Chern-Simons Term for BF Theory and Gravity as a Generalized Topological Field Theory in Four Dimensions

Han-Ying Guo[1,2],* Yi Ling[2,3],† Roh-Suan Tung[2,4],‡ and Yuan-Zhong Zhang[1,2]§
[1] *CCAST (World Laboratory), P.O. Box 8730, Beijing 100080, China*
[2] *Institute of Theoretical Physics, Chinese Academy of Sciences P.O.Box 2735, Beijing 100080, China*
[3] *Center for Gravitational Physics and Geometry, Department of Physics, Pennsylvania State University, University Park, PA 16802, USA and*
[4] *Department of Physics, University of California, Davis, CA 95616, USA*
(Dated: April 25, 2020)

A direct relation between two types of topological field theories, Chern-Simons theory and BF theory, is presented by using "Generalized Differential Calculus", which extends an ordinary $p-$form to an ordered pair of p and $(p+1)-$form. We first establish the generalized Chern-Weil homomormism for generalized curvature invariant polynomials in general even dimensional manifolds, and then show that BF gauge theory can be obtained from the action which is the generalized second Chern class with gauge group G. Particularly when G is taken as $SL(2,C)$ in four dimensions, general relativity with cosmological constant can be derived by constraining the topological BF theory.

PACS numbers: 04.60.Ds,04.20.Gz

I. INTRODUCTION

In the last few years one of the most important progress in quantum gravity is that deep relations between gravity and the gauge fields have been further disclosed. In string theory one seminal work is the proposed AdS/CFT correspondence[1], which implies there might exist a general duality between gravity theory and Yang-Mills theory. On the side of loop quantum gravity, the replacing of geometrodynamics by connection dynamics has also shed light on the analogy of gravitational fields and gauge fields[2]. In particular, there has been recently much interest in the relations between topological field theory and gravity[3, 4]. Besides the well known fact that three dimensional gravity is simply a topological field theory without local degrees of freedom[5], recent progress shows that even in higher dimensions Einstein's general relativity and supergravity in Ashtekar formalism may also be written as topological field theories with extra constraints[6–10][22]. Since no pre-exiting metric or other geometric structure of spacetime is needed in the context of topological field theory, the advantages of taking this framework as a starting point to explore the background-independent quantum theory of gravity have been revealed in the context of loop quantum gravity from many aspects. Two remarkable programmes are "state sum model" advocated by Crane and Yetter[3] and "spin foam model" by Rovelli and Reisenberger[11]. In both programmes such connections between gravity and topological field theories allow one to apply the elegant quantization methods in topological quantum field theory to quantum gravity. Furthermore, this formulation provides several features to realize the holographic principle proposed by 'tHooft and Susskind at quantum mechanical level[9, 10].

Two typical sorts of topological field theories are Chern-Simons theory and BF theory[12, 13]. Both of them are of the Schwarz type topological field theories and have important applications in quantum gravity. In particular one finds once Einstein-Hilbert action is written as a BF theory with extra constraints, the behavior of gravitational field on the boundary of spacetime can be described by Chern-Simons theories after imposing appropriate boundary conditions. This interesting intersection raises the question of whether these two kinds of topological field theories have closer geometric relations.

In this paper, we propose an answer to this question by using a "*Generalized Differential Calculus*" (see Appendix)[14, 15]. The main assumption is to generalize the ordinary differential $p-$form to an ordered pair of p and $(p+1)-$form, and then treat the gauge fields and their topological properties in the framework of Generalized Differential Calculus. Using this Generalized Differential Calculus, we first establish the generalized Chern-Weil homomormism for generalized curvature invariant polynomials in general even dimensional manifolds (Section 2). Then we obtain the generalized Chern-Simons term for BF theory in four dimensions. This leads to a close relation between these two topological field theories of the Schwarz type. We also rederive the geometric properties for both $P(M^4,G)$ and pseudo-Riemannian spacetime manifolds from the generalized topological field theory of BF type (Sections 3 and 4). In other words, we obtain both BF gauge theories or BF gravity without matter from the action as the generalized second Chern class with gauge group G or $SL(2,C)$ respectively. As a consequence, we find that

*Electronic address: hyguo@itp.ac.cn
†Electronic address: ling@phys.psu.edu
‡Electronic address: tung@dirac.ucdavis.edu
§Electronic address: yzhang@itp.ac.cn

GR with cosmological constant in the absence of matter can be derived by a constrained topological field theory (Section 5) related to holographic formulation [9].

II. CHERN-WEIL HOMOMORMISM IN GENERALIZED DIFFERENTIAL CALCULUS

Let us consider a principle bundle $P(M,G)$ and introduce the generalized gauge fields in Generalized Differential Calculus on it. For the semisimple gauge group G with Lie algebra g, a generalized g-valued connection 1-form field $\mathcal{A}$ is defined as a g-valued pairing of a 1-form and a 2-form

$$\mathcal{A} = (A^p, B^p)T_p = (A, B), \quad T_p \in g, \tag{1}$$

where A is the ordinary g-valued connection 1-form and B the g-valued 2-form which is assumed as gauge covariant under the gauge transformations in order to introduce a generalized gauge covariant curvature $\mathcal{F}$

$$\begin{aligned}\mathcal{F} &= \mathbf{d}\mathcal{A} + \mathcal{A}\wedge\mathcal{A}\\ &= (dA + A\wedge A + kB, \quad dB + A\wedge B - B\wedge A)\\ &= (F + kB,\ DB).\end{aligned} \tag{2}$$

It satisfies the generalized Bianchi identity:

$$\begin{aligned}\mathcal{D}\mathcal{F} &= \mathbf{d}\mathcal{F} + \mathcal{A}\wedge\mathcal{F} - \mathcal{F}\wedge\mathcal{A}\\ &= (DF,\ D^2B) \equiv 0.\end{aligned} \tag{3}$$

In order to consider the topological invariants in the framework of this Generalized Differential Calculus, let us first briefly remind the properties of the curvature invariant symmetric polynomials on $P(M,G)$.

Taking a connection 1-form A on the bundle, the curvature 2-form is $F = dA + A\wedge A$. The curvature invariant symmetric polynomial, say, for simplicity, $P(F^m)$ is a $2m$-form on M

$$P(F^m) = P(\underbrace{F, \cdots, F}_{m}) \tag{4}$$

satisfying (a) $P(F)$ is closed, i.e.,

$$dP(F^m) = 0, \tag{5}$$

(b) $P(F^m)$ has topologically invariant integrals. Namely, it satisfies the Chern-Weil homomormism formula:

$$P(F_1^m) - P(F_0^m) = dQ(A_0, A_1), \tag{6}$$

where

$$Q(A_0, A_1) = \int_0^1 P(A_1 - A_0, F_t^{m-1})dt, \tag{7}$$

where A_0 and A_1 are two connection 1-forms, F_0 and F_1 the corresponding curvature 2-forms,

$$A_t = A_0 + t\eta, \quad \eta = A_1 - A_0, \quad (0 \le t \le 1), \tag{8}$$

the interpolation between A_0 and A_1,

$$F_t = dA_t + A_t \wedge A_t. \tag{9}$$

Since $P(F_1^m)$ and $P(F_0^m)$ differ by an exact form, their integrals over manifolds without boundary give the same results and $Q(A_0, A_1)$ is called the secondary topological class.

For the generalized connection $\mathcal{A}$ and curvature $\mathcal{F}$ in Generalized Differential Calculus, it can be proved that the generalized curvature invariant symmetric polynomial, say, for simplicity, $\mathcal{P}(\mathcal{F}^m)$ also satisfies the similar closed condition and the generalized Chern-Weil homomormism formula:

$$(i) \qquad \mathbf{d}\mathcal{P}(\mathcal{F}^m) = 0, \tag{10}$$

$$(ii) \qquad \mathcal{P}(\mathcal{F}_1^m) - \mathcal{P}(\mathcal{F}_0^m) = \mathbf{d}\mathcal{Q}(\mathcal{A}_0, \mathcal{A}_1). \tag{11}$$

Let us now sketch the proof. For proving (i), it is a straightforward consequence by using the generalized Bianchi identity (3).

To prove (ii), let us take two distinct generalized connections $\mathcal{A}_0$, $\mathcal{A}_1$ and the corresponding curvatures $\mathcal{F}_0$, $\mathcal{F}_1$ on the bundle. Let

$$\mathcal{A}_t = \mathcal{A}_0 + t\boldsymbol{\eta}, \qquad \boldsymbol{\eta} = \mathcal{A}_1 - \mathcal{A}_0, \quad (0 \le t \le 1), \tag{12}$$

and the corresponding curvature is

$$\mathcal{F}_t = \mathbf{d}\mathcal{A}_t + \mathcal{A}_t \wedge \mathcal{A}_t. \tag{13}$$

It is easy to see that

$$\frac{d}{dt}\mathcal{F}_t = \mathbf{d}\boldsymbol{\eta} + \mathcal{A}_t\wedge\boldsymbol{\eta} + \boldsymbol{\eta}\wedge\mathcal{A}_t \equiv \mathcal{D}_t\boldsymbol{\eta}. \tag{14}$$

Hence

$$\begin{aligned}\frac{d}{dt}\mathcal{P}(\mathcal{F}_t^m) &= m\mathcal{P}(\mathcal{D}_t\boldsymbol{\eta}, \mathcal{F}_t^{m-1})\\ = m\mathcal{D}_t\mathcal{P}(\boldsymbol{\eta}, \mathcal{F}_t^{m-1}) &= m\mathbf{d}\mathcal{P}(\boldsymbol{\eta}, \mathcal{F}_t^{m-1}),\end{aligned} \tag{15}$$

Thus

$$\begin{aligned}\mathcal{P}(\mathcal{F}_1^m) - \mathcal{P}(\mathcal{F}_0^m) &= \mathbf{d}\int_0^1 \mathcal{P}(\mathcal{A}_1 - \mathcal{A}_0, \mathcal{F}_t^{m-1})dt\\ &= \mathbf{d}\mathcal{Q}(\mathcal{A}_0, \mathcal{A}_1).\end{aligned} \tag{16}$$

This shows that the generalized characteristic polynomials with respect to different connections only differ by an exact form in Generalized Differential Calculus. Namely, they are also homomormism. $\mathcal{Q}(\mathcal{A}_1, \mathcal{A}_0)$ is called the generalized Chern-Simons secondary class.

Thus, we have established the generalized Chern-Weil homomormism for generalized curvature invariant polynomials in any even dimensional manifolds. But, their topological meaning should be as same as before.

III. GENERALIZED CHERN-SIMONS TERM FOR BF THEORY

Consider an action on the base manifold of $P(M^4,G)$ of the form :

$$\mathcal{S}_T = \int_{M^4} \mathcal{L}_T = \int_{M^4} Tr(\mathcal{F}\wedge\mathcal{F}). \quad (17)$$

The Lagrangian 4-form $\mathcal{L}_T$ can be given by taking $\mathcal{A}_1 = \mathcal{A}$ and $\mathcal{A}_0 = 0$ in (11), then

$$Tr(\mathcal{F}\wedge\mathcal{F}) = \mathbf{d}\mathcal{Q}_{CS}, \quad (18)$$

$\mathcal{Q}_{CS}$ is the generalized local Chern-Simons 3-form, i.e., the pairing of a 3-form and a 4-form

$$\begin{aligned}\mathcal{Q}_{CS} &= Tr(\mathcal{A}\wedge\mathcal{F} - \tfrac{1}{3}\mathcal{A}\wedge\mathcal{A}\wedge\mathcal{A}) \\ &= (Tr(A\wedge F - \tfrac{1}{3}A\wedge A\wedge A + kA\wedge B), \\ &\qquad Tr(A\wedge DB + B\wedge F + kB\wedge B)). \quad (19)\end{aligned}$$

In the pairing, the 3-form is the usual Chern-Simons term up to a $kTr(A\wedge B)$ term.

On the other hand, the generalized Lagrangian 4-form in (17) is a pairing of a 4-form and a 5-form:

$$\begin{aligned}\mathcal{L}_T &= Tr((F\wedge F + 2kB\wedge F + k^2 B\wedge B), \\ &\qquad 2\,(F\wedge DB + kB\wedge DB)). \quad (20)\end{aligned}$$

Using the Bianchi identity, we can rearrange the 5-form so that

$$\begin{aligned}\mathcal{L}_T &= Tr((F\wedge F + 2kB\wedge F + k^2 B\wedge B), \\ &\qquad d(B\wedge F + kB\wedge B)). \quad (21)\end{aligned}$$

The first term is just the BF Lagrangian up to an $F\wedge F$ term, the second term is a total derivative of the BF Lagrangian.

Thus the pairing of the action (17) shows a relation between two types of topological field theories, the Chern-Simons type and the BF type in four dimensions :

$$\begin{aligned}\mathcal{S}_T[\mathcal{A}] &= \int_{M^4} \mathcal{L}_T = \int_{M^4} \mathbf{d}\mathcal{Q}_{CS} \\ &= \int_{M^4} Tr\left(F\wedge F + 2kB\wedge F + k^2 B\wedge B\right) (22)\end{aligned}$$

We can obtain the field equations by varying the Lagrangian with respect to the generalized gauge potentials, i.e., the g-valued generalized connection 1-form. With these generalized gauge potentials fixed at the boundary, the field equations are

$$D(F + 2kB) = 0, \quad (23)$$
$$k(F + kB) = 0. \quad (24)$$

The second equation gives $F = -kB$ which can be substituted into the first equation and leads to the Bianchi identity. Note that we start up with any 2-form B in the generalized connection. Then with the action given by the generalized second Chern class, it gives $B = -(1/k)F$. Therefore, the action (17) does not give any dynamics.

However, it should be noted that with $k = -1$ all geometrical properties of the bundle $P(M^4,G)$ are rederived from a generalized topological field theory of the BF type with a generalized Chern-Simons term associated in Generalized Differential Calculus.

IV. GENERALIZED CONNECTION WITH $SL(2,C)$ GAUGE GROUP ON PSEUDO-RIEMANNIAN MANIFOLD M^4

Let us consider the tangent bundle $T(M^4) \simeq P(M^4, SL(2,C))$ on the base manifold (M^4,g) as the pseudo-Riemannian spacetime manifold with signature $sign(g) = -2$. The $sl(2,C)$ algebraic relation reads [23]:

$$[M_{AB}, M_{CD}] = \epsilon_{C(A}M_{B)D} + \epsilon_{D(A}M_{B)C}, \quad (25)$$

where $\epsilon_{C(A}M_{B)D} = \frac{1}{2}(\epsilon_{CA}M_{BD} + \epsilon_{CB}M_{AD})$. The Cartan-Killing metric $\eta_{pq} = \mathrm{diag}(\eta_{(AB)(MN)})$ is given by

$$\eta_{(AB)(MN)} = \tfrac{1}{2}(\epsilon_{AM}\epsilon_{BN} + \epsilon_{AN}\epsilon_{BM}). \quad (26)$$

Since $SL(2,C)$, the covering of the Lorentz group $SO(3,1)$, is the gauge group of the bundle, we may introduce an $sl(2,C)$-valued generalized connection 1-form in the framework of Generalized Differential Calculus

$$\mathcal{A} = (\omega^{AB}, B^{AB})M_{AB}, \quad (27)$$

where ω^{AB} is an ordinary $sl(2,C)$-valued connection 1-form on the bundle and B^{AB} is an $SL(2,C)$-gauge covariant 2-form. Given the connection $\mathcal{A}$, the generalized curvature ($\mathcal{F} = \mathcal{F}^p T_p = \mathcal{F}^{AB}M_{AB}$) is given by

$$\mathcal{F}^{AB} = (R^{AB} + kB^{AB},\ DB^{AB}). \quad (28)$$

where $R^{AB} = d\omega^{AB} + \omega^A{}_C\wedge\omega^{CB}$ is the $SL(2,C)$ curvature 2-form. The generalized Bianchi identity is given by

$$\mathcal{D}\mathcal{F}^{AB} = (DR^{AB}, D^2B^{AB}) \equiv 0. \quad (29)$$

A simple generalized Lagrangian 4-form in (17) using this connection $\mathcal{A}$ is

$$\begin{aligned}\mathcal{S}_{SL(2,C)}[\mathcal{A}] &= \int_{M^4} \mathcal{L}_{SL(2,C)} = \int_{M^4} \mathcal{F}^{AB}\wedge\mathcal{F}_{AB} \\ &= \int_{M^4} R^{AB}\wedge R_{AB} + 2kR^{AB}\wedge B_{AB} + k^2 B^{AB}\wedge B_{AB}. \quad (30)\end{aligned}$$

The field equations are obtained by varying the Lagrangian with respect to the $sl(2,C)$-valued generalized connection 1-form with fixed value at the boundary. This leads to the field equations

$$D(R^{AB} + 2kB^{AB}) = 0, \quad (31)$$
$$k(R^{AB} + kB^{AB}) = 0. \quad (32)$$

The second equation gives $R^{AB} = -kB^{AB}$, which can be substituted into the first and leads to the Bianchi identity. Therefore, as expected, the action $\mathcal{S}_{SL(2,C)}$ does not give any dynamics. Moreover, when $k = -1$, all properties in the pseudo-Riemannian geometry on (M^4, g) are recovered by the generalized topological field theories of BF type in four dimensions.

V. GRAVITY AS A GENERALIZED TOPOLOGICAL FIELD THEORY

Let us now consider pure gravity in four dimensions. As in the previous section, the spacetime manifold (M^4, g) is pseudo-Riemannian with signature $sign(g) = -2$ and the gauge group now is $SL(2, C)$.

Note that the 4-form Bianchi identity (29), $D^2B^{AB} = 0$, looks similar to the identity $D^2(e^{AA'} \wedge e^B{}_{A'}) = 0$, where $e^{AA'}$ is the frame 1-form. Thus we may introduce an ansatz $B^{AB} = l^{-2}\, e^{AA'} \wedge e^B{}_{A'}$, where l is a dimensional constant. The $sl(2, C)$-valued generalized connection 1-form (27) now becomes[24]

$$\mathcal{A} = (\omega^{AB}, \ \ \frac{1}{l^2} e^{AA'} \wedge e^B{}_{A'}) M_{AB} + c.c. \quad (33)$$

We shall show that given the above generalized connection 1-form (33), the action of generalized topological field theory type (30) becomes the Hilbert action with cosmological constant plus an ordinary topological term. If we vary this generalized topological field theory action with respect to the introduced new variable $e^{AA'}$, it yields the Einstein equation with a cosmological constant term.

Using the generalized connection 1-form (33), the generalized curvature is given by

$$\mathcal{F} = (R^{AB} + \frac{k}{l^2} e^{AA'} \wedge e^B{}_{A'}, \ \ \frac{1}{l^2} D(e^{AA'} \wedge e^B{}_{A'})) M_{AB} + c.c. \quad (34)$$

In terms of this generalized curvature, the action (30), with a change of variational variables from B^{AB} to $e^{AA'}$,

$$\begin{aligned} \mathcal{S}[\omega^{AB}, \omega^{A'B'}, e^{AA'}] &= \int_{M^4} Tr(\mathcal{F} \wedge \mathcal{F}) + c.c. \\ = \int_{M^4} (R^{AB} \wedge R_{AB} &+ \frac{2k}{l^2} R^{AB} \wedge e^{AA'} \wedge e^B{}_{A'} \\ &+ \frac{k^2}{l^4} e^{AA'} \wedge e^B{}_{A'} \wedge e_A{}^{C'} \wedge e_{BC'}) + c.c., \end{aligned} \quad (35)$$

gives the Einstein-Hilbert action with the cosmological constant plus a topological term. Namely, General Relativity in the absence of matter is formulated as a generalized topological field theory.

By varying with respect to ω^{AB}, we obtain

$$D(e^{AA'} \wedge e^B{}_{A'}) = 0, \quad (36)$$

which gives the equation for torsion-free. While by varying with respect to $e^{AA'}$, we obtain

$$R^{AB} \wedge e_B{}^{A'} + \frac{k}{l^2} e^{AB'} \wedge e^B{}_{B'} \wedge e_B{}^{A'} + c.c. = 0, \quad (37)$$

which is the Einstein equation with a cosmological constant Λ if we set Λ by $\Lambda = \frac{k}{l^2}$.

Alternatively, we can consider adding a constraint on B^{AB} as in [9]

$$\begin{aligned} \mathcal{S}[\mathcal{A}^p, \lambda_{AB}, e^{AA'}] &= \int_{M^4} \mathcal{F}^{AB} \wedge \mathcal{F}_{AB} \\ &+ \lambda_{AB} \wedge (\frac{1}{l^2} e^{AA'} \wedge e^B{}_{A'} - B^{AB}) + c.c. \\ = \int_{M^4} & R^{AB} \wedge R_{AB} + 2kR^{AB} \wedge B_{AB} + k^2 B^{AB} \wedge B_{AB} \\ &+ \lambda_{AB} \wedge (\frac{1}{l^2} e^{AA'} \wedge e^B{}_{A'} - B^{AB}) + c.c., \end{aligned} \quad (38)$$

where λ_{AB} is the Lagrangian multiplier. The variational principle leads to the same equation as (37).

In the last part of this section we note that as the cosmological constant, namely the constant k goes to zero, the equation (37) does not necessarily hold any more. It implies our generalization perhaps is only valid for the cases with non-zero cosmological constant.

VI. DISCUSSION

In this paper, we have generalized the Chern-Weil homomormism in Generalized Differential Calculus, associated the generalized Chern-Simons Lagrangian to BF theories and re-derived geometrical properties of $P(M^4, G)$ from a generalized topological field theory of the BF type on four dimensions. We have also recovered the properties of the pseudo-Riemannian manifold M^4 from a generalized topological field theory of the BF type and reformulated GR in the absence of matter as either a generalized topological field theory or a constrained one.

For General Relativity, our approach starts with an $sl(2, C)$- valued generalized connection which includes the 2-form B fields. In a sense, this approach is similar to that of MacDowell-Mansouri [17], in which General Relativity is found as a consequence of breaking the $Sp(4)$ symmetry of a topological field theory down to Lorentz group's covering group $SL(2, C)$ and introduce the tetrad 1-form fields $e^{AA'}$ to parameterize the coset $Sp(4)/SL(2, C)$. However, our approach differs from theirs. Instead of breaking the symmetry of a topological field theory, we start with the Lorentz group and redefined a new generalized $sl(2, C)$-valued connection in Generalized Differential Calculus. This directly leads to a BF theory.

In order to include the matter, there might be at least two possibilities. The first may link with the Kaluza-Klein formalism, since the gauge theories can be formulated as Kaluza-Klein theories on Minkowski spacetime. Therefore, it might be possible to deal with gauge theories as generalized topological field theories in our approach. Of course, there should be certain restriction to the dimensions of the gauge groups. Furthermore, this approach might be generalized to fermions and Higgs [16]. On the other hand, to generalize it to supergravity might be another possibility. For instance, the Generalized Differential Calculus may be generalized to supersymmetric cases. Then, supergravity can be obtained by gauging the $OSp(1,4)$ group with the generalized connection.

It is interesting to see that the present formulation only works for 4-dimensional BF theories with B a 2-form field. On one hand, it is reasonable to establish the relation between the Donaldson-Witten invariants in four dimensions and the topological field theory such as the BF type. On the other hand, however, especially for the GR with cosmological constant, it seems amazing that the dimensions of our nature is also four. If this formulation could not be generalized to arbitrary higher dimensions, whether this dimension four has more profound meaning rather than just a coincidence. This question has to be left for further study and inspiration.

Appendix: Generalized Differential Calculus

A generalized p-form [14][15], $\overset{p}{\mathbf{a}}$, is defined to be an ordered pair of an ordinary p-form $\overset{p}{\alpha}$ and an ordinary $(p+1)$-form $\overset{p+1}{\alpha}$ on an n-dimensional manifold M, that is

$$\overset{p}{\mathbf{a}} \equiv (\overset{p}{\alpha}, \overset{p+1}{\alpha}) \in \Lambda^p \times \Lambda^{p+1}, \tag{39}$$

where $-1 \leq p \leq n$. The minus one-form is defined to be an ordered pair

$$\overset{-1}{\mathbf{a}} \equiv (0, \overset{0}{\alpha}), \tag{40}$$

where $\overset{0}{\alpha}$ is a function on M. The product and derivatives are defined by

$$\overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}} \equiv (\overset{p}{\alpha} \wedge \overset{q}{\beta}, \overset{p}{\alpha} \wedge \overset{q+1}{\beta} + (-1)^q \overset{p+1}{\alpha} \wedge \overset{q}{\beta}), \tag{41}$$

$$\mathbf{d} \overset{p}{\mathbf{a}} \equiv (d \overset{p}{\alpha} + (-1)^{p+1} k \overset{p+1}{\alpha}, d \overset{p+1}{\alpha}), \tag{42}$$

where k is a constant. These exterior products and derivatives of generalized forms satisfy the standard rules of exterior algebra

$$\overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}} = (-1)^{pq} \overset{q}{\mathbf{b}} \wedge \overset{p}{\mathbf{a}} \tag{43}$$

$$\mathbf{d}(\overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}}) = \mathbf{d} \overset{p}{\mathbf{a}} \wedge \overset{q}{\mathbf{b}} + (-1)^p \overset{p}{\mathbf{a}} \wedge \mathbf{d} \overset{q}{\mathbf{b}}, \tag{44}$$

and $\mathbf{d}^2 = 0$.

For a generalized p-form $\overset{p}{\mathbf{a}} = (\overset{p}{\alpha}, \overset{p+1}{\alpha})$, the integration on M^p can be defined as usual by

$$\int_{M^p} \overset{p}{\mathbf{a}} = \int_{M^p} (\overset{p}{\alpha}, \overset{p+1}{\alpha}) = \int_{M^p} \overset{p}{\alpha}. \tag{45}$$

Acknowledgments

We would like to thank Professors Y.F. Liu, Q.K. Lu, K. Wu, Z. Xu and M. Yu for valuable discussions. RST thanks Prof S. Carlip for helpful discussions. This project is in part supported by NNSFC under Grants Nos. 90103004, 10175070, 10047004, 19835040 and also by NKBRSF G19990754.

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[22] This idea is similar to that of MacDowell-Mansouri [17, 18] in which general relativity is obtained by breaking the $SO(3,2)$ symmetry of a topological field theory down to $SO(3,1)$. The earlier work to present general relativity as a constrained topological field theory can also be seen in[19]

[23] The upper-case Latin letters $A, B, ... = 0, 1$ denote two component spinor indices, which are raised and lowered with the constant symplectic spinors $\epsilon_{AB} = -\epsilon_{BA}$ together with its inverse and their conjugates according to the conventions $\epsilon_{01} = \epsilon^{01} = +1$, $\lambda^A := \epsilon^{AB}\lambda_B$, $\mu_B := \mu^A\epsilon_{AB}$ [20].

[24] The formulation can be written with purely unprimed spinors by defining spinor 1-forms $\varphi^A = e^{A0'}$ and $\chi^A = e^{A1'}$ [21]. In terms of these spinor 1-forms, the purely unprimed $sl(2, C)$-valued generalized connection 1-form is $\mathcal{A}^+ = \left(\omega^{AB},\ \ (2/l^2)\ \chi^{(A} \wedge \varphi^{B)}\right) M_{AB}$.