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Anomaly Cancellation: A Retrospective From a Modern Perspective

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Abstract

The mechanism by which gauge and gravitational anomalies cancel in certain string theories is reviewed. The presentation is aimed at theorists who do not necessarily specialize in string theory.

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1 Introduction

As is well-known, in the early 1980s it appeared that superstrings could not describe parity-violating theories, because of quantum mechanical inconsistencies due to anomalies. The discovery that in certain cases the anomalies could cancel[1] was important for convincing many theorists that string theory is a promising approach to unification. In the 17 years that have passed since then, string theory has been studied intensively, and many issues are understood much better now. This progress enables me to describe the original anomaly cancellation mechanism in a more elegant way than was originally possible. The improvements that are incorporated in the following discussion include an improved understanding of the association of specific terms with specific string world-sheets as well as improved tricks for manipulating the relevant characteristic classes.

When a symmetry of a classical theory is broken by radiative corrections, the symmetry is called "anomalous." In this case there is no choice of local counterterms that can be added to the low energy effective action to restore the symmetry. Anomalies arise from divergent Feynman diagrams, with a classically conserved current attached, that do not admit a regulator compatible with conservation of the current. Anomalies only arise at one-loop order (Adler-Bardeen theorem) in diagrams with a chiral fermion or boson going around the loop. Their origin can be traced (Fujikawa) to the behavior of Jacobian factors in the path-integral measure.

There are two categories of anomalies. The first category consists of anomalies that break a global symmetry. An example is the axial part of the flavor $SU(2) \times SU(2)$ symmetry of QCD. These anomalies are "good" in that they do not imply any inconsistency. Rather, they make it possible to carry out certain calculations to high precision. The classic example is the rate for the decay $\pi^0 \to \gamma\gamma$. The second category of anomalies consists of ones that break a local gauge symmetry. These are "bad", in that they imply that the quantum theory is inconsistent. They destroy unitarity, causality, and other related sacred principles. In the remainder of this talk, I will only be concerned with this second category of anomalies. Either they cancel, or the theory in question is inconsistent.

Chiral fields only exist in spacetimes with an even dimension. If the dimension is $D = 2n$, then anomalies can occur in diagrams with one current and n gauge fields attached to a chiral field circulating around the loop. In four dimensions these are triangle diagrams and in ten dimensions these are hexagon diagrams. The resulting nonconservation of the current J^μ

takes the form

$$\partial_\mu J^\mu \sim \epsilon^{\mu_1\mu_2\cdots\mu_{2n}} F_{\mu_1\mu_2} \cdots F_{\mu_{2n-1}\mu_{2n}} \tag{1}$$

In string theory there are various world-sheet topologies that correspond to one-loop diagrams. In the case of type II or heterotic theories it is a torus. For the type I superstring theory it can be a torus, a Klein bottle, a cylinder or a Möbius strip. However, the anomaly analysis can be carried out entirely in terms of a low-energy effective field theory, which is what I will do. Still it is interesting to interpret the Type I result in terms of string world sheets. The torus turns out not to contribute to the anomaly. For the other world sheet topologies, it is convenient to imagine them as made by piecing together boundary states $|B\rangle$ and cross-cap states $|C\rangle$. (Cross-caps can be regarded as boundaries that have opposite points identified.) In this way $\langle B|B\rangle$ represents a cylinder, $\langle B|C\rangle$ and $\langle C|B\rangle$ represent a Möbius strip, and $\langle C|C\rangle$ represents a Klein bottle. The correct relative weights are encoded in the combinations

$$(\langle B| + \langle C|) \times (|B\rangle + |C\rangle). \tag{2}$$

We will interpret the consistency of the $SO(32)$ type I theory as arising from a cancellation between the boundary and cross-cap contributions. It should also be pointed out that the modern interpretation of the boundary state is in terms of a world-sheet that ends on a *D-brane*, whereas the cross-cap state corresponds to a world-sheet that ends on an *orientifold plane*.

2 Anomaly Analysis

2.1 Chiral Fields

The kinds of chiral fields that can exist depend on the number of space and time dimensions with a pattern that repeats every 8 dimensions. For example, it is true in both four and ten dimensions that a fermion field can be Majorana (real in a suitable representation of the Dirac algebra) or Weyl (definite handedness). However, a significant difference between the two cases is that, unlike the case of four dimensions, in ten dimensions the two conditions are compatible, so that it is possible for a fermion field to be simultaneously Majorana and Weyl. Indeed, the basic (irreducible) spinors in ten dimensions are Majorana–Weyl. These statements depend on the signature as well as the dimension. I am assuming Lorentzian signature throughout.

Another difference between four and ten dimensions is that in ten dimensions it is also possible to have chiral bosons! To be specific, consider a fourth rank antisymmetric tensor field $A_{\mu\nu\rho\lambda}$, which is conveniently represented as a four-form A. Then the five-form field strength $F = dA$ has a gauge invariance analogous to that of the Maxwell field. Moreover, one can covariantly eliminate half of the degrees of freedom associated with this field by requiring that it is self-dual (or anti-self dual). The resulting degrees of freedom are not reflection invariant, and they therefore describe a chiral boson. The self-duality condition of the free theory is deformed by interaction terms. This construction in ten dimensions is consistent with Lorentzian signature, whereas in four dimensions a two-form field strength can be self-dual for Euclidean signature (instantons).

2.2 Differential Forms and Characteristic Classes

To analyze anomalies it is extremely useful to use differential forms and characteristic classes. In modern times this kind of mathematics has become part of the standard arsenal of theoretical physicists. For example, Yang–Mills fields are Lie-algebra-valued one-forms:

$$A = \sum_{\mu,a} A_\mu^a(x)\lambda^a dx^\mu. \tag{3}$$

Here the λ^a are matrices in a convenient representation (call it ρ) of the Lie algebra $\mathcal{G}$. The field strengths are Lie-algebra-valued two-forms:

$$F = \frac{1}{2}\sum_{\mu\nu} F_{\mu\nu} dx^\mu \wedge dx^\nu = dA + A \wedge A. \tag{4}$$

Under an infinitesimal gauge transformation

$$\delta_\Lambda A = d\Lambda + [A, \Lambda], \tag{5}$$

$$\delta_\Lambda F = [F, \Lambda]. \tag{6}$$

Λ is an infinitesimal Lie-algebra-valued zero-form.

Gravity (in the vielbein formalism) is described in an almost identical manner. The *spin connection* one-form

$$\omega = \sum_{\mu,a} \omega_\mu^a(x)\lambda^a dx^\mu. \tag{7}$$

is a gauge field for local Lorentz symmetry. The λ^a are chosen to be in the fundamental representation of the Lorentz algebra ($D \times D$ matrices). The curvature two-form is

$$R = d\omega + \omega \wedge \omega. \tag{8}$$

Under an infinitesimal local Lorentz transformation (with infinitesimal parameter Θ)

$$\delta_\Lambda \omega = d\Theta + [\omega, \Theta], \tag{9}$$

$$\delta_\Lambda R = [R, \Theta]. \tag{10}$$

Characteristic classes are differential forms, constructed out of F and R, that are closed and gauge invariant. Thus $X(F,R)$ is a characteristic class provided that $dX = 0$ and $\delta_\Lambda X = \delta_\Theta X = 0$. Some examples are

$$\mathrm{tr}(F \wedge \ldots \wedge F) \equiv \mathrm{tr}(F^k), \tag{11}$$

$$\mathrm{tr}(R \wedge \ldots \wedge R) \equiv \mathrm{tr}(R^k), \tag{12}$$

as well as polynomials built out of these building blocks using wedge products.

2.3 Characterization of Anomalies

Yang–Mills and local Lorentz anomalies in $D = 2n$ dimensions are encoded in a characteristic class that is a $2n+2$ form, denoted I_{2n+2}. You can't really antisymmetrize $2n+2$ indices in $2n$ dimensions, so these expressions are a bit formal, though they can be given a precise mathematical justification. In any case, the physical anomaly is characterized by a $2n$ form G_{2n}, which certainly does exist. The precise formula is

$$\delta S_{\text{eff}} = \int G_{2n}. \tag{13}$$

The formulas for G_{2n} are rather ugly and subject to the ambiguity of local counterterms and total derivatives, whereas by pretending that there are two extra dimensions one uniquely encodes the anomalies in beautiful formulas I_{2n+2}. Moreover, any G_{2n} that is deduced from an I_{2n+2} by the formulas that follow, is guaranteed to satisfy the Wess–Zumino consistency conditions.

The anomaly G_{2n} is obtained from I_{2n+2} (in a coordinate patch) by the descent equations $I_{2n+2} = d\omega_{2n+1}$ and $\delta\omega_{2n+1} = dG_{2n}$. Here δ represents a combined gauge transformation (i.e., $\delta = \delta_\Lambda + \delta_\Theta$). The ambiguities in the determination of the Chern–Simons form ω_{2n+1} and the anomaly form G_{2n} from these equations are just as they should be and do not pose a problem. The total anomaly is a sum of contributions from each of the chiral fields in the theory, and it can be encoded in a characteristic class

$$I_{2n+2} = \sum_\alpha I^{(\alpha)}_{2n+2} \tag{14}$$

The formulas for every possible anomaly contribution $I^{(\alpha)}_{2n+2}$ were worked out by Alvarez-Gaumé and Witten.[2] Dropping an overall normalization factor (which we can do, because we are interested in achieving cancellation) their results are as follows:

• A left-handed Weyl fermion belonging to the ρ representation of the Yang-Mills gauge group contributes

$$I_{1/2}(R,F) = \left(\hat{A}(R)\,\mathrm{tr}_\rho e^{iF}\right)_{2n+2}. \tag{15}$$

The notation $(\cdots)_{2n+2}$ means that one should extract the $(2n+2)$-form part of the enclosed expression. The Dirac roof genus $\hat{A}(R)$ is given by

$$\hat{A}(R) = \prod_{i=1}^{n}\left(\frac{\lambda_i/2}{\sinh\lambda_i/2}\right), \tag{16}$$

where the λ_i are the "eigenvalue two-forms" of the curvature:

$$R \sim \begin{pmatrix} 0 & \lambda_1 & & & & & & \\ -\lambda_1 & 0 & & & & & & \\ & & 0 & \lambda_2 & & & & \\ & & -\lambda_2 & 0 & & & & \\ & & & & \cdot & & & \\ & & & & & \cdot & & \\ & & & & & & 0 & \lambda_n \\ & & & & & & -\lambda_n & 0 \end{pmatrix}. \tag{17}$$

• A left-handed Weyl gravitino, which is always a singlet of any Yang-Mills groups, gives a contribution denoted $I_{3/2}(R)$. In the following, we will circumvent the need for the explicit formula.

• A self-dual tensor gives a contribution denoted $I_A(R)$. It is related to the Hirzebruch L-function

$$L(R) = \prod_{i=1}^{n}\frac{\lambda_i}{\tanh\lambda_i} \tag{18}$$

by $I_A(R) = -\frac{1}{8}L(R)$.

In each case a chiral field of the opposite chirality (right-handed instead of left-handed) gives an anomaly contribution of the opposite sign. Later we will utilize the identity[3]

$$\hat{A}(R/2) = \sqrt{L(R/4)\hat{A}(R)}, \tag{19}$$

which is an immediate consequence of eqs. (16) and (18).

2.4 The Type IIB Theory

Type IIB superstring theory is a ten-dimensional parity-violating theory, whose massless chiral fields consist of two left-handed Majorana–Weyl gravitinos (or, equivalently, one Weyl gravitino), two right-handed Majorana–Weyl spinors (or "dilatinos") and a self-dual boson. Thus the total anomaly is given by the 12-form part of

$$I = I_{3/2}(R) - I_{1/2}(R) + I_A(R). \tag{20}$$

An important result of the Alvarez-Gaumé and Witten paper [2] is that this 12-form vanishes, so that this theory is anomaly-free. The proof requires showing that the expression

$$\left(\prod_{i=1}^{5} \frac{\lambda_i/2}{\sinh\lambda_i/2}\right)\left(2\sum \cosh\lambda_i - 2\right) - \frac{1}{8}\prod_{i=1}^{5}\frac{\lambda_i}{\tanh\lambda_i} \tag{21}$$

contains no terms of sixth order in the λ_i. This involves three nontrivial cancellations. The relevance of this fact to the type I theory, to which we turn next, is that we can represent $I_{3/2}(R)$ by $I_{1/2}(R) - I_A(R)$. This is only correct for the 12-form part, but that is all that we need.

3 Type I Superstring Theory

Type I superstring theory has 16 conserved supercharges, which form a Majorana–Weyl spinor in ten dimensions. The massless fields of type I superstring theory consist of a supergravity multiplet in the closed string sector and a super Yang–Mills multiplet in the open string sector. The supergravity multiplet contains three bosonic fields: the metric (35), a two-form (28), and a scalar dilaton (1). The parenthetical numbers are the number of physical polarization states represented by these fields. None of these is chiral. It also contains two fermionic fields: a left-handed Majorana–Weyl gravitino (56) and a right-handed Majorana–Weyl dilatino (8). These are chiral and contribute an anomaly given by

$$I_{\rm sugra} = \frac{1}{2}\left(I_{3/2}(R) - I_{1/2}(R)\right)_{12} = -\frac{1}{2}\left(I_A(R)\right)_{12} = \frac{1}{16}\left(L(R)\right)_{12}. \tag{22}$$

The super Yang–Mills multiplet contains the gauge fields and left-handed Majorana–Weyl fermions (gauginos), each of which belongs to the adjoint representation of the gauge group. Classically, the gauge group can be any orthogonal or symplectic group. In the following we only consider the case of $SO(N)$, since it is the one of most interest. In this case the adjoint representation corresponds to antisymmetric $N \times N$ matrices, and has dimension $N(N-1)/2$. Adding the anomaly contribution of the gauginos to the supergravity contribution given above yields

$$I_{12} = \left(\frac{1}{2}\hat{A}(R)\,\mathrm{Tr}e^{iF} + \frac{1}{16}L(R)\right)_{12} \tag{23}$$

The symbol Tr is used to refer to the adjoint representation, whereas the symbol tr is used (later) to refer to the N-dimensional fundamental representation.

Next we use the Chern character property

$$\mathrm{tr}_{\rho_1\times\rho_2}e^{iF} = \left(\mathrm{tr}_{\rho_1}e^{iF}\right)\left(\mathrm{tr}_{\rho_2}e^{iF}\right) \tag{24}$$

to deduce that for $SO(N)$

$$\mathrm{Tr}e^{iF} = \frac{1}{2}\left(\mathrm{tr}e^{iF}\right)^2 - \frac{1}{2}\mathrm{tr}e^{2iF} = \frac{1}{2}\left(\mathrm{tr}\cos F\right)^2 - \frac{1}{2}\mathrm{tr}\cos 2F. \tag{25}$$

In the last step we have used the fact that the trace of an odd power of F vanishes.

Substituting eq. (25) into eq. (23) gives the anomaly as the 12-form part of

$$\frac{1}{4}\hat{A}(R)\left(\mathrm{tr}\cos F\right)^2 - \frac{1}{4}\hat{A}(R)\mathrm{tr}\cos 2F + \frac{1}{16}L(R). \tag{26}$$

Since this is of 6th order in R's and F's, the following expression has the same 12-form part:

$$I' = \frac{1}{4}\hat{A}(R)\left(\mathrm{tr}\cos F\right)^2 - 16\hat{A}(R/2)\mathrm{tr}\cos F + 256L(R/4). \tag{27}$$

Moreover, using eq. (19), this can be recast as a perfect square

$$I' = \left(\frac{1}{2}\sqrt{\hat{A}(R)}\mathrm{tr}\cos F - 16\sqrt{L(R/4)}\right)^2. \tag{28}$$

There is no choice of N for which $I'_{12} = I_{12}$ vanishes. However, as will be explained later, it is possible to introduce a local counterterm that cancels the anomaly if I_{12} factorizes into a product of a 4-form and an 8-form. We have shown that $I' = Y^2$, where

$$Y = \frac{1}{2}\sqrt{\hat{A}(R)}\,\mathrm{tr}\cos F - 16\sqrt{L(R/4)}. \tag{29}$$

A priori, this is a sum of forms $Y_0 + Y_4 + Y_8 + \ldots$. However, if the constant term vanishes ($Y_0 = 0$), then

$$I_{12} = (Y_4 + Y_8 + \ldots)^2_{12} = 2Y_4Y_8 \tag{30}$$

as required. To examine the constant term in Y, we use the fact that L and $\hat{A}$ are equal to 1 plus higher order forms and that $\mathrm{tr}\cos F = N + \ldots$ to deduce that $Y_0 = (N-32)/2$. Thus, the desired factorization only works for the choice $N = 32$ in which case the gauge algebra is $SO(32)$.

Let us express Y as a sum of two terms $Y_B + Y_C$, where

$$Y_B = \frac{1}{2}\sqrt{\hat{A}(R)}\,\mathrm{tr}\cos F \tag{31}$$

and

$$Y_C = -16\sqrt{L(R/4)}. \tag{32}$$

This decomposition has a simple interpretation in terms of string world-sheets. Y_B is the boundary – or D-brane – contribution. It carries all the dependence on the gauge fields. Y_C is the cross-cap – or orientifold plane – contribution. Note that

$$I' = Y^2 = Y_B^2 + 2Y_B Y_C + Y_C^2 \tag{33}$$

displays the anomaly contributions arising from distinct world-sheet topologies: the cylinder, the Möbius strip, and the Klein bottle.

In order to cancel the anomaly, what we need is a local counterterm, S_C, with the property that

$$\delta S_C = -\int G_{10}, \tag{34}$$

where G_{10} is the anomaly 10-form that follows, via the descent equations, from $I_{12} = 2Y_4Y_8$. As we mentioned earlier, there are ambiguities in the determination of G_{10} from I_{12}. A convenient choice in the present case is

$$G_{10} = 2G_2Y_8, \tag{35}$$

where G_2 is a two-form that is related to Y_4 by the descent equations $Y_4 = d\omega_3$ and $\delta\omega_3 = dG_2$. This works because Y_8 is closed and gauge invariant.

Recall that the type I supergravity multiplet contains a two form field, which we denote C_2. (Parenthetically, we note that in the lingo of the RNS superstring description it belongs to the Ramond-Ramond sector of the spectrum.) In terms of its index structure, it would seem that the field C should be invariant under Yang–Mills gauge transformations and local Lorentz transformations. However, it does transform nontrivially under each of them in just such a way as to cancel the anomaly.[1] Specifically, writing the counterterm as

$$S_C = \mu\int C_2Y_8, \tag{36}$$

eq. (34) is satisfied provided that

$$\mu\delta C_2 = -2G_2. \tag{37}$$

The coefficient μ is a parameter whose value depends on normalization conventions that we are not specifying here.

One consequence of the nontrivial gauge transformation properties of the field C_2 is that the naive kinetic term $\int |dC_2|^2$ must be modified to give gauge invariance. The correct choice is $\int |H_3|^2$, where

$$H_3 = dC_2 + 2\mu^{-1}\omega_3. \tag{38}$$

Note that ω_3 contains both Yang–Mills and Lorentz Chern–Simons terms. Only the former is present in the classical supergravity theory.

4 Concluding Remarks

The techniques that I have described for analyzing and cancelling anomalies in the type I $SO(32)$ theory can be used to analyze more complicated examples. Recently, in work carried out with Witten,[4] a number of other examples were analyzed. These included the following

- The type IIB theory with n spacetime-filling D-brane anti-D-brane pairs and gauge symmetry $U(n) \times U(n)$.[5] This model has tachyon fields associated to open strings connecting D-branes to anti-D-branes.
- The type I theory with n additional spacetime-filling D-brane anti-D-brane pairs and gauge symmetry $SO(32+n) \times SO(n)$.[6]
- A tachyon-free ten-dimensional string model (originally due to Sugimoto[6]) with spontaneously broken supersymmetry and gauge group $Sp(16)$.
- A nonsupersymmetric and tachyon-free ten-dimensional string model (originally due to Sagnotti[7]) with gauge group $U(32)$.
- Various six-dimensional string models.

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