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Hierarchies of Spin Models related to Calogero-Moser Models

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Abstract

The universal formulation of spin exchange models related to Calogero-Moser models implies the existence of integrable hierarchies, which have not been explored. We show the general structures and features of the spin exchange model hierarchies by taking as examples the well-known Heisenberg spin chain with the nearest neighbour interactions. The energy spectra of the second member of the hierarchy belonging to the models based on the A_r root systems ($r = 3, 4, 5$) are explicitly and *exactly* evaluated. They show many many interesting features and in particular, much higher degree of degeneracy than the original Heisenberg model, as expected from the integrability.

In a previous paper [1] we presented universal Lax pairs for the spin Calogero-Moser models [2, 3, 4] and spin exchange models [5, 6, 7, 8] which give rise to enough number of conserved quantities. As in the continuous Calogero-Moser models [9, 10, 11], these conserved quantities are shared by all members of the integrable models belonging to the same *hierarchy*. However, in the context of spin exchange models which we will mainly discuss in this paper, the notion of integrable hierarchies is rather unfamiliar and virtually

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unexplored in contradistinction with the soliton hierarchies or Calogero-Moser hierarchies. The purpose of the present paper is to call attention to this interesting and potentially useful subject in mathematical/statistical physics by examining the second member of the integrable hierarchy of the simple and well-known Heisenberg spin chain with the nearest neighbour interactions. The energy eigenvalues of these models are evaluated explicitly and exactly for relatively small numbers of the spins and remarkably high degree of *degeneracies* of the eigenvalues is observed. We believe that this fact is closely related to the existence of a large number of conserved quantities shared by the hierarchy.

The hierarchy of the spin exchange models [1] consists of models associated with a fixed root system Δ and the totality of irreducible representations of the finite reflection (or Coxeter or Weyl) group G_Δ. In the case of the Heisenberg spin chain the root system is A_r, which corresponds to the Lie algebra of $su(r+1)$. In this case the Weyl group of A_r is the well-known symmetric group $\mathcal{S}_{r+1}$. First let us rephrase various concepts and dynamical quantities of the well-known Heisenberg spin chain in terms of the languages of the weights and roots and the associated reflections, which are useful for the universal description of the Calogero-Moser models [12, 13, 15, 16] and the spin exchange models [1]. If the A_r root system is embedded in $\mathbf{R}^{r+1}$, it has the following simple representation in terms of an orthonormal basis $\{\mathbf{e}_j\}$ of $\mathbf{R}^{r+1}$:

$$A_r = \{\mathbf{e}_j - \mathbf{e}_k |\ j,k = 1,\ldots,r+1\}. \tag{1}$$

The original Heisenberg model is based on the *vector* ($\mathbf{V}$) representation of A_r corresponding to the simplest Young diagram $\square$. It consists of the following *weights*:

$$\square: \qquad \mathbf{V} = \{\mathbf{e}_j + \gamma |\ j = 1,\ldots,r+1\}, \tag{2}$$

in which γ is orthogonal to all the A_r roots and it will be omitted hereafter. At each "site" $\mathbf{e}_j$ (or simply the site j) a spin space is attached. Throughout this paper we consider the simplest case of the 1/2 spins of $su(2)$, that is ups ($\uparrow$) and downs ($\downarrow$). In other words, the state space of the original Heisenberg chain is a 2^{r+1} dimensional vector space. The Hamiltonian now reads

$$\mathcal{H} = J \sum_{\rho \in \Pi^{(1)}} \hat{\mathcal{P}}_\rho, \quad J \in \mathbf{R}, \tag{3}$$

in which J is the coupling constant and $\Pi^{(1)}$ is the set of *affine simple roots* of A_r:

$$\Pi^{(1)} = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \ldots, \mathbf{e}_r - \mathbf{e}_{r+1}, \mathbf{e}_{r+1} - \mathbf{e}_1\}, \tag{4}$$

and $\hat{\mathcal{P}}_\rho$ is the spin exchange operator corresponding to the root ρ. In the present case $\hat{\mathcal{P}}_{\mathbf{e}_j-\mathbf{e}_{j+1}}$ exchanges spins at the sites j and $j+1$ (to be more precise the sites $\mathbf{e}_j$ and $\mathbf{e}_{j+1}$) and keeps the other spins intact. Let us denote the Pauli spin matrix acting on the site $\mathbf{e}_j$ simply by $\vec{\sigma}_j$, then $\hat{\mathcal{P}}_\rho$ has a well-known realisation:

$$\rho = \mathbf{e}_j - \mathbf{e}_k \Longrightarrow \hat{\mathcal{P}}_\rho = (1 + \vec{\sigma}_j \cdot \vec{\sigma}_k)/2. \tag{5}$$

Thus we find that the above Hamiltonian (3) describes the nearest neighbour Heisenberg spin chain and the last root in (4) provides a periodic lattice.

It is worth remarking that the above Hamiltonian (3) can be obtained as a special limit of a more general spin exchange model with elliptic exchange functions [6], and the conserved quantities can be constructed [8] as shown by one of the authors. The above Hamiltonian (3) and its conserved quantities are shared by all the members of the same integrable hierarchy. Each member of the hierarchy, to be called a model $\mathcal{R}$ for short, is specified by an irreducible representation $\mathcal{R}$ of the Weyl group of A_r. The elements of $\mathcal{R}$, to be called "sites", too, are vectors in $\mathbf{R}^{r+1}$ which form a single Weyl orbit:

$$\mu \in \mathcal{R} \Longrightarrow s_\rho(\mu) \in \mathcal{R}, \quad \forall \rho \in \Delta, \tag{6}$$

in which s_ρ reflects a vector $\xi \in \mathbf{R}^{r+1}$ in a hyperplane perpendicular to the root ρ:

$$\xi \to s_\rho(\xi) = \xi - (\rho^\vee \cdot \xi)\rho, \quad \rho^\vee \equiv 2\rho/\rho^2, \quad \rho \in \Delta. \tag{7}$$

As above, an $su(2)$ spin, an up ($\uparrow$) or a down ($\downarrow$) is attached to each "site" μ. Thus the state space of the model $\mathcal{R}$ is a 2^D dimensional vector space in which D is the total number of sites in $\mathcal{R}$. The action of the spin exchange operator $\hat{\mathcal{P}}_\rho$ is now

$$\hat{\mathcal{P}}_\rho : \quad \text{spin at } \mu \Leftrightarrow \text{spin at } s_\rho(\mu), \quad \forall \mu \in \mathcal{R} \tag{8}$$

simultaneously. If a site μ is orthogonal to a given root ρ, $\rho \cdot \mu = 0$, then $\mu = s_\rho(\mu)$. That is, $\hat{\mathcal{P}}_\rho$ affects only those spins at the sites μ such that $\rho \cdot \mu \neq 0$. In this way the action of the Hamiltonian (3) in the model $\mathcal{R}$ is completely defined.

The simplest example of the hierarchy is given by the *anti-symmetric tensor* representation (**AT**) of A_r corresponding to the Young diagram $\begin{array}{|c|}\hline\ \\ \hline\ \\ \hline\end{array}$. It consists of $D = r(r+1)/2$ weights of the form

$$\begin{array}{|c|}\hline\ \\ \hline\ \\ \hline\end{array}: \qquad \mathbf{AT} = \{\mathbf{e}_j + \mathbf{e}_k |\ k > j = 1, \ldots, r+1\}, \tag{9}$$

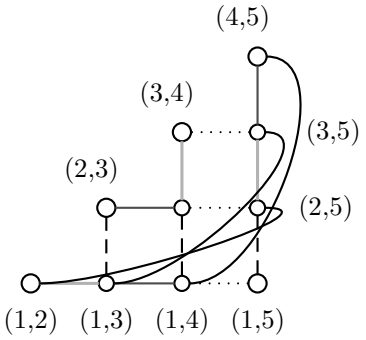

Figure 1: Structure of the second member of the $A_4^{(1)}$ Heisenberg spin model hierarchy with nearest neighbour interactions. The spins on the sites which are connected by the same bonds are exchanged simultaneously.

in which as before (2) we have omitted a constant vector δ which is orthogonal to all the roots. For notational simplicity, we express the site $\mathbf{e}_j + \mathbf{e}_k$ by a symmetric pair of integers $(j,k) = (k,j)$ and the Pauli spin matrix on the site by $\vec{\sigma}_{(j,k)}$. Obviously the sites $\{(j,k)\}$'s form a two dimensional orthogonal lattice. A generic site (j,k) is connected to *four* nearest neighbouring sites:

$$(j,k) \Longleftrightarrow (j-1,k),\ (j+1,k),\ (j,k-1),\ (j,k+1). \tag{10}$$

Those sites on the "boundary"

$$\text{boundary of } \square = \{(1,2),\ldots,(j,j+1),\ldots,(r,r+1),(r+1,1)\} \tag{11}$$

are connected to *two* other sites. The spin exchange operator $\hat{\mathcal{P}}_\rho$ for $\rho = \mathbf{e}_j - \mathbf{e}_k$ exchanges the spins in the same *row* or *column*, i.e. $(j,\ell) \Leftrightarrow (k,\ell)$, $\forall \ell \neq j,k$, *simultaneously*:

$$\rho = \mathbf{e}_j - \mathbf{e}_k, \quad \hat{\mathcal{P}}_\rho : \qquad \mathbf{e}_j + \mathbf{e}_\ell \Leftrightarrow \mathbf{e}_k + \mathbf{e}_\ell, \quad \ell \neq j,k. \tag{12}$$

Thus for the *anti-symmetric tensor* (**AT**) model, the same Hamiltonian (3) no longer describes the nearest neighbour interactions in the ordinary sense. As in the original Heisenberg model (5) the spin exchange operator has a simple realisation in terms of the Pauli spin matrices:

$$\rho = \mathbf{e}_j - \mathbf{e}_k \Longrightarrow \hat{\mathcal{P}}_\rho = \prod_{\ell \neq j,k} [(1 + \vec{\sigma}_{(j,\ell)} \cdot \vec{\sigma}_{(k,\ell)})/2]. \tag{13}$$

It should be emphasised that the two realisations (5) and (13) of the spin exchange operator $\hat{\mathcal{P}}_\rho$ (and any other realisations) satisfy the same commutation relations

$$\hat{\mathcal{P}}_\rho \hat{\mathcal{P}}_\sigma \hat{\mathcal{P}}_\rho = \hat{\mathcal{P}}_{s_\rho(\sigma)}, \quad \hat{\mathcal{P}}_\rho^2 = 1, \quad \hat{\mathcal{P}}_{-\rho} = \hat{\mathcal{P}}_\rho, \tag{14}$$

which are a direct consequence of the well-known commutation relations of the reflections

$$s_\rho s_\sigma s_\rho = s_{s_\rho(\sigma)}, \quad s_\rho^2 = 1, \quad s_{-\rho} = s_\rho. \tag{15}$$

Since the Hamiltonian commutes with the total spin operator

$$\vec{S} = \frac{1}{2}\sum_{\mu\in\mathcal{R}} \vec{\sigma}_\mu, \tag{16}$$

the operators S^2 and S_z define good quantum numbers. In the absence of the external magnetic field, the energy eigenstates consist of $su(2)$ multiplets of spin S ($2S+1$ fold degenerate) states as in the original Heisenberg spin chain. The vector (**V**), anti-symmetric tensor (**AT**) and the other higher rank anti-symmetric tensor representations of A_r are special examples of the *minimal* representations which have played important roles in the Calogero-Moser Lax pairs [13, 14, 15, 16, 17]. The Calogero-Moser Lax pairs in these representations were first presented by D'Hoker and Phong [13]. Minimal representations are characterised by the condition (see [14, 15])

$$\rho^\vee \cdot \mu = 0, \ \pm 1, \quad \forall \mu \in \mathcal{R}, \quad \forall \rho \in \Delta. \tag{17}$$

It is expected that the spin exchange models for these $\mathcal{R}$'s have some simpler features than others.

In the rest of this paper we show the energy eigenvalues of the anti-symmetric tensor representation (**AT**) model for lower rank ($r \leq 5$) cases. We evaluate the Hamiltonian (3) in each $S_z = k$, $-r/2 \leq k \leq r/2$ sector. In each sector, the Hamiltonian is a non-negative symmetric matrix with integer entries. In both **V** and **AT** models or any other models in the hierarchy, the highest eigenvalue is $r+1$ which is the number of spin exchange operators $\hat{\mathcal{P}}_\rho$ in the Hamiltonian (3). The lowest possible eigenvalue is $-(r+1)$. These two facts are based on the relation $\hat{\mathcal{P}}_\rho^2 = 1$ (14) implying that its eigenvalues are ± 1. The highest eigenvalue $r+1$ occurs in every S_z sector but the lowest possible eigenvalue $-(r+1)$ appears only in smaller S_z sector which have larger number of states. In particular, in the original Heisenberg (**V**) model, the lowest possible eigenvalue $-(r+1)$ never occurs, see (21), (23), (27) and (31). Once the lowest possible eigenvalue $-(r+1)$ is reached in an $S_z = k$ sector, then in all the $|S_z| < k$ sectors no new eigenvalues will appear. The energy eigenvalues of the vector representation (**V**) models are also listed for comparison. (The coupling constant J is put to unity for simplicity.) It is easy to see that the energy eigenvalues of the *single*

spin down (*up*) sector of the (**V**) model are given by

$$\square\,(\downarrow): \quad \mathcal{E}_j = r - 1 + 2\cos\frac{2j\pi}{(r+1)}, \quad j = 0, 1, \ldots, r, \tag{18}$$

which are related to the eigenvalues of the Cartan matrix of $A_r^{(1)}$ and thus related to the mass spectrum of the affine Toda field theory based on $A_r^{(1)}$ [18]. All these eigenvalues appear in other sectors because of the $su(2)$ invariance. It is also easy to see that for proper choices of the bases the Hamiltonian of the *double spin down* (*up*) sector of the (**V**) model is exactly the same as that of *single spin down* (*up*) sector of the (**AT**) model and so on:

$$\square\,(\downarrow\downarrow) \;= \begin{array}{|c|}\hline\;\;\\\hline\;\;\\\hline\end{array}\,(\downarrow), \quad \square\,(\downarrow\downarrow\downarrow) \;= \begin{array}{|c|}\hline\;\;\\\hline\;\;\\\hline\;\;\\\hline\end{array}\,(\downarrow), \quad \cdots, . \tag{19}$$

<u>A_2 **case**</u> In this case the vector rep. $\square$ (**V**) model and the anti-symmetric tensor rep. $\begin{array}{|c|}\hline\;\;\\\hline\;\;\\\hline\end{array}$ (**AT**) model are both 3-dimensional and these two models are equivalent from the spin model point of view. The total number of states is $8 = 2^3$ and the energy eigenvalues together with (`multiplicity`) are

$$3\ (4), \quad 0\ (4). \tag{20}$$

The former consists of one $S = 3/2$ multiplet, whereas the zero energy states consist of two $S = 1/2$ states.

<u>A_3 **case**</u> The vector rep. $\square$ is 4-dimensional whereas the anti-symmetric tensor rep. $\begin{array}{|c|}\hline\;\;\\\hline\;\;\\\hline\end{array}$ is 6-dimensional. The spectrum of the $\square$ (**V**) model is

$$\square: \quad 2^4 = 16 \text{ states} \quad 4\ (5), \quad 2\ (7), \quad 0\ (3), \quad -2\ (1). \tag{21}$$

The maximal eigenvalue $\mathcal{E} = 4$ states are one $S = 2$ multiplet and the $\mathcal{E} = 2$ states consist of two $S = 1$ multiplets and one $S = 0$ singlet. The $\mathcal{E} = 0$ states are one $S = 1$ multiplet and the lowest energy $\mathcal{E} = -2$ state is a singlet. The spectrum of the $\begin{array}{|c|}\hline\;\;\\\hline\;\;\\\hline\end{array}$ (**AT**) model is

$$\begin{array}{|c|}\hline\;\;\\\hline\;\;\\\hline\end{array}: \quad 2^6 = 64 \text{ states} \quad 4\ (11), \quad 2\ (26), \quad 0\ (12), \quad -2\ (14), \quad -4\ (1). \tag{22}$$

The maximal eigenvalue $\mathcal{E} = 4$ states are one $S = 3$, $S = 1$ multiplet each and a singlet. The $\mathcal{E} = 2$ states are three $S = 2$ and three $S = 1$ multiplets and two singlets. The zero energy $\mathcal{E} = 0$ states are one $S = 2$ multiplet, two $S = 1$ multiplets and one singlet. The $\mathcal{E} = -2$ states have one $S = 2$ and three $S = 1$ multiplets. The lowest energy $\mathcal{E} = -4$ state is a singlet.

$\underline{A_4\ \textbf{case}}$ The vector rep. $\square$ is 5-dimensional whereas the anti-symmetric tensor rep. $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ is 10-dimensional. The spectrum of the $\square$ (**V**) model is

$$\square: \quad 2^5 = 32 \text{ states} \quad 5\ (\mathtt{6}), \quad \frac{5\pm\sqrt{5}}{2}\ (\mathtt{8}), \quad 1\pm\sqrt{5}\ (\mathtt{4}), \quad 1\ (\mathtt{2}). \tag{23}$$

The maximal eigenvalue $\mathcal{E}=5$ states are one $S=5/2$ multiplet. The pair of energy levels $\mathcal{E}=(5\pm\sqrt{5})/2$ are both two $S=3/2$ multiplets. Another pair $\mathcal{E}=1\pm\sqrt{5}$ of states, one of which is the lowest energy, have each two $S=1/2$ multiplets. The $\mathcal{E}=1$ states are one $S=1/2$ multiplet. The spectrum of the $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ (**AT**) model is

$$\begin{aligned}\begin{smallmatrix}\square\\\square\end{smallmatrix}: \quad 2^{10} \;=\; & 1024 \text{ states} \quad 5\ (\mathtt{34}), \quad \frac{5\pm\sqrt{5}}{2}\ (\mathtt{112}), \quad 1\pm\sqrt{5}\ (\mathtt{116}), \quad -1\pm\sqrt{5}\ (\mathtt{76}), \\ & \frac{-5\pm\sqrt{5}}{2}\ (\mathtt{32}), \quad \pm\sqrt{5}\ (\mathtt{36}), \quad 1\ (\mathtt{58}), \quad 0\ (\mathtt{144}), \quad -1\ (\mathtt{38}), \quad -5\ (\mathtt{6}).\end{aligned} \tag{24}$$

This is a highly degenerate spectrum. The average multiplicity per energy level is $1024/15\approx 68$ in contrast to $1024/139\approx 7.3$ in the vector rep. of A_9 case (31) which has the same number of sites as in this case. The highest $\mathcal{E}=5$ states consist of $\mathtt{1}$ $S=5$ (11-dimensional) , $\mathtt{1}$ $S=3$ (7-dimensional) , $\mathtt{2}$ $S=2$ (5-dimensional) and $\mathtt{2}$ $S=1$ (3-dimensional) multiplets, which we denote as follows:

$$\mathcal{E}=5 \quad (\mathtt{34}): \quad \mathtt{1}\times\mathbf{11}, \qquad \mathtt{1}\times\mathbf{7}, \quad \mathtt{2}\times\mathbf{5}, \quad \mathtt{2}\times\mathbf{3}. \tag{25}$$

Similarly we have the following degeneracy pattern:

$$\begin{array}{rcrlllll}
\mathcal{E} &=& (5\pm\sqrt{5})/2 & (\mathtt{112}): & \mathtt{2}\times\mathbf{9}, & \mathtt{4}\times\mathbf{7}, & \mathtt{8}\times\mathbf{5}, & \mathtt{8}\times\mathbf{3}, \quad \mathtt{2}\times\mathbf{1}, \\
\mathcal{E} &=& 1\pm\sqrt{5} & (\mathtt{116}): & \mathtt{2}\times\mathbf{9}, & \mathtt{4}\times\mathbf{7}, & \mathtt{8}\times\mathbf{5}, & \mathtt{8}\times\mathbf{3}, \quad \mathtt{6}\times\mathbf{1}, \\
\mathcal{E} &=& -1\pm\sqrt{5} & (\mathtt{76}): & & \mathtt{4}\times\mathbf{7}, & \mathtt{4}\times\mathbf{5}, & \mathtt{8}\times\mathbf{3}, \quad \mathtt{4}\times\mathbf{1}, \\
\mathcal{E} &=& (-5\pm\sqrt{5})/2 & (\mathtt{32}): & & & \mathtt{4}\times\mathbf{5}, & \mathtt{4}\times\mathbf{3}, \\
\mathcal{E} &=& \pm\sqrt{5} & (\mathtt{36}): & & \mathtt{1}\times\mathbf{7}, & \mathtt{3}\times\mathbf{5}, & \mathtt{4}\times\mathbf{3}, \quad \mathtt{2}\times\mathbf{1}, \\
\mathcal{E} &=& 1 & (\mathtt{58}): & \mathtt{1}\times\mathbf{9}, & \mathtt{2}\times\mathbf{7}, & \mathtt{4}\times\mathbf{5}, & \mathtt{4}\times\mathbf{3}, \quad \mathtt{3}\times\mathbf{1}, \\
\mathcal{E} &=& 0 & (\mathtt{144}): & & \mathtt{4}\times\mathbf{7}, & \mathtt{12}\times\mathbf{5}, & \mathtt{16}\times\mathbf{3}, \quad \mathtt{8}\times\mathbf{1}, \\
\mathcal{E} &=& -1 & (\mathtt{38}): & & \mathtt{2}\times\mathbf{7}, & \mathtt{2}\times\mathbf{5}, & \mathtt{4}\times\mathbf{3}, \quad \mathtt{2}\times\mathbf{1}, \\
\mathcal{E} &=& -5 & (\mathtt{6}): & & & \mathtt{1}\times\mathbf{5}, & \mathtt{1}\times\mathbf{1}.
\end{array} \tag{26}$$

In the A_4 models all the multiplicities are even numbers.

$\underline{A_5\ \textbf{case}}$ The vector rep. $\square$ is 6-dimensional whereas the anti-symmetric tensor rep. $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ is 15-dimensional. The spectrum of the $\square$ (**V**) model is

$$\square: \quad 2^6 = 64 \text{ states} \qquad 6\ (\mathtt{7}=\mathtt{1}\times\mathbf{7}), \quad 5\ (\mathtt{10}=\mathtt{2}\times\mathbf{5}), \quad 4\ (\mathtt{3}=\mathtt{1}\times\mathbf{3}), \quad 3\ (\mathtt{10}=\mathtt{2}\times\mathbf{5}),$$

$$2\ (7 = 2\times\mathbf{3} + 1\times\mathbf{1}),\quad 1\ (6 = 2\times\mathbf{3}),\quad 0\ (1),\qquad (27)$$
$$(5\pm\sqrt{17})/2\ (6 = 2\times\mathbf{3}),\quad 1\pm\sqrt{5}\ (3 = 1\times\mathbf{3}),\quad 1\pm\sqrt{13}\ (1).$$

The spectrum of the $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ (**AT**) model is

$$\begin{aligned}&\begin{smallmatrix}\square\square\\\square\end{smallmatrix}: 2^{15} = 32768 \text{ states}\quad 6\ (156),\quad 5\ (776),\quad 4\ (1060),\quad 3\ (776),\ 2\ (2756),\quad 1\ (4224)\\ &0\ (1304),\ -1\ (3600),\quad -2\ (1988),\quad -3\ (296),\quad -4\ (628),\quad -5\ (296),\quad -6\ (28),\\ &(5\pm\sqrt{17})/2\ (1200),\quad (-5\pm\sqrt{17})/2\ (576),\quad 1\pm\sqrt{5}\ (600),\quad -1\pm\sqrt{5}\ (288),\\ &1\pm\sqrt{13}\ (264),\quad -1\pm\sqrt{13}\ (240),\quad \pm c_1, \pm c_2, \pm c_3\ (1424),\end{aligned}\qquad (28)$$

in which

$$c_j = \left(1 - 10\cos[(\phi - 2j\pi)/3]\right)/3,\quad j = 1, 2, 3,\quad \cos\phi = 7/250.\qquad (29)$$

These six eigenvalues always appear with the same multiplicities in each S_z sector. This is a highly degenerate spectrum. The degeneracy pattern is as follows:

$\mathcal{E}$	total Mult.	**16**	**14**	**12**	**10**	**8**	**6**	**4**	**2**
6	156	1		1	3	4	6	6	3
5	776		2	4	10	24	34	36	30
4	1060		1	4	13	30	50	61	42
3	776		2	4	10	24	34	36	30
2	2756		1	8	33	76	135	168	113
1	4224		2	12	46	118	210	256	182
0	1304			3	15	34	66	85	55
-1	3600			6	34	100	186	234	168
-2	1988			2	19	52	104	136	95
-3	296				2	8	16	20	18
-4	628				5	17	33	45	32
-5	296				2	8	16	20	18
-6	28					1	2	1	2
$(5\pm\sqrt{17})/2$	1200		2	6	16	34	54	64	44
$(-5\pm\sqrt{17})/2$	576				4	16	30	42	30
$1\pm\sqrt{5}$	600		1	3	8	17	27	32	22
$-1\pm\sqrt{5}$	288				2	8	15	21	15
$1\pm\sqrt{13}$	264			1	4	6	13	17	9
$-1\pm\sqrt{13}$	240			1	3	6	12	15	9
$\pm c_1,\ \pm c_2,\ \pm c_3$	1424			4	14	40	74	86	64.

(30)

The average multiplicity per energy level is $32768/31 \approx 1057$. We believe this is a rule for the higher members of the spin model hierarchies.

$\underline{A_9}$ **case** The vector rep. $\square$ is 10-dimensional which has the same number of sites as the anti-symmetric tensor rep. $\begin{smallmatrix}\square\\\square\end{smallmatrix}$ of A_4. There are 139 different energy levels among 1024 states. Here we list the spectrum (numerical values only) and (`multiplicity`) for comparison.

$\square$: $2^{10} = 1024$ states

10 (11), 9.618 (18), 9.345 (14), 9.173 (10), 9.118 (1), 9.099 (6), 9.064 (7), 8.982 (3),
8.800 (10), 8.678 (14), 8.666 (5), 8.618 (18), 8.613 (10), 8.250 (6), 8.191 (14),
8 (7), 7.978 (3), 7.963 (2), 7.934 (14), 7.855 (10), 7.800 (6), 7.594 (6), 7.421 (2),
7.382 (18), 7.361 (10), 7.346 (10), 7.303 (10), 7.206 (10), 7.169 (6), 7.120 (14),
6.893 (6), 6.730 (6), 6.695 (7), 6.677 (1), 6.653 (14), 6.540 (2), 6.501 (10),
6.382 (18), 6.347 (6), 6.341 (3), 6.212 (1), 6.141 (6), 6.131 (6), 6.112 (10), 6.061 (10),
6 (26), 5.902 (10), 5.899 (10), 5.708 (14), 5.627 (10), 5.618 (6), 5.459 (14), 5.364 (2),
5.357 (14), 5.213 (10), 5.198 (6), 5.167 (10), 5.145 (10), 5.132 (6), 5.107 (6), 5.093 (2),
5.018 (10), 4.929 (6), 4.902 (1), 4.848 (14), 4.666 (10), 4.648 (3), 4.646 (2), 4.618 (6),
4.599 (5), 4.591 (6), 4.561 (10), 4.514 (10), 4.421 (14), 4.390 (10), 4.356 (6), 4.341 (2),
4.232 (14), 4.171 (6), 4.163 (6), 4 (12), 3.939 (6), 3.697 (10), 3.566 (10), 3.563 (2),
3.488 (2), 3.431 (2), 3.382 (6), 3.284 (10), 3.275 (6), 3.202 (6), 3.199 (10), 3.189 (6),
3.091 (14), 3.027 (6), 3.025 (1), 2.964 (14), 2.869 (1), 2.747 (6), 2.694 (6), 2.687 (3),
2.517 (10), 2.471 (10), 2.382 (6), 2.241 (7), 2.208 (10), 2.137 (10), 2.096 (3), 1.923 (2),
1.916 (2), 1.904 (10), 1.696 (6), 1.687 (5), 1.646 (2), 1.565 (6), 1.523 (6), 1.454 (10),
1.140 (6), 1.106 (6), 1.078 (6), 0.962 (1), 0.942 (6), 0.808 (1), 0.744 (10), 0.359 (10),
0.332 (3), 0.122 (3), 0.025 (10), −0.029 (2), −0.535 (2), −0.561 (6), −0.773 (2),
−0.782 (6), −0.952 (5), −1.492 (6), −2.087 (6), −2.541 (1), −3.184 (3), −4.031 (1).

(31)

Some remarks and comments are in order.

One might feel that the *simultaneous exchange* of the spins is artificial. Rather it is quite natural within the context of the conventional spin exchange models, if we consider any member of the conserved quantities as a new Hamiltonian but keeping the sites of the spin the same as in the original model. As shown in the previous paper [1] the conserved quantities

consist of the linear combination of ordered products of the spin exchange operators. Each product term of the conserved quantity specifies simultaneous exchange of spins. The only exception is the original spin exchange models based on A_r root systems and the vector representation $\mathbf{V}$. For the other root systems, even for the classical ones B_r, C_r or D_r, the single spin exchange operator implies a multiple exchange as shown explicitly in (3.21) of [1].

It would be interesting to determine the energy eigenvalues of the Hamiltonian (3) for a different choice of the set of affine simple roots, for example $D_r^{(1)}$:

$$\Pi^{(1)} = \{\mathbf{e}_1 - \mathbf{e}_2, \mathbf{e}_2 - \mathbf{e}_3, \ldots, \mathbf{e}_{r-1} - \mathbf{e}_r, \mathbf{e}_{r-1} + \mathbf{e}_r, -(\mathbf{e}_1 + \mathbf{e}_2)\}, \tag{32}$$

and for the D_r vector representations or (anti-) spinor representations. The former would give a D-series analog of the nearest neighbour Heisenberg model.

Another well-known method for spin models is the use of Yang-Baxter equations [19]. It would be interesting to compare the fusion procedure for the R-matrices with the present method of generating hierarchies.

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