Lattice Models with $\mathcal{N}=2$ Supersymmetry

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We introduce lattice models with explicit $\mathcal{N}$=2 supersymmetry. In these interacting models, the supersymmetry generators $Q^{\pm}$ yield the Hamiltonian $H=\{Q^+,Q^-\}$ on any graph. The degrees of freedom can be described as either fermions with hard cores, or as quantum dimers. The Hamiltonian of our simplest model contains a hopping term and a repulsive potential, as well as the hard-core repulsion. We discuss these models from a variety of perspectives: using a fundamental relation with conformal field theory, via the Bethe ansatz, and using cohomology methods. The simplest model provides a manifestly-supersymmetric lattice regulator for the supersymmetric point of the massless 1+1-dimensional Thirring (Luttinger) model. We discuss the ground-state structure of this same model on more complicated graphs, including a 2-leg ladder, and discuss some generalizations.

Supersymmetry is an exceptionally powerful theoretical tool. It often allows exact computations in field theory and string theory, even when interactions are strong. In this paper we introduce $\mathcal{N}=2$ supersymmetric lattice models describing interacting fermions and monomer-dimer systems. We show that the continuum limit of the simplest of these models, defined on a one-dimensional lattice, is a well-known $1+1$-dimensional quantum field theory with $\mathcal{N}=2$ superconformal symmetry.

Our definition of $\mathcal{N}$=2 supersymmetry is that the Hamiltonian H is built from two nilpotent fermionic generators denoted Q^+ and $Q^-=(Q^+)^\dagger$ [1]. It is

$$H=\{Q^+,Q^-\}. \tag{1}$$

The fact that Q^+ and Q^- commute with H follows from the nilpotency $(Q^+)^2=(Q^-)^2=0$. Our models also have a fermion-number symmetry generated by F with $[F,Q^{\pm}]=\pm Q^{\pm}$. We shall show how, in at least some cases, this lattice supersymmetry extends to a space-time super(conformal) symmetry in the field theory describing the continuum limit. Lattice models with a symmetry involving fermionic generators, such as the t-J model at J=$\pm 2t$, are often called "supersymmetric" in the condensed-matter literature, but do not have a Hamiltonian of the form eq. (1).

All eigenvalues E of the Hamiltonian eq. (1) satisfy $E\geq 0$. All eigenstates form either singlet or doublet representations of the supersymmetry algebra. All states $|g\rangle$ with $E=0$ must be singlets: $Q^+|g\rangle=Q^-|g\rangle=0$ [1]. Conversely, all singlets must have $E=0$. All the other eigenstates of H can be decomposed into doublets under the supersymmetry, and conversely any doublet representation is an eigenstate. This is simple to prove: a doublet consists of two states $|s\rangle, Q^+|s\rangle$, where $Q^-|s\rangle=0$. It follows from the definition of H and the nilpotency of $Q^{\pm}$ that both of these states are eigenstates of H with the same eigenvalue. All eigenstates can be decomposed into doublets: the four-dimensional representation $(|s'\rangle, Q^-|s'\rangle, Q^+|s'\rangle, Q^+Q^-|s'\rangle)$ is reducible. Let

$$|s\rangle \equiv |s'\rangle - \frac{1}{E_s}Q^+Q^-|s'\rangle$$

where $E_s>0$ is defined by $H|s'\rangle=E_s|s'\rangle$. Then $Q^-|s\rangle=0$, and $(|s\rangle, Q^+|s\rangle)$ and $(Q^-|s'\rangle, Q^+Q^-|s'\rangle)$ form two irreducible doublets.

The models we introduce can be defined on any lattice (or actually, any graph) in any dimension. The simplest model involves a single species of fermion c_i, placed at any site i of the lattice. The fermion obeys the usual anti-commutator $\{c_i,c_j^\dagger\}=\delta_{ij}$, and the operator $F=\sum_i c_i^\dagger c_i$ counts the number of fermions. We impose the restriction that the fermions have hard cores, meaning that fermions are not allowed on neighboring sites. A hard-core fermion is created by $c_i^\dagger\mathcal{P}_{<i>}$, where the projection operator $\mathcal{P}_{<i>}$ requires all sites neighboring i to be empty:

$$\mathcal{P}_{<i>}=\prod_{j\text{ next to }i}(1-c_j^\dagger c_j)\ . \tag{2}$$

On this space of states, the supersymmetry operators are defined by

$$Q^+=\sum_i c_i^\dagger\mathcal{P}_{<i>} \qquad\qquad Q^-=\sum_i c_i\mathcal{P}_{<i>}. \tag{3}$$

It is easy to verify that $(Q^+)^2=(Q^-)^2=0$. The Hamiltonian is therefore

$$H=\sum_i\sum_{j\text{ next to }i}\mathcal{P}_{<i>}c_i^\dagger c_j\mathcal{P}_{<j>}+\sum_i\mathcal{P}_{<i>}. \tag{4}$$

The first term in the Hamiltonian allows fermions to hop to neighboring sites, with the projectors maintaining the hard-core repulsion. The second term favors having more fermions, as long as they are more than two sites from each other. Thus one can view it as a repulsive potential for fermions, in addition to the hard core.

There are two key questions to try to answer. The first is: what properties can be computed exactly using the supersymmetry? We have already noted the positive energy and the pairing in the excited-state spectrum, but these are just the simplest consequences of the supersymmetry. The second question is: what (if any) field theory describes the model in the continuum limit?

To illustrate the power of supersymmetry, we find the ground states for a chain of six sites and periodic boundary conditions. First, we count all the states. There is one state $|0\rangle$ with f=0, and six states $c_i^\dagger|0\rangle$ with f=1, while because of the hard cores, there are nine states with f=2, and two with f=3. The vacuum obeys $Q^-|0\rangle = 0$ and $Q^+|0\rangle = \sum_{i=1}^6 c_i^\dagger|0\rangle$, so $(|0\rangle, Q^+|0\rangle)$ make up a doublet. The remaining five states with f=1 are all annihilated by Q^-, and Q^+ acts non-trivially on them. There are thus five doublets with $(f, f+1) = (1,2)$. The states with f=3 are both annihilated by Q^+, and Q^- acts non-trivially on both, giving two doublets with $(f, f+1) = (2,3)$. This accounts for all the states in the theory, except for two states with f=2. These two cannot form a doublet, because they have same fermion number. They therefore must be singlets, so there are two E=0 ground states in this theory, both with f=2. With a little more work, one finds that they have eigenvalues $\exp(\pm i\pi/3)$ under translation by one site.

A basic quantity in a supersymmetric theory is the Witten index [1]

$$W = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right]. \tag{5}$$

Because the two states in a doublet have the same energy, their contribution to W cancels, leaving the trace only over ground states and W independent of β. W can thus be found by evaluating (5) in the $\beta \to 0$ limit, where all states contribute with weight $(-1)^F$. For example, for the six-site chain discussed above, we confirm that $W = 1 - 6 + 9 - 2 = 2$. For our model on the cube, one finds $W = 1 - 8 + 16 - 8 + 2 = 3$.

Computing W for the model eq. (4) on a general graph poses a fascinating combinatorial problem. Cohomology theory is a powerful tool to compute the number of ground states at any fermion number, and therefore also W. The supersymmetry generator Q^+ satisfies $(Q^+)^2 = 0$, and the $E = 0$ ground states are precisely the states $|s\rangle$ that satisfy $Q^+|s\rangle = 0$ and that cannot be written in the form $|s\rangle = Q^+|s'\rangle$. Those states form what is called the cohomology of the operator Q^+ and for its computation a variety of techniques are available. These include the 'spectral sequence' technique, which can be applied in this context as follows: one splits the lattice in two sublattices, with corresponding fermion number operators F_1 and F_2, so that $F = F_1 + F_2$. Q^+ can also be split as $Q_1^+ + Q_2^+$, so that Q_i^+ increases F_i by one. The two operators Q_1^+ and Q_2^+ are nilpotent and anti-commute. The first step in the spectral sequence is to compute the cohomology of Q_1^+. Q_2^+ becomes an operator acting on this cohomology, and the second step is to compute the cohomology of Q_2^+ on this subspace. Often the process terminates here, and the result is the cohomology of Q^+. In general the procedure continues for a finite number of additional steps, see e.g. [2] for details. A similar procedure exists for a decomposition of a lattice into n sublattices, and in particular when every sublattice consists of a single site. Applying this to the N-site periodic chain, with F_1 consisting of every third site, we find ground states solely at fermion number $f = \mathrm{int}((N+1)/3)$. For a chain with $N = 3p$ with p integer, we find two ground states (so $W = 2(-1)^p$), while for $N = 3p \pm 1$, there is a single ground state (so $W = (-1)^p$).

We now address our questions in the one-dimensional case, where the Hamiltonian eq. (4) is on an N-site chain:

$$H = \sum_{i=1}^{N} \left[P_{i-1} \left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i \right) P_{i+2} + P_{i-1} P_{i+1} \right] \tag{6}$$

where $P_i \equiv 1 - c_i^\dagger c_i$ is the projector on a single site. We take periodic boundary conditions, so indices are defined mod N. The translation operator T commutes with both H and F, and its eigenvalues t satisfying $t^N = 1$ characterize the eigenstates of H. The Hamiltonian eq. (6) resembles a lattice version of the Thirring model, a $1+1$-dimensional field theory with a four-fermion interaction term. Below we will make the connection precise.

Before we apply the Bethe ansatz, we indicate how the space of states $\mathcal{H}_N$ of the model can be obtained by applying a systematic 'finitization' procedure [3] to the chiral spectrum of a specific $\mathcal{N} = 2$ superconformal field theory (SCFT) with central charge $c = 1$. This construction follows two steps. In the first step, the full chiral Hilbert space of the SCFT is written in a 'quasi-particle' basis, with the fundamental quasi-particles forming a supersymmetry doublet with charges $(1/3, -2/3)$. In the second step, the momenta of the individual quasi-particles are constrained to a maximum value in the order of N, corresponding to a discretization of the space direction of the SCFT with spacing of the order $1/N$. This then leads to a truncated or 'finitized' partition sum $Q_N(q, w)$, which is closely related to the partition sum $Z_N(q, w) = \mathrm{Tr}(q^{\frac{N}{2\pi i} \log T}\, w^{(N-3F)})$, which keeps track of the eigenvalues of the translation operator T and the fermion number F of the lattice model eq. (6). This entire construction, to be detailed elsewhere, respects the supersymmetry. This gives a rationale for the existence of supersymmetry generators on the space $\mathcal{H}_N$, and it provides an independent way of determining the quantum numbers t and f of the supersymmetric ground states. We remark that an analogous construction based on the spinon basis of the simplest $SU(2)$-invariant CFT naturally leads to the space of states of the Heisenberg and Haldane-Shastry models for N spin-1/2 degrees of freedom. Clearly, many generalizations are possible.

Having discussed the model eq. (6) from different points of view, we now give some of the results of a Bethe ansatz computation [4]. This computation lets us identify the continuum limit of the theory. An eigenstate of H with f fermions is of the form

$$\phi^{(f)} = \sum_{\{i_k\}} \varphi(i_1, i_2, \dots i_f) c^\dagger_{i_1} c^\dagger_{i_2} \dots c^\dagger_{i_f} |0\rangle \tag{7}$$

where we order $1 \le i_1 < i_2 - 1 < i_3 - 2 \dots$. Bethe's ansatz for the eigenstates of H is [5]

$$\varphi(i_1, i_2, \dots i_f) = \sum_P A_P\, \mu_{P1}^{i_1-1} \mu_{P2}^{i_2-1} \dots \mu_{Pf}^{i_f-1}. \tag{8}$$

for some numbers $\{\mu_1, \dots, \mu_f\}$ and A_P; the sum is over permutations P of the set $(1, 2, \dots, f)$. By construction, the translation operator T has eigenvalue $t = \prod_{k=1}^f \mu_k^{-1}$.

The next step is to find highest-weight states under the symmetries of the model. In Bethe's case, the symmetry is the $O(3)$ of the Heisenberg spin chain; here it is the supersymmetry. Here, $Q^- \phi^{(f)} = 0$ requires

$$t^{-1} (\mu_k)^{N-f} = \prod_{j=1}^f \frac{\mu_k \mu_j + 1 - \mu_k}{\mu_k \mu_j + 1 - \mu_j} \tag{9}$$

for all $k = 1 \dots f$. These are very similar to the Bethe equations for the antiferromagnetic XXZ spin chain at $\Delta = \pm 1/2$ [6]. The only difference in (9) is in the left-hand-side, which in the XXZ case reads $(\mu_k)^N \tau$, where $\tau \neq 1$ corresponds to twisted boundary conditions.

Demanding that a state be an eigenstate of a continuous symmetry fixes all the free parameters in the Bethe ansatz. The miracle of the ansatz is that in the Heisenberg and other integrable models, this state is also an eigenstate of the Hamiltonian. In our case, supersymmetry provides the miracle. Any Bethe ansatz state obeying $Q^- |s\rangle = 0$ is either a singlet or part of a doublet, and so it must be an eigenstate of H. Its energy is

$$E = N - 2f + \sum_{k=1}^f \left[\mu_i + \frac{1}{\mu_i} \right]. \tag{10}$$

The supersymmetry doublets appear naturally within the Bethe ansatz. If $(\mu_1, \dots \mu_f)$ satisfies the Bethe equations, then the set $(1, \mu_1, \dots \mu_f)$ also must satisfy the Bethe equations, and moreover, both sets have the same energy E. It is straightforward to check that the states associated with these two sets are related by $\phi^{(f+1)} = Q^+ \phi^{(f)}$.

The Bethe equations are f coupled polynomial equations of order N. They cannot be solved in closed form, and to make further progress, one usually needs to take N large. In our case, however, the supersymmetry allows us to derive more results from the Bethe ansatz for finite N. Precisely, we define w_k in terms of μ_k as $w_k = (\mu_k - q)/(q\mu_k - 1)$, where $q \equiv \exp(-i\pi/3)$. Then Baxter's Q-function [6] $\mathcal{Q}(w) \equiv \prod_{i=1}^f (w - w_k)$ has zeroes at $w = w_k$. Defining $\mathcal{R}(w) \equiv \mathcal{Q}(w)(1+w)^{N-f}$, we find that for the w_k giving the ground state,

$$\mathcal{R}(q^{-2} w) = t q^{-N} \mathcal{R}(w) + t^{-1} q^N \mathcal{R}(q^2 w). \tag{11}$$

We derive an explicit expression for $\mathcal{R}(w)$ in the sequel [4], but from (11) directly we can rederive f and t for the ground state(s). When $N = 3p$ with p an integer, there are non-trivial solutions to (11) only when $f = N/3$ and $t = (-1)^N \exp(\pm i\pi/3)$. For $N \neq 3p$, one has only a single solution with $f = \mathrm{int}((N+1)/3)$, and $t = (-1)^{N-1}$.

We can see heuristically why the ground state has $f = \mathrm{int}((N+1)/3)$. The potential term in (6) alone is minimized by the state with a fermion on every third site; adding any more fermions forces fermions to be two sites away and raises the energy. The hopping term alone also discourages fermions from being only two sites away, because it has negative eigenvalues when fermions can hop to an adjacent site and back again, and the hard cores prevent this if there is another fermion two sites away. The state with a fermion on every third site,

$$\cdots \bullet \circ \circ \bullet \circ \circ \bullet \circ \circ \bullet \circ \circ \bullet \circ \circ \bullet \circ \circ \bullet \circ \circ \bullet \circ \circ \bullet \circ \circ \dots$$

resembles a Néel state for a Heisenberg anti-ferromagnet. It is not an eigenstate of the Hamiltonian: the full ground state is disordered. However, like the Néel state, we expect this state to be a part of the ground state. Our derivations of $f_{GS} = \mathrm{int}((N+1)/3)$ confirm this intuition. This heuristic picture also gives the fermion numbers of the low-lying excited states. The excitations include defects (domain walls) in the Néel-like state, such as

$$\cdots \bullet \circ \circ \bullet \circ \bullet \circ \circ \bullet \circ \circ \bullet \circ \bullet \circ \circ \bullet \circ \circ \bullet \circ \bullet \circ \circ \bullet \circ \circ \dots$$

The fermion number of this configuration is just one higher than that of the Néel-like state, and it has three identical defects. Since defects can be moved arbitrarily far apart with no change in the potential, it is natural to treat each defect as a quasiparticle with charge $1/3$. The existence of fractional charge in $1+1$ dimensions is an old story; this was first discovered in field theory [7].

Finally, we give the field theory describing the continuum limit of (6). When taking N large, one can rewrite the Bethe equations in terms of densities of roots [5], and then derive integral equations (known as thermodynamic Bethe ansatz equations) yielding the free energy. Our model has the same thermodynamic equations as the XXZ chain at $\Delta = 1/2$, so the two models coincide in the continuum limit. The continuum limit of the XXZ chain is described by the massless Thirring model [8], or equivalently a free massless boson Φ with action [9]

$$S = \frac{2g}{\pi}\int dx\, dt\ \left[(\partial_t\Phi)^2 - (\partial_x\Phi)^2\right].$$

The continuum limit of the $\Delta = 1/2$ model has $g = 2/3$; this is the simplest field theory with $\mathcal{N} = (2,2)$ superconformal symmetry [9]. The $(2,2)$ means that there are two left and two right-moving supersymmetries: in the continuum limit the fermion decomposes into left- and right-moving components over the Fermi sea. The boson also can be decoupled into left and right pieces, so that $\Phi = \Phi_L + \Phi_R$, while its dual $\widetilde{\Phi} = g(\Phi_L - \Phi_R)$. The states of the field theory are given by the vertex operators $V_{m,n} = \exp(im\Phi + in\widetilde{\Phi})$, of conformal dimensions $h_{L,R} = (m \pm gn)^2/(4g)$. The four components of the Dirac fermion in the Thirring model are $V_{\pm1,\pm1/2}$, while the supersymmetry generators are $Q^{\pm}_L = V_{\pm1,\pm3/2}$ and $Q^{\pm}_R = V_{\pm1,\mp3/2}$. In a finite size $\mathcal{L}$, the lowest-energy state is in the Neveu-Schwarz sector, where the Thirring fermion has anti-periodic boundary conditions. This state has $E_{NS} = -\pi/(6\mathcal{L})$. The lowest-energy states in the Ramond (periodic boundary conditions) sector are given by $|\pm\rangle_R = V_{0,\pm1/2}|0\rangle_{NS}$; both have energy zero. States in this conformal field theory can be built up by operating with the "spinons" $V_{\pm1/3,\pm1/2}$.

Comparing this superconformal field theory with the lattice model on $N = 3p$ sites, we identify our two $E = 0$ ground states with the two Ramond vacua; all other states of the lattice model are in the Ramond sector as well. The $U(1)$ quantum number m corresponds to $f - N/3$ (the fermion number relative to the ground state). The spinons here have charge $\pm1/3$, so it is natural to identify these with the fractionally-charged excitations in the lattice model.

One can define a quantum dimer model by placing the variables $c_i^\dagger$ on the links instead of the sites of a lattice. The product in $\mathcal{P}_{<i>}$ is then over links which meet the link i, and the projections are precisely such that overlapping dimers are avoided. On the $N = 3p$-site chain, the ground states have dimer number $N_d = N/3$, which is 2/3 of the value for close-packed dimers. A more involved example is that of the supersymmetric dimer model on a 2-leg ladder with $N+1$ rungs. Using cohomology techniques, we found that the number N_{GS} of ground states grows quickly with N, and that not all ground states have the same dimer number N_d, leading to (partial) cancellations in the Witten index W. For example, for $N = 7$ one finds five $E = 0$ ground states with $N_d = 5, 5, 5, 5, 6$, so $W = -3$. Our results, asymptotically precise for N large, are $N_{GS} \sim (1.395)^N$ and $W \sim (1.356)^N$. These huge degeneracies suggest additional symmetries in the model. It will be interesting to see if supersymmetry can be of use in other quantum dimer models [10].

Following the ideas above, one can obtain other supersymmetric models by including more projectors in (3), or by including several species of fermions. Consider a two-species model, with fermions labeled by $+$ and $-$, and with the conditions that (i) a single site may not be occupied by two particles and (ii) same-type particles may not occupy nearest neighbor sites. On a periodic chain with $4n$ sites, the Witten index of this model turns out to be $W = 3$, and we have strong indications that the continuum limit of this theory is the second model (at $c = 3/2$) of the series of $\mathcal{N}$=2 minimal superconformal field theories. A typical ground state pattern is

$$\ldots \circ + \circ - \circ + \circ - \circ + \circ - \circ \ldots$$

and one recognizes the possibility of domain walls of charge $\pm1/2$ ($\circ + - \circ$) and neutral defects ($+ \circ +$). The $+/-$ pattern indicates an Ising substructure in the model, in accord with the fact that the $c = 3/2$ $\mathcal{N} = 2$ minimal model can be written in terms of a Majorana fermion and a free boson.

We finally remark that in our models, the space of states is made up solely of fermions on which the supersymmetry acts non-linearly. (This does not preclude a linear realization on bosons and fermions in the continuum theory.) Such realizations of $\mathcal{N} = 2$ lattice supersymmetry seem very different from other known realizations, for example in lattice gauge theory [11].

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[1] E. Witten, Nucl. Phys. **B202** (1982) 253.
[2] R. Bott and L.W. Tu, *Differential Forms in Algebraic Topology*, GTM 82, (Springer Verlag, New York, 1982).
[3] K. Schoutens, Phys. Rev. Lett. **79** (1997) 2608 [cond-mat/9706166]; P. Bouwknegt and K. Schoutens, Nucl. Phys. B **547** (1999) 501 [hep-th/9810113].
[4] P. Fendley and K. Schoutens, to appear.
[5] H. Bethe, Z. Phys. **71** (1931) 205.
[6] R.J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, 1982) and references therein.
[7] R. Jackiw and C. Rebbi, Phys. Rev. D **13** (1976) 3398; J. Goldstone and F. Wilczek, Phys. Rev. Lett. **47** (1981) 986.
[8] H. B. Thacker, Rev. Mod. Phys. **53** (1981) 253.
[9] D. Friedan and S. H. Shenker, in C. Itzykson, H. Saleur and J.B. Zuber, *Conformal invariance and applications to statistical mechanics* (World Scientific, 1988).
[10] R. Moessner, S.L. Sondhi, E. Fradkin, Phys. Rev. B **65** (2002) 024504 [cond-mat/0103396], and references therein.
[11] A. Feo, arXiv:hep-lat/0210015 and references therein.