Holography and Entropy Bounds in Gauss-Bonnet Gravity

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Abstract

We discuss the holography and entropy bounds in Gauss-Bonnet gravity theory. By applying a Geroch process to an arbitrary spherically symmetric black hole, we show that the Bekenstein entropy bound always keeps its form as $S_{\rm B} = 2\pi ER$, independent of gravity theories. As a result, the Bekenstein-Verlinde bound also remains unchanged. Along the Verlinde's approach, we obtain the Bekenstein-Hawking bound and Hubble bound, which are different from those in Einstein gravity. Furthermore, we note that when $HR = 1$, the three cosmological entropy bounds become identical as in the case of Einstein gravity. But, the Friedmann equation in Gauss-Bonnet gravity can no longer be cast to the form of cosmological Cardy formula.

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1 Introduction

According to the holographic principle [1], within a given volume V the number of degrees of freedom is bounded by a quantity proportional to the surface area A of the volume. This is obtained from the idea that the maximal entropy inside the volume is given by the largest black hole that just fits inside the volume, while the entropy of the latter obeys the Bekenstein-Hawking entropy formula $A/4G$, where G is the Newton constant. Thus, the holographic principle gives an entropy bound on matter inside the volume

$$S \leq \frac{A}{4G},\tag{1.1}$$

which is called the holographic bound.

Fischler and Susskind [2] were the first to consider entropy bound in the cosmological setting. In a closed universe, the holographic bound in its naive form (1.1) is not applicable because there is no boundary in the closed universe. On the other hand, the argument leading to (1.1) assumes that it is possible to form a black hole filling the whole volume. This is no longer valid in the universe since the expansion rate H of the universe and the total energy in the universe restrict the maximal size of black hole [3]. Following Fischler and Susskind, it was argued that the maximal entropy inside the universe is produced by black holes with size of Hubble horizon [4]. The usual holographic arguments lead to the result that the total entropy should be less than or equal to the Bekenstein-Hawking entropy of a Hubble horizon-sized black hole times the number of Hubble regions in the universe. That is, one has $S \leq \beta HV/G$, where V represents the volume of the universe and β is a pure coefficient. This coefficient is fixed by Verlinde [3] by using a local version of holographic bound [2, 5]. This bound is called the Hubble entropy bound, which has the form

$$S_{\mathrm{H}} = (n-1)\frac{HV}{4G},\tag{1.2}$$

where n stands for spatial dimensions of the universe. The Hubble bound is valid for a strongly self-gravitating universe ($HR \geq 1$). Except for the Hubble bound, Verlinde introduced other two entropy bounds [3]:

$$\begin{aligned} \mathrm{Bekenstein-Verlinde\ bound:} &\quad S_{\mathrm{BV}} = \frac{2\pi}{n}ER \\ \mathrm{Bekenstein-Hawking\ bound:} &\quad S_{\mathrm{BH}} = (n-1)\frac{V}{4G_{n+1}R}. \end{aligned}\tag{1.3}$$

Here E is the total energy of the matter filling the universe and R is the scale factor. The Bekenstein-Verlinde bound S_{BV} is the counterpart of the Bekenstein entropy bound [6] in the cosmological setting [7], which is believed to hold for a weakly self-gravitating universe ($HR \leq 1$). The Bekenstein-Hawking entropy bound does not serve as an entropy bound, but acts as a criterion whether the universe is in a weakly self-gravitating phase ($HR \leq 1$)

or in a strongly self-gravitating phase ($HR \geq 1$) [3]. The Friedmann equation of a ($n+1$)-dimensional, closed Friedmann-Robertson-Walker (FRW) universe is

$$H^2 = \frac{16\pi G}{n(n-1)} \frac{E}{V} - \frac{1}{R^2},$$ (1.4)

from which one can see that $S_{\mathrm{BV}} \leq S_{\mathrm{BH}}$ for $HR \leq 1$, while $S_{\mathrm{BV}} \geq S_{\mathrm{BH}}$ for $HR \geq 1$. Clearly one has $S_{\mathrm{BV}} = S_{\mathrm{BH}} = S_{\mathrm{H}}$ at the critical point $HR = 1$. Furthermore, Verlinde found that with the three cosmological entropy bounds, the Friedmann equation (1.4) can be cast to

$$S_{\mathrm{H}} = \sqrt{S_{\mathrm{BH}}(2S_{\mathrm{BV}} - S_{\mathrm{BH}})},$$ (1.5)

the cosmological Cardy formula. This formula (1.5) has a close relation to the Cardy-Verlinde formula describing the entropy of conformal field theories. For more discussions, see [3].

We note that those discussions on the entropy bounds crucially depend on the area entropy formula of black holes (1.1). However, it is well-known that the area entropy formula of black holes holds only in Einstein gravity. If some higher derivative curvature terms appear, for example, one has to include some additional terms to the area entropy formula of black holes [8]. Therefore it would be interesting to see how those entropy bounds get modified in higher derivative gravity theories. In this note we will discuss entropy bounds in the Gauss-Bonnet gravity, which belongs to a special class of higher derivative gravity theories in the sense that the equation of motion for the Gauss-Bonnet gravity contains no more than second derivatives of metric.

2 Bekenstein bound and Bekenstein-Verlinde bound

Bekenstein was the first to consider the issue of maximal entropy for a macroscopic system. He argued that for a closed system with total energy E, which fits in a sphere with radius R in three spatial dimensions, there exists an upper bound on the entropy of the system

$$S \leq S_{\mathrm{B}} = 2\pi ER,$$ (2.1)

which is called the Bekenstein entropy bound. This bound is believed to be valid for a system with limited self-gravity, which means that the gravitational self-energy is negligibly small compared to its total energy. However, it is interesting to note that this bound gets saturated for a ($3+1$) dimensional Schwarzschild black hole, which of course is a strongly self-gravitating object. Furthermore it was found that the form (2.1) is independent of spatial dimensionality. That is, the form (2.1) keeps unchanged for any dimensional object. This is obtained by considering a Geroch process in an arbitrary

dimensional Schwarzschild black hole and the generalized second law of black hole thermodynamics [9]. It is easy to show that for a higher ($n+1 > 4$) dimensional Schwarzschild black hole, the Bekenstein entropy bound still holds, but it is not saturated.

In deriving the Bekenstein entropy bound [6, 9], black hole thermodynamics is used. And the thermodynamics of black holes is dependent of gravity theories under consideration. In fact, we show here that the Bekenstein entropy bound is independent of gravity theories. As a result, the Bekenstein-Verlinde bound has also the same feature of independence of gravity theories. Consider an arbitrary, ($n + 1$)-dimensional spherically symmetric black hole solution

$$ds^2 = -e^{2\delta(r)} \left(1 - \frac{2m(r)}{r^{n-2}}\right) dt^2 + \left(1 - \frac{2m(r)}{r^{n-2}}\right)^{-1} dr^2 + r^2 d\Omega_{n-1}^2, \qquad (2.2)$$

where δ and m are two continuous functions of r, It is assumed that $e^{2\delta(r)} \neq 0$ in the whole spacetime. The black hole horizon r_+ is determined by equation $1 - 2m(r_+)/r_+^{n-2} = 0$. The Hawking temperature T associated with the horizon is

$$T = \frac{e^{\delta(r_+)}}{4\pi} \left(\frac{n-2}{r_+} - \frac{2m'(r_+)}{r_+^{n-2}}\right), \qquad (2.3)$$

where a prime denotes derivative with respect to r. We denote by M the mass of the black hole. According to the first law of black hole thermodynamics, which always holds because a black hole behaves as a thermodynamic system, one has the entropy variation $\triangle S$ when the mass gets increase by a small amount $\triangle M$,

$$\triangle S = T^{-1} \triangle M. \qquad (2.4)$$

Let us consider a Geroch process in the black hole background (2.2). Suppose that one has a thermodynamic system with energy E and R being the radius of the smallest ($n-1$)-sphere circumscribing the system. Now move this system from infinity to a place just outside the horizon of the black hole (2.2), and drop the matter into the black hole.

The mass added to the black hole is given by the energy E of the system, which gets redshifted according to the position of the center of mass at the drop-off point, at which the circumscribing sphere almost touches the horizon. The center of mass can be brought to within a proper distance R from the horizon, while all parts of the system still remain outside the horizon. Thus one needs to calculate the redshift factor at a proper distance R from the horizon [9].

Let x be the radial coordinate distance from the horizon $x = r - r_+$. The redshift factor near the horizon is given by

$$\chi^2(x) = e^{2\delta(r_+)} \left(\frac{n-2}{r_+} - \frac{2m'(r_+)}{r_+^{n-2}}\right) x \qquad (2.5)$$

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up to the leading order of x. Near the horizon, the proper distance R has a relation to the coordinate distance x,

$$R = 2\sqrt{\frac{x}{(n-2)/r_+ - 2m'(r_+)/r_+^{n-2}}}.$$ (2.6)

Hence the absorbed mass is $\triangle M = E\chi(x)$. Substituting this into (2.4), we find that the increased entropy of the black hole is

$$\triangle S = T^{-1}E\chi(x) = 2\pi ER.$$ (2.7)

According to the generalized second law of black hole thermodynamics [10], which says that the total entropy of black hole and matter outside the black hole never decreases in any physical process, we can immediately obtain the maximal entropy of the system,

$$S_m \leq 2\pi ER.$$ (2.8)

This is just the Bekenstein entropy bound (2.1). From the above one can see that we have neither specified what the black hole solution (2.2) is, nor in which gravity theory it is. Hence the resulting conclusion (2.8) is independent of gravity theories. This is an expected result since as stated above, the Bekenstein bound is valid only for systems with limited self-gravity, which implies that gravity effect is negligible. In addition, the Bekenstein-Verlinde bound S_{BV} in (1.3) is the counterpart of the Bekenstein bound in the cosmological setting. Therefore we conclude that the Bekenstein-Verlinde bound is also independent of gravity theories.

3 Hubble bound and Bekenstein-Hawking bound in Gauss-Bonnet gravity

Now we consider the so-called Gauss-Bonnet gravity theory by adding the Gauss-Bonnet term to the Einstein-Hilbert action,

$$S = \frac{1}{16\pi G}\int d^{n+1}x\sqrt{-g}\left(\mathcal{R} + \alpha(\mathcal{R}_{\mu\nu\gamma\sigma}\mathcal{R}^{\mu\nu\gamma\sigma} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2)\right),$$ (3.1)

where α is a constant. Here we exclude the case of $n = 3$ since in that case the Gauss-Bonnet term is a topological term. The static spherically symmetric black hole solutions in (3.1) have been found in [11, 12]. The entropy of the black holes has the expression [13, 14]

$$S = \frac{A}{4G}\left(1 + \frac{n-1}{n-3}\frac{2\tilde{\alpha}}{R^2}\right),$$ (3.2)

where $\tilde{\alpha} = (n-2)(n-3)\alpha$, A represents the horizon area of the black hole and R the horizon radius. Following Verlinde [3], in this section we "derive" the Hubble bound for a closed FRW universe in the Gauss-Bonnet theory.

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In [3] Verlinde used a version of holographic bound proposed by Fischler and Susskind [2] and subsequently developed by Bousso [5], which gives a restriction of entropy flow S through a contracting light sheet: the entropy flow S is less than or equal to $A/4G$, where A is the area of the surface from which the light sheet originates. The infinitesimal version of the holographic bound plays a crucial role in the "derivation" by Verlinde. According to the infinitesimal version, for every $(n-1)$-dimensional surface at time $t+dt$ with area $A+dA$, one has $dS \leq dA/4G$. Here dS represents the entropy flow through the infinitesimal light sheets originating at the surface at $t+dt$ and extending back to time t, and dA denotes the increase in area between t and $t+dt$. Obviously the holographic bound is based on the area entropy formula of black holes. In our case, the black hole entropy is given by (3.2). The infinitesimal version is then changed to

$$dS \leq \frac{1}{4G} d\left[A\left(1 + \frac{n-1}{n-3}\frac{2\tilde{\alpha}}{R^2} \right) \right]. \tag{3.3}$$

In a $(n+1)$ dimensional closed FRW universe, for a surface which is fixed in comoving coordinates, the area A changes as a result of the expansion of the universe by an amount

$$dA = (n-1)HA dt, \tag{3.4}$$

where the relation $A \sim R^{n-1}$ has been used. Choose one of two past light sheets that originate at the surface: the inward or the outward going. The entropy flow through this light sheet between t and $t+dt$ is given by the entropy density $s = S/V$ times the infinitesimal volume $A dt$ swept out by the light sheet. That is, one has

$$dS = \frac{S}{V}A dt. \tag{3.5}$$

Applying $A \sim R^{n-1}$ to (3.3), and then substituting (3.3) and (3.4) into (3.5), we obtain

$$S \leq S_{\mathrm{H}} = (n-1)\frac{HV}{4G}\left(1 + \frac{2\tilde{\alpha}}{R^2} \right), \tag{3.6}$$

which is the Hubble entropy bound in the Gauss-Bonnet gravity. This is the main result of ours. If a cosmological constant is added to the action (3.1), the entropy of black holes still has the expression (3.2) [14]. Hence the Hubble entropy bound still takes the form (3.6) even if a cosmological constant is present in the Gauss-Bonnet gravity.

On the other hand, the Friedmann equation for the Gauss-Bonnet gravity (3.1) is

$$H^2 + \frac{1}{R^2} + \tilde{\alpha}\left(H^2 + \frac{1}{R^2} \right)^2 = \frac{16\pi G}{n(n-1)}\frac{E}{V}, \tag{3.7}$$

from which we see that when $HR = 1$, the Bekenstein-Verlinde bound $S_{\mathrm{BV}} = 2\pi ER/n$ equals the Hubble bound S_{H} given by (3.6). This is a good check for our "derivation"

of the Hubble bound. Furthermore, from the Friedmann equation (3.7), we find that the Bekenstein-Hawking bound has the form

$$S_{\mathrm{BH}} = (n-1)\frac{V}{4GR}\left(1 + \frac{2\tilde{\alpha}}{R^2}\right). \tag{3.8}$$

As in the case of Einstein theory [3], this Bekenstein-Hawking bound (3.8) was obtained by identifying $S_{\mathrm{BH}} = S_{\mathrm{BV}}$ via the Friedmann equation (3.7) at the critical point $HR = 1$. Hence at the critical point the property that three cosmological entropy bounds become identical in Einstein theory persists in the Gauss-Bonnet gravity. Inspecting the Friedmann equation (3.8), however, we find that it can no longer be rewritten in the form (1.5), which might be related to that the black hole entropy in higher derivative theories cannot be cast to the Cardy-Verlinde formula [15].

4   Conclusion

In summary this paper has initiated the study of holography in gravity theories with higher derivative curvature terms. As a concrete model, we have considered the Gauss-Bonnet theory. We have shown that as expected, the Bekenstein bound and the Bekenstein-Verlinde bound keep the same forms as in Einstein theory, while the Hubble bound and Bekenstein-Hawking bound get modified. Along the Verlinde's approach, we have obtained expressions of the Hubble bound and Bekenstein-Hawking bound. When the universe undergoes a transition from a weakly self-gravitating phase ($HR \leq 1$) to a strongly self-gravitating phase ($HR \geq 1$), the three cosmological entropy bounds get matched at the critical point $HR = 1$, as in the case of Einstein gravity. However, the Friedmann equation of the Gauss-Bonnet gravity cannot be rewritten in the form of the cosmological Cardy formula.

Acknowledgment

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