Nonlocal braneworld action: an alternative to Kaluza-Klein description

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Abstract

We construct the nonlocal braneworld action in the two-brane Randall-Sundrum model in a holographic setup alternative to Kaluza-Klein description: the action is written as a functional of the two metric and radion fields on the branes. This action effectively describes the dynamics of the gravitational field both on the branes and in the bulk in terms of the brane geometries directly accessible for observations. Its nonlocal form factors incorporate the cumulative effect of the bulk Kaluza-Klein modes. We also consider the reduced version of this action obtained by integrating out the fields on the negative-tension brane invisible from the viewpoint of the Planckian brane observer. This effective action features a nontrivial transition (AdS flow) between the local and nonlocal phases of the theory associated with the limits of small and large interbrane separation. Our results confirm a recently proposed braneworld scenario with diverging (repulsive) branes and suggest possible new implications of this phase transition in brane cosmology.

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I. INTRODUCTION

Recent developments in string theory [1] and the attempts to resolve the hierarchy problem [2] suggest that the observable world can be a brane embedded in a higher-dimensional spacetime with a certain number of noncompact dimensions. Moreover, string-inspired field theories imply the existence of several branes interacting and propagating in the multi-dimensional bulk. Their dynamics manifests itself for the observer as an effective four-dimensional theory that, in the cosmological context [3, 4], should explain the origin of structure in the Universe by means of an inflationary or some other scenario [5, 6], explain its particle phenomenology, and shed light on problems such as a possibly observable cosmological acceleration [7].

The efficient way of description for the braneworld scenario is the method of effective action. Generally, this notion is very ambiguous, because its precise meaning ranges from the generating functional of one-particle irreducible diagrams in local field theory to the low-energy effective action of target spacetime fields in string theory. What is, apparently, in common for all these definitions is that the effective action manifestly describes the dynamics of the observable variables and, simultaneously, incorporates in implicit form the effect of invisible degrees of freedom that are integrated (or traced) out. This type of description becomes actually indispensable in string-theoretical, Kaluza-Klein and braneworld contexts when the observable variables turn out to be very different from the fundamental degrees of freedom whose dynamics underlies the effective "visible" dynamics.

This situation is characteristic of the old Kaluza-Klein and new (braneworld) pictures, because the "visible" fields $\phi(x)$ are four-dimensional in contrast to the fundamental fields $\Phi(x, y)$ in the multi-dimensional spacetime depending on the visible (four-dimensional) coordinates x and the coordinates of extra dimensions y. This is also the case in the widely celebrated by string physicists duality relations of the Anti-de Sitter/conformal-field-theory correspondence (AdS/CFT-correspondence) between the bulk and boundary theories [8]. However, the four-dimensional fields $\phi(x)$ originate from $\Phi(x, y)$ in these two cases by two different procedures and their effective actions essentially differ.

In the Kaluza-Klein formalism $\phi(x) = \{\phi_n(x)\}$ arises as an infinite tower of Kaluza-Klein modes — the coefficients of the expansion of $\Phi(x, y)$ in a certain complete set of harmonics on the y-space. Correspondingly, its effective action is just the original action of the fundamental field $S[\,\Phi(x, y)\,]$ rewritten in terms of $\{\phi_n(x)\}$. This type of action was built for the two-brane Randall-Sundrum scenario (cf. [9]) in [10], and a good review of its particle phenomenology can be found in [11]. This action is, however, very often not helpful in the braneworld context, because it does not convey a number of its important features like non-compactness of extra dimensions [12, 13], recovery of the four-dimensional Einstein gravity [14], and its interpretation in terms of the AdS/CFT-correspondence [15–17].

In the holographic formalism of the AdS/CFT-correspondence, $\phi(x)$ roughly turns out to be the value of $\Phi(x, y)$ at the boundary, $\phi(x) = \Phi(x, y_{\rm bound})$[1]. In the braneworld Randall-

[1] Here we disregard the subtleties of this identification associated with the asymptotic properties of the AdS-spacetime boundary and the conformal structure of bulk and brane operators, see [18, 19].

Sundrum model, where the boundary is associated with the brane Σ located at $y_{\rm brane}$, this identification has led to the understanding that the recently observed localization of the graviton zero mode [13] and the recovery of the four-dimensional Einstein gravity on the brane [13, 14, 20] can be interpreted in terms of the AdS/CFT-correspondence [15–17]. This conclusion was reached in the language of the effective action of the brane field — the four-dimensional metric induced on the brane from the bulk geometry. Thus, generically, a natural variable to describe a braneworld scenario becomes the value of the field at the brane in question, $\phi(x) = \Phi(x, y_{\rm brane})$. Unlike in a Kaluza-Klein reduction, its effective action $S_{\rm eff}[\,\phi(x)\,]$ is obtained from the fundamental action $S[\,\Phi\,]$ by a less trivial procedure — by substituting in $S[\,\Phi\,]$ a solution of the classical equations of motion for $\Phi(x,y)$ in the bulk, $\Phi = \Phi[\,\phi(x)\,]$, parametrized by their boundary values on the branes, that is $S_{\rm eff}[\,\phi(x)\,] = S\big[\,\Phi[\phi(x)\,]\,\big]$. This construction obviously generalizes to the case of several branes Σ_I enumerated by the index I and the set of brane fields $\phi^I = \Phi(\Sigma_I)$.

Such a definition corresponds to the tree-level approximation for the quantum effective action

$$\exp\Big(iS_{\rm eff}[\,\phi\,]\Big) = \int D\Phi\,\exp\Big(iS[\,\Phi\,]\Big)\,\Big|_{\,\Phi(\Sigma)\,=\,\phi}\,, \tag{1}$$

where the functional integration over the bulk fields runs subject to the brane boundary conditions. The scope of this formula is very large, because it arises in very different contexts. In particular, its Euclidean version $(iS \to -S_{\rm Euclid})$ underlies the construction of the no-boundary wavefunction in quantum cosmology [21]. Semiclassically, in the braneworld scenario, it represents a Hamilton-Jacobi functional, and its evolutionary equations of motion in the "fifth time" y can be interpreted as renormalization-group equations [22]. It also underlies the effective action formulation of the AdS/CFT-correspondence principle between supergravity theory on an $AdS_5\times S^5$ background and the superconformal field theory on its infinitely remote boundary [8, 16, 17, 23, 24].

Here we construct the braneworld effective action of the above type for the two-brane Randall-Sundrum model as a functional of two induced metrics $g^{\pm}_{\mu\nu}(x)$ and radion fields $\psi^{\pm}(x)$ on its branes $\Sigma_{\pm}$. We obtain it perturbatively on the background of the Randall-Sundrum solution to quadratic order in brane curvatures $R^{\pm}_{\mu\nu}$ and radions. Current interest in this action can be explained by the attempts to solve the hierarchy problem and generate cosmological scenarios incorporating the dynamics of either colliding [5, 25–27] or diverging [28] branes. In particular, here we justify the result for the braneworld action previously obtained in [28] by a simplified method disregarding curvature perturbations on the invisible brane and, thus, confirm the scenario of [28] corresponding to diverging branes.

Another motivation for the two-field braneworld action comes from the papers [29] and, especially [30], occupying a somewhat intermediate position between the Kaluza-Klein setting and the setting of (1). The authors of [29, 30] use an ansatz in which the two brane metrics $g^{\pm}_{\mu\nu}(x)$ are conformally equivalent, $g^{+}_{\mu\nu}(x) \sim g^{-}_{\mu\nu}(x)$, and differ only by the conformal (warp) factors at the branes. Thus, the braneworld action of [30] depends on one metric and two radion fields associated with the warp factors. This restriction of the total configuration space of the model does not leave room for important degrees of freedom which are in the focus of this paper.

The organization of the paper is as follows. In Sec. II we focus on the definition of the effective action in a braneworld setup and explain the method of its calculation. In Sec. III we present the final result for this action in the two-brane Randall-Sundrum model, which we advocate here. This action is obtained as a quadratic form in Ricci curvatures and radion fields on two branes, which is covariant with respect to two independent diffeomorphisms associated with these branes. Sec. IV is devoted to the derivation of this result. The derivation begins with the construction of the effective equations of motion for the two-brane Randall-Sundrum model. Then we recover the action which generates these equations by a variational procedure. The nonlocal form factors of the braneworld action are explicitly constructed in Sec. V. We show that their zeros generate the tower of Kaluza-Klein modes and then consider the form factors in the few lowest orders of the derivative ($\Box$) expansion. This expansion incorporates the massless (graviton) mode of the Kaluza-Klein tower, which is responsible for the recovery of the four-dimensional Einstein theory on the positive-tension brane. In the same section we consider the limit of large interbrane distance which turns out to be the high-energy limit on the negative-tension brane and discuss the properties of the relevant nonlocal operators. In Sec. VI we build the reduced (one-field) effective action corresponding to the on-shell reduction in the sector of fields on the negative-tension brane. In this way we confirm the result of [28], where the braneworld action took the form of a Brans-Dicke type theory with the radion field non-minimally coupled to curvature. In the limit of large brane separation we confirm the realization of the AdS/CFT-correspondence principle. We also discuss setting Hartle boundary conditions at the AdS horizon and the problem of analytic continuation to the Euclidean spacetime. The concluding section discusses the transition between the local and nonlocal phases of the braneworld action. This transition is associated with the renormalization type AdS flow from small to large interbrane distances, which is likely to be realized dynamically due to a repulsive interbrane potential. We briefly discuss possible implications of this transition in the diverging branes model of [28].

II. EFFECTIVE ACTION OF BRANE-LOCALIZED FIELDS AND THE METHODS OF ITS CALCULATION

In the definition (1) the effective action by construction depends on the four-dimensional fields associated with brane(s). The number of these fields equals the number of branes, geometrically each field being carried by one of the branes in the system. In the generalized Randall-Sundrum setup, the braneworld effective action is generated by the path integral of the type (1),

$$\int DG \exp\Big(iS[\,G,g,\phi\,]\Big)\bigg|_{{}^4G(\Sigma)=g} = \exp\Big(iS_{\rm eff}[\,g,\phi\,]\Big), \tag{2}$$

where the integration over bulk metrics runs subject to fixed induced metrics on the branes — the arguments of $S_{\rm eff}[\,g,\phi\,]$. Here $S[\,G,g,\phi\,]$ is the action of the five-dimensional gravitational field with the metric $G = G_{AB}(x,y)$, $A = (\mu,5)$, $\mu = 0,1,2,3$, propagating in the bulk spacetime ($x^A = (x, y)$, $x = x^\mu$, $x^5 = y$), and matter fields ϕ are confined to the branes Σ_I — four-dimensional timelike surfaces embedded in the bulk,

$$S[\,G, g, \phi\,] = S_5[\,G\,] + \sum_I \int_{\Sigma_I} d^4x \left(L_{\rm m}(\phi, \partial\phi, g) - g^{1/2}\sigma_I + \frac{1}{8\pi G_5}[K] \right), \tag{3}$$

$$S_5[\,G\,] = \frac{1}{16\pi G_5} \int\limits_{M^5} d^5x\, G^{1/2} \left({}^5R(G) - 2\Lambda_5 \right). \tag{4}$$

The branes are enumerated by the index I and carry induced metrics $g = g_{\mu\nu}(x)$ and matter field Lagrangians $L_{\rm m}(\phi, \partial\phi, g)$. The bulk part of the action contains the five-dimensional gravitational and cosmological constants, G_5 and Λ_5, while the brane parts have four-dimensional cosmological constants σ_I. The bulk cosmological constant Λ_5 is negative and, therefore, is capable of generating the AdS geometry, while the brane cosmological constants play the role of brane tensions σ_I and, depending on the model, can be of either sign. The Einstein-Hilbert bulk action (4) is accompanied by the brane 'Gibbons-Hawking' terms containing the jump of of the extrinsic curvature trace $[K]$ associated with both sides of each brane [31][2].

In the tree-level approximation the path integral (2) is dominated by the stationary point of the action (3). Its variation is given as a sum of five- and four-dimensional integrals,

$$\begin{aligned} \delta S[\,G, g, \phi\,] = &-\frac{1}{16\pi G_5} \int d^4x\, dy\, G^{1/2} \left({}^5R^{AB} - \frac{1}{2}\, {}^5R\, G^{AB} + \Lambda_5 G^{AB} \right) \delta G_{AB}(x, y) \\ &+ \sum_I \int_{\Sigma_I} d^4x\, g^{1/2} \left(-\frac{1}{16\pi G_5}[K^{\mu\nu} - g^{\mu\nu}K] + \frac{1}{2}(T^{\mu\nu} - g^{\mu\nu}\sigma) \right) \delta g_{\mu\nu}(x), \end{aligned} \tag{5}$$

where $[K^{\mu\nu} - g^{\mu\nu}K]$ denotes the jump of the extrinsic curvature terms across the brane, and $T^{\mu\nu}(x)$ is the corresponding four-dimensional stress-energy tensor of matter fields on the branes,

$$T^{\mu\nu}(x) = \frac{2}{g^{1/2}} \frac{\delta S_{\rm m}[g, \phi]}{\delta g_{\mu\nu}(x)}, \tag{6}$$

$$S_{\rm m}[\,g, \phi\,] = \sum_I \int_{\Sigma_I} d^4x\, L_{\rm m}(\phi, \partial\phi, g) \tag{7}$$

(we use a collective notation g, ϕ and σ for the induced metrics, matter fields and tensions on all branes Σ). The action is stationary when the integrands of both integrals in (5) vanish, which gives rise to Einstein equations in the bulk,

$$\frac{\delta S[\,G, g, \phi\,]}{\delta G_{AB}(x, y)} \equiv -\frac{1}{16\pi G_5} G^{1/2} \left({}^5R^{AB} - \frac{1}{2} G^{AB}\, {}^5R + \Lambda_5 G^{AB} \right) = 0, \tag{8}$$

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[2] The extrinsic curvature $K_{\mu\nu}$ is defined as a projection on the brane of the tensor $\nabla_A n_B$ with the outward unit normal n_B, i.e. the normal pointing from the bulk to the brane. With this definition the normals on the two sides of the brane are oppositely oriented and the extrinsic curvature jump $[K_{\mu\nu}]$ actually equals the sum of the so-defined curvatures on both sides of the brane.

which are subject to (generalized) Neumann type boundary conditions — the well-known Israel junction conditions —

$$\frac{\delta S[\,G, g, \phi\,]}{}\delta g_{\mu\nu}(x) \equiv -\frac{1}{16\pi G_5} g^{1/2}\left[K^{\mu\nu} - g^{\mu\nu}K\right] + \frac{1}{2} g^{1/2}(T^{\mu\nu} - g^{\mu\nu}\sigma) = 0, \tag{9}$$

or to Dirichlet type boundary conditions corresponding to fixed (induced) metrics on the branes, with $\delta g_{\mu\nu} = 0$ in the variation (5),

$$ {}^4G_{\mu\nu}\Big|_{\Sigma} = g_{\mu\nu}(x)\ . \tag{10}$$

The solution of the latter, Dirichlet, problem is obviously a functional of brane metrics, $G_{AB} = G_{AB}[\,g_{\mu\nu}(x)\,]$, and it enters the tree-level approximation for the path integral (2). $S_{\rm eff}[\,g, \phi\,]$ in this approximation reduces to the original action (3)–(4) calculated on this solution $G_{AB}[g_{\mu\nu}(x)]$, $S_{\rm eff}[\,g, \phi\,] = S[\,G[\,g\,], g, \phi\,] + O(\hbar)$. With this definition, the matter part of effective action coincides with the original action Eq. (7)

$$S_{\rm eff}[\,g, \phi\,] = S_4[\,g\,] + S_{\rm m}[\,g, \phi\,], \tag{11}$$

while all non-trivial dependence on g arising from the functional integration is contained in $S_4[\,g\,]$.

The Dirichlet problem (8), (10) can be regarded as an intermediate stage in solving the problem (8), (9). Indeed, given the action (11) as a result of solving the Dirichlet problem (8), (10), one can further apply the variational procedure, now with respect to the induced metric $g_{\mu\nu}$, to get the effective equations

$$\frac{\delta S_{\rm eff}[\,g, \phi\,]}{\delta g_{\mu\nu}(x)} = \frac{\delta S_4[\,g\,]}{\delta g_{\mu\nu}(x)} + \frac{1}{2} g^{1/2} T^{\mu\nu}(x) = 0, \tag{12}$$

which are equivalent to the Israel junction conditions — a part of the full system of the bulk-brane equations of motion (8), (9) (cf. the appendix of [32]). The procedure of solving this system of equations is split into two stages. First we solve it in the bulk subject to *Dirichlet* boundary conditions on the branes and substitute the result into the bulk action to get the *off-shell* brane effective action. Stationarity of the latter with respect to the four-dimensional metric comprises the remaining set of equations to be solved at the second stage.

This observation suggests two equivalent methods of recovering the braneworld effective action. One method is straightforward — the direct substitution of the solution $G_{AB} = G_{AB}[\,g_{\mu\nu}(x)\,]$ of the Dirichlet problem (8), (10) into the five-dimensional action. The other method, which we choose to pursue in the following, is less direct, but technically is simpler. We will recover the effective action from the effective equations (12). Their left-hand side considered as a variational derivative with respect to the brane metric(s) $g_{\mu\nu}$ can be functionally integrated to yield $S_{\rm eff}[\,g, \phi\,]$. Therefore we will, first, obtain these effective equations by solving the bulk part of the equations of motion and by explicitly rewriting the Israel matching conditions in terms of $g_{\mu\nu}$. The crucial point in the further functional integration of the latter is the recovery of a correct integrating factor. This will be based on the simple observation that the stress tensor of matter fields always enters the variational derivative of the effective action with the algebraic coefficient $(1/2)g^{1/2}$ of Eq. (12). In a subsequent publication [33] it will be shown that this derivation of $S_{\rm eff}[\,g,\phi]$ is equivalent to the first method based on the solution of the Dirichlet boundary value problem, (8), (10).

III. TWO-BRANE RANDALL-SUNDRUM MODEL: THE FINAL ANSWER FOR THE TWO-FIELD BRANEWORLD ACTION

The action of the two-brane Randall-Sundrum model [9] is given by Eq. (3) in which the index $I = \pm$ enumerates two branes with tensions $\sigma_\pm$. The fifth dimension has the topology of a circle labelled by the coordinate y, $-d < y \leq d$, with an orbifold Z_2-identification of points y and $-y$. The branes are located at antipodal fixed points of the orbifold, $y = y_\pm$, $y_+ = 0$, $|y_-| = d$. When they are empty, $L_{\rm m}(\phi, \partial\phi, g_{\mu\nu}) = 0$, and their tensions are opposite in sign and fine-tuned to the values of Λ_5 and G_5,

$$\Lambda_5 = -\frac{6}{l^2},\;\; \sigma_+ = -\sigma_- = \frac{3}{4\pi G_5 l}, \tag{13}$$

this model admits a solution with an AdS metric in the bulk (l is its curvature radius),

$$ds^2 = dy^2 + e^{-2|y|/l}\eta_{\mu\nu}dx^\mu dx^\nu, \tag{14}$$

$0 = y_+ \leq |y| \leq y_- = d$, and with a flat induced metric $\eta_{\mu\nu}$ on both branes [9]. The metric on the negative tension brane is rescaled by the warp factor $\exp(-2d/l)$ providing a possible solution for the hierarchy problem [9]. With the fine tuning (13) this solution exists for arbitrary brane separation d — two flat branes stay in equilibrium. Their flatness is the result of compensation between the bulk cosmological constant and brane tensions.

Now consider the Randall-Sundrum model with small matter sources for metric perturbations $h_{AB}(x,y)$ on the background of this solution [13, 14, 16, 20],

$$ds^2 = dy^2 + e^{-2|y|/l}\eta_{\mu\nu}dx^\mu dx^\nu + h_{AB}(x,y)\,dx^A dx^B, \tag{15}$$

such that this five-dimensional metric *induces* on the branes two four-dimensional metrics of the form

$$g^\pm_{\mu\nu}(x) = a_\pm^2\,\eta_{\mu\nu} + h^\pm_{\mu\nu}(x). \tag{16}$$

Here the scale factors $a_\pm = a(y_\pm)$ can be expressed in terms of the interbrane distance

$$a_+ = 1,\;\; a_- = e^{-2d/l} \equiv a\;, \tag{17}$$

and $h^\pm_{\mu\nu}(x)$ are the perturbations by which the brane metrics $g^\pm_{\mu\nu}(x)$ differ from the (conformally) flat metrics of the Randall-Sundrum solution (14)[3].

[3] It is needless to emphasize that $h^\pm_{\mu\nu}(x) \neq h_{\mu\nu}(x,y_\pm)$ because the induced brane metrics are non-trivially related to (15) via brane embedding functions.

The main result of this paper is the braneworld effective action (11) calculated for the boundary conditions (10) of this perturbed form (16). We calculate it in the approximation quadratic in perturbations, so that it represents the quadratic form in terms of the two-dimensional columns of fields $h^{\pm}_{\mu\nu}(x)$,

$$\mathbf{h}_{\mu\nu} = \left[\begin{array}{c} h^{+}_{\mu\nu}(x) \\ h^{-}_{\mu\nu}(x) \end{array} \right]. \tag{18}$$

It should be emphasized here that, in contrast to [30] and other papers on two-brane scenarios, the metric perturbations $h^{+}_{\mu\nu}(x)$ and $h^{-}_{\mu\nu}(x)$ are independent, which makes the configuration space of the theory much richer and results in additional degrees of freedom responsible for interbrane interaction.

On the other hand, the braneworld effective action is invariant under the four-dimensional diffeomorphisms acting on the branes. In the linearized approximation they reduce to the transformations of metric perturbations,

$$h^{\pm}_{\mu\nu} \to h^{\pm}_{\mu\nu} + f^{\pm}_{\mu\,,\,\nu} + f^{\pm}_{\nu\,,\,\mu} \tag{19}$$

with two *independent* local vector field parameters $f^{\pm}_{\mu} = f^{\pm}_{\mu}(x)$. Therefore, rather than in terms of metric perturbations themselves, the action in question is expressible in terms of the tensor invariants of these transformations — linearized Ricci tensors of $h_{\mu\nu} = h^{\pm}_{\mu\nu}(x)$,

$$R_{\mu\nu} = \frac{1}{2}\left(-\Box h_{\mu\nu} + h^{\lambda}_{\nu,\lambda\mu} + h^{\lambda}_{\mu,\lambda\nu} - h_{,\mu\nu}\right), \tag{20}$$

on *flat* four-dimensional backgrounds of both branes[4]. Commas denote partial derivatives, raising and lowering of braneworld indices here and everywhere throughout the paper is performed with the aid of the flat four-dimensional metric $\eta_{\mu\nu}$, $h^{\lambda}_{\nu} \equiv \eta^{\lambda\sigma}h_{\sigma\nu}$, $h \equiv \eta^{\mu\nu}h_{\mu\nu}$, $R = \eta^{\mu\nu}R_{\mu\nu}$, and $\Box$ denotes the flat spacetime d'Alembertian

$$\Box = \eta^{\mu\nu}\partial_{\mu}\partial_{\nu}. \tag{21}$$

Finally, we have to describe the variables which determine the embedding of branes into the bulk. Due to metric perturbations the branes no longer stay at fixed values of the fifth coordinate. Up to four-dimensional diffeomorphisms (19), their embedding variables consist of two four-dimensional scalar fields — the radions $\psi^{\pm}(x)$ — and, according to the mechanism discussed in [28], the braneworld action can depend on these scalars. Their geometrically invariant meaning is revealed in a special coordinate system where the bulk metric perturbations $h_{AB}(x,y)$ of Eq. (15) satisfy the so called Randall-Sundrum gauge conditions, $h_{A5} = 0$, $h_{\mu\nu}{}^{,\nu} = h^{\mu}_{\mu} = 0$. In this coordinate system the brane embeddings are defined by the equations

$$\Sigma_{\pm}: \;\; y = y_{\pm} + \frac{l}{a^2_{\pm}}\psi^{\pm}(x), \;\; y_{+} = 0, \, y_{-} = d. \tag{22}$$

[4] Strictly speaking, $R^{-}_{\mu\nu}$ is the linearized Ricci tensor of the artificial metric $\eta_{\mu\nu} + h^{-}_{\mu\nu}$. It differs from the linearized Ricci curvature of the second brane, $R_{\mu\nu}(a^2\eta + h^{-}) = R^{-}_{\mu\nu}/a^2$, by a factor of a^2.

In the approximation linear in perturbation fields and vector gauge parameters (in this approximation all these quantities are of the same order of magnitude, $h^{\pm}_{\mu\nu}(x) \sim \psi^{\pm}(x) \sim f^{\pm}_{\mu}(x)$), these radion fields are invariant under the action of diffeomorphisms (19).

The answer for the braneworld effective action, which we advocate here, and which will be derived in the following two sections, is given in terms of the invariant fields of the above type, $(R^{\pm}_{\mu\nu}(x), \psi^{\pm}(x))$, by the following spacetime integral of a 2×2 quadratic form,

$$S_4\,[\,g^{\pm}_{\mu\nu},\psi^{\pm}] = \frac{1}{16\pi G_4}\int d^4x\,\Big[\,\mathbf{R}^T_{\mu\nu}\,\frac{2\mathbf{F}(\Box)}{l^2\Box^2}\,\mathbf{R}^{\mu\nu} + \frac{1}{6}\,\mathbf{R}^T\,\frac{\mathbf{K}(\Box)-6\mathbf{F}(\Box)}{l^2\Box^2}\,\mathbf{R} \\ -3\Big(\Box\mathbf{\Psi}+\frac{1}{6}\mathbf{R}\Big)^T\,\frac{\mathbf{K}(\Box)}{l^2\Box^2}\,\Big(\Box\mathbf{\Psi}+\frac{1}{6}\mathbf{R}\Big)\Big]. \tag{23}$$

Here G_4 is an effective four-dimensional gravitational coupling constant,

$$G_4 = \frac{G_5}{l}, \tag{24}$$

$(\mathbf{R}^{\mu\nu}, \mathbf{\Psi})$ and $(\mathbf{R}^T_{\mu\nu}, \mathbf{\Psi}^T)$ are the two-dimensional columns

$$\mathbf{R}_{\mu\nu} = \begin{bmatrix} R^+_{\mu\nu}(x) \\ R^-_{\mu\nu}(x) \end{bmatrix}, \quad \mathbf{\Psi} = \begin{bmatrix} \psi^+(x) \\ \psi^-(x) \end{bmatrix} \tag{25}$$

and rows

$$\mathbf{R}^T_{\mu\nu} = \begin{bmatrix} R^+_{\mu\nu}(x) & R^-_{\mu\nu}(x) \end{bmatrix}, \quad \mathbf{\Psi}^T = \begin{bmatrix} \psi^+(x) & \psi^-(x) \end{bmatrix}, \tag{26}$$

of two sets of curvature perturbations and radion fields, associated with two branes (T denotes the matrix and vector transposition). The indices here are raised as above by the flat spacetime metric $\eta_{\mu\nu}$ and $\mathbf{R} \equiv \eta^{\mu\nu}\mathbf{R}_{\mu\nu}$. The kernels of the quadratic forms in (23) are nonlocal operators — non-polynomial functions of the flat-space d'Alembertian (21). They are both built in terms of the fundamental 2×2 matrix-valued operator

$$\mathbf{F}(\Box) = \begin{bmatrix} F_{++}(\Box) & F_{+-}(\Box) \\ F_{-+}(\Box) & F_{--}(\Box) \end{bmatrix} \tag{27}$$

and powers of the D'Alembertian[5]. In particular, the operator $\mathbf{K}(\Box)$ reads

$$\mathbf{K}(\Box) = 2\,\mathbf{F}(\Box) + \begin{bmatrix} -1 & 0 \\ 0 & 1/a^2 \end{bmatrix} l^2\Box. \tag{28}$$

The fundamental operator $\mathbf{F}(\Box)$ is the inverse of the operator-valued matrix $\mathbf{G}(\Box)$,

$$\mathbf{F}(\Box)\mathbf{G}(\Box) = \mathbf{I}, \tag{29}$$

[5] Nonlocalities require the prescription of boundary conditions which depend on the type of physical problem one is solving. We will assume that they are specified by a particular type of analytic continuation from the Euclidean spacetime in which these boundary conditions are trivial — Dirichlet boundary conditions at the Euclidean braneworld infinity, $|x| \to \infty$. This continuation will be discussed below in Sec. VI B.

which is determined by the Green's function of the following five-dimensional differential operator with Neumann boundary conditions,

$$\left(\frac{d^2}{dy^2} - \frac{4}{l^2} + \frac{\Box}{a^2(y)}\right) G(x, y\,|\,x', y') = \delta^{(4)}(x, x')\,\delta(y - y'), \tag{30}$$

$$\left(\frac{d}{dy} + \frac{2}{l}\right) G(x, y\,|\,x', y')\Big|_{y=y_\pm} = 0. \tag{31}$$

The kernel of this Green's function rewritten in x-space as the operator function of $\Box$ parametrically depends on y and y',

$$G(x, y\,|\,x', y') = l\,G(y, y'|\,\Box)\,\delta(x, x'). \tag{32}$$

Then, the dimensionless elements of the matrix $\mathbf{G}(\Box) = G_{IJ}(\Box)$, $I, J = \pm$, in (29) are

$$G_{IJ}(\Box) = G(y_I, y_J|\,\Box). \tag{33}$$

In the next two sections we derive the braneworld effective action (23) and then consider its interpretation and applications in braneworld physics. In particular, the concrete form of the fundamental operator $\mathbf{F}(\Box)$ will be given in Sec. V–VI where various energy limits of this nonlocal form factor will be considered in much detail[6]. scite

IV. DERIVATION OF THE BRANEWORLD EFFECTIVE ACTION

A. The effective equations of motion

The effective equations, as variations of the action with respect to brane metrics, see (12), are equivalent to Israel junction conditions with the five-dimensional metric coefficients $G_{AB}(x, y)$ (entering extrinsic curvatures of the branes) expressed in terms of $g^{\pm}_{\mu\nu}(x)$ and $T^{\mu\nu\,\pm}(x)$. So to disentangle effective four-dimensional equations we have to solve the bulk equations of motion for $G_{AB}(x, y)$ in terms of $g^{\pm}_{\mu\nu}(x)$ and $T^{\mu\nu\,\pm}(x)$ and substitute the result in the junction conditions. Implicitly this has been done at the exact level in [4] where all unknown terms were isolated in form of the five-dimensional Weyl tensor and its derivatives. For the linearized theory this can be done explicitly along the lines of [14].

We start by linearizing the five-dimensional Einstein equations (8) in terms of metric perturbations $h_{AB}(x, y)$ on the background of the Randall-Sundrum solution (15). One can

[6] Eq. (23) gives the action in the approximation quadratic in curvature perturbations and radion fields $(\mathbf{R}_{\mu\nu}, \mathbf{\Psi})$ with all higher order terms discarded. In this approximation it is sufficient to keep the nonlocal form factors of the quadratic form in (23) as flat-space ones and build the d'Alembertian in cartesian coordinates in terms of partial derivatives. In particular, in these coordinates there is no need to write the covariant density weights of $g^{1/2}(x)$ which become non-trivial only in higher orders of curvature expansion. In an arbitrary curvilinear coordinate system the expression (23) should be appropriately covariantized by the technique of the covariant perturbation theory of [34, 35].

always go to the coordinate system with $h_{55} = h_{\mu 5} = 0$, $h^{\nu}_{\mu\,,\nu} = 0$, $h^{\mu}{}_{\mu} = 0$ [16, 20][7], in which the linearized equations in the bulk simplify to a single equation for transverse-traceless $h_{\mu\nu}$,

$$\left(\frac{d^2}{dy^2} - \frac{4}{l^2} + \frac{\Box}{a^2(y)}\right) h_{\mu\nu}(x,y) = 0, \tag{34}$$

The Randall-Sundrum coordinate system is, however, not Gaussian normal relative to $\Sigma_\pm$ — both branes in these coordinates are *not* located at constant values of the fifth coordinate $y_\pm$. Therefore, let us introduce two coordinate systems which are Gaussian normal relative to their respective branes and mark them by $(\pm)$ [20]. Metric perturbations in these coordinate systems, denoted correspondingly by $h^{(\pm)}_{\mu\nu}(x,y)$, also have only $\mu\nu$-components, Σ_+ is located at constant value y_+ in the $(+)$-coordinates, while Σ_- has in these coordinates a non-trivial embedding. Vice versa, in $(-)$-coordinates Σ_- is located at fixed value y_-, while Σ_+ is embedded non-trivially. Metric perturbations $h^{(\pm)}_{\mu\nu}(x,y)$ in $(\pm)$-coordinates are related to those of the Randall-Sundrum coordinate system, $h_{\mu\nu}(x,y)$, by remnant coordinate transformations — the transformations that preserve the conditions $h_{5A}(x,y) = 0$. Every such transformation is parametrized by one four-dimensional scalar field $\xi(x)$ and one four-dimensional vector field $\xi_\mu(x)$ and has a fixed dependence on the fifth coordinate [14]. These transformations to two Gaussian normal coordinate systems read

$$h^{(\pm)}_{\mu\nu}(x,y) = h_{\mu\nu}(x,y) + l\xi^{\pm}_{,\mu\nu}(x) + \frac{2}{l}\eta_{\mu\nu}a^2(y)\,\xi^{\pm}(x) + a^2(y)\,\xi^{\pm}_{(\mu,\nu)}(x) \tag{35}$$

and, thus, give rise to two radion fields $\xi^{\pm}(x)$ and two four-dimensional vector fields $\xi^{\pm}_{\mu}(x)$.

In the $(\pm)$ Gaussian normal coordinates, five-dimensional metrics $g^{(\pm)}_{\mu\nu}(x,y)$ are given by Eq. (15) with perturbations $h^{(\pm)}_{\mu\nu}(x,y)$,

$$ds^2 = dy^2 + e^{-2|y|/l}\eta_{\mu\nu}dx^\mu dx^\nu + h^{(\pm)}_{\mu\nu}(x,y)\,dx^\mu dx^\nu. \tag{36}$$

The extrinsic curvatures of $\Sigma_\pm$ in these coordinates simplify to $K^{\pm}_{\mu\nu} = \mp\frac{1}{2}dg^{(\pm)}_{\mu\nu}(x,y)/dy\,|_{y=y_\pm}$, where the sign factor originates from different orientation of outward normals to the corresponding branes. Therefore, the linearized boundary conditions (9) take the form

$$\left(\frac{d}{dy} + \frac{2}{l}\right)h^{(\pm)}_{\mu\nu}(x,y)\Big|_{y=y_\pm} = \mp 8\pi G_5\Big(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T\Big)^{\pm}, \tag{37}$$

where[8]

$$T^{\pm}_{\mu\nu} \equiv a^4_\pm\eta_{\mu\alpha}\eta_{\nu\beta}T^{\alpha\beta}_{\pm} = 2\eta_{\mu\alpha}\eta_{\nu\beta}\frac{\delta S_{\rm m}}{\delta h^{\pm}_{\alpha\beta}}, \quad T^{\pm} \equiv \eta^{\mu\nu}T^{\pm}_{\mu\nu}. \tag{38}$$

[7] Usually this is called the Randall-Sundrum gauge. However, this is a combination of the gauge-fixing procedure and the use of the non-dynamical (constraint) part of linearized Einstein equations in the bulk.

[8] In the definition of $T_{\mu\nu}$ we have to deviate from the general rule of lowering the indices with the flat metric $\eta_{\mu\nu}$ because in the exact theory, the covariant stress tensor is related to the contravariant one via the full metric $g^{\pm}_{\mu\nu} = a^2_\pm\eta_{\mu\nu} + O(h_{\mu\nu})$.

Here we took into account that in view of the orbifold Z_2-symmetry the curvature jump is $[K_{\mu\nu}]^{\pm} = 2K^{\pm}_{\mu\nu}$. Using (35) in (37) and taking into account that $(d/dy + 2/l)a^2(y) = 0$ we finally find

$$\Big(\frac{d}{dy} + \frac{2}{l}\Big)h_{\mu\nu}(x,y)\Big|_{y=y_\pm} = \mp 8\pi G_5 \Big(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T\Big)^{\pm} - 2\xi^{\pm}_{,\mu\nu}. \tag{39}$$

These are the linear boundary conditions on perturbations $h_{\mu\nu}(x,y)$ in the Randall-Sundrum gauge[9].

By taking the trace of this equation one finds on account of tracelessness of $h_{\mu\nu}$ the dynamical four-dimensional equation for radion fields [14],

$$\Box\xi^{\pm}(x) = \pm\frac{8\pi G_5}{6}T^{\pm}(x). \tag{40}$$

Now the system of equations and the boundary conditions becomes complete. Eq. (40) determines the radions, while the boundary value problem (34) and (39) determines the perturbations in the bulk. The solution to this problem can be given in terms of the Green's function of the equation (34) with homogeneous Neumann boundary conditions. This Green's function satisfies (30)–(31). The desired solution reads

$$\begin{aligned} &h_{\mu\nu}(x,y) = l\,G(y,y_+|\,\Box)\,w^{+}_{\mu\nu}(x) - l\,G(y,y_-|\,\Box)\,w^{-}_{\mu\nu}(x), \\ &w^{\pm}_{\mu\nu} = \mp 8\pi G_5\left(T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T\right)^{\pm} - 2\xi^{\pm}{}_{,\mu\nu}\,, \end{aligned} \tag{41}$$

where for brevity we have denoted the right-hand sides of the (inhomogeneous) boundary conditions (39) by $w^{\pm}_{\mu\nu}$ and used the shorthand notation for the kernel of the five-dimensional Green's function (32), which allows one to omit the x-integration signs.

We will be interested in the effective dynamics of the observable fields only — the induced metric perturbations on the branes $h^{\pm}_{\mu\nu}(x)$. In the notation of Sec. III they form the column (18), and the Green's function of Eq. (41) can be regarded as the 2×2-matrix (33), acting in the space of such columns. With these notations the observable metric perturbations in the Randall-Sundrum gauge read as the following (nonlocal) linear combination of stress tensors and radion fields,

$$\begin{bmatrix} h_{\mu\nu}(x,y_+) \\ h_{\mu\nu}(x,y_-) \end{bmatrix} = -8\pi G_5 l\,\mathbf{G}(\Box)\begin{bmatrix} T^{+}_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T^{+} \\ T^{-}_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T^{-} \end{bmatrix} - 2l\,\mathbf{G}(\Box)\begin{bmatrix} \xi^{+}{}_{,\mu\nu} \\ -\xi^{-}{}_{,\mu\nu} \end{bmatrix}. \tag{42}$$

These perturbations do not, however, coincide with those of the *induced* metrics on branes, $h^{\pm}_{\mu\nu}$. The latter coincide with the metric coefficients in two respective Gaussian normal coordinate systems, $h^{\pm}_{\mu\nu}(x) \equiv h^{(\pm)}_{\mu\nu}(x,y_\pm)$. Therefore, $h^{\pm}_{\mu\nu}(x)$ is related to $h_{\mu\nu}(x,y_\pm)$ by

[9] Strictly speaking, in the Randall-Sundrum coordinate system the branes are shifted from the constant values $y_\pm$. However, this displacement is of first order of magnitude in $\xi^{\pm}(x) \sim h^{\pm}_{\mu\nu}(x)$ and, therefore, it contributes to (39) only in higher orders of perturbation theory.

Eq. (35) and read in column notations of Eq.(18)

$$\mathbf{h}_{\mu\nu}(x) = -8\pi G_5 l\,\mathbf{G}(\Box)\Big(\mathbf{T}_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}\mathbf{T}\Big) + \frac{2}{l}\eta_{\mu\nu}\begin{bmatrix}\xi^+\\ a^2\xi^-\end{bmatrix} + l\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} - 2\mathbf{G}(\Box)\right)\begin{bmatrix}\xi^+_{,\mu\nu}\\ -\xi^-_{,\mu\nu}\end{bmatrix} + \begin{bmatrix}\xi^+_{(\mu,\nu)}\\ a^2\xi^-_{(\mu,\nu)}\end{bmatrix}, \quad (43)$$

where we took into account that $a_+ = 1$ and $a_- \equiv a$, Eq. (17).

From the four-dimensional viewpoint, the last two terms in this expression represent two diffeomorphisms on the branes that can be gauged away, so that the induced metric perturbations (we denote them in the new gauge by $H^{\pm}_{\mu\nu}$) finally read

$$\mathbf{H}_{\mu\nu} = -8\pi G_5 l\,\mathbf{G}(\Box)\Big(\mathbf{T}_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}\mathbf{T}\Big) + \frac{2}{l}\eta_{\mu\nu}\begin{bmatrix}\xi^+\\ a^2\xi^-\end{bmatrix}. \quad (44)$$

The gauge conditions for $H^{\pm}_{\mu\nu}(x)$ are unknown and differ from those of $h^{\pm}_{\mu\nu}(x)$ (also unknown ones), but they will be determined later when covariantizing the effective equations of motion in terms of four-dimensional curvatures.

In Eq. (44) we have a typical five-dimensional combination of the stress tensor and its trace, $T_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T$. This can be rewritten as the four-dimensional combination $T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T$ plus the contribution of the trace $\frac{1}{6}\eta_{\mu\nu}T$ which can be expressed in terms of ξ according to (40). Then Eq. (44) takes the form

$$\mathbf{H}_{\mu\nu} = -8\pi G_5 l\,\mathbf{G}(\Box)\Big(\mathbf{T}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathbf{T}\Big) + l\eta_{\mu\nu}\Box\mathbf{G}_\xi(\Box)\begin{bmatrix}\xi^+\\ \xi^-\end{bmatrix}, \quad (45)$$

where the new operator $\mathbf{G}_\xi(\Box)$ reads in terms of the brane separation parameter a as

$$\mathbf{G}_\xi(\Box) \equiv \mathbf{G}(\Box)\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix} + \frac{2}{l^2\,\Box}\begin{bmatrix}1 & 0\\ 0 & a^2\end{bmatrix}. \quad (46)$$

As we will see below, such a rearrangement is very illuminating when recovering the effective four-dimensional Einstein theory in the low-energy approximation.

The next step consists in the covariantization of these equations. Thus far they are written in terms of brane metric perturbations $h^{\pm}_{\mu\nu} = H^{\pm}_{\mu\nu}$ in a particular gauge corresponding to omission of gauge transformation terms on the right-hand side of (43). It is always useful to have the dynamical equations in gauge independent form. This form can be easily attained by rewriting them in terms of the curvature. For this purpose we first determine explicitly the gauge conditions for the perturbations $H^{\pm}_{\mu\nu}$ and then express these perturbations in terms of the linearized Ricci tensors of brane metrics. The gauge for $H^{\pm}_{\mu\nu}$ can be found by going back to the original transverse-traceless perturbations $h^{\pm}_{\mu\nu}(x, y_\pm)$. However, it is much easier to recover these gauge conditions directly from the equations of motion (44). By applying the first-order differential operator of harmonic gauge conditions to $H^{\pm}_{\mu\nu}$ in (44) and using the conservation law for matter stress tensors we find that

$$\mathbf{H}^{\nu}_{\mu\,,\,\nu} - \frac{1}{2}\mathbf{H}_{,\mu} = -l\Box\mathbf{G}_\xi(\Box)\begin{bmatrix}\xi^+_{,\mu}\\ \xi^-_{,\mu}\end{bmatrix}. \quad (47)$$

These equations serve as a set of generalized harmonic gauge conditions on brane metrics $H^{\pm}_{\mu\nu}$ and radions $\xi^{\pm}$. Now, one can use them in Eq. (20) for the linearized Ricci tensor in order to obtain the equation for $H_{\mu\nu}$ that can be solved by iterations in powers of $R_{\mu\nu}$ and ξ [34]. In the linear approximation this solution reads

$$\mathbf{H}_{\mu\nu} = -\frac{2}{\Box}\mathbf{R}_{\mu\nu} - 2l\,\mathbf{G}_{\xi}(\Box)\left[\begin{array}{c}\xi^{+}_{,\mu\nu}\\ \xi^{-}_{,\mu\nu}\end{array}\right]. \tag{48}$$

Using (48) in Eq. (44), we rewrite the latter in terms of the linearized Einstein tensors $E^{\pm}_{\mu\nu} = R^{\pm}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R^{\pm}$ of the brane metrics,

$$-\frac{2}{\Box}\mathbf{E}_{\mu\nu} + 8\pi G_5\,l\,\mathbf{G}(\Box)\mathbf{T}_{\mu\nu} - 2l\,(\nabla_\mu\nabla_\nu - \eta_{\mu\nu}\Box)\,\mathbf{G}_{\xi}(\Box)\left[\begin{array}{c}\xi^{+}\\ \xi^{-}\end{array}\right] = 0. \tag{49}$$

Now these equations are covariant and equally valid in terms of brane metrics $g^{\pm}_{\mu\nu} = a^2_{\pm}\eta_{\mu\nu} + h^{\pm}_{\mu\nu}$ with perturbations $h^{\pm}_{\mu\nu}$ taken in *any* gauge, not necessarily coinciding with that of $H^{\pm}_{\mu\nu}$.

As it was discussed earlier, we have to find the action that generates these equations by the variational procedure. For this purpose we must find the integrating factor — the overall matrix valued coefficient — with which the left-hand side of (49) enters the variational derivative of the action, see (12). To find it, we note that matter fields living on branes are directly coupled only to $g^{\pm}_{\mu\nu}$ via their stress tensors (6) and, moreover, the matter action additively enters the full effective action (11). Therefore, in the linear approximation the overall coefficients of $T^{\mu\nu}_{\pm}$ in the variational derivatives of the action should be the local numericals $\frac{1}{2}a^4_{\pm}$ (remember that in cartesian coordinates $g^{1/2}_{\pm} = a^4_{\pm} + O(h_{\mu\nu})$). To achieve this we raise the indices in (49) with $\eta^{\mu\nu}$ and act upon it by the operator $\mathbf{F}(\Box)/16\pi G_5 l$, where $\mathbf{F}(\Box)$ is the inverse of the Green's function $\mathbf{G}(\Box)$, Eq. (29). Then, the left hand side of eq. (49) as a metric variational derivative takes the form ($\delta/\delta\mathbf{g}_{\mu\nu}$ denotes the column of derivatives with respect to $g^{\pm}_{\mu\nu}$)

$$\begin{aligned}\frac{\delta}{\delta\mathbf{g}_{\mu\nu}}S_{\rm eff}\left[\,g^{\pm},\xi^{\pm},\phi^{\pm}\,\right] = &-\frac{1}{8\pi G_5 l}\frac{\mathbf{F}(\Box)}{\Box}\,\mathbf{E}^{\mu\nu} + \frac{1}{2}\mathbf{g^{1/2}T}^{\mu\nu}\\ &-\frac{1}{8\pi G_5}\,(\nabla^\mu\nabla^\nu - \eta^{\mu\nu}\Box)\,\mathbf{F}(\Box)\,\mathbf{G}_{\xi}(\Box)\left[\begin{array}{c}\xi^{+}\\ \xi^{-}\end{array}\right].\end{aligned} \tag{50}$$

Here we take into account that in view of the definition (38) $\eta^{\mu\alpha}\eta^{\nu\beta}T^{\pm}_{\alpha\beta} = a^4_{\pm}T^{\mu\nu}_{\pm} = g^{1/2}_{\pm}T^{\mu\nu}_{\pm}$.

Our next goal is to rewrite the equations for the radions (40) also as variational equations for the braneworld effective action. Again, matter cannot be directly coupled to radions because the matter action additively enters the full action. Therefore, the stress tensor part of (40) should be reexpressed in terms of the curvature. By taking the trace of (50) and using (40) one can exclude the stress tensor traces in terms of $R^{\pm}$ and $\xi^{\pm}$, so that the dynamical equations for the latter reduce, in view of (46), to

$$\left[\begin{array}{c}R^{+}\\ R^{-}\end{array}\right] + \frac{6}{l}\left[\begin{array}{cc}1 & 0\\ 0 & a^2\end{array}\right]\Box\left[\begin{array}{c}\xi^{+}\\ \xi^{-}\end{array}\right] = 0. \tag{51}$$

Actually, Eq. (51) has a simple physical interpretation. It is the linearised equation of motion for worldsheet perturbations of the brane coupled to bulk perturbations [36]. In the next section we recover the braneworld effective action that generates the full set of equations (50) and (51).

B. The recovery of the effective action

From the structure of (50) it is obvious that the graviton-radion part of the full braneworld action $S_4\,[\,g^\pm_{\mu\nu},\xi^\pm]$ is given by the sum of the purely gravitational part, the non-minimal coupling of the radions to the curvatures and the radion action itself,

$$S_4\,[\,g^\pm_{\mu\nu},\xi^\pm] = S_{\rm grav}[\,g^\pm_{\mu\nu}] + S_{\rm n-m}[\,g^\pm_{\mu\nu},\xi^\pm] + S_{\rm rad}\,[\,g^\pm_{\mu\nu},\xi^\pm]. \tag{52}$$

Here $S_{\rm grav}$ and $S_{\rm n-m}$ give as contributions to the variational derivative of the action the first and the third terms on the right-hand side of (50), while $S_{\rm rad}[\,g^\pm_{\mu\nu},\xi^\pm]$ still has to be found by comparing its radion variational derivative with (51).

Due to the linearity of (51) in the metric perturbations, $S_{\rm grav}+S_{\rm n-m}$ can be obtained from the right-hand side of (50) by contracting it with the row of metric perturbations $\mathbf{H}^T_{\mu\nu}$ and integrating over x. Using (48), one can convert the result to the final form quadratic in Ricci curvatures

$$S_{\rm grav}[\,g^\pm] = \frac{1}{8\pi G_5 l}\int d^4x\,\mathbf{R}^T_{\mu\nu}\,\frac{\mathbf{F}(\Box)}{\Box^2}\,\mathbf{E}^{\mu\nu}, \tag{53}$$

$$S_{\rm n-m}[\,g^\pm,\xi^\pm] = -\,\frac{1}{8\pi G_5}\int d^4x\,\Big[\,\xi^+\;\;\xi^-\Big]\,\mathbf{F}(\Box)\mathbf{G}_\xi(\Box)\,\mathbf{R}. \tag{54}$$

With the action (54) for the non-minimal coupling the equations of motion for the radions read

$$\begin{bmatrix}\delta/\delta\xi^+\\ \delta/\delta\xi^-\end{bmatrix}\Big(S_{\rm rad}+S_{\rm n-m}\Big) = \begin{bmatrix}\delta/\delta\xi^+\\ \delta/\delta\xi^-\end{bmatrix}\,S_{\rm rad} - \frac{1}{8\pi G_5}\mathbf{G}^T_\xi(\Box)\mathbf{F}^T(\Box)\,\mathbf{R} = 0. \tag{55}$$

The comparison with (51) shows that

$$\begin{bmatrix}\delta/\delta\xi^+\\ \delta/\delta\xi^-\end{bmatrix}\,S_{\rm rad} = -\frac{6}{8\pi G_5 l^3}\begin{bmatrix}1 & 0\\ 0 & a^2\end{bmatrix}\mathbf{K}(\Box)\begin{bmatrix}1 & 0\\ 0 & a^2\end{bmatrix}\begin{bmatrix}\xi^+\\ \xi^-\end{bmatrix}, \tag{56}$$

$$\mathbf{K}(\Box) \equiv \begin{bmatrix}1 & 0\\ 0 & 1/a^2\end{bmatrix}\mathbf{G}^T_\xi(\Box)\mathbf{F}^T(\Box)\,l^2\Box\;. \tag{57}$$

As follows from the definition of $\mathbf{G}_\xi(\Box)$, Eq. (46), the matrix $\mathbf{K}(\Box)$ is given by Eq. (28). Thus, it is symmetric in view of the symmetry of $\mathbf{F}(\Box)$. Eq. (56) is therefore integrable and the radion action acquires the quadratic form with this new matrix valued operator $\mathbf{K}(\Box)$ as a kernel. The equations become simpler if we introduce instead of $\xi^\pm$ the new *dimensionless* radion fields

$$\psi^+ = \frac{\xi^+}{l},\quad \psi^- = a^2\,\frac{\xi^-}{l}, \tag{58}$$

in terms of which the non-minimal coupling and radion actions read

$$S_{\rm n-m}[\,g^{\pm},\xi^{\pm}] = -\,\frac{l}{8\pi G_5}\int d^4x\,\mathbf{R}^T\,\frac{\mathbf{K}(\Box)}{l^2\Box}\,\mathbf{\Psi}, \tag{59}$$

$$S_{\rm rad}[\,g^{\pm},\xi^{\pm}] = -\,\frac{3l}{8\pi G_5}\int d^4x\,\mathbf{\Psi}^T\,\frac{\mathbf{K}(\Box)}{l^2}\,\mathbf{\Psi}\;. \tag{60}$$

Thus, the sum of gravitational (53), non-minimal (59) and radion (60) parts form the full braneworld effective action in the approximation quadratic in fields. Collecting the radion and non-minimal parts one can easily recover the last term of (23) — the 2×2 quadratic form in $(\Box\mathbf{\Psi}+\mathbf{R}/6)$ — plus an extra term quadratic in the Ricci scalars on the branes $\mathbf{R}$. When combined with (53), the latter gives rise to the first two terms of (23). This accomplishes the derivation of the basic result advocated in Sec. III. The nonlocal kernels in the quadratic forms of (23) are built in terms of 2×2-matrix-valued operators $\mathbf{F}(\Box)$ and $\mathbf{K}(\Box)$ which we analyze in the next section.

V. NONLOCAL FORM FACTORS OF THE BRANEWORLD ACTION

To construct the operators $\mathbf{F}(\Box)$ and $\mathbf{K}(\Box)$ we need the Green's function of the boundary value problem (30)–(31). This problem simplifies to the Bessel's equation in terms of the new variable z,

$$z = l\exp\frac{y}{l},\quad a(y)=\frac{l}{z}. \tag{61}$$

For the dimensionless function $\bar G(z,z'\,|\,\Box)\equiv G(y,y'\,|\,\Box)$ defined by Eq. (32) it reads as

$$\left(\frac{d}{dz}z\frac{d}{dz}+z\Box-\frac{4}{z}\right)\bar G(z,z'\,|\,\Box)=\delta(z-z'), \tag{62}$$

$$\frac{d}{dz}z^2\bar G(z,z'\,|\,\Box)\,\Big|_{z=z_\pm}=0. \tag{63}$$

This Green's function can be built in terms of the basis functions $u_\pm(z)=u_\pm(z|\,\Box)$ of Eq. (62) — two linearly independent solutions of the homogeneous Bessel equation satisfying Neumann boundary conditions at z_+ and z_-, respectively,

$$\left(\frac{d}{dz}z\frac{d}{dz}+z\Box-\frac{4}{z}\right)u_\pm(z)=0,\quad z_+\le z\le z_-, \tag{64}$$

$$\frac{d}{dz}z^2u_\pm(z)\,\Big|_{z=z_\pm}=0. \tag{65}$$

They are given by linear combinations of Bessel and Neumann functions of the second order, $Z_2(z\sqrt{\Box})=(J_2(z\sqrt{\Box}),Y_2(z\sqrt{\Box}))$, with the coefficients easily derivable from the boundary conditions on account of the relation $(d/dx)x^2Z_2(x)=x^2Z_1(x)$,

$$u_\pm(z)=Y_1^\pm J_2(z\sqrt{\Box})-J_1^\pm Y_2(z\sqrt{\Box}), \tag{66}$$

$$J_1^\pm\equiv J_1(z_\pm\sqrt{\Box}),\quad Y_1^\pm\equiv Y_1(z_\pm\sqrt{\Box}). \tag{67}$$

In what follows we introduce the abbreviation for the Bessel function of any order $Z_\nu^\pm \equiv Z_\nu(z_\pm\sqrt{\Box})$ to avoid excessive use of their different arguments.

In terms of $u_\pm(z)$, the Green's function has a well known representation,

$$\bar{G}(z,z'|\,\Box) = \theta(z-z')\frac{u_-(z)u_+(z')}{\Delta} + \theta(z'-z)\frac{u_+(z)u_-(z')}{\Delta}, \tag{68}$$

where $\theta(z)$ is the step function, $\theta(z)=1,\ z\geq 0$, $\theta(z)=0,\ z<0$, and Δ is the conserved Wronskian inner product of basis functions in the space of solutions of Eq. (64),

$$\Delta \equiv z\Big(u_+(z)\frac{d}{dz}u_-(z) - u_-(z)\frac{d}{dz}u_+(z)\Big) = \frac{2}{\pi}\left(J_1^+Y_1^- - Y_1^+J_1^-\right). \tag{69}$$

The calculation of the 2×2-matrix Green's function (33) on the basis of (68) requires the knowledge of $u_+(z_\pm)$ and $u_-(z_\pm)$. Some of these simplify to elementary functions, $u_\pm(z_\pm) = Y_1^\pm J_2^\pm - J_1^\pm Y_2^\pm = 2/\pi z_\pm\sqrt{\Box}$. Using this result together with (69) in (68), one finds the following exact expression for $\mathbf{G}(\Box)$,

$$\mathbf{G}(\Box) = \frac{1}{\Box z_+z_-}\frac{1}{J_1^+Y_1^- - J_1^-Y_1^+}\begin{bmatrix} \sqrt{\Box}z_-u_-(z_+) & \dfrac{2}{\pi} \\ \dfrac{2}{\pi} & \sqrt{\Box}z_+u_+(z_-)\end{bmatrix}. \tag{70}$$

The Green's functions in $\mathbf{G}(\Box)$ have first been calculated in [37].

Interestingly, its inverse $\mathbf{F}(\Box)$ can be represented, in essence, in a form dual to this expression. Indeed, for the inversion of the matrix (70) we need its determinant which can be shown to equal[10]

$$\det\mathbf{G}(\Box) = -\frac{1}{z_+z_-\Box}\frac{Y_2^-J_2^+ - Y_2^+J_2^-}{Y_1^-J_1^+ - Y_1^+J_1^-}, \tag{71}$$

so that $\mathbf{F}(\Box)$ assumes a form structurally similar to (70),

$$\mathbf{F}(\Box) = -\frac{1}{J_2^+Y_2^- - J_2^-Y_2^+}\begin{bmatrix} \sqrt{\Box}z_+u_+(z_-) & -\dfrac{2}{\pi} \\ -\dfrac{2}{\pi} & \sqrt{\Box}z_-u_-(z_+)\end{bmatrix}. \tag{72}$$

This exact expression for $\mathbf{F}(\Box)$ will be important for us in what follows when considering the low-energy approximation, because a direct expansion of (72) in powers of $\Box$ turns out to be much simpler than the expansion of (70) with its subsequent inversion.

A. The low-energy limit — recovery of Einstein theory

The Green's function (70) and its inverse operator (72) are nonlinear functions of $\Box$. This fact corresponds to the essentially nonlocal nature of the effective four-dimensional

[10] To derive this representation for the determinant one should explicitly use the bilinear in Bessel functions expression for $u_\pm(z_\pm)$ in the off-diagonal elements of (70), and then notice that the numerator of the determinant factorizes into the product $(Y_2^-J_2^+ - Y_2^+J_2^-)(J_1^+Y_1^- - J_1^-Y_1^+)$.

theory induced on the branes from the bulk. A remarkable property of the one-brane Randall-Sundrum model is however that the low-energy approximation of the effective four-dimensional theory is (quasi)local and corresponds to Einstein gravity minimally coupled to matter fields on the branes. In the two–brane model the situation is more complicated — the low-energy theory belongs to the Brans-Dicke type with a non-minimal curvature coupling of the extra scalar field that mediates the interaction of the metric with the trace of the matter stress tensor [14].

We begin considering the low-energy effective theory in the leading order approximation by taking the limit $\Box \to 0$ of the Green's function $\mathbf{G}(\Box)$, (70). By using the well-known asymptotics of small argument for Bessel and Neumann functions one finds the low-energy behavior of various ingredients of Eqs. (70) and (72) as expansions in integer and half-integer powers of $\Box$ and, thus, obtains the leading order approximation in $\Box \to 0$

$$\mathbf{G}(\Box) = \frac{2}{l^2\Box}\,\frac{1}{1-a^2}\left[\begin{array}{cc} 1 & a^2 \\ a^2 & a^4 \end{array}\right] + O(l^2\Box), \tag{73}$$

$$\mathbf{G}_\xi(\Box) = -\frac{2}{l^2\Box}\,\frac{a^2}{1-a^2}\left[\begin{array}{cc} 1 & -1 \\ 1 & -1 \end{array}\right] + O(l^2\Box), \tag{74}$$

where a is the brane separation parameter, $a = e^{-d/l}$.

With these Green's functions the effective equations (49) take the form of the following two (linearized) Einstein equations,

$$E^+_{\mu\nu} = 8\pi\Big(G^+T^+_{\mu\nu} + G^-T^-_{\mu\nu}\Big) + \frac{1}{l}\,\frac{e^{-d/l}}{\sinh(d/l)}\,(\nabla_\mu\nabla_\nu - \eta_{\mu\nu}\Box)(\xi^+ - \xi^-), \tag{75}$$

$$E^-_{\mu\nu} = 8\pi e^{-2d/l}\Big(G^+T^+_{\mu\nu} + G^-T^-_{\mu\nu}\Big) + \frac{1}{l}\,\frac{e^{-d/l}}{\sinh(d/l)}\,(\nabla_\mu\nabla_\nu - \eta_{\mu\nu}\Box)(\xi^+ - \xi^-). \tag{76}$$

Matter sources on both branes are coupled to gravity with the effective four-dimensional gravitational constants $G^\pm$ [14] depending on the brane separation,

$$G^\pm = \frac{G_5}{l}\,\frac{e^{\pm d/l}}{2\sinh(d/l)}, \tag{77}$$

and there is also a non-minimal curvature coupling to a particular combination of radion fields $(\xi^+ - \xi^-)$ which, obviously, describes dynamical disturbances of the interbrane distance. Thus, the low-energy theory reduces to the generalized Brans-Dicke model — the fact that was first observed in [14] (see also [38] on the realization of this property in braneworld scenarios with bulk scalar fields).

For large distance between the branes, $a = e^{-d/l} \ll 1$, the metric field on the positive-tension brane decouples from all fields on the other brane and from radions because

$$G^+ = \frac{G_5}{l}\,\frac{1}{1-e^{-2d/l}} \to G_4, \;\; G^- \simeq G_4 e^{-2d/l} \to 0, \;\; d \to \infty, \tag{78}$$

and the low-energy theory on this brane (usually called Planckian) reduces to Einstein gravity with the four-dimensional gravitational constant (24). This is a manifestation of the so-called graviton zero-mode localization in the one-brane Randall-Sundrum model [13] or the AdS/CFT-correspondence [15, 17, 23]. The recovery of Einstein theory on the Planckian brane is the result of a non-trivial cancellation between the contributions of the stress tensor trace and the radion in the right-hand side of the effective equations of motion (44). This cancellation leads to Eq. (45) with an exponentially small matrix $\mathbf{G}_\xi(\Box) \sim a^2 \to 0$, (74), of the non-minimal coupling to radions.

For finite interbrane distance, both stress tensors $T^\pm_{\mu\nu}$ contribute to the right-hand sides of Einstein's equations (75)–(76). They contribute to $E^\pm_{\mu\nu}$ with different strengths (their contribution to the negative tension brane metric is $e^{-2d/l}$ times weaker), but in one and the same combination $G^+T^+_{\mu\nu} + G^-T^-_{\mu\nu}$. This, maybe physically obvious, fact has a crucial consequence for the structure of the off-shell extension of the effective braneworld theory. Mathematically this property manifests itself in the degeneracy of the leading-order matrix Green's function (73) and results, as we will see below, in the presence of a massive graviton mode. Another degeneration that occurs in the low-energy limit is the fact that among two radion fields, that were introduced on a kinematical ground as independent entities, only their combination $(\xi^+ - \xi^-)$ is dynamical. Apparently, this is the explanation why only one radion field is usually considered as a dynamically relevant variable in the two-brane Randall-Sundrum model (see [30] where this property was explained by the homogeneity of the AdS background). Such a degeneration, as we see, is not fundamental, but turns out to be an artifact of the adopted low-energy approximation scheme.

The braneworld effective action requires the knowledge of $\mathbf{F}(\Box)$ — the inverse of the Green's function $\mathbf{G}(\Box)$. The latter, as we have just seen, is degenerate in the lowest order of the $\Box$-expansion (73), which is why we have to go beyond this approximation. In this expansion one can single out two distinctly different physical regimes characterized by the values of the dimensionless parameters $l^2\Box$ and a. One regime corresponds to small energies compared to the AdS-scale, $\Box \ll 1/l^2$ or $l^2\Box \ll 1$, and small or finite interbrane distance $a = O(1)$, such that $l^2\Box/a^2 \ll 1$. The last requirement implies that the physical energy range is small also at the negative tension brane, because the physical energy at this brane is determined by $\sqrt{g^{\mu\nu}_-\partial_\mu\partial_\nu} = \sqrt{\Box}/a$.

Another regime corresponds to small energies on the positive tension brane and large energies on the negative tension one, $l^2\Box \ll 1$ and $l^2\Box/a^2 \gg 1$, when $a \ll 1$. This limit includes, in particular, the situation when the second brane is pushed to infinity of the fifth coordinate (the AdS horizon) and qualitatively is equivalent to the one-brane situation[11].

B. Low-energy derivative expansion

Here we consider the derivative expansion of the nonlocal form factors in the first of the low-energy regimes, corresponding to small or finite interbrane distance. In this regime the

[11] The situation with an infinitely remote second brane is not entirely equivalent to the one-brane model, because the Israel junction condition on the second brane is different from the usually assumed Hartle boundary conditions on the AdS horizon, see Sec. VI B.

arguments of both sets of Bessel and Neumann functions in (70)–(72) are small

$$l\sqrt{\Box} \ll 1, \quad \frac{l\sqrt{\Box}}{a} \ll 1. \tag{79}$$

Therefore, one can expand $\mathbf{G}(\Box)$ to higher than zeroth order in $\Box$ and, thus, make it explicitly invertible. Since $\mathbf{G}(\Box)$ is degenerate in the zeroth order, its matrix determinant is at least $O(1/\Box)$ rather than $O(1/\Box^2)$ and, therefore, one should expect that its inverse $\mathbf{F}(\Box)$ will be a massive operator — its expansion in powers of $\Box$ will start with the mass matrix $O(\Box^0)$. This degeneracy leads also to an additional difficulty — in order to achieve the kinetic term in $\mathbf{F}(\Box)$ linear in $\Box$ one would have to calculate the Green's function $\mathbf{G}(\Box)$ to $O(\Box^2)$ inclusive. Fortunately, instead of inverting the Green's function expansion, we have the exact expression (72) for $\mathbf{F}(\Box)$ that can be directly expanded to a needed order. Thus, using the small argument expansions of Bessel functions in (72) we get

$$\frac{\mathbf{F}(\Box)}{l^2} = -\mathbf{M}_F + \mathbf{D}_F\Box + \mathbf{F}^{(2)}l^2\Box^2 + O(\Box^3), \tag{80}$$

$$\mathbf{M}_F = \frac{1}{l^2}\,\frac{4}{1-a^4}\begin{bmatrix} a^4 & -a^2 \\ -a^2 & 1 \end{bmatrix}, \tag{81}$$

$$\mathbf{D}_F = \frac{1-a^2}{6(1+a^2)^2}\begin{bmatrix} a^2+3 & 2 \\ 2 & 3+a^{-2} \end{bmatrix}, \tag{82}$$

where the components of the matrix $\mathbf{F}^{(2)}$ are rather lengthy (but we will need them below because they will qualitatively effect the low-energy behaviour in the radion sector),

$$\begin{aligned}
F^{(2)}_{++} &= \frac{(1-a^2)^3(3+a^2)}{72a^2(1+a^2)^3} - \frac{\ln a}{4(1-a^4)^2} - \frac{4+5a^2-4a^4+a^6}{96a^2(1-a^4)}, \\
F^{(2)}_{--} &= \frac{(1-a^2)^3(3a^2+1)}{72a^4(1+a^2)^3} - \frac{a^4\ln a}{4(1-a^4)^2} - \frac{1-4a^2+5a^4+4a^6}{96a^4(1-a^4)}, \\
F^{(2)}_{+-} &= \frac{(1-a^2)^3}{36a^2(1+a^2)^3} + \frac{a^2\ln a}{4(1-a^4)^2} - \frac{1-8a^2+a^4}{96a^2(1-a^4)}.
\end{aligned} \tag{83}$$

The matrix coefficients of this $\Box$-expansion are non-diagonal and, therefore, nontrivially entangle the fields on both branes. An important property of the lowest-order coefficient — the mass matrix $\mathbf{M}_F$ — is that it is degenerate and has rank one. As we will see below, this fact will guarantee the presence of one massless graviton in the spectrum of the braneworld action.

The low-energy expansion for $\mathbf{K}(\Box)$ follows from that of the operator $\mathbf{F}(\Box)$, (80)–(83),

$$\frac{\mathbf{K}(\Box)}{l^2} = -\mathbf{M}_K + \mathbf{D}_K\Box + \mathbf{K}^{(2)}\,l^2\Box^2 + O(\Box^3)\,, \tag{84}$$

$$\mathbf{M}_K = 2\mathbf{M}_F = \frac{8}{l^2}\,\frac{1}{1-a^4}\begin{bmatrix} a^4 & -a^2 \\ -a^2 & 1 \end{bmatrix}, \tag{85}$$

$$\mathbf{D}_K = \frac{2}{3(1+a^2)^2}\begin{bmatrix} -4a^2-2a^4 & 1-a^2 \\ 1-a^2 & 4+2a^{-2} \end{bmatrix}, \tag{86}$$

where $\mathbf{K}^{(2)} = 2\mathbf{F}^{(2)}$. Similarly to the graviton operator $\mathbf{F}(\Box)$, the mass matrix $\mathbf{M}_K$ is degenerate. Moreover, the matrix determinant of the radion operator is at least quadratic in $\Box$ — the property responsible for the dipole-ghost nature of the radion field (see Sect.VI),

$$\det \mathbf{K}(\Box) = \frac{4\ln a}{1-a^4}\,(l^2\Box)^2 + O[\,(l^2\Box)^3]. \tag{87}$$

C. Particle spectrum of the braneworld action and Kaluza-Klein modes

The quadratic approximation for the action and its nonlocal formfactors obviously determines the spectrum of excitations in the theory. Here we show that in the graviton sector this spectrum corresponds to the tower of Kaluza-Klein modes well-known from a conventional Kaluza-Klein setup. The graviton sector arises when one decomposes metric perturbations on both branes into irreducible components — transverse-traceless tensor, vector and scalar parts,

$$h^{\pm}_{\mu\nu} = \gamma^{\pm}_{\mu\nu} + \varphi^{\pm}\eta_{\mu\nu} + f_{\mu,\nu} + f_{\nu,\mu}\,, \quad \gamma_{\mu\nu}{}^{,\nu} = \eta^{\mu\nu}\gamma_{\mu\nu} = 0. \tag{88}$$

On substituting this decomposition in the linearized curvatures of (23) one finds that the vector parts do not contribute to the action, and the latter reduces to the sum of the graviton and scalar sectors,

$$S_4\,[\,g^{\pm}_{\mu\nu}, \psi^{\pm}] = S_{\rm graviton}[\gamma^{\pm}_{\mu\nu}] + S_{\rm scalar}[\,\varphi^{\pm}, \psi^{\pm}]. \tag{89}$$

The graviton part is entirely determined by the operator $\mathbf{F}(\Box)$ and reads

$$S_{\rm graviton}[\gamma^{\pm}_{\mu\nu}] = \frac{1}{16\pi G_4}\int d^4x\,\frac{1}{2}\,[\,\gamma^{+}_{\mu\nu}\ \ \gamma^{-}_{\mu\nu}\,]\,\frac{\mathbf{F}(\Box)}{l^2}\left[\begin{array}{c}\gamma^{\mu\nu}_{+}\\ \gamma^{\mu\nu}_{-}\end{array}\right], \tag{90}$$

while the scalar sector consists of the radion fields of Eq. (25) and the doublets of the trace (or conformal) parts of the metric perturbations $\varphi^{\pm}$,

$$\mathbf{\Phi} = \left[\begin{array}{c}\varphi^{+}(x)\\ \varphi^{-}(x)\end{array}\right], \quad \mathbf{\Phi}^T = \Big[\,\varphi^{+}(x)\ \ \varphi^{-}(x)\Big]. \tag{91}$$

Their action diagonalizes in terms of the conformal modes and the (redefined) radion modes $2\mathbf{\Psi}-\mathbf{\Phi}$,

$$\begin{aligned} S_{\rm scalar}[\,\varphi^{\pm}, \psi^{\pm}] = \frac{3}{32\pi G_4}\int d^4x\,\Big(-\varphi^{+}\Box\varphi^{+} + \frac{1}{a^2}\,\varphi^{-}\Box\varphi^{-}\Big)&\\ -\frac{3}{16\pi G_4}\int d^4x\,\big(2\mathbf{\Psi}-\mathbf{\Phi}\big)^T\frac{\mathbf{K}(\Box)}{l^2}\big(2\mathbf{\Psi}-\mathbf{\Phi}\big).& \end{aligned} \tag{92}$$

Note that the sector of conformal modes is entirely local, and the last 2×2 radion quadratic form here corresponds to the last term of (23) quadratic in the left-hand side of the radion equations of motion (51), $\Box\mathbf{\Psi} + \mathbf{R}/6 = \Box(2\mathbf{\Psi}-\mathbf{\Phi})/2$.

Excitations in the graviton sector are the transverse-traceless basis functions $\mathbf{v}(x) = \mathbf{v}_{\mu\nu}(x)$ of the operator $\mathbf{F}(\Box)$,

$$\mathbf{F}(\Box)\,\mathbf{v}_n(x) = 0, \quad \mathbf{v}_n(x) = \left[\begin{array}{c}v^{+}_n(x)\\ v^{-}_n(x)\end{array}\right]. \tag{93}$$

The existence condition for zero-vectors of the 2×2 matrix operator $\mathbf{F}(\Box)$,

$$\det \mathbf{F}(\Box) = 0, \tag{94}$$

serves as the equation for the masses of these propagating modes m_n,

$$(\Box - m_n^2)\,\mathbf{v}_n(x) = 0, \tag{95}$$

determined by the roots $\Box = m_n^2$ of (94). In view of Eq. (71) these roots coincide with the zeros of the following combination of Bessel functions

$$\Box\left(Y_1^- J_1^+ - Y_1^+ J_1^-\right) = 0. \tag{96}$$

But the same equation (96) determines the spectrum of the Kaluza-Klein modes — the zeros of the Wronskian of basis functions $u_\pm(z)$, Eq. (69), entering the Neumann Green's function (68). Therefore, a conventional tower of massive Kaluza-Klein modes is contained in the spectrum of the braneworld effective action.

The derivative expansion of the previous section allows one to get a reliable description for the massless sector of the spectrum, $m_0^2 = 0$. In this sector Eq. (93) reduces to $\mathbf{M}_F\mathbf{v}_0(x) = 0$ and implies that $\mathbf{v}_0(x)$ is a zero mode of the mass matrix $\mathbf{M}_F$ — the property guaranteed by the degeneracy of this matrix, mentioned above. The non-diagonal nature of (81) implies that this mode is still the collective excitation of tensor perturbations $v_0^\pm(x)$ on both branes, but $v_0^-(x) = a^2 v_0^+(x) < v_0^+(x)$ and for large interbrane separation, $a \to 0$, the negative brane component tends to zero, $v_0^-(x) \to 0$, so that the massless graviton is essentially localized on the positive-tension brane. This is, certainly, another manifestation of the recovery of Einstein theory on this brane for the two-brane Randall-Sundrum model.

The attempt to describe massive modes within the derivative expansion for $\mathbf{F}(\Box)$ turns out to be inconsistent. Indeed, the truncation of the series (80) on the second term, for the goal of finding the first massive level, leads to the equation $\det(-\mathbf{M}_F + \mathbf{D}_F\Box) = 0$ instead of (94). It implies the simultaneous diagonalization of both mass and kinetic term matrices in the basis of the first two (massless and massive) propagating modes (which is possible in view of the positive-definiteness of the kinetic matrix $\mathbf{D}_F$). But, unfortunately, the massive root of this equation, $\Box = M^2 \equiv 24a^2(1+a^2)/l^2(1-a^2)^2 \gg a^2/l^2$, strongly violates the second of the low-energy conditions (79). The excited massive graviton mode always turns out to be in the physical high-energy domain on the negative-tension brane, and the low-energy description fails. For this reason in this paper we restrict ourselves to the massless sector of the theory — the effects of massive modes will be considered elsewhere [39].

D. Large interbrane distance

In view of the discussion of the previous section, it is instructive to consider in the low-energy approximation on the positive-tension brane the case of large brane separation, when $a \ll 1$ and

$$l\sqrt{\Box} \ll 1,\ \ \frac{l\sqrt{\Box}}{a} \gg 1. \tag{97}$$

This range of coordinate distances $1/\sqrt{\Box}$ corresponds to the long-distance approximation on the Σ_+-brane and to the physical *short*-distance limit, $a/\sqrt{\Box} \ll l$, on the Σ_--brane. Now one should use the asymptotic expressions of large arguments of the Bessel functions (J_ν^-, Y_ν^-), $\nu = 1, 2$,

$$J_\nu^- \simeq \sqrt{\frac{2a}{\pi l \Box^{1/2}}} \cos\left(\frac{l\sqrt{\Box}}{a} - \frac{\pi}{4} - \frac{\pi\nu}{2}\right), \quad Y_\nu^- \simeq \sqrt{\frac{2a}{\pi l \Box^{1/2}}} \sin\left(\frac{l\sqrt{\Box}}{a} - \frac{\pi}{4} - \frac{\pi\nu}{2}\right), \tag{98}$$

and as before the small-argument expansions for (J_ν^+, Y_ν^+). Then, in the leading order the operator $\mathbf{F}(\Box)$ reads

$$\mathbf{F}(\Box) \simeq \begin{bmatrix} \dfrac{l^2\Box}{2} & \dfrac{l^2\Box}{2J_2^-} \\ & \\ \dfrac{l^2\Box}{2J_2^-} & -\dfrac{J_1^-}{J_2^-}\dfrac{l}{a}\sqrt{\Box} \end{bmatrix}. \tag{99}$$

In contrast to the case of small brane separation, the short-distance corrections to this matrix operator contain a nonlocal $\Box^2 \ln \Box$-term. Here we present it for the $F_{++}(\Box)$-element,

$$F_{++}(\Box) = \frac{l^2\Box}{2} + \frac{(l^2\Box)^2}{2} k_2(\Box) + O\Big[(l^2\Box)^3\Big], \tag{100}$$

$$k_2(\Box) = \frac{1}{4}\left(\ln\frac{4}{l^2\Box} - 2\mathbf{C} + \pi\frac{Y_2^-}{J_2^-}\right) \tag{101}$$

(the meaning of the subscript in $k_2(\Box)$ will become clear below). This is a manifestation of the well-known phenomenon of AdS/CFT-correspondence [8, 15, 40] when typical quantum field theoretical effects in four-dimensional theory can be generated from the classical theory in the bulk. The AdS/CFT-duality exists for the boundary theory in the AdS bulk when the brane, treated as a boundary of the AdS spacetime, tends to infinity. In the wording of the two-brane Randall-Sundrum model [15, 28] this situation qualitatively corresponds to the case of the negative tension brane tending to the horizon of the AdS spacetime, $y \to \infty$, or $a \ll 1$.

The non-logarithmic term of Eq. (101) include the functions with an infinite series of poles at $l^2\Box \simeq \pi^2 a^2 (n + 3/4)^2$, $n = 0, 1, ...$, contained in the ratio

$$\frac{Y_2^-}{J_2^-} \simeq -\frac{J_1^-}{J_2^-} \simeq \tan\left(\frac{l\sqrt{\Box}}{a} - \frac{\pi}{4}\right). \tag{102}$$

Interestingly, these poles arise in the wave operator of the theory, rather than only in its Green's function. This is an artifact of the nonlocality, when both the propagator and its inverse are nonlocal. The poles are separated by the sequence of roots on the positive real axis of the $\Box$-plane, $\Box = m_n^2 \simeq (\pi^2 a^2/l^2)(n + 3/4)^2/l^2(1 + \pi^2 a^2 (n + 3/4)/2 + ...)$, which correspond to the tower of massive Kaluza-Klein modes in the energy range (97)[12]. They

[12] In this range the above expressions for poles and zeros of $\mathbf{F}(\Box)$ are valid for large n satisfying $1 \ll \pi(n + 3/4) \ll 1/a$.

are excited for large brane separation, $a \ll 1$, because a small energy range at the positive tension brane turns out to be the high energy range at the second brane.

The presence of both zeros and poles in the wave operator $\mathbf{F}(\Box)$ can, apparently, be explained by the duality relation between the Dirichlet and Neumann Green's functions in braneworld physics [33]. According to this relation the Neumann Green's function in the bulk when restricted to branes (which is exactly $\mathbf{F}(\Box)$) is the inverse of the bulk Dirichlet Green's function properly differentiated with respect to its two arguments and also restricted to branes. Therefore, the necessarily existing poles of the Dirichlet Green's function generate zeros of the Neumann one and vicy versa. The presence of zeros, each of which being located between a relevant pair of neighbouring poles of $\mathbf{F}(\Box)$, is actually very important, because this guarantees the positivity of residues of all poles in $\mathbf{G}(\Box) = \mathbf{F}^{-1}(\Box)$ or the normal non-ghost nature of all massive modes.

VI. THE REDUCED EFFECTIVE ACTION

If we take the usual viewpoint of the braneworld framework, that our visible world is one of the branes embedded in a higher-dimensional bulk, then the fields living on other branes are not directly observable. In this case the effective dynamics should be formulated in terms of fields on the visible brane. In the two-brane Randall-Sundrum model this is equivalent to constructing the reduced action — an action with on-shell reduction for the invisible fields in terms of the visible ones. This reduction is nothing but the tree-level procedure of tracing (or integrating) out the unobservable variables. It implies that in the two-brane action we have to exclude the fields on the invisible brane in terms of those on the visible one. As the latter we choose the positive-tension brane. One of the reasons for such a choice is that the low-energy dynamics on this brane is closest to four-dimensional Einstein theory, while that of the negative-tension brane is encumbered with the problems discussed above — intrusion into the high-energy domain, excitation of massive modes, etc.

We perform the reduction of the action to the Σ_+-fields separately in the graviton and scalar sectors. In the graviton sector (90) the on-shell reduction — the exclusion of $\gamma^-_{\mu\nu}$-perturbations in terms of $\gamma^+_{\mu\nu} = \gamma_{\mu\nu}$ (in what follows we omit the label $+$, because only one field remains) — corresponds to the replacement of the original action by the new one,

$$S_{\rm graviton}[\,\gamma^{\pm}_{\mu\nu}] \;\Rightarrow\; S^{\rm red}_{\rm graviton}[\,g_{\mu\nu}] = \frac{1}{16\pi G_4}\int d^4x\,\sqrt{g}\,\gamma^+_{\mu\nu}\frac{F_{\rm red}(\Box)}{2\,l^2}\gamma^{\mu\nu}_+\,, \tag{103}$$

with the original kernel $\mathbf{F}(\Box)$ going over to the new one-component kernel $F_{\rm red}(\Box)$ according to the following simple prescription

$$\mathbf{F}(\Box) \;\Rightarrow\; F_{\rm red}(\Box) = F_{++}(\Box) - F_{+-}(\Box)\frac{1}{F_{--}(\Box)}F_{-+}(\Box). \tag{104}$$

It is useful to rewrite (103) back to the covariant form in terms of (linearized) Ricci curvatures on a single visible brane,

$$S^{\rm red}_{\rm graviton}[g_{\mu\nu}] = \frac{1}{8\pi G_4}\int d^4x\,\sqrt{g}\left(R_{\mu\nu}\frac{F_{\rm red}(\Box)}{l^2\,\Box^2}\,R^{\mu\nu} - \frac{1}{3}\,R\frac{F_{\rm red}(\Box)}{l^2\,\Box^2}\,R\right)\,. \tag{105}$$

A similar reduction in the scalar sector implies omitting in the first integral of (92) the negative-tension term and replacing the 2×2 quadratic form in the second integral by the quadratic form in ψ^+ with the reduced operator

$$K_{\rm red}(\Box) = K_{++}(\Box) - K_{+-}(\Box)\frac{1}{K_{--}(\Box)}K_{-+}(\Box) = \frac{\det \mathbf{K}(\Box)}{K_{--}(\Box)} \,. \tag{106}$$

Finally, we express the conformal mode in terms of the (linearized) Ricci scalar $\varphi^+ = -(1/3\Box)R$, and denote the radion by $\psi^+ = \psi$. Then the combination of the reduced scalar sector together with the graviton part (105) yields the reduced action in its covariant form

$$\begin{aligned} S_{\rm red}[g_{\mu\nu},\psi] = \frac{1}{16\pi G_4}\int d^4x\,\sqrt{g}\,\Big[\,R_{\mu\nu}\frac{2F_{\rm red}}{l^2\Box^2}\Big(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\Big) \\ -\frac{1}{6}R\left(\frac{1}{\Box}-\frac{2F_{\rm red}}{l^2\Box^2}\right)R - 6l^2\left(\Box\psi+\frac{R}{6}\right)\frac{2K_{\rm red}}{l^2\Box^2}\left(\Box\psi+\frac{R}{6}\right)\,\Big]. \end{aligned} \tag{107}$$

Here we have deliberately singled out the term bilinear in the Ricci and Einstein tensors, because this form will be useful in comparing the low-energy approximation of this action with the Einstein action. Below we analyze this result in the two low-energy regimes (79) and (97).

A. Small interbrane distance

In the regime of small or finite brane separation (79), the calculation of the reduced operator (104) gives, on using (80)–(82), a very simple result,

$$F_{\rm red}(\Box) = \frac{l^2\Box}{2}(1-a^2) + \frac{(l^2\Box)^2}{2}\kappa_1(a) + O[\,(l^2\Box)^3], \tag{108}$$

$$\kappa_1(a) = \frac{1}{4}\left[\,\ln\frac{1}{a^2} - (1-a^2) - \frac{1}{2}(1-a^2)^2\,\right]. \tag{109}$$

The reduced operator turns out to be massless, and this is a corollary of the degenerate nature of the mass matrix (81), $\det \mathbf{M}_F = 0$, because $F_{\rm red}(\Box) = \det \mathbf{F}(\Box)/F_{--}(\Box) = O(\Box)$. Similary, in view of (87), the reduced operator in the radion sector (106) is at least quadratic in $\Box$,

$$K_{\rm red}(\Box) = \kappa_2(a)(l^2\Box)^2 + O[\,(l^2\Box)^3], \tag{110}$$

$$\kappa_2(a) = \frac{1}{4}\ln\frac{1}{a^2}, \tag{111}$$

so that the low-energy radion turns out to be a dipole ghost.

After substituting (108) into (107), the first term bilinear in $R_{\mu\nu}$ and $R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R$ seems to remain nonlocal. However, this term is nothing but the part of the local Einstein action, which is quadratic in metric perturbations $\delta g_{\mu\nu} = h_{\mu\nu}$ on a flat spacetime background. To see this, note that, up to diffeomorphism with some vector field parameter f_μ, this perturbation can be nonlocally rewritten in terms of the (linearized) Ricci tensor, $h_{\mu\nu} =$

$-(2/\Box)R_{\mu\nu}+f_{\mu\,,\nu}+f_{\nu\,,\mu}+O[\,R^2_{\mu\nu}\,]$. When substituted into the quadratic part of the Einstein action it takes an explicitly nonlocal form in terms of Ricci curvatures,

$$\frac{1}{2}\,\delta^2\!\int d^4x\,\sqrt{g}R=\int d^4x\,\sqrt{g}\left(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\right)\frac{1}{\Box}R_{\mu\nu}+O[\,R^3_{\mu\nu}\,], \tag{112}$$

which is exactly the first term of (107) with the first-order in $\Box$ approximation for $F_{\rm red}(\Box)$, (108). The part of the Einstein action linear in perturbations is a total divergence, which we disregard here, and the zeroth order term is identically vanishing. Therefore, this term can be rewritten as the local Einstein action linear in scalar curvature with the a-dependent four-dimensional gravitational constant (78), $G_4(a)=G_4/(1-a^2)=G^+$. The second term of (107) with $F_{\rm red}(\Box)$ taken in the same approximation stays nonlocal although this nonlocality is suppressed by the factor $a^2<1$.

The quadratic in $\Box$ contribution to (108) generates the local part of the action quadratic in curvatures which enter in a special combination $R^2_{\mu\nu}-R^2/3$. This combination differs from the square of the Weyl tensor $C^2_{\mu\nu\alpha\beta}$ by the density of the Gauss-Bonnet invariant $E=R^2_{\mu\nu\alpha\beta}-4R^2_{\mu\nu}+R^2$, $2(R^2_{\mu\nu}-R^2/3)=C^2_{\mu\nu\alpha\beta}-E$, which can be omitted under the integral sign (because it is topologically invariant and in the quadratic order in metric perturbations explicitly reduces to the total surface term [34, 35]).

Thus, collecting different contributions together, we get the reduced braneworld action in the low-energy regime of finite interbrane distance

$$\begin{aligned}S_{\rm red}[\,g_{\mu\nu},\psi\,]=\frac{1}{16\pi G_4}\int d^4x\,\sqrt{g}\,\Big[\,&(1-a^2)R-\frac{a^2}{6}\,R\frac{1}{\Box}R\\&-6l^2\kappa(a)\Big(\Box\psi+\frac{R}{6}\Big)^2+\frac{l^2}{2}\,\kappa_1(a)\,C^2_{\mu\nu\alpha\beta}\Big].\end{aligned} \tag{113}$$

The first term here confirms the recovery of Einstein theory on the positive tension brane with the well-known expression for the effective gravitational constant $G_4(a)$ [14, 20]. The higher derivative nature of the radion does not really imply physical instability, because ψ can hardly be treated as non-gauge variable[13]. Its equation of motion, $\Box\,(\Box\psi+R/6)=0$, implies that the on-shell restriction of (113) leaves us with the first two metric-field terms.

In the low-energy regime the first three terms dominate over the local short-distance Weyl-squared part. They form the action that was derived in [28] by a simplified (and, strictly speaking, not very legitimate) method — by just freezing to zero all field perturbations on the invisible brane. Here this derivation is justified within a consistent scheme accounting for the fact that, even without matter sources on Σ_-, the field perturbations on the visible

[13] Indeed, in Sec. IV A radions were introduced as gauge variables relating the Randall-Sundrum coordinate system to two Gaussian systems associated with two branes. Then it was demanded that the kinematical relations between radions and stress tensor traces (40) should be generated as dynamical equations from the braneworld action (23). This has led to the last term in (23) quadratic in $(\Box\mathbf{\Psi}+\mathbf{R}/6)$. Thus, this term can be regarded as the result of the off-shell extension in the radion sector. The on-shell reduction simply corresponds to the exclusion of radions in terms of the metric fields or, equivalently, to omitting the last term in (23).

brane induce nontrivial fields on the invisible one, and they contribute to the full effective action. Interestingly, however, the result turns out to be the same as in [28].

The second term of (113) is nonlocal. However, according to the discussion in [28], it can be localized in terms of an extra scalar field. Actually, this extra field can be identified with the radion itself up to some nonlocal reparametrization. Indeed, the following reparametrization from ψ to the new field φ,

$$\varphi = \sqrt{\frac{3}{4\pi G_4}}\left[a\Big(1-\frac{1}{6\Box}R\Big) - l\sqrt{\frac{\kappa_2(a)}{-\Box}}\,\Big(\Box\psi + \frac{1}{6}R\Big)\right] \tag{114}$$

converts the action (113) to the local form

$$S_{\rm red}[\,g_{\mu\nu},\varphi\,] = \int d^4x\,\sqrt{g}\,\left[\,\left(\frac{1}{16\pi G_4} - \frac{1}{12}\varphi^2\right)R + \frac{1}{2}\varphi\Box\varphi + \frac{l^2}{32\pi G_4}\,\kappa_1(a)\,C^2_{\mu\nu\alpha\beta}\right] \tag{115}$$

whose first three (low-derivative) terms were derived in [28] by a simplified procedure. The field φ introduced here by the formal transformation (114) directly arose in [28] as a local redefinition of the radion field relating the Randall-Sundrum coordinates to the Gaussian normal coordinates associated with the positive tension brane. It is non-minimally coupled to the curvature, and in [28] it was used to play the role of the inflaton generating inflation in the presence of a small detuning between the values of the brane tensions $\sigma_\pm$ from their Randall-Sundrum values (13). Initial conditions for inflation in [28] were suggested within the tunneling wavefunction scheme [41] modified according to the braneworld creation framework [17, 23, 29]. In this framework the Lorentzian spacetime arises as a result of analytic continuation from the Euclidean space describing the classically forbidden (underbarrier) state of the gravitational field. Note, in connection with this, that the transformation (114) is well-defined only in Euclidean spacetime with the negative-definite operator $\Box$. Thus, the justification of the off-shell reparametrization between the actions (113) and (115) comes from the Euclidean version of the theory, which underlies the braneworld creation scheme[14]. In the next section we shall extend this justification even further by resorting to Hartle boundary conditions on the AdS horizon.

B. Large interbrane distance and Hartle boundary conditions

In the limit $a \to 0$ the nonlocal and correspondingly non-minimal terms of (113) and (115) vanish and the low-energy model seems to reproduce the Einstein theory. However, this limit corresponds to another energy regime (97) in which one should use the expressions (99)–(101) in order to obtain the reduced operator (104). Then the latter, up to quadratic in $\Box$ terms inclusive, reads as

$$F_{\rm red}(\Box) = \frac{l^2\Box}{2} + \frac{(l^2\Box)^2}{8}\left(\ln\frac{4}{l^2\Box} - 2\mathbf{C}\right) + \frac{(l^2\Box)^2}{2}\left(\frac{\pi}{4}\frac{Y_2^-}{J_2^-} + \frac{a}{2l\sqrt{\Box}J_1^-J_2^-}\right). \tag{116}$$

[14] The transformation (114) is complex-valued for timelike momenta, but its on-shell restriction in the radion sector is real.

As in (100) it involves the logarithmic nonlocality (101) in $\Box^2$-terms. Moreover, the last term here simplifies to the ratio of the first order Bessel functions Y_1^-/J_1^-, so that $F_{\rm red}(\Box)$ takes a form very similar to that of the large interbrane separation (100),

$$F_{\rm red}(\Box) = \frac{l^2\Box}{2} + \frac{(l^2\Box)^2}{8}\left[\,\ln\frac{4}{l^2\Box} - 2\mathbf{C} + \pi\frac{Y_1^-}{J_1^-}\right] + O\Big[\,(l^2\Box)^3\Big]\,. \tag{117}$$

The calculation of the radion operator (106) with $\mathbf{K}(\Box)$ following from (99) for $l^2\Box/a^2 \gg 1$ results in

$$K_{\rm red}(\Box) = (l^2\Box)^2\, k_2(\Box), \tag{118}$$

where $k_2(\Box)$ is defined by (101).

Thus, $F_{\rm red}(\Box)$ and $K_{\rm red}(\Box)$ are given by the following two nonlocal operators,

$$k_\nu(\Box) = \frac{1}{4}\left[\,\ln\frac{4}{l^2\Box} - 2\mathbf{C} + \pi\frac{Y_\nu^-}{J_\nu^-}\right], \quad \nu = 1, 2, \tag{119}$$

and the reduced (one-brane) action finally reads

$$S_{\rm red}[g_{\mu\nu},\psi] = \frac{1}{16\pi G_4}\int d^4x\,\sqrt{g}\,\Big[\,R + \frac{l^2}{2}\,C_{\mu\nu\alpha\beta}\,k_1(\Box) C^{\mu\nu\alpha\beta} \\ -6l^2\left(\Box\psi + \frac{R}{6}\right)k_2(\Box)\left(\Box\psi + \frac{R}{6}\right)\Big]\,. \tag{120}$$

Here terms quadratic in curvature (which we again rewrote in terms of the Weyl squared combination in view of integration by parts) represent short distance corrections with form factors whose logarithmic parts have an interpretation in terms of the AdS/CFT-correspondence [8, 15, 28, 40]. Their Bessel-function parts are more subtle for interpretation and less universal. When taken literally they give rise to the massive resonances discussed above in Sec. V D. However, with the usual Wick rotation prescription $\Box \to \Box + i\varepsilon$ these ratios tend to

$$\frac{Y_\nu^-}{J_\nu^-} \simeq \tan\left(\frac{l}{a}\sqrt{\Box + i\varepsilon} - \frac{\pi}{4} - \frac{\pi\nu}{2}\right) \to i, \quad a \to 0, \tag{121}$$

and both form factors (119) for $\Box < 0$ (Euclidean or spacelike momenta) become real and can be expressed in terms of one Euclidean form factor as

$$k_\nu(\Box + i\varepsilon)\Big|_{a\to 0} = k(\Box) \equiv \frac{1}{4}\left(\,\ln\frac{4}{l^2(-\Box)} - \mathbf{C}\,\right). \tag{122}$$

This Wick rotation after moving the second brane to the AdS horizon imposes a special choice of vacuum or special boundary conditions at the AdS horizon. The Hartle boundary conditions corresponding to this type of analytic continuation imply that the basis function $u_-(z)$ instead of (66) is given by the Hankel function, $u_-(z) = H_2^{(1)}(z\sqrt{\Box}) = J_2(z\sqrt{\Box}) + iY_2(z\sqrt{\Box})$, and thus corresponds to ingoing waves at the horizon [16, 18, 19]. This is equivalent to the replacement $Y_1^-, Y_2^- \to 1$, $J_1^-, J_2^- \to -i$, in (119) and, thus, justifies the Wick rotation of the above type. Hartle boundary conditions and the Euclidean form factor (122) naturally arise when the Lorentzian AdS spacetime is viewed as the analytic continuation from the Euclidean AdS (EAdS) via Wick rotation in the complex plane of time. Under this continuation the AdS horizon is mapped to the inner regular point of EAdS, the coordinate z playing a sort of an inverse radius, and the regularity of fields near this point is equivalent to the exponential decay of the Euclidean basis function $u_-(z) = 2iK_2(z\sqrt{-\Box})/\pi \to 0$, $z \to \infty$ (remember that $\Box < 0$ in Euclidean spacetime). Such an analytic continuation models the mechanism of cosmological creation via no-boundary [21] or tunneling [41] prescriptions extended to the braneworld context [17, 23, 28, 29]. In particular, it determines (otherwise ambiguous) nonlocal operations in Lorentzian spacetime from their uniquely defined Euclidean counterparts.

VII. CONCLUSIONS

To summarize, we have constructed the braneworld effective action in the two-brane Randall-Sundrum model to quadratic order in curvature perturbations and radion fields on both branes. We have obtained the exact nonlocal form factors in this two-field action and their low-energy approximation. The zeros of the form factors reproduce the spectrum of Kaluza-Klein modes. Thus, this description is intrinsically equivalent to the Kaluza-Klein setup. However, it explicitly features two fields rather than the infinite tower of local Kaluza-Klein modes. The price one pays for this is that the two-field action is essentially nonlocal and its nonlocality is a cumulative effect of the Kaluza-Klein modes.

We have also considered the reduced version of the action — the functional of the fields associated with only one positive-tension brane. A physical motivation for this reduction is the fact that, if this brane with its metric and other fields is regarded as the only visible one, then one has to trace out in the whole two-brane system the fields on the second brane[15]. In the tree-level approximation this procedure is equivalent to excluding the invisible fields via their equations of motion in terms of the fields on the positive-tension brane. In the low-energy approximation the result turns out to be very simple — the action is dominated by the Einstein term with the four-dimensional gravitational constant explicitly depending on the brane separation. This is a manifestation of the well-known localization of the graviton zero mode on the brane or recovery of the low-energy Einstein theory.

However, for finite interbrane distance the action does not get completely localized even in the long-distance approximation — in the conformal sector a certain nonlocality survives. Nevertheless, the latter can be localized in terms of the additional scalar field non-minimally interacting with the brane curvature. Interestingly, the result coincides with the braneworld action constructed in [28] by a simplified method disregarding the metric perturbations on the negative-tension brane. In contrast to [28], the radion field in (113) has a dipole ghost nature, but on its mass shell the action (113) concides with that of [28], given by (115). Moreover, off-shell both actions can be identically transformed into one another by the

[15] This may also lead to decoherence for the visible fields, cf. [42].

nonlocal reparametrization (114).

Thus, our results justify the conclusions of [28]. These conclusions were used to generate inflation by means of the radion field (114) playing the role of an inflaton. For this purpose, the theory was generalized to the case when the tension on the visible brane is slightly detuned from the flat brane Randall-Sundrum value. Then the action (115) rewritten in the Einstein frame would acquire a nontrivial inflaton potential, such that its slow-roll dynamics corresponds to branes diverging under the action of a repulsive interbrane force — a scenario qualitatively different from the models of colliding branes [5, 25, 26]. However, it suffers from an essential drawback. This is the necessity to introduce by hand a four-dimensional cosmological constant — the brane tension detuning of the above type.

Quite interestingly, the present results suggest the mechanism of interbrane repulsion based on the presence of the Weyl-squared term in (113) and (120). When the brane Universe is filled with the graviton radiation this term is nonzero, $C^2_{\mu\nu\alpha\beta} > 0$, and for small brane separation it forms the interbrane potential $-(l^2/2)\kappa_1(a)C^2_{\mu\nu\alpha\beta}$. It has a maximum at the point of coinciding branes $a = 1$ because the coefficient $\kappa_1(a)$ given by (109) is strictly positive. The repelling force is very small, though, and identically vanishes at $a = 1$, because of the behaviour of $\kappa_1(a)$ at the brane collision point[16], $\kappa_1(a) \sim (1 - a^2)^3/12$. Unfortunately, this potential is strictly negative, because $\kappa_1(a) \geq 0$, and, therefore, cannot maintain inflation (for recent studies of models with a negative cosmological constant, see [43]). Rather, it can serve as a basis of brane models with AdS_4 geometry embedded in AdS_5 [44]. It can also be useful in the model of "thick" branes in the Big Crunch/Big Bang transitions [45] of ekpyrotic and cyclic cosmologies [5, 27] and provide an alternative to the mechanisms of repulsion caused by matter fields on branes [46].

Thus, the renormalization flow in a (AdS flow) can, in principle, be realized in our model at the dynamical level. This flow interpolating between short and long interbrane distances is very interesting. It features the transition from the phase of the local action (115) to the action (120) with logarithmic nonlocal form factors in the Weyl-squared term. At the initial stage the local logarithm in $\kappa_1(a)$, Eq. (109), structurally resembles the logarithmic behaviour of the Coleman-Weinberg potential of the field (114), $\varphi \sim \sqrt{3/4\pi G_4}\, a$, $\kappa_1(a) = -(1/4)\ln(\varphi^2/m_P^2) + ...$, $m_P^2 = 1/G_4$ (although it multiplies instead of the local power of φ the square of the Weyl tensor). When extrapolated to small a (big interbrane distances) it, instead of a naive infinite growth, enters another energy domain (97), $a^2 \sim l^2\Box$, where it gets saturated by the characteristic scale of the Weyl tensor (or the energy scale of the graviton radiation contained in the model). Therefore, the local coefficient $\kappa_1(a)$ gets replaced by the form factor $k_1(\Box)$, Eq. (119), the dominant logarithmic term of $\kappa_1(a) \sim (1/4)\ln(1/a^2)$ going over into the logarithm of $k_1(\Box) \sim (1/4)\ln(4/l^2\Box)$. Thus, this renormalization AdS flow leads to the delocalization of the initial radion condensate $\kappa_1(a)C^2_{\mu\nu\alpha\beta}$ to nonlocal (but short-distance) corrections $C_{\mu\nu\alpha\beta}\, k_1(\Box)C^{\mu\nu\alpha\beta}$ characteristic of the AdS/CFT-correspondence principle in the limit of large interbrane distance.

Concrete implications of these phase transitions in cosmology still have to be worked out.

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[16] Interestingly, the expression (109) represents the logarithm $\ln(1/a^2)$ with exactly the first two terms of its Taylor series in $(1 - a^2)$ subtracted.

Here we only make a final remark that they can comprise an essential point of departure from the scenario of diverging branes of [28] for large interbrane distances (97). The corresponding action (120) is different from the one in (115) (or equivalently (113)) that was extrapolated in [28] to late stages of the brane runaway. In contrast to (115), the action (120) does not have any non-minimal curvature coupling of the modulus a, which together with the brane tension detuning served as a basis for the acceleration stage in [28]. The diverging-branes scenario of [28] also admitted this stage as a sequel to the slow-roll inflation, but the inflation and acceleration stages overlapped there and, thus, caused unsurmountable difficulties for reheating [43]. Thus, it would be interesting to observe the effect of the nonlocal short-distance corrections (replacing the non-minimal curvature coupling of (115)) on the late time behavior in the scenario of diverging branes. Since these corrections are dominated by curvature-squared terms, their effect can be equivalent to the R^2-inflation model [47] (see also [48] for the same conjecture on the realization of the Starobinsky model in braneworld scenarios). We hope to deal with these issues in a future publication.

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