USCHEP/02-ib5 hep-th/0211131
UT-02-58 November, 2002

String Amplitudes from Moyal String Field Theory

I. Bars[a,*], I. Kishimoto[b,†] and Y. Matsuo[b,‡],

[a)] *Department of Physics and Astronomy, University of Southern California, Los Angeles, CA 90089-0484, USA*

[b)] *Department of Physics, Faculty of Science, University of Tokyo Hongo 7-3-1, Bunkyo-ku, Tokyo 113-0033, Japan*

Abstract

We illustrate a basic framework for analytic computations of Feynman graphs using the Moyal star formulation of string field theory. We present efficient methods of computation based on (a) the monoid algebra in noncommutative space and (b) the conventional Feynman rules in Fourier space. The methods apply equally well to perturbative string states or nonperturbative string states involving D-branes. The ghost sector is formulated using Moyal products with fermionic (b,c) ghosts. We also provide a short account on how the purely cubic theory and/or VSFT proposals may receive some clarification of their midpoint structures in our regularized framework.

[*]e-mail address: bars@usc.edu
[†]e-mail address: ikishimo@hep-th.phys.s.u-tokyo.ac.jp
[‡]e-mail address: matsuo@phys.s.u-tokyo.ac.jp

1 Introduction

During the past two years there has been a remarkable conceptual and technical progress in string field theory (SFT) which was stimulated by its application [1] to tachyon condensation, and the prospect of further applications to more general physics of D-branes. The numerical computation of the D-brane tension, for example, has reached a rather accurate estimate [2].

The role of SFT [3] as a method to analyze non-perturbative string phenomena has by now become rather evident. Consequently, efficient computational tools to achieve analytic understanding of non-perturbative string physics are now needed. Toward this goal, a new computational technique has been developing over the past two years, starting with the discovery [4] of a direct connection between Witten's star product and the usual Moyal star product that is well known in noncommutative geometry. The new Moyal star $\star$ is applied on string fields $A(\bar{x}, x_e, p_e)$ in the phase space of *even* string modes, independently for each $e.$ The product is local in the string midpoint $\bar{x}.$ Some basic numerical infinite matrices $T_{eo}, R_{oe}, w_e, v_o,$

$$T_{eo} = \frac{4o\,(i)^{o-e+1}}{\pi\,(e^2 - o^2)},\ R_{oe} = \frac{4e^2\,(i)^{o-e+1}}{\pi o\,(e^2 - o^2)},\ w_e = \sqrt{2}\,(i)^{-e+2}\,,\ v_o = \frac{2\sqrt{2}}{\pi}\frac{(i)^{o-1}}{o} \tag{1}$$

labeled by even/odd integers ($e = 2, 4, 6, \cdots$, and $o = 1, 3, 5, \cdots$) were needed to disentangle the Witten star into independent Moyal stars for each mode $e.$ These matrices enter in a fundamental way in all string computations in the Moyal star formulation of string field theory (MSFT).

In subsequent work [5, 6] MSFT was developed into as a precise definition of string field theory, by resolving all midpoint issues, formulating a consistent cutoff method in the number of string modes $2N$, and developing a monoid algebra as an efficient and basic computational tool.

Computations in MSFT are based only on the use of the Moyal star product. The new star provides an alternative to the oscillator tool or the conformal field theory tool as a method of computation. In particular, cumbersome Neumann coefficients or conformal maps that appear in the other approaches to SFT are not needed, since they follow correctly from the Moyal star [6][1].

A cutoff is needed in all formulations of SFT to resolve associativity anomalies [5]. The cutoff consists of working with a finite number of string modes $n = 1, 2, \cdots, 2N$ that have oscillator frequencies κ_n, and introducing finite $N \times N$ matrices T_{eo}, R_{oe}, w_e, v_o that are uniquely determined as functions of a diagonal matrix $\kappa = diag\,(\kappa_e, \kappa_o)$ which represents arbitrary frequencies. The $\kappa_n = (\kappa_e, \kappa_o)$ are any reasonable functions of $n = (e, o)$, including the possible choice of the usual oscillator frequencies $\kappa_n = n$, even at finite N. The finite matrices T, R, w, v are introduced through the following defining relations (a bar means transpose)

$$R = (\kappa_o)^{-2}\,\bar{T}\,(\kappa_e)^2\,, \quad R = \bar{T} + v\bar{w}, \quad v = \bar{T}w, \quad w = \bar{R}v\,. \tag{2}$$

[1] Subsequent proposals of Moyal star products equivalent to the one in [4] have appeared [7][8][9]. They all become discrete and well defined with the same cutoff method, and remain related to the $\star$ which we use here.

The same relations are satisfied by the infinite matrices in Eq.(1) that have the usual frequencies $\kappa_n = n$ and $N \to \infty$. These equations were uniquely solved in terms of arbitrary κ_n, N [6];

$$T_{eo} = \frac{w_e v_o \kappa_o^2}{\kappa_e^2 - \kappa_o^2}, \quad R_{oe} = \frac{w_e v_o \kappa_e^2}{\kappa_e^2 - \kappa_o^2}, \tag{3}$$

$$w_e = i^{2-e} \frac{\prod_{o'} \left|\kappa_e^2/\kappa_{o'}^2 - 1\right|^{\frac{1}{2}}}{\prod_{e' \neq e} \left|\kappa_e^2/\kappa_{e'}^2 - 1\right|^{\frac{1}{2}}}, \quad v_o = i^{o-1} \frac{\prod_{e'} \left|1 - \kappa_o^2/\kappa_{e'}^2\right|^{\frac{1}{2}}}{\prod_{o' \neq o} \left|1 - \kappa_o^2/\kappa_{o'}^2\right|^{\frac{1}{2}}}. \tag{4}$$

For $\kappa_n = n$ and $N = \infty$, these reduce to the expressions in Eq.(1). Although the finite matrices are given quite explicitly, most computations are done by using simple matrix relations among them without the need for their explicit form. The following matrix relations are derived [6] from Eq.(2):

$$\begin{aligned} &TR = 1_e, \quad RT = 1_o, \quad \bar{R}R = 1 + w\bar{w}, \quad \bar{T}T = 1 - v\bar{v}, \\ &T\bar{T} = 1 - \frac{w\bar{w}}{1 + \bar{w}w}, \quad Tv = \frac{w}{1 + \bar{w}w}, \quad \bar{v}v = \frac{\bar{w}w}{1 + \bar{w}w}, \\ &Rw = v(1 + \bar{w}w), \quad R\bar{R} = 1 + v\bar{v}\,(1 + \bar{w}w)\,. \end{aligned} \tag{5}$$

It is important to emphasize that in our formalism computing with arbitrary frequencies κ_n and finite number of modes $2N$, is as easy as working directly in the limit[2].

For example, as a test of MSFT, Neumann coefficients for any number of strings were computed in [6] with arbitrary oscillator frequencies κ_n and cutoff N. The cutoff version of Neumann coefficients $N_{mn}^{rs}(t)\,, N_{0n}^{rs}(t,w)\,, N_{00}^{rs}(t,w)\,,$ were found to be simple analytic expressions that depend on a single $N \times N$ matrix $t_{eo} = \kappa_e^{1/2} T_{eo} \kappa_o^{-1/2}$ and an N-vector w_e. These explicitly satisfy the Gross-Jevicki nonlinear relations for any κ_n, N [6]. It is then evident that T and w (which follow from Eq.(2) as functions of κ) are more fundamental than the Neumann coefficients. As a corollary of this result, by diagonalizing the matrix t [6] one can easily understand at once why there is a Neumann spectroscopy for the 3-point vertex [11] or more generally the n-point vertex [6].

Such explicit analytic results, especially at finite N, are new, and not obtained consistently in any other approach. At finite N the MSFT results could be used in numerical as well as analytic computations as a more consistent method than level truncation.

In this paper, we give a brief report on explicit computations of string Feynman diagrams in MSFT. Related work, but in the oscillator formalism, is pursued in [12]. Our formalism, with the finite N regularization, has the advantage that it applies in a straightforward manner when the external states are either perturbative string states or non-perturbative D-brane type states. So we can perform computations with the same ease when nonperturbative states are involved. Our

[2] The infinite matrices in Eq.(1) have well defined products when multiplied two at a time, e.g. $TR = 1_e$, $T\bar{T} = 1_e$, etc.. However they give ambiguous results in multiple matrix products due to associativity anomalies [5] that arise from marginally convergent infinite sums. For example $(RT)v = v$, but $R\,(Tv) = 0$. The unregulated Neumann coefficients suffer from the same anomaly [10][6]. The finite matrices resolve all ambiguities. One can follow how the anomaly occurs by noting from Eq.(1) that $\bar{w}w \to \infty$ as $N \to \infty$. For example, the zero in $Tv = w\,(1 + \bar{w}w)^{-1} \to 0$ gets multiplied by an infinity that comes from the product $Rw = v\,(1 + \bar{w}w) \to \infty$. A unique answer is obtained for any association, $RTv = v$, by doing all computations at finite N and taking the limit only at the end.

regularization plays a role similar to that of lattice regularization in defining nonperturbative QCD. Any string amplitude is analytically defined in this finite scheme. Furthermore, we emphasize that to recover correctly the usual string amplitudes in the large N limit, it is essential that associativity anomalies are resolved in the algebraic manipulations of T, R, w, v in these computations[2]. In this paper, we only present the basic ideas and the important steps of the computation. The details will appear in a series of related publications [13].

The organization of this paper is as follows. In section 2, we define the regularized action for Witten's string field theory. In this paper, we will work in the Siegel gauge where explicit realization of the finite N regularization is possible. In section 3, we present Feynman diagram computations in coordinate representation in noncommutative space. This is an effective framework closely related to the methods in [6]. In section 4, we define systematically Feynman rules in the Fourier basis. This is useful to see the connection with the conventional Feynman rules in quantum field theory in noncommutative space [14]. We present a few examples of scattering amplitudes computed in both frameworks. In section 5, we consider a re-organization of Feynman rules in Fourier space to give a direct relation with the computations in section 3. In section 6, we briefly outline the definition of Moyal product for the (fermionic) ghost system. In section 7, we consider a possible relation with vacuum string field theory (VSFT).

2 Regularized action

The starting point of our study is Witten's action [3] for the open bosonic string, taken in the Siegel gauge, and rewritten in the Moyal basis

$$S=\int d^d\bar{x}\, Tr\left(\frac{1}{2}A\star(L_0-1)A+\frac{1}{3}A\star A\star A\right)\ . \tag{6}$$

The field $A\left(\bar{x},\xi\right)$ depends on the noncommutative coordinates $\xi_i^\mu=\left(x_2^\mu,x_4^\mu\cdots,x_{2N}^\mu,p_2^\mu,p_4^\mu\cdots,p_{2N}^\mu\right)$. The ξ_i^μ may include ghosts in either the bosonic or the fermionic version. The bosonized ghost was discussed in [6] as a 27'th bosonic coordinate $\xi_i^{27}=(\phi_e,\pi_e)$, while the fermionic case will be discussed in a later section in this paper. In the following sections, however, we concentrate basically on the matter sector for the simplicity of argument. The Moyal product $\star$ and the trace Tr are defined at fixed $\bar{x}$ as,[3]

$$(A\star B)(\bar{x},\xi)=A(\bar{x},\xi)\ e^{\frac{1}{2}\eta_{\mu\nu}\overleftarrow{\partial}_\mu^i\sigma_{ij}\overrightarrow{\partial}_\nu^j}B(\bar{x},\xi),\ Tr\left(A\left(\bar{x}\right)\right)=\int\frac{(d\xi)}{\left(\det 2\pi\sigma\right)^{d/2}}A\left(\bar{x},\xi\right). \tag{7}$$

The string field lives in the direct product of the Moyal planes, with $\left[\xi_i^\mu,\xi_j^\nu\right]_\star=\sigma_{ij}\,\eta^{\mu\nu}$, where

$$\sigma_{ij}=i\theta\begin{pmatrix}0 & 1_e\\ -1_e & 0\end{pmatrix}. \tag{8}$$

[3] In the following, we denote $d^d\xi_1\cdots d^d\xi_{2N}$ as $(d\xi)$.

The parameter θ absorbs units and could be mapped to 1 by a rescaling of the units of p_e.

The kinetic term is given by the Virasoro operator L_0 which was computed in Moyal space in [6]. Here we rewrite it in the form of a differential operator

$$L_0 = \frac{1}{2}\beta_0^2 - \frac{d}{2}Tr\left(\tilde{\kappa}\right) - \frac{1}{4}\bar{D}_\xi \left(M_0^{-1}\tilde{\kappa}\right) D_\xi + \bar{\xi}\left(\tilde{\kappa}M_0\right)\xi, \tag{9}$$

where $\beta_0 = -il_s\frac{\partial}{\partial \bar{x}}$, $D_\xi = \left(\left(\frac{\partial}{\partial x_e} - i\frac{\beta_0}{l_s}w_e\right), \frac{\partial}{\partial p_e}\right)$, and l_s is the string length. The $(2N)\times(2N)$ matrices

$$\tilde{\kappa} = \begin{pmatrix} \kappa_e & 0 \\ 0 & T\kappa_o R \end{pmatrix}, \; M_0 = \begin{pmatrix} \frac{\kappa_e}{2l_s^2} & 0 \\ 0 & \frac{2l_s^2}{\theta^2}T\kappa_o^{-1}\bar{T} \end{pmatrix}, \tag{10}$$

give the block diagonal forms $M_0^{-1}\tilde{\kappa} = diag\left(2l_s^2 1_e, \frac{\theta^2}{2l_s^2}\kappa_e^2\right)$ and $\tilde{\kappa}M_0 = diag\left(\frac{\kappa_e^2}{2l_s^2}, \frac{2l_s^2}{\theta^2}\left(T\bar{T}\right)_{ee'}\right)$ after using Eqs.(2,5). We note that $T\bar{T}$ in $\tilde{\kappa}M_0$ is almost diagonal, since $T\bar{T} = 1 - \frac{w\bar{w}}{1+\bar{w}w}$, and the second term becomes naively negligible in the large N limit since $\bar{w}w \to \infty$. A major simplification would occur if one could neglect this term. However, with this simplification one cannot recover the string one-loop amplitude or other quantities correctly, as we will see below in Eq.(46), because of the anomalies discussed in footnote (2). The lesson is that one should not take the large N limit at the level of the Lagrangian[4]. One should do it only after performing all the algebraic manipulations that define the string diagram. Consequently all of the following expressions are at finite N unless specified otherwise.

The string field that represents the perturbative vacuum is given by the gaussian $A_0 \sim \exp\left(-\bar{\xi}M_0\xi\right)$ (for any κ_n, N). The on-shell tachyon state $\left(L_0 - 1\right)A_t = 0$ is given by $A_0 e^{ik\cdot x_0}$, which is

$$A_t\left(\bar{x},\xi\right) = \mathcal{N}_0 e^{-\bar{\xi}M_0\xi - \bar{\xi}\lambda_0} e^{ik\cdot\bar{x}}, \; \left(\lambda_0\right)_i^\mu = -ik^\mu\left(w_e, 0\right), \; \mathcal{N}_0 = \left(\det 4\sigma M_0\right)^{d/4}, \; l_s^2k^2 = 2\,. \tag{11}$$

The form of $\left(\lambda_0\right)_i^\mu$ for the tachyon follows from $e^{ik\cdot x_0}$ after rewriting the center of mass coordinate x_0 in terms of the midpoint $\bar{x}$, i.e. $x_0 = \bar{x} + w_e x_e$. The norm $\mathcal{N}_0$ is fixed by requiring $Tr\left(A_t^* \star A_t\right) = 1$.

All perturbative string states with definite center of mass momentum k^μ are represented by polynomials in ξ multiplying the tachyon field. All of them can be obtained from the following generating field by taking derivatives with respect to a general λ

$$A\left(\bar{x},\xi\right) = \mathcal{N}e^{-\bar{\xi}M_0\xi - \bar{\xi}\lambda}\; e^{ik\cdot\bar{x}}, \tag{12}$$

and setting $\lambda \to \lambda_0 = -ik^\mu\left(w_e, 0\right)$ at the end.

Nonperturbative string fields that describe D-branes involve projectors in VSFT conjecture [1]. A general class is [6]

$$A_{D,\lambda}\left(\xi\right) = \mathcal{N}\exp\left(-\bar{\xi}D\xi - \bar{\xi}\lambda\right), \; \mathcal{N} = 2^{dN}\exp\left(-\frac{1}{4}\bar{\lambda}\sigma D\sigma\lambda\right), \; D = \begin{pmatrix} a & ab \\ ba & \frac{1}{a\theta^2} + bab \end{pmatrix}. \tag{13}$$

[4] This term in L_0 was missed in [7] in their attempt to compare the discrete Moyal $\star_e$ of [4] to the continuous Moyal $\star_\kappa$ directly at $N = \infty$, and erroneously concluded that there was a discrepancy. In fact, there is full agreement.

For any λ, and symmetric a, b, these satisfy $A \star A = A$, and $Tr A = 1$. For a D-brane the components of λ parallel to the brane vanish, $\lambda^{\parallel} = 0$, while those perpendicular to the brane are nonzero as a function of the midpoint $\lambda^{\perp}(\bar{x}_{\perp}) \neq 0$. Examples such as the sliver field, butterfly field, etc., are special cases of these formulas with specific forms of the matrix D [6].

It appears that for all computations of external states of interest we should consider the field configurations that contain the general parameters $\mathcal{N}, M_{ij}, \lambda_i^{\mu}, k^{\mu}$

$$A_{\mathcal{N},M,\lambda,k} = \mathcal{N} \exp\left(-\bar{\xi} M \xi - \bar{\xi}\lambda + ik \cdot \bar{x}\right), \tag{14}$$

where for perturbative states $\mathcal{N}$ is a constant, but for D-brane states it may depend on $\bar{x}$. These fields form a closed algebra under the star

$$\begin{aligned} & \left(\mathcal{N}_1 \exp\left(-\bar{\xi} M_1 \xi - \bar{\xi}\lambda_1 + ik_1 \cdot \bar{x}\right)\right) \star \left(\mathcal{N}_2 \exp\left(-\bar{\xi} M_2 \xi - \bar{\xi}\lambda_2 + ik_2 \cdot \bar{x}\right)\right) && (15) \\ = & \left(\mathcal{N}_{12} \exp\left(-\bar{\xi} M_{12} \xi - \bar{\xi}\lambda_{12} + i\left(k_1 + k_2\right) \cdot \bar{x}\right)\right) && (16) \end{aligned}$$

where the structure of $\mathcal{N}_{12}, (M_{12})_{ij}, (\lambda_{12})_i^{\mu}$ is given as (define $m_1 = M_1\sigma,\ m_2 = M_2\sigma,\ m_{12} = M_{12}\sigma$)

$$\begin{aligned} m_{12} &= \left(m_1 + m_2 m_1\right)\left(1 + m_2 m_1\right)^{-1} + \left(m_2 - m_1 m_2\right)\left(1 + m_1 m_2\right)^{-1}, && (17) \\ \lambda_{12} &= \left(1 - m_1\right)\left(1 + m_2 m_1\right)^{-1}\lambda_2 + \left(1 + m_2\right)\left(1 + m_1 m_2\right)^{-1}\lambda_1, && (18) \\ \mathcal{N}_{12} &= \frac{\mathcal{N}_1 \mathcal{N}_2}{\det\left(1 + m_2 m_1\right)^{d/2}} e^{\frac{1}{4}\left(\left(\bar{\lambda}_1 + \bar{\lambda}_2\right)\sigma\left(m_1 + m_2\right)^{-1}\left(\lambda_1 + \lambda_2\right) - \bar{\lambda}_{12}\sigma\left(m_{12}\right)^{-1}\lambda_{12}\right)}. && (19) \end{aligned}$$

This algebra is a monoid, which means it is associative, closed, and includes the identity element (number 1). It is short of being a group since some elements (in particular projectors) do not have an inverse, although the generic element does have an inverse. The trace of a monoid is given through Eq.(7) (assuming a decaying exponential in ξ)

$$Tr\left(A_{\mathcal{N},M,\lambda,k}\right) = \frac{\mathcal{N} e^{ik\cdot\bar{x}} e^{\frac{1}{4}\bar{\lambda} M^{-1}\lambda}}{\det\left(2M\sigma\right)^{d/2}}. \tag{20}$$

Building on the computations in [6], this monoid algebra will be used as a basic computational tool in evaluating string field theory diagrams.

3 Feynman graphs in ξ basis

In this section we discuss Feynman graphs in the noncommutative ξ basis and in the next section we formulate them in the Fourier transformed basis.

In a Feynman diagram an external string state will be represented by a monoid $A(\bar{x}, \xi)$ that corresponds to a perturbative string state or nonperturbative D-brane state as discussed in the previous section. This corresponds to a boundary condition in the language of a worldsheet representation of the Feynman diagram. The propagator is given as an integral using a Schwinger parameter $(L_0-1)^{-1} = \int_0^\infty d\tau e^\tau \exp(-\tau L_0)$. This corresponds to the free propagation of the string as represented by the worldsheet between the boundaries.

To evaluate Feynman diagrams we will need the τ-evolved monoid element

$$e^{-\tau L_0}\left(\mathcal{N}e^{-\bar{\xi}M\xi-\bar{\xi}\lambda}e^{ip\cdot\bar{x}}\right)=\mathcal{N}(\tau)\ e^{-\bar{\xi}M(\tau)\xi-\bar{\xi}\lambda(\tau)}e^{ip\cdot\bar{x}}. \tag{21}$$

Both sides must be annihilated by the Schrödinger operator $(\partial_\tau+L_0)$. The result of the computation is given by

$$M(\tau)=\left[\sinh\tau\tilde{\kappa}+\left(\sinh\tau\tilde{\kappa}+M_0M^{-1}\cosh\tau\tilde{\kappa}\right)^{-1}\right]\left(\cosh\tau\tilde{\kappa}\right)^{-1}M_0, \tag{22}$$

$$\lambda(\tau)=\left[\left(\cosh\tau\tilde{\kappa}+MM_0^{-1}\sinh\tau\tilde{\kappa}\right)^{-1}(\lambda+iwp)\right]-iwp, \tag{23}$$

$$\mathcal{N}(\tau)=\frac{\mathcal{N}\ e^{-\frac{1}{2}l_s^2p^2\tau}\ \exp\left[\frac{1}{4}\left(\bar{\lambda}+ip\bar{w}\right)\left(M+\coth\tau\tilde{\kappa}\ M_0\right)^{-1}(\lambda+iwp)\right]}{\det\left(\frac{1}{2}\left(1+MM_0^{-1}\right)+\frac{1}{2}\left(1-MM_0^{-1}\right)e^{-2\tau\tilde{\kappa}}\right)^{d/2}}. \tag{24}$$

For the tachyon in Eq.(11) this simplifies greatly $M_0(\tau)=M_0$, $\lambda_0(\tau)=\lambda_0$, $\mathcal{N}_0(\tau)=\mathcal{N}_0e^{-\tau}$. Note that even in the general case the evolved monoid is also a monoid that can be star multiplied easily with other monoids.

The diagrams below will be given as a function of the Schwinger parameters τ_i. The function should be integrated using the measure $\int_0^\infty d\tau_i e^{\tau_i}$ for each propagator. We now give some examples of tree diagrams

- The diagram $1-2$ for two external states A_1, A_2 joined by a propagator is given by

$$\begin{aligned}&\int d^d\bar{x}\,Tr\left(A_1\star e^{-\tau L_0}A_2\right)\\ =\ &\frac{\mathcal{N}_1\mathcal{N}_2(\tau)\exp\left(\frac{1}{4}\left(\bar{\lambda}_1+\bar{\lambda}_2(\tau)\right)\left(M_1+M_2(\tau)\right)^{-1}\left(\lambda_1+\lambda_2(\tau)\right)\right)}{\left(\det\left(2\left(M_1+M_2(\tau)\right)\sigma\right)\right)^{d/2}}(2\pi)^d\,\delta^d(p)\end{aligned} \tag{25}$$

  where $p^\mu=k_1^\mu+k_2^\mu$. To evaluate it we used Eqs.(21-24), and the trace in Eq.(20). For tachyon external states of Eq.(11) this expression collapses to just $e^{-\tau}(2\pi)^d\,\delta^d(p)$, which is the expected result. For more general states our formula provides an explicit analytic result.

- The 4 point function is computed from the diagram for ${}^2_1{>}-{<}^3_4$ and its various permutations of $(1,2,3,4)$. The MSFT expression for this diagram is

$${}_{12}A_{34}=\int d^d\bar{x}\,Tr\left(e^{-\tau L_0}\left(A_1\star A_2\right)\star A_3\star A_4\right). \tag{26}$$

  The two external lines $(1,2)$ are joined to the resulting state by the product $A_{12}=A_1\star A_2$, which is a monoid as given in Eq.(16). This monoid is propagated to $A_{12}(\tau)=e^{-\tau L_0}(A_1\star A_2)$ by using Eqs.(21-24), and then traced with the monoid $A_{34}=A_3\star A_4$. Then the computation of the four point function is completed by using the formula in Eq.(25). That is, replace the monoid A_1 by the monoid A_{34}, and similarly A_2 by A_{12}, and apply Eqs.(22-24) for the monoid

A_{12}. The remainder of the computation is straightforward algebra and those details will be published in a future paper.[13] We emphasize that the external states can be nonperturbative. For the case of perturbative tachyon scattering, for *off shell tachyons*, the result is

$$
\begin{aligned}
{}_{12}A_{34} &= \frac{\det\left(2m_0\right)^{d/2}}{\det\left(1+m_0^2\right)^d}(2\pi)^d\delta^d(p_1+p_2+p_3+p_4) \\
&\times \frac{e^{-\frac{1}{2}l_s^2(p_1+p_2)^2(\tau+\alpha(\tau))}\ e^{l_s^2(p_1+p_3)^2\beta(\tau)}\ e^{\frac{1}{2}l_s^2\sum_{i=1}^4 p_i^2\gamma(\tau)}}{(\det(2G_e(\tau)e^{\tau\kappa_e}))^{-d/2}(\det\left(2G_o(\tau)e^{\tau\kappa_o}\right))^{-d/2}}
\end{aligned}
\tag{27}
$$

where

$$
\begin{aligned}
\alpha\left(\tau\right) &= \bar{z}\left[\bar{t}G_e\left(E_e-1\right)t+G_o\left(E_o-1\right)\right]z, &(28)\\
\beta\left(\tau\right) &= \bar{z}G_oz,\ \ \gamma\left(\tau\right)=\bar{z}G_o\left(E_o-1\right)z-\bar{z}(1+\bar{t}t)z &(29)
\end{aligned}
$$

are given in terms of the following definitions

$$
\begin{aligned}
&z=\left(1+\bar{t}t\right)^{-1}\bar{t}\kappa_e^{-1/2}w,\ \ t=\kappa_e^{1/2}T\kappa_o^{-1/2},\ \ m_0=M_0\sigma\,, &(30)\\
&E_e\left(\tau\right)=\cosh\left(\kappa_e\tau\right)+\frac{2}{1+\bar{t}t}\sinh(\kappa_e\tau),\quad E_o\left(\tau\right)=\cosh\left(\kappa_o\tau\right)+\frac{2}{1+\bar{t}t}\sinh(\kappa_o\tau)\,, &(31)\\
&S_{e,o}\left(\tau\right)=\sinh(\kappa_{e,o}\tau),\quad G_{e,o}\left(\tau\right)=2S_{e,o}\left(E_{e,o}^2-1\right)^{-1}. &(32)
\end{aligned}
$$

Before integrating with the measure $\int_0^\infty d\tau e^\tau$ we also need to multiply this expression by the ghost contribution, which will appear in our future paper. This should reproduce the Veneziano formula when all tachyons are put on shell $l_s^2p_i^2=2$, and we take the large N limit with $\kappa_n=n$.

- Similarly, the diagram ${}^2_1\!> -^{3}| -<^4_5$ involves

$$
\int d^d\bar{x}\,Tr\left(A_{12}\left(\tau_1\right)\star A_3\star A_{45}\left(\tau_2\right)\right)=\int d^d\bar{x}\,Tr\left(e^{-\tau_1L_0}\left(A_1\star A_2\right)\star A_3\star e^{-\tau_2L_0}\left(A_4\star A_5\right)\right). \tag{33}
$$

- The diagram ${}^2_1\!> -^{3}| -^{4}| - <^5_6$ involves

$$
\int d^d\bar{x}\,Tr\left(A_{12}\left(\tau_1\right)\star A_3\star e^{-\tau_2L_0}\left(A_4\star A_{56}\left(\tau_3\right)\right)\right). \tag{34}
$$

Next we consider loop diagrams. We start from an expression given above for a tree diagram and then identify any two external lines to make a closed loop. Suppose the external legs that are identified were represented by the fields A_i, A_j in the tree diagram. In the loop these fields are replaced by

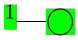
$$
A_i\rightarrow e^{-\tau_iL_0}\left(e^{i\bar{\xi}\eta}e^{ip\cdot\bar{x}}\right),\ \ A_j\rightarrow e^{-i\bar{\xi}\eta}e^{-ip\cdot\bar{x}} \tag{35}
$$

and the integral over η is performed (the Fourier basis is a complete set of states to sum over in the propagation). Here τ_i is the modulus of the new propagator and p^μ becomes the momentum flowing in this propagator by momentum conservation. Some examples of loops follow

- The one loop diagram  with no external legs is obtained from the 2-point vertex $Tr\left(A_1 \star A_2\right)$ by identifying legs 1,2. This leads to the integral

$$\int d^d \bar{x} \int \frac{d^d p}{(2\pi)^d} \frac{(d\eta)}{(2\pi)^{2dN}} Tr\left(\left(e^{-i\bar{\xi}\eta} e^{-ip\cdot\bar{x}}\right) \star \left(e^{-\tau L_0}\left(e^{i\bar{\xi}\eta} e^{ip\cdot\bar{x}}\right)\right)\right). \tag{36}$$

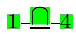
  It is a simple exercise to compute it by using the methods above. The result is given below in Eq.(46) where it is in agreement with our next method of computation in Fourier space. This computation illustrates the importance of the correct treatment of the anomaly[2] as will be emphasized following Eq.(46).

- The tadpole diagram 

is obtained from the 3-point vertex $Tr\left(A_1 \star A_2 \star A_3\right)$ by identifying legs 2,3. This leads to the expression

$$\int d^d \bar{x} \int \frac{d^d p}{(2\pi)^d} \frac{(d\eta)}{(2\pi)^{2dN}} Tr\left(A_1 \star \left(e^{-i\bar{\xi}\eta} e^{-ip\cdot\bar{x}}\right) \star \left(e^{-\tau L_0}\left(e^{i\bar{\xi}\eta} e^{ip\cdot\bar{x}}\right)\right)\right)$$

  which is again straightforward to evaluate.

- The one loop correction to the propagator attached to external states 

is obtained by identifying legs 2,3 in the 4-point function above. This leads to

$$\int d^d \bar{x} \int \frac{d^{dp}}{(2\pi)^d} \frac{(d\eta)}{(2\pi)^{2dN}} Tr\left(e^{-\tau_1 L_0}\left(A_1 \star \left(e^{-i\bar{\xi}\eta} e^{-ip\cdot\bar{x}}\right)\right) \star \left(e^{-\tau_2 L_0}\left(e^{i\bar{\xi}\eta} e^{ip\cdot\bar{x}}\right)\right) \star A_4\right)$$
$$= (2\pi)^d \delta^d (k_1 + k_4) \int \frac{d^{dp}}{(2\pi)^d} \frac{(d\eta)}{(2\pi)^{2dN}} Tr\left(\left(e^{-\tau_1 L_0(k_1-p)}\left(A_1 \star e^{-i\bar{\xi}\eta}\right)\right) \star \left(e^{-\tau_2 L_0(p)} e^{i\bar{\xi}\eta}\right) \star A_4\right)$$

  where in the last line the momentum dependent part of A_1, A_4 has already been peeled off, and the $\bar{x}$ integral performed. Then β_0 in $L_0(\beta_0)$ has been replaced by $k_1 - p$ and p as appropriate for the propagator with the corresponding momentum.

These examples are sufficient to illustrate our approach to such computations.

4 Feynman rules in the Fourier basis

The definition of the action in section 2 is enough to define the Feynman rules for the open string diagram. Note that in the absence of the last term in L_0 of Eq.(9) the kinetic term becomes basically the same as the conventional Lagrangian of the ϕ^3 theory on the non-commutative plane [14].

Vertex The remark above implies that if we take the plane waves $e^{i\bar{\eta}\xi}$ as the basis to expand the noncommutative field, then the n-string interaction vertex in this basis is,

$$Tr\left(e^{i\bar{\eta}_1\xi} \star \cdots \star e^{i\bar{\eta}_n\xi}\right) = \left(\det \frac{\sigma}{2\pi}\right)^{-d/2} \exp\left(-\frac{1}{2}\sum_{i<j} \bar{\eta}_i \sigma \eta_j\right) \delta^{2dN}(\eta_1 + \cdots + \eta_n), \tag{37}$$

which is identical to the interaction vertex for non-commutative field theory.

Propagator The simplification of the vertex is compensated by the complication of the propagator. Now L_0 is not diagonal. It is still easily computable, however, by using Eqs.(21-24) for $M=0$, $\mathcal{N}=1$, $\lambda=-i\eta'$, and inserting them in Eq.(25)

$$\Delta(\eta,\eta',\tau,p) \equiv \int \frac{(d\xi)}{(2\pi)^{2dN}} e^{-i\bar{\eta}\xi} e^{-\tau L_0} e^{i\bar{\eta}'\xi} \tag{38}$$

$$= g(\tau,p)\exp\left(-\bar{\eta}F(\tau)\eta - \bar{\eta}'F(\tau)\eta' + 2\bar{\eta}G(\tau)\eta' + (\bar{\eta}+\bar{\eta}')H(\tau,p)\right), \tag{39}$$

where

$$g(\tau,p) = \left(\frac{\theta}{2\pi}\right)^{dN} (1+\bar{w}w)^{\frac{d}{4}} \left(\prod_{e>0}(1-e^{-2\tau\kappa_e})\prod_{o>0}(1-e^{-2\tau\kappa_o})\right)^{-\frac{d}{2}} e^{-\left(\frac{\tau}{2}+\bar{w}\frac{\tanh(\frac{\tau\kappa_e}{2})}{\kappa_e}w\right)l_s^2p^2} \tag{40}$$

$$F(\tau) = \frac{1}{4}M_0^{-1}(\tanh(\tau\tilde{\kappa}))^{-1} = \begin{pmatrix} \frac{l_s^2}{2\kappa_e}(\tanh(\tau\kappa_e))^{-1} & 0 \\ 0 & \frac{\theta^2}{8l_s^2}\bar{R}\kappa_o(\tanh(\tau\kappa_o))^{-1}R \end{pmatrix}, \tag{41}$$

$$G(\tau) = \frac{1}{4}M_0^{-1}(\sinh(\tau\tilde{\kappa}))^{-1} = \begin{pmatrix} \frac{l_s^2}{2\kappa_e}(\sinh(\tau\kappa_e))^{-1} & 0 \\ 0 & \frac{\theta^2}{8l_s^2}\bar{R}\kappa_o(\sinh(\tau\kappa_o))^{-1}R \end{pmatrix}, \tag{42}$$

$$H(\tau,p) = \frac{\tanh(\tau\kappa_e/2)}{\kappa_e} w l_s^2 p. \tag{43}$$

A critical difference from the conventional propagator in momentum representation is that the propagator depends on the momentum variables at both ends of the propagator in a nontrivial fashion (because momentum is not conserved due to the potential term in L_0). Therefore in the Feynman diagram computation, the momentum integration $d\eta$ is performed at both ends of each propagator.

External State We note also that the external state in Eq.(14) is not diagonal in the momentum basis. We need its Fourier transform

$$\tilde{A}_{\mathcal{N},M,\lambda} = \tilde{\mathcal{N}} e^{-\frac{1}{4}\bar{\eta}M^{-1}\eta + \frac{i}{2}\bar{\lambda}M^{-1}\eta} e^{ip\cdot\bar{x}}, \quad \tilde{\mathcal{N}} = \mathcal{N}(4\pi)^{-dN}(\det M)^{-d/2} e^{\frac{1}{4}\bar{\lambda}M^{-1}\lambda}. \tag{44}$$

For comparison to the oscillator approach, such monoids corresponds to shifted squeezed states $\exp\left(-\frac{1}{2}a^\dagger\mathcal{S}a^\dagger - \mu a^\dagger\right)|p\rangle$, with momentum p^μ. For the general case the relation between oscillator and MSFT parameters is given in Eqs.(3.4-3.6) of [6]. For perturbative states with $\mathcal{S}$=0 these reduce to coherent states $\exp(\sum_n \mu_n\alpha_{-n})|p\rangle$, with a corresponding Moyal field that contains the M_0 of Eq.(10) and $\mathcal{N},\lambda$ given by

$$\mathcal{N} = (\det(4\kappa_o)/\det(\kappa_e))^{d/4}\; e^{\frac{1}{2}(\bar{\mu}_e\mu_e - \bar{\mu}_o\mu_o)}, \quad \lambda = i\begin{pmatrix} -\frac{\sqrt{2}}{l_s}\sqrt{\kappa_e}\mu_e - w_e p \\ \frac{2\sqrt{2}il_s}{\theta}T\kappa_o^{-1/2}\mu_o \end{pmatrix}. \tag{45}$$

To summarize all, the Feynman diagram computation of MSFT reduces to the following simple prescriptions. As in the conventional field theory, we decompose the string diagram into the vertex, the propagator and the external states. We need to perform the momentum integrations attached to each junction of the components. All integrals are gaussian. Therefore the computation of any string amplitude reduces to the computation of the determinant and the inverse of the large matrix which describes the connections among three components (the external states, propagator, vertex). Since the matrices which appear in our computation are explicitly given and finite dimensional, we obtain a finite and well-defined quantity for any string diagram.

In order to illustrate our Feynman rule, we present some examples of the string amplitudes restricted to the matter sector contribution[5].

- 1-loop vacuum amplitude

  One of the simplest graphs is the 1-loop vacuum amplitude. It can be computed directly from Eqs.(21-24), which amounts to integrating $\Delta(\eta, \eta', \tau, p)$ (38):

$$\int d^d p \, Tr(e^{-\tau L_0(p)}) = \int d^d p \int (d\eta) \Delta(\eta, \eta, \tau, p) \tag{46}$$

$$= (2\pi)^{\frac{d}{2}} l_s^{-d} \tau^{-\frac{d}{2}} \prod_{e>0} (1 - e^{-\tau\kappa_e})^{-d} \prod_{o>0} (1 - e^{-\tau\kappa_o})^{-d}, \tag{47}$$

  We see that the correct spectrum (κ_e, κ_o) is read off from the 1-loop graph at any κ_n, N. By taking $\kappa_e = e, \kappa_o = o$ and $N = \infty$, we reproduce the standard perturbative string spectrum! Although this graph does not include any interaction, the coincidence of the spectrum implies the correctness of our propagator. It is essential to *keep* the term $(1 + \bar{w}w)^{-1} \left(\sum_{e>0} w_e p_e \right)^2$ in L_0 which converts κ_e into κ_o.[6] In fact, if one takes the $\bar{w}w = \infty$ limit *first*, this term drops out and one ends up with the wrong spectrum (κ_e, κ_e) instead of (κ_e, κ_o), as happened in Ref. [7][4].

- 4-tachyon amplitude at the tree level: $^{2}_{1}\rangle\!-_5\!-\!-\!-_6\!-\!\langle^{3}_{4}$

  As a simple example including interaction, we consider the perturbative 4-tachyon amplitude $^{2}_{1}\rangle\!-_5\!-\!-\!-_6\!-\!\langle^{3}_{4}$ that we discussed in Eq.(26). Following our Feynman rules in the Fourier basis, we assign the momentum variable η_i ($i = 1, 2, \cdots, 6$) to each junctions of the components. The amplitude is represented by

$$\int (d\eta_1) \cdots (d\eta_6) e^{-\frac{1}{2}(\bar{\eta}_1\sigma\eta_2 + \bar{\eta}_2\sigma\eta_5 + \bar{\eta}_1\sigma\eta_5) - \frac{1}{2}(\bar{\eta}_3\sigma\eta_4 + \bar{\eta}_4\sigma\eta_6 + \bar{\eta}_3\sigma\eta_6)}$$
$$\times \delta^{2dN}(\eta_1 + \eta_2 + \eta_5) \delta^{2dN}(\eta_3 + \eta_4 + \eta_6) \Delta(\eta_5, -\eta_6, \tau, p) \tilde{A}_{p_1}(\eta_1) \tilde{A}_{p_2}(\eta_2) \tilde{A}_{p_3}(\eta_3) \tilde{A}_{p_4}(\eta_4),$$
$$\tilde{A}_{p_i}(\eta) = \mathcal{N}_0 (4\pi)^{-dN} (\det M_0)^{-\frac{d}{2}} e^{ip_i\bar{x}} e^{-\frac{1}{4}(\bar{\eta} - p_i\bar{w}) M_0^{-1} (\eta - p_i w)}, \tag{48}$$

  where τ is the length of the propagator and $p = p_1 + p_2 - p_3 - p_4$ is the zero mode transfer momentum. We first perform the momentum integrations over η_5, η_6 to cancel the delta functions which represent the momentum conservations. The remaining integrations are gaussian

[5]Although these amplitudes are similar to some of those in Ref. [12], our formulas are more general since they all contain the κ_e, κ_o which are arbitrary frequencies at finite N. Furthermore, we can apply them to nonperturbative external states as they stand, with no more effort. To obtain the ordinary perturbative string amplitude, we set $\kappa_e = e, \kappa_o = o$ and take $N \to \infty$ limit (which corresponds to $\bar{w}w \to \infty$) at the last stage of computations.

[6]This contribution comes from the off-diagonal part of TT in Eq.(9).

and the above expression reduces to

$$\mathcal{N}_0^4 \det M_0{}^{-2d} \left(\frac{2\pi}{4^4\theta}\right)^{dN} g(\tau,p)\cdot(\det\mathcal{A})^{-\frac{d}{2}}\cdot e^{\frac{1}{4}\bar{\mathcal{B}}\mathcal{A}^{-1}\mathcal{B}} e^{-\frac{1}{4}(p_1^2+p_2^2+p_3^2+p_4^2)\bar{w}M_0^{-1}w}, \tag{49}$$

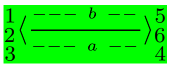
with

$$\mathcal{A} = \begin{pmatrix} F(\tau)+\frac{1}{4}M_0^{-1} & F(\tau)+\frac{1}{4}\sigma & G(\tau) & G(\tau) \\ F(\tau)-\frac{1}{4}\sigma & F(\tau)+\frac{1}{4}M_0^{-1} & G(\tau) & G(\tau) \\ G(\tau) & G(\tau) & F(\tau)+\frac{1}{4}M_0^{-1} & F(\tau)+\frac{1}{4}\sigma \\ G(\tau) & G(\tau) & F(\tau)-\frac{1}{4}\sigma & F(\tau)+\frac{1}{4}M_0^{-1} \end{pmatrix}, \tag{50}$$

$$\bar{\mathcal{B}} = \left(-\bar{H}+\frac{1}{2}p_1\bar{w}M_0^{-1}, -\bar{H}+\frac{1}{2}p_2\bar{w}M_0^{-1}, \bar{H}+\frac{1}{2}p_3\bar{w}M_0^{-1}, \bar{H}+\frac{1}{2}p_4\bar{w}M_0^{-1}\right). \tag{51}$$

In the matrix $\mathcal{A}$, M_0 comes from the external tachyons, F, G from the propagator and σ from the Moyal $\star$ product. On the other hand $\bar{\mathcal{B}}$ originates from zero-mode momentum-dependent terms. The appearances of F at the off-diagonal parts in $\mathcal{A}$ come from the mixing induced from the momentum integrations over $\eta_{5,6}$.

This formula looks more complicated than the expression given in Eq.(27) since it involves the matrices of larger size. The reduction to (27) can be proved by the reorganization of the gaussian integrations which is outlined in the next section.

- 2-loop vacuum amplitude: 

As a next example, we consider one of the 2-loop vacuum graphs: Two 3-string vertices are connected by three propagators which have momenta p_a, p_b, p_c and lengths τ_a, τ_b, τ_c. We assign the momentum variables η_i ($i = 1, \cdots, 6$) as depicted in the figure.

$$\begin{aligned} &\int (d\eta_1)\cdots(d\eta_6) e^{-\frac{1}{2}(\bar{\eta}_1\sigma\eta_2+\bar{\eta}_1\sigma\eta_3+\bar{\eta}_2\sigma\eta_3)-\frac{1}{2}(\bar{\eta}_4\sigma\eta_6+\bar{\eta}_4\sigma\eta_5+\bar{\eta}_6\sigma\eta_5)} \\ &\quad \times \delta^{2dN}(\eta_1+\eta_2+\eta_3)\delta^{2dN}(\eta_4+\eta_5+\eta_6) \\ &\quad \times \Delta(\eta_3,-\eta_4,\tau_a,p_a)\Delta(\eta_1,-\eta_5,\tau_b,p_b)\Delta(\eta_2,-\eta_6,\tau_c,p_c) \\ = \;& g(\tau_a,p_a)g(\tau_b,p_b)g(\tau_c,p_c)\left(\det\left(\pi^{-1}\mathcal{M}\right)\right)^{-\frac{d}{2}} \int \frac{(d\chi_1)}{(2\pi)^{2dN}}\frac{(d\chi_2)}{(2\pi)^{2dN}} e^{\frac{1}{4}\bar{\mathcal{K}}\mathcal{M}^{-1}\mathcal{K}} \end{aligned} \tag{52}$$

where, after denoting $H(\tau_a, p_a)$ as H_a, the quantities $\mathcal{M}, \mathcal{K}$ are given by

$$\mathcal{M} = \begin{pmatrix} F(\tau_b) & \frac{1}{4}\sigma & \frac{1}{4}\sigma & 0 & G(\tau_b) & 0 \\ -\frac{1}{4}\sigma & F(\tau_c) & \frac{1}{4}\sigma & 0 & 0 & G(\tau_c) \\ -\frac{1}{4}\sigma & -\frac{1}{4}\sigma & F(\tau_a) & G(\tau_a) & 0 & 0 \\ 0 & 0 & G(\tau_a) & F(\tau_a) & \frac{1}{4}\sigma & \frac{1}{4}\sigma \\ G(\tau_b) & 0 & 0 & -\frac{1}{4}\sigma & F(\tau_b) & -\frac{1}{4}\sigma \\ 0 & G(\tau_c) & 0 & -\frac{1}{4}\sigma & \frac{1}{4}\sigma & F(\tau_c) \end{pmatrix}, \tag{53}$$

$$\bar{\mathcal{K}} = \left(i\bar{\chi}_1+\bar{H}_b, i\bar{\chi}_1+\bar{H}_c, i\bar{\chi}_1+\bar{H}_a, i\bar{\chi}_2-\bar{H}_a, i\bar{\chi}_2-\bar{H}_b, i\bar{\chi}_2-\bar{H}_c\right). \tag{54}$$

In passing from the first to the second line in Eq.(52), we rewrote the delta functions at the vertices by,

$$\delta^{2dN}(\eta_1+\eta_2+\eta_3)=\int\frac{(d\chi_1)}{(2\pi)^{2dN}}e^{i\bar{\chi}_1(\eta_1+\eta_2+\eta_3)}\,, \tag{55}$$

and performed the η_i integrations.

We are now ready to write down the explicit form of the scattering amplitude for any Feynman diagram. We use the trick Eq.(55) to convert the delta functions at the vertices to the gaussian integration over χs. The general formula after η integrations is given by

$$\prod_{a\in\{\text{propagators}\}} g(\tau_a,p_a)\cdot\prod_{u\in\{\text{external states}\}}\tilde{\mathcal{N}}_u e^{ip_u\bar{x}}\cdot\left(\det\left(\pi^{-1}\mathcal{M}\right)\right)^{-\frac{d}{2}}\cdot\int\prod_{i\in\{\text{vertices}\}}\frac{(d\chi_i)}{(2\pi)^{2dN}}e^{\frac{1}{4}\bar{\mathcal{K}}\mathcal{M}^{-1}\mathcal{K}} \tag{56}$$

where the matrix $\mathcal{M}$ is

$$\mathcal{M}_{II} = \left\{\begin{array}{cl} F(\tau_a) & I\in\text{vertex}(i)\cap\text{propagator}(a)\\ \frac{1}{4}M_u^{-1} & I\in\text{vertex}(i)\cap\text{external state}(u)\\ F(\tau_a)+\frac{1}{4}M_u^{-1} & I\in\text{propagator}(a)\cap\text{external state}(u)\end{array}\right., \tag{57}$$

$$\mathcal{M}_{IJ} = \left\{\begin{array}{cl} \pm\frac{1}{4}\sigma & I,J\in\text{vertex}(i)\\ \pm G(\tau_a) & I,J\in\text{propagator}(a)\\ 0 & \text{otherwise}\end{array}\right.\qquad (I\neq J)\,, \tag{58}$$

and vector $\mathcal{K}$ is

$$\mathcal{K}_I=\left\{\begin{array}{cl} i\chi_i\pm H(\tau_a,p_a) & I\in\text{vertex}(i)\cap\text{propagator}(a)\\ i\chi_i+\frac{i}{2}M_u^{-1}\lambda_u & I\in\text{vertex}(i)\cap\text{external state}(u)\\ 0 & \text{otherwise}\end{array}\right.. \tag{59}$$

The indices $I,J,\cdots$ in the matrix $\mathcal{M}$ and $\mathcal{K}$ represent the junctions between the arbitrary combinations of the basic components (propagators, external states and vertices). In Eqs.(57,58), vertex(i) (propagator(a), external state (u)) represents the set of the boundaries of the i^{th} vertex (a^{th} propagator, u^{th} external state). We specify the junction I by taking the common set among them as indicated by a Feynman graph. In the example of the two-loop amplitude, all the indices $I=1,\cdots,6$ describe the junction between a vertex and a propagator. Applying the rules we obtain Eqs.(52-54). Similarly, in the example of the 4-tachyon scattering, the indices $I=1,2,3,4$ describe the vertex-external state junctions while the indices $I=5,6$ describe the vertex-propagator junctions. In this example, after applying these rules we first construct a 6×6 matrix $\mathcal{M}$ and a corresponding $\mathcal{K}$. After integrating over $\chi_{1,2}$ we obtain the 4×4 matrix $\mathcal{A}$ and the corresponding $\mathcal{B}$ given in Eqs.(50, 51). An example of propagator-external state junction appears in the Feynman graph considered in Eq.(25) in the previous setion[7].

[7]Strictly speaking there are other possibilities for the junctions, for instance, propagator-propagator, vertex-vertex. However, since they can be obtained by taking the appropriate limit (for example infinitely short propagator), we do not write them explicitly.

The plus or minus signs in Eqs.(58,59) are determined by the relative positions of the labels I in the diagram. For example, the sign of $\mathcal{M}_{IJ} \to \pm\frac{1}{4}\sigma$ at a vertex is $+(-)$ if I is positioned after (before) J when going clockwise around the vertex. To figure out the signs of G, H systematically, the diagram may be decorated with arrows for all momenta directed into each vertex. Recall that each propagator has different momenta at each end, therefore a propagator with ends (I, J) will have momenta $(+\eta_I, -\eta_J)$. The sign in front of G is given by the product of the signs of the momenta at the two ends of the propagator, times (-1). So if the arrows are drawn as suggested, the sign is $+G$. Finally the sign in front of H is determined by the sign of η_I at each end of the propagator. If the direction of the arrows is changed according to some other convention the signs on G, H will flip accordingly.

5 Reorganization of gaussian integration

Our computation of Feynman diagrams in Fourier basis reduces to the computation of the determinant and the inverse of the large matrices which connect all the external states. It is somehow obscure how such computation is related to the computation in ξ basis presented in section 3. To see the correspondence more explicitly, it is illuminating to carry out some of the momentum integrals.

For that purpose, we dissect all the propagators which connect two vertices. In the following, we carry out the momentum integrations associated with each vertex attached to the dissected propagators. More explicitly, we consider the following integration,

$$
\begin{aligned}
V_n(\tau, \epsilon) \;\; &\equiv \;\; \int (d\eta_1) \cdots (d\eta_n)\, \delta^{2dN}(\eta_1 + \cdots + \eta_n)\, e^{-\frac{1}{2}\sum_{i<j} \bar{\eta}_i \sigma \eta_j} \\
&\times \prod_{i=1}^{n} \left(g(\tau_i, p_i)\, e^{-\bar{\eta}_i F(\tau_i)\eta_i + \bar{\epsilon}_i G(\tau_i)\eta_i + (\bar{\eta}_i + \bar{\epsilon}_i) H(\tau_i, p_i) - \bar{\epsilon}_i F(\tau_i)\epsilon_i} \right) \,. \qquad (60)
\end{aligned}
$$

The first line comes from the definition of the vertex in the momentum basis and the second line comes from the propagator. We note that the second line may be written as, $\prod_i \tilde{A}_{\tilde{\mathcal{N}}_i, \tilde{M}_i, \tilde{\lambda}_i}$ where

$$
\tilde{A}_{\tilde{\mathcal{N}}_i, \tilde{M}_i, \tilde{\lambda}_i} \equiv \tilde{\mathcal{N}}_i e^{-\bar{\eta}\tilde{M}_i \eta - \bar{\eta}\tilde{\lambda}_i}, \quad \tilde{\mathcal{N}}_i = g(\tau_i, p_i) e^{\bar{\epsilon}_i H(\tau_i, p_i) - \bar{\epsilon}_i F(\tau_i)\epsilon_i},\; \tilde{M}_i = F(\tau_i),\; \tilde{\lambda}_i = -G(\tau_i)\epsilon_i - H(\tau_i, p_i)\,. \qquad (61)
$$

Actually this is nothing but the trace of the Moyal product of n gaussian functions,

$$
\begin{aligned}
V_n(\tau, \epsilon) \;\; &= \;\; Tr\left(A_{\mathcal{N}_1, M_1, \lambda_1}(\tau_1, \epsilon_1) \star \cdots \star A_{\mathcal{N}_n, M_n, \lambda_n}(\tau_n, \epsilon_n) \right), \qquad (62) \\
M_i \;\; &= \;\; (4F(\tau_i))^{-1} \;\;, \quad \lambda_i = -\frac{i}{2} F(\tau_i)^{-1} (G(\tau_i)\epsilon_i + H(\tau_i, p_i)), \qquad (63) \\
\mathcal{N}_i \;\; &= \;\; g(\tau_i, p_i) \pi^{dN} (\det F(\tau_i))^{\frac{d}{2}}\, e^{\bar{\epsilon}_i H(\tau_i, p_i) - \bar{\epsilon}_i F(\tau_i)\epsilon_i + \frac{1}{4}(\bar{H}(\tau_i, p_i) + \bar{\epsilon}_i G(\tau_i)) F(\tau_i)^{-1} (H(\tau_i, p_i) + G(\tau_i)\epsilon)}.
\end{aligned}
$$

While it looks complicated, the explicit evaluation of such expressions are given in [6].

We note that once this expression is evaluated, one may write down any string amplitude schematically as,

$$\mathcal{A} \sim \int (d\epsilon) \prod_{\text{vertices}} V_{n^{(i)}}(\tau^{(i)}, \epsilon^{(i)}) \prod_{\text{external legs}} \tilde{A}_{\tilde{\mathcal{N}}_i, \tilde{M}_i, \tilde{\lambda}_i} \prod_{\text{connection}} \delta^{2dN}(\epsilon_i + \epsilon_j) \tag{64}$$

where the final factor describes the momentum integrations for each dissection point of the propagators. The second factor comes from the external states. As an example, the two-loop vacuum amplitude is now neatly written as,

$$\int (d\epsilon_1)(d\epsilon_2)(d\epsilon_3) V_3(\tau_1, \tau_2, \tau_3; \epsilon_1, \epsilon_2, \epsilon_3) V_3(\tau_1', \tau_2', \tau_3'; -\epsilon_1, -\epsilon_2, -\epsilon_3). \tag{65}$$

Other amplitudes can be written as easily as this one.

6 Moyal formulation of bc-ghost sector

In Ref.[6], the Moyal formulation of *bosonized* ghost was discussed and was shown to be almost the same as a matter boson. In certain explicit computations in the ghost sector, it is often convenient to use the fermionic b, c-ghosts. Because $b(\sigma), c(\sigma)$ have $\cos n\sigma$ as well as $\sin n\sigma$ modes, we need to develop a regularized version of half string formulation for sine modes.[8]. In the ordinary split string formulation, we find the infinite matrix

$$\tilde{R}_{oe} = \frac{4}{\pi} \int_0^{\frac{\pi}{2}} d\sigma \sin o\sigma \sin e\sigma = \frac{4e(i)^{e-o+1}}{\pi(e^2 - o^2)}. \tag{66}$$

The inverse matrix is its transpose: $\tilde{R}\bar{\tilde{R}} = 1_o, \bar{\tilde{R}}\tilde{R} = 1_e$. However, $\bar{\tilde{R}}$ has a zero mode $\tilde{w}_o = \sqrt{2}(i)^{o-1}$, namely $\sum_{o=1}^{\infty} \tilde{w}_o \tilde{R}_{oe} = 0$. As emphasized in footnote (2), this situation causes an associativity anomaly of infinite matrices, which leads to ambiguities in computation as was discussed in Ref.[5]. To perform well-defined computations, we construct a finite $N \times N$ matrix $\tilde{R}$ using arbitrary spectrum κ_e, κ_o as we did for T, R, w, v in [6]

$$\tilde{R}_{oe} := \frac{w_e v_o \kappa_e \kappa_o}{\kappa_e^2 - \kappa_o^2}. \tag{67}$$

The original $\tilde{R}$ (66) is recovered by putting $\kappa_e = e, \kappa_o = o$ and taking $N \to \infty$.

We now follow a procedure parallel to that in [4]. Using the finite version of $T, R, \tilde{R}, v, w$, we define half string modes for $b(\sigma), c(\sigma)$, and perform the Fourier transform with respect to the even modes of the full string:[9]

$$A(\xi_0, x_o, p_o, y_o, q_o) := \int dc_0 \prod_{e>0} (dx_e dy_e)\, e^{-\xi_0 c_0 + (\xi_0 + \frac{2}{g} q_o v)\bar{w} y_e + \frac{2}{g} p_o \tilde{R} x_e + \frac{2}{g} q_o \bar{T} y_e} \langle c_0, x_n, y_n | \Psi \rangle. \tag{68}$$

[8]The cosine modes were developed in Ref.[5][6]. This was enough to discuss matter and bosonized ghost sector. The half-string formulation of b, c ghost was developed in [15].

[9]In the matter sector the Moyal product could be formulated in either the even or odd sectors [4][9], in the bc sector it is more natural to formulate it using odd modes.

$$\langle c_0, x_n, y_n| = \langle\Omega|\hat{c}_{-1}\hat{c}_0 \exp\left(c_0\hat{b}_0 + \sum_{n>0}\left(-\hat{c}_n\hat{b}_n - i\sqrt{2}\hat{c}_n x_n + \sqrt{2}y_n\hat{b}_n + iy_n x_n\right)\right). \tag{69}$$

The *Grassmann odd* version of the Moyal product among them[10]:

$$A \star B = A \exp\left(\frac{g}{2}\sum_{o>0}\left(\frac{\overleftarrow{\partial}}{\partial x_o}\frac{\overrightarrow{\partial}}{\partial p_o} + \frac{\overleftarrow{\partial}}{\partial y_o}\frac{\overrightarrow{\partial}}{\partial q_o} + \frac{\overleftarrow{\partial}}{\partial p_o}\frac{\overrightarrow{\partial}}{\partial x_o} + \frac{\overleftarrow{\partial}}{\partial q_o}\frac{\overrightarrow{\partial}}{\partial y_o}\right)\right) B \tag{70}$$

corresponds to (anti-)overlapping condition of Witten's star product: $b^{(r)}(\sigma) - b^{(r-1)}(\pi - \sigma) = 0,\ c^{(r)} + c^{(r-1)}(\pi - \sigma) = 0$. Here $x_n, y_n, p_o, q_o, c_0, \xi_0$ are Grassmann odd variables. A ghost zero mode ξ_0 also enters in reproducing Witten's star along with the above Moyal $\star$ product. Through the oscillator formalism [19] we establish the link between Witten's product and our Moyal product as follows

$$\int d\xi_0^{(1)} d\xi_0^{(2)} d\xi_0^{(3)}\ Tr\left[A^{(1)}(\xi_0^{(1)}, \xi) \star A^{(2)}(\xi_0^{(2)}, \xi) \star A^{(3)}(\xi_0^{(3)}, \xi)\right] \sim \langle\Psi^{(1)}|\langle\Psi^{(2)}|\langle\Psi^{(3)}|V_3\rangle, \tag{71}$$

where

$$\xi = (x_o, p_o, y_o, q_o), \quad Tr A(\xi) := \int \prod_{o>0}(dx_o dp_o dy_o dq_o)\, A(\xi). \tag{72}$$

In fact, by substituting the coherent states and their Fourier transform for $\Psi^{(i)}, A^{(i)}$, we have verified that the Neumann coefficients in the above identification coincide with the ones which were defined using 6-string vertex in *matter* sector in Ref.[6]. This provides a successful test of the ghost zero mode part.[11] This coincidence holds for arbitrary κ_e, κ_o, N. As usual, we reproduce the ordinary Neumann coefficients of Witten's string field theory by putting $\kappa_e = e, \kappa_o = o$ and taking $N \to \infty$ in the last stage of computations.

7 Comment on relation with VSFT

So far, we have established the utility of MSFT for computing Feynman graphs with perturbative as well as nonperturbative external states. To make further progress with nonperturbative effects, it will be important to understand how MSFT could be used in vacuum string field theory. In this section, we make some remarks in this direction.

[10] The Moyal formulation of the ghost system was discussed in Ref.[16] in the context of the continuous basis. The author also defined the discrete Moyal star product using *even* modes starting from the continuous product. While our switch to odd modes appears trivial, there are some important differences between our treatment and that of [16]. A critical point is the treatment of the midpoint mode $b(\pi/2)$ which is nontrivial in our case but vanishes in [16]. This has an important consequence for producing the correct Neumann coefficients. Actually we also developed the *even* mode formulation as [16] but with the proper treatment of the midpoint. It is, however, more complicated than the odd mode formulation presented here. Another important difference is on the regularization of the infinite matrices where our setup holds for arbitrary κ_e, κ_o, N. The details will be given in [13]. Also for more comments on midpoint issues related to the continuous basis, see [9].

[11] The coincidence of the Neumann coefficients for nonzero mode implies that our Moyal $\star$ product is essentially the same as the *reduced* product which was introduced in the Siegel gauge [17, 18].

The second order differential operator $L_0(\beta_0)$, including the matter and ghost sectors, can be rewritten using the Moyal $\star$ product as follows

$$L_0 A = \mathcal{L}_0(\beta_0) \star A + A \star \mathcal{L}_0(-\beta_0) + \gamma A,$$

where

$$\begin{aligned}
\mathcal{L}_0(\beta_0) &= \sum_{e>0}\left(\frac{l_s^2}{\theta^2}p_e^2 + \frac{\kappa_e^2}{4l_s^2}x_e^2 - \frac{l_s}{\theta}w_e p_e \beta_0\right) + \frac{1}{4}(1+\bar{w}w)\beta_0^2 - \frac{d-2}{4}\sum_{n>0}\kappa_n \\
&\quad + i\sum_{o>0}\kappa_o\left(\frac{1}{2}x_o y_o + \frac{2}{g^2}p_o q_o\right), \qquad (73)\\
\gamma &= -\frac{1}{1+\bar{w}w}\frac{2l_s^2}{\theta^2}\left(\sum_{e>0} w_e p_e\right)^2 + \frac{4i}{g^2}(1+\bar{w}w)\left(\sum_{o>0}\kappa_o v_o p_o\right)\left(\sum_{o>0} v_o q_o\right). \qquad (74)
\end{aligned}$$

The $\mathcal{L}_0$ terms are star products with a field without involving explicit derivatives with respect to ξ. However, the γ term is an ordinary product, not a star product. It involves $\sum_{e>0} p_e^\mu w_e$ which can be rewritten as

$$\sum_{e>0} p_e^\mu w_e = \sum_{e,o>0} p_o^\mu R_{oe} w_e = (1+\bar{w}w)\sum_{o>0} p_o^\mu v_o = (1+\bar{w}w)\,\tilde{p}^\mu, \qquad (75)$$

where the mode $\tilde{p}^\mu \equiv \sum_{o>0} v_o p_o^\mu$ was discussed in [5] as being closely related to associativity anomalies in string field theory. Also note that in the large N limit $\tilde{p}^\mu$ becomes the *unpaired* zero momentum mode in the continuum Moyal representation of ref.[7]: $(\lim_{k\to 0} p^\mu(k)) \sim \tilde{p}^\mu$. The ghost part has a similar structure. Evidently, these bits are closely connected to midpoint anomalies.

If we could neglect the γ term, $L_0 A$ would be given by left $\star$ multiplication with $\mathcal{L}_0(\beta_0)$ and right star multiplication with $\mathcal{L}_0(-\beta_0)$. The left-right splitting of the kinetic term reminds us of the situation of the purely cubic string field theory [20] where the BRST operator Q_B was decomposed into the left and right star multiplication of the string fields $Q_L I$ or $Q_R I$. In this sense, purely cubic theory is essentially the matrix analog of Witten's string field theory. In our MSFT framework $Q_B, Q_L I$ correspond to L_0 and $\mathcal{L}_0$ respectively because we are in the Siegel gauge. However, we have now seen that this structure must be corrected with our γ term.

One of the lessons we learned in this paper is that we can not neglect the gamma term because it is indispensable to reproduce the correct spectrum in the computation of 1-loop vacuum amplitude, and other quantities, in both matter and ghost sectors, even if the coefficients in front of them appear to vanish naively in the large N limit.

The origin of the γ term is a non-vanishing energy-momentum tensor at the midpoint. While the integration measure is zero, it still gives a finite contribution. The situation is similar in the gauge covariant BRST formulation, namely the BRST current does not vanish at the midpoint. Our observation here indicates that one might need a more careful analysis of the midpoint BRST operator in the purely cubic theory and/or in VSFT.

We are currently in the process of solving the classical equations of motion of the original theory[12] and hope to establish a careful connection between VSFT and the original theory in the context of MSFT. We expect that the more careful treatment of this term would clarify the midpoint structure of VSFT. The finite N regularization which is used in this paper will be essential in this viewpoint.

8 Outlook

In this paper, we have restricted ourselves to computations in the Siegel gauge, and demonstrated the utility of MSFT.

Ideally, for further insights, we would like to aim for a more gauge invariant approach. To achieve this carefully it is necessary to construct the nilpotent BRST operator. We can do this in the infinite N limit, but not yet in the finite N case. The reason is that the Virasoro algebra does not close with a finite number of modes. Therefore one needs to find a finite dimensional algebra that closes at finite N, and becomes the Virasoro algebra at infinite N. With such an algebra one can construct a BRST operator and a gauge invariant Lagrangian at finite N. This would be the analog of lattice gauge theory for QCD. The cutoff theory we have described so far would correspond to the gauge fixed lattice gauge theory. It is possible that in this process the (κ_e, κ_o) that have remained arbitrary so far in our formalism would be fixed as a function of (e, o) at finite N.

On the other hand, if the VSFT proposal is valid, we can easily construct the midpoint nilpotent BRST operator from only midpoint degrees of freedom. In that context a gauge invariant theory is easily constructed without ever encountering a restriction on the (κ_e, κ_o) at finite N.

The clarification of such issues will be critical for the future development of string field theory.

Acknowledgments

I.B. is supported in part by a DOE grant DE-FG03-84ER40168. I.K. is supported in part by JSPS Research Fellowships for Young Scientists. Y.M. is supported in part by Grant-in-Aid (# 13640267) from the Ministry of Education, Science, Sports and Culture of Japan.

References

[1] L. Rastelli, A. Sen and B. Zwiebach, Adv. Theor. Math. Phys. **5**, 353 (2002) [arXiv:hep-th/0012251]; arXiv:hep-th/0106010;
D. Gaiotto, L. Rastelli, A. Sen and B. Zwiebach, arXiv:hep-th/0111129;

[12] There are some attempts to solve it in different scheme [21].

V. A. Kostelecky and R. Potting, Phys. Rev. D **63**, 046007 (2001) [arXiv:hep-th/0008252];
D. J. Gross and W. Taylor, JHEP **0108**, 009 (2001) [arXiv:hep-th/0105059]; JHEP **0108**, 010 (2001) [arXiv:hep-th/0106036].
H. Hata and T. Kawano, JHEP **0111**, 038 (2001) [arXiv:hep-th/0108150];
K. Okuyama, JHEP **0201**, 027 (2002) [arXiv:hep-th/0201015]; JHEP **0203**, 050 (2002) [arXiv:hep-th/0201136];
Y. Okawa, JHEP **0207**, 003 (2002) [arXiv:hep-th/0204012].

[2] A. Sen and B. Zwiebach, JHEP **0003**, 002 (2000) [hep-th/9912249];
N. Moeller and W. Taylor, Nucl. Phys. **B583**, 105 (2000) [hep-th/0002237];
W. Taylor, arXiv:hep-th/0208149.;
D. Gaiotto and L. Rastelli, arXiv:hep-th/0211012.

[3] E. Witten, Nucl. Phys. B **268** (1986) 253.

[4] I. Bars, Phys. Lett. B **517**, 436 (2001) [arXiv:hep-th/0106157].

[5] I. Bars and Y. Matsuo, Phys. Rev. D **65**, 126006 (2002) [arXiv:hep-th/0202030].

[6] I. Bars and Y. Matsuo, Phys. Rev. D **66**, 066003 (2002) [arXiv:hep-th/0204260].

[7] M. R. Douglas, H. Liu, G. Moore and B. Zwiebach, JHEP **0204**, 022 (2002) [arXiv:hep-th/0202087].

[8] I. Y. Arefeva and A. A. Giryavets, arXiv:hep-th/0204239;
C. S. Chu, P. M. Ho and F. L. Lin, JHEP **0209**, 003 (2002) [arXiv:hep-th/0205218];
D. M. Belov and A. Konechny, JHEP **0210**, 049 (2002) [arXiv:hep-th/0207174].

[9] I. Bars, " MSFT - Moyal star formulation of string field theory", talk at the Sakharov conference, June 2002, in preparation.

[10] H. Hata and S. Moriyama, JHEP **0201**, 042 (2002) [arXiv:hep-th/0111034];
H. Hata, S. Moriyama and S. Teraguchi, JHEP **0202**, 036 (2002) [arXiv:hep-th/0201177].

[11] L. Rastelli, A. Sen and B. Zwiebach, JHEP **0203**, 029 (2002) [arXiv:hep-th/0111281].

[12] W. Taylor, arXiv:hep-th/0207132.

[13] I. Bars, I. Kishimoto and Y. Matsuo, in preparation.

[14] T. Filk, Phys. Lett. B **376** (1996) 53;
N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, Nucl. Phys. B **573** (2000) 573 [arXiv:hep-th/9910004];
S. Minwalla, M. Van Raamsdonk and N. Seiberg, JHEP **0002** (2000) 020 [arXiv:hep-th/9912072].

[15] A. Abdurrahman, F. Anton and J. Bordes, Nucl. Phys. B **411**, 693 (1994).

[16] T. G. Erler, arXiv:hep-th/0205107.

[17] I. Kishimoto, JHEP **0112**, 007 (2001) [arXiv:hep-th/0110124].

[18] K. Okuyama, JHEP **0201**, 043 (2002) [arXiv:hep-th/0111087].

[19] D. J. Gross and A. Jevicki, Nucl. Phys. B **287**, 225 (1987).

[20] G. T. Horowitz, J. Lykken, R. Rohm and A. Strominger, Phys. Rev. Lett. **57**, 283 (1986); Y. Matsuo, Mod. Phys. Lett. A **16** (2001) 1811 [arXiv:hep-th/0107007]; I. Kishimoto and K. Ohmori, JHEP **0205** (2002) 036 [arXiv:hep-th/0112169].

[21] J. Kluson, JHEP **0204**, 043 (2002) [arXiv:hep-th/0202045]; arXiv:hep-th/0209255; T. Takahashi and S. Tanimoto, JHEP **0203**, 033 (2002) [arXiv:hep-th/0202133]; I. Kishimoto and T. Takahashi, Prog. Theor. Phys. **108**, 591 (2002) [arXiv:hep-th/0205275].