The Schwinger Model and the Physical Projector:
a Nonperturbative Quantization without Gauge Fixing

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Abstract

Based on the physical projector approach, a nonpertubative quantization of the massless Schwinger model is considered which does not require any gauge fixing. The spectrum of physical states, readily identified following a diagonalization of the operator algebra, is that of a massive pseudoscalar field, namely the electric field having acquired a mass proportional to the gauge coupling constant. The physical spectrum need not be identified with confined bound fermion-antifermion pairs, an interpretation which one is otherwise led to given whatever gauge fixing procedure but which is not void of gauge fixing artefacts.

1   Introduction

Ever since its inception,[1] the massless Schwinger model, namely 1+1 dimensional massless quantum electrodynamics, has been a favourite theoretical laboratory of mathematical physics in which to test our understanding and methods of interacting relativistic quantum field theories. This model is one of the very few instances of an integrable quantum field theory possessing an exact nonperturbative solution.[1, 2, 3, 4] It also displays most of the nonperturbative features of the theory for the strong interactions of quarks, namely quantum chromodynamics (QCD) based on the nonabelian $SU(3)_C$ colour gauge symmetry, leading to the still open issues of confinement and spontaneous dynamical chiral symmetry breaking with its cortège of characteristic consequences for hadronic interactions. Hence, it is not suprising that whenever a new nonperturbative phenomenon is suggested to occur in quantum field theory, or a new methodology towards such issues is designed, the Schwinger model is brought into the arena to test out new ideas. The present contribution is no exception. Given the recent proposal[5] of the physical projector approach to the quantization of gauge invariant systems (and more generally, constrained ones),[6] it thus seems timely to investigate the potential new insight which that approach may bring to the nonperturbative issues characteristic of the Schwinger model. The efficacy of the physical projector has already been demonstrated in the case of gauge invariant systems in 0+1 dimensions,[7] as well as topological quantum field theories in 2+1 dimensions,[8] with their own specific circumstances. It would thus be welcome to demonstrate that the physical projector approach is also of relevance with regards to the nonperturbative quantization of interacting field theories.

Within a continuum spacetime, all previous approaches[1, 2, 3, 4] to the Schwinger model (SchM), in one way or another, are based in an essential manner on a choice of gauge fixing of the U(1) local symmetry. In effect, this leads to the reduction of the gauge boson degrees of freedom expressed in terms of the fermionic ones, except for some gauge invariant zero mode in the case of a compact space topology, namely a circle. The latter choice is made[3, 4] in order to render manifest the topological features of the SchM. The conclusion is then that the spectrum of gauge invariant states is that of a massive scalar field, whose quanta are the confined states of fermion-antifermion pairs bound to one another by an electric flux line. Indeed, the one-dimensional Gauss law in vacuum implies a constant electric field, hence a linearly rising electric potential leading to a confining interaction. However, such an intuitive picture raises some difficulties of interpretation. For instance, given two point particles of opposite charges and momenta forming a scalar bound state on the circle, on which "side" of the circle should the electric field binding these two charges be set-up? As a matter of fact, other such artefacts arise as a consequence of gauge fixing, which tend to confuse the physical understanding and interpretation of the nonperturbative dynamics of the model.

Within the physical projector approach which does not require any gauge fixing, such issues are avoided altogether.[6, 9] Physical states are nothing but the quanta of the electric field,[10] which is a pseudoscalar field in 1+1 dimensions, indeed the only spacetime local construct based on the original degrees of freedom which is genuinely gauge invariant. The fermionic degrees of freedom contribute in a gauge invariant manner to the physical spectrum only nonlocally through their bosonized representation and only in combination with the local gauge dependent degrees of freedom of the gauge boson sector. Even though the same physical spectrum is of course recovered as in any of the gauge fixed quantizations, we find such a physical picture much more appealing than the usual one, the more so since within the Hamiltonian formulation of the SchM, the electric field readily finds its rightful place and is obviously a gauge invariant field whose quanta must belong to the physical spectrum whatever the status of the other degrees of freedom.

A full account of the analysis is not presented,[10] restricting the discussion to its most salient

features. Sect.2 introduces the model and our notations. Sect.3 describes its system of constraints and basic Hamiltonian formulation. A nonperturbative quantization is developed in Sect.4, based on the bosonization of the fermion degrees of freedom and the operator diagonalization of the model. In Sect.5, the physical spectrum is identified through the physical projector. Finally, some conclusions are presented in Sect.6.

2   The Massless Schwinger Model

The topology of the 1+1 dimensional spacetime is taken to be $\mathbb{R} \times S$, $\mathbb{R}$ standing for the time coordinate, t, and S for the spatial one, x, with the geometry of a circle of radius R and circumference $L = 2\pi R$. The Minkowski metric is $\eta_{\mu\nu} = \text{diag}\,(+-)$ $(\mu, \nu = 1, 2)$, while units such that $\hbar = 1 = c$ are used throughout. The antisymmetric tensor $\epsilon^{\mu\nu}$ is such that $\epsilon^{01} = +1$. For the Clifford-Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, we shall work with the chiral representation given by

$$\gamma^0 = \sigma^1 \quad , \quad \gamma^1 = i\sigma^2 \quad , \quad \gamma_5 = \gamma^0\gamma^1 = -\sigma^3 \, , \tag{1}$$

σ^i $(i = 1, 2, 3)$ being the usual Pauli matrices.

The field degrees of freedom of the model are, on the one hand, a single massless Dirac spinor $\psi(x^\mu)$, and on the other hand, the real U(1) gauge vector field $A_\mu(x^\mu)$. The Dirac spinor decomposes into two complex Weyl spinors of opposite chiralities, $\psi = \psi_+ + \psi_-$, such that $\gamma_5\psi_\pm = \mp\psi_\pm$. The index "$\pm$" refers to the left- or right-moving character, respectively, of these two Weyl spinors in the absence of any interaction. Furthermore, the following choice of periodic and twisted boundary conditions on the circle is assumed,

$$A_\mu(t, x + L) = A_\mu(t, x) \quad , \quad \psi_\pm(t, x + L) = -e^{2i\pi\alpha_\pm}\,\psi_\pm(t, x) \, , \tag{2}$$

$\alpha_\pm$ being arbitrary real quantities defined modulo any integers, which, in fact, have to be equal modulo integers to ensure parity invariance, $\alpha_+ = \alpha = \alpha_-$ (mod $\mathbb{Z}$).

The dynamics of the model derives from the local Lagrangian density (the summation convention is implicit throughout)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \; + \; \frac{1}{2}i\,\bar{\psi}\gamma^\mu\partial_\mu\psi - \frac{1}{2}i\,\partial_\mu\bar{\psi}\gamma^\mu\psi \; - \; e\bar{\psi}\gamma^\mu A_\mu\psi \, , \tag{3}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the gauge field strength, and e the U(1) gauge coupling constant. In 1+1 dimensions and in units of mass, the gauge field A_μ is dimensionless, the spinors $\psi_\pm$ have dimension $1/2$, and e dimension unity. In covariant form, the equations of motion are

$$\gamma^\mu(i\partial_\mu - eA_\mu)\psi = 0 \quad , \quad \partial^\nu F_{\nu\mu} = e\bar{\psi}\gamma_\mu\psi = e\,J_\mu \, . \tag{4}$$

When written out in chiral components, the Weyl spinors $\psi_\pm(t \pm x)$ are seen to be left- and right-moving, respectively, in the absence of any interaction, $e = 0$.

On account of its antisymmetry, the field strength $F_{\mu\nu}$ represents a single field, F_{01}, which coincides with the electric field, indeed a pseudoscalar field in 1+1 dimensions,

$$E = -\partial_1 A^0 - \partial_0 A^1 = F_{01} = -F^{01} \, . \tag{5}$$

In particular, Gauss' law reads $\partial_1 E = e\psi^\dagger\psi$.

By construction, the system is invariant under the following global symmetries. Besides Poincaré invariance in the spacetime manifold, one also has the vector U(1) and axial U(1)$_A$ symmetries whose Noether currents are, respectively,

$$J^\mu = \bar{\psi}\gamma^\mu\psi \quad , \quad J_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi \, , \tag{6}$$

with the duality relations $J_5^\mu = -\epsilon^{\mu\nu} J_\nu$ and $J^\mu = -\epsilon^{\mu\nu} J_{5\nu}$. The space integral of the time component of each of these currents defines the corresponding Noether charges Q and Q_5, respectively. At the classical level, both are conserved quantities, thus generating a symmetry of the dynamics. For the quantized system however, the axial charge is no longer conserved due to quantum effects.

The U(1) vector symmetry is of course the gauge symmetry of the system, acting as

$$\psi'(t,x) = e^{-ie\epsilon(t,x)}\psi(t,x) \quad, \quad A'_\mu(t,x) = A_\mu(t,x) + \partial_\mu \epsilon(t,x) \ , \tag{7}$$

with $\epsilon(t,x)$ an arbitrary function of spacetime of the general form

$$\epsilon(t,x) = \epsilon_0(t,x) + \frac{2\pi k}{eL}x \ , \tag{8}$$

$\epsilon_0(t, x+L) = \epsilon_0(t,x)$ being an arbitrary periodic function on the spatial circle, defined modulo $2\pi/|e|$ and parametrizing small U(1) gauge transformations, while the nonperiodic contribution proportional to x represents the homotopy class of large U(1) gauge transformations of winding number k, k being an arbitrary integer which labels the corresponding element of the gauge modular group $\mathbb{Z}$. These two classes of gauge transformations are to be included into the construction of the physical projector and the identification of physical states.

3   The Basic Hamiltonian Formulation

Being straightforward enough, the Hamiltonian analysis of constraints is not detailed here. Let us only remark that the above Lagrangian density is already in Hamiltonian form in the fermionic sector,[6, 11] while A^0 is necessarily the Lagrange multiplier for a first-class constraint, in fact the sole first-class constraint of the system, namely the generator of small U(1) gauge transformations which is nothing but Gauss' law.[6] Hence only A^1 needs to have its conjugate momentum included in order to identify the basic Hamiltonian formulation[6] of the SchM. In terms of the corresponding first-order action on phase space, this formulation is defined by

$$S = \int dt \int_0^L dx \left\{ \partial_0 A^1 \pi_1 + \frac{1}{2} i\psi^\dagger \partial_0 \psi - \frac{1}{2} i\partial_0 \psi^\dagger \psi - \mathcal{H} - A^0 \phi + \partial_1(A^0 \pi_1) \right\} \ , \tag{9}$$

where $\mathcal{H}$ is the first-class Hamiltonian density,

$$\mathcal{H} = \frac{1}{2}\pi_1^2 - \frac{1}{2} i\psi^\dagger \gamma_5 (\partial_1 - ieA^1)\psi + \frac{1}{2} i(\partial_1 + ieA^1)\psi^\dagger \gamma_5 \psi \ , \tag{10}$$

while ϕ is the first-class constraint of the system,

$$\phi = \partial_1 \pi_1 + e\psi^\dagger \psi \ , \tag{11}$$

whose Lagrange multiplier is nothing else than the time component A^0 of the gauge field. This first-order action encodes the fundamental Grassmann graded Poisson bracket structure given by $(\alpha, \beta = +, -)$,

$$\{A^1(t,x), \pi_1(t,y)\} = \delta(x-y) \quad, \quad \{\psi_\alpha(t,x), \psi_\beta^\dagger(t,y)\} = -i\delta_{\alpha\beta}\delta(x-y) \ . \tag{12}$$

In fact, the momentum π_1 conjugate to A^1 is nothing but the electric field up to a sign, $\pi_1 = F_{10} = -E$, so that the generator ϕ of small Hamiltonian U(1) gauge transformations coincides with Gauss' law, one of the two independent equations of motion in the gauge sector, namely $\partial^\nu F_{\nu 0} = e\psi^\dagger \psi$. Manifest gauge invariance of the formulation readily follows from the brackets

$$\{\phi(t,x), \mathcal{H}(t,y)\} = 0 \quad, \quad \{\phi(t,x), \phi(t,y)\} = 0 \ . \tag{13}$$

Given the total Hamiltonian density $\mathcal{H}_T = \mathcal{H} + A^0\phi - \partial_1(A^0\pi_1)$, the Hamiltonian equations of motion read

$$\partial_0\psi = -\gamma_5(\partial_1 - ieA^1)\psi - ieA^0\psi \;,\; \partial_0 A^1 = \pi_1 - \partial_1 A^0 \;,\; \partial_0\pi_1 = e\psi^\dagger\gamma_5\psi \;, \qquad (14)$$

together with the constraint $\phi = \partial_1\pi_1 + e\psi^\dagger\psi = 0$. One thus recovers the Lagrangian equation of motion for the Dirac fermion; the equation for A^1 leads back to the relation defining the conjugate momentum $\pi_1 = -E$ in terms of the field A_μ; and finally the equation for π_1 reproduces the second equation of motion in the gauge sector, namely $\partial^\nu F_{\nu 1} = e\psi^\dagger\gamma_5\psi$, Gauss' law being reproduced through the first-class constraint $\phi = 0$. The dynamics of the system is thus equally well represented through the basic Hamiltonian formulation. In particular, upon substitution of the expression for π_1 into (9), one recovers the original Lagrangian density, inclusive of the explicit surface term in (9).

Infinitesimal small U(1) gauge transformations are generated by the first-class constraint ϕ through Poisson brackets. Extended to finite transformations, one has

$$
\begin{aligned}
\psi'(t,x) &= e^{-ie\epsilon_0(t,x)}\psi(t,x) \;, \\
A^{1'}(t,x) &= A^1(t,x) - \partial_1\epsilon_0(t,x) \;, \\
\pi_1'(t,x) &= \pi_1(t,x) \;, \\
A^{0'}(t,x) &= A^0(t,x) + \partial_0\epsilon_0(t,x) \;,
\end{aligned}
\qquad (15)
$$

$\epsilon_0(t,x)$ being an arbitrary function of periodicity L in x and defined modulo $2\pi/|e|$. These transformations leave (9) exactly invariant, and coincide with the small gauge transformations of the Lagrangian formulation of the system.

Even though large U(1) gauge transformations $\epsilon(t,x) = (2\pi kx)/(eL)$ are not generated by the first-class constraint ϕ, nevertheless they also define an invariance of the basic Hamiltonian action (9), without any surface term being induced. The phase space transformations are then given by

$$
\begin{aligned}
\psi'(t,x) &= e^{-ie\epsilon(t,x)}\psi(t,x) = e^{-i\frac{2\pi k}{L}x}\psi(t,x) \;, \\
A^{1'}(t,x) &= A^1(t,x) - \partial_1\epsilon(t,x) = A^1(t,x) - \frac{2\pi k}{eL} \;, \\
\pi_1'(t,x) &= \pi_1(t,x) \;, \\
A^{0'}(t,x) &= A^0(t,x) + \partial_0\epsilon(t,x) = A^0(t,x) \;.
\end{aligned}
\qquad (16)
$$

From these expressions, it follows that in terms of a Fourier series expansion of the field $A^1(t,x)$ on the circle, all its nonzero modes may always be gauged away entirely through an appropriate small finite gauge transformation which in any case leaves its zero mode invariant, whereas large gauge transformations affect only its zero mode which is thus defined modulo $(2\pi)/(|e|L)$. Hence only the x-independent mode of $A^1(t,x)$ is an actual dynamical gauge invariant physical degree of freedom taking its values on a circle of radius $1/(|e|L)$.[3, 4] As we shall see, this topological feature translates at the quantum level in terms of θ-vacua. In contradistinction, the momentum π_1 conjugate to A^1 is totally gauge invariant and thus physical. Even though the A^1 zero mode is not exatly gauge invariant except modulo $(2\pi)/(|e|L)$ whereas the π_1 zero mode is genuinely gauge invariant, one may suspect that these zero modes could be conjugate to one another. However, since the nonzero modes of A^1 are pure gauge, which are the gauge invariant physical degrees of freedom of the system that are conjugate to the gauge invariant nonzero modes of π_1?

4 Nonperturbative Canonical Quantization

The operator quantization of the model proceeds from its basic Hamiltonian formulation. Thus, the space of quantum states must provide a representation space for the commutation and anticommutation relations (only the nonvanishing (anti)commutators are given throughout)

$$[A^1(x),\pi_1(y)] = i\delta(x-y) \;,\; \{\psi_\alpha(x),\psi_\beta^\dagger(y)\} = \delta_{\alpha\beta}\delta(x-y) \;, \qquad (17)$$

where it is understood that one is working within the Schrödinger picture considered, say, at $t = 0$, an argument which henceforth is no longer displayed. Quantum dynamics of the system is generated through the first-class Hamiltonian density (10) and constraint (11). However, being composite operators, these first-class quantities require a choice of operator ordering which must remain anomaly free and gauge invariant, namely such that the gauge commutation relations

$$[\phi(x), \mathcal{H}(y)] = 0 \quad , \quad [\phi(x), \phi(y)] = 0 \tag{18}$$

are still obtained at the operator level. In the case of the SchM, it is possible to meet all these requirements and construct an explicit exact diagonalization of the quantum Hamiltonian. This is achieved through a bosonization of the fermionic sector of degrees of freedom.[3, 4]

4.1 Bosonization of the fermionic degrees of freedom

By lack of space, we do not provide details of this construction, but only its basic ingredients.[10] A quantization of the system amounts to the specification of a quantum representation space for the commutation and anticommutation relations (17). We shall obtain the above field degrees of freedom $A^1(x)$, $\pi_1(x)$ and $\psi_\pm(x)$ as composite operators of an alternative set of degrees of freedom whose commutation relations define the space of quantum states. For this purpose, let us introduce the following mode algebras,

$$[\varphi_n, \varphi_m^\dagger] = \delta_{n,m} \quad , \quad [Q_\pm, p_\pm] = i \quad , \quad [A_{\pm,n}, A_{\pm,m}^\dagger] = \delta_{n,m} \ , \tag{19}$$

where for the φ_n and $\varphi_m^\dagger$ modes the indices n and m take all integer values, while for the operators $A_{\pm,n}$ and $A_{\pm,m}^\dagger$ they only take strictly positive integer values. We also have $Q_\pm^\dagger = Q_\pm$ and $p_\pm^\dagger = p_\pm$. Furthermore, the eigenvalue spectrum of the $Q_\pm$ operators is constrained to lie within the interval $[0, 2\pi]$, namely $Q_\pm$ stands for the coordinate of a circle of radius unity. Consequently, the eigenvalue spectra of their conjugate operators $p_\pm$ for the Heisenberg algebras $[Q_\pm, p_\pm] = i$ are quantized as $(n_\pm + \lambda_\pm)$, $n_\pm$ being any integers and $\lambda_\pm$ (mod $\mathbb{Z}$) being the U(1) holonomies labelling all inequivalent representations of the Heisenberg algebra on the circle of radius unity.[12] It is clear that the complete representation space of the commutation relations (19) is the tensor product of these two Heisenberg algebra representations characterized by $\lambda_\pm$ together with Fock space representations for each of the independent modes φ_n and $A_{\pm,n}$ and their adjoint operators $\varphi_n^\dagger$ and $A_{\pm,n}^\dagger$, the former being annihilation and the latter creation operators.

Given the fundamental algebra (19), let us now introduce the following operators, for all strictly positive integer values $n \geq 1$,

$$
\begin{aligned}
q_\pm &= Q_\pm + N_0(\varphi_0 + \varphi_0^\dagger) \ , \\
a_{+,n} &= A_{+,n} + N_n\sqrt{n}\left(\varphi_n^\dagger + \varphi_{-n}\right) \ , \\
a_{+,n}^\dagger &= A_{+,n}^\dagger + N_n\sqrt{n}\left(\varphi_n + \varphi_{-n}^\dagger\right) \ , \\
a_{-,n} &= A_{-,n} + N_n\sqrt{n}\left(\varphi_n + \varphi_{-n}^\dagger\right) \ , \\
a_{-,n}^\dagger &= A_{-,n}^\dagger + N_n\sqrt{n}\left(\varphi_n^\dagger + \varphi_{-n}\right) \ ,
\end{aligned}
\tag{20}
$$

N_n being some normalization factor to be specified presently, as well as the bosonic fields (in the

Schrödinger picture),

$$
\begin{aligned}
\varphi(x) &= A \sum_{n=-\infty}^{+\infty} N_n \left[\varphi_n\, e^{\frac{2i\pi}{L}nx} + \varphi_n^\dagger\, e^{-\frac{2i\pi}{L}nx} \right] , \\
\pi_\varphi(x) &= \frac{-i}{2AL} \sum_{n=-\infty}^{+\infty} \frac{1}{N_n} \left[\varphi_n\, e^{\frac{2i\pi}{L}nx} - \varphi_n^\dagger\, e^{-\frac{2i\pi}{L}nx} \right] , \\
\Phi_\pm(x) &= Q_\pm \pm \frac{2\pi}{L} p_\pm x + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[A_{\pm,n}^\dagger\, e^{\pm\frac{2i\pi}{L}nx} + A_{\pm,n}\, e^{\mp\frac{2i\pi}{L}nx} \right] , \\
\phi_\pm(x) &= q_\pm \pm \frac{2\pi}{L} p_\pm x + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[a_{\pm,n}^\dagger\, e^{\pm\frac{2i\pi}{L}nx} + a_{\pm,n}\, e^{\mp\frac{2i\pi}{L}nx} \right] ,
\end{aligned}
\tag{21}
$$

the factor A being yet another normalization constant in terms of which N_n is expressed as

$$
N_n = \frac{1}{|A|} \frac{1}{\sqrt{L}\sqrt{2\omega_n}} , \quad \omega_n = \sqrt{\mu^2 + \left(\frac{2\pi}{L}n\right)^2} ,
\tag{22}
$$

where $\mu > 0$ is a given mass parameter. When applied to the SchM, the quantities A and μ are related to the gauge coupling constant e. Note that we also have $\phi_\pm(x) - \Phi_\pm(x) = \varphi(x)/A$. Only the fields $\varphi(x)$, $\pi_\varphi(x)$ and $\Phi_\pm(x)$ define the independent set of basic phase space degrees of freedom.

A straightforward analysis then establishes that the only nonvanishing commutation relations for these fields are

$$
\begin{aligned}
{}[\varphi(x), \pi_\varphi(y)] &= i\delta(x-y) , \\
[\Phi_\pm(x), \pi_\varphi(y)] = 0 \quad &, \quad [\Phi_\pm(x), \partial_1\Phi_\pm(y)] = \pm 2i\pi\delta(x-y) , \\
[\phi_\pm(x), \pi_\varphi(y)] = \tfrac{i}{A}\delta(x-y) \quad &, \quad [\phi_\pm(x), \partial_1\phi_\pm(y)] = \pm 2i\pi\delta(x-y) .
\end{aligned}
\tag{23}
$$

In other words, $\varphi(x)$ and its conjugate momentum $\pi_\varphi(x)$ define the degrees of freedom of a scalar field which is periodic in x on the circle of radius R and takes its values in the real line, while $\Phi_\pm(x)$ are two independent chiral bosons each taking their values in the circle of radius unity and obeying quantum boundary conditions such that $\Phi_\pm(x + L) = \Phi_\pm(x) + 2\pi(n_\pm + \lambda_\pm)$, $(n_\pm + \lambda_\pm)$ being the eigenvalues of the momentum zero modes $p_\pm$, namely corresponding to twisted chiral bosons on the circle of radius unity when $\lambda_\pm \neq 0$ (mod $\mathbb{Z}$). These nontrivial topology properties of the chiral boson zero modes turn out to be crucial for the correct representations of the gauge invariance properties of the system under both small and large U(1) gauge transformations.

In order to make contact with the fermionic degrees of freedom, let us now introduce the following field operators,

$$
\begin{aligned}
\psi_\pm(x) = \quad & \frac{1}{\sqrt{L}}\, e^{\pm\frac{i\pi}{L}x}\, e^{i\frac{\pi}{2}\eta p_\mp}\, e^{\pm i\lambda q_\pm}\, e^{\frac{2i\pi\lambda}{L}p_\pm x} \\
& \times \prod_{n=1}^{\infty} e^{\pm\frac{i\lambda}{\sqrt{n}} a_{\pm,n}^\dagger e^{\pm\frac{2i\pi}{L}nx}} \prod_{n=1}^{\infty} e^{\pm\frac{i\lambda}{\sqrt{n}} a_{\pm,n} e^{\mp\frac{2i\pi}{L}nx}} ,
\end{aligned}
\tag{24}
$$

where η and λ are two real constants such that $\eta^2 = 1 = \lambda^2$. Except for the first three factors, this operator is nothing but the exponential $e^{\pm i\lambda\phi_\pm(x)}$ with an operator ordering such that all the $q_\pm$ and $a_{\pm,n}^\dagger$ are to the left of all the $p_\pm$ and $a_{\pm,n}$ operators. These fields possess the following holonomy properties

$$
\psi_\pm(x + L) = -e^{2i\pi\lambda\lambda_\pm}\, \psi_\pm(x) ,
\tag{25}
$$

on account of the topology properties of the $Q_\pm$ and $p_\pm$ zero modes of the chiral fields $\Phi_\pm(x)$ and $\phi_\pm(x)$. That the fields $\psi_\pm(x)$ are indeed fermions in 1+1 dimensions follows from their anticommutation relations obtained from the fundamental algebra in (19),

$$
\{\psi_\alpha(x), \psi_\beta^\dagger(y)\} = \delta_{\alpha\beta}\delta(x-y) , \quad \alpha, \beta = +,- .
\tag{26}
$$

In other words, the bosonic fields $\phi_\pm(x)$ and their quantum states also provide a representation of the fermionic algebra in (17), irrespective of the choice of sign factors $\eta = \pm 1$ and $\lambda = \pm 1$.

Hence, the fermionic matter sector of the SchM may be bosonized in this manner in terms of the above bosonic field degrees of freedom, provided the U(1) holonomies $\lambda_\pm$ are chosen such that $\lambda_\pm = \lambda\alpha_\pm$ (mod $\mathbb{Z}$). Note that in order to obtain the anticommutation relations in (17) on their own, so far it is only through the combinations $\phi_\pm(x) = \Phi_\pm(x) + \varphi(x)/A$ that the fields $\varphi(x)$ and $\Phi_\pm(x)$ are involved in this construction. However, when considered together with the bosonic commutations relations for the gauge field in (17), the whole set of degrees of freedom associated to each of the independent fields $\varphi(x)$, $\pi_\varphi(x)$ and $\Phi_\pm(x)$ separately is essential to establish the complete operator diagonalization of the model.

4.2 Gauge invariant point-splitting regularization

Given the above bosonization of the fermionic fields, one must now define the composite operators associated to the first-class Hamiltonian density (10) and constraint (11). Since gauge invariance must be preserved at all steps, a gauge invariant point-splitting regularization of short-distance singularities is a relevant choice. Namely, associated for example to the product $\psi^\dagger_\pm(x)\psi_\pm(x)$ which is ill-defined as such as a composite operator, one considers the gauge invariant quantity

$$\psi^\dagger_\pm(y)\, e^{ie\int_x^y du A^1(u)}\, \psi_\pm(x) \tag{27}$$

in the limit where y goes to x, and subtracts any divergent term that arises. Since the Wilson line phase factor is inserted in this point-split product of the two fields, the result is manifestly gauge invariant. This example applies directly to the definition of the vector and axial current operators. A similar approach is used for the products of fermionic fields, now including the U(1) covariant derivative, that contribute to the first-class Hamiltonian.

In terms of the bosonized representation of the fermion fields, one then obtains the following definitions for composite quantum operators,

$$\psi^\dagger_\pm\psi_\pm = -\frac{\lambda}{2\pi}\left(\partial_1\phi_\pm \mp e\lambda A^1\right)\;, \tag{28}$$

so that the first-class constraint reads,

$$\phi = \partial_1\pi_1 + e\psi^\dagger\psi = \partial_1\pi_1 - \frac{e\lambda}{2\pi}\partial_1(\phi_+ + \phi_-)\;, \tag{29}$$

while the fermion contributions to the first-class Hamiltonian density $\mathcal{H}$ are

$$\frac{1}{2}i\psi^\dagger_\pm\left(\partial_1 - ieA^1\right)\psi_\pm - \frac{1}{2}i\left(\partial_1 + ieA^1\right)\psi^\dagger_\pm\psi_\pm = \pm\frac{1}{4\pi}\left(\partial_1\phi_\pm \mp e\lambda A^1\right)^2 \mp \frac{\pi}{12L^2}\;. \tag{30}$$

Note that the vector and axial currents as well as the first-class constraint, which are composite quantities in terms of the fermionic fields, are local noncomposite operators in terms of the bosonic degrees of freedom, a property not shared, though, by the first-class Hamiltonian whose definition thus still requires the specification of some choice of operator ordering.

4.3 The diagonalized operator basis

Let us now turn to the definition of the first-class Hamiltonian operator in correspondence with (10). Given the above bosonization and gauge invariant point-splitting regularization, one finds

$$\mathcal{H} = \frac{1}{2}\pi_1^2 + \frac{1}{4\pi}\left(\partial_1\phi_+ - e\lambda A^1\right)^2 + \frac{1}{4\pi}\left(\partial_1\phi_- + e\lambda A^1\right)^2\;, \tag{31}$$

where the Casimir energy contribution $-\pi/(6L^2)$ has been ignored since the energy is defined up to an arbitrary constant anyway. However, completing the sum of squares as follows

$$\mathcal{H} = \tfrac{1}{2}\pi_1^2 + \tfrac{1}{8\pi}\left(\partial_1\phi_+ + \partial_1\phi_- - \tfrac{2\pi}{e\lambda}\partial_1\pi_1 + \tfrac{2\pi}{e\lambda}\partial_1\pi_1\right)^2$$
$$+ \tfrac{1}{8\pi}\left(\partial_1\phi_+ - \partial_1\phi_- - 2e\lambda A^1\right)^2 , \tag{32}$$

one recognizes the explicit appearance of the first-class constraint ϕ in the form (29), so that one may also write

$$\mathcal{H} = \tfrac{1}{2}\pi_1^2 + \tfrac{\pi}{2e^2}(\partial_1\pi_1)^2 + \tfrac{e^2}{2\pi}\left(A^1 - \tfrac{1}{2e\lambda}\partial_1(\phi_+ - \phi_-)\right)^2$$
$$+ \tfrac{\pi}{2e^2}\phi^2 - \tfrac{\pi}{e^2}\phi(\partial_1\pi_1) . \tag{33}$$

Given this expression, let us then introduction the following definitions,

$$\varphi = -\frac{1}{\mu}\pi_1 \ , \quad \pi_\varphi = \mu\left[A^1 - \frac{1}{2e\lambda}\partial_1(\phi_+ - \phi_-)\right] , \tag{34}$$

μ being the mass parameter

$$\mu = \frac{|e|}{\sqrt{\pi}} . \tag{35}$$

In particular, note that except for the mass factor μ, the field φ is nothing else than the electric field, $\varphi = E/\mu$. The first-class Hamiltonian then reads

$$\mathcal{H} = \frac{1}{2}\pi_\varphi^2 + \frac{1}{2}(\partial_1\varphi)^2 + \frac{1}{2}\mu^2\varphi^2 + \frac{1}{2}\left(\frac{\phi}{\mu}\right)^2 + \left(\frac{\phi}{\mu}\right)(\partial_1\varphi) , \tag{36}$$

a most suggestive result indeed!

To confirm the fact that the fields φ and π_φ are conjugate to one another, one need only compute their algebra given the quantization rules (17), leading to the canonical commutation relations,

$$[\varphi(x), \pi_\varphi(y)] = i\delta(x - y) . \tag{37}$$

Furthermore, the remaining commutation relations are

$$[\phi_\pm(x), \pi_\varphi(y)] = -\frac{i\pi\mu}{e\lambda}\delta(x - y) \ , \quad [\phi_\pm(x), \partial_1\phi_\pm(y)] = \pm 2i\pi\delta(x - y) . \tag{38}$$

This set of commutations relations is thus equivalent, upon bosonization, to the original set in (17). Furthermore, it is seen to coincide exactly with the relations in (23) provided only one sets $A = -e\lambda/(\pi\mu)$ and identifies the mass scale μ with the quantity $\mu = |e|/\sqrt{\pi}$, so that $N_n = \sqrt{\pi}/(\sqrt{L}\sqrt{2\omega_n})$.

Consequently, all quantities must now be expressed in terms of the mode degrees of freedom $Q_\pm$, $p_\pm$, φ_n, $A_{\pm,n}$ and their adjoint operators, or equivalently in terms of the fields $\varphi(x)$, $\pi_\varphi(x)$ and $\Phi_\pm(x)$. Thus, the first-class constraint is finally given by

$$\phi = -\frac{e\lambda}{2\pi}\partial_1\left[\Phi_+ + \Phi_-\right] , \tag{39}$$

namely,

$$\phi(x) = -\frac{e\lambda}{L}\Big\{(p_+ - p_-) + $$
$$+ i\sum_{n=1}^{\infty}\sqrt{n}\left((A_{+,n}^\dagger + A_{-,n})e^{\frac{2i\pi}{L}nx} - (A_{-,n}^\dagger + A_{+,n})e^{-\frac{2i\pi}{L}nx}\right)\Big\} , \tag{40}$$

while the first-class Hamiltonian reads

$$H = \sum_{n=-\infty}^{+\infty}\omega_n\varphi_n^\dagger\varphi_n + \int_0^L dx\left[\frac{1}{2}\left(\frac{\phi}{\mu}\right)\left(\frac{\phi}{\mu}\right) + \left(\frac{\phi}{\mu}\right)(\partial_1\varphi)\right] , \tag{41}$$

8

where the contribution of the (φ, π_φ) degrees of freedom is displayed explicitly in terms of their creation and annihilation operators following the usual normal ordering prescription for that sector. For the gauge degrees of freedom contribution to H though, since the operators $\phi(x)$ and $\varphi(x)$ commute with themselves and one another, these operators are multiplied as shown, without any normal ordering prescription in terms of the modes $Q_\pm$, $p_\pm$, $A_{\pm,n}$ and $A_{\pm,n}^\dagger$ being applied. The reason for this choice is that these contributions to the total Hamiltonian then vanish identically for all physical states, while otherwise some infinite normal ordering constant would arise. Finally, the vector and axial charge operators are given by

$$
\begin{array}{rcl}
Q & = & \int_0^L dx\, \psi^\dagger \psi = -\lambda(p_+ - p_-) \ , \\
Q_5 & = & \int_0^L dx\, \psi^\dagger \gamma_5 \psi = -\frac{i\lambda\sqrt{2\mu}}{2\sqrt{\pi}\sqrt{L}} \left[\varphi_0 - \varphi_0^\dagger \right] \ .
\end{array}
\tag{42}
$$

In conclusion, the SchM and its dynamics has been quantized in terms of the decoupled set of basic bosonic fields, $\varphi(x)$ and $\pi_\varphi(x)$ on the one hand, and $\Phi_\pm(x)$ on the other hand, whose commutation relations (19) are those of a periodic scalar field taking its values in the real line and two twisted chiral bosons taking their values in a circle of radius unity. These two sectors of operators, which commute with one another, see (23), diagonalize the commutator and anticommutator algebra (17) defining the quantum system as well as its first-class Hamiltonian H and constraint $\phi(x)$ operators, (40) and (41). In particular, the field $\varphi(x)$ coincides, up to the mass scale μ, with the electric field, whose nonperturbative quantum dynamics is that of a pseudoscalar field whose quanta have a mass μ proportional to the U(1) gauge coupling e of the original gauge field A_μ to the original fermionic degrees of freedom. The appearance of this screening length $1/\mu$ for the electric field is a purely quantum effect, and is one of the consequences within the physical sector of the system of its underlying fermionic one even though the latter degrees of freedom are not gauge invariant except through specific combinations. The structure of the physical ground state is also dependent on gauge invariant topology features of the fermionic matter excitations. Furthermore, the physical spectrum of U(1) gauge invariant states consists only of superpositions of the massive free quanta of the electric field. These results are described next.

5   The Physical Projector and Spectrum

Having defined a quantization of the model, different issues may be addressed. First, U(1) gauge invariance remains a symmetry of the quantum dynamics, namely the first-class algebra $[\phi(x), \phi(y)] = 0$ and $[H, \phi(x)] = 0$ is preserved for the quantum operators. Likewise, the Poincaré algebra remains satisfied,[10] namely manifest spacetime covariance is maintained through quantization on the space of quantum states, as well as on the subspace of physical states to be constructed hereafter since the Poincaré algebra commutes with the U(1) generator ϕ.

Switching to the Heisenberg picture (as indicated by the subscript "H" in the relations hereafter), it is possible to establish the operator equations of motion generated by the total Hamiltonian $H_T = H + \int_0^L dx \left[A^0 \phi + \partial_1(A^0 \varphi)/\mu \right]$. For the operators of interest, one finds

$$
\begin{array}{rcl}
[\partial_0^2 - \partial_1^2 + \mu^2]\varphi_H & = & \partial_1\left(\frac{\phi_H}{\mu}\right) \ , \\
\partial_0\left(\frac{\phi_H}{\mu}\right) & = & 0 \ , \\
\partial_\mu J_H^\mu = 0 & , & \frac{d}{dt}Q_H = 0 \ , \\
\partial_\mu J_{5H}^\mu = \frac{\mu^3}{e}\varphi_H = \frac{e}{\pi}E_H & , & \frac{d}{dt}Q_{5H} = \frac{e}{\pi}\int_0^L dx\, E_H \ .
\end{array}
\tag{43}
$$

Note that the axial current J_{5H}^μ and charge Q_{5H} are no longer conserved at the quantum level as they are at the classical one, showing that this classical global symmetry is explicitly broken by

quantum properties of the system. This chiral anomaly is directly proportional to the electric field, namely the actual physical configuration space of the system. In contradistinction, the local U(1) vector symmetry is explicitly preserved in the quantized system on account of the gauge invariant regularization through point splitting.

Finally, let us turn to the issue of gauge invariance. It is clear that among the basic fields, the electric field sector (φ, π_φ) is explicitly gauge invariant, while the chiral bosons $\Phi_\pm(x)$ are not. Given any L-periodic function $\epsilon_0(x)$ and an operator $\mathcal{O}(x)$, the corresponding finite small gauge transformation generated by the first-class constraint $\phi(x)$ is given by the adjoint action of the following unitary operator

$$\mathcal{O}'(x) = U(\epsilon_0)\,\mathcal{O}(x)\,U^{-1}(\epsilon_0) \quad , \quad U(\epsilon_0) = e^{i\int_0^L dx\epsilon_0(x)\phi(x)} \ . \tag{44}$$

For the basic fields, one finds

$$\varphi'(x) = \varphi(x) \quad , \quad \pi'_\varphi(x) = \pi_\varphi(x) \quad , \quad \Phi'_\pm(x) = \Phi_\pm(x) \mp e\lambda\epsilon_0(x) \ . \tag{45}$$

Since the latter relation for $\Phi_\pm(x)$ implies that the fields $\phi_\pm(x)$ also transform with the same rules, the correct phase transformation of the fermionic fields (24) is reproduced. Given a mode expansion of the gauge parameter,

$$\epsilon_0(x) = \sum_{n=-\infty}^{+\infty} \epsilon_n^{(0)}\, e^{\frac{2i\pi}{L}nx} \ , \tag{46}$$

the above rules translate into the gauge transformation for chiral modes,

$$Q'_\pm = Q_\pm \mp e\lambda\epsilon_0^{(0)} \quad , \quad A'_{\pm,n} = A_{\pm,n} \mp e\lambda\sqrt{n}\,\epsilon_{\mp n}^{(0)} \quad , \quad A^{\dagger\prime}_{\pm,n} = A^\dagger_{\pm,n} \pm e\lambda\sqrt{n}\,\epsilon_{\pm n}^{(0)} \ , \tag{47}$$

showing that the chiral momentum zero modes $p_\pm$ are invariant under small gauge transformations, as are of course also all the modes φ_n and $\varphi_n^\dagger$ of the fields $\varphi(x)$ and $\pi_\varphi(x)$. Furthermore, recall that the gauge parameter $\epsilon_0(x)$ is defined modulo $2\pi/|e|$, hence so is its zero mode $\epsilon_0^{(0)}$. Consequently, the actual reason for our previous assumption that the chiral coordinate zero modes $Q_\pm$ take their values on a circle of radius unity, namely that they are defined modulo 2π, stems directly from their properties under small U(1) gauge transformations.

Consider now a large gauge transformation of homotopy class k, thus associated to the parameter $\epsilon(x) = (2\pi kx/eL)$. Since in this case the fermion fields $\psi_\pm$ transform with the phase factor $e^{-2i\pi kx/L}$ while the gauge field $A^1(x)$ is shifted by the quantity $-\partial_1\epsilon(x) = -2\pi k/(eL)$, it follows that among all the modes of the basic fields $\varphi(x)$, $\pi_\varphi(x)$ and $\Phi_\pm(x)$, only the chiral momentum zero modes $p_\pm$ must transform under large gauge transformations according to the following rule

$$p'_\pm = p_\pm - \lambda k \ . \tag{48}$$

This transformation is generated by a unitary operator of the form

$$U_k = C_k\, e^{ik\lambda(Q_++Q_-)} \quad , \quad p'_\pm = U_k\, p_\pm\, U_k^{-1} \ , \tag{49}$$

where C_k is a cocycle factor introduced in order to obtain the following group composition law for any two large gauge transformations of U(1) homotopy classes k and ℓ,

$$U_k\, U_\ell = U_{k+\ell} \ . \tag{50}$$

The general solution to this condition is $C_k = e^{i\theta k\lambda}$ where θ is an arbitrary real constant defined modulo 2π. Hence, the operator representation of large U(1) gauge transformations of homotopy class k is simply $U_k = e^{i\theta k\lambda}\, e^{ik\lambda(Q_++Q_-)}$, up to any transformation $U(\epsilon_0)$.

10

These gauge transformations also answer a question raised previously. The gauge invariant degrees of freedom which are conjugate to the gauge invariant electric field are those of the momentum $\pi_\varphi(x)$, which is indeed a specific combination of the original gauge dependent degrees of freedom which is gauge invariant under both small and large gauge transformations. Namely, $\pi_\varphi(x)$ is essentially the difference of the original gauge field $A^1(x)$ with the gradient $\partial_1(\Phi_+ - \Phi_-)(x)$ of the difference of the chiral bosons. In particular, because of this gradient contribution, the zero mode of the electric field, which is proportional to $(\varphi_0 + \varphi_0^\dagger)$, is conjugate to the combination $(A_0^1 - \pi(p_+ + p_-)/(e\lambda L))$, which is proportional to $(\varphi_0 - \varphi_0^\dagger)$, rather than to the $A^1(x)$ zero mode A_0^1 on its own as might have been anticipated. This situation is in contradistinction with results obtained within a gauge fixed formulation.[3, 4] Moreover, the fact that this zero mode A_0^1 is defined modulo $(2\pi)/(|e|L)$ because of its properties under large gauge transformations, is now seen to correspond to the analogous property for the chiral momentum zero modes $p_\pm$ which are defined modulo integers for the same reason, since the combination $(\varphi_0 - \varphi_0^\dagger)$ is invariant under both small and large gauge transformations. Note that the actual phase space topology of the twisted chiral zero mode sector $(Q_\pm, p_\pm)$ is that of a two-torus for each chiral boson.

Having identified the operators $U(\epsilon_0)$ and U_k which generate all small and large gauge transformations on the space of quantum states, the physical projector is readily constructed as the sum of all these transformations,[5] in order to retain only the totally gauge invariant components of any state,[6] namely the actual physical states of the system. Hence, the physical projector is given by

$$\mathbb{E} = \sum_{k=-\infty}^{+\infty} \int_0^{\frac{2\pi}{|e|}} d\epsilon_0^{(0)} \prod_{n=1}^{\infty} \int_{-\infty}^{\infty} d(\mathrm{Re}\,\epsilon_n^{(0)}) \int_{-\infty}^{\infty} d(\mathrm{Im}\,\epsilon_n^{(0)})\, U_k\, U(\epsilon_0)\ , \tag{51}$$

where the absolute normalizations of these integrals are not specified at this stage, the operator $\mathbb{E}$ being a non-normalizable projector density.

In order to describe the physical spectrum, let us now introduce a basis for the representation space of the algebra (19). In the electric field sector, we have the normalized Fock vacuum $|0>_\varphi$ which is annihilated by the modes φ_n and which is acted on with the creation operators $\varphi_n^\dagger$ to generate the basis of normalized states,

$$|\{k_p\}>_\varphi = \prod_{p=-\infty}^{+\infty} \frac{1}{\sqrt{k_p!}} \left(\varphi_p^\dagger\right)^{k_p} |0>_\varphi\ , \tag{52}$$

in which $k_p \geq 0$ quanta of integer momentum $-\infty < p < +\infty$ are excited. In the chiral boson sector, the creation operators $A_{\pm,n}^\dagger$ act on a normalized Fock vacuum $|0>_\Phi$ which is annihilated by the operators $A_{\pm,n}$, to also generate the full Fock basis of quantum excitations of the chiral fields. Finally, the chiral zero mode sector $(Q_\pm, p_\pm)$ is represented in the orthonormalized momentum eigenbasis $|n_+, n_->$, $<n'_+, n'_-|n_+, n_-> = \delta_{n'_+, n_+} \delta_{n'_-, n_-}$, such that,

$$p_\pm|n_+, n_-> = (n_\pm + \lambda_\pm)|n_+, n_->\ . \tag{53}$$

From the explicit expression for the physical projector, it is then possible to work out its action on this basis of the quantum space of states.[10] One then finds

$$\mathbb{E} = \mathbb{1}_\varphi \otimes |\Omega_\theta><\Omega_\theta|\ , \tag{54}$$

where the tensor product refers to that between the electric and chiral sectors of degrees of freedom, while the gauge invariant vacuum $|\Omega_\theta>$ in the sector of gauge dependent degrees of freedom is given by the non-normalizable state

$$|\Omega_\theta> = \sum_{\ell=-\infty}^{+\infty} e^{i\ell\theta} e^{-\sum_{n=1}^\infty A_{+,n}^\dagger A_{-,n}^\dagger} |n_+ = \ell, n_- = \ell; 0>_\Phi\ , \tag{55}$$

11

which is thus seen to correspond to a θ-vacuum associated to a coherent superposition, characterized by the parameter θ, of the chiral momentum eigenstates. In fact, gauge invariance enforces the constraint $p_+ = p_-$, which not only requires identical eigenvalues $n_+ = \ell = n_-$ but also identical values modulo integers for the U(1) holonomies $\lambda_+ = \lambda_-$ (mod $\mathbb{Z}$), namely the parity invariant condition $\alpha_+ = \alpha = \alpha_-$ (mod $\mathbb{Z}$) since $\alpha_\pm = \lambda\lambda_\pm$ (mod $\mathbb{Z}$).

Consequently, all physical states of the system are spanned by the free massive quanta of the electric field, namely the multiparticle states

$$|\{k_p\} >_\varphi \otimes|\Omega_\theta > , \tag{56}$$

which are all invariant under the action of both small and large U(1) gauge transformations, $U(\epsilon_0)$ and U_k. In particular, the first-class constraint operator $\phi(x)$ annihilates any physical state, since $\phi(x)|\Omega_\theta >= 0$. This basis of states also diagonalizes the total Hamiltonian H_T of the system, which thus coincides with the first-class one H on the subspace of physical states, leading to the physical energy spectrum

$$E(\{k_p\}) = \sum_{p=-\infty}^{+\infty} k_p \omega_p \ , \quad \omega_p = \sqrt{\left(\frac{2\pi}{L}p\right)^2 + \mu^2} \ , \quad \mu = \frac{|e|}{\sqrt{\pi}} \ , \tag{57}$$

whose spacetime momentum value is also

$$P^1(\{k_p\}) = \sum_{p=-\infty}^{+\infty} k_p \left(\frac{2\pi p}{L}\right) \ . \tag{58}$$

Finally, note that the non-normalizable character of physical states resides only within the gauge invariant contribution of the gauge dependent degrees of freedom, namely within the factorized single gauge invariant θ-vacuum $|\Omega_\theta >$.

For the sake of illustration, let us also consider the expectation values of some quantities of interest. Given any multiparticle state, the expectation value of the electric field is readily seen to vanish identically at all times, with nonvanishing fluctuations around that mean value however,

$$< E(x) >= 0 \ , \quad <: E^2(x) :>= \frac{1}{L} \sum_{p=-\infty}^{+\infty} \frac{\mu \, k_p}{\sqrt{(\frac{2\pi}{L}p)^2 + \mu^2}} \ . \tag{59}$$

On the other hand, the expectation value of the gauge potential $A^1(x)$ within the same multiparticle states reads,

$$< A^1(x) >= \frac{2\pi}{e\lambda L}\lambda_\pm = \frac{2\pi}{eL}\alpha_\pm \ . \tag{60}$$

Even though the operator $A^1(x)$ is not gauge invariant, its expectation value for gauge invariant states should be; nonetheless, it need not necessarily vanish for that reason. Indeed, first note that on account of translation invariance in x, the quantity $< A^1(x) >$ measures only the expectation value of the zero mode of $A^1(x)$, which is thus certainly invariant under small U(1) gauge transformations. Furthermore, because of large U(1) gauge transformations, this zero mode is actually defined only modulo $(2\pi)/(|e|L)$. This is indeed also the property shared by the above result for $< A^1(x) >$, given that the U(1) holonomies $\lambda_\pm$, which are directly related to the choice of boundary conditions of the fermionic fields $\psi_\pm(x)$ in x, are defined modulo integers. Consequently, even though nonvanishing, the expectation value $< A^1(x) >$ is perfectly gauge invariant, a possibility which arises since the zero mode of that operator is defined on a circle of radius $1/(|e|L)$ rather than the real line. Note that consistency of this result requires once again the two U(1) holonomies $\lambda_\pm$, or the two factors

$\alpha_\pm$, to be equal to one another modulo integers. It is interesting to remark that the expectation value $< A^1(x) >$ is also a measure of the fermionic boundary conditions. In contradistinction, the expectation value of the other original gauge dependent fields, namely the Weyl spinors $\psi_\pm(x)$, always vanishes for multiparticle states, $< \psi_\pm(x) >= 0$, which is again a result perfectly consistent with gauge invariance. Since these degrees of freedom vary continuously for small as well as large U(1) gauge transformations in contradistinction to the zero mode of $A^1(x)$, being gauge invariant necessarily their expectation value for gauge invariant states much vanish identically.

Note that the U(1) charge operator Q, see (42), vanishes identically for any physical state, on account of gauge invariance under small gauge transformations of these states, $Q|\{k_p\} >_\varphi \otimes|\Omega_\theta >= 0$. None of the physical states thus carries any electric charge. This conclusion includes the physical ground state $|0 >_\varphi \otimes|\Omega_\theta >$, showing that the exact U(1) vector gauge symmetry of the quantized system is not spontaneously broken. Had the global axial symmetry U(1)$_A$ not be broken explicitly by a quantum anomaly, the issue of the Wigner or Goldstone mode realization of that symmetry could have been raised as well. However, the quantum axial charge Q_5, see (42), does not commute with the quantum Hamiltonian H, so that its action does not conserve energy even though it maps physical states into physical states since it commutes with the first-class constraint operator $\phi(x)$. For instance, acting on the physical ground state, one finds

$$Q_5|0 >_\varphi \otimes|\Omega_\theta >= i\frac{\lambda\sqrt{2\mu}}{2\sqrt{\pi}\sqrt{L}} \ |k_0 = 1 >_\varphi \otimes|\Omega_\theta > \ , \qquad (61)$$

resulting in a zero momentum 1-particle excitation of the electric field. For the same reason, the expectation value of the chiral charge vanishes identically for any multiparticle state, including the physical ground state, $< Q_5 >= 0$. Furthermore, any finite axial transformation acts as follows on the physical ground state,

$$e^{i\alpha Q_5} |0 >_\varphi \otimes|\Omega_\theta >= |z_0 >_\varphi \otimes|\Omega_\theta > \qquad \text{with} \qquad z_0 = -\frac{\alpha\lambda\sqrt{2\mu}}{2\sqrt{\pi}\sqrt{L}} \ , \qquad (62)$$

where $|z_0 >_\varphi$ stands for the holomorphic zero momentum coherent state in the $(\varphi(x), \pi_\varphi(x))$ sector defined by

$$|z_0 >_\varphi= e^{-\frac{1}{2}|z_0|^2} e^{z_0\varphi_0^\dagger} |0 >_\varphi \ , \qquad (63)$$

z_0 being an arbitrary complex variable. Consequently, axial transformations act by producing coherent zero momentum excitations of the electric field. However, they do not induce a change in the θ parameter of the θ-vacuum $|\Omega_\theta >$ in the sector of gauge dependent degrees of freedom. This conclusion, as well as some of the above results, should be contrasted with those available in the literature following any gauge fixing procedure,[2, 3, 4] some of which are thus seen to be actually artefacts due to gauge fixing.

6   Concluding Remarks

Through quantization but without any gauge fixing procedure whatsoever, the original gauge and fermionic degrees of freedom of the massless Schwinger model have been reorganized into a diagonalized set of basic gauge invariant and gauge dependent operators. The gauge invariant ones are those of the electric field whose conjugate momentum variable is the single gauge invariant combination of the vector field A^1 with the bosonized Weyl spinors. The gauge dependent ones are those of the specific combinations of the bosonized Weyl spinors with the electric field which are decoupled from the gauge invariant sector. The gauge invariant dynamics and physical content of the system is that of a free pseudoscalar field of mass $\mu = |e|/\sqrt{\pi}$, namely the electric field, which may be excited in

any multiparticle state. The gauge dependent degrees of freedom contribute only through a uniquely defined gauge invariant combination which is the θ-vacuum $|\Omega_\theta>$. This state is comprised, see (55), on the one hand, of a condensate which is the coherent superposition of pairs of nonzero mode excitations of the chiral bosons $\Phi_+(x)$ and $\Phi_-(x)$ of vanishing total momentum, and on the other hand, of a θ-vacuum coherent superposition of the chiral boson zero mode momentum eigenstates sharing a common eigenvalue.

The latter feature stems from the gauge invariant topological properties of the zero modes of the gauge dependent sector under both small and large U(1) gauge transformations. The θ variable, defined modulo 2π, parametrizes the sole freedom left at the quantum level in the gauge invariant physical contribution of the gauge dependent sector. Up to that parameter, which labels all possible distinct quantum representations of the modular group $\mathbb{Z}$ of large U(1) gauge transformations, there exists only a single gauge invariant quantum state within the sector of gauge dependent degrees of freedom, namely the θ-vacuum $|\Omega_\theta>$. All the gauge invariant topological features of the system thus reside in that θ-vacuum for the gauge invariant combination of its zero modes.

The quantization and solution of the model is nonperturbative in that taking the limit of a vanishing gauge coupling constant e leads to a singular behaviour. There is no analytic deformation possible of the set of field operators that diagonalize the interaction-free system into that which diagonalizes the interacting system, thus precluding any perturbation expansion. Note also that the compactification of the spatial coordinate into a circle is essential in bringing to the fore all the topology features of the system within its zero mode sector which are implied by gauge invariance under both small and large U(1) gauge transformations. The radius R or circumference L of that circle provides a regularization of the system which allows for a discrete spectrum of quantum states free of infrared divergences in 1+1 dimensions, and thus in particular a clear separation of the zero mode contributions and their topological properties.

The physical projector, free of the necessity of any gauge fixing, is thus able to unambiguously identify, through a nonperturbative quantization which diagonalizes the dynamics of the massless Schwinger model, and in agreement with results established following whatever gauge fixing procedure at least in as far as the physical spectrum is concerned, the actual physical content of the model, without suffering, however, from artefacts resulting from gauge fixing which render the physical interpretation sometimes confusing. The physical projector has also achieved the quantization of a compact phase space, namely the product of the two two-tori corresponding to the zero mode sectors of the two gauge dependent chiral bosons, without having recourse to any geometric quantization methods beyond the usual rules of canonical quantization.

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