Fermionic determinant with a linear domain wall in 2 + 1 dimensions

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Abstract

We consider a Dirac field in 2 + 1 Euclidean dimensions, in the presence of a linear domain wall defect in its mass, and a constant electromagnetic field. We evaluate the exact fermionic determinant for the situation where the defect is assumed to be rectilinear, static, and the gauge field is minimally coupled to the fermions. We discuss the dependence of the result on the (unique) independent geometrical parameter of this system, namely, the relative orientation of the wall and the direction of the external field. We apply the result for the determinant to the evaluation of the vacuum energy.

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1   Introduction

In general, the fermionic action S is (or may be transformed into) a quadratic expression in the Grassmann fields, which is in turn determined by a Dirac operator $\mathcal{D}$. This Dirac operator, except for some trivial cases, carries a dependence on some parameters and fields. At some stage in the resolution of a dynamical problem, it may be useful to regard those fields as 'external', either because they don't have a proper dynamics, or because they have not yet been quantized.

The fermionic determinant is formally given by the product of the eigenvalues of the Dirac operator. The explicit calculation of those eigenvalues is, however, impossible to achieve for an arbitrarily general situation, and one has to resort to some form of approximate expansion. Nevertheless, there are many highly symmetrical external field configurations where this explicit diagonalization can be successfully carried out. In particular, the constant $F_{\mu\nu}$ case in several spacetime dimensions, allows for the determination of the exact eigenvalues, since the diagonalization of $\mathcal{D}$ reduces in this case to finding the spectrum of a one dimensional harmonic oscillator [1]. Many interesting results have been found in this area, and they have application, for example, to the determination of effective actions [2].

In most of these cases, the method of resolution amounts to expressing first the Dirac operator $\mathcal{D}$ as a quadratic form in the operators x_μ, p_μ [1] (as in the Fock-Schwinger method [1]). The diagonalization procedure is then equivalent to defining a transformation from x_μ and p_μ to a new set of canonical operators, and thus may in general be expressed as a Bogoliubov transformation in some suitable creation and annihilation operators. The original problem becomes then an algebraic one, with a complexity which depends of course on the kind of situation considered.

In this article we will evaluate the fermionic determinant corresponding to a Dirac field in $2 + 1$ dimensions, coupled to an external uniform electromagnetic field $F_{\mu\nu}$, and with a mass term which is a linear functional of the coordinates. This kind of mass term is a simple example of a situation where a domain wall like defect (of constant slope) is present, in this case in the parity-breaking mass term. We note that, if the electromagnetic field were parallel to the defect, the situation would fall into the well-known Callan-Harvey phenomenon [3], by which a chiral fermionic zero mode is induced on the defect. We will, however, consider situations where the field has a different orientation.

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[1]We work with Euclidean coordinates: $x_\mu = (x_0, x_1, x_2)$, where x_0 denotes the Euclidean time.

We will show that the modulus of the determinant can be found exactly for these configurations, what represents a non-trivial generalization of the well-known situation corresponding to a uniform electromagnetic field and a *constant* mass term.

The Euclidean action S for this system is

$$S = \int d^3x \, \bar{\Psi} \mathcal{D} \Psi \, , \tag{1}$$

where $\mathcal{D} = \partial\!\!\!/ + i \, A\!\!\!/ + M$, and we have absorbed the electric charge into the definition of the gauge field A. The Hermitian γ-matrices are in the irreducible 2×2 representation of the Dirac algebra:

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \, . \tag{2}$$

The domain wall, defined as the region where the mass changes sign, will in our case have only two defining properties: its location and its slope. The latter determines the localization (or not) of the defect, and is quantitatively measured by the normal derivative of the mass along the curve of the defect. The situation we consider in this paper may be considered as an approximate treatment of localized defects; a general discussion on the localization of modes for this system may be found in [4].

The Dirac operator $\mathcal{D}$ appearing in (1) is not the more suitable one to deal with the eigenvalue problem, since it is not Hermitian. It is convenient to consider, instead, a related operator $\mathcal{H}$ which is always Hermitian, has a complete set of eigenstates, and its eigenvalues are the squares of the Dirac operator's eigenvalues. This operator $\mathcal{H}$ is of course

$$\mathcal{H} = \mathcal{D}^\dagger \mathcal{D} \, , \tag{3}$$

which is indeed the operator defining the positive Hermitian part of $\mathcal{D}$ in its polar decomposition. We can see from the structure of $\mathcal{H}$, that for a mass linear in the coordinates, and with a constant electromagnetic potential, it will be a quadratic expression in x_μ and p_μ. The mass will be regarded as a linear functional of the coordinates.

This kind of configuration for the mass and the electromagnetic field can be characterized by two vectors in Euclidean spacetime. For the mass, vanishing along a line that passes through the origin (what is always possible by a proper choice of coordinates), can be defined in terms of a vector η_μ,

$$M = \eta_\mu x_\mu \, , \tag{4}$$

so that η_μ points in the direction normal to the plane that defines the defect hyperplane. Regarding the electromagnetic potential A_μ, a linear function

of the coordinates suffices to generate a constant electromagnetic field $F_{\mu\nu}$, since the gauge potential can be written in a symmetric gauge as follows:

$$A_\mu \;=\; -\frac{1}{2}F_{\mu\nu}x_\nu \;. \tag{5}$$

We have found it convenient to use, rather than the antisymmetric tensor, $F_{\mu\nu}$, its dual $\widetilde{F}_\mu$,

$$\widetilde{F}_\mu \;=\; \frac{1}{2}\epsilon_{\mu\nu\rho}F_{\nu\rho}, \tag{6}$$

where $\epsilon_{\mu\nu\rho}$ is the totally antisymmetric tensor in 3 dimensions. This puts both the defect and the gauge field on an equal footing, simplifying the study of the dependence of the fermionic determinant on the geometric invariants that may be built out of those external fields.

Writing A_μ in terms of the dual of $F_{\mu\nu}$, $A_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho}\widetilde{F}_\rho x_\nu$, one can present the Dirac operator parametrized by two constant vectors, thus summarizing all its dependence on the external parameters:

$$\mathcal{D} \;=\; \gamma\cdot\partial - \frac{i}{2}\gamma\cdot x\times\widetilde{F} + \eta\cdot x \;, \tag{7}$$

where we have used the notation

$$(a\times b)_\mu \;\equiv\; \epsilon_{\mu\nu\rho}a_\nu b_\rho \;, \qquad a\cdot b \;\equiv\; a_\mu b_\mu \;. \tag{8}$$

The relative orientation between η_μ and $\widetilde{F}_\mu$ covers all the freedom to describe the different geometrical configurations allowed within this model for the Dirac operator. Different particular cases will describe rather different situations, both from the physical and the mathematical points of view. The simplest situation corresponds to η parallel to $\tilde{F}$, as in the Callan and Harvey mechanism, since the gauge field is then entirely contained in the defect worldsheet. Then the determinant of the three dimensional system immediately factorizes into the product of an infinite number of (decoupled) determinants in $2+1$ dimensions [4]. Our presentation begins, in section 2, with the more interesting case $\widetilde{F}\cdot\eta = 0$. Namely, the electromagnetic field is normal to the defect hypersurface. The interest in this case stems from the fact that it takes into account the competition between the localizing effect due to the defect and the effect of the electromagnetic field, which will tend to 'drag' any charge distribution in a direction normal to the defect. An application to the calculation of the vacuum energy, for this case, is presented in section 3. Finally, we conclude by deriving the eigenvalues of the equivalent diagonal operator for the general case in section 4.

2 Fermionic determinant for $\widetilde{F} \cdot \eta = 0$

In this situation, $\widetilde{F}_\mu$ and η_μ define two orthogonal directions under the scalar Euclidean product defined in (8). Thus, in terms of these two vector fields, we may construct two families of curves in the Euclidean 3-dimensional space that may be used to define coordinates, varying along their integral lines. A third family of curves, orthogonal to the two previous ones, can then be uniquely defined by selecting a right-handed orientation. We define $\hat{t}_\mu$ to be the unit vector in the direction of $\widetilde{F}_\mu$. It is obvious from equation (7) that in the particular case we are dealing with, the operator $\mathcal{D}$ does not depend on the coordinate x_t (which parametrizes the integral curves of $\hat{t}_\mu$). We see that it is then sufficient to diagonalize the part of $\mathcal{H}$ which depends on the other two coordinates. Indeed, using equation (7), we can express $\mathcal{H}$ in terms of the external parameters,

$$\mathcal{H} \;=\; -\partial^2 + \frac{1}{4}[x^2\widetilde{F}^2 - (x \cdot \widetilde{F})^2] + (x \cdot \eta)^2 \,-\, i(x \times \partial) \cdot \widetilde{F} \,+\, \gamma \cdot (\widetilde{F} - \eta) \;, \quad (9)$$

which, in the coordinates defined by the unit vectors

$$\hat{\eta} \equiv \frac{\eta}{\|\eta\|}, \quad \hat{t} \equiv \frac{\widetilde{F}}{\|\widetilde{F}\|}, \quad \hat{b} \equiv \hat{\eta} \wedge \hat{t} \;, \quad (10)$$

becomes,

$$\mathcal{H} \;=\; -\partial_\eta^2 - \partial_t^2 - \partial_b^2 + (\frac{\widetilde{F}^2}{4} + \eta^2)x_\eta^2 + \frac{\widetilde{F}^2}{4}x_b^2 - i\widetilde{F}(x_b\partial_\eta - x_\eta\partial_b) + \gamma \cdot (\widetilde{F} - \eta) \;. \quad (11)$$

In this expression, $\mathcal{H}$ is invariant under translations in x_t, thus the diagonalization of $\mathcal{H}$ amounts to solving the reduced problem defined by a different Hamiltonian H acting on functions depending on the two coordinates x_η and x_b,

$$\mathcal{H} \;\equiv\; -\partial_t^2 \,+\, 2H, \quad (12)$$

where

$$H \;=\; \frac{1}{2}p_\eta^2 + \frac{1}{2}p_b^2 + \frac{\omega_n^2}{2}x_\eta^2 + \frac{\omega_b^2}{2}x_b^2 + \frac{\widetilde{F}}{2}(x_bp_\eta - x_\eta p_b) + \frac{1}{2}\gamma \cdot (\widetilde{F} - \eta) \;. \quad (13)$$

In (13) we have introduced the constants

$$\omega_\eta^2 = \frac{\widetilde{F}^2}{4} + \eta^2, \qquad \omega_b^2 = \frac{\widetilde{F}^2}{4} \;. \quad (14)$$

To study the diagonalization of H, it is first convenient to define suitable creation and annihilation operators for each coordinate,

$$
\begin{aligned}
a_j &= \frac{1}{\sqrt{2}}(\omega_j^{1/2}x_j + i\omega_j^{-1/2}p_j), \\
a_j^\dagger &= \frac{1}{\sqrt{2}}(\omega_j^{1/2}x_j - i\omega_j^{-1/2}p_j), \quad j = \eta, b
\end{aligned}
\tag{15}
$$

(no sum over j), since the expression for H becomes more symmetric, and its diagonalization can be done by a generalization of the usual procedure for a quadratic Hamiltonian.

In terms of these operators, H contains both bilinear and constant terms:

$$
\begin{aligned}
H &= \omega_\eta a_\eta^\dagger a_\eta + \omega_b a_b^\dagger a_b + \frac{\|\widetilde{F}\|}{4i}\left(\sqrt{\frac{\omega_\eta}{\omega_b}} - \sqrt{\frac{\omega_b}{\omega_\eta}}\right)(a_\eta a_b - a_\eta^\dagger a_b^\dagger) \\
&\quad + \frac{\|\widetilde{F}\|}{4i}\left(\sqrt{\frac{\omega_\eta}{\omega_b}} + \sqrt{\frac{\omega_b}{\omega_\eta}}\right)(a_\eta a_b^\dagger - a_\eta^\dagger a_b) \\
&\quad + \frac{1}{2}[\omega_\eta + \omega_b + \gamma \cdot (\widetilde{F} - \eta)] \,,
\end{aligned}
\tag{16}
$$

which has the important property of containing terms which are quadratic in the creation and annihilation operators, and not just the standard one with creation and annihilation operators mixed by a Hermitian matrix. This makes the diagonalization more cumbersome. Indeed, to diagonalize H, we first introduce some 2×2 matrices, defined by

$$
H = a_j^\dagger A_{jl} a_l + \frac{i}{2}a_j B_{jl} a_l - \frac{i}{2}a_j^\dagger B_{jl} a_l^\dagger + \frac{1}{2}[\omega_\eta + \omega_b + \gamma \cdot (\widetilde{F} - \eta)] \,,
\tag{17}
$$

where A is Hermitian while B is real and symmetric

$$
A = \begin{pmatrix} \omega_\eta & -iu \\ iu & \omega_b \end{pmatrix} \qquad B = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix},
\tag{18}
$$

with the constants u and t defined by

$$
u = \frac{\|\widetilde{F}\|}{4i}\left(\sqrt{\frac{\omega_\eta}{\omega_b}} + \sqrt{\frac{\omega_b}{\omega_\eta}}\right), \quad t = \frac{\|\widetilde{F}\|}{4i}\left(\sqrt{\frac{\omega_\eta}{\omega_b}} - \sqrt{\frac{\omega_b}{\omega_\eta}}\right).
\tag{19}
$$

The diagonalization is non trivial due to the presence of the B matrix, which only vanishes for $\eta = 0$. To diagonalize the bilinear term in equation (17), we shall need to define new creation and annihilation operators, which will be linear combinations of the ones defined in Eq. 15. We first note that the $a_j, a_j^\dagger$

operators have been defined in such a way that they verify the commutation relations:

$$[a_j^\dagger, a_l^\dagger] = [a_j, a_l] = 0,$$
$$[a_j, a_l^\dagger] = \delta_{jl},$$
$$[H, a_l^\dagger] = a_j^\dagger A_{jl} + i a_j B_{jl},$$
$$[H, a_l] = -a_j A_{jl}^* + i a_j^\dagger B_{jl}, \tag{20}$$

where A^* is the complex conjugate of A. We require the new operators $b_j, b_j^\dagger$ to be also canonical and to diagonalize the bilinear form of (17). Following [5], we require them to satisfy the commutation relations,

$$[b_j^\dagger, b_l^\dagger] = [b_j, b_l] = 0,$$
$$[b_j, b_l^\dagger] = \delta_{jl},$$
$$[H, b_l^\dagger] = b_l^\dagger E_l,$$
$$[H, b_l] = -b_l E_l \tag{21}$$

(no sum over l). We can write the new operators in terms of the old ones by using a general Bogoliubov transformation of the following kind:

$$b_l^\dagger = a_j^\dagger U_{jl} - a_j V_{jl},$$
$$b_l = -a_j^\dagger V_{jl}^* + a_j U_{jl}^*, \tag{22}$$

where, to fulfill the commutation relations above, the coefficients U and V must verify the matrix equations

$$U^\dagger U - V^\dagger V = I \ , \tag{23}$$

$$U^t V - V^t U = 0 \ , \tag{24}$$

$$AU - iBV = UE \ , \tag{25}$$

and

$$iBU + A^* V = -VE \ , \tag{26}$$

E being the diagonal matrix whose elements are the eigenvalues of H and $U^\dagger$, as usual, stands for the adjoint of U. Similar redefinitions, although for a different case, where used in [5] and [6].

The system of equations involving the coefficient matrices defines a generalized diagonalization problem, which involves matrices rather than vectors. The next step is to obtain E explicitly in terms of the parameters of the theory. We first note that not all the equations that define this eigensystem

are independent. To determine E we may, for example, use Equation (25) to express the coefficients V_{jl} in terms of U_{jl}, the parameters of the theory and the eigenvalues E_l. This yields

$$V = \begin{pmatrix} r_{11}U_{11} + s_{11}U_{21} & r_{12}U_{12} + s_{12}U_{22} \\ r_{21}U_{21} + s_{21}U_{11} & r_{22}U_{22} + s_{22}U_{12} \end{pmatrix}, \qquad (27)$$

where the coefficients r_{jl} and s_{jl} are defined by

$$r = \begin{pmatrix} \frac{u}{t} & \frac{u}{t} \\ -\frac{u}{t} & -\frac{u}{t} \end{pmatrix}, \qquad s = \begin{pmatrix} \frac{i(E_1-\omega_b)}{t} & \frac{i(E_2-\omega_b)}{t} \\ \frac{i(E_1-\omega_\eta)}{t} & \frac{i(E_2-\omega_\eta)}{t} \end{pmatrix}. \qquad (28)$$

Introducing this result into Eq. (26) yields two equations, now involving U_{11} and U_{21}, plus two others involving U_{12} and U_{22},

$$\begin{aligned} U_{21} &= U_{11} \frac{2i\omega_\eta u}{E_1^2 + t^2 - E_1\omega_b - u^2 + E_1\omega_\eta - \omega_\eta\omega_b} \\ &= iU_{11} \frac{E_1^2 + t^2 - E_1\omega_\eta - u^2 + E_1\omega_b - \omega_\eta\omega_b}{2\omega_b u}, \end{aligned} \qquad (29)$$

and

$$\begin{aligned} U_{12} &= -U_{22} \frac{2i\omega_b u}{E_2^2 + t^2 - E_2\omega_\eta - u^2 + E_2\omega_b - \omega_\eta\omega_b} \\ &= -iU_{22} \frac{E_2^2 + t^2 - E_2\omega_b - u^2 + E_2\omega_\eta - \omega_\eta\omega_b}{2\omega_\eta u}. \end{aligned} \qquad (30)$$

Demanding equations (29) and (30) to be consistent, implies a set of constraints for the eigenvalues E_l. After some algebra, those constraints may be written in terms of just one equation, that determines the possible values of E_l in terms of the external parameters. There are 4 solutions to this equation: besides the double eigenvalue $E_1 = 0$, we find:

$$E_2 = -E_3 = \sqrt{\omega_\eta^2 + 3\omega_b^2}. \qquad (31)$$

This set of solutions contains the true eigenvalues; however, to discard the spurious ones, we should check whether they are consistent with (23) and (24), which guarantee the canonical commutation relations for the 'new' operators. In terms of the variables defined in Eq.(27), (24) can be recast in the form,

$$pq(s_{22} - s_{11}) + q(r_{22} - r_{21}) + p(r_{12} - r_{11}) + (s_{12} - s_{21}) = 0, \qquad (32)$$

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where p and q are defined by:

$$p = -i \frac{E_j^2 + t^2 - E_j \omega_b - u^2 + E_j \omega_\eta - \omega_\eta \omega_b}{2 \omega_\eta u}, \qquad (33)$$

and

$$q = i \frac{2 \omega_\eta u}{E_k^2 + t^2 - E_k \omega_b - u^2 + E_k \omega_\eta - \omega_\eta \omega_b} \qquad (34)$$

where the labels j and k stand for the two different eigenvalues that can be chosen from the four possibilities in Eq.(31).

Eq.(23) splits into three algebraic equations, one corresponding to the non-diagonal elements,

$$p + q^* - (r_{11} + q s_{11})^*(r_{12} p + s_{12}) - (r_{21} q + S_{21})^*(r_{22} + p s_{22}) = 0, \qquad (35)$$

plus two others for the diagonal terms,

$$|U_{11}|^2[1 + |q|^2 - |r_{11} + q s_{11}|^2 - |r_{21} q + s_{21}|^2] = 1, \qquad (36)$$

$$|U_{22}|^2[1 + |p|^2 - |r_{12} p + s_{12}|^2 - |r_{22} + p s_{22}|^2] = 1 . \qquad (37)$$

Eqs. (32) and (35) do not contain the coefficients of the matrices U and V, and thus they are equations for the eigenvalues E_l. In terms of the original parameters, we obtain for the eigenvalues of H,

$$E_1 = 0, \quad E_2 = \sqrt{\omega_\eta^2 + 3\omega_b^2} = \sqrt{\widetilde{F}^2 + \eta^2} . \qquad (38)$$

Note that (36) and (37) only fix the moduli of U_{11} and U_{22}; the phase of these coefficients is not fixed by the basis choice, and on the other hand no physical magnitude will depend on it.

We note that one of the eigenvalues vanishes, what implies $|U_{11}| \to \infty$. This means that, in the direction corresponding to the operators b_1 and $b_1^\dagger$, there is no harmonic mode, but rather a free motion. Thus we define the corresponding conjugate x_1 and p_1 variables, such that $\mathcal{H}$ in (11) becomes

$$\mathcal{H} = p_t^2 + p_1^2 + 2E_2 b_2^\dagger b_2 + \omega_\eta + \omega_b + \gamma \cdot (\widetilde{F} - \eta) . \qquad (39)$$

We have thus obtained the diagonal form for the operator $\mathcal{H}$. From this expression, we can obtain the modulus of the determinant of the Dirac operator $\mathcal{D}$.

3 Vacuum energy in a constant electromagnetic field

In this section we evaluate the vacuum energy for the fermionic system studied in the previous section (i.e., when $\widetilde{F}\cdot\eta = 0$); we shall see that this physical magnitude is entirely determined by (9).

To that end we first consider the vacuum energy density for a fermionic system in the presence of an external electromagnetic field. The vacuum to vacuum transition probability amplitude is given by the $\mathcal{S}$ matrix expectation value between vacuum states, which depends on the external potential A,

$$\mathcal{S}_0(A) = < 0 \; in|\mathcal{S}|0 \; in > \; . \tag{40}$$

This object is usually normalized with respect to the transition amplitude in the absence of external fields, $\mathcal{S}_0(A)$:

$$|\mathcal{S}_0'(A)|^2 \equiv \frac{|\mathcal{S}_0(A)|^2}{|\mathcal{S}_0(0)|^2} \; . \tag{41}$$

In terms of the normalized transition amplitude $\mathcal{S}_0'(A)$, one then defines a local function $w(x)$ [8]

$$|\mathcal{S}_0'(A)|^2 \equiv exp[- \int d^n x \; w(x)] \; , \tag{42}$$

where n denotes the spacetime dimension. In the Euclidean formulation we can interpret $w(x)$ as half the vacuum energy density. To see this we consider the Euclidean evolution operator in the interaction representation $U(\beta, \beta_0)$, which obeys the equations [9],

$$\lim_{\beta \to \infty}[Tr \, U(\frac{\beta}{2}, -\frac{\beta}{2})] = \langle 0, in|U(\infty, -\infty)|0, in\rangle = S_0'(A) \tag{43}$$

and

$$\lim_{\beta \to \infty}[-\frac{1}{\beta} \ln Tr \, U(\frac{\beta}{2}, -\frac{\beta}{2})] = E_0, \tag{44}$$

where E_0 is the vacuum energy. From these equations and the definition (42) we see that the integral of $w(x)$ over the spatial coordinates is half the vacuum energy. In order to obtain $w(x)$, we first have to evaluate $\mathcal{S}_0(A)$. In the functional integral representation, we may write

$$\mathcal{S}_0(A) = |\mathcal{N}|^2 \int D\bar{\Psi} D\Psi e^{- \int d^3 x \bar{\Psi} \mathcal{D}[A] \Psi}, \tag{45}$$

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where the notation $\mathcal{D}[A]$ indicates the Dirac operator dependence on the external field A. Since the integral is over Grassmannian variables, we write

$$\mathcal{S}_0(A) = |\mathcal{N}|^2 \det(\mathcal{D}[A]), \tag{46}$$

where det stands for the determinant over both spinor and functional spaces. Inserting (46) into the definition (41), we see that

$$|\mathcal{S}'_0(A)|^2 = exp\{Tr \ \ln\mathcal{H}[A] - Tr \ \ln\mathcal{H}[0]\}. \tag{47}$$

Now, we use the Frullani's identity:

$$\ln\frac{a}{b} = \lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s}(e^{-sb} - e^{-sa}) \tag{48}$$

to rewrite equation (47) in a particularly convenient integral representation:

$$\log(|\mathcal{S}'_0(A)|^2) = Tr\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s}(e^{-s\mathcal{H}[A]} - e^{-s\mathcal{H}[0]}), \tag{49}$$

whence we can obtain the vacuum energy density as:

$$w(x) = tr\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s}(e^{-s\mathcal{H}[A]} - e^{-s\mathcal{H}[0]}), \tag{50}$$

where now the trace only affects the spinor space indices.

We evaluate now the vacuum energy, the integral of $w(x)$ over all the Euclidean space, for the previously described case. The operators $\mathcal{H}[A]$ and $\mathcal{H}[0]$ are respectively given by,

$$\mathcal{H}[A] = p_t^2 + p_1^2 + 2b_2^\dagger b_2 E_2 + \omega_n + \omega_b + \gamma\cdot(\widetilde{F} - \eta) \tag{51}$$

and

$$\mathcal{H}[0] = p^2 + M^2 - \gamma\cdot\eta \ . \tag{52}$$

We can write $\mathcal{H}[0]$ in terms of some creation and annihilation operators c and $c^\dagger$, defined as in equation (15), with frequency $\omega = \|\eta\|$, as follows:

$$\mathcal{H}[0] = p_t^2 + p_b^2 + \|\eta\|(2c^\dagger c + 1) - \gamma\cdot\eta \ . \tag{53}$$

Evaluating the trace, we see that:

$$w(x) = 2\lim_{\epsilon\to 0}\int_\epsilon^\infty \frac{ds}{s}\left\{\langle x|e^{-s[p_t^2 + p_1^2 + 2b_2^\dagger b_2 + \omega_n + \omega_b]}(e^{sE_2} - e^{-sE_2})|x\rangle\right.$$

$$\left. - e^{-s[p_t^2 + p_b^2 + 2c^\dagger c + \|\eta\|]}(e^{s\|\eta\|} - e^{-s\|\eta\|})\}|x\rangle\right\} \ . \tag{54}$$

The system we are considering is not translation invariant, and moreover, we may write the explicit coordinate dependence of $w(x)$ by using the completeness of the eigenstates of the operators $b_2^\dagger b_2$ and $c^\dagger c$ for the respective Hilbert spaces. Denoting by $\langle x_2|m_2\rangle$ and $\langle x_\eta|m_\eta\rangle$ the respective eigenstates of the operators $b_2^\dagger b_2$ and $c^\dagger c$, which are as usual labeled by a non-negative integer $m = 0, 1, \ldots, \infty$, then we may write $w(x)$ as

$$w(x) = 2\lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{ds}{s} \sum_{m=0}^\infty \left\{ |\langle x_2|m_2\rangle|^2 \langle x_\eta x_b | e^{-s[p_t^2 + p_1^2 + 2m + \omega_\eta + \omega_b]} |x_\eta x_b\rangle \right.$$

$$\left. (e^{sE_2} - e^{-sE_2}) - |\langle x_\eta|m_\eta\rangle|^2 \langle x_t x_b | e^{-s[p_t^2 + p_b^2 + 2m + \|\eta\|]} |x_t x_b\rangle (e^{s\|\eta\|} - e^{-s\|\eta\|}) \right\} .$$

$$(55)$$

It should be obvious then that each term in $w(x)$ in in fact a function of only one coordinate, x_2. Indeed, one may even check that each term in the series above may be regarded as a $1 + 1$ dimensional determinant times the square of the amplitude for the corresponding harmonic oscillator mode.

The vacuum energy density in terms of the eigenstates of harmonic modes $b_2^\dagger b_2$ and $c^\dagger c$. To obtain the total vacuum energy, W, we have to integrate over the phase space. After this step the sum over m can be performed and we obtain,

$$W = \frac{L^2}{2\pi} \lim_{\epsilon \to 0} w_\epsilon \equiv \frac{L^2}{2\pi} \lim_{\epsilon \to 0} \int_\epsilon^\infty \frac{ds}{s^2} [e^{-s(\omega_\eta + \omega_b - E_2)} \coth(sE_2) - \coth(s\|\eta\|)].$$

$$(56)$$

The L^2 factor appears because we have considered the space volume to be a square box. This equation expresses the vacuum energy of a fermionic system with a domain wall that can be approximated linearly near the defect, and in the presence of a constant electromagnetic field, for a particular configuration ($\widetilde{F} \cdot \eta = 0$). We can see that in the case where the electromagnetic field is absent the vacuum energy vanishes, which is consistent with our non interacting fermions normalization condition.

The regularized expression for W diverges as ϵ goes to zero. The divergences in Eq.(56) should be isolated in some terms as a first step and afterwards suppressed by some renormalization procedure. To determine the divergent parts in ϵ, we use the expansion,

$$\coth(u) = \frac{1}{u} + \sum_{k=1}^\infty \frac{2^{2k} B_{2k}}{(2k)!} u^{2k-1}, \tag{57}$$

valid for $u^2 < \pi^2$, B_{2k}, being the Bernoulli numbers. We obtain,

$$W_\epsilon = \frac{L^2}{2\pi} \{ \frac{A^2}{E_2} \left[\Gamma(-2, \epsilon A) - \Gamma(-2, \frac{\pi A}{E_2}) \right] + B_2 E_2 \left[\Gamma(0, \epsilon A) - \Gamma(0, \frac{\pi A}{E_2}) \right]$$

$$+ \sum_{k=2}^{\infty} \frac{2^{2k} B_{2k} E_2^{2k-1} A^{2-2k}}{(2k)!} \left[\Gamma(2k-2, \epsilon A) - \Gamma(2k-2, \frac{\pi A}{E_2}) \right] - \frac{|\eta|}{2\pi^2}$$

$$+ \quad \frac{1}{2|\eta|\epsilon^2} - \log \left[\frac{\pi}{|\eta|} \right]^{-B_2|\eta|} + B_2|\eta| \log \epsilon - \sum_{k=2}^{\infty} \frac{2^{2k} B_{2k} \pi^{2k-2} |\eta|}{(2k)!}$$

$$+ \quad F_1(w_\eta, w_b, E_2) + F_2(|\eta|) \} , \tag{58}$$

where

$$F_1(w_\eta, w_b, E_2) = \int_{\frac{\pi}{E_2}}^{\infty} ds s^{-2} e^{-sA} \coth(sE_2), \ F_2(|\eta|) = \int_{\frac{\pi}{|\eta|}}^{\infty} ds s^{-2} \coth(s|\eta|) ,$$
$$\tag{59}$$

$A = w_\eta + w_b - E_2$ and $\Gamma(\alpha, x)$ is the incomplete Gamma-function. In the expression (58) singularities are still present in the terms $\Gamma(-2, \epsilon A)$ and $\Gamma(0, \epsilon A)$. To isolate them we use the formula,

$$\Gamma(\alpha, x) = e^{-x} x^\alpha \sum_{n=0}^{\infty} \frac{L_n^\alpha}{n+1}, \tag{60}$$

where L_n^α are the Laguerre polynomials. We get at the limit $\epsilon \to 0$, $\Gamma(-2, \epsilon A) \to \frac{1}{2A^2\epsilon^2}$ and $\Gamma(0, \epsilon A) \to \sum_0^\infty \frac{1}{n+1}$. This last series diverges logarithmically. After some rearrangements, we obtain,

$$
\begin{aligned}
W_\epsilon \ = \ & \frac{L^2}{2\pi} \{ (\frac{1}{2E_2} - \frac{1}{2|\eta|}) \frac{1}{\epsilon^2} + (B_2 E_2 \sum_{n=0}^{\infty} \frac{1}{n+1} + B_2|\eta| \log \epsilon) \\
& - \ \frac{A^2}{E_2} \Gamma(-2, \frac{\pi A}{E_2}) - B_2 E_2 \Gamma(0, \frac{\pi A}{E_2}) - \log(\frac{\pi}{|\eta|})^{-B_2|\eta|} \\
& + \ \frac{A^2}{E_2} \sum_{k=2}^{\infty} \frac{B_{2k}}{(2k)!} (\frac{2E_2}{A})^{2k} \left[\Gamma(2k-2, \epsilon A) - \Gamma(2k-2, \frac{\pi A}{E_2}) \right] \\
& - \ \frac{|\eta|}{\pi^2} (-\frac{1}{2} + \sum_{k=2}^{\infty} \frac{(2\pi)^{2k} B_{2k}}{(2k)!}) + F_1(w_\eta, w_b, E_2) + F_2(|\eta|) \}.
\end{aligned}
$$
$$\tag{61}$$

The two first terms in (61), contain quadratic and logarithmic divergencies which may be suppressed by a renormalization procedure. It may be, at first sight, shocking to see that the divergent part of W_ϵ is in fact not a finite degree polynomial in the external field $F_{\mu\nu}$ and its derivatives, as the usual divergence theorems for a standard Quantum Field Theory imply [8]. The reason for this seemingly contradictory result is that the hypothesis for those theorems do not hold in the present case. Firstly, the mass term of the 'free'

fermionic field is not a constant, thus the inhomogeneity of space already changes the main assumptions, like power counting behaviour. Note that in our case there is no unique large momentum behaviour for the propagator. And secondly, the would be external field $\tilde{F}$ appears (after diagonalization) in such a way that it plays a similar role to the inhomogeneous mass. Then the external field is more likely to appear in a similar way to a mass in a standard divergent expression, which is indeed a non-polynomial dependence. In spite of this, a renormalization prescription may indeed be used for this quantity. For example, we may realize that W_ϵ, if regarded as a function of E_2, has divergences where E_2 appear as a pole, a constant or a linear term in E_2. Thus we may cancel all the divergences of this system by including three counterterms, namely, by adding to W_ϵ the 1-loop counterterm action δW_ϵ defined by:

$$\delta W_\epsilon \;=\; \alpha^{-1}\,(E_2)^{-1} \;+\; \alpha_0 \;+\; \alpha_1 E_2 \tag{62}$$

where the α_j's are divergent constants. These three constants require the use of some renormalization conditions to fix them; in our case one could of course use the Laurent expansion of the actually measured W in order to fix those coefficients. It is amusing to note that this procedure requires the knowledge of the full dependence of W_ϵ on $\tilde{F}$, since the divergent part is not just a single polynomial in $\tilde{F}$.

4   The general case

Let us call $\widetilde{F}_\parallel$ and $\widetilde{F}_\perp$ respectively the projections of $\widetilde{F}$ onto the direction of η and onto a direction orthogonal to η (with respect to the scalar product defined in section 2) Then the square modulus of the Dirac operator for general configuration of the external vectors, can be written in the form,

$$\mathcal{H} = -\partial^2 + \frac{1}{4}\{x^2(\widetilde{F}_\perp^2 + \widetilde{F}_\parallel^2) - [x.(\widetilde{F}_\perp + \widetilde{F}_\parallel)]^2\} + (x.\eta)^2 - ix\times\partial.(\widetilde{F}_\perp + \widetilde{F}_\parallel) + \gamma.(\widetilde{F} - \eta). \tag{63}$$

In the same way as before we define versors $\hat{\eta}$, $\hat{t}$ and $\hat{b}$, where now $\hat{t}_\mu$ is a unit vector in the direction of $\widetilde{F}_\perp$, i. e. in the direction

$$\widetilde{F}_\mu - \widetilde{F} \cdot \hat{\eta}\,\hat{\eta}_\mu. \tag{64}$$

We can now define creation and destruction operators a_α and $a_\alpha^\dagger$, analogously as it has been done in Eq. (15), with $\alpha = \eta, t, b$, and ω_α defined by

$$\omega_\eta^2 = \frac{\widetilde{F}_\perp^2}{4} + \eta^2, \qquad \omega_t^2 = \frac{\widetilde{F}_\parallel^2}{4}, \qquad \omega_b^2 = \frac{\widetilde{F}_\perp^2 + \widetilde{F}_\parallel^2}{4}. \tag{65}$$

In terms of these, $\mathcal{H}$ can be written as,

$$
\begin{aligned}
\mathcal{H} = \;& \sum_\alpha \omega_\alpha (2a_\alpha^\dagger a_\alpha + 1) - 4\left(\frac{\omega_t(\omega_b^2 - \omega_t^2)}{\omega_n}\right)^{1/2}(a_t a_\eta + a_t a_\eta^\dagger + a_t^\dagger a_\eta + a_t^\dagger a_\eta^\dagger) \\
& -i(\omega_b^2 - \omega_t^2)^{1/2}\left(\sqrt{\frac{\omega_\eta}{\omega_b}} - \sqrt{\frac{\omega_b}{\omega_\eta}}\right)(a_b a_\eta - a_b^\dagger a_\eta^\dagger) \\
& -i(\omega_b^2 - \omega_t^2)^{1/2}\left(\sqrt{\frac{\omega_\eta}{\omega_b}} + \sqrt{\frac{\omega_b}{\omega_\eta}}\right)(a_b^\dagger a_\eta - a_b a_\eta^\dagger) \\
& -i\omega_t\left(\sqrt{\frac{\omega_b}{\omega_t}} - \sqrt{\frac{\omega_t}{\omega_b}}\right)(a_t a_b - a_t^\dagger a_b^\dagger) - i\omega_t\left(\sqrt{\frac{\omega_b}{\omega_t}} + \sqrt{\frac{\omega_t}{\omega_b}}\right)(a_t^\dagger a_b - a_t a_b^\dagger).
\end{aligned}
\tag{66}
$$

We see that $\mathcal{H}$ is again a bilinear form in the creation and destruction operators a_α and $a_\alpha^\dagger$:

$$H = a_j^\dagger A_{jl} a_l - \frac{i}{2} a_j B_{jl} a_l + \frac{i}{2} a_j^\dagger B_{jl}^* a_l^\dagger + \text{constant}, \tag{67}$$

where A_{jl} and B_{jl} are components of 3×3 matrices, A Hermitian and B symmetric:

$$A = \begin{pmatrix} 2\omega_\eta & -t & iv \\ -t & 2\omega_t & -ir \\ -iv & ir & 2\omega_b \end{pmatrix} \quad B = \begin{pmatrix} 0 & -it & u \\ -it & 0 & s \\ u & s & 0 \end{pmatrix}, \tag{68}$$

with the constants defined by

$$r = \left(\sqrt{\frac{\omega_b}{\omega_t}} + \sqrt{\frac{\omega_t}{\omega_b}}\right)\omega_t \quad s = \left(\sqrt{\frac{\omega_b}{\omega_t}} - \sqrt{\frac{\omega_t}{\omega_b}}\right)\omega_t \quad t = 4\frac{\omega_t}{\omega_\eta}\sqrt{\omega_b^2 - \omega_t^2}$$

$$u = \left(\sqrt{\frac{\omega_\eta}{\omega_b}} - \sqrt{\frac{\omega_b}{\omega_\eta}}\right)\sqrt{\omega_b^2 - \omega_t^2} \quad v = \left(\sqrt{\frac{\omega_\eta}{\omega_b}} + \sqrt{\frac{\omega_b}{\omega_\eta}}\right)\sqrt{\omega_b^2 - \omega_t^2}. \tag{69}$$

We also see that in this case it is possible to use a transformation, defined in terms of 3×3 matrices U and V, such that the Hamiltonian is diagonalized. The properties of this transformation can be summarized by the equations:

$$U^\dagger U - V^\dagger V = I \quad U^t V - V^t U = 0 \tag{70}$$

15

and

$$AU + iB^*V = UE \quad -iBU + A^*V = -VE \; . \tag{71}$$

As in the simpler 2×2 case, one may obtain linear equations that determine the transformation matrices U and V. In order for those equations to have a non-trivial solution, it is necessary to demand the condition:

$$\det\left[A^*(B^*)^{-1}A - B + ((B^*)^{-1}A - A^*(B^*)^{-1})\mathcal{E} - (B^*)^{-1}\mathcal{E}^2\right] = 0 \tag{72}$$

where $\mathcal{E}$ are the 'energies', i.e., the numbers appearing in the diagonal of the matrix E. Equation (72) may be explicitly solved, what yields 6 solutions. Of these 6 solutions, we should eliminate 3 spurious ones, since they are not consistent with the algebraic relations that are verified by U and V. This leads to the following three solutions. One of them vanishes: $\mathcal{E}_1 = 0$, and the other two have a rather cumbersome expression, which may be written in terms of the previously defined ω parameters as follows:

$$\mathcal{E}_2 = \sqrt{2}\left\{\omega_\eta^2 + \omega_t^2 + 3\omega_b^2 - (\omega_\eta^{-1}[(\omega_b^2 - \omega_t^2)(64\omega_t^3 + 96\omega_t^2\sqrt{\omega_\eta\omega_t})]\right.$$

$$\left. +\omega_\eta^4 - 14\omega_\eta^2\omega_t^2 + 13\omega_t^4 + 6(\omega_\eta^2 - \omega_t^2)\omega_b^2 + 9\omega_b^4)]\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}} \tag{73}$$

$$\mathcal{E}_3 = \sqrt{2}\left\{\omega_\eta^2 + \omega_t^2 + 3\omega_b^2 + (\omega_\eta^{-1}[(\omega_b^2 - \omega_t^2)(64\omega_t^3 + 96\omega_t^2\sqrt{\omega_\eta\omega_t})]\right.$$

$$\left. +\omega_\eta^4 - 14\omega_\eta^2\omega_t^2 + 13\omega_t^4 + 6(\omega_\eta^2 - \omega_t^2)\omega_b^2 + 9\omega_b^4)]\right)^{\frac{1}{2}}\right\}^{\frac{1}{2}} \; . \tag{74}$$

¿From these solutions the Hamiltonian operator can be written in diagonal form and the determinant of the Dirac operator can be obtained for general $\widetilde{F}$ and η. Calculations in this case are more involved, but following along the same lines as we have done in section 3, it should be possible to obtain the vacuum energy in the general case. Progress on this subject will be reported elsewhere.

Acknowledgments

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