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Algebraic Aspects of the Background Field Method

Ruggero Ferrari[1], Marco Picariello[2] and Andrea Quadri[3]

Dipartimento di Fisica, Università di Milano via Celoria 16, 20133 Milano, Italy
and INFN, Sezione di Milano

Abstract

The background field method allows the evaluation of the effective action by exploiting the (background) gauge invariance, which in general yields Ward identities, i.e. linear relations among the vertex functions.
In the present approach an extra gauge fixing term is introduced right at the beginning in the action and it is chosen in such a way that BRST invariance is preserved. The background effective action is considered and it is shown to satisfy both the Slavnov-Taylor (ST) identities and the Ward identities. This allows the proof of the background equivalence theorem with the standard techniques. In particular we consider a BRST doublet where the background field enters with a non-zero BRST transformation.
The rational behind the introduction of an extra gauge fixing term is that of removing the singularity of the Legendre transform of the background effective action, thus allowing the construction of the connected amplitudes generating functional $W_{\rm bg}$. By using the relevant ST identities we show that the functional $W_{\rm bg}$ gives the same physical amplitudes as the original one we started with. Moreover we show that $W_{\rm bg}$ cannot in general be derived from a classical action by the Gell-Mann-Low formula.
As a final point of the paper we show that the BRST doublet generated from the background field does not modify the anomaly of the original underlying gauge theory. The proof is algebraic and makes no use of arguments based on power-counting.

Key words: Background, Renormalization, BRST.

[1]ruggero.ferrari@mi.infn.it
[2]marco.picariello@mi.infn.it
[3]andrea.quadri@mi.infn.it

1 Introduction

The background field method [1, 2, 3] allows the evaluation of the effective action by exploiting the (background) gauge invariance, which in general yields Ward identities, i.e. linear relations among the vertex functions. This is a noticeable technical advantage in comparison with the Slavnov-Taylor identities.

Thus the construction of the background effective action turns out to be simpler than the quantization of the ordinary underlying gauge theory, since the structure of counterterms can be greatly simplified by using the symmetry requirement of background gauge invariance. For this reason the background field method has been advantageously applied to gravity and supergravity computations [4] and to the calculation of the β function in Yang-Mills theory [2, 5]. More recently, it has been applied to the quantization of the Standard Model [6, 7].

Recent applications of the background field method in the context of the renormalization group flow [8] suggest to construct first a regularized background-gauge invariant effective action and then to recover the Slavnov-Taylor identities (broken by the regularization procedure) by fine-tuning the free parameters of the model.

The quantization of gauge theories in perturbative quantum field theory requires the introduction of a gauge-fixing term. The gauge-invariance of the effective action is thus spoiled even at the classical level. Gauge invariance is only recovered at the very end of the calculations, when physical amplitudes are considered.

The classical gauge-fixed action is however symmetric under the BRST transformations and for non-anomalous theories, the corresponding quantum action satisfies the Slavnov-Taylor (ST) identities.

The introduction of the background field method quantization allows to define a modified effective action, the background effective action, depending on classical background gauge-fields and matter-fields, which, in the non-anomalous cases, can be chosen to be symmetric under suitably defined linear background gauge transformations.

Already in early papers on the background field method, as for instance in Ref. [3] and more recently in Refs. [9, 11, 12] it has been pointed out that a further gauge fixing term is needed beside the usual background gauge. The seminal discussion of [3] has been improved in [12], where the problem of the extra gauge-fixing term, needed to construct connected amplitudes, has been clarified. In fact the construction of the connected amplitudes by using the effective action, obtained with the background field method, fails in the case of the two-point function, since in this case the vertex function has no inverse. In the present approach this new gauge fixing term is introduced right at the beginning in the action and it is chosen in such a way that BRST invariance is preserved, although the gauge invariance of the background effective action is lost. This difficulty is overcome by defining a new background effective action by subtracting a harmless functional linear in the quantized fields. Linearity is important in order to avoid the introduction of a *ad hoc* composite operator.

The interplay of the background field symmetry with Slavnov-Taylor identities has first been considered in Ref. [3], where the equivalence of the background gauge field quantization and the ordinary perturbative quantization of gauge theories was proven by using a Ward identity derived from BRST invariance of the theory. It turns out [9] to be particularly useful to allow the background fields to vary under the BRST transformation s, so that the background fields enter into the BRST transformations as doublets, together with a suitably defined set of classical ghosts Ω. For instance the transformation of the background gauge-fields V_a^μ are

$$sV_a^\mu = \Omega_a^\mu\,, \qquad s\Omega_a^\mu = 0\,. \tag{1}$$

The introduction of the doublet (V_a^μ, Ω_a^μ) allows an elegant derivation of the Slavnov-Taylor identity for the connected amplitudes used in the background equivalence theorem. It is however necessary to show that the extension of the BRST transformations given by eq. (1) does not alter the anomaly content of the theory. In fact the removal of the doublet has to be explicitly checked in the background field method, as it is done in Ref. [9] for $SU(n)$ and with emphasis on the Standard Model group in Ref. [7, 10]. In the proof presented in this paper we do not assume power-counting: only locality of the breaking term of the Slavnov-Taylor identity is necessary.

The doublet (V_a^μ, Ω_a^μ) is not essential for the construction of the theory. In fact the physical content of the theory is insensitive to the BRST transformation properties of V_a^μ. As an example, consider the alternative approach where $sV_a^\mu = 0$ and compare with the case described by eq.(1) for the local term $F_a^{\mu\nu}(V)^2$. In the first case this term is a ST-invariant and therefore can be removed from the action by a suitable normalization condition [13, 14], while in the second case it is removed by the requirement of ST-invariance.

In the previous approaches [3, 12] emphasis was put on the independence from the background gauge of the physical elements of the S-matrix. For this reason the discussion has been limited [12] to an infrared-safe theory. In fact the result can be extended to quite a general situation where one considers any expectation value of quasi-local observable operator (therefore BRST invariant objects), thus including the physical S-matrix elements of a massive theory. Thus we consider physical observables given by the equivalent classes of ST invariant quasi-local operators, where the equivalence relation is given by

$$\mathcal{O} \sim \mathcal{O}' + sX \tag{2}$$

for some X.

In the present paper we consider also the Legendre transform of the background effective action. Thus we get the functional $W_{\rm bg}$, which generates the connected amplitudes. By this procedure one needs to derive the free propagator of the gauge field from the background effective action, which exists only if an extra gauge fixing term, beside the background one, as it has been stressed previously [12]. By using the relevant ST identities we show that the functional $W_{\rm bg}$ gives the same physical amplitudes as the original one we started with. Moreover we show that $W_{\rm bg}$ in general cannot be derived from a classical action by the Gell-Mann-Low formula. In fact the Feynman rules for the vertices inside a 1PI amplitude and for those containing a connecting line are different in general (not for abelian gauge theories).

The paper is organized as follows. Section 2 is devoted to a short summary of the background field method. Here the BRST is extended to the background gauge-fields V as in previous discussions (see Refs. [9, 11, 12]). This provides a technical advantage in the derivation of the necessary Slavnov-Taylor identities. Section 3 shows the independence of the physical amplitudes from the background field. Section 4 extends the theorem of equivalence for the background field method to the general case of vacuum expectation value of physical observables, and deals with a thorough discussion of the background field method construction of physical amplitudes. Section 5 shows the properties of the Legendre transform of the background effective action and the physical equivalence of the background functional for the connected amplitudes. Section 6 contains an improved proof that the extended BRST transformation on the background gauge field V (eq.(1)) does not change the anomaly properties of the Slavnov-Taylor identity. The conclusions in Section 7 provide a summary of the background field method and the improvements of the method obtained in the present paper.

2 The generating functional for the Feynman amplitudes

We consider a generic gauge theory; however matter fields will not be displayed in the notations. The fields are $A_{a\mu}$ (gauge fields), B_a (Lagrange multipliers in the gauge fixing), $V_{a\mu}$ (the background fields), $\Omega_{a\mu}$ (the BRST partner of $V_{a\mu}$), $\bar\theta_a, \theta_a$ (the Faddeev-Popov fields). The action is

$$\Gamma^{(0)}[A,V] = \int d^4x \left\{ -\frac{1}{4} F^2_{\mu\nu}(A) + \alpha\ s \left[\bar\theta_a \left(\frac{B_a}{2} - f_a \right) \right] - \alpha' s\ \left[\bar\theta_a \partial^\mu V_{a\mu} \right] \right\} \quad (3)$$

where s is the BRST operator and

$$f_a = \partial_\mu (A-V)^\mu_a + f_{abc} V^\mu_b (A-V)_{c\mu} \equiv D^\mu(V)_{ac}(A-V)_{c\mu} \quad (4)$$

is the background gauge fixing in the notation

$$D^\mu(V)_{ac} \equiv \delta_{ac}\partial^\mu + f_{abc} V^\mu_b. \quad (5)$$

The BRST transformations are

$$\begin{array}{ll} s\, A_{a\mu} = D_\mu(A)_{ab}\theta_b & s\,\theta_a = -\frac{1}{2} f_{abc}\theta_b\theta_c \\ s\,\bar\theta_a = B_a & s\, B_a = 0 \\ s\, V_{a\mu} = \Omega_{a\mu} & s\,\Omega_{a\mu} = 0. \end{array} \quad (6)$$

The α' term is introduced in order to deal with the degenerate case $A = V$ (i.e. $f = 0$). This will be discussed later on in Section 5.

By using the transformations (6) the action becomes

$$\begin{aligned} \Gamma^{(0)}[A,V] = \int d^4x \Big\{ & -\frac{1}{4} F^2_{\mu\nu}(A) \\ & +\alpha \left(\frac{B^2}{2} - BD(V)(A-V) + \bar\theta \left(D(V)D(A)\theta - D(A)\Omega \right) \right) \\ & -\alpha' \left(B\partial^\mu V_\mu - \bar\theta \partial^\mu \Omega_\mu \right) + \bar\theta^* B + A^* D(A)\theta - \frac{1}{2}\theta^*_a f_{abc}\theta_b\theta_c \Big\}. \end{aligned} \quad (7)$$

The anti-fields $A^*, \bar\theta^*, \theta^*$ are the external fields coupled to the BRST-transforms ($D_\mu(A)_{ab}\theta_b$, B_a, $-\frac{1}{2} f_{abc}\theta_b\theta_c$) of the fundamental fields.

Now we consider the generating functional of the Green functions associated to the above action, with external currents $J, K, \eta, \bar{\eta}$ coupled to the quantized fields $A, B, \bar{\theta}, \theta$:

$$
\begin{aligned}
Z[J, V, \ldots] \;&=\; \exp\left(iW\right) \equiv \int \mathcal{D}A\mathcal{D}B\mathcal{D}\bar{\theta}\mathcal{D}\theta \exp i \int d^4x \left\{ -\frac{1}{4} F^2_{\mu\nu}(A) \right. \\
&\quad + \alpha \left(\frac{B^2}{2} - Bf + \bar{\theta}\left(D(V)D(A)\theta - D(A)\Omega\right) \right) \\
&\quad - \alpha' \left(B\partial^\mu V_\mu - \bar{\theta}\partial^\mu \Omega_\mu \right) \\
&\quad \left. + JA + KB - \eta\bar{\theta} - \bar{\eta}\theta + A^* D(A)\theta - \frac{1}{2}\theta^*_a f_{abc}\theta_b\theta_c \right\} . \qquad (8)
\end{aligned}
$$

The invariance under the BRST transformations corresponds to require the validity of the Slavnov-Taylor identity

$$
\begin{aligned}
\mathcal{S}W[J, V, \ldots] \;&=\; \int d^4x \left(-J\frac{\delta}{\delta A^*} - \eta\frac{\delta}{\delta\bar{\theta}^*} - \bar{\eta}\frac{\delta}{\delta\theta^*} + \Omega\frac{\delta}{\delta V} \right) W[J, V, \ldots] \\
&=\; 0 \qquad (9)
\end{aligned}
$$

for the generating functional of the (connected) Green functions. The anti-field $\bar{\theta}^*_a$ is the external field coupled to the BRST-transforms B of the fundamental field $\bar{\theta}$, then the current K_a can be identified with $\bar{\theta}^*_a$. The Feynman rules of the perturbative expansion can be read from eq.(8).

2.1 Background gauge symmetry

For $\alpha' = 0$ the action in eq.(7) is invariant under the *background gauge transformations.*

$$
\begin{array}{ll}
\delta\, A = D(A)\omega & \delta\, V = D(V)\omega \\
\delta\, \phi_a = -f_{abc}\omega_b\phi_c & \text{with } \phi_a \in \{\theta_a, \bar{\theta}_a, B_a, \Omega_a, A^*_{a\mu}, \theta^*_a\}
\end{array}
\qquad (10)
$$

where ω_a is a group parameter. Consequently the functional

$$
\tilde{Z}[J, V, \ldots] = \exp\left(i\tilde{W}\right) \equiv Z[J, V, \ldots]\; \exp\left\{ -i \int d^4x V J \right\} \qquad (11)
$$

is, for $\alpha' = 0$, invariant under the transformation

$$
\begin{array}{ll}
\delta\, V = D(V)\omega & \delta\, \Omega_{a\mu} = -f_{abc}\omega_b\Omega_{c\mu} \\
\delta\, \zeta_a = -f_{abc}\omega_b\zeta_c & \text{with } \zeta_a \in \{J_{a\mu}, K_a, \bar{\eta}_a, \eta_a, A^*_{a\mu}, \theta^*_a\}.
\end{array}
\qquad (12)
$$

The corresponding Ward identity

$$
\begin{aligned}
\mathcal{G}_a(x)\tilde{W} \;&\equiv\; \left\{ -D_\mu(V)_{ab}\frac{\delta}{\delta V_{b\mu}(x)} - f_{abc}\left(\Omega_{b\mu}(x)\frac{\delta}{\delta\Omega_{c\mu}(x)} + J_{b\mu}(x)\frac{\delta}{\delta J_{c\mu}(x)} \right.\right. \\
&\quad + K_b(x)\frac{\delta}{\delta K_c(x)} + \eta_b(x)\frac{\delta}{\delta\eta_c(x)} + \bar{\eta}_b(x)\frac{\delta}{\delta\bar{\eta}_c(x)} + A^*_{b\mu}(x)\frac{\delta}{\delta A^*_{c\mu}(x)} \\
&\quad \left.\left. + \theta^*_b(x)\frac{\delta}{\delta\theta^*_c(x)} \right) \right\} \tilde{W} = 0 \qquad (13)
\end{aligned}
$$

can be used in the renormalization procedure together with the ST (derived from eq.(9))

$$\begin{aligned}\mathcal{S}\tilde{W}[J,V,\ldots] &= \int d^4x \left\{ \left(-J\frac{\delta}{\delta A^*} - \eta\frac{\delta}{\delta\bar{\theta}^*} - \bar{\eta}\frac{\delta}{\delta\theta^*} + \Omega\frac{\delta}{\delta V} \right) \tilde{W}[J,V,\ldots] - J\Omega \right\} \\ &= 0\,, \end{aligned} \tag{14}$$

since the corresponding operators commute

$$[\mathcal{S}, \mathcal{G}_a] = 0\,. \tag{15}$$

If $\alpha' \neq 0$ the breaking of the *background gauge symmetry* is harmless since it is due to a gauge fixing term linear in the quantized fields. The associated Ward identity for the functional of the connected amplitudes $\tilde{W}$ is then modified by a term proportional to α'

$$\begin{aligned}\mathcal{G}_a^{(\alpha')}(x)\tilde{W} \equiv & \left\{ -D_\mu(V)_{ab}\frac{\delta}{\delta V_{b\mu}(x)} - f_{abc}\left(\Omega_{b\mu}(x)\frac{\delta}{\delta \Omega_{c\mu}(x)} + J_{b\mu}(x)\frac{\delta}{\delta J_{c\mu}(x)} \right.\right. \\ & + K_b(x)\frac{\delta}{\delta K_c(x)} + \eta_b(x)\frac{\delta}{\delta \eta_c(x)} + \bar{\eta}_b(x)\frac{\delta}{\delta \bar{\eta}_c(x)} + A^*_{b\mu}(x)\frac{\delta}{\delta A^*_{c\mu}(x)} \\ & \left. + \theta^*_b(x)\frac{\delta}{\delta \theta^*_c(x)} \right) + \alpha'\partial^\mu \left(D_\mu(V)_{ab}\frac{\delta}{\delta K_b(x)} - f_{abc}\Omega_{b\mu}\frac{\delta}{\delta \eta_c(x)} \right) \Big\} \tilde{W} = 0\,. \end{aligned} \tag{16}$$

By explicit computation one can verify that also

$$\left[\mathcal{S}, \mathcal{G}_a^{(\alpha')} \right] = 0\,. \tag{17}$$

Then the renormalization program can be performed by using both conditions expressed by the ST identity in eq.(14) and the Ward identity given in eq.(13) or eq.(16).

2.2 Gauge invariant effective action

It is worthwhile to illustrate from a different point of view the properties of the functionals $\tilde{Z}$. By a simple change of variable in the functional integral one gets

$$\begin{aligned}\tilde{Z}[J,V,\ldots] &= \exp(iW) = \int \mathcal{D}\tilde{A}\mathcal{D}B\mathcal{D}\bar{\theta}\mathcal{D}\theta \exp i \int d^4x \Big\{ -\frac{1}{4}F^2_{\mu\nu}(\tilde{A}+V) \\ & +\alpha \left(\frac{B^2}{2} - BD^\mu(V)\tilde{A}_\mu + \bar{\theta}(D(V)D(\tilde{A}+V)\theta - D(\tilde{A}+V)\Omega) \right) \\ & -\alpha' \left(B\partial^\mu V_\mu - \bar{\theta}\partial^\mu \Omega_\mu \right) \\ & +J\tilde{A} + KB - \eta\bar{\theta} - \bar{\eta}\theta + A^* D(\tilde{A}+V)\theta - \frac{1}{2}\theta^*_a f_{abc}\theta_b\theta_c \Big\}. \end{aligned} \tag{18}$$

This expression gives the Feynman rules for any Green function involving any number of derivatives with respect to V. In fact the functional dependence of $\tilde{Z}$ from V should be understood as given by the ensemble of all possible derivatives of $\tilde{Z}$ with respect to V. Now we can introduce the generating functional for the 1PI Green functions associated to $\tilde{Z}$, i.e. we set

$$\tilde{A} = \frac{\delta}{\delta J}\tilde{W} \tag{19}$$

and perform the Legendre transform

$$\tilde{\Gamma}[\tilde{A}, V] = \tilde{W} - \int d^4x J\tilde{A} - \dots \tag{20}$$

In eq.(20) $\tilde{\Gamma}$ is the full Legendre transform of $\tilde{W}$. Dots indicate the remaining conjugate variables besides $(J, \tilde{A})$. By using eq.(11) one gets

$$\tilde{A} = \frac{\delta}{\delta J} W - V = A - V\,, \tag{21}$$

and therefore

$$\begin{aligned}\tilde{\Gamma}[\tilde{A}, V] &= \tilde{W} - \int d^4x J\tilde{A} - \dots \\ &= W - \int d^4x J(\tilde{A} + V) - \dots \\ &= \Gamma[A, V]_{|A=\tilde{A}+V}\,.\end{aligned} \tag{22}$$

The lowest order $\tilde{\Gamma}$ is given by (see eq.(7))

$$\begin{aligned}\tilde{\Gamma}^{(0)}[\tilde{A}, V] = & \int d^4x \Big\{ -\frac{1}{4} F^2_{\mu\nu}(\tilde{A} + V) \\ & +\alpha \left(\frac{B^2}{2} - BD(V)\tilde{A} + \bar{\theta}(D(V)D(\tilde{A} + V)\theta - D(\tilde{A} + V)\Omega) \right) \\ & -\alpha' \left(B\partial^\mu V_\mu - \bar{\theta}\partial^\mu\Omega_\mu \right) + A^* D(\tilde{A} + V)\theta - \frac{1}{2}\theta^*_a f_{abc}\theta_b\theta_c \Big\}.\end{aligned} \tag{23}$$

The generating functional of the 1PI $\tilde{\Gamma}$ associated to $\tilde{W}$ satisfies the corresponding Ward identity in eq.(16)

$$\begin{aligned}&\mathcal{G}^{(\alpha')}_a(x)\tilde{\Gamma}[\tilde{A}, V] = \Big\{ - D_\mu(V)_{ab}\frac{\delta}{\delta V_{b\mu}(x)} - f_{abc}\Big[\Omega_{b\mu}(x)\frac{\delta}{\delta\Omega_{c\mu}(x)} \\ &+\tilde{A}_{b\mu}(x)\frac{\delta}{\delta\tilde{A}_{c\mu}(x)} + B_b(x)\frac{\delta}{\delta B_c(x)} + \bar{\theta}_b(x)\frac{\delta}{\delta\bar{\theta}_c(x)} + \theta_b(x)\frac{\delta}{\delta\theta_c(x)} \\ &+A^*_{b\mu}(x)\frac{\delta}{\delta A^*_{c\mu}(x)} + \theta^*_b(x)\frac{\delta}{\delta\theta^*_c(x)}\Big]\Big\}\tilde{\Gamma}[\tilde{A}, V] \\ &+\alpha'\partial^\mu \left(D_\mu(V)_{ab}B_b + f_{abc}\bar{\theta}_b\Omega_{c\mu} \right) = 0\end{aligned} \tag{24}$$

and the ST identity

$$\begin{aligned}\int d^4x \quad & \Big\{\Omega\frac{\delta\tilde{\Gamma}[\tilde{A}, V]}{\delta V} + B\frac{\delta\tilde{\Gamma}[\tilde{A}, V]}{\delta\bar{\theta}} + \frac{\delta\tilde{\Gamma}[\tilde{A}, V]}{\delta\theta^*}\frac{\delta\tilde{\Gamma}[\tilde{A}, V]}{\delta\theta} \\ & - \left(-\frac{\delta\tilde{\Gamma}[\tilde{A}, V]}{\delta A^*} + \Omega \right) \frac{\delta\tilde{\Gamma}[\tilde{A}, V]}{\delta\tilde{A}} \Big\} = 0\,.\end{aligned} \tag{25}$$

The α' part in eq.(24) can be easily removed by introducing the functional

$$\Gamma_{\rm bg}[\tilde{A}, V] \equiv \tilde{\Gamma}[\tilde{A}, V] + \alpha' \int d^4x \Big\{ B\partial^\mu V_\mu - \bar{\theta}\partial^\mu\Omega_\mu \Big\}. \tag{26}$$

It is straightforward to show that $\Gamma_{\rm bg}[\tilde{A},V]$ satisfies the Ward identities

$$\mathcal{G}_a(x)\Gamma_{\rm bg}[\tilde{A},V] = \Big\{ - D_\mu(V)_{ab}\frac{\delta}{\delta V_{b\mu}(x)} - f_{abc}\Big[\Omega_{b\mu}(x)\frac{\delta}{\delta \Omega_{c\mu}(x)}$$
$$+\tilde{A}_{b\mu}(x)\frac{\delta}{\delta \tilde{A}_{c\mu}(x)} + B_b(x)\frac{\delta}{\delta B_c(x)} + \bar{\theta}_b(x)\frac{\delta}{\delta \bar{\theta}_c(x)} + \theta_b(x)\frac{\delta}{\delta \theta_c(x)}$$
$$+A^*_{b\mu}(x)\frac{\delta}{\delta A^*_{c\mu}(x)} + \theta^*_b(x)\frac{\delta}{\delta \theta^*_c(x)}\Big]\Big\}\Gamma_{\rm bg}[\tilde{A},V] = 0\ . \tag{27}$$

These Ward identities follow from the invariance of $\Gamma_{\rm bg}[\tilde{A},V]$ under the transformation

$$\begin{array}{ll} \delta\,\tilde{A}_{a\mu} = -f_{abc}\omega_b\tilde{A}_{c\mu} & \delta\,V = D(V)\omega \\ \delta\,\phi_a = -f_{abc}\omega_b\phi_c & \text{with } \phi_a \in \{\theta_a,\bar{\theta}_a,B_a,\Omega_a,A^*_{a\mu},\theta^*_a\}, \end{array} \tag{28}$$

while the generating functional of the 1PI $\tilde{\Gamma}$ is not invariant under the *background gauge transformations* due to the breaking term proportional to α'.

2.3 Further algebra

There is an interesting limit: take $\tilde{A}\to 0$. Then from eq.(20) one gets

$$\tilde{\Gamma}[0,V,\ldots] = \tilde{W}[J,V,\ldots] - \int d^4x\,\big(KB - \eta\bar{\theta} - \bar{\eta}\theta\big) \tag{29}$$

where J is given by eq.(19)

$$\tilde{A} = \frac{\delta}{\delta J}\tilde{W}[J,V,\ldots] = 0. \tag{30}$$

Moreover one gets

$$\tilde{\Gamma}[0,V] = \Gamma[V,V]. \tag{31}$$

This very interesting equation looks very simple and innocuous. In fact the LHS is given by the 1PI amplitudes constructed with the Feynman rules given by the action in eq.(23), while the RHS is given by the 1PI amplitudes provided by the action used to define the generating functional Z in eq.(8).

3 Independence of the physical amplitudes from the background field

Physical amplitudes should be independent from the background field $V_{a\mu}$, from the source K_a of the field B_a and from the gauge parameters α and α'. This can be proved by using the Slavnov-Taylor identities (9) introduced in Section 2.

The physical amplitudes can be obtained by introducing a set of external sources $\beta_i(x)$, $\quad i = 1,\ldots$ coupled to local or quasi-local BRST invariant quantities. Both $A_{a\mu}$ and B_a cannot be physical fields, therefore $J_{a\mu}$ and K_a cannot belong to the set of β

sources. Let us consider the dependence of the physical amplitudes on V, by taking the derivative of eq.(9) with respect to Ω and then putting $\eta = \bar{\eta} = \Omega = 0$

$$\Big\{ - \int d^4y J(y) \frac{\delta^2}{\delta\Omega(x)\delta A^*(y)} + \frac{\delta}{\delta V(x)} \Big\} W[J, V, \ldots]|_{\eta=\bar{\eta}=\Omega=0} = 0\,. \tag{32}$$

Notice that the insertion of the operator coupled to $\Omega_{b\mu}$

$$-\alpha D_\mu(A)_{bc}\bar{\theta}_c + \alpha' \partial_\mu \bar{\theta}_b \tag{33}$$

is just (for $\alpha' = 0$) the composite operator appearing in eq.(2.10) of Ref. [3]. Eq.(32) is the starting point to prove the independence of physical amplitudes from V. Any number of derivatives can be taken with respect to $\beta_i(x)$ and moreover one has to put $J = 0$ being conjugated to an unphysical field. The result is independent from the value of V.

A similar argument can be used in order to prove that the derivative with respect to the source K of any physical amplitude is zero. In fact by taking the functional derivative with respect to η of eq.(9) one gets

$$\Big\{ - \int d^4y J(y) \frac{\delta^2}{\delta\eta(x)\delta A^*(y)} - \frac{\delta}{\delta\bar{\theta}^*(x)} \Big\} W[J, V, \ldots]|_{\eta=\bar{\eta}=\Omega=0} = 0 \tag{34}$$

and the derivative with respect to $\bar{\theta}^*$ takes down a B field. Thus there is no K-dependence for the physical amplitudes even in presence of the external fields V.

The independence of the physical amplitudes from α and α' can be established according to the standard arguments [15]. The proof follows the same pattern as the one outlined above.

4 The background equivalence theorem

This section reports a well known result and it is included only to make the discussion self contained.

The results of the previous Sections allow us to formulate the theorem of equivalence for the background field method in a rather simple way:

Theorem of equivalence: The construction of the connected amplitudes functional can be equivalently performed (i.e. giving the same physical amplitudes) either by using the 1PI vertex amplitudes generated by the functional $\Gamma[A, V]$ or by using the 1PI vertex amplitudes generated by $\tilde{\Gamma}[0, V]|_{V=A}$.

We remark that the functional $\tilde{\Gamma}[0, V]$ gives the same 1-PI functions as the gauge invariant functional $\Gamma[0, V]_{\rm bg}$ except for the two-point functions.

The proof is as follows [3]. The starting theory is described by the classical action in eq.(3). Correspondingly, the functionals one has to compute are $\Gamma[A, V]$ and $W[J, V]$. If $\Gamma[A, V]$ is known then the connected amplitudes, generated by $W[J, V]$, can be obtained by gluing together the 1PI vertex amplitudes with the connected two-point functions provided by the inverse of the relevant part of $-\Gamma[A, 0]|_{\beta=0}$.

We now restrict ourselves to physical quantities. Eq.(32) says that

$$\left.\frac{\delta^{(n+1)}W}{\delta V \delta\beta_1 \ldots \delta\beta_n}\right|_{J=\eta=\bar{\eta}=0} = \left.\frac{\delta^{(n+1)}\Gamma}{\delta V \delta\beta_1 \ldots \delta\beta_n}\right|_{\frac{\delta\Gamma}{\delta A}=\frac{\delta\Gamma}{\delta\bar{\theta}}=\frac{\delta\Gamma}{\delta\theta}=0} = 0\,, \tag{35}$$

then, in constructing the connected amplitude for physical processes, one can add to any 1PI vertex $\Gamma_{A_{a_1\mu_1}\dots A_{a_n\mu_n}}[A,V,\beta]|_{A=0}$ all those, where some derivatives with respect to $A_{b\nu}$ are replaced by the corresponding derivatives with respect to $V_{b\nu}$. This procedure amounts to replace $\Gamma[A,V]$ with $\Gamma[A,A]$ i.e. with $\tilde{\Gamma}[0,V]|_{V=A}$ according to eq.(31). In general the resulting connected amplitudes will be different from those generated by W; however the physical amplitudes will coincide. This concludes the proof.

We would like to stress that the connected two-point functions used in the above construction are the ones generated by $\Gamma[A,V]$ expanded in powers of V and not the ones generated by $\tilde{\Gamma}[0,V]|_{V=A} = \Gamma[A,A]$. Thus the background equivalence theorem is valid also in the case $\alpha' = 0$.

5 The background gauge functional for connected amplitudes

In the construction outlined in sect.4 the 1-PI vertex amplitudes depending on $V_{a\mu}$ are joined by the connected two-point functions generated by $\Gamma[A,0]$. This is the choice adopted in [2, 3, 12].

However, this is not strictly necessary [12]. I.e. the physical amplitudes can be obtained from the connected Green functions where the 1-PI vertex amplitudes $\tilde{\Gamma}[0,V]|_{V=A}$ are glued together by using connected two-point functions different from those used in sect.4. One interesting possibility is provided by introducing the Legendre transform of the vertex functional $\tilde{\Gamma}[0,V]|_{V=A}$. In this case one needs the extra gauge fixing parameter $\alpha' \neq 0$ in order that the bilinear part of $\tilde{\Gamma}[0,V]|_{\beta=0}$ possesses the inverse.

5.1 Legendre transform of the background effective action

Then one can introduce a new functional $W_{\rm bg}$ defined by

$$\begin{cases} W_{\rm bg}[J,\beta,\dots] = \tilde{\Gamma}[0,V,\beta,\dots] + \displaystyle\int d^4x\, J\, V + \int d^4x\, \left(KB - \eta\bar{\theta} - \bar{\eta}\theta\right) \\ J = -\dfrac{\delta\tilde{\Gamma}[0,V]}{\delta V} \\ K = -\dfrac{\delta\tilde{\Gamma}[0,V]}{\delta B}\, , \eta = -\dfrac{\delta\tilde{\Gamma}[0,V]}{\delta\bar{\theta}}\, , \bar{\eta} = -\dfrac{\delta\tilde{\Gamma}[0,V]}{\delta\theta}\, . \end{cases} \tag{36}$$

The functional derivatives of $W_{\rm bg}$ with respect to the sources β reproduce in the on-shell limit $J = \beta = 0$ the on-shell physical connected amplitudes of the original theory. We will report a proof of this fact in sect.5.2.

The construction of $W_{\rm bg}$ is possible only for $\alpha' \neq 0$, since, otherwise, the bilinear part of $\tilde{\Gamma}[0,V]$ does not possess the inverse. It is important to notice that in general $W_{\rm bg}$ will not correspond to a field theory. In fact the Feynman rules for vertices involving only internal legs in the 1PI graphs ($\tilde{A}$ vertices in eq.(23)) do not coincide with those, where some V are involved. See the rules given by the free action in eq.(23).

As a consequence of these facts one cannot derive identities for $W_{\rm bg}$ and for $\tilde{\Gamma}[0,V]$ from invariance properties of the *classical* action (action principle). We have to revert to the properties of $\tilde{\Gamma}[A,V]$ and $\tilde{W}$ and use eqs.(14) and (24). From eq.(24)

$$\mathcal{G}_a^{(\alpha')}(x)\tilde{\Gamma}[0,V] = \Big\{ - D_\mu(V)_{ab}\frac{\delta}{\delta V_{b\mu}(x)} - f_{abc}\Big[\Omega_{b\mu}(x)\frac{\delta}{\delta\Omega_{c\mu}(x)} + B_b(x)\frac{\delta}{\delta B_c(x)} + \bar{\theta}_b(x)\frac{\delta}{\delta\bar{\theta}_c(x)} + \theta_b(x)\frac{\delta}{\delta\theta_c(x)} + A^*_{b\mu}(x)\frac{\delta}{\delta A^*_{c\mu}(x)} + \theta^*_b(x)\frac{\delta}{\delta\theta^*_c(x)}\Big]\Big\}\tilde{\Gamma}[0,V] + \alpha'\left(D_\mu(V)_{ab}B_b + f_{abc}\bar{\theta}_b\Omega_{c\mu}\right) = 0\,, \tag{37}$$

which corresponds to setting $\tilde{A}_{a\mu} = 0$ in the transformations in eq.(28).

In eq.(14) one can put $J = \bar{J}[\tilde{A},V]$, the solution of eq.(21)

$$\begin{aligned} \mathcal{S}\tilde{W}[\bar{J},V,\ldots] &= \int d^4x \left(-\bar{J}\frac{\delta}{\delta A^*} - \eta\frac{\delta}{\bar{\eta}^*} - \bar{\eta}\frac{\delta}{\delta\theta^*} + \Omega\frac{\delta}{\delta V}\right)\tilde{W}[\bar{J},V,\ldots] \\ &\quad - \int d^4x \bar{J}[\tilde{A},V]\Omega = 0 \end{aligned} \tag{38}$$

which can be written in terms of $\tilde{\Gamma}[0,V]$ setting $\tilde{A} = 0$:

$$\int d^4x \left\{\left(-\bar{J}[0,V]\frac{\delta}{\delta A^*} - \bar{\eta}\frac{\delta}{\delta\theta^*} + \Omega\frac{\delta}{\delta V}\right)\tilde{\Gamma}[0,V] - \eta B - \bar{J}[0,V]\Omega\right\} = 0\,. \tag{39}$$

The above equation shows that the ST identity does not close, in fact from eq.(22) we have

$$\bar{J}[0,V] = -\frac{\delta\tilde{\Gamma}[\tilde{A},V]}{\delta\tilde{A}}|_{\tilde{A}=0}\,, \tag{40}$$

and finally we get

$$\begin{aligned} &\int d^4x \left\{\Omega\frac{\delta\tilde{\Gamma}[0,V]}{\delta V} + B\frac{\delta\tilde{\Gamma}[0,V]}{\delta\bar{\theta}} + \frac{\delta\tilde{\Gamma}[0,V]}{\delta\theta^*}\frac{\delta\tilde{\Gamma}[0,V]}{\delta\theta}\right\} = \\ &\qquad \int d^4x \left(-\frac{\delta\tilde{\Gamma}[0,V]}{\delta A^*} + \Omega\right)\left(\frac{\delta\tilde{\Gamma}[\tilde{A},V]}{\delta\tilde{A}}\right)_{\tilde{A}=0}\,. \end{aligned} \tag{41}$$

Therefore in general the ST identity cannot be satisfied by $\tilde{\Gamma}[0,V]$.

5.2 Physical equivalence of the connected background functional

The physical equivalence of the connected background functional $W_{\rm bg}$ has been proved by Becchi and Collina [12]. Here we provide a shorter proof.

We comment on the relationship between $W_{\rm bg}$ in eq.(36) and W in eq.(8). From the definition of $\tilde{W}$ in eq.(11) and by using eq.(29) we obtain:

$$\begin{aligned} W[\bar{J}[0,V,\beta],V,\beta,\ldots] &= \tilde{\Gamma}[0,V,\beta,\ldots] + \int d^4x\,\bar{J}[0,V,\beta]V \\ &\quad + \int d^4x\left(K[0,V,\beta]\;B - \eta[0,V,\beta]\;\bar{\theta} - \bar{\eta}[0,V,\beta]\;\theta\right)\,. \end{aligned} \tag{42}$$

Notice that (K, B), $(\eta, \bar{\theta})$, $(\bar{\eta}, \theta)$ are conjugate variables for the functional $\tilde{W}$ in eq.(29). Moreover, since $\tilde{A} = 0$ we get from eq.(21):

$$A = \frac{\delta W[J, V, \ldots]}{\delta J}|_{J=\bar{J}[0,V,\beta]} = V \ . \tag{43}$$

Now we differentiate both sides of eq.(42) with respect to V. By using eq.(43) we obtain

$$-\frac{\delta\tilde{\Gamma}[0, V, \beta]}{\delta V} \quad = \quad -\frac{\delta W[J, V, \beta]}{\delta V}|_{J=\bar{J}[0,V,\beta]} + \bar{J}[0, V, \beta] \,. \tag{44}$$

Now, by using eq.(43), it follows from eq.(42) by explicit computation that

$$\frac{\delta^{(n)} W[J, V, \beta, \ldots]}{\delta\beta_{i_1}(x_1) \ldots \delta\beta_{i_n}(x_n)}|_{J=\bar{J}[0,V,\beta]} = \frac{\delta^{(n)}\tilde{\Gamma}[0, V, \beta, \ldots]}{\delta\beta_{i_1}(x_1) \ldots \delta\beta_{i_n}(x_n)} \,. \tag{45}$$

From the properties of the Legendre transform one also gets:

$$\frac{\delta^{(n)}\tilde{\Gamma}[0, V, \beta, \ldots]}{\delta\beta_{i_1}(x_1) \ldots \delta\beta_{i_n}(x_n)} = \frac{\delta^{(n)} W_{\rm bg}[J, \beta, \ldots]}{\delta\beta_{i_1}(x_1) \ldots \delta\beta_{i_n}(x_n)} \qquad \text{for } J = -\frac{\delta\tilde{\Gamma}[0, V, \beta, \ldots]}{\delta V} \tag{46}$$

and finally, by combining eqs.(45) and (46),

$$\frac{\delta^{(n)} W[J, V, \beta, \ldots]}{\delta\beta_{i_1}(x_1) \ldots \delta\beta_{i_n}(x_n)}|_{J=\bar{J}[0,V,\beta]} = \frac{\delta^{(n)} W_{\rm bg}[J, \beta, \ldots]}{\delta\beta_{i_1}(x_1) \ldots \delta\beta_{i_n}(x_n)}\,|_{J=\frac{\delta\tilde{\Gamma}[0,V,\beta,\ldots]}{\delta V}} \,. \tag{47}$$

We now set $\beta = 0$. We can always set $\bar{J}[0, V, \beta]|_{\beta=0} = 0$ (by choosing $V = 0$).

By the arguments of Section 3

$$\left.\frac{\delta W[J, V]}{\delta V}\right|_{J=0} = 0 \tag{48}$$

and from eq.(44) this implies $\frac{\delta\tilde{\Gamma}}{\delta V}[0, V, \beta]|_{V=0=\beta} = 0$. Thus from eq.(47)

$$\left.\frac{\delta^{(n)} W[0, V, \beta, \ldots]}{\delta\beta_{i_1}(x_1) \ldots \delta\beta_{i_n}(x_n)}\right|_{\beta=0} = \left.\frac{\delta^{(n)} W_{\rm bg}[J, \beta, \ldots]}{\delta\beta_{i_1}(x_1) \ldots \delta\beta_{i_n}(x_n)}\right|_{J=0=\beta} \,, \tag{49}$$

for all values of V. This is the formal proof that on-shell physical connected Green functions generated by W can be obtained from the connected amplitudes generated by $W_{\rm bg}$.

6 Removal of doublets

This section deals with a rather technical but otherwise important issue connected with the introduction of the BRST transformation of $V_{a\mu}$. In the present approach $V_{a\mu}$ forms a "doublet" together with $\Omega_{a\mu}$.

The problem we want to discuss can be formulated in various ways. Roughly speaking one can ask whether the introduction of the new field Ω might modify the Slavnov-Taylor identities thus depriving the theory of the invariance necessary for the accomplishment of the renormalization procedure and, in the present case, of the essential tool for the proof of the background equivalence theorem.

Fortunately one can prove that the anomaly of the Slavnov-Taylor is not modified in a substantial way by the introduction of one or more doublets. The assumptions which guarantee the validity of this results will be put in evidence during the proof. The removal of the doublets is a standard procedure [16] and is reported here for completeness. Actually our formulation is a pure algebraic one and therefore no use of power-counting is made. Our proof has the virtue to emphasize that the removal of doublets (V, Ω) only requires a linear gauge-fixing term such that the full dependence of the classical action on the antifields A^* is through the combination:

$$A^*_{a\mu} + \alpha D_\mu(V)_{ab}\bar{\theta}_b \,. \tag{50}$$

In particular no further assumption on the gauge group structure is required.

Let us assume that the ST identities have been reestablished up to order $n-1$. By the Quantum Action Principle, the n-order ST breaking term $\Delta^{(n)} \equiv \mathcal{S}(\Gamma)^{(n)}$ is a local functional of the fields and the external sources and of their derivatives. Moreover it obeys the Wess-Zumino consistency condition

$$\mathcal{S}_0(\Delta^{(n)}) = 0 \,. \tag{51}$$

$\mathcal{S}_0$ denotes the linearized classical ST operator given by

$$\begin{aligned} \mathcal{S}_0 \;=\; & \int d^4x \left(D_\mu(A)_{ai}\theta_i \frac{\delta}{\delta A_{a\mu}} - \frac{1}{2} f_{aij}\theta_i\theta_j \frac{\delta}{\delta\theta_a} + B_a \frac{\delta}{\delta\bar{\theta}_a} + \Omega_{a\mu}\frac{\delta}{\delta V_{a\mu}} \right. \\ & \left. + \frac{\delta\Gamma^{(0)}}{\delta A_{a\mu}} \frac{\delta}{\delta A^*_{a\mu}} + \frac{\delta\Gamma^{(0)}}{\delta\theta_a}\frac{\delta}{\delta\theta^*_a} \right) . \end{aligned} \tag{52}$$

If the ST identities are not restored at lower orders, $\Delta^{(n)}$ turns out to be a non-local functional and the Wess-Zumino consistency condition is modified with respect to eq.(51) by non-local contributions [17].

In order to study the dependence of the cohomology of $\mathcal{S}_0$ on the doublets $(V_{a\mu}, \Omega_{a\mu})$ we perform the following change of variables. We replace the antifields $A^*_{a\mu}$ with

$$\hat{A}^*_{a\mu} \equiv A^*_{a\mu} + \alpha D_\mu(V)_{ab}\bar{\theta}_b \tag{53}$$

while we keep all other fields and antifields unchanged. This transformation is invertible. We introduce the operator $\mathcal{T}$ such that

$$\hat{X}(A^*_{a\mu}, \phi) \equiv \mathcal{T}X(A^*_{a\mu}, \phi) = X(A^*_{a\mu} + \alpha D_\mu(V)_{ab}\bar{\theta}_b, \phi) = X(\hat{A}^*_{a\mu}, \phi) \,.$$

Then we define $\hat{\mathcal{S}}_0 = \mathcal{T}\mathcal{S}_0\mathcal{T}^{-1}$, and we find

$$\begin{aligned} \hat{\mathcal{S}}_0 \;=\; & \int d^4x \left(D_\mu(A)_{ai}\theta_i \frac{\delta}{\delta A_{a\mu}} - \frac{1}{2} f_{aij}\theta_i\theta_j \frac{\delta}{\delta\theta_a} + B_a \frac{\delta}{\delta\bar{\theta}_a} + \Omega_{a\mu}\frac{\delta}{\delta V_{a\mu}} \right. \\ & \left. + \left[\frac{\delta\Gamma^{(0)}}{\delta A_{a\mu}} + \alpha\partial_\mu B_a + \alpha f_{aij}\Omega_{i\mu}\bar{\theta}_j + \alpha f_{aij}V_{i\mu}B_j \right] \frac{\delta}{\delta\hat{A}^*_{a\mu}} + \frac{\delta\Gamma^{(0)}}{\delta\theta_a}\frac{\delta}{\delta\theta^*_a} \right) . \end{aligned} \tag{54}$$

Notice that eq.(51) tells us that

$$\hat{\mathcal{S}}_0\mathcal{T}\Delta^{(n)} = \mathcal{T}\mathcal{S}_0\Delta^{(n)} = 0\,. \tag{55}$$

We introduce the operator

$$\mathcal{K} = \int_0^1 dt\ V_{\mu a}\ \lambda_t\ \frac{\delta}{\delta\Omega_{a\mu}} \tag{56}$$

where the action of λ_t on a generic functional $X(V,\Omega,\hat{A}^*,\phi)$ is given by

$$\lambda_t\Big(X(V,\Omega,\hat{A}^*,\phi)\Big) = X(tV,t\Omega,\hat{A}^*,\phi) \tag{57}$$

being ϕ any other field or source [4]. By explicit computation one verifies that

$$\begin{aligned}\{\hat{\mathcal{S}}_0,\mathcal{K}\} &= \int_0^1 dt\ \left(V\lambda_t\frac{\delta}{\delta V} + \Omega\lambda_t\frac{\delta}{\delta\Omega}\right) + \\ &\int_0^1 dt\ V_b\lambda_t\left(\frac{\delta^2\Gamma^{(0)}}{\delta A_a\delta\Omega_b} + \alpha f_{abc}\bar{\theta}_c\right)\frac{\delta}{\delta\hat{A}^*_a} + \int_0^1 dt\ V_b\lambda_t\frac{\delta^2\Gamma^{(0)}}{\delta\theta_a\delta\Omega_b}\frac{\delta}{\delta\theta^*_a}\,,\end{aligned} \tag{58}$$

and due to the fact that

$$\frac{\delta^2\Gamma^{(0)}}{\delta\theta_a\delta\Omega_b} = 0 \quad \text{and} \quad \frac{\delta^2\Gamma^{(0)}}{\delta A_a\delta\Omega_b} = -\alpha f_{abc}\bar{\theta}_c \tag{59}$$

we obtain

$$\{\hat{\mathcal{S}}_0,\mathcal{K}\} = \int_0^1 dt\ \left(V\lambda_t\frac{\delta}{\delta V} + \Omega\lambda_t\frac{\delta}{\delta\Omega}\right)\,. \tag{60}$$

When applied to the functional $\Delta^{(n)}(V,\Omega,\hat{A}^*,\phi)$ the last equation gives:[5]

$$\{\hat{\mathcal{S}}_0,\mathcal{K}\}\Delta^{(n)}(V,\Omega,\hat{A}^*,\phi) \;=\; \Delta^{(n)}(V,\Omega,\hat{A}^*,\phi) - \Delta^{(n)}(0,0,\hat{A}^*,\phi)\,. \tag{61}$$

Then

$$\begin{aligned}\Delta^{(n)}(V,\Omega,\hat{A}^*,\phi) &= \Delta^{(n)}(0,0,\hat{A}^*,\phi) + \hat{\mathcal{S}}_0\mathcal{K}\Delta^{(n)}(V,\Omega,\hat{A}^*,\phi) \\ &\quad + \mathcal{K}\hat{\mathcal{S}}_0\Delta^{(n)}(V,\Omega,\hat{A}^*,\phi)\end{aligned} \tag{62}$$

The last term is vanishing according to eq.(55).

By applying $\mathcal{T}^{-1}$ to eq.(62) we obtain:

$$\Delta^{(n)}(V,\Omega,A^*,\phi) \;=\; \Delta^{(n)}(0,0,A^*_{a\mu} + \alpha D_\mu(V)_{ab}\bar{\theta}_b,\phi) + \mathcal{S}_0[\mathcal{T}^{-1}\mathcal{K}\mathcal{T}\Delta^{(n)}] \tag{63}$$

The dependence on $\Omega_{a\mu}$ is confined to the cohomologically trivial term

$$\mathcal{S}_0[\mathcal{T}^{-1}\mathcal{K}\mathcal{T}\Delta^{(n)}]\,,$$

[4]Notice that $X(tV,t\Omega,\hat{A}^*,\phi) \neq \hat{X}(tV,t\Omega,A^*,\phi)$, i.e. $[\lambda_t,\mathcal{T}] \neq 0$.

[5]Notice that $X(0,0,\hat{A}^*,\phi) \neq \hat{X}(0,0,A^*,\phi)$.

and $\Delta^{(n)}$ depends non-trivially on $V_{a\mu}$ only through the dependence on $\hat{A}^*_{a\mu}$.

By the same technique we show that the non-trivial dependence of $\Delta^{(n)}$ on $\bar{\theta}$ is only through the dependence on $\hat{A}^*_{a\mu}$. Moreover, there is no non-trivial dependence on B_a.

For this purpose we introduce the new homotopy

$$\mathcal{K}_B = \int_0^1 dt\ \bar{\theta}_a\ \lambda_t\ \frac{\delta}{\delta B_a} \tag{64}$$

where now λ_t refers to the doublet $(\bar{\theta}_a, B_a)$:

$$\lambda_t X(\bar{\theta}, B, \hat{A}^*, \phi) = X(t\bar{\theta}, tB, \hat{A}^*, \phi) \tag{65}$$

being ϕ any other field or source. A computation analogous to eq.(58) now yields:

$$\begin{aligned}\{\hat{\mathcal{S}}_0, \mathcal{K}_B\} \ &= \ \int_0^1 dt\ \left(\bar{\theta}\lambda_t \frac{\delta}{\delta\bar{\theta}} + B\lambda_t \frac{\delta}{\delta B}\right) + \\ &\quad \int_0^1 dt\ \bar{\theta}_b \lambda_t \left(\frac{\delta^2\Gamma^{(0)}}{\delta A_a \delta B_b} + \alpha f_{abc} V_c\right) \frac{\delta}{\delta \hat{A}^*_a} + \int_0^1 dt\ \bar{\theta}_b \lambda_t \frac{\delta^2\Gamma^{(0)}}{\delta\theta_a \delta B_b} \frac{\delta}{\delta\theta^*_a}\,,\end{aligned} \tag{66}$$

and due to the fact that

$$\frac{\delta^2\Gamma^{(0)}}{\delta\theta_a\delta B_b} = 0 \quad \text{and} \quad \frac{\delta^2\Gamma^{(0)}}{\delta A_a \delta B_b} = -\alpha f_{abc} V_c \tag{67}$$

we obtain

$$\{\hat{\mathcal{S}}_0, \mathcal{K}_B\} = \int_0^1 dt\ \left(\bar{\theta}\lambda_t \frac{\delta}{\delta\bar{\theta}} + B\lambda_t \frac{\delta}{\delta B}\right)\,. \tag{68}$$

When applied to the functional $\Delta^{(n)}(V, \Omega, \hat{A}^*, \phi)$ the last equation gives:

$$\{\hat{\mathcal{S}}_0, \mathcal{K}_B\}\Delta^{(n)}(V, \Omega, \hat{A}^*, \phi) \ = \ \Delta^{(n)}(\bar{\theta}, B, \hat{A}^*, \phi) - \Delta^{(n)}(0, 0, \hat{A}^*, \phi)\,, \tag{69}$$

and we get

$$\begin{aligned}\Delta^{(n)}(\bar{\theta}, B, \hat{A}^*, \phi) \ &= \ \Delta^{(n)}(0, 0, \hat{A}^*, \phi) + \hat{\mathcal{S}}_0\mathcal{K}_B\Delta^{(n)} + \mathcal{K}_B\hat{\mathcal{S}}_0\Delta^{(n)} \\ &= \ \Delta^{(n)}(0, 0, \hat{A}^*, \phi) + \hat{\mathcal{S}}_0\mathcal{K}_B\Delta^{(n)}\,,\end{aligned} \tag{70}$$

which gives the announced result.

Notice that the same dependence on $B, \bar{\theta}$ and A^* for $\Delta^{(n)}$ is obtained if one requires that the quantum effective action Γ fulfills the ghost equations:

$$G_a(\Gamma) \equiv \left(\frac{\delta}{\delta\bar{\theta}_a} - \alpha D_\mu(V)_{ab} \frac{\delta}{\delta A^*_{b\mu}}\right)\Gamma = (\alpha' - \alpha)(\partial^\mu\Omega_{a\mu}) - \alpha f_{abc} A^\mu_b \Omega_{c\mu}\,. \tag{71}$$

Eq.(71) implies for $n \geq 1$ that

$$G_a(\Gamma^{(n)}) = 0\,. \tag{72}$$

Moreover, by explicit computation one gets:

$$
\begin{aligned}
\{\mathcal{S}_0, G_a\} &= -\alpha f_{aij}\Omega^\mu_i \frac{\delta}{\delta A^*_{j\mu}} + \int d^4x \left(\left(G_a \Gamma^{(0)}_{A^\mu_i} \right) \frac{\delta}{\delta A^*_{i\mu}} + \left(G_a \Gamma^{(0)}_{\theta_i} \right) \frac{\delta}{\delta \theta^*_i} \right) \\
&= -\alpha f_{aij}\Omega^\mu_i \frac{\delta}{\delta A^*_{j\mu}} + \int d^4x \left(\frac{\delta}{\delta A^\mu_i} \left(G_a \Gamma^{(0)} \right) \frac{\delta}{\delta A^{*\mu}_i} - \frac{\delta}{\delta \theta_i} \left(G_a \Gamma^{(0)} \right) \frac{\delta}{\delta \theta^*_i} \right) \\
&\quad + \int d^4x \left(\left[G_a, \frac{\delta}{\delta A^\mu_i} \right] \Gamma^{(0)} \frac{\delta}{\delta A^*_{i\mu}} + \left\{ G_a, \frac{\delta}{\delta \theta_i} \right\} \Gamma^{(0)} \frac{\delta}{\delta \theta^*_i} \right) \\
&= -\alpha f_{aij}\Omega^\mu_i \frac{\delta}{\delta A^*_{j\mu}} + \alpha f_{aij}\Omega^\mu_i \frac{\delta}{\delta A^*_{j\mu}} = 0 \,, \qquad (73)
\end{aligned}
$$

since

$$
\left[G_a, \frac{\delta}{\delta A^\mu_i} \right] = \left\{ G_a, \frac{\delta}{\delta \theta_i} \right\} = 0 \,. \qquad (74)
$$

Then $G_a(\Gamma^{(n)}) = 0$ implies $G_a(\Delta^{(n)}) = 0$. Notice that the converse is not true: $\Delta^{(n)}$ is not modified if one adds to $\Gamma^{(n)}$ an action-like term of the form

$$
\Xi^{(n)} \equiv \mathcal{S}_0 \left(\int d^4x \, \bar\theta_a H^a \right) ,
$$

where H^a is a FP-charge 0 Lorentz-invariant polynomial in the fields and the external sources of the model with dimension ≤ 2. However, $\Xi^{(n)}$ could spoil the ghost equations at order n (take for example $H_a = \partial_\mu V^\mu_a$).

One is thus led to study the following consistency condition

$$
S_0 \Delta^{(n)}(A^*_{a\mu} + \alpha D_\mu(V)_{ab}\bar\theta_b, A_{a\mu}, \theta_a, \theta^*_a) = 0 \,. \qquad (75)
$$

This can be explicitly recast as:

$$
\begin{aligned}
0 &= \int d^4x \left(D_\mu(A)_{ai}\theta_i \frac{\delta}{\delta A_{a\mu}} - \frac{1}{2} f_{aij}\theta_i\theta_j \frac{\delta}{\delta \theta_a} + B_a \frac{\delta}{\delta \bar\theta_a} + \Omega_{a\mu} \frac{\delta}{\delta V_{a\mu}} \right. \\
&\quad \left. + \frac{\delta \Gamma^{(0)}}{\delta A_{a\mu}} \frac{\delta}{\delta A^*_{a\mu}} + \frac{\delta \Gamma^{(0)}}{\delta \theta_a} \frac{\delta}{\delta \theta^*_a} \right) \Delta^{(n)}(A^*_{a\mu} + \alpha D_\mu(V)_{ab}\bar\theta_b) \\
&= \int d^4x \left(\left[\frac{\delta \Gamma^{(0)}}{\delta A_{a\mu}} + \alpha \partial_\mu B_a + \alpha f_{aij}\Omega_{i\mu}\bar\theta_j + \alpha f_{aij} V_{i\mu} B_j \right] \frac{\delta}{\delta \hat A^*_{a\mu}} \right. \\
&\quad \left. + D_\mu(A)_{ai}\theta_i \frac{\delta}{\delta A_{a\mu}} - \frac{1}{2} f_{aij}\theta_i\theta_j \frac{\delta}{\delta \theta_a} + \frac{\delta \Gamma^{(0)}}{\delta \theta_a} \frac{\delta}{\delta \theta^*_a} \right) \Delta^{(n)}(\hat A^*_{a\mu}) \qquad (76) \\
&= \int d^4x \left(\left[\frac{\delta}{\delta A_{a\mu}} \Gamma^{(0)}(\hat A^*_{a\mu}, A_{a\mu}, \theta_a, \theta^*_a)|_{B=0,\Omega=0} \right] \frac{\delta}{\delta \hat A^*_{a\mu}} \right. \\
&\quad \left. + D_\mu(A)_{ai}\theta_i \frac{\delta}{\delta A_{a\mu}} - \frac{1}{2} f_{aij}\theta_i\theta_j \frac{\delta}{\delta \theta_a} + \frac{\delta \Gamma^{(0)}}{\delta \theta_a} \frac{\delta}{\delta \theta^*_a} \right) \Delta^{(n)}(\hat A^*_{a\mu}) \,. \qquad (77)
\end{aligned}
$$

In the second line we have used the fact that the dependence of $\Delta^{(n)}$ on $V_{a\mu}, \bar\theta_a$ is only through $\hat A_{a\mu}$, while in the last line we have used the fact that the term in the square bracket is actually independent of B_a and Ω_a. Also notice that, as a consequence of eq.(71), $\Gamma^{(0)}$ evaluated at $B = 0, \Omega = 0$ depends on $V_{a\mu}, \bar{\theta}_a$ through the antifield $\hat{A}^*_{a\mu}$ only.

The solution $\Delta^{(n)}(\hat{A}^*_{a\mu}, A_{a\mu}, \theta_a, \theta^*_a)$ to eq.(77), evaluated at

$$\hat{A}^*_{a\mu} = A^*_{a\mu} + \alpha D_\mu(V = 0)_{ab}\bar{\theta}_b \,,$$

is the solution to the Wess-Zumino consistency condition of the original theory without the background fields [15]. We obtain the anomalous functional for the theory where $V \neq 0$ by evaluating $\Delta^{(n)}(\hat{A}^*_{a\mu}, A_{a\mu}, \theta_a, \theta^*_a)$ at $\hat{A}^*_{a\mu} = A^*_{a\mu} + \alpha D_\mu(V)_{ab}\bar{\theta}_b$.

Note that if one considers the solution to eq.(77) belonging to a subspace of the local functionals in $(\hat{A}^*_{a\mu}, A_{a\mu}, \theta_a, \theta^*_a)$ of a given dimension, one can revert to the anomalous functional for the original theory depending on $(A^*_{a\mu}, V, \bar{\theta}, A_{a\mu}, \theta_a, \theta^*_a)$ of the same dimension by applying the transformation in eq.(53), since this transformation preserves power-counting. This means that in this case the algebraic procedure is consistent with the use of power-counting arguments.

7 Conclusions

The background field method quantization turns out to be a very powerful tool in deriving physical predictions of gauge theories.

In the background field gauge both the connected amplitude functional and 1PI vertex function satisfy STI associated to the BRST invariance and Ward identities associated to the background gauge transformations. This can be used to prove the validity of the formal change of variables on the gauge fields and proves the independence of the physical amplitude from the background field. The whole procedure remains valid even if one introduces an extra gauge fixing term $(-\alpha' s\,\left[\bar{\theta}_a \partial^\mu V_{a\mu}\right])$ right at the beginning.

Part of the background effective action for $\tilde{A} \neq 0$ has to be computed if one wants to go beyond the 1-loop approximation. The renormalization of these amplitudes must be performed by requiring the validity of the ST identities [18]. On the other side the ST identities have been the essential tool in the proof of the background equivalence theorem (see Ref. [3] and Section 4).

The Legendre transform for the background effective action can be constructed only in presence of the extra gauge fixing term and it is shown to provide the same physical amplitude as the original one. However it can not be in general associated to a field theory since there is no match between the Feynman rules for vertices inside the 1PI amplitudes and the vertices connected by the linking propagators.

The present approach has the virtue to allow a complete proof of the background equivalence theorem for all physical amplitudes, including any expectation value of quasi-local observable operator (therefore BRST invariant objects) and in particular the physical S-matrix elements of any gauge theory.

We finally would like to remark that the present approach further clarifies the rôle of the gauge-fixing condition in perturbative quantum field theory.

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References

[1] B. S. DeWitt, Phys. Rev. **162** (1967) 1195; G. 't Hooft, *Acta Universitatis Wratislavensis* **368** (1976) 345; H. Kluberg-Stern and J. Zuber, Phys. Rev. **D12** (1975) 482 and 3159; D. G. Boulware, Phys. Rev. **D23** (1981) 389; C. F. Hart, Phys. Rev. **D28** (1983) 1993.

[2] L.F. Abbott, Nucl. Phys. **B185** (1981) 189.

[3] L.F. Abbott, M.T. Grisaru and R.K. Schaefer. Nucl. Phys. **B229** (1983) 372.

[4] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, "Superspace Or One Thousand And One Lessons In Supersymmetry," *Reading, Usa: Benjamin/Cummings (1983) 548 P. (Frontiers In Physics, 58)* and references therein.

[5] S. Ichinose and M. Omote, Nucl. Phys. **B203** (1982) 221; D. M. Capper and A. MacLean, Nucl. Phys. **B203** (1982) 413.

[6] A. Denner, G. Weiglein and S. Dittmaier, Nucl. Phys. **B440** (1995) 95 [hep-ph/9410338]; A. Denner, S. Dittmaier and G. Weiglein, Acta Phys. Polon. **B27** (1996) 3645 [hep-ph/9609422].

[7] P. A. Grassi, Nucl. Phys. **B560** (1999) 499 [hep-th/9908188].

[8] C. Becchi, Lezioni tenute a Parma (2000).

[9] P. A. Grassi, Nucl. Phys. **B462** (1996) 524 [hep-th/9505101].

[10] P. A. Grassi, T. Hurth and M. Steinhauser, "Practical Algebraic Renormalization", hep-ph/9907426.

[11] P. A. Grassi, Nucl. Phys. **B537** (1999) 527 [hep-th/9804013].

[12] C. Becchi and R. Collina, Nucl. Phys. **B562** (1999) 412 [hep-th/9907092].

[13] R. Ferrari and P.A. Grassi, Phys. Rev. **D60**, 65010 (1999) [hep-th/9807191].

[14] R. Ferrari, P.A. Grassi and A. Quadri, Phys. Lett. **B472**, 346-356 (2000) [hep-th/9905192].

[15] O. Piguet and S. P. Sorella, "Algebraic renormalization: Perturbative renormalization, symmetries and anomalies," *Berlin, Germany: Springer (1995) (Lecture notes in physics: m28).*

[16] B. Zumino, UCB-PTH-83/16 *Lectures given at Les Houches Summer School on Theoretical Physics, Les Houches, France, Aug 8 - Sep 2, 1983.*

[17] M. Picariello and A. Quadri, Phys. Lett. **B**497:91-100,2001 [hep-th/0001174].

[18] R. Stora, "Remarks on perturbative renormalization in background gauges", Desy 4-5 Nov 1999, unpublished.