Quantum stability of defects for a Dirac field coupled to a scalar field in $2 + 1$ dimensions

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Abstract

We study the Euclidean effective action and the full fermion propagator for a Dirac field in the presence of a scalar field with a domain wall defect, in $2 + 1$ dimensions. We include quantum effects due to both fermion and scalar field fluctuations, in a one-loop approximation. The results are interpreted in terms of the quantum stability of the zero mode solution. We also study, for this system, the induced 'inertial' electric field for the fermions on the defect, due to the quantum fluctuations of the scalar field.

1 Introduction

The subject of fermionic fields in the background of defects presents itself in many different areas of physics, from textures in superfluid phases of He_3 [1] to cosmic strings [2]. Those defects are generally defined as some field configurations having a nontrivial topological content, and one of the most important features of these systems is the existence of localized fermionic zero modes. The most interesting property of the zero modes, when transport properties are considered, is the fact that they are perfect conductors (namely, they have linear dispersion relations for small values of the momenta).

The most widely known example of this phenomenon is, perhaps, the so called Callan-Harvey mechanism [3] by which a Fermi field (either 'fundamental' or arising as an effective variable in an approximate description of a physical system) living in an odd-dimensional spacetime, is coupled to a domain wall like defect. This defect is defined as a mass term that changes sign on a one-dimensional curve in the constant-time plane. A general result for this kind of system is that the fermionic field spectrum has, besides the usual continuum part, an extra zero-mass bound-state located precisely on the defect. This zero mode corresponds then to a one-dimensional fermion which, because of its linear dispersion relation, is capable of carrying electric currents.

When the Fermi field is also minimally coupled to an Abelian gauge field, the resulting Chern-Simons current, proportional to the sign of the fermionic mass, has an apparent 'anomaly' on the defect, due to the vanishing of the mass. This 'bulk' anomaly is, however, exactly compensated by an equal and opposite (chiral) anomaly for the fermionic current at the defect, which arises because the zero modes are described by a $1 + 1$ dimensional *chiral* theory. A similar phenomenon is found at the edge of a droplet of a non-relativistic electron gas in two spatial dimensions, in the presence of a magnetic field, both in the regime of the fractional and integer quantum Hall effects [4]. Here, the Chern-Simons (Hall) current is of course also present, and its apparent 'bulk' anomaly on the borders is canceled by the gauge anomaly of the chiral edge states [5, 6]. Indeed, this anomaly-matching mechanism is usually a guiding principle for the construction of the models describing the matter fields on the borders, since in two spacetime dimensions the form of the anomaly is very closely constrained, though no completely determined, by the matter content.

While for many different physical situations static defects are sufficient to describe the relevant physics, in many others (particularly in condensed matter physics) it is of current interest to understand the possible effects due to the *dynamics* of the defects. In particular, it is interesting to investigate the interaction of the fermion zero modes with the fluctuating geometry induced by the dynamics of the defect. In [7] we generalized the results of [8], which dealt with static defects in $2+1$ dimensions, to the case of an arbitrary, space *and time* dependent defect, showing that an analog of the Callan and Harvey mechanism [3] still holds. Namely, there exists a chiral zero mode, and it is localized, this time on the defect *worldsheet*. Besides, an important new phenomenon is that (even in the absence of any external gauge field) if the defect is accelerated, there is an induced fermionic current on the defect. This effect is caused by the gravitational chiral anomaly of the fermions localized on the defect. This current can be thought of as due to the existence of an effective 'inertial' electric field, which can be entirely defined in terms of the geometry of the defect, and may also be interpreted in terms of the gravitational anomaly for the fermions living on the defect worldsheet [9].

In this paper, we shall consider a related situation: the defect will be allowed to fluctuate, but the dynamics of these fluctuations will be, in the present case, defined by a scalar field action. To make the system tractable, however, we shall deal with configurations where the 'topological charge' of the configurations is fixed, corresponding to just *one* defect. The excursions around these configurations will be first taken into account by performing a loop expansion around a static defect background. Although the expansion itself is a standard procedure, it will demand here the use of a suitable set of coordinates, compatible with the (non translation invariant) configuration of the system. After showing that the domain wall plus zero mode configuration is a consistent classical configuration, we shall show that they are stable under quantum fluctuations of the fermionic and scalar fields.

When the fermion field is a Majorana rather than a Dirac spinor, the model becomes supersymmetric. We show that a similar mechanism occurs in such a case, although the model itself is less interesting from the phenomenological point of view, since there is no 'electric' current associated to the Majorana fermion.

The organization of this paper is as follows: in section 2, we define the system and explore some of its fundamental properties. In section 3 we discuss some non-trivial aspects of the loop expansion for the effective action, applying this to the derivation of the full fermion propagator including one-loop corrections in section 4. This result is applied to the study of the stability of the zero mode solution under the fluctuations.

The effective inertial electric field, due to the accelerating defect is discussed in section 5.

The supersymmetric version of the model is presented, for the sake of completeness, in Appendix A. Section 6 presents our conclusions.

2 The system

The system we shall deal with may be conveniently defined in terms of an Euclidean generating functional:

$$\mathcal{Z}(j,\bar{\eta},\eta) \;=\; \int \mathcal{D}\phi\mathcal{D}\psi\mathcal{D}\bar{\psi}\; \exp[-S(\phi,\bar{\psi},\psi) \,+\, \int d^3x(j\phi \,+\, \bar{\eta}\psi \,+\, \bar{\psi}\eta)] \quad (1)$$

where the (Euclidean) action $S(\phi,\bar{\psi},\psi)$ can be written as:

$$S(\phi,\bar{\psi},\psi) \;=\; S_B(\phi) \,+\, S_F(\bar{\psi},\psi,\phi) \quad (2)$$

with

$$S_B(\phi) \;=\; \int d^3x \, [\frac{1}{2}(\partial\cdot\phi)^2 \,+\, V(\phi)] \quad (3)$$

and

$$S_F(\bar{\psi},\psi,\phi) \;=\; \int d^3x \, \bar{\psi} \, [\not\partial \,+\, g\phi(x)] \, \psi \; . \quad (4)$$

In Equation (3), $V(\phi)$ denotes a local potential, which should of course be consistent with the existence of a domain wall configuration for ϕ, and whose form will be made more explicit later on. The coupling constant g controls the strength of the interaction between the fermions and the scalar field.

The Euclidean Dirac matrices γ_μ have been chosen according to the convention

$$\gamma_0 \;=\; \sigma_3 \quad \gamma_1 \;=\; \sigma_1 \quad \gamma_2 \;=\; \sigma_2 \; , \quad (5)$$

where the σ_α's denote the usual Pauli matrices.

In the absence of external sources, the classical equations of motion for this system are:

$$[\not\partial + g\phi(x)]\psi \;=\; 0 \quad (6)$$

and

$$-\partial^2\phi(x) \,+\, V'(\phi(x)) \,+\, g\bar{\psi}(x)\psi(x) \;=\; 0 \; , \quad (7)$$

where $V' \equiv \frac{dV}{d\phi}$. In order to have a fermionic zero mode, one should demand the existence of a domain wall like configuration for ϕ. This, in turn, suggests that the potential V must be non quadratic, with the simplest choice being a quartic function. Thus the domain wall would correspond to an (x_1) translation invariant kink solution. This, of course, amounts to an infinite energy configuration in the infinite volume case. We have in mind, however, a finite system which can nevertheless be regarded as infinite (for calculational simplicity) in most of the objects we shall be interested in.

Let us, for the sake of simplicity, consider a static configuration of this type; i.e., $\phi = \phi(x_2)$. Then (6) and (7) reduce to:

$$[\gamma_0\partial_0 + \gamma_1\partial_1 + \gamma_2\partial_2 + g\phi(x_2)]\psi \;=\; 0 \quad (8)$$

and

$$-\frac{d^2\phi(x_2)}{dx_2^2} + V'(\phi(x_2)) + g\bar{\psi}(x_2)\psi(x_2) = 0 , \qquad (9)$$

respectively. To see that a zero mode is indeed a possible consistent solution for this coupled system of equations, let us consider a field configuration $\varphi_k(x_2)$ verifying

$$-\frac{d^2\varphi_k(x_2)}{dx_2^2} + V'(\varphi_k(x_2)) = 0 , \qquad (10)$$

with a potential V, such that $\varphi_k(x_2)$ is kink-like, changing sign at $x_2 = 0$, say (the 'center' of the kink). Inserting this configuration φ_k in the place of ϕ in (8) yields, by the standard Callan-Harvey mechanism, a chiral zero mode solution for the fermion field, and this in turn implies the vanishing of $\bar{\psi}\psi$, since this term involves the two chiralities. This means that $\varphi_k(x_2)$ is also an exact solution of the full system of coupled equations (8) and (9).

Let us now consider the more interesting issue of the quantum dynamics corresponding to the above configurations. To that end, we first introduce $W[j, \bar{\eta}, \eta]$, the generating functional of connected Green's functions, which is defined by

$$W[j, \bar{\eta}, \eta] = \ln \mathcal{Z}[j, \bar{\eta}, \eta] \qquad (11)$$

and the effective action Γ, the Legendre transform of W:

$$\Gamma[\varphi, \bar{\chi}, \chi] + W[j, \bar{\eta}, \eta] = \int d^3x [j(x)\varphi(x) + \bar{\eta}(x)\chi(x) + \bar{\chi}(x)\eta(x)] \qquad (12)$$

where φ, $\bar{\chi}$ and χ are the 'classical fields':

$$\varphi(x) = \frac{\delta W}{\delta j(x)} \quad \chi(x) = \frac{\delta W}{\delta \bar{\eta}(x)} \quad \bar{\chi}(x) = -\frac{\delta W}{\delta \eta(x)} \qquad (13)$$

and we adopted the convention that functional derivatives of Grassmann objects always act *from the left of the corresponding object.*

3 Loop expansion

A standard one-loop calculation of the effective action Γ yields [10]

$$\Gamma(\varphi, \bar{\chi}, \chi) = S(\varphi, \bar{\chi}, \chi) + \Gamma_1(\varphi, \bar{\chi}, \chi) + \ldots \qquad (14)$$

where S is the classical action and Γ_1 denotes the order-$\hbar$ correction, given explicitly by

$$\Gamma_1(\varphi, \bar{\chi}, \chi) = \Gamma_1^a(\varphi) + \Gamma_1^b(\varphi, \bar{\chi}, \chi) \qquad (15)$$

where:

$$\Gamma_1^a \;=\; -\mathrm{Tr}\ln\mathcal{D} \tag{16}$$

and

$$\Gamma_1^b \;=\; \frac{1}{2}\mathrm{Tr}\ln\left[\frac{\delta^2 S_B(\varphi)}{\delta\phi(x_1)\delta\phi(x_2)} \;-\; 2\,g^2\,\bar{\chi}(x_1)\mathcal{D}^{-1}(x_1,x_2)\chi(x_2)\right]\;, \tag{17}$$

with the definition:

$$\mathcal{D} \;=\; \not{\partial} + g\varphi \;\;. \tag{18}$$

The two terms contributing to Γ_1 have a different physical interpretation: while Γ_1^a corresponds to the energy of the 'distorted' Dirac vacuum for the fermionic field in the presence of the scalar field φ, Γ_1^b takes into account the bosonic fluctuations around the classical kink background. We note that, besides the standard fluctuation operator $\frac{\delta^2 S_B}{\delta\phi(x_1)\delta\phi(x_2)}$ corresponding to a self-interacting scalar field, there is also a correction due to the fermion-boson interaction.

As we have already pointed out in the introduction, the field φ will be assumed to fluctuate around a background defect. Rather than considering arbitrary configurations for that defect, we shall concentrate on those corresponding to a sufficiently smooth, space dependent step-like defect. Moreover, the choice of the configuration will also be dictated by the practical requirement of making the calculations simpler. Our first simplifying assumption is to consider a rectilinear defect, with φ being a function of only one of the spatial coordinates, say x_2, and changing sign along the straight line $x_2 = 0$. This will introduce in (15) an explicit dependence on x_2 in both terms. This dependence is obviously carried by the operator $\mathcal{D}$ defined in (18) but also an x_2 dependence will be induced in the double functional derivative of S_B appearing in Γ_1^b, since it has to be evaluated on the classical configuration for φ. To simplify the task, taking advantage of the translation invariance on the $x_0 x_1$ plane, one should be able to disentangle the dynamics of the fermions in two pieces: one depending only on x_2, and the other having support exclusively in the plane $x_2 = 0$. To this end, we note that the operator $\mathcal{D}$ may be rewritten as:

$$\mathcal{D} \;=\; (a + \hat{\not{\partial}})\mathcal{P}_L + (a^\dagger + \hat{\not{\partial}})\mathcal{P}_R \tag{19}$$

where $a, a^\dagger$ are operators acting on functions of x_2,

$$a \;=\; \partial_2 + g\varphi(x_2) \;\;,\;\; a^\dagger \;=\; -\partial_2 + g\varphi(x_2)\;, \tag{20}$$

$\mathcal{P}_{L,R}$ are projectors along the eigenspaces of the 'chirality' matrix γ_2:

$$\mathcal{P}_L \;=\; \frac{1}{2}(1+\gamma_2) \quad \mathcal{P}_R \;=\; \frac{1}{2}(1-\gamma_2)\;, \tag{21}$$

and $\hat{\not\partial}$ is the two-dimensional Euclidean Dirac operator corresponding to the two coordinates x_0 and x_1, which we denote collectively by $\hat{x}$, namely

$$\hat{\not\partial} \;=\; \hat{\gamma} \cdot \hat{\partial} \;=\; \gamma_\alpha \partial_\alpha \quad , \quad \alpha = 0, 1 \,. \tag{22}$$

Expression (19) suggests that one could get rid of the x_2 dependence of the fields by a suitable expansion in the modes of some operator, thus obtaining a 'dimensional reduction' from the three dimensional spacetime to the two dimensional one described by $\hat{x}$. As $\mathcal{D}$ itself is not Hermitian, we may use instead any of the positive Hermitian operators $\mathcal{H}$ or $\widetilde{\mathcal{H}}$:

$$\mathcal{H} \;=\; \mathcal{D}^\dagger \mathcal{D} \;\; , \;\; \widetilde{\mathcal{H}} \;=\; \mathcal{D} \mathcal{D}^\dagger \tag{23}$$

to define the modes.

Using the expression (19) for $\mathcal{D}$, leads to the explicit form for $\mathcal{H}$ and $\widetilde{\mathcal{H}}$:

$$\mathcal{H} \;=\; -\partial^2 + g^2 \varphi^2 - g \gamma_2 \partial_2 \varphi \;\; , \;\; \tilde{\mathcal{H}} \;=\; -\partial^2 + g^2 \varphi^2 + g \gamma_2 \partial_2 \varphi \,. \tag{24}$$

These expressions enable us to write the contribution Γ_1^a to the effective action either in terms of $\mathcal{H}$ or $\widetilde{\mathcal{H}}$:

$$\Gamma_1^a(\varphi) \;=\; -\frac{1}{2} \mathrm{Tr} \ln \mathcal{H} \;=\; -\frac{1}{2} \mathrm{Tr} \ln \widetilde{\mathcal{H}} \,. \tag{25}$$

Taking the trace over Dirac indices, and using any of the two previous representations, we end up with:

$$\Gamma_1^a(\varphi) \;=\; -\frac{1}{2} \mathrm{Tr} \ln(-\partial^2 + g^2 \varphi^2 + g \partial_2 \varphi) - \frac{1}{2} \mathrm{Tr} \ln(-\partial^2 + g^2 \varphi^2 - g \partial_2 \varphi) \tag{26}$$

where the trace is now meant only on functional space.

On the other hand, in Γ_1^b we may factorize a bosonic fluctuation determinant, so that

$$\Gamma_1^b \;=\; \frac{1}{2} \mathrm{Tr} \ln \left[\frac{\delta^2 S_B}{\delta\varphi(x_1) \delta\varphi(x_2)} \right]$$
$$+ \frac{1}{2} \mathrm{Tr} \ln \left[\delta(x_1 - x_2) - 2g^2 \int_y \Delta_\varphi(x_1, y) \bar{\chi}(y) \mathcal{D}^{-1}(y, x_2) \chi(x_2) \right] \,, \tag{27}$$

where:

$$\int_y \Delta_\varphi(x_1, y) \frac{\delta^2 S_B}{\delta\varphi(y) \delta\varphi(x_2)} = \int_y \frac{\delta^2 S_B}{\delta\varphi(x_1) \delta\varphi(y)} \Delta_\varphi(y, x_2) = \delta(x_1 - x_2) \,. \tag{28}$$

With equation (27) in mind, we note that the one-loop effective action, Γ_1, may also be written as:

$$\Gamma_1 \;=\; \Gamma_1^{(0)} \,+\, \Gamma_1^{(I)} \,, \tag{29}$$

where $\Gamma_1^{(0)}$ represents the contribution that would correspond to a system consisting of a free fermion in a non-trivial φ background plus the vacuum energy due to the self-interacting scalar field φ, with no term taking into account the interaction energy between scalar and fermion fields:

$$\begin{aligned}\Gamma_1^{(0)} &= -\frac{1}{2}\,\mathrm{Tr}\ln\left[-\hat{\partial}^2 - \partial_2^2 + g^2\varphi^2(x_2) - g\partial_2\varphi(x_2)\right] \\ &\quad -\frac{1}{2}\,\mathrm{Tr}\ln\left[-\hat{\partial}^2 - \partial_2^2 + g^2\varphi^2(x_2) + g\partial_2\varphi(x_2)\right] \\ &\quad +\frac{1}{2}\,\mathrm{Tr}\ln\left[-\hat{\partial}^2 - \partial_2^2 + V''(\varphi(x_2))\right] \; . \end{aligned} \tag{30}$$

The $\Gamma_1^{(I)}$ term, instead, is a measure of the interaction energy between bosons and fermions, vanishing when $g \to 0$:

$$\Gamma_1^{(I)} = \frac{1}{2}\mathrm{Tr}\ln\left[\delta(x_1 - x_2) - 2g^2\int_y \Delta_\varphi(x_1,y)\bar{\chi}(y)\mathcal{D}^{-1}(y,x_2)\chi(x_2)\right] \; . \tag{31}$$

In what follows, we shall discuss the contributions $\Gamma_1^{(0)}$ and $\Gamma_1^{(I)}$ separately. We first make the observation that the 'vacuum' term $\Gamma_1^{(0)}$ may be conveniently put in the following form:

$$\Gamma_1^{(0)} = -\frac{1}{2}\mathrm{Tr}\ln\left[-\hat{\partial}^2 + h_f\right] - \frac{1}{2}\mathrm{Tr}\ln\left[-\hat{\partial}^2 + \tilde{h}_f\right] + \frac{1}{2}\mathrm{Tr}\ln\left[-\hat{\partial}^2 + h_\varphi\right] \; , \tag{32}$$

where the dependence on the domain wall profile $\varphi(x_2)$ and on the potential $V(\varphi(x_2))$ is encoded in the operators h_f, $\tilde{h}_f$ and h_φ, which are defined by

$$h_f = a^\dagger a \;\; , \;\; \tilde{h}_f = a a^\dagger \;\; , \;\; h_\varphi = -\partial_2^2 + V''(\varphi(x_2)) \; . \tag{33}$$

These operators play the role of one dimensional quantum mechanical 'Hamiltonians', whose eigenvalues appear as parameters in the corresponding functional determinants. The eigenvalues and eigenvectors of these operators are defined by the equations

$$\begin{aligned} h_f\psi_n(x_2) &= \lambda_n^2\psi_n(x_2) \qquad \tilde{h}_f\tilde{\psi}_n(x_2) = \lambda_n^2\tilde{\psi}_n(x_2) \\ h_\varphi\xi_n(x_2) &= \mu_n^2\xi_n(x_2) \; , \end{aligned} \tag{34}$$

where all the eigenvalues are written as squares (of real numbers) to emphasize the fact that the corresponding operators are positive.

We now start to consider some simplifying assumptions regarding the shape of the defect. Noting that

$$\Gamma_1^a(\varphi) = -\frac{1}{2}\mathrm{Tr}\ln(-\hat{\partial}^2 + h) - \frac{1}{2}\mathrm{Tr}\ln(-\hat{\partial}^2 + \tilde{h}) \tag{35}$$

where $h = a^\dagger a$, $\tilde{h} = aa^\dagger$ and $a = \partial_2 + g\varphi$, we assume that only h has a zero mode $|0\rangle$, which of course verifies the first order equation $a|0\rangle = 0$. The rest of the h and $\tilde{h}$ spectra coincide, and we demand it to be a continuum, separated by a finite gap $\kappa^2 > 0$ from the zero mode. The resulting equations uniquely fix the form of φ to be:

$$\varphi(x_2) \;=\; \frac{\kappa}{g} \tanh(\kappa x_2) \;, \tag{36}$$

where the constant κ remains arbitrary. The first order equation for the zero mode can be integrated by quadratures, yielding:

$$\psi_0(x_2) \;=\; \langle x_2|0\rangle \;=\; \sqrt{\frac{\kappa}{2}}\,[\cosh(\kappa x_2)]^{-1} \;, \tag{37}$$

which has been normalized to $\langle 0|0\rangle = 1$. For the previous φ field profile, we see that the explicit forms of the operators h_f and $\tilde{h}_f$ become

$$h_f \;=\; -\partial_2^2 + \kappa^2[2\tanh^2(\kappa x_2) - 1] \qquad \tilde{h}_f \;=\; -\partial_2^2 + \kappa^2 \;. \tag{38}$$

Although it is not immediately evident from (38), the spectra of h_f and $\tilde{h}_f$ coincide, except for the zero in h_f. Indeed, any non-zero mode of $\tilde{h}_f$ may be used to generate an eigenvector of h_f (with identical eigenvalue), by application of $a^\dagger$, and reciprocally, applying a to a non-zero mode of h_f produces an eigenvector of $\tilde{h}_f$. On the other hand, by assumption, only h_f has a zero mode. Taking this into account, we see that Γ_1^a may be written as

$$\Gamma_1^a \;=\; -\mathrm{Tr}_{2+1}\ln(-\hat{\partial}^2 - \partial_2^2 + \kappa^2) - \frac{1}{2}\mathrm{Tr}_{1+1}\ln(-\hat{\partial}^2) \tag{39}$$

where the dimension of the spacetime where the functional trace is defined has been explicitly shown. Of course, we may also go back to a 'spinorial' representation:

$$\Gamma_1^a \;=\; -\mathrm{Tr}_{2+1}\ln(\not\partial + \kappa) \;-\; \mathrm{Tr}_{1+1}\ln(\hat{\not\partial}\mathcal{P}_L) \;, \tag{40}$$

where the zero mode contribution has been singled out, and the trace affects spinorial indices also.

Fixing (36) to be the form of φ, we now need to make sure that it is a minimum of S_B. This may be achieved by a judicious choice of the potential V. As already advanced, the simplest choice is to use a quartic potential; indeed, assuming the form of $V(\varphi)$ to be:

$$V(\varphi) \;=\; \frac{\lambda}{2}\,(\varphi^2 - \mu^2)^2 \;, \tag{41}$$

then there is a classical kink solution

$$\varphi(x_2) \; = \; \mu \, \tanh(\sqrt{\lambda}\, \mu x_2) \; . \tag{42}$$

In order to match the configuration (36) (something we require in order to simplify the spectra of h and $\tilde{h}$), we should demand:

$$\mu \; = \; \frac{\kappa}{g} \;\; , \;\; \sqrt{\lambda}\mu \; = \; \kappa \tag{43}$$

so that the scalar potential is:

$$V(\varphi) \; = \; \frac{g^2}{2} \left(\varphi^2 - (\frac{\kappa}{g})^2 \right)^2 \; . \tag{44}$$

With V as given by (44), we see that h_φ is

$$h_\varphi \; = \; -\partial_2^2 \, + \, 6\kappa^2 \tanh^2(\kappa x_2) - 2\kappa^2 \; . \tag{45}$$

The spectrum of h_φ can also be found by algebraic means. We note that h_φ can be factorized similarly to h_f:

$$h_\varphi \; = \; b^\dagger b \tag{46}$$

where we introduced:

$$b \; = \; \partial_2 + 2\kappa \tanh(\kappa x_2) \quad , \quad b \; = \; -\partial_2 + 2\kappa \tanh(\kappa x_2) \; . \tag{47}$$

This already shows that there will be a zero mode ξ_0 for h_φ, which verifies $b\xi_0 = 0$, and is given by:

$$\xi_0(x_2) \; = \; \sqrt{\frac{3\kappa}{4}} \, [\cosh(\kappa x_2)]^{-2} \; , \tag{48}$$

when normalized to $\langle \xi_0 | \xi_0 \rangle \; = \; 1$. Regarding the spectrum above the zero mode, we may take advantage of the fact that the operator $\tilde{h}_\varphi \equiv b b^\dagger$ verifies:

$$\tilde{h}_\varphi \; = \; h_f \, + \, 3\kappa^2 \tag{49}$$

to infer that h_φ will also have another bound state (with eigenvalue $3\kappa^2$), and then a continuum starting at $4\kappa^2$. The form of these extra states may of course be found by applying the operator $b^\dagger$ to the corresponding eigenstates of h_f.

Thus the spectrum of h_φ contains two discrete states of eigenvalues 0 and $3\kappa^2$, plus a continuum starting at $4\kappa^2$. Using this information in combination with the already known form of Γ_1^a, we see that

$$\Gamma_1^{(0)} \;=\; -\frac{1}{2}\mathrm{Tr}_{1+1}\ln(-\hat{\partial}^2) \,+\, \frac{1}{2}\mathrm{Tr}_{1+1}\ln(-\hat{\partial}^2) \,+\, \frac{1}{2}\mathrm{Tr}_{1+1}\ln(-\hat{\partial}^2+3\kappa^2)$$

$$-\mathrm{Tr}_{2+1}\ln(-\hat{\partial}^2-\partial_2^2+\kappa^2) \,+\, \frac{1}{2}\mathrm{Tr}_{2+1}\ln(-\hat{\partial}^2-\partial_2^2+4\kappa^2)\;, \qquad (50)$$

where all the terms traces affect only functional space. Note that the first two terms, which correspond to the fermionic and bosonic zero modes, cancel out exactly.

4 The full fermion propagator

The full fermion propagator $\mathcal{S}_F$ may be determined by evaluating the inverse of the second functional derivative of the (one-loop corrected) effective action, evaluated at zero fermionic field, namely,

$$\mathcal{S}_F \;=\; \tilde{\mathcal{D}}^{-1} \qquad (51)$$

where

$$\tilde{\mathcal{D}}(x,y) \;=\; -\frac{\delta^2\Gamma}{\delta\bar{\chi}(x)\delta\chi(y)}|_{\chi=\bar{\chi}=0} \;=\; \mathcal{D}(x,y) \;+\; \mathcal{D}_1(x,y) \qquad (52)$$

with

$$\mathcal{D}_1(x,y) \;=\; g^2\,\Delta_\varphi(y,x)\,\mathcal{D}^{-1}(x,y)\;. \qquad (53)$$

It is evident that $\tilde{\mathcal{D}}$ plays the role of a ‘corrected Dirac operator’, since it includes a one-loop contribution. Contrary to what happens for the translation invariant case, we see that the use of a standard Fourier representation for all the coordinates does not simplify the treatment. Rather, a mixed expansion using plane waves for the $\hat{x}$ coordinates and suitable eigenfunctions for the dependence on x_2 is more adequate. Indeed, in terms of these eigenfunctions and eigenvalues, we may formulate the expansion for the propagator $\mathcal{D}^{-1}$ as follows

$$\langle x|\mathcal{D}^{-1}|y\rangle \;=\; \int\frac{d^2\hat{p}}{(2\pi)^2}\,e^{i\hat{p}\cdot(\hat{x}-\hat{y})}\Big\{\langle x_2|\psi_0\rangle\langle\psi_0|y_2\rangle\frac{-i\;\hat{\not p}}{\hat{p}^2}\mathcal{P}_R$$

$$+\sum_{n=1}^{\infty}[\langle x_2|\psi_n\rangle\langle\psi_n|y_2\rangle\frac{-i\;\hat{\not p}}{\hat{p}^2+\lambda_n^2}\mathcal{P}_R+\;\langle x_2|\tilde{\psi}_n\rangle\langle\tilde{\psi}_n|y_2\rangle\frac{-i\;\hat{\not p}}{\hat{p}^2+\lambda_n^2}\mathcal{P}_L$$

$$+\sum_{n=1}^{\infty}[\langle x_2|\tilde{\psi}_n\rangle\langle\psi_n|y_2\rangle\frac{\lambda_n}{\hat{p}^2+\lambda_n^2}\mathcal{P}_R+\;\langle x_2|\psi_n\rangle\langle\tilde{\psi}_n|y_2\rangle\frac{\lambda_n}{\hat{p}^2+\lambda_n^2}\mathcal{P}_L]\Big\}\;. \qquad (54)$$

We are using a discrete sum notation here for the sake of simplicity; it should be noted, however, that with our assumptions on the potential, there is only one discrete state, ψ_0. We shall take care of this, using a more explicit notation when necessary.

On the other hand, for Δ_φ we use instead the expansion:

$$\langle x|\Delta_\varphi|y\rangle \;=\; -\int \frac{d^2\hat{k}}{(2\pi)^2}\, e^{i\hat{k}\cdot(\hat{x}-\hat{y})} \sum_{n=0}^{\infty} \langle x_2|\xi_n\rangle\langle\xi_n|y_2\rangle \frac{1}{\hat{k}^2+\mu_n^2} \;. \tag{55}$$

Of course, the same remark on the number of discrete modes (now there are two of them), applies here.

In terms of the kernels defined above, we see that

$$\mathcal{D}_1(x,y) \;=\; -\frac{\delta^2\Gamma_1}{\delta\bar{\chi}(x)\delta\chi(y)} \;=\; g^2\Delta_\varphi(y,x)\,\langle x|\mathcal{D}^{-1}|y\rangle \;, \tag{56}$$

whose explicit form in the mixed Fourier representation becomes:

$$\mathcal{D}_1(x,y) \;=\; \int \frac{d^2\hat{k}}{(2\pi)^2}\, e^{i\hat{k}\cdot(\hat{x}-\hat{y})} \gamma_k(x_2,y_2) \tag{57}$$

with

$$\begin{aligned}
\gamma_k(x_2,y_2) \;=\; -g^2 \Bigg\{ &\sum_{n=0}^{\infty} \xi_n(x_2)\psi_0(x_2)\xi_n^\dagger(y_2)\psi_0^\dagger(y_2)\, J(\hat{k};\mu_n,0)\mathcal{P}_R \\
&+ \sum_{n=0,m=1}^{\infty} [\xi_n(x_2)\psi_m(x_2)\xi_n^\dagger(y_2)\psi_m^\dagger(y_2)\, J(\hat{k};\mu_n,\lambda_m)\mathcal{P}_R] \\
&+ \sum_{n=0,m=1}^{\infty} \xi_n(x_2)\tilde{\psi}_m(x_2)\xi_n^\dagger(y_2)\tilde{\psi}_m^\dagger(y_2)\, J(\hat{k};\mu_n,\lambda_m)\mathcal{P}_L \\
&+ \sum_{n=0,m=1}^{\infty} \xi_n(x_2)\tilde{\psi}_m(x_2)\xi_n^\dagger(y_2)\psi_m^\dagger(y_2)\, I(\hat{k};\mu_n,\lambda_m)\mathcal{P}_R \\
&+ \sum_{n=0,m=1}^{\infty} \xi_n(x_2)\psi_m(x_2)\xi_n^\dagger(y_2)\tilde{\psi}_m^\dagger(y_2)\, I(\hat{k};\mu_n,\lambda_m)\mathcal{P}_L \Bigg\} \;,
\end{aligned} \tag{58}$$

where we used a discrete notation for the sum over eigenvalues, but of course an integral over the continuous part of the spectrum is implicitly assumed. More precisely, we only have one discrete zero mode ($\lambda_0 = 0$) for the fermionic part, plus two discrete modes with $\mu_0^2 = 0$ and $\mu_1^2 = 3\kappa^2$ for the bosonic part of the sums.

The functions I and J appearing in (58) may be interpreted as $1+1$ dimensional loop integrals. They are explicitly given by

$$I(\hat{k};M_1,M_2) \;=\; \int \frac{d^2\hat{p}}{(2\pi)^2} \frac{M_2}{[(\hat{k}-\hat{p})^2+M_1^2](\hat{p}^2+M_2^2)} \tag{59}$$

and

$$J(\hat{k};M_1,M_2) \;=\; \int \frac{d^2\hat{p}}{(2\pi)^2} \frac{-i\;\hat{\not{p}}}{[(\hat{k}-\hat{p})^2+M_1^2](\hat{p}^2+M_2^2)} \;. \tag{60}$$

These (UV convergent) functions may be evaluated by using standard methods, what yields

$$I(\hat{k};M_1,M_2) \;=\; \frac{M_2}{4\pi\sqrt{(\hat{k}^2+4M^2)(k^2+\Delta^2)}}$$

$$\ln\left[\frac{\hat{k}^2+M_1^2+M_2^2+\sqrt{(\hat{k}^2+4M^2)(k^2+\Delta^2)}}{\hat{k}^2+M_1^2+M_2^2-\sqrt{(\hat{k}^2+4M^2)(k^2+\Delta^2)}}\right] \;, \tag{61}$$

and

$$J(\hat{k};M_1,M_2) \;=\; -\frac{i\;\hat{\not{k}}}{4\pi k^2(\alpha^+-\alpha^-)}\left[\alpha^+\ln(\frac{\alpha^+-1}{\alpha^+})-\alpha^-\ln(\frac{\alpha^--1}{\alpha^-})\right] \tag{62}$$

where

$$\alpha^{\pm} \;=\; \frac{1}{2\hat{k}^2}\left[\hat{k}^2+2M\Delta\mp\sqrt{(\hat{k}^2+4M^2)(k^2+\Delta^2)}\right] \;. \tag{63}$$

and we used the notation: $M=\frac{M_1+M_2}{2}$, $\Delta=M_1-M_2$.

We now consider the effect of including this correction into the Dirac operator. The corrected Dirac operator $\tilde{\mathcal{D}}$, equation (52), in a representation where the first two components have been Fourier transformed, will have a kernel

$$\langle x_2|\tilde{\mathcal{D}}|y_2\rangle = \langle x_2|(a+i\hat{\not{k}})|y_2\rangle\mathcal{P}_L + \langle x_2|(a^\dagger+i\hat{\not{k}})|y_2\rangle\mathcal{P}_R + \gamma_K(x_2,y_2) \tag{64}$$

where the 'free' part is of course local and translation invariant, since

$$\langle x_2|a|y_2\rangle = a(x_2)\delta(x_2-y_2) \tag{65}$$

and the analogous relation for $a^\dagger$. The correction γ_k, in turn, will be non local, with an structure which may be described as follows:

$$\gamma_k(x_2,y_2) \;=\; u_k(x_2,y_2)\mathcal{P}_L + \tilde{u}_k(x_2,y_2)\mathcal{P}_R$$

$$+v_k(x_2,y_2)i\hat{\not k}\mathcal{P}_L+\tilde{v}_k(x_2,y_2)i\hat{\not k}\mathcal{P}_R\,, \tag{66}$$

where the functions $u_k, \tilde{u}_k, v_k, \tilde{v}_k$ have a rather complicated expression, as may be inferred from (58).

To study the stability of the chiral zero mode solution, we note that the zero mode $|\Psi_0\rangle$ of the 'free' Dirac operator, $\mathcal{D}$, may be written as: $|\Psi_0\rangle = \mathcal{P}_L\psi_0(x_2)|k\rangle$, were $|k\rangle$ is a zero mass spinor, solution of the $1+1$ dimensional Dirac equation. If we apply now the corrected Dirac operator $\tilde{\mathcal{D}}$ to the same state $|\Psi_0\rangle$, we see that:

$$\langle x_2|\tilde{\mathcal{D}}|\Psi_0\rangle \;=\; 0+\int_{-\infty}^{+\infty} dy_2\,\gamma_k(x_2,y_2)\langle y_2|\Psi_0\rangle\;, \tag{67}$$

and let us now quantify the effect of the rhs. From the explicit form of γ_k, we see that only the terms proportional to $\mathcal{P}_L$ may contribute. Moreover, the integrals over the y_2 coordinate will be of the form:

$$\int_{-\infty}^{+\infty} dy_2\,\xi_n^\dagger(y_2)\tilde{\psi}_m^\dagger(y_2)\psi_0(y_2) \;=\; \xi_{nm}\;. \tag{68}$$

The explicit forms of the functions appearing in the integral are

$$\psi_0(y_2) \;=\; \sqrt{\frac{\kappa}{2}}\,\mathrm{sech}(\kappa y_2) \tag{69}$$

for ψ_0, and

$$\tilde{\psi}^\dagger_{m\neq 0}(y_2) \;=\; \frac{1}{\sqrt{2\pi}}\,e^{i\sqrt{\lambda_m^2-\kappa^2}y_2} \tag{70}$$

since $\tilde{h}_f$ is a (shifted) free particle one dimensional Hamiltonian. Of course, $\lambda_n^2\geq\kappa^2$. Regarding ξ_n, it has different profiles depending on whether $n=0$, $n=1$ or belongs to the continuum:

$$\xi_0(y_2) \;=\; \sqrt{\frac{3\kappa}{4}}\,\mathrm{sech}^2(\kappa y_2)\;, \tag{71}$$

$$\xi_1^\dagger(y_2) \;=\; b^\dagger\psi_0(y_2) \;=\; N(-\partial_2+2\kappa\tanh(\kappa y_2))\cosh(\kappa y_2) \;=\; \sqrt{\frac{3\kappa}{2}}\,\frac{\sinh(\kappa y_2)}{\cosh^2(\kappa y_2)} \tag{72}$$

and for states in the continumm,

$$\xi_n^\dagger(y_2) \;=\; \frac{1}{\sqrt{2\pi}}\,(-i\sqrt{\mu_n^2-4\kappa^2}+2\kappa\tanh(\kappa y_2))\,e^{-i\sqrt{\mu_n^2-4\kappa^2}y_2}\;, \tag{73}$$

where $\mu_n^2\geq 4\kappa^2$.

We now show that $\xi_{n,m}$ is exponentially small. When $n=0$, we have:

$$\int_{-\infty}^{+\infty} dy_2\, \xi_0^\dagger(y_2)\tilde{\psi}_m^\dagger(y_2)\psi_0(y_2) \;=$$

$$\frac{\kappa}{2}\sqrt{\frac{3}{\pi}}\int_0^{+\infty} dy_2\, \cos[\sqrt{\lambda_m^2-\kappa^2}y_2]\mathrm{sech}^3(\kappa y_2)$$

$$= \frac{3\sqrt{\pi}}{8}\,(\frac{\lambda_m}{\kappa})^2\,\mathrm{sech}[\frac{\pi}{2}\sqrt{(\frac{\lambda_m}{\kappa})^2-1}] \tag{74}$$

which is exponentially decreasing with λ_m. For the $n=1$ case, we have instead:

$$\int_{-\infty}^{+\infty} dy_2\, \xi_1^\dagger(y_2)\tilde{\psi}_m^\dagger(y_2)\psi_0(y_2) \;=$$

$$= -i\frac{1}{4}\sqrt{\frac{3\pi}{2}}\,((\frac{\lambda_m}{\kappa})^2-1)\,\mathrm{sech}[\frac{\pi}{2}\sqrt{(\frac{\lambda_m}{\kappa})^2-1}] \tag{75}$$

which is also decreases exponentially.

When n belongs to the continuum, a very similar inequality shows that $\xi_{n,m}$ is also exponentially small, and indeed much smaller than in the $n=0$ case.

All this shows that the free zero mode $|\Psi_0\rangle$ is still a zero mode of the corrected Dirac operator, modulo exponentially small corrections. It is noteworthy that the parameter that regulates the rate of decrease of the exponentials is κ. Thus, the sharper the domain wall, the more negligible the departures shall be. Of course, the fact that the correction is so small suggests the use of the free zero mode as a starting point for a perturbative evaluation of the corrected one, by repeated application of the corrected Dirac operator.

It is interesting to see what happens when the gap between the zero modes and the higher modes is very large, at the level of the interaction part of the effective action $\Gamma_1^{(I)}$, equation (31). When κ becomes very large, we may approximate the propagators as follows:

$$\mathcal{D}(x,y) \;\sim\; \psi_0(x_2)\psi_0(y_2)\,\hat{\mathcal{D}}(\hat{x},\hat{y}) \quad , \quad \hat{\mathcal{D}}(\hat{x},\hat{y}) \;=\; -\langle\hat{x}|\hat{\not{\partial}}^{-1}\mathcal{P}_R|\hat{y}\rangle$$

$$\Delta_\varphi(x,y) \;\sim\; \xi_0(x_2)\xi_0(y_2)\,\hat{\Delta}_\varphi(\hat{x},\hat{y}) \quad , \quad \hat{\Delta}_\varphi(\hat{x},\hat{y}) \;=\; -\,\langle\hat{x}|\hat{\partial}^{-2}|\hat{y}\rangle\;. \tag{76}$$

Also, by a similar argument to the one used for the fermionic propagator, we see that the 'classical fields' χ appearing in (31) may be replaced by the projections:

$$\chi(x) \sim \psi_0(x_2)\hat{\chi}(\hat{x}) \tag{77}$$

where $\hat{\chi}(\hat{x})$ is an arbitrary two dimensional spinor. Using these approximations, we find that

$$\Gamma_1^{(I)} \sim \frac{1}{2}\mathrm{Tr}\ln\left[\delta(x-y) - 2g^2\sqrt{\frac{\kappa}{3}}\xi_0(x_2)\phi_0^2(x_2)\,G(\hat{x},\hat{y})\right] \tag{78}$$

where

$$G(\hat{x},\hat{y}) = \int_{\hat{z}} \hat{\Delta}_\varphi(\hat{x},\hat{z})\bar{\hat{\chi}}(\hat{z})\hat{\mathcal{D}}(\hat{z},\hat{y})\hat{\chi}(\hat{y}) . \tag{79}$$

On the other hand, from the relation between ψ_0 and ξ_0, we note that

$$\xi_0(x_2)\,\psi_0^2(x_2) = \sqrt{\frac{\kappa}{3}}\langle x_2|\xi_0\rangle\langle\xi_0|y_2\rangle , \tag{80}$$

and finally, this implies:

$$\Gamma_1^{(I)} \sim \frac{1}{2}\mathrm{Tr}\ln\left[\delta(\hat{x}-\hat{y}) - 2g^2\sqrt{\frac{\kappa}{3}}\,G(\hat{x},\hat{y})\right] \tag{81}$$

where the x_2 dependence has disappeared.

5 Induced electric field

Let us discuss here the application of some general results [7] regarding the induced electric field seen by the fermionic zero mode, to show that, for this kind of configuration, and in this approximation, that electric field vanishes on the average.

The effective electric field will only be non-zero when the defect is accelerated, and this can only happen, in our situation, when we include the fluctuations in the scalar field. We recall that, when the fluctuations are small and the defect is rectilinear, the formula that yields the effective electric field E_{eff} for the fermionic zero mode [7] in a given scalar field configuration is:

$$E_{eff} = \frac{1}{6}\left[\partial_0^2\eta\partial_1^2\eta - (\partial_0\partial_1\eta)^2\right] \tag{82}$$

where $\eta(x_0, x_1)$ is the fluctuation of φ around its static kink configuration with center in $x_2 = 0$. Expression (82) is the value of the electric field for a given φ configuration; it is of course interesting to consider its quantum average $\langle E_{eff}\rangle$, the average defined by the Euclidean action of the model. Before considering the specific form of the average, we note that the *integral* of E_{eff} necessarily vanishes, since

$$\int dx_0 dx_1\, E_{eff} = \frac{1}{6}\int dx_0 dx_1 \left[\partial_0^2\eta\partial_1^2\eta - (\partial_0\partial_1\eta)^2\right] = 0 \tag{83}$$

by an integration by parts. This has consequences for the spacetime average of $\langle E_{eff} \rangle$, but it leaves room for the existence of a locally non-zero $\langle E_{eff} \rangle$. The form of this local effective electric field can be written, in our approximation, as follows:

$$\langle E_{eff}(x_0, x_1) \rangle \;=\; \frac{1}{6} \left[\frac{\partial^2}{\partial x_0^2} \frac{\partial^2}{\partial y_1^2} - \frac{\partial^2}{\partial x_0 \partial x_1} \frac{\partial^2}{\partial y_0 \partial y_1} \right] \Delta_\varphi(x_0, x_1; y_0, y_1)|_{y_\mu \to x_\mu} \;, \tag{84}$$

where Δ_φ is the 'fluctuation operator' defined in (28). Now it is evident, since Δ_φ is translation invariant in x_0 and x_1, that (84) vanishes. Indeed, translation invariance implies that Δ_φ can only depend on the differences of its arguments, and this makes it possible to show that the two terms on the right hand side of (84) cancel out.

This negative answer for the quantum average might of course be expected on symmetry grounds. However, it can be used the other way around, to look for the existence of a non-vanishing average by considering different scalar field configurations violating the invariance under translations in more than one coordinate.

6 Conclusions

We have studied a system consisting of a Dirac fermion interacting with a real scalar field, to understand the effect of quantum fluctuations on the zero modes due to the Callan and Harvey mechanism, to one-loop order. We have shown that the effect of the fluctuations on the zero mode is exponentially suppressed, and hence negligible, what signals the perturbative stability of the classical zero mode solution under quantum fluctuations.

The effective inertial electric field induced on the fermions vanishes to this order; and this also points to the direction of stability, since an strong electric field on the zero mode may have meant that an extra interaction should have been included.

Based on these results, we conclude that instabilities may very likely arise from different configurations, possible with more than one defect, and with translation invariance along only one spacetime coordinate. Also, it remains an open question whether a nonperturbative treatment for the quauntum coupled system is possible. Nonperturbative results do indeed exist for this system, although they deal with the external field problem only [11].

Results on these systems, as well as on numerical nonperturbative solutions of the coupled system, will be reported elsewhere.

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Appendix A: Supersymmetric model

We discuss here some particularities of the (simplest) supersymmetric version of the model discussed previously. Taking into account the fact that the bosonic part of the model has to be a real scalar [1], and we want a non-trivial interaction between boson and fermion fields, we are lead to consider a $2+1$ dimensional Wess-Zumino model supplemented by a mass term and an interaction term.

Following the notation and conventions of [12], we consider the action

$$S \;=\; S_{WZ} \,+\, S_m \,+\, S_g \tag{85}$$

where S_{WZ} is the free, massless, Wess-Zumino action

$$S_{WZ} \;=\; \int d^3x \left[-\frac{1}{2}\partial^\mu \phi \partial_\mu \phi - \frac{1}{2}\bar{\psi} \; \partial\!\!\!/ \psi + \frac{\alpha}{2} F^2 \right] \tag{86}$$

where we used a Minkowski space metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1)$, and ψ is a (real) Majorana fermion. The actions S_m and S_g are mass and interaction terms, respectively:

$$\begin{aligned} S_m \;&=\; \int d^3x \, \frac{\kappa}{2} \left(F\phi - \frac{1}{2}\bar{\psi}\psi \right) \\ S_g \;&=\; \int d^3x \; g \left(F\phi^2 - \bar{\psi}\psi\phi \right) \; . \end{aligned} \tag{87}$$

As usual, the role of the auxiliary field F is to make the action S of-shell invariant under the supersymmetry transformations, which, as shown in [12], are given by:

$$\begin{aligned} \delta\phi \;&=\; \bar{\epsilon}\psi \\ \delta F \;&=\; \bar{\epsilon} \; \partial\!\!\!/ \psi \\ \delta\psi \;&=\; \partial\!\!\!/ \phi\epsilon \,+\, F\epsilon \; . \end{aligned} \tag{88}$$

[1] This must be so in order to have domain wall like solutions and fermionic zero modes.

Supersymmetry is preserved, although now on-shell, if the auxiliary field F is 'integrated out'. This yields the equivalent action

$$S \;=\; \int d^3x \left[-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi \;-\; \frac{1}{2}\bar{\psi}\;\partial\!\!\!/\psi \;-\; \frac{1}{2}(\frac{\kappa}{2}\phi + g\phi^2)^2 \;-\; \frac{1}{2}(\frac{\kappa}{2} + 2g\phi)\bar{\psi}\psi \right] \;. \tag{89}$$

In order to compare with the non supersymmetric action, it is convenient to make the field redefinition $\phi(x) \to \varphi(x)$, with $\varphi(x) = \phi(x) + \frac{\kappa}{4g}$ so that, in the new variables, the action is:

$$S \;=\; \int d^3x \left[-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}\bar{\psi}\;\partial\!\!\!/\psi - \frac{g^2}{2}(\varphi^2 - (\frac{\kappa}{g})^2)^2 - g\varphi\bar{\psi}\psi \right] \;. \tag{90}$$

This action is almost identical to its non-supersymmetric version, differing only in their fermionic sectors.

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