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Representation of $SU(\infty)$ Algebra for Matrix Models

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Abstract

We investigate how the matrix representation of $SU(N)$ algebra approaches that of the Poisson algebra in the large N limit. In the adjoint representation, the $(N^2-1)\times(N^2-1)$ matrices of the $SU(N)$ generators go to those of the Poisson algebra in the large N limit. However, it is not the case for the $N\times N$ matrices in the fundamental representation.

1 Introduction

In these years, matrix models have been studied to investigate string theories [1]-[4]. It is important to clarify the limit of the size of matrices going to infinity to define those theories. We also expect that there are such field theories that are properly regularized by matrices and hence they are useful to study string theories since we know a lot of techniques to examine field theories. It was pointed out long ago that the generators of $SU(N)$ approach those of the Poisson algebra when N becomes large, and the commutators of the $SU(N)$ algebra are close to the Poisson bracket [5]-[8]. Recently it was argued in ref.[9] that the group of area-preserving diffeomorphisms of any Riemann surface, or connected, compact and orientable world-sheet, is equivalent to $SU(\infty)$. This implies that matrix models could be formulated on the world-sheet with the area-preserving diffeomorphisms. On the other hand, there are some investigations that the bosonic part of the IIB matrix model, which is naively a matrix regularization of the Schild model [10], do not agree with the Schild model in the large N limit [11, 12].

In this paper, we study how the large N limit of the generators of $SU(N)$ go to those of the Poisson algebra in some representations. In the next section, we show that in the adjoint representation, all the matrix elements of the $SU(N)$ generators approach to those of the Poisson algebra in the large N limit. On the other hand, fundamental representation is usually used in defining large N matrix models instead of the adjoint representation. However, the similar large N limit of the $N\times N$ matrices of the $SU(N)$ generators are divergent, while the matrix elements of the generators of the Poisson algebra with the same basis are finite. These points are discussed in section 3. The last section is devoted to summary and discussion.

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2 Representation with functions on world-sheet

Lie algebra of infinitesimal area-preserving diffeomorphism is represented by the operators over functions on a surface, which we call world-sheet,

$$X_{f(\sigma)} \equiv \omega^{ab}\partial_a f(\sigma)\partial_b, \tag{1}$$

where ω is the symplectic two-form of the world-sheet and it satisfies

$$\omega^{ad}\partial_d\omega^{bc} + \omega^{bd}\partial_d\omega^{ca} + \omega^{cd}\partial_d\omega^{ab} = 0\,. \tag{2}$$

Actually we can straightforwardly show the following relation,

$$[X_{f(\sigma)}, X_{g(\sigma)}] = X_{f(\sigma)}X_{g(\sigma)} - X_{g(\sigma)}X_{f(\sigma)} = X_{\{f(\sigma),g(\sigma)\}}\,, \tag{3}$$

where $\{f(\sigma), g(\sigma)\}$ is the Poisson bracket defined by

$$\{f(\sigma), g(\sigma)\} \equiv \omega^{ab}\partial_a f(\sigma)\cdot\partial_b g(\sigma)\,. \tag{4}$$

Let us consider the case that the world-sheet is a two-dimensional torus, $\sigma \in [0,1)\times[0,1)$, and hence $\omega^{ab} = \epsilon^{ab}$. Functions on the torus can be expanded in Fourier series and the operator (1) is represented with the basis as

$$(\,X_{\mathbf{m}}\,)_{\mathbf{kl}} \equiv \int d^2\sigma\, e^{-2\pi i\mathbf{k}\cdot\sigma}\, X_{e^{2\pi i\mathbf{m}\sigma}}\, e^{2\pi i\mathbf{l}\cdot\sigma} = -4\pi^2\mathbf{m}\times\mathbf{l}\,\delta_{\mathbf{m}\,(\mathbf{k}-\mathbf{l})}\,, \tag{5}$$

where $\mathbf{m}\times\mathbf{n} \equiv \epsilon^{ab}m_a n_b$. Then eq.(3) is represented by

$$(\,[X_{\mathbf{m}}, X_{\mathbf{n}}]\,)_{\mathbf{kl}} = \sum_{\mathbf{j}}\Big[(\,X_{\mathbf{m}}\,)_{\mathbf{kj}}\,(\,X_{\mathbf{n}}\,)_{\mathbf{jl}} - (\,X_{\mathbf{n}}\,)_{\mathbf{kj}}\,(\,X_{\mathbf{m}}\,)_{\mathbf{jl}}\Big] = -4\pi^2\mathbf{m}\times\mathbf{n}\,\left(\,X_{\mathbf{m}+\mathbf{n}}\,\right)_{\mathbf{kl}}\,. \tag{6}$$

This means that $(\,X_{\mathbf{m}}\,)_{\mathbf{kl}}$ in (5) are the matrix elements for the adjoint representation of the Poisson algebra. It seems natural to use fields over the world-sheet with the Poisson bracket instead of the operators (1) as representation of the Poisson algebra, however, the fields on the world-sheet themselves cannot represent a matrix algebra. On the other hand, with the operators (1) we have the Poisson algebra in (3) and the matrix algebra (6) as well. Since the fields on the world-sheet have one-to-one correspondence (up to constant functions) with the operators (1), we can take either of them to consider the representation of the Poisson algebra. Our aim is to study the large N limit of matrix models, and then it is more useful to take those operators.

Next we consider the matrix representation of $SU(N)$ algebra in the adjoint representation and its large N limit. To construct the representation, however, let us first consider $SU(N)$ algebra in the fundamental representation. We can choose the generators of $SU(N)$ in the following form [11, 13, 7, 14],[1]

$$Y^{SU(N)}_{(m_1,m_2)} = e^{i\frac{2\pi}{N}m_1 m_2}\,U^{m_1}\,V^{m_2}, \tag{7}$$

[1] We assume that N is odd for definiteness. Transformation to the Cartan-Weyl basis is shown in [7].

where U and V are $N \times N$ clock and shift matrices, respectively,

$$U = \begin{pmatrix} 1 & & & 0 \\ & e^{i\frac{4\pi}{N}} & & \\ & & \ddots & \\ 0 & & & e^{i\frac{4\pi}{N}(N-1)} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & & & 0 \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & 0 & & 0 & 1 \\ 1 & & & & 0 \end{pmatrix}, \tag{8}$$

which satisfy $U^N = V^N = 1$ and hence $Y^{SU(N)}_{(m_1+k_1N, m_2+k_2N)} = Y^{SU(N)}_{(m_1,m_2)}$ $(k_i \in \mathbb{Z})$. The commutation relations are given by

$$[\,Y^{SU(N)}_{\mathbf{m}}, Y^{SU(N)}_{\mathbf{n}}\,] = -2i \sin\left(\frac{2\pi}{N}\mathbf{m}\times\mathbf{n}\right) Y^{SU(N)}_{\mathbf{m}+\mathbf{n}}. \tag{9}$$

Thus the matrix elements of $SU(N)$ generators in the adjoint representation can be taken as[2]

$$(Z^N_{\mathbf{m}})_{\mathbf{kl}} \equiv -2i \sin\left(\frac{2\pi}{N}\mathbf{m}\times\mathbf{l}\right) \delta_{\mathbf{m}(\mathbf{k}-\mathbf{l})}\,. \tag{10}$$

Actually, we have

$$\begin{aligned} \left([Z^N_{\mathbf{m}}, Z^N_{\mathbf{m}}]\right)_{\mathbf{kl}} &= \sum_{\mathbf{i}} \left[(Z^N_{\mathbf{m}})_{\mathbf{ki}}\ (Z^N_{\mathbf{n}})_{\mathbf{il}} - (Z^N_{\mathbf{n}})_{\mathbf{ki}}\ (Z^N_{\mathbf{m}})_{\mathbf{il}}\right] \\ &= -2i \sin\left(\frac{2\pi}{N}\mathbf{m}\times\mathbf{n}\right)\ (Z^N_{\mathbf{m}+\mathbf{n}})_{\mathbf{kl}}\,, \end{aligned} \tag{11}$$

that is, the generators of $SU(N)$ are realized by the matrices $(Z^N_{\mathbf{n}})_{\mathbf{kl}}$.

Now we consider the following large N limit of the algebra. In fact, multiplying eq.(9) by N^2, we have

$$[\,N\,Y^{SU(N)}_{\mathbf{m}}, N\,Y^{SU(N)}_{\mathbf{n}}\,] = -2i \left\{N \sin\left(\frac{2\pi}{N}\mathbf{m}\times\mathbf{n}\right)\right\} N\,Y^{SU(N)}_{\mathbf{m}+\mathbf{n}}. \tag{12}$$

Since the structure constant, $N\sin(2\pi\mathbf{m}\times\mathbf{n}/N)$, becomes $2\pi\mathbf{m}\times\mathbf{n}$ for fixed $\mathbf{m}$ and $\mathbf{n}$ when $N \to \infty$, one may naively expect that $X_{\mathbf{m}} = \lim_{N\to\infty}[N\,Y^{SU(N)}_{\mathbf{m}}]$ will satisfy $[X_{\mathbf{m}}, X_{\mathbf{n}}] = -4\pi i\,\mathbf{m}\times\mathbf{n}\,X_{\mathbf{m}+\mathbf{n}}$, however, one should be careful with the limit in the defining equation of $X_{\mathbf{m}}$, that is, $N\,Y^{SU(N)}_{\mathbf{m}}$ could diverge and hence the limit does not exist. Let us consider eq.(10), the $SU(N)$ generators in the adjoint representation. In the large N limit, they agree with those of the Poisson algebra in eq.(5) when we rescale $(Z^N_{\mathbf{m}})_{\mathbf{kl}}$ as[3]

$$i\pi N (Z^N_{\mathbf{m}})_{\mathbf{kl}} \underset{N\to\infty}{\longrightarrow} (X_{\mathbf{m}})_{\mathbf{kl}}\,. \tag{13}$$

We shall give the operators with functions on the torus whose matrix representation coincides with eq.(10). They are given by

$$Z^N_f(\sigma) = f(\sigma)\left(\exp\left[-\frac{i}{2\pi N}\boldsymbol{\partial}(\ln f)\times\boldsymbol{\partial}\right] - \exp\left[\frac{i}{2\pi N}\boldsymbol{\partial}(\ln f)\times\boldsymbol{\partial}\right]\right). \tag{14}$$

[2]We have, of course, hermitian combinations, $Z^N_{\mathbf{m}} + Z^N_{-\mathbf{m}}$ and $i(Z^N_{\mathbf{m}} - Z^N_{-\mathbf{m}})$, so that $Z^N_{\mathbf{m}}$ are the generators of $SU(N)$.

[3]Notice that for fixed $\mathbf{k}, \mathbf{l}$ and $\mathbf{m}$, the matrix element of $N\,(Z^N_{\mathbf{m}})_{\mathbf{kl}}$ is finite in the $N \to \infty$ limit. This is not the case in the fundamental representation, as will be seen in the next section.

In fact, the matrix elements of the operators with $f(\sigma) = f_{\mathbf{m}}(\sigma)\ (\equiv \exp(i2\pi\mathbf{m}\cdot\sigma)\ )$ are given by

$$(Z^N_{\mathbf{m}})_{\mathbf{kl}} \equiv \int d^2\sigma\, f^*_{\mathbf{k}}(\sigma)\, Z^N_{\mathbf{m}}(\sigma)\, f_{\mathbf{l}}(\sigma) = -2i\sin\left(\frac{2\pi}{N}\mathbf{m}\times\mathbf{l}\right)\delta_{\mathbf{m}\,(\mathbf{k}-\mathbf{l})}\,. \tag{15}$$

Furthermore, we can straightforwardly take $N\to\infty$ limit of $NZ^N_f(\sigma)$ as

$$\begin{aligned} NZ^N_f(\sigma) &= Nf(\sigma)\left(\exp\left[-\frac{i}{2\pi N}\boldsymbol{\partial}(\ln f)\times\boldsymbol{\partial}\right] - \exp\left[\frac{i}{2\pi N}\boldsymbol{\partial}(\ln f)\times\boldsymbol{\partial}\right]\right) \\ &= -Nf\left(\frac{i}{\pi N}\boldsymbol{\partial}(\ln f)\times\boldsymbol{\partial} + O(\frac{1}{N^2})\right) \\ &\to -\frac{i}{\pi}\boldsymbol{\partial} f\times\boldsymbol{\partial}\,, \end{aligned} \tag{16}$$

which coincides with X_f.

We realized $SU(N)$ algebra in the infinite dimensional operator space and we can realize the finite dimensional representation of those operators by restricting fields due to the periodicity in eq.(10). Since $(Z^N_{\mathbf{m}})_{\mathbf{kl}}$ in eq.(15) are $(N^2-1)\times(N^2-1)$ matrices, we can hardly regard that this representation is the large N limit of the $N\times N$ matrix representation of $SU(N)$ algebra. So we need other ways to represent $SU(N)$ and the Poisson algebra simultaneously to formulate the large N limit of matrix models.

3 Representation with a and $a^\dagger$

We need the fundamental representation of the Poisson algebra, which could be regarded as the large N limit of $SU(N)$ algebra, to construct the large N matrix model. It is natural to think that the representation is embedded in an infinite dimensional representation, and we could relate the large N limit of $SU(N)$ and the Poisson algebra as in the adjoint representation case in eq.(13). As we show below, these algebras can be embedded in the operator space generated by the creation and annihilation operators and its representation space,

$$[\,a, a^\dagger\,] = 1, \quad a|\,0\,\rangle = 0, \quad |\,n\,\rangle \equiv \frac{1}{\sqrt{n!}}(a^\dagger)^n|\,0\,\rangle\,. \tag{17}$$

Since the commutator of a and $a^\dagger$ is the same as the one between the coordinates in a non-commutative space, functions of a are regarded as those on a non-commutative space.

We show how the generators of the algebras for the fundamental representation are embedded in the operator space. First, let us consider the following operators [8],

$$T^{N,\eta}_{\mathbf{m}} = \exp\left[2i\sqrt{\frac{\pi}{N}}\left(\eta m_1 a + i\frac{m_2}{\eta}a^\dagger\right)\right], \tag{18}$$

where η is an arbitrary parameter. The commutators of $T^{N,\eta}_{\mathbf{m}}$ are easily calculated as

$$[\,T^{N,\eta}_{\mathbf{m}}, T^{N,\eta}_{\mathbf{n}}\,] = -2i\sin\left(\frac{2\pi}{N}\mathbf{m}\times\mathbf{n}\right)T^{N,\eta}_{\mathbf{m}+\mathbf{n}}\,, \tag{19}$$

which are certainly commutation relations for $SU(N)$ algebra as in eq.(9), although $T^{N,\eta}_{\mathbf{m}}$ do not satisfy the periodicity, $T^{N,\eta}_{\mathbf{m}+N\mathbf{k}} \neq T^{N,\eta}_{\mathbf{m}}$.

We construct $N \times N$ dimensional representation of (18) with the following bases of the vector space and its dual space, respectively,

$$|\,k;\eta\,)_N \;\equiv\; \exp\left[2\sqrt{\frac{\pi}{N}}\,\frac{ka^\dagger}{\eta}\right]|\,0\,\rangle\,, \tag{20}$$

$${}_N(\,k;\eta\,| \;\equiv\; \frac{1}{N}\langle\,0\,|\sum_{m=0}^{N-1}\exp\left[-i\frac{4\pi}{N}km\right]\,\exp\left[2i\sqrt{\frac{\pi}{N}}\,\eta ma\right]\,, \tag{21}$$

where $k = 0,\cdots,N-1$. The inner products are ${}_N(\,k;\eta\,|\,l;\eta\,)_N = \delta_{k\,l}$. Thus the following operator,

$$P^\eta_N \equiv \sum_{k=0}^{N-1}|\,k;\eta\,)_N\;{}_N(\,k;\eta\,|, \tag{22}$$

is a projection operator to a N dimensional Fock subspace. Then the matrix elements of $T^N_{\mathbf{m}}$ are given by

$$\begin{aligned}{}_N(\,k;\eta\,|T^{N,\eta}_{\mathbf{m}}|\,l;\eta\,)_N &= \frac{1}{N}\sum_{n=0}^{N-1} e^{-i\frac{2\pi}{N}m_1m_2}e^{4\pi i\frac{n(l-k)}{N}}\,e^{-4\pi i\left(\frac{n\eta m_2}{\eta N}-\frac{\eta m_1 l}{\eta}\right)}\\ &= e^{-i\frac{2\pi}{N}m_1m_2}e^{-i\frac{4\pi}{N}m_1k}\delta_{l\,[k+m_2]_N}\,,\end{aligned} \tag{23}$$

where we have introduced the notation that $[k]_N \equiv k \bmod N$,i.e., $0 \le [k]_N < N$. Note that the above matrix elements are exactly the same as those of the generators of $SU(N)$ in eq.(7) with the clock and the shift matrices in eq.(8) and the periodicity is realized on the representation, ${}_N(\,k;\eta\,|T^{N,\eta}_{(m_1+Nk_1,m_2+Nk_2)}|\,l;\eta\,)_N = {}_N(\,k;\eta\,|T^{N,\eta}_{\mathbf{m}}|\,l;\eta\,)_N$. This is a $N\times N$ matrix representation of $SU(N)$ algebra, so that this operator space will be suitable to formulate the large N limit of matrix models.

On the other hand, the generators of the Poisson algebra are constructed as

$$T^{(P,\xi)}_{\mathbf{m}} = \left\{1-2i\sqrt{\pi}\left(\xi m_1 a + i\frac{m_2}{\xi}a^\dagger\right)\right\}\exp\left[2i\sqrt{\pi}\left(\xi m_1 a + i\frac{m_2}{\xi}a^\dagger\right)\right], \tag{24}$$

where $m_i, n_i \in \mathbb{Z}$ and ξ is an arbitrary constant. In fact, their commutation relations are

$$[\,T^{(P,\xi)}_{\mathbf{m}}, T^{(P,\xi)}_{\mathbf{n}}\,] = 4\pi i\,\mathbf{m}\times\mathbf{n}\,T^{(P,\xi)}_{\mathbf{m}+\mathbf{n}}\,, \tag{25}$$

whose proof are given in the appendix. It is natural to expect that we can construct the fundamental representation of the Poisson algebra with suitable bases of a vector space and its dual space as in $SU(N)$ case in eqs.(20) and (21), and the large N limits of the $SU(N)$ generators in the fundamental representation coincide with those of the Poisson algebra as in the adjoint representation case in eq.(13) due to ref.[9]. However, this seems to be incorrect. In the above arguments, we expect implicitly that the large N limit of the vector space spanned by $|\,k;\eta\,)_N$ in eq.(20) and its dual space in eq.(21) coincide with the whole Fock space and its dual, respectively, which means that the large N limit of the projection operator P^η_N (22) is the identity on the Fock space. However, we can see that $\lim_{N\to\infty}\langle\,0\,|\,[a,P^\eta_N]\,|\,1\,\rangle$ is divergent by elementary calculations. This means that since $\lim_{N\to\infty} P_N^\eta$, if it exists, is the projection operator, P_N^η cannot be an identity operator even in $N\to\infty$ and the large N limit of the vector space (or its dual space) is not the whole Fock space but its subspace. This fact implies that the generators of the Poisson algebra in eq.(24) cannot be represented with the vector space in eq.(20) and its dual space in eq.(21), or we need the whole Fock space to represent the Poisson algebra. The matrix elements $\langle\, k\,|T_{\mathbf{m}}^{(P,\xi)}|\, l\,\rangle \quad (k,l=0,1,\cdots)$ can be straightforwardly calculated as[4]

$$\begin{aligned}\langle\, k\,|T_{\mathbf{m}}^{(P,\xi)}|\, l\,\rangle \\ = \begin{cases} \sqrt{\dfrac{l!}{k!}}\,\lambda_2^{k-l}\left[(1-z)L_l^{(k-l)}(z)+zL_l^{(k-l+1)}(z)-(l+1)L_{l+1}^{(k-l-1)}(z)\right]\,, & (k>l) \\ (1-z)L_k^{(0)}(z)+2zL_{k+1}^{(0)}(z)\,, & (k=l) \\ \sqrt{\dfrac{k!}{l!}}\,\lambda_1^{l-k}\left[(1-z)L_k^{(l-k)}(z)+zL_k^{(l-k+1)}(z)-(k+1)L_{k+1}^{(l-k-1)}(z)\right]\,, & (k<l)\end{cases}\end{aligned} \tag{26}$$

where

$$z=4\pi i m_1 m_2\,,\qquad \lambda_1=2i\sqrt{\pi}m_1\xi\,,\qquad \lambda_2=-\frac{2\sqrt{\pi}m_2}{\xi}\,, \tag{27}$$

and $L_n^{(\alpha)}(x)$ are the generalized Laguerre polynomials. The commutation relations of the Poisson algebra are expressed with the matrix elements[5],

$$\sum_{p=0}^{\infty}\left[\langle\, k\,|T_{\mathbf{m}}^{(P,\xi)}|\, p\,\rangle\,\langle\, p\,|T_{\mathbf{n}}^{(P,\xi)}|\, l\,\rangle-\langle\, k\,|T_{\mathbf{n}}^{(P,\xi)}|\, p\,\rangle\,\langle\, p\,|T_{\mathbf{m}}^{(P,\xi)}|\, l\,\rangle\right]=4\pi i\,\mathbf{m}\times\mathbf{n}\,\langle\, k\,|T_{\mathbf{m}+\mathbf{n}}^{(P,\xi)}|\, l\,\rangle\,. \tag{28}$$

Using these coefficients $\langle\, k\,|T_{\mathbf{m}}^{(P,\xi)}|\, l\,\rangle\equiv W_{\mathbf{m}}^{kl}$, we can give another expression of the operators for the Poisson algebra,

$$\begin{aligned} V_{\mathbf{m}} &\equiv \sum_{k,l=0}^{\infty} W_{\mathbf{m}}^{kl}\,|\, k\,\rangle\langle\, l\,| \\ &= \sum_{k,l}\frac{W_{\mathbf{m}}^{kl}}{\sqrt{k!\,l!}}\,:(a^\dagger)^k\,e^{-a^\dagger a}\,a^l:\,, \end{aligned} \tag{29}$$

where "$:\cdots:$" stands for the normal product. Furthermore, using the operator $|\,k\,\rangle\langle\,l\,|$ we can construct the generators of $SU(N)$ as

$$\begin{aligned} V_{\mathbf{m}}^{SU(N)} &= e^{\frac{2\pi i}{N}m_1m_2}\left[\sum_{k=0}^{N-1-m_2}e^{\frac{4\pi i}{N}km_1}\,|\,k\,\rangle\langle\,k+m_2\,|+\sum_{k=N-m_2}^{N-1}e^{\frac{4\pi i}{N}km_1}\,|\,k\,\rangle\langle\,k+m_2-N\,|\right] \\ &= e^{\frac{2\pi i}{N}m_1m_2}\sum_{k=0}^{N-1}e^{\frac{4\pi i}{N}km_1}\,:(a^\dagger)^k\,e^{-a^\dagger a}\,a^{[k+m_2]_N}:\,, \end{aligned} \tag{30}$$

[4] For fixed k,l and $\mathbf{m}$, the matrix elements are finite.

[5] For fixed $k,l,\mathbf{m}$ and $\mathbf{n}$, the summation over p is well-defined.

We can easily see that these $V_{\mathbf{m}}^{SU(N)}$ have the periodicity, $V_{(m_1+Nk_1,m_2+Nk_2)}^{SU(N)} = V_{(m_1,m_2)}^{SU(N)}$, and they satisfy the same commutation relation in eq.(19). Hence $V_{\mathbf{m}}^{SU(N)}$ have the $N\times N$ matrix representation in the N-dimensional vector space with basis $\{|\,0\,\rangle, |\,1\,\rangle, \cdots, |\,N-1\,\rangle\}$ and its dual with $\{\langle\,0\,|, \langle\,1\,|, \cdots, \langle\,N-1\,|\}$,

$$\begin{aligned} (\,V_{\mathbf{m}}^{SU(N)}\,)_{kl} &= \langle\,k\,|V_{\mathbf{m}}^{SU(N)}|\,l\,\rangle \\ &= e^{-i\frac{2\pi}{N}m_1m_2}\,e^{-i\frac{4\pi}{N}m_1k}\,\delta_{l[k+m_2]_N}\Big(=\ {}_N(\,k;\eta\,|T_{\mathbf{m}}^{N,\eta}|\,l;\eta\,)_N\Big)\,. \end{aligned} \tag{31}$$

However, we shall see that the $N\times N$ matrices, $N\left(V_{\mathbf{m}}^{SU(N)}\right)$, do not have the well-defined $N\to\infty$ limit[6] and hence, contrary to eq.(13), $N\left(V_{\mathbf{m}}^{SU(N)}\right)$ cannot become $\left(V_{\mathbf{m}}\right)$ in the large N limit.

4 Summary and discussion

We have studied whether the large N limit of $SU(N)$ algebra coincide with the Poisson algebra. In the adjoint representation, we have shown their coincidence by comparing their matrix elements as in eq.(13), while they do not coincide with each other in the fundamental representation. In fact, the matrix elements of the $SU(N)$ generators can be written by eq.(31) (or eq.(23)) and the rescaled matrix which is multiplied by $\mathcal{O}(N)$ constant (cf. eq.(13)), is divergent in the $N\to\infty$ limit. In other words, we cannot give a sequence of $N\times N$ matrix representation of $SU(N)$ algebra with the structure constant which is proportional to $N\sin(2\pi\mathbf{m}\times\mathbf{n}/N)$ that goes to the representation of the Poisson algebra.

Let us consider N^2 operators, $V_{\mathbf{m}}^N \equiv \sum_{k,l=0}^{N-1} W_{\mathbf{m}}^{kl}\,|\,k\,\rangle\langle\,l\,| \quad (0\le|m_a|\le(N-1)/2)$, which give another regularization of the Poisson algebra. They go to $V_{\mathbf{m}}$ in the $N\to\infty$ limit, however, we can see that $V_{\mathbf{m}}^N$ are linear dependent and hence $V_{\mathbf{m}}^N$ (or $V_{\mathbf{m}}^N/N$) do not satisfy $SU(N)$ algebra. On the other hand, another set of N^2 operators, $V_{\mathbf{m}}^N \quad (1\le\pm m_a\le(N\pm 1)/2)$, seems to be linear independent[7] and they satisfy the $SU(N)$, actually $U(N)$, algebra whose structure constants are different from those in eq.(19). For finite N, $V_{(m_1,0)}^N$ and $V_{(0,m_2)}^N$ are given by the linear combinations of $V_{\mathbf{m}}^N \quad (1\le\pm m_a\le(N\pm1)/2)$, respectively, while all $V_{\mathbf{m}}$ are independent. Then, a sequence of the set of $(N+1)^2$ operators, $V_{\mathbf{m}}^N \quad (0\le\pm m_a\le(N\pm1)/2)$, which are linear dependent for finite N, will go to $V_{\mathbf{m}}$ in the $N\to\infty$ limit. This implies that the $N\times N$ matrix models may not be suitable to regularize theories with area-preserving diffeomorphisms[8]. How the dependent set of matrices becomes independent ones, or the algebra for the $(N+1)^2$ matrices, is deserved to be investigated further to understand the matrix models.

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[6] Even for fixed k,l and $\mathbf{m}$, some matrix elements are divergent when $N\to\infty$.

[7] We have checked linear independence for small N's but we have not given a general proof.

[8] In this case the Poisson algebra is realized with the creation and the annihilation operators. Then the arguments in ref.[9] will be inapplicable since the generators in eq.(24) or eq.(29) are *not* associated with functions on any ordinary compact world-sheet.

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A Calculation of eq.(25)

Once we first notice the following identities, eq.(25) can be shown straightforwardly,

$$e^{2i\sqrt{\pi}\left(\xi m_1 a+i\frac{m_2}{\xi}a^\dagger\right)}\, e^{2i\sqrt{\pi}\left(\xi n_1 a+i\frac{n_2}{\xi}a^\dagger\right)} = e^{2i\sqrt{\pi}\left[\xi(m_1+n_1)a+i\frac{m_2+n_2}{\xi}a^\dagger\right]}. \tag{32}$$

Actually we have

$$[\,2i\sqrt{\pi}\,(\xi m_1 a+i\frac{m_2}{\xi}a^\dagger),\, 2i\sqrt{\pi}\,(\xi n_1 a+i\frac{n_2}{\xi}a^\dagger)\,] = -4\pi i\,(m_1 n_2 - m_2 n_1) \in 4\pi i\,\mathbb{Z}, \tag{33}$$

so that the phase factor in eq.(32), which comes from eq.(33), are trivial. To complete the calculation, we must evaluate cross terms as

$$[\,2i\sqrt{\pi}\,(\xi m_1 a+i\frac{m_2}{\xi}a^\dagger),\, e^{2i\sqrt{\pi}\left(\xi n_1 a+i\frac{n_2}{\xi}a^\dagger\right)}\,] = \,-\,4\pi i\,(m_1 n_2 - m_2 n_1)\, e^{2i\sqrt{\pi}\left(\xi n_1 a+i\frac{n_2}{\xi}a^\dagger\right)}. \tag{34}$$

Then eqs.(32, 33, 34) lead to eq.(25).

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