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Quantization of Noncommutative Scalar Solitons at finite θ

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Abstract

We start by discussing the classical noncommutative (NC) Q-ball solutions near the commutative limit, then generalize the virial relation. Next we quantize the NC Q-ball canonically. At very small θ quantum correction to the energy of the Q-balls is calculated through summation of the phase shift. UV/IR mixing terms are found in the quantum corrections which cannot be renormalized away. The same method is generalized to the NC GMS soliton for the smooth enough solution. UV/IR mixing is also found in the energy correction and UV divergence is shown to be absent. In this paper only $(2+1)$ dimensional scalar field theory is discussed.

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1 Introduction

Noncommutative (NC) field theory has been considered as certain limit of the effective action of open string modes on the brane [1, 2]. Since the discovery of NC solitons in NC scalar field theory by Gopakumar, Minwalla and Strominger (GMS) [3], the soliton solutions have been explicitly constructed in different gauge theories with or without matter [4, 14]. Such NC solitons can be interpreted as lower dimensional D-branes in string field theory [5, 6].

GMS solitons, which exist in $(2+1)$ dimensional NC scalar field theory, while classically stable, cease to exist at sufficiently small NC parameter θ , due to the nonexistence theorem of Derrick [7] in the commutative limit($\theta \to 0$). In this commutative limit, however, time dependent nontopological solitons, or Q-balls exist in all space dimensions [9, 10]. In this paper we'll study the NC generalization of time dependent Q-balls, NC Q-balls first. In particular, we found all the stable soliton solution family, which depend smoothly on the NC parameter θ and have a closed form in the NC limit ($\theta \to \infty$) [11], sustain for arbitrary small θ . Classically they all reduce to the ordinary Q-ball solution in the commuative limit. Thus it becomes interesting to investigate the quantum properties of NC solitons near the commutative limit. In the perturbative NC field theory UV/IR mixing occurs in renormalization [12]. We discuss the similar NC effects in the quantization of the nonperturbative NC field theory. The same method is further generalized to the GMS soliton case at finite θ . In this paper we deal, for simplicity, only with $(2+1)$ dimensional NC scalar field theory.

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2 Classical Noncommutative Q-ball Solution

2.1 Hamiltonian and Equation of motion

In this section we derive the equation of motion for NC Q-ball solutions, following brief introduction of NC scalar field theory. The form of the solution is already given in [13]. We discuss the existence and stablity of the solutions, and show that in the commutative limit NC Q-balls just reduce to the commutative Q-balls.

Consider a NC scalar field theory action with global $U(1)$ phase invariance,

$$S = -\int dt d^2x \left[\partial_\mu \bar{\phi} \partial^\mu \phi + V(\frac{1}{2}\{\bar{\phi}, \phi\}) \right] \ , \tag{1}$$

where the space-time metric is $(-,+,+)$, and the fields are multiplied by NC star product, generally made implicit in this paper, and $\{A\ , B\} \equiv A \star B + B \star A$. The potential V has a global minimum at the origin, with the scaling property

$$V(\bar{\phi}\phi) = g^{-2} V(g^2 \bar{\phi}\phi) \ . \tag{2}$$

g is then the coupling constant assumed to be small. The commutative limit of this action is where ordinary Q-balls have already been constructed [9, 10]. The NC star product is defined to be,

$$(\phi \star \psi)(x) \equiv \exp(i\frac{\theta}{2}\epsilon_{jk}\frac{\partial}{\partial x_j}\frac{\partial}{\partial y_k})\phi(x)\psi(y)\Big|_{y=x} \ , \tag{3}$$

where $j, k = 1, 2$. The NC algebra of the functions with the star product defined above is well known to be isomorphic to an algebra of operators on a one particle Hilbert space [8],

$$[\hat{x}^1\ ,\ \hat{x}^2] = i\theta \ . \tag{4}$$

In this isomorphism, also known as Weyl transform, the star product is just the operator product,

$$\phi(x) \star \psi(x) \longleftrightarrow \hat{\phi}(\hat{x})\hat{\psi}(\hat{x}) \ , \tag{5}$$

and

$$\int \phi(x) d^2x = 2\pi\theta \mathrm{Tr}\hat{\phi}(\hat{x}) \ , \tag{6}$$

where $\hat{\phi}(\hat{x}) \leftrightarrow \phi(x)$ are the Weyl transforms of each other. The derivative are written as

$$\frac{\partial}{\partial x^i}\phi(x) \longleftrightarrow \frac{i}{\theta}\epsilon_{ij}[\hat{x}^j, \hat{\phi}(\hat{x})] \ . \tag{7}$$

Define creation and annihilation operators

$$a = \frac{1}{\sqrt{2\theta}}(\hat{x}^1 + i\hat{x}^2) \ , \qquad a^\dagger = \frac{1}{\sqrt{2\theta}}(\hat{x}^1 - i\hat{x}^2) \ , \tag{8}$$

with $[a, a^\dagger] = 1$ as usual. The field $\phi(x)$ or $\hat{\phi}(\hat{x})$ can be expanded in the orthonormal basis $f_{nm}(x)$ or $|n><m|$ [14].

In the operator formalism the action integral (1) becomes

$$S[\hat{\phi}, \hat{\bar{\phi}}] = \int dt 2\pi\theta \mathrm{Tr}\left(\partial_0 \hat{\bar{\phi}} \partial_0 \hat{\phi} + \frac{1}{\theta}([a, \hat{\bar{\phi}}][a^\dagger, \hat{\phi}] + [a^\dagger, \hat{\bar{\phi}}][a, \hat{\phi}]) - V(\hat{\bar{\phi}}\hat{\phi}) \right) . \tag{9}$$

The equation of motion is

$$\partial_0^2 \hat{\phi} + \frac{2}{\theta}[a, [a^\dagger, \hat{\phi}]] + \hat{\phi} V(\hat{\bar{\phi}}\hat{\phi}) = 0 \ . \tag{10}$$

The action has global $U(1)$ phase invariance, which yields a conserved charge

$$Q[\hat{\phi}, \hat{\bar{\phi}}] = \int d^2x j^0 = i2\pi\theta \mathrm{Tr}(\hat{\bar{\phi}}\partial_0\hat{\phi} - \partial_0\hat{\bar{\phi}}\hat{\phi}) \ . \tag{11}$$

Q is interpreted as particle number in the physical system. A particular system always exists with fixed particle number $N = Q[\hat{\phi}, \hat{\bar{\phi}}]$. To find nondissipative soliton solutions [9, 10] under this constraint, we write Hamiltonian

$$H = 2\pi\theta \mathrm{Tr}\left(\partial_0\hat{\bar{\phi}}\partial_0\hat{\phi} - \frac{1}{\theta}([a,\hat{\bar{\phi}}][a^\dagger,\hat{\phi}] + [a^\dagger,\hat{\bar{\phi}}][a,\hat{\phi}]) + V(\hat{\bar{\phi}}\hat{\phi})\right) + \omega(N - Q[\hat{\phi},\hat{\bar{\phi}}]) \ , \tag{12}$$

with the constraint applied before the Poisson bracket is worked out [15]. The minimum energy solution occurs at

$$\left.\frac{\delta H}{\delta(\partial_0\hat{\bar{\phi}})}\right|_N = \partial_0\hat{\phi} + i\omega\hat{\phi} = 0 \ , \tag{13}$$

which yields

$$\hat{\phi} = \frac{1}{\sqrt{2}}\hat{\sigma}(\hat{x})e^{-i\omega t}. \tag{14}$$

Assuming hermitian $\hat{\sigma}(\hat{x})$ or real $\sigma(x)$, H becomes

$$H = 2\pi\theta \mathrm{Tr}\left(-\frac{1}{2}\omega^2\hat{\sigma}^2 - \frac{1}{\theta}[a,\hat{\sigma}][a^\dagger,\hat{\sigma}] + V(\frac{1}{2}\hat{\sigma}^2)\right) + \omega N \ , \tag{15}$$

with the particle number

$$N = 2\pi\theta\omega \mathrm{Tr}(\hat{\sigma}^2) \ . \tag{16}$$

and the equation of motion (10)

$$\frac{2}{\theta}[a,[a^\dagger,\hat{\sigma}]] - \omega^2\hat{\sigma} + \hat{\sigma}V'(\frac{1}{2}\hat{\sigma}^2) = 0 \ . \tag{17}$$

Note the equation of motion (17) also follows from $(\delta H/\delta\hat{\sigma})|_N = 0$, which means that the solution $\hat{\sigma}$ has the same form as the static GMS soliton solution in the potential

$$U(\hat{\sigma}) = V(\frac{1}{2}\hat{\sigma}^2) - \frac{1}{2}\omega^2\hat{\sigma}^2 \ . \tag{18}$$

Consider spherically symmetric solution [3] expanded in terms of the projection operators,

$$\hat{\sigma}(\hat{x}) = \sum_{n=0}^{\infty}\lambda_n P_n \ , \tag{19}$$

where $P_n \equiv |n><n|$. Replace $\hat{\sigma}$ in (15), (16), and the equation of motion (17),

$$H = 2\pi\sum_n[(n+1)(\lambda_{n+1}-\lambda_n)^2 + \theta U(\lambda_n)] + \omega N \ , \tag{20}$$

$$N = 2\pi\theta\omega\sum_n \lambda_n^2 \ , \tag{21}$$

$$(n+1)(\lambda_{n+1}-\lambda_n) - n(\lambda_n - \lambda_{n-1}) = \frac{\theta}{2}U(\lambda_n) \ . \tag{22}$$

Sum the equation of motion from $n = 0$ to $n = K$

$$\lambda_{K+1} - \lambda_K = \frac{\theta}{2(K+1)}\sum_{n=0}^{K}U'(\lambda_n) \ , \tag{23}$$

where $K \geq 0$ is an arbitrary integer. A particular set of λ_n's defines a solution. Many properties of the solution can still be derived from Eqn. (20-23) though a closed form has not been constructed. For example, because of the finiteness of both the energy H and the particle number N , we have,

$$\lambda_{n+1} = \lambda_n \ , \qquad \lambda_n = 0 \ , \quad \text{for} \ \ n \to \infty \ , \tag{24}$$

2.2 Q-ball solutions

In static GMS soliton theory, the global minimum of the potential is generally assumed to be at the origin, and the core of the soliton is localized at the local minimum of the potential (false bubble solution). It is the noncommutativity that forbids the classical decay of the solitons. The corresponding commutative potential does not have nontrivial topological structures, and hence yields no soliton solutions. Therefore NC GMS solitons are genuine NC effects and they disappear at small enough θ , where the commutative limit is approached.

This is not the case with Q-ball solutions. The existence and stablity of Q-ball solution rely on the conservation of the charge Q as the consequence of the global symmetry. The potential for Q-ball solutions does not have nontrivial topological structure. Therefore NC Q-balls are expected to exist even for very small θ . We'll show that such NC Q-ball solutions would smoothly reduce to the Q-ball solution in the commutative limit.

In the following we discuss the existence of NC Q-ball solutions in a typical potential form,

$$U(\sigma) = V(\sigma^2) - \frac{1}{2}\omega^2\sigma^2 = a\sigma^2 - b\sigma^4 + c\sigma^6 \ , \tag{25}$$

where the coefficients b and c are larger than zero, and $a = \frac{1}{2}(m^2 - \omega^2)$.

$U(\sigma)$ varies for different ω . If $\omega^2 > m^2$ or $a < 0$, $U(\sigma)$ has a local maximum at the origin. In the commutative limit there is only a plane wave solution. Here similar plane wave solution in NC limit can also be constructed. Since for a stabe soliton solution λ_n would have to take values between s and the origin and monotonically decrease in n [11], a simple argument can show that solitons cannot exist. There is a constraint that $\sum_{n=0}^{K} U'(\lambda_n)$ converges to zero as K goes to infinity, which cannot be satisfied in this case. To prove this constraint, suppose that

$$\sum_{n=0}^{\infty} U'(\lambda_n) \sim v \neq 0 \ , \tag{26}$$

Sum Eqn. (23) from a particular $K = q$ sufficiently large to a point p close to infinity,

$$\lambda_p - \lambda_q \sim \sum_{K=q}^{p} \frac{v}{K} \ . \tag{27}$$

It's then easy to see λ_p will not converge to zero as p goes to infinity.

When $\omega^2 < \nu^2$, $\nu^2 = m^2 - b^2/2c$, $U(\sigma)$ has only a global minimum at the origin. Even though in the commutative theory no soliton solutions exist, for NC theory at sufficiently large θ , there are GMS type solitons exist. It has been shown that there is a critical lower bound on θ for the existence of NC soliton [16]. Similar bounds would be expected to exist for NC Q-ball for $0 < \omega < \nu$ as well.

As $\nu^2 < \omega^2 < m^2$, $U(\sigma)$ has a local mininum at the origin, a global minimum at s ($U(s) < 0$) and a zero $w = [(b - \sqrt{b^2 - 4ac})/2a]^{1/2}$ between 0 and s ($0 < w < s$). In the commutative case such potential form enables the existence of Q- ball solutions. In the NC case it is expected that NC-Q ball solutions exist even for small θ. In the following we take the continuum limit of Eqn. (23) for very small θ , and show that all the solitons sP_K exist at $\theta \to \infty$ converges to the commutative Q-ball solution as $\theta \to 0$.

For very small θ, all λ_n's can be considered as sufficiently close. Therefore λ_n can be approximated by a continuous function $\lambda(u)$ [2]. Let $u = K\theta$, and $\lambda_K = \lambda(K\theta) = \lambda(u)$. Eqn. (23) becomes

$$\lambda'(u) = \frac{1}{2(u+\theta)} \int_0^u U'[\lambda(s)]ds + \mathcal{O}(\theta) \ . \tag{28}$$

Ignore $\mathcal{O}(\theta)$ term, we have

$$\frac{d\lambda}{du} + u\frac{d^2\lambda}{du^2} = \frac{1}{2}\frac{dU}{d\lambda} \ . \tag{29}$$

Let $u = \frac{1}{2}v^2$,

$$\frac{d^2\lambda}{dv^2} + \frac{1}{v}\frac{d\lambda}{dv} = \frac{dU(\lambda)}{d\lambda} \ . \tag{30}$$

[2]Thanks to Dr. Shabnov for helping on this point

This is exactly the equation of motion for the commutative Q-ball solution $\lambda(v)$, with v identified as radius r . This can be explained as the following. The Weyl transform of $\frac{1}{2}r^2$ is $a^\dagger a$, and $a^\dagger a$ has the eigenvalue $n\theta$ on the state $|n>$. As θ gets smaller, the eigenvalues $n\theta$ gets closer, and eventually becomes continuous as $\frac{1}{2}r^2$ in the commuative limit. The coefficient λ_n just becomes the field $\lambda(r)$ in this limit. In this description the commutative Q-ball can be considered as the analytical continuation of the NC Q-balls in θ .

The formula for the energy and particle number in the commutative limit can also be recovered by taking the continuum limit of Eqn. (20) and (21) ,

$$H = 2\pi \int_0^\infty du \left[u(\frac{d\lambda}{du})^2 + U(\lambda) + \omega N \right] = 2\pi \int_0^\infty vdv \left[\frac{1}{2}(\frac{d\lambda}{dv})^2 + U(\lambda) + \omega N \right] , \tag{31}$$

$$N = 2\pi\omega \int_0^\infty du \lambda^2(u) = 2\pi\omega \int_0^\infty vdv\lambda^2(v) . \tag{32}$$

The existence of the commutative Q-ball solution are proved by considering an analogous problem in which a classical particle moves in the one-dimensional potential $-U(\lambda)$ [10]. The field configuration $\lambda(v)$ of the Q-ball starts from a unique value $\rho = \lambda(0)$ between the zero w and the global minimum s, then monotonically decreases in v , and approaches 0 when $v \to \infty$. This property of $\lambda(v)$ is consistent with those of the general stable NC soliton solutions λ_n at finite θ . In [11], it's found that there exist smooth θ families of spherically symmetric solutions in which λ_n is monotonically decreasing in n . In the infinite θ limit such solution is just sP_K . As θ decrease, $\lambda_0, \cdots, \lambda_K$ decrease from s , while other $\lambda_n (n > K)$ starts to move away from the origin towards s , but the whole λ_n series remain monotonically decrease in n . Since in the commutative limit λ_n just becomes $\lambda(v)$, one can conclude that as θ decreases from ∞ to zero, λ_0 will decrease from s and eventually to ρ at the commutative limit.

2.3 Virial Relation

The Hamiltonian (15) in the function formalism is

$$H[\sigma] = 2\pi \int rdr \left(\frac{1}{2}(\partial_i \sigma)^2 + U(\theta\ , \sigma) + \omega N \right) , \tag{33}$$

where the potential U has explicit dependence on θ through star product. Suppose $\sigma(x)$ is the Q-ball solution, $H[\sigma(x/a)]$ must be stationary at $a = 1$. A change of the integration variables shows that

$$H[\sigma(x/a)] = 2\pi \int rdr \left(\frac{1}{2}(\partial_i \sigma(x))^2 + a^2 U(\frac{\theta}{a^2}\ , \sigma(x)) + \omega N \right) , \tag{34}$$

and

$$\frac{d}{da} H[\sigma(x/a)] \Big|_{a=1} = 2\pi \int rdr \left(2U(\theta\ , \sigma) - 2\frac{\partial}{\partial\theta} U(\theta\ , \sigma) \right) = 0 . \tag{35}$$

Unlike the Virial theorem for $d = 1$ space dimension, here the kinetic energy is scale invariant. Scaling dependence of the energy includes two seperate terms from the potential and from its dependence on θ through the star products. The significance of Eqn. (35) is more explicit in GMS soliton case, where the potential energy

$$2\pi \int rdr U(\sigma) = \theta \sum_{n=0}^\infty U(\lambda_n) > 0 . \tag{36}$$

The scaling variable a can be thought of as the size of the NC soliton. While the positive potential energy favors shrinking of the soliton, but the NC star products keep it from decay.

3 Quantization of noncommutative Q-ball

Solitons are extended objects exist in field theory, the properties of which receive quantum corrections as the fields are quantized. In this section we follow very closely to the canonical quantization procedure proposed in [17, 9]. Then we evaluate the ultraviolet divergences in the quantum corrections to the soliton energy at very small θ .

3.1 Canonical Quantization

The general procedure to investigate the properties of the solitons is to expand the fields around the classical solution. Because the momentum and particle number are conserved in the system, we'll have to impose the corresponding constraints to erase the zero-frequency modes in the expansion.

We start by making a point canonical transformation of $\phi(x)$,

$$\phi = \frac{1}{\sqrt{2}} e^{-i\beta(t)} [\sigma(x - X(t)) + \chi(x - X(t), t)] \ , \tag{37}$$

$$\chi(x - X(t), t) \equiv \chi_R(x - X(t), t) + i\chi_I(x - X(t), t) \ , \tag{38}$$

where $\beta(t)$ and $X^i(t)$ are the collective coordinates represent the over-all phase and the center of mass position. Impose the constraints on χ to ensure the above transformation is a canonical transformation with equal number of degrees of freedom before and after,

$$\int \sigma\chi_I = 0 \ , \qquad \int \chi_R \partial_i \sigma = 0 \ , \tag{39}$$

where $i = 1, 2$. The integral sign denotes two dimensional integrations over x. The star product is suppressed. Unless indicated otherwise, from now on the differential d^2x and the NC star product are implied wherever applicable. The above constraints also remove the perturbative zero mode solutions in the meson field χ . Let

$$\chi_R(x, t) \ = \ \sum_{a=3}^{\infty} q_{Ra}(t) f_a(x) \ , \tag{40}$$

$$\chi_I(x, t) \ = \ \sum_{\dot{a}=2}^{\infty} q_{I\dot{a}}(t) g_{\dot{a}}(x) \ , \tag{41}$$

where $f_a(x)$ and $g_{\dot{a}}(x)$ are the real normal functions satisfy

$$\int f_a f_b \ = \ \delta_{ab} \ , \tag{42}$$

$$\int g_{\dot{a}} g_{\dot{b}} \ = \ \delta_{\dot{a}\dot{b}} \ , \tag{43}$$

$$\tag{44}$$

and under the constraints,

$$\int \partial_i \sigma f_a \ = \ 0 \ , \tag{45}$$

$$\int \sigma g_{\dot{a}} \ = \ 0 \ , \tag{46}$$

where $a = 3, 4, \ldots$ and $\dot{a} = 2, 3, \ldots$ always in this paper.

Rewrite the Lagrangian (1) with (37),

$$L = \frac{1}{2} \dot{q}^T \mathcal{M} \dot{q} + \mathcal{V}(q), \tag{47}$$

where $q^T = (X_1 \ , X_2 \ , \beta \ , q_{R3} \ , \ldots \ , q_{I2} \ , \ldots)$ and T denotes matrix transpose, and

$$\mathcal{V}(q) \equiv \int (\partial_i \bar{\phi} \partial_i \phi + V(\frac{1}{2}\{\bar{\phi} \ , \phi\})) \tag{48}$$

The matrix elements of the symmetric $\mathcal{M}$ are

$$\mathcal{M}_{ij} \ = \ M_0 \delta_{ij} + \int (2\partial_i \sigma \partial_j \chi_R + \partial_i \chi_R \partial_j \chi_R + \partial_i \chi_I \partial_j \chi_I) \ , \tag{49}$$

$$
\begin{aligned}
\mathcal{M}_{\beta\beta} &= I + \int (2\sigma\chi_R + \chi_R^2 + \chi_I^2) \ , & (50)\\
\mathcal{M}_{\beta i} &= \int (-2\partial_i\sigma\chi_I + \chi_R\partial_i\chi_I - \chi_I\partial_i\chi_R) \ , & (51)\\
\mathcal{M}_{ia} &= -\int f_a\partial_i\chi_R \ , & (52)\\
\mathcal{M}_{i\dot{a}} &= -\int g_{\dot{a}}\partial_i\chi_I \ , & (53)\\
\mathcal{M}_{\beta a} &= \int f_a\chi_I \ , & (54)\\
\mathcal{M}_{\beta\dot{a}} &= -\int g_{\dot{a}}\chi_R \ , & (55)\\
\mathcal{M}_{ab} &= \delta_{ab} \ , & (56)\\
\mathcal{M}_{\dot{a}\dot{b}} &= \delta_{\dot{a}\dot{b}} \ . & (57)
\end{aligned}
$$

where $M_0 \equiv \frac{1}{2}\int(\partial_i\sigma)^2$ and $I \equiv \int \sigma^2$. The conjugate momentum of q is

$$p = M\dot{q} \equiv (P_1 \ , P_2 \ , N \ , p_{R3} \ , \dots \ , p_{I2} \ , \dots)^T \ . \tag{58}$$

The particle number N and the total momentum P_i are conserved since the Lagrangian (47) is independent of the collective coordinates β and X^i. Quantize the new canonical coordinates,

$$
\begin{aligned}
[X^i \ , P^j] &= i\delta^{ij} \ , & (59)\\
[\beta \ , N] &= i \ , & (60)\\
[q_{Ra} \ , p_{Rb}] &= i\delta_{ab} \ , & (61)\\
[q_{I\dot{a}} \ , p_{I\dot{b}}] &= i\delta_{\dot{a}\dot{b}} \ . & (62)
\end{aligned}
$$

The Hamiltonian

$$H = \frac{1}{2}J^{-1}p^T\mathcal{M}^{-1}Jp + \mathcal{V}(q) \ , \tag{63}$$

where $J = \sqrt{\det \mathcal{M}}$ because the operator ordering in H is unambiguously determined by the ordinary quantized Hamiltonian with the coordinates ϕ [17].

The quantum states can be labelled as $|P^1 \ , P^2 \ , N \ , q_{Ra} \ , q_{I\dot{a}} >$. One can solve the Schrödinger Equation perturbatively around the one soliton ground state $|P^1 = P^2 = 0 \ , N = I\omega \ , 0 >$. In this state P^i and N are the momentum and particle number of the classical solution σ, which can be obtained by letting $\phi = \sigma$ in Eqn. (21) and $P^i = \int \dot{\bar{\phi}}\partial^i\phi + \partial^i\bar{\phi}\dot{\phi}$ [19]. 0 labels the lowest energy state with the given P^i and N value.

We can then treat χ_R and χ_I as perturbative degrees of freedom, and expand the Hamiltonian perturbatively around the one soliton ground state order by order in the weak coupling constant g , defined in Eqn. (2).

σ is at the order of g^{-1} as a soliton solution. M_0 and I are g^{-2} order. Then P_i and N are at the g^{-2} order, while p_{Ra} and $p_{I\dot{a}}$ at g^0 order. Since J commute with P_i and N , and at the leading g^{-3} order, $J = M_0\sqrt{I}$ is a constant or $[p \ , J] = 0$, one can check that J would not be a factor in the Hamiltonian up to the order g^0 .

The Hamiltonian can be expanded order by order, $H = H_0 + H_1 + H_2$, with the expansion relation,

$$\mathcal{M}^{-1} = \mathcal{M}_0^{-1} + \mathcal{M}_0^{-1}\Delta\mathcal{M}_0^{-1} + \mathcal{M}_0^{-1}\Delta\mathcal{M}_0^{-1}\Delta\mathcal{M}_0^{-1} + \cdots \ , \tag{64}$$

where $\mathcal{M} = \mathcal{M}_0 + \Delta$, and $\mathcal{M}_0$ has only nonzero diagonal elements, $\mathcal{M}_0^{qq} = (M_0 \ , M_0 \ , I \ , 1 \ , 1)$.

H_0 , equal to the energy of the classical solution, is at the order of g^{-2} ,

$$H_0 = M_0 + \frac{1}{2}I\omega^2 + V(\frac{1}{2}\sigma^2) \ , \tag{65}$$

H_1 , linear in χ , vanishes due to the fixed N and P_i , which ensures that χ_R and χ_I are at the order of g^0 .

The term quadratic in χ is at the g^0 order,

$$H_2 = \frac{1}{2}(p_{Ra} - \omega \int f_a \chi_I)^2 + \frac{1}{2}(p_{I\dot{a}} + \omega \int g_{\dot{a}} \chi_R)^2 + 2\frac{\omega^2}{M_0}(\int \partial_i \sigma \chi_I)^2 + 2\frac{\omega^2}{I}(\int \sigma \chi_R)^2 + \mathcal{V}_2(q) \ , \tag{66}$$

where

$$\begin{aligned} \mathcal{V}_2(q) \quad = \quad & \int \Big\{ \frac{1}{2}[(\partial_i \chi_R)^2 + (\partial_i \chi_I)^2] - \frac{1}{2}\omega^2(\chi_R^2 + \chi_I^2) \\ & + \frac{1}{2}(\chi_R^2 + \chi_I^2) V'(1/2\sigma^2) + V\left(\frac{1}{2}\sigma^2 \ , \frac{1}{2}\{\chi_R, \sigma\} \ , \frac{1}{2}\{\chi_R, \sigma\}\right) \Big\} \ , \end{aligned} \tag{67}$$

where $V\left(\frac{1}{2}\sigma^2 \ , \frac{1}{2}\{\chi_R, \sigma\} \ , \frac{1}{2}\{\chi_R, \sigma\}\right)$ represents the terms from the expansion of the potential V quadratic in χ_R .

3.2 Energy Corrections at Very Small θ

The Hamiltonian is seperated into two parts, described by the baryon degrees of freedom $(P^i \ , N)$ and meson degrees of freedom $(\chi_R \ , \chi_I)$ respectively. The sum of the frequencies of the meson excitations is the zero-point energy of H_2,

$$< P^1 = P^2 = 0 \ , N = I\omega \ , 0 | \ H_2 \ | P^1 = P^2 = 0 \ , N = I\omega \ , 0 > \ , \tag{68}$$

which, subtracted by the vacuum energy $E_{\text{vac}} = \int d^2k/(2\pi)^2 \sqrt{k^2 + m^2}$, gives the quantum corrections to the soliton energy.

$\mathcal{V}_2(q)$ is the perturbative expansion of the effective potential $\mathcal{V}(q) - \omega Q[\phi, \bar{\phi}]$, Eqn. (48) and (11), around the solution σ . It's easy to check that $\chi_R = \partial_i \sigma$ and $\chi_I = \sigma$ are the eigenmodes of $\mathcal{V}_2(q)$ with eigenfrequency 0, due to the translational and rotational invariance of the potential. Therefore we can define the normal functions f_a and $g_{\dot{a}}$ to be the eigenmodes of $\mathcal{V}_2$, or

$$\mathcal{V}_2(q) = \frac{1}{2}\Omega_{Ra}^2 q_{Ra}^2 + \frac{1}{2}\Omega_{I\dot{a}}^2 q_{I\dot{a}}^2 \ , \tag{69}$$

with the frequencies Ω_{Ra} and $\Omega_{I\dot{a}}$. The potential $\mathcal{V}_2$ are highly nonlocal since the fields are multiplied by NC star product. In the commutative Q-ball case, $\mathcal{V}_2$ has been shown to have only one s-wave eigenmode in χ_R sector with imaginary frequency Ω_{R3} [9]. In last section, we have shown that as NC parameter θ is taken to be small enough, the NC soliton solution will reduce arbitrary close to its commutative analog. Therefore close to the commutative limit $\mathcal{V}_2$ is expected to have the similar eigenvalues and eigenmodes as its commutative analog. NC Q-ball is also expected to be stable as its commutative analog. We'll assume θ is chosen to be such a small value in evaluating the quantum effects of the noncommutativity.

Define $f_i = 1/\sqrt{M_0}\partial_i \sigma$ and $g_1 = 1/\sqrt{I}\sigma$, Rewrite the Hamiltonian H_2 , (66), in the matrix form,

$$H_2 = \frac{1}{2}(\mathcal{P}^T - \omega \mathcal{Q}^T \Upsilon^T)(\mathcal{P} - \omega \Upsilon \mathcal{Q}) + 2\omega^2 \mathcal{Q}^T \Xi \mathcal{Q} + \frac{1}{2}\mathcal{Q}^T \Omega^2 \mathcal{Q} \ , \tag{70}$$

where the matrices are defined in the following,

$$\mathcal{P}^T \equiv (p_{Ra} \ , p_{I\dot{a}}) \ , \qquad \mathcal{Q}^T \equiv (q_{Ra} \ , q_{I\dot{a}}) \ , \tag{71}$$

$$\Upsilon \equiv \begin{pmatrix} 0 & \Gamma_{a\dot{a}} \\ -\Gamma^T_{\dot{a}a} & 0 \end{pmatrix} \ , \qquad \Xi \equiv \begin{pmatrix} \mathcal{F}_{ab} & 0 \\ 0 & \mathcal{G}_{\dot{a}\dot{b}} \end{pmatrix} \ , \qquad \Omega \equiv \begin{pmatrix} \Omega_{Ra} & 0 \\ 0 & \Omega_{I\dot{a}} \end{pmatrix} \ , \tag{72}$$

where

$$\Gamma_{a\dot{a}} \equiv \int f_a g_{\dot{a}} \ , \qquad \mathcal{F}_{ab} \equiv \int g_1 f_a \int g_1 f_b \ , \qquad \mathcal{G}_{\dot{a}\dot{b}} \equiv \int f_i g_{\dot{a}} \int f_i g_{\dot{b}} \ . \tag{73}$$

The equation of motion,

$$\dot{\mathcal{Q}} = \frac{\partial H_2}{\partial \mathcal{P}} \ , \qquad \dot{\mathcal{P}} = -\frac{\partial H_2}{\partial \mathcal{Q}} \ , \tag{74}$$

give

$$\dot{\mathcal{Q}} = \mathcal{P} - \omega\Upsilon\mathcal{Q} \ , \tag{75}$$
$$\dot{\mathcal{P}} = \omega\Upsilon^T(\mathcal{P} - \Upsilon\mathcal{Q}) - 4\omega^2\Xi\mathcal{Q} - \Omega^2\mathcal{Q} \ . \tag{76}$$

Therefore,

$$\ddot{\mathcal{Q}} + 2\omega\Upsilon\dot{\mathcal{Q}} + 4\omega^2\Xi\mathcal{Q} + \Omega^2\mathcal{Q} = 0 \ . \tag{77}$$

Let the real normal eigenmodes of $\mathcal{Q}$ be

$$\mathcal{Q}^A = (\xi^A_{Ra} \ , \xi^A_{I\dot{a}})^T, \tag{78}$$

where ${\mathcal{Q}^A}^T\mathcal{Q}^B = \delta^{AB}$. Replace $\mathcal{Q} = \mathcal{Q}^A \exp(-i\Lambda_A t)$ (Index A is not summed over) in the above equation. Since ${\mathcal{Q}^A}^T\Upsilon\mathcal{Q}^A = 0$,

$$\Lambda_A = \sqrt{{\mathcal{Q}^A}^T(4\omega^2\Xi + \Omega^2)\mathcal{Q}^A} \ . \tag{79}$$

Introduce creation and annihilation operators, $[\mathcal{C}_A \ , \mathcal{C}^\dagger_B] = \delta_{AB}$, $\mathcal{Q}$ can then be quantized as,

$$\mathcal{Q} = \sum_A \frac{\mathcal{Q}^A}{\sqrt{2\Lambda_A}}(\mathcal{C}_A e^{-i\Lambda_A t} + \mathcal{C}^\dagger_A e^{i\Lambda_A t}) \ . \tag{80}$$

Use this equation and Eqn. (76) and (70), one can define the one soliton ground state,

$$\mathcal{C}_A | P^1 = P^2 = 0 \ , N = I\omega \ , 0 >= 0 \ , \tag{81}$$

then the zero-point energy of H_2 (68) is

$$\frac{1}{2}\sum_A \Lambda_A = \frac{1}{2}\mathrm{Tr}\{\Lambda\} \ , \tag{82}$$

where the matrix Λ is diagonal with the eigenvalues Λ_A .

In the commutative theory the zero-point energy contains the divergences even after subtraction of the vacuum energy. The finiteness of the soliton energy is recovered by starting from the renormalized form of the action (1), which induces the counter terms also contain the divergences [20].

Work in the specific form of the ϕ^6 potential (25),

$$V(\frac{1}{2}\{\bar{\phi},\phi\}) = m^2(\frac{1}{2}\{\bar{\phi},\phi\}) - bm^2g^2(\frac{1}{2}\{\bar{\phi},\phi\})^2 + cm^2g^4(\frac{1}{2}\{\bar{\phi},\phi\})^3 \ . \tag{83}$$

At the g^0 order, or the one-loop order, the general formula for the soliton energy is

$$E_{\text{soliton}} \equiv < P^1 = P^2 = 0 \ , N = I\omega \ , 0| \ H \ |P^1 = P^2 = 0 \ , N = I\omega \ , 0 > \tag{84}$$
$$= H_0 + \frac{1}{2}\mathrm{Tr}\{\Lambda\} - E_{\text{vac}} + \frac{1}{2}\delta m^2 \int \sigma^2 - bm^2\delta g^2_{(4)} \int \sigma^4 \ , \tag{85}$$

where δm^2 and $\delta g^2_{(4)}$ are the counter terms for the mass and the ϕ^4 coupling respectively. The ϕ^6 coupling doesn't receive loop corrections. The ϕ^4 coupling terms yield the right coefficients and can be renormalized [25].

The loop integration in the NC field theory generally contains phase factors which yield the interesting UV-IR phenomenon upon renormalization [12]. In the following we evaluate the quantum correction from the zero-point energy of H_2 in Eqn. (82) and show that it contains the same phase factors as those appears in the counter terms δm^2 and $\delta g^2_{(4)}$.

We start by arguing that only $1/2\mathrm{Tr}\{\Omega\}$ is needed in evaluating the leading divergence. In Eqn. (79), it's easy to see ${\mathcal{Q}^A}^T\Xi\mathcal{Q}^A$ is finite,

$$\begin{aligned}
{\mathcal{Q}^A}^T\Xi\mathcal{Q}^A &= (\int g_1 f_a \xi_{Ra})^2 + (\int f_i g_{\dot{a}} \xi_{I\dot{a}})^2 \\
&\leq [\int g_1^2 + \int (f_a\xi_{Ra})^2]^2 + [\int f_1^2 + \int (g_{\dot{a}}\xi_{I\dot{a}})^2]^2 + [\int f_2^2 + \int (g_{\dot{a}}\xi_{I\dot{a}})^2]^2 \\
&= (1 + \xi^2_{Ra})^2 + 2(1 + \xi^2_{I\dot{a}})^2 \ \leq \ 12 \ .
\end{aligned} \tag{86}$$

As we'll see that the eigenvalues Ω_{Ra} and $\Omega_{I\dot{a}}$ behave like $\sqrt{k^2-\omega^2+m^2}$ at very large k . The leading divergence of $\mathrm{Tr}\{\Lambda\}$ will be determined by $\mathrm{Tr}\{\Omega\}$.

Ω_{Ra} and $\Omega_{I\dot{a}}$, eigenfrequencies of $\mathcal{V}_2(q)$ in Eqn. (67), satisfy the linear equations,

$$
\begin{aligned}
&(-\partial_i^2-\omega^2+m^2)\chi_I-\frac{1}{2}bm^2g^2\{\sigma^2,\chi_I\}+\frac{3}{8}cm^2g^4\{\sigma^4,\chi_I\}=\Omega_{I\dot{a}}^2\chi_I\ , \qquad (87)\\
&(-\partial_i^2-\omega^2+m^2)\chi_R-bm^2g^2(\{\sigma^2,\chi_R\}+\sigma\chi_R\sigma)+\\
&\qquad\frac{3}{4}cm^2g^4(\{\sigma^4,\chi_R\}+\sigma^2\chi_R\sigma^2+\sigma\{\sigma,\chi_R\}\sigma)=\Omega_{Ra}^2\chi_R\ . \qquad (88)
\end{aligned}
$$

The above equations are just time independent Schrödinger equations. In particular phase shifts from the central potential have been used in calculating the soliton energy correction [18]. The basic idea is that in the central potential for each partial wave, the difference of the density of the states between the scattered wave and the free wave is related to the derivative of the phase shift,

$$
\rho_l(k)-\rho(0)=\frac{1}{\pi}\frac{d\delta_l(k)}{dk}\ , \qquad (89)
$$

where l goes from $-\infty$ to ∞ . The finiteness of the particle number, $N=\omega\int\sigma^2$, determines that $\sigma\to o$ as $r\to\infty$. Therefore the NC potential in Eqn. (87) and (88) is radial symmetric and vanishes at ∞ . For the most general potential term $\mathcal{W}_F(r)\star\chi\star\mathcal{W}_B(r)$,

$$
[\mathcal{W}_F(r)\star\chi\star\mathcal{W}_B(r)\ ,L]=\mathcal{W}_F(r)\star[\chi\ ,L]\star\mathcal{W}_B(r)\ , \qquad (90)
$$

where $L=-i\epsilon^{ij}x^i\partial^j$ is the angular momentum. The star product is made explicit here and in the rest of the section. This formula can be easily proved in the Weyl transforms of the fields. Going to the momentum space, one can generalize the result in [22] and show that

$$
\mathcal{W}_F(x)\star\chi(x)\star\mathcal{W}_B(x)=\int\frac{d^2p_f}{(2\pi)^2}\frac{d^2p_b}{(2\pi)^2}\widetilde{\mathcal{W}}_F(p_f)\widetilde{\mathcal{W}}_B(p_b)e^{i\mathbf{p}_f(\mathbf{x}+\frac{i\theta}{2}\tilde{\partial})}e^{i\mathbf{p}_b(\mathbf{x}-\frac{i\theta}{2}\tilde{\partial})}\chi(x)\ , \qquad (91)
$$

where $\tilde{\partial}^i\equiv\epsilon^{ij}\partial^j$.

Using Eqn. (89), consider only the leading divergence, we have [21],

$$
\begin{aligned}
\frac{1}{2}\mathrm{Tr}\{\Lambda\}-E_{\mathrm{vac}} &\sim \frac{1}{2}\mathrm{Tr}\{\Omega\}-E_{\mathrm{vac}}\\
&\sim \frac{1}{2\pi}\int d\sqrt{k^2+m^2}\sum_l[\delta_{Il}(k)+\delta_{Rl}(k)]\ , \qquad (92)
\end{aligned}
$$

where $\delta_{Il}(k)$ and $\delta_{Rl}(k)$ are the phase shifts for χ_I and χ_R . The sum of the phase shifts can be evaluated through Born approximation. In the commutative case, this leads to the cancellation of the tadpole diagram [18].

Eqn. (87) and (88) have the Jost solution form at large r ,

$$
\chi\sim h_l^*(kr)+e^{2i\delta_l}h_l(kr)\ . \qquad (93)
$$

Considering the asymptotic ($r\to\infty$) behaviour of the solution, the standard procedure [23] leads to the scattering amplitude,

$$
f(\mathbf{k}',\mathbf{k})=f(\phi)=\sum_l f_l(k)e^{il\phi}=\frac{1}{\sqrt{k}}\sum_l e^{i\delta_l}\sin\delta_l e^{il\phi}\ , \qquad (94)
$$

where $k'=k$ and ϕ is the angle between $\mathbf{k}'$ and $\mathbf{k}$. At large l , or $\delta_l\approx 0$, we can see

$$
\sum_l\delta_l\approx\sqrt{k}f(\phi=0) \qquad (95)
$$

$f(\mathbf{k}',\mathbf{k})$ can also be calculated through Born approximation, replacing χ by $e^{-i\mathbf{kx}}$ in the potential form (91),

$$\begin{aligned} f(\mathbf{k}',\mathbf{k}) &= -\frac{1}{4\sqrt{k}}\int d^2x e^{-i\mathbf{k}'\mathbf{x}}\sum_{\mathrm{i}}\mathcal{W}_F^{(\mathrm{i})}\star e^{-i\mathbf{kx}}\star\mathcal{W}_B^{(\mathrm{i})} \\ &= -\frac{1}{4\sqrt{k}}\int d^2x e^{-i(\mathbf{k}'-\mathbf{k})\mathbf{x}}\sum_{\mathrm{i}}\mathcal{W}_F^{(\mathrm{i})}(x-\frac{\theta}{2}\tilde{k})\mathcal{W}_B^{(\mathrm{i})}(x+\frac{\theta}{2}\tilde{k}) \end{aligned} \tag{96}$$

where i labels the potential terms in Eqn. (87) and (88), and $\tilde{k}^i\equiv\epsilon^{ij}k^j$. Therefore

$$\begin{aligned} \sum_l \delta_l &= -\frac{1}{4}\int d^2x\sum_{\mathrm{i}}\mathcal{W}_F^{(\mathrm{i})}(x-\frac{\theta}{2}\tilde{k})\mathcal{W}_B^{(\mathrm{i})}(x+\frac{\theta}{2}\tilde{k}) \\ &= -\frac{1}{4}\int\frac{d^2p}{(2\pi)^2}\sum_{\mathrm{i}}\widetilde{\mathcal{W}}_F^{(\mathrm{i})}(p)\widetilde{\mathcal{W}}_B^{(\mathrm{i})}(-p)e^{-i\theta\mathbf{p}\tilde{\mathbf{k}}}\ . \end{aligned} \tag{97}$$

The right hand side only depends on the magnitude k due to the central potential $\mathcal{W}(r)$.

Now we are ready to evaluate Eqn. (92),

$$\begin{aligned} &\frac{1}{2\pi}\int d\sqrt{k^2+m^2}\sum_l[\delta_{Il}(k)+\delta_{Rl}(k)] \\ =& -\frac{1}{8\pi}\int d\sqrt{k^2+m^2}\int\frac{d^2p}{(2\pi)^2}\sum_{\mathrm{i}}\widetilde{\mathcal{W}}_F^{(\mathrm{i})}(p)\widetilde{\mathcal{W}}_B^{(\mathrm{i})}(-p)e^{-i\theta\mathbf{p}\tilde{\mathbf{k}}} \\ =& -\frac{1}{2}\int\frac{d^2p}{(2\pi)^2}\int\frac{d^2k}{(2\pi)^2}\frac{e^{-i\theta\mathbf{p}\tilde{\mathbf{k}}}}{2\sqrt{k^2+m^2}}\sum_{\mathrm{i}}\widetilde{\mathcal{W}}_F^{(\mathrm{i})}(p)\widetilde{\mathcal{W}}_B^{(\mathrm{i})}(-p)\ . \end{aligned} \tag{98}$$

The integration over k is exactly part of the tadpole diagram belongs to δm^2 [25], and it contains the UV/IR divergence ($\Lambda\to\infty$ and $p\to 0$) evaluated with the cutoff Λ [12],

$$\int\frac{d^2k}{(2\pi)^2}\frac{e^{-i\theta\mathbf{p}\tilde{\mathbf{k}}}}{2\sqrt{k^2+m^2}}=\frac{2}{(4\pi)^{3/2}}(m\Lambda_p)^{1/2}K_{\frac{1}{2}}\left(\frac{2m}{\Lambda_p}\right)=\frac{\Lambda_p}{8\pi}+\mathcal{O}(1)\ , \tag{99}$$

where $\Lambda_p\equiv(\theta^2p^2/4+1/\Lambda^2)^{-1/2}$. Notice that the above UV/IR divergence from Eqn. (98) occurs only when both $\mathcal{W}_F$ and $\mathcal{W}_B$ exist. In other words, only the terms $\sigma\chi_R\sigma$,$\sigma^2\chi_R\sigma^2$ and $\sigma\{\sigma,\chi_R\}\sigma$ in Eqn. (87) and (88) yield UV/IR divergence. All other potential terms only give the normal UV divergence where the phase factor is absent. Since the counter term δm^2 and δg^2 do not include UV/IR divergence, we are certain that the Q-ball energy correction includes UV/IR divergence. Cancellation of the UV divergences is not obvious because the exact value of the eigenfrequencies Λ_A is unknown.

4 Finite θ and Noncommutative GMS Solitons

The above calculation assumes that the NC parameter θ is sufficiently small so that the NC potential will generate the Jost solution form as in the commutative case. Let's consider the effects of the NC potential (91) in the case that θ is not small.

Since

$$[x^i\pm\frac{i\theta}{2}\tilde{\partial}^i,x^j\pm\frac{i\theta}{2}\tilde{\partial}^j]=\pm i\theta\epsilon^{ij}\ , \tag{100}$$

the effective scattering potential for the NC interaction $\mathcal{W}_F(x)\star\chi\star\mathcal{W}_B(x)$ (91) is just

$$\widehat{\mathcal{W}}_F(x+\frac{i\theta}{2}\tilde{\partial})\widehat{\mathcal{W}}_B(x-\frac{i\theta}{2}\tilde{\partial})\ , \tag{101}$$

multiplication of the Weyl transforms of $\mathcal{W}_F(x)$ and $\mathcal{W}_B(x)$. Notice $\widehat{\mathcal{W}}_F$ and $\widehat{\mathcal{W}}_B$ commute since $[x^i+i\theta/2\tilde{\partial}^i,x^j-i\theta/2\tilde{\partial}^j]=0$.

Now considering

$$\widehat{\mathcal{W}}_F(x) = \int \frac{d^2p_f}{(2\pi)^2}\widetilde{\mathcal{W}}_F(p_f)e^{i\mathbf{p}_f(\mathbf{x}+\frac{i\theta}{2}\tilde{\partial})} \ , \tag{102}$$

the noncommutativity,

$$[p_f^1(x^1+\frac{i\theta}{2}\tilde{\partial}^1) \ , p_f^2(x^2+\frac{i\theta}{2}\tilde{\partial}^2)] = i\theta p_f^1 p_f^2 \ , \tag{103}$$

can be suppressed even if θ is not small, as long as $\mathcal{W}(x)$ is smooth enough or $\widetilde{\mathcal{W}}(p_f) \to 0$ at large p_f . Notice small p_f is also the IR limit we discussed in the last section. Under this assumption we can write

$$\widehat{\mathcal{W}}_F(x) \approx \mathcal{W}_F(x+\frac{i\theta}{2}\tilde{\partial}) = \mathcal{W}_F(|\mathbf{x}+\frac{i\theta}{2}\tilde{\partial}|^2) = \mathcal{W}_F(r^2-\frac{\theta}{2}L+\partial_i^2) \ . \tag{104}$$

Acting on the field $\chi(x) = u_l(kr)e^{il\phi}$, the effective potential becomes $\mathcal{W}_F(r^2-\theta l/2+\partial_i^2)$. Similar calculation applied to $\widehat{\mathcal{W}}_B(x)$ yields $\mathcal{W}_B(r^2+\theta l/2+\partial_i^2)$. Therefore at large k and large r, we can treat the scattering potential perturbatively as in the commutative case. The phase shift evaluation of the energy of the soliton could still apply provided that $\mathcal{W}$ or the soliton solution σ are smooth enough.

NC GMS soliton only exists at finite θ . Based on the above arguments, we can still evaluate its quantum corrections with the phase shift method in the last section.

Quantization of GMS soliton and Q-ball share a lot of similarities. To get the GMS soliton theory, we make the replacements $(\phi \ , \bar{\phi} \to 1/\sqrt{2}\Phi)$ in the previous complex scalar field theory (1). With the potential (83), the Lagrangian becomes,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu\Phi)^2 + V(\Phi) = -\frac{1}{2}(\partial_\mu\Phi)^2 + \frac{1}{2}m^2\Phi^2 - \frac{1}{4}bm^2g^2\Phi^4 + \frac{1}{8}cm^2g^4\Phi^6 \ , \tag{105}$$

where Φ is multiplied by the star product. The renormalizability of the theory is proved in [24]. Let $\omega = 0$ because there is no conserved charge Q (11) in the theory. When $b^2-4c<0$, the potential in (105) has a local minimum at $\sqrt{b}g/2$ besides the global minimum at the origin, and the GMS soliton solution σ exists. Replace the expansion (37) by $\Phi = \sigma+\chi$, where $\chi = \chi_R$ is real. As a result, the meson degrees of freedom are only χ_R or χ . Upon quantization, the soliton energy is still given by Eqn. (85). Since $\omega = 0$ and $\Lambda = \Omega$ are the exact eigenfrequencies of H_2 (66), we have

$$(-\partial_i^2+m^2)\chi_R - bm^2g^2(\{\sigma^2,\chi_R\}+\sigma\chi_R\sigma) + \frac{3}{4}cm^2g^4(\{\sigma^4,\chi_R\}+\sigma^2\chi_R\sigma^2+\sigma\{\sigma,\chi_R\}\sigma) = \Lambda_a^2\chi_R \ . \tag{106}$$

Eqn. (92) describes the exact ultraviolet divergences in $1/2\mathrm{Tr}\{\Lambda\}-E_{\mathrm{vac}}$. Therefore we are able to check the cancellation of the divergences in Eqn. (85) . As mentioned in the previous section, in the above equation, the terms $\sigma\chi_R\sigma$, $\sigma^2\chi_R\sigma^2$ and $\sigma\{\sigma,\chi_R\}\sigma$ yield UV/IR divergences, while the rest terms yield UV divergence. A critical observation is that those terms yield UV/IR divergence have one to one correspondence with the contractions of the fields yield Nonplanar Feynman diagrams [12], and those terms yield UV divergence correspond exactly to the planar digrams. We can just spare the details of counting the divergences. Since the counter terms δm^2 and $\delta g_{(4)}^2$ cancel exactly the UV divergence part, we conclude that the soliton energy (85) is UV finite, but includes all the UV/IR divergences.

5 Conclusion and Discussion

In this paper we discuss the quantization of NC solitons in (2+1) dimensional scalar field theory. In particular, classical solutions and quantization of the NC Q-balls at very small θ are investigated in detail. Classically NC Q-balls reduce to the commutative Q-ball as θ goes to zero. Quantum mechanically, because loop integrations in the NC field thoery have different ultraviolet structure from those in the commutative theory, i. e. UV/IR mixing, quantum corrections to the NC soliton energy necessarily include the UV/IR divergent terms which cannot be renormalized away. The existence of such terms in the energy is demonstrated through the phase shift summation. The same method is futher generalized to NC GMS solitons which exist only at finite θ . In the small momentum limit, or for the suffciently smooth soliton solutions, divergence structure of the soliton energy can be calculated exactly. In this case the energy is found to contain no UV divergence but all the UV/IR divergences. In [26] quantum corrections to the NC soliton energy are also calculated but at very large θ, where no UV/IR divergence is found. We believe that's because at large θ, the noncommutativity (103) is not small and cannot be ignored, and the potential term is the dominant term instead of a perturbative one. In this case the phase shift sum is not a good approximation to the energy correction.

An interpretation to the UV/IR divergence is given in [27], where new light degrees of freedom are introduced in the Wilsonian effective action. UV/IR divergence can be reproduced by integrating out those new degrees of freedom, which are then interpreted as closed string modes with channel duality. It will be interesting to further consider the NC solitons in the gauge theory, where they are interpreted as D-branes [5, 6] and D-brane action is properly recovered. If such NC solitons are quantized one might be able to recover the effective interaction between D-branes and closed strings .

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