SUPERSYMMETRY IN QUANTUM THEORY OVER A GALOIS FIELD

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Abstract:

As shown in our previous papers (hep-th/0209001 and reference therein), quantum theory based on a Galois field (GFQT) possesses a new symmetry between particles and antiparticles, which has no analog in the standard approach. In the present paper it is shown that this symmetry (called the AB one) is also compatible with supersymmetry. We believe this is a strong argument in favor of our assumption that the AB symmetry is a fundamental symmetry in the GFQT (and in nature if it is described by quantum theory over a Galois field). We also consider operatorial formulations of space inversion and X inversion in the GFQT. It is shown in particular that the well known fact, that the parity of bosons is real and the parity of fermions is imaginary, is a simple consequence of the AB symmetry.

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1 Introduction

In papers [1] we have proposed an approach to quantum theory where the wave functions of the system under consideration are described by elements of a linear space over a Galois field, and the operators of physical quantities - by linear operators in this space. A detailed discussion of this approach has been given in a recent paper [2]. In particular, it has been shown that when the characteristic p of the Galois field is very large, there exists a correspondence between such a description and the standard one (in terms of states and operators in the Hilbert space). In other words, there exist situations when quantum theory based on a Galois field (GFQT) predicts results which are practically indistinguishable from those given by the standard theory. It has also been argued that the description of quantum systems in terms of Galois fields is more natural than the standard description in terms of complex numbers.

The first obvious conclusion about the GFQT is as follows: since any Galois field has only a finite number of elements, divergencies in this theory cannot exist in principle and all operators are automatically well defined. It is also natural to expect that, since arithmetic of Galois field differs from the standard one, the GFQT has some properties which have no analog in the standard theory.

In particular, as shown in Ref. [3], the GFQT possesses a new symmetry between particles and antiparticles called AB symmetry. The existence of this symmetry is a consequence of the fact that in the GFQT an irreducible representation (IR) of the symmetry algebra describes a particle and its antiparticle simultaneously. This automatically explains the existence of antiparticles and shows that a particle and its antiparticle are different states of the same object. As argued in Refs. [2, 3], such an explanation is more natural than the standard one, where negative energy solutions of covariant equations are used and a particle and its antiparticle are described by independent IRs of the symmetry algebra.

If the AB symmetry is combined with the Pauli spin-statistics theorem [4] then one comes to the conclusions [3] that in the GFQT any neutral particle (in particular the photon) cannot be elementary but only composite, and any interaction can contain only an even number of creation/annihilation operators.

These results are in striking disagreement with our experience but as argued in Ref. [3] this by no means excludes a possibility that future quantum physics will be based on a Galois field.

In the present paper a supersymmetric version of the GFQT is considered. We assume that the symmetry algebra is the Galois field analog of the superalgebra osp(1,4), which is a supersymmetric generalization of the anti de Sitter (AdS) algebra so(2,3). Therefore our first goal is to construct Galois field analogs of IRs of this superalgebra. This is done in Sect. 4 by analogy with the construction of Ref. [5] where all positive energy IRs of the osp(1,4) superalgebra in Hilbert spaces have been found. In preparatory Sects. 2 and 3 we describe the most important properties of Galois field analogs of IRs of the sp(2) and so(2,3) algebras. In Sect. 5 explicit formulas for matrix elements of fermionic operators of IR of the osp(1,4) algebra are given. In Sect. 6 it is shown how the representation operators in the space of IR can be related to the corresponding operators in the Fock space. The construction is used in Sect. 7 where the main result of the paper is proved: the AB symmetry is compatible with supersymmetry. We believe this is a strong argument in favor of our assumption (made in Ref. [2]) that the AB symmetry is a fundamental symmetry in the GFQT (and in nature if it is described by quantum theory over a Galois field). Finally, in Sects. 8 and 9 we consider operatorial formulation of space inversion and X inversion in the GFQT. It is shown in particular that the well known result, that the parity of bosons is real and the parity of fermions is imaginary, is a direct consequence of the AB symmetry.

For reading the present paper, only very elementary knowledge of Galois fields is needed. Although the notion of the Galois field is extremely simple and elegant, the majority of physicists is not familiar with this notion. For this reason, in Ref. [2] an attempt has been made to explain the basic facts about Galois fields in a simplest possible way (and using arguments which, hopefully, can be accepted by physicists). The readers who are not familiar with Galois fields can also obtain basic knowledge from standard textbooks (see e.g. Refs. [6]).

2 Modular IRs of the sp(2) algebra

If a conventional quantum theory has a symmetry group or algebra, then there exists a unitary representation of the group or a representation of the algebra by Hermitian operators in the Hilbert space describing the quantum system under consideration. In the latter case the representation is also often called unitary meaning that it can be extended to the unitary representation of the corresponding group.

Let p be a prime number and F_{p^2} be a Galois field containing p^2 elements. This field has only one nontrivial automorphism $a \to \bar{a}$ (see e.g. Refs. [6, 2]) which is the analog of complex conjugation in the field of complex numbers. The automorphism can be defined as $a \to \bar{a} = a^p$ [6].

In the GFQT, unitary representations in Hilbert spaces are replaced by representations in spaces over F_{p^2}. Representations in spaces over fields of nonzero characteristics are called modular representations. A review of the theory of modular IRs can be found e.g. in Ref. [7]. In the present paper we do not need a general theory since modular IRs in question can be constructed explicitly.

A modular analog of the Hilbert space is a linear space V over F_{p^2} supplied by a scalar product $(...,...)$ such that for any $x, y \in V$ and $a \in F_{p^2}$, $(x, y) \in F_{p^2}$ and the following properties are satisfied:

$$(x,y) = \overline{(y,x)}, \quad (ax,y) = \bar{a}(x,y), \quad (x,ay) = a(x,y) \tag{1}$$

By analogy with usual notations, we use * to denote the Hermitian conjugation in spaces over F_{p^2}. This means that if A is an operator in V then A^* is the operator satisfying $(Ax, y) = (x, A^*y)$ for all $x, y \in V$.

By definition, a particle is called elementary if it is described by an IR of the symmetry group or algebra in the given theory. Since we assume that the role of the symmetry algebra is played by the superalgebra osp(1,4), this implies that elementary particles are described by modular IRs of this superalgebra. For representations in Hilbert spaces, all unitary positive energy IRs of the osp(1,4) superalgebra have been found in Ref. [5]. By analogy with this construction, we will seek modular IRs of the osp(1,4) superalgebra by investigating their decompositions into modular IRs of the so(2,3) algebra. In turn, the key role in constructing such modular IRs is played by modular IRs of the sp(2) subalgebra. The latter are also very important for understanding the AB symmetry. On the one hand, modular IRs of the sp(2) algebra are very simple while on the other they clearly demonstrate the main difference between the standard and modular cases: in contrast to the standard case, where unitary IRs are necessarily infinite dimensional, all modular IRs are finite dimensional (this statement has been proved in the general case by Zassenhaus [8]).

Representations of the sp(2) algebra are described by a set of operators $(a', a", h)$ satisfying the commutation relations

$$[h, a'] = -2a' \quad [h, a"] = 2a" \quad [a', a"] = h \tag{2}$$

The modular analogs of unitary representations of the sp(2) algebra are characterized by the conditions that $a^{'*} = a^{"}$ and $h^* = h$.

The Casimir operator of the second order for the algebra (2) has the form

$$K = h^2 - 2h - 4a"a' = h^2 + 2h - 4a'a" \tag{3}$$

We will consider representations with the vector e_0, such that

$$a'e_0 = 0, \quad he_0 = q_0e_0, \quad (e_0, e_0) = 1 \tag{4}$$

One can easily prove [1, 2] that q_0 is "real", i.e. $q_0 \in F_p$ where F_p is the residue field modulo p: $F_p = Z/Zp$ where Z is the ring of integers. The field F_p consists of p elements and represents the simplest possible Galois field.

Denote $e_n = (a")^n e_0$. Then it follows from Eqs. (3) and (4), that for any $n = 0, 1, 2, ...$

$$he_n = (q_0 + 2n)e_n, \quad Ke_n = q_0(q_0 - 2)e_n, \tag{5}$$

$$a'a"e_n = (n + 1)(q_0 + n)e_n \tag{6}$$

$$(e_{n+1}, e_{n+1}) = (n + 1)(q_0 + n)(e_n, e_n) \tag{7}$$

The case $q_0 = 0$ is trivial and corresponds to zero representation, so we assume that $q_0 \neq 0$. Then we have the case when ordinary and modular representations considerably differ each other. Consider first the ordinary case when q_0 is any real positive number. Then IR is infinite-dimensional, e_0 is a vector with a minimum eigenvalue of the operator h (minimum weight) and there are no vectors with the maximum weight. This is in agreement with the well known fact that unitary IRs of noncompact groups are infinite dimensional. However in the modular case q_0 is one of the numbers $1, ...p-1$. The set $(e_0, e_1, ...e_N)$ will be a basis of IR if $a"e_i \neq 0$ for $i < N$ and $a"e_N = 0$. These conditions must be compatible with $a'a"e_N = 0$. Therefore, as follows from Eq. (6), N is defined by the condition $q_0 + N = 0$ in F_p. As a result, $N = p - q_0$ and the dimension of IR is equal to $p - q_0 + 1$.

One might say that e_0 is the vector with the minimum weight while e_N is the vector with the maximum weight. However, the notions of "less than" or "greater than" have only a limited sense in F_p, as well as the notion of positive and negative numbers in F_p. If q_0 is positive in this sense (see Ref. [2] for details), then Eqs. (6) and (7) indicate that the modular IR under consideration can be treated as the modular analog of IR with "positive energies". However, it is easy to see that e_N is the eigenvector of the operator h with the eigenvalue $-q_0$ in F_p, and the same IRs can be treated as the modular analog of IRs with "negative energies" (see Ref. [2] for details).

3 Modular IRs of the so(2,3) algebra

The standard AdS group is ten-parametric, as well as the Poincare group. However, in contrast to the Poincare group, all the representation generators are angular momenta. In Ref. [2] we explained the reason why for our purposes it is convenient to work with the units $\hbar/2 = c = 1$. Then the representation generators are dimensionless, and the commutation relations for them can be written in the form

$$[M^{ab}, M^{cd}] = -2i(g^{ac}M^{bd} + g^{bd}M^{cd} - g^{ad}M^{bc} - g^{bc}M^{ad}) \qquad (8)$$

where a, b, c, d take the values 0,1,2,3,5, and the operators M^{ab} are antisymmetric. The diagonal metric tensor has the components $g^{00} = g^{55} = -g^{11} = -g^{22} = -g^{33} = 1$. In these units the spin of fermions is odd, and the spin of bosons is even. If s is the particle spin then the corresponding IR of the su(2) algebra has the dimension $s+1$. Note that if s is interpreted in such a way then it does not depend on the choice of units (in contrast to the maximum eigenvalue of the z projection of the spin operator).

For analyzing IRs implementing Eq. (8), it is convenient to work with another set of ten operators. Let $(a_j', a_j", h_j)$ $(j = 1,2)$ be two independent sets of operators satisfying the commutation relations for the sp(2) algebra

$$[h_j, a_j'] = -2a_j' \quad [h_j, a_j"] = 2a_j" \quad [a_j', a_j"] = h_j \tag{9}$$

The sets are independent in the sense that for different j they mutually commute with each other. We denote additional four operators as $b', b", L_+, L_-$. The meaning of L_+, L_- is as follows. The operators $L_3 = h_1 - h_2, L_+, L_-$ satisfy the commutation relations of the su(2) algebra

$$[L_3, L_+] = 2L_+ \quad [L_3, L_-] = -2L_- \quad [L_+, L_-] = L_3 \tag{10}$$

while the other commutation relations are as follows

$$\begin{gathered}
[a_1', b'] = [a_2', b'] = [a_1", b"] = [a_2", b"] = \\
[a_1', L_-] = [a_1", L_+] = [a_2', L_+] = [a_2", L_-] = 0 \\
[h_j, b'] = -b' \quad [h_j, b"] = b" \quad [h_1, L_\pm] = \pm L_\pm, \\
[h_2, L_\pm] = \mp L_\pm \quad [b', b"] = h_1 + h_2 \\
[b', L_-] = 2a_1' \quad [b', L_+] = 2a_2' \quad [b", L_-] = -2a_2" \\
[b", L_+] = -2a_1", \quad [a_1', b"] = [b', a_2"] = L_- \\
[a_2', b"] = [b', a_1"] = L_+, \quad [a_1', L_+] = [a_2', L_-] = b' \\
[a_2", L_+] = [a_1", L_-] = -b"
\end{gathered} \tag{11}$$

At first glance these relations might seem to be rather chaotic but in fact they are very natural in the Weyl basis of the so(2,3) algebra.

The relation between the above sets of ten operators is as follows

$$M_{10} = i(a_1" - a_1' - a_2" + a_2') \quad M_{15} = a_2" + a_2' - a_1" - a_1'$$
$$M_{20} = a_1" + a_2" + a_1' + a_2' \quad M_{25} = i(a_1" + a_2" - a_1' - a_2')$$
$$M_{12} = L_3 \quad M_{23} = L_+ + L_- \quad M_{31} = -i(L_+ - L_-)$$
$$M_{05} = h_1 + h_2 \quad M_{35} = b' + b" \quad M_{30} = -i(b" - b') \qquad (12)$$

In addition, if $L_+^* = L_-$, $a_j'^* = a_j"$, $b'^* = b"$ and $h_j^* = h_j$ then the operators M^{ab} are Hermitian (we do not discuss the difference between selfadjoined and Hermitian operators). In modular IRs of the so(2,3) algebra, the commutation relations (9-11) are realized in spaces over F_{p^2} and the Hermitian conjugation is understood as explained above.

There exists a vast literature on ordinary IRs of the so(2,3) algebra in Hilbert spaces. The representations relevant to elementary particles in the AdS space have been constructed for the first time in Refs. [9, 10], while modular representations of algebra (9-11) have been investigated for the first time by Braden [11]. In Refs. [1, 2] we have reformulated his investigation in such a way that the correspondence between modular and ordinary IRs are straightforward. Our construction is described below.

We use the basis in which the operators (h_j, K_j) $(j = 1, 2)$ are diagonal. Here K_j is the Casimir operator (3) for algebra $(a_j', a_j", h_j)$. For constructing IRs we need operators relating different representations of the sp(2)×sp(2) algebra. By analogy with Refs. [10, 11], one of the possible choices is as follows (see the discussion in Refs. [2, 3])

$$B^{++} = b" - a_1"L_-(h_1 - 1)^{-1} - a_2"L_+(h_2 - 1)^{-1} +$$
$$a_1"a_2"b'[(h_1 - 1)(h_2 - 1)]^{-1} \quad B^{+-} = L_+ - a_1"b'(h_1 - 1)^{-1}$$
$$B^{-+} = L_- - a_2"b'(h_2 - 1)^{-1} \quad B^{--} = b' \qquad (13)$$

We consider the action of these operators only on the space of "minimal" sp(2)×sp(2) vectors, i.e. such vectors x that $a_j'x = 0$ for $j = 1, 2$, and x is the eigenvector of the operators h_j. It is easy to see that if x is a minimal vector such that $h_jx = \alpha_jx$ then $B^{++}x$ is the minimal eigenvector of the operators h_j with the eigenvalues α_j+1, $B^{+-}x$ - with the eigenvalues (α_1+1, α_2-1), $B^{-+}x$ - with the eigenvalues (α_1-1, α_2+1), and $B^{--}x$ - with the eigenvalues α_j-1.

By analogy with the construction of ordinary representations with positive energy [9, 10], we require the existence of the vector e_0 satisfying the conditions

$$
\begin{aligned}
a'_j e_0 = b' e_0 = L_+ e_0 = 0 \quad h_j e_0 = q_j e_0 \\
(e_0, e_0) \neq 0 \quad (j = 1, 2)
\end{aligned}
\tag{14}
$$

It is well known that $M^{05} = h_1 + h_2$ is the AdS analog of the energy operator, since $M^{05}/2R$ becomes the usual energy when the AdS group is contracted to the Poincare one (here R is the radius of the AdS space while the notion of contraction has been developed in Ref. [12]). As follows from Eqs. (9) and (11), the operators (a'_1, a'_2, b') reduce the AdS energy by two units. Therefore in the conventional theory e_0 is the state with the minimum energy. In this theory the spin in our units is equal to the maximum value of the operator $L_3 = h_1 - h_2$ in the "rest state". For these reasons we use s to denote $q_1 - q_2$ and m to denote $q_1 + q_2$.

In the standard classification [9, 10], the massive case is characterized by the conditions $q_1 \geq q_2,\ q_2 > 1$ while the massless one — by $q_1 \geq q_2,\ q_2 = 1$ (see also Ref. [2]). There also exist two exceptional IRs discovered by Dirac [13] and called Dirac singletons. They are characterized by the values of (q_1, q_2) equal to (1/2,1/2) and (3/2,1/2), respectively, i.e. the values of the mass and spin are equal to (1,0) and (2,1), respectively. The quantity 1/2 in F_p is a very big number if p is big (since 1/2=$(p-1)/2$ in F_p) but nevertheless, the modular analog of the singleton IRs can be investigated easily [1] and we will not dwell on this. As explained above, in the modular case the notion of 'greater than' is not so straightforward. Nevertheless, for IRs related to elementary particles it is possible to formulate an analog of these conditions [2].

As follows from the above remarks, the elements

$$
e_{nk} = (B^{++})^n (B^{-+})^k e_0 \tag{15}
$$

represent the minimal sp(2)×sp(2) vectors with the eigenvalues of the operators h_1 and h_2 equal to $Q_1(n,k) = q_1 + n - k$ and $Q_2(n,k) = q_2 + n + k$, respectively.

We use $a(n,k)$ and $b(n,k)$ to denote the following quantities:

$$a(n,k) = \frac{(n+1)(m+n-2)(q_1+n)(q_2+n-1)}{(q_1+n-k-1)(q_2+n+k-1)} \tag{16}$$

$$b(n,k) = [(k+1)(s-k)(q_2+k-1)]/(q_2+n+k-1) \tag{17}$$

Then it can be shown by a direct calculation (see Ref. [2] for details) that

$$B^{--}B^{++}e_{nk} = a(n,k)e_{n,k} \quad (e_{n+1,k}, e_{n+1,k}) = a(n,k)(e_{nk}, e_{nk}) \tag{18}$$

$$B^{-+}B^{++}e_{nk} = b(n,k)e_{nk} \quad (e_{n,k+1}, e_{n,k+1}) = b(n,k)(e_{nk}, e_{nk}) \tag{19}$$

In the massive case, as follows from Eqs. (17) and (19), k can assume only the values $0, 1, ...s$, as well as in the ordinary case. At the same time, it follows from Eqs. (16) and (18), that, in contrast to the ordinary case where $n = 0, 1, ...\infty$, in the modular one $n = 0, 1, ...n_{max}$. where $n_{max} = p + 2 - m$. Hence the space of minimal vectors has the dimension $(s+1)(n_{max}+1)$, and IR turns out to be finite-dimensional and even finite since the field F_{p^2} is finite. In the massless case, when $q_2 = 1$, the above expressions contain ambiguities 0/0. The problem of their correct treatment in the modular case has been discussed in Refs. [2, 3] by analogy with the consideration in Ref. [10]. The result is that the values of k are in the same range, for $k = 0$ and $k = s$, n takes the values $0, 1, ..., p+1-s$ while for the values of k in the range $1 \leq k \leq s-1$ (such values of k exist if $s \geq 2$) n can take only the value $n = 0$.

In this paper we describe in detail calculations in the massive case. The derivation in the massless and singleton cases is much simpler and the corresponding results are reported in Sect. 7.

The full basis of the representation space can be chosen in the form

$$e(n_1 n_2 n k) = (a_1")^{n_1}(a_2")^{n_2} e_{nk} \tag{20}$$

where, as follows from the results of this and preceding sections,

$$n_1 = 0, 1, ...N_1(n, k) \quad n_2 = 0, 1, ...N_2(n, k)$$
$$N_1(n, k) = p - q_1 - n + k \quad N_2(n, k) = p - q_2 - n - k \tag{21}$$

As follows from Eqs. (7) and (20), the quantity

$$Norm(n_1 n_2 nk) = (e(n_1 n_2 nk), e(n_1 n_2 nk)) \tag{22}$$

can be represented as

$$Norm(n_1 n_2 nk) = F(n_1 n_2 nk) G(nk) \tag{23}$$

where

$$F(n_1 n_2 nk) = n_1!(Q_1(n, k) + n_1 - 1)! n_2!(Q_2(n, k) + n_2 - 1)! \tag{24}$$

$$G(nk) = (e_{nk}, e_{nk})/[(Q_1(n, k) - 1)!(Q_2(n, k) - 1)!] \tag{25}$$

By using Eqs. (16-19) and the definitions of $Q_1(n, k)$ and $Q_2(n, k)$, one can show by a direct calculation that

$$G(nk) = [n!k!s!(e_0, e_0)][(s - k)!(q_1 + n - k - 1)!$$
$$(q_2 + n + k - 1)]^{-1} \prod_{l=1}^{n} \{[(m + l - 3)(q_1 + l - 1)(q_2 + l - 2)]$$
$$[(q_1 + l - k - 2))(q_2 + l + k - 2)]^{-1}\} \tag{26}$$

In standard Poincare and AdS theories there also exist IRs with negative energies. They are not used in the standard approach and instead, for describing antiparticles one is using negative energy solutions of the corresponding covariant equation. THe negative energy IRs can be constructed by analogy with positive energy ones. Instead of Eq. (14) one can require the existence of the vector e_0' such that

$$a_j" e_0' = b" e_0' = L_- e_0' = 0 \quad h_j e_0' = -q_j e_0'$$
$$(e_0', e_0') \neq 0 \quad (j = 1, 2) \tag{27}$$

where the quantities q_1, q_2 are the same as for positive energy IRs. It is obvious that positive and negative energy IRs are fully independent since the spectrum of the operator M^{05} for such IRs is positive and negative, respectively. At the same time, as shown in Refs. [2, 3], *the modular analog of a positive energy IR characterized by q_1, q_2 in Eq. (14), and the modular analog of a negative energy IR characterized by the same values of q_1, q_2 in Eq. (27) represent the same modular IR.*

4 Modular IRs of the osp(1,4) superalgebra

Representations of the osp(1,4) superalgebra have several interesting distinctions from representations of the Poincare superalgebra. For this reason we first briefly mention some well known facts about the latter representations (see e.g Ref. [14] for details).

Representations of the Poincare superalgebra are described by 14 generators. Ten of them are the well known representation generators of the Poincare group — four momentum operators and six representation generators of the Lorentz group, which satisfy the well known commutation relations. In addition, there also exist four fermionic generators. The anticommutators of the fermionic generators are linear combinations of the momentum operators, and the commutators of the fermionic generators with the Lorentz group generators are linear combinations of the fermionic generators. In addition, the fermionic generators commute with the momentum operators.

From the formal point of view, representations of the osp(1,4) superalgebra are also described by 14 generators — ten representation generators of the so(2,3) algebra and four fermionic operators. There are three types of relations: the generators of the so(2,3) algebra commute with each other as usual (see the preceding section), anticommutators of the fermionic generators are linear combinations of the so(2,3) generators and commutators of the latter with the fermionic generators are their linear combinations. However, in fact representations of the osp(1,4) superalgebra can be described exclusively in terms of the fermionic generators. The matter is as follows. In the general case the anticommutators of four operators form ten independent linear combinations. Therefore, ten bosonic generators can be expressed in terms of fermionic ones. This is not the case for the Poincare superalgebra since the Poincare group generators are obtained from the so(2,3) ones by contraction. One can say that the representations of the osp(1,4) superalgebra is an implementation of the idea that the supersymmetry is the extraction of the square root from the usual symmetry (by analogy with the well known treatment of the Dirac equation as a square root from the Klein-Gordon one).

We denote the fermionic generators of the osp(1,4) superalgebra as (d_1, d_2, d_1^*, d_2^*) where the * means the Hermitian conjugation as usual. They should satisfy the following relations. If (A, B, C) are any fermionic generators, [...,...] is used to denote a commutator and $\{..., ...\}$ to denote an anticommutator then

$$[A, \{B, C\}] = F(A, B)C + F(A, C)B \tag{28}$$

where the form $F(A, B)$ is skew symmetric, $F(d_j, d_j^*) = 1$ $(j = 1, 2)$ and the other independent values of $F(A, B)$ are equal to zero. The fact that the representation of the osp(1,4) superalgebra is fully defined by Eq. (28) and the properties of the form $F(.,.)$, shows that osp(1,4) is a special case of the superalgebra.

We can now **define** the so(2,3) generators as follows:

$$\begin{gathered} b' = \{d_1, d_2\} \quad b" = \{d_1^*, d_2^*\} \quad L_+ = \{d_2, d_1^*\} \quad L_- = \{d_1, d_2^*\} \\ a_j' = (d_j)^2 \quad a_j" = (d_j^*)^2 \quad h_j = \{d_j, d_j^*\} \quad (j = 1, 2) \end{gathered} \tag{29}$$

Then by using Eq. (28) and the properties of the form $F(.,.)$, one can show by a direct calculations that so defined operators satisfy the commutation relations (9-11). This result can be treated as a fact that the representation generators of the so(2,3) algebra are not fundamental, only the fermionic generators are.

By analogy with the construction of IRs of the osp(1,4) superalgebra in the conventional theory [5], we require the existence of the cyclic vector e_0 satisfying the conditions (compare with Eq. (14)):

$$d_j e_0 = L_+ e_0 = 0 \quad h_j e_0 = q_j e_0 \quad (e_0, e_0) \neq 0 \quad (j = 1, 2) \tag{30}$$

The full representation space can be obtained by successively acting by the fermionic generators on e_0 and taking all possible linear combinations of such vectors.

We use E to denote an arbitrary linear combination of the vectors $(e_0, d_1^* e_0, d_2^* e_0, d_2^* d_1^* e_0)$. Our next goal is to prove a statement analogous to that in Ref. [5]:

Statement 1: Any vector from the representation space can be represented as a linear combination of the elements $O_1O_2...O_nE$ where $n = 0, 1, , ,$ and O_i is a generator of the so(2,3) algebra.

The first step is to prove a simple

Lemma: If D is any fermionic generator then DE is a linear combination of elements E and OE where O is a generator of the so(2,3) algebra.

The proof is by a straightforward check using Eqs. (28-30). For example,

$$d_1^*(d_2^*d_1^*e_0) = \{d_1^*, d_2^*\}d_1^*e_0 - d_2^*a_1"e_0 = b"d_1^*e_0 - a_1"d_2^*e_0 \ etc.$$

To prove Statement 1 we define the height of a linear combination of the elements $O_1O_2...O_nE$ as the maximum sum of powers of the fermionic generators in this element. For example, since every generator of the so(2,3) algebra is composed of two fermionic generators, the height of the element $O_1O_2...O_nE$ is equal to $2n+2$ if E contains $d_2^*d_1^*e_0$, is equal to $2n+1$ if E does not contain $d_2^*d_1^*e_0$ but contains either $d_1^*e_0$ or $d_2^*e_0$ and is equal to $2n$ if E contains only e_0.

We can now prove Statement 1 by induction. The elements with the heights 0,1 and 2 obviously have the required form since, as follows from Eq. (29), $d_1^*d_2^*e_0 = b"e_0 - d_2^*d_1^*e_0$. Let us assume that Statement 1 is correct for all elements with the heights $\leq N$. Every element with the height $N+1$ can be represented as Dx where x is an element with the height N. If $x = O_1O_2...O_nE$ then by using Eq. (28) we can represent Dx as $Dx = O_1O_2...O_nDE + y$ where the height of the element y is equal to $N-1$. As follows from the induction assumption, y has the required form, and, as follows from Lemma, DE is a linear combination of the elements E and OE. Therefore Statement 1 is proved.

As follows from Eqs. (28) and (29),

$$\begin{aligned}
&[d_j, h_j] = d_j \quad [d_j^*, h_j] = -d_j^* \quad (j = 1, 2) \\
&[d_j, h_l] = [d_j^*, h_l] = 0 \quad (j, l = 1, 2\ j \neq l)
\end{aligned} \tag{31}$$

It follows from these expressions that if x is such that $h_jx = \alpha_jx$ ($j = 1, 2$) then d_1^*x is the eigenvector of the operators h_j with the eigenvalues

(α_1+1,α_2), d_2^*x - with the eigenvalues (α_1,α_2+1), d_1x - with the eigenvalues (α_1-1,α_2), and d_2x - with the eigenvalues α_1,α_2-1.

Let us assume that $q_2 \geq 1$ and $q_1 \geq q_2$ in the sense explained in the preceding section. We again use m to denote q_1+q_2 and s to denote q_1-q_2.

Statement 1 obviously remains valid if we now assume that E contains linear combinations of (e_0,e_1,e_2,e_3) where

$$\begin{aligned} e_1 = d_1^*e_0 \quad e_2 = d_2^*e_0 - \tfrac{1}{s+1}L_-e_1 \\ e_3 = (d_2^*d_1^*e_0 - \tfrac{q_1-1}{m-2}b" + \tfrac{1}{m-2}a_1"L_-)e_0 \end{aligned} \tag{32}$$

We assume for simplicity that $(e_0,e_0)=1$. Then it can be shown by direct calculations using Eqs. (28-30) that

$$(e_1,e_1)=q_1 \quad (e_2,e_2)=\frac{s(q_2-1)}{s+1} \quad (e_3,e_3)=\frac{q_1(q_2-1)(m-1)}{m-2} \tag{33}$$

As follows from Eqs. (28-31), e_0 satisfies Eq. (14) and e_1 satisfies the same condition with q_1 replaced by q_1+1. We see that the representation of the osp(1,4) superalgebra defined by Eq. (30) necessarily contains at least two IRs of the so(2,3) algebra characterized by the values of the mass and spin (m,s) and $(m+1,s+1)$, and the cyclic vectors e_0 and e_1, respectively.

As follows from Eqs. (28-31), the vectors e_2 and e_3 satisfy the conditions

$$\begin{aligned} h_1e_2 = q_1e_2 \quad h_2e_2 = (q_2+1)e_2 \quad h_1e_3 = (q_1+1)e_3 \\ h_2e_3 = (q_2+1)e_3 \quad a_1'e_j = a_2'e_j = b'e_j = L_+e_j = 0 \end{aligned} \tag{34}$$

$(j=2,3)$ and therefore (see Eq. (14)) they are candidates for being cyclic vectors of IRs of the so(2,3) algebra if their norm is not equal to zero. As follows from Eq. (33), $(e_2,e_2)\neq 0$ if $s\neq 0$ and $q_2\neq 1$. Therefore, if these conditions are satisfied, e_2 is the cyclic vector of IR of the so(2,3) algebra characterized by the values of the mass and spin $(m+1,s-1)$. Analogously, if $q_2\neq 1$ then e_3 is the cyclic vector of IR of the so(2,3) algebra characterized by the values of the mass and spin $(m+2,s)$.

As already mentioned, our considerations are similar to those in Ref. [5]. Therefore modular IRs of the osp(1,4) can be characterized in the same way as conventional IRs [5]:

- If $q_2 > 1$ and $s \neq 0$ (massive IRs), the osp(1,4) supermultiplets contain four IRs of the so(2,3) algebra characterized by the values of the mass and spin
$$(m, s), (m+1, s+1), (m+1, s-1), (m+2, s).$$
- If $q_2 > 1$ and $s = 0$ (collapsed massive IRs), the osp(1,4) supermultiplets contain three IRs of the so(2,3) algebra characterized by the values of the mass and spin
$$(m, s), (m+1, s+1), (m+2, s).$$
- If $q_2 = 1$ (massless IRs) the osp(1,4) supermultiplets contains two IRs of the so(2,3) algebra characterized by the values of the mass and spin
$$(2+s, s), (3+s, s+1)$$
- Dirac supermultiplet containing two Dirac singletons (see the preceding section).

The first three cases have well known analogs of IRs of the super-Poincare algebra (see e.g. Ref. [14]) while there is no super-Poincare analog of the Dirac supermultiplet.

In this paper we describe details of calculation only for the first case while the results for other cases are described in Sect. 7.

We will say that the massive supermultiplet consists of particle 0, particle 1, particle 2 and particle 3, where particle i ($i = 0, 1, 2, 3$) is described by the IR of the so(2,3) algebra with the cyclic vector e_i. The set of minimal sp(2)×sp(2) vectors in the multiplet will be characterized by the elements $e(nk; i)$ where the range of (n, k) for each i is defined by the values of the corresponding mass and spin (m_i, s_i). As follows from the above classification,

$$m_0 = m \quad s_0 = s \quad m_1 = m+1 \quad s_1 = s+1 \quad m_2 = m+1$$
$$s_2 = s-1 \quad m_3 = m+2 \quad s_3 = s \tag{35}$$

The full basis of the supermultiplet consists of elements $e(n_1n_2nk;i)$ where the range of (n_1n_2nk) for each i can be defined as described in the preceding section. We also use (q_{1i}, q_{2i}) to denote the eigenvalues of the vectors e_i with respect to h_1 and h_2, i.e. $h_je_i = q_{ji}e_i$ $(j = 1, 2)$. It is easy to see that

$$q_{10} = q_{12} = q_1 \; q_{11} = q_{13} = q_1 + 1 \; q_{20} = q_{21} = q_2 \; q_{22} = q_{23} = q_2 + 1 \quad (36)$$

We now define

$$Norm(n_1n_2nk;i) = (e(n_1n_2nk;i), e(n_1n_2nk;i)) \quad (i = 0, 1, 2, 3) \quad (37)$$

Then one can show by analogy with Eqs. (24-26) that

$$\begin{gathered} Norm(n_1n_2nk;i) = F(n_1n_2nk;i)G(nk;i) \quad where \\ F(n_1n_2nk;i) = n_1!(q_{1i} + n - k + n_1 - 1)!n_2!(q_{2i} + n + k + n_2 - 1)! \\ G(nk;i) = [n!k!s_i!(e_i, e_i)][(s_i - k)!(q_{1i} + n - k - 1)! \\ (q_{2i} + n + k - 1)]^{-1} \prod_{l=1}^{n}\{[(m_i + l - 3)(q_{1i} + l - 1)(q_{2i} + l - 2)] \\ [(q_{1i} + l - k - 2))(q_{2i} + l + k - 2)]^{-1}\} \end{gathered} \quad (38)$$

5 Matrix elements of fermionic operators

As already noted, the main goal of the present paper is to prove that the representation operators of the osp(1,4) superalgebra are compatible with the AB symmetry. Since the representation operators of the so(2,3) algebra are bilinear in fermionic operators (see Eq. (29)), it is sufficient to prove the compatibility for the latter. Moreover, it is sufficient to prove the compatibility only for the operators d_1^* and d_2^* since d_1 and d_2 are adjoint to d_1^* and d_2^*, respectively.

By analogy with the way of finding matrix elements in the so(2,3) case (see Sect. 3), we can first investigate operators acting in the subspace of minimal sp(2)×sp(2) vectors. Let us define

$$B_j^- = d_j \quad B_j^+ = d_j^* - a_j"d_j(h_j - 1)^{-1} \quad (j = 1, 2) \quad (39)$$

By using Eqs. (28) and (29), it is easy to show that if x is a minimal vector such that $h_jx = \alpha_jx$ then B_1^+x is the minimal eigenvector of the operators (h_1, h_2) with the eigenvalues (α_1+1, α_2), B_2^+x - with the eigenvalues (α_1, α_2+1), B_1^-x - with the eigenvalues (α_1-1, α_2), and B_2^-x - with the eigenvalues (α_1, α_2-1). By using also Eq. (13), one can show by a direct calculation that

$$[B_j^+, B^{++}] = [B_1^+, B^{+-}] = [B_2^+, B^{-+}] = [B_j^-, B^{--}] = \\ [B_1^-, B^{-+}] = [B_2^-, B^{+-}] = 0 \quad (j=1,2) \tag{40}$$

In addition, if x and y are any minimal sp(2)×sp(2) vectors then

$$(x, B_j^+y) = (B_j^-x, y) \quad (j=1,2) \tag{41}$$

This is a consequence of the definition (39) and the fact that $(a_j")^*x = a_j'x = 0$.

By using these expressions, one can proceed as follows. First one can compute $B_j^+e(0k;i)$ and $B_j^-e(0k;i)$ by using Eqs. (13), (15), (28), (29) and the fact that, as follows from Eq. (14), $e(0k;i) = (L_-)^k e(00;i)$. Then, by using Eq. (40) and (41), one can compute $B_j^+e(nk;i)$ and $B_j^-e(nk;i)$. When these calculations are done, one can compute $d_j^*e(n_1n_2nk;i)$ by using Eqs. (20), (39) and the fact that the d_j^* commute with $a_j"$:

$$d_j^*e(n_1n_2nk;i) = (a_1")^{n_1}(a_2")^{n_2}d_j^*e(nk;i) = \\ (a_1")^{n_1}(a_2")^{n_2}[B_j^+ + a_j"B_j^-(h_j-1)^{-1}]e(nk;i) \quad (j=1,2) \tag{42}$$

The final result is as follows.

$$d_1^*e(n_1n_2nk;0) = \frac{s+1-k}{s+1}e(n_1n_2nk;1) - \\ ke(n_1n_2,n,k-1;2) + \\ \frac{n(q_2+n-2)}{(q_1+n-k-1)(q_1+n-k-2)(s+1)}e(n_1+1,n_2,n-1,k+1;1) + \\ \frac{n(q_1+n-1)}{(q_1+n-k-1)(q_1+n-k-2)}e(n_1+1,n_2,n-1,k;2) \tag{43}$$

$$d_1^*e(n_1n_2nk;1) = \frac{k(q_2+k-2)}{m-2}e(n_1n_2,n+1,k-1;0) - \\ ke(n_1n_2n,k-1;3) + \\ \frac{(m+n-2)(q_1+n)(q_1-k-1)}{(m-2)(q_1+n-k)(q_1+n-k-1)}e(n_1+1,n_2nk;0) + \\ \frac{n(q_1+n)}{(q_1+n-k)(q_1+n-k-1)}e(n_1+1,n_2,n-1,k;3) \tag{44}$$

$$
\begin{gathered}
d_1^* e(n_1 n_2 n k; 2) = \tfrac{(s-k)(q_2+k+1)}{(s+1)(m-2)} e(n_1 n_2, n+1, k; 0) - \\
\tfrac{s-k}{s+1} e(n_1 n_2 n k; 3) - \\
\tfrac{(q_2+n-1)(m+n-2)(q_1-k-2)}{(q_1+n-k-1)(q_1+n-k-2)((s+1)(m-2)} e(n_1+1, n_2, n, k+1; 0) - \\
\tfrac{n(q_2+n-1))}{(q_1+n-k-1)(q_1+n-k-2)(s+1)} e(n_1+1, n_2, n-1, k+1; 3)
\end{gathered} \tag{45}
$$

$$
\begin{gathered}
d_1^* e(n_1 n_2 n k; 3) = \tfrac{(q_2+k-1)(s+1-k)}{(m-2)(s+1)} e(n_1 n_2, n+1, k; 1) - \\
\tfrac{k(q_2+k-1)}{m-2} e(n_1 n_2, n+1, k-1; 2) - \\
\tfrac{(m+n-1)(q_1-k-1)(q_2+n-1)}{(m-2)(q_1+n-k)(q_1+n-k-1)(s+1)} e(n_1+1, n_2, n, k+1; 1) - \\
\tfrac{(m+n-1)(q_1-k-1)(q_1+n)}{(m-2)(q_1+n-k)(q_1+n-k-1)} e(n_1+1, n_2, n, k; 2)
\end{gathered} \tag{46}
$$

$$
\begin{gathered}
d_2^* e(n_1 n_2 n k; 0) = e(n_1 n_2 n k; 2) + \tfrac{1}{s+1} e(n_1 n_2 n, k+1; 1) + \\
\tfrac{n(s+1-k)(q_2+n-2)}{(s+1)(q_2+n+k-1)(q_2+n+k-2)} e(n_1, n_2+1, n-1, k; 1) - \\
\tfrac{kn(q_1+n-1)}{(q_2+n+k-1)(q_2+n+k-2)} e(n_1, n_2+1, n-1, k-1; 2)
\end{gathered} \tag{47}
$$

$$
\begin{gathered}
d_2^* e(n_1 n_2 n k; 1) = \tfrac{q_1-k-1}{m-2} e(n_1 n_2, n+1, k; 0) + e(n_1 n_2 n k; 3) + \\
\tfrac{k(q_1+n)(m+n-2)(q_2+k-2)}{(m-2)(q_2+n+k-1)(q_2+n+k-2)} e(n_1, n_2+1, n, k-1; 0) - \\
\tfrac{kn(q_1+n)}{(q_2+n+k-1)(q_2+n+k-2)} e(n_1, n_2+1, n-1, k-1; 3)
\end{gathered} \tag{48}
$$

$$
\begin{gathered}
d_2^* e(n_1 n_2 n k; 2) = -\tfrac{q_1-k-2}{(s+1)(m-2)} e(n_1, n_2, n+1, k+1; 0) - \\
-\tfrac{1}{s+1} e(n_1 n_2 n, k+1; 3) + \\
\tfrac{(s-k)(m+n-2)(q_2+n-1)(q_2+k-1)}{(s+1)(m-2)(q_2+n+k)(q_2+n+k-1)} e(n_1, n_2+1, nk; 0) - \\
\tfrac{n(s-k)(q_2+n-1)}{(s+1)(q_2+n+k)(q_2+n+k-1)} e(n_1, n_2+1, n-1, k; 3)
\end{gathered} \tag{49}
$$

$$
\begin{gathered}
d_2^* e(n_1 n_2 n k; 3) = -\tfrac{q_1-k-1}{m-2} e(n_1, n_2, n+1, k; 2) - \\
\tfrac{q_1-k-1}{(m-2)(s+1)} e(n_1 n_2, n+1, k+1; 1) + \\
\tfrac{(m+n-1)(q_2+k-1)}{(m-2)(q_2+n+k-1)(q_2+n+k)} [\tfrac{(q_2+n-1)(s+1-k)}{s+1} e(n_1, n_2+1, nk; 1) - \\
k(q_1+n) e(n_1, n_2+1, n, k-1; 2)]
\end{gathered} \tag{50}
$$

6 Second quantization of representation operators

In conventional quantum theory the operators of physical quantities act in the Fock space of the system under consideration. Suppose, for example, that the system consists of free superparticles and their antiparticles. Since $(n_1n_2nk;i)$ is the full set of quantum numbers characterizing the superparticles and their antiparticles, one can define creation and annihilation operators for them. Let

$$a(n_1n_2nk;i),\quad a(n_1n_2nk;i)^*,\quad b(n_1n_2nk;i)\quad b(n_1n_2nk;i)^*$$

be the operators having the meaning of annihilation operator for particle i in the state with the quantum numbers (n_1n_2nk), creation operator for particle i in the state with these quantum numbers, annihilation operator for antiparticle i in the state with these quantum numbers, and creation operator for antiparticle i in the state with these quantum numbers, respectively. The basis of the Fock space consists of the vacuum vector Φ_0 and linear combinations of vectors obtained by acting on Φ_0 by the creation operators for particles and antiparticles with all possible quantum numbers.

In the standard approach, the necessity to have separate annihilation and creation operators for particles and antiparticles, respectively, is related to the fact that they are described by independent IRs of the symmetry algebra. On the contrary, in the GFQT one IR describes a particle and its antiparticle simultaneously. For this reason there is no need to introduce (b, b^*) operators since the set of all possible operators (a, a^*) describes annihilation and creation of both particles and antiparticles. This question has been already discussed in detail in Refs. [2, 3].

The problem of second quantization of representation operators can now be formulated as follows. Let $(A_1, A_2 ....A_n)$ be representation generators describing IR of an algebra or superalgebra. One should replaced them by operators acting in the Fock space such that the commutation-anticommutation relations between the new operators are the same.

Let A be a representation generator of the osp(1,4) algebra. Its matrix elements are defined as

$$Ae(n_1n_2nk;i)=\sum_{n_1'n_2'n'k'i'}A(n_1'n_2'n'k';i'|n_1n_2nk;i)e(n_1'n_2'n'k';i') \quad (51)$$

If B is the operator adjoint to A then, as follows from the relation $(x,Ay)=(Bx,y)$, the matrix elements of these operators are related to each other as

$$B(n_1'n_2'n'k';i'|n_1n_2nk;i)=[Norm(n_1n_2nk;i)/Norm(n_1'n_2'n'k';i')] \overline{A(n_1n_2nk;i|n_1'n_2'n'k';i')} \quad (52)$$

This relation differs from the usual one since our basis elements are not normalized to one.

We assume that the operators $a(n_1n_2nk;i)$ and $a(n_1n_2nk;i)^*$ satisfy either the anticommutation relation

$$\{a(n_1n_2nk;i),a(n_1'n_2'n'k';i')^*\}=Norm(n_1n_2nk;i)\delta_{n_1n_1'}\delta_{n_2n_2'}\delta_{nn'}\delta_{kk'}\delta_{ii'} \quad (53)$$

or the commutation relation

$$[a(n_1n_2nk;i),a(n_1'n_2'n'k';i')^*]=Norm(n_1n_2nk;i)\delta_{n_1n_1'}\delta_{n_2n_2'}\delta_{nn'}\delta_{kk'}\delta_{ii'} \quad (54)$$

In that case $\{a,a'\}=\{a^*,a^{'*}\}=0$ for any pair of a or a^* operators in the case of Eq. (53) and $[a,a']=[a^*,a^{'*}]=0$ in the case of Eq. (54). Then the secondly quantized form of the operator A is

$$A=\sum A(n_1'n_2'n'k';i'|n_1n_2nk;i)a(n_1'n_2'n'k';i')^* a(n_1n_2nk;i)/Norm(n_1n_2nk;i) \quad (55)$$

where the sum is taken over all possible values of $(n_1'n_2'n'k'i'n_1n_2nki)$. The operator adjoint to A in the Fock space is obviously

$$A^*=\sum \overline{A(n_1'n_2'n'k';i'|n_1n_2nk;i)}a(n_1n_2nk;i)^* a(n_1'n_2'n'k';i')/Norm(n_1n_2nk;i) \quad (56)$$

At the same time, if B is the operator adjoint to A in the space of IR then its secondly quantized form is

$$B = \sum B(n'_1 n'_2 n' k'; i' | n_1 n_2 n k; i) a(n'_1 n'_2 n' k'; i')^* \\ a(n_1 n_2 n k; i) / Norm(n_1 n_2 n k; i) \tag{57}$$

As follows from Eq. (52), the operators given by Eqs. (56) and (57), are equal to each other and therefore the correspondence between the operators in the space of IR and the Fock space is compatible with the Hermitian conjugation in these spaces.

Suppose now that A and B are arbitrary operators in the space of IR, and Eqs. (55) and (57) represent their secondly quantized forms, respectively. Then, if A and B satisfy some commutation relation in the space of IR, the corresponding secondly quantized operators satisfy the same commutation relation regardless whether the (a, a^*) operators satisfy Eq. (53) or Eq. (54). This means that for representations of ordinary Lie algebras and their modular analogs, the above quantization procedure by itself does not impose any restrictions on the type of statistics.

Let now A and B be two fermionic operators in IR of the osp(1,4) superalgebra and Eqs. (55) and (57) be their secondly quantized forms. Since the anticommutator of A and B is a linear combination of generators of the so(2,3) algebra, the question arises when the same relation is valid for the corresponding secondly quantized operators. In other words, if C is an operator in the space of IR, $\{A, B\} = C$ in this space, and the secondly quantized form of C is

$$C = \sum C(n'_1 n'_2 n' k'; i' | n_1 n_2 n k; i) a(n'_1 n'_2 n' k'; i')^* \\ a(n_1 n_2 n k; i) / Norm(n_1 n_2 n k; i) \tag{58}$$

then the question arises when $\{A, B\} = C$ for the secondly quantized operators A, B and C given by Eqs. (55), (57) and (58), respectively.

It is easy to see that this relation cannot be satisfied if the (a, a^*) operators satisfy either Eq. (53) or Eq. (54) for all the values of i and i'. This means that supersymmetry indeed combines particles with different type of statistics into one supermultiplet. For a short time, we renumerate the particles as follows: particles 0 and 2 will refer to bosons while particles 1 and 3 - to fermions. Assuming that bosonic and fermionic operators commute with each other, we can write the relations between all the (a, a^*) operators as follows

$$\begin{aligned} a(n_1n_2nk;i)a(n_1'n_2'n'k';i')^* = Norm(n_1n_2nk;i)\delta_{n_1n_1'}\delta_{n_2n_2'}\delta_{nn'}\delta_{kk'}\delta_{ii'} \\ +(-1)^{ii'}a(n_1'n_2'n'k';i')^*a(n_1n_2nk;i) \\ a(n_1n_2nk;i)^*a(n_1'n_2'n'k';i')^* = (-1)^{ii'}a(n_1'n_2'n'k';i')^*a(n_1n_2nk;i)^* \\ a(n_1n_2nk;i)a(n_1'n_2'n'k';i') = (-1)^{ii'}a(n_1'n_2'n'k';i')a(n_1n_2nk;i) \end{aligned} \quad (59)$$

Then one can directly verify the validity of the following

Statement 2: if A, B and C are operators in the representation space of the osp(1,4) algebra such that $\{A, B\} = C$, then the secondly quantized forms of these operators given by Eqs. (56-58) satisfy the same relation in the Fock space if and only if the operators A and B have nonzero matrix elements only for transitions *boson* $\rightarrow$ $fermion$ and $fermion \rightarrow$ *boson*.

Returning to the initial way of enumerating the particles in the supermultiplet (see Sect. 4), we conclude that, as a consequence of Eqs. (43-50)): if particle 0 is a boson then particles 1 and 2 are fermions and particle 3 is a boson while if particle 0 is a fermion then particles 1 and 2 are bosons and particle 3 is a fermion. As follows from these expressions and the discussed rule of constructing secondly quantized operators, the secondly quantized forms of the operators d_1^* and d_2^* can be written as follows:

$$d_1^* = d_1^{*(0)} + d_1^{*(1)} + d_1^{*(2)} + d_1^{*(3)} \quad d_2^* = d_2^{*(0)} + d_2^{*(1)} + d_2^{*(2)} + d_2^{*(3)} \quad (60)$$

where

$$\begin{aligned} d_1^{*(0)} = \sum\{\tfrac{s+1-k}{s+1}a(n_1n_2nk;1)^* - ka(n_1n_2,n,k-1;2)^* + \\ \tfrac{n(q_2+n-2)}{(q_1+n-k-1)(q_1+n-k-2)(s+1)}a(n_1+1,n_2,n-1,k+1;1)^* + \\ \tfrac{n(q_1+n-1)}{(q_1+n-k-1)(q_1+n-k-2)}a(n_1+1,n_2,n-1,k;2)^*\} \\ a(n_1n_2nk;0)/Norm(n_1n_2nk;0) \end{aligned} \quad (61)$$

$$d_1^{*(1)} = \sum\{\tfrac{k(q_2+k-2)}{m-2}a(n_1n_2,n+1,k-1;0)^* -$$
$$ka(n_1n_2n,k-1;3)^* +$$
$$\tfrac{(m+n-2)(q_1+n)(q_1-k-1)}{(m-2)(q_1+n-k)(q_1+n-k-1)}a(n_1+1,n_2nk;0)^* +$$
$$\tfrac{n(q_1+n)}{(q_1+n-k)(q_1+n-k-1)}a(n_1+1,n_2,n-1,k;3)^*\}$$
$$a(n_1n_2nk;1)/Norm(n_1n_2nk;1) \tag{62}$$

$$d_1^{*(2)} = \sum\{\tfrac{(s-k)(q_2+k+1)}{(s+1)(m-2)}a(n_1n_2,n+1,k;0)^* - \tfrac{s-k}{s+1}a(n_1n_2nk;3)^* -$$
$$\tfrac{(q_2+n-1)(m+n-2)(q_1-k-2)}{(q_1+n-k-1)(q_1+n-k-2)((s+1)(m-2)}a(n_1+1,n_2,n,k+1;0)^* -$$
$$\tfrac{n(q_2+n-1))}{(q_1+n-k-1)(q_1+n-k-2)(s+1)}a(n_1+1,n_2,n-1,k+1;3)\}$$
$$a(n_1n_2nk;2)/Norm(n_1n_2nk;2) \tag{63}$$

$$d_1^{*(3)} = \sum\{\tfrac{(q_2+k-1)(s+1-k)}{(m-2)(s+1)}a(n_1n_2,n+1,k;1)^* -$$
$$\tfrac{k(q_2+k-1)}{m-2}a(n_1n_2,n+1,k-1;2)^* -$$
$$\tfrac{(m+n-1)(q_1-k-1)(q_2+n-1)}{(m-2)(q_1+n-k)(q_1+n-k-1)(s+1)}a(n_1+1,n_2,n,k+1;1)^* -$$
$$\tfrac{(m+n-1)(q_1-k-1)(q_1+n)}{(m-2)(q_1+n-k)(q_1+n-k-1)}a(n_1+1,n_2,n,k;2)^*\}$$
$$a(n_1n_2nk;3)/Norm(n_1n_2nk;3) \tag{64}$$

$$d_2^{*(0)} = \sum\{a(n_1n_2nk;2)^* + \tfrac{1}{s+1}a(n_1n_2n,k+1;1)^* +$$
$$\tfrac{n(s+1-k)(q_2+n-2)}{(s+1)(q_2+n+k-1)(q_2+n+k-2)}a(n_1,n_2+1,n-1,k;1)^* -$$
$$\tfrac{kn(q_1+n-1)}{(q_2+n+k-1)(q_2+n+k-2)}a(n_1,n_2+1,n-1,k-1;2)^*\}$$
$$a(n_1n_2nk;0)/Norm(n_1n_2nk;0) \tag{65}$$

$$d_2^{*(1)} = \sum\{\tfrac{q_1-k-1}{m-2}e(n_1n_2,n+1,k;0) + a(n_1n_2nk;3)^* +$$
$$\tfrac{k(q_1+n)(m+n-2)(q_2+k-2)}{(m-2)(q_2+n+k-1)(q_2+n+k-2)}a(n_1,n_2+1,n,k-1;0)^* -$$
$$\tfrac{kn(q_1+n)}{(q_2+n+k-1)(q_2+n+k-2)}a(n_1,n_2+1,n-1,k-1;3)^*\}$$
$$a(n_1n_2nk;1)/Norm(n_1n_2nk;1) \tag{66}$$

$$d_2^{*(2)} = \sum\{-\tfrac{q_1-k-2}{(s+1)(m-2)}a(n_1,n_2,n+1,k+1;0)^* -$$
$$\tfrac{1}{s+1}a(n_1n_2n,k+1;3)^* +$$
$$\tfrac{(s-k)(m+n-2)(q_2+n-1)(q_2+k-1)}{(s+1)(m-2)(q_2+n+k)(q_2+n+k-1)}a(n_1,n_2+1,nk;0)^* -$$
$$\tfrac{n(s-k)(q_2+n-1)}{(s+1)(q_2+n+k)(q_2+n+k-1)}a(n_1,n_2+1,n-1,k;3)^*\}$$
$$a(n_1n_2nk;2)/Norm(n_1n_2nk;2) \qquad (67)$$

$$d_2^{*(3)} = \sum\{-\tfrac{q_1-k-1}{m-2}a(n_1,n_2,n+1,k;2)^* -$$
$$\tfrac{q_1-k-1}{(m-2)(s+1)}a(n_1n_2,n+1,k+1;1)^* +$$
$$\tfrac{(m+n-1)(q_2+k-1)}{(m-2)(q_2+n+k-1)(q_2+n+k)}[\tfrac{(q_2+n-1)(s+1-k)}{s+1}a(n_1,n_2+1,nk;1)^* -$$
$$k(q_1+n)a(n_1,n_2+1,n,k-1;2)]\}$$
$$a(n_1n_2nk;3)/Norm(n_1n_2nk;3) \qquad (68)$$

where the sum in each expression is taken over all possible values of the quantities (n_1n_2nk). Using these equations one can easily write down (if necessary) the secondly quantized forms of the operators d_1 and d_2.

The reader can notice that the results of this and preceding sections are obtained without explicitly using the fact that we consider the theory over a Galois field. The analogous results take place in the standard theory as well. The difference between the two approaches is that in the standard one the quantities (n_1n_2n) take the infinite number of values while in the GFQT - only the finite ones.

7 AB symmetry of fermionic operators

One can reformulate the result (21) as follows. For each i, n and k, the quantities n_1 and n_2 take the values

$$n_1 = 0,1,...N_1(n,k;i) \quad n_2 = 0,1,...N_2(n,k;i)$$
$$N_1(n,k;i) = p - q_{1i} - n + k \quad N_2(n,k;i) = p - q_{2i} - n - k \qquad (69)$$

Then, as follows from the results of Ref. [3], the secondly quantized so(2,3) generators for particle i are invariant under the AB transformation

$$
\begin{aligned}
a(n_1 n_2 nk;i)^* &\to (-1)^{n_1+n_2+n}\alpha_i \\
a(N_1(n,k;i)-n_1&, N_2(n,k;i)-n_2, nk;i) \\
F(N_1(n,k;i)-n_1&, N_2(n,k;i)-n_2, nk;i)^{-1} \\
a(n_1 n_2 nk;i) &\to (-1)^{n_1+n_2+n}\bar{\alpha}_i \\
a(N_1(n,k;i)-n_1&, N_2(n,k;i)-n_2, nk;i)^* \\
F(N_1(n,k;i)-n_1&, N_2(n,k;i)-n_2 nk;i)^{-1}
\end{aligned}
\tag{70}
$$

if the constant α_i, which can be called the AB parity of particle i, satisfies the condition

$$\alpha_i \bar{\alpha}_i = \pm(-1)^{s_i} \tag{71}$$

with the plus sign for fermions and the minus sign for bosons.

If the spin-statistics theorem is applicable to the GFQT (i.e. we have a property that the fermions have an odd spin in our units and the bosons - the even one) then Eq. (71) becomes

$$\alpha_i \bar{\alpha}_i = -1 \tag{72}$$

for both, fermions and bosons. Such a relation is impossible in the standard theory but is possible if $\alpha_i \in F_{p^2}$. Indeed, we can use the fact that any Galois field is cyclic with respect to multiplication [6]. Let r be a primitive root of F_{p^2}. This means that any element of F_{p^2} can be represented as a power of r. As mentioned in Sect. 2, F_{p^2} has only one nontrivial automorphism which is defined as $\alpha \to \bar{\alpha} = \alpha^p$. Therefore if $\alpha = r^k$ then $\alpha\bar{\alpha} = r^{(p+1)k}$. On the other hand, since $r^{(p^2-1)} = 1$, we conclude that $r^{(p^2-1)/2} = -1$. Therefore there exists at least a solution with $k = (p-1)/2$.

Now the following remarks are in order. We assume that the AB symmetry is the fundamental symmetry in the GFQT. This means in particular that in the supersymmetric version of the GFQT the generators obtained after the transformation (70), satisfy the same relations as the original generators. This property is obviously satisfied if all the generators, including the fermionic ones, are invariant under this transformation. However, the relations (28) also remain unchanged if the AB transformation transforms (d_1, d_2, d_1^*, d_2^*) into $(-d_1, -d_2, -d_1^*, -d_2^*)$. In other words, while the generators of the so(2,3) algebra should necessarily have the AB parity equal to 1, the fermionic generators can have the AB parity equal to either 1 or -1. At this stage it is not clear which of the possibilities should be accepted. For this reason we investigate for definiteness the conditions when the fermionic generators are invariant under the AB transformation, i.e. their AB parity is equal to 1. It will be clear that the results can be easily reformulated if the AB parity of the fermionic generators should be equal to -1.

By using the famous Wilson theorem in number theory that $(p-1)! = -1\ (mod\ p)$ in F_p (see e.g. Ref. [6]) and Eqs. (38) and (69), one can easily show that

$$F(N_1(n,k;i)-n_1, N_2(n,k;i)-n_2, nk;i)F(n_1, n_2nk;i) = (-1)^{s_i} \quad (73)$$

This fact is very important in investigating the AB symmetry.

Consider the first term in Eq. (61)

$$d_{1(1)}^{*(0)} = \sum \frac{s+1-k}{s+1} a(n_1n_2nk;1)^* a(n_1n_2nk;0)/Norm(n_1n_2nk;0) \quad (74)$$

As follows from the results of Sect. 3, the quantities (n_1n_2nk) in this sum are in the range

$$n_1 \in [0, N_1(n,k;1)]\ n_2 \in [0, N_2(n,k;0)]\ n \in [0, p-3-m]\ k \in [0,s] \quad (75)$$

since $q_{11} = q_1 + 1$, $q_{21} = q_2$, $s_1 = s+1$ and $m_1 = m+1$.

Now we apply the transformation (70). Assuming that the bosonic and fermionic operators commute with each other and using Eqs. (38) and (69), we obtain that the transformed form of this expression is

$$\begin{gathered}(d_{1(1)}^{*(0)})^{AB} = \alpha_1\bar{\alpha}_0 \sum [(s+1-k)/(s+1)] \\ a(N_1(n,k;0)-n_1, N_2(n,k;0)-n_2, nk;0)^* \\ a(N_1(n,k;1)-n_1, N_2(n,k;1)-n_2, nk;1) \\ [F(N_1(n,k;1)-n_1, N_2(n,k;1)-n_2, nk;1) \\ F(N_1(n,k;0)-n_1, N_2(n,k;0)-n_2, nk;0) \\ F(n_1n_2nk;0)G(nk;0)]^{-1}\end{gathered} \quad (76)$$

By using Eq. (73), we can rewrite this expression as

$$
\begin{aligned}
(d^{*(0)}_{1(1)})^{AB} = (-1)^s\alpha_1\bar{\alpha}_0 \sum[(s+1-k)/(s+1)]\\
a(N_1(n,k;0)-n_1, N_2(n,k;0)-n_2, nk;0)^*\\
a(N_1(n,k;1)-n_1, N_2(n,k;1)-n_2, nk;1)\\
[F(N_1(n,k;1)-n_1, N_2(n,k;1)-n_2, nk;1)G(nk;0)]^{-1}
\end{aligned}
\tag{77}
$$

As follows from Eq. (38)

$$
G(nk;0) = G(nk;1)\frac{(s+1-k)(m-2)(q_1+n-k)(q_1+n-k-1)}{(s+1)(m+n-2)(q_1+n)(q_1-k-1)}
\tag{78}
$$

By using this relation, changing the summation variables in Eq. (77) as $n_1 \to N_1(n,k;1)-n_1$, $n_2 \to N_2(n,k;1)-n_2$ and using the relations $N_1(n,k;0) = N_1(n,k;1)+1$ and $N_2(n,k;0) = N_2(n,k;1)$ (see Eq. (69)) we arrive at the following conclusion: the AB transformation of the first term in Eq. (61) is equal to the third term in Eq. (62) if $(-1)^s\alpha_1\bar{\alpha}_0 = 1$.

The investigation of the other terms in Eqs. (61-68) can be carried out analogously, and the result is as follows. The fermionic generators are invariant under the AB transformation if

$$
\alpha_3 = -\alpha_1 \;\; \alpha_2 = \alpha_1 \;\; (-1)^s\alpha_1\bar{\alpha}_0 = (-1)^s\alpha_0\bar{\alpha}_1 = 1
\tag{79}
$$

Consider the last relation in Eq. (79). Since the automorphism in F_{p^2} is defined as $\alpha \to \alpha^p$ (see Sect. 2), we conclude that if $\alpha_1 = \alpha_0\kappa$ then κ^{p-1} =1. This relation shows that κ is an element of F_p. Assuming that the spin-statistics relation is satisfied (see Eq. (72)), our final conclusion can be formulated in the form of

Statement 3: The fermionic generators in the massive supermultiplet are invariant under the AB transformation if and only if the AB parities of the particles in the supermultiplet satisfy the relations

$$
\alpha_3 = -\alpha_0 \quad \alpha_2 = \alpha_1 \quad \alpha_1 = \alpha_0\kappa \quad \kappa = (-1)^{s+1}
\tag{80}
$$

The fermionic generators change their sign under the AB transformation if the last relation is replaced by $\kappa = (-1)^s$.

The investigation of the collapsed supermultiplet, massless supermultiplet and Dirac supermultiplet (see Sect. 4) can be carried out analogously and simpler. In the collapsed supermultiplet, particle 2 is absent and the remaining relations between the AB parities are the same as in Eq. (80). In the massles and Dirac supermultiplets, particles 2 and 3 are absent, and the relation between the AB parities of particles 0 and 1 is the same as in Eq. (80).

8 Space inversion in GFQT

In terms of representation generators of the so(2,3) algebra, the space inversion is defined as a transformation $M_{ab} \to M^P_{ab}$ such that

$$M^P_{ik} = M_{ik} \quad M^P_{i0} = -M_{i0} \quad M^P_{i5} = -M_{i5} \quad M^P_{05} = M_{05} \quad (i,k = 1,2,3) \tag{81}$$

Our first task is to find a transformation of the (a, a^*) operators resulting in Eq. (81). It is easy to see that if the transformation is defined as

$$\begin{aligned} a(n_1n_2nk;i)^* &\to (-1)^{n_1+n_2+n}\bar{\eta}_i a(n_1n_2nk;i)^* \\ a(n_1n_2nk;i) &\to (-1)^{n_1+n_2+n}\eta_i a(n_1n_2nk;i) \end{aligned} \tag{82}$$

where the parity η_i of particle i is such that $\eta_i\bar{\eta}_i = 1$ then the transformed generators for particle i indeed satisfy Eq. (81). Indeed, as follows from the construction described in Sect. 3 (see also Ref. [3]) for details), the operators $(a'_j, a_j", b', b")$ $(j = 1, 2)$ have only such nonzero matrix elements for which the values of $n_1 + n_2 + n$ in the initial and final states differ by ± 1. Therefore these operators change their sign under transformation (82). At the same time, the operators $L_\pm, h_j$ have nonzero matrix elements only for transitions where the values of $n_1 + n_2 + n$ in the initial and final states are the same. Then the result (81) follows from Eq. (12).

A well known result in the conventional theory is that fermions have imaginary parity (see e.g. Ref. [14]). We will show soon that in the GFQT this result is a consequence of the AB symmetry. However, we will first discuss a well known mathematical problem of how the field F_p can be extended to F_{p^2} (see e.g. Ref. [6]).

By analogy with the field of complex numbers, we can try to define F_{p^2} as a set of p^2 elements $a + bi$ where $a, b \in F_p$ and i is a formal element such that $i^2 = 1$. The question arises whether so defined F_{p^2} is a field, i.e. we can define all the four operations excepting division by zero. The definition of addition, subtraction and multiplication in so defined F_{p^2} is obvious and, by analogy with the field of complex numbers, we can try to define division as $1/(a+bi) = a/(a^2+b^2) - ib/(a^2+b^2)$. This definition can be meaningful only if $a^2 + b^2 \neq 0$ in F_p for any $a, b \in F_p$ i.e. $a^2 + b^2$ is not divisible by p. Therefore the definition is meaningful only if p *cannot* be represented as a sum of two squares and is meaningless otherwise. Since the prime number p in question is necessarily odd, we have two possibilities: the value of $p\,(mod\,4)$ is either 1 or 3. The well known result of the number theory (see e.g. the textbooks [6]) is that a prime number p can be represented as a sum of two squares only in the former case and cannot in the latter one. Therefore the above construction of the field F_{p^2} is correct only if $p\,(mod\,4) = 3$. In that case it is easy to verify that if $z = a + ib$ then the automorphism $z \to \bar{z} = z^p$ is the usual complex conjugation.

In the general case we can extend F_p to F_{p^2} as follows. Let the equation $\kappa^2 = -a_0$ $(a_0 \in F_p)$ have no solutions in F_p. Then F_{p^2} can be formally described as a set of elements $a + b\kappa$, where $a, b \in F_p$ and κ satisfies the condition $\kappa^2 = -a_0$. The actions in F_{p^2} are defined in the natural way. The condition that the equation $\kappa^2 = -a_0$ has no solutions in F_p is important in order to ensure that any nonzero element from F_{p^2} has an inverse. Indeed, the definition $(a + b\kappa)^{-1} = (a - b\kappa)/(a^2 + b^2 a_0)$ is correct since the denominator can be equal to zero only if both, a and b are equal to zero.

If $p = 3\,(mod\,4)$ then a possible choice of a_0 is $a_0 = 1$ and then one can use the usual notation i for κ. If $p = 1\,(mod\,4)$ then the choice $a_0 = 1$ is impossible but other possible choices exist and $\overline{a + b\kappa} = a - b\kappa$ (see Ref. [2] for details). In the both cases it is possible to prove the correspondence between the GFQT and the standard approach (see Ref. [2]). In other words, at present one *cannot* unambigously conclude that the realistic value of p is only such that $p = 3\,(mod\,4)$.

We now return to the problem of fermion parity. Apply transformation (82) to the both parts of the first (or second) expression in Eq. (70). Then, as follows from Eq. (69), $\bar{\eta}_i = (-1)^{s_i}\eta_i$. Therefore, in the both cases, $p = 1\,(mod\,4)$ and $p = 3\,(mod\,4)$, the parity of fermions is "imaginary" and the parity of bosons is "real". However, the relation $\eta_i\bar{\eta}_i = 1$ can be satisfied only in the second case.

Consider now when the transformation defined by Eq. (82) is compatible with supersymmetry. Suppose that, as a result of this transformation,

$$d_j \rightarrow \eta d_j \quad d_j^* \rightarrow \bar{\eta} d_j^* \quad (j = 1, 2) \tag{83}$$

Then, as follows from Eq. (29), the operators $(a_j', a_j", b', b")$ will indeed change their sign if $\eta^2 = -1$, and the operators $L_\pm, h_j$ will remain unchanged if $\eta\bar{\eta} = 1$. Therefore, in the standard theory the quantity η should be imaginary, and the same is valid in the GFQT if $p = 3\,(mod\,4)$ (see the above discussion). Now, as follows from Eqs. (61-68) and (82), the property (83) can be satisfied if and only if $\eta_2 = \eta_1$, $\eta_3 = -\eta_0$ (in the standard theory these properties are well known [14]) and $\bar{\eta} = \bar{\eta}_1\eta_0 = -\bar{\eta}_0\eta_1$. The last property is again satisfied if the parity of fermions is "imaginary" and the parity of bosons is "real", in full analogy with the standard theory.

9 X inversion

The relations (28) defining a representation of the osp(1,4) algebra are obviously invariant under the transformation $d_1 \leftrightarrow d_2$, $d_1^* \leftrightarrow d_2^*$. As follows from Eq. (29), in this case $a_1' \leftrightarrow a_2'$, $a_1" \leftrightarrow a_2"$, $h_1 \leftrightarrow h_2$ and the operators $(L_\pm, b', b")$ remain unchanged. Therefore, as follows from Eq. (12), if $a, b = 0, 2, 3, 5$ then $M^{1a} \rightarrow -M^{1a}$ and $M^{ab} \rightarrow M^{ab}$. One can therefore treat the above transformation as that corresponding to the inversion of the x axis. Since the conventional space inversion can be obtained by successively applying the inversions of the x, y and z axis, it is reasonable to think that X inversion is more general than the space one.

We will show that in terms of the (a, a^*) operators the X transformation can be defined as

$$\begin{aligned} a(n_1 n_2 n k; i) &\to \beta_i \tfrac{k!}{(s_i-k)!} a(n_2 n_1 n, s_i - k; i) \\ a(n_1 n_2 n k; i)^* &\to \beta_i \tfrac{k!}{(s_i-k)!} a(n_2 n_1 n, s_i - k; i)^* \end{aligned} \quad (84)$$

where β_i is the X parity. The fact that β_i is "real" easily follows from the AB symmetry (see Eq. (70)). Note that, as follows from Eq. (69)

$$N_1(n, s_i - k; i) = N_2(n, k; i) \quad N_2(n, s_i - k; i) = N_1(n, k; i) \quad (85)$$

As follows from Eq. (38)

$$\begin{aligned} G(n, s_i - k; i) &= [\tfrac{k!}{(s_i-k)!}]^2 G(nk; i) \\ F(n_1 n_2 n k; i) &= F(n_2 n_1 n, s_i - k; i) \end{aligned} \quad (86)$$

For this reason the anticommutation or commutation relations (53) or (54) are invariant under the transformation (84) if $(\beta_i)^2 = 1$.

Consider again the first term in Eq. (61), which, as noted above, can be written in the form of Eq. (74). After applying the transformation (84), this expression becomes

$$\begin{aligned} (d^{*(0)}_{1(1)})^X = \beta_0 \beta_1 \sum \tfrac{1}{s+1} [\tfrac{k!}{(s-k)!}]^2 a(n_2 n_1 n, s+1-k; 1)^* \\ a(n_2 n_1 n, s-k; 0)[F(n_1 n_2 n k; 0) G(nk; 0)]^{-1} \end{aligned} \quad (87)$$

since $s_1 = s + 1$. By using Eq. (85), (86) and changing the summation variables as $n_1 \leftrightarrow n_2$, $k \to (s - k)$, it is easy to demonstrate that this expression becomes the first term in Eq. (65) if $\beta_1 = \beta_0$. One can consider the other terms analogously and come to

Statement 4: The transformation (84) is a symmetry if

$$\beta_1 = -\beta_2 = -\beta_3 = \beta_0 \quad (88)$$

We see that the X parities of particles 2 and 3 are opposite to those for particles 0 and 1.

10 Conclusion

The main result of the present paper, proved in Sect. 7, is that the AB symmetry (discussed in detail in Refs [2, 3]) is compatible with supersymmetry. Let us discuss why this result is very important (in our opinion). First of all we briefly describe the meaning of the AB symmetry and explain why it has no analog in the standard theory.

In quantum theory based on a Galois field (GFQT), the field of complex numbers is replaced by a Galois field F_{p^2} containing p^2 elements. In particular, the ring of integers Z is replaced by a simplest Galois field $F_p = Z/Zp$ — the residue field modulo p. While the elements from Z are in the range $-\infty, ... - 2, -1, 0, 1, 2, ...\infty$, one can treat F_p as a set of elements in the range $-A, ... - 2, -1, 0, 1, 2, ...A$ where $A = (p-1)/2$. One might think that if p is very big, there should not be a big difference between the theories based on Z and F_p. However, since arithmetic of Galois fields differs from the standard one, it is reasonable to expect that the GFQT has its own specific features. As already noted, in the GFQT a particle and its antiparticle belong to the same IR of the symmetry algebra, in contrast with the standard theory where they belong to independent IRs. The AB symmetry relates particles and antiparticles within their common IR, and that's why it has no analog in the standard theory.

The existence of the AB symmetry imposes considerable restrictions on the structure of the GFQT [3]. Since any osp(1,4) supermultiplet contains 2, 3 or 4 particles described by modular IRs of the so(2,3) algebra, the conclusions of Ref. [3] remain unchanged in the presence of supersymmetry. The fact that the AB supersymmetry passes the supersymmetry test is a strong argument in favor of our assumption that it plays a fundamental role. If this is accepted then in the presence of supersymmetry it is possible to impose additional restrictions on the structure of interactions. In particular, the AB parities of particles in a supermultiplet are not independent (see Sect. 7).

In Sects. 8 and 9 we have considered space inversion and X inversion in the GFQT. In particular, we have reproduced the well known fact that the parities of bosons are real and the parities of fermions are imaginary [14]. This result is interesting for two reasons. First, in the GFQT it is a direct consequence of the AB symmetry. In addition (see the discussion in Sect. 8), it gives an indication that $p = 3$ $(mod\ 4)$ rather than $p = 1$ $(mod\ 4)$ since only in the former case F_{p^2} contains an element with the same properties as the imaginary unity i. If one accepts that the GFQT is fundamental, the result can be reinterpreted in this way: quantum theory involves complex numbers since (for some reasons) $p = 3$ $(mod\ 4)$.

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