y-DEFORMED BPS Dp- BRANES ON A SURFACE IN A CALABI-YAU THREEFOLD

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ABSTRACT

Using y-deformed algebraic geometric techniques the y-deformed Mukay vector of RR-charges of the y-deformed BPS Dp-branes localized on a sufarce in a Calabi-Yau threefold. The formules that are obtained here are generalizations of the formulaes of the fourth section of the preprint hep-th/0007243

# 1 Introduction:y-deformed BPS Dp-branes on a Calabi-Yau threefold

A BPS D-brane on a Calabi-Yau threefold X can be represented using a coherent $O_X$-module G. The RR charge of G is given by the Mukai vector[1]:

$$\boldsymbol{v_X(G) = ch(G)\sqrt{Todd(T_X)} \in H_{2*}(X;Q) := \oplus_{i=0}^{3} H_{2i}(X;Q)}$$

where $ch(G) = \sum_{i=0}^{3} ch_i(G)$ is the Chern character with $ch_i(G) \in H_{6-2i}(X;Q)$, which can be computed by the homology-cohomology duality[1]: always one can to have a resolution of G by locally free sheaves $(V_*)$,in such way that one can to set that $ch(G) := \sum_{i=0}^{3}(-1)^i ch(V_i)$,and these result does not depend on the choise of the resolution. Finally $Todd(T_X) = [X] + \frac{\mathrm{c_1[X]}}{2} + \frac{\mathrm{c_2[X]+c_1[X]^2}}{12} + \frac{\mathrm{c_2[X]c_1[X]}}{24}$. Now when X is a Calabi-Yau threefold one has $c_1[X] = 0$ and then one obtains: $Todd(T_X) = [X] + \frac{\mathrm{c_2[X]}}{12}$. From these the effect of the square root of the Todd Class on the RR charges,is to say the geometric version of the Witten effect is given by:

$$\boldsymbol{\sqrt{Todd(T_X)} = [X] + \frac{\mathrm{c_2[X]}}{24}}$$

For the investigation of the topological aspects of D-branes is of the great importance to obtain several basic invariants of BPS D-Branes. One of these invariants is the RR charge of the D-brane. Other invariant is the intersection form on D-branes on X [1]. This invariant for intersections of two Dp-branes is obtained by multiplication of the Mukay vectors of RR charges corresponding to the intersecting Dp-branes and is given by: [1]

$$\boldsymbol{I_X(G_1,G_2) = [v_X(G_1)^v.v_X(G_2)]_X =}$$

$$\boldsymbol{[(ch(G_1)\sqrt{Todd(T_X)})^v.ch(G_2)\sqrt{Todd(T_X)}]_X =}$$

$$\boldsymbol{[ch(G_1)^v.ch(G_2)Todd(T_X)]_X}$$

where $[...]_X$ evaluates the degree of $H_0(X;Q)\cong Q$ component, and $v^\vee$ flips the sign of $H_0(X)\oplus H_4(X)$ part of the Mukay vector v . In particular, if G itself is locally free, then $ch(G)^\vee = ch(G^\vee)$, where $G^\vee = Hom_X(G, 0_X)$ is the dual sheaf. Finally is easy to check that: $I_X(G_1,G_2) = -I_X(G_2,G_1)$. On other hand the invariant of intersection between D-branes is an application of the Hirzebruch-Riemann-Roch and for then you can write[1]

$$\boldsymbol{I_X(G_1,G_2) = \textstyle\sum_{i=0}^{3}(-1)^i dimExt_X^i(G_1,G_2)}$$

For this reason the skew-symmetric property $I_X(G_1,G_2) = -I_X(G_2,G_1)$ of the intersection form $I_X$ for the intersection of two Dp-branes may be attributed to the Serre duality: $Ext_X^i(G_1,G_2)\cong Ext_X^{3-i}(G_1,G_2)^\vee$ [1]. Another interesting comentary is that from the integrality theorems for diferential and complex manifolds the formula H.R.R. is an integer and this assures that $I_X$ takes values in Z. [1],[2].

Now the result that this work presents is about the y-deformed Dp-branes on a Calabi Yau threefold. A y-deformed BPS Dp-brane on a Calabi-yau X can be represented by a y-deformed $O_X - moduloG$. The y-deformed RR charge of G is given by the y-deformed Mukai vector:

$$\boldsymbol{v_{X,y}(G) = ch_y(G)\sqrt{\chi_y(T_X)}\in(H_{2*}(X;Q)\otimes Q[y]) :=}$$
$$\boldsymbol{\oplus_{i=0}^{3}(H_{2i}(X;Q)\otimes Q[y])}$$

where $\chi_y$ is the y-chi-genus which is a generalization of the Todd class [2,3] and $ch_y(G)$ is the y-deformed Chern Character. the total Chern Class for $T_X$ has the following sumarization:

$$\boldsymbol{c(T_X) = \textstyle\sum_{j=0}^{3} c_j(T_X)}$$

also, the total Chern Class for the such bundle has the following factorization:

$$c(T_X) = \prod_{i=1}^{3}(1 + x_i)$$

The CHI-y- genus for $T_X$ has the following formal factorisation:

$$\chi_y(T_X) = \prod_{i=1}^{3} \frac{(1+y\exp(-(y+1)x_i))x_i}{1-\exp(-(y+1)x_i)}$$

The CHI-y- genus for $T_X$ has the following formal sumarisation in terms of the y-deformed Todd polynomials which are formed from the corresponding Chern classes and from the polynomials on y :

$$\chi_y(T_X) = \sum_{j=0}^{\infty} T_j(c_1(T_X), ..., c_j(T_X), y)$$

The y-Todd polynomials are given by:

$$T_0(c_0(T_X), y) = T_0(1, y) = 1$$

$$T_1(c_1(T_X), y) = \frac{(1-y)c_1(T_X)}{2}$$

$$T_2(c_1(T_X), c_2(T_X), y) = \frac{(y+1)^2 c_1(T_X)^2 + (y^2-10y+1)c_2(T_X)}{12}$$

$$T_3(c_1(T_X), c_2(T_X), c_3(T_X), y) = \frac{-(y+1)^2(y-1)c_1(T_X)c_2(T_X) + 12y(y-1)c_3(T_X)}{24}$$

.

Then one has:

$$\chi_y(T_X) = 1 + \frac{(1-y)c_1(T_X)}{2} + \frac{(y+1)^2 c_1(T_X)^2 + (y^2-10y+1)c_2(T_X)}{12} + \frac{-(y+1)^2(y-1)c_1(T_X)c_2(T_X) + 12y(y-1)c_3(T_X)}{24}$$

.

When X is a Calabi-Yau threefold then the chi-y-genus is given by

$$\chi_y(T_X) = 1 + \frac{(y^2-10y+1)c_2(T_X)}{12} + \frac{12y(y-1)c_3(T_X)}{24}$$

.

From this one can to write the following formula for the y-deformed geometric version of the Witten effect:

$$\sqrt{\chi_y(T_X)} = [X] + \frac{(y^2-10y+1)c_2[X]}{24} + \frac{y(y-1)c_3[X]}{4}$$

when y=0 one obtains the usual Witten effect:

$$\sqrt{\chi_0(T_X)} = [X] + \frac{(0^2-0+1)c_2[X]}{24} + \frac{0(0-1)c_3[X]}{4} = [X] + \frac{c_2[X]}{24}$$

For the other hand the y-deformed Chern Character $ch_y(G)$ is given by: $ch_y(G) = \sum_{i=0}^{3} ch_{i,y}(G)$ with $ch_{i,y}(G) \in (H_{6-2i}(X;Q) \otimes Q[y])$, which can be computed using y-deformed homology-cohomology duality: always one can to have a y-deformed resolution of G by y-deformed locally free sheaves $(V_*)$,in such way that one can to set that $ch_y(G) := \sum_{i=0}^{3}(-1)^i ch_y(V_i)$,and these result does not depend on the choise of the y-deformed resolution. The total Chern Class for G has the following sumarization:

$$c(G) = \sum_{j=0}^{q} c_j(G)$$

also, the total Chern Class for G has the following factorization:

$$c(G) = \prod_{i=1}^{q}(1 + z_i)$$

The total Chern character of G is defined by:

$$ch(G) = \sum_{j=1}^{q} e^{z_i}$$

The total y-deformed Chern character for G has the following sumarization:

$$ch_y(G) = \sum_{j=1}^{q} e^{(1+y)z_i}$$

The total y-deformed Chern character for G has the following expantion in terms of the Chern class of G and polynomials for y:

$$ch_y(G) = rk(G) + (y+1)c_1(G) + (y+1)^2\left(\frac{c_1(G)^2 - c_2(G)}{2}\right) + (y+1)^3\left(\frac{c_1(G)^3 - 3c_1(G)c_2(G) + 3c_3(G)}{6}\right)$$

It is easy to see that when y=0, one obtains the usual expantion for the usual Chern character. For the investigation of the topological aspects of the y-deformed D-branes is of the great importance to obtain several basic y-deformed invariants of y-deformed BPS D-Branes. One of these y-deformed invariants is the y-deformed RR charge of the y-deformed D-brane. Other y-deformed invariant is the y-deformed intersection form on y-deformed D-branes on X . This y-deformed invariant for intersections of two y-deformed Dp-branes is obtained by multiplication of the y-deformed Mukay vectors of the y-deformed RR charges corresponding to the intersecting y-deformed Dp-branes and is given by:

$$\boldsymbol{I_{X,y}(G_1,G_2) = [v_{X,y}(G_1)^v . v_{X,y}(G_2)]_X =}$$

$$\boldsymbol{[(ch(G_1)\sqrt{\chi_y(T_X)})^v . ch(G_2)\sqrt{\chi_y(T_X)}]_X = [ch(G_1)^v . ch(G_2)\chi_y(T_X)]_X}$$

where $[...]_{X,y}$ evaluates the degree of $(H_0(X;Q)\otimes Q[y])\cong(Q\otimes Q[y])$ component, and $v^\vee$ flips the sign of $(H_0(X)\otimes Q[y])\oplus(H_4(X)\otimes Q[y]))$ y-deformed part of the y-deformed Mukay vector v . In particular, if G itself is locally free, then $ch_y(G)^\vee = ch_y(G^\vee)$, where $G^\vee = Hom_X(G, 0_X)$ is the y-deformed dual sheaf. Finally is easy to check that: $I_{X,y}(G_1,G_2) = -I_{X,y}(G_2,G_1)$.

On other hand the y-deformed invariant of intersection between y-deformed D-branes is an application of the y-deformed Hirzebruch-Riemann-Roch and for then you can write:

$$\boldsymbol{I_{X,y}(G_1,G_2) = \sum_{i=0}^{3}(-1)^i dim Ext^i_{X,y}(G_1,G_2)}$$

For this reason the skew-symmetric property $I_{X,y}(G_1,G_2) = -I_{X,y}(G_2,G_1)$ of the intersection form $I_{X,y}$ for the intersection of two y-deformed Dp-branes may be attributed to the y-deformed Serre duality: $Ext^i_{X,y}(G_1,G_2)\cong Ext^{3-i}_{X,y}(G_1,G_2)^\vee$. Another interesting comentary is that from the y-deformed integrality theorems for diferential and complex manifolds the y-deformed formula H.R.R. is an polynomial on y and this assures that $I_{X,y}$ takes values in Q[y].

Now let $J_{X,y}\in(H_4(X;R)\otimes R[y])$ be a y-deformed Kahler form on X, whis is here identified with an y-deformed R-extended ample divisor. The y-deformed classical expression of the y-deformed central charge of the y-deformed D-brane G is then given by [1]:

$$\boldsymbol{Z^d_{J_{X,y}}(G) = -[e^{-J_{X,y}} . v_{X,y}(G)]_X = -\sum_{k=0}^{3}\frac{(-1)^k}{k!}[J^k_{X,y} . v_{X,y,k}(G)]_X}$$

where $v_{X,y,k}$ is the $H_{2k}(X)\otimes Q[y]$ component of $v_{X,y}\in(H_{2*}(X;Q)\otimes Q[y])$.

In such way we obtain the three y-deformed invariants: y-deformed RR charge, y-deformed central charge and y-deformed intersections pairings of two y-deformed BPS Dp-branas. With this aid of some algebraic geometry-topology techniques we can to begin the study of topological aspects of y-deformed BPS Dp-branes bounded on a proyective algebraic surface in a Calabi-Yau threefold X.

# 2 y-deformed BPS Dp-branes localized on a surface in a Calabi-Yau threefold

Let f be an embedding of a proyective algebraic surface S in a Calabi-Yau threefold X. In the limit of infinite elliptic fiber, the y-deformed BPS Dp-branes for which the y-deformed central charge remains finite are those y-deformed BPS Dp-branes which are confined to the algebraic surface S. The physical and topological propertis of the y-deformed BPS D-p-branes localized on the algebraic surface S then dependen not on the details of the global model X, but only on the intrinsic y-deformed geometry of S ans its y-deformed normal bundle $N_{S,y} = N_{S|X,y}$ which is isomorphic to the y-deformed canonical line bundle $K_{S,y}$. In particular, this means that we can compute the y-deformed central charges of y-deformed BPS D-p-branes using y-deformed local mirror symmetry principle on S.

In a elementary physical configuration you have a y-deformed BPS Dp-brane sticking to S. Such y-deformed D-brane sticking to S can be described mathematically by a y-deformed $O_S - moduleE$. For this configuration an important y-deformed topological invariant is the y-deformed Euler number of E (the Euler y-polynomial for E) which is defined by $\chi_y(E) = \sum_{j=0}^{2}(-1)^i h^i(S,E,y)$, where $h^i(S,E,y) = dim(H^i(S,E))_y$. For to obtain the y-deformed Euler number of E or the Euler polynomial of E the first thing that one needs is the y-deformed Todd class of S or $\chi_y$ class of S:

$$\boldsymbol{\chi_y(T_S) = [S] + \frac{(1-y)c_1(S)}{2} + \frac{(y+1)^2c_1(S)^2+(y^2-10y+1)c_2(S)}{12}}$$

this expansion can be writen as:

$$\boldsymbol{\chi_y(T_S) = [S] + \frac{(1-y)c_1(S)}{2} + \chi_y(O_S)[pt]}$$

where:

$$\boldsymbol{\chi_y(O_S) = [\frac{(y+1)^2c_1(S)^2+(y^2-10y+1)c_2(S)}{12}]_S}$$

The second thing for to do is to apply the y-deformed H.R.R formula, and then one get:

$$\boldsymbol{\chi_y(E) = [ch_y(E)\chi_y(T_S)]_S = [ch_y(E)([S] + \frac{(1-y)c_1(S)}{2} + \chi_y(O_S)]_S =}$$

$$\boldsymbol{[(rk(E) + (y+1)c_1(E) + (y+1)^2(\frac{c_1(E)^2-c_2(E)}{2}))([S] + \frac{(1-y)c_1(S)}{2} +}$$

$$\boldsymbol{\chi_y(O_S)]_S =}$$

$$\boldsymbol{rk(E)\chi_y(O_S) + [(y+1)^2(\frac{c_1(E)^2-c_2(E)}{2})) + \frac{(y+1)(1-y)c_1(S).c_1(E)}{2}]_S}$$

From the other side, there is y-deformed canonical push-forward homomorphism $f_*$ from $H_{2*}(S;Q)\otimes Q[y]$ to $H_{2*}(X;Q)\otimes Q[y]$, which maps a y-deformed cycle on S that on X. Also, on can define the y-deformed coherent sheaf $f_!E$ on X by extending E by zero to X/S. Now using the y-deformation of the celebrated Grothendieck-Riemman-Roch formula for the embeding f od S in X, one can to relate the y-deformed chern characters of E and $f_!E$ as follows:

$$\boldsymbol{ch_y(f_!E) = f_*(ch_y(E)\frac{1}{chi_y(N_S)})}$$

Multiplying the boht sides of the y-deformed GRR formula by $\sqrt{\chi_y(T_X)}$ , one has:

$$\boldsymbol{ch_y(f_!E)\sqrt{\chi_y(T_X)} = f_*(ch_y(E)\sqrt{\frac{chi_y(T_S))}{chi_y(N_S)}})}$$

where we have used the y-deformed proyection formula:

$$f_*(a.f^*b) = f_*a.b$$

with $a\in(H_{2*}(S;Q)\otimes Q[y]), b\in(H_{2*}(X;Q)\otimes Q[y])$

and $f^*chi_y(T_X) = chi_y(T_S).chi_y(N_S)$ , which follows from the y-deformed short exact sequence of bundles on S: $0 - - - - - - > T_S - - - - > f^*T_X - - - - > N_S - - - - > 0$, combined with the multiplicative property of the chi-y-genus.

Now the y-deformed BPS Dp-brane on a Calabi-Yau threefold X is represented by G and y-deformed BPS Dp-brane sticking to S can be described by E then one has $G = f_!E$ and following formula for the y-deformed Mukai vector of the y-deformed RR charges of $G = f_!E$

$$v_{X,y}(f_!E) = ch_y(f_!E)\sqrt{\chi_y(T_X)}\in(H_{2*}(X;Q)\otimes Q[y]) := \oplus_{i=0}^{3}(H_{2i}(X;Q)\otimes Q[y])$$

The you have:

$$v_{X,y}(f_!E) = f_*(ch_y(E)\sqrt{\frac{\mathrm{chi_y(T_S))}}{\mathrm{chi_y(N_S)}}}) = f_*(v_{S,y}(E))$$

In such way the y-deformed RR charge of the y-deformed BPS Dp-brane represented by E on S regarded as a y-deformed BPS Dp-brane on X can written in the following intrinsic description (of the y-deformed RR charge on S):

$$v_{S,y}(E) = ch_y(E)\sqrt{\frac{\mathrm{chi_y(T_S))}}{\mathrm{chi_y(N_S)}}} = ch_y(E)\sqrt{\frac{\mathrm{chi_y(T_S))}}{\mathrm{chi_y(K_S)}}}$$

The y-deformed gravitational correction factor for S admits the following expansion:

$$\sqrt{\frac{\mathrm{chi_y(T_S))}}{\mathrm{chi_y(K_S)}}} =$$

$$[S] + \frac{\mathrm{(1-y)c_1(S)}}{\mathrm{2}} + \frac{\mathrm{(-10y+1+y^2)c_2(S)+3(y-1)^2c_1(S)^2}}{\mathrm{24}}\in(H_{2*}(S;Q)\otimes Q[y])$$

As a simple exercise one can to compute the y-deformed RR charge of a y-deformed sheaf on S. For this let i: C—¿ S be an embedding of a smooth genus g algebraic curve in S with the normal bundle $N_C = N_{C§}$. Then from a lin bundle $L_C$ on C, one obtains a y-deformed torsion sheaf $i_! L_C$ on S and $ch_y(i_! L_C)$ can be computed from the y-deformed G.R.R. formula:

$$\boldsymbol{ch_y(i_! L_C) = i_*(ch_y(L_C)\frac{1}{\mathrm{chi_y(N_C)}}) = i_*((rk(L_C) + (y+1)c_1(L_C)(1 +}$$

$$\boldsymbol{\frac{\mathrm{(y-1)c_1(N_C)}}{2})) = i_*[C] + ((y+1)c_1(L_C) + \frac{\mathrm{(y-1)c_1(N_C)}}{2})[pt] =}$$

$$\boldsymbol{i_*[C] + ((y+1)deg(L_C) + \frac{\mathrm{(y-1)deg(N_C)}}{2})[pt]}$$

where $deg(L) := [c_1(L)]_C$ for a line bundle on C. Then y-deformed RR charge of the y-deformed BPS Dp-brane bounded on S represented by the y-deformed $O_S - module\ i_! L_C$ can be computed as follows:

$$\boldsymbol{v_{S,y}(i_! L_C) = ch_y(i_! L_C)\sqrt{\frac{\mathrm{chi_y(T_C))}}{\mathrm{chi_y(K_C)}}} =}$$

$$\boldsymbol{(i_*[C] + ((y+1)deg(L_C) + \frac{\mathrm{(y-1)deg(N_C)}}{2})[pt])([C] + \frac{\mathrm{(1-y)c_1(C)}}{2}) =}$$

$$\boldsymbol{(i_*[C] + ((y+1)deg(L_C) + (1-y)c_1(C))[pt]) \in \oplus (H_0(S) \otimes Q[y]))}$$

I now turn again to intersection pairings of the y-deformed BPS Dp-branes one has the question about the what is the most appropiate intersection for on y-deformed D-branes on S. Here we will describe only y-deformed candite.

The y-deformed candidate uses the intrinsic y-deformed Mukay vector $v_{S,y}$ and defines a y-deformed symmetric form:

$$\boldsymbol{I_{S,y}(E_1, E_1) = -[v_{S,y}(E_1)^v . v_{S,y}(E_2)]_S =}$$

$$\boldsymbol{\frac{\mathrm{r_1 r_2 (y^2 - 10y + 1) chi(S)}}{12}) + [r_1 ch_2(E_2) + r_2 ch_2(E_1) - c_1(E_1).c_1(E_2)]_S}$$

where $ch(E) = r[S] + c_1(E) + ch_2(E)$, $\chi(S) = [c_2(S)]_S$ Is THE euler number, and $v_y^\vee = -v_{0,y} + v_{1,y} - v_{2,y}$ with $v_{i,y}$ being the y-deformed $(H_{2i}(S) \otimes Q[y])$ componente of the y-deformed vector $v_y$.

In constrast with $I_X$ that have values in $Q[y]$ and when y=0 then takes values in Z, now $I_S$ also have values in $Q[y]$ but in this case when y=0 $I_S$ is not Z-valued in general.

# 3 References

[1] hep-th/0007243

[2] F. Hirzebruch, Topological Methods in Algebraic Geometry, 1978