

Bulk versus boundary quantum states

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Abstract

An explicit holographic correspondence between AdS bulk and boundary quantum states is found in the form of a one to one mapping between scalar field creation/annihilation operators. The mapping requires the introduction of arbitrary energy scales and exhibits an ultraviolet-infrared duality: a small regulating mass in the boundary theory corresponds to a large momentum cutoff in the bulk. In the massless (conformal) limit of the boundary theory the mapping covers the whole field spectrum of both theories. The mapping strongly depends on the discretization of the field spectrum of compactified AdS space in Poincare coordinates.

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The holographic principle asserts that a quantum system with gravity can be represented by a theory on the corresponding boundary[1, 2, 3]. This principle was inspired by the result that the black hole entropy is proportional to its horizon area[4, 5]. A realization of that principle was proposed by Maldacena in the form of a conjecture[6] on the equivalence (or duality) of the large N limit of $SU(N)$ superconformal field theories in n dimensions with supergravity (as a limit of superstring theory) defined in $n + 1$ dimensional anti de Sitter spacetime times a compact manifold (AdS/CFT correspondence). Prescriptions for realizing this conjecture, using Poincaré coordinates in the AdS bulk, were established by Gubser, Klebanov and Polyakov [7] and Witten [8]. In their approach, the AdS solutions play the role of classical sources for the boundary field correlators (for a review and a wide list of references see[9, 10]). The relation between the holographic mapping and the renormalization group flow was discussed in [11]. Further, the recent model of Randall and Sundrum[12] that proposes a solution to the hierarchy problem also presents holographic mapping between AdS bulk and boundary[13].

The isomorphism between the Hilbert spaces of the AdS string theory and the boundary CFT was established in [14, 15, 16, 17]. However, in this context it is difficult to find an explicit one to one mapping between bulk and boundary quantum states. Besides the involving string structure, one source of difficulty for an explicit mapping is the different dimensionality of the spaces. So it would be interesting to have an example of a one to one mapping between bulk and boundary quantum states. We show that this is possible considering a simple model with scalar fields for bulk and boundary. Scalar fields in the AdS bulk has already been discussed in[7, 8], although the associated boundary field in the AdS/CFT correspondence would be composite as can be seen from its conformal dimension.

In this letter we find an explicit one to one relation between the creation-annihilation operators of scalar fields in AdS spacetime and on its boundary. This implies a direct relation between the corresponding quantum states. This mapping is possible because of a discretization of the field spectrum in the AdS bulk as discussed previously in refs.[18, 19]. A fundamental ingredient is that canonical commutation relations in both theories are preserved. This is a realization of the holographic principle. One remarkable fact is that there is an ultraviolet-infrared duality. Starting with boundary fields with some small

mass μ (that can be interpreted as some infrared regulator) we find that the bulk field has an ultraviolet cut off behaving as $1/\mu$. Also remarkable is the fact that the mapping completely covers both theories in the conformal (massless) limit of the boundary field.

In order to consistently define a quantum field theory in AdS space one actually needs a compactification of this space. This way one is able to impose appropriate boundary conditions and avoid the loss or gain of information at spatial infinity in finite times and thus have a well defined Cauchy problem. This was established in [20, 21] in the context of global coordinates (these coordinates have finite ranges).

Anti-de Sitter spacetime of $n + 1$ dimensions can be represented[9, 10] as the hyperboloid $X_0^2 + X_{n+1}^2 - \sum_{i=1}^n X_i^2 = \Lambda^2$ with $\Lambda = \text{constant}$ embedded in a flat $n + 2$ dimensional space with metric $ds_{n+2}^2 = -dX_0^2 - dX_{n+1}^2 + \sum_{i=1}^n dX_i^2$. The so called Poincaré coordinates z, \vec{x}, t are introduced by

$$\begin{aligned} X_0 &= \frac{1}{2z} (z^2 + \Lambda^2 + \vec{x}^2 - t^2) , \\ X_i &= \frac{\Lambda x^i}{z} , \quad X_{n+1} = \frac{\Lambda t}{z} , \\ X_n &= -\frac{1}{2z} (z^2 - \Lambda^2 + \vec{x}^2 - t^2) , \end{aligned} \tag{1}$$

where $\vec{x} = (x^1, x^2, \dots, x^{n-1})$ with $-\infty < x^i < \infty$, $-\infty < t < \infty$ and $0 \leq z < \infty$. In this case the AdS_{n+1} measure with Lorentzian signature reads

$$ds^2 = \frac{\Lambda^2}{z^2} (dz^2 + (d\vec{x})^2 - dt^2) . \tag{2}$$

In recent articles [18, 19] we investigated the quantization of scalar fields in the AdS bulk in terms of Poincare coordinates, taking into account the need of compactification of the space. The AdS boundary corresponds to the region $z = 0$ described by usual Minkowski coordinates \vec{x}, t plus a “point” at infinity ($z \rightarrow \infty$). This point belongs to the boundary in global coordinates and must be added to the space in order to find the appropriate compactification. As discussed in [18, 19] this compactified AdS space can not be completely represented in just one set of Poincaré coordinates. So one needs to introduce two coordinate charts in order to represent the compactified (in the axial z direction) AdS space. Each chart stops at some value of its z coordinate. The necessity of cutting this axial coordinate has the non trivial consequence that the field spectrum is

discrete in the z direction as one should expect from a compact dimension. This reduces the dimensionality of the bulk space of states and makes it possible to find a one to one mapping into the boundary states. Note that one chart can be taken arbitrarily large in order to describe as much of the AdS space as wanted.

Let us consider a massive scalar field Φ in the AdS_{n+1} spacetime described by these coordinates with action

$$I[\Phi] = \frac{1}{2} \int d^{n+1}x \sqrt{g} \left(\partial_\xi \Phi \partial^\xi \Phi + m^2 \Phi^2 \right), \quad (3)$$

where we take $x^0 \equiv z$, $x^{n+1} \equiv t$, $\sqrt{g} = (x^0)^{-n-1}$ and $\xi = 0, 1, \dots, n+1$.

We consider a Poincare chart in AdS_{n+1} with¹ $n \geq 3$ given by $0 \leq z \leq R$, where we will take R to be arbitrarily large (but finite) in order to take as much of the AdS space as we want. The solutions of the classical equations of motion implied by the action (3) can be used to construct quantum fields in this region giving[18, 19]

$$\Phi(z, \vec{x}, t) = \sum_{p=1}^{\infty} \int \frac{d\vec{k}}{(2\pi)^{n-1}} \frac{z^{n/2} J_\nu(u_p z)}{R w_p(\vec{k}) J_{\nu+1}(u_p R)} \{ \mathbf{a}_p(\vec{k}) e^{-i w_p(\vec{k}) t + i \vec{k} \cdot \vec{x}} + c.c. \}, \quad (4)$$

where $\vec{k} = (k_1, \dots, k_{n-1})$, $w_p(\vec{k}) = \sqrt{u_p^2 + \vec{k}^2}$, u_p are such that $J_\nu(u_p R) = 0$ with $\nu = \frac{1}{2}\sqrt{n^2 + m^2}$ and $c.c.$ means complex conjugate. The operators \mathbf{a}_p , \mathbf{a}_p^\dagger satisfy the commutation relations

$$[\mathbf{a}_p(\vec{k}), \mathbf{a}_{p'}^\dagger(\vec{k}')] = 2(2\pi)^{n-1} w_p(\vec{k}) \delta_{pp'} \delta^{n-1}(\vec{k} - \vec{k}'). \quad (5)$$

On the n dimensional boundary $z = 0$ we consider quantum scalar fields with a mass μ :

$$\Theta_\mu(\vec{x}, t) = \frac{1}{(2\pi)^{n-1}} \int_{-\infty}^{\infty} \frac{d\vec{K}}{2w(\vec{K})} \{ \mathbf{b}(\vec{K}) e^{-i w(\vec{K}) t + i \vec{K} \cdot \vec{x}} + c.c. \}, \quad (6)$$

where $\vec{K} = (K_1, \dots, K_{n-1})$, $w(\vec{K}) = \sqrt{\vec{K}^2 + \mu^2}$ and the creation-annihilation operators satisfy the canonical algebra

$$[\mathbf{b}(\vec{K}), \mathbf{b}^\dagger(\vec{K}')] = 2(2\pi)^{n-1} w(\vec{K}) \delta(\vec{K} - \vec{K}'). \quad (7)$$

Note that \vec{K} and \vec{k} have the same dimensionality once we separate the component u_p of the bulk momentum which is discrete.

¹The AdS_3 case has some peculiarities and should be discussed separately.

In order to establish a correspondence between these two theories we use generalized spherical coordinate systems for representing both boundary and bulk momentum variables $\vec{K} = (K, \tilde{\phi}, \tilde{\theta}_\ell)$ and $\vec{k} = (k, \phi, \theta_\ell)$ respectively where $K = |\vec{K}|$, $k = |\vec{k}|$ and $\ell = 1, \dots, n-3$. So we rewrite the phase space volume elements as

$$\begin{aligned} d\vec{K} &= K^{n-2} dK d\tilde{\Omega}^{n-1} \\ d\vec{k} &= k^{n-2} dk d\Omega^{n-1}, \end{aligned} \quad (8)$$

where $d\tilde{\Omega}^{n-1}$, $d\Omega^{n-1}$ are the infinitesimal elements of solid angle in $n-1$ dimensions for respectively boundary and bulk.

Now, using this spherical coordinate representation, we introduce a sequence of energy scales $\epsilon_1, \epsilon_2, \dots$ and split the operator Θ_μ as

$$\begin{aligned} \Theta_\mu(\vec{x}, t) &= \frac{1}{(2\pi)^{n-1}} \int_0^{\epsilon_1} \frac{K^{n-2} dK}{2w(K)} \int d\tilde{\Omega}^{n-1} \{ \mathbf{b}(\vec{K}) e^{-iw(K)t + i\vec{K} \cdot \vec{x}} + c.c. \} \\ &+ \frac{1}{(2\pi)^{n-1}} \int_{\epsilon_1}^{\epsilon_2} \frac{K^{n-2} dK}{2w(K)} \int d\tilde{\Omega}^{n-1} \{ \mathbf{b}(\vec{K}) e^{-iw(K)t + i\vec{K} \cdot \vec{x}} + c.c. \} \\ &+ \dots \end{aligned} \quad (9)$$

Then with a suitable mapping one can relate each of the Θ_μ integrals above with the integral of the bulk field Φ , eq.(4), over $d\vec{k}$ for a fixed u_p . Considering first the interval $0 \leq K \leq \epsilon_1$ and $p=1$ we introduce relations between the creation-annihilation operators of both theories. We assume that k is some function of K and that the angular part of the mapping is trivial so that the same set of angular coordinates are used for bulk and boundary momenta. We choose

$$\begin{aligned} K^{\frac{n-2}{2}} \mathbf{b}(K, \phi, \theta_\ell) &= k^{\frac{n-2}{2}} \mathbf{a}_1(k, \phi, \theta_\ell) \\ K^{\frac{n-2}{2}} \mathbf{b}^\dagger(K, \phi, \theta_\ell) &= k^{\frac{n-2}{2}} \mathbf{a}_1^\dagger(k, \phi, \theta_\ell), \end{aligned} \quad (10)$$

where the moduli of the momenta are mapped onto each other through

$$k = g_1(K, \mu). \quad (11)$$

Requiring that the canonical commutation relations (5,7) are consistent with the above relations we find that the function g_1 is of the form:

$$g_1(K, \mu) = \frac{1}{2} \frac{u_1^2 C_1(\mu)}{(K + \sqrt{K^2 + \mu^2})} - \frac{1}{2} \frac{K + \sqrt{K^2 + \mu^2}}{C_1(\mu)}, \quad (12)$$

where $C_1(\mu)$ is an arbitrary integration constant, for a given μ . In order to have $k \geq 0$ we put

$$C_1(\mu) = \frac{\epsilon_1 + \sqrt{\epsilon_1^2 + \mu^2}}{u_1} \quad (13)$$

so that the maximum value of $k = g_1(K, \mu)$ corresponds to $K = 0$ and is given by

$$\lambda_1 = \frac{1}{2}u_1 \left(\frac{\epsilon_1 + \sqrt{\epsilon_1^2 + \mu^2}}{\mu} - \frac{\mu}{\epsilon_1 + \sqrt{\epsilon_1^2 + \mu^2}} \right). \quad (14)$$

Then, for the other intervals $\epsilon_{i-1} < K \leq \epsilon_i$, that we put in correspondence with u_i , we introduce similarly the relations

$$\begin{aligned} \mathbf{b}(K, \phi, \theta_\ell) &= \left[\frac{K^2 + \mu^2}{(K - \epsilon_{i-1})^2 + \mu^2} \right]^{\frac{1}{4}} \left[\frac{g_i(K, \mu)}{K} \right]^{\frac{n-2}{2}} \mathbf{a}_i(g_i(K, \mu), \phi, \theta_\ell) \\ \mathbf{b}^\dagger(K, \phi, \theta_\ell) &= \left[\frac{K^2 + \mu^2}{(K - \epsilon_{i-1})^2 + \mu^2} \right]^{\frac{1}{4}} \left[\frac{g_i(K, \mu)}{K} \right]^{\frac{n-2}{2}} \mathbf{a}_i^\dagger(g_i(K, \mu), \phi, \theta_\ell), \end{aligned} \quad (15)$$

with $k = g_i(K, \mu)$ and again we impose that the canonical relations (5,7) are preserved, finding

$$g_i(K, \mu) = \frac{u_i}{2} \left[\frac{\Delta\epsilon_i + \sqrt{(\Delta\epsilon_i)^2 + \mu^2}}{K - \epsilon_{i-1} + \sqrt{(K - \epsilon_{i-1})^2 + \mu^2}} - \frac{K - \epsilon_{i-1} + \sqrt{(K - \epsilon_{i-1})^2 + \mu^2}}{\Delta\epsilon_i + \sqrt{(\Delta\epsilon_i)^2 + \mu^2}} \right], \quad (16)$$

where $\Delta\epsilon_i = \epsilon_i - \epsilon_{i-1}$, so that $g_i(\epsilon_i, \mu) = 0$. The maximum for $g_i(K, \mu)$ happens for $K = \epsilon_{i-1}$ and is given by

$$\lambda_i = \frac{1}{2}u_i \left(\frac{\Delta\epsilon_i + \sqrt{(\Delta\epsilon_i)^2 + \mu^2}}{\mu} - \frac{\mu}{\Delta\epsilon_i + \sqrt{(\Delta\epsilon_i)^2 + \mu^2}} \right). \quad (17)$$

Note that the λ_i are different in general depending on u_i and $\Delta\epsilon_i$. The u_i are related to the zeros of the Bessel functions and obey the ordering $u_i > u_{i-1}$, but the intervals $\Delta\epsilon_i$ are of arbitrary size by construction. This mapping between the momenta K and k is illustrated in. Fig. 1.

So we have established a correspondence between the states of scalar fields in *AdS* bulk (massive or not) with massive scalar fields on its boundary.

An important feature of this correspondence is that the boundary theory (which has an infrared cutoff μ) is mapped into a bulk theory with ultraviolet cutoffs λ_i given by eq. (17) for each value of u_i . Note that a small μ corresponds to large λ_i (with a leading

order term $\sim 1/\mu$). So that we find explicitly a duality of the regimes UV-IR in the bulk/boundary mapping.

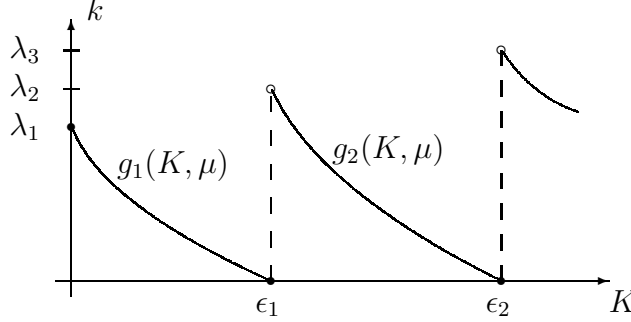


Figure 1: *The mapping between the boundary momentum K and bulk momentum k for the case of a massive boundary theory. Finite intervals on K are mapped into intervals for k with cutoffs λ_i for each u_i .*

The mapping of massive boundary fields into bulk scalar fields implies a direct relation between the corresponding quantum states

$$|\vec{K}_i, \mu\rangle \leftrightarrow |\vec{k}, u_i\rangle, \quad (18)$$

with $\vec{K}_i = (K_i, \phi, \theta_\ell)$ being a momentum with modulus $\epsilon_{i-1} < K_i \leq \epsilon_i$ and $\vec{k} = (k, \phi, \theta_\ell)$ with modulus $0 \leq k < \lambda_i$. This is a realization of the Holographic principle in terms of quantum states and it exhibits the Ultraviolet-Infrared duality expected from the bulk boundary correspondence[3].

Now we focus on the important limiting case of a massless (conformal) boundary theory. First we note that the first interval will be taken as $0 < K \leq \epsilon_1$, excluding the state of $K = 0$ that is not physically relevant in this massless case. The other intervals are taken again as $\epsilon_{i-1} < K \leq \epsilon_i$ as in the massive case. Here we find ($\epsilon_0 \equiv 0$)

$$g_i(K) = \frac{u_i}{2} \left[\frac{\Delta\epsilon_i}{K - \epsilon_{i-1}} - \frac{K - \epsilon_{i-1}}{\Delta\epsilon_i} \right], \quad (19)$$

such that $g_i(\epsilon_i) = 0$. Note that the maximum for $g_i(K)$ also happens for $K = \epsilon_{i-1}$ and is given by

$$\lambda_i = \frac{1}{2}u_i \left(\frac{\Delta\epsilon_i}{\mu} - \frac{\mu}{\Delta\epsilon_i} \right) \Big|_{\mu \rightarrow 0} \longrightarrow \infty. \quad (20)$$

So we find out that when the boundary theory is conformal the whole phase space of the bulk (without any UV cutoff) is mapped in the whole phase space of the boundary (with the exception of the state of zero momentum that has no Physical content). This one to one mapping between the momenta K and k is represented in Fig. 2.

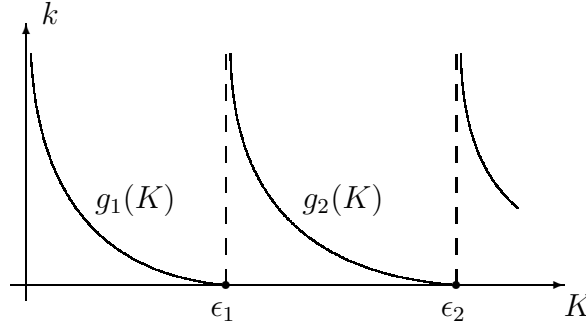


Figure 2: *The mapping between the boundary momentum K and bulk momentum k for the case of a conformal boundary theory. Every finite interval on K is mapped into an infinite interval for k (corresponding to each value of u_i), so that the bulk phase space is completely covered by this mapping.*

In the conformal case we found a direct mapping of boundary/bulk quantum states:

$$|\vec{K}_i\rangle \leftrightarrow |\vec{k}, u_i\rangle, \quad (21)$$

where again $\epsilon_{i-1} < K_i \leq \epsilon_i$ but now $0 \leq k < \infty$ without any ultraviolet cutoff.

The correlation functions for the conformal boundary theory can be calculated [22] directly from the boundary fields, eq. (6)

$$\langle \Theta_0(x) \Theta_0(x') \rangle \sim \frac{1}{(x - x')^{2d}}, \quad (22)$$

where $x = (\vec{x}, t)$, $x' = (\vec{x}', t')$ and $d = (n - 2)/2$ is the conformal dimension for the scalar field $\Theta_0 \equiv \Theta_{\mu=0}$ defined in the boundary of AdS_{n+1} .

Let us now comment on the differences between our approach and that of the $\text{AdS}_{n+1}/\text{CFT}_n$ correspondence [6, 7, 8]. In that case bulk scalars of mass m are mapped into boundary composite operators of conformal dimension $(n + \sqrt{n^2 + 4m^2})/2$. It is interesting to note that m^2 can be negative subjected to a lower bound $m^2 \geq -n^2/4$ [21, 8], so that the conformal dimension is $\geq n/2$. This dimension will not match that of our boundary field because we considered a simpler situation of bulk and boundary scalar theories. However with this simple model we found a direct one to one mapping between quantum states.

We expect that our mapping could be generalized to other fields if one starts with appropriate expansion for the boundary operators. This would enlarge the mechanism proposed here possibly allowing the inclusion of composite operators. In that case the relation between such a mapping and the AdS/CFT correspondence would be closer.

Finally we point out that once established a one to one mapping between bulk and boundary quantum states it is possible to relate their entropies in the same way. So the entropy area law would be a consequence of this mapping, at least for the system of scalar fields analyzed here.

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