LOCAL FIELD THEORY ON κ-MINKOWSKI SPACE, STAR PRODUCTS AND NONCOMMUTATIVE TRANSLATIONS*

P. Kosiński
Institute of Physics, University of Lódź,
ul. Pomorska 149/53 90–236 Lódź, Poland

J. Lukierski
Institute of Theoretical Physics, University of Wroclaw
pl. M. Borna 9, 50-205 Wrocław, Poland

P. Maślanka
Institute of Physics, University of Lódź,
ul. Pomorska 149/53 90–236 Lódź, Poland

Abstract

We consider local field theory on κ-deformed Minkowski space which is an example of solvable Lie-algebraic noncommutative structure. Using integration formula over κ-Minkowski space and κ-deformed Fourier transform we consider for deformed local fields the reality conditions as well as deformation of action functionals in standard Minkowski space. We present explicite formulas for two equivalent star products describing CBH quantization of field theory on κ-Minkowski space. We express also via star product technique the noncommutative translations in κ-Minkowski space by commutative translations in standard Minkowski space.

1 Introduction

Recently the idea that the commuting space-time coordinates x_μ are replaced by algebra of noncommuting generators $\hat{x}_\mu$ became quite popular (see e.g. [1-5]). The noncommutative Minkowski space on one side represents algebraically the quantum gravity corrections [1,2], on other hand describes the end points of strings coupled to antisymmetric two-index gauge field $B_{\mu\nu}$ (in $D = 10$) [3,5] and boundary strings for membranes coupled to the antisymmetric three-index gauge field $C_{\mu\nu\rho}$ (in $D = 11$). In general the noncommutativity in "new" string

*To be published in Proceedings of Colloqium on Quantum Groups and Integraable Systems, Prague, June 2000 (Czech. J. Phys. **50**, 2000, in press

theory occurs for the $(p+1)$ world-volume coordinates of Dirichlet p-branes and follows from the modification of standard quantization rules of the classical $(p+1)$-dimensional p-brane fields in the presence of external generalized gauge potentials.

In most of the papers dealing with noncommutative field theory one considers the Dopplicher–Fredenhagen–Roberts (DFR) deformation [1] of space–time coordinates

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu} \qquad \theta_{\mu\nu} = -\theta_{\nu\mu} \text{ constant}. \tag{1}$$

In the presence of DFR noncommutativity the relativistic invariance remains classical and the translations $x_\mu \to \hat{x}'_\mu = \hat{x}_\mu + a_\mu$ preserving (1) are commutative, i.e. $[a_\mu, a_\nu] = 0$. This feature is not valid if $\theta_{\mu\nu}$ depends on $\hat{x}_\mu$, in particular if we assume the linear $\hat{x}_\mu$–dependence, i.e.

$$[\hat{x}_\mu, \hat{x}_\nu] = ic^\lambda_{\mu\nu}\hat{x}_\lambda \,. \tag{2}$$

In such a case the translations

$$\hat{x}'_\mu = \hat{x}_\mu + \hat{v}_\mu \,, \tag{3}$$

are not commutative, commute with κ-Minkowski coordinates ($[\hat{x}_\mu, \hat{v}_\mu] = 0$) and provide second copy of the algebra (2)

$$[\hat{v}_\mu, \hat{v}_\nu] = ic^\lambda_{\mu\nu}\hat{v}_\lambda \,. \tag{4}$$

One can describe (3) as a coproduct $\Delta(\hat{x}_\mu) = x_\mu \otimes 1 + 1 \otimes x_\mu$, where $\hat{x}_\mu \otimes 1 \equiv \hat{x}_\mu$ and $\hat{v}_\mu = 1 \otimes \hat{x}_\mu$. It appears that for particular choices of the structure constants $c^\lambda_{\mu\nu}$ the noncommutative translations (4) can be incorporated as translation sector of the quantum–deformed Poincaré group, with noncommutative group parameters[1]

It should be stressed that a natural mathematical framework providing noncommutative space-time is given by quantum groups describing the deformations of space-time symmetries. The formalism of quantum groups (see e.g. [9]) provides the modified symmetries in the form of a dual pair of Hopf algebras $A \otimes A^\star$ (in the applications discussed here A describes the deformation of Poincaré group A and $A^\star$ - the deformation of Poincaré algebra) where the noncommutative coordinates $\hat{x}_\mu \in A$ and the fourmomenta $\hat{p}_\mu \in A^\star$. In particular coproducts of $\hat{x}_\mu$ and $\hat{p}_\mu$ describe the addition laws of noncommutative coordinates and noncommutative momenta.

In this talk we will consider a particular example of so-called κ-deformation of space-time symmetries [11–15] with noncommutativity of space-time coordinates described by the following solvable Lie-algebraic relations:

$$[\hat{x}_0, \hat{x}_i] = \frac{i}{\kappa}\hat{x}_i \,, \qquad [\hat{x}_l, \hat{x}_j] = 0 \,. \tag{5}$$

The primitive coproducts lead to the addition law (3) for the noncommutative coordinates and the dual fourmomentum generators are described by <u>commuting</u> fourmomenta p_μ, with coproducts generating the following nonabelian addition law

$$p_0^{(1+2)} = p_0^{(1)} + p_0^{(2)} \,, \qquad p_i^{(1+2)} = p_i^{(1)} \mathrm{e}^{-\frac{\mathrm{p}_0^{(2)}}{\kappa}} + \mathrm{p}_\mathrm{i}^{(2)} \,. \tag{6}$$

[1]For classification of quantum Poincaré groups see [10].

In this lecture we shall describe some new aspects of the deformation of local relativistic $D = 4$ field theory on noncommutative space-time (5), with taking into consideration the relations (6). In particular we shall consider new κ–deformed Fourier transform of local fields on $D = 4$ κ–Minkowski space, the reality conditions and action integrals for complex κ–deformed local fields. Our next result here is the explicite CBH formula for star-product quantization of κ–deformed field theory[2] Also we shall show that the noncommutative translations (3) of field arguments can be expressed via star product technique in terms of classical, commutating translation parameters.

It should be recalled that the κ–deformed local interaction vertices lead to modified Feynman diagram rules with deformed fourmomentum conservation laws [18,19]. It is interesting to mention that the effects of nonconservation of momenta has been observed recently also in the noncommutative field theory constructed without any reference to Hopf–algebraic structures (see e.g. [20]).

2 κ-Deformed Fourier Transform and Reality Conditions

The duality structure provides natural definition of Fourier transform mapping the functions (fields) on noncommutative space- time $\hat{x}_\mu$ into the functions of dual fourmomentum space [9].

In κ-deformed Minkowski space one can write the following formulae[3]

$$\Phi(\hat{x}) = \frac{1}{(2\pi)^2} \int \mathrm{d}^4\mathrm{p}\ \widetilde{\Phi}_\kappa(\mathrm{p})\ : \mathrm{e}^{\mathrm{i}\mathrm{p}\hat{\mathrm{x}}} : = \widehat{\mathrm{F}}_\kappa \widetilde{\Phi}(\mathrm{p})\,, \tag{7}$$

$$\widetilde{\Phi}(p) = \frac{1}{(2\pi)^2} \iint \mathrm{d}^4\ \hat{\mathrm{x}}\Phi(\hat{\mathrm{x}})\ : \mathrm{e}^{-\mathrm{i}\mathrm{p}\hat{\mathrm{x}}} : = \widehat{\mathrm{F}}_\kappa^{-1}\Phi(\hat{\mathrm{x}})\,, \tag{8}$$

where $p\hat{x} = \vec{p}\vec{x} - p_0x_0$ and

$$: \mathrm{e}^{\mathrm{i}\mathrm{p}\hat{\mathrm{x}}} := \mathrm{e}^{-\mathrm{i}\mathrm{p}_0\hat{\mathrm{x}}_0}\mathrm{e}^{\mathrm{i}\vec{\mathrm{p}}\vec{\mathrm{x}}}, \tag{9}$$

$$\widetilde{\Phi}_\kappa(p) = \mathrm{e}^{\frac{3\mathrm{p}_0}{\kappa}}\,\widetilde{\Phi}\left(\mathrm{e}^{\frac{\mathrm{p}_0}{\kappa}}\vec{\mathrm{p}}, \mathrm{p}_0\right)\,. \tag{10}$$

It should be pointed out that the κ-deformed exponential has been chosen in the way providing the κ-deformed addition rule of fourmomenta (see also (6))

$$: \mathrm{e}^{\mathrm{i}\mathrm{p}^{(1)}\hat{\mathrm{x}}} :: \mathrm{e}^{\mathrm{i}\mathrm{p}^{(2)}\hat{\mathrm{x}}} :=: \mathrm{e}^{\mathrm{i}\mathrm{p}^{(1+2)}\hat{\mathrm{x}}} : \tag{11}$$

and the κ-deformed integration over κ-Minkowski space is generated by the formula

$$\frac{1}{(2\pi)^4} \iint \mathrm{d}^4\ \hat{\mathrm{x}}\ : \mathrm{e}^{\mathrm{i}\mathrm{p}\hat{\mathrm{x}}} : = \delta^4(\mathrm{p})\,. \tag{12}$$

[2]For general Campbell–Baker-Hausdorff quantization formula describing quantization of linear Poisson brackets see e.g. [16,17].

[3]For the formulae in $1 + 1$ Minkowski space see [21]. It should be mentioned that we introduce here modified formula (7) in comparison with the one presented in [18]. In the formulae (7–8) the Fourier transform (8) is inverse to the one given by (7).

The formulae (11-12) imply that

$$\frac{1}{(2\pi)^2}\iint \mathrm{d}^4\,\hat{\mathrm{x}}\,\Phi(\hat{\mathrm{x}}) = \widetilde{\Phi}_\kappa(0) = \widetilde{\Phi}(0)\,, \tag{13}$$

$$\begin{aligned}\iint \mathrm{d}^4\,\hat{\mathrm{x}}\,\Phi^2(\hat{\mathrm{x}}) &= \int \mathrm{d}^4\mathrm{p}\;\mathrm{d}^4\mathrm{q}\;\widetilde{\Phi}\left(\mathrm{e}^{\frac{\mathrm{p}_0}{\kappa}}\vec{\mathrm{p}},\mathrm{p}_0\right)\widetilde{\Phi}\left(\mathrm{e}^{\frac{\mathrm{q}_0}{\kappa}}\vec{\mathrm{q}},\mathrm{q}_0\right)\\ &\quad\cdot\,\delta\left(p_0+q_0\right)\delta^{(3)}\left(\vec{p}\mathrm{e}^{-\frac{\mathrm{q}_0}{\kappa}}+\vec{\mathrm{q}}\right)\\ &= \int \mathrm{d}^4\mathrm{p}\;\widetilde{\Phi}\left(-\vec{\mathrm{p}},-\mathrm{p}_0\right)\widetilde{\Phi}\left(\mathrm{e}^{\frac{\mathrm{p}_0}{\kappa}}\vec{\mathrm{p}},\mathrm{p}_0\right)\,.\end{aligned} \tag{14}$$

Let us introduce now the Hermitean conjugation of the field (7). Defining

$$\begin{aligned}\Phi^+(\hat{x}) &= \frac{1}{(2\pi)^2}\int \mathrm{d}^4\mathrm{p}\;\widetilde{\Phi}\,{}^*_\kappa(\mathrm{p})\mathrm{e}^{-\mathrm{i}\vec{\mathrm{p}}\vec{\mathrm{x}}}\mathrm{e}^{\mathrm{i}\mathrm{p}_0\hat{\mathrm{x}}_0}\\ &= \frac{1}{(2\pi)^2}\int \mathrm{d}^4\mathrm{p}\;\widetilde{\Phi}\,{}^+_\kappa(\mathrm{p}) : \mathrm{e}^{\mathrm{i}\mathrm{p}\hat{\mathrm{x}}} :\end{aligned} \tag{15}$$

and using the formula

$$\mathrm{e}^{-\mathrm{i}\vec{\mathrm{p}}\vec{\mathrm{x}}}\mathrm{e}^{\mathrm{i}\mathrm{p}_0\mathrm{x}_0} = \mathrm{e}^{\mathrm{i}\mathrm{p}_0\hat{\mathrm{x}}_0}\mathrm{e}^{-\mathrm{i}\mathrm{e}^{\frac{\mathrm{p}_0}{\kappa}}\vec{\mathrm{p}}\vec{\mathrm{x}}}\,, \tag{16}$$

one gets

$$\Phi^+_\kappa(p) = \widetilde{\Phi}\,{}^\star(-p)\,. \tag{17}$$

Let us observe that the reality condition $\Phi^+(\hat{x}) = \Phi(\hat{x})$ gives

$$\Phi_\kappa(p) = \Phi^+_\kappa(p) \leftrightarrow \Phi_\kappa(p) = \widetilde{\Phi}\,{}^\star(-p)\,. \tag{18}$$

For real noncommutative fields one gets, using (18)

$$\begin{aligned}\iint \mathrm{d}^4\,\hat{\mathrm{x}}\,\Phi^2(\hat{\mathrm{x}}) &= \int \mathrm{d}^4\mathrm{p}\;\mathrm{e}^{\frac{3\mathrm{p}_0}{\kappa}}\,\widetilde{\Phi}(\mathrm{p})\,\widetilde{\Phi}\,{}^\star(\mathrm{p})\\ &= \int \mathrm{d}^4\mathrm{p}\;\mathrm{e}^{-\frac{3\mathrm{p}_0}{\kappa}}\,\widetilde{\Phi}^\star_\kappa(\mathrm{p})\widetilde{\Phi}_\kappa(\mathrm{p})\,.\end{aligned} \tag{19}$$

For complex noncommutative fields one can write

$$\iint \mathrm{d}^4\,\hat{\mathrm{x}}\,\Phi(\hat{\mathrm{x}})\Phi^+(\hat{\mathrm{x}}) = \int \mathrm{d}^4\mathrm{p}\;\widetilde{\Phi}(\mathrm{p})\widetilde{\Phi}\,{}^\star(\mathrm{p})\,, \tag{20a}$$

$$\iint \mathrm{d}^4\,\hat{\mathrm{x}}\,\Phi^+(\hat{\mathrm{x}})\Phi(\hat{\mathrm{x}}) = \int \mathrm{d}^4\mathrm{p}\;\mathrm{e}^{\frac{3\mathrm{p}_0}{\kappa}}\,\widetilde{\Phi}\,{}^\star(\mathrm{p})\widetilde{\Phi}(\mathrm{p})\,. \tag{20b}$$

It should be added that we obtain alternative definition of Fourier transform if in the formula (8) the κ–deformed exponential is located on left side of the field $\Phi(\hat{x})$.

3 Star Product for κ–Deformed Field Theory

It is known [16,17,4,22] that the multiplication of two group elements with Lie algebra generators $\hat{x}_\mu$ described by relation (2) is given by CBH formula

$$e^{i\alpha^\mu \hat{x}_\mu} e^{i\beta^\mu \hat{x}_\mu} = e^{i\gamma^\mu(\alpha,\beta)\hat{x}_\mu} , \tag{21}$$

where

$$\begin{aligned}\gamma^\mu(\alpha,\beta) &= \alpha^\mu + \beta^\mu + c^\mu_{\rho\tau}\alpha^\rho\beta^\tau \\ &+ \frac{1}{12} c^\mu_{\rho\tau} c^\rho_{\lambda\nu}\left(\alpha^\tau\alpha^\lambda\beta^\nu + \beta^\tau\beta^\lambda\alpha^\nu\right) + \ldots .\end{aligned} \tag{22}$$

The noncommutative group multiplication (21) is translated into the framework of star products by substitution $\hat{x}_\mu \to x_\mu$, where x_μ are classical Minkowski coordinates and copying the relation (21)

$$e^{i\alpha^\mu x_\mu} \star e^{i\beta^\mu x_\mu} = e^{i\gamma^\mu(\alpha,\beta)x_\mu} . \tag{23}$$

The formula (23) describes the CBH star product and the associativity of group multiplication leads to the associativity of CBH star–multiplication. From relation (23) follows CBH star product for arbitrary two classical fields $\phi(x), \chi(x)$ on Minkowski space

$$\phi(x) \star \chi(x) = \phi(y) \exp i x_\mu \widetilde{\gamma}^{\,\mu}\left(\frac{\overleftarrow{\partial}}{\partial y}, \frac{\overrightarrow{\partial}}{\partial z}\right) \chi(z)\Big|_{y=z=x} , \tag{24}$$

where $\gamma^\mu(\alpha,\beta) = \alpha^\mu + \beta^\mu + \widetilde{\gamma}^{\,\mu}(\alpha,\beta)$ (see (22)).

It can be shown that the formula (24) is an example of Kontsevich formula [23] describing star product for arbitrary Poisson structure, which is written in particular basis.

In the case of solvable algebra (5) the formulae (21) and (23) can be written in closed form as follows:

$$e^{ik^\mu \hat{x}_\mu} \cdot e^{il^\mu \hat{x}_\mu} = e^{ir^\mu(k,l)\hat{x}_\mu} , \tag{25}$$

where $f_\kappa(\alpha) = \frac{\kappa}{\alpha}\left(1 - e^{-\frac{\alpha}{\kappa}}\right)$,

$$\begin{aligned} r^0(k,l) &= k^0 + l^0 , \\ r^i(k,l) &= \frac{f_\kappa(k^0) e^{\frac{l^0}{\kappa}} k^i + f_\kappa(l^0) l^i}{f_\kappa\left(k^0 + l^0\right)} ,\end{aligned} \tag{26}$$

and

$$e^{ik^\mu x_\mu} \star e^{il^\mu x_\mu} = e^{ir^\mu(k,l)x_\mu} . \tag{27}$$

Differentiating (27) twice and putting $k^\mu = l^\mu = 0$ one obtains two relations

$$\begin{aligned} x_0 \star x_i &= x_0 x_i + \frac{i}{2\kappa} x_i , \\ x_i \star x_0 &= x_0 x_i - \frac{i}{2\kappa} x_i , \end{aligned} \tag{28}$$

in consistency with (5). In general one gets

$$\phi(x) \star \chi(x) = \phi\left(\frac{1}{i}\frac{\partial}{\partial k^\mu}\right)\chi\left(\frac{1}{i}\frac{\partial}{\partial l^\mu}\right) e^{ir^\mu(k,l)x_\mu}\Big|_{k^\mu=l^\mu=0} . \qquad (29)$$

The quantization obtained by the multiplication of ordered exponentials (9) describes other reparametrization of the CBH star product. Indeed, using the formula (see also [24])

$$e^{ik^\mu \hat{x}_\mu} =: e^{ip^\mu \hat{x}_\mu} := e^{-ip_0\hat{x}_0} e^{i\vec{p}\,\vec{x}} , \qquad (30)$$

where

$$p_0 = k_0 , \qquad p_i = \frac{\kappa}{k_0}\left(1 - e^{-\frac{k_0}{\kappa}}\right) k_i = f_\kappa(-k_0) k_i , \qquad (31)$$

one arrives at the formula (11). The corresponding new star product $\circledast$ takes the form

$$e^{ip^\mu x_\mu} \circledast e^{i\tilde{p}^\mu x_\mu} = e^{i(p^0+\tilde{p}^0)x_0 + i\left(p^i e^{-\frac{\tilde{p}_0}{\kappa}} + \tilde{p}^i\right)x_i} . \qquad (32)$$

Differentiating twice relation (32) one gets

$$\begin{aligned} x^0 \circledast x^i &= x^0 x^i , \\ x^i \circledast x^0 &= x^0 x^i - \frac{i}{\kappa} x^i , \end{aligned} \qquad (33)$$

and we reproduce again relations (5).

It should be pointed out that the star product (32) is more physical, because the parameters p^μ can be interpreted as the fourmomenta with the addition law derived from the κ–deformation of relativistic symmetries [11–15].

4 Noncommutative translational Invariance

Let us observe that the κ–Minkowski space-time coordinates $\hat{x}_\mu$ and noncommutative translations $\hat{v}_\mu$ satisfy the same Lie–algebraic relations (see (2) and (4)). If we consider the noncommutative fields $\Phi(\hat{x},\hat{v}), \chi(\hat{x},\hat{v})$, these two commuting algebraic structures can be represented by the following extension of CBH star product (24):

$$\begin{aligned} \phi(x,v) \star \chi(x,v) &= \phi(y,u) \qquad (34) \\ &\quad \cdot \exp i\left\{ x_\mu \tilde{\gamma}^{\,\mu}\left(\frac{\overleftarrow{\partial}}{\partial y}, \frac{\overrightarrow{\partial}}{\partial z}\right) + v_\mu \tilde{\gamma}^{\,\mu}\left(\frac{\overleftarrow{\partial}}{\partial u}, \frac{\overrightarrow{\partial}}{\partial w}\right)\right\} \chi(z,w)\Big|_{\substack{y=z=x \\ u=w=v}} . \end{aligned}$$

In particular if $\Phi(\hat{x},\hat{v}) \equiv \Phi(\hat{x}+\hat{v})$ then one can write

$$\phi(x+v) \star \chi(x+v) = \phi(y) \exp i\,(x_\mu + v_\mu) \;\; \cdot \tilde{\gamma}^{\,\mu}\left(\frac{\overleftarrow{\partial}}{\partial y}, \frac{\overrightarrow{\partial}}{\partial z}\right)\chi(z)\Big|_{y=z=x} . \qquad (35)$$

If we introduce the notation stressing the nonlocal character of CBH star product

$$\phi(x) \star \chi(x) = \int \mathrm{d^4x_1\, d^4x_2 \hat{K}\,(x; x_1, x_2)}\, \phi(\mathrm{x_1})\chi(\mathrm{x_2}) , \qquad (36)$$

then from (35) follows that the noncommutative translations in κ–Minkowski space imply the shift of standard space–time coordinates x_μ by commuting translations v_μ i.e. in the formula (36) one should replace $\hat{K}(x;x_1,x_2) \to \hat{K}(x+v;x_1,x_2)$. We obtain

$$\begin{aligned} \iint \mathrm{d}^4\hat{\mathrm{x}}\,\Phi(\hat{\mathrm{x}}+\hat{\mathrm{v}})\chi(\hat{\mathrm{x}}+\hat{\mathrm{v}}) &= \int \mathrm{d}^4\mathrm{x}\,\mathrm{d}^4\mathrm{x}_1\,\mathrm{d}^4\mathrm{x}_2\hat{\mathrm{K}}(\mathrm{x}+\mathrm{v};\mathrm{x}_1,\mathrm{x}_2)\phi(\mathrm{x}_1)\chi(\mathrm{x}_2) \\ &= \int \mathrm{d}^4\mathrm{x}\,\mathrm{d}^4\mathrm{x}_1\,\mathrm{d}^4\mathrm{x}_2\hat{\mathrm{K}}(\mathrm{x};\mathrm{x}_1,\mathrm{x}_2)\phi(\mathrm{x}_1)\chi(\mathrm{x}_2) \\ &= \iint \mathrm{d}^4\hat{\mathrm{x}}\,\Phi(\hat{\mathrm{x}})\chi(\hat{\mathrm{x}})\,. \end{aligned} \tag{37}$$

For the arguments presented above it is essential linear form of the Poisson structure. It appears however, that one can extend the κ–Minkowski star product in such a way that general transformations of κ-deformed Poincaré group, with quadratic relations describing commutator of Lorentz group parameters and noncommutative translations, can be also considered [25].

5 Final Remarks

The discussion presented in Sect. 2-4 (action integrals for complex noncommutative fields, star products, translational invariance) can be extended to higher local powers, describing e.g. the interaction Lagrangeans in noncommutative local field theories (for $\lambda\Phi^4$ theory see [18]). These results will be also applied to the description of Abelian and nonAbelian gauge theories on κ–deformed Minkowski space [24].

References

[1] S. Dopplicher, K. Fredenhagen and J. Roberts, Phys. Lett. **B331**, 39 (1994); Comm. Math. Phys. **172**, 187 (1995).

[2] L.J. Garay, Int. Journ. Mod. Phys. **A10**, 145 (1995).

[3] N. Seiberg, E. Witten, JHEP 9909: 032 (1999).

[4] J. Madore, S. Schraml, P. Schupp and J. Wess, hep–th/0001203.

[5] N. Seiberg, L. Susskind and N. Toumbas, hep–th/0005040.

[6] S. Kawamoto and N. Sasakura, hep–th/0005123.

[7] Ali H. Chamseddine, hep–th/0005222.

[8] L. Alvarez–Gaumé and J.L.F. Barbon, hep– th/0006209.

[9] S. Majid, Foundations of Quantum Group Theory, Cambridge Univ. Press, 1995.

[10] P. Podleś and S.L. Woronowicz, Commun. Math. Phys. **178**, 61 (1996).

[11] J. Lukierski, A. Nowicki, H. Ruegg and V.N. Tolstoy, Phys. Lett. **B264**, 331 (1991).

[12] S. Giller, P. Kosiński, J. Kunz, M. Majewski and P. Maślanka, Phys. Lett. **B286**, 57 (1992).

[13] S. Zakrzewski, Journ. Phys. **A27**, 2075 (1994).

[14] S. Majid, H. Ruegg, Phys. Lett. **B334**, 384 (1994).

[15] J. Lukierski, H. Ruegg and W.J. Zakrzewski, Ann. of Phys. **243**, 90 (1995).

[16] V. Kathotia, math.QA/9811174.

[17] H. Garcia-Compean, J.F. Plebański, hep– th/9907183.

[18] P. Kosiński, J. Lukierski and P. Maślanka, Phys. Rev. **D62** (2000) 025004; hep–th/9902037.

[19] G. Amelino-Camelia, S. Majid, hep–th/9907110.

[20] S Imai and N. Sasakura, hep–th/0005178.

[21] S. Majid and R. Oeckl, Comm. Math. Phys. **205**, 617 (1999).

[22] B. Jurco, S. Schraml, P. Schupp and J. Wess, hep– th/0006246.

[23] M. Kontsevich, q–alg/9709040.

[24] P. Kosiński, J. Lukierski, P. Maślanka, A. Sitarz, Czech. Journ. Phys. **48**, 1407 (1998).

[25] P. Kosiński, J. Lukierski and P. Maślanka, in preparation.