

The coefficients of the Seiberg-Witten prepotential as intersection numbers (?)^{*}

R. Flume[†]

R. Poghossian^{†,‡}

H. Storch[†]

[†]*Physikalisches Institut der Universität Bonn
Nußallee 12, D-53115 Bonn, Germany*

Abstract

The n -instanton contribution to the Seiberg-Witten prepotential of $\mathcal{N}=2$ supersymmetric $d=4$ Yang Mills theory is represented as the integral of the exponential of an equivariantly exact form. Integrating out an overall scale and a $U(1)$ angle the integral is rewritten as $(4n-3)$ fold product of a closed two form. This two form is, formally, a representative of the Euler class of the Instanton moduli space viewed as a principal $U(1)$ bundle, because its pullback under bundle projection is the exterior derivative of an angular one-form. We comment on a recent speculation of Matone concerning an analogy linking the instanton problem and classical Liouville theory of punctured Riemann spheres.

^{*} To be published in the collection “From Integrable Models to Gauge Theories” (World Scientific, Singapore, 02) to honour Sergei Matinyan at the occasion of his 70’th birthday.

[‡] on leave of absence from Yerevan Physics Institute, Armenia

e-mail: flume@th.physik.uni-bonn.de

poghos@th.physik.uni-bonn.de

hst@mosaic-ag.com

1 Introduction

One of the key events in the field of theoretical high energy physics in the last decade has been the proposal by Seiberg and Witten (SW) [1], [2] of an exact formula for the so called prepotential of $d=4$, $N=2$ supersymmetric Yang-Mills quantum field theories. The repercussions of the SW proposal were manifold:

- (i) It is for the first time that exact statements on the dynamics of an interacting quantum field theory in four dimensions (beyond asymptotic freedom) are available;
- (ii) The findings of SW have lead to new insights into the rich phase structure of supersymmetric Yang-Mills theories ([3], [4] and references therein);
- (iii) Soon after its discovery it became clear [5], [6], that the SW prepotential appears naturally in string theory as well as in field theory providing therewith a bridge between the two disciplines;
- (iv) With the advent of Maldacena's proposal of the AdS/CFT correspondence [7] instanton calculations, both stringy [8], [9] and field theoretical, [10] became one of the mayor tools to check the proposal. The instanton calculations in the field theoretical context can be regarded as a continuation of the endeavor verifying the SW prepotential.

Such a central result as that of SW deserves for a rigorous proof. We hope to provide with the present article some modest steps into this direction.

We will constrain ourselves to a discussion of the simplest possible setting for the appearance of an $N=2$ prepotential. That is, we choose $SU(2)$ as underlying gauge group (spontaneously broken to $U(1)$) and restrict the field content to an $N=2$ vector multiplet, denoted in the following by Ψ ¹. The SW proposal $\mathcal{F}(\Psi)$ as a function of Ψ amounts under this circumstances to the ansatz for \mathcal{F} as inverse of the modular elliptic function \mathcal{J} [11] with the following representation in terms of a power series expansion

$$\mathcal{F}(\Psi) = \frac{i}{2\pi} \log \Psi^2 \log \frac{\Psi^2}{\Lambda^2} - \frac{i}{\pi} \sum_{n=1}^{\infty} \mathcal{F}_n \left(\frac{\Lambda}{\Psi} \right)^{4n} \Psi^2, \quad (1)$$

where Λ denotes a dynamical scale enforced by ultraviolet divergencies. The first (logarithmic) term on the r.h.s. of eq. (1) comprises the perturbative contributions which are the classical prepotential $\propto \Psi^2$ and the one-loop contribution $\propto \Psi^2 \log \Psi^2$. A non-renormalization theorem [12] implies that there are no other perturbative (higher loop) contributions to $\mathcal{F}(\Psi)$. The second term in the r.h.s. of eq. (1) with numerical coefficients \mathcal{F}_n is of nonperturbative

¹ Ψ is assumed to be composed of reduced $U(1)$ -degrees of freedom due to symmetry breaking. Ψ incorporates the breaking parameter v : $\Psi \equiv v + \dots$.

origin due to instanton configurations. The perturbative and non-perturbative pieces add up to the inverse elliptic modular function reflecting therewith a dynamically realized (S-) duality. For the arguments leading to the ansatz, (1), which might be called macroscopic - including in particular duality and a certain minimality assumption (which might be coined as the assumption of minimal analytical consistency) [13] - we refer to [1] and [2].

Our purpose in the present article is to reexamine the structure of the instanton calculations, that is, the “microscopical” approach towards a determination of the coefficients \mathcal{F}_n . We recall in Sec. 2 part of the pioneering work by Dorey, Khoze and Mattis (DKM) [14], [15], [16], leading to the determination of the one- and two-instanton coefficients [14] (for one instanton calculations see also [17] - [19]) and also providing a suggestive ansatz for the measure in the n -instanton moduli space [16]. It will be noted that the DKM measure has to be supplemented by the specification of a domain of integration, since the space of integration proposed by DKM containing redundant degrees of freedom is not orientable. In Sec. 3 we show that \mathcal{F}_n can be represented as integral of the top form of the exponential of an equivariantly exact form. Results equivalent to those in this section have been obtained in [20] and [21] within a technically different framework. Sec. 4 contains our main result, the representation of \mathcal{F}_n as integral of a $(4n-3)$ fold product of a closed two-form. The two form is formally a representative of the Euler class associated with the ADHM moduli space viewed as a (principal) $U(1)$ bundle. In Sec. 5 we relate our findings to an interesting speculation by Matone [22] for a tentative analogy between classical Liouville theory on punctured Riemann spheres and the $d=4$ instanton problem.

2 The DKM results and the necessity of $SO(n)$ gauge fixing

The coefficients \mathcal{F}_n are given in terms of certain “reduced” matrix elements, that is, expressions in which the parameters corresponding to overall translation invariance are eliminated together with their supersymmetry partners. These matrix elements are represented by integrals over the supersymmetrized moduli space of n -instanton configurations - the Atiyah, Drinfeld, Manin (ADHM) [23] parameters -

$$\mathcal{F}_n = \int d\mu^{(n)} e^{-S^{(n)}}. \quad (2)$$

$d\mu^{(n)}$ denotes here a particular measure on the $\mathcal{N}=2$ supersymmetrised ADHM moduli space and $S^{(n)}$ stands for an effective $\mathcal{N}=2$ action. We will quote the results of DKM concerning $d\mu^{(n)}$ and S^n referring for detailed explanations to the articles [14], [15] and [16].

The algebraic ADHM construction of selfdual $SU(2)$ Yang-Mills fields in four-dimensional Euclidean space - of integer valued topological charge n - is based on an $(n+1) \times n$ matrix

$$a = \begin{pmatrix} w_1 & \dots & w_n \\ a'_{11} & \dots & a'_{1n} \\ & \dots & \\ a'_{n1} & \dots & a'_{nn} \end{pmatrix}, \quad a'_{ij} = a'_{ji} \quad (3)$$

with quaternion valued entries w_i , and $a'_{1 \leq i, j \leq n}$. We will enumerate the rows of the matrix a by the integers $0, \dots, n$, so that $a_{0i} \equiv w_i$ and $a_{ij} \equiv a'_{ij}$. a is required to satisfy the quadratic constraint

$$\begin{aligned} \Im(\bar{a}a)_{ik} &\equiv \frac{1}{2} \sum_{\lambda=0}^n (\bar{a}_{\lambda i} a_{\lambda k} - \bar{a}_{\lambda k} a_{\lambda i}) \\ &\equiv \frac{1}{2} \left\{ \bar{w}_i w_k - \bar{w}_k w_i + \sum_{j=1}^n (\bar{a}'_{ji} a_{jk} - \bar{a}'_{jk} a_{ji}) \right\} = 0, \end{aligned} \quad (4)$$

where $\bar{}$ denotes quaternionic conjugation (for a matrix of quaternion this implies in addition also transposition)². Eq.'s (3) and (4) together with the rank condition

$$\begin{aligned} \det \bar{\Delta}(x) \Delta(x) &\neq 0, \\ \Delta(x) &\equiv \begin{pmatrix} w_1, \dots, w_n \\ a'_{ij} + x \delta_{ij} \end{pmatrix} \end{aligned} \quad (5)$$

for arbitrary quaternion x , provide the basis for an algebraic construction of (all) $SU(2)$ n -instanton configurations [23].

Chiral Weil zero-modes in the selfdual Yang-Mills background are constructed out of two $(n+1) \times n$ matrices \mathcal{M}, \mathcal{N}

$$\mathcal{M} = \begin{pmatrix} \mu_1 & \dots & \mu_n \\ \mathcal{M}'_{11} & \dots & \mathcal{M}'_{1n} \\ & \dots & \\ \mathcal{M}'_{n1} & \dots & \mathcal{M}'_{nn} \end{pmatrix}, \quad \mathcal{N} = \begin{pmatrix} \nu_1 & \dots & \nu_n \\ \mathcal{N}'_{11} & \dots & \mathcal{N}'_{1n} \\ & \dots & \\ \mathcal{N}'_{n1} & \dots & \mathcal{N}'_{nn} \end{pmatrix} \quad (6)$$

²Our conventions concerning quaternions are the following: an arbitrary quaternion $x_{\alpha\beta}$ can be decomposed in a basis of unit quaternions $x_{\alpha\beta} = \sum_{\mu=1}^4 x_{\mu} (e^{\mu})_{\alpha\beta}$, x_{μ} real, $\{e^{\mu}\} = -i\sigma^{\mu}$, $\mu = 1, 2, 3$; $\sigma^4 = \mathbf{I}$; $(\bar{x})^{\beta\alpha} = \epsilon^{\beta\gamma} \epsilon^{\alpha\delta} x_{\delta\gamma}$; $\epsilon^{12} = \epsilon^{12} = 1$. The notions of imaginary (\Im) and real (\Re) parts, which we will use further on, should be understood with respect to this conjugation.

with the (two component Weil-) spinor valued entries ($\mu_i = \{\mu_i^\alpha; \alpha = 1, 2\}$ and similarly for other entries) obeying the constraints

$$\mathcal{M}'_{ij} = \mathcal{M}'_{ji}, \quad \mathcal{N}'_{ij} = \mathcal{N}'_{ji} \quad (7)$$

$$\bar{a}\mathcal{M} - (\bar{a}\mathcal{M})^T = 0, \quad \bar{a}\mathcal{N} - (\bar{a}\mathcal{N})^T = 0. \quad (8)$$

The undotted $\mathbf{2}$ -index of the quaternion \bar{a} is paired here with the spinor label of \mathcal{M} (\mathcal{N}). For the subsequent discussion it is important to note that the data (3)-(8) are redundant in the following sense:

Eq.'s (4),(8) are invariant under $O(n)$ transformations

$$X \cdot a = \begin{pmatrix} w \cdot X \\ X^{-1} \cdot a' \cdot X \end{pmatrix}; \quad (9)$$

$$X \cdot \mathcal{M} = \begin{pmatrix} \mu \cdot X \\ X^{-1} \cdot \mathcal{M}' \cdot X \end{pmatrix}; \quad X \cdot \mathcal{N} = \begin{pmatrix} \nu \cdot X \\ X^{-1} \cdot \mathcal{N}' \cdot X \end{pmatrix}, \quad (10)$$

$$X \in O(n) \quad (X \cdot X^T = I_{n \times n}, \quad X \text{ real})$$

and $O(n)$ related data a and $X \cdot a$ etc. give rise to the same vector potentials and Weil zero modes.

The field content of $d=4$, $N=2$ extended supersymmetric Yang-Mills field theory without matter fields can be read off from the Lagrangian

$$\mathcal{L}_{N=2} = tr \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + 2\mathcal{D}_\mu \Phi^* \mathcal{D}^\mu \Phi + \sum_a (i\bar{\lambda}_a \bar{\sigma}_\mu \mathcal{D}^\mu \lambda^a + g\Phi^* [\lambda_a, \lambda^a] + g\Phi [\bar{\lambda}_a, \bar{\lambda}^a]) + 2g^2 [\Phi^*, \Phi]^2 \right\}, \quad (11)$$

which consists of a complex scalar field Φ , two Weil spinors λ_1, λ_2 together with their chiral conjugates and a vector field V_μ , all of which transform under the adjoint representation of the gauge group. The parts of (11) contributing to $S^{(n)}$ (eq. (2)) are the kinetic energies of the bosonic fields and the chiral Yukawa interaction:

$$S^{(n)} = \int d^4x \, tr \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + 2\mathcal{D}_\mu \Phi^* \mathcal{D}^\mu \Phi + g\Phi^* [\lambda_a, \lambda^a] \right\}. \quad (12)$$

The fields appearing on the r.h.s. are supposed to satisfy a reduced set of equations of motion

$$F_{\mu\nu} = \tilde{F}_{\mu\nu}; \quad (\tilde{F} \text{ denotes the dual of } F) \quad (13)$$

$$\bar{\sigma}_\mu \mathcal{D}^\mu \lambda^a = 0; \quad (14)$$

$$\mathcal{D}_\mu \mathcal{D}^\mu \Phi = -g [\lambda_a, \lambda^a] \quad (15)$$

and the symmetry breaking ($SU(2) \rightarrow SU(1)$) boundary condition

$$\lim_{|x| \rightarrow \infty} \Phi(x) = v \quad (16)$$

with \mathbf{u} being a constant quaternion with vanishing real part but otherwise arbitrary. The terms, which have been omitted in the transition from eq. (11) to eq.'s (12) and (13) -(15), are of higher order in the coupling constant and are believed not to contribute to \mathcal{F}_n .³ The general solution of the equations (13) -(15) has been constructed by DKM in ref. [14]. One has to insert this into $S^{(n)}$, eq. (12), and to write terms involving the scalar fields as a boundary contribution

$$\begin{aligned} \int d^4x \text{tr} \left(\mathcal{D}_\mu \Phi^* \mathcal{D}^\mu \Phi - g\sqrt{2} [\lambda, \lambda] \Phi^\dagger \right) \\ = \int \partial_\mu \text{tr} (\Phi^* \mathcal{D}^\mu \Phi) \\ = \int dS^\mu \text{tr} (\Phi^* \mathcal{D}_\mu \Phi), \end{aligned}$$

where for the first equality use has been made of the equation of motion (15). The integral in the last line is to be evaluated on the three sphere formally bounding Euclidean four-space. One obtains finally

$$\pi^{-2} S^{(n)} = \frac{8n}{g^2} + 16|v|^2 \sum_{k=1}^n |w_k|^2 - 8 \text{tr}_n \bar{\Lambda} \mathbf{L}^{-1} (\Lambda + \Lambda_f) + 4\sqrt{2} \sum_{k=1}^n \bar{\mu}_k \bar{\nu}_k, \quad (17)$$

where Λ and Λ_f are $n \times n$ antisymmetric matrices

$$\Lambda_{kl} = \bar{w}_k v w_l - \bar{w}_l v w_k; \quad (18)$$

$$\Lambda_f = \frac{1}{2\sqrt{2}} (\bar{\mathcal{M}} \mathcal{N} - \bar{\mathcal{N}} \mathcal{M}), \quad (19)$$

and \mathbf{L} is a real operator acting on any skew symmetric $n \times n$ matrix X by

$$\mathbf{L} \cdot X = \frac{1}{2} \{X, W\} + \bar{a}' [a', X] - [\bar{a}', X] a', \quad (20)$$

where W is the real valued symmetric $n \times n$ matrix

$$W_{kl} = \bar{w}_k w_l + \bar{w}_l w_k. \quad (21)$$

It can be shown that \mathbf{L} is invertible for a' and \mathbf{u} 's satisfying the constraints (4), (5).

³A conceptually clear cut argument for this fact has not been given as far as we know.

The measure $d\mu^{(n)}$ is according to DKM [16] uniquely determined by the requirements of supersymmetry and the cluster property to be of the form

$$d\mu^{(n)} = \frac{C_n}{Vol(O(n))} \prod_{i=1}^n d^4 w_i d^2 \mu_i d^2 \nu_i \prod_{i \leq j} d^4 a'_{ij} d^2 \mathcal{M}'_{ij} d^2 \mathcal{N}'_{ij} \prod_{i < j} d(\mathcal{A}_{tot})_{ij} \\ \times \prod_{i < j} \delta \left((\mathbf{L} \cdot \mathcal{A}_{tot} - \Lambda - \Lambda_f)_{ij} \right) \delta^{(3)} \left(\frac{1}{2} \left((\bar{a}a)_{ij} - (\bar{a}a)_{ji} \right) \right) \\ \times \delta^{(2)} \left((\bar{a}\mathcal{M})_{ij} - (\bar{a}\mathcal{M})_{ji} \right) \delta^{(2)} \left((\bar{a}\mathcal{N})_{ij} - (\bar{a}\mathcal{N})_{ji} \right). \quad (22)$$

The factor $1/Vol(O(n))$ takes care of the fact that the redundant $O(n)$ degrees of freedom have not been discarded from the integral. The normalization constant C_n is fixed through the cluster condition and the normalization of the one-instanton coefficient. One easily recognizes in the last three groups of delta-functions on the r.h.s. of (22) the bosonic and fermionic constraints of eq.'s (4), (8). The auxiliary variables \mathcal{A}_{tot} can be integrated out leaving behind a factor $1/\det \mathbf{L}$. But the bouquet of variables (including \mathcal{A}_{tot}) chosen in (22) allows for a more direct verification of supersymmetry of the measure. (There should be, to start with, as many bosonic as fermionic differentials and δ -functions for the sake of manifest supersymmetry.)

The DKM representation, eq. (2), of \mathcal{F}_n together with the specifications in eq.'s (17), (22) has to be supplemented by an $O(n)$ gauge fixing condition since the larger space including redundant $O(n)$ degrees of freedom turns out to be non-orientable. To see the necessity of such a gauge fixing procedure we choose a gauge and verify a posteriori that the restriction is unavoidable to obtain a non-vanishing result for \mathcal{F}_n . We may consider for this purpose any representation built from the variables w and a and impose on this gauge fixing conditions. To concretize the ideas let us consider the real symmetric matrix

$$Y_{ik} = \Re e (\bar{a}_{ij} a_{jk})$$

which transforms under the adjoint representation of $O(n)$. Y can be brought through an $O(n)$ transformation into diagonal form with the diagonal elements arranged in increasing order of their absolute values⁴. There will not appear in this procedure a non-trivial Faddeev-Popov determinant because of supersymmetry. (The assertion will become evident from the deductions in the next section where we identify fermions with one-forms.)

The gauge fixing condition degenerates at places where two or more than two eigenvalues of Y coincide. There part of the $O(n)$ group is restored. In the generic case of two coinciding

⁴This condition does not fix completely the gauge since some discrete transformations are still possible. Dropping the factor $1/Vol(O(n))$ from (22) one has to take care of the latter leftover redundancies.

eigenvalues an $O(2)$ sub-group is revived. It means that we hit a Gribov horizon [24]. Points at the two sides of the horizon of codimension one are related by a permutation (considered as an element of the group $O(n)$). This implies that the corresponding volume elements have opposite orientations what confirms the above claim ⁵.

3 Simplifications

To evaluate for general n the integral (2) with the measure (22) and the induced action, eq. (17), appears to be a formidable task. The evaluation of \mathcal{I}_2 , achieved by DKM, [14], already seems, at least at first sight, to be miraculous. We propose in the following two subsections simplifications of the integrals, which we hope, will give some insight into the algebraic-geometric nature of the problem. The results of these two subsections have also been derived by Bellisai, Bruzzo, Fucito, Tanzini and Travaglini [20], [21], who use a technical approach different from our.

3.1 Fermions as differential forms

It is a well known fact that fermion zero modes in selfdual Yang-Mills backgrounds are in correspondence with the fluctuation modes of the vector fields which makes it appearing natural to identify part or all of the cotangent space of the ADHM moduli with the Grassmann valued fermion zero modes. For the case of the $\mathcal{N}=2$ supersymmetry the correspondence of fermion modes and cotangent space turns out to be one-to-one.

To start with we combine the Grassmannian spinor valued matrices \mathcal{M}, \mathcal{N} into a quaternion valued matrix denoted by \mathcal{P} . Let A^α, B^α be some two component ($\alpha = 1, 2$) c-number spinors satisfying

$$\begin{aligned} A \cdot B &\equiv A^\alpha B_{\dot{\alpha}} \equiv A^\alpha B^{\dot{\beta}} \epsilon_{\dot{\alpha}\dot{\beta}} = 1 \\ B_{\dot{\alpha}} &= \epsilon_{\dot{\alpha}\dot{\beta}} B^{\dot{\beta}}, \epsilon_{1\dot{2}} = -\epsilon_{2\dot{1}} = 1, \end{aligned}$$

and define

$$(\mathcal{P})_{\alpha\dot{\alpha}} = \mathcal{M}_\alpha B_{\dot{\alpha}} - \mathcal{N}_\alpha A_{\dot{\alpha}} \quad (23)$$

⁵DKM avoid in [14] the horizon problem by integrating the *modulus* of the integrand and the measure over the larger space.

with the inverse relation

$$\mathcal{M}_\alpha = (\mathcal{P})_{\alpha\dot{\alpha}} A^{\dot{\alpha}}; \quad \mathcal{N}_\alpha = (\mathcal{P})_{\alpha\dot{\alpha}} B^{\dot{\alpha}}. \quad (24)$$

ADHM matrix labels are here suppressed. The constraints (8) read in terms of the new variables as

$$\bar{a}\mathcal{P} - (\bar{a}\mathcal{P})^T = 0, \quad (25)$$

and the fermionic \mathbf{f} -functions appearing in the measure (22) read as

$$\prod_{i < j} \delta^{(4)} \left((\bar{a}\mathcal{P})_{ij} - (\bar{a}\mathcal{P})_{ji} \right). \quad (26)$$

It is easily seen that the imaginary parts of the fermionic constraints (25) are automatically fulfilled if one substitutes for \mathcal{P} any $O(n)$ -covariant derivative of \mathbf{a} ,

$$\begin{aligned} \mathcal{P} &= (d + X) a \equiv \mathcal{D}a; \\ X \cdot a &\equiv X \cdot \begin{pmatrix} w \\ a' \end{pmatrix} = \begin{pmatrix} -w \cdot X \\ [X, a'] \end{pmatrix} \end{aligned} \quad (27)$$

with \mathbf{X} being any real skewsymmetric $n \times n$ matrix of one-forms. Indeed, one has the identities:

$$\begin{aligned} \Im m \left(\bar{a} \mathcal{D}a - (\bar{a} \mathcal{D}a)^T \right) &= d \Im m (\bar{a}a), \\ \Im m \left(\bar{a} (X \cdot a) - (\bar{a} (X \cdot a))^T \right) &= [X, \Im m (\bar{a}a)], \end{aligned} \quad (28)$$

so that both the ordinary exterior differential as well as the $O(n)$ connection part lead to vanishing contributions inserted into the imaginary part of eq. (25) as long as \mathbf{a} satisfies the ADHM constraint (4). For an arbitrary $O(n)$ Lie algebra valued one form \mathbf{X} holds

$$\Re e \left(\bar{a} (X \cdot a) - (\bar{a} (X \cdot a))^T \right) = -\mathbf{L} \cdot X \quad (29)$$

with \mathbf{L} as introduced in eq. (20). This allows to choose a connection s.t. also the remaining real part of (25) vanishes which is found to be given by

$$X = \mathbf{L}^{-1} \cdot \Re e \left(\bar{a} da - (\bar{a} da)^T \right). \quad (30)$$

We assume now that, the bosonic ADHM moduli \mathbf{a} satisfy the constraints (4), (5) and the gauge fixing condition

$$Y_{ik} \equiv \Re e (\bar{a}a)_{ik} = 0; \quad \text{for } i \neq k. \quad (31)$$

The fermionic variables are determined by eq.'s (27) and (30) and satisfy therefore the constraints (25). The Jacobian factors arising from the integration of the bosonic \mathbf{b} -functions in eq. (22) and the imaginary projections of the fermionic \mathbf{f} -functions, eq. (26), cancel each other

as a consequence of the relations (28). From the real part of the fermionic \mathbf{g} -functions survives the $SO(n)$ Haar measure (multiplied by $\mathbf{1}$)

$$\int_{g \in SO(n)} \prod_{i < j} (\mathbf{L}(g^{-1}dg))_{ij} = Vol(SO(n)) \det \mathbf{L}.$$

The factor $\det \mathbf{L}$ from the last integration drops out together with the $1/\det \mathbf{L}$ factor of the integration of the auxiliary variables \mathcal{A}_{tot} in (22). Viewing the induced action $S^{(n)}$ as a mixed differential form

$$S^{(n)} = S^{(n)}(a, \mathcal{D}a)$$

we are all in all lead to rewrite eq. (2) as

$$\mathcal{F}_n \simeq C_n \frac{Vol(SO(n))}{Vol(O(n))} \int_{\mathcal{M}_n} e^{-S^{(n)}(a, \mathcal{D}a)} \quad (32)$$

with \mathcal{M}_n denoting the n -instanton moduli space. The exponential under the integral has to be expanded s.t. the top form on \mathcal{M}_n is reached.

3.2 $S^{(n)}$ as an equivariantly exact form

The integrand on the r.h.s. of eq. (32) is also invariant under a $U(1)$ symmetry (besides its $O(n)$ invariance), the remainder of the original $SU(2)$ gauge group. We want to introduce an equivariant calculus with respect to this $U(1)$ symmetry, (for a detailed account of the equivariant differential calculus one may consult chapter 7 of [26]).

Let \mathbf{M} denote a manifold with a continuous group \mathbf{G} acting on it. Vectors \mathbf{X} of the Lie algebra \mathfrak{g} of \mathbf{G} are mapped to vector fields L_X on \mathbf{M} . With i_X we denote the nilpotent operation of contraction of the vector field L_X with differential forms on \mathbf{M} . Following Cartan [27] one introduces the “equivariant” external differential

$$d_X = d - i_X. \quad (33)$$

d_X is nilpotent on \mathbf{G} -invariant differential forms since one has

$$d_X^2 = -(d \circ i_X + i_X \circ d) = \mathcal{L}_X \quad (34)$$

with \mathcal{L}_X being identified according to Cartan’s homotopy formula with the Lie derivative along the vector field L_X . The extension of the formalism to the setting of a vector bundle over \mathbf{M} is straightforward.

Let $\mathcal{D} = d + A$ denote a covariant derivative acting on sections of such a bundle (in the case under consideration a bundle associated to $O(n)$). The equivariant operation \mathcal{D}_X is defined in analogy to eq. (33) by

$$\mathcal{D}_X = \mathcal{D} - i_X. \quad (35)$$

This gives rise to the notion of equivariant curvature

$$\mathcal{F}_X(\cdot) = (\mathcal{D}_X^2 + \mathcal{L}_X)(\cdot) \quad (36)$$

satisfying the equivariant Bianchi identity

$$\mathcal{D}_X \mathcal{F}_X = 0. \quad (37)$$

The $U(1)$ symmetry in question is represented infinitesimally on the bosonic ADHM parameters w, a' by

$$\delta w \sim vw, \quad \delta \bar{w} \sim -\bar{w}v, \quad \delta a' = 0, \quad (38)$$

v being the breaking parameter introduced above. Let L_v denote the vector field on the moduli space corresponding to the transformation laws (38). The contraction operation associated to this vector field is given by

$$i_v \cdot dw = vw, \quad i_v \cdot d\bar{w} = -\bar{w}v, \quad i_v \cdot da' = 0. \quad (39)$$

A simple calculation reveals that the induced action $S^{(n)}$ can be represented as equivariant external exterior derivative of an $U(1)$ invariant one form denoted by ω

$$\begin{aligned} S^{(n)} &= \mathcal{D}_v \cdot \omega \equiv d_v \omega \\ \mathcal{D}_v &= d + X - i_v, \\ d_v &= d - i_v, \\ \omega &= - \sum_{i=1}^n 2\Re e(\bar{w}_i \bar{v} \mathcal{D} w_i) \end{aligned} \quad (40)$$

$$\omega = - \sum_{i=1}^n 2\Re e(\bar{w}_i \bar{v} \mathcal{D} w_i) \quad (41)$$

The two alternative representations of $S^{(n)}$ ⁶ in (40) are a consequence of the fact that the one-form ω is $O(n)$ invariant.

So, \mathcal{F}_n can be represented as the integral of an exponentially exact form (superseding eq. (32))

$$\mathcal{F}_n \simeq \int_{\mathcal{M}_n} e^{-d_v \omega}. \quad (42)$$

⁶ $S^{(n)}$ in eq. (40) differs by an irrelevant overall normalization factor from the DKM action, eq. (17).

To demonstrate the naturalness of the equivariant calculus in the present context (apart from the concise appearance of the action (40), (41)) we quote the antichiral supersymmetry transformations induced from $\mathbf{N} = 2$ field theory to the ADHM supermoduli (as shown in [15])⁷:

$$\begin{aligned} \delta a_{\alpha\dot{\beta}} &= (\mathcal{D}_v a)_{\alpha\dot{\beta}} \quad (\equiv \mathcal{D} a_{\alpha\dot{\beta}}); \\ \delta (\mathcal{D}_v a)_{\alpha\dot{\beta}} &= \mathcal{D}_v^2 a_{\alpha\dot{\beta}} = \mathcal{F}_v a_{\alpha\dot{\beta}} \equiv \begin{pmatrix} -\omega_{\alpha\dot{\beta}} \mathcal{F}_v \\ [\mathcal{F}_v, a'_{\alpha\dot{\beta}}] \end{pmatrix}. \end{aligned} \quad (43)$$

The closure of this representation of supersymmetry transformations - modulo $O(n)$ transformations - is a consequence of the equivariant Bianchi identity, eq (37).

4 \mathcal{F}_n as a formal intersection number

Standard localization theory for exact equivariant forms [26], [28] might suggest that the integral (42) can be localized at the set of critical points of the vector field L_v , that is , at $w_1 = \dots w_n = 0$. This turns out not to be the case, as the residuum in question at the locus $w = 0$ vanishes. It should also be noted that the standard theory, tailored for compact manifolds without boundaries, does not obviously apply to our problem since there are at least three potential obstacles

- (i) The variables w_i and a'_{ij} reach out to infinity;
- (ii) We cannot ignore a Gribov horizon, as noted above, which supplies one type of boundary.
- (iii) The other type of boundary, the Donaldson - Uhlenbeck boundary [29], [30], appears at places where the rank condition (5) is violated.

To deal with item (i) we introduce a scaling variable by setting $a = R\hat{a}$, i.e.

$$w = R\hat{w}, \quad a'_{ij} = R\hat{a}'_{ij} \quad (44)$$

s.t. holds⁸

$$\sum_{i=1} |\hat{w}_i|^2 + \sum_{i,j=1}^n |\hat{a}'_{ij}|^2 = 1 \quad (45)$$

$S^{(n)}$ reads in terms of the variables R, \hat{a} as

$$S^{(n)} = dR \hat{\omega}^{(n)} + R d_v \hat{\omega}^{(n)} \quad (46)$$

⁷we quote here a special supersymmetry transformation. The general transformations are of the form $Q_{\alpha}^i a_{\alpha\dot{\beta}} = \epsilon_{\alpha\dot{\beta}} (\mathcal{D} a)_{\alpha}^i$, $i = 1, 2$ etc.

⁸The appearance of the quadratic form on the l.h.s. of eq. (45) is irrelevant as long as it is non-degenerate.

with the notation $\hat{\omega} \equiv \omega(\hat{a}, d\hat{a})$. The scaling variable R can be integrated straightforwardly,

$$\mathcal{F}_n \simeq \int_{\mathcal{M}_n} e^{-dR\hat{\omega} - Rd_v\hat{\omega}} = - \int_{\hat{\mathcal{M}}_n} \int_0^\infty dR \hat{\omega} e^{-Rd_v\hat{\omega}} = \int_{\hat{\mathcal{M}}_n} \frac{\hat{\omega}}{d_v\hat{\omega}}. \quad (47)$$

($\hat{\mathcal{M}}_n$ is the manifold of the rescaled moduli \hat{a})

Using the notation $\rho = \hat{\omega}/i_v \cdot \hat{\omega}$ we rewrite the previous equation in the form

$$\mathcal{F}_n \simeq \int_{\hat{\mathcal{M}}_n} \rho (d\rho)^{4n-3}. \quad (48)$$

To verify the equality of (47) and (48) one has to expand the denominator in the integrand of (47) and to take into account the identities

$$\frac{\hat{\omega}}{i_v \cdot \hat{\omega}} \left(\frac{d\hat{\omega}}{i_v \cdot \hat{\omega}} \right)^k = \frac{\hat{\omega}}{i_v \cdot \hat{\omega}} \left(d \frac{\hat{\omega}}{i_v \cdot \hat{\omega}} \right)^k \equiv \rho (d\rho)^k \quad (49)$$

for any nonnegative integer k . The most important properties of the forms ρ , $d\rho$ are contained in the equations

$$i_v \cdot \rho = 1; \quad (50)$$

$$i_v \cdot d\rho = 0, \quad (51)$$

the latter equality being a consequence of the relations

$$\rho = \frac{\hat{\omega}}{i_v \cdot \hat{\omega}}, \quad \mathcal{L}_v \hat{\omega} \equiv (i_v \circ d + d \circ i_v) \hat{\omega} = 0. \quad (52)$$

One may view $\hat{\mathcal{M}}_n$ as a S^1 bundle vis-à-vis the action of the $U(1)$ symmetry. Eq. (51) means that ρ contains with coefficient $1/2|v|$ the differential of an angle parameterizing the $U(1)$ group orbits. $2|v|\rho$ is, with other words, an angular one-form of the S^1 -bundle and $2|v|d\rho$ would have to be identified with the Euler class of that bundle were there no boundaries. The differential of the $U(1)$ angle, call it φ , only shows up in ρ but not, according to eq. (51), in $d\rho$. We may therefore substitute ρ in the integrand on the r.h.s. of eq. (48) by $2|v|d\varphi$ (other parts of ρ lead to vanishing contributions). Integrating out φ we arrive at

$$\mathcal{F}_n \simeq \int_{\hat{\mathcal{M}}'_n} (d\rho)^{4n-3}, \quad (53)$$

where $\hat{\mathcal{M}}'_n$ denotes the base of the S^1 -bundle of which $\hat{\mathcal{M}}_n$ is the total bundle space. \mathcal{F}_n as represented in eq. (53), appears as an intersection number, i.e. the integral of the $(4n-3)$ -fold product of $d\rho$.

Remarks:

1) The concrete coordinatization of the $U(1)$ group requires the choice of a concrete, complex scalar $U(1)$ representation. It seems advisable to choose an $O(n)$ invariant combination of variables forming such a representation. (Only in this case the $U(1)$ angle will not be changed along $O(n)$ orbits and, hence, can be naturally defined in the space $\hat{\mathcal{M}}_n$.) One may, for example, combine the quaternionic components of

$$w = (w_1, \dots, w_n) \quad (54)$$

$$w_i = \sum_{\mu=0}^3 e_\mu \cdot w_i^\mu, \quad (55)$$

into complex eigenvectors ψ_i, χ_i of the $U(1)$ transformations

$$\begin{aligned} \psi_i &= w_i^0 + iw_i^3; \\ \chi_i &= w_i^2 + iw_i^1 \end{aligned}$$

and build with them $O(n)$ -invariant variables

$$\begin{aligned} Z &= \sum_{i=1}^n \psi_i^2 \\ Z' &= \sum_{i=1}^n \chi_i^2. \end{aligned}$$

The real and imaginary part of Z (Z') are of the type of doublets we are looking for. Suppose we parameterize the Z -plane with a radial and an angular variable. The latter may be identified with the above variable φ . One is now confronted with three kinds of boundaries, the Gribov horizon, the Donaldson-Uhlenbeck (DU) boundary and the submanifold given by $Z=0$, where the chosen coordinatization of the $U(1)$ symmetry breaks down. The Gribov horizon does not give a contribution because there at least one $O(2)$ subgroup of $O(n)$ is restored as symmetry and this is not covered by the locally $O(n)$ invariant combinations of differential forms showing up in $S^{(n)}$. The vanishing of the contributions at the DU boundary is the consequence of simple estimates. We conclude that the integral (53) is localized at $Z=0$

$$\mathcal{F}_n \simeq \int_{\mathcal{M}'_n \cap \{Z=0\}} \rho(d\rho)^{4n-4}. \quad (56)$$

We can iterate this procedure choosing as new $U(1)$ variable, e.g., the argument of the complex variable Z' and afterwards possibly other combinations of variables. Unfortunately we have not been able to execute the recursion efficiently enough to reach a determination of \mathcal{F}_n for general n .

2) We want to emphasize that the representation (53) for \mathcal{F}_n is by no means unique because of the quasi-cohomological character of the problem. To illustrate the point we show that the result for \mathcal{F}_n is to a large extent independent of the choice of an $O(n)$ -connection.

Let $\mathbf{L}(\alpha, \beta)$ denote the operator

$$\mathbf{L}(\alpha, \beta)(\cdot) = \alpha \{W, \cdot\} + \beta (\bar{a}' [a', \cdot] - [\bar{a}', \cdot] a') \quad (57)$$

with α and β being positive real numbers. $\mathbf{L}(\alpha, \beta)$ is invertible on $\hat{\mathcal{M}}_n$. We may consider the modified equivariant connection

$$\begin{aligned} \mathcal{D}_v^{(\alpha, \beta)} &= d + A^{(\alpha, \beta)} - i_v, \\ A^{(\alpha, \beta)} &= \frac{1}{\mathbf{L}(\alpha, \beta)} \Re e \left\{ \alpha \bar{w} dw + \beta \bar{a}' da' - (\dots)^T \right\} \end{aligned}$$

and the modified action

$$S^{(n)(\alpha, \beta)} = -d_v \cdot \Re e \left(\bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w \right) \quad (58)$$

and

$$\mathcal{F}_n^{(\alpha, \beta)} \simeq \int_{\mathcal{M}_n} e^{-S^{(n)(\alpha, \beta)}}. \quad (59)$$

We want to show that $\mathcal{F}_n^{(\alpha, \beta)} = \mathcal{F}_n$. In fact we have

$$\begin{aligned} \mathcal{F}_n - \mathcal{F}_n^{(\alpha, \beta)} &= \int_{\mathcal{M}_n} \left(e^{-S^{(n)}} - e^{-S^{(n)(\alpha, \beta)}} \right) = \\ &= \int_{\hat{\mathcal{M}}_n} \left(\frac{\Re e (\bar{w} \bar{v} \mathcal{D}_v w)}{d_v \cdot \Re e (\bar{w} \bar{v} \mathcal{D}_v w)} - \frac{\Re e (\bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w)}{d_v \cdot \Re e (\bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w)} \right). \end{aligned}$$

The topform (t.f.) of the difference of terms in the last integral is exact, as can be inferred from the following chain of identities:

$$\begin{aligned} & \left\{ \frac{\Re e (\bar{w} \bar{v} \mathcal{D}_v w)}{d_v \cdot \Re e (\bar{w} \bar{v} \mathcal{D}_v w)} - \frac{\Re e (\bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w)}{d_v \cdot \Re e (\bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w)} \right\} \Big|_{t.f.} = \\ & \frac{\Re e (\bar{w} \bar{v} \mathcal{D}_v w) d_v \Re e (\bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w) - d_v \Re e (\bar{w} \bar{v} \mathcal{D}_v w) \Re e (\bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w)}{d_v \Re e (\bar{w} \bar{v} \mathcal{D}_v w) \cdot d_v \Re e (\bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w)} \Big|_{t.f.} = \\ & d_v \cdot \frac{\Re e (\bar{w} \bar{v} \mathcal{D}_v w) \Re e (\bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w)}{d_v \Re e (\bar{w} \bar{v} \mathcal{D}_v w) \cdot d_v \Re e (\bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w)} \Big|_{t.f.} = \\ & d \cdot \frac{\Re e (\bar{w} \bar{v} \mathcal{D}_v w) \Re e (\bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w)}{d_v \Re e (\bar{w} \bar{v} \mathcal{D}_v w) \cdot d_v \Re e (\bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w)} \Big|_{t.f.}, \quad (60) \end{aligned}$$

where for the last identity use has been made of the fact that the equivariant and ordinary exterior derivatives coincide if evaluated on the top form. One concludes from arguments as used above in remark (i) that there are no boundary contributions to the integral of the exact form (60) and hence $\mathcal{F}_n = \mathcal{F}_n^{(\alpha, \beta)}$.

5 Punctured Riemann spheres and instantons

Matone, [33], has pointed to an interesting analogy between the recursive determination of the Weil-Peterson volume of the moduli space of punctured Riemann sphere and the calculation of the $\mathcal{N}=2$ prepotential. The analogy has a “macroscopical” (in the sense used in the introduction) and a microscopical aspect. While the former is very neat the later is so far of a purely speculative nature. Let us start to describe the macroscopic approach towards the determination of the SW coefficients by Matone [33] ⁹.

Let u denote the $SU(2)$ invariant order parameter of the $\mathcal{N}=2$ Higgs-Yang-Mills theory:

$$u = \langle \text{tr} \Phi^2 \rangle.$$

Superconformal Ward identities can be used to derive, [38], the following relation between the prepotential and u , both being considered as functions of u :

$$\left\{ \mathcal{F}(v) - \frac{1}{2} v \frac{\partial}{\partial v} \mathcal{F}(v) \right\} = \frac{1}{2} u(v).$$

The variable u and its dual

$$v_D \equiv \frac{\partial \mathcal{F}(v)}{\partial v}$$

appear in the construction of SW as period integrals of a hyperelliptic Riemann surface:

$$v_D = \frac{\sqrt{2}}{\pi} \int_1^u \frac{dx \sqrt{x-u}}{\sqrt{x^2-1}}, \quad v = \frac{\sqrt{2}}{\pi} \int_{-1}^1 \frac{dx \sqrt{x-u}}{\sqrt{x^2-1}};$$

u and v_D may alternatively be represented as solution system of a Fuchs differential equation (a hypergeometric equation in the case at hand)

$$\left\{ (1-u^2) \partial_u^2 - \frac{1}{4} \right\} v_D = 0. \tag{61}$$

⁹A more general one-term recursion relation, which is exceedingly complicated, has been found by Chang and d'Hoker [37]. We believe that a microscopical reconstruction of this recursion is more remote than that of Matone's.

Inverting the functional dependence Matone [33] derived a non-linear differential equation for \mathcal{G} as function of v ,

$$(1 - \mathcal{G}^2) \mathcal{G}'' + \frac{1}{4} a g'^3 = 0 \quad (62)$$

with the notation

$$\mathcal{G} = \frac{-\pi i}{2} u \equiv \pi i \left(\mathcal{F}(v) - \frac{1}{2} v \frac{\partial}{\partial v} \mathcal{F}(v) \right).$$

Inserting the power series

$$\mathcal{G}(v) = \sum_{i=1}^{\infty} \mathcal{G}_n v^n$$

into (62) one obtains a three term recursion relation for the coefficients \mathcal{G}_n ;

$$\mathcal{G}_{n+1} = \frac{1}{2(n+1)^2} \left\{ (2n-1)(4n-1)\mathcal{G}_n + \sum_{k=0}^{n-1} c(k, n) \mathcal{G}_{n-k} \mathcal{G}_{k+1} - 2 \sum_{j=0}^{n-1} \sum_{k=0}^{j+1} d(j, k, n) \mathcal{G}_{n-j} \mathcal{G}_{j+1-k} \mathcal{G}_k \right\} \quad (63)$$

($c(k, n) = 2k(n-k-1) + n-1$, $d(j, k, n) = [2(n-j)-1][2n-3j-1+2k(j-k+1)]$) and therewith also a recursion relation for the coefficients $\mathcal{F}_n = \mathcal{G}_n / 2\pi i n$.

The moduli space of a Riemann sphere with n punctures

$$\Sigma_{0,n} = \hat{C} \setminus \{z_1, \dots, z_n; z_i \neq z_j \text{ for } i \neq j\},$$

with \hat{C} denoting the compactified complex plane, is the space of isomorphism classes of punctured spheres

$$\mathcal{M}_{0,n} = \{z_1, \dots, z_n; z_i \neq z_j \text{ for } i \neq j\} / \{Symm(n) \times PSL(2, C)\},$$

Let $\omega_{WP}^{(n)}$ be the two-form on $\mathcal{M}_{0,n}$ associated with the Weil-Petersson metric giving rise to a finite W-P-volume

$$Vol_{WP}(\mathcal{M}_{0,n}) = \frac{1}{(n-3)!} \int_{\mathcal{M}_{0,n}} \omega_{WP}^{n-3}. \quad (64)$$

Zograf proves [34] that the quantities

$$v_n = \frac{(n-3)!}{\pi^{2(n-3)}} Vol_{WP}(\mathcal{M}_{0,n})$$

obey for $n \geq 4$ the recursion relation

$$v_n = \sum_{j=1}^{n-3} \frac{j(n-j-2)}{n-1} \binom{n-4}{j-1} \binom{n}{j+1} v_{j+2} v_{n-j}; \quad v_3 = 1. \quad (65)$$

The origin of the two-term structure of (65) is easy to understand within Zograf's approach to the problem. He is executing the integral of one of the $n-3$ two forms in (64). The homology cycles separates the punctures on \tilde{C} into two groups leading to a factorization into the product of the volumes of two spheres, each with a smaller number of punctures, as a result of the localization of the two-dimensional integration on the Deligne, Knudsen, Mumford boundary [39] of $\mathcal{M}_{0,n}$. This concludes our rough sketch of the microscopic part of the problem.

To find the analogues of eq.'s (61), (62) we introduce, following Matone [32], the generating function

$$g(x) = \sum a_k x^{k-1};$$

$$a_k = \frac{v_k}{k-1} ((k-3)!)^2.$$

The recursion relation for the coefficient g_n is equivalent to the non-linear differential equation for the function g , [32]

$$x(x-g)g'' = xg'^2 + (x-g)g'. \quad (66)$$

With an appropriate change of variables one achieves a functional inversion of (66), [35], [36] with the linear substitute for (66), a Bessel differential equation,

$$y \frac{d^2 x}{dy^2} + x = 0. \quad (67)$$

The analogies between (64) and (53), (67) and (61), (66) and (62) are obvious and striking. What is missing for a perfect analogy is a transposition of the above quoted work of Zograf to the instanton problem. It is tempting to speculate that that the integral for \mathcal{F}_n eq. (53), becomes localized on (some part of) the Donaldson-Uhlenbeck boundary. But this is at least for the representation given by eq.(48) not the case. The residua in question do vanish.

6 Discussion

Keeping the above impressive analogies in mind we have nevertheless to emphasize that the topological complexity of the instanton manifold is by no means to be compared with the clear situation encountered in the case of punctured Riemann spheres. The ADHM moduli manifold may be considered as a non-trivial principal $O(n)$ - bundle (apart from being a principal $U(1)$ - bundle which was of prime importance above). The non-triviality is reflected in the fact that an $O(n)$ gauge fixing condition has to be chosen together with a domain of integration

bounded by a Gribov horizon. Configurations of clusters of “instantons of small size” on the Donaldson-Uhlenbeck boundary of which one might speculate that they give rise to the recursion mentioned in the preceeding section, are orbifold points of the $O(n)$ principal bundle. We have not been able to verify the latter speculation on the basis of the representation given by eq. (53) since the relevant residua vanish. A crucial point for any further progress in this matter will be - to our opinion - that one finds a (so far missing) idea as to which geometrical fact the three-term recursion could be attached. (In the case of punctured spheres it is the effectiveness of a lasso as a tool for catching whatever one wants to catch in two dimensions.) For that purpose a detailed knowledge of second cohomology classes H^2 of the space $\hat{\mathcal{M}}_n$ would be of prime importance. The complete analogy between punctured Riemann spheres and the SW prepotential in four dimensions, enthusiastically announced in [32] has still to be found. We hope that our representation (53) of the coefficients \mathcal{F}_n as formal intersection numbers may be a useful step into this direction.

Acknowledgments

R.F. owes his initiation to the subject of the Seiberg-Witten potentials to the late Lochlainn O’Raifeartaigh and to Ivo Sachs. This is gratefully acknowledged. R.P. acknowledges the partial financial support of INTAS 00-561 and of the Alexander von Humboldt foundation of Germany.

References

- [1] N. Seiberg, E. Witten, Nucl.Phys. B426 (1994) 19-52; Erratum-ibid. B430 (1994) 485-486, hep-th/9407087.
- [2] N. Seiberg, E. Witten, Nucl.Phys. B431 (1994) 484-550, hep-th/9408099.
- [3] K. Intriligator, N. Seiberg, Nucl.Phys.Proc.Suppl. 45BC (1996) 1-28, hep-th/9509066
- [4] S.A.Abel, J.L.F.Barbon, I.I.Kogan, E.Rabinovici: Some Thermodynamical Aspects of String Theory, hep-th/9911004.

- [5] M. Bershadsky, A. Johansen, T. Pantev, V. Sadov, C. Vafa, Nucl.Phys. B505 (1997) 153-164, hep-th/9612052.
- [6] W. Lerche, Nucl.Phys.Proc.Suppl. 55B (1997) 83-117, hep-th/9611190.
- [7] J.Maldacena, Adv.Theor.Math.Phys. 2 (1998) 231-252, hep-th/9711200.
- [8] M. Green, M. Gutperle, Phys.Rev. D58 (1998) 46007, hep-th/9804123.
- [9] M. Bianchi, M. Green, S. Kovacs, G. Rossi, JHEP 9808 (1998) 013, hep-th/9807033.
- [10] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis, S. Vandoren, Nucl.Phys. B552 (1999) 88-168, hep-th/9901128.
- [11] H. Bateman, A. Erdélyi, Higher Transcendental Functions, Vol. 3 (McGraw-Hill, New York, 1953).
- [12] N. Seiberg, Phys.Lett. B318 (1993) 469-475, hep-ph/9309335
- [13] R. Flume, M. Magro, L. O’Raifeartaigh, I. Sachs, O. Schnetz, Nucl.Phys. B494 (1997) 331-345, hep-th/9611123
- [14] N. Dorey, V.V. Khoze, M.P. Mattis, Phys.Rev. D54 (1996) 2921-2943, hep-th/9603136.
- [15] N. Dorey, V.V. Khoze, M. Mattis, Phys.Rev. D54 (1996) 7832-7848, hep-th/9607202.
- [16] N. Dorey, V.V. Khoze, M.P. Mattis, Nucl.Phys. B513 (1998) 681-708, hep-th/9708036.
- [17] D. Finnell (SLAC), P. Pouliot, Nucl.Phys. B453 (1995) 225-239, hep-th/9503115.
- [18] K. Ito, N. Sasakura, Phys.Lett. B382 (1996):95-103, hep-th/9602073.
- [19] A. Yung, Nucl.Phys. B485 (1997) 38-62, hep-th/9605096.
- [20] D. Bellisai, F. Fucito, A. Tanzini, G. Travaglini, JHEP 0007 (2000) 017, hep-th/0003272.
- [21] U. Bruzzo, F. Fucito, A. Tanzini, G. Travaglini, Nucl.Phys. B611 (2001) 205-226, hep-th/0008225.
- [22] M. Matone, JHEP 0104 (2001) 041, hep-th/0103246.
- [23] M. Atiyah, V. Drinfeld, N. Hitchin, Yu. Manin, Phys. Lett. A65 (1978) 185.
- [24] V. Gribov, Nucl. Phys. B139 (1978) 1.

- [25] H. Osborn, *Ann. of Physics* 135 (1981) 373.
- [26] N. Berline, E. Getzler and M. Vergne, *Heat Kernels and Dirac Operators*, Springer, Berlin, 1996.
- [27] H. Cartan, La transgression dans un groupe de Lie et dans un espace fibré principal, in *Colloque de Topologie*, C. B. R. M., Bruxelles, 1950, pp 57-71.
- [28] A. Schwarz, O. Zaboronsky, *Commun.Math.Phys.* 183 (1997) 463-476, hep-th/9511112.
- [29] S. K. Donaldson, P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford University Press, 1990.
- [30] D. S. Freed and K. K. Uhlenbeck, *Instantons and Four-Manifolds*, Springer, Berlin, 1984
- [31] R. Bott and L. Tu, *Differential Forms in Algebraic Geometry*, Springer - verlag, New York, 1982.
- [32] M. Matone, *J.Geom.Phys.* 21 (1997) 381-398, hep-th/9402081.
- [33] M. Matone, *Phys.Lett.* B357 (1995) 342-348, hep-th/9506102.
- [34] P. Zograf, *Contemporary Mathematics*, v. 150 (1993) 367-372.
- [35] M. Kontsevich, Yu. Manin, *Invent. Math.* 124 (1996)313.
- [36] M. Kontsevich, Yu. Manin and D. Zagier, *Comm. Math. Phys.* 181 (1996)763.
- [37] G. Chan, E. D'Hoker, *Nucl.Phys.* B564 (2000) 503-516, hep-th/9906193.
- [38] P. S. Howe, P. C. West, *Nucl.Phys.* B486 (1997) 425-442, hep-th/9607239.
- [39] P. Deligne and D. Mumford, *J. H. E. S. Publ. Math.* 36(1969)75;
F. Knudsen and D. Mumford, *Math. Scand.* 39 (1976) 19.