hep-th/0103111
Imperial/TP/0-01/12
Napoli/10/2001
QMUL-PH-01-03

Holographic Renormalisation and Anomalies

Jussi Kalkkinen[a], Dario Martelli[b] and Wolfgang Mück[c1]

[a] *Blackett Laboratory, Imperial College*
Prince Consort Road, London SW7 2BZ, U.K.

[b] *Queen Mary & Westfield College, University of London*
Mile End Road, London E1 4NS, U.K.

[c] *Dipartimento di Scienze Fisiche, Università di Napoli "Federico II"*
Via Cintia, 80126 Napoli, Italy

Abstract

The Weyl anomaly in the Holographic Renormalisation Group as implemented using Hamilton–Jacobi language is studied in detail. We investigate the breakdown of the descent equations in order to isolate the Weyl anomaly of the dual field theory close to the (UV) fixed point. We use the freedom of adding finite terms to the renormalised effective action in order to bring the anomalies in the expected form. We comment on different ways of describing the bare and renormalised schemes, and on possible interpretations of the descent equations as describing the renormalisation group flow non-perturbatively. We find that under suitable assumptions these relations may lead to a class of c-functions.

[1]E-mail addresses: `j.kalkkinen@ic.ac.uk` (J. K.), `d.martelli@qmw.ac.uk` (D. M.), `wolfgang.mueck@na.infn.it` (W. M.)

1 Introduction

The AdS/CFT correspondence [1, 2, 3] has provided a remarkable example of holographic duality [4, 5]. According to this duality a theory containing gravity, for which it is at least very difficult to define local observables, is described by a local quantum field theory living on the boundary of space-time, and, *vice versa*, physical quantities of the boundary field theory can be obtained from the bulk theory. Specifically, the AdS/CFT correspondence relates theories living on a $(d+1)$–dimensional anti-de Sitter (AdS) space to conformal field theories (CFTs) living on the d–dimensional boundary.

Among the several proposed generalisations, the idea of implementing a holographic version of the renormalisation group has attracted considerable attention [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16]. This can be realised by considering string theory vacua in which some scalar field acquires a non-trivial dependence along one direction, and the bulk space-time is no longer AdS. Such backgrounds are supposed to represent the running of the couplings in terms of the so-called holographic coordinate, which in turn is interpreted as an energy scale of the dual boundary field theory. Several such supergravity solutions have been found explicitly [17, 18, 19] (see [20] for a review), and some correlation functions of the boundary theories dual to these backgrounds have been computed [21, 22, 11, 9, 23, 12]. Even if most of the solutions available display some kind of singularity in the IR, here we are interested in considering non-singular solutions that smoothly interpolate between two fixed point CFT's. We will attempt to find non-perturbative relations that should be valid along these flows, and appropriately reduce to the fixed point results.

According to the general prescription of the holographic duality, one must solve the equations of motion of the bulk theory and evaluate the (suitably renormalised) on-shell action in order to obtain the generating functional of the boundary field theory. The only exceptions to this effort are anomalies (or holographic one-point functions), which can be evaluated without knowledge of the full bulk solutions. Here, an analysis of the asymptotic behaviour of the bulk fields towards the boundary is sufficient [24, 25, 26, 27].

Another interesting approach to the study of the holographic renormalisation group, using the Hamilton–Jacobi formulation of the bulk theory, has been proposed by de Boer, Verlinde and Verlinde (dBVV) [28, 29, 30, 31] and studied further in [32, 33, 34, 35, 36, 37]. The Hamilton equations of motion resemble the flow of couplings with the cut-off scale, and a holographic beta function can be defined. Moreover, the boundary counter terms necessary to renormalise the on-shell action are elegantly obtained using a derivative expansion and solving the resulting descent equations. Holographic anomalies occur whenever a descent equation breaks.

Despite the formal agreement of the results of dBVV's method with those of the asymptotic analysis, the exact relation between dBVV's interpretation of the holographic renormalisation group and the standard implementation of holography has remained somewhat obscure. In this paper, we shall try to elaborate this point by a systematic study of dBVV's descent equations up to level four. Our results for the anomalies in the vicinity of a given fixed point agree with those of [25], but are slightly more general. An interpretation of the descent equations along the full renormalisation group flow proves more difficult. We find that a holographic c-function for $d=2$ can be defined from the level two descent equations, but such an interpretation becomes problematic in higher dimensions.

As a simple generic model, we consider the action for scalar fields coupled to gravity in $d+1$ dimensions,[2]

$$S = \int d^{d+1}x\sqrt{\tilde{g}}\left[-\tilde{R} + \frac{1}{2}\tilde{g}^{ab}\partial_a\phi^I G_{IJ}(\phi)\partial_b\phi^J - V(\phi)\right] + 2\int d^dx\sqrt{g}\,H\ . \qquad (1)$$

The second term on the right hand side of eqn. (1) is the Gibbons–Hawking boundary term including the extrinsic curvature H of the boundary.

Assuming that the potential V has a local extremum at $\bar{\phi}$ with $V(\bar{\phi}) > 0$, there exists a solution to the classical equations of motion of (1) with $\phi = \bar{\phi}$ and AdS geometry with cosmological constant $-2\Lambda = V(\bar{\phi}) = d(d-1)/l^2$. Such backgrounds give rise to boundary CFTs, and the AdS/CFT correspondence has been well-studied in the past years. In particular, the holographic Weyl anomaly, derived first by Henningson and Skenderis [24], is, for $d = 2$ and $d = 4$, respectively,

$$\langle T \rangle = -c_2 R\ , \qquad (2)$$

$$\langle T \rangle = c_4 I_4 - a_4 E_4\ . \qquad (3)$$

(I_4 and E_4 are the square of the Weyl tensor and the Euler density in four dimensions, respectively.) The coefficients $c_2 = l$, $c_4 = l^3/8$ and $a_4 = l^3/8$ are identified as central charges of the boundary CFTs. Taking into account the coupling with nontrivial scalar fields gives rise to additional terms, that generically mix with curvature terms. Some of these terms were computed in [39] in the context of AdS/CFT. We will illustrate how such terms can be obtained for bare fields in the vicinity of the UV fixed point by performing Hamilton–Jacobi analysis, and eventually translated to renormalised quantities. We will also see that a non-perturbative extrapolation of the results towards the IR fixed point proves more difficult.

One can use the following arguments [38] to identify UV and IR fixed points. Scalar fluctuations around the constant scalar background have a mass $m^2 = -\partial^2 V/\partial\bar{\phi}^2$ and act as sources of primary conformal operators of dimension $\Delta = d/2 + \sqrt{d^2/4 + m^2 l^2}$. Thus, at a local minimum of V, tachyons are present in the bulk theory, which act as sources of relevant operators ($\Delta < d$) in the boundary CFT so that the field theory is unstable at this point (UV fixed point). *Vice versa*, at a local maximum of V, there are no tachyons in the bulk and only irrelevant operators in the CFT (IR fixed point). For two neighbouring extrema of V, we have $V_{\mathrm{UV}} < V_{\mathrm{IR}}$, $l_{\mathrm{UV}} > l_{\mathrm{IR}}$, and thus generically $c_{\mathrm{UV}} > c_{\mathrm{IR}}$ for the central charges.

Let us conclude this section with an outline of the rest of the paper. In Sec. 2 we briefly review dBVV's method. The descent equations are analysed in the vicinity of a fixed point in Sec. 3, and the possible anomalies are derived. In Sec. 4 we give conclusions and try to extend the meaning of the descent equations to the full renormalisation group flow. The lengthy expressions of level four are listed in the appendix.

[2] Our conventions for the curvature tensor are $R^\mu{}_{\nu\rho\lambda} = \partial_\rho\Gamma^\mu{}_{\nu\lambda} + \Gamma^\mu{}_{\rho\sigma}\Gamma^\sigma{}_{\nu\lambda} - (\rho \leftrightarrow \lambda)$, $R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}$. Here we have adorned with a tilde quantities belonging to $(d+1)$ space to distinguish them from those of the d-dimensional hypersurfaces.

2 Hamilton–Jacobi Approach

In this section, we briefly recall the Hamilton–Jacobi approach to the holographic renormalisation group. We shall put special emphasis on the differences with the standard implementation of holography in order to establish a suitable connection in later sections. The reader is referred to [28, 29, 30, 33, 34] for details of the method.

The bulk field theory with the action (1) is treated in ADM formalism. The constraints in phase space are

$$\mathcal{H} = \pi^{\mu\nu}\pi_{\mu\nu} - \frac{1}{d-1}\pi^2 + \frac{1}{2}\pi_I G^{IJ}\pi_J + R - \frac{1}{2}g^{\mu\nu}\partial_\mu\phi^I G_{IJ}\partial_\nu\phi^J + V \approx 0 \ , \tag{4}$$

$$\mathcal{H}^\nu = \nabla_\mu \pi^{\mu\nu} - \frac{1}{2}\pi_I \nabla^\nu \phi^I \approx 0 \ . \tag{5}$$

We will choose lapse and shift functions such that the bulk metric is

$$ds^2 = \tilde{g}_{ab}dx^a dx^b = dr^2 + g_{\mu\nu}dx^\mu dx^\nu \tag{6}$$

$(a, b = 0, \ldots, d,\ \mu, \nu = 1, \ldots, d)$ and the space-like foliation parameter is $x^0 = r$, to which we will refer as the cut-off.

For a classical trajectory the on-shell action, henceforth denoted by S, is a functional of the values $g_{\mu\nu}(r, x)$ and $\phi^I(r, x)$ at the cut-off (boundary values). The canonical momenta at the cut-off are given by the functional derivative of S with respect to the boundary data,

$$\pi_I = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta \phi^I} \ , \qquad\qquad \pi^{\mu\nu} = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}} \ , \tag{7}$$

so that both the constraints (4) and (5) become the Hamilton–Jacobi equations for S. Moreover, (one half of) the Hamilton equations of motion yield the flow equations,

$$\dot{\phi}^I = G^{IJ}\pi_J \ , \tag{8}$$

$$\dot{g}_{\mu\nu} = 2\pi_{\mu\nu} - \frac{2}{d-1}g_{\mu\nu}\pi \ . \tag{9}$$

According to the "AdS/CFT master formula" [3, 2], the on-shell action S is identified (up to possible local counter terms) with the generating functional of the boundary field theory, where the (suitably rescaled) boundary data play the role of the sources coupling to the boundary field theory operators. De Boer, Verlinde and Verlinde [28] proposed to use a derivative expansion to systematically determine the counter terms and wrote the on-shell action S as

$$S = [S]_0 + [S]_2 + [S]_4 + \cdots + \Gamma \ , \tag{10}$$

where

$$[S]_0 = \int d^d x \sqrt{g}\, U(\phi) \ , \tag{11}$$

$$[S]_2 = \int d^d x \sqrt{g} \left[\Phi(\phi)R + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi^I M_{IJ}(\phi)\partial_\nu\phi^J\right], \quad \text{etc.,} \tag{12}$$

are the local counter terms involving $0, 2, \ldots$ derivatives. Γ, which is generically non-local, is identified with the generating functional of the boundary field theory, so that

$$\langle \mathcal{O}_I \rangle = \frac{1}{\sqrt{g}} \frac{\delta \Gamma}{\delta \phi^I} \ , \qquad \qquad \langle T^{\mu\nu} \rangle = \frac{2}{\sqrt{g}} \frac{\delta \Gamma}{\delta g_{\mu\nu}} \ . \tag{13}$$

Such relations hold both for the bare $\phi(x, r)$ and renormalised $\hat{\phi}(x)$ fields [28]. We define the bare fields as solutions of full bulk equations of motion with the asymptotic boundary conditions at $r \to \infty$,

$$g_{\mu\nu}(x, r) = \mathrm{e}^{2r/l} \hat{g}_{\mu\nu}(x) + \cdots \ , \tag{14}$$

$$\phi(x, r) = \mathrm{e}^{(\Delta - d)r/l} \hat{\phi}(x) + \cdots \ . \tag{15}$$

The fields $\hat{g}_{\mu\nu}$ and $\hat{\phi}$ are the boundary field theory sources coupling to the energy momentum operator, $\hat{T}^{\mu\nu}$, and some scalar operator, $\hat{\mathcal{O}}$, respectively. Because in the Hamilton–Jacobi theory one naturally works in terms of bare couplings, we need to give a procedure for finding renormalised correlators. As the two sets of fields are related by the equations of motion, one finds

$$\frac{\delta}{\delta \hat{\phi}(x)} = \frac{\delta \phi(x, r)}{\delta \hat{\phi}(x)} \frac{\delta}{\delta \phi(x, r)} \equiv \mathcal{A}(x, r) \frac{\delta}{\delta \phi(x, r)} \tag{16}$$

where $\mathcal{A}(x, r)$ acts as a bulk–boundary propagator. This operator is in general a complicated non-local operator, whose explicit form is not known. Nevertheless, its expansion for a single free scalar in fixed gravitational background is known [25]: There, it turns out that apart from the scaling factor already explicit in (15), $\mathcal{A}(x, r)$ is a series in the boundary d'Alembertians. For the purposes of the present analysis, these corrections coming from the Jacobian (16) will give rise to higher order derivative corrections, and it turns out that they can be consistently neglected (cf. Sec. 3).

Consistency of perturbation theory in the bulk field theory requires $\Delta < d$, *i.e.*, we are dealing with operators at the UV fixed point.

The constraint (5) is satisfied by each of the local terms $[S]_{2n}$, whereas for Γ it represents the Ward identity for diffeomorphisms in the boundary field theory,[3]

$$\nabla_\mu \langle T^{\mu\nu} \rangle - \langle \mathcal{O}_I \rangle \nabla^\nu \phi^I = 0 \ . \tag{17}$$

The constraint (4) is used to determine the unknown functions U, Φ, M_{IJ}, etc. These equations do not determine a unique solution. Actually, to completely fix one it is necessary to impose still more boundary conditions by comparing with the asymptotic AdS solution, as done in Sec. 3.

In the original proposal, de Boer, Verlinde and Verlinde included all possible local terms $[S]_{2n}$ for $2n < d$ in the derivative expansion. Fukuma, Matsuma and Sakai [33] pointed out that adding terms $[S]_d$ is physically irrelevant, since it corresponds to adding

[3]This turns out to be the case for scalar fields interacting with gravity, whereas the analogous equation in the presence of interacting fermion fields obtains unusual contributions [35].

total derivative terms to the anomaly, or terms proportional to the beta function that therefore vanish at the fixed point. We shall show here that these terms in $[S]_d$ are needed to cancel spurious terms in the anomaly in the vicinity of the fixed point.

The terms of the Hamiltonian (4) stemming from the local $[S]_{2n}$ can be split into expressions of level $2n$, where $2n$ is the number of space-time derivatives. Thus, one obtains for level zero

$$[\mathcal{H}]_0 = \frac{U}{2(d-1)}\left[-\frac{d}{2}U + \frac{U}{4(d-1)}\beta^I G_{IJ}\beta^J + \frac{2(d-1)}{U}V\right], \tag{18}$$

where β^I denotes the holographic beta function and is defined as

$$\beta^I = -G^{IJ}\frac{2(d-1)}{U}\partial_J U \ . \tag{19}$$

Notice that this actually takes into account only the lowest order in derivatives of the full beta function that follows from varying the local part of S. This is in fact what appears in the Callan–Symanzik equation [28].

At level two, one finds

$$[\mathcal{H}]_2 = \frac{U}{2(d-1)}\left\{-c_2 R - \lambda_{IJ}\partial^\mu\phi^I\partial_\mu\phi^J + \nabla_\mu\left[\left(2(d-1)\partial_I\Phi + M_{IJ}\beta^J\right)\partial^\mu\phi^I\right]\right\}, \tag{20}$$

where

$$c_2 = (d-2)\Phi - \frac{2(d-1)}{U} + \beta^I\partial_I\Phi \ , \tag{21}$$

$$\lambda_{IJ} = \frac{d-2}{2}M_{IJ} + \frac{d-1}{U}G_{IJ} + \frac{1}{2}\left(M_{IK}\partial_J\beta^K + M_{JK}\partial_I\beta^K + \partial_K M_{IJ}\beta^K\right). \tag{22}$$

In addition to the $[\mathcal{H}]_{2n}$ coming from the local terms in S, there are terms involving the non-local Γ. The first such term contains the anomalies,

$$[\mathcal{H}]_\Gamma = -\frac{U}{2(d-1)}\left(\langle T\rangle + \beta^I\langle\mathcal{O}_I\rangle\right). \tag{23}$$

The Hamiltonian constraint (4) is incorporated by setting each term in $[\mathcal{H}]_{2n}$ to zero for $2n < d$ and trying to solve this system of descent equations for the unknowns U, Φ, M_{IJ}, etc. For $n = 0$, this gives an equation for U in terms of V,

$$\frac{1}{2d(d-1)}\beta^I G_{IJ}\beta^J = 1 - \frac{4(d-1)V}{dU^2} \ , \tag{24}$$

after substituting eqn. (19). We must observe here that eqn. (24) does not determine U uniquely. In fact, we will revert to AdS/CFT results in order to make sure that the lowest orders in the Taylor expansion of U correspond to the leading local terms of the on-shell action. As we are interested in interpolating renormalisation group trajectories, we should also assume that U has another critical point in the IR. This boils down to a nonperturbative boundary condition on the U we are considering, and determines it completely [32].

For $n=1$ and $d>2$, we have the three equations

$$2(d-1)\partial_I\Phi + \beta^J M_{IJ} = 0 \ , \tag{25}$$

$$c_2 = 0 \ , \tag{26}$$

$$\lambda_{IJ} = 0 \ . \tag{27}$$

Although these are three equations for only two unknowns, the system is not over–determined, since the equations are not functionally independent. In fact, eqn. (25) ensures that

$$(d-1)\partial_I c_2 = -\lambda_{IJ}\beta^J \ , \tag{28}$$

so that eqn. (26) implies eqn. (27).

Writing down $[\mathcal{H}]_{2n}$ becomes increasingly involved for $n>2$. We list $[\mathcal{H}]_4$ in the appendix for later use.

If one could solve the descent equations to any level, starting from level zero, one would obtain all counter terms $[S]_{2n}$. However, typically, the descent equations will break at some level n, in which case one must revert to $[\mathcal{H}]_{2n}+[\mathcal{H}]_\Gamma=0$, which yields the holographic anomaly of the boundary field theory. Moreover, the anomaly involves also the unknown functions of $[S]_d$, which should reflect the ambiguity of choosing a renormalisation scheme. We will use the freedom to add these finite terms to bring the anomaly to a standard form close to the fixed point.

3 Anomalies in the Vicinity of a Fixed Point

In this section we analyse the vicinity around a given fixed point by expansion of the descent equations of level zero to four in powers of the scalar field. To a certain extent, this has been done in [33], but we repeat the analysis here to be self-contained.

For simplicity, we shall consider the case of a single scalar field with $G_{II}=1$. Furthermore, without loss of generality, let us assume that the fixed point is at $\phi=0$. For the holographic beta function, we shall substitute

$$\beta = (\Delta - d)\phi \ . \tag{29}$$

We are also neglecting higher derivative terms of the full beta function, as discussed below eqn. (19). Generically, our notation will be

$$A(\phi) = A^{(0)} + A^{(1)}\phi + \frac{1}{2}A^{(2)}\phi^2 + \cdots \ , \tag{30}$$

although for most quantities some of the terms are absent.

Let us start with the expansion of U, which is essentially the input at the top of the descent equations. The constant $U^{(0)}$ is known from pure gravity calculations in AdS/CFT

[39, 40], and the quadratic term follows directly from eqns. (19) and (29), while there is no linear term,

$$U(\phi) = -\frac{2(d-1)}{l} + \frac{\Delta - d}{2l}\phi^2 + \cdots \ . \tag{31}$$

Thus, we have made sure that U is the leading term of the on-shell action. Now, writing the potential V as

$$V(\phi) = V_0 - \frac{1}{2}m^2\phi^2 + \cdots \tag{32}$$

and setting $[\mathcal{H}]_0 = 0$, yields

$$V_0 = \frac{d(d-1)^2}{l} \quad \text{and} \quad m^2 l^2 = \Delta(\Delta - d) \ , \tag{33}$$

reproducing the AdS cosmological constant at the fixed point and the AdS/CFT mass formula. To quadratic order in ϕ, all terms of the level zero constraint can be solved, so that no anomaly is obtained at this level. This comes somewhat as a surprise, since from AdS/CFT one should expect an anomaly proportional to ϕ^2 for operators of dimension $\Delta = d/2$ [25]. However, the absence of the anomaly can be explained by the fact that here we are dealing with bare fields. We will show at the end of this section how it can be recovered when translating to renormalised quantities.

Let us proceed to the level two equations starting with the coefficient λ_{IJ}, given in eqn. (22). One finds

$$\lambda = M^{(0)}\left(\Delta - \frac{d}{2} - 1\right) - \frac{l}{2} + \mathcal{O}(\phi) \ . \tag{34}$$

Obviously, for operators of dimension $\Delta \neq d/2 + 1$, $\lambda = 0$ can be solved to lowest order with the result

$$M^{(0)} = \frac{l}{2(\Delta - d/2 - 1)} \ . \tag{35}$$

In contrast, for operators of dimension $\Delta = d/2 + 1$, λ contributes a term $l/2\, \partial_\mu\phi\partial^\mu\phi$ to the anomaly, just as expected. Moreover, in this anomalous case the value $M^{(0)}$ remains undetermined.

Given $M^{(0)}$, eqn. (25) can always be solved and gives

$$\Phi^{(2)} = -M^{(0)}\frac{\Delta - d}{2(d-1)} \ . \tag{36}$$

Last, eqn. (21) is expanded as

$$c_2 = (d-2)\Phi^{(0)} + l + \phi^2\left[\left(\Delta - \frac{d}{2} - 1\right)\Phi^{(2)} + \frac{(\Delta - d)l}{4(d-1)}\right] + \cdots \ . \tag{37}$$

For operators of dimension $\Delta \neq d/2 + 1$, the quadratic term vanishes by virtue of eqns. (36) and (35), while, for $\Delta = d/2 + 1$, c_2 contributes a term $(d-2)l/[8(d-1)]\phi^2 R$ to the anomaly. Moreover, in dimensions $d \neq 2$, the constant part of c_2 can be solved and gives

$$\Phi^{(0)} = -\frac{l}{d-2} \ , \tag{38}$$

while, for $d=2$, we obtain the standard Weyl anomaly $-lR$.

Our treatment of the level two descent equations illustrates the occurrence of anomalies in the Hamilton–Jacobi approach to holography. In summary, we have found the following level two anomalies in the neighbourhood of a given fixed point,

$$\langle T\rangle + \beta\langle\mathcal{O}\rangle = \begin{cases} \frac{l}{2}\partial_\mu\phi\partial^\mu\phi + \frac{(d-2)l}{8(d-1)}\phi^2 R + \mathcal{O}(\phi^3)\ , & \text{if } \Delta = d/2+1, \\ -lR + \mathcal{O}(\phi^3)\ , & \text{if } d=2. \end{cases} \tag{39}$$

Let us now come to the anomalies in the neighbourhood of the fixed point stemming from the level four descent equations. These will occur, if $d=4$, or if $\Delta = d/2+2$. We have listed the level four terms of the Hamiltonian in appendix A. It must be stressed that the level four equations can only be used as they stand, if the level two equations can be solved, *i.e.*, if there is no level two anomaly. In these cases terms like $R_{\mu\nu}\langle T^{\mu\nu}\rangle$ contribute to level four. This means, that the following arguments are valid only, if $d\neq 2$ and $\Delta \neq d/2+1$.

Starting with the term in $[\mathcal{H}]_4$ proportional to R^2, we have from eqn. (62)

$$\begin{aligned}\psi_1 = &-(d-4)\Psi_1^{(0)} + \frac{dl}{4(d-1)}\left(\Phi^{(0)}\right)^2 - \phi^2\left\{\left(\Delta - \frac{d}{2} - 2\right)\Psi_1^{(2)}\right. \\ &\left. -\frac{dl}{4(d-1)}\left[\frac{\Delta-d}{4(d-1)}\left(\Phi^{(0)}\right)^2 + \Phi^{(0)}\Phi^{(2)}\right] + \frac{l}{2}\left(\Phi^{(2)}\right)^2\right\} + \cdots\ ,\end{aligned} \tag{40}$$

where eqn. (31) has been used. Thus, if $d=4$, we cannot solve for $\Psi_1^{(0)}$, which results in a contribution to the anomaly, $\psi_1 = l/3\left(\Phi^{(0)}\right)^2 = l^3/12$. If $\Delta = d/2+2$, $\Psi_1^{(2)}$ remains undetermined, and we have

$$\psi_1 = -\frac{(d-4)l^3\phi^2}{128(d-1)^2(d-2)^2}(d^3 - 4d^2 + 16d - 16)\ .$$

Similarly, from eqn. (63) we find

$$\begin{aligned}\psi_2 = &-(d-4)\Psi_2^{(0)} - l\left(\Phi^{(0)}\right)^2 \\ &-\phi^2\left\{\left(\Delta - \frac{d}{2} - 2\right)\Psi_2^{(2)} + l\left[\frac{\Delta-d}{4(d-1)}\left(\Phi^{(0)}\right)^2 + \Phi^{(0)}\Phi^{(2)}\right]\right\} + \cdots\ .\end{aligned} \tag{41}$$

Again, for $d=4$, we find the contribution $\psi_2 = -l\left(\Phi^{(0)}\right)^2 = -l^3/4$, whereas, for $\Delta = d/2+2$, we have

$$\psi_2 = \frac{(d-4)l^3\phi^2}{8(d-2)^2}\ .$$

It is easy to see from eqn. (64) that there is no impediment to setting $\psi_3 = 0$ to quadratic order, which therefore does not generate anomaly terms. Similarly, to quadratic order in ϕ, one can solve $\chi = \xi = \zeta = \kappa = 0$.

The next equation that generates an anomaly term is (69). Indeed, we have

$$\sigma = -2\left(\Delta - \frac{d}{2} - 2\right)L^{(1)}\phi + lM^{(0)}\Phi^{(2)}\phi + \cdots\ , \tag{42}$$

so that, for $\Delta = d/2+2$,

$$\sigma = \frac{(d-4)l^3\phi}{16(d-1)} + \cdots \ .$$

The remaining descent equations, (70), (71), (72), (73) and (74), generate further anomaly terms, but, in contrast to the anomaly terms obtained so far, they are not uniquely determined. This non-uniqueness reflects the possibility of having various renormalisation schemes, and the results should be physically equivalent. In the following, we shall give one possible choice. Let us start by setting eqn. (74) to zero, which is achieved at lowest order in ϕ by $(d-2)N^{(0)} = 4(d-1)K^{(0)}$. Then, from eqn. (73), we have

$$\epsilon = -2\left(\Delta - \frac{d}{2} - 2\right) A^{(0)} - \frac{l}{2} M^{(0)} + \cdots \ , \tag{43}$$

so that, for $\Delta = d/2+2$, $\epsilon = -l^3/8$.

Next, eqn. (70) becomes

$$\gamma = -2\left(\Delta - \frac{d}{2} - 2\right) K^{(0)} - \frac{(d-2)l}{4(d-1)} \Phi^{(0)} M^{(0)} + \cdots \ , \tag{44}$$

and the contribution to the anomaly for $\Delta = d/2+2$ is $\gamma = l^3/[8(d-1)]$.

Finally, there is no difficulty to set either ν or α to zero to lowest order in ϕ, and we shall choose $\alpha = 0$. Then, eqns. (71) and (72) yield

$$\nu = \phi \left[2\left(\Delta - \frac{d}{2} - 2\right) N^{(0)} + l\Phi^{(0)} M^{(0)} \right] + \cdots \ . \tag{45}$$

Hence, for $\Delta = d/2+2$, we find $\nu = -\phi l^3/[2(d-2)]$.

Let us now summarise the level four anomalies. For $d = 4$, we have

$$\langle T \rangle + \beta \langle \mathcal{O} \rangle = \frac{l^3}{4}\left(\frac{1}{3}R^2 - R_{\mu\nu}R^{\mu\nu}\right) + \mathcal{O}(\phi^3) \ , \tag{46}$$

whereas, collecting all the terms for $\Delta = d/2+2$, we have

$$\begin{aligned} \langle T \rangle + \beta \langle \mathcal{O} \rangle = & -\frac{l^3\phi^2(d-4)}{8(d-2)^2}\left(\frac{d^3-4d^2+16d-16}{16(d-1)^2}R^2 - R_{\mu\nu}R^{\mu\nu}\right) \\ & + \frac{l^3(d-4)}{16(d-1)} R\phi\nabla^2\phi + \frac{l^3}{8(d-1)} R\partial_\mu\phi\partial^\mu\phi \\ & - \frac{l^3}{8}\nabla^2\phi\nabla^2\phi - \frac{l^3}{2(d-2)}\left(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R\right)\phi\nabla_\mu\partial_\nu\phi + \mathcal{O}(\phi^3) \ . \end{aligned} \tag{47}$$

Next, let us translate the bare anomalies (39), (46) and (47) to renormalised ones, where the asymptotic fields $\hat{g}_{\mu\nu}$ and $\hat{\phi}$ are the field theory sources. This amounts to taking into account the Jacobian $\mathcal{A}(x,r)$ discussed around eqn. (16). As the anomaly is homogeneous in derivatives, the derivative corrections will contribute in higher orders and can therefore be neglected here. Notice that had we had level two anomalies, they would have contributed on level four through the first correction in $\mathcal{A}$. In a general anomaly higher order derivative terms do appear together with appropriately suppressed powers of $\mathrm{e}^{-r/l}$, thus not contributing to the fixed point result.

What will be important, however, is the r-dependent scaling in $\mathcal{A}$. Writing

$$\langle\hat{\mathcal{O}}\rangle = \frac{1}{\sqrt{\hat{g}}}\frac{\delta\Gamma}{\delta\hat{\phi}} \ , \qquad\qquad \langle\hat{T}^{\mu\nu}\rangle = \frac{2}{\sqrt{\hat{g}}}\frac{\delta\Gamma}{\delta\hat{g}_{\mu\nu}} \ , \tag{48}$$

we find, using eqns. (14) and (15),

$$\langle T\rangle = \mathrm{e}^{-dr/l}\langle\hat{T}\rangle + \cdots \ , \tag{49}$$

$$\langle\mathcal{O}\rangle = \mathrm{e}^{-\Delta r/l}\langle\hat{\mathcal{O}}\rangle + \cdots \ . \tag{50}$$

Moreover,

$$\beta = (\Delta - d)\mathrm{e}^{(\Delta-d)r/l}\hat{\phi} + \cdots = \mathrm{e}^{(\Delta-d)r/l}\hat{\beta} + \cdots \ , \tag{51}$$

so that

$$\langle T\rangle + \beta\langle\mathcal{O}\rangle = \mathrm{e}^{-dr/l}\left(\langle\hat{T}\rangle + \hat{\beta}\langle\hat{\mathcal{O}}\rangle\right) + \cdots \ . \tag{52}$$

It is easy to see that the right hand sides of eqns. (39), (46) and (47) scale in the same manner when written in terms of the hatted fields. Therefore, our results directly translate into the anomalies of the boundary field theories with properly identified sources.

Finally, let us return to the $\Delta = d/2$ anomaly mentioned earlier below eqn. (33). In this case, the leading asymptotic behaviour of the field ϕ is not given by eqn. (15), but rather by

$$\phi(x,r) = r\mathrm{e}^{-dr/(2l)}\hat{\phi}(x) + \cdots \tag{53}$$

as follows from the equations of motion in the asymptotic AdS background. Hence,

$$\dot{\phi} = \mathrm{e}^{-dr/(2l)}\hat{\phi} + \frac{(\Delta-d)}{l}\phi + \cdots \ .$$

This behaviour should be described by the flow equation (8), which means that the momentum π_I must receive non-negligible contributions from the one-point function, $\pi_I = \partial_I U + \langle\mathcal{O}_I\rangle + \cdots$, which gives

$$\langle\mathcal{O}\rangle = \mathrm{e}^{-dr/(2l)}\hat{\phi} + \cdots \ .$$

Thus, after going to renormalised quantities, we find the level zero contribution

$$\langle\hat{T}\rangle + \hat{\beta}\langle\hat{\mathcal{O}}\rangle = -\frac{d}{2}\hat{\phi}^2 + \mathcal{O}(\hat{\phi}^3) \ , \tag{54}$$

which is the anomaly of the expected form.

4 Discussion

We have found that dBVV's Hamilton–Jacobi approach to the holographic renormalisation group can be successfully applied in order to derive anomalies in the vicinity of a given fixed point. Regarding the boundary data at the cut-off as the bare sources of the boundary field theory, the anomalies are formally obtained, except in the case $\Delta = d/2$, and the results are easily translated to the proper renormalised sources.

We should point out that our method of renormalising the sources differs from that of dBVV's in that they considered the boundary data at some renormalisation scale r_R as the renormalised sources [28]. In contrast, we, in agreement with the standard method for calculating holographic correlation functions [2, 3, 8, 23], consider the asymptotically rescaled boundary data—hatted fields in eqns. (14) and (15)—as the renormalised sources. This renders the interpretation of the descent equations rather difficult, if the cut-off boundary data are far away from the UV fixed point. In other words, even if we can write down an expansion for the bulk–boundary propagator $\mathcal{A}$ near a UV fixed point, the full expression would be needed for a nonperturbative analysis of the interpolating trajectory.

In this paper we have observed that the assumption that the flow of the bulk fields is essentially determined by the leading local term $[S]_0$—implicit in eqn. (19)—becomes invalid. We have seen at the end of the last section that a non-local term had to be taken into account for operators $\Delta = d/2$ even close to the fixed point. Moreover, there is no guarantee that one can solve the descent equations up to an arbitrary level $2n$. Consider, for example, operators of dimension $\Delta = 3$ in $d = 4$. Here, the level two equations are broken already close to the UV fixed point. This means that there is a level two anomaly, which contributes via terms like $R_{\mu\nu}\langle T^{\mu\nu}\rangle$ to the level four equations.

The level two descent equations seem to be the only ones which can be analysed globally. If one takes the point of view that the cut-off boundary data are the bare sources of the boundary field theory at the cut-off, one is able to identify a class of c-functions in $d = 2$. Consider again the level two terms of the Hamiltonian, eqn. (20), and $d = 2$. There is no obvious difficulty to set the total derivative term to zero globally, *i.e.*, we let

$$-2\partial_I \Phi = \beta^J M_{IJ} \ , \tag{55}$$

and obtain the anomaly

$$\langle T\rangle + \beta^I \langle \mathcal{O}_I\rangle = -c_2 R - \lambda_{IJ}\partial_\mu \phi^I \partial^\mu \phi^J + \cdots \ , \tag{56}$$

where the ellipses stand for terms with more derivatives or curvatures, which would arise from higher levels. As shown in Sec. 3, setting $M_{IJ} = \Phi = 0$ gives unexpected anomalies in the vicinity of a fixed point except for marginal operators ($\Delta = d = 2$). The coefficients c_2 and λ_{IJ} are functions of ϕ at the cut-off. Moreover, eqn. (55) ensures that eqn. (28) is still valid, *i.e.*,

$$\partial_I c_2 = -\lambda_{IJ}\beta^J \ . \tag{57}$$

It is interesting to notice that the flow-derivative of c_2 is given by

$$\dot{c}_2 = \frac{U}{2}\beta^I\beta^J\lambda_{IJ} \ . \tag{58}$$

Therefore assuming λ_{IJ} positive definite, c_2 is monotonous and can actually be interpreted as a c-function of the holographic renormalisation group.

The presence of an anomaly means that the functions M_{IJ} and Φ cannot be adjusted as to make c_2 and λ_{IJ} vanish identically. We may therefore try to use the freedom, already seen in Sec. 3, to choose these functions in such a way that c_2 becomes monotonous. Nevertheless we must also make sure that $\lambda_{IJ} = 0$ at the fixed points without marginal operators. Generically, then, there will be ambiguities in the definition of the c-function, as also expected from the field theory analysis [11].

On the other hand, starting from the UV fixed point, for non-marginal operators ($\Delta \neq 2$) one could also attempt to solve $\lambda_{IJ} = 0$ globally. Then it would follow from eqn. (57) that c_2 is constant. This would be a "non-renormalisation" theorem for the c-function as it would be independent of both the cut-off and the couplings. However, since this would lead to an M_{IJ} that is singular at the IR fixed point, one should conclude that the perturbative renormalisation scheme we are using cannot be extrapolated towards the IR, and therefore a global choice of a regular M_{IJ} is not possible.

It would be interesting to compare our results, where the sources are the asymptotically r-independent parts of the coupling, to those of dBVV, where the relationship between bare and renormalised quantities is somewhat different [28]—there for a given field configuration the field at the cut-off is the source. Here asymptotically rescaled fields are cut-off independent for any given field configuration, but the cut-off boundary is still needed to regularise the on-shell action. This radial behaviour appears to be responsible for some of the expected QFT features observed above. But, we leave the investigation of the meaning of this open for further studies.

Acknowledgements

We would like to thank M. Bertolini, M. Bianchi, A. Karch, A. Miemiec, A. C. Petkou, and H. S. Reall for discussions and useful comments. We also thank the organisers of the RTN conference "Latest developments in M-Theory", Paris, February 1–8, 2001, during which this work was essentially completed. The work was supported by the European Commission RTN programme HPRN-CT-2000-00131, in which J. K. is associated with Imperial College London and W. M. with INFN, Sezione di Frascati. D. M. acknowledges partial support from PPARC through SPG#613.

A Level Four Hamiltonian

Here, we give a summary of the level four terms in the Hamiltonian. The local terms $[S]_4$ are

$$[S]_4 = \int d^d x \sqrt{g} \Bigg[\Psi_1 R^2 + \Psi_2 R_{\mu\nu} R^{\mu\nu} + \Psi_3 R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} + L_I R \nabla^2 \phi^I \\ + K_{IJ} R \partial_\mu \phi^I \partial^\mu \phi^J + N_{IJ} \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \partial_\mu \phi^I \partial_\nu \phi^J + A_{IJ} \nabla^2 \phi^I \nabla^2 \phi^J \\ + B_{IJK} \nabla_\mu \nabla_\nu \phi^I \partial^\mu \phi^J \partial^\nu \phi^K + C_{IJKL} \partial_\mu \phi^I \partial^\mu \phi^J \partial_\nu \phi^K \partial^\nu \phi^L \Bigg] . \tag{59}$$

According to the general procedure, this leads to

$$[\mathcal{H}]_4 = \frac{U}{2(d-1)} \left(-2[\pi]_4 - \beta^I [\pi_I]_4 \right) + [\pi_{\mu\nu}]_2 [\pi^{\mu\nu}]_2 - \frac{1}{d-1} \left([\pi]_2 \right)^2 + \frac{1}{2} [\pi_I]_2 G^{IJ} [\pi_J]_2 \tag{60}$$

$$= \frac{U}{2(d-1)} \Bigg[R^2 \psi_1 + R_{\mu\nu} R^{\mu\nu} \psi_2 + R_{\mu\nu\lambda\rho} R^{\mu\nu\lambda\rho} \psi_3 + \nabla^2 (R\chi) + \nabla_\mu (R \partial^\mu \phi^I \xi_I) \\ + \nabla^2 (\nabla^2 \phi^I \zeta_I) + \nabla_\mu (\nabla^2 \phi^I \partial^\mu \phi^J \kappa_{IJ}) + R \nabla^2 \phi^I \sigma_I + R \partial_\mu \phi^I \partial^\mu \phi^J \gamma_{IJ} \\ + \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \nabla_\mu \partial_\nu \phi^I \nu_I + \left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R \right) \partial_\mu \phi^I \partial_\nu \phi^J \alpha_{IJ} \\ + \nabla^2 \phi^I \nabla^2 \phi^J \epsilon_{IJ} + \nabla_\mu \partial_\nu \phi^I \nabla^\mu \partial^\nu \phi^J \tau_{IJ} + \nabla^2 \phi^I \partial_\mu \phi^J \partial^\mu \phi^K \rho_{IJK} \\ + \nabla_\mu \partial_\nu \phi^I \partial^\mu \phi^J \partial^\nu \phi^K \theta_{IJK} + \partial_\mu \phi^I \partial^\mu \phi^J \partial_\nu \phi^K \partial^\nu \phi^L \omega_{IJKL} \Bigg] . \tag{61}$$

The coefficients in eqn. (61) are given by

$$\psi_1 = -(d-4)\Psi_1 - \beta^I\partial_I\Psi_1 - \frac{d}{2U}\Phi^2 + \frac{d-1}{U}\partial_I\Phi G^{IJ}\partial_J\Phi, \tag{62}$$

$$\psi_2 = -(d-4)\Psi_2 - \beta^I\partial_I\Psi_2 + \frac{2(d-1)}{U}\Phi^2, \tag{63}$$

$$\psi_3 = -(d-4)\Psi_3 - \beta^I\partial_I\Psi_3, \tag{64}$$

$$\chi = 4(d-1)\Psi_1 + d\Psi_2 + 4\Psi_3 - \beta^I L_I, \tag{65}$$

$$\xi_I = (d-2)L_I + 2L_J\partial_I\beta^J + 2K_{IJ}\beta^J, \tag{66}$$

$$\zeta_I = 2(d-1)L_I - 2A_{IJ}\beta^J, \tag{67}$$

$$\kappa_{IJ} = 4(d-1)K_{IJ} + 2(d-2)A_{IJ} + 4A_{IK}\partial_J\beta^K - 2\beta^K(B_{KIJ} - B_{IJK}), \tag{68}$$

$$\sigma_I = -(d-4)L_I - \beta^J\partial_J L_I - L_J\partial_I\beta^J - \frac{2(d-1)}{U}M_{IJ}G^{JK}\partial_K\Phi, \tag{69}$$

$$\begin{aligned}\gamma_{IJ} = &-\frac{d-2}{2}N_{IJ} + (d+2)K_{IJ} - \beta^K\partial_K K_{IJ} - K_{IK}\partial_J\beta^K - K_{JK}\partial_I\beta^K \\ &- L_K\partial_I\partial_J\beta^K + \frac{1}{2}\beta^K(B_{IJK} + B_{JIK} - B_{KIJ}) + \frac{d-2}{2U}\Phi M_{IJ} \\ &- \frac{d-1}{U}\partial_L\Phi G^{LK}(\partial_J M_{IK} + \partial_I M_{JK} - \partial_K M_{IJ}),\end{aligned} \tag{70}$$

$$\nu_I = 2(d-2)\partial_I\Psi_2 + 8\partial_I\Psi_3 + 2\beta^J N_{IJ} - \frac{4(d-1)}{U}\Phi\partial_I\Phi, \tag{71}$$

$$\begin{aligned}\alpha_{IJ} = &\, 2(d-2)\partial_I\partial_J\Psi_2 + 8\partial_I\partial_J\Psi_3 - 2(d-3)N_{IJ} + 4(d-1)K_{IJ} \\ &+ \beta^K(\partial_I N_{JK} + \partial_J N_{IK} - \partial_K N_{IJ} + B_{IJK} + B_{JIK} - B_{KIJ}) \\ &+ \frac{2(d-1)}{U}\Phi(M_{IJ} - 2\partial_I\partial_J\Phi),\end{aligned} \tag{72}$$

$$\begin{aligned}\epsilon_{IJ} = &\,(d-2)N_{IJ} - 4(d-1)K_{IJ} - (d-4)A_{IJ} - \beta^K\partial_K A_{IJ} - A_{IK}\partial_J\beta^K - A_{JK}\partial_I\beta^K \\ &- \beta^K(B_{IJK} + B_{JIK} - B_{KIJ}) - \frac{2(d-1)}{U}\partial_I\Phi\partial_J\Phi + \frac{d-1}{U}M_{IK}G^{KL}M_{LJ},\end{aligned} \tag{73}$$

$$\tau_{IJ} = -(d-2)N_{IJ} + 4(d-1)K_{IJ} + \beta^K(B_{IJK} + B_{JIK} - B_{KIJ}) + \frac{2(d-1)}{U}\partial_I\Phi\partial_J\Phi. \tag{74}$$

The remaining three coefficients ρ_{IJK}, θ_{IJK} and ω_{IJKL} will not be needed.

References

[1] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998), `hep-th/9711200`.

[2] E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998), `hep-th/9802150`.

[3] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Phys. Lett. B* **428**, 105 (1998), `hep-th/9802109`.

[4] G. 't Hooft, gr-qc/9310026.

[5] L. Susskind, *J. Math. Phys.* **36**, 6377 (1995), hep-th/9409089.

[6] E. T. Akhmedov, *Phys. Lett. B* **442**, 152 (1998), hep-th/9806217.

[7] B. Balasubrasmanian and P. Kraus, *Phys. Rev. Lett.* **83**, 3605 (1999), hep-th/9903190.

[8] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, hep-th/9904017.

[9] O. DeWolfe and D. Z. Freedman, hep-th/0002226.

[10] E. Alvarez and C. Gómez, *Nucl. Phys. B* **541**, 441 (1999), hep-th/9807226.

[11] D. Anselmi, L. Girardello, M. Porrati and A. Zaffaroni, *Phys. Lett. B* **481**, 346 (2000), hep-th/0002066.

[12] M. Bianchi, O. DeWolfe, D. Z. Freedman and K. Pilch, *J. High Energy Phys.* **01**, 021 (2001), hep-th/0009156.

[13] N. Hambli, hep-th/0010054.

[14] K. Graham, I. Runkel and G. M. T. Watts, hep-th/0010082.

[15] N. P. Warner, hep-th/0011207.

[16] N. Evans, hep-th/0012097.

[17] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, *Nucl. Phys. B* **569**, 451 (2000), hep-th/9909047.

[18] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, *J. High Energy Phys.* **07**, 038 (2000), hep-th/9906194.

[19] A. Brandhuber and K. Sfetsos, *Adv. Theor. Math. Phys.* hep-th/9906201, to appear.

[20] S. S. Gubser, hep-th/0002160.

[21] I. Chepelev and R. Roiban, *Phys. Lett. B* **462**, 74 (1999), hep-th/9906224.

[22] R. C. Rashkov and K. S. Viswanathan, *Phys. Rev. D* **62**, 046009 (2000), hep-th/9911160.

[23] G. Arutyunov, S. Frolov and S. Theisen, *Phys. Lett. B* **484**, 295 (2000), hep-th/0003116.

[24] M. Henningson and K. Skenderis, *J. High Energy Phys.* **07**, 023 (1998), hep-th/9806087.

[25] S. de Haro, K. Skenderis and S. N. Solodukhin, hep-th/0002230.

[26] A. Schwimmer and S. Theisen, *J. High Energy Phys.* **08**, 032 (2000), hep-th/0008082.

[27] R. Manvelyan, R. Mkrtchyan and H. J. W. Müller-Kirsten, hep-th/0103082.

[28] J. de Boer, E. Verlinde and H. Verlinde, *J. High Energy Phys.* **08**, 003 (2000), hep-th/9912012.

[29] E. Verlinde and H. Verlinde, *J. High Energy Phys.* **05**, 034 (2000), `hep-th/9912018`.

[30] E. Verlinde, *Class. Quant. Grav.* **17**, 1277 (2000), `hep-th/9912058`.

[31] J. de Boer, `hep-th/0101026`.

[32] V. L. Campos, G. Ferretti, H. Larsson, D. Martelli and B. E. W. Nilsson, *J. High Energy Phys.* **0006**, 023 (2000), `hep-th/0003151`.

[33] M. Fukuma, S. Matsuura and T. Sakai, *Prog. Theor. Phys.* **104**, 1089 (2000), `hep-th/0007062`.

[34] M. Fukuma and T. Sakai, *Mod. Phys. Lett. A* **15**, 1703 (2000), `hep-th/0007200`.

[35] J. Kalkkinen and D. Martelli, *Nucl. Phys. B* **596**, 415 (2001), `hep-th/0007234`.

[36] S. Nojiri, S. D. Odintsov and S. Ogushi, *Phys. Lett. B* **500**, 199 (2001), `hep-th/0011182`.

[37] J. Kalkkinen and D. Martelli, `hep-th/0101028`.

[38] D. Z. Freedman and P. Henry-Labordère, `hep-th/0011086`.

[39] H. Liu and A. A. Tseytlin, *Nucl. Phys. B* **533**, 88 (1998), `hep-th/9804083`.

[40] G. E. Arutyunov and S. A. Frolov, *Nucl. Phys. B* **544**, 576 (1999), `hep-th/9806216`.