Radiation reaction and renormalization in classical electrodynamics of point particle in any dimension

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Abstract

The effective equations of motion for a point charged particle taking account of radiation reaction are considered in various space-time dimensions. The divergencies steaming from the pointness of the particle are studied and the effective renormalization procedure is proposed encompassing uniformly the cases of all even dimensions. It is shown that in any dimension the classical electrodynamics is a renormalizable theory if not multiplicatively beyond $d = 4$. For the cases of three and six dimensions the covariant analogs of the Lorentz-Dirac equation are explicitly derived.

1 Introduction

The problem of accounting for the radiation back-reaction to the relativistic motion of a point charge has been the subject of intensive studies since Dirac's seminal paper [1]. The equation of motion proposed by Dirac coincides in nonrelativistic limit to the zero-size limit of the Lorentz model for an electron [2] and for this reason it is often referred to as the Lorenz-Dirac (LD) equation. For modern review and further references see [3, 4, 5, 6, 7]. Apart from the pure theoretical interest, the LD equation finds applications in physics of accelerators and astrophysics [8].

In this paper we formulate the general framework for deriving the LD equation in arbitrary dimension space-time. The main problem one comes up against when trying to consistently derive the effective equation of motion for a point charge is the inevitable infinities arising due to the "pointness" of the particle. The elimination procedure for these "classical" divergencies is relied on the same renormalization philosophy which is used in the quantum field theory and in this case one may put it to a rigor mathematical framework. The point is that, as far as the classical particle motion is concerned, one has to regularize the equations of motion which are linear in the field. This may require to remove the infinities only from a single Green function (that determines retarded Liénard and Wiechert potentials), no products of such functions are needed. We show that in this linear situation the regularization problem is resolved by the standard tools of the classical functional analysis, without invoking to the more powerful but less rigor machinery of the quantum renormalization theory. As a result we establish the general structure of the Lorentz-Dirac equations for any even d as well as the counterterms needed to compensate all the divergences. In the particular case of $d = 6$ the results of our analysis are in good agreement with those of work [9] where six-dimensional LD equation was obtained on basis of energy conservation and reparametrization invariance arguments.

When this work had been completed, we learned about the paper [10] where the question was discussed of the radiation reaction in various dimensions. In this paper, the distinctions are noticed between even and odd dimensions and the radiation reaction force is derived in $d = 3$. It was also argued that d=3,4 are the only dimensions where all the divergencies are removed by the mass renormalization, that checks well with our analysis and previous studies of the paper [9]. It should be noted, however, that the form of $3d$ integro-differential LD equation proposed in [10] differs in appearance from that derived in the present paper.

The paper is organized as follows. In Section 2 we set notations and discuss a difference between the retarded Green functions in odd and even dimensions. The regularization procedure for the singular linear functionals relevant to our problem is detailed in Section 3. In Section 4, this technique is applied to deriving the LD equation in various dimensions. Starting with case of even dimensions we develop a convenient regularization scheme for deriving the four-dimensional LD equation and yet allowing to generate its higher dimensional analogs by mere expansion in a regularization parameter. The case of odd dimensions seems to be less interesting as it leads to the *nonlocal* (integro-differential) LD equation, so after the discussion of general structure of the LD force we restrict ourselves by considering a simple example of $2 + 1$ particle. The main result of this section is that in any dimension the infinities coming from the particle's self-action can be compensated by a finite number of counterterms added to the original action functional, that means the renormalizability of classical electrodynamics. In concluding section we summarize the results and outline the prospects of further investigations.

2 Equations of motion and Green function

In this section we remind some basic formulas concerning the one-particle problem of classical electrodynamics formulated in an arbitrary dimensional space-time. The detailed treatment of the subject can be found, for example, in [11].

So, let $\mathbb{R}^{d-1,1}$ be d-dimensional Minkowski space with coordinates x^μ, $\mu = 0, ..., d-1$, and signature $(+-\cdots-)$. Consider a scalar point particle of mass m and charge e coupled to electromagnetic field. The dynamics of the whole system (field)+(particle) is governed by the action functional

$$S = -\frac{N_d}{4}\int d^dx F_{\mu\nu}F^{\mu\nu} + e\int d\tau A_\mu \dot{x}^\mu - m\int d\tau\sqrt{\dot{x}^2} \ , \qquad N_d = \frac{\pi^{\frac{1-d}{2}}}{2}\Gamma\left(\frac{d-1}{2}\right) \ , \tag{1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is strength tensor of electromagnetic field A_μ and dot means the derivative with respect to the particle's proper time τ[1]. Hereafter we use the natural system of units ($c = 1, \hbar = 1$) so that $[m] = -[x] = 1$ and $[e] = 1 - [A] = (4-d)/2$.

The variation of the action (1) results in a coupled system of the Maxwell and Lorentz equations describing both the motion of a charged point particle in response to electromagnetic field and the propagation of electromagnetic field produced by the moving charge. In the Lorenz gauge $\partial^\mu A_\mu = 0$ the equations take form

$$\Box A_\mu(x) = -N_d^{-1} j_\mu(x), \qquad D^2x^\mu = \mathcal{F}^\mu \tag{2}$$

where the right-hand sides of equations are given by the electric current of the point particle moving along the world-line $x^\mu(\tau)$ and the Lorentz force:

$$j^\mu(x) = e\int d\tau\delta(x - x(\tau))\dot{x}^\mu(\tau), \qquad \mathcal{F}_\mu = \frac{e}{m}F_{\mu\nu}(x)Dx^\nu \ . \tag{3}$$

Hereafter we use an invariant derivative

$$D = \frac{1}{\sqrt{\dot{x}^2}}\frac{d}{d\tau} \tag{4}$$

whose repeat action $D^n x^\mu(\tau)$ on a trajectory remains intact under reparametrization: $\tau \to \tau'(\tau)$.

It is well known that character of propagation of the electromagnetic waves depends strongly on the space-time dimension d, and especially on its parity. Mathematically, this manifests in quite different expressions for the *retarded* Green function $G = \Box^{-1}$ associated to the D'Alambert operator:

$$G(x) = \begin{cases} \dfrac{1}{2}\pi^{\frac{2-d}{2}}\theta(x_0)\delta^{(d/2-2)}(x_0^2 - \mathbf{x}^2) & \text{for} \quad \mathrm{d} = 4, 6, 8, \ldots \\[2ex] \dfrac{(-1)^{\frac{d-3}{2}}}{2}\pi^{-\frac{d}{2}}\Gamma\left(\dfrac{d-2}{2}\right)\theta(x_0 - |\mathbf{x}|)(x_0^2 - \mathbf{x}^2)^{\frac{2-d}{2}} & \text{for} \quad \mathrm{d} = 3, 5, 7, \ldots \end{cases} , \tag{5}$$

[1]Strictly speaking, the parameter m entering to the Lagrangian is, so called, *bare mass*. The physical mass of the particle will be introduced bellow within renormalization procedure.

where $\mathbf{x} = (x^1, \ldots x^{d-1})$. In an even-dimensional space-time the Green function is localized on a forward light-cone with the vertex at the origin, while for the case of odd dimensions its support extends to the interior of light-cone. These distinctions have a crucial physical consequence which can be illustrated by the following gedanken experiment. Suppose a source of light is turned on at an initial time t and is turned then off at a time t'. If the number of space-time dimensions is even, an observer, being located at some distance from the source, will see the light signal with clear-cut forward and backward wave fronts separated by the time interval $t - t'$. In so doing, the magnitude of the signal observed will not vary during the time provided the source works with a constant intensity. This is in a good agreement with our daily experience. Another picture would be observed in odd-dimensional space-time. Of course, there will be a definite instant of time when the observer finds the source to be turned on, it is the time point when the forward wave front reaches his eyes, but thereafter he will see the source to go slowly out and no sharply definite backward front will be observed. Thus, in odd-dimensional Universe the light source being ones turned on can never be turned off! Sometimes this phenomenon is mentioned as a failure of the Huygens principle for odd-dimensions.

In the context of present work, the distinction just drawn will lead to essentially different forms for the Lorentz-Dirac equation: it will be given by a finite-order differential equation for even d's and by an integro-differential one for the odd ones.

Return now to the equations of motion in the one-particle problem. The general solution is given to the field equation (2) by the sum of its particular solution and the general solution of the homogeneous equations

$$\Box A_\mu(x) = 0\,, \qquad \partial^\mu A_\mu(x) = 0\,, \tag{6}$$

describing free electromagnetic waves incident on the particle. Using the Green function (5) and the charge-current density vector (3) we can construct the particular solution as the retarded potentials of Liénard and Wiechert,

$$A_\mu(x) = -N_d^{-1} \int G(x-y) j_\mu(y) d^n y = -N_d^{-1} e \int G(x - x(\tau)) \dot{x}_\mu d\tau\,. \tag{7}$$

Thus we arrive at unambiguously determined decomposition of the electromagnetic potential into the exterior field (6) and the field created by the particle (7). The combined action of these fields on the particle is described by the Lorentz force (3), which is composed of exterior electromagnetic field, if any, and an inevitable action of the particle upon itself, i.e. Lorentz-Dirac force,

$$\mathcal{F}_\mu = \mathcal{F}_\mu^{ext} + \mathcal{F}_\mu^{LD}$$

The main technical and conceptual difficulty one faces with when calculating $\mathcal{F}^{LD}$ is the divergence of the integral for retarded field (7) taken at the points of particle's trajectory

- the fact is of no great surprise if one bears in mind the singularity of the Coulomb potential associated to a point charge. This problem is closely related to another one, namely, the problem of electromagnetic mass of an electron and sometimes the both are even identified.

In the next section we extend Dirac's result to an arbitrary dimensional space-time renormalizing Green function. We will show that in higher dimensions the elimination of infinities is not exhausted by renormalization of mass parameter m only but brings about new renormalization constants having no analogs in the original theory (1).

3 Regularization

We start with some general remarks concerning the mathematical status of the retarded Green function G. The Green function (5) being a kernel of the inverse wave-operator (7) is well-defined when acting on a smooth compactly supported source functions $j_\mu(x)$. The difficulties may arise, however, in attempting to apply the operator to a singular current like that produced by a point charge. The problem is twofold. First, except for the case of $d=3$, the Green functions involve products of generalized functions, namely, the derivatives of δ-function multiplied by θ-function. Such the products are ill-defined when one treats them to be generalized functions of one variable, say x^0, considering the other variables as parameters. It is the problem we will face with by restricting the argument of Green function onto particle's world-line $x^\mu(\tau)$. The second point concerns the geometry of the domain the Green function is supported within. Depending of dimension, the support coincides with, or bounded by, the light-cone surface

$$x^2 = 0\,, \qquad x_0 \geq 0\,, \tag{8}$$

which is not differentiable at $x^\mu = 0$. As a result, $G(x)$ is singular at the vertex of light-cone and ill-defined even in a generalized sense.

Let us illustrate this point by a simple example which, however, will play a significant role in subsequent analysis. Namely, consider the generalized function $F(s) = \delta(s^2)$ defined on the real half-line $s \geq 0$ [2]. Relations $s^2 = 0$, $s > 0$ are the one-dimensional analog of (8). We put by definition

$$\delta[f] = \int_0^\infty \delta(s) f(s) ds = f(0) \tag{9}$$

[2]When working with generalized functions one has to fix the dual space of basic functions, and in fact the latter is an element of definition for the former. The both are in a "dialectical" relation to each other: extending the space of basic function one narrows, at the same time, that of the generalized ones and vice versa. The particular choice of a basic functional space is mainly dictated by a problem to be considered. Hereafter one may always thought of a basic function as any smooth function on real half-line. In so doing, all the derivatives at zero are understood as the derivatives on the right.

Then the integral associated to linear functional F reads

$$F[f] = \int_0^\infty \delta(s^2) f(s) ds = \frac{1}{2}\int_0^\infty \delta(s) \left(\frac{f(\sqrt{s})}{\sqrt{s}} \right) ds = \frac{1}{2} \lim_{s\to+0} \frac{f(s)}{s}\,, \tag{10}$$

where $f(x)$ is a test function. In general, this integral diverges, so that functional F is ill-defined. Note, however, that integral (10) does have meaning when evaluating on basic functions vanishing at zero. The question is how to extend the functional F, in a consistent way, from subspace of functions vanishing at zero to the whole functional space. We will call a solution to this problem as a regularization of generalized function $F(s)$ and denote it by $\mathrm{reg}F(s)$. For example, the following expression solves the problem:

$$\mathrm{reg}F[f] = \int_0^\infty \delta(s^2)(f(s) - f(0))ds + a_0 f(0) = \frac{1}{2} f'(0) + a_0 f(0) \tag{11}$$

a_0 being arbitrary constant. Indeed, the functional (11) is well-defined for any basic function $f(x)$ and coincides with (10) if $f(0) = 0$. In fact, (11) is the general solution to our problem since the complementary space to that of vanishing-at-zero functions (i.e. subspace on which F comes to infinity) is one dimensional and spanned, for example, by constant function $f(s) = 1$. Regularizing linear functional F we just replace the infinite value $F[1] = \infty$ by any finite constant $\mathrm{reg}F[1] = a_0$.

This procedure can be straightforwardly extended to derivatives of δ-function. Consider the generalized function

$$F^n(s) = \delta^{(n)}(s^2) = \left(\frac{d}{ds^2}\right)^n \delta(s^2)\,, \qquad n = 1, 2, \ldots\,. \tag{12}$$

Then we put

$$\mathrm{reg}F^n[f] = \int_0^\infty \delta^{(n)}(s^2) \left(f(s) - f(0) - sf'(0) - \ldots - \frac{s^{2n}}{2n!} f^{(2n)}(0) \right) ds + \tag{13}$$

$$+ a_0 f(0) + a_1 f'(0) + \ldots + a_{2n} f^{(2n)}(0) =$$

$$= \int_0^\infty \delta(s) \frac{1}{2s} \left(-\frac{d}{2sds} \right)^n \left(f(s) - \sum_{k=0}^{2n} \frac{s^k}{k!} f^{(k)}(0) \right) ds + \sum_{k=0}^{2n} a_k f^{(k)}(0)\,,$$

that yields

$$\mathrm{reg}F^n[f] = \frac{f^{(2n+1)}(0)}{(n+1)!} + \sum_{k=0}^{2n} a_k f^{(k)}(0)\,. \tag{14}$$

As before, the functional (12), as it stands, is defined only on functions which are $o(s^{2n+1})$ as $s \to +0$. Subtracting from a function f several first terms of its Taylor expansion we just project f onto the subspace of such functions. The value of functional on the complementary $(2n + 1)$-dimensional subspace is fixed by the arbitrary chosen constants $a_0, \ldots, a_{2n}$.

The formula (14) is a particular example of a quite general mathematical procedure known as the regularization of divergent integrals or generalized functions. The procedure is applicable to a wide class of singular functionals, in particular, to functionals with polynomial singularities [12].

For our purposes it is instructive to re-derive expression (14) in another, perhaps more familiar for physicists, way. Namely, using the sequential approach to generalized functions one may represent δ-function (9) as

$$\delta(s) = \lim_{a\to+0} \frac{e^{-\frac{s}{a}}}{a} \,. \tag{15}$$

Substituting this representation to rel. (10) we get a one-parametric family of well-defined functionals until $a \neq 0$:

$$F_a[f] = \int_0^\infty \frac{e^{-\frac{s^2}{a}}}{a} f(s)ds \tag{16}$$

In so doing, $F[f] = \lim_{a\to 0} F_a[f]$. For this reason we refer to a as the regularization parameter. Given the basic function $f(s)$, the integral (16) defines a meromorphic function in $\sqrt{a}$ given by the Laurent series

$$F_a[f] = a^{-\frac{1}{2}} \int_0^\infty e^{-t^2} f(t\sqrt{a})dt = \sum_{n=0}^{\infty} a^{\frac{n-1}{2}} \frac{f^{(n)}(0)}{n!} \int_0^\infty e^{-t^2} t^n dt = \tag{17}$$

$$= \sum_{n=0}^{\infty} a^{\frac{n-1}{2}} \frac{f^{(n)}(0)}{n!} \Gamma\left(\frac{n+1}{2}\right)$$

In the limit $a \to 0$ of switching regularization off the singularity of functional $F[f]$ appears as a simple pole, while the non-vanishing term of the regular part of (17) coincides with (11) if take $a_0 = 0$. Similarly, the delta-shaped sequence (15) defines a holomorphic function in a with coefficients being the generalized functions in s,

$$\delta_a(s) = \frac{e^{-\frac{s}{a}}}{a} = \sum_{n=0}^{\infty} a^n \delta^{(n)}(s) \,. \tag{18}$$

Substitution of the last expression to (16) gives

$$F^n[f] = \lim_{a\to 0} \frac{1}{n!} \frac{d^n F_a[f]}{da^n} \tag{19}$$

We see that the regularization (14) of functionals F^n corresponds to replacement

$$a^{-\frac{k+1}{2}} \to a_k \,, \qquad k = 0, 1, \ldots, n \,, \tag{20}$$

of all the poles in expression (19) by arbitrary finite constants.

Thus the sequential approach to the problem leads to the same finite part for the functional to be regularized and contains the same ambiguity in the final definition as the scheme based on the direct subtraction.

4 Renormalization and the Lorentz-Dirac force.

Now we are going to explicitly calculate the Lorentz-Dirac force $\mathcal{F}^{LD}$. As the results drastically differ in odd and even dimensions, we consider these cases separately.

4.1 Even-dimensional space-time.

The retarded Green functions are given by the first line in (5). We start with the case of $d = 4$. The expression for LD force follows from (7) and (3):

$$\mathcal{F}^{LD}_\mu(s) = 4e^2 Dx^\nu(s) \int \theta(x^0(s) - x^0(\tau))\delta'((x(s) - x(\tau))^2)(x(s) - x(\tau))_{[\nu}\dot{x}_{\mu]}(\tau)d\tau \qquad (21)$$

where square brackets mean antisymmetrization. Since for the massive particle, equation $(x(s) - x(\tau))^2 = 0$ means $x^\mu(s) = x^\mu(\tau)$ that in turn implies $s = \tau$ the integrand is supported at the point $s = \tau$. Changing the integration variable $\tau \to s - \tau$ we get a singular integral of the form

$$\mathcal{F}^{LD}_\mu(s) = 4e^2 Dx^\nu(s) \int_0^\infty \delta'((x(s) - x(s-\tau))^2)(x(s) - x(s-\tau))_{[\nu}\dot{x}_{\mu]}(s-\tau)d\tau \qquad (22)$$

and the singularity comes from the derivative of delta function. Following to the general regularization prescription we replace δ' by an appropriate sequence of smooth functions. For example, from (18) it follows that

$$\delta'(s) = \lim_{a\to+0} \frac{\partial}{\partial a}\frac{e^{-\frac{s}{a}}}{a}, \qquad s \geq 0\,. \qquad (23)$$

This leads to the regular expression for LD force

$$\mathcal{F}^{LD}_\mu(s,a) = 4e^2 Dx^\nu(s)\frac{\partial}{\partial a}\int_0^\infty e^{-\frac{(x(s)-x(s-t))^2}{a}}(x(s) - x(s-t))_{[\nu}\dot{x}_{\mu]}(s-t)\frac{dt}{a} \qquad (24)$$

$$\mathcal{F}^{LD}_\mu(s) = \lim_{a\to+0}\mathcal{F}^{LD}_\mu(s,a)$$

Evaluating this integral by the Laplace method we get an asymptotic expansion for $\mathcal{F}^{LD}_\mu(s,a)$ in half-integer powers of a. The actual calculations are considerably simplified if one notes that form of the integral is invariant under reparametrizations. So, we may assume the proper time τ to satisfy the additional normalization condition $\dot{x}^\mu\dot{x}_\mu = 1$. In this gauge, many of terms vanish in the Taylor expansion for the exponential and pre-exponential factors in (24), leaving us with

$$(x(s)-x(s-\tau))^2 = \dot{x}^2\tau^2 - \frac{1}{12}(D^2x)^2\tau^4 + \frac{1}{12}D^2x\cdot D^3x\tau^5 + \left(\frac{1}{45}(D^3x)^2 + \frac{1}{40}D^2x\cdot D^4x\right)\tau^6 + O(\tau^7)\,,$$

$$(x(s) - x(s-\tau))_{[\nu}\dot{x}_{\mu]}(s-\tau) = -\frac{1}{2}D^2x_{[\mu}Dx_{\nu]}\tau^2 + \frac{1}{3}D^3x_{[\mu}Dx_{\nu]}\tau^3 - \qquad (25)$$

$$-\left(\frac{1}{8}D^4x_{[\mu}Dx_{\nu]}+\frac{1}{12}D^3x_{[\mu}D^2x_{\nu]}\right)\tau^4+\left(\frac{1}{30}D^5x_{[\mu}Dx_{\nu]}+\frac{1}{24}D^4x_{[\mu}D^2x_{\nu]}\right)\tau^5+O(\tau^6)\,.$$

Substituting these expressions back to the eq.(24) we get an integral of the form (17). The result of integration reads:

$$\mathcal{F}^{LD}_\mu(a)=2e^2\dot{x}^2\sum_{k=2}^{\infty}(-1)^{k+1}\frac{k-1}{k!}\Gamma\left(\frac{k+1}{2}\right)f^{(k)}_\mu(Dx,...,D^kx)a^{\frac{k-3}{2}}\,. \tag{26}$$

The explicit expressions for the several first terms are given by

$$f^\mu_{(2)}=D^2x^\mu,$$

$$f^\mu_{(3)}=D^3x^\mu-(D^2x)^2Dx^\mu$$

$$f^\mu_{(4)}=D^4x^\mu-\frac{3}{2}(D^2x)^2D^2x^\mu-3D^2x\cdot D^3xDx^\mu$$

$$f^\mu_{(5)}=D^5x^\mu-\frac{5}{2}(D^2x)^2f^\mu_{(3)}-\frac{15}{2}D^2x\cdot D^3xD^2x^\mu-(4D^2x\cdot D^4x+3(D^3x)^2)Dx^\mu$$

$$\cdots$$

$$f^\mu_{(k)}=D^kx^\mu+\cdots$$

$$\cdots$$

Notice that in view of reparametrization invariance of the model the vector of the regularized LD force is transverse to the particle's velocity, i.e. $\dot{x}^\mu\mathcal{F}_\mu(s,a)=0$. Upon removing the regularization only the first two terms of the series (26) survive - one singular and one finite - that leads to the well-known LD equation for a $d=4$ point charge interacting with its own field

$$\left(m+\frac{e^2}{2}\sqrt{\frac{\pi}{a}}\right)D^2x^\mu=\frac{2}{3}e^2(D^3x^\mu-(D^2x)^2Dx^\mu)\,,\qquad a\to 0\,. \tag{27}$$

The r.h.s. of this equation describes a back-reaction of the particle upon radiating of electromagnetic waves and this is more than just an interpretation. Notice that our calculations, being as rigorous as possible, contain, however, an apparent ambiguity related to the definition of Dirac's δ-function on half-line (9). Indeed, starting with an arbitrary delta-shaped sequence defined on the whole real line one may restrict the respective functional to test functions vanishing identically at $s<0$. The different sequences will then lead to the different results distinguished from each other by an overall constant factor. For example, any symmetric (with respect to zero) approximation for $\delta(s)$ will give the additional $1/2$ multiplier in the r.h.s. of rel. (9). However, requiring the total energy of the system to conserve one has to equate the work of the LD force to the energy of the electromagnetic field *radiated* by accelerating particle that immediately leads to our convention for δ-function on half-line. It is the energy conservation argument which is frequently used to derive the LD equation in four dimensions (see e.g. [1], [4], [6]).

The physical interpretation for the infinite contribution in the l.h.s. of eq. (27) is also obvious: its appearance reflects the infinite energy, or mass, of the field *adjunct* to the particle. Within the paradigm of renormalization theory this singularity is removed by a simple redefinition of the particle's mass: one just replace the sum of unobservable bare mass m and the infinite contribution due to the interaction by a finite (experimental) value,

$$m_{exp} = m + \frac{e^2}{2}\sqrt{\frac{\pi}{a}}\,, \tag{28}$$

so that renormalized action of $d = 4$ particle takes the form

$$S^{(4)}_{renorm} = \left(m_{exp} - \frac{e^2}{2}\sqrt{\frac{\pi}{a}}\right)\int d\tau\sqrt{\dot{x}^2} \tag{29}$$

This means that the classical electrodynamics in four dimensions is a multiplicatively renormalizable theory.

The next task is to try to extend this result to the higher even dimensions using the explicit expressions for the retarded Green functions (5). To do this one has no need to repeat all the calculations from the very beginning. In view of the relations (18) and (19) the desired expressions for the LD force can be derived for any even d by successive differentiation of the universal series (26) obtained for $d = 4$. More precisely,

$$\mathcal{F}^{(d)}_\mu = \frac{\sqrt{\pi}}{2\left(\frac{d-4}{2}\right)!}\Gamma\left(\frac{d-1}{2}\right)\left(\frac{\partial}{\partial a}\right)^{\frac{d-4}{2}}\mathcal{F}^{LD}_\mu(a)|_{a=0} \tag{30}$$

In the case of six dimensions the respective equation of motion reads

$$\begin{gathered}\left(m - a^{-\frac{3}{2}}\frac{e^2\sqrt{\pi}}{6}\right)D^2x^\mu = \frac{4e^2}{45}\left(D^5x^\mu - \frac{5}{3}(D^2x)^2[D^3x^\mu - (D^2x)^2Dx^\mu] - \right.\\ \left. -\frac{15}{3}D^2x\cdot D^3xD^2x^\mu - [4D^2x\cdot D^4x + 3(D^3x)^2]Dx^\mu\right) - \\ - a^{-\frac{1}{2}}\frac{e^2\sqrt{\pi}}{16}\left(D^4x^\mu - \frac{3}{2}(D^2x)^2D^2x^\mu - 3D^2x\cdot D^3xDx^\mu\right).\end{gathered} \tag{31}$$

Besides the infinite mass term we observe a new type divergence involving fourth-order derivatives. It is interesting to note that both the divergences are Lagrangian, i.e. they can be canceled out by adding appropriate counterterms to the initial Lagrangian (1), so that the renormalized action reads

$$S^{(6)}_{renorm} = -\left(m_{exp} + a^{-\frac{3}{2}}\frac{e^2\sqrt{\pi}}{6}\right)\int d\tau\sqrt{\dot{x}^2} - a^{-\frac{1}{2}}\frac{e^2\sqrt{\pi}}{32}\int d\tau\sqrt{\dot{x}^2}(D^2x)^2 \tag{32}$$

Contrary to this, the finite part of the LD force containing fifth derivatives of x's cannot be represented as the variation of a Lorentz-invariant functional. These results generally agree with the previous analysis of work [9], where the explicit expressions for $6d$ LD force and the counterterm in (32) were obtained from requirements of energy conservation and reparametrization invariance [3].

This situation is general and nothing changes this picture as the number of dimensions increases. Namely, for any even d the finite part of the LD force is given by a polynomial function in the derivatives of $x^\mu(\tau)$ up to $(d-1)$ order inclusive. Note that the higher (odd) derivative $D^{d-1}x^\mu$ enters linearly to LD force and therefore no variation principle for the LD equation exists. This reflects the dissipative character of the system losing the energy due to the radiation. Besides, there are $d/2-1$ divergent terms "the most singular of which" corresponds to the infinite electromagnetic mass of the particle, while the interpretation for the other terms is not so simple as the structures of such types are lacking in the original theory. By analogy with the four and six dimensions one may expect that all the singularities are Lagrangian and can be removed by adding appropriate counterterms to the action (1). This appears to be the case. Indeed, since the Maxwell equations (2) are linear we may resolve them in a general form (7) and, substituting result back to the action functional (1), get a functional of particle's trajectory only,

$$S = -m\int d\tau\sqrt{\dot{x}^2} - \frac{1}{2}N_d^{-1}\int d^dx\int d^dy j^\mu(x)G(x-y)j_\mu(y) \tag{33}$$

$$= -m\int d\tau\sqrt{\dot{x}^2} - \frac{e^2}{2}N_d^{-1}\int ds\int d\tau\, \dot{x}^\mu(s)G(x(s)-x(\tau))\dot{x}_\mu(\tau)\,.$$

The first term is the usual action functional of free scalar particle and the second describes self-action. Since the retarded Green function is localized on the light-cone we can explicitly perform (after a suitable regularization) one integration in the double integral and get the Lagrangian model for the relativistic particle with higher derivatives. It is not hard to check that in the cases of $d=4,6$ the regularized self-action term in (33) exactly reproduces the counterterms in the corresponding action functionals (29), (32). As to the LD force, it can not be obtained from the higher derivative model since only symmetric part of the retarded Green function actually enters to the nonlocal action (33). It seems very likely that the same situation takes place in the higher dimension as well.

To summarize, for even dimension $d>4$ one force to extend the original Lagrangian of the free relativistic particle by the addition of $d/2-2$ extra (higher derivative) terms in order to get a renormalizable theory, so that whole renormalization procedure involves, together with the physical mass, $d/2-1$ arbitrary constants.

[3]The radiation reaction force (31) differs from that of [9] by an overall coefficient. This seems because of minor inaccuracy in the normalization of $6d$ Liénard-Wiechert potentials accepted in [9].

4.2 Odd-dimensional space-time

The Green function is given by a product of $\theta-$ function and a singular analytical expression (5). According to this, the gradient of Green function entering to the strength tensor $F_{\mu\nu}$ of adjunct electromagnetic field consists of structures proportional to θ - and δ - functions. This allows us to decompose the respective LD force onto the local and nonlocal parts and both of these parts are singular in general. To remove the singularities one may apply the direct subtraction scheme discussed above. The obtaining of explicit expressions requires, however, a great amount of computations, so we confine ourselves by example of $2+1$ particle to illustrate how this technique works in the most simple case.

The local and nonlocal parts of the LD force are given by the integrals

$$\mathcal{F}^{\mu}_{nonlocal} = e^2 Dx_\nu(s)\int_{-\infty}^{s} d\tau \left(\frac{(x(s)-x(\tau))^{[\mu}\dot{x}(\tau)^{\nu]}}{|x(s)-x(\tau)|^3} \right) \tag{34}$$

$$\mathcal{F}^{\mu}_{local} = -2e^2 Dx_\nu(s)\int_{-\infty}^{s} d\tau \left(\frac{\delta((x(s)-x(\tau))^2)(x(s)-x(\tau))^{[\mu}\dot{x}^{\nu]}(\tau)}{|x(s)-x(\tau)|} \right) \tag{35}$$

In near the coincidence limit $s\to\tau$ the numerator and denominator of the integrand (34) behave like $(s-\tau)^2$ and $(s-\tau)^3$, respectively, so that the integral diverges logarithmically, as one would expect for the two-dimensional Coulomb potential. Using asymptotic equality

$$2(x(s)-x(\tau))^{[\mu}\dot{x}^{\nu]}(\tau) \sim D^2x^{[\nu}(s)Dx^{\mu]}(s)(x(s)-x(\tau))^2\sqrt{\dot{x}^2(\tau)}\,, \quad s\to\tau\,, \tag{36}$$

we can extract the infinity as follows:

$$\mathcal{F}^{\mu}_{nonlocal} = e^2 Dx_\nu(s)\int_{-\infty}^{s} d\tau \left(\frac{(x(s)-x(\tau))^{[\mu}\dot{x}(\tau)^{\nu]}}{|x(s)-x(\tau)|^3} - \frac{D^2x^{[\nu}(s)Dx^{\mu]}(s)\sqrt{\dot{x}^2(\tau)}}{2|x(s)-x(\tau)|} \right) + \tag{37}$$

$$+\,\delta m D^2x^\mu(s),$$

where

$$\delta m = \frac{e^2}{2}\int_{-\infty}^{s} \frac{d\tau\sqrt{\dot{x}^2(\tau)}}{|x(s)-x(\tau)|} = \infty\,. \tag{38}$$

Now the first integral is regular and it describes the nonlocal self-action of the point charge, whereas the second term gives rise to the infinite mass renormalization. In fact, the form of counterterm is uniquely determined by reasons of reparametrization invariance, right physical dimension and short-distance behavior.

As to the local part of LD force (35) it turns out to be finite and proportional to the free equations of motion,

$$\mathcal{F}^{\mu}_{local} = \frac{e^2}{2}D^2x^\mu\,, \tag{39}$$

Note that here we may not worry about the “right” definition for the δ-function on half-line as, in any case, this contribution is absorbed by the renormalization of mass. It would be interesting to compare the work produced by the renormalzed force (37) with the loss of energy due to the radiation.

The main lesson to be learned from this consideration is that, despite the nonlocal character of the self-action force in $2+1$ dimensions, the divergent part of the LD force is local and even Lagrangian. There is no doubt that the same conclusion remains true for any higher odd dimension.

5 Concluding remarks

In this paper we have considered derivation of the Lorentz-Dirac equation for a point charge in various dimensions. In any even d the radiation reaction is given by the general formula (30) implying only formal differentiation of $4d$ LD force by regularization parameter. It has been shown that contrary to the quantum electrodynamics in $d>4$, the classical electrodynamics of point particles is a renormalizable theory, although non-multiplicatively, beyond $d=4$. The necessity of extra counterterms (involving higher derivatives) in addition to that responsible for the mass renormalization (attributed naturally to the infinite energy of field surrounding a point charge) seems rather counterintuitive but this is a must for obtaining a reasonable theory.

The results of the paper can be extended at least in two directions. First one may consider a supersymmetric or higher-spin generalizations for the relativistic particle. The models with N-extended world-line supersymmetry have been studied in refs. [13, 14]. It has been shown that after quantization these models can be consistently interpreted as relativistic spinning particles of spin $N/2$. By analogy with the quantum field theory one may expect that inclusion of supersymmetry will result in correction of some singularities or will even lead to finite models. If this is the case, the account of spin can give rise to a fully consistent theory of charged point particles. The consistent interactions of massive arbitrary-spin particles to exterior fields including higher dimensions have been constructed in [16, 17]. Although the spin induced radiation is known, its back-reaction remains an open question even in $d=4$.

The second option is to extend the above analysis to p-brain system universally coupled to a $(p+1)$-form field and other background fields. In this set up one may rise a question of classical stability for such a system. At the linear level the problem was studied in ref. [15], where a full set of constraints on masses and couplings was established for 0-brain minimally coupled to a multiplet of vector and scalar fields. It is anticipated that generalizing this analysis to case of extended object we get a certain restriction on background fields in a form of local equations of motion. If so, this may shed new light upon the origin of low-energy effective field equations in the string and brane world.

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