RELATIVISTIC TOP DEVIATION EQUATION AND GRAVITATIONAL WAVES

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Abstract

By using the relativistic top theory, we derive a relativistic top deviation equation. This equation turns out to be a generalization of the geodesic deviation equation for a pair of nearby point particles. In fact, we show that when the spin angular momentum tensor associated to the top vanishes, such a relativistic top deviation equation reduces to the geodesic deviation equation for spinless point particles. Just as the geodesic deviation equation for spinless particles can be used to investigate the detection of gravitational waves, our generalized formula for a relativistic top can be used to study the gravitational wave background. Our formulation may be of special interest to detect the inflationary gravitational waves via the polarization of the cosmic background radiation.

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I. INTRODUCTION

It is well recognized that the geodesic deviation equation (GDE) for spinless particles plays an important role in the search of gravitational wave detectors [1-2]. Indeed, it is known that all of the projects currently used to detect gravitational waves, including LIGO [3], VIRGO [4] and LISA [5], have among their root physical bases such an equation. However, some years ago, in references [6] and [7], it was proposed to use the relativistic top equations of motion (RTEM) [8] (see also Refs. [9] and [10], and references there in) instead of the GDE for the same purpose. The main motivation for this alternative emerged from the observation that the motion of a relativistic top is influenced by a gravitational force involving the Riemann tensor just in a similar way as two spinless particles in the GDE involve the Riemann tensor. In Ref. [6] general solutions of the RTEM for the case of a top interacting with a gravitational wave were investigated, while in Ref. [7] such solutions were applied to the specific case of considering pulsars as gravitational wave detectors. Although the alternative method proposed in Refs. [6] and [7] to detect gravitational waves is interesting by itself, it has to be extended with the purpose of comparing, in a direct way, the results derived from the RTEM with those obtained from the method based on the GDE. Hence, instead of treating the two alternatives as two independent methods, in this work we combine them and we obtain a generalization of the GDE formalism. Specifically, we derive a relativistic top deviation equation (RTDE) which is reduced to the GDE when the spin tensor associated to the top vanishes.

It is known that binary pulsar systems may be important sources of gravitational waves [11]. Here, as an application of the RTDE we shall argue that binary pulsars can also be used as detectors of gravitational waves. For this effect to be viable it is necessary that the binary pulsar system has a companion source of gravitational waves. A similar but not quite the same idea has already been considered by Laguna and Welszczan [12]. These authors consider a rotating black hole as the companion of the binary pulsar and investigate the Shapiro time delate [13] due to the Kerr-Newmann curvature produced by such a black hole. Instead of focusing the attention on the black hole curvature we think about the black hole gravitational waves as being the responsible of the timing effect on the binary pulsars. Our work may be useful, among other things, to distinguish these two possibilities. Actually, our formulation is so general that the companion of the binary pulsar can be any other source of gravitational waves such as supernovae or vibrating neutron stars.

Also, this work may be of special interest in connection with other related works. Recently, Wang *et al.* [14] made the proposal to construct a 50m radio telescope to measure pulsar timings. The Wang *et al.* considerations are based on the idea proposed some time ago by Detweiler [15] who showed that measurements of signal arrival time from a pulsar may be used to search for stochastic gravitational wave background (SGWB). The main strategy of these authors to detect of the SGWB is to consider a number of pulsars separated at different parts in the sky. It is clear then that our RTDE formulation may be useful for the project of these authors. Also recently, Kessari *et al.* [16] (see also Refs. [17]and [18]) extended the idea of Ref. [7] to include electromagnetic waves. Since the polarization vector of an electromagnetic wave can be described by the spin tensor of a massless top, it seems reasonable to think that the RTDE approach may also be of special interest in this direction.

Finally, the RTDE may also have an interesting application in connection with the so called inflationary gravitational waves (see [19] and references there in). As it is known, the polarization of the cosmic microwave radiation [20] may solve the problem of detecting the gravitational waves produced during the inflationary scenario. Just before the universe became transparent to radiation, the plasma motion caused by the gravitational waves may have different sources. In particular, the effect predicted by the RTDE may be of particular interest in this scenario.

Moreover, inflationary gravitational waves are predicted by higher dimensional theories such as string/M theories [21]-[24] and supergravity. An attractive scenario with extra dimensions is the brane worlds cosmology. In this case, our 3+1 dimensional spacetime is the dynamical 3d brane embedded in the higher dimensional space. The information about the bulk/brane geometry can be obtained through the gravitational waves. All particles in the standard model are confined to the brane and they cannot move in the higher dimensional space, with the sole exception of gravity. Therefore, if we identify the internal angular momentum of the top with the spin tensor of a fundamental particle, then the RTDE provides an interaction between brane world gravitational waves and the spin of the particles in the plasma contained in the brane. Thus, the RTDE system may be useful to get information through gravitational excitations.

The plan of this work is as follows. In section II, we briefly review one of the possible mechanisms to obtain the GDE and in section III we apply similar techniques to obtain the RTDE formulation. In section IV, we explain how the RTDE can be applied to the detection of gravitational waves. Finally, in section V, we make some final comments.

II. GEODESIC DEVIATION EQUATION

Several methods can be used to obtain the GDE. Some of them are, in fact, quite brief. For our purpose, however, it turns out to be more convenient to follow the one in reference [25], emphasizing that our computations are more specific than those of [25].

Consider a point particle whose trajectory is described by the coordinates $x^\mu(\tau)$, where τ is the proper time parameter. The geodesic equation is

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}(x)\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0. \tag{1}$$

Here, $\Gamma^\mu_{\alpha\beta}(x)$ stands for the Christoffel symbols.

Equation (1) can be written in a more compact form as

$$\frac{D^2x^\mu}{D\tau^2} = 0, \tag{2}$$

where $\frac{D}{D\tau}$ denotes covariant derivative with respect to τ and $\frac{D^2x^\mu}{D\tau^2} = \frac{D}{D\tau}(\frac{Dx^\mu}{D\tau})$. Note that, since the coordinates x^μ are scalars fields, one has $\frac{Dx^\mu}{D\tau} = \frac{dx^\mu}{d\tau}$.

A nearby point particle must also satisfy the geodesic equation. If we use the coordinates $x'^\mu(\tau)$ to describe the position of such a nearby point particle we have

$$\frac{d^2x'^\mu}{d\tau^2} + \Gamma'^\mu_{\alpha\beta}(x')\frac{dx'^\alpha}{d\tau}\frac{dx'^\beta}{d\tau} = 0. \tag{3}$$

By nearby we mean that the coordinates $x'^\mu(\tau)$ can be written as

$$x'^\mu = x^\mu + \xi^\mu(x), \tag{4}$$

with $\mid \xi^\mu \mid$ being a very small quantity.

To first order in ξ^μ we have

$$\Gamma'^\mu_{\alpha\beta}(x+\xi) = \Gamma^\mu_{\alpha\beta}(x) + \Gamma^\mu_{\alpha\beta},_\lambda\,\xi^\lambda, \tag{5}$$

with $\Gamma^\mu_{\alpha\beta},_\lambda = \frac{\partial\Gamma^\mu_{\alpha\beta}}{\partial x^\lambda}$. Thus, using (4) and (5) we find that equation (3) becomes

$$\frac{d^2x^\mu}{d\tau^2} + \frac{d^2\xi^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} + 2\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\xi^\beta}{d\tau} + \Gamma^\mu_{\alpha\beta},_\lambda\,\xi^\lambda\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0. \tag{6}$$

By virtue of (1), we see that the first and the third term of (6) can be dropped. Hence, the expression (6) is reduced to

$$\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\xi^\beta}{d\tau} + \Gamma^\mu_{\alpha\beta},_\lambda\,\xi^\lambda\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0. \tag{7}$$

Our goal is now to write this equation in a covariant form. For this purpose we first write (7) in the form

$$\frac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\xi^\beta}{d\tau} = -\Gamma^\mu_{\alpha\beta},_\lambda\,\xi^\lambda\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}. \tag{8}$$

Now, by adding to both sides of (8) the expression

$$\Gamma^\mu_{\alpha\lambda},_\beta\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\xi^\lambda + \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\alpha\lambda}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\xi^\lambda - \Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\xi^\lambda. \tag{9}$$

we observe that (8) becomes

$$\begin{gathered}\tfrac{d^2\xi^\mu}{d\tau^2} + 2\Gamma^\mu_{\alpha\beta}\tfrac{dx^\alpha}{d\tau}\tfrac{d\xi^\beta}{d\tau} + \Gamma^\mu_{\alpha\lambda},_\beta\tfrac{dx^\alpha}{d\tau}\tfrac{dx^\beta}{d\tau}\xi^\lambda\\ +\Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\alpha\lambda}\tfrac{dx^\alpha}{d\tau}\tfrac{dx^\beta}{d\tau}\xi^\lambda - \Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\alpha\beta}\tfrac{dx^\alpha}{d\tau}\tfrac{dx^\beta}{d\tau}\xi^\lambda = -R^\mu_{\alpha\lambda\beta}\tfrac{dx^\alpha}{d\tau}\xi^\lambda\tfrac{dx^\beta}{d\tau},\end{gathered} \tag{10}$$

where $R^\mu_{\alpha\lambda\beta}$ is the usual curvature Riemann tensor,

$$R^\mu_{\alpha\lambda\beta} = \Gamma^\mu_{\alpha\beta},_\lambda - \Gamma^\mu_{\alpha\lambda},_\beta + \Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\alpha\beta} - \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\alpha\lambda}. \tag{11}$$

By considering (1), we note that the fifth term of the left hand side of (10) can be written as

$$-\Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\xi^\lambda = \Gamma^\mu_{\sigma\lambda}\frac{d^2x^\sigma}{d\tau^2}\xi^\lambda. \tag{12}$$

Substituting this result into (10), it is not difficult to see that (10) can be written in the covariant form

$$\frac{D^2\xi^\mu}{D\tau^2} = -R^\mu_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}\xi^\lambda\frac{dx^\beta}{d\tau}, \tag{13}$$

which is, of course, the famous geodesic deviation equation (GDE) for a pair of nearby freely falling particles in a gravitational field background.

III. RELATIVISTIC TOP DEVIATION EQUATION

The equations of motion of a relativistic top moving in a gravitational field background are

$$\frac{D^2x^\mu}{D\tau^2} = -\frac{1}{2}R^\mu_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}S^{\lambda\beta}, \tag{14}$$

and

$$\frac{DS^{\mu\nu}}{D\tau} = 0. \tag{15}$$

Here $S^{\mu\nu} = -S^{\nu\mu}$ is the internal angular momentum (or the spin tensor) per unit mass of the top satisfying the Pirani constraint [26] $S^{\mu\nu}\frac{dx_\nu}{d\tau} = 0$. It is worth mentioning that the formulae (14) and (15) can be derived from a number of different methods [8] (see also Refs. [10], and references there in). Perhaps two of the most interesting are the Lagrangian formulation due to Rietdijk–Holten [9] and Galvao and Teitelboim [27] (see Refs. [28] and [29] for early works), and Hojman [30]. In the Rietdijk-Holten-Galvao-Teitelboim (RHGT) approach the spinning top is described by the variables $x^\mu(\tau)$ and $\theta^\mu(\tau)$, where the $\theta^\mu(\tau)$ are anticommuting variables (Grassmann coordinates), while in the Hojman formalism the rotation of the top is described by the four vectors $e^\mu_{(\alpha)}(\tau)$. Specifically, in the case of RHGT the spin tensor $S^{\mu\nu}$ is given by $S^{\mu\nu} = i\theta^\mu\theta^\nu$, while in the Hojman's approach $S^{\mu\nu}$ is the canonical momentum associated to the angular velocity $\sigma^{\mu\nu} = e^{\mu(\alpha)}\frac{D}{D\tau}e^\nu_{(\alpha)}$. In both cases the corresponding Lagrangian is taken to be a Poincaré invariant. One of the advantages of the Lagrangian formulation is that it allows the use of a variational principle to insure the consistence of the equations of motion. However, the main motivation to develop the Lagrangian formalism for the top is the desire to use the Dirac´s constraint Hamiltonian method to quantize the system. An important aspect of this construction is that it is more convenient to consider the Tulczyjew constraints [31] $S^{\mu\nu}P_\nu = 0$, with P_μ the linear momentum of the top, rather than the Pirani constraint. As a result the linear momentum P_μ turns out to be non parallel to the velocity $u^\mu = \frac{dx^\mu}{d\tau}$. However this difference is very slight and gauge dependent. Therefore, for practical purposes the Tulczyjew constraint and the Pirani constraint are the same and the reduced equations of motion look like (14) and (15). A final observation is that (14) can be understood as the analogue of the geodesic equation (1) and in fact it reduces to (1) when the spin tensor $S^{\mu\nu}$ vanishes.

After comparing equations (13) and (14) we observe a great similarity. But in fact they are very different in the sense that while equation (13) refers to a pair of nearby point particles, (14) is associated with just one physical system: a relativistic top. Nevertheless, this similarity was used as an inspiration to propose that just as (13) is used to detect gravitational waves, equation (14) can be used for the same purpose. In order to better understand the real differences between the two point particles system and the relativistic top it is necessary to derive the analogue of (13) for a pair of nearby relativistic tops. For this purpose let us closely follow the method of section II, but now using the relativistic top equation of motion (14) instead of the formula (1).

A nearby top must satisfy the corresponding equation of motion

$$\frac{d^2x'^{\mu}}{d\tau^2} + \Gamma'^{\mu}_{\alpha\beta}(x')\frac{dx'^{\alpha}}{d\tau}\frac{dx'^{\beta}}{d\tau} = -\frac{1}{2}R'^{\mu}_{\alpha\lambda\beta}(x')\frac{dx'^{\alpha}}{d\tau}S'^{\lambda\beta}. \tag{16}$$

Consider now a perturbation of the form

$$x'^{\mu} = x^{\mu} + \xi^{\mu}(x). \tag{17}$$

and

$$S'^{\mu\nu} = S^{\mu\nu} + S^{\mu\nu}{}_{,\alpha}\,\xi^{\alpha}(x). \tag{18}$$

We are interested in developing the formula

$$\begin{aligned} &\tfrac{d^2(x^{\mu}+\xi^{\mu})}{d\tau^2} + (\Gamma^{\mu}_{\alpha\beta}(x) + \Gamma^{\mu}_{\alpha\beta},_{\lambda}\,\xi^{\lambda})\tfrac{d(x^{\alpha}+\xi^{\alpha})}{d\tau}\tfrac{d(x^{\beta}+\xi^{\beta})}{d\tau} \\ &= -\tfrac{1}{2}(R^{\mu}_{\alpha\lambda\beta}(x) + R^{\mu}_{\alpha\lambda\beta},_{\sigma}\,\xi^{\sigma})(\tfrac{d(x^{\alpha}+\xi^{\alpha})}{d\tau})(S^{\lambda\beta} + S^{\lambda\beta},_{\gamma}\,\xi^{\gamma}) \end{aligned} \tag{19}$$

to first order in ξ^{μ}. We find

$$\begin{aligned} &\tfrac{d^2x^{\mu}}{d\tau^2} + \tfrac{d^2\xi^{\mu}}{d\tau^2} + \Gamma^{\mu}_{\alpha\beta}\tfrac{dx^{\alpha}}{d\tau}\tfrac{dx^{\beta}}{d\tau} + 2\Gamma^{\mu}_{\alpha\beta}\tfrac{dx^{\alpha}}{d\tau}\tfrac{d\xi^{\beta}}{d\tau} + \Gamma^{\mu}_{\alpha\beta},_{\lambda}\,\xi^{\lambda}\tfrac{dx^{\alpha}}{d\tau}\tfrac{dx^{\beta}}{d\tau} \\ &= -\tfrac{1}{2}[R^{\mu}_{\alpha\lambda\beta}\tfrac{dx^{\alpha}}{d\tau}S^{\lambda\beta} + R^{\mu}_{\alpha\lambda\beta}\tfrac{d\xi^{\alpha}}{d\tau}S^{\lambda\beta} + R^{\mu}_{\alpha\lambda\beta}\tfrac{dx^{\alpha}}{d\tau}S^{\lambda\beta},_{\gamma}\,\xi^{\gamma} + R^{\mu}_{\alpha\lambda\beta},_{\sigma}\,\xi^{\sigma}\tfrac{dx^{\alpha}}{d\tau}S^{\lambda\beta}]. \end{aligned} \tag{20}$$

By using (14), this formula is simplified to

$$\begin{aligned} &\tfrac{d^2\xi^{\mu}}{d\tau^2} + 2\Gamma^{\mu}_{\alpha\beta}\tfrac{dx^{\alpha}}{d\tau}\tfrac{d\xi^{\beta}}{d\tau} + \Gamma^{\mu}_{\alpha\beta},_{\lambda}\,\xi^{\lambda}\tfrac{dx^{\alpha}}{d\tau}\tfrac{dx^{\beta}}{d\tau} \\ &= -\tfrac{1}{2}[R^{\mu}_{\alpha\lambda\beta}\tfrac{d\xi^{\alpha}}{d\tau}S^{\lambda\beta} + R^{\mu}_{\alpha\lambda\beta}\tfrac{dx^{\alpha}}{d\tau}S^{\lambda\beta},_{\gamma}\,\xi^{\gamma} + R^{\mu}_{\alpha\lambda\beta},_{\sigma}\,\xi^{\sigma}\tfrac{dx^{\alpha}}{d\tau}S^{\lambda\beta}]. \end{aligned} \tag{21}$$

It is more convenient to write this equation as

$$
\begin{array}{c}
\frac{d^2\xi^\mu}{d\tau^2}+2\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\xi^\beta}{d\tau}=-\Gamma^\mu_{\alpha\beta},_\lambda\,\xi^\lambda\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\\
-\frac{1}{2}[R^\mu_{\alpha\lambda\beta}\frac{d\xi^\alpha}{d\tau}S^{\lambda\beta}+R^\mu_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}S^{\lambda\beta},_\gamma\,\xi^\gamma+R^\mu_{\alpha\lambda\beta},_\sigma\,\xi^\sigma\frac{dx^\alpha}{d\tau}S^{\lambda\beta}],
\end{array}
\tag{22}
$$

The next step is to write this formula in a covariant form.

For this purpose, as in section II, we add to both sides of (22) the expression

$$
\Gamma^\mu_{\alpha\lambda},_\beta\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\xi^\lambda+\Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\alpha\lambda}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\xi^\lambda-\Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\xi^\lambda. \tag{23}
$$

We get

$$
\begin{array}{c}
\frac{d^2\xi^\mu}{d\tau^2}+2\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{d\xi^\beta}{d\tau}+\Gamma^\mu_{\alpha\lambda},_\beta\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\xi^\lambda+\Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\alpha\lambda}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\xi^\lambda-\Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\xi^\lambda\\
=-R^\mu_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}\xi^\lambda\frac{dx^\beta}{d\tau}-\frac{1}{2}[R^\mu_{\alpha\lambda\beta}\frac{d\xi^\alpha}{d\tau}S^{\lambda\beta}+R^\mu_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}S^{\lambda\beta},_\gamma\,\xi^\gamma+R^\mu_{\alpha\lambda\beta},_\sigma\,\xi^\sigma\frac{dx^\alpha}{d\tau}S^{\lambda\beta}],
\end{array}
\tag{24}
$$

where we have used the definition of the Riemann tensor $R^\mu_{\alpha\lambda\beta}$.

In contrast with (12), using (14) we now find that the fifth term in the left hand side of (24) leads to

$$
-\Gamma^\mu_{\sigma\lambda}\Gamma^\sigma_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}\xi^\lambda=\Gamma^\mu_{\sigma\lambda}\frac{d^2x^\sigma}{d\tau^2}\xi^\lambda+\frac{1}{2}\Gamma^\mu_{\sigma\gamma}\xi^\gamma R^\sigma_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}S^{\lambda\beta}. \tag{25}
$$

Substituting (25) into (24), we learn that (24) can be written as

$$
\begin{array}{c}
\frac{D^2\xi^\mu}{D\tau^2}+\frac{1}{2}\Gamma^\mu_{\sigma\gamma}\xi^\gamma R^\sigma_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}S^{\lambda\beta}=-R^\mu_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}\xi^\lambda\frac{dx^\beta}{d\tau}\\
-\frac{1}{2}[R^\mu_{\alpha\lambda\beta}\frac{d\xi^\alpha}{d\tau}S^{\lambda\beta}+R^\mu_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}S^{\lambda\beta},_\gamma\,\xi^\gamma+R^\mu_{\alpha\lambda\beta},_\sigma\,\xi^\sigma\frac{dx^\alpha}{d\tau}S^{\lambda\beta}].
\end{array}
\tag{26}
$$

It is not difficult to show that

$$
\begin{array}{c}
-\frac{1}{2}[R^\mu_{\alpha\lambda\beta}\frac{d\xi^\alpha}{d\tau}S^{\lambda\beta}+R^\mu_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}S^{\lambda\beta},_\gamma\,\xi^\gamma+R^\mu_{\alpha\lambda\beta},_\sigma\,\xi^\sigma\frac{dx^\alpha}{d\tau}S^{\lambda\beta}]\\
=-\frac{1}{2}[R^\mu_{\alpha\lambda\beta}\frac{D\xi^\alpha}{D\tau}S^{\lambda\beta}+R^\mu_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}S^{\lambda\beta};_\gamma\,\xi^\gamma+R^\mu_{\alpha\lambda\beta};_\sigma\,\xi^\sigma\frac{dx^\alpha}{d\tau}S^{\lambda\beta}]\\
+\frac{1}{2}\Gamma^\mu_{\sigma\gamma}\xi^\gamma R^\sigma_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}S^{\lambda\beta}.
\end{array}
\tag{27}
$$

Here, for any contravariant tensor A^μ we define the covariant derivative as $A^\mu;_\sigma=A^\mu,_\sigma+\Gamma^\mu_{\sigma\gamma}A^\gamma$. Although the result (27) seems to be evident, it is not a trivial one since for its obtainment many terms were cancelled.

We finally discover that using (27), the expression (26) becomes

$$\frac{D^2\xi^\mu}{D\tau^2} = -R^\mu_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}\xi^\lambda\frac{dx^\beta}{d\tau}$$
$$-\frac{1}{2}[R^\mu_{\alpha\lambda\beta}\frac{D\xi^\alpha}{D\tau}S^{\lambda\beta} + R^\mu_{\alpha\lambda\beta}\frac{dx^\alpha}{d\tau}S^{\lambda\beta};_\gamma\xi^\gamma + R^\mu_{\alpha\lambda\beta};_\sigma\xi^\sigma\frac{dx^\alpha}{d\tau}S^{\lambda\beta}], \tag{28}$$

which is the covariant form of the relativistic top deviation equation (RTDE). Clearly, (28) reduces to (13) when the spin tensor $S^{\lambda\beta}$ vanishes. Therefore, (28) is an extension of (13). (It is worth mentioning that in Ref. [32] appears similar equations to those in (28).) One of the attractive features of (28) is that the spin angular momentum $S^{\lambda\beta}$ of the top is coupled to gravity via the curvature Riemann tensor $R^\mu_{\alpha\lambda\beta}$ and the gradient of this. It seems reasonable to think that this characteristic can provide a better description of the properties of the underlying curvature geometry. In particular, we shall see in the next section that the RTDE may be used to study different properties of a gravitational wave background.

IV. THE RELATIVISTIC TOP DEVIATION EQUATION AND GRAVITATIONAL WAVES

In this section we will investigate the consequences of equation (28) to the case of gravitational waves. But for completeness we will start by reviewing briefly how the formula (13) is used for this particular case.

Consider a gravitational wave in a flat background. For this case, the metric $g_{\mu\nu}$ can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{29}$$

where $\eta_{\mu\nu}$ is the Minkowski metric and $\mid h_{\mu\nu} \mid \ll 1.$ In the transverse-traceless gauge

$$h_{0\mu} = 0,$$
$$h_{ij,j} = 0, \tag{30}$$
$$h^\mu_\mu = 0,$$

with the indices $i, j, ..., etc$ running from 1 to 3. The Einstein gravitational field equations imply that h_{ij} satisfies the wave equation

$$\Box^2 h_{ij} = 0, \tag{31}$$

where $\Box^2 = \partial^\mu \partial_\mu$ is the D'Alambertian. In the gauge given in (30), the space-time components of the Riemann tensor R_{i0j0} have the simple form

$$R_{i0j0} = -\frac{1}{2} h_{ij,00}\,. \tag{32}$$

By considering (30), one discovers that h_{ij} can be written as

$$h_{ij} = A_+ e^+_{ij} + A_\times e^\times_{ij}, \tag{33}$$

where A_+ and $A_\times$ are two independent dimensionless amplitudes and e^+_{ij} and $e^\times_{ij}$ are polarization tensors. For a wave traveling in the $z-$direction the only nonvanishing components of e_{ij} are

$$\begin{aligned} e^+_{xx} &= -e^+_{yy}, \\ e^\times_{xy} &= e^\times_{yx} \end{aligned} \tag{34}$$

and in this case A_+ and $A_\times$ turn out to be functions that depend only on $t-z$.

In a proper reference frame we have $x^0 = \tau$, $x^i = 0$, so that $\frac{dx^0}{d\tau} = 1$ and $\frac{dx^i}{d\tau} = 0$. In this reference frame we find that (13) becomes

$$\frac{d^2\xi^i}{dt^2} = -R^i_{0j0}\xi^j, \tag{35}$$

By using (32), we find that equation (35) can be rewritten as

$$\frac{d^2\xi^i}{dt^2} = \frac{1}{2} h^i_{j,00}\,\xi^j, \tag{36}$$

with $h^i_j = \delta^{ik} h_{kj}$, and for a wave propagating in the $z-$direction, (36) implies

$$\frac{d^2\xi^z}{dt^2} = 0, \tag{37}$$

as well as

$$\frac{d^2\xi^a}{dt^2} = \frac{1}{2} h^a_{b,00}\,\xi^b, \tag{38}$$

where now, the latin indices $a, b, .., etc$ run from 1 to 2. The formula (38) tells us that only separations between two nearby point particles in the transverse direction are meaningful.

Let us now address the problem at hand, namely, we are interested in applying a similar method as the one above to the case of a system with two nearby relativistic tops. For this purpose let us consider the formula (28) in a proper reference frame. We have

$$\frac{d^2\xi^\mu}{dt^2} = -R^\mu_{0k0}\xi^k - \frac{1}{2}[R^\mu_{\alpha kl}\frac{d\xi^\alpha}{d\tau}S^{kl} + R^\mu_{0kl}S^{kl},_\gamma \xi^\gamma + R^\mu_{0kl},_\sigma \xi^\sigma S^{kl}], \tag{39}$$

where we used the fact that $S^{0\mu} = 0$ due to the Twlczyjew-Pirani constraint $S^{\mu\nu}\frac{dx_\nu}{d\tau} = 0$.

Using the symmetries of the Riemann curvature tensor we find that the time component of (39) is

$$\frac{d^2\xi^0}{dt^2} = -\frac{1}{2}R^0_{jkl}\frac{d\xi^j}{dt}S^{kl}, \tag{40}$$

while the space components become

$$\begin{aligned} \tfrac{d^2\xi^i}{dt^2} &= -R^i_{0k0}\xi^k - \tfrac{1}{2}[R^i_{0kl}\tfrac{d\xi^0}{dt}S^{kl} + R^i_{0kl}S^{kl},_0 \xi^0 + R^i_{0kl},_0 \xi^0 S^{kl}] \\ &\quad -\tfrac{1}{2}[R^i_{jkl}\tfrac{d\xi^j}{dt}S^{kl} + R^i_{0kl}S^{kl},_j \xi^j + R^i_{0kl},_j \xi^j S^{kl}]. \end{aligned} \tag{41}$$

We shall now apply (40) and (41) for the particular case of a gravitational plane wave propagating in the $z-$direction. It is convenient to write the indices $i, j...etc$ as (a, z). With this notation (40) can be written as

$$\frac{d^2\xi^0}{dt^2} = -R^0_{azb}\frac{d\xi^a}{dt}S^{zb}, \tag{42}$$

where we used the fact that the only nonvanishing components of the Riemann curvature tensor are

$$R_{zazb} = R_{0a0b} = -R_{0azb} = -\frac{1}{2}h_{ab,00}. \tag{43}$$

While (41) yields

$$\begin{aligned}
\frac{d^2\xi^i}{dt^2} &= -R^i_{0b0}\xi^b - [R^i_{0zb}\frac{d\xi^0}{dt}S^{zb} + R^i_{0zb}S^{zb}{}_{,0}\,\xi^0 + R^i_{0zb}{}_{,0}\,\xi^0 S^{zb}] \\
&-\frac{1}{2}[R^i_{0ab}\frac{d\xi^0}{dt}S^{ab} + R^i_{0ab}S^{ab}{}_{,0}\,\xi^0 + R^i_{0ab}{}_{,0}\,\xi^0 S^{ab}] \\
&-[R^i_{zbz}\frac{d\xi^z}{dt}S^{bz} + R^i_{0bz}S^{bz}{}_{,z}\,\xi^z + R^i_{0bz}{}_{,z}\,\xi^z S^{bz}] \\
&-\frac{1}{2}[R^i_{zab}\frac{d\xi^z}{dt}S^{ab} + R^i_{0ab}S^{ab}{}_{,z}\,\xi^z + R^i_{0ab}{}_{,z}\,\xi^z S^{ab}] \\
&-[R^i_{abz}\frac{d\xi^a}{dt}S^{bz} + R^i_{0bz}S^{bz}{}_{,a}\,\xi^a + R^i_{0bz}{}_{,a}\,\xi^a S^{bz}] \\
&-\frac{1}{2}[R^i_{abc}\frac{d\xi^a}{dt}S^{bc} + R^i_{0bc}S^{bc}{}_{,a}\,\xi^a + R^i_{0bc}{}_{,a}\,\xi^a S^{bc}]
\end{aligned} \tag{44}$$

The z component of (44) is

$$\frac{d^2\xi^z}{dt^2} = -R^z_{azb}\frac{d\xi^a}{dt}S^{zb}, \tag{45}$$

where we used (43), and the x and y components are

$$\begin{aligned}
\frac{d^2\xi^a}{dt^2} &= -R^a_{0b0}\xi^b - [R^a_{0bz}\frac{d\xi^0}{dt}S^{bz} + R^a_{0bz}S^{bz}{}_{,0}\,\xi^0 + R^a_{0bz}{}_{,0}\,\xi^0 S^{bz}] \\
&-[R^a_{zbz}\frac{d\xi^z}{dt}S^{bz} + R^a_{0bz}S^{bz}{}_{,z}\,\xi^z + R^a_{0bz}{}_{,z}\,\xi^z S^{bz}] - R^a_{0bz}S^{bz}{}_{,a}\,\xi^a.
\end{aligned} \tag{46}$$

Note that the S^{ab} component of the spin angular momentum does not appear in (42), (45) and (46) and that only the S^{zb} component remains. To better understand this let us define

$$S^i = \frac{1}{2}\varepsilon^{ijk}S_{jk}, \tag{47}$$

where ε^{ijk} is the Levi-Civita tensor with $\varepsilon^{xyz} = 1$. From (47) we see that the z component of the intrinsic angular momentum does not play any role in equations (42), (45) and (46). This means that no effect is expected when the top is oriented along the direction of propagation of the gravitational wave .

The second interesting observation is that if S^{zb} is nonvanishing then $\frac{d^2\xi^0}{dt^2} \neq 0$ and $\frac{d^2\xi^z}{dt^2} \neq 0$, in contrast with the case of a pair of nearby point particles in which both of these terms vanish. The third important observation is that the $\frac{d^2\xi^a}{dt^2}$ equation contains a large number of terms in addition to the usual one $R^a_{0b0}\xi^b$. Clearly, this means that the solution of (46) will not be as simple as in the case of nearby point particles.

Let us focus on the terms in (46) not involving derivatives of S^{zb} and the Riemann tensor, which presumably represent small order corrections. In this case (46) is reduced to

$$\frac{d^2\xi^a}{dt^2} = -R^a_{0b0}\xi^b - R^a_{0bz}\frac{d\xi^0}{dt}S^{bz} - R^a_{zbz}\frac{d\xi^z}{dt}S^{bz}. \tag{48}$$

By means of (43), we find that (42), (45) and (48) become

$$\frac{d^2\xi^0}{dt^2} = -\frac{1}{2}h_{ab,00}\frac{d\xi^a}{dt}S^{zb}, \tag{49}$$

$$\frac{d^2\xi^z}{dt^2} = \frac{1}{2}h_{ab,00}\frac{d\xi^a}{dt}S^{zb} \tag{50}$$

and

$$\frac{d^2\xi^a}{dt^2} = \frac{1}{2}h^a_b{}_{,00}\,\xi^b - \frac{1}{2}h^a_b{}_{,00}\,\frac{d\xi^0}{dt}S^{bz} + \frac{1}{2}h^a_b{}_{,00}\,\frac{d\xi^z}{dt}S^{bz}, \tag{51}$$

respectively. From (49) and (50) we discover that we can set $\xi^0 = -\xi^z + cte.$ Therefore (51) becomes

$$\frac{d^2\xi^a}{dt^2} = \frac{1}{2}h^a_b{}_{,00}\,\xi^b - h^a_b{}_{,00}\,\frac{d\xi^z}{dt}S^{bz}. \tag{52}$$

Observe that if at an initial time the spin of the top S^{bz} has an orientation such that

$$\xi^b - 2\frac{d\xi^z}{dt}S^{bz} = 0, \tag{53}$$

then there is not transverse motion $\xi^a = const.$ and therefore (50) admits the solution

$$\xi^z = const. \tag{54}$$

What this means is that if the spin of the top is oriented along the vector separation ξ^a of the two tops, then the gravitational wave does not produce any effect on the system. This appears to be a new interesting result since in the ordinary case of GDE the wave is always transverse in its physical effects.

Let us now look for a solution of (52) of the form

$$\xi^a = \xi^a_0 + \frac{1}{2}h^a_b\xi^b_0 - h^a_b(\frac{d\xi^z}{dt}\,|_0)S^{bz}. \tag{55}$$

where $\xi_0^a = const$. We observe that $\frac{d\xi^a}{dt} = \frac{1}{2}h^a_b{}_{,0}\,\xi_0^b - h^a_b{}_{,0}\,(\frac{d\xi^z}{dt}\mid_0)S^{bz}$. Substituting (55) into (50) we find that to first order in h (50) becomes

$$\frac{d^2\xi^z}{dt^2} \approx 0, \tag{56}$$

Therefore if initially $\frac{d\xi^z}{dt}\mid_0 = 0$ to first order of approximation the solution (55) reduces to the ordinary case of a pair nearby point particles.

V. COMMENTS

We have been pursuing the possibility of using relativistic tops as detectors of gravitational waves. In two previous works (Refs. [6] and [7]) an isolated top in a gravitational field was considered, making difficult to compare the results with the case of DGE. In order to overcome this difficulty and to gain further progress towards our goal, in this article we have derived the RTDE equation for a pair of nearby tops. We have shown that the RTDE reduces to the GDE when the spin tensor vanishes.

By considering a plane gravitational wave we find the solution of the RTDE for two simple cases. In the first case we discover that if the internal angular momentum of the top is oriented along the vector separation of the two tops, the gravitational wave does not produce any effect on the physical system. In the second more general case, we find that the nearby top will have an effect different from the case of GDE only to a second order of approximation in the perturbation h_{ab}. At first sight, it could seem that these two cases show that even though the RTDE formulation it may be theoretically interesting it does not offer a promising route for experiments. However, rough estimates can show that this is not the case.

Let us write (55) in the form

$$\xi^a = \xi_0^a + \frac{1}{2}h^a_b\xi_0^b + \eta^a, \tag{57}$$

where

$$\eta^a = h^a_b(\frac{d\xi^z}{dt}\mid_0)S^{bz}. \tag{58}$$

Here, we are interested in making a rough estimate for η^a. The result $\xi^0 = -\xi^z + const.$ allows to set $(\frac{d\xi^z}{dt}\mid_0) \sim 1.$ Thus, (58) is reduced to

$$\eta^a \sim h^a_b\frac{S^{bz}}{m_0 c}, \tag{59}$$

where, in order to have the correct units, we restored the constants m_0 and c. (Here, m_0 is the mass of the top and c denotes the light velocity.) In an order of magnitude, the expression (59) can be written in the form

$$\eta \sim h\frac{S}{m_0 c}. \tag{60}$$

Roughly we can write $S \sim m_0 r^2 \omega$, so (60) becomes

$$\eta \sim h\frac{r^2\omega}{c}. \tag{61}$$

For typical milli second (ms) pulsars $\omega \sim 10^3 Hz$ and $r \sim 10^6 cm$. Therefore, $\frac{r^2\omega}{c} \sim 10^{15} cm$ and from (61) we find that

$$\eta \sim h(10^5 cm). \tag{62}$$

If we consider the best expected sensitivities of the wave amplitude h over the earth, roughly $h \sim 10^{-18}$, we find that $\eta \sim 10^{-13} cm.$ This value for η is, of course, too small for current research detections. However, this estimate is based on the value $h \sim 10^{-18}$, which corresponds to gravitational radiation sources at distances of Mpc from the Earth, but it may not necessarily correspond to some places in the interstellar medium where the arrival of gravitational waves can have a stronger effect. For instance, a pulsar may be close enough to a strong source of gravitational waves companion such as another pulsar, black hole or supernova. If this is the case, from the formula

$$h_{jk} = \frac{2}{r}\frac{G}{c^2}\frac{d^2 Q_{jk}}{dt^2}, \tag{63}$$

where Q_{jk} is the mass quadrupole moment, one finds that the expected value for h should be increased by several orders of magnitude. For example, it is known that some globular clusters may be considered as kitchens for making several ms pulsars. In such a scenario, from the rough estimate of (63)

$$h \sim \frac{r_{sch}}{r}\frac{r_{sch}}{R}, \tag{64}$$

where r_{sch} is the Schwarzschild radius and R is the radius of the source system, one finds that the value of h could be, for instance, of the order 10^{-1}, which leads to a sensitivity of $\eta \sim 10^4 cm.$ This result appears to be large enough to be detected from the arrival radio signal coming from a pulsar.

A possible interesting extension of this work is to apply a similar method to the one used to obtain the RTDE in the case of the nongeodesic equations of motion of spinning particles in a teleparallel gravitational background

(GETGB) [33]-[34]. The GETGB equations are an extension of the RTEM in the sense that they include torsion interactions in addition to the gravity spin interaction. Recently, Garcia [35] has revisited the motion of spinless particles in a teleparallel gravitational waves background and proposed an experimental mechanism to detect the torsion via the GETGB model. The complete picture should be to detect both gravitational waves and torsional waves and therefore it may be interesting for further research to find the non-deviation geodesic equations associated to the GETGB model.

Finally, we would like to comment about the possibility of using the RTDE in connection with the inflationary gravitational waves scenario and brane new world cosmology. In the first case, just before the universe became transparent, the gravitational waves, produced during inflation, interacted with the plasma producing polarization patterns of the cosmic microwave background (CMB). This kind of phenomena is especially interesting since recently some observations have shown the variation of the polarization pattern of the CMB. The RTDE model can be interesting in this direction if spin tensor $S^{\lambda\beta}$ is identified as the fermionic spin ($S^{\mu\nu} = i\theta^{\nu}\theta^{\mu}$) of elementary particles. According to the RTDE, a gravitational wave should cause two kinds of motions on the constituent fermionic particles of such a plasma. The first one is the motion of the particles caused by the forces due to GDE and the second one is the motion on the plasma caused by the spin-gravity interaction. Therefore, one should expect that the spin-gravity interaction may also leave a "print" in the polarization pattern of the CMB.

In the case of the world brane universes, only gravity can move in the extra dimensions, all the matter and other forces are confined to the branes. Only gravitational waves (or gravitons) travel from brane to brane carrying some energy information away from the branes. Presumably, such gravitational waves affect objects held together by gravity, such as stars and galaxies, by distances shorter than millimeters. But, according to our above rough estimate such short distance changes are also predicted by the RTDE model.

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