DIRAC-KÄHLER EQUATION[1]

S.I. Kruglov [2]

International Educational Centre, 2727 Steeles Ave. W, # 202, Toronto, Ontario, Canada M3J 3G9

Abstract

Tensor, matrix and quaternion formulations of Dirac-Kähler equation for massive and massless fields are considered. The equation matrices obtained are simple linear combinations of matrix elements in the 16-dimensional space. The projection matrix-dyads defining all the 16 independent equation solutions are found. A method of computing the traces of 16-dimensional Petiau-Duffin-Kemmer matrix product is considered. We show that the symmetry group of the Dirac-Kähler tensor fields for charged particles is $SO(4,2)$. The conservation currents corresponding this symmetry are constructed. We analyze transformations of the Lorentz group and quaternion fields. Supersymmetry of the Dirac-Kähler fields with tensor and spinor parameters is investigated. We show the possibility of constructing a gauge model of interacting Dirac-Kähler fields where the gauge group is the noncompact group under consideration.

1 Introduction

The important problems of particle physics are the confinement of quarks and the chiral symmetry breaking (CSB) [1]. Both problems can not be solved within perturbative quantum chromodynamics (QCD).

One of the promising methods in the infra-red limit of QCD is lattice QCD. Lattice QCD takes into account both nonperturbative effects - CSB and the confinement of quarks, and provides computational hadronic characteristics with good accuracy. A natural framework of the lattice fermion formulation and some version of Kogut-Suskind fermions [2, 3] are Dirac-Kähler fermions [4-11]. The interest in this theory is due to the possibility of applying the Dirac-Kähler equation for describing fermion fields with spin $1/2$ on the lattice [4-11].

Recently much attention has been paid to the study of the Dirac-Kähler field in the framework of differential forms [11-24]. The author [12] considered an equation for inhomogeneous differential forms what is equivalent to

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[2]E-mail: krouglov@sprint.ca

introducing a set of antisymmetric tensor fields of arbitrary rank. It implies the simultaneous consideration of fields with different spins.

Kähler [12] showed that the Dirac equation for particles with spin $1/2$ can be constructed from inhomogeneous differential forms. Now such fields are called Dirac-Kähler fields. Using the language of differential forms, Dirac-Kähler's equation in four dimensional space-time is given by

$$(d - \delta + m)\,\Phi = 0, \tag{1}$$

where d denotes the exterior derivative, $\delta = - \star d \star$ turns $n-$forms into $(n-1)-$form; $\star$ is the Hodge operator which connects a $n-$form to a $(4-n)-$form so, that $\star^2 = 1$, $d^2 = \delta^2 = 0$. The Laplacian is given by

$$(d - \delta)^2 = -\left(d\delta + \delta d\right) = \partial_\mu \partial^\mu .$$

So, the operator $(d - \delta)$ is the analog of the Dirac operator $\gamma_\mu \partial_\mu$. The inhomogeneous differential form Φ can be expanded as

$$\begin{gathered}\Phi = \varphi(x) + \varphi_\mu(x) dx^\mu + \frac{1}{2!}\varphi_{\mu\nu}(x) dx^\mu \wedge dx^\nu + \\ + \frac{1}{3!}\varphi_{\mu\nu\rho}(x) dx^\mu \wedge dx^\nu \wedge dx^\rho + \frac{1}{4!}\varphi_{\mu\nu\rho\sigma}(x) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma ,\end{gathered} \tag{2}$$

where $\wedge$ is the exterior product. The form Φ includes scalar $\varphi(x)$, vector $\varphi_\mu(x)$, and antisymmetric tensor fields $\varphi_{\mu\nu}(x)$, $\varphi_{\mu\nu\rho}(x)$, $\varphi_{\mu\nu\rho\sigma}(x)$. The antisymmetric tensors of the third and forth ranks $\varphi_{\mu\nu\rho}(x)$, $\varphi_{\mu\nu\rho\sigma}(x)$ define a pseudovector and pseudoscalar, respectively:

$$\widetilde{\varphi}_\mu(x) = \frac{1}{3!}\epsilon_\mu^{\ \nu\rho\sigma}\varphi_{\nu\rho\sigma}(x), \qquad \widetilde{\varphi}(x) = \frac{1}{4!}\epsilon^{\mu\nu\rho\sigma}\varphi_{\mu\nu\rho\sigma}(x), \tag{3}$$

where $\epsilon^{\mu\nu\rho\sigma}$ is an antisymmetric tensor Levy-Civita. In fact, the Dirac-Kähler equation (1) describes scalar, vector, pseudoscalar and pseudovector fields. Some authors (see e. g. [6, 18-21]) showed that Eq. (1) is equivalent to four Dirac equations

$$\left(\gamma_\mu \partial_\mu + m\right) \psi^{(b)}(x) = 0, \qquad b = 1, 2, 3, 4. \tag{4}$$

The mapping between equations (1) and (4) makes it possible to describe fermions with spin $1/2$ with the help of Eq. (1), i.e., boson fields ! As we have already mentioned, this possibility is used in the lattice formulation of QCD and for describing fermions with spin-$1/2$.

It should be noted that Ivanenko and Landau [25] considered (in 1928) an equation for the set of antisymmetric tensor fields which is equivalent to the Dirac-Kähler equation (1). Similar equations were discussed after [26-32] long before the appearance of [12], and later [33-42]. The author [34, 37] found the internal symmetry group $SO(4,2)$ (or locally isomorphic group $SU(2,2)$) of the Dirac-Kähler action and the corresponding conserving currents. The Lorentz covariance of the Dirac-Kähler equation are also shown. The transformations of the $SO(4,2)$ group mix the fields with different spins and do not commute with the Lorentz transformations. Later others [36, 39, 18] also paid attention to this symmetry. The transformations of the Lorentz and internal symmetry groups discussed do not commute each other. So, parameters of the group are tensors but not scalars as in (the more common) gauge theories. This kind of symmetry is also different from supersymmetry where group parameters are spinors. The difference is that in our case the algebra of generators of the symmetry is closed without adding the generators of the Poincaré group. However the indefinite metric should be introduced here. The localization of parameters of the internal symmetry group leads to the gauge fields and field interactions.

The paper is organized as follows. In Section 2 we investigate the tensor and matrix formulations of the Dirac-Kähler equation for massive and massless fields. The tensor and spinor representations of the Lorentz group are analized. All independent solutions of equations are found in the form of matrix-dyads. It is shown in Section 3 that the internal symmetry group of the Dirac-Kähler tensor fields is $SO(4,2)$. For the case of spinor fields we come to the $U(4)$-group of symmetry. In Section 4 within the framework of the quaternion approach, the six-parameter internal-symmetry subgroup and the Lorentz covariance of the Dirac-Kähler equation are considered.

The quantization of Dirac-Kähler's fields is carried out in Section 5 by using an indefinite metric. It is shown in Section 6 that in the field theory including Dirac-Kähler fields it is possible to analyze supersymmetry groups with tensor and spinor parameters without including coordinate transformations at the same time. We show in Section 7 the possibility of constructing a gauge model of interacting Dirac-Kähler fields where the gauge group is the noncompact group $SO(4,2)$ under consideration. Section 8 contains a conclusion. A method of computing the traces of $16-$dimensional Petiau-Duffin-Kemmer matrix products is considered in Appendix A. Appendix B devoted to the Lorentz transformations and the quaternion algebra.

We use the system of units $\hbar = c = 1$, $\alpha = e^2/4\pi = 1/137$, $e > 0$, and Euclidean metrics, so that the squared four-vector is $v_\mu^2 = \mathbf{v}^2 + v_4^2 = \mathbf{v}^2 - v_0^2$ ($\mathbf{v}^2 = v_1^2 + v_2^2 + v_3^2$, $v_4 = iv_0$).

2 Tensor and matrix formulations of Dirac-Kähler equation

It is easy to show that Dirac-Kähler equation (1) with definitions (2), (3) is equivalent to the following tensor equations

$$\partial_\nu \varphi_{\mu\nu} - \partial_\mu \varphi + m^2 \varphi_\mu = 0, \qquad \partial_\nu \widetilde{\varphi}_{\mu\nu} - \partial_\mu \widetilde{\varphi} + m^2 \widetilde{\varphi}_\mu = 0, \tag{5}$$

$$\partial_\mu \varphi_\mu - \varphi = 0, \qquad \partial_\mu \widetilde{\varphi}_\mu - \widetilde{\varphi} = 0, \tag{6}$$

$$\varphi_{\mu\nu} = \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu - \varepsilon_{\mu\nu\alpha\beta} \partial_\alpha \widetilde{\varphi}_\beta, \tag{7}$$

where

$$\widetilde{\varphi}_{\mu\nu} = \frac{1}{2} \varepsilon_{\mu\nu\alpha\beta} \varphi_{\alpha\beta} \tag{8}$$

is the dual tensor, $\varepsilon_{\mu\nu\alpha\beta}$ is an antisymmetric tensor Levy-Civita; $\varepsilon_{1234} = -i$. It should be noted that Eq. (7) is the most general representation for the antisymmetric tensor of second rank in accordance with the Hodge theorem [43] (see also [44-46]).

If the φ, $\widetilde{\varphi}$, φ_μ, $\widetilde{\varphi}_\mu$ and $\varphi_{\mu\nu}$ are complex values, Eqs. (5)-(7) describe the charged vector fields. These equations are the tensor form of the Dirac-Kähler equation (1) which was written in differential forms [12].

Now we show that in matrix form equations (5)-(7) can be represented as the Dirac-like equation with 16×16 -dimensional Dirac matrices. The projection matrix-dyads defining all the 16 independent equation solutions will be constructed [35].

The matrix form facilitates the investigation of the general group of internal symmetry. To obtain the matrix form for both the massive and massless cases, generalized equations are introduced:

$$\begin{gathered}
\partial_\mu \widetilde{\psi}_\mu + m_2 \widetilde{\psi}_0 = 0, \\
\partial_\nu \psi_{[\mu\nu]} + \partial_\mu \psi_0 + m_1 \psi_\mu = 0, \\
\partial_\nu \psi_\mu - \partial_\mu \psi_\nu - e_{\mu\nu\alpha\beta} \partial_\alpha \widetilde{\psi}_\beta + m_2 \psi_{[\mu\nu]} = 0, \\
\partial_\mu \psi_\mu + m_2 \psi_0 = 0.
\end{gathered} \tag{9}$$

With $m_1 = m_2 = m$, $\psi_0 = -\varphi$, $\psi_\mu = m\varphi_\mu$, $\psi_{[\mu\nu]} = \varphi_{\mu\nu}$, $\widetilde{\psi}_\mu = im\widetilde{\varphi}_\mu$, $\widetilde{\psi}_0 = -i\widetilde{\varphi}$, $e_{\mu\nu\alpha\beta} = i\varepsilon_{\mu\nu\alpha\beta}$ ($e_{1234} = 1$), we arrive at the Dirac-Kähler equations (5)-(7). In the case $m_1 = 0$, $m_2 \neq 0$ (where m_2 is the dimension parameter), Eqs. (9) are the generalized Maxwell equations in the dual-symmetric form [47]. Let us introduce the $16-$component wave function

$$\Psi(x)=\{\psi_A\}\,,\qquad\qquad A=0,\mu,[\mu\nu],\widetilde{\mu},\widetilde{0}, \tag{10}$$

where $\psi_{\widetilde{\mu}}\equiv\widetilde{\psi}_\mu$, $\psi_{\widetilde{0}}\equiv\widetilde{\psi}_0$. It is convenient to introduce the matrix $\varepsilon^{A,B}$ [48] with dimension 16×16; its elements consist of zeroes and only one element is unity where row A and column B cross. Thus the multiplication and matrix elements of these matrices are

$$\varepsilon^{A,B}\varepsilon^{C,D}=\varepsilon^{A,D}\delta_{BC},\qquad\qquad \left(\varepsilon^{A,B}\right)_{CD}=\delta_{AC}\delta_{BD}, \tag{11}$$

where indexes $A,B,C,D=1,2,...,16$. Using the elements of the entire algebra $\varepsilon^{A,B}$, equations (9) take the form

$$\Bigg\{\partial_\nu\Bigg[\varepsilon^{\mu,[\mu\nu]}+\varepsilon^{[\mu\nu],\mu}+\varepsilon^{\nu,0}+\varepsilon^{0,\nu}+\varepsilon^{\widetilde{\nu},\widetilde{0}}+\varepsilon^{\widetilde{0},\widetilde{\nu}}+$$
$$+\frac{1}{2}e_{\mu\nu\rho\omega}\left(\varepsilon^{\widetilde{\mu},[\rho\omega]}+\varepsilon^{[\rho\omega],\widetilde{\mu}}\right)\Bigg]_{AB}+ \tag{12}$$
$$+\left[m_1\left(\varepsilon^{\mu,\mu}+\varepsilon^{\widetilde{\mu},\widetilde{\mu}}\right)+m_2\left(\varepsilon^{0,0}+\frac{1}{2}\varepsilon^{[\mu\nu],[\mu\nu]}+\varepsilon^{\widetilde{0},\widetilde{0}}\right)\right]_{AB}\Bigg\}\Psi_B(x)=0.$$

Let us introduce the projection matrices

$$\overline{P}=\varepsilon^{\mu,\mu}+\varepsilon^{\widetilde{\mu},\widetilde{\mu}},\qquad\qquad P=\varepsilon^{0,0}+\frac{1}{2}\varepsilon^{[\mu\nu],[\mu\nu]}+\varepsilon^{\widetilde{0},\widetilde{0}} \tag{13}$$

with the properties $P\overline{P}=\overline{P}P=0$, $P+\overline{P}=I_{16}$; the I_{16} is the unit $16\times16-$matrix and

$$\Gamma_\nu=\varepsilon^{\mu,[\mu\nu]}+\varepsilon^{[\mu\nu],\mu}+\varepsilon^{\nu,0}+\varepsilon^{o,\nu}+\varepsilon^{\widetilde{\nu},\widetilde{0}}+\varepsilon^{\widetilde{0},\widetilde{\nu}}+\frac{1}{2}e_{\mu\nu\rho\omega}\left(\varepsilon^{\widetilde{\mu},[\rho\omega]}+\varepsilon^{[\rho\omega],\widetilde{\mu}}\right). \tag{14}$$

Then Eq. (12) takes the form of the relativistic wave equation

$$\left(\Gamma_\nu\partial_\nu+m_1\overline{P}+m_2P\right)\Psi(x)=0 \tag{15}$$

which includes both the massive and massless cases. The $16\times16-$matrix Γ_ν can be represented in the form

$$\Gamma_\nu=\beta_\nu^{(+)}+\beta_\nu^{(-)},\qquad \beta_\nu^{(+)}=\beta_\nu^{(1)}+\beta_\nu^{(\widetilde{0})},\qquad \beta_\nu^{(-)}=\beta_\nu^{(\widetilde{1})}+\beta_\nu^{(0)},$$
$$\beta_\nu^{(1)}=\varepsilon^{\mu,[\mu\nu]}+\varepsilon^{[\mu\nu],\mu},\qquad \beta_\nu^{(\widetilde{1})}=\frac{1}{2}e_{\mu\nu\rho\omega}\left(\varepsilon^{\widetilde{\mu},[\rho\omega]}+\varepsilon^{[\rho\omega],\widetilde{\mu}}\right), \tag{16}$$
$$\beta_\nu^{(\widetilde{0})}=\varepsilon^{\widetilde{\nu},\widetilde{0}}+\varepsilon^{\widetilde{0},\widetilde{\nu}},\qquad \beta_\nu^{(0)}=\varepsilon^{\nu,0}+\varepsilon^{0,\nu}.$$

Matrices $\beta_\nu^{(1)}$, $\beta_\nu^{(\widetilde{1})}$ and $\beta_\nu^{(0)}$, $\beta_\nu^{(\widetilde{0})}$ realize $10-$ and $5-$ dimensional irreducible representations of the Petiau-Duffin-Kemmer [49-51] algebra

$$\beta_\mu^{(1,0)}\beta_\nu^{(1,0)}\beta_\alpha^{(1,0)} + \beta_\alpha^{(1,0)}\beta_\nu^{(1,0)}\beta_\mu^{(1,0)} = \delta_{\mu\nu}\beta_\alpha^{(1,0)} + \delta_{\alpha\nu}\beta_\mu^{(1,0)} \tag{17}$$

and $\beta_\nu^{(+)}$, $\beta_\nu^{(-)}$ are $16-$dimensional reducible representations of the Petiau-Duffin-Kemmer algebra [28-31]. These matrices obey the Petiau-Duffin-Kemmer algebra (17) and the matrix Γ_ν is a $16 \times 16-$Dirac matrix with the algebra:

$$\Gamma_\nu\Gamma_\mu + \Gamma_\mu\Gamma_\nu = 2\delta_{\mu\nu}. \tag{18}$$

For the massive case when $m_1 = m_2 = m$, Eq. (15) becomes

$$\left(\Gamma_\nu\partial_\nu + m\right)\Psi(x) = 0. \tag{19}$$

The $16-$component wave equation in the form of the first-order Eq. (19) was also studied in [33]. Now we find all independent solutions of Eq. (19) in the form of matrix-dyads. In the momentum space Eq. (19) becomes

$$-i\widehat{p}\Psi_p = \varepsilon m\Psi_p, \tag{20}$$

where $\widehat{p} = \Gamma_\nu p_\nu$, and parameter $\varepsilon = \pm 1$ corresponds to two values of the energy. From the property of the Dirac matrices, Eq. (18), we find the minimal equation for the operator $\widehat{p}$:

$$\left(i\widehat{p} + m\right)\left(i\widehat{p} - m\right) = 0. \tag{21}$$

According to the general method [52, 53], the projection operator extracting the states with definite energy (for particle or antiparticle) is given by

$$M_\varepsilon = \frac{m - i\varepsilon\widehat{p}}{2m}. \tag{22}$$

This operator has virtually the same form as in the Dirac theory of particles with spin $1/2$. This is because the algebra of the matrices (18) coincides with the algebra of the Dirac matrices γ_μ. However, here we have the wave function $\Psi(x)$ which is transformed in the tensor representation of the Lorentz group. It is also possible to use equation (19) to describe spinor particles. In this case the wave function $\Psi(x)$ will be a spinor representation of the Lorentz group and equation (19) is the direct sum of four Dirac equations (see (4)). This case is used for fermions on the lattice [4-11].

Now we consider the bosonic case. The generators of the Lorentz group representation in the $16-$dimensional space of the wave functions $\Psi(x)$ are given by (see [28-31, 33])

$$J_{\mu\nu} = \frac{1}{4}\left(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu + \overline{\Gamma}_\mu\overline{\Gamma}_\nu - \overline{\Gamma}_\nu\overline{\Gamma}_\mu\right), \tag{23}$$

where the matrices $\overline{\Gamma}_\nu$ also obey the Dirac algebra (18) and have the form (see (16)):

$$\overline{\Gamma}_\nu = \beta_\nu^{(+)} - \beta_\nu^{(-)}. \tag{24}$$

It may be verified that the matrices Γ_μ and $\overline{\Gamma}_\nu$ commute each other, i.e.

$$\left[\Gamma_\mu, \overline{\Gamma}_\nu\right] = 0. \tag{25}$$

The spin projection operator here is given by

$$\sigma_p = -\frac{i}{2 \mid \mathbf{p} \mid}\epsilon_{abc}p_a J_{bc} = -\frac{i}{4 \mid \mathbf{p} \mid}\epsilon_{abc}p_a \left(\Gamma_b\Gamma_c + \overline{\Gamma}_b\overline{\Gamma}_c\right) \tag{26}$$

which satisfies the following equation

$$\sigma_p \left(\sigma_p - 1\right)\left(\sigma_p + 1\right) = 0. \tag{27}$$

In accordance with [52, 53] the corresponding projection operators are given by

$$\widehat{S}_{(\pm 1)} = \frac{1}{2}\sigma_p \left(\sigma_p \pm 1\right), \qquad \widehat{S}_{(0)} = 1 - \sigma_p^2. \tag{28}$$

Operators $\widehat{S}_{(\pm 1)}$ correspond to the spin projections $s_p = \pm 1$ and $\widehat{S}_{(0)}$ to $s_p = 0$. It is easy to verify that the required commutation relations hold: $\widehat{S}^2_{(\pm 1)} = \widehat{S}_{(\pm 1)}$, $\widehat{S}_{(\pm 1)}\widehat{S}_{(0)} = 0$, $\widehat{S}^2_{(0)} = \widehat{S}_{(0)}$. The squared Pauli-Lubanski vector σ^2 is given by

$$\sigma^2 = \left(\frac{1}{2m}\varepsilon_{\mu\nu\alpha\beta}p_\nu J_{\alpha\beta}\right)^2 = \frac{1}{m^2}\left(J^2_{\mu\nu}p^2 - J_{\mu\sigma}J_{\nu\sigma}p_\mu p_\nu\right). \tag{29}$$

It may be verified that this operator obeys the minimal equation

$$\sigma^2\left(\sigma^2 - 2\right) = 0, \tag{30}$$

so that eigenvalues of the squared spin operator σ^2 are $s(s+1) = 0$ and $s(s+1) = 2$. This confirms that the considered fields describe the superposition of two spins $s = 0$ and $s = 1$. To separate these states we use the projection operators

$$S^2_{(0)} = 1 - \frac{\sigma^2}{2}, \qquad S^2_{(1)} = \frac{\sigma^2}{2} \tag{31}$$

having the properties $S^2_{(0)}S^2_{(1)} = 0$, $\left(S^2_{(0)}\right)^2 = S^2_{(0)}$, $\left(S^2_{(1)}\right)^2 = S^2_{(1)}$, $S^2_{(0)} + S^2_{(1)} = 1$, where $1 \equiv I_{16}$ is the unit matrix in $16-$ dimensional space. In accordance with the general properties of the projection operators, the matrices

$S^2_{(0)}$, $S^2_{(1)}$ acting on the wave function extract pure states with spin 0 and 1, respectively. Here there is a doubling of the spin states of fields because we have scalar ψ_0, pseudoscalar $\widetilde{\psi}_0$, vector ψ_μ, and pseudovector $\widetilde{\psi}_\mu$, fields. To separate these states it is necessary to introduce additional projection operators. We use the following projection operator

$$\overline{M}_{\bar{\varepsilon}} = \frac{m - i\bar{\varepsilon}\overline{p}}{2m} \tag{32}$$

which has the same structure as Eq. (22) but with the matrix $\overline{p} = \overline{\Gamma}_\nu p_\nu$ and an additional quantum number $\bar{\varepsilon} = \pm 1$. Following the procedure [52, 53], 16 independent solutions in the form of projection matrix-dyads are given by

$$\Delta_{\varepsilon,\pm 1,\bar{\varepsilon}} = \frac{\sigma^2}{2} \cdot \frac{m - i\varepsilon\widehat{p}}{2m} \cdot \frac{m - i\bar{\varepsilon}\overline{p}}{2m} \cdot \frac{1}{2}\sigma_p\left(\sigma_p \pm 1\right) = \Psi_{\varepsilon,\pm 1,\bar{\varepsilon}} \cdot \overline{\Psi}_{\varepsilon,\pm 1,\bar{\varepsilon}},$$

$$\Delta^{(1)}_{\varepsilon,\bar{\varepsilon}} = \frac{\sigma^2}{2} \cdot \frac{m - i\varepsilon\widehat{p}}{2m} \cdot \frac{m - i\bar{\varepsilon}\overline{p}}{2m} \cdot \left(1 - \sigma_p^2\right) = \Psi_{\varepsilon,\bar{\varepsilon}} \cdot \overline{\Psi}_{\varepsilon,\bar{\varepsilon}}, \tag{33}$$

$$\Delta^{(0)}_{\varepsilon,\bar{\varepsilon}} = \left(1 - \frac{\sigma^2}{2}\right) \cdot \frac{m - i\varepsilon\widehat{p}}{2m} \cdot \frac{m - i\bar{\varepsilon}\overline{p}}{2m} \cdot \left(1 - \sigma_p^2\right) = \Psi^{(0)}_{\varepsilon,,\bar{\varepsilon}} \cdot \overline{\Psi}^{(0)}_{\varepsilon,\bar{\varepsilon}},$$

where operators $\Delta_{\varepsilon,\pm 1,\bar{\varepsilon}}$, $\Delta^{(1)}_{\varepsilon,\bar{\varepsilon}}$ correspond to states with spin 1 and spin projections ± 1 and 0, respectively and the projection operator $\Delta^{(0)}_{\varepsilon,\bar{\varepsilon}}$ extracts spin 0. The wave function Ψ_p ($\Psi_{\varepsilon,\pm 1,\bar{\varepsilon}}$ or $\Psi_{\varepsilon,\bar{\varepsilon}}$ or $\Psi^{(0)}_{\varepsilon,,\bar{\varepsilon}}$) is the eigenvector of the equations

$$\begin{aligned} -i\widehat{p}\Psi_p &= \varepsilon m\Psi_p, & -i\overline{p}\Psi_p &= \bar{\varepsilon} m\Psi_p, \\ \sigma_p\Psi_p &= s_p\Psi_p, & \sigma^2\Psi_p &= s(s+1)\Psi_p, \end{aligned} \tag{34}$$

where the spin projections are $s_p = \pm 1, 0$ and the spin is $s = 1, 0$. The Hermitianizing matrix, η, in $16-$dimensional space is given by:

$$\eta = \Gamma_4\overline{\Gamma}_4. \tag{35}$$

This matrix obeys the equations

$$\eta\Gamma_i = -\Gamma_i\eta, \qquad (i = 1, 2, 3), \qquad \eta\Gamma_4 = \Gamma_4\eta$$

which guarantee the existence of a relativistically invariant bilinear form [48, 53]

$$\overline{\Psi}\Psi = \Psi^+\eta\Psi, \tag{36}$$

where $\overline{\Psi}_p = \Psi^+\Gamma_4\overline{\Gamma}_4$, and Ψ^+ is the Hermitian-conjugate wave function.

In the spinor case, when Eq. (12) is the direct sum of four Dirac equations, generators of the Lorentz group in $16-$dimensional space are

$$J_{\mu\nu}^{(1/2)}=\frac{1}{4}\left(\Gamma_\mu\Gamma_\nu-\Gamma_\nu\Gamma_\mu\right), \tag{37}$$

and the Hermitianizing matrix is $\eta_{1/2}=\Gamma_4$. Using the unitary transformation we can find the representation $\Gamma'_\mu=I_4\otimes\gamma_\mu$, where I_4 is $4\times 4-$unit matrix, γ_μ are the Dirac matrices and $\otimes$ means direct product. In this basis the matrices $\overline{\Gamma}_\mu$ become $\overline{\Gamma}'_\mu=\gamma_\mu\otimes I_4$.

It is convenient also to use equations

$$(m-i\varepsilon\widehat{p})\left(m-i\overline{\varepsilon p}\right)=2ip^{(\pm)}\left(ip^{(\pm)}-\varepsilon m\right), \tag{38}$$

$$\sigma_p=\sigma_p^{(+)}=-\frac{i}{\mid\mathbf{p}\mid}\epsilon_{abc}p_a\beta_b^{(+)}\beta_c^{(+)}=\sigma_p^{(-)}=-\frac{i}{\mid\mathbf{p}\mid}\epsilon_{abc}p_a\beta_b^{(-)}\beta_c^{(-)}, \tag{39}$$

where the sign ($+$) in Eq. (38) corresponds to the equality $\varepsilon=\overline{\varepsilon}$, and sign ($-$) to $\varepsilon=-\overline{\varepsilon}$. With the help of Eqs. (38), (39), the projection operators (33) are rewritten as

$$\Delta_{\varepsilon,\pm1,\overline{\varepsilon}}=\frac{\sigma^2}{8m^2}ip^{(\pm)}\left(ip^{(\pm)}-\varepsilon m\right)\sigma_p^{(\pm)}\left(\sigma_p^{(\pm)}\pm1\right)=\Psi_{\varepsilon,\pm1}^{(\pm)}\cdot\overline{\Psi}_{\varepsilon,\pm1}^{(\pm)},$$

$$\Delta_{\varepsilon,\overline{\varepsilon}}^{(1)}=\frac{\sigma^2}{4m^2}ip^{(\pm)}\left(ip^{(\pm)}-\varepsilon m\right)\left(1-\sigma_p^{(\pm)2}\right)=\Psi_{\varepsilon,\overline{\varepsilon}}^{(\pm)}\cdot\overline{\Psi}_{\varepsilon,\overline{\varepsilon}}^{(\pm)}, \tag{40}$$

$$\Delta_{\varepsilon,\overline{\varepsilon}}^{(0)}=\frac{1}{2m^2}\left(1-\frac{\sigma^2}{2}\right)ip^{(\pm)}\left(ip^{(\pm)}-\varepsilon m\right)\left(1-\sigma_p^{(\pm)2}\right)=\Psi_{\varepsilon,,\overline{\varepsilon}}^{(\pm)}\cdot\overline{\Psi}_{\varepsilon,\overline{\varepsilon}}^{(\pm)},$$

where $p^{(\pm)}=p_\mu\beta_\mu^{(\pm)}$. Projection matrix-dyads (40) extract solutions $\Psi_p^{(\pm)}$ which are the solutions of the equations

$$-ip^{(+)}\Psi_p^{(+)}=\frac{1}{2}\left(\varepsilon+\overline{\varepsilon}\right)m\Psi_p^{(+)}, \tag{41}$$

$$-ip^{(-)}\Psi_p^{(-)}=\frac{1}{2}\left(\varepsilon-\overline{\varepsilon}\right)m\Psi_p^{(-)}, \tag{42}$$

$$\sigma_p^{(\pm)}\Psi_p^{(\pm)}=s_p\Psi_p^{(\pm)},\qquad\qquad \sigma^2\Psi_p^{(\pm)}=s(s+1)\Psi_p^{(\pm)}. \tag{43}$$

For the bosonic case with $\varepsilon=\overline{\varepsilon}$, Eq. (41) describes the superposition of vector and pseudoscalar fields, and Eq. (42) with $\varepsilon=-\overline{\varepsilon}$ describes the superposition of pseudovector and scalar fields (see [28-31]).

Let us investigate the case of massless fields; Eq. (15) with $m_1=0$ becomes

$$\left(\Gamma_\nu\partial_\nu+m_2P\right)\Psi(x)=0. \tag{44}$$

In the momentum space, the field function Ψ_k is the solution to the equation

$$B\Psi_k = 0, \qquad B = i\widehat{k} + m_2 P, \tag{45}$$

where $\widehat{k} = \Gamma_\mu k_\mu$, $k_\mu^2 = 0$ and the matrix B obeys the minimal equation

$$B\left(B - m_2\right) = 0. \tag{46}$$

The projection operator which extracts the solution to Eq. (45) is

$$\alpha = \frac{m_2 - B}{m_2} \tag{47}$$

with the equality $\alpha^2 = \alpha$ required by a projection operator. The corresponding spin operators are given by

$$\sigma_k = -\frac{i}{k_0}\epsilon_{abc}k_a\beta_b^{(\pm)}\beta_c^{(\pm)}. \tag{48}$$

We have mentioned that the theory under consideration involves the doubling of the spin states of particles, because there are vector and pseudovector fields. To separate these states in the massless case we can use the following projection operators

$$\Lambda_\epsilon = \frac{1}{2}\left(1 + \epsilon\overline{\Gamma}_5\right),$$

where $\overline{\Gamma}_5 = \overline{\Gamma}_1\overline{\Gamma}_2\overline{\Gamma}_3\overline{\Gamma}_4$, $\epsilon = \pm 1$. The matrix Λ_ϵ commutes with the matrix B of Eq. (45) and with spin operators (48), and possesses the required relation $\Lambda_\epsilon^2 = \Lambda_\epsilon$. As a result, the projection matrix-dyads, corresponding to the generalized Maxwell field after extracting spin 0 and spin projections ± 1, take the form

$$\begin{aligned}\Pi_\epsilon^{(0)} &= \frac{1}{m_2}\left(1 - \frac{\sigma^2}{2}\right)\left(m_2 - B\right)\Lambda_\epsilon = \Psi_\epsilon^{(0)} \cdot \overline{\Psi}_\epsilon^{(0)},\\ \Pi_\epsilon^{(\pm 1)} &= \frac{1}{2m_2}\sigma_k\left(\sigma_k \pm 1\right)\left(m_2 - B\right)\Lambda_\epsilon = \Psi_\epsilon^{(\pm)} \cdot \overline{\Psi}_\epsilon^{(\pm)}.\end{aligned} \tag{49}$$

Let us consider the case of spinor particles when the wave function $\Psi(x)$ realizes the spinor representation of the Lorentz group with the generators (37) and Hermitianizing matrix $\eta_{1/2} = \Gamma_4$. In this case the wave function $\Psi(x)$ represents the direct sum of four bispinors and the variables ψ_0, ψ_μ, $\psi_{[\mu\nu]}$, $\widetilde{\psi}_\mu$, $\widetilde{\psi}_0$, which comprise $\Psi(x)$ (10), are connected with components of spinors. Under the Lorentz transformations with generators (37) these variables do not transform as tensors. Thus the equations for the eigenvalues and the spin operator are:

$$- i\widehat{p}\Psi_p^{(1/2)} = \varepsilon m \Psi_p^{(1/2)}, \qquad \sigma_p^{(1/2)}\Psi_p^{(1/2)} = s_p \Psi_p^{(1/2)},$$
$$\sigma_p^{(1/2)} = -\frac{i}{4 \mid \mathbf{p} \mid}\epsilon_{abc} p_a \Gamma_b \Gamma_c, \tag{50}$$

where $s_p = \pm 1/2$. As there is a degeneracy of states due to the 16 -dimensionality we should use additional equations with the corresponding quantum numbers. Taking into account Eq. (25) we can use the following additional equations to separate states of spinor fields:

$$- i\overline{p}\Psi_p^{(1/2)} = \overline{\varepsilon} m \Psi_p^{(1/2)}, \qquad \overline{\sigma}_p^{(1/2)}\Psi_p^{(1/2)} = \overline{s}_p \Psi_p^{(1/2)},$$
$$\overline{\sigma}_p^{(1/2)} = -\frac{i}{4 \mid \mathbf{p} \mid}\epsilon_{abc} p_a \overline{\Gamma}_b \overline{\Gamma}_c \tag{51}$$

with $\overline{s}_p = \pm 1/2$. We can treat the additional quantum number $\overline{s}_p$ as the "internal spin" because matrices (24) obey the Dirac algebra

$$\overline{\Gamma}_\nu \overline{\Gamma}_\mu + \overline{\Gamma}_\mu \overline{\Gamma}_\nu = 2\delta_{\mu\nu}. \tag{52}$$

Thus it is easy to find all independent solutions of equations (50), (51) in the form of matrix-dyads:

$$\Delta_{\varepsilon,\overline{\varepsilon},s_p,\overline{s}_p} = \frac{m - i\varepsilon\widehat{p}}{2m} \cdot \frac{m - i\overline{\varepsilon}\overline{p}}{2m} \cdot \left(\frac{1}{2} + 2s_p\sigma_p^{(1/2)}\right)\left(\frac{1}{2} + 2\overline{s}_p\overline{\sigma}_p^{(1/2)}\right)$$
$$= \Psi_{\varepsilon,\overline{\varepsilon},s_p,\overline{s}_p} \cdot \overline{\Psi}_{\varepsilon,\overline{\varepsilon},s_p,\overline{s}_p}, \tag{53}$$

where $\overline{\Psi}_{\varepsilon,\overline{\varepsilon},s_p,\overline{s}_p} = \Psi^+_{\varepsilon,,\overline{\varepsilon},s_p,\overline{s}_p}\Gamma_4$, $\varepsilon = \pm 1$, $s_p = \pm 1/2$, and we introduce two additional internal quantum numbers $\overline{\varepsilon} = \pm 1$, $\overline{s}_p = \pm 1/2$. In [33] the author also used a similar construction for the solutions of the field equations but without the dyad representation (53). The dyad representation is essential as all quantum electrodynamic calculations can only be done using matrix-dyads [52, 53]. The necessary method of computing the traces of 16$-$dimensional matrix products is considered in Appendix A.

3 The Lorentz covariance and symmetry group $O(4.2)$ of charged vector fields

Let us consider the Lorentz group transformations of coordinates:

$$x'_\mu = L_{\mu\nu} x'_\nu, \tag{54}$$

where the Lorentz matrix $L=\{L_{\mu\nu}\}$ obeys the equation

$$L_{\mu\alpha}L_{\nu\alpha}=\delta_{\mu\nu}. \tag{55}$$

Under the Lorentz coordinates transformations (54) the wave function (10) transforms as follows

$$\Psi'(x')=T\Psi(x), \tag{56}$$

where $16\times 16-$matrix T realizes the tensor or spinor representations of the Lorentz group. Then the wave equation of the first order (19) is converted into

$$\left(\Gamma_\mu\partial'_\mu+m\right)\Psi'(x')=\left(\Gamma_\mu L_{\mu\nu}\partial_\nu+m\right)T\Psi(x)=0. \tag{57}$$

We took into account that at the Lorentz transformations (54) the derivatives ∂_μ become $\partial'_\mu=L_{\mu\nu}\partial_\nu$. The Lorentz covariance of the Dirac-Kähler equation (57) occures if the equation

$$\Gamma_\mu TL_{\mu\nu}=T\Gamma_\nu \tag{58}$$

is valid. The infinitesimal Lorentz transformations (54) are given by the matrix

$$L_{\mu\nu}=\delta_{\mu\nu}+\varepsilon_{\mu\nu}, \qquad\qquad \varepsilon_{\mu\nu}=-\varepsilon_{\nu\mu}, \tag{59}$$

where six parameters $\varepsilon_{\mu\nu}$ define three rotations and boosts. At the same time the matrix T at the infinitesimal transformations (59) can be written as

$$T=I_{16}+\frac{1}{2}\varepsilon_{\mu\nu}J_{\mu\nu}, \tag{60}$$

where I_{16}, $J_{\mu\nu}$ are the unit matrix and generators of the Lorentz group in $16-$dimensional space, respectively. With the help of equations (59), (60) and using the smallness of parameters $\varepsilon_{\mu\nu}$ we arrive from Eq. (58) at (see [54])

$$\Gamma_\mu J_{\alpha\nu}-J_{\alpha\nu}\Gamma_\mu=\delta_{\alpha\mu}\Gamma_\nu-\delta_{\nu\mu}\Gamma_\alpha. \tag{61}$$

It is easy to verify that generators (23) for bosonic fields and generations (37) for fernionic fields obey Eq. (61). This means that Eq. (19) is covariant $16-$dimensional Dirac-like wave equation which can describe as bosons as fermions. For the bosonic case the wave function $\Psi(x)$ is given by Eq.(10) but for the fermion case it is a direct sum of four bispinors (see Eq. (4)). At the finite Lorentz transformations the wave function transforms according to Eq. (56) with the matrix

$$T=\exp\left(\frac{1}{2}\varepsilon_{\mu\nu}J_{\mu\nu}\right). \tag{62}$$

We will show that the group $SO(4,2)$ is the symmetry group of the Dirac-Kähler charged vector fields. This will be obtained using the Dirac matrix algebra and the minimality of the electromagnetic interaction [37,36].

The interaction with electromagnetic field is introduced by the substitution $\partial_\mu \to D_\mu^{(-)} = \partial_\mu - ieA_\mu$, where A_μ is the vector -potential of the electromagnetic field. Consider the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\left[\overline{\Psi}(x)\left(\Gamma_\mu \overrightarrow{D_\mu^{(-)}} + m\right)\Psi(x) - \overline{\Psi}(x)\left(\Gamma_\mu \overleftarrow{D_\mu^{(+)}} - m\right)\Psi(x)\right] - \frac{1}{4}\mathcal{F}_{\mu\nu}, \tag{63}$$

where $D_\mu^{(+)} = \partial_\mu + ieA_\mu$, $\overline{\Psi} = \Psi^+\Gamma_4\overline{\Gamma}_4$, and the arrows above the $D_\mu^{(\pm)}$ show the direction in which these operators act; $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength tensor of the electromagnetic field. From the variation of the Lagrangian (63) on wave functions Ψ and $\overline{\Psi}$, we find equations for Dirac-Kähler vector fields in the external electromagnetic fields

$$\left(\Gamma_\mu D_\mu^{(-)} + m\right)\Psi(x) = 0, \qquad \overline{\Psi}(x)\left(\Gamma_\mu \overleftarrow{D_\mu^{(+)}} - m\right) = 0. \tag{64}$$

From the variation of Eq. (63) on the vector-potential A_μ we get Maxwell equations, in which the source is the electromagnetic current $J_\mu^{el} = ie\overline{\Psi}\Gamma_\mu\Psi$.

Let us consider the set of 16 linear independent matrices:

$$\begin{gathered} I = I_{16}, \qquad I_\mu = \overline{\Gamma}_\mu, \qquad I_{\mu\nu} = \frac{1}{4}\overline{\Gamma}_{[\mu}\overline{\Gamma}_{\nu]}, \\ \widetilde{I} = \overline{\Gamma}_5 = \overline{\Gamma}_1\overline{\Gamma}_2\overline{\Gamma}_3\overline{\Gamma}_4, \qquad \widetilde{I}_\mu = \overline{\Gamma}_5\overline{\Gamma}_\mu. \end{gathered} \tag{65}$$

They commute with the operators $\left(\Gamma_\mu D_\mu^{(-)} + m\right)$, $\left(\Gamma_\mu \overleftarrow{D_\mu^{(+)}} - m\right)$ of Eqs. (64) and generate the algebra of the symmetry of Eqs. (64). The algebra of the generators (65) is isomorphic to the Clifford algebra with the commutation relations:

$$\begin{gathered} [I_{\alpha\beta}, I_{\mu\nu}] = \delta_{\beta\mu}I_{\alpha\nu} + \delta_{\alpha\nu}I_{\beta\mu} - \delta_{\beta\nu}I_{\alpha\mu} - \delta_{\alpha\mu}I_{\beta\nu}, \\ [I_\mu, I_{\alpha\beta}] = \delta_{\mu\alpha}I_\beta - \delta_{\mu\beta}I_\alpha, \\ \left[\widetilde{I}_\mu, I_{\alpha\beta}\right] = \delta_{\mu\alpha}\widetilde{I}_\beta - \delta_{\mu\beta}\widetilde{I}_\alpha, \\ \left[\widetilde{I}_\mu, I_\nu\right] = 2\widetilde{I}\delta_{\mu\nu}, \qquad [I_\mu, I_\nu] = 4I_{\mu\nu}, \qquad \left[\widetilde{I}_\mu, \widetilde{I}_\nu\right] = -4I_{\mu\nu}, \\ \left[I_\mu, \widetilde{I}\right] = -2\widetilde{I}_\mu, \qquad \left[\widetilde{I}_\mu, \widetilde{I}\right] = -2I_\mu, \qquad \left[I_{\alpha\beta}, \widetilde{I}\right] = 0. \end{gathered} \tag{66}$$

Let us introduce the anti-Hermitian generators [55]:

$$I_0 = iI_{16}, \qquad I_{56} = -I_{65} = \frac{i}{2}\widetilde{I},$$

$$I_{6\mu} = -I_{\mu 6} = \frac{1}{2}\widetilde{I}_\mu, \qquad I_{5\mu} = -I_{\mu 5} = \frac{i}{2}I_\mu. \tag{67}$$

With the help of Eqs. (67), the commutation relations (66) take the form

$$[I_{AB}, I_{CD}] = \delta_{BC}I_{AD} + \delta_{AD}I_{BC} - \delta_{AC}I_{BD} - \delta_{BD}I_{AC},$$
$$[I_{AB}, I_0] = 0, \qquad A, B, C, D, = 1, 2, ..., 6. \tag{68}$$

The algebra (68) corresponds with the direct product of the group of $6-$dimensional rotation $SO(6)$ and the unitary group $U(1)$ (for real group parameters). This group is isomorphic to the $U(4)$ group. The transformations of the corresponding group are given by

$$\Psi'(x) = U\Psi(x),$$
$$U = \exp\left(I\alpha + I_\mu\beta_\mu + I_{\mu\nu}\omega_{\mu\nu} + \widetilde{I}_\mu\delta_\mu + \widetilde{I}\xi\right), \tag{69}$$

where α, β_μ, $\omega_{\mu\nu}$, δ_μ, ξ are the group parameters; if these parameters are complex, we have the $GL(4,c)$ group. For the neutral Dirac-Kähler fields, the transformations (60) should leave real components as real components with the conditions: $\alpha^* = \alpha$, $\beta^*_m = \beta_m$, $\beta^*_4 = -\beta_4$, $\omega^*_{mn} = \omega_{mn}$, $\omega^*_{m4} = -\omega_{m4}$, $\delta^*_m = -\delta_m$, $\delta^*_4 = \delta_4$, $\xi^* = -\xi$ corresponding to the $SO(3,3)\otimes GL(1,R)$ group. Such a contraction of the $GL(4,c)$ group is a consequence of charged fields being described by complex fields having more degrees of freedom.

The requirement that the Lagrangian (63) is invariant under the transformations (69) leads to the constraints: $\alpha^* = -\alpha$, $\beta^*_m = \beta_m$, $\beta^*_4 = -\beta_4$, $\omega^*_{mn} = \omega_{mn}$, $\omega^*_{m4} = -\omega_{m4}$, $\delta^*_m = \delta_m$, $\delta^*_4 = -\delta_4$, $\xi^* = \xi$ which corresponds to contraction of the $SO(4,2) \otimes U(1)$ group with 16 parameters [36,37]. This occurs only for charged Dirac-Kähler fields. The subgroup $U(1)$ is the known group of gauge transformations: $\Psi'(x) = \exp(I\alpha)\Psi(x)$ ($\alpha^* = -\alpha$) which gives the conservation law of four-current. For neutral fields, the group leaving the Lagrangian invariant under the transformation is $SO(3,2)$ with generators $\widetilde{I}_\mu$, $I_{\mu\nu}$ and corresponding parameters $\omega^*_{mn} = \omega_{mn}$, $\omega^*_{m4} = -\omega_{m4}$, $\delta^*_m = -\delta_m$, $\delta^*_4 = \delta_4$. The generators $I_{\mu\nu}$ (see (66)) with parameters $\omega^*_{mn} = \omega_{mn}$, $\omega^*_{m4} = -\omega_{m4}$ correspond to the subgroup $SO(3,1)$. The one-parameter subgroup of the Larmor transformations with the generator $\widetilde{I}$ was mentioned in [28-31]. Only generators of Larmor and gauge transformations commute with the Lorentz group generators (23). This means that in the general case, the transformations of the group with internal symmetry $SO(4,2)$ do not commute with the Lorentz transformations. The transformations of the Lorentz group realize the operation of internal automorphism with respect to the elements of the group considered. As a consequence, the parameters of this group are tensors. This is the main difference between the group being considered and the usual groups of internal symmetry where parameters are scalars.

As the Lagrangian (63) is invariant under the $SO(4,2)\otimes U(1)$ group we find in accordance with Noether's theorem that the variation of the action is

$$\delta S = \int d^4x \partial_\mu \left(\overline{\Psi}(x)\Gamma_\mu \delta\Psi(x) - \delta\overline{\Psi}(x)\Gamma_\mu \Psi(x)\right) = 0. \tag{70}$$

As parameters of transformations (69) are independent we find from (70) the conservation tensors:

$$J_\mu = i\overline{\Psi}(x)\Gamma_\mu\Psi(x), \qquad K_\mu = \overline{\Psi}(x)\Gamma_\mu\overline{\Gamma}_5\Psi(x), \qquad R_{\mu\alpha} = \overline{\Psi}(x)\Gamma_\mu\overline{\Gamma}_5\overline{\Gamma}_\alpha\Psi(x),$$
$$C_{\mu\alpha} = \overline{\Psi}(x)\Gamma_\mu\overline{\Gamma}_\alpha\Psi(x), \qquad \Theta_{\mu[\alpha\beta]} = \overline{\Psi}(x)\Gamma_\mu\overline{\Gamma}_{[\alpha}\overline{\Gamma}_{\beta]}\Psi(x). \tag{71}$$

These conservation currents were also constructed in [28-31] without consideration of the corresponding internal symmetry. Conservation of the currents (71) follows from the symmetry-group $SO(4,2)\otimes U(1)$ of the Lagrangian (63) (Noether's theorem). For the neutral Dirac-Kähler fields, there is a conservation of tensors $C_{\mu\alpha}$ and $\Theta_{\mu[\alpha\beta]}$ corresponding to the symmetry-subgroup $SO(3,2)$. Using the matrices Γ_μ, $\overline{\Gamma}_\mu$ (16), (24) and wave function (10) $\Psi(x)$, $\overline{\Psi}(x) = \Psi^+(x)\Gamma_4\overline{\Gamma}_4$, it is easy to verify that in this case (of neutral fields), the currents J_μ, K_μ and $R_{\mu\alpha}$ are identically zero.

For the spinor case with the generators (37), the Lagrangian (63) with the conjugate function $\overline{\Psi}(x) = \Psi^+(x)\Gamma_4$ is invariant under the $U(4)$ - group transformation (69) with the parameter constraints: $\alpha^* = -\alpha$, $\beta^*_\mu = -\beta_\mu$, $\omega^*_{\mu\nu} = \omega_{\mu\nu}$, $\delta^*_\mu = \delta_\mu$, $\xi^* = -\xi$. In this case the transformation (69) commutes with the Lorentz transformations (see (37). The existence of the additional quantum numbers $\overline{s}_p$ and $\overline{\varepsilon}$ is connected here with the presence of the group $SU(4)$. The subgroup $U(1)$ with the parameter α is the well known group of gauge transformations giving the conservation of the electric current $J_\mu = i\overline{\Psi}(x)\Gamma_\mu\Psi(x)$.

4 Quaternion form of Dirac-Kähler's fields and symmetry

Dirac-Kähler equations (5)-(7) may be written in quaternion form [56, 57]. Within the framework of the quaternion approach, the six-parameter internal-symmetry subgroup of the Dirac-Kähler equation is considered. It is shown that the Lorentz group is the automorphism group of the group under consideration. The possibility is investigated to reproduce the potentials and field transformations induced by the coordinate transformation in the complex space-time [58].

Let us introduce the following biquaternions (see Appendix B)

$$\nabla = e_\mu \partial_\mu, \qquad F = F_\mu e_\mu, \qquad G = G_\mu e_\mu,$$
$$F_m = H_m - iE_m, \qquad F_4 = -\varphi - i\widetilde{\varphi}, \qquad G_\mu = \left(\varphi_\mu + i\widetilde{\varphi}_\mu\right), \tag{72}$$
$$H_m = \frac{1}{2}\epsilon_{mnk}\varphi_{nk}, \qquad E_m = i\varphi_{m4}.$$

Using the algebra of quaternions (see Appendix B) we represent Eqs. (5)-(7) as

$$\nabla F + m^2 G = 0, \qquad F = -\overline{\nabla} G, \tag{73}$$

where $\overline{\nabla} = \overline{e}_\mu \partial_\mu$; $\overline{e}_\mu = (e_4, -e_m)$ are the conjugated quaternion elements. The Lagrangian (63) with the help of the basis elements of the quaternion algebra takes the form

$$\mathcal{L} = -\frac{1}{2}\left(F\overline{F} + m^2 G\overline{G} + F^*\overline{F}^* + m^2 G^*\overline{G}^*\right). \tag{74}$$

Equations (73) preserve their form under the following transformations of the field variables:

$$G \to G' = GD, \qquad F \to F' = FD. \tag{75}$$

In the general case, transformations (75) define the $SL(2,c)$ group as the quaternion algebra isomorphic to the algebra of Pauli's matrices (see Appendix B). The Lagrangian (74) is invariant under the transformations (75) if the biquaternion D satisfies the equality $D\overline{D} = 1$. This condition defines a $6-$parameter group of internal symmetry $SO(3,1)$, which is a subgroup of $SO(4,2)$ investigated in the previous Section. We can use the following parametrization of the biquaternion D:

$$D = \exp \mathbf{n}, \tag{76}$$

where $\mathbf{n}$ is the vector biquaternion with six parameters. The complex quaternion (76) obeys the equation $D\overline{D} = 1$. The finite transformations (75) with the biquaternion D (76) define subgroups $SU(1,1)$ and $SU(2)$ for the real and complex parameter n_i, respectively. It is easy to verify that transformations (75) corresponds to (69) at $\alpha = \beta_\mu = \delta_\mu = \xi = 0$, $\omega_{[m4]} = -\omega_{[4m]} = \mathrm{Im}\ n_m$, $\omega_{[mn]} = \epsilon_{mnk}\mathrm{Re}\ n_k$.

Under the transformations of the Lorentz group, the quaternions G, F, ∇ and $\overline{\nabla}$ are changed as follows

$$G^L = \overline{L}^* G L, \qquad F^L = \overline{L} F L,$$

$$\nabla^L = \overline{L}^*\nabla L, \qquad \overline{\nabla}^L = \overline{L}\overline{\nabla}L^* \tag{77}$$

with the condition $L\overline{L} = \overline{L}L = 1$. It is obvious that the field equations (73) are invariant under the Lorentz transformations (77) which guarantee the relativistic invariance. Now let us consider the transformations of the Lorentz group L and $D-$transformations (75). We have

$$G^{DL} = \overline{L}^*GDL, \qquad G^{LD} = \overline{L}^*GLD,$$
$$F^{DL} = \overline{L}FDL, \qquad F^{LD} = \overline{L}FLD. \tag{78}$$

It follows from Eqs. (78) that the Lorentz group does not commute with the group of the internal symmetry (75) as $G^{DL} \neq G^{LD}$, $F^{DL} \neq F^{LD}$. Under the Lorentz group transformations, the biquaternion D transforms as

$$D^L = \overline{L}DL. \tag{79}$$

It is seen from Eq. (79) that parameters of the group (75) are transformed under the tensor representation of the Lorentz group. Eq. (79) also denotes that the Lorentz group is the automorphism group of the group under consideration.

Let us consider the complex four-dimensional space where coordinates are complex potentials $G_\mu = \varphi_\mu + i\widetilde{\varphi}_\mu$. The group of transformations of four-dimensional rotations in this space is homomorphic to the group $SO(4,c)$; it leaves the quadratic form G_μ^2 invariant. We suppose that the space-time coordinates x_μ are not transformed here. The transformations of this group are given by (Appendix B)

$$G \to G' = SGR, \tag{80}$$

where $S\overline{S} = R\overline{R} = 1$. Because coordinates are not changed under this transformation, the quaternions ∇ and $\overline{\nabla}$ are also not changed. Transformation (80) leaves Eqs. (73) invariant. Indeed,

$$F' = -\overline{\nabla}SGR,$$
$$\nabla F' + m^2G' = \nabla\left(-\overline{\nabla}SGR + m^2SGR\right) = S\left(\nabla F + m^2G\right)R = 0. \tag{81}$$

This is extracted three subgroups $SL(2,c)$ from $SO(4,c)$:

$$S = 1, \qquad R\overline{R} = 1, \qquad G' = GR, \qquad F' = FR, \tag{82}$$

$$R = 1, \qquad S\overline{S} = 1, \qquad G' = SG, \qquad F' = -\overline{\nabla}SG, \tag{83}$$

$$S = \overline{R}^*, \qquad R\overline{R} = 1, \qquad G' = \overline{R}^* G R, \qquad F' = -\overline{\nabla}\overline{R}^* G R. \tag{84}$$

It is obvious that the group (82) coincides with the group (75).

Now we discuss the possibility of inducing the transformations (75) by the Lorentz transformations in the complex space-time (see Appendix B). Transformations of the complex coordinates $z' = LzR$ of the complex Lorentz group $SO(4,c)$ induce the following transformations of biquaternions ∇, $\overline{\nabla}$, G and F:

$$\begin{gathered} \nabla' = L\nabla R, \qquad \overline{\nabla}' = \overline{R}\overline{\nabla}\overline{L}, \\ G' = LGR, \qquad F' = \overline{R}FR, \end{gathered} \tag{85}$$

with the constraints: $R\overline{R} = 1$, $L\overline{L} = 1$. At the transition to the complex space-time, transformations of the complex Lorentz group $SO(4,c)$ and induced transformations of biquaternions ∇, $\overline{\nabla}$, G and F Eq. (85) retain the invariant form of Eqs. (73). In the case of the ordinary Lorentz group $SO(3,1)$, we should set $L = \overline{R}^*$. The transformations of the group of the internal symmetry of potentials G, Eq.(75), and the transformations of G, Eq. (85), from the complex Lorentz group at $L = 1$ have the same form; i. e. they are not different. But transformations of the field biquaternion F in Eq. (75) and (85) are different. Therefore the transformations of the potentials φ_μ , $\widetilde{\varphi}_\mu$ can be induced by the transformations of the complex Lorentz group $SO(4,c)$ but the fields $\varphi_{\mu\nu}$, φ, $\widetilde{\varphi}$ cannot.

The possibility of considering transformations (75) for the field variables is based on the fact that $F_4 = -\varphi - i\widetilde{\varphi} \neq 0$ but transformations (85) are valid also for $F_4 = 0$, i. e. for equations without additional scalar and pseudoscalar fields. At $\varphi = \widetilde{\varphi} = 0$, $\widetilde{\varphi}_\mu = 0$ Eqs. (73) represent the quaternion form of the Proca equations. For them the transformations (75) are not possible.

It should be noted that the transition to complex space-time is used in the investigation of some general problems of quantum field theory; e. g. the solution of some specific problems in electrodynamics [59-60].

5 Quantization of fields

The quantization of Dirac-Kähler's fields will be carried out by using an indefinite metric. It will be shown that the renormalization procedure is carried out in the same manner as in quantum electrodynamics [61, 62].

The Lagrangian of charged Dirac-Kähler fields (63) within four-divergences can be written (when electromagnetic fields are absent) as

$$\mathcal{L} = -\overline{\Psi}(x)\left(\Gamma_\mu \partial_\mu + m\right)\Psi(x), \tag{86}$$

where $\overline{\Psi}(x)=\Psi(x)^{+}\Gamma_{4}\overline{\Gamma}_{4}$ corresponds to the tensor representation of the Lorentz group, where the 16-component wave function Ψ describes scalar, pseudoscalar, vector and pseudovector fields. In the case of spinor representation of the Lorentz group, Ψ is the direct sum of four Dirac bispinors, $\overline{\Psi}(x)=\Psi(x)^{+}\Gamma_{4}$, and the quantizing procedure is similar to the Dirac theory.

Now we will consider the case of the boson fields. Using the canonical quantization, one arrives at the commutators

$$\left[\Psi_{M}(x),\overline{\Psi}_{N}(x')\right]_{t=t'}=\left(\Gamma_{4}\right)_{MN}\delta(\mathbf{x}-\mathbf{x}'), \tag{87}$$

where M, $N=1$,2,...,16. It follows from (87) that it is necessary to introduce the indefinite metric, as for finite-dimensional equations (with the diagonal matrix Γ_4) which describe fields with integer spins; only fields obeying the Petiau-Duffin-Kemmer equation have positive energy [63]. With the help of Eqs. (10), (14) we get the following commutation relations for the tensor fields:

$$[\varphi(x),\varphi_{0}^{*}(x')]_{t=t'}=i\delta(\mathbf{x}-\mathbf{x}'),\qquad [\widetilde{\varphi}(x),\widetilde{\varphi}_{0}^{*}(x')]_{t=t'}=-i\delta(\mathbf{x}-\mathbf{x}'),$$
$$[\widetilde{\varphi}_{k}(x),\varphi_{mn}^{*}(x')]_{t=t'}=i\epsilon_{kmn}\delta(\mathbf{x}-\mathbf{x}'), \tag{88}$$

$$[\varphi_{k}(x),\varphi_{n4}^{*}(x')]_{t=t'}=\delta_{kn}\delta(\mathbf{x}-\mathbf{x}'),$$

plus complex conjugated relations. In the momentum space, the equation of motion of fields with spins 0 and 1, takes the form

$$\left(m\pm i\widehat{p}\right)\Psi^{\pm}(\mathbf{p})=0, \tag{89}$$

where $\widehat{p}=p_{\mu}\Gamma_{\mu}$, $\Psi^{\pm}(\mathbf{p})$ are positive ($\Psi^{+}(\mathbf{p})$) and negative ($\Psi^{-}(\mathbf{p})$) frequency parts of the wave function corresponding positive $p_0>0$ and negative $p_0<0$ energies of particles, respectively. For each value of the energy, there are eight solutions with definite spin, spin projection and addition quantum number. Wave functions $\Psi^{\pm}(\mathbf{p})$, $\overline{\Psi}^{\pm}(\mathbf{p})$ can be expanded in spin states as follows

$$\Psi_{M}^{\pm}(\mathbf{p})=a_{s}^{\mp}(\mathbf{p})v_{M}^{s,\mp}(\mathbf{p}),\qquad \overline{\Psi}_{M}^{\pm}(\mathbf{p})=a_{s}^{*\mp}(\mathbf{p})v_{N}^{*s,\mp}(\mathbf{p})\left(\Gamma_{4}\overline{\Gamma}_{4}\right)_{NM}, \tag{90}$$

where index $s=0,m,\widetilde{n},\widetilde{0}$ (m, $n=1,2,3$), and operator $a_{s}^{*+}(\mathbf{p})$ is the creation operator of a particle in the scalar state ($s=0$), pseudoscalar state ($s=\widetilde{0}$), vector state ($s=m$), pseudovector state ($s=\widetilde{n}$), and $a_{s}^{-}(\mathbf{p})$ is the annihilation operator of a particle. The normalization conditions for solutions (90) are different from the Dirac bispinors case, and are given by

$$v_{N}^{*s,\pm}(\mathbf{p})\left(\overline{\Gamma}_{4}\right)_{NM}v_{M}^{r,\mp}(\mathbf{p})=\pm\varepsilon_{s}\delta_{sr}, \tag{91}$$

$$\overline{v}^{s,\pm}(\mathbf{p})v^{r,\mp}(\mathbf{p}) = \varepsilon_s\delta_{sr}\frac{m}{p_0}, \qquad \left(\overline{v}^{s,\pm}(\mathbf{p}) = v^{*s,\pm}(\mathbf{p})\Gamma_4\overline{\Gamma}_4\right), \tag{92}$$

$$v^{*s,\pm}(\mathbf{p})\overline{\Gamma}_4 v^{r,\pm}(-\mathbf{p}) = 0, \qquad \left(v^{s,\pm}(\mathbf{p})\right)^* = v^{*s,\mp}(\mathbf{p}), \tag{93}$$

and $\varepsilon_s = 1$ at $s = \widetilde{0}$, m, and $\varepsilon_s = -1$ at $s = 0$, $\tilde{n}$. We use here the normalization on the charge and in the right hand sides of Eqs. (91), (92) there is no summation on indexes s . The summation formula on indexes s corresponding to the normalization conditions, has the form

$$\sum_s \varepsilon_s v_M^{s,\pm}(\mathbf{p})\overline{v}_N^{s,\mp}(\mathbf{p}) = \left(\frac{m \pm i\widehat{p}}{2p_0}\right)_{MN}. \tag{94}$$

If we take the trace of the matrix (94) and compare it with the expression (92) summed over all states s, we will get the equality. Multiplying Eq. (94) into the matrix Γ_4, and then calculating the trace of both sides of the matrix equality, we arrive at Eq. (91).

The appearance of the coefficient $\varepsilon_s = \pm 1$ in the right hand side of Eq. (92) reflects the fact that the energy of vector and pseudoscalar states is positive, and the energy of pseudovector and scalar states is negative. We can also come to this conclusion using the expression for the energy-momentum tensor

$$T_{\mu\nu} = -\overline{\Psi}(x)\Gamma_\mu\partial_\nu\Psi(x) \tag{95}$$

which follows from the Lagrangian (86). Taking into account the expansions for wave functions:

$$\begin{aligned}\Psi(x) &= (2\pi)^{-3/2}\int\left[\Psi^+(\mathbf{p})e^{ipx} + \Psi^-(\mathbf{p})e^{-ipx}\right]d^3p,\\ \overline{\Psi}(x) &= (2\pi)^{-3/2}\int\left[\overline{\Psi}^+(\mathbf{p})e^{ipx} + \overline{\Psi}^-(\mathbf{p})e^{-ipx}\right]d^3p\end{aligned} \tag{96}$$

found from Eq. (95), the energy-momentum-vector

$$P_\mu = -i\int\overline{\Psi}(x)\Gamma_4\partial_\mu\Psi(x)d^3x, \tag{97}$$

and Eqs. (90)-(93), we arrive at the following expression

$$P_\mu = \int p_\mu\sum_s\varepsilon_s\left(a_s^{*+}(\mathbf{p})a_s^-(\mathbf{p}) + a_s^{*-}(\mathbf{p})a_s^+(\mathbf{p})\right)d^3p. \tag{98}$$

To have the positive energy in accordance with the Eq. (98), we should use the following commutation relations for creation and annihilation operators:

$$\left[a_s^{*-}(\mathbf{p}), a_r^+(\mathbf{p})\right] = \left[a_s^-(\mathbf{p}), a_r^{*+}(\mathbf{p})\right] = \varepsilon_s\delta_{sr}\delta(\mathbf{p} - \mathbf{p}'). \tag{99}$$

With the help of Eqs. (90), (94) and (99) we find

$$\begin{aligned}\left[\Psi^-_M(x),\overline{\Psi}^+_N(y)\right]&=-(2\pi)^{-3}\int\frac{(m+i\widehat{p})_{MN}}{2p_0}e^{-ip(x-y)}d^3p\\&=\left(\Gamma_\mu\frac{\partial}{\partial x_\mu}-m\right)_{MN}\Delta_-(x-y),\end{aligned}\tag{100}$$

$$\begin{aligned}\left[\Psi^+_M(x),\overline{\Psi}^-_N(y)\right]&=(2\pi)^{-3}\int\frac{(m-i\widehat{p})_{MN}}{2p_0}e^{ip(x-y)}d^3p\\&=-\left(\Gamma_\mu\frac{\partial}{\partial x_\mu}-m\right)_{MN}\Delta_+(x-y).\end{aligned}\tag{101}$$

It follows from Eqs. (100), (101) (by taking into account the definition of the Pauli-Jordan function [54, 64] $\Delta_0=i\left(\Delta_+(x)-\Delta_-(x)\right)$) that

$$\begin{aligned}&\left[\Psi(x),\overline{\Psi}(y)\right]=S(x-y),\qquad S(x-y)=S^+(x-y)+S^-(x-y),\\&S^\pm(x-y)=\mp\left(\Gamma_\mu\frac{\partial}{\partial x_\mu}-m\right)\Delta_\pm(x-y),\end{aligned}\tag{102}$$

where the function $S(x-y)$ satisfies the following equations

$$\left(\Gamma_\mu\frac{\partial}{\partial x_\mu}+m\right)S(x-y)=i\left(\frac{\partial^2}{\partial x_\mu^2}-m^2\right)\Delta_0(x-y)=0.\tag{103}$$

From Eqs. (102), at $t=t'$, and using Eqs. (100), (101) we arrive at the commutator (87). With the help of the relationships (102), we can find the chronological pairing of the operators:

$$\begin{aligned}&\langle T\Psi_M(x)\overline{\Psi}_N(y)\rangle_0=S^c_{MN}(x-y)\\&=\Theta(x_0-y_0)S^+_{MN}(x-y)-\Theta(y_0-x_0)S^-_{MN}(x-y)\\&=\frac{i}{(2\pi)^4}\int\frac{(i\widehat{p}-m)_{MN}}{p^2+m^2-i\varepsilon}e^{ip(x-y)}d^4p\end{aligned}\tag{104}$$

which has formally the same form as in quantum electrodynamics (QED). Here $\Theta(x)$ is the well known theta-function [64] and x_0 is the time.

It is seen from Eq. (104), that the Feynman rules for particles with spins $0,1$ interacting with the electromagnetic field, eventually are the same as in QED. We should not, however, here use the QED factor $\eta=(-1)^l$, where l is the number of loops in the diagram due to different statistics [64]. The difference is in the number of spin states of the charged particle, and in the dimension of matrices Γ_μ. As the propagator (104) formally coincides with the electron propagator of QED, so all divergences can be cancelled by the standard procedure, i. e., we have here a renormalizable theory. All matrix elements of quantum processes describing the interaction of particles with multispin $0, 1$ coincide eventually with the corresponding elements in QED. The difference is in the density matrix $\Psi \cdot \overline{\Psi}$ which we found in Sec. 2.

The commutation relations (99) with sign ($-$) in the right hand side (at $\varepsilon_s = -1$) require the introduction of the indefinite metric. The space of states is divided into two substates: H_p and H_n with positive (H_p) and negative (H_n) square norm. The vector and pseudoscalar states correspond to a positive square norm, and pseudovector and scalar states $-$ to a negative square norm. The total space is the direct sum of the two subspaces H_p and H_n.

6 Supersymmetry of Dirac-Kähler's fields

For the Dirac-Kähler fields, it will be shown that in field theory it is possible to analyze transformation groups with tensor and spinor parameters without including coordinate transformations at the same time [65].

A graduated Lie algebra must be absolutely related to transformations involving space-time coordinates [66], but it seems perfectly obvious that this is always the case if we are dealing with transformations whose generators are of neither a tensor nor spinor nature (see [67, 68], for example).

A theory of Dirac-Kähler's fields, however, raises the possibility of constructing a transformation group with tensor and spinor generators which does not at the same time include coordinate transformations.

Let us consider the field equations

$$\left(\gamma_\mu \partial_\mu + \frac{1}{2}\left(m_1 + m_2\right)\right) G(x) + \frac{1}{2}\left(m_2 - m_1\right) \gamma_5 G(x) \gamma_5 = 0, \tag{105}$$

where γ_μ are the Dirac matrices, and the matrix $G(x)$ is

$$G(x) = \psi_0(x) I_4 - \psi_\mu(x)\gamma_\mu + \frac{1}{2}\psi_{[\mu\nu]}(x)\gamma_{[\mu}\gamma_{\nu]} + i\widetilde{\psi}_\mu(x)\gamma_\mu\gamma_5 - i\widetilde{\psi}_0(x)\gamma_5. \tag{106}$$

The quantities $\psi_0(x)$, $\psi_\mu(x)$, $\psi_{[\mu\nu]}(x)$, $\widetilde{\psi}_\mu(x)$, and $\widetilde{\psi}_0(x)$ in Eq. (106) are, respectively, a scalar, a vector, an antisymmetric tensor, a pseudovector and a pseudoscalar; under the Lorentz group, $G(x)$ transforms as follows:

$$G(x) \to G^L(x) = SG(x)S^{-1}, \qquad S = \exp\left(\frac{1}{4}\varepsilon_{\mu\nu}\gamma_{[\mu}\gamma_{\nu]}\right), \tag{107}$$

where $\varepsilon_{\mu\nu}$ are the Lorentz group parameters. Multiplying Eq. (105) successively by the Clifford-algebra elements γ_A: iI_4, γ_μ, $(1/2)\gamma_{[\mu}\gamma_{\nu]}$, $\gamma_\mu\gamma_5$

and γ_5, and taking the trace, we find the tensor equations which coincide with Eqs. (9) including the massive and massless cases. The case of a massless field corresponds to the choice $m_1 = 0$, while a massive field (at $m_1 = m_2 = m$) is described by an equation of the type

$$(\gamma_\mu \partial_\mu + m)\, G(x) = 0. \tag{108}$$

The matrix equation (108) is equivalent to the massive Dirac-Kähler equation. It is obvious that Eq. (108) describes spinor particles when the $4\times4-$matrix $G(x)$ represents four bispinors, and that it describes scalar, vector, antisymmetric tensor, pseudovector and pseudoscalar fields when $G(x)$ is expanded by (106). For the spinor case, however, $G(x)$ transforms as follows (in this case we use the index $1/2$):

$$G_{1/2}(x) \to G^L_{1/2}(x) = SG_{1/2}(x), \qquad S = \exp\left(\frac{1}{4}\varepsilon_{\mu\nu}\gamma_{[\mu}\gamma_{\nu]}\right). \tag{109}$$

The Lagrangian corresponding to Eq. (108) is

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}\left[\overline{G}(x)\gamma_\mu\partial_\mu G(x) - \overline{G}(x)\gamma_\mu\overleftarrow{\partial_\mu} G(x) + 2m\overline{G}(x)G(x)\right], \tag{110}$$

where $\overline{G}(x) = \gamma_4 G(x)\gamma_4$, the arrow specifies the direction in which the differential operator acts and tr means the trace of matrices. After taking the trace in Eq. (110), we arrive at the Lagrangian which is equivalent to Eq. (86).

Equation (108) is invariant under the following transformations of the matrix quantity $G(x)$:

$$G(x) \to G'(x) = G(x)D. \tag{111}$$

The relativistic invariance of Eq. (108) is retained if the matrix D transforms under the Lorentz group as

$$D(x) \to D^L(x) = SDS^{-1}, \tag{112}$$

i. e., if the generators of the transformation group (111) are of a tensor nature with respect to the Lorentz group. In the spinor case, all parameters of transformation (111) are scalars. Therefore the Lorentz transformations (109) and the transformations of the internal symmetry (111) commute each other. If we make the Lorentz (109) and symmetry (111) transformations for the case of spin $1/2$, one gets

$$G_{1/2}(x) \to G^{L\prime}_{1/2}(x) = SG_{1/2}(x)D. \tag{113}$$

The commutation of these two groups of transformations is obvious from Eq. (113). If we put $D = S^{-1}$ in Eq. (113) we arrive at the law of the transformation of the tensor fields (107). That is why it is possible to describe the spinor particles with spin $1/2$ by the tensor fields using the expansion (106). The same conclusion follows from the formalism of Sec. 2.

The requirement that the Lagrangian (110) be invariant under transformations (111) leads to the condition

$$D\overline{D} = 1, \qquad \overline{D}(x) = \gamma_4 D^+ \gamma_4. \tag{114}$$

Writing D in the form

$$D = \exp\left(i\alpha I_4 + \beta_\mu \gamma_\mu + \frac{1}{2}\Omega_{\mu\nu}\gamma_{[\mu}\gamma_{\nu]} + \delta_\mu \gamma_\mu \gamma_5 + \rho\gamma_5\right), \tag{115}$$

where iI_4, γ_μ, $\frac{1}{2}\gamma_{[\mu}\gamma_{\nu]}$, $\gamma_\mu\gamma_5$ and γ_5 are the generators of the group $GL(4, C)$, we find from condition (114) a restriction on the parameters: $\alpha^* = \alpha$, $\beta_m^* = \beta_m$, $\beta_4^* = -\beta_4$, $\Omega_{mn}^* = \Omega_{mn}$, $\Omega_{m4}^* = -\Omega_{m4}$, $\delta_m^* = \delta_m$, $\delta_4^* = -\delta_4$, $\rho^* = \rho$, in accordance with a singling out of the $SO(4,2) \otimes U(1)$ subgroup (or locally isomorphic to $U(2,2)$). The subgroup $U(1)$ corresponds to the gauge transformations that conserve electric current. So the symmetry group of equation (108) is $GL(4, C)$ and the corresponding Lagrangian $-$ $SO(4,2) \otimes U(1)$ for a case of tensor fields. In the case of spinor fields, the symmetry group is $U(4)$, in accordance with conclusions of Sec. 3.

From the invariance of the Lagrangian (110) under transformations (111) ((114) is taken into account) we find conservation laws for quantities of the type

$$\Theta_{\mu A} = \frac{1}{2}\mathrm{tr}\left(\gamma_\mu G(x)\gamma_A \overline{G}(x) - \gamma_\mu G(x)\gamma_4 \gamma_A^+ \gamma_4 \overline{G}(x)\right), \tag{116}$$

where $\gamma_A = iI_4$, γ_μ, $(1/2)\gamma_{[\mu}\gamma_{\nu]}$, $\gamma_\mu\gamma_5$, γ_5 and γ_A^+ are the complex conjugated Clifford-algebra elements. The conservation currents (116) were found in Sec. 3 in another formalism. From the physical standpoint, the appearance of this symmetry results from a mass degeneracy of the spin states of the particle which are mixed by transformations (111). The symmetry is preserved in a nonlinear generalization of equation (108) (equations of the Heisenberg type):

$$(\gamma_\mu \partial_\mu + m)\, G(x) + lG(x)\overline{G}(x)G(x) = 0, \tag{117}$$

where l is the coupling constant.

The matrix $G(x)$ corresponds to a second-rank bispinor G_α^β. If the bispinor satisfies the Dirac equations with respect to each index simultaneously, we find a system of Bargmann-Wigner equations [69] which describe a particle with spin 0 and system of particles with spin 1.

By jointly analyzing Eqs. (108) and the Dirac equation

$$(\gamma_\mu \partial_\mu + m)\, \Psi(x) = 0 \tag{118}$$

which is invariant under phase transformations of wave function $\Psi(x)$ [$\Psi(x) \to \Psi'(x) = \exp(i\theta)\Psi(x)$], we can construct a symmetry group which incorporates these phase transformations and transformation (111) as a subgroup. The system (108), (118) is invariant under the following transformations:

$$\begin{gathered} G'(x) = G(x)D + \Psi(x) \cdot \overline{\zeta}, \\ \Psi'(x) = \Psi(x)\lambda + G(x)\xi, \end{gathered} \tag{119}$$

where $\Psi(x) \cdot \overline{\zeta} = \left(\Psi_\alpha(x) \cdot \overline{\zeta}^\beta\right)$ is the matrix-dyad, and $\overline{\zeta}$, ξ are bispinor-parameters, and λ is the complex number-parameter. Transformations (119) form a group with the following parameter composition law:

$$\begin{gathered} D'' = D'D + \xi' \cdot \overline{\zeta}, \qquad \overline{\zeta}'' = \lambda' \overline{\zeta} + \overline{\zeta}' D, \\ \lambda'' = \lambda' \lambda + \overline{\zeta}' \xi, \qquad \xi'' = \xi' \lambda + D' \xi. \end{gathered} \tag{120}$$

Under the Lorentz group, the parameters $\overline{\zeta}$ and ξ transform as bispinors $\overline{\Psi}$ and Ψ, respectively and are constant quantities, independent of the space-time coordinates. In order to preserve the relationship between the spin and the statistics, we must require that the parameters ξ and $\overline{\zeta}$ anticommute: $\{\xi_\alpha, \xi_\beta\} = \left\{\overline{\zeta}^\alpha, \overline{\zeta}^\beta\right\} = 0$. The need for this condition can be seen directly from the commutation relations for boson and fermion fields and from the explicit form of transformations (119).

To establish the group structure of transformations (119) it is convenient to use a $20-$component column function $\Phi(x)$ whose first components are formed by the elements of the lines of the matrix $G_\alpha^\beta(x)$ (in the order of an alternation of lines and of the elements in them). The other four components correspond to the wave function $\Psi(x),$ so

$$\Phi(x) = \begin{pmatrix} G_\alpha^\beta(x) \\ \Psi_\alpha(x) \end{pmatrix}. \tag{121}$$

A direct check confirms the following form for writing transformations (119):

$$\Phi'(x) = (I_4 \otimes B)\, \Phi(x), \qquad B = \begin{pmatrix} D^T & \overline{\zeta}\downarrow \\ \overrightarrow{\xi} & \lambda \end{pmatrix}, \tag{122}$$

where $\overline{\zeta}\downarrow$ is a column, and $\overrightarrow{\xi}$ is a row and $\left(D^T\right)_{\alpha\beta} = D_{\beta\alpha}$. Under the condition $\mathrm{tr}\left(D^T\right) = \lambda$, the transformations in (122) correspond to the graduated Lie algebra $SL(4 \mid 1)$, in the notation of [70]. The form in (122) for the transformations (119) is of a standard type [71], where D^T and λ are even elements of a Grassmann algebra, and ξ and $\overline{\zeta}$ are odd elements (i.e., anticommuting elements).

A fundamental distinction between this symmetry and the "ordinary" supersymmetry, with tensor and spinor generators, is that the corresponding superalgebra is closed without appealing to the generators of a Poincaré group.

By analogy with the description of fields with a maximum spin of 1, fields with a maximum spin of $s/2$ can be described by the equations

$$(\gamma_\mu \partial_\mu + m)\, G_{\alpha_1\alpha_2\ldots\alpha_s} = 0. \tag{123}$$

Dirac equation (118) and Eq. (108) constitute a particular case of Eq. (123), with $(G_{\alpha_1}) = \Psi(x)$ and $(G_{\alpha_1\alpha_2}) = G(x)$. A particle with a maximum spin of $3/2$ and a rest mass is described, for example, by the function $G_{\alpha_1\alpha_2\alpha_3}$. These internal-symmetry transformations can be generalized immediately to the case of particles with maximum spins of 1 and $3/2$, which are the particles of the greatest physical interest.

What possible physical applications could this symmetry have? In an analysis of systems consisting of two (mesons) and three (baryons) quarks in the s state, and in the absence of a spin-spin interaction, the hadrons may be described as multispin particles with a rest mass. In this case the symmetry under consideration holds rigorously, and the hadron interactions can be associated with the internal-symmetry group $SO(3,1) \otimes SU(3)$, where $SO(3,1)$ is the internal symmetry group, which forms with the Poincaré group a semidirect product. As has been mentioned in the literature [72] the latter property is a necessary feature of strong-interaction symmetry groups with incorporated quark spin. From this standpoint the considered supersymmetry corresponds to an internal symmetry of a field theory which incorporates, along with composite particles, their structural components. A further study of this symmetry will be required, of course, to take into account its possible extension to interacting fields.

7 Non-Abelian tensor gauge theory

We will show the possibility of constructing a gauge model of interacting Dirac-Kähler fields where the gauge group is the noncompact group $SO(4,2)$ under consideration [73].

The starting point in the introduction of Yang-Mills fields is the localization of parameters of the symmetry group, the transformations of which do not affect the space-time coordinates. We consider fields $\Psi(x)$ that possess certain transformation properties under the Lorentz group and that may be transformed under a certain representation of the internal symmetry group (usually compact and semisimple of type $SU(n)$). For example, QCD considers the fermionic fields of spin $1/2$, which are transformed under the fundamental representation of $SU(3)$, in which the colored quarks are the principal objects. In other words the concept that there exist some internal quantum numbers (isospin, colour, etc.) is a requisite physical element of non-Abelian gauge theory. However, it has been a long-standing problem to describe particles without involving internal ("isotopic") spaces (for example, the works of Heisenberg, Dürr [74, 75] and Budini [76]), but using instead an adequate generalization of the known relativistic wave equations (RWE) (see reference [77] and other references below). It becomes imperative today, for an approach of this kind, to imply the concept of a non–Abelian gauge field$-$the carrier of interactions. The possibility of constructing the gauge theory is inherent in the theory of RWE. The theory might be based on the concept of a multispin or, equivalently, of a particle having several spin states. The equation remains coupled$-$i. e., it describes, as the ordinary Dirac equation does, a particle (and antiparticle) having a set of states, rather than a set of particles, as is the case with, for example, the equation corresponding to the direct sum of the Dirac equations. Transformations of internal symmetry groups result in a mixture of states related to different values of the spin squared operator. Their localization leads to non-Abelian gauge fields having multispin $0, 1, 2$. In this case the dynamics of the interaction of particles with multispin $0, 1$ are associated with the change of their state through the exchange of particles with maximum spin 2. The theory constructed in this manner represents a space-time analogue of gauge theory with internal symmetry.

An attractive possibility is to describe quarks by the Dirac-Kähler equations. The authors [4-11] used lattice version of the Dirac-Kähler equations describing fermions by inhomogeneous differential forms. This is equivalent to introducing a set of antisymmetric tensor fields of arbitrary rank for describing the fermion matter fields. As shown in the Introduction, we arrive at the Dirac-Kähler formulation which includes a scalar, a vector, an antisymmetric tensor, a pseudovector and a pseudoscalar field. Here we consider the continuum case of the equations and introduce non-Abelian tensor gauge fields (gluon fields) for interacting quarks. In this point of view, quarks possess the multispin $0, 1$. As we have already shown, in the continuum case the equation for the $16-$component Dirac equation can be reduced to four independent $4-$component Dirac equations. We proceed to use the language of tensor fields to formulate non-Abelian tensor gauge theory.

The requirement that the Lagrangian (63) be invariant under local transformations (69) leads to the necessity of introducing a compensating field A_μ^B, where the index B is "internal" (in our case it represents a set of tensor indices specifying a scalar, a four-vector, a skew second-rank tensor, an axial four-vector and a pseudoscalar). The gauge invariant Lagrangian has the known form:

$$\mathcal{L} = -\overline{\Psi}(x)\left[\Gamma_\mu\left(\partial_\mu - ieA_\mu - gA_\mu^B I^B\right) + m\right]\Psi(x) - \frac{1}{4}\mathcal{F}_{\mu\nu}^2 - \frac{1}{4}\left(F_{\mu\nu}^B\right)^2, \quad (124)$$

where

$$\begin{aligned} \mathcal{F}_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu, \\ F_{\mu\nu}^B &= \partial_\mu A_\nu^B - \partial_\nu A_\mu^B - \frac{1}{2}g c_{CD}^B A_{[\mu}^C A_{\nu]}^D \end{aligned} \quad (125)$$

and c_{CD}^B are the structure constants of $SO(4,2)$ group; $\mathcal{F}_{\mu\nu}$, $F_{\mu\nu}^B$ are the strengths of the electromagnetic and "gluon" fields, respectively; g is the "gluon" coupling constant and $B = \{\mu, [\alpha\beta], \widetilde{\mu}, \widetilde{0}\}$. The localization of the $U(1)$ group produced the electromagnetic field with four-potential A_μ.

The corresponding wave equations which follow from the Lagrangian (124) are

$$\partial_\nu \mathcal{F}_{\mu\nu} = J_\mu, \quad (126)$$

$$\partial_\nu F_{\mu\nu}^B + g c_{DC}^B A_\nu^C F_{\mu\nu}^D = J_\mu^B, \quad (127)$$

$$\left[\Gamma_\mu\left(\partial_\mu - ieA_\mu - gA_\mu^B I^B\right) + m\right]\Psi(x) = 0, \quad (128)$$

where $J_\mu = ie\overline{\Psi}\Gamma_\mu\Psi$, $J_\mu^B = g\overline{\Psi}\Gamma_\mu I^B\Psi$; $I^B = \overline{\Gamma}_\mu$, $(1/4)\overline{\Gamma}_{[\mu}\overline{\Gamma}_{\nu]}$, $\overline{\Gamma}_5$, $\overline{\Gamma}_5\overline{\Gamma}_\mu$ (see Eq. (66)). The conservation current for the non-Abelian fields is

$$\widehat{J}_\mu^B = J_\mu^B - g c_{DC}^B A_\nu^C F_{\mu\nu}^D. \quad (129)$$

In the general case the gauge field multiplets A_ν^C include a second-rank tensors A_ν^α, $A_\nu^{\widetilde{\alpha}}$, a third-rank tensor antisymmetric over two indices $A_\nu^{[\mu\nu]}$, and an axial four-vector $A_\nu^{\widetilde{0}}$. The structure constants c_{CD}^B transform in the tensor representation of the Lorentz group. The gauge fields carry the maximal spin 2. Indeed, the second-rank tensor A_ν^α is transformed on the following superposition of the irreducible representations of the Lorentz group:

$$\left(\frac{1}{2},\frac{1}{2}\right) \otimes \left(\frac{1}{2},\frac{1}{2}\right) = (0,0) \oplus (0,1) \oplus (1,0) \oplus (1,1) \quad (130)$$

corresponding to the fields with the spins $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$. The third-rank tensor $A_{\nu}^{[\mu\nu]}$ realizes the representations

$$\left(\frac{1}{2},\frac{1}{2}\right)\otimes[(0,1)\oplus(1,0)]=\left(\frac{1}{2},\frac{1}{2}\right)\oplus\left(\frac{1}{2},\frac{3}{2}\right)\oplus\left(\frac{3}{2},\frac{1}{2}\right)\oplus\left(\frac{1}{2},\frac{1}{2}\right) \tag{131}$$

which also contain fields with spins $\mathbf{0}$, $\mathbf{1}$, $\mathbf{2}$; hence we come to the same conclusion about the spin of the gauge fields.

We can also consider the localization of some subgroups of the total group $SO(4,2)$. Then the Lagrangian (124) will be invariant under the local transformations of this subgroup. The main requirement is to extract the subgroup in a relativistic manner. We suggest the following subgroups and their corresponding generators

$$\begin{gathered} SO(3,1)-\left\{I_{[\mu\nu]}\right\}, \qquad SO(3,2)-\left\{I_{[\mu\nu]}, I_{\alpha}\right\}, \\ SO(4,1)-\left\{I_{[\mu\nu]}, \widetilde{I}_{\alpha}\right\}, \qquad GL(1,R)-\left\{\widetilde{I}\right\}. \end{gathered} \tag{132}$$

The possibility of constructing a dynamic theory with all the main properties of gauge theories, but based upon notions of space-time rather than on new internal quantum numbers, is of evident interest.

The absolute group symmetry G corresponds to the semidirect multiplication of the Poincaré group P on the internal symmetry group D ($SO(4,2)$ group):

$$G=P\cdot D, \tag{133}$$

and the transformations of the symmetry group ($D-$group) commute with the transformations of the subgroup of four-translations T_4. Following [78], it is possible to define the "auxiliary" Poincaré group P', which is isomorphic to P, by the relationships:

$$P'=\left\{L'_{\mu\nu}\right\}\cdot T_4, \qquad L'_{\mu\nu}=L_{\mu\nu}-I_{\mu\nu}, \tag{134}$$

where $I_{\mu\nu}$ are the generators of the internal symmetry group $SO(3,1)$ (see (66), (69)). As a result we have

$$G=P\cdot D=P'\otimes D, \tag{135}$$

i. e., the absolute group symmetry G represents the direct product of the auxiliary Poincaré group P' and the internal symmetry group D , taking into account that $\left[L'_{\mu\nu}, I_{\mu\nu}\right]=[T_4, I_{\mu\nu}]=0$. In our case of the Dirac-Kähler equation, $I_{\mu\nu}=(1/4)(\overline{\Gamma}_{\mu}\overline{\Gamma}_{\nu}-\overline{\Gamma}_{\nu}\overline{\Gamma}_{\mu})$ and

$$L'_{\mu\nu}=\frac{1}{4}\left(\Gamma_{\mu}\Gamma_{\nu}-\Gamma_{\nu}\Gamma_{\mu}\right). \tag{136}$$

The generators of the "auxiliary" Lorentz group, $L'_{\mu\nu}$, commute with the generators of the internal symmetry group, I_{AB} (67), and the wave function transforms in the spinor representation of the group $\left\{L'_{\mu\nu}\right\}$. This confirms that the Dirac-Kähler equation describing a set of antisymmetric tensor fields by the inhomogeneous differential forms, can describe spin $1/2$ particles (pseudoscalar and pseudovector fields are equivalent to the antisymmetric tensors of fourth and third ranks, respectively). The non-Abelian gauge theory under consideration is an analogy to the ordinary non-Abelian gauge theory of spin $1/2$ particles interacting via gluon fields with internal symmetry group $SO(6)$ (or $SU(4)$). However in our case we have noncompact gauge group $SO(4,2)$ (or $SU(2,2)$) which requires the introduction of an indefinite metric.

8 Conclusion

We have considered Dirac-Kähler equations which can be represented as the direct sum of four Dirac equations. The main feature of such scheme is the presence of the additional symmetry associated with non-compact group in the Minkowski space. The transformations of this group mix fields with different spins and they do not commute with the Lorentz transformations. As a result, the group parameters realize tensor representations of the Lorentz group. This kind of symmetry differs from the colour and flavour symmetries of QCD and supersymmetry. At the same time it was shown that the Dirac-Kähler fields allow us to introduce graduated groups with tensor and spinor parameters without including coordinate transformations.

The field scheme considered allow us to construct gauge theories with different non-compact groups $O(3,1)$, $O(4,2)$, $O(3,3)$, where "gluon" fields carry spins 0, 1, 2. Some of these sup-groups become compact groups in the Euclidean space-time. The theory constructed represents a space-time analogue of gauge theory with internal symmetry but there is a difficulty arising from the presence of an indefinite metric. The field schemes considered can be applied to a construction of quark models or for the classification of hadrons by non-compact groups (see [79]), and possibly, for studying subquark matter.

The calculated density matrices (matrix-dyads) for fields and the method of computing the traces of $16-$dimensional Petiau-Duffin-Kemmer matrix products allow us to make evaluations of different physical quantities in a covariant manner.

APPENDIX A

A method of computing the traces of $16-$dimensional Petiau-Duffin-Kemmer matrix products will be considered [80].

In order to compare the $16-$component model of vector fields (including scalar states) with the Proca and Petiau-Duffin-Kemmer theories we consider here the density matrices in the form of matrix-dyads (40) for pure spin states. Taking into account Eqs. (16) it is possible to have the simpler expressions by using equalities

$$\begin{aligned} &\frac{\sigma^2}{2}p^{(+)} = p^{(+)}\frac{\sigma^2}{2} = p^{(1)}, \qquad \frac{\sigma^2}{2}p^{(-)} = p^{(-)}\frac{\sigma^2}{2} = p^{(\widetilde{1})}, \\ &\left(1-\frac{\sigma^2}{2}\right)p^{(+)} = p^{(\widetilde{0})}, \qquad \left(1-\frac{\sigma^2}{2}\right)p^{(-)} = p^{(0)}, \end{aligned} \tag{137}$$

where $p^{(1)} = p_\mu\beta_\mu^{(1)}$, $p^{(\widetilde{1})} = p_\mu\beta_\mu^{(\widetilde{1})}$, $p^{(0)} = p_\mu\beta_\mu^{(0)}$, $p^{(\widetilde{0})} = p_\mu\beta_\mu^{(\widetilde{0})}$. The relationships (137) are obtained by using Eqs. (16), (24) and the expression for the squared spin operator:

$$\sigma^2 = \left[\frac{i}{2m}\epsilon_{\mu\nu\alpha\beta}\frac{1}{4}\left(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu + \overline{\Gamma}_\mu\overline{\Gamma}_\nu - \overline{\Gamma}_\mu\overline{\Gamma}_\nu\right)\right]^2. \tag{138}$$

Taking into account relations (137), the projection matrix-dyads (40) are transformed into

$$\Delta^{(1)} = \frac{1}{4m^2}ip^{(1)}\left(ip^{(1)} - \varepsilon m\right)\sigma_p^{(1)}\left(\sigma_p^{(1)} + s_p\right) = \Psi^{(1)}\cdot\overline{\Psi}^{(1)}, \tag{139}$$

$$\Delta^{(\widetilde{1})} = \frac{1}{4m^2}ip^{(\widetilde{1})}\left(ip^{(\widetilde{1})} - \varepsilon m\right)\sigma_p^{(\widetilde{1})}\left(\sigma_p^{(\widetilde{1})} + s_p\right) = \Psi^{(\widetilde{1})}\cdot\overline{\Psi}^{(\widetilde{1})}, \tag{140}$$

$$\Delta_0^{(1)} = \frac{1}{2m^2}ip^{(1)}\left(ip^{(1)} - \varepsilon m\right)\left(1 - \sigma_p^{(1)2}\right) = \Psi_0^{(1)}\cdot\overline{\Psi}_0^{(1)}, \tag{141}$$

$$\Delta_0^{(\widetilde{1})} = \frac{1}{2m^2}ip^{(\widetilde{1})}\left(ip^{(\widetilde{1})} - \varepsilon m\right)\left(1 - \sigma_p^{(\widetilde{1})2}\right) = \Psi_0^{(\widetilde{1})}\cdot\overline{\Psi}_0^{(\widetilde{1})}, \tag{142}$$

$$\begin{aligned} \Delta^{(0)} &= \frac{1}{2m^2}ip^{(0)}\left(ip^{(0)} - \varepsilon m\right)\left(1 - \sigma_p^{(0)2}\right) \\ &= \frac{1}{2m^2}ip^{(0)}\left(ip^{(0)} - \varepsilon m\right) = \Psi^{(0)}\cdot\overline{\Psi}^{(0)}, \end{aligned} \tag{143}$$

$$\begin{aligned} \Delta^{(\widetilde{0})} &= \frac{1}{2m^2}ip^{(\widetilde{0})}\left(ip^{(\widetilde{0})} - \varepsilon m\right)\left(1 - \sigma_p^{(\widetilde{0})2}\right) \\ &= \frac{1}{2m^2}ip^{(\widetilde{0})}\left(ip^{(\widetilde{0})} - \varepsilon m\right) = \Psi^{(\widetilde{0})}\cdot\overline{\Psi}^{(\widetilde{0})}, \end{aligned} \tag{144}$$

where we also used here the equalities:

$$p^{(0)}\left(1 - \sigma_p^{(0)2}\right) = p^{(0)}, \qquad p^{(1)}\sigma_p = p^{(1)}\sigma_p^{(1)} = -\frac{i}{\mid\mathbf{p}\mid}p^{(1)}\epsilon_{abc}p_a\beta_b^{(1)}\beta_c^{(1)},$$

and so on.

The matrix-dyads (139), (141) correspond to the vector state, (140), (142) - to the pseudovector state, (143) - to the scalar state, and (144) - to the pseudoscalar state. Eventually Eqs. (139)-(144) coincide with solutions of $10-$component (for vector and pseudovector states) and $5-$component (for scalar and pseudoscalar states) free Petiau-Duffin-Kemmer equations.

It is known that cross-sections for the scattering processes are summed to evaluate the transition probabilities for a particle going from the initial to the final states. These probabilities are proportional to the squared module's of the matrix elements which can be written as (see for example [52, 53]):

$$\mid M \mid^2 = e^2 \mathrm{tr}\left\{ Q \Pi_1 \overline{Q} \Pi_2 \right\}, \tag{145}$$

where Q is the vertex operator, $\overline{Q} = \eta Q^{+} \eta$ ($\eta = \Gamma_4 \overline{\Gamma}_4$ is the Hermitianizing matrix, Q^{+} is the Hermite conjugated operator), the matrix-dyads Π_1, Π_2 correspond to initial and final states, respectively. Therefore we need the traces of $16-$dimensional Petiau-Duffin-Kemmer matrix products to calculate some processes with the presence of vector and scalar fields.

It is easy to verify that the property of traces of the $16 \times 16-$ Petiau-Duffin-Kemmer matrices $\beta_\mu^{(\pm)}$ (16)

$$\mathrm{tr}\left\{ \beta_{\mu_1} \beta_{\mu_2} ... \beta_{\mu_n} \right\} = \mathrm{tr}\left\{ \left(P \beta_{\mu_n} \beta_{\mu_1} P \right) \left(P \beta_{\mu_2} \beta_{\mu_3} P \right) ... \left(P \beta_{\mu_{n-2}} \beta_{\mu_{n-1}} P \right) \right\}$$
$$+ \mathrm{tr}\left\{ \left(P \beta_{\mu_1} \beta_{\mu_2} P \right) \left(P \beta_{\mu_3} \beta_{\mu_4} P \right) ... \left(P \beta_{\mu_{n-1}} \beta_{\mu_n} P \right) \right\} \tag{146}$$

is valid, where $P = \varepsilon^{0,0} + \varepsilon^{\widetilde{0},\widetilde{0}} + (1/2)\varepsilon^{[\mu\nu],[\mu\nu]}$ is the projection operator. The analogous identity was derived in [48] for the $5 \times 5-$ and $10 \times 10-$Petiau-Duffin-Kemmer matrices. The trace of the odd-numbered matrices $\beta_\mu^{(\pm)}$ is equal to zero. Eq. (146) is valid for any Petiau-Duffin-Kemmer matrix given by (16). Using the properties of the entire matrix algebra $\varepsilon^{A,B}$ we find

$$P\beta_\mu^{(-)}\beta_\nu^{(+)}P = \varepsilon^{0,[\mu\nu]} + \frac{1}{2} e_{\nu\mu\rho\omega} \varepsilon^{[\rho\omega],\widetilde{0}},$$
$$P\beta_\mu^{(+)}\beta_\nu^{(-)}P = \varepsilon^{[\mu\nu],0} + \frac{1}{2} e_{\mu\nu\rho\omega} \varepsilon^{\widetilde{0},[\rho\omega]},$$
$$P\beta_\mu^{(+)}\beta_\nu^{(+)}P = \delta_{\mu\nu}\varepsilon^{\widetilde{0},\widetilde{0}} + \varepsilon^{[\rho\mu],[\rho\nu]},$$
$$P\beta_\mu^{(-)}\beta_\nu^{(-)}P = \delta_{\mu\nu}\varepsilon^{0,0} + \frac{1}{4} e_{\lambda\mu\rho\omega} e_{\lambda\nu\sigma\alpha} \varepsilon^{[\rho\omega][\sigma\alpha]}, \tag{147}$$
$$P\beta_\mu^{(0)}\beta_\nu^{(+)}P = P\beta_\mu^{(-)}\beta_\nu^{(1)}P = \varepsilon^{0,[\mu\nu]},$$

$$P\beta_\mu^{(+)}\beta_\nu^{(0)}P = P\beta_\mu^{(1)}\beta_\nu^{(-)}P = \varepsilon^{[\nu\mu],0},$$

$$P\beta_\mu^{(0)}\beta_\nu^{(-)}P = P\beta_\mu^{(-)}\beta_\nu^{(0)}P = \delta_{\mu\nu}\varepsilon^{0,0}.$$

The relations (146), (147) with the help of the equalities tr$\left\{\varepsilon^{A,B}\right\} = \delta_{A,B}$, $\delta_{[\mu\nu][\rho\sigma]} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}$, $\varepsilon^{A,B}\varepsilon^{C,D} = \delta_{BC}\varepsilon^{A,D}$, $\left(\varepsilon^{A,B}\right)_{CD} = \delta_{AC}\delta_{BD}$ allow us to find traces of any Petiau-Duffin-Kemmer matrices.

APPENDIX B

We analize here the Lorentz transformations in the framework of the quaternion algebra. The quaternion algebra is defined by four basis elements $e_\mu = (e_k, e_4)$ (see, for example [81]) with the multiplication properties:

$$e_4^2 = 1, \qquad e_1^2 = e_2^2 = e_3^2 = -1, \qquad e_1e_2 = e_3,$$

$$e_2e_1 = -e_3, \qquad e_2e_3 = e_1, \qquad e_3e_2 = -e_1, \tag{148}$$

$$e_3e_1 = e_2, \qquad e_1e_3 = -e_2, \qquad e_4e_m = e_me_4 = e_m,$$

where $m = 1, 2, 3$ and $e_4 = 1$ is the unit element.

The complex quaternion (or biquaternion) q is

$$q = q_\mu e_\mu = q_m e_m + q_4 e_4, \tag{149}$$

where the q_μ are complex numbers ($q_\mu =$Re $q_\mu + i$Im q_μ, $i^2 = -1$). Using the laws of multiplication (148), we find that the product of two arbitrary quaternions, q, q', is defined by:

$$qq' = (q_4q_4' - q_mq_m')\, e_4 + (q_4'q_m + q_4q_m' + \epsilon_{mnk}q_nq_k')\, e_m. \tag{150}$$

It is convenient to represent the arbitrary quaternion as $q = q_4 + \mathbf{q}$ (so $q_4e_4 \to q_4$, $q_me_m \to \mathbf{q}$), where q_4 and $\mathbf{q}$ are the scalar and vector parts of the quaternion, respectively. With the help of this notation, Eq. (150) can be rewritten as

$$qq' = q_4q_4' - q_mq_m' + q_4'\mathbf{q} + q_4\mathbf{q}' + [\mathbf{q}, \mathbf{q}']\,. \tag{151}$$

Thus the scalar $(\mathbf{q}, \mathbf{q}') = q_mq_m'$, and vector $[\mathbf{q}, \mathbf{q}']$ products are parts of the quaternion multiplication. It is easy to verify the combined law for three quaternions:

$$(q_1q_2)\, q_3 = q_1\,(q_2q_3)\,. \tag{152}$$

The operation of quaternion conjugation (hyperconjugation) denotes the transition to

$$\overline{q} = q_4e_4 - q_me_m \equiv q_4 - \mathbf{q}, \tag{153}$$

so that the equalities

$$\overline{q_1 + q_2} = \overline{q}_1 + \overline{q}_2, \qquad \overline{q_1 q_2} = \overline{q}_2 \overline{q}_1 \tag{154}$$

are valid for two arbitrary quaternions q_1 and q_2. The modulus of the quaternion $\mid q \mid$ is defined by

$$\mid q \mid = \sqrt{q\overline{q}} = \sqrt{q_\mu^2}. \tag{155}$$

This formula allows us to divide one quaternions by another, and thus the quaternion algebra includes this division. For example, the simple equations and their solutions are given by

$$\begin{aligned} q_2 x = q_1, \qquad x_L = \frac{\overline{q}_2 q_1}{\mid q_2 \mid^2}, \\ x q_2 = q_1, \qquad x_R = \frac{q_1 \overline{q}_2}{\mid q_2 \mid^2}. \end{aligned} \tag{156}$$

Quaternions are a generalization of the complex numbers and we can consider quaternions as a doubling of the complex numbers. They are convenient for investigating the symmetry of fields and relativistic kinematics. In particular, the finite transformations of the Lorentz eigengroup are given by [81]:

$$x' = Lx\overline{L}^*, \tag{157}$$

where $x = x_4 + \mathbf{x}$ is the quaternion of the coordinates ($x_4 = it$, t is the time, x_m are the spatial coordinates), L is the quaternion of the Lorentz group with the constraint $L\overline{L} = 1$, $\overline{L}^* = L_4^* - \mathbf{L}^*$ and $*$ means the complex conjugation. The biquaternion L with the constraint $L\overline{L} = 1$ is defined by six independent parameters which characterize the Lorentz transformations. The squared four-vector of coordinates, x_μ^2, is invariant under the transformations (157):

$$x_\mu'^2 = x'\overline{x}' = Lx\overline{L}^* L^* \overline{x}\overline{L} = x\overline{x} = x_\mu^2, \tag{158}$$

as $\overline{L}^* L^* = L^* \overline{L}^* = 1$. Eq. (158) shows that the $6-$parameter transformations (157) belong to the Lorentz group $SO(3,1)$.

The complex Lorentz group $SO(4,c)$ (see [82]) acts in complex space-time with the coordinates $z_\mu = (z_m, iz_0)$. The transformations of the complex Lorentz group are defined by

$$z' = LzR, \tag{159}$$

where independent biquaternions L and R satisfy the requirement that $L\overline{L} = 1$, $R\overline{R} = 1$. As a result there are 12 independent parameters defining the $SO(4,c)$ group in which remains z_μ^2 invariant. Indeed,

$$z_\mu'^2 = z'\overline{z}' = LzR\overline{R}\overline{z}\overline{L} = z\overline{z} = z_\mu^2. \tag{160}$$

In the case of the ordinary Lorentz group we should put $R = \overline{L}^*$.

It should be noted that quaternion algebra can be realized using the Pauli $2\times 2-$matrices τ_k ($k = 1, 2, 3$). Setting $e_4 = I_2$, $e_k = i\tau_k$ and using the properties of Pauli's matrices:

$$\begin{gathered} \tau_m\tau_n = i\epsilon_{mnk}\tau_k + \delta_{mn}, \\ \tau_i\tau_k + \tau_k\tau_i = 2\delta_{ik}, \end{gathered} \tag{161}$$

we come to the quaternion algebra (148).

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