# Symplectic Symmetry of the Neutrino Mass For Many Neutrino Flavors

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**Abstract**

The algebraic structure of the neutrino mass Hamiltonian is presented for two neutrino flavors considering both Dirac and Majorana mass terms. It is shown that the algebra is $Sp(8)$ and also discussed how the algebraic structure generalizes for the case of more than two neutrino flavors.

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## 1 Introduction

Recently, the algebraic structure of the most general neutrino mass Hamiltonian has been discussed by Balantekin and Öztürk [1] and it has been found that the algebra is $Sp(4)$ for a single neutrino flavor. Basically, one can consider the mass term in the Hamiltonian as a generalized pairing problem and for a single neutrino (four Dirac components) the most general pairing algebra is $SO(8)$ (see, for example, [2]). The authors in [1] showed that for the neutrino mass Lorentz invariance constrains this algebra to be $Sp(4)$.

The Pauli-Gürsey transformation [3, 4] which relates a Dirac spinor to its charge conjugation corresponds to an $SU(2)$ rotation which is embedded in the associated $Sp(4)$ Lie group. A particular Pauli-Gürsey transformation generates the seesaw mechanism [5] which explains why neutrino masses are much lighter than those of the quarks and leptons. In the literature, the Pauli-Gürsey transformation plays a crucial role in constructing the diquark charges in grand unified theories (see, for example, [6]).

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It has also been mentioned in [1] that for three neutrino flavors one can introduce three commuting copies of the $Sp(4)$ algebra and similar arguments follow. It is possible to introduce a single $Sp(4)$ algebra for three flavors if the individual masses of the mass eigenstates are equal up to phases. This case seems to be too restrictive for model building if one considers the recent observations.

The purpose of the present work is to investigate the algebraic structure for two neutrino flavors in detail and to discuss whether the algebraic structure gives any constraint on the number of flavors of neutrinos.

# 2 The Algebraic Structure For Two Neutrino Flavors

For simplicity if one starts with 2-neutrino flavors, say $(\nu_e, \nu_\mu)$, and considers the Lorentz invariant mass terms; the Dirac mass Hamiltonian can be written as [7]

$$H_m^D = \int d^3x \Big[ \big( m_{ee}^D \bar{\nu}_{eL} \nu_{eR} + h.c. \big) + \big( m_{\mu\mu}^D \bar{\nu}_{\mu L} \nu_{\mu R} + h.c. \big) + \big( m_{\mu e}^D ( \bar{\nu}_{\mu L} \nu_{eR} + \bar{\nu}_{eL} \nu_{\mu R} ) + h.c. \big) \Big] \tag{1}$$

and the left- and right-handed Majorana mass Hamiltonian as [7]

$$H_m^L = \int d^3x \Big[ \frac{1}{2} \big( m_{ee}^L \bar{\nu}_{eL} (\nu_{eL})^c + h.c. \big) + \frac{1}{2} \big( m_{\mu\mu}^L \bar{\nu}_{\mu L} (\nu_{\mu L})^c + h.c. \big) + \big( m_{\mu e}^L \bar{\nu}_{eL} (\nu_{\mu L})^c + h.c. \big) \Big] \tag{2}$$

$$H_m^R = \int d^3x \Big[ \frac{1}{2} \big( m_{ee}^R \bar{\nu}_{eR} (\nu_{eR})^c + h.c. \big) + \frac{1}{2} \big( m_{\mu\mu}^R \bar{\nu}_{\mu R} (\nu_{\mu R})^c + h.c. \big) + \big( m_{\mu e}^R \bar{\nu}_{eR} (\nu_{\mu R})^c + h.c. \big) \Big]. \tag{3}$$

In these equations $m^D$, $m^L$ and $m^R$ are the Dirac, left- and right-handed Majorana masses. The last parenthesis in each Hamiltonian represents the mass-mixing terms. As has been done in [1], each mass term can be defined as a generator and the commutations relations between the generators lead to a closed algebraic structure.

Starting with non-mixing terms (the first two parenthesis in Eqs. (1), (2) and (3)) one can write down two commuting $Sp(4)$ algebras:

$$
\begin{array}{ll}
D^e_+ = \int d^3x(\bar\nu_{eL}\nu_{eR}) & D^\mu_+ = \int d^3x(\bar\nu_{\mu L}\nu_{\mu R}) \\
D^e_- = \int d^3x(\bar\nu_{eR}\nu_{eL}) & D^\mu_- = \int d^3x(\bar\nu_{\mu R}\nu_{\mu L}) \\
D^e_0 = \frac{1}{2}(\nu^\dagger_{eL}\nu_{eL} - \nu^\dagger_{eR}\nu_{eR}) & D^\mu_0 = \frac{1}{2}(\nu^\dagger_{\mu L}\nu_{\mu L} - \nu^\dagger_{\mu R}\nu_{\mu R}) \\
L^e_+ = \frac{1}{2}\int d^3x[\bar\nu_{eL}(\nu_{eL})^c] & L^\mu_+ = \frac{1}{2}\int d^3x[\bar\nu_{\mu L}(\nu_{\mu L})^c] \\
L^e_- = \frac{1}{2}\int d^3x[\overline{(\nu_{eL})^c}\nu_{eL}] & L^\mu_- = \frac{1}{2}\int d^3x[\overline{(\nu_{\mu L})^c}\nu_{\mu L}] \\
L^e_0 = \frac{1}{4}\int d^3x(\nu^\dagger_{eL}\nu_{eL} - \nu_{eL}\nu^\dagger_{eL}) & L^\mu_0 = \frac{1}{4}\int d^3x(\nu^\dagger_{\mu L}\nu_{\mu L} - \nu_{\mu L}\nu^\dagger_{\mu L}) \\
R^e_+ = \frac{1}{2}\int d^3x[\overline{(\nu_{eR})^c}\nu_{eR}] & R^\mu_+ = \frac{1}{2}\int d^3x[\overline{(\nu_{\mu R})^c}\nu_{\mu R}] \\
R^e_- = \frac{1}{2}\int d^3x[\bar\nu_{eR}(\nu_{eR})^c] & R^\mu_- = \frac{1}{2}\int d^3x[\bar\nu_{\mu R}(\nu_{\mu R})^c] \\
R^e_0 = \frac{1}{4}\int d^3x(\nu_{eR}\nu^\dagger_{eR} - \nu^\dagger_{eR}\nu_{eR}) & R^\mu_0 = \frac{1}{4}\int d^3x(\nu_{\mu R}\nu^\dagger_{\mu R} - \nu^\dagger_{\mu R}\nu_{\mu R}) \\
A^e_+ = \int d^3x(-\nu^T_{eL}C\gamma_0\nu_{eR}) & A^\mu_+ = \int d^3x(-\nu^T_{\mu L}C\gamma_0\nu_{\mu R}) \\
A^e_- = \int d^3x(\nu^\dagger_{eR}\gamma_0 C\nu^T_{eL}{}^\dagger) & A^\mu_- = \int d^3x(\nu^\dagger_{\mu R}\gamma_0 C\nu^T_{\mu L}{}^\dagger) \\
A^e_0 = \frac{1}{2}\int d^3x(\nu_{eL}\nu^\dagger_{eL} - \nu^\dagger_{eR}\nu_{eR}) & A^\mu_0 = \frac{1}{2}\int d^3x(\nu_{\mu L}\nu^\dagger_{\mu L} - \nu^\dagger_{\mu R}\nu_{\mu R})
\end{array}
\tag{4}
$$

The $D$'s, $L$'s, $R$'s and $A$'s in each column satisfy an $SU(2)$ algebra, suppressing the $e, \mu$ indices above, one has

$$[D_+, D_-] = 2D_0 \quad , \quad [D_0, D_+] = D_+ \quad , \quad [D_0, D_-] = -D_- \tag{5}$$

$$[L_+, L_-] = 2L_0 \quad , \quad [L_0, L_+] = L_+ \quad , \quad [L_0, L_-] = -L_- \tag{6}$$

$$[R_+, R_-] = 2R_0 \quad , \quad [R_0, R_+] = R_+ \quad , \quad [R_0, R_-] = -R_- \tag{7}$$

$$[A_+, A_-] = 2A_0 \quad , \quad [A_0, A_+] = A_+ \quad , \quad [A_0, A_-] = -A_- \tag{8}$$

and the $D^e_0, A^e_0, D^\mu_0$ and $A^\mu_0$ are not independent generators:

$$
\begin{array}{l}
D^e_0 \equiv L^e_0 + R^e_0 \quad , \quad D^\mu_0 \equiv L^\mu_0 + R^\mu_0, \\
A^e_0 \equiv R^e_0 - L^e_0 \quad , \quad A^\mu_0 \equiv R^\mu_0 - L^\mu_0.
\end{array}
\tag{9}
$$

The ten independent generators in each column in Eq. (4) form an $Sp(4)$ algebra, say $Sp(4)^e$ and $Sp(4)^\mu$, respectively.

Similarly defining each mixing-mass term as a generator in the following, one can easily show that the ten generators $D^M_+$, $D^M_-$, $L^M_+$, $L^M_-$, $L^M_0$, $R^M_+$, $R^M_-$, $R$

$$
\begin{array}{rcl}
D^M_+ &=& \int d^3x\,(\bar\nu_{\mu L}\nu_{eR}+\bar\nu_{eL}\nu_{\mu R}) \equiv D^{1M}_+ + D^{2M}_+ \\
[0.3cm]D^M_- &=& \int d^3x\,(\bar\nu_{eR}\nu_{\mu L}+\bar\nu_{\mu R}\nu_{eL}) \equiv D^{1M}_- + D^{2M}_- \\
[0.3cm]D^M_0 &=& \frac{1}{2}\int d^3x(\nu^\dagger_{eL}\nu_{eL}+\nu^\dagger_{\mu L}\nu_{\mu L}-\nu^\dagger_{eR}\nu_{eR}-\nu^\dagger_{\mu R}\nu_{\mu R}) \equiv L^M_0 + R^M_0 \\
[0.3cm]L^M_+ &=& \int d^3x[\bar\nu_{eL}(\nu_{\mu L})^c] \\
[0.3cm]L^M_- &=& \int d^3x[\overline{(\nu_{\mu L})^c}\nu_{eL}] \\
[0.3cm]L^M_0 &=& \frac{1}{2}\int d^3x(\nu^\dagger_{eL}\nu_{eL}-\nu_{\mu L}\nu^\dagger_{\mu L}) \equiv L^e_0 + L^\mu_0 \\
[0.3cm]R^M_+ &=& \int d^3x[\overline{(\nu_{\mu R})^c}\nu_{eR}] \\
[0.3cm]R^M_- &=& \int d^3x[\bar\nu_{eR}(\nu_{\mu R})^c] \\
[8pt]R^M_0 &=& \frac{1}{2}\int d^3x(-\nu^\dagger_{eR}\nu_{eR}+\nu_{\mu R}\nu^\dagger_{\mu R}) \equiv R^e_0 + R^\mu_0 \\
[0.3cm]A^M_+ &=& \int d^3x(-\nu^T_{eL}C\gamma_0\nu_{eR}-\nu^T_{\mu L}C\gamma_0\nu_{\mu R}) \equiv A^e_+ + A^\mu_+ \\
[0.3cm]A^M_- &=& \int d^3x(\nu^\dagger_{eR}\gamma_0 C\nu^T_{eL}{}^\dagger+\nu^\dagger_{\mu R}\gamma_0 C\nu^T_{\mu L}{}^\dagger) \equiv A^e_- + A^\mu_- \\
[0.3cm]A^M_0 &=& \frac{1}{2}\int d^3x(\nu_{eL}\nu^\dagger_{eL}+\nu_{\mu L}\nu^\dagger_{\mu L}-\nu^\dagger_{eR}\nu_{eR}-\nu^\dagger_{\mu R}\nu_{\mu R}) \equiv R^M_0 - L^M_0
\end{array}
\qquad (10)
$$

The $D^M$'s, $L^M$'s, $R^M$'s and $A^M$'s satisfy an $SU(2)$ algebra:

$$[D^M_+, D^M_-] = 2D^M_0 \quad , \quad [D^M_0, D^M_+] = D^M_+ \quad , \quad [D^M_0, D^M_-] = -D^M_- \qquad (11)$$

$$[L^M_+, L^M_-] = 2L^M_0 \quad , \quad [L^M_0, L^M_+] = L^M_+ \quad , \quad [L^M_0, L^M_-] = -L^M_- \qquad (12)$$

$$[R^M_+, R^M_-] = 2R^M_0 \quad , \quad [R^M_0, R^M_+] = R^M_+ \quad , \quad [R^M_0, R^M_-] = -R^M_- \qquad (13)$$

$$[A^M_+, A^M_-] = 2A^M_0 \quad , \quad [A^M_0, A^M_+] = A^M_+ \quad , \quad [A^M_0, A^M_-] = -A^M_- \qquad (14)$$

Note that $D^M_0$, $L^M_0$, $R^M_0$, $A^M_0$ can be expressed in terms of the other generators as in Eq. (10).

To find a closed algebraic structure, one has to calculate all the commutation relations between the generators of $Sp(4)^e$, $Sp(4)^\mu$ and $Sp(4)^M$. They give the following new generators:

$$
\begin{aligned}
N_+ &= \int d^3x(\nu^\dagger_{\mu L}\nu_{eL} - \nu^\dagger_{eR}\nu_{\mu R}) \equiv N^1_+ + N^2_+ \\
[0.3cm]N_- &= \int d^3x(\nu^\dagger_{eL}\nu_{\mu L} - \nu^\dagger_{\mu R}\nu_{eR}) \equiv N^1_- + N^2_- \\
[0.3cm]N_0 &= \frac{1}{2}\int d^3x(\nu^\dagger_{\mu L}\nu_{\mu L} - \nu^\dagger_{eL}\nu_{eL} + \nu^\dagger_{eR}\nu_{eR} - \nu^\dagger_{\mu R}\nu_{\mu R}) \\
[0.3cm]M_+ &= \int d^3x(\nu^T_{eL}\gamma_0 C\nu_{\mu R}) \\
[0.3cm]M_- &= \int d^3x(-\nu^\dagger_{\mu R}C\gamma_0\nu^{T\dagger}_{eL}) \\
[0.3cm]M_0 &= \frac{1}{2}\int d^3x(\nu_{eL}\nu^\dagger_{eL} - \nu^\dagger_{\mu R}\nu_{\mu R}) \\
[0.3cm]T_+ &= \int d^3x(-\nu^T_{eR}\gamma_0 C\nu_{\mu L}) \\
[0.3cm]T_- &= \int d^3x(-\nu^\dagger_{\mu L}C\gamma_0\nu^{T\dagger}_{eR}) \\
[0.3cm]T_0 &= \frac{1}{2}\int d^3x(\nu_{\mu L}\nu^\dagger_{\mu L} - \nu^\dagger_{eR}\nu_{eR})
\end{aligned}
\qquad (15)
$$

The $N$'s, $M$'s and $T$'s satisfy an $SU(2)$ algebra:

$$[N_+, N_-] = 2N_0 \quad , \quad [N_0, N_+] = N_+ \quad , \quad [N_0, N_-] = -N_- \qquad (16)$$

$$[M_+, M_-] = 2M_0 \quad , \quad [M_0, M_+] = M_+ \quad , \quad [M_0, M_-] = -M_- \qquad (17)$$

$$[T_+, T_-] = TN_0 \quad , \quad [T_0, T_+] = T_+ \quad , \quad [T_0, T_-] = -T_- \qquad (18)$$

also note that $N_0 \equiv M_0 - T_0$. The commutation relations between the new generators in Eq. (15) and all the others in Eqs. (4), (10) don't give any other new generator. As a result, the 36 independent generators in the following form a Lie algebra which is the symplectic algebra $Sp(8)$.

$$D^e_+, D^e_-, L^e_+, L^e_-, L^e_0, R^e_+, R^e_-, R^e_0 \quad \text{(8 generators)}$$

$$D^\mu_+, D^\mu_-, L^\mu_+, L^\mu_-, L^\mu_0, R^\mu_+, R^\mu_-, R^\mu_0 \quad \text{(8 generators)}$$

$$D^{1M}_+, D^{2M}_+, D^{1M}_-, D^{2M}_-, L^M_+, L^M_-, R^M_+, R^M_-, A^M_+, A^M_- \quad \text{(10 generators)}$$

$$N^1_+, N^2_+, N^1_-, N^2_-, M_+, M_-, M_0, T_+, T_-, T_0 \quad \text{(10 generators)}$$

In order to illustrate the $Sp(8)$ algebra structure 282 commutation relations have been calculated, due to space limitations these relations are not given here. Fortunately, after presenting the algebraic structure for two neutrino flavors in detail it is straightforward to construct the algebra for 3, 4, ... neutrino flavors, one obtains $Sp(12)$, $Sp(16)$, ... respectively.

# 3 Conclusion

After having discussed the algebraic structure for a single neutrino flavor in [1], it has been shown in the present work that the symplectic algebra structure can be extended to many flavors of neutrinos. The detailed calculations are given for the two neutrino flavors and the algebra is found to be the $Sp(8)$ symplectic algebra.

For $n$ neutrino flavors the algebraic structure can be generalized to $Sp(4n)$. Since one does not obtain a non-closed algebra by increasing the number of flavors, it may be useful to emphasize that the approach based on algebraic structure does not give any constraint on the number of neutrino flavors. Nevertheless, it would be interesting to investigate if the symplectic algebra facilitates the computation of physical neutrino parameters, work along this direction is in progress.

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