

Doubly Special Relativity theories as different bases of κ -Poincaré algebra

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Abstract

Doubly Special Relativity (DSR) theory is a theory with two observer-independent scales, of velocity and mass (or length). Such a theory has been proposed by Amelino–Camelia as a kinematic structure which may underline quantum theory of relativity. Recently another theory of this kind has been proposed by Magueijo and Smolin. In this paper we show that both these theories can be understood as particular bases of the κ -Poincaré theory based on quantum (Hopf) algebra. This observation makes it possible to construct the space-time sector of Magueijo and Smolin DSR. We also show how this construction can be extended to the whole class of DSRs. It turns out that for all such theories the structure of space-time commutators is the same. This results lead us to the claim that physical predictions of properly defined DSR theory should be independent of the choice of basis.

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1 Introduction

About a year ago, in two seminal papers [1], [2] (see also [3]) G. Amelino-Camelia proposed a theory with two observer-independent kinematical scales: of velocity c and of mass m_P ¹. There are two major motivations for such extension of special relativity. The first stems from the quest for quantum gravity in which the Planck length is supposed to play a fundamental role. For example, in loop quantum gravity the area and volume operators have discrete spectra, with minimal value proportional to the square and cube of Planck length, respectively [4]. The basic observation is that if one regards the Planck length as a fundamental, intrinsic characteristic of the space-time structure, this length should be the same for all observers and thus one immediately finds himself in conflict with FitzGerald-Lorentz contraction. The only solution of this problem is to modify the principles of special relativity so as to incorporate the existence of this second observer-independent scale. It was indicated [1], [2] that a possible candidate for such a theory might be a theory based on deformed Poincaré symmetry, for example the κ -Poincaré theory [5, 6, 7, 8, 9]. This claim was then proved to be correct: it has been explicitly shown in the papers [10], [11] that the κ -Poincaré theory in the so-called bicrossproduct basis indeed predicts the existence of an observer-independent maximal mass. The second motivation comes from some puzzling observations of ultra-high energy cosmic rays, whose existence seems to contradict the standard understanding of astrophysical process like $e^+e^- \rightarrow \pi^0\pi^0$ production in pp collisions and the photopion production in the scattering of high energy protons with soft photons. It turns out that both these effects can be explained if one assumes that the threshold condition becomes deformed, possibly as a result of existence of a new fundamental length (or mass) scale (see [12] and references therein). It is exciting to note that it is likely that this purely quantum gravitational effect will be a subject of numerous experimental tests in the near future.

Most of the works on the relativity theory with two observed independent kinematical scales, dubbed “Doubly Special Relativity” (DSR), has been done in the framework of (or strongly motivated by) algebraic construction based on the quantum (Hopf) κ -Poincaré algebra, being a deformation of the standard Poincaré algebra of special relativity. Recently however Magueijo and Smolin [13] have proposed a seemingly completely different DSR. The relation between these two theories was thoroughly analyzed in [14]. The existence of two DSR’s raises an obvious question how many theories of this kind may exist. In this paper we show that from the quantum algebraic point of view both above mentioned DSR theories are in fact completely equivalent, and might be considered as representations of the same κ -Poincaré algebra in different bases. Moreover, one can easily construct different yet representations of this algebra, each of whose would correspond to a different DSR theory. Therefore, in what follows we would use the notion of different basis instead of different DSR.

The plan of this paper is the following. In the next section we present three bases of κ -Poincaré algebra: the bicrossproduct one, lying behind the

¹In his papers Amelino-Camelia uses the scale of length ℓ_P instead of the scale of mass m_P , however since the construction presented there describes the energy-momentum sector of the theory, it seems more natural to use the scale of mass. It should be also noted that it was the scale of mass (more precisely of momentum, p_P), and not of length, that has been shown explicitly to be observer-independent.

first DSR construction, the Magueijo–Smolin one, and the classical one, whose algebraic sectors is identical with the standard Poincaré algebra, and we derive the transformations relating them. In section 3 we make use of the co-algebraic sector of these bases to derive the space-time non-commutative structure and to extend it to the whole of the phase space. Section 4 is devoted to physical interpretation of a picture resulting from mathematical constructions presented in sections 2 and 3.

2 Lorentz algebra and energy–momentum sector

One of the main assumption in construction of DSR is that Lorentz subalgebra of the κ -Poincaré algebra is not to be deformed. This assumption is motivated by the fact that one wants to work with Lorentz structure that integrates to a group, and not to a quasigroup. It restricts the possible choices of bases severely (note that this postulate is not satisfied by the so-called standard basis of the κ -Poincaré algebra [5].) We therefore assume that the three rotation generators M_i and three boost generators N_i satisfy

$$[M_i, M_j] = i \epsilon_{ijk} M_k, \quad [M_i, N_j] = i \epsilon_{ijk} N_k, \quad [N_i, N_j] = i \epsilon_{ijk} M_k. \quad (1)$$

One also assumes that the action of rotations is not deformed and that generators of momenta commute. Taking these postulates as a starting point, we can define the (deformed) action of the Lorentz algebra on energy–momentum sector. Our starting point would be the bicrossproduct basis in which the resulting algebra is a quantum algebra, i.e., in addition to the algebra of commutators (which is usually non-linear) it possess additional structures: co-product Δ and antipode S . However one should remember that quantum algebra structure is built on an enveloping algebra, which means that one is entitled to make any transformations among generators (and not only the linear one as in the case of Lie algebras.) This leaves, of course, a lot of freedom in the choice of energy and momentum generators.

2.1 The bicrossproduct basis

Since, as said above the action of rotations is standard it is sufficient to write down only the commutators of deformed boost generators with momenta. One gets [6]

$$[N_i, p_j] = i \delta_{ij} \left(\frac{\kappa}{2} \left(1 - e^{-2p_0/\kappa} \right) + \frac{1}{2\kappa} \vec{p}^2 \right) - i \frac{1}{\kappa} p_i p_j, \quad (2)$$

and

$$[N_i, p_0] = i p_i. \quad (3)$$

One can easily check that the first Casimir operator of the algebra (1) – (3) reads²

$$m^2 = \left(2\kappa \sinh \left(\frac{p_0}{2\kappa} \right) \right)^2 - \vec{p}^2 e^{p_0/\kappa}. \quad (4)$$

²It turns out that this form of the Casimir, which was used in all the papers devoted to κ -Poincaré algebra and its application is not physical, see section 4 for detailed discussion.

It follows that for positive κ the three-momentum is bounded from above $\vec{p}^2 < \kappa^2$ and the maximal value of momentum corresponds to infinite energy [10], [11].

The quantum algebra (Hopf) structure in this basis is provided by the following co-products

$$\begin{aligned}\Delta(M_i) &= M_i \otimes \mathbb{1} + \mathbb{1} \otimes M_i, \\ \Delta(N_i) &= N_i \otimes \mathbb{1} + e^{-p_0/\kappa} \otimes N_i + \frac{1}{\kappa} \epsilon_{ijk} p_j \otimes M_k, \\ \Delta(p_i) &= p_i \otimes \mathbb{1} + e^{-p_0/\kappa} \otimes p_i, \\ \Delta(p_0) &= p_0 \otimes \mathbb{1} + \mathbb{1} \otimes p_0,\end{aligned}\tag{5}$$

and the antipodes

$$\begin{aligned}S(M_i) &= -M_i, \\ S(N_i) &= -e^{p_0/\kappa} N_i + \frac{1}{\kappa} \epsilon_{ijk} e^{p_0/\kappa} p_j M_k, \\ S(p_0) &= -P_0, \\ S(P_i) &= -e^{p_0/\kappa} P_i.\end{aligned}\tag{6}$$

Taking these formulas as a starting point, we can now turn to analysis of another bases.

2.2 Magueijo–Smolin basis

In the recent paper [13] Magueijo and Smolin proposed another DSR theory, whose boost generators were constructed as a linear combination of the standard Lorentz generators and the generator of dilatation (but in such a way that the algebra (2) holds.) In this basis the commutators of four-momenta P_μ and boosts have the following form

$$[N_i, P_j] = i \left(\delta_{ij} P_0 - \frac{1}{\kappa} P_i P_j \right),\tag{7}$$

and

$$[N_i, P_0] = i \left(1 - \frac{P_0}{\kappa} \right) P_i.\tag{8}$$

It is easy to check that the Casimir for this algebra has the form

$$M^2 = \frac{P_0^2 - \vec{P}^2}{\left(1 - \frac{P_0}{\kappa}\right)^2}.\tag{9}$$

The question arises as to if this basis is equivalent to the bicrossproduct basis above. The answer is affirmative, indeed one easily checks that the relation between variables P_μ and p_μ is given by

$$p_i = P_i\tag{10}$$

$$p_0 = -\frac{\kappa}{2} \log \left(1 - \frac{2P_0}{\kappa} + \frac{\vec{P}^2}{\kappa^2} \right), \quad P_0 = \frac{\kappa}{2} \left(1 - e^{-2p_0/\kappa} + \frac{\vec{p}^2}{\kappa^2} \right). \quad (11)$$

Let us note by passing that the formula above shows that the maximal momentum in the bicrossproduct basis ($\vec{p}^2 = \kappa^2$, $p_0 = \infty$) corresponds to the maximal energy in the Magueijo–Smolin basis, $P_0 = \kappa$.

Using formulas above one can without difficulty promote this algebra to the quantum algebra. This amounts only in using the homomorphisms (10), (11) to define the new co-products and antipodes. They read

$$\Delta(P_i) = P_i \otimes 1 + \left(1 - \frac{2P_0}{\kappa} + \frac{\vec{P}^2}{\kappa^2} \right)^{1/2} \otimes P_i \quad (12)$$

$$\begin{aligned} \Delta(P_0) = & P_0 \otimes 1 + 1 \otimes P_0 - \frac{2}{\kappa} P_0 \otimes P_0 + \frac{1}{\kappa^2} \vec{P}^2 \otimes P_0 + \\ & + \frac{1}{\kappa} \left(1 - \frac{2P_0}{\kappa} + \frac{\vec{P}^2}{\kappa^2} \right)^{1/2} \sum P_i \otimes P_i \end{aligned} \quad (13)$$

$$S(P_0) = \left(1 - \frac{2P_0}{\kappa} + \frac{\vec{P}^2}{\kappa^2} \right)^{-1} \left(\frac{\vec{P}^2}{2\kappa} - \frac{\kappa}{2} \right) + \frac{\kappa}{2} \quad (14)$$

$$S(P_i) = -P_i \left(1 - \frac{2P_0}{\kappa} + \frac{\vec{P}^2}{\kappa^2} \right)^{-1/2} \quad (15)$$

2.3 The classical basis

There is yet another basis which we will present here for comparison (this basis was first described in [15], [17], [16], [8].) In this basis, which we call the classical one, the boosts–momenta commutators together with the Lorentz sector form the classical Poincaré algebra, to wit

$$[N_i, \mathcal{P}_j] = i \delta_{ij} \mathcal{P}_0, \quad [N_i, \mathcal{P}_0] = i \mathcal{P}_i. \quad (16)$$

The Casimir for this basis equals, of course the one of special relativity, to wit

$$\mathcal{M}^2 = \mathcal{P}_0^2 - \vec{\mathcal{P}}^2 \quad (17)$$

The classical generators \mathcal{P}_μ are related to the bicrossproduct basis generators by the formulas

$$\mathcal{P}_0 = \kappa \sinh \frac{p_0}{\kappa} + e^{p_0/\kappa} \frac{\vec{p}^2}{2\kappa}, \quad (18)$$

$$\mathcal{P}_i = e^{p_0/\kappa} p_i \quad (19)$$

and one can easily compute the expression for co-product

$$\begin{aligned} \Delta(\mathcal{P}_0) = & \frac{\kappa}{2} (K \otimes K - K^{-1} \otimes K^{-1}) + \\ & + \frac{1}{2\kappa} (K^{-1} \vec{\mathcal{P}}^2 \otimes K + 2K^{-1} \mathcal{P}_i \otimes \mathcal{P}_i + K^{-1} \otimes K^{-1} \vec{\mathcal{P}}^2), \end{aligned} \quad (20)$$

$$\Delta(\mathcal{P}_i) = \mathcal{P}_i \otimes K + \mathbb{1} \otimes \mathcal{P}_i \quad (21)$$

where

$$K = e^{p_0/\kappa} = \frac{1}{\kappa} \left[\mathcal{P}_0 + \left(\mathcal{P}_0^2 - \vec{\mathcal{P}}^2 + \kappa^2 \right)^{1/2} \right],$$

and the antipode

$$S(\mathcal{P}_0) = -\mathcal{P}_0 + \frac{1}{\kappa} \vec{\mathcal{P}}^2 K^{-1} \quad (22)$$

$$S(\mathcal{P}_i) = -\mathcal{P}_i K^{-1} \quad (23)$$

3 The space-time non-commutativity

In the preceding section we investigated the energy-momentum algebras. Now it is time to explain the relevance of the co-product (quantum) structure of these algebras. Briefly, this structure makes it possible to extend an energy-momentum algebra to the whole of a phase space, i.e., the space describing both energy-momentum and space-time sectors in self consistent way. One should stress that this is the only way to interconnect the space-time and energy-momentum sectors in a systematic way. It turns out that the co-algebra of the energy-momentum sector is in one to one correspondence with algebraic sector of space-time algebra and vice versa. Observe that in order to construct such a correspondence we need one more dimensionful parameter, which we identify with the Planck constant (in the following we will use the convention in which $\hbar = 1$.) We will comment on this point in the next section.

There is a general procedure how to construct the space-time commutator algebra from energy-momentum co-algebra, which consists of the following steps [6], [18]:

1. One defines the bracket $\langle \star, \star \rangle$ between momentum variables p, q and position variables x, y in a natural way as follows

$$\langle p_\mu, x_\nu \rangle = -i\eta_{\mu\nu}, \quad \eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1). \quad (24)$$

2. This bracket is to be consistent with the co-product structure in the following sense

$$\langle p, xy \rangle = \langle p_{(1)}, x \rangle \langle p_{(2)}, y \rangle, \quad \langle pq, x \rangle = \langle p, x_{(1)} \rangle \langle q_{(2)}, x_{(2)} \rangle, \quad (25)$$

where we use the natural notation for co-product

$$\Delta t = \sum t_{(1)} \otimes t_{(2)}.$$

It should be also noted that by definition

$$\langle \mathbb{1}, \mathbb{1} \rangle = 1.$$

One sees immediately that the fact that momenta commute translates to the fact that positions co-commute

$$\Delta x_\mu = \mathbb{1} \otimes x_\mu + x_\mu \otimes \mathbb{1}. \quad (26)$$

Then the first equation in (25) along with (24) can be used to deduce the form of the space-time commutators.

3. It remains only to derive the cross relations between momenta and positions. These can be found from the definition of the so-called Heisenberg double (see [18]) and read

$$[p, x] = x_{(1)} < p_{(1)}, x_{(2)} > p_{(2)} - xp \quad (27)$$

As an example let us perform these steps in the bicrossproduct basis [6], [18]. It follows from (25) that

$$< p_i, x_0 x_j > = -\frac{1}{\kappa} \delta_{ij}, \quad < p_i, x_j x_0 > = 0,$$

from which one gets

$$[x_0, x_i] = -\frac{i}{\kappa} x_i. \quad (28)$$

Let us now make use of (27) to get the standard relations

$$[p_0, x_0] = i, \quad [p_i, x_j] = -i \delta_{ij}. \quad (29)$$

However it turns out that this algebra contains one more non-vanishing commutator, namely

$$[p_i, x_0] = -\frac{i}{\kappa} p_i. \quad (30)$$

Of course, the algebra (28–30) satisfies the Jacobi identity.

3.1 Magueijo–Smolin basis

To find the non-commutative structure of space time in Magueijo–Smolin basis we start again with eq. (24)

$$< P_\mu, X_\nu > = -i\eta_{\mu\nu}.$$

Let us now turn to the next step, eq. (25). It is easy to see that the only terms in (12), (13), which are relevant for our computations are the bilinear ones, so we can write

$$\Delta(P_i) = \mathbb{1} \otimes P_i + P_i \otimes \mathbb{1} - \frac{1}{\kappa} P_0 \otimes P_i + \dots$$

$$\Delta(P_0) = \mathbb{1} \otimes P_0 + P_0 \otimes \mathbb{1} - \frac{2}{\kappa} P_0 \otimes P_0 + \frac{1}{\kappa} \sum P_i \otimes P_i + \dots$$

It follows immediately that the only non-vanishing commutators in the position sector are

$$[X_0, X_i] = -\frac{i}{\kappa} X_i. \quad (31)$$

Now we can use eq. (27) to derive the form of the remaining commutators. Since this computation is a bit tricky, let us present the necessary steps.

$$\begin{aligned} [P_0, X_i] &= \sum_j \left\langle \frac{1}{\kappa} \sqrt{1 - \frac{2P_0}{\kappa} + \frac{\vec{P}^2}{\kappa^2}} P_j, X_i \right\rangle P_j + X_i < \mathbb{1}, \mathbb{1} > P_0 - X_i P_0 = \\ &= \frac{1}{\kappa} \sum_j < P_j, X_i > P_j \end{aligned}$$

(we made use of the fact that the only terms linear in momenta have non-vanishing bracket with positions) from which it follows immediately that

$$[P_0, X_i] = -\frac{i}{\kappa} P_i \quad (32)$$

and by employing the same procedure we obtain the remaining commutators

$$[P_0, X_0] = i \left(1 - \frac{2P_0}{\kappa} \right) \quad (33)$$

$$[P_i, X_j] = -i \delta_{ij} \quad (34)$$

$$[P_i, X_0] = -\frac{i}{\kappa} P_i. \quad (35)$$

Of course, as it is easy to check, the algebra above satisfies the Jacobi identity.

3.2 The classical basis

We again start with the duality relation

$$\langle \mathcal{P}_\mu, \mathcal{X}_\nu \rangle = -i\eta_{\mu\nu},$$

and to get the commutators in the position sector as above we take the part of the co-product up to the bilinear terms

$$\Delta(\mathcal{P}_i) = \mathbb{1} \otimes \mathcal{P}_i + \mathcal{P}_i \otimes \mathbb{1} + \frac{2}{\kappa} \mathcal{P}_0 \otimes \mathcal{P}_i + \frac{1}{\kappa} \mathcal{P}_i \otimes \mathcal{P}_0 + \dots$$

$$\Delta(\mathcal{P}_0) = \mathbb{1} \otimes \mathcal{P}_0 + \mathcal{P}_0 \otimes \mathbb{1} + \frac{1}{\kappa} \sum \mathcal{P}_i \otimes \mathcal{P}_i + \dots$$

which leads again to

$$[\mathcal{X}_0, \mathcal{X}_i] = -\frac{i}{\kappa} \mathcal{X}_i. \quad (36)$$

Then by employing the same method as in the preceding subsection we find the space-time commutators

$$[\mathcal{P}_0, \mathcal{X}_0] = \frac{i}{2} \left(K + K^{-1} - \frac{1}{\kappa^2} \vec{\mathcal{P}}^2 K^{-1} \right) \quad (37)$$

$$[\mathcal{P}_0, \mathcal{X}_i] = -\frac{i}{\kappa} \mathcal{P}_i \quad (38)$$

$$[\mathcal{P}_i, \mathcal{X}_j] = -i K \delta_{ij} \quad (39)$$

$$[\mathcal{P}_i, \mathcal{X}_0] = 0. \quad (40)$$

Let us summarize the results of obtained so far. We found the transformations in the energy-momentum sector relating the bicrossproduct, Magueijo-Smolín, and classical bases of κ -Poincaré algebra. Next we extended this construction to the space-time sector. This examples strongly suggest that the following two claims hold under mild and physically acceptable assumption

concerning the change of bases (one must assume that in any basis an algebra becomes the standard Poincare algebra in the leading order in κ expansion)

(a) Any algebra consisting of the undeformed Lorentz sector with the standard action of rotations and generic action of boosts on commuting four-momenta can be equipped with the κ -Poincare quantum algebraic structure and

(b) This structure can be extended to the whole of the phase space, and the non-commutative structure of the space-time sector is always (i.e., in any basis) given by

$$[x_0, x_i] = -\frac{i}{\kappa} x_i. \quad (41)$$

The precise formulation of these statements and their proof will be presented in forthcoming paper.

4 Physical interpretation

Till now we have been dealing with mathematics, it is time therefore to turn to physics. In view of remarks at the end of preceding section the situation is as follows. Instead of having a number of distinct DSR theories we have to do with a whole class of such theories, which are related to each other by redefinition of momenta

$$p_0 \rightarrow f(p_0, \vec{p}^2), \quad p_i \rightarrow g(p_0, \vec{p}^2) p_i \quad (42)$$

restricted only by the requirement that the action of rotations is preserved, each of whose can be interpreted as being a particular basis of κ -Poincare quantum algebra. If we are to base a physical theory on such a mathematical structure, we have clearly two choices either to find some physical and/or mathematical motivation to single out one particular basis, or to define the physical theory in such a way that its physical predictions are independent of the choice of a basis. This last possibility reminds very much the postulate of general coordinate invariance of general relativity and seems worth pursuing. From this perspective the fact that one of the most fundamental prediction of the theory, namely the non-commutativity in the space-time sector (41) is invariant under the change of basis is very encouraging. However one should note that in such a case, the “momentum” variables do not have a direct physical meaning. This does not seem very surprising after all. Indeed, in special relativity as well as in non-relativistic mechanics energy and momentum are defined so as to be conserved in physical process and this conservation is regarded as one of the most important physical properties of nature. On the other hand energy/momentum conservation is directly related to the homogeneity of space-time. Here we have to do with non-commutative space-time and the presence of length scale strongly suggest that space-time is homogeneous only at scales much larger than this scale. It is not clear how to define the energy/momentum conservation (and thus to answer the question what is the *physical* energy and momentum) for a non-commutative space-time, but it is certain that any such definition must include information concerning the space-time non-commutativity.

Let us turn to another point. It is clear that one of the most important physical characteristic of a particle is its rest mass. This mass should be, as in the special relativity and non-relativistic mechanics equal to the Casimir operator of the theory at hands. It is reasonable therefore to adopt the following

definition of physical rest mass

$$\frac{1}{m_0} = \lim_{p \rightarrow 0} \frac{1}{p} \frac{dE}{dp}, \quad E \equiv p_0 \quad (43)$$

It is easy to check that this definition works in special relativity, where $m^2 = E^2 - p^2$ and in non-relativistic mechanics, where $m = p^2/2E$. Let us now check whether the Casimirs in bases discussed above are physical masses. This is clearly the case in the classical basis. In the Magueijo–Smolin basis one gets

$$M_0^2 = \frac{E^2}{\left(1 - \frac{E}{\kappa}\right)^2} = M^2$$

and thus the Magueijo–Smolin Casimir (9) indeed equals the physical rest mass.

The situation in the bicrossproduct basis is different, however. One easily finds that in this basis

$$m_0^2 = \frac{\kappa^2}{4} \left(1 - e^{-2p_0/\kappa}\right)^2 = \frac{\kappa^2}{4} \left(1 - \left(-\frac{m}{2\kappa} + \sqrt{\frac{m^2}{4\kappa^2} + 1}\right)^4\right)^2, \quad (44)$$

where m^2 is given by eq. (4). This result is of importance because it suggests that the form of dispersion relation used in the literature so far might be incorrect. This suggests in particular that one should rethink the status of κ -Poincaré as a possible explanation of the threshold anomalies in cosmic rays astrophysics (cf. [12].) We will return to this question in separate paper.

It is also interesting to note the following relation between physical masses in bicrossproduct (m_0), Magueijo–Smolin (M_0), and classical (M_0) bases

$$\frac{1}{M_0} = \frac{1}{m_0} - \frac{1}{\kappa}, \quad \frac{1}{M_0^2} = \frac{1}{M_0^2} - \frac{1}{\kappa^2}. \quad (45)$$

This suggests that an appropriately defined physical mass might be invariant under change of basis and therefore a candidate for the second „observable” of DSR if this theory could be consistently constructed in the basis-independent way.

Another point to be stressed is the following. In the DSR theory proposed in [1], [2] one has to do with two scales of velocity and of mass. It should be noted that here we have three scales (in our notation two of them κ and ℓ_P were put equal ℓ_P): of speed κ , which relates time and space components of physical quantities, of mass κ , which governs the deformation of the energy–momentum sector, and, finally, the Planck constant ℓ_P , which makes it possible to relate the energy–momentum and space–time sectors. This means that in fact we have to do with Triply not Doubly Special Relativity. Observe that these three scales must be present in the theory to make the construction presented above possible. Of course, the question arises as to what is the physical status of these scale: are they observer-independent quantities or just coupling constants. To answer this question one must find out what is an operational definition of each of them.

5 Conclusions

Let us summarize the basic results of this paper. We showed that DSR theories analyzed so far can be understood as different bases of the κ -Poincaré theory. This observation made it possible to construct the space-time sector of these theories. This construction, in turn, led us to the claim that the DSR theory is perhaps based on principle similar to diffeomorphism invariance of general relativity, namely that the physical predictions of DSR should be independent of the choice of basis.

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