\pi^\bullet, \mathscr{H}^\bullet)$ *be the representation of the associated restricted particle weight. If the underlying quantum field theory satisfies the Compactness Condition of Fredenhagen and Hertel, then* ν*-almost all of the irreducible representations* $(\pi_\xi, \mathscr{H}_\xi)$ *occurring in the spatial disintegration* (2.9a) *of* $(\pi^\bullet, \mathscr{H}^\bullet)$ *by course of Theorem 2.4 comply with this condition as well, relation* (2.9a) *still being valid with* X *replaced by the appropriate* ν*-measurable non-null subset* X_0.

The central result of the present section is the perception that, under the above assumptions on the structure of phase space, the representations $(\pi_w, \mathscr{H}_w)$ and $(\pi^\bullet, \mathscr{H}^\bullet)$ of the quasi-local C^*-algebras $\mathfrak{A}$ and $\mathfrak{A}^\bullet$, respectively, as well as ν-almost all of the irreducible representations $(\pi_\xi, \mathscr{H}_\xi)$ of $\mathfrak{A}^\bullet$ occurring in the direct integral decomposition of the latter, are locally normal. This means that for arbitrary bounded regions $\mathscr{O}$ the restriction of the representation $(\pi_w, \mathscr{H}_w)$ to the local algebra $\mathfrak{A}(\mathscr{O})$ is continuous with respect to the relative σ-weak topologies of $\mathfrak{A}(\mathscr{O}) \subseteq \mathscr{B}(\mathscr{H})$ as well as of $\pi_w(\mathfrak{A}(\mathscr{O})) \subseteq \mathscr{B}(\mathscr{H}_w)$. In the case of representations of $\mathfrak{A}^\bullet$ the corresponding formulation uses bounded regions in the countable collection $\mathscr{R}^c$. Having established local normality, the representations of $\mathfrak{A}^\bullet$ can be continuously extended to all of $\mathfrak{A}$ in such a way that the disintegration formula (2.9a) stays valid when X is replaced by the non-null set X_0 occurring in Proposition 3.4.

Theorem 3.5 (Local Normality of Representations)**.** *Given the presumptions formulated in Proposition 3.4, the following assertions hold:*

(i) The GNS-representation $(\pi_w, \mathscr{H}_w)$ of the quasi-local algebra $\mathfrak{A}$ is locally normal.

(ii) The representation $(\pi^\bullet, \mathscr{H}^\bullet)$ of the quasi-local algebra $\mathfrak{A}^\bullet$ is locally normal. The same applies to the irreducible representations $(\pi_\xi, \mathscr{H}_\xi)$ occurring in the spatial disintegration of $(\pi^\bullet, \mathscr{H}^\bullet)$ when the indices ξ are astricted to X_0.

(iii) The representations $(\pi^\bullet, \mathscr{H}^\bullet)$ and $(\pi_\xi, \mathscr{H}_\xi)$, $\xi \in \mathrm{X}_0$, allow for unique locally normal extensions to the whole of the original quasi-local algebra $\mathfrak{A}$ designated $(\overline{\pi}^\bullet, \mathscr{H}^\bullet)$ and $(\overline{\pi}_\xi, \mathscr{H}_\xi)$, respectively, which are related by

$$(\overline{\pi}^\bullet, \mathscr{H}^\bullet) \simeq \int_{\mathrm{X}_0}^{\oplus} d\nu(\xi)\, (\overline{\pi}_\xi, \mathscr{H}_\xi), \tag{3.2}$$

where the representations $(\overline{\pi}_\xi, \mathscr{H}_\xi)$ are again irreducible.

Theorem 3.5 shows that, given sensible phase space restrictions, no information on a physical system described by a normal state of bounded energy, $\omega \in \mathscr{S}(\Delta)$, gets lost in the entirety of constructions presented in [25, Section 3] and Section 2 of the present article. These lead from ω via an associated particle weight with representation $(\pi_w, \mathscr{H}_w)$ of the quasi-local algebra $\mathfrak{A}$ to the induced restricted particle weight with representation $(\pi^\bullet, \mathscr{H}^\bullet)$ of the algebra $\mathfrak{A}^\bullet$ allowing for a disintegration in terms of a field of irreducible representations $\{(\pi_\xi, \mathscr{H}_\xi) : \xi \in \mathrm{X}_0\}$. According to the preceding theorem, this disintegration is again extendible in a unique fashion to one in terms of locally normal representations of the original algebra $\mathfrak{A}$ as expressed by (3.2). Now, due to the explicit construction of $(\pi^\bullet, \mathscr{H}^\bullet)$ from $(\pi_w, \mathscr{H}_w)$ in Theorem 2.1, the local normality of both these representations implies that, actually, $\overline{\pi}^\bullet$ coincides with the restriction of π_w to the subspace $\mathscr{H}^\bullet$ of $\mathscr{H}_w$. Thus we arrive at a partial reconstruction of the GNS-representation $(\pi_w, \mathscr{H}_w)$ which only depends on the initial choice of a subspace of the Hilbert space $\mathscr{H}_w$. Moreover, by Theorem 3.5, this entails a spatial disintegration of Δ-bounded particle weights $\langle \,.\, | \,.\, \rangle$ according to the following reformulation of (3.2):

$$(\pi_w, \mathscr{H}^\bullet) \simeq \int_{\mathrm{X}_0}^{\oplus} d\nu(\xi)\, (\overline{\pi}_\xi, \mathscr{H}_\xi). \tag{3.3}$$

4 Proof of the Disintegration Theorem

Remark. The concepts occurring in the theory of direct integrals of Hilbert spaces (standard Borel space, decomposable and diagonalisable operators, and the like) are expounded in [3, Chapter 3], [12, Part II] and likewise [28, Section IV.8 and Appendix].

Theorem 2.4. The presuppositions of this theorem meet the requirements for an application of [13, Theorem 8.5.2]. This supplies us with

- a standard Borel space $\overline{\mathrm{X}}$;
- a bounded positive measure $\overline{\nu}$ on $\overline{\mathrm{X}}$;
- a $\overline{\nu}$-measurable field $\xi \mapsto (\pi_\xi, \mathscr{H}_\xi)$ on $\overline{\mathrm{X}}$ consisting of irreducible representations π_ξ of the C^*-algebra $\mathfrak{A}^\bullet$ on the Hilbert spaces $\mathscr{H}_\xi$;
- an isomorphism (a linear isometry) $\overline{W}$ from $\mathscr{H}^\bullet$ onto the direct integral of these Hilbert spaces such that

$$\overline{W} : \mathscr{H}^\bullet \to \int_{\overline{\mathrm{X}}}^{\oplus} d\overline{\nu}(\xi)\, \mathscr{H}_\xi \tag{4.1a}$$

transforms $\pi^\bullet$ into the direct integral of the representations π_ξ according to

$$\overline{W}\pi^\bullet(A)\overline{W}^* = \int_{\overline{X}}^{\oplus} d\overline{\nu}(\xi)\, \pi_\xi(A), \quad A \in \mathfrak{A}^\bullet, \tag{4.1b}$$

and the maximal abelian von Neumann algebra $\mathfrak{M}$ can be identified with the algebra of diagonalisable operators via

$$\overline{W}T\overline{W}^* = \int_{\overline{X}}^{\oplus} d\overline{\nu}(\xi)\, g_T(\xi)\,\mathbf{1}_\xi, \quad T \in \mathfrak{M}, \tag{4.1c}$$

with an appropriate function $g_T \in L^\infty\big(\overline{X}, d\overline{\nu}(\xi)\big)$.

At first sight, the different statements of [13, Theorem 8.5.2] listed above seem to cover almost all of the assertions of the present Theorem 2.4, but one must not forget that the disintegration is to be expressed in terms of a field of *restricted particle weights*. So we are left with the task to establish their defining properties in the irreducible representations $(\pi_\xi, \mathscr{H}_\xi)$ supplied by standard disintegration theory. Simultaneously, relation (2.9b) is to be satisfied presenting the following problem: In general the isomorphism $\overline{W}$ connects a given vector $\Psi \in \mathscr{H}^\bullet$ not with a unique vector field $\{\Psi_\xi : \xi \in \overline{X}\}$ but rather with an equivalence class of such fields, characterized by the fact that its elements differ pairwise at most on $\overline{\nu}$-null sets. In contrast to this, (2.9b) associates the vector field $\{|L\rangle_\xi : \xi \in X\}$ with $|L\rangle^\bullet$ for any $L \in \mathfrak{L}^c \cup \overline{\mathfrak{L}_0}$, leaving no room for any ambiguity. In particular, the algebraic relations prevailing in the set $\mathfrak{L}^c \cup \overline{\mathfrak{L}_0}$ which carry over to $|\,.\,\rangle$ have to be observed in defining each of the mappings $|\,.\,\rangle_\xi$. The contents of the theorem quoted above, important as they are, can therefore only serve as the starting point for the constructions carried out below, in the course of which again and again $\overline{\nu}$-null sets have to be removed from $\overline{X}$ to secure definiteness of the remaining components in the disintegration of a given vector. In doing so, one has to be cautious not to apply this procedure uncountably many times; for, otherwise, by accident the standard Borel space $X \subseteq \overline{X}$ arising in the end could happen to be itself a $\overline{\nu}$-null set, $\overline{\nu}(X) = 0$, in contradiction to the disintegration (2.9a) of the *non-zero* representation $(\pi^\bullet, \mathscr{H}^\bullet)$.

(i) The task set by the first item in Theorem 2.1 is to establish the existence of $(\mathbb{Q} + i\mathbb{Q})$-linear mappings $|\,.\,\rangle_\xi$ from $\mathfrak{L}^c$ onto countable dense subspaces $\mathscr{H}_\xi^c$ in each of the component Hilbert spaces $\mathscr{H}_\xi$ supplied by [13, Theorem 8.5.2] such that

$$\pi_\xi(A)|L\rangle_\xi = |AL\rangle_\xi, \quad A \in \overline{\mathfrak{A}^c}, \quad L \in \mathfrak{L}^c. \tag{4.2}$$

By relation (4.1a), there exists to each $L \in \mathfrak{L}^c$ an equivalence class of vector fields on $\overline{X}$ which corresponds to the element $|L\rangle^\bullet$ in $\mathscr{H}^\bullet$. The assumed $(\mathbb{Q} + i\mathbb{Q})$-linearity of the mapping $|\,.\,\rangle^\bullet : \mathfrak{L}^c \to \mathscr{H}^\bullet$ carries over to these equivalence classes, *not* to their representatives. This means that, if we pick out one representative of the vector $|L\rangle^\bullet$ for every L in the denumerable set $\mathfrak{L}^c$ and designate it as $\{|L\rangle_\xi : \xi \in \overline{X}\}$, all of the countably many relations that constitute $(\mathbb{Q} + i\mathbb{Q})$-linearity are satisfied only for $\overline{\nu}$-almost all of the components. Upon selection of these representatives, the desired linearity of $|\,.\,\rangle_\xi$ is thus automatically valid for all ξ in a Borel subset of $\overline{X}$ which is left by dismissing an appropriate $\overline{\nu}$-null set. The same reasoning can be applied to the disintegration of vectors of the form $|AL\rangle^\bullet = \pi^\bullet(A)|L\rangle^\bullet$ with $A \in \overline{\mathfrak{A}^c}$ and $L \in \mathfrak{L}^c$. Again with (2.9b) in mind, the number of relations (4.2) to be satisfied is countable so that in view of relation (4.1b) the removal of another appropriate $\overline{\nu}$-null set leaves only those indices ξ behind for which the mappings $|\,.\,\rangle_\xi$ indeed have the desired property (4.2). According to [12, Section II.1.6, Proposition 8], the fact that the set $\{|L\rangle^\bullet : L \in \mathfrak{L}^c\}$ is total in $\mathscr{H}^\bullet$ implies that the corresponding property holds for $\overline{\nu}$-almost all ξ in the disintegration. As a result there exists a non-null Borel set $X_1 \subseteq \overline{X}$, such that the mappings $|\,.\,\rangle_\xi$, $\xi \in X_1$, are not only $(\mathbb{Q} + i\mathbb{Q})$-linear and satisfy (4.2) but also map $\mathfrak{L}^c$ onto a dense subset of $\mathscr{H}_\xi$. In this way, all of the characteristics presented in the first item of Theorem 2.1 are implemented, and additionally we have

$$\overline{W}\,|L\rangle^{\bullet} = \int_{\overline{X}_1}^{\oplus} d\overline{\nu}(\xi)\,|L\rangle_{\xi}, \quad L \in \mathfrak{L}^c. \tag{4.3}$$

(ii) Next, the mappings $|\,.\,\rangle_{\xi}$ constructed above have to be extended to the set $\overline{\mathfrak{L}_0}$ of all Poincaré transforms of operators from $\mathfrak{L}_0^c$ in such a way that the mapping

$$\mathsf{P}_+^{\uparrow} \ni (\Lambda, x) \mapsto \big|\alpha^{\bullet}_{(\Lambda,x)}(L')\big\rangle_{\xi} \in \mathscr{H}_{\xi}, \qquad L' \in \overline{\mathfrak{L}_0}, \tag{4.4}$$

is continuous. Here the special selection of $\mathfrak{L}_0^c$ as consisting of compactly regularized vacuum annihilation operators comes into play in combination with the invariance of this set under transformations $(\Lambda, x) \in \overline{\mathsf{P}}^c$. Based on the differentiability properties of the operators in question, one has to take care in the extension not to impose uncountably many conditions on the mappings $|\,.\,\rangle_{\xi}$ to ensure that only a $\overline{\nu}$-null subset of $\overline{X}_1$ gets lost, the remaining ones sharing the claimed extension property.

Consider a covering of the Poincaré group $\mathsf{P}_+^{\uparrow}$ by a sequence of open sets V_i with compact closures C_i contained in corresponding open charts (U_i, ϕ_i) such that the sets $\phi_i(\mathsf{C}_i) \subseteq \mathbb{R}^{d_{\mathsf{P}}}$ are convex (d_{P} denotes the dimension of $\mathsf{P}_+^{\uparrow}$). Select one of these compacta C_k, say, and fix $\hat{L}_0 \in \mathfrak{L}_0^c$ that, by assumption, is the regularization of an element $L_0 \in \mathfrak{L}_0^c$ with an infinitely often differentiable function F having compact support $\mathsf{S}_F \subseteq \mathsf{P}_+^{\uparrow}$:

$$\hat{L}_0 = \alpha_F(L_0) \doteq \int_{\mathsf{S}_F} d\mu(\Lambda, x)\, F(\Lambda, x)\, \alpha_{(\Lambda,x)}(L_0). \tag{4.5a}$$

According to [25, Lemma 5.4] the mapping $|\,.\,\rangle$ commutes with this integral so that

$$|\hat{L}_0\rangle = \int_{\mathsf{S}_F} d\mu(\Lambda, x)\, F(\Lambda, x)\, \big|\alpha_{(\Lambda,x)}(L_0)\big\rangle \in \mathscr{H}_w. \tag{4.5b}$$

The same equation holds for the Poincaré transforms of the operator $\hat{L}_0$. Thus, invariance of the Haar measure on $\mathsf{P}_+^{\uparrow}$ in connection with the compact support of F implies for arbitrary $(\Lambda_0, x_0) \in \mathsf{C}_k$:

$$\begin{aligned} \big|\alpha_{(\Lambda_0,x_0)}(\hat{L}_0)\big\rangle &= \int_{\mathsf{S}_F} d\mu(\Lambda, x)\, F(\Lambda, x)\, \big|\alpha_{(\Lambda_0,x_0)(\Lambda,x)}(L_0)\big\rangle \\ &= \int_{\mathsf{C}_k \cdot \mathsf{S}_F} d\mu(\Lambda, x)\, F\big((\Lambda_0, x_0)^{-1}(\Lambda, x)\big)\, \big|\alpha_{(\Lambda,x)}(L_0)\big\rangle. \end{aligned} \tag{4.5c}$$

The derivatives of the mapping $(\Lambda_0, x_0) \mapsto \big|\alpha_{(\Lambda_0,x_0)}(\hat{L}_0)\big\rangle$ on the neighbourhood $\mathsf{V}_k \subseteq \mathsf{C}_k$ are then explicitly seen to be expressible in terms of derivatives of the functions

$$F^{(\Lambda,x)} : \mathsf{V}_k \to \mathbb{C} \qquad (\Lambda_0, x_0) \mapsto F^{(\Lambda,x)}(\Lambda_0, x_0) \doteq F\big((\Lambda_0, x_0)^{-1}(\Lambda, x)\big).$$

Let (Λ_1, x_1) and (Λ_2, x_2) be a pair of Poincaré transformations lying in the common neighbourhood V_k; then an application of the Mean Value Theorem yields, in terms of the coordinates from $\phi_k(\mathsf{V}_k)$,

$$\begin{aligned} &\big|\alpha_{(\Lambda_1,x_1)}(\hat{L}_0) - \alpha_{(\Lambda_2,x_2)}(\hat{L}_0)\big\rangle \\ &= \int_0^1 d\vartheta \int_{\mathsf{C}_k \cdot \mathsf{S}_F} d\mu(\Lambda, x) \sum_i \big[\partial_i (F^{(\Lambda,x)} \circ \phi_k^{-1})(\boldsymbol{t} + \vartheta(\boldsymbol{s} - \boldsymbol{t}))\big]\,(s_i - t_i)\, \big|\alpha_{(\Lambda,x)}(L_0)\big\rangle, \end{aligned} \tag{4.5d}$$

where $\boldsymbol{s} \doteq \phi_k(\Lambda_1, x_1)$ and $\boldsymbol{t} \doteq \phi_k(\Lambda_2, x_2)$ belong to the compact and *convex* set $\phi_k(\mathsf{C}_k)$ and ∂_i denotes the partial derivative with respect to the i-th coordinate component. This vector defines a positive functional on the algebra $\mathscr{B}(\mathscr{H}_w)$, and we want to show that it can be majorized by a positive normal functional in $\mathscr{B}(\mathscr{H})_*$. The integrals in (4.5d) exist in the uniform topology of $\mathscr{H}_w$ so that they commute with every bounded linear operator. Setting

$$\big|\Psi\big(\vartheta;(\Lambda,x)\big)\big\rangle \doteq \sum_i \big[\partial_i(F^{(\Lambda,x)}\circ\phi_k^{-1})(\boldsymbol{t}+\vartheta(\boldsymbol{s}-\boldsymbol{t}))\big](s_i-t_i)\,\big|\alpha_{(\Lambda,x)}(L_0)\big\rangle, \tag{4.6a}$$

we thus get for *positive* $B\in\mathscr{B}(\mathscr{H}_w)$

$$\begin{aligned}
&\big\langle \alpha_{(\Lambda_1,x_1)}(\hat{L}_0)-\alpha_{(\Lambda_2,x_2)}(\hat{L}_0)\big|B\big|\alpha_{(\Lambda_1,x_1)}(\hat{L}_0)-\alpha_{(\Lambda_2,x_2)}(\hat{L}_0)\big\rangle\\
&\quad=\iint\limits_{[0,1]\times[0,1]} d\vartheta\,d\vartheta' \iint\limits_{\mathsf{C}_k\cdot\mathsf{S}_F\times\mathsf{C}_k\cdot\mathsf{S}_F} d\mu(\Lambda,x)\,d\mu(\Lambda',x')\,\big\langle\Psi\big(\vartheta';(\Lambda',x')\big)\big|B\big|\Psi\big(\vartheta;(\Lambda,x)\big)\big\rangle\\
&\qquad\leqslant \mu(\mathsf{C}_k\cdot\mathsf{S}_F)\int_0^1 d\vartheta\int_{\mathsf{C}_k\cdot\mathsf{S}_F} d\mu(\Lambda,x)\,\big\langle\Psi\big(\vartheta;(\Lambda,x)\big)\big|B\big|\Psi\big(\vartheta;(\Lambda,x)\big)\big\rangle.
\end{aligned}\tag{4.6b}$$

Here use was made of the fact that the second line is invariant with respect to an exchange of primed and unprimed integration variables and that the integrand can thus be estimated according to the following relation that holds for arbitrary vectors Ψ and Φ in $\mathscr{H}_w$ and positive $B\in\mathscr{B}(\mathscr{H}_w)$: $\langle\Psi|B|\Phi\rangle+\langle\Phi|B|\Psi\rangle\leqslant\langle\Psi|B|\Psi\rangle+\langle\Phi|B|\Phi\rangle$. In view of (4.6a) the integrand of (4.6b) is the product of $\big\langle\alpha_{(\Lambda,x)}(L_0)\big|B\big|\alpha_{(\Lambda,x)}(L_0)\big\rangle$ and a continuous function of $\boldsymbol{s}$, $\boldsymbol{t}$, ϑ and (Λ,x), which is therefore bounded on the respective compact domains $\phi_k(\mathsf{C}_k)$, $[0,1]$ and $\mathsf{C}_k\cdot\mathsf{S}_F$ by $C(F;\mathsf{C}_k)^2\,|\boldsymbol{s}-\boldsymbol{t}|^2$ with a suitable constant $C(F;\mathsf{C}_k)$. As a consequence, we finally arrive at

$$\begin{aligned}
&\big\langle \alpha_{(\Lambda_1,x_1)}(\hat{L}_0)-\alpha_{(\Lambda_2,x_2)}(\hat{L}_0)\big|B\big|\alpha_{(\Lambda_1,x_1)}(\hat{L}_0)-\alpha_{(\Lambda_2,x_2)}(\hat{L}_0)\big\rangle\\
&\qquad\leqslant C(F;\mathsf{C}_k)^2\,|\boldsymbol{s}-\boldsymbol{t}|^2\mu(\mathsf{C}_k\cdot\mathsf{S}_F)\int_{\mathsf{C}_k\cdot\mathsf{S}_F} d\mu(\Lambda,x)\,\big\langle\alpha_{(\Lambda,x)}(L_0)\big|B\big|\alpha_{(\Lambda,x)}(L_0)\big\rangle,
\end{aligned}\tag{4.6c}$$

where the right-hand side defines the aspired positive normal functional on $\mathscr{B}(\mathscr{H}_w)$ majorizing the vector functional corresponding to $\big|\alpha_{(\Lambda_1,x_1)}(\hat{L}_0)-\alpha_{(\Lambda_2,x_2)}(\hat{L}_0)\big\rangle$.

Let $P^\bullet$ denote the orthogonal projection from $\mathscr{H}_w$ onto the subspace $\mathscr{H}^\bullet$. Then the integral in (4.6c) defines a positive normal functional on the preselected maximal abelian von Neumann algebra $\mathfrak{M}$ through

$$\varphi^{[\hat{L}_0;\mathsf{C}_k]}(T)\doteq\int_{\mathsf{C}_k\cdot\mathsf{S}_F} d\mu(\Lambda,x)\,\big\langle\alpha_{(\Lambda,x)}(L_0)\big|P^\bullet T P^\bullet\big|\alpha_{(\Lambda,x)}(L_0)\big\rangle,\quad T\in\mathfrak{M}, \tag{4.7a}$$

which, by [28, Proposition IV.8.34] in connection with (4.1c), corresponds to a unique integrable field $\Big\{\varphi_\xi^{[\hat{L}_0;\mathsf{C}_k]}:\xi\in\overline{\mathsf{X}}\Big\}$ of positive normal functionals on the von Neumann algebras $\mathbb{C}\cdot\mathbf{1}_\xi$ from the direct integral decomposition of $\mathfrak{M}$. Explicitly,

$$\varphi^{[\hat{L}_0;\mathsf{C}_k]}(T)=\int_{\overline{\mathsf{X}}} d\overline{\nu}(\xi)\,g_T(\xi)\,\varphi_\xi^{[\hat{L}_0;\mathsf{C}_k]}(\mathbf{1}_\xi) \tag{4.7b}$$

with an appropriate function $g_T\in L^\infty\big(\overline{\mathsf{X}},d\overline{\nu}(\xi)\big)$. On the other hand, specializing to transformations (Λ_1,x_1) and (Λ_2,x_2) in the countable subgroup P^{c}, the unique disintegration of $\big|\alpha^\bullet_{(\Lambda_1,x_1)}(\hat{L}_0)-\alpha^\bullet_{(\Lambda_2,x_2)}(\hat{L}_0)\big\rangle^\bullet=P^\bullet\big|\alpha_{(\Lambda_1,x_1)}(\hat{L}_0)-\alpha_{(\Lambda_2,x_2)}(\hat{L}_0)\big\rangle$ is given by equation (4.3)

$$\overline{W}\,\big|\alpha^\bullet_{(\Lambda_1,x_1)}(\hat{L}_0)-\alpha^\bullet_{(\Lambda_2,x_2)}(\hat{L}_0)\big\rangle^\bullet=\int^\oplus_{\overline{\mathsf{X}}_1} d\overline{\nu}(\xi)\,\big|\alpha^\bullet_{(\Lambda_1,x_1)}(\hat{L}_0)-\alpha^\bullet_{(\Lambda_2,x_2)}(\hat{L}_0)\big\rangle_\xi. \tag{4.7c}$$

Making use of the decomposition (4.1c) of the operator $T\in\mathfrak{M}$, its expectation value in the corresponding vector state is, since $\overline{\mathsf{X}}$ and $\overline{\mathsf{X}}_1$ differ only by a $\overline{\nu}$-null set:

$$\begin{aligned}
&{}^\bullet\big\langle\alpha^\bullet_{(\Lambda_1,x_1)}(\hat{L}_0)-\alpha^\bullet_{(\Lambda_2,x_2)}(\hat{L}_0)\big|T\big|\alpha^\bullet_{(\Lambda_1,x_1)}(\hat{L}_0)-\alpha^\bullet_{(\Lambda_2,x_2)}(\hat{L}_0)\big\rangle^\bullet\\
&\quad=\int^\oplus_{\overline{\mathsf{X}}_1} d\overline{\nu}(\xi)\,g_T(\xi)\;{}_\xi\big\langle\alpha^\bullet_{(\Lambda_1,x_1)}(\hat{L}_0)-\alpha^\bullet_{(\Lambda_2,x_2)}(\hat{L}_0)\big|\alpha^\bullet_{(\Lambda_1,x_1)}(\hat{L}_0)-\alpha^\bullet_{(\Lambda_2,x_2)}(\hat{L}_0)\big\rangle_\xi.
\end{aligned}\tag{4.7d}$$

Specializing to *positive* $\boldsymbol{T}$, these results in combination with (4.6c) yield

$$\int_{\mathsf{X}_1} d\overline{\nu}(\xi)\, g_T(\xi)\; {}_\xi\big\langle \alpha^\bullet_{(\Lambda_1,x_1)}(\hat{L}_0) - \alpha^\bullet_{(\Lambda_2,x_2)}(\hat{L}_0) \big| \alpha^\bullet_{(\Lambda_1,x_1)}(\hat{L}_0) - \alpha^\bullet_{(\Lambda_2,x_2)}(\hat{L}_0) \big\rangle_\xi \leqslant C(F;\mathsf{C}_k)^2\, |\boldsymbol{s}-\boldsymbol{t}|^2 \mu(\mathsf{C}_k \cdot \mathsf{S}_F) \int_{\mathsf{X}_1} d\overline{\nu}(\xi)\, g_T(\xi)\, \varphi_\xi^{[\hat{L}_0;\mathsf{C}_k]}(\mathbf{1}_\xi). \tag{4.8a}$$

For arbitrary measurable subsets M of X_1 corresponding to orthogonal projections $P_{\mathsf{M}} \in \mathfrak{M}$ and thus to characteristic functions $\chi_{\mathsf{M}} \in L^\infty\big(\mathsf{X}_1, d\overline{\nu}(\xi)\big)$ relation (4.8a) reads

$$\int_{\mathsf{M}} d\overline{\nu}(\xi)\, \big\| \big| \alpha^\bullet_{(\Lambda_1,x_1)}(\hat{L}_0) - \alpha^\bullet_{(\Lambda_2,x_2)}(\hat{L}_0) \big\rangle_\xi \big\|^2 \leqslant C(F;\mathsf{C}_k)^2\, |\boldsymbol{s}-\boldsymbol{t}|^2 \mu(\mathsf{C}_k \cdot \mathsf{S}_F) \int_{\mathsf{M}} d\overline{\nu}(\xi)\, \varphi_\xi^{[\hat{L}_0;\mathsf{C}_k]}(\mathbf{1}_\xi). \tag{4.8b}$$

Due to arbitrariness of $\mathsf{M} \subseteq \mathsf{X}_1$, we then infer, making use of elementary results of integration theory [19, Chapter V, viz. § 25, Theorem D], that for $\overline{\nu}$-almost all $\xi \in \mathsf{X}_1$

$$\big\| \big| \alpha^\bullet_{(\Lambda_1,x_1)}(\hat{L}_0) - \alpha^\bullet_{(\Lambda_2,x_2)}(\hat{L}_0) \big\rangle_\xi \big\|^2 \leqslant \big| \phi_k(\Lambda_1,x_1) - \phi_k(\Lambda_2,x_2) \big|^2 C(F;\mathsf{C}_k)^2 \mu(\mathsf{C}_k \cdot \mathsf{S}_F) \cdot \varphi_\xi^{[\hat{L}_0;\mathsf{C}_k]}(\mathbf{1}_\xi), \tag{4.8c}$$

where the points $\boldsymbol{s}$ and $\boldsymbol{t}$ from coordinate space were replaced by their pre-images (Λ_1,x_1) and (Λ_2,x_2) in $\mathsf{V}_k \cap \overline{\mathsf{P}}^c$. The important thing to notice at this point is that, apart from the factor $\big|\phi_k(\Lambda_1,x_1) - \phi_k(\Lambda_2,x_2)\big|$, the terms on the right-hand side of (4.8c) only hinge upon the operator $\hat{L}_0$ and on the neighbourhood V_k with compact closure C_k containing (Λ_1,x_1), $(\Lambda_2,x_2) \in \overline{\mathsf{P}}^c$. Therefore, this estimate also holds for any other pair of Lorentz transformations in $\mathsf{V}_k \cap \overline{\mathsf{P}}^c$; of course, in each of the resulting countably many relations one possibly loses a further $\overline{\nu}$-null subset of X_1. The reasoning leading up to this point can then be applied to any combination of an operator in the denumerable selection $\mathfrak{L}_0^c$ with an open set from the countable cover of $\mathsf{P}_+^\uparrow$ to produce in each case a relation of the form of (4.8c) for the respective Poincaré transformations in $\overline{\mathsf{P}}^c$. Simultaneously, the domain of indices ξ, for which *all* of these inequalities are valid, shrinks to an appropriate $\overline{\nu}$-measurable non-null subset X_2 of X_1.

Consider an arbitrary Poincaré transformation (Λ_0,x_0), belonging to at least one set V_j, with approximating sequence $\big\{(\Lambda_n,x_n)\big\}_{n\in\mathbb{N}} \subseteq \overline{\mathsf{P}}^c \cap \mathsf{V}_j$. It is a Cauchy sequence in the initial topology of $\mathsf{P}_+^\uparrow$, and, due to (4.8c), each corresponding sequence for $\xi \in \mathsf{X}_2$

$$\big\{ \big| \alpha^\bullet_{(\Lambda_n,x_n)}(\hat{L}_0) \big\rangle_\xi \big\}_{n\in\mathbb{N}} \subseteq \mathscr{H}_\xi, \quad \hat{L}_0 \in \mathfrak{L}_0^c, \tag{4.9a}$$

likewise has the Cauchy property with respect to the Hilbert space norms. Their limits exist in the complete spaces $\mathscr{H}_\xi$ and are obviously independent of the approximating sequence of Lorentz transformations from $\overline{\mathsf{P}}^c$. According to the notion of measurability for vector fields [14, Definition II.4.1], the one that arises as the pointwise limit of measurable vector fields,

$$\mathsf{X}_2 \ni \xi \mapsto \lim_{n\to\infty} \big| \alpha^\bullet_{(\Lambda_n,x_n)}(\hat{L}_0) \big\rangle_\xi \in \mathscr{H}_\xi, \tag{4.9b}$$

is itself measurable with respect to the restriction of $\overline{\nu}$ to X_2 and turns out to be a representative of the vector $\big|\alpha^\bullet_{(\Lambda_0,x_0)}(\hat{L}_0)\big\rangle^\bullet \in \mathscr{H}^\bullet$ [12, Section II.1.5, Proof of Proposition 5(ii)]. The obvious next step therefore is to *define* the vector $\big|\alpha^\bullet_{(\Lambda_0,x_0)}(\hat{L}_0)\big\rangle_\xi \in \mathscr{H}_\xi$ by the right-hand side of (4.9b) to implement relation (2.9b). But first and foremost this limit depends on $\hat{L}_0$ and on (Λ_0,x_0) separately, so one has to ensure that different combinations that represent the same operator in $L' \in \overline{\mathfrak{L}_0}$ give rise to coinciding limits. Let $\hat{L}_1$, $\hat{L}_2 \in \mathfrak{L}_0^c$ and let (Λ_1, x_1), $(\Lambda_2, x_2) \in \mathsf{P}_+^\uparrow$ with $L' = \alpha^\bullet_{(\Lambda_1,x_1)}(\hat{L}_1) = \alpha^\bullet_{(\Lambda_2,x_2)}(\hat{L}_2)$. Then, according to the constructions of Section 2.1, $(\Lambda_1, x_1)^{-1}(\Lambda_2, x_2)$ belongs to $\overline{\mathsf{P}}^c$ and $\hat{L}_1 = \alpha^\bullet_{(\Lambda_1,x_1)^{-1}(\Lambda_2,x_2)}(\hat{L}_2)$ so that for any sequence $\{(\Lambda_{1,n}, x_{1,n})\}_{n\in\mathbb{N}} \subseteq \overline{\mathsf{P}}^c$ approximating (Λ_1, x_1)

$$\alpha^\bullet_{(\Lambda_{1,n},x_{1,n})}(\hat{L}_1) = \alpha^\bullet_{(\Lambda_{1,n},x_{1,n})(\Lambda_1,x_1)^{-1}(\Lambda_2,x_2)}(\hat{L}_2). \tag{4.9c}$$

Since the product of transformations on the right-hand side constitutes a sequence in $\overline{\mathsf{P}}^c$ with limit (Λ_2, x_2) allowing for passage to the limit of (4.9b), this relation establishes the independence of these limits from the selected representation of L'. The only problem that is still left open with respect to an unambiguous definition of vectors of the form $|L'\rangle_\xi$, $L' \in \overline{\mathfrak{L}_0}$, occurs when the vacuum annihilation operator L' happens to be an element of $\mathfrak{L}^c$ so that its components in the Hilbert spaces $\mathscr{H}_\xi$ have already been fixed in the initial step. But, as $\mathfrak{L}^c$ is a denumerable set, such a coincidence will be encountered at most countably often and can thus be redressed by exclusion of an appropriate $\overline{\nu}$-null subset from $\overline{\mathsf{X}}_2$. For all ξ in the resulting non-null set $\overline{\mathsf{X}}_3$ we can therefore define

$$|L'\rangle_\xi \doteq \lim_{n\to\infty} \left|\alpha^\bullet_{(\Lambda_{1,n},x_{1,n})}(\hat{L}_1)\right\rangle_\xi, \quad L' = \alpha^\bullet_{(\Lambda_1,x_1)}(\hat{L}_1) \in \overline{\mathfrak{L}_0}, \tag{4.9d}$$

such that

$$\overline{W}\,|L'\rangle^\bullet = \int^\oplus_{\overline{\mathsf{X}}_3} d\overline{\nu}(\xi)\; |L'\rangle_\xi. \tag{4.9e}$$

The extension of $|\,.\,\rangle_\xi$ to the set $\overline{\mathfrak{L}_0}$ being given by (4.9d) for $\xi \in \overline{\mathsf{X}}_3$, continuity of the mappings (4.4) has to be established (cf. relation (2.5) in Theorem 2.2). But this is immediate by a 3ε-argument from the very definition (4.9d) (involving Poincaré transformations from $\overline{\mathsf{P}}^c$) in connection with (4.8c).

(iii) The last property of restricted particle weights to be established is the existence of unitary representations $x \mapsto U_\xi(x)$ which satisfy relations (2.6) in each $(\pi_\xi, \mathscr{H}_\xi)$, respectively. First, select one element L of the countable space $\mathfrak{L}^c$ together with a single spacetime translation y in the denumerable dense subgroup T^c of $\mathbb{R}^{s+1}$. By assumption (2.8), operators in the von Neumann algebra $\mathfrak{M}$ commute with $\{U^\bullet(x) : x \in \mathbb{R}^{s+1}\}$, which means that for any measurable subset M of $\overline{\mathsf{X}}_3$ with associated orthogonal projection $P_\mathsf{M} \in \mathfrak{M}$ there holds the equation

$$\int_\mathsf{M} d\overline{\nu}(\xi)\; \left\| |\alpha^\bullet_y(L)\rangle_\xi \right\|^2 = \left\| P_\mathsf{M} U^\bullet(y) |L\rangle^\bullet \right\|^2 = \left\| P_\mathsf{M} |L\rangle^\bullet \right\|^2 = \int_\mathsf{M} d\overline{\nu}(\xi)\; \left\| |L\rangle_\xi \right\|^2. \tag{4.10a}$$

This result being valid for arbitrary measurable sets M, we infer by [19, Chapter V, § 25, Theorem E] that for $\overline{\nu}$-almost all $\xi \in \overline{\mathsf{X}}_3$

$$\left\| |\alpha^\bullet_y(L)\rangle_\xi \right\| = \left\| |L\rangle_\xi \right\|. \tag{4.10b}$$

Performed for any of the *countable* number of combinations of elements in $\mathfrak{L}^c$ and T^c, the above derivation implies that (4.10b) is true in all of these cases when the domain of ξ is restricted to a $\overline{\nu}$-measurable set $\overline{\mathsf{X}}_4$, differing from $\overline{\mathsf{X}}_3$ only by a null set. For $\xi \in \overline{\mathsf{X}}_4$ and arbitrary $y \in \mathsf{T}^c$ we can then define the following mappings on the countable dense subspaces $\mathscr{H}_\xi^c \subseteq \mathscr{H}_\xi$, the images of $\mathfrak{L}^c$ under $|\,.\,\rangle_\xi$:

$$\overline{U}_\xi(y) : \mathscr{H}_\xi^c \to \mathscr{H}_\xi^c \qquad \overline{U}_\xi(y)|L\rangle_\xi \doteq |\alpha^\bullet_y(L)\rangle_\xi. \tag{4.10c}$$

These are determined unambiguously according to (4.10b). By the same relation, they are norm-preserving and, moreover, turn out to be $(\mathbb{Q} + i\mathbb{Q})$-linear operators on $\mathscr{H}_\xi^c$.

Definition (4.10c) is to be extended in two respects: All spacetime translations y and all vectors from $\mathscr{H}_\xi$ shall be permissible. Now, let L be an arbitrary element of $\mathfrak{L}^c$,

$$L = \sum_{i=1}^{N} A_i L_i, \quad A_i \in \overline{\mathfrak{A}^c}, \quad L_i \in \overline{\mathfrak{L}_0^c}, \tag{4.11a}$$

and consider the limit $x \in \mathbb{R}^{s+1}$ of the sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathsf{T}^c$. Then, by definition (4.10c) in connection with property (4.2), for $\xi \in \mathsf{X}_4$ the translates of the vectors $|L\rangle_\xi$ by x_k and x_l are subject to the following relation:

$$\begin{aligned}\overline{U}_\xi(x_k)|L\rangle_\xi - \overline{U}_\xi(x_l)|L\rangle_\xi &= \sum_{i=1}^{N} \pi_\xi\big(\alpha^\bullet_{x_k}(A_i)\big)\big|\alpha^\bullet_{x_k}(L_i)\big\rangle_\xi - \sum_{i=1}^{N} \pi_\xi\big(\alpha^\bullet_{x_l}(A_i)\big)\big|\alpha^\bullet_{x_l}(L_i)\big\rangle_\xi \\ &= \sum_{i=1}^{N} \pi_\xi\big(\alpha^\bullet_{x_k}(A_i) - \alpha^\bullet_{x_l}(A_i)\big)\big|\alpha^\bullet_{x_k}(L_i)\big\rangle_\xi + \sum_{i=1}^{N} \pi_\xi\big(\alpha^\bullet_{x_l}(A_i)\big)\big(\big|\alpha^\bullet_{x_k}(L_i)\big\rangle_\xi - \big|\alpha^\bullet_{x_l}(L_i)\big\rangle_\xi\big).\end{aligned} \tag{4.11b}$$

As the group of automorphisms $\big\{\alpha_{(\Lambda,x)} : (\Lambda,x) \in \mathsf{P}_+^\uparrow\big\}$ is strongly continuous and X_4 is a subset of X_2, so that relation (4.8c) holds, the sequences of operators $\big\{\pi_\xi\big(\alpha^\bullet_{x_k}(A_i)\big)\big\}_{k\in\mathbb{N}}$ and of vectors $\big\{\big|\alpha^\bullet_{x_k}(L_i)\big\rangle_\xi\big\}_{k\in\mathbb{N}}$ both possess the Cauchy property in their respective topologies and are thus convergent as well as bounded. Therefore, the right-hand side of (4.11b) can be made arbitrarily small for all pairs k, $l \in \mathbb{N}$ exceeding a certain number. The sequences $\big\{\overline{U}_\xi(x_n)|L\rangle_\xi\big\}_{n\in\mathbb{N}}$ built from the terms appearing on the left-hand side of inequality (4.11b) thus turn out to be Cauchy sequences that converge in the Hilbert spaces $\mathscr{H}_\xi$. The arising limits are independent of the sequence in T^c approximating x, as can be seen by anew applying the above reasoning. So the following relation unambiguously defines the mappings $\overline{U}_\xi(x)$ for arbitrary $x \in \mathbb{R}^{s+1}$, $L \in \mathfrak{L}^c$ and $\xi \in \mathsf{X}_4$:

$$\overline{U}_\xi(x)|L\rangle_\xi \doteq \lim_{n\to\infty} \overline{U}_\xi(x_n)|L\rangle_\xi = \lim_{n\to\infty} \big|\alpha^\bullet_{x_n}(L)\big\rangle_\xi. \tag{4.11c}$$

Again these mappings act as $(\mathbb{Q}+i\mathbb{Q})$-linear operators on the spaces $\mathscr{H}_\xi^c$ and preserve the Hilbert space norm. As a consequence, they can, by the standard procedure used for completions of uniform spaces, be continuously extended in a unique fashion to all of the Hilbert spaces since their countable domain constitutes a dense subset of $\mathscr{H}_\xi$ according to part (i) of this proof. Changing the notation from $\overline{U}_\xi$ to U_ξ for these extensions, their definition on arbitrary vectors $\Psi_\xi \in \mathscr{H}_\xi$ approximated by a sequence $\big\{\big|L^{(l)}\big\rangle_\xi\big\}_{l\in\mathbb{N}} \subseteq \mathscr{H}_\xi^c$ then reads for any $x \in \mathbb{R}^{s+1}$ and $\xi \in \mathsf{X}_4$

$$U_\xi(x)\Psi_\xi \doteq \lim_{l\to\infty} \overline{U}_\xi(x)\big|L^{(l)}\big\rangle_\xi \tag{4.11d}$$

and is again independent of the selected sequence. For any $L \in \mathfrak{L}^c$ the associated vector field $\big\{U_\xi(x)|L\rangle_\xi : \xi \in \mathsf{X}_4\big\}$, being the pointwise limit of a sequence of measurable vector fields by (4.11c) and hence itself measurable according to [14, Definition II.4.1], corresponds to the limit $\big|\alpha^\bullet_x(L)\big\rangle^\bullet \in \mathscr{H}^\bullet$ (where we neglect the $\overline{\nu}$-null difference between X and X_4):

$$\overline{W}\, U^\bullet(x)|L\rangle^\bullet = \overline{W}\,\big|\alpha^\bullet_x(L)\big\rangle^\bullet = \int_{\mathsf{X}_4}^{\oplus} d\overline{\nu}(\xi)\; U_\xi(x)|L\rangle_\xi. \tag{4.11e}$$

We now have to check that the families of mappings $\big\{U_\xi(x) : x \in \mathbb{R}^{s+1}\big\} \subseteq \mathscr{B}(\mathscr{H}_\xi)$, $\xi \in \mathsf{X}_4$, obey (2.6). Their $\mathbb{C}$-linearity is an immediate consequence of the way in which they were introduced above; the same holds true for the property of norm-preservation.

Another property readily checked by use of relations (4.11d) and (4.11c) (in connection with (4.10b)) is the fact that for arbitrary x, $y \in \mathbb{R}^{s+1}$

$$U_\xi(x) \cdot U_\xi(y) = U_\xi(x+y). \tag{4.12}$$

As evidently $U_\xi(0) = \mathbf{1}_\xi$, each operator $U_\xi(x)$ thus has the inverse $U_\xi(-x)$ and proves to be an isometric isomorphism of $\mathscr{H}_\xi$. Hence the family of these operators indeed turns out to be a unitary representation of spacetime translations in $\mathscr{B}(\mathscr{H}_\xi)$. Its strong continuity is easily seen: Consider the representation (4.11a) of $L \in \mathfrak{L}^c$ and two sequences $\{x_k\}_{k\in\mathbb{N}}$, $\{y_l\}_{l\in\mathbb{N}}$ in T^c converging to x and y, respectively. Equation (4.11b) holds with y_l replacing x_l and passing to the limit in compliance with (4.11c) yields

$$\begin{aligned}\big\|\overline{U}_\xi(x)|L\rangle_\xi - \overline{U}_\xi(y)|L\rangle_\xi\big\| \\ \leqslant \sum_{i=1}^N \big\|\alpha_x^\bullet(A_i) - \alpha_y^\bullet(A_i)\big\| \big\||\alpha_x^\bullet(L_i)\rangle_\xi\big\| + \sum_{i=1}^N \|A_i\| \big\||\alpha_x^\bullet(L_i)\rangle_\xi - |\alpha_y^\bullet(L_i)\rangle_\xi\big\|.\end{aligned} \tag{4.13}$$

This inequality shows that the right-hand side can be made arbitrarily small for all y in an appropriate neighbourhood of x; as regards the first term this is brought about by strong continuity of the automorphism group $\{\alpha_{(\Lambda,x)} : (\Lambda,x) \in \mathsf{P}_+^\uparrow\}$, whereas for the second term it is a consequence of continuity of (4.4) demonstrated above. Strong continuity of the group in question is thus established for vectors in the dense subset $\mathscr{H}_\xi^c$ and can, by use of a 3ε-argument, be readily extended to all of $\mathscr{H}_\xi$.

Before considering the support of the spectral measure $E_\xi(\ .\ )$ associated with this strongly continuous unitary representation, we mention a result on the interchange of integrations with respect to Lebesgue measure on $\mathbb{R}^{s+1}$ and the bounded positive measure $\overline{\nu}$ on $\overline{\mathsf{X}}_4$. This is necessary as Fubini's Theorem does not apply. Let g be a continuous bounded function in $L^1(\mathbb{R}^{s+1}, d^{s+1}x)$, then $x \mapsto g(x)\ {}_\xi\langle L_1|U_\xi(x)|L_2\rangle_\xi$ is an integrable mapping for any L_1, $L_2 \in \mathfrak{L}^c$ and $\xi \in \overline{\mathsf{X}}_4$. Moreover, it is Riemann integrable over any compact $(s+1)$-dimensional interval K, and this integral is the limit of a Riemann sequence (cf. [20, Kapitel XXIII, Abschnitt 197 and Lebesgueсches Integrabilitätskriterium 199.3]):

$$\int_K d^{s+1}x\, g(x)\ {}_\xi\big\langle L_1\big|U_\xi(x)\big|L_2\big\rangle_\xi = \lim_{i\to\infty} \sum_{m=1}^{n_i} \big|\mathcal{Z}_m^{(i)}\big|\, g\big(x_m^{(i)}\big)\ {}_\xi\big\langle L_1\big|U_\xi\big(x_m^{(i)}\big)\big|L_2\big\rangle_\xi, \tag{4.14a}$$

where $\big\{\mathcal{Z}_m^{(i)} : m = 1,\ldots,n_i\big\}$ denotes the i-th subdivision of K, $\big|\mathcal{Z}_m^{(i)}\big|$ are the Lebesgue measures of these sets, and $x_m^{(i)} \in \mathcal{Z}_m^{(i)}$ are corresponding intermediate points. The sums on the right-hand side of this equation turn out to be $\overline{\nu}$-measurable and so is the limit on the left-hand side. This property is preserved in passing to the limit $K \nearrow \mathbb{R}^{s+1}$:

$$\overline{\mathsf{X}}_4 \ni \xi \mapsto \int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x)\ {}_\xi\big\langle L_1\big|U_\xi(x)\big|L_2\big\rangle_\xi \in \mathbb{C}$$

is $\overline{\nu}$-measurable and, in addition, integrable since

$$\begin{aligned}\int_{\overline{\mathsf{X}}_4} d\overline{\nu}(\xi) \left| \int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x)\ {}_\xi\big\langle L_1\big|U_\xi(x)\big|L_2\big\rangle_\xi \right| \\ \leqslant \|g\|_1 \int_{\overline{\mathsf{X}}_4} d\overline{\nu}(\xi)\ \||L_1\rangle_\xi\|\||L_2\rangle_\xi\| \leqslant \|g\|_1\ \||L_1\rangle^\bullet\|\||L_2\rangle^\bullet\|.\end{aligned} \tag{4.14b}$$

The counterpart of (4.14a) is valid in $\mathscr{H}^\bullet$, too, and, if M denotes a measurable subset of

X_4 with associated projection $P_{\mathsf{M}} \in \mathfrak{M}$, we get, by use of (4.11e) and (4.1c),

$$\int_{K} d^{s+1}x\, g(x) \langle L_1 | P_{\mathsf{M}} U^{\bullet}(x) | L_2 \rangle = \lim_{i\to\infty} \sum_{m=1}^{n_i} \big| Z_m^{(i)} \big|\, g\big(x_m^{(i)}\big) \langle L_1 | P_{\mathsf{M}} U^{\bullet}\big(x_m^{(i)}\big) | L_2 \rangle$$
$$= \lim_{i\to\infty} \int_{\mathsf{M}} d\bar{\nu}(\xi) \sum_{m=1}^{n_i} \big| Z_m^{(i)} \big|\, g\big(x_m^{(i)}\big)\, {}_{\xi}\big\langle L_1 \big| U_{\xi}\big(x_m^{(i)}\big) \big| L_2 \big\rangle_{\xi}$$
$$= \int_{\mathsf{M}} d\bar{\nu}(\xi) \int_{K} d^{s+1}x\, g(x)\, {}_{\xi}\big\langle L_1 \big| U_{\xi}(x) \big| L_2 \big\rangle_{\xi}. \tag{4.14c}$$

In the last equation, use was made of Lebesgue's Dominated Convergence Theorem taking into account that the integrable function $\xi \mapsto \|g\|_1 \||L_1\rangle_{\xi}\| \||L_2\rangle_{\xi}\|$ majorizes both sides of (4.14a). Again, equation (4.14c) stays true in passing to the limit $K \nearrow \mathbb{R}^{s+1}$, resulting in the announced statement on commutability of integrations:

$$\int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x) \langle L_1 | P_{\mathsf{M}} U^{\bullet}(x) | L_2 \rangle = \int_{\mathsf{M}} d\bar{\nu}(\xi) \int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x)\, {}_{\xi}\big\langle L_1 \big| U_{\xi}(x) \big| L_2 \big\rangle_{\xi}. \tag{4.14d}$$

The support of the spectral measure $E_{\xi}(\,.\,)$ associated with the generators P_{ξ} of $x \mapsto U_{\xi}(x)$ can now be investigated as in the proof of [25, Theorem 3.12]. Note that the complement of the closed set $\overline{V}_+ - q \subseteq \mathbb{R}^{s+1}$ can be covered by an increasing sequence $\{\Gamma_N\}_{N\in\mathbb{N}}$ of compact subsets, each admitting an infinitely often differentiable function $\tilde{g}_N$ with support in $\complement(\overline{V}_+ - q)$ that has the property $0 \leqslant \chi_{\Gamma_N} \leqslant \tilde{g}_N$. As before, let M be a measurable subset of X_4 with associated orthogonal projection $P_{\mathsf{M}} \in \mathfrak{M}$, then, by assumption on the spectral support of the unitary representation implementing spacetime translations in the underlying particle weight,

$$\int_{\mathbb{R}^{s+1}} d^{s+1}x\, g_N(x) \langle L_1 | P_{\mathsf{M}} U^{\bullet}(x) | L_2 \rangle = 0 \tag{4.15a}$$

for any $N \in \mathbb{N}$ and any pair of vectors $|L_1\rangle$ and $|L_2\rangle$, where L_1, $L_2 \in \mathfrak{L}^c$. Hence, by (4.14d) and arbitrariness of M, we conclude once more that for $\bar{\nu}$-almost all $\xi \in \mathsf{X}_4$

$$\int_{\mathbb{R}^{s+1}} d^{s+1}x\, g_N(x)\, {}_{\xi}\big\langle L_1 \big| U_{\xi}(x) \big| L_2 \big\rangle_{\xi} = 0. \tag{4.15b}$$

This equation holds for any element of the countable set of triples $\big(g_N, |L_1\rangle_{\xi}, |L_2\rangle_{\xi}\big)$ if ξ belongs to an appropriate non-null set $\mathsf{X}_5 \subseteq \mathsf{X}_4$ and even stays valid for these $\xi \in \mathsf{X}_5$ if the special vectors $|L_1\rangle_{\xi}$ and $|L_2\rangle_{\xi}$ are replaced by arbitrary ones. Stone's Theorem then implies (cf. [25, equation (5.23)]) that $\tilde{g}_N(P_{\xi}) = 0$ and therefore, since $\tilde{g}_N$ majorizes χ_{Γ_N}, we have $E_{\xi}(\Gamma_N) = \chi_{\Gamma_N}(P_{\xi}) = 0$ for any $N \in \mathbb{N}$. As the spectral measure $E_{\xi}(\,.\,)$ is regular, passing to the limit $N \to \infty$ yields the desired result

$$E_{\xi}\big(\complement(\overline{V}_+ - q)\big) = 0, \qquad \xi \in \mathsf{X}_5. \tag{4.15c}$$

By definition (4.10c) in connection with (4.2), one has for arbitrary $A' \in \mathfrak{A}^c$ and $L \in \mathfrak{L}^c$ and for any translation $x' \in \mathsf{T}^c$

$$\pi_{\xi}\big(\alpha^{\bullet}_{x'}(A')\big)|L\rangle_{\xi} = \big|\alpha^{\bullet}_{x'}(A')L\big\rangle_{\xi} = \overline{U}_{\xi}(x')\big|A'\alpha^{\bullet}_{-x'}(L)\big\rangle_{\xi} = \overline{U}_{\xi}(x')\pi_{\xi}(A')\overline{U}_{\xi}(x')^{*}|L\rangle_{\xi}, \tag{4.16a}$$

and, since the vectors $|L\rangle_{\xi} \in \mathscr{H}^c_{\xi}$, $L \in \mathfrak{L}^c$, constitute a dense subset of $\mathscr{H}_{\xi}$,

$$\pi_{\xi}\big(\alpha^{\bullet}_{x'}(A')\big) = \overline{U}_{\xi}(x')\pi_{\xi}(A')\overline{U}_{\xi}(x')^{*}. \tag{4.16b}$$

This equation readily extends to all translations x in $\mathbb{R}^{s+1}$ and, by uniform density of $\mathfrak{A}^c$ in $\mathfrak{A}^{\bullet}$, also to any operator A in the C^*-algebra $\mathfrak{A}^{\bullet}$, thus proving the counterpart of equation (2.6a):

$$\pi_{\xi}\big(\alpha^{\bullet}_{x}(A)\big) = U_{\xi}(x)\pi_{\xi}(A)U_{\xi}(x)^{*}, \quad A \in \mathfrak{A}^{\bullet}, \quad x \in \mathbb{R}^{s+1}. \tag{4.16c}$$

The action of the unitary operators $\{U_\xi(x) : x \in \mathbb{R}^{s+1}\}$ on the vectors $\{|L'\rangle_\xi : L' \in \overline{\mathfrak{L}_0}\}$ according to (2.6b) is an immediate consequence of the defining relations (4.11c) and (4.11d) in combination with (4.9d) and the continuity statement (4.4). In the present setting, we thus have

$$U_\xi(x)|L'\rangle_\xi \doteq \big|\alpha^\bullet_x(L')\big\rangle_\xi, \quad L' \in \overline{\mathfrak{L}_0}. \tag{4.17}$$

Let $L \in \mathfrak{L}^c$ have energy-momentum transfer Γ_L. Defined as the support of the Fourier transform of an operator-valued distribution, Γ_L is a closed Borel set so that the reasoning that led to (4.15c) can again be applied with Γ_L in place of $\overline{V}_+ - q$ and L instead of L_1 and L_2. Here the consequence of the counterpart of (4.15b) is that the relation $E_\xi\big(\complement\Gamma_L\big)|L\rangle_\xi = 0$ holds for $\bar{\nu}$-almost all $\xi \in \overline{\mathsf{X}}_5$. By countability, this result is valid for arbitrary $L \in \mathfrak{L}^c$ if a $\bar{\nu}$-measurable non-null set $\overline{\mathsf{X}}_6 \subseteq \overline{\mathsf{X}}_5$ is appropriately selected. The complementary statement presents a restricted version of the counterpart of (2.6c):

$$E_\xi(\Gamma_L)|L\rangle_\xi = |L\rangle_\xi, \quad L \in \mathfrak{L}^c, \quad \xi \in \overline{\mathsf{X}}_6. \tag{4.18a}$$

Now, let $\hat{L}_0 \in \overline{\mathfrak{L}^c_0}$ have energy-momentum transfer $\Gamma_{\hat{L}_0}$, then that of its Poincaré transform $\alpha^\bullet_{(\Lambda,x)}(\hat{L}_0) \in \overline{\mathfrak{L}^c_0} \subseteq \mathfrak{L}^c$ is $\Lambda\Gamma_{\hat{L}_0}$ implying, according to (4.18a),

$$E_\xi(\Lambda\Gamma_{\hat{L}_0})\big|\alpha^\bullet_{(\Lambda,x)}(\hat{L}_0)\big\rangle_\xi = \big|\alpha^\bullet_{(\Lambda,x)}(\hat{L}_0)\big\rangle_\xi, \quad \xi \in \overline{\mathsf{X}}_6. \tag{4.18b}$$

This result can be applied to investigate generic elements of $\overline{\mathfrak{L}_0}$. For $(\Lambda_0, x_0) \in \mathsf{P}_+^\uparrow$ approximated by the sequence $\{(\Lambda_n, x_n)\}_{n\in\mathbb{N}} \subseteq \mathsf{P}^c$ we have, by virtue of (4.9d),

$$\big|\alpha^\bullet_{(\Lambda_0,x_0)}(\hat{L}_0)\big\rangle_\xi = \lim_{n\to\infty} \big|\alpha^\bullet_{(\Lambda_n,x_n)}(\hat{L}_0)\big\rangle_\xi,$$

and Lebesgue's Dominated Convergence Theorem in connection with Stone's Theorem yields for any function $g \in L^1\big(\mathbb{R}^{s+1}, d^{s+1}x\big)$ and any $\xi \in \overline{\mathsf{X}}_6$

$$\begin{aligned}\int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x)\, {}_\xi\big\langle \alpha^\bullet_{(\Lambda_0,x_0)}(\hat{L}_0)\big|U_\xi(x)\big|\alpha^\bullet_{(\Lambda_0,x_0)}(\hat{L}_0)\big\rangle_\xi \\ = (2\pi)^{(s+1)/2} \lim_{n\to\infty} {}_\xi\big\langle \alpha^\bullet_{(\Lambda_n,x_n)}(\hat{L}_0)\big|\tilde{g}(P_\xi)\big|\alpha^\bullet_{(\Lambda_n,x_n)}(\hat{L}_0)\big\rangle_\xi.\end{aligned} \tag{4.18c}$$

In the limit of large n one finds the energy-momentum transfer $\Lambda_n\Gamma_{\hat{L}_0}$ of $\alpha^\bullet_{(\Lambda_n,x_n)}(\hat{L}_0)$ in a small ε-neighbourhood of $\Lambda_0\Gamma_{\hat{L}_0}$. Therefore, in view of (4.18b), the right-hand side of (4.18c) vanishes for all n exceeding a certain $N \in \mathbb{N}$ if g is chosen in such a way that $\operatorname{supp}\tilde{g} \subseteq \complement(\Lambda_0\Gamma_{\hat{L}_0})$. Thus, the distribution $x \mapsto {}_\xi\big\langle \alpha^\bullet_{(\Lambda_0,x_0)}(\hat{L}_0)\big|U_\xi(x)\big|\alpha^\bullet_{(\Lambda_0,x_0)}(\hat{L}_0)\big\rangle_\xi$ has a Fourier transform supported by $\Lambda_0\Gamma_{\hat{L}_0}$. Hence

$$E_\xi(\Lambda_0\Gamma_{\hat{L}_0})\big|\alpha^\bullet_{(\Lambda_0,x_0)}(\hat{L}_0)\big\rangle_\xi = \big|\alpha^\bullet_{(\Lambda_0,x_0)}(\hat{L}_0)\big\rangle_\xi, \quad \xi \in \overline{\mathsf{X}}_6, \tag{4.18d}$$

which is the formulation of (4.18b) for *arbitrary* operators in $\overline{\mathfrak{L}_0}$. Equations (4.18a) and (4.18d) are readily generalized, making use of the order structure of spectral projections reflecting the inclusion relation of Borel subsets of $\mathbb{R}^{s+1}$. Thus operators from $\mathfrak{L}^c \cup \overline{\mathfrak{L}^c_0}$ having energy-momentum transfer in the Borel set Δ' satisfy

$$E_\xi(\Delta')|L\rangle_\xi = |L\rangle_\xi, \tag{4.18e}$$

so that the counterpart of (2.6c) is established for the remaining $\xi \in \overline{\mathsf{X}}_6$.

The above construction has supplied us with a measurable subset $\mathsf{X} \doteq \overline{\mathsf{X}}_6$ of the standard Borel space $\overline{\mathsf{X}}$ introduced at the outset (emerging from an application of [13, Theorem 8.5.2]) in such a way that, as care has been taken to ensure properties (2.4) through (2.6), to each $\xi \in \mathsf{X}$ there corresponds a restricted particle weight. Moreover, X is a non-null set and is itself a standard Borel space (cf. the definition in [3, Section 3.3]) carrying the bounded positive measure $\nu \doteq \bar{\nu} \upharpoonright \mathsf{X}$. What remains to be done now is a verification of the properties listed in (2.9).

(i) Arising as the restriction to a measurable subset in $\overline{X}$ of a field of irreducible representations, the field $\xi \mapsto (\pi_\xi, \mathscr{H}_\xi)$ on X is obviously ν-measurable and its components inherit the feature of irreducibility.

(ii) As X and $\overline{X}$ only differ by a $\overline{\nu}$-null set, one has

$$\int_{\overline{X}}^{\oplus} d\overline{\nu}(\xi)\, \mathscr{H}_\xi \simeq \int_X^{\oplus} d\nu\, \mathscr{H}_\xi, \tag{4.19}$$

and the relations (4.1) can be reformulated, using the right-hand side of (4.19) and an isomorphism W consisting of the composition of $\overline{W}$ with the isometry implementing (4.19). As an immediate consequence of (4.1a) and (4.1b), we get the equivalence assertion of (2.9a). Moreover, by (4.3) and (4.9e), the operator W connects vector fields $\{|L\rangle_\xi : \xi \in X\}$ with vectors $|L\rangle^\bullet$ for $L \in \mathfrak{L}^c \cup \overline{\mathfrak{L}_0}$ as asserted in (2.9b).

(iii) (2.9c) is a mere reformulation of (4.1c) in terms of X and W.

(iv) The mappings $\xi \mapsto {}_\xi\langle L_1|U_\xi(x)|L_2\rangle_\xi$, ξ restricted to X and L_1 as well as L_2 taken from $\mathfrak{L}^c$, are measurable for all vectors $|L_1\rangle_\xi$ and $|L_2\rangle_\xi$ in the dense subsets $\mathscr{H}_\xi^c$ (cf. the argument preceding (4.11e)), and this suffices, by [12, Section II.2.1, Proposition 1], to establish measurability of the field $\xi \mapsto U_\xi(x)$ for arbitrary $x \in \mathbb{R}^{s+1}$. Moreover, this bounded field of operators defines a bounded operator on $\mathscr{H}^\bullet$ which is given by (2.9d) as an immediate consequence of (4.11e), bearing in mind that X and $\overline{X}_4$ only differ by a $\overline{\nu}$-null set. To demonstrate (2.9e), first assume that the Borel set Δ in question is open so that we can take advantage of the regularity of spectral measures. According to [14, Definition II.8.2], construct a sequence of compact subsets $\{\Gamma_N\}_{N\in\mathbb{N}}$ and of infinitely differentiable functions $\{\tilde{g}_N\}_{N\in\mathbb{N}}$ with support in Δ such that $0 \leqslant \chi_{\Gamma_N} \leqslant \tilde{g}_N \leqslant \chi_\Delta$ and

$$ {}_\xi\big\langle L\big|E_\xi(\Delta)\big|L\big\rangle_\xi = \lim_{N\to\infty} {}_\xi\big\langle L\big|E_\xi(\Gamma_N)\big|L\big\rangle_\xi = \lim_{N\to\infty} {}_\xi\big\langle L\big|\tilde{g}_N(P_\xi)\big|L\big\rangle_\xi, \tag{4.20a}$$

$$ \big\langle L\big|E^\bullet(\Delta)\big|L\big\rangle = \lim_{N\to\infty} \big\langle L\big|E^\bullet(\Gamma_N)\big|L\big\rangle = \lim_{N\to\infty} \big\langle L\big|\tilde{g}_N(P^\bullet)\big|L\big\rangle \tag{4.20b}$$

for any $L \in \mathfrak{L}^c$. By use of Stone's Theorem and the method applied on page 16 f., it can be seen that the sequence appearing on the right-hand side of (4.20a) consists of ν-measurable functions of ξ, hence its limit function on the left-hand side is ν-measurable, too. Another application of Stone's Theorem in connection with (4.14d) formulated in terms of X and ν shows that

$$\begin{aligned}(2\pi)^{(s+1)/2}\big\langle L\big|\tilde{g}_N(P^\bullet)\big|L\big\rangle &= \int_{\mathbb{R}^{s+1}} d^{s+1}x\, g_N(x)\, \big\langle L\big|U^\bullet(x)\big|L\big\rangle \\ &= \int_X d\nu(\xi) \int_{\mathbb{R}^{s+1}} d^{s+1}x\, g_N(x)\, {}_\xi\big\langle L\big|U_\xi(x)\big|L\big\rangle_\xi = (2\pi)^{(s+1)/2} \int_X d\nu(\xi)\, {}_\xi\big\langle L\big|\tilde{g}_N(P_\xi)\big|L\big\rangle_\xi,\end{aligned} \tag{4.20c}$$

and, passing to the limit by application of Lebesgue's Dominated Convergence Theorem, entails, according to (4.20a) and (4.20b),

$$\big\langle L\big|E^\bullet(\Delta)\big|L\big\rangle = \int_X d\nu(\xi)\, {}_\xi\big\langle L\big|E_\xi(\Delta)\big|L\big\rangle_\xi. \tag{4.20d}$$

This formula, as yet valid only for open Borel sets Δ, is readily generalized to closed Borel sets and from there to arbitrary ones, since by regularity their spectral measures can be approximated by a sequence in terms of compact subsets. By polarization and the fact that ket vectors with entries from $\mathfrak{L}^c$ are dense in $\mathscr{H}^\bullet$ and $\mathscr{H}_\xi$, respectively, we first conclude with [12, Section II.2.1, Proposition 1] that all fields $\xi \mapsto E_\xi(\Delta)$ are measurable for arbitrary Borel sets Δ and then pass from (4.20d) to (2.9e).

(v) According to (2.8), the unitary operators $V^\bullet(x)$, $x \in \mathbb{R}^{s+1}$, defined in (2.7b) belong to the von Neumann algebra $\mathfrak{M}$ and are thus diagonalisable in the form

$$W V^\bullet(x) W^* = \int_X^\oplus d\nu(\xi)\ \exp(i p_\xi x) \mathbf{1}_\xi \tag{4.21a}$$

which can be reformulated in terms of the canonical unitary representation (2.7a):

$$W U^\bullet_{can}(x) W^* = \int_X^\oplus d\nu(\xi)\ \exp(i p_\xi x) U_\xi(x). \tag{4.21b}$$

The definition

$$U^{can}_\xi(x) \doteq \exp(i p_\xi x) U_\xi(x), \quad x \in \mathbb{R}^{s+1}, \quad \xi \in X, \tag{4.22}$$

then provides the asserted canonical choice of a strongly continuous unitary representation of spacetime translations on each Hilbert space $\mathscr{H}_\xi$. Its spectral properties are derived from those of the representation $x \mapsto U^\bullet_{can}(x)$ by the methods that have already been used repeatedly above. Possibly a further ν-null subset of X gets lost by this procedure. □

5 Proofs for Section 3

Proposition 3.3. (i) The assumed Δ-boundedness of the particle weight (cf. relation (3.1)) implies that a finite cover of $T^{\mathscr{O}}_{\overline{\Delta}}(\mathfrak{A}_r(\mathscr{O})) = E(\overline{\Delta})\mathfrak{A}_r(\mathscr{O})E(\overline{\Delta})$, $\mathfrak{A}_r(\mathscr{O})$ the r-ball in $\mathfrak{A}(\mathscr{O})$, by sets of diameter less than a given $\delta > 0$, existent on account of precompactness, induces a corresponding cover of $E_w(\Delta')\pi_w(\mathfrak{A}_r(\mathscr{O}))E_w(\Delta')$ by sets with diameter smaller than $c \cdot \delta$, c the parameter occurring in (3.1), thereby establishing total boundedness of this subset of $\mathscr{B}(\mathscr{H}_w)$. By arbitrariness of Δ' as well as of the bounded region $\mathscr{O}$, the representation $(\pi_w, \mathscr{H}_w)$ is thus seen to satisfy the Compactness Criterion of Fredenhagen and Hertel in the sense of precompactness of all mappings

$$T^{\mathscr{O}}_{w,\Delta'} : \mathfrak{A}(\mathscr{O}) \to \mathscr{B}(\mathscr{H}_w) \qquad A \mapsto T^{\mathscr{O}}_{w,\Delta'}(A) \doteq E_w(\Delta')\pi_w(A)E_w(\Delta').$$

(ii) According to the construction of $(\pi^\bullet, \mathscr{H}^\bullet)$ from $(\pi_w, \mathscr{H}_w)$ as explained in the proof of Theorem 2.1, both these representations are related by the inequality

$$\|E^\bullet(\Delta')\pi^\bullet(A)E^\bullet(\Delta')\| \leqslant \|E_w(\Delta')\pi_w(A)E_w(\Delta')\| \tag{5.1a}$$

which holds for any $A \in \mathfrak{A}^\bullet$. Therefore, Δ-boundedness of the underlying particle weight again implies the existence of a bounded Borel set $\overline{\Delta} \supseteq \Delta + \Delta'$ such that

$$\|E^\bullet(\Delta')\pi^\bullet(A)E^\bullet(\Delta')\| \leqslant c \cdot \|E(\overline{\Delta})AE(\overline{\Delta})\|. \tag{5.1b}$$

This replaces (3.1) in the proof of the first part, and we conclude that indeed $(\pi^\bullet, \mathscr{H}^\bullet)$ inherits the precompactness properties of the underlying quantum field theory in the sense that all the sets $E^\bullet(\Delta')\pi^\bullet(\mathfrak{A}^\bullet_r(\mathscr{O}_k))E^\bullet(\Delta') \subseteq \mathscr{B}(\mathscr{H}^\bullet)$ are totally bounded for any $r > 0$ whenever Δ' is an arbitrary bounded Borel set and $\mathscr{O}_k$ is one of the countably many localization regions in $\mathscr{R}^c$. This suffices to establish that the Fredenhagen-Hertel Compactness Condition is satisfied in the restricted setting for local quantum physics introduced in Section 2.1. □

Proposition 3.4. Select a dense sequence $\{A_k\}_{k\in\mathbb{N}}$ in the norm-separable C^*-algebra $\mathfrak{A}^\bullet$ and consider the countable set of compact balls Γ_N of radius N around the origin of $\mathbb{R}^{s+1}$. The corresponding operators $E^\bullet(\Gamma_N)\pi^\bullet(A_k)E^\bullet(\Gamma_N) \in \mathscr{B}(\mathscr{H}^\bullet)$ are decomposable according to Theorem 2.4:

$$W E^\bullet(\Gamma_N)\pi^\bullet(A_k)E^\bullet(\Gamma_N) W^* = \int_X^\oplus d\nu(\xi)\ E_\xi(\Gamma_N)\pi_\xi(A_k)E_\xi(\Gamma_N), \tag{5.2a}$$

and [12, Section II.2.3, Proposition 2] tells us that the respective norms are related by

$$\|E^\bullet(\Gamma_N)\pi^\bullet(A_k)E^\bullet(\Gamma_N)\| = \nu\text{-ess sup}\big\{\|E_\xi(\Gamma_N)\pi_\xi(A_k)E_\xi(\Gamma_N)\| : \xi \in \mathrm{X}\big\}. \tag{5.2b}$$

With regard to the countably many combinations of A_k and Γ_N we thus infer the existence of a measurable non-null subset X_0 of X such that for all k, N and all $\xi \in \mathrm{X}_0$

$$\|E_\xi(\Gamma_N)\pi_\xi(A_k)E_\xi(\Gamma_N)\| \leqslant \|E^\bullet(\Gamma_N)\pi^\bullet(A_k)E^\bullet(\Gamma_N)\|. \tag{5.3}$$

Now, let Δ' be an arbitrary bounded Borel set contained in the compact ball Γ_{N_0} and note that, by continuity of the representations π_ξ and $\pi^\bullet$, the inequality (5.3) extends to arbitrary operators $A \in \mathfrak{A}^\bullet$. Therefore,

$$\|E_\xi(\Delta')\pi_\xi(A)E_\xi(\Delta')\| \leqslant \|E_\xi(\Gamma_{N_0})\pi_\xi(A)E_\xi(\Gamma_{N_0})\| \leqslant \|E^\bullet(\Gamma_{N_0})\pi^\bullet(A)E^\bullet(\Gamma_{N_0})\| \tag{5.4a}$$

which, by (5.1b), implies the existence of a bounded Borel set $\overline{\Delta} \supseteq \Delta + \Delta'$ so that

$$\|E_\xi(\Delta')\pi_\xi(A)E_\xi(\Delta')\| \leqslant c \cdot \|E(\overline{\Delta})AE(\overline{\Delta})\|. \tag{5.4b}$$

The arguments given in the proof of Proposition 3.3 can then again be applied to the present situation to show that for $\xi \in \mathrm{X}_0$ the irreducible representations $(\pi_\xi, \mathscr{H}_\xi)$ altogether meet the requirements of the Fredenhagen-Hertel Compactness Condition. $\square$

Theorem 3.5. (i) Let $\overline{\Delta}$ be a bounded Borel set and suppose that ρ is a normal functional on $\mathscr{B}(\mathscr{H})$. Then so is the functional $\rho_{\overline{\Delta}}(\,.\,) \doteq \rho\big(E(\overline{\Delta})\,.\,E(\overline{\Delta})\big)$, and therefore

$$T_{\overline{\Delta}} : \mathfrak{A} \to \mathscr{B}(\mathscr{H}) \qquad A \mapsto T_{\overline{\Delta}}(A) \doteq E(\overline{\Delta})AE(\overline{\Delta})$$

is continuous with respect to the relative σ-weak topology of $\mathfrak{A}$. Now, according to the Compactness Condition, $T_{\overline{\Delta}} \upharpoonright \mathfrak{A}(\mathscr{O}) = T^{\mathscr{O}}_{\overline{\Delta}}$ maps the unit ball $\mathfrak{A}_1(\mathscr{O})$ of the local C^*-algebra $\mathfrak{A}(\mathscr{O})$ onto the relatively compact set $E(\overline{\Delta})\mathfrak{A}_1(\mathscr{O})E(\overline{\Delta})$. The restriction of $T^{\mathscr{O}}_{\overline{\Delta}}$ to $\mathfrak{A}_1(\mathscr{O})$ is obviously continuous with respect to the relative σ-weak topologies, a statement that can be tightened up in the following sense: The relative σ-weak topology, being Hausdorff and coarser than the relative norm topology, and the relative uniform topology itself coincide on the compact norm closure of $E(\overline{\Delta})\mathfrak{A}_1(\mathscr{O})E(\overline{\Delta})$ due to a conclusion of general topology [22, Chapter One, § 3, 2.(6)]. Therefore $T^{\mathscr{O}}_{\overline{\Delta}}$ is still continuous on $\mathfrak{A}_1(\mathscr{O})$ when its image is furnished with the norm topology instead. Now, suppose that Δ' is an arbitrary bounded Borel set and that (3.1) holds for $\overline{\Delta} \supseteq \Delta + \Delta'$. Then the linear mapping

$$E(\overline{\Delta})AE(\overline{\Delta}) \mapsto E_w(\Delta')\pi_w(A)E_w(\Delta') \tag{5.5}$$

is well-defined and continuous with respect to the uniform topologies of both domain and image. As a consequence of the previous discussion, we infer that the composition of this map with $T_{\overline{\Delta}}$

$$\pi_{w,\Delta'} : \mathfrak{A} \to \mathscr{B}(\mathscr{H}_w) \qquad A \mapsto \pi_{w,\Delta'}(A) \doteq E_w(\Delta')\pi_w(A)E_w(\Delta'), \tag{5.6}$$

is continuous when restricted to $\mathfrak{A}_1(\mathscr{O})$ endowed with the σ-weak topology and the range furnished with the relative norm topology. Now, let η denote a σ-weakly continuous functional on $\mathscr{B}(\mathscr{H}_w)$, then so is $\eta_{\Delta'}(\,.\,) \doteq \eta\big(E_w(\Delta')\,.\,E_w(\Delta')\big)$, and, given a σ-weakly convergent net $\{A_\iota : \iota \in J\} \subseteq \mathfrak{A}_1(\mathscr{O})$ with limit $A \in \mathfrak{A}_1(\mathscr{O})$, we conclude from the above continuity result that

$$\eta_{\Delta'}\big(\pi_w(A_\iota - A)\big) = \eta\big(\pi_{w,\Delta'}(A_\iota - A)\big) \xrightarrow[\iota \in J]{} 0. \tag{5.7}$$

Moreover, due to strong continuity of the spectral measure, η is the uniform limit of the net of functionals $\eta_{\Delta'}$ for $\Delta' \nearrow \mathbb{R}^{s+1}$. Therefore, the right-hand side of the estimate

$$\begin{aligned}\big|\eta \circ \pi_w(A_\iota - A)\big| &\leqslant \big|\eta\big(\pi_w(A_\iota - A)\big) - \eta_{\Delta'}\big(\pi_w(A_\iota - A)\big)\big| + \big|\eta_{\Delta'}\big(\pi_w(A_\iota - A)\big)\big| \\ &\leqslant \|\eta - \eta_{\Delta'}\|\|\pi_w(A_\iota - A)\| + \big|\eta_{\Delta'}\big(\pi_w(A_\iota - A)\big)\big| \leqslant 2\|\eta - \eta_{\Delta'}\| + \big|\eta_{\Delta'}\big(\pi_w(A_\iota - A)\big)\big|\end{aligned} \tag{5.8}$$

can, by selection of a suitable bounded Borel set Δ' and, depending on it, an appropriate index ι_0, be made arbitrarily small for $\iota \succ \iota_0$. This being true for any σ-weakly continuous functional η on $\mathscr{B}(\mathscr{H}_w)$ and arbitrary nets $\{A_\iota : \iota \in J\}$ in $\mathfrak{A}_1(\mathscr{O})$ converging to $A \in \mathfrak{A}_1(\mathscr{O})$ with respect to the σ-weak topology of $\mathscr{B}(\mathscr{H})$, we infer, in view of the left-hand side, that the restrictions of the representation π_w to each of the unit balls $\mathfrak{A}_1(\mathscr{O})$ are σ-weakly continuous. According to [21, Lemma 10.1.10], this assertion extends to the entire local C^*-algebra $\mathfrak{A}(\mathscr{O})$ so that π_w indeed turns out to be locally normal.

(ii) *Mutatis mutandis*, the above reasoning concerning π_w can be transferred literally to the representations $\pi^\bullet$ and π_ξ, $\xi \in \mathrm{X}_0$, where the relations (5.1b) and (5.4b) established in the proofs of Propositions 3.3 and 3.4 substitute (3.1) used in the first part.

(iii) Complementary to the statements of the second part, [21, Lemma 10.1.10] exhibits that $\pi^\bullet$ and π_ξ, $\xi \in \mathrm{X}_0$, allow for unique σ-weakly continuous extensions $\overline{\pi}^\bullet$ and $\overline{\pi}_\xi$, respectively, onto the weak closures $\mathfrak{A}^\bullet(\mathscr{O}_k)''$ [6, Corollary 2.4.15] of the local algebras which, due to the Bicommutant Theorem [6, Theorem 2.4.11], coincide with the strong closures and thus, by the very construction of $\mathfrak{A}^\bullet(\mathscr{O}_k)$, $\mathscr{O}_k \in \mathscr{R}^c$, expounded in Section 2.1, contain the corresponding local C^*-algebras $\mathfrak{A}(\mathscr{O}_k)$ of the underlying quantum field theory. Now, due to the net structure of $\mathscr{O}_k \mapsto \mathfrak{A}(\mathscr{O}_k)$, the quasi-local C^*-algebra $\mathfrak{A}$ is its C^*-inductive limit, i. e., the norm closure of the *-algebra $\bigcup_{\mathscr{O}_k \in \mathscr{R}^c} \mathfrak{A}(\mathscr{O}_k)$. As the representations $\overline{\pi}^\bullet$ and $\overline{\pi}_\xi$, $\xi \in \mathrm{X}_0$, are altogether uniformly continuous on this *-algebra [23, Theorem 1.5.7], they can be continuously extended in a unique way to its completion $\mathfrak{A}$ [22, Chapter One, § 5, 4.(4)], and these extensions, again denoted $\overline{\pi}^\bullet$ and $\overline{\pi}_\xi$, respectively, are easily seen to be compatible with the algebraic structure of $\mathfrak{A}$. $(\overline{\pi}^\bullet, \mathscr{H}^\bullet)$ and $(\overline{\pi}_\xi, \mathscr{H}_\xi)$ are thus representations of this quasi-local algebra, evidently irreducible in the case of $\overline{\pi}_\xi$ and altogether locally normal. This last property applies, since, by construction, the representations are σ-weakly continuous when restricted to local algebras $\mathfrak{A}(\mathscr{O}_k)$ pertaining to the countable subclass of regions $\mathscr{O}_k \in \mathscr{R}^c$, and each arbitrary local algebra $\mathfrak{A}(\mathscr{O})$ is contained in at least one of these. Furthermore, the extensions are uniquely characterized by their local normality, as they are singled out being σ-weakly continuous on $\mathfrak{A}(\mathscr{O}_k)$, $\mathscr{O}_k \in \mathscr{R}^c$.

To establish (3.2), first note that any $B \in \mathfrak{A}(\mathscr{O}_k)$ is the σ-weak limit of a *sequence* $\{B_n\}_{n\in\mathbb{N}}$ in $\mathfrak{A}^\bullet_r(\mathscr{O}_k)$ with $r = \|B\|$. This statement in terms of *nets* in $\mathfrak{A}^\bullet_r(\mathscr{O}_k)$ is a consequence of Kaplansky's Density Theorem [28, Theorem II.4.8] in connection with [28, Lemma II.2.5] and the various relations between the different locally convex topologies on $\mathscr{B}(\mathscr{H})$. The specialization to sequences is justified by [28, Proposition II.2.7] in view of the separability of $\mathscr{H}$. Now, the operators $L \in \mathfrak{L}^c$ define fundamental sequences of measurable vector fields $\{|L\rangle_\xi : \xi \in \mathrm{X}_0\}$ [12, Section II.1.3, Definition 1] and, as the operators $\pi^\bullet(B_n)$ are decomposable, all the functions $\xi \mapsto {}_\xi\big\langle L_1\big|\pi_\xi(B_n)\big|L_2\big\rangle_\xi$ are measurable for arbitrary L_1, $L_2 \in \mathfrak{L}^c$. By [14, II.1.10], the same holds true for their pointwise limits on X_0, the functions $\xi \mapsto {}_\xi\big\langle L_1\big|\overline{\pi}_\xi(B)\big|L_2\big\rangle_\xi$, and, according to [12, Section II.2.1, Proposition 1], this suffices to demonstrate that $\big\{\overline{\pi}_\xi(B) : \xi \in \mathrm{X}_0\big\}$ is a measurable field of operators. As, by assumption, the sequence $\big\{\pi^\bullet(B_n)\big\}_{n\in\mathbb{N}}$ converges σ-weakly to $\overline{\pi}^\bullet(B)$ and, moreover, $\nu(\mathrm{X}_0)$ is finite and the family of operators $\big\{\pi_\xi(B_n) : \xi \in \mathrm{X}_0\big\}$ is bounded by $\|B\|$ for any n, we conclude with Lebesgue's Dominated Convergence Theorem applied to the decompositions of $\pi^\bullet(B_n)$ with respect to X_0 that

$$\begin{aligned}\big\langle L_1\big|\pi^\bullet(B_n)\big|L_2\big\rangle &= \int_{\mathrm{X}_0} d\nu(\xi)\ {}_\xi\big\langle L_1\big|\pi_\xi(B_n)\big|L_2\big\rangle_\xi \\ &\xrightarrow[n\to\infty]{} \int_{\mathrm{X}_0} d\nu(\xi)\ {}_\xi\big\langle L_1\big|\overline{\pi}_\xi(B)\big|L_2\big\rangle_\xi = \big\langle L_1\big|\overline{\pi}^\bullet(B)\big|L_2\big\rangle. \end{aligned} \tag{5.9}$$

Let W_0 denote the isometry that implements the unitary equivalence (2.9a) in terms of X_0 instead of X and shares all the properties of the original operator W introduced in Theorem 2.4, then, by density of the set $\{|L\rangle^\bullet : L \in \mathfrak{L}^c\}$ in $\mathscr{H}^\bullet$, we infer from (5.9)

$$W_0 \,\overline{\pi}^\bullet(B)\, W_0^* = \int_{X_0}^{\oplus} d\nu(\xi)\, \overline{\pi}_\xi(B), \quad B \in \mathfrak{A}(\mathscr{O}_k). \tag{5.10a}$$

To get rid of the limitation of (5.10a) to operators from $\mathfrak{A}(\mathscr{O}_k)$, note that it is possible to reapply the above reasoning in the case of an arbitrary element A of the quasi-local algebra $\mathfrak{A}$ which can be approximated uniformly by a sequence $\{A_n\}_{n\in\mathbb{N}}$ from $\bigcup_{\mathscr{O}_k\in\mathscr{R}^c} \mathfrak{A}(\mathscr{O}_k)$. In this way, (5.10a) is extended to all of $\mathfrak{A}$ and we end up with

$$W_0 \,\overline{\pi}^\bullet(A)\, W_0^* = \int_{X_0}^{\oplus} d\nu(\xi)\, \overline{\pi}_\xi(A), \quad A \in \mathfrak{A}, \tag{5.10b}$$

a reformulation of (3.2). □

6 Conclusions

This article establishes the existence of a (spatial) disintegration theory for generic particle weights in terms of pure components associated with irreducible representations. These pure particle weights can be assigned mass and spin even in an infraparticle situation (cf. [10, 17] and [25]), a result due to Buchholz which is to be thoroughly formulated and proved elsewhere. As shown in Subsections 2.1 and 2.2, one first has to give a separable reformulation of particle weights in order to have the standard results of disintegration theory at one's disposal. In Section 3, these restrictions were seen to be inessential for theories complying with the Fredenhagen-Hertel Compactness Condition. As mentioned there, a couple of criteria have been proposed to effectively control the structure of phase space. Compactness and nuclearity criteria of this kind (cf. [9] and references therein) have proved useful to single out quantum field theoretic models that allow for a complete particle interpretation.

Some initial steps have been taken to implement the alternative Choquet approach to disintegration theory (cf. [1] and [24]) with respect to the positive cone of all particle weights [27], again making use of the Compactness Condition of Fredenhagen and Hertel. It is hoped that the separability assumptions, in the present context necessary to formulate the spatial disintegration, finally turn out to be incorporated in physically reasonable requirements of this kind on the structure of phase space. Presumably, both the spatial disintegration and the Choquet decomposition will eventually prove to be essentially equivalent, revealing relations similar to those encountered in the disintegration theory of states on C^*-algebras [6, Chapter 4]. Further studies have to disclose the geometrical structure of the positive cone of particle weights, as the particle content of a theory appears to be encoded in this information.

A A Lemma on Norm-Separable C^*-Algebras

The following result is an adaptation of [21, Lemma 14.1.17] to our needs.

Lemma A.1. *Let $\mathfrak{A}$ be a unital C^*-subalgebra of $\mathscr{B}(\mathscr{H})$, the algebra of bounded linear operators on a separable Hilbert space $\mathscr{H}$. Then there exists a norm-separable C^*-subalgebra $\mathfrak{A}^0$ containing the unit element $\mathbb{1}$ which lies strongly dense in $\mathfrak{A}$.*

Proof. Let $\mathfrak{M} \doteq \mathfrak{A}''$ denote the von Neumann algebra generated by $\mathfrak{A}$. According to von Neumann's Density Theorem, $\mathfrak{M}$ coincides with the strong closure $\mathfrak{A}^-$ of the algebra $\mathfrak{A}$ which, containing $\mathbb{1}$, acts non-degenerately on $\mathscr{H}$ (cf. [12, Section I.3.4], [6, Corollary 2.4.15]). Let furthermore $\{\phi_n\}_{n\in\mathbb{N}}$ be a dense sequence of non-zero vectors in $\mathscr{H}$.

First, assume the existence of a separating vector for $\mathfrak{M}$; then any normal functional on $\mathfrak{M}$ is of the form $\omega_{\psi,\psi'} \upharpoonright \mathfrak{M}$ with ψ, $\psi' \in \mathscr{H}$ [21, Proposition 7.4.5 and Corollary 7.3.3]. Due to Kaplansky's Density Theorem [23, Theorem 2.3.3], it is possible to choose operators $A_{j,k} \in \mathfrak{A}_1$ for any pair (ϕ_j, ϕ_k) such that the normal functional ω_{ϕ_j,ϕ_k} satisfies the relation $\omega_{\phi_j,\phi_k}(A_{j,k}) \geqslant \|\omega_{\phi_j,\phi_k} \upharpoonright \mathfrak{M}\| - \delta$ with $0 < \delta < 1$ arbitrary but fixed. Let $\mathfrak{A}^0$ denote the norm-separable C^*-algebra generated by the unit element $\mathbf{1}$ together with all these operators. We now assume the existence of a normal functional $\omega_{\xi,\theta}$ on $\mathfrak{M}$ such that $\|\omega_{\xi,\theta} \upharpoonright \mathfrak{A}^0\| = 0$ and $\|\omega_{\xi,\theta} \upharpoonright \mathfrak{M}\| > 0$ and establish a contradiction. Without loss of generality, assume $\|\omega_{\xi,\theta} \upharpoonright \mathfrak{M}\| = 1$. To any $\varepsilon > 0$ there exist vectors ϕ_j, ϕ_k in the dense sequence rendering $\|\phi_j - \xi\|$ and $\|\phi_k - \theta\|$ small enough to ensure $\|(\omega_{\xi,\theta} - \omega_{\phi_j,\phi_k}) \upharpoonright \mathfrak{M}\| < \varepsilon$. Combining all this, we get the estimate

$$\begin{aligned}\varepsilon > \|(\omega_{\xi,\theta} - \omega_{\phi_j,\phi_k}) \upharpoonright \mathfrak{M}\| \geqslant \|(\omega_{\xi,\theta} - \omega_{\phi_j,\phi_k})(A_{j,k})\| \\ = \|\omega_{\phi_j,\phi_k}(A_{j,k})\| \geqslant \|\omega_{\phi_j,\phi_k} \upharpoonright \mathfrak{M}\| - \delta,\end{aligned}$$

and thence

$$\|\omega_{\xi,\theta} \upharpoonright \mathfrak{M}\| \leqslant \|(\omega_{\xi,\theta} - \omega_{\phi_j,\phi_k}) \upharpoonright \mathfrak{M}\| + \|\omega_{\phi_j,\phi_k} \upharpoonright \mathfrak{M}\| < 2\varepsilon + \delta.$$

In contradiction to the assumed normalization of $\omega_{\xi,\theta}$ on $\mathfrak{M}$, this implies, by arbitraryness of ε, that $\|\omega_{\xi,\theta} \upharpoonright \mathfrak{M}\| \leqslant \delta < 1$. Thus, $\omega_{\xi,\theta} \upharpoonright \mathfrak{A}^0 = 0$ implies $\omega_{\xi,\theta} \upharpoonright \mathfrak{M} = 0$, i. e., any normal functional on $\mathfrak{M}$ annihilating $\mathfrak{A}^0$ annihilates $\mathfrak{M}$ as well. Now, since the C^*-algebra $\mathfrak{A}^0$ acts non-degenerately on $\mathscr{H}$, von Neumann's Density Theorem tells us that its strong and σ-weak closures coincide, $\mathfrak{A}^{0\prime\prime} = \mathfrak{A}^{0-}$. The latter in turn is equal to the von Neumann algebra $\mathfrak{M}$, for the existence of an element $A \in \mathfrak{M}$ not contained in $\mathfrak{A}^{0-}$ would, by the Hahn-Banach-Theorem, imply existence of a σ-weakly continuous (normal) functional that vanishes on $\mathfrak{A}^0$ but not on $A \in \mathfrak{M} \setminus \mathfrak{A}^{0-}$ in contradiction to the above result.

Now suppose that there does not exist a separating vector for the von Neumann algebra $\mathfrak{M} = \mathfrak{A}^-$. Then the sequence

$$\Big((n\|\phi_n\|)^{-1}\phi_n\Big)_{n\in\mathbb{N}} \subseteq \underline{\mathscr{H}} \doteq \bigoplus_{n=1}^{\infty} \mathscr{H}$$

is a vector of this kind for the von Neumann algebra $\underline{\mathfrak{M}} \doteq \left(\bigoplus_{n=1}^{\infty} \iota\right)(\mathfrak{M})$, ι denoting the identity representation of $\mathfrak{M}$ in $\mathscr{H}$. The result of the preceding paragraph thus applies to the C^*-algebra $\underline{\mathfrak{A}} \doteq \left(\bigoplus_{n=1}^{\infty} \iota\right)(\mathfrak{A})$ of operators on the separable Hilbert space $\underline{\mathscr{H}}$. This algebra is weakly dense in $\underline{\mathfrak{M}}$: $\underline{\mathfrak{A}}^- = \underline{\mathfrak{M}}$. We infer that there exists a norm-separable C^*-subalgebra $\underline{\mathfrak{A}}^0$ of $\underline{\mathfrak{A}}$ including its unit $\underline{\mathbf{1}} \doteq (\mathbf{1})_{n\in\mathbb{N}}$, which is strongly dense in $\underline{\mathfrak{A}}$. Now, $\underline{\iota} \doteq \bigoplus_{n=1}^{\infty} \iota$ is a faithful representation of $\mathfrak{A}$ on $\underline{\mathscr{H}}$, and its inverse $\underline{\iota}^{-1} : \underline{\mathfrak{A}} \to \mathfrak{A}$ is a faithful representation of $\underline{\mathfrak{A}}$ on $\mathscr{H}$ which is continuous with respect to the strong topologies of $\underline{\mathfrak{A}}$ and $\mathfrak{A}$. Therefore $\mathfrak{A}^0 \doteq \underline{\iota}^{-1}(\underline{\mathfrak{A}}^0)$ is a norm-separable C^*-subalgebra of $\mathfrak{A}$, containing the unit element $\mathbf{1}$ and lying strongly dense in $\mathfrak{A}$. □

Acknowledgements

The results presented above owe much to discussions with and support by Detlev Buchholz during the preparation of my thesis. Bernd Kuckert has given valuable advice in editing the final version of this article. Financial support by Deutsche Forschungsgemeinschaft is gratefully acknowledged which I obtained from the Graduiertenkolleg "Theoretische Elementarteilchenphysik" at the II. Institut für Theoretische Physik of the University of Hamburg.

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