The Interface of Noncommutative Geometry and Physics

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Abstract

As a mathematical theory per se, noncommutative geometry (NCG) is by now well established. From the beginning, its progress has been crucially influenced by quantum physics: we briefly review this development in recent years.

The Standard Model of fundamental interactions, with its central role for the Dirac operator, led to several formulations culminating in the concept of a real spectral triple. String theory then came into contact with NCG, leading to an emphasis on Moyal-like algebras and formulations of quantum field theory on noncommutative spaces. Hopf algebras have yielded an unexpected link between the noncommutative geometry of foliations and perturbative quantum field theory.

The quest for a suitable foundation of quantum gravity continues to promote fruitful ideas, among them the spectral action principle and the search for a better understanding of "noncommutative spaces".

1 Introduction

About 20 years ago, the mathematical theory nowadays known as Noncommutative Geometry (NCG) began taking shape. A landmark paper of Connes (1980) ushered in a differential geometric treatment of the noncommutative torus [1] (further developed and classified by Rieffel [2]), which remains the paradigm of a noncommutative space. Its differential calculus was put in a more general framework at the Oberwolfach meeting in September–October 1981, where Connes unveiled a "homology of currents for operator algebras" [3], which soon became known as cyclic cohomology [4]. This was developed in detail in his "Noncommutative Differential Geometry" [5], in preprint form around Christmas 1982; the related periodic cyclic cohomology is a precise generalization, in algebraic language, of the de Rham homology of smooth manifolds.

The same algebraic approach, applied to the theory of foliations [6], led Connes to emphasize the notion of Fredholm module, which is a cornerstone of his work with Karoubi on

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canonical quantization [7]. A key observation here is that anomalous commutators form a cyclic 1-cocycle [8], so that in the noncommutative approach to quantum field theory, the Schwinger terms are built in.

Noncommutative geometry, then, is an operator-algebraic reformulation of the foundations of geometry, extending to noncommutative spaces. It allows consideration of "singular spaces", erasing the distinction between the continuous and the discrete. On the mathematical side, current topics of interest include index theory and groupoids, mathematical quantization, the Baum–Connes conjectures on the K-theory of group algebras, locally compact quantum groups, second quantization in the framework of spectral triples, and the Riemann hypothesis. Our focus here, however, is on its interface with physics.

2 NCG and the Standard Model

We say interface because one should not speak of the "application" of NCG to physics, but rather of mutual intercourse. Indeed, the first use of noncommutative geometry in physics did not attempt to derive the laws of physics from some NCG construct, but simply and humbly, to learn from the mainstream physical theories —concretely, the Standard Model (SM) of fundamental interactions— what the (noncommutative) geometry of the world could be.

The crucial concepts of the SM are those of gauge fields and of chiral fermions: they correspond to two basic notions of NCG, namely connections and Dirac operators. Indeed, the algebraic definition of linear connection is imported verbatim into NCG. Chiral fermions, for their part, are acted on by Dirac and Dirac–Weyl operators.

Dirac operators are a source of NCG: any complex spinor bundle on a smooth manifold $S \to M$ gives rise to a generalized Dirac operator D on the spinor space $L^2(M, S)$, whose sign operator $F = D|D|^{-1}$ determines a Fredholm module; its K-homology class $[F] \in K_\bullet(M)$ depends only on the underlying spin^c-structure [9, 10]. Since the spin^c structure determines the orientation of the manifold, this fundamental class —sometimes called a K-orientation [11]— is a finer invariant than the usual fundamental class in homology.

The approach to the SM by Connes and Lott [12] used a noncommutative algebra to describe the electroweak sector, plus a companion algebra to incorporate colour symmetries (see [13] and [14] for reviews of this preliminary approach). Later on [15], a better understanding of the role of the charge conjugation allowed this pair of algebras to be replaced by a *single* algebra acting bilaterally.

The gauge potentials appearing in the SM may be collected into a single package of differential forms:

$$\mathbb{A}' = i(B, W, A),$$

where

$$B = -\tfrac{i}{2} g_1 \mathbf{B}_\mu \, dx^\mu, \quad W = -\tfrac{i}{2} g_2 \, \boldsymbol{\tau} \cdot \mathbf{W}_\mu \, dx^\mu \quad \text{and} \quad A = -\tfrac{i}{2} g_3 \, \boldsymbol{\lambda} \cdot \mathbf{A}_\mu \, dx^\mu,$$

with $\mathbf{B}$, $\mathbf{W}$ and $\mathbf{A}$ denoting respectively the hypercharge, weak isospin and colour gauge potentials; W is to be regarded as a quaternion-valued 1-form. Thus, $\mathbb{A}'$ is an element of

$\Lambda^1(M)\otimes\mathcal{A}_F$, where the noncommutative algebra $\mathcal{A}_F := \mathbb{C}\oplus\mathbb{H}\oplus M_3(\mathbb{C})$, that we have called the "Eigenschaften algebra" [16], plays the crucial role.

We next collect all chiral fermion fields into a multiplet Ψ and denote by J the charge conjugation; then the fermion kinetic term is rewritten as follows:

$$I(\Psi, \mathbb{A}', J) = \langle \Psi \mid (i\partial\!\!\!/ + \mathbb{A}' + J\mathbb{A}'J^\dagger)\Psi\rangle.$$

To incorporate the Yukawa part of the SM Lagrangian, let ϕ be a Higgs doublet with vacuum expectation value $v/\sqrt{2}$, normalized by setting $\Phi := \sqrt{2}\,\phi/v$. We need both

$$\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \quad \text{and} \quad \widetilde{\Phi} := \begin{pmatrix} -\bar{\Phi}_2 \\ \bar{\Phi}_1 \end{pmatrix}.$$

The Higgs may be properly regarded as a quaternion-valued field; by introducing $q_\Phi = \begin{pmatrix} \bar{\Phi}_1 & \bar{\Phi}_2 \\ -\Phi_2 & \Phi_1 \end{pmatrix}$, where $\langle q_\Phi\rangle = 1$, we may write, schematically for a right-left splitting of the fermion multiplets:

$$\mathbb{A}'' = \begin{pmatrix} & M^\dagger(q_\Phi - 1) \\ (q_\Phi - 1)^\dagger M & \end{pmatrix},$$

where M denotes the mass matrix for quarks (including the Cabibbo–Kobayashi–Maskawa parameters) and leptons. Denoting by $\mathcal{D}_F$ the Yukawa operator which relates the left- and right-handed chiral sectors in the space of internal degrees of freedom, the Yukawa terms for both particles and antiparticles (for the first generation) can now be written as

$$\begin{aligned} I(\Psi, \mathbb{A}'', J) &:= \langle \Psi \mid (\mathcal{D}_F + \mathbb{A}'' + J\mathbb{A}''J^\dagger)\Psi\rangle \\ &= \bar{q}_L\Phi\, m_d\, d_R + \bar{q}_L\widetilde{\Phi}\, m_u\, u_R + q_R\bar{\Phi}\,\bar{m}_d\,\bar{d}_L + q_R\bar{\widetilde{\Phi}}\,\bar{m}_u\,\bar{u}_L \\ &\qquad + \bar{\ell}_L\Phi\, m_e\, e_R + \bar{\ell}_L\widetilde{\Phi}\, m_\nu\, \nu_R + \ell_R\bar{\Phi}\,\bar{m}_e\,\bar{e}_L + \ell_R\bar{\widetilde{\Phi}}\,\bar{m}_\nu\,\bar{\nu}_L + \text{h.c.} \end{aligned}$$

Altogether, we get a Dirac–Yukawa operator $\mathcal{D} = i\partial\!\!\!/ \oplus \mathcal{D}_F$. With $\mathbb{A} := \mathbb{A}' \oplus \mathbb{A}''$, the *whole* fermionic sector of the SM is recast as

$$I(\Psi, \mathbb{A}, J) = \langle \Psi \mid (\mathcal{D} + \mathbb{A} + J\mathbb{A}J^\dagger)\Psi\rangle.$$

The upshot is that the ordinary gauge fields and the Higgs are combined as entries of a *generalized gauge potential*. The Yukawa terms come from the minimal coupling recipe applied to the gauge field in the internal space. The Dirac–Yukawa operator is seen to contain in NCG all the relevant information pertaining to the SM.

This Connes–Lott reconstruction of the SM gave rise to two "predictions". (At that time, the top quark had not yet been seen, and the best estimates for its mass ranged around 130 GeV.) The NCG model sort of explains why the masses of the top quark, the W and Z particles and the Higgs particle should be of the same order, and gave right away

$$m_{\text{top}} \geq \sqrt{3}\, m_W \approx 139\,\text{GeV}.$$

With a bit of renormalization group running [17], it fell right on the mark. On the other hand, the "prediction" for the Higgs mass from Connes' NCG has remained stuck around 200 GeV, while the current phenomenological prejudice is that it should be much lower.

A major limitation of the Connes–Lott approach is that the fermion mass matrix must be taken as an input. A different though less ambitious proposal, put forward about the same time, was the Mainz–Marseille scheme, based on organizing the (W, B) forms and the Higgs field components as a 3×3 matrix in the Lie superalgebra $\mathfrak{su}(2|1)$. The known families of quarks and leptons can then be fitted into (reducible but indecomposable) $\mathfrak{su}(2|1)$ representations, and some relations among the quark masses and CKM mixing parameters emerge [18]; this analysis applies likewise to lepton masses and neutrino mixing.

This "bottom-up" interaction between physics and NCG yielded an important dividend. The clarification of the role of J as a "Tomita conjugation" [19] ostered the emergence of the concept of a *real spectral triple* —the word "real" being taken in the sense of Atiyah's "Real K-theory" [20]— which led to a construction of noncommutative spin manifolds [21]. This construction explained in our *Elements of Noncommutative Geometry* [22]. Thus, we now know how to put fermion fields on a noncommutative manifold.

A *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ consists of a (unital) *algebra* $\mathcal{A}$ represented on a Hilbert space $\mathcal{H}$, plus a selfadjoint operator D on $\mathcal{H}$, such that $[D, a]$ is bounded for all $a \in \mathcal{A}$ and D^{-1} is compact. It is *even* if there is a grading operator χ (or "γ_5") on H wrt which $\mathcal{A}$ is even and D is odd. It is *real* if there is an antiunitary operator J on H such that $J^2 = \pm 1$, $JD = \pm DJ$ and $J\chi = \pm \chi J$ (even case); the signs depend on a certain dimension mod8. From these data, by imposing a few extra conditions, spin manifolds can be reconstructed [23].

3 The spectral action principle

The early Connes–Lott models did not take account of gravity. To remedy that, Connes and Chamseddine [24] proposed a universal formula for an action associated with a non-commutative spin geometry, modelled by a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$. The action $S(D) = B_\phi[D] + \langle \Psi \mid D\Psi \rangle$ is based on the spectrum of the Dirac operator and is a geometric invariant. Automorphisms of the algebra $\mathcal{A}$ combine ordinary diffeomorphisms with internal symmetries which alter the metric by $D \mapsto D + A + JAJ^\dagger$.

The bosonic part of the action functional is $B_\phi[D] = \operatorname{Tr} \phi(D^2)$, where ϕ is an "arbitrary" positive function (a regularized cutoff) of D. Chamseddine and Connes argue that B_ϕ has an asymptotic expansion

$$B_\phi[D/\Lambda] \sim \sum_{n=0}^{\infty} f_n \, \Lambda^{4-2n} \, a_n(D^2) \quad \text{as } \Lambda \to \infty,$$

where the a_n are the coefficients of the heat kernel expansion for D^2 and $f_0 = \int_0^\infty x\phi(x)\, dx$, $f_1 = \int_0^\infty \phi(x)\, dx$, $f_2 = \phi(0)$, $f_3 = -\phi'(0)$, and so on: this is in fact a Cesàro asymptotic development [25]. On computing this expansion for the Dirac–Yukawa operator of the Standard Model, they found all terms in the bosonic part of the SM action, plus unavoidable gravity couplings. That is to say, the spectral action for the Standard Model unifies with gravity at a very high energy scale.

Recently, Wulkenhaar [26] has conjectured that on θ-deformed spacetime, the spectral action may have the necessary additional symmetries to renormalize gauge theories. In this regard, Langmann [27] has managed to prove that the effective action of fermions coupled to a Yang–Mills field contains the usual Yang–Mills bosonic action. To check the conjecture, one first needs to extend the spectral action to the context of noncompact NC manifolds.

By "noncompact noncommutative spin geometry" we understand a real spectral triple $(\mathcal{A}, \mathcal{H}, D, J)$ where $\mathcal{A}$ is a *nonunital* algebra, where $[D, a]$ is bounded and $a|D|^{-1}$ is compact for all $a \in \mathcal{A}$. Geometries of this type are discussed in [28, 29, 43]: the analytic toolbox of NC spin geometries [22] extends to the noncompact case if suitable multiplier algebras are employed.

4 Noncommutative field theory

The next phase of the dialogue between NCG and the physics of fundamental interactions was characterized by a "top-down" approach. An important precursor is the 1947 paper by Snyder, "Quantized space-time" [30], where it was first suggested that coordinates x^μ may be noncommuting operators; the six commutators are of the form $[x^\mu, x^\nu] = (ia^2/\hbar)\, L^{\mu\nu}$ where a is a basic unit of length and the $L^{\mu\nu}$ are generators of the Lorentz group; throughout, Lorentz covariance is maintained. Then as now, noncommuting coordinates were used to describe spacetime in the hope of improving the renormalizability of QFT and of coming to terms with the nonlocality of physics at the Planck scale.

In a similar vein, Doplicher, Fredenhagen and Roberts [31] have considered a model with commutation relations

$$[x^\mu, x^\nu] = i\, Q^{\mu\nu},$$

where the $Q^{\mu\nu}$ are the components of a tensor, but commute among themselves and with each x^μ. Thus in their formalism, Lorentz invariance is also explicitly kept.

String theorists have recently revived this top-down approach. In their most popular model, the commutation relations are simply of the form

$$[x^\mu, x^\nu] = i\, \theta^{\mu\nu}, \tag{1}$$

where the $\theta^{\mu\nu}$ are c-numbers, breaking Lorentz invariance. As anticipated by Sheikh-Jabbari [32] and plausibly argued by Seiberg and Witten [33], open strings with allowed endpoints on 2D-branes in a B-field background act as electric dipoles of the abelian gauge field of the brane; the endpoints live on the noncommutative space determined by (1), as pointed out by [34].

Slightly before, Connes, Douglas and Schwarz [35] had shown that compactification of M-theory, in the context of dimensionally reduced gauge theory actions, leads to spaces with embedded noncommutative tori. See also [36] for the relation between noncommutative geometry and strings.

An important feature of [33] is the "Seiberg–Witten map" in gauge theory, which relates gauge fields and gauge variations in a noncommutative theory with their commutative counterparts. In the NC theory, multiplication is replaced by the Moyal product $\star_\theta$ with parameter $\theta = [\theta^{\mu\nu}]$; in order to preserve gauge equivalence (whenever A and A' are equivalent gauge fields, so should be the NC gauge fields $\widehat{A}$ and $\widehat{A}'$), Seiberg and Witten found θ-dependent formulas for the latter. As explained by Jackiw and Pi [37] (see also [38]), these formulas correspond to an infinitesimal 1-cocycle for a projective representation of the underlying gauge group in the Moyal algebra.

The Moyal product which appears here is nonperturbatively defined, for nondegenerate skewsymmetric θ, as

$$f \star_\theta g(u) := (\pi\theta)^{-4} \int_{\mathbb{R}^4 \times \mathbb{R}^4} f(u+s)g(u+t)\, e^{2is\theta^{-1}t}\, d^4s\, d^4t,$$

and this gives rise to the commutation relations (1). Those are just the commutation relations of quantum mechanics, when $\hbar$ replaces θ! The precise relation of this integral formula to the asymptotic development usually put forward as the Moyal product was spelled out some time ago in [39]. This product is the basis of the Weyl–Wigner–Moyal or phase-space approach to quantum mechanics [40], which already had a long history when (a version of) the Moyal product was rediscovered by string theorists. It should be said that many of the recent papers which purport to use this product in string theory or NC field theory are rather careless; some are unaware of the mathematical properties of the Moyal product, which are outlined, for instance, in our [41, 42] or in [43].

It is worth pointing out that noncommutative field theory can be developed independently of its string theory motivation, and indeed preexisted the Seiberg–Witten paper. Quantum field theory has an algebraic core which is independent of the nature of spacetime. From the representation theory of the infinite dimensional orthogonal group (or an appropriate subgroup), with the input of a one-particle space, one can derive all Fock space quantities of interest: nothing really changes if the "matter field" evolves on a noncommutative space. That is to say, one can apply the canonical quantization machinery to a noncommutative kind of one-particle space [44]. The long-standing hope, that giving up *locality* in the interaction of fields would be rewarded with a better ultraviolet behaviour, was now amenable to rigorous scrutiny, and it is not borne out. QFT on noncommutative manifolds also requires renormalization. This, in some sense the first result of NCFT, was proved in general by Gracia-Bondía and myself in [44], using a cohomological argument internal to noncommutative geometry.

Of course, one can prove the same in the context of a *particular* NCG model, by writing down the integral corresponding to a Feynman diagram, and finding it to be divergent. That had been shown previously by Filk [45], for the scalar Lagrangian theory associated to the Moyal product algebra. Filk made the point that the momentum integrals for planar Feynman graphs are identical to those in the commutative theory, and the contributions from nonplanar graphs cannot cancel them. The same basic point had been made much earlier in [46], with regard to the continuum limit of a reduced model of large N field theory.

Figure 1: Planar and nonplanar tadpole diagrams in NC ϕ^4 theory

The distinction between planar and nonplanar Feynman diagrams is an essential feature of NC field theory. Consider, for instance, the theory given by the action functional

$$S = \int d^4x \left(\frac{1}{2} \frac{\partial \phi}{\partial x^\mu} \frac{\partial \phi}{\partial x_\mu} + \frac{1}{2} m^2 \phi^2 + \frac{g}{4!} \phi \star_\theta \phi \star_\theta \phi \star_\theta \phi \right).$$

The Feynman rules yield the same propagators as in the commutative theory, but the vertices get in momentum space an extra factor proportional to

$$\exp\left(-\frac{i}{2} \sum_{k<l} p_{k\mu} \theta^{\mu\nu} p_{l\nu} \right),$$

where $p_1, \dots, p_r$ are the momenta incoming on the vertex, in cyclic order. Planar diagrams get overall phase factors depending only on external momenta; for nonplanar diagrams, the phase factors may also depend on loop variables, and the corresponding integrals may become convergent. For the tadpole diagrams of Figure 1, we get amplitudes of the form

$$\Gamma_{\rm pl}(p) \propto \int \frac{d^4k}{k^2 + m^2}, \qquad \Gamma_{\rm npl}(p) \propto \int \frac{d^4k}{k^2 + m^2}\, e^{-ip\theta k},$$

and the second integral is finite for $p \neq 0$.

However, nonplanar diagrams may become divergent again for particular values of the momenta (try $p = 0$ in the previous example). For complicated diagrams with subdivergences, this dependence of the amplitude behaviour on p is troublesome, because such diagrams may unexpectedly become divergent again. This is the notorious UV/IR mixing [47], which tends to spoil renormalizability. For Moyal NC Yang–Mills theory, this happens already at the 2-loop level.

It was pointed out by Gomis and Mehen [48] that whenever there is timelike noncommutativity $\theta^{0i} \neq 0$, one encounters a violation of unitarity of the S-matrix. However, Bahns et al [49] have argued that, if the above Lagrangian approach is replaced by a Hamiltonian approach to NC field theory, the apparent failure of unitarity disappears. The change in viewpoint concerns only the nonplanar diagrams. This should not really be surprising: even for ordinary Yang–Mills theories, the full equivalence of both approaches has never been established [50]. Actually, the indications seem to be that they are *not* equivalent in noncommutative field theory [51, 52].

The literature on NC field theories is already very large, and of uneven quality; we cannot really do it justice here. For recent extensive reviews, see [53] and [54].

Finally, it is now possible to combine these NC field theories with the Connes–Lott approach, by taking the tensor product $\mathcal{A}_{NC} \otimes \mathcal{A}_F$ of a noncommutative spacetime algebra and the Eigenschaften algebra $\mathcal{A}_F$. This has been done by Morita [55], Chaichian et al [56] and the München group [57], in various ways (and with different outcomes). However, there is some doubt as to whether these models are anomalous [58], and therefore nonrenormalizable [59].

5 Noncommutative spaces

In an eventual noncommutative approach to quantum gravity, one must be able to sum over families of noncommutative spaces. This, together with the need for good examples, has inspired a search for noncommutative manifolds. Connes [60] has suggested that "NC spheres" may be obtained from two homological conditions: (1) the Chern character form vanishes in all intermediate degrees; and (2) the metric may vary while keeping the volume form fixed. In even dimensions $n = 2m$, this comes down to setting

$$\mathrm{ch}_k(e) \equiv (-1)^k \frac{(2k)!}{k!} \operatorname{tr}((e - \tfrac{1}{2})\,(de\,de)^k) = 0 \quad \text{for} \quad k = 0, 1, \dots, m-1, \tag{2a}$$

$$\pi_D(\mathrm{ch}_m(e)) = \chi, \tag{2b}$$

where $e = e^2 = e^* \in M_{2^m}(\mathcal{A})$ is an orthogonal projector, χ is the chiral grading operator on $\mathcal{H}$, and $\pi_D(a_0\,da_1 \dots da_n) := a_0\,[D, a_1] \dots [D, a_n]$; the domain of π_D is the universal graded differential algebra over $\mathcal{A}$, which may be regarded as the space of chains for Hochschild homology. Briefly, one finds that (2a) makes $\mathrm{ch}_m(e)$ a Hochschild cycle, and the (2b) says that this cycle gives the desired volume form [23]. Now (2) becomes a system of equations which impose severe restrictions on the algebra $\mathcal{A}$ to which the matrix elements of e belong.

For $n = 2$, there is only the commutative solution [60], $\mathcal{A} = C^\infty(\mathbb{S}^2)$. For $n = 4$, Connes and Landi [61] found "θ-twisted 4-spheres" $\mathbb{S}^4_\theta$ with embedded copies of the NC 2-torus $\mathbb{T}^2_\theta$. Later, Connes and Dubois-Violette [62] showed that there is a 3-parameter family of NC 3-spheres, including a θ-twisted subfamily $\mathbb{S}^3_\theta$. These θ-twisted spheres can all be described as quantum homogeneous spaces [62, 63]: in fact, we can construct M_θ by twisting whenever $M = G/H$ is a quotient of compact Lie groups with $\operatorname{rank} H \geq 2$. The noncommutative algebra $C^\infty(M_\theta)$ is simply $C^\infty(M)$ equipped with a periodic version of the Moyal product [64], and the symmetry group G is correspondingly deformed to a quantum group [63]. If M is spin, then so is M_θ; it carries a NC spin geometry obtained by isospectral deformation from that of M [61]. Noncommutative twistors can also be obtained in this manner.

One motivation for constructing such examples of noncommutative spaces is to come back to quantum gravity by (a) allowing for metric fluctuations with fixed volume; and eventually (b) relaxing the Hochschild condition to incorporate "virtual" NC manifolds, whereby the condition (2b) would appear as the signal of a "true" manifold [65].

6 The Connes–Kreimer Hopf Algebras

Bogoliubov's renormalization scheme in dimensional regularization can be summarized as follows. Let Γ be a one-particle irreducible (1PI) graph (i.e., a connected graph which cannot be disconnected by removing a single line), with amplitude $f(\Gamma)$; if Γ is primitive (i.e., has no subdivergences), set

$$C(\Gamma) := -T(f(\Gamma)), \quad \text{and then} \quad R(\Gamma) := f(\Gamma) + C(\Gamma),$$

where $C(\Gamma)$ is the *counterterm*, $R(\Gamma)$ is the desired finite value, and T projects on the pole part: in other words, for primitive graphs, one simply removes the pole part. We recursively define Bogoliubov's $\overline{R}$-operation by setting

$$\overline{R}(\Gamma) = f(\Gamma) + \sum_{\emptyset \subsetneq \gamma \subsetneq \Gamma} C(\gamma)\, f(\Gamma/\gamma),$$

where $C(\gamma_1 \dots \gamma_r) := C(\gamma_1) \dots C(\gamma_r)$, whenever $\gamma = \gamma_1 \dots \gamma_r$ is a disjoint union of several pieces. Then we remove the pole part of the previous expression: $C(\Gamma) := -T(\overline{R}(\Gamma))$ and $R(\Gamma) := \overline{R}(\Gamma) + C(\Gamma)$. Overall,

$$C(\Gamma) := -T\Big[f(\Gamma) + \sum_{\emptyset \subsetneq \gamma \subsetneq \Gamma} C(\gamma)\, f(\Gamma/\gamma)\Big], \tag{3a}$$

$$R(\Gamma) := f(\Gamma) + C(\Gamma) + \sum_{\emptyset \subsetneq \gamma \subsetneq \Gamma} C(\gamma)\, f(\Gamma/\gamma). \tag{3b}$$

Now let Φ stand for any particular QFT. There is an associated Hopf algebra H_Φ [66,67] which is, first of all, a commutative algebra generated by the 1PI graphs Γ of Φ. The product is given by the disjoint union of graphs. The counit ε is defined on generators by $\varepsilon(\Gamma) := 0$ unless Γ is empty, and $\varepsilon(\emptyset) := 1$; and the unit map η is determined by $\eta(1) := \emptyset$. The *coproduct* Δ is given by

$$\Delta\Gamma := \sum_{\emptyset \subseteq \gamma \subseteq \Gamma} \gamma \otimes \Gamma/\gamma,$$

where the sum ranges over all subgraphs γ which are divergent and proper (i.e., removing one internal line cannot make more connected components); γ itself need not be connected. The terms for $\gamma = \emptyset$ and $\gamma = \Gamma$ in the sum are $\Gamma \otimes 1 + 1 \otimes \Gamma$. The notation Γ/γ denotes the (connected, 1PI) graph obtained from Γ by replacing each component of γ by a single vertex. One checks that Δ is coassociative [66], so H_Φ is a bialgebra.

Here are some coproducts for $\Phi = \varphi^4_4$, taken from [68]:

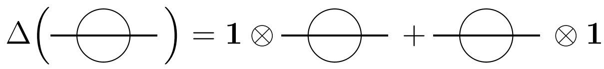

Figure 2: The "setting sun": a primitive diagram[1]

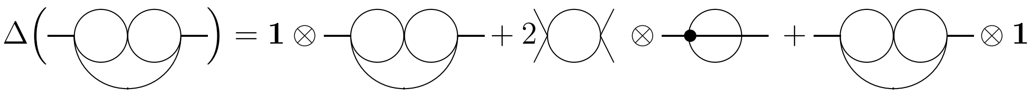

Figure 3: The "double ice cream in a cup" (depth = 2)

A grading, which ensures that H_Φ is a Hopf algebra, is provided by *depth* [59]: a graph Γ has depth k (or is "k-primitive") if

$$P^{\otimes(k+1)}(\Delta^k\Gamma) = 0 \quad \text{and} \quad P^{\otimes k}(\Delta^{k-1}\Gamma) \neq 0.$$

where P is the projection $\eta\varepsilon - \mathrm{id}$. Depth measures the maximal length of the inclusion chains of subgraphs appearing in the Bogoliubov recursion. In dimensional regularization, a graph

[1] Note: the setting sun diagram is primitive in position space, and is usually not considered primitive when working in momentum space. Of course, the associated amplitude must not depend on this description; and it does not [69]. This piece of wisdom seems less well known than it should be.

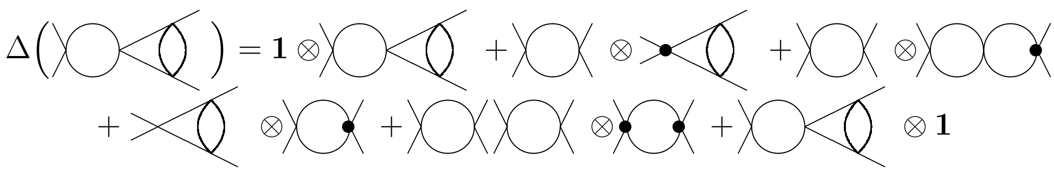

Figure 4: The “rag-doll” (depth = 3)

of depth l is expected to display a pole of order l. The antipode S can now be defined as the inverse of $\mathrm{id} = \eta\varepsilon - P$ for the convolution; if Γ_l is a graph of depth l, one finds

$$S(\Gamma_l) := \sum_{k=1}^{l} P^{*k}\,\Gamma_l = -\Gamma_l + \sum_{\emptyset \subsetneq \gamma \subsetneq \Gamma_l} S(\gamma)\,\Gamma_l/\gamma. \tag{4}$$

As it stands, the Hopf algebra H_Φ corresponds to a formal manipulation of graphs. These formulas can be matched to expressions for numerical values, as follows. The Feynman rules for the unrenormalized theory prescribe an algebra homomorphism

$$f : H_\Phi \to \mathcal{A}$$

into some commutative algebra $\mathcal{A}$; that is, f is linear and $f(\Gamma_1\Gamma_2) = f(\Gamma_1)\,f(\Gamma_2)$. In dimensional regularization, $\mathcal{A}$ is an algebra of Laurent series in a complex parameter ε, and $\mathcal{A}$ is the direct sum of two *subalgebras*:

$$\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_-,$$

where $\mathcal{A}_+$ is the holomorphic subalgebra of Taylor series and $\mathcal{A}_-$ is the subalgebra of polynomials in $1/\varepsilon$ without constant term. The projection $T\colon \mathcal{A} \to \mathcal{A}_-$, with $\ker T = \mathcal{A}_+$, picks out the pole part, in a minimal subtraction scheme. Now T is not a homomorphism, but the property that both its kernel and image are subalgebras is reflected in a “multiplicativity constraint”:

$$T(ab) + T(a)\,T(b) = T(T(a)\,b) + T(a\,T(b)).$$

The equation (3a) means that “the antipode delivers the counterterm”: one replaces S in the calculation (4) by C to obtain the right hand side, before projection with T. From the definition of the coproduct in H_Φ, (3b), which extracts the finite value, is a *convolution* in $\mathrm{Hom}(H_\Phi, \mathcal{A})$, namely, $R = C * f$. To show that R is multiplicative, it is enough to verify that the counterterm map C is multiplicative: convolution of homomorphisms is a homomorphism because $\mathcal{A}$ is commutative. The multiplicativity of C follows from the constraint on T, as shown by Connes and Kreimer in [66]. See also [70] and [71] in regard to these convolution formulas.

The previous discussion is logically independent of NCG, but there is an important historical link. The Hopf algebra approach to renormalization theory arose in parallel with the Connes–Moscovici noncommutative theory of foliations. In that theory, a foliation is described by a noncommutative algebra of functions twisted by local diffeomorphisms, $\mathcal{A} = C_c^\infty(F) \rtimes \Gamma$; horizontal and vertical vector fields on the frame bundle $F \to M$ are represented on $\mathcal{A}$ by the action of a certain Hopf algebra H_{CM} which provides a way to compute a local index formula in NCG [72]. One can map H_{CM} into an extension of the *Hopf algebra of rooted trees*, a precursor of the Connes–Kreimer graphical Hopf algebras which is described in detail in [73] and [22]. On extending the Hopf algebra H_Φ of graphs by incorporating operations of insertion of subgraphs, one obtains a noncommutative Hopf algebra of the H_{CM} type, which gives a supplementary handle on the combinatorial structure of H_Φ [74].

7 Outlook

Noncommutative Geometry has had, for many years now, a mutually rewarding conversation with quantum physics. The underlying motif of this conversation can be said to be the belief that Quantum Field Theory encodes the true geometry of the world, and that the mathematical task is to elucidate this geometrical structure. The payback to physics takes the form of new tools and methods; and the work is far from over. For the biggest challenge, that of understanding quantum gravity, there is a long way yet to travel.

Just as the effort to understand the gauge symmetries of the Standard Model led, in due time, to the introduction of real structures for spectral triples and from there to a noncommutative understanding of spin geometries, we may likewise expect that the NC approach to gravity will help to clarify our still imperfect understanding of the nature of noncommutative manifolds. The story continues ...

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