On the Gauss Law and Global Charge for QCD

J. Kijowski
Center for Theoretical Physics, Polish Academy of Sciences
al. Lotników 32/46, 02-668 Warsaw, Poland

G. Rudolph
Institut für Theoretische Physik, Universität Leipzig
Augustusplatz 10/11, 04109 Leipzig, Germany

April 25, 2020

Abstract

The local Gauss law of quantum chromodynamics on a finite lattice is investigated. It is shown that it implies a gauge invariant, additive law giving rise to a gauge invariant $\mathbb{Z}_3$-valued global charge in QCD. The total charge contained in a region of the lattice is equal to the flux through its boundary of a certain $\mathbb{Z}_3$-valued, additive quantity. Implications for continuous QCD are discussed.

1 Introduction

Quantum chromodynamics (QCD) is one of the basic building blocks of the standard model for describing elementary particle interactions. During the last decades QCD was quite successful, e.g. in describing deep inelastic scattering processes within the framework of perturbation theory at one hand and "measuring" certain types of observables using (nonperturbative) lattice approximation techniques on the other hand. Nonetheless, we are still facing the basic challenge, which consists in constructing an effective microscopic theory of interacting hadrons out of this gauge theory. For this purpose, nonperturbative methods for describing the low energy regime should be developed. In particular, the observable algebra and the superselection structure of this theory should be investigated. The present paper is a step in this direction.

We stress that standard methods from algebraic quantum field theory for models which do not contain massless particles, see [2], do not apply here. Some progress towards an implementation of similar ideas for theories with massless particles has been made, for the case of quantum electrodynamics (QED) see [3] and [4] and further references therein. In particular, Buchholz developed the concept of so called charge classes and found a criterion for distinguishing between the electric charge (carried by massive particles) and additional superselection sectors corresponding to inequivalent asymptotic infrared clouds of photons. For some attempts to deal with the nonabelian case we refer to papers by Strocchi and Wightman, see [5] and [6].

In QED, the notion of *global* (electric) charge is easy to understand. This is due to the fact, that in this theory we have a *local* Gauss law, which is built from gauge invariant operators and which is linear. Thus, one can "sum up" the local Gauss laws over all points of a given (spacelike) hyperplane in space time yielding the following gauge invariant conservation law: The global electric charge is equal to the electric flux through a 2-sphere at infinity. On the contrary, in QCD the local Gauss law is neither built from gauge invariant operators nor is it linear. The main point of the present paper is to show that it is possible to extract from the local Gauss equation of QCD a gauge invariant, additive law for operators with eigenvalues in $\mathbb{Z}_3$, the center of $SU(3)$. This implies – as in QED – a gauge invariant conservation law: The global $\mathbb{Z}_3$-valued charge is equal to a $\mathbb{Z}_3$-valued gauge invariant quantity obtained from the color electric flux at infinity. There is a paper by Fredenhagen and Marcu, see [7], dealing with a $\mathbb{Z}_2$-gauge theory (with $\mathbb{Z}_2$-valued matter fields) on the lattice. These authors were able to construct the ground state and charged states of this model. For some regions in the space of coupling constants, the thermodynamic limit of charged states was controlled. It is likely that similar techniques will be relevant for studying $\mathbb{Z}_3$-valued colour charged states of QCD.

We stress that our main observation is universal in the following sense: Suppose somebody had constructed a nonperturbative version of QCD rigorously. Then our construction of the global charge would apply. Since, however, such a theory is not at our disposal, for a rigorous discussion we have to restrict ourselves to the case of QCD approximated on a finite lattice. For basic notions concerning lattice gauge theories (including fermions) we refer to [1] and references therein. Thus, we consider QCD on a finite, regular cubic lattice in the Hamiltonian approach. Our starting point is the notion of the algebra of field operators. Since, up to our knowledge, this has never been published, we discuss it in some detail. By imposing the local Gauss law and gauge invariance we obtain the observable algebra $\mathcal{O}(\Lambda)$ for QCD on the lattice. For an analogous investigation of the superselection structure of QED on the lattice we refer to [8] and for a (rather heuristic) discussion of continuum QCD within the functional integral approach (using only gauge invariant quantities) we refer to [9]. For QED, the mathematical structure of $\mathcal{O}(\Lambda)$ has been completely clarified, see [10]. A similar investigation for QCD seems to be a very difficult problem, which will be addressed in the future. In this paper, we concentrate on a careful analysis of the local Gauss law and its consequences for the notion of global charge in QCD.

Our paper is organized as follows: In Sections 2 and 3 we discuss the algebra of fields and the observable algebra for QCD on a finite lattice. In Section 4 we analyse the local Gauss law and invent the gauge invariant notion of the local charge densitiy. Using this notion, in Section 5 the global charge is defined and the flux law for QCD is discussed. Here also some heuristic remarks concerning the continuum case are added.

2 The Field Algebra for Lattice QCD

We consider QCD in the Hamiltonian framework on a finite, regular 3-dimensional lattice Λ. We denote the set of oriented, i-dimensional elements of Λ by Λ^i, $i = 0, 1, 2, 3$. Such elements are (in increasing order of i) called sites, links, plaquettes and cubes. The set of non-oriented i-dimensional elements will be denoted by $|\Lambda|^i$. If, for instance, $(x, y) \in \Lambda^1$ is an oriented link, then by $|(x, y)| \in |\Lambda|^1$ we mean the corresponding non-oriented link. Instead of using a concrete Hilbert space representation (e.g. the Schrödinger representation), we give an abstract definition of the field algebra in terms of generators and defining relations.

The basic fields of lattice QCD are quarks living at lattice sites and gluons living on links. The field algebra is thus, by definition, the tensor product of local fermionic and bosonic algebras:

$$\mathcal{F}(\Lambda) := \bigotimes_{x\in|\Lambda|^0} \mathcal{F}_x \bigotimes_{|(x,y)|\in|\Lambda|^1} \mathcal{F}_{|(x,y)|} \,. \tag{2.1}$$

We impose locality of the lattice quantum fields by postulating that the local algebras corresponding to different elements of Λ commute with each other.

The fermionic field algebra $\mathcal{F}_x$ associated with a lattice site x is the enveloping algebra generated from the complex Lie super-algebra $\mathcal{L}_x$ of canonical anticommutation relations of quarks. In terms of coordinates, the quark field is given by

$$|\Lambda|^0 \ni x \to (\psi^{aA}(x)) \in \mathcal{L}_x \,, \tag{2.2}$$

where a stands for bispinorial and (possibly) flavour degrees of freedom and $A = 1, 2, 3$ is the colour index corresponding to the fundamental representation of $SU(3)$. The conjugate quark field is denoted by $\psi^{*a}{}_A(x)$, where we raise and lower indices by the help of the canonical hermitian metric tensor g_{AB} in $\mathbb{C}^3$. Finally, the (nontrivial) canonical anticommutation relations for elements of $\mathcal{L}_x$ read:

$$[\psi^{*a}{}_A(x), \psi^{bB}(x)]_+ = \delta^B{}_A \delta^{ab} \,. \tag{2.3}$$

Passing to self-adjoint generators (real and imaginary parts of ψ), we observe that $\mathcal{F}_x$ is a Clifford algebra.

The bosonic field algebra $\mathcal{F}_{|(x,y)|}$ associated with the non-oriented link $|(x,y)|$ will be constructed in terms of its equivalent copies $\mathcal{F}_{(x,y)}$ and $\mathcal{F}_{(y,x)}$, corresponding to the two orientations of the link (x,y). We will see that there is a natural identification of these two algebras, induced from the vector character of both the gluonic potential A and the colour electric field E of the underlying continuum theory: under the change of orientation of the k'th axis both A_k and E^k change their signs.

The bosonic field algebra $\mathcal{F}_{(x,y)}$ is, by definition, the tensor product

$$\mathcal{F}_{(x,y)} := \tilde{\mathcal{E}}_{(x,y)} \otimes \mathcal{A}_{(x,y)} \,, \tag{2.4}$$

where $\tilde{\mathcal{E}}_{(x,y)}$ is the enveloping algebra of the (real) Lie algebra $\mathcal{E}_{(x,y)} \cong su(3)$ and $\mathcal{A}_{(x,y)} \cong C^\infty(SU(3))$ is the commutative $*$-algebra of smooth functions on the Lie group $SU(3)$. We note that the above tensor product is naturally endowed with the structure of a crossed product of Hopf algebras, given by the action of generators of $\tilde{\mathcal{E}}_{(x,y)}$ on functions, see [11]. We identify the tensor product of elements of $\tilde{\mathcal{E}}_{(x,y)}$ and functions with their product as differential operators on $SU(3)$. This way, $\mathcal{F}_{(x,y)}$ gets identified with the algebra of differential operators on the group manifold. For our purposes, we endow $\mathcal{F}_{(x,y)}$ with a Lie algebra structure. Thus, we have to define the commutator between a generator $e \in \mathcal{E}_{(x,y)}$ of $\tilde{\mathcal{E}}_{(x,y)}$ and a function $f \in \mathcal{A}_{(x,y)}$,

$$[e, f] := e^R(f) \,, \tag{2.5}$$

where $e^R(f)$ denotes the derivative of f with respect to the right-invariant vector field e^R generated by e,

$$e^R(f)(g) := \frac{d}{ds}\Big|_{s=0} f(\exp(se) \cdot g) \,, \;\; g \in SU(3) \,. \tag{2.6}$$

Now we give an explicit description of $\mathcal{F}_{(x,y)}$ in terms of generators and defining relations. The algebra $\mathcal{A}_{(x,y)}$ is generated by matrix elements of the gluonic gauge potential on the link (x,y),

$$\Lambda^1 \ni (x,y) \to U^A{}_B(x,y) \in \mathcal{A}_{(x,y)} \,, \tag{2.7}$$

where $A, B = 1, 2, 3$ are colour indices. Being functions on $SU(3)$, these generators have to fulfil the following relations:

$$(U^A{}_B(x,y))^* U_A{}^C(x,y) = \delta^C{}_B \, \mathbf{1} \,, \tag{2.8}$$

$$\epsilon_{ABC} \, U^A{}_D(x,y) U^B{}_E(x,y) U^C{}_F(x,y) = \epsilon_{DEF} \, \mathbf{1} \,. \tag{2.9}$$

The algebra $\tilde{\mathcal{E}}_{(x,y)}$ is generated by colour electric fields, spanning the Lie algebra $\mathcal{E}_{(x,y)}$. Choosing a basis $\{t_i\}$, $i = 1, \ldots, 8$, of $su(3)$ we denote by $\{E_i(x,y)\}$ the corresponding basis of $\mathcal{E}_{(x,y)}$,

$$\Lambda^1 \ni (x,y) \to E_i(x,y) := t_i \in \mathcal{E}_{(x,y)} \,. \tag{2.10}$$

These generators are self-adjoint (real), $E_i^* = E_i$. In the sequel we take as the basis the hermitean, traceless Gell-Mann matrices ${t_i}^A{}_B$, normalized as follows:

$$\sum_i {t_i}^A{}_B {t_i}^C{}_D = \delta^A_D \delta^C_B - \frac{1}{3} \delta^A_B \delta^C_D \,, \tag{2.11}$$

or, equivalently, ${t_i}^A{}_B \, {t_j}^B{}_A = \delta_{ij}$. We will also use the following traceless matrix built from these fields

$$E^A{}_B(x,y) = \sum_i E_i(x,y) {t_i}^A{}_B \,. \tag{2.12}$$

Since the coefficients ${t_i}^A{}_B$ are complex, these fields are no longer self-adjoint:

$$(E^A{}_B(x,y))^* \;=\; E_B{}^A(x,y) \,. \tag{2.13}$$

The commutation relations between elements of $su(3)$ translated to the language of these fields read

$$[E^A{}_B(x,y), E^C{}_D(x,y)] = \delta^C{}_B E^A{}_D(x,y) - \delta^A{}_D E^C{}_B(x,y) \,, \tag{2.14}$$

whereas the commutation relations (2.5) between Lie algebra elements and functions, rewritten in terms of generators, take the following form:

$$[E^A{}_B(x,y), U^C{}_D(x,y)] = \delta^C{}_B U^A{}_D(x,y) - \frac{1}{3} \delta^A{}_B U^C{}_D(x,y) \,. \tag{2.15}$$

Observe that, for every link (x,y), we have a model of quantum mechanics with configuration space being the group manifold $SU(3)$. The matrix elements of the gluonic potential play the role of functions of position variables, whereas the colour electric fields play the role of (non-commuting) canonically conjugate momenta. Formula (2.15) is the analog of the canonical commutation relation $[p,q] = -i$.

Now, we describe the transformation law of these objects under the change of the link orientation. This law is an isomorphism between bosonic field algebras:

$$\mathcal{I}_{(x,y)} : \mathcal{F}_{(x,y)} \to \mathcal{F}_{(y,x)} \,. \tag{2.16}$$

The vector character of the continuum gauge potential implies that the (classical) $SU(3)$-valued parallel transporter $g(x,y)$ on (x,y) transforms to $g^{-1}(x,y)$ under the change of orientation. Thus, the change of orientation on (x,y) induces the transformation

$$SU(3) \ni g \to i(g) = g^{-1} \in SU(3) \tag{2.17}$$

on configuration space. This transformation lifts naturally to the field algebra $\mathcal{F}_{(x,y)}$. Indeed, for elements of $\mathcal{A}_{(x,y)}$ it implies the following transformation law

$$\mathcal{A}_{(x,y)} \ni f \to \breve{f} \in \mathcal{A}_{(y,x)} \ , \tag{2.18}$$

where $\breve{f}(g) := f(g^{-1})$. We thus put $\mathcal{I}_{(x,y)}(0,f) := (0,\breve{f})$. It may be easily proved that the unique $*$-isomorphism, which fulfils this requirement is given by

$$\mathcal{I}_{(x,y)}(e,f) := (e^L, \breve{f}) \ , \tag{2.19}$$

where e^L is the left invariant vector field on $SU(3)$ generated by $-e$. It acts on functions in the following way:

$$e^L(f)(g) = \frac{d}{ds}\Big|_{s=0} f(g \cdot \exp(-se)) \ . \tag{2.20}$$

Observe that e^L is not an element of $\mathcal{E}_{(y,x)}$, but it can be expanded with respect to right-invariant vector fields with coefficients being functions on the group. Actually, the following formula is easily proved:

$$e^L = -\sum_{i=1}^{8} \mathrm{Tr}(geg^{-1}t_i)t_i^R \ , \tag{2.21}$$

where t_i is any orthonormal basis of $su(3)$.

To prove that $\mathcal{I}_{(x,y)}$ is an isomorphism indeed, we observe that it is obviously a bijective mapping between generators. It extends to an isomorphism of field algebras if we prove that it preserves the commutator. This is obvious for elements, which either belong both to $\mathcal{A}_{(x,y)}$ or to $\mathcal{E}_{(x,y)}$. Thus, it is sufficient to consider the commutator of a Lie algebra element e with a function f. Using the identity

$$e^R(f)(g) = \frac{d}{ds}\Big|_{s=0} f(\exp(se)\cdot g) = \frac{d}{ds}\Big|_{s=0} \breve{f}(g^{-1}\cdot \exp(-se)) = e^L(\breve{f})(g^{-1}) \ , \tag{2.22}$$

and applying $\mathcal{I}_{(x,y)}$ to the functions on both sides, we obtain due to (2.5) and (2.18):

$$\mathcal{I}_{(x,y)}\left([e,f]\right) = \mathcal{I}_{(x,y)}(e^R(f)) = e^L(\breve{f}) = \left[\mathcal{I}_{(x,y)}(e), \mathcal{I}_{(x,y)}(f)\right] \ . \tag{2.23}$$

This shows that $\mathcal{I}_{(x,y)}$ is an isomorphism, indeed. We add that, in more abstract terms, it is given by $\mathcal{I}_{(x,y)}(e,f) = (-i_*(e), i^*(f))$.

Field configurations, which are related under $\mathcal{I}_{(x,y)}$ will be identified as different representations of the same object. Thus, the bosonic field algebra $\mathcal{F}_{|(x,y)|}$, associated with the non-oriented link $|(x,y)|$, is defined as the subalgebra of $\mathcal{F}_{(x,y)} \times \mathcal{F}_{(y,x)}$, obtained by this identification:

$$\mathcal{F}_{|(x,y)|} \ := \ \left\{ (l_{(x,y)}, k_{(y,x)}) \in \mathcal{F}_{(x,y)} \times \mathcal{F}_{(y,x)} \ : \ k_{(y,x)} = \mathcal{I}_{(x,y)}(l_{(x,y)}) \right\} \ . \tag{2.24}$$

Projection onto the first (resp. second) component gives us an isomorphism of $\mathcal{F}_{|(x,y)|}$ with $\mathcal{F}_{(x,y)}$ (resp. $\mathcal{F}_{(y,x)}$).

The above transformation law yields relations between generators of the two algebras. For functions on $SU(3)$, formula (2.8) enables us to rewrite (2.18) as follows:

$$U^A{}_B(y,x) = (U_B{}^A(x,y))^* \ . \tag{2.25}$$

For Lie algebra elements, formula (2.21) applied to $e = t_i$ reads:

$$E_j(y,x) = -\sum_{i=1}^{8} U^A{}_B(x,y) t_j{}^B{}_C U^C{}_D(y,x) t_i{}^D{}_A \ E_i(x,y) \,, \tag{2.26}$$

or, in terms of generators $E^A{}_B(x,y)$,

$$E^A{}_B(y,x) \quad = \quad -U^A{}_D(y,x) U^C{}_B(x,y) E^D{}_C(x,y) \ . \tag{2.27}$$

To summarize, the field algebra $\mathcal{F}(\Lambda)$ is the $*$-algebra, generated by elements

$$\left\{ \psi^{aA}(x) \, , \, U^A{}_B(x,y) \, , \, E^A{}_B(x,y) \right\} ,$$

fulfilling relations (2.8), (2.9), (2.13), (2.25) and (2.27), together with canonical (anti-) commutation relations

$$\begin{aligned}
[\psi^{*a}{}_A(x), \psi^{bB}(y)]_+ &= \delta(x-y)\delta^B{}_A\delta^{ab} \,, \\
\left[E^A{}_B(x,y), E^C{}_D(u,z)\right] &= \delta(x-u)\delta(y-z) \left(\delta^C{}_B E^A{}_D(x,y) - \delta^A{}_D E^C{}_B(x,y)\right) \,, \\
\left[E^A{}_B(x,y), U^C{}_D(u,z)\right] &= +\,\delta(x-u)\delta(y-z) \left(\delta^C{}_B U^A{}_D(x,y) - \frac{1}{3}\delta^A{}_B U^C{}_D(x,y)\right) \\
&\quad -\,\delta(x-z)\delta(y-u) \left(\delta^A{}_D U^C{}_B(y,x) - \frac{1}{3}\delta^A{}_B U^C{}_D(y,x)\right) \,.
\end{aligned}$$

3 The Observable Algebra

The observable algebra $\mathcal{O}(\Lambda)$ is defined by imposing the local Gauss law and gauge invariance.

The group of local gauge transformations acts on $\mathcal{F}(\Lambda)$ by automorphisms $x \to g^A{}_B(x)$ as follows:

$$\psi^{aA}(x) \ \rightarrow \ g^A{}_B(x)\psi^{aB}(x) \ , \tag{3.28}$$

$$U^A{}_B(x,y) \ \rightarrow \ g^A{}_C(x) U^C{}_D(x,y) (g^{-1})^D{}_B(y) \ , \tag{3.29}$$

$$E^A{}_B(x,y) \ \rightarrow \ g^A{}_C(x) E^C{}_D(x,y) (g^{-1})^D{}_B(x) \ . \tag{3.30}$$

It is easy to check that these transformations are generated by

$$\mathcal{G}^A{}_B(x) := \rho^A{}_B(x) - \sum_y E^A{}_B(x,y)\,, \tag{3.31}$$

where

$$\rho^A{}_B(x) = \sum_a \left(\psi^{*aA}(x)\psi^a{}_B(x) - \frac{1}{3}\delta^A_B \psi^{*aC}(x)\psi^a{}_C(x) \right) \tag{3.32}$$

is the local matter charge density. Observe that $\rho^A{}_A(x) = 0$.

To implement gauge invariance we have to take those elements of $\mathcal{F}(\Lambda)$, which commute with all generators $\mathcal{G}^A{}_B(x)$. Thus, the subalgebra of gauge invariant fields is, by definition, the commutant $\mathcal{A}(\mathcal{G})'$ of the algebra $\mathcal{A}(\mathcal{G})$, generated by the set $\{\mathcal{G}^A{}_B(x)\}$.

The local Gauss law at $x \in \Lambda^0$ has the form

$$\sum_y E^A{}_B(x,y) = \rho^A{}_B(x)\,, \tag{3.33}$$

with the sum taken over all points y adjacent to x. Imposing it on the subalgebra $\mathcal{A}(\mathcal{G})'$ of gauge invariant fields means factorizing the latter with respect to $\mathcal{I}(\Lambda) \cap \mathcal{A}(\mathcal{G})'$, where $\mathcal{I}(\Lambda)$ is the ideal generated by (3.33). Thus, the observable algebra is defined as follows:

$$\mathcal{O}(\Lambda) := \mathcal{A}(\mathcal{G})'/\{\mathcal{I}(\Lambda) \cap \mathcal{A}(\mathcal{G})'\}\,. \tag{3.34}$$

Obviously, every element of $\mathcal{O}(\Lambda)$ is represented by a gauge invariant element of $\mathcal{F}(\Lambda)$ and $\mathcal{O}(\Lambda)$ can be viewed as a $*$-subalgebra of $\mathcal{F}(\Lambda)$ generated by gauge invariant bosonic combinations of U and E and by gauge invariant combinations of ψ and ψ^* of mesonic and baryonic type, with the Gauss law inducing some identities between those generators.

It will be shown in the next section that there is a (gauge invariant), *additive* law, obtained by combining these two equations, which characterizes the local colour charge density carried by the lattice quantum fields. Additivity will allow us to obtain the global colour charge by adding up the local Gauss laws. In a subsequent paper we will show that the irreducible representations of $\mathcal{O}(\Lambda)$ are labeled by this global charge. This way, we get the physical Hilbert space $\mathcal{H}^{phys}$ as a direct sum of colour charge superselection sectors. The similar problem for QED on the lattice was solved in [8].

4 The Local Charge density

In this section we are going to analyze the local Gauss law (3.33). Suppose, for that purpose, that we are given a collection of operators $F^A{}_B$ in a Hilbert space $\mathcal{H}$, fulfilling $F^A{}_A = 0$ and $(F^A{}_B)^* = F_B{}^A$, realizing the canonical commutation relations for the Lie algebra $su(3)$:

$$[F^A{}_B, F^C{}_D] = \delta^C{}_B F^A{}_D - \delta^A{}_D F^C{}_B\,. \tag{4.1}$$

The field algebra $\mathcal{F}(\Lambda)$ provides us with two basic examples of this type: the electric field $E^A{}_B(x,y)$ on each lattice link, see (2.14), and the charge operator $\rho^A{}_B(x)$ at each lattice site, given by formula (3.32). Indeed, due to canonical anticommutation relations (2.3), the latter fulfills (4.1). Thus, the operators occurring on both sides of the local Gauss law fulfill (4.1).

Throughout this paper, we assume integrability of the Lie algebra representations under consideration. This means that for each F there exists a unitary representation of the group $SU(3)$

$$SU(3) \ni g \to \bar{F}(g) \in B(\mathcal{H}) \, , \tag{4.2}$$

associated with F.

It is easy to check that if F and G are two commuting representations of $su(3)$ then also $F+G$ is. Indeed, if $\bar{F}(g)$ and $\bar{G}(g)$ are representations of $SU(3)$ corresponding to F and G, then $F+G$ may be obtained by differentiating the representation $SU(3) \ni g \to \bar{F}(g)\bar{G}(g) \in B(\mathcal{H})$, where $B(\mathcal{H})$ denotes the C^*-algebra of bounded operators on $\mathcal{H}$. Moreover, $-F^*$ is also a representation of $su(3)$, corresponding to the following representation of $SU(3)$: $SU(3) \ni g \to (\bar{F}(g^{-1}))^* \in B(\mathcal{H})$.

Such a collection of operators is an *operator domain* in the sense of Woronowicz (see [12]). We are going to construct an operator function on this domain, i. e. a mapping $F \to \varphi(F)$ which satisfies $\varphi(UFU^{-1}) = U\varphi(F)U^{-1}$ for an arbitrary isometry U. We are going to prove that this function has the following properties:

$$\varphi(-F^*) = -\varphi(F) \, , \tag{4.3}$$

$$\varphi(F+G) = \varphi(F) + \varphi(G) \, , \tag{4.4}$$

for commuting F and G. This function will be built from the two gauge-invariant, self-adjoint and commuting (Casimir) operators K_2 and K_3 of F:

$$K_2 = F^A{}_B F^B{}_A \tag{4.5}$$

$$K_3 = \frac{1}{2}\left(F^A{}_B F^B{}_C F^C{}_A + F^A{}_B F^C{}_A F^B{}_C\right) . \tag{4.6}$$

The Hilbert space $\mathcal{H}$ splits into the direct sum of subspaces $\mathcal{H}_\alpha$ on which F acts irreducibly. Each of these subspaces is a common eigenspace of K_2 and K_3. Denoting the highest weight characterizing a given irreducible representation by (m,n), with m and n being nonnegative integers, the eigenvalues k_2 and k_3 of K_2 and K_3 are given by:

$$k_2 = \frac{2}{3}(m^2 + mn + n^2 + 3m + 3n) \, , \tag{4.7}$$

$$k_3 = \frac{1}{9}(m-n)(3+2m+n)(3+m+2n) \, . \tag{4.8}$$

It is easy to check that the above formulae may be uniquely solved with respect to m and

n yielding:

$$
\begin{aligned}
m = M(k_2,k_3) \quad &:= \quad \sqrt{\tfrac{2}{3}(k_2+2)}\Bigg(\cos\left(\frac{1}{3}\arccos\frac{\sqrt{6}k_3}{\sqrt{(k_2+2)^3}}+\frac{2}{3}\pi\right)+ \\
&\quad 2\cos\left(\frac{1}{3}\arccos\frac{\sqrt{6}k_3}{\sqrt{(k_2+2)^3}}\right)\Bigg)-1\,, \qquad (4.9)\\
n = N(k_2,k_3) \quad &:= \quad -\sqrt{\tfrac{2}{3}(k_2+2)}\Bigg(2\cos\left(\frac{1}{3}\arccos\frac{\sqrt{6}k_3}{\sqrt{(k_2+2)^3}}+\frac{2}{3}\pi\right)+ \\
&\quad \cos\left(\frac{1}{3}\arccos\frac{\sqrt{6}k_3}{\sqrt{(k_2+2)^3}}\right)\Bigg)-1\,. \qquad (4.10)
\end{aligned}
$$

Using these functions we may define a function with values in $\mathbb{Z}_3$, which may be identified with the center $\mathcal{C}$ of the gauge group $SU(3)$:

$$f(k_2,k_3) := (M(k_2,k_3)-N(k_2,k_3)) \,\mathbf{mod}\, 3\,. \tag{4.11}$$

For our purposes it is convenient to use the parametrization $\mathbb{Z}_3=(-1,0,1)$. Since K_2 and K_3 are commuting and self-adjoint, there exists an operator-valued function:

$$\varphi(F)=f(K_2(F),K_3(F))\,. \tag{4.12}$$

This means that $\varphi(F)$ may take eigenvalues $-1,0,1$ and that every irreducible subspace $\mathcal{H}_\alpha$ is an eigenspace of $\varphi(F)$ with eigenvalue $m-n$ **mod** 3.

To prove property (4.3) observe that $K_2(-F^*)=K_2(F)$, whereas $K_3(-F^*)=-K_3(F)$. Consequently, due to (4.7) and (4.8), we have $M(k_2,-k_3)=N(k_2,k_3)$ and $N(k_2,-k_3)=M(k_2,k_3)$ which implies (4.3). It remains to prove property (4.4). Let there be given two commuting representations, F and G, of $su(3)$ in $\mathcal{H}$. Denote the irreducible components of F by $\{\mathcal{H}^F_\alpha\}$ and of G by $\{\mathcal{H}^G_\beta\}$. The irreducible spaces may be chosen in such a way that $\mathcal{H}$ decomposes as follows:

$$\mathcal{H}=\bigoplus_{\alpha,\beta}\mathcal{H}^F_\alpha\cap\mathcal{H}^G_\beta\,. \tag{4.13}$$

Take $0\neq x\in\mathcal{H}^F_\alpha\cap\mathcal{H}^G_\beta$ and consider the space $\mathcal{H}^{F+G}_x\subset\mathcal{H}$ generated by vectors $\{\bar{F}(g)\bar{G}(g)x\}$, $g\in SU(3)$, where $\bar{F}$ and $\bar{G}$ are the corresponding ("integrated") representations of $SU(3)$. By construction, $\mathcal{H}^{F+G}_x$ carries an irreducible representation of $\bar{F}\bar{G}$. There exists a canonical embedding

$$\mathcal{H}^{F+G}_x\ni y\to T(y)\in\mathcal{H}^F_\alpha\otimes\mathcal{H}^G_\beta\,, \tag{4.14}$$

given by

$$T\left(\bar{F}(g)\bar{G}(g)x\right):=\bar{F}(g)x\otimes\bar{G}(g)x\,, \tag{4.15}$$

intertwining the representation $\bar{F}\bar{G}$ with $\bar{F}\otimes\bar{G}$. This means that $\bar{F}\bar{G}$, acting on $\mathcal{H}_x^{F+G}$, is equivalent to one of the irreducible components of $\bar{F}\otimes\bar{G}$. Passing again to representations of the Lie algebra $su(3)$, we conclude that $F+G$, acting on $\mathcal{H}_x^{F+G}$, is equivalent to one of the irreducible components of $F\otimes\mathbf{1}+\mathbf{1}\otimes G$, acting on $\mathcal{H}_\alpha^F\otimes\mathcal{H}_\beta^G$. Now, property (4.4) follows from the following

Lemma 1 *Let there be given two irreducible representations* (m,n) *and* (m',n') *of* $su(3)$*, together with the decomposition of their tensor product into irreducible components,*

$$(m,n)\otimes(m',n')=(m_1,n_1)\oplus\ldots\oplus(m_p,n_p)\,. \tag{4.16}$$

Then we have

$$(m-n)\,\mathbf{mod}\,3\,+(m'-n')\,\mathbf{mod}\,3\,=(m_i-n_i)\,\mathbf{mod}\,3\,, \tag{4.17}$$

for every $i=1,\ldots,p$.

Proof: We use the classification of irreducible representations in terms of Young-tableaux. For $su(3)$, the following two equations hold:

1. The number of boxes constituting the Young-tableau of (m_i,n_i) equals the sum of boxes of the tableaux corresponding to (m,n) and (m',n') minus $3p$, where p is a nonnegative integer.

2. The number of boxes of an arbitrary irreducible representation of $su(3)$ is equal to $m+2n\equiv m-n+3n$.

Taking the first equation modulo three and using the second equation yields the thesis.

Applying the operator function φ to the local Gauss law (3.33) and using additivity (4.4) of φ we obtain:

$$\sum_y\varphi(E(x,y))=\varphi(\rho(x))\,. \tag{4.18}$$

This is a gauge invariant equation for operators with eigenvalues in $\mathbb{Z}_3$, valid at every lattice site x. The quantity on the right hand side is the (gauge invariant) local colour charge density carried by the quark field.

5 The Global Charge and the Flux Law

Using the commutation relation between E and U, transformation law (2.27) for $E(x,y)$ may be rewritten in three equivalent ways:

$$\begin{aligned}
E^A{}_B(y,x) &= U^A{}_D(y,x)E^D{}_C(x,y)U^C{}_B(x,y)+\frac{8}{3}\delta^A_B && (5.1)\\
&= -U^C{}_B(x,y)E^D{}_C(x,y)U^A{}_D(y,x)-\frac{8}{3}\delta^A_B && (5.2)\\
&= -E^D{}_C(x,y)U^A{}_D(y,x)U^C{}_B(x,y)\,. && (5.3)
\end{aligned}$$

These equations imply that

$$K_2(E(x,y)) = K_2(E(y,x)) \,, \qquad K_3(E(x,y)) = -K_3(E(y,x)) \,. \tag{5.4}$$

Hence, we have:

$$\varphi(E(x,y)) + \varphi(E(y,x)) = 0 \,, \tag{5.5}$$

for every lattice bond (x,y).

Now we take the sum of equations (4.18) over all lattice sites $x \in \Lambda$. Due to the above identity, all terms on the left hand side cancel, except for contributions coming from the boundary. This way we obtain the total flux through the boundary $\partial\Lambda$ of Λ:

$$\Phi_{\partial\Lambda} := \sum_{x\in\partial\Lambda} \varphi(E(x,\infty)) \,, \tag{5.6}$$

where by $E(x,\infty)$ we denote the colour electric charge along the external link, connecting the point x on the boundary of Λ with the "rest of the world". On the right hand side we get the (gauge invariant) global colour charge, carried by the matter field

$$Q_\Lambda = \sum_{x\in\Lambda} \varphi(\rho(x)) \,. \tag{5.7}$$

Both quantities appearing in the global Gauss law

$$\Phi_{\partial\Lambda} = Q_\Lambda \,, \tag{5.8}$$

take values in the center $\mathbb{Z}_3$ of $SU(3)$. The "sum modulo three" is the composition law in $\mathbb{Z}_3$.

In the above discussion we have admitted non-zero values of $E(x,\infty)$ at boundary points $x \in \partial\Lambda$. In the remainder of this section we make some remarks on the nature of these objects. Our discussion will be rather heuristical, some points will be made precise in a subsequent paper.

1. One may treat Λ as a piece of a bigger lattice $\widetilde{\Lambda}$. Then the boundary flux operators $E(x,\infty)$ belong to $\mathcal{F}(\widetilde{\Lambda})$ and commute with $\mathcal{F}(\Lambda)$, (and also with $\mathcal{O}(\Lambda)$). They are external from the point of view of $\mathcal{F}(\Lambda)$ and measure the "violation of the local Gauss law" on the boundary $\partial\Lambda$:

   $$E(x,\infty) := \rho(x) - \sum_y E(y,x) \,. \tag{5.9}$$

   Non-vanishing of this element is equivalent to gauge dependence of quantum states under the action of boundary gauges $g(x) \in SU(3)$, $x \in \partial\Lambda$. Let us discuss this point in more detail. Every irreducible representation of $SU(3)$ is equivalent to some tensor representation. More precisely, denote by $T^m_n(\mathbb{C}^3)$ the space of m-contravariant, n-covariant tensors over $\mathbb{C}^3$, endowed with the natural scalar product induced by the scalar product on $\mathbb{C}^3$. Let $\mathbb{T}^m_n(\mathbb{C}^3) \subset T^m_n(\mathbb{C}^3)$ be the subspace of *irreducible*, i. e. completely symmetric and traceless tensors. These tensors form a Hilbert space $\mathcal{T}(\mathbb{C}^3)$ defined as the direct sum

$$\mathcal{T}(\mathbb{C}^3) := \bigoplus_{m,n} \mathbb{T}^m_n(\mathbb{C}^3) \,. \tag{5.10}$$

Under gauge transformations at $x \in \partial\Lambda$, physical states of QCD on Λ behave like elements of $\mathcal{T}(\mathbb{C}^3)$, whereas the subspaces $\mathbb{T}^m_n(\mathbb{C}^3)$ correspond to eigenspaces of the invariant operators $N(E(x,\infty))$ and $M(E(x,\infty))$, constructed from external fluxes (5.9). If one wants to include all these gauge invariant operators into an axiomatic formulation, as given in Section 2, one has different options. The remarks at the beginning of this point suggest to postulate that these operators commute with all elements of the observable algebra $\mathcal{O}(\Lambda)$. This corresponds to treating external fluxes as purely *classical objects*, describing extra superselection rules (cf. [4], [5], [6]).

2. Our results concerning the charge superselection structure of QED on a finite lattice [8] suggest, however, a second option. In [8] all irreducible representations of the observable algebra were classified in terms of the global electric charge Q_Λ contained in Λ. Representations differing only by the local electric flux distribution over the boundary $\partial\Lambda$, but having the same value of the global flux $\Phi_{\partial\Lambda}$ were proved to be equivalent. The redistribution of fluxes is obtained by the action of certain unitary operators, see [8], acting on the quantum state under consideration. Such a *redistribution operator* has the following (heuristic) counterpart in continuum QED:

$$U(n) := \exp\left(\frac{i}{\hbar}\int_\Sigma n(x)\cdot A(x)\; d^3x\right) \;, \tag{5.11}$$

where $n = (n^k)$ is a divergence-free (i. e. fulfilling $\partial_k n^k \equiv 0$) vector-density on $\Sigma \subset \mathbb{R}^3$. Formally, we have:

$$\tilde{E}^k(x) := \left(U^*(n)E(x)U(n)\right)^k = E^k(x) + n^k(x) \;. \tag{5.12}$$

It is obvious that replacing the field E by $\tilde{E}$ and leaving all other observables unchanged gives an equivalent representation of the observable algebra. Nevertheless, the flux field on the boundary $\partial\Sigma$ of the domain Σ is changed by $n^\perp(x)$, where "$\perp$" denotes the component orthogonal to $\partial\Sigma$.

In a subsequent paper we are going to present a similar result for lattice QCD. We shall prove that all irreducible representations of the observable algebra $\mathcal{O}(\Lambda)$ of QCD on a finite lattice are classified by the value of the global colour charge

Q_Λ, yielding three different superselection sectors labeled by elements of $\mathbb{Z}_3$. However, the local distribution of the (gauge invariant) gluon and anti-gluon fluxes $M(E(x,\infty))$ and $N(E(x,\infty))$ over the boundary $\partial\Lambda$ may be arbitrarily changed within one sector. The redistribution of fluxes is obtained by the following procedure. Take an arbitrary pair of points $\xi, \eta \in \partial\Lambda$ at the boundary and a path (collection of lattice links) $\gamma = \{(\xi, x_1), (x_1, x_2), \ldots, (x_k, \eta)\}$, connecting them. Define the following operator-valued matrix $U(\gamma) = (U^A{}_B(\gamma))$, where

$$U^A{}_B(\gamma) := \frac{1}{\sqrt{3}} \; U^A{}_{C_1}(\xi, x_1) U^{C_1}{}_{C_2}(x_1, x_2) \ldots U^{C_k}{}_B(x_k, \eta) \quad . \tag{5.13}$$

The action of $U(\gamma)$ on a quantum state ψ is, by definition, a collection $(U^A{}_B(\gamma)\psi)$ with an extra contravariant index A at ξ and an extra covariant index B at η. In general, the new state does not belong to any irreducible representation of $SU(3)$ at ξ and η, even if ψ did. This means that $U(\gamma)\psi$ *is not* an eigenstate of operators $M(E(\xi,\infty))$, $N(E(\xi,\infty))$, $M(E(\eta,\infty))$ and $N(E(\eta,\infty))$, even if ψ was. Decomposing it into irreducible representations, we observe, however, that the value of $\varphi(E(\xi,\infty))$ has been changed by plus one and the value of $\varphi(E(\eta,\infty))$ has been changed by minus one by this procedure. This suggests, that these objects should be rather treated as *quantum* and not as classical quantities. Only their sum, the global flux $\Phi_{\partial\Lambda}$, is a classical object proportional to the identity on every superselection sector. This point of view was strongly advocated by Staruszkiewicz already a decade ago (see [13]). We also refer to Giulini [14], who discussed decoherence phenomena in QED in terms of *quantum* fluxes at infinity.

3. Finally, we stress that it does not make sense to attribute any physical meaning to the external gluon or anti-gluon fluxes M and N themselves. It is only the quantity $(M - N)$ "modulo three" which makes sense. Heuristically, this again can be made transparent for the continuum theory: Indeed, if we want to assign the value of the flux through a piece $S \subset \partial\Sigma$ of the boundary of a domain Σ, we must be able to integrate E over S. Suppose, for that purpose, that S has been divided into small portions, $S = \bigcup_\alpha S_\alpha$. We have

$$\int_S E = \sum_\alpha \int_{S_\alpha} E \, . \tag{5.14}$$

The functions M, N or even $(M - N)$ *are not* additive and, therefore, cannot be applied to the left hand side in a way which is compatible with the Riemann sums arising in the integration. On the other hand, the function φ is additive. This enables us, in principle, to define the local $\mathbb{Z}_3$-valued flux through S as the sum of fluxes corresponding to its small portions S_α. Hence, if (one day in the future) continuum QCD will be constructed as an appropriate limit of lattice theories, also these surface fluxes will be defined as limits of appropriate Riemann sums, corresponding to these lattice approximations. We conclude that the only additive, gauge invariant aspect of both the colour charge density ρ and the surface fluxes E is carried by the $\mathbb{Z}_3$-valued quantities $\varphi(\rho)$ and $\varphi(E)$.

Acknowledgments

The authors are very much indebted to J. Dittmann, C. Śliwa, M. Schmidt and I.P. Volobuev for helpful discussions and remarks. This work was supported in part by the Polish KBN Grant Nr. 2 P03A 047 15. One of the authors (J. K.) is grateful to Professor E. Zeidler for his hospitality at the Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany.

References

[1] E. Seiler, *Gauge Theories as a Problem of Constructive Quantum Field Theory and Statistical Mechanics*, Lecture Notes in Phys., vol. 159, Springer (1982)
E. Seiler, "Constructive Quantum Field Theory: Fermions", in *Gauge Theories: Fundamental Interactions and Rigorous Results*, eds. P. Dita, V. Georgescu, R. Purice

[2] S. Doplicher, R. Haag and J. Roberts, Commun. Math. Phys. 23 (1971) 199

[3] J. Fröhlich, Commun. Math. Phys. 66 (1979), 223

[4] D. Buchholz, Commun. Math. Phys. 85 (1982) 49
D. Buchholz, Phys. Lett. B174 (1986) 331

[5] F. Strocchi and A. Wightman, J. Math. Phys. 15 (1974) 2198

[6] F. Strocchi, Phys. Rev. D 17 (1978) 2010

[7] K. Fredenhagen, M. Marcu, Commun. Math. Phys. 92 (1983) 81

[8] J. Kijowski, G. Rudolph and A. Thielman, Commun. Math. Phys. 188 (1997) 535

[9] J. Kijowski, G. Rudolph and M. Rudolph, Ann. Inst. H. Poincaré Vol. 68, n. 3, (1998) 285

[10] J. Kijowski, G. Rudolph and C. Sliwa, Lett. Math. Phys. 43 (1998) 299

[11] A. Klimyk, K. Schmüdgen, *Quantum Groups and Their Representations*, Springer 1997

[12] S. L. Woronowicz, *Duality in the C^*-algebra theory*, Proc. Intern. Congr. Mathematicians Warsaw 1983, pp. 1347-1356, PWN Warszawa.
S. L. Woronowicz, Rev. Math. Phys. 7 (1995) 481

[13] A. Staruszkiewicz, Ann. Phys. (N.Y.) 190 (1989) 354
A. Staruszkiewicz, Acta Phys. Pol. B23 (1992) 959
A. Staruszkiewicz, Acta Phys. Pol. B23 (1992) 927

[14] D. Giulini, *States, Symmetries and Superselection*, in Springer Lecture Notes in Physics "Decoherence: Theoretical, Experimental and Conceptual Problems", Springer Verlag 2000.