Galois currents and the projective kernel in Rational Conformal Field Theory

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Abstract

The notion of Galois currents in Rational Conformal Field Theory is introduced and illustrated on simple examples. This leads to a natural partition of all theories into two classes, depending on the existence of a non-trivial Galois current. As an application, the projective kernel of a RCFT, i.e. the set of all modular transformations represented by scalar multiples of the identity, is described in terms of a small set of easily computable invariants.

1 Introduction

A most important characteristic of a (Rational) Conformal Field Theory is the associated modular representation [1]. Not only does it prescribe the transformation of the genus one characters of the primary fields under the modular group $\Gamma(1) = SL(2,\mathbb{Z})$, but it does also determine the central charge (modulo 8), the fractional part of the conformal weights, the fusion rules via Verlinde's formula, etc.[2, 3]. All these properties are linked to the fact that the modular representation of an RCFT provides part of the defining data of a modular tensor category [4, 5, 6].

A fundamental property of the modular representation, conjectured by many authors over the years [7, 8, 9, 10, 11] and finally proved in [12], is the congruence subgroup property: there exists a natural number N (which turns out to equal the order of the matrix T representing the Dehn-twist $\tau \mapsto \tau+1$), such that the kernel of the modular representation contains the principal congruence subgroup

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \,|\, a,d \equiv 1 \pmod{N},\ b,c \equiv 0 \pmod{N} \right\}.$$

In other words all modular transformations which belong to $\Gamma(N)$ are represented by the identity. This result has far reaching implications both in theory and practice, e.g. it is the basis of powerful algorithms to compute arbitrary modular matrices.

Once we know that the kernel is a congruence subgroup, we can look for a complete description of it in terms of a small set of easy-to-compute invariants. As it turns out, this won't work for the kernel, but the reason is that the kernel is not the right notion to look at. It is the projective kernel, i.e. the set of modular transformations represented by scalar matrices, that is the natural object, and for it one does indeed get a simple description. Along the way emerges the notion of Galois currents, which are simple currents [13, 14, 15] related to the Galois action [16, 17]. The basic result about Galois currents is uniqueness: a model can have at most one non-trivial Galois current. This leads at once to the partition of all RCFTs into two classes, according to whether they have a non-trivial Galois current or not. As we shall see, while the latter are the generic ones, many important examples of RCFTs - like the Ising model - fall into the first class.

This paper aims to give a brief survey of the above results, concentrating on the conceptual issues, and leaving the technical details to a separate publication. Besides introducing the relevant notions and stating the main results, a couple of simple examples are included in order to illustrate the general theory.

2 Modular matrices and the Galois action

As indicated in the introduction, to each rational CFT is associated a finite dimensional representation D of the modular group $\Gamma(1)$. This representation is most conveniently described by a pair of matrices S and T, which represent the generators $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ of the modular group. These matrices have to satisfy the defining relations

$$\begin{aligned} STS &= T^{-1}ST^{-1}, \\ S^4 &= 1 \end{aligned}$$

of the modular group. It should be noted that the matrices S and T carry much more information than a mere linear representation of $\Gamma(1)$, for the individual matrix elements have an invariant meaning in themselves, in other words the representation comes equipped with a distinguished basis formed by the genus 1 characters of the primary fields. The properties of the matrix elements with respect to this distinguished basis are summarized in Verlinde's theorem [2, 3]:

1. The matrix T is diagonal and of finite order;
2. The matrix S is symmetric;
3. The fusion rules of the primaries are given by Verlinde's formula

   $$N_{pqr} = \sum_s \frac{S_p^s S_q^s S_r^s}{S_0^s},$$

   where 0 labels the vacuum (the conformal block of the identity).

One may show that the modular representation is quite special in many respects, the basic result being the congruence subgroup property mentioned in the introduction, i.e. that the kernel $\ker D$ of the modular representation is a congruence subgroup of level N [12]. One may also show that the *conductor* N, which is equal to the order of the matrix T, is bounded by a universal function of the dimension (= number of primary fields). Moreover, if we denote by K the projective order of T, i.e. the smallest positive integer such that T^K is a scalar matrix, then the integer $e = \frac{N}{K}$ is a divisor of 12 [12]. The integers e and K are important characteristics of the model.

Because the individual matrix elements have an invariant meaning, it is meaningful to look at their arithmetic properties, which is the content of the theory of the Galois action [16, 17]. The basic idea is to look at the field F obtained by adjoining to the rationals the matrix elements of all modular matrices (it is enough to adjoin those of the generators S and T). It turns out that the field F equals the cyclotomic field $\mathbb{Q}[\zeta_N]$, where N is the conductor of the model. By a well known result of algebraic number theory, the Galois group of the cyclotomic field $\mathbb{Q}[\zeta_N]$ over $\mathbb{Q}$ is isomorphic to the group $\mathcal{G}_N = (\mathbb{Z}/N\mathbb{Z})^*$ of prime residues modulo N, where to the prime residue $l \in \mathcal{G}_N$ corresponds the map σ_l sending ζ_N to its l-th power ζ_N^l. As σ_l maps F into itself, it is meaningful to consider its action on the matrix elements of S and T. The basic result [16, 17], which follows from Verlinde's theorem, is

$$\sigma_l\left(S_q^p\right) = \varepsilon_l(q)\, S_{\pi_l q}^p,$$

where π_l is a permutation of the primaries - the *Galois permutation* associated to $l \in \mathcal{G}_N$ -, while $\varepsilon_l(q) = \pm 1$ is a sign. The above Galois permutations define a permutation action of $\mathcal{G}_N$ on the set of primaries, because they satisfy $\pi_{lm} = \pi_l \pi_m$.

3 Galois currents

According to the traditional definition [13, 14, 15], a simple current is a primary field α whose quantum dimension is 1, i.e. such that $S_0^\alpha = S_0^0$. From this definition follows that the fusion product of a simple current α with a primary p is again a primary field, denoted αp, i.e. simple currents induce permutations of the primaries. Of course, the permutations arising this way are quite special, and it is a challenging problem to characterize those permutation groups that may correspond to the action of simple currents on the primary fields of some RCFT. In any case, this induced permutation action allows us to consider for each simple current α the set

$$L_\alpha = \{l \in \mathcal{G}_N \,|\, \alpha p = \pi_l p\},$$

i.e. the set of prime residues mod N such that the Galois permutation π_l is the same as the permutation induced by α (this set may be empty). We call a simple current α a *Galois current* when $L_\alpha \neq \emptyset$. The set of Galois currents may be shown to be closed under the fusion product, in other words they form a group.

Note that the vacuum is always a Galois current, as it is a simple current for which the corresponding permutation is the identity, consequently $1 \in L_0$. The interesting question is whether there exists models with non-trivial simple currents, so let's give a couple of simple examples.

Our first example is the Ising model, the Virasoro minimal model $\mathcal{M}(4,3)$ of central charge $c = \frac{1}{2}$. As it is well known, this model has 3 primaries: the vacuum $\mathbf{0}$, the energy operator ε, and the spin operator σ. One has $K = 16$ and $e = 3$ in this model. A simple computation reveals that $L_\varepsilon = \{5, 11, 13, 19, 29, 35, 37, 43\}$, i.e. ε is a non-trivial Galois current. More generally, one may show that the only Virasoro minimal models with a non-trivial Galois current are those of the form $\mathcal{M}(4,p)$ with p odd - the Ising model corresponding to $p = 3$ -, in which case the primary field with Kac-label $(3,1)$ is the non-trivial Galois current (for $p > 3$).

There are many more examples of theories with non-trivial Galois currents, e.g. some of the Ashkin-Teller models (= $\mathbb{Z}_2$ orbifolds of the compactified boson, the Galois current being the marginal perturbation), many WZNW models, products and orbifolds of the above, etc. As these examples show, Galois currents appear in many types of RCFTs. Closer examination of the above examples leads to the following observations:

- the group of Galois currents is small (in all of the above examples its order is either 1 or 2);
- theories with non-trivial Galois currents are sparse: in two-parameter families of models they form one parameter subfamilies.

As it turns out, the above observations reflect general properties of RCFT, for one may show - the proof is quite lengthy and technical - that

1. If there is a non-trivial Galois current, then it is unique, i.e. the group of Galois currents has either 1 or 2 elements;
2. If a theory has a non-trivial Galois current, then e is odd and K is a multiple of 16.

With respect to this last result, we note that the reverse implication is by no means true, for there exist examples of RCFTs with odd e and K a multiple of 16, but no non-trivial Galois current. On the other hand, in those classes of RCFTs for which one has an explicit expression for e and K, those with odd e and K a multiple of 16 are indeed sparse, in accord with the second observation above.

As to the first result, it exhibits an important dichotomy for RCFTs: a model has either one non-trivial Galois current, or it has none. As we shall see in the next section, the existence of a non-trivial Galois current has important consequences regarding the structure of the modular representation, so the above mentioned dichotomy is not only meaningful, but also relevant. In this respect one might say that the existence of a non-trivial Galois current for the Ising model is a good signal of its special nature.

4 Ordinary vs projective kernel

Given a linear representation of a group, it is natural to look at the kernel of the representation, i.e. the set of group elements that are represented by the identity operator. As we have mentioned in section 2, in case of the modular representation associated to a RCFT, the kernel is a congruence subgroup whose level equals the order N of T. Knowing this fundamental result, the next natural step is to look for a simple description of the kernel. As it turns out the natural object to look at is not the kernel itself, but rather the *projective kernel* $\mathbb{P}\mathcal{K} = \{m \in \Gamma(1) \mid D(m) = \xi(m) 1\}$, which consists of those modular transformations that are represented by scalar matrices, where $\xi : \mathbb{P}\mathcal{K} \to \mathbb{C}$ is a linear character of $\mathbb{P}\mathcal{K}$, which we term the *central character.*. Let's explain why this is so.

Consider a composite system denoted $\mathcal{C}_1 \otimes \mathcal{C}_2$, which is made up of two independent and non-interacting subsystems $\mathcal{C}_1$ and $\mathcal{C}_2$. Note that if both subsystems $\mathcal{C}_1$and $\mathcal{C}_2$ are RCFTs, then so is their composite $\mathcal{C}_1 \otimes \mathcal{C}_2$. As the subsystems are completely independent, one expects that it should be possible to determine unambiguously the value of any "natural" quantity for the composite system from the knowledge of the corresponding values for the subsystems.

In this respect, the partition function and all correlators are "natural", for they factorize in the composite system into a product of corresponding quantities for the subsystems. Similarly, the modular representation is "natural", for the modular representation associated to $\mathcal{C}_1 \otimes \mathcal{C}_2$ is the tensor product of the modular representations associated to $\mathcal{C}_1$ and $\mathcal{C}_2$. On the other hand, the kernel is not "natural", for one cannot determine the kernel of a tensor product from the sole knowledge of the kernels of the factors. The "natural" notion is that of the projective kernel, because the projective kernel of a tensor product is simply the intersection of the projective kernels of the factors. This is the conceptual explanation for looking at the projective kernel instead of the ordinary one. We note that the same "naturality" argument explains why it is the parameter K and not the conductor N that enters most of our results, for the former is "natural" in the above sense, while the later is not.

After these preliminaries, let's describe the structure of the projective kernel. To this end, let's first introduce the notation

$$\Gamma(K, \mathfrak{g}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid a, d \in \mathfrak{g},\ b, c \equiv 0 \pmod{K} \right\}$$

for a subgroup $\mathfrak{g} < \mathcal{G}_K$. Clearly, $\Gamma(K, \{1\})$ equals the principal congruence subgroup $\Gamma(K)$, from which follows that $\Gamma(K, \mathfrak{g})$ is a congruence subgroup for any $\mathfrak{g} < \mathcal{G}_K$.

Remember the Galois permutations π_l from section 2, and consider the kernel of the permutation action π, i.e. the set $\ker \pi = \{l \in \mathcal{G}_N \mid \pi_l = 1\}$. Clearly, $\ker \pi$ is a subgroup of $\mathcal{G}_N$, and reducing all elements of $\ker \pi$ modulo K we get a subgroup $\mathfrak{h}$ of $\mathcal{G}_K$. It does follow from general properties of the Galois action that the exponent of $\mathfrak{h}$ is 2, in other words $l^2 \equiv 1 \pmod{K}$ for all $l \in \mathfrak{h}$ [12]. The basic result about the projective kernel is that $\Gamma(K, \mathfrak{h})$ is a normal subgroup of $\mathbb{P}\mathcal{K}$ - in particular $\mathbb{P}\mathcal{K}$ is a congruence subgroup of level K -, and there is an explicit isomorphism between the factor group

$$\mathbb{P}\mathcal{K} / \Gamma(K, \mathfrak{h})$$

and the group of Galois currents! This means that the knowledge of K, the subgroup $\mathfrak{h}$ and the group of Galois currents does completely determine the projective kernel. Note that the index of the principal congruence subgroup $\Gamma(K)$ in $\mathbb{P}\mathcal{K}$, which equals the number of Galois currents times the order of $\mathfrak{h}$, is always a power of 2 by the above, a result that seems fairly non-trivial.

As an example, let's consider once again the Ising model $\mathcal{M}(4,3)$. As discussed in section 3, one has $e = 3$ and $K = 16$, and there is a non-trivial Galois current (the energy operator ε). In this case, the group $\mathfrak{h}$ consists of the prime residues $\{1, 7, 9, 15\}$ modulo 16. As to the projective kernel, it has the coset decomposition

$$\mathbb{P}\mathcal{K} = \Gamma(K, \mathfrak{h}) \cup \begin{pmatrix} 3 & 8 \\ 8 & 11 \end{pmatrix} \Gamma(K, \mathfrak{h})$$

5 Discussion

As we have seen, the notion of Galois currents is not only meaningful in the sense that there exists non-trivial examples, but also relevant to the analysis of the properties of RCFTs. While we have concentrated on their impact on the structure of the projective kernel, it is quite plausible that they play an important role for other aspects of the theory as well, e.g. fusion rules and modular invariants. Elucidating these connections seems to be a rewarding task for the future.

The simple characterization of the projective kernel described in section 4 should be regarded only as a first step. It should be followed by an understanding of the central character, the map $\xi : \mathbb{P}\mathcal{K} \to \mathbb{C}$ appearing in the definition of the projective kernel. Another interesting point would be to find out which subgroups $\mathfrak{h} < \mathcal{G}_K$ are allowed for a given value of K.

Finally, one might speculate on the relevance of the above for the classification of RCFTs. The knowledge of the projective kernel might be an important ingredient of classification attempts. From another point of view, one may hope that the special properties of RCFTs with non-trivial Galois currents could lead to a classification of this special class of model.

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