

# An exact universal amplitude ratio for percolation.

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## Abstract

The universal amplitude ratio  $\tilde{R}_\xi^+$  for percolation in two dimensions is determined exactly using results for the dilute A model in regime 1, by way of a relationship with the  $q$ -state Potts model, with  $q \leq 4$ .

Percolation is a subject which can be described simply and yet has many important physical applications (see [1] and references therein). In statistical mechanics, one natural arena in which to study its inherent critical phenomena, percolation is related by way of the random cluster model [2] to the  $q$ -state Potts model with  $q = 1$ .

The *critical exponents* of the two-dimensional Potts model for  $q \leq 4$ , determined from numerical and renormalization group studies [3, 4], led to the identification [5] of the Potts model (in the scaling limit) for  $q = 2, 3, 4$  with the  $\phi_{2,1}$  perturbation of the minimal unitary series  $\mathcal{M}(t, t+1)$  by way of

$$\sqrt{q} = 2 \sin \left( \frac{\pi(t-1)}{2(t+1)} \right). \quad (1)$$

With care, one can extend the identification to  $q = 1$ .

On the other hand, for the associated *critical amplitudes* (except in the Ising case  $q = 2$  where exact results are known [6]) the series and Monte Carlo results can vary widely [7], making it difficult to obtain reliable estimates for the universal amplitude ratios [8]. Recently numerical values for various amplitude ratios of the Potts model (including percolation) have been determined [9] by a technique of perturbed conformal field theory, specifically by the two-kink approximation in the form factor approach to the  $S$ -matrix of [10].

The only one of the amplitude ratios determined in [9] which will be considered in this letter is the percolation amplitude [8, 9]

$$\tilde{R}_\xi^+ = \lim_{q \rightarrow 1} \left[ \frac{\alpha(2-\alpha)(1-\alpha)\mathcal{A}_f}{(q-1)} \right]^{\frac{1}{2}} \xi_0^+. \quad (2)$$

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Here  $\xi_0^+$  is the leading term amplitude of the correlation length (above the critical temperature)

$$\xi \simeq \xi_0^+ \tau^{-\nu},$$

and  $\mathcal{A}_f$  and  $\alpha$  come from the singular part of the free energy,

$$-f_s \simeq \mathcal{A}_f \tau^{2-\alpha}.$$

The percolation analogue of the free energy is the mean number of clusters, and of  $\xi$ , the pair connectivity. That  $\tilde{R}_\xi$  is universal, i.e. independent of metric factors associated with the reduced temperature  $\tau \propto T - T_c$ , follows from the scaling relation  $2 - \alpha = d\nu$  in  $d = 2$  dimensions.

There is an exactly solvable model, the dilute  $A_L$  model [13], which has also been identified with the  $\phi_{2,1}$  perturbation of the minimal unitary series [14]. It is an  $L$  state, interaction-round-a-face model, solved in four regimes, two of which provide off-critical extensions of the unitary minimal conformal field theories [14]. While regime 2 of the model is associated with the perturbation  $\phi_{1,2}$ , regime 1 of the model realizes (in the scaling limit) the perturbation  $\phi_{2,1}$  of the minimal unitary series  $\mathcal{M}(L+1, L+2)$ .

That they are both identified with the same perturbation  $\phi_{2,1}$  suggests a relationship at least between certain members of the dilute  $A_L$  hierarchy and the Potts models. In general the Potts model and the dilute A model represent different universality classes associated with  $\phi_{2,1}$ , just as in the field theory context there may be more than one  $S$ -matrix for a given perturbation. One simple way to see that the models have different internal symmetries is to count ground states: for the Potts model there are  $q$  of them, and for the dilute A model the number grows linearly with  $L$ , but the connection between  $L$  and  $q$  implied by setting  $t = L + 1$  in (1) is certainly not linear.

Nevertheless, on the basis of this relationship between models, weaker than shared universality class, it is still possible to determine one universal amplitude ratio for the Potts model, including percolation, from the dilute A model. In the language of perturbed conformal field theory, the free energy and the correlation length are related directly to the coupling constant  $g$  of the perturbation and the associated conformal weight  $\Delta$ :

$$f_s \sim g^{\frac{d}{d-2\Delta}} \quad \xi \sim g^{\frac{-1}{d-2\Delta}}.$$

Thus when attention is confined to the correlation length amplitude ratio (2), the required quantities  $\mathcal{A}_f$ ,  $\xi_0$  and  $\alpha$  (or equivalently  $\Delta$ ) relate solely to the perturbing operator. This operator is  $\phi_{2,1}$  for both dilute A in regime 1 and the Potts model, and any universal observable associated only to it should be common [15].

From (1), percolation corresponds to  $t = 2$ , so the member of the dilute A hierarchy to be considered is  $L = 1$ . Since  $L$  labels the number of states in the dilute  $A_L$  model, it is properly an integer  $L \geq 2$ . However, it has long been realized that  $q$  can be treated as a continuous variable in an appropriate formulation of the Potts model [2] and it is in this spirit that we now take  $L \rightarrow 1$  in the free energy and correlation length expressions of the dilute A model.

The leading term of the correlation length [16, 17] of the dilute  $A_L$  model in regime 1 is

$$\xi^{-1} \simeq 4\sqrt{3} p^{2(L+1)/3L}.$$

There is good reason to believe that this applies to the high temperature regime for all  $L$ , although it was actually determined from  $L$  odd alone.

Although the singular part of the free energy of the dilute A model has been determined both by the inversion relation [14] and exact perturbative [16, 17] approaches, the coefficient has not previously been explicitly given; apart from  $L = 2$ , in regime 1 it may be written

$$-f_s \simeq \frac{4\sqrt{3} \sin(2\pi(L-1)/3L)}{\sin(\pi(L-2)/3L)} p^{4(L+1)/3L}. \quad (3)$$

This correctly gives the critical exponent  $\alpha = -2/3$  for percolation by setting  $L = 1$  in

$$\alpha = 2 - \frac{4(L+1)}{3L} = \frac{2(L-2)}{3L}.$$

The coefficient in (3) vanishes at  $L = 1$ , which agrees with the  $q = 1$  Potts model amplitude given by Kaufman and Andelman [18].

The universal amplitude ratio  $\tilde{R}_\xi^+$ , however, is finite and non-zero. Using trigonometric identities in (1) it is possible to write

$$q - 1 = 4 \sin\left(\frac{\pi(2t-1)}{3(t+1)}\right) \sin\left(\frac{\pi(t-2)}{3(t+1)}\right),$$

in which we now set  $t = L + 1$ . Using the last four expressions in (2), and taking the limit  $L \rightarrow 1$  gives the algebraic expression for percolation:

$$\tilde{R}_\xi^+ = \left[ \frac{(L-2)(L+1)(L+4)(L+2)}{27\sqrt{3}L^4 \sin\left(\frac{\pi(2L+1)}{3(L+2)}\right) \sin\left(\frac{\pi(L-2)}{3L}\right)} \right]_{L=1}^{\frac{1}{2}} = \left[ \frac{40}{27\sqrt{3}} \right]^{\frac{1}{2}}. \quad (4)$$

To compare this result with the numerical field theory result of [9], it is to four figures

$$\tilde{R}_\xi^+ = 2^{\frac{3}{2}} 3^{-\frac{7}{4}} 5^{\frac{1}{2}} = 0.9248 \dots$$

Delfino and Cardy [9] quote a previous best estimate  $\tilde{R}_\xi^+ \approx 1.1$  based on earlier series [11] and Monte Carlo [12] results. By the technique they adopt, they obtain for comparison the numerical value  $\tilde{R}_\xi^+ = 0.926$ . Both of these values use the second moment correlation length which differs (only slightly for  $q = 1$ ) from the true correlation length used in this letter, though in [9] the expression (4) could also be obtained. Numerical values for other universal amplitude ratios are also found in [9]; unlike (4), these do not all appear to be accessible exactly or from existing solvable models. Thus the value of knowing the reliability of the field theory approach, specifically the form factor approach at the two kink level, is clear (see [19]).

A more detailed explanation of the relationship between the  $q$ -state Potts model and the dilute A model, together with the calculation of the universal correlation length amplitude ratio  $\tilde{R}_\xi^+$  for the  $q$ -state Potts model for all integer  $1 \leq q \leq 4$ , will be given elsewhere [20].

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