Towards a General Theory of Quantized Fields on the Anti-de Sitter Space-Time.

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Abstract

We propose a general framework for studying quantum field theory on the anti-de-Sitter space-time, based on the assumption of positivity of the spectrum of the possible energy operators. In this framework we show that the n-point functions are analytic in suitable domains of the complex AdS manifold, that it is possible to Wick rotate to the Euclidean manifold and come back, and that it is meaningful to restrict AdS quantum fields to Poincaré branes. We give also a complete characterization of two-point functions which are the simplest example of our theory. Finally we prove the existence of the AdS-Unruh effect for uniformly accelerated observers on trajectories crossing the boundary of AdS at infinity, while that effect does not exist for all the other uniformly accelerated trajectories.

1 Introduction

Quantum Field Theory (QFT) on the anti-de Sitter (AdS) space-time has come today to the general attention because of the role that AdS geometry plays at several places in modern theoretical physics [M, RS]. AdS QFT is also believed to provide an infrared regularization [CW] that can be useful for instance to understand the longstanding problems of gauge QFT's. From a more conceptual viewpoint, this regain of interest has led some authors to realize the need for a deeper setting of AdS QFT and to investigate the consequences of general principles which, in spite of the difficulties generated by the peculiar geometry of AdS, might lead to a reasonable approach to the interacting fields on this spacetime; such an approach should of course include the case of AdS free fields, whose preliminary versions were given in early works [AIS] [F]. In this spirit three recent works on general AdS QFT can be mentioned. [BFS] uses the framework of observable algebras, but without assuming local commutativity from the outset. Instead the property of "passivity" (in the sense of [PW]) is postulated for the vacuum, and a remarkable proportion of the more standard properties is shown to follow. Indeed we argue in our final remarks that, heuristically speaking, their assumptions imply those of the present paper. Two other works [Re] and [BBMS] (motivated by [M]) independently exhibit explicit relationships of a general type between AdS QFT and Conformal QFT in the Minkowskian boundary of ADS: while [Re] also pertains to the framework of local observable algebras, [BBMS] relies on a limited use of analytic n-point functions of local fields in a Wightman-type approach (see [SW]). It is the complete setting of such a Wightman-type framework for AdS QFT and the derivation of a number of general results for interacting fields belonging to that framework which are the purpose of the present paper.

It may be useful to start by recalling how AdS QFT is rendered difficult by the lack of the global hyperbolicity property of the underlying manifold. This manifests itself in two ways: there exist closed timelike curves on the AdS manifold (in particular geodesics) and there is a "boundary" at spacelike infinity. The first problem is commonly avoided by considering the covering of the manifold; the second fact implies that the standard procedure of canonical quantization for free fields (see e.g. [BD, W]) cannot be used, since there does not exist a global Cauchy surface and information can flow in from spacelike infinity. To construct a viable QFT under these circumstances one may need to specify suitable boundary conditions at infinity. To this end, the nice idea in [AIS] was to use the conformal embedding of the AdS manifold in the Einstein Static Universe, which is a globally hyperbolic space-time. It has been then possible (with some restrictions) to produce a class of boundary conditions that render the resulting AdS QFT well defined. Unfortunately the procedure is very special and tricky, and can work at best for free field theories.

Generally speaking, a well known problem when studying QFT on gravitational backgrounds is the absence of a criterion to select the physically meaningful states. There are indeed infinitely many inequivalent representations of the same field algebra and it is in general impossible to characterize the physically relevant vacuum states as, for instance, fundamental states for the energy operator, since the very concept of energy as a global quantity is in general not defined. However, an important aspect of the energy concept which keeps its full value is the notion of energy "relative to an observer whose world-line is an orbit of a one-parameter group of isometries of the spacetime": for such observers, the usual quantum notion of energy, represented by the generator of time-translations, is in fact applicable with respect to the proper-time parameter. Therefore the properties of energy-positivity and temperature (i.e. the notions of ground state and KMS-state) are meaningful *relatively to such observers* and technically characterized by relevant analyticity properties of the correlation functions of the fields in the corresponding complexified orbits. Of course, this approach of the energy concept remains particularly simple because we have to deal with curved spacetimes of holomorphic type, such as the de Sitter and AdS quadrics.

In a previous work dealing with QFT on de Sitter spacetime ([BEM]), we had shown that the absence of global energy operators could *in that case* be successfully replaced by appropriate global analyticity properties of the n-point functions of the fields in "natural" tuboidal domains: the latter played the same role as the tube domains resulting from energy-positivity in the case of Minkowskian fields. Moreover a thermal interpretation of these QFT for all geodesic (as well as uniformly accelerated) observers could then be proved as a byproduct of these global analyticity properties, thus providing an extension of the Bisognano-Wichmann analyticity property obtained in the Minkowskian case [BW], or in physical terms of the "Unruh effect".

Surprisingly, in the AdS case, the situation concerning energy operators turns out to be more fortunate than in the de Sitter case, and in fact quite favourable ! This is because there exist *two classes* of time-like orbits of *"generic" one-parameter isometry groups*, namely the class of elliptic orbits and the class of hyperbolic orbits, also supplemented by a "boundary-type" class of parabolic orbits. The hyperbolic orbits of AdS represent uniformly accelerated motions similar to *all* the geodesic or uniformly accelerated motions of de Sitter spacetime: the corresponding complexified orbits, which are complex hyperbolae, are thus also "plagued" by a natural periodicity in the *imaginary part* of their proper-time parameter, which forbids the positivity of the corresponding energy operator (but can support at best a KMS-condition whose temperature is related to the radius of the hyperbola and thereby to the acceleration). On the contrary, the (complexified) elliptic orbits, whose class contains geodesic as well as uniformly accelerated motions, only present the peculiarity of periodicity in the *real part* of their proper-time parameter: this pathology already mentioned above (namely the "time loops") is cured by considering QFT on the *universal covering* of AdS; in fact in all the following, this covering space will appear as much more natural than AdS itself for the setting of interacting fields. At any rate, since there is no geometrical periodicity in the imaginary part of the proper-time of the elliptic orbits and since the corresponding one-parameter groups have no orbits of other type, their generators can be considered as genuine global energy generators: nothing forbids one to postulate the positivity of the corresponding energy operators, expressed technically by relevant analyticity properties in half-planes of the proper-time parameter. So, as in the Wightman axiomatics of Minkowskian QFT, it is still natural here to postulate the spectral condition for all the generators of time-like elliptic orbits, the latter playing the same role as the time-like straight-lines (or uniform motions) in the Minkowskian spacetime. We note that such a condition has been proposed in Fronsdal's group-theoretical study [F] of Klein-Gordon AdS QFT. In this connection one must also quote [DL] whose authors point out the discrepancy between the elliptic and hyperbolic AdS trajectories and give arguments for attributing to them respectively a zero-temperature and a finite temperature specified in terms of the acceleration.

Another postulate which will play a crucial role in our approach is an adaptation of the property of local commutativity (or microcausality). It still appears as natural for the covering of AdS space-time, in spite of its lack of global-hyperbolicity, while it is harder to justify on the "pure" AdS space-time itself, because of the time-loop phenomenon. However we may regard a field theory on the pure AdS space-time as just a special case of one defined on its covering. Another justification is also provided by [BFS].

After having set the relevant geometrical notions in Sections 2 and 3, we shall propose in Section 4 a plausible set of hypotheses for an interacting AdS QFT, among which the positivity of the spectrum of the above mentioned energy operators, AdS-covariance and an adaptation of microcausality. The spectral hypothesis readily implies that the n-point correlation functions admit analytic continuations in tuboidal domains in the Cartesian product of n copies of (the covering of) the complexified AdS manifold. These analyticity properties of the correlation functions parallel as closely as possible what happens in Wightman QFT [SW], where the analyticity of the correlation functions in similar tubular domains is analogously obtained from the positivity of the spectrum of the energy. These domains are described in Section 3. Unfortunately the n-point tuboids are rather complicated geometrical objects and, at present, can be given a simple description only in the case $n = 1$, namely for the two-point functions. Note however that for general n they contain "flat domains" corresponding to all points moving on orbits of *the same one-parameter isometry group*, whose description is identical with those of the corresponding sets in complex Minkowskian spacetime; in [BBMS], these flat domains have played a useful role in the construction of the "asymptotic forms" of AdS QFT, recognized as Lüscher-Mack-type theories [LM] on the asymptotic cone of AdS, in correspondence with conformal QFT in Minkowskian spacetime of one dimension less. At the end of our Section 3 (subsection 3.5), we have tried to give a summary of some analogies and discrepancies between the n-point tuboids of complex AdS and the corresponding ones of complex Minkowski space.

One of the interesting points of the AdS geometry is that there exist families of submanifolds that can be identified with Minkowski space-times in one dimension less (branes): as a matter of fact, these submanifolds contain all the two-plane sections of parabolic-type of the AdS-quadric mentioned above as the third class of timelike orbits. This very fact has raised recently a large interest [M, RS]. Our construction guarantees the possibility of considering restrictions of AdS quantum fields to these "Poincaré branes" and obtaining this way completely well-defined Minkowskian QFT's. It is perhaps worthwhile to stress that this result, described in Section 5, is not as obvious as the well-known restrictibility of Minkowskian theories to lower dimensionality space-times because of the more complicated geometry. From a geometrical viewpoint, the conformal theories obtained in [BBMS] under asymptotic scaling assumptions then also appear as limits of the previous Minkowskian QFT's when the corresponding parabolic sections tend to infinity.

In any approach of QFT, the case of two-point functions deserves a special study. This is why we give in Section 6 and in Appendix A a complete characterization of the AdS two-point functions; the latter are actually maximally analytic, exactly as in the Minkowski [SW] and de Sitter [BM, BEM] cases. As usual this permits the constructions of generalized free fields and their Wick powers which fulfill all the hypotheses. This study also provides the opportunity of displaying some strange implications of the postulate of microcausality in the "pure AdS" case. In fact, the discrepancy between the pure AdS spacetime and the covering of AdS appears in a characteristic way in the classification of the two-point functions; it is revealed by the property of uniformity (or nonuniformity) of these functions in their analyticity domain $\mathbf{C} \setminus [-1, +1]$ in the complex plane of the cosine of the AdS invariant distance. These phenomena introduce the more general problem of characterizing the interacting QFT's on the pure AdS spacetime with respect to those on the covering, which will be briefly discussed in our outlook (Section 9).

In Section 7 we derive from our postulates the property of Bisognano-Wichmann analyticity in all the hyperbolic orbits corresponding to the class of uniformly accelerated motions mentioned earlier, or in other words the "AdS-Unruh effect": an accelerated observer of the AdS world whose world-line belongs to that class will perceive a thermal bath of particles with inverse temperature equal to 2π times the radius of the corresponding hyperbolic orbit. We note that this result, first described in a special free-field theory in [DL], has been also justified in the general framework of algebraic QFT in [BFS], by taking the principle of passivity of [PW] as a starting point. We also note that, in spite of the peculiarities of the global geometry of AdS, this result is similar to the one proved in [BW] for Minkowskian QFT and in [BEM] for its extension to de Sitterian QFT.

Section 8 is devoted to the AdS version of the "Euclidean" field theory and to the corresponding Osterwalder-Schrader reconstruction on the covering of AdS.

Section 9 contains some final remarks among which a brief discussion of the relationship of [BFS] and the present work.

2 Preliminaries

We start with some notations and some well-known facts. Let E_{d+1} (resp $E_{d+1}^{(c)}$) denote $\mathbf{R}^{d+1}$ (resp. $\mathbf{C}^{d+1}$) equipped with the scalar product

$$\begin{aligned}(x,y) &= x^0y^0 + x^dy^d - x^1x^1 - \ldots x^{d-1}x^{d-1} \\ &= x^0y^0 + x^dy^d - \vec{x}\cdot\vec{y} \\ &= x^\mu \eta_{\mu\nu} x^\nu \ , \end{aligned} \tag{2.1}$$

where $\vec{x}$ denotes $(x^1,\ \ldots,\ x^{d-1})$. A vector x in E_{d+1} is called timelike, spacelike or lightlike according to whether (x,x) is positive, negative or equal to zero. We also use the notation $||x||^2 = \sum_{\mu=0}^{d} |x^\mu|^2$ for $x \in E_{d+1}$ or $x \in E_{d+1}^{(c)}$ and we introduce the corresponding orthonormal basis of vectors e_μ in E_{d+1} ($e_\mu^\nu = \delta_{\mu\,\nu}$). If A is a linear operator in E_{d+1} or $E_{d+1}^{(c)}$ we put $||A|| = \sup\{||Ax||\ : ||x|| = 1\}$.

We denote G (resp. $G^{(c)}$) the group of real (resp. complex) "AdS transformations", i.e. the set of real (resp. complex) linear transformations of E_{d+1} (resp $E_{d+1}^{(c)}$) which preserve the scalar product (2.1), G_0 and $G_0^{(c)}$ the connected components of the identity in these groups, $\widetilde{G}_0$ and $\widetilde{G}_0^{(c)}$ the correponding covering groups. An element of G_0 (resp. $G_0^{(c)}$) will be called a proper AdS transformation (resp. a proper complex AdS transformation).

2.1 "Pure AdS", complexified and "Euclidean" AdS, coverings

The real (resp. complex) *"pure" anti-de-Sitter space-time* X_d (resp. $X_d^{(c)}$) of radius R is defined as the submanifold of E_{d+1} (resp $E_{d+1}^{(c)}$) consisting of the points x such that $(x,x) = (x^0)^2 + (x^d)^2 - \vec{x}^2 = R^2$. Except for the thermal considerations in Section 7, we shall always take for simplicity $R = 1$ throughout this paper. The group G_0 (resp. $G_0^{(c)}$) acts transitively on X_d (resp. $X_d^{(c)}$).

By changing z^μ to iz^μ for $0 < \mu < d$, the complex quadric $X_d^{(c)}$ becomes the complex unit sphere in $\mathbf{C}^{d+1}$, which has the same homotopy type as the real unit sphere S^d. In particular $\pi_1(X_1^{(c)}) = \mathbf{Z}$, $\pi_1(X_d^{(c)}) = 0$ (i.e. $X_d^{(c)}$ is simply connected) for $d \geq 2$. It follows that for $d \geq 2$ the *covering space* of $X_d^{(c)}$ is $X_d^{(c)}$ itself . However, as seen below, X_d admits a nontrivial *covering space* $\widetilde{X}_d$ whose "physical" role is that it suppresses the time-loops of pure AdS; its construction will also imply the existence of nontrivial coverings of important domains of $X_d^{(c)}$ (although the full space $X_d^{(c)}$ itself has a trivial covering).

It is possible to introduce in $X_d^{(c)}$ an analog of the so-called Euclidean spacetime in complex Minkowski space (where space is real and time purely imaginary): we choose it to be the connected real submanifold $X_d^{(\mathcal{E})}$ of $X_d^{(c)}$ defined by putting $z^0 = iy^0,\ x^1, \ldots, x^d$ real, $x^d > 0$. This sheet ($x^d > 0$) of the two-sheeted hyperboloid with equation $(x^d)^2 - (y^0)^2 - \vec{x}^2 = 1$, equipped with the Riemannian metric induced by the ambient quadratic form (2.1), will be called *"Euclidean" AdS spacetime*. This choice singles out the "base point" $e^d = (0, \ldots, 0, 1)$ as the analog of the origin in Minkowski spacetime.

A concrete way of representing the "Euclidean" spacetime $X_d^{(\mathcal{E})}$ together with X_d *and its covering* $\widetilde{X}_d$ is to introduce the diffeomorphism χ of $S^1 \times \mathbf{R}^{d-1}$ onto X_d given by

$$(t,\ \vec{x}) \mapsto (\sqrt{1+\vec{x}^2}\ \sin t,\ \vec{x},\ \sqrt{1+\vec{x}^2}\ \cos t) \tag{2.2}$$

(where S^1 is identified to $\mathbf{R}/2\pi\mathbf{Z}$).

$X_d^{(\mathcal{E})}$ is obtained by changing t into is in this representation, namely in the extension $\chi^{(c)}$ of χ to $(S^1)^{(c)} \times \mathbf{R}^{d-1}$. This yields the following parametrization of $X_d^{(\mathcal{E})}$:

$$(s,\ \vec{x}) \mapsto (i\sqrt{1+\vec{x}^2}\,\mathrm{sh}\ s,\ \vec{x},\ \sqrt{1+\vec{x}^2}\,\mathrm{ch}\ s) \tag{2.3}$$

The diffeomorphism $\widetilde{\chi}$, defined by lifting χ on the covering $\mathbf{R}^d$ of $S^1 \times \mathbf{R}^{d-1}$ provides a global coordinate system on $\widetilde{X}_d$. There also exists an extension $\widetilde{\chi}^{(c)}$ of $\widetilde{\chi}$ to $\mathbf{C} \times \mathbf{R}^{d-1}$, whose image is a partial complexification of the covering $\widetilde{X}_d$ of AdS; this complexified covering contains the same "Euclidean" spacetime $X_d^{(\mathcal{E})}$ as $X_d^{(c)}$. It is clear that since G_0 acts transitively on X_d the diffeomorphism χ can be transported by any transformation of the group G_0; it follows that $\widetilde{G}_0$ also acts transitively on $\widetilde{X}_d$.

The Schwartz space $\mathcal{S}(X_d^n)$ of test-functions on X_d^n will be defined as the space of functions on X_d^n which admit extensions in $\mathcal{S}(E_{d+1}^n)$. A $\mathcal{C}^\infty$ function f on $\widetilde{X}_d^n$ belongs to $\mathcal{S}(\widetilde{X}_d^n)$ if every derivative of f with respect to the ambient coordinates decreases faster than any power of the geodesic distance in the Riemannian geometry induced by $||x||$. Equivalently $f \circ \widetilde{\chi} \in \mathcal{S}(\mathbf{R}^d)$.

2.2 The Lorentzian structure of AdS

The restriction to X_d of the pseudo-Riemannian metric $\eta_{\mu\nu}dx^\mu dx^\nu$ is locally Lorentzian with signature $(+,\ -,\ \ldots,\ -)$. An elementary description follows from the fact that G_0 acts transitively on X_d: it is sufficient to look at the situation in the tangent hyperplane to the base point $x = e_d$ (i.e. $\{x;\ x_d = 1\}$) whose intersection with X_d is the light-cone $\{e_d + y\ :\ (y^0)^2 - \vec{y}^2 = 0,\ y^d = 0\}$. At the base point e_d, the future (resp. past) cone is then defined by $(y^0)^2 - \vec{y}^2 > 0,\ y^0 > 0$ (resp. $y^0 < 0$). At any point $x \in X_d$, the tangent hyperplane to X_d is $\{x + y\ :\ (x, y) = 0\}$. Its intersection with X_d is the light-cone with apex at x, $\{x + y \in X_d\ :\ (y, y) = 0\}$. We say that a tangent vector y at any point x is time-like, light-like or space-like according to whether $(y, y) > 0$, $(y, y) = 0$ or $(y, y) < 0$, which is consistent with the situation at the base point. Defining the local future (resp. past) cone at each point x can also be done by using the transitive action of G_0 (or simply by continuity) starting from e_d, but a more explicit characterization can be given by using the Lie algebra of G (see below). It can be easily seen that the circular sections of X_d by the planes parallel to (e_0, e_d), parametrized by t in the representation (2.2) have time-like tangents whose future is in the direction of increasing t, for all points $x = \chi(t, \vec{x})$; this clearly exhibits the phenomenon of closed time-loops announced earlier and its suppression by going to the covering of AdS.

From a global viewpoint, two events $x_1,\ x_2$ of X_d are space-like separated if $(x_1 - x_2)^2 < 0$, i.e. $(x_1, x_2) > 1$. The acausal set of the base point e_d, i.e. the set of points x which are space-like with respect to e_d, is then given by

$$\Gamma_a(e_d) = \{x \in X_d; x^d > 1\}. \tag{2.4}$$

Let us introduce the causal set of e_d as the complement in X_d of the closure of $\Gamma_a(e_d)$:

$$\Gamma_c(e_d) = \{x \in X_d\ ;\ x^d \leq 1\}. \tag{2.5}$$

$\Gamma_c(e_d)$ can be decomposed into three sets $\Gamma_c(e_d) = \Gamma_+(e_d) \cup \Gamma_-(e_d) \cup \Gamma_{ex}(e_d)$:

$$\begin{aligned} \Gamma_+(e_d) &= \{x \in X_d \ ; \ -1 < x^d < 1,\, x^0 > 0\}, \\ \Gamma_-(e_d) &= \{x \in X_d \ ; \ -1 < x^d < 1,\, x^0 < 0\}, \\ \Gamma_{ex}(e_d) &= \{x \in X_d \ ; \ x^d \le -1, \}. \end{aligned} \tag{2.6}$$

The regions $\Gamma_+(e_d)$ and $\Gamma_-(e_d)$ can be conventionally called future and past of e_d. Similar regions $\Gamma_a(x)$, $\Gamma_c(x)$, $\Gamma_\pm(x)$, $\Gamma_{ex}(x)$ can be associated with any point x of X_d (again by the transitive action of G_0).

The geodesics of X_d are conic sections by 2-planes containing the origin of the ambient space; if x_1 and x_2 are two points on a connected branch of geodesic γ the distance $d(x_1, x_2)$ is defined by $d(x_1, x_2) = \theta(x_1, x_2)$, where θ is the angle under which the arc (x_1, x_2) of γ is seen from the origin; otherwise stated, it is the parameter of the isometry which transforms x_1 into x_2 in the minimal subgroup of G admitting γ as an orbit. θ is an ordinary angle if γ is an ellipse, and a hyperbolic angle if γ is a branch of hyperbola. The former case is interpreted as "normal" time-like separation if $|\theta| \le \pi$ (future and past being distinguished by the sign of θ) and one has $(x_1, x_2) = \cos d(x_1, x_2)$; $d(x_1, x_2)$ is interpreted as the interval of proper time elapsed between x_1 and x_2 for the geodesic observer sitting on γ. This case is typically realized by the geodesic with equations $(x^0)^2 + (x^d)^2 = 1, \vec{x} = 0$ (i.e. the curve $x = \chi(t, \vec{0})$ in (2.2)). The second case corresponds to space-like separation ($(x_1 - x_2)^2 < 0$) and one has $(x_1, x_2) = \operatorname{ch} d(x_1, x_2) > 1$. It is typically realized by taking the section of X_d by the plane (e_0, e_1) (or (e_d, e_1)).

The AdS space-time is not geodesically convex: indeed for any point x, all the temporal geodesics which contain x also contain the antipodal point $-x$, and the proper time interval beetwen x and $-x$ is π on each of these geodesics. Furthermore, the temporal geodesics emerging from an event x do not cover the full causal region $\Gamma_c(x)$ but only $\Gamma_+(x) \cup \Gamma_-(x)$. In order to go from x to a point y in the region $\Gamma_{ex}(x)$, one needs to follow at least two arcs of temporal geodesics, which implies a "boost" (i.e. some "interaction") at the junction of these two arcs. Similar remarks apply as regards causality and geodesics on the covering $\widetilde{X}_d$.

2.3 Three types of planar trajectories in AdS and its covering

Two-planes of E_{d+1} containing the origin can always be spanned by a pair of linearly independent orthogonal vectors (a, b) and classified by considering the possibility for each vector a and b to be timelike, spacelike or lightlike, which gives six different cases. To each pair (a, b) there corresponds an element M of the Lie algebra of G, an isometry group e^{tM} which is a one-parameter subgroup of G_0 and a family of parallel two-planes whose sections by AdS are conics (possibly degenerated into straight lines) invariant under this subgroup; it follows that each connected component of these sections is either a timelike or a spacelike curve (or a lightlike straightline in the degenerate case). Being interested by the classification of the *timelike* curves (or trajectories), we retain three possible cases which reduce to simple models in terms of the basis $\{e_\mu\}$, by using again the fact that G_0 acts transitively on X_d.

i) *The elliptic trajectories*: this is the case when $a^2 > 0$ and $b^2 > 0$. Since this entails that (for all α, β) $(\alpha a + \beta b)^2 > 0$, all the corresponding parallel sections of AdS (in a family specified by (a, b)) are timelike. Their models are the circular sections by all two-planes parallel to (e_0, e_d), described above in (2.2): the isometry subgroup associated with them is the rotation group with parameter t. There is one and only one geodesic in each such family (in the unique plane of the family containing the origin).

ii) *The hyperbolic trajectories*: this is the case when $a^2 > 0$ and $b^2 < 0$. In each family of parallel sections of AdS specified by (a, b), there are two subfamilies. One of them is composed of spacelike branches of hyperbolae and contains the unique geodesic of the family. The other one is composed of timelike branches of hyperbolae, interpreted as uniformly accelerated motions: there is no timelike geodesic of that type. The isometry subgroup associated with such a family is a group of pure Lorentz transformation. A model for this family is given by the sections parallel to the $(e_0, e_1)-$plane. Since it will be used repeatedly (in particular in Section 7) we introduce the following notations. For each

$\lambda \in \mathbf{C} \setminus \{0\}$, we denote $[\lambda]$ the special Lorentz transformation such that

$$([\lambda]x)^0 = \frac{\lambda + \lambda^{-1}}{2}x^0 + \frac{\lambda - \lambda^{-1}}{2}x^1, \quad ([\lambda]x)^1 = \frac{\lambda - \lambda^{-1}}{2}x^0 + \frac{\lambda + \lambda^{-1}}{2}x^1 \ , \tag{2.7}$$

the other components of x remaining unchanged. In other words

$$([e^s]x)^0 = x^0 \mathrm{ch}\, s + x^1 \mathrm{sh}\, s, \quad ([e^s]x)^1 = x^0 \mathrm{sh}\, s + x^1 \mathrm{ch}\, s \ . \tag{2.8}$$

The corresponding subfamily of timelike orbits on AdS is characterized by the following condition to be satisfied by the other components: $\rho(x)^2 = (x^d)^2 - (x^2)^2 - \cdots - (x^{d-1})^2 - 1 > 0$. The corresponding orbits are then branches of "hyperbolae with radius $|\rho|$", namely with equations

$$x^0 = \rho \mathrm{sh} t, \ x^1 = \rho \mathrm{ch} t. \tag{2.9}$$

iii) *The parabolic trajectories*: this is the case when $a^2 > 0$ and $b^2 = 0$: since this entails that (for all α, β) $(\alpha a + \beta b)^2 \geq 0$, all the corresponding parallel sections of AdS (in a family specified by (a, b)) are timelike or exceptionally lightlike. Their models are the parabolic sections by all two-planes parallel to $(e_0, \ e_{d-1} - e_d)$, admitting the following representation in terms of a translation group parameter t:

$$x^0 = \sigma e^v t, \quad x^{d-1} = \sigma(\mathrm{sh} v + \frac{1}{2}e^v t^2), \quad x^d = \sigma(\mathrm{ch} v - \frac{1}{2}e^v t^2), \tag{2.10}$$

where $\sigma^2 = 1 + (x^1)^2 + \cdots + (x^{d-2})^2$. The complete description of the corresponding subgroup of G_0 admitting these parabolic orbits will be given and used in Sec 5 (see Eq. (5.7)). These timelike sections admit as a limiting case (for v tending to $-\infty$) the lightlike section by the plane with equations $x^{d-1} + x^d = 0, x^1 = \cdots = x^{d-2} = 0$. This lightlike section gives the only geodesics in the family.

We note that only the elliptic trajectories have nontrivial liftings into the covering of X_d. The hyperbolic and parabolic trajectories belong to "a single sheet of $\widetilde{X}_d$". This is connected with the following difference between these families of trajectories: while each family of elliptic trajectories (resp. their liftings) covers the *whole space* X_d (resp. $\widetilde{X}_d$), each family of the two other types is decomposed into two subfamilies which cover disjoint domains of X_d, namely *wedge-shaped regions* of the form $X_d \cap \{x; \ \pm x^1 > |x^0|\}$ for the hyperbolic case and *halves of AdS* of the form $X_d \cap \{x; \ \pm(x^{d-1} + x^d) > 0\}$ for the parabolic case.

It is also interesting to note that by putting $t = i\tau$ in the representations of the *three families of trajectories* (2.2),(2.9),(2.10), one obtains curves inside the "Euclidean" AdS spacetime $X_d^{(\mathcal{E})}$, namely respectively, branches of hyperbolae, circles and parabolae. The orbits of the second case, which are associated with purely imaginary Lorentz transfomations and exhibit periodicity in the corresponding imaginary time parameter are to be compared with those occurring as well in the Euclidean space of complex Minkowski space as in the ("Euclidean") hypersphere of the complex de Sitter spacetime (see e.g. ([BEM])): as shown below in Sec 7, they correspond to the existence of an Unruh effect as in these other two spacetimes. On the other hand, the other two classes of orbits are rather to be compared with those of the time-translation groups of Minkowskian space (although not corresponding to geodesical motions on X_d generically), as far as they allow a topologically equivalent "Wick-rotation" procedure to be performed (see Sec 8 and 5).

Interpretation of the planar trajectories as uniformly accelerated motions:

In this paragraph we reintroduce the radius R of the AdS spacetime. If $x = x(t)$ denotes an arbitrary AdS trajectory parametrized by its *proper time* t, the corresponding velocity-vector $u(t) = \frac{d}{dt}x(t)$ at the point $x(t)$ satisfies the following relations (in terms of scalar products in the ambient space E_{d+1}): $(u(t), u(t)) = 1$, $(x(t), u(t)) = 0$. It follows that the *ambient acceleration-vector* $w(t) = \frac{d}{dt}u(t)$ satisfies the relation $(w(t), x(t)) = -1$. The latter together with the AdS equation $(x(t), x(t)) = R^2$ then imply that the *AdS acceleration-vector* $\gamma(t)$, defined as the E_{d+1}–orthogonal projection of $w(t)$ onto the tangent hyperplane to AdS at the point $x(t)$, is given by the following formula: $\gamma(t) = w(t) + \frac{1}{R^2}x(t)$. In view of the previous relations, this entails that $(\gamma(t), \gamma(t)) = (w(t), w(t)) - \frac{1}{R^2}$. *Motions with constant acceleration* are those for which $(\gamma(t), \gamma(t))$ or equivalently $(w(t), w(t))$ is independent of t.

Consider any planar trajectory of AdS; if it is either elliptic or hyperbolic, we can represent it by the equation $x(t) = \hat{x}(t) + Rc$, where $\hat{x}(t)$ varies in a two-plane Π and the vector c can always be chosen orthogonal to Π. If $c^2 < 0$, the trajectory is elliptic (this is the case described by (2.2), with $c = (0, \vec{x}, 0)$). If $c^2 > 1$, the trajectory is hyperbolic (this is the case described by (2.9), with $c = (0, 0, x^2, \ldots, x^d)$, $c^2 = (1 + \rho^2)$). Since $w(t)$ as well as $u(t)$ remain in Π, and since $(w(t), u(t)) = (\hat{x}(t), u(t)) = 0$, there holds the colinearity condition $w(t) = \lambda(t)\hat{x}(t)$ and the condition $(w(t), x(t)) = (w(t), \hat{x}(t)) = -1$ then yields $\lambda(t) = -\frac{1}{(\hat{x}(t), \hat{x}(t))} = \frac{1}{R^2(c^2-1)}$. It then follows that $(w(t), w(t)) = -\frac{1}{R^2(c^2-1)}$. is independent of t. This motion is therefore uniformly accelerated with the following value of the AdS acceleration: $\mathrm{a} = \sqrt{-(\gamma(t), \gamma(t))} = \frac{1}{R}\sqrt{\frac{c^2}{c^2-1}}$. This leads us to state

Lemma 2.1 *All the planar trajectories of the AdS spacetime correspond to uniformly accelerated motions, with the following specifications:*

i) If the trajectory is elliptic, the corresponding acceleration $\mathrm{a} = \frac{1}{R}\sqrt{\frac{|c^2|}{|c^2|+1}}$ *takes any value between* 0 *(i.e.the geodesic case obtained for* $c = 0$*) and* $\frac{1}{R}$ *(i.e. the case* $c \to \infty$ *corresponding to ellipses in far-away two-planes).*

ii) If the trajectory is hyperbolic, the corresponding acceleration $\mathrm{a} = \frac{1}{R}\sqrt{\frac{c^2}{c^2-1}}$ *takes any value between* $\frac{1}{R}$ *(i.e. the case* $c \to \infty$ *corresponding to hyperbolae in far-away two-planes) and* $+\infty$ *(i.e. the case* $c^2 \to 1$ *corresponding to degenerate hyperbolae (or "bifurcate horizons")).*

iii) If the trajectory is parabolic, the corresponding acceleration is $\mathrm{a} = \frac{1}{R}$

To complete the proof, we just have to treat the case of parabolic trajectories, whose prototype is described by Eqs (2.10). Introducing the proper time parameter and the radius R of AdS, the latter can be rewritten:

$x^0 = t,\ x^{d-1} = \sigma \mathrm{sh} v + \frac{t^2}{2\sigma e^v},\quad x^d = \sigma \mathrm{ch} v - \frac{t^2}{2\sigma e^v}$, with $\sigma^2 = R^2 + (x^1)^2 + \cdots + (x^{d-2})^2$.

The ambient acceleration-vector is $w(t) = (0, \ldots, \frac{1}{\sigma e^v}, -\frac{1}{\sigma e^v})$, which is such that $(w(t), w(t)) = 0$ and therefore the AdS acceleration-vector $\gamma(t) = w(t) + \frac{1}{R^2}x(t)$ is such that $(\gamma(t), \gamma(t)) = -\frac{1}{R^2}$.

2.4 The Lie algebra of G, time-like and isotropic two-planes

The Lie algebra $\mathcal{G}$ of G can be identified with the real vector space of the linear operators A on E_{d+1} such that $A^\mu{}_\nu = A^{\mu\rho}\eta_{\rho\nu}$ with $A^{\mu\rho} = -A^{\rho\mu}$. Hence there is a canonical linear bijection ℓ of the space of antisymmetric 2-tensors over E_{d+1} onto $\mathcal{G}$. A basis of $\mathcal{G}$ is provided by $\{M_{\mu\nu}\ :\ 0 \leq \mu < \nu \leq d\}$, with $(M_{\mu\nu})^\rho{}_\sigma = e^\rho_\mu e_{\nu\sigma} - e^\rho_\nu e_{\mu\sigma}$; using the standard notation $a \wedge b = a \otimes b - b \otimes a$, we write: $M_{\mu\nu} = \ell(e_\mu \wedge e_\nu)$.

In particular

$$M_{0d} = \begin{pmatrix} 0 & \ldots & 1 \\ \vdots & 0 & \vdots \\ -1 & \ldots & 0 \end{pmatrix}, \qquad e^{tM_{0d}} = \begin{pmatrix} \cos t & \ldots & \sin t \\ \vdots & 1 & \vdots \\ -\sin t & \ldots & \cos t \end{pmatrix} \tag{2.11}$$

and the diffeomorphism χ (see Eq. (2.2)) can be rewritten as follows

$$(t, \vec{x}) \mapsto \exp(tM_{0d})(0,\ \vec{x},\ \sqrt{1 + \vec{x}^2}), \tag{2.12}$$

with $t \in S^1$. The same formula can also be used for describing the lifting $\widetilde{\chi}$ of χ on the covering spaces, provided t is allowed to vary in $\mathbf{R}$, and $A \mapsto \exp(A)$ is understood as the exponential map of $\mathcal{G}$ into $\widetilde{G}_0$, and $(0,\ \vec{x},\ \sqrt{1 + \vec{x}^2})$ is identified with a point of one of the fundamental domains of $\widetilde{X}_d$. The same remark applies to the formulae

$$\chi(t + s,\ \vec{x}) = \exp(sM_{0d})\chi(t,\ \vec{x}), \quad \widetilde{\chi}(t + s,\ \vec{x}) = \exp(sM_{0d})\widetilde{\chi}(t,\ \vec{x}). \tag{2.13}$$

For simplicity, we keep the same notation, and let the context decide on its interpretation.

It is also easy to verify that, with the notations of (2.8), $[e^s] = \exp sM_{10}$.

The scalar product (2.1) naturally induces a G-invariant scalar product in the space of the contravariant tensors of order p, namely $(A,B) = A^{\mu_1\cdots\mu_p} g_{\mu_1\nu_1}\dots g_{\mu_p\nu_p} B^{\nu_1\cdots\nu_p}$. This satisfies

$$(a_1 \otimes \ldots \otimes a_p,\ b_1 \otimes \ldots \otimes b_p) = (a_1, b_1)\ldots(a_p, b_p)\ . \tag{2.14}$$

In particular, given $a, b \in E_{d+1}$,

$$\frac{1}{2}(a\wedge b,\ a\wedge b) = (a,a)(b,b) - (a,b)^2\ , \tag{2.15}$$

$$\frac{1}{2}(e_0\wedge e_d,\ a\wedge b) = a_0 b_d - a_d b_0\ . \tag{2.16}$$

The square of this 2-dimensional determinant is a 2-dimensional Gramian :

$$\begin{aligned}(a^0b^d - a^db^0)^2 &= \left[(a^0)^2 + (a^d)^2\right]\left[(b^0)^2 + (b^d)^2\right] - (a^0b^0 + a^db^d)^2 \\ &= ((a,a) + \vec{a}^2)((b,b) + \vec{b}^2) - ((a,b) + \vec{a}\cdot\vec{b})^2 \\ &= (a,a)(b,b) - (a,b)^2 + \vec{a}^2\vec{b}^2 - (\vec{a}\cdot\vec{b})^2 \\ &\quad + (a,a)\vec{b}^2 + (b,b)\vec{a}^2 - 2(a,b)\vec{a}\cdot\vec{b}\ .\end{aligned} \tag{2.17}$$

Supposing $(a,a) > 0$, $(b,b) > 0$, and $(a,a)(b,b) - (a,b)^2 > 0$, we find $(a^0b^d - a^db^0)^2 > 0$. If a and b are continuously varied while (a,a), (b,b), (a,b) are kept constant with the above inequalities satisfied, the sign of $(e_0\wedge e_1, a\wedge b)$ remains constant. Under the same conditions, a can be brought to the form a^0e_0, $a^0 > 0$, by a transformation in G_0. Then, denoting $b' = (0,\ b^1,\ldots,b^d)$, we have $(a^0)^2(b',b') > 0$, i.e. b' is time-like in the Minkowski space $\{x\ :\ x^0 = 0\}$. It can be brought to the form $b'^d e_d$ by a Lorentz transformation acting in the same space, after which $a = a^0e_0$, $b = b^0e_0 + b^de_d$, $a^0b^d - a^db^0 = a^0b^d$. In particular if a and b are orthonormal, the necessary and sufficient condition for $a = \Lambda e_0$, $b = \Lambda e_d$, $\Lambda \in G_0$, is that the above scalar product (2.16) be positive. Suppose a and b are as above and $(e_0\wedge e_d, a\wedge b) > 0$. Then the two dimensional real vector subspace spanned by a and b, and any parallel 2-plane, equipped with the metric induced by (2.1), is a Euclidean space (with positive metric). Conversely given a 2-plane with strictly positive induced metric, the parallel two dimensional real vector subspace has an orthonormal base $(a,\ b)$ and the scalar product $(e_0\wedge e_d,\ a\wedge b)$ can be made positive by changing b to $-b$ if necessary. We call such a 2-plane time-like, and always regard it as oriented by the 2-form $d(a,x)\wedge d(b,x)$. The one-parameter subgroup of G_0 defined by $t\mapsto \exp t\ell(a\wedge b)$ leaves this 2-plane invariant, and every orbit of this subgroup is contained in a parallel 2-plane. This subgroup is conjugated in G_0 to $\exp tM_{0d}$. It follows that

$$\begin{aligned}\exp t\ell(a\wedge b)\,a &= \cos(t)\,a - \sin(t)\,b\ , \\ \exp t\ell(a\wedge b)\,b &= \sin(t)\,a + \cos(t)\,b\ , \\ \exp t\ell(a\wedge b)\,x &= x \ \text{ if }\ (a,x) = (b,x) = 0\ .\end{aligned} \tag{2.18}$$

We denote $\mathcal{C}_1$ the subset of $\mathcal{G}$ consisting of all elements of the form $\ell(a\wedge b)$ with $(a,a) = (b,b) = 1$, $(a,b) = 0$, and $(e_0\wedge e_d,\ a\wedge b) > 0$. Equivalently,

$$\mathcal{C}_1 = \{\Lambda\, M_{0d}\,\Lambda^{-1}\ :\ \Lambda\in G_0\}\ . \tag{2.19}$$

We denote $\mathcal{C}_+$ the cone generated in $\mathcal{G}$ by $\mathcal{C}_1$, i.e. $\mathcal{C}_+ = \bigcup_{\rho>0}\rho\mathcal{C}_1$.

We note that for all the elements M of $\mathcal{C}_1$ the corresponding group elements $\exp tM$ can be considered as belonging either to G_0 or to $\widetilde{G}_0$ according to whether exp is regarded as the exponential map of one or the other group. As mentioned at the beginning of this section, we always identify G_0 (resp. $G_0^{(c)}$) with a subgroup of $SL(d+1,\ \mathbf{R})$ (resp. $SL(d+1,\ \mathbf{C})$), and its exponential map with the matrix exponential.

While in the real Minkowski space the isotropic subspaces are one-dimensional, the maximal isotropic subspaces in E_{d+1} are two-dimensional when $d > 2$.

Lemma 2.2 *Let $a,\ b \in E_{d+1}$ be linearly independent and satisfy $(a,a) = (b,b) = (a,b) = 0$. Then $d+1 \geq 4$ and there is a $\Lambda \in G_0$ such that $\Lambda a = e_0 + e_1$ and $\Lambda b = \pm(\varepsilon e_{d-1} + e_d)$, where $\varepsilon = \pm 1$. Any $x \in E_{d+1}$ such that $(x,a) = (x,b) = (x,x) = 0$ is a linear combination of a and b.*

Proof. If $d = 2$, E_{d+1} has a metric opposite to a Minkowskian metric and the isotropic subspaces are one-dimensional. We assume that $d \geq 3$. The vector a must have a non-vanishing projection in the 2-plane spanned by e_0 and e_d, and can be brought by some $\exp(\theta M_{0d})$ to have $a^d = 0$, $a^0 > 0$. In the Minkowski space $\{x\ :\ x^d = 0\}$, there is a Lorentz transformation in the connected component of the identity which brings a to be equal to $e_0 + e_1$. After these transformations b satisfies $b^0 = b^1 = c$. Hence the projection of b into the Minkowski space orthogonal to e_0 and e_1 is light-like and cannot vanish, since then b would be colinear to a. In particular $b^d \neq 0$. Assuming e.g. $b^d > 0$, there exists, in the Minkowski space generated by $\{e_2, \ldots, e_d\}$, a Lorentz transformation in the connected component of the identity which brings b' to the form $\varepsilon e_{d-1} + e_d$ ($\varepsilon = \pm 1$) without affecting the subspace generated by e_0 and e_1. At this point, $a = e_0 + e_1$, $b = c(e_0 + e_1) + \varepsilon e_{d-1} + e_d$. We now apply the transformation $A(\varepsilon c) \in G_0$ given, in the space generated by $\{e_0,\ e_1,\ e_{d-1},\ e_d\}$, by

$$A(\varepsilon c) = \begin{pmatrix} 1+\frac{c^2}{2} & -\frac{c^2}{2} & -\varepsilon c & 0 \\ \frac{c^2}{2} & 1-\frac{c^2}{2} & -\varepsilon c & 0 \\ -\varepsilon c & \varepsilon c & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \exp \varepsilon c \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{2.20}$$

This brings a and b to the forms $e_0 + e_1$ and $\varepsilon e_{d-1} + e_d$, respectively. Suppose now that $x \in E_{d+1}$ satisfies $(x,x) = (x,a) = (x,b) = 0$. Then $x^0 = x^1$, $x^{d-1} = \varepsilon x^d$ and the projection of x into the subspace orthogonal to $\{e_0,\ e_1,\ e_{d-1},\ e_d\}$ has to vanish. Thus $x = x^0 a + x^d b$. $\square$

We finally give an explicit characterization of the local future cone V_x^+ at any point x of X_d. We define V_x^+ as the connected component of $\{y\ :\ (y,x) = 0,\quad (y,y) > 0\}$ which contains the timelike vector $M_{0d}\, x$. In the case when $x = e_d$, $M_{0d}\, e_d = e_0$ and any $y \in V_{e_d}^+$ can be written as $\Lambda \rho e_0$, where $\Lambda \in G_0$ belongs to the stabilizer of e_d: $\Lambda e_d = e_d$ and $\rho > 0$. Hence

$$\begin{aligned} V_{e_d}^+ &= \{\Lambda\, \rho M_{0d}\, \Lambda^{-1}\, e_d\ :\ \Lambda \in G_0, \quad \Lambda e_d = e_d, \quad \rho > 0\} , \\ &= \{\rho\, M e_d\ :\ M = \ell(a \wedge e_d), \quad (a,a) = 1, \quad (a, e_d) = 0, \\ &\qquad (e_0 \wedge e_d,\ a \wedge e_d) > 0, \quad \rho > 0\} , \end{aligned} \tag{2.21}$$

and therefore, for any $x \in X_d$

$$\begin{aligned} V_x^+ = \{\rho\, Mx\ :\ M = \ell(a \wedge x), \quad (a,a) = 1, \quad (a,x) = 0, \\ (e_0 \wedge e_d,\ a \wedge x) > 0, \quad \rho > 0\} . \end{aligned} \tag{2.22}$$

3 Tuboids

3.1 Future and past tuboids of $X_d^{(c)}$

We denote

$$\mathbf{C}_+ = \{\zeta \in \mathbf{C}\ :\ \mathrm{Im}\, \zeta > 0\} = -\mathbf{C}_- . \tag{3.1}$$

For $z = x + iy \in E_{d+1}^{(c)}$, we define

$$\epsilon(z) = \frac{1}{2}(e_0 \wedge e_d,\ y \wedge x) = y^0 x^d - x^0 y^d , \tag{3.2}$$

Definition 3.1 *The future tuboid $\mathcal{Z}_{1+}$ and the past tuboid $\mathcal{Z}_{1-}$ of $X_d^{(c)}$ are defined by*

$$\mathcal{Z}_{1+} = \{\exp(\tau M)\, c\ :\ M \in \mathcal{C}_1, \quad c \in X_d, \quad \tau \in \mathbf{C}_+\} = \mathcal{Z}_{1-}^* \tag{3.3}$$

($\mathcal{C}_1$ being defined by Eq. (2.19)). The tuboids $\widetilde{\mathcal{Z}}_{1\pm}$ are defined as the universal coverings of $\mathcal{Z}_{1\pm}$. They are given by the same formula where $\exp(\tau M)$ *is now understood as an element of $\widetilde{G}_0^{(c)}$, τ varies in $\mathbf{C}_+$ and c_1 as an arbitrary element of $\widetilde{X}_d$.*

Lemma 3.1 *The "Euclidean" AdS spacetime $X_d^{(\mathcal{E})}$ is contained in $X_d \cup \mathcal{Z}_{1+} \cup \mathcal{Z}_{1-}$*

Proof. Since $X_d^{(\mathcal{E})}$ is represented by (2.12) where t is changed into it, all the complex points of this manifold are seen to be obtained as points of $\mathcal{Z}_{1+} \cup \mathcal{Z}_{1-}$ by taking $M = M_{0d}$, $\tau = it$ and $c = (0, \vec{x}, \sqrt{1+\vec{x}^2})$ in (3.3).

Lemma 3.2 *$\mathcal{Z}_{1+}$ has the following properties:*

$$\begin{aligned} &(i) && \mathcal{Z}_{1+} = \{z = x+iy \in E_{d+1}^{(c)} \ : \ (x,x)-(y,y) = 1, \quad (x,y) = 0, \\ &&& \qquad (y,y) > 0, \quad \epsilon(z) > 0\}. && (3.4) \\ &(ii) && \mathcal{Z}_{1+} = \{\Lambda \exp(itM_{0d})\, e_d \ : \ \Lambda \in G_0, \quad t > 0\}\ . && (3.5) \\ &(iii) && \textit{If } z = x+iy \in \mathcal{Z}_{1+} \textit{ then } z^0 \neq 0, \quad z^d \neq 0, \quad \textit{and } \mathrm{Im}\,(z^0/z^d) > 0. \end{aligned}$$

Proof. Denote A_1 the rhs of (3.4), and A_2 the rhs of (3.5). It is clear that $\mathcal{Z}_{1+}$, A_1, A_2 are invariant under G_0, and that $A_2 \subset \mathcal{Z}_{1+}$. We first prove that $\mathcal{Z}_{1+} \subset A_1$. Suppose that $z = \exp(\tau M)\, c$ with $c \in X_d$, $M \in \mathcal{C}_1$, and $t = \mathrm{Im}\,\tau > 0$. Let $\Lambda_1 \in G_0$ be such that $\Lambda_1 M \Lambda_1^{-1} = M_{0d}$. Then $\Lambda_1 z = \exp((s+it)M_{0d})c'$, where $c' \in X_d$, $c'^0 = 0$, $c'^d > 0$. Thus $z = \Lambda z'$ with $\Lambda = \Lambda_1^{-1}\exp(sM_{0d})$ and $z' = x'+iy' = \exp(itM_{0d})c'$, i.e. $x'^0 = 0$, $x'^d = c'^d \mathrm{ch}\,(t)$, $y' = c'^d \mathrm{sh}\,(t) e_0$, hence $\epsilon(z') > 0$. It follows that z', and therefore also z, belong to A_1.
We now show that $A_1 \subset A_2$. Suppose that a point $z = x+iy$ belongs to A_1. There exists a $\Lambda \in G_0$ and a real $t > 0$ such that $y = \Lambda\, \mathrm{sh}\,(t)\, e_0$ and $x = \Lambda\, \mathrm{ch}\,(t)\, e_d$, i.e. $\Lambda^{-1}(x+iy) = \sin(it)\, e_0 + \cos(it)\, e_d = \exp(itM_{0d})\, e_d$. This can be rewritten as

$$\begin{aligned} z &= \exp(is\,\ell(y\wedge x))\, x/\sqrt{(x,x)}, \\ s &= \log(\sqrt{(x,x)} + \sqrt{(y,y)})/\sqrt{(x,x)(y,y)}\ . \end{aligned} \tag{3.6}$$

To prove *(iii)*, we suppose that $x+iy \in A_1$. Then $x^d = y^d = 0$ implies that both x and y belong to a "usual" Minkowski space, hence they cannot be both time-like as well as orthogonal. The same argument excludes $z^0 = 0$. Finally $\epsilon(z) = |z^d|^2 \mathrm{Im}\,(z^0/z^d)$. $\square$

The form (3.4) makes it obvious that $\mathcal{Z}_{1+}$ and therefore $\mathcal{Z}_{1-}$ are open subsets of $X_d^{(c)}$ while the definition makes it obvious that they are connected. $\mathcal{Z}_{1+}$ and $\mathcal{Z}_{1-}$ are disjoint since every point z of $\mathcal{Z}_{1-}$ satisfies $\epsilon(z) < 0$.

Lemma 3.3 *Let $a+ib \in \overline{\mathcal{Z}_{1+}}$. Then for every $M \in \mathcal{C}_1$ and $\tau \in \mathbf{C}_+$, the point* $\exp(\tau M)(a+ib)$ *is in $\mathcal{Z}_{1+}$.*

Proof. By the invariance of $\mathcal{Z}_{1+}$ under G_0, it suffices to prove the statement in the case when $M = M_{0d}$ and $\tau = it$ with real $t > 0$. We suppose that $(b,b) \geq 0$, $(a,a)-(b,b) = 1$, $(a,b) = 0$, and $\epsilon(a+ib) \geq 0$. This implies $(a^0)^2 + (a^d)^2 \geq (a,a) \geq 1$. Let $x+iy = \exp(itM_{0d})(a+ib)$. A simple calculation shows that

$$\begin{aligned} (y,y)-(b,b) &= \left[(a^0)^2+(a^d)^2+(b^0)^2+(b^d)^2\right]\mathrm{sh}^2 t \\ &+ 2(b^0a^d - b^da^0)\mathrm{sh}\,t\,\mathrm{ch}\,t > 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} y^0x^d - y^dx^0 &= \left[(a^0)^2+(a^d)^2+(b^0)^2+(b^d)^2\right]\mathrm{sh}\,t\,\mathrm{ch}\,t \\ &+ (b^0a^d - b^da^0)(\mathrm{ch}^2 t + \mathrm{sh}^2 t) > 0. \end{aligned} \tag{3.8}$$

Lemma 3.4 *(i) The image of the domain $\mathcal{Z}_{1+}$ (or $\mathcal{Z}_{1-}$) under the coordinate map $z \mapsto z^d = (e_d, z)$ is the cut-plane $\Delta = \mathbf{C} \setminus [-1, 1]$.*

(ii) The image of the domain $\mathcal{Z}_{1-} \times \mathcal{Z}_{1+}$ (or $\mathcal{Z}_{1+} \times \mathcal{Z}_{1-}$) by the (scalar product) mapping $z_1, z_2 \mapsto (z_1, z_2)$ is the cut-plane Δ.

(iii) The map $z_1, z_2 \mapsto (z_1, z_2)$ of $\mathcal{Z}_{1-} \times \mathcal{Z}_{1+}$ onto Δ can be lifted to a map of $\widetilde{\mathcal{Z}}_{1-} \times \widetilde{\mathcal{Z}}_{1+}$ onto the covering $\widetilde{\Delta}$ of the cut-plane Δ.

Proof.
(i) If $z \in \mathcal{Z}_{1+}$, then, by Lemma 3.2, it can be written as $z = \Lambda \exp(itM_{0d})e_d$, with $\Lambda \in G_0$ and $t > 0$. Hence $(e_d,\ z) = (a,\ \exp(itM_{0d})e_d)$ where $a \in X_d$ can be written as

$$a = (\sqrt{1 + \vec{a}^2}\,\sin s,\ \vec{a},\ \sqrt{1 + \vec{a}^2}\,\cos s) = \exp(sM_{0d})\,(0,\ \vec{a},\ \sqrt{1 + \vec{a}^2})\ , \tag{3.9}$$

hence $(e_d,\ z) = \sqrt{1 + \vec{a}^2}\,\cos(it - s) \in \Delta$. Conversely any $\zeta \in \Delta$ can be written as $\zeta = \cos(u + iv)$, $v > 0$, i.e. $\zeta = (e_d,\ \exp((u + iv)M_{0d})e_d)$ is in the image of $\mathcal{Z}_{1+}$.

(ii) Let $z_1 \in \mathcal{Z}_{1-}$, $z_2 \in \mathcal{Z}_{1+}$. Then $z_1 = \exp(-\tau_1 M_1)c_1$, $z_2 = \exp(\tau_2 M_2)c_2$, with $M_j \in \mathcal{C}_1$, $\tau_j \in \mathbf{C}_+$, $c_j \in X_d$, $j = 1,\ 2$. Hence

$$(z_1,\ z_2) = (c_1,\ \exp(\tau_1 M_1)\exp(\tau_2 M_2)c_2) = (e_d,\ z'), \tag{3.10}$$

where $z' = \Lambda \exp(\tau_1 M_1)\exp(\tau_2 M_2)c_2$ (for some $\Lambda \in G_0$) belongs to $\mathcal{Z}_{1+}$ by Lemma 3.3. Therefore $(z_1,\ z_2) \in \Delta$ by *(i)*. Conversely any $\zeta \in \Delta$ can be written as $\zeta = \cos(u + iv)$, $v > 0$, i.e. $\zeta = (\exp(-(u + iv)M_{0d}/2)e_d,\ \exp((u + iv)M_{0d}/2)e_d)$ is in the image of $\mathcal{Z}_{1-} \times \mathcal{Z}_{1+}$.

(iii) It is easy to see that elements of $\mathcal{Z}_{1-} \times \mathcal{Z}_{1+}$ such that $z_1 = \exp(t_1 + is_1)M_{0d}e_d$, $s_1 > 0$, and $z_2 = \exp(t_2 + is_2)M_{0d}e_d$, $s_2 < 0$, have a scalar product $(z_1, z_2) = \cos(t_1 - t_2 + i(s_1 - s_2))$ which runs on an elliptic path with foci $-1, +1$ in the infinite-sheeted domain $\widetilde{\Delta}$ when t_1 and t_2 vary in $\mathbf{R}$ at fixed s_1, s_2. These elliptic paths remain in $\widetilde{\Delta}$, which they fully recover, when s_1 and s_2 vary in their ranges. Similar elliptic paths homotopic to the latter would be generated by starting from more general elements of $\mathcal{Z}_{1-} \times \mathcal{Z}_{1+}$. See also Appendix A for an alternative argument based on a more global representation of the tuboids. □

Lemma 3.5 *The domain $\mathcal{Z}_{1+}$ is a domain of holomorphy.*

Proof. As proved in Appendix A, there exists a biholomorphic map of $\mathcal{Z}_{1+}$ onto the domain $\mathcal{T}_+ \setminus \{z\ :\ z^1 = 0\}$ in the usual d-dimensional complex Minkowski space. This also exhibits the topoplogy of $\mathcal{Z}_{1+}$. □

3.2 The minimal n-point tuboids

We define the minimal n-point future tuboid $\mathcal{Z}_{n+}$ as the subset of $X_d^{(c)n}$ consisting of all points $(z_1,\ \ldots,\ z_n)$ such that

$$z_1 = e^{\tau_1 M_1}\,c_1,\ \ z_2 = e^{\tau_1 M_1}\,e^{\tau_2 M_2}\,c_2,\quad \ldots,\quad z_n = e^{\tau_1 M_1}\,\ldots e^{\tau_n M_n}\,c_n, \tag{3.11}$$

where, for $1 \le j \le n$, $\tau_j \in \mathbf{C}_+$, $M_j \in \mathcal{C}_1$, and $c_j \in X_d$. Equivalently

$$\begin{aligned}\mathcal{Z}_{n+} \quad = \quad & \{(z_1,\ \ldots,\ z_n) \in X_d^{(c)n}\ :\ \forall j = 1, \ldots, n,\quad z_j = e^{it_1 M_1}\ldots e^{it_j M_j}\,c_j\ ,\\ & t_j > 0,\quad M_j \in \mathcal{C}_1,\quad c_j \in X_d\}\ .\end{aligned} \tag{3.12}$$

Indeed if z is of the form (3.11) with $\tau_j = s_j + it_j$ then

$$\begin{aligned} z_j &= e^{it_1 M_1'} \dots e^{it_j M_j'}\, c_j' \ , \\ M_j' &= e^{s_1 M_1} \dots e^{s_{j-1} M_{j-1}}\, M_j\, e^{-s_{j-1} M_{j-1}} \dots e^{-s_1 M_1} \ , \\ c_j' &= e^{s_1 M_1} \dots e^{s_j M_j}\, c_j \ . \end{aligned} \tag{3.13}$$

We define the minimal n-point past tuboid as $\mathcal{Z}_{n-} = \mathcal{Z}_{n+}^*$. We denote $\widetilde{\mathcal{Z}}_{n\pm}$ the universal coverings of these sets. However, we shall write simply $\mathcal{Z}_{n\pm}$ when no confusion arises.

Lemma 3.6 *The set $\mathcal{Z}_{n+}$ is open in $X_d^{(c)n}$.*

Proof. We prove, by induction on n, the following more detailed statement (St_n): *Let $z = (z_1,\ \dots,\ z_n) \in X_d^{(c)n}$ be such that, for $0 \le j \le n$,*

$$z_j = e^{it_1 M_1} \dots e^{it_j M_j}\, c_j\ , \quad t_j > 0, \quad M_j \in \mathcal{C}_1\ , \quad c_j \in X_d\ . \tag{3.14}$$

For every neighborhood W of $(t_1, \dots, t_n,\ M_1, \dots, M_n,\ c_1, \dots, c_n)$ in $(0,\ \infty)^n \times \mathcal{C}_1^n \times X_d^n$ there is a neighborhood V of z in $X_d^{(c)n}$ such that for every $z' \in V$ there exists a $(t',\ M',\ c') \in W$ such that $z_j' = e^{it_1' M_1'} \dots e^{it_j' M_j'}\, c_j'$ for every $j = 1, \dots, n$.

We start with the case $n = 1$, for which we give a detailed proof. The cases $n > 1$ will then be more sketchily treated. Since the statement St_1 is invariant under G_0, it suffices to consider the case when $z = z_1 = x + iy = \exp(itM_{0d})\, c$ with $t > 0$ and $c_0 = 0$, $c_d > 0$. Thus

$$x + iy = (ic_d \mathrm{sh}\,(t),\ \vec{c},\ c_d \mathrm{ch}\,(t)), \quad t > 0, \quad c_d > 0\ . \tag{3.15}$$

Let $R = 1 + ||z||$. If $z'' \in X_d^{(c)}$ is of the form $(iy_0'',\ \vec{x''},\ x_d'')$, with $x_d'' > 0$, then

$$\begin{aligned} z'' = x'' + iy'' = (ic_d'' \mathrm{sh}\,(t''),\ \vec{c''},\ c_d'' \mathrm{ch}\,(t'')) = \exp(it'' M_{0d}) c''\ , \\ \vec{c''} = \vec{x''}, \quad c_d'' = ((x_d'')^2 - (y_0'')^2)^{1/2}, \quad \mathrm{th}\,(t'') = y_0''/x_d''\ , \end{aligned} \tag{3.16}$$

where c'' and t'' are continuous functions of z''. Hence given $\varepsilon \in (0,\ 1)$, there is a $\delta_1 > 0$ such that if z'' is of the form $(iy_0'',\ \vec{x''},\ x_d'')$, and $||z'' - z|| < \delta_1$, then (3.16) holds with $|t'' - t| < \varepsilon$ and $||c'' - c|| < \varepsilon/4$, and $||z''|| < R(1 + \varepsilon/4)$. There is a $\delta > 0$ such that, for every $z' \in X_d^{(c)}$ such that $||z' - z|| < \delta$, there exists a $\Lambda \in G_0$, satisfying $||\Lambda - 1|| < \varepsilon/4R$, $||\Lambda^{-1} - 1|| < \varepsilon/4R$, and such that $z'' = \Lambda z'$ satisfies $||z'' - z|| < \delta_1$ and $z'' = (iy_0'',\ \vec{x''},\ x_d'')$, $x_d'' > 0$. Then (3.16) holds, and $z' = \exp(it'' M')c'$, with $M' = \Lambda^{-1} M_{0d} \Lambda$, $c' = \Lambda^{-1} c''$. Since $||M_{0d}|| = 1$, $||c|| \le ||z||$, this implies $||M' - M_{0d}|| < \varepsilon$ and $||c' - c|| < \varepsilon$.

We now assume that the statement St_m has been proved for all $m \le n - 1 \ge 1$. Let $z = (z_1,\ \dots,\ z_n)$ satisfy (3.14) and let $z' = (z_1',\ \dots,\ z_n')$ be sufficiently close to z. By St_1, $z_1' = \exp(it_1' M_1') c_1'$ with t_1', M_1', c_1' respectively close to t_1, M_1, c_1. The point

$$(z_2'', \dots, z_n'') = (\exp(-it_1' M_1') z_2', \dots, \exp(-it_1' M_1') z_n') \tag{3.17}$$

is close, in $X_d^{(c)(n-1)}$, to the point

$$(e^{it_2 M_2} c_2\ , \dots ,\ e^{it_2 M_2} \dots e^{it_n M_n} c_n)\ . \tag{3.18}$$

By St_{n-1} the point (3.17) can be rewritten as

$$(e^{it,_2 M_2'} c_2'\ , \dots ,\ e^{it_2' M_2'} \dots e^{it_n' M_n'} c_n')\ , \tag{3.19}$$

where, for $2 \le j \le n$, t_j', M_j', c_j' are respectively close to t_j, M_j, c_j. This proves St_n. $\square$

3.3 The n-point future and past tuboids

Definition 3.2 *We denote G_0^+ (resp. $\widetilde{G}_0^+$) the subset of $G_0^{(c)}$ (resp. $\widetilde{G}_0^{(c)}$) consisting of all elements of the form $\exp(\tau_1 M_1)\dots\exp(\tau_N M_N)$, where $N \in \mathbf{N}$, $\tau_j \in \mathbf{C}_+$, $M_j \in \mathcal{C}_+$ for all $j = 1, \dots, N$. We denote $G_0^- = \{\Lambda\ :\ \Lambda^{-1} \in G_0^+\}$. This is also the complex conjugate of G_0^+. We define the n-point future and past tuboids $\mathcal{T}_{n+}$ and $\mathcal{T}_{n-}$ by*

$$\begin{aligned}\mathcal{T}_{n+} &= \{(z_1,\ \dots,\ z_n) \in X_d^{(c)n}\ :\ \forall j = 1, \dots, n,\\ &\qquad z_j = \Lambda_1 \dots \Lambda_j\, c_j\ ,\quad \Lambda_j \in G_0^+\ ,\quad c_j \in X_d\ \}\ ,\\ \mathcal{T}_{n-} &= \mathcal{T}_{n+}^*\ .\end{aligned} \tag{3.20}$$

Note that if $\Lambda_1,\ \Lambda_2 \in G_0^+$, then $\Lambda_1\Lambda_2 \in G_0^+$. If $\Lambda \in G_0$ then $\Lambda G_0^+ \Lambda^{-1} = G_0^+$. Similar properties hold for $\widetilde{G}_0^+$.

Lemma 3.7
(i) G_0^+ is open in $G_0^{(c)}$.
(ii) $G_0^+ = G_0 G_0^+ = G_0^+ G_0$.

This lemma is proved in Appendix B. Similar properties hold for $\widetilde{G}_0^+$.

The tuboids $\mathcal{T}_{n\pm}$ are invariant under G_0, i.e. if $\Lambda \in G_0$ and $(z_1, \dots, z_n) \in \mathcal{T}_{n+}$, then $(\Lambda z_1, \dots, \Lambda z_n) \in \mathcal{T}_{n+}$. Obviously $\mathcal{Z}_{n\pm} \subset \mathcal{T}_{n\pm}$. As a consequence of Lemma 3.3, $\mathcal{Z}_{1+} = \mathcal{T}_{1+}$.

Lemma 3.8 *For any $n \in \mathbf{N}$, $\mathcal{T}_{n+}$ is open.*

Proof. As in the proof of Lemma 3.6, we prove, by induction on n,
(P_n): *Let $z = (z_1,\ \dots,\ z_n) \in X_d^{(c)n}$ be such that, for $0 \le j \le n$,*

$$z_j = \Lambda_1 \dots \Lambda_j\, c_j, \quad \Lambda_j \in G_0^+\ ,\quad c_j \in X_d\ . \tag{3.21}$$

Then, for any $z' \in X_d^{(c)n}$ sufficiently close to z, there exist $\Lambda'_1, \dots, \Lambda'_n \in G_0^+$ and $c'_1, \dots, c'_n \in X_d$, respectively close to $\Lambda_1, \dots, \Lambda_n$ and $c_1, \dots, c_n$, such that $z'_j = \Lambda'_1 \dots \Lambda'_j\, c'_j$.

We start with $n = 1$ and suppose, without loss of generality in view of G_0-invariance, that $z = \Lambda\, c$ with $c \in X_d$, and $\Lambda = \exp(it_1 M_1)\dots\exp(it_L M_L)$, $t_k > 0$ for all $k = 1, \dots, L$. Assume e.g. $L > 1$ and let $z' \in X_d^{(c)}$ be sufficiently close to z. Then $z'' = \exp(-it_{L-1}M_{L-1})\dots\exp(-it_1 M_1)z'$ is close to $\exp(it_L M_L)c$ and it can, by the proof of Lemma 3.6, be written as $z'' = \exp(it'_L M'_L)c'$, where $t'_L > 0$, $M'_L \in \mathcal{C}_1$ and $c' \in X_d$ are respectively close to $t_L > 0$, M_L, and c. Thus $z' = \Lambda' c'$ with $\Lambda' = \exp(it_1 M_1)\dots\exp(it_{L-1}M_{L-1})\exp(it'_L M'_L)$.

The inductive proof of P_n for all $n > 1$ follows the same line as in the proof of Lemma 3.6. □

Note that $\mathcal{T}_{n+} \subset (\mathcal{T}_{1+})^n$. Another proof of Lemma 3.8 can be based on Lemma 3.7.

Definition 3.3 *The tuboids $\widetilde{\mathcal{T}}_{n\pm}$ are defined as the universal covering spaces of $\mathcal{T}_{n\pm}$.*

Remark 3.1 The transformation $[-1] = e^{i\pi M_{10}}$ belongs to G and $G_0^{(c)}$ but not to G_0. Indeed $[-1]M_{0d}[-1] = -M_{0d}$. If $L = \ell(u \wedge v) \in \mathcal{C}_1$, and $L' = [-1]L[-1] = \ell(u' \wedge v')$ we find $(M_{0d},\ L') = -(M_{0d},\ L)$, i.e. $u'_0 v'_d - u'_d v'_0 < 0$ hence $L' \in -\mathcal{C}_1$. As a consequence if $(z_1, \dots,\ z_n) \in \mathcal{T}_{n+}$ (resp. $\mathcal{Z}_{n+}$) then $([-1]z_1, \dots,\ [-1]z_n) \in \mathcal{T}_{n+}^*$ (resp. $\mathcal{Z}_{n+}^*$).

3.4 Permuted tuboids

For $n \ge 2$ and for any permutation π of $\{1, \dots,\ n\}$, the permuted tuboid $\mathcal{T}_{n,\pi}$ is defined by

$$\mathcal{T}_{n,\pi} = \{z \in X_d^{(c)n}\ :\ (z_{\pi(1)}, \dots,\ z_{\pi(n)}) \in \mathcal{T}_{n+}\}\ . \tag{3.22}$$

An analogous definition is used for $\widetilde{\mathcal{T}}_{n,\pi}$. Two permuted tubes $\mathcal{T}_{n,\pi}$ and $\mathcal{T}_{n,\pi'}$ are called adjacent if $\pi'\pi^{-1}$ is the transposition of two consecutive indices. An interesting peculiarity of the AdS space-time is that (in contrast to the Minkowskian situation) adjacent permuted tuboids are not disjoint. The simplest example is

Lemma 3.9 *Let c_1 and c_2 be real and $|(c_1, c_2)| > 1$. Then,*
(i) for each $A \in G_0^+$, $(Ac_1,\ Ac_2) \in \mathcal{T}_{2+}$.
(ii) for each $A \in G_0^+$, $(Ac_1,\ Ac_2) \in \mathcal{T}_{2+} \cap \mathcal{T}_{2,(2,1)}$.

Here $\mathcal{T}_{2,(2,1)} = \{z\ :\ (z_2,\ z_1) \in \mathcal{T}_{2+}\}$.
Proof. It is obvious that (ii) follows from (i), since the hypotheses are invariant under the exchange of c_1 and c_2. To prove (i), we may assume that, in the coordinates 0, 1, d (all others kept equal to 0),

$$c_1 = e_d = (0,\ 0,\ 1), \quad c_2 = (0,\ \mathrm{sh}\, u,\ \varepsilon \mathrm{ch}\, u), \quad \varepsilon = \pm 1, \quad u \neq 0. \tag{3.23}$$

Let s be real and satisfy $|s| \in (0,\ \pi)$ and $\varepsilon u s > 0$. Then

$$\begin{aligned} e^{isM_{10}}\, c_2 &= (i\mathrm{sh}\, u \sin s,\ \mathrm{sh}\, u \cos s,\ \varepsilon \mathrm{ch}\, u) & (3.24)\\ &= (ib^d \mathrm{sh}\, t,\ b^1,\ b^d \mathrm{ch}\, t) \ = \ e^{itM_{0d}}\, b & (3.25)\end{aligned}$$

$$\mathrm{th}\, t = \varepsilon \sin s\, \mathrm{th}\, u \in (0,\ 1), \quad b^0 = 0, \quad b^1 = \cos s\, \mathrm{sh}\, u, \quad b^d = \varepsilon \mathrm{ch}\, u / \mathrm{ch}\, t\ . \tag{3.26}$$

Hence $e^{isM_{10}}\, c_2 \in \mathcal{T}_{1+}$. Let $A \in G_0^+$ and $z_1 = Ac_1$, $z_2 = Ac_2$. For sufficiently small $|s| > 0$, since G_0^+ is open, we have $Ae^{-isM_{10}} \in G_0^+$, and

$$z_1 = (Ae^{-isM_{10}})\, c_1, \quad z_2 = (Ae^{-isM_{10}})\, (e^{isM_{10}}\, c_2) = (Ae^{-isM_{10}})\, e^{itM_{0d}}\, b, \tag{3.27}$$

so that $(z_1,\ z_2) \in \mathcal{T}_{2+}$. Similarly $(z_2,\ z_1) \in \mathcal{T}_{2+}$. $\square$

Remark 3.2 Denote

$$\mathcal{R}_2 = \{(c_1,\ c_2) \in X_d^2\ :\ (c_1, c_2) > 1\}, \qquad \mathcal{R}_2' = \{(c_1,\ c_2) \in X_d^2\ :\ (c_1, c_2) < -1\}\ . \tag{3.28}$$

then the sets $G_0^+\mathcal{R}_2 = \{(Ac_1,\ Ac_2)\ :\ (c_1,\ c_2) \in \mathcal{R}_2,\ \ A \in G_0^+\}$ and $G_0^+\mathcal{R}_2'$ are disjoint. If $d > 2$, each of them is connected.

Lemma 3.9 can be generalized to n-point tubes as follows:

Lemma 3.10 *Let $(c_1, \ldots,\ c_j,\ c_{j+1}, \ldots,\ c_n) \in X_d^n$ be such that $|(c_j,\ c_{j+1})| > 1$. Then for any choice of $\Lambda_k \in G_0^+$, $1 \le k < j$ or $j+1 < k \le n$, and of $A \in G_0^+$, the point z such that*

$$\begin{aligned} z_k &= \Lambda_1 \ldots \Lambda_k\, c_k \quad \text{for } k < j,\\ z_j &= \Lambda_1 \ldots \Lambda_{j-1} A\, c_j, \quad z_{j+1} \ = \ \Lambda_1 \ldots \Lambda_{j-1} A\, c_{j+1},\\ z_k &= \Lambda_1 \ldots \Lambda_{j-1} A \Lambda_{j+2} \ldots \Lambda_k\, c_k \quad \text{for } k > j+1, \end{aligned} \tag{3.29}$$

belongs to $\mathcal{T}_{n+}$.

Therefore points of the form (3.29) belong to the intersection of $\mathcal{T}_{n+}$ with the permuted tuboid $\mathcal{T}_{n,(j+1,j)}$ obtained by exchanging the indices j and $j+1$. Let $\mathcal{R}_{j,k} = \{x \in X_d^n\ :\ (x_j, x_k) > 1\}$. This is an open subset of X_d^n which has two connected components if $d = 2$, only one otherwise. As a result of Lemma 3.10, the intersection $\mathcal{T}_{n+} \cap \mathcal{T}_{n,(j+1,j)}$ is an open tuboid which has connected components bordered by those of $\mathcal{R}_{j,j+1}$.

Proof of Lemma 3.10. We may again assume that

$$c_j = e_d = (0,\ 0,\ 1), \quad c_{j+1} = (0,\ \mathrm{sh}\, u,\ \varepsilon \mathrm{ch}\, u), \quad \varepsilon = \pm 1, \quad u \neq 0. \tag{3.30}$$

For sufficiently small $|s| > 0$, and $\varepsilon us > 0$, let t and $b = (0,\ b^1,\ b^d)$ be given by (3.26), and let $\Lambda_j = Ae^{-isM_{01}} \in G_0^+$, $\Lambda_{j+1} = e^{itM_{0d}} \in G_0^+$. Then

$$z_j = \Lambda_1 \dots \Lambda_{j-1}\Lambda_j c_j, \quad z_{j+1} = \Lambda_1 \dots \Lambda_{j-1}\Lambda_j\Lambda_{j+1} b. \tag{3.31}$$

Furthermore

$$A\Lambda_{j+2} = \Lambda_j\Lambda_{j+1}\Lambda'_{j+2}, \quad \Lambda'_{j+2} = e^{-itM_{0d}}e^{isM_{01}}\Lambda_{j+2} \in G_0^+ \ . \ \Box \tag{3.32}$$

On the other hand "opposite" tuboids such as $\mathcal{T}_{n+}$ and $\mathcal{T}_{n-}$ do not intersect since they are respectively contained in $\mathcal{T}_{1+}^n$ and $\mathcal{T}_{1-}^n$.

3.5 Comparing the n-point tuboids of $X_d^{(c)}$ and their coverings with the tubes of complex Minkowski space

The family of one-parameter subgroups $e^{\tau M}$ of $\widetilde{G}_0$ whose generator M belongs to the set $\mathcal{C}_1$ of $\mathcal{G}$ can be considered as the analog of the family of timelike translation groups $e^{\tau a}$ acting on Minkowskian spacetime, whose generator a belongs to the unit hyperboloid shell $H_1 = \{a \in V^+; a^2 = 1\}$, but of course there is a major difference: while in the latter case, this family of one-parameter groups form a commutative subgroup T_+ of the group of spacetime translations, the groups of the former family are all mutually noncommutative. Nevertheless, the analogy between the two families has provided us with the basic idea for defining the n-point tuboids in the AdS case. In fact, if in the definitions of $\mathcal{Z}_{n+}$ (resp. $\mathcal{T}_{n+}$) one replaces the Lie elements $M_j \in \mathcal{C}_1$ by $a_j \in H_1$ (resp. $\Lambda_j \in G_0^+$ by $g_j \in T_+$) and the points c_j on AdS by points in Minkowskian spacetime, one exactly reobtains the usual n-point tubes of complex Minkowski space as they are defined in [SW]. The most obvious (and unpleasant) effect of noncommutativity in the AdS case is that for $n \geq 2$ the description of the tuboids remains very implicit, since the defining conditions involve group elements in a heavy way instead of being characterized directly by equations on the AdS manifold; in particular it is not clear whether the "minimal tuboids" $\mathcal{Z}_{n+}$ are really smaller than the ("complete") n-point tuboids $\mathcal{T}_{n+}$, which will be used in a natural way for expressing the spectral condition of AdS quantum fields (see below in Sec. 4). We wish however to emphasize a simple result which displays a reassuring analogy between the AdS tuboids and the Minkowskian tubes, namely the inclusion of regions of the corresponding "Euclidean" spacetimes in these domains. In fact, one has the following property, in which $\widetilde{\chi}^{(c)}$ denotes the extension to $\mathbf{C} \times \mathbf{R}^{d-1}$ of the diffeomorphism $\widetilde{\chi}$ (see subsection 2.1 after formulae (2.2) and (2.3)).

Lemma 3.11 *For each n, the tuboid $\widetilde{\mathcal{Z}}_{n+}$ contains the image by $\widetilde{\chi}^{(c)n}$ of the following "flat tube": $\{(\tau_1, \vec{x}_1) \dots, (\tau_n, \vec{x}_n) \in \mathbf{C}^n \times \mathbf{R}^{(d-1)n};\ 0 < \mathrm{Im}\,\tau_1 < \dots < \mathrm{Im}\,\tau_n\}$. In particular $\widetilde{\mathcal{Z}}_{n+}$ contains the following open subset of $X_d^{(\mathcal{E})n}$:*

$$\begin{aligned}
&\{(z_1, \dots, z_n) \in X_d^{(c)n} \ : \\
&\quad z_j = (i\sqrt{1+\vec{x}_j^2}\ \mathrm{sh}\ s_j,\ \vec{x}_j,\ \sqrt{1+\vec{x}_j^2}\ \mathrm{ch}\ s_j), \\
&\quad 1 \leq j \leq n;\ 0 < s_1 < \dots < s_n\}
\end{aligned} \tag{3.33}$$

The proof is readily obtained by putting $M_1 = \dots = M_n = M_{0d}$ and $c_j = (0, \vec{x}_j, \sqrt{1+\vec{x}_j^2}),\ (1 \leq j \leq n)$ in (3.11) and changing the sequence $(\tau_1, \tau_1 + \tau_2, \dots, \tau_1 + \dots + \tau_n),\ \tau_j \in \mathbf{C}_+$ into $(\tau_1, \tau_2, \dots, \tau_n)$. The subset (3.33) of $\widetilde{\mathcal{Z}}_{n+}$ which is then exhibited is clearly (in view of (2.3)) a subset of $X_d^{(\mathcal{E})n}$.

This result will entail the validity of a "Wick-rotation procedure" for the *universal covering* of AdS spacetime (see Sec. 8). In fact, we note that the flat tubes of lemma 3.11 represent domains in the complexification (in the time variable) of Minkowski space as well as $\widetilde{X}_d$: these domains are isomorphic. In the pure AdS spacetime, this isomorphism does not exist since one has to consider quotients of the previous flat tubes corresponding to the geometric periodicity conditions under the transformations $\tau_j \to \tau_j + 2\pi$ as described in (3.11).

Another close analogy between the tuboids $\mathcal{Z}_{n\pm}$ and the corresponding Minkowskian n-point tubes will be displayed in subsection 7.2: it will be proved that there exists a special subset of real points in X_d^n enjoying the same property as the Jost points of Minkowski space, namely all the complex points obtained by the action of appropriate *one-parameter complex subgroups* of $G_0^{(c)}$ on these "Jost points" are contained in $\mathcal{Z}_{n+} \cup \mathcal{Z}_{n-}$ and this is the starting point of analytic completions (of the type of Glaser-Streater's theorem) which are crucial for proving the Bisognano-Wichmann property (see Section 7).

We shall now mention two interesting discrepancies between the tuboids $\mathcal{Z}_{n+}$ and the corresponding n-point tubes of complex Minkowski space. The first one, which is developed in Section 5, concerns the parabolic trajectories already introduced in Section 2; the corresponding complexified curves turn out to exhibit sections of the tuboids $\mathcal{Z}_{n+}$ which lie in the complexified "Poincaré sections of AdS". These peculiarities are at the origin of the fact that QFT's in these Poincaré sections can be generated as restrictions of QFT's on AdS. No analogs of such families of complexified trajectories in isotropic (or "lightlike") hyperplanes exist in complex Minkowski space.

Finally, we have exhibited above in subsection 3.4 a property of pairs of permuted tuboids in X_d which is definitely new with respect to the corresponding pairs in complex Minkowski space. In the latter case, such tubes always have an empty intersection and the property of a common analytic continuation for pairs of functions analytic in "adjacent permuted tubes" necessitates the application of the edge-of-the-wedge theorem to the boundary values from these two domains through an appropriate "coincidence region". In the present case, the situation is somewhat simpler, since according to lemma 3.9 it is a general fact that *adjacent pairs of permuted tuboids in $X_d^{(c)}$ have nonempty intersections.*

4 QFT on X_d and $\widetilde{X}_d$

As usual, it is possible to formulate the main assumptions in terms of distributions (the test-functions having then compact supports) or tempered distributions. We denote $\mathcal{B}_n$ the space of test-functions on X_d^n or $\widetilde{X}_d^n$. This may be either $\mathcal{D}(X_d^n)$ (resp. $\mathcal{D}(\widetilde{X}_d^n)$) or $\mathcal{S}(X_d^n)$ (resp. $\mathcal{S}(\widetilde{X}_d^n)$). The Borchers algebra $\mathcal{B}$ on X_d (resp. $\widetilde{X}_d$) is the complex vector space of terminating sequences of test-functions $f = (f_0, f_1(x_1), \dots, f_n(x_1, \dots, x_n),\ \dots)$, where $f_0 \in \mathbf{C}$ and $f_n \in \mathcal{B}_n$ for all $n \geq 1$, the product and $\star$ operations being given by

$$(fg)_n = \sum_{\substack{p,\ q \in \mathbf{N} \\ p+q=n}} f_p \otimes g_q, \qquad (f^\star)_n(x_1, \dots,\ x_n) = \overline{f_n(x_n, \dots,\ x_1)}, \tag{4.1}$$

The action of $\Lambda \in G_0$ (resp. $\widetilde{G}_0$) on $\mathcal{B}$ is defined by $f \mapsto f_{\{\Lambda\}}$, where

$$\begin{aligned} f_{\{\Lambda\}} &= (f_0, f_{1\{\Lambda\}}, \dots, f_{n\{\Lambda\}}, \dots), \\ f_{n\{\Lambda\}}(x_1, \dots, x_n) &= f_n(\Lambda^{-1}x_1, \dots, \Lambda^{-1}x_n). \end{aligned} \tag{4.2}$$

It will also be useful to denote

$$f_{n\{\Lambda_1,\dots,\Lambda_n\}}(x_1, \dots, x_n) = f_n(\Lambda_1^{-1}x_1, \dots, \Lambda_n^{-1}x_n), \tag{4.3}$$

where $f_n \in \mathcal{B}_n$ and $\Lambda_1, \dots, \Lambda_n$ belong to G_0 or $\widetilde{G}_0$. If π is a permutation of $(1, \dots, n)$, and $f_n \in \mathcal{B}_n$ we define the function ${}_\pi f_n \in \mathcal{B}_n$ by

$${}_\pi f_n(x_{\pi(1)}, \dots, x_{\pi(n)}) = f_n(x_1, \dots, x_n)\ . \tag{4.4}$$

The theory of a single scalar quantum field theory on X_d (resp. $\widetilde{X}_d$) is specified by a continuous linear functional $\mathcal{W}$ on $\mathcal{B}$, i.e. by a sequence $\{\mathcal{W}_n \in \mathcal{B}'_n\}_{n\in\mathbf{N}}$ (resp. $\{\mathcal{W}_n \in \mathcal{B}'_n\}_{n\in\mathbf{N}}$), called Wightman functions, with the following properties:

1. **Covariance:** Each $\mathcal{W}_n$ is invariant under the group G_0 (resp. $\widetilde{G}_0$), i.e. for all
$$\langle \mathcal{W}_n,\ f_{n\{\Lambda\}}\rangle = \langle \mathcal{W}_n,\ f_n\rangle \tag{4.5}$$
for all $\Lambda \in G_0$ (resp. $\widetilde{G}_0$).

2. **Locality:**
$$\mathcal{W}_n(x_1,\ldots,x_j,x_{j+1},\ldots,x_n) = \mathcal{W}_n(x_1,\ldots,x_{j+1},x_j,\ldots,x_n) \tag{4.6}$$
if x_j and x_{j+1} are space-like separated.

3. **Positive Definiteness:** For each $f \in \mathcal{B}$, $\mathcal{W}(f^\star f) \geq 0$. Explicitly, given $f_0 \in \mathbf{C}, f_1 \in \mathcal{B}_1,\ldots,$ $f_k \in \mathcal{B}_k$, then
$$\sum_{n,m=0}^{k} \langle \mathcal{W}_{n+m},\ f_n^\star \otimes f_m\rangle \geq 0. \tag{4.7}$$

If these conditions are satisfied the GNS construction (see [Bo, J]) provides a Hilbert space $\mathcal{H}$, a continuous unitary representation $\Lambda \mapsto U(\Lambda)$ of G_0 (resp. $\widetilde{G}_0$) and a representation $f \mapsto \mathbf{\Phi}(f)$ (by unbounded operators) as well as a unit vector $\Omega \in \mathcal{H}$, invariant under U, such that $\mathcal{W}(f) = (\Omega,\ \mathbf{\Phi}(f)\Omega)$ for all $f \in \mathcal{B}$. As a special case the field operator ϕ is the operator valued distribution over X_d (resp. $\widetilde{X}_d$) such that $\phi(f_1) = \mathbf{\Phi}(f)$ where $f = (0,\ f_1,\ 0,\ \ldots)$. In addition the construction provides vector valued distributions $\Phi_n^{(b)}$ such that
$$\begin{aligned}\langle \Phi_n^{(b)},\ f_n\rangle &= \mathbf{\Phi}(f)\Omega \\ &= \int f_n(x_1,\ldots,\ x_n)\,\phi(x_1)\ldots\phi(x_n)\,\Omega\,d\sigma(x_1)\ldots d\sigma(x_n)\end{aligned} \tag{4.8}$$
where $f = (0,\ \ldots,\ 0,\ f_n,\ 0,\ \ldots)$. As usual, for every $\Lambda \in G_0$ (resp. $\widetilde{G}_0$),
$$U(\Lambda)\,\mathbf{\Phi}(f)\,U(\Lambda)^{-1} = \mathbf{\Phi}(f_{\{\Lambda\}}), \qquad U(\Lambda)\,\Omega = \Omega\ . \tag{4.9}$$
The details of this construction are completely analogous to those of the Minkowskian case. To every element M of the Lie algebra $\mathcal{G}$ we can associate the one-parameter subgroup $t \mapsto \exp tM$ of G_0 (resp. $\widetilde{G}_0$) and a self-adjoint operator $\widehat{M}$ acting in $\mathcal{H}$ such that $\exp it\widehat{M} = U(\exp tM)$ for all $t \in \mathbf{R}$. With these notations, we postulate the following

4. **Strong Spectral Condition:** For every $M \in \mathcal{C}_+$, every $\Psi \in \mathcal{H}$, and every $\mathcal{C}^\infty$ function $\widetilde{\varphi}$ with compact support contained in $(-\infty,\ 0)$,
$$\int_{\mathbf{R}} \left(\int \widetilde{\varphi}(p)\,e^{-itp}\,dp\right) U(\exp tM)\,\Psi\,dt = 0\ . \tag{4.10}$$
Equivalently $\widehat{M}$ has its spectrum contained in $\mathbf{R}_+$.

Using the spectral decomposition of $\widehat{M}$, this implies that, for every $\Psi \in \mathcal{H}$, $t \mapsto \exp(it\widehat{M})\,\Psi = U(\exp tM)\,\Psi$ extends to a function, again denoted $z \mapsto \exp(iz\widehat{M})\,\Psi$, continuous on $\overline{\mathbf{C}_+}$ and holomorphic in $\mathbf{C}_+$, and bounded in norm by $||\Psi||$. For any finite sequence $\{M_j\}_{1\leq j\leq N}$ of elements of $\mathcal{C}_+$, the function
$$(z_1,\ldots,z_N) \mapsto e^{iz_1\widehat{M}_1}\ldots e^{iz_N\widehat{M}_N}\,\Psi \tag{4.11}$$
is therefore continuous and bounded in norm by $||\Psi||$ on the "flattened tube"
$$\bigcup_{j=1}^{N}\{(z_1,\ldots,z_N) \in \mathbf{C}^N\ :\ z_j \in \overline{\mathbf{C}_+},\quad z_k \in \mathbf{R}\quad \forall k \neq j\}\ . \tag{4.12}$$

It is holomorphic in z_j in $\mathbf{C}_+$ when the other z_k are kept real. By the flattened tube (Malgrange-Zerner) theorem, this function extends to a continuous function on $\overline{\mathbf{C}_+}^N$, holomorphic in $\mathbf{C}_+^N$, and bounded in norm by $||\Psi||$. Thus the function $\Lambda \mapsto U(\Lambda)\Psi$ extends to a bounded holomorphic function on G_0^+ (resp. $\widetilde{G}_0^+$) with continuous boundary value on G_0 (resp. $\widetilde{G}_0$).

We now consider

$$\begin{aligned}
&(\tau_1, \ldots, \tau_n) \mapsto \\
&\qquad \int (\Omega,\ e^{i\tau_1 \widehat{M}_1}\phi(x_1)\ldots e^{i\tau_n \widehat{M}_n}\phi(x_n)\,\Omega)\, f_n(x_1,\ldots,x_n)\, d\sigma(x_1)\ldots d\sigma(x_n) \\
&= \quad \int (\Omega,\ \phi(e^{\tau_1 M_1}x_1)\,\phi(e^{\tau_1 M_1}e^{\tau_2 M_2}x_2)\ldots\phi(e^{\tau_1 M_1}\ldots e^{\tau_n M_n}x_n), \Omega) \\
&\qquad\qquad\qquad\qquad f_n(x_1,\ldots,x_n)\, d\sigma(x_1)\ldots d\sigma(x_n) \\
&\stackrel{\text{def}}{=} \quad \langle \mathcal{W}_n,\ f_{n\{\Lambda_1,\ldots,\Lambda_n\}}\rangle \ ,
\end{aligned} \tag{4.13}$$

where, for $1 \le j \le n$, $\tau_j \in \mathbf{R}$, $M_j \in \mathcal{C}_+$, and $\Lambda_j = e^{\tau_1 M_1}\ldots e^{\tau_j M_j}$. Suppose that $f_n = g_1 \otimes \ldots \otimes g_n$ where $g_j \in \mathcal{B}_1$. Then (4.13) extends to a function of $\tau_1, \ldots, \tau_n$ which is $\mathcal{C}^\infty$ on the flattened tube

$$\bigcup_{j=1}^{n} \{(\tau_1, \ldots, \tau_n)\ :\ \operatorname{Im}\tau_j \ge 0, \quad \operatorname{Im}\tau_k = 0 \quad \forall k \ne j\}\ , \tag{4.14}$$

and holomorphic in τ_j in $\mathbf{C}_+$ when the other τ_k are kept real. For every $K > 0$, the restriction of this function to

$$\bigcup_{j=1}^{n} \{(\tau_1, \ldots, \tau_n)\ :\ |\tau_j| < K, \quad \operatorname{Im}\tau_j \ge 0, \quad |\tau_k| < K, \ \operatorname{Im}\tau_k = 0 \quad \forall k \ne j\} \tag{4.15}$$

is bounded in modulus by

$$C(K)\prod_{j=1}^{n} ||g_j||_{m(K)}\ , \tag{4.16}$$

where $||g_j||_{m(K)}$ is one of the seminorms defining the topology of $\mathcal{B}_1$. The envelope of holomorphy of the set (4.15) contains the topological product

$$H_n(K) = \prod_{j=1}^{n} \{\tau_j \in \mathbf{C}_+\ :\ |\tau_j| < K \mathrm{tg}\,(\pi/4n)\}. \tag{4.17}$$

Therefore the function (4.13) extends to a $\mathcal{C}^\infty$ function on $(\overline{\mathbf{C}_+})^n$, holomorphic in $(\mathbf{C}_+)^n$ and bounded in modulus by (4.16) on the set $H_n(K)$ for every $K > 0$. For every $\tau \in H_n(K)$, the value of (4.13) defines a continuous n-linear form on $\mathcal{B}_1^n$, hence, by the nuclear theorem, a unique continuous linear functional on $\mathcal{B}_n$. We conclude that, for a general $f_n \in \mathcal{B}_n$, the function (4.13) extends to a $\mathcal{C}^\infty$ function on $(\overline{\mathbf{C}_+})^n$, holomorphic in $(\mathbf{C}_+)^n$. By standard arguments it follows that there exists a function W_n, holomorphic in $\mathcal{Z}_{n+}$, having $\mathcal{W}_n$ as its boundary value in the sense of distributions. The same proof (but with a more cumbersome notation) shows that W_n is holomorphic in $\mathcal{T}_{n+}$.

In the remainder of this paper we will require the Wightman functions $\mathcal{W}_n$ to be tempered distributions on X_d^n (resp. $\widetilde{X}_d^n$), i.e. we will take $\mathcal{B}_n = \mathcal{S}(X_d^n)$ (resp. $\mathcal{S}(\widetilde{X}_d^n)$), and we will assume, instead of the "strong spectral condition", that the following holds:

5. **Tempered Spectral Condition:** For each pair of integers $m \ge 0$ and $n \ge 0$, $\mathcal{W}_{m+n}(w_m, \ldots, w_1,\ z_1, \ldots, z_n)$ is the boundary value, in the sense of tempered distributions, of a function $W_{m,n}$ of $(w,\ z)$ holomorphic and of tempered growth in $\mathcal{T}_{m+}^* \times \mathcal{T}_{n+} = \mathcal{T}_{m-} \times \mathcal{T}_{n+}$. (In particular $\mathcal{W}_n(z_1, \ldots, z_n)$ is the boundary value of a function W_n holomorphic and of tempered growth in $\mathcal{T}_{n+}$.) Moreover for any $f_n \in \mathcal{B}_n$ and every choice of $M_1, \ldots, M_n \in \mathcal{C}_1$ the function defined by (4.13) is $\mathcal{C}^\infty$ and at most of polynomial growth in $\overline{\mathbf{C}_+^n}$.

If the positive definiteness condition holds, the tempered spectral condition implies the strong spectral condition. However the tempered spectral condition makes sense even if the positive definiteness condition does not hold.

The following lemma follows from arguments given in [BEM] (Sect. 5), using as the main tool a theorem of V. Glaser [G1].

Lemma 4.1 *Let $\mathcal{W}$ be a Wightman functional satisfying the conditions of covariance and locality and the tempered spectral condition. Suppose in addition that there is a real open set $V \in X_d$ such that $\mathcal{W}(f^\star f) \geq 0$ for all $f \in \mathcal{B}$ with support in V (i.e. such that f_n has support in V^n for all $n \in \mathbf{N}$). Then there exists, for each $n \in \mathbf{N}$, a vector valued function Φ_n, holomorphic in $\mathcal{T}_{n+}$, and with tempered growth at infinity and near the boundaries, having $\Phi_n^{(b)}$ as a boundary value in the sense of tempered distributions. In particular $\mathcal{W}$ satisfies the unrestricted positive definiteness condition, and the Reeh-SchliederTheorem holds.*

The permuted Wightman functions $\mathcal{W}_{n,\pi}$ defined, as usual, by $\langle \mathcal{W}_{n,\pi}\,,\ f_n\rangle = \langle \mathcal{W}_n\,,\ _\pi f_n\rangle$, are boundary values of functions $W_{n,\pi}$ holomorphic in the permuted tuboids $\mathcal{T}_{n,\pi} = \{z\ :\ (z_{\pi(1)}, \ldots, z_{\pi(n)}) \in \mathcal{T}_{n+}\}$. Owing to local commutativity, they are branches of a single holomorphic function.

Remark 4.1 If a set of Wightman functions satisfies the tempered spectral condition but only a part of the locality condition, i.e. if it is assumed that $\mathcal{W}_n$ and $\mathcal{W}_{n,(j+1,j)}$ coincide in an open subset of $\mathcal{R}_{j,j+1}$ (necessarily symmetric under the exchange of j and $j+1$), then it follows from Lemma 3.10 that they coincide in the whole of $\mathcal{R}_{j,j+1}$. Similar extension theorems are well-known in the Minkowskian case (see e.g. [SW]) due to phenomena of analytic completion. It is remarkable that no completion is needed in the AdS case.

If the conditions 1-5 hold, one may want to assume

6. **Uniqueness of the vacuum:** The invariant subspace of $\mathcal{H}$, $\{\Psi \in \mathcal{H}\ :\ U(\Lambda)\Psi = \Psi\ \forall\Lambda \in G_0 \text{ (resp. } \widetilde{G}_0)\}$ is one-dimensional, i.e. it is equal to $\mathbf{C}\Omega$.

In the Minkowskian case this condition is equivalent to a clustering property of the Wightman functions, namely the truncated Wightman functions tend to zero when a proper subset of their arguments tend to space-like infinity while the others remain bounded. (The truncated Wightman functions have the same inductive definition as in the Minkowskian case ([J] p. 66). They have the same linear properties as the Wightman functions.) A similar equivalence holds in the anti-de Sitter case. In Sect. 5, we discuss some properties of this type, which are equivalent to the uniqueness of the vacuum in the presence of positivity.

5 Parabolic (Poincaré) sections

A convenient chart of a part of X_d (resp. $X_d^{(c)}$) is provided by the parabolic coordinates $\mathbf{z},\ v \mapsto z(\mathbf{z},\ v)$ given by

$$\mathbf{z}, v \to z(\mathbf{z}, v) = \left\{ \begin{array}{lcl} z^\mu & = & e^v \mathbf{z}^\mu \\ z^{d-1} & = & \operatorname{sh} v + \frac{1}{2} e^v \mathbf{z}^2 \\ z^d & = & \operatorname{ch} v - \frac{1}{2} e^v \mathbf{z}^2 \end{array} \right. . \tag{5.1}$$

In this equation $\mu = 0, 1, ..., d-2$, $\mathbf{z}^0, ..., \mathbf{z}^{d-2}$ are the coordinates of an arbitrary event in a real (resp. complex) $(d-1)$-dimensional Minkowski space-time with metric[1] $ds_M^2 = d\mathbf{z}^{0\,2} - d\mathbf{z}^{1\,2} - \ldots - d\mathbf{z}^{d-2\,2}$, $\mathbf{z}^2 = \mathbf{z}^0\mathbf{z}^0 - \sum_{j=1}^{d-2} \mathbf{z}^j\mathbf{z}^j$ and $v \in \mathbf{R}$ (resp. $\mathbf{C}$).

[1]Here and in the following where it appears, an index M stands for Minkowski.

This explains why the coordinates $\mathbf{z}, v$ of the parametrization (5.1) are also called Poincaré coordinates. As $\mathbf{z},\ v$ vary in $\mathbf{R}^d$, the image of this map is $\{x \in X_d\ :\ x^{d-1}+x^d > 0\}$. The scalar product and the AdS metric can then be rewritten as follows:

$$(z,z') = \mathrm{ch}\,(v-v') - \frac{1}{2}e^{v+v'}\left(\mathbf{z}-\mathbf{z}'\right)^2, \tag{5.2}$$

$$\mathrm{d}s^2_{AdS} = e^{2v}\mathrm{d}s^2_M - \mathrm{d}v^2. \tag{5.3}$$

Eq. (5.2) implies that

$$(z(\mathbf{z},\ v) - z'(\mathbf{z}',\ v'))^2 = e^{v+v'}(\mathbf{z}-\mathbf{z}')^2 - 2\mathrm{ch}\,(v-v') + 2\ . \tag{5.4}$$

For a given real v we denote $\mathcal{M}_v$ the parabolic section

$$\mathcal{M}_v = \{z \in X_d \text{ (resp. } X_d^{(c)})\ :\ (z,\ e_d - e_{d-1}) = z^{d-1} + z^d = e^v\}. \tag{5.5}$$

The subgroup $G_{(d-1)d}$ (resp. $G^{(c)}_{(d-1)d}$) of G_0 (resp. $G_0^{(c)}$) which fixes $e_d - e_{d-1}$, and therefore leaves $\mathcal{M}_v$ globally invariant, is isomorphic to the real (resp. complex) Poincaré group operating on the $(d-1)$-Minkowski space. In particular, for $b = (b^0,\dots,b^{d-2})$, the transformation

$$\exp(b^\mu L_\mu) = \begin{pmatrix} 1 & \dots & 0 & b^0 & b^0 \\ 0 & \dots & 0 & b^1 & b^1 \\ \vdots & & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & b^{d-2} & b^{d-2} \\ b_0 & \dots & b_{d-2} & 1+\frac{(b,b)}{2} & \frac{(b,b)}{2} \\ -\,b_0 & \dots & -b_{d-2} & -\frac{(b,b)}{2} & 1-\frac{(b,b)}{2} \end{pmatrix} \tag{5.6}$$

operates in the Minkowski leaf as $\mathbf{z} \mapsto \mathbf{z} + b$. If $b = (\tau,\ \vec{0},\ 0)$, this transformation leaves the coordinates $z^1,\dots,z^{d-2}$ unchanged and, in the 3-dimensional space of the coordinates $z^0,\ z^{d-1},\ z^d$, is given by

$$e^{\tau L_0} = \begin{pmatrix} 1 & \tau & \tau \\ \tau & 1+\frac{\tau^2}{2} & \frac{\tau^2}{2} \\ -\,\tau & -\frac{\tau^2}{2} & 1-\frac{\tau^2}{2} \end{pmatrix}\ . \tag{5.7}$$

Here

$$L_0 = \ell(e_0 \wedge (e_d - e_{d-1})) = M_{0d} - M_{0(d-1)} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\ , \tag{5.8}$$

Similarly $L_\mu = \ell(e_\mu \wedge (e_d - e_{d-1}))$ where $0 \le \mu \le d-2$. With these notations,

$$z(\mathbf{z},\ v) = \exp(\mathbf{z}^\mu L_\mu)\exp(vM_{(d-1)d})\, e_d\ . \tag{5.9}$$

We denote

$$\begin{aligned} \Gamma_{(d,d-1)} &= \{b^\mu L_\mu \in \mathcal{G}\ :\ b^0 > |\vec{b}|\} \\ &= \{\Lambda\, tL_0\, \Lambda^{-1}\ :\ t>0,\ \ \Lambda \in G_0,\ \ \Lambda e_d = e_d,\ \ \Lambda e_{d-1} = e_{d-1}\}\ . \end{aligned} \tag{5.10}$$

If c is a real point of $\mathcal{M}_v$, i.e. $c = z(\mathbf{c},\ v)$ for some real $\mathbf{c}$ and v, and $Q = b^\mu L_\mu \in \Gamma_{(d,d-1)}$ then $\exp(iQc) = z(\mathbf{c}+ib,\ v)$ is a point of the future tube in the complexified $\mathcal{M}_v$ considered as a Minkowski space. Conversely any point of the future tube is of this form.

More generally, we define the n-point forward tuboid $\mathcal{F}_{d,d-1,n}$ as

$$\begin{aligned} \mathcal{F}_{d,d-1,n} = \{&(z_1,\dots,z_n) \in X_d^{(c)n}\ :\ \forall j = 1,\dots,n \\ &z_j = \exp(iQ_1)\cdots\exp(iQ_j)\,c_j\,,\quad Q_j \in \Gamma_{(d,d-1)}\,, \\ &c_j \in X_d,\ \ c_j^{d-1} + c_j^d > 0\}\ . \end{aligned} \tag{5.11}$$

The intersection of this set with $\mathcal{M}_v^n$ is the set obtained by restricting the c_j to lie in $\mathcal{M}_v$ in the above definition. This intersection is just the n-point forward tube in the Minkowskian variables $z_1, \ldots, z_n$. The main point of this section is

Lemma 5.1 *For all n, $\mathcal{F}_{d,d-1,n} \subset \mathcal{Z}_{n+}$.*

The proof of this lemma consists of Lemmas 5.2 and 5.3 below. We begin with the following remark.

Remark 5.1 Let

$$\Lambda_u = \exp(u\, M_{d(d-1)}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \operatorname{ch} u & -\operatorname{sh} u \\ 0 & -\operatorname{sh} u & \operatorname{ch} u \end{pmatrix} . \tag{5.12}$$

Then $\Lambda_u e_\mu = e_\mu$ for $0 \le \mu \le d-2$, and

$$\Lambda_u e_{d-1} = (\operatorname{ch} u) e_{d-1} - (\operatorname{sh} u) e_d, \qquad \Lambda_u e_d = (\operatorname{ch} u) e_d - (\operatorname{sh} u) e_{d-1} . \tag{5.13}$$

Therefore

$$\lim_{u\to\pm\infty} 2e^{-|u|}\Lambda_u e_{d-1} = e_{d-1} \mp e_d , \qquad \lim_{u\to\pm\infty} 2e^{-|u|}\Lambda_u e_d = e_d \mp e_{d-1} , \tag{5.14}$$

and, for $0 \le \mu \le d-2$,

$$\Lambda_u M_{\mu d}\Lambda_u^{-1} = \ell(e_\mu \wedge \Lambda_u e_d) = (\operatorname{ch} u) M_{\mu d} - (\operatorname{sh} u) M_{\mu(d-1)} \tag{5.15}$$

$$\lim_{u\to\pm\infty} 2e^{-|u|}\Lambda_u M_{\mu d}\Lambda_u^{-1} = M_{\mu d} \mp M_{\mu(d-1)} . \tag{5.16}$$

$$\lim_{u\to\pm\infty} 2e^{-|u|}\Lambda_u M_{\mu(d-1)}\Lambda_u^{-1} = M_{\mu(d-1)} \mp M_{\mu d} . \tag{5.17}$$

Lemma 5.2 *Let $c \in \mathcal{M}_v$ and $L \in \Gamma_{(d,d-1)}$, and let $z = \exp(iL)c$. Then*
(i) There is an $M \in \mathcal{C}_+$, and a $c' \in X_d$, arbitrarily close to L and c, respectively, such that $z = \exp(iL)c = \exp(iM)c'$.
(ii) For every neighborhood W of $(L,\ c)$ in $\mathcal{G} \times X_d$, there is a neighborhood V of $z = \exp(iL)c$ in $X_d^{(c)}$ such that every $z'' \in V$ can be written as $z'' = \exp(iM'')\, c''$, with $M'' \in \mathcal{C}_+$ and $(M'',\ c'') \in W$.

Proof. (i) It suffices to prove the statement in case $L = sL_0$ for some $s > 0$, and $c^0 = 0$. In the coordinates $(z^0,\ z^{d-1},\ z^d)$, for any $\tau \in \mathbf{C}$, $c \in \mathcal{M}_v$,

$$\exp(\tau L_0)\, c = \begin{pmatrix} c^0 + e^v\tau \\ c^0\tau + c^{d-1} + e^v\frac{\tau^2}{2} \\ -\,c^0\tau + c^d - e^v\frac{\tau^2}{2} \end{pmatrix} . \tag{5.18}$$

As expected the two last components add up to e^v. We now set $c^0 = 0$ and $\tau = is$ ($s > 0$) in (5.18). We look for M in the form

$$M = \Lambda_u\, tM_{0d}\, \Lambda_u^{-1}, \tag{5.19}$$

with Λ_u as in (5.12) and $u > 0$ large. Then

$$\Lambda(u,\ it) = \Lambda_u \exp(itM_{0d})\, \Lambda_u^{-1} = \\ \begin{pmatrix} \operatorname{ch} t & i\operatorname{sh} u \operatorname{sh} t & i\operatorname{ch} u \operatorname{sh} t \\ i\operatorname{sh} u \operatorname{sh} t & \operatorname{ch}^2 u - \operatorname{sh}^2 u \operatorname{ch} t & \operatorname{ch} u \operatorname{sh} u\, (1-\operatorname{ch} t) \\ -\, i\operatorname{ch} u \operatorname{sh} t & -\operatorname{ch} u \operatorname{sh} u\, (1-\operatorname{ch} t) & -\operatorname{sh}^2 u + \operatorname{ch}^2 u \operatorname{ch} t \end{pmatrix} . \tag{5.20}$$

The set of all vectors $z = (iy^0,\ \vec{x},\ x^{d-1},\ x^d)$ with pure imaginary 0-component and all other components real is mapped into itself by $\Lambda(u,\ it)$ for all real u and t. As u tends to $+\infty$,

$$\Lambda_u\, tM_{0d}\, \Lambda_u^{-1} = (\operatorname{ch} u) tM_{0d} - (\operatorname{sh} u) tM_{0(d-1)} \tag{5.21}$$

tends to L_0 provided $t \approx 2e^{-u}$, and $\Lambda(u,\ it)$ tends to $\exp(isL_0)$ provided $t \approx 2se^{-u}$. For fixed real $c = (0,\ \vec{c},\ c^{d-1},\ c^d)$, satisfying $c^{d-1}+c^d = e^v$, and real $s > 0$, we wish to find a real $c' = (0,\ \vec{c},\ c'^{d-1},\ c'^d)$ and $t > 0$ such that $\Lambda(u,\ it)c' = \exp(isL_0)c$, i.e. $c' = \Lambda(u,\ -it)\exp(isL_0)c$. The condition that $c'^0 = 0$ gives

$$\mathrm{th}\, t = \frac{se^v}{c^{d-1}\mathrm{sh}\, u + c^d\mathrm{ch}\, u + (s^2/2)e^v e^{-u}} \approx 2se^{-u}\ . \tag{5.22}$$

The other components of c' are then real and tend to those of c as $u \to \infty$. This proves (i).
(ii) Using (i), $z = \exp(iM)c'$, where $M \in \mathcal{C}_+$ and $c' \in X_d$ can be chosen arbitrarily close to L and c respectively. If $z'' \in X_d^{(c)}$ is sufficiently close to z, we may apply St_1 of the proof of Lemma 3.6, which yields the conclusion of (ii). ▭

The condition that $c \in \mathcal{M}_v$ in Lemma 5.2 can be replaced by the condition $c^d + c^{d-1} > 0$, and in fact by $c^d + c^{d-1} \neq 0$ since the case $c^d + c^{d-1} < 0$ can be dealt with by changing c to $-c$. We note also that $(c^d + c^{d-1})^2 = (L_0c,\ L_0c)$. We define

$$\begin{aligned}\Gamma_+ &= \{\Lambda\rho L_0\Lambda^{-1}\ :\ \Lambda \in G_0,\quad \rho > 0\} \\ &= \{a \wedge b\ :\ (a,a) = 1,\quad (b,b) = (a,b) = 0,\quad a^0b^d - a^db^0 > 0\}\end{aligned} \tag{5.23}$$

(the same calculations as in (2.17) show that if $(a,a) = 1$, $(b,b) = (a,b) = 0$, then $(a^0b^d - a^db^0)^2 \geq \vec{b}^2 = b_0^2 + b_d^2$). The following lemma extends Lemma 5.2 to the case of n points. It may be useful to spell it out in some detail.

Lemma 5.3 *let $z = (z_1,\ \dots,\ z_n) \in X_d^{(c)n}$ be such that, for $1 \leq j \leq n$,*

$$z_j = e^{iQ_1}\dots e^{iQ_j}\, c_j\ ,\quad Q_j \in \Gamma_{(d,d-1)}\ ,\quad c_j \in X_d,\quad (Q_jc_j, Q_jc_j) > 0\ . \tag{5.24}$$

For each $\varepsilon > 0$, there exists a $\delta(n,\ \varepsilon,\ Q,\ c) > 0$ such that, for any $z' = (z'_1, \dots, z'_n) \in X_d^{(c)n}$ satisfying $||z_j - z'_j|| < \delta(n,\ \varepsilon,\ Q,\ c)$ for all $j = 1, \dots, n$, there exist $M_1, \dots, M_n \in \mathcal{C}_+$ and $c'_1, \dots, c'_n \in X_d$ such that $||M_j - Q_j|| < \varepsilon$, $||c_j - c'_j|| < \varepsilon$, and $z'_j = \exp(iM_1)\, c'_1\ \dots \exp(iM_j)\, c'_j$ for all $j = 1, \dots, n$.

Proof. Let St'_n denote the statement of the lemma. St'_1 follows from Lemma 5.2 in view of the preceding remarks. Assume that St'_m has been proved for all $m \leq n-1$ and let $z = (z_1, \dots, z_n)$ be as in (5.24). Let $\varepsilon \in (0,\ 1)$, $R = 1 + \sup_{1\leq j\leq n} ||z_j||$. Denote $\widetilde{z} = (z_2, \dots, z_n) \in X_d^{(c)(n-1)}$, $\widetilde{Q} = (Q_2, \dots, Q_n)$, $\widetilde{c} = (c_2, \dots, c_n)$, and

$$\delta_2 = \frac{1}{4R}\delta\left(n-1,\ \varepsilon,\ \widetilde{Q},\ \widetilde{c}\right). \tag{5.25}$$

By St'_1, there exists a $\delta_1 > 0$ such that for any z'_1 with $z'_1 - z_1 < \delta_1$, there exist $M_1 \in \mathcal{C}_+$ and $c'_1 \in X_d$ such that $z'_1 = \exp(iM_1)c'_1$, $||c'_1 - c_1|| < \varepsilon$, $||M_1 - Q_1|| < \varepsilon$, and $||\exp(iM_1) - \exp(iQ_1)|| < \delta_2$. Let $z'_1, \dots, z'_n \in X_d^{(c)n}$ satisfy $||z'_j - z_j|| < \min\{R/4,\ \delta_1,\ \delta_2/(1+||\exp(-iQ_1)||)\}$, and let M_1, c'_1 be as above. Then, for $2 \leq j \leq n$,

$$\begin{aligned}||\exp(-iM_1)z'_j - \exp(-iQ_1)z_j|| &\leq ||\exp(-iM_1) - \exp(-iQ_1)||\, ||z'_j|| \\ &+ ||\exp(-iQ_1)||\, ||z'_j - z_j|| \\ &\leq \delta(n-1,\ \varepsilon,\ \widetilde{Q},\ \widetilde{c}).\end{aligned} \tag{5.26}$$

By St'_{n-1}, there exist $M_2, \dots, M_n \in \mathcal{C}_+$ and $c'_2, \dots, c'_n \in X_d$ such that, for $2 \leq j \leq n$, $||M_2 - Q_2|| < \varepsilon$, $||c'_2 - c_2|| < \varepsilon$, and $\exp(-iM_1)z'_j = \exp(iM_2)\dots\exp(iM_j)c'_j$. This proves St'_n. ▭

We digress at this point and take advantage of the above calculations to recall the proof of the following lemma (which appears in [BB]).

Lemma 5.4 (Borchers-Buchholz) *Let U be a continuous unitary representation of G_0 in a Hilbert space $\mathcal{H}$. Let $\Psi \in \mathcal{H}$ be such that $U(\exp(u\,M_{d(d-1)}))\Psi = \Psi$ for all real u. Then $U(\Lambda)\Psi = \Psi$ for all $\Lambda \in G_0$.*

Obviously $M_{d(d-1)}$ can be replaced, in the statement of Lemma 5.4, by any of its conjugates under G_0, e.g. M_{01} etc.

Proof of Lemma 5.4. By Remark 5.1

$$\begin{aligned}\lim_{u\to\pm\infty} \quad & \exp(uM_{d(d-1)})\,\exp(2re^{-|u|}M_{\mu d})\,\exp(-uM_{d(d-1)}) \\ & = \exp(r(M_{\mu d} \mp M_{\mu(d-1)})), \quad 0 \le \mu \le d-2. \end{aligned} \tag{5.27}$$

Here r is any real. Suppose that $\Psi \in \mathcal{H}$ is such that $U(\exp(uM_{d(d-1)}))\Psi = \Psi$ for all real u. Then for all real u and t, $0 \le \mu \le d-2$,

$$||U(e^{uM_{d(d-1)}}\,e^{tM_{\mu d}}\,e^{-uM_{d(d-1)}})\,\Psi - \Psi|| = ||U(e^{tM_{\mu d}})\,\Psi - \Psi||\ . \tag{5.28}$$

We set $t = 2re^{-|u|}$ for some fixed real r. As $u \to \pm\infty$, t tends to 0, so that the rhs of this equation tends to 0. By the continuity of U and (5.27), the lhs tends to $||U(\exp(r(M_{\mu d} \mp M_{\mu(d-1)})))\Psi - \Psi||$ hence this quantity is equal to 0. By the continuity of U, the subgroup $N_\Psi = \{\Lambda \in G_0\ :\ U(\Lambda)\Psi = \Psi\}$ of G_0 is closed, hence it is a Lie subgroup (see e.g. [St], pp. 228 ff). Its Lie algebra contains $M_{d(d-1)}$ and $M_{\mu d} \mp M_{\mu(d-1)}$ for all $\mu = 0,\dots,d-2$, hence all $M_{\mu d}$ for $\mu = 0,\dots,d-1$. These elements generate the whole of $\mathcal{G}$ since $[M_{\mu d},\ M_{\nu d}] = M_{\nu\mu}$ for $0 \le \mu < \nu \le d-1$. Hence $N_\Psi = G_0$. $\square$

Lemma 5.1 follows, as announced, from Lemmas 5.2 and 5.3. If $\{\mathcal{W}_n\}_{n\in\mathbf{N}}$ is a sequence of Wightman functions satisfying the conditions of Sect. 4, but not necessarily the positive definiteness condition, the tempered distribution $\mathcal{W}_n$ can be restricted to $\mathcal{M}_v^n$, and more generally to $\mathcal{M}_{v_1} \times \cdots \times \mathcal{M}_{v_n}$. The distributions $\mathcal{W}_n(z(\mathbf{z}_1,\ v_1),\ \dots,\ z(\mathbf{z}_n,\ v_n))$ have the linear properties of the n-point Wightman functions for a set of Minkowskian fields on $\mathbf{R}^{d-1}$, $A(\mathbf{z},\ v) = \phi(z(\mathbf{z},\ v))$, labelled by a real parameter v and depending in a $\mathcal{C}^\infty$ manner on v. The usual Minkowskian covariance and analyticity properties are satisfied by virtue of Lemma 5.1. The local commutativity is inherited from that postulated for the $\mathcal{W}_n$ in view of the formula (5.4). If the $A(.,\ v)$ satisfy the positivity condition for all v in a non-empty open interval $(a,\ b)$, then by Lemma 4.1 (Sect. 4), the original $\mathcal{W}_n$ satisfy the positivity condition on the whole of X_d.

In case the positive definiteness condition holds, we also have

Lemma 5.5 *Let $Q \in \Gamma_+$ and let $\widehat{Q}$ denote the self-adjoint operator on $\mathcal{H}$ such that $\exp(it\widehat{Q}) = U(\exp(tQ))$ for all $t \in \mathbf{R}$. Then the spectrum of $\widehat{Q}$ is contained in $\mathbf{R}_+$. More precisely, for every $\Psi \in \mathcal{H}$, and every $\mathcal{C}^\infty$ function $\widetilde{\varphi}$ on $\mathbf{R}$ with compact support contained in $(-\infty,\ 0)$,*

$$\int_{\mathbf{R}} \left(\int \widetilde{\varphi}(p)\,e^{-itp}\,dp\right) U(\exp tQ)\,\Psi\,dt = 0\ . \tag{5.29}$$

Proof. Let $\varphi(t) = \int \widetilde{\varphi}(p)\,e^{-itp}\,dp$. We may assume that $\int |\varphi(t)|dt \le 1$ and $||\Psi|| = 1$. Given $\varepsilon > 0$, let $T > 0$ be such that $\int_{|t|>T} |\varphi(t)|dt < \varepsilon/3$. Let V be a neighborhood of the identity in G_0 (or $\widetilde{G}_0$) such that $||(U(\Lambda) - 1)\Psi|| < \varepsilon/3$ for all $\Lambda) \in V$. It is possible to choose $M \in \mathcal{C}_+$ such that $\exp(-tQ)\exp(tM) \in V$ for all $t \in [-T,\ T]$. Then

$$\begin{aligned} ||\int \varphi(t)U(e^{tQ})\Psi\,dt|| &\le ||\int \varphi(t)U(e^{tM})\Psi\,dt|| + ||\int_{|t|>T} \varphi(t)U(e^{tM})\Psi\,dt|| \\ &+ ||\int_{|t|>T} \varphi(t)U(e^{tQ})\Psi\,dt|| + ||\int_{|t|\le T} \varphi(t)U(e^{tQ})\left(1 - U(e^{-tQ}e^{tM})\right)\,\Psi\,dt|| \\ &< \varepsilon \end{aligned} \tag{5.30}$$

since the first term in the rhs is zero. $\square$

Under the same hypotheses, we note that if $\Psi \in \mathcal{H}$ is invariant under $U(G_{(d-1)d})$ then it is in particular invariant under $U(\exp(\mathbf{R}M_{01}))$, so that, by Lemma 5.4, it is invariant under $U(G_0)$. Let again $A(\mathbf{z},\ v) = \phi(z(\mathbf{z},\ v))$, which we consider as an operator valued distribution on the Minkowski space $\mathbf{R}^{d-1}$ depending smoothly on the real parameter v. By the Reeh-Schlieder theorem for the field ϕ, the vacuum is cyclic for the set of fields $\{A(.,\ v)\ :\ v \in (a,\ b)\}$, where $(a,\ b)$ is any non-empty open interval in $\mathbf{R}$. As a consequence, the uniqueness of the vacuum for the original theory is equivalent to the uniqueness of the vacuum for the fields A, hence to the cluster property for their Wightman functions, which is also a certain cluster property for the Wightman functions of ϕ. We note that the Borchers-Buchholz Lemma also provides the following characterization of the uniqueness of the vacuum in terms of the Wightman functions.

Lemma 5.6 *Assume that the conditions 1-5 of Sect. 4 hold (this includes the positivity condition). Then the condition of uniqueness of the vacuum is equivalent to*

$$\forall f, g \in \mathcal{B}, \quad \lim_{T\to+\infty} \frac{1}{T}\int_0^T \Big(\langle \mathcal{W},\ f^\star\, g_{\{\exp(tM_{d(d-1)})\}} \rangle - \langle \mathcal{W},\ f^\star \rangle \langle \mathcal{W},\ g \rangle \Big)\, dt = 0 \tag{5.31}$$

Proof. If all the conditions 1-5 of Sect. 4 hold, (5.31) is equivalent to

$$\begin{aligned} \forall f, g \in \mathcal{B}, \qquad & \lim_{T\to+\infty} \frac{1}{T}\int_0^T (\mathbf{\Phi}(f)\Omega,\ \exp(it\widehat{M}_{d(d-1)})\mathbf{\Phi}(g)\Omega)\, dt \\ = \quad & (\mathbf{\Phi}(f)\Omega,\ \Omega)(\Omega,\ \mathbf{\Phi}(g)\Omega)\ . \end{aligned} \tag{5.32}$$

By the mean ergodic theorem, the limit in the lhs is equal to $(\mathbf{\Phi}(f),\ E\,\mathbf{\Phi}(g))$, where E is the projector on the subspace of all vectors invariant under $\exp(it\widehat{M}_{d(d-1)})$. By Lemma 5.4 this is the subspace of all vectors invariant under $U(G_0)$. Therefore (by the density of $\{\mathbf{\Phi}(f)\Omega\ :\ f \in \mathcal{B}\}$) the condition (5.31) is equivalent to the fact that the subspace of all vectors invariant under $U(G_0)$ is $\mathbf{C}\Omega$. $\square$

6 Two-point functions

We now consider the two-point function of a scalar field theory on $\widetilde{X}_d$ which satisfies the general requirements described in the previous sections:

$$\mathcal{W}_2(x_1, x_2) = \mathcal{W}(x_1, x_2) = (\Omega, \phi(x_1)\phi(x_2)\Omega)\ . \tag{6.1}$$

By the tempered spectral condition, this is the boundary value on $\widetilde{X}_d^2$, in the sense of tempered distributions, of a function $W_+(z_1, z_2)$, holomorphic and of tempered growth in $\widetilde{\mathcal{T}}_{1-} \times \widetilde{\mathcal{T}}_{1+}$. The permuted Wightman function $\mathcal{W}(x_2, x_1)$ is the boundary value of $W_-(z_1, z_2) = W_+(z_2, z_1)$, holomorphic in $\widetilde{\mathcal{T}}_{1+} \times \widetilde{\mathcal{T}}_{1-}$. The two permuted Wightman functions are invariant under $\widetilde{G}_0$ and coincide in the real open subset of $\widetilde{X}_d^2$ in which x_1 and x_2 are space-like separated. Therefore $W_\pm$ are branches of a single holomorphic function $W(z_1, z_2)$. Extensions of standard arguments (see Appendix A) show that there exists a function w, holomorphic on the universal covering $\widetilde{\Delta}$ of $\Delta = \mathbf{C} \setminus [-1,\ 1]$, such that $W_+(z_1, z_2) = w((z_1, z_2))$ when z_1 and z_2 belong to $\widetilde{\mathcal{T}}_{1-}$ and $\widetilde{\mathcal{T}}_{1+}$, respectively. Theories on the *covering* of Anti-de Sitter spacetime are thus in this respect closely similar to Minkowski [SW] and de Sitter [BM] field theories.

In the case of a field theory on $\widetilde{X}_d$, the commutator function can be written (non uniquely) as the difference of a retarded and an advanced "function" with supports in $\{(x_1,\ x_2) \in \widetilde{X}_d^2\ :\ x_j = \widetilde{\chi}(s_j,\ \vec{x_j}),\ \ \pm(s_1 - s_2) \geq 0\}$, respectively (see (2.2)).

In the case of a field theory on the *pure* AdS spacetime X_d, w is actually a function holomorphic on Δ, and, in particular, $W_+(x_1, x_2)$ and $W_-(x_1, x_2)$ coincide not only on $\mathcal{R}_2 = \{x \in X_d^2\ :\ (x_1, x_2) > 1\}$, but also on the "exotic region" $\mathcal{R}_2' = \{x \in X_d^2\ :\ (x_1, x_2) < -1\}$. In this case the support of the commutator function $\mathcal{W}(x_1,x_2)-\mathcal{W}(x_2,x_1)$ is contained in $X_d^2 \setminus \mathcal{R}_2 \cup \mathcal{R}_2'$. This is the union of the two closed sets $\{x \in X_d^2\ :\ |(x_1,x_2)| \leq 1,\ (x_2 \wedge x_1, e_0 \wedge e_d) \geq 0\}$ and $\{x \in X_d^2\ :\ |(x_1,x_2)| \leq 1,\ (x_2 \wedge x_1, e_0 \wedge e_d) \leq 0\}$, and the commutator function can be written as the difference of an advanced and retarded function with supports in these two sets (respectively). This splitting is, as usual, not unique.

Any two-point function with the above mentioned properties is the two-point function of a generalized free field on $\widetilde{X}_d$ or X_d, which satisfies the positive definiteness condition if and only if, for any $\mathcal{C}^\infty$ function φ with compact support in $\widetilde{X}_d$ (resp. X_d),

$$\langle \mathcal{W},\ \overline{\varphi} \otimes \varphi \rangle \geq 0. \tag{6.2}$$

Wick powers of such a field are well-defined: their Wightman functions can be obtained by the standard formulae, using W, as boundary values of holomorphic functions in the relevant tuboids. They satisfy all the requirements of Sect. 4 with the possible exception of positive definiteness, which holds if and only if (6.2) holds. Note that two Wick powers of generalized free fields on X_d share the property of commuting when their arguments are in $\mathcal{R}_2 \cup \mathcal{R}_2'$.

6.1 The simplest example: Klein-Gordon fields.

Let us consider fields satisfying the AdS Klein-Gordon equation on $\widetilde{X}_d$:

$$\Box \phi + m^2 \phi = 0. \tag{6.3}$$

There is a preferred choice of solutions of that equation that display the simplest example of the previous analytic structure. The corresponding two-point functions are expressed in terms of generalized Legendre functions [BV]

$$Q_\lambda^{(d+1)}(\zeta) = \frac{\Gamma\left(\frac{d}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2}\right)} \int_1^\infty (\zeta + it\sqrt{\zeta^2-1})^{-\lambda-d+1}(t^2-1)^{\frac{d-3}{2}}\,dt \tag{6.4}$$

by the following formula:

$$\begin{aligned} W_{\lambda+\frac{d-1}{2}}(z_1,z_2) &= w_{\lambda+\frac{d-1}{2}}(\zeta) \\ &= \frac{e^{-i\pi d}}{\pi^{\frac{d-1}{2}}}\Gamma\left(\frac{d+1}{2}\right) h_{d+1}(\lambda) Q_\lambda^{(d+1)}(\zeta), \quad \zeta = (z_1,z_2), \end{aligned} \tag{6.5}$$

where the parameter λ is related to the mass by the formula

$$m^2 = \lambda(\lambda + d - 1). \tag{6.6}$$

The normalization can be obtained by imposing the local Hadamard condition, that gives

$$h_{d+1}(\lambda) = \frac{(2\lambda+d-1)}{\Gamma(d)} \frac{\Gamma(\lambda+d-1)}{\Gamma(\lambda+1)}. \tag{6.7}$$

It can be checked directly that the functions $w_{\lambda+\frac{d-1}{2}}((z_1,z_2))$ defined by Eqs (6.5), (6.4), (6.7) are holomorphic in Δ or $\widetilde{\Delta}$ according to whether λ is an integer or not.

It is useful to display also an equivalent expression for the Wightman functions (6.5) in terms of the usual associate Legendre's function of the second-kind[2] [B]:

$$W_{\lambda+\frac{d-1}{2}}(z_1,z_2) = W_\nu(z_1,z_2) = w_\nu(\zeta) = \frac{e^{-i\pi\frac{d-2}{2}}}{(2\pi)^{\frac{d}{2}}}(\zeta^2-1)^{-\frac{d-2}{4}} Q_{\nu-\frac{1}{2}}^{\frac{d-2}{2}}(\zeta), \tag{6.8}$$

[2] Note the slightly different notation with the generalized Legendre's function defined in Eq. (6.4) with the upper index in parentheses. This is the way these Wightman functions were first written in [F] for the four dimensional case ($d = 4$). Their identification with second–kind Legendre functions is worth being emphasized again, in place of their less specific (although exact) introduction under the general label of hypergeometric functions, used in recent papers. In fact Legendre functions are basically linked to the geometry of the dS and AdS quadrics from both group–theoretical and complex analysis viewpoints [BV, BM, V]

where we introduced the parameter $\nu = \lambda + \frac{d-1}{2}$ such that

$$\nu^2 = \frac{(d-1)^2}{4} + m^2. \tag{6.9}$$

Theories with $\nu > -1$ are acceptable in the sense that they satisfy all the axioms of Section 4 including the positive-definiteness property, which we shall prove below. There are in fact two regimes [BF]:

i) for $\nu > 1$ the axioms uniquely select one field theory for each given mass;

ii) for $|\nu| < 1$ there are two acceptable theories for each given mass.

The case $\nu = 1$ is a limiting case. Eq. (6.8) shows that the difference between the theories parametrized by opposite values of ν is in their large distance behavior. More precisely, in view of Eq. (3.3.1.4) of [B], we can write:

$$w_{-\nu}(\zeta) = w_\nu(\zeta) + \frac{\sin \pi\nu}{(2\pi)^{\frac{d+1}{2}}} \Gamma\left(\frac{d}{2} - \nu\right) \Gamma\left(\frac{d}{2} + \nu\right) (\zeta^2 - 1)^{-\frac{d-1}{4}} P^{-\frac{d-1}{2}}_{-\frac{1}{2}-\nu}(\zeta). \tag{6.10}$$

The last term in this relation is *regular on the cut* $\zeta \in [-1, 1]$ and therefore does not contribute to the commutator. By consequence the two theories represent the same algebra of local observables at short distances. But since the last term in the latter relation grows the faster the larger is $|\nu|$ (see [B] Eqs. (3.9.2)), the two theories drastically differ by their long range behaviors.

Let us discuss for now the positive definiteness of the Wightman function (6.8). To this end, following the remarks in Section 5 we can consider the restriction

$$W_{M,v,v'}(\mathbf{z}, \mathbf{z}') = W_\nu \left(\mathrm{ch}\,(v - v') - \frac{1}{2} \exp(v + v')(\mathbf{z} - \mathbf{z}')^2 \right) \tag{6.11}$$

of the two-point function W_ν to $\mathcal{M}_v \times \mathcal{M}_{v'}$ defined for v, v' real and $\mathbf{z} = \mathbf{x} + i\mathbf{y}$, $\mathbf{z}' = \mathbf{x}' + i\mathbf{y}'$ such that $\mathbf{y}^2 > 0$, $\mathbf{y}^0 < 0$, $\mathbf{y}'^2 > 0$ and $\mathbf{y}'^0 > 0$ (see eq. (5.5)). The results of Section 5 imply that this restriction defines a local and (Poincaré) covariant two-point function which satisfies the condition of positivity of the energy spectrum [SW].

Let us consider now the following Hankel-type transform of $W_{M,v,v'}(\mathbf{z}, \mathbf{z}')$ w.r.t. the brane coordinate v':

$$H(\lambda', v, (\mathbf{z} - \mathbf{z}')^2) = \int \mathrm{d}v'\, e^{(d-3)v'} \theta_{\lambda'}(v') W_{M,v,v'}(\mathbf{z}, \mathbf{z}') \tag{6.12}$$

with

$$\theta_\lambda(v) = \frac{1}{\sqrt{2}} e^{-\frac{d-1}{2}v} J_\nu\left(\sqrt{\lambda}\, e^{-v}\right) . \tag{6.13}$$

Inversion gives

$$\begin{aligned} W_\nu(z(v, \mathbf{z}), z'(v', \mathbf{z})) \quad = \quad & \frac{1}{2(2\pi)^{\frac{d-2}{2}}} \int_0^\infty \frac{\mathrm{d}\lambda}{2}\, \lambda^{\frac{d-3}{4}} e^{-\frac{d-1}{2}(v+v')} \\ & J_\nu(\sqrt{\lambda}\, e^{-v}) J_\nu(\sqrt{\lambda}\, e^{-v'})\ \delta^{\frac{d-3}{2}} K_{\frac{d-3}{2}}(\sqrt{\lambda}\ \delta) \end{aligned} \tag{6.14}$$

where $\delta^2 = -(\mathbf{z} - \mathbf{z}')^2$ (see [B2], p. 64, formula (12), and [BBGMS] for more details).

Eq. (6.14) together with Lemma 4.1 show that the Wightman functions 6.8 are positive-definite.

6.2 Källén-Lehmann-type representations

We postpone to a further paper the proof of Källén-Lehmann-type representations which will be based on an appropriate Laplace-type transformation, in a spirit similar to that given in [BM] for the case of two-point functions on de Sitter spacetime.

For two-point functions of general fields on $\widetilde{X}_d$:

$$W(z_1, z_2) = \int_{-\frac{d+1}{2}}^{=\infty} \rho(\lambda)\ W_{\lambda+\frac{d-1}{2}}(z_1, z_2)\ d\lambda, \tag{6.15}$$

For two-point functions of general fields on X_d:

$$W(z_1, z_2) = \sum_{\ell > -\frac{d+1}{2}} \rho_\ell\ W_{\ell+\frac{d-1}{2}}(z_1, z_2). \tag{6.16}$$

In these equations, $\rho(\lambda)$ (resp. $\{\rho_\ell\}$) represents a positive measure (resp. sequence), as a consequence of the positive-definiteness property.

7 Bisognano-Wichmann analyticity and "AdS-Unruh effect"

In this section we prove that our assumptions of locality and tempered spectral condition imply a KMS-type analyticity property of the Wightman functions in the complexified orbits of one-parameter Lorentz subgroups of G_0: the physical interpretation of this property can be called an "AdS-Unruh effect", since the relevant hyperbolic-type orbits we are considering are trajectories of uniformly accelerated motions. In fact, this property is similar to the analyticity property in the complexified orbits of the Lorentz boosts proved by Bisognano and Wichmann [BW] in Minkowskian theory. However, in contrast with the method used in [BW], our method does not rely at all on the positivity properties of the theory but only makes use of an analytic completion procedure of geometrical type which has been presented in [BEM] for the case of de Sitter spacetime (noting there that it applied similarly to the Minkowskian case) and which can be adapted in a straightforward way to the AdS case as we show below. Throughout this section, the distinction between the cases of pure AdS and covering of AdS will be irrelevant because all the regions and corresponding analyticity properties considered will always take place in a *single sheet* of $\widetilde{X}_d$; so we can speak here of AdS without caring about that distinction; in particular the notation $\mathcal{Z}_{n\pm}$ will denote here as well the corresponding tuboids $\widetilde{\mathcal{Z}}_{n\pm}$ if the spacetime considered is $\widetilde{X}_d$.

To be specific, let us consider once for all the Lorentz subgroup of G_0 whose orbits are parallel to the two-plane of coordinates x_0, x_1, with the notations of Eqs (2.7),(2.8) (all the other Lorentz subgroups of G_0 acting on AdS being conjugate of the latter by the action of G_0 itself). The following "wedge-shaped" region of AdS, which is invariant under that subgroup is then distinguished:

$$W_R(e_d) = \{x \in X_d\ : x^1 > |x^0|,\ x^d > 0\}\ . \tag{7.1}$$

(Note that this region of AdS admits the base point e_d as an exposed boundary point). Considering n-point test-functions $f_n \in \mathcal{B}_n$ and Lorentz transformations $[\lambda] = [e^s] \in \mathbf{R} \setminus \{0\}$ we abbreviate $f_{n\{[\lambda]\}}$ into $f_{n\lambda}$, i.e.

$$f_{n\lambda}(x_1, \dots,\ x_n) = f_n([\lambda^{-1}]x_1, \dots,\ [\lambda^{-1}]x_n). \tag{7.2}$$

Then the main result of this section is the following

Theorem 7.1 *If a set of Wightman functions satisfies the locality and tempered spectral conditions, then for all $m,\ n \in \mathbf{N}$, $f_m \in \mathcal{D}(W_R(e_d)^m)$ and $g_n \in \mathcal{D}(W_R(e_d)^n)$, there is a function $G_{(f_m,g_n)}$ holomorphic on $\mathbf{C} \setminus \mathbf{R}_+$ with continuous boundary values $G^{\pm}_{(f_m,g_n)}$ on $\mathbf{R}_+ \setminus \{0\}$ from the upper and lower half-planes such that, for all $\lambda \in \mathbf{R}_+$,*

$$G^+_{(f_m,g_n)}(\lambda) = \langle \mathcal{W}_{m+n},\ f_m \otimes g_{n\lambda}\rangle, \qquad G^-_{(f_m,g_n)}(\lambda) = \langle \mathcal{W}_{m+n},\ g_{n\lambda} \otimes f_m\rangle\ . \tag{7.3}$$

The thermal interpretation of this theorem could be presented with all the details of Theorem 3 of [BEM], but here we shall only dwell on the following remarks, specific to the geometry of AdS.

Consider an AdS spacetime with radius R (i.e. with equation $(x,x) = R^2$ and base point $R\ e_d = (0,\ldots,0,\ R)$) and assume that all the test-functions $f_m,\ g_n$ considered in the statement of Theorem 7.1 have their supports contained respectively in sets of the form $\mathcal{V}_a^m,\ \mathcal{V}_a^n$, where $\mathcal{V}_a$ is a neighbourhood in X_d of a certain point $x(a)$ in the wedge $W_R(e_d)$ which we take of the following form:

$$x(a) = (0,\ Ra,\ 0\ldots,0,\ R\sqrt{a^2+1}), \tag{7.4}$$

a being a positive number. The Lorentzian orbits followed by the points in the supports of f_m and g_n are in a neighbourhood of the orbit of $x(a)$ which is the branch of hyperbola with equations

$$[e^{\frac{t}{Ra}}]x(a) = (Ra\,\mathrm{sh}\,\frac{t}{Ra},\ Ra\,\mathrm{ch}\,\frac{t}{Ra},\ 0\ldots,0,\ R\sqrt{a^2+1})\ . \tag{7.5}$$

In the latter, the normalization of the group parameter $\frac{t}{Ra}$ has been chosen such that t is the *proper time* along the trajectory of $x(a)$ with respect to the metric of the AdS spacetime of radius R. Expressed in this parameter t such that $\lambda = e^{\frac{t}{Ra}}$, the analyticity property stated in the theorem corresponds to the analyticity in a KMS-type domain, namely a β-periodic cut-plane generated by the strip $\{t;\ 0 < \mathrm{Im}\,t < \beta\}$, with the period $i\beta = 2i\pi\ aR$ (accompanied by the relevant conditions of KMS-type for the boundary values at t real and $t - i\beta$ real in terms of the products of field-observables). Thus an observer living on the trajectory (7.5), and whose measurements are therefore supposed to be performed in terms of field observables whose support at $t = 0$ lies in the neighbourhood $\mathcal{V}_a$ of $x(a)$, will perceive a thermal bath of particles at the temperature $T = \beta^{-1} = \frac{1}{2\pi\ Ra}$. In view of lemma 2.1, the motion of such an observer is uniformly accelerated and its acceleration a is related to the "radius" Ra of the hyperbolic trajectory by the following formula (noting that the parameter c^2 of lemma 2.1 is $c^2 = a^2+1$): $\mathrm{a} = \frac{1}{R}\ \sqrt{\frac{c^2}{c^2-1}} = \frac{1}{R}\ \sqrt{1+\frac{1}{a^2}}$. Therefore one has:

Lemma 7.1 *the temperature T perceived by an AdS-Unruh observer living on the trajectory described by (7.5) is given in terms of his (or her) acceleration* a *by*

$$T = \frac{1}{2\pi}\sqrt{\mathrm{a}^2 - \frac{1}{R^2}}. \tag{7.6}$$

The proof of Theorem 7.1 contains two steps (as the similar Theorem 2 of [BEM]). In subsection 7.1 we introduce special sets of points in X_d^n which play the role of the Jost points of the Minkowskian case, because all the complex Lorentz transformations $[\lambda]$ transport them into $\mathcal{Z}_{n+} \cup \mathcal{Z}_{n-}$. This is the starting point of a standard analytic completion procedure which generates the analyticity of the Wightman functions in domains obtained by the action of all complex Lorentz transformations on the tuboids $\mathcal{Z}_{n+}$ (or $\mathcal{Z}_{n-}$) as in Lemma 2 of [BEM]. Here also, there is (for each pair (m,n) considered in the statement of Theorem 7.1) a quartet of Wightman functions analytic respectively in the tuboids $\mathcal{Z}_{n\pm}$ and in two other tuboids $\mathcal{Z}'_{n\pm}$ which are involved in that procedure. This is fully explained in subsection 7.2.

7.1 A special set of Jost points

Let $a \in X_d$ be given by

$$a = (0,\ \mathrm{sh}\,u,\ \vec{0},\ \mathrm{ch}\,u), \quad u > 0\ . \tag{7.7}$$

With the notations of (2.7) and (2.8), and real $s \in (0,\ \pi)$,

$$[e^{is}]\,a = e^{isM_{10}}\,a = (i\mathrm{sh}\,u\sin s,\ \mathrm{sh}\,u\cos s,\ \vec{0},\ \mathrm{ch}\,u)\ . \tag{7.8}$$

This can be reexpressed as $\exp(itM_{0d})b$ with $b = (0,\ \mathrm{sh}\, u \cos s,\ \vec{0},\ r)$, i.e.

$$\begin{aligned} [e^{is}]\, a &= (ir\mathrm{sh}\, t,\ \mathrm{sh}\, u \cos s,\ \vec{0},\ r\mathrm{ch}\, t), \\ \mathrm{th}\, t &= \mathrm{th}\, u \sin s, \\ r &= \sqrt{(\mathrm{ch}\, u)^2 - (\mathrm{sh}\, u)^2 (\sin s)^2} \end{aligned} \tag{7.9}$$

Let $\mathcal{J}_{n0}$ be the set of points $(a_1, \dots, a_n) \in X_d^n$ such that

$$\begin{aligned} &a_j = (0,\ \mathrm{sh}\, u_j,\ \vec{0},\ \mathrm{ch}\, u_j), \quad j = 1, \dots, n, \\ &0 < u_1 < \dots < u_n\ . \end{aligned} \tag{7.10}$$

Let $(a_1, \dots, a_n) \in X_d^n$ be of the form (7.10). Then for $0 < s < \pi$,

$$\begin{aligned} &[e^{is}]\ a_j = \exp(it_j M_{0d}) b_j, \\ &\mathrm{th}\, t_j = \mathrm{th}\, u_j \sin s, \quad j = 1, \dots, n, \\ &0 < t_1 < \dots < t_n\ , \end{aligned} \tag{7.11}$$

with $b_j \in X_d$. Since t_j can be rewritten as $\theta_1 + \dots + \theta_j$ with $\theta_k > 0$, the point $[e^{is}]a$ is in $\mathcal{Z}_{n+}$. By the invariance of $\mathcal{Z}_{n+}$ under G_0 (resp. $\widetilde{G}_0$) it follows that $[\lambda]a \in \mathcal{Z}_{n+}$ for all $\lambda \in \mathbf{C}_+$. Obviously $[\lambda]a \in \mathcal{Z}_{n-}$ for all $\lambda \in \mathbf{C}_-$, since $\mathcal{Z}_{n-} = \mathcal{Z}_{n+}^*$.

7.2 Derivation of Bisognano-Wichmann analyticity

This subsection closely follows the treatment given in [BEM]. We refer the reader to that reference for more details. Let

$$W_R = \{x \in E_{d+1}\ :\ x^1 > |x^0|\}\ . \tag{7.12}$$

If $x \in W_R \cap X_d$ and $x^d > 0$, we denote (consistently with the notation $W_R(e_d)$ introduced in (7.1))

$$W_R(x) = \{y \in X_d\ : y - x \in W_R, \quad y^d > 0\}\ . \tag{7.13}$$

Let

$$\mathcal{K}_{n+} = \{(x_1, \dots, x_n) \in X_d^n\ :\ x_1 \in W_R(e_d), \quad x_j \in W_R(x_{j-1}) \quad \forall j = 2, \dots, n\}\ . \tag{7.14}$$

$\mathcal{K}_{n-}$ is defined by reflecting $\mathcal{K}_{n+}$ across the hyperplane $\{x \in E_{d+1}\ :\ x^1 = 0\}$ or equivalently $\mathcal{K}_{n-} = [-1]\mathcal{K}_{n+}$ with $[-1] = [e^{i\pi}]$ as in (2.8). If $x \in \mathcal{K}_{n+}$, then x_j and x_k are space-like separated whenever $1 \le j < k \le n$. Note that $\mathcal{J}_{n0} \subset \mathcal{K}_n$. If $(z_1, \dots, z_n)$ belongs to X_d^n or $X_d^{(c)n}$, we denote $z_{\leftarrow} = (z_n, \dots, z_1)$.

Besides $\mathcal{Z}_{n+}$ and $\mathcal{Z}_{n-}$ we shall also use the other two tuboids

$$\begin{aligned} \mathcal{Z}'_{n+} &= \{z \in X_d^{(c)n}\ :\ z_{\leftarrow} \in \mathcal{Z}_{n-}\}, \\ \mathcal{Z}'_{n-} &= \{z \in X_d^{(c)n}\ :\ z_{\leftarrow} \in \mathcal{Z}_{n+}\} = {\mathcal{Z}'_{n+}}^*\ . \end{aligned} \tag{7.15}$$

We fix $m \ge 0$, $n \ge 1$ and a function $f_m \in \mathcal{D}(X_d^m)$ with support in $W_R(e_d)^m$. There exist two functions $z \mapsto F_+(f_m;\ z)$ and $z \mapsto F_-(f_m;\ z)$, respectively holomorphic in $\mathcal{Z}_{n+}$ and $\mathcal{Z}_{n-} = \mathcal{Z}_{n+}^*$, having boundary values $F_{\pm}^{(b)}(f_m;\ x)$ on X_d^n (resp. $\widetilde{X}_d^n$) in the sense of distributions, such that for every $g \in \mathcal{B}_n$ with compact support,

$$\begin{aligned} &\int_{X_d^n} F_+^{(b)}(f_m;\ x_1, \dots,\ x_n)\, g_n(x_1, \dots,\ x_n)\, d\sigma(x_1) \dots d\sigma(x_n) = \\ &\int_{X_d^{m+n}} \mathcal{W}_{m+n}(w_1, \dots,\ w_m,\ x_1, \dots,\ x_n) f_m(w_1, \dots,\ w_m)\, g_n(x_1, \dots,\ x_n) \\ &\qquad d\sigma(w_1) \dots d\sigma(w_m) d\sigma(x_1) \dots d\sigma(x_n)\ , \end{aligned} \tag{7.16}$$

$$\int_{X_d^n} F_-^{(b)}(f_m;\ x_1,\dots,\ x_n)\,g_n(x_1,\dots,\ x_n)\,d\sigma(x_1)\dots d\sigma(x_n) =$$
$$\int_{X_d^{m+n}} \mathcal{W}_{m+n}(x_n,\dots,\ x_1,\ w_1,\dots,\ w_m)\,f_m(w_1,\dots,\ w_m)\,g_n(x_1,\dots,\ x_n)$$
$$d\sigma(w_1)\dots d\sigma(w_m)d\sigma(x_1)\dots d\sigma(x_n)\ , \tag{7.17}$$

(with $\widetilde{X}_d$ instead of X_d when needed). The functions $z \mapsto F'_+(f_m;\ z) = F_-(f_m;\ z_\leftarrow)$ and $z \mapsto F'_-(f_m;\ z) = F_+(f_m;\ z_\leftarrow)$ are respectively holomorphic in $\mathcal{Z}'_{n_+}$ and $\mathcal{Z}'_{n_-}$. Their boundary values at real points, in the sense of distributions, are ${F'_+}^{(b)}(f_m;\ x) = F_-^{(b)}(f_m;\ x_\leftarrow)$ and ${F'_-}^{(b)}(f_m;\ x) = F_+^{(b)}(f_m;\ x_\leftarrow)$.

In the sense of distributions, $F_+^{(b)}(f_m;\ x)$ and $F_-^{(b)}(f_m;\ x)$ coincide for $x \in \mathcal{K}_{n-}$ by virtue of local commutativity. Hence, by the edge-of-the-wedge theorem, $z \mapsto F_+(f_m;\ z)$ and $z \mapsto F_-(f_m;\ z)$ have a common holomorphic extension $z \mapsto F(f_m;\ z)$ in $\Delta_n = \mathcal{Z}_{n+} \cup \mathcal{Z}_{n-} \cup \mathcal{V}_n$, where $\mathcal{V}_n$ is a complex neighborhood of $\mathcal{K}_{n-}$ such that $[\lambda]\mathcal{V}_n = \mathcal{V}_n$ for all $\lambda > 0$. Let $a \in \mathcal{J}_{n0}$. As we have noted $[e^{is}]a \in \mathcal{Z}_{n+}$ for all $s \in (0,\ \pi)$. Moreover $[e^{i\pi}]a$ is in $\mathcal{V}_n$. Hence there is an $\varepsilon > 0$ (depending on a) such that $[e^{is}]a$ is in Δ_n for all $s \in (0,\ \pi+\varepsilon)$. This also means that, denoting $a' = [e^{i\varepsilon/2}]a \in \mathcal{Z}_{n+}$, all points of the compact set $\{z = [e^{is}]a'\ :\ 0 \le s \le \pi\}$ belong to Δ_n. Since the latter is open, there exists a $\rho_a > 0$ such that the open ball $B_{a'}(\rho_a) = \{z \in X_d^{(c)n}\ :\ ||z - a'|| < \rho_a\}$ is contained in $\mathcal{Z}_{n+}$, and $[e^{is}]B_{a'}(\rho_a) \subset \Delta_n$ for all $s \in [0,\ \pi]$ hence $[\lambda]B_{a'}(\rho_a) \subset \Delta_n$ for all $\lambda \in \overline{\mathbf{C}_+}\setminus\{0\}$. The function $(z,\ \lambda) \mapsto G(f_m;\ z,\ \lambda) = F(f_m;\ [\lambda]z)$ is holomorphic in

$$\{(z,\ \rho e^{i\theta}) \in \mathcal{Z}_{n+} \times \mathbf{C}\ :\ \rho > 0,\quad |\sin\theta| < \alpha(z)\}, \tag{7.18}$$

where $\alpha(z) > 0$ for all $z \in \mathcal{Z}_{n+}$. Moreover G is also holomorphic in $B_{a'}(\rho_a) \times \mathbf{C}_+$. By Lemma 3 of [BEM] (Appendix A), G extends to a function holomorphic in $\mathcal{Z}_{n+} \times \mathbf{C}_+$. However we wish to prove that the boundary values of G as z tends to the reals (in the sense of distributions) are also analytic in λ in $\mathbf{C}_+$. As a consequence of the temperedness assumption, for real λ, $G(f_m;\ z,\ \lambda)$ defines a holomorphic function of tempered behavior in $\mathcal{Z}_{n+}$ with values in the functions of λ bounded by some power of $(|\lambda| + |\lambda|^{-1})$. If $z \in B_{a'}(\rho_a)$, $\lambda \mapsto G(f_m;\ z,\ \lambda)$ extends to a function analytic in $\mathbf{C}_+$ and also bounded there by some power of $(|\lambda| + |\lambda|^{-1})$. Let $\varphi(t) = \int \widetilde{\varphi}(p)\,e^{-itp}\,dp$, where $\widetilde{\varphi}$ is a $\mathcal{C}^\infty$ function with compact support contained in $(-\infty,\ 0)$. For $z \in B_{a'}(\rho_a)$, and sufficiently large $N \in \mathbf{N}$,

$$\int_{\mathbf{R}} \varphi(\lambda)\,\lambda^N\,G(f_m;\ z,\ \lambda)\,d\lambda = 0\ . \tag{7.19}$$

The l.h.s of this equation is holomorphic and of tempered behavior in $\mathcal{Z}_{n+}$, hence it vanishes together with its boundary values. Therefore the boundary value $G^{(b)}(f_m;\ x,\ \lambda)$ extends to a function of λ holomorphic in $\mathbf{C}_+$. We note that $G^{(b)}(f_m;\ x,\ \lambda) = F_+^{(b)}(f_m;\ [\lambda]x)$ for $\lambda > 0$ and $G^{(b)}(f_m;\ x,\ \lambda) = F_-^{(b)}(f_m;\ [\lambda]x)$ for$\lambda < 0$.

In the same way $F'_+(f_m;\ z)$ and $F'_-(f_m;\ z)$ have a common extension $F'(f_m;\ z)$ holomorphic in $\Delta'_n = \mathcal{Z}'_{n_+} \cup \mathcal{Z}'_{n_-} \cup \mathcal{V}'_n$ with $\mathcal{V}'_n = \{x\ :\ x_\leftarrow \in \mathcal{V}_n\}$. Note that the domain Δ'_n is equal to $\{z\ :\ z^*_\leftarrow \in \Delta_n\}$. Hence $G'(f_m;\ z,\ \lambda) = F'(f_m;\ [\lambda]z)$ extends to a function holomorphic in $\mathcal{Z}'_{n_+} \times \mathbf{C}_-$, and its boundary value ${G'}^{(b)}(f_m;\ x,\ \lambda)$ has properties that mirror those of $G^{(b)}(f_m;\ x,\ \lambda)$. Finally suppose that $x \in W_R(e_d)^n$. Then, by local commutativity, for $\lambda < 0$, $F_+^{(b)}(f_m;\ [\lambda]x)$ coincides with $F_-^{(b)}(f_m;\ [\lambda]x_\leftarrow) = {F'}_+^{(b)}(f_m;\ [\lambda]x)$. Hence if $x \in W_R(e_d)^n$, $\lambda \mapsto G^{(b)}(f_m;\ x,\ \lambda)$ and $\lambda \mapsto {G'}^{(b)}(f_m;\ x,\ \lambda)$ have a common analytic continuation in $\mathbf{C} \setminus \mathbf{R}_+$. This ends the proof of Theorem 7.1.

Remark 7.1 The tuboids $\mathcal{Z}_{n\pm}$ and $\mathcal{Z}'_{n\pm}$ can be replaced in the above discussion by $\mathcal{T}_{n\pm}$ and $\mathcal{T}'_{n\pm}$ (similarly defined).

7.3 "CTP"

In the proof of the preceding subsection, if we assume that f_m has support in the left, instead of the right wedge, we find that $(z,\ \lambda) \mapsto G(f_m;\ z,\ \lambda)$ extends to a function holomorphic in $\mathcal{T}_{n+} \times \mathbf{C}_-$ instead of $\mathcal{T}_{n+} \times \mathbf{C}_+$. Thus in case $m = 0$ it is easy to obtain the following special case of the Glaser-Streater theorem:

Lemma 7.2 *If a set of Wightman functions satisfies the locality and tempered spectral conditions, then for all integer $n \geq 1$, there exists a function $(z,\ \lambda) \mapsto G_n(z,\ \lambda)$, holomorphic and of tempered growth in $\mathcal{T}_{n+} \times (\mathbf{C} \setminus \{0\})$, such that*

$$\begin{aligned} G_n(z_1, \ldots,\ z_n,\ \lambda) &= \mathcal{W}_n([\lambda]z_1, \ldots,\ [\lambda]z_n) \quad \text{for } \lambda > 0, \\ G_n(z_1, \ldots,\ z_n,\ \lambda) &= \mathcal{W}_n([\lambda]z_n, \ldots,\ [\lambda]z_1) \quad \text{for } \lambda < 0. \end{aligned} \tag{7.20}$$

If we now assume the covariance condition (4.5) holds, then G_n is actually independent of λ and we obtain

Lemma 7.3 *If a set of Wightman functions satisfies the locality, covariance, and tempered spectral conditions, then for all integer $n \geq 1$, and all $z \in \mathcal{T}_{n+}$,*

$$\mathcal{W}_n(z_1, \ldots,\ z_n) = \mathcal{W}_n([-1]z_n, \ldots,\ [-1]z_1). \tag{7.21}$$

If positivity holds this implies, as usual, the existence of an anti-unitary operator θ such that $\theta\phi(x)\theta^{-1} = \phi([-1]x)^*$. In [BFS], the existence of this operator is a non-trivial step in the derivation of commutativity for opposite wedges. Note that the above proof of Lemma 7.3, also valid in Minkowski space, does not require the Bargmann-Hall-Wightman Lemma.

8 Wick rotations and Osterwalder-Schrader reconstruction on the covering of AdS

In this section, we consider QFT's on the *covering* of AdS which satisfy the *tempered spectral condition* and we wish to show that such theories can be formulated equivalently in terms of theories on the "Euclidean" AdS spacetime $X_d^{(\mathcal{E})}$ in a way which is reminiscent of the Wick rotation in complex Minkowski space. In fact, the simple geometrical fact which allows the latter to hold is the property of the tuboids $\widetilde{\mathcal{Z}}_{n+}$ described in lemma 3.11 and obtained by making use of the complexification in τ of the map

$$(\tau,\ \vec{x}) \mapsto \widetilde{\chi}(\tau,\ \vec{x}) = \exp(\tau M_{0d})(0,\ \vec{x},\ \sqrt{\vec{x}^2 + 1}) \tag{8.1}$$

of $\mathbf{R} \times \mathbf{R}^{d-1}$ into $\widetilde{X}_d$. As a matter of fact, making τ and $\vec{x}$ complex yield charts for a certain complexification denoted $[\widetilde{X}_d]^{(c)}$ of $\widetilde{X}_d$: it is defined as the image of the extension of $\widetilde{\chi}$ to the set $\{\tau \in \mathbf{C}\} \times \{\vec{z} \in \mathbf{C}^{d-2}\ :\ \vec{z}^2 + 1 \notin \mathbf{R}_-\}$. (Of course this set $[\widetilde{X}_d]^{(c)}$ is the covering of a complex region which is not the full space $X_d^{(c)}$, since $\widetilde{X}_d^{(c)} \equiv X_d^{(c)}$).

In the sequel we omit the symbol $\widetilde{\chi}$ and identify a point $(\tau,\ \vec{x})$ with its image $\widetilde{\chi}(\tau,\ \vec{x})$. According to lemma 3.11, all points $z = ((\tau_1,\ \vec{x}_1), \ldots,\ (\tau_n,\ \vec{x}_n)) \in \left([\widetilde{X}_d]^{(c)}\right)^n$ such that $\vec{x}_j$ is real and $0 < \mathrm{Im}\,\tau_1 < \cdots < \mathrm{Im}\,\tau_n$, are in $\widetilde{\mathcal{Z}}_{n+}$. Therefore (using the global invariance under $\widetilde{G}_0$)

$$\mathcal{W}_n((\tau_1,\ \vec{x}_1), \ldots,\ (\tau_n,\ \vec{x}_n)) \tag{8.2}$$

extends to a function of $(\tau_1, \ldots, \tau_n)$, holomorphic in the tube $\{(\tau_1, \ldots, \tau_n)\ : \mathrm{Im}\,\tau_1 < \ldots < \mathrm{Im}\,\tau_n\}$, of tempered growth at infinity and at the boundaries, with values in the tempered distributions (in fact the polynomially bounded $\mathcal{C}^\infty$ functions) in $\vec{x}_1, \ldots, \vec{x}_n$. (This function depends only on the differences of the τ_j variables, but this does not, of course, hold for the $\vec{x}_j$.) Together with the other permuted Wightman functions, this defines a function of $(\tau_1, \ldots, \tau_n)$, holomorphic in $\{(\tau_1, \ldots, \tau_n)\ : \mathrm{Im}\,(\tau_j - \tau_k) \neq 0\ \ \forall j \neq k\}$.

We denote

$$S_n(s_1,\ \vec{x}_1, \ldots,\ s_n,\ \vec{x}_n) = \mathcal{W}_n((is_{\pi(1)},\ \vec{x}_{\pi(1)}), \ldots,\ (is_{\pi(n)},\ \vec{x}_{\pi(n)})) \tag{8.3}$$

for real $s_1, \ldots, s_n$ such that $s_{\pi(1)} < \cdots < s_{\pi(n)}$, for all permutations π of $(1, \ldots, n)$. Standard analytic completions, using the edge-of-the-wedge theorem and Bremermann's Continuity Theorem, show that S_n is actually analytic at all non-coinciding points of $\mathbf{R}^{nd}$. Geometrically, in view of the representation (2.3) of $X_d^{(\mathcal{E})}$, this means that *the functions $S_n(z_1, \ldots, z_n)$ appear as n-point Schwinger functions defined at all noncoinciding points of the "Euclidean" AdS spacetime $X_d^{(\mathcal{E})}$*, namely

$$\{(z_1, \ldots, z_n) \in X_d^{(\mathcal{E})n};\ z_i \neq z_j \text{ for all } i, j,\ i \neq j\} \tag{8.4}$$

By construction, $S_n(s_1,\ \vec{x}_1, \ldots,\ s_n,\ \vec{x}_n) = S_n(s_{\pi(1)},\ \vec{x}_{\pi(1)}, \ldots,\ s_{\pi(n)},\ \vec{x}_{\pi(n)})$ for every permutation π. In case the positive definiteness condition holds, the sequence $\{S_n\}$ has the Osterwalder-Schrader positivity property: if $f_0 \in \mathbf{C}$ and, for $1 \leq n \leq N$, $f_n \in \mathcal{S}(\mathbf{R}^{nd})$ has its support in $\{(s_1,\ \vec{x}_1, \ldots,\ s_n,\ \vec{x}_n)\ :\ 0 < s_1 < \cdots < s_n\}$, then, with the convention $S_0 = 1$,

$$\begin{aligned} \sum_{m,n=0}^{N} \int \overline{f_m(s'_1,\ \vec{x}'_1, \ldots,\ s'_m,\ \vec{x}'_m)}\, f_n(s_1,\ \vec{x}_1, \ldots,\ s_n,\ \vec{x}_n) \\ S_{m+n}(-s'_m,\ \vec{x}'_m, \ldots, -s'_1,\ \vec{x}'_1, s_1,\ \vec{x}_1, \ldots,\ s_n,\ \vec{x}_n) \\ ds'_1\, d\vec{x}'_1 \ldots ds_n\, d\vec{x}_n \geq 0\ . \end{aligned} \tag{8.5}$$

This follows from the existence of the vector-valued holomorphic functions Φ_n (see Sec. 4). Note that the analytic completion mentioned above applies to these vector-valued functions.

Conversely, let $\{S_n\}$ be a sequence of functions defined at all noncoinciding points of $X_d^{(\mathcal{E})}$, namely on the images of the sets $\{((s_1,\ \vec{x}_1), \ldots, (s_n,\ \vec{x}_n)) \in \mathbf{R}^{nd}\ :\ s_j \neq s_k\ \vec{x}_j \neq \vec{x}_k\ \ \forall\, j \neq k\}$, symmetric, invariant under a common translation of the variables s_j, and satisfying the Osterwalder-Schrader positivity property (8.5) when the supports of the $\{f_n\}$ are as above. Then it is possible, by the same method as in the "flat" case ([OS1, OS2, G2]), to construct a Hilbert space $\mathcal{H}$, $\Omega \in \mathcal{H}$, and a sequence of functions $\{\Phi_n\}_{n\in\mathbf{N}}$ with values in $\mathcal{H}$ on $\{((\tau_1,\ \vec{x}_1), \ldots, (\tau_n,\ \vec{x}_n))\ :\ 0 < \operatorname{Im}\tau_1 < \cdots < \operatorname{Im}\tau_n,\ \ \vec{x}_j \in \mathbf{R}^{d-1}\}$, holomorphic in the τ_j and $\mathcal{C}^\infty$ in the $\vec{x}_j$, such that

$$\begin{aligned} S_{m+n}(-s'_m,\ \vec{x}'_m, \ldots - s'_1,\ \vec{x}'_1, s_1,\ \vec{x}_1, \ldots,\ s_n,\ \vec{x}_n) = \\ (\Phi_m((is'_1,\ \vec{x}'_1), \ldots, (is'_m,\ \vec{x}'_m)),\ \Phi_n((is_1,\ \vec{x}_1), \ldots, (is_n,\ \vec{x}_n))) \end{aligned} \tag{8.6}$$

whenever $0 < s'_1 < \cdots < s'_m$ and $0 < s_1 < \cdots < s_n$. The temperedness of the Φ_n at the real points, hence the existence of boundary values, can be obtained if, as we shall assume, the sequence $\{S_n\}$ satisfies suitable growth conditions, as in [OS1, OS2, G2]. Note that the boundary values of the scalar products of the Φ_n, namely the distributions $\mathcal{W}_n$ thus obtained are automatically defined on the covering space $\widetilde{X}_d^n$ of X_d^n, since the analytic continuation in the complex variables $\tau_j \in \mathbf{C}$ corresponds to travelling in the covering space $\left([\widetilde{X}_d]^{(c)}\right)^n$ (no periodicity condition in the variables τ_j being produced in general).

If the functions $\widetilde{S}_n$ are invariant under the transformations of $\widetilde{G}_0^{(c)}$ which preserve $\{z \in \widetilde{X}_d^{(c)n}\ :\ z^0 \in i\mathbf{R}^n,\ \ \vec{z} \in \mathbf{R}^{n(d-1)}\}$, then differential operators representing $\mathcal{G}$ can be used as in [OS1] to show that the $\mathcal{W}_n$ are invariant under $\widetilde{G}_0$. Finally the analyticity of the $\mathcal{W}_n$ shows that the operator $\widehat{M_{0d}}$ representing M_{0d} in $\mathcal{H}$ has positive spectrum and so does the representative $\widehat{M}$ of any $M \in \mathcal{C}_1$ by invariance under $\widetilde{G}_0$.

9 Final remarks

First of all, we wish to emphasize the peculiar thermal aspects of "generic" field theories on AdS or its covering, which are strongly related to the geometry of the AdS quadric, namely to the existence of uniformly accelerated motions on three types of trajectories: while the elliptic and parabolic observers perceive a world at zero temperature (satisfying a condition of energy-positivity) in spite of the fact that the acceleration can take all values between zero and $\frac{1}{R}$, the hyperbolic observers perceive an Unruh-type temperature effect growing from zero to infinity when the acceleration grows from $\frac{1}{R}$ to infinity.

Our subsequent remarks concern some important peculiarities of QFT's on the *pure* AdS spacetime with respect to those on its covering, which appeared in the study of general two-point functions (see Section 6), namely:

i) The vanishing of the commutator vacuum expectation value in the region of spacelike separation $(x_1 - x_2)^2 < 0$ implies its vanishing in the region of time-like separation $(x_1 + x_2)^2 < 0$, which we called the "region of exotic causal separation",

ii) the periodicity condition on time-like geodesics implies that the two-point function is a holomorphic function of (z_1, z_2) in the domain $\mathbf{C}\setminus[-1, +1]$ itself instead of its covering. In the free-field case, this selects the Legendre functions $Q_\lambda^{(d+1)}((z_1, z_2))$ for which $\lambda = \ell$ integer, which results in a quantization of the mass parameter $m^2 = \ell(\ell + d - 1)$. More generally all the fields on the pure AdS spacetime have two-point functions which are (positive) superpositions of the free-field functions $Q_\ell^{(d+1)}((z_1, z_2))$ with integer indices ℓ.

These peculiarities suggest respectively the following questions which may be linked together and deserve to be studied in further works.

a) Does local commutativity, formulated as the vanishing of the commutator in the region of spacelike separation implies its vanishing in the region of exotic causal separation (or "exotic causality") ?

b) Do the axioms of Section 4 applied to the pure AdS spacetime exclude the existence of non-trivial interacting QFT ?

It has in fact been noted in [BFS] that the region of spacelike separation does not really deserve that name in the pure AdS case, since pairs of points in that region can also be connected by classes of *non-geodesic* timelike paths; as it is stated, the condition of local commutativity therefore appears as a strong constraint which would force the interactions to respect the time-periodicity of the geometry. One can then also say that the condition of exotic locality would represent a constraint of similar nature acting at odd number of half-periods of time, and that is why one could possibly expect it to be a consequence of local commutativity.

We know that, besides generalized free fields, their Wick polynomials are allowed to exist, since their n-point functions, which are combinations of products of two-point functions, do satisfy the required (periodic) analyticity properties in the relevant tubes $\mathcal{T}_{n\pm}$ of the pure AdS spacetime. However, if we think of nontrivial interacting fields in terms of perturbation theory and consider Feynman-type convolution products of free two-point functions involving not only products but integrals on the AdS spacetime, one can show that the constraints of periodic analyticity cannot be maintained in general: in particular the action of retarded propagators can be seen to propagate the interaction in the covering tubes $\widetilde{\mathcal{T}}_{n\pm}$. So one could formulate as a conjecture, that there might be no other QFT's on the pure AdS spacetime than the ones previously mentioned, which would then also entail (as a byproduct) a positive answer to our question a). However, there is at the moment no genuine proof relying on the axioms of Section 4 that such a conjecture is valid.

We end these remarks with a brief comparison of the formalism of this paper with that proposed by [BFS]. A difficulty is of course the usual gap between a formulation based on fields and one based on local algebras. This difficulty is compounded in the AdS case by the lack of a translation group. In [BFS] the pure AdS, and not its covering space, is considered. From the principle of passivity of the vacuum, the authors succeed in deriving the positivity of the energy, the existence of a CTP operator (both in the same sense as here), and the commutativity of operators localized in opposite wedges. If we suppose that, in terms of fields, these results correspond to the tempered spectral, covariance, and positivity conditions, and a portion of local commutativity, then, by Remark 4.1 (Section 4), this implies full local commutativity, and thus all our axioms.

A Appendix. More on two-point functions

A.1 The case of X_d

We begin by considering a complex function w_+ holomorphic and of polynomial growth on $\mathcal{T}_{1-} \times \mathcal{T}_{1+}$. The function w_- defined by $w_-(z_1,\ z_2) = w_+(z_2,\ z_1)$ is holomorphic in $\mathcal{T}_{1+} \times \mathcal{T}_{1-}$. We denote $w_\pm^{(b)}$ the boundary value of $w_\pm$ on X_d in the sense of tempered distributions. We suppose that $w_+^{(b)}$ and $w_-^{(b)}$ coincide on the real open subset $\mathcal{R}_2$ of X_d^2 defined by

$$\mathcal{R}_2 = \{x \in X_d^2\ :\ (x_1 - x_2,\ x_1 - x_2) < 0\} \quad = \{x \in X_d^2\ :\ (x_1, x_2) > 1\}\ . \tag{A.1}$$

Since we are ultimately interested in the case when $w_\pm(z_1,\ z_2) = w_\pm(\Lambda z_1,\ \Lambda z_2)$ for all $\Lambda \in G_0$, we make the simplifying assumption that $w_\pm(x_1 + iy_1,\ x_2 + iy_2)$ has a boundary value which is $\mathcal{C}^\infty$ in x_1 and of tempered growth in $z_2 = x_2 + iy_2$ when y_1 tends to 0 while $z_2 \in \mathcal{T}_{1\pm}$. Actually the general case can be reduced to the simplified one by considering $\int_{G_0} \varphi(\Lambda) w_\pm(\Lambda z_1,\ \Lambda z_2)\, d\Lambda$ for suitable test-functions φ. The functions $f_\pm$ defined by $f_\pm(z) = w_\pm(e_d,\ z)$ are respectively holomorphic and of tempered growth in $\mathcal{T}_{1\pm}$, and have boundary values $f^{(b)}$ on X_d in the sense of tempered distributions. These boundary values coincide in the real open subset $\mathcal{R}$ of X_d defined by

$$\mathcal{R} = \{x \in X_d\ :\ (x - e_d)^2 < 0\} \quad = \{x \in X_d\ :\ x^d > 1\}\ . \tag{A.2}$$

We also define

$$\mathcal{R}' = \{x \in X_d\ :\ (x + e_d)^2 < 0\} \quad = \{x \in X_d\ :\ x^d < -1\}\ . \tag{A.3}$$

Let H (resp. $H^{(c)}$) denote the subgroup of G_0 (resp. $G_0^{(c)}$) which leaves e_d unchanged. This is just the connected real (resp. complex) Lorentz group for the d-dimensional Minkowski space $\Pi_0 = \{x \in E_{d+1}\ :\ x^d = 0\}$ (resp. $\Pi_0^{(c)} = \{z \in E_{d+1}^{(c)}\ :\ z^d = 0\}$). The properties mentioned above for $f_\pm$ are invariant under H. Let

$$\mathcal{T}'_{1,d} = H^{(c)}\, \mathcal{T}_{1+} = H^{(c)}\, \mathcal{T}_{1-}\ . \tag{A.4}$$

The last equality is due to the fact that $I_{01} \in H^{(c)}$, where $I_{01}e_0 = -e_0$, $I_{01}e_1 = -e_1$, $I_{01}e_\mu = e_\mu$ for $1 < \mu \le d$, and that I_{01} is a bijection of $\mathcal{T}_{1+}$ onto $\mathcal{T}_{1-}$. Obviously $\mathcal{T}'_{1,d} = H^{(c)}\, \mathcal{T}'_{1,d}$. We also denote:

$$\Pi_\pm = \{x \in E_{d+1}\ :\ x^d = \pm 1\}, \quad \Pi_\pm^{(c)} = \{z \in E_{d+1}^{(c)}\ :\ z^d = \pm 1\}\ , \tag{A.5}$$

$$Q_\pm = \{z \in E_{d+1}^{(c)}\ :\ (z \mp e_d)^2 = 0\}\ , \tag{A.6}$$

$$Q_0 = Q_\pm \cap \Pi_0^{(c)} = \{Z \in \Pi_0^{(c)}\ :\ (Z, Z) = -1\}\ . \tag{A.7}$$

The intersection of X_d with the light-cone with apex at $-e_d$ is contained in the hyperplane Π_-, in fact

$$X_d^{(c)} \cap Q_- = X_d^{(c)} \cap \Pi_-^{(c)}\ . \tag{A.8}$$

Note also that $\mathcal{T}_{1\pm} \cap \Pi_\pm^{(c)} = \emptyset$. Indeed we know that the image of $\mathcal{T}_{1\pm}$ under the map $z \mapsto z^d$ is $\mathbf{C} \setminus [-1,\ 1]$. As a consequence $\mathcal{T}'_{1,d} \cap \Pi_\pm^{(c)} = \emptyset$. Indeed the image of $\mathcal{T}'_{1,d}$ under $z \mapsto z^d = (e_d, z)$ is the same as that of $\mathcal{T}_{1\pm}$ since $H^{(c)}e_d = \{e_d\}$ by definition. We will prove:

Lemma A.1 *Let $f_\pm$ be functions respectively holomorphic and of tempered growth in $\mathcal{T}_{1\pm}$, $f_\pm^{(b)}$ their boundary values, in the sense of tempered distributions, on X_d. We suppose that $f_+^{(b)}$ and $f_-^{(b)}$ coincide in $\mathcal{R}$. Then:*
(i) There exists a function f holomorphic on $\mathcal{T}'_{1,d}$ whose restriction to $\mathcal{T}_{1\pm}$ is $f_\pm$.
(ii) $\mathcal{T}'_{1,d} = \{z \in X_d^{(c)}\ :\ z^d \in \Delta = \mathbf{C} \setminus [-1,\ 1]\}$. In particular $\mathcal{T}'_{1,d}$ contains $\mathcal{R}$ and $\mathcal{R}'$.
(iii) If $f_\pm^{(b)}$ is invariant under H, i.e. if $f_\pm^{(b)}(x) = f_\pm^{(b)}(\Lambda\, x)$ in the sense of distributions for all $\Lambda \in H$, then there exists a function h holomorphic on Δ such that $f(z) = h(z^d)$ for all $z \in \mathcal{T}'_{1,d}$.

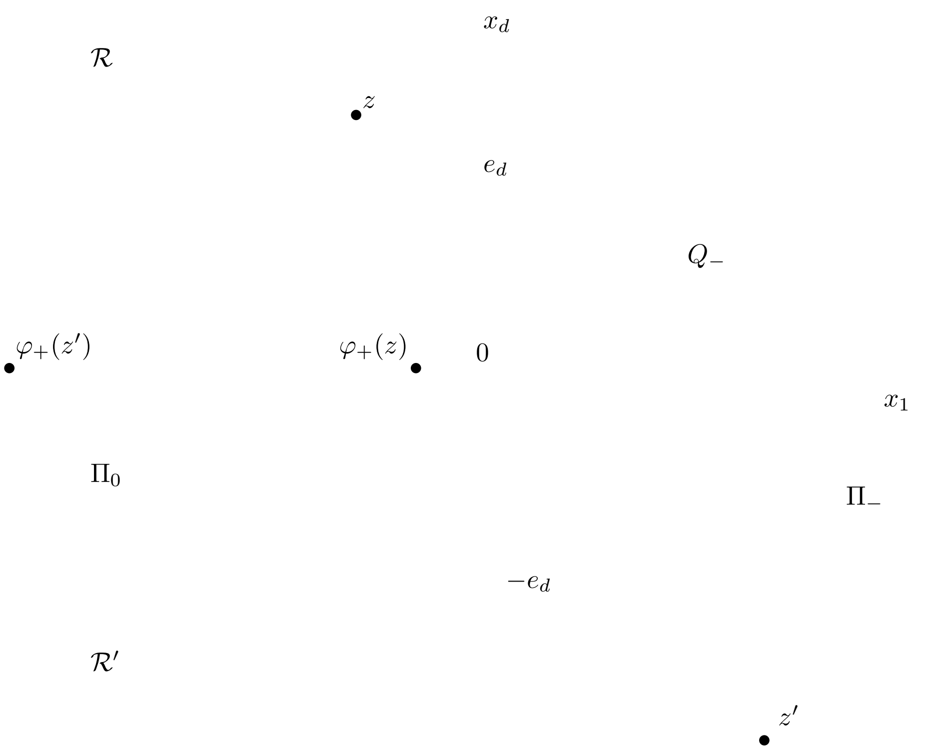

Figure 1: Schematic representation of the map φ_+ in projection onto the $(1, d)$ plane.

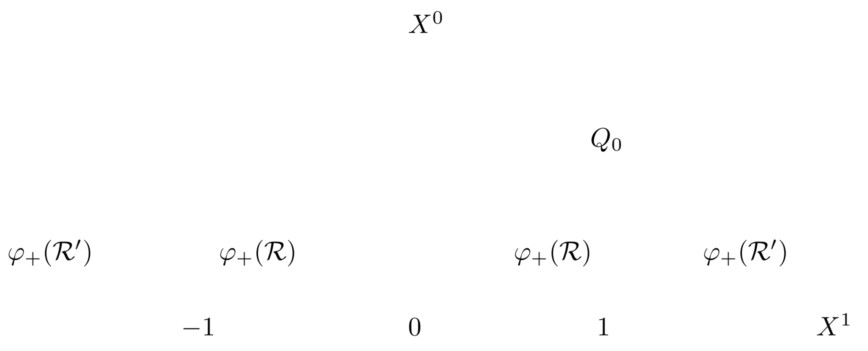

Figure 2: In the subspace Π_0 the region $\varphi_+(\mathcal{R})$ (light gray) and the region $\varphi_+(\mathcal{R}')$ (dark gray) in the case $d = 2$.

Proof
(i) We will reduce this statement to a simple case of the Glaser-Streater Theorem [Sr, J] by using the inversion (stereographic projection)

$$z \mapsto \varphi_+(z) = Z, \quad Z + e_d = 2\,\frac{z+e_d}{(z+e_d)^2}, \quad z + e_d = 2\,\frac{Z+e_d}{(Z+e_d)^2}\ , \tag{A.9}$$

$$(Z+e_d)^2 = \frac{4}{(z+e_d)^2}\ . \tag{A.10}$$

φ_+ is a holomorphic involution of $E_{d+1}^{(c)} \setminus Q_-$ onto itself and commutes with every element of $H^{(c)}$. If $z = x + iy \in X_d^{(c)} \setminus Q_- = X_d^{(c)} \setminus \Pi_-^{(c)}$, and $Z = X + iY = \varphi_+(z)$, we find

$$Z + e_d = 2\,\frac{z+e_d}{(z+e_d)^2} = \frac{z+e_d}{z^d+1}, \tag{A.11}$$

which imply $Z^d = 0$. Conversely if $Z \in \Pi_0^{(c)} \setminus Q_0$, it follows from (A.9) that $(z,z) = 1$. Therefore φ_+ restricted to $X_d^{(c)} \setminus \Pi_-^{(c)}$ is a holomorphic bijection onto $\Pi_0^{(c)} \setminus Q_0$. Moreover (if $z \in X_d^{(c)} \setminus \Pi_-^{(c)}$)

$$(Z+e_d)^2 = (Z,Z) + 1 = \frac{2}{z^d+1}, \quad z + e_d = 2\,\frac{Z+e_d}{(Z,Z)+1}\ , \tag{A.12}$$

$$Y = \frac{(x^d+1)y - y^d(x+e_d)}{(x^d+1)^2 + y_d^2}, \quad (Y,Y) = \frac{(y,y)}{(x^d+1)^2+y_d^2}\ , \tag{A.13}$$

$$Y^0 = \frac{y^0(x^d+1) - y^d x^0}{(x^d+1)^2 + y_d^2}\ . \tag{A.14}$$

Assume that $z \in \mathcal{T}_{1+}$. Then $(y,y) > 0$, hence $(Y,Y) > 0$. Moreover $(x,x) = (y,y) + 1 > 1$, $(x,y) = 0$, $y^0x^d - y^dx^0 > 0$, and

$$\begin{aligned}(y^0x^d - y^dx^0)^2 &= ((y,y) + \vec{y}^2)((x,x) + \vec{x}^2) - (\vec{x}\cdot\vec{y})^2 \\ &\geq ((y,y)+\vec{y}^2)(x,x) > (y^0)^2,\end{aligned} \tag{A.15}$$

hence $Y^0 > 0$. Thus Z belongs to $\mathcal{T}_+$, the future tube in the complex Minkowski space $\Pi_0^{(c)}$, and in fact $Z \in \mathcal{T}_+ \setminus Q_0$. Similarly φ_+ maps $\mathcal{T}_{1-}$ into the past tube of $\Pi_0^{(c)}$. Conversely suppose that $Z = X + iY \in \mathcal{T}_+ \setminus Q_0$. Then $Z = \varphi_+(z)$ where $z = x + iy$ is given by the last equality in (A.9). It satisfies $(y,y) > 0$ by (A.13), and $y^0x^d - y^dx^0 > 0$ since otherwise we would have $Y^0 < 0$ by (A.14).

For real $x \in X_d \setminus \Pi_-$ and $X = \varphi_+(x) \in \Pi_0 \setminus Q_0$, the preceding formulae specialize to

$$(X,X) = \frac{1-x_d}{1+x_d}\ . \tag{A.16}$$

As a consequence

$$\begin{aligned}\varphi_+(\mathcal{R}) &= \{X \in \Pi_0\ :\ -1 < (X,X) < 0\}, && \text{(A.17)}\\ \varphi_+(\mathcal{R}') &= \{X \in \Pi_0\ :\ (X,X) < -1\}. && \text{(A.18)}\end{aligned}$$

The extended tube $\mathcal{T}'$ in $\Pi_0^{(c)}$ is defined as usual as $\mathcal{T}' = H^{(c)}\mathcal{T}_+ = H^{(c)}\mathcal{T}_-$. Obviously $\varphi_+(\mathcal{T}'_{1,d}) = \mathcal{T}' \setminus Q_0$ (recall that $\mathcal{T}'_{1,d} \cap \Pi_-^{(c)} = \emptyset$). As it is well-known $\varphi_+(\mathcal{R}) \subset \mathcal{T}'$ hence $\varphi_+(\mathcal{R}) \subset \mathcal{T}' \setminus Q_0$, and also $\varphi_+(\mathcal{R}') \subset \mathcal{T}' \setminus Q_0$. Furthermore

$$\mathcal{T}' = \{Z \in \Pi_0^{(c)}\ :\ (Z,Z) \notin \mathbf{R}_+\}\ , \tag{A.19}$$

Let $g_\pm = f_\pm \circ \varphi_+^{-1}$. Then $g_\pm$ is holomorphic and of tempered growth in $\mathcal{T}_\pm \setminus Q_0$, where $g_\pm(Z) = f_\pm(z)$, $z \in \mathcal{T}_{1\pm} \setminus \Pi_-^{(c)}$ being given by (A.12). The boundary values if $g_\pm$ coincide in $\varphi_+(\mathcal{R})$ so that, by the edge-of-the-wedge theorem, $g_\pm$ have a common holomorphic extension in $(\mathcal{T}_+ \cup \mathcal{T}_- \setminus Q_0) \cup \mathcal{N}$, where $\mathcal{N}$ is a complex open neighborhood of $\varphi + (\mathcal{R})$ invariant under H. By following the steps of the proof of the Glaser-Streater Theorem as presented e.g. in [BEG] it is immediate to see that all the arguments apply to the present case, owing to the fact that Q_0 is invariant under $H^{(c)}$. We therefore conclude that $g_\pm$ have a common extension g holomorphic in $\mathcal{T}' \setminus Q_0$. Note in particular that g is holomorphic in a neighborhood of $\varphi_+(\mathcal{R}')$, hence that the boundary values of $g_\pm$ coincide there. Since φ_+ is a bijection of $\mathcal{T}'_{1,d}$ onto $\mathcal{T}' \setminus Q_0$, setting $f(z) = g(\varphi_+(z))$ proves part(i).
(ii) follows from (A.19) and from the fact that for $z \in X_d^{(c)} \setminus \Pi_-^{(c)}$, $Z = \varphi_+(z)$,

$$(Z, Z) = \frac{1 - z^d}{1 + z^d}, \qquad z^d = \frac{1 - (Z, Z)}{1 + (Z, Z)} \ . \tag{A.20}$$

This implies that $\mathcal{R} = \{x \in X_d \ : \ x^d > 1\} \subset \mathcal{T}'_{1,d}$, and similarly $\mathcal{R}' \subset \mathcal{T}'_{1,d}$.
(iii) In addition to the preceding hypotheses we now assume that $f_\pm$, and therefore $g_\pm$, are invariant under the real Lorentz group H. Applying the Bargmann-Hall-Wightman Theorem [SW, J], we find that there exists a function $\widehat{g}$, holomorphic in $\mathbf{C} \setminus \mathbf{R}_+ \setminus \{-1\}$ such that $g(Z) = \widehat{g}(Z^2)$ for all $Z \in \mathcal{T}' \setminus Q_0$. Setting $h(\zeta) = \widehat{g}((1-\zeta)/(1+\zeta))$ we obtain part (iii). $\Box$

Remark A.1 Another proof of part (i) can be given by using the temperedness of $f_\pm$. Indeed there is an integer $M > 0$ such that the functions $Z \mapsto g'_\pm(Z) = ((Z, Z) + 1)^M g_\pm(Z)$ are holomorphic in $\mathcal{T}_\pm$ (respectively) and coincide on $\varphi_+(\mathcal{R})$. By the "double-cone theorem" [Bo2, p. 68], their region of coincidence extends to $\{X \in \Pi_0 \ : \ (X, X) < 0\}$ and the standard Glaser-Streater Theorem can be applied.

We return to the functions $w_\pm$, which we now assume invariant under G_0, and apply Lemma A.1 to $f_\pm(z) = w_\pm(e_d, \ z)$: these functions have a common holomorphic extension to $\mathcal{T}'_{1,d}$, and there exists a function h, holomorphic on Δ, such that $w_\pm(e_d, \ z) = h(z^d)$. For $z_1 \in \mathcal{T}_{1-}$ and $z_2 \in \mathcal{T}_{1+}$, let $w'_+(z_1, \ z_2) = h((z_1, z_2))$. For real $x_1 \in X_d$ and $z_2 \in \mathcal{T}_{1+}$, we have $w'_+(x_1, \ z_2) = w_+(x_1, \ z_2)$. Indeed we can write $x_1 = \Lambda e_d$, $z_2 = \Lambda z'$ for some $\Lambda \in G_0$ and $z' \in \mathcal{T}_{1+}$, so that $w_+(x_1, \ z_2) = w_+(e_d, \ z') = h(z'^d) = w'_+(x_1, \ z_2)$. Therefore $w_+(z_1, \ z_2)$ coincides with $w'_+(z_1, \ z_2)$' i.e. with $h((z_1, z_2))$ for all $z_1 \in \mathcal{T}_{1-}$ and $z_2 \in \mathcal{T}_{1+}$. Similarly $w_-(z_1, \ z_2)$ coincides with $h((z_1, z_2))$ on $\mathcal{T}_{1+} \times \mathcal{T}_{1-}$.

We have proved:

Lemma A.2 *With the preceding assumptions on $w_\pm$, and assuming in addition that these functions are invariant under G_0, there exists a function h, holomorphic on Δ, such that $w_\pm$ coincide with $(z_1, \ z_2) \mapsto h((z_1, z_2))$ in their domains of definition.*

Remark A.2 This implies that $w_\pm^{(b)}$ coincide not only on $\mathcal{R}_2$, but also on the "exotic region" $\mathcal{R}'_2$:

$$\mathcal{R}'_2 = \{x \in X_d^2 \ : \ (x_1 + x_2, \ x_1 + x_2) < 0\} \ \ = \{x \in X_d^2 \ : \ (x_1, x_2) < -1\} \ . \tag{A.21}$$

Our proof shows that this also holds without assuming that $w_\pm$ are invariant under G_0.

A.2 The case of $\widetilde{X}_d$

The topology of $\mathcal{T}_{1\pm}$ (and hence of $\widetilde{\mathcal{T}}_{1\pm}$) is made clear by the holomorphic bijection φ_+, which maps $\mathcal{T}_{1\pm}$ onto $\mathcal{T}_\pm \setminus Q_0$ in the complex Minkowski space $\Pi_0^{(c)}$. To make things even clearer, one can use the map ψ defined in $\Pi_0^{(c)}$ by

$$\psi(Z) + e_{d-1} = -2\frac{Z + e_{d-1}}{(Z + e_{d-1})^2} \ . \tag{A.22}$$

This is a holomorphic bijection of $\{Z \in \Pi_0^{(c)} \ : \ (Z + e_{d-1})^2 \neq 0\}$ onto itself, which maps $\mathcal{T}_+$ onto itself (i.e. in particular $(Z + e_{d-1})^2 \neq 0$ for $Z \in \mathcal{T}_\pm$). It maps $\{Z \in \Pi_0^{(c)} \ : \ (Z, Z) = -1, \quad (Z + e_{d-1})^2 \neq 0\}$ onto $\{Z \in \Pi_0^{(c)} \ : \ Z^{d-1} = 0, \quad (Z + e_{d-1})^2 \neq 0\}$ and therefore it is a holomorphic bijection of $\mathcal{T}_+ \setminus Q_0$ onto $\mathcal{T}_+ \cap \{Z \in \Pi_0^{(c)} \ : \ Z^{d-1} \neq 0\}$ (and similarly for $\mathcal{T}_- \setminus Q_0$). We shall however continue to work with $\mathcal{T}_\pm \setminus Q_0$ which has the advantage of being invariant under H. We denote

$$\mathcal{L} = \{z \in \Pi_0^{(c)} \ : \ (z, z) + 1 \in \mathbf{R}_-\} \ . \tag{A.23}$$

This is an analytic hypersurface containing Q_0.

Lemma A.3 *The open set $\mathcal{T}_+ \setminus \mathcal{L}$ is connected and simply connected.*

Proof. The set $A = \mathcal{T}_+ \setminus \mathcal{L}$ is star-shaped with respect to 0, i.e. $\rho A \subset A$ for every $\rho \in (0,\ 1)$ (but $0 \notin A$). For every $z \in \Pi_0^{(c)}$, $|(z, z)| \leq ||z||^2$. hence if B denotes the open ball $B = \{z \in \Pi_0^{(c)} \ : \ ||z||^2 < 1/2\}$, we have $A \cap B = \mathcal{T}_+ \cap B$ and this intersection is convex. We can define a map $\sigma(t, z) = ((1-t) + t||2z||^{-1})z$ of $[0,\ 1] \times A$ into A such that $\sigma(0,\ z) = z$ and $z \mapsto \sigma(1,\ z)$ sends A into $A \cap B$. Hence any two points in A can be connected by a continuous arc, and every closed curve in A is homotopic to 0. ▭

We now suppose given a pair of functions $g_\pm$ respectively holomorphic and of tempered growth in the covering of $\mathcal{T}_\pm \setminus Q_0$. This is equivalent to giving a pair of sequences $\{g_{n\pm} \ : \ n \in \mathbf{Z}\}$, with the following properties:
(1) for each $n \in \mathbf{Z}$, $g_{n\pm}$ is holomorphic in $\mathcal{T}_\pm \setminus \mathcal{L}$;
(2) every $z \in \mathcal{L} \cap (\mathcal{T}_+ \setminus Q_0)$ has an open complex neighborhood V_z such that $g_{n+}|V_z \cap \{Z \ : \ \mathrm{Im}\,(Z, Z) > 0\}$ and $g_{(n+1)+}|V_z \cap \{Z \ : \ \mathrm{Im}\,(Z, Z) < 0\}$ have a common holomorphic extension in V_z;
(3) Similarly for g_{n-} in $\mathcal{T}_-$.
In addition we suppose that
(4) the boundary values of $g_{0\pm}$ coincide in $\varphi_+(\mathcal{R})$.

The proof of the Glaser-Streater Theorem again applies to show that $g_{0\pm}$ have a common holomorphic extension g_0 in $\mathcal{T}' \setminus \mathcal{L}$, in particular that, the map $(\Lambda,\ z) \mapsto g_{0\pm}(\Lambda z)$ of $H \times (\mathcal{T}_\pm \setminus \mathcal{L})$ into $\mathbf{C}$ extends to a holomorphic function on $H^{(c)} \times (\mathcal{T}_\pm \setminus \mathcal{L})$. From this it follows that $(\Lambda,\ z) \mapsto g_{n\pm}(\Lambda z)$ also extends to a holomorphic function on $H^{(c)} \times (\mathcal{T}_\pm \setminus \mathcal{L})$. The Bargmann-Hall-Wightman Lemma again shows that each of the functions $g_{n\pm}$ extends to a function holomorphic in $\mathcal{T}' \setminus \mathcal{L}$. Moreover these two functions coincide, and we denote $g_n = g_{n\pm}$. This can be seen, for $n \geq 0$, by induction on n. Supposing it to hold up to $n - 1$, we consider points of the form iy for $y \in V_+$ and $(y, y) > 1$, which belong to $\mathcal{T}_+ \cap \mathcal{L}$. At such a point

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} g_{n+}((i - \varepsilon)y) \quad &= \quad \lim_{\varepsilon \downarrow 0} g_{(n-1)+}((i + \varepsilon)y) \\ = \lim_{\varepsilon \downarrow 0} g_{(n-1)-}((i + \varepsilon)y) \quad &= \quad \lim_{\varepsilon \downarrow 0} g_{n-}((i - \varepsilon)y) \ . \end{aligned} \tag{A.24}$$

Note however that if in addition the "hermiticity condition" $g_{0-}(z) = g_{0+}(z^*)^*$ holds, this extends to $g_{-n-}(z) = g_{n+}(z^*)^*$ for all $n \in \mathbf{Z}$. There is no general reason for $g_{n-}(z)$ and $g_{n+}(z^*)^*$ to coincide for $n \neq 0$.

If we suppose that $g_{0\pm}$ are invariant under H, then for each $n \in \mathbf{Z}$ the functions g_n are locally Lorentz invariant and, again by the Bargmann-Hall-Wightman Lemma, there is a function h_n, holomorphic in $\mathbf{C} \setminus R_+ \setminus (-1 + \mathbf{R}_-)$, such that $g_n(z) = h_n((z, z))$. Moreover $h_{n+1}(t - i0) = h_n(t + i0)$ for all $t \in (-\infty,\ -1)$ and all $n \in \mathbf{Z}$. Therefore there exists a function $\widetilde{h}$, holomorphic on $\mathbf{C} \setminus \bigcup_{n \in \mathbf{Z}}(2i\pi n + \mathbf{R}_+)$ such that, for each $n \in \mathbf{Z}$, $h_n(e^w - 1) = \widetilde{h}(w + 2in\pi)$ in $\{w \in \mathbf{C} \setminus \mathbf{R}_+ \ : \ -i\pi < \mathrm{Im}\, w < i\pi\}$

Let now $w_\pm$ be functions holomorphic and of tempered growth on $\widetilde{\mathcal{T}}_{1-} \times \widetilde{\mathcal{T}}_{1+}$ and $\widetilde{\mathcal{T}}_{1+} \times \widetilde{\mathcal{T}}_{1-}$, respectively. We suppose that $w_\pm(\Lambda z_1,\ \Lambda z_2) = w_\pm(z_1,\ z_2)$ for all $\Lambda \in \widetilde{G}_0$ and all z in the relevant domain, and that the boundary values of $w_\pm$ on $\widetilde{X}_d$ coincide at space-like separated arguments. As in the case of X_d, we assume that the boundary values can be extended to $\mathcal{C}^\infty$ functions of a real first argument, holomorphic and of tempered growth in the second argument in $\widetilde{\mathcal{T}}_{1\pm}$ (respectively) and define $f_\pm(z) = w_\pm(e_d,\ z)$, $g_\pm(Z) = f_\pm(\varphi_+^{-1}(Z))$. Then $g_\pm$ satisfy the conditions (1)-(4) mentioned above, $g_{n\pm}$ are H invariant, and the preceding remarks provide the functions g_n, h_n and $\widetilde{h}$. We can transport back these properties to the functions $w_\pm$.

Remark A.3 It is worth noting the following formula, where $z_1,\ z_2 \in X_d^{(c)} \setminus \Pi_-^{(c)}$ and $Z_1 = \varphi_+(z_1)$, $Z_2 = \varphi_+(z_2)$:

$$(z_1, z_2) - 1 = \frac{-2(Z_1 - Z_2)^2}{(Z_1^2+1)(Z_2^2+1)}\ , \tag{A.25}$$

B Appendix. Proof of Lemma 3.7

We start with

Lemma B.1 *In any neighborhood of M_{0d} in $\mathcal{G}$, one can find a basis $\{h_{\mu,\nu} \in \mathcal{C}_+\ :\ 0 \le \mu < \nu \le d\}$ of $\mathcal{G}$ such that $M_{0d} = 2\sum_{0\le\mu<\nu\le d} h_{\mu,\nu}/d(d+1)$.*

Proof. Let $\kappa > 0$ be sufficiently small. We denote $u_j = \kappa e_j$ for all $j = 1, \dots, d-1$. For $0 \le \mu < \nu \le d$, we define $f_{\mu,\nu} \in \bigwedge_2(E_{d+1})$ as follows (j and k are integers in $[1,\ d-1]$):

$$\begin{aligned}
f_{0,d} &= \left(e_0 + \sum_{j=1}^{d-1} u_j\right) \wedge \left(e_d + \sum_{k=1}^{d-1} k\, u_k\right) \\
&= e_0\wedge e_d + \sum_{k=1}^{d-1} k\, e_0 \wedge u_k + \sum_{j=1}^{d-1} u_j \wedge e_d + \sum_{1\le j<k\le d-1} (k-j)\, u_j \wedge u_k\ , \\
f_{j,k} &= (e_0 + (k-j)\, u_j) \wedge (e_d - u_k) \quad \text{for } 1 \le j < k \le d-1 \\
&= e_0 \wedge e_d - e_0 \wedge u_k + (k-j)\, u_j \wedge e_d - (k-j)\, u_j \wedge u_k\ , \\
f_{0,j} &= e_0 \wedge (e_d - a_j u_j), \quad f_{j,d} = (e_0 - b_j u_j) \wedge e_d \quad \text{for } 1 \le j \le d-1\ .
\end{aligned} \tag{B.1}$$

This gives

$$\begin{aligned}
\sum_{0\le\mu<\nu\le d} f_{\mu,\nu} &= \frac{(d+1)d}{2} e_0 \wedge e_d \\
&+ \sum_{k=1}^{d-1} \left(k - \sum_{1\le j<k} 1 - a_k\right) e_0 \wedge u_k \\
&+ \sum_{j=1}^{d-1} \left(1 + \sum_{j<k\le d-1} (k-j) - b_j\right) u_j \wedge e_d\ .
\end{aligned} \tag{B.2}$$

We adjust a_j and b_j so that only the first term survives in the rhs of this identity, i.e

$$a_j = 1, \quad b_j = 1 + (d-1-j)(d-j)/2 \quad \text{for } 1 \le j \le d-1\ . \tag{B.3}$$

With these values, $e_0 \wedge u_j$ and $u_j \wedge e_d$ can be recovered from $f_{0,j}$ and $f_{d,j}$, respectively, then $u_j \wedge u_k$ can be recovered from $f_{j,k}$. Thus the $\{f_{\mu,\nu}\}_{0\le\mu<\nu\le d}$ form a basis of $\bigwedge_2(E_{d+1})$. We obtain a basis $\{h_{\mu,\nu}\}_{0\le\mu<\nu\le d}$ of $\mathcal{G}$ by setting $h_{\mu,\nu} = \ell(f_{\mu,\nu})$. Clearly, given a neigborhood V of $M_{0d} = \ell(e_0 \wedge e_d)$, κ can be chosen so small that $h_{\mu,\nu} \in V$ for all μ and ν. $\square$

Corollary B.1 *The convex cone* $\widehat{\mathcal{C}}_+$ *generated by* $\mathcal{C}_1$ *is open. Also*

$$G_0 = \{\exp(s_1 M_1)\ldots\exp(s_N M_N)\ :\ N \in \mathbf{N},\quad s_j \in \mathbf{R},\quad M_j \in \mathcal{C}_1\ \ \forall j\}. \tag{B.4}$$

Corollary B.2 G_0^+ *is open in* $G_0^{(c)}$.

Proof. We first show that, if $\tau \in \mathbf{C}_+$ has sufficiently small modulus, $\exp(\tau M_{0d})$ is an interior point of G_0^+. Denoting $L = d(d+1)/2$, let $(LA_1, \ldots, LA_L) = (h_{0,1}, \ldots,\ h_{d-1,d})$, be the basis of $\mathcal{G}$ constructed in the proof of Lemma B.1. In particular $M_{0d} = (A_1 + \cdots + A_L)$. We consider the two holomorphic maps of $\mathbf{C}^L$ into $G_0^{(c)}$

$$\begin{aligned} h_1(z_1, \ldots, z_L) &= \exp(z_1 A_1)\cdots\exp(z_L A_L), \\ h_2(z_1, \ldots, z_L) &= \exp(z_1 A_1 + \cdots + z_L A_L)\ . \end{aligned} \tag{B.5}$$

These maps are tangent at 0, and, for sufficiently small $\varepsilon > 0$, both are biholomorphic maps of the polycylinder $\{z \in \mathbf{C}^L\ :\ |z_j| < \varepsilon\ \forall j\}$ into $G_0^{(c)}$. In particular the subset $V_1 = h_1(\{z \in \mathbf{C}^L\ :\ |z_j| < \varepsilon,\quad \mathrm{Im}\, z_j > 0\ \forall j\})$ of $G_0^{(c)}$ is open and contained in G_0^+. For sufficiently small $|\tau|$ the curve $\tau \mapsto h_1^{-1} \circ h_2(\tau, \ldots, \tau)$ exists and is tangent to $\tau \mapsto (\tau, \ldots, \tau)$. Therefore there exists an $\eta > 0$ such that $|\tau| < \eta$ and $\mathrm{Im}\,\tau > 0$ imply $h_2(\tau, \ldots, \tau) \in V_1$, i.e. $\exp(\tau M_{0d}) \in V_1$. We now consider a point $\Lambda \in G_0^+$ of the form $\exp(\tau_1 M_1)\cdots\exp(\tau_N M_N)\exp\tau M_0 d$, where $\tau_1, \ldots,\ \tau_N,\ \tau \in \mathbf{C}_+$ and $M_1, \ldots, M_N \in \mathcal{C}_1$. This point can be rewritten as $\Lambda = \Lambda_\theta \exp(\theta\tau M_{0d})$ where θ is arbitrary in $(0,\ 1)$ and $\Lambda_\theta \in G_0^+$. For sufficiently small θ, $\exp(\theta\tau M_{0d}) \in V_1$ hence $\Lambda \in \Lambda_\theta V_1$, an open set contained in G_0^+. Since G_0^+ is invariant under conjugations from G_0, this proves the corollary. ▭

Corollary B.3

$$G_0^+ = G_0\, G_0^+ = G_0^+\, G_0\ . \tag{B.6}$$

Proof. Let $\Lambda \in G_0^+$ and $M \in \mathcal{C}_+$. For any real s and t,

$$e^{sM}\,\Lambda = e^{(s+it)M}(e^{-itM}\Lambda)\ . \tag{B.7}$$

Since G_0^+ is open, for sufficiently small $t > 0$, $e^{-itM}\Lambda \in G_0^+$ and $e^{(s+it)M} \in G_0^+$ hence $e^{sM}\,\Lambda \in G_0^+$. Any $S \in G_0$ can be written as a finite product $\exp(s_1 M_1)\ldots\exp(s_N M_N)$ where $s_j \in \mathbf{R}$ and $M_j \in \mathcal{C}_+$, hence $S\Lambda \in G_0^+$ for any $\Lambda \in G_0^+$. Thus $G_0^+ = G_0\, G_0^+$, and since $SG_0^+ S^{-1} = G_0^+$ for all $S \in G_0$, this implies $G_0^+ = G_0^+\, G_0$. ▭

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