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Holography in Radiation-dominated Universe with a Positive Cosmological Constant

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Abstract

We discuss the holographic principle in a radiation-dominated, closed Friedmann-Robertson-Walker (FRW) universe with a positive cosmological constant. By introducing a cosmological D-bound on entropy of matter in the universe, we can write the Friedmann equation governing evolution of the universe in the form of Cardy formula. When the cosmological D-bound is saturated, the Friedmann equation coincides with the Cardy-Verlinde formula describing entropy of radiation in the universe. As a concrete model, we consider the brane cosmology in the background of Schwarzschild-de Sitter black holes. It is found that the cosmological D-bound is saturated when the brane crosses the black hole horizon of the background. At that moment, the Friedmann equation coincides with the Cardy-Verlinde formula describing the entropy of radiation matter on the brane.

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1 Introduction

The holographic principle is perhaps one of fundamental principles of nature, which relates a theory with gravity in D dimensions to a theory without gravity in lower dimensions [1]. Although we do not yet completely understand how the hologram of gravity is realized, some beautiful examples as the realization of the principle have been found, through the AdS/CFT correspondence [2].

In a seminar paper [3], E. Verlinde found a quite interesting holographic relation between the Friedmann equation describing a radiation-dominated, closed Friedmann-Robertson-Walker (FRW) universe and Cardy formula [4] describing the entropy of matter filling the universe. The radiation can be represented by a conformal field theory (CFT) with a large central charge, while the entropy for the latter can be expressed in terms of the so-called Cardy-Verlinde formula [3], a generalized form of Cardy formula in any dimension. The Cardy-Verlinde formula was proved to hold at least for CFTs with AdS gravity dual, for instance, for CFTs dual to Schwarzschild-AdS black holes [3], Kerr-AdS black holes [5], hyperbolic AdS black holes and charged AdS black holes [6], Taub-Bolt AdS instanton solutions [7], and Kerr-Newman-AdS black holes [8]. Verlinde found that the Friedmann equation can be rewritten in the form of Cardy-Verlinde formula with the help of three cosmological entropy bounds, and that when the Hubble entropy bound is saturated, the Friedmann equation coincides with the Cardy-Verlinde formula. This observation is very interesting in the sense that the Friedmann equation is a dynamic one describing geometry evolution of the universe while the Cardy-Verlinde formula is just a formula describing the number of degrees of freedom of matter in the universe. Therefore, the Verlinde's observation is indeed an interesting manifestation of holographic principle in the cosmological setting.

By considering a moving brane universe in the background of Schwarzschild-AdS black holes in arbitrary dimensions, the holographic connection between the geometry and matter can be realized. A radiation-dominated closed FRW universe appears as an induced metric on the brane embedded in the bulk background. Savonije and Verlinde [9] interpreted the radiation as the thermal CFT dual to the bulk Schwarzschild-AdS black hole. Further they observed that when the brane crosses the bulk black hole horizon, (i) the entropy and temperature of the universe can be simply expressed in terms of the Hubble parameter and its time derivative; (ii) the entropy formula (Cardy-Verlinde formula) of CFTs in any dimension coincides with the Friedmann equation; and (iii) the Hubble entropy bound is just saturated by the entropy of bulk black holes.

Since then a lot of studies have been done focusing on the generalization of [3, 9] to various bulk geometries. In this paper, we are interested in the case where a positive cosmological constant is present in the universe. Namely we will discuss the holography for a radiation-dominated closed FRW universe with a positive cosmological constant. This is motivated partially by the de Sitter (dS)/CFT correspondence [10], and by recent astronomy observations on supernova indicating that our universe is accelerating [11], which can be interpreted that there might be a positive cosmological constant in our universe. We will start with a brief review on the case without the cosmological constant [3] in the next section, and then discuss the holography in the case with a positive cosmological constant by introducing a cosmological D-bound on entropy of matter in the universe. As an example, in Sec. 3 we investigate the holographic connection between the Friedmann equation and Cardy-Verlinde formula in the brane cosmology in the background of Schwarzschild-de Sitter black holes. We end this paper in Sec. 4 with some conclusions and discussions.

2 Holography in a radiation-dominated closed universe with a positive cosmological constant

Let us consider a $(n+1)$-dimensional closed FRW universe

$$ds^2 = -d\tau^2 + R^2 d\Omega_n^2, \tag{2.1}$$

where R is the scale factor of the universe and $d\Omega_n^2$ denotes the line element of a n-dimensional unit sphere. The evolution of the universe is determined by the FRW equations

$$\begin{aligned} H^2 &= \frac{16\pi G_{n+1}}{n(n-1)}\frac{E}{V} - \frac{1}{R^2} + \frac{1}{l_{n+1}^2}, \\ \dot{H} &= -\frac{8\pi G_{n+1}}{n-1}\left(\frac{E}{V} + p\right) + \frac{1}{R^2}, \end{aligned} \tag{2.2}$$

where H represents the Hubble parameter with the definition $H = \dot{R}/R$ and the overdot stands for derivative with respect to the cosmic time τ, E is the energy of matter filling the universe, p is the pressure, V is the volume of the universe, $V = R^n\Omega_n$ with Ω_n being the volume of a n-dimensional unit sphere, and G_{n+1} is the Newton constant in $(n+1)$ dimensions. In addition, l_{n+1}^2 is related to the cosmological constant $\Lambda_{n+1} = n(n-1)/2l_{n+1}^2$ in $(n+1)$ dimensions.

2.1 The case without cosmological constant

In [3] Verlinde introduced three entropy bounds[1] :

$$\begin{aligned} \text{Bekenstein} - \text{Verlinde bound}: \quad & S_{\rm BV} = \frac{2\pi}{n} ER \\ \text{Bekenstein} - \text{Hawking bound}: \quad & S_{\rm BH} = (n-1)\frac{V}{4G_{n+1}R} \\ \text{Hubble bound}: \quad & S_{\rm H} = (n-1)\frac{HV}{4G_{n+1}}. \end{aligned} \tag{2.3}$$

The Bekenstein-Verlinde bound is supposed to hold for a weakly self-gravitating universe ($HR \leq 1$), while the Hubble entropy bound works when the universe is in the strongly self-gravitating phase ($HR \geq 1$). In the case without the cosmological constant, the Friedmann equation [the first equation in (2.2)] can be rewritten as

$$S_{\rm H} = \sqrt{S_{\rm BH}(2S_{\rm BV} - S_{\rm BH})}, \tag{2.4}$$

in terms of three entropy bounds. Expression (2.4) is similar to the Cardy formula [4], an entropy formula of CFTs in two dimensions. It is interesting to note that when $HR = 1$, one has $S_{\rm BV} = S_{\rm BH} = S_{\rm H}$.

Let us define a quantity $E_{\rm BH}$ which corresponds to energy needed to form a black hole with size of the whole universe through the relation: $S_{\rm BH} = (n-1)V/4G_{n+1}R \equiv 2\pi ER/n$. With this quantity, the Friedmann equation (2.4) can be further cast to

$$S_{\rm H} = \frac{2\pi R}{n}\sqrt{E_{\rm BH}(2E - E_{\rm BH})}, \tag{2.5}$$

which takes the same form as the Cardy-Verlinde formula [3]

$$S = \frac{2\pi R}{n}\sqrt{E_c(2E - E_c)}. \tag{2.6}$$

This formula is supposed to describe the entropy S of a CFT living on a n-dimensional sphere with radius R. Here E is the total energy of the CFT and E_c stands for the Casimir energy of the system, the non-extensive part of the total energy. Now we suppose that the entropy of the radiation matter in the FRW universe can be described by the Cardy-Verlinde formula. Comparing (2.5) with (2.6), one can easily see that if $E_{\rm BH} = E_c$, $S_{\rm H}$ and S must be equal. In other words, the Hubble entropy bound is saturated by the

[1] In [3] the first bound is called the Bekenstein bound. In fact this bound is slightly different from the original Bekenstein bound [12] by a numerical factor $1/n$. So we call this the Bekenstein-Verlinde bound. This bound could be viewed as the counterpart of the Bekenstein bound in the cosmological setting [13].

entropy of radiation matter in the universe if the Casimir energy E_c is just enough to form a black hole with the size of the universe. At that moment, equations (2.5) and (2.6) coincide with each other. This implies that the Friedmann equation somehow knows the entropy formula of radiation-matter filling the universe [3]. Considering a moving brane universe in the background of Schwarzschild-AdS black holes, Savonije and Verlinde [9] found that when the brane crosses the black hole horizon, the Hubble entropy bound is saturated by the entropy of black holes in the bulk.

In (2.6) the Casimir energy E_c is defined as [3]

$$E_c = n(E + pV - TS), \tag{2.7}$$

where T stands for the temperature of the thermal CFT. Further Verlinde found that except for the similarity between the Friedmann equation (2.5) and the Cardy-Verlinde formula (2.6), there is also a similarity between the second equation in (2.2) concerning the time derivative of Hubble parameter and the equation (2.7) about the Casimir energy of CFTs. Let us define a (limiting) temperature

$$T_{\rm H} = -\frac{\dot{H}}{2\pi H}. \tag{2.8}$$

Here the minus sign is necessary to get a positive result. In addition, it is assumed that we are in the strongly self-gravitating phase with $HR \geq 1$ so that $H \neq 0$ and $T_{\rm H}$ is well-defined. With this temperature, the second equation in (2.2) can be rewritten as

$$E_{\rm BH} = n(E + pV - T_{\rm H} S_{\rm H}). \tag{2.9}$$

Thus we see that when the Hubble bound $S_{\rm H}$ is saturated by the matter entropy S, the (limiting) temperature $T_{\rm H}$ equals to the thermodynamic temperature T of matter filling the universe. Note that like the Hubble entropy bound, the (limiting) temperature $T_{\rm H}$ is a geometric quantity determined by the Hubble parameter and its time derivative.

2.2 The case with a positive cosmological constant

Now we turn to the case with a nonvanishing cosmological constant, and generalize those interesting observations to a radiation-dominated closed FRW universe with a positive cosmological constant. We will argue that three cosmological entropy bounds in (2.3) keep the same forms even though a cosmological constant is present.

To write down a formula like (2.4), let us first discuss three entropy bounds in (2.3). The Bekenstein-Verlinde bound $S_{\rm BV}$ is the counterpart of the Bekenstein entropy bound [12] in the cosmological setting [13]. It is supposed to hold for systems with limited self-gravity, which means that the gravitational self-energy of the system is small compared to the total energy E. Namely, the gravitational effect on the bound can be neglected. Therefore this bound is independent of gravity theories. It is also independent of whether the gravity theory under consideration includes or not a cosmological constant. In other words, the form of the Bekenstein-Verlinde bound should keep unchanged in any gravitational theory[2]. Hence even when a positive cosmological constant is present, the Bekenstein-Verlinde bound still takes the form in (2.3). As a result, for a radiation-dominated FRW universe with a cosmological constant, the Bekenstein-Verlinde bound is a constant because of $E \sim R^{-1}$. Thus once this bound is satisfied at one time, it will be always satisfied at all times if the entropy S of matter does not change.

As for the Bekenstein-Hawking bound in (2.3), it can be viewed as the holographic Bekenstein-Hawking entropy of a black hole with the size of the universe [3]. Indeed, it varies like an area instead of the volume. And for a closed universe it is the closest one that can lead to the usual area formula of black hole entropy $A/4G$. We know from the thermodynamics of black holes that in Einstein gravity the entropy of a black hole is always proportional to its horizon area in spite of whether the gravitational theory includes or not a cosmological constant [15]. Further, as argued by Verlinde [3], the role of $S_{\rm BH}$ is not to serve as a bound on the total entropy, but rather on a sub-extensive component of the entropy that is associated with the Casimir energy of CFTs. The above leads to the conclusion that the Bekenstein-Hawking bound should remain unchanged in its form and implication as in the case without the cosmological constant.

Finally we consider the Hubble entropy bound, which is an entropy bound for matter in a strongly self-gravitating universe ($HR \geq 1$). In such a strongly self-gravitating universe, black holes might occur. As argued in [16, 17], the maximal entropy inside the universe is produced by black holes with size of the Hubble horizon. The usual holographic argument shows that the total entropy should be less than or equal to the Bekenstein-Hawking entropy of a Hubble-horizon-sized black hole times the number of Hubble regions in the universe. In [3], by using a local holographic bound due to Fischler and Susskind [16] and Bousso [18], see also [19], Verlinde "derived" the Hubble entropy bound in (2.3). It is worth noting that in the "derivation" of the Hubble bound, Verlinde used mainly the idea that the entropy flow S through a contracting light sheet is less than or equal to $A/4G$, where A is the area of the surface from which the light sheet originates. Hence we insist that the cosmological constant will not affect the form of the Hubble

[2]In [14] we show that the Bekenstein entropy bound always has the form $S_{\rm B} = 2\pi ER$ independent of gravity theories by applying a Geroch process to an arbitrary black hole.

bound (see also [14]). This conclusion is based on the fact that even if a cosmological constant is present, it will not occurs explicitly in the "derivation" of Hubble entropy bound.

We conclude that in a closed FRW universe with a cosmological constant, three bounds introduced in (2.3) are still applicable. That is, their forms and implications keep unchanged even if the cosmological constant is present. However, we see that the cosmological constant indeed affects the evolution of the universe. Is there a similar relation between the Friedmann equation in (2.2) and the Cardy-Verlinde formula (2.6) as in the case without the cosmological constant? Our key observation is that the positive cosmological constant provides an additional entropy measure. When the cosmological constant occurs, not the Hubble bound, but a new cosmological D-bound plays the role as the Hubble bound does in the case without the cosmological constant.

Let us go to the details. We know that in a de Sitter universe, there is a cosmological horizon for an inertial observer. Like a black hole horizon, the cosmological horizon has a Hawking temperature and an associated entropy [20]. The entropy is proportional to the area of the cosmological horizon. It is a geometric quantity although it has a statistical origin in quantum gravity. In an asymptotically de Sitter space, the cosmological horizon shrinks. According to the second law of thermodynamics, *entropy of matter in de Sitter space* is bounded by the difference (D) between the entropy of de Sitter space and that of the asymptotically de Sitter space:

$$S_{\rm m} \leq \frac{1}{4G}\left(A_0 - A\right), \tag{2.10}$$

where A_0 and A are areas of cosmological horizons for de Sitter and asymptotically de Sitter spaces, respectively. This is the so-called D-bound proposed by Bousso in [21]. The D-bound is closely related to the Bekenstein bound which applies in flat backgrounds [21, 13].

In our present context, the occurrence of the cosmological constant does not guarantee that the universe approaches to a de Sitter phase. Like the case without the cosmological constant, in general the universe starts from a big bang, reaches a maximal radius and then recollapses with a big crunch. From (2.2), however, we see that for an empty flat universe[3], the Hubble radius is just the cosmological horizon size l_{n+1} of de Sitter space. It implies that the cosmological constant provides a new entropy measure in the universe. By analogy with the Hubble entropy bound [16, 17, 3], we define a quantity

$$S_\Lambda = (n-1)\frac{V}{4G_{n+1}l_{n+1}}. \tag{2.11}$$

[3] In that case, $E = p = 0$, and the term $1/R^2$ will be also absent.

which is the entropy of a de Sitter horizon times the number of the regions with the size of the de Sitter horizon in the universe. Like the Hubble entropy bound, it is also a geometric quantity. Together with the three entropy bounds in (2.3), the Friedmann equation in (2.2) can be rewritten as

$$S_{\rm H}^2 - S_{\Lambda}^2 = S_{\rm BH}(2S_{\rm BV} - S_{\rm BH}). \tag{2.12}$$

Further we note that the cosmological horizon in the asymptotically de Sitter spaces is always less that of corresponding de Sitter spaces, but one can see from (2.2) that the Hubble radius H^{-1} is not always less than the cosmological horizon l_{n+1} of de Sitter spaces. As a result, the left-hand side of equation (2.12) is not always positive. In addition, we stress that in the case without the cosmological constant, the Hubble bound in (2.5) is a geometry quantity, which gives an entropy bound of matter in the universe when the universe is in the strongly self-gravitating phase. Considering the D-bound (2.10) of matter in de Sitter spaces, we can define a cosmological entropy bound in the universe with a positive cosmological constant[4]

$$S_{\rm D} = \sqrt{|S_{\rm H}^2 - S_{\Lambda}^2|}. \tag{2.13}$$

We call it the cosmological D-bound, which can be viewed as the counterpart of the D-bound in the cosmology setting. Note that the cosmological D-bound is a square root of the difference between two geometric quantity squares, while the D-bound in de Sitter spaces is the difference between two geometric quantities.

On the analogy of the (limiting) temperature $T_{\rm H}$ in (2.8), we further define a new geometric temperature in our case

$$T_{\rm D} = -\frac{\dot{H}}{2\pi\sqrt{|1/l_{n+1}^2 - H^2|}}. \tag{2.14}$$

Note that this is also a geometric quantity like $T_{\rm H}$ for the case without the cosmological constant. With this, the second equation in (2.2) can be expressed as

$$E_{\rm BH} = n(E + pV - T_{\rm D}S_{\rm D}). \tag{2.15}$$

Here the definition of $E_{\rm BH}$ is the same as the one in (2.5).

Now we turn to the Cardy-Verlinde formula (2.6). In the form (2.6) it is implicitly assumed that one has $2E - E_c \geq 0$ for any CFT. In fact, in some circumstances, this

[4]Since a black hole larger than the cosmological horizon cannot form, one therefore should have $S_{\rm H} \geq S_{\Lambda}$. As a result, if $S_{\rm H} < S_{\Lambda}$, a cosmological singularity might occur during the evolution of the universe.

condition does not get satisfied. For example, in the CFT description dual to the thermodynamics of Schwarzschild-de Sitter black hole horizon, this quantity is negative [22]. In that case, the Cardy-Verlinde formula should be changed to

$$S = \frac{2\pi R}{n}\sqrt{E_c(E_c - 2E)}, \tag{2.16}$$

where the definition of E_c is still the same as the one (2.7).

In summary, for a radiation-dominated closed FRW universe with a positive cosmological constant the dynamic equations can be rewritten as

$$\begin{aligned} S_{\rm D} &= \frac{2\pi R}{n}\sqrt{E_{\rm BH}(2E - E_{\rm BH})}, \\ E_{\rm BH} &= n(E + pV - T_{\rm D}S_{\rm D}), \end{aligned} \tag{2.17}$$

when $S_{\rm H} \geq S_\Lambda$, while the entropy of the radiation can be expressed as

$$\begin{aligned} S &= \frac{2\pi R}{n}\sqrt{E_c(2E - E_c)}, \\ E_c &= n(E + pV - TS). \end{aligned} \tag{2.18}$$

On the other hand, when $S_{\rm H} \leq S_\Lambda$, the dynamic equations can be rewritten as

$$\begin{aligned} S_{\rm D} &= \frac{2\pi R}{n}\sqrt{E_{\rm BH}(E_{\rm BH} - 2E)}, \\ E_{\rm BH} &= n(E + pV - T_{\rm D}S_{\rm D}), \end{aligned} \tag{2.19}$$

and the entropy expressions are

$$\begin{aligned} S &= \frac{2\pi R}{n}\sqrt{E_c(E_c - 2E)}, \\ E_c &= n(E + pV - TS). \end{aligned} \tag{2.20}$$

When the cosmological D-bound is saturated by the entropy S of radiation matter, both sets of equations (2.17) [or (2.19)] and (2.18) [or (2.20)] coincide with each other, just like the case without the cosmological constant. Further, as the D-bound in (2.10) does, the cosmological D-bound in (2.13) provides an entropy bound for matter filling the universe when the universe is in the strongly self-gravitating phase ($HR \geq 1$)[5]. Namely, the cosmological D-bound plays the same role as the Hubble bound does in the case without cosmological constant. Furthermore we note from (2.2) that one has $S_{\rm D} = S_{\rm BV} = S_{\rm BH}$ when $HR = 1$.

[5]When the universe is in the weakly self-gravitating phase ($HR \leq 1$), as argued in the above, the Bekenstein-Verlinde bound still works well.

3 Brane cosmology in the background of Schwarzschild-de Sitter black holes

3.1 Thermodynamics of Schwarzschild-de Sitter black holes

Consider a $(n+2)$-dimensional Schwarzschild-de Sitter black hole, whose line element is

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\Omega_n^2. \tag{3.1}$$

Here

$$f(r) = 1 - \frac{\omega_n M}{r^{n-1}} - \frac{r^2}{l_{n+2}^2}, \quad \omega_n = \frac{16\pi G_{n+2}}{n\Omega_n},$$

M stands for the mass of the Schwarzschild-de Sitter black hole in the definition due to Abbott and Deser [23], G_{n+2} denotes the $(n+2)$-dimensional Newton constant, and l_{n+2} represents the cosmological radius of the $(n+2)$-dimensional de Sitter universe. When $M=0$, the solution (3.1) reduces to a de Sitter space with a cosmological horizon at $r_c = l_{n+2}$. When M increases from $M=0$, a black hole horizon appears and grows, while the cosmological horizon shrinks. Finally the black hole horizon $r_{\rm BH}$ touches the cosmological horizon $r_{\rm CH}$ when

$$M = M_N \equiv \frac{2}{\omega_n(n+1)}\left(\frac{n-1}{n+1}l_{n+2}^2\right)^{(n-1)/2}.$$

This is the Nariai black hole, the maximal black hole in de Sitter space. When $M > M_N$, both the two horizons disappear and the solution describes a naked singularity. When $M < M_N$, the equation $f(r)=0$ has two real roots, the larger one is the cosmological horizon, while the smaller one is the black hole horizon.

The Hawking temperature $T_{\rm HK}$ and entropy S associated with the black hole horizon are [22]

$$T_{\rm HK} = \frac{1}{4\pi r_{\rm BH}}\left((n-1) - (n+1)\frac{r_{\rm BH}^2}{l_{n+2}^2}\right), \qquad S = \frac{r_{\rm BH}^n \Omega_n}{4G_{n+2}}. \tag{3.2}$$

With the identification $E = M$ and the definition $E_c = (n+1)E - nT_{\rm HK}S$ [6], one can easily obtain

$$E_c = \frac{2n r_{\rm BH}^{n-1}\Omega_n}{16\pi G_{n+2}}, \quad 2E - E_c = -\frac{2n r_{\rm BH}^{n+1}\Omega_n}{16\pi G_{n+2} l_{n+2}^2}. \tag{3.3}$$

Clearly the entropy (3.2) can be expressed by the Cardy-Verlinde formula [22]

$$S = \frac{2\pi l_{n+2}}{n}\sqrt{E_c(E_c - 2E)}. \tag{3.4}$$

[6] Here it is assumed that the thermodynamics of the Schwarzschild-de Sitter black hole can be described in terms of a CFT.

If one rescales the energies by a factor R/l_{n+2}, the above equation is changed to

$$S = \frac{2\pi R}{n}\sqrt{E_c(E_c - 2E)}. \tag{3.5}$$

Note that this expression is exactly the same as the entropy formula in (2.20).

3.2 Brane dynamics in the background of Schwarzschild-de Sitter black holes

Let us introduce a $(n+1)$-dimensional brane with tension σ moving in the background of the Schwarzschild-de Sitter black holes (3.1). Its dynamics is determined by the following action [9, 24]

$$S_{\rm brane} = \frac{1}{8\pi G_{n+2}}\int_{\partial M} d^{n+1}x\sqrt{-h}K + \frac{1}{8\pi G_{n+2}}\int_{\partial M} d^{n+1}x\sqrt{-h}\sigma. \tag{3.6}$$

Here the brane is viewed as boundary of the bulk spacetime (3.1), K is the extrinsic curvature for the boundary with the induced metric h_{ab}. The equation of motion for the brane is

$$K_{ab} = \frac{\sigma}{n}h_{ab}. \tag{3.7}$$

The brane cosmology in the Schwarzschild-de Sitter black holes has been first considered in [25]. The holography in brane cosmology in various asymptotically de Sitter spaces has also been discussed in [25, 26][7]. Now let us specify the location of the brane as $r = r(t)$. We introduce a cosmic time τ so that $t = t(\tau)$ and $r = r(\tau)$ and require

$$f(r)\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{f(r)}\left(\frac{dr}{d\tau}\right)^2 = 1, \tag{3.8}$$

which implies that the brane moves along a radial time-like geodesic in the background (3.1)[8]. In that case, the induced metric h_{ab} on the brane becomes

$$ds^2 = -d\tau^2 + R^2(\tau)d\Omega_n^2, \tag{3.9}$$

which is just a $(n+1)$-dimensional closed FRW universe metric (2.1) with scale factor $R(\tau) = r(\tau)$.

[7]However, our philosophy in understanding the holography in the case with a cosmological constant is different from those in [25, 26]. We will discuss this point at the end of this paper.

[8]The dynamics of brane along a radial space-like geodesic in various asymptotically de Sitter backgrounds has also been discussed in [25, 26].

Calculating the extrinsic curvature for the brane and then from the equation (3.7), we have

$$\frac{dt}{d\tau} = \frac{\sigma R}{n f(R)}. \tag{3.10}$$

Substituting into (3.8) yields

$$H^2 = \frac{\omega_n M}{R^{n+1}} - \frac{1}{R^2} + \frac{1}{l_{n+2}^2} + \frac{\sigma^2}{n^2}, \tag{3.11}$$

The time derivative of the Hubble parameter is

$$\dot{H} = -\frac{(n+1)\omega_n M}{2R^{n+1}} + \frac{1}{R^2}. \tag{3.12}$$

These equations just describe a radiation-dominated closed FRW universe with a positive cosmological constant $\Lambda_{n+1} = n(n-1)/2l_{n+1}^2$ with

$$\frac{1}{l_{n+1}^2} = \frac{1}{l_{n+2}^2} + \frac{\sigma^2}{n^2}, \tag{3.13}$$

from which we see $l_{n+1}^2 < l_{n+2}^2$. Now we consider the solution of (3.11). As an example, let us discuss the special case of $n = 3$. The generalization to other dimensions is straightforward. In that case, the solution has been found in three different cases depending on the parameter $\omega_4 M/l_{n+1}^2$ in [27], where the authors discussed the dynamics of a noncritical brane in the Schwarzschild-AdS black hole. Defining $x = R^2$, we can rewrite (3.11) as

$$\dot{x}^2 = \frac{4}{l_4^2}(x - x_+)(x - x_-), \tag{3.14}$$

where

$$x_\pm = \frac{l_4^2}{2}\left(1 \pm \sqrt{1 - 4\omega_4 M/l_4^2}\right). \tag{3.15}$$

Note that when $x = x_\pm$, one has $H = 0$. Actually, $x_\pm$ are turning points of the brane.

(1) When $4\omega_4 M = l_4^2$, one has $x_+ = x_-$. In this case, the brane has only one turning point. However, the solution has two branches:

- $x \in [x_+, \infty)$. In this case, the solution is given by

$$R^2 = x(\tau) = \frac{l_4^2}{2}\left(1 + e^{2\tau/l_4}\right), \quad \tau \in (-\infty, \infty). \tag{3.16}$$

  And the Hubble parameter takes the expression

$$H = \frac{1}{l_4}\frac{e^{2\tau/l_4}}{1 + e^{2\tau/l_4}}. \tag{3.17}$$

  Clearly in this case one has $0 < H < 1/l_4$.

- $x \in (0, x_+]$. In this case the solution is

$$R^2(\tau) = \frac{l_4^2}{2}\left(1 - e^{-2\tau/l_4}\right), \quad \tau \in [0, \infty). \tag{3.18}$$

Accordingly the Hubble parameter reads

$$H = \frac{1}{l_4}\frac{e^{-2\tau/l_4}}{1 - e^{-2\tau/l_4}}. \tag{3.19}$$

In this branch, $0 \le H < \infty$. Since the black hole horizon $x_{\rm BH}$ falls in the range of $0 < x_{\rm BH} < x_+$, we find $H < 1/l_4$ in the range of $x_{\rm BH} < x < x_+$.

(2) When $4\omega_4 M > l_4^2$, the brane has no turning point. Namely, there is no point which has $H = 0$ along the geodesic of the brane. In this case one has the solution

$$R^2(\tau) = \frac{l_4^2}{2}\left(1 + \sqrt{4\omega_4 M/l_4^2 - 1}\,\sinh(2\tau/l_4)\right), \tag{3.20}$$

where $\tau_0 \le \tau < \infty$ with

$$\sinh(2\tau_0/l_4) = -\left(4\omega_4 M/l_4^2 - 1\right)^{-1/2}.$$

The evolution of the Hubble parameter is given by

$$H = \frac{1}{l_4}\frac{\sqrt{4\omega_4 M/l_4^2 - 1}\,\cosh(2\tau/l_4)}{1 + \sqrt{4\omega_4 M/l_4 - 1}\,\sinh(2\tau/l_4)}. \tag{3.21}$$

We find that outside the black hole horizon, $H < 1/l_4$. Actually, there exists another solution for the equation (3.14). But this solution describes the same movement of the brane as the solution (3.20) does. So we do not present it here.

(3) When $4\omega_4 M < l_4^2$, the brane has two turning points $x_\pm$. The range $x \in [x_-, x_+]$ is not allowed since in which $H^2 \le 0$. As a result, solution of equation (3.14) has two branches:

- $x \in (0, x_-]$. The solution is

$$R^2(\tau) = \frac{l_4^2}{2}\left(1 - \sqrt{1 - 4\omega_4 M/l_4^2}\,\cosh(2\tau/l_4)\right), \tag{3.22}$$

where τ takes value in the range $-\tau_c \le \tau \le \tau_c$ with

$$\cosh(2\tau_c/l_4) = (1 - 4\omega_4 M/l_4)^{-1/2}. \tag{3.23}$$

When $\tau = \pm\tau_c$, one has $R = 0$ and $H = \infty$. The Hubble parameter

$$H = \frac{1}{l_4}\frac{\sqrt{1 - 4\omega_4 M/l_4^2}\sinh(2\tau/l_4)}{1 - \sqrt{1 - 4\omega_4 M/l_4^2}\cosh(2\tau/l_4)}, \tag{3.24}$$

from which we see that $H = 0$ when $\tau = 0$. Note that because of the relation (3.13), it is easy to see that the black hole horizon $x_{\rm BH} \in (0, x_-)$, while the cosmological horizon $x_{\rm CH} \in (x_-, x_+)$. Thus we find that in this branch one has $H < 1/l_4$ as the brane stays outside the black hole horizon of the bulk background.

- $x \in [x_+, \infty)$. In this branch the solution is

$$R^2(\tau) = \frac{l_4^2}{2}\left(1 + \sqrt{1 - 4\omega_4 M/l_4^2}\cosh(2\tau/l_4)\right), \quad \tau \in (-\infty, \infty). \tag{3.25}$$

  And the Hubble parameter is given by

$$H = \frac{1}{l_4}\frac{\sqrt{1 - 4\omega_4 M/l_4^2}\sinh(2\tau/l_4)}{1 + \sqrt{1 - 4\omega_4 M/l_4^2}\cosh(2\tau/l_4)}. \tag{3.26}$$

  At $\tau = 0$, one has $H = 0$.

We conclude that the evolution of the brane depends on value of the parameter $4\omega_4 M/l_4^2$ and its initial position. Since we are interested in a radiation-dominated cosmology beginning with a big bang, so in addition to the solution in case (2), the solutions in the branch $x \in (0, x_+]$ of case (1) and in the branch $x \in (0, x_-]$ of case (3) are suitable, respectively, for our purpose. Inspecting the three appropriate solutions, we find that $H < 1/l_4$ always holds when the brane stays outside the bulk black hole horizon. Further we mention that in the above discussions, the condition $4\omega_4 M < l_5^2$ is assumed to hold, which implies that the black hole horizon is always present.

3.3 Holography in the brane cosmology

In brane world scenario with an AdS bulk, the tension of the brane can be adjusted to result in a so-called critical brane on which the effective cosmological constant vanishes [9]. In the present case, one can see from (3.13) that it is impossible to obtain a vanishing cosmological constant on the brane. Now we set

$$\sigma = n/l_{n+2}. \tag{3.27}$$

In that case the Newton constant on the brane has the relation

$$G_{n+1} = \frac{n-1}{l_{n+2}}G_{n+2}, \tag{3.28}$$

to the Newton constant in the bulk. This relation is the same as that for a critical brane in AdS bulk [9]. Furthermore the parameter M in the solution (3.1) is the black hole mass measured in the bulk coordinates [23]. According to the relation (3.10), the holographic energy E measured on the brane is[9]

$$E = \frac{l_{n+2}}{R} M. \tag{3.29}$$

Substituting (3.27), (3.28) and (3.29) into (3.11) and (3.12), we have

$$\begin{aligned} H^2 &= \frac{16\pi G_{n+1}}{n(n-1)} \frac{E}{V} - \frac{1}{R^2} + \frac{1}{l_{n+1}^2}, \\ \dot{H} &= -\frac{8\pi G_{n+1}}{n-1} \left(\frac{E}{V} + p \right) + \frac{1}{R^2}, \end{aligned} \tag{3.30}$$

with $l_{n+1}^2 = l_{n+2}^2/2$ and $p = E/nV$. These two equations are the same as the ones in (2.2). The equation of state $p = E/nV$ is just the one for radiation matter (or more general CFTs). As a result, the discussions on holography in Sec. 2 are applicable here.

Suppose the brane moves between the bulk black hole horizon and cosmological horizon[10]. Since the brane is viewed as the boundary of the bulk spacetime, the entropy of holographic matter (radiation) on the brane is just the entropy of black hole horizon, which is a constant during the evolution of the brane universe. However, the entropy density varies with time as

$$s \equiv \frac{S}{V} = \frac{r_{\rm BH}^n}{4G_{n+2}R^n} = \frac{(n-1)r_{\rm BH}^n}{4G_{n+1}l_{n+2}R^n}, \tag{3.31}$$

and the energy density of radiation-matter

$$\rho \equiv \frac{E}{V} = \frac{n r_{\rm BH}^{n-1} l_{n+2}}{16\pi G_{n+2} R^{n+1}} \left(1 - \frac{r_{\rm BH}^2}{l_{n+2}^2} \right), \tag{3.32}$$

in terms of the black hole horizon radius $r_{\rm BH}$. Further from the scaling relation (3.29), the temperature T on the brane is given by

$$T = \frac{l_{n+2}}{R} T_{\rm HK} = \frac{l_{n+2}}{4\pi r_{\rm BH} R} \left((n-1) - (n+1)\frac{r_{\rm BH}^2}{l_{n+2}^2} \right). \tag{3.33}$$

[9]Due to existence of the cosmological horizon in the bulk, this relation is not justified well as the case for the AdS bulk [9]. Even for the latter case, there exists a different viewpoint, for example, see [28], which argued that this relation holds only near the boundary of AdS space. However, we note that the rescaling (3.29) indeed gives a scale relation for a radiation matter in universe. Further, the relation (3.29) holds at least for small black holes as in the AdS case.

[10]In case (2) the brane can cross the cosmological horizon. We have not yet well understood the holographic connection when the brane crosses the bulk cosmological horizon. In case (1) and (3) the brane is always inside the cosmological horizon for branches of interest.

Applying the first law of thermodynamics to the radiation matter in the brane universe, one has

$$Tds = d\rho + n(\rho + p - Ts)\frac{dR}{R}. \tag{3.34}$$

Following [9] and defining

$$\gamma = \frac{n}{2}(\rho + p - Ts)R^2, \tag{3.35}$$

we have

$$\gamma = \frac{n r_{\rm BH}^{n-1} l_{n+2}}{16\pi G_{n+2} R^{n-1}} = \frac{n(n-1) r_{\rm BH}^{n-1}}{16\pi G_{n+1} R^{n-1}}. \tag{3.36}$$

Then the entropy density (3.31) can be expressed as

$$s = \frac{4\pi}{n}\sqrt{\gamma\left(\frac{\gamma}{R^2} - \rho\right)}. \tag{3.37}$$

Now we consider a special moment that the brane crosses the bulk black hole horizon. In that time, one has $R = r_{\rm BH}$. From (3.11) we see

$$H^2 = \frac{1}{l_{n+2}^2}. \tag{3.38}$$

Note that we have taken $\sigma^2/n^2 = 1/l_{n+2}^2$. At that moment the cosmological D-bound in (2.13) turns out to be

$$S_{\rm D} = \frac{(n-1)\Omega_n}{4G_{n+1} l_{n+2}} = \frac{r_{\rm BH}^n \Omega_n}{4G_{n+2}}, \tag{3.39}$$

which is just the black hole horizon entropy (3.2). That is, when the brane crosses the bulk black hole horizon, the cosmological D-bound is saturated by the entropy of the bulk black hole. At that time, the geometric temperature in (2.14) is given by

$$T_{\rm D} = \frac{l_{n+2}}{4\pi r_{\rm BH}^2}\left((n-1) - (n+1)\frac{r_{\rm BH}^2}{l_{n+2}^2}\right), \tag{3.40}$$

which equals the temperature (3.33) of radiation filling the universe. Furthermore, at that moment the FRW equations (2.19) coincide with the equations (3.37) and (3.36) which describe the entropy of radiation matter in the brane universe with a positive cosmological constant. Thus we reach the same conclusion as in the case without the cosmological constant [9].

4 Conclusion and discussion

We have discussed the holography in a radiation-dominated, closed FRW universe with a positive cosmological constant. By introducing the cosmological D-bound (2.13) on entropy of matter in the universe, the Friedmann equation describing the evolution of the universe can be rewritten in the form of Cardy-Verlinde formula which describes the degree of freedom of radiation matter filling the universe. When the cosmological D-bound is saturated by the entropy of matter, these two equations coincide with each other. Thus we have successfully generalized interesting observations by Verlinde [3, 9] on the holographic connection between the Friedmann equation and Cardy-Verlinde formula to the case with a positive cosmological constant. By considering brane cosmology in the background of Schwarzschild-de Sitter black holes, we have found that the cosmological D-bound is saturated when the brane crosses the black hole horizon in the bulk background. At that moment, the Friedmann equation and Cardy-Verlinde formula coincide with each other, and the introduced geometric temperature $T_{\rm D}$ in (2.14) equals the thermodynamic temperature T of the radiation matter.

We stress that when discussing the holographic connection in the brane cosmology in the Schwarzschild-de Sitter black holes, we have taken a special tension (3.27), which is the same as in the case for critical brane in the AdS bulk. Only in that case, the Friedmann equation and Cardy-Verlinde formula coincide with each other very well. We point out here that if the brane tension is arbitrary, the Friedmann equation still has a form as the Cardy-Verlinde formula, but a factor $\sigma l_{n+2}/n$ will appear in (3.37). Furthermore, for the radiation matter dual to the black holes in de Sitter spaces, we see from (3.37) that $\rho - \gamma/R^2 < 0$. If considering a noncritical brane cosmology in the Schwarzschild-AdS black holes, one will see that in that case $\rho - \gamma/R^2 > 0$. This case corresponds to the holographic connection described by equations (2.17) and (2.18). In addition, if the cosmological constant becomes negative, the minus sign in front of S_Λ has to be changed to plus. The quantity (2.11) then will lose its interpretation, but all formulas will still work well.

We have noticed that the holography was discussed in many literatures in the case with a cosmological constant, for example, see [25, 26, 28, 29, 30, 31, 32]. However, our understanding is different from those in existing literatures: in some papers the cosmological constant term is incorporated to the Bekenstein-Verlinde entropy bound; in some papers this term is kept as an independent term. In those literatures the Friedmann equation and the Cardy-Verlinde formula have not a same form, and when the Hubble bound is saturated, these two formulas do not get matched. Finally we point out that at the end of paper [3], Verlinde mentioned that when the cosmological constant does not vanish, the Hubble entropy bound needs to be modified by replacing H with the square root of $H^2 - 1/l_{n+1}^2$. But, as argued in this paper, three entropy bounds: Bekenstein-Verlinde bound, Bekenstein-Hawking bound and Hubble bound in (2.3) still have the same forms as the case without the cosmological constant, even when the cosmological constant is present. The cosmological D-bound introduced in this work provides a new entropy bound of matter in the strongly self-gravitating universe ($HR > 1$) with a positive cosmological constant and makes all formulas work so nicely as the case without the cosmological constant.

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