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Gauge Fields on Tori and T-duality

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Abstract

We discuss gauge fields on tori in diverse dimensions, mainly in two and four dimensions. We construct various explicit gauge fields which have some topological charges and find the Dirac zero modes in the background of the gauge fields. By using the zero mode, we give new gauge fields on the dual torus, which is a gauge theoretical description of T-duality transformation of the corresponding D-brane systems including D$\bar{\rm D}$ systems. From the transformation, we can easily see the duality expected from the index theorem. It is also mentioned that, for each topological charges, the corresponding constant curvature bundle can be constructed and their duality transformation can be performed in terms of Heisenberg modules.

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1 Introduction

In the study of non-perturbative aspects of field theories, solitons and instantons have played crucial roles. Especially, gauge theories on tori possess various topological characters which comes from the non-triviality of the topology of the tori, and have been studied intensively, for example, in order to explain quark confinement [1, 2].

Gauge theories on four-dimensional tori T^4 have the mysterious duality, that is, *Fourier-Mukai-Nahm duality*, which is one-to-one correspondence (more strongly hyperKähler isometry) between instanton moduli space on the torus T^4 and that on the dual torus $\hat{T}^4$ [3, 4, 5, 6]. The duality transformation is called *Nahm transformation* which originated in the application of ADHM construction [7] to BPS monopoles [8]. One of the interesting points is that under the Nahm transformation, the rank of the gauge group and the instanton number are interchanged. Physically, the transformation is just the T-duality transformation.

Gauge theoretical analysis are often strong to study D-brane dynamics, such as tachyon condensation [9]. Nahm transformation is taken for the gauge fields on D-brane and hence is expected to describe more in detail in some points than usual T-duality transformation and to reveal new aspects of D-brane dynamics.

In the present paper, we study gauge theories on tori in diverse dimensions and take explicit Nahm-like transformations via the Dirac zero modes in the background of the gauge fields. We construct various explicit gauge fields which have constant curvatures. This specifies the corresponding D-brane systems. We find the explicit Dirac zero modes in the background of the gauge fields which show consistency with the family index theorem [10]. By using the zero mode, we give new gauge fields on the dual torus in canonical ways, which is consistent with the T-duality transformation of the corresponding D-brane systems including D$\bar{\text{D}}$ systems. We focus on two- and four-dimensional tori. The higher-dimensional extension is straightforward.

This paper is organized as follows. In section 2, we discuss gauge theories on two-dimensional tori and T-duality transformation of D0-D2 brane systems. In section 3, we discuss gauge theories on four-dimensional tori and T-duality transformation of D0-D4 brane systems, which is Nahm transformation. In section 4, we discuss the higher-dimensional extensions. Finally in section 5, we give conclusions and some comments on noncommutative extensions of the present discussion.

2 Gauge fields on T^2 and T-duality

In this section, we discuss gauge fields on two-tori T^2 and the gauge theoretical descriptions of T-duality transformations of D0-D2 brane systems. We first give a general scheme of two-dimensional version of Nahm transformation and then apply it to some explicit gauge fields. The key point in the transformation is to find the Dirac zero mode. The result shows a beautiful duality and is consistent with T-duality. We mainly treat a flux solution where the rank of the gauge field is N and the first Chern number[1] (the magnetic flux) is k which corresponds to k D0-branes and N D2-branes on T^2. Finally we comment on the generalization to non-flux solutions including D0-$\bar{\text{D}}$0 branes on D2-branes on T^2.

For simplicity, we set the both periods of the torus T^2 2π.

First of all, we shall recall general results of Atiyah-Singer family index theorem. The detailed discussion is given later soon.

Main theorem is:

$$\mathrm{ch}(\mathrm{ind}\not{D}_\xi) = \int_{T^{2n}} \mathrm{ch}(\mathcal{P})\mathrm{ch}(E), \tag{1}$$

where $\mathrm{ch}(E) := \mathrm{Tr}\exp(F/2\pi i)$ represents the Chern character of the vector bundle E over a $2n$-dimensional torus and F is the curvature two form of E. The i-th Chern character $\mathrm{ch}_i(E)$ is concretely represented as

$$\mathrm{ch}_1(E) = \frac{i}{2\pi}\,\mathrm{Tr}\,F, \quad \mathrm{ch}_2(E) = -\frac{1}{8\pi^2}\,\mathrm{Tr}\,F\wedge F, \cdots .$$

$\mathcal{P}$ are called the Poincaré bundle. On even dimensional tori, the Dirac operator $\not{D}_\xi$ is decomposed into two Weyl components $\mathcal{D}_\xi$ and $\bar{\mathcal{D}}_\xi$. $\mathrm{ind}\not{D}_\xi$ is then defined as $\mathrm{ind}\not{D}_\xi := \ker\bar{\mathcal{D}}_\xi - \ker\mathcal{D}_\xi$, which belong to K_0-group over the dual torus $\hat{T}^{2n}$. If the dimension of either $\ker\bar{\mathcal{D}}_\xi$ or $\ker\mathcal{D}_\xi$ is constant with respect to ξ, the index theorem implies the dimensions of both $\ker\bar{\mathcal{D}}_\xi$ and $\ker\mathcal{D}_\xi$ are constant, and then $\ker\bar{\mathcal{D}}_\xi$ and $\ker\mathcal{D}_\xi$ are vector bundles over dual torus. Physically this is just the D$\bar{\text{D}}$ system. In particular if $\ker\mathcal{D}_\xi$ is trivial, the index bundle $\mathrm{ind}\not{D}_\xi$ becomes so called zero-mode bundle $\ker\bar{\mathcal{D}}_\xi =: \hat{E}$ over the dual torus.

Now we define the two-dimensional Nahm transformation explicitly. Suppose that the gauge group is $U(N)$ and the first Chern number is k. The coordinates of torus T^2 and the dual torus are denoted by (x_1, x_2) and (ξ_1, ξ_2), respectively. The indices run as follows: space indices $\mu, \nu = 1, 2$; D2 indices $u, v = 1, \ldots, N$; D0 indices $p, q = 1, \ldots, k$.

[1]In the present paper, we use the word "the n-th Chern number" as the integral of the n-th Chern character.

Let us consider two-dimensional (massless) Dirac operator on T^2:

$$\not{D} := \gamma_\mu \otimes D_\mu = \gamma_\mu \otimes (\partial_\mu + A_\mu),$$

where $\gamma_\mu = -i\sigma_\mu,\ \mu = 1, 2$. ($\sigma_i$ are usual Pauli matrices.) The Dirac operator act on the section of the tensor product of the vector bundle E and spinor bundle S on T^2.

Now we take the tensor product of (the pull-back of) E and the Poincaré bundle P on $T^2 \times \hat{T}^2$ The connection of the Poincaré bundle is given by $-i\xi_\mu$. Hence the connection of this tensor bundle is $D_\mu(\xi) = \partial_\mu + A_\mu - i\xi_\mu$ and the Dirac operator is as follows:

$$\begin{aligned} \not{D}_\xi &:= \gamma_\mu \otimes D_\mu(\xi) = \begin{pmatrix} 0 & \mathcal{D}_\xi \\ \bar{\mathcal{D}}_\xi & 0 \end{pmatrix} \\ \not{D}_\xi^2 &= \begin{pmatrix} \mathcal{D}_\xi \bar{\mathcal{D}}_\xi & 0 \\ 0 & \bar{\mathcal{D}}_\xi \mathcal{D}_\xi \end{pmatrix}. \end{aligned}$$

Let us suppose that the (chiral-decomposed) Dirac operator $\mathcal{D}_\xi$ has no smooth solution. Then the (family) index theorem says that the Dirac operator $\bar{\mathcal{D}}_\xi$ has independent normalized k solutions:

$$\bar{\mathcal{D}}_\xi \psi = 0, \tag{2}$$

where $\psi(x, \xi)$ is $N \times k$ matrix in which the each row vector ψ^p is the independent solution and is called the Dirac zero mode. The gauge field on the dual torus can be constructed by the orthonormal projection onto the zero-mode bundle: $\hat{D}_\mu = P\hat{\partial}_\mu = (\psi\psi^\dagger)\hat{\partial}_\mu$, or explicitly,

$$\hat{A}_\mu(\xi) = \int_{T^2} d^2x\ \psi^\dagger \hat{\partial}_\mu \psi. \tag{3}$$

The dual gauge field is on the dual torus and anti-Hermite. Hence the gauge group is actually $U(k)$. According to the family index theorem (1) :

$$\mathrm{rank}(\hat{E}) = C_1(E), \quad \hat{C}_1 = \mathrm{rank}(E).$$

This means that the two-dimensional Nahm transformation exchanges the rank of gauge group and the first Chern number, which is consistent with T-duality transformation of D0-D2 brane system.

In the general arguments above, we assumed the existence of vector bundles with $\ker \mathcal{D}_\xi = 0$. Such a vector bundle actually exists for each topological charge with positive first Chern number. It can be described by a simple solution which describes k D0-branes as flux on N D2-branes:

$$\Omega_1 = e^{ikx_2/N}U \ , \quad \Omega_2 = V \ , \quad UV = e^{-2\pi ik/N}VU, \tag{4}$$

$$A_1 = 0, \quad A_2 = -\frac{i}{2\pi}\frac{k}{N}x_1, \quad F_{12} = -\frac{i}{2\pi}\frac{k}{N}, \quad C_1 = \frac{i}{2\pi}\int_{T^2} d^2x \ F_{12} = k. \tag{5}$$

Here the matrices U and V are $N \times N$ matrices defined by

$$U_{uv} = \delta_{uv}e^{\frac{2\pi iku}{N}}, \quad V_{uv} = \delta_{u+1,v} + \delta_{uN}\delta_{v1}. \tag{6}$$

The patching matrices (transition functions) $\Omega_1(x_1, x_2), \ \Omega_2(x_1, x_2)$ specify the topology of the bundle and act on the section ψ in the fundamental representation of the gauge group as

$$\Omega_1(x_1, x_2)\psi(x_1, x_2) = \psi(x_1 + 2\pi, x_2), \quad \Omega_2(x_1, x_2)\psi(x_1, x_2) = \psi(x_1, x_2 + 2\pi). \tag{7}$$

In fact, the patching matrices (4) are defined so that the cocycle condition for the vector bundle holds

$$\Omega_1^{-1}(x_1 + 2\pi, x_2)\Omega_2^{-1}(x_1 + 2\pi, x_2 + 2\pi)\Omega_1(x_1, x_2 + 2\pi)\Omega_2(x_1, x_2) = 1,$$

and the covariant derivatives (5) also satisfy the following compatibility conditions:

$$\begin{aligned}(\partial_\mu + A_\mu)(x_1 + 2\pi, x_2) &= \Omega_1(x_1, x_2)(\partial_\mu + A_\mu)(x_1, x_2)\Omega_1^{-1}(x_1, x_2), \\ (\partial_\mu + A_\mu)(x_1, x_2 + 2\pi) &= \Omega_2(x_1, x_2)(\partial_\mu + A_\mu)(x_1, x_2)\Omega_2^{-1}(x_1, x_2)\end{aligned}$$

for $\mu = 1, 2$. The general form of the section is given by [11, 12, 13]

$$\psi_u(x_1, x_2) = \sum_{s\in\mathbb{Z}}\sum_{p=1}^{k} \exp\left[ix_1\left\{\frac{k}{N}\left(\frac{x_2}{2\pi} + u + Ns\right) + p\right\}\right]\phi^p\left(\frac{x_2}{2\pi} + u + Ns + \frac{N}{k}p\right). \tag{8}$$

One can confirm that this actually satisfies Eq. (7) for each u.

This section actually has no $\ker \mathcal{D}_\xi$. Now let us solve the Dirac equation (2) and give the form of the dual bundle. The zero mode have to have the general form (8) and satisfies the Dirac equation and the normalized condition. The solution is

$$\begin{aligned}\psi_u^p(\xi, x) \quad = \quad & \left(\frac{N}{2\pi k}\right)^{\frac{1}{4}} \sum_{s\in\mathbb{Z}} \exp\left[ix_1\left\{\frac{k}{N}\left(\frac{x_2}{2\pi} + u + Ns\right) + p\right\}\right] \\ & \times \exp\left[-2\pi i\xi_2\left\{\frac{x_2}{2\pi} + u + Ns - \frac{N}{k}(\xi_1 - p)\right\}\right] \\ & \times \exp\left[-\pi\frac{k}{N}\left\{\frac{x_2}{2\pi} + u + Ns - \frac{N}{k}(\xi_1 - p)\right\}^2\right].\end{aligned} \tag{9}$$

This solution have several interesting points. First, the summation over p in the general section (8) is dropped out in the zero-mode because of the normalization condition and hence label p runs just from 1 to k, which suggests the index theorem on the number of Dirac zero mode. Second, if we take the opposite sign for the first Chern number, then the Gaussian factor in the third line of Eq. (9) diverges and there is no normalized zero-mode of Dirac operator $\bar{\mathcal{D}}$. Instead in this case, there is k normalized zero mode of $\mathcal{D}$, which also clearly suggests the index theorem.

Finally we get the dual gauge field from Eq. (3):

$$\hat{A}_1 = 2\pi i \frac{N}{k}\xi_2, \quad \hat{A}_2 = 0, \quad \hat{F}_{12} = -2\pi i \frac{N}{k}, \quad \hat{C}_1 = \frac{i}{2\pi}\int_{\hat{T}^2} d^2\xi \ \hat{F}_{12} = N.$$

where the patching matrices are

$$\hat{\Omega}_1 = \hat{V} \ , \quad \hat{\Omega}_2 = e^{-2\pi i N\xi_1/k}\hat{U}. \tag{10}$$

where $\hat{U}, \hat{V}$ are $k \times k$ matrices like (6) such as $\hat{U}\hat{V} = e^{-2\pi i \frac{N}{k}}\hat{V}\hat{U}$. We also note that the result is already expected at the stage of the zero mode (9). It is in fact compatible with the dual patching matrices (10) except for the factor e^{ix_2} which disappears in the integration over torus T^2.

Thus, we can show that the vector bundles defined by Eq. (4) and (5) are transformed to those of the same type. In particular, the transformation exchange the rank of gauge group and the first Chern number beautifully. As stated above, one can also begin with vector bundles with negative first Chern numbers. In this case, one obtains some systems on $\bar{\text{D}}$2-branes. Only the trivial bundle, that is, the vector bundle with zero first Chern number, can not be transformed in the context of these differential geometry. In fact, from the index theorem the rank of the dual 'bundle' is zero. This is a coherent sheaf on the dual torus and can be treated in the framework of the algebraic geometry.

There are some simple generalizations. Let $E_{(N,k)}$ and $\hat{E}_{(k,N)}$ be the bundle in question on torus T^2 and on the dual torus $\hat{T}^2$, respectively. We can construct the product bundle $E_{(N_1,k_1)} \oplus \cdots \oplus E_{(N_n,k_n)}$. In this case, the curvature is not a scalar matrix in general and the corresponding D0-brane is not regarded as a flux. Especially if $k_i > 0, \ k_j < 0$ for some i, j, then the D-brane configuration contains k_i D0-branes and k_j anti-D0-brane. In fact, when one consider the fluctuation around the solution, the fluctuation of the fields between the D-branes $E_{(N_i,k_i)}$ and $E_{(N_j,k_j)}$ have negative mass square, which implies that the fluctuation includes tachyon modes and the system is unstable. [2] It is interesting that we can take the explicit T-duality transformation of such D$\bar{\text{D}}$ systems in the context of the gauge theories on D-branes in the present way. (See also [14].)

[2] The fluctuation spectrum is studied in detail in, for example, [15, 16]. Similar situations are discussed in noncommutative cases in [17].

3 Gauge fields on T^4 and T-duality

In this section, we discuss anti-self-dual gauge fields on four-tori T^4 and gauge theoretical descriptions of T-duality transformation of D0-D4 brane systems which is known as Nahm transformation. General scheme of the transformation is the same as that of D0-D2. The important point is that this transformation preserves the self-duality of the gauge fields.

In this case, we have to consider usual four-dimensional theory. As the previous section, the Dirac operator with real parameter is defined as follows:

$$\begin{aligned} \not{D}_\xi &:= \gamma_\mu \otimes D_\mu(\xi) = \begin{pmatrix} 0 & \mathcal{D}_\xi \\ \bar{\mathcal{D}}_\xi & 0 \end{pmatrix} \\ \not{D}_\xi^2 &= \begin{pmatrix} \mathcal{D}_\xi \bar{\mathcal{D}}_\xi & 0 \\ 0 & \bar{\mathcal{D}}_\xi \mathcal{D}_\xi \end{pmatrix} = \begin{pmatrix} D^2 + \eta^{(-)\mu\nu} F_{\mu\nu} & 0 \\ 0 & D^2 + \eta^{(+)\mu\nu} F_{\mu\nu} \end{pmatrix}, \end{aligned}$$

where $D_\mu(\xi) = \partial_\mu + A_\mu - i\xi_\mu$ and the gamma matrices are

$$\gamma_\mu := \begin{pmatrix} 0 & e_\mu \\ \bar{e}_\mu & 0 \end{pmatrix}.$$

The Euclidean 4-dimensional Pauli matrices are defined by

$$e_\mu := (-i\sigma_i, 1), \quad \bar{e}_\mu = (i\sigma_i, 1)$$

and satisfy the following relations:

$$\bar{e}_\mu e_\nu = \delta_{\mu\nu} + i\eta^{i(+)}_{\mu\nu}\sigma_i, \quad e_\mu \bar{e}_\nu = \delta_{\mu\nu} + i\eta^{i(-)}_{\mu\nu}\sigma_i.$$

Here $\eta^{i(\pm)}_{\mu\nu}$ are called 't Hooft symbol and are anti-symmetric and (anti-)self-dual.

First let us consider four-dimensional Dirac operator with real parameter ξ_μ. Suppose that the gauge group is $U(N)$ and the gauge field is anti-self-dual whose instanton number C_2 (the second Chern number) is k.

Here we suppose that the Dirac operator $\mathcal{D}_\xi$ has no smooth solution. Then index theorem implies that the Dirac operator $\bar{\mathcal{D}}_\xi$ has independent normalized k solutions:

$$\bar{\mathcal{D}}_\xi \psi = 0, \tag{11}$$

where the each column corresponds to the independent k solution and hence ψ can be considered as $2N \times k$ matrix. Then we can construct the gauge field on the dual torus $\hat{T}^4$ just as in two-dimensional case:

$$\hat{A}_\mu = \int_{T^4} dx^4 \ \psi^\dagger \hat{\partial}_\mu \psi.$$

The second Chern number of the gauge field $\hat{A}$ is N. This transformation is in fact one-to-one and called Nahm transformation.

In the following, we construct various (anti-)self-dual gauge field and take the Nahm transformation by solving the Dirac equations.

The first example is very simple. It is essentially given by the tensor product of the previous example on two-dimensional torus.

The anti-self-dual gauge field is given as follows:

$$A_1 = 0, \quad A_2 = -\frac{i}{2\pi}\frac{k}{N}x_1 \otimes \mathbf{1}_{N\times N}, \quad A_3 = 0, \quad A_4 = \mathbf{1}_{N\times N} \otimes \frac{i}{2\pi}\frac{k}{N}x_3. \tag{12}$$

The curvature is

$$F_{12} = -F_{34} = -\frac{i}{2\pi}\frac{k}{N}\mathbf{1}_{N\times N} \otimes \mathbf{1}_{N\times N}.$$

and the second Chern number is k^2. This bundle is constructed as tensor-like product $E_{(N,k)} \otimes E_{(N,-k)}$. Because of the opposite twist, self-duality of the gauge field are realized. The gauge group is considered as $U(N^2)$.

The Dirac zero mode is essentially the product of two-dimensional case:

$$\begin{aligned}\psi^{pp'}_{uu'}(\xi, x) \quad &= \quad \left(\frac{N}{2\pi k}\right)^{\frac{1}{2}} \sum_{s,t\in\mathbb{Z}} e^{ix_1(\frac{k}{N}(\frac{x_2}{2\pi}+u+Ns)+p)} e^{-2\pi i\xi_2(\frac{x_2}{2\pi}+u+Ns-\frac{N}{k}(\xi_1-p))} e^{-\frac{\pi k}{N}(\frac{x_2}{2\pi}+u+Ns-\frac{N}{k}(\xi_1-p))^2} \\ &\qquad \times e^{-ix_3(\frac{k}{N}(\frac{x_4}{2\pi}+u'+Nt)+p')} e^{2\pi i\xi_4(\frac{x_4}{2\pi}+u'+Nt-\frac{N}{k}(\xi_3-p'))} e^{-\frac{\pi k}{N}(\frac{x_4}{2\pi}+u'+Nt-\frac{N}{k}(\xi_3-p'))^2}.\end{aligned} \tag{13}$$

Then the dual gauge field becomes

$$\hat{A}_1 = 2\pi i\frac{N}{k}\xi_2 \otimes \mathbf{1}_{k\times k}, \quad \hat{A}_2 = 0, \quad \hat{A}_3 = \mathbf{1}_{k\times k} \otimes -2\pi i\frac{N}{k}\xi_4, \quad \hat{A}_4 = 0.$$

whose second Chern number is N^2 and gauge group is $U(k^2)$. So D0-brane charge and D4-brane charge are exchanged.

As discussed previously, there is a simple generalization of this: the Nahm transformation of $(E_{(N_1,k_1)} \otimes E_{(N_1,-k_1)}) \oplus \cdots \oplus (E_{(N_n,k_n)} \otimes E_{(N_n,-k_n)})$ where the rank of the gauge group is $\sum_{i=1}^n n_i^2$ and the second Chern number is $\sum_{i=1}^n k_i^2$. If we take $\sum_{i=1}^n k_i = 0$, then the first Chern number becomes all zero and D2-brane charges disappear. The discussion in the $n = 2, N_1 = 1, N_2 = 1, k_1 = 1, k_2 = -1$ case coincides with that in [18]. Alternatively, if we consider a self dual configuration as $E_{(N_1,k_1)} \otimes E_{(N_1,k_1)}$, it is transformed to the systems on $\bar{\text{D}}4$-brane. Moreover, if one consider the direct sum of a anti-self-dual configuration and a self dual configuration, one can see that the fluctuations corresponding to the modes between them are negative and include the tachyonic modes similarly as discussed on the two-dimensional tori.

The previous example has the separate form between 1-2 direction and 3-4 direction. But we can treat the mixed version as follows. For simplicity, we focus on $G = U(1), C_2 = 2$.

The gauge field is given by

$$A_1 = 0, \quad A_2 = -\frac{i}{2\pi}(x_1 - x_3), \quad A_3 = 0, \quad A_4 = \frac{i}{2\pi}(x_1 + x_3).$$

The field strength is

$$F_{12} = -F_{34} = \frac{i}{2\pi}, \quad F_{14} = -F_{23} = \frac{i}{2\pi}.$$

Now F_{14} part contributes more to the second Chern number. Hence the number becomes two.

In this case, the corresponding section is of the form

$$\begin{aligned} \psi(x_1, x_2, x_3, x_4) \quad &= \sum_{s,t\in\mathbb{Z},p\in\mathbb{Z}_2} \exp\left[i(x_1 - x_3)\left(\frac{x_2}{2\pi} + s + \frac{p}{2}\right) - i(x_1 + x_3)\left(\frac{x_4}{2\pi} + t + \frac{p}{2}\right)\right] \\ &\quad \times \phi^p\left(\frac{x_2}{2\pi} + s + \frac{p}{2}, \frac{x_4}{2\pi} + t + \frac{p}{2}\right) \ . \end{aligned} \tag{14}$$

This can also be written as

$$\begin{aligned} \psi(x_1, x_2, x_3, x_4) \quad &= \sum_{s,t\in\mathbb{Z},p\in\mathbb{Z}_2} \exp\left[ix_1\left(\frac{x_2 - x_4}{2\pi} + s'\right) - ix_3\left(\frac{x_2 + x_4 + p}{2\pi} + t'\right)\right] \\ &\quad \times \phi^p\left(\frac{x_2}{2\pi} + s + \frac{p}{2}, \frac{x_4}{2\pi} + t + \frac{p}{2}\right) \ . \end{aligned}$$

where $s' := s - t$ and $t' := s + t + p$ run over $\mathbb{Z}^2$ densely. The fact explains the reason why p is necessary in Eq. (14). In the process of solving the Dirac equation $\bar{\mathcal{D}}_\xi \psi = 0$, one can see that the solutions are of the form

$$\begin{aligned} \psi^p = \begin{pmatrix} \psi^p_+ \\ \psi^p_- \end{pmatrix} \quad &= \begin{pmatrix} 1 \\ c \end{pmatrix} \sum_{s,t\in\mathbb{Z}} \exp\left[i(x_1 - x_3)\frac{x'_2}{2\pi} - i(x_1 + x_3)\frac{x'_4}{2\pi}\right] \\ &\quad \times \exp\left[-\left(a_2(x'_2)^2 + a_4(x'_4)^2 + a_{24}x'_2x'_4 + b_2x'_2 + b_4x'_4\right)\right] \end{aligned}$$

where $x'_2 := x_2 + 2\pi(s + p/2)$, $x'_4 := x_4 + 2\pi(t + p/2)$, $c, a_2, a_4, a_{24}, b_2, b_4 \in \mathbb{C}$ and especially $\mathrm{Re}(a_2) > 0$, $\mathrm{Re}(a_4) > 0$. Solving the differential equations leads quadratic equations. Especially, the quadratic equation for c is obtained:

$$c^2 + 2c - 1 = 0.$$

This equation has the two solutions: $c = -1 \pm \sqrt{2}$. On the other hand, both a_2 and a_4 are positive only when $c = -1 + \sqrt{2}$ because of $a_2 + a_4 = (1 + c)/2\pi$. For fixed c, the rest variables $a_2, a_4, a_{24}, b_2, b_4$ are determined uniquely. Thus we can confirm that there are two orthogonal Dirac zero-modes (ψ_+^p, ψ_-^p) for $p = 0, 1$.

Next, the ξ dependence is determined by normalizing these zero-modes. We get

$$\begin{aligned}\begin{pmatrix}\psi_+^p \\ \psi_-^p\end{pmatrix} &\propto \begin{pmatrix}1 \\ c\end{pmatrix} \sum_{s,t\in\mathbb{Z}} \exp\left[i(x_1 - x_3)\frac{x_2'}{2\pi} - i(x_1 + x_3)\frac{x_4'}{2\pi}\right] \\ &\times \exp\left[-\frac{1}{2}\mathrm{Re}(X^t)\begin{pmatrix}2a_2 & a_{24} \\ a_{24} & 2a_4\end{pmatrix}\mathrm{Re}(X) - i\mathrm{Re}(X^t)\begin{pmatrix}2a_2 & a_{24} \\ a_{24} & 2a_4\end{pmatrix}\mathrm{Im}(X)\right]\end{aligned}$$

where

$$X = \begin{pmatrix}x_2' \\ x_4'\end{pmatrix} + 2\pi\begin{pmatrix}1+c & 1-c \\ 1-c & 1+c\end{pmatrix}^{-1}\begin{pmatrix}-(\xi_1 + i\xi_2) + c(\xi_3 + i\xi_4) \\ (\xi_3 - i\xi_4) + c(\xi_1 - i\xi_2)\end{pmatrix} .$$

Then the dual gauge field is obtained as

$$A_1 = \frac{2\pi i}{2}(-\xi_2 + \xi_4)\mathbf{1}_{2\times 2}, \quad A_2 = 0, \quad A_3 = -\frac{2\pi i}{2}(\xi_2 + \xi_4)\mathbf{1}_{2\times 2}, \quad A_4 = 0 .$$

One can then confirm that the topological numbers agree with the ones expected from the T-duality or the family index theorem (1), in particular the rank is two and $\hat{C}_2 = 1$.

4 Generalizations

We can easily generalize the previous discussions to higher-dimensional case. The family index theorem (1) holds in every even dimensions and the zero-mode bundle can be constructed in fact even if we suppose that there is no kernel of $\not{D}$. Hence we can perform the explicit higher-dimensional Nahm transformation for the constant curvature gauge field in [19, 16]. The zero mode can be constructed as the product of the two-dimensional zero-mode (9) as in four dimensional one (13).

Note that we can construct any other constant curvature bundles in terms of Heisenberg modules [20] and take their Nahm transformations for arbitrary dimensional tori. Heisenberg modules are known as projective modules (noncommutative analogue of vector bundles) over noncommutative tori with constant curvature connections [20], but they can also be used in commutative cases. Recall the constant curvature bundles we have used in the body of this paper. The forms of their sections, for example Eq. (8) and Eq. (14), are essentially determined by ϕ^p for fixed Chern characters. The ϕ^p's are nothing but the Heisenberg modules. Generally, the Heisenberg modules over n dimensional (noncommutative) tori are described as functions on $\mathbb{R}^d \times \mathbb{Z}^{d'} \times F$ where F is a finite group and $2d + d' = n$. They are specified by defining the action of the generators of $C^\infty(T^n)$. Here $\mathbb{Z}^{d'}$ part corresponds to that the bundle is trivial for the corresponding d' directions of the torus. This means that its Nahm dual module is not described by a vector bundle. Hence we consider the case $\mathbb{R}^d \times F$ where $n = 2d$. Then one can see that the ϕ^p in Eq. (8) is the case $d = 1, F = \mathbb{Z}_k$. Similarly, the example (12) corresponds to the case $d = 2, F = \mathbb{Z}_k \times \mathbb{Z}_k$. ϕ^p in Eq. (14) and its dual bundle are then the case $d = 2, F = 1$ and $d = 2, F = \mathbb{Z}_2$, respectively. Any Heisenberg module has a constant curvature. Moreover, when one construct the twisted bundles corresponding to the Heisenberg module as in our examples, the constant curvature on the Heisenberg module is compatible with that on the twisted bundle. Namely, our examples of the Nahm transformations are essentially those accomplished by using Heisenberg modules and the extensions to any other constant curvature bundles can also be done in terms of Heisenberg modules.

5 Conclusions and Discussions

We studied the transformations of gauge fields which correspond to the T-duality transformations on even dimensional tori. On two-dimensional tori, after expanding general arguments of the transformations, we presented the explicit example, where we considered a constant curvature bundle for any topological number and took the Nahm-like transformation. The dual bundle was confirmed to be the same type as the original constant curvature bundle, and the result certainly agrees with the T-duality transformations. On four-dimensional tori, we considered the instanton configurations of gauge fields and their transformations, which are just the Nahm transformations. Applying the results to the two-dimensional situations, we constructed various constant curvature bundle and took their Nahm transformations explicitly. One can also consider the direct sum of these constant curvature bundle and their Nahm-like transformations. We saw that these situations generally express the T-duality transformations of D$\bar{\text{D}}$ systems. Finally we commented about higher dimensional extensions. In particular, for any topological number the corresponding constant curvature bundle and their Nahm-like transformations can be obtained in terms of Heisenberg modules.

One of the future direction is noncommutative extensions of the Nahm transformation. A noncommutative Nahm transformation is discussed formally in [21], however the noncommutative Nahm transformation have not ever performed. From the viewpoint along this paper, one may deal with the noncommutative version of the twisted bundles discussed in [12, 13]. However, it is difficult to solve the Dirac equation consistently since it is a differential equation on noncommutative space. One may also consider Heisenberg modules. However, one arrives at the same problem if one tries to obtain the Dirac zero modes by solving a differential equation. Another approach is to define the tensor product between two Heisenberg modules directly like as in [22, 23]. Also, noncommutative versions of index theorems may be useful. Anyway, this problem seems to be relevant to the Morita equivalence [20] of D$\bar{\text{D}}$ system on noncommutative tori [24, 25] from physical viewpoints. We would like to report such directions elsewhere.

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