Instantons on General Noncommutative $\mathbf{R^4}$

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Abstract

We study the $U(1)$ and $U(2)$ instanton solutions of gauge theory on general noncommutative $\mathbf{R^4}$. In all cases considered we obtain explicit results for the projection operators. In some cases we computed numerically the instanton charge and found that it is an integer, independent of the noncommutative parameters $\theta_{1,2}$.

1 Introduction

The study of exact solutions in field theory is a very important subject. It is often the first step to learn something nonperturbatively about a given theory.

In recent years the study of noncommutative field theory becomes an active research area, mostly due to its relevance with string theory [1]. Perturbative analysis of noncommutative field theory reveals an interesting inter-relation between infrared and ultraviolet divergences [2]. This is due to the presence of a short distance like cutoff in the noncommutative space on which the theory is formulated. Other peculiar features include the exact soliton solution in pure scalar field theory [3] and a vast of other interesting exact solutions in noncommutative field theories [4].

Instantons are exact solutions in gauge field theory. These solutions are also interesting in mathematics [5]. Recently, noncommutative instantons [6] become one of the great interests in theoretical physics.

In this paper we will study instantons in noncommutative gauge theory. In particular we focus our study on how the involved quantities vary with the noncommutative parameters θ^{mn}. We note that the usual treatment of setting $\theta_1 = \pm\theta_2$ (see below for our notations) by rescaling the coordinates x^m on general noncommutative $\mathbf{R^4}$ (or in brief, $\mathbf{R^4_{\rm NC}}$) is not allowed because it will change the metric and so the (anti-)self-dual equations. In this paper we will keep this arbitrariness and study in detail the various explicit solutions in noncommutative $U(1)$ and $U(2)$ gauge theory. We note that these solutions are studied for the case $\theta_1 = \pm\theta_2$ in [7, 8, 9, 10, 16].

2 $\mathbf{R^4_{\rm NC}}$ and the (anti-)self-dual equations

First let us recall briefly the noncommutative $\mathbf{R^4}$ and set our notations[1]. For a general noncommutative $\mathbf{R^4}$ we mean a space with (operator) coordinates x^m, $m = 1, \cdots, 4$, which satisfy the following relations:

$$[x^m, x^n] = i\theta^{mn}, \tag{1}$$

where θ^{mn} are real constants. If we assume the standard (Euclidean) metric for the noncommutative $\mathbf{R^4}$, we can use the orthogonal transformation with positive determinant to change θ^{mn} into the following standard form:

$$(\theta^{mn}) = \begin{pmatrix} 0 & \theta^{12} & 0 & 0 \\ -\theta^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta^{34} \\ 0 & 0 & -\theta^{34} & 0 \end{pmatrix}, \tag{2}$$

where $\theta^{12} > 0$ and $\theta^{12} + \theta^{34} \geq 0$. By using this form of θ^{mn}, the only nonvanishing commutators are as follows:

$$[x^1, x^2] = i\theta^{12}, \qquad [x^3, x^4] = i\theta^{34}, \tag{3}$$

[1]For general reviews on noncommutative geometry and field theory, see, for example, [9, 11, 4, 12, 13].

and other twos obtained by using the anti-symmetric property of the commutators. Introducing complex coordinates:

$$\begin{array}{ll} z_1 = x^2 + ix^1, & \bar{z}_1 = x^2 - ix^1, \\ z_2 = x^4 + ix^3, & \bar{z}_2 = x^4 - ix^3, \end{array} \tag{4}$$

the non-vanishing commutation relations are

$$[\bar{z}_1, z_1] = 2\theta^{12} \equiv \theta_1, \quad [\bar{z}_2, z_2] = 2\theta^{34} \equiv \theta_2. \tag{5}$$

By a noncommutative gauge field A_m we mean an operator valued field. The (anti-hermitian) field strength F_{mn} is defined similarly as in the commutative case:

$$F_{mn} = \hat{\partial}_{[m} A_{n]} + A_{[m} A_{n]} \equiv \hat{\partial}_m A_n - \hat{\partial}_n A_m + [A_m, A_n], \tag{6}$$

where the derivative operator $\hat{\partial}_m$ is defined as follows:

$$\hat{\partial}_m f \equiv -i\theta_{mn}[x^n, f], \tag{7}$$

where θ_{mn} is the inverse of θ^{mn}. For our standard form (2) of θ^{mn} we have

$$\hat{\partial}_1 A = \frac{i}{\theta^{12}}[x^2, A], \qquad \hat{\partial}_2 A = -\frac{i}{\theta^{12}}[x^1, A], \tag{8}$$

which can be expressed by the complex coordinates (4) as follows:

$$\partial_1 A \equiv \hat{\partial}_{z_1} A = \frac{1}{\theta_1}[\bar{z}_1, A], \qquad \bar{\partial}_1 A \equiv \hat{\partial}_{\bar{z}_1} A = -\frac{1}{\theta_1}[z_1, A], \tag{9}$$

and similar relations for $x^{3,4}$ and $z_2, \bar{z}_2$.

For a general metric g_{mn} the instanton equations are

$$F_{mn} = \pm \frac{\epsilon^{pqrs}}{2\sqrt{g}} g_{mp} g_{nq} F_{rs}, \tag{10}$$

and the solutions are known as self-dual (SD, for "+" sign) and anti-self-dual (ASD, for "−" sign) instantons. Here ϵ^{pqrs} is the totally anti-symmetric tensor ($\epsilon^{1234} = 1$ etc.) and g is the metric. We will take the standard metric $g_{mn} = \delta_{mn}$ and take the noncommutative parameters $\theta_{1,2}$ as free parameters. We also note that the notions of self-dual and anti-self-dual are interchanged by a parity transformation. A parity transformation also changes the sign of θ^{mn}. In the following discussion we will consider only the ASD instantons. So we should not restrict θ_2 to be positive.

3 Instantons in Noncommutative Gauge Theory

3.1 ADHM construction for ordinary gauge theory

For ordinary gauge theory all the (ASD) instanton solutions are obtained by ADHM (Atiyah-Drinfeld-Hitchin-Manin) construction [14]. In this construction we introduce the following ingredients (for $U(N)$ gauge theory with instanton number k):

- complex vector spaces V and W of dimensions k and N,

- $k \times k$ matrix $B_{1,2}$, $k \times N$ matrix I and $N \times k$ matrix J,
- the following quantities:

$$\begin{aligned} \mu_r &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + I\, I^\dagger - J^\dagger J, && (11) \\ \mu_c &= [B_1, B_2] + I\, J. && (12) \end{aligned}$$

The claim of ADHM is as follows:

- Given $B_{1,2}$, I and J such that $\mu_r = \mu_c = 0$, an ASD gauge field can be constructed;
- All ASD gauge fields can be obtained in this way.

It is convenient to introduce a quaternionic notation[2] for the 4-dimensional Euclidean space-time indices:

$$x \equiv x^n \sigma_n, \qquad \bar{x} \equiv x^n \bar{\sigma}_n, \tag{13}$$

where $\sigma_n = (i\vec{\tau}, 1)$ and τ^c, $c = 1, 2, 3$ are the three Pauli matrices, and the conjugate matrices $\bar{\sigma}_n = \sigma_n^\dagger = (-i\vec{\tau}, 1)$. In terms of the complex coordinates (4) we have

$$(x_{\alpha\dot{\alpha}}) = \begin{pmatrix} z_2 & z_1 \\ -\bar{z}_1 & \bar{z}_2 \end{pmatrix}, \qquad (\bar{x}^{\dot{\alpha}\alpha}) = \begin{pmatrix} \bar{z}_2 & -z_1 \\ \bar{z}_1 & z_2 \end{pmatrix}. \tag{14}$$

Then the basic object in the ADHM construction is the $(N + 2k) \times 2k$ matrix Δ which is linear in the space-time coordinates:

$$\Delta = a + b\bar{x}, \tag{15}$$

where the constant matrices

$$a = \begin{pmatrix} I^\dagger & J \\ B_2^\dagger & -B_1 \\ B_1^\dagger & B_2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{16}$$

Consider the conjugate operator of Δ:

$$\Delta^\dagger = a^\dagger + x b^\dagger = \begin{pmatrix} I & B_2 + z_2 & B_1 + z_1 \\ J^\dagger & -B_1^\dagger - \bar{z}_1 & B_2^\dagger + \bar{z}_2 \end{pmatrix}. \tag{17}$$

It is easy to check that the ADHM equations (11) and (12) are equivalent to the so-called factorization condition:

$$\Delta^\dagger \Delta = \begin{pmatrix} f^{-1} & 0 \\ 0 & f^{-1} \end{pmatrix}, \tag{18}$$

where $f(x)$ is a $k \times k$ hermitian matrix. From the above condition we can construct a hermitian projection operator P as follows:

$$\begin{aligned} P &= \Delta f \Delta^\dagger, \\ P^2 &= \Delta f f^{-1} f \Delta^\dagger = P. && (19) \end{aligned}$$

[2]We follow closely the notation of [10].

Obviously, the null-space of $\Delta^\dagger(x)$ is of N dimension for generic x. The basis vector for this null-space can be assembled into an $(N+2k)\times N$ matrix $U(x)$:

$$\Delta^\dagger U = 0, \tag{20}$$

which can be chosen to satisfy the following orth-normalization condition:

$$U^\dagger U = 1. \tag{21}$$

The above orth-normalization condition guarantees that $UU^\dagger$ is also a hermitian projection operator. Now it can be proved that the completeness relation

$$P + UU^\dagger = 1 \tag{22}$$

holds if U contains the whole null-space of $\Delta^\dagger$. In other words, this completeness relation requires that U consists of all the zero modes of $\Delta^\dagger$. The proof is sketched as follows: the two projection operator P and $UU^\dagger$ are orthogonal to each other, and so $1-P-UU^\dagger$ is also a hermitian projection operator. Now this can always be written as the form $VV^\dagger$; then V must consist of some zero modes of $\Delta^\dagger$ other than those in U because Δ and f are both of maximum rank (this notion is ambiguous in the infinite-dimensional case, but some other notions can be used instead) and $PVV^\dagger = 0$. This conclusion is in conflict with the assumption that U contains all the zero modes of $\Delta^\dagger$.

The (anti-hermitian) gauge potential is constructed from U by the following formula:

$$A_m = U^\dagger \partial_m U. \tag{23}$$

Substituting this expression into (6), we get the following field strength:

$$\begin{aligned} F_{mn} &= \partial_{[m}(U^\dagger\partial_{n]}U) + (U^\dagger\partial_{[m}U)(U^\dagger\partial_{n]}U) = \partial_{[m}U^\dagger(1-UU^\dagger)\partial_{n]}U \\ &= \partial_{[m}U^\dagger\Delta f\Delta^\dagger\partial_{n]}U = U^\dagger\partial_{[m}\Delta f\partial_{n]}\Delta^\dagger U = U^\dagger b\bar\sigma_{[m}\sigma_{n]}fb^\dagger U \\ &= 2i\bar\eta^c_{mn}U^\dagger b(\tau^c f)b^\dagger U. \end{aligned} \tag{24}$$

Here $\bar\eta^a_{ij}$ is the standard 't Hooft η-symbol, which is anti-self-dual:

$$\frac{1}{2}\epsilon_{ijkl}\bar\eta^a_{kl} = -\bar\eta^a_{ij}. \tag{25}$$

3.2 Noncommutative ADHM construction

The above construction has been extended to noncommutative gauge theory [6]. We recall this construction briefly here. By introducing the same data as above but considering the z_i's as noncommutative we see that the factorization condition (18) still gives $\mu_c = 0$, but μ_r no longer vanishes. It is easy to check that the following relation holds:

$$\mu_r = \zeta \equiv \theta_1 + \theta_2. \tag{26}$$

In this case the two ADHM equations (11) and (12) can be combined into one [10]:

$$\tau^{c\dot\alpha}{}_{\dot\beta}(\bar a^{\dot\beta} a_{\dot\alpha})_{ij} = \delta_{ij}\delta^{c3}\zeta. \tag{27}$$

As studied mathematically by various people (see, for example, the lectures by H. Nakajima [15]), the moduli space of the noncommutative instantons is better behaved than their commutative counterpart. In the noncommutative case the operator $\Delta^\dagger\Delta$ always has maximum rank, i.e., it has no zero modes (see [11]).

Though there is no much difference between the noncommutative ADHM construction and the commutative one, we should study the noncommutative case in more detail. One important problem is that the instanton charge is not evidently integer. In order to study the instanton solution precisely, we use a Fock space representation as follows ($n_1, n_2 \geq 0$):

$$z_1|n_1, n_2\rangle = \sqrt{\theta_1}\sqrt{n_1+1}|n_1+1, n_2\rangle, \tag{28}$$

$$\bar{z}_1|n_1, n_2\rangle = \sqrt{\theta_1}\sqrt{n_1}|n_1-1, n_2\rangle, \tag{29}$$

by using the commutation relation (5). Similar expressions for z_2 and $\bar{z}_2$ also apply (but paying a little attention to the sign of θ_2 which is not restricted to be positive). In this representation the z_i's are infinite-dimensional matrices, and so are the operator Δ, $\Delta^\dagger$ etc. Because of infinite dimensions are involved we can not determine the dimension of null-space of $\Delta^\dagger$ straightforwardly from the difference of the numbers of its rows and columns. But it turns out that $\Delta^\dagger$ also has infinite number of zero modes, and they can be arranged into an $(N+2k)\times N$ matrix with entries from the (noncommutative) algebra generated by the coordinates, which resembles the commutative case.

In the following sections, we will study in full detail the ASD 1-instanton and 2-instanton solutions of $U(1)$ theory and 1-instanton solutions of $U(2)$ theory on general $\mathbf{R}^4_{\mathrm{NC}}$. For each of them, the two distinct cases, $\theta_2 > 0$ and $\theta_2 < 0$, are considered separately and all the details of the zero modes are worked out. The instanton charge is numerically computed to be integer in the $U(1)$ 1-instanton cases.

4 $U(1)$ 1-instanton solution

In this case, the ADHM matrix (15) which satisfies (27) is given by

$$\Delta = \begin{pmatrix} \sqrt{\zeta} & 0 \\ \bar{z}_2 & -z_1 \\ \bar{z}_1 & z_2 \end{pmatrix}, \quad \Delta^\dagger = \begin{pmatrix} \sqrt{\zeta} & z_2 & z_1 \\ 0 & -\bar{z}_1 & \bar{z}_2 \end{pmatrix}, \tag{30}$$

($\zeta = 2\theta^{12} + 2\theta^{34} \geq 0$ for our assumption) when the center of mass collective coordinates set to zero. It is straightforward to obtain

$$f = (Z_1 + Z_2 + \zeta)^{-1} \tag{31}$$

where $Z_1 \equiv z_1\bar{z}_1$ and $Z_2 \equiv z_2\bar{z}_2$.

Now we construct the matrix U. It is easy to find a general solution U_0:

$$U_0 = \begin{pmatrix} Z_1 + Z_2 \\ -\sqrt{\zeta}\bar{z}_2 \\ -\sqrt{\zeta}\bar{z}_1 \end{pmatrix}, \quad \Delta^\dagger U_0 = 0. \tag{32}$$

But the problem is that this U_0 is either over-complete or incomplete for each θ_2 cases. We will solve this problem in the following.

4.1 $\theta_2 > 0$ case

In this case $\bar{z}_1$ and $\bar{z}_2$ are annihilation operators and U_0 above obviously annihilates the vacuum, i.e.

$$U_0|0,0\rangle = 0. \tag{33}$$

In other words, as an infinite-dimensional matrix in the Fock space representation, U_0 has a redundant column with all its elements vanishing. This column can be removed by a shift operator $u^\dagger$ which projects out the vacuum:

$$u^\dagger u = p \equiv 1 - p_0, \quad uu^\dagger = 1, \tag{34}$$

where $p_0 = |0,0\rangle\langle 0,0|$. By using this projection the normalized U which satisfies the completeness relation (22) can be obtained as follows:

$$U = \tilde{U}_0\beta, \quad U^\dagger U = 1, \tag{35}$$

where $\tilde{U}_0 = U_0 u^\dagger$ is the exact set of all zero modes and

$$\beta = (\tilde{U}_0^\dagger \tilde{U}_0)^{-1/2} = [u(Z_1+Z_2)(Z_1+Z_2+\zeta)u^\dagger]^{-1/2} \equiv u\beta_p u^\dagger \tag{36}$$

is a normalization factor.

Now we compute the field strength. It is given by (24) which turns out to be as follows:

$$F_{mn} = 2i\bar{\eta}^c_{mn} U^\dagger b(\tau^c f) b^\dagger U, \tag{37}$$

and so can be written as

$$F = \zeta\beta u[(z_2 f\bar{z}_2 - z_1 f\bar{z}_1)(dz_1 d\bar{z}_1 - dz_2 d\bar{z}_2) + 2z_1 f\bar{z}_2 d\bar{z}_1 dz_2 + 2z_2 f\bar{z}_1 d\bar{z}_2 dz_1]u^\dagger\beta. \tag{38}$$

The topological instanton charge is then

$$Q = -\frac{1}{8\pi^2}\int F\wedge F = -\frac{\zeta^2}{\pi^2}\int dx^4 uTu^\dagger = -\zeta^2|\theta_1\theta_2|\mathrm{Tr}_{\mathcal{H}}(uTu^\dagger) \tag{39}$$

where the expression T is only defined on $p\mathcal{H}$:

$$\begin{aligned} T &= [\beta_p(z_2 f\bar{z}_2 - z_1 f\bar{z}_1)\beta_p]^2 + 2\beta_p z_1 f\bar{z}_2\beta_p^2 z_2 f\bar{z}_1\beta_p + 2\beta_p z_2 f\bar{z}_1\beta_p^2 z_1 f\bar{z}_2\beta_p \\ &= \tfrac{1}{(Z_1+Z_2)(Z_1'+Z_2')}\Big[(\tfrac{Z_2}{Z_1'+Z_2} - \tfrac{Z_1}{Z_1+Z_2'})^2\tfrac{1}{(Z_1+Z_2)(Z_1'+Z_2')} \\ &\quad + \tfrac{2Z_1Z_2'}{(Z_1+Z_2')^2(Z_1+Z_2'-\theta_1)(Z_1+Z_2'+\theta_2)} + \tfrac{2Z_2Z_1'}{(Z_1'+Z_2)^2(Z_1'+Z_2-\theta_2)(Z_1'+Z_2+\theta_1)}\Big], \end{aligned} \tag{40}$$

where Z_1' means $Z_1+\theta_1$, Z_2' means $Z_2+\theta_2$.

Notice that Z_1 and Z_2 have eigenvalues $n_1\theta_1, n_1 \geq 0$ and $n_2\theta_2, n_2 \geq 0$ on $\mathcal{H}$ respectively, we get then

$$\mathrm{Tr}_{\mathcal{H}}(uTu^\dagger) = \sum_{n_1+n_2>0} T|_{Z_1=n_1\theta_1, Z_2=n_2\theta_2}. \tag{41}$$

Unlike the familiar $\theta_1 = \theta_2$ case, the expression in (41) seems too complicated to be easily worked out by analytic method. We didn't try hard to sum them analytically. A simple numerical calculation should be sufficient to convince us what is the final result. For reasonable $\theta_{1,2}$, the series converge quite fast. For example, for $\theta_1 = 1.6, \theta_2 = 0.4$ we have

$$Q(n_1, n_2 \leq 200)|_{\theta_1=1.6,\theta_2=0.4} = -0.999895, \tag{42}$$

by using the popular software Mathematica. When θ_2 tends to 0 (fixing θ_1), the series (41) seems to blow up and its convergency decreases rapidly. In such cases, we should properly adjust the range of summation and still we get the satisfying result $Q=-1$. For example

$$Q(n_1\leq 20, n_2\leq 2000)|_{\theta_1=1.99,\theta_2=0.01}=-0.998137. \tag{43}$$

These numerical results strongly convince us that $Q=-1$ should be the right answer.

4.2 $\theta_2<0$ case

In this case $\bar{z}_1$ is an annihilation operator and $\bar{z}_2$ is a creation operator. The matrix U_0 can be directly normalized:

$$\tilde{U}=U_0\beta, \quad \tilde{U}^\dagger\tilde{U}=1 \tag{44}$$

where

$$\beta=(U_0^\dagger U_0)^{-1/2}=[(Z_1+Z_2)(Z_1+Z_2+\zeta)]^{-1/2}. \tag{45}$$

But the problem is that $\tilde{U}$ does not satisfy the completeness relation. Explicitly we have

$$(\Delta f\Delta^\dagger+\tilde{U}\tilde{U}^\dagger)\begin{pmatrix}0\\|0,0\rangle\\0\end{pmatrix}=0\neq\begin{pmatrix}0\\|0,0\rangle\\0\end{pmatrix}. \tag{46}$$

So $\tilde{U}$ is not the right answer. In fact, $\tilde{U}$ contains almost all the zero modes of $\Delta^\dagger$ except one. We can simply add an extra column to $\tilde{U}$ to make it complete:

$$U=\begin{pmatrix}0\\p_0\\0\end{pmatrix}+\tilde{U}u \tag{47}$$

where u is the same shift operator introduced in the last subsection. It is straightforward to check that the completeness relation (22) is now satisfied, and so the ASD instanton solution in the $\theta_2<0$ case can been deduced.

The field strength F is again given by (37), but we will not go on to give the lengthy expression of its explicit form here and in the following sections. The topological instanton charge is

$$Q=-\frac{1}{8\pi^2}\int F\wedge F=-|\theta_1\theta_2|\mathrm{Tr}_{\mathcal{H}}T \tag{48}$$

where

$$\begin{aligned}T\ &=[(\sqrt{\zeta}u^\dagger\beta z_2-p_0)f(\sqrt{\zeta}\bar{z}_2\beta u-p_0)-\zeta u^\dagger\beta^2 z_1 f\bar{z}_1 u]^2\\&\quad+2\zeta u^\dagger\beta^2 z_1 f^2(\zeta\bar{z}_2\beta^2 z_2+p_0)\bar{z}_1 u\\&\quad+2\zeta(\sqrt{\zeta}u^\dagger\beta z_2-p_0)f^2\bar{z}_1\beta^2 z_1(\sqrt{\zeta}\bar{z}_2\beta u-p_0),\end{aligned} \tag{49}$$

and so

$$\begin{aligned}\mathrm{Tr}_{\mathcal{H}}T\ &=\textstyle\sum_{n_1\geq 0,n_2\geq 1}\{\zeta^2\beta^4[Z_2(Z_1'+Z_2)^{-1}-Z_1(Z_1+Z_2')^{-1}]^2\\&\quad+2\zeta\beta^2 Z_1 Z_2'(Z_1+Z_2')^{-2}(Z_1+Z_2'-\theta_1)^{-1}(Z_1+Z_2'+\theta_2)^{-1}\\&\quad+2\zeta\beta^2 Z_2 Z_1'(Z_1'+Z_2)^{-2}(Z_1'+Z_2-\theta_2)^{-1}(Z_1'+Z_2+\theta_1)^{-1}\}\\&\quad+\theta_1^{-2}+2\zeta\theta_1^{-2}(\theta_1-\theta_2)^{-1},\end{aligned} \tag{50}$$

The same numerical calculations give the following results:

$$Q(n_1, n_2 \leq 200)|_{\theta_1=2.4,\theta_2=-0.4} = -0.999895, \tag{51}$$

$$Q(n_1 \leq 20, n_2 \leq 2000)|_{\theta_1=2.01,\theta_2=-0.01} = -0.998141, \tag{52}$$

and so we get $Q = -1$ which agrees with the result obtained by the well-known argument.

5 $U(1)$ 2-instanton solution

The moduli space of 2-instanton is much more complicated than that of 1-instanton. In the limit of coincident instantons, which attract the interests of many theoretical and mathematical physicist recently and can be given in explicit form, the ADHM data (15) can be written as follows:

$$\Delta = \begin{pmatrix} 0 & \sqrt{2\zeta} & 0 & 0 \\ \bar{z}_2 & 0 & -z_1 & -\sqrt{\zeta} \\ 0 & \bar{z}_2 & 0 & -z_1 \\ \bar{z}_1 & 0 & z_2 & 0 \\ \sqrt{\zeta} & \bar{z}_1 & 0 & z_2 \end{pmatrix}, \tag{53}$$

$$\Delta^\dagger = \begin{pmatrix} 0 & z_2 & 0 & z_1 & \sqrt{\zeta} \\ \sqrt{2\zeta} & 0 & z_2 & 0 & z_1 \\ 0 & -\bar{z}_1 & 0 & \bar{z}_2 & 0 \\ 0 & -\sqrt{\zeta} & -\bar{z}_1 & 0 & \bar{z}_2 \end{pmatrix}. \tag{54}$$

By using this ADHM matrix, we have

$$f = \begin{pmatrix} (Z_1'' + Z_2' + \zeta)(Z''\,')^{-1} & -\sqrt{\zeta}(Z''\,')^{-1}\bar{z}_1 \\ -\sqrt{\zeta}z_1(Z''\,')^{-1} & (Z_1' + Z_2' - \theta_1)(Z'\,')^{-1} \end{pmatrix}. \tag{55}$$

where $Z''\,'$ means $(Z_1''+Z_2')^2 - \theta_1 Z_1'' + \theta_2 Z_2'$, $Z'\,'$ means $(Z_1'+Z_2')^2 - \theta_1 Z_1' + \theta_2 Z_2'$.

By following the same strategy as used in the 1-instanton case, we find a general solution as follows:

$$U_0 = \begin{pmatrix} (2\zeta)^{-1/2} Z \\ \sqrt{\zeta}\bar{z}_2\bar{z}_1 \\ -\bar{z}_2(Z_1 + Z_2 + \theta_2) \\ \sqrt{\zeta}\bar{z}_1\bar{z}_1 \\ -\bar{z}_1(Z_1 + Z_2 - \theta_1) \end{pmatrix} \tag{56}$$

where $Z = (Z_1+Z_2)^2 - \theta_1 Z_1 + \theta_2 Z_2$. Now we discuss the two θ_2 cases separately.

5.1 The $\theta_2 > 0$ case

In this case U_0 annihilates two states: $|0,0\rangle$ and $|1,0\rangle$ and so we must introduce one more shift operator $\tilde{u}^\dagger$ which satisfies the following relations:

$$\tilde{u}^\dagger\tilde{u} = 1 - p_0 - p_1, \quad \tilde{u}\tilde{u}^\dagger = 1, \tag{57}$$

where $p_1 = |1,0\rangle\langle 1,0|$. The correct U is again given by:

$$U = \tilde{U}_0\beta, \quad \tilde{U}_0 = U_0\tilde{u}^\dagger, \tag{58}$$

where β is a normalization factor:

$$\beta = (\tilde{U}_0^\dagger\tilde{U}_0)^{-1/2} = (2\zeta)^{1/2}(\tilde{u}ZZ'\,'\tilde{u}^\dagger)^{-1/2}. \tag{59}$$

5.2 The $\theta_2 < 0$ case

There is a subtlety in this case. The matrix element $f_{22} = (Z_1' + Z_2' - \theta_1)(Z'\,')^{-1}$ of the matrix f is not well-defined on the vacuum $|0,0\rangle$. We remedy this arbitrariness by the following definition:

$$f_{22}|0,0\rangle = (2\theta_1 + \theta_2)^{-1}|0,0\rangle. \tag{60}$$

With this definition one easily show that $f^{-1}f = ff^{-1} = 1$. What is f^{-1}.

In the $\theta_2 < 0$ case U_0 still annihilates $|0,0\rangle$. Moreover, it turns out that we need three extra zero modes of $\Delta^\dagger$ to make U_0 complete. The matrix U is constructed as follow:

First, it is easy to find the following zero mode:

$$\Delta^\dagger \begin{pmatrix} 0 \\ 0 \\ |0,0\rangle \\ 0 \\ 0 \end{pmatrix} = 0 \tag{61}$$

is not included in U_0 and it is orthogonal to all columns of U_0. So we can replace the redundant column of U_0 with this extra zero mode and get

$$\tilde{U}_0 = \begin{pmatrix} 0 \\ 0 \\ p_0 \\ 0 \\ 0 \end{pmatrix} + U_0, \tag{62}$$

which can be normalized as

$$\tilde{U} = \tilde{U}_0\beta, \quad \tilde{U}^\dagger\tilde{U} = 1 \tag{63}$$

where β is a normalization factor:

$$\beta = (\tilde{U}_0^\dagger\tilde{U}_0)^{-1/2} = (2\zeta)^{1/2}(ZZ'\,' + 2\zeta p_0)^{-1/2}. \tag{64}$$

Now we insert the other two (normalized) extra zero modes into $\tilde{U}$ and get

$$U = \begin{pmatrix} \sqrt{\frac{-\theta_2}{2\theta_1+\theta_2}}|0,0\rangle \\ 0 \\ -\sqrt{\frac{2\zeta}{2\theta_1+\theta_2}}|0,1\rangle \\ 0 \\ 0 \end{pmatrix} \langle 0,0| + \begin{pmatrix} 0 \\ \sqrt{\frac{\theta_1}{2\theta_1+\theta_2}}|0,0\rangle \\ -\sqrt{\frac{\zeta}{2\theta_1+\theta_2}}|1,0\rangle \\ 0 \\ 0 \end{pmatrix} \langle 1,0| + \tilde{U}\tilde{u}, \tag{65}$$

which is the required U and can be directly checked to satisfy the completeness relation (22).

6 $U(2)$ 1-instanton solution

We will be brief here and rely on early results in the $\theta_1 = \theta_2$ case. The instanton positioned at the origin and sized ρ is determined from the following ADHM matrix:

$$\Delta = \begin{pmatrix} \sqrt{\zeta+\rho^2} & 0 \\ 0 & \rho \\ \bar{z}_2 & -z_1 \\ \bar{z}_1 & z_2 \end{pmatrix}, \quad \bar{\Delta} = \begin{pmatrix} \sqrt{\zeta+\rho^2} & 0 & z_2 & z_1 \\ 0 & \rho & -\bar{z}_1 & \bar{z}_2 \end{pmatrix}. \tag{66}$$

The factorization relation is satisfied and we have

$$f = (Z_1 + Z_2 + \zeta + \rho^2)^{-1}. \tag{67}$$

It is not difficult to find the general solution:

$$U_0 = \begin{pmatrix} Z_1 + Z_2 & 0 \\ 0 & Z_1 + Z_2 + \zeta \\ -\sqrt{\zeta+\rho^2}\bar{z}_2 & \rho z_1 \\ -\sqrt{\zeta+\rho^2}\bar{z}_1 & -\rho z_2 \end{pmatrix} \tag{68}$$

which should be modified to lead to the correct U.

6.1 The $\theta_2 > 0$ case

It is easy to see that the first column of U_0 annihilates the vacuum. So we can introduce an operator

$$v^\dagger = \begin{pmatrix} u^\dagger & 0 \\ 0 & 1 \end{pmatrix} \tag{69}$$

to remove the redundance of U_0 and get the normalized U:

$$U = \tilde{U}_0 \beta, \quad \tilde{U}_0 = U_0 v^\dagger \tag{70}$$

where

$$\beta = (\tilde{U}_0^\dagger \tilde{U}_0)^{-1/2} = \begin{pmatrix} [u(Z_1+Z_2)f^{-1}u^\dagger]^{-1/2} & 0 \\ 0 & [(Z_1+Z_2+\zeta)f^{-1}]^{-1/2} \end{pmatrix}. \tag{71}$$

This matrix U now satisfies the completeness relation (22).

6.2 The $\theta_2 < 0$ case

As in the $U(1)$ theory, U_0 is directly normalizable:

$$\tilde{U} = U_0 \beta, \quad \tilde{U}^\dagger \tilde{U} = 1 \tag{72}$$

where

$$\beta = (U_0^\dagger U_0)^{-1/2} = \begin{pmatrix} [(Z_1+Z_2)f^{-1}]^{-1/2} & 0 \\ 0 & [(Z_1+Z_2+\zeta)f^{-1}]^{-1/2} \end{pmatrix}. \tag{73}$$

Again one more zero mode should be added to $\tilde{U}$ by using the shift operator v and the right U is given by:

$$U = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ p_0 & 0 \\ 0 & 0 \end{pmatrix} + \tilde{U} v. \tag{74}$$

One can again check that this U satisfies the completeness relation (22).

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