Abelian duality in three dimensions*

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Abstract

Abelian duality on the closed three-dimensional Riemannian manifold $\mathcal{M}^3$ is discussed. Partition functions for the ordinary $U(1)$ gauge theory and circle-valued scalar field theory on $\mathcal{M}^3$ are explicitly calculated and compared.

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1 Introduction

At present, Abelian duality is a classical theme in (quantum) field theory. In four-dimensional case, there are two well-known papers on electric-magnetic (or S-duality) in Abelian gauge theory, viz. [1,2]. In three dimensions, one

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could cite the other two papers, [3] and [4], but unfortunately, neither is fully comprehensive nor satisfactory. The first ([3]) lacks explicit formulas for the partition functions and, in a sense, is purely "classical", whereas the second one ([4]) lacks sufficient generality and topological aspects are there largely ignored. Moreover, the both are seemingly contradictory: [3] explicitly shows duality between Abelian gauge field and scalar field, whereas the corresponding partition functions calculated in [4] markedly differ.

The aim of this short work is to fill the gap and clarify some points (the reader is advised to consult the above-mentioned papers for more details which are not repeated here). In particular, we will explicitly calculate the both partition functions and discuss their mutual duality.

2 Partition function of the Abelian theory

We consider the connected, orientable three-dimensional Riemannian (of Euclidean signature) manifold $\mathcal{M}^3$ with torsionless (co)homology throughout. The action of $U(1)$ gauge theory is defined by

$$S[A] = \frac{1}{4\pi e^2}\int_{\mathcal{M}^3} F_A \wedge *F_A = \frac{1}{8\pi e^2}\int_{\mathcal{M}^3} d^3x\sqrt{g}\, F_{ij}F^{ij}, \quad i,j=1,2,3, \tag{1}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$ $(F_A = dA)$, e is the coupling constant $(e > 0)$, and g_{ij} is the metric tensor. A standard form of the partition function is

$$Z_{U(1)} = \int \mathcal{D}A\, e^{-S[A]}, \tag{2}$$

where $\mathcal{D}A$ denotes a formal integration measure with respect to gauge non-equivalent field configurations, i.e. the Faddeev–Popov procedure is applied.

The partition function consists of several factors: (1) Z_{det}—product of determinants, (2) Z_{class}—sum over the classical saddle points, (3) Z_{vol}—the volume of the space of classical minima, (4) Z_0—the contribution from zero modes.

The form of $Z_{\rm det}$, coming from Gaussian integration with respect to gauge fields and ghosts, is rather standard, i.e.

$$Z_{\rm det} = \frac{\det{}'\Delta_0}{(\det{}'\Delta_1)^{\frac{1}{2}}}, \tag{3}$$

where the prime denotes removal of zero modes, Δ_0 is a Laplacian acting on zero-forms (Faddeev–Popov ghosts), and Δ_1 is a Laplacian acting on one-forms (gauge fields). Obviously, $Z_{\rm det}$ is independent of the coupling constant e^2 and it corresponds to "quantum fluctuations".

The classical part represents the sum over classical saddle points,

$$Z_{\rm class} = \sum e^{-S[A_{\rm class}]}, \tag{4}$$

where $A_{\rm class}$ are local minima of the action (1) corresponding to different line bundles. For non-trivial second homology of $\mathcal{M}^3$ we have field configurations with non-zero flux,

$$\int_{\Sigma_I} F = 2\pi m^I, \quad m^I \in \mathbb{Z}, \tag{5}$$

with $I = 1, ..., b_2 = \dim H_2(\mathcal{M}^3)$. Then the solutions of the field equations can be represented by the sum

$$F = 2\pi \sum_I m^I \omega_I, \tag{6}$$

where ω_I span an integral basis of harmonic two-forms with normalization

$$\int_{\Sigma_I} \omega_J = \delta^I_J. \tag{7}$$

Inserting the expansion (6) into the partition function (4) we obtain

$$Z_{\rm class} = \sum_{m^I} e^{-S[m^I]}, \tag{8}$$

where

$$S[m^I] = \frac{\pi}{e^2} \sum_{I,J} G_{IJ} m^I m^J, \tag{9}$$

with

$$G_{IJ} = \frac{1}{2} \int_{\mathcal{M}^3} d^3x \sqrt{g}\, \omega_{Iij} \omega_J^{ij}. \tag{10}$$

The volume of the space of classical minima is a torus of dimension $b_1(\mathcal{M}^3)$ [1], that is, in our case,

$$Z_{\rm vol} = {\rm vol}(T^{b_1}) = (2\pi)^{b_1}. \tag{11}$$

Finally, the contribution from zero modes is

$$(2\pi e)^{b_0 - b_1} = (2\pi e)^{1-b_1}, \tag{12}$$

where b_0 corresponds to Faddeev–Popov ghosts, whereas b_1 is related to Abelian gauge fields. Actually, we should divide this by 2π to obtain an agreement with "a direct calculation" of the path integral (this is explicitly demonstrated by considering a simple finite-dimensional example in Appendix of [5]). Thus,

$$Z_0 = \frac{1}{2\pi} (2\pi e)^{1-b_1}. \tag{13}$$

Collecting all the above terms, i.e. Eq.(3), Eq.(8), Eq.(11) and Eq.(13), we obtain an explicit form of the partition function for three-dimensional $U(1)$ gauge theory,

$$Z_{U(1)} = e^{1-b_1} \frac{\det{}' \Delta_0}{(\det{}' \Delta_1)^{\frac{1}{2}}} \sum_{m^I} e^{-S[m^I]}, \tag{14}$$

with $S[m^I]$ defined by Eq.(9).

3 Partition function of the scalar theory

In this section we will calculate the partition function of the three-dimensional (single-component) scalar field theory on $\mathcal{M}^3$. In principle, we have the two possibilities: we could consider ordinary scalar field assuming values in $\mathbb{R}$ or we could analyze circle-valued scalar field with values in $\mathbb{S}$. A short reflection suggests the second possibility. First of all, for ordinary scalar field we would not have a chance to reproduce the reach topological sector of the form (8), (9). Besides, in [3] Witten has shown duality between the Abelian theory and circle-valued scalar theory. Anyway, real-valued theory can be considered as a topologically trivial sector in circle-valued one.

The action of the scalar theory is of the form

$$S[\phi] = \frac{e^2}{4\pi} \int_{\mathcal{M}^3} E_\phi \wedge *E_\phi = \frac{e^2}{4\pi} \int_{\mathcal{M}^3} d^3x \sqrt{g}\, E_i E^i, \tag{15}$$

where $E_i = \partial_i \phi \ (E_\phi = d\phi)$, with $\phi \in [0, 2\pi)$. The rest of the notation is analogous to the one used in the previous section. The partition function for the scalar theory,

$$Z_\phi = \int D\phi \, e^{-S[\phi]}, \tag{16}$$

can be calculated according to the scheme proposed for $Z_{U(1)}$.

The determinantal part consists of only one determinant coming from non-zero modes of scalar field,

$$Z_{\rm det} = (\det{}' \Delta_0)^{-\frac{1}{2}}. \tag{17}$$

As far as the classical part is concerned we can rewrite Eq.(4) for ϕ field,

$$Z_{\rm class} = \sum e^{-S[\phi_{\rm class}]}, \tag{18}$$

where, analogously, the sum is taken over the classical saddle points. But this time, for non-trivial first homology of $\mathcal{M}^3$ we have field configurations with non-zero circulation

$$\int_{C_I} E, \tag{19}$$

with $I = 1, ..., b_1 =$dim$H_1(\mathcal{M}^3)$. Since for orientable three-dimensional manifold $\mathcal{M}^3$ we have $b_1 = b_2$, we assume the same type of indices for Σ and C (see Eq.(5)). Then the solution of the field equations can be expressed by the sum

$$E = 2\pi \sum_I n^I \alpha_I, \tag{20}$$

where α_I span a basis of harmonic one-forms. The issue of normalization of α_I is postponed to the next section. Inserting the expansion (20) into the partition function (18) we obtain

$$Z_{\rm class} = \sum_{n^I} e^{-\widetilde{S}[n^I]}, \tag{21}$$

where this time

$$\widetilde{S}[n^I] = \pi e^2 \sum_{I,J} H_{IJ} n^I n^J, \tag{22}$$

with

$$H_{IJ} = \int_{\mathcal{M}^3} d^3x \sqrt{g}\, \alpha_{Ii} \alpha^i_J, \tag{23}$$

The volume of the space of classical minima which is the target space for this model (the circle S) is

$$Z_{\rm vol} = {\rm vol}(S) = 2\pi, \tag{24}$$

and the zero-mode contribution (from a single zero mode of scalar field)

$$Z_0 = \frac{e}{2\pi}. \tag{25}$$

The final shape of the partition function as the product of Eq.(17), Eq.(21), Eq.(24), and Eq.(25) assumes the form

$$Z_\phi = e \, \frac{1}{(\det{}' \Delta_0)^{\frac{1}{2}}} \sum_{n^I} e^{-\widetilde{S}[n^I]}, \tag{26}$$

with $\widetilde{S}[n^I]$ defined by Eq.(22).

4 Duality and final discussion

Since we hope for duality, at least in some limited sense, of $U(1)$ gauge theory and circle-valued scalar field theory, we could expect exact duality of classical parts of the partition functions, Eq.(8) and Eq.(18). The classical duality takes place if the metric G_{IJ} on the space of second cohomology and the metric H_{IJ} on the space of first cohomology are equal,

$$G_{IJ} = H_{IJ}. \tag{27}$$

The equality (27) follows from the duality in de Rham cohomology, i.e.

$$\begin{aligned} G_{IJ} = (\omega_I, \omega_J) = \int_{\mathcal{M}^3} \omega_I \wedge * \, \omega_J = \int_{\mathcal{M}^3} \omega_I \wedge \alpha_J \\ = \int_{\mathcal{M}^3} * \, \alpha_I \wedge \alpha_J = (\alpha_I, \alpha_J) = H_{IJ}, \end{aligned} \tag{28}$$

where by definition

$$\alpha_I = * \, \omega_I. \tag{29}$$

Eq.(29) imposes the lacking normalization condition for one-forms α_I, announced after Eq.(20), and crucial for Eq.(27). This way, the both "classical parts" are exactly dual in the sense of "e^2 goes to $1/e^2$" equivalence.

The "quantum parts" are definitely different. As far as the determinants are concerned, the possible equality of Eq.(3) and Eq.(17) would imply

$$\frac{\det{}'\Delta_0}{(\det{}'\Delta_1)^{\frac{1}{2}}}\,\frac{1}{(\det{}'\Delta_0)^{-\frac{1}{2}}} = 1. \tag{30}$$

But the LHS of Eq.(30) is exactly the inverse of the Ray–Singer analytic torsion [6] which is not equal to 1, in general. On the other hand treating the LHS of Eq.(30) as a formal ratio of volumes of spaces of forms [7] gives 1, which is in accordance with the RHS of Eq.(30), and is in favor of duality. Then analytic torsion following from (zeta) regularization of determinants can be considered as a "volume ratio anomaly".

Another kind of regularization yields different powers of the coupling constant e in front of the both partition functions. More precisely, the regularized partition function (2) of $U(1)$ gauge theory should read

$$Z_{U(1)}(e) = (2\pi e)^{B_0 - B_1} \int \mathcal{D}A\, e^{-S[A]}, \tag{31}$$

whereas the partition function (16) of S-valued scalar theory is of the form

$$Z_\phi(e) = \left(\frac{e}{2\pi}\right)^{B_0} \int D\phi\, e^{-S[\phi]}, \tag{32}$$

where B_k is the (infinite) dimension of the space of k-forms on $\mathcal{M}^3$. Upon procedure described earlier we obtain the desired explicit forms of the partition functions, Eq.(13) and Eq.(25).

One could easily generalize our considerations to the case of Abelian $U(1)^n$ gauge theory. The dual scalar model would assume values in n-dimensional torus T^n, or we could speak on (linear) σ-model with target space T^n. Obviously, the whole analysis would be analogous to the above one.

References

[1] E. Witten, 1995 *On S-Duality In Abelian Gauge Theory*, e-print archive: hep-th/9505186.

[2] E. Verlinde, 1995 *Global Aspects Of Electric-Magnetic Duality*, Nucl. Phys. B **455**, 211–228.

[3] E. Witten, 1999 *Lecture II-7: Abelian Duality* in *Quantum Fields and Strings: A Course For Mathematicians*, vol. 1,2. ed. P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, D.R. Morrison, E. Witten, AMS, Providence, http://www.math.ias.edu/QFT/spring/witten7.ps

[4] E.M. Prodanov, S. Sen, 2000 *Abelian Duality*, Phys. Rev. D **62**, 045009–045013.

[5] L. Rozansky, 1995 *A Large K Asymptotics of Wiiten's Invariant of Seifert Manifolds*, Commun. Math. Phys. **171** 279–322.

[6] D. Birmingham, M. Blau, M. Rakowski, G. Thompson, 1991 *Topological Field Theory*, Phys. Rept. **209** 129–340.

[7] D.H. Adams, E.M. Prodanov, 2000 *A Remark on Schwarz's Topological Field Theory*, Lett. Math. Phys. **51** 249–255.