DIRAC OSCILLATOR VIA R-DEFORMED HEISENBERG ALGEBRA

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Abstract

The complete energy spectrum for the Dirac oscillator via R-deformed Heisenberg algebra is investigated.

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1 Introduction

The relativistic Dirac oscillator proposed by Moshinsky-Szczepaniac [1] is a spin $\frac{1}{2}$ object with the Hamiltonian which in the non-relativistic limit leads to that of a 3-dimensional isotropic oscillator shifted by a constant term plus a $\vec{L}\cdot\vec{S}$ coupling term for both signs of energy. There they construct a Dirac Hamiltonian, linear in the momentum $\vec{p}$ and position $\vec{r}$, whose square leads to the ordinary harmonic oscillator in the non-relativistic limit. The Dirac oscillator have been investigated in several context [2].

The R-deformed Heisenberg algebra or Wigner-Heisenberg algebraic technique [3] was recently super-realized for the SUSY isotonic oscillator [4, 5]. The R-Heisenberg algebra has also been investigated for the three-dimensional non-canonical oscillator to generate a representation of the orthosympletic Lie superalgebra $osp(3/2)$ [6].

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The R-Heisenberg algebra has been found relevant in the context of integrable models [7], and the Calogero interaction [8, 9]. Recently it has been employed for bosonization of supersymmetry in quantum mechanics [10], and the discrete space structure for the 3D Wigner quantum oscillator has been investigated [12]. In this work, we obtain the complete energy spectrum for the Dirac oscillator via R-deformed Heisenberg (RDH) algebra.

2 3D Wigner Oscillator

In this Section, we provide a three dimension presentation of the Wigner system with its bosonic sector to be the 3D isotropic oscillator (assumed to be of spin-$\frac{1}{2}$, to aid factorization).

The R-deformed Heisenberg (or Wigner-Heisenberg) algebra is given by following (anti-)commutation relations ($[A, B]_+ \equiv AB + BA$ and $[A, B]_- \equiv AB - BA$) :

$$H = \frac{1}{2}[a^-, a^+]_+, \quad [H, a^\pm]_- = \pm a^\pm, \quad [a^-, a^+]_- = 1 + cR, \quad [R, a^\pm]_+ = 0, \quad R^2 = 1, \tag{1}$$

where c is a real constant associated to the Wigner parameter [4]. Note that when $c = 0$ we have the standard Heisenberg algebra.

It is straightforward, following the analogy with the Ref. [4], to define the super-realizations for the ladder operators $a^\mp(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ for $H_W \equiv H(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ taking the explicitly forms

$$a^\mp = a^\mp(\vec{\sigma}\cdot\vec{L}+\mathbf{1}) = \frac{1}{\sqrt{2}}\left\{\mp\Sigma_1\left(\frac{\partial}{\partial r}+\frac{1}{r}\right) \pm \frac{1}{r}(\vec{\sigma}\cdot\vec{L}+\mathbf{1})\Sigma_1\Sigma_3 - \Sigma_1 r\right\} \tag{2}$$

which satisfy together with $H_W \equiv H(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ all the algebraic relations of the RDH algebra with the constant $\frac{c}{2}$ replaced by $(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ and $R = \Sigma_3$. Note that $(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ commutes with all the basic elements ($a^\mp$ and H_W) of the RDH algebra.

It may be observed that the RDH algebra that gets defined here is in fact three dimensional (one dimension for r and two for $(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$) and is identically satisfied on any arbitrary three dimensional wave function.

On the eigenspaces of the operator $(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$, the 3D Wigner algebra gets reduced to a 1D from with $(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ replaced by its eigenvalue $\mp(\ell+1)$, $\ell = 0, 1, 2, \cdots$, where ℓ is the orbital angular momentum quantum number. The eigenfuncitons of $(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ for the eigenvalues $(\ell+1)$ and $-(\ell+1)$ are respectivaly given by the well known spin-spherical harmonic $y_\mp$.

Now, considering simultaneous eigenfuncitons of the mutually commuting H_W and $(\vec{\sigma}\cdot\vec{L}+\mathbf{1})$ by

$$\psi_{W,+} = \begin{pmatrix} \tilde{R}_{1,+}(r) \\ \tilde{R}_{2,+}(r) \end{pmatrix} y_+, \quad (\vec{\sigma} \cdot \vec{L} + \mathbf{1})\psi_{W,+} = (\ell + 1)\psi_{W,+}, \tag{3}$$

$$\psi_{W,-} = \begin{pmatrix} \tilde{R}_{1,-}(r) \\ \tilde{R}_{2,-}(r) \end{pmatrix} y_-, \quad (\vec{\sigma} \cdot \vec{L} + \mathbf{1})\psi_{W,-} = -(\ell + 1)\psi_{W,-}, \tag{4}$$

(where the use of the subscript $+(-)$ indicates association with $[y_+(y_-)]$, we observe that the positive semi-definite form of H_W the ladder relations and the form of H_W dictat that the ground state energy $E_w^{(0)}(\vec{\sigma} \cdot \vec{L} + \mathbf{1}) \geq 0$, where $E_W(\vec{\sigma} \cdot \vec{L} + \mathbf{1})$ indicates a function of $\vec{\sigma} \cdot \vec{L} + \mathbf{1}$, is determined by the annihilation condition which reads as two cases.

3 The Dirac Oscillator Model

Adding an "anomalous momentum" in the form of a (nonlocal) linear and hermitian interaction, $\vec{\alpha}.\vec{\pi} \equiv -iM\omega\beta\vec{\alpha}.\vec{r} = (\vec{\alpha}.\vec{\pi})^\dagger$, in the (noncovariant) Dirac free particle equation with mass M and spin-$\frac{1}{2}$, in the natural sistem of units,

$$i\frac{\partial\psi}{\partial t} = (\vec{\alpha}.\vec{p} + M\beta)\psi, \tag{5}$$

one obtains the equation for the Dirac oscillator [1]:

$$i\frac{\partial\psi}{\partial t} = \{\vec{\alpha}.(\vec{p} + \vec{\pi}) + M\beta\}\psi, \tag{6}$$

where M and ω are, respectively, the mass of the particle and the frequency of the oscillator, and the matrices $(\vec{\alpha}, \beta)$ satisfy the following properties:

$$[\alpha_i, \beta]_+ = 0, \quad [\alpha_i, \alpha_j]_+ = 2\delta_{ij}\mathbf{1}, \quad \beta^2 = \mathbf{1} = \alpha_i^2, \quad (i, j = 1, 2, 3). \tag{7}$$

Writing the Dirac spinor in terms of the upper and lower components, respectively, ψ_1 and ψ_2, $\Psi(\vec{r}, t) = \exp(-\mathrm{iEt}) \begin{bmatrix} \psi_1(\vec{r}) \\ \psi_2(\vec{r}) \end{bmatrix}$ the standard representation of the matrices $\vec{\alpha}$ and β.

4 The Dirac oscillator via RDH algebra

In this section, we implement a new realization of the Dirac oscillator in terms of elements of the R-deformed Heisenberg algebra. To solve the equation Dirac, following the usual procedure, we consider the second order differential equation,

$$\tilde{H}_D\psi(\vec{r}) = E\psi(\vec{r}), \tag{8}$$

where $\tilde{H}_D$ is a second order Hamiltonian, $\tilde{H}_D = H_D^2 + M^2\mathbf{1}$, $\quad \tilde{E} = \frac{E^2-M^2}{2M}$. In the spherical polar coordinate system, we obtain the non-relativistic form of the Hamiltonian Ui [11], for an isotropic 3D SUSY harmonic oscillator with spin-$\frac{1}{2}$.

We consider a unitary operator in terms of the radial projection of the spin,

$$U = \begin{bmatrix} 1 & 0 \\ 0 & \sigma_r \end{bmatrix} = U^{-1} = U^{\dagger}, \tag{9}$$

to obtain the following relation between the transformed Dirac Hamiltonian, $\tilde{H}_D$, the 3D Wigner Hamiltonian, H_W, and the SUSY Hamiltonian, H_{SUSI} [11]:

$$H_{\mathrm{SUSI}} = U\tilde{H}_D U^{\dagger} = H_W - \frac{1}{2}\{1 + 2(\sigma \cdot \vec{L} + \mathbf{1})\Sigma_3\}\omega\Sigma_3. \tag{10}$$

4.1 The energy spectrum of the Dirac oscillator

The energy spectra of the operators $\tilde{H}_D$ and H_{SUSY} are identical, since these operators are related by a unitary transformation. However, the relation between the principal quantum number N and the angular momentum (ℓ) is different, in each case. Obviously, the energy spectrum associated with the two types of eigenspaces belonging to the eigenvalues $\pm(\ell+1)$:

Case(i) $\rightarrow \vec{\sigma} \cdot \vec{L} + \mathbf{1} \rightarrow \ell + 1 = j + \frac{1}{2}, \quad j = \ell + \frac{1}{2}$

$$\tilde{E}_{N\ell} = \frac{E^2 - M^2}{2M} = \begin{cases} 2m\omega = \tilde{E}^{+}_{N(\ell+1)}, \\ 2(m+1)\omega = \tilde{E}^{-}_{N\ell}, \end{cases} \tag{11}$$

where $m = 0, 1, 2, \ldots$.

Case(ii) $\rightarrow \vec{\sigma} \cdot \vec{L} + \mathbf{1} \rightarrow -(\ell + 1) = -(j + \frac{1}{2}), \quad j = (\ell + 1) - \frac{1}{2}$:

$$\tilde{E}_{N\ell} = \frac{E^2-M^2}{2M} = \begin{cases} (N + j + 3/2)\omega = \tilde{E}^{+}_{N\ell}, \quad N = j - \frac{1}{2}, j + 3/2, j + 7/2, \ldots, \\ (N + j + 5/2)\omega = \tilde{E}^{-}_{N(\ell+1)}, \quad N = j + \frac{1}{2}, j + 5/2, \ldots. \end{cases}$$

5 Conclusion

In this work we investigate the Dirac oscillator with the help of techniques of super-realization of the R-deformed Heisenberg algebra.

The Dirac oscillator with different interactions has been treated by Castaños *et al.* and by Dixit *et al.* [2]. These works motivate the construction of a new linear Hamiltonian in terms of the momentum, position and mass coordinates, through a set of seven mutually anticommuting 8x8-matrices yielding a representation of the Clifford algebra $C\ell_7$. The seven elements of the Clifford algebra $C\ell_7$ generate the three linear momentum components, the three position coordinates components and the mass, and their squares are the 8x8-identity matrix $\mathbf{I}_{8X8}$. Results of our analysis on Dirac oscillator via the Clifford algebra $C\ell_7$ are in preparation.

In a forthcoming paper we show that the Dirac oscillator equation can be resolved algebrically without having to transform it into a second order diferential equation. Therefore, the important connection for the Dirac 3D-isotropic oscillator with the linear ladder operators of the R-deformed Heisenberg algebra, satisfying the concomitant general oscillator quantum rule of Wigner, have explicited in this work.

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