Type IIA Supergravity Excitations
in Plane-Wave Background

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Abstract

We study the low-lying excitations of Type IIA superstring theory in a plane wave background with 24 supersymmetries. In the light-cone gauge, the superstring action has $\mathcal{N} = (4, 4)$ supersymmetry and is exactly solvable, since it is quadratic in superstring coordinates. We obtain explicitly the spectrum of the Type IIA supergravity fluctuation modes in the plane wave background and give its correspondence with the spectrum of string states from the zero-mode sector of the light-cone superstring Hamiltonian.

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1 Introduction

Recently, much attention has been paid to the M theory on the maximally supersymmetric eleven-dimensional pp-wave background after the work by Berenstein, Maldacena and Nastase [1]. The formulation of the theory has been given in the framework of Matrix model and the resulting Matrix model has turned out to have intriguing properties [1, 2]. One may take two basic properties, which may be regarded as the sources of the other. One is the removal of flat directions because of the presence of bosonic mass terms and another is the time dependent supersymmetry making the bosons and fermions have different masses. The Matrix model with these properties shows a peculiar perturbative aspect such as the protected multiplet and has various kinds of non-perturbative BPS states [2]-[8]. The discussions on other aspects of the model may be found in [9]-[15].

In addition to the Matrix model description of the M theory, there is another possible way of studying the M theory by going down to the ten dimensional Type IIA superstring theory through the strong-weak duality between M and Type IIA superstring theory. In this approach, there is an advantage that one may use the rather well-developed machinery of string theory compared to that of Matrix theory.

To obtain the IIA superstring theory from the M theory, one should pick up an isometry direction on which the M theory is compactified. As for the eleven-dimensional maximally supersymmetric pp-wave geometry, there are various spacelike isometries, along which the M theory can be compactified [16]. One choice of the compact direction leads to the following ten-dimensional pp-wave background [12, 17, 18]:

$$ds^2 = -2dx^+ dx^- - A(x^I)(dx^+)^2 + \sum_{I=1}^{8}(dx^I)^2 \ ,$$

$$\bar{F}_{+123} = \mu \ , \quad \bar{F}_{+4} = -\frac{\mu}{3} \ , \tag{1.1}$$

where the quantities with bar mean that they are background and we have defined

$$A(x^I) = \sum_{i=1}^{4} \frac{\mu^2}{9}(x^i)^2 + \sum_{i'=5}^{8} \frac{\mu^2}{36}(x^{i'})^2 \ . \tag{1.2}$$

It has been shown in [12, 17] that this background admits only 24 Killing spinors.

The various aspects of the Type IIA Green-Schwarz (GS) superstring theory have been studied in [17, 18, 19, 20]. In particular, in the light-cone gauge formulation, the Type IIA GS superstring action in this background has been shown to be quadratic in terms of the string coordinates indicating the exact solvability of the theory. Furthermore, the light-cone gauge superstring action has the interesting linearly-realized worldsheet supersymmetry identified as $\mathcal{N} = (4,4)$ [17]. The situation is similar to the Type IIB case except that the IIB

background is maximally supersymmetric [21, 22]. We note that rather general discussions on the superstring in the pp-wave background have been given in [23]-[27] .

Having the solvable superstring theory, it may be a natural step to investigate the spectrum of quantized string states and their dynamics. In this paper, we concentrate on the low-lying perturbative string states and study their spectrum in the field theoretic way. That is, we obtain the physical excitation modes of the Type IIA supergravity in the pp-wave background, (1.1), and give their correspondence with the low-lying string states from the zero-mode sector of the string theory. As alluded above, the study of IIA string theory is motivated by a hope to understand the M theory in a controllable way. In the situation that we have the Matrix model as another way of studying the M theory, it may be expected that the low-energy perturbative study of the IIA string theory is helpful in uncovering the physics related to the perturbative spectrum of the Matrix model in the pp-wave. The work in this paper may be regarded as the first step in this direction of study.

The organization of this paper is as follows. In section 2, following Refs. [17, 19], we review the derivation of Type IIA GS superstring action in the pp-wave background, (1.1), which is given in the light-cone gauge formulation, and then the quantization of the superstring. After the review, we give the low-lying spectrum of string states. In section 3, we consider the fields of Type IIA supergravity in the pp-wave background and obtain the physical supergravity excitation modes around the background. We shall see how the spectrum of the supergravity modes corresponds to that of low-lying string states obtained in section 2. Finally, the conclusion and discussion follow in section 4.

2   Type IIA superstring in plane-wave background

In this section, we review the light-cone gauge fixed Type IIA GS superstring action in the pp-wave background and its quantization following [17, 19].

It is very complicate to get the full expression of the GS superstring action in the general background (see, for example, [28, 29]). However, in the case at hand, we can use the fact that eleven-dimensional pp-wave geometry can be thought as a special limit of $AdS_4 \times S^7$ geometry on which the full supermembrane action is constructed using coset method [30]. The full IIA GS superstring action on this geometry can be obtained by the double dimensional reduction [31] of the supermembrane action of [30] in the Penrose limit [32]. The superstring action is simplified drastically in the light-cone gauge chosen as

$$\Gamma^+ \theta = 0 \ , \quad X^+ = \alpha' p^+ \tau \ , \tag{2.1}$$

where p^+ is the total momentum conjugate to X^- and τ is the worldsheet time coordinate.

In this light-cone gauge, IIA string action is given by[1]

$$S_{LC} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[\eta^{mn}\partial_m X^I \partial_n X^I + \frac{m^2}{9}(X^i)^2 + \frac{m^2}{36}(X^{i'})^2 \right.$$

$$\left. + \bar{\theta}\Gamma^-\partial_\tau\theta + \bar{\theta}\Gamma^{-9}\partial_\sigma\theta - \frac{m}{4}\bar{\theta}\Gamma^-\left(\Gamma^{123} + \frac{1}{3}\Gamma^{49}\right)\theta \right] , \qquad (2.2)$$

where

$$m \equiv \mu\alpha' p^+ \qquad (2.3)$$

is a mass parameter which characterizes the masses of the worldsheet fields, and the Majorana fermion θ is the combination of Majorana-Weyl fermions θ^1 and θ^2 with opposite ten dimensional $SO(1,9)$ chiralities, that is, $\theta = \theta^1 + \theta^2$. (1 (2) is for positive (negative) chirality.) Therefore the light-cone gauge-fixed action S_{LC} is quadratic in bosonic as well as fermionic fields and thus describes a free theory much the same as in the IIB string theory [21] on the pp-wave geometry [3].

The characteristic feature of IIA string theory in pp-wave background is the structure of worldsheet supersymmetry. Sixteen spacetime supersymmetries with transformation parameter ϵ satisfying $\Gamma^+\epsilon = 0$ are non-linearly realized on the worldsheet action. As is typical in light-cone GS superstring, the remaining eight spacetime supersymmetries, combined with appropriate kappa transformations, turn into worldsheet (4,4) supersymmetry of Yang-Mills type [17]. In order to see this more clearly, we rewrite the action S_{LC} in the 16 component spinor notation. We should first introduce the representation for $SO(1,9)$ gamma matrices which we take as

$$\Gamma^0 = -i\sigma^2 \otimes \mathbf{1}_{16} , \quad \Gamma^{11} = \sigma^1 \otimes \mathbf{1}_{16} , \quad \Gamma^I = \sigma^3 \otimes \gamma^I ,$$

$$\Gamma^9 = -\sigma^3 \otimes \gamma^9 , \quad \Gamma^\pm = \frac{1}{\sqrt{2}}(\Gamma^0 \pm \Gamma^{11}) , \qquad (2.4)$$

where σ's are Pauli matrices, and $\mathbf{1}_{16}$ the 16×16 unit matrix. γ^I are the 16×16 symmetric real gamma matrices satisfying the spin(8) Clifford algebra $\{\gamma^I, \gamma^J\} = 2\delta^{IJ}$, which are reducible to the $\mathbf{8_s} + \mathbf{8_c}$ representation of spin(8). We note that, since the pp-wave background (1.1) has been obtained by compactifying the eleven dimensional pp-wave along the x^9 direction [17], Γ^9 is the $SO(1,9)$ chirality operator and γ^9 becomes $SO(8)$ chirality operator,

$$\gamma^9 = \gamma^1 \cdots \gamma^8 . \qquad (2.5)$$

Then, with the spinor notation $\theta^A = \frac{1}{2^{1/4}} \begin{pmatrix} 0 \\ \psi^A \end{pmatrix}$ satisfying the light-cone gauge (2.1)

---

[1]η^{mn} is the flat worldsheet metric with m, n taking values of τ, σ.

(Superscript A denotes the $S(1,9)$ chirality.), the action S_{LC} becomes

$$S_{LC} = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[\eta^{mn}\partial_m X^I \partial_n X^I + \frac{m^2}{9}(X^i)^2 + \frac{m^2}{36}(X^{i'})^2 \right.$$

$$\left. -i\psi_+^1 \partial_+ \psi_+^1 - i\psi_-^1 \partial_+ \psi_-^1 - i\psi_+^2 \partial_- \psi_+^2 - i\psi_-^2 \partial_- \psi_-^2 + 2i\frac{m}{3}\psi_+^2 \gamma^4 \psi_-^1 - 2i\frac{m}{6}\psi_-^2 \gamma^4 \psi_+^1 \right] ,$$

$$(2.6)$$

where $\partial_\pm = \partial_\tau \pm \partial_\sigma$. Here the sign of subscript in $\psi_\pm^A$ represents the eigenvalue of γ^{1234}. In our convention, fermion has the same $SO(1,9)$ and $SO(8)$ chirality measured by Γ^9 and γ^9, respectively.

Thus, among sixteen fermionic components in total, eight with $\gamma^{12349} = 1$ have the mass of $m/6$ and the other eight with $\gamma^{12349} = -1$ the mass of $m/3$, which are identical with the masses of bosons. Therefore the theory contains two supermultiplets $(X^i, \psi_-^1, \psi_+^2)$ and $(X^{i'}, \psi_+^1, \psi_-^2)$ of worldsheet (4,4) supersymmetry with the masses $m/3$ and $m/6$, respectively.

Let us now turn to the quantization of closed string in the pp-wave background [19] and consider the low-lying string states constructed by acting the zero-mode creation operators on the vacuum which should correspond to the Type IIA supergravity excitations in the pp-wave background.

We first consider the bosonic sector of the theory. The equations of motion for the bosonic coordinates X^I are read off from the action (2.6) as

$$\eta^{mn}\partial_m \partial_n X^i - \left(\frac{m}{3}\right)^2 X^i = 0 , \quad \eta^{mn}\partial_m \partial_n X^{i'} - \left(\frac{m}{6}\right)^2 X^{i'} = 0 , \qquad (2.7)$$

where the fields are subject to the periodic boundary condition,

$$X^I(\tau, \sigma + 2\pi) = X^I(\tau, \sigma) . \qquad (2.8)$$

The solutions of the above equations are given in the form of mode expansion and found to be

$$X^i(\tau, \sigma) = x^i \cos\left(\frac{m}{3}\tau\right) + \alpha' p^i \frac{3}{m} \sin\left(\frac{m}{3}\tau\right) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{\omega_n}(\alpha_n^i \phi_n(\tau, \sigma) + \tilde{\alpha}_n^i \tilde{\phi}_n(\tau, \sigma)) ,$$

$$X^{i'}(\tau, \sigma) = x^{i'} \cos\left(\frac{m}{6}\tau\right) + \alpha' p^{i'} \frac{6}{m} \sin\left(\frac{m}{6}\tau\right) + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{1}{\omega_n'}(\alpha_n^{i'} \phi_n'(\tau, \sigma) + \tilde{\alpha}_n^{i'} \tilde{\phi}_n'(\tau, \sigma)) ,$$

$$(2.9)$$

where x^I and p^I are center-of-mass variables defined in the usual manner, coefficients for the zero-modes, and α_n^I and $\tilde{\alpha}_n^I$ are the expansion coefficients for the non-zero modes. The basis functions for the non-zero modes are given by

$$\phi_n(\tau, \sigma) = e^{-i\omega_n \tau - in\sigma} , \quad \tilde{\phi}_n(\tau, \sigma) = e^{-i\omega_n \tau + in\sigma} , \qquad (2.10)$$

$$\phi_n'(\tau, \sigma) = e^{-i\omega_n' \tau - in\sigma} , \quad \tilde{\phi}_n'(\tau, \sigma) = e^{-i\omega_n' \tau + in\sigma} , \qquad (2.11)$$

with the wave frequencies

$$\omega_n = \text{sign}(n)\sqrt{\left(\frac{m}{3}\right)^2 + n^2} \ , \quad \omega'_n = \text{sign}(n)\sqrt{\left(\frac{m}{6}\right)^2 + n^2} \ . \tag{2.12}$$

We note that the reality of X^I requires that $\alpha_n^{I\dagger} = \alpha_{-n}^I$ and $\tilde{\alpha}_n^{I\dagger} = \tilde{\alpha}_{-n}^I$.

We promote the expansion coefficients in the mode expansions (2.9) to operators. By using the canonical equal time commutation relations for the bosonic fields,

$$[X^I(\tau, \sigma), \mathcal{P}^J(\tau, \sigma')] = i\delta^{IJ}\delta(\sigma - \sigma') \ , \tag{2.13}$$

where $\mathcal{P}^J = \partial_\tau X^J/2\pi\alpha'$ is the canonical conjugate momentum of X^J, we have the following commutation relations between mode operators:

$$[x^I, p^J] = i\delta^{IJ} \ , \quad [\alpha_n^i, \alpha_m^j] = \omega_n \delta^{ij}\delta_{n+m,0} \ , \quad [\alpha_n^{i'}, \alpha_m^{j'}] = \omega'_n \delta^{i'j'}\delta_{n+m,0} \ . \tag{2.14}$$

Let us next turn to the fermionic sector of the theory. The fermionic fields are split into two parts according to the $(4, 4)$ supersymmetry; (ψ_-^1, ψ_+^2) and (ψ_+^1, ψ_-^2). We first consider the former case. The equations of motion for ψ_-^1 and ψ_+^2 are obtained as

$$\partial_+\psi_-^1 + \frac{m}{3}\gamma^4\psi_+^2 = 0 \ , \quad \partial_-\psi_+^2 - \frac{m}{3}\gamma^4\psi_-^1 = 0 \ . \tag{2.15}$$

The non-zero mode solutions of these equations are given by using the modes, (2.10). For the zero mode part of the solution, we impose a condition that, at $\tau = 0$, the solution behaves just as that of massless case. The mode expansions for the fermionic coordinates are then

$$\begin{aligned}
\psi_-^1(\tau, \sigma) &= c_0\tilde{\psi}_0 \cos\left(\frac{m}{3}\tau\right) - c_0\gamma^4\psi_0 \sin\left(\frac{m}{3}\tau\right) \\
&\quad + \sum_{n\neq 0} c_n\left(\tilde{\psi}_n\tilde{\phi}_n(\tau, \sigma) - i\frac{3}{m}(\omega_n - n)\gamma^4\psi_n\phi_n(\tau, \sigma)\right) \ , \\
\psi_+^2(\tau, \sigma) &= c_0\psi_0 \cos\left(\frac{m}{3}\tau\right) + c_0\gamma^4\tilde{\psi}_0 \sin\left(\frac{m}{3}\tau\right) \\
&\quad + \sum_{n\neq 0} c_n\left(\psi_n\phi_n(\tau, \sigma) + i\frac{3}{m}(\omega_n - n)\gamma^4\tilde{\psi}_n\tilde{\phi}_n(\tau, \sigma)\right) \ ,
\end{aligned} \tag{2.16}$$

where $\gamma^{1234}\psi_n = \psi_n$ and $\gamma^{1234}\tilde{\psi}_n = -\tilde{\psi}_n$ for all n, and c_n are the normalization constants given by

$$c_0 = \sqrt{\alpha'} \ , \quad c_n = \frac{\sqrt{\alpha'}}{\sqrt{1 + \left(\frac{3}{m}\right)^2 (\omega_n - n)^2}} \ .$$

Promoting the expansion coefficients to operators and using the canonical equal time anti-commutation relations,

$$\{\psi_\pm^A(\tau, \sigma), \psi_\pm^B(\tau, \sigma')\} = 2\pi\alpha'\delta^{AB}\delta(\sigma - \sigma') \ , \tag{2.17}$$

the following anti-commutation relations between mode operators are obtained.

$$\{\psi_n, \psi_m\} = \delta_{n+m,0} \ , \quad \{\tilde{\psi}_n, \tilde{\psi}_m\} = \delta_{n+m,0} \ . \tag{2.18}$$

The quantization of fermionic coordinates in the other $(4,4)$ supermultiplet proceed along the same way. The equations of motion for ψ^1_+ and ψ^2_- are respectively

$$\partial_+ \psi^1_+ - \frac{m}{6}\gamma^4 \psi^2_- = 0 \ , \quad \partial_- \psi^2_- + \frac{m}{6}\gamma^4 \psi^1_+ = 0 \ , \tag{2.19}$$

whose solutions are found to be

$$
\begin{aligned}
\psi^1_+(\tau,\sigma) &= c'_0 \tilde{\psi}'_0 \cos\left(\frac{m}{6}\tau\right) + c'_0 \gamma^4 \psi'_0 \sin\left(\frac{m}{6}\tau\right) \\
&\quad + \sum_{n\neq 0} c'_n \left(\tilde{\psi}'_n \tilde{\phi}'_n(\tau,\sigma) + i\frac{6}{m}(\omega'_n - n)\gamma^4 \psi'_n \phi'_n(\tau,\sigma) \right) \ , \\
\psi^2_-(\tau,\sigma) &= c'_0 \psi'_0 \cos\left(\frac{m}{6}\tau\right) - c'_0 \gamma^4 \tilde{\psi}'_0 \sin\left(\frac{m}{6}\tau\right) \\
&\quad + \sum_{n\neq 0} c'_n \left(\psi'_n \phi'_n(\tau,\sigma) - i\frac{6}{m}(\omega'_n - n)\gamma^4 \tilde{\psi}'_n \tilde{\phi}'_n(\tau,\sigma) \right) \ , \tag{2.20}
\end{aligned}
$$

where $\gamma^{1234}\psi'_n = -\psi'_n$, $\gamma^{1234}\tilde{\psi}'_n = \tilde{\psi}'_n$, and

$$c'_0 = \sqrt{\alpha'} \ , \quad c'_n = \frac{\sqrt{\alpha'}}{\sqrt{1 + \left(\frac{6}{m}\right)^2 (\omega'_n - n)^2}} \ .$$

Then the equal time anti-commutation relations, (2.17), lead us to have

$$\{\psi'_n, \psi'_m\} = \delta_{n+m,0} \ , \quad \{\tilde{\psi}'_n, \tilde{\psi}'_m\} = \delta_{n+m,0} \ . \tag{2.21}$$

We now consider the light-cone Hamiltonian of the theory, which is written as[2]

$$H_{LC} = \int_0^{2\pi} d\sigma P^- = \frac{2\pi}{p^+}\int_0^{2\pi} d\sigma \mathcal{H} \ . \tag{2.22}$$

The $\mathcal{H}$ is the Hamiltonian density obtained from Eq. (2.6) as

$$
\begin{aligned}
\mathcal{H} &= \frac{1}{2}(\mathcal{P}^I)^2 + \frac{1}{2}(\partial_\sigma X^I)^2 + \frac{1}{2}\left(\frac{m}{3}\right)^2 (X^i)^2 + \frac{1}{2}\left(\frac{m}{6}\right)^2 (X^{i'})^2 \\
&\quad - \frac{i}{2}\psi^1_- \partial_\sigma \psi^1_- + \frac{i}{2}\psi^2_+ \partial_\sigma \psi^2_+ + i\frac{m}{3}\psi^2_+ \gamma^4 \psi^1_- \\
&\quad - \frac{i}{2}\psi^1_+ \partial_\sigma \psi^1_+ + \frac{i}{2}\psi^2_- \partial_\sigma \psi^2_- - i\frac{m}{6}\psi^2_- \gamma^4 \psi^1_+ \ . \tag{2.23}
\end{aligned}
$$

By plugging the mode expansions for the fields, Eqs. (2.9), (2.16), and (2.20), into Eq. (2.22), the light-cone Hamiltonian becomes

$$H_{LC} = E_0 + E + \tilde{E} \ , \tag{2.24}$$

---

[2]From now on, we set $2\pi\alpha' = 1$ for notational convenience.

where E_0 is the zero mode contribution and E, $\tilde{E}$ are the contributions of the non-zero modes:

$$
\begin{aligned}
E_0 &= \frac{2\pi^2}{p^+}\left(\left(\frac{p^I}{2\pi}\right)^2 + \left(\frac{m}{3}\right)^2(x^i)^2 + \left(\frac{m}{6}\right)^2(x^{i'})^2 - \frac{i}{\pi}\frac{m}{3}\tilde{\psi}_0\gamma^4\psi_0 + \frac{i}{\pi}\frac{m}{6}\tilde{\psi}_0'\gamma^4\psi_0'\right) , \\
E &= \frac{\pi}{p^+}\sum_{n\neq 0}(\alpha_{-n}^I\alpha_n^I + \omega_n\psi_{-n}\psi_n + \omega_n'\psi_{-n}'\psi_n') , \\
\tilde{E} &= \frac{\pi}{p^+}\sum_{n\neq 0}(\tilde{\alpha}_{-n}^I\tilde{\alpha}_n^I + \omega_n\tilde{\psi}_{-n}\tilde{\psi}_n + \omega_n'\tilde{\psi}_{-n}'\tilde{\psi}_n') .
\end{aligned}
\tag{2.25}
$$

In the quantized version, the modes in the expression of Hamiltonian become operators with the commutation relations, (2.14), (2.18), and (2.21), and should be properly normal ordered. For the string oscillator contributions, E and $\tilde{E}$, we place operator with negative mode number to the left of operator with positive mode number as in the flat case. The normal ordered expressions of them are then given by

$$
\begin{aligned}
E &= \frac{2\pi}{p^+}\sum_{n=1}^{\infty}(\alpha_{-n}^I\alpha_n^I + \omega_n\psi_{-n}\psi_n + \omega_n'\psi_{-n}'\psi_n') , \\
\tilde{E} &= \frac{2\pi}{p^+}\sum_{n=1}^{\infty}(\tilde{\alpha}_{-n}^I\tilde{\alpha}_n^I + \omega_n\tilde{\psi}_{-n}\tilde{\psi}_n + \omega_n'\tilde{\psi}_{-n}'\tilde{\psi}_n') .
\end{aligned}
\tag{2.26}
$$

Here we note that there is no zero-point energy because bosonic contributions are exactly canceled by those of fermions.

The zero mode contribution is the Hamiltonian for the simple harmonic oscillators and massive fermions. For the bosonic part, we introduce the usual creation and annihilation operators as

$$
\begin{aligned}
a^{i\dagger} &= \sqrt{\frac{3\pi}{m}}\left(\frac{p^i}{2\pi} + i\frac{m}{3}x^i\right) , \quad a^i = \sqrt{\frac{3\pi}{m}}\left(\frac{p^i}{2\pi} - i\frac{m}{3}x^i\right) , \\
a^{i'\dagger} &= \sqrt{\frac{6\pi}{m}}\left(\frac{p^{i'}}{2\pi} + i\frac{m}{6}x^{i'}\right) , \quad a^{i'} = \sqrt{\frac{6\pi}{m}}\left(\frac{p^{i'}}{2\pi} - i\frac{m}{6}x^{i'}\right) ,
\end{aligned}
\tag{2.27}
$$

whose commutation relations are read as, from Eq. (2.14),

$$
[a^I, a^{J\dagger}] = \delta^{IJ} .
\tag{2.28}
$$

As for the fermionic creation and annihilation operators, we take the following combination of modes.

$$
\begin{aligned}
\chi^\dagger &= \frac{1}{\sqrt{2}}(\psi_0 - i\gamma^4\tilde{\psi}_0) , \quad \chi = \frac{1}{\sqrt{2}}(\psi_0 + i\gamma^4\tilde{\psi}_0) , \\
\chi'^\dagger &= \frac{1}{\sqrt{2}}(\psi_0' + i\gamma^4\tilde{\psi}_0') , \quad \chi' = \frac{1}{\sqrt{2}}(\psi_0' - i\gamma^4\tilde{\psi}_0') ,
\end{aligned}
\tag{2.29}
$$

where $\gamma^{12349}\chi = -\chi$ and $\gamma^{12349}\chi' = \chi'$. From Eqs. (2.18) and (2.21), the anti-commutation relations between these operators become

$$\{\chi, \chi^\dagger\} = 1 , \quad \{\chi', \chi'^\dagger\} = 1 . \tag{2.30}$$

In terms of the operators introduced above, Eqs. (2.27) and (2.29), the normal ordered zero mode contribution to the light-cone Hamiltonian is then given by

$$E_0 = \frac{\mu}{6}(2a^{i\dagger}a^i + a^{i'\dagger}a^{i'} + 2\chi^\dagger\chi + \chi'^\dagger\chi') , \tag{2.31}$$

which has vanishing zero-point energy as in the case of string oscillator contributions.

Though the string physics is described by the light-cone Hamiltonian, (2.24), there is a constraint constraining the string states, which is the usual Virasoro constraint imposing the invariance under the translation in σ direction. In the light-cone gauge, the Virasoro constraint is given by

$$\int_0^{2\pi} d\sigma \left(-\frac{1}{2\pi} p^+ \partial_\sigma X^- + \mathcal{P}^I \partial_\sigma X^I + \frac{i}{2}\psi_+^A \partial_\sigma \psi_+^A + \frac{i}{2}\psi_-^A \partial_\sigma \psi_-^A \right) = 0 . \tag{2.32}$$

The integration of the first integrand vanishes trivially since p^+ is constant, and the remaining parts give us the following constraint.

$$N = \tilde{N} , \tag{2.33}$$

where N and $\tilde{N}$ are defined as

$$
\begin{aligned}
N &= \sum_{n=1}^{\infty} n \left(\frac{1}{\omega_n} \alpha_{-n}^i \alpha_n^i + \frac{1}{\omega_n'} \alpha_{-n}^{i'} \alpha_n^{i'} + \psi_{-n}\psi_n + \psi_{-n}'\psi_n' \right) , \\
\tilde{N} &= \sum_{n=1}^{\infty} n \left(\frac{1}{\omega_n} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i + \frac{1}{\omega_n'} \tilde{\alpha}_{-n}^{i'} \tilde{\alpha}_n^{i'} + \tilde{\psi}_{-n}\tilde{\psi}_n + \tilde{\psi}_{-n}'\tilde{\psi}_n' \right) .
\end{aligned}
\tag{2.34}
$$

The normal ordered expressions Eqs. (2.26) and (2.31) now constitute the quantum light-cone Hamiltonian, which implicitly defines the vacuum $|0\rangle$ of the quantized theory as a state annihilated by string oscillation operators with positive mode number, that is $n \geq 1$, and zero mode operators a^I, χ, and χ' defined in Eqs. (2.27) and (2.29). Actually, the vacuum defined in this paper, especially the vacuum state in the zero mode sector, is not unique but one of the possible Clifford vacua, since our theory is massive and there can be various definitions for the creation and annihilation operators. This is also the case for the IIB superstring in pp-wave background and has been discussed in [22]. However, considering the regularity of states at $\tau \to i\infty$ that has been pointed out in [27], our definition is a natural one.

The low-lying string states are obtained by acting the fermionic and bosonic zero-mode creation operators on the vacuum $|0\rangle$ and correspond to the excitation modes of Type

e_0	Bosonic states	N_B	Fermionic states	N_F
0	(0,0)	1		0
1		0	(0,1)	4
2	(0,2)	6	(1,0)	4
3	(1,1)	16	(0,3)	4
4	(2,0) (0,4)	7	(1,2)	24
5	(1,3)	16	(2,1)	24
6	(2,2)	36	(3,0) (1,4)	8
7	(3,1)	16	(2,3)	24
8	(4,0) (2,4)	7	(3,2)	24
9	(3,3)	16	(4,1)	4
10	(4,2)	6	(3,4)	4
11		0	(4,3)	4
12	(4,4)	1		0
		128		128

Table 1: Low-lying string states constructed by acting fermionic zero-mode operators on the vacuum; $(\chi^\dagger)^n(\chi'^\dagger)^{n'}|0\rangle$ with $n, n' = 0, 1, 2, 3, 4$. The states are characterized by (n, n'). $N_B(N_F)$ is the number of bosonic (fermionic) degrees of freedom. e_0 is the light-cone energy in units of $\mu/6$, that is, $E_0 = \mu e_0/6$.

IIA supergravity fields expanded near the plane-wave background. Among the string states, those constructed by using only the fermionic zero-modes, that is, $(\chi^\dagger)^n(\chi'^\dagger)^{n'}|0\rangle$ with $n, n' = 0, 1, 2, 3, 4$, correspond to the supergravity excitation modes with the minimal light-cone energies, and the string states obtained by acting the bosonic zero-modes on them are related to the supergravity modes with higher light-cone energies. In table 1, we list the states $(\chi^\dagger)^n(\chi'^\dagger)^{n'}|0\rangle$ according to their light-cone energy H_{LC}, which is simply E_0 of Eq. (2.31), in units of $\mu/6$. We shall see in the next section how these string states correspond to the supergravity excitation modes.

3 Supergravity excitation spectrum

The equations of motion for the Type IIA supergravity fields expanded to linear order in fluctuations on the plane-wave background can be used to determine the light-cone energy spectrum of the fluctuating fields. The equations for fluctuations in the plane-wave background (1.1) has the following typical form

$$(\Box + i\alpha\partial_-)\varphi = 0 \tag{3.1}$$

with

$$\Box \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,\bar{g}^{\mu\nu}\partial_\nu\right) = -2\partial_+\partial_- + A(x^I)\partial_-^2 + \partial_I^2,$$

where $\bar{g} \equiv \det \bar{g}_{\mu\nu} = -1$ and α is an arbitrary constant. In the light-cone description where x^+ is the evolution parameter of the system, the fluctuation field φ can be expressed by using the Fourier transformation as follows

$$\varphi(x^+, x^-, x^I) = \int \frac{dp^+ d^8 p^I}{(2\pi)^{9/2}} e^{i(-x^- p^+ + x^I p^I)} \tilde{\varphi}(x^+, p^+, p^I), \tag{3.2}$$

where $\tilde{\varphi}$ satisfies

$$\left[2p^+ P^- + (p^+)^2 \left(\sum_{i=1}^{4} \frac{\mu^2}{9} \partial_{p^i}^2 + \sum_{i'=5}^{8} \frac{\mu^2}{36} \partial_{p^{i'}}^2 \right) - p_I^2 + \alpha p^+ \right] \varphi = 0. \tag{3.3}$$

From the Eq. (3.3), we obtain the light-cone Hamiltonian,

$$H \equiv i\partial_+ = P^- = \frac{1}{2p^+} \left((p^I)^2 - m_1^2 \sum_{i=1}^{4} \partial_{p^i}^2 - m_2^2 \sum_{i'=5}^{8} \partial_{p^{i'}}^2 \right) - \frac{\alpha}{2}, \tag{3.4}$$

where $m_1 = \frac{\mu}{3} p^+$, $m_2 = \frac{\mu}{6} p^+$. To obtain the light-cone energy spectrum, we introduce two sets of creation and annihilation operators,

$$a^{i\dagger} \equiv \frac{1}{\sqrt{2m_1}} (p^i - m_1 \partial_{p^i}), \qquad a^i \equiv \frac{1}{\sqrt{2m_1}} (p^i + m_1 \partial_{p^i}), \qquad [a^i, a^{\dagger j}] = \delta^{ij}, \tag{3.5}$$

$$a^{i'\dagger} \equiv \frac{1}{\sqrt{2m_2}} (p^{i'} - m_2 \partial_{p^{i'}}), \quad a^{i'} \equiv \frac{1}{\sqrt{2m_2}} (p^{i'} + m_2 \partial_{p^{i'}}), \quad [a^{i'}, a^{\dagger j'}] = \delta^{i'j'} . \tag{3.6}$$

Then the normal ordered expression of the light-cone Hamiltonian is given by

$$H = \frac{\mu}{3} \sum_{i=1}^{4} a^{i\dagger} a^i + \frac{\mu}{6} \sum_{i'=1}^{4} a^{i'\dagger} a^{i'} + \mu - \frac{\alpha}{2}. \tag{3.7}$$

From this relation (3.7), we see that the fluctuation field which satisfies the Eq. (3.1) has the minimal light-cone energy $\mathcal{E}_0$ defined as

$$\frac{\mu}{6} \mathcal{E}_0 = \mu - \frac{\alpha}{2} , \tag{3.8}$$

which will be used to characterize the excitation modes of the IIA supergravity in the pp-wave background.

3.1   Bosonic excitations

In the bosonic sector of the Type IIA supergravity, we have five fields, which are dilaton Φ, graviton $g_{\mu\nu}$, NS-NS two-form gauge field $B_{\mu\nu}$, and two R-R gauge fields A_μ and $A_{\mu\nu\rho}$. The

equations of motion for these bosonic fields are, in the Einstein frame,

$$\nabla^2 \Phi = -\frac{1}{12} e^{-\Phi} H_{\mu\nu\rho} H^{\mu\nu\rho} + \frac{3}{8} e^{\frac{3}{2}\Phi} F_{\mu\nu} F^{\mu\nu} + \frac{1}{96} e^{\frac{1}{2}\Phi} \tilde{F}_{\mu\nu\rho\sigma} \tilde{F}^{\mu\nu\rho\sigma} , \tag{3.9}$$

$$\nabla_\mu \left(e^{\frac{3}{2}\Phi} F^{\mu\nu} \right) = -\frac{1}{6} e^{\frac{1}{2}\Phi} H_{\mu\rho\sigma} \tilde{F}^{\mu\nu\rho\sigma} , \tag{3.10}$$

$$\nabla_\mu \left(e^{-\Phi} H^{\mu\nu\rho} + e^{\frac{1}{2}\Phi} A_\sigma \tilde{F}^{\sigma\mu\nu\rho} \right) = \frac{1}{2 \cdot (4!)^2} \frac{\epsilon^{\nu\rho\mu_1\cdots\mu_8}}{\sqrt{-g}} F_{\mu_1\cdots\mu_4} F_{\mu_5\cdots\mu_8} , \tag{3.11}$$

$$\nabla_\mu \left(e^{\frac{1}{2}\Phi} \tilde{F}^{\mu\nu\rho\sigma} \right) = -\frac{1}{6 \cdot 4!} \frac{\epsilon^{\mu\nu\rho\sigma\mu_1\cdots\mu_6}}{\sqrt{-g}} H_{\mu\mu_1\mu_2} F_{\mu_3\cdots\mu_6}, \tag{3.12}$$

$$\begin{aligned} R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = &-\frac{1}{4} g_{\mu\nu} \nabla_\rho \Phi \nabla^\rho \Phi + \frac{1}{2} \nabla_\mu \Phi \nabla_\nu \Phi \\ &-\frac{1}{24} g_{\mu\nu} e^{-\Phi} H_{\rho\sigma\lambda} H^{\rho\sigma\lambda} + \frac{1}{4} e^{-\Phi} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} \\ &-\frac{1}{2} g_{\mu\nu} \left(\frac{1}{4} e^{\frac{3}{2}\Phi} F_{\rho\sigma} F^{\rho\sigma} + \frac{1}{48} e^{\frac{1}{2}\Phi} \tilde{F}_{\rho\sigma\lambda\kappa} \tilde{F}^{\rho\sigma\lambda\kappa} \right) \\ &+\frac{1}{2} e^{\frac{3}{2}\Phi} F_{\mu\rho} F_\nu{}^\rho + \frac{1}{12} e^{\frac{1}{2}\Phi} \tilde{F}_{\mu\rho\sigma\lambda} \tilde{F}_\nu{}^{\rho\sigma\lambda}, \end{aligned} \tag{3.13}$$

where $\epsilon^{\mu_1\cdots\mu_{10}}$ is the Levi-Civita symbol chosen by $\epsilon^{+-12\cdots8} = 1$, and the field strengths are defined as

$$\begin{aligned} F_{\mu\nu} &= 2\partial_{[\mu} A_{\nu]}, \\ H_{\mu\nu\rho} &= 3\partial_{[\mu} B_{\nu\rho]}, \\ F_{\mu\nu\rho\sigma} &= 4\partial_{[\mu} A_{\nu\rho\sigma]}, \\ \tilde{F}_{\mu\nu\rho\sigma} &= 4\partial_{[\mu} A_{\nu\rho\sigma]} + 4A_{[\mu} H_{\nu\rho\sigma]}. \end{aligned}$$

To obtain the linearized equations of motion for the fluctuation fields, we expand the fields near the pp-wave background given in Eq. (1.1) as follows

$$\begin{aligned} \Phi &= \phi, \\ g_{\mu\nu} &= \bar{g}_{\mu\nu} + h_{\mu\nu} \rightarrow R_{\mu\nu} = \bar{R}_{\mu\nu} + r_{\mu\nu}, \\ A_\mu &= \bar{A}_\mu + a_\mu \rightarrow F_{\mu\nu} = \bar{F}_{\mu\nu} + f_{\mu\nu}, \\ B_{\mu\nu} &= b_{\mu\nu} \rightarrow H_{\mu\nu\rho} = h_{\mu\nu\rho}, \\ A_{\mu\nu\rho} &= \bar{A}_{\mu\nu\rho} + a_{\mu\nu\rho} \rightarrow F_{\mu\nu\rho\sigma} = \bar{F}_{\mu\nu\rho\sigma} + f_{\mu\nu\rho\sigma}, \end{aligned} \tag{3.14}$$

where the fields with bar denote the background fields. And we shall choose the usual light-cone gauge for the fluctuations, $a_\mu, b_{\mu\nu}, a_{\mu\nu\rho}, h_{\mu\nu}$ such as

$$a_- = b_{-I} = a_{-IJ} = h_{-I} = 0. \tag{3.15}$$

The linearized form of the equation of motion (3.9) for dilaton is the following coupled equation

$$\Box \phi = \frac{\mu}{2} \partial_- (a_4 - a_{123}). \tag{3.16}$$

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And the equations of motion (3.10), (3.11) and (3.12) for the gauge fields have the following linearized forms respectively,

$$\partial_\mu \left(\frac{3}{2}\phi\bar{F}^{\mu\nu} + f^{\mu\nu} - h^{\mu\mu'}\bar{F}_{\mu'}{}^\nu - \bar{F}^\mu{}_{\nu'}h^{\nu'\nu} \right) = -\frac{1}{3!}h_{\mu\rho\sigma}\bar{F}^{\mu\nu\rho\sigma} ,\tag{3.17}$$

$$\partial_\mu \left(h^{\mu\nu\rho} + a_\sigma\bar{F}^{\sigma\mu\nu\rho} + \bar{A}_\sigma f^{\sigma\mu\nu\rho} + 4\bar{A}_\sigma\bar{A}^{[\sigma}h^{\mu\nu\rho]} \right) = \frac{\mu}{4!}\epsilon^{+123\nu\rho\mu_1\cdots\mu_4}f_{\mu_1\cdots\mu_4} ,\tag{3.18}$$

$$\partial_\mu \left(\frac{1}{2}\phi\bar{F}^{\mu\nu\rho\sigma} + f^{\mu\nu\rho\sigma} + 4\bar{A}^{[\mu}h^{\nu\rho\sigma]} - h^{\mu\mu'}\bar{F}_{\mu'}{}^{\nu\rho\sigma} \right.$$

$$\left. + h^{\nu\nu'}\bar{F}_{\nu'}{}^{\mu\rho\sigma} - h^{\rho\rho'}\bar{F}_{\rho'}{}^{\mu\nu\sigma} + h^{\sigma\sigma'}\bar{F}_{\sigma'}{}^{\mu\nu\rho} \right) = \frac{\mu}{3!}\epsilon^{+123\nu\rho\sigma\mu_1\mu_2\mu_3}h_{\mu_1\mu_2\mu_3} ,$$

$$\tag{3.19}$$

where the raising and lowering of the Lorentz indices are performed by the plane-wave metric $\bar{g}_{\mu\nu}$.

The $(+),(+I),(+IJ)$ components of the Eqs. (3.17), (3.18) and (3.19) give constraints implying that the modes a_+, b_{+I} and a_{+IJ} are non-dynamical. We can express these non-dynamical modes in terms of physical modes a_I, b_{IJ}, a_{IJK} as follows,

$$a_+ = \frac{1}{\partial_-}\partial_I a_I,\tag{3.20}$$

$$b_{I+} = \frac{1}{\partial_-}\partial_J b_{IJ},\tag{3.21}$$

$$a_{IJ+} = \frac{1}{\partial_-}\left(\partial_K a_{IJK} + \bar{A}_+\partial_- b_{IJ} \right),\tag{3.22}$$

where $\bar{A}_+ = \frac{\mu}{3}x_4$.

The linearized form of Einstein equation (3.13) has rather complicated form,

$$r_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}r = -\frac{\mu}{6}\bar{g}_{\mu\nu}\partial_- a_4 + \frac{\mu}{2}\bar{g}_{\mu\nu}\partial_- a_{123}$$

$$+ \frac{3}{4}\phi\bar{F}_{\mu\rho}\bar{F}_\nu{}^\rho + \frac{1}{24}\phi\bar{F}_{\mu\rho\sigma\lambda}\bar{F}_\nu{}^{\rho\sigma\lambda} + \frac{1}{2}\left(\bar{F}_{\mu\rho}f_\nu{}^\rho + f_{\mu\rho}\bar{F}_\nu{}^\rho \right)$$

$$+ \frac{1}{12}\left[\bar{F}_\mu{}^{\rho\sigma\lambda}\left(f_{\nu\rho\sigma\lambda} + 4\bar{A}_{[\nu}h_{\rho\sigma\lambda]} \right) + \left(f_{\mu\rho\sigma\lambda} + 4\bar{A}_{[\mu}h_{\rho\sigma\lambda]} \right)\bar{F}_\nu{}^{\rho\sigma\lambda} \right]$$

$$- \frac{1}{2}\bar{F}_{\mu\rho}\bar{F}_{\nu\rho'}h^{\rho\rho'} - \frac{1}{4}\bar{F}_{\mu\rho\sigma\lambda}\bar{F}_{\nu\rho'}{}^{\sigma\lambda}h^{\rho\rho'}\tag{3.23}$$

with

$$r_{\mu\nu} = \frac{1}{2}\left(-\bar{\nabla}^2 h_{\mu\nu} + \bar{\nabla}_\mu\bar{\nabla}^\rho h_{\rho\nu} + \bar{\nabla}_\nu\bar{\nabla}^\rho h_{\rho\mu} - \bar{\nabla}_\mu\bar{\nabla}_\nu h^\rho_\rho \right.$$

$$\left. + 2\bar{R}_{\mu\rho\sigma\nu}h^{\rho\sigma} + \bar{R}_{\mu\rho}h^\rho_\nu + \bar{R}_{\nu\rho}h^\rho_\mu \right) ,\tag{3.24}$$

$$r \equiv \bar{g}^{\mu\nu}r_{\mu\nu},\tag{3.25}$$

where the covariant derivative $\bar{\nabla}_\mu$ is defined by using the background pp-wave metric $\bar{g}_{\mu\nu}$ and the non-vanishing connection and curvature quantities are given by

$$\Gamma^-_{I+} = \frac{1}{2}\partial_I A, \quad \Gamma^I_{++} = \frac{1}{2}\partial_I A, \quad R_{+IJ+} = \frac{1}{2}\partial_I\partial_J A, \quad R_{++} = \frac{1}{2}\partial_I^2 A .\tag{3.26}$$

The $(--)$ component of the Eq. (3.23) gives the following zero-trace condition for the transverse modes of the graviton

$$h_{II} = 0. \tag{3.27}$$

And the $(-I)$ components of the Eq. (3.23) lead to the expressions for non-dynamical modes h_{+I},

$$h_{+I} = \frac{1}{\partial_-} \partial_J h_{IJ}. \tag{3.28}$$

Under the conditions (3.27) and (3.28), the trace for the space-time indices of the Eq. (3.23) gives an additional condition,

$$h_{++} = \frac{1}{(\partial_-)^2} \partial_I \partial_J h_{IJ} + \frac{\mu}{4} \frac{1}{\partial_-} (a_4 - a_{123}). \tag{3.29}$$

Now let us consider the linearized equations for physical modes which determine the light-cone energy spectrum. There are four sets of decoupled modes and seven sets of coupled ones which we need to diagonalize to determine the light-cone energy spectrum for the physical modes. The linearized equation for dilaton, (4)-component of the Eq. (3.17), (123)-component of the Eq. (3.19), the trace for $SO(3)$ indices and (44)-component of the Eq. (3.23) are coupled each other and given by[3]

$$\Box\phi - \frac{\mu}{2}\partial_- a_4 + \frac{\mu}{2}\partial_- a_{123} = 0 \ , \quad \Box a_4 + \frac{\mu}{2}\partial_-\phi - \frac{\mu}{3}\partial_- h_{44} = 0,$$

$$\Box a_{123} - \frac{\mu}{2}\partial_-\phi + \mu\partial_- h_{ii} = 0 \ , \quad \Box h_{ii} - \frac{\mu}{4}\partial_- a_4 - \frac{15\mu}{4}\partial_- a_{123} = 0 \ ,$$

$$\Box h_{44} + \frac{7\mu}{12}\partial_- a_4 + \frac{3\mu}{4}\partial_- a_{123} = 0. \tag{3.30}$$

These coupled equations form $SO(3) \times SO(4)$ scalar multiplet. Then we obtain the following diagonalized equations

$$\Box\phi_0 = 0 \ ,$$
$$\Box\phi_2 + i\frac{2\mu}{3}\partial_-\phi_2 = 0 \ , \quad \Box\bar{\phi}_2 - i\frac{2\mu}{3}\partial_-\bar{\phi}_2 = 0 \ ,$$
$$\Box\phi_6 + 2i\mu\partial_-\phi_6 = 0 \ , \quad \Box\bar{\phi}_6 - 2i\mu\partial_-\bar{\phi}_6 = 0, \tag{3.31}$$

where we define

$$\phi_0 \equiv \phi + \frac{1}{3}h_{ii} + h_{44} \ ,$$
$$\phi_2 \equiv \phi + \frac{4}{3}ia_4 - \frac{2}{3}h_{44} \ , \quad \bar{\phi}_2 \equiv \phi - \frac{4}{3}ia_4 - \frac{2}{3}h_{44} \ ,$$
$$\phi_6 \equiv \phi - 4ia_{123} - 2h_{ii} \ , \quad \bar{\phi}_6 \equiv \phi + 4ia_{123} - 2h_{ii} \ . \tag{3.32}$$

According to Eqs. (3.1) and (3.8), these equations in Eq. (3.31) mean that the minimal light-cone energies of the fields in Eq. (3.32) are given by

$$\mathcal{E}_0\left(\phi_0\right) = 6 \ , \quad \mathcal{E}_0\left(\phi_2\right) = 4 \ , \quad \mathcal{E}_0\left(\bar{\phi}_2\right) = 8 \ , \quad \mathcal{E}_0\left(\phi_6\right) = 0 \ , \quad \mathcal{E}_0\left(\bar{\phi}_6\right) = 12 \ . \tag{3.33}$$

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[3]From now one, the index of the type i is taken to run from 1 to 3, while the range of i' is not changed.

There are two sets of $SO(3)$ vector multiplets. One is decoupled multiplet and the other is coupled one. The decoupled one comes from the $(i4)$-components of the Eq. (3.18) and given by

$$\Box \beta_{0i} = 0, \tag{3.34}$$

where $\beta_{0i} \equiv b_{4i}$ and their minimal light-cone energies are

$$\mathcal{E}_0 \left(\beta_{0i} \right) = 6 . \tag{3.35}$$

The coupled $SO(3)$ vector multiplets come from the (i), (ij), $(4ij)$ and $(i4)$-components of the Eqs. (3.17), (3.18), (3.19) and (3.23) respectively and written by

$$\Box a_i - \frac{\mu}{3}\partial_- h_{4i} - \frac{\mu}{2}\epsilon_{ijk}\partial_- b_{jk} = 0 , \quad \Box b_{ij} + \mu\epsilon_{ijk}\partial_- a_k - \frac{\mu}{3}\partial_- a_{4ij} = 0,$$

$$\Box a_{4ij} + \frac{\mu}{3}\partial_- b_{ij} + \mu\epsilon_{ijk}\partial_- h_{4k} = 0 , \quad \Box h_{4i} + \frac{\mu}{3}\partial_- a_i - \frac{\mu}{2}\epsilon_{ijk}\partial_- a_{4jk} = 0 . \tag{3.36}$$

These coupled equations are diagonalized as

$$\Box\beta_{2i} + i\frac{2\mu}{3}\partial_-\beta_{2i} = 0 , \quad \Box\bar{\beta}_{2i} - i\frac{2\mu}{3}\partial_-\bar{\beta}_{2i} = 0 ,$$

$$\Box\beta_{4i} + i\frac{4\mu}{3}\partial_-\beta_{4i} = 0 , \quad \Box\bar{\beta}_{4i} - i\frac{4\mu}{3}\partial_-\bar{\beta}_{4i} = 0 , \tag{3.37}$$

where we define the diagonalized physical modes as

$$\beta_{2i} \equiv a_i - ih_{4i} + \frac{i}{2}\epsilon_{ijk}b_{jk} + \frac{1}{2}\epsilon_{ijk}a_{4jk}, \quad \bar{\beta}_{2i} \equiv a_i + ih_{4i} - \frac{i}{2}\epsilon_{ijk}b_{jk} + \frac{1}{2}\epsilon_{ijk}a_{4jk},$$

$$\beta_{4i} \equiv a_i + ih_{4i} + \frac{i}{2}\epsilon_{ijk}b_{jk} - \frac{1}{2}\epsilon_{ijk}a_{4jk}, \quad \bar{\beta}_{4i} \equiv a_i - ih_{4i} - \frac{i}{2}\epsilon_{ijk}b_{jk} - \frac{1}{2}\epsilon_{ijk}a_{4jk}, \tag{3.38}$$

whose minimal light-cone energies are given by

$$\mathcal{E}_0 \left(\beta_{2i} \right) = 4 , \quad \mathcal{E}_0 \left(\bar{\beta}_{2i} \right) = 8 , \quad \mathcal{E}_0 \left(\beta_{4i} \right) = 2 , \quad \mathcal{E}_0 \left(\bar{\beta}_{4i} \right) = 10 . \tag{3.39}$$

There are two kinds of coupled multiplets in $SO(4)$ vector multiplets. The (i') and $(4i')$-components of Eqs. (3.17) and (3.18) form one set of coupled equations

$$\Box a_{i'} - \frac{\mu}{3}\partial_- h_{4i'} = 0, \quad \Box h_{4i'} + \frac{\mu}{3}\partial_- a_{i'} = 0, \tag{3.40}$$

and the $(4i')$ and $(i'j'k')$-components of Eqs. (3.18) and (3.19) give the other set

$$\Box b_{4i'} + \frac{\mu}{6}\epsilon_{i'j'k'l'}\partial_- a_{j'k'l'} = 0, \quad \Box a_{i'j'k'} + \mu\epsilon_{i'j'k'l'}\partial_- b_{4l'} = 0, \tag{3.41}$$

where $\epsilon_{i'j'k'l'}$ are Levi-Civita symbols and we choose $\epsilon_{5678} = 1$. By introducing two sets of complex $SO(4)$ vectors

$$\beta_{1i'} \equiv a_{i'} + ih_{4i'}, \quad \bar{\beta}_{1i'} \equiv a_{i'} - ih_{4i'}, \tag{3.42}$$

$$\beta_{3i'} \equiv b_{4i'} - \frac{i}{6}\epsilon_{i'j'k'l'}a_{j'k'l'}, \quad \bar{\beta}_{3i'} \equiv b_{4i'} + \frac{i}{6}\epsilon_{i'j'k'l'}a_{j'k'l'}, \tag{3.43}$$

we can diagonalize the Eqs. (3.40) and (3.41) as

$$\Box\beta_{1i'} + i\frac{\mu}{3}\partial_-\beta_{1i'} = 0 , \quad \Box\bar{\beta}_{1i'} - i\frac{\mu}{3}\partial_-\bar{\beta}_{1i'} = 0 , \tag{3.44}$$

$$\Box\beta_{3i'} + i\mu\partial_-\beta_{3i'} = 0 , \quad \Box\bar{\beta}_{3i'} - i\mu\partial_-\bar{\beta}_{3i'} = 0 . \tag{3.45}$$

Thus we obtain the minimal light-cone energies of the diagonalized modes as

$$\mathcal{E}_0(\beta_{1i'}) = 5 , \quad \mathcal{E}_0(\bar{\beta}_{1i'}) = 7 , \quad \mathcal{E}_0(\beta_{3i'}) = 3 , \quad \mathcal{E}_0(\bar{\beta}_{3i'}) = 9 . \tag{3.46}$$

The (ij')-components of the Eq. (3.18) and the $(4ij')$-components of the Eq. (3.19) form a coupled set of equations as follows,

$$\Box b_{ij'} - \frac{\mu}{3}\partial_-a_{4ij'} = 0, \qquad \Box a_{4ij'} + \frac{\mu}{3}\partial_-b_{ij'} = 0. \tag{3.47}$$

By defining the complex tensors

$$\beta_{1ij'} \equiv b_{ij'} + ia_{4ij'}, \qquad \bar{\beta}_{1ij'} \equiv b_{ij'} - ia_{4ij'}, \tag{3.48}$$

the equations in Eq. (3.47) are diagonalized as

$$\Box\beta_{1ij'} + i\frac{\mu}{3}\partial_-\beta_{1ij'} = 0 , \quad \Box\bar{\beta}_{1ij'} - i\frac{\mu}{3}\partial_-\bar{\beta}_{1ij'} = 0 , \tag{3.49}$$

so that the minimal light-cone energies are

$$\mathcal{E}_0(\beta_{1ij'}) = 5 , \quad \mathcal{E}_0(\bar{\beta}_{1ij'}) = 7 . \tag{3.50}$$

The $(i'j')$-components of the Eq. (3.18) and $(4i'j')$-ones of the Eq. (3.19) are coupled also and form anti-symmetric 2-form field multiplets. The coupled equations are

$$\Box b_{i'j'} - \frac{\mu}{3}\partial_-a_{4i'j'} + \frac{\mu}{2}\epsilon_{i'j'k'l'}\partial_-a_{4k'l'} = 0,$$
$$\Box a_{4i'j'} + \frac{\mu}{3}\partial_-b_{i'j'} - \frac{\mu}{2}\epsilon_{i'j'k'l'}\partial_-b_{k'l'} = 0, \tag{3.51}$$

and these are diagonalized by defining the anti-symmetric complex 2-form fields,

$$\beta_{2i'j'} \equiv a^+_{i'j'} - i\tilde{a}^+_{i'j'}, \qquad \bar{\beta}_{2i'j'} \equiv a^+_{i'j'} + i\tilde{a}^+_{i'j'},$$
$$\beta_{4i'j'} \equiv a^-_{i'j'} + i\tilde{a}^-_{i'j'}, \qquad \bar{\beta}_{4i'j'} \equiv a^-_{i'j'} - i\tilde{a}^-_{i'j'}, \tag{3.52}$$

where we used the (anti-)self-dual tensors which are irreducible tensors of the $SO(4)$ algebra and defined by

$$a^\pm_{i'j'} \equiv b_{i'j'} \pm \frac{1}{2}\epsilon_{i'j'k'l'}b_{k'l'}, \qquad \tilde{a}^\pm_{i'j'} \equiv a_{4i'j'} \pm \frac{1}{2}\epsilon_{i'j'k'l'}a_{4k'l'}.$$

Then the equations in Eq. (3.51) are diagonalized by

$$\Box\beta_{2i'j'} + i\frac{2\mu}{3}\partial_-\beta_{2i'j'} = 0 , \quad \Box\bar{\beta}_{2i'j'} - i\frac{2\mu}{3}\partial_-\bar{\beta}_{2i'j'} = 0 ,$$
$$\Box\beta_{4i'j'} + i\frac{4\mu}{3}\partial_-\beta_{4i'j'} = 0 , \quad \Box\bar{\beta}_{4i'j'} - i\frac{4\mu}{3}\partial_-\bar{\beta}_{4i'j'} = 0 , \tag{3.53}$$

thus the minimal light-cone energies are

$$\mathcal{E}_0(\beta_{2i'j'}) = 4 \ , \quad \mathcal{E}_0(\bar{\beta}_{2i'j'}) = 8 \ , \quad \mathcal{E}_0(\beta_{4i'j'}) = 2 \ , \quad \mathcal{E}_0(\bar{\beta}_{4i'j'}) = 10 \ . \tag{3.54}$$

There is remaining one mixed multiplet. The (ijk')-components of the Eq. (3.19) and (ij')-ones of the Eq. (3.23) give

$$\Box a_{ijk'} + \mu \epsilon_{ijl} \partial_- h_{lk'} = 0, \qquad \Box h_{ij'} - \frac{\mu}{2} \epsilon_{ikl} \partial_- a_{klj'} = 0. \tag{3.55}$$

By defining the complex tensors

$$\beta_{3ijk'} \equiv a_{ijk'} - i \epsilon_{ijk} h_{kk'}, \qquad \bar{\beta}_{3ijk'} \equiv a_{ijk'} + i \epsilon_{ijk} h_{kk'}, \tag{3.56}$$

we can diagonalize the equations in Eq. (3.55) as

$$\Box \beta_{3ijk'} + i\mu \partial_- \beta_{3ijk'} = 0, \qquad \Box \bar{\beta}_{3ijk'} - i\mu \partial_- \bar{\beta}_{3ijk'} = 0, \tag{3.57}$$

so that the minimal energies are

$$\mathcal{E}_0(\beta_{3ijk'}) = 3 \ , \quad \mathcal{E}_0(\bar{\beta}_{2ijk'}) = 9 \ . \tag{3.58}$$

The $(ij'k')$-components of the Eq. (3.19) are decoupled and the linearized equations and minimal energies are given by

$$\Box \beta_{0ij'k'} = 0 \ , \quad \mathcal{E}_0(\beta_{0ij'k'}) = 6 \ , \tag{3.59}$$

where $\beta_{0ij'k'} \equiv a_{ij'k'}$.

From (ij) and $(i'j')$-components of the Eq. (3.23), we can extract two sets of traceless gravitons which belong to $SO(3)$ and $SO(4)$ graviton multiplets respectively. Then the equations and minimal light-cone energies are given by

$$\Box h^{\perp}_{ij} = 0, \qquad \Box h^{\perp}_{i'j'} = 0 \ , \quad \mathcal{E}_0(h^{\perp}_{ij}) = \mathcal{E}_0(h^{\perp}_{i'j'}) = 6 \ , \tag{3.60}$$

where we have defined

$$h^{\perp}_{ij} \equiv h_{ij} - \frac{1}{3} \delta_{ij} h_{kk}, \qquad h^{\perp}_{i'j'} \equiv h_{i'j'} - \frac{1}{4} \delta_{i'j'} h_{k'k'}.$$

3.2   Fermionic excitations

The fermionic fields of Type IIA supergravity are spin-1/2 dilatino Λ and spin-3/2 gravitino Ψ_μ, each of which has real 32 components and is decomposed into two pieces of opposite $SO(1,9)$ chiralities. Though it is usual to decompose the fermionic fields based on $SO(1,9)$ chiralities, it will be convenient to take a different decomposition in this paper in a way that the $SO(3) \times SO(4)$ symmetry structure of the pp-wave background manifests.

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If we do not take any decomposition for a while, the equations of motion for Λ and Ψ_μ are as follows:

$$
\begin{aligned}
\Gamma^\mu D_\mu \Lambda &= -\frac{\sqrt{2}}{4}\Gamma^9\Gamma^\mu\Gamma^\nu\Psi_\mu\partial_\nu\Phi - \frac{1}{192}e^{\Phi/4}\left(\frac{1}{\sqrt{2}}\Gamma^9\Gamma^\mu\Gamma^{\rho\sigma\lambda\kappa}\Psi_\mu - \frac{3}{2}\Gamma^{\rho\sigma\lambda\kappa}\Lambda\right)\tilde{F}_{\rho\sigma\lambda\kappa} \\
&\quad + \frac{\sqrt{2}}{48}e^{-\Phi/2}\Gamma^\mu\Gamma^{\rho\sigma\lambda}\Psi_\mu H_{\rho\sigma\lambda} - \frac{1}{16}e^{3\Phi/4}\left(\frac{3}{\sqrt{2}}\Gamma^\mu\Gamma^{\rho\sigma}\Psi_\mu + \frac{5}{2}\Gamma^9\Gamma^{\rho\sigma}\Lambda\right)F_{\rho\sigma} \\
&\quad + \cdots ,
\end{aligned}
\tag{3.61}
$$

$$
\begin{aligned}
\Gamma^{\mu\nu\rho}D_\nu\Psi_\rho &= \frac{\sqrt{2}}{4}\Gamma^\nu\Gamma^\mu\Gamma^9\Lambda\partial_\nu\Phi \\
&\quad - \frac{1}{192}e^{\Phi/4}\left(2\Gamma^{\mu\nu\rho\sigma\lambda\kappa}\Psi_\nu + 24g^{\mu\rho}\Gamma^{\sigma\lambda}\Psi^\kappa + \frac{1}{\sqrt{2}}\Gamma^{\rho\sigma\lambda\kappa}\Gamma^\mu\Gamma^9\Lambda\right)\tilde{F}_{\rho\sigma\lambda\kappa} \\
&\quad + \frac{1}{48}e^{-\Phi/2}\left(2\Gamma^9\Gamma^{\mu\nu\rho\sigma\lambda}\Psi_\nu - 12g^{\mu\rho}\Gamma^9\Gamma^\sigma\Psi^\lambda + \sqrt{2}\Gamma^{\rho\sigma\lambda}\Gamma^\mu\Lambda\right)H_{\rho\sigma\lambda} \\
&\quad + \frac{1}{16}e^{3\Phi/4}\left(2\Gamma^9\Gamma^{\mu\nu\rho\sigma}\Psi_\nu + 4g^{\mu\rho}\Gamma^9\Psi^\sigma - \frac{3}{\sqrt{2}}\Gamma^{\rho\sigma}\Gamma^\mu\Lambda\right)F_{\rho\sigma} + \cdots \\
&\equiv J^\mu + \cdots ,
\end{aligned}
\tag{3.62}
$$

where the dots on the right hand sides represent the terms of cubic in fermionic fields, which describe the interactions between excitations and hence are not our concern, and the covariant derivative for spinor is given by[4]

$$
D_\mu = \partial_\mu + \frac{1}{4}\omega_\mu^{rs}\Gamma_{rs} ,
\tag{3.63}
$$

whose explicit expression for the pp-wave background, (1.1), is

$$
D_+ = \partial_+ - \frac{1}{4}\partial_I A\Gamma^{+I} , \quad D_- = \partial_- , \quad D_I = \partial_I ,
\tag{3.64}
$$

under the following choice of the zehnbein

$$
e^+ = dx^+ , \quad e^- = dx^- + \frac{1}{2}A(x^I)dx^+ , \quad e^I = dx^I .
\tag{3.65}
$$

For the study of the linearized equation of motion for gravitino, we note that it is convenient to rewrite the Eq. (3.62) as

$$
\Gamma^\nu D_\nu \Psi_\mu - D_\mu \Psi = J_\mu - \frac{1}{8}\Gamma_\mu\Gamma_\nu J^\nu ,
\tag{3.66}
$$

where $\Psi \equiv \Gamma^\mu\Psi_\mu$ and we have ignored the interaction terms represented by dots in Eq. (3.62). For the consideration of the physical modes, we should impose the gauge condition for the gravitino, which we take as the following light-cone gauge,

$$
\Psi_- = 0 .
\tag{3.67}
$$

---

[4]The indices $r, s, \ldots$ represent the flat tangent space indices. The gamma matrices Γ^r satisfy the $SO(1,9)$ Clifford algebra, $\{\Gamma^r, \Gamma^s\} = 2\eta^{rs}$, where η^{rs} is the ten dimensional flat metric. We note that all the indices of gamma matrices, which are not denoted by Greek characters, are flat indices.

In the light-cone formulation, the fermionic fields are decomposed into the dynamical (physical) and non-dynamical modes explicitly by using $\{\Gamma^+, \Gamma^-\} = 2\eta^{+-}$. For the spin-1/2 field, we have the decomposition,

$$\Lambda = \lambda + \eta ,\tag{3.68}$$

where

$$\lambda \equiv -\frac{1}{2}\Gamma^-\Gamma^+\Lambda , \quad \eta \equiv -\frac{1}{2}\Gamma^+\Gamma^-\Lambda ,\tag{3.69}$$

while for the spin-3/2 gravitino field,

$$\Psi_\mu = \psi_\mu + \rho_\mu ,\tag{3.70}$$

where

$$\psi_\mu \equiv -\frac{1}{2}\Gamma^-\Gamma^+\Psi_\mu , \quad \rho_\mu \equiv -\frac{1}{2}\Gamma^+\Gamma^-\Psi_\mu .\tag{3.71}$$

As we will see, η and ρ_μ correspond to non-dynamical fields expressed in terms of the physical fields λ and ψ_μ respectively.

Having the decomposition of the fermionic fields, Eqs. (3.68) and (3.70), we first consider the equation of motion for the spin-1/2 field Λ, Eq. (3.61). The linearized form of the equation of motion is

$$\begin{aligned}\Gamma^+(\partial_+ &- \frac{1}{2}A\partial_-)\lambda + \Gamma^-\partial_-\eta + \Gamma^I\partial_I(\lambda + \eta) \\ &= \frac{\mu}{48}\Gamma^+\Gamma^{49}(5 - 9\Gamma^{12349})\lambda - \frac{\mu}{8\sqrt{2}}\Gamma^+\Gamma^I\Gamma^4(1 - \Gamma^{12349})\psi_I .\end{aligned}\tag{3.72}$$

We see that the field η does not have the dependence on the light-cone time x^+ and is expected to be non-dynamical. Indeed, this equation leads to the expression of η in terms of λ as

$$\eta = \frac{1}{2\partial_-}\Gamma^+\Gamma^I\partial_I\lambda .\tag{3.73}$$

Taking into account this expression, the linearized equation of motion for the physical mode λ becomes

$$\Box\lambda = -\frac{\mu}{24}\Gamma^{49}(5 - 9\Gamma^{12349})\partial_-\lambda + \frac{\mu}{4\sqrt{2}}\Gamma^I\Gamma^4(1 - \Gamma^{12349})\partial_-\psi_I .\tag{3.74}$$

We see that the gravitino appears in the equation of motion for the dilatino, (3.74). This lets us turn to the equation of motion for the gravitino and pick up the physical modes before going on further analysis for the dilatino. The explicit expressions for the 'current' J_μ are first needed in the light-cone gauge, (3.67). The light-cone components are obtained as $J_+ = \Gamma^+(\dots)$ and $J_- = 0$, where the detail of J_+ is not required in this paper except that

it is proportional to Γ^+, and, for the spatial components, we have

$$
\begin{aligned}
J_i &= -\frac{\mu}{4}\Gamma^+\left(\Gamma^{123}(\delta^{ij} - \Gamma^i\Gamma^j)\psi_j - \frac{1}{3}\Gamma^{49}\psi_i + \frac{1}{3}\Gamma^{i9}\psi_4 + \frac{1}{2\sqrt{2}}\Gamma^i\Gamma^4(1 - \Gamma^{12349})\lambda\right) , \\
J_4 &= -\frac{\mu}{4}\Gamma^+\left(\Gamma^{i'}\Gamma^{1234}\psi_{i'} - \frac{1}{2\sqrt{2}}(1 - \Gamma^{12349})\lambda\right) , \\
J_{i'} &= \frac{\mu}{4}\Gamma^+\left(\Gamma^{123}(\delta^{i'j'} - \Gamma^{i'}\Gamma^{j'})\psi_{j'} + \frac{1}{3}\Gamma^{49}\psi_{i'} + \Gamma^{i'}\left(\Gamma^{1234} - \frac{1}{3}\Gamma^9\right)\psi_4\right. \\
&\quad \left. -\frac{1}{2\sqrt{2}}\Gamma^{i'}\Gamma^4(1 + \Gamma^{12349})\lambda\right) .
\end{aligned}
\tag{3.75}
$$

We note that the property of J_μ in the light-cone gauge is that $\Gamma^\mu J_\mu = \Gamma^I J_I$, where the index of Γ^I is the flat one.

Then the $\mu = -$ component of Eq. (3.66) gives

$$
\Psi = \Gamma^+\Psi_+ + \Gamma^I\Psi_I = 0 ,
\tag{3.76}
$$

which implies $\Gamma^+\Gamma^I\Psi_I = 0$ and states that Ψ_+ is non-dynamical. For the $\mu = I$ component, we have

$$
\Gamma^+(\partial_+ - \frac{1}{2}A\partial_-)\psi_I + \Gamma^-\partial_-\rho_I + \Gamma^J\partial_J(\psi_I + \rho_I) = J_I - \frac{1}{8}\Gamma_I\Gamma^J J_J ,
\tag{3.77}
$$

with the decomposition, (3.70), and the expressions of J_I, (3.75). Obviously, the ρ_I field is non-dynamical, and is given in terms of ψ_I as

$$
\rho_I = \frac{1}{2\partial_-}\Gamma^+\Gamma^J\partial_J\psi_I .
\tag{3.78}
$$

This relation leads us to the following linearized equation of motion for the physical modes ψ_I:

$$
\Box\psi_I - \Gamma^-\partial_-(J_I - \frac{1}{8}\Gamma_I\Gamma^J J_J) = 0 .
\tag{3.79}
$$

The remaining vector component of the gravitino is Ψ_+, which is non-dynamical. From Eqs. (3.66) and (3.76), we have

$$
\psi_+ = \frac{1}{2\partial_-}(\delta^{IJ} + \Gamma^{IJ})\partial_J\psi_I , \quad \rho_+ = \frac{1}{2\partial_-}(\Gamma^+\Gamma^I\partial_I\psi_+ + \frac{1}{4}\Gamma^I J_I) .
\tag{3.80}
$$

As alluded at the beginning of this subsection, we now decompose the physical modes to reflect the symmetry structure $SO(3) \times SO(4)$ of the pp-wave background. However, to avoid some complexity, we are concerned only about the $SO(4)$ structure. The case of $SO(3)$ will be required at later stage. Since the decomposition is performed in the transverse eight dimensional space and the physical mode has 16 independent spinor components, it is natural to use the 16×16 gamma matrices, γ^I, of Eq. (2.4). In addition to this, from our convention and the Eqs. (3.69) and (3.71), we make replacements $\lambda \to \begin{pmatrix} \lambda \\ 0 \end{pmatrix}$ and

$\psi_I \rightarrow \begin{pmatrix} \psi_I \\ 0 \end{pmatrix}$ to specify that the physical fields have 16 components. The decomposition of physical fields is then

$$\lambda = \lambda^+ + \lambda^- , \quad \psi_I = \psi_I^+ + \psi_I^- \tag{3.81}$$

where the sign of superscript indicates the $SO(4)$ chirality in the space spanned by $x^5, ..., x^8$, that is, $\gamma^{5678}\lambda^\pm = \pm\lambda^\pm$ and $\gamma^{5678}\psi_I^\pm = \pm\psi_I^\pm$. We note that $\gamma^{5678} = \gamma^{12349}$, which will be useful in the following formulation.

For each $SO(4)$ chirality, the dilatino equation of motion, (3.74), leads us to have

$$(\Box - \frac{7\mu}{12}\gamma^{49}\partial_-)\lambda^+ - \frac{\mu}{2\sqrt{2}}\gamma^{i4}\partial_-\psi_i^+ - \frac{\mu}{2\sqrt{2}}\partial_-\psi_4^+ = 0 , \tag{3.82}$$

$$(\Box + \frac{\mu}{6}\gamma^{49}\partial_-)\lambda^- - \frac{\mu}{2\sqrt{2}}\gamma^{i'4}\partial_-\psi_{i'}^+ = 0 . \tag{3.83}$$

For the case of gravitino, we first split the vector components as $\psi_I^\pm = (\psi_i^\pm, \psi_4^\pm, \psi_{i'}^\pm)$, and decompose the modes $\psi_i^\pm$ and $\psi_{i'}^\pm$ into the transverse and parallel parts with respect to the γ^I matrices as follows:

$$\psi_{\perp i}^\pm \equiv (\delta_{ij} - \frac{1}{3}\gamma^i\gamma^j)\psi_j^\pm , \qquad \psi_\parallel^\pm \equiv \gamma^i\psi_i^\pm ,$$

$$\psi_{\perp i'}^\pm \equiv (\delta_{i'j'} - \frac{1}{4}\gamma^{i'}\gamma^{j'})\psi_{j'}^\pm , \qquad \psi_\parallel'^\pm \equiv \gamma^{i'}\psi_{i'}^\mp . \tag{3.84}$$

From Eq. (3.79), we see that the equations of motion for the transverse parts do not include other modes and are given by

$$(\Box - \frac{\mu}{3}\gamma^{123}\partial_-)\psi_{\perp i}^+ = 0 , \quad (\Box - \frac{2\mu}{3}\gamma^{123}\partial_-)\psi_{\perp i}^- = 0 ,$$

$$(\Box + \frac{2\mu}{3}\gamma^{123}\partial_-)\psi_{\perp i'}^+ = 0 , \quad (\Box + \frac{\mu}{3}\gamma^{123}\partial_-)\psi_{\perp i'}^- = 0 . \tag{3.85}$$

γ^{123} in these equations now requires the modes to have definite $SO(3)$ structure. Since $(\gamma^{123})^2 = -1$, its eigenvalues are $\pm i$, the $SO(3)$ chirality. For a generic spinor Θ, we introduce the following notation.

$$\Theta^{\pm\pm} , \tag{3.86}$$

where the first superscript represents the SO(3) chirality and the second the $SO(4)$ chirality. Then the equations for the transverse modes, (3.85), become

$$(\Box - i\frac{\mu}{3}\partial_-)\psi_{\perp i}^{++} = 0 , \quad (\Box + i\frac{\mu}{3}\partial_-)\psi_{\perp i}^{-+} = 0 ,$$

$$(\Box - i\frac{2\mu}{3}\partial_-)\psi_{\perp i}^{+-} = 0 , \quad (\Box + i\frac{2\mu}{3}\partial_-)\psi_{\perp i}^{--} = 0 ,$$

$$(\Box + i\frac{2\mu}{3}\partial_-)\psi_{\perp i'}^{++} = 0 , \quad (\Box - i\frac{2\mu}{3}\partial_-)\psi_{\perp i'}^{-+} = 0 ,$$

$$(\Box + i\frac{\mu}{3}\partial_-)\psi_{\perp i'}^{+-} = 0 , \quad (\Box - i\frac{\mu}{3}\partial_-)\psi_{\perp i'}^{--} = 0 , \tag{3.87}$$

which state that, according to (3.8), the minimal light-cone energy for the respective physical gravitino modes are

$$\mathcal{E}_0(\psi_{\perp i}^{--}, \psi_{\perp i'}^{++}) = 4 \;, \quad \mathcal{E}_0(\psi_{\perp i}^{-+}, \psi_{\perp i'}^{+-}) = 5 \;, \quad \mathcal{E}_0(\psi_{\perp i}^{++}, \psi_{\perp i'}^{--}) = 7 \;, \quad \mathcal{E}_0(\psi_{\perp i}^{+-}, \psi_{\perp i'}^{-+}) = 8 \;. \quad (3.88)$$

The equations of motion for the other physical modes of gravitino are grouped into two sets according to the $SO(4)$ chirality. As for the positive chirality, we have

$$(\Box + \frac{\mu}{4}\gamma^{123}\partial_-)\psi_4^+ - \frac{3\mu}{8}\gamma^{1234}\partial_-\psi_{\parallel}'^+ + \frac{5\mu}{8\sqrt{2}}\partial_-\lambda^+ = 0 \;,$$

$$(\Box + \frac{7\mu}{6}\gamma^{123}\partial_-)\psi_{\parallel}'^+ + \frac{5\mu}{3}\gamma^9\partial_-\psi_4^+ + \frac{\mu}{2\sqrt{2}}\gamma^4\partial_-\lambda^+ = 0 \;, \quad (3.89)$$

while for the negative chirality

$$\Box\psi_4^- - \frac{3\mu}{8}\gamma^{1234}\partial_-\psi_{\parallel}'^- + \frac{\mu}{4\sqrt{2}}\partial_-\lambda^- = 0 \;,$$

$$(\Box + \frac{5\mu}{6}\gamma^{123}\partial_-)\psi_{\parallel}'^- - \frac{4\mu}{3}\gamma^9\partial_-\psi_4^- - \frac{\mu}{\sqrt{2}}\gamma^4\partial_-\lambda^- = 0 \;. \quad (3.90)$$

We note that the modes $\psi_{\parallel}^{\pm}$ are not dynamical because $\gamma^I\psi_I^{\pm} = 0$ as can be seen from Eq. (3.76). Thus the equations of motion for them have not been considered.

We see that the physical modes λ^+, ψ_4^+, and $\psi_{\parallel}'^+$ are linked with each other through their equations of motion, Eqs. (3.82) and (3.89). This is also the case for λ^-, ψ_4^-, and $\psi_{\parallel}'^-$ from Eqs. (3.83) and (3.90). Diagonalization procedure is thus required to obtain the spectrum of normal excitation modes. In order to do that, it is convenient to introduce the following definitions:

$$\lambda^{\pm} \equiv -\sqrt{2}\gamma^9\hat{\lambda}^{\pm} \;, \quad \psi_4^{\pm} \equiv \gamma^4\hat{\psi}_4^{\pm} \;. \quad (3.91)$$

For the modes with positive $SO(4)$ chirality, we then have from Eqs. (3.82) and (3.89)

$$(\Box - \frac{7\mu}{12}\gamma^{123}\partial_-)\hat{\lambda}^+ + \frac{\mu}{2}\gamma^{123}\partial_-\hat{\psi}_4^+ + \frac{\mu}{4}\gamma^{123}\partial_-\psi_{\parallel}'^+ = 0 \;,$$

$$(\Box - \frac{\mu}{4}\gamma^{123}\partial_-)\hat{\psi}_4^+ + \frac{5\mu}{8}\gamma^{123}\partial_-\hat{\lambda}^+ + \frac{3\mu}{8}\gamma^{123}\partial_-\psi_{\parallel}'^+ = 0 \;,$$

$$(\Box + \frac{7\mu}{6}\gamma^{123}\partial_-)\psi_{\parallel}'^+ + \frac{\mu}{2}\gamma^{123}\partial_-\hat{\lambda}^+ + \frac{5\mu}{3}\gamma^{123}\partial_-\hat{\psi}_4^+ = 0 \;. \quad (3.92)$$

The diagonalization of these equations is done by defining

$$\chi_1^+ \equiv -\frac{1}{2}\hat{\lambda}^+ - \frac{3}{4}\hat{\psi}_4^+ + \psi_{\parallel}'^+ \;,$$

$$\chi_3^+ \equiv \frac{3}{2}\hat{\lambda}^+ - \frac{7}{4}\hat{\psi}_4^+ + \psi_{\parallel}'^+ \;,$$

$$\chi_5^+ \equiv \frac{1}{6}\hat{\lambda}^+ + \frac{1}{4}\hat{\psi}_4^+ + \psi_{\parallel}'^+ \;, \quad (3.93)$$

whose equations of motion are then obtained as

$$(\Box - \frac{\mu}{3}\gamma^{123}\partial_-)\chi_1^+ = 0 \;, \quad (\Box - \mu\gamma^{123}\partial_-)\chi_3^+ = 0 \;, \quad (\Box + \frac{5\mu}{3}\gamma^{123}\partial_-)\chi_5^+ = 0 \;. \quad (3.94)$$

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If we split the modes in a way that respects the $SO(3)$ structure and follow the notation of Eq. (3.86), these equations lead to

$$(\Box - i\frac{\mu}{3}\partial_-)\chi_1^{++} = 0 , \quad (\Box + i\frac{\mu}{3}\partial_-)\chi_1^{-+} = 0 ,$$
$$(\Box - i\mu\partial_-)\chi_3^{++} = 0 , \quad (\Box + i\mu\partial_-)\chi_3^{-+} = 0 ,$$
$$(\Box + i\frac{5\mu}{3}\partial_-)\chi_5^{++} = 0 , \quad (\Box - i\frac{5\mu}{3}\partial_-)\chi_5^{-+} = 0 , \tag{3.95}$$

from which we have the following minimal light-cone energy spectrum for the diagonalized physical modes:

$$\mathcal{E}_0(\chi_1^{++}) = 7 , \quad \mathcal{E}_0(\chi_1^{-+}) = 5 ,$$
$$\mathcal{E}_0(\chi_3^{++}) = 9 , \quad \mathcal{E}_0(\chi_3^{-+}) = 6 ,$$
$$\mathcal{E}_0(\chi_5^{++}) = 1 , \quad \mathcal{E}_0(\chi_5^{-+}) = 11 . \tag{3.96}$$

We now turn to the modes with negative $SO(4)$ chirality, that is, λ^-, ψ_4^- and $\psi_\parallel'^-$. With the redefinitions, (3.91), Eqs. (3.83) and (3.90) lead to

$$(\Box - \frac{\mu}{6}\gamma^{123}\partial_-)\hat{\lambda}^- + \frac{\mu}{4}\gamma^{123}\partial_-\psi_\parallel'^- = 0 ,$$
$$\Box\hat{\psi}_4^- - \frac{\mu}{4}\gamma^{123}\partial_-\hat{\lambda}^- + \frac{3\mu}{8}\gamma^{123}\partial_-\psi_\parallel'^- = 0 ,$$
$$(\Box + \frac{5\mu}{6}\gamma^{123}\partial_-)\psi_\parallel'^- + \mu\gamma^{123}\partial_-\hat{\lambda}^- + \frac{4\mu}{3}\gamma^{123}\partial_-\hat{\psi}_4^- = 0 . \tag{3.97}$$

The diagonalized equations of motion are obtained as

$$\Box\chi_0^- = 0 , \quad (\Box - \frac{2\mu}{3}\gamma^{123}\partial_-)\chi_2^- = 0 , \quad (\Box + \frac{4\mu}{3}\gamma^{123}\partial_-)\chi_4^- = 0 , \tag{3.98}$$

where the normal excitation modes are defined by

$$\chi_0^- \equiv \frac{3}{2}\hat{\lambda}^- - \frac{7}{4}\hat{\psi}_4^- + \psi_\parallel'^- ,$$
$$\chi_2^- \equiv -\frac{1}{2}\hat{\lambda}^- - \frac{3}{4}\hat{\psi}_4^- + \psi_\parallel'^- ,$$
$$\chi_4^- \equiv \frac{1}{6}\hat{\lambda}^- + \frac{1}{4}\hat{\psi}_4^- + \psi_\parallel'^- . \tag{3.99}$$

After taking into account the $SO(3)$ structure with the notation of Eq. (3.86), the above set of equations, (3.98), leads us to have

$$\Box\chi_0^{+-} = 0 , \quad \Box\chi_0^{--} = 0 ,$$
$$(\Box - i\frac{2\mu}{3}\partial_-)\chi_2^{+-} = 0 , \quad (\Box + i\frac{2\mu}{3}\partial_-)\chi_2^{--} = 0 ,$$
$$(\Box + i\frac{4\mu}{3}\partial_-)\chi_4^{+-} = 0 , \quad (\Box - i\frac{4\mu}{3}\partial_-)\chi_4^{--} = 0 . \tag{3.100}$$

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$\mathcal{E}_0$	Bosonic mode ($N_{d.o.f.}$)	N_B	Fermionic mode ($N_{d.o.f.}$)	N_F
0	$\phi_6(1)$	1		0
1		0	$\chi_5^{++}(4)$	4
2	$\beta_{4i}(3)\ \beta_{4i'j'}(3)$	6	$\chi_4^{+-}(4)$	4
3	$\beta_{3i'}(4)\ \beta_{3ijk'}(12)$	16	$\chi_3^{-+}(4)$	4
4	$\phi_2(1)\ \beta_{2i}(3)\ \beta_{2i'j'}(3)$	7	$\chi_2^{--}(4)\ \psi_{\perp i}^{--}(8)\ \psi_{\perp i'}^{++}(12)$	24
5	$\beta_{1i'}(4)\ \beta_{1ij'}(12)$	16	$\chi_1^{-+}(4)\ \psi_{\perp i}^{-+}(8)\ \psi_{\perp i'}^{+-}(12)$	24
6	$\phi_0(1)\ \beta_{0i}(3)\ \beta_{0ij'k'}(18)\ h_{ij}^\perp(5)\ h_{i'j'}^\perp(9)$	36	$\chi_0^{+-}(4)\ \chi_0^{--}(4)$	8
7	$\bar\beta_{1i'}(4)\ \bar\beta_{1ij'}(12)$	16	$\chi_1^{++}(4)\ \psi_{\perp i}^{++}(8)\ \psi_{\perp i'}^{--}(12)$	24
8	$\bar\phi_2(1)\ \bar\beta_{2i}(3)\ \bar\beta_{2i'j'}(3)$	7	$\chi_2^{+-}(4)\ \psi_{\perp i}^{+-}(8)\ \psi_{\perp i'}^{-+}(12)$	24
9	$\bar\beta_{3i'}(4)\ \bar\beta_{3ijk'}(12)$	16	$\chi_3^{++}(4)$	4
10	$\bar\beta_{4i}(3)\ \bar\beta_{4i'j'}(3)$	6	$\chi_4^{--}(4)$	4
11		0	$\chi_5^{-+}(4)$	4
12	$\bar\phi_6(1)$	1		0
		128		128

Table 2: Type IIA supergravity excitation modes. $N_{d.o.f}$ means the number of degrees of freedom. N_B (N_F) is the number of bosonic (fermionic) degrees of freedom at each light-cone energy.

According to Eq. (3.8), we then have the minimal light-cone energy for the modes as

$$\mathcal{E}_0(\chi_0^{+-}) = 6 \ , \quad \mathcal{E}_0(\chi_0^{--}) = 6 \ ,$$
$$\mathcal{E}_0(\chi_2^{+-}) = 8 \ , \quad \mathcal{E}_0(\chi_2^{--}) = 4 \ ,$$
$$\mathcal{E}_0(\chi_4^{+-}) = 2 \ , \quad \mathcal{E}_0(\chi_4^{--}) = 10 \ . \tag{3.101}$$

4 Conclusion and discussion

We have obtained the Type IIA supergravity excitation modes and their spectrum in the pp-wave background, which are summarized in table 2. The supergravity modes have been arranged such that they respect the $SO(3) \times SO(4)$ symmetry structure of the background. We see that there is mismatch between the bosonic and fermionic degrees of freedom at each light-cone energy, basically due to the fact that the supersymmetry preserved by the pp-wave background is time-dependent. (More precisely, 16 among 24 supersymmetries depend on the light-cone time x^+ [17].)

The result we have obtained shows how the low-lying string states listed in table 1 correspond to the supergravity modes of table 2. This implies that we can associate the vertex operator for a certain low-lying string state to a definite supergravity excitation mode in the pp-wave background. It may be expected that the study of string amplitudes with such vertex operators gives useful insight into the structure of the M theory in the maximally supersymmetric eleven-dimensional pp-wave background.

Concerning the eleven dimensional origin of the ten dimensional supergravity excitation spectrum, it would be interesting to uncover the structure of the spectrum obtained in this paper. The eleven dimensional perturbative spectrum from the Matrix model in the eleven dimensional pp-wave background has intriguing features such as the protected multiplet and the indication of the presence of the transverse five-brane [7, 33]. Since the Type IIA supergravity inherits most of its features through the dimensional reduction, the structure in eleven dimensions would be encoded in the spectrum of ten dimensional supergravity. Thus the eleven dimensional perturbative spectrum would lead us to have the deeper understanding of the ten dimensional spectrum. What we would get from the physics related to the transverse five-brane is particularly interesting. We hope to return to this issue in the near future [34].

Acknowledgments

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