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The covariant form of the gauge anomaly on noncommutative $\mathbb{R}^{2n}$

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The covariant form of the non-Abelian gauge anomaly on noncommutative $\mathbb{R}^{2n}$ is computed for $U(N)$ groups. Its origin and properties are analyzed. Its connection with the consistent form of the gauge anomaly is established. We show along the way that bi-fundamental $U(N) \times U(M)$ chiral matter carries no mixed anomalies, and interpret this result as a consequence of the half-dipole structure which characterizes the charged non-commutative degrees of freedom.

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1 Introduction

Field theories of fermions with chiral couplings to gauge fields on commutative manifolds play a prominent role –at least up to a few Tev– in the description of Nature. Field theories over noncommutative space-time [1, 2] may turn out to be phenomenologically relevant at the Tev scale and above [3, 4, 5, 6, 7, 8, 9, 10, 11]. It is therefore a must to understand the properties of quantum field theories of fermions chirally coupled to gauge fields on noncommutative manifolds. See refs. [12, 13, 14, 15] for the mathematics of noncommutative manifolds.

The chief feature of quantum field theories of chiral fermions interacting with gauge fields is that they are liable to carry gauge anomalies –other types of anomalies such the conformal anomaly [16] will not be discussed here. It is a well established fact [17, 18, 19, 20, 21, 22, 23] that if space-time is commutative, a gauge anomaly comes in either of two guises, namely, its consistent form or its covariant form. The consistent form of the anomaly satisfies the Wess-Zumino consistency conditions [24], the covariant form does not. One can retrieve either form of the anomaly from the other by adding to the corresponding current a polynomial of the gauge fields and its derivatives. For noncommutative $\mathbb{R}^4$ the consistent form of the gauge anomaly has been obtained in a number of papers [25, 26, 27] –see also ref. [28, 29, 30] for general analysis of chiral anomalies on noncommutative spaces and ref. [31, 32] for explicit computations. As for its covariant form, there is as yet no thorough discussion of the gauge anomaly for noncommutative space-time –although some results have been issued in ref. [33]. The purpose of this paper is to remedy this situation. First, by using path integral techniques, we shall compute explicitly the form of the gauge anomaly on noncommutative $\mathbb{R}^{2n}$ for $U(N)$ groups. In so doing, we shall see that the covariant form of the gauge anomaly is associated with a given definition of the path integral. This definition being a $*$-deformation of the ordinary one as given in ref. [18, 22, 23]. Then, we shall show that the covariant form of the gauge anomaly can be turned into the consistent form of it, by adding to the covariant current a $*$-polynomial of the gauge field and the field strength; this $*$-polynomial been a $*$-deformation of the polynomial for the commutative $\mathbb{R}^{2n}$ case. Finally, we shall analyze the transformation properties, under gauge transformations of the gauge field, of both the both the consistent and covariant currents. We shall thus show that in the presence of the gauge anomaly the consistent current –that which can be obtained by functional differentiation of the effective action– cannot transform covariantly; whereas, the covariant current does transform covariantly and, hence, it cannot be the functional derivative of the effective action with respect to the gauge field. For commutative $\mathbb{R}^{2n}$, these properties of the currents were established in refs. [19, 20, 22, 23].

2 The covariant form of the gauge anomaly: fundamental matter

Let $\psi_{\rm R}^j$ denote a right handed fermion, $\psi_{\rm R}^j = {\rm P}_+\psi^j$, ${\rm P}_+ = (1+\gamma_{2n+1})/2$, carrying the fundamental representation of the group $U(N)$. The matrix γ_{2n+1} is given by $\gamma_{2n+1} = (-i)^n \prod_{\mu=1}^{2n} \gamma^\mu$, where the gamma matrices γ^μ, $\mu = 1,\cdots,2n$, are Hermitian matrices which satisfy $\{\gamma^\mu,\gamma^\nu\} = 2\delta^{\mu\nu}$. The physics of $\psi_{\rm R}^j$ interacting with a background $U(N)$ gauge field on noncommutative $\mathrm{I\!R}^{2n}$ is ruled by the classical action

$$
{\rm S} \;=\; \int d^{2n}x\;\; \bar{\psi}_i\, i\hat{D}({\rm A})^i{}_j\, \psi^j. \tag{1}
$$

The operator $i\hat{D}({\rm A})$, which acts on the Dirac spinor ψ^j as follows $i\hat{D}({\rm A})^i{}_j\,\psi^j = i(\partial\!\!\!/\psi^i + {\rm A}^i_{\mu\,j}\star\gamma^\mu{\rm P}_+\psi^j)$, is not an Hermitian operator, but it is an elliptic operator. The Dirac spinor ψ^j carries the fundamental representation of $U(N)$ and the complex matrix ${\rm A}^i_{\mu\,j}$, with $({\rm A}^i_{\mu\,j})^* = -{\rm A}^j_{\mu\,i}$, is the $U(N)$ gauge field. The indices i,j run from 1 to N. The previous action is invariant under the following chiral gauge transformations:

$$
\begin{aligned}
\big(\delta_\omega {\rm A}_\mu\big)^i{}_j &= -\partial_\mu\omega^i_{\;j} \,-\, {\rm A}^i_{\mu\,k}\star\omega^k_{\;j} \,+\, \omega^i_{\;k}\star{\rm A}^k_{\mu\,j},\\
(\delta_\omega\psi)^i = \omega^i_{\;j}\star{\rm P}_+\,\psi^j, &\quad (\delta_\omega\bar{\psi})_k = -\bar{\psi}_k\star\omega^k_{\;i}{\rm P}_-,
\end{aligned} \tag{2}
$$

where ${\rm P}_- = (1-\gamma_{2n+1})/2$. The complex functions $\omega^i_{\;j} = -\omega^{*\,j}{}_i$, $i,j = 1,\cdots,N$, are the infinitesimal gauge transformation parameters. The symbol $\star$ denotes the Moyal product of functions on $\mathrm{I\!R}^{2n}$. The Moyal product is given by $(f\star g)(x) = \int\frac{d^{2n}p}{(2\pi)^{2n}}\int\frac{d^{2n}q}{(2\pi)^{2n}}\;\; e^{i(p+q)_\mu x^\mu}\;\; e^{-\frac{i}{2}\theta^{\mu\nu}p_\mu q_\nu}\;\;\hat{f}(p)\hat{g}(q)$. Here, $\theta^{\mu\nu}$ is an anti-symmetric real matrix either of magnetic type or light-like type. It is for these choices of matrix θ that a unitary theory exists at the quantum level [34, 35, 36, 37].

To define the partition function,

$$
{\rm Z[A]} \;\equiv\; \int d\bar{\psi}d\,\psi \quad e^{-{\rm S[A]}}, \tag{3}
$$

of the quantum theory with classical action given in eq. (1), we shall follow Fujikawa [18] and use the the set of eigenvalues and the set of eigenfunctions of the Hermitian operators

$\left(i\hat{D}(\mathrm{A})\right)^{\dagger} i\hat{D}(\mathrm{A})$ and $i\hat{D}(\mathrm{A})\left(i\hat{D}(\mathrm{A})\right)^{\dagger}$. These sets are defined by the following equations:

$$
\begin{aligned}
&\left(i\hat{D}(\mathrm{A})\right)^{\dagger} i\hat{D}(\mathrm{A})\varphi_m \;=\; \lambda_m^2\,\varphi_m, \quad i\hat{D}(\mathrm{A})\left(i\hat{D}(\mathrm{A})\right)^{\dagger}\phi_m \;=\; \lambda_m^2\,\phi_m,\\
&\phi_m \;=\; \frac{1}{\lambda_m}\, i\hat{D}(\mathrm{A})\varphi_m, \ \text{ if } \lambda_m \neq 0, \quad \text{and } \ i\hat{D}(\mathrm{A})\varphi_m \;=\; 0, \ \text{ if } \lambda_m = 0,\\
&\varphi_m \;=\; \frac{1}{\lambda_m}\left(i\hat{D}(\mathrm{A})\right)^{\dagger}\phi_m, \ \text{ if } \lambda_m \neq 0, \quad \text{and } \ \left(i\hat{D}(\mathrm{A})\right)^{\dagger}\phi_m \;=\; 0, \ \text{ if } \lambda_m = 0,\\
&\int d^{2n}x\ \varphi_m^{\dagger}(x)\,\varphi_{m'}(x) \;=\; \delta_{mm'}, \quad \int d^{2n}x\ \phi_m^{\dagger}(x)\,\phi_{m'}(x) \;=\; \delta_{mm'}.
\end{aligned}
\tag{4}
$$

We take without loss of generality $\lambda_m \geq 0$. We next define the fermionic measure as follows

$$
d\bar{\psi} d\,\psi \quad = \prod_m d\bar{b}_m da_m. \tag{5}
$$

Here, a_m and $\bar{b}_m$ are Grassmann variables defined by the expansions $\psi = \sum_m a_m\varphi_m$ and $\bar{\psi} = \sum_m \bar{b}_m\phi_m^{\dagger}$. Notice that this definition of $d\bar{\psi}d\,\psi$ is the obvious generalization to the noncommutative framework of the definition in refs. [22, 23]. Then,

$$
\int d\bar{\psi} d\,\psi \quad e^{-\mathrm{S}[\mathrm{A},\psi,\bar{\psi}]} \equiv \int \prod_m d\bar{b}_m\, da_m\; e^{-\sum_m \ \lambda_m\,\bar{b}_m\,a_m};
$$

which after Grassmann integration leads to $\mathrm{Z}[\mathrm{A}] \equiv \prod_m \lambda_m$. Notice that we have taken into account eq. (3). Hence, we have formally defined the partition function of the theory, $\mathrm{Z}[\mathrm{A}]$, as the determinant of the square root of the operator $\left(i\hat{D}(\mathrm{A})\right)^{\dagger} i\hat{D}(\mathrm{A})$.

Now, it is not difficult to show that if $g = 1+\omega$ is an infinitesimal gauge transformation, one has $(1+\omega\mathrm{P}_+)\left(i\hat{D}(\mathrm{A}^g)\right)^{\dagger} i\hat{D}(\mathrm{A}^g)(1-\omega\mathrm{P}_+) = \left(i\hat{D}(\mathrm{A})\right)^{\dagger} i\hat{D}(\mathrm{A}) + 0(\omega^2)$. Hence, $\lambda_m(\mathrm{A}^g) = \lambda_m(\mathrm{A}) + O(\omega^2)$, so that $\mathrm{Z}[\mathrm{A}]$, as defined above, is formally gauge invariant under infinitesimal gauge transformations. We can make this statement rigorous by using Pauli-Villars regularization or zeta function regularization. It is thus clear that for our definition of partition function the chiral gauge anomaly cannot be interpreted as the lack of invariance of $\mathrm{W}[\mathrm{A}]$ ($\mathrm{W}[\mathrm{A}] = -\ln \mathrm{Z}[\mathrm{A}]$) under infinitesimal gauge transformations. An interpretation which holds true for the consistent form of the anomaly [17, 19]. Let us show next that, as in the ordinary case [18], the covariant form of the anomaly comes from the lack of invariance under infinitesimal chiral gauge transformations of the fermionic measure defined above –see eq. (5). Let $\psi' = \psi + \delta_\omega\psi$ and $\bar{\psi}' = \bar{\psi} + \delta_\omega\bar{\psi}$, where δ_ω is given in eq. (2). Let $\{a'_m\}_m$ and $\{\bar{b}'_m\}_m$ be given by the expansions $\psi' = \sum_m a'_m\varphi_m$ and $\bar{\psi}' = \sum_m \bar{b}'_m\phi_m^{\dagger}$. Then, the identity

$$
\int d\bar{\psi} d\,\psi \quad e^{-\mathrm{S}[\mathrm{A},\psi,\bar{\psi}]} \equiv \int d\bar{\psi}' d\,\psi' \quad e^{-\mathrm{S}[\mathrm{A},\psi',\bar{\psi}']},
$$

leads to

$$\int d^{2n}x \; \omega^i{}_j(x) \left(D_\mu[\mathrm{A}]\, \mathcal{J}_\mu^{(cov)}\right)^j{}_i(x) \;=\; -\,\delta\, J \;\equiv\; \mathcal{A}[\omega, \mathrm{A}]^{(cov)}. \tag{6}$$

$\delta\, J$, which is defined by the equation $\prod_m\, d\bar{b}'_m\, da'_m - \prod_m\, d\bar{b}_m\, da_m = \delta\, J + O(\omega^2)$, is equal to

$$\int d^{2n}x\, \sum_m \{\phi_m^\dagger \star \omega \star P_- \phi_m - \varphi_m^\dagger \star \omega \star P_+ \varphi_m\}$$

and the current $\mathcal{J}_\mu^{(cov)}$ is defined by the identity

$$\left(\mathcal{J}_\mu^{(cov)}\right)^j{}_i(x) = -i\langle\left(\psi^i_\beta \star \bar{\psi}_{\alpha\, j}\right)(x)\left(\gamma_\mu \mathrm{P}_+\right)_{\alpha\beta}\rangle.$$

$\langle\cdots\rangle$ denotes the vacuum expectation value as given by

$$\langle\cdots\rangle \;=\; \frac{\int d\bar{\psi} d\,\psi \quad \langle\cdots\rangle \quad e^{-\mathrm{S[A]}}}{\int d\bar{\psi} d\,\psi \quad e^{-\mathrm{S[A]}}},$$

with the fermionic measure $d\bar{\psi} d\,\psi$ as defined by eq. (5).

As it stands, the right hand side of eq. (6) is ill-defined; we shall obtain a well-defined object out of it by using the Gaussian cut-off furnished by the eigenvalues λ_m^2 in eq. (4). This well-defined object, which we shall denote by $\mathcal{A}[\omega, \mathrm{A}]^{(cov)}$, is the covariant form of the gauge anomaly:

$$\begin{aligned}
\mathcal{A}[\omega, \mathrm{A}]^{(cov)} &= \lim_{\Lambda\to\infty}\; \int d^{2n}x\, \sum_m\; e^{-\frac{\lambda_m^2}{\Lambda^2}} \Big\{\varphi_m^\dagger \star \omega \star \mathrm{P}_+ \varphi_m - \phi_m^\dagger \star \omega \star \mathrm{P}_- \phi_m\Big\} \\
&= \lim_{\Lambda\to\infty}\; \int d^{2n}x\, \sum_m\; \omega \star \Big\{\big(\mathrm{P}_+ e^{-\frac{\lambda_m^2}{\Lambda^2}} \varphi_m\big) \star \varphi_m^\dagger - \big(\mathrm{P}_- e^{-\frac{\lambda_m^2}{\Lambda^2}} \phi_m\big) \star \phi_m^\dagger\Big\} \\
&= \lim_{\Lambda\to\infty}\; \int d^{2n}x\, \sum_m\; \omega \star \Big\{\big(\mathrm{P}_+ e^{-\frac{(i\hat{D}(\mathrm{A}))^\dagger i\hat{D}(\mathrm{A})}{\Lambda^2}} \varphi_m\big) \star \varphi_m^\dagger - \big(\mathrm{P}_- e^{-\frac{i\hat{D}(\mathrm{A})(i\hat{D}(\mathrm{A}))^\dagger}{\Lambda^2}} \phi_m\big) \star \phi_m^\dagger\Big\} \\
&= \lim_{\Lambda\to\infty}\; \int d^{2n}x\, \mathrm{Tr}\; \omega \star \int \frac{d^{2n}p}{(2\pi)^{2n}}\;\; \mathrm{tr}\; \Big\{\Big(\gamma_{2n+1}\, e^{-\frac{(i\not{D}(\mathrm{A}))^2}{\Lambda^2}}\; e^{ipx}\Big) \star e^{-ipx}\Big\}.
\end{aligned}$$

The last line of the previous equation is obtained by changing to a plane-wave basis. In this last line $i\not{D}(\mathrm{A}) = i(\not{\partial} + \not{\mathrm{A}}\star)$ denotes the Dirac operator, and Tr and tr stand for traces over the $U(N)$ and Dirac matrices, respectively.

Let $\mathrm{D(A)}_\mu = \partial_\mu + \mathrm{A}_\mu\star$. Taking into account that $(i\not{D}(\mathrm{A}))^2 = -(\mathrm{D(A)}^\mu \mathrm{D(A)}_\mu + \frac{1}{2}\gamma^\mu\gamma^\nu \mathrm{F}_{\mu\nu})$ and that $\mathrm{D(A)}^\mu \mathrm{D(A)}_\mu (f \star e^{ipx}) = ((ip_\mu + \partial_\mu + \mathrm{A}_\mu\star)^2\, f) \star e^{ipx}$, and that the Moyal product is associative, one can show that

$$\lim_{\Lambda\to\infty}\;\int d^{2n}x\,\mathrm{Tr}\;\omega\star\int\frac{d^{2n}p}{(2\pi)^{2n}}\;\;\mathrm{tr}\;\Big\{\Big(\gamma_{2n+1}\,e^{-\frac{(i\not{D}(\mathrm{A}))^2}{\Lambda^2}}\,e^{ipx}\Big)\star e^{-ipx}\Big\}=$$

$$\int d^{2n}x\;\;\mathrm{Tr}\;\omega\star\sum_{m=0}^{\infty}\Big\{\lim_{\Lambda\to\infty}\frac{1}{m!\Lambda^{2m}}\int\frac{d^{2n}p}{(2\pi)^{2n}}\;\;e^{-\frac{p^2}{\Lambda^2}}$$

$$\mathrm{tr}\Big\{\gamma_{2n+1}\,\big[(\partial_\mu+\mathrm{A}_\mu\star)^2+2ip_\mu(\partial_\mu+\mathrm{A}_\mu\star)+\tfrac{1}{2}\gamma^\mu\gamma^\nu\mathrm{F}_{\mu\nu}\star\big]^m\,\mathrm{I\!I}\Big\}\star e^{ipx}\star e^{-ipx}\Big\}.$$

Putting it all together, we conclude that

$$\mathcal{A}[\omega,\mathrm{A}]^{(cov)}\;=\;\frac{i^n}{(4\pi)^n\,n!}\,\varepsilon^{\mu_1\cdots\mu_{2n}}\,\mathrm{Tr}\;\int d^{2n}x\;\omega\star\mathrm{F}_{\mu_1\mu_2}\star\cdots\star\mathrm{F}_{\mu_{2n-1}\mu_{2n}}(x).$$

It is clear that $\mathcal{A}[\omega,\mathrm{A}]^{(cov)}$ does not satisfy the Wess-Zumino consistency conditions

$$\delta_{\omega_1}\mathcal{A}(\omega_2,\mathrm{A})\;-\;\delta_{\omega_2}\mathcal{A}(\omega_1,\mathrm{A})\;=\;\mathcal{A}([\omega_1,\omega_2],\mathrm{A}).$$

Hence, $\mathcal{J}_\mu^{(con)}(x)$ cannot be expressed as the derivative of effective action $\mathrm{W}[\mathrm{A}]\;=\;-\ln\mathrm{Z}[\mathrm{A}]$ with respect to the gauge field.

3 The covariant form of the gauge anomaly: bi-fundamental and adjoint chiral matter

We shall consider a bi-fundamental [38] chiral fermion $\psi^i_{R\,j}\;=\;\mathrm{P}_+\psi^i_{\;j}$, $i\;=\;1,\cdots,N$ and $j=1,\cdots,M$ coupled to a $U(N)$ gauge field, say, A_μ, and a $U(M)$ gauge field, say, B_μ. The classical action of this theory reads

$$\mathrm{S}\;=\;\int d^{2n}x\;\;\bar{\psi}^j_{\;i}\star(i\hat{D}(\mathrm{A},\mathrm{B})\psi)^i_{\;j}. \tag{7}$$

The elliptic operator $i\hat{D}(\mathrm{A},\mathrm{B})$ acts on the bi-fundamental Dirac spinor $\psi^i_{\;j}$ as follows

$$(i\hat{D}(\mathrm{A},\mathrm{B})\psi)^{i_1}_{\;j_1}\;=\;i\big(\not{\partial}\,\delta^{i_1}_{\;i_2}\delta^{\;j_2}_{j_1}+\mathrm{A}^{i_1}_{\mu\,i_2}\star\,\delta^{\;j_2}_{j_1}\gamma^\mu\mathrm{P}_+-\delta^{i_1}_{\;i_2}\star\mathrm{B}^{j_2}_{\mu\,j_1}\gamma^\mu\mathrm{P}_+\big)\psi^{i_2}_{\;j_2}.$$

We are using the following notation with regard to the $\star$-product: $\mathrm{A}_\mu\star\big)\psi\equiv\mathrm{A}_\mu\star\psi$ and $\star\mathrm{B}_\mu\big)\psi\equiv\psi\star\mathrm{B}_\mu$. Throughout this section, the i-indices run from 1 to N and the j-indices run from 1 to M. A_μ and B_μ are anti-Hermitian matrices.

The action in eq. (7) is invariant under the following infinitesimal gauge transformations

$$
\begin{aligned}
(\delta_{(\omega,\chi)}\psi)^{i_1}_{\;j_1} &= \Big(\omega^{i_1}_{\;i_2}\star \mathrm{P}_+\psi^{i_2}_{\;j_1} - \mathrm{P}_+\psi^{i_1}_{\;j_2}\star\chi^{j_2}_{\;j_1}\Big),\\
(\delta_{(\omega,\chi)}\bar\psi)^{j_1}_{\;i_1} &= -\Big(\bar\psi^{j_1}_{\;i_2}\star\omega^{i_2}_{\;i_1}\mathrm{P}_- - \chi^{j_1}_{\;j_2}\star\bar\psi^{j_2}_{\;j_1}\mathrm{P}_-\Big),\\
\big(\delta_\omega \mathrm{A}_\mu\big)^{i_1}_{\;i_2} &= -\partial_\mu\omega^{i_1}_{\;i_2} - \mathrm{A}^{i_1}_{\mu\, i_3}\star\omega^{i_3}_{\;i_2} + \omega^{i_1}_{\;i_3}\star\mathrm{A}^{i_3}_{\mu\, i_2},\\
\big(\delta_\chi \mathrm{B}_\mu\big)^{j_1}_{\;j_2} &= -\partial_\mu\chi^{j_1}_{\;j_2} - \mathrm{B}^{j_1}_{\mu\, j_3}\star\chi^{j_3}_{\;j_2} + \chi^{j_1}_{\;j_3}\star\mathrm{B}^{j_3}_{\mu\, j_2},
\end{aligned}
\tag{8}
$$

where $\omega^{i_1}_{\;i_2} = -\omega^{*\,i_2}_{\;\;\;i_1}$, $i_1,i_2 = 1,\cdots,N$, and $\chi^{j_1}_{\;j_2} = -\chi^{*\,j_2}_{\;\;\;j_1}$, $j_1,j_2 = 1,\cdots,M$, are the infinitesimal gauge transformation parameters.

Following the strategy developed in the previous section, we obtain

$$
\mathrm{Z}[\mathrm{A},\mathrm{B}] \;\equiv\; \int d\bar\psi d\,\psi \quad e^{-\mathrm{S}[\mathrm{A},\mathrm{B}]} \;=\; \int \prod_m d\bar b_m da_m\; e^{-\sum_n\lambda_m\bar b_m a_m} \;=\; \prod_m \lambda_m[\mathrm{A},\mathrm{B}].
$$

The Grassmann variables, a_m and $\bar b_m$, are given now by the expansions $\psi = \sum_m a_m\varphi_m$ and $\bar\psi = \sum_m \bar b_m\phi^\dagger_m$; φ_m and ϕ_m being the eigenvectors solving the eigenvalue problems and satisfying the identities that one gets by replacing in eq. (4) $i\hat D(\mathrm{A})$ with $i\hat D(\mathrm{A},\mathrm{B})$. $\lambda^2_m[\mathrm{A},\mathrm{B}]$ are the eigenvalues of the problems so obtained. These eigenvalues are invariant under the infinitesimal gauge transformations of eq. (8). Hence, the zeta regularization version of $\mathrm{Z}[\mathrm{A},\mathrm{B}]$ above is gauge invariant.

Proceeding as in the previous section, one obtains the covariant form the gauge anomaly equation for bi-fundamental chiral matter:

$$
\int d^{2n}x\,\Big[\omega^{i_1}_{\;i_2}\star\,\big(D_\mu[\mathrm{A}]\,\mathcal{J}^{(\mathrm{A},\,cov)}_\mu\big)^{i_2}_{\;\;i_1} + \chi^{j_1}_{\;j_2}\star\,\big(D_\mu[\mathrm{B}]\,\mathcal{J}^{(\mathrm{B},\,cov)}_\mu\big)^{j_2}_{\;\;j_1} \;=\; \mathcal{A}[\omega,\chi;\mathrm{A},\mathrm{B}]^{(cov)}. \tag{9}
$$

Here the currents $\mathcal{J}^{(\mathrm{A},\,cov)}_\mu$ and $\mathcal{J}^{(\mathrm{B},\,cov)}_\mu$ are defined, respectively, by the identities

$$
\begin{aligned}
\big(\mathcal{J}^{(\mathrm{A},\,cov)}_\mu\big)^{i_1}_{\;\;i_2}(x) &= -i\langle\big(\psi^{i_1}_{\beta\, j_1}\star\bar\psi^{j_1}_{\alpha\, i_2}\big)(x)\big(\gamma_\mu\mathrm{P}_+\big)_{\alpha\beta}\rangle,\\
\big(\mathcal{J}^{(\mathrm{B},\,cov)}_\mu\big)^{j_1}_{\;\;j_2}(x) &= -i\langle\big(\bar\psi^{j_1}_{\alpha\, i_1}\star\psi^{i_1}_{\beta\, j_2}\big)(x)\big(\gamma_\mu\mathrm{P}_+\big)_{\alpha\beta}\rangle;
\end{aligned}
\tag{10}
$$

and $\mathcal{A}[\omega,\chi;\mathrm{A},\mathrm{B}]^{(cov)}$ is given by

$$
\begin{aligned}
\mathcal{A}[\omega,\chi;\mathrm{A},\mathrm{B}]^{(cov)} = \lim\nolimits_{\Lambda\to\infty}\,\int d^{2n}x\,\sum\nolimits_m\; e^{-\frac{\lambda^2_m[\mathrm{A},\mathrm{B}]}{\Lambda^2}}\Big\{\varphi^\dagger_m\star\omega\star\mathrm{P}_+\varphi_m - \phi^\dagger_m\star\omega\star\mathrm{P}_-\phi_m\Big\} -\\
\lim\nolimits_{\Lambda\to\infty}\,\int d^{2n}x\,\sum\nolimits_m\; e^{-\frac{\lambda^2_m[\mathrm{A},\mathrm{B}]}{\Lambda^2}}\Big\{\chi\star\varphi^\dagger_m\star\mathrm{P}_+\varphi_m - \chi\star\phi^\dagger_m\star\mathrm{P}_-\phi_m\Big\}.
\end{aligned}
\tag{11}
$$

Let $i\not{D}(\mathrm{A},\mathrm{B}) = i(1_{N\times N}\otimes 1_{M\times M}\,\not{\partial} + 1_{M\times M}\otimes \not{\mathrm{A}}\star + \star\not{\mathrm{B}}\otimes 1_{N\times N}\,)$ the Dirac operator acting on the bi-fundamental Dirac spinor $\psi^i_{\;j}$, where $1_{N\times N}$ and $1_{M\times M}$ denote the unit matrices. It can be shown that the right hand side of eq. (11) can be written as follows

$$\begin{aligned}
&\lim_{\Lambda\to\infty}\int d^{2n}x\;\mathrm{Tr}_{\mathbb{M}_{N\times N}}\,\mathrm{Tr}_{\mathbb{M}_{M\times M}}\;\omega\star\int\frac{d^{2n}p}{(2\pi)^{2n}}\;\mathrm{tr}\left\{\left(\gamma_{2n+1}\,e^{-\frac{(i\not{D}(\mathrm{A},\mathrm{B}))^2}{\Lambda^2}}\;e^{ipx}\right)\star e^{-ipx}\right\}-\\
&\lim_{\Lambda\to\infty}\int d^{2n}x\;\mathrm{Tr}_{\mathbb{M}_{N\times N}}\,\mathrm{Tr}_{\mathbb{M}_{M\times M}}\;\chi\star\int\frac{d^{2n}p}{(2\pi)^{2n}}\;\mathrm{tr}\left\{e^{-ipx}\star\left(\gamma_{2n+1}\,e^{-\frac{(i\not{D}(\mathrm{A},\mathrm{B}))^2}{\Lambda^2}}\;e^{ipx}\right)\right\}.
\end{aligned}\tag{12}$$

tr denotes the trace over the gamma matrices, and $\mathrm{Tr}_{\mathbb{M}_{N\times N}}$ and $\mathrm{Tr}_{\mathbb{M}_{N\times N}}$ stand for the trace over $N\times N$ and $M\times M$ complex matrices, respectively. It is not difficult to see that

$$\begin{aligned}
(\not{D}(\mathrm{A},\mathrm{B}))^2 &= (1_{N\times N}\otimes\, 1_{M\times M}\,\partial_\mu\, +\, 1_{M\times M}\otimes\;\mathrm{A}_\mu\star\, -\, \star\,\mathrm{B}_\mu\otimes 1_{N\times N})^2+\\
&\tfrac{1}{2}\gamma^\mu\gamma^\nu\,(1_{M\times M}\otimes F_{\mu\nu}[\mathrm{A}]\star\, -\, \star\, F_{\mu\nu}[\mathrm{B}]\otimes 1_{N\times N}).
\end{aligned}$$

In the previous equation $\mathrm{A}_\mu\star$, $\star\mathrm{B}_\mu$, $F_{\mu\nu}[\mathrm{A}]\star$ and $\star F_{\mu\nu}[\mathrm{B}]$ are to be understood as operators acting on a appropriate matrix valued functions f and g as follows: $\mathrm{A}_\mu\star)f = \mathrm{A}_\mu\star f$, $\star\mathrm{B}_\mu)g = g\star\mathrm{B}_\mu$, $F_{\mu\nu}[\mathrm{A}]\star)f = F_{\mu\nu}[\mathrm{A}]\star f$, $\star F_{\mu\nu}[\mathrm{B}])g = g\star F_{\mu\nu}[\mathrm{B}]$. f takes values on $N\times N$ complex matrices and g takes values on the $M\times M$ complex matrices. $F_{\mu\nu}[\mathrm{A}]$ and $F_{\mu\nu}[\mathrm{B}]$ are the field strengths of A and B, respectively. $1_{N\times N}$ and $1_{M\times M}$ are, respectively, the identity matrices of rank N and M. Now, taking into account that $e^{ipx}\star f(x)\star e^{-ipx} = f(x+\theta p)$, with $(\theta p)^\mu = \theta^{\mu\nu}p_\nu$, it is not difficult to show that eq. (12) can be cast into the form

$$\begin{aligned}
&\int d^{2n}x\;\mathrm{Tr}_{\mathbb{M}_{N\times N}}\mathrm{Tr}_{\mathbb{M}_{M\times M}}\;\omega\star\sum_{m=0}^{\infty}\Bigg\{\lim_{\Lambda\to\infty}\frac{1}{m!\Lambda^{2m}}\;\int\frac{d^{2n}p}{(2\pi)^{2n}}\;\;e^{-\frac{p^2}{\Lambda^2}}\\
&\mathrm{tr}\Big\{\gamma_{2n+1}\,\Big[\big[1_{N\times N}\otimes 1_{M\times M}\,\partial_\mu + 1_{M\times M}\otimes A_\mu(x)\star - \star\,\mathrm{B}_\mu(x+\theta p)\;\otimes 1_{N\times N}\big]^2+\\
&\qquad 2ip_\mu\big[1_{N\times N}\otimes 1_{M\times M}\,\partial_\mu + 1_{M\times M}\otimes\mathrm{A}_\mu(x)\star - \star\,\mathrm{B}_\mu(x+\theta p)\;\otimes 1_{N\times N}\big]+\\
&\qquad \tfrac{1}{2}\gamma^\mu\gamma^\nu\big[1_{M\times M}\otimes\mathrm{F}_{\mu\nu}[\mathrm{A}](x)\star - \star\,\mathrm{F}_{\mu\nu}[\mathrm{B}](x+\theta p)\;\otimes 1_{N\times N}\big]\Big]^m\,\mathbb{I}\Big\}\star e^{ipx}\star e^{-ipx}\Bigg\}-\\
&\int d^{2n}x\;\mathrm{Tr}_{\mathbb{M}_{N\times N}}\mathrm{Tr}_{\mathbb{M}_{M\times M}}\;\chi\star\sum_{m=0}^{\infty}\Bigg\{\lim_{\Lambda\to\infty}\frac{1}{m!\Lambda^{2m}}\;\int\frac{d^{2n}p}{(2\pi)^{2n}}\;\;e^{-\frac{p^2}{\Lambda^2}}\\
&\mathrm{tr}\Big\{\gamma_{2n+1}\,e^{-ipx}\star e^{ipx}\star\Big[\big[1_{N\times N}\otimes 1_{M\times M}\,\partial_\mu + 1_{M\times M}\otimes A_\mu(x-\theta p)\star - \star\,\mathrm{B}_\mu(x)\;\otimes 1_{N\times N}\big]^2+\\
&\qquad 2ip_\mu\big[1_{N\times N}\otimes 1_{M\times M}\,\partial_\mu + 1_{M\times M}\otimes\mathrm{A}_\mu(x-\theta p)\star - \star\,\mathrm{B}_\mu(x)\;\otimes 1_{N\times N}\big]+\\
&\qquad \tfrac{1}{2}\gamma^\mu\gamma^\nu\big[1_{M\times M}\otimes\mathrm{F}_{\mu\nu}[\mathrm{A}](x-\theta p)\star - \star\,\mathrm{F}_{\mu\nu}[\mathrm{B}](x)\;\otimes 1_{N\times N}\big]\Big]^m\,\mathbb{I}\Big\}\Bigg\}.
\end{aligned}\tag{13}$$

$\mathbb{I}$ is the symbol for the unit function on $\mathbb{R}^{2n}$. Let us now recall that $\mathrm{tr}\,(\gamma_{2n+1}\gamma^{\mu_1}\cdots\gamma^{\mu_k})=0$, if $k<2n$; and that for a given value of m the limit $\lim_{\Lambda\to\infty}$ above vanishes, if the number of powers of Λ turns negative upon rescaling p to Λp. Keeping these two results in mind, one can show that the expression in the previous equation is equal to

$$\begin{aligned}
&\int d^{2n}x\ \mathrm{Tr}_{\mathbb{M}_{N\times N}}\mathrm{Tr}_{\mathbb{M}_{M\times M}}\ \omega\star\Big\{\lim_{\Lambda\to\infty}\tfrac{i^n}{n!}\int\tfrac{d^{2n}p}{(2\pi)^{2n}}\ e^{-p^2}\varepsilon^{\mu_1\cdots\mu_{2n}}\\
&\big[\mathrm{F}_{\mu_1\mu_2}[\mathrm{A}](x)\star-\star\,\mathrm{F}_{\mu_1\mu_2}[\mathrm{B}](x+\Lambda\,\theta p)\big]\cdots\big[\mathrm{F}_{\mu_{2n-1}\mu_{2n}}[\mathrm{A}](x)\star-\star\,\mathrm{F}_{\mu_{2n-1}\mu_{2n}}[\mathrm{B}](x+\Lambda\,\theta p)\big]\,\mathbb{I}\Big\}-\\
&\int d^{2n}x\ \mathrm{Tr}_{\mathbb{M}_{N\times N}}\mathrm{Tr}_{\mathbb{M}_{M\times M}}\ \chi\star\Big\{\lim_{\Lambda\to\infty}\tfrac{i^n}{n!}\int\tfrac{d^{2n}p}{(2\pi)^{2n}}\ e^{-p^2}\varepsilon^{\mu_1\cdots\mu_{2n}}\\
&\big[\mathrm{F}_{\mu_1\mu_2}[\mathrm{A}](x-\Lambda\,\theta p)\star-\star\,\mathrm{F}_{\mu_1\mu_2}[\mathrm{B}](x)\big]\cdots\big[\mathrm{F}_{\mu_{2n-1}\mu_{2n}}[\mathrm{A}](x-\Lambda\,\theta p)\star-\star\,\mathrm{F}_{\mu_{2n-1}\mu_{2n}}[\mathrm{B}](x)\big]\,\mathbb{I}\Big\}.
\end{aligned}\tag{14}$$

Generally speaking, in noncommutative quantum field theory, the limits $\Lambda\to\infty$ and $\theta p\to 0$ do not commute as a consequence of the intriguing UV/IR mixing [39]. Then, to define the renormalized theory, one has make a choice regarding the order of these limits. One would like to obtain the renormalized noncommutative theory at $\theta p=0$ by taking the limit $\theta p\to 0$ of renormalized one at $\theta p\neq 0$. Hence, we shall take the limit $\Lambda\to\infty$ first and then take the limit $\theta p\to 0$. Now, the gauge fields A_μ and B_μ satisfy the boundary conditions $\mathrm{F}_{\mu\nu}[\mathrm{A}](y)\to 0$ and $\mathrm{F}_{\mu\nu}[\mathrm{B}](y)\to 0$ as $|y|\to\infty$. It is thus plain that (14) is equal to

$$\begin{aligned}
&M\,\tfrac{i^n}{(4\pi)^n\,n!}\,\varepsilon^{\mu_1\cdots\mu_{2n}}\,\mathrm{Tr}_{\mathbb{M}_{N\times N}}\int d^{2n}x\ \omega\star\mathrm{F}_{\mu_1\mu_2}[\mathrm{A}]\star\cdots\star\mathrm{F}_{\mu_{2n-1}\mu_{2n}}[\mathrm{A}](x)-\\
&N\,\tfrac{(-i)^n}{(4\pi)^n\,n!}\,\varepsilon^{\mu_1\cdots\mu_{2n}}\,\mathrm{Tr}_{\mathbb{M}_{M\times M}}\int d^{2n}x\ \omega\star\mathrm{F}_{\mu_1\mu_2}[\mathrm{B}]\star\cdots\star\mathrm{F}_{\mu_{2n-1}\mu_{2n}}[\mathrm{B}](x).
\end{aligned}\tag{15}$$

Putting it all together (see (9)–(15)), we conclude that the covariant form of the anomaly $\mathcal{A}[\omega,\chi;\mathrm{A},\mathrm{B}]^{(cov)}$ reads thus

$$\begin{aligned}
\mathcal{A}[\omega,\chi;\mathrm{A},\mathrm{B}]^{(cov)}\ =\ &M\,\tfrac{i^n}{(4\pi)^n\,n!}\,\varepsilon^{\mu_1\cdots\mu_{2n}}\,\mathrm{Tr}_{\mathbb{M}_{N\times N}}\int d^{2n}x\ \omega\star\mathrm{F}_{\mu_1\mu_2}[\mathrm{A}]\star\cdots\star\mathrm{F}_{\mu_{2n-1}\mu_{2n}}[\mathrm{A}](x)-\\
&N\,\tfrac{(-i)^n}{(4\pi)^n\,n!}\,\varepsilon^{\mu_1\cdots\mu_{2n}}\,\mathrm{Tr}_{\mathbb{M}_{M\times M}}\int d^{2n}x\ \omega\star\mathrm{F}_{\mu_1\mu_2}[\mathrm{B}]\star\cdots\star\mathrm{F}_{\mu_{2n-1}\mu_{2n}}[\mathrm{B}](x).
\end{aligned}\tag{16}$$

Notice that as in the consistent case [40, 41] there are no mixed anomalies. Also notice that if in (14) we set $\theta=0$ before sending Λ to ∞, ie, we go to commutative space, the mixed anomalies pop-up back; and that it is the characteristic half-dipole structure of the charged degrees of freedom of the noncommutative field theories –see the $\mathrm{F}_{\mu_i\mu_{i+1}}[\mathrm{A}](x-\Lambda\,\theta p)$ and

$\mathrm{F}_{\mu_i\mu_{i+1}}[\mathrm{B}](x+\Lambda\,\theta p)$ terms in the mixed contributions of (14)– which is responsible for the lack of mixed anomalies in the noncommutative arena. That charged degrees of freedom have a half-dipole structure rather than a dipole structure [42, 43] was unveiled in ref. [44].

The consistent form of the gauge anomaly for an adjoint right-handed fermion can be obtained by setting $\mathrm{A}=\mathrm{B}$ in eq. (16). We thus conclude that if $D=4m$ (D is the space-time dimension) there is no gauge anomaly, but if $D=4m+2$, the anomaly is $2N$ times the anomaly in the fundamental representation.

4 Redefinition of currents

In this section we shall show that there exists a $*$-polynomial, $\mathcal{X}^\mu$, of A and F, such that

$$\mathcal{J}_\mu^{(con)}(x)=\mathcal{J}_\mu^{(cov)}(x)\,+\,\mathcal{X}_\mu(x). \tag{17}$$

Here $\mathcal{J}_\mu^{(con)}(x)$ denotes the consistent gauge current for a fundamental right handed fermion –see ref. [25, 26, 27]– and $\mathcal{J}_\mu^{(cov)}(x)$ stands for the corresponding covariant gauge current. In view of the results presented in the previous section, the generalization of the analysis we are about to begin to bi-fundamental and/or adjoint right handed fermions is trivial.

To compute $\mathcal{X}^\mu$ we shall adapt to the case at hand the techniques of ref. [20]. To do so we shall employ the formalism of differential forms and BRST cohomology introduced in ref. [26].

Let $\mathcal{J}^{(con)}$ and $\mathcal{J}^{(cov)}$ be the dual currents

$$\begin{aligned}\mathcal{J}^{(con)}&=\tfrac{1}{(2n-1)!}\,\varepsilon^{\mu_1}{}_{\mu_2\cdots\mu_{2n}}\,\mathcal{J}_{\mu_1}^{(con)}\,dx^{\mu_2}\cdots dx^{\mu_{2n}},\\ \mathcal{J}^{(cov)}&=\tfrac{1}{(2n-1)!}\,\varepsilon^{\mu_1}{}_{\mu_2\cdots\mu_{2n}}\,\mathcal{J}_{\mu_1}^{(cov)}\,dx^{\mu_2}\cdots dx^{\mu_{2n}}.\end{aligned} \tag{18}$$

These currents are $(2n-1)$-differential forms in the sense of ref. [26]. Let C be the ghost zero-form introduced through the BRST transformations: $sA=DC=d\mathrm{C}+[\mathrm{A},\mathrm{C}]$, $sc=\mathrm{C}\star\mathrm{C}$. s is the BRST operator, d is the exterior derivative and $\mathrm{A}=\mathrm{A}_\mu\,dx^\mu$. s and d satisfy $s^2=d^2=sd+ds=0$. We introduce next the two-form field-strength $\mathrm{F}=\frac{1}{2}\mathrm{F}_{\mu\nu}\,dx^\mu dx^\nu=d\mathrm{A}+\mathrm{A}\star\mathrm{A}$. Then, $\mathcal{J}^{(con)}$ and $\mathcal{J}^{(cov)}$ are defined so that they satisfy, respectively, the consistent form and the covariant form of gauge anomaly equation:

$$\int\mathrm{Tr}\;D\mathrm{C}\star\mathcal{J}^{(con)}\,=\,\mathcal{A}(\mathrm{C},\mathrm{A})^{(con)}\qquad\text{and}\qquad\int\mathrm{Tr}\;D\mathrm{C}\star\mathcal{J}^{(cov)}\,=\,\mathcal{A}(\mathrm{C},\mathrm{A})^{(cov)}, \tag{19}$$

where

$$\mathcal{A}(\mathrm{C},\mathrm{A})^{(con)} \;=\; \frac{i^n}{(2\pi)^n\,(n+1)!}\int \mathcal{Q}^1{}_{2n}\,(\mathrm{C},\mathrm{A},\mathrm{F}) \tag{20}$$

and

$$\mathcal{A}(\mathrm{C},\mathrm{A})^{(cov)} \;=\; \frac{i^n}{(2\pi)^n\,n!}\int [\![\mathrm{Tr}\;\mathrm{C}\star\mathrm{F}^n]\!]. \tag{21}$$

$\mathcal{Q}^1_{2n}\,(\mathrm{C},\mathrm{A},\mathrm{F})$, which can be obtained by solving the descent equations, reads

$$\mathcal{Q}^1{}_{2n}\,(\mathrm{C},\mathrm{A},\mathrm{F}) \;=\; (n+1)\int_0^1 dt\,(1-t)\sum_{k=0}^{n-1}[\![\mathrm{Tr}\;\mathrm{C}\star d(\mathrm{F}_t^k\star\mathrm{A}\star\mathrm{F}_t^{n-1-k})]\!], \tag{22}$$

with $\mathrm{F}_t \;=\; d\mathrm{A}_t+\mathrm{A}_t^2$ and $\mathrm{A}_t \;=\; t\mathrm{A}$. F^k denotes the k-th power of F with respect to the Moyal product. An expression like $[\![\mathrm{Tr}\;(\mathrm{E}_1\star\mathrm{E}_2\star\cdots\star\mathrm{E}_m)]\!]$ denotes the equivalence class obtained by imposing on the space of objects of the type $\mathrm{Tr}\,(\mathrm{E}_1\star\mathrm{E}_2\star\cdots\star\mathrm{E}_m)$ the relationship $\mathrm{Tr}\;(\mathrm{E}_1\star\mathrm{E}_2\star\cdots\star\mathrm{E}_m)\equiv(-1)^{k_m(k_1+\cdots+k_{m-1})}\mathrm{Tr}\;(\mathrm{E}_m\star\mathrm{E}_1\star\cdots\star\mathrm{E}_{m-1})$. E_i denotes a form of degree k_i. See ref. [26] for further details.

To find $\mathcal{X}$,

$$\mathcal{X}=\frac{1}{(2n-1)!}\,\varepsilon^{\mu_1}{}_{\mu_2\cdots\mu_{2n}}\,\mathcal{X}_{\mu_1}\,dx^{\mu_2}\cdots dx^{\mu_{2n}}, \tag{23}$$

such that $\mathcal{X}_\mu$ satisfies eq. (17), we shall first show that $\mathcal{Q}^1{}_{2n}\,(\mathrm{C},\mathrm{A},\mathrm{F})$ in eq. (22) is also given by the following equation

$$\begin{aligned}\mathcal{Q}^1{}_{2n}\,(\mathrm{C},\mathrm{A},\mathrm{F}) \;=\; &(n+1)[\![\mathrm{Tr}\;\mathrm{C}\star\mathrm{F}^n]\!]-\\ &(n+1)\textstyle\int_0^1 dt\,[\![\mathrm{Tr}\;\mathrm{C}\star D(\sum_{k=0}^{n-1}\,\mathrm{F}_t^k\star\mathrm{A}_t\star\mathrm{F}_t^{n-1-k})]\!].\end{aligned} \tag{24}$$

It can be shown that the right hand side of eq. (22) is equal to

$$(n+1)\int_0^1 dt\,[\![\mathrm{Tr}\;\mathrm{C}\star\mathrm{F}_t^n]\!]\;+\;(n+1)\int_0^1 dt\sum_{k=0}^{n-1}(t-1)\,[\![\mathrm{Tr}\;(\mathrm{F}_t^k\star\mathrm{A}_t\star\mathrm{F}_t^{n-1-k}\star[\mathrm{A},\mathrm{C}])]\!]. \tag{25}$$

Now, taking into account that $(D_t-D)\mathrm{C}\;=\;(t-1)[\mathrm{A},\mathrm{C}]$ and that $[\![\mathrm{Tr}D\mathcal{O}]\!]\;=\;d[\![\mathrm{Tr}\mathcal{O}]\!]$ for a form, $\mathcal{O}$, of even degree; one readily shows that eq. (25) can be turned into the following expression

$$\begin{aligned}&(n+1)\Big(\textstyle\int_0^1 dt\,[\![\mathrm{Tr}\;\mathrm{C}\star\mathrm{F}_t^n]\!]\;-\;d\int_0^1 dt\,\sum_{k=0}^{n-1}[\![\mathrm{Tr}\;(\mathrm{F}_t^k\star\mathrm{A}_t\star\mathrm{F}_t^{n-1-k}\star\mathrm{C})]\!]+\\ &\textstyle\int_0^1 dt\,\sum_{k=0}^{n-1}[\![\mathrm{Tr}\;(\mathrm{F}_t^k\star D_t\mathrm{A}_t\star\mathrm{F}_t^{n-1-k}\star\mathrm{C})]\!]\;-\;\int_0^1 dt\,\sum_{k=0}^{n-1}[\![\mathrm{Tr}\;(\mathrm{F}_t^k\star\mathrm{A}_t\star\mathrm{F}_t^{n-1-k}\star D\mathrm{C})]\!]\Big).\end{aligned}$$

Upon employing that $D_t\mathrm{A}_t = t\partial_t\mathrm{F}_t$ and that $\partial_t\mathrm{F}_t^n = \sum_{k=0}^{n-1}\mathrm{F}_t^k\star\partial_t\mathrm{F}_t\star\mathrm{F}_t^{n-1-k}$, the previous equation can be converted into the following one

$$(n+1)\Big(\int_0^1 dt\,[\![\mathrm{Tr}\;\mathrm{C}\star\mathrm{F}_t^n]\!] \;-\; d\int_0^1 dt\,\sum_{k=0}^{n-1}[\![\mathrm{Tr}\;(\mathrm{F}_t^k\star\mathrm{A}_t\star\mathrm{F}_t^{n-1-k}\star\mathrm{C})]\!]\,+$$
$$\int_0^1 dt\,t\,\partial_t\,[\![\mathrm{Tr}\;(\mathrm{F}_t^n\star\mathrm{C})]\!] \;-\; \int_0^1 dt\,\sum_{k=0}^{n-1}[\![\mathrm{Tr}\;(\mathrm{F}_t^k\star\mathrm{A}_t\star\mathrm{F}_t^{n-1-k}\star D\mathrm{C})]\!]\Big).$$

Partial integration yields then

$$(n+1)\Big([\![\mathrm{Tr}\;\mathrm{C}\star\mathrm{F}^n]\!] \;-\; \int_0^1 dt\,\sum_{k=0}^{n-1}[\![\mathrm{Tr}\;(\mathrm{F}_t^k\star\mathrm{A}_t\star\mathrm{F}_t^{n-1-k}\star D\mathrm{C})]\!]\,-$$
$$d\int_0^1 dt\,\sum_{k=0}^{n-1}[\![\mathrm{Tr}\;(\mathrm{F}_t^k\star\mathrm{A}_t\star\mathrm{F}_t^{n-1-k}\star\mathrm{C})]\!]\Big).$$

This equation and

$$d\,[\![\mathrm{Tr}\;(\mathrm{F}_t^k\star\mathrm{A}_t\star\mathrm{F}_t^{n-1-k}\star\mathrm{C})]\!] \;=\; [\![\mathrm{Tr}\;(D(\mathrm{F}_t^k\star\mathrm{A}_t\star\mathrm{F}_t^{n-1-k})\star\mathrm{C})]\!] - [\![\mathrm{Tr}\;(\mathrm{F}_t^k\star\mathrm{A}_t\star\mathrm{F}_t^{n-1-k}\star D\mathrm{C})]\!]$$

finally lead to eq. (24).

We are now ready to compute $\mathcal{X}$ so that eqs. (17) and (23) hold:

$$\begin{aligned}\int\mathrm{Tr}\;\mathrm{C}\star D\mathcal{X} \;&=\; \int\mathrm{Tr}\;\mathrm{C}\star D\mathcal{J}^{(con)} \;-\; \int\mathrm{Tr}\;\mathrm{C}\star D\mathcal{J}^{(cov)}\\ &=\; \tfrac{i^n}{(2\pi)^n\,(n+1)!}\int\Big(\mathcal{Q}^1{}_{2n}(\mathrm{C},\mathrm{A},\mathrm{F}) \;-\;(n+1)\,[\![\mathrm{Tr}\;\mathrm{C}\star\mathrm{F}^n]\!]\Big)\\ &=\; \tfrac{i^{(n+2)}}{(2\pi)^n\,n!}\int\int_0^1 dt\,[\![\mathrm{Tr}\;\mathrm{C}\star D(\textstyle\sum_{k=0}^{n-1}\mathrm{F}_t^k\star\mathrm{A}_t\star\mathrm{F}_t^{n-1-k})]\!].\end{aligned}\tag{26}$$

To obtain the previous array of identities, eqs. (19)– (24) are to be taken into account. In view of eq. (26), we conclude that the following choice of $\mathcal{X}$,

$$\mathcal{X} \;=\; \frac{i^{(n+2)}}{(2\pi)^n\,n!}\int_0^1 dt\,\sum_{k=0}^{n-1}\mathrm{F}_t^k\star\mathrm{A}_t\star\mathrm{F}_t^{n-1-k},\tag{27}$$

would do the job. Notice that the result we have obtained is the naive $\star$-deformation of the ordinary expression without symmetrization.

5 Currents and gauge transformations

In this section we shall study the behaviour under gauge transformations of the consistent and covariant dual currents $-\mathcal{J}^{(con)}$ and $\mathcal{J}^{(cov)}$ in eq. (18), respectively. We shall employ the techniques of ref. [20] and show that, when there is an anomaly, the following equation does not hold

$$s\,\mathcal{J}^{(con)}\;=\;[\mathrm{C},\mathcal{J}^{(con)}],$$

but the following equation does

$$s\,\mathcal{J}^{(cov)}\;=\;[\mathrm{C},\mathcal{J}^{(cov)}]. \tag{28}$$

The consistent current is obtained from the effective action $\mathrm{W[A]}$ by functional differentiation of the latter, ie,

$$\delta\,\mathrm{W[A]}\;=\;\int\,\mathrm{Tr}\,\delta\mathrm{A}\,\star\,\mathcal{J}^{(con)}.$$

The operator δ is given by $\int\,\delta\mathrm{A}\star\frac{\delta}{\delta\mathrm{A}}$. The BRST variation of infinitesimal one-form $\delta\mathrm{A}$ is defined to be $s\,\delta\mathrm{A}\;=\;[\delta\mathrm{A},\mathrm{C}]$. It can be readily seen that $s\delta=\delta s$. We next introduce the anti-derivation l as follows $l\mathrm{A}\;=\;0$, $l\mathrm{F}\;=\;\delta\mathrm{A}$. It can be shown that $l\,d\,+\,d\,l\;=\;\delta$, on the space of polynomials of A and F with respect to the Moyal product.

The consistent form of the gauge anomaly equation runs thus in terms of $\mathrm{W[A]}$:

$$s\mathrm{W[A]}\;=\;\frac{i^n}{(2\pi)^n\,(n+1)!}\int\,\mathcal{Q}^1{}_{2n}\,(\mathrm{C},\mathrm{A},\mathrm{F}).$$

$\mathcal{Q}^1{}_{2n}\,(\mathrm{C},\mathrm{A},\mathrm{F})$ is given in eq. (22). Acting with δ on both sides of the previous equation, one obtains

$$\begin{aligned}\tfrac{i^n}{(2\pi)^n\,(n+1)!}\int\,\delta\,\mathcal{Q}^1{}_{2n}\,(\mathrm{C},\mathrm{A},\mathrm{F})\;&=\;\delta\,s\mathrm{W[A]}\;=\;s\,\delta\mathrm{W[A]}\;=\;s\,\int\,\mathrm{Tr}\,\delta\mathrm{A}\,\star\,\mathcal{J}^{(con)}\\&=\;s\,\int\,\mathrm{Tr}\,\delta\mathrm{A}\,\star\,\mathcal{J}^{(con)}\;=\;-\int\,\mathrm{Tr}\,\delta\mathrm{A}\star\big\{s\mathcal{J}^{(con)}\;-\;[\mathrm{C},\mathcal{J}^{(con)}]\big\}.\end{aligned} \tag{29}$$

Hence, the existence of the gauge anomaly $-\mathcal{Q}^1{}_{2n}\,(\mathrm{C},\mathrm{A},\mathrm{F})$ does not vanish– prevents $\mathcal{J}^{(con)}$ from transforming covariantly. Notice that eq. (29) tell us that the behaviour of $\mathcal{J}^{(con)}$ under gauge transformations is given by the anomaly.

Let us finally show that the covariant current, $\mathcal{J}^{(cov)}$, obtained by subtracting $\mathcal{X}$ in eq. (27) from $\mathcal{J}^{(con)}$, deserves that name indeed, ie, it satisfies eq. (28). In view of eq. (29), we just have to show that

$$\int\,\mathrm{Tr}\,\delta\mathrm{A}\star\big\{\,-\,s\mathcal{X}\;+\;[\mathrm{C},\mathcal{X}]\big\}\;=\;\frac{i^n}{(2\pi)^n\,(n+1)!}\int\,\delta\,\mathcal{Q}^1{}_{2n},\,(\mathrm{C},\mathrm{A},\mathrm{F}).$$

with $\mathcal{X}$ given in eq. (23). Taking into account the (graded) cyclicity of $\int \mathrm{Tr}\, E_{k_1} \star \cdots E_{k_n}$, and that $t\delta \mathrm{A} = l\, \mathrm{F}_t$ and that $l\, \mathrm{F}_t^n = \sum_{k=0}^n \mathrm{F}_t^{n-1-k} \star l \mathrm{F}_t \star \mathrm{F}_t^k$, with $\mathrm{F}_t = d\mathrm{A}_t + \mathrm{A}_t^2$; one shows that

$$\int \mathrm{Tr}\, \delta\mathrm{A} \star \big\{ -s\mathcal{X} + [\mathrm{C}, \mathcal{X}] \big\} = s \int \mathrm{Tr}\, \delta\mathrm{A} \star \mathcal{X}$$
$$- \tfrac{i^n}{(2\pi)^n\,(n+1)!} \int s\, l \big\{ (n+1) \int_0^1 dt\ \mathrm{Tr} \mathrm{A} \star \mathrm{F}_t^n \big\}.$$

It is plain that

$$\int s\, l \big\{ (n+1) \int_0^1 dt\ \mathrm{Tr} \mathrm{A} \star \mathrm{F}_t^n \big\} = \int s\, l \big\{ (n+1) \int_0^1 dt\ [\![\mathrm{Tr} \mathrm{A} \star \mathrm{F}_t^n]\!] \big\}.$$

Now, the descent equation formalism of ref. [26] leads to $s\mathcal{Q}^0{}_{2n+1}(\mathrm{A}, \mathrm{F}) = d\mathcal{Q}^1{}_{2n}(\mathrm{C}, \mathrm{A}, \mathrm{F})$ where $\mathcal{Q}^0{}_{2n+1}(\mathrm{A}, \mathrm{F}) = (n+1) \int_0^1 dt\ [\![\mathrm{Tr}\mathrm{A} \star \mathrm{F}_t^n]\!]$ and $\mathcal{Q}^1{}_{2n}(\mathrm{C}, \mathrm{A}, \mathrm{F})$ is given in eq. (22). Putting it all together, we get

$$\int \mathrm{Tr}\, \delta\mathrm{A} \star \big\{ -s\mathcal{X} + [\mathrm{C}, \mathcal{X}] \big\} = -\tfrac{i^n}{(2\pi)^n\,(n+1)!} \int s\, l\, \mathcal{Q}^0{}_{2n+1}(\mathrm{A}, \mathrm{F}) =$$
$$\tfrac{i^n}{(2\pi)^n\,(n+1)!} \int l\, s\, \mathcal{Q}^0{}_{2n+1}(\mathrm{A}, \mathrm{F}) = \tfrac{i^n}{(2\pi)^n\,(n+1)!} \int l\, d\, \mathcal{Q}^1{}_{2n}(\mathrm{C}, \mathrm{A}, \mathrm{F}) =$$
$$\tfrac{i^n}{(2\pi)^n\,(n+1)!} \int (-dl + \delta)\, \mathcal{Q}^1{}_{2n}(\mathrm{C}, \mathrm{A}, \mathrm{F}) = \tfrac{i^n}{(2\pi)^n\,(n+1)!} \int \delta\, \mathcal{Q}^1{}_{2n}(\mathrm{C}, \mathrm{A}, \mathrm{F}).$$

6 Conclusions

In this paper we have considered the noncommutative gauge anomaly for $U(N)$ gauge groups. We have shown that the covariant form of gauge anomaly on noncommutative $\mathbb{R}^{2n}$ can be understood as the lack of invariance of the fermionic measure under chiral gauge transformations of the fermion fields. This lack of invariance is given by a non-trivial Jacobian, which when defined by using an appropriate regularization yields the covariant form of the anomaly. By using these path integral techniques, we have finally computed the convariant form of the gauge anomaly on $\mathbb{R}^{2n}$ to show that it is given by a $\star$-polynomial of the gauge field strength. The covariant form of the gauge anomaly on even dimensional space is thus seen to be given by an appropriate $\star$-deformation of the ordinary expression.

We have proved that one can trade the covariant form of the gauge anomaly for the consistent one by adding to the covariant current a $\star$-polynomial of the gauge field and the gauge field strength. We have computed this polynomial explicitly. Gauge anomalies are thus given by local expressions in the sense of noncommutative geometry –of course, these expressions are non-local from the ordinary quantum field theory point of view.

We have seen that the gauge transformation properties of the consistent current are given by the consistent form of the gauge the anomaly. The existence of the gauge anomaly prevents the consistent current from transforming covariantly, but allows the covariant current to transform covariantly under gauge transformations of the gauge field.

It is worth stressing that in the course of our path integral computations we have proved that on noncommutative $\mathbb{R}^{2n}$ the covariant form of the gauge anomaly for bi-fundamental chiral matter carries no mixed anomaly. For the noncommutative theory, the absence of these mixed anomalies is interpreted –see ref. [41] for an interpretation in terms of the Green-Schwarz mechanism– as a consequence of the half-dipole structure [44] which is characteristic of the noncommutative charged fields. From the covariant form of the gauge anomaly for bi-fundamental chiral matter, one readily obtains the covariant form of the anomaly form adjoint chiral fermions. If D denotes the espace dimension, our results -for adjoint chiral right-handed fermion in the continuum– run thus: there is no anomaly at $D=4m$; at $D=4m+2$, the anomaly is $2N$ times the fundamental anomaly. Interestingly enough one can formulate such theories on the lattice in an anomaly free manner for any even integer D [45].

It is an interesting task to try to extend the results presented here to groups other than the $U(N)$ groups. New techniques and ideas such us the ones introduced in refs. [46, 47] will be unavoidably needed, if one is to succeed.

Finally, as we were writing the closing sentences in this paper, we became aware of ref. [48]. The results discussed above are in complete harmony with the results presented in this last reference.

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