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WDVV Equations as Functional Relations

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We discuss the associativity or WDVV equations and demonstrate that they can be rewritten as certain functional relations between the *second* derivatives of a single function, similar to the dispersionless Hirota equations. The properties of these functional relations are further discussed.

1 Introduction

The associativity or Witten-Dijkgraaf-Verlinde-Verlinde (WDVV) equations [1] have been widely discussed in connection with various problems of mathematical physics for over ten years. They have arisen in the context of quantum cohomologies and mirror symmetry, and also with regard to multidimensional supersymmetric gauge field theories. In their most general form these equations may be expressed as [2]

$$\mathsf{F}_i \cdot \mathsf{F}_j^{-1} \cdot \mathsf{F}_k = \mathsf{F}_k \cdot \mathsf{F}_j^{-1} \cdot \mathsf{F}_i \qquad \forall i, j, k. \tag{1}$$

Here the matrices $\|\mathsf{F}_i\|_{jk} \equiv \mathcal{F}_{ijk}$ are constructed from the third derivatives of the function $\mathcal{F}(\mathbf{a})$,

$$\mathcal{F}_{ijk} = \frac{\partial^3 \mathcal{F}}{\partial a^i \, \partial a^j \, \partial a^k}. \tag{2}$$

When $\mathcal{F}$ is a function of one or two variables then (1) are empty: that is they are satisfied by any function. For more variables however, despite the simplicity of their compact matrix form, these equations form a highly nontrivial overdetermined system of nonlinear partial differential equations satisfied by the function $\mathcal{F}$.

In their original two-dimensional topological field theoretic setting the WDVV equations (1) were supplemented by a further constraint: there was a distinguished coordinate such that the matrix $\mathcal{F}_{1jk}$ was a constant. This came from the existence of a distinguished operator, the identity operator Φ_1, and the third derivatives of $\mathcal{F}$ being related to three-point functions via $\langle \Phi_i \Phi_j \Phi_k \rangle = \mathcal{F}_{ijk}$. This distinguished coordinate meant their was a preferred "metric", F_1. Associated with this class of solutions to (1) – together with a quasi-homogeneity condition – Dubrovin introduced the notion of Frobenius manifold [3]. However, there are physically interesting solutions of (1) that fail to possess these extra properties and for these the notion of Frobenius manifold fails to encapsulate their geometry.

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This more general class of solutions to (1) includes some prepotentials arising from low-energy effective actions of $\mathcal{N}=2$ supersymmetric gauge theories in four dimensions via Seiberg-Witten theory. Supersymmetry typically restricts the possible *geometries* encountered in field theories. Thus sigma models have target spaces that are Riemannian, or possibly Kähler or hyper-Kähler depending on the number of supersymmetry generators. For $\mathcal{N}=2$ SUSY gauge theories the moduli space of vector multiplets is a special[3] Kähler manifold [4, 5]. This means that on the moduli space there exists a single *holomorphic* function $\mathcal{F}(\mathbf{a})$ whose second derivatives

$$T_{ij} = \frac{\partial^2 \mathcal{F}}{\partial a^i \, \partial a^j} \tag{3}$$

determines the metric via

$$ds^2 = g_{i\bar{j}} \, da^i d\overline{a^j}, \qquad g_{i\bar{j}} = \frac{1}{2}(\mathrm{Im}T)_{ij}. \tag{4}$$

Here a^i are complex coordinates on the Kähler manifold (and $\overline{a^j}$ their complex conjugates). The connection with Seiberg-Witten theory is that the second derivatives T_{ij} coincide with the matrix elements of the period matrix of the Seiberg-Witten auxiliary curve and $\mathcal{F}$ is the prepotential. In [8] it was argued that this auxiliary curve could be identified with the spectral curve of a completely integrable system. Indeed, the phase space of (appropriate algebraically) completely integrable systems can be identified as a toric fibration (the angles x^i) over a special Kähler manifold (the actions y_j). The Kähler form is then

$$\omega = dx^i \wedge dy_i = \frac{\sqrt{-1}}{2} \, (\mathrm{Im}T)_{ij} \, da^i \wedge d\overline{a^j}, \tag{5}$$

and the real coordinates $\{x^i, y_j\}$ are related to complex coordinates by

$$\frac{\partial}{\partial a^i} = \frac{1}{2}\left(\frac{\partial}{\partial x^i} - T_{ij} \, \frac{\partial}{\partial y_j} \right).$$

The Kähler potential here is $K(\mathbf{a}, \bar{\mathbf{a}}) = \frac{1}{2}\left(\overline{a^i} \, \dfrac{\partial \mathcal{F}}{\partial a^i} \right)$. A special Kähler manifold has two natural connections associated with it. There is the Levi-Civita connection D, with Christoffel symbols $\Gamma_{\bar{i}jk} = -\frac{\sqrt{-1}}{4}\mathcal{F}_{ijk}$, and there is also a flat torsionfree connection $\nabla = D + A$ such that $\nabla dx^i = 0 = \nabla dy_j$. Here the connection A,

$$A\left(\frac{\partial}{\partial a^i} \right) = \left(A_j\right)^{\bar{l}}{}_{i} \, da^j \otimes \frac{\partial}{\partial \overline{a^l}} = \frac{\sqrt{-1}}{4} \left(g^{\bar{l}m} \mathcal{F}_{jim} \right) da^j \otimes \frac{\partial}{\partial \overline{a^l}}, \tag{6}$$

satisfies $0 = \partial^D A = \partial A + \Gamma \wedge A - A \wedge \Gamma$. Because the Levi-Civita connection connection is torsionfree we may also write this as $D_i A^{\bar{l}}_{jk} = D_j A^{\bar{l}}_{ik}$ with $A^{\bar{l}}_{jk} = \left(A_j\right)^{\bar{l}}{}_{k}$. We refer to [9, 10] for further material on the connections between Seiberg-Witten theory and integrable systems.

Following the work of [8], it seemed to be very important that the function $\mathcal{F}$ satisfied some well-known nonlinear integrable differential equations. The connection between

[3] In the original supergravity context such a manifold was called "rigid special Kähler" with "special Kähler" referring to the local (supergravity) setting; since [6] the local setting is often known by "projective special Kähler". See [7] for a survey of these manifolds.

Seiberg-Witten theory and the WDVV equations is that *some* prepotentials lead to solutions of (1). Certainly not all prepotentials yield solutions and a characterisation of what special Kähler manifolds give solutions is still sought. Now the geometric origin of the WDVV equations in Seiberg-Witten theory appears to be completely different from that of Frobenius manifolds. Seiberg-Witten theory lacks the analogue of the identity operator and constant "topological metric". Indeed there appears no obvious connection between the (nonholomorphic) metric of special Kähler geometry, built out of second derivatives of the prepotential, and the "topological metric" F_1 of the Frobenius manifolds which is a third derivative. Further, the WDVV equations were shown to be covariant with respect to arbitrary symplectomorphisms in [11], which is quite natural in the special Kähler setting, but such do *not* preserve the Frobenius manifold structure.

The geometric origin underlying the WDVV equations has yet be clarified. In this note we will reformulate these equations as functional relations. In particular, they will be in terms of second derivatives. These functional relations are reminiscent of the Hirota equations and we will give an example making this clearer.

2 WDVV and dispersionless Hirota equations

The WDVV equations usually follow from the crossing relations

$$\sum_k C^k_{\ ij} C^n_{\ kl} = \sum_k C^k_{\ il} C^n_{\ kj} \tag{7}$$

for the structure constants of some algebra

$$\phi_i \circ \phi_j = \sum_k C^k_{\ ij} \phi_k. \tag{8}$$

It is useful also to write (7) in the matrix form

$$\mathsf{C}_i \cdot \mathsf{C}_j = \mathsf{C}_j \cdot \mathsf{C}_i, \qquad \forall i, j \qquad (\mathsf{C}_i)^l_{\ m} = C^l_{\ im}. \tag{9}$$

Equations (7) are algebraic relations and they turn into the WDVV system of nonlinear differential equations after expressing the structure constants in terms of the third derivatives of some function $\mathcal{F}(a^1, \ldots, a^N)$

$$C^k_{\ ij} = \eta^{kl} \mathcal{F}_{ijl}. \tag{10}$$

Generically the matrix η can be an arbitrary linear combination with time dependent coefficients $\eta = \sum_j \alpha^j(\mathbf{a}) \mathsf{F}_j$, but for simplicity we will mostly consider $\eta = \mathsf{F}_1$. Then $\eta_{rs} = \iota_\alpha dT_{rs}$.

Various rewritings of the WDVV equations are possible. In the Frobenius manifold setting there is a pencil of flat connections, while the general equations (1) are equivalent [12] to the commuting of matrix-valued vector fields:

$$[\partial_i - (\mathsf{F}_j)^{-1} \mathsf{F}_i \, \partial_j, \partial_k - (\mathsf{F}_j)^{-1} \mathsf{F}_k \, \partial_j] = 0$$

(with fixed j). For our purposes one may also rewrite (7) as

$$\sum_k C^k_{\ ij} \mathcal{F}_{kln} = \sum_k C^k_{\ il} \mathcal{F}_{kjn}. \tag{11}$$

With one form $C = \eta^{-1} dT$ (i.e. $C^k_{\ ij} da^i = \eta^{kl} \partial_i T_{jl} da^i = \eta^{kl} \mathcal{F}_{ijl} da^i$) we have

$$[\mathsf{C}_i, \mathsf{C}_j] = 0 \Longleftrightarrow C \wedge C = 0 \Longleftrightarrow dT \wedge \eta^{-1} dT = 0. \tag{12}$$

Unfortunately the connection C is unrelated to the various connections arising in the special Kähler geometry because of the (arbitrary) appearance of η which is unrelated to the special Kähler metric. In general the curvature of C is nonvanishing. (The curvature will, for example, vanish as a consequence of the WDVV equations if there is a symmetry $\mathcal{L}_\alpha dT = \mu dT$, for some matrix μ.) The final form of (12) is reminiscent of the Hirota bilinear equations which may be interpreted in terms of Plücker formulae, but again the appearance of η means the bilinear form in such formulae is not constant. Whatever, the final form shows that the WDVV equations impose relations on the period matrices T, and consequently restrict the associated Seiberg-Witten curves that can yield solutions of (1).

An important observation was made recently in [13] where it was shown that in the case of dispersionless integrable hierarchies the WDVV equations maybe derived directly by differentiating corresponding Hirota algebraic relations. In particular, the solutions associated with the Landau-Ginzburg topological models can be constructed in this way. Below, we will reverse the arrow of this implication and demonstrate that the WDVV equations, if rewritten in terms of second derivatives, can be simply viewed as certain Hirota-like functional relations.

The dispersionless Hirota relations (see [13] and references therein about the details and their explicit form) for the second derivatives, which can be written as

$$\mathcal{F}_{ij} = T_{ij}(\boldsymbol{\varphi}) \tag{13}$$

where T_{ij} are some *known* functions and $\{\varphi_i\}$ denote some restricted set (of cardinality N, the size of the matrices under consideration in the finite-dimensional situation) second derivatives, for example

$$\varphi_i = \mathcal{F}_{1i}(\mathbf{a}) \tag{14}$$

Without specifying form of the functions T_{ij} in (13) this is just a *dimensional* statement that the *matrix* of second derivatives of any function can be expressed in terms of only a *vector* of variables.

Consider the "period" matrix (3) which for any integrable system is a function of only N variables. This means that in the "Siegel upper half-space" $\{T_{ij}\}$ of dimension $\dfrac{N(N+1)}{2}$ we have a "submanifold" of dimension N

$$T_{ij} = T_{ij}(a^1, \ldots, a^N) \tag{15}$$

or codimension $\dfrac{N(N-1)}{2}$ given by defining equations

$$f_A(T_{ij}) = 0 \quad A = 1, \ldots, \frac{N(N-1)}{2} \tag{16}$$

Of course, on this submanifold (under a generic assumption of nonsingularity) one may choose new variables, say

$$\varphi_i = T_{1i}(\mathbf{a}) \tag{17}$$

Consequently there exist functional relations

$$T_{ij} = T_{ij}(\varphi_1, \ldots, \varphi_N) = T_{ij}(\mathbf{a}(\varphi_1, \ldots, \varphi_N)) \tag{18}$$

between the matrix elements of the whole period matrix, exactly as one has in dispersionless Hirota's relations. In the case of (17) all of the matrix elements of the period matrix would be expressed in terms of a single row or column. (We give an example of this in section 4.)

More generally, let us assume we can find coordinates φ_k so that

$$dT_{ij} = C^k{}_{ij}\, d\varphi_k. \tag{19}$$

Using the definition (10) this means

$$d\varphi_k = \eta_{kl} da^l. \tag{20}$$

Both (19) and (20) have the common integrability condition

$$d\eta_{kl} \wedge da^l = 0 \Longleftrightarrow M^i_s \mathcal{F}_{irk} = M^i_r \mathcal{F}_{isk}, \qquad M^i_r = \frac{\partial \alpha^i}{\partial a^r}, \tag{21}$$

where the α^i's were the coefficients defining η above. These are certainly satisfied for constant α^i. Thus the choice (17) would correspond to $\eta = \mathsf{F}_1$. Supposing the integrability condition (21) holds, our assumption (19) means that the structure constants can be *integrated*, i.e. defined as derivatives of the functions (13)

$$C^k{}_{ij} = \frac{\partial T_{ij}}{\partial \varphi_k} \tag{22}$$

and that

$$T_{ij}(\boldsymbol{\varphi}) = \int \sum_k C^k_{ij} d\varphi_k, \tag{23}$$

i.e. the structure constants determine the form of the Hirota-like relations (13).

3 The associativity equations as cocycle conditions

We shall now derive some consequences of (19). For a fixed index a

$$d\varphi_k = dT_{ai}\, (\mathsf{C}_a^{-1})^i{}_k$$

whence for any b

$$d\varphi_k = dT_{ai}\, (\mathsf{C}_a^{-1})^i{}_k = dT_{bj}\, (\mathsf{C}_b^{-1})^j{}_k.$$

From these it follows that

$$dT_{ai} = dT_{bj} \left(\mathsf{C}_b^{-1} \cdot \mathsf{C}_a\right)^j{}_i = dT_{bj} \left(\mathsf{C}_j^{-1} \cdot \mathsf{C}_i\right)^b{}_a, \tag{24}$$

where the last equality follows from the symmetry of the period matrices. For fixed a and b we may view these as giving a change of variables

$$T_{ai} = T_{ai}(T_{bj}), \qquad \forall\; i, j, \tag{25}$$

and

$$\frac{\partial T_{ai}}{\partial T_{bj}} = \left(\mathsf{C}_b^{-1} \cdot \mathsf{C}_a\right)^j{}_i = \left(\mathsf{C}_j^{-1} \cdot \mathsf{C}_i\right)^b{}_a. \tag{26}$$

These equations may simply be viewed as the chain-rule

$$\frac{\partial T_{ai}}{\partial T_{bj}} = \sum_s \frac{\partial T_{ai}}{\partial \varphi_s} \frac{\partial \varphi_s}{\partial T_{bj}} = \left(\mathsf{C}_b^{-1} \cdot \mathsf{C}_a\right)^j{}_i\,, \tag{27}$$

where we understand (for fixed b) $\frac{\partial \varphi_s}{\partial T_{bj}} = \left(\mathsf{C}_b^{-1}\right)^j{}_s$ in terms of the change of variables $\varphi_s \leftrightarrow T_{bj}$. For future comparison we also record the identity (for fixed a, b, c)

$$\frac{\partial T_{ai}}{\partial T_{bj}} \frac{\partial T_{ck}}{\partial T_{ai}} \frac{\partial T_{bl}}{\partial T_{ck}} = (\mathsf{C}_b^{-1}\mathsf{C}_a)^j{}_i (\mathsf{C}_a^{-1}\mathsf{C}_c)^i{}_k (\mathsf{C}_c^{-1}\mathsf{C}_b)^k{}_l = (\mathsf{C}_b^{-1}\mathsf{C}_a\mathsf{C}_a^{-1}\mathsf{C}_c\mathsf{C}_c^{-1}\mathsf{C}_b)^j{}_l = \delta_l^j. \tag{28}$$

Thus far we have only used change of variables and the fact that the period matrix (3) can be expressed in terms of only genus $g = N$ number of variables. Such will hold for *any* integrable system, but this does not mean the associativity equations (1) or (9) must hold. Lets now see the meaning of these. From (26) we have

$$\frac{\partial T_{ai}}{\partial T_{bj}} \frac{\partial T_{bk}}{\partial T_{ci}} \frac{\partial T_{cl}}{\partial T_{ak}} = (\mathsf{C}_b^{-1}\mathsf{C}_a)^j{}_i (\mathsf{C}_c^{-1}\mathsf{C}_b)^i{}_k (\mathsf{C}_a^{-1}\mathsf{C}_c)^k{}_l = (\mathsf{C}_b^{-1}\mathsf{C}_a\mathsf{C}_c^{-1}\mathsf{C}_b\mathsf{C}_a^{-1}\mathsf{C}_c)^j{}_l. \tag{29}$$

Now from (9) it follows also that the matrices of structure constants satisfy (for all a and b)

$$\mathsf{C}_a \cdot \mathsf{C}_b^{-1} = \mathsf{C}_b^{-1} \cdot \mathsf{C}_a. \tag{30}$$

Substituting (30) into (29) we conclude that

$$\sum_{i,k} \frac{\partial T_{ai}}{\partial T_{bj}} \frac{\partial T_{bk}}{\partial T_{ci}} \frac{\partial T_{cl}}{\partial T_{ak}} = \delta_l^j. \tag{31}$$

Equally from (31) we deduce (1). Thus the WDVV equations are equivalent to the Hirota-like functional relations (31). The relation (31) is our main statement.

4 Example: perturbative Seiberg-Witten prepotentials

Here we shall consider one of the simplest examples of solutions to the WDVV equations coming from Seiberg-Witten theory. These are related to $SU(N+1)$ perturbative prepotentials [2] and the corresponding Riemann surface is degenerate. With $a_{ij} \equiv a_i - a_j$ the perturbative prepotential is given by

$$\mathcal{F} = \frac{1}{2} \sum_{i<j}^{N} a_{ij}^2 \log \frac{a_{ij}}{\Lambda} + \frac{1}{2} \sum_{i=1}^{N} a_i^2 \log \frac{a_i}{\Lambda}. \tag{32}$$

The second derivatives (3) (for a special choice of Λ) are

$$T_{ii} = \log a_i + \sum_{k \neq i} \log a_{ik}; \qquad T_{ij} = -\log a_{ij}\,, \quad i<j; \qquad T_{ij} = T_{ji}, \quad i>j. \tag{33}$$

Using (33) one may reexpress all matrix elements T_{ij} through a given row explicitly. For example, in the first nontrivial case corresponding to $SU(4)$ one finds

$$T = \begin{pmatrix} x & y & z \\ y & \log\left(e^{x+y+z} - e^{-y}\right)\left(e^{-z} - e^{-y}\right) - y & -\log\left(e^{-z} - e^{-y}\right) \\ z & -\log\left(e^{-z} - e^{-y}\right) & \log\left(e^{x+y+z} - e^{-z}\right)\left(e^{-z} - e^{-y}\right) - z \end{pmatrix}$$

$$= \begin{pmatrix} \log\left(e^{\tilde{x}+\tilde{y}+\tilde{z}} + e^{-\tilde{x}}\right)\left(e^{-\tilde{x}} + e^{-\tilde{z}}\right) - \tilde{x} & \tilde{x} & -\log\left(e^{-\tilde{x}} + e^{-\tilde{z}}\right) \\ \tilde{x} & \tilde{y} & \tilde{z} \\ -\log\left(e^{-\tilde{x}} + e^{-\tilde{z}}\right) & \tilde{z} & \log\left(e^{\tilde{x}+\tilde{y}+\tilde{z}} + e^{-\tilde{z}}\right)\left(e^{-\tilde{x}} + e^{-\tilde{z}}\right) - \tilde{z} \end{pmatrix}$$

$$= \begin{pmatrix} \log\left(e^{\hat{x}+\hat{y}+\hat{z}} + e^{-\hat{x}}\right)\left(e^{-\hat{x}} - e^{-\hat{y}}\right) - \hat{x} & -\log\left(e^{-\hat{x}} - e^{-\hat{y}}\right) & \hat{x} \\ -\log\left(e^{-\hat{x}} - e^{-\hat{y}}\right) & \log\left(e^{\hat{x}+\hat{y}+\hat{z}} + e^{-\hat{y}}\right)\left(e^{-\hat{x}} - e^{-\hat{y}}\right) - \hat{y} & \hat{y} \\ \hat{x} & \hat{y} & \hat{z} \end{pmatrix}. \tag{34}$$

Here we have exhibited the different dependence on the rows which are taken as independent variables: $(x, y, z) = (T_{11}, T_{12}, T_{13})$, $(\tilde{x}, \tilde{y}, \tilde{z}) = (T_{21}, T_{22}, T_{23})$ and $(\hat{x}, \hat{y}, \hat{z}) = (T_{31}, T_{32}, T_{33})$.

In the $SU(4)$ perturbative Seiberg-Witten case equation (31) has essentially only the one nontrivial relation

$$\sum_{i,k} \frac{\partial T_{2i}}{\partial T_{1j}} \frac{\partial T_{1k}}{\partial T_{3i}} \frac{\partial T_{3l}}{\partial T_{2k}} = \delta_l^j. \tag{35}$$

The corresponding matrices may be straightforwardly computed using (33). It is easy to check, that (35) holds provided the Hirota relations (cf. with [14]) are satisfied:

$$\mathrm{sign}(j-i)e^{-T_{ij}} + \mathrm{sign}(k-j)e^{-T_{jk}} + \mathrm{sign}(i-k)e^{-T_{ki}} = 0, \qquad i \neq j \neq k, \tag{36}$$

together with

$$e^{T_{11}+T_{12}+T_{13}-T_{23}} + e^{T_{13}+T_{23}+T_{33}-T_{12}} - e^{T_{12}+T_{22}+T_{23}-T_{13}} = 0, \tag{37}$$

and

$$e^{T_{12}+T_{22}-T_{13}} - e^{T_{13}+T_{33}-T_{12}} = 0. \tag{38}$$

These may be seen to be satisfied upon utilising (33).

5 Discussion

Although the WDVV equations arise in several different physical settings no unifying geometry as yet underpins them. With the additional restrictions of topological field theory Frobenius manifolds successfully encode the geometry, but the restrictions are too severe to allow other interesting examples coming from Seiberg-Witten theory and the dispersionless limits of solutions to the Hirota equations. In this note we have pursued this link between the WDVV equations and the algebraic relations arising from the dispersionless Hirota equations [13]. The key idea was to focus on the second derivatives of the prepotential and the connections between them. First we noted a rather general phenomenon, independent of the associativity equations. Any "generalized period matrix" (3) of an integrable system implies the existence of certain subspace of matrix elements of the period matrices. As such, there are certain functional relations between the matrix elements. When we additionally impose the associativity or WDVV equations we obtained a set of equivalent functional relations (31). From this perspective we can easily understand why not every Seiberg-Witten curve, or the spectral curve of every integrable system, will lead to solutions of the WDVV equations: only very special subspaces lead to solutions. The WDVV impose bilinear relations (12) on the differentials of the period matrix restricted to this subspace.

Whereas in the case of dispersionless hierarchies the Hirota relations (13) have a rather simple form (they are algebraic, or the functions $T_{ij}(\boldsymbol{\varphi})$ are polynomials) in the general setting this is not the case, and that is why we prefer to call them *functional* Hirota relations. Our functional relations (31) depend on a choice of three different indices a, b and c. These three cycles are similar to the three points usually chosen when one writes down the conventional Hirota equations (see, for example [14]). In the setting where T_{ij} plays the role of the period matrix of a Riemann surface (of genus g), this choice of a, b and c corresponds to a choice of three different *cycles* on the corresponding Riemann surface. Here the Schottky relations of the period matrices reduce the dimension of $g(g+1)/2$ symmetric matrices to a space of dimension $3g-3$; the constraint coming from integrability reduces this still further (to in general g, with further restriction from the WDVV-functional relations. Our final example illustrated the functional relations for a perturbative Seiberg-Witten solution.

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