

# Berry's phase in noncommutative spaces

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**1 Abstract.** We introduce the perturbative aspects of noncommutative quantum mechanics. Then we study the Berry's phase in the framework of noncommutative quantum mechanics. The results show deviations from the usual quantum mechanics which depend on the parameter of space/space noncommutativity.

## 1 Introduction.

Noncommutative quantum mechanics have received a wide attention once it was realized that they could be obtained as low energy limit of string theory in the presence of a  $B$  field [1,2]. In field theories the noncommutativity is introduced by replacing the standard product by the star product. for a manifold parameterized by the coordinates  $x_i$ , the noncommutative relation can be written as :

$$[\hat{x}_i, \hat{x}_j] = i\theta_{ij} \quad [\hat{x}_i, \hat{p}_i] = i\delta_{ij} \quad [\hat{p}_i, \hat{p}_j] = 0. \quad (1)$$

Many physical problems have been studied in the framework of the noncommutative quantum mechanics (NCQM), see e.g. [6-19]. NCQM is formulated in the same way as the standard quantum mechanics SQM (quantum mechanics in commutative spaces), that is in terms of the same dynamical variables represented by operators in a Hilbert space and a state vector that evolves according to the Schroedinger equation :

$$i\frac{d}{dt}|\psi\rangle = H_{nc}|\psi\rangle, \quad (2)$$

we have taken in to account  $\hbar = 1$ .  $H_{nc} \equiv H_\theta$  denotes the Hamiltonian for a given system in the noncommutative space. In the literatures two approaches

have been considered for constructing the NCQM :

- a)  $H_\theta = H$ , so that the only difference between SQM and NCQM is the presence of a nonzero  $\theta$  in the commutator of the position operators i.e. equ.(1).
- b) By deriving the Hamiltonian from the moyal analog of the standard Schroedinger equation :

$$i\frac{\partial}{\partial t}\psi(x, t) = H(p = \frac{1}{i}\nabla, x) * \psi(x, t) \equiv H_\theta\psi(x, t), \quad (3)$$

where  $H(p, x)$  is the same Hamiltonian as in the standard theory, and as we observe the  $\theta$  - dependence enters now through the star product [17]. In [19], it has been shown that these two approaches lead to the same physical theory. For the Hamiltonian of the type :

$$H(\hat{p}, \hat{x}) = \frac{\hat{p}^2}{2m} + V(\hat{x}). \quad (4)$$

The modified Hamiltonian  $H_\theta$  can be obtained by a shift in the argument of the potential [6,17] :

$$x_i = \hat{x}_i + \frac{1}{2}\theta_{ij}\hat{p}_j \quad \hat{p}_i = p_i. \quad (5)$$

which lead to

$$H_\theta = \frac{p^2}{2m} + V(x_i - \frac{1}{2}\theta_{ij}p_j). \quad (6)$$

The variables  $x_i$  and  $p_i$  now, satisfy in the same commutation relations as the usual case :

$$[x_i, x_j] = [p_i, p_j] = 0 \quad [x_i, p_j] = \delta_{ij}. \quad (7)$$

Topological effects are among the most important quantum phenomena discovered since the creation of the quantum mechanics. The Aharonov-Bohm (AB) effect [3], was the first such effect which changed basic assumptions about the role of the potentials in physics. There is also a geometric effect discovered by Berry [4], almost two decades later. Berry investigated quantum systems that undergo an adiabatic evolution goverened by several changing parameters. When these parameters return to their initial values, quantum systems return to their initial states up to a phase. Berry found that the phase contains a geometric part, which depends on the path in the space of the parameters, but not on the rate evolution of the system along the path. Aharonov and Anandan subsequently generalized Berry's phase to nonadiabatic evolution [5]. The AB effect in noncommutative spaces has been studied in [7] and [8]. In this paper we study the Berry's phase in noncommutative spaces.

## 2 Perturbation aspects of noncommutative dynamics.

In this section we discuss the "perturbation aspects of noncommutative dynamics". The perturbation aspects of  $\theta$ -deformed dynamics in one dimensional

$\theta$ -spaces has been studied in [20] and [21]. Using

$$U(x + \Delta x) = U(x) + \sum_{n=1}^{\infty} \frac{U^{(n)}(x)}{n!} (\Delta x)^n, \quad (8)$$

and equ.(6) for small  $\theta$  we have :

$$H_{nc} = \frac{p^2}{2m} + V(x_i) + \sum_{n=1}^{\infty} \frac{V^{(n)}(x_i)}{n!} (\Delta x_i)^n, \quad (9)$$

where  $\Delta x_i = -\frac{1}{2}\theta\epsilon_{ij}p_j$  and  $H = \frac{p^2}{2m} + V(x)$  is the Hamiltonian in ordinary(commutative) space. To the first order we have :

$$H_{nc} = \frac{p^2}{2m} + V(x_i) + \Delta x_i \frac{\partial V}{\partial x_i} = H + \Delta x_i \frac{\partial V}{\partial x_i} = H + \theta H_I. \quad (10)$$

We can use perturbation theory to obtain the eigenvalues and eigenfunctions of  $H_{nc}$  :

$$E_n = E_n^0 + \Delta E_n^0 = E_n^0 + \theta E_n^{(1)} + \theta^2 E_n^{(2)} + \dots \quad (11)$$

$$\psi_n = \phi_n + \sum_{k \neq n} C_{nk}(\theta) \phi_k. \quad (12)$$

where :

$$C_{nk}(\theta) = \theta C_{nk}^{(1)} + \theta^2 C_{nk}^{(2)} + \dots \quad (13)$$

To the first order in perturbation theory we have :

$$\theta E_n^{(1)} = \langle \phi_n | \theta H_I | \phi_n \rangle, \quad (14)$$

$$\psi_n = \phi_n + \theta \sum_{k \neq n} C_{nk}^{(1)} \phi_k, \quad (15)$$

$$\theta C_{nk}^{(1)} = \frac{\langle \phi_k | \theta H_I | \phi_n \rangle}{E_n^0 - E_k^0}, \quad (16)$$

where  $E_n^0$  and  $\phi_n$  are the  $n$ th eigenvalue and eigenfunction of the Hamiltonian  $H$ .  $E_n$  and  $\psi_n$  are the  $n$ th eigenvalue and eigenfunction of  $H_{nc}$ . For example for the Hydrogen atom :

$$V(r) = -\frac{ze^2}{r} = -\frac{ze^2}{\sqrt{\hat{x}\hat{x}}}. \quad (17)$$

and we have :

$$\theta H_I = -\frac{1}{2}\theta\epsilon_{ij}p_j \left( \frac{x^i z e^2}{r^3} \right) = -ze^2 \theta \frac{x_i \epsilon_{ij} p_j}{2r^3}. \quad (18)$$

which is in agreement with the result of Ref. [6]. Using (15) and (16) we can find the eigenfunctions for the Hydrogen atom in noncommutative spaces.

$$\psi_n^{Hydrogen} = \phi_n + \sum_{k \neq n} \frac{\langle \phi_k | \theta H_I | \phi_n \rangle}{E_n^0 - E_k^0} \phi_k, \quad (19)$$

where  $\partial H_t$  is given by equ.(18).

## 2 Berry's phase in noncommutative spaces.

Now the Hamiltonian  $H_{nc}$ , its eigenfunctions  $\psi_n$  and eigenvalues  $E_n$  are known (eqs. (11)-(13)) and we can calculate the Berry's phase in noncommutative case. suppose a quantum system is in  $n$ th eigenstate :

$$H_{nc}|\psi_n\rangle = E_n\psi_n. \quad (20)$$

If  $H_{nc}$  is slowly changed, then according to the adiabatic theorem the system remains in the  $n$ th eigenstate of the slowly-changing Hamiltonian  $H_{nc}$ . If at time  $\tau$  it returns to its initial form  $H_{nc}(\tau) = H_{nc}(t=0)$ , it follows that the system must return to its initial state up to a phase factor, and we have :

$$|\psi_n(\tau)\rangle = e^{-i \int_0^\tau E_n(t') dt'} e^{i\eta_n(\tau)} |\psi_n\rangle. \quad (21)$$

From the Schroedinger equation we have :

$$i \frac{\partial |\psi_n\rangle}{\partial t} = E_n(t) |\psi_n(t)\rangle. \quad (22)$$

Differentiating eq.(21) at time  $\tau$ , and plugging in to the Schroedinger equation, we obtain :

$$\frac{\partial}{\partial t} |\psi_n(t)\rangle + i \frac{d\eta_n(t)}{dt} |\psi_n(t)\rangle = 0. \quad (23)$$

Multiplying on the left by  $\langle \psi_n(t) |$ , we have :

$$\eta_n(\tau) = i \int_0^\tau \langle \psi_n(t) | \frac{\partial}{\partial t} |\psi_n(t)\rangle dt. \quad (24)$$

If the time dependence of the Hamiltonian arises because we are changing some parameters  $R$  with time, then we may write

$$\eta_n(\tau) = i \int_0^\tau \langle \psi_n(t) | \frac{\partial}{\partial t} |\psi_n(t)\rangle dt = i \oint dR \langle \psi_n(R) | \nabla_R \psi_n(R) \rangle. \quad (25)$$

By substituting for  $\psi_n(t)$  for a given time  $t$  :

$$|\psi_n(t)\rangle = |\phi_n(t)\rangle + \sum_{k \neq n} C_{nk}(\theta) |\phi_k(t)\rangle. \quad (26)$$

we have :

$$\langle \psi_n(t) | \frac{\partial}{\partial t} |\psi_n(t)\rangle =$$

$$[\langle \phi_n(t) | + \sum_{k \neq n} C_{nk}(\theta) \langle \phi_k(t) |] \frac{\partial}{\partial t} [|\phi_n(t)\rangle + \sum_{\ell \neq n} C_{n\ell}(\theta) |\phi_\ell(t)\rangle]. \quad (27)$$

which leads to :

$$\begin{aligned} \eta_n(\tau) = i \int_0^\tau dt < \phi_n(t) | \frac{\partial}{\partial t} | \phi_n(t) > + \\ i \sum_{i=1}^\infty \theta^i \int_0^\tau dt \{ \sum_{\ell \neq n} C_{n\ell}^{(i)} < \phi_n(t) | \frac{\partial}{\partial t} | \phi_\ell(t) > + \sum_{k \neq n} C_{nk}^{(i)} < \phi_k(t) | \frac{\partial}{\partial t} | \phi_n(t) > \\ + \sum_{r=1}^\infty \sum_{\ell \neq n} \sum_{k \neq n} \theta^r C_{n\ell}^{(i)} C_{nk}^{(r)} < \phi_\ell(t) | \frac{\partial}{\partial t} | \phi_k(t) > \}. \end{aligned} \quad (28)$$

or :

$$\begin{aligned} \eta_n = i \oint dR < \phi_n(R) | \nabla_R \phi_n(R) > + \\ i \sum_{i=1}^\infty \theta^i \oint dR \{ \sum_{\ell \neq n} C_{n\ell}^{(i)} < \phi_n(R) | \nabla_R \phi_\ell(R) > + \sum_{k \neq n} C_{nk}^{(i)} < \phi_k(R) | \nabla_R \phi_n(R) > \\ + \sum_{r=1}^\infty \sum_{\ell \neq n} \sum_{k \neq n} \theta^r C_{n\ell}^{(i)} C_{nk}^{(r)} < \phi_\ell(R) | \nabla_R \phi_k(R) > \}. \end{aligned} \quad (29)$$

The first term is Berry's phase in commutative space and the second term gives the correction to the Berry's phase due to the noncommutativity of space. If the parameter  $\theta$  is small, the higher order terms (higher powers in  $\theta$ ) are very small and we have :

$$\begin{aligned} \eta_n = i \oint dR < \phi_n(R) | \nabla_R \phi_n(R) > + \\ i\theta \oint dR \{ \sum_{\ell \neq n} C_{n\ell}^{(1)} < \phi_n(R) | \nabla_R \phi_\ell(R) > + \sum_{k \neq n} C_{nk}^{(1)} < \phi_k(R) | \nabla_R \phi_n(R) > \} + O(\theta^2). \end{aligned} \quad (30)$$

As we mentioned above the argument can be generalized to nonadiabatic evolution.

Let us consider the case of a spin- $\frac{1}{2}$  particle in a magnetic field. If the magnetic field  $\mathbf{B}(t)$  initially lie along the  $\hat{z}$ -direction  $\mathbf{B}(t=0) = B\hat{z}$ , then the initial Hamiltonian is given by :

$$H(t=0) = -\mathbf{M} \cdot \mathbf{B} = \frac{eg}{4mc} \sigma \cdot \mathbf{B} = \frac{eg}{2mc} B S_z. \quad (31)$$

Assume that the particle initially starts out in an eigenstate of  $S_z$ , we want to determine what happens to the particle if the direction of the  $\mathbf{B}$ -field is rotated in some matter such that at time  $t = \tau$ , the field again lines up with the  $\hat{z}$ -axis. If the  $\mathbf{B}$ -field doesn't depend on the coordinates i.e. it is constant in space, then  $H_I = 0$  and we have :

$$H_{nc} = H. \quad (32)$$

This means that for a spin- $\frac{1}{2}$  particle at the presence of constant magnetic field in a noncommutative space, the correction to the Berry's phase vanishes.

### Acknowledgment.

I would like to thank Professor Masud Chaichian for his warm hospitality during my visit to the university of Helsinki. I am very grateful to M. M. Sheikh-Jabbari for his useful comments. I acknowledge S. M. Harun-or-Rashid for his help on preparation of the manuscript. This work is partially supported by the Ministry of Science, Research and Technology of Iran.

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