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STAR PRODUCT FOR SECOND CLASS CONSTRAINT SYSTEMS FROM A BRST THEORY

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Abstract. We explicitly quantize the general second-class constrained system at the level of deformation quantization such that the quantization is covariant with respect to local coordinates on the phase space. The approach is based on constructing the effective first-class constraint (gauge) system equivalent to the original second-class one and can also be understood as a far-going generalization of the Fedosov quantization. The effective gauge system is quantized by the BFV–BRST procedure. The star product for the Dirac bracket is explicitly constructed as the quantum multiplication of BRST observables. We introduce and explicitly construct a Dirac bracket counterpart of the symplectic connection, called the *Dirac connection*. We identify a particular star product associated with the Dirac connection for which the constraints are in the center of the respective star-commutator algebra; when reduced to the constraint surface, this star product can be recognized as a Fedosov one.

1. Introduction

In the previous paper [15] by two of the present authors it was shown that the Fedosov deformation quantization [11] can be understood as a BFV-BRST quantization [1] (see [16] for a review) of the appropriately constructed effective gauge system. It was demonstrated that an arbitrary symplectic manifold can be embedded as a second-class constraint surface into the appropriately extended phase space. By converting these second class constraints into the first-class ones by introducing additional degrees of freedom one arrives at the effective first-class constraint (gauge) theory that is equivalent to the original symplectic manifold. The BRST quantization of the system reproduces the Fedosov star product as a quantum multiplication of BRST observables.

An advantage of this approach is that it can be naturally generalized to the case of constraint systems on arbitrary symplectic manifolds. The point is that one can treat the constraints responsible for the embedding of the phase space as a second-class surface and the original constraints present in the model on equal footing [3]. From the BRST theory standpoint, this implies that the respective BRST charge is (roughly speaking) a sum of two parts, each part corresponding to the respective subset of the constraints. In this paper, we present a quantization scheme for a general second-class constraint system on an arbitrary symplectic manifold.

The main ingredient of the second-class system quantization (at the level of deformation quantization) is that of finding a star-product that in the first order in $\hbar$ coincides with the respective *Dirac bracket*. From the mathematical standpoint this implies quantization of Poisson manifolds, with the Poisson structure being a Dirac bracket associated to the second-class constraints.

The existence of deformation quantization for an arbitrary Poisson manifold has been established in a remarkable paper [17] by Kontsevich. The quantization formula in [17] has also been given an interesting physical explanation in [8]. However, this formula is not explicitly covariant[1] w.r.t. the phase space coordinates, and seems too involved to apply to the case of regular Poisson structures (the Dirac bracket for a second-class system is a regular Poisson bracket). In the case of regular Poisson manifolds, one can also use an appropriate generalization of the Fedosov quantization method [12]. However, this scheme requires an explicit separation of symplectic leaves of the Poisson bracket, which in the constraint theory case implies solving the constraints.

Unlike the system with degenerate bracket, observables of the second-class system are functions on the constraint surface. Quantization of the system implies thus not only constructing a star product for the Dirac bracket but also an appropriate specification of the space of quantum observables.

It turns out that within the BRST theory approach developed in this paper one can find an explicitly covariant quantization of a second-class system with the phase space being an arbitrary symplectic manifold. The correct space of observables is described as a ghost number zero cohomology of the appropriately constructed BRST charge. This approach produces, in particular, a covariant star-product for the respective Dirac bracket.

Constructing the phase space covariant quantization requires introducing an appropriate connection on the phase space, which is compatible with the Poisson structure to be quantized. In the case of an unconstraint system on a symplectic manifold an appropriate connection, called the *symplectic connection*, is a symmetric connection compatible with the symplectic form. Together with the symplectic form on the phase space, the symplectic connection determines a *Fedosov structure* [14], which is a basic starting point of the Fedosov quantization.

We show in this paper that by developing the BRST description of the second-class system quantization one naturally finds a proper Dirac counterpart of the Fedosov structure. Namely, the symplectic structure of the extended phase space of the effective gauge system naturally involves a symmetric connection compatible with the Dirac bracket. This connection, called the *Dirac connection*, is explicitly constructed in terms of the constraint functions and an arbitrary symplectic connection on the phase space.

An essential feature of the Dirac connection is that it determines a symplectic connection on the constraint surface considered as a symplectic manifold. Using the Dirac connection allows us to identify a star product compatible with the constraints in the sense that the constraints are the central functions of the respective star commutator. We also develop a reduction procedure which shows that when reduced to the constraint surface this star product can be identified as the Fedosov one, constructed by the restriction of the Dirac connection to the surface.

This paper is organized as follows: In Section **2** we recall the basics of the second-class system theory and introduce the notion of the Dirac connection. We also collect there some general geometrical facts that we need in what follows. In Section **3** we construct at the classical level the effective gauge system equivalent to the original second-class system. Quantization of the effective system is obtained in Section **4**. A quantum deformation of the Dirac bracket is also constructed and analyzed there. In Section **5** we develop the reduction of the extended phase space of the effective gauge system to that constructed for the original constraint surface. Finally, in Section **6** we propose an alternative

[1] After this paper was finished we became aware of recent paper [9] where a globally defined version of the Kontsevich quantization had been proposed.

approach to the second-class system (quantization), which can be thought of as an alternative form of the conversion of the second-class constraints into the first-class ones.

2. Preliminaries

We begin with recalling basic facts concerning the second class constraint systems on general symplectic manifolds. In addition to the standard facts, we propose a Dirac bracket counterpart of the Fedosov geometry. This includes constructing a symmetric connection compatible with the Dirac bracket that can be reduced to the constraint surface, thereby giving a symmetric symplectic connection on the surface. These geometrical structures are essential for constructing covariant star product for the Dirac bracket.

2.1. **General phase space.** We start with the properties of the phase space of a general second-class system. The phase space is a symplectic manifold $\mathcal{M}$ with the symplectic form ω (closed and nondegenerate 2-form). In local coordinates $x^i, i = 1, \ldots, 2N$ on $\mathcal{M}$ the coefficients of ω are

$$\omega_{ij} = \omega(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})\,. \tag{2.1}$$

The Poisson bracket induced by the symplectic form reads as

$$\{f, g\}_{\mathcal{M}} = \frac{\partial f}{\partial x^i}\omega^{ij}\frac{\partial g}{\partial x^j}\,, \qquad \omega^{il}\omega_{lj} = \delta^i_j\,. \tag{2.2}$$

Because we are interested in quantizing of $\mathcal{M}$ we specify a symmetric symplectic connection Γ on $\mathcal{M}$. The compatibility condition reads as

$$\nabla\omega = 0\,, \qquad \partial_j\omega_{ik} - \Gamma_{ijk} + \Gamma_{kji} = 0\,, \tag{2.3}$$

where the coefficients Γ_{jik} are introduced as

$$\Gamma_{jik} = \omega_{jl}\Gamma^l_{ik}\,, \qquad \Gamma^j_{ik}\frac{\partial}{\partial x^j} = \nabla_i\frac{\partial}{\partial x^k}\,. \tag{2.4}$$

Given a symplectic form and a compatible symmetric connection on $\mathcal{M}$ one says that $\mathcal{M}$ is a *Fedosov manifold* [14]. It is well known that a symmetric symplectic connection exists on every symplectic manifold, each symplectic manifold is therefore a Fedosov one. However, unlike the Riemannian connection, a symmetric symplectic connection is not unique; the arbitrariness is that of adding a completely symmetric tensor field to the coefficients Γ_{jik}; the Fedosov structure is not completely determined by the symplectic one.

2.2. **Second-class constraint system.** Let now $\mathcal{M}$ be a phase space of a second-class constraint system. The system is specified by the set of second-class constraints

$$\theta_\alpha = 0\,, \qquad \alpha = 1, \ldots, 2M\,. \tag{2.5}$$

We assume θ_α to be globally defined functions on $\mathcal{M}$. Let $\Sigma \subset \mathcal{M}$ be the respective constraint surface. The Dirac matrix reads as

$$\Delta_{\alpha\beta} = \{\theta_\alpha, \theta_\beta\} \tag{2.6}$$

and is assumed to be invertible. Its inverse is denoted by

$$\Delta^{\alpha\beta}\,, \qquad \Delta_{\alpha\gamma}\Delta^{\gamma\beta} = \delta^\beta_\alpha\,.$$

Invertibility of the Dirac matrix implies that Σ is a smooth submanifold in $\mathcal{M}$.

Observables of the second-class system are functions on the constraint surface Σ. The algebra of inequivalent observables of the second-class system on $\mathcal{M}$ is a Poisson algebra of functions on the constraint surface, with the Poisson structure corresponding to the symplectic form $\omega^\Sigma \equiv \omega\big|_\Sigma$ (the 2-form $\omega\big|_\Sigma$ denotes the restriction of to $\Sigma \subset \mathcal{M}$). At the classical level, two constraint systems are said equivalent iff their algebras of inequivalent observables are isomorphic as Poisson algebras.

2.3. **Dirac bracket.** The goal of the Dirac bracket approach to the second-class system is to represent the algebra of inequivalent observables as a quotient of the algebra of functions on $\mathcal{M}$ modulo the ideal generated by the constraints. The point is that $\mathcal{M}$ can be equipped with a Poisson bracket called the *Dirac bracket* which is well defined on this quotient. Given second-class constraints θ_α the respective Dirac bracket is given explicitly by

$$\{f,g\}^D_\mathcal{M} = \{f,g\}_\mathcal{M} - \{f,\theta_\alpha\}_\mathcal{M}\,\Delta^{\alpha\beta}\,\{\theta_\beta,g\}_\mathcal{M}\,. \tag{2.7}$$

We recall the basic properties of a Dirac bracket

$$\begin{aligned}\{f,g\}^D_\mathcal{M} + \{g,f\}^D_\mathcal{M} &= 0\\ \{f,gh\}^D_\mathcal{M} - \{f,g\}^D_\mathcal{M}\,h - g\,\{f,h\}^D_\mathcal{M} &= 0\\ \Big\{f,\,\{g,h\}^D_\mathcal{M}\Big\}^D_\mathcal{M} + \mathrm{cycle}(f,g,h) &= 0\\ \{f,\theta_\alpha\}^D_\mathcal{M} &= 0\,.\end{aligned} \tag{2.8}$$

The first three lines say that $\{,\}^D_\mathcal{M}$ is a Poisson bracket. In what follows we use the notation D^{ij} for the components of the Poisson bivector of the Dirac bracket:

$$D^{ij} = \left\{x^i,x^j\right\}^D_\mathcal{M}\,. \tag{2.9}$$

The last line in (2.8) implies that θ_α are characteristic (Casimir) functions of the bracket $\{,\}^D_\mathcal{M}$. This, in turn, implies that $\{,\}^D_\mathcal{M}$ is well-defined on the quotient algebra of $\mathcal{C}^\infty(\mathcal{M})$ modulo functions of the form $F^\alpha\theta_\alpha$ (i.e. modulo the ideal of functions vanishing on Σ). This quotient can be naturally identified with the algebra $\mathcal{C}^\infty(\Sigma)$ of functions on Σ. Thus $\mathcal{C}^\infty(\Sigma)$ is a Poisson algebra. We refer to the Poisson bracket in $\mathcal{C}^\infty(\Sigma)$ as to the restriction of a Dirac bracket to Σ.

The constraint surface Σ is in fact a symplectic manifold, with the symplectic form $\omega^\Sigma \equiv \omega\big|_\Sigma$. The Poisson bracket on Σ corresponding to the symplectic form ω^Σ coincides with the restriction $\{,\}^D_\mathcal{M}\big|_\Sigma$ of a Dirac bracket to Σ. Thus description of a second-class system in terms of a Dirac bracket is equivalent to that based on the reduction to the constraint surface.

From the geometrical viewpoint, θ_α give a maximal set of independent characteristic functions of the Dirac bracket. This implies that $\mathcal{M}$ is a regular Poisson manifold. Each surface of the constant values of the functions θ_α is therefore a symplectic leaf of the Dirac bracket and is a symplectic manifold. In particular, constraint surface Σ is a symplectic leaf. As we have seen the symplectic form on Σ (on each symplectic leaf) is the restriction $\omega|_\Sigma$ of the symplectic form ω on $\mathcal{M}$ to $\Sigma \subset \mathcal{M}$ (respectively the symplectic leaf).

2.4. **Dirac connection.** At the level of quantum description we also need the *Dirac connection* that is a symmetric connection compatible with the Dirac bracket. Given a symmetric symplectic connection

Γ on $\mathcal{M}$ there exists a Dirac connection $\overline{\Gamma}^0$ whose coefficients are

$$(\overline{\Gamma}^0)_{ij}^{\;k} = \omega^{kl}(\Gamma_{lij} + \partial_l\theta_\beta\Delta^{\beta\alpha}\nabla_i\nabla_j\theta_\alpha)\,, \qquad \nabla_i\nabla_j\theta_\alpha = \partial_i\partial_j\theta_\alpha - \Gamma_{ij}^m\partial_m\theta_\alpha\,, \tag{2.10}$$

where ∇_i is the covariant derivative w.r.t. Γ. It is a matter of direct observation that $\overline{\Gamma}^0$ preserves the Dirac bivector. Moreover, one can check that

$$\overline{\nabla}_i^0\partial_j\theta_\alpha = 0\,, \tag{2.11}$$

where the notation $\overline{\nabla}^0$ is introduced for the covariant differentiation determined by the connection $\overline{\Gamma}^0$. The curvature of the Dirac connection is given explicitly by

$$\begin{aligned}(\overline{R}^0)_{ij;\,l}^m \;&=\; D^{mk}\overline{R}^0_{ij;\,kl} \;= \\ &=\; D^{mk}\Big(R_{ij\,;kl} + (\nabla_i\nabla_k\theta_\alpha)\Delta^{\alpha\beta}(\nabla_j\nabla_l\theta_\beta) - (\nabla_j\nabla_k\theta_\alpha)\Delta^{\alpha\beta}(\nabla_i\nabla_l\theta_\beta)\Big)\,,\end{aligned} \tag{2.12}$$

where $R_{ij\,;kl} = \omega_{km}R^m_{ij;\,l}$ is the curvature of Γ.

An important point is that the Dirac connection $\overline{\Gamma}^0$ on $\mathcal{M}$ determines a connection on the constraint surface Σ. This implies that the parallel transport along Σ carries vectors tangent to Σ to tangent ones and thus determines a connection on Σ. To show that $\overline{\Gamma}^0$ does restrict to Σ let Y, Z be vector fields on $\mathcal{M}$ tangent to Σ. This implies that there exist functions $Y^\alpha_\beta, Z^\alpha_\beta$ such that

$$Y\theta_\alpha = Y_\alpha^\beta\theta_\beta\,, \qquad Z\theta_\alpha = Z_\alpha^\beta\theta_\beta\,. \tag{2.13}$$

Let us show that $\overline{\nabla}^0_Y Z$ is also tangent to Σ. Indeed,

$$\begin{aligned}(\overline{\nabla}^0_Y Z)\theta_\alpha &= (Y^i\partial_i Z^j)\partial_j\theta_\alpha + Y^i(\overline{\Gamma}^0)_{ik}^{\;j}Z^k\partial_j\theta_\alpha = \\ &= YZ\theta_\alpha - Y^iZ^j\partial_i\partial_j\theta_\alpha + Y^iZ^k\Gamma^j_{ik}\partial_j\theta_\alpha + Y^iZ^k\omega^{jn}\partial_n\theta_\gamma\Delta^{\gamma\beta}\nabla_i\nabla_k\theta_\beta\partial_j\theta_\alpha = \\ &= YZ\theta_\alpha = (YZ_\alpha^\beta)\theta_\beta + Z_\alpha^\beta Y_\beta^\gamma\theta_\gamma\,.\end{aligned} \tag{2.14}$$

The last expression obviously vanishes on Σ. The vector field $\overline{\nabla}^0_Y Z$ is therefore tangent to Σ. This implies that connection $\overline{\Gamma}^0$ determines a connection $(\overline{\Gamma}^0)^\Sigma$ on Σ. Since $\overline{\Gamma}^0$ preserves Dirac bivector $D^{ij}\frac{\partial}{\partial x^i}\wedge\frac{\partial}{\partial x^j}$, reduced connection $(\overline{\Gamma}^0)^\Sigma$ preserves its restriction to Σ. Because the Poisson bracket corresponding to the symplectic form $\omega^\Sigma = \omega|_\Sigma$ is the restriction of the Dirac bracket to Σ , $(\overline{\Gamma}^0)^\Sigma$ is a symmetric symplectic connection on Σ. Thus any second-class constraint surface in the Fedosov manifold is also a Fedosov manifold, with the symplectic structure and a compatible symmetric connection being the restrictions to Σ of the symplectic structure and the Dirac connection on $\mathcal{M}$ respectively.

Let us also find a coordinate description for the reduction of a Dirac connection to the constraint surface. To this end, we note that expression (2.10) for the Dirac connection can be equivalently rewritten as:

$$(\overline{\Gamma}^0)_{ij}^{\;k} = D^{kl}\Gamma_{lij} + \omega^{kl}\partial_l\theta_\beta\Delta^{\beta\alpha}\partial_i\partial_j\theta_\alpha\,. \tag{2.15}$$

Further, let us pick functions $\overline{x}^a\,,\ a = 1,\ldots,2N-2M$ such that $\overline{x}^a$ and $\overline{x}^\alpha = \theta_\alpha\,,\ \alpha = 1,\ldots,2M$ form a local coordinate system on $\mathcal{M}$. This coordinate system is adopted to the embedded submanifold $\Sigma\subset\mathcal{M}$ in the sense that locally Σ is singled out by equations $\overline{x}^\alpha = 0$. It follows that the functions

$\overline{x}^a|_\Sigma$, $a = 1, \ldots, 2N-2M$ form a local coordinate system on Σ. In the coordinate system $\overline{x}^a, \overline{x}^\alpha$ the expression $\partial_i\partial_j\theta_\alpha$ obviously vanishes and the Dirac connection becomes

$$(\overline{\Gamma}^0)^{\,k}_{ij} = D^{kl}\Gamma_{lij}\,, \tag{2.16}$$

where we use i (or j, k, l) as a collective notations for either a or α. Because $D^{i\alpha} = \{\overline{x}^i, \overline{x}^\alpha\}^D_{\mathcal{M}} = 0$, the respective components of the Dirac connection vanishes:

$$(\overline{\Gamma}^0)^{\alpha}_{ij} = 0\,. \tag{2.17}$$

Then, for a vector field tangent to Σ one has

$$\nabla_{\frac{\partial}{\partial \overline{x}^a}} \frac{\partial}{\partial \overline{x}^b} = (\overline{\Gamma}^0)^{l}_{ab}\frac{\partial}{\partial \overline{x}^l} = (\overline{\Gamma}^0)^{c}_{ab}\frac{\partial}{\partial \overline{x}^c}\,, \tag{2.18}$$

in view of (2.17). When restricted to Σ the equation implies that the functions $(\overline{\Gamma}^0)^a_{bc}\big|_\Sigma$ are the coefficients of a symmetric symplectic connection $(\overline{\Gamma}^0)^\Sigma$ on Σ w.r.t. the coordinate system $\overline{x}^a\big|_\Sigma$ on Σ.

2.5. **Modified Poisson bracket on the cotangent bundle over a symplectic manifold.** The phase space of the system under consideration is a general symplectic manifold. Even without constraints on $\mathcal{M}$ the quantization problem for $\mathcal{M}$ can not be solved directly and requires using specific quantization methods. Since we are interested in quantizing a constraint system the most suitable method is that based on the representation of $\mathcal{M}$ itself as a second-class constraint system; the method was proposed in [2] and generalized to the case of arbitrary symplectic manifolds in [15]. It was also shown [15] to provide the (BRST form of the) constraint theory description of the Fedosov quantization.

To represent $\mathcal{M}$ as a second-class constraint surface let us consider first an appropriate generalization of the canonical Poisson bracket on a cotangent bundle. Let $T^*\mathcal{M}$ be a cotangent bundle over $\mathcal{M}$. Let x^i, p_j be natural local coordinates on the base and the fibres respectively. Let also $\pi : T^*\mathcal{M} \to \mathcal{M}$ be a bundle projection which sends a 1-form at point m to m (points of $T^*\mathcal{M}$ are pairs $(m,a) : m \in \mathcal{M},\, a \in T^*_m\mathcal{M}$) and $\pi^* : \Lambda(\mathcal{M}) \to \Lambda(T^*\mathcal{M})$ is the pullback map associated with π. Here we use notation $\Lambda(\mathcal{M}) = \oplus_{k=0}^{\dim(\mathcal{M})}\Lambda^k(\mathcal{M})$ for the space of differential forms on $\mathcal{M}$).

Cotangent bundle $T^*\mathcal{M}$ is equipped with the canonical symplectic structure:

$$\omega^{T^*\mathcal{M}} = 2\, dp_i \wedge dx^i$$

The corresponding Poisson bracket also has standard form:

$$\{x^i, p_j\}_{T^*\mathcal{M}} = \delta^i_j\,.$$

Let $\omega = \omega_{ij}dx^i \wedge dx^j$ be a closed 2-form on $\mathcal{M}$. Let us introduce a modified symplectic structure on $T^*\mathcal{M}$ by:

$$\omega^{mod} = \omega^{T^*\mathcal{M}} + \pi^*\omega = 2\, dp_i \wedge dx^j + \omega_{ij}dx^i \wedge dx^j\,. \tag{2.19}$$

This 2-form is obviously nondegenerate and closed. Hereafter we denote $T^*\mathcal{M}$ by $\mathcal{T}^*_\omega\mathcal{M}$ indicating the fact that it is equipped with the modified symplectic structure. Symplectic structure ω^{mod} determines a Poisson bracket on $\mathcal{T}^*_\omega\mathcal{M}$, with the basic Poisson bracket relations given by:

$$\{x^i, p_j\}_{\mathcal{T}^*_\omega\mathcal{M}} = \delta^i_j\,, \qquad \{p_i, p_j\}_{\mathcal{T}^*_\omega\mathcal{M}} = \omega_{ij}\,. \tag{2.20}$$

The Jacobi identity for this bracket holds provided 2-form ω is closed.

One can easily check that at least locally one can bring the bracket to the standard (canonical) form by means of the following transformation $p_i \to p_i - \rho_i(x)$, with $\rho_i(x)$ being a symplectic potential for the symplectic structure ω:

(2.21) $$\omega = 2\,d\rho\,, \qquad \omega_{ij} = \partial_i \rho_j - \partial_j \rho_i\,.$$

As the Poisson brackets of the coordinate functions p_i form an invertible matrix, constraints

(2.22) $$\theta_i = 0\,, \qquad \theta_i \equiv -p_i$$

(we choose minus sign for convenience) are second-class ones, with the respective constraint surface being $\mathcal{M}$. In general, these constraints are not the globally defined functions on $\mathcal{T}^*_\omega\mathcal{M}$. This, in particular, implies that the Dirac bracket associates with the constraints θ_i is not a globally defined Poisson bracket on $\mathcal{T}^*_\omega\mathcal{M}$. However, this Dirac bracket is well defined on the constraint surface. The constraint surface is $\mathcal{M}$ (considered as zero section of $\mathcal{T}^*_\omega\mathcal{M}$) and the restriction of the Dirac bracket to $\mathcal{M} \subset \mathcal{T}^*_\omega\mathcal{M}$ coincides with the Poisson bracket $\{,\}_\mathcal{M}$ on $\mathcal{M}$. In this way one can represent an arbitrary Hamiltonian system on the symplectic manifold as the second class constraint system.

2.6. **Symplectic structure.** In what follows we also need a specific Poisson bracket on the appropriately extended cotangent bundle over a symplectic manifold. Given an arbitrary symplectic vector bundle $\mathbf{W}(\mathcal{N}) \to \mathcal{N}$ over a manifold $\mathcal{N}$ let e_A be a local frame (locally defined basic sections of $\mathbf{W}(\mathcal{M})$). Let also $\mathcal{D}$ be the symplectic form on the fibres of $\mathbf{W}(\mathcal{M})$. The components of $\mathcal{D}$ w.r.t. e_A are determined by $\mathcal{D}_{AB} = \mathcal{D}(e_A, e_B)$.

It is well known (see e.g. [12]) that any symplectic vector bundle admits a symplectic connection. Let $\mathbf{\Gamma}$ and ∇ denotes a symplectic connection and the corresponding covariant differential in $\mathbf{W}(\mathcal{N})$. The compatibility condition reads as

(2.23) $$\nabla\mathcal{D} = 0\,, \qquad \partial_i\mathcal{D}_{AB} - \mathbf{\Gamma}^C_{iA}\mathcal{D}_{CB} - \mathbf{\Gamma}^C_{iB}\mathcal{D}_{AC} = 0\,,$$

where the coefficients $\mathbf{\Gamma}^C_{iA}$ of $\mathbf{\Gamma}$ are determined as:

(2.24) $$\nabla e_A = dx^i\mathbf{\Gamma}^C_{iA}e_C\,.$$

It is useful to introduce the following connection 1-form:

(2.25) $$\mathbf{\Gamma}_{AB} = dx^i\mathbf{\Gamma}_{AiB}\,, \qquad \mathbf{\Gamma}_{AiB} = \mathcal{D}_{AC}\mathbf{\Gamma}^C_{iB}\,.$$

Then compatibility condition (2.23) rewrites as

(2.26) $$d\mathcal{D}_{AB} = \mathbf{\Gamma}_{AB} - \mathbf{\Gamma}_{BA}\,, \qquad \partial_i\mathcal{D}_{AB} - \mathbf{\Gamma}_{AiB} + \mathbf{\Gamma}_{BiA} = 0\,.$$

As a consequence of the condition one arrives at the following property of the connection 1-form $\mathbf{\Gamma}_{AB}$:

(2.27) $$d\mathbf{\Gamma}_{AB} = d\mathbf{\Gamma}_{BA}\,.$$

Consider the following direct sum of vector bundles:

(2.28) $$\boldsymbol{\mathcal{E}} = \mathbf{W}(\mathcal{N}) \oplus \mathcal{T}^*_\omega\mathcal{N}\,,$$

where for generality we assume that $\mathcal{N}$ is equipped with a closed 2-form ω and $\mathcal{T}^*_\omega\mathcal{N}$ is a modified cotangent bundle over $\mathcal{N}$. Let x^i, p_j and Y^A are standard local coordinates on $\boldsymbol{\mathcal{E}}$ (x^i are local coordinates on $\mathcal{N}$, p_j are standard coordinates on the fibres of $\mathcal{T}^*_\omega\mathcal{N}$, and Y^A are coordinates on the fibres of $\mathbf{W}(\mathcal{N})$ corresponding to the local frame e_A).

We claim that considered as a manifold $\mathcal{E}$ is equipped with the following symplectic structure

$$\begin{aligned} \omega^{\mathcal{E}} &= \pi^*\omega + 2dp_i \wedge dx^i + \\ &+ \mathcal{D}_{AB} dY^A \wedge dY^B + Y^A Y^B d\boldsymbol{\Gamma}_{AB} - 2Y^A \boldsymbol{\Gamma}_{AB} \wedge dY^B \,, \end{aligned} \tag{2.29}$$

where $\pi^*\omega$ is the 2-form ω on $\mathcal{N}$ pulled back by bundle projection $\pi : \mathcal{E} \to \mathcal{N}$. One can directly check that 2-form (2.29) is well defined. That it is closed follows from $d\omega = 0$, condition (2.26), and Eq. (2.27).

The Poisson bracket on $\mathcal{E}$ corresponding to the symplectic form (2.29) is determined by the following relations:

$$\begin{aligned} \{x^i, p_j\}_{\mathcal{E}} &= \delta^i_j \,, \\ \{Y^A, Y^B\}_{\mathcal{E}} &= \mathcal{D}^{AB} \,, \\ \{Y^A, p_i\}_{\mathcal{E}} &= -\boldsymbol{\Gamma}^A_{iB} Y^B \,, \\ \{p_i, p_j\}_{\mathcal{E}} &= \omega_{ij} + \frac{1}{2} \boldsymbol{R}_{ij;AB} Y^A Y^B \,, \end{aligned} \tag{2.30}$$

with all the others vanishing. Here, $\boldsymbol{R}_{ij;\,AB}$ denotes the curvature of $\boldsymbol{\Gamma}$:

$$\begin{aligned} \boldsymbol{R}_{ij;\,AB} = \mathcal{D}_{AC} \boldsymbol{R}^C_{ij\,B} &= \\ = \mathcal{D}_{AC} \left(\partial_i \boldsymbol{\Gamma}^C_{jB} - \partial_j \boldsymbol{\Gamma}^C_{iB} + \boldsymbol{\Gamma}^C_{iD} \boldsymbol{\Gamma}^D_{jB} - \boldsymbol{\Gamma}^C_{jD} \boldsymbol{\Gamma}^D_{iB} \right) &= \\ = \partial_i \boldsymbol{\Gamma}_{AjB} - \partial_j \boldsymbol{\Gamma}_{AiB} + \boldsymbol{\Gamma}_{CiA} \mathcal{D}^{CD} \boldsymbol{\Gamma}_{DjB} - \boldsymbol{\Gamma}_{CjA} \mathcal{D}^{CD} \boldsymbol{\Gamma}_{DiB} \,. \end{aligned} \tag{2.31}$$

The last equality follows from nondegeneracy of $\mathcal{D}_{AB}$ and compatibility condition (2.26).

3. Conversion – classical description

In this section we construct the BFV-BRST description of the second-class constraint system on $\mathcal{M}$ in terms of an equivalent effective first-class constraint (gauge) system. This includes explicit construction of the gauge system, its BRST charge, and observables.

3.1. **Unification of constraints θ_α and θ_i.** Given a second-class constraint system on $\mathcal{M}$ we first embed $\mathcal{M}$ as a zero section in $\mathcal{T}^*_\omega\mathcal{M}$. According to Section **2.5** the constraints $\theta_i \equiv -p_i = 0$ represent $\mathcal{M}$ as a second-class constraint surface in $\mathcal{T}^*_\omega\mathcal{M}$ and thus determine on $\mathcal{T}^*_\omega\mathcal{M}$ the constraint system which is equivalent to the original symplectic manifold $\mathcal{M}$ (i.e. an unconstraint system on $\mathcal{M}$).

In order to describe the constraint system on $\mathcal{M}$ specified by the second-class constraints θ_α one considers a constraint system on $\mathcal{T}^*_\omega\mathcal{M}$, with the constraints being θ_i and θ_α. We treat these constraints on equal footing and introduce a unified notation:

$$\Theta_A = (\theta_i\,, \theta_\alpha)\,, \qquad A = 1, \ldots, 2N + 2M\,. \tag{3.1}$$

Their Dirac matrix is

$$D_{AB} = \{\Theta_A, \Theta_B\}_{\mathcal{T}^*_\omega\mathcal{M}}\,. \tag{3.2}$$

Explicitly, D_{AB} is given by

$$D = \begin{pmatrix} D_{ij} & D_{i\beta} \\ D_{\alpha j} & D_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \omega_{ij} & \partial_i\theta_\beta \\ -\partial_j\theta_\alpha & 0 \end{pmatrix} \tag{3.3}$$

One can check that $det(D) = det(\omega_{ij})det(\Delta_{\alpha\beta})$. The matrix $D^{AB} \equiv (D^{-1})^{AB}$ that is inverse to D_{AB} reads as

$$D^{-1} = \begin{pmatrix} D^{ij} & D^{i\beta} \\ D^{\alpha j} & D^{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \omega^{ij} - \omega^{ik}\omega^{jl}\partial_k\theta_\alpha\partial_l\theta_\beta\Delta^{\alpha\beta} & -\omega^{il}\partial_l\theta_\gamma\Delta^{\gamma\beta} \\ \omega^{jk}\partial_k\theta_\gamma\Delta^{\gamma\alpha} & \Delta^{\alpha\beta} \end{pmatrix} \tag{3.4}$$

Note that the left upper block of the matrix is nothing but the Poisson bivector of the Dirac bracket on $\mathcal{M}$ associated with the second-class constraints θ_α.

Among the constraints Θ_A there are constraints θ_i which are not the globally defined functions on $\mathcal{T}^*_\omega\mathcal{M}$; they transform as the components of a 1-form on $\mathcal{M}$. Consequently the Dirac bracket associated with Θ_A is not a globally defined Poisson bracket on $\mathcal{T}^*_\omega\mathcal{M}$. However, it is well defined for p_i-independent functions.

Proposition 3.1. *Let f and g be arbitrary functions on $\mathcal{M}$. Let π^* is the pullback associated with the bundle projection $\pi : \mathcal{T}^*_\omega\mathcal{M} \to \mathcal{M}$. Then*

$$\{\pi^* f, \pi^* g\}^D_{\mathcal{T}^*_\omega\mathcal{M}} = \pi^*(\{f, g\}^D_{\mathcal{M}} , \tag{3.5}$$

where the Dirac bracket in the L.H.S. is taken w.r.t. the constraints Θ_A and in the R.H.S. w.r.t. θ_α only. In particular, these Dirac brackets are identical on $\Sigma \subset \mathcal{M}$.

Because each physical observable of the constraint system on $\mathcal{T}^*_\omega\mathcal{M}$ can be taken as a p_i-independent function, the Proposition implies that the original constraint system on $\mathcal{M}$ (determined by the constraints θ_α) is equivalent with the constraint system on $\mathcal{T}^*_\omega\mathcal{M}$ (determined by the constraints Θ_A). A direct way to check the equivalence of these constraint systems is to observe that the constraints Θ_A on $\mathcal{T}^*_\omega\mathcal{M}$ and θ_α on $\mathcal{M}$ determine the same constraint surface Σ and the respective Dirac brackets $\{,\}^D_{\mathcal{T}^*_\omega\mathcal{M}}$ and $\{,\}^D_{\mathcal{M}}$ coincide on Σ.

3.2. **Symplectic connection.** In order to convert the second-class constraints θ_A into the first-class ones we introduce the conversion variables Y^A associated to the second-class constraints. We treat the variables Y^A as coordinates on the fibres of the vector bundle $\mathbf{W}(\mathcal{M})$ associated to the constraints Θ_A. This means that variables Y^A have the transformation properties dual to those of Θ_A. Because constraints Θ_A are split into θ_i and θ_α, the vector bundle $\mathbf{W}(\mathcal{M})$ is a direct sum

$$\mathbf{W}(\mathcal{M}) = T\mathcal{M} \oplus \mathbf{V}(\mathcal{M}) = T\mathcal{M} \times V , \tag{3.6}$$

where $\mathbf{V}(\mathcal{M})$ is the vector bundle associated with the constraints θ_α. Since the constraints $\theta_\alpha, \ \alpha = 1, \ldots, 2M$ are globally defined functions on $\mathcal{M}$, the vector bundle $\mathbf{V}(\mathcal{M})$ is a direct product of $\mathcal{M}$ and $2M$-dimensional vector space V.

The Dirac matrix D_{AB} obviously determines a symplectic form on each fiber of $\mathbf{W}(\mathcal{M})$, making $\mathbf{W}(\mathcal{M})$ into the symplectic vector bundle. To convert the second-class constraints Θ_A into the first-class ones one has to extend the Poisson bracket on $\mathcal{T}^*_\omega\mathcal{M}$ to the phase space $\mathcal{E}_0$ of the extended system. This in turn requires to introduce a symplectic connection in $\mathbf{W}(\mathcal{M})$.

Each symplectic vector bundle admits a symplectic connection. However, it is instructive to find the explicit form of the symplectic connection in $\mathbf{W}(\mathcal{M}) = T\mathcal{M} \oplus \mathbf{V}(\mathcal{M})$ compatible with the symplectic form D. Moreover, in what follows we need the specific symplectic connection $\overline{\Gamma}^0$ in $\mathbf{W}(\mathcal{M})$ which is constructed below.

Introducing coefficients of $\overline{\Gamma}$ with lowered indices as

$$\overline{\Gamma}_{AiB} = D_{AC}\overline{\Gamma}^C_{iB}\,, \tag{3.7}$$

we write equation (2.26) in components:

$$\begin{aligned} \frac{\partial}{\partial x^i}\omega_{jk} + \overline{\Gamma}_{kij} - \overline{\Gamma}_{jik} = 0\,,\\ \frac{\partial^2}{\partial x^i \partial x^j}\theta_\alpha + \overline{\Gamma}_{\alpha ij} - \overline{\Gamma}_{ji\alpha} = 0\,,\\ \overline{\Gamma}_{\alpha i\beta} - \overline{\Gamma}_{\beta i\alpha} = 0\,, \end{aligned} \tag{3.8}$$

The first equation is that for a symplectic connection on $\mathcal{M}$. Thus it is natural to chose a particular solution to the first equation as

$$\overline{\Gamma}^0_{kij} = \omega_{kl}\Gamma^l_{ij}\,, \tag{3.9}$$

where Γ^l_{ij} are coefficients of the fixed symmetric symplectic connection on $\mathcal{M}$. Further, under the change of coordinates on $\mathcal{M}$ the coefficients $\overline{\Gamma}_{ij\alpha}$ transform as the the components of a tensor field. Thus the condition $\overline{\Gamma}_{ij\alpha} = 0$ is the invariant one and one can choose a particular solution of the second equation as:

$$\overline{\Gamma}^0_{ij\alpha} = 0\,, \qquad \overline{\Gamma}^0_{\alpha ij} = -\partial_i\partial_j\theta_\alpha\,. \tag{3.10}$$

Finally, we choose $\overline{\Gamma}^0_{\alpha i\beta} = 0$. Thus we obtain the particular solution $\overline{\Gamma}^0_{AiB}$ for equations (3.8). A general solution is obviously given by

$$\overline{\Gamma}_{AiB} = \overline{\Gamma}^0_{AiB} + T_{AiB}\,, \qquad T_{AiB} - T_{BiA} = 0 \tag{3.11}$$

where T_{AiB} is an arbitrary 1-form taking values in rank-2 symmetric tensors in $\mathbf{W}(\mathcal{M})$.

An explicit expression for the non-vanishing coefficients of $\overline{\Gamma}^0$ read as:

$$\begin{aligned} (\overline{\Gamma}^0)^i_{jk} = D^{iA}\overline{\Gamma}^0_{Ajk} = D^{il}\overline{\Gamma}^0_{ljk} + D^{i\alpha}\overline{\Gamma}^0_{\alpha jk} = \omega^{il}(\Gamma_{ljk} + \partial_l\theta_\beta\Delta^{\beta\alpha}\nabla_j\nabla_k\theta_\alpha)\,,\\ (\overline{\Gamma}^0)^\alpha_{jk} = D^{\alpha A}\overline{\Gamma}^0_{Ajk} = D^{\alpha l}\overline{\Gamma}^0_{ljk} + D^{\alpha\beta}\overline{\Gamma}^0_{\beta jk} = -\Delta^{\alpha\beta}\nabla_j\nabla_k\theta_\beta \end{aligned} \tag{3.12}$$

One can see that the coefficients $(\overline{\Gamma}^0)^i_{jk}$ coincide with those of the Dirac connection on $\mathcal{M}$ given by (2.10). Indeed, D^{ij} is nothing but the Poisson bivector of the Dirac bracket on $\mathcal{M}$; compatibility condition (2.23) then implies:

$$\overline{\nabla}^0_i D^{jk} = \partial_i D^{jk} + (\overline{\Gamma}^0)^j_{im}D^{mk} + (\overline{\Gamma}^0)^k_{im}D^{jm} = 0\,, \tag{3.13}$$

since the coefficients $(\overline{\Gamma}^0)^j_{i\alpha}$ vanish. This allows one to consider the Dirac connection on $\mathcal{M}$ as the restriction of $\overline{\Gamma}^0$ in $\mathbf{W}(\mathcal{M})$ to $T\mathcal{M} \subset \mathbf{W}(\mathcal{M})$.

Let us also write in components the Riemann curvature of $\overline{\Gamma}^0$. The Riemann tensor with lowered indices is determined by

$$\begin{aligned} \overline{R}^0_{ij;AB} \;=\; D_{AC}(R^0)^C_{ij\,B} \;=\\ = \partial_i\overline{\Gamma}^0_{AjB} - \partial_j\overline{\Gamma}^0_{AiB} + \overline{\Gamma}^0_{CiA}D^{CD}\overline{\Gamma}^0_{DjB} - \overline{\Gamma}^0_{CjA}D^{CD}\overline{\Gamma}^0_{DiB}\,. \end{aligned} \tag{3.14}$$

The only non-vanishing components of $R^0_{ij;AB}$ are given by:

$$\begin{aligned}\overline{R}^0_{ij;kl} \;&=\; \partial_i\overline{\Gamma}^0_{kjl} - \partial_j\overline{\Gamma}^0_{kil} + \overline{\Gamma}^0_{Aik}D^{AB}\overline{\Gamma}^0_{Bjl} - \overline{\Gamma}^0_{Ajk}D^{AB}\overline{\Gamma}^0_{Bil} \;= \\ &=\; R_{ij;\,kl} + (\nabla_i\nabla_k\theta_\alpha)\Delta^{\alpha\beta}(\nabla_j\nabla_l\theta_\beta) - (\nabla_j\nabla_k\theta_\alpha)\Delta^{\alpha\beta}(\nabla_i\nabla_l\theta_\beta)\,,\end{aligned} \tag{3.15}$$

where $R_{ij;\,kl}$ is the Riemannian curvature of the symplectic connection Γ from (3.9).

In what follows it goes without saying that the vector bundle associated with constraints Θ_A is a direct sum $T\mathcal{M}\oplus\mathbf{V}(\mathcal{M}) = T\mathcal{M}\times V$ and is equipped with the symplectic connection $\overline{\Gamma}$. We also reserve notations $\overline{\Gamma}^0$ and $\overline{R}^0$ for the specific symplectic connection given by (3.12) and its curvature.

3.3. **Extended phase space.** Recall (see subsection **2.6**) that given a symplectic vector bundle $\mathbf{W}(\mathcal{N})$ endowed with a symplectic connection Γ and a symplectic form D one can equip $\mathbf{W}(\mathcal{N})\oplus\mathcal{T}^*_\omega\mathcal{N}$ with the symplectic structure. Specifying this general construction to the vector bundle $\mathbf{W}(\mathcal{M}) = T\mathcal{M}\times V$ equipped with the symplectic form D and the connection $\overline{\Gamma}$ one arrives at the following symplectic structure on the phase space $\mathcal{E}_0 = \mathcal{T}^*_\omega\mathcal{M}\oplus\mathbf{W}(\mathcal{M})$:

$$\begin{aligned}\omega^{\mathcal{E}_0} \;&=\; \pi^*\omega + 2dp_i\wedge dx^i \;+ \\ &\qquad + \;D_{AB}dY^A\wedge dY^B \;+\; Y^AY^Bd\overline{\Gamma}_{AB} \;-\; 2Y^A\overline{\Gamma}_{AB}\wedge dY^B\,.\end{aligned} \tag{3.16}$$

Our aim is to construct an effective first-class constraint theory on $\mathcal{E}_0$ by converting the second-class constraints Θ_A into the first-class ones. At the classical level, the conversion is achieved by continuation of the constraints Θ_A into the new constraints T_A defined on $\mathcal{E}_0$ such that

$$\{T_A, T_B\}_{\mathcal{E}_0} = 0\,, \qquad T_A|_{Y^A=0} = \Theta_A\,. \tag{3.17}$$

The constraints T_A are understood as formal power series in Y^A

$$T_A = \sum_{s=0}^{\infty} T^s_A\,, \qquad T^0_A = \Theta_A\,, \quad T^s_A = T^s_{A\,B_1,\dots,B_s}(x)Y^{B_1},\dots,Y^{B_s}\,, \tag{3.18}$$

where the coefficients $T^s_{AB_1,\dots,B_s}$ are assumed to be p_i-independent functions.

In spite of the fact that the constraints T_A are Abelian, it is useful to proceed within the BFV-BRST approach. Accordingly, we introduce the ghost variables $\mathcal{C}^A$ and $\mathcal{P}_A\,, A = 1\,,\dots\,,2M+2N$ associated to the constraints T_A. The variables $\mathcal{C}^A$ and $\mathcal{P}_A$ are Grassmann odd ones. The ghost number grading is introduced by

$$\mathrm{gh}(\mathcal{C}^A) = 1\,, \qquad \mathrm{gh}(\mathcal{P}_A) = -1\,, \tag{3.19}$$

with all the others variables carrying vanishing ghost number.

It is natural to consider the ghost variables $\mathcal{C}$ and $\mathcal{P}$ as coordinates on the fibres of the respective vector bundle $\Pi\mathbf{W}(\mathcal{M})$ and $\Pi\mathbf{W}^*(\mathcal{M})$. Here, Π denotes parity reversing operation; when applied to a vector bundle it transform the bundle into the super vector bundle with the same base manifold and the transition functions and the fibres being the Grassmann odd vector superspaces.

In the BFV-BRST quantization one needs to extend the Poisson structure on the phase space to the ghost variables, with the variables $\mathcal{C}^A$ and $\mathcal{P}_A$ being canonically conjugated w.r.t. the bracket. To this end we consider the following extension of the phase space $\mathcal{E}_0$

$$\mathcal{E} = \mathcal{T}^*_\omega(\Pi\mathbf{W}(\mathcal{M}))\oplus\mathbf{W}(\Pi\mathbf{W}(\mathcal{M}))\,, \tag{3.20}$$

where $\mathbf{W}(\Pi\mathbf{W}(\mathcal{M}))$ is the vector bundle $\mathbf{W}(\mathcal{M})$ pulled back by the projection $\rho : \Pi\mathbf{W}(\mathcal{M}) \to \mathcal{M}$ and $\oplus$ denotes the direct sum of vector bundles over $\Pi\mathbf{W}(\mathcal{M})$. Note also that construction of modified cotangent bundle over $\Pi\mathbf{W}(\mathcal{M})$ involves the closed 2-form $\rho^*\omega$ defined on $\Pi\mathbf{W}(\mathcal{M})$, which is the symplectic form on $\mathcal{M}$ pulled back by the bundle projection ρ. Identifying $\mathcal{C}^A$ with the coordinates on the fibres of $\Pi\mathbf{W}(\mathcal{M})$ and $\mathcal{P}_A$ with their conjugate momenta one can indeed see that when $\mathcal{C}^A = \mathcal{P}_A = 0$ extended phase space $\mathcal{E}$ reduces to $\mathcal{E}_0 = \mathcal{T}^*_\omega\mathcal{M} \oplus \mathbf{W}(\mathcal{M})$. Symplectic structure (3.16) can be easily extended to $\mathcal{E}$ by

$$
\begin{aligned}
\omega^{\mathcal{E}} &= \pi^*\omega + 2dp_i \wedge dx^i + 2d\mathcal{P}_A \wedge d\mathcal{C}^A + \\
&+ \ D_{AB}dY^A \wedge dY^B + Y^AY^B d\overline{\Gamma}_{AB} - 2Y^A\overline{\Gamma}_{AB} \wedge dY^B \,.
\end{aligned}
\tag{3.21}
$$

where π^* is the pullback associated with $\pi : \mathcal{E} \to \mathcal{M}$. The respective Poisson bracket relations are as follows

$$
\begin{aligned}
\{x^i, p_j\}_{\mathcal{E}} &= \delta^i_j \,, \\
\{Y^A, Y^B\}_{\mathcal{E}} &= D^{AB} \,, \\
\{Y^A, p_i\}_{\mathcal{E}} &= -\overline{\Gamma}^A_{iB} Y^B \,, \\
\{\mathcal{C}^A, \mathcal{P}_B\}_{\mathcal{E}} &= \delta^A_B \,, \\
\{p_i, p_j\}_{\mathcal{E}} &= \omega_{ij} + \frac{1}{2} R_{ij;\,AB} Y^A Y^B \,,
\end{aligned}
\tag{3.22}
$$

with all the others vanishing. This Poisson bracket can be thought of as that given by (2.30), with $\mathcal{N} = \Pi\mathbf{W}(\mathcal{M})$. This shows that (3.22) determines a globally defined Poisson bracket on $\mathcal{E}$.

To complete the description of the extended phase space we specify a class of functions on this space. Instead of smooth ($\mathcal{C}^\infty(\mathcal{E})$) functions we consider those wich are formal power series in $Y, \mathcal{C}, \mathcal{P}$ and polynomial in p with coefficients in smooth functions on $\mathcal{M}$. The reason is that variables Y serve as the *conversion variables* and one should allow formal power series in Y. As for the ghost variables $\mathcal{C}$ and $\mathcal{P}$, each function is always a polynomial in $\mathcal{C}$ and $\mathcal{P}$ since they are Grassmann odd. Let us note, however, that in the case where $\mathcal{M}$ is a supermanifold one should allow formal power series in respective ghost variables. In what follows it goes without saying that under the algebra $\mathcal{F}(\mathcal{E})$ of "functions" on $\mathcal{E}$ we mean the algebra of the power series described above. $\mathcal{F}(\mathcal{E})$ is thus the algebra of sections of an appropriate bundle over $\mathcal{M}$. One can check that Poisson bracket (3.22) is well defined in $\mathcal{F}(\mathcal{E})$.

3.4. **Conversion – classical description.** Now we are in position to proceed with the conversion of the second-class constraints Θ_A within the BRST formalism. A conversion is to be understood as the solution of the master equation

$$
\{\Omega, \Omega\} = 0 \,, \qquad \mathrm{p}(\Omega) = 1 \,, \quad \mathrm{gh}(\Omega) = 1 \,,
\tag{3.23}
$$

(with $\mathrm{p}(\Omega)$ denoting Grassmann parity of Ω) subjected to the boundary condition

$$
\Omega|_{Y=0} = \Omega^0 = \mathcal{C}^A\Theta_A = -\mathcal{C}^i p_i + \mathcal{C}^\alpha\theta_\alpha \,.
\tag{3.24}
$$

Let us expand Ω into the sum of homogeneous components w.r.t. Y^A

$$
\Omega = \sum_{s=0}^{\infty} \Omega^s \,,
\tag{3.25}
$$

and assume that Ω^r do to depend on momenta p_i and ghost momenta $\mathcal{P}^A$ for $r \geq 1$. To construct solution iteratively it is useful to fix also the first-order term by

$$\Omega^1 = -\,C^A D_{AB} Y^B\,, \tag{3.26}$$

and to introduce a nilpotent operator δ [6, 11, 15]:

$$\delta f = \mathcal{C}^A \frac{\partial}{\partial Y^A} f\,, \qquad \delta^2 = \delta\delta = 0\,. \tag{3.27}$$

If f doesn't depend on the momenta p_i and $\mathcal{P}_A$ then

$$\delta f = -\,\{\Omega^1, f\}_{\mathcal{E}}\,, \qquad f = f(x, Y, \mathcal{C})\,. \tag{3.28}$$

An operator δ^* is introduced by its action on the homogeneous functions of the form

$$f_{pq} = f_{A_1,\dots,A_p\,;\,B_1,\dots,B_q}(x)\,Y^{A_1}\dots Y^{A_p}\mathcal{C}^{B_1}\dots\mathcal{C}^{B_q} \tag{3.29}$$

by means of

$$\begin{aligned} &\delta^* f_{pq} = \frac{1}{p+q} Y^A \frac{\partial}{\partial \mathcal{C}^A} f_{pq}\,, \qquad p+q \neq 0 \\ &\delta^* f_{00} = 0\,. \end{aligned} \tag{3.30}$$

The operator δ is in some sense inverse to δ and serve as a contracting homotopy for δ. Indeed,

$$f|_{Y=\mathcal{C}=0} + \delta\delta^* f + \delta^*\delta f = f\,. \tag{3.31}$$

Theorem 3.2. *Given the second-class constraints θ_α on $\mathcal{M}$ and symplectic connection $\overline{\Gamma}$ in $\mathbf{W}(\mathcal{M})$ there exists solution to the Eq.* (3.23) *satisfying boundary conditions* (3.24) *and* (3.26)*. If in addition one requires $\delta^*\Omega^r = 0$ and $\Omega^r = \Omega^r(x, Y, \mathcal{C})$ for $r \geq 2$ the solution is unique.*

Proof. In the zeroth order in Y equation (3.23) implies

$$\{\Omega^0, \Omega^0\}_{\mathcal{E}}|_{Y=0} + \{\Omega^1, \Omega^1\}_{\mathcal{E}} = 0\,. \tag{3.32}$$

This holds provided the boundary conditions (3.24) and (3.26) are compatible. In the r-th ($r \geq 1$) order in Y (3.23) implies:

$$\delta\Omega^{r+1} = B^r\,, \tag{3.33}$$

where the quantity B^r is given by

$$\begin{aligned} &B^1 = \{\Omega^0, \Omega^1\}_{\mathcal{E}}\,, \\ &B^2 = \{\Omega^0, \Omega^2\}_{\mathcal{E}} + \frac{1}{2}\{\Omega^2, \Omega^2\}_{\mathcal{E}} + \frac{1}{4}\mathcal{C}^i\mathcal{C}^j\overline{R}_{ij;\,AB}Y^A Y^B\,, \\ &B^r = \{\Omega^0, \Omega^r\}_{\mathcal{E}} + \frac{1}{2}\sum_{s=0}^{r-2}\{\Omega^{s+2}, \Omega^{r-s}\}_{\mathcal{E}}\,, \quad r \geq 3\,, \end{aligned} \tag{3.34}$$

and we have assumed that $\Omega^r = \Omega^r(x, Y, \mathcal{C})$ for $r \geq 2$. The necessary and sufficient condition for Eq. (3.33) to have a solution is $\delta B^r = 0$. Let us first show explicitly that $\delta B^1 = 0$. Indeed, in view of the zeroth order equation

$$\begin{aligned} \delta B^1 = \delta\,\{\Omega^0, \Omega^1\} = -\,\{\Omega^1,\,\{\Omega^1, \Omega^0\}\,\} = -\frac{1}{2}\{\,\{\Omega^1, \Omega^1\}_{\mathcal{E}}, \Omega^0\}_{\mathcal{E}} = \\ = \frac{1}{2}\{\,\{\Omega^0, \Omega^0\}_{\mathcal{E}}|_{Y=0}, \Omega^0\}_{\mathcal{E}} = 0\,. \end{aligned} \tag{3.35}$$

Then a particular solution for Ω^2 is

$$\Omega^2 = \delta^* \left\{\Omega^0, \Omega^1\right\} . \tag{3.36}$$

The proof of the statement goes further along the standard induction procedure [6, 15]: one can first check that $\delta B^s = 0$ provided Ω^r satisfy (3.33) for $r \leq s-1$; one then finds:

$$\Omega^{s+1} = \delta^* B^s . \tag{3.37}$$

Finally, one can check that $\Omega^{s+1} = \delta^* B^s$ is a unique solution of Eq. (3.33) for $r = s$ provided the additional condition $\delta^*\Omega^{s+1} = 0$ is imposed and $\mathrm{gh}(\Omega^{s+1}) = 1$. □

The statement implies that we have arrived at the first-class constraint theory whose extended phase space is $\mathcal{E}$. Since under the additional condition $\delta^*\Omega^r = 0\,,\, r \geq 2$ classical BRST charge is unique and is obviously linear in $\mathcal{C}$, this first class constraint system is an Abelian one.

3.5. **BRST cohomology.** By definition, an observable of the BFV-BRST system determined by the BRST charge Ω is a function f on the extended phase space satisfying

$$\{\Omega, f\}_{\mathcal{E}} = 0\,, \qquad \mathrm{gh}(f) = 0\,. \tag{3.38}$$

Two observables f and g are said equivalent iff their difference is BRST exact, i.e. $f - g = \{\Omega, h\}_{\mathcal{E}} = 0$ for some function $h \in \mathcal{F}(\mathcal{E})$. The space of inequivalent observables of the system is thus the BRST cohomology with ghost number zero (i.e. quotient of the BRST closed functions with zero ghost number modulo exact ones). Let us now investigate the structure of the BRST cohomology of the BFV system on $\mathcal{E}$ determined by the BRST charge Ω constructed in **3.2**.

Proposition 3.3. (1) *Let f_0 be an arbitrary Y-independent function (i.e. a function of $x, p, \mathcal{C}, \mathcal{P}$ only) of a nonnegative ghost number. Then there exist function f on $\mathcal{E}$ such that*

$$\{\Omega, f\}_{\mathcal{E}} = 0\,, \quad f\big|_{Y=0} = f_0\,, \qquad \mathrm{gh}(f) = \mathrm{gh}(f_0)\,. \tag{3.39}$$

Moreover, if $\tilde{f}$ also satisfies (3.39) *then*

$$\tilde{f} - f = \{\Omega, g\} \tag{3.40}$$

for some function g on $\mathcal{E}$.

(2) *If in addition one requires f to satisfy $\delta^*(f - f_0) = 0$ then f is the unique solution to* (3.39)*.*

Proof. Let us expand the adjoint action $\{\Omega, \cdot\}_{\mathcal{E}}$ of the BRST charge and the function f into the sum of homogeneous components w.r.t. Y

$$\{\Omega, \cdot\}_{\mathcal{E}} = -\delta + \sum_{s=0}^{\infty} \delta_s\,, \qquad f = \sum_{s=0}^{\infty} f_s\,, \tag{3.41}$$

with $\delta = \mathcal{C}^A \dfrac{\partial}{\partial Y^A}$. In the r-th order in Y equation (3.39) then becomes

$$\delta f_{r+1} = B_r\,, \tag{3.42}$$

where B_r is given by

$$B_r = \sum_{s=0}^{r} \delta_s f_{r-s}\,. \tag{3.43}$$

The consistency condition for equation (3.42) is $\delta B_r = 0$. Let us show that $\delta B_p = 0$ provided (3.42) is fulfilled for any $r \leq p$ and f_0 carries nonnegative ghost number. Indeed, in the zeroth order in Y (3.42) rewrites as

$$\delta f_1 = \delta_0 f_0 \,. \tag{3.44}$$

The consistency condition is obviously fulfilled since

$$\delta B_0 = \delta\delta_0 f_0 = -\delta_0 \delta f_0 = 0 \,. \tag{3.45}$$

The later equality follows because f_0 is independent of Y. Assume that f_r are given for $r \leq p$ and equation (3.42) is fulfiled for $r \leq p-1$. Consider then the identity:

$$\left\{\Omega, \left\{\Omega, \sum_{s=0}^{p} f_s\right\}_{\mathcal{E}}\right\}_{\mathcal{E}} = (-\delta + \sum_{q=0}^{\infty}\delta_q)(-\delta + \sum_{t=0}^{\infty}\delta_t)\sum_{s=0}^{p} f_s = 0 \,. \tag{3.46}$$

One can see that

$$(-\delta + \sum_{t=0}^{\infty}\delta_t)\sum_{s=0}^{p} f_s = B_p + \ldots \tag{3.47}$$

were $\ldots$ denote terms of order higher than p. In the $p-1$-th order in Y Eq. (3.46) then implies: $\delta B_p = 0$.

That $\delta B_p = 0$ allows one to construct solution iteratively:

$$f_{p+1} = \delta^* B_p \,. \tag{3.48}$$

One can indeed check that

$$\delta f_{p+1} = \delta\delta^* B_p = \delta\delta^* B_p + \delta^*\delta B_p + B_p\big|_{\mathcal{C}=Y=0} = B_p \,, \tag{3.49}$$

since $\delta B_p = 0$ and $B_p\big|_{\mathcal{C}=Y=0} = 0$. The later equality is obvious for $p \geq 1$; the fact that $B_0\big|_{\mathcal{C}=Y=0} = (\delta_0 f_0)\big|_{\mathcal{C}=Y=0} = 0$ follows because f_0 has a nonnegative ghost number. Thus the first part of the statement is proved.

Further, let f and $\tilde{f}$ satisfy (3.39). Assume that $f_s = \tilde{f}_s$ for any $s \leq r$. In the r-th order in Y Eq. (3.39) then implies:

$$\delta f_{r+1} = B_r \,, \qquad \delta \tilde{f}_{r+1} = B_r \,. \tag{3.50}$$

It follows that $\delta(\tilde{f}_r - f_r) = 0$. Introducing $g_r = \delta^*(\tilde{f}_r - f_r)$ one can see that

$$\tilde{f}_r - f_r = \delta g_r \,, \tag{3.51}$$

since g_r is homogeneous in Y of order $r+1$ and $\mathrm{gh}(g_r) = \mathrm{gh}(f) - 1$. Thus

$$\overline{f} = f + \{\Omega, g_r\}_{\mathcal{E}} \tag{3.52}$$

satisfies (3.39) and coincides with $\tilde{f}$ up to to the terms of order higher than $r+1$. Iteratively applying this procedure one can construct function g such that

$$\tilde{f} = f + \{\Omega, g\}_{\mathcal{E}} \,. \tag{3.53}$$

Finally, let f and $\tilde{f}$ satisfy (3.39) and additional condition $\delta^*(f - f_0) = \delta^*(\tilde{f} - f_0) = 0$. For $d_{r+1} = f_{r+1} - \tilde{f}_{r+1}$ one then has:

$$\delta d_{r+1} = \delta^* d_{r+1} = 0 \,.$$

This implies that $d_{r+1} = 0$ because d_{r+1} is at least linear in Y. This proves second item. $\square$

Lemma 3.4. *Let f_0 be an arbitrary Y-independent function of a nonnegative ghost number. Let also f be a BRST invariant extension of f_0 obtained by Proposition* **3.3**. *Then*

$$f = \{\Omega, h\}_{\mathcal{E}} \,, \tag{3.54}$$

for some function h on $\mathcal{E}$ if and only if

$$f_0\big|_{\mathcal{C}^A=\theta_\alpha=p_i=0} = 0 \,. \tag{3.55}$$

Proof. It is useful to introduce new coordinate functions

$$\overline{p}_i = p_i - \mathcal{P}_k \Gamma^k_{ij} \mathcal{C}^j \,, \tag{3.56}$$

where Γ^k_{ij} are the coefficients of an arbitrary symmetric connection on $\mathcal{M}$. The reason is that unlike p_i that have inhomogeneous transformation properties, the coordinate functions $\overline{p}_i$ transform as the coefficients of a 1-form on $\mathcal{M}$. The functions $\overline{p}, x, Y, \mathcal{C}$ and $\mathcal{P}$ also form a local coordinate system on $\mathcal{E}$; the conditions $\mathcal{C}^A = \theta_\alpha = p_i = 0$ and $\mathcal{C}^A = \theta_\alpha = \overline{p}_i = 0$ are obviously equivalent.

Any BRST exact function f (i.e. a function that can be represented as $f = \{\Omega, h\}$) evidently vanishes when $Y^A = \mathcal{C}^A = \theta_\alpha = \overline{p}_i = 0$. Conversely, assume that f_0 vanishes when $\mathcal{C}^A = \theta_\alpha = \overline{p}_i = 0$ and carries nonnegative ghost number. Then it can be represented as

$$f_0 = \mathcal{C}^A f_A + \theta_\alpha f^\alpha + \overline{p}_i f^i \,, \tag{3.57}$$

where functions f^i can be taken in the form: $f^i = f^i(x, p)$. One can also choose f_A and f^i such that they transform as the components of a section of $\mathbf{W}^*(\mathcal{M})$ and components of a vector field on $\mathcal{M}$ respectively. We introduce

$$\begin{aligned}
\overline{f}^\alpha &= f^\alpha + Y^l \partial_l f^\alpha \,, \\
\overline{f}^i &= f^i + Y^l \,(\partial_l f^i + \Gamma^i_{lk} f^k - \overline{p}_j \Gamma^j_{lk} \frac{\partial}{\partial \overline{p}_k} f^i + \frac{1}{2} \mathcal{C}^j R^m_{ljk} \mathcal{P}_m \frac{\partial}{\partial \overline{p}_k} f^i) \,,
\end{aligned} \tag{3.58}$$

where R is the curvature of Γ. Note that $\overline{f}^i$ transform as components of a vector field on $\mathcal{M}$; in particular, $\overline{f}^i \mathcal{P}_i$ is the globally defined function on $\mathcal{E}$. Picking h_0 as

$$h_0 = -Y^A f_A + \mathcal{P}_\alpha \overline{f}^\alpha - \mathcal{P}_i \overline{f}^i \,, \tag{3.59}$$

one can indeed check that

$$f_0 = (\{\Omega, h_0\}_{\mathcal{E}})\,|_{Y=0} \,. \tag{3.60}$$

Finally, it follows from Proposition **3.3** that there exists function h_1 such that $f = \{\Omega, h_0 + h_1\}_{\mathcal{E}}$. □

Putting together Proposition **3.3** and Lemma **3.4** we arrive at

Theorem 3.5. *In nonnegative ghost number the BRST cohomology of the BRST charge Ω constructed in Theorem* **3.2** *is:*

$$\begin{aligned}
H^n &= 0 \,, \quad n \geq 1 \\
H^n &= \mathcal{C}^\infty(\Sigma) \,, \quad n = 0 \,.
\end{aligned} \tag{3.61}$$

where H^n denotes cohomology with ghost number n and $\mathcal{C}^\infty(\Sigma)$ is an algebra of smooth functions on the constraint surface $\Sigma \subset \mathcal{M}$.

It follows from the theorem that at least at the classical level, the original second-class constraint system on $\mathcal{M}$ is equivalent to the constructed BFV-BRST system on $\mathcal{E}$.

4. Quantum description and star-product

In this section we quantize the constructed BFV-BRST system. This includes constructing quantum BRST charge, quantum BRST observables and evaluating quantum BRST cohomology. The star product for the Dirac bracket on $\mathcal{M}$ is constructed as the quantum multiplication of BRST observables.

4.1. **Quantization of the extended phase space.** The extended phase space $\mathcal{E}$ of the BFV-BRST system is in general a non-flat manifold and thus can not be quantized directly. Fortunately, all physical observables as well as the generators of the BRST algebra (the BRST charge Ω and the ghost charge $G = \mathcal{C}^A\mathcal{P}_A$) can be chosen as elements of a certain subalgebra $\mathfrak{A} \subset \mathcal{F}(\mathcal{E})$. In its turn $\mathfrak{A}$ is closed w.r.t. Poisson bracket and can be explicitly quantized. This implies that one can equip $\mathfrak{A}$ with the quantum multiplication satisfying the standard correspondence principle.

The construction of $\mathfrak{A}$ is a direct generalization of that from [15] and we present it very brief here. Let $\mathfrak{A}_0$ be a subalgebra of functions on $\mathcal{E}$, which do not depend on the momenta p_i and the ghost momenta $\mathcal{P}_i$. A general element of this algebra is then given by

$$a = a(x^i, Y^A, \mathcal{C}^A, \mathcal{P}_\alpha)\,. \tag{4.1}$$

In invariant terms $\mathfrak{A}_0$ is a tensor product of algebra generated by $\mathcal{P}_\alpha$ and algebra of functions on $\Pi W(\mathcal{M}) \oplus W(\mathcal{M})$ pulled back by the projection $\mathcal{E} \to \Pi W(\mathcal{M}) \oplus W(\mathcal{M})$. $\mathfrak{A}_0$ is a Poisson algebra, i.e. it is closed w.r.t. the ordinary multiplication and the Poisson bracket.

At the quantum level it is useful to consider the algebra

$$\hat{\mathfrak{A}}_0 \equiv \mathfrak{A}_0 \otimes [[\hbar]]\,, \tag{4.2}$$

where $[[\hbar]]$ denotes formal power series in $\hbar$. $\hat{\mathfrak{A}}_0$ is also a Poisson algebra. It is easy to obtain a deformation quantization of $\hat{\mathfrak{A}}_0$ considered as the Poisson algebra. Indeed the Weyl star multiplication works well. Namely, for any $a, b \in \hat{\mathfrak{A}}_0$ one postulates

$$\begin{aligned}(a \star b)(x, Y, \mathcal{C}, \mathcal{P}, \hbar) \; &= \\ = \; \{(a(x, Y_1, \mathcal{C}_1, \mathcal{P}_2, \hbar) exp(-\frac{i\hbar}{2}(D^{AB}&\frac{\overleftarrow{\partial}}{\partial Y_1^A}\frac{\partial}{\partial Y_2^B} + \frac{\overleftarrow{\partial}}{\partial \mathcal{C}_1^\alpha}\frac{\partial}{\partial \mathcal{P}_\alpha^2} + \frac{\overleftarrow{\partial}}{\partial \mathcal{P}_\alpha^1}\frac{\partial}{\partial \mathcal{C}_2^\alpha})) \\ &b(x, Y_2, \mathcal{C}_2, \mathcal{P}_2, \hbar)\}\big|_{Y_1=Y_2=Y,\, \mathcal{C}_1=\mathcal{C}_2=\mathcal{C},\, \mathcal{P}_1=\mathcal{P}_2=\mathcal{P}}\,,\end{aligned} \tag{4.3}$$

where $\mathcal{P}$ stands for the dependence on $\mathcal{P}_\alpha$ only.

Subalgebra $\hat{\mathfrak{A}}$ ($\mathfrak{A}$) is an extension of $\hat{\mathfrak{A}}_0$ (respectively $\mathfrak{A}_0$) by elements $\mathbf{P} = -\mathcal{C}^i p_i$, $\mathbf{G} = \mathcal{C}^i\mathcal{P}_i$. A general homogeneous element $A \in \mathfrak{A}$ is then given by

$$a = \mathbf{P}^r\mathbf{G}^s a_0\,, \quad r = 0, 1\,, \quad s = 0, 1, \ldots, 2N\,, \quad a_0 \in \hat{\mathfrak{A}}_0\,. \tag{4.4}$$

Weyl star product (4.3) can be easily extended from $\hat{\mathfrak{A}}_0$ to $\hat{\mathfrak{A}}$. Explicit construction of the star-product in $\hat{\mathfrak{A}}$ is an obvious generalization of that presented in [15]. Here we write explicitly only the multiplication table for $\mathbf{G}$-independent elements; it turns out that this is sufficient for present considerations:

$$\mathbf{P} \star a = \mathbf{P}a\,, \qquad a \star \mathbf{P} = a\mathbf{P} + (-1)^{\mathrm{p}(a)}\frac{i\hbar}{2}\overline{\nabla}a\,, \qquad \mathbf{P} \star \mathbf{P} = -i\hbar(\overline{R} + \omega)\,, \tag{4.5}$$

where a is an arbitrary element of $\hat{\mathfrak{A}}_0$, $\overline{\nabla} = \mathcal{C}^i\partial_i - \mathcal{C}^i\overline{\Gamma}^A_{iB}Y^B\frac{\partial}{\partial Y^A}$, and functions $\overline{R}$ and ω are the "generating functions" for the curvature and symplectic form:

$$\overline{R} = \frac{1}{4}\mathcal{C}^i\mathcal{C}^j\overline{R}_{ij;\,AB}Y^AY^B\,, \qquad \omega = \frac{1}{2}\mathcal{C}^i\omega_{ij}\mathcal{C}^j\,. \qquad \overline{R}\,,\omega \in \mathfrak{A} \subset \hat{\mathfrak{A}}\,. \tag{4.6}$$

In what follows we treat $\hat{\mathfrak{A}}$ as an associative algebra with the product determined by (4.3) and (4.5) and extended to $\mathbf{G}$-dependent elements as in [15]. Let us introduce a useful grading in $\hat{\mathfrak{A}}$. Namely, we prescribe the following degrees to the variables

$$\begin{aligned} \deg(x^i) = \deg(\mathcal{C}^A) = 0\,, \qquad \deg(p_i) = \deg(\mathcal{P}_A) = 2\,,\\ \deg(Y^A) = 1\,, \qquad \deg(\hbar) = 2\,. \end{aligned} \tag{4.7}$$

The star-commutator in $\hat{\mathfrak{A}}$ obviously preserves the degree.

4.2. **Quantum BRST charge.** Since the classical BRST charge from Theorem **3.2** belongs to $\hat{\mathfrak{A}}$ it is natural to a define quantum BRST charge $\hat{\Omega}$ as a solution to the quantum master equation

$$[\hat{\Omega},\hat{\Omega}]_\star \equiv 2\,\hat{\Omega}\star\hat{\Omega} = 0\,, \qquad \hat{\Omega}\in\hat{\mathfrak{A}}\,, \qquad \mathrm{p}(\hat{\Omega}) = 1\,,\ \mathrm{gh}(\hat{\Omega}) = 1\,. \tag{4.8}$$

In analysis of the master equation it is useful to expand $\hat{\Omega}$ w.r.t. degree introduced in (4.7)

$$\hat{\Omega} = \sum_{r=0}^{\infty}\hat{\Omega}^r\,, \qquad \deg(\hat{\Omega}^r) = r\,. \tag{4.9}$$

A proper quantum counterpart of the classical boundary condition chosen in **3.2** reads as:

$$\Omega^0 = \mathcal{C}^\alpha\theta_\alpha\,, \qquad \Omega^1 = -\mathcal{C}^AD_{AB}Y^B\,, \qquad \Omega^2 = -\mathcal{C}^ip_i + \frac{i}{\hbar}\delta^*([\hat{\Omega}^0,\hat{\Omega}^1]_\star)\,. \tag{4.10}$$

Theorem 4.1. *Equations* (4.8) *has a solution* $\hat{\Omega}$ *satisfying boundary condition* (4.10). *If in addition one requires* $\hat{\Omega}$ *to satisfy* $\delta^*\hat{\Omega}^r = 0$ *and* $\hat{\Omega}^r = \hat{\Omega}^r(x,Y,\mathcal{C},\hbar)$ *for* $r \geq 3$ *the solution is unique.*

Proof. The proof is a direct generalization of that of the analogous statement from [15]. Assume that $\hat{\Omega}^r \in \hat{\mathfrak{A}}_0$ for $r \geq 3$ and $\hat{\Omega}$ doesn't depend on $\mathcal{P}$. Then the solution can be constructed iteratively in the form

$$\hat{\Omega}^{r+1} = \delta^*\hat{B}^r\,, \qquad r \geq 2\,,$$

where $\deg(\hat{\Omega}^a) = a$ and $\hat{B}^r$ is given by

$$\hat{B}^r = \frac{i}{2\hbar}\sum_{s=0}^{r-2}[\hat{\Omega}^{s+2},\hat{\Omega}^{r-s}]_\star\,, \qquad r \geq 2\,. \tag{4.11}$$

□

4.3. **Quantum BRST observables and star-product.** At the classical level each physical observable (element of zero ghost number BRST cohomology) can be considered as an element of $\hat{\mathfrak{A}}$. It is then natural to define a quantum BRST observable as a function f satisfying:

$$[\hat{\Omega}, f]_\star = 0\,, \qquad f \in \hat{\mathfrak{A}}\,. \tag{4.12}$$

Two observables are said equivalent iff their difference can be represented as $\frac{i}{\hbar}[\hat{\Omega}, g]_\star$ for some $g \in \hat{\mathfrak{A}}$. Inequivalent observables is thus a zero ghost number cohomology of $\frac{i}{\hbar}[\hat{\Omega}, \cdot]_\star$.

Let us consider first observables in $\hat{\mathfrak{A}}_0$. It turns out that any function $f_0(x,\mathcal{C})$ admits a BRST invariant extension f satisfying (4.12) and the boundary condition

$$f\big|_{Y=0} = f_0 \,. \tag{4.13}$$

If f_0 has a definite ghost number we also require: $\mathrm{gh}(f) = \mathrm{gh}(f_0)$.

Proposition 4.2. *Given a function $f_0(x,\mathcal{C})$ there exists solution $f \in \hat{\mathfrak{A}}_0$ to the equation* (4.12) *satisfying boundary condition* (4.13)*. If in addition one requires f to satisfy $\delta^*(f - f_0) = 0$ and $f = f(x, Y, \mathcal{C}, \hbar)$ then the solution is unique.*

Proof. The proof is a direct generalization of that of the analogous statement from [15]. Let us expand f as

$$f = f_0 + \sum_{s=1}^{\infty} f_s\,, \qquad \deg(f_s) = s\,. \tag{4.14}$$

The solution is constructed iteratively in the form

$$f_{r+1} = \delta^* B_r\,,$$

with B^r being

$$B_r = \frac{i}{2\hbar}\sum_{s=0}^{r-2}[\hat{\Omega}^{s+2}, f_{r-s}]_\star\,.$$

In particular, for the function $f_0 = f_0(x)$ we have explicitly

$$f = f_0 + Y^i \partial_i f_0 + \dots\,, \tag{4.15}$$

where $\dots$ denotes terms of higher order in Y and $\hbar$. ☐

Because the BRST invariant extension determined by the statement is obviously a linear map it can be extended to functions depending formally on $\hbar$.

By means of the statement we obtain a one-to-one correspondence between $\mathcal{C}^\infty(\mathcal{M}) \otimes [[\hbar]]$ and the BRST invariant functions depending on x, Y and $\hbar$ only. The space of these functions is obviously closed w.r.t. the quantum multiplication (4.3) in $\hat{\mathfrak{A}}_0$. This multiplication determines thus a star product on $\mathcal{M}$, giving a deformation quantization of the Dirac bracket on $\mathcal{M}$. Namely, given functions f_0 and g_0 on $\mathcal{M}$ one has

$$f_0 \star_{\mathcal{M}} g_0 = (f \star g)\big|_{Y=0} = f_0 g_0 - \frac{i\hbar}{2}\{f_0, g_0\}^D_{\mathcal{M}} + \dots\,, \tag{4.16}$$

where $f = f(x, Y, \hbar)$ and $g = g(x, Y, \hbar)$ are the unique quantum BRST invariant extensions of f_0 and g_0 obtained by Proposition **4.2**, $\star$ is a Weyl product given by (4.3), and $\dots$ denote terms of higher order in $\hbar$.

4.4. **BRST cohomology at the quantum level.** To complete description of the constructed quantum gauge system let us calculate cohomology of the quantum BRST charge obtained in Theorem **4.1**. Since we do not have a star product in the algebra $\mathcal{F}(\mathcal{E})$ of functions on the entire $\mathcal{E}$ but only in the subalgebra $\hat{\mathfrak{A}}$ we are interested in the cohomology of $\hat{\Omega}$ evaluated in $\hat{\mathfrak{A}}$.

Let us first note that cohomology of the classical BRST charge Ω evaluated in $\hat{\mathfrak{A}}$ (considered as a Poisson algebra) coincides with that calculated in $\mathcal{F}(\mathcal{E}) \otimes [[\hbar]]$. Indeed, appropriate modifications of Proposition **3.3** and Lemma **3.4** show this.

Instead of formulating quantum counterparts of **3.3** and **3.4** we construct the quantum BRST invariant elements as the quantum deformations of classical ones.

Proposition 4.3. (1) *Let $f \in \mathfrak{A}$ has a nonnegative ghost number and satisfies*

$$\left\{\hat{\Omega}\big|_{\hbar=0}, f\right\}_{\mathcal{E}} = 0\,, \tag{4.17}$$

where $\hat{\Omega}$ is the quantum BRST charge obtained in Theorem **4.1**. *Then there exist the quantum corrections $f_s\,, s \geq 1$ such that $f_s \in \hat{\mathfrak{A}}$, $\mathrm{gh}(f_s) = \mathrm{gh}(f)$, and $\hat{f} = f - i\hbar f_1 + (-i\hbar)^2 f_2 + \ldots$ satisfies*

$$[\hat{\Omega}, \hat{f}]_\star = 0\,. \tag{4.18}$$

(2) *Let $\hat{f} \in \hat{\mathfrak{A}}$ carries a nonnegative ghost number and satisfies* (4.18). *Then there exists $\hat{g} \in \hat{\mathfrak{A}}$ such that $\hat{f} + \frac{i}{\hbar}[\hat{\Omega}, \hat{g}]_\star$ is the function of $x, Y, \mathcal{C}$ and $\hbar$ only.*

Proof. Function f_0 is a classical BRST observable, because of $\Omega \equiv \hat{\Omega}\big|_{\hbar=0}$ is obviously a classical BRST charge from **3.2**. In the $r+2$-th order in $\hbar$ Eq. (4.18) implies

$$\{\Omega, f_{r+1}\}_{\mathcal{E}} + \mathcal{B}_r = 0\,, \tag{4.19}$$

with $\mathcal{B}_r$ given by

$$\mathcal{B}_r = \sum_{s+t+u=r+2} [\hat{\Omega}^s, f_t]^u_\star\,, \qquad s, u \geq 1\,, \quad t \geq 0\,, \tag{4.20}$$

where we have expanded $\hat{\Omega}$ and $[\,,]_\star$ in $\hbar$ according to

$$\hat{\Omega} = \Omega + \sum_{s=1}^{\infty} (-i\hbar)^s \hat{\Omega}^s\,, \qquad [\cdot\,,\cdot]_\star = \sum_{s=1}^{\infty} (-i\hbar)^s\, [\cdot\,,\cdot]^s_\star\,. \tag{4.21}$$

Arguments similar to those of the proof for Proposition **3.3** show that $\{\Omega, \mathcal{B}^p\}_{\mathcal{E}} = 0$ provided Eq. (4.19) holds for all $r \leq p-1$. Since cohomology of a classical BRST charge vanish in strictly positive ghost number and $\mathrm{gh}(\mathcal{B}_{r+1}) \geq 1$ we see that Eq. (4.19) has a solution for $r = p$. Moreover, f_{r+1} can be taken to belong to $\hat{\mathfrak{A}}$.

The second part of the statement is trivial when $\hat{f}$ carries strictly positive ghost number, since classical BRST cohomology vanish in this case. Let $\hat{f} \in \hat{\mathfrak{A}}$ satisfies (4.18) and $\mathrm{gh}(\hat{f}) = 0$. Let us show that there exists $\hat{g}$ such that $\hat{f} + \frac{i}{\hbar}[\hat{\Omega}, \hat{g}]_\star$ depends on x, Y and $\hbar$ only. This is obvious for the zero order term f in the expansion $\hat{f} = f - i\hbar f_1 + (-i\hbar)^2 f_2 + \ldots$ of $\hat{f}$ in powers of $\hbar$. Assume that this is the case for all f_s, $s \leq r$. In the $r+2$-th order in $\hbar$ Eq. (4.18) then takes the form (4.19). Because f_s is a function of only x, Y for $s \leq r$ then $\mathcal{B}_r$ in (4.19) has the form $\mathcal{C}^A b_A(x, Y)$; one can then find $\tilde{f}_{r+1}(x, Y)$ such that

$$\left\{\Omega, \tilde{f}_{r+1}\right\}_{\mathcal{E}} + \mathcal{B}_r = 0\,. \tag{4.22}$$

Since $\left\{\Omega, f_{r+1} - \tilde{f}_{r+1}\right\}_{\mathcal{E}} = 0$ there exist functions $\overline{f}_{r+1}(x, Y)$ and $g_{r+1} \in \hat{\mathfrak{A}}$ such that

$$f_{r+1} = \overline{f}_{r+1} - \{\Omega, g_{r+1}\}_{\mathcal{E}}\,. \tag{4.23}$$

It follows that $\overline{f}_{r+1}(x, Y)$ is a $r+1$-th order term in the expansion of $\hat{f} + \frac{i}{\hbar}[\hat{\Omega}, (-i\hbar)^{r+1} g_{r+1}]_\star$ in powers of $\hbar$. Proceeding further by induction one can find $\hat{g}$ required in item 2. $\square$

Proposition **4.2** establishes a one-to-one correspondence between $\mathcal{C}^\infty(\mathcal{M}) \otimes [[\hbar]]$ and the quantum BRST invariant functions depending on x, Y and $\hbar$ only. Among these functions those vanishing when $Y = \theta = 0$ are BRST trivial (this can be checked directly or by means of the arguments similar to those in the proof of item 2 of Proposition above) while those which do not vanish when $Y = \theta = 0$ are obviously nontrivial. In this way we arrive at the quantum counterpart of Theorem **3.5**:

Theorem 4.4. *In nonnegative ghost number the quantum BRST cohomology of $\hat{\Omega}$ evaluated in $\hat{\mathfrak{A}}$ is given by*

$$\begin{aligned} \hat{H}^n &= 0\,, \quad n \geq 1 \\ \hat{H}^n &= \mathcal{C}^\infty(\Sigma) \otimes [[\hbar]]\,, \quad n = 0\,. \end{aligned} \tag{4.24}$$

It follows from the theorem that the constructed gauge system is equivalent to the original second-class system on $\mathcal{M}$ at the quantum level as well. This, in particular, implies that the quantum multiplication of inequivalent quantum BRST observables determines a star-product on Σ.

Indeed, the quantum multiplication in $\mathfrak{A}_0$ determines a quantum multiplication in $\hat{H}^0$; the isomorphism $\mathcal{C}^\infty(\Sigma) \otimes [[\hbar]] \to \hat{H}^0$ as well as the inverse map is given explicitly by the Proposition **4.2**. Namely, each function $f_0 \in \mathcal{C}^\infty(\Sigma) \otimes [[\hbar]]$ can be represented as $f|_\Sigma$ for some function $f \in \mathcal{C}^\infty(\mathcal{M}) \otimes [[\hbar]]$. Thus the isomorphism map $\mathcal{C}^\infty(\Sigma) \otimes [[\hbar]] \to \hat{H}^0$ is given explicitly by the quantum BRST invariant extension of f determined by Proposition **4.2**. The inverse map is obviously given by the restriction to Σ. The star product on Σ is then given by

$$f_0 \star_\Sigma g_0 = (f \star_{\mathcal{M}} g)\big|_\Sigma = (\hat{f} \star \hat{g})\big|_\Sigma = f_0 g_0 - \frac{i}{2\hbar}\{f_0, g_0\}_\Sigma + \dots\,, \qquad f|_\Sigma = f_0,\ g|_\Sigma = g_0\,, \tag{4.25}$$

where $\hat{f}$ and $\hat{g}$ are the quantum BRST invariant extensions of functions f and g respectively, $\{,\}_\Sigma$ denotes the Poisson bracket on Σ (restriction of the Dirac bracket $\{,\}^D_{\mathcal{M}}$ to Σ), and $\dots$ denote higher order terms in $\hbar$. It follows that the star product (4.25) is well defined in the sense that it doesn't depend on the choice of representatives $f, g : f|_\Sigma = f_0\,,\ g|_\Sigma = g_0$ of functions f_0, g_0 on Σ. In fact, it also doesn't depend on the choice of a BRST invariant extensions for f and g.

4.5. **Quantization with the Dirac connection and the center of the star-commutator algebra.** Let us make some general observation on the structure of the BRST charge and the BRST invariant observables in the case where the symplectic connection $\overline{\Gamma}$ entering the symplectic structure of $\mathcal{E}$ is the specific connection $\overline{\Gamma}^0$ given by (3.12). Although the proposed quantization scheme works well with an arbitrary symplectic connection in $\mathbf{W}(\mathcal{M})$, it turns out that when one uses the specific connection $\overline{\Gamma}^0$ the star product on $\mathcal{M}$ respects the center of the Dirac bracket algebra. Let us consider first the structure of the quantum BRST charge $\hat{\Omega}$.

Proposition 4.5. *Let the symplectic connection entering the symplectic structure be the specific connection $\overline{\Gamma}^0$ given by* (3.12)*. Let also $\hat{\Omega}_0$ be the unique solution to the master equation* (4.8) *obtained by Theorem* **4.1** *with the boundary conditions* (4.10) *and the additional conditions $\delta^* \hat{\Omega}^r_0 = 0$ and $\hat{\Omega}^r_0 = \hat{\Omega}^r_0(x, Y, \mathcal{C}, \hbar)$ for $r \geq 3$. Then $\hat{\Omega}_0$ satisfies*

$$\frac{\partial}{\partial \mathcal{C}^\alpha}\hat{\Omega}^r_0 = \frac{\partial}{\partial Y^\alpha}\hat{\Omega}^r_0 = 0\,, \qquad r \geq 3\,, \qquad \deg(\hat{\Omega}^r_0) = r\,. \tag{4.26}$$

The classical BRST charge Ω_0 can be obtained as $\Omega_0 = (\hat{\Omega}_0)\big|_{\hbar=0}$ and is then also Y^α and $\mathcal{C}^\alpha$ independent (except the terms of zero and first order in Y).

Proof. For the degree 2 term one has $\hat{\Omega}_0^2 = -\mathcal{C}^i p_i$, since $[\hat{\Omega}_0^0, \hat{\Omega}_0^1]_\star$ vanishes. Assume that (4.26) holds for $r \leq s$. Then $\hat{\Omega}_0^{s+1}$ is also $\mathcal{C}^\alpha$ and Y^α independent. Indeed, for $\hat{\Omega}_0^{s+1}$ one has

$$\hat{\Omega}_0^{s+1} = \delta^* \hat{B}^s \,, \tag{4.27}$$

with $\hat{B}^s$ given by (4.11). The space of $\mathcal{C}^\alpha$ and Y^α independent functions is closed w.r.t. the star commutator in $\mathfrak{A}$, since the respective components of the connection $\overline{\Gamma}^0$ and its curvature vanish and δ^* also preserves this space. Thus $\hat{\Omega}_0^{s+1}$ satisfies (4.26) and the statement follows by induction. Up to the terms of degree higher than three $\hat{\Omega}_0$ reads as

$$\hat{\Omega}_0 = \mathcal{C}^\alpha \theta_\alpha - \mathcal{C}^i p_i - \mathcal{C}^A D_{AB} Y^B - \frac{1}{8}\mathcal{C}^i \overline{R}^0_{ij;\,kl} Y^j Y^k Y^l + \ldots \,. \tag{4.28}$$

□

The same concerns the BRST invariant extension of functions from $\mathcal{M}$ both quantum and classical, which is obtained w.r.t. quantum BRST charge $\hat{\Omega}_0$ and classical BRST charge Ω_0 respectively.

Proposition 4.6. *Let $f_0(x)$ be a function on $\mathcal{M}$ and $\hat{\Omega}_0$ be the BRST charge from Proposition* **4.5***. Let also $f(x, Y, \hbar) \in \hat{\mathfrak{A}}_0$ be the unique solution to the equation*

$$[\hat{\Omega}_0, f]_\star = 0 \,, \tag{4.29}$$

obtained by Proposition **4.2***, with the boundary condition $f\big|_{Y=0} = f_0$ and the additional conditions $\delta^* f = 0$ and $f = f(x, Y, \mathcal{C}, \hbar)$. Then f doesn't depend on Y^α. The unique classical BRST invariant extension of f_0 can be obtained as $f|_{\hbar=0}$ and thus is also Y^α independent.*

Proof. The proof goes in the same way as that of Proposition **4.5** with taking **4.5** into account. □

Let us write down explicitly a few first terms of the BRST invariant extension of $f_0(x)$ obtained by Proposition **4.6**:

$$f = f_0 + Y^i \partial_i f_0 + \frac{1}{2} Y^i Y^j \overline{\nabla}^0_i \overline{\nabla}^0_j f_0 + \frac{1}{6} Y^i Y^j Y^k \overline{\nabla}^0_i \overline{\nabla}^0_j \overline{\nabla}^0_k f_0 - \frac{1}{24}\mathcal{C}^i \overline{R}^0_{ij;\,kl} Y^k Y^l D^{jm} \partial_m f_0 + \ldots \,, \tag{4.30}$$

The expression coincides with that in the Fedosov quantization with the Poisson bivector ω^{ij} substituted by the Dirac bivector D^{ij} and the symplectic connection replaced by the Dirac connection $\overline{\Gamma}^0$.

Taking as f_0 a constraint function θ_α one arrives at the following expression for the unique BRST invariant extension $\overline{\theta}_\alpha$ of θ_α:

$$\overline{\theta}_\alpha = \theta_\alpha + Y^i \partial_i \theta_\alpha \,. \tag{4.31}$$

All the higher order terms in the expansion vanish since $D^{ij}\partial_j \theta_\alpha = 0$ and $\partial_j \partial_i \theta_\alpha - (\overline{\Gamma}^0)^k_{ij} \partial_k \theta_\alpha = 0$. Consequently one arrives at:

Theorem 4.7. *Let the star product $\star_\mathcal{M}$ on $\mathcal{M}$ is constructed by means of $\hat{\Omega}_0$ from Proposition* **4.5***. Then for any function f_0 on $\mathcal{M}$ one has:*

$$f_0 \star_\mathcal{M} \theta_\alpha = \theta_\alpha \star_\mathcal{M} f_0 = f_0 \theta_\alpha \,. \tag{4.32}$$

The statement implies that for the star product constructed by means of $\hat{\Omega}_0$, the central subalgebra of the Dirac bracket algebra is also a central subalgebra of the respective star commutator algebra.

Thus we obtain the explicit construction of the deformation quantization of an arbitrary Dirac bracket on a general symplectic manifold, thereby giving a deformation quantization of respective second-class constraint system.

5. Reduction to the constraint surface

The purpose of this section is to establish an explicit relation between the quantization scheme developed in this paper and the approach based on directly quantizing the respective second-class constraint surface. The goal of the previous sections is to quantize general second-class system on $\mathcal{M}$ in terms of the original constraints θ_α and general coordinates on $\mathcal{M}$. In this way we have arrived at the star product for the Dirac bracket on $\mathcal{M}$, which in turn determines a star product on the constraint surface Σ.

On the other hand, one can find quantization of Σ as a symplectic manifold (e.g. by means of the Fedosov approach). Although explicit reduction to the constraint surface Σ is a huge task in the realistic physical models it is instructive to trace the correspondence between the reduced phase space quantization and the approach developed above. Fortunately, it turns out that this approach reproduces not only the Fedosov star-product on Σ but also all the ingredients (including the extended phase space, the BRST charge, and the BRST invariant extensions of functions on Σ) of the BRST description for the Fedosov quantization of Σ. This, in particular, proves equivalence of the respective approaches to the constraint system quantization.

5.1. **The extended phase space for Σ.** Let us consider the constraint surface Σ as a symplectic manifold, with the symplectic form $\omega^\Sigma \equiv \omega\big|_\Sigma$. Equivalently, the respective Poisson bracket on Σ is the restriction of the Dirac bracket $\{\,,\}^D_\mathcal{M}$ to Σ. We equip Σ with the symmetric symplectic connection $(\overline{\Gamma}^0)^\Sigma$, which is the restriction of the Dirac connection $\overline{\Gamma}^0$ on $\mathcal{M}$ to Σ (see Section **2.4**). Then one can apply to Σ the quantization method of Sections **3** and **4**, with Σ considered as a phase space of the unconstraint system (in this case this method reduces to that of [15] which in turn provides a BRST formulation of the Fedosov quantization). Accordingly, the extended phase space for Σ is given by:

$$\mathcal{E}_\Sigma = \mathcal{T}^*_\omega(\Pi T\Sigma) \oplus T(\Pi T\Sigma)\,. \tag{5.1}$$

$\mathcal{E}_\Sigma$ is equipped with the symplectic form

$$\begin{aligned}\omega^{\mathcal{E}_\Sigma} \;=\; \pi^*(\omega\big|_\Sigma) + 2dp_a\wedge dx^a \;+\; 2d\mathcal{C}^a\wedge d\mathcal{P}_a \;+\\ +\;(\omega\big|_\Sigma)_{ab}\,dY^a\wedge dY^B \;+\; Y^aY^b d\overline{\Gamma}^0_{ab} \;-\; 2Y^a\overline{\Gamma}^0_{ab}\wedge dY^b\,,\end{aligned} \tag{5.2}$$

where we have introduced local coordinates

$$x^a, Y^a, p_a, \mathcal{C}^a, \mathcal{P}_a\,, \tag{5.3}$$

with x^a being a local coordinates on Σ, considered as function on $\mathcal{E}_\Sigma$, and $Y^a, p_a, \mathcal{C}^a, \mathcal{P}_a$ introduced according to Section **3.3**. Along the line of Sections **3** and **4** one can also identify subalgebras $\hat{\mathfrak{A}}^\Sigma$ and $\hat{\mathfrak{A}}^\Sigma_0$, construct the quantum BRST charge, the quantum BRST observables and find a covariant star product on Σ. We will see that all this structures can be obtained by the reduction of the respective structures from $\mathcal{E}$.

5.2. **Constraints on the extended phase space and reduction to $\mathcal{E}_\Sigma$ at the classical level.** A crucial point is that $\mathcal{E}_\Sigma$ can be embedded into the extended phase space $\mathcal{E}$ of Sections **3** and **4**. Indeed, assume that a connection $\overline{\Gamma}$ entering the symplectic structure on $\mathcal{E}$ is the specific connection $\overline{\Gamma}^0$ given by (3.12) and consider a submanifold of the entire extended phase space $\mathcal{E}$ determined by the following constraints:

$$
\begin{array}{rclcrcl}
\theta_\alpha & = & 0\,, & \qquad & Y_\alpha & \equiv & Y^i\partial_i\theta_\alpha = 0\,,\\
Y^\alpha & = & 0\,, & & p_\alpha & \equiv & p_i\omega^{ij}\partial_i\theta_\alpha = 0\,,\\
\mathcal{C}^\alpha & = & 0\,, & & \mathcal{P}_\alpha & = & 0\,,\\
\mathcal{C}_\alpha & \equiv & \mathcal{C}^i\partial_i\theta_\alpha = 0\,, & & \mathcal{P}^2_\alpha & \equiv & \mathcal{P}_i\omega^{ij}\partial_j\theta_\beta = 0\,.
\end{array}
\tag{5.4}
$$

This submanifold can be naturally identified with $\mathcal{E}_\Sigma$. Moreover, the symplectic form (5.2) on $\mathcal{E}_\Sigma$ coincides with the restriction of the symplectic form $\omega^{\mathcal{E}}$ defined on $\mathcal{E}$ to $\mathcal{E}_\Sigma$

A useful way to work with $\mathcal{E}_\Sigma$ is to consider $\mathcal{E}_\Sigma$ as a second-class constraint surface in $\mathcal{E}$. Indeed, a Poisson bracket matrix of these constraints reads as:

$$
\begin{array}{c|cccccccc}
 & \theta_\beta & \mathcal{C}_\beta & \mathcal{C}^\beta & Y_\beta & Y^\beta & \mathcal{P}_\beta & \mathcal{P}^2_\beta & p_\beta \\
\hline
\theta_\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{\alpha\beta}\\
\mathcal{C}_\alpha & 0 & 0 & 0 & 0 & 0 & 0 & \Delta_{\alpha\beta} & \cdot\\
\mathcal{C}^\alpha & 0 & 0 & 0 & 0 & 0 & \delta^\alpha_\beta & \cdot & \cdot\\
Y_\alpha & 0 & 0 & 0 & 0 & -\delta^\beta_\alpha & \cdot & \cdot & \cdot\\
Y^\alpha & 0 & 0 & 0 & \delta^\alpha_\beta & \cdot & \cdot & \cdot & \cdot\\
\mathcal{P}_\alpha & 0 & 0 & \delta^\alpha_\beta & \cdot & \cdot & \cdot & \cdot & \cdot\\
\mathcal{P}^2_\alpha & 0 & \Delta_{\alpha\beta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot\\
p_\alpha & -\Delta_{\alpha\beta} & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
\tag{5.5}
$$

where "dots" denote the possibly non-vanishing blocks whose explicit structure, however, is not needed below. This matrix is obviously invertible. In what follows we also need an explicit form of the matrix inverse to (5.5). It is given by:

$$
\begin{array}{c|cccccccc}
 & \theta_\beta & \mathcal{C}_\beta & \mathcal{C}^\beta & Y_\beta & Y^\beta & \mathcal{P}_\beta & \mathcal{P}^2_\beta & p_\beta \\
\hline
\theta_\alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -\Delta^{\alpha\beta}\\
\mathcal{C}_\alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \Delta^{\alpha\beta} & 0\\
\mathcal{C}^\alpha & \cdot & \cdot & \cdot & \cdot & \cdot & \delta^\beta_\alpha & 0 & 0\\
Y_\alpha & \cdot & \cdot & \cdot & \cdot & \delta^\alpha_\beta & 0 & 0 & 0\\
Y^\alpha & \cdot & \cdot & \cdot & -\delta^\beta_\alpha & 0 & 0 & 0 & 0\\
\mathcal{P}_\alpha & \cdot & \cdot & \delta^\beta_\alpha & 0 & 0 & 0 & 0 & 0\\
\mathcal{P}^2_\alpha & \cdot & \Delta^{\alpha\beta} & 0 & 0 & 0 & 0 & 0 & 0\\
p_\alpha & \Delta^{\alpha\beta} & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}
\tag{5.6}
$$

Proposition 5.1. *Let $\{,\}^D_{\mathcal{E}}$ be a Dirac bracket associated to the constraints* (5.4)*. Then the BRST charge Ω_0 from Proposition* **4.5** *satisfies a "weak" master equation:*

$$\left(\{\Omega_0, \Omega_0\}^D_{\mathcal{E}}\right)\big|_{\mathcal{E}_\Sigma} = 0\,. \tag{5.7}$$

Proof. Let us write down explicitly the following terms:

$$\begin{aligned}
&\{\Omega_0, \theta_\alpha\}_{\mathcal{E}} = \mathcal{C}^i \partial_i \theta_\alpha = \mathcal{C}_\alpha \\
&\{\Omega_0, \mathcal{C}_\alpha\}_{\mathcal{E}} = 0 \\
&\{\Omega_0, \mathcal{C}^\alpha\}_{\mathcal{E}} = 0 \\
&\{\Omega_0, Y_\alpha\}_{\mathcal{E}} = -\mathcal{C}_\alpha\,.
\end{aligned} \tag{5.8}$$

The first three equalities are trivial. The last one follows from Proposition **4.5**.

Further, it follows from the explicit form of the matrix (5.6) entering the Dirac bracket $\{,\}^D_{\mathcal{E}}$ that each non-vanishing term in $\{\Omega_0, \Omega_0\}^D_{\mathcal{E}}$ is proportional to (5.8) and thereby vanishes on $\mathcal{E}_\Sigma$. □

Similar arguments lead to the following statement:

Proposition 5.2. *Let function f on $\mathcal{E}$ be such that*

$$\{\Omega_0, f\}_{\mathcal{E}} = 0\,. \tag{5.9}$$

Let also f satisfies

$$\begin{aligned}
(\{f, \theta_\alpha\}_{\mathcal{E}})\big|_{\mathcal{E}_\Sigma} &= 0\,, \qquad & (\{f, Y_\alpha\}_{\mathcal{E}})\big|_{\mathcal{E}_\Sigma} &= 0\,, \\
(\{f, \mathcal{C}_\alpha\}_{\mathcal{E}})\big|_{\mathcal{E}_\Sigma} &= 0\,, \qquad & (\{f, \mathcal{C}^\alpha\}_{\mathcal{E}})\big|_{\mathcal{E}_\Sigma} &= 0\,.
\end{aligned} \tag{5.10}$$

Then,

$$(\{\Omega_0, f\}^D_{\mathcal{E}})\big|_{\mathcal{E}_\Sigma} = 0\,. \tag{5.11}$$

In particular, if $f(x, Y)$ is the unique BRST invariant extension of the function $f_0(x)$, obtained by Proposition **4.6***, then f satisfies* (5.11)*.*

As a simple consequence of Propositions **5.1** and **5.2** we see that

$$\{\Omega_\Sigma, \Omega_\Sigma\}_{\mathcal{E}_\Sigma} = 0\,, \qquad \{\Omega_\Sigma, f_\Sigma\}_{\mathcal{E}_\Sigma} = 0\,, \tag{5.12}$$

where we have introduced separate notations Ω_Σ and f_Σ for the restrictions of the BRST charge Ω_0 and BRST invariant function f to $\mathcal{E}_\Sigma$. One can also see that if $f = f(x, Y)$ is a BRST invariant extension of function $f_0(x)$, then $f_\Sigma = f\big|_{\mathcal{E}_\Sigma}$ is a BRST invariant (w.r.t. Ω_Σ) extension of $f_0\big|_\Sigma$.

Thus, at the classical level, our scheme reproduces all the basic structures of the BRST formulation of the Fedosov quantization for the constraint surface Σ. In particular, the extended phase space for Σ is $\mathcal{E}_\Sigma$, the Poisson bracket therein is the restriction of the Dirac bracket $\{,\}^D_{\mathcal{E}}$ to $\mathcal{E}_\Sigma$, the BRST charge is $\Omega_\Sigma = \Omega_0\big|_{\mathcal{E}_\Sigma}$, and the BRST invariant extension of a function $f_0\big|_\Sigma \in \mathcal{C}^\infty(\Sigma)$ is the restriction $f_\Sigma = f\big|_{\mathcal{E}_\Sigma}$ of the BRST extension f of $f_0 \in \mathcal{C}^\infty(\mathcal{M})$.

5.3. **Quantum reduction and relation with the Fedosov quantization of Σ.** Now we are going to show that the results of the previous subsection can be restated at the quantum level. Let $\hat{\mathfrak{A}}^\Sigma$ and $\hat{\mathfrak{A}}_0^\Sigma$ are the subalgebras of the algebra of functions on $\mathcal{E}_\Sigma$ constructed as in Section **4.1** for the unconstraint system on Σ. Let also $\star_\Sigma$ denotes the quantum multiplication (see Sec. **4.1**) in $\hat{\mathfrak{A}}^\Sigma$ and $\hat{\mathfrak{A}}_0^\Sigma$.

Theorem 5.3. *Let $\hat{\Omega}_\Sigma$ be the restriction of the quantum BRST charge $\hat{\Omega}_0$ from Proposition* **4.5** *to $\mathcal{E}_\Sigma$. Let also f_Σ be a restriction to $\mathcal{E}_\Sigma$ of the quantum BRST invariant extension $f(x, Y, \hbar)$ obtained by Proposition* **4.6** *for a function f_0 on $\mathcal{M}$. Then $\hat{\Omega}_\Sigma$ and f_Σ belong to $\hat{\mathfrak{A}}^\Sigma$ and $\hat{\mathfrak{A}}_0^\Sigma$ respectively and satisfy:*

$$[\hat{\Omega}_\Sigma, \hat{\Omega}_\Sigma]_{\star_\Sigma} = 0\,, \qquad [\hat{\Omega}_\Sigma, f_\Sigma]_{\star_\Sigma} = 0\,. \tag{5.13}$$

Proof. Consider the following subset of the constraints (5.4):

$$Y_\alpha \equiv Y^i \partial_i \theta_\alpha = 0\,, \qquad Y^\alpha = 0\,, \qquad \mathcal{C}^\alpha = 0\,, \qquad \mathcal{P}_\alpha = 0\,. \tag{5.14}$$

These constraints are also second-class ones. It is easy to write down respective Dirac bracket; the non-vanishing basic Dirac bracket relations are given by:

$$\begin{array}{rclcrcl} \{x^i, p_j\}_{\mathcal{E}} & = & \delta^i_j\,, & & \{Y^i, Y^j\}_{\mathcal{E}} & = & D^{ij}\,, \\ \{Y^j, p_i\}_{\mathcal{E}} & = & -(\overline{\Gamma}^0)^j_{ik} Y^k\,, & & \{\mathcal{C}^i, \mathcal{P}_j\}_{\mathcal{E}} & = & \delta^i_j\,, \\ & & \{p_i, p_j\}_{\mathcal{E}} \;=\; \omega_{ij} + \frac{1}{2}\overline{R}^0_{ij;\,kl}\, Y^k Y^l\,. & & & & \end{array} \tag{5.15}$$

It follows from (5.15) that algebras $\hat{\mathfrak{A}}_0$ and $\hat{\mathfrak{A}}$ are closed w.r.t. the Dirac bracket (5.15). Let us introduce the Weyl star product $\star_D$ in $\hat{\mathfrak{A}}_0$ build by the restriction of the Dirac bracket (5.15) to $\hat{\mathfrak{A}}_0$:

$$(a \star_D b)(x, Y, \mathcal{C}, \mathcal{P}, \hbar) = exp(-\frac{i\hbar}{2} D^{ij} \frac{\partial}{\partial Y_1^i} \frac{\partial}{\partial Y_2^j}) a(x, Y_1, \mathcal{C}, \mathcal{P}, \hbar) b(x, Y_2, \mathcal{C}, \mathcal{P}, \hbar)\big|_{Y_1 = Y_2 = Y}\,, \tag{5.16}$$

where $\mathcal{P}$ stands for dependence on $\mathcal{P}_\alpha$ only.

Lemma 5.4. (1) *Let $f, g \in \hat{\mathfrak{A}}_0$ do not depend on Y^α and $\mathcal{P}_\alpha$. Then,*

$$f \star_D g = f \star g\,, \tag{5.17}$$

where $\star$ in the R.H.S. denotes Weyl star product (4.3).

(2) *For any $a \in \hat{\mathfrak{A}}_0$ its restriction $a\big|_{\mathcal{E}_\Sigma}$ belongs to $\hat{\mathfrak{A}}_0^\Sigma$. The star multiplications $\star_D$ determines a star multiplication in $\hat{\mathfrak{A}}_0^\Sigma$ that is identical with with $\star_\Sigma$:*

$$(a\big|_{\mathcal{E}_\Sigma}) \star_\Sigma (b\big|_{\mathcal{E}_\Sigma}) = (a \star_D b)\big|_{\mathcal{E}_\Sigma}\,. \tag{5.18}$$

Proof. The only nontrivial is the second statement. It is easy to see that restriction $f\big|_{\mathcal{E}_\Sigma}$ of an arbitrary element $f \in \hat{\mathfrak{A}}_0$ belongs to $\hat{\mathfrak{A}}^0_\Sigma$. That $\star_D$ restricts to $\hat{\mathfrak{A}}_0^\Sigma$ follows from the following properties of $\star_D$:

$$\begin{array}{rclclcrclcl} f \star_D Y^\alpha & = & Y^\alpha \star_D f & = & f\, Y^\alpha\,, & & f \star_D Y_\alpha & = & Y_\alpha \star_D f & = & f\, Y_\alpha\,, \\ f \star_D \mathcal{C}^\alpha & = & \pm \mathcal{C}^\alpha \star_D f & = & f\, \mathcal{C}^\alpha\,, & & f \star_D \mathcal{C}_\alpha & = & \pm \mathcal{C}_\alpha \star_D f & = & f\, \mathcal{C}_\alpha\,, \\ & & & & f \star_D \mathcal{P}_\alpha = \pm \mathcal{P}_\alpha \star_D f = f\, \mathcal{P}_\alpha\,, & & & & & & \end{array} \tag{5.19}$$

for any $f \in \hat{\mathfrak{A}}_0$. Indeed, $\hat{\mathfrak{A}}_0^\Sigma$ (as an algebra w.r.t. ordinary commutative product) can be identified with the quotient of $\hat{\mathfrak{A}}_0$ modulo ideal generated (by ordinary commutative product) by elements $Y^\alpha, Y_\alpha, \mathcal{C}^\alpha, \mathcal{C}_\alpha, \mathcal{P}_\alpha \in \hat{\mathfrak{A}}_0$. Then, (5.19) implies that this ideal is also an ideal in $\hat{\mathfrak{A}}_0$ w.r.t. the $\star_D$-product. Thus $\star_D$ determines a star product in $\hat{\mathfrak{A}}_0^\Sigma$. One can easily check that in $\hat{\mathfrak{A}}_0^\Sigma$ the product coincides with $\star_\Sigma$. ∎

Let us now rewrite the master equation (4.8) for $\hat{\Omega}_0$ in the "Fedosov" form:

$$-\delta r + \overline{\nabla}^0 r + \frac{i}{2\hbar}[r, r]_\star = \overline{R}^0 , \tag{5.20}$$

where

$$r = \sum_{s=3}^{\infty} \hat{\Omega}_0^s , \qquad \overline{\nabla}^0 = \mathcal{C}^i \partial_i - \mathcal{C}^i (\overline{\Gamma}^0)^A_{iB} Y^B \frac{\partial}{\partial Y^A} , \qquad \delta = \mathcal{C}^A \frac{\partial}{\partial Y^A} , \tag{5.21}$$

and

$$\overline{R}^0 = \frac{1}{4} \mathcal{C}^i \mathcal{C}^j \overline{R}^0_{ij;\,AB} Y^A Y^B = \frac{1}{4} \mathcal{C}^i \mathcal{C}^j \overline{R}^0_{ij;\,kl} Y^k Y^l . \tag{5.22}$$

Since r doesn't depend on Y^α the master equation can be equivalently rewritten as that w.r.t. the star product $\star_D$ and the Dirac connection on $\mathcal{M}$:

$$-\delta r + (\mathcal{C}^i \partial_i - \mathcal{C}^i (\overline{\Gamma}^0)^k_{ij} Y^j \frac{\partial}{\partial Y^j}) r + \frac{i}{2\hbar}[r, r]_{\star_D} = \overline{R}^0 . \tag{5.23}$$

Taking into account the second item of lemma **5.4** one arrives at

$$-\delta_\Sigma (r|_{\mathcal{E}_\Sigma}) + \overline{\nabla}^0{}_\Sigma (r|_{\mathcal{E}_\Sigma}) + \frac{i}{2\hbar}[r|_{\mathcal{E}_\Sigma}, r|_{\mathcal{E}_\Sigma}]_{\star_\Sigma} = \overline{R}^0|_{\mathcal{E}_\Sigma} . \tag{5.24}$$

where $\overline{\nabla}^0_\Sigma$ and δ_Σ are restrictions of $\overline{\nabla}^0$ and δ defined on $\mathcal{E}$ to $\mathcal{E}_\Sigma$. In the coordinates $x^a, p_a, Y^a, \mathcal{C}^a, \mathcal{P}_a$ on $\mathcal{E}_\Sigma$ one has

$$\overline{\nabla}^0{}_\Sigma = \mathcal{C}^a \partial_a - \mathcal{C}^a (\overline{\Gamma}^0)^c_{ab} Y^b \frac{\partial}{\partial Y^c} , \qquad \delta_\Sigma = \mathcal{C}^a \frac{\partial}{\partial Y^a} , \tag{5.25}$$

where $(\overline{\Gamma}^0)^c_{ab}$ are coefficients of a connection $(\overline{\Gamma}^0)^\Sigma$ on Σ w.r.t. the coordinates x^a (recall that Dirac connection $(\overline{\Gamma}^0)$ on $\mathcal{M}$ restricts to Σ).

Finally, one can observe that for

$$\hat{\Omega}_\Sigma \equiv \hat{\Omega}_0|_{\mathcal{E}_\Sigma} = -\mathcal{C}^a p_a - \mathcal{C}^a \omega^\Sigma_{ab} y^b + r|_{\mathcal{E}_\Sigma} \tag{5.26}$$

Eq. (5.24) implies:

$$[\hat{\Omega}_\Sigma, \hat{\Omega}_\Sigma]_{\star_\Sigma} = 0 . \tag{5.27}$$

Similar arguments show the rest of the statement. □

It follows from the theorem that the star product on Σ obtained in Section **4.4** as a quantum multiplication of the nonequivalent quantum observables can be identified with the Fedosov star product on Σ, provided the symplectic connection entering the Poisson bracket on $\mathcal{E}$ is the specific symplectic connection $\overline{\Gamma}^0$; the Fedosov star product on Σ corresponds then to the connection $(\overline{\Gamma}^0)^\Sigma$ on Σ obtained by the restriction of the Dirac connection $\overline{\Gamma}^0$ defined on $\mathcal{M}$ to Σ.

6. An alternative formulation

An interesting understanding of the quantization scheme proposed in this paper is provided by the identification of the symplectic form D in the vector bundle $\mathbf{W}(\mathcal{M})$ as the symplectic form on the appropriate symplectic manifold. Namely, let us consider the vector bundle $\mathbf{V}(\mathcal{M}) = \mathcal{M} \times V$ associated with the constraints θ_α (see the beginning of Section **3.2**. Let η^α be coordinates on V. Considered as a manifold $\mathbf{V}(\mathcal{M})$ is equipped with the following 2-form:

$$\overline{D} = \pi^* \omega - d\theta_\alpha \wedge d\eta^\alpha . \tag{6.1}$$

It is useful to write respective matrix:

$$\overline{D} = \begin{pmatrix} \overline{D}_{ij} & \overline{D}_{i\beta} \\ \overline{D}_{\alpha j} & \overline{D}_{\alpha\beta} \end{pmatrix} = \begin{pmatrix} \omega_{ij} & \partial_i\theta_\beta \\ -\partial_j\theta_\alpha & 0 \end{pmatrix} \tag{6.2}$$

The 2-form $\overline{D}$ is obviously closed and is nondegenerate provided the respective Dirac matrix $\Delta_{\alpha\beta} = \{\theta_\alpha, \theta_\beta\}_{\mathcal{M}}$ is invertible. Introducing a unified notation x^A for the coordinates x^i and η^α it is easy to see that coefficients $\overline{D}_{AB}$ of the symplectic form $\overline{D}$ coincides with the coefficients of the symplectic form D on the fibres of $\mathbf{W}(\mathcal{M})$ from Section **3** (it is assumed that the coefficients corresponds to the same coordinate system on $\mathcal{M}$ and the same basis of constraints). Speaking geometrically, the fibres of the symplectic vector bundle $\mathbf{W}(\mathcal{M})$ is identified with the fibres of the tangent bundle over $\mathbf{V}(\mathcal{M})$.

The Poisson bracket corresponding to the symplectic form (6.1) reads as

$$\begin{aligned} \{x^i, x^j\}_{\mathbf{V}(\mathcal{M})} &= \omega^{ij} - \omega^{il}(\partial_l\theta_\alpha)\Delta^{\alpha\beta}(\partial_k\theta_\beta)\omega^{kj}\,, \\ \{x^i, \eta^\alpha\}_{\mathbf{V}(\mathcal{M})} &= -\omega^{il}(\partial_l\theta_\beta)\Delta^{\beta\alpha}\,, \\ \left\{\eta^\alpha, \eta^\beta\right\}_{\mathbf{V}(\mathcal{M})} &= \Delta^{\alpha\beta}\,. \end{aligned} \tag{6.3}$$

Note that this Poisson bracket coincides with the Dirac bracket on $\mathcal{M}$ when evaluated on η-independent functions.

6.1. **First class constraint system on $\mathbf{V}(\mathcal{M})$.** Let us consider functions θ_α as the constraints on $\mathbf{V}(\mathcal{M})$. It is easy to see that these constraints are the first class ones. Moreover, the first-class system determined by θ_α is Abelian. Indeed,

$$\{\theta_\alpha, \theta_\beta\}_{\mathbf{V}(\mathcal{M})} = \{\theta_\alpha, \theta_\beta\}^D_{\mathcal{M}} = 0\,, \tag{6.4}$$

where $\{,\}^D_{\mathcal{M}}$ is a Dirac bracket on $\mathcal{M}$ associated with the second-class constraints θ_α. As a matter of simple analysis this first-class system is equivalent to the original second-class system on $\mathcal{M}$.

This representation of the second-class system on $\mathcal{M}$ allows one to develop an alternative quantization procedure based on quantizing the first-class system on $\mathbf{V}(\mathcal{M})$. However, to quantize this first-class system one should first find quantization of $\mathbf{V}(\mathcal{M})$ considered as a symplectic manifold.

6.2. **BRST quantization of $\mathbf{V}(\mathcal{M})$.** The quantization of $\mathbf{V}(\mathcal{M})$ is rather standard and can be obtained within the quantization scheme of Section **4** applied to the unconstraint system on $\mathbf{V}(\mathcal{M})$ (in this case the approach reduces to that proposed in [15] and results in the Fedosov star-product on $\mathbf{V}(\mathcal{M})$).

According to the scheme one should fix a symmetric symplectic connection in the tangent bundle $T\mathbf{V}(\mathcal{M})$. In fact we already have this connection. Indeed, because $T\mathbf{V}(\mathcal{M})$ can be identified with the vector bundle $\mathbf{W}(\mathcal{M})$ from Section **3**, pulled back by the projection $\mathbf{V}(\mathcal{M}) \to \mathcal{M}$, the connection in $T\mathbf{V}(\mathcal{M})$ can be obtained from that in $\mathbf{W}(\mathcal{M})$. Namely, let us consider a connection on $\mathbf{V}(\mathcal{M})$ determined by

$$\tilde{\overline{\Gamma}}{}^A_{iB} = \overline{\Gamma}{}^{0A}_{\ \ iB}\,, \qquad \tilde{\overline{\Gamma}}{}^A_{\alpha B} = 0\,, \tag{6.5}$$

where coefficients $\overline{\Gamma}{}^{0A}_{\ \ iB}$ are given by (3.12). One can easily check that $\tilde{\overline{\Gamma}}$ is indeed a symmetric symplectic connection on $\mathbf{V}(\mathcal{M})$. Note that coefficients of $\tilde{\overline{\Gamma}}$ in the coordinate system x^i, η^α on $\mathbf{V}(\mathcal{M})$ do not depend on η^α.

Let $\overline{\mathcal{E}}$ be the extended phase space of the unconstraint system on $\mathbf{V}(\mathcal{M})$, which is constructed according to Section **3.3**. Let also p_A, Y^A, $\mathcal{C}^A$ and $\mathcal{P}^A$ denote momenta, conversion variables, ghost variables, and ghost momenta respectively. In this setting Poisson bracket (3.22) reads as

$$\begin{aligned} \{x^A, p_B\}_{\overline{\mathcal{E}}} &= \delta^A_B\,, & \{Y^A, Y^B\}_{\overline{\mathcal{E}}} &= \overline{D}^{AB}\,,\\ \{Y^A, p_B\}_{\overline{\mathcal{E}}} &= -\tilde{\overline{\Gamma}}^A_{BC} Y^B\,, & \{\mathcal{C}^A, \mathcal{P}_B\}_{\overline{\mathcal{E}}} &= \delta^A_B\,,\end{aligned}$$
$$\{p_A, p_B\}_{\overline{\mathcal{E}}} = \overline{D}_{AB} + \frac{1}{2}\tilde{\overline{R}}_{AB;CD} Y^C Y^D\,, \tag{6.6}$$

where $\tilde{\overline{R}}$ denotes a curvature of the connection $\tilde{\overline{\Gamma}}$ on $\mathbf{V}(\mathcal{M})$.

Specifying the constructions of Section **4** to the extended phase space $\overline{\mathcal{E}}$ one obtains a unique quantum BRST charge $\hat{\overline{\Omega}} \in \hat{\mathfrak{A}}^{\overline{\mathcal{E}}}$ satisfying

$$[\hat{\overline{\Omega}}, \hat{\overline{\Omega}}]_\star = 0\,, \qquad \mathrm{p}(\hat{\overline{\Omega}}) = 1\,, \quad \mathrm{gh}(\hat{\overline{\Omega}}) = 1\,, \tag{6.7}$$

the boundary condition

$$\hat{\overline{\Omega}}^0 = 0\,, \qquad \hat{\overline{\Omega}}^1 = -\mathcal{C}^A D_{AB} Y^B\,, \qquad \hat{\overline{\Omega}}^2 = -\mathcal{C}^A p_A\,, \tag{6.8}$$

and the additional conditions conditions $\hat{\overline{\Omega}}^r \in \hat{\mathfrak{A}}^{\overline{\mathcal{E}}}_0$ and $\delta^* \hat{\overline{\Omega}}^r = 0$ for $r \geq 3$.

A unique quantum BRST extension of a function $f_0(x^A)$ is the solution to the equation

$$[\hat{\overline{\Omega}}, f]_\star = 0\,, \qquad f|_{Y=0} = f_0\,, \quad f \in \hat{\mathfrak{A}}^{\overline{\mathcal{E}}}_0\,, \quad \mathrm{gh}(f) = \mathrm{gh}(f_0)\,, \tag{6.9}$$

subjected to the additional condition $\delta^* f = 0$.

In this way one can find the star product $\star_{\mathbf{V}(\mathcal{M})}$ on $\mathbf{V}(\mathcal{M})$, giving a deformation quantization of the Poisson bracket (6.3) on $\mathbf{V}(\mathcal{M})$. An important point is that $\hat{\overline{\Omega}}^r$ doesn't depend on the variables η^α and p_α for $r \geq 3$. The same holds for the unique BRST invariant extension of a function $f_0(x^i)$. Since the quantum multiplication in $\hat{\mathfrak{A}}^{\overline{\mathcal{E}}}$ obviously preserve the space of η^α and p_α independent elements, the star product $\star_{\mathbf{V}(\mathcal{M})}$ preserves the space of η-independent functions, giving thus a deformation quantization of the Dirac bracket on $\mathcal{M}$. Given η-independent functions $f_0(x)$ and $g_0(x)$ one has

$$f_0 \star_{\mathcal{M}} g_0 = f \star g|_{Y=0} = f_0 g_0 - \frac{i\hbar}{2}\{f_0, g_0\}^D_{\mathcal{M}} + \dots\,, \tag{6.10}$$

where f and g are the unique quantum BRST extensions of f_0 and g_0, $\star$ is the Weyl product in $\hat{\mathfrak{A}}^{\overline{\mathcal{E}}}_0$, and $\dots$ denote higher order terms in $\hbar$.

6.3. **The total BRST charge.** In spite of the fact that the BRST charge $\hat{\overline{\Omega}}$ constructed above allows one to find a quantum deformation of the Dirac bracket on $\mathcal{M}$, the BFV-BRST theory determined by $\hat{\overline{\Omega}}$ is not equivalent to the original second-class system on $\mathcal{M}$. The matter is that the original first-class constraints θ_α on $\mathbf{V}(\mathcal{M})$ have not been taken into account.

A way to incorporate the original first-class constraints is well known [3, 6]. To this end one should find BRST invariant extensions of the original first-class constraints and then incorporate them into the appropriately extended BRST charge with their own ghost variables.

Specifying the construction to the case at hand let $\overline{\mathcal{C}}^\alpha$ and $\overline{\mathcal{P}}_\alpha$ be the ghosts and their conjugate momenta associated to the first-class constraints. A total BRST charge $\hat{\overline{\Omega}}_{total}$ is then given by

$$\hat{\overline{\Omega}}_{total} = \hat{\overline{\Omega}} + \overline{\mathcal{C}}^\alpha \overline{\theta}_\alpha = \hat{\overline{\Omega}} + \overline{\mathcal{C}}^\alpha \theta_\alpha + \overline{\mathcal{C}}^\alpha Y^i \partial_i \theta_\alpha\,. \tag{6.11}$$

Because $\bar{\theta}_\alpha$ is the BRST invariant extension of θ_α, $\hat{\bar{\Omega}}_{total}$ is obviously nilpotent. Finally, one can check that $\hat{\bar{\Omega}}_{total}$ determines the correct spectrum of observables.

6.4. **Equivalence to the standard approach.** To complete the description of the alternative formulation we show that the star product (6.10) coincides with that obtained in Section **4.5**.

Note, that the extended phase space $\mathcal{E}$ constructed in Section **3.3** can be identified with the submanifold in $\bar{\mathcal{E}}$ determined by $\eta^\alpha = p_\beta = 0$. Since $\hat{\bar{\Omega}}^r$ do not depend on the variables η^α and p_β for $r \geq 3$, they can can be considered as functions on $\mathcal{E}$.

Proposition 6.1. *Let* $\hat{\bar{\Omega}}$ *be the unique quantum BRST charge of the unconstraint system on* $\mathbf{V}(\mathcal{M})$ *obtained in Section* **6.2** *and* $\hat{\Omega}_0$ *be the unique quantum BRST charge of the second class system on* $\mathcal{M}$ *obtained in Proposition* **4.5**. *Then*

$$\hat{\bar{\Omega}}^r = \hat{\Omega}_0^r\,, \qquad r \geq 3\,. \tag{6.12}$$

where $\hat{\bar{\Omega}}^r$ *and* $\hat{\Omega}_0^r$ *are the respective terms in the expansions of* $\hat{\bar{\Omega}}$ *and* $\hat{\Omega}$ *w.r.t. degree:* $\deg(\hat{\Omega}_0^r) = \deg(\hat{\bar{\Omega}}^r) = r$.

Proof. The statement of the theorem can be explicitly checked for $r = 3$. Further, assuming that (6.12) holds for all $r \leq p$ one can see that the respective quantities $\hat{B}^p$ and $\hat{\bar{B}}^p$ (see the proof of Theorem **4.1**) do coincide. Since the operators δ and δ^* are precisely the same in both cases one observes that $\hat{\bar{\Omega}}^{p+1} = \hat{\Omega}^{p+1}$. The statement then follows by induction. □

It follows from the theorem that for an arbitrary function f depending on x^i, Y^i and $\hbar$) only one has $[\hat{\bar{\Omega}}, f]_\star = [\hat{\Omega}_0, f]_\star$. This implies that the unique BRST invariant extensions of functions from $\mathcal{M}$, determined by $\hat{\bar{\Omega}}$ and $\hat{\Omega}_0$ do coincide. This, in turn, implies that the star product on $\mathcal{M}$ given by (6.10) coincides with that obtained in Section **4.5** using the BRST charge $\hat{\Omega}_0$.

As a final remark we note that the equivalence statement can also be generalized to the case where connection $\bar{\Gamma}$ entering the symplectic structure on $\mathcal{E}$ is an arbitrary symplectic connection in $\mathbf{W}(\mathcal{M})$. In this setting, however, one should equip $T\mathbf{V}(\mathcal{M})$ with the symplectic connection appropriately build by $\bar{\Gamma}$.

7. Conclusion

We summarize the results of this paper. For a second-class constraint system on an arbitrary symplectic manifold $\mathcal{M}$, we have constructed an effective first-class constraint (gauge) system equivalent to the original second-class one. The construction is based on representing the symplectic manifold as a second-class surface in the cotangent bundle $\mathcal{T}^*_\omega\mathcal{M}$ equipped with a modified symplectic structure. The second-class system on $\mathcal{T}^*_\omega\mathcal{M}$ determined by the constraints reponsible for the embedding of $\mathcal{M}$ and the original second-class constraints are converted into an effective first-class system by applying a globally defined version of the standard conversion procedure. Namely, the conversion variables are introduced as coordinates on the fibres of the vector bundle $\mathbf{W}(\mathcal{M}) = T\mathcal{M} \oplus \mathbf{V}(\mathcal{M})$ associated with the complete set of constraints, with the symplectic form given by the respective Dirac matrix. The phase space of the effective system is equipped with the specific symplectic structure build with the help of a symplectic connection in $\mathbf{W}(\mathcal{M})$. We present an explicit form of the particular symplectic connection in $\mathbf{W}(\mathcal{M})$, which is in some sense a minimal one. Remarkably, this connection reduces to the Dirac connection on $\mathcal{M}$, i.e., to a symmetric connection compatible with the Dirac bracket on $\mathcal{M}$.

The effective gauge system thus constructed is quantized by the BFV–BRST procedure and the algebra of quantum observables is explicitly constructed. The star product for the Dirac bracket on $\mathcal{M}$ is obtained as the quantum multiplication of BRST observables.

In the case where the effective gauge system is constructed by the particular (Dirac) connection, the original second-class constraints are shown to be in the center of the respective star-commutator algebra. When restricted to the constraint surface, this star product is also shown to coincide with the Fedodov product constructed by restricting the Dirac connection to the surface.

The proposed quantization method is explicitly phase space covariant (i.e., covariant with respect to the change of local coordinates on the phase space). An interesting problem is to construct its generalization that is also covariant under changing the basis of constraints.

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