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Schwinger Pair Production via Instantons in Strong Electric Fields

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In the space-dependent gauge, each mode of the Klein-Gordon equation in a strong electric field takes the form of a time-independent Schrödinger equation with a potential barrier. We propose that the single- and multi-instantons of quantum tunnelling may be related with the single- and multi-pair production of bosons and the relative probability for the no-pair production is determined by the total tunnelling probability via instantons. In the case of a static uniform electric field, the instanton interpretation recovers exactly the well-known pair production rate for bosons and when the Pauli blocking is taken into account, it gives the correct fermion production rate. The instanton is used to calculate the pair-production rate even in an inhomogeneous electric field. Furthermore, the instanton interpretation confirms the fact that bosons and fermions cannot be produced by a static magnetic field only.

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I. INTRODUCTION

Strong electromagnetic fields lead to two physically important phenomena: the pair-production and vacuum polarization. A strong electric field makes the quantum electrodynamics (QED) vacuum unstable which decays by emitting significantly boson or fermion pairs [1–3]. The vacuum fluctuations of an external electromagnetic field also result in an effective action of the nonlinear Maxwell equations [2–4]. As its long history, there have been developed many different methods such as the proper time method [3, 5], canonical method [5], etc., to derive the QED effective action in external electromagnetic fields. Also there have been applications to various physical problems [6]. The proper time method by Schwinger [3] and DeWitt [5] has widely been employed to compute the effective action. The real part of the effective action leads to the vacuum polarization and the imaginary part to the pair-production. Though that method is conceptually well-defined and technically rigorous, it is sometimes difficult to apply the method to some concrete physical problems such as inhomogeneous electromagnetic fields and others. On the other hand, the canonical method [5] proves quite efficient in calculating the pair-production rate of bosons and fermions by static or time-dependent uniform electric fields in many physical contexts.

In canonical approach the most frequently used gauge for the electromagnetic potential is the time-dependent gauge. In that gauge the Klein-Gordon equation for bosons or the Dirac equation for fermions in a uniform electric field, when appropriately mode-decomposed, takes the form of time-dependent Schrödinger equations. Now the pair-production by the external electric field is analogous to the particle production by a time-dependent metric of a curved spacetime [7–9]. In both problems one imposes the same boundary condition that an incident, positive frequency component in the past infinity is scattered by a potential barrier into a superposition of positive and negative frequency components in the future infinity. It is the complex conjugate of the boundary condition for scattering problems in quantum mechanics. The coefficients determine the Bogoliubov transformation and, in particular, the coefficient of the negative frequency component gives the number created bosons or fermions per mode. The pair-production rates were calculated for time-varying electric fields [10–17]. Using both canonical and path integral methods, the pair-production in a uniform electric field was studied in the time-dependent gauge [18, 19] and in Rindler coordinates [20]. The pair-production was also studied for a uniform electric field confined to a finite region, an inhomogeneous field [21–23]

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A shortcoming of the time-dependent gauge is that except for uniform fields, the gauge potential and thereby the Klein-Gordon equation involve both the space and time coordinates at the same time. So it is technically difficult to apply the Bogoliubov transformation for inhomogeneous fields. On the other hand, in the space-dependent (Coulomb) gauge for a static electric field, each mode of the Klein-Gordon equation for bosons or the Dirac equation for fermions takes the form of a time-independent Schrödinger equation for quantum tunnelling through a potential barrier. In that space-dependent gauge there is no direct interpretation of wave components in terms of positive and negative frequencies. However, in the case of the static uniform electric field, Brezin and Itzykson explained the dominant contribution to the pair-production rate by quantum tunnelling through the potential barrier [10, 24], and Casher *et al* rederived the Schwinger's pair-production rate by semiclassical tunnelling calculation [25–27]. Nikishov found the pair-production rate in scattering matrix formalism for a uniform field and an inhomogeneous field of Sauter type gauge potential [28]. Hansen and Ravndal showed that the transmission probability through the barrier of a uniform electric field gives the probability for pair-production for bosons and fermions [29], solving the Klein paradox [1, 30, 31]. Also Padmanabhan [19] suggested that the reflection probability of the scattering problem gives the correct relative probability for the vacuum-to-vacuum transition for bosons. The role of tunnelling solutions for pair-production was also noticed in Refs.[12, 32–35].

The purpose of this paper is to interpret and derive the boson or fermion pair-production rate by strong static uniform or inhomogeneous electric fields in terms of instantons through potential barriers in the space-dependent gauge in any spacetime dimensions. This formula in terms of the instanton action may provide a simple way to estimate the pair-production rate by inhomogeneous electric fields, for instance, from charged black holes, neutron stars, or astrophysical objects [36, 37]. For these static inhomogeneous fields it is easier to apply the space-dependent gauge than the time-dependent gauge. We propose that the single- and multi-instantons for quantum tunnelling determine somehow the single- and multi-pair production. In particular, we show that all the contributions from multi-instantons and anti-instantons yield exactly the total tunnelling probability for the static uniform electric field, and thereby determine the relative vacuum-to-vacuum transition and the boson pair-production rate. We further show that the instanton interpretation together with the Pauli blocking gives correctly the fermion production rate by the static uniform electric field. Using the formula in terms of the instanton action, we find the pair-production rates for bosons and fermions which are asymptotically valid for extremely strong electric fields. Also the pair-production rates for bosons and fermions by a static inhomogeneous electric field are calculated using WKB (adiabatic) approximation for the instantons. Finally we show that according to the instanton interpretation a static localized magnetic field does not lead to any pair-production, confirming the result from the proper time method.

The organization of this paper is as follows. In Sec. II we show that the tunnelling probability by instantons gives correctly the pair-production rates for bosons and fermions by a static uniform electric field. We calculate the pair-production rates in any spacetime dimensions and find their asymptotic form for extremely strong field and compare them with those from other methods. In Sec. III we extend the instanton interpretation of pair-production to an inhomogeneous electric field and find the pair-production rates in terms of the instanton action. In Sec. IV we apply the idea to a static magnetic field to show that any pair of boson or fermion are not produced. This resolves some of the puzzling issue in the canonical method on the pair-production by a static localized magnetic field.

II. UNIFORM ELECTRIC FIELD

We consider a charged boson in a static uniform electric field in a $(d+1)$-dimensional Minkowski spacetime. It satisfies the Klein-Gordon equation (in units of $\hbar = c = 1$)

$$\left[\eta^{\mu\nu}\left(\frac{\partial}{\partial x^\mu} + iqA_\mu\right)\left(\frac{\partial}{\partial x^\nu} + iqA_\nu\right) + m^2\right]\Phi(t, \mathbf{x}) = 0, \tag{1}$$

where q is the charge and m the mass of the boson. In the space-dependent (Coulomb) gauge, the vector potential for the uniform electric field in the $x_\parallel$-direction is given by

$$A_\mu(t, \mathbf{x}) = (-E_0 x_\parallel, 0, \cdots, 0). \tag{2}$$

Each Fourier-mode of the boson field

$$\Phi(t, \mathbf{x}) = e^{i(\mathbf{k}_\perp \cdot \mathbf{x}_\perp - \omega t)}\phi_{\omega, \mathbf{k}_\perp}(x_\parallel), \tag{3}$$

satisfies the one-dimensional equation

$$\left[-\frac{1}{2}\frac{d^2}{dx_\parallel^2} - \frac{1}{2}\left(\omega + qE_0 x_\parallel\right)^2\right]\phi_{\omega, \mathbf{k}_\perp}(x_\parallel) = -\frac{1}{2}(m^2 + \mathbf{k}_\perp^2)\phi_{\omega, \mathbf{k}_\perp}(x_\parallel). \tag{4}$$

Now one may interpret Eq. (4) as a Schrödinger-like equation for a unit mass moving in the inverted harmonic potential with the center at $x_{\parallel,c} = -\omega/(qE_0)$ and the energy $\epsilon = -(m^2 + \mathbf{k}_\perp^2)/2$. As the energy is negative $(\epsilon < 0)$, Eq. (4) indeed describes a tunnelling problem for all transverse momenta $\mathbf{k}_\perp$. The wave function describing the tunnelling process is given by the complex parabolic cylindrical function [38]

$$\phi_{\omega,\mathbf{k}_\perp}(\xi) = cE(a_{\mathbf{k}_\perp}, \xi), \tag{5}$$

where c is a complex number, and

$$\xi = \sqrt{\frac{2}{qE_0}}(\omega + qE_0 x_\parallel), \quad a_{\mathbf{k}_\perp} = \frac{m^2 + \mathbf{k}_\perp^2}{2qE_0}. \tag{6}$$

It has the asymptotic forms in two regimes

$$\begin{aligned} \phi_{\omega,\mathbf{k}_\perp}(\xi) &= A\varphi_{\omega,\mathbf{k}_\perp}(\xi) - B\varphi^*_{\omega,\mathbf{k}_\perp}(\xi), \quad (\xi \ll -2\sqrt{a_{\mathbf{k}_\perp}}), \\ \phi_{\omega,\mathbf{k}_\perp}(\xi) &= C\varphi^*_{\omega,\mathbf{k}_\perp}(\xi), \quad (\xi \gg 2\sqrt{a_{\mathbf{k}_\perp}}), \end{aligned} \tag{7}$$

where

$$\varphi_{\omega,\mathbf{k}_\perp}(\xi) = \sqrt{\frac{2}{|\xi|}} e^{-\frac{i}{4}\xi^2}. \tag{8}$$

Here the coefficients are given by

$$A = ic\sqrt{1 + e^{2\pi a_{\mathbf{k}_\perp}}}, \quad B = -ice^{\pi a_{\mathbf{k}_\perp}}, \quad C = c. \tag{9}$$

In the region $\xi \ll -2\sqrt{a_{\mathbf{k}_\perp}}$, the components $\varphi_{\omega,\mathbf{k}_\perp} e^{-i\omega t}$ describes an incoming particle, from and $\varphi^*_{\omega,\mathbf{k}_\perp} e^{-i\omega t}$ an outgoing particle, to $\xi = -\infty$, whereas in the region $\xi \gg 2\sqrt{a_{\mathbf{k}_\perp}}$ the component $\varphi^*_{\omega,\mathbf{k}_\perp} e^{-i\omega t}$ describes an incoming anti-particle from $\xi = +\infty$. Hansen and Ravndal showed that the transmission probability $|C/A|^2$ gives the probability for one-pair production [29]. Also Padmanabhan suggested that the reflection probability $|B/A|^2$ gives the relative probability for the vacuum-to-vacuum transition [18, 19]. His interpretation implies that $\varphi_{\omega,\mathbf{k}_\perp}(-\infty)e^{-i\omega t}$ and $\varphi^*_{\omega,\mathbf{k}_\perp}(-\infty)e^{-i\omega t}$ correspond to the incoming and outgoing vacuum state, respectively. Extending their arguments to any static field, we further propose that the single- and multi-pair production of bosons are related to the single- and multi-instantons of potential barrier in such a way that the tunnelling probability $P^{\rm t}$ gives the probability for the pair-production and therefore the relative probability for the vacuum-to-vacuum transition is given by the probability for the no-pair production $P^{\rm n-p} = 1 - P^{\rm t}$. Here and from now on we restrict the *tunnelling probability* to the transmission probability through potential barrier but exclude any nonzero transmission probability above a potential barrier or a potential well. Further we shall assume that the tunnelling probability is accurately given by the instanton action or with its higher corrections.

To see how the instanton interpretation works for the uniform electric field, we calculate the tunnelling probability from the asymptotic form (7) and compare it with the result from the instanton calculation. As the negative energy for all momenta $\mathbf{k}_\perp$ is below the potential barrier in Eq. (4), the tunnelling probability is given by the transmission probability

$$P^{\rm b.t}_{\mathbf{k}_\perp} = \left|\frac{C}{A}\right|^2 = \frac{1}{e^{2\pi a_{\mathbf{k}_\perp}} + 1}. \tag{10}$$

Likewise, the probability for the no-pair production, *i.e.*, the vacuum-to-vacuum transition, given by the reflection probability

$$P^{\rm b.n-p}_{\mathbf{k}_\perp} = 1 - P^{\rm b.t}_{\mathbf{k}_\perp} = \frac{1}{1 + e^{-2\pi a_{\mathbf{k}_\perp}}} = \left|\frac{B}{A}\right|^2, \tag{11}$$

as a consequence of the flux conservation. Hence, what is needed in finding the probability for the no-pair production (vacuum-to-vacuum transition) even in a general electric field is the corresponding total tunnelling probability via the single- and multi-instantons.

Now let us interpret the tunnelling probability (10) in terms of multi-instantons and anti-instantons of tunnelling process. In instanton physics [39], the leading contribution to the tunnelling probability

$$P^{\rm t}_{\mathbf{k}_\perp} = e^{-2S_{\mathbf{k}_\perp}}, \tag{12}$$

is determined by the single-instanton action

$$S_{\mathbf{k}_\perp} = \int_{x_-}^{x^+} dx_\parallel \sqrt{m^2 + \mathbf{k}_\perp^2 - \left(\omega + qE_0 x_\parallel\right)^2} = \pi a_{\mathbf{k}_\perp}, \tag{13}$$

where $x_\pm = \pm\sqrt{m^2 + \mathbf{k}_\perp^2} - \omega$ are the classical turning points. We propose that the single-instanton and multi-instantons may be related in a certain way with one-pair and multi-pair production, whereas multi-anti-instantons with the annihilation of created boson pairs. As there is no limitation from the Pauli blocking for the multi-pair production of bosons, the correct total tunnelling probability should take into account both multi-instantons and anti-instantons

$$P_{\mathbf{k}_\perp}^{\mathrm{b.t}} = \sum_{n=1}^{\infty} (-1)^{n+1} e^{-2nS_{\mathbf{k}_\perp}} = \frac{1}{e^{2S_{\mathbf{k}_\perp}} + 1}, \tag{14}$$

where instantons contribute positively and anti-instantons negatively. Similarly, the relative probability for the no-pair production (vacuum-to-vacuum transition) is given by

$$P_{\mathbf{k}_\perp}^{\mathrm{b.n-p}} = \sum_{n=0}^{\infty} (-1)^{n} e^{-2nS_{\mathbf{k}_\perp}} = \frac{1}{1 + e^{-2S_{\mathbf{k}_\perp}}}, \tag{15}$$

These results agree with Eqs. (10) and (11). The physical interpretation of the alternating signs is that only the instantons of even repeated periodic motions in the inverted potential contribute positively (creating pairs) to the tunnelling probability, whereas the anti-instantons of odd repeated periodic motions contribute negatively (annihilating created-pairs) to the tunnelling probability.

The vacuum means the absence of any particle for possible physical states. So the vacuum-to-vacuum transition, *i.e.*, the vacuum persistence, is the total relative probability for the no-pair production:

$$|\langle 0, \mathrm{out}|0, \mathrm{in}\rangle|^2 = \prod_{\text{all states}} P_{\mathbf{k}_\perp}^{\mathrm{b.n-p}} = \exp\left[-\sum_{\text{all states}} \ln(1 + e^{-2S_{\mathbf{k}_\perp}})\right]. \tag{16}$$

On the other hand, the vacuum-to-vacuum transition is given by the imaginary part of the effective action for boson

$$|\langle 0, \mathrm{out}|0, \mathrm{in}\rangle|^2 = \exp\left[-2VT\,\mathrm{Im}\mathcal{L}_{\mathrm{eff.}}^{\mathrm{b}}\right], \tag{17}$$

where V and T are the relevant volume and the duration of time. Therefore, the pair-production rate per unit time per unit volume is twice of the imaginary part of the effective action:

$$w^{\mathrm{b}} = 2\mathrm{Im}\mathcal{L}_{\mathrm{eff.}}^{\mathrm{b}} = \frac{1}{VT} \sum_{\text{all states}} \ln(1 + e^{-2S_{\mathbf{k}_\perp}}). \tag{18}$$

Then the pair-production rate for bosons is explicitly given by

$$\begin{aligned} w^{\mathrm{b}} &= \frac{(2s+1)V_\perp}{V} \int \frac{d\omega d\mathbf{k}_\perp^{d-1}}{(2\pi)^d} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} e^{-\frac{\pi n}{qE_0}\mathbf{k}_\perp^2} e^{-\frac{\pi m^2}{qE_0} n} \\ &= \frac{(2s+1)}{(2\pi)^d} \sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{qE_0}{n}\right)^{(d+1)/2} e^{-\frac{\pi m^2}{qE_0} n}, \end{aligned} \tag{19}$$

where s is the spin of the boson. Here we used $\int d\omega = (qE_0)V_\parallel$, where $V_\parallel$ is the longitudinal extension of the field, and $V = V_\perp V_\parallel$, $V_\perp$ being the transverse volume [28]. It should be noted that Eq. (19) recovers the standard result for the boson pair-production in any dimension in Ref. [40].

The fermion pair-production can be understood similarly. The created fermion pair blocks the multi-pair production. So the total tunnelling probability for the fermion pair-production per each mode is just

$$P_{\mathbf{k}_\perp}^{\mathrm{f.t}} = e^{-2S_{\mathbf{k}_\perp}}. \tag{20}$$

Therefore, the relative probability for the no-pair production of fermions is now given by

$$P^{\mathrm{f.n-p}}_{\mathbf{k}_\perp} = 1 - e^{-2S_{\mathbf{k}_\perp}}. \tag{21}$$

Finally, the fermion pair-production rate per unit time per unit volume is found to be

$$w^{\mathrm{f}} = 2\mathrm{Im}\mathcal{L}^{\mathrm{f}}_{\mathrm{eff.}} = -\frac{1}{VT}\sum_{\text{all states}} \ln(1 - e^{-2S_{\mathbf{k}_\perp}}), \tag{22}$$

and takes the form

$$\begin{aligned} w^{\mathrm{f}} &= \frac{(2s+1)V_\perp}{V}\int \frac{d\omega d\mathbf{k}_\perp^{d-1}}{(2\pi)^d}\sum_{n=1}^{\infty}\frac{1}{n}e^{-\frac{\pi n}{qE_0}\mathbf{k}_\perp^2}e^{-\frac{\pi m^2}{qE_0}n} \\ &= \frac{(2s+1)}{(2\pi)^d}\sum_{n=1}^{\infty}\left(\frac{qE_0}{n}\right)^{(d+1)/2} e^{-\frac{\pi m^2}{qE_0}n}. \end{aligned} \tag{23}$$

Also Eq. (23) recovers the standard result for the fermion pair-production in Ref. [40].

Though the production rate (19) for bosons and (23) for fermions are well defined for all electric fields, the series converge strongly for weak electric fields because all higher terms are exponentially suppressed. But for extremely strong electric fields the exponential terms approach to unity and the series are approximated by the Riemann eta function $\eta(2)$ for bosons and the Riemann zeta function $\zeta(2)$ for fermions. So, for strong electric fields, instead of using a special resummation of the series, we adopt directly the pair-production formula (18) and (22) and evaluate properly the integrals suitable for strong fields. In four dimensions $(d = 3)$, the boson pair-production rate (18) becomes

$$\begin{aligned} w^{\mathrm{b}} &= \frac{(2s+1)}{(2\pi)^3}(qE_0)\int_0^\infty (2\pi)dk_\perp k_\perp \ln\left(1 + e^{-\frac{\pi(m^2+k_\perp^2)}{qE_0}}\right) \\ &= \frac{(2s+1)}{(2\pi)^3}(qE_0)^2\left\{\int_0^\infty dy\ln(1+e^{-y}) - \int_0^{\pi m^2/qE_0} dy\ln(1+e^{-y})\right\}, \end{aligned} \tag{24}$$

where

$$y = \frac{\pi}{qE_0}k_\perp^2 + \frac{\pi m^2}{qE_0}. \tag{25}$$

Using the integral [41]

$$\int_0^\infty dy\ln(1+e^{-y}) = \frac{\pi^2}{12}, \tag{26}$$

and expanding the exponential and then the logarithmic function to any desired order

$$\ln(1+e^{-y}) = \ln 2 - \frac{1}{2}y + \frac{1}{8}y^2 + \frac{1}{96}y^4 + \mathcal{O}(y^5), \tag{27}$$

we obtain the pair-production rate

$$\begin{aligned} w^{\mathrm{b}} &= \frac{(2s+1)}{(2\pi)^3}\left\{\frac{\pi^2}{12}(qE_0)^2 - (\ln 2)\pi m^2 qE_0 + \frac{1}{4}(\pi m^2)^2\right. \\ &\left. - \frac{1}{24}\frac{(\pi m^2)^3}{qE_0} - \frac{1}{480}\frac{(\pi m^2)^5}{(qE_0)^3} + \mathcal{O}\left(\frac{(\pi m^2)^6}{(qE_0)^4}\right)\right\}. \end{aligned} \tag{28}$$

Similarly, the fermion pair-production rate (22) for strong fields takes the form

$$\begin{aligned} w^{\mathrm{f}} &= -\frac{(2s+1)}{(2\pi)^3}(qE_0)\int_0^\infty (2\pi)dk_\perp k_\perp \ln\left(1 - e^{-\frac{\pi(m^2+k_\perp^2)}{qE_0}}\right) \\ &= \frac{(2s+1)}{(2\pi)^3}(qE_0)^2\left\{\int_0^\infty dy\ln(1-e^{-y}) - \int_0^{\pi m^2/qE_0} dy\ln(1-e^{-y})\right\}. \end{aligned} \tag{29}$$

Using the integral [41]

$$\int_0^\infty dy \ln(1-e^{-y}) = -\frac{\pi^2}{6}, \tag{30}$$

and expanding the exponential function and then the logarithmic function

$$\ln(1-e^{-y}) = \ln y - \frac{1}{2}y + \frac{1}{24}y^2 + \frac{11}{720}y^4 + \mathcal{O}(y^5), \tag{31}$$

we finally obtain the fermion pair-production rate

$$\begin{aligned} w^{\rm f} &= \frac{(2s+1)}{(2\pi)^3}\Bigg\{\frac{\pi^2}{6}(qE_0)^2 - \pi m^2 qE_0\Bigg(\ln\Bigg(\frac{qE_0}{\pi m^2}\Bigg)+1\Bigg) - \frac{1}{4}(\pi m^2)^2 \\ &+ \frac{1}{72}\frac{(\pi m^2)^3}{qE_0} + \frac{11}{3600}\frac{(\pi m^2)^5}{(qE_0)^3} + \mathcal{O}\Bigg(\frac{(\pi m^2)^6}{(qE_0)^4}\Bigg)\Bigg\}. \end{aligned} \tag{32}$$

The fermion pair-production rate (32) for strong electric fields confirms the result obtained from different methods in Refs. [42–44].

A comment is in order. The Schwinger pair-production by a static uniform electric field is an ideal calculation in which one neglects the pair-production due to the interactions of the created pairs with the electric field background and among the created pairs. For instance, a single-pair can produce another pair through the interaction with the electric field, whose rate is proportional to $(qE_0/m^2)^2(q/m)^2$ [45] and can be larger than the multi-pair production rate, $e^{-(\pi m^2 n)/(qE_0)}(qE_0)^2/n^2$, from multi-instantons for all sufficiently large n even for an extremely strong electric field E_0. However, we shall not consider this complicated real situation but rather focus on the ideal calculation without the back-reaction of produced pairs.

III. INHOMOGENEOUS ELECTRIC FIELDS

We now consider the pair-production by a static inhomogeneous electric field. Without loss of generality, the electric field is assumed to be localized in the $x_\parallel$-direction and to have the gauge potential

$$A_\mu(t,\mathbf{x}) = (A_0(x_\parallel), 0, \cdots, 0), \tag{33}$$

where $E(x_\parallel) = -dA_0(x_\parallel)/dx_\parallel$. We restrict only to the case where all produced particles $(q>0)$ and antiparticles reach the asymptotic regions $x=+\infty$ and $x=-\infty$, respectively, without being bounded by the electric field. This requires that $qA_0(-\infty) - qA_0(+\infty) \geq 2m$. The mode-decomposed Klein-Gordon equation then takes the form

$$\Bigg[-\frac{1}{2}\frac{d^2}{dx_\parallel^2} - \frac{1}{2}\Bigg(\omega - qA_0(x_\parallel)\Bigg)^2\Bigg]\phi_{\omega,\mathbf{k}_\perp}(x_\parallel) = -\frac{1}{2}(m^2+\mathbf{k}_\perp^2)\phi_{\omega,\mathbf{k}_\perp}(x_\parallel). \tag{34}$$

We can still interpret Eq. (34) as a one-dimensional quantum system of a unit mass with the potential $-(\omega - qA_0(x_\parallel))^2/2$ and the energy $\epsilon = -(m^2+\mathbf{k}_\perp^2)/2$. In the WKB (adiabatic) approximation the asymptotic form for the tunnelling probability for each mode $\mathbf{k}_\perp$ is given by [46–48]

$$P^{\rm b.t}_{\mathbf{k}_\perp} = \frac{1}{e^{2S_{\mathbf{k}_\perp}}+1}, \tag{35}$$

where

$$S_{\mathbf{k}_\perp} = \sum_{n=0}^{\infty} S^{(2n)}_{\mathbf{k}_\perp}. \tag{36}$$

Here the leading contribution to $S_{\mathbf{k}_\perp}$ is given by the instanton action

$$S^{(0)}_{\mathbf{k}_\perp} = \int_{x_-}^{x_+} dx_\parallel \Bigg[Q_{\mathbf{k}_\perp}(x)\Bigg]^{1/2}, \tag{37}$$

and the next-to-leading term by

$$S^{(2)}_{\mathbf{k}_\perp} = \int_{x_-}^{x_+} dx_\parallel \left[\frac{1}{8}\frac{Q''_{\mathbf{k}_\perp}(x)}{Q^{3/2}_{\mathbf{k}_\perp}(x)} - \frac{5}{32}\frac{Q'^2_{\mathbf{k}_\perp}(x)}{Q^{5/2}_{\mathbf{k}_\perp}(x)}\right], \tag{38}$$

where

$$Q_{\mathbf{k}_\perp}(x) = m^2 + \mathbf{k}_\perp^2 - \Big(\omega - qA_0(x_\parallel)\Big)^2. \tag{39}$$

Hence the relative probability for the no-pair production (vacuum-to-vacuum transition) of bosons is given by

$$P^{\rm b.n-p}_{\mathbf{k}_\perp} = \frac{1}{1+e^{-2S_{\mathbf{k}_\perp}}}, \tag{40}$$

and for fermions by

$$P^{\rm f.n-p}_{\mathbf{k}_\perp} = 1 - e^{-2S_{\mathbf{k}_\perp}}. \tag{41}$$

A few comments are in order. First, if the electric field extends over all the space as in the uniform field case and has the potential $|A_0(\pm\infty)| = \infty$, then the potential barrier decreases indefinitely at both $\pm\infty$. Therefore, there are always instantons for all $\mathbf{k}_\perp$. Second, if the electric field is localized or has finite values of the potential at $\pm\infty$, then pairs are produced only when $\omega - qA_0(+\infty) \geq m$ and $\omega - qA_0(-\infty) \leq -m$. So there is a change of the sign of $\omega - qA_0(x_\parallel)$ implying a potential barrier. Thus only those modes belonging to $|\mathbf{k}_\perp| \leq k_{\perp,\rm max}$ have finite instantons and lead to pair-production, where the upper limit is given by the minimum of two asymptotic values

$$k^2_{\perp,\rm max} = {\rm Min}\Big\{\Big(\omega - qA_0(+\infty)\Big)^2 - m^2,\ \Big(\omega - qA_0(-\infty)\Big)^2 - m^2\Big\}. \tag{42}$$

In the inhomogeneous electric field, we obtain the boson pair-production rate per unit time per unit volume

$$\begin{aligned} w^{\rm b} &= 2{\rm Im}\mathcal{L}^{\rm b}_{\rm eff.} \\ &= \frac{(2s+1)}{VT}\sum_{\text{all allowed states}} \ln\Big(1+e^{-2S_{\mathbf{k}_\perp}}\Big) \\ &= \frac{(2s+1)}{(2\pi)^d V_\parallel}\frac{(d-1)\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\int_{qA_0(+\infty)+m}^{qA_0(-\infty)-m} d\omega \int_0^{k_{\perp,\rm max}} dk_\perp k_\perp^{d-2} e^{-2nS_{\mathbf{k}_\perp}}, \end{aligned} \tag{43}$$

and the fermion pair-production rate

$$\begin{aligned} w^{\rm f} &= 2{\rm Im}\mathcal{L}^{\rm f}_{\rm eff.} \\ &= -\frac{(2s+1)}{2VT}\sum_{\text{all allowed states}} \ln\Big(1-e^{-2S_{\mathbf{k}_\perp}}\Big) \\ &= \frac{(2s+1)}{2(2\pi)^d V_\parallel}\frac{(d-1)\pi^{(d-1)/2}}{\Gamma(\frac{d+1}{2})}\sum_{n=1}^{\infty}\frac{1}{n}\int_{qA_0(+\infty)+m}^{qA_0(-\infty)-m} d\omega \int_0^{k_{\perp,\rm max}} dk_\perp k_\perp^{d-2} e^{-2nS_{\mathbf{k}_\perp}}. \end{aligned} \tag{44}$$

As an exactly solvable model we consider a localized electric field $E(x_\parallel) = E_0{\rm sech}^2(x_\parallel/L)$ with the Sauter type gauge potential [1, 28]

$$A_0(x_\parallel) = -E_0 L \tanh(\frac{x_\parallel}{L}). \tag{45}$$

In the limit of $L \to \infty$ the gauge potential (45) reduces to the uniform electric field in Sec. II. Since the gauge potential (45) is a more general case including the uniform field as a special case, it is worthy to apply the instanton interpretation to the pair-production and compare the result with the exact one. Bosons gain an additional contribution to momenta from the acceleration by the localized electric field and have asymptotic values at $x_\parallel \to \pm\infty$:

$$k^2_\parallel(\infty) = (qE_0L + \omega)^2 - m^2 - \mathbf{k}^2_\perp, \quad k^2_\parallel(-\infty) = (qE_0L - \omega)^2 - m^2 - \mathbf{k}^2_\perp. \tag{46}$$

In the large L limit the instanton action (37) is given by

$$S_{\mathbf{k}_\perp} = \pi \frac{m^2+\mathbf{k}_\perp^2}{2qE_0}\left[1+\frac{\omega^2}{(qE_0L)^2}+\frac{m^2+\mathbf{k}_\perp^2}{4(qE_0L)^2}+\mathcal{O}(\frac{1}{L^4})\right]. \tag{47}$$

The exact wave function describing the tunnelling process is found

$$\phi_{\omega,\mathbf{k}_\perp}(x_\parallel) = Ce^{-\mu\frac{x_\parallel}{L}}\mathrm{sech}^\nu(\frac{x_\parallel}{L})F(\alpha,\beta;\gamma;\zeta), \tag{48}$$

where F is the hypergeometric function and

$$\begin{aligned}\mu &= -i\frac{L}{2}\Big(k_\parallel(\infty)+k_\parallel(-\infty)\Big), \quad \nu = -i\frac{L}{2}\Big(k_\parallel(\infty)-k_\parallel(-\infty)\Big),\\ \alpha &= \nu+\frac{1}{2}+i\sqrt{(qE_0L^2)^2-\frac{1}{4}}, \quad \beta = \nu+\frac{1}{2}-i\sqrt{(qE_0L^2)^2-\frac{1}{4}},\\ \gamma &= 1-iLk_\parallel(\infty), \quad \zeta = \frac{1}{2}\Big(1-\tanh(\frac{x_\parallel}{L})\Big).\end{aligned} \tag{49}$$

In the limit of $x_\parallel \gg L$, Eq. (48) has the asymptotic form

$$\phi_{\omega,\mathbf{k}_\perp}(x_\parallel) = 2^\nu Ce^{ik_\parallel(\infty)x_\parallel}. \tag{50}$$

It describes a wave function after tunnelling (an incoming anti-particle). In the limit of $x_\parallel \ll -L$, we may use another form for Eq. (48)

$$\begin{aligned}\phi_{\omega,\mathbf{k}_\perp}(x_\parallel) = Ce^{-\mu\frac{x_\parallel}{L}}\mathrm{sech}^\nu(\frac{x_\parallel}{L})&\Bigg[\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}F(\alpha,\beta;\gamma;1-\zeta)\\ &+\frac{\Gamma(\gamma)\Gamma(\alpha+\beta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)}(1-\zeta)^{\gamma-\alpha-\beta}F(\alpha,\beta;\gamma;1-\zeta)\Bigg].\end{aligned} \tag{51}$$

In this limit the first term of Eq. (51) describes an incident wave (an incoming particle) having the asymptotic form

$$\phi_{\omega,\mathbf{k}_\perp}(x_\parallel) = 2^\nu C\frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}e^{ik_\parallel(-\infty)x_\parallel}. \tag{52}$$

Therefore, from Eqs. (50) and (52) we can find the probability for tunnelling

$$\begin{aligned}P^{\mathrm{b.t}}_{\mathrm{k}_\perp} &= \frac{k_\parallel(-\infty)}{k_\parallel(\infty)}\left|\frac{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}\right|^2\\ &= \frac{\sinh\pi\Big(Lk_\parallel(\infty)\Big)\sinh\pi\Big(Lk_\parallel(-\infty)\Big)}{\cosh\pi\Big(\frac{L}{2}(k_\parallel(\infty)+k_\parallel(-\infty))+Q\Big)\cosh\pi\Big(\frac{L}{2}(k_\parallel(\infty)+k_\parallel(-\infty))-Q\Big)},\end{aligned} \tag{53}$$

where $Q=\sqrt{(qE_0L^2)^2-1/4}$. In the large L limit we obtain approximately the probability for tunnelling

$$P^{\mathrm{b.\ tun.}}_{\mathrm{k}_\perp} = \frac{1}{1+e^{2S_{\mathbf{k}_\perp}}}. \tag{54}$$

Here, we used the binomial expansion

$$\begin{aligned}k_\parallel(\pm\infty) &= (qE_0L\pm\omega)\left[1-\frac{m^2+\mathbf{k}_\perp^2}{(qE_0L)^2\Big(1\pm\frac{\omega}{(qE_0L)^2}\Big)^2}\right]^{1/2}\\ &= qE_0L\pm\omega-\frac{m^2+\mathbf{k}_\perp^2}{2qE_0L}\left[1\mp\frac{\omega}{qE_0L}+\frac{\omega^2}{(qE_0L)^2}\right]\\ &\quad -\frac{(m^2+\mathbf{k}_\perp^2)^2}{8(qE_0L)^3}\left[1\mp\frac{3\omega}{qE_0L}+\frac{6\omega^2}{(qE_0L)^2}\right]+\cdots.\end{aligned} \tag{55}$$

Therefore, using instanton action (47) we are able to obtain the pair-production rate for bosons and fermions according to Eqs. (43) and (44). Thus we have shown that in the space-dependent gauge the instanton interpretation for wave function gives correctly the pair-production rates for bosons and fermions for two exactly solvable models.

IV. MAGNETIC FIELDS

A static uniform magnetic field leads only to a real effective action and thus implies no-pair production [3]. Recently it has also been shown that any static magnetic field, having no imaginary part, does not lead to the pair-production [17, 49, 50]. On the other hand, in canonical method, each mode of the Klein-Gordon or Dirac equation has a nonzero reflection probability for static magnetic fields. This issue has been raised and discussed to interpret the reflection probability as the pair-production by static localized magnetic fields in Ref. [49]. In this section we resolve this issue from the view point of the instanton interpretation.

Let us consider a static magnetic field in a 4-dimensional spacetime with the gauge potential

$$A_\mu(t, \mathbf{x}) = (0, A_1(x_2), 0, 0). \tag{56}$$

The magnetic field is given by $\mathbf{B} = (dA_1(x_2)/dx_2)\hat{\mathbf{x}}_3$. The Klein-Gordon equation has the form

$$\left[\frac{\partial^2}{\partial t^2} - \left(\frac{\partial}{\partial x_1} + iqA_1(x_2)\right)^2 - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} + m^2\right]\Phi(t, \mathbf{x}) = 0. \tag{57}$$

As in the case of the electric field, each mode of the field

$$\Phi(t, \mathbf{x}) = e^{i(k_1 x_1 + k_3 z_3 - \omega t)}\phi_{\omega, k_1, k_3}(x_2) \tag{58}$$

leads to a Schrödinger-like equation

$$\left[-\frac{1}{2}\frac{d^2}{dx_2^2} + \frac{1}{2}\left(k_1 - qA_1(x_2)\right)^2\right]\phi_{\omega, k_1, k_3}(x_2) = \frac{1}{2}(\omega^2 - m^2 - k_3^2)\phi_{\omega, k_1, k_3}(x_2). \tag{59}$$

As a one-dimensional quantum system, Eq. (59) has the potential $(k_1 - qA_1(x_2))^2/2$ and the energy $(\omega^2 - m^2 - k_3^2)/2$.

In the case of a uniform magnetic field, the gauge potential $A_1(x_2) = -B_0 x_2$ is indefinitely unbounded at $x_2 = \pm\infty$. Then the potential of Eq. (59) is exactly that of a harmonic oscillator and the energy is quantized

$$\epsilon_n = qB_0(2n + 1). \tag{60}$$

The quantized energy has been used to calculate the effective action in the uniform magnetic field [2]. From the view point of instanton interpretation, there is no pair-production since there are no finite instantons at all. All would-be instantons from one spatial infinity to another are infinite and do not contribute to the tunnelling probability. This result agrees with that obtained from the proper time and other methods.

We now consider a static localized magnetic field $\mathbf{B}(x_2) = B_0 \mathrm{sech}^2(x_2/L)\hat{\mathbf{x}}_3$. The gauge potential is given by $A_1(y) = -B_0 L \tanh(x_2/L)$. The gauge potential in Eq. (59) has two asymptotic values $(k_1 \pm qB_0 L)^2/2$ at $x_2 = \pm\infty$, respectively, and a minimum value in-between. There is a potential well instead of a potential barrier, so the reflection probability may be nonzero, though the tunnelling probability via instantons is zero. Therefore, according to the instanton interpretation, there is no pair-production. The instanton interpretation thus resolves the contradiction between the effective action and the canonical method raised in Ref. [49]. Our result also agrees with that by Dunne and Hall, who showed that the imaginary part of the effective action vanishes for the static magnetic field considered above and therefore neither bosons nor fermions are produced in pairs [50, 51].

V. DISCUSSION AND CONCLUSION

In this paper we have studied the pair-production of bosons and fermions by static uniform or inhomogeneous electric field. For these fields we used the space-dependent (Coulomb) gauge and solved the Klein-Gordon equation. For strong electric fields the mode-decomposed Klein-Gordon equations have potential barriers from the gauge potential. The set of wave functions describing pair-production in quantum field theory is the same of the standard scattering problem through potential barriers in quantum mechanics [18, 19, 29], in contract with the time-dependent gauge. Together with the fact that the most dominant contribution to pair-production rate is the single instanton [10, 24–27], we further propose that all multi-instantons contribute to the pair-production and anti-multi-instantons to the annihilation of created pairs and that the total tunnelling probability from all multi-instantons and anti-multi-instantons is related with pair-production and the probability for the vacuum-to-vacuum transition is the probability for no-pair production. Based on this we derived the pair-production formula for bosons (18) and fermions (22).

This instanton interpretation means that the single-instanton is related in a certain way with the single-pair production, multi-instantons with the multi-pair production and anti-multi-instantons with the annihilation of the created pairs. In fact, when the instanton action is large, the single-instanton is the dominant contribution to one-pair production and multi-instantons are the dominant contribution to the multi-pair production. Also it implies the no-pair production when there is not any tunnelling instanton. In the case of a uniform electric field, when all the contributions from multi-instantons and anti-instantons are taken into account, the pair-production rate for bosons calculated according to the instanton interpretation recovers the well-known result from the proper time method. By taking the Pauli-blocking into account the pair-production rate for fermions is found also to agree with the standard result. Further the pair-production formula from instanton action yields the correct forms (28) and (32) for extremely strong electric fields, confirming the consistency of the formula with other methods. Using the instantons obtained in the WKB (adiabatic) approximation we are also able to provide the formula for the pair-production rate of bosons and fermions by inhomogeneous electric fields.

As a by-product we are able to show that any static (localized) magnetic field cannot produce pairs of bosons or fermions. In the case of magnetic fields the space-dependent gauge reduces the Klein-Gordon equation to time-independent Schrödinger equations with potential wells instead of potential barriers of the electric field case. Since potential wells cannot have finite instantons and possible infinite instantons from either side of potential wells give the zero probability for pair-production, the pair-production of bosons or fermions cannot proceed. As the nonzero transmission probability through potential wells is the result without any finite instanton, the instanton interpretation excludes the possibility of pair-production by a static localized magnetic field in the canonical method raised in Ref. [49]. Therefore we conclude that any static magnetic field does not lead to pair-production and the canonical method equipped with the instanton interpretation is compatible with the effective action method [17, 49, 50].

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