Comment on "Supersymmetry in the half-oscillator - revisited"

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We point out the flaw in the analysis of Gangopadhyaya and Mallow, hep-th/0206133, where it is claimed that supersymmetry is broken in the SUSY half-oscillator, even with a regularization respecting supersymmetry.

In an earlier paper [1], we had shown that supersymmetry, in quantum mechanical theories with singular potentials or nontrivial boundaries, is preserved if the system is regularized in a manner respecting supersymmetry. This is in contrast to the earlier claims [2–4] that supersymmetry is broken in such systems. We had shown, in particular, that if the superpotential is regularized, as opposed to the conventional wisdom of regularizing the potential, supersymmetry is maintained in such systems in a natural manner. The reason for this is quite clear. A regularized superpotential leads to a pair of supersymmetric Hamiltonians for every value of the regularization parameter whereas a conventionally regularized potential does not lead to a pair of supersymmetric Hamiltonians for any value of the regularization parameter.

In a recent paper, Gangopadhyaya and Mallow [5] claim that of the examples treated in [1], the supersymmetric half-oscillator does not possess supersymmetry in spite of using the supersymmetric regularizations proposed in [1]. In fact, their analysis is exactly the same as given in [1]. However, their final solution of the boundary condition and, therefore, their conclusion is incorrect for a very simple reason that we wish to point out.

Let us recall that the SUSY half-oscillator was first studied in [3] where the authors concluded that supersymmetry is broken. Physically, their result can be understood as follows. If we require the wavefunctions to vanish at the origin (as these authors do and as would be the case when one conventionally regularizes the potential), only odd solutions of the harmonic oscillator are allowed. On the other hand, supersymmetry requires the partner wavefunctions to have opposite parity and since this is not allowed by the boundary conditions (which follows from the particular regularization used), supersymmetry must be broken. The problem with this conclusion, as pointed out in [1], is that such a treatment of the system corresponds to regularizing the potential in a manner which itself breaks supersymmetry and, therefore, the result of such an analysis can be an artifact of the regularization used. In stead, we had proposed to regularize the superpotential directly as

$$W(x) = -\omega x\theta(x) + c\theta(-x) \tag{1}$$

where c is the regularization parameter and we take $c \to +\infty$ at the end. This superpotential leads to the regularized supersymmetric potentials:

$$\begin{aligned} V_{+}(x) &= \frac{1}{2}[(\omega^2 x^2 - \omega)\theta(x) + c^2\theta(-x) - c\delta(x)] \\ V_{-}(x) &= \frac{1}{2}[(\omega^2 x^2 + \omega)\theta(x) + c^2\theta(-x) + c\delta(x)] \end{aligned} \tag{2}$$

The crucial difference between this and the earlier treatment of this problem [3] is the appearance of Dirac delta functions in the potentials. The delta function is attractive for one of the Hamiltonians while it is repulsive for the other. It is clear, therefore, that, in the presence of an attractive delta function potential, the usual argument for the non existence of even solutions is not automatic. In fact, we had shown explicitly that supersymmetry is maintained in the limit $c \to +\infty$ in this theory. We had also proposed an alternate regularization in [1] that avoids the appearance of delta functions, and had shown that the end result is the same if the regularization maintains supersymmetry.

It is surprising, therefore, that Gangopadhyaya and Mallow [5], who carry out exactly our analysis would arrive at a different conclusion, namely, that supersymmetry is broken in this system. Let us note that the boundary condition (eq. (23) in [1] and eq. (10) in [5]) is given by (we choose $\omega = \frac{1}{2}$ for simplicity)

$$-\sqrt{2}\,\frac{\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(-\epsilon)} = \sqrt{c^2 - 2\epsilon} - c \tag{3}$$

where ϵ represents the energy of the eigenstates. Gangopadhyaya and Mallow claim that $\epsilon = 0$ is not a solution of the above equation, in which case supersymmetry will be broken. It is, of course, obvious that $\epsilon = 0$ is a solution of

(3) for any value of c and, in particular, for $c \to +\infty$, but let us look at this more carefully to understand the flaw in the analysis of [5].

Let us define, for simplicity,

$$F(\epsilon) \equiv -\sqrt{2}\,\frac{\Gamma(\frac{1}{2}-\epsilon)}{\Gamma(-\epsilon)} \qquad \text{and} \qquad G(\epsilon,c) \equiv \sqrt{c^2-2\epsilon}-c \quad , \tag{4}$$

so that we can represent the boundary condition (3) as

$$F(\epsilon) = G(\epsilon,c) \tag{5}$$

In a quantum theory, the boundary condition (5) may not hold for all values of ϵ. The fact that energy is quantized means that it can be satisfied only for some discrete values of $\epsilon = \epsilon_n$. Let us, in particular, investigate whether $\epsilon = 0$ is a solution of the equation, and if it is, whether it remains a solution in the limit $c \to +\infty$.

Let us note that both $F(\epsilon)$ and $G(\epsilon,c)$, as complex functions of ϵ, are analytic around $\epsilon = 0$. For $F(\epsilon)$, we can see this from the fact that the function $\frac{1}{\Gamma(-\epsilon)}$ is entire in the whole ϵ plane with simple zeroes at the points $\epsilon = n$, $(n = 0, 1, 2, ...)$, and the function $\Gamma(\frac{1}{2}-\epsilon)$ is also analytic around $\epsilon = 0$, with $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. For $G(\epsilon,c) = \sqrt{c^2-2\epsilon}-c$, the same is also true.

Since both these functions are analytic around $\epsilon = 0$, a Taylor expansion around this value leads to,

$$\begin{aligned} F(\epsilon) &\simeq 0 + \sqrt{2\pi}\,\epsilon + \mathcal{O}(\epsilon^2) & (6)\\ G(e,c) &\simeq 0 - \frac{1}{c}\,\epsilon + \mathcal{O}(\epsilon^2) & (7) \end{aligned}$$

The fact that $\epsilon = 0$ is a solution of equation (5) is a consequence of the fact that the zeroth order terms in the Taylor expansion in equations (6) and (7) coincide for that value. Note that this result is independent of c and, therefore, also holds in the limit $c \to +\infty$.

The mistake in the analysis of Gangopadhyaya and Mallow lies in the fact that, for whatever reason, they impose that the coefficient of the linear terms in the Taylor expansion should match, in addition to the zeroth order terms. Indeed, the equation below equation (11) (for $\omega = \frac{1}{2}$) in [5] is precisely the requirement of the equality for the linear terms in equations (6) and (7). This imposition, however, is unjustified, since that would tantamount to requiring

$$F(0) = G(0,c), \qquad \text{and} \qquad F'(0) = G'(0,c) \tag{8}$$

The boundary condition (5) does not have to hold for all values of ϵ, not even in a neighborhood of $\epsilon = 0$. It should hold only for discrete values of ϵ. For the case $\epsilon = 0$, this translates into the fact that only the zeroth order terms in the Taylor expansion of F and G should be equal, but no extra conditions must be imposed on higher order terms. The incorrect conclusion in [5] results from the imposition of this additional condition. Supersymmetry is, in fact, unbroken in this theory, when analyzed properly.

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