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Penrose Limit and String Theories on Various Brane Backgrounds

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Abstract

We investigate the Penrose limit of various brane solutions including Dp-branes, NS5-branes, fundamental strings, (p, q) fivebranes and (p, q) strings. We obtain special null geodesics with the fixed radial coordinate (critical radius), along which the Penrose limit gives string theories with constant mass. We also study string theories with time-dependent mass, which arise from the Penrose limit of the brane backgrounds. We examine equations of motion of the strings in the asymptotic flat region and around the critical radius. In particular, for (p, q) fivebranes, we find that the string equations of motion in the directions with the B field are explicitly solved by the spheroidal wave functions.

1 Introduction

The Penrose limits [1, 2] of backgrounds of the M theory and type II superstring theories are useful for studying the holography between string theories and conformal field theories [3, 4]. This is based on the fact that the type IIB Green-Schwarz superstring on the pp-wave background is solved exactly in the light-cone gauge [5, 6].

Superstring theories on various pp-wave backgrounds have been studied. It would be quite interesting to study nonconformal or nonlocal field theories in this framework [7, 8, 9, 10, 11, 12, 13]. Recently, the Penrose limit of the Pilch-Warner solution [14, 15, 16] and the Dp-brane background in the near-horizon limit [15] has been investigated from the viewpoint of the holographic RG flow. In order to study string theories in such nontrivial backgrounds, one must solve string theory in a time-dependent background. Previous attempts to study the string theories on the solvable plane wave background can be seen in refs. [17, 18, 19, 20, 21]. Recently, a solvable model based on $N = 2$ supersymmetric sine-Gordon model was proposed by Maldacena and Maoz [22].

The purpose of this paper is to investigate the solvable string theories in the pp-wave background with time-dependent mass. We study the Penrose limit of various brane backgrounds including Dp ($p \leq 6$) branes, the NS5-branes and the fundamental strings. We also investigate the Penrose limit of the (p, q) fivebranes and the (p, q) strings, which are obtained by applying the $SL(2, \mathbf{Z})$ symmetry of type IIB superstrings.

In the present work, we do not restrict our analysis to the near-horizon limit. The string theories on the Penrose limit of brane backgrounds include some interesting features. When we consider the Penrose limit along the generic null geodesic, we have the flat space at infinity. String theories on this background have time-dependent masses, which vanish in the asymptotic region. We are also able to find special null geodesic for each brane solution such that the radial transverse coordinate from the brane is fixed. The Penrose limit along this geodesic with such a critical radius gives rise to the Cahen-Wallach space, on which string theories have constant masses in the light-cone gauge. This type of solution has been recently studied by Oz and Sakai [23] in the case of near-horizon limit of (p, q) fivebranes.

We study the equations of motion of bosonic strings in the plane wave background with time-dependent masses, which reduce to the second-order ordinary differential equations with respect to the radial transverse coordinate. We examine these differential equations for various brane solutions. In the near horizon limit, a large class of backgrounds are shown to be solved by the Bessel functions. When we do not take the near horizon limit, the equations are difficult to solve in general. In this paper, we study the case of (p, q) fivebrane backgrounds in detail, since we can solve the equations of motion in two regions: near the critical radius and far from the origin. The solutions are given by the Mathieu functions for both regions. Furthermore, we are able to find the non-trivial exact solution in the directions with the B-field by using the spheroidal wave functions.

This paper is organized as follows: In section 2, the Penrose limit of various p-brane solutions will be studied. The metric and the $(p+2)$-form field strength are calculated as functions of the radial transverse coordinate. In section 3, we study special null geodesics with the critical radius, along which the Penrose limit gives the Cahen-Wallach space. In section 4, we study the closed bosonic string theories on the Penrose limit of various brane backgrounds. In particular, we will consider the theories at the critical radius and in the near horizon limit. In section 5, we will study the string theories on the (p, q) fivebrane background. We examine equations of motion of the strings in the asymptotic flat region and around the critical radius. We solve the string equation of motion in the directions with the B-field using the spheroidal wave function. In the Appendix, the string theory which is solvable in terms of the Bessel functions is discussed in some detail.

2 Penrose limits of brane solutions

The Penrose limit is defined as a specific scaling of the metric and supergravity fields along a null geodesic. In this section, we will review the Penrose limits of the p-brane solutions and give explicit forms of various brane solutions.

2.1 Review of the Penrose limit

We will briefly review the Penrose limit of the p-brane solution [3]. The typical p-brane solution for a D-dimensional supergravity is

$$ds^2 = A^2(r)ds^2(\mathbf{E}^{1,p}) + B^2(r)ds^2(\mathbf{E}^{D-p-1}),$$

$$
\begin{aligned}
&F_{p+2} = dC_{p+1} = d\mathrm{vol}(\mathbf{E}^{1,p}) \wedge dC(r), \\
&\phi = \phi(r)
\end{aligned}
\tag{2.1}
$$

where r is the radial transverse coordinate, $\mathbf{E}^{1,p}$ is the world-volume of the p-brane and $\mathbf{E}^{D-p-1}$ is the transverse space. C_{p+1} is the $(p+1)$-form potential coupled to the brane. A, B, C and scalar field ϕ depend on r. The p-brane metric is rewritten as follows.

$$
ds^2 \;\; = \;\; A^2(r)(-dt^2 + ds^2(\mathbf{E}^p)) + B^2(r)\left(dr^2 + r^2(d\psi^2 + (\sin\psi)^2 d\Omega^2_{D-p-3})\right) \tag{2.2}
$$

where ψ is colatitude of $D-p-2$-dimensional sphere and $d\Omega^2_{D-p-3}$ is the metric for $D-p-3$ dimensional sphere.

We will consider the null geodesic in (t, r, ψ) space. With the metric

$$
ds^2_{(3)} = -A^2 dt^2 + B^2 dr^2 + B^2 r^2 d\psi^2. \tag{2.3}
$$

There are two Killing vectors ∂_t and ∂_ψ. The corresponding conserved quantities are

$$
E \equiv -g_{tt}\dot{t} = A^2\dot{t}, \quad J \equiv g_{\psi\psi}\dot{\psi} = B^2 r^2 \dot{\psi} \tag{2.4}
$$

where the dot is the derivative with respect to the affine parameter λ along the geodesic. E and J are energy and angular momentum respectively. To normalize the energy $E = 1$, we define an affine parameter $u \equiv E\lambda$. The equations (2.4) become

$$
A^2 \frac{\partial t}{\partial u} = 1, \quad B^2 r^2 \frac{\partial \psi}{\partial u} = J/E. \tag{2.5}
$$

The remaining free parameter which determines the geodesic is $\ell \equiv J/E$. The condition for a null geodesic is

$$
g_{tt}\left(\frac{\partial t}{\partial u}\right)^2 + g_{rr}\left(\frac{\partial r}{\partial u}\right)^2 + g_{\psi\psi}\left(\frac{\partial \psi}{\partial u}\right)^2 = B^2\left(\frac{\partial r}{\partial u}\right)^2 - \left(\frac{1}{A^2} - \frac{\ell^2}{B^2 r^2}\right) = 0. \tag{2.6}
$$

For a fixed ℓ, r must satisfy $\ell \leq rB/A$. We will examine this inequality for each brane solution in section 3. When $\ell = 0$, the null geodesic exists in (t, r) space. This is called radial null geodesic.

We introduce the coordinate transformation $(t, r, \psi) \to (u, v, \tilde{z})$

$$
u = u(r), \quad v = t + \ell\psi + a(r), \quad \tilde{z} = \psi + b(r), \tag{2.7}
$$

such that the metric (2.3) becomes

$$ds_{(3)}^2 = 2dudv + Kdv^2 + Ldvd\tilde{z} + Md\tilde{z}^2. \tag{2.8}$$

In this metric, the tangent vector ∂_u is a null geodesic vector. The conditions for the null geodesic (2.5) and (2.6) lead to

$$\begin{aligned}
&\frac{da}{dr} = \pm\sqrt{\frac{B^2}{A^2} - \frac{\ell^2}{r^2}}, \quad \frac{db}{dr} = \mp\frac{\ell/r^2}{\sqrt{\frac{B^2}{A^2} - \frac{\ell^2}{r^2}}}, \\
&\frac{du}{dr} = \pm\frac{B^2}{\sqrt{\frac{B^2}{A^2} - \frac{\ell^2}{r^2}}} \equiv Q(r).
\end{aligned} \tag{2.9}$$

Then the functions K, L and M in the metric (2.8) are given by

$$K = -A^2, \quad L = 2\ell A^2, \quad M = B^2r^2 - \ell^2A^2. \tag{2.10}$$

We rescale the coordinates, the metric g, the $(p+2)$-form field strength F_{p+2} and the scalar field ϕ.

$$\begin{aligned}
&u \to u, \quad v \to \Omega^2 v, \quad \tilde{z} \to \Omega\tilde{z}, \quad \tilde{x}^a \to \Omega\tilde{x}^a, \quad \tilde{y}^l \to \Omega\tilde{y}^l \\
&g \to \Omega^{-2}g, \quad F_{p+2} \to \Omega^{-p}F_{p+2}, \quad \phi \to \phi
\end{aligned} \tag{2.11}$$

where $\tilde{x}^a$ $(a = 1, \cdots, p)$ are the spatial coordinates of the p-branes and $\tilde{y}^l$ $(l = 1, \cdots, D - p - 3)$ are coordinates of S^{D-p-3}. The Penrose limit is the limit where $\Omega \to 0$. Then the D-dimensional metric and the $(p+2)$-form field strength F_{p+2} become

$$\begin{aligned}
ds^2 &= 2dudv + (B^2r^2 - \ell^2A^2)dz^2 + A^2ds^2(\mathbf{E}^p) + B^2r^2(\sin b)^2 ds^2(\mathbf{E}^{D-p-3}), \\
F_{p+2} &= \pm\frac{dC(r)}{dr}\frac{\ell}{B}\sqrt{\frac{1}{A^2} - \frac{\ell^2}{B^2r^2}}du \wedge d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^p \wedge d\tilde{z}.
\end{aligned} \tag{2.12}$$

This coordinate is called the Rosen coordinate.

We transform the Rosen coordinate $(u, v, \tilde{x}^a, \tilde{y}^l)$ to the Brinkman coordinate $(x^\pm, x^a, y^l)$

$$\begin{aligned}
u &= x^+, \\
v &= x^- + \frac{\partial_+ A(x^+)}{2A(x^+)}\sum_a (x^a)^2 + \frac{\partial_+(r(x^+)B(x^+)\sin b(x^+))}{2r(x^+)B(x^+)\sin b(x^+)}\sum_l (y^l)^2 \\
&\quad + \frac{\partial_+\sqrt{B(x^+)^2r(x^+)^2 - \ell^2A(x^+)^2}}{2\sqrt{B(x^+)^2r(x^+)^2 - \ell^2A(x^+)^2}}z^2,
\end{aligned}$$

$$
\begin{aligned}
\tilde{x}^a &= \frac{x^a}{A(x^+)}, \\
\tilde{y}^l &= \frac{y^l}{r(x^+)B(x^+)}, \\
\tilde{z} &= \frac{z}{\sqrt{B^2(x^+)r^2(x^+) - \ell^2 A^2(x^+)}}.
\end{aligned} \tag{2.13}
$$

In the Brinkman coordinate, the solution (2.12) becomes

$$
\begin{aligned}
ds^2 &= 2dx^+dx^- + \left(m_x^2 \sum_{a=1}^{p} x_a^2 + m_y^2 \sum_{l=1}^{D-p-3} y_l^2 + m_z^2 z^2 \right) (dx^+)^2 \\
&\quad + ds^2(\mathbf{E}^p) + ds^2(\mathbf{E}^{D-p-3}) + dz^2, \\
m_x^2(x^+) &= \frac{\partial_+^2 A}{A}, \quad m_y^2(x^+) = \frac{\partial_+^2 (rB\sin b)}{rB\sin b}, \\
m_z^2(x^+) &= \frac{\partial_+^2 \sqrt{B^2 r^2 - \ell^2 A^2}}{\sqrt{B^2 r^2 - \ell^2 A^2}}, \\
F_{p+2} &= \pm \frac{dC(r)}{dr} \frac{\ell}{rA^{p+1}B^2} dx^+ \wedge dx^1 \wedge \cdots \wedge dx^p \wedge dz
\end{aligned} \tag{2.14}
$$

where $ds^2(\mathbf{E}^p) = \sum_{a=1}^{p} (dx^a)^2$ and $ds^2(\mathbf{E}^{D-p-3}) = \sum_{l=1}^{D-p-3} (dy^l)^2$.

From (2.9) and (2.14), m_i are written as functions of the coordinate r;

$$
\begin{aligned}
m_x^2 &= \frac{-Q^{-3}\partial_r Q \partial_r A + Q^{-2}\partial_r^2 A}{A}, \\
m_y^2 &= \frac{-Q^{-3}\partial_r Q \partial_r (Br\sin b) + Q^{-2}\partial_r^2 (Br\sin b)}{Br\sin b}, \\
m_z^2 &= \frac{-Q^{-3}\partial_r Q \partial_r \sqrt{B^2 r^2 - \ell^2 A^2} + Q^{-2}\partial_r^2 \sqrt{B^2 r^2 - \ell^2 A^2}}{\sqrt{B^2 r^2 - \ell^2 A^2}}.
\end{aligned} \tag{2.15}
$$

2.2 Plane wave geometry for various brane solutions

We have obtained the plane wave metric for typical p-brane solution. In this subsection, we will write down the explicit forms of the plane wave metric for the Dp-brane ($p \leq 6$), fundamental string, NS5-brane, (p, q) string and (p, q) fivebrane solutions. We also consider the near horizon limit. Since we are interested in the string theory on these backgrounds, the metrics which are discussed below are those in the string frame.

(1) Dp-brane solution

First we discuss the Penrose limit of the Dp-branes. In string frame, the Dp-brane solution takes the form [24]

$$
\begin{aligned}
ds^2 &= H^{-1/2} ds^2(\mathbf{E}^{1,p}) + H^{1/2} ds^2(\mathbf{E}^{9-p}), \\
F_{p+2} &= d\mathrm{vol}(\mathbf{E}^{1,p}) \wedge dH^{-1}, \\
e^{2\phi} &= H^{\frac{3-p}{2}}.
\end{aligned}
\tag{2.16}
$$

$H(r)$ is the harmonic function on $\mathbf{E}^{9-p}$

$$
H = 1 + \frac{Q_p}{r^{7-p}}, \quad Q_p = c_p g_{YM}^2 N (\alpha')^{5-p} \tag{2.17}
$$

where $c_p = 2^{7-2p} \pi^{\frac{9-3p}{2}} \Gamma(\frac{7-p}{2})$ and $g_{YM}^2 = (2\pi)^{p-2} g_s (\alpha')^{\frac{p-3}{2}}$. See e.g. ref. [26].

In this case, $A(r) = H^{-1/4}$, $B(r) = H^{1/4}$ and $C(r) = H^{-1}$. Plugging these into (2.9), we obtain[1]

$$
\begin{aligned}
\frac{db}{dr} &= -\frac{\ell}{r\sqrt{r^2 - \ell^2 + Q_p r^{-5+p}}}, \\
Q(r) = \frac{du}{dr} &= \left(\frac{r^{7-p} + Q_p}{r^{7-p} + Q_p - \ell^2 r^{5-p}} \right)^{\frac{1}{2}}.
\end{aligned}
\tag{2.18}
$$

From (2.14), we find the Penrose limit of the metric and the $(p+2)$-form field strength

$$
\begin{aligned}
ds^2 &= 2dx^+ dx^- + \left(m_x^2 \sum_{a=1}^{p} x_a^2 + m_y^2 \sum_{l=1}^{7-p} y_l^2 + m_z^2 dz^2 \right) (dx^+)^2 + ds^2(\mathbf{E}^8), \\
F_{p+2} &= \ell(7-p) Q_p r^{9-p} (1 + Q_p r^{7-p})^{\frac{p-9}{4}} dx^+ \wedge dx^1 \wedge \cdots \wedge dx^p \wedge dz.
\end{aligned}
\tag{2.19}
$$

The explicit forms of m_i are given by (2.15)

$$
\begin{aligned}
m_x^2 = (7-p) Q_p \Big[&(3-p) Q_p^2 + (3p-13) Q_p \ell^2 r^{5-p} - (29-3p) Q_p r^{7-p} \\
&+ 4(9-p) \ell^2 r^{12-2p} - 4(8-p) r^{14-2p} \Big] \Big/ 16 r^2 (r^{7-p} + Q_p)^3, \\
m_y^2 = (7-p) Q_p \Big[&(3-p) Q_p^2 - (p+1) Q_p \ell^2 r^{5-p} + (27-5p) Q_p r^{7-p} \\
&- 4(9-p) \ell^2 r^{12-2p} + 4(6-p) r^{14-2p} \Big] \Big/ 16 r^2 (r^{7-p} + Q_p)^3,
\end{aligned}
$$

[1] In the following we will choose the upper signs in (2.9).

$$m_z^2 = (7-p)Q_p\Big[(3-p)Q_p^2 + (3p-13)Q_p\ell^2 r^{5-p} + (27-5p)Q_p r^{7-p}$$
$$+4(9-p)\ell^2 r^{12-2p} + 4(6-p)r^{14-2p}\Big]\Big/16r^2(r^{7-p}+Q_p)^3. \qquad (2.20)$$

In the near horizon limit, these m_i become

$$\begin{aligned} m_z^2 = m_x^2 &= \frac{(7-p)\left[(3-p)Q_p + (3p-13)\ell^2 r^{5-p}\right]}{16r^2 Q_p}, \\ m_y^2 &= \frac{(7-p)\left[(3-p)Q_p + (-1-p)\ell^2 r^{5-p}\right]}{16r^2 Q_p}, \end{aligned} \qquad (2.21)$$

and the $(p+2)$-form field strength becomes

$$F_{p+2} = \ell(7-p)Q_p^{\frac{p-5}{4}} r^{(p-3)(p-9)/4} dx^+ \wedge dx^1 \wedge \cdots \wedge dx^p \wedge dz, \qquad (2.22)$$

which agree with the result in ref. [15].

(2) NS5-brane solution

Next we will discuss the Penrose limit of the NS5-branes. The NS5-brane solution is [25]

$$\begin{aligned} & ds^2 = ds^2(\mathbf{E}^{1,5}) + H ds^2(\mathbf{E}^4), \\ & F_7 = d\mathrm{vol}(\mathbf{E}^{1,5}) \wedge dH^{-1}, \quad e^{2\phi} = H, \\ & H = 1 + \frac{Q_5}{r^2}. \end{aligned} \qquad (2.23)$$

Since we are working in string frame, the dual NS-NS 3-form field strength $H_3 = dB_2$ is

$$H_3 = e^{2\phi} * F_7 = 2Q_5 d\mathrm{vol}(S^3) \qquad (2.24)$$

where S^3 is in the transverse space $\mathbf{E}^4$.

In this case, since $A(r) = 1$, $B(r) = H^{1/2}$ and $C(r) = H^{-1}$, du/dr becomes

$$\frac{du}{dr} = Q(r) = \frac{r^2 + Q_5}{r\sqrt{r^2 + Q_5 - \ell^2}}. \qquad (2.25)$$

The Penrose limit of the metric and the NS-NS 3-form field strength H_3 are

$$\begin{aligned} ds^2 &= 2dx^+dx^- + \left(m_x^2 \sum_{a=1}^{5} x_a^2 + m_y^2 \sum_{l=1}^{2} y_l^2 + m_z^2 dz^2\right)(dx^+)^2 + ds^2(\mathbf{E}^8), \\ H_3 &= \frac{2\ell Q_5}{(r^2+Q_5)^2} dx^+ \wedge dy^1 \wedge dy^2 \end{aligned} \qquad (2.26)$$

where

$$
\begin{aligned}
m_x^2 &= 0, \\
m_y^2 &= \frac{Q_5(-Q_5\ell^2 + 2r^2 Q_5 - 4r^2\ell^2 + 2r^4)}{(r^2+Q_5)^4}, \\
m_z^2 &= \frac{2Q_5 r^2}{(r^2+Q_5)^3}.
\end{aligned}
\tag{2.27}
$$

In the near horizon limit, m_i become constants

$$
m_x^2 = 0, \quad m_y^2 = -\frac{\ell^2}{Q_5^2}, \quad m_z^2 = 0,
\tag{2.28}
$$

and the NS-NS 3-form field strength is

$$
H_3 = \frac{2\ell}{Q_5} dx^+ \wedge dy^1 \wedge dy^2.
\tag{2.29}
$$

For $\ell \neq 0$, this plane wave geometry becomes the Nappi-Witten geometry [27] as noted in refs. [28, 29]. Taking the Penrose limit along the radial null geodesic $\ell = 0$, all of m_i become zero and the resulting string theory is linear dilaton theory [3].

(3) Fundamental string solution

The fundamental string solution is [30]

$$
\begin{aligned}
& ds^2 = H^{-1} ds^2(\mathbf{E}^{1,1}) + ds^2(\mathbf{E}^8), \\
& F_3 = d\mathrm{vol}(\mathbf{E}^{1,1}) \wedge dH^{-1}, \quad e^{2\phi} = H^{-1}, \\
& H = 1 + \frac{Q_1}{r^6}.
\end{aligned}
\tag{2.30}
$$

Since $A = H^{-1/2}$, $B = 1$ and $C = H^{-1}$ in this case, we obtain

$$
\frac{du}{dr} = Q(r) = \frac{r^3}{\sqrt{r^6 + Q_1 - \ell^2 r^4}}.
\tag{2.31}
$$

The Penrose limit of the metric and the NS-NS 3-form field strength F_3 are

$$
\begin{aligned}
& ds^2 = 2dx^+ dx^- + \left(m_x^2 x^2 + m_y^2 \sum_{l=1}^{6} y_l^2 + m_z^2 dz^2 \right) (dx^+)^2 + ds^2(\mathbf{E}^8), \\
& F_3 = \frac{6\ell Q_1}{r^2(r^6+Q_1)} dx^+ \wedge dx \wedge dz
\end{aligned}
\tag{2.32}
$$

where m_i are

$$\begin{aligned} m_x^2 &= -3\frac{Q_1(Q_1^2+Q_1\ell^2r^4+8Q_1r^6-8\ell^2r^{10}+7r^{12})}{r^8(r^6+Q_1)^2}, \\ m_y^2 &= -3\frac{Q_1}{r^8}, \\ m_z^2 &= -3\frac{Q_1(Q_1^2+Q_1\ell^2r^4+2Q_1r^6-8\ell^2r^{10}+r^{12})}{r^8(r^6+Q_1)^2}. \end{aligned} \tag{2.33}$$

In the near horizon limit, m_i become

$$m_x^2 = -3\frac{Q_1+\ell^2r^4}{r^8}, \quad m_y^2 = -3\frac{Q_1}{r^8}, \quad m_z^2 = -3\frac{Q_1+\ell^2r^4}{r^8}, \tag{2.34}$$

and the NS-NS 3-form field strength is

$$F_3 = \frac{6\ell}{r^2}dx^+ \wedge dx \wedge dz. \tag{2.35}$$

(4) (p,q) fivebrane solution

Since type IIB superstring theory has S-duality, there is a (p,q) fivebrane solution. The metric and the B-field for the (p,q) fivebranes are derived from those of the NS5-brane solution by acting $SL(2,\mathbf{Z})$ duality transformation in Einstein frame [31, 32]. By pulling back to the string frame, the metric, NS-NS 3-form field strength H_3, RR 3-form field strength F_3, dilaton ϕ and axion χ become

$$\begin{aligned} & ds^2 = h^{-1/2}(-dt^2+\sum_{a=1}^{5}dx_a^2+f(dr^2+r^2d\Omega_3^2)), \\ & f = 1+\frac{\tilde{Q}_5}{r^2}, \quad h^{-1} = \sin^2\gamma f^{-1}+\cos^2\gamma, \\ & H_3 = 2\cos\gamma\tilde{Q}_5 d\mathrm{vol}(S^3), \quad F_3 = 2\sin\gamma\tilde{Q}_5 d\mathrm{vol}(S^3), \\ & e^{2\phi} = fh^{-2}, \quad \chi = h\sin\gamma\cos\gamma(f^{-1}-1) \end{aligned} \tag{2.36}$$

where γ is defined by

$$\cos\gamma = \frac{p}{\sqrt{p^2+q^2}}, \quad \sin\gamma = \frac{q}{\sqrt{p^2+q^2}}. \tag{2.37}$$

When $\gamma = 0$ and $\gamma = \pi/2$, this solution reduces to that of the NS5-branes and the D5-branes, respectively.

In this case, $A(r) = h^{-1/4}$ and $B(r) = h^{-1/4} f^{1/2}$. The equation in (2.9) leads to

$$\frac{du}{dr} = Q(r) = \frac{\sqrt{r^2 + \tilde{Q}_5}\sqrt{r^2 + \cos^2\gamma \tilde{Q}_5}}{r\sqrt{r^2 + \tilde{Q}_5 - \ell^2}}. \tag{2.38}$$

Taking the Penrose limit, the resulting metric and the 3-form field strengths are

$$\begin{aligned} ds^2 &= 2dx^+ dx^- + \left(m_x^2 \sum_{a=1}^{5} x_a^2 + m_y^2 \sum_{l=1}^{2} y_l^2 + m_z^2 dz^2 \right) (dx^+)^2 + ds^2(\mathbf{E}^8), \\ H_3 &= \frac{2\ell(\cos\gamma)\tilde{Q}_5}{(r^2 + \tilde{Q}_5)(r^2 + \tilde{Q}_5 \cos^2\gamma)} dx^+ \wedge dy^1 \wedge dy^2, \\ F_3 &= \frac{2\ell(\sin\gamma)\tilde{Q}_5}{(r^2 + \tilde{Q}_5)(r^2 + \tilde{Q}_5 \cos^2\gamma)} dx^+ \wedge dy^1 \wedge dy^2 \end{aligned} \tag{2.39}$$

where m_i become

$$\begin{aligned} m_x^2(r) = &\frac{\sin^2\gamma \tilde{Q}_5 r^2}{4(r^2 + \tilde{Q}_5)^3 (r^2 + \cos^2\gamma\, \tilde{Q}_5)^3} \Big[4\cos^2\gamma\, \tilde{Q}_5^2 (\tilde{Q}_5 - \ell^2) \\ &+ \{(-1 + 3\cos^2\gamma\,)\tilde{Q}_5^2 + (1 + 3\cos^2\gamma\,)\tilde{Q}_5 \ell^2\} r^2 \\ &+ \{-(7 + \cos^2\gamma\,)\tilde{Q}_5 + 8\ell^2\} r^4 - 6r^6 \Big], \end{aligned} \tag{2.40}$$

$$\begin{aligned} m_y^2(r) = &\frac{\tilde{Q}_5}{4(r^2 + \tilde{Q}_5)^3 (r^2 + \cos^2\gamma\, \tilde{Q}_5)^3} \Big[-4\cos^4\gamma \tilde{Q}_5^3 \ell^2 \\ &+ \{4\cos^2\gamma\,(1 + \cos^2\gamma\,)\tilde{Q}_5^3 - 12\cos^2\gamma\,(1 + \cos^2\gamma\,)\tilde{Q}_5^2 \ell^2\} r^2 \\ &+ \{(-1 + 20\cos^2\gamma\, + 5\cos^4\gamma)\tilde{Q}_5^2 - 3(1 + 10\cos^2\gamma\, + \cos^4\gamma)\tilde{Q}_5 \ell^2\} r^4 \\ &+ \{(1 + 22\cos^2\gamma\, + \cos^4\gamma)\tilde{Q}_5 - 8(1 + \cos^2\gamma\,)\ell^2\} r^6 + 2(1 + 3\cos^2\gamma\,) r^8 \Big], \end{aligned} \tag{2.41}$$

$$\begin{aligned} m_z^2(r) = &\frac{\tilde{Q}_5 r^2}{4(r^2 + \tilde{Q}_5)^3 (r^2 + \cos^2\gamma\, \tilde{Q}_5)^3} \Big[4\cos^2\gamma\,(1 + \cos^2\gamma\,)\tilde{Q}_5^3 - 4\cos^2\gamma\, \sin^2\gamma \tilde{Q}_5^2 \ell^2 \\ &+ \{(-1 + 20\cos^2\gamma\, + 5\cos^4\gamma)\tilde{Q}_5^2 + \sin^2\gamma(1 + 3\cos^2\gamma\,)\tilde{Q}_5 \ell^2\} r^2 \\ &+ \{(1 + 22\cos^2\gamma\, + \cos^4\gamma)\tilde{Q}_5 + 8\sin^2\gamma \ell^2\} r^4 + 2(1 + 3\cos^2\gamma\,) r^6 \Big]. \end{aligned} \tag{2.42}$$

It is easily checked that they reduce to m_i of the NS5-branes and the D5-branes when $\cos\gamma = 1$ and $\cos\gamma = 0$, respectively.

The near horizon limit of the (p, q) fivebrane solution is given by replacing $f = 1 + \tilde{Q}_5/r^2$ with $\tilde{Q}_5/r^2$ in (2.36). Taking the Penrose limit of this background, m_i become

$$m_x^2 \;=\; m_z^2 = \frac{1}{4} \frac{r^2 \sin^2\gamma(\tilde{Q}_5 - \ell^2)(-r^2 \sin^2\gamma + 4\tilde{Q}_5 \cos^2\gamma)}{\tilde{Q}_5 (r^2 \sin^2\gamma + \tilde{Q}_5 \cos^2\gamma)^3},$$

$$m_y^2 = -\frac{1}{4}\frac{4\tilde{Q}_5^2\ell^2\cos^4\gamma - 4\tilde{Q}_5\sin^2\gamma\cos^2\gamma(\tilde{Q}_5 - 3\ell^2)r^2 + \sin^4\gamma(\tilde{Q}_5 + 3\ell^2)r^4}{\tilde{Q}_5(r^2\sin^2\gamma + \tilde{Q}_5\cos^2\gamma)^3}, \tag{2.43}$$

and the 3-form field strengths become

$$H_3 = \frac{2\ell\cos\gamma}{\tilde{Q}_5\cos^2\gamma + r^2\sin^2\gamma}dx^+ \wedge dy^1 \wedge dy^2,$$
$$F_3 = \frac{2\ell\sin\gamma}{\tilde{Q}_5\cos^2\gamma + r^2\sin^2\gamma}dx^+ \wedge dy^1 \wedge dy^2. \tag{2.44}$$

(5) (p,q) string solution

Finally we will consider the (p,q) strings. By $SL(2,\mathbf{Z})$ symmetry of type IIB superstrings, the metric and the B-field for the (p,q) strings are also derived from those of the fundamental string solution in Einstein frame [33]. In string frame, the metric, NS-NS 3-form field strength F_3, RR 3-form field strength H_3, dilaton ϕ and axion χ are

$$\begin{aligned}
&ds^2 = h^{-1/2}(f^{-1}(-dt^2 + dx^2) + dr^2 + r^2 d\Omega_7^2),\\
&f = 1 + \frac{\tilde{Q}_1}{r^6}, \quad h^{-1} = \sin^2\gamma f + \cos^2\gamma,\\
&F_3 = \cos\gamma d\mathrm{vol}(\mathbf{E}^{1,1}) \wedge df^{-1},\ H_3 = \sin\gamma d\mathrm{vol}(\mathbf{E}^{1,1}) \wedge df^{-1},\\
&e^{2\phi} = f^{-1}h^{-2}, \quad \chi = h\sin\gamma\cos\gamma(f-1)
\end{aligned} \tag{2.45}$$

where γ is defined by (2.37) as the (p,q) fivebranes. The charge $\tilde{Q}_1$ is related with the original fundamental string charge Q_1 by $\tilde{Q}_1 = \sqrt{p^2+q^2}Q_1$. When $\gamma = 0$ and $\gamma = \pi/2$, this solution reduces to the fundamental string and the D1-brane solution, respectively.

Since $A(r) = h^{-1/4}f^{-1/2}$ and $B(r) = h^{-1/4}$, we obtain

$$\frac{du}{dr} = Q(r) = \frac{\sqrt{r^6 + \sin^2\gamma\tilde{Q}_1}}{\sqrt{r^6 - \ell^2 r^4 + \tilde{Q}_1}}. \tag{2.46}$$

The Penrose limit of this solution is

$$\begin{aligned}
&ds^2 = 2dx^+dx^- + \left(m_x^2 x^2 + m_y^2\sum_{l=1}^{6} y_l^2 + m_z^2 dz^2\right)(dx^+)^2 + ds^2(\mathbf{E}^8),\\
&F_3 = \frac{6\cos\gamma\ell\tilde{Q}_1 r^4}{(r^6+\tilde{Q}_1)(r^6+\tilde{Q}_1\sin\gamma)}dx^+ \wedge dx \wedge dz,\\
&H_3 = \frac{6\sin\gamma\ell\tilde{Q}_1 r^4}{(r^6+\tilde{Q}_1)(r^6+\tilde{Q}_1\sin\gamma)}dx^+ \wedge dx \wedge dz
\end{aligned} \tag{2.47}$$

where we used $C = \cos\gamma f^{-1}$ and $C = \sin\gamma f^{-1}$ in F_3 and H_3, respectively. In the above equations, m_i are

$$
\begin{aligned}
m_x^2(r) = &-\frac{3\tilde{Q}_1}{4r^2(r^6+\tilde{Q}_1)^2(r^6+\sin^2\gamma\tilde{Q}_1)^3}\Big[-\sin^4\gamma\tilde{Q}_1^4+5\sin^4\gamma\tilde{Q}_1^3\ell^2r^4\\
&-\sin^2\gamma(-11+23\cos^2\gamma)\tilde{Q}_1^3r^6+6\sin^2\gamma(-1+5\cos^2\gamma)\tilde{Q}_1^2\ell^2r^{10}\\
&+(39-60\cos^2\gamma+25\cos^4\gamma)\tilde{Q}_1^2r^{12}+(-27+30\cos^2\gamma+\cos^4\gamma)\tilde{Q}_1\ell^2r^{16}\\
&+(41-10\cos^2\gamma+\cos^4\gamma)\tilde{Q}_1r^{18}-16(1+\cos^2\gamma)\ell^2r^{22}+14(1+\cos^2\gamma)r^{24}\Big],\\
m_y^2(r) = &-\frac{3\tilde{Q}_1}{4r^2(r^6+\sin^2\gamma\tilde{Q}_1)^3}\Big[-\sin^4\gamma\tilde{Q}_1^2+\sin^4\gamma\tilde{Q}_1\ell^2r^4-\sin^2\gamma(11+\cos^2\gamma)\tilde{Q}_1r^6\\
&+16\sin^2\gamma\ell^2r^{10}+(-10+14\cos^2\gamma)r^{12}\Big],\\
m_z^2(r) = &-\frac{3\tilde{Q}_1}{4r^2(r^6+\tilde{Q}_1)^2(r^6+\sin^2\gamma\tilde{Q}_1)^3}\Big[-\sin^4\gamma\tilde{Q}_1^4+5\sin^4\gamma\tilde{Q}_1^3\ell^2r^4\\
&+\sin^2\gamma(-13+\cos^2\gamma)\tilde{Q}_1^3r^6+6\sin^2\gamma(-1+5\cos^2\gamma)\tilde{Q}_1^2\ell^2r^{10}\\
&+(-33+36\cos^2\gamma+\cos^4\gamma)\tilde{Q}_1^2r^{12}+(-27+30\cos^2\gamma+\cos^4\gamma)\tilde{Q}_1\ell^2r^{16}\\
&+(-31+38\cos^2\gamma+\cos^4\gamma)\tilde{Q}_1r^{18}-16(1+\cos^2\gamma)\ell^2r^{22}\\
&+(-10+14\cos^2\gamma)r^{24}\Big].
\end{aligned}
\tag{2.48}
$$

It is also checked that they become the Penrose limit of the fundamental string solution and the D1-brane solution when $\cos\gamma = 1$ and $\cos\gamma = 0$, respectively.

In the near horizon limit, which is obtained by the replacement $f = 1+\tilde{Q}_2/r^6 \to \tilde{Q}_2/r^6$ in (2.45), m_i become

$$
\begin{aligned}
m_x^2 = m_z^2 = &-\frac{3}{4}\frac{1}{r^2(r^6\cos^2\gamma+\tilde{Q}_1\sin^2\gamma)^3}\Big[4\ell^2r^{16}\cos^4\gamma+4\tilde{Q}_1r^{12}\cos^4\gamma\\
&+24\tilde{Q}_1\ell^2r^{10}\cos^2\gamma\sin^2\gamma-12\tilde{Q}_1r^6\cos^2\gamma\sin^2\gamma\\
&+5\tilde{Q}_1^2\ell^2r^4\sin^4\gamma-\tilde{Q}_1^3\sin^4\gamma\Big],\\
m_y^2 = &-\frac{3}{4}\frac{1}{r^2(r^6\cos^2\gamma+\tilde{Q}_1\sin^2\gamma)^3}\tilde{Q}_1\Big[4r^{12}\cos^4\gamma+16\ell^2r^{10}\cos^2\gamma\sin^2\gamma\\
&-12\tilde{Q}_1r^6\cos^2\gamma\sin^2\gamma+\tilde{Q}_1\ell^2r^4\sin^4\gamma-\tilde{Q}_1^2\sin^4\gamma\Big],
\end{aligned}
\tag{2.49}
$$

and the 3-form field strengths are

$$
F_3 = \frac{6\ell r^4\cos\gamma}{r^6\cos^2\gamma+\tilde{Q}_1\sin^2\gamma}dx^+\wedge dx\wedge dz,
$$

$$H_3 = \frac{6\ell r^4 \sin\gamma}{r^6\cos^2\gamma + \tilde{Q}_1 \sin^2\gamma} dx^+ \wedge dx \wedge dz. \tag{2.50}$$

3 Null geodesics and critical radius

In the previous section, we have obtained the Penrose limits of various brane solutions. In this section, we examine the allowed region for the radial coordinate r of a null geodesic with a given parameter ℓ for each brane solution. In particular, we point out the existence of special null geodesics which stay at a fixed ('critical') radius. The Penrose limits along these geodesics are of interest, since, as we will see in section 4, the string theories on these backgrounds have constant masses in the light-cone gauge and are solvable. We will present explicit forms of the plane wave geometry resulting from such Penrose limits.

We begin with the discussion on the geodesics for the Dp-brane solutions. The relation between the affine parameter u and the radial coordinate r is given by (2.18) for the Dp-branes. Since du/dr is necessarily real, the region of r must obey

$$f(r) \equiv r^{7-p} + Q_p - \ell^2 r^{5-p} \geq 0. \tag{3.1}$$

We shall first consider the condition that a null geodesic starting from $r = \infty$ reach the origin $r = 0$. It is given by $f(r) > 0$ throughout $0 \leq r \leq \infty$, that is,

$$f(r_{min}) > 0 \tag{3.2}$$

where $f(r_{min})$ is the minimum of $f(r)$ in the region $0 \leq r \leq \infty$. When $p \leq 4$, as we can see from (3.1), the minimum of $f(r)$ is at

$$r_{min} = \left(\frac{5-p}{7-p}\right)^{1/2} \ell. \tag{3.3}$$

Thus, (3.2) amounts to

$$\ell < (cQ_p)^{\frac{1}{7-p}} \quad \text{where} \quad c = \frac{7-p}{2}\left(\frac{7-p}{5-p}\right)^{\frac{5-p}{2}} \qquad (p \leq 4). \tag{3.4}$$

In the cases of $p = 5, 6$ the minimum of $f(r)$ is at $r = 0$. For $p = 5$, $f(0) = Q_5 - \ell^2$ and the condition (3.2) becomes

$$Q_p > \ell^2. \tag{3.5}$$

For $p = 6$, the condition is

$$\ell = 0. \tag{3.6}$$

If (3.2) is not the case, the equation $f(r) = 0$ have positive root(s). Let us first consider the $p \leq 4$ cases. When (3.4) is not satisfied, there are two positive roots $r_{0\pm}$ ($r_{0+} \geq r_{0-}$). Explicit forms of $r_{0\pm}$ for $p = 1, 3, 4$ are found by solving second or third order equations and given as follows. For $p = 1$, and when $27Q_1/4 \leq \ell^6$, we have

$$\begin{aligned} r_{0\pm} = \Bigg[\frac{\ell^2}{3} - \Bigg\{ \left(1 \pm i\sqrt{3}\right) \left(2\ell^6 - 27Q_1 - i3\sqrt{3}\sqrt{4\ell^6 Q_1 - 27Q_1^2}\right)^{1/3} \\ + \left(1 \mp i\sqrt{3}\right) \left(2\ell^6 - 27Q_1 + i3\sqrt{3}\sqrt{4\ell^6 Q_1 - 27Q_1^2}\right)^{1/3} \Bigg\} \Big/ (6 \cdot 2^{1/3}) \Bigg]^{1/2}. \end{aligned}$$

For $p = 3$, when $4Q_3 \leq \ell^4$,

$$r_{0\pm} = \sqrt{\frac{\ell^2 \pm \sqrt{\ell^4 - 4Q_3}}{2}}.$$

For $p = 4$, when $27Q_4^2/4 \leq \ell^6$,

$$\begin{aligned} r_{0\pm} = \Bigg\{ \left(1 \mp i\sqrt{3}\right) \left(-27Q_4 + i3\sqrt{12\ell^6 - 81Q_4^2}\right)^{1/3} \\ + \left(1 \pm i\sqrt{3}\right) \left(-27Q_4 - i3\sqrt{12\ell^6 - 81Q_4^2}\right)^{1/3} \Bigg\} \Big/ (6 \cdot 2^{1/3}). \end{aligned}$$

For the cases under consideration, a null geodesic which starts at $r = \infty$ can reach only to $r = r_{0+}$, but no nearer to the origin. We may also consider a null geodesic which remain in the region near the origin $0 \leq r \leq r_{0-}$ where $f(r) > 0$. In addition, there are special null geodesics which stay at a certain fixed radius $r = r_{0+}$ or $r = r_{0-}$, since $dr/du = 0$ at these points[2].

In the cases of $p = 5, 6$, on the other hand, the equation $f(r) = 0$ can have only one positive root r_0. For $p = 5$, when $\ell^2 \leq Q_5$, the root is

$$r_0 = \sqrt{\ell^2 - Q_5}. \tag{3.7}$$

[2]To see the behavior of the geodesic around the radius where $dr/du = 0$ more precisely, we need to study the geodesic equations and examine the quantities such as the second derivative with respect to the affine parameter.

and for $p = 6$, when $\ell \neq 0$,

$$r_0 = \frac{\sqrt{Q_6^2 + 4\ell^2} - Q_6}{2}. \tag{3.8}$$

In these cases, a geodesic which starts at $r = \infty$ can reach only to $r = r_0$. We can also consider a geodesic which stays at $r = r_0$. However, there is no geodesic which remain within finite radius near the origin, in contrast to the $p \leq 4$ case.

The conditions for the allowed region of r for the NS and (p, q) fivebranes are the same as the one for the D5-branes (with Q_5 replaced by $\tilde{Q}_5$). It is because the reality conditions of du/dr for these backgrounds are the same as the one for the D5-branes, i.e. (3.1) with $p = 5$, as we can see from (2.25) and (2.38). For the same reason, the conditions for the allowed region of r for the fundamental and (p, q) strings are the same as the one for the D1-branes (with Q_1 replaced by $\tilde{Q}_1$).

When we consider the near-horizon geometries of the brane solutions, there are slight modifications to the above discussion. The derivative du/dr becomes in the near-horizon limit

$$\frac{du}{dr} = \left(\frac{Q_p}{Q_p - \ell^2 r^{5-p}} \right)^{\frac{1}{2}}.$$

For $p \neq 5$, there is one solution of $dr/du = 0$ ($du/dr = \infty$):

$$r_0 = \left(\frac{Q_p}{\ell^2} \right)^{\frac{1}{5-p}} \qquad (p \neq 5). \tag{3.9}$$

The allowed region of the geodesic for $p \leq 4$ is $0 \leq r \leq r_0$. On the other hand, for $p = 6$, the allowed region is $r_0 \leq r$, and the geodesic does not reach the origin (if $\ell \neq 0$). We may also consider a special geodesic which stays at $r = r_0$ for the $p \neq 5$ cases. For $p = 5$, the reality condition for du/dr is $\ell^2 \leq Q_5$, which is independent of the radius. If $\ell^2 < Q_5$, the geodesic covers the whole region of r. If $\ell^2 = Q_5$, the geodesic stays at a fixed radius (which can take an arbitrary value), since we have $dr/du = 0$.

We have seen that there are special null geodesics which stay at a fixed radius, for all the backgrounds in discussion under certain conditions on ℓ and the charge Q_p (or $\tilde{Q}_5$, or $\tilde{Q}_1$). We call the point where $dr/du = 0$ the 'critical radius' and denote by r_0. Precisely, r_0 is defined as the point which satisfy $r^{7-p} + Q_p - \ell^2 r^{5-p} = 0$ for the case of p-brane solutions; and as the point which satisfy $Q_p - \ell^2 r^{5-p} = 0$ when we are considering the near-horizon geometry of p-brane solutions. (r_0 means either r_{0+} or r_{0-} when there are two critical radii.) If we take the Penrose limit along such a geodesic with a fixed radius, the geometry becomes independent of x^+. In other words, the resulting background is the Cahen-Wallach space [34].

We will present explicit forms of the geometries in the Penrose limit along a null geodesic at the critical radius for each brane solution. For the Dp-branes, m_i in (2.20) at the critical radius can be written in the form

$$\begin{aligned} m_x^2 &= \frac{(7-p)Q_p\{(p-7)Q_p + 2\ell^2 r_0^{5-p}\}}{8\ell^4 r_0^{12-2p}}, \\ m_y^2 &= \frac{(7-p)Q_p\{(7-p)Q_p - 6\ell^2 r_0^{5-p}\}}{8\ell^4 r_0^{12-2p}}, \\ m_z^2 &= \frac{(7-p)Q_p\{(-35+5p)Q_p + (30-4p)\ell^2 r_0^{5-p}\}}{8\ell^4 r_0^{12-2p}}. \end{aligned} \tag{3.10}$$

In the near-horizon limit, evaluating m_i in (2.21) with the values of r_0 for $p \neq 5$ given in (3.9), we have

$$m_x^2 = m_z^2 = \frac{(7-p)(p-5)}{8\left(\frac{Q_p}{\ell^2}\right)^{\frac{2}{5-p}}}, \quad m_y^2 = \frac{(7-p)(1-p)}{8\left(\frac{Q_p}{\ell^2}\right)^{\frac{2}{5-p}}}. \tag{3.11}$$

For the D5-brane solution, $dr/du = 0$ if $\ell^2 = Q_5$. In this case, m_i become

$$m_x^2 = m_z^2 = 0, \quad m_y^2 = -\frac{1}{\tilde{r}_0^2} \tag{3.12}$$

where $\tilde{r}_0$ is a constant which can take an arbitrary value.

Next we will discuss the NS5-branes. The critical radius is $r_0 = \sqrt{\ell^2 - Q_5}$ and at this radius, m_i and H_3 are evaluated as

$$\begin{aligned} & m_x^2 = 0, \quad m_y^2 = \frac{Q_5(Q_5 - 2\ell^2)}{\ell^6}, \quad m_z^2 = \frac{2Q_5(\ell^2 - Q_5)}{\ell^6}, \\ & H_3 = \frac{2Q_5}{\ell^3} dx^+ \wedge dy^1 \wedge dy^2. \end{aligned} \tag{3.13}$$

In the near horizon limit, $dr/du = 0$ when $Q_5 = \ell^2$ as in the D5-brane case. In this case, we have $m_x^2 = m_z^2 = 0$, $m_y^2 = -1/\ell^2$ and $H_3 = \frac{2}{\ell} dx^+ \wedge dy^1 \wedge dy^2$.

For the fundamental strings, m_i at the critical radius are given as

$$m_x^2 = -\frac{3Q_1(3Q_1 - \ell^2 r_0^4)}{\ell^2 r_0^{12}}, \quad m_y^2 = -\frac{3Q_1}{r_0^8}, \quad m_z^2 = -\frac{3Q_1(9Q_1 - 7\ell^2 r_0^4)}{\ell^2 r_0^{12}}. \tag{3.14}$$

In the near horizon limit, evaluating m_i at the critical radius $r_0 = (Q_1/\ell^2)^{1/4}$, we obtain $m_x^2 = m_z^2 = -6\ell^4/Q_1$ and $m_y^2 = -3\ell^4/Q_1$. The NS-NS 3-form field strength becomes $F_3 = \frac{6\ell^2}{Q_1^{1/2}} dx^+ \wedge dx \wedge dz$.

For the (p,q) fivebrane solution, m_i at the critical radius $r_0 = \sqrt{\ell^2 - \tilde{Q}_5}$ become

$$
\begin{aligned}
m_x^2 &= \frac{(\ell^2 - \tilde{Q}_5)^2 \tilde{Q}_5 \sin^2\gamma}{2\ell^4(\ell^2 - \tilde{Q}_5 \sin^2\gamma)^2}, \\
m_y^2 &= -\frac{\tilde{Q}_5(7\ell^4 - 6\ell^2\tilde{Q}_5 + \tilde{Q}_5^2 + (\ell^4 + 2\ell^2\tilde{Q}_5 - \tilde{Q}_5^2)\cos 2\gamma)}{4\ell^4(\ell^2 - \tilde{Q}_5\sin^2\gamma)^2}, \\
m_z^2 &= -\frac{(\ell^2 - \tilde{Q}_5)\tilde{Q}_5(-9\ell^2 + 5\tilde{Q}_5 + (\ell^2 - 5\tilde{Q}_5)\cos 2\gamma)}{4\ell^4(\ell^2 - \tilde{Q}_5\sin^2\gamma)^2}.
\end{aligned}
\tag{3.15}
$$

In the near-horizon limit, $dr/du = 0$ when $\tilde{Q}_5 = \ell^2$. The resulting geometry is given by

$$
\begin{aligned}
& m_x^2 = m_z^2 = 0, \quad m_y^2 = -\frac{1}{\tilde{Q}_5\cos^2\gamma + \tilde{r}_0^2\sin^2\gamma}, \\
& H_3 = \frac{2\sqrt{\tilde{Q}_5}\cos\gamma}{\tilde{Q}_5\cos^2\gamma + \tilde{r}_0^2\sin^2\gamma} dx^+ \wedge dy^1 \wedge dy^2
\end{aligned}
\tag{3.16}
$$

where $\tilde{r}_0$ is an arbitrary constant. This case was studied in ref. [23]. If we choose $\tilde{r}_0^2 = \tilde{Q}_5$, the above background agree with the one given in ref. [23].

In the case of the (p,q) string solution, m_i at the critical radius become

$$
\begin{aligned}
m_x^2 &= -\frac{3\tilde{Q}_1 r_0^4}{2\ell^2(r_0^6 + \tilde{Q}_1\sin^2\gamma)^3}\Big(-\tilde{Q}_1\ell^2\cos^2\gamma(-9 + 7\cos^2\gamma) \\
&\quad + 2(3\tilde{Q}_1\cos^4\gamma + \ell^6\sin^4\gamma)r_0^2 - \ell^4(3 - 3\cos^2\gamma + 2\cos^4\gamma)r_0^4\Big), \\
m_y^2 &= \frac{3\tilde{Q}_1(-3 + \cos^2\gamma)r_0^4}{2(r_0^6 + \tilde{Q}_1\sin^2\gamma)^2}, \\
m_z^2 &= -\frac{3\tilde{Q}_1 r_0^4}{2\ell^2(r_0^6 + \tilde{Q}_1\sin^2\gamma)^3}\Big(-\tilde{Q}_1\ell^2\cos^2\gamma(-33 + 19\cos^2\gamma) \\
&\quad + 2(9\tilde{Q}_1\cos^4\gamma + \ell^6\sin^4\gamma)r_0^2 - \ell^4(15 - 3\cos^2\gamma + 2\cos^4\gamma)r_0^4\Big).
\end{aligned}
\tag{3.17}
$$

In the near horizon limit, the critical radius is $r_0 = (\tilde{Q}_1/\ell^2)^{1/4}$ and m_i are given by

$$
\begin{aligned}
m_x^2 &= m_z^2 = -\frac{3\tilde{Q}_1^3\left(2\tilde{Q}_1\cos^4\gamma + 3\ell^4 r_0^2\cos^2\gamma\sin^2\gamma + \ell^6\sin^4\gamma\right)}{\ell^6 r_0^2(r_0^6\cos^2\gamma + \tilde{Q}_1\sin^2\gamma)^3}, \\
m_y^2 &= -\frac{3\tilde{Q}_1^3\cos^2\gamma\left(\tilde{Q}_1\cos^2\gamma + \ell^4 r_0^2\sin^2\gamma\right)}{\ell^6 r_0^2(r_0^6\cos^2\gamma + \tilde{Q}_1\sin^2\gamma)^3}.
\end{aligned}
\tag{3.18}
$$

4 Strings on the Penrose limits of various branes

Having obtained the Penrose limits of the various brane solutions, we shall now study string theories on these backgrounds. In the present paper, we concentrate on the analysis of the bosonic sector. We plan to report on the analysis of the fermionic sector and on the related issues such as the condition on the background for the realization of the world-sheet supersymmetry in the light-cone gauge in a future work. In section 4.1, we give a brief review on string theory on the plane wave background. We derive the equations of motion for the bosonic string in the light-cone gauge, and see that we have massive theories with time-dependent masses in general. Then in section 4.2, we start the study of the equations of motion for each case. We discuss the structures of the singular points of the differential equations, especially. In section 4.3, we point out that in certain circumstances, the equations of motion generically take simple forms. The first example is the theory on the critical radius, for which the masses become constant. Another example is the near-horizon limit. In this limit, the equations of motion for a large class of backgrounds can be solved by the Bessel functions. The case of the fivebranes, for which the exact solutions exist, will be discussed separately in detail in section 5.

4.1 Light-cone string theory on the plane wave background

The sigma-model action for the bosonic part of string theory is given by

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(\sqrt{-h} h^{\alpha\beta} g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu + \epsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \right) \tag{4.1}$$

where $h_{\alpha\beta}$ $(\alpha, \beta = \tau, \sigma)$ is the world-sheet metric and $\epsilon^{\tau\sigma} = +1$. $g_{\mu\nu}$, $B_{\mu\nu}$ $(\mu, \nu = 0, \ldots, 9)$ are the background metric and NS-NS two-form potential, respectively. The RR backgrounds do not couple to the bosonic sector.

The plane wave metrics obtained in section 2 are of the form (2.14):

$$ds^2 = 2dx^+dx^- + \left(m_x^2(x^+) x_a^2 + m_y^2(x^+) y_l^2 + m_z^2(x^+) z^2 \right) (dx^+)^2 + dx_a^2 + dy_l^2 + dz^2.$$

The directions represented by x_a $(a = 1, \ldots, p)$ are the ones which were originally the spatial directions of the world-volume of the p-branes, and those represented by y_l $(l = 1, \ldots, 7 - p)$ come from the angular coordinates of the transverse $(7 - p)$-sphere. The coordinate z comes from the angular coordinate which appears in (2.3). In the following, we frequently denote those x-, y- and z- directions collectively as x^i ($i = 1, \ldots, 8$).

For the fundamental and (p, q) string and the NS and (p, q) fivebrane solutions, there are non-vanishing NS-NS B-fields. As we have seen, the only non-vanishing component of the field strength is F_{+ij}. We take the non-vanishing component of the B-field in the form

$$B_{ij} = B_{ij}(x^+)$$

by choosing the gauge.

We adopt the conformal gauge ($\sqrt{-h}h^{\alpha\beta} = \eta^{\alpha\beta}$) for the world-sheet metric, and substitute the plane wave background into the action. Then, (4.1) becomes

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'} \int d^2\sigma \Big(2\partial_\alpha X^+ \partial^\alpha X^- + \partial_\alpha X^i \partial^\alpha X^i \\ &\qquad + m_i^2(X^+) \partial_\alpha X^+ \partial^\alpha X^+ (X^i)^2 + \epsilon^{\alpha\beta} B_{ij}(X^+) \partial_\alpha X^i \partial_\beta X^j \Big). \end{aligned} \tag{4.2}$$

We can take the light-cone gauge $X^+ = \tau$, since it is consistent with the equation of motion derived from (4.2) [18]. In the light-cone gauge, the fields in the transverse directions X^i are effectively described by the action

$$S = \frac{1}{4\pi\alpha'} \int d^2\sigma \left(-\partial_\tau X^i \partial_\tau X^i + \partial_\sigma X^i \partial_\sigma X^i - m_i^2(\tau)(X^i)^2 + \epsilon^{\alpha\beta} B_{ij}(\tau) \partial_\alpha X^i \partial_\beta X^j \right),$$

which gives the equation of motion

$$\partial_\tau^2 X^i - \partial_\sigma^2 X^i - m_i^2(\tau) X^i - \sum_j \partial_\tau B_{ij}(\tau) \partial_\sigma X^j = 0. \tag{4.3}$$

We have obtained a massive theory whose mass is dependent on the world-sheet time. Note that in our sign convention, $m_i^2 < 0$ is massive and $m_i^2 > 0$ is tachyonic.

Expanding X^i in the Fourier components with respect to σ ($0 \le \sigma \le 2\pi$)

$$X^i(\tau, \sigma) = \sum_{n=-\infty}^{\infty} \left(\alpha_n^i \varphi_n^i(\tau) + \tilde{\alpha}_{-n}^i \tilde{\varphi}_{-n}^i(\tau) \right) e^{in\sigma}, \tag{4.4}$$

we obtain the equations of motion for the functions φ_n^i:

$$\frac{d^2}{d\tau^2} \varphi_n^i + n^2 \varphi_n^i - m_i^2(\tau) \varphi_n^i - in \sum_j \partial_\tau B_{ij}(\tau) \varphi_n^j = 0. \tag{4.5}$$

$\tilde{\varphi}^i_{-n}$ satisfy the same equations of motion.

In the remaining part of the paper, we analyze the equation of motion (4.5) for the plane wave background for each brane solution. Though the main aim of our paper is to find solutions of the equation, we would like to make a few comments on the quantization of the string here. By imposing the canonical commutation relations on the fields X^i and the conjugate momenta, we obtain the oscillator commutation relations for α^i_n and $\tilde{\alpha}^i_n$, *provided* that the functions φ_n and $\tilde{\varphi}_n$ are the solutions of the equation of motion. In contrast to the string theory on the flat background, the light-cone Hamiltonian does not commute with the number operator, for the case of the time-dependent plane wave background. This situation will be illustrated in the appendix when we give an example of the mode expansion using Hankel functions as a basis.

We should also comment on the dilaton coupling. That coupling is not taken into account in this paper, that is, we study the propagation of a free string. When we are considering the brane solutions which have finite dilaton background throughout the spacetime, and if we set the dilaton v.e.v. at infinity to a small value, this gives a good description. However, when we discuss the branes with the dilaton which diverges at the origin (the Dp-branes with $p \leq 2$, the (p,q) fivebranes other than the D5-branes, and (p,q) strings other than the fundamental strings), care is needed. If we consider a geodesic which passes the origin in such backgrounds, the effect of the interaction will become important.

4.2 Equations of motion for each brane solution

The equations of motion of the string on the Penrose limits of the brane solutions are given by substituting m_i^2 and B_{ij} obtained in section 2 into the general form (4.5).

(1) Dp-branes

We first discuss the case of the Dp-branes. We have $B_{ij} = 0$, and the equation of motion is given by

$$\left(\frac{d^2}{d\tau^2} + n^2 - m_i^2(\tau) \right) \varphi_n = 0. \tag{4.6}$$

The masses $m_i^2(\tau)$ are functions of τ through the radial coordinate $r(\tau)$. The relation between r and τ is given in the differential form as

$$\frac{d\tau}{dr} \equiv Q(r) = \left(\frac{r^{7-p} + Q_p}{r^{7-p} + Q_p - \ell^2 r^{5-p}} \right)^{\frac{1}{2}}. \tag{4.7}$$

Note that this is obtained from (2.18) by simply replacing u with τ, because we are working in the light-cone gauge $\tau = x^+ = u$.

To find solutions of the differential equation, it is convenient to take r as an independent variable instead of τ. The equation (4.6) is rewritten as

$$\left(\frac{d^2}{dr^2} - \frac{1}{Q(r)} \frac{dQ(r)}{dr} \frac{d}{dr} + (n^2 - m_i^2(r)) Q^2(r) \right) \varphi_n = 0. \tag{4.8}$$

For the case of the Dp-branes with odd p ($p = 2k+1$ where $k = 0, 1, 2$), we will use $x = r^2$ as the independent variable. The equation of motion is transformed into

$$\left(\frac{d^2}{dx^2} + p(x) \frac{d}{dx} + q(x) \right) \varphi_n = 0 \tag{4.9}$$

where

$$p(x) = \frac{1}{2x} - \frac{1}{Q} \frac{dQ}{dx}, \tag{4.10}$$

$$q(x) = \frac{1}{4x} (n^2 - m_i^2) Q^2. \tag{4.11}$$

Let us examine the structure of the singular points of the differential equation for the odd p case (4.9). A singular point x_0 is the (complex) value of x where $p(x)$ or $q(x)$ diverge as $x \to x_0$. If $p(x) = O((x - x_0)^{-1})$ and $q(x) = O((x - x_0)^{-2})$, x_0 is called a regular singular point. If this is not the case, x_0 is called an irregular singular point. For our case, the coefficient $p(x)$ is obtained using (4.7) as

$$p(x) = \frac{1}{2x} - \frac{(3-k)x^{2-k}}{2(x^{3-k} + Q_p)} + \frac{(3-k)x^{2-k} - (2-k)\ell^2 x^{1-k}}{2(x^{3-k} + Q_p - \ell^2 x^{2-k})}. \tag{4.12}$$

The coefficient $q(x)$ for the cases of the x-, y- or z-directions are calculated by substituting m_x^2, m_y^2 or m_z^2 in (4.11), respectively.

For the D5-branes ($k = 2$), (4.9) has four singular points. As we can see from the denominators of $p(x)$ and $q(x)$, we have regular singular points at $x = 0, Q_p, \ell^2 - Q_p$. There is an irregular singular point at $x = \infty$, which can be seen by transforming the variable

x to $1/x$. As k decreases, the number of the singular points increases. For the D3-branes ($k=1$), we have regular singular points at $x = 0, \pm i\sqrt{Q_p}, (\ell^2 \pm \sqrt{\ell^2 - 4Q_p})/2$. For the D1-branes ($k=0$), we have regular singular points at 0, the three roots of $x^3 + Q_p = 0$, and the three roots of $x^3 + Q_p - \ell^2 x^2 = 0$. The irregular singular point at $x = \infty$ is present for all the cases.

For Dp-branes with even p, use of the above x does not lead to rational expressions, and we use the variable r and examine the differential equation (4.8). The coefficient of $d\varphi_n/dr$ is given by

$$-\frac{1}{Q(r)}\frac{dQ(r)}{dr} = -\frac{(7-p)r^{6-p}}{2(r^{7-p}+Q_p)} + \frac{(7-p)r^{6-p} - (5-p)\ell^2 r^{4-p}}{2(r^{7-p}+Q_p-\ell^2 x^{5-p})}. \tag{4.13}$$

Generically, there are regular singular points at 0, the roots of $r^{7-p} + Q_p = 0$, and the roots of $r^{7-p} + Q_p - \ell^2 r^{5-p} = 0$, and an irregular singular point at $r = \infty$.

In general, differential equations having fewer number of singular points are more tractable. (See e.g. ref.[35].) For the differential equations for the D5-branes, we can find non-trivial solutions as we see in section 5. It should also be noted that it may be possible to reduce the number of the singular points by the change of variables or by factoring out some function from φ_n. Indeed, the equation in the y-directions for the D5-branes can be transformed to the one which has only two regular singular points at $x = 0, \ell^2 - Q_p$ and an irregular singular point at $x = \infty$, as we discuss in section 5.

We comment here that the differential equations in the y- and z-directions for the D6-branes in the $\ell = 0$ case can be transformed into the form which is found in the literature (eq. VII, p. 503 of ref. [35]). The properties of the solution of that equation have been investigated to some extent [36], but we do not discuss this case any further in this paper.

(2) NS5-branes and (p,q) fivebranes

In these cases, we have non-vanishing NS-NS B-fields in the two y-directions. Let us consider the NS5-brane case. The equations of motion (4.3) for the y-directions are coupled equations

$$\begin{aligned} &\partial_\tau^2 \varphi_n^1 + n^2 \varphi_n^1 - m_y^2(\tau)\varphi_n^1 - in\partial_\tau B(\tau)\varphi_n^2 = 0, \\ &\partial_\tau^2 \varphi_n^2 + n^2 \varphi_n^2 - m_y^2(\tau)\varphi_n^2 + in\partial_\tau B(\tau)\varphi_n^1 = 0, \end{aligned} \tag{4.14}$$

where $B(\tau) \equiv B_{12} = -B_{21}$ and the indices $1, 2$ denote the two y-directions. Explicit form of $\partial_\tau B$ for the NS5-branes is given by

$$\partial_\tau B(\tau) = \partial_{x^+} B_{12}(x^+) = H_{+12} = \frac{2\ell Q_5}{(r^2 + Q_5)^2}. \tag{4.15}$$

Note that we have used the light-cone gauge condition $\tau = x^+$ in the first equality. The equation (4.14) is diagonalized by the combination $\varphi_n^{(\pm)} = \varphi_n^1 \mp i\varphi_n^2$:

$$\Big(\partial_\tau^2 + n^2 - m_y^2(\tau) \pm n\partial_\tau B(\tau)\Big)\varphi_n^{(\pm)} = 0. \tag{4.16}$$

We use x as the independent variable and analyze the differential equation of the form (4.9). The coefficient $p(x)$ which is given by the formula (4.10) is evaluated as

$$p(x) = \frac{1}{x} + \frac{-1}{x + Q_5} + \frac{\frac{1}{2}}{x + Q_5 - \ell^2} \tag{4.17}$$

using $Q(r)$ obtained in (2.25). The formulae for $q(x)$ for $\varphi_n^{(\pm)}$ are modified to

$$q(x) = \frac{1}{4x}(n^2 - m_y^2 \pm \partial_\tau B)Q^2. \tag{4.18}$$

It is evaluated using m_y^2 and $\partial_\tau B$ given in (2.27) and (4.15), respectively. As for the x- and z-directions, the differential equation is given by (4.6). By using x, we obtain the differential equation with $p(x)$ given by (4.17), $q(x)$ given by (4.11) with the substitution of m_x^2 and m_z^2 which are found in (2.27). Note that the equation of motion for x-directions is the one for the flat background, since $m_x^2 = 0$. The differential equations have three regular singular points at $x = 0, -Q_5, \ell^2 - Q_5$, and an irregular singular point at $x = \infty$.

The (p, q) fivebrane case can be analyzed exactly in the same way as for the NS5-brane case. We have the coupled equations (4.14) where

$$\partial_\tau B(\tau) = \frac{2\tilde{Q}_5 \ell \cos\gamma}{(r^2 + \tilde{Q}_5)(r^2 + \tilde{Q}_5 \cos^2\gamma)}. \tag{4.19}$$

The coefficient $p(x)$ is evaluated using $Q(r)$ in (2.38) and is given by

$$p(x) = \frac{1}{x} + \frac{-\frac{1}{2}}{x + \tilde{Q}_5} + \frac{\frac{1}{2}}{x + \tilde{Q}_5 - \ell^2} + \frac{-\frac{1}{2}}{x + \cos^2\gamma \tilde{Q}_5}. \tag{4.20}$$

The differential equations have four regular singular points at $x = 0, -\tilde{Q}_5, \ell^2 - \tilde{Q}_5, -\cos^2\gamma\tilde{Q}_5$, and an irregular singular point at $x = \infty$. The differential equations for the y-directions for the NS5-branes (or the (p,q) fivebranes) can be transformed to the ones which have only two regular singular points $x = 0, \ell^2 - Q_5$ (or $\ell^2 - \tilde{Q}_5$) and an irregular singular point at $x = \infty$, as in the case of the D5-branes.

(3) fundamental strings and (p,q) strings

The non-vanishing component of the B-field is B_{xz} in these cases. The equations of motion in the x- and z-directions are coupled equations

$$\begin{aligned} &\partial_\tau^2 \varphi_n^x + n^2 \varphi_n^x - m_x^2(\tau)\varphi_n^x - in\partial_\tau B(\tau)\varphi_n^z = 0, \\ &\partial_\tau^2 \varphi_n^z + n^2 \varphi_n^z - m_z^2(\tau)\varphi_n^z + in\partial_\tau B(\tau)\varphi_n^x = 0, \end{aligned} \tag{4.21}$$

where $B(\tau) \equiv B_{xz} = -B_{zx}$. Note that $m_x^2 \neq m_z^2$ for the present cases.

The B-field for the fundamental strings is given by

$$\partial_\tau B(\tau) = \partial_+ B_{xz}(x^+) = F_{+xz} = \frac{6\ell Q_1}{r^2(r^6 + Q_1)} \tag{4.22}$$

and the masses m_i^2 are given in (2.33). Diagonalizing (4.21), the equations of motion in the x- and z-directions become

$$\left(\partial_\tau^2 + n^2 - \bar{m}^2(\tau) \pm \Delta m^2\right)\varphi_n^{(\pm)} = 0 \tag{4.23}$$

where

$$\bar{m}^2 \equiv \frac{m_x^2 + m_z^2}{2} = -3\frac{Q_1(Q_1^2 + Q_1\ell^2 r^4 + 5Q_1 r^6 - 8\ell^2 r^{10} + 4r^{12})}{r^8(r^6 + Q_1)^2}, \tag{4.24}$$

$$\Delta m^2 \equiv \frac{3Q_1}{r^2(r^6 + Q_1)}\sqrt{4n^2\ell^2 + 9}. \tag{4.25}$$

The differential equations can be analyzed using x as the independent variable. Number of the singular points is the same as the one for the D1-brane case.

The B-field for the (p,q) strings is given by

$$\partial_\tau B(\tau) = \frac{6\ell\tilde{Q}_1 \cos\gamma r^4}{(r^6 + \tilde{Q}_1)(r^6 + \sin^2\gamma\tilde{Q}_1)} \tag{4.26}$$

and the masses are given in (2.48). Diagonalization of the coupled equations is performed as in the case of the fundamental string and gives (4.23) with

$$\bar{m}^2 \equiv \frac{m_x^2 + m_z^2}{2}, \tag{4.27}$$

$$\Delta m \equiv \frac{3\tilde{Q}_1 r^4}{(r^6 + \tilde{Q}_1)(r^6 + \sin^2\gamma\tilde{Q}_1)}\sqrt{4n^2\ell^2\cos^2\gamma + 9}. \tag{4.28}$$

The differential equations for the (p, q) strings have extra singular points at the roots of $x^3 + \sin^2\gamma\tilde{Q}_1 = 0$, compared to the case of the fundamental strings.

4.3 Analysis of the simply solvable theories

As we mentioned in section 3, it is possible to consider a geodesic for which r is kept on a fixed radius $r = r_0$. The Penrose limit along such a geodesic gives the string theory with constant mass. The equation of motion (for the $B_{ij} = 0$ case)

$$\left(\frac{d^2}{d\tau^2} + n^2 - m_i^2(r_0)\right)\varphi_n = 0 \tag{4.29}$$

is easily solved by

$$\begin{aligned} &\varphi_n = e^{i\omega_n\tau}, \qquad \tilde{\varphi}_{-n} = e^{-i\omega_n\tau}, \\ &\omega_n = \sqrt{n^2 - m_i^2(r_0)}. \end{aligned} \tag{4.30}$$

When $B_{ij} \neq 0$, we can start from the diagonalized equation of the forms (4.16) and (4.21) and find similar solutions.

Quantization of the strings should be performed following the standard procedure. (See e.g. ref.[6].) We note here that there are some cases for which $m_i^2(r_0) > 0$. It seems that tachyonic states are present in the resulting theories. Whether we can construct physically meaningful string theories for such cases is not clear to us at present.[3]

In the near-horizon limit

$$r \ll Q_p^{\frac{1}{7-p}}, \tag{4.31}$$

the string theory also becomes simple. The masses in the near-horizon limit for the case of the Dp-branes are given in (2.21), while the relation between τ and r becomes

$$\frac{d\tau}{dr} = \left(\frac{Q_p}{Q_p - \ell^2 r^{5-p}}\right)^{\frac{1}{2}}. \tag{4.32}$$

Firstly, for the case of D3-branes, the masses become constant, as mentioned in section 2. Next, for $p = 5$, (4.32) states that r is proportional to τ, and the masses become

$$m_x^2 = m_z^2 = \frac{1}{4\tau^2},$$

[3]Some discussions on the tachyonic mass from the standpoint of the holographic renormalization group are given in ref.[15].

$$m_y^2 = -\frac{3\ell^2 + Q_p}{Q_p - \ell^2}\frac{1}{\tau^2}. \tag{4.33}$$

The equations of motion take the form

$$\left(\frac{d^2}{d\tau^2} + n^2 - \frac{C_1}{\tau^2}\right)\varphi_n = 0 \tag{4.34}$$

where C_1 is a constant which can be read off from (4.33). The equation (4.34) can be transformed to the Bessel equation (when $n \neq 0$)

$$\left(\frac{d^2}{d\tau'^2} + \frac{1}{\tau'}\frac{d}{d\tau'} + 1 - \frac{\nu^2}{\tau'^2}\right)\tau'^{-\frac{1}{2}}\varphi_n = 0 \tag{4.35}$$

where $\tau' \equiv n\tau$ and $\nu^2 \equiv C_1 + 1/4$.

For the cases $p = 0, 1, 2, 4$, we take the limit

$$r \ll \left(\frac{Q_p}{\ell^2}\right)^{\frac{1}{5-p}} \qquad (p = 0, 1, 2, 4), \tag{4.36}$$

in addition to the near-horizon limit. Then, the second terms in the denominator of (2.21) can be neglected, and the relation between τ and r (4.32) become $\tau = r$. Note that if we take the radial null geodesic ($\ell = 0$), (4.36) is always satisfied. In this limit, the masses become

$$m_x^2 = m_y^2 = m_z^2 = \frac{(7-p)(3-p)}{16\tau^2} \tag{4.37}$$

and we obtain the equations of motion of the form (4.34).

In the appendix, we present a preliminary study for the canonical quantization of the string on this kind of plane wave background. Using the solutions of the equation of motion as a basis, we construct oscillators and obtain the light-cone Hamiltonian.

5 Solutions for the fivebrane backgrounds

In this section, we study string theories on the Penrose limit of the D5-brane, NS5-brane and (p, q) fivebrane solutions. These backgrounds are of particular interest, because we can solve the equations of motion of strings in the two regions: near the critical radius, and far from the origin. We study the former limit in section 5.1 and the latter limit in section 5.2. Furthermore, for a particular component, the equations of motion can be solved exactly. In section 5.3, we describe the exact solutions which are written by a special function called the spheroidal wave function.

5.1 Solutions near the critical radius

We firstly consider the D5-brane case. As discussed in section 4, the equation of motion of the string leads to the Schrödinger type equation of the form

$$\left(\frac{d^2}{d\tau^2} + n^2 - m_i(\tau)^2\right)\varphi_n(\tau) = 0. \tag{5.1}$$

Explicit forms of m_i^2 are given by the $p = 5$ case of (2.20):

$$m_x^2 = -\frac{1}{4}\frac{Q_5(Q_5^2 - Q_5\ell^2 + (7Q_5 - 8\ell^2)r^2 + 6r^4)}{r^2(r^2 + Q_5)^3}, \tag{5.2}$$

$$m_y^2 = -\frac{1}{4}\frac{Q_5(Q_5^2 + 3Q_5\ell^2 + (-Q_5 + 8\ell^2)r^2 - 2r^4)}{r^2(r^2 + Q_5)^3}, \tag{5.3}$$

$$m_z^2 = -\frac{1}{4}\frac{Q_5(Q_5^2 - Q_5\ell^2 - (Q_5 + 8\ell^2)r^2 - 2r^4)}{r^2(r^2 + Q_5)^3}. \tag{5.4}$$

The relation between τ and r is given from (2.18) as

$$\frac{d\tau}{dr} = Q(r) = \left(\frac{r^2 + Q_5}{r^2 + Q_5 - \ell^2}\right)^{\frac{1}{2}}. \tag{5.5}$$

As we have seen in section 3, if $\ell^2 > Q_5$, the geodesic in the fivebrane background stops at a critical radius $r_0 = \sqrt{\ell^2 - Q_5}$. We consider this kind of geodesics here, and study the theory near $r = r_0$.

Taking the leading term in the expansion of (5.5) around $r = r_0$, we note that the relation between τ and r becomes

$$\tau = \int dr \frac{\ell}{\sqrt{r^2 - r_0^2}} = \ell \,\mathrm{arccosh}\frac{r}{r_0},$$
$$r = r_0 \cosh\frac{\tau}{\ell}. \tag{5.6}$$

Now we expand the masses around $r = r_0$ and take the leading correction to the constant mass

$$m_i^2(r) = m_i^2(r_0) + \frac{\partial m_i^2(r_0)}{\partial r^2}(r^2 - r_0^2). \tag{5.7}$$

Substituting this form of mass term and using the relation (5.6), we find that the equation of motion (5.1) takes the form of the (modified) Mathieu equation [37]

$$\left(\frac{d^2}{dz^2} - h + 2\theta \cosh 2z\right)\varphi_n = 0. \tag{5.8}$$

where $z \equiv \tau/\ell$. The parameters h and θ are given by

$$\begin{aligned} h &= -\ell^2 \left(n^2 - m_i^2(r_0) + \frac{r_0^2}{2} \frac{\partial m_i^2(r_0)}{\partial r^2} \right), \\ \theta &= -\frac{r_0^2 \ell^2}{4} \frac{\partial m_i^2(r_0)}{\partial r^2}. \end{aligned} \tag{5.9}$$

When evaluated using the mass terms (5.2)-(5.4), they become

$$\begin{aligned} & h = -\ell^2 n^2 + \frac{Q_5(16\ell^2 - 11Q_5)}{8\ell^4}, && \theta = \frac{Q_5(12\ell^2 - 11Q_5)}{16\ell^4} && (x\text{-directions}), \\ & h = -\ell^2 n^2 - \frac{Q_5(32\ell^4 - 29Q_5\ell^2 + 9Q_5^2)}{8\ell^4 r_0^2}, && \theta = -\frac{Q_5(20\ell^4 - 25Q_5\ell^2 + 9Q_5^2)}{16\ell^4 r_0^2} && (y\text{-directions}), \\ & h = -\ell^2 n^2 + \frac{3Q_5(16\ell^2 - 9Q_5)}{8\ell^4}, && \theta = \frac{Q_5(28\ell^2 - 27Q_5)}{16\ell^4} && (z\text{-direction}). \end{aligned} \tag{5.10}$$

Next we consider the NS5-branes. The equations of motion in the x- and z-directions are given by (5.1) and the ones for the y-directions are

$$\left(\partial_\tau^2 + n^2 - m_y(\tau)^2 \pm n\partial_\tau B(\tau) \right) \varphi_n^{(\pm)} = 0. \tag{5.11}$$

The relation between τ and r is

$$\frac{d\tau}{dr} = \frac{r^2 + Q_5}{r\sqrt{r^2 + Q_5 - \ell^2}}, \tag{5.12}$$

which leads to

$$r = r_0 \cosh \frac{r_0 \tau}{\ell^2} \tag{5.13}$$

in the $r \to r_0$ limit. Expanding the masses and $\partial_\tau B$ in (2.27) and (4.15) around $r = r_0$, and substituting them into the equations of motion, we obtain the modified Mathieu equation (5.8) with parameters:

$$\begin{aligned} & h = -\frac{\ell^4 n^2}{r_0^2} - \frac{Q_5(6\ell^4 - 8Q_5\ell^2 + 3Q_5^2 \mp 2n\ell^3(2\ell^2 - Q_5))}{\ell^4 r_0^2}, \quad \theta = -\frac{Q_5(4\ell^2 - 3Q_5 \mp 2n\ell^3)}{2\ell^4} \\ & \qquad \text{for } \varphi_n^{(\pm)} \ (y\text{-directions}), \\ & h = -\frac{\ell^4 n^2}{r_0^2} + \frac{Q_5(4\ell^2 - 3Q_5)}{\ell^4}, \quad \theta = \frac{Q_5(2\ell^2 - 3Q_5)}{2\ell^4} \qquad (z\text{-direction}) \end{aligned} \tag{5.14}$$

Since $m_x^2 = 0$, the equations in the x-directions are the same as the ones for the flat space. The equations of motion for the (p, q) fivebrane case can be brought to the modified Mathieu equation, following the same steps as above.

The solutions to the equations of motion is given by the Mathieu functions. Since the theory which we are considering is an approximation around a point in the spacetime, it is not clear what kind of boundary conditions should be imposed on the solutions, as it stands. It would be interesting to study the solutions which match the ones in the $r \to \infty$ region which are given in the next subsection. It should enable us to investigate the whole region of the geodesic. This problem will be discussed elsewhere.

5.2 Asymptotic solutions at infinity

We shall now study the $r \to \infty$ limit. First consider the D5-brane example. As we see from (2.20), the mass terms for all the directions are proportional to $1/r^4$ asymptotically

$$m_i^2 \to -\frac{C_2}{r^4} \tag{5.15}$$

where the constant C_2 is given by

$$C_2 = \frac{3}{2}Q_5 \quad (x\text{-directions}), \qquad C_2 = -\frac{1}{2}Q_5 \quad (y\text{- and } z\text{-directions}). \tag{5.16}$$

Moreover, we have $\tau = r$, as we can see from the $r \to \infty$ limit of (2.18).

By using $x = r^2 = \tau^2$ as the independent variable, the equation of motion (5.1) reads

$$\left(\frac{d^2}{dx^2} + \frac{1}{2x}\frac{d}{dx} + \frac{n^2}{4x} + \frac{C_2}{4x^3} \right) \varphi_n = 0. \tag{5.17}$$

This differential equation has two irregular singular points at $x = 0$ and $x = \infty$. This equation is transformed to the standard form [35] by the redefinitions $\psi_n = x^{-1/4}\varphi_n$ and $z = nx/\sqrt{C_2}$:

$$\left(z^2\frac{d^2}{dz^2} + z\frac{d}{dz} - \frac{1}{4}\{h - \theta(z + \frac{1}{z})\} \right) \psi_n = 0, \tag{5.18}$$

where

$$h = \frac{1}{4}, \qquad \theta = \sqrt{C_2}n. \tag{5.19}$$

By the substitution $z = e^{2iw}$, (5.18) becomes the Mathieu equation [37]

$$\left(\frac{d^2}{dw^2} + h - 2\theta \cos 2w \right) \psi_n = 0. \tag{5.20}$$

For the cases of the NS5-branes or the (p,q) fivebranes, the discussion is essentially the same. Asymptotic behaviors of the masses m_x^2 and m_z^2 for the (p,q) fivebranes are given by (5.15) with

$$C_2 = \frac{3}{2}\sin^2\gamma \tilde{Q}_5 \quad (x\text{-directions}), \qquad C_2 = -\frac{1}{2}(1+3\cos^2\gamma)\tilde{Q}_5 \quad (z\text{-directions}). \quad (5.21)$$

The differential equations for these directions are brought to the Mathieu equations with the C_2 defined above.

We also note that the asymptotic forms of m_y^2 and $\partial_\tau B$ are

$$m_y^2 \to \frac{(1+3\cos^2\gamma)\tilde{Q}_5}{2r^4}, \qquad \partial_\tau B \to \frac{2\ell\cos\gamma \tilde{Q}_5}{r^4}. \quad (5.22)$$

The equations of motion for $\varphi_n^{(\pm)}$ in the y-directions are given by replacing m_y^2 with $m_y^2 \mp n\partial_\tau B$. In this case, we obtain the Mathieu equations with

$$C_2 = -\frac{1}{2}(1+3\cos^2\gamma)\tilde{Q}_5 \pm 2n\ell\cos\gamma\tilde{Q}_5 \qquad \text{for } \varphi_n^{(\pm)} \ (y\text{-directions}). \quad (5.23)$$

The NS5-brane case is given by setting $\cos\gamma = 1$ (and replacing $\tilde{Q}_5$ with Q_5) in the above expressions.

5.3 Exact solutions

For the equation of motion in the y-directions, we can find the exact solutions. First, we consider the D5-brane case. The equation of motion is given by (4.9)-(4.11) with m_y^2 in (5.3) and $Q(r)$ in (5.5). With the substitution $\varphi_n(x) = f(x)\phi_n(x)$, we obtain

$$\left(\frac{d^2}{dx^2} + \tilde{p}(x)\frac{d}{dx} + \tilde{q}(x)\right)\phi_n(x) = 0, \quad (5.24)$$

where

$$\begin{aligned}\tilde{p}(x) &= p(x) + \frac{2f'(x)}{f(x)}, \\ \tilde{q}(x) &= q(x) + \frac{f'(x)}{f(x)}p(x) + \frac{f''(x)}{f(x)}.\end{aligned} \quad (5.25)$$

Since $p(x)$ and $q(x)$ have the singular points at $x = 0$, $-Q_5$ and $-Q_5 + \ell^2$, we will use $f(x) = x^\alpha(x+Q_5)^\beta(x+Q_5-\ell^2)^\gamma$. Then $\tilde{q}(x)$ takes the form $q_1(x)/(x^2(x+Q_5)^2(x+Q_5-$

$\ell^2)^2)$, where $q_1(x)$ is the fifth order polynomial in x. We choose the exponents (α, β, γ) such that $q_1(x) = 0$ for $x = 0$, $-Q_5$ and $-Q_5 + \ell^2$. These conditions determine the exponents as follows:

$$\begin{aligned}
\alpha &= \alpha_\pm \equiv \frac{1}{4} \pm \frac{1}{2}\left(\frac{\ell^2}{\ell^2 - Q_5}\right)^{\frac{1}{2}}, \\
\beta &= \frac{1}{4}, \frac{5}{4}, \\
\gamma &= 0, \frac{1}{2}.
\end{aligned} \tag{5.26}$$

The choice $(\alpha, \beta, \gamma) = (\alpha_+, \frac{1}{4}, 0)$ takes the functions $\tilde{p}(x)$ and $\tilde{q}(x)$ into simpler forms:

$$\begin{aligned}
\tilde{p}(x) &= \frac{\frac{1}{2} + 2\alpha_+}{x} + \frac{\frac{1}{2}}{x + Q_5 - \ell^2}, \\
\tilde{q}(x) &= \frac{1}{4}\frac{n^2 Q_5(\ell^2 - Q_5) + \ell^2 + \sqrt{\ell^2(\ell^2 - Q_5)} + n^2(\ell^2 - Q_5)x}{x(x + Q_5 - \ell^2)(\ell^2 - Q_5)}.
\end{aligned} \tag{5.27}$$

We notice that the present form of the differential equation (5.24) with (5.27) is nothing but the associated Mathieu equation [35]. In fact, by the substitution $x = (Q_5 - \ell^2)(z - 1)$, we get

$$\frac{d^2\phi_n}{dz^2} + \left\{\frac{\frac{1}{2}}{z} + \frac{1 - r}{z - 1}\right\}\frac{d\phi_n}{dz} - \frac{a + k^2 z}{4z(z-1)}\phi_n = 0, \tag{5.28}$$

where

$$\begin{aligned}
r &= \frac{1}{2} - 2\alpha_+ = -\left(\frac{\ell^2}{\ell^2 - Q_5}\right)^{\frac{1}{2}}, \\
k^2 &= n^2(\ell^2 - Q_5), \qquad (5.29) \\
a &= \frac{-n^2\ell^2(\ell^2 - Q_5) - \ell^2 - \sqrt{\ell^2(\ell^2 - Q_5)}}{\ell^2 - Q_5}. \qquad (5.30)
\end{aligned}$$

It is known that the associated Mathieu equation (5.28) is equivalent to the differential equation for the spheroidal wave function [37] by the transformation $w = z^{\frac{1}{2}}$ and $\phi_n = (1 - w^2)^{\frac{r}{2}} f_n$:

$$(1 - w^2)\frac{d^2 f_n}{dw^2} - 2w\frac{df_n}{dw} + \left(\lambda + 4\theta(1 - w^2) - \frac{\mu^2}{1 - w^2}\right)f_n = 0, \tag{5.31}$$

where

$$\lambda = a + r(r-1) + k^2 = -n^2 Q_5,$$

$$\begin{aligned}\theta &= -\frac{k^2}{4} = -\frac{n^2(\ell^2 - Q_5)}{4}, \\ \mu &= r = -\sqrt{\frac{\ell^2}{\ell^2 - Q_5}}.\end{aligned} \tag{5.32}$$

The regular singular points $x = \ell^2 - Q_5$ and $x = 0$ correspond to $w = 0$ and $w = \pm 1$, respectively. The irregular singular point at $x = \infty$ corresponds to $w = \infty$.

The spheroidal wave function with order μ has been investigated in refs. [37, 38]. With the replacement $\zeta = 2\theta^{1/2}w$, (5.31) becomes

$$(\zeta^2 - 4\theta)\frac{d^2 f_n}{d\zeta^2} + 2\zeta\frac{df_n}{d\zeta} + \left(\zeta^2 - \lambda - 4\theta - \frac{4\theta\mu^2}{\zeta^2 - 4\theta}\right) f_n = 0. \tag{5.33}$$

In the $\theta = 0$ limit, we obtain the Bessel equation which have two independent solutions:

$$\psi_\nu^{(1)}(\zeta) = \left(\frac{\pi}{2\zeta}\right)^{1/2} J_{\nu+\frac{1}{2}}(\zeta), \quad \psi_\nu^{(2)}(\zeta) = \left(\frac{\pi}{2\zeta}\right)^{1/2} Y_{\nu+\frac{1}{2}}(\zeta), \tag{5.34}$$

where $\lambda = \nu(\nu + 1)$. $J_{\nu+1/2}(\zeta)$ is the Bessel function and $Y_{\nu+1/2}(\zeta)$ is the Neumann function. One may also introduce the Hankel functions by

$$\psi_\nu^{(3)} = \psi_\nu^{(1)} + i\psi_\nu^{(2)} = \left(\frac{\pi}{2\zeta}\right)^{1/2} H^{(1)}_{\nu+\frac{1}{2}}(\zeta), \quad \psi_\nu^{(4)} = \psi_\nu^{(1)} - i\psi_\nu^{(2)} = \left(\frac{\pi}{2\zeta}\right)^{1/2} H^{(2)}_{\nu+\frac{1}{2}}(\zeta). \tag{5.35}$$

The spheroidal wave functions are given by the expansions of the form

$$S_\nu^{\mu(j)}(w, \theta) = (1 - w^{-2})^{-\mu/2} s_\nu^\mu(\theta) \sum_{r=-\infty}^{\infty} a^\mu_{\nu,r}(\theta)\psi^{(j)}_{\nu+2r}(2\theta^{1/2}w), \quad (j = 1, 2, 3, 4) \tag{5.36}$$

where $s_\nu^\mu(\theta)$ is a normalization constant and $a^\mu_{\nu,r}(\theta)$ obey a recursion relation:

$$\begin{aligned}&\frac{(\nu + 2r - \mu)(\nu + 2r - \mu - 1)}{(\nu + 2r - 3/2)(\nu + 2r - 1/2)}\theta a^\mu_{\nu,r-1}(\theta) + \frac{(\nu + 2r + \mu + 2)(\nu + 2r + \mu + 1)}{(\nu + 2r + 3/2)(\nu + 2r + 5/2)}\theta a^\mu_{\nu,r+1}(\theta) \\ &+ \left[\lambda - (\nu + 2r)(\nu + 2r + 1) + \frac{(\nu + 2r)(\nu + 2r + 1) + \mu^2 - 1}{(\nu + 2r - 1/2)(\nu + 2r + 3/2)} 2\theta\right] a^\mu_{\nu,r}(\theta) = 0\end{aligned} \tag{5.37}$$

Asymptotic behavior of $S^{(j)}$ as $w \to \infty$ has been determined by Meixner [37, 38]. If we choose the normalization factor $s_\nu^\mu(\theta)$ as

$$s_\nu^\mu(\theta) = \frac{1}{\sum_{r=-\infty}^{\infty}(-1)^r a^\mu_{\nu,r}(\theta)}, \tag{5.38}$$

the asymptotic forms may be written as

$$S_\nu^{\mu(j)}(w,\theta) \sim \psi_\nu^{(j)}(2\theta^{1/2}w), \quad w \to \infty, \quad (j = 1,2,3,4) \tag{5.39}$$

where $|\arg(\theta^{1/2}w)| < \pi$. In particular, using the asymptotic expansion of the Hankel functions $H^{(i)}_{\nu+1/2}$ $(i = 1,2)$, the asymptotic expansions of $S_\nu^{\mu(3)}(w,\theta)$ and $S_\nu^{\mu(4)}(w,\theta)$ for large $|w|$ are given by

$$\begin{aligned} S_\nu^{\mu(3)}(w,\theta) &= \frac{1}{2\theta^{1/2}w} e^{i(2\theta^{1/2}w - \nu\pi/2 - \pi/2)} \left[1 + O(|w|^{-1})\right], \quad w \to \infty, -\pi < \arg(\theta^{1/2}w) < 2\pi \\ S_\nu^{\mu(4)}(w,\theta) &= \frac{1}{2\theta^{1/2}w} e^{-i(2\theta^{1/2}w - \nu\pi/2 - \pi/2)} \left[1 + O(|w|^{-1})\right], \quad w \to \infty, -2\pi < \arg(\theta^{1/2}w) < \pi. \end{aligned} \tag{5.40}$$

Let us go back to the equation (5.1) in the y-direction. We are interested in the two solutions $\varphi_n(\tau)$ and $\tilde{\varphi}_{-n}(\tau)$ whose asymptotic behaviors for $\tau \to \infty$ take the form

$$\begin{aligned} \varphi_n(\tau) &\sim e^{in\tau}, \\ \tilde{\varphi}_{-n}(\tau) &\sim e^{-in\tau}. \end{aligned} \tag{5.41}$$

Since $\theta = \sqrt{n^2(Q_5 - \ell^2)}/2$ and $w = \left(1 + \frac{x}{Q_5 - \ell^2}\right)^{\frac{1}{2}}$, we find

$$2\theta^{1/2}w = \sqrt{n^2(x - \ell^2 + Q_5)} \sim |n|\tau, \quad (\tau \to \infty) \tag{5.42}$$

where we have used $x = r^2 \sim \tau^2$ as $r \to \infty$. Therefore using the spheroidal wave functions $S_\nu^{\mu(3)}(w,\theta)$ and $S_\nu^{\mu(4)}(w,\theta)$, we express the solutions of (5.1) as

$$\varphi_n(\tau) = \tilde{\varphi}_n(\tau) = \begin{cases} A_n x^{1/4}(x+Q_5)^{1/4} S_\nu^{\mu(3)}(\sqrt{n^2(x-\ell^2+Q_5)},\,\theta) & (n > 0), \\ A_n^* x^{1/4}(x+Q_5)^{1/4} S_\nu^{\mu(4)}(\sqrt{n^2(x-\ell^2+Q_5)},\,\theta) & (n < 0) \end{cases} \tag{5.43}$$

where

$$A_n = |n| e^{i(\nu\pi/2 + \pi/2)}. \tag{5.44}$$

Similar analysis can be done in the case of the (p,q) fivebranes as well as the NS5-branes. The string equation of motion in the y-directions are given by

$$\left(\partial_\tau^2 + n^2 - m_y(\tau)^2 \pm n\partial_\tau B(\tau)\right)\varphi_n^{(\pm)} = 0 \tag{5.45}$$

where $m_y^2(\tau)$ is in (2.41) and $\partial_\tau B(\tau)$ is in (4.19). As in the case of the D5 branes, we change variable τ into $x = r^2$ and $\varphi_n^{(\pm)}(\tau) = f(x)\phi_n^{(\pm)}(x)$, where $f(x) = x^\alpha(x + \tilde{Q}_5)^\beta(x +$

$\tilde{Q}_5-\ell^2)^\delta(x+\tilde{Q}_5\cos^2\gamma)^\eta$. The choice of the exponents $(\alpha,\beta,\delta,\eta)=(\alpha_+^{(\pm)},\frac{1}{4},0,\frac{1}{4})$ simplify the differential equation (5.45),where

$$\alpha_+^{(\pm)}=\frac{1}{2}\frac{\ell\mp n\tilde{Q}_5\cos\gamma}{\sqrt{\ell^2-\tilde{Q}_5}}. \tag{5.46}$$

$\phi_n^{(\pm)}(x)$ satisfy the equation (5.24) with

$$\begin{aligned}
\tilde{p}(x) &= \frac{1+2\alpha_+^{(\pm)}}{x}+\frac{\frac{1}{2}}{x+\tilde{Q}_5-\ell^2} && (5.47)\\
\tilde{q}(x) &= -\frac{1}{4x(x+\tilde{Q}_5-\ell^2)(\ell^2-\tilde{Q}_5)}\Big(n^2\tilde{Q}_5(\tilde{Q}_5-\ell^2-\cos^2\gamma\ell^2)-\ell^2\\
&\quad \pm 2n\tilde{Q}_5\ell\cos\gamma-\sqrt{\ell^2-\tilde{Q}_5}(\ell\mp n\tilde{Q}_5\cos\gamma)-n^2(\ell^2-\tilde{Q}_5)x\Big). && (5.48)
\end{aligned}$$

This differential equation also becomes the associated Mathieu equation (5.28) by the change of the variables $x=(\tilde{Q}_5-\ell^2)(z-1)$, where

$$\begin{aligned}
r &= -2\alpha_+^{(\pm)}=\frac{\ell\mp n\tilde{Q}_5\cos\gamma}{\sqrt{\ell^2-\tilde{Q}_5}},\\
k^2 &= n^2(\ell^2-\tilde{Q}_5),\\
a &= \frac{1}{\ell^2-\tilde{Q}_5}\Big(n^2\ell^2(\tilde{Q}_5-\ell^2-\tilde{Q}_5\cos^2\gamma)-\ell^2\pm 2n\tilde{Q}_5\ell\cos\gamma\\
&\quad -\sqrt{\ell^2-\tilde{Q}_5}(\ell\mp n\tilde{Q}_5\cos\gamma)\Big). && (5.49)
\end{aligned}$$

By the transformation $w=z^{1/2}$ and $\phi_n^{(\pm)}=(1-w^2)^{r/2}f_n^{(\pm)}$, we obtain the differential equation for the spheroidal wave functions (5.31) with parameters

$$\begin{aligned}
\lambda &= -n^2\tilde{Q}_5(1+\cos^2\gamma),\\
\theta &= -\frac{n^2(\ell^2-\tilde{Q}_5)}{4},\\
\mu &= \frac{\ell\mp n\tilde{Q}_5\cos\gamma}{\sqrt{\ell^2-\tilde{Q}_5}}. && (5.50)
\end{aligned}$$

By using the spheroidal wave function $S_\nu^{\mu(j)}(w,\theta)$ with $\lambda=\nu(\nu+1)$, we may solve the wave functions $\varphi_n^{(\pm)}(\tau)$ and $\tilde{\varphi}_{-n}^{(\pm)}(\tau)$ with asymptotic behavior (5.41). These are expressed as

$$\varphi_n^{(\pm)}(\tau)=\tilde{\varphi}_n^{(\pm)}(\tau)$$

$$= \begin{cases} A_n(x+Q_5)^{1/4}(x+\tilde{Q}_5\cos^2\gamma)^{1/4}S_\nu^{\mu(3)}(\sqrt{n^2(x-\ell^2+\tilde{Q}_5)},\,\theta) & (n>0), \\ A_n^*(x+Q_5)^{1/4}(x+\tilde{Q}_5\cos^2\gamma)^{1/4}S_\nu^{\mu(4)}(\sqrt{n^2(x-\ell^2+\tilde{Q}_5)},\,\theta) & (n<0) \end{cases} \tag{5.51}$$

where the constant A_n is the same as (5.44).

The solution at $w = 0$ corresponds to the theory at the critical radius. The expansion of the spheroidal wave functions in the region $|w \pm 1| < 2$ are written in terms of the Legendre polynomials [37, 38], but in this paper, we do not discuss this expansion.

6 Conclusions and discussion

In this paper, we have studied string theories on the Penrose limit of various brane solutions of type II supergravity. Firstly, we obtained the plane wave geometry for the Dp-brane, NS and (p,q) fivebrane, fundamental and (p,q) string solutions, not restricting ourselves to the near-horizon limit. We have thoroughly investigated the equations of motion of the light-cone bosonic string on these backgrounds, which have time-dependent masses.

We have found several types of the solutions. The simplest class of the solutions are given when the masses become constant. These arise when we consider the Penrose limit along a geodesic which stays at a fixed radius. We have analyzed the condition for the existence of this kind of geodesics, and gave the formula for the mass terms. In the near-horizon limit, we have various examples in which the equations of motion can be solved by the Bessel functions. We presented a preliminary analysis for the canonical quantization of the string for the latter case.

We found non-trivial solutions for the case of fivebrane backgrounds. Firstly, the equations of motion can be solved by the Mathieu functions in the region near the critical radius (the point on the geodesic which is nearest to the origin). In the asymptotic region at infinity, we found solutions also in terms of the Mathieu functions. Furthermore, for a particular component, exact solutions exist which is written by using the spheroidal wave functions. The above solutions exist for the fivebranes with general (p,q).

For physical applications, it is important to extend our analysis to the fermionic sector of the superstring. Green-Schwarz action for the superstring can be written in arbitrary background in principle [39]. It is a non-trivial question whether the equation of motion for fermionic sectors derived from that action can be solved as we have done for the bosonic sectors. It is also interesting to attempt a study in the NSR formalism. In some of our plane wave backgrounds (for NS branes, or for the case of radial null geodesics), RR backgrounds are not present. For these cases, the latter formalism may be more useful.

Most interesting problem for the future study is the quantization of the string on the plane wave backgrounds for the brane solutions. As discussed briefly in the appendix, performing the mode expansion using the solutions of the equations of motion as the basis, and following the procedure of the canonical quantization, we can obtain the oscillators and construct the Fock space. For time-dependent plane wave backgrounds, such states do not diagonalize the light-cone Hamiltonian in general. In other words, an asymptotic state will evolve into another due to the influence of the brane background. For the fivebrane backgrounds we can cover the whole region of the geodesics either by matching the solutions obtained in the two limits, or by using the exact solution. It should enable us to study the scattering or absorption of string states by the fivebranes. We hope to report progress in this direction in a future publication.

It would also be interesting to apply our results to the holographic correspondence with gauge theories. As mentioned in the introduction, six-dimensional gauge theory is investigated from the perspective of the Penrose limit of the NS5-brane background [9, 23]. The results of this paper such as the classification of the geodesics and the exact solutions for the fivebrane backgrounds may give further information on the gauge theory.

Holographic dualities for general (dilatonic) Dp-branes have been proposed by Itzhaki et.al. [26], but they are less understood than the AdS_5/CFT_4 correspondence. There are few quantitative studies on those dualities. (See refs. [40, 41] for such an attempt for the case of the D0-branes.) It would be nice if it is possible to calculate the gauge theory correlators from string theory on the Penrose limit of the brane backgrounds. Our solution of the string equations of motion in the near-horizon limit of Dp-branes (with $p \leq 4$) may be helpful in this regard. We have found the solutions taking an additional limit (4.36), but further analysis without that restriction would be important.

Holographic renormalization group for nonconformal field theories is also an interesting subject for future studies. Analysis of the holographic RG for the pp-wave backgrounds has been done in refs. [14, 15, 16], where the radial coordinate plays the role of the scale of the dual gauge theories. It is an interesting question how to interpret the string theories on the critical radius in this context.

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Appendix: Mode expansion using the Hankel functions

In this appendix, we give a preliminary analysis for the canonical quantization of the string on a particular plane wave background. We expand X^i using the solutions of the equations of motion as the basis and impose the canonical commutation relation. We demonstrate that we can construct the oscillators, and present the form of the light-cone Hamiltonian.

Consider the string equation of motion of the form

$$\left(\frac{d^2}{d\tau^2} + n^2 - \frac{C_1}{\tau^2}\right)\varphi_n = 0. \tag{A.1}$$

This equation is obtained when we study the near-horizon limit of various branes, as we have seen in section 4.3. The constant C_1 depends on the model. We shall solve (A.1) with the following boundary condition for $\tau \to \infty$:

$$\varphi_n(\tau) \to e^{in\tau}, \quad \tilde{\varphi}_{-n}(\tau) \to e^{-in\tau}. \tag{A.2}$$

Note that (A.1) approaches the equation of motion in the flat background asymptotically (as $\tau \to \infty$). The above boundary condition is the one which match the mode expansion in the flat background[4].

The solutions for the non-zero modes are given by

$$\varphi_n(\tau) = \tilde{\varphi}_n(\tau) = \begin{cases} \sqrt{\frac{\pi}{2}} e^{\frac{i}{2}(\nu+\frac{1}{2})\pi} (n\tau)^{\frac{1}{2}} H_\nu^{(1)}(n\tau) & (n > 0) \\ \sqrt{\frac{\pi}{2}} e^{-\frac{i}{2}(\nu+\frac{1}{2})\pi} (-n\tau)^{\frac{1}{2}} H_\nu^{(2)}(-n\tau) & (n < 0) \end{cases} \tag{A.3}$$

[4]Since the present theory is obtained by taking the near-horizon limit and is not valid for $r \to \infty$ (or $\tau \to \infty$), whether we should choose this boundary condition is not clear.

where $\nu^2 \equiv C_1 + 1/4$ and $H_\nu^{(1)}$ and $H_\nu^{(2)}$ are the Hankel functions which are related to the Bessel functions of the first kind J_ν and the second kind Y_ν by

$$H_\nu^{(1)}(x) = J_\nu(x) + iY_\nu(x), \quad H_\nu^{(2)}(x) = J_\nu(x) - iY_\nu(x).$$

The two independent solutions for the zero-modes are

$$\varphi_0 = \tau^{\nu+\frac{1}{2}}, \qquad \tilde{\varphi}_0 = \tau^{-\nu+\frac{1}{2}}. \tag{A.4}$$

We perform the quantization of the string using the solutions found above as the basis functions. Remember that X^i is given as

$$X^i(\tau,\sigma) = \sum_n \left(\alpha_n^i \varphi_n(\tau) + \tilde{\alpha}_{-n}^i \tilde{\varphi}_{-n}(\tau)\right) e^{in\sigma}.$$

The reality condition for X^i leads to

$$\tilde{\alpha}_n^{i\,\dagger} = \tilde{\alpha}_{-n}^i, \quad \alpha_n^{i\,\dagger} = \alpha_{-n}^i, \tag{A.5}$$

where we have used $H^{(1)*}(x) = H^{(2)}(x)$ for a real x.

Now we impose the canonical commutation relation for X^i

$$[X^i(\tau,\sigma), \partial_\tau X^j(\tau,\sigma')] = -i2\pi\alpha'\delta^{ij}\delta(\sigma-\sigma'). \tag{A.6}$$

This implies the following commutation relations for the oscillators

$$\begin{aligned}
[\tilde{\alpha}_n^i, \tilde{\alpha}_{n'}^j] &= \frac{1}{2n}\delta^{ij}\delta_{n,-n'} \qquad (n \neq 0), \\
[\alpha_n^i, \alpha_{n'}^j] &= \frac{1}{2n}\delta^{ij}\delta_{n,-n'} \qquad (n \neq 0), \\
[\alpha_0^i, \tilde{\alpha}_0^j] &= i\delta^{ij},
\end{aligned} \tag{A.7}$$

where commutators among other components are zero. The consistency between (A.6) and (A.7) is guaranteed by the following equation (for the non-zero mode part)

$$\begin{aligned}
\varphi_n\partial_\tau\varphi_{-n} - \tilde{\varphi}_{-n}\partial_\tau\tilde{\varphi}_n &= \varphi_n\partial_\tau\varphi_{-n} - \varphi_{-n}\partial_\tau\varphi_n \qquad \text{(A.8)}\\
&= \frac{\pi n\tau}{2}\left(H_\nu^{(1)}(n\tau)\partial_\tau H_\nu^{(2)}(n\tau) - H_\nu^{(2)}(n\tau)\partial_\tau H_\nu^{(1)}(n\tau)\right) = -2ni
\end{aligned}$$

where the last equality is due to the Lommel's formula for the Hankel functions.

Note that

$$\varphi_n \partial_\tau \varphi_{-n} - \varphi_{-n} \partial_\tau \varphi_n = 2ni \tag{A.9}$$

is the Wronskian of the two independent solutions φ_n and φ_{-n} of (A.1). We can easily prove that the Wronskian for our differential equation is τ-independent, by taking its derivative and using (A.1). Thus, (A.9) is given by the value evaluated at $\tau \to \infty$, which is the r.h.s. of that equation. Moreover, the Wronskian for a differential equation of the form (4.6) is independent of τ for arbitrary $m_i^2(\tau)$. Thus, if we consider an equation with $m_i^2(\tau) \to 0$ as $\tau \to \infty$, and if we take the solutions with the boundary condition (A.2), the value of the Wronskian agrees with (A.9). This fact suggests that we can get the commutation relations for the oscillators (A.7) from the canonical commutation relation (A.6), for those generic plane wave backgrounds.

The light-cone Hamiltonian for the model of (A.1) is given by

$$H = \frac{1}{2\pi\alpha' P^+} \int d\sigma \left(\frac{1}{2}(\partial_\tau X^i)^2 + \frac{1}{2}(\partial_\sigma X^i)^2 + \frac{C_1}{2\tau^2}(X^i)^2 \right). \tag{A.10}$$

Substituting the mode expansion, we obtain

$$\begin{aligned} H &= \frac{1}{2\alpha' P^+} \sum_n \Big((\partial_\tau \varphi_n \partial_\tau \varphi_{-n} + (n^2 + \frac{C_1}{\tau^2}) \varphi_n \varphi_{-n}) \alpha_n^i \alpha_{-n}^i \\ &\qquad + (\partial_\tau \tilde{\varphi}_n \partial_\tau \tilde{\varphi}_{-n} + (n^2 + \frac{C_1}{\tau^2}) \tilde{\varphi}_n \tilde{\varphi}_{-n}) \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i \\ &\qquad + 2(\partial_\tau \varphi_n \partial_\tau \tilde{\varphi}_n + (n^2 + \frac{C_1}{\tau^2}) \varphi_n \tilde{\varphi}_n) \alpha_n^i \tilde{\alpha}_n^i \Big). \end{aligned} \tag{A.11}$$

The Hamiltonian is time dependent, and especially, it does not commute with the number operator. The first two terms conserve the particle number, but the last term does not. Note that the case of the flat background is given by substituting $C_1 = 0$ and $\varphi_n = \tilde{\varphi}_n = e^{in\tau}$ into (A.11). In this case, the last term of (A.11) vanish, but for the time-dependent plane wave backgrounds, that term should be present in general.

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