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Asymptotic symmetries of AdS_2 Branes

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ABSTRACT

I analyze the asymptotic symmetries of a theory of gravity in a background consisting of two patches of AdS_3 spacetime glued together along an AdS_2 brane. These are generated by a single Virasoro algebra, as expected from the conjectured dual description in terms of a scale-invariant interface separating two conformal field theories. Contributed to the proceedings of the Francqui Colloquium 2001: 'Strings and Gravity: Tying the Forces Together .'

1 Introduction

A long time ago, Brown and Henneaux [1] proved that a theory of gravity in a three-dimensional anti-de-Sitter (AdS_3) background has an infinite number of asymptotic symmetries, generated by two (a left and a right) Virasoro algebras with central charge

$$c = \frac{3\ell}{2G_N} \ . \tag{1.1}$$

Here ℓ is the radius of AdS_3, and G_N the 3D Newton's constant. This is a remarquable result, which shows how an effect normally thought to be 'quantum' – the central extension of the Virasoro algebras – can arise from a classical calculation. The result anticipated the general AdS_{n+1}/CFT_n correspondence [2], according to which the theory of gravity can be described by a dual conformal theory defined at the boundary of spacetime. It has been rederived by different methods more recently [3, 4, 5, 6].

In this short note I want to extend the Brown-Henneaux argument to a situation in which two different patches of AdS_3 spacetime are glued together along a AdS_2 brane. One can glue together, more generally, two patches of AdS_{n+1} spacetime along a AdS_n brane. A related geometry was studied, for $n = 4$, by Karch and Randall [7] as a model for localized gravity that did not require perfect fine tuning of the brane tension (see also [8, 9, 10, 11]). The setup can, furthermore, be embedded in string theory [12, 13], though it is still unclear whether 'realistic models' of localized gravity can be truly obtained in this way.[1] Quantum gravity in the above background is, in any case, believed to admit a dual description where a scale-invariant interface separates two (a priori different) conformal theories [13, 15, 16].[2] For $n = 2$ there are no degrees of freedom on the interface, which preserves one (non-chiral) Virasoro symmetry. What I will show here is how to realize this symmetry in terms of non-trivial bulk diffeomorphisms.

[1] See for instance [14] for a potential obstruction.

[2] Keeping only one AdS_{n+1} patch, and treating the AdS_n as part of the boundary, leads to a different holographic picture [7, 17]. I thank Massimo Porrati for pointing this out.

2 AdS_2 brane in AdS_3

We consider two patches of AdS_3 spacetime glued together along an AdS_2 brane. The metric in conformally-flat coordinates is

$$ds^2 = f^{-2}(dv^2 + dy^2 - dt^2) \ , \tag{2.1}$$

where

$$f(v,y) = \begin{cases} (v + a_1 y)/\ell & \text{for} \quad y > 0 \\ \\ (v + a_2 y)/\ell & \text{for} \quad y < 0 \ . \end{cases} \tag{2.2}$$

The coordinates range over all values such that $0 < f < +\infty$. The surface $f = 0$ is the spacetime boundary, while $f = \infty$ is a coordinate horizon. The brane sits at $y = 0$ and has radius ℓ. The radii of the bulk AdS_3 geometry on either side of the brane are

$$\ell_r = \ell \cos\theta_r \ , \quad \text{where} \quad \theta_r = \arctan a_r \quad \text{for} \ \ r = 1,2 \ . \tag{2.3}$$

One can check this claim by rotating the coordinates (v, y) by an angle $\theta_{1(2)}$, so as to put the upper (lower) AdS_3 patch in standard Poincaré form. Note that $\ell_1 \neq \ell_2$ in general, so that the 3D cosmological constant jumps discontinuously at the position of the brane. This does not happen if $a_1 = -a_2$, in which case $y \to -y$ is a Z_2 isometry, or if $a_1 = a_2$ in which case there is no (back-reacting) brane. The metric and all its derivatives are continuous at $y = 0$, in the latter case.

The metric (2.1–2.2) solves the variational equations that are derived from the bulk plus brane action

$$S = S_1^{\text{bulk}} + S_2^{\text{bulk}} + S_1^{\text{boundary}} + S_2^{\text{boundary}} + S^{\text{brane}} \ , \tag{2.4}$$

where

$$S_r^{\text{bulk}} = -\frac{1}{16\pi G_N}\int_{\mathcal{M}_r} \sqrt{-g}\,(R - \frac{2}{{\ell_r}^2}) \tag{2.5}$$

is the Einstein-Hilbert action with cosmological term(s),

$$S_r^{\text{boundary}} = -\frac{1}{8\pi G_N}\int_{\partial\mathcal{M}_r} \sqrt{-\gamma}\,(K + \frac{1}{\ell_r}) \tag{2.6}$$

is the Gibbons-Hawking boundary term required to eliminate second derivatives of the metric [18], and

$$S^{\rm brane} = T \int_\Sigma \sqrt{-\hat g} \ \ - \ \frac{1}{8\pi G_N} \int_\Sigma \sqrt{-\hat g}\ [K]\ . \tag{2.7}$$

Σ in the above expression is the worldvolume of the brane, treated as a thin shell that separates spacetime in two disjoint regions $\mathcal{M}_r$. The boundary of spacetime has been decomposed as $\partial\mathcal{M}_1 \cup \partial\mathcal{M}_2$. The tension of the brane is denoted T, $\hat g$ is the induced metric on the brane, and γ the metric on the boundary. Furthermore K is the trace of the extrinsic curvature tensor, and $[K]$ its discontinuity across the brane.

The boudary and brane actions (2.6) and (2.7) require some further explanation. First, the boundaries $\partial\mathcal{M}_1$ and $\partial\mathcal{M}_2$ should be placed at a finite cutoff value $f = \epsilon$, which will be taken in the end to zero. Following references [3, 4, 5], I have included in (2.6) a counterterm required to keep the energy-momentum tensor finite in this limit. Secondly, I have added Gibbons-Hawking surface terms on the brane wordvolume Σ, which is the common boundary of $\mathcal{M}_1$ and $\mathcal{M}_2$. These should be thought as arising from the integral of the Einstein-Hilbert action over the (infinitesimal) thickness of the brane. They do not, therefore, correspond to a matter source, even though I include them for convenience in the brane action.

Extremizing the total action (2.4) leads to the Israel junction conditions [19, 20] (here Latin indices stand for worldvolume directions)

$$[K_{ab}] - \hat g_{ab}[K] \ = \ -8\pi G_N T \hat g_{ab}\ . \tag{2.8}$$

The extrinsic curvature tensor is the covariant derivative of the outward-pointing unit normal vector $\hat n_\mu dx^\mu = -f^{-1}dy$, projected on the worldvolume of the brane. A straightforward calculation gives

$$K^{(r)}_{ab} \equiv -e^\mu_a e^\nu_b\ \nabla_\mu \hat n_\nu = \frac{a_r}{\ell}\ \hat g_{ab}\ , \tag{2.9}$$

where e^μ_a is a basis of tangent vectors, and

$$\hat g_{ab}\ dx^a dx^b = \frac{\ell^2}{v^2}\ (dv^2 - dt^2) \tag{2.10}$$

is the induced brane metric. As the reader can verify easily, the junction conditions are indeed satisfied, provided that

$$a_1 - a_2 = 8\pi G_N \ell \, T \ . \tag{2.11}$$

The three free parameters of the metric (2.1–2.2) can thus be expressed, via equations (2.3) and (2.11) , in terms of the parameters ℓ_r, T and G_N that enter in the action.

3 Asymptotic symmetries

In order to discuss the asymptotic symmetries, it is convenient to make the following coordinate change :

$$u \equiv v + y \ \tan\theta_r \quad \text{and} \quad x \equiv y/\cos\theta_r \ , \quad \text{in region} \ \ \mathcal{M}_r \ . \tag{3.1}$$

The new coordinates range over $0 < u < \infty$, and $-\infty < x < \infty$. The brane sits at $x = 0$, and the spacetime boundary is at $u = 0$. As usual, we introduce an ultraviolet cutoff in the dual conformal theory by placing the boundary at a finite value $u = \epsilon > 0$.

Although the reparametrization (3.1) is continuous, its Jacobian has a step-function discontinuity at the position of the brane. Therefore the metric has a corresponding step-function jump :

$$ds^2 = \frac{\ell^2}{u^2} \ \Big(du^2 + dx^2 - dt^2 - 2 \ \sin\theta_r \ dx \ du \Big) \quad \text{in region} \ \ \mathcal{M}_r \ . \tag{3.2}$$

Since this discontinuity is a coordinate artifact, it is not problematic so long as one treats it with the required care.

The metric (3.2) has a SL(2,R) group of isometries, which include time translations, and the global rescalings $(x,t,u) \to \lambda \ (x,t,u)$. There is however a larger set of coordinate transformations, which only leave invariant the asymptotic form of the metric. They form a Virasoro algebra, and act on the Hilbert space of states at the boundary. The relevant infinitesimal transformations (in region $\mathcal{M}_r$) are:

$$x^{\pm} \to x^{\pm} - \xi^{\pm} + \frac{u}{2} \ \sin\theta_r \ (\partial_+ \xi^+ - \partial_- \xi^-) - \frac{u^2}{2} \ \partial_{\mp}^2 \xi^{\mp} \ , \tag{3.3}$$

and

$$\frac{1}{u} \to \frac{1}{u} + \frac{1}{2u}\,(\partial_+\xi^+ + \partial_-\xi^-) - \frac{\sin\theta_r}{2}\,(\partial_+^2\xi^+ - \partial_-^2\xi^-)\ . \tag{3.4}$$

Here $x^\pm = t \pm x$ are light-cone coordinates, and $\xi^\pm = f(x^\pm)$ with f an arbitrary infinitesimal function. The reader will have no problems verifying that the above transformations : (a) act on the boundary, at $u = 0$, as conformal mappings which preserve the $x = 0$ interface, and (b) that they reduce to the Brown–Henneaux transformations [1] in the case of pure anti-de-Sitter spacetime, i.e. for $\theta_r = 0$.

To check that they are asymptotic symmetries, we need to calculate the transformed metric. A lengthy but straightforward computation gives the following variation in region $\mathcal{M}_r$:

$$\begin{aligned} 2\,ds^2 \to\ \ & 2\,ds^2 + \ell^2\cos^2\theta_r\ [\partial_-^3\xi^-(dx^-)^2 + \partial_+^3\xi^+(dx^+)^2] \\ & \qquad + \ell^2\sin^2\theta_r\ (\partial_+^3\xi^+ + \partial_-^3\xi^-)\ dx^+dx^- \\ & \qquad + \ell^2\sin\theta_r\ (\partial_+^3\xi^+dx^+ - \partial_-^3\xi^-dx^-)\ du\ . \end{aligned} \tag{3.5}$$

We see that the metric variation is down by two powers of u. The transformations (3.3–3.4) correspond therefore to asymptotic symmetries, if we supplement the definition of the theory with the boundary conditions $\delta g_{\mu\nu} \sim o(1)$ near $u \to 0$.[3]

It is worth stressing that $\xi^+(t) = \xi^-(t)$ is required by continuity of the coordinate transformations at the location of the brane, at $x = 0$. The same condition also forces the infinitesimal function f to be the same in the regions $\mathcal{M}_1$ and $\mathcal{M}_2$. The asymptotic symmetries therefore depend on a single arbitrary function. This agrees with the holographic interpretation, in which only a single Virasoro symmetry survives [15].

[3] Our boundary conditions look superficially weaker than the ones of Brown and Henneaux, who require the off-diagonal components $g_{u\,\pm}$ to vanish linearly. This can, however, be always arranged, so long as $\sin\theta_r \neq 0$, with the help of additional (subleading) coordinate transformations: $\delta(1/u) \sim o(u)$ and $\delta t \sim o(u^3)$.

4 Energy-momentum tensor

To further elucidate the meaning of the asymptotic symmetries, we will compute their effect on the energy-momentum tensor. Following [21, 4] we define this latter as the variation of the total action with respect to the metric at the boundary. For solutions of the classical equations the bulk terms in the action don't contribute, so we find

$$T_{ab} \;=\; \frac{2}{\sqrt{-\gamma}}\frac{\delta S}{\delta \gamma^{ab}} \;=\; \frac{1}{8\pi G_N}\left[K_{ab} - K\gamma_{ab} - \frac{1}{\ell_r}\gamma_{ab}\right] \;. \tag{4.1}$$

Latin indices here refer to the two boundary coordinates x and t, and they are raised and lowered with γ. They should not be confused with the brane-worldvolume indices of section 2.

It is convenient to write the 3D metric in ADM form

$$ds^2 = \gamma_{ab}(dx^a + N^a du)(dx^b + N^b du) + (N du)^2 \;. \tag{4.2}$$

The extrinsic curvature can then be expressed in terms of the lapse and shift functions as follows [20] :

$$K_{ab} = -\frac{1}{2N}\left[\partial_a N_b + \partial_b N_a - 2N^c \Gamma_{c|ab} - \frac{\partial \gamma_{ab}}{\partial u}\right] \;, \tag{4.3}$$

with

$$\Gamma_{c|ab} = \frac{1}{2}(\partial_a \gamma_{bc} + \partial_b \gamma_{ac} - \partial_c \gamma_{ab}) \;. \tag{4.4}$$

For the metric (3.2) we have

$$\gamma_{ab} = \frac{\ell^2}{u^2}\,\eta_{ab} \;, \quad N^x = -\mathrm{sin}\theta_r \;, \quad \text{and} \quad N = \frac{\ell_r}{u} \;. \tag{4.5}$$

A simple calculation then gives

$$K_{ab} = -\frac{1}{\ell_r}\,\gamma_{ab} \quad \Longrightarrow \quad T_{ab} = 0 \;. \tag{4.6}$$

This is in acoordance with the fact that the vacuum expectation value of the energy-momentum tensor in the dual CFT should vanish.

To calculate the transformed energy-momentum tensor, we first put (3.5) in ADM form. The variation $\delta\gamma_{ab}$ can be read off directly from the transformed metric. The variation of the lapse and shift functions can be derived easily with the result:

$$\delta N^{\pm} = \pm u^2 \sin\theta_r \left[\frac{3}{2}\cos^2\theta_r \, \partial^3_{\mp}\xi^{\mp} - \frac{1}{2}\sin^2\theta_r \, \partial^3_{\pm}\xi^{\pm}\right] , \tag{4.7}$$

and

$$\delta N = \frac{u}{2} \, \ell_r \, \sin^2\theta_r \, (\partial^3_+\xi^+ + \partial^3_-\xi^-) \ . \tag{4.8}$$

From expression (4.3) and the fact that $\delta\gamma_{ab}$ is independent of u, we find the following variation of the extrinsic curvature:

$$\delta K_{ab} = -\frac{\delta N}{N} K_{ab} - \frac{1}{2N} \left(\gamma_{ac} \, \partial_b \delta N^c + \gamma_{bc} \, \partial_a \delta N^c + N^c \partial_c \delta\gamma_{ab}\right) \ . \tag{4.9}$$

Inserting the expressions for $\delta\gamma_{ab}$, $\delta N^{\pm}$ and δN leads to:

$$\delta K_{+-} = -\frac{\ell^2 \sin^2\theta_r}{4\ell_r} \, (\partial^3_+\xi^+ + \partial^3_-\xi^-) \ , \quad \delta K = 0 \ , \tag{4.10}$$

and

$$\delta K_{\pm\pm} = \mp\frac{u\ell^2}{2\ell_r} \, \sin\theta_r \, \cos^2\theta_r \, \partial^4_{\pm}\xi^{\pm} \ . \tag{4.11}$$

Furthermore from equation (4.1) we have :

$$8\pi G_N \, \delta T_{ab} = \delta K_{ab} + \frac{1}{\ell_r} \delta\gamma_{ab} \ . \tag{4.12}$$

Substituting our results for δK_{ab} and $\delta\gamma_{ab}$, and dropping terms that vanish at $u = 0$, leads to our final expression valid in the region $\mathcal{M}_r$:

$$\delta T_{\pm\pm} = \frac{\ell_r}{16\pi G_N} \, \partial^3_{\pm}\xi^{\pm} \ , \quad \text{and} \quad \delta T_{+-} = 0 \ . \tag{4.13}$$

Recall the usual transformation of the energy-momentum tensor under a 2d conformal map :

$$\delta T_{\pm\pm} = -(2\partial_{\pm}\xi^{\pm} + \xi^{\pm}\partial_{\pm}) T_{\pm\pm} + \frac{c}{24\pi} \partial^3_{\pm}\xi^{\pm} \ . \tag{4.14}$$

Comparing with (4.13) we conclude that the central charge,

$$c_r = \frac{3\ell_r}{2G_N} \quad \text{in} \ \ \mathcal{M}_r \ , \tag{4.15}$$

must jump discontinuously at $x = 0$. This is again consistent with the dual description, where two (a priori) distinct conformal theories are glued together along a common interface.

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