KUNS-1712
hep-th/0103018

Fuzzy Sphere and Hyperbolic Space from Deformation Quantization

Isao KISHIMOTO*

Department of Physics, Kyoto University, Kyoto 606-8502, Japan

March, 2001

Abstract

We explicitly construct noncommutative $*$ products on circularly symmetric two dimensional space by using the technique of Fedosov's deformation quantization. Especially, on constant curvature spaces i.e., S^2 and H^2, we get $su(2)$ and $su(1,1)$ algebra respectively. These are candidates of $*$ products applicable to noncommutative field theories or noncommutative gauge theories on spaces with nontrivial symplectic structure.

*ikishimo@gauge.scphys.kyoto-u.ac.jp

1 Introduction

Since the relation between string theory and noncommutative geometry was discussed in [1], noncommutative field theories and noncommutative gauge theories have been investigated enthusiastically from various viewpoints.

Many authors use the Moyal product[1] as noncommutative associative $*$ product for explicit calculations. It corresponds to a constant NS-NS B-field background in flat space in the context of string theory. On the other hand, at least formally, more general $*$ products which may correspond to string theory on nonconstant B-field background in curved space are defined by some authors[2]. However, explicit form of $*$ products other than the Moyal product has been scarcely discussed in physical context[3].

In this paper, we use the technique of Fedosov's deformation quantization [3] to get explicit forms of $*$ products on nontrivial backgrounds. For simplicity, we investigate $*$ products on circularly symmetric two dimensional spaces. Specifically, we focus on constant curvature spaces S^2, H^2 and $\mathbb{R}^2$, and explicitly construct $*$ products which are different from the Moyal product. We also discuss some physical applications of our $*$ products.

2 Construction of $*$ product

Here we review the construction of Fedosov's $*$ product very briefly[4], and apply this procedure to circularly symmetric two dimensional spaces.

First, for a given symplectic manifold (M, Ω_0), we define the Weyl algebra bundle W which has $\circ$ product of the Moyal type and its Abelian connection D with some input parameter. For $\mathrm{Ker}D \subset W$ (which is called flat section W_D), we get a one to one correspondence with $C^\infty(M)[[\hbar]]$, where $\hbar$ is the deformation parameter. We denote the map from $C^\infty(M)[[\hbar]]$ to W_D as Q, and its inverse map as σ. Then Fedosov's $*$ product on $C^\infty(M)[[\hbar]]$ is defined by

$$a_0 * b_0 := \sigma(Q(a_0) \circ Q(b_0)), \quad a_0, b_0 \in C^\infty(M)[[\hbar]]. \tag{1}$$

This is a solution of the problem of deformation quantization, i.e.,

[1]Here we call $* = \exp\left(\frac{i}{2}\frac{\overleftarrow{\partial}}{\partial x^i}\theta^{ij}\frac{\overrightarrow{\partial}}{\partial x^j}\right)$ with constant $\theta^{ij} = -\theta^{ji}$ the Moyal product.

[2][2],[3], for example.

[3]In [4], nonassociative star product which generalizes [2],[3] is discussed to describe D-brane in curved backgrounds.

[4]See [3],[5] for details.

$*$ is associative and its commutator $[\ ,\ ]_*$ is expanded as

$$[\ ,\ ]_* = i\hbar\{\ ,\ \} + \mathcal{O}(\hbar^2) \tag{2}$$

where $\{\ ,\ \}$ is the Poisson bracket with respect to the symplectic form Ω_0.

Now, we apply this procedure to a two dimensional space M with metric

$$ds^2 = e^{\Phi(r)}(dr^2 + r^2 d\theta^2), \tag{3}$$

where $\Phi(r)$ is some function of r only (i.e. circularly symmetric space) for simplicity. Its volume form is given by

$$\Omega_0 = e^{\Phi(r)} r dr \wedge d\theta, \tag{4}$$

and we identify it with symplectic form. Using Fedosov's procedure with the input[5]

$$\begin{aligned}
&\Omega_0 = \theta^1 \wedge \theta^2 = -\frac{1}{2}\omega_{ij}\theta^i \wedge \theta^j, \\
&\theta^1 = e^{\Phi(r)} dr, \quad \theta^2 = r d\theta, \quad \omega_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
&\Omega_1 = 0,\ \nabla = d, \\
&\mu = \frac{1}{3} e^{-\Phi(r)} r^{-1} (y^1)^2 y^2,
\end{aligned} \tag{5}$$

we get an Abelian connection D as

$$\begin{aligned}
&Da = da - \delta a + \frac{i}{\hbar}(\mathrm{r} \circ a - a \circ \mathrm{r}), \quad a \in W, \\
&\mathrm{r} = e^{-\Phi(r)} r^{-1} y^1 y^2 \theta^1, \\
&\circ := \exp\left(-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial y^i}\omega^{ij}\frac{\overrightarrow{\partial}}{\partial y^j}\right), \quad \omega^{ij} := (\omega^{-1})^{ij}.
\end{aligned} \tag{6}$$

For this Abelian connection D, we solve the equation $Da = 0$ and get the map $Q : C^\infty(M)[[\hbar]] \to W_D$ as

$$a = Q(a_0(r,\theta)) = a_0\left(G(r, y^1), \theta + \frac{y^2}{r}\right), \tag{7}$$

where $G(r, y^1)$ is given by

$$\int_r^{G(r,y^1)} e^{\Phi(r')} r' dr' = y^1 r. \tag{8}$$

Then we can define a $*$ product on M by eq.(1).

[5]See [3],[5] for the meaning of $\nabla, \Omega_1, \mu, \delta$. Here we choose these parameters in such a way that the iteration formula (eq.(21) of [5]) which gives an Abelian connection is satisfied trivially, i.e., $\nabla \mathrm{r} + \frac{i}{\hbar}\mathrm{r} \circ \mathrm{r} = 0$. Then we get $\mathrm{r} = \delta\mu + \delta^{-1}(d(\omega_{ij} y^i \theta^j) - \Omega_1)$ for the input (5).

3 S^2 case

In this section we apply the result of §2 to the case $M = S^2$. We consider 2-sphere S^2 with radius R, which is defined as two dimensional surface embedded in $\mathbb{R}^3$:

$$(X^1)^2 + (X^2)^2 + (X^3)^2 = R^2. \tag{9}$$

We parametrize the coordinate $X^i, i = 1, 2, 3$ on S^2 as

$$\begin{aligned} &X^1 = \frac{2R^2 r}{r^2 + R^2}\cos\theta,\ X^2 = \frac{2R^2 r}{r^2 + R^2}\sin\theta,\ X^3 = R\frac{r^2 - R^2}{r^2 + R^2}, \\ &r \geq 0,\ 0 \leq \theta \leq 2\pi. \end{aligned} \tag{10}$$

Then the metric of S^2, $ds^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2$, is given by

$$ds^2 = \frac{4R^4}{(r^2 + R^2)^2}(dr^2 + r^2 d\theta^2), \tag{11}$$

and the conformal factor e^Φ of eq.(3) is identified as

$$e^{\Phi(r)} = \frac{4R^4}{(r^2 + R^2)^2}. \tag{12}$$

From eqs. (12), (7) and (1), we get the explicit form of our $*$ product on S^2:

$$\begin{aligned} &a_0(r,\theta) * b_0(r,\theta) \\ = &\left(a_0\left(\sqrt{\frac{r^2 + \frac{y^1}{2R^2}r(r^2+R^2)}{1 - \frac{y^1}{2R^4}r(r^2+R^2)}}, \theta + \frac{y^2}{r}\right) \exp\left(-\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial y^1}\frac{\overrightarrow{\partial}}{\partial y^2} - \frac{\overleftarrow{\partial}}{\partial y^2}\frac{\overrightarrow{\partial}}{\partial y^1}\right)\right)\right. \\ &\left.\cdot b_0\left(\sqrt{\frac{r^2 + \frac{y^1}{2R^2}r(r^2+R^2)}{1 - \frac{y^1}{2R^4}r(r^2+R^2)}}, \theta + \frac{y^2}{r}\right)\right)_{y^1=y^2=0}. \end{aligned} \tag{13}$$

By using this definition, we can calculate $*$ product of the S^2 coordinate X^i (10). In particular, we have

$$[X^i, X^j]_* = i\frac{\hbar}{R}\varepsilon^{ijk}X^k, \tag{14}$$

$$X^1 * X^1 + X^2 * X^2 + X^3 * X^3 = R^2\left(1 - \frac{\hbar^2}{4R^4}\right), \tag{15}$$

where ε^{ijk} is the antisymmetric tensor with $\varepsilon^{123} = +1$. Eq.(14) means that the commutators of X^i's form $su(2)$ algebra which is known as fuzzy sphere algebra, and eq.(15) means that its radius is given by $R\sqrt{1 - \frac{\hbar^2}{4R^4}}$ which is deformed by $\mathcal{O}(\hbar^2)$ from the original radius R of commutative S^2 (9). Namely, we have obtained a fuzzy sphere by deforming S^2 using the $*$ product (13).

4 H^2 case

In this section we apply the result of §2 to the case $M = H^2$. Calculation is quite similar to the S^2 case (§3). We consider two dimensional hyperbolic space H^2 with radius R, which is defined as two dimensional surface embedded in $\mathbb{R}^{1,2}$:

$$-(Y^0)^2 + (Y^1)^2 + (Y^2)^2 = -R^2, \quad Y^0 > 0. \tag{16}$$

We parametrize the coordinates $Y^i, i = 0, 1, 2$ on H^2 as

$$\begin{aligned} &Y^0 = R\frac{R^2 + r^2}{R^2 - r^2},\ Y^1 = \frac{2R^2 r}{R^2 - r^2}\cos\theta,\ Y^2 = \frac{2R^2 r}{R^2 - r^2}\sin\theta, \\ &0 \le r \le R,\ 0 \le \theta \le 2\pi. \end{aligned} \tag{17}$$

Then, the metric of H^2, $ds^2 = -(dY^0)^2 + (dY^1)^2 + (dY^2)^2$, and the conformal factor are given respectively by

$$ds^2 = \frac{4R^4}{(R^2 - r^2)^2}(dr^2 + r^2 d\theta^2), \tag{18}$$

$$e^{\Phi(r)} = \frac{4R^4}{(R^2 - r^2)^2}. \tag{19}$$

From eqs. (19), (7) and (1), we get the explicit form of our $*$ product on H^2:

$$\begin{aligned} & a_0(r,\theta) * b_0(r,\theta) \\ = & \left(a_0\left(\sqrt{\frac{r^2 + \frac{y^1}{2R^2}r(R^2 - r^2)}{1 + \frac{y^1}{2R^4}r(R^2 - r^2)}}, \theta + \frac{y^2}{r}\right) \exp\left(-\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial y^1}\frac{\overrightarrow{\partial}}{\partial y^2} - \frac{\overleftarrow{\partial}}{\partial y^2}\frac{\overrightarrow{\partial}}{\partial y^1}\right)\right) \right. \\ & \left. \cdot b_0\left(\sqrt{\frac{r^2 + \frac{y^1}{2R^2}r(R^2 - r^2)}{1 + \frac{y^1}{2R^4}r(R^2 - r^2)}}, \theta + \frac{y^2}{r}\right)\right)_{y^1 = y^2 = 0}. \end{aligned} \tag{20}$$

By using this definition, we obtain the following $*$ products of the H^2 coordinate Y^i (17):

$$[Y^0, Y^1]_* = i\frac{\hbar}{R}Y^2, \quad [Y^2, Y^0]_* = i\frac{\hbar}{R}Y^1, \quad [Y^1, Y^2]_* = -i\frac{\hbar}{R}Y^0, \tag{21}$$

$$-Y^0 * Y^0 + Y^1 * Y^1 + Y^2 * Y^2 = -R^2\left(1 - \frac{\hbar^2}{4R^4}\right). \tag{22}$$

Eq.(21) means that commutators of Y^i's form $su(1,1)$ algebra which corresponds to isometry of H^2, and eq.(22) means that its radius is given by $R\sqrt{1 - \frac{\hbar^2}{4R^4}}$ which is deformed by $\mathcal{O}(\hbar^2)$ from the original radius R of commutative H^2 (16). Namely, we get fuzzy hyperbolic space by deforming H^2 using the $*$ product (20).

5 Large R limit and $\mathbb{R}^2$

Here we consider large radius limit of the results of §3 and §4. The sectional curvature of S^2 (9) (H^2 (16)) is $\frac{1}{R^2}$ ($-\frac{1}{R^2}$), which tends to $+0$ (-0) in the limit $R \to \infty$. Therefore they approach the flat space $\mathbb{R}^2$ in the large R limit in the usual commutative picture. How about it from the noncommutative viewpoint?

For comparison, we construct a $*$ product on $\mathbb{R}^2$ following the method of §2. We adopt as its flat metric

$$ds^2 = 4(dr^2 + r^2 d\theta^2) \tag{23}$$

with its front factor 4 chosen so that (23) coincides with the large R limit of (11) and (18). With $e^\Phi = 4$, we get the explicit form of our $*$ product on $\mathbb{R}^2$:

$$\begin{aligned}
&a_0(r,\theta) * b_0(r,\theta) \\
&= \left(a_0\left(\sqrt{r^2 + \frac{y^1 r}{2}}, \theta + \frac{y^2}{r}\right) \exp\left(-\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial y^1}\frac{\overrightarrow{\partial}}{\partial y^2} - \frac{\overleftarrow{\partial}}{\partial y^2}\frac{\overrightarrow{\partial}}{\partial y^1}\right)\right) \right. \\
&\left. \cdot b_0\left(\sqrt{r^2 + \frac{y^1 r}{2}}, \theta + \frac{y^2}{r}\right)\right)_{y^1=y^2=0}.
\end{aligned} \tag{24}$$

Then, we can calculate the $*$ products of the complex coordinate $z := re^{i\theta}$, $\bar{z} := re^{-i\theta}$:

$$\begin{aligned}
&z * z = \sqrt{r^4 - \frac{\hbar^2}{16}} e^{2i\theta} = \overline{\bar{z} * \bar{z}}, \quad z * \bar{z} = r^2 - \frac{\hbar}{4}, \quad \bar{z} * z = r^2 + \frac{\hbar}{4}, \\
&[z, \bar{z}]_* = -\frac{\hbar}{2}.
\end{aligned} \tag{25}$$

The commutator $[z, \bar{z}]_*$ coincides with that of the usual Moyal product for Cartesian coordinates on $\mathbb{R}^2$, but $*$ product itself is different from the Moyal product. This difference comes from ambiguity of deformation quantization.

We can calculate the commutator $[z, \bar{z}]_*$ also in the S^2 and H^2 cases. For S^2, from eq.(13) we get

$$[z, \bar{z}]_* = \frac{-\frac{\hbar}{2R^4}(r^2 + R^2)^2}{1 - \left(\frac{\hbar}{4R^4}(r^2 + R^2)\right)^2} = -\frac{\hbar}{2R^4}(R^2 + z * \bar{z})(R^2 + \bar{z} * z). \tag{26}$$

And for H^2, from eq.(20) we get

$$[z, \bar{z}]_* = \frac{-\frac{\hbar}{2R^4}(R^2 - r^2)^2}{1 - \left(\frac{\hbar}{4R^4}(R^2 - r^2)\right)^2} = -\frac{\hbar}{2R^4}(R^2 - z * \bar{z})(R^2 - \bar{z} * z). \tag{27}$$

Both eqs.(26) and (27) are reduced to $[z, \bar{z}]_* = -\frac{\hbar}{2}$ (25) as $R \to \infty$. In other words, the $*$ product which we obtained in §2 connects $su(2)$ algebra (or fuzzy S^2) with $su(1,1)$ algebra (or fuzzy H^2) through $R = \infty$.

6 An application

In the previous sections, we explicitly calculated $*$ products by using Fedosov's formulation. They are candidates of $*$ product for defining noncommutative field theory or noncommutative gauge theory on fuzzy S^2, H^2 and $\mathbb{R}^2$.

As an example, we discuss four dimensional noncommutative $U(1)$ gauge theory with one scalar field which is given by the action[6]

$$S = \mathrm{Tr}\left(\frac{1}{4}G^{IJ}G^{KL}F_{IK} * F_{JL} + \frac{1}{2}G^{IJ}D_I\phi * D_J\phi\right). \tag{28}$$

We assume that only two dimensional space is noncommutative (1,2 direction), and use a general formulation of noncommutative gauge theory of [5]:

$$\begin{aligned}
&G^{IJ} = \delta^{IJ},\ I, J = 1, \cdots, 4, \\
&F_{IJ} = \partial_I A_J - \partial_J A_I - i[A_I, A_J]_* - \frac{J_{IJ}}{\hbar}, \quad J_{12} = -J_{21} = 1, \text{others} = 0, \\
&\partial_I = \frac{i}{\hbar}[-J_{IJ}\tilde{\phi}^J,\ ]_*,\ I = 1, 2, \quad \partial_3 = \frac{\partial}{\partial x^3}, \partial_4 = \frac{\partial}{\partial x^4} \\
&D_I\phi = \partial_I\phi - i[A_I, \phi]_*,
\end{aligned} \tag{29}$$

Here, $\tilde{\phi}^I$ is the "canonical" noncommutative coordinate satisfying

$$\frac{i}{\hbar}[\tilde{\phi}^1, \tilde{\phi}^2]_* = 1. \tag{30}$$

Its explicit form is

$$\tilde{\phi}^1 = \frac{2Rr}{\sqrt{r^2 + R^2}}\cos\theta,\ \tilde{\phi}^2 = \frac{2Rr}{\sqrt{r^2 + R^2}}\sin\theta \tag{31}$$

for fuzzy S^2 (13),

$$\tilde{\phi}^1 = \frac{2Rr}{\sqrt{R^2 - r^2}}\cos\theta,\ \tilde{\phi}^2 = \frac{2Rr}{\sqrt{R^2 - r^2}}\sin\theta \tag{32}$$

for fuzzy H^2 (20), and

$$\tilde{\phi}^1 = 2r\cos\theta,\ \tilde{\phi}^2 = 2r\sin\theta \tag{33}$$

for fuzzy $\mathbb{R}^2$ (24). The action (28) is invariant under noncommutative $U(1)$ gauge transformation:

$$\delta_\lambda A_I = \partial_I\lambda - i[A_I, \lambda]_*, \qquad \delta_\lambda\phi = -i[\phi, \lambda]_*. \tag{34}$$

The equations of motion of (28) are

$$D^I F_{IJ} = -i[\phi, D_J\phi]_*,\ D^I D_I\phi = 0, \tag{35}$$

[6]The symbol Tr is trace for the $*$ product satisfying $\mathrm{Tr} f * g = \mathrm{Tr} g * f$ [3], but we can discuss equations of motion without using the explicit form of the trace.

and we obtain a solution by solving the $U(1)$ noncommutative BPS equation:

$$B_I = D_I\phi, I=1,2,3, \quad \partial_4 = 0, A_4 = 0, \quad B_I := \frac{1}{2}\varepsilon^{IJK}\left(F_{JK} + \frac{J_{JK}}{\hbar}\right). \tag{36}$$

Under the ansatz

$$\begin{aligned} &A_1 + iA_2 = if_A(l,x^3)(\tilde{\phi}^1 + i\tilde{\phi}^2), \qquad A_3 = 0, \\ &\phi = f(l,x^3), \qquad l := \sqrt{(\tilde{\phi}^1)^2 + (\tilde{\phi}^2)^2 + (x^3)^2}, \end{aligned} \tag{37}$$

eq.(36) can be rewritten as

$$\begin{aligned} \partial_3 G^{(m)} - 4\partial_L f^{(m)} &= \sum_{\substack{2n+k=m,\\ n\geq 1}} \frac{4\partial_L^{2n+1} f^{(k)}}{(2n+1)!} + \sum_{\substack{2n+k+k'\\ =m-1}} \frac{4G^{(k')}\partial_L^{2n+1} f^{(k)}}{(2n+1)!}, \\ \partial_3 f^{(m)} - \partial_L(LG^{(m)}) &= \sum_{\substack{2n+k=m,\\ n\geq 1}} \frac{\partial_L^{2n+1}(LG^{(k)})}{(2n+1)!} \end{aligned} \tag{38}$$

with

$$L := (\tilde{\phi}^1)^2 + (\tilde{\phi}^2)^2, \quad f = \sum_{k=0}^{\infty} \hbar^k f^{(k)}, \quad \left(\frac{1}{\hbar} + f_A\right)^2 = \frac{1}{\hbar^2} + \frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k G^{(k)}. \tag{39}$$

We can solve eq.(38) order by order in $\hbar$, and we get

$$\begin{aligned} f &= \frac{g}{l} + \hbar g^2\left(\frac{2x^3}{l^4} - \frac{1}{l^3}\right) + \hbar^2\left(\frac{-8g^3x^3}{l^6} - \frac{g}{4l^5} + \left(\frac{5g}{8} + 10g^3\right)\frac{(x^3)^2}{l^7}\right) + \mathcal{O}(\hbar^3), \\ f_A &= \frac{g}{l(l+x^3)} + \hbar g^2\left(\frac{2}{l^4} - \frac{1}{l^3(l+x^3)} - \frac{1}{2l^2(l+x^3)^2}\right) \\ &+\hbar^2\left(\frac{-8g^3}{l^6} + \frac{4g^3}{l^5(l+x^3)} + \frac{g^3}{l^4(l+x^3)^2} + \frac{g^3}{2l^3(l+x^3)^3} - \left(\frac{5g}{8} + 10g^3\right)\frac{x^3}{l^7}\right) + \mathcal{O}(\hbar^3), \end{aligned} \tag{40}$$

as a solution such that it becomes the $U(1)$ Dirac monopole in the commutative limit (i.e., $\hbar \to 0$). In the fuzzy $\mathbb{R}^2$ case (33), the $\mathcal{O}(\hbar)$ terms coincide with those in [6] which solved the equations of motion with the usual Moyal product.

7 Conclusion and discussion

In this paper we have presented explicit construction of $*$ products on two dimensional constant curvature spaces S^2, H^2 and $\mathbb{R}^2$. We have found that the algebras of the $*$ products represent fuzzy S^2, H^2 and $\mathbb{R}^2$ because the commutators of the $*$ product form $su(2), su(1,1)$

and Heisenberg algebra respectively. The commutators $[z,\bar{z}]_*$ for fuzzy S^2 and H^2 are reduced to that of fuzzy $\mathbb{R}^2$ in the large R limit. In this sense, fuzzy S^2 and H^2 approach to fuzzy $\mathbb{R}^2$ as $R \to \infty$. This is consistent with usual commutative picture.

In §6 we applied explicit form of our $*$ products to $U(1)$ noncommutative BPS equation (36), and obtained its solution to $\mathcal{O}(\hbar^2)$. In eq.(36) the $*$ product appears only in the commutator $[\ ,\ ]_*$. Therefore, eq.(36) is solved unifiedly for fuzzy S^2, H^2 and $\mathbb{R}^2$ by using "canonical" noncommutative coordinate $\tilde{\phi}^I$ (30). In other words, we can get a solution of eq.(36) even if the definition of $*$ is different as long as we use "canonical" noncommutative coordinate $\tilde{\phi}^I$ for the $*$ product.

To study the effects of the difference of $*$ products themselves, we should consider noncommutative equations containing "bare" $*$ products. Its typical example is $\phi * \phi = \phi$ which is essentially the equation for noncommutative soliton [7]. Even for the $\mathbb{R}^2$ case, the $*$ product which we get here is different from the usual Moyal product, and hence $\phi \sim \exp(-r^2)$ is *not* a solution[7] of $\phi * \phi = \phi$. It is a future problem to find an explicit solution of it and to investigate its meaning.

For fuzzy S^2, $*$ product is usually defined by using representation matrix of $su(2)$ and spherical harmonic function, and depends on the size of matrix. On the other hand our $*$ product depends on the deformation parameter $\hbar$, so they are very different in appearance. It is also a future problem to study an explicit relation between them. If the relation becomes clear, our $*$ product may give some suggestions to string theory in the literature [8] for example.

Acknowledgements

We would like to thank T. Asakawa, S. Goto, H. Hata, H. Kawai, S. Moriyama and S. Terashima for valuable discussions and comments. This work is supported in part by the Grant-in-Aid for Scientific Research (#9858) from the Ministry of Education, Science, Sports and Culture of Japan.

References

[1] N. Seiberg and E. Witten, *"String Theory and Noncommutative Geometry," JHEP* **9909** (1999) 032. hep-th/9908142.

[7] In the case of the Moyal product, this is a solution.

[2] M. Kontsevich, *“Deformation quantization of Poisson manifolds, I,”* `q-alg/9709040`.

[3] B.V. Fedosov, *“Deformation quantization and index theory,” Berlin, Germany: Akademie-Verl.* (1996).

[4] L. Cornalba and R. Schiappa, *“Nonassociative Star Product Deformations for D-brane Worldvolumes in Curved Backgrounds,”* `hep-th/0101219`.

[5] T. Asakawa and I. Kishimoto, *“Noncommutative Gauge Theories from Deformation Quantization,” Nucl. Phys.* **B591** (2000) 611-635. `hep-th/0002138`.

[6] K. Hashimoto and T. Hirayama, *“Branes and BPS Configurations of Non-Commutative/Commutative Gauge Theories,” Nucl. Phys.* **B587** (2000) 207-227. `hep-th/0002090`.

[7] R. Gopakumar, S. Minwalla and A. Strominger, *“Noncommutative Solitons,” JHEP* **0005** (2000) 020. `hep-th/0003160`.

[8] A. Yu. Alekseev, A. Recknagel and V. Schomerus, *“Brane Dynamics in Background Fluxes and Non-commutative Geometry,” JHEP* **0005** (2000) 010. `hep-th/0003187`.