Time Evolution as a Gauge Transformation: x^5-dependent Cosmological Solution in 5d Kaluza-Klein theory*

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Abstract

We discuss a new feature of the 5d Kaluza-Klein cosmology. For that purpose, we obtain the simplest x^5-dependent solution which, in the reduced description, is associated with a radiation-dominated Robertson-Walker universe, and also can be regarded as an extension of the Schwarzschild solution. This solution enables us to deduce an important result that an evolving universe is related with a static universe by the gauge transformation, i.e., they are gauge equivalent. This means that having a different universe simply corresponds to choosing a different gauge.

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Birkhoff's theorem states that every spherically symmetric solution - regardless of whether it is static or dynamic - to the vacuum Einstein equations is essentially the Schwarzschild solution[1]. So, it is not surprising that the Schwarzschild solution has a cosmological interpretation. The 5d Schwarzschild solution has indeed a cosmological interpretation describing time evolution of the 4d isotropic, homogeneous (Robertson-Walker) universe [2]. One of reasons for considering such an embedding is because it may provide the possibility of avoiding cosmological singularities arising in the conventional Robertson-Walker cosmology. It is known that certain genuine singularities in four dimensions can be resolved by simply going to higher dimensions [3].

Apart from this, there has recently been proposed a new mechanism [4] for solving the hierarchy problem based on the assumption that the conventional Planck scale $M_{pl} \sim 10^{19} GeV$ is essentially not the fundamental scale in nature; M_{pl} is simply an effective constant determined by the Electroweak scale M_{EW}(which is assumed to be the only short-distance fundamental scale) and the volume (or curvature [5]) of the extra dimensions. This assumption then leads to the result that the hierarchy between M_{pl} and M_{EW} can be eliminated by taking the extra dimensions to be very large. The important consequence of this is that Kaluza-Klein excitations are not insensible anymore, and their effects become important in the theory. This in turn implies that the dependence of the metric fields on the extra dimensions is not to be neglected ; rather, they become crucial [6]. In this letter, we are examining the simplest x^5- dependent (x^5 being the fifth coordinate) cosmological solu-

[1]This is not quite true in the Kaluza-Klein theory. See for instance ref.[1].

tion of the standard 5d Kaluza-Klein theory, which can be also regarded as an extension of the 5d Schwarzschild solution, then we end up with a dramatic result that the time evolution of the effective (or observed) universe in the reduced description is in fact a kind of gauge transformation.

The simplest way of obtaining a 5d cosmological solution (in the absence of matter) is to make a coordinate transformation

$$t = \int \frac{dR}{[-k(1 - R_0^2/R^2)]^{1/2}} \tag{1}$$

in the 5d Schwarzschild solution

$$ds^2 = [k(1 - \frac{R_0^2}{R^2})]^{-1} dR^2 + R^2 d\Omega_k^2 - [k(1 - \frac{R_0^2}{R^2})] d\tau^2, \tag{2}$$

where R_0 is some constant, and $d\Omega_k^2$ is the line element of the 3d volume with constant curvature[2] $k = 1, -1$. Upon renaming $\tau \to x^5$, we then obtain

$$ds^2 = -dt^2 + [R_0^2 - k(t - t_0)^2] d\Omega_k^2 + \frac{(t - t_0)^2}{[R_0^2 - k(t - t_0)^2]} (dx^5)^2, \tag{3}$$

the 5d version of the Robertson-Walker metric. Equation(3) coincides (for $k = 1$) with the solution found in ref.[7] provided the integral constant t_0 is identified with R_0. With this identification the metric(3) describes after obvious dimensional reduction a closed universe which starts to expand from the initial (big-bang) singularity at $t = 0$, reaches maximum radius R_0 at $t = R_0$, then collapses to a point(crunch singularity) as $t \to 2R_0$. So in this case both big-bang and crunch singularities are present. For $k = -1$, on the other hand, the absence of the big-bang singularity is manifest. In this case, R_0 represents the minimum radius instead of maximum on the contrary

[2]Here we omit the case $k = 0$ for convenience. This, however, dose not ruin the generality of our discussion.

to the case $k=1$. The expansion starts from R_0 at $t=t_0$, then continues forever. Thus, the initial singularity does not exist as long as R_0 is a non-zero constant.

Though the metric (2) is a vacuum solution in itself, the effective cosmology immanent in it is not trivial. As discussed in ref.[7] metric (3) describes (in the 4d sector) a radiation-dominated universe; the dynamics of the fifth dimension contributes effective radiation source. In fact the analogy between (3) and the standard Robertson-Walker cosmology becomes clear once we compare the scale factor $R^2(t) \equiv R_0^2 - k(t-t_0)^2$ in (3) with the one in standard cosmology which was found to be [8]

$$R^2(t) = \begin{cases} (\sqrt{\kappa A/3})^2 - [t-(\tilde{t_0}+\sqrt{\kappa A/3})]^2 & \text{for } k\text{=1} \\ -(\sqrt{\kappa A/3})^2 + [(t-\tilde{t_0})+\sqrt{\kappa A/3}]^2 & \text{for } k\text{=-1,} \end{cases} \tag{4}$$

where $\kappa \equiv 8\pi G$, $\tilde{t_0}$ is an arbitrary constant, and A is related with energy density of radiation by $A=\rho R^4 =$constant. From(4) one immediately see that for $k=1$ both $R(t)$'s are identical if we identify $R_0 \leftrightarrow \sqrt{\kappa A/3}$, and $t_0 \leftrightarrow \tilde{t_0}+\sqrt{\kappa A/3}$. But for $k=-1$ the situation is a little bit different. $R(t)$ in (4) starts from the initial singularity at $t=t_0$ instead of starting from R_0. Indeed for $k=-1$ those $R(t)$'s have some different time dependence near the big-bang as we can see from (3) and (4). However, it should be noted that both have the same asymptotic behavior, $R(t)\sim t$ as $t\to\infty$, which is typical of radiation-dominated open universe.

Now we get into the main point. First, as an extension of (3) we introduce an ansatz

$$ds^2 = -dt^2 + R^2(t,x^5)d\Omega_k^2 + e^{\mu(t,x^5)}(dx^5)^2. \tag{5}$$

Equation(5) is just the (five dimensional) Tolman metric with the radial coordinate r in the conventional Tolman metric replaced by the extra coordinate x^5. These two metrics (i.e., the metric in eq.(5) and the conventional Tolman metric) have the same mathematical form, but their physical contents are quite different: while the latter describes collapse phase of a dust, the former a family of evolving universes each of which is parametrized by x^5. Since we are considering an extension of the vacuum solution the appropriate field equations to be worked out are the vacuum Einstein equations $R_{AB} = 0$, which reduce after some algebra to a set of independent field equations

$$2\dot{R}' - \dot{\mu}R' = 0, \tag{6}$$

$$R\ddot{R} + \dot{R}^2 + k - e^{-\mu}R'^2 = 0, \tag{7}$$

$$(\frac{\dot{R}}{R})^2 + \frac{1}{2}\dot{\mu}\frac{\dot{R}}{R} + \frac{k}{R^2} - e^{-\mu}[\frac{R''}{R} + (\frac{R'}{R})^2 - \frac{1}{2}\mu'\frac{R'}{R}] = 0. \tag{8}$$

Here, "primes" and "overdots" denote $\prime \equiv \partial/\partial x^5$ and $\cdot \equiv \partial/\partial t$, respectively, and other equations are simply combinations of these three equations. The procedure for solving the above field equations is well known [9]: eq.(6) is integrated to give

$$e^{\mu} = \frac{R'^2}{k + f(x^5)}, \tag{9}$$

$f(x^5)$ being an arbitrary function of x^5 alone; substituting (9) in (7) then gives[3]

$$\dot{R}^2 = f(x^5) + \frac{g(x^5)}{R^2}, \tag{10}$$

$g(x^5)$ being another arbitrary function of x^5 alone; finally by virtue of (9) and (10) the last equation (8) reduces to

$$\frac{1}{R^3}\frac{g'}{R'} = 0, \tag{11}$$

[3]In the case of ordinary 4d Tolman solution the 2nd. term of the r.h.s. of (10) appears to be $\sim 1/R$ (instead of $\sim 1/R^2$).

whose obvious solution is

$$g(x^5) = constant \equiv g_0. \tag{12}$$

The procedure then goes differently from here. The most general solution to eq.(10) is found to be

$$R^2(t, x^5) = R_0^2(x^5) + f(x^5)[t - \hat{t_0}(x^5)]^2 \tag{13}$$

with $R_0(x^5)$ defined by

$$R_0^2(x^5) \equiv -\frac{g(x^5)}{f(x^5)}, \tag{14}$$

and where it is important to note that $\hat{t_0}(x^5)$, which has been introduced as an integral constant, is actually not simply a constant; it is most generally a function of x^5. The scale factor (9), on the other hand, is now written as

$$e^{\mu(t,x^5)} = \frac{1}{(k+f)} \frac{[R_0 R_0' + (f'/2)^2 (t-\hat{t_0})^2 - f\hat{t_0}'(t-\hat{t_0})]^2}{[R_0^2 + f(t-\hat{t_0})^2]}, \tag{15}$$

which, however, diverges as $t \to \infty$ unless f' is zero. At this point we recall that the size of the compact extra dimensions in Kaluza-Klein theories is closely related with observed constants (electric charge or Newton's constant for instance) of the 4d effective theory [1]. So it is customary in Kaluza-Klein cosmology to assume that the radius of the compact dimension should be constant at least asymptotically; we therefore require: $e^{\mu(t,x^5)} \to constant$ as $t \to \infty$. This requirement then immediately implies

$$f(x^5) = constant \equiv f_0, \tag{16}$$

$$\hat{t_0}'(x^5) = constant \equiv -\alpha, \tag{17}$$

and, by (12) and (16), $R_0^2(x^5)$ in (14) now becomes

$$R_0^2(x^5) = -\frac{g_0}{f_0} = constant \equiv R_0^2. \tag{18}$$

In particular this constant R_0 is a turning point in the classical motion described by (10); note that the kinetic energy term $\dot{R}^2$ vanishes when $R^2(t, x^5) = -g/f = R_0^2$. Thus the constant R_0 in (18) is identified with R_0 in the metric(3). After all this we find the solution which meets the given requirement to be written as

$$ds^2 = -dt^2 + R^2(t, x^5) d\Omega_k^2 + \phi(t, x^5)(dx^5)^2 \tag{19}$$

with

$$R^2(t, x^5) = R_0^2 + f_0(t - \hat{t_0})^2, \tag{20}$$

$$\phi(t, x^5) \equiv e^{\mu(t,x^5)} = \frac{f_0^2(t - \hat{t_0})^2}{R_0^2 + f_0(t - \hat{t_0})^2}. \tag{21}$$

Again R_0 and f_0 in (20) and (21) are constants, but $\hat{t_0}$ is a function of x^5 :

$$\hat{t_0}(x^5) = t_0 - \alpha x^5 \ , \ (t_0 = constant), \tag{22}$$

where the constant α has been taken to be

$$\alpha = \pm(k + f_0)^{1/2}. \tag{23}$$

Obviously eq.(19) is a generalization of the metric (3); one can see that (19) reduces to (3) for $f_0 \to -k$. In fact the solution(19) is cosmologically the unique extension of (3). The constant α, on the other hand, is real for $f_0 \geq -k$, but it is imaginary for $f_0 < -k$ and in this case we obtain 'two time physics' [10] by replacing $x^5 \to i\tau$ (τ being the extra time).

The extended solution (19) has a remarkable feature. It is gauge equivalent to the static (in the 4d sense) solution. Let us recall that in 5d Kaluza-Klein theory a local $U(1)$ gauge transformation takes the form of the special case of the general coordinate transformation

$$\begin{aligned} \tilde{x}^\alpha &= x^\alpha, \\ \tilde{x}^5 &= x^5 + \kappa\Lambda(x^\alpha). \end{aligned} \tag{24}$$

Indeed the line element of the 5d Kaluza-Klein theory is assumed to take the form

$$\begin{aligned} ds_5^2 &= {}^5\mathrm{g}_{MN}(x^A)dx^M dx^N \\ &= {}^4\mathrm{g}_{\mu\nu}(x^\alpha, x^5)dx^\mu dx^\nu + \phi(x^\alpha, x^5)[dx^5 + \kappa A_\mu(x^\alpha, x^5)dx^\mu]^2, \end{aligned} \tag{25}$$

and using the standard transformation law for the metric tensor ${}^5\mathrm{g}_{MN}(x^A)$ one can show that the 4d field quantities transform, respectively, as

$$\begin{aligned} {}^4\tilde{\mathrm{g}}_{\mu\nu}(x^\alpha, \tilde{x}^5) &= {}^4\mathrm{g}_{\mu\nu}(x^\alpha, x^5), \\ \tilde{A}_\mu(x^\alpha, \tilde{x}^5) &= A_\mu(x^\alpha, x^5) - \partial_\mu\Lambda(x^\alpha), \\ \tilde{\phi}(x^\alpha, \tilde{x}^5) &= \phi(x^\alpha, x^5) \end{aligned} \tag{26}$$

under (24), which leads us to believe that the transformation (24) is associated with the local $U(1)$ gauge transformation, and consequently $A_\mu(x^\alpha, x^5)$ can be interpreted as a $U(1)$gauge boson after dimensional reduction . Also one can readily see that the line element (25) is invariant under (24) and (26), meaning that the transformation (24) is really a symmetry of the theory. But it should be mentioned that the translation along the fifth coordinate is not an isometry transformation here because $\phi(x^\alpha, x^5)$ and other fields depend on x^5; i.e., the space structure is not quite cylindrical, and the 'cylindricity' condition fails to hold in this case [11]. But still, (24) is a symmetry of the theory. Now let us apply the above gauge transformation to the metric (19). If we take

$$\Lambda(x^\alpha) = \frac{1}{\alpha\kappa}(t - t_0), \tag{27}$$

then $t - \hat{t}_0$ becomes simply $\alpha\tilde{x}^5$, and (19) reduces to

$$d\tilde{s}^2 = -dt^2 + \tilde{R}^2(\tilde{x}^5)d\Omega_k^2 + \tilde{\phi}(\tilde{x}^5)[d\tilde{x}^5 + \kappa\tilde{A}_0(\tilde{x}^5)dt]^2 \tag{28}$$

with

$$\begin{aligned} \tilde{R}^2(\tilde{x}^5) &= R_0^2 + \alpha^2 f_0(\tilde{x}^5)^2, \\ \tilde{\phi}(\tilde{x}^5) &= \frac{\alpha^2 f_0^2(\tilde{x}^5)^2}{R_0^2 + \alpha^2 f_0(\tilde{x}^5)^2}, \\ \tilde{A}_0(\tilde{x}^5) &= -\frac{1}{\alpha\kappa} = constant. \end{aligned} \tag{29}$$

This result is remarkable. Being time independent the gauge transformed metric (28) is associated with a static universe, while ds^2 in (19) describes an evolving universe in the reduced description, which means that the evolving universe is gauge equivalent to the static universe with constant gauge potential. This result is analogous to the Higgs mechanism where the massless Goldstone boson is eaten up (by the gauge transformation)by photon which, as a result, acquires a mass. In our case the 'time' (or time degree of freedom) is eaten up (or absorbed into the fifth dimension) by the gauge transformation, and as a result the universe acquires a pure gauge potential. The fact that the acquired potential is a pure gauge is not difficult to understand because the metric (19) does not contain any gauge potential term;

i.e., $A_\mu(x^\alpha, x^5) = 0$ in (26), so $\tilde{A}_\mu(x^\alpha, \tilde{x}^5)$ has to be a pure gauge. In fact the appearance of the pure gauge is more manifest in the Schwarzschild form of the metric (19)(or equivalently, the metric (5)). Using (9) and (10) together with (20) and (21) we obtain from (5):

$$ds^2 = (k + f_0\frac{R_0^2}{R^2})^{-1}dR^2 + R^2 d\Omega_k^2 - (k + f_0\frac{R_0^2}{R^2})[dx^5 + A_R(R)dR]^2 \tag{30}$$

with

$$A_R(R) = -\frac{\alpha}{[f_0(1 - R_0^2/R^2)]^{1/2}(k + f_0 R_0^2/R^2)}. \tag{31}$$

The potential $A_R(R)$ is a pure gauge since $\vec{\nabla} \times \vec{A}$ is obviously zero. Also note that eq.(30) is not exactly the Schwarzschild metric, but it goes to (2) as $f_0 \to -k$, so it is an extension of the Schwarzschild metric.

To proceed, we now ask how each universe described by (19) and its gauge transformed version (28) looks to the low-energy physicists who can not make enough power to probe the fifth dimension. For those physicists the observed result will be simply the average over x^5 (or $\tilde{x}^5$); i.e. they can only observe $n = 0$ massless mode in the expansion

$$g_{MN}(x^\alpha, x^5) = \sum_{-\infty}^{\infty} g_{MN}^{(n)}(x^\alpha)e^{inx^5/R_c}, \tag{32}$$

and similarly for $\tilde{g}_{MN}(x^\mu, \tilde{x}^5)$. In eq.(32), R_c represents the compactication radius of the fifth dimension, and x^5 (and $\tilde{x}^5$) is periodic with period $2\pi R_c$; i.e., $0 \leq x^5 \leq 2\pi R_c$, $x^5 \sim x^5 + 2\pi R_c$, similarly for $\tilde{x}^5$. The reduced line elements of (19) and (28), which retain only $n = 0$ mode, are found, respectively, to be

$$ds_{(0)}^2 = -dt^2 + [R^{(0)}(t)]^2 d\Omega_k^2 + \phi^{(0)}(t)(dx^5)^2 \tag{33}$$

with

$$[R^{(0)}(t)]^2 = [R_0^2 + \frac{f_0}{3}(\alpha\pi R_c)^2] + f_0[(t-t_0) + (\alpha\pi R_c)]^2, \tag{34}$$

$$\begin{aligned} \phi^{(0)}(t) &= f_0 - f_0\frac{R_0}{2\pi R_c\alpha\sqrt{f_0}}\{\tan^{-1}[\frac{2\pi R_c\alpha\sqrt{f_0}}{R_0} + \frac{\sqrt{f_0}}{R_0}(t-t_0)] \\ &\quad - \tan^{-1}[\frac{\sqrt{f_0}}{R_0}(t-t_0)]\}, \end{aligned} \tag{35}$$

and

$$d\tilde{s}^2_{(0)} = -dt^2 + [\tilde{R}^{(0)}]^2 d\Omega_k^2 + \tilde{\phi}^{(0)}[d\tilde{x}^5 + \kappa\tilde{A}_0^{(0)}dt]^2 \tag{36}$$

with

$$[\tilde{R}^{(0)}]^2 = [R^{(0)}(t_0)]^2 = R_0^2 + \frac{4}{3}f_0(\alpha\pi R_c)^2 = constant, \tag{37}$$

$$\tilde{\phi}^{(0)} = \phi^{(0)}(t_0) = f_0 - f_0\frac{R_0}{2\pi R_c\alpha\sqrt{f_0}}\tan^{-1}\frac{2\pi R_c\alpha\sqrt{f_0}}{R_0} = constant, \tag{38}$$

$$\tilde{A}_0^{(0)} = -\frac{1}{\alpha\kappa} = constant. \tag{39}$$

The universe described by (33) is still a radiation-dominated universe (in the 4d sector), but a little different[4] from that described by (3). In particular the constant R_0^2 in (34) admits an addition term $f_0(\alpha\pi R_c)^2/3$. This is of interest particularly in the case $k=-1$. Recall that the constant f_0 must satisfy $f_0 \geq -k$ in order for α to be real in eq.(23). So f_0, and consequently the additional term $f_0(\alpha\pi R_c)^2/3$ should be positive for $k=-1$, which implies that the big-bang singularity does not exist even in the case where the free parameter R_0 is set equal to zero.

As mentioned above the line elements (33) and (36) describe the effective universes the low-energy physicists can observe, and we see that those

[4]But reader can easily check that the line element (33) reduces to (3) for $f_0 \to -k$ (or $\alpha \to 0$).

universes are entirely distinct from one another; one is static, another is evolving. From this one can deduce an important result that two universes mutually gauge equivalent at the level of full theory retaining all $n \neq 0$ modes can appear to be totally different universes to the low-energy physicists. In other words, an evolving universe observed at low energy level may be a different expression of the static universe with non-zero gauge potential. That is, choosing a different gauge corresponds to having a different universe with a different time evolution and a different gauge potential. For the universe described by (33), the 'time' comes into play in the gauge $A_0 = 0$. This is a mystery. Why has a certain observed universe had to choose the corresponding particular gauge? Or, what has made it choose that particular gauge? Unfortunately, we do not know the answer. The only thing we can say is that the result obtained in this paper is clearly implicated in the existence of the Kaluza-Klein excitations, and can not be deduced without considering x^5-dependent solutions.

References

[1] D. J. Gross and M. J. Perry, Nucl. Phys. **B 226** (1983) 29.

[2] K. Behrndt and S. Förste, Nucl. Phys. **B 430** (1994) 441; Phys. Lett. **B 320** (1994) 253; F. Larsen and F.Wilczek, Phys. Rev. **D 55** (1997) 4591.

[3] G. W. Gibbons, G. T. Horowitz and P. K. Townsend, Class. Quantum Grav. **12** (1995) 297.

[4] N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B 429** (1998) 263, I. Antoniadis, N. Arkani-Hamed, S. Dimopoulos and G. Dvali, Phys. Lett. **B 436** (1998) 257, R. Sundrum, Phys. Rev. **D59** (1999) 0805009; Phys. Rev. **D 59** (1999) 085010.

[5] L.Randall and R. Sundrumm, Phys. Rev. Lett. **83** (1999) 4690.

[6] S. B. Giddings, E. Katz and L. Randall, JHEP **03** (2000) 023.

[7] R. A. Matzner and A. Mezzacappa, Phys. Rev. **D 32** (1985) 3114.

[8] H. Stephani, General Relativity:An introduction to the theory of the gravitational field 2nd edition, editied by J. Stewart (Cambridge University Press, Cambridge, 1990).

[9] See for instance M. Carmeli, Classical Fields: General Relativity and Gauge theory (Wiley and Sons, New York,1982).

[10] I. Bars and C. Kounnas, Phys. Lett. **B 402** (1997) 25; I. Bars, Phys. Rev. **D 59** (1999) 045019.

[11] A. Einstein and P. Bergmann, Ann. of Math. **39** (1938) 683.