REDUCED SPIN-STATISTICS THEOREM

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Abstract:

As argued in our previous papers, it would be more natural to modify the standard approach to quantum theory by requiring that i) one unitary irreducible representation (UIR) of the symmetry algebra should describe a particle and its antiparticle simultaneously. This would automatically explain the existence of antiparticles and show that a particle and its antiparticle are different states of the same object. If i) is adopted then among the Poincare, so(2,3) and so(1,4) algebras only the latter is a candidate for constructing elementary particle theory. We extend our analysis in hep-th/0210144 and prove that: 1) UIRs of the so(1,4) algebra can indeed be interpreted in the framework of i) and cannot be interpreted in the framework of the standard approach; 2) as a consequence of a new symmetry (called AB one) between particles and antiparticles for UIRs satisfying i), elementary particles described by UIRs of the so(1,4) algebra can be only fermions; 3) as a consequence of the AB symmetry, the vacuum condition can be consistent only for particles with the half-integer spin (in conventional units) and therefore only such particles can be elementary. In our approach the well known fact that fermions have imaginary parity is a consequence of the AB symmetry.

PACS: 11.30Cp, 11.30.Ly

1 Introduction

1.1 Motivation

The phenomenon of local quantum field theory (LQFT) has no analogs in the history of science. There is no branch of science where so impressive agreements between theory and experiment have been achieved. At the same time, the level of mathematical rigor in the LQFT is very poor and, as a consequence, the LQFT has several well known difficulties and inconsistencies. The absolute majority of physicists believes that agreement with experiment is much more important than the lack of mathematical rigor, but not all of them think so. For example, Dirac wrote in Ref. [1]: ′The agreement with observation is presumably by coincidence, just like the original calculation of the hydrogen spectrum with Bohr orbits. Such coincidences are no reason for turning a blind eye to the faults of the theory. Quantum electrodynamics is rather like Klein-Gordon equation. It was built up from physical ideas that were not correctly incorporated into the theory and it has no sound mathematical foundation.′

One could agree or disagree with this statement, but in any case, the majority of physicists believes that the LQFT should be treated [2] ′in the way it is′, but at the same time it is [2] a ′low energy approximation to a deeper theory that may not even be a field theory, but something different like a string theory′.

The main problem of course is the choice of strategy for constructing a new quantum theory. Since nobody knows for sure what strategy is the best one, different approaches should be investigated. Dirac's advice given in Ref. [1] is as follows: ′I learned to distrust all physical concepts as a basis for a theory. Instead one should put one's trust in a mathematical scheme, even if the scheme does not appear at first sight to be connected with physics. One should concentrate on getting an interesting mathematics.′

Typically the LQFT starts from a local Lagrangian for which, by using the canonical Noether formalism, one can determine a set of conserving physical observables. After quantizing the local fields in question, these observables become quantum operators and the quantum Lagrangian obtained in such a way contains products of local quantum fields at coinciding points. However, interacting field operators can be treated only as operator valued distributions [3] and therefore their products at coinciding points are not well defined. Although there exists a wide literature on this problem, a universal solution has not been found yet.

There exist two essentially different approaches to quantum theory — the standard operator approach and the path integral approach. We accept the operator approach. In this case, to be consistent, we should assume that *any* physical quantity is described by a selfadjoint operator in the Hilbert space of states for the system under consideration (we will not discuss the difference between selfadjoint and Hermitian operators). Then the first question which immediately arises is that, even in the nonrelativistic quantum mechanics, there is no operator corresponding to time [4]. It is also well known that when quantum mechanics is combined with relativity, then there is no operator satisfying all the properties of the spatial position operator (see e.g. Ref. [5]) . For these reasons the quantity x in the Lagrangian density $L(x)$ is not the coordinate but a parameter which becomes the coordinate in the classical limit.

These facts were well known already in 30th of the 20th century. As a result of evolution of these ideas, it was widely believed in 50th and 60th that spacetime is a rudimentary notion which will disappear in the ultimate quantum theory. Since that time, no arguments questioning those ideas have been given, but in view of the great success of gauge theories in 70th and 80th , such ideas became almost forgotten.

At present, the predictions of the standard model are in agreement with experiment with an unprecedented accuracy. Nevertheless, the difficulties of the canonical LQFT have not been overcome. For this reason there exist several approaches, the goal of which is to avoid the product of interacting quantum fields at the same spacetime point. In addition to the well known theories (string theory, noncommutative quantum theory etc.) we also would like to mention a very interesting approach by Saniga where the classical spacetime is replaced by a Galois field [6].

In view of this situation, the problem arises of how one should define the notion of elementary particles.

Although particles are observable and fields are not, in the spirit of the LQFT, fields are more fundamental than particles, and a possible definition is as follows [7]: 'It is simply a particle whose field appears in the Lagrangian. It does not matter if it's stable, unstable, heavy, light — if its field appears in the Lagrangian then it's elementary, otherwise it's composite'.

Another approach has been developed by Wigner in his investigations of unitary irreducible representations (UIRs) of the Poincare group [8]. In view of this approach, one might postulate that in the general case, a particle is elementary if the space of its wave functions is the space of UIR of the symmetry group in the given theory.

In standard well-known theories (QED, electroweak theory and QCD) the above approaches coincide although the problem arises of whether the second definition is compatible with confinement. However, when the symmetry group is not the Poincare one, additional problems arise. For example, in view of modern approaches to the LQFT in curved spacetime, the de Sitter group SO(1,4) cannot be the symmetry group since, from the standpoint of any local observer, the vacuum has a finite temperature and admits particle destruction and creation (see e.g. Refs. [9, 10]). We discuss this problem in Sect. 8.

Although the Wigner approach is well defined in the framework of standard mathematics, the following problem arises. The symmetry group is usually chosen as the group of motions of some classical manifold. How does this agree with the above discussion that quantum theory in the operator formulation should not contain spacetime? A possible answer is as follows. One can notice that for computing observables (e.g. the spectrum of the Hamiltonian) we need in fact not a representation of the group but a representation of its Lie algebra by Hermitian operators. After such a representation has been constructed, we have only operators acting in the Hilbert space and this is all we need in the operator approach. The representation operators of the group are needed only if it is necessary to calculate some macroscopic transformation, e.g. time evolution. In the approximation when classical time is a good approximate parameter, one can calculate evolution, but nothing guarantees that this is always the case. Let us also note that in the stationary formulation of scattering theory, the S-matrix can be defined without any mentioning of time (see e.g. Ref. [11]). For these reasons we can assume that on quantum level the symmetry algebra is more fundamental than the symmetry group (see also the discussion in Sect. 8).

In other words, instead of saying that some operators satisfy commutation relations of a Lie algebra A because spacetime X has a group of motions G such that A is the Lie algebra of G, we say that there exist operators satisfying commutation relations of a Lie algebra A such that: for some operator functions $\{O\}$ of them, the classical limit is a good approximation, a set X of the eigenvalues of the operators $\{O\}$ represents a classical manifold with the group of motions G and its Lie algebra is A (see also Sect. 2). This is not of course in the spirit of famous Klein's Erlangen program [12] or LQFT.

Summarizing our discussion, we assume that, *by definition*, on quantum level a Lie algebra is the symmetry algebra if there exist physical observables such that their operators satisfy the commutation relations characterizing the algebra. Then, a particle is called elementary if the space of its wave functions is a space of irreducible representation of this algebra by Hermitian operators. In the literature such representations usually also are called UIRs meaning that the representation of the algebra can be extended to an UIR of the corresponding Lie group.

The approach we accept is in the spirit of that considered by Dirac in Ref. [13]. Let us also note that although we treat the description of quantum systems in terms of representations of algebras as more fundamental, this does not mean that for investigating properties of algebra representations we cannot use mathematical results on group representations.

In our papers [14], we discussed an approach when quantum states are described by elements of a linear space over a Galois field, and the operators of physical quantities — by operators in this space. It has been argued that such an approach is more natural than the standard one and the future quantum physics will be based on Galois fields. However, in the present paper we work in the framework of the standard quantum theory based on complex numbers.

1.2 Statement of the problem

In standard Poincare or anti de Sitter (AdS) invariant theories, the field theoretical and Wigner definitions of elementary particles (see the preceding subsection) do not contradict each other. Therefore, each elementary particle can be described by using an UIR of the Poincare or AdS group or algebra. In addition, it also can be described by using a Poincare or AdS covariant equation. In these theories the existence of antiparticles is explained as follows. For each values of the mass and spin, there exist two UIRs - with positive and negative energies, respectively. At the same time, the corresponding covariant equation has solutions with both positive and negative energies. As noted by Dirac (see e.g. his Nobel lecture [15]), the existence of the negative energy solutions represents a difficulty which should be resolved. In the standard approach, the solution is given in the framework of quantization such that the creation and annihilation operators for the antiparticle have the usual meaning but they enter the quantum Lagrangian with the coefficients representing the negative energy solutions.

Such an approach has lead to impressive success in describing various experimental data. However, as noted by Weinberg [7], 'this is our aim in physics, not just to describe nature, but to explain nature'. From this point of view, it seems unnatural that the covariant equation describes the particle and antiparticle simultaneously while the UIRs for them are fully independent of each other. Moreover, the UIRs with negative energies are not used at all.

The necessity to have negative energy solutions is related to the implementation of the idea that the creation or annihilation of an antiparticle can be treated, respectively, as the annihilation or creation of the corresponding particle with the negative energy. However, since negative energies have no direct physical meaning in the standard theory, this idea is implemented implicitly rather than explicitly.

The above program cannot be implemented if the de Sitter (dS) group SO(1,4) is chosen as the symmetry group or the dS algebra so(1,4) is chosen as the symmetry algebra. Some of the reasons have been already indicated in the preceding subsection. Also, it is well known that in UIRs of the dS algebra, the dS Hamiltonian is not positive definite and has the spectrum in the interval $(-\infty, +\infty)$ see e.g. Refs. [16, 17, 18, 19, 20]). Note also that in contrast to the AdS algebra so(2,3), the dS one does not have a supersymmetric generalization. For this and other reasons it was believed that the dS group or algebra were not suitable for constructing elementary particle theory. Although our approach considerably differs from that in Refs. [9, 10] and references therein, we come to the same conclusion (see Sect. 3) that in the standard approach the dS group cannot be a symmetry group. However, it is possible to modify the standard approach in such a way (see below) that theories with the dS symmetry become consistent.

It is well known that the group SO(1,4) is the symmetry group of the four-dimensional manifold in the five-dimensional space, defined by the equation

$$x_0^2 - x_1^2 - x_2^2 - x_3^2 - x_4^2 = -R^2 \tag{1}$$

where a constant R has the dimension of length. The quantity R^2 is often written as $R^2 = 3/\Lambda$ where Λ is the cosmological constant. The nomenclature is such that $\Lambda < 0$ for the AdS symmetry while $\Lambda > 0$ - for the dS one. The recent astronomical data show that, although Λ is very small, it is probably positive (see e.g. Ref. [21]). For this reason the interest to dS theories has increased. Nevertheless, the existing difficulties have not been overcome (see e.g. Ref. [22]).

As shown in Ref. [14], in quantum theory based on a Galois field, Galois field analogs of UIRs of the AdS algebra so(2,3) have a property that a particle and its antiparticle are described by the same irreducible representation of the symmetry algebra. This automatically explains the existence of antiparticles and shows that a particle and its antiparticle represent different states of the same object. As argued in Ref. [14], the description of quantum theory in terms of Galois fields is more natural than the standard description based on the field of complex numbers. However, in the present paper we consider only the standard approach based on complex numbers, but with the following modification. Instead of saying that UIRs describe elementary particles, we assume that

Supposition 1: Any UIR of the symmetry algebra should describe a particle and its antiparticle simultaneously.

With such a requirement, among the Poincare, AdS and dS algebras, only the latter can be a candidate for constructing the elementary particle theory. As shown in Ref. [23], UIRs of the dS algebra are indeed compatible with Supposition 1. By quantizing such UIRs and requiring that the energy should be positive definite in the Poincare limit, it has been shown that only fermions can be elementary.

In the present paper we analyze UIRs of the so(1,4) algebra not only in the Poincare limit but in the general case as well. In Sect. 2 we describe well known results on the derivation of explicit expressions for the representation generators of the dS group. In Sect. 3 the Poincare limit is discussed. It is explained why UIRs of the dS algebra can be treated in the framework of Supposition 1 and cannot be treated in the framework of the standard approach. In Sect. 4 we describe in detail a basis of UIRs such that all the quantum numbers are discrete, and in Sect. 5 the generators are explicitly written down in the quantized form. In Sect. 6 we discuss the implementation of Supposition 1 in the case when the dS algebra is the exact symmetry algebra. In Sect. 7 it is shown that UIRs of the dS algebra possess a new symmetry between particles and antiparticles. Following Ref. [14], we call this symmetry the AB one. It is shown that the AB symmetry is compatible only with the anticommutation relations and therefore only fermions can be elementary. As shown in Sect. 8, the vacuum condition is consistent only for particles with the half-integer spin (in conventional units), and therefore only such particles can be elementary. Finally, in Sect. 9 we argue that neutral elementary particles cannot exist and show that the well known fact that fermions have imaginary parity follows from the AB symmetry.

2 UIRs of the SO(1,4) group

As already noted, the de Sitter group SO(1,4) is the symmetry group of the four-dimensional manifold defined by Eq. (1). Elements of a map of the point $(0,0,0,0,R)$ (or $(0,0,0,0,-R)$) can be parametrized by the coordinates (x_0,x_1,x_2,x_3). If R is very large then such a map proceeds to Minkowski space and the action of the dS group on this map — to the action of the Poincare group.

In the present paper it will be convenient for us to work with the units $\hbar/2=c=1$. Then the spin of any particle is always an integer. For the normal relation between spin and statistics, the spin of fermions is odd and the spin of bosons is even. In this system of units the representation generators of the SO(1,4) group M^{ab} ($a,b=0,1,2,3,4$, $M^{ab}=-M^{ba}$) should satisfy the commutation relations

$$[M^{ab},M^{cd}]=-2i(\eta^{ac}M^{bd}+\eta^{bd}M^{as}-\eta^{ad}M^{bc}-\eta^{bc}M^{ad}) \tag{2}$$

where η^{ab} is the diagonal metric tensor such that $\eta^{00}=-\eta^{11}=-\eta^{22}=-\eta^{33}=-\eta^{44}=1$.

An important observation is as follows. If we accept that the symmetry on quantum level means that proper commutation relations are satisfied (see the preceding section) then Eq. (2) can be treated as the *definition* of the dS symmetry on quantum level. In our system of units, all the operators M^{ab} are dimensionless, in contrast to the situation with the Poincare algebra, where the representation generators of the Lorentz group are dimensionless while the momentum operators have the dimension $(length)^{-1}$. For this reason it is natural to think that the dS or AdS symmetries are more fundamental that the Poincare symmetry. Note that such a definition does not involve the cosmological constant at all. It appears only if one is interested in interpreting results in terms of the dS spacetime or in the Poincare limit.

If one assumes that spacetime is fundamental then in the spirit of General Relativity it is natural to think that the empty space is flat, i.e. that the cosmological constant is equal to zero. This was the subject of the well-known dispute between Einstein and de Sitter. In the modern approach to the LQFT, the cosmological constant is given by a contribution of vacuum diagrams, and the problem is to explain why it is so small. On the other hand, if we assume that symmetry on quantum level in our formulation is more fundamental, then the problem of the cosmological constant does not exist at all. Instead we have a problem of why nowadays the Poincare symmetry is so good approximate symmetry. It seems natural to involve the anthropic principle for the explanation of this phenomenon (see e.g. Ref. [24] and references therein).

There exists a wide literature devoted to UIRs of the dS group and algebra (see e.g. Refs. [25, 26, 27, 28, 29, 17, 18, 16, 30, 31, 19, 20]). In particular the first complete mathematical classification of the UIRs has been given in Ref. [25], three well-known realizations of the UIRs have been first considered in Ref. [26] and their physical context has been first discussed in Ref. [27].

It is well known that for classification of UIRs of the dS group, one should, strictly speaking, consider not the group SO(1,4) itself but its universal covering group. The investigation carried out in Refs. [25, 26, 27, 28, 18] has shown that this involves only replacement of the SO(3) group by its universal covering group SU(2). Since this procedure is well known then for illustrations we will work with the SO(1,4) group itself and follow a very elegant presentation for physicists in terms of induced representations, given in the book [17] (see also Refs. [16, 33, 28]). The elements of the SO(1,4) group can be described in the block form

$$g = \left\| \begin{array}{ccc} g_0^0 & \mathbf{a}^T & g_4^0 \\ \mathbf{b} & r & \mathbf{c} \\ g_0^4 & \mathbf{d}^T & g_4^4 \end{array} \right\| \tag{3}$$

where

$$\mathbf{a} = \left\| \begin{array}{c} a^1 \\ a^2 \\ a^3 \end{array} \right\| \quad \mathbf{b}^T = \| \, b_1 \;\; b_2 \;\; b_3 \, \| \quad r \in SO(3) \tag{4}$$

(the subscript ^{T} means a transposed vector).

UIRs of the SO(1,4) group are induced from UIRs of the subgroup H defined as follows [28, 17, 16]. Each element of H can be uniquely represented as a product of elements of the subgroups SO(3), A and $\mathbf{T}$: $h = r\tau_A \mathbf{a_T}$ where

$$\tau_A = \left\| \begin{array}{ccc} cosh(\tau) & 0 & sinh(\tau) \\ 0 & 1 & 0 \\ sinh(\tau) & 0 & cosh(\tau) \end{array} \right\| \quad \mathbf{a_T} = \left\| \begin{array}{ccc} 1+\mathbf{a}^2/2 & -\mathbf{a}^T & \mathbf{a}^2/2 \\ -\mathbf{a} & 1 & -\mathbf{a} \\ -\mathbf{a}^2/2 & \mathbf{a}^T & 1-\mathbf{a}^2/2 \end{array} \right\| \tag{5}$$

The subgroup A is one-dimensional and the three-dimensional group $\mathbf{T}$ is the dS analog of the conventional translation group (see e.g. Ref. [17]). We hope it should not cause misunderstandings when 1 is used in its usual meaning and when to denote the unit element of the SO(3) group. It should also be clear when r is a true element of the SO(3) group or belongs to the SO(3) subgroup of the SO(1,4) group.

Let $r \to \Delta(r; \mathbf{s})$ be a UIR of the group SO(3) with the spin $\mathbf{s}$ and $\tau_A \to exp(i\mu\tau)$ be a one-dimensional UIR of the group A, where μ is a real parameter. Then UIRs of the group H used for inducing to the SO(1,4) group, have the form

$$\Delta(r\tau_A\mathbf{a_T}; \mu, \mathbf{s}) = exp(i\mu\tau)\Delta(r; \mathbf{s}) \tag{6}$$

We will see below that μ has the meaning of the dS mass and therefore UIRs of the SO(1,4) group are defined by the mass and spin, by analogy with UIRs in Poincare invariant theory.

Let G=SO(1,4) and $X = G/H$ be a factor space (or coset space) of G over H. The notion of the factor space is well known (see e.g. Refs. [32, 16, 28, 17, 33]). Each element $x \in X$ is a class containing the elements $x_G h$ where $h \in H$, and $x_G \in G$ is a representative of the class x. The choice of representatives is not unique since if x_G is a representative of the class $x \in G/H$ then $x_G h_0$, where h_0 is an arbitrary element from H, also is a representative of the same class. It is well known that X can be treated as a left G space. This means that if $x \in X$ then the action of the group G on X can be defined as follows: if $g \in G$ then gx is a class containing gx_G (it is easy to verify that such an action is correctly defined). Suppose that the choice of representatives is somehow fixed. Then $gx_G = (gx)_G(g,x)_H$ where $(g,x)_H$ is an element of H. This element is called a factor.

As noted in the preceding section, although we can use well known facts about group representations, our final goal is the construction of the generators. The explicit form of the generators M^{ab} depends on the choice of representatives in the space G/H. As explained in several papers devoted to UIRs of the SO(1,4) group (see e.g. Ref. [17]), to obtain the possible closest analogy between UIRs of the SO(1,4) and Poincare groups, one should proceed as follows. Let $\mathbf{v}_L$ be a representative of the Lorentz group in the factor space SO(1,3)/SO(3) (strictly speaking, we should consider $SL(2,c)/SU(2)$). This space can be represented as the well known velocity hyperboloid with the Lorentz invariant measure

$$d\rho(\mathbf{v}) = d^3\mathbf{v}/v_0 \tag{7}$$

where $v_0 = (1+\mathbf{v}^2)^{1/2}$. Let $I \in SO(1,4)$ be a matrix which formally has the same form as the metric tensor η. One can show (see e.g. Ref. [17] for details) that $X = G/H$ can be represented as a union of three spaces, X_+, X_- and X_0 such that X_+ contains classes $\mathbf{v}_L h$, X_- contains classes $\mathbf{v}_L I h$ and X_0 is of no interest for UIRs describing elementary particles since it has measure zero relative to the spaces X_+ and X_-.

As a consequence of these results, the space of UIR of the SO(1,4) algebra can be implemented as follows. If s is the spin of the particle under consideration, then we use $||...||$ to denote the norm in the space of UIR of the su(2) algebra with the spin s. Then the space of UIR in question is the space of functions $\{f_1(\mathbf{v}), f_2(\mathbf{v})\}$ on two Lorentz hyperboloids with the range in the space of UIR of the su(2) algebra with the spin s and such that

$$\int [||f_1(\mathbf{v})||^2 + ||f_2(\mathbf{v})||^2] d\rho(\mathbf{v}) < \infty \tag{8}$$

We see that, in contrast with UIRs of the Poincare algebra (and AdS one), where UIRs are implemented on one Lorentz hyperboloid, UIRs of the dS algebra can be implemented only on two Lorentz hyperboloids, X_+ and X_-. Even this fact (which is well known) is a strong indication that UIRs of the dS algebra might have a natural interpretation in the framework of Supposition 1.

In the case of the Poincare and AdS algebras, the positive energy UIRs are implemented on an analog of X_+ and negative energy UIRs - on an analog of X_-. Since the Poincare and AdS groups do not contain elements transforming these spaces to one another, the positive and negative energy UIRs are fully independent. At the same time, the dS group contains such elements (e.g. I [17, 16, 30]) and for this reason its UIRs cannot be implemented only on one hyperboloid.

In Ref. [23] we have described all the technical details needed for computing the explicit form of the generators M^{ab}. In our system of units the results are as follows. The action of the generators on functions with the supporter in X_+ has the form

$$\mathbf{M}^{(+)} = 2l(\mathbf{v}) + \mathbf{s}, \quad \mathbf{N}^{(+)} == -2iv_0\frac{\partial}{\partial \mathbf{v}} + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1},$$
$$\mathbf{B}^{(+)} = \mu\mathbf{v} + 2i[\frac{\partial}{\partial \mathbf{v}} + \mathbf{v}(\mathbf{v}\frac{\partial}{\partial \mathbf{v}}) + \frac{3}{2}\mathbf{v}] + \frac{\mathbf{s}\times\mathbf{v}}{v_0+1},$$
$$M_{04}^{(+)} = \mu v_0 + 2iv_0(\mathbf{v}\frac{\partial}{\partial \mathbf{v}} + \frac{3}{2}) \tag{9}$$

where $\mathbf{M} = \{M^{23}, M^{31}, M^{12}\}$, $\mathbf{N} = \{M^{01}, M^{02}, M^{03}\}$, $\mathbf{B} = -\{M^{14}, M^{24}, M^{34}\}$, $\mathbf{s}$ is the spin operator, and $\mathbf{l}(\mathbf{v}) = -i\mathbf{v}\times\partial/\partial\mathbf{v}$. At the same time, the action of the generators on functions with the supporter in X_- is given by

$$\mathbf{M}^{(-)} = 2l(\mathbf{v}) + \mathbf{s}, \quad \mathbf{N}^{(-)} == -2iv_0\frac{\partial}{\partial \mathbf{v}} + \frac{\mathbf{s}\times\mathbf{v}}{}v_0 + 1,$$
$$\mathbf{B}^{(-)} = -\mu\mathbf{v} - 2i[\frac{\partial}{\partial \mathbf{v}} + \mathbf{v}(\mathbf{v}\frac{\partial}{\partial \mathbf{v}}) + \frac{3}{2}\mathbf{v}] - \frac{\mathbf{s}\times\mathbf{v}}{v_0+1},$$
$$M_{04}^{(-)} = -\mu v_0 - 2iv_0(\mathbf{v}\frac{\partial}{\partial \mathbf{v}} + \frac{3}{2}) \tag{10}$$

In view of the fact that SO(1,4)=SO(4)AT and H=SO(3)AT, there also exists a choice of representatives which is probably even more natural than that described above [17, 16, 18]. Namely, we can choose as representatives the elements from the coset space SO(4)/SO(3). Since the universal covering group for SO(4) is SU(2)$\times$SU(2) and for SO(3) — SU(2), we can choose as representatives the elements of the first multiplier in the product SU(2)$\times$SU(2). Elements of SU(2) can be represented by the points $u = (\mathbf{u}, u_4)$ of the three-dimensional sphere S^3 in the four-dimensional space as $u_4 + i\boldsymbol{\sigma}\mathbf{u}$ where $\boldsymbol{\sigma}$ are the Pauli matrices and $u_4 = \pm(1-\mathbf{u}^2)^{1/2}$ for the upper and lower hemispheres, respectively. Then the calculation of the generators is similar to that described above. Since such a form of generators will be needed only for illustrative purposes, we will not discuss technical details and describe only the result.

The Hilbert space for such a choice of representatives is the space of functions $\varphi(u)$ on S^3 with the range in the space of the UIR of the su(2) algebra with the spin $\mathbf{s}$ and such that

$$\int ||\varphi(u)||^2 du < \infty \tag{11}$$

where du is the SO(4) invariant volume element on S^3. The explicit calculation shows that the generators for this realization have the form

$$\mathbf{M} = 2l(\mathbf{u}) + \mathbf{s}, \quad \mathbf{B} = 2\imath u_4\frac{\partial}{\partial \mathbf{u}} - \mathbf{s},$$
$$\mathbf{N} = 2\imath[\frac{\partial}{\partial \mathbf{u}} - \mathbf{u}(\mathbf{u}\frac{\partial}{\partial \mathbf{u}})] - (\mu + 3\imath)\mathbf{u} + \mathbf{u}\times\mathbf{s} - u_4\mathbf{s},$$
$$M_{04} = (\mu + 3\imath)u_4 + 2\imath u_4\mathbf{u}\frac{\partial}{\partial \mathbf{u}} \tag{12}$$

Since Eqs. (8-10) on the one hand and Eqs. (11) and (12) on the other are the different realization of one and the same representation, there exists a unitary operator transforming functions $f(v)$ into $\varphi(u)$ and operators (9,10) into operators (12). For example in the spinless case the operators (9) and (12) are related to each other by a unitary transformation

$$\varphi(u) = exp(-\frac{\imath}{2}\mu\, lnv_0)v_0^{3/2}f(v) \tag{13}$$

where $\mathbf{u} = -\mathbf{v}/v_0$.

In view of this relation, the sphere S^3 is usually interpreted in the literature as the velocity space (see e.g. Refs. [17, 16, 18, 30]). As argued in Refs. [19, 20, 34], there also exist reasons to interpret S^3 as the coordinate space. However, the behavior of a particle in the dS space is rather unusual (see e.g. Refs. [17, 30, 9]). For this reason the standard physical intuition is expected to work only for elements of the full Hilbert space which become physical states in the Poincare limit. Unitary transformations similar to those in Eq. (13) transform such states in such a way that the standard contraction to the Poincare group is impossible for them. For these reasons, S^3 probably does not have a universal interpretation (see also Sect. 8).

3 Poincare limit

A general notion of contraction has been developed in Ref. [35]. In our case it can be performed as follows. Let us assume that $\mu > 0$ and denote $m = \mu/2R$, $\mathbf{P} = \mathbf{B}/2R$ and $E = M_{04}/2R$. Then, as follows from Eq. (9), in the limit when $R \to \infty$, $\mu \to \infty$ but μ/R is finite, one obtains a standard representation of the Poincare algebra for a particle with the mass m such that $\mathbf{P} = m\mathbf{v}$ is the particle momentum and $E = mv_0$ is the particle energy. In that case the generators of the Lorentz algebra have the same form for the Poincare and dS algebras. Analogously the operators given by Eq. (10) are contracted to ones describing negative energy UIRs of the Poincare algebra.

In the standard interpretation of UIRs it is assumed that each element of the full representation space represents a possible physical state for the elementary particle in question. It is also well known (see e.g. Ref. [16, 17, 18, 30]) that the dS group contains elements (e.g. $\boldsymbol{I}$) such that the corresponding representation operator transforms positive energy states to negative energy ones and *vice versa*. Are these properties compatible with the fact that in the Poincare limit there exist states with negative energies?

One might say that the choice of the energy sign is only a matter of convention. For example, in the standard theory we can define the energy not as $(m^2 + \mathbf{p}^2)^{1/2}$ but as $-(m^2 + \mathbf{p}^2)^{1/2}$. However, let us consider, for example, a system of two free noninteracting particles. The fact that they do not interact means mathematically that the representation describing the system is the tensor product of single-particle UIRs. The generators of the tensor product are equal to sums of the corresponding single-particle generators. In the Poincare limit the energy and momentum can be chosen diagonal. If we assume that both positive and negative energies are possible then a system of two free particles with the equal masses can have the same quantum numbers as the vacuum (for example, if the first particle has the energy E and momentum $\mathbf{p}$ while the second one has the energy $-E$ and the momentum $-\mathbf{p}$) what obviously contradicts experiment. For this and other reasons it is well known that in the Poincare invariant theory all the particles in question should have the same energy sign.

We conclude that UIRs of the dS algebra cannot be interpreted in the standard way since such an interpretation is physically meaningless even in the Poincare limit. Although our approach considerably differs from that in LQFT in curved spacetime, this conclusion is in agreement with that in Refs. [9, 10] and references therein (see Sect. 1).

In the framework of Supposition 1, one could try to interpret the operators (10) as those describing a particle while the operators (11) as those describing the corresponding antiparticle. Such a program has been implemented in Ref. [23]. If one requires that the dS Hamiltonian should be positive definite in the Poincare limit, then, as shown in Ref. [23], the annihilation and creation operators for the particle and antiparticle in question can satisfy only anticommutation relations, i.e. the particle and antiparticle can be only fermions.

If one assumes that the dS algebra is the symmetry algebra in the elementary particle theory then one has to consider not only the limit when the contraction to the Poincare algebra is possible, but the general case as well. This is just the goal of the present paper.

Concluding this section, let us note the following. As assumed by Mensky [17], UIRs of the dS group could be a basis for new approaches to the CPT theorem. We believe that Supposition 1 is in the spirit of Mensky's idea. Indeed, a comparison of Eqs. (9) and (10) shows that the operators M_{ab} in these expressions not containing the subscript 4 are the same while those containing this subscript have different signs (the operator M_{44} is of no interest since it is identical zero). If the coordinates x^μ ($\mu = 0, 1, 2, 3$) are inversed (i.e. one applies the PT transformation) then the operators $M_{\mu 4}$ should change their sign while the other operators remain unchanged. In particular, a positive definite Hamiltonian becomes negative definite. To avoid such an undesirable behavior we can relate the new Hamiltonian to antiparticles and quantize it in a proper way. In other words, the PT transformation should be necessarily accompanied by transition from particles to antiparticles and *vice versa*, i.e. the PT transformation should be replaced by the CPT one.

4 UIRs in the su(2)$\times$su(2) basis

Proceeding from the method of su(2)$\times$su(2) shift operators, developed by Hughes [36] for constructing UIRs of the group SO(5), and following Ref. [34], we now give a pure algebraic description of UIRs of the so(1,4) algebra. It will be convenient for us to deal with the set of operators $(\mathbf{J}', \mathbf{J}", R_{ij})$ ($i, j = 1, 2$) instead of M^{ab}. Here $\mathbf{J}'$ and $\mathbf{J}"$ are two independent su(2) algebras (i.e. $[\mathbf{J}', \mathbf{J}"] = 0$). In each of them one chooses as the basis the operators (J_+, J_-, J_3) such that in our system of units $J_1 = J_+ + J_-$, $J_2 = -\imath(J_+ - J_-)$ and the commutation relations have the form

$$[J_3, J_+] = 2J_+, \quad [J_3, J_-] = -2J_-, \quad [J_+, J_-] = J_3 \tag{14}$$

The commutation relations of the operators $\mathbf{J}'$ and $\mathbf{J}"$ with R_{ij} have the form

$$\begin{aligned} &[J_3', R_{1j}] = R_{1j}, \quad [J_3', R_{2j}] = -R_{2j}, \quad [J_3", R_{i1}] = R_{i1}, \\ &[J_3", R_{i2}] = -R_{i2}, \quad [J_+', R_{2j}] = R_{1j}, \quad [J_+", R_{i2}] = R_{i1}, \\ &[J_-', R_{1j}] = R_{2j}, \quad [J_-", R_{i1}] = R_{i2}, \quad [J_+', R_{1j}] = \\ &[J_+", R_{i1}] = [J_-', R_{2j}] = [J_-", R_{i2}] = 0, \end{aligned} \tag{15}$$

and the commutation relations of the operators R_{ij} with each other have the form

$$\begin{aligned} &[R_{11}, R_{12}] = 2J_+', \quad [R_{11}, R_{21}] = 2J_+", \\ &[R_{11}, R_{22}] = -(J_3' + J_3"), \quad [R_{12}, R_{21}] = J_3' - J_3" \\ &[R_{11}, R_{22}] = -2J_-", \quad [R_{21}, R_{22}] = -2J_-' \end{aligned} \tag{16}$$

The relation between the sets $(\mathbf{J}', \mathbf{J}", R_{ij})$ and M^{ab} is given by

$$\begin{aligned} &\mathbf{M} = \mathbf{J}' + \mathbf{J}", \quad \mathbf{B} = \mathbf{J}' - \mathbf{J}", \quad M_{01} = \imath(R_{11} - R_{22}), \\ &M_{02} = R_{11} + R_{22}, \quad M_{03} = -i(R_{12} + R_{21}), \\ &M_{04} = R_{12} - R_{21} \end{aligned} \tag{17}$$

Then it is easy to see that Eq. (2) follows from Eqs. (15-17) and *vice versa*.

Consider the space of maximal $su(2) \times su(2)$ vectors, i.e. such vectors x that $J_+'x = J_+"x = 0$. Then from Eqs. (15) and (16) it follows that the operators

$$\begin{aligned} &A^{++} = R_{11}, \quad A^{+-} = R_{12} - J_-"R_{11}(J_3" + 1)^{-1}, \\ &A^{-+} = R_{21} - J_-'R_{11}(J_3' + 1)^{-1}, \\ &A^{--} = -R_{22} + J_-"R_{21}(J_3" + 1)^{-1} + \\ &J_-'R_{12}(J_3' + 1)^{-1} - J_-'J_-"R_{11}[(J_3' + 1)(J_3" + 1)]^{-1} \end{aligned} \tag{18}$$

act invariantly on this space. The notations are related to the property that if x^{kl} ($k,l > 0$) is the maximal su(2)$\times$su(2) vector and simultaneously the eigenvector of operators J_3' and $J_3"$ with the eigenvalues k and l, respectively, then $A^{++}x^{kl}$ is the eigenvector of the same operators with the values $k+1$ and $l+1$, $A^{+-}x^{kl}$ - the eigenvector with the values $k+1$ and $l-1$, $A^{-+}x^{kl}$ - the eigenvector with the values $k-1$ and $l+1$ and $A^{--}x^{kl}$ - the eigenvector with the values $k-1$ and $l-1$.

As follows from Eq. (14), the vector $x_{ij}^{kl} = (J_-')^i (J_-")^j x^{kl}$ is the eigenvector of the operators J_3' and $J_3"$ with the eigenvalues $k-2i$ and $l-2j$, respectively. Since

$$\mathbf{J}^2 = J_3^2 - 2J_3 + 4J_+J_- = J_3^2 + 2J_3 + 4J_-J_+$$

is the Casimir operator for the $\mathbf{J}$ algebra, and the Hermiticity condition can be written as $J_-^* = J_+$, it follows in addition that

$$\mathbf{J}'^2 x_{ij}^{kl} = k(k+2)x_{ij}^{kl}, \quad \mathbf{J}"^2 x_{ij}^{kl} = l(l+2)x_{ij}^{kl} \tag{19}$$

$$J_+' x_{ij}^{kl} = i(k+1-i)x_{i-1,j}^{kl}, \quad J_+" x_{ij}^{kl} = j(l+1-j)x_{i,j-1}^{kl} \tag{20}$$

$$(x_{ij}^{kl}, x_{ij}^{kl})) = \frac{i!j!k!l!}{(k-i!)(l-j!)}(x^{kl}, x^{kl}) \tag{21}$$

where $(...,...)$ is the scalar product in the representation space. From these formulas it follows that the action of the operators $\mathbf{J}'$ and $\mathbf{J}"$ on x^{kl} generates a space with the dimension $(k+1)(l+1)$ and the basis x_{ij}^{kl} ($i = 0, 1, ...k$, $j = 0, 1, ...l$). Note that the vectors x_{ij}^{kl} are orthogonal but we deliberately do not normalize them to unity since the normalization (21) will be convenient below.

The Casimir operator of the second order for the algebra (2) can be written as

$$I_2 = -\frac{1}{2}\sum_{ab} M_{ab}M^{ab} =$$
$$4(R_{22}R_{11} - R_{21}R_{12} - J_3') - 2(\mathbf{J}'^2 + \mathbf{J}"^2) \tag{22}$$

A direct calculation shows that for the generators given by Eqs. (9), (10) and (12), I_2 has the numerical value

$$I_2 = w - s(s+2) + 9 \tag{23}$$

where $w = \mu^2$. As noted in the preceding section, $\mu = 2mR$ where m is the conventional mass. If $m \neq 0$ then μ is very big since R is very big. We conclude that for massive UIRs the quantity I_2 is a big positive number.

The basis in the representation space can be explicitly constructed assuming that there exists a vector e^0 which is the maximal su(2)$\times$su(2) vector such that

$$J_3' e_0 = n_1 e_0 \quad J_3" e_0 = n_2 e_0 \tag{24}$$

and n_1 is the minimum possible eigenvalue of J_3' in the space of the maximal vectors. Then e_0 should also satisfy the conditions

$$A^{--} e_0 = A^{-+} e_0 = 0 \tag{25}$$

We use $\tilde{I}$ to denote the operator $R_{22}R_{11} - R_{21}R_{12}$. Then as follows from Eqs. (15), (16), (18), (22), (24) and (25),

$$\tilde{I} n_1 e_0 = 2n_1(n_1+1)e_0.$$

Therefore, if $n_1 \neq 0$ the vector e_0 is the eigenvector of the operator $\tilde{I}$ with the eigenvalue $2(n_1+1)$ and the eigenvector of the operator I_2 with the eigenvalue

$$-2[(n_1+2)(n_2-2) + n_2(n_2+2)].$$

The latter is obviously incompatible with Eq. (23) for massive UIRs. Therefore the compatibility can be achieved only if $n_1 = 0$. In that case we use s to denote n_2 since it will be clear soon that the value of n_2 indeed has the meaning of spin. Then, as follows from Eqs. (23) and (24), the vector e_0 should satisfy the conditions

$$\begin{aligned} &\mathbf{J}' e^0 = J_+" e^0 = 0, \quad J_3" e^0 = s e^0, \\ &I_2 e^0 = [w - s(s+2) + 9] e^0 \end{aligned} \tag{26}$$

where $w, s > 0$ and s is an integer.

Define the vectors

$$e^{nr} = (A^{++})^n (A^{+-})^r e^0 \tag{27}$$

Then a direct calculation taking into account Eqs. (14)-(16), (18), (19), (22), (25) and (26) gives

$$A^{++}e^{nr} = e^{n+1,r} \quad A^{+-}e^{nr} = \frac{s-r+1}{n+s-r+1}e^{n,r+1} \tag{28}$$

$$A^{--}e^{nr} = -\frac{n(n+s+1)[w+(2n+s+1)^2]}{4(n+r+1)(n+s-r+1)}e^{n-1,r} \tag{29}$$

$$A^{-+}e^{nr} = -\frac{r(s+1-r)[w+(s+1-2r)^2]}{4(n+r+1)(s+2-r)}e^{n,r-1} \tag{30}$$

As follows from Eqs. (29) and (30), the possible values of n are $n = 0, 1, 2, ...$ while r can take only the values of $0, 1, ..., s$ (and therefore s indeed has the meaning of the particle spin). Since e^{nr} is the maximal $su(2)\times su(2)$ vector with the eigenvalues of the operators $\mathbf{J}'$ and $\mathbf{J}$" equal to $n+r$ and $n+s-r$, respectively, then as a basis of the representation space one can take the vectors $e^{nr}_{ij} = (J'_-)^i(J_-")^j e^{nr}$ where, for the given n and s, the quantity i can take the values of $0, 1, ...n+r$ and j - the values of $0, 1, ...n+s-r$.

A direct calculation shows that $Norm(i,j,n,r) = (e^{nr}_{ij}, e^{nr}_{ij})$ can be represented as

$$Norm(i,j,n,r) = G(n,r)F(i,j,n,r) \tag{31}$$

where

$$G(n,r) = \frac{(s+1-r)r!n!(n+1+s)!(s+1-r)!}{}4^{(n+r)}(s+1)(s+1)!(n+r+$$

$$\{1/[w+(s+1)^2]\}\prod_{l=-r}^{n}[w+(s+1+2l)^2]$$

$$F(i,j,n,r) = i!j!/[n+r-i)!(n+s-r-j)!] \tag{33}$$

It is obvious that for the allowed values of $(ijnr)$ the norm defined by Eq. (31) is positive definite.

One can show that the basis discussed in this section is an implementation of the generators (12) but not (9) and (10). In particular, the elements e^{0r} correspond to functions $\varphi(u)$ not depending on u. If the elements e^{0r} are interpreted as rest states then S^3 could be interpreted as the coordinate space since the wave function of the rest space does not depend on coordinates. However, let us stress again, that the generators (12) do not allow a direct contraction to the Poincare group, and the standard intuition does not apply.

5 Quantization of representation generators

In standard approach to quantum theory, the operators of physical quantities act in the Fock space of the system under consideration. Suppose that the system consists of free particles and their antiparticles. Strictly speaking, in our approach it is not clear yet what should be treated as a particle or antiparticle. The considered UIRs of the dS algebra describe objects such that $(ijnr)$ is the full set of their quantum numbers. Therefore we can introduce the annihilation and creation operators $(a(i,j,n,r), a(i,j,n,r)^*)$ for these objects. Taking into account the fact that the elements e_{ij}^{nr} are not normalized to one (see Eq. (31)), we require that in the case of anticommutation relations the operators $(a(i,j,n,r), a(i,j,n,r)^*)$ satisfy the conditions

$$\{a(i,j,n,r), a(i',j',n',r')^*\} = Norm(i,j,n,r)\delta_{ii'}\delta_{jj'}\delta_{nn'}\delta_{rr'} \qquad (34)$$

while in the case of commutation relations

$$[a(i,j,n,r), a(i',j',n',r')^*] = Norm(i,j,n,r)\delta_{ii'}\delta_{jj'}\delta_{nn'}\delta_{rr'} \qquad (35)$$

In the first case, any two a-operators or any two a^* operators anticommute with each other while in the second case they commute with each other.

Eqs. (34) and (35) have a clear physical meaning if a is the annihilation operator and a^* is the creation operator. A discussion whether such an interpretation is consistent will be given in Sect. 8.

The problem of quantization of representation generators can now be formulated as follows. One should construct a linear map F from the Lie algebra of representation generators to a Lie algebra of operators acting in the Fock space. If $F(M_{ab})$ is the image of the operator M_{ab} in the Fock space and $F(M_{cd})$ is the image of the operator M_{cd} then the image of $[M_{ab}, M_{cd}]$ should be equal to $[F(M_{ab}), F(M_{cd})]$. In other words, we should have a homomorphism of Lie algebras of operators acting in the space of UIR and in the Fock space. We can also require that our map should be compatible with the Hermitian conjugation in both spaces. In what follows it will be always clear whether an operator acts in the space of UIR or in the Fock space. For this reason we will not use the notation $F(M_{ab})$ and will simply write M_{ab} instead.

The matrix elements of the operator M_{ab} in the space of UIR can be defined as

$$M_{ab}e_{ij}^{nr} = \sum_{i'j'n'r'} M_{ab}(i',j',n',r';i,j,n,r)e_{i'j'}^{n'r'} \tag{36}$$

where the sum is taken over all possible values of $(i'j'n'r')$ for the UIR in question. Then one can verify that if the image of the operator M_{ab} in the Fock space is defined as

$$\begin{aligned} M_{ab} = &\sum_{i'j'n'r'ijnr} M_{ab}(i',j',n',r';i,j,n,r) \\ &a(i',j',n',r')^* a(i,j,n,r)/Norm(i,j,n,r) \end{aligned} \tag{37}$$

then the images of any two representation generators will properly commute with each other if the operators $(a(i,j,n,r), a(i,j,n,r)^*)$ satisfy either Eq. (34) or Eq. (35).

Our next goal is to write down explicit expressions for M_{ab} in quantized form.

Since the elements e_{ij}^{nr} are the eigenvectors of the operators J_3' and $J"_3$ with the eigenvalues $n+r-2i$ and $n+s-r-2j$, respectively, the form of the matrix elements of these operators is obvious. Then, as follows from Eq. (37), the operators J_3' and $J_3"$ in quantized form can be written as

$$J_3' = \sum_{ijnr} (n+r-2i)a(i,j,n,r)^* a(i,j,n,r)/Norm(i,j,n,r) \tag{38}$$

$$J_3" = \sum_{ijnr}(n+s-r-2j)a(i,j,n,r)^* a(i,j,n,r)/Norm(i,j,n,r) \quad (39)$$

Since $J'_- e^{nr}_{ij} = e^{nr}_{i+1,j}$ and $J_-" e^{nr}_{ij} = e^{nr}_{i,j+1}$, the quantized form of the operators J'_- and $J_-"$ is as follows

$$J'_- = \sum_{ijnr} a(i+1,j,n,r)^* a(i,j,n,r)/Norm(i,j,n,r) \quad (40)$$

$$J_-" = \sum_{ijnr} a(i,j+1,n,r)^* a(i,j,n,r)/Norm(i,j,n,r) \quad (41)$$

The quantized form of the operators J'_+ and $J_+"$ easily follows from Eq. (21):

$$J'_+ = \sum_{ijnr} i(n+r+1-i)a(i-1,j,n,r)^* a(i,j,n,r)/Norm(i,j,n,r) \quad (42)$$

$$J_+" = \sum_{ijnr} j(n+s-r+1-j)a(i,j-1,n,r)^* a(i,j,n,r)/Norm(i,j,n,r) \quad (43)$$

The expressions for the quantized form of the operators $R_{\alpha\beta}$ ($\alpha, \beta = 1,2$) can be obtained as follows. First one can use the fact that $e^{nr}_{ij} = (J'_-)^i (J_-")^j e^{nr}$. Therefore, by using the commutation relations (15) one can express the action of $R_{\alpha\beta}$ on e^{nr}_{ij} in terms of the action of $R_{\alpha\beta}$ on e^{nr}. Since e^{nr} is the maximal su(2)$\times$su(2) vector, then by using Eq. (18) one can express the action of $R_{\alpha\beta}$ on e^{nr} in terms of the action of the A-operators on e^{nr}. The final expressions for the matrix elements can be obtained by using Eqs. (28-30) and then the final expressions in the quantized form — by using Eq. (37). The result of the calculations is as follows.

$$\begin{aligned} R_{11} = \sum_{ijnr} \{ & 4(n+r+1-i)(n+s-r+1-j)a(i,j,n+1,r)^* - \\ & 4j(s+1-r)(n+r+1-i)a(i,j-1,n,r+1)^* + \\ & ir(s+1-r)(n+s-r+1-j)[w+(s+1-2r)^2] \\ & a(i-1,j,n,r-1)^*/(s+2-r) + \\ & ijn(n+s+1)[w+(2n+s+1)^2]a(i-1,j-1,n-1,r)^* \} \\ & a(i,j,n,r)/[4(n+r+1)(n+s-r+1)Norm(i,j,n,r)] \end{aligned} \quad (44)$$

$$R_{21} = \sum_{ijnr} \{-jn(n+s+1)[w+(2n+s+1)^2]$$
$$a(i,j-1,n-1,r)^* - (n+s-r+1-j)r(s+1-r)$$
$$[w+(s+1-2r)^2]a(i,j,n,r-1)^*/(s+2-r) -$$
$$4j(s+1-r)a(i+1,j-1,n,r+1)^* +$$
$$4(n+s-r+1-j)a(i+1,j,n+1,r)^*\}$$
$$a(i,j,n,r)/[4(n+r+1)(n+s-r+1)Norm(i,j,n,r)] \quad (45)$$

$$R_{12} = \sum_{ijnr} \{-in(n+s+1)[w+(2n+s+1)^2]$$
$$a(i-1,j,n-1,r)^* + ir(s+1-r)[w+(s+1-2r)^2]$$
$$a(i-1,j+1,n,r-1)^*/(s+2-r) +$$
$$4(s+1-r)(n+r+1-i)a(i,j,n,r+1)^* +$$
$$4(n+r+1+1-i)a(i,j+1,n+1,r)^*\}$$
$$a(i,j,n,r)/[4(n+r+1)(n+s-r+1)Norm(i,j,n,r)] \quad (46)$$

$$R_{22} = \sum_{ijnr} \{n(n+s+1)[w+(2n+s+1)^2]$$
$$a(i,j,n-1,r)^* - r(s+1-r)[w+$$
$$(s+1-2r)^2]a(i,j+1,n,r-1)^*/(s+2-r) +$$
$$4(s+1-r)a(i+1,j,n,r+1)^* + 4a(i+1,j+1,n+1,r)^*\}$$
$$a(i,j,n,r)/[4(n+r+1)(n+s-r+1)Norm(i,j,n,r)] \quad (47)$$

6 Problem of physical and nonphysical states

Consider now the following question. As noted in Sect. 3, in the Poincare limit the operator $\mathbf{B}$ is such that $\mathbf{P} = \mathbf{B}/2R$ becomes the ordinary momentum. In this limit the energy $E = M_{04}/2R$ commutes with $\mathbf{P}$, and the sign of the energy is a good criterion for distinguishing physical and nonphysical states. However, if the basis is chosen as in Eq. (12) or Sect. 4 then the operator $\mathbf{B}$ has no longer such a simple meaning and the *standard* contraction is impossible. Nevertheless there should exist conditions when the Poincare algebra is an approximate symmetry algebra. What are these conditions?

Note that if $\mathbf{p}$ is the ordinary momentum and $p = |\mathbf{p}|$ then in the Poincare limit $|\mathbf{B}|$ is of order pR, i.e. is much greater than the ordinary angular momentum. We can assume that at the conditions when the Poincare algebra is the approximate symmetry algebra, the value of $|\mathbf{B}|$ is still very big. As follows from Eq. (17), $|\mathbf{B}|$ is much greater than $|\mathbf{M}|$ if $\mathbf{J}'$ and $\mathbf{J}''$ have approximately the same magnitude and are approximately anticollinear. Since in the state e_{ij}^{nr} the magnitude of $\mathbf{J}'$ is $n+r$, the magnitude of $\mathbf{J}''$ is $n+s-r$ and $r=0,1,...s$, then it is natural to think that in the Poincare limit the quantity n is very big and much greater than s and r.

Let us calculate the dS energy operator M_{04} in the limit when n is much greater than s and r, and i and j are of order n. It will be convenient for this purpose to normalize the basis vectors not to $Norm(i,j,n,r)$, but to one. Accordingly, the operators (a,a^*) should now satisfy not the condition (34) but

$$\{a(i,j,n,r),a(i',j',n',r')^*\} = \delta_{ii'}\delta_{jj'}\delta_{nn'}\delta_{rr'} \tag{48}$$

and analogously in the case of Eq. (35). This can be achieved if the operators $a(i,j,n,r)$ in Eqs. (38-47) are replaced by $Norm(i,j,n,r)^{1/2}a(i,j,n,r)$. Then a direct calculation using Eqs. (31-33) gives that in the above approximation Eq. (45) becomes

$$R_{21} = \sum_{ijnr}\{[i(n-j)]^{1/2}a(i+1,j,n+1,r)^* - [j(n-i)]^{1/2}$$
$$a(i,j-1,n-1,r)^*\}(w+4n^2)^{1/2}a(i,j,n,r)/2n \tag{49}$$

and Eq. (46) becomes.

$$R_{12} = \sum_{ijnr}\{[j(n-i)]^{1/2}a(i,j+1,n+1,r)^* - [i(n-j)]^{1/2}$$
$$a(i-1,j,n-1,r)^*\}(w+4n^2)^{1/2}a(i,j,n,r)/2n \tag{50}$$

Now, as follows from Eqs. (17), (49) and (50), in this approximation e_{ij}^{nr} is the eigenvector of the operator M_{04} with the eigenvalue

$$E = (w+4n^2)^{1/2}\{[j(n-i)]^{1/2} - [i(n-j)]^{1/2}\}/n.$$

It is obvious that $E > 0$ if $j > i$ and $E < 0$ if $j < i$.

The element e_{ij}^{nr} is the eigenvector of the operator $B^3 = J_3' - J_3"$ with the eigenvalue $B^3(i,j,r) = 2(r+j-i)-s$. In the above approximation $B^3(i,j,r) = 2(j-i)$ and therefore the condition $B^3(i,j,r) > 0$ is equivalent to $E > 0$ and the condition $B^3(i,j,r) < 0$ is equivalent to $E < 0$. Note also that if $i \neq j$ then $|E| > 2|j-i|$ if $w \geq 0$, by analogy with the standard case. At the same time, in contrast with the standard case, the quantities E and $2(j-i)$ always have the same sign.

Let us recall (see Sect. 4) that the quantity i can take the values $0, 1, ...N_1(n,r)$ and j can take the values $0, 1, ...N_2(n,r)$ where $N_1(n,r) = n+r$ and $N_2 = n+s-r$. The vector e_{ij}^{nr} is the eigenvector of the operator J_3' with the eigenvalue $n+r-2i$ and the eigenvector of the operator $J_3"$ with the eigenvalue $n+s-r-2j$. Therefore when $i \to N_1(n,r)-i$ and $j \to N_2(n,r)-j$, the eigenvalues change their signs and $B^3(i,j,r) \to -B^3(i,j,r)$. In view of this observation, the following question arises. If e_{ij}^{nr} is a physical state then is $e_{N_1(n,r)-i,N_2(n,r)-j}^{nr}$ a physical state too?

As follows from the above discussion, if n is big, e_{ij}^{nr} and $e_{N_1(n,r)-i,N_2(n,r)-j}^{nr}$ are also eigenvectors of the operator M_{04} with the opposite eigenvalues. If the both states are treated as physical, we can consider a system of two particles with the equal mass and spin, such that the first particle is in the state e_{ij}^{nr} and the second one - in the state $e_{N_1(n,r)-i,N_2(n,r)-j}^{nr}$. Such a system is the eigenstate of the operators $(J_3' J_3" M_{04})$ with all the eigenvalues equal to zero.

How can we investigate the system in the general case, when n is arbitrary? One of the possibilities is to use the theory of decomposition of the tensor product of two induced UIRs into UIRs [32, 37, 33]. As follows from the results of Chapt. 18 in Ref. [33], in the given case of two equal masses and spins one has to induce the tensor product of two UIRs $\Delta(r; \mathbf{s})$ in SO(1,4) and decompose it into UIRs. The latter task is not easy from the technical point of view. Another possible way is as follows. We can use the result of Sect. 5 in Ref. [19] where the mass operator of a system of two dS particles has been explicitly calculated assuming that all the states are physical. As a consequence of this result we have

Statement 1: The decomposition of the state vector of the system of two free particles with the equal mass and spin, and such that the first particle is in the state e_{ij}^{nr} and the second one - in the state $e_{N_1(n,r)-i,N_2(n,r)-j}^{nr}$, contains a state with zero values of mass and spin.

Therefore, we have a situation analogous to that discussed in Sect. 3 when the state of a system of two particles contains a state with the same quantum numbers as the vacuum. Since such a situation is unacceptable, we should conclude that the states e_{ij}^{nr} and $e_{N_1(n,r)-i,N_2(n,r)-j}^{nr}$ cannot be physical simultaneously. This is another proof that UIRs of the dS group cannot be treated in the standard way.

In the framework of Supposition 1, the problem posed by Statement 1 can be resolved if we accept

Supposition 2: If e_{ij}^{nr} is a physical state then $e_{N_1(n,r)-i,N_2(n,r)-j}^{nr}$ is a nonphysical state and *vice versa*.

In the general case the operator M_{04} does not commute with J_3' and J_3", and e_{ij}^{nr} is not the eigenvector of M_{04}. Therefore, the sign of the dS energy cannot be used for distinguishing physical and nonphysical states. However, a reasonable assumption is that the sign of $B^3(i,j,r)$ can be used for this purpose. Indeed, when n is big, this is the case since the sign of $B^3(i,j,r)$ is the same as the sign of M_{04}, and in the general case $B^3(i,j,r)$ satisfies the condition $B^3(N_1(n,r)-i, N_2(n,r)-j,n,r) = -B^3(i,j,n,r)$.

In what follows we will use only Supposition 2 and no explicit criterion for distinguishing physical and nonphysical states will be used.

Supposition 2 can be consistent only if the relations $i = N_1(n,r) - i$ and $j = N_2(n,r) - j$ *cannot* be satisfied simultaneously. This question is discussed in Sect. 8.

7 AB symmetry

In the standard approach, where a particle and its antiparticle are described by independent UIRs, Eq. (37) describes either the quantized field for particles or antiparticles. In the standard theory the notations are such that the operators $a(i,j,n,r)$ and $a(i,j,n,r)^*$ are related to particles while the operators $b(i,j,n,r)$ and $b(i,j,n,r)^*$ satisfy the analogous commutation or anticommutation relations and describe the annihilation and creation of antiparticles. Then the operators of the quantized particle-antiparticle field are given by

$$\begin{gathered} M^{ab}_{standard} = \sum_{i'j'n'r'ijnr} M^{ab}_{particle}(i',j',n'r';i,j,n,r) \\ a(i',j',n',r')^* a(i,j,n,r)/Norm(i,j,n,r) + \\ \sum_{i'j'n'r'ijnr} M^{ab}_{antiparticle}(i',j',n',r';i,j,n,r) \\ b(i',j',n',r')^* b(i,j,n,r)/Norm(i,j,n,r) \end{gathered} \tag{51}$$

where the quantum numbers in each sum take the values allowable for the corresponding UIR.

In contrast to the standard approach, Eq. (37) describes the quantized field for particles and antiparticles simultaneously. More precisely, it describes the quantized field for some object such that a particle and its antiparticle are different states of this object. The problem arises how to interpret Eq. (37) in usual terms, i.e. in terms of particles and antiparticles.

As noted in Sect. 3, in the limit when the dS algebra can be contracted to the Poincare one, we can use the following procedure (see also Ref. [23]). The states for which the energy is positive can be treated as physical states describing the particle and for them the (a, a^*) operators have the usual meaning. On the other hand, the states for which the energy is negative are nonphysical and for them the operators (a, a^*) cannot have the usual meaning. In this case we can use the idea (see Sect. 1) that annihilation of the particle with the negative energy can be treated as the creation of the antiparticle with the positive energy, and creation of the particle with the negative energy can be treated as the annihilation of the antiparticle with the positive energy. As noted in Sect. 1, in the standard approach this idea is implemented implicitly while in our approach it can be implemented explicitly.

When the contraction to the Poincare algebra is not possible, we can use Statement 2 and generalize the approach of Ref.

[23] as follows. If e_{ij}^{nr} is a physical state then we assume as usual that $a(i,j,n,r)$ is the operator annihilating this state while $a(i,j,n,r)^*$ is the operator creating this state. However, if e_{ij}^{nr} is a nonphysical state then it should be related to the antiparticle as follows. Since the state $e_{N_1(n,r)-i,N_2(n,r)-j}^{nr}$ is now physical, we can **define** the antiparticle annihilation and creation operators (b,b^*) such that $b(N_1(n,r)-i,N_2(n,r)-j,n,r)$ should be proportional to $a(i,j,n,r)^*$ and $b(N_1(n,r)-i,N_2(n,r)-j,n,r)^*$ should be proportional to $a(i,j,n,r)$.

We define the b-operators as follows.

$$\begin{aligned} &a(i,j,n,r)^* = \eta(i,j,n,r)F(i,j,n,r) \\ &b(N_1(n,r)-i,N_2(n,r)-j,n,r) \\ &a(i,j,n,r) = \eta(i,j,n,r)^*F(i,j,n,r) \\ &b(N_1(n,r)-i,N_2(n,r)-j,n,r)^* \end{aligned} \tag{52}$$

where $\eta(i,j,n,r)$ is some function. We assume that Eq. (52) defines the (b,b^*) operators regardless of whether the state e_{ij}^{nr} is physical or nonphysical. Then, as follows from the above remarks, the (b,b^*) operators have the usual meaning of antiparticle operators only for such $(ijnr)$ that e_{ij}^{nr} is physical.

The (b,b^*) operators should satisfy either

$$\{b(i,j,n,r),b(i',j',n',r')^*\} = Norm(i,j,n,r)\delta_{ii'}\delta_{jj'}\delta_{nn'}\delta_{rr'} \tag{53}$$

in the case of anticommutators, and

$$[b(i,j,n,r),b(i',j',n',r')^*] = Norm(i,j,n,r)\delta_{ii'}\delta_{jj'}\delta_{nn'}\delta_{rr'} \tag{54}$$

in the case of commutators.

As follows from Eq. (33)

$$F(N_1(n,r)-i,N_2(n,r)-j,nr)F(i,j,n,r) = 1 \tag{55}$$

Therefore, as follows from Eq. (31) and (52), Eq. (53) can be satisfied only if

$$\eta(i,j,n,r)\eta(i,j,n,r)^* = 1 \tag{56}$$

i.e. $\eta(i,j,n,r)$ is a phase factor. At the same time, Eq. (54) obviously cannot be satisfied for any choice of $\eta(i,j,n,r)$.

In general, we can conclude that if b is proportional to a^* and b^* is proportional to a then the (b,b^*) operators can satisfy only anticommutation relations.

Consider now the following question. If we take the generators in the quantized form and replace the (a,a^*) operators in them by the (b,b^*) operators using Eq. (52) then is it possible that the generators in terms of (b,b^*) will have the same form as in (a,a^*)? If such a property is satisfied and the (b,b^*) operators satisfy the same anticommutation or commutation relations as the (a,a^*) operators then, following Ref. [14], we will say that the AB symmetry takes place.

For testing the AB symmetry one needs the following property of the matrix elements:

$$\sum_{ijnr} M_{ab}(i,j,n,r;i,j,n,r) = 0 \tag{57}$$

i.e. the trace of any generator in the space of UIR is equal to zero. For the diagonal operators J_3' and J_3" the proof follows from the fact that the l.h.s. of Eq. (57) for them is the sum of all eigenvalues, and for any UIR of the su(2) algebra the sum of all eigenvalues is equal to zero. Therefore for any fixed (nr) the sum of all eigenvalues is equal to zero. For the remaining nondiagonal operators Eq. (57) also is satisfied since the nondiagonal operators necessarily change at least one quantum number $(ijnr)$.

Let us now take Eqs. (38-47) and replace the (a,a^*) operators by the (b,b^*) operators using Eq. (52). Then a direct calculation using Eqs (31-33) and (57) shows that the generators in terms of (b,b^*) have the same form as in terms of (a,a^*) if and only if

$$\eta(i,j,n,r) = \alpha(-1)^{i+j+n+r} \tag{58}$$

where α is a constant such that $\alpha\alpha^* = 1$.

Let us note that the AB symmetry has no analog in the standard theory where the sets (a,a^*) and (b,b^*) are fully independent.

They are defined only on physical states and are related to each other by the CPT transformation in Schwinger's formulation (see e.g. Refs. [38, 2]). On the contrary, Eq. (52) represents not a transformation but a definition relating the operators in physical and nonphysical states.

8 Vacuum condition

The results of the preceding section are based on the assumption that the (a, a^*) operators satisfy either Eq. (34) or Eq. (35). This is the case if a has the meaning of annihilation operator and a^* has the meaning of creation operator. Analogously, the (b, b^*) operators satisfy either Eq. (53) or Eq. (54) if b has the meaning of annihilation operator and b^* has the meaning of creation operator. However, no arguments have been given yet that these operators indeed can be treated in such a way.

By analogy with the standard approach, one might define the vacuum vector Φ_0 such that

$$a(i,j,n,r)\Phi_0 = b(i,j,n,r)\Phi_0 = 0 \quad \forall\ (i,j,n,r) \tag{59}$$

Then the elements

$$\Phi_+(i,j,n,r) = a(i,j,n,r)^*\Phi_0 \quad \Phi_-(i,j,n,r) = b(i,j,n,r)^*\Phi_0 \tag{60}$$

might be treated as one-particle states for particles and antiparticles, respectively.

However, if one requires the condition (59) then it is obvious from Eqs. (52) that the elements defined by Eq. (60) are null vectors. We can therefore try to modify Eq. (59) as follows. Let S_+ be a set of elements $(ijnr)$ such that e_{ij}^{nr} is a physical state and S_- be a set of elements $(ijnr)$ such that e_{ij}^{nr} is a nonphysical state. Then if Supposition 2 is consistent, S_+ and S_- do not intersect and each element $(ijnr)$ belongs either to S_+ or S_-. Now instead of the condition (59) we require

$$a(i,j,n,r)\Phi_0 = b(i,j,n,r)\Phi_0 = 0 \quad \forall\ (i,j,n,r) \in S_+ \tag{61}$$

In that case the elements defined by Eq. (60) will indeed have the meaning of one-particle states for $(i,j,n,r) \in S_+$.

In our approach there is no problem with the stability of the vacuum, at least in the absence of interactions. At the same time, as noted in Sect. 1, in modern LQFT in curved spacetime [9] the vacuum in the SO(1,4) invariant theory is unstable. Let us discuss this question in greater details. Consider particles described by the operators (9). By analogy with the nonrelativistic quantum mechanics, one can define the position operator as $i/(m\partial \mathbf{v})$ and time can be defined by the condition that the dS Hamiltonian is the operator describing evolution. Then one can show that, in the quasiclassical limit, the motion of the particles is in agreement with the classical motion in the dS space. The proof is especially simple for nonrelativistic particles (see e.g. Refs. [19, 20, 34]). Therefore, in our approach the classical dS space exists only in the approximation when there are quasiclassical particles. If the system is in the state described by the vacuum vector Φ_0 then, in our approach, there are no particles and no classical dS space exists (in the spirit of Mach's principle). On the contrary, in the approach of Ref. [9], the classical dS space always exists and one can consider properties of quantum states from the standpoint of a geodesic observer. It is well known that in Copenhagen formulation of quantum theory, the presence of a classical observer is always assumed. It is not clear whether the formulation is universal, e.g. whether it applies at the very early stage of the Universe. We will not dwell on the discussion of this problem since it has been extensively discussed in the literature.

We now return to the discussion of quantum states in our approach. Regardless of how the sets S_+ and S_- are defined, it is clear that the above construction can be consistent only if there are no such values of $(ijnr)$ that the equalities $i = N_1(n,r) - i$ and $j = N_2(n,r) - j$ are satisfied simultaneously. Indeed, suppose that there exist such elements $(ijnr)$ that the both equalities are satisfied simultaneously. Then these elements belong to both S_+ and S_- (see Sect. 6). Since $b(i,j,n,r)$ is proportional to $a(N_1(n,r) - i, N_2(n,r) - j, n, r)^*$ then, as follows from Eq. (52), if $i = N_1(n,r) - i$, $j = N_2(n,r) - j$ and Φ_0 is annihilated by both $a(i,j,n,r)$ and $b(i,j,n,r)$, it is also annihilated by both $a(i,j,n,r)$ and $a(i,j,n,r)^*$. However this contradicts Eqs. (34) and (35).

Since $N_1(n,r) = n+r$ and $N_2(n,r) = n+s-r$, it is obvious that if s is even then $N_1(n,k)$ and $N_2(n,k)$ are either both even or both odd. Therefore in that case we will necessarily have a situation when for some values of (nr), $N_1(n,k)$ and $N_2(n,k)$ are both even. In that case $i = N_1(n,r) - i$ and $j = N_2(n,r) - j$ necessarily takes place for $i = N_1(n,k)/2$ and $j = N_2(n,k)/2$. Moreover, since for each (nr) the number of all possible values of $(ijnr)$ is equal to $(N_1(n,r) + 1)(N_2(n,r)+1)$, this number is odd (therefore one cannot divide the set of all possible values into the equal nonintersecting parts S_+ and S_-).

On the other hand, if s is odd then for all the values of (nr) we will necessarily have a situation when either $N_1(n,r)$ is even and $N_2(n,r)$ is odd or $N_1(n,r)$ is odd and $N_2(n,r)$ is even. Therefore for each value of (nr) the case when the equalities $i = N_1(n,r) - i$ and $j = N_2(n,r) - j$ are satisfied simultaneously is impossible, and the number of all possible values of $(ijnr)$ is even.

We conclude that the condition (61) is consistent only if s is odd, or in other words, if the particle spin in usual units is half-integer.

In Sect. 6 we argued that the sign of $B^3(i,j,r)$ is a good criterion for distinguishing physical and nonphysical states. If $i = N_1(n,r) - i$ and $j = N_2(n,r) - j$ are satisfied simultaneously, then it is obvious that $B^3(i,j,r) = 0$. Therefore Supposition 2 will be always consistent if $B^3(i,j,r) = 0$ is impossible. This is automatically satisfied if s is odd, since $B^3(i,j,r) = 2(r+j-i)-s$. Let us stress, however, that the conclusion of this section that s should be odd, does not depend on the explicit way of breaking the set of elements $(ijnr)$ into S_+ and S_-.

9 Neutral particles, AB^2 parity and space inversion

Suppose that the particle in question is neutral, i.e. the particle coincides with its antiparticle. On the language of the operators (a, a^*) and (b, b^*) this means that these sets are the same, i.e. $a(i, j, n, r) = b(i, j, n, r)$ and $a(i, j, n, r)^* = b(i, j, n, r)^*$. Then as follows from Eqs. (55) and (58), Eq. (52) is consistent only if s is even. This means that in our approach neutral elementary particles with the half-integer spin (in conventional units) cannot exist. At the same time, as shown in the preceding section, the integer spin in conventional units is incompatible with the vacuum condition. For this reason we conclude that in our approach there can be no neutral elementary particles. As argued in Ref. [14], this conclusion is natural in view of the following observation. If one irreducible representation describes a particle and its antiparticle simultaneously, the energy operator necessarily contains the contribution of the both parts of the spectrum, corresponding to the particle and its antiparticle. If a particle were the same as its antiparticle then the energy operator would contain two equal contributions and thus the value of the energy would be twice as big as necessary.

Consider now the following question. In Sect. 7 the AB symmetry has been formulated as the condition that the (b, b^*) operators satisfy the same anticommutation or commutation relations as the (a, a^*) operators and the representation generators have the same form in terms of (a, a^*) and (b, b^*). In that case the operators (b, b^*) are defined in terms of (a, a^*) by Eqs. (52). A desire to have operators which can be interpreted as those relating separately to particles and antiparticles is natural in view of our experience in the standard approach. However, in the spirit of our approach, there is no need to have separate operators for particles and antiparticles since they are different states of the same object. For this reason the operators (b, b^*) are stricly speaking redundant. We can therefore reformulate the AB symmetry as follows. Instead of Eqs. (52), we consider a *transformation* defined as

$$\begin{array}{l} a(i,j,n,r)^* \to \eta(i,j,n,r)F(i,j,n,r) \\ a(N_1(n,r)-i, N_2(n,r)-j,n,r) \\ a(i,j,n,r) \to \eta(i,j,n,r)^*F(i,j,n,r) \\ a(N_1(n,r)-i, N_2(n,r)-j,n,r)^* \end{array} \tag{62}$$

Then the AB symmetry can be formulated as a requirement that the anticommutation or commutation relations and operators related to physical quantities should be invariant under this transformation.

The results of Sect. 7 can now be reformulated in such a way that the representation generators are compatible with this new formulation of the AB symmetry (strictly speaking, the name "AB symmetry" is not appropriate anymore but we retain it for "backward compatibility").

Let us now apply the AB transformation twice. Then, as follows from Eqs. (55) and (58),

$$a(i,j,n,r)^* \to (-1)^s a(i,j,n,r)^* \quad a(i,j,n,r) \to (-1)^s a(i,j,n,r) \tag{63}$$

Since only an odd value of s is compatible with the vacuum condition, we can formulate this result by saying that the AB^2 parity of each elementary particle is equal to -1. Therefore, as a consequence of the AB symmetry, any interaction can involve only an even number of creation and annihilation operators. Since in our approach only fermions with the half-integer spin (in conventional units) can be elementary, this result is obvious.

The results of Sect. 8 can be easily reformulated for the case when only the (a,a^*) operators are used. In this case $a(i,j,n,r)$ can be treated as annihilation operator when $(ijnr)\in S_+$ and as creation one when $(ijnr)\in S_-$. Analogously, $a(i,j,n,r)^*$ can be treated as creation operator when $(ijnr)\in S_+$ and as annihilation one when $(ijnr)\in S_-$. The consistency condition is the requirement that there should be no such $(ijnr)$ that $i=N_1(n,r)-i$ and $j=N_2(i,j)-j$. As shown in Sect.

8, this condition can be satisfied only for particles with the half-integer spin (in conventional units).

Finally, consider the space inversion in our approach. We define the space inversion as the transformation

$$a(i,j,n,r)^* \to \eta_P^*(-1)^n[G(r)/G(s-r)]^{1/2}a(j,i,n,s-r)^*$$
$$a(i,j,n,r) \to \eta_P(-1)^n[G(r)/G(s-r)]^{1/2}a(j,i,n,s-r) \quad (64)$$

where η_P is the spatial parity and

$$G(r) = [(s+1-r)/2^r]^2 \prod_{l=1}^{r}[w+(s+1-2l)^2] \quad (65)$$

Then, as follows from Eqs. (31-33), the anticommutation or commutation relations, (34) or (35), are invariant under the transformation (64) if $|\eta_P| = 1$, i.e. η_P is a phase factor. If we apply the transformation (64) to Eqs. (38-47) then a direct calculation using Eqs. (17), (31-33) and (65) shows that $\mathbf{M} \to \mathbf{M}$, $\mathbf{B} \to -\mathbf{B}$, $\mathbf{N} \to -\mathbf{N}$ and $M_{04} \to M_{04}$. Therefore Eq. (64) indeed has the meaning of the space inversion.

Consider now whether the space inversion is compatible with the AB symmetry. Let us first apply the transformation (64) to the second relation in Eq. (62). Taking into account Eq. (58) we obtain

$$\eta_P a(j,i,n,s-r) \to (-1)^{i+j+n+r}(\alpha\eta_P)^* F(i,j,n,r)$$
$$a(n+s-r-j,n+r-i,n,s-r) \quad (66)$$

On the other hand, by applying the transformation (62) to $a(j,i,n,s-r)$ we obtain

$$a(j,i,n,s-r) \to (-1)^{i+j+n+s-r}\alpha^* F(j,i,n,s-r)$$
$$a(n+s-r-j,n+r-i,n,s-r) \quad (67)$$

Now, as follows from Eq. (33), Eqs. (66) and (67) are compatible with each other if and only if

$$\eta_P^* = (-1)^s \eta_P \quad (68)$$

We have obtained a well known result (see e.g. Ref. [39]) that particles with the half-integer spin (in conventional units) have imaginary parity. In our approach this result is a direct consequence of the AB symmetry.

10 Discussion

In the present paper we have reformulated the standard approach to quantum theory as follows. Instead of requiring that each elementary particle is described by its own UIR of the symmetry algebra, we require that one UIR should describe a particle and its antiparticle simultaneously. In that case, among the Poincare, AdS and dS algebras, only the latter can be a candidate for constructing elementary particle theory.

Although our approach considerably differs from that in LQFT in curved spacetime, our results confirm the conclusion of Refs. [9, 10] and references therein that the dS group cannot be a symmetry group in the standard approach (see Sects. 1 and 3). In contrast with the standard approach, the space of UIR in our one contains two sets — physical and nonphysical states. As a consequence, the dS algebra can be a symmetry algebra if one accepts Supposition 1.

In Ref. [10], which appeared after the original version of the present paper, problems with the dS spacetime have been discussed in the framework of the thermofield theory, which was originally invented in many-body theory (see Ref. [10] for references). The existence of physical and nonphysical states in our approach looks similar to the analogous feature in the thermofield theory, but the interpretation of such states in our approach fully differs from that in the thermofield theory. Nevertheless, this example confirms that similar ideas can be invented in approaches which considerably differ each other.

For the physical states the operators (a, a^*) have the usual meaning while for nonphysical states a becomes the creation operator while a^* - the annihilation one. It is obvious that only anticommutation relations are consistent with such an interchange. This simple observation immediately explains why in our approach only fermions can be elementary (see Sect. 7).

The fact that the vacuum condition is consistent only for particles with the half-integer spin (in conventional units) has been proved in Sect. 8. A simple explanation is as follows. Since there should be equal numbers of physical and nonphysical states and they should not intersect with each other, the number of states in each su(2)$\times$su(2) multiplet should be even. These conditions are satisfied only if the spin is half-integer.

Our results can be summarized as follows. *If the de Sitter algebra so(1,4) is the symmetry algebra in elementary particle theory then only fermions can be elementary and they can have only the half-integer spin.*

In the standard theory there exists the well-known Pauli spin-statistics theorem [40] stating that elementary particles with the half-integer spin are fermions while the particles with the integer spin — bosons. The proof has been given in the framework of the standard local quantum field theory. After the original Pauli proof, many authors investigated more general approaches to the spin-statistics theorem (see e.g. Ref. [41] and references cited therein). On the other hand, in the framework of the Skyrme model [42] and its generalizations (see e.g. Refs. [43]) fermions can be built of bosons.

Our result has been proved only for free particles (as well as the original proof [40]). It follows only from general properties of representations of groups and algebras in Hilbert spaces and does not involve any locality conditions.

In Ref. [23], where the Poincare limit has been considered, we have proved that only fermions can be elementary but no restriction on spin has been obtained. The reason is that in the Poincare limit the physical and nonphysical states are fully disjoint (they have supporters on the upper and lower Lorentz hyperboloids, respectively), and the number of all possible states is always even for any spin. On the other hand, as noted in Sect. 1, in the Poincare limit, the Wigner approach to elementary particles is compatible with that in the LQFT.

In our opinion, the very possibility that only fermions with the half-integer spin can be elementary, is very attractive from the aesthetic point of view. Indeed, what was the reason for nature to create elementary fermions and bosons if the latter can be built of the former? A well known historical analogy is that before the discovery of the Dirac equation, it was believed that nothing could be simpler than the Klein-

Gordon equation for spinless particles. However, it has turned out that the spin 1/2 particles are simpler since the covariant equation for them is of the first order, not the second one as the Klein-Gordon equation. A very interesting possibility (which has been probably considered first by Heisenberg) is that only spin 1/2 particles are elementary.

In our recent series of papers [14] it has been argued that quantum theory based on a Galois field (GFQT) is more natural than the standard one. In the GFQT the property that one IR simultaneously describes a particle and its antiparticle, is satisfied automatically, and it was the main reason for the present investigation. In other words, in the GFQT there are no IRs describing only particles without antiparticles. It has been proved in Ref. [14] that in the GFQT only a half-integer spin is compatible with the vacuum condition, and the proof given in Sect. 8 of the present paper is similar. At the same time, in Ref. [14] we have not succeeded in proving that only fermions are elementary. Roughly speaking, the reason is that in Galois fields the quantity $\alpha\alpha^*$ is not necessary greater than zero when $\alpha \neq 0$. It has been also noted that if only fermions are elementary, then the actual infinity is not present in the theory in any form, and for each sort of elementary particles their number in the Universe cannot be greater than p^3 where p is the characteristic of the Galois field.

Acknowledgements: The author is grateful to V. Dobrev, L. Koyrakh, M.B. Mensky, M.A. Olshanetsky, E. Pace, G. Salme and M. Saniga for fruitful discussions.

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