Doubly Special Relativity and de Sitter space

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Abstract

In this paper we recall the construction of Doubly Special Relativity (DSR) as a theory with energy-momentum space being the four dimensional de Sitter space. Then the bases of the DSR theory can be understood as different coordinate systems on this space. We investigate the emerging geometrical picture of Doubly Special Relativity by presenting the basis independent features of DSR that include the non-commutative structure of space-time and the phase space algebra. Next we investigate the relation between our geometric formulation and the one based on quantum κ-deformations of the Poincaré algebra. Finally we re-derive the five-dimensional differential calculus using the geometric method, and use it to write down the deformed Klein-Gordon equation and to analyze its plane wave solutions.

1 Introduction

Without doubts the quest for a theory of quantum gravity is one of the most important challenges of the high energy physics. In the recent years however

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the developments on this field have undertake a sudden turned. Contrary to the earlier expectations it turned out that we are likely to being close to discover "quantum gravity signals" in experiments, if we do not see them already in the anomalous behavior of ultra-high-energy cosmic rays and TeV photons (for up to date review of the "quantum gravity phenomenology" see e.g. [1] and references therein.)

These developments have led to the opening of the new and rapidly growing field of research. The idea behind these research is that since the observable effects are due not to the extremely strong gravitational fields but rather to cumulation of minute effects one should look for the traces of quantum gravity in weak gravitational field regime, something that might be called "quantum special relativity", being a limit of quantum gravity in a similar way Special Relativity is a limit of General Relativity.

Doubly Special Relativity [2], [3] is a proposal of such a theory. The idea is that there exist in nature two observer-independent scales, of velocity, identified with the speed of light, and of mass, which is expected to be of order of Planck mass. The appearance of the second scale is to be understood as a trace of quantum structure of space-time, that is present even if the gravitational field is switched off. The introduction of the second observer-independent scale makes it necessary, of course, to modify the rules that govern transformations from one inertial observer to another. In the algebraic language this is equivalent to replacing the standard Poincaré algebra of Special Relativity by another, deformed algebra, tending to the former in the limit of large mass scale $\kappa \to \infty$. Indeed such an example of Doubly Special Relativity has been found and described in some details in [4], [5]. It has been shown further [9], [15] that the structure of this theory, called nowadays DSR1 or bicrossproduct (basis) DSR, with non-commutative space-time, is identical with the quantum deformations of Poincaré symmetry, known as κ-Poincaré algebra [6], [7].

DSR1 is of course not the only Doubly Special Relativity. Another theory of this kind has been constructed by Magueijo and Smolin [8], and it was this paper that became a main motivation to investigate the whole class of DSR theories, instead of particular examples. Such investigations have been undertaken in [9], [10] and the results of these papers can be summarized as follows:

1. There exists a whole class of Doubly Special Relativity Theories, for whose the Lorentz symmetry algebra is not deformed. This means that,

contrary to the statements one can sometimes find in the literature, in DSR theories we do not have to do with Lorentz symmetry breaking, i.e., there exist a subgroup of the group of symmetries of the theory, whose algebra is exactly the Lorentz algebra of Special Relativity.

2. The algebra of Lorentz transformations and momenta is deformed, though. All such deformations of Poincaré algebra (i.e., all DSR theories) are related to each other by reparametrizations of momentum variables [9], [10]. In this paper, for illustrative purposes we will employ the DSR1 theory, described by the κ-Poincaré algebra in the bicrossproduct basis [6], [7].

3. Each of the deformed algebras of Lorentz transformation and momenta can be provided with the quantum algebra structure, in particular with the one identical with κ-Poincaré. This structure makes it possible in turn to construct the the phase space algebra [11], [12], i.e., the set of commutators between Lorentz generators, momenta and positions. It turns out also that positions do not commute, with either κ-Minkowski [7], or Snyder [13] type of non-commutativity, so that in DSR we have to do with the Lie type, non-commutative space-time structure [15], [10].

4. Since DSR is to be a theory of particle kinematics, one cannot restrict oneself to the energy momentum space only (as it was implicit at the early stages of the development of DSR programme.) Instead a DSR theory is defined by the phase space algebra, which in addition to the energy momentum sector contains the noncommutative space-time structure, and a set of nontrivial cross commutators of energy/momenta with space/time positions.

It has been shown in the recent paper [14] that all these properties can be understood if one employs a simple geometrical language. Namely, any DSR theory can be understood as a particular coordinate system on four dimensional de Sitter space of momenta imbedded in five dimensional Minkowski space. In this picture the Lorentz transformations are identified with the $SO(3,1)$ subgroup of the $SO(4,1)$ group of the symmetries of de Sitter space, while the (non-commutative) positions with the remaining four generators, being "translations" in energy-momentum space. The present paper is devoted to investigations of the geometric picture associated with DSR theories in more details. In Section 2 we present some basic information concerning de Sitter space of momenta and its symmetries, in Section 3 we discuss the geometric picture of the DSR theories in more details. Section 4 is devoted to investigating relation between geometric de Sitter space picture and the quantum algebraic approach to DSR. In Section 5 we describe the construction of the covariant differential calculus, and use it to write down the deformed Klein-Gordon equation as well as to derive the plane wave solutions of the this equation. Section 6 is devoted to explaining relation of the de Sitter space formalism and some other techniques employed in investigations of DSR theories.

Throughout this paper we will try to formulate some results of general nature. For the illustrative purposes however we employ the DSR1 theory. In this theory the (commutative) momenta p_μ transform under action of boosts N_i, and rotations M_i as follows [7]

$$[N_i, p_j] = i\,\delta_{ij}\left(\frac{\kappa}{2}\left(1 - e^{-2p_0/\kappa}\right) + \frac{1}{2\kappa}\vec{p}^{\,2}\right) - i\,\frac{1}{\kappa}p_i p_j, \quad [N_i, p_0] = ip_i, \tag{1}$$

$$[M_i, p_j] = i\,\epsilon_{ijk}\,p_k, \quad [M_i, p_0] = 0. \tag{2}$$

There are of course many other DSR theories, and we list the basic properties of some of them in the Appendix.

2 De Sitter space of momenta

Doubly Special Relativity was initially formulated [2], [3], [4], [5] as a theory of nonlinear realization of the Lorentz symmetry on the energy-momentum space. It has been obvious however that one should extend it to the formulation based on the complete phase space. This has been done in [10] with the help of the quantum κ-Poincaré algebra structure and the so-called Heisenberg double procedure [11], [12]. In the paper [14] it has been shown that an equivalent, geometric method of deriving the phase space algebra exists. This method is based on appropriate interpretation of ten symmetries of the four dimensional space of momenta, which contrary to the standard case is assumed to be a maximally symmetric space of constant positive curvature, the de Sitter space. Let us now turn to the description of this space and its symmetries.

Consider the five dimensional space of variables η_A, $A = 0, \ldots, 4$ of dimension of momentum equipped with the Minkowski metric

$$ds^2 = g^{AB} d\eta_A d\eta_B = -d\eta_0^2 + d\eta_i^2 + d\eta_4^2. \tag{3}$$

This metric is invariant under the action of the group $SO(4,1)$, whose generators are rotations M_i, boosts N_i and four remaining generators which we will denote X_μ, and call “positions” (this terminology will become clear later on.) The Lorentz generators form the $\mathcal{SO}(3,1)$ subalgebra of the $\mathcal{SO}(4,1)$ algebra and satisfy

$$[M_i, M_j] = i\,\epsilon_{ijk} M_k, \quad [M_i, N_j] = i\,\epsilon_{ijk} N_k,$$
$$[N_i, N_j] = -i\,\epsilon_{ijk} M_k, \tag{4}$$

These operators act on the first four variables η_μ in a standard way, to wit

$$[M_i, \eta_j] = i\epsilon_{ijk}\eta_k, \quad [N_i, \eta_j] = i\,\delta_{ij}\,\eta_0, \quad [N_i, \eta_0] = i\,\eta_i, \tag{5}$$

and leave the variable η_4 invariant. The commutational relations between these generators and positions are defined by decomposing the $\mathcal{SO}(4,1)$ into $\mathcal{SO}(3,1)$ and the remainder. In the case of Cartan decomposition, we have the relations

$$[X_0, X_i] = -\frac{i}{\kappa^2}\,N_i, \quad [X_i, X_j] = \frac{i}{\kappa^2}\,\epsilon_{ijk}\,M_k \tag{6}$$

Since we will identify X_μ with position operators, and for that reason we have re-scaled them so that they have the dimension of length, the relations (6) correspond to space-time non-commutativity of Snyder’s type [13].

Another natural decomposition is possible, leading to the commutators

$$[x_0, x_i] = -\frac{i}{\kappa}\,x_i, \quad [x_i, x_j] = 0., \tag{7}$$

This quite different type of non-commutativity corresponds to the κ-Minkowski space-time [7]. Note for example that contrary to (6), (7) describe the Lie-type non-commutativity of space-time. To emphasize this difference we will denote positions in Snyder’s non-commutative space-time by capital letters, and those in κ-Minkowski space-time by small ones. The remaining $\mathcal{SO}(4,1)$ commutators read then

$$[N_i, X_0] = iX_i, \quad [N_i, X_j] = i\,\delta_{ij}\,X_0 \tag{8}$$

and

$$[N_i, x_0] = i x_i - \frac{i}{\kappa} N_i, \quad [N_i, x_j] = i\delta_{ij} x_0 - \frac{i}{\kappa} \epsilon_{ijk} M_k. \tag{9}$$

Note that the relation between κ-Minkowski and Snyder's position variables x_μ and X_μ reads

$$x_0 = X_0, \quad x_i = X_i + \frac{1}{\kappa} N_i, \tag{10}$$

so that indeed these variables are related by simple rearrangement of the basis of the Lie algebra $\mathcal{SO}(4,1)$ of symmetries of de Sitter space.

Knowing the algebra, we can write down the action of $\mathcal{SO}(4,1)$ generators on variables η_A. The Lorentz boosts and rotations act in the standard way

$$[M_i, \eta_j] = i\epsilon_{ijk}\, \eta_k, \quad [M_i, \eta_0] = [M_i, \eta_4] = 0 \tag{11}$$

$$[N_i, \eta_j] = i\, \delta_{ij}\, \eta_0, \quad [N_i, \eta_0] = i\eta_i, \quad [N_i, \eta_4] = 0 \tag{12}$$

The cross commutators of κ-Minkowski positions and η_A have the form

$$[x_0, \eta_4] = \frac{i}{\kappa}\, \eta_0, \quad [x_0, \eta_0] = \frac{i}{\kappa}\, \eta_4, \quad [x_0, \eta_i] = 0, \tag{13}$$

$$[x_i, \eta_4] = [x_i, \eta_0] = \frac{i}{\kappa}\, \eta_i, \quad [x_i, \eta_j] = \frac{i}{\kappa}\, \delta_{ij}(\eta_0 - \eta_4), \tag{14}$$

while for the Snyder's position variables one finds

$$[X_0, \eta_4] = \frac{i}{\kappa}\, \eta_0, \quad [X_0, \eta_0] = \frac{i}{\kappa}\, \eta_4, \quad [X_0, \eta_i] = 0, \tag{15}$$

$$[X_i, \eta_4] = \frac{i}{\kappa}\, \eta_i, \quad [X_i, \eta_0] = 0 \quad [X_i, \eta_j] = -\frac{i}{\kappa}\, \delta_{ij}\eta_4, \tag{16}$$

As it will be shown in the next section equations (11)–(16) can be used to reconstruct the whole of the phase space of any particular DSR theory.

3 DSR theories as coordinates on de Sitter space

It is well known that the algebra $\mathcal{SO}(4,1)$ discussed in the previous section is an algebra of symmetries of de Sitter space defined as a following surface in the five-dimensional Euclidean space of Minkowski signature (3)

$$-\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = \kappa^2, \tag{17}$$

Let us define an origin $\mathcal{O}$ of de Sitter space as a point invariant under action of the $SO(3,1)$ subgroup of this symmetry group, cf. (11), (12); thus the point $\mathcal{O}$ has the coordinates $(0,0,0,0,\kappa)$. Let us define coordinates p_0, p_i (physical momenta) on the surface (17) by

$$\eta_0 = \eta_0(p_0, \vec{p}^2), \quad \eta_i = p_i\, \eta(p_0, \vec{p}^2), \quad \eta_4 = \sqrt{\kappa^2 + \eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2}, \quad (18)$$

where we explicitly assumed that the coordinates p_0, p_i transform under rotations in the standard way, and such that $\eta_0(0, \vec{0}) = 0$. Thus the origin $\mathcal{O}$ corresponds to the zero value of the physical momenta.

Of course, the coordinates p_0, p_i are defined only up to an arbitrary redefinition, i.e., general coordinate transformation, leaving the origin $\mathcal{O}$ invariant, and may or may not cover the whole of the space (17).

As an example take the following coordinates

$$\begin{aligned}
\eta_0 &= \kappa\, \sinh\frac{p_0}{\kappa} + \frac{\vec{p}^2}{2\kappa}\, e^{\frac{p_0}{\kappa}} \\
\eta_i &= p_i\, e^{\frac{p_0}{\kappa}} \\
\eta_4 &= \kappa\, \cosh\frac{p_0}{\kappa} - \frac{\vec{p}^2}{2\kappa}\, e^{\frac{p_0}{\kappa}}.
\end{aligned} \quad (19)$$

Using the expressions (12) and the Leibniz rule one easily finds that these coordinates correspond to the transformation rules of DSR in the bicrossproduct basis (1). Thus the prescription (19) provides the geometric definition of the DSR1. Similar prescriptions are possible for other DSR models, and we present them in the Appendix.

A natural question arises as to whether there is a freedom in choosing the coordinates on de Sitter space, corresponding to the given DSR transformation rules, i.e., given commutators of Lorentz boosts and physical momenta of a particular DSR model. To answer it, let us consider the most general DSR boost transformations (assuming, as usual, the standard action of rotations)

$$[N_i, p_j] = i\, \delta_{ij}\alpha + i\, p_i p_j \beta \quad (20)$$

(the term of form $\sim \epsilon_{ijk} p_k$ is excluded by Jacobi identity) and

$$[N_i, p_0] = i\, p_i\, \gamma, \quad (21)$$

where α, β, γ are functions of p_0, $\vec{p}^2$ only, i.e., they are scalars with respect to rotations, and are restricted by Jacobi identity

$$\frac{\partial\alpha}{\partial p_0}\gamma + 2\frac{\partial\alpha}{\partial\vec{p}^2}(\alpha + \vec{p}^2\beta) - \alpha\beta = 1 \tag{22}$$

Take now some η_μ transforming according to (12). Because of the undeformed action of rotations with no loss of generality we can make the ansatz

$$\eta_0 = \eta_0(p_0, \vec{p}^2), \quad \eta_i = p_i\,\eta(p_0, \vec{p}^2) \tag{23}$$

Assume now that the momenta of (23) transform under boost according to (20), (21). Using (12) and the Leibniz rule we get a system of differential equations

$$\frac{\partial\eta_0}{\partial p_0}[N_i, p_0] + 2\frac{\partial\eta_0}{\partial\vec{p}^2}p_j\,[N_i, p_j] = ip_i\,\eta \tag{24}$$

$$[N_i, p_j]\,\eta + \frac{\partial\eta}{\partial p_0}[N_i, p_0]\,p_j + 2\frac{\partial\eta}{\partial\vec{p}^2}p_j\,p_k\,[N_i, p_k] = i\,\delta_{ij}\eta_0. \tag{25}$$

From eq. (25) we have

$$\alpha\,\eta = \eta_0 \tag{26}$$

and

$$\beta\,\eta + \frac{\partial\eta}{\partial p_0}\gamma + 2\frac{\partial\eta}{\partial\vec{p}^2}\left(\alpha + \vec{p}^2\,\beta\right) = 0 \tag{27}$$

It can be checked that for any solution of eqs. (26), (27) eq. (24) is satisfied identically due to the Jacobi identity (22).

As an example consider again the DSR1, for whose

$$\alpha = \frac{\kappa}{2}\left(1 - e^{-2p_0/\kappa}\right) + \frac{\vec{p}^2}{2\kappa}, \quad \beta = -\frac{1}{\kappa}, \quad \gamma = 1.$$

Eq. (27) can be then solved explicitly, giving

$$\eta = e^{p_0/\kappa}\,f\left[\kappa\cosh\left(\frac{p_0}{\kappa}\right) - \frac{\vec{p}^2}{2\kappa}\,e^{p_0/\kappa}\right], \tag{28}$$

where f is an arbitrary function of the Casimir $\mathcal{C}$ (30). In particular, for $f = 1$ we get the expression (19).

Let us observe that the argument of the function f above is nothing but η_4 in (19) and this suggests that the general solution of eq. (27) is of the form

$$\eta(p_0, \vec{p}^{\,2}) = A(p_0, \vec{p}^{\,2})\, f(\mathcal{C}), \tag{29}$$

where $A(p_0, \vec{p}^{\,2})$ is a particular solution of eq. (27) ($A(p_0, \vec{p}^{\,2}) = e^{p_0/\kappa}$ for DSR1), and

$$\mathcal{C} \equiv \eta_0^2 - \vec{\eta}^{\,2} = \left(2\kappa \sinh\left(\frac{p_0}{2\kappa}\right)\right)^2 - \vec{p}^{\,2}\, e^{p_0/\kappa} = m^2 \tag{30}$$

is the Casimir of the algebra (20), (21). It can be easily shown that any solution of eq. (27) is of this form. However we have not be able to prove that any solution of this equation has the form (29) in general case, i.e., for arbitrary choice of α, β, and γ.

Let us now turn to eqs. (13), (14) in order to get the remaining commutators of the phase space. Let us first note that

$$\eta_4 = \sqrt{\kappa^2 + \eta_0^2 - \vec{\eta}^{\,2}} = \sqrt{\kappa^2 + \eta^2(\alpha^2 - \vec{p}^{\,2})} \tag{31}$$

Then one plugs this formula along with the expressions $\eta_0 = \alpha\, \eta$, $\eta_i = p_i\, \eta$ to eqs. (13), (14) and reads the phase space commutators using the Leibniz rule. For example in the case of the bicrossproduct basis with $f = 1$, one gets

$$[p_0, x_0] = -i, \quad [p_i, x_0] = \frac{i}{\kappa}\, p_i,$$
$$[p_i, x_j] = i\, \delta_{ij}\, e^{-2p_0/\kappa} - \frac{i}{\kappa^2}\left(\vec{p}^{\,2}\, \delta_{ij} - 2p_i p_j\right), \quad [p_0, x_i] = -\frac{2i}{\kappa}\, p_i \tag{32}$$

Let us note that in addition to the freedom of de Sitter coordinates corresponding to given Lorentz transformation rules, there exists an additional freedom in construction of the phase space. Indeed it is clear from eq. (12) that the action of Lorentz generators is unaffected by the replacement $\eta_\mu \to -\eta_\mu$, $\eta_4 \to \eta_4$, while such replacement clearly changes the position–momentum commutators. For example if we replace the bicrossproduct basis above with

$$\begin{aligned}
\eta_0 &= -\kappa \sinh\frac{P_0}{\kappa} - \frac{\vec{P}^{\,2}}{2\kappa}\, e^{\frac{P_0}{\kappa}} \\
\eta_i &= -P_i\, e^{\frac{P_0}{\kappa}} \\
\eta_4 &= \kappa \cosh\frac{P_0}{\kappa} - \frac{\vec{P}^{\,2}}{2\kappa}\, e^{\frac{P_0}{\kappa}},
\end{aligned} \tag{33}$$

(which corresponds to $f = -1$ in (29),) and make use of eqs. (13), (14), we find

$$[P_0, x_0] = i, \quad [P_i, x_0] = -\frac{i}{\kappa} P_i, \quad [P_i, x_j] = -i\,\delta_{ij}, \quad [P_0, x_i] = 0. \tag{34}$$

This is nothing but the κ-Minkowski phase space, well known from the literature on quantum κ-Poincaré (quantum) algebra and κ-Poincaré (quantum) group (see, e.g., [7].) Let us therefore investigate the relation between quantum algebraic and geometric approaches in more details.

4 De Sitter vs. quantum algebraic approach to DSR

The phase space of DSR has been first derived not within the geometric picture presented in the preceding section, but in the framework of quantum algebras. Let us recall this construction. The starting point is the extension of the algebra (1), (2) to the Hopf algebra, by introducing the additional structures: co-product Δ and antipode S, as follows [6]

$$\begin{aligned}
\Delta(M_i) &= M_i \otimes \mathbb{1} + \mathbb{1} \otimes M_i\,, \\
\Delta(N_i) &= N_i \otimes e^{-P_0/\kappa} + \mathbb{1} \otimes N_i - \frac{1}{\kappa}\epsilon_{ijk} M_j \otimes P_k\,, \\
\Delta(P_i) &= P_i \otimes \mathbb{1} + e^{-P_0/\kappa} \otimes P_i\,, \\
\Delta(P_0) &= P_0 \otimes \mathbb{1} + \mathbb{1} \otimes P_0\,,
\end{aligned} \tag{35}$$

$$\begin{aligned}
S(P_i) &= e^{-\frac{P_0}{\kappa}} P_i \qquad S(P_0) = -P_0 \\
S(M_i) &= -M_i \qquad S(N_i) = -e^{\frac{P_0}{\kappa}} N_i + \frac{1}{\kappa}\varepsilon_{ijk} e^{\frac{P_0}{\kappa}} P_j M_k\,.
\end{aligned} \tag{36}$$

In equations above we used capital letter P_μ to denote momenta. The difference between these variables and the variables p_μ used in the previous section will be explained in a moment.

As explained in [11], [12], [10] the co-product encodes information concerning the phase space of the theory. To disclose this information one uses the procedure called “Heisenberg double”, which consists of the following:

1. One defines the bracket $<\star,\star>$ between momentum variables P, Q and position variables X, Y in a natural way as follows

$$< P_\mu, x_\nu >= -i\eta_{\mu\nu}, \quad \eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1). \tag{37}$$

2. This bracket is to be consistent with the co-product structure in the following sense

$$\begin{aligned} &< P, xy >=< P_{(1)}, x >< P_{(2)}, y >, \\ &< PQ, x >=< P, x_{(1)} >< Q_{(2)}, x_{(2)} >, \end{aligned} \tag{38}$$

where we use the natural ("Sweedler") notation for co-product

$$\Delta\mathcal{T} = \sum \mathcal{T}_{(1)} \otimes \mathcal{T}_{(2)}.$$

It should be also noted that by definition

$$< 1\!\!1, 1\!\!1 >= 1.$$

One sees immediately that the fact that momenta commute translates to the fact that positions co-commute

$$\Delta x_\mu = 1\!\!1 \otimes x_\mu + x_\mu \otimes 1\!\!1. \tag{39}$$

Then the first equation in (38) along with (37) can be used to deduce the form of the space-time commutators.

3. It remains only to derive the cross relations between momenta and positions. These can be found from the definition of the so-called Heisenberg double (see [11]) and read

$$[P, x] = x_{(1)} < P_{(1)}, x_{(2)} > P_{(2)} - xP \tag{40}$$

As an example let us perform these steps in the DSR1 It follows from (38) that

$$< P_i, x_0 x_j >= -\frac{1}{\kappa}\,\delta_{ij}, \quad < P_i, x_j x_0 >= 0,$$

from which one gets

$$[x_0, x_i] = -\frac{i}{\kappa}\, x_i. \tag{41}$$

Let us now make use of (40) to get the standard relations

$$[P_0, x_0] = -i, \quad [P_i, x_j] = i\,\delta_{ij}. \tag{42}$$

It turns out that this algebra contains one more non-vanishing commutator, namely

$$[P_i, x_0] = -\frac{i}{\kappa}\,P_i. \tag{43}$$

Thus starting from the κ-Poincaré co-product (35) using the Heisenberg double prescription one derives the phase space (34).

It is then obvious that the existence of the second phase space (32) must be related to the existence of another co-product on the same deformed κ-Poincaré algebra (1), (2). This co-product can be computed by a rather tedious procedure and we have been able to construct it only up to linear terms in the first factor, and therefore we will not present it here explicitly. Using this co-product and the Heisenberg double procedure one obtains the phase space relation (32).

Similarly one deduces the commutators of boost an position operators. One starts with the pairing

$$< N_i, x_j >= i\delta_{ij}x_0, \quad < N_i, x_0 >= ix_i. \tag{44}$$

Then, after simple computations, making use of eq. (35) we get

$$[N_i, x_j] = i\delta_{ij}x_0 - \frac{i}{\kappa}\,\epsilon_{ijk}M_k, \quad [N_i, x_0] = ix_i - \frac{i}{\kappa}\,N_i. \tag{45}$$

This relations are of course identical with the ones obtained by geometrical method (9). One should note at this point that, as proved in [10] the commutators (45) are basis independent, because in the Heisenberg double procedure they picked up contributions from the first order terms in co-product, that are left unchanged if one turns from one DSR basis to another. This of course in perfectly consistent with the universal role the $\mathcal{SO}(4,1)$ algebra of symmetries of de Sitter space plays in the geometric formulation of DSR.

5 Differential calculus and plane waves

As emphasized above the space-time of DSR theories is a non-commutative space (with Lie type non-commutativity, contrary to the "central-charge-type" non-commutativity considered in string theory.) It is therefore a highly non-trivial exercise to establish the differential calculus on this space-time. Such a calculus in turn is necessary to give meaning to derivative and other differential operators, needed to formulate field theory.

On κ-Minkowski space-time one can construct two distinct differential calculi. Chronologically the first one is the calculus bi-covariant under action of the full DSR algebra; however it is necessarily five dimensional [17]. Such a calculus (in four space-time dimensions) was derived in [18], [19] using the methods of quantum groups theory. Here we present an alternative (though equivalent) derivation of this form of differential calculus.

On κ-Minkowski space-time there also exists a four-dimensional translational invariant calculus proposed by Majid and Oeckl [20], which is however not covariant under the action of the full DSR algebra (generated by momenta and Lorentz generators). For this reason we will not consider it here. It should be noted that this calculus has been used in [22] to derive properties of the plane waves in DSR1 theory.

The problem we are to solve here is the following. In commutative space-time positions commute with differentials (one forms). However here we are working with non-commutative space-time, and thus we cannot assume a priori that positions commute with one forms. Instead, let us take a basis of one forms. This basis should include differentials dx_μ, but it turns out that in order to obtain a consistent covariant differential calculus one must add one more one-form, which we will denote ϕ. Let us therefore denote the elements of the basis of one forms by $\chi_A = (dx_\mu, \phi)$, $A = 0, \ldots, 4$. In the next step we must postulate the form of the commutator $[x_\mu, \chi_A]$ and we assume that it is proportional to the linear combination of the basic one-forms, to wit

$$[x_\mu, \chi_A] = f^B{}_{\mu A}\, \chi_B. \tag{46}$$

Since positions form Lie type algebra, $[x_0, x_i] = -\frac{i}{\kappa}\, x_i$, taking the commutator of both sides of this equation with χ_A and using the Jacobi identity and (46) we get

$$f^B{}_{0A}\, f^C{}_{iB} - f^B{}_{iA}\, f^C{}_{0B} = \frac{i}{\kappa}\, f^C{}_{iA}. \tag{47}$$

Next we apply the exterior derivative to both sides of $[x_0, x_i] = -\frac{i}{\kappa}\, x_i$ and we use the Leibnitz rule to obtain

$$[x_0, dx_i] + [dx_0, x_i] = -\frac{i}{\kappa}\, dx_i$$

that is

$$f^B{}_{0i} - f^B{}_{i0} = -\frac{i}{\kappa}\,\delta^B{}_i. \tag{48}$$

Similarly, using $[x_i, x_j] = 0$ we find

$$f^B{}_{ij} - f^B{}_{ji} = 0. \tag{49}$$

Note that by taking exterior derivative once again and using the fact that $d^2 = 0$ we find

$$[dx_\mu, dx_\nu] = 0 \tag{50}$$

We need to append these conditions by covariance requirement, i.e., the condition that both sides of (46) transform in the same way under action of rotations and boosts. Using Jacobi identity we find (cf. (9)

$$-[x_0, [\chi_A, M_i]] = f^B{}_{0A}\,[M_i, \chi_B], \tag{51}$$

$$i\epsilon_{ijk}\, f^B{}_{kA}\,\chi_B - [x_j, [\chi_A, M_i]] = f^B{}_{jA}\,[M_i, \chi_B], \tag{52}$$

$$if^B{}_{iA}\,\chi_B + \frac{i}{\kappa}[\chi_A, N_i] - [x_0, [\chi_A, N_i]] = f^B{}_{0A}\,[N_i, \chi_B], \tag{53}$$

$$i\delta_{ij}\, f^B{}_{0A}\,\chi_B + \frac{i}{\kappa}\,\epsilon_{ijk}\,[\chi_A, M_k] - [x_j, [\chi_A, N_i]] = f^B{}_{jA}\,[N_i, \chi_B], \tag{54}$$

Any solution of eqs. (47) – (54) determines the first order covariant calculus of differential forms. It is now clear why the four dimensional covariant differential calculus does not exist. To see this recall that the Lorentz generators form together with the positions the algebra $SO(4,1)$. Eqs. (47) – (54) together with the eqs. (5), (7), (9) would therefore define a Lie algebra of the semidirect product of $SO(4,1)$ with R^4, but such algebra could exist only if x_μ commutes with dx_μ. Therefore we could conclude that the minimal dimension of the covariant differential calculus equals 5. The same conclusion can be reached, of course, by tedious analysis of eqs. (47) – (54).

Thus we see that the basis of one-forms of the covariant differential calculus is indeed $\chi_A = (dx_\mu, \phi)$. Since ϕ does not carry the space-time index, it must be invariant under action of the Lorentz generators

$$[M_i, \phi] = [N_i, \phi] = 0, \tag{55}$$

since invariance is the only covariant behavior of scalars. Moreover, since the action of rotations is assumed to be classical, we have

$$[M_i, dx_0] = 0, \quad [M_i, dx_j] = i\epsilon_{ijk}\, dx_k. \tag{56}$$

Next, since the action of boosts must also transform one forms into one forms, we have that

$$[N_i, dx_0] = i dx_i, \quad [N_i, dx_j] = i\delta_{ij}\, dx_0. \tag{57}$$

(In principle one can add a term proportional to $i\delta_{ij}\,\phi$ to the right hand side of the second equation, but such term can be absorbed into redefinition of dx_0.) Then one can solve eqs. (47) – (54) to obtain

$$[x_\mu, \phi] = \frac{i}{\kappa}\, dx_\mu,$$

$$[x_0, dx_0] = \frac{i}{\kappa}\, \phi, \quad [x_0, dx_i] = 0,$$

$$[x_i, dx_0] = \frac{i}{\kappa}\, dx_i, \quad [x_i, dx_j] = \frac{i}{\kappa}\, \delta_{ij}\, (dx_0 - \phi)\, . \tag{58}$$

Compare now the expressions (58) with eqs. (13), (14). They are identical if we substitute $dx_\mu = \eta_\mu$, $\phi = \eta_4$. Thus the η_A variables are nothing but the basis of one forms.

Knowing the basis of one forms, one can try to understand the meaning of the differential of function. To this end, one must define a particular ordering of polynomials, which we assume to be "x_0-to-the-left" one, and denote it by $:\ast:$. Then such an ordering can be, at least formally, extended to any analytic function of positions. Given such a function, its differential is defined to be

$$df = \partial_\mu f\, dx^\mu + \partial f\, \phi, \tag{59}$$

which in turn defines the left partial derivatives ∂_μ, ∂. Let us now derive the explicit expression for partial derivatives. Using equation above, after tedious computations one finds the general expression for the differential [19]

$$d\ :f(x):=:\left(\kappa \sin(\left(\frac{\partial_0}{\kappa}\right) + \frac{i}{2\kappa}\, e^{i\partial_0/\kappa}\, \frac{\partial^2}{\partial x_i \partial x_i}\right) f:\, dx_0 + :e^{i\partial_0/\kappa}\, \frac{\partial}{\partial x_i} f:\, dx_i$$

$$-i:\left(1 - \cos\left(\frac{\partial_0}{\kappa}\right) - \frac{1}{2}\, e^{i\partial_0/\kappa}\, \frac{\partial^2}{\partial x_i \partial x_i}\right) f:\, \phi. \tag{60}$$

and partial derivatives [19]

$$\partial_0\ :f:=:\left(\kappa \sin(\left(\frac{\partial_0}{\kappa}\right) + \frac{i}{2\kappa}\, e^{i\partial_0/\kappa}\, \frac{\partial^2}{\partial x_i \partial x_i}\right) f: \tag{61}$$

$$\partial_i\ :f:=:e^{i\partial_0/\kappa}\,\frac{\partial}{\partial x_i}\,f: \tag{62}$$

$$\partial\ :f:=\ -i:\left(1-\cos\left(\frac{\partial_0}{\kappa}\right)-\frac{1}{2}\,e^{i\partial_0/\kappa}\,\frac{\partial^2}{\partial x_i\partial x_i}\right)\,f: \tag{63}$$

for ordered functions.

Note the remarkable fact that differential operators in eqs. (60)–(63) are just expressions (19) or (33) with momenta replaced by appropriate derivatives. This property can be easily understood if one notices that for the DSR1 momenta with the phase space (34) one has [19]

$$F(P_\mu)\ :f(x):=:F\left(i\frac{\partial}{\partial x_\mu}\right)\,f(x): \tag{64}$$

We can conclude that this property distinguishes the DSR1 theory among other DSR theories, but, of course, one could reach the same conclusion by observing that the phase space structure (34) is the simplest possible one, compatible with κ-Minkowski type non-commutativity.

Let us also stress that in derivation of the (60)–(63) formulae we used only the definition of covariant differentials and the κ-Minkowski non-commutativity.

Let us now turn to the issue of definition of plane waves on κ-Minkowski space-time, using the differential calculus presented above. The plane waves, as in the standard case, are defined to be fundamental solutions of an appropriate operator, which defines standard dynamics (the non-commutative analog of the Klein-Gordon operator).

Let us observe now that the operator ∂ defined in (63) is a natural candidate for such an operator. Indeed, it is clearly Lorentz invariant, and is by construction closely related to the Casimir operator (recall that $\eta_4=\sqrt{\kappa^2+\eta_0^2-\vec{\eta}^{\,2}}=\sqrt{\kappa^2+m^2}$.) It is now sufficient to observe that the dynamical (deformed Klein-Gordon) operator takes the form [19]

$$0=\partial\,\Psi=\left(\partial_0^2-\vec{\partial}^{\,2}\right)\,\Psi, \tag{65}$$

where the partial derivatives are defined by eqs. (61), (62). But then it follows immediately that for the wave

$$\Psi=e^{iP_0x_0}\,e^{iP_ix_i} \tag{66}$$

the on-shell condition takes the form

$$\kappa^2 \cosh \frac{p_0}{\kappa} - \frac{\vec{p}^2}{2}\, e^{p_0/\kappa} = 0 \tag{67}$$

Thus (66) represents the on-shell massless excitation moving in the κ-Minkowski space-time. Let us note at this point that one can transform the plane wave solution (66) to any other DSR basis, given by $\mathcal{P}_0, \mathcal{P}_i$ by simply making use of the transformation $P_0 \to P_0(\mathcal{P}_0, \mathcal{P}_i)$, $P_i \to P_i(\mathcal{P}_0, \mathcal{P}_i)$.

At this point we face the following problem. In the analysis reported in [22] the authors prove that group velocity for the wave packet composed from the waves of the form (66)[1] is $v_g = \frac{\partial p_0}{\partial |\vec{p}|}$. Yet in the hamiltonian analysis reported in [21] one finds that for all DSR theories, the velocity of massless particles equals 1, the universal velocity of light[2]. It seems that the only way out of this dilemma is that wave packets constructed as a linear combination of plane waves do not represent localized particles states in the DSR theories. One should recall at this point that even in Special Relativity the three velocity obtained from hamiltonian analysis is given by $v_i^{(H)} = u_i/u_0$, while the group velocity of wave packet is given by the derivative of energy with respect to momenta calculated on-shell. However it is an accidental property of Special Relativistic kinematics that these velocities turn out to be equal. Of course, the proper understanding of the concept of velocity in DSR theories is the question of central importance for the whole DSR programme and deserves urgent studies.

6 Relation to other formalisms

In this section we would like to discuss relation between de Sitter space formalism developed in the preceding sections and some other techniques used in the context of Doubly Special Relativity. More specifically, we will discuss the the use of classical fourmomenta variables for description of energy-momentum conservation laws [15], [24] and the method of $\mathcal{U}$ operator employed by Magueijo and Smolin [8], [16].

[1]In the paper [22] the authors make use of the different, Majid and Oeckl four dimensional differential calculus, which further strengthens the result obtained here.

[2]This result agrees with another calculation of group velocity of wave packets moving in κ-Minkowski space-time reported in [23]. The method of obtaining this result employed in this paper has been however criticized in [22].

Classical variables variables and energy-momentum conservation

In the papers [15],[24], in order to solve the outstanding problems of addition of momenta for composition system and conservation laws in DSR, Lukierski and Nowicki and Judes and Visser proposed as a simple solution to introduce auxiliary "classical" variables $\mathcal{P}_\mu$, related to the physical momenta p_μ in a given basis in such a way that $\mathcal{P}_\mu$ transform under Lorentz transformation precisely in the way momenta in the standard Special Relativity do. The authors of these papers claim that in order to compute total momentum of the system one must simply use the standard rules Special Relativistic linear addition rule for $\mathcal{P}_\mu$, and then just transform back the result to the physical variables.

It is a rather trivial observation that in the geometric language adopted here the classical variables are nothing but the de Sitter coordinates η_μ, the latter having the required transformation rules under action of Lorentz transformation. However, as shown in [21], these variables have physical interpretation of four velocities. More specifically, if we have a point particle carrying energy/momentum (p_0, p_i) and if we compute the four-velocity of the particle using the standard hamiltonian method, and taking care of the non-trivial phase space structure of DSR, in *any* DSR theory we find $u_\mu = \eta_\mu$. This provides a relation between four velocity (and three velocity as well) of *one* particle and the energy/momentum it carries, however a priori does n ot tell anything about total energy/momentum of a system composed of *many* particles.

One should also stress that the construction of classical variables is not as straightforward as it may apparently seem to be. The problem is that, as shown in Section 3, the construction of these variables is not unique. There are many functions $\eta_\mu(p_\mu)$ (or in the language of [15],[24] $\mathcal{P}_\mu(p_\mu)$) that correspond to given Lorentz transformation rules for p_μ. It might be argued that this freedom is in fact severely restricted, since from the results of Section 3 we know that various choices of $\eta_\mu(p_\mu)$ are controlled by an arbitrary function of the Casimir. However one still faces the problem, which choice should be made for the mapping from the physical variables to the classical ones and its inverse, in particular should both of them be the same. Moreover, one should note that every choice of this mapping leads to different phase space, and it seems hard to believe that energy/momentum composition law is to be independent of the form of the phase space of the system under consideration.

We would like to stress it once again that a DSR theory does not consist only of the prescription of how Lorentz generators act on momenta, but of the whole of the phase space in which dynamics of the system takes place. If DSR was a just the momentum space Special Relativity in a nonlinear disguise, it would not be of much interest, because as observed in [15] most likely would be physically indistinguishable from Special Relativity. But Doubly Special Relativity is built, as shown above, on a highly non-trivial non-commutative space-time sector, along with strongly deformed phase space commutators.

To conclude, it seems the only relevance of the classical variables seem to be that they are just identical with de Sitter variables.

Magueijo–Smolin operator

In order to construct their DSR theory (called nowadays DSR2) Magueijo and Smolin [8] (the formulation of the DSR2 theory in the de Sitter space language is presented in the Appendix) started from the non-linear realization of the Lorentz generators $\mathcal{L}$ of the form

$$\mathcal{L} = \mathcal{U}^{-1}\, L\, \mathcal{U}, \tag{68}$$

where $\mathcal{U}$ is some one-to-one mapping, and L denotes the standard Lorentz generators.

In the framework of the geometric de Sitter space approach to DSR, the role of $\mathcal{U}$ mapping is easy to understand. This mapping is just a one-to-one map from a subset of momentum de Sitter space, on which the physical momenta p_μ are defined as coordinates to a subset of the five-dimensional Minkowski space, with coordinates η_μ. For example, in the case of the DSR1 (33) we have

$$\begin{aligned} \eta_0 &= \mathcal{U}(P_0, \vec{P}) \circ P_0 = -\kappa\, \sinh\frac{P_0}{\kappa} - \frac{\vec{P}^{\,2}}{2\kappa}\, e^{\frac{P_0}{\kappa}} \\ \eta_i &= \mathcal{U}(P_0, \vec{P}) \circ P_i = -P_i\, e^{\frac{P_0}{\kappa}}. \end{aligned} \tag{69}$$

As it was in the case of the original construction presented by Magueijo and Smolin [8] it is useful to represent $\mathcal{U}(p_0, \vec{p})$ as an exponent of a linear operator acting on p_μ. Then the meaning of eq. (68) is clear: one goes from de Sitter to Minkowski, performs Lorentz transformation and that goes back to de Sitter. This is of course, a direct counterpart of the Leibnitz rule procedure employed in Section III.

7 Conclusions

In this paper we investigated in details the de Sitter geometric picture of Doubly Special Relativity theories. This picture is equivalent the quantum algebras approach to these theories, initiated by investigations of the κ-Poincaré algebra [6] and then generalized to incorporate this quantum algebraic structure to all DSR theories in [9], [10]. In particular we show that the geometric approach makes it possible to construct the phase space of DSR (which agrees with the one obtained by the Heisenberg double" prescription), the Lorentz transformation rules for both momenta and positions, as well as the covariant differential calculus, which is the first step in construction of the DSR-covariant field theory.

Many outstanding problems remain still to be solved, of course. Let us finish this paper with (a partial) list of the ones that we feel are most urgent

- The theory we present in this paper is to be understood as a theory of kinematics of one particle systems. In particular we do not know yet how the proper description of the many particle states should look like. In the course of transition from the description one particle to many particles system one would have to define, among others the consistent notion of the total momentum and energy, and the conservation rules. The fact that our theory can be at best invariant under non-commutative translations indicates that the conservation rules would differ from the ones well known from the classical (relativistic and non-relativistic) mechanics and field theory.

- Keeping in mind that very sensitive time-of-flight experiments [1] are expected to be performed in a near future, the question of what velocity of physical particles is, is perhaps the most urgent one in the field of DSR phenomenology.

- The algebras we have been dealing with are understood to be commutator algebras of quantum operators. Therefore to formulate a complete theory one investigate the functional analysis of these operators. In particular it would be interesting to see if these algebras lead to the appearance of the minimal length.

- Understanding quantum mechanics would make it possible to built quantum field theory with Doubly Special Relativity playing the role analogous to the one played by Poincaré symmetry in the standard QFT.

- Last but not least, if indeed, as claimed in the Introduction, Doubly Special Relativity can be understood as a "Quantum Special Relativity" the complete understanding of DSR would without doubt be an important step in our quest for the theory of quantum gravity.

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Appendix. Another DSR bases

In this Appendix we present the derivation of phase spaces for a class of "classical bases" in which momenta transform under Lorentz transformations in the same way as in Special relativity, and some bases with transformation law of Magueijo and Smolin.

The classical bases

According to our investigations of Section 3, the most general form of η variables in the classical basis reads

$$\eta_\mu = p_\mu f(p_0^2 - \vec{p}^2)$$

$$\eta_4 = \left(\kappa^2 + (p_0^2 - \vec{p}^2) f^2(p_0^2 - \vec{p}^2)\right)^{1/2} \equiv \sqrt{\kappa^2 + m^2 f^2},$$

where we used abbreviation $m^2 \equiv p_0^2 - \vec{p}^2$. The functions f are restricted only by the requirement that in the limit $\kappa \to \infty$, $f \to 1$.

Let us assume now that

$$[x_0, p_i] = ip_i A, \quad [x_0, p_0] = iB$$

where A and B are rotational invariant functions. It follows from

$$[x_0, \eta_i] = 0$$

that

$$2f' p_0\, B + A(f - 2\vec{p}^{\,2}\, f') = 0, \tag{70}$$

where f' denotes derivative of f with respect to its argument. From

$$[x_0, \eta_0] = \frac{i}{\kappa}\eta_4$$

we find

$$(f + 2p_0^2 f')\, B - 2p_0\vec{p}^{\,2}\, f'\, A = \frac{1}{\kappa}\sqrt{\kappa^2 + m^2\, f^2} \tag{71}$$

Note that equations (70), (71) become degenerate for $f' = 0$ (the equation $f - 2\vec{p}^{\,2}\, f' = 0$ does not have any solutions.) With no loss of generality we can assume that $f = 1$. Below we will consider the degenerate case, and the generic non-degenerate one.

In the case $f = 1$ we have to do simply with the algebra (13), (14) with η_4 replaced with $\sqrt{\kappa^2 + p_0^2 - \vec{p}^{\,2}}$:

$$[x_0, p_0] = \frac{i}{\kappa}\sqrt{\kappa^2 + p_0^2 - \vec{p}^{\,2}}, \quad [x_0, p_i] = 0, \tag{72}$$

$$[x_i, p_0] = \frac{i}{\kappa}\, p_i, \quad [x_i, p_j] = \frac{i}{\kappa}\, \delta_{ij}(p_0 - \sqrt{\kappa^2 + p_0^2 - \vec{p}^{\,2}}), \tag{73}$$

In the generic case, we have

$$[x_0, p_0] = \frac{i}{\kappa}\frac{\left(\kappa^2 + m^2\, f^2\right)^{1/2}(f + 2\vec{p}^{\,2}\frac{\partial f}{\partial \vec{p}^{\,2}})}{f\left(f - 2m^2\, \frac{\partial f}{\partial \vec{p}^{\,2}}\right)} \tag{74}$$

$$[x_0, p_i] = ip_i\, \frac{1}{\kappa}\frac{\left(\kappa^2 + m^2\, f^2\right)^{1/2} 2p_0\frac{\partial f}{\partial \vec{p}^{\,2}}}{f\left(f - 2m^2\, \frac{\partial f}{\partial \vec{p}^{\,2}}\right)} \tag{75}$$

The remaining commutators have the form

$$[x_i, p_j] = i(\delta_{ij}B + p_ip_jC), \quad [x_i, p_0] = ip_i(\frac{1}{\kappa} + p_0C) \tag{76}$$

where

$$B = \frac{1}{\kappa}\frac{p_0 f - \left(\kappa^2 + m^2\, f^2\right)^{1/2}}{f},$$

$$C = \frac{1}{\kappa}\frac{2\frac{\partial f}{\partial \vec{p}^2}(p_0 - \kappa B)}{f - 2m^2\,\frac{\partial f}{\partial \vec{p}^2}},$$

where, as above $m^2 \equiv p_0^2 - \vec{p}^2$.

Magueio–Smolin basis (DSR2)

The Magueio–Smolin basis (DSR2) [8] is defined by

$$\eta_\mu = \frac{P_\mu}{1 - P_0/\kappa} \tag{77}$$

Let us derive the phase space commutators. By making use of

$$[x_0, \eta_0] = \frac{i}{\kappa}\eta_4$$

we find

$$[x_0, P_0] = \frac{i}{\kappa}(\kappa^2(1 - P_0/\kappa)^2 + P_0^2 - \vec{P}^2)^{1/2}(1 - P_0/\kappa) \tag{78}$$

and from

$$[x_0, \eta_i] = 0$$

we obtain

$$[x_0, P_i] = -\frac{i}{\kappa}(\kappa^2(1 - P_0/\kappa)^2 + P_0^2 - \vec{P}^2)^{1/2}. \tag{79}$$

Next, from

$$[x_i, \eta_0] = \frac{i}{\kappa}\eta_i$$

it follows that

$$[x_i, P_0] = iP_i(1 - P_0/\kappa), \tag{80}$$

while from

$$[x_i, \eta_j] = \frac{i}{\kappa}\delta_{ij}(\eta_0 - \eta_4)$$

$$[x_i, P_j] = \frac{i}{\kappa}\left(\delta_{ij}\left(P_0 - (\kappa^2(1 - P_0/\kappa)^2 + P_0^2 - \vec{P}^2)^{1/2}\right) - P_iP_j\right) \tag{81}$$

In the paper [10] we made use of the another form of the DSR2 basis, given by

$$\eta_0 = -p_0 \left(1 - \frac{2p_0}{\kappa} + \frac{\vec{p}^2}{\kappa^2}\right)^{-1/2} \tag{82}$$

$$\eta_4 = (\kappa - p_0) \left(1 - \frac{2p_0}{\kappa} + \frac{\vec{p}^2}{\kappa^2}\right)^{-1/2} \tag{83}$$

$$\eta_i = -p_i \left(1 - \frac{2p_0}{\kappa} + \frac{\vec{p}^2}{\kappa^2}\right)^{-1/2} \tag{84}$$

with the phase space

$$[p_0, x_i] = -\frac{i}{\kappa} p_i \tag{85}$$

$$[p_0, x_0] = i \left(1 - \frac{2p_0}{\kappa}\right) \tag{86}$$

$$[p_i, x_j] = -i\, \delta_{ij} \tag{87}$$

$$[p_i, x_0] = -\frac{i}{\kappa}\, P_i. \tag{88}$$

The relation between these bases reads

$$P_\mu = -\frac{p_\mu}{(1 - 2p_0/\kappa + \vec{p}^{\,2}/\kappa^2)^{1/2} - p_0/\kappa}. \tag{89}$$

References

[1] G. Amelino-Camelia, "Quantum-gravity phenomenology: Status and prospects," Mod. Phys. Lett. A **17**, 899 (2002) [arXiv:gr-qc/0204051].

[2] G. Amelino-Camelia, "Testable scenario for relativity with minimum-length," Phys. Lett. B **510**, 255 (2001) [arXiv:hep-th/0012238].

[3] G. Amelino-Camelia, "Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale," Int. J. Mod. Phys. D **11**, 35 (2002) [arXiv:gr-qc/0012051].

[4] J. Kowalski-Glikman, "Observer independent quantum of mass," Phys. Lett. A **286** (2001) 391 [arXiv:hep-th/0102098].

[5] N. R. Bruno, G. Amelino-Camelia and J. Kowalski-Glikman, "Deformed boost transformations that saturate at the Planck scale," Phys. Lett. B **522** (2001) 133 [arXiv:hep-th/0107039].

[6] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoi, "Q deformation of Poincare algebra," Phys. Lett. B **264** (1991) 331.

[7] S. Majid and H. Ruegg, "Bicrossproduct structure of kappa Poincare group and noncommutative geometry," Phys. Lett. B **334** (1994) 348 [arXiv:hep-th/9405107]; J. Lukierski, H. Ruegg and W. J. Zakrzewski, "Classical quantum mechanics of free kappa relativistic systems," Annals Phys. **243** (1995) 90 [arXiv:hep-th/9312153].

[8] J. Magueijo and L. Smolin, "Lorentz invariance with an invariant energy scale," Phys. Rev. Lett. **88** (2002) 190403 [arXiv:hep-th/0112090].

[9] J. Kowalski-Glikman and S. Nowak, "Doubly special relativity theories as different bases of kappa-Poincare algebra," Phys. Lett. B **539** (2002) 126 [arXiv:hep-th/0203040].

[10] J. Kowalski-Glikman and S. Nowak, "Non-commutative space-time of doubly special relativity theories," Int. J. Mod. Phys. D **12** (2003) 299 [arXiv:hep-th/0204245].

[11] A. Nowicki, `math.QA/9803064`.

[12] J. Lukierski and A. Nowicki, Proceedings of Quantum Group Symposium at Group 21, (July 1996, Goslar) Eds. H.-D. Doebner and V.K. Dobrev, Heron Press, Sofia, 1997, p. 186.

[13] H. S. Snyder, "Quantized Space-Time," Phys. Rev. **71** (1947) 38.

[14] J. Kowalski-Glikman, "De Sitter space as an arena for doubly special relativity," Phys. Lett. B **547** (2002) 291 [arXiv:hep-th/0207279].

[15] J. Lukierski and A. Nowicki, "Doubly Special Relativity versus κ-deformation of relativistic kinematics," Int. J. Mod. Phys. A **18** (2003) 7 [arXiv:hep-th/0203065].

[16] J. Magueijo and L. Smolin, "Generalized Lorentz invariance with an invariant energy scale," arXiv:gr-qc/0207085.

[17] A. Sitarz, “Noncommutative differential calculus on the kappa Minkowski space,” Phys. Lett. B **349** (1995) 42 [arXiv:hep-th/9409014].

[18] J.A. de Azcarraga and J.C. Pérez Bueno, “Relativistic and Newtonian κ-spacetimes”, J. Math. Phys. **36** (1995) 6879 [arXive:q-alg/9505004]; P. Kosinski, P. Maslanka, and J. Sobczyk, “The bicovariant differential calculus on the κ-Poincaré group and on the κ-Minkowski space” [arXive:q-alg/9508021].

[19] S. Giller, C. Gonera, P. Kosinski, and P. Maslanka, “A note on geometry of κ-Minkowski space”, Acta Phys. Polon. B **27** (1996) 2171 [arXive:q-alg/9602006]; P. Kosinski, and P. Maslanka, J. Lukierski, and A. Sitarz, “Towards κ-deformed D=4 relativistic field theory”, Chech. J. Phys. **48** (1998) 1407.

[20] R. Oeckl, “Classification of differential calculi on $U_q(b_+)$, classical limit and duality” J. Math. Phys. **40** (1999) 3588-3604 [arXiv:math.QA/9807097]; S. Majid, *Foundation of Quantum Group Theory*, Cambridge University Press, 1995.

[21] M. Daszkiewicz, K. Imilkowska, and J. Kowalski–Glikman, “Velocity of particles in Doubly Special Relativity”, arXiv:hep-th/0304027.

[22] G. Amelino-Camelia, F. D’Andrea and G. Mandanici, “Group velocity in noncommutative spacetime,” arXiv:hep-th/0211022.

[23] T. Tamaki, T. Harada, U. Miyamoto, and T. Torii, “Have we already detected astrophysical symptoms of space-time non-commutativity?”, Phys. Rev. D **65** (2002) 083003 [arXive:gr-qc/0111056].

[24] S. Judes and M. Visser “Conservation Laws in Doubly Special Relativity,” arXiv:gr-qc/0205067.