DO NEUTRAL ELEMENTARY PARTICLES EXIST?

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Abstract:

We consider massless elementary particles in quantum theory based on a Galois field (GFQT). In our recent paper hep-th/0206078 where the GFQT has been discussed in detail, it has been shown that the theory possesses a new symmetry between particles and antiparticles, which has no analog in the standard approach. In the present paper we prove that the symmetry is compatible with all the operators describing such particles. As a consequence, the existence of massless neutral elementary particles in the GFQT is incompatible with the usual relation between spin and statistics (in particular, this implies that in the GFQT, the photon and the graviton can be only composite particles). We argue that the same statement is also valid for neutral particles with arbitrary masses.

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1 Introduction

At high energies any particle can be created and annihilated by other particles in different reactions. For this reason the property of a particle to be elementary or composite has no clear experimental meaning. However in theory this property is well defined. By definition, a particle is called elementary if the full set of its wave functions forms a space of irreducible representation (IR) for the symmetry group or algebra in the given theory. Such an approach has been first proposed by Wigner in Ref. [1] where unitary IRs of the Poincare group have been constructed.

In the standard approach to quantum theory each elementary particle either has or does not have the corresponding antiparticle with the same mass and spin. The latter case can also be treated in such a way that the particle and its antiparticle are the same. Elementary particles with such a property are called neutral.

Let us briefly discuss how the standard theory explains the existence of antiparticles. Consider, for example, the electron and the positron which are the antiparticles for each other. The explanation is based on the fact that the famous Dirac equation has solutions with both positive and negative energies. As noted by Dirac (see e.g. his Nobel lecture [2]), the existence of the negative energy solutions represents a difficulty which should be resolved. In the standard approach the solution is given in the framework of second quantization such that the creation and annihilation operators for the positron have the usual meaning but they enter the quantum Lagrangian with the coefficients representing the negative energy solutions. This is an implementation of the idea that the creation or annihilation of an antiparticle can be treated respectively as the annihilation or creation of the corresponding particle with the negative energy. However, since negative energies have no direct physical meaning in the standard theory this idea is implemented implicitly rather than explicitly. Note also that the electron and the positron are described by unitary IRs with positive energies, but these representations are fully independent.

In papers [3] we have proposed an approach to quantum theory where the wave functions of the system under consideration are described by elements of a linear space over a Galois field, and the operators of physical quantities - by linear operators in this space. A detailed discussion of this approach has been given in a recent paper [4]. In particular, it has been shown that at some conditions such a description gives the same predictions as the standard approach. It has also been argued that the description of quantum systems in terms of Galois fields is more natural than the standard description in terms of complex numbers.

The first obvious conclusion about quantum theory based on a Galois field (GFQT) is as follows: since any Galois field has only a finite number of elements, then in the GFQT, divergencies cannot exist in principle, and all operators are automatically well defined. It is also natural to expect that, since arithmetic of Galois field differs from the standard one, the GFQT has some properties which have no analog in the standard theory.

In particular, as shown in Ref. [4], in contrast to the standard approach, where a particle and its antiparticle are described by independent IRs of the symmetry group, in the GFQT a particle and its antiparticle are described by the same IR of the symmetry algebra. This automatically explains the existence of antiparticles and shows that a particle and its antiparticle represent different states of the same object. As a consequence, the GFQT possesses a new symmetry between particles and antiparticles, which has no analog in the standard quantum theory.

In Ref. [4] it has been shown that this symmetry is compatible with a subalgebra of the full symmetry algebra, but the compatibility with the full algebra has not been proved. Nevertheless we argued that the existence of neutral elementary particles is incompatible with the standard relation between spin and statistics.

It is obvious that the problem of existence of neutral elementary particles is of greatest interest for massless particles, e.g. for the photon and the graviton. For this reason, in the present paper (see Sects. 2 and 3) we consider the massless case. In Sect. 4 the new symmetry is described in detail, and in Sect. 5 we prove that it is compatible with all the representation operators for massless particles. In Sect. 6 it is shown that, as a consequence, in the massless case the existence of massless neutral elementary particles in the GFQT is incompatible with the standard relation between spin and statistics. We although argue that the same conclusion is valid for neutral particles with arbitrary masses.

For reading the present paper, only very elementary knowledge of Galois fields is needed. Although the notion of the Galois field is extremely simple and elegant, the majority of physicists is not familiar with this notion. For this reason, in Ref. [4] an attempt has been made to explain the basic facts about Galois fields in a simplest possible way (and using arguments which, hopefully, can be accepted by physicists). The readers who are not familiar with Galois fields can also obtain basic knowledge from the standard textbooks (see e.g. Refs. [5, 6]).

2 Representation operators of the anti de Sitter algebra

If a conventional quantum theory has a symmetry group (or algebra), then there exists a unitary representation of the group (or a representation of the algebra by Hermitian operators) in the Hilbert space describing the quantum system under consideration. In the present paper we assume that the symmetry algebra is the modular analog of the anti de Sitter (AdS) algebra so(2,3), and quantum systems are described by modular representations of this algebra (see Ref. [4] for details). The standard AdS group is ten-parametric, as well as the Poincare group. However, in contrast to the Poincare group, all the representation generators are angular momenta. In Ref. [4] we explained the reason why for our purposes it is convenient to work with the units $\hbar/2 = c = 1$. Then the representation generators are dimensionless, and the commutation relations for them can be written in the form

$$[M^{ab}, M^{cd}] = -2i(g^{ac}M^{bd} + g^{bd}M^{cd} - g^{ad}M^{bc} - g^{bc}M^{ad}) \qquad (1)$$

where a, b, c, d take the values 0,1,2,3,5, and the operators M^{ab} are antisymmetric. The diagonal metric tensor has the components $g^{00} = g^{55} = -g^{11} = -g^{22} = -g^{33} = 1$. In these units the spin of fermions is odd, and the spin of bosons is even. If s is the particle spin then the corresponding IR of the su(2) algebra has the dimension $s + 1$.

Note that if s is interpreted in such a way then it does not depend on the choice of units (in contrast to the maximum eigenvalue of the z projection of the spin operator).

For analyzing IRs implementing Eq. (1), it is convenient to work with another set of ten operators. Let $(a'_j, a_j", h_j)$ $(j = 1, 2)$ be two independent sets of operators satisfying the commutation relations

$$[h_j, a'_j] = -2a'_j \quad [h_j, a_j"] = 2a_j" \quad [a'_j, a_j"] = h_j \tag{2}$$

The sets are independent in the sense that for different j they mutually commute with each other. We denote additional four operators as $b', b", L_+, L_-$. The meaning of L_+, L_- is as follows. The operators $L_3 = h_1 - h_2, L_+, L_-$ satisfy the commutation relations of the su(2) algebra

$$[L_3, L_+] = 2L_+ \quad [L_3, L_-] = -2L_- \quad [L_+, L_-] = L_3 \tag{3}$$

while the other commutation relations are as follows

$$\begin{gathered}
[a'_1, b'] = [a'_2, b'] = [a_1", b"] = [a_2", b"] = \\
[a'_1, L_-] = [a_1", L_+] = [a'_2, L_+] = [a_2", L_-] = 0 \\
[h_j, b'] = -b' \quad [h_j, b"] = b" \quad [h_1, L_\pm] = \pm L_\pm, \\
[h_2, L_\pm] = \mp L_\pm \quad [b', b"] = h_1 + h_2 \\
[b', L_-] = 2a'_1 \quad [b', L_+] = 2a'_2 \quad [b", L_-] = -2a_2" \\
[b", L_+] = -2a_1", \quad [a'_1, b"] = [b', a_2"] = L_- \\
[a'_2, b"] = [b', a_1"] = L_+, \quad [a'_1, L_+] = [a'_2, L_-] = b' \\
[a_2", L_+] = [a_1", L_-] = -b"
\end{gathered} \tag{4}$$

At first glance these relations might seem to be rather chaotic but in fact they are very natural in the Weyl basis of the so(2,3) algebra.

The relation between the above sets of ten operators is as follows

$$\begin{gathered}
M_{10} = i(a_1" - a'_1 - a_2" + a'_2) \quad M_{15} = a_2" + a'_2 - a_1" - a'_1 \\
M_{20} = a_1" + a_2" + a'_1 + a'_2 \quad M_{25} = i(a_1" + a_2" - a'_1 - a'_2) \\
M_{12} = L_3 \quad M_{23} = L_+ + L_- \quad M_{31} = -i(L_+ - L_-) \\
M_{05} = h_1 + h_2 \quad M_{35} = b' + b" \quad M_{30} = -i(b" - b')
\end{gathered} \tag{5}$$

In addition, if * is used to denote the Hermitian conjugation, $L_+^* = L_-$, $a_j^{'*} = a_j$", $b^{'*} = b$" and $h_j^* = h_j$ then the operators M^{ab} are Hermitian (we do not discuss the difference between selfadjoined and Hermitian operators).

Let p be a prime number, and F_{p^2} be a Galois field containing p^2 elements. This field has only one nontrivial automorphism $a \to \bar{a}$ (see e.g. Refs. [5, 6, 4]) which is the analog of complex conjugation in the field of complex numbers. The automorphism can be defined as $a \to \bar{a} = a^p$ [5, 6]. Our goal is to find IRs, implementing the commutation relations (2-4) in spaces over F_{p^2}. Representations in spaces over fields of nonzero characteristics are called modular representations. A review of the theory of modular IRs can be found e.g. in Ref. [7]. In the present paper we do not need a general theory since modular IRs in question can be constructed explicitly. A modular analog of the Hilbert space is a linear space V over F_{p^2} supplied by a scalar product (...,...) such that for any $x, y \in V$ and $a \in F_{p^2}$, $(x, y) \in F_{p^2}$ and the following properties are satisfied:

$$(x,y) = \overline{(y,x)}, \quad (ax,y) = \bar{a}(x,y), \quad (x,ay) = a(x,y) \tag{6}$$

In the modular case * is used to denote the Hermitian conjugation, such that $(Ax, y) = (x, A^*y)$.

Eq. (2) defines the commutation relations for representations of the sp(2) algebra. These representations play an important role in constructing modular IRs of the so(2,3) algebra. For this reason, following Refs. [3, 4], we describe below modular IRs of the sp(2) algebra such that the representation generators are denoted as $a', a", h$.

The Casimir operator of the second order for the algebra (2) has the form

$$K = h^2 - 2h - 4a"a' = h^2 + 2h - 4a'a" \tag{7}$$

We will consider representations with the vector e_0, such that

$$a'e_0 = 0, \quad he_0 = q_0 e_0, \quad (e_0, e_0) = 1 \tag{8}$$

One can easily prove [3, 4] that q_0 is "real", i.e. $q_0 \in F_p$ where F_p is the residue field modulo p: $F_p = Z/Zp$ where Z is the ring of integers.

The field F_p consists of p elements and represents the simplest possible Galois field.

Denote $e_n = (a")^n e_0$. Then it follows from Eqs. (7) and (8), that for any $n = 0, 1, 2, ...$

$$he_n = (q_0 + 2n)e_n, \quad Ke_n = q_0(q_0 - 2)e_n, \tag{9}$$

$$a'a"e_n = (n+1)(q_0 + n)e_n \tag{10}$$

$$(e_{n+1}, e_{n+1}) = (n+1)(q_0 + n)(e_n, e_n) \tag{11}$$

The case $q_0 = 0$ is trivial and corresponds to zero representation, so we assume that $q_0 \neq 0$. Then we have the case when ordinary and modular representations considerably differ each other. Consider first the ordinary case when q_0 is any real positive number. Then IR is infinite-dimensional, e_0 is a vector with a minimum eigenvalue of the operator h (minimum weight) and there are no vectors with the maximum weight. This is in agreement with the well known fact that unitary IRs of noncompact groups are infinite dimensional. However in the modular case q_0 is one of the numbers $1, ...p-1$. The set $(e_0, e_1, ...e_N)$ will be a basis of IR if $a"e_i \neq 0$ for $i < N$ and $a"e_N = 0$. These conditions must be compatible with $a'a"e_N = 0$. Therefore, as follows from Eq. (11), N is defined by the condition $q_0 + N = 0$ in F_p. As a result, $N = p - q_0$ and the dimension of IR is equal to $p - q_0 + 1$.

One might say that e_0 is the vector with the minimum weight while e_N is the vector with the maximum weight. However the notions of "less than" or "greater than" have only a limited sense in F_p, as well as the notion of positive and negative numbers in F_p. If q_0 is positive in this sense (see Ref. [4] for details), then Eqs. (8) and (9) indicate that the modular IR under consideration can be treated as the modular analog of IR with "positive energies". However it is easy to see that e_N is the eigenvector of the operator h with the eigenvalue $-q_0$ in F_p, and the same IRs can be treated as the modular analog of IRs with "negative energies" (see Ref. [4] for details).

3 Massless modular representations of the AdS algebra

There exists a vast literature on ordinary IRs of the so(2,3) algebra in Hilbert spaces. The representations relevant for elementary particles in the AdS space have been constructed for the first time in Refs. [8, 9], while modular representations of algebra (2-4) have been investigated for the first time by Braden [10]. In Refs. [3, 4] we have reformulated his investigation in such a way that the correspondence between modular and ordinary IRs are straightforward. Our construction is described below.

We use the basis in which the operators (h_j, K_j) $(j = 1, 2)$ are diagonal. Here K_j is the Casimir operator (7) for algebra $(a'_j, a_j", h_j)$. By analogy with Refs. [9, 10] we introduce the operators

$$B^{++} = b" - a_1"L_-(h_1-1)^{-1} - a_2"L_+(h_2-1)^{-1} +$$
$$a_1"a_2"b'[(h_1-1)(h_2-1)]^{-1} \quad B^{+-} = L_+ - a_1"b'(h_1-1)^{-1}$$
$$B^{-+} = L_- - a_2"b'(h_2-1)^{-1} \quad B^{--} = b' \tag{12}$$

and consider their action only on the space of "minimal" sp(2)×sp(2) vectors, i.e. such vectors x that $a'_j x = 0$ for $j = 1, 2$, and x is the eigenvector of the operators h_j.

It is easy to see that if x is a minimal vector such that $h_j x = \alpha_j x$ then $B^{++}x$ is the minimal eigenvector of the operators h_j with the eigenvalues $\alpha_j + 1$, $B^{+-}x$ - with the eigenvalues $(\alpha_1 + 1, \alpha_2 - 1)$, $B^{-+}x$ - with the eigenvalues $(\alpha_1 - 1, \alpha_2 + 1)$, and $B^{--}x$ - with the eigenvalues $\alpha_j - 1$.

By analogy with the construction of ordinary representations with positive energy [8, 9], we require the existence of the vector e_0 satisfying the conditions

$$a'_j e_0 = b'e_0 = L_+ e_0 = 0 \quad h_j e_0 = q_j e_0$$
$$(e_0, e_0) = 1 \quad (j = 1, 2) \tag{13}$$

In the ordinary case the massless IRs are characterized by the condition

$q_2 = 1$. In the modular case we have the same condition but now $q_2 \in F_p$.

It is well known that $M^{05} = h_1 + h_2$ is the AdS analog of the energy operator, since $M^{05}/2R$ becomes the usual energy when the AdS group is contracted to the Poincare one (here R is the radius of the AdS space). As follows from Eqs. (2) and (4), the operators (a_1', a_2', b') reduce the AdS energy by two units. Therefore in the conventional theory e_0 is the state with the minimum energy. In this theory the spin in our units is equal to the maximum value of the operator $L_3 = h_1 - h_2$ in the "rest state". For this reason we use s to denote $q_1 - q_2$. In our units $s = 2$ for the photon, and $s = 4$ for the graviton. Note that in contrast to the Poincare invariant theories, massless particles in the AdS case do have states which can be treated as rest ones (see below).

The problem arises how to define the action of the operators B^{++} and B^{-+} on e_0 which is the eigenvector of the operator h_2 with the eigenvalue $q_2 = 1$. A possible way to resolve ambiguities 0/0 in matrix elements is to write q_2 in the form $q_2 = 1 + \epsilon$ and take the limit $\epsilon \to 0$ at the final stage of computations. This confirms a well known fact that analytical methods can be very useful in problems involving only integers. At the same time, one can justify the results by using only integers (or rather elements of the Galois field in question), but we will not go into details.

By using the above prescription, we require that

$$B^{++}e_0 = [b" - a_1" L_-(h_1 - 1)^{-1}]e_0 \quad B^{-+}e_0 = L_-e_0 \tag{14}$$

if $s \neq 0$ (and thus $h_1 \neq 1$), and

$$B^{++}e_0 = b"e_0 \quad B^{+-}e_0 = B^{-+}e_0 = 0 \tag{15}$$

if $s = 0$. As follows from the previous remarks, so defined operators transform minimal vectors to minimal ones, and therefore the element

$$e_{nk} = (B^{++})^n (B^{-+})^k e_0 \tag{16}$$

is the minimal sp(2)×sp(2) vector with the eigenvalues of the operators h_1 and h_2 equal to $q_1 + n - k$ and $q_2 + n + k$, respectively.

One can directly verify that, as follows from Eqs. (2-4)

$$B^{-+}B^{++}(h_1-1) = B^{++}B^{-+}(h_1-2)$$
$$B^{+-}B^{++}(h_2-1) = B^{++}B^{+-}(h_2-2), \qquad (17)$$

and, in addition, as follows from Eq. (13) (see Ref. [4] for details)

$$B^{--}e_{nk} = a(n,k)e_{n-1,k} \quad B^{+-}e_{nk} = b(n,k)e_{n,k-1} \qquad (18)$$

where

$$a(n,k) = \frac{n(q_1+q_2+n-3)(q_1+n-1)(q_2+n-2)}{(q_1+n-k-2)(q_2+n+k-2)}$$
$$b(n,k) = \frac{k(s+1-k)(q_2+k-2)}{q_2+n+k-2} \qquad (19)$$

As follows from these expressions, the elements e_{nk} form a basis in the space of minimal sp(2)×sp(2) vectors, and our next goal is to determine the range of the numbers n and k.

Consider first the quantity $b(0,k) = k(s+1-k)$ and let k_{max} be the maximum value of k. For consistency we should require that if $k_{max} \neq 0$ then $k = k_{max}$ is the greatest value of k such that $b(0,k) \neq 0$ for $k = 1,...k_{max}$. We conclude that k can take only the values of $0,1,..s$.

Let now $n_{max}(k)$ be the maximum value of n at a given k. For consistency we should require that if $n_{max}(k) \neq 0$ then $n_{max}(k)$ is the greatest value of n such that $a(n,k) \neq 0$ for $n = 1,...n_{max}(k)$. As follows from Eq. (19), in the massless case (when $q_2 = 1$) $a(1,k) = 0$ for $k = 1,..s-1$ if such values of k exist (i.e. when $s \geq 2$), and $a(n,k) = n(s+n)$ if $k = 0$ or $k = s$. We conclude that at $k = 1,...s-1$, the quantity n can take only the value $n = 0$ while at $k = 0$ or $k = s$, the possible values of n are $0,1,...n_{max}$ where $n_{max} = p-s-1$ (in contrast to the standard theory where $n = 0,1,...\infty$).

The full basis of the representation space can be chosen in the form

$$e(n_1n_2nk) = (a_1")^{n_1}(a_2")^{n_2}e_{nk} \qquad (20)$$

where, as follows from the results of the preceding section,

$$n_1 = 0,1,...N_1(n,k) \quad n_2 = 0,1,...N_2(n,k)$$
$$N_1(n,k) = p-q_1-n+k \quad N_2(n,k) = p-q_2-n-k \qquad (21)$$

We conclude that, in contrast to the standard theory where IRs of Lie algebras by Hermitian operators are necessarily infinite-dimensional, massless modular IRs are finite-dimensional and even finite since the field F_{p^2} is finite. It has been proved by Zassenhaus [11] that any modular IR is finite-dimensional. This does not imply however that modular IRs cannot be used in physics. If the value of p is big then the standard and modular IRs contain big subsets of elements for which the action of representation operators are the same. For states constructed from such elements, the standard theory and the GFQT are experimentally indistinguishable (see Refs. [3, 4] for details) but it is clear that these theories cannot be fully identical.

Let us now discuss why IRs in question can be treated as massless. It is easy to see that

$$\begin{aligned} h_1 e(n_1 n_2 nk) &= (q_1 + n - k + 2n_1) e(n_1 n_2 nk) \\ h_2 e(n_1 n_2 nk) &= (q_2 + n + k + 2n_2) e(n_1 n_2 nk) \\ M^{05} e(n_1 n_2 nk) &= (q_1 + q_2 + 2n + 2n_1 + 2n_2) e(n_1 n_2 nk) \end{aligned} \tag{22}$$

Therefore in the standard AdS theory the corresponding IR is characterized by the minimum AdS energy equal to $q_1 + q_2 = 2q_2 + s$. Since in the usual case the mass is treated as the minimum energy, and the conventional energy is equal to $M^{05}/2R$, the conventional mass becomes zero when $q_2 = 1$ and $R \to \infty$. However this observation is still insufficient to conclude that $q_2 = 1$ is distinguished among other values of q_2 since $(2q_2 + s)/2R \to 0$ when $R \to \infty$ if q_2 is any finite number (say 2, 3 etc.). Let us recall that massless particles in conventional theory do not have "rest states", and for this reason the value of s does not characterize the number of states in the corresponding IR of the su(2) algebra. Instead, massless particles are characterized by helicity which can have only two values: s or $-s$. The AdS analog of this situation is that at $q_2 = 1$ and $n > 0$ there exist only the elements e_{nk} with $k = 0$ and $k = s$. Only if $n = 0$, there exist the elements e_{nk} with $k = 0, 1 ... s$. When the AdS algebra is contracted to the Poincare one (the meaning of contruction is well known [12]), the discrete spectrum becomes the continuous one, and the probability for a particle to have zero energy is negligible.

Taking into consideration the above remarks, in the literature the massless case is often characterized not only by the condition $q_2 = 1$, but also by the condition $s \geq 2$, since only in that case $1 \leq s-1$. In the present paper we also assume that this condition is valid since this is the simplest and most interesting case.

The above results about the action of the B operators on minimal sp(2)×sp(2) vectors can be summarized in the expressions

$$\begin{aligned} B^{++}e_{nk} &= e_{n+1,k} \quad (k=0,s;\ n=0,1...n_{max}-1) \\ B^{--}e_{nk} &= n(s+n)e_{n-1,k} \quad (k=0,s;\ n=1,...n_{max}) \\ B^{+-}e_{0k} &= k(s+1-k)e_{0,k-1} \quad (k=1,...s) \\ B^{-+}e_{0k} &= e_{0,k+1} \quad (k=0,1,..s-1) \end{aligned} \tag{23}$$

while at other values of n and k the action of these operators on e_{nk} is equal to zero.

Our next task is to compute the quantities

$$Norm(n_1n_2nk) = (e(n_1n_2nk), e(n_1n_2nk)).$$

Consider, for example $(B^{++}e_{nk}, B^{++}e_{nk})$. As follows from Eq. (12), the conditions $(a_j")^* = a'_j$ and the fact that the a'_j annihilate the minimal sp(2)×sp(2) vectors,

$$(B^{++}e_{nk}, B^{++}e_{nk}) = (e_{nk}, B^{--}B^{++}e_{nk}).$$

Analogously

$$(B^{-+}e_{0k}, B^{-+}e_{0k}) = (e_{0k}, B^{+-}B^{-+}e_{0k}).$$

Then, by using Eqs. (11) and (23) we obtain

$$\begin{aligned} Norm(n_1n_2nk) &= F(n_1n_2nk)G(k), \quad where \\ F(n_1n_2nk) &= n_1!n_2!(n_1+n+s-k)!(n+n_2+k)! \\ G(k) &= s!/[(s-k)!]^2 \end{aligned} \tag{24}$$

In standard Poincare and AdS theories there also exist IRs with negative energies which are (indirectly) related to antiparticles.

They can be constructed by analogy with positive energy IRs. Instead of Eq. (13) one can require the existence of the vector e_0' such that

$$a_j"e_0' = b"e_0' = L_-e_0' = 0 \quad h_je_0' = -q_je_0'$$
$$(e_0', e_0') \neq 0 \quad (j = 1, 2) \tag{25}$$

where the quantities q_1, q_2 are the same as for positive energy IRs. It is obvious that positive and negative energy IRs are fully independent since the spectrum of the operator M^{05} for such IRs is positive and negative, respectively. At the same time, as shown in Ref. [4], *the modular analog of a positive energy IR characterized by q_1, q_2 in Eq. (13), and the modular analog of a negative energy IR characterized by the same values of q_1, q_2 in Eq. (25) represent the same modular IR.* Since this is the crucial difference between the standard quantum theory and the GFQT, we give below the proof (which slightly differs from that in Ref. [4]).

Let e_0 be a vector satisfying Eq. (13). Denote $N_1 = p - q_1$ and $N_2 = p - q_2$. We will prove that the vector $x = (a_1")^{N_1}(a_2")^{N_2}e_0$ satisfies the conditions (25), i.e. x can be identified with e_0'.

As follows from Eq. (9), the definition of N_1, N_2 and the results of the preceding section, the vector x is the eigenvector of the operators h_1 and h_2 with the eigenvalues $-q_1$ and $-q_2$, respectively, and, in addition, it satisfies the conditions $a_1"x = a_2"x = 0$.

Let us now prove that $b"x = 0$. Since $b"$ commutes with the $a_j"$, we can write $b"x$ in the form

$$b"x = (a_1")^{N_1}(a_2")^{N_2}b"e_0 \tag{26}$$

As follows from Eqs. (4) and (13), $a_2'b"e_0 = L_+e_0 = 0$ and $b"e_0$ is the eigenvector of the operator h_2 with the eigenvalue $q_2 + 1$. Therefore, $b"e_0$ is the minimal vector of the sp(2) representation which has the dimension $p - q_2 = N_2$. Therefore $(a_2")^{N_2}b"e_0 = 0$ and $b"x = 0$.

The next stage of the proof is to show that $L_-x = 0$. As follows from Eq. (4) and the definition of x,

$$L_-x = (a_1")^{N_1}(a_2")^{N_2}L_-e_0 - N_1(a_1")^{N_1-1}(a_2")^{N_2}b"e_0 \tag{27}$$

We have already shown that $(a_2")^{N_2}b"e_0 = 0$, and therefore it suffice to prove that the first term in the r.h.s. of Eq. (27) is equal to zero. As follows from Eqs. (4) and (13), $a'_2 L_- e_0 = b'e_0 = 0$, and $L_- e_0$ is the eigenvector of the operator h_2 with the eigenvalue $q_2 + 1$. Therefore $(a_2")^{N_2} L_- e_0 = 0$ and we have proved that $L_- x = 0$.

The fact that $(x, x) \neq 0$ immediately follows from the definition of the vector x and the results of the preceding section. Therefore the vector x can be indeed identified with e'_0 and the above statement is proved.

The fact that the same modular IR can be treated as the modular analog of ordinary positive energy and negative energy IRs simultaneously, does not mean of course that modular IRs of the so(2,3) algebra contradict experiment. As shown in Ref. [4], at one end of the spectrum there exists the correspondence between the modular IR and the ordinary positive energy IR while at the other end - the correspondence between the same modular IR and ordinary negative energy IR.

In the standard theory, negative energies have no direct physical meaning but they are associated with antiparticles in the formalism of second quantization. In the next section this question is discussed in detail for both, the standard theory and the GFQT.

The matrix elements of the operator M^{ab} are defined as

$$M^{ab} e(n_1 n_2 nk) = \sum_{n'_1 n'_2 n' k'} M^{ab}(n'_1 n'_2 n' k', n_1 n_2 nk) e(n'_1 n'_2 n' k') \tag{28}$$

In the modular case the trace of each operator M^{ab} is equal to zero. For the operators $(a'_j, a_j", L_\pm, b', b")$ this is clear immediately: since they do not contain nonzero diagonal elements at all, they necessarily change one of the quantum numbers $(n_1 n_2 nk)$. The proof for the diagonal operators h_1 and h_2 is as follows. For each IR of the sp(2) algebra with the minimal weight q_0 and the dimension $N + 1$, the eigenvalues of the operator h are $(q_0, q_0 + 2, ...q_0 + 2N)$. The sum of these eigenvalues is equal to zero in F_p since $q_0 + N = 0$ in F_p (see the preceding section).

Therefore we conclude that

$$\sum_{n_1 n_2 n k} M^{ab}(n_1 n_2 n k, n_1 n_2 n k) = 0 \tag{29}$$

This property is very important for investigating a new symmetry between particles and antiparticles in the GFQT (see the next section).

4 New symmetry between particles and antiparticles in GFQT

Since $(n_1 n_2 n k)$ is the complete set of quantum numbers for the elementary particle in question, we can define operators describing annihilation and creation of the particle in the states with such quantum numbers. Let $a(n_1 n_2 n k)$ be the operator of particle annihilation in the state described by the vector $e(n_1 n_2 n k)$. Then the adjoint operator $a(n_1 n_2 n k)^*$ has the meaning of particle creation in that state. Since we do not normalize the states $e(n_1 n_2 n k)$ to one (see the discussion in Ref. [4]), we require that the operators $a(n_1 n_2 n k)$ and $a(n_1 n_2 n k)^*$ should satisfy either the anticommutation relations

$$\begin{gathered}\{a(n_1 n_2 n k), a(n_1' n_2' n' k')^*\} = \\ Norm(n_1 n_2 n k)\delta_{n_1 n_1'}\delta_{n_2 n_2'}\delta_{n n'}\delta_{k k'}\end{gathered} \tag{30}$$

or the commutation relation

$$\begin{gathered}[a(n_1 n_2 n k), a(n_1' n_2' n' k')^*] = \\ Norm(n_1 n_2 n k)\delta_{n_1 n_1'}\delta_{n_2 n_2'}\delta_{n n'}\delta_{k k'}\end{gathered} \tag{31}$$

Then, taking into account the fact that the matrix elements satisfy the proper commutation relations, it is easy to demonstrate that the operators M^{ab} in the secondly quantized form

$$\begin{gathered}M^{ab} = \sum M^{ab}(n_1' n_2' n' k', n_1 n_2 n k) \\ a(n_1' n_2' n' k')^* a(n_1 n_2 n k)/Norm(n_1 n_2 n k)\end{gathered} \tag{32}$$

satisfy the commutation relations in the form (1) or (2-4).

In the standard theory, where the particle and its antiparticle are described by independent IRs, Eq. (32) describes either the quantized field for particles or antiparticles. To be precise, let us assume that the operators $a(n_1n_2nk)$ and $a(n_1n_2nk)^*$ are related to particles while the operators $b(n_1n_2nk)$ and $b(n_1n_2nk)^*$ satisfy the analogous commutation relations and describe the annihilation and creation of antiparticles. Then in the standard theory the operators of the quantized particle-antiparticle field are given by

$$\begin{gathered} M^{ab}_{standard} = \sum M^{ab}_{particle}(n_1'n_2'n'k', n_1n_2nk) \\ a(n_1'n_2'n'k')^* a(n_1n_2nk)/Norm(n_1n_2nk) + \\ \sum M^{ab}_{antiparticle}(n_1'n_2'n'k', n_1n_2nk) \\ b(n_1'n_2'n'k')^* b(n_1n_2nk)/Norm(n_1n_2nk) \end{gathered} \tag{33}$$

where the quantum numbers (n_1n_2nk) in each sum take the values allowable for the corresponding IR (in the general case, IRs for a particle and its antiparticle can be not the same in the standard theory (say in the case of neutrino)).

In contrast to the standard theory, Eq. (32) describes the quantized field for particles and antiparticles simultaneously. When the values of (n_1n_2n) are much less than p, the contribution of such values correctly describes particles (see Ref. [4]) for details). The problem arises whether this expression correctly describes the contribution of antiparticles in the GFQT. Indeed, when the AdS energy is negative, the operator $a(n_1n_2nk)$ cannot be treated as the annihilation operator and $a(n_1n_2nk)^*$ cannot be treated as the creation operator.

Let us recall (see Sect. 3) that at any fixed values of n and k, the quantities n_1 and n_2 can take only the values $0, 1...N_1(n,k)$ and $0, 1...N_2(n,k)$, respectively (see Eq. (21)). We use $Q_1(n,k)$ and $Q_2(n,k)$ to denote q_1+n-k and q_2+n+k, respectively. Then, as follows from Eq. (22), the element $e(n_1n_2nk)$ is the eigenvector of the operators h_1 and h_2 with the eigenvalues $Q_1(n,k)+2n_1$ and $Q_2(n,k)+2n_2$, respectively. As follows from the results of Sect. 2, the first IR of the sp(2) algebra has the dimension $N_1(n,k)+1$ and the second IR has the dimension $N_2(n,k)+1$. If $n_1 = N_1(n,k)$ then it follows from Eq. (22) that the first eigenvalue is equal to $-Q_1(n,k)$ in F_p, and if $n_2 = N_2(n,k)$ then the second eigenvalue is equal to $-Q_2(n,k)$ in F_p. We use $\tilde{n}_1$ to denote $N_1(n,k) - n_1$ and $\tilde{n}_2$ to denote $N_2(n,k) - n_2$. Then it follows from Eq. (22) that $e(\tilde{n}_1\tilde{n}_2 nk)$ is the eigenvector of the operator h_1 with the eigenvalue $-(Q_1(n,k) + 2n_1)$ and the eigenvector of the operator h_2 with the eigenvalue $-(Q_2(n,k) + 2n_2)$.

In the GFQT the operators $b(n_1n_2nk)$ and $b(n_1n_2nk)^*$ cannot be independent on $a(n_1n_2nk)$ and $a(n_1n_2nk)^*$. The meaning of the operators $b(n_1n_2nk)$ and $b(n_1n_2nk)^*$ should be such that if the values of (n_1n_2n) are much less than p, these operators can be interpreted as those describing the annihilation and creation of antiparticles. Therefore it is reasonable to think that the operator $b(n_1n_2nk)$ should be defined in such a way that it is proportional to $a(\tilde{n}_1, \tilde{n}_2, n, k)^*$ and $b(n_1n_2nk)^*$ should be defined in such a way that it is proportional to $a(\tilde{n}_1, \tilde{n}_2, n, k)$. In this way we can directly implement the idea that the creation of the antiparticle with the positive energy can be described as the annihilation of the particle with the negative energy, and the annihilation of the antiparticle with the positive energy can be described as the creation of the particle with the negative energy. As noted in Sect. 1, in the standard theory this idea is implemented implicitly.

As follows from the well known Wilson theorem $(p-1)! = -1$ in F_p (see e.g. [5, 6]) and Eq. (24)

$$F(n_1n_2nk)F(\tilde{n}_1\tilde{n}_2nk) = (-1)^s \tag{34}$$

We now **define** the b-operators as follows.

$$a(n_1n_2nk)^* = \eta(n_1n_2nk)b(\tilde{n}_1\tilde{n}_2nk)/F(\tilde{n}_1\tilde{n}_2nk) \tag{35}$$

where $\eta(n_1n_2nk)$ is some function. Note that in the standard theory the CPT-transformation in Schwinger's formulation also transforms a^* to b [13, 14], but in that case the both operators refer only to positive energies, in contrast to Eq. (35).

As a consequence of this definition,

$$
\begin{aligned}
a(n_1n_2nk) &= \bar{\eta}(n_1n_2nk)b(\tilde{n}_1\tilde{n}_2nk)^*/F(\tilde{n}_1\tilde{n}_2nk)\\
b(n_1n_2nk)^* &= a(\tilde{n}_1\tilde{n}_2nk)F(n_1n_2nk)/\bar{\eta}(\tilde{n}_1\tilde{n}_2nk)\\
b(n_1n_2nk) &= a(\tilde{n}_1\tilde{n}_2nk)^*F(n_1n_2nk)/\eta(\tilde{n}_1\tilde{n}_2nk)
\end{aligned}
\tag{36}
$$

Eqs. (35) and (36) define a transformation when the set (a, a^*) is replaced by the set (b, b^*). Let us call this transformation as AB. To understand whether it is a new symmetry, we should investigate when so defined (b, b^*) operators satisfy the same commutation or anticommutation relations as the (a, a^*) operators, and whether the operators M^{ab} written in terms of (b, b^*) have the same form as in terms of (a, a^*).

As follows from Eqs. (30) and (31), the b-operators should satisfy either

$$
\begin{gathered}
\{b(n_1n_2nk), b(n_1'n_2'n'k')^*\} = \\
Norm(n_1n_2nk)\delta_{n_1n_1'}\delta_{n_2n_2'}\delta_{nn'}\delta_{kk'}
\end{gathered}
\tag{37}
$$

in the case of anticommutators or

$$
\begin{gathered}
[b(n_1n_2nk), b(n_1'n_2'n'k')^*] = \\
Norm(n_1n_2nk)\delta_{n_1n_1'}\delta_{n_2n_2'}\delta_{nn'}\delta_{kk'}
\end{gathered}
\tag{38}
$$

in the case of commutators.

Now, as follows from Eqs. (24), (30), (34-36), Eq. (37) is satisfied if

$$\eta(n_1n_2nk)\bar{\eta}(n_1, n_2, nk) = (-1)^s \tag{39}$$

At the same time, in the case of commutators it follows from Eqs. (24), (31) and (34-36) that Eq. (38) is satisfied if

$$\eta(n_1n_2nk)\bar{\eta}(n_1, n_2, nk) = (-1)^{s+1} \tag{40}$$

We now represent $\eta(n_1n_2nk)$ in the form

$$\eta(n_1n_2nk) = \alpha f(n_1n_2nk) \tag{41}$$

where $f(n_1n_2nk)$ should satisfy the condition

$$f(n_1n_2nk)\bar{f}(n_1, n_2, nk) = 1 \tag{42}$$

Then α should be such that

$$\alpha\bar{\alpha} = \pm(-1)^s \tag{43}$$

where the plus sign refers to anticommutators and the minus sign to commutators, respectively. If the normal relations between spin and statistics is satisfied, i.e. we have anticommutators for odd values of s and commutators for even ones (this is the well known Pauli theorem in local quantum field theory [15]) then the r.h.s. of Eq. (43) is equal to -1.

Eq. (43) is a consequence of the fact that our basis is not normalized to one (see Ref. [4] for discussion). In the standard theory such a relation is impossible but if $\alpha \in F_{p^2}$, a solution of Eq. (43) exists. Indeed, we can use the fact that any Galois field is cyclic with respect to multiplication [5, 6]. Let r be a primitive root of F_{p^2}. This means that any element of F_{p^2} can be represented as a power of r. As mentioned in Sect. 2, F_{p^2} has only one nontrivial automorphism which is defined as $\alpha \to \bar{\alpha} = \alpha^p$. Therefore if $\alpha = r^k$ then $\alpha\bar{\alpha} = r^{(p+1)k}$. On the other hand, since $r^{(p^2-1)} = 1$, we conclude that $r^{(p^2-1)/2} = -1$. Therefore there exists at least a solution with $k = (p-1)/2$.

In the next section we discuss when the transformation defined by Eqs. (35) and (36) is compatible with Eq. (32).

5 Compatibility of the AB transformation with representation operators

Let us consider the operators (32) and use the fact that in the modular case the trace of the operators M^{ab} is equal to zero (see Eq. (29)). Therefore, as follows from Eqs. (30) and (31), we can rewrite Eq. (32) as

$$\begin{aligned} M^{ab} = \mp \sum M^{ab}(n_1'n_2'n'k', n_1n_2nk) \\ a(n_1n_2nk)a(n_1'n_2'n'k')^*/Norm(n_1n_2nk) \end{aligned} \tag{44}$$

where the minus sign refers to anticommutators and the plus sign - to commutators. Then by using Eqs. (34-36) and (41-43), we obtain in the both cases

$$M^{ab} = -\sum M^{ab}(n_1'n_2'n'k', n_1n_2nk)f(n_1'n_2'n'k')\bar{f}(n_1n_2nk)$$
$$b(\tilde{n}_1\tilde{n}_2nk)^*b(\tilde{n}_1'\tilde{n}_2'n'k')/[F(\tilde{n}_1'\tilde{n}_2'n'k')G(k)] =$$
$$-\sum M^{ab}(\tilde{n}_1\tilde{n}_2nk, \tilde{n}_1'\tilde{n}_2'n'k')f(\tilde{n}_1\tilde{n}_2nk)\bar{f}(\tilde{n}_1'\tilde{n}_2'n'k')$$
$$b(n_1'n_2'n'k')^*b(n_1n_2nk)/[F(n_1n_2nk)G(k')] \quad (45)$$

We first consider the AdS energy operator which is diagonal. As follows from Eq. (22), in the massless case the matrix elements of the M^{05} operator are given by

$$M^{05}(n_1'n_2'n'k'n_1n_2nk) = (2+s+2n+2n_1+2n_2)\delta_{n_1n_1'}\delta_{n_2n_2'}\delta_{nn'}\delta_{kk'} \quad (46)$$

Therefore the operator (32) in this case can be written as

$$M^{05} = \sum_{n_1n_2nk}[s+2(n+n_1+n_2+1)]a(n_1n_2nk)^* \times$$
$$a(n_1n_2nk)/Norm(n_1n_2nk) \quad (47)$$

At the same time, as follows from Eqs. (42), (45), (46) and the definition of the transformation $n_1 \to \tilde{n}_1,\ n_2 \to \tilde{n}_2$ (see Sect. 4)

$$M^{05} = \sum_{n_1n_2nk}[s+2(n+n_1+n_2+1)]b(n_1n_2nk)^* \times$$
$$b(n_1n_2nk)/Norm(n_1n_2nk) \quad (48)$$

In Eqs. (47) and (48), the sum is taken over all the values of (n_1n_2nk) relevant to the particle modular IR. At the same time, for the correspondence with the standard case, we should consider only the values of the (n_1n_2n) which are much less than p (see Refs. [3, 4]). The derivation of Eq. (48) demonstrates that the contribution of those (n_1n_2n) originates from such a contribution of (n_1,n_2) to Eq. (47) that $(\tilde{n}_1,\tilde{n}_2)$ are much less than p. In this case the (n_1,n_2) are comparable to p. Therefore, if we consider only such states that the (n_1n_2n) in the a and b operators are much less than p then the AdS Hamiltonian can be written in the form

$$M^{05} = {\sum}'_{n_1n_2nk}[s+2(n+n_1+n_2+1)][a(n_1n_2nk)^* \times$$
$$a(n_1n_2nk) + b(n_1n_2nk)^*b(n_1n_2nk)]/Norm(n_1n_2nk) \quad (49)$$

where $\sum'_{n_1n_2nk}$ means that the sum is taken only over the values of the (n_1n_2nk) which are much less than p. In this expression the contributions of particles and antiparticles are written explicitly and the corresponding standard AdS Hamiltonian is positive definite.

The above results show that as far as the operator M^{05} is concerned, Eq. (35) indeed defines a new symmetry since M^{05} has the same form in terms of (a, a^*) and (b, b^*) (compare Eqs. (47) and (48)). Note that we did not assume that the theory is C-invariant (in the standard theory C-invariance can be defined as the transformation

$$a(n_1n_2nk) \leftrightarrow b(n_1n_2nk)).$$

It is well known that C-invariance is not a fundamental symmetry. In the standard theory only CPT-invariance is fundamental since, according to the famous CPT-theorem [16], any local Poincare invariant theory is automatically CPT-invariant. Our assumption is that Eq. (35) defines a fundamental symmetry in the GFQT. To understand its properties one has to investigate not only M^{05} but other representation generators as well.

By analogy with the case of the operator M^{05}, it is easy to show that at the same conditions, the operators h_1 and h_2 have the same form in terms of (a, a^*) and (b, b^*).

Consider now the operator a_1" (see Sect. 2). As follows from its definition, its matrix elements are given by

$$a_1"(n_1'n_2'n'k'n_1n_2nk) = \delta_{n_1,n_1'-1}\delta_{n_2n_2'}\delta_{nn'}\delta_{kk'} \qquad (50)$$

and therefore, as follows from Eq. (32), the secondly quantized form of a_1" is

$$a_1" = \sum_{n_1=0}^{N_1-1}\sum_{n_2nk} a(n_1+1, n_2nk)^* a(n_1n_2nk)/Norm(n_1n_2nk) \qquad (51)$$

We have to prove that in terms of (b, b^*) this operator has the same form, i.e.

$$a_1" = \sum_{n_1=0}^{N_1-1}\sum_{n_2nk} b(n_1+1, n_2nk)^* b(n_1n_2nk)/Norm(n_1n_2nk) \qquad (52)$$

As follows from Eqs. (45) and (50), Eq. (52) is indeed valid if

$$f(n_1n_2nk)\bar{f}(n_1-1,n_2nk)=-1 \tag{53}$$

Since the action of the operator a_1' can be written as

$$a_1'e(n_1n_2nk)=a_1'a_1"e(n_1-1,n_2nk)$$

then, as follows from Eq. (10), the matrix elements of the operator a_1' are given by

$$a_1'(n_1'n_2'n'k'n_1n_2nk)=n_1(Q_1(n,k)+n_1-1)\delta_{n_1,n_1'+1}\delta_{n_2n_2'}\delta_{nn'}\delta_{kk'} \tag{54}$$

Therefore, as follows from Eq. (32), the secondly quantized form of this operator is

$$a_1'=\sum_{n_1=1}^{N_1}\sum_{n_2nk}n_1(Q_1(n,k)+n_1-1)a(n_1-1,n_2nk)^*$$
$$a(n_1n_2nk)/Norm(n_1n_2nk) \tag{55}$$

By analogy with the proof of Eq. (52), one can prove that in terms of (b,b^*) this operator has the same form, i.e.

$$a_1'=\sum_{n_1=1}^{N_1}\sum_{n_2nk}n_1(Q_1(n,k)+n_1-1)b(n_1-1,n_2nk)^*$$
$$b(n_1n_2nk)/Norm(n_1n_2nk) \tag{56}$$

if

$$f(n_1n_2nk)\bar{f}(n_1+1,n_2nk)=-1 \tag{57}$$

Note that in the process of derivation, n_1 transforms to $N_1(n,k)+1-n_1$ and therefore

$$n_1(Q_1(n,k)+n_1-1)\to(N_1(n,k)+1-n_1)(Q_1(n,k)+$$
$$N_1(n,k)-n_1)=n_1(Q_1(n,k)+n_1-1) \tag{58}$$

in F_p since $N_1(n,k)+Q_1(n,k)=0$ in F_p. This derivation clearly has no analog in the standard theory.

Analogously we can prove that the secondly quantized operators $a_2"$ and a_2' also have the same form in terms of (a,a^*) and (b,b^*) if

$$f(n_1n_2nk)\bar{f}(n_1,n_2+1,nk)=-1$$
$$f(n_1n_2nk)\bar{f}(n_1,n_2-1,nk)=-1 \tag{59}$$

As follows from Eqs. (42), (53, (57) and (59), the function $f(n_1n_2nk)$ necessarily has the form

$$f(n_1n_2nk) = (-1)^{n_1+n_2} f(n,k) \tag{60}$$

where the function $f(n,k)$ should satisfy the condition

$$f(n,k)\bar{f}(n,k) = 1 \tag{61}$$

The next step is to investigate whether the remaining operators $(b', b", L_+, L_-)$ have the same form in terms of (a, a^*) and (b, b^*). We discuss in detail the operator b' since computations with the other operators are analogous (and simpler).

As follows from Eq. (4) and (20),

$$\begin{gathered} b'e(n_1n_2nk) = b'(a_1")^{n_1}(a_2")^{n_2}e_{nk} = (a_1")^{n_1}b'(a_2")^{n_2}e_{nk} + \\ n_1(a_1")^{n_1-1}L_+(a_2")^{n_2}e_{nk} = (a_1")^{n_1}(a_2")^{n_2}b'e_{nk} + \\ n_2(a_1")^{n_1}(a_2")^{n_2-1}L_-e_{nk} + n_1(a_1")^{n_1-1}(a_2")^{n_2}L_+e_{nk} + \\ n_1n_2(a_1")^{n_1-1}(a_2")^{n_2-1}b"e_{nk} \end{gathered} \tag{62}$$

By using Eq. (12) we can express the action of the $(b', b", L_+, L_-)$ operators on the minimal vectors in terms of the B operators:

$$\begin{gathered} b" = B^{++} + a_1"B^{-+}(h_1-1)^{-1} + a_2"B^{+-}(h_2-1)^{-1} + \\ a_1"a_2"B^{--}[(h_1-1)(h_2-1)]^{-1} \quad L_+ = B^{+-} + a_1"B^{--}(h_1-1)^{-1} \\ L_- = B^{-+} + a_2"b'(h_2-1)^{-1} \quad b' = B^{--} \end{gathered} \tag{63}$$

and then use Eq. (23).

In such a way we can explicitly compute $b'e(n_1n_2nk)$ in Eq. (62), find the matrix elements of b' by using Eq. (28) and write the operator b' in the secondly quantized form by using Eq. (32). The result is

$$\begin{gathered} b' = \sum_{n=0}^{n_{max}-1} \sum_{k=0,s} \sum_{n_1=1}^{N_1} \sum_{n_2=1}^{N_2} (n_1 n_2) \\ a(n_1-1, n_2-1, n+1, k)^* a(n_1 n_2 n k)/Norm(n_1 n_2 n k) + \\ \sum_{n=1}^{n_{max}} \sum_{k=0,s} \sum_{n_1=0}^{N_1} \sum_{n_2=0}^{N_2} (n+s-k+n_1)(n+k+n_2) \\ a(n_1 n_2, n-1, k)^* a(n_1 n_2 n k)/Norm(n_1 n_2 n k) + \\ \sum_{k=0}^{s-1} \sum_{n_1=0}^{N_1} \sum_{n_2=1}^{N_2} [n_2(s-k+n_1)/(s-k)] \\ a(n_1, n_2-1, 0, k+1)^* a(n_1 n_2 0 k)/Norm(n_1 n_2 0 k) + \\ \sum_{k=1}^{s} \sum_{n_1=1}^{N_1} \sum_{n_2=0}^{N_2} n_1(n_2+k)(s+1-k) \\ a(n_1-1, n_2, 0, k-1)^* a(n_1 n_2 0 k)/Norm(n_1 n_2 0 k) \end{gathered} \tag{64}$$

where (see Sect. 3) $n_{max} = p-1-s$, $N_1 = N_1(n,k)$ and $N_2 = N_2(n,k)$.

The next step is to express the (a, a^*) operators in terms of the (b, b^*) operators by using Eqs. (35) and (36), and use Eqs. (34), (41), (43) and (60). The results are as follows. In terms of the operators (b, b^*) the first term in Eq. (64) has the same form as the second term in terms of (a, a^*) if

$$f(n,k)\bar{f}(n-1,k) = -1 \tag{65}$$

at $k = 0, s$. The second term in Eq. (64) has the same form in terms of (b, b^*) as the first term in terms of (a, a^*) if

$$f(n,k)\bar{f}(n+1,k) = -1 \tag{66}$$

The third term in Eq. (64) has the same form in terms of (b, b^*) as the fourth term in terms of (a, a^*) if

$$f(0,k)\bar{f}(0,k-1) = 1 \tag{67}$$

and finally the fourth term in Eq. (64) has the same form in terms of (b, b^*) as the third term in terms of (a, a^*) if

$$f(0,k)\bar{f}(0,k+1) = 1 \tag{68}$$

Analogous computations for the operators (b" $L_+ L_-$) give the same conditions (65-68). Then by using Eq. (61) we conclude that the only solution for $f(n,k)$ is such that it does not depend on k and can be represented as

$$f(n,k) = c(-1)^n \tag{69}$$

where c is any constant such that $c|c| = 1$. Therefore, as follows from Eqs. (41) and (60), the final solution for $\eta(n_1n_2nk)$ is

$$\eta(n_1n_2nk) = \alpha f(n_1n_2nk) \quad f(n_1n_2nk) = (-1)^{n_1+n_2+n} \tag{70}$$

where α satisfies Eq. (43).

We have proved that the AB transformation defined by Eq. (35) is indeed a fundamental symmetry in the GFQT (at least for massless elementary particles).

6 Problem of existence of neutral elementary particles

As already noted, the above discussion does not involve C-invariance. The question arises whether we can apply the C-transformations to the both parts of Eqs. (35) and (36). If it were possible then we would be able to obtain new restrictions on the function $\eta(n_1n_2nk)$. However the requirement of C-invariance means that the C-transformations can be applied only to the operators of observable quantities which are bilinear in (a,a^*) or (b,b^*). Suppose now that the particle in question is neutral, i.e. the particle coincides with its antiparticle. On the language of the operators (a,a^*) and (b,b^*) this means that these sets are the same, i.e. $a(n_1n_2nk) = b(n_1n_2nk)$ and $a(n_1n_2nk)^* = b(n_1n_2nk)^*$. As a consequence, Eq. (35) has now the form

$$a(n_1n_2nk)^* = \eta(n_1n_2nk)a(\tilde{n}_1\tilde{n}_2nk)/F(\tilde{n}_1\tilde{n}_2nk) \tag{71}$$

and therefore

$$a(n_1n_2nk) = \bar{\eta}(n_1n_2nk)a(\tilde{n}_1\tilde{n}_2nk)^*/F(\tilde{n}_1\tilde{n}_2nk) \tag{72}$$

As follows from Eqs. (41) and (43), these expressions are compatible with each other only if

$$f(n_1n_2nk)\bar{f}(\tilde{n}_1,\tilde{n}_2,nk) = \pm 1 \tag{73}$$

where the plus sign refers to anticommutators and the minus sign to commutators, respectively. Therefore the problem arises whether Eqs. (70) and (73) are compatible with each other. As follows from Eq. (21), (60), (61) and the definition of the transformations $n_j \to \tilde{n}_j$ (see Sect. 4)

$$f(n_1 n_2 nk)\bar{f}(\tilde{n}_1 \tilde{n}_2 nk) = (-1)^s \tag{74}$$

By comparing Eqs. (73) and (74) we conclude that they are incompatible with each other if the normal relation between spin and statistics takes place. Therefore massless particles in the GFQT cannot be elementary but only composite.

Note that, although we have proved the compatibility of the AB transformation with all the representation generators, our conclusion is based only on Eqs. (60) and (61), i.e. on the fact that the AB transformation is compatible with the representation operators of the sp(2)×sp(2) subalgebra. It is easy to see that the compatibility with the sp(2)×sp(2) algebra does not depend on the (nk) content of IR in question, and in Ref. [4] the compatibility with the sp(2)×sp(2) subalgebra has been proved in the general case. Therefore the above conclusion is valid for particles with arbitrary masses, although, strictly speaking, one has to prove the compatibility with the full so(2,3) algebra.

7 Discussion

In the present paper we have considered massless IRs in quantum theory based on a Galois field (GFQT). One of the crucial differences between the GFQT and the standard theory is that in the GFQT a particle and its antiparticle represent different states of the same object. As a consequence, the annihilation and creation operators for a particle and its antiparticle can be directly expressed in terms of each other. This imposes additional restrictions on the structure of the theory. In particular, Eq. (35) defines a new symmetry which has no analog in the standard theory. We have shown in Sect. 5 that this is indeed a symmetry in the massless case since the representation operators have the same form in terms of annihilation and creation operators for particles and antiparticles. As a consequence, as shown in Sect. 6, the existence of massless neutral elementary particles in the GFQT is incompatible with the normal relation between spin and statistics. It has been also argued that this conclusion is valid for particles with arbitrary masses.

We can reformulate the problem in the following way: is it natural that the requirement about the normal relation between spin and statistics excludes the existence of neutral elementary particles? Suppose for a moment that there is no restriction about the relation between spin and statistics. Then we cannot exclude the existence of neutral elementary particles in the GFQT, but such an existence seems to be rather unnatural. Indeed, since one modular IR simultaneously describes a particle and its antiparticle, the AdS energy operator necessarily contains the contribution of the both parts of the spectrum, corresponding to the particle and its antiparticle (see Eq. (49)). If a particle were the same as its antiparticle then Eq. (49) would contain two equal contributions and thus the value of the AdS energy would be twice as big as necessary.

The Pauli theorem about the relation between spin and statistics [15] has been proved in the framework of local quantum field theory. We believe that the results of the paper sheds a new light on the nature of the theorem. Although the theorem is already very powerful in the standard theory, it is probably even more powerful than it is usually believed. Our proof indicates that the Pauli theorem has a deep relation to the famous Wilson theorem in number theory (see Sect. 4-6).

Although the conclusion about the nonexistence of neutral elementary particles has been made for both bothons and fermions it is obvious that the case of bothons is of greater importance. A possibility that the photon is composite has been already discussed in the literature. For example, in Refs. [17, 18, 19] a model where the photon is composed of two Dirac singletons [20] has been investigated. However, in the framework of the standard theory, the compositeness of the photon is only a possible (and attractive) scenario while in the GFQT this is inevitable.

It is well known that the standard local quantum field theory (LQFT) has achieved very impressive success in comparing theory and experiment. In particular, quantum electrodynamics and the electroweak theory are based on the assumption that the photon is the elementary particle. For this reason one might doubt whether our conclusion has any relevance to physics. At the same time the LQFT has several well known drawbacks and inconsistencies. The majority of physicists believes that [14] the LQFT should be treated ′in the way it is′, but at the same time it is [14] a ′low energy approximation to a deeper theory that may not even be a field theory, but something different like a string theory′.

In Refs. [3, 4] we argued that the future quantum physics will be based on a Galois field. In that case the theory does not contain actual infinity, all operators are well defined, divergencies cannot exist in principle etc. We believe however that not only this makes the GFQT very attractive.

For centuries, scientists and philosophers have been trying to understand why mathematics is so successful in explaining physical phenomena (see e.g. Ref. [21]). However, such a branch of mathematics as number theory and, in particular, Galois fields, have practically no implications in physics. Historically, every new physical theory usually involved more complicated mathematics. The standard mathematical tools in modern quantum theory are differential and integral equations, distributions, analytical functions, representations of Lie algebras in Hilbert spaces etc. At the same time, very impressive results of number theory about properties of natural numbers (e.g. the Wilson theorem) and even the notion of primes are not used at all! The reader can easily notice that the GFQT involves only arithmetic of Galois fields (which are even simpler than the set of natural numbers). The very possibility that the future quantum theory could be formulated in such a way, is of indubitable interest.

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