On (2+1)Dimensional Topologically Massive Non-linear Electrodynamics

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Abstract

The (2+1) dimensional non-linear electrodynamics, the so called Pagels–Tomboulis electrodynamics, with the Chern–Simons term is considered. We obtain "generalized self–dual equation" and find the corresponding generalized massive Chern–Simons Lagrangian. Similar results for (2+1) massive dilaton electrodynamics have been obtained.

1 The Pagels–Tomboulis model

Among many gauge theories the Pagels–Tomboulis model [1]

$$L = -\frac{1}{4}\left(\frac{F^a_{\mu\nu}F^{a\mu\nu}}{\Lambda^4}\right)^{\delta-1} F^a_{\mu\nu}F^{a\mu\nu} \tag{1}$$

has a special place. Here

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \epsilon^{abc} A^b_\mu A^c_\nu$$

is the standard field tensor, δ is a dimensionless parameter and Λ is a dimensional constant. The gauge field is $SU(2)$ type i.e. $a = 1, 2, 3$. One can check that this Lagrangian has positively defined energy for $\delta \geq \frac{1}{2}$.

The model was originally found as an effective model for the low energy (3+1) QCD [1]. In fact, it was shown that electrical sources are confined for

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$\delta \geq \frac{3}{2}$. Energy of the electric field generated by the external charge is infinite due to the divergence at large distance. Moreover, the dipole-like external source gives a finite energy field configuration. The energy $\mathcal{E}$ behaves like

$$\mathcal{E} = c_0 |q|^{\frac{2\delta}{2\delta-1}} \Lambda^{\frac{4\delta-4}{2\delta-1}} R^{\frac{2\delta-3}{2\delta-1}}, \tag{2}$$

where c_0 is a numerical constant, q is an external charge and R is the distance between charges [2]. It is very interesting that for $\delta = \frac{5}{2}$ one obtains the $\sqrt{R}$ behaviour of the energy, which is in agreement with a phenomenological potential found in fits to spectra of heavy quarkonia [3]. The standard linear potential appears in the limit $\delta \to \infty$.

However, recently the Pagels–Tomboulis model has been also considered as an example of the field theory with the vanishing trace of the energy–momentum tensor. In case of any $(n+1)$-dimensional gauge theory defined by a Lagrangian

$$L = L(F),$$

where $F = F^a_{\mu\nu} F^{a\mu\nu}$, the trace $T = T_\mu^{\ \mu}$ has the following form

$$T = 4 \frac{dL}{dF} F - (n+1)L. \tag{3}$$

One can easily find that for any $(n+1)$-dimensional space–time there exists the unique Lagrangian (up to the multiplicative constant), which gives the vanishing trace of the energy–momentum tensor. Namely,

$$L = -\frac{1}{4} (F^a_{\mu\nu} F^{a\mu\nu})^{\frac{n+1}{4}}. \tag{4}$$

Only in $(3+1)$-dimensional space–time the Lagrangian is a linear function of F. For instance, in $(2+1)$ dimension the pertinent model takes the form

$$L_{2+1} = -\frac{1}{4} (F^a_{\mu\nu} F^{a\mu\nu})^{\frac{3}{4}}. \tag{5}$$

Such particular Lagrangian, in its Abelian version, have been used as a source in the Einstein equations. Many static, spherically symmetric solutions have been obtained [4].

This incomplete list of applications of the Pagels–Tomboulis model in various areas of theoretical physics shows that the model is very interesting and has rich mathematical structure. Unfortunately, in contradistinction to other non-linear gauge theories (for example Born–Infeld theory [5]) it has not been considered in the systematic way.

In the present paper we focus on the (2+1) Abelian Pagels–Tomboulis model with the additional topological term - the Chern–Simons term

$$L = -\frac{1}{4}(F_{\mu\nu}F^{\mu\nu})^{\delta} + \frac{m}{4}\epsilon^{\mu\nu\rho}A_{\mu}F_{\nu\rho}. \tag{6}$$

Here, for simplicity, the dimensional constant Λ has been neglected. This model is the natural generalization of the non–linear electrodynamics (5) considered in [4]. It is well known that the Chern–Simons part of the Lagrangian (6) does not enter explicitly to the expression for the energy. It is due to the fact that this term is metric independent. The energy–momentum tensor is unchanged. It has been shown using the field equations that in the Maxwell limit i.e. for $\delta = 1$ the gauge field from (6) is proportional to the dual strength tensor [6], [7]

$$A_{\mu} = \frac{1}{2m}\epsilon_{\mu\nu\rho}F^{\nu\rho}. \tag{7}$$

This self–dual equation can be derived also from the massive Chern–Simons Lagrangian [8]

$$L_{mass} = \frac{1}{m^2}A_{\mu}A^{\mu} - \frac{m}{4}\epsilon^{\mu\nu\rho}A_{\mu}F_{\nu\rho}. \tag{8}$$

In fact, it was shown that these Lagrangians are equivalent. Let us now generalize these results for all $\delta > \frac{1}{2}$. The pertinent equations of motion read

$$\partial_{\nu}\left[(F_{\sigma\rho}F^{\sigma\rho})^{\delta-1}F^{\nu\mu}\right] + \frac{m}{2\delta}\epsilon^{\mu\nu\rho}F_{\nu\rho} = 0. \tag{9}$$

The solution of the second order equations (9) has the generalized form of the self–dual equation (7)

$$A_{\mu} = \frac{\delta}{2m}(F_{\sigma\lambda}F^{\sigma\lambda})^{\delta-1}\epsilon^{\mu\nu\rho}F_{\nu\rho}. \tag{10}$$

It is immediately seen that after differentiation of both side of generalized self–dual equation and multiplication by $\epsilon^{\alpha\beta\gamma}$ we obtain (9). As in the Maxwell case, the generalized self–dual equation emerges as a field equation from generalized massive Chern–Simons Lagrangian

$$L_{mass} = \frac{1}{4}(f_{\mu}f^{\mu})^{\frac{\delta}{2\delta-1}} - \frac{D\delta}{2\delta-1}\epsilon^{\mu\nu\rho}f_{\mu}\partial_{\nu}f_{\rho}, \tag{11}$$

where we have distinguished the $U(1)$ gauge field f_{μ} from generalized massive Chern–Simons and from the original Pagels–Tomboulis Lagrangian. The field equations have the following form

$$\epsilon^{\mu\nu\rho}\partial_{\nu}f_{\rho} - \frac{1}{4D}(f_{\nu}f^{\nu})^{\frac{1-\delta}{2\delta-1}}f^{\mu} = 0. \tag{12}$$

In order to establish the generalized self–dual equation for new gauge field f_μ one has to rewrite (12) as

$$f_{\mu\nu}f^{\mu\nu}=\frac{1}{8D^2}(f_\nu f^\nu)^{\frac{1}{2\delta-1}}. \tag{13}$$

Here $f_{\mu\nu}=\partial_\mu f_\nu-\partial_\nu f_\mu$. Then we express $f_\mu f^\mu$ in terms of the corresponding field strength tensor and substitute this into the field equation (12). We obtain that

$$f_\mu=2\cdot 8^{\delta-1}D^{2\delta-1}\left(f_{\nu\rho}f^{\nu\rho}\right)^{\delta-1}\epsilon^{\mu\lambda\sigma}f_{\lambda\sigma}. \tag{14}$$

This generalized self-dual equation becomes identical to (10) if the constant D reads

$$D=\left(\frac{2\delta}{m8^\delta}\right)^{\frac{1}{2\delta-1}}. \tag{15}$$

One can see that equations (11) and (14) are in agreement with the results presented in [9], where the case of $\delta=\frac{p}{q}$, $p,q\in Z$ was considered. The relations between the topological massive Pagels–Tomboulis model and the generalized massive Chern–Simons model (11) become clearly visible when we observe that they have the common origin, that is they follow from a single Lagrangian

$$L_M=\frac{\alpha}{4}(f_\mu f^\mu)^{\frac{\delta}{2\delta-1}}-\beta\epsilon^{\mu\nu\rho}f_\mu\partial_\nu A_\rho+\frac{m}{2}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho \tag{16}$$

where the constants are

$$\frac{\alpha}{\delta}\left(\frac{2\beta^2}{\alpha^2}\right)^\delta=1,\quad \frac{\beta^2}{\alpha}=2m\left(\frac{2\delta}{m8^\delta}\right)^{\frac{1}{2\delta-1}}, \tag{17}$$

and the fields A_μ and f_μ are treated independently. Indeed, after variation of (16) with respect to f_μ one can use obtained equation to eliminate this field from the Lagrangian and get (6). In the same way the gauge field A_μ can be expressed in terms of f_μ. As a result we get the generalized massive Chern–Simons model. The Lagrangian (16) gives in the limit $\delta=1$ the so called master Lagrangian [6].

2 The dilaton model

Let us now find the analogous dual structure for the dilaton-like Lagrangian (it is possible to add a potential term for the scalar field but it does not change the result obtained below)

$$L=-\frac{\sigma(\phi)}{4}F_{\mu\nu}F^{\mu\nu}+\frac{m}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}+\frac{1}{2}(\partial_\mu\phi)^2 \tag{18}$$

In fact, as it was shown in [1] the models (6) and (18) share many features (especially in the context of the low energy QCD where the topological term is omitted). It emerges from the fact that they can be understood as the usual electrodynamics in rather an unusual medium. In the other words both models have the form $L = \epsilon F_{\mu\nu}F^{\mu\nu} + ...$ where the dielectric function ϵ is a function of $F_{\mu\nu}F^{\mu\nu}$ in the Pagels–Tomboulis model or ϕ in the dilaton model. In particular, in (18) $\sigma(\phi) = \phi^{\delta-1}$ plays the same role as $\epsilon(F_{\mu\nu}F^{\mu\nu}) = (F_{\mu\nu}F^{\mu\nu})^{\delta-1}$ in (6) (see e.g. [2], [12]).

On the other hand, the Lagrangian (18) appears in the natural way as a part of the topological generalization of the $(2+1)$ dilaton-Maxwell-Einstein theory [10]. This theory has been treated as the toy model of the quantum gravitation. There have been found exact solutions describing the formation of a black hole by collapsing matter. The Hawking radiation can be also described in the frame of this model. The particular form of σ function for this model is motivated by the string theory and usually reads

$$\sigma = e^{a\phi},$$

where a is a dimensionless constant. However, some other forms of σ have been also under consideration [11].
The equations of motion are as follows

$$\partial_\mu(\sigma F^{\mu\nu}) + \frac{m}{2}\epsilon^{\nu\mu\rho}F_{\mu\rho} = 0 \tag{19}$$

and

$$\partial_\mu\partial^\mu\phi + \frac{1}{4}\sigma' F_{\mu\nu}F^{\mu\nu} = 0, \tag{20}$$

where prime denotes the differentiation with respect to the scalar field. It is easy to notice that the solution of the equation (19) has self–dual-like form

$$A_\mu = \frac{\sigma(\phi)}{2m}\epsilon^{\mu\nu\rho}F_{\nu\rho}. \tag{21}$$

The corresponding massive Chern–Simons-like Lagrangian is found to be

$$L_{mass} = \frac{m^2}{2}\cdot\frac{1}{\sigma(\phi)}f_\mu f^\mu - \frac{m}{2}\epsilon^{\mu\nu\rho}f_\mu\partial_\nu f_\rho + \frac{1}{2}(\partial_\mu\phi)^2. \tag{22}$$

The pertinent field equations read

$$f_\mu = \frac{1}{2m}\sigma(\phi)\epsilon^{\mu\nu\rho}f_{\nu\rho} \tag{23}$$

and

$$\partial_\mu\partial^\mu\phi + \frac{m^2}{2}\cdot\frac{\sigma'}{\sigma^2}f_\mu f^\mu = 0. \tag{24}$$

It is immediately seen that the self–dual equation (21) for the massive Chern–Simons-dilaton model (22) is just the equation of motion. Moreover, using (23) one eliminates the field f_μ from the second field equation. After that equation (24) takes the from

$$\partial_\mu \partial^\mu \phi + \frac{1}{4}\sigma' f_{\mu\nu} f^{\mu\nu} = 0. \tag{25}$$

As we have expected both Lagrangian (18) and (22) give the same equation of motion. Additionally we see mutual duality of these models. The strong coupling sector of the one theory is interchanged with the weak coupling sector in the other one.
At least at the theoretical level one can consider a model where the dielectric function depends on $U(1)$ gauge invariant $F_{\mu\nu}F^{\mu\nu}$ as well as on the scalar function ϕ:

$$L = -\frac{\sigma}{4}(F_{\mu\nu}F^{\mu\nu})^\delta + \frac{m}{4}\epsilon^{\mu\nu\rho} A_\mu F_{\nu\rho} + \frac{1}{2}(\partial_\mu \phi)^2. \tag{26}$$

We see that the Pagels–Tomboulis and the dilaton model are included in this Lagrangian and can be derived in the particular limits. It is easy to check that the corresponding generalized self-dual equation has the form

$$A_\mu = \frac{\delta \sigma(\phi)}{2m} \epsilon^{\mu\nu\rho} F_{\nu\rho} (F_{\sigma\lambda}F^{\sigma\lambda})^{\delta-1}, \tag{27}$$

whereas the massive Chern–Simons-like Lagrangian

$$L_{mass} = A\sigma^{\frac{1}{1-2\delta}} (f_\mu f^\mu)^{\frac{\delta}{2\delta-1}} - \frac{D\delta}{2\delta-1}\epsilon^{\mu\nu\rho} f_\mu \partial_\nu f_\rho + \frac{1}{2}(\partial_\mu \phi)^2. \tag{28}$$

Here the constants read

$$\frac{D}{A} = \left(\frac{\delta 2^\delta}{2m}\right)^{\frac{1}{1+2\delta}}, \quad A\left(\frac{D}{A}\right)^{2\delta} = 2^{\delta-2}(2\delta-1) \tag{29}$$

3 Conclusions

In the present paper we have considered the (2+1) Pagels–Tomboulis electrodynamics with the topological term. The generalized version of the self–dual equations and the corresponding massive Chern–Simons-like Lagrangian have been found. Moreover, we have proved that both models can be derived from the generalized master equation (16).

The dual structure has been also obtained in case of the (2+1) topological dilaton–Maxwell model. There are two equivalent Lagrangians (18) and (22) consisting of scalar field and $U(1)$ field. It seems to be interesting that the strong coupling regime in one theory is related to the weak coupling sector in the second. The non–perturbative effects in one model can be reformulated as the perturbative effects in the other one and solved applying standard methods. Knowing that $(2+1)$ topological dilaton–Maxwell model is a part of the $(2+1)$ gravity we believe that this feature can give us possibility to find some new gravitational solutions.

Similar duality has been observed in the combined Pagels–Tomboulis–dilaton model as well.

There are two obvious directions in which the present work can be continued. First of all, as it was mentioned before, the full $(2+1)$ topological dilaton–Maxwell–Einstein theory should be considered. Secondly, because of the fact that the Pagels–Tomboulis Lagrangian is mostly considered in its non–Abelian version it seems to be important to analyze the non–Abelian generalization of the results obtained here. Then the topological term takes the form of the well-known $SU(2)$ Chern–Simons invariant. Very interesting, new results, concerning the standard $\delta = 1$ case, have been recently obtained [14].

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