# Classical and Quantum Strings in plane waves, shock waves and space-time singularities : synthesis and new results

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**Abstract :**

Key issues and essential features of classical and quantum strings in gravitational plane waves, shock waves and spacetime singularities are synthetically understood. This includes the string mass and mode number excitations, energy-momentum tensor, scattering amplitudes, vacuum polarization and wave-string polarization effect. The role of the real pole singularities characteristic of the tree level string spectrum (real mass resonances) and that of the spacetime singularities is clearly exhibited. This throws light on the issue of singularities in string theory which can be thus classified and fully physically characterized in two different sets : *strong* singularities (poles of order $\geq 2$, and black holes) where the string motion is *collective* and non oscillating in time, outgoing states and scattering sector do not appear, the string *does not* cross the singularities, and *weak* singularities (poles of order $< 2$, (Dirac $\delta$ belongs to this class) and conic/orbifold singularities) where the whole string motion is oscillatory in time, outgoing and scattering states exist, and the string *crosses* the singularities.
Common features of strings in singular wave backgrounds and in inflationary backgrounds are explicitely exhibited.
The string dynamics and the scattering/excitation through the singularities (whatever their kind : strong or weak) is fully physically consistent and meaningful.

**CONTENTS**



# 1 Introduction

Classical and quantum strings propagating in gravitational plane waves, shock waves and space-time singularities and its classification, were investigated 10 years ago by de Vega and Sanchez refs.[1] - [6] and by de Vega, Ramon Medrano and Sanchez refs.[7] - [9], including the exact computation of the string mass spectrum, number operators, scattering amplitudes, string energy-momentum tensor, string vacuum polarization and string/wave polarization effects.
These results are relevant to answer questions as "Singularities in String Theory", and the boson-fermion and fermion-boson transmutations induced by superstring backgrounds.
The subject has subtle points and subtle physical interpretation which were overlocked in refs.[10], [11].

A lot attention has been payed recently to the subject [19], [20].
In this paper we provide a synthetic understanding to the problem and throw light on the issue of string singularities and its classification.

## 2 Understanding and Results

We consider [1] strings propagating in gravitational plane-wave spacetimes described by the metric

$$dS^2 = F(U,X,Y)dU^2 - dUdV + dX^j dX^j, \tag{1}$$

where

$$F(U,X,Y) = W(U)(X^2 - Y^2) \quad and \quad 2 \le j \le D-3. \tag{2}$$

$W(U)$ is an arbitrary function; $U$ and $V$ are light-cone variables. (These are sourceless solutions of the Einstein equations, i.e., vacuum spacetime). The only nonvanishing components of the Riemann tensor are

$$R_{UiVj} = \frac{1}{2}\partial_i\partial_j F(U,X,Y)$$

i.e.

$$R_{UXVX} = -R_{UYVY} = W(U)$$

Thus, there will be spacetime singularities if $W(U)$ is singular.

We are interested in functions $W(U)$ nonzero in a bounded interval $-T < U < T$ and which have a singular behavior for $U \to 0$, such as

$$W(U) \sim \frac{\alpha}{|U|^\beta} \quad \text{for} \quad U \to 0, \tag{3}$$

$\alpha$ and $\beta$ being positive contants. (A change of sign in $\alpha$ is just equivalent to exchanging $X$ with $Y$.)

The essential features in this problem depend on the function W(U) (and not on the (X,Y) dependence). Moreover, our results hold for any function F(U,X,Y) = W(U) F(X,Y) (see below). The form of F(X,Y) determines whether or not the problem is exactly solvable but it is enough to fix the asymptotic behaviour of F(X,Y) to solve (asymptotically) the dynamics, to determine the behaviour near the space time singularities, and to determine whether $\langle M^2 \rangle$ and $\langle N \rangle$ will be (or will be not) finite.

The string equations in the class of backgrounds eqs (1), (2) are linear and exactly solvable. In the light-cone gauge $U = \alpha' p \tau$ and after Fourier expansion in the world-sheet coordinate $\sigma$, the Fourier components $X_n(\tau)$ and $Y_n(\tau)$ satisfy a one-dimensional Schrödinger-type equation but with $\tau$ playing the role of the spatial coordinate and $p^2 W(\alpha' p \tau)$ as the potential [2]. (Here $p$ stands for the $U$ component of the string momentum).

The propagation of the string when it approaches the singularity at $U = 0$ depends on the singularity exponent $\beta$ . We find different behaviors depending on whether $\beta < 2$ or $\beta \geq 2$ :

(i) For $\beta < 2$, the string coordinates $X$ and $Y$ are regular everywhere; that is, the string propagates *smoothly* through the gravitational singularity $U = 0$.

(ii) For a strong enough singularity ($\beta \geq 2$), the string goes off to $X = \infty$ grazing the singularity plane $U = 0$. This means that the string *does not go across* the gravitational wave; that is, the string cannot reach the $U > 0$ region. For particular initial configurations, the string remains trapped at the point $X = Y = 0$ in the gravitational-wave singularity $U = 0$.

(iii) The case in which $\beta = 2$ and then $W(U) = \alpha / U^2$ for all $|U| < T$ is explicitly solved in terms of Bessel functions.

The string propagation in these singular spacetimes has common features with the fall of a point particle into a singular attractive potential $-\alpha / x^{\beta}$. In both cases, the falling takes place when $\beta \geq 2$.
The behavior in $\tau$ of the string coordinates $X^A(\sigma, \tau)$ is analogous to the behavior of the Schrödinger wave function $\Psi(x)$ of a point particle. *However, the physical content is different.*
The string coordinates $X^A(\sigma, \tau)$ are dynamical variables and not wave functions. Moreover, our analysis also holds for the quantum propagation of the string : the behavior in $\tau$ is the same as in the classical evolution with the coefficients being quantum operators. At the classical as well as at the quantum level, the string propagates or does not propagate through the gravitational wave depending on whether $\beta < 2$ or $\beta \geq 2$, respectively. In other words, the tunnel effect *does not take place* in this string problem.

It must be noticed that for $\tau \to 0^-$, i.e., $U \to 0^-$, the behavior of the string solutions is *non oscillatory* in $\tau$, whereas for $\tau \to \infty$, the string oscillates. This new type of behavior in $\tau$ is analogous to that found for strings in cosmological inflationary backgrounds.

For $\beta = 2$, we express the coefficients ($B_n^A$) characterizing the solution for $\tau \to 0$ in terms of the oscillator operators for $\tau \to \infty$. We label with the indices $<$ and $>$ the operators in the region $U < -T$ and $U > T$, respectively (i.e., before and after the collision with the singularity plane $U = 0$).

For $\beta < 2$ we compute the total mass squared $\langle M_>^2 \rangle$ and the total number of modes $\langle N_> \rangle$ after the string propagates through the singularity plane $U = 0$ and reaches the flat space-time region $U > T$. *This has a meaning only for* $\beta < 2$.

For $\beta \geq 2$, the string *does not reach* the $U > 0$ region and hence there are no outgoing operators ($>$). In particular no mass squared $M_>^2$ and total number $N_>$ operator can be defined for $\beta \geq 2$.

For $\beta < 2$, $\langle M_>^2 \rangle$ and $\langle N_> \rangle$ are given by [1].

$$\langle M_>^2 \rangle = m_0^2 + \frac{2}{\alpha'} \sum_{n=1}^{\infty} n(|B_n^x|^2 + |B_n^y|^2), \tag{4}$$

$$\langle N_> \rangle = 2 \sum_{n=1}^{\infty} (|B_n^x|^2 + |B_n^y|^2), \tag{5}$$

where

$$B_n^x = -B_n^y \sim \left[\frac{2n}{\alpha' p}\right]^{\beta-2} \quad \text{for} \quad n \to \infty \tag{6}$$

($|0_<\rangle$ stands for the ingoing ground state, so that $\alpha_{n<}|0_<\rangle = 0$ for all $n$).

Therefore, $\langle M_>^2 \rangle$ is *finite* for $\beta < 1$ but *diverges* for $1 \leq \beta < 2$. $\langle N_> \rangle$ is finite for $\beta < \frac{3}{2}$ and diverges for $\frac{3}{2} \leq \beta < 2$.

We analyze below the origin and physical meaning of these infinities.
Notice that the exponent $\beta$ in Eq.(6) is just minus the scaling dimension of $W(U)$ for small $U$.

# 3 Sourceless shock waves

The sourceless shock-wave case

$$W(U) = \alpha \delta(U),$$

is particularly useful in order to understand the physical origin of the divergences in $\langle M^2_> \rangle$ when $1 \leq \beta < 2$ (and in $\langle N_> \rangle$ when $\frac{3}{2} \leq \beta < 2$).

For a sourceless skock wave with a metric function

$$F(X, Y) = \alpha \delta(U)(X^2 - Y^2) \tag{7}$$

In this $\delta$-function case, the $B_n$ coefficients are easy to compute exactly with the result

$$B_n^x = -B_n^y = \frac{\alpha p \alpha'}{2in}$$

Comparing the $B_n$ cofficients Eqs.(6), (8), we see that the large-$n$ behavior of $B_n$ for $W(U) = \alpha \delta(U)$, is the same as that for $W(U) = \alpha |U|^{-\beta}$ with $\beta = 1$. This is related to the fact that both functions $W(U)$ have the same scaling dimension.

In this case the string propagation is formally like a Schrödinger equation with a Dirac $\delta$ potential : the string passes across the singularity at $U = 0$ and tunnel effect is present.
This is a weak singularity, the behaviour of the string is oscillatory in time, and in, out scattering states can be defined.
The string scattering in this sourceless shock wave is very similar to the string scattering by a shock wave with a nonzero source density [2, 3, 6 ].
We also compute $\langle M^2_> \rangle$ and $\langle N_> \rangle$ for a metric function

$$F(U, X, Y) = \alpha \delta(U)(X^2 - Y^2)\theta(\rho_0^2 - X^2 + Y^2), \tag{8}$$

where $\theta$ is the step function and $\rho_0$ gives the transverse size of the shock wave front. This $F$ belongs to the shock-wave class with a density source we have treated in refs.[2, 3, 6].
We find that $\langle M^2_> \rangle$ is *finite as long as* $\rho_0$ *is finite*. This divergence in $\langle M^2_> \rangle$ is due to the infinite transverse extent of the wave front and **not** to the short-distance singularity $\delta(U)$ at $U = 0$.
The gravitational forces in the transverse directions $X, Y$ transfer to the string a finite amount of energy when the transverse size of the shock wave ($\rho_0$) is finite. When $\rho_0 = \infty$, the energy transferred by the shock wave to the string produce large elongation amplitudes in the $X$ and $Y$ directions which are responsible for the divergence of $\langle M^2_> \rangle$.

More generally, for a string propagating in a shock-wave spacetime with the generic profile

$$F(U, X, Y) = \delta(U)f(X, Y) \tag{9}$$

we have found the **exact** expression of $\langle M^2_> \rangle$ and $\langle N_> \rangle$ [6]. If $f(X,Y)$ is nonzero at $X = Y = \infty$, then $\langle M^2_> \rangle = \infty$.The divergence of ($M^2_>$) is due to the nonvanishing of $f(X,Y)$ at large distances and **not** to the singularity at U = 0 of the gravitational wave. When $f(X,Y)$ has an infinite range, the gravitational forces in the $X, Y$ directions have the possibility to transfer an infinite amount of energy to the string modes.

# 4 String Energy Momentum Tensor

We also compute the string energy-momentum tensor $\mathcal{T}^{AB}$. It is convenient to define it in the present context as an integral over a spatial volume completely enclosing the string at a time $X^0$, as we have proposed in Ref. 6.

For *all* $\beta > 0$, we find that the $\mathcal{T}^{AB}$ components can be grouped according to their $\tau \to 0$ behavior into four sets :
(i) $\mathcal{T}^{VV}$ diverges for $\beta > 0$.
(ii) $\mathcal{T}^{VX}$, $\mathcal{T}^{VY}$ diverge for $\beta > 1$ and tend to a finite constant for $\beta < 1$.
(iii) $\mathcal{T}^{UV}\mathcal{T}^{V}j(3 \leq j \leq D-1)$ and $\mathcal{T}^{XX}$ are finite and nonzero. $\mathcal{T}^{XX}$ vanishes for $\beta < 2$.
(iv) The other components vanish.

The explicit expressions were obtained in ref. [1]. The energy density $\mathcal{T}^{00}$, the energy flux in the propagation direction of the gravitational wave $\mathcal{T}^{01}$, as well as the stress in this direction ($\mathcal{T}^{11}$), strongly diverge for $\tau \to 0$. For $\beta \leq 2$, they diverge as a negative power of $\tau$. For $\beta > 2$, the divergence is exponential.

We compute the ground-state expectation values in the illustrative case $\beta = 2$. The expectation values of **all** the (operator) coefficients of the powers of $\tau$ turn out to be **finite**.

It can be noticed that the spatial proper length of the string grows indefinitely for $\tau \to 0$ when the string approaches the singularity plane. This phenomenon is analogous to that found for strings in cosmological inflationary backgrounds.

In conclusion, for $\beta \geq 2$ and $\beta < 1$ the propagation of classical and quantum strings through the singular spacetimes Eqs. (1)-(3) is **physically meaningful** and has the physical features described above.

For $1 \leq \beta < 2$, we find that the expectation value $\langle M^2_> \rangle$ diverges. However, this divergence is due to the infinite transverse extent of the wave front and **not** to the short-distance singularity of $W(U)$ at $U = 0$. In support of this analysis, we found that the large-$n$ behavior of the transmission coefficients (determining whether $\langle M^2_> \rangle$ is finite or infinite), is **the same** for $W(U) = \alpha/U$ as for $W(U) = \alpha\delta(U)$.
Therefore, the divergences of $\langle M^2_> \rangle$ in shock-wave spacetimes and in singular plane waves, have the same origin : the infinite transverse extension of the wave front.

# 5 String Mass and Mode Number in Singular Plane-Wave Backgrounds

As we have shown in ref.[1], the string performs strong oscillations in $X$ and $Y$ when it enters into the gravitational wave. These oscillations are stronger as the singularity at $\tau = 0$ is stronger.

For $\beta \geq 2$, the string *does not cross* the $U = 0$ singularity and it escapes to $X = \infty$ grazing the singularity plane $U = 0$.

For $\beta < 2$, the string *crosses* the gravitational wave and reaches the flat space-time region $U > T$.
The expectation value of the string mass-square operator after the propagation through the gravitational wave has been computed in ref.[1].

The convergence of the series depends on the behaviour of Bogoliubov coefficients $\mathrm{B}_n$ for large n. In this regime they can be computed in the Born approximation, with the result

$$\mathrm{B_n} \sim -2i\alpha \left(\frac{2n}{\alpha' p}\right)^{\beta-2} C_\beta \quad \text{for} \quad n \to \infty \quad 0 < \beta < 1 \tag{10}$$

Therefore

$$n|B_n|^2 \sim n^{2\beta-3}$$

Hence,

$$\langle M^2_> \rangle = \begin{cases} \quad\quad\quad finite \quad for \quad \beta < 1 \\ divergent \quad for \quad 1 \leq \beta < 2 \end{cases} \tag{11}$$

Recall that, when $\beta \geq 2$, the string does not reach the region $U > 0$ and hence, there are no $>$ operators in the $\beta \geq 2$ case.

For the expectation value of the number operator $\langle N_> \rangle$ we have

$$\langle N_> \rangle = \begin{cases} finite \quad for \quad \beta < \frac{3}{2}, \\ divergent \quad for \quad \frac{3}{2} \leq \beta < 2. \end{cases} \tag{12}$$

The divergence appearing in $\langle M_>^2 \rangle$ for $1 \leq \beta < 2$ is due to the infinite transverse extent of the wave front and **not** to the short-distance singularity of $W(U)$ at $U = 0$.

The gravitational forces in the transverse directions ($X, Y$) transfer to the string a finite amount of energy when the size of the shock-wave front ($p_0$) is finite. The strong gravitational forces impart in the transverse directions ($X, Y$) large elongation amplitudes which are responsible for the divergent $\langle M_>^2 \rangle$ when the size of the wave front is infinite.

The divergent $\langle M_>^2 \rangle$ for $W(U) = \alpha |U|^{-\beta}$ when $1 \leq \beta < 2$ have the same explanation. (And similarly, those of $\langle N_> \rangle$ when $\frac{3}{2} \leq \beta < 2$).

More generally, for a string propagating in a wave type spacetime with generic profile

$$F(U, X, Y) = W(U) f(X, Y),$$

if $f(X, Y)$ does not vanish at $X = Y = \infty$, then $\langle M_>^2 \rangle = \infty$.

# 6 Polarized plane waves and String Polarization

The above properties can be generalized to singular gravitationaal waves with two arbitrary singular profile functions $W_1(U) = \alpha_1/|U|^{\beta_1}$, $W_2(U) = \alpha_2/|U|^{\beta_2}$.

The dynamics is exactly and explicitely solvable [7]. The string time evolution is fully determined by the background geometry, whereas the overall $\sigma$-dependence is fixed by the initial string state.

The proper length stretches infinitely at the singularities when $\beta_1 \geq 2$ and / or $\beta_2 \geq 2$ (strong singularities)

When $\beta_1 \geq 2$ and / or $\beta_2 \geq 2$, the string does not cross the singularity (U = 0) but goes off to infinity in a given direction $\alpha$ which depends on the polarization of the gravitational wave. The string escape situation is the following :
(i) for $\beta_1 > \beta_2$ (and $\beta_1 \geq 2$), then the angle $\alpha = 0$ and the string goes off to infinity in the X-direction. In this case, the singularity of $W_1$ dominates over that of $W_2$ and we recover the previous situation of ref. [1].

(ii) for $\beta_2 > \beta_1$ (and $\beta_2 \geq 2$), then

$$\alpha = (\pi/4) \quad \text{sgn} \quad \alpha_2, \tag{13}$$

and

(iii) for $\beta_1 = \beta_2 \geq 2$, then

$$\tan\alpha = \frac{\alpha_2}{\alpha_1 + \sqrt{\alpha_1^2 + \alpha_2^2}}$$

that is

$$\tan 2\alpha = \alpha_2/\alpha_1.$$

If $\alpha_1 > 0 \quad (\alpha_1 < 0)$, the string escape directions are within the cone $|\alpha| < \pi/4 \quad (|\alpha - \pi/2| < \pi/4)$.
In addition to escaping to infinity, the string oscillates in the (X, Y) plane, perpendicularly to the escape direction, and with vanishing amplitude for $U \to 0$.
The string behaviour near the singularity expresses naturally in terms of the null variable $\hat{U}$.

$$\hat{U}(\sigma,\tau) = \begin{cases} \ln(-U) & \beta = 2 \\ \frac{(-U)^{1-(\beta/2)}}{1-(\beta/2)} & \beta > 2 \end{cases}$$

For instance, the oscillatory modes in the (X, Y) plane are not harmonic in $U$ but in $\hat{U}$. The variable $\hat{U}$ is like the cosmic time of strings in cosmological backgrounds (in terms of which the string oscillates), whereas $U$ is like the conformal time. Here, for simplicity, $\beta_1 = \beta_2 = \beta$ but this case is actually generic.

For $\beta_1 < 2$ and $\beta_2 < 2$ (weak singularities), the string passes smoothly through the space-time singularity and reaches the outgoing region, and, the string behaviour is oscillatory in time. In this case, outgoing operators make sense and can be explicitely related to the in-operators.

For the particles described by the quantum string states, this implies two types of effects :
(i) rotation of spin polarization in the (X, Y) plane, and
(ii) transmutation between different particles.

The expectation values of the outgoing mass ($M^2_>$) operator and of the mode-number operator $N_>$, in the ingoing ground state $|O_<\rangle$ are different from the ingoing expectation values $M^2_<$ and $N_<$. This difference is due to the excitation of the string modes after crossing the spacetime singularity. In other words, the string state is not an eigenstate of $M^2_>$, but an infinity superposition of one-particle states with different masses. This is a consequence of the particle transmutation which allows particle masses different from the initial one ($m_0^2$).

# 7 Generic features. Common features with strings in inflationary backgrounds

The string evolution near the spacetime singularity is a *collective motion* governed by the nature of the gravitational field. The state of the string fixes the overall $\sigma$-dependent coefficients whereas the $\tau$-dependence is fully determined by the spacetime geometry. In other words, the $\tau$-dependence is the same for all modes $n$.
In some directions, the string collective propagation turns to be an infinite motion (the escape direction), whereas in other directions, the motion is oscillatory, but with a fixed ($n$-independent) frequency. In fact, these features are not restricted to singular gravitational waves, but are *generic* to strings in strong gravitational fields .

Moreover, it is interesting to compare the string behaviour for $\tau \to 0$ in the inflationary backgrounds for which we have ref.[12]

$$ds^2 = (dX^0)^2 - R^2(X^0)(dX^i)^2$$

with $R(\tau) = -(H\tau)^{-\beta/2}$

$$X^0(\sigma,\tau)_{\tau\to 0} = \begin{cases} \text{const} & \beta < 2\,(\text{superinflationary}) \\ -H^{-1}\ln[H\tau L(\sigma)] & \beta = 2\,(\text{de Sitter, inflationary}) \\ -\frac{(\tau L(\sigma))^{1-(\beta/2)}}{1-(\beta/2)} & \beta > 2(\text{power type, inflationary}) \end{cases}$$

.

$$X^i(\sigma,\tau)_{\tau\to 0} = \begin{cases} A^i(\sigma) + D^i(\sigma)\frac{\tau^2}{2} + \tau^{1+\beta}F^i_{(\sigma)} & \beta \neq 2 \\ A^i(\sigma) + \tilde{D}^i(\sigma)\frac{\tau^2}{2} + \frac{\tau^2}{2}\ln\tau\tilde{F}^i_{(\sigma)} & \beta = 2. \end{cases}$$

The $A^i(\sigma)$ are arbitrary functions of $\sigma$, $D^i(\sigma)$, $\tilde{D}^i(\sigma)$, $F^i(\sigma)$, and $\tilde{F}^i(\sigma)$ are fixed by the constraints. Here $X^0$ is the cosmic time, while $\eta \approx \tau$ is the conformal time.
For the singular plane waves we have

$$\hat{U}(\sigma,\tau)_{\tau\to 0} = \begin{cases} \text{const} & \beta < 2 \\ \ln(-U) & \beta = 2 \\ \frac{(-U)^{1-(\beta/2)}}{1-(\beta/2)} & \beta > 2. \end{cases}$$

The behaviour of the string time coordinates are *regular* and *non-oscillating* for $\tau \to 0$ ('frozen'), while for singular plane waves, in addition, one of the transverse coordinates is non-oscillating and singular for $\tau \to 0$.

The above behaviours for $X^0$ and $U$ are characteristic of strings in strong gravitational fields. Notice that the inflationary backgrounds are non- singular, whereas the plane waves (equations (1)-(2)) are singular spacetimes. This is connected to the fact the string coordinates $X$, $Y$ exhibit divergences in these plane-wave spacetimes.

# 8 Generic wave profiles. The non-linear in-out unitary transformation

In the case of plane or shock waves with profile functions ($X^2 \pm Y^2$), the string equations became *exactly linear*, and the transformation between in-going ($\langle$) and outgoing operators ($\rangle$) is *linear*, ie a Bogoliubov transformation.

For generalized profile functions of plane waves and shock waves, the string equations are non linear and the ingoing-outgoing unitary transformation describing the scattering/excitation of the string is **non-linear**. The linear (Bogoliubov) approximation holds only for large impact parameters. One must use the **exact** in-out transformation to include **all** impact parameters. In ref [6] we succeeded in computing the **exact** transformation. This throws light on both computations and interpretation, and in particular on the rôle played by the spacetime background. We do all the treatement for any function $f(X^i)$, i.e. for any wave profile. We express the nonlinear transformation between ingoing and outgoing zero modes and oscillators in terms of a hermitean operator

$$G = \frac{p_U}{8\pi}\int_0^{2\pi} d\sigma \int d\mathbf{p}^{D-2} \,:\, e^{\mathbf{ip}-\mathbf{x}(\sigma,\tau=0)} \,:\, \varphi(\mathbf{p}),$$

where : : stands for normal ordering with respect to the ingoing state $|0_<\rangle$ and $\varphi(\mathbf{p})$ is the Fourier transform of the wave profile.

This operator acts on the Fock space operators and Fock space states and it is suitable to express the relevant expectation values after the collision of the string with the wave background.
The ingoing-outgoing ground-state transition amplitude $\langle 0_<|0_>\rangle$ expresses as

$$\sum_n \frac{i^n}{n!}\langle 0_<|G^n|0_<\rangle .$$

We interpret these terms as a $n$-leg scalar amplitude with vertex operators inserted at $\tau = 0$ (a line of pinchs at the intersection of the world sheet with the wave), and find an integral representation for these quantities.

The integrands possess equally spaced real pole singularities typical of string models in flat space-time. The presence and structure of these poles is not at all related to the structure of the space-time geometry (which may or may not be singular). We give a sense to these integrals by taking the principal value prescription, yielding for $\langle 0_<|G^n|0_<\rangle$ a well defined *finite* and real result.

The exact expressions for the total outgoing mode operator $\langle N_>\rangle$ and mass square $\langle M_>^2\rangle$ operator in the ingoing ground state $|0_<\rangle$ can be thus computed. We find that the contribution from each $n$-level, that is $M_n^2$ for $n\to\infty$, decreases like

$$\alpha'\tilde{p}^2 n^{-1}(2\alpha'/\pi \log n)^{1-D/2} \tag{14}$$

The large $n$ behaviour of $\langle M_n^2\rangle$ depends on the density matter $\tilde{\rho}$ only through its total energy $\tilde{p} = \int d^{D-2}\mathbf{X}\,\tilde{\rho}(\mathbf{X})$.

All the shock wave geometries exhibit the same large $n$ behaviour of $M_n^2$, irrespective of the structure of the localized source.
The sum over $n$ in both $\langle N_>\rangle$, and $\langle M_>^2\rangle = (D-2)/(12\alpha') + \sum_{n=1}^{\infty}\langle M_N^2\rangle$ converges and the total values are exactly computed.

The contribution of the excited states is suppressed by a factor $\langle 0_<|0_<\rangle = (L/2\pi)^{D-2}$ with respect to the direct (initial state) contribution $\mu^2 = -(D-2)/(12\alpha')$, $L$ being a typical large box size.

We find for $\langle N_>\rangle$ and $\langle M_>^2\rangle$ integral representations which exhibit a similar structure to $\langle G^2 \rangle$, that is, the integrands factorizes into two pieces : $|\phi(\mathbf{p})|^2$ which characterizes the wave geometry and the function

$$\mathrm{tg}(\alpha'\pi\mathbf{p}^2)\Gamma(\alpha'\mathbf{p}^2)/\Gamma\left(\frac{1}{2}+\alpha'\mathbf{p}^2\right) ,$$

which depends only on the string. These integrands possess real singularities (poles) like the tree level string spectrum. This is exactly what happens in the tree amplitudes of string models. Like the tree level string spectrum, these poles correspond to equally spaced real resonances in the mass spectrum. As is usually expected, loop corrections provide a width to these resonances and will therefore shift the poles away from the integration path, leading to finite results. The physical interpretation of such poles is that they correspond to all higher string states which become excited after the collision through the wave singularities.
In conclusion, singularities in string theory are fully, mathematically and **physically consistent**.

# 9 Clarification

Some aspects of strings in these gravitational-wave backgrounds have been studied by the authors of refs.[10], [11]. However, this problem has subtle points which were overlooked in refs. [10], [11].
The analysis done in refs. [10], [11] by analogy with the Schrödinger equation is not careful enough. The mass and number operators are expressed in terms of the transmission coefficient $B_n$. The cases in refs. [10], [11] in which $B_n = \infty$ mean that there is no transmission to the region $U > 0$, and therefore that there is no outgoing mass nor outgoing number operator (since there is no string) in that region. This is the situation of string falling to $U = 0$ for $\beta \geq 2$, the string *does not cross* the singularity, the string scapes to infinity *grazing* the singularity plane U = 0.
Therefore,the outgoing ($>$) operators as $M^2_>$ and $N_>$ make sense only for $\beta < 2$ and any statement about $M^2_>$ and $N_>$ for $\beta \geq 2$ is **meaningless**, (since there is no string in the $>$ region for $\beta \geq 2$).

In the cases in which $\langle M^2_> \rangle$ is divergent, such infinity is **not** related at all to the spacetime singularity at $U = 0$. This happens for $\beta < 2$ when the transverse size (i.e., perpendicular to the propagation direction) of the gravitational wave front is infinity. Then, the gravitational wave carries an infinite energy which transfers to the string according to the behavior of $W(U)$. This may lead to a finite or infinite value for $\langle M^2_> \rangle$ as we have seen above.

In conclusion, the propagation of classical and quantum strings through the singular space-times is **physically meaningful** and follows the evolution, properties and classification summarized here.

# 10 Conclusions and Classification

**(i)** As is known, in the context of point particle QFT, vacuum polarization effects do **not** arise in plane wave and shock wave backgrounds since ingoing $<$ and outgoing $>$ operators **do not** get mixed in this context. Therefore no particle creation effects takes place for point particle field theories in these geometries. On the contrary, particle transmutations as well as vacuum polarization effects on the energy- momentum tensor **do appear** for strings in these wave space-times. These effects can be traced back to the mixing of creation and annihilation $<$ and $>$ string oscillators.

**(ii)** Pole-type curvature singularities with $\beta < 2$, $\delta(U)$ singularities conical and orbifold singularities are **weak** singularities : the string motion is oscillatory in time, string crosses smoothly the singularities, ingoing and outgoing states can be defined and so all the scattering sector. Mass, number and energy-momentum of the string are well defined and **finite**. The string oscillators get excited in the scattering and crossing the singularities. The string proper length is finite. The detailed classical and quantum string dynamics in conical space times with general deficit angles

$$\delta\phi = 2\pi(1-\alpha) = 8\pi G\mu$$

$$0 \leq \phi < 2\pi\alpha,$$

was treated in refs [13], [14]. In the simpler case of orbifolds, which is a particular case of the above for deficit angles $2\pi(1-1/N)$, the scattering is trivial.

**(iii)** Pole-type singularities with $\beta \geq 2$ and black holes (r = 0) are **strong** singularities; string does not pass across the singularity (eventually it can gets trapped); outgoing operators can not be defined. Ingoing mass, number and energy momentum tensor of the string are well defined and **finite**.
The proper string length stretches indefinitely.

**(iv)** The features of the strong singularity case are **generic** to strings in strong gravitational fields : the string evolution near the strong spacetime singularity is a **collective motion** governed by the nature of the gravitational field. The state of the string fixes the overall $\sigma$-dependence. The time dependence is fully determined by the spacetime geometry, that is, the same for all modes $n$. In some directions, the string collective propagation is an infinite motion (the escape direction); in other directions, the motion is oscillatory, but with a fixed ($n$)- independent frequency.
This happens too for strings in inflationary backgrounds (which are not singular). In the inflationary cases, all transverse spatial coordinates are **non oscillating** in time, but they are **regular** ("frozen") for $\tau \to 0$, while for strong singular plane waves some of the transverse coordinates are singular.
For the string time coordinate the plane wave case $\beta < 2$ is like super inflation, the case $\beta = 2$ is like de Sitter inflation, and the case $\beta > 2$ is like power type inflation.

**(v)** Tests strings do propagate **consistently** in singular space-times : Klein-Gordon equation (for a point particle) is ill defined, whereas the string equations are well behaved. This includes strings in plane wave backgrounds, shock wave backgrounds (sourceless shock waves and of Aichelburg-Sexl type, and with generic profiles), as well as in the black hole geometries, where the string behaviour is **regular** at the horizon and near the r = 0 sigularity refs.[15], [16], [17]. That is, strings feel the space time singularities much less than point particles.
Furthermore, we would not be surprised by the presence of space-time singularities in string theory as long as one sticks to a geometry description using a metric tensor $G_{AB}(X)$ in spite of the fact it fulfills the exact or corrected string equations. We do not expect that a space-time description in terms of a Riemannian manifold with local coordinates $X^A$ will be meaningful at the Planck scale.

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