Poisson Structure and Moyal Quantisation of the Liouville Theory

George Jorjadze[a] [1] and Gerhard Weigt[b] [2]

[a]Razmadze Mathematical Institute,

M.Aleksidze 1, 380093, Tbilisi, Georgia

[b]DESY Zeuthen, Platanenallee 6,

D-15738 Zeuthen, Germany

Abstract

The symplectic and Poisson structures of the Liouville theory are derived from the symplectic form of the $SL(2,\mathbb{R})$ WZNW theory by gauge invariant Hamiltonian reduction. Causal non-equal time Poisson brackets for a Liouville field are presented. Using the symmetries of the Liouville theory, symbols of chiral fields are constructed and their $*$-products calculated. Quantum deformations consistent with the canonical quantisation result, and a non-equal time commutator is given.

1 Introduction

Wess-Zumino-Novikov-Witten(WZNW) models [1] are fascinating two-dimensional integrable conformal field theories which turn up in many areas of physics and mathematics. A rich variety of additional integrable theories is given by their cosets. Hamiltonian reduction [2, 3] provides a complete description of the relationship between these theories. In particular, the symplectic and Poisson structures of this theory [4, 5] and its cosets can be derived from the general solution of the equations of motions of the WZNW model. Moreover, Hamiltonian reduction also proves to be a general method for the integration of gauged WZNW theories. It might be worthwhile to illustrate this by considering the $SL(2,\mathbb{R})$ WZNW theory together with the Liouville theory [6]-[8] and the $SL(2,\mathbb{R})/U(1)$ black hole model [9, 3].

Our basic aim is to quantise the coset theories. Even for these well-studied cases fundamental problems remain. As an alternative to standard canonical quantisation [6, 7, 8, 3] it seems to be advantageous to consider a Moyal formalism [10]-[12]. Such methods have

[1]email: `jorj@rmi.acnet.ge`

[2]email: `weigt@ifh.de`

recently been variously applied to field theories with non-commutative geometry. In our case, however, such a formalism arises quite naturally as a consequence of a transitive symmetry group acting on the phase space. This symmetry may prove useful in the explicit construction of Liouville correlation functions.

In this paper we shall restrict our attention to Liouville theory. As well as being the simplest coset model, there is an extensive literature on quantum Liouville theory [6]-[8] against which we can test our Moyal approach. Before we do the quantisation, a full description of the classical Hamiltonian reduction will be given. The classical analogue of the exchange algebra is derived, and from this we deduce the general causal non-equal time Poisson brackets for a Liouville exponential. Such non-equal time Poisson structures may play a role in other interacting field theories.

In chapter 2 we derive the basic Poisson brackets of the $SL(2,\mathbb{R})$ WZNW theory from its symplectic form and discuss the symmetry properties of this theory. Chapter 3 describes the nilpotent reduction of these structures to the Liouville theory and presents in particular the symmetries on the phase space and the causal non-equal time Poisson brackets. After the introduction of a symbol calculus, $*$-products of chiral fields and a non-equal time commutator are calculated. Chapter 5 summarises the results, and some technical details are provided in four appendices.

2 The $SL(2,\mathbb{R})$ WZNW theory

The WZNW theory has a chiral structure and its investigation can essentially be reduced to the analysis of the chiral or anti-chiral part. The general solution of the dynamical equation gives the $SL(2,\mathbb{R})$ field $g(\tau,\sigma) = g(z)\,\bar{g}(\bar{z})$ as a product of the chiral and anti-chiral fields $g(z)$ and $\bar{g}(\bar{z})$, where $z = \tau + \sigma$, $\bar{z} = \tau - \sigma$ are light cone coordinates. The symplectic form and the Hamiltonian have a chiral structure as well.

2.1 The basic Poisson brackets

We consider periodic boundary conditions in σ for the $SL(2,\mathbb{R})$ field $g(\tau,\sigma)$, and the chiral fields have a monodromy $g(z+2\pi) = g(z)M$ with $M \in SL(2,\mathbb{R})$. The chiral part of the symplectic form of the $SL(2,\mathbb{R})$ WZNW theory is then [4, 5]

$$\omega = -\frac{1}{\gamma^2}\int_{\tau}^{\tau+2\pi} \langle (g^{-1}(z)\,\delta g(z))' \wedge g^{-1}(z)\,\delta g(z)\rangle\, dz \\ - \frac{1}{\gamma^2}\langle \delta M\, M^{-1} \wedge g^{-1}(\tau)\delta g(\tau)\rangle. \tag{2.1}$$

$\gamma > 0$ is a coupling constant, $'$ denotes differentiation, and the normalised trace is defined by $\langle\cdot\rangle = -\frac{1}{2}\mathrm{tr}(\cdot)$.

Let us take the basis of the $sl(2,\mathbb{R})$ algebra

$$t_0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad t_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad t_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2}$$

It satisfies the relations $t_m\, t_n = -\eta_{mn}\, I + \epsilon_{mn}{}^{l}\, t_l$, where I is the unit matrix, $\eta_{mn} = \mathrm{diag}(+,-,-)$ the metric tensor of a $3d$ Minkowski space, and $\epsilon_{012} = 1$. For the matrices t_n one has $\langle t_m\, t_n\rangle = \eta_{mn}$, $\langle t_l\, t_m\, t_n\rangle = \epsilon_{lmn}$ and for any $a \in sl(2,\mathbb{R})$ $t^n\,\langle t_n\, a\rangle = a$. We shall also use the nilpotent elements of the algebra $t_\pm = t_1 \pm t_0$.

In general, the 2-form (2.1) is singular and one has to quotient the space of the chiral fields by $SL(2,\mathbb{R})$ transformations $g(z) \mapsto g(z)N$ [4] under which $M \mapsto N^{-1}MN$. The monodromy M can be so transformed into an abelian subgroup, and we shall choose

$$M = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix}, \quad \text{with} \quad \lambda < 0, \tag{2.3}$$

in order to get a regular Liouville field after Hamiltonian reduction. The 2-form (2.1) then becomes symplectic and its inversion gives the Poisson brackets (see Appendix A for the explicit calculations)

$$\{2\lambda,\, g(z)\} = \gamma^2\, g(z)\, t_2, \tag{2.4}$$

$$\{\, g_{ab}(z),\, g_{cd}(y)\,\} = \frac{\gamma^2}{4}\,[\,(\,g(z)\,t_2\,)_{ab}\,(\,g(y)\,t_2\,)_{cd}\,\epsilon(z-y) \\ +(\,g(z)\,t_-\,)_{ab}\,(\,g(y)\,t_+\,)_{cd}\theta_{-2\lambda}(z-y) \\ +\,(\,g(z)\,t_+\,)_{ab}\,(\,g(y)\,t_-\,)_{cd}\theta_{2\lambda}(z-y)]. \tag{2.5}$$

Here $\epsilon(z)$ is the stair-step function $\epsilon(z) = 2n+1$ for $2\pi n < z < 2\pi(n+1)$, and

$$\theta_{2\lambda}(z-y) = \frac{e^{\lambda\epsilon(z-y)}}{2\,\sinh\lambda} \tag{2.6}$$

is the Green's function [6] which inverts the operator ∂_z on functions $A(z)$ with the monodromy $A(z+2\pi) = e^{2\lambda} A(z)$ for $\lambda \neq 0$. Since in the interval $z \in (-2\pi, 2\pi)$ $\epsilon(z) = sign\,(z)$, we have

$$\theta_{2\lambda}(z) = \frac{\cosh\lambda}{2\sinh\lambda} + \frac{1}{2}\,\epsilon(z). \tag{2.7}$$

Equations (2.4), (2.5) are the basic Poisson brackets of the $SL(2,\mathbb{R})$ WZNW theory. The corresponding relations of the coset theories can be derived through the Hamiltonian reduction.

2.2 The Kac-Moody and conformal symmetries

As a consequence of the basic algebra (2.4), (2.5) the components of the (periodic) Kac-Moody currents

$$J_n(z) = \frac{1}{\gamma^2}\,\langle\, t_n\, g\,'(z)\, g^{-1}(z)\,\rangle \tag{2.8}$$

commute with λ and satisfy

$$\{J_n(z),\, g(y)\} = -\frac{1}{2}(t_n\, g(y))\,\delta(z-y), \tag{2.9}$$

which means that the currents $J_n(z)$ are generators of left multiplications

$$R(f):\; g(z) \;\mapsto\; f(z)\, g(z) \qquad (f(z) \in SL(2,\mathbb{R})). \tag{2.10}$$

Since the symplectic form (2.1) contains the 1-forms $g^{-1}(z)\,\delta g(z)$ which are invariant under left multiplications, (2.10) are symmetry transformations of the system. The $SL(2,\mathbb{R})$ Kac-Moody algebra follows directly from (2.8) and (2.9) as

$$\{J_m(z),\, J_n(y)\} = \epsilon_{mn}{}^{\,l}\, J_l(z)\,\delta(z-y) + \frac{1}{2\gamma^2}\,\eta_{mn}\,\delta\,'(z-y), \tag{2.11}$$

and for the chiral Sugawara energy momentum tensor

$$T_g(z) = -\gamma^2\, J_n(z)\, J^n(z) \tag{2.12}$$

we get the Poisson bracket relations

$$\{T_g(z),\, g(y)\} = g\,'(y)\,\delta(z-y), \tag{2.13}$$

$$\{T_g(z),\, J_n(y)\} = J_n'(y)\,\delta(z-y) - J_n(y)\delta\,'(z-y), \tag{2.14}$$

$$\{T_g(z),\, T_g(y)\} = T_g\,'(y)\,\delta(z-y) - 2T_g(y)\delta\,'(z-y). \tag{2.15}$$

We see that $g(z)$, $J_n(z)$ and $T_g(z)$ have the conformal weights 0, 1 and 2, respectively. $T_g(z)$ generates the conformal transformations $g(z) \mapsto g(\xi(z))$, which leave (2.1) invariant too. The symmetry of the $SL(2,\mathbb{R})$ theory is therefore given by the semi-direct product of the conformal and the Kac-Moody groups.

In order to complete the description of the $SL(2,\mathbb{R})$ WZNW structures by the anti-chiral part, we have to replace the left invariant 1-form $g^{-1}(z)\,\delta g(z)$ of (2.1) by the right invariant $\delta\bar{g}(\bar{z})\,\bar{g}^{-1}(\bar{z})$. So we get the anti-chiral symplectic form of the theory with a similar symmetry structure. The monodromy of the anti-chiral field is given by $\bar{g}(\bar{z}-2\pi) = \bar{M}\bar{g}(\bar{z})$ and the periodicity of the WZNW field $g(\tau,\sigma)$ requires $\bar{M} = M^{-1}$.

3 The Liouville theory

The Liouville theory can be obtained by nilpotently gauging the $SL(2,\mathbb{R})$ WZNW theory, imposing constraints for the chiral and anti-chiral fields separately [2]. For the chiral part the constraint is

$$J_+(z) + \rho = 0, \qquad \text{where} \qquad J_+(z) = \frac{1}{\gamma^2}\langle t_+\, g'(z)\, g^{-1}(z)\rangle. \tag{3.1}$$

$\rho > 0$ is a fixed parameter and t_+ the nilpotent element of the $sl(2,\mathbb{R})$ algebra. $J_+(z)+\rho = 0$ is a first class constraint which generates gauge transformations.

In fact, we apply a gauge invariant version of the Hamiltonian reduction of [3] to the nilpotent case.

3.1 The exchange algebra

Let us use for the variables and Poisson brackets of the reduced system the same notation as for the $SL(2,\mathbb{R})$ WZNW theory. Anticipating the Liouville theory we rename the gauge invariant components of the chiral field $g_{11}(z) = \psi(z)$ and $g_{12}(z) = \chi(z)$. Due to gauge invariance the form of the Poisson brackets between ψ and χ is obviously covariant under the reduction. Thus we can simply read off the result directly from the basic Poisson brackets of the $SL(2,\mathbb{R})$ WZNW theory (2.5) as

$$\{\psi(z), \psi(y)\} = \frac{\gamma^2}{4}\psi(z)\,\psi(y)\epsilon(z-y), \tag{3.2}$$

$$\{\chi(z), \chi(y)\} = \frac{\gamma^2}{4}\chi(z)\,\chi(y)\epsilon(z-y), \tag{3.3}$$

$$\{\psi(z), \chi(y)\} = -\frac{\gamma^2}{4}\psi(z)\,\chi(y)\epsilon(z-y) + \gamma^2\,\chi(z)\,\psi(y)\theta_{2\lambda}(z-y). \tag{3.4}$$

This is the basic Poisson algebra of the reduced system. The corresponding quantum commutation relations are nothing but the well-known exchange algebra [7]. Below we shall show that eqs. (3.2)-(3.4) together with the similar anti-chiral relations, provide locality and canonicity of the Liouville field.

3.2 The nilpotent reduction of the $SL(2,\mathbb{R})$ theory

From the constrained Kac-Moody current

$$J(z) = \begin{pmatrix} -J_2(z) & \rho \\ -J_-(z) & J_2(z) \end{pmatrix}, \qquad (J_- = J_1 - J_0), \tag{3.5}$$

and the relation

$$g'(z) = \gamma^2 J(z) g(z), \tag{3.6}$$

we easily find the reduced chiral field

$$g(z) = \begin{pmatrix} \psi(z) & \chi(z) \\ \beta^{-1}\psi'(z) + \rho^{-1}J_2(z)\psi(z) & \beta^{-1}\chi'(z) + \rho^{-1}J_2(z)\chi(z) \end{pmatrix}, \tag{3.7}$$

where $\beta = \rho\gamma^2$. The gauge freedom of $g(z)$ is given by $J_2(z)$. Since $\det\, g = 1$, $\psi(z)$ and $\chi(z)$ have a constant Wronskian

$$\psi(z)\chi'(z) - \psi'(z)\chi(z) = \beta, \tag{3.8}$$

and (2.3) defines their monodromy

$$\psi(z+2\pi) = e^{\lambda}\,\psi(z), \qquad \chi(z+2\pi) = e^{-\lambda}\,\chi(z). \tag{3.9}$$

To get a regular Liouville field we assume $\psi(z) > 0$. The relation (3.8) can then be integrated by using (3.9), and $\chi(z)$ becomes

$$\chi(z) = \beta\,\psi(z)\int_0^{2\pi} dy\,\frac{\theta_{-2\lambda}(z-y)}{\psi^2(y)}, \tag{3.10}$$

which is positive since $\lambda < 0$ (see (2.3)). So the phase space of the reduced system can be parameterised by the field $\psi(z)$ only. Inserting the field (3.7) into the symplectic form

(2.1), the reduced symplectic form can be written in terms of gauge invariant variables only

$$\omega = \frac{1}{\gamma^2} \int_{\tau}^{\tau+2\pi} \frac{1}{\psi^2(z)}\, \delta\psi'(z) \wedge \delta\psi(z)\, dz + \frac{1}{\gamma^2}\, \delta\lambda \wedge \frac{\delta\psi(\tau)}{\psi(\tau)}. \tag{3.11}$$

Since $\psi(z) > 0$, we shall write $\psi(z) = e^{-\gamma\phi(z)}$, and (3.11) gets in terms of $\phi(z)$ the free-field form

$$\omega = \int_{\tau}^{\tau+2\pi} \delta\phi'(z) \wedge \delta\phi(z)\, dz + \frac{\delta p_0}{2} \wedge \delta\phi(\tau), \tag{3.12}$$

with $p_0 = -2\lambda/\gamma$ ($p_0 > 0$). Eq. (3.9) requires that this chiral field $\phi(z)$ has the standard monodromy $\phi(z + 2\pi) = \phi(z) + p_0/2$ of a free field. Using the Fourier mode expansion

$$\phi(z) = q_0 + \frac{p_0 z}{4\pi} + \frac{i}{\sqrt{4\pi}} \sum_{n \neq 0} \frac{a_n}{n}\, e^{-inz}, \tag{3.13}$$

(3.12) yields canonical Poisson brackets for the modes

$$\{p_0,\, q_0\} = 1, \qquad \{a_m,\, a_n\} = im\delta_{m+n,0}. \tag{3.14}$$

The classical vacuum configuration corresponds to $a_n = 0$, whereas the zero modes live on the half-plane $p_0 > 0$. The vacuum is thus $(q,\, p)$-dependent and the corresponding fields $\psi(z)$ and $\chi(z)$ are

$$\psi_0(z,p) = e^{-\gamma\left(q + \frac{pz}{4\pi}\right)}, \qquad \chi_0(z,p) = \frac{2\pi\beta}{\gamma p}\, e^{\gamma\left(q + \frac{pz}{4\pi}\right)}. \tag{3.15}$$

In order to prove that the nilpotent gauging discussed here indeed leads to the Liouville theory, we have to add the corresponding anti-chiral part requiring the constraint $\bar{J}_+(\bar{z}) = \bar{\rho}$ with $\bar{\rho} > 0$. $\bar{J}_+(\bar{z})$ is defined similarly to (3.1) by the right Kac-Moody current. The gauge invariant components are now $\bar{\psi}(\bar{z}) = \bar{g}_{12}(\bar{z})$ and $\bar{\chi}(\bar{z}) = \bar{g}_{22}(\bar{z})$. They have the monodromy

$$\bar{\psi}(\bar{z} - 2\pi) = e^{-\bar{\lambda}}\, \bar{\psi}(\bar{z}), \qquad \bar{\chi}(\bar{z} - 2\pi) = e^{\bar{\lambda}}\, \bar{\chi}(\bar{z}), \tag{3.16}$$

and on the constrained surface they are related by the Wronskian condition $\bar{\psi}(\bar{z})\bar{\chi}'(\bar{z}) - \bar{\psi}'(\bar{z})\bar{\chi}(\bar{z}) = \bar{\beta}$, where $\bar{\beta} = \gamma^2\bar{\rho}$. Assuming again $\bar{\psi}(\bar{z}) > 0$ and 'bosonise' $\bar{\psi}(\bar{z}) = e^{-\gamma\bar{\phi}(\bar{z})}$ the reduced part is described correspondingly by the anti-chiral free field $\bar{\phi}(\bar{z})$ with the zero modes $\bar{p}_0,\, \bar{q}_0,$ and the oscillators $\bar{a}_n$.

The periodicity of the $SL(2,\mathbb{R})$ field $g(\tau,\sigma)$ requires $\bar{\lambda} = \lambda$ which for the zero modes leads to the constraint $p_0 - \bar{p}_0 = 0$. The free field $\phi(\tau,\sigma) = \phi(z) + \bar{\phi}(\bar{z})$ then becomes periodic as well and its canonical zero modes are given by $q = q_0 + \bar{q}_0$ and $p = p_0 = \bar{p}_0 > 0$. Note that this constrained zero mode system which we got by reduction just corresponds to the old-fashioned 'Fubini-Veneziano trick' to treat the dual model zero modes conveniently. Indeed, we shall use this trick henceforth. Then the chiral fields can be treated as independent fields with closed Poisson brackets.

For the periodic field $g(\tau,\sigma) = g(z)\ \bar{g}(\bar{z})$ only the one component $g_{12}(\tau,\sigma) = \psi(z)\bar{\psi}(\bar{z}) + \chi(z)\bar{\chi}(\bar{z})$ remains gauge invariant, and it is the Liouville field $\varphi(\tau,\sigma)$ which is expected to parameterise it

$$\psi(z)\bar{\psi}(\bar{z}) + \chi(z)\bar{\chi}(\bar{z}) = e^{-\gamma\varphi(\tau,\sigma)}. \tag{3.17}$$

From the constant Wronskian we obtain for $\varphi(\tau,\sigma)$, indeed, the Liouville equation

$$\left(\partial_\tau^2 - \partial_\sigma^2\right)\varphi(\tau,\sigma) + \frac{4m^2}{\gamma}e^{2\gamma\varphi(\tau,\sigma)} = 0, \qquad \text{with} \qquad m^2 = \beta\bar{\beta}, \tag{3.18}$$

and eqs. (3.17) and (3.10) provide its general solution in the standard parametrisation

$$e^{2\gamma\varphi(\tau,\sigma)} = \frac{A'(z)\bar{A}'(\bar{z})}{[1 + m^2\, A(z)\bar{A}(\bar{z})]^2}, \tag{3.19}$$

where

$$A(z) = \int_0^{2\pi} dy\ \theta_{\gamma p}\,(z-y)\, e^{2\gamma\phi(y)}. \tag{3.20}$$

$\varphi(\tau,\sigma)$ is periodic, and one can show that this description covers the class of all regular periodic Liouville fields.

It is worth mentioning that the integration of both the Liouville and the $SL(2,\mathbb{R})/U(1)$ theory [3] by Hamiltonian reduction is a strong indication that this approach can be generalised to any gauged WZNW theory. The standard Lax-pair method [13] applies to the nilpotent gaugings only.

3.3 The symmetries of the Liouville theory

The symmetry properties of the chiral fields $\psi(z)$ and $\chi(z)$ will play an important role to implement the Moyal deformation quantisation. Putting (3.5) and (3.7) into (3.6), comparison of the components yields

$$\frac{\psi''(z)}{\psi(z)} = \frac{\chi''(z)}{\chi(z)} = \gamma^2(T_g(z) - J_2'(z)). \tag{3.21}$$

$T_g(z) = \gamma^2(J_2^2(z) - \rho J_-(z))$ is the reduced energy-momentum tensor of (2.12) which becomes gauge invariant only through the 'improvement' $T_g(z) \mapsto T(z) = T_g(z) - J_2'(z)$ [14], we have got in eq. (3.21). From (2.13)-(2.15) we easily find

$$\{T(z), \psi(y)\} = \psi\,'(y)\,\delta(z-y) + \frac{1}{2}\psi(y)\delta\,'(z-y), \tag{3.22}$$

$$\{T(z), \chi(y)\} = \chi\,'(y)\,\delta(z-y) + \frac{1}{2}\chi(y)\delta\,'(z-y), \tag{3.23}$$

$$\begin{aligned}\{T(z), T(y)\} = {} & T\,'(y)\,\delta(z-y) - 2T(y)\delta\,'(z-y) \\ & + \frac{1}{2\gamma^2}\delta\,'''(z-y).\end{aligned} \tag{3.24}$$

We see that both $\psi(z)$ and $\chi(z)$ have the same conformal weight $-\frac{1}{2}$ and that the algebra of the improved tensor is deformed by a central extension only. Thus we have, as expected, conformal invariance of the Liouville theory, and the gauge invariant $T(z)$ will therefore be identified with its energy-momentum tensor.

The relation (3.21) corresponds to the Schrödinger equation, well-known in Liouville theory, for both $\psi(z)$ and $\chi(z)$ with the solutions $\psi(z) = e^{-\gamma\phi(z)}$ and (3.10). It allows us to transform the energy-momentum tensor into a free-field form with the typical improvement term

$$T(z) = \phi'^{\,2}(z) - \frac{1}{\gamma}\,\phi''(z). \tag{3.25}$$

Our deformation quantisation will rely heavily on the following important observation. The symplectic form (3.11) is invariant under $\psi(z) \mapsto e^{-\gamma\rho(z)}\psi(z)$ with periodic $\rho(z)$. These transformations are generated by $\phi'(z)$

$$\{\phi'(z),\, \psi(y)\} = -\frac{\gamma}{2}\delta(z-y)\psi(y), \tag{3.26}$$

and for the chiral free field this corresponds to the translations $\phi(z) \mapsto \phi(z) + \rho(z)$. These translations and the conformal transformations represent together the symmetry group of the free-field symplectic form (3.12). The corresponding Lie algebra is given by (3.24) and

$$\{\phi'(z),\, \phi'(y)\} = -\frac{1}{2}\delta\,'(z-y), \tag{3.27}$$

$$\{T(z),\, \phi'(y)\} = \phi\,''(y)\delta(z-y) - \phi\,'(y)\delta\,'(z-y) + \frac{1}{2\gamma}\delta\,''(z-y). \tag{3.28}$$

Comparing this with (2.10), (2.11) and (2.14), we see that the $\phi'(z)$ and the Kac-Moody current obviously play a similar role for the Liouville and $SL(2,\mathbb{R})$ WZNW theories, respectively.

Although the fields $\psi(z)$ and $\chi(z)$ have the same conformal weight their transformations with respect to the translation group is different. In distinction to (3.26) the infinitesimal translation of $\chi(z)$

$$\{\phi'(z),\, \chi(y)\} = -\frac{\gamma}{2}\delta(z-y)\chi(y) + \gamma\beta\,\theta_{\gamma p}(y-z)\, e^{-\gamma\phi(y)}e^{2\gamma\phi(z)} \tag{3.29}$$

produces the bilocal field

$$B(y,z) = \theta_{\gamma p}(y-z)\, e^{-\gamma\phi(y)}e^{2\gamma\phi(z)}\,. \tag{3.30}$$

This bilocal field transforms linearly with respect to the translation

$$B(y,x) \mapsto e^{-\gamma\rho(y)}\, e^{2\gamma\rho(x)}\, B(y,x), \tag{3.31}$$

and its x-integration gives

$$\chi(y) = \beta \int_0^{2\pi} dx\, B(y,x). \tag{3.32}$$

Since $p > 0$, there is no global translation symmetry in the p direction of the half-plane. In order to have a group with a transitive action on the phase space at hand, we introduce the dilatations $p \mapsto e^{-\varepsilon}p$, $q \mapsto e^{\varepsilon}q$ generated by $K = pq$. The vacuum configuration (3.15) transforms for $q = 0$ under these dilatations as

$$\psi_0(z, e^{-\varepsilon}\, p) = \psi_0(e^{-\varepsilon}\, z, p), \qquad \chi_0(z, e^{-\varepsilon}\, p) = e^{\varepsilon}\, \chi_0(e^{-\varepsilon}\, z, p). \tag{3.33}$$

Adding this one-parameter group to the translations $\phi(z) \mapsto \phi(z) + \rho(z)$ the new group $G(\rho(z), \varepsilon)$ is defined. The Moyal quantisation of the Liouville theory is based just on this symmetry group.

3.4 The non-equal time Poisson structure of Liouville fields

In this section we calculate non-equal time Poisson brackets for the exponential of the Liouville field (3.17) denoted here by

$$u(z,\bar{z}) = e^{-\gamma\varphi(\tau,\sigma)}. \tag{3.34}$$

Using the exchange algebra (3.2)-(3.4) together with the similar anti-chiral relations, we easily find

$$\begin{aligned}\{u(z,\bar{z}),\, u(y,\bar{y})\} = {} & \gamma^2\,\Theta_{\gamma p}\,\psi(z)\chi(y)\bar{\chi}(\bar{z})\bar{\psi}(\bar{y}) + \gamma^2\,\Theta_{-\gamma p}\,\chi(z)\psi(y)\bar{\psi}(\bar{z})\bar{\chi}(\bar{y}) \\ & + \frac{\gamma^2}{2}\,\Theta\,[\psi(z)\psi(y)\bar{\psi}(\bar{z})\bar{\psi}(\bar{y}) + \chi(z)\chi(y)\bar{\chi}(\bar{z})\bar{\chi}(\bar{y}) \\ & - \psi(z)\chi(y)\bar{\psi}(\bar{z})\bar{\chi}(\bar{y}) - \chi(z)\psi(y)\bar{\chi}(\bar{z})\bar{\psi}(\bar{y})].\end{aligned} \tag{3.35}$$

Here we have introduced the notation

$$\Theta_{\gamma p} = \theta_{\gamma p}(z-y) + \theta_{-\gamma p}(\bar{z}-\bar{y}), \qquad \Theta = \frac{1}{2}[\epsilon(z-y) + \epsilon(\bar{z}-\bar{y})]. \tag{3.36}$$

With the relation

$$\begin{aligned} u(z,\bar{z})\, u(y,\bar{y}) - u(z,\bar{y})\, u(y,\bar{z}) = \psi(z)\chi(y)\bar{\psi}(\bar{z})\bar{\chi}(\bar{y}) + \chi(z)\psi(y)\bar{\chi}(\bar{z})\bar{\psi}(\bar{y}) \\ - \psi(z)\chi(y)\bar{\chi}(\bar{z})\bar{\psi}(\bar{y}) - \chi(z)\psi(y)\bar{\psi}(\bar{z})\bar{\chi}(\bar{y}), \end{aligned} \tag{3.37}$$

which follows from (3.17) and the definition (3.34), the Poisson bracket (3.35) becomes

$$\begin{aligned} \{u(z,\bar{z}),\, u(y,\bar{y})\} = \gamma^2\,(\Theta_{\gamma p} - \Theta)\psi(z)\chi(y)\bar{\chi}(\bar{z})\bar{\psi}(\bar{y}) \\ + \gamma^2\,(\Theta_{-\gamma p} - \Theta)\,\chi(z)\psi(y)\bar{\psi}(\bar{z})\bar{\chi}(\bar{y}) \\ + \frac{\gamma^2}{2}\Theta[2u(z,\bar{y})\, u(y,\bar{z}) - u(z,\bar{z})\, u(y,\bar{y})]. \end{aligned} \tag{3.38}$$

In the fundamental interval $z - y \in (-2\pi,\, 2\pi)$ and $\bar{z} - \bar{y} \in (-2\pi,\, 2\pi)$, where (2.7) gives $\Theta_{\gamma p} = \Theta = \Theta_{-\gamma p}$, (3.38) reduces to the following non-equal time bracket relation

$$\{u(z,\bar{z}),\, u(y,\bar{y})\} = \frac{\gamma^2}{4}[\epsilon(z-y) + \epsilon(\bar{z}-\bar{y})]\,[2u(z,\bar{y})\, u(y,\bar{z}) - u(z,\bar{z})\, u(y,\bar{y})]. \tag{3.39}$$

To our knowledge this result is not in the existing literature. The step character of the function $\epsilon(z)$ provides causality of this Poisson bracket, and in particular its equal time form vanishes. For the Liouville theory on the line where we do not have zero modes eq. (3.39) is valid in general. It shows that the Poisson brackets of the Liouville field at two different space-time points $(z,\bar{z}),\ (y,\bar{y})$ is expressed by the field in these two and two other points $(z,\bar{y}),\ (y,\bar{z})$ obtained by exchanging the light-cone coordinates.

When $z - y$ and $\bar{z} - \bar{y}$ are out of the fundamental domain we still can get a closed form of (3.38) in terms of the field u . For this purpose we introduce two new space time points $(z + 2\pi, \bar{y} + 2\pi)$ and $(y + 2\pi,\, \bar{z} + 2\pi)$ which are shifted by 2π (in time) with respect to $(z,\, \bar{y})$ and $(y,\, \bar{z})$, respectively. The monodromies of ψ and χ give

$$u(z + 2\pi, \bar{y} + 2\pi) = e^{-\gamma p}\psi(z)\bar{\psi}(\bar{y}) + e^{\gamma p}\chi(z)\bar{\chi}(\bar{y}), \tag{3.40}$$

which lead to

$$\begin{aligned} \psi(z)\bar{\psi}(\bar{y}) = \frac{e^{\gamma p}\, u(z,\bar{y}) - u(z + 2\pi, \bar{y} + 2\pi)}{2\sinh \gamma p}, \\ \chi(z)\bar{\chi}(\bar{y}) = \frac{u(z + 2\pi, \bar{y} + 2\pi) - e^{-\gamma p}\, u(z,\bar{y})}{2\sinh \gamma p}. \end{aligned} \tag{3.41}$$

Similarly the monodromy of $u(y,\bar z)$ defines the products $\psi(y)\bar\psi(\bar z)$ and $\chi(y)\bar\chi(\bar z)$. Inserting these formulas into (3.38) we obtain the non-equal time Poisson bracket for $u(z,\bar z)$ in general. It now relates quadratic terms of u at six different space-time points.
In the space-time coordinates

$$\tau = \frac{1}{2}(z+\bar z), \qquad \sigma = \frac{1}{2}(z-\bar z), \qquad \tau_1 = \frac{1}{2}(y+\bar y), \qquad \sigma_1 = \frac{1}{2}(y-\bar y), \tag{3.42}$$

the fundamental interval corresponds to the time-interval $|\tau-\tau_1| < 2\pi$, where (3.39) is valid. Outside of it we have $|\tau-\tau_1| > 2\pi$, and the additional terms of (3.38) contribute due to the spatial periodicity.
Differentiating (3.39) with respect to τ and putting $\tau_1 = \tau$ we obtain the equal time relation

$$\{\partial_\tau e^{-\varphi(\tau,\sigma)},\, e^{-\varphi(\tau,\sigma_1)}\} = \delta(\sigma-\sigma_1)\, e^{-2\varphi(\tau,\sigma)}, \tag{3.43}$$

which is equivalent to the canonical Poisson bracket $\{\dot\varphi(\tau,\sigma),\, \varphi(\tau,\sigma_1)\} = \delta(\sigma-\sigma_1)$.
With these results we could also calculate the Poisson bracket for the physically more interesting arbitrary exponential of $\varphi(\tau,\sigma_1)$, the Liouville vertex function.

4 A Moyal quantisation of the Liouville theory

Following the ideas of deformation quantisation (for a general treatment see [15, 16]), we describe the quantum Liouville theory not in the operator formalism but rather by functionals on the phase space. In our case the formalism is based on the symmetry group $G(\rho(z),\varepsilon)$ of Chapter 3.3. Let us introduce the elements we shall need for our alternative quantisation of the Liouville theory, discussing first a single oscillator.

4.1 The Moyal formalism for the oscillator and zero modes

We consider the Hamiltonian

$$H = \frac{1}{2}\left(P^2 + \nu^2 Q^2\right), \tag{4.1}$$

and recall some well-known facts of quantum mechanics [17]. The coherent states $|\mathbf{a}\rangle = |P,Q;\nu\rangle$ with given frequency $\nu > 0$ are labelled by the points of the phase space. The state $|\mathbf{a}\rangle$ is related to the vacuum state $|\mathbf{0}\rangle$ by the Weyl group transformation

$$|\mathbf{a}\rangle = \exp\frac{i}{\hbar}\left(\hat Q\, P - \hat P\, Q\right)|\mathbf{0}\rangle = \exp\frac{1}{\hbar\nu}\left(\hat a^+\, a - \hat a\, a^*\right)|\mathbf{0}\rangle, \tag{4.2}$$

which means that the translations of the coherent states are given up to a phase factor by the classical law. $\hat{a}^+$, $\hat{a}$ are creation and annihilation operators with the canonical commutation relation $[\hat{a}, \hat{a}^+] = \nu\hbar$, and

$$a = \frac{P - i\nu Q}{\sqrt{2}}, \qquad a^* = \frac{P + i\nu Q}{\sqrt{2}} \tag{4.3}$$

are the corresponding classical variables. The coherent states are eigenstates of the annihilation operator

$$\hat{a}\,|\mathbf{a}\rangle = a\,|\mathbf{a}\rangle, \tag{4.4}$$

and they are complete

$$\int |\mathbf{a}\rangle\, d^2\mathbf{a}\,\langle\mathbf{a}| = \hat{I}, \qquad \text{with} \quad d^2\mathbf{a} = (2\pi\hbar)^{-1}\,dp\,dq. \tag{4.5}$$

As a consequence of the Baker-Hausdorf formula the scalar product of the coherent states becomes

$$\langle\mathbf{b}\,|\,\mathbf{a}\rangle = \exp\left(-\frac{|a|^2 + |b|^2 - 2b^*\,a}{2\hbar\nu}\right). \tag{4.6}$$

The time evolution of the coherent states also follows the classical law

$$e^{-\frac{i}{\hbar}\hat{H}\,t}\,|a,\ a^*\rangle = e^{-\frac{i}{2}\nu t}\,|e^{-i\nu t}a,\ e^{i\nu t}a^*\rangle, \tag{4.7}$$

It is an important observation that there is a one-to-one correspondence between an operator $\hat{A}$ on the Hilbert space and a function $\breve{A}(P,Q)$ on the phase space. We specify such a map by the normal ordering prescription

$$\breve{A}(a^*, a) = \langle\mathbf{a}|\hat{A}|\mathbf{a}\rangle, \qquad \hat{A} =: \breve{A}(\hat{a}^*, \hat{a}) : \tag{4.8}$$

The function $\breve{A}(a^*, a)$ is known as the normal (Wick) or Berezin symbol of the operator $\hat{A}$ [12]. The non-diagonal matrix elements are given by

$$\langle\mathbf{b}\,|\hat{A}|\,\mathbf{a}\rangle = \breve{A}(b^*, a)\,\langle\mathbf{b}\,|\,\mathbf{a}\rangle. \tag{4.9}$$

The non-commutativity of quantum mechanics is introduced through the Moyal $*$-product of symbols [11], which is a symbol of the product of the corresponding two operators

$$\breve{A} * \breve{B} = \langle\mathbf{a}|\hat{A}\,\hat{B}|\mathbf{a}\rangle. \tag{4.10}$$

Using the completeness relation (4.5) as well as (4.6) and (4.9) we find the useful integral representation

$$\breve{A} * \breve{B} = \int d\mu(\xi)\, e^{-\frac{|\xi|^2}{\nu}}\ \breve{A}(a^*,\, a + \sqrt{\hbar}\, \xi)\, \breve{B}(a^* + \sqrt{\hbar}\xi^*,\, a), \tag{4.11}$$

where we have first made a shift of the integration variables (4.5) and then a dilatation by $\sqrt{\hbar}$. $d\mu(\xi) = (2\pi)^{-1} dxdy$ is the corresponding normalised measure and $\xi = (x + i\nu y)/\sqrt{2}$. Expanding the integrand in powers of $\hbar$, after Gaussian integration, the $*$-product is easily seen to be a deformation of the product of two ordinary functions

$$\breve{A} * \breve{B} = \breve{A} \cdot \breve{B} + \hbar\nu \frac{\partial \breve{A}}{\partial a}\frac{\partial \breve{B}}{\partial a^*} + \frac{1}{2!}(\hbar\nu)^2 \frac{\partial^2 \breve{A}}{\partial a^2}\frac{\partial^2 \breve{B}}{\partial a^{*\,2}} + \cdots. \tag{4.12}$$

The key object which corresponds to the commutator is the Moyal $*$-bracket of two symbols

$$\{\breve{A},\, \breve{B}\}_* = \frac{i}{\hbar}(\breve{A} * \breve{B} - \breve{B} * \breve{A}). \tag{4.13}$$

Since the $*$-product (4.12) is associative, the Moyal bracket obeys the Jacobi identity. Equation (4.12) gives the practically useful expansion

$$\{\breve{A},\, \breve{B}\}_* = \{\breve{A},\, \breve{B}\} + i\hbar \frac{\nu^2}{2!}\left(\frac{\partial^2 \breve{A}}{\partial a^2}\frac{\partial^2 \breve{B}}{\partial a^{*\,2}} - \frac{\partial^2 \breve{B}}{\partial a^2}\frac{\partial^2 \breve{A}}{\partial a^{*\,2}}\right) + \cdots, \tag{4.14}$$

where $\{\breve{A},\, \breve{B}\}$ is the standard Poisson bracket.

Applying these quantisation rules to the generators of the Weyl transformations $\hat{P}$, $\hat{Q}$, and to the Hamiltonian $\hat{H} = \hat{a}^+\hat{a}$ the belonging symbols obviously coincide with their classical counterparts, and their $*$-brackets with an arbitrary function $\breve{A}(P,Q)$ are simply the Poisson brackets. Such non-deformation properties remain valid in general for the generators of symmetry groups which define the coherent states [17].

For the Liouville theory the chiral part of the Hamiltonian

$$H = \int_0^{2\pi} dz\, \tilde{T}(z) = \frac{p^2}{8\pi} + \sum_{n>0} |a_n|^2 \tag{4.15}$$

has for the non-zero modes the discussed oscillator structure. These modes have therefore the standard coherent states with the frequency parameter $\nu_n = n$ and the symbol calculus can be applied as before.

But the treatment of the zero modes is essentially different since they are given on the half-plane $p > 0$ only. Here the Weyl symmetry fails and the corresponding coherent states and symbols have to be constructed separately (see Appendix B). On the half-plane the coordinate representation does not exist, but we can still make use of the momentum representation on the Hilbert space $L^2(\mathbb{R}_+)$. The operator $\hat{p}$ then acts as a multiplication on a wave function $\Psi(p) \in L^2(\mathbb{R}_+)$, and the generator of dilatations is $\hat{K} = i\hbar p\partial_p + i\hbar/2$. The operator $\hat{q} = i\hbar\partial_p$ is not self-adjoint on $L^2(\mathbb{R}_+)$, its exponent $e^{\beta\hat{q}}$ acts by $e^{\beta\hat{q}}\,\Psi(p) = \Psi(p + i\hbar\beta)$ which leads to $A(\hat{p})\,e^{\alpha\hat{q}} = e^{\alpha\hat{q}}\,A(\hat{p} - i\hbar\alpha)$. We shall associate the function $e^{2\alpha q}A(p)$ with the symbol of the operator

$$\hat{A} = e^{\alpha\hat{q}}A(\hat{p})e^{\alpha\hat{q}}. \tag{4.16}$$

Such symbols are just characteristic for our chiral fields. The $*$-product of two symbols then becomes

$$\check{A} * \check{B} = e^{2(\alpha+\beta)q}\; A(p - i\hbar\beta)\; B(p + i\hbar\alpha). \tag{4.17}$$

Together with (4.11), this defines the $*$-product of symbols for the Liouville theory. From (4.17), it is easy to see that the $*$-brackets of p and p^2 with any $\check{A}$ are the Poisson brackets. The same is expected for the dilatation $K = pq$. Although K does not belong to the class $e^{2\beta q}B(p)$ its $*$-bracket can be inferred by differentiating the $*$-bracket $\{e^{2\beta q}p, \check{A}\}_*$ with respect to β and putting $\beta = 0$. Indeed, this leads to $\{\check{A},\, K\}_* = \{\check{A},\, K\}$. In this manner the $*$-brackets of other polynomials can be derived as well.

4.2 The Moyal formalism for the Liouville theory

For the Liouville theory the chiral symbols are functionals of the chiral field $\phi(z)$, $\check{A} = \check{A}[\phi(z)]$. In order to specify them we make use of the decomposition (3.13)

$$\phi(z) = q + \frac{pz}{4\pi} + \phi^+(z) + \phi^-(z), \qquad \text{where} \qquad \phi^\pm(z) = \pm\frac{i}{\sqrt{4\pi}}\sum_{n>0}\frac{a_{\pm n}}{n}e^{\mp inz}. \tag{4.18}$$

We shall treat here symbols containing the q zero mode as exponentials only

$$\check{A}[\phi(z)] = e^{2\alpha q}\, A_0[p,\, \phi^-(z), \phi^+(z)], \tag{4.19}$$

and associate it with the operator

$$\hat{A} = e^{\alpha\hat{q}}\; : A_0[p,\, \hat{\phi}^-(z), \hat{\phi}^+(z)] : e^{\alpha\hat{q}}. \tag{4.20}$$

Here $: \ :$ denotes normal ordering of the oscillator modes of A_0. Then the $*$-product of two symbols becomes

$$\check{A} * \check{B} = e^{2(\alpha+\beta)q} \int \prod_{n>0} d\mu(\xi_n)\, e^{-\frac{|\xi_n|^2}{n}} \times$$
$$A_0[p - i\hbar\beta,\, \phi^-(z),\, \phi^+(z) + \sqrt{\hbar}\, \xi^+(z)]\ B_0[p + i\hbar\alpha,\, \phi^-(z) + \sqrt{\hbar}\, \xi^-(z),\, \phi^+(z)], \quad (4.21)$$

where $d\mu(\xi_n) = (2\pi)^{-1}\, dp_n\, dq_n$ is the Liouville measure on a plane, ξ_n are the complex coordinates on it

$$\xi_n = \frac{p_n - inq_n}{\sqrt{2}}, \qquad \text{and} \qquad \xi^{\pm}(z) = \pm\frac{i}{\sqrt{4\pi}} \sum_{n>0} \frac{\xi_{\pm n}}{n}\, e^{\mp inz}. \quad (4.22)$$

Expanding the $*$-product (4.21) in powers of $\hbar$ we find

$$\check{A} * \check{B} = \check{A} \cdot \check{B} - \frac{i\hbar}{2}\left(\frac{\partial \check{A}}{\partial p}\frac{\partial \check{B}}{\partial q} - \frac{\partial \check{A}}{\partial q}\,\frac{\partial \check{B}}{\partial p} + 2i \sum_{n>0} n\, \frac{\partial \check{A}}{\partial a_n}\, \frac{\partial \check{B}}{\partial a_n^*}\right)$$
$$- \frac{\hbar^2}{8}\left(\frac{\partial^2 \check{A}}{\partial p^2}\, \frac{\partial^2 \check{B}}{\partial q^2} + \frac{\partial^2 \check{A}}{\partial q^2}\, \frac{\partial^2 \check{B}}{\partial p^2} - 2\frac{\partial^2 \check{A}}{\partial p \partial q}\, \frac{\partial^2 \check{B}}{\partial p \partial q} - 4 \sum_{n,m>0} mn\, \frac{\partial^2 \check{A}}{\partial a_m\, \partial a_n}\, \frac{\partial^2 \check{B}}{\partial a_m^*\, \partial a_n^*}\right)$$
$$- \frac{i\hbar^2}{2} \sum_{n>0} n \left(\frac{\partial^2 \check{A}}{\partial p\, \partial a_n}\, \frac{\partial^2 \check{B}}{\partial q \partial a_n^*} - \frac{\partial^2 \check{A}}{\partial q\, \partial a_n}\, \frac{\partial^2 \check{B}}{\partial p\, \partial a_n^*}\right) + \cdots \quad (4.23)$$

This expansion leads to the $*$-bracket

$$\{\check{A},\, \check{B}\}_* = \frac{i}{\hbar}(\check{A} * \check{B} - \check{B} * \check{A}) = \{\check{A},\, \check{B}\} + \hbar\, X_1(\check{A}, \check{B}) + \hbar^2 X_2(\check{A}, \check{B}) + \cdots, \quad (4.24)$$

which includes now the zero mode contributions

$$X_1(\check{A}, \check{B}) = \frac{i}{2} \sum_{n,m>0} mn \left(\frac{\partial^2 \check{A}}{\partial a_m\, \partial a_n}\, \frac{\partial^2 \check{B}}{\partial a_m^*\, \partial a_n^*} - \frac{\partial^2 \check{B}}{\partial a_m\, \partial a_n}\, \frac{\partial^2 \check{A}}{\partial a_m^*\, \partial a_n^*}\right) +$$
$$\frac{1}{2} \sum_{n>0} n \left(\frac{\partial^2 \check{A}}{\partial p\, \partial a_n}\, \frac{\partial^2 \check{B}}{\partial q\, \partial a_n^*} - \frac{\partial^2 \check{B}}{\partial p\, \partial a_n}\, \frac{\partial^2 \check{A}}{\partial q\, \partial a_n^*} + \frac{\partial^2 \check{A}}{\partial p\, \partial a_n^*}\, \frac{\partial^2 \check{B}}{\partial q\, \partial a_n} - \frac{\partial^2 \check{B}}{\partial p\, \partial a_n^*}\, \frac{\partial^2 \check{A}}{\partial q\, \partial a_n}\right) \quad (4.25)$$

$X_2(\check{A}, \check{B})$ contains third derivatives of $\check{A}$ and $\check{B}$, *etc.*

The symbols of the Hamiltonian (4.15) and the generators of the translations $\phi'(z)$ and dilatations $K = pq$ remain undeformed

$$\check{\phi}'(z) = \phi'(z), \qquad \check{K} = K, \qquad \check{H} = H, \quad (4.26)$$

and their $*$-brackets with any $\check{A}$ then coincide, as expected, with the Poisson brackets

$$\{\check{A},\,\phi'(z)\}_* = \{\check{A},\,\phi'(z)\}, \qquad \{\check{A},\,K\}_* = \{\check{A},\,K\}, \qquad \{\check{A},\,H\}_* = \{\check{A},\,H\}. \tag{4.27}$$

This gives the realisation of the translation and dilatation symmetries for the symbols. But we still have to include the generators of the conformal symmetry. We assume that the commutation relations corresponding to (3.24), (3.28) are deformed at most by a central term. Using (3.27) and the Jacobi identity, the $*$-bracket for (3.28) is determined up to a constant C_0

$$\{\check{T}(z),\,\phi'(y)\}_* = \phi''(y)\,\delta(z-y) - \phi'(y)\,\delta'(z-y) + C_0\delta''(z-y). \tag{4.28}$$

Because its left-hand side coincides with the Poisson bracket, a first order linear variational equation for $\check{T}(z)$ arises which leads by integration to

$$\check{T}(z) = \phi'^{\,2}(z) - C_0\,\phi''(z) + C_1(p,z), \tag{4.29}$$

where the integration 'constant' C_1 depends on p and z only. Since (3.24) can be deformed by a central term only, $C_1 = 0$, and the symbol of the energy-momentum tensor is

$$\check{T}(z) = \phi'^{\,2}(z) - C_0\,\phi''(z). \tag{4.30}$$

It might be suggestive not to deform the generators of the conformal symmetry [7, 8], as it is the case for the group $G(\rho(z),\epsilon)$, but this is not a necessary requirement for the energy-momentum tensor [6, 8] and we take $\eta = \gamma C_0$ as a deformation parameter for (3.25)

$$\check{T}(z) = \phi'^{\,2}(z) - \frac{\eta}{\gamma}\,\phi''(z). \tag{4.31}$$

Its relation to the undeformed expression is simply given by $\gamma \mapsto \gamma\eta$, but for the time being η is not determined.

Since $\check{T}(z)$ has quadratic and linear terms in $\phi(z)$ only its $*$-bracket with any $\check{A}$ does not have terms of order higher than $\hbar$

$$\{\check{T}(z),\,\check{A}\}_* = \{\check{T}(z),\,\check{A}\} + \hbar\,X_1(\check{T}(z),\check{A}), \tag{4.32}$$

where according to (4.25)

$$X_1(\check{T}(z),\check{A}) = \frac{i}{4\pi}\sum_{n,m>0} mn\left(e^{-i(n+m)z}\,\frac{\partial^2\check{A}}{\partial a_m^*\,\partial a_n^*} - e^{i(n+m)z}\,\frac{\partial^2\check{A}}{\partial a_m\,\partial a_n}\right) + \frac{1}{(4\pi)^{3/2}}\sum_{n>0} n\left(e^{-inz}\,\frac{\partial^2\check{A}}{\partial q\,\partial a_n^*} + e^{inz}\,\frac{\partial^2\check{A}}{\partial q\,\partial a_n}\right). \tag{4.33}$$

Taking now $\check{A} = \check{T}(y)$ we obtain

$$X_1(\check{T}(z), \check{T}(y)) = \frac{i\hbar}{8\pi^2} \sum_{k\neq 0,\pm 1} e^{-ik(z-y)} \sum_{m=1}^{k-1} m\,(k-m)\,, \tag{4.34}$$

and using

$$\sum_{m=1}^{k-1} m\,(k-m) = \frac{k^3-k}{6}\,, \tag{4.35}$$

the well-known Virasoro algebra results

$$\begin{aligned} \{\check{T}(z), \check{T}(y)\}_* &= \check{T}'(y)\,\delta(z-y) - 2\check{T}(y)\delta'(z-y) \\ &\quad + \left(\frac{\eta^2}{2\gamma^2} + \frac{\hbar}{24\pi}\right)\delta'''(z-y) + \frac{\hbar}{24\pi}\delta'(z-y). \end{aligned} \tag{4.36}$$

This completes the discussion of the elements we need for the application of the Moyal formalism to the Liouville theory.

4.3 The construction of symbols for chiral fields

In general a symbol differs from its classical counterpart. The symbols of fields will be constructed by their transformation properties under the symmetries of the theory. This principle also operates for canonical quantisation .

Let us first consider the chiral field $\check{\psi}(z)$. The $*$-bracket relation which corresponds to (3.26) is a first order linear homogeneous variational equation for $\check{\psi}(z)$, which leads to $\check{\psi}(z) = C(p,z)\psi(z)$, where $C(p,z)$ is the integration 'constant'. Commutation with the Hamiltonian $\{H, \check{\psi}(z)\}_* = \check{\psi}'(z)$ yields $\partial_z C(p,z) = 0$. Thus,

$$\check{\psi}(z) = C(p)\,\psi(z) = C(p)\,e^{-\gamma\phi(z)}. \tag{4.37}$$

The conformal weight $\Delta(\check{\psi})$ of $\check{\psi}(z)$ is defined by its commutation with $\check{T}(z)$ via (4.32), (4.33)

$$\{\check{T}(z), \check{\psi}(y)\}_* = \check{\psi}'(y)\,\delta(z-y) + \frac{1}{2}\left(\eta + \frac{\hbar\gamma^2}{4\pi}\right)\check{\psi}(y)\delta'(z-y), \tag{4.38}$$

and we read off

$$\Delta(\check{\psi}) = -\frac{1}{2}\,(\eta + \alpha)\,, \tag{4.39}$$

where

$$\alpha = \frac{\hbar\gamma^2}{4\pi}. \tag{4.40}$$

An arbitrary exponential $e^{\mu\gamma\phi}$ is also primary with

$$\Delta(e^{\mu\gamma\phi}) = \frac{1}{2}(\eta\mu - \alpha\mu^2)\,. \tag{4.41}$$

The construction of the symbol $\check{\chi}(z)$ requires more labour. Since the field $\chi(z)$ is given by integration of the bilocal field (3.30) which linearly transforms under the translations (3.31), it is convenient first to construct $\check{B}(y,x)$. Using the commutation relation

$$\{\phi'(z), \check{B}(y,x)\}_* = -\frac{\gamma}{2}\delta(z-y)\,\check{B}(y,x) + \gamma\delta(z-x)\,\check{B}(y,x), \tag{4.42}$$

we find by the same technique as given before that $\check{B}(y,x) = f(y,x;p)B(y,x)$ with an arbitrary function $f(y,x;p)$. The commutation of $\check{B}$ with the Hamiltonian $\{H, \check{B}(y,x)\}_* = \partial_y\check{B}(y,x) + \partial_x\check{B}(y,x)$ leads to $f(y,x;p) = f(y-x;p)$. To find the function $f(y-x,p)$ we commute $\check{T}(z)$ with

$$\check{\chi}(y) = e^{-\gamma\phi(y)} \int_0^{2\pi} dy\, f(y-x,p)\,\theta_{\gamma p}\,(y-x)\,e^{2\gamma\phi(x)}\,. \tag{4.43}$$

For the detailed calculations we refer to Appendix C. Eqs. (4.32) and (4.33) provide for $\check{A} = \check{\chi}(y)$ the result (C.4) with anomalies. We require cancellation of these anomalies in order to have a primary $\check{\chi}(y)$

$$\{\check{T}(z), \check{\chi}(y)\}_* = \check{\chi}'(y)\,\delta(z-y) + \frac{1}{2}\,(\eta+\alpha)\,\check{\chi}(y)\delta'(z-y). \tag{4.44}$$

This procedure determines uniquely the deformation parameter of the energy-momentum tensor (4.31)

$$\eta = 1 + 2\alpha. \tag{4.45}$$

For undeformed $\check{T}(z)$, given by $\gamma \to \gamma\eta$, the η is defined by the known quadratic equation. Furthermore, the function $f(y-x,p)$ has to fulfil the first order differential equation

$$\partial_y f(y,p) = \alpha\cot(y/2)\,f(y,p), \tag{4.46}$$

which has the solution

$$f_\alpha(y) = \left(4\sin^2\frac{y}{2}\right)^\alpha. \tag{4.47}$$

The field $\check{\chi}(y)$ then becomes

$$\check{\chi}(y) = S(p)\,e^{-\gamma\phi(y)} \int_0^{2\pi} dy\; e^{\frac{\gamma p}{2}\epsilon(y-x)}\,f_\alpha(y-x)\,e^{2\gamma\phi(x)}\;, \tag{4.48}$$

with the still arbitrary function $S(p)$. The conformal weights of $\check{\chi}(z)$ and $\check{\psi}(z)$ have quantum corrections, but their weights remain equal

$$\Delta(\check{\psi}) = \Delta(\check{\chi}) = -\frac{1}{2}\,(1+3\alpha)\,. \tag{4.49}$$

This provides primariness of the Liouville field too.

The symbol of the integral $A(z)$ (3.20) can be constructed similarly and we obtain

$$\check{A}(z) = A(z), \qquad\qquad \Delta(\check{A}) = 0, \tag{4.50}$$

which is consistent with $\Delta(e^{2\gamma\phi}) = 1$.

In Appendix D we will argue that the functions $C(p)$ of (4.37) and $S(p)$ of (4.48) can be calculated from dilatation properties of normalised vacuum Berezin symbols.

Then the results found so far by the Moyal quantisation are in agreement with those of the canonical quantisation of the Liouville theory [6, 8].

4.4 The $*$-product exchange algebra of chiral fields

In this section we calculate the $*$-products of the chiral fields $\check{\psi}(z)$ and $\check{\chi}(z)$ to get the quantum exchange algebra. The symbols of $\psi(z)$ and $\chi(z)$ have the useful property that the non-zero modes appear only linearly in the exponent. The $*$-products of these symbols can therefore be calculated directly by Gaussian functional integration of (4.21). These integrations give the results

$$\check{\psi}(z) * \check{\psi}(y) = \psi(z)\,\psi(y)\,C_+(p)C_-(p)\,e^{-i\pi\alpha\epsilon^+(z-y)}\,, \tag{4.51}$$

$$\check{\chi}(z) * \check{\chi}(y) = \psi(z)\,\psi(y)\,I_0(z,y)\,e^{-i\pi\alpha\epsilon^+(z-y)}\,, \tag{4.52}$$

$$\check{\psi}(z) * \check{\chi}(y) = \psi(z)\,\psi(y)\,I_1(z,y)\,e^{i\pi\alpha\epsilon^-(z-y)}\,, \tag{4.53}$$

$$\check{\chi}(z) * \check{\psi}(y) = \psi(z)\,\psi(y)\,I_2(z,y)\,e^{i\pi\alpha\epsilon^-(z-y)}\,, \tag{4.54}$$

where $C_\pm(p) = C(p \pm \hbar\gamma/2)$, and I_0, I_1, and I_2 are the integrals

$$\begin{aligned} I_0(z,y) = S_+(p)\,S_-(p) \int_0^{2\pi}\!\!\int_0^{2\pi} dx\,dv\, f_\alpha^{-2}(x-v)\, f_\alpha(z-x)\, f_\alpha(y-v) \\ \times\, f_\alpha(z-v)\, f_\alpha(y-x)\; e^{\frac{\gamma p}{2}\epsilon(z-x)} e^{\frac{\gamma p}{2}\epsilon(y-v)}\; e^{2\gamma\phi(x)+2\gamma\phi(v)}, \end{aligned} \tag{4.55}$$

$$I_1(z,y) = R_-(p) \int_0^{2\pi} dx\;\, f_\alpha(z-x)\, f_\alpha(y-x)\, e^{\frac{\gamma p}{2}\epsilon(y-x)}\; e^{2\gamma\phi(x)}\; e^{-i\pi\alpha\kappa(z,y,x)}\,, \tag{4.56}$$

$$I_2(z,y) = R_+(p) \int_0^{2\pi} dx\;\, f_\alpha(z-x)\, f_\alpha(y-x)\; e^{\frac{\gamma p}{2}\epsilon(z-x)}\; e^{2\gamma\phi(x)}\; e^{-i\pi\alpha\kappa(z,y,x)}\,, \tag{4.57}$$

with

$$\begin{aligned} &S_\pm(p) = S(p \pm \hbar\gamma/2), \qquad R_\pm(p) = C_\pm(p) S_\pm(p), \\ &\kappa(z,y,x) = \epsilon(z-y) + \epsilon(y-x) + \epsilon(x-z), \end{aligned} \tag{4.58}$$

and

$$\epsilon^\pm(z) = \frac{z}{2\pi} \pm \frac{i}{\pi} \sum_{n>0} \frac{e^{\pm inz}}{n}. \tag{4.59}$$

The $\epsilon^\pm(z)$ functions are the positive and negative frequency parts of the stair-step function $\epsilon(z) = \epsilon^+(z) + \epsilon^-(z)$. They are related by $\epsilon^+(-z) = -\epsilon^-(z)$ and contain the short distance singularities

$$\epsilon^\pm(z) = \frac{1}{2}\epsilon(z) \mp \frac{i}{2\pi} \log\left(4\sin^2\frac{z}{2}\right). \tag{4.60}$$

We should mention here that the integration over the singularity f_α^{-2} of (4.55) is defined only if $\alpha < 1/4$, which restricts the coupling constant $\gamma^2 < \pi/\hbar$.

The bilocal objects $I_0(z,y)$, $I_1(z,y)$ and $I_2(z,y)$ have the following symmetry properties under the exchange of the z, y coordinates

$$I_0(y,z) = I_0(z,y), \tag{4.61}$$

$$I_1(y,z) = a(p) I_2(z,y) + b(p) I_1(z,y) e^{\frac{\gamma p}{2}\epsilon(z-y)}, \tag{4.62}$$

$$I_2(y,z) = c(p) I_2(z,y) + d(p) I_1(z,y) e^{-\frac{\gamma p}{2}\epsilon(z-y)}. \tag{4.63}$$

Equation (4.61) is obvious. To get (4.62) we take into account that the function $\kappa(z,y,x)$ has in the integration region $x \in [0, 2\pi]$ the two values $\kappa = \pm 1$ only. Then the equality of the integrands on the left and right hand sides define the coefficients $a(p)$ and $b(p)$

$$a(p) = \frac{R_-(p)\,\sinh\left(\frac{1}{2}\gamma p - 2i\pi\alpha\right)}{R_+(p)\,\sinh\frac{1}{2}\gamma p}, \qquad b(p) = \frac{i\sin 2\pi\alpha}{\sinh\frac{1}{2}\gamma p}, \tag{4.64}$$

The derivation of (4.63) is similar and it yields

$$c(p) = \frac{R_+(p)\,\sinh\left(\frac{1}{2}\gamma p + 2i\pi\alpha\right)}{R_-(p)\,\sinh\frac{1}{2}\gamma p}, \qquad d(p) = -\frac{i\sin 2\pi\alpha}{\sinh\frac{1}{2}\gamma p}. \tag{4.65}$$

The function $R(p) = C(P)\,S(p)$ is given by (D.12)

$$R(p) = R_0 \left(\sinh^2\frac{\gamma p}{2} + \sin^2\pi\alpha\right)^{-\frac{1}{2}}, \tag{4.66}$$

which determines $a(p)$ and $c(p)$ uniquely

$$a(p) = c(p) = \left(1 + \frac{\sin^2(2\pi\alpha)}{\sinh^2 \frac{\gamma p}{2}}.\right)^{\frac{1}{2}} \tag{4.67}$$

The quantum exchange algebra [7, 8] follows then immediately from (4.51)-(4.52) and (4.62)-(4.61)

$$e^{i\pi\alpha\epsilon(z-y)}\,\check\psi(z) * \check\psi(y) = \check\psi(y) * \check\psi(z), \tag{4.68}$$

$$e^{i\pi\alpha\epsilon(z-y)}\,\check\chi(z) * \check\chi(y) = \check\chi(y) * \check\chi(z), \tag{4.69}$$

$$e^{-i\pi\alpha\epsilon(z-y)}\,\check\psi(z) * \check\chi(y) = \check\chi(y) * \check\psi(z)\,a(p) - 2i\sin(2\pi\alpha)\,\check\psi(y) * \check\chi(z)\,\theta_{-\gamma p}(z-y), \tag{4.70}$$

$$e^{-i\pi\alpha\epsilon(z-y)}\,\check\chi(z) * \check\psi(y) = \check\psi(y) * \check\chi(z)\,c(p) - 2i\sin(2\pi\alpha)\,\check\chi(y) * \check\psi(z)\,\theta_{\gamma p}(z-y). \tag{4.71}$$

Because $\check\psi(y) * \check\chi(z)$ does not depend on q its product with $a(p)$, $c(p)$ and $\theta_{\pm\gamma p}$ is here an ordinary one. Expanding (4.68)-(4.71) in powers of $\hbar$ the zero and first order terms in $\hbar$ reproduce the classical exchange algebra (3.2)-(3.4).

4.5 The non-equal time $*$-brackets of Liouville fields

The symbol calculus for the anti-chiral part is similar. Combining the chiral and anti-chiral fields we construct the symbol of the Liouville exponential (3.17)

$$\check u(z,\bar z) = \check E(z,\bar z) + \check K(z,\bar z), \tag{4.72}$$

where

$$\check E(z,\bar z) = \check\psi(z)\check{\bar\psi}(\bar z), \qquad\qquad \check K(z,\bar z) = \check\chi(z)\check{\bar\chi}(\bar z). \tag{4.73}$$

For the calculation of the $*$-product we treat the chiral and anti-chiral fields independently by using again the Fubini-Veneziano trick and put $p_0 = \bar p_0 = p$ and $q_0 + \bar q_0 = q$ afterwards

$$A(p,p)e^{\alpha q} * B(p,p)e^{\beta q} = A(p_0,\bar p_0)e^{\alpha(q_0+\bar q_o)} * B(p_0,\bar p_0)e^{\beta(q_0+\bar q_o)}|_{p_0=\bar p_0=p, q_0+\bar q_0=q}. \tag{4.74}$$

This relation obviously follows from (4.17). The eqs. (4.51)-(4.52) provide then

$$\check E(z,\bar z) * \check E(y,\bar y) = E(z,\bar z)E(y,\bar y)\,C_+^2(p)C_-^2(p)e^{-i\pi\alpha[\epsilon^+(z-y)+\epsilon^+(\bar z-\bar y)]}\,, \tag{4.75}$$

$$K(z,\bar z) * K(y,\bar y) = E(z,\bar z)E(y,\bar y)I_0(z,y)\bar I_0(\bar z,\bar y)\,e^{-i\pi\alpha[\epsilon^+(z-y)+\epsilon^+(\bar z-\bar y)]}\,, \tag{4.76}$$

$$E(z,\bar z) * K(y,\bar y) = E(z,\bar z)E(y,\bar y)I_1(z,y)\bar I_1(\bar z,\bar y)\,e^{i\pi\alpha[\epsilon^-(z-y)+\epsilon^-(\bar z-\bar y)]}\,, \tag{4.77}$$

$$K(z,\bar z) * E(y,\bar y) = E(z,\bar z)E(y,\bar y)I_2(z,y)\bar I_2(\bar z,\bar y)\,e^{i\pi\alpha[\epsilon^-(z-y)+\epsilon^-(\bar z-\bar y)]}\,. \tag{4.78}$$

Our aim is to find the quantum realisation of the non-equal time Poisson brackets discussed in subsection 3.4. We consider here for simplicity the case (3.39) only. One expects the quantum relation

$$\check{u}(z,\bar{z}) * \check{u}(y,\bar{y}) = C[\check{u}(z,\bar{y}) * \check{u}(y,\bar{z}) + \check{u}(y,\bar{z}) * \check{u}(z,\bar{y})] + D\check{u}(y,\bar{y}) * \check{u}(z,\bar{z}), \tag{4.79}$$

where C and D are functions of space-time coordinates only. In order to determine C and D we use the fact that the product $E(z,\bar{z})E(y,\bar{y})$ is invariant under the exchanges of z and y or $\bar{z}$ and $\bar{y}$. The comparison of the coefficients of $E * E$ on the left and right hand sides of (4.79) gives the condition

$$\left(e^{i\pi\alpha\epsilon} + e^{i\pi\alpha\epsilon}\right) C + e^{i\pi\alpha\Theta}\, D = 1. \tag{4.80}$$

Here we use the shorthand notation $\epsilon = \epsilon(z-y)$, $\bar{\epsilon} = \epsilon(\bar{z}-\bar{y})$, and Θ is defined by (3.36). Due to the symmetry (4.61) the same condition also holds for the coefficients of $K * K$. For the term $E*K+K*E$ we use the exchange relations (4.62) and (4.63) and rewrite the left and right hand sides of (4.79) as a linear combination of the four independent terms $I_1(z,y)\bar{I}_1(\bar{z},\bar{y})$, $I_2(z,y)\bar{I}_2(\bar{z},\bar{y})$, $I_1(z,y)\bar{I}_2(\bar{z},\bar{y})$ and $I_2(z,y)\bar{I}_1(\bar{z},\bar{y})$, which are multiplied by p-dependent coefficients. In this manner we find four equations but due to (4.67) only two of them are independent

$$\left(e^{-i\pi\alpha\epsilon} + e^{-i\pi\alpha\bar{\epsilon}}\right) C + 2i\sin(2\pi\alpha)\,\Theta\, e^{-2i\pi\alpha\Theta} D = 0, \tag{4.81}$$

$$i\sin(2\pi\alpha)\left[\epsilon e^{-i\pi\alpha\epsilon} + \bar{\epsilon}\, e^{-i\pi\alpha\bar{\epsilon}}\right] C + [1 - \sin^2(2\pi\alpha)(1+\epsilon\bar{\epsilon})] e^{-2i\pi\alpha\Theta}\, D = 1. \tag{4.82}$$

Equations (4.80) and (4.81) determine

$$\begin{aligned} C &= \frac{1}{e^{i\pi\alpha\epsilon} + e^{i\pi\alpha\bar{\epsilon}}}\, \frac{2i\sin(2\pi\alpha)\Theta}{2i\sin(2\pi\alpha)\Theta - e^{2i\pi\alpha\Theta}}\ , \\ D &= -\frac{1}{2i\sin(2\pi\alpha)\Theta - e^{2i\pi\alpha\Theta}}\ , \end{aligned} \tag{4.83}$$

which satisfy (4.82) as well. The quantum non-equal time relation becomes

$$\begin{aligned} \left[e^{2i\pi\alpha\Theta} - \left(e^{2i\pi\alpha} - e^{-2i\pi\alpha}\right)\Theta\right] \check{u}(z,\bar{z}) * \check{u}(y,\bar{y}) &= \check{u}(y,\bar{y}) * \check{u}(z,\bar{z}) \\ &- \Theta\, \frac{e^{2i\pi\alpha} - e^{-2i\pi\alpha}}{e^{i\pi\alpha\epsilon} + e^{i\pi\alpha\bar{\epsilon}}} [\check{u}(z,\bar{y}) * \check{u}(y,\bar{z}) + \check{u}(y,\bar{z}) * \check{u}(z,\bar{y})]. \end{aligned} \tag{4.84}$$

It provides for $\Theta = 0$ obviously the causality condition $\{\check{u}(z,\bar{z}), \check{u}(y,\bar{y})\}_* = 0$. For $\Theta = \pm 1$ (4.84) relates quadratic combinations of the field u at four different space-time points, which can also be written as a $*$-bracket, and combining it with the case $\Theta = 0$ we get the final result

$$\{\check{u}(z,\bar{z}),\, \check{u}(y,\bar{y})\}_* = \frac{1}{\hbar}\sin(\hbar\gamma^2/4)\,[\epsilon(z-y)+\epsilon(\bar{z}-\bar{y})]\times$$
$$\left[\check{u}(z,\bar{y})*\check{u}(y,\bar{z})+\check{u}(y,\bar{z})*\check{u}(z,\bar{y})-\frac{\check{u}(z,\bar{z})*\check{u}(y,\bar{y})+\check{u}(y,\bar{y})*\check{u}(z,\bar{z})}{2\cos(\hbar\gamma^2/4)}\right]. \tag{4.85}$$

Much in the same manner as for the derivation of (3.43) we can easily construct from it a symbol of the local field $e^{-2\gamma\phi(\tau,\sigma)}$.
Its expansion in powers of $\hbar$ reproduces the classical Poisson bracket (3.39).

5 Summary

In this paper the Liouville theory was revisited, classically and quantum mechanically. Its Poisson and symmetry structures are defined from the $SL(2,\mathbb{R})$ WZNW theory by gauge invariant Hamiltonian reduction, and for quantisation a Moyal formalism is applied.
The classical form of the exchange algebra arises as the basic Poisson algebra, from which causal non-equal time Poisson brackets for a Liouville exponential are derived. We observed a transitive symmetry group acting on the phase space, and we have shown that Hamiltonian reduction is a suitable method for the integration of gauged WZNW theories. Following the ideas of geometric quantisation, coherent states have been defined via this transitive symmetry group and a symbol calculus developed. The symbols of fields are constructed through the symmetries of the Liouville theory. We reproduce results of the canonical quantisation, which includes the deformed exchange algebra. In addition, the deformed causal non-equal time commutators of a Liouville exponential is calculated. From this rich structure other symbols and $*$-products can be derived.
We presume that vacuum Berezin symbols provide a natural definition for Liouville correlation functions.

Acknowledgements

We would like to thank Chris Ford for many helpful discussions. We also thank Martin Reuter for interesting discussions about the Moyal formalism. G.J. is grateful to DESY

Zeuthen for hospitality. His research was supported by grants from the DFG, GSRT INTAS and RFBR.

Appendix A

For Poisson bracket calculations the technique of symplectic geometry is applied [18]. If a symplectic form is given by $\omega = \omega_{mn}(\xi)\,\delta\xi^m \wedge \delta\xi^n$, then for an observable $F(\xi)$ one has

$$-\,\delta F = 2\omega_{mn}\,(\xi)\,\{\,F, \xi^m\,\}\,\delta\xi^n, \tag{A.1}$$

which for non-degenerate $\omega_{mn}\,(\xi)$ defines the Poisson brackets $\{\,F, \xi^m\,\}$.
We make use of (A.1) for the symplectic form (2.1) (for a more general treatment see [19]). First we parametrise the chiral field $g(z)$ by

$$g(z) = f(z)\,M(z), \qquad M(z) = \exp\left(\frac{\lambda z}{2\pi}\,t_2\right), \tag{A.2}$$

where $f(z)$ is a $SL(2,\mathbb{R})$ valued periodic field. This parametrisation assumes the monodromy matrix $M = \exp\lambda t_2$. Then, from (2.1) we get

$$\begin{aligned}\omega = -\frac{1}{\gamma^2}\int_0^{2\pi} \langle (f^{-1}(\sigma)\,\delta f(\sigma))' \wedge f^{-1}(\sigma)\,\delta f(\sigma)\rangle\, d\sigma \\ -\frac{\lambda}{2\pi\gamma^2}\int_0^{2\pi} d\sigma\,\langle [f^{-1}(\sigma)\delta f(\sigma), t]\,\wedge\, f^{-1}(\sigma)\delta f(\sigma)\rangle \\ -\frac{1}{\pi\gamma^2}\,\delta\lambda\wedge\langle t\int_0^{2\pi} d\sigma\; f^{-1}(\sigma)\delta f(\sigma)\rangle.\end{aligned} \tag{A.3}$$

For $F = \lambda$ we read off from (A.3) the equations

$$A'(\sigma) + \frac{\lambda}{2\pi}[A(\sigma), t_2] = 0 \quad \text{and} \quad \frac{1}{\pi\gamma^2}\int_0^{2\pi} d\sigma\,\langle t_2\; A(\sigma)\rangle = -1,$$

where $A(\sigma) = f^{-1}(\sigma)\,\{\lambda, f(\sigma)\}$. Due to the periodicity of $f(\sigma)$ we get the unique solution

$$\{\lambda, f(\sigma)\} = \frac{\gamma^2}{2}\,f(\sigma)\,t_2. \tag{A.4}$$

In order to write down the Poisson brackets of group elements it is convenient to introduce the notation [4]

$$\{\,f_{\alpha_1\beta_1}(\sigma_1),\; f_{\alpha\beta}(\sigma)\,\} = \{\,f(\sigma_1)\otimes f(\sigma)\,\}_{\alpha_1\alpha,\,\beta_1\beta}\,. \tag{A.5}$$

For $F = f_{\alpha_1\beta_1}(\sigma_1)$, the Poisson brackets of $f(\sigma)$'s can then be written as

$$\{ f(\sigma_1), \otimes f(\sigma) \} = A^{(n)}(\sigma_1, \sigma) \otimes (f(\sigma)\, t_n), \tag{A.6}$$

where the t_n are given by (2.2), and the $A^{(n)}$ satisfy the equations

$$\begin{aligned} \partial_\sigma A^{(n)}(\sigma_1, \sigma) + \frac{\lambda}{\pi}\, \epsilon^n{}_{m2}\, A^{(m)}(\sigma_1, \sigma) - \frac{\gamma^2}{4\pi}\, \delta^n_2\, (f(\sigma_1)\, t_2) = \\ \frac{\gamma^2}{2}\, (f(\sigma_1)\, t^n)\, \delta(\sigma_1 - \sigma), \end{aligned} \tag{A.7}$$

and

$$\int_0^{2\pi} d\sigma\, A^{(2)}(\sigma_1, \sigma) = 0. \tag{A.8}$$

Eq. (A.7) splits into independent equations

$$\partial_\sigma A^{(2)}(\sigma_1, \sigma) = -\frac{\gamma^2}{2}\, (f(\sigma_1)\, t_2) \left(\delta(\sigma_1 - \sigma) - \frac{1}{2\pi} \right), \tag{A.9}$$

$$\partial_\sigma A^{(\pm)}(\sigma_1, \sigma) \pm \frac{\lambda}{\pi} A^{(\pm)}(\sigma_1, \sigma) = -\frac{\gamma^2}{2}\, (f(\sigma_1)\, t_\mp)\, \delta(\sigma_1 - \sigma), \tag{A.10}$$

where $A^{(\pm)} = A^1 \pm A^0$. Eq. (A.8)-(A.10) have an unique solution for $\lambda \neq 0$, which inverts the symplectic form. Indeed, from (A.9) and (A.8) we find

$$A^{(2)}(\sigma_1, \sigma) = \frac{\gamma^2}{4}\, (f(\sigma_1)\, t_2)\; h(\sigma_1 - \sigma), \tag{A.11}$$

and integration of (A.10) gives

$$A^{(\pm)}(\sigma_1, \sigma) = \frac{\gamma^2}{2}\, (f(\sigma_1)\, t_\mp)\; h_{\mp\lambda}(\sigma_1 - \sigma), \tag{A.12}$$

where

$$h_\lambda(z) = \frac{e^{\lambda h(z)}}{2\, \sinh \lambda}. \tag{A.13}$$

Note that for $\lambda \neq 0$ the operator $\partial_\sigma + \lambda/\pi$ has an inverse on the class of periodic functions, and $h_\lambda(\sigma - \sigma_1)$ is its kernel.

Now using (A.6) and (A.11), (A.12) we obtain

$$\begin{aligned} \{ f(\sigma_1) \otimes f(\sigma) \} = \frac{\gamma^2}{4} [\, (f(\sigma_1)\, t_2) \otimes (f(\sigma)\, t_2)\; h(\sigma_1 - \sigma) \\ + (f(\sigma_1)\, t_-) \otimes (f(\sigma)\, t_+)\; h_{-\lambda}(\sigma_1 - \sigma) \\ + (f(\sigma_1)\, t_+) \otimes (f(\sigma)\, t_-)\; h_{+\lambda}(\sigma_1 - \sigma)]. \end{aligned} \tag{A.14}$$

With this result we are able to calculate the Poisson brackets of the chiral WZNW fields

$$\begin{aligned}\{\, g(z_1) \otimes g(z) \,\} &= \{\, f(z_1) \otimes f(z) \,\} \cdot (\, M(z_1) \otimes M(z) \,) \\ &+ (\, I \otimes f(z) \,) \cdot \{\, f(z_1) \otimes M(z) \,\} \cdot (\, M(z_1) \otimes I \,) \\ &+ (\, f(z_1) \otimes I \,) \cdot \{\, M(z_1) \otimes f(z) \,\} \cdot (\, I \otimes M(z) \,)\end{aligned} \tag{A.15}$$

explicitly. From (A.4) follows

$$\begin{aligned}\{\, f(z_1) \otimes M(z) \,\} &= -\frac{\gamma^2 z}{4\pi} \,(\, f(z_1)\, t_2 \,) \otimes (\, M(z)\, t_2 \,), \\ \{\, M(z_1) \otimes f(z) \,\} &= \frac{\gamma^2 z_1}{4\pi} \,(\, M(z_1)\, t_2 \,) \otimes (\, f(z)\, t_2 \,),\end{aligned} \tag{A.16}$$

and it is easy to check that

$$M^{-1}(z)\, t_\pm \, M(z) \,=\, t_\pm \ \exp\left(\pm\frac{\lambda z}{\pi}\right). \tag{A.17}$$

Putting (A.14)-(A.17) into (A.15) we get our final result (2.5).

Another remark is in order: because of (A.4) the Kac-Moody currents (2.8) have zero Poisson brackets with λ. This simplifies the calculation of $\{J_n(z_1),\, g(z)\}$ from (2.1) (with $M = \exp(\lambda t_2)$). Eq. (A.1) gives for $F = J_n(z_1)$

$$-\,\delta J_n(z_1) = \frac{2}{\gamma^2} \int_\tau^{\tau+2\pi} \langle (g^{-1}(z)\,\delta g(z))'\, g^{-1}(z)\,\{\, J_n(z_1), g(z)\,\}\rangle\, dz, \tag{A.18}$$

since the term proportional to $\delta\lambda$ is cancelled by partial integration. The variation of (2.8)

$$\delta J_n(z) = \frac{1}{\gamma^2}\,\langle\, t_n\, g(z)\, ((g^{-1}(z)\,\delta g(z))'\, g^{-1}(z)\,\rangle,$$

and (A.18) then provide (2.9).

Appendix B

The coherent states on the half-plane $p > 0$ are transformed into each other by an irreducible representation of the group of translations and dilatations

$$q \mapsto q + b; \qquad p \mapsto e^{-\varepsilon p}, \qquad q \mapsto e^{\varepsilon} q. \tag{B.1}$$

In the complex coordinates

$$\zeta = \frac{1}{p} + i\nu q, \qquad \zeta^* = \frac{1}{p} - i\nu q, \tag{B.2}$$

these transformations become a holomorphic map $\zeta \mapsto e^{-\varepsilon}\zeta - i\nu b$ of the half-plane onto itself. (B.2) is the analogue of (4.3) for the half-plane and ν is a squeezing parameter. The coherent state $\Psi_{\zeta_1}(p)$ is defined as an eigenstate of the operator $\hat{\zeta} = 1/p - \hbar\nu/2p - \hbar\nu\partial_p$

$$\left(\frac{1-\hbar\nu/2}{p} - \hbar\nu\partial_p\right)\Psi_{\zeta_1}(p) = \zeta_1\Psi_{\zeta_1}(p), \tag{B.3}$$

with the eigenvalue $\zeta_1 = 1/p_1 + i\nu q_1$. We use here the standard measure dp for the scalar product of wave functions

$$\langle\Psi_{\zeta_1}|\Psi_{\zeta_2}\rangle = \int dp\, \Psi^*_{\zeta_1}(p)\Psi_{\zeta_2}(p), \tag{B.4}$$

and the term $\hbar\nu/2p$ of the operator $\hat{\zeta}$ is necessary in order to get the correct normalisation and completeness of the coherent states. This term can also be justified with in geometric quantisation [18] but it will be cancelled if we pass to the dilatation invariant measure dp/p.

The solution of (B.3) with $\beta = 1/\hbar\nu$ is

$$\Psi_{\zeta_1}(p) = C_\beta(p_1, q_1)\,\frac{1}{\sqrt{p}}\,p^\beta\,e^{-\beta\zeta_1 p} \tag{B.5}$$

and the integration constant $C_\beta(p_1, q_1)$ can be defined by the conditions

$$\hat{p}\Psi_{\zeta_1}(p) = i\hbar\partial_{q_1}\Psi_{\zeta_1}(p), \qquad \text{and} \qquad \hat{K}\Psi_{\zeta_1}(p) = i\hbar(q_1\partial_{q_1} - p_1\partial_{p_1})\Psi_{\zeta_1}(p), \tag{B.6}$$

which implies that $\hat{p} = p$ and $\hat{K} = i\hbar(p\partial_p + 1/2)$ are the translation and dilatation operators respectively. From (B.6) follows $C_\beta(p_1, q_1) = c_\beta\, p_1^{-\beta}$, and the normalisation condition $\langle\Psi_{\zeta_1}|\Psi_{\zeta_1}\rangle = 1$ yields $c_\beta^2\,\Gamma(2\beta) = (2\beta)^{2\beta}$. Thus we get the wave functions

$$\Psi_{\zeta_1}(p) = (2\beta)^\beta\Gamma^{-1/2}(2\beta)\frac{1}{\sqrt{p}}\left(\frac{p}{p_1}\right)^\beta e^{-\beta\zeta_1 p}. \tag{B.7}$$

which have the scalar product

$$\langle\Psi_{\zeta_1}|\Psi_{\zeta_2}\rangle = (p_1p_2)^{-\beta}\left(\frac{\zeta_1 + \zeta_2^*}{2}\right)^{-2\beta}, \tag{B.8}$$

and satisfy the completeness relation

$$\left(1-\frac{\hbar\nu}{2}\right)\int\frac{dp\,dq}{2\pi\hbar}\Psi^*_\zeta(p_1)\Psi_\zeta(p_2)=\delta(p_1-p_2). \tag{B.9}$$

This structure defines the the Berezin symbol calculus for

$$\check{A}(p,q)=\langle\Psi_\zeta|\hat{A}|\Psi_\zeta\rangle. \tag{B.10}$$

At $\nu=0$ the coherent states become non-normalisable, like the standard coherent states on the plane. In particular for $\nu\to 0$ we have

$$|\Psi_{\zeta_1}(p)|^2\to\delta(p-p_1). \tag{B.11}$$

Appendix C

In this appendix we describe the cancellation of anomalies needed to get a primary $\check{\chi}(y)$. First we calculate the $*$-bracket of $\check{T}(z)$ with the bilocal field

$$\check{B}(y,x)=F(y-x,p)\,e^{-\gamma\phi(y)}\,e^{2\gamma\phi(x)} \tag{C.1}$$

taking into account (4.38), and (4.41) for $\mu=2$. Then eqs. (4.32), (4.33) provide

$$\begin{aligned}\{\check{T}(z),\,\check{B}(y,x)\}_* &= \left(e^{-\gamma\phi(y)}\right)'\,F(y-x,p)e^{2\gamma\phi(x)}\,\delta(z-y)\\ &\quad+\frac{1}{2}\,(\eta+\alpha)\,e^{-\gamma\phi(y)}\,F(y-x,p)\,e^{2\gamma\phi(x)}\,\delta'(z-y)+\\ e^{-\gamma\phi(y)}\,F(y-x,p)\,&\left(e^{2\gamma\phi(x)}\right)'\,\delta(z-x)\;-(\eta-2\alpha)\,e^{-\gamma\phi(y)}\,F(y-x,p)\,e^{2\gamma\phi(x)}\,\delta'(z-x)\\ &\quad+\alpha\,e^{-\gamma\phi(y)}\,F(y-x,p)\,e^{2\gamma\phi(x)}\cot\frac{1}{2}(y-z)\,[\delta(z-y)-\delta(z-x)],\end{aligned} \tag{C.2}$$

where we have used

$$\begin{aligned}\frac{i}{2\pi}\sum_{k\geq 2}e^{-ik(z-y)}\sum_{m=1}^{k-1}e^{-im(y-x)}+\frac{i}{4\pi}\sum_{k\geq 1}\left(e^{-ik(z-y)}+e^{-im(y-x)}\right)+c.c.=\\ \frac{1}{2}\cot\frac{1}{2}(y-z)\,[\delta(z-y)-\delta(z-x)].\end{aligned} \tag{C.3}$$

The integration of (C.3) over x gives the result

$$\begin{aligned}\{\check{T}(z),\check{\chi}(y)\}_* &= \check{\chi}'(y)\,\delta(z-y)+\frac{1}{2}\,(\eta+\alpha)\,\check{\chi}(y)\delta'(z-y)\\ &\quad+(1-\eta+2\alpha)\,e^{-\gamma\phi(y)}F(y-z,p)\,\left(e^{2\gamma\phi(z)}\right)'\\ &\quad+(\eta-2\alpha)\,e^{-\gamma\phi(y)}F'(y-z,p)\,e^{2\gamma\phi(z)}-\alpha\,e^{-\gamma\phi(y)}\cot\frac{1}{2}(y-z)\,F(y-z,p)\,e^{2\gamma\phi(z)}\\ &\quad-\left[e^{-\gamma\phi(y)}\int_0^{2\pi}dx\,\left(F'(y-x,p)-\alpha\cot\frac{1}{2}(y-x)\,F(y-x,p)\right)\,e^{2\gamma\phi(x)}\right]\delta(z-y),\end{aligned} \tag{C.4}$$

which has in addition to the standard terms undesirable anomalies. The cancellation of the anomalies uniquely yields (4.45) and (4.46).

Appendix D

Here we present some heuristic considerations and argue that the equation (4.66) can be derived by means of the dilatation properties of the vacuum configuration (3.33). For this purpose let us define a new (s-)symbol $A_s(p)e^{2\alpha q}$ with $A_s(p) > 0$ and associate it with the operator

$$\hat{A} = \sqrt{A_s(\hat{p})}\; F_{2\alpha}(\hat{p} + i\hbar\alpha)\, e^{2\alpha\hat{q}}\; \sqrt{A_s(\hat{p})}. \tag{D.1}$$

The function $F_{2\alpha}(p)$ is included here in order to get a Hermitian operator $\hat{A}$. Such a symbol is related to the vacuum expectation value of $\hat{A}$, and it can be obtained by a normalised limiting procedure $\nu \to 0$ from the Berezin symbols (B.10). Comparing the two different forms (4.16) and (D.1) of an operator $\hat{A}$, the functions $A(p)$ and $A_s(p)$ are related by

$$A(p) = \sqrt{A_s(p + i\hbar\alpha)\, A_s(p - i\hbar\alpha)}\; F_{2\alpha}(p). \tag{D.2}$$

This relationship will now be used to specify the undetermined functions $C(p)$ and $S(p)$ of (4.37) and (4.48). The dilatation properties of the vacuum configuration (3.33) give for the s-symbol

$$\check{\psi}_s(z)|_{q=0,\phi^\pm=0} = c\, e^{-\frac{\gamma p z}{4\pi}}, \qquad \check{\chi}_s(z)|_{q=0,\phi^\pm=0} = \frac{s}{p}\, e^{\frac{\gamma p z}{4\pi}}, \tag{D.3}$$

where c and s are constants, and (D.2) leads to

$$\check{\psi}(z)|_{q=0,\phi^\pm=0} = c\, F_{-\gamma}(p)\, e^{-\frac{\gamma p z}{4\pi}}, \tag{D.4}$$

$$\check{\chi}(z)|_{q=0,\phi^\pm=0} = s\, F_{\gamma}(p)\, \left[p^2 + (\hbar\gamma/2)^2\right]^{-\frac{1}{2}}\; e^{\frac{\gamma p z}{4\pi}}. \tag{D.5}$$

But the vacuum configurations of (4.37) and (4.48) are also given directly

$$\check{\psi}(z)|_{q=0,\phi^\pm=0} = C(p)\, e^{-\frac{\gamma p z}{4\pi}}, \tag{D.6}$$

$$\check{\chi}(z)|_{q=0,\phi^\pm=0} = S(p)\, \frac{A_\alpha\, e^{\frac{\gamma p z}{4\pi}}}{\Gamma(1+\alpha+i\frac{\gamma p}{2\pi})\Gamma(1+\alpha-i\frac{\gamma p}{2\pi})}. \tag{D.7}$$

To get (D.7) have used the integral

$$\int_0^\pi dx\, (\sin x)^{2\alpha}\, e^{\frac{\gamma p x}{\pi}} = A_\alpha\, \frac{e^{\frac{\gamma p}{2}}}{\Gamma(1+\alpha+i\frac{\gamma p}{2\pi})\;\Gamma(1+\alpha-i\frac{\gamma p}{2\pi})}, \tag{D.8}$$

where A_α is a p-independent constant. The comparison of (D.4)-(D.7) relates the unknown functions

$$C(p)\,S(p) = \frac{F_{-\gamma}(p)\,F_\gamma(p)\,\Gamma(1+\alpha-i\frac{\gamma p}{2\pi})\,\Gamma(1+\alpha+i\frac{\gamma p}{2\pi})}{[p^2+(\frac{\hbar\gamma}{2})^2]^{\frac{1}{2}}}, \tag{D.9}$$

where R_0 is a p-independent constant. The functions F_γ can be calculated, in principle, by a normalised limiting procedure of Berezin symbols which we have not done yet. Instead, we make here a suggestion which is guided by locality consideration [6, 8]

$$F_{-\gamma}(p)\,F_\gamma(p) = \left(\frac{\Gamma(1-\alpha-i\frac{\gamma p}{2\pi})\,\Gamma(1-\alpha+i\frac{\gamma p}{2\pi})}{\Gamma(1+\alpha-i\frac{\gamma p}{2\pi})\,\Gamma(1+\alpha+i\frac{\gamma p}{2\pi})}\right)^{\frac{1}{2}}. \tag{D.10}$$

The identity

$$\Gamma(1-a-ib)\Gamma(1-a+ib)\Gamma(1+a-ib)\Gamma(1+a+ib) = \frac{(\pi a)^2+(\pi b)^2}{\sinh^2\pi a+\sin^2\pi b}, \tag{D.11}$$

provides finally

$$C(p)\,S(p) = R_0\left(\sinh^2\frac{\gamma p}{2}+\sin^2\pi\alpha\right)^{-\frac{1}{2}}. \tag{D.12}$$

This result defines the coefficients of the quantum exchange algebra, and it leads to the causal commutation relations (4.85).

References

[1] J. Wess, B. Zumino, *Phys. Lett.* **B37** (1971) 95; S. P. Novikov, *Usp. Math. Nauk.* **37** (1982) 3; E. Witten, *Comm. Math. Phys.* **92** (1984) 455.

[2] J. Balog, L. Feher, P. Forgacs, L. O'Raifeartaigh, A. Wipf, *Phys.Lett.* **B227** (1989) 214; *Annals Phys.* **203** (1992) 269.

[3] C. Ford, G. Jorjadze, G. Weigt, Integration of the $SL(2,\mathbb{R})/U(1)$ Gauged WZNW Theory by Reduction and Quantum Parafermions, hep-th/0003246, to be published in Theor. Math. Phys.

[4] M.F. Chu, P. Goddard, I Halliday, D. Olive and A Schwimmer, *Phys. Lett.*, **B266** (1991) 71.

[5] K. Gawedzki, *Comm. Math. Phys.* **139** (1991) 201.

[6] E. Braaten, T Curtright, and C. Thorn, *Ann. Phys.* (NY) **147** (1983) 365.

[7] J.-L. Gervais and A. Neveu, *Nucl. Phys.***B238** (1984) 125; J. Schnittger, Quantum Group Structure and Local Fields in the Algebraic Approach to 2-d Gravity, *Theor. Math. Phys.* **104** (1996) 892.

[8] H.J. Otto and G. Weigt, *Z. Phys.* **C31** (1986) 219; G. Weigt, Canonical Quantisation of the Liouville Theory, Quantum Group Structures, and Correlation Functions, Pathways to Fundamental Theories, World Scientific Publishing Co. Pte. Ltd., 1993, p. 227.

[9] K. Bardakci, M. Crescimanno, E. Rabinovici, *Nucl. Phys.* **B344** (1990) 344.

[10] H. Weyl, *Z. Phys.* **46** 1; E. Wigner, *Phys. Rev.* **40** (1932) 153.

[11] J.E. Moyal, *Proc. Cambridge Phil. Soc.* **45** (1949) 99.

[12] F.A. Berezin, *Comm. Math. Phys.* **40** (1975) 411.

[13] A. N. Leznov, M. V. Saveliev, *Acta Appl. Math.* **16** (1989) 1.

[14] G. P. Jorjadze, A. K. Pogrepkov, M. C. Polivanov *Teor. Mat. Fiz.* **40** n.2 (1979) 221.

[15] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer, *Ann. Phys.* **111** (1978) 61 and 111.

[16] E. Gozzi, M. Reuter, *Int.J. Mod. Phys.* **A9** (1994) 2191.

[17] A.M. Perelomov, Generalised Coherent States and their Applications, Springer-Verlag, Berlin, 1986.

[18] N.M.J. Woodhouse, Geometric Quantisation, Clarendon, Oxford, 1992.

[19] J. Balog, L. Feher, L. Palla, *Nucl. Phys.* **B568** (2000) 503.