Riemann surfaces, separation of variables and classical and quantum integrability.

O. Babelon [1] and M. Talon [1]

Laboratoire de Physique Théorique et Hautes Energies. [2]

Abstract

We show that Riemann surfaces, and separated variables immediately provide classical Poisson commuting Hamiltonians. We show that Baxter's equations for separated variables immediately provide quantum commuting Hamiltonians. The construction is simple, general, and does not rely on the Yang–Baxter equation.

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[1]Member of CNRS.

[2]L.P.T.H.E. Universités Paris VI–Paris VII (UMR 7589), Boîte 126, Tour 16, 1er étage, 4 place Jussieu, F-75252 PARIS CEDEX 05

1 Introduction.

We know since Liouville that integrability means commuting Hamiltonians. It is the primary role of Lax matrices and the Yang-Baxter equation to provide non trivial such Hamiltonians. In the classical theory, additional benefits are the spectral curve Γ and the ability to separate variables by considering $g = \text{genus}(\Gamma)$ points on it [1].

In the quantum theory, the analog construction is Sklyanin's method of separation of variables and Baxter's equations [2, 3]. Despite the beauty of this result, the route from a Yang–Baxter defined quantum integrable model to the separated variables is usually long and difficult, especially in the non hyperelliptic case.

Here, we show that we can reverse the strategy. We start from separated variables and consider Baxter's equations as equations for the Hamiltonians. We then prove that these Hamiltonians commute under very general hypothesis.

By its generality, its simplicity and its close analogy to the classical case, this result could provide a good starting point to build a theory of quantum integrable systems.

2 The main theorem.

Consider a curve in C^2

$$\Gamma(\lambda, \mu) \equiv R_0(\lambda, \mu) + \sum_{j=1}^{g} R_j(\lambda, \mu) H_j = 0 \tag{1}$$

where the H_i are the only dynamical moduli, so that $R_0(\lambda, \mu)$ and $R_i(\lambda, \mu)$ do not contain any dynamical variables. If things are set up so that Γ is of genus g and there are exactly g Hamiltonian H_j (see below for realizations of this setup), then the curve is completely determined by requiring that it passes through g points (λ_i, μ_i), $i = 1, \cdots, g$. Indeed, the moduli H_j are determined by solving the linear system

$$\sum_{j=1}^{g} R_j(\lambda_i, \mu_i) H_j + R_0(\lambda_i, \mu_i) = 0, \quad i = 1, \cdots, g \tag{2}$$

whose solution is

$$H = -B^{-1}V \tag{3}$$

where

$$H = \begin{pmatrix} H_1 \\ \vdots \\ H_i \\ \vdots \\ H_g \end{pmatrix}, \quad B = \begin{pmatrix} R_1(\lambda_1, \mu_1) & \cdots & R_g(\lambda_1, \mu_1) \\ \vdots & & \vdots \\ R_1(\lambda_i, \mu_i) & \cdots & R_g(\lambda_i, \mu_i) \\ \vdots & & \vdots \\ R_1(\lambda_g, \mu_g) & \cdots & R_g(\lambda_g, \mu_g) \end{pmatrix}, \quad V = \begin{pmatrix} R_0(\lambda_1, \mu_1) \\ \vdots \\ R_0(\lambda_i, \mu_i) \\ \vdots \\ R_0(\lambda_g, \mu_g) \end{pmatrix}$$

Here, of course, we assume that generically $\det B \neq 0$.

Theorem 1 *Suppose that the variables (λ_i, μ_i) are separated i.e. they Poisson commute for $i \neq j$:*

$$\{\lambda_i, \lambda_j\} = 0, \quad \{\mu_i, \mu_j\} = 0, \quad \{\lambda_i, \mu_j\} = p(\lambda_i, \mu_i)\delta_{ij} \tag{4}$$

Then the Hamiltonians H_i, $i = 1 \cdots g$, defined by eq.(3) Poisson commute

$$\{H_i, H_j\} = 0$$

<u>Proof</u>. Let us compute

$$
\begin{aligned}
B_1 B_2 \{(B^{-1}V)_1, (B^{-1}V)_2\} &= \{B_1, B_2\}(B^{-1}V)_1(B^{-1}V)_2 \\
&\quad -\{B_1, V_2\}(B^{-1}V)_1 - \{V_1, B_2\}(B^{-1}V)_2 + \{V_1, V_2\}
\end{aligned}
$$

Taking the matrix element i, j of this expression, we get

$$
\begin{aligned}
\left(B_1 B_2 \{(B^{-1}V)_1, (B^{-1}V)_2\}\right)_{ij} &= \delta_{ij} \sum_{k,l} \{B_{ik}, B_{il}\}(B^{-1}V)_k(B^{-1}V)_l \\
&\quad -\delta_{ij} \sum_k \{B_{ik}, V_i\}(B^{-1}V)_k - \delta_{ij} \sum_l \{V_i, B_{il}\}(B^{-1}V)_l + \delta_{ij}\{V_i, V_i\} = 0
\end{aligned}
$$

where δ_{ij} occurs because the variables are separated. ∎

It can hardly be simpler. The only thing we use is that the Poisson bracket vanishes between different lines of the matrices, and then the antisymmetry. We did not even need to specify the Poisson bracket between λ_i and μ_i. The Hamiltonian are in involution whatever this Poisson bracket is. This is the root of the multihamiltonian structure of integrable systems.

Can we make it quantum? Let us consider a set of separated variables

$$[\lambda_i, \lambda_j] = 0, \quad [\mu_i, \mu_j] = 0, \quad [\lambda_i, \mu_j] = p(\lambda_i, \mu_i)\delta_{ij}$$

We want Baxter's equation, so we start from the linear system

$$\sum_j R_j(\lambda_i, \mu_i) H_j + R_0(\lambda_i, \mu_i) = 0 \tag{5}$$

Here the H_j are on the right, and in $R_j(\lambda_i, \mu_i)$, $R_0(\lambda_i, \mu_i)$, we assume some order between λ_i, μ_i, but the coefficients in these functions are non dynamical. Hence we start from the linear system

$$BH = -V \tag{6}$$

We notice that we can define unambiguously the left inverse of B. First, the determinant D of B is well defined because it never involves a product of elements on the same line. The same is true for the cofactor Δ_{ij} of the element B_{ij} (we include the sign $(-1)^{i+j}$ in the definition of Δ_{ij}). Define

$$B_{ij}^{-1} \equiv (B^{-1})_{ij} = D^{-1}\Delta_{ji}$$

We have

$$(B^{-1}B)_{ij} = \sum_k D^{-1}\Delta_{ki}B_{kj}$$

But Δ_{ki} does *not* contain any element B_{kl}, hence the product $\Delta_{ki}B_{kj}$ is commutative, and the usual construction of the inverse of B is still valid. Since the left and right inverse coincide in an associative algebra with unit, we have the identities

$$(BB^{-1})_{ij} = \sum_k B_{ik}B_{kj}^{-1} = \sum_k B_{ik}D^{-1}\Delta_{jk} = \delta_{ij} \tag{7}$$

We write the solution of eq.(6) as

$$H = -B^{-1}V \tag{8}$$

Theorem 2 *The quantities H_i defined by eq.(8), which solve Baxter's equations eqs.(5), are all commuting*

$$[H_i, H_j] = 0$$

<u>Proof</u>. Using that V_k and V_l commute, $[V_k, V_l] = 0$, we compute

$$
\begin{aligned}
[H_i, H_j] &= \sum_{k,l} [B_{ik}^{-1} V_k, B_{jl}^{-1} V_l] \tag{9}\\
&= \sum_{k,l} [B_{ik}^{-1}, B_{jl}^{-1}] V_k V_l - B_{ik}^{-1} [B_{jl}^{-1}, V_k] V_l + B_{jl}^{-1} [B_{ik}^{-1}, V_l] V_k
\end{aligned}
$$

Using

$$[A^{-1}, B^{-1}] = A^{-1} B^{-1} [A, B] B^{-1} A^{-1} = B^{-1} A^{-1} [A, B] A^{-1} B^{-1}$$

so that

$$
\begin{aligned}
[B_{ik}^{-1}, B_{jl}^{-1}] &= \sum_{rs,r's'} B_{ir}^{-1} B_{jr'}^{-1} [B_{rs}, B_{r's'}] B_{s'l}^{-1} B_{sk}^{-1}\\
&= \sum_{rs,r's'} B_{jr'}^{-1} B_{ir}^{-1} [B_{rs}, B_{r's'}] B_{sk}^{-1} B_{s'l}^{-1}
\end{aligned}
$$

the first term can be written

$$
\begin{aligned}
\sum_{k,l} [B_{ik}^{-1}, B_{jl}^{-1}] V_k V_l &= \sum \frac{1}{2} B_{ir}^{-1} B_{jr'}^{-1} [B_{rs}, B_{r's'}] \Big(B_{s'l}^{-1} B_{sk}^{-1} + B_{s'k}^{-1} B_{sl}^{-1} \Big) V_k V_l\\
&= \sum \frac{1}{2} B_{jr'}^{-1} B_{ir}^{-1} [B_{rs}, B_{r's'}] \Big(B_{sk}^{-1} B_{s'l}^{-1} + B_{sl}^{-1} B_{s'k}^{-1} \Big) V_k V_l
\end{aligned}
$$

Using that $[B_{rs}, B_{r's'}] = \delta_{rr'} [B_{rs}, B_{rs'}]$ and is therefore antisymmetric in ss', and setting

$$K_{ss'} = \sum_{k,l} \Big(B_{s'l}^{-1} B_{sk}^{-1} + B_{s'k}^{-1} B_{sl}^{-1} - B_{sl}^{-1} B_{s'k}^{-1} - B_{sk}^{-1} B_{s'l}^{-1} \Big) V_k V_l$$

we get

$$
\begin{aligned}
\sum_{k,l} [B_{ik}^{-1}, B_{jl}^{-1}] V_k V_l &= \sum_{rss'} \frac{1}{4} B_{ir}^{-1} B_{jr}^{-1} [B_{rs}, B_{rs'}] K_{ss'}\\
&= -\sum_{rss'} \frac{1}{4} B_{jr}^{-1} B_{ir}^{-1} [B_{rs}, B_{rs'}] K_{ss'}\\
&= \sum_{rss'} \frac{1}{8} [B_{ir}^{-1}, B_{jr}^{-1}] [B_{rs}, B_{rs'}] K_{ss'}
\end{aligned}
$$

The last two terms in eq.(9) are simpler, we get

$$\sum_{k,l} B_{jl}^{-1} [B_{ik}^{-1}, V_l] V_k - B_{ik}^{-1} [B_{jl}^{-1}, V_k] V_l = \sum_{rsk} [B_{ir}^{-1}, B_{jr}^{-1}] [B_{rs}, V_r] B_{sk}^{-1} V_k$$

4

The quantities H_i will commute if

$$[B_{ir}^{-1}, B_{jr}^{-1}] = 0, \quad \forall i, j, r \tag{10}$$

This is true as shown in the next Lemma. ∎

The condition eq.(10) says that the elements on the same column of B^{-1} commute among themselves. In a sense this is a condition dual to the one on B. It is true semiclassically because

$$\{B_{ir}^{-1}, B_{jr}^{-1}\} = \sum_{a,a',b,b'} B_{ia}^{-1} B_{ja'}^{-1} \{B_{ab}, B_{a'b'}\} B_{br}^{-1} B_{b'r}^{-1} = \sum_{a,b,b'} B_{ia}^{-1} B_{ja}^{-1} \{B_{ab}, B_{ab'}\} B_{br}^{-1} B_{b'r}^{-1} = 0$$

where in the last step we use the antisymmetry of the Poisson bracket. We show that it is also true quantum mechanically[3].

Lemma 1 *Let B be a matrix whose elements commute if they do not belong to the same line*

$$[B_{ik}, B_{jl}] = 0 \quad \text{if } i \neq j$$

Then the inverse B^{-1} of B is defined without ambiguity and moreover elements on a same column of B^{-1} commute

$$[B_{ir}^{-1}, B_{jr}^{-1}] = 0$$

<u>Proof</u>. We want to show that

$$\Delta_{ri} B_{jr}^{-1} = \Delta_{rj} B_{ir}^{-1}$$

denote by $\beta_i^{(r)}$ the vector with components B_{ki}, $k \neq r$. Then we have (with $j > i$)

$$
\begin{aligned}
\Delta_{ri} B_{jr}^{-1} &= (-1)^{r+i} \beta_1^{(r)} \wedge \beta_2^{(r)} \wedge \cdots \widehat{\beta_i^{(r)}} \wedge \cdots \beta_j^{(r)} \wedge \cdots \beta_g^{(r)} B_{jr}^{-1} \\
&= (-1)^{r+i+g-j} \beta_1^{(r)} \wedge \beta_2^{(r)} \wedge \cdots \widehat{\beta_i^{(r)}} \wedge \cdots \widehat{\beta_j^{(r)}} \wedge \cdots \beta_g^{(r)} \wedge \beta_j^{(r)} B_{jr}^{-1} \\
&= (-1)^{r+i+g-j+1} \beta_1^{(r)} \wedge \beta_2^{(r)} \wedge \cdots \widehat{\beta_i^{(r)}} \wedge \cdots \widehat{\beta_j^{(r)}} \wedge \cdots \beta_g^{(r)} \wedge \sum_{k \neq j} \beta_k^{(r)} B_{kr}^{-1} \\
&= (-1)^{r+i+g-j+1} \beta_1^{(r)} \wedge \beta_2^{(r)} \wedge \cdots \widehat{\beta_i^{(r)}} \wedge \cdots \widehat{\beta_j^{(r)}} \wedge \cdots \beta_g^{(r)} \wedge \beta_i^{(r)} B_{ir}^{-1} \\
&= (-1)^{r+j} \beta_1^{(r)} \wedge \beta_2^{(r)} \wedge \cdots \beta_i^{(r)} \wedge \cdots \widehat{\beta_j^{(r)}} \wedge \cdots \beta_g^{(r)} B_{ir}^{-1} \\
&= \Delta_{rj} B_{ir}^{-1}
\end{aligned}
$$

where in the third line we used eq.(7). In the above manipulations, we never have two operators B_{ij} on the same line so we can use the usual properties of the wedge product. Moreover it is important that the line r is absent in the definition of $\beta^{(r)}$. Remark that this equation can also be written $\Delta_{ri} D^{-1} \Delta_{rj} = \Delta_{rj} D^{-1} \Delta_{ri}$ which is a Yang–Baxter type equation. ∎

With this Lemma, we have completed the proof of our theorem. It is remarkable that, again, only the separated nature of the variables λ_i, μ_i is used in this construction, but the precise commutation relations between λ_i, μ_i does not even need to be specified. This is the origin of the multi Hamiltonian structure of integrable systems, here extended to the quantum domain.

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[3]The same Lemma already appeared in [11]. Our two proofs are different and independent however. We thank B. Enriquez for drawing our attention to his work.

3   Choosing the right number of dynamical moduli.

Let us explain how one can set up things in order that the number of dynamical moduli is equal to the genus of the Riemann surface. To understand the origin of the conditions we will write, let us explain first what happens in the setting of general rational Lax matrices described in [4, 9]. Quite generally, a Lax matrix $L(\lambda)$ depending rationally on a spectral parameter λ, with poles at points λ_k can be written as

$$L(\lambda) = L_0 + \sum_k L_k(\lambda) \tag{11}$$

where $L_0 = \mathrm{Diag}(a_1, \cdots, a_N)$ is a constant diagonal matrix and $L_k(\lambda)$ is the polar part of $L(\lambda)$ at λ_k, ie. $L_k(\lambda) = \sum_{r=-n_k}^{-1} L_{k,r}(\lambda - \lambda_k)^r$. In order to have a good phase space to work with, we assume that $L_k(\lambda)$ lives in a coadjoint orbit of the group of $N \times N$ matrix regular in the vicinity of $\lambda = \lambda_k$, i.e.

$$L_k = \left(g_k A_k g_k^{-1} \right)_-$$

Here $A_k(\lambda)$ is a diagonal matrix with a pole of order n_k at $\lambda = \lambda_k$, and g_k has a regular expansion at $\lambda = \lambda_k$. The notation $()_-$ means taking the singular part at $\lambda = \lambda_k$. This singular part only depends on the singular part $(A_k)_-$ and the first n_k coefficients of the expansion of g_k in powers of $(\lambda - \lambda_k)$. The matrix $(A_k)_-$ is an orbit invariant which specifies the coadjoint orbit, and is *not* a dynamical variable. It is in the center of the Kirillov bracket which as shown in [4] induces the Poisson bracket eq.(4), with $p(\lambda_i, \mu_i) = 1$, on the separated variables. The physical degrees of freedom are contained in the first n_k coefficients of $g_k(\lambda)$. Note however that since A_k commutes with diagonal matrices one has to take the quotient by $g_k \to g_k d_k$ where $d_k(\lambda)$ is a regular diagonal matrix, in order to correctly describe the dynamical variables on the orbit. The dimension of the orbit of L_k is thus $N(N-1)n_k$ so that $L(\lambda)$ depends on $\sum_k N(N-1)n_k$ degrees of freedom. Finally, the form and analyticity properties of $L(\lambda)$ are invariant under conjugation by constant matrices. To preserve the normalization, L_0, at ∞ these matrices have to be diagonal (if all the a_i's are different). Generically, these transformations reduce the dimension of the phase space by $2(N-1)$, yielding:

$$\dim \mathcal{M} = (N^2 - N) \sum_k n_k - 2(N-1)$$

The spectral curve is

$$\Gamma \quad : \quad R(\lambda, \mu) \equiv \det(L(\lambda) - \mu \, \mathbf{1}) = (-\mu)^N + \sum_{q=0}^{N-1} r_q(\lambda)\mu^q = 0 \tag{12}$$

The coefficients $r_q(\lambda)$ are polynomials in the matrix elements of $L(\lambda)$ and therefore have poles at λ_k. The curve is naturally presented as a N–sheeted covering of the λ-plane. We call $\mu_j(\lambda)$ the N branches over λ. Using the Riemann–Hurwitz formula, we can compute the genus of Γ [4]:

$$g = \frac{N(N-1)}{2} \sum_k n_k - N + 1 = \frac{1}{2} \dim \mathcal{M}$$

It is important to observe that the genus is half the dimension of phase space. So the number of action variables occurring as independent parameters in the eq.(12) should also be equal to g. Let

us verify it. Since $r_j(\lambda)$ is the symmetrical function $\sigma_j(\mu_1, \cdots, \mu_N)$, it is a rational function of λ. It has a pole of order jn_k at $\lambda = \lambda_k$. Its value at $\lambda = \infty$ is known since $\mu_j(\lambda) \to a_j$. Hence it can be expressed on $j \sum_k n_k$ parameters namely the coefficients of all these poles. Altogether we have $\frac{1}{2}N(N+1)\sum_k n_k$ parameters. They are not all independent however. Above $\lambda = \lambda_k$ the various branches can be written:

$$\mu_j(\lambda) = \sum_{n=1}^{n_k} \frac{c_n^{(j)}}{(\lambda - \lambda_k)^n} + \text{regular} \tag{13}$$

where all the coefficients $c_1^{(j)}, \cdots, c_{n_k}^{(j)}$ are fixed and non–dynamical because they are the matrix elements of the diagonal matrices $(A_k)_-$, while the regular part is dynamical. This implies on $r_j(\lambda)$ that the coefficients of its n_k highest order pole terms are fixed. Summing over j, we get Nn_k constraints and we are left with $\frac{1}{2}N(N-1)\sum_k n_k$ parameters, that is $g + N - 1$ parameters. It remains to take the quotient by the action of constant diagonal matrices. The generators of this action are the Hamiltonians $H_n = (1/n)\,\mathrm{res}_{\lambda=\infty}\mathrm{Tr}\,(L^n(\lambda))\,d\lambda$, i.e. the term in $1/\lambda$ in $\mathrm{Tr}\,(L^n(\lambda))$. Setting

$$\mu_j(\lambda) = a_j + \frac{b_j}{\lambda} + \cdots \tag{14}$$

around the point $Q_j = (\infty, a_j)$, we have $H_n = \sum_j a_j^{n-1} b_j$. After Hamiltonian reduction these quantities are to be set to fixed (non–dynamical) values. So, both a_i (by definition) and b_i are non dynamical. On the functions $r_j(\lambda)$ this implies that their expansion at infinity starts as $r_j(\lambda) = r_j^{(0)} + \frac{r_j^{(-1)}}{\lambda} + \cdots$, with $r_j^{(0)}$ and $r_j^{(-1)}$ non dynamical. Hence when the system is properly reduced we are left with exactly g action variables.

The constraints eqs.(13, 14) can be summarized in a very elegant way [5, 9]. Introduce the differential δ with respect to the dynamical moduli. Then our constraints mean that the differential $\delta\mu d\lambda$ is regular everywhere on the spectral curve because the coefficients of the various poles being non dynamical, they are killed by δ:

$$\delta\mu \, d\lambda = \text{ holomorphic}$$

Since the space of holomorphic differentials is of dimension g, the right hand side of the above equation is spanned by g parameters which are the g independent action variables we were looking for. Notice that these action variables are coefficients in the pole expansions of the functions $r_j(\lambda)$, and thus appear linearly in the equation of Γ. Hence eq.(12) can be written in the form eq.(1). Clearly, these considerations can be adapted by considering more general conditions such as

$$\frac{\delta\mu}{\mu^n} \frac{d\lambda}{\lambda^m} = \text{ holomorphic}$$

4   Examples.

Let us show how well known models fit into our scheme. For the hyperelliptic ones, things are so simple that we can directly check the commutation of the Hamiltonians.

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4.1 Neumann model.

The spectral curve can be written in the form [9]:

$$\mu^2 = \frac{\prod_{i=1}^{N-1}(\lambda - b_i)}{\prod_{i=1}^{N}(\lambda - a_i)} = \frac{P(\lambda)}{Q(\lambda)} \tag{15}$$

Performing the birational transformation $s = \mu Q(\lambda)$, we get:

$$s^2 = Q(\lambda)P(\lambda) \tag{16}$$

which is an hyperelliptic curve of genus $g = N - 1$. The polynomial $Q(\lambda)$ is non dynamical. We have $(N - 1)$ independent dynamical quantities, namely the $(N - 1)$ symmetrical functions of the b_i, coefficients of $P(\lambda)$. We have

$$\delta\mu \, d\lambda = \frac{\delta P(\lambda)}{2\mu Q(\lambda)} d\lambda = \frac{\delta P(\lambda)}{2s} d\lambda = \text{ holomorphic}$$

Asking that a curve of the form eq.(15) passes through the g points (λ_i, μ_i) determines the polynomial $P(\lambda)$. The solution of Baxter's equations

$$P(\lambda_i) = Q(\lambda_i)\mu_i^2$$

simplifies in this case because the matrix B depends only on the λ_i. It is equivalent to Lagrange interpolation formula:

$$P(\lambda) = P^{(0)}(\lambda) + P^{(2)}(\lambda)$$

with

$$P^{(0)}(\lambda) = \prod_i(\lambda - \lambda_i), \quad P^{(2)}(\lambda) = \sum_j S_j(\lambda)Q(\lambda_j)\mu_j^2, \quad S_j(\lambda) = \frac{\prod_{k\neq j}(\lambda - \lambda_j)}{\prod_{k\neq j}(\lambda_j - \lambda_k)}$$

Introducing the canonical commutation relations

$$[\mu_j, \lambda_k] = i\hbar\delta_{jk}$$

so that

$$[\mu_j, f(\lambda_j)] = i\hbar\partial_{\lambda_j}f(\lambda_j), \quad [\mu_j^2, f(\lambda_j)] = 2i\hbar\partial_{\lambda_j}f(\lambda_j)\mu_j + (i\hbar)^2\partial_{\lambda_j}^2 f(\lambda_j)$$

We can check that $[P(\lambda), P(\lambda')] = 0$ is a consequence of $\partial_{\lambda_j}P^{(0)}(\lambda) = -\prod_{k\neq j}(\lambda_j - \lambda_k)S_j(\lambda)$, and the identities

$$S_j(\lambda)\partial_{\lambda_j}^n S_i(\lambda') - S_j(\lambda')\partial_{\lambda_j}^n S_i(\lambda) = 0, \quad \forall n > 0$$

These identities follow from the remark that, if we define the translation operators $t_j\lambda_i = \lambda_j + \sigma\delta_{ij}$, then

$$S_j(\lambda)t_j S_i(\lambda') - S_j(\lambda')t_j S_i(\lambda) = \frac{\prod_{k\neq i,j}(\lambda - \lambda_k)\prod_{k\neq i,j}(\lambda' - \lambda_k)}{\prod_{k\neq j}(\lambda_j - \lambda_k)\prod_{k\neq i}(\lambda_i - \lambda_k)}(\lambda_i - \lambda_j)(\lambda - \lambda') \tag{17}$$

is independent of σ.

4.2 Toda Chain.

The spectral curve can be written in the form [8, 9]:

$$\mu + \mu^{-1} = 2P(\lambda) \tag{18}$$

where $2P(\lambda) = \lambda^{n+1} - \sum_{i=1}^{n+1} p_i \lambda^n + \cdots$ is a polynomial of degree $(n+1)$. The spectral curve is hyperelliptic since it can be written as

$$s^2 = P^2(\lambda) - 1, \quad \text{with} \quad s = \mu - P(\lambda) \tag{19}$$

The polynomial $P^2(\lambda)$ is of degree $2(n+1)$ so the genus of the curve is $g = n$. The number of dynamical moduli is $g = n$ in the center of mass frame $\sum_{i=1}^{n+1} p_i = 0$. We have

$$\frac{\delta\mu}{\mu}\, d\lambda = \frac{2\delta P(\lambda)}{\mu - \mu^{-1}} d\lambda = \frac{\delta P(\lambda)}{s} d\lambda = \text{ holomorphic}$$

Asking that the curve eq.(18) passes through the n points (λ_i, μ_i), we get Baxter's equations.

$$2P(\lambda_i) = \mu_i + \mu_i^{-1}$$

Their solution is again given by Lagrange interpolation formula:

$$2P(\lambda) = P^{(0)}(\lambda) + P^{(1)}(\lambda)$$

where

$$P^{(0)}(\lambda) = (\lambda + \sum_i \lambda_i)\prod_{i=1}^{n}(\lambda - \lambda_i), \quad P^{(1)}(\lambda) = \sum_i S_i(\lambda)(\mu_i + \mu_i^{-1})$$

The polynomial $P^{(0)}(\lambda)$ is of degree $n + 1$, vanishes for $\lambda = \lambda_i$ and has no λ^n term. Let the commutation relations of the separated variables be given by:

$$\mu_j \lambda_j = q\lambda_j \mu_j, \quad \mu_j \lambda_i = \lambda_i \mu_j, \quad i \neq j$$

Then again $[P(\lambda), P(\lambda')] = 0$ as a result of eq.(17), where t_j is interpreted as $t_j \lambda_j = q\lambda_j$, and the facts that

$$\begin{aligned} S_j(\lambda)t_j^{\pm 1}P^{(0)}(\lambda') - S_j(\lambda')t_j^{\pm 1}P^{(0)}(\lambda) &= P^{(0)}(\lambda')S_j(\lambda) - P^{(0)}(\lambda)S_j(\lambda') \\ &= \frac{\prod_{k\neq j}(\lambda - \lambda_k)\prod_{k\neq j}(\lambda' - \lambda_k)}{\prod_{k\neq j}(\lambda_j - \lambda_k)}(\lambda + \lambda' + \sum_{i\neq j}\lambda_i)(\lambda' - \lambda) \end{aligned}$$

4.3 A non–hyperelliptic model.

We consider the model studied in [2, 6, 7] . The spectral curve can be written in the form:

$$R(\lambda, \mu) \equiv \mu^N + t^{(1)}(\lambda)\mu^{N-1} + \cdots t^{(N)}(\lambda) = 0 \tag{20}$$

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The polynomials $t^{(k)}(\lambda)$ are such that degree $t^{(k)}(\lambda) \leq kn - 1$ and degree $t^{(N)}(\lambda) = Nn - 1$ for some integer n. The genus of this curve is

$$g = \frac{1}{2}(N-1)(Nn-2)$$

Assuming that there is no singular point at finite distance, the homomorphic differentials are

$$\omega_{kl} = \frac{\mu^l \lambda^k}{\partial_\mu R(\lambda, \mu)} d\lambda, \quad 0 \leq l < N - 1, \quad 0 \leq k < (N - l - 1)n - 1$$

We have

$$\delta\mu \frac{d\lambda}{\lambda} = -\sum_{k=1}^{N} \frac{\delta t^{(k)}(\lambda)\mu^{N-k}}{\partial_\mu R(\lambda, \mu)} \frac{d\lambda}{\lambda}$$

This will be holomorphic if $\delta t^{(1)}(\lambda) = 0$ and

$$\delta t^{(k)}(\lambda) = \delta H_1^{(k)}\lambda + \cdots + \delta H_{(k-1)n-1}^{(k)} \lambda^{(k-1)n-1}$$

Baxter's equations and the commutation of the Hamiltonians where proved in this case, starting from the definition of the quantum model through it Lax matrix and the Yang–Baxter equation. Our approach gives a very simple proof of this result.

5 Conclusion.

We have shown that starting from the separated variables, one can give an easy definition of a quantum integrable system. The next step is to reconstruct the Lax matrix and the original dynamical variables of the model. While this is a well understood problem in the classical theory [1, 9], (it is the essence of the classical inverse scattering method), its quantum counterpart will require a deeper understanding of the quantum affine Jacobian [10].

References

[1] B.A. Dubrovin, I.M. Krichever, S.P. Novikov, *Integrable Systems I.* Encyclopedia of Mathematical Sciences, Dynamical systems IV. Springer (1990) p.173–281.

[2] E.K. Sklyanin, *Separation of variables in quantum integrable models related to the Yangian $Y[sl(3)]$.* hep-th/9212076 (1992)

[3] E.K. Sklyanin, *Separation of variables.* Prog. Theor. Phys. (suppl), 185 (1995) p.35.

[4] O.Babelon, M. Talon, *The symplectic structure of rational Lax pair systems.* solv-int/9812009 Physics Letters A 257 (1999) p.139-144.

[5] I.M. Krichever, D.H. Phong, *On the integrable geometry of soliton equations and $N = 2$ supersymmetric gauge theories.* J. Diff. Geom. 45 (1997) p.349–389.

[6] F.A. Smirnov, *Separation of variables for quantum integrable models related to $U_q(\hat{sl}_N)$.* math-ph/0109013

[7] F.A. Smirnov, V. Zeitlin, *Affine Jacobians of spectral Curves and Integrable Models.* math-ph/0203037

[8] L. Faddeev, L. Takhtajan, **Hamiltonian Methods in the Theory of Solitons.** Springer 1986.

[9] O.Babelon, D. Bernard, M. Talon, **Introduction to Classical Integrable Systems.** Cambridge University Press (to appear).

[10] F.A. Smirnov, *Dual Baxter equations and quantization of Affine Jacobian.* math-ph/0001032

[11] B. Enriquez, V. Rubtsov, *Commuting families in skew fields and quantization of Beauville's fibration.* math.AG/0112276.