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Quantum Field Theory on Pseudo-Complex Spacetime

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Abstract

The pseudo-complex Poincaré group encodes both a universal speed and a maximal acceleration, which can be viewed as the kinematics of Born-Infeld electrodynamics. The irreducible representations of this group are constructed, providing the particle spectrum of a relativistic quantum theory that also respects a maximal acceleration. One finds that each standard relativistic particle is associated with a 'pseudo'-partner of equal spin but generically different mass. These pseudo-partners act as Pauli-Villars regulators for the other member of the doublet, as is found from the explicit construction of quantum field theory on pseudo-complex spacetime. Conversely, a Pauli-Villars regularised quantum field theory on real spacetime possesses a field phase space with integrable pseudo-complex structure, which gives rise to a quantum field theory on pseudo-complex spacetime.

This equivalence between *(i)* maximal acceleration kinematics, *(ii)* pseudo-complex quantum field theory, and *(iii)* Pauli-Villars regularisation rigorously establishes a conjecture on the regularising property of the maximal acceleration principle in quantum field theory, by Nesterenko, Feoli, Lambiase and Scarpetta [*Phys. Rev.* **D60**, 065001].

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I. INTRODUCTION

Standard quantum field theory is built on the assumption that fundamental particles are irreducible representations of the spacetime Poincaré group. The physical rationale for this is the assumption that the symmetries of Maxwell electrodynamics, associated with an invariant speed c, are kinematical, i.e., all fundamental theories must possess them. However, quantum field theory is plagued with divergences, as is Maxwell electrodynamics of point particles. Indeed, the problem of infinite electric field energy of a charged point particle led to the formulation of Born-Infeld electrodynamics [1]. The latter parameterises the energy divergence, but preserves Lorentz invariance. This raises the question of whether replacing the kinematics of Maxwell theory, i.e., special relativity, by extended kinematics extracted from Born-Infeld theory, might also regulate the divergences encountered in quantum field theory. In this paper we show that this indeed is the case.

It was shown in [2, 3] that the kinematisation of the symmetries associated with the maximal field strength b^{-1} of Born-Infeld theory,

$$L_{BI} = \det{}^{\frac{1}{2}} \left(\eta_{\mu\nu} + bF_{\mu\nu}\right), \tag{1}$$

leads to the pseudo-complexified Lorentz group $SO_{\mathbb{P}}(1,3)$ on pseudo-complexified Minkowski spacetime $\mathbb{P}^{1,3}$, where the ring of pseudo-complex numbers is defined as

$$\mathbb{P} \equiv \{a + Ib | a, b \in \mathbb{R}, I^2 = +1, I \notin \mathbb{R}\}. \tag{2}$$

Resulting corrections to symmetry-sensitive calculations in relativity, such as the Thomas precession [4], yield an upper bound [2] for the Born-Infeld parameter,

$$b \leq 10^{-11} C N^{-1}, \tag{3}$$

or, equivalently, a lower bound

$$\mathfrak{a} \geq 10^{22} m s^{-2} \tag{4}$$

on the maximal acceleration of an electrically charged particle coupled to Born-Infeld theory, in order to be in accordance with data from high precision experiments [5].

The real Lorentz group is a subgroup of $SO_{\mathbb{P}}(1,3)$. Therefore, Born-Infeld kinematics, encoded in the pseudo-complex Lorentz group, extends special relativity *without* breaking

Lorentz invariance, in contrast to other approaches to theories with two fundamental constants [6, 7]. Most importantly, the theory can be consistently extended to curved spaces [8] and thus allows for the inclusion of gravity [2, 3].

In [9], Nesterenko, Feoli, Lambiase and Scarpetta quantise a submaximally accelerated classical point particle in Minkowski spacetime,

$$L_{\mathrm{NFLS}} = \frac{m}{\mathfrak{a}}\sqrt{\mathfrak{a}^2 + \ddot{x}^2}\sqrt{\dot{x}^2}, \tag{5}$$

where $\mathfrak{a}$ presents a fundamental upper bound on admissible accelerations. Because of the higher order derivatives in (5), the transition to the Hamiltonian formulation is performed using the Ostrogradski formalism. The authors construct 'field equations' of a corresponding second quantised theory by imposing the classical Noether conservation laws as operator equations on a spacetime function. The Green's function of the arising field equation is of sixth order, which leads the authors to the conjecture that a maximal acceleration principle might possibly regularise quantum field theory. It is recognised, however, that for the rigorous establishment of such a claim, one needs a field theory based on maximal acceleration kinematics from the outset. In particular, fields must belong to well-defined irreducible representations of an appropriate symmetry group.

In this paper, we show the equivalence of

(i) quantum field theory on pseudo-complex spacetime,

(ii) Pauli-Villars regularised quantum field theory on real spacetime, and

(iii) a finite upper bound on admissible particle accelerations.

In particular, a quantum field theory based on the pseudo-complexified Poincaré group, gives rise to Pauli-Villars regularised propagators, with a cutoff determined by the maximal acceleration parameter $\mathfrak{a}$. One understands this result directly from the representation theory of the pseudo-complex Poincaré group: Each standard particle is found to have as its 'pseudo-partner' a Weyl ghost of equal spin, but generically different mass.

The organisation of the paper is as follows. We start by reviewing the pseudo-complex techniques needed for an understanding of Born-Infeld kinematics in flat spacetime. In section III, we find the irreducible representations of the pseudo-complex Poincaré group, providing a proper definition of quantum mechanical particles with submaximal acceleration. The existence of degenerate representations requires the exclusion of certain fields from the particle spectrum, which can be achieved by an adaptation of the action principle for pseudo-complex quantum theories, as explained in section IV. Dynamics for the free scalar field representation are explicitly constructed in section V. This leads to a pseudo-complex propagator, whose projection to real spacetime is shown be Pauli-Villars regularised. Conversely, in section VI, we demonstrate that the field phase space of a Pauli-Villars regularised theory carries an integrable pseudo-complex structure, giving rise to a pseudo-complex field theory. Section VII deals with the extension of the explicit constructions to spinor and non-abelian vector fields. In section VIII, we summarise and conclude.

II. PSEUDO-COMPLEXIFIED MINKOWSKI SPACE

We briefly review the pseudo-complex formulation of Born-Infeld kinematics, as developed in [2, 3]. The commutative unit ring of pseudo-complex numbers over the field $F \in \{\mathbb{R}, \mathbb{C}\}$,

$$\mathbb{P}_F \equiv \{Q_1 + IQ_2 \mid Q_1, Q_2 \in F\}\,, \tag{6}$$

where $I \notin F$, $I^2 = +1$, possesses zero divisors

$$\mathbb{P}^0_\pm \equiv \{\lambda(1 \pm I) \mid \lambda \in F\}\,, \tag{7}$$

for which no multiplicative inverses exist in $\mathbb{P}_F$. For notational convenience, we use the shorthand $\mathbb{P} \equiv \mathbb{P}_\mathbb{R}$. The zero divisors will play a crucial rôle later on, and it is often useful to decompose a pseudo-complex number $Q = Q_1 + IQ_2 \in \mathbb{P}_F$ into its zero divisor components

$$Q_\pm \equiv Q_1 \pm Q_2 \quad \in F, \tag{8}$$

via the multiplicatively acting projectors

$$\sigma_\pm \equiv \frac{1}{2}(1 \pm I) \in \mathbb{P}^0_\pm, \tag{9}$$

so that

$$Q = \sigma_+ Q_+ + \sigma_- Q_-. \tag{10}$$

A function $f : \mathbb{P}_F \longrightarrow \mathbb{P}_F$ is called *pseudo-complex differentiable*, if it is, understood as a map $\tilde{f} : F^2 \longrightarrow F^2$, F-differentiable and satisfies the pseudo-Cauchy-Riemann equations

$$\partial_1 \tilde{f}_1 = \partial_2 \tilde{f}_2, \tag{11a}$$

$$\partial_2 \tilde{f}_1 = \partial_1 \tilde{f}_2. \tag{11b}$$

These allow to re-identify $D\tilde{f} : F^2 \longrightarrow F^{2\times 2}$ with $Df : \mathbb{P}_F \longrightarrow \mathbb{P}_F$, where the pseudo-complex differential operator D can hence be written

$$D = \frac{1}{2}(\partial_1 + I\partial_2). \tag{12}$$

The pseudo-complexification of a finite-dimensional real vector space $M \equiv \mathbb{R}^{1+n}$,

$$M_{\mathbb{P}} \equiv \left\{ X \equiv x + I\mathfrak{a}^{-1}u | x, u \in M \right\} = \mathbb{P}^{1+n}, \tag{13}$$

where $\mathfrak{a}$ will assume the rôle of a fundamental finite upper bound on accelerations later on, is a (free) module over $\mathbb{P}$. Equipping $M_{\mathbb{P}}$ with a metric η of signature $(1, n)$ leads to a metric module $\mathbb{P}^{1,n}$. A basis $\{e_{(\mu)}\}$ of $M_{\mathbb{P}}$ where the metric takes the diagonal form

$$\eta(e_{(\mu)}, e_{(\nu)}) = \eta_{\mu\nu} \equiv \mathrm{diag}(1, -1, \ldots, -1), \tag{14}$$

is called a uniform frame. In such, the symmetry group of $(M_{\mathbb{P}}, \eta)$ is generated by the pseudo-complexified Lorentz algebra

$$so_{\mathbb{P}}(1, n) \equiv \{\omega_{\mu\nu} M^{\mu\nu} | \omega_{\mu\nu} \in \mathbb{P}, M^{\mu\nu} \in so_{\mathbb{R}}(1, n)\}. \tag{15}$$

Clearly, $so_{\mathbb{R}}(1, n) \subset so_{\mathbb{P}}(1, n)$ is a subalgebra, and hence pseudo-complex Lorentz-invariant theories do not break Lorentz invariance.

A curve $X : \mathbb{R} \longrightarrow \mathbb{P}^{1,n}$ is called an orbit, if there exists a uniform frame where

$$X^\mu = x^\mu + I\mathfrak{a}^{-1}\frac{dx^\mu}{d\tau}, \tag{16}$$

with $d\tau^2 \equiv dx^\mu dx_\mu$ being the real Minkowskian line element. Such uniform frames are called inertial frames for the orbit X.

For an orbit $X = x + I\mathfrak{a}^{-1}u$ in an arbitrary uniform frame, the relation $u = dx/d\tau$ does not generally hold, but is seen [3] to be weakened to the orthogonality

$$\eta(dx, du) = 0. \tag{17}$$

Therefore, for an orbit X, the $SO_{\mathbb{P}}(1,n)$-invariant line element $d\omega$, defined by

$$d\omega^2 \equiv dX^\mu \, dX_\mu, \tag{18}$$

is always real-valued. This allows the following definition: An orbit X is called *submaximally accelerated*, if

$$\eta(dX, dX) > 0. \tag{19}$$

For the real spacetime projection $x : \mathbb{R} \longrightarrow \mathbb{R}^{1,n}$ of a submaximally accelerated orbit $X = x + I\mathfrak{a}^{-1}u$, one finds an upper bound on the scalar acceleration,

$$-\frac{d^2x^\mu}{d\tau^2}\frac{d^2x_\mu}{d\tau^2} < \mathfrak{a}^2, \tag{20}$$

so that x is a spacetime trajectory of Minkowski curvature less than $\mathfrak{a}$, justifying the above terminology. Using the $SO_{\mathbb{P}}(1,n)$-invariant line element (18), one can combine (17) and (19) into the single scalar condition

$$\eta(\frac{dX}{d\omega}, \frac{dX}{d\omega}) = 1, \tag{21}$$

for a submaximally accelerated orbit X. Thus $SO_{\mathbb{P}}(1,3)$ manifestly preserves the orthogonality (17) and the maximal acceleration. The symmetry group $SO_{\mathbb{P}}(1,n)$ contains the real Lorentz transformations as a subgroup, and further the transformations to uniformly submaximally accelerated frames, as shown in [3].

Classical Lagrangian dynamics with $SO_{\mathbb{P}}(1,n)$ symmetry give rise to submaximally accelerated orbits [3], which we interpret as classical point particles in $n+1$ dimensions, respecting the maximal acceleration. In quantum theory, however, the particle spectrum is given by all irreducible representations of the underlying symmetry group. Therefore, we study the representation theory of the pseudo-complexified Poincaré group in the next section.

III. REPRESENTATION THEORY OF THE PSEUDO-COMPLEX POINCARÉ GROUP

We show that in a quantum theory with pseudo-complex Poincaré invariance, the standard relativistic particle spectrum is doubled, providing each real particle with a pseudo-partner of generically different mass, but equal spin.

The pseudo-complexified Poincaré algebra $\mathcal{P}_{\mathbb{P}}$ in $3+1$ dimensions is generated by

$$\mathcal{P}_{\mathbb{P}} \equiv \left\langle P^{\mu}, M^{\alpha\beta} \right\rangle_{\mathbb{P}}, \tag{22}$$

where, if acting on spacetime functions, $P_{\mu} \equiv iD_{\mu}$ are the generators of translations in $\mathbb{P}^{1,3}$, and $M^{\alpha\beta} \equiv X^{\alpha}D^{\beta} - X^{\beta}D^{\alpha}$ are the generators of $SO_{\mathbb{P}}(1,3)$. Decomposition into zero-divisor components (10) immediately yields two decoupled real Poincaré algebras

$$\begin{aligned} \mathcal{P}_{\mathbb{P}} &= \sigma_{+} \left\langle P_{+}^{\mu}, M_{+}^{\alpha\beta} \right\rangle_{\mathbb{R}} \oplus \sigma_{-} \left\langle P_{-}^{\mu}, M_{-}^{\alpha\beta} \right\rangle_{\mathbb{R}} \\ &= \sigma_{+}\mathcal{P}_{\mathbb{R}} \oplus \sigma_{-}\mathcal{P}_{\mathbb{R}}, \end{aligned} \tag{23}$$

where the sum is direct because $\sigma_{+}\sigma_{-} = 0$. Thus, *generically*, a pseudo-complex particle is labelled by two independent real masses and two independent spins or helicities, as follows from the well-known representation theory of the real Poincaré group.

However, we want to realise the representations by pseudo-complex fields $\phi = \sigma_{+}\phi_{+} + \sigma_{-}\phi_{-}$. Clearly, the subalgebras $\sigma_{\pm}\mathcal{P}_{\mathbb{R}}$ act independently on the (real) zero-divisor components $\phi_{\pm}$. Hence, ϕ_{+} and ϕ_{-} must belong to real representations of equal spin, as otherwise the real and pseudo-imaginary parts of ϕ, i.e., $\phi_{+} \pm \phi_{-}$, would not be algebraically defined. We will see this constraint at work more explicitly below.

As $P_{\pm}^{2}$ are Casimir operators of the respective real Poincaré algebras $\sigma_{\pm}\mathcal{P}_{\mathbb{R}}$, the pseudo-complex operator

$$P^{2} = \sigma_{+}P_{+}^{2} + \sigma_{-}P_{-}^{2} \tag{24}$$

is a Casimir of $\mathcal{P}_{\mathbb{P}}$, and its value $M^{2} \in \mathbb{P}$ is called the *pseudo-complex mass* of the representation. We further define the pseudo-complex Pauli-Ljubanski vector

$$W_{\mu} = \frac{1}{2}\epsilon_{\mu\gamma\alpha\beta}P^{\gamma}M^{\alpha\beta}, \tag{25}$$

whose zero-divisor components

$$W_{\mu\pm} = \frac{1}{2}\epsilon_{\mu\gamma\alpha\beta}P_{\pm}^{\gamma}M_{\pm}^{\alpha\beta} \tag{26}$$

present the real Pauli-Ljubanski vectors of the two real Poincaré algebras $\sigma_{\pm}\mathcal{P}_{\mathbb{R}}$.

Representations of the pseudo-complex Poincaré algebra fall into three different classes, which we call *massive*, *almost massless*, and *massless*, according to whether the pseudo-complex mass is no zero-divisor, a zero-divisor, or zero. We now discuss these cases in turn.

A. Massive case ($M^2 \notin \mathbb{P}^0$)

As in this case $M_\pm^2 > 0$, the Casimirs of the respective real Poincaré algebras are given by the squared Pauli-Ljubanski vectors $W_\pm^2 = M_\pm^2 J_\pm^2$, with $J_{\pm i} \equiv \frac{1}{2}\epsilon_{ijk} M_\pm^{jk}$. Clearly, the squared pseudo-complex Pauli-Ljubanski vector W^2 is then a Casimir of $\mathcal{P}_\mathbb{P}$, and one observes that the pseudo-complex spin operator

$$S^2 \equiv \frac{W^2}{M^2} \tag{27}$$

can be written

$$S^2 = \sigma_+ J_+^2 + \sigma_- J_-^2, \tag{28}$$

using the identity

$$\sigma_\pm \frac{R}{Q} = \sigma_\pm \frac{R_\pm}{Q_\pm} \tag{29}$$

for $R, Q \in \mathbb{P}$ and $Q \notin \mathbb{P}^0$.

For $\phi = \sigma_+\phi_+ + \sigma_-\phi_-$ to be algebraically defined, we need that J_+^2 and J_-^2 both yield the same real spin. Then, we see from (28) that S^2, when acting on a pseudo-complex field representation, is always real, so that for $S^2 = s(s+1)$, the spin eigenvalues are half-integer,

$$s \in \frac{1}{2}\mathbb{N}_0. \tag{30}$$

A massive pseudo-complex field therefore gives rise to two real particles of generically different non-zero masses $M_\pm$, but equal half-integer spins s,

$$|M, s\rangle_\mathbb{P} = |M_+, s, +\rangle_\mathbb{R} \oplus |M_-, s, -\rangle_\mathbb{R}\,. \tag{31}$$

We have included, into the labelling of the real representations, the zero divisor branch on which the real particles take their values. This is necessary, because the pseudo-imaginary unit I presents a Casimir operator, distinguishing the two real representations $\sigma_\pm \mathcal{P}_\mathbb{R}$, because

$$I\sigma_\pm = \pm\sigma_\pm. \tag{32}$$

Later, from the explict construction of a pseudo-complex quantum field theory, we will see that $|+\rangle$ indicates a proper real particle, and $|-\rangle$ a Weyl ghost.

B. Almost Massless case ($0 \neq M^2 \in \mathbb{P}^0$)

Let, without loss of generality, $P_+^2 = M_+^2 > 0$ and $P_-^2 = M_-^2 = 0$. Then $W_+^2 = M_+^2 J_+^2$. From $W_-^2 = P_-^2 = P_- W_- = 0$, it follows that $W_- = \lambda P_-$ for some helicity $\lambda \in \mathbb{R}$. The helicity operator can be expressed as

$$\lambda = -\frac{\mathbf{P}_-.\mathbf{W}_-}{\|\mathbf{P}_-\|^2} = \frac{\mathbf{P}_-.\mathbf{J}_-}{\|\mathbf{P}_-\|}, \tag{33}$$

boldface symbols denoting the spatial parts of the respective four-vectors. The spin S_+^2 and the helicity λ act separately on ϕ_+ and ϕ_-, respectively. Again, for a pseudo-complex field ϕ to be defined, we need that the helicity of ϕ_- equals the spin of ϕ_+.

Thus all physically observed massless particles, possessing *half-integer*-valued helicities, can be accommodated in the almost massless pseudo-complex field representation. In general, an almost massless pseudo-complex particle gives rise to a real doublet

$$|M_\pm \sigma_\pm, s\rangle_{\mathbb{P}} = |M_\pm, s, \pm\rangle_{\mathbb{R}} \oplus |0, \lambda = s, \mp\rangle_{\mathbb{R}}\,. \tag{34}$$

Again, $|-\rangle$ will be seen to indicate a Weyl ghost, and $|+\rangle$ a proper particle, according to the eigenvalue ± 1 of I.

C. Massless case ($M^2 = 0$)

Here we are left with two continuous real helicities $\lambda_\pm$ from

$$W_\pm = \lambda_\pm P_\pm. \tag{35}$$

Particles with continuous helicity are not observed in experiment, and we hence exclude the massless case from the physical particle spectrum.

The representation theory of the pseudo-complex Poincaré algebra has shown that the experimentally observed physical particles occur as the massive and almost massless representations. A pseudo-complex particle gives rise to a doublet of real particles of equal spins, but generically different real masses. From the explicit construction of pseudo-complex quantum field theory, and its spacetime projection, we will find in the following sections that the $|+\rangle$ particles are proper real particles, for which their $|-\rangle$ pseudo-partners act as Pauli-Villars regulators.

IV. TRIVIAL FIELDS

From the decomposition (23) of the pseudo-complex Poincaré algebra, it is clear that for a field representation ϕ that takes values only in the zero-divisor branches $\mathbb{P}_{F}{}^{0}_{+}$ or $\mathbb{P}_{F}{}^{0}_{-}$, the doublet of real particles collapses to just *one* real particle. We must exclude such fields from the particle spectrum, if we do not want to get back standard quantum field theory on spacetime as a sector of the pseudo-complex theory. The way to render solutions of a dynamical theory meaningless, is to devise equations of motion that are trivially solved by them. Hence, the appropriate formulation of the action principle for pseudo-complex quantum field theory is

$$\delta S \in \mathbb{P}_F^0, \tag{36}$$

where S is the action of the pseudo-complex theory at hand. This adaptation is particularly natural from the algebraic point of view, given that the zero divisors of a ring play very much the rôle of the zero in a field. However, the avoidance of a breakdown of the particle doublets into the standard singlets provides the compelling physical reason for requiring (36).

Note that for purely F-valued field theory, (36) reduces to the standard action principle $\delta S = 0$. We will see that for non-trivial fields, i.e., $\phi \notin \mathbb{P}_{F}{}^{0}_{\pm}$, one can rewrite (36) as an equation.

V. PSEUDO-COMPLEX SCALAR FIELD

Now having a clear definition of quantum particles with submaximal acceleration at our disposal, we can formulate dynamical equations for the free field representations. This is achieved in a standard manner by imposing classical constraints as operator equations on the appropriate fields.

The massive and almost massless scalar representations of $\mathcal{P}_{\mathbb{P}}$ have Casimirs

$$W^2 = 0, \tag{37a}$$

$$P^2 = M^2, \qquad M \neq 0. \tag{37b}$$

To devise a field equation, we impose these as constraints on a field ϕ on pseudo-complex spacetime. Condition (37a) implies that the field is a function

$$\phi : \mathbb{P}^{1,3} \longrightarrow \mathbb{P}_F. \tag{38}$$

Applying the operator equation (37b) to the Fourier transformation $\tilde{\phi}$ of the field ϕ, gives

$$(P^2 - M^2)\tilde{\phi} = 0, \tag{39}$$

where for $X \equiv X_{(1)} + IX_{(2)}$,

$$\tilde{\phi}(P) \equiv \int \mathrm{d}^4 X_{(1)}\, d^4 X_{(2)}\, \phi(X) \exp(-iP^\mu X_\mu). \tag{40}$$

Thus, in position space, we find the pseudo-complex Poincaré invariant field equation

$$(D^2 + M^2)\phi(X) = 0. \tag{41}$$

However, following our reasoning in section IV, on the exclusion of trivial fields $\phi \in {\mathbb{P}_F}^0$ from the spectrum of meaningful dynamical solutions, we replace (41) by

$$(D^2 + M^2)\phi(X) \in \mathbb{P}_F^0. \tag{42}$$

The Green's function $G(X - Y)$ for the operator $D^2 + M^2$ is defined by

$$(D^2 + M^2)G(X - Y) = (2\pi)^8 I \delta^{(8)}(X - Y), \tag{43}$$

so that, in momentum space representation,

$$\tilde{G}(P) = \frac{I}{P^2 - M^2}. \tag{44}$$

Observing that $I\sigma_\pm = \pm\sigma_\pm$, and using the identity (29), one gets

$$\tilde{G}(P) = \sigma_+ \frac{1}{P_+^2 - M_+^2} - \sigma_- \frac{1}{P_-^2 - M_-^2}. \tag{45}$$

Note that $\tilde{G}$ is generically pseudo-complex valued.

In order to compare this result to standard quantum field theory on real spacetime, we project the field $\phi : \mathbb{P}^{1,3} \longrightarrow \mathbb{P}_F$ to a spacetime field $\varphi + I\pi : \mathbb{R}^{1,3} \longrightarrow \mathbb{P}_F$. Clearly, this must be done for an observer in an inertial frame, in order to have a well-defined vacuum for the projected field theory [10]. Such an inertial projection is obviously given by mapping the pseudo-imaginary part of the momentum P to zero,

$$P_{(2)} \mapsto 0. \tag{46}$$

The action of this projection on fields can therefore be implemented straightforwardly on the Fourier transform

$$\tilde{\phi} : \mathbb{P}^{1,3} \longrightarrow \mathbb{P}_F, \tag{47}$$

such that the projection to a spacetime field is given by

$$\tilde{\varphi}(P_1) + I\tilde{\pi}(P_1) \equiv \tilde{\phi}(P_1) : \mathbb{R}^{1,3} \longrightarrow \mathbb{P}_F. \tag{48}$$

This projection clearly breaks the $\mathcal{P}_{\mathbb{P}}$-symmetry down to $\mathcal{P}_{\mathbb{R}}$. However, we will see in section VI that the pseudo-complex structure *survives* as the geometry of the field phase space (φ, π) of the projected field

$$\varphi(x) \equiv \int \mathrm{d}^4p\, \tilde{\varphi}(p) \exp(ip^\mu x_\mu). \tag{49}$$

The projection $\tilde{\phi} \mapsto \tilde{\varphi}$ is well-defined under real Poincaré transformations, as the diagram

$$\begin{tikzcd}
\tilde{\phi}(p+If) \arrow[r, "\Lambda \in SO_{\mathbb{R}}(1,3)"] \arrow[d] & \tilde{\phi}(\Lambda^{-1}(p+If)) \arrow[d] \\
\tilde{\varphi}(p) \arrow[r, "\Lambda \in SO_{\mathbb{R}}(1,3)"] & \tilde{\varphi}(\Lambda^{-1}p)
\end{tikzcd}$$

commutes.

For non-trivial $\tilde{\phi}$, relation (42) can be rewritten as an equation,

$$[P_+^2 - M_+^2][P_-^2 - M_-^2]\tilde{\phi}(P) = 0. \tag{50}$$

Application of the inertial frame projection $\tilde{\phi} \mapsto \tilde{\varphi}$, with $P \mapsto P_{(1)}$, yields

$$[P_{(1)}^2 - M_+^2][P_{(1)}^2 - M_-^2]\tilde{\varphi}(P_{(1)}) = 0, \tag{51}$$

revealing higher order dynamics for the projected field φ. The corresponding real spacetime propagator reads

$$\tilde{g}(p) = \frac{1}{[p^2 - M_+^2][p^2 - M_-^2]}, \tag{52}$$

writing $p \equiv P_{(1)}$ for short. This can be brought to the form

$$\tilde{g}_M(p) \equiv (M_+^2 - M_-^2)\tilde{g}(p) = \frac{1}{p^2 - M_+^2} - \frac{1}{p^2 - M_-^2}. \tag{53}$$

In the special relativity limit $\mathfrak{a} \longrightarrow \infty$, the propagator $\tilde{g}_M(p)$ must reproduce the standard Klein-Gordon propagator for a scalar real particle of mass m. Therefore, we require that

$$M_+^2 \stackrel{\mathfrak{a}\to\infty}{\longrightarrow} m^2, \qquad M_-^2 \stackrel{\mathfrak{a}\to\infty}{\longrightarrow} \infty. \tag{54}$$

In the almost massless case, $M \in \mathbb{P}^0_-$, say, this correspondence principle fixes the pseudo-complex mass, up to field redefinitions, to

$$M = \sigma_- \mathfrak{a}, \tag{55}$$

as $\mathfrak{a}$ is the only massive parameter available. We therefore *adopt*, in the massive case $M \notin \mathbb{P}^0$, the pseudo-complex mass

$$M = \sigma_+ m + \sigma_- \mathfrak{a}, \tag{56}$$

where m is the mass of the real ϕ_+ particle. With this motivated choice, the propagator for the projected spacetime field φ,

$$\tilde{g}_M(p) = \frac{1}{p^2 - m^2} - \frac{1}{p^2 - \mathfrak{a}^2}, \tag{57}$$

is seen to be Pauli-Villars regularised, with the cutoff determined by the maximal acceleration parameter $\mathfrak{a}$. In particular, note that the real representation $|M_+, 0, +\rangle$ is a proper particle, while $|M_-, 0, -\rangle$ is a Weyl ghost.

We conclude that quantum field theory on pseudo-complex spacetime gives rise, after an inertial projection to real spacetime, to a Pauli-Villars regularised quantum field theory on real spacetime. This proves the implication $(i) \Rightarrow (ii)$ stated in the introduction. The next section will show that the converse also holds.

VI. SCALAR FIELD PHASE SPACE

It is worthwile to investigate the symmetries of the projected spacetime theory (51). To this end, consider Pauli-Villars regularised scalar field theory on real spacetime,

$$\mathcal{L} = -\frac{1}{(M_+^2 - M_-^2)} \varphi(\Box + M_+^2)(\Box + M_-^2)\varphi, \tag{58}$$

where $M_+ \ll M_-$ are the masses of the particle and the regulator, respectively. De Urries and Julve [11] developed a Lorentz-covariant version of the Ostrogradski formalism for higher-derivative scalar field theories, and proved its equivalence with the standard non-covariant approach. Defining $[\![\pm]\!] \equiv (\Box + M_\pm^2)$ and $\langle +- \rangle \equiv M_+^2 - M_-^2$, the Lagrangian (58) can be cast into the form

$$\begin{aligned} \mathcal{L} &= -\frac{1}{\langle +- \rangle}\varphi[\![+]\!][\![-]\!]\varphi \\ &\cong -\frac{1}{\langle +- \rangle}[\![+]\!]\varphi[\![+]\!]\varphi + \varphi[\![+]\!]\varphi, \end{aligned} \tag{59}$$

observing that $[\![-]\!] = [\![+]\!] - \langle +- \rangle$, and discarding surface terms. Hence, it suffices to consider derivatives of the form $[\![+]\!]\varphi$. Defining the canonical momentum density

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial [\![+]\!]\varphi}, \tag{60}$$

and solving for $[\![+]\!]\varphi$ in terms of φ and π,

$$[\![+]\!]\varphi = -\frac{\langle +- \rangle}{2}(\pi - \varphi), \tag{61}$$

one obtains the positive definite Hamiltonian density

$$\begin{aligned} \mathcal{H}_1 &\equiv \pi[\![+]\!]\varphi - \mathcal{L}(\varphi, [\![+]\!]\varphi(\varphi, \pi)) \\ &= -\frac{\langle +- \rangle}{4}(\pi - \varphi)^2. \end{aligned} \tag{62}$$

It is shown in [11] that the evolution in field phase space (φ, π) is then governed by the Hamiltonian equations

$$[\![+]\!]\varphi = \frac{\partial \mathcal{H}_1}{\partial \pi}, \tag{63a}$$

$$[\![+]\!]\pi = \frac{\partial \mathcal{H}_1}{\partial \varphi}, \tag{63b}$$

exhibiting manifestly the *almost pseudo-complex* structure of the field phase space of a fourth-order Lagrangian field theory. In case the above pair of equations can be combined into one single pseudo-complex equation, we speak of a *pseudo-complex* structure.

We now identify the necessary and sufficient condition for an almost pseudo-complex phase space structure (63) to be pseudo-complex. Assume that there exists a real-valued function $\mathcal{H}_2(\varphi, \pi)$, such that

$$\mathcal{H} \equiv \mathcal{H}_1 + I\mathcal{H}_2 \tag{64}$$

satisfies the pseudo-Cauchy-Riemann equations (11). In this case, we call $\mathcal{H}$ a pseudo-complex extension of $\mathcal{H}_1$. Combining the field and its canonical momentum into one pseudo-complex valued field on real spacetime,

$$\phi : \mathbb{R}^{1,3} \longrightarrow \mathbb{P}, \tag{65}$$

$$\phi(x) \equiv \varphi(x) + I\pi(x), \tag{66}$$

one can write (63) as

$$[\![+]\!]\phi = I\frac{D\mathcal{H}}{D\phi}, \tag{67}$$

if, and only if, $\mathcal{H}$ is a pseudo-complex extension of $\mathcal{H}_1$. This shows that the field phase space of any fourth order Lagrangian (scalar) theory possesses a pseudo-complex structure, if, and only if, the corresponding Hamiltonian has a pseudo-complex extension.

For the particular Hamiltonian (62), describing a Pauli-Villars regularised field, such extensions exist and are unique up to an arbitrary pseudo-complex constant C,

$$\mathcal{H} = (1 - I)\mathcal{H}_1 + C = -\frac{\langle +- \rangle}{2}\sigma_-\phi^2 + C, \tag{68}$$

as can be seen directly from the integration of the pseudo-Cauchy-Riemann equations for (64).

The dynamics of a field theory with pseudo-complex phase space structure can be captured within the single equation (67), involving only one field degree of freedom. Therefore, this equation can be obtained from a Lagrangian,

$$\mathcal{L} = \frac{1}{2}\phi(\Box + M_+^2)\phi - I\mathcal{H}(\phi). \tag{69}$$

For the special case of a Pauli-Villars regularised spacetime theory, the potential (68) is a mass term, which can be absorbed into the free Lagrangian,

$$\mathcal{L} = \frac{1}{2}\phi(x)(\Box + M^2)\phi(x), \tag{70}$$

with a pseudo-complex mass

$$M = \sigma_+ M_+ + \sigma_- M_-. \tag{71}$$

Note that (70) describes a pseudo-complex valued field ϕ defined on real, rather than pseudo-complex spacetime. However, in an inertial frame, this is equivalent to the fully pseudo-complex Poincaré invariant dynamics

$$\mathcal{L} = \frac{1}{2}\phi(X)(D^2 + M^2)\phi(X) \tag{72}$$

on pseudo-complex spacetime. This is most easily seen starting from the Fourier transform of (67),

$$-(p^2 - M_+^2)\tilde{\phi}(p) = \langle +-\rangle \sigma_- \tilde{\phi}(p), \tag{73}$$

where we have used (68). In an inertial frame, the pseudo-complex extension $p \mapsto P = p + If$ does not change this equation, because $f = 0$, so that

$$(P^2 - M^2)\tilde{\phi}(P) = 0. \tag{74}$$

This is recognised as the Fourier transform (40) of the equation of motion derived from the manifestly pseudo-complex Poincaré invariant Lagrangian (72).

We conclude that a Pauli-Villars regularised scalar theory on real spacetime gives rise to a scalar field theory on pseudo-complex spacetime, due to the integrability of the almost pseudo-complex structure. This proves the implication $(ii) \Rightarrow (i)$ stated in the introduction. Together with the results from section V, we have thus explicitly shown the equivalence of quantum field theory on pseudo-complex spacetime and Pauli-Villars regulated quantum field theory on real spacetime, in the case of a scalar field. The constructions can be extended to higher tensor and spinor fields, whose pseudo-complexification we will discuss in the next section.

VII. SPINOR AND VECTOR FIELDS

It is straightforward to apply the pseudo-complexification procedure to spinor or higher tensor fields. The pseudo-complex Dirac Lagrangian for an $SO_{\mathbb{P}}(1,3)$-spinor ψ reads

$$\bar{\psi}(i\gamma^\mu D_\mu - M)\psi, \tag{75}$$

with pseudo-complex mass $M \neq 0$, and standard Dirac gamma matrices. An almost massless, abelian $SO_{\mathbb{P}}(1,3)$-vector field A^μ is governed by the pseudo-complexified Proca Lagrangian

$$-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}M^2\sigma_- A^\mu A_\mu, \tag{76}$$

where $F_{\mu\nu} \equiv D_\mu A_\nu - D_\nu A_\mu$, and we assume, without loss of generality, $M \in \mathbb{P}^0_-$.

Pauli-Villars regularisation of a vector field in standard quantum field theory on real spacetime requires the introduction of a non-zero regulating mass, which breaks gauge invariance. We now demonstrate that, in contrast, gauge invariance is fully preserved in an *almost massless* pseudo-complex non-abelian gauge theory, and only broken by the projection (48) to spacetime.

Consider the pseudo-complexified Dirac Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu D_\mu - M)\Psi \tag{77}$$

for an N-multiplet of spinor fields

$$\Psi : \mathbb{P}^{1,3} \longrightarrow \mathbb{P}_\mathbb{C}, \tag{78}$$

with pseudo-complex mass $M \neq 0$. Let Ψ belong to an irreducible representation of a simple compact Lie group G, with generators $\mathbf{t}_a$ satisfying the algebra

$$[\mathbf{t}^a, \mathbf{t}^b] = i f^{abc}\mathbf{t}^c. \tag{79}$$

The theory (77) possesses the global gauge symmetry

$$\Psi \mapsto \exp(i\alpha^a \mathbf{t}^a)\Psi. \tag{80}$$

Now we promote the α^a to fields $\alpha^a : \mathbb{P}^{1,3} \longrightarrow \mathbb{P}$, and require (80) to be a local symmetry. Define the gauge covariant derivative

$$\nabla_\mu \equiv D_\mu - igA^a_\mu \mathbf{t}^a, \tag{81}$$

where the $A^a : \mathbb{P}^{1,3} \longrightarrow \mathbb{P}^{1,3}$ are taken to be almost massive vector fields with pseudo-complex mass $N \in \mathbb{P}^0_-$. The free field dynamics for the multiplet A is correspondingly given by the Proca-Lagrangian

$$\mathcal{L}_A = -\frac{1}{4}F^{a\,\mu\nu}F^a_{\mu\nu} + \frac{1}{2}N^2\sigma_- A^{a\,\mu}A^a_\mu, \tag{82}$$

where $F^a_{\mu\nu} \equiv D_\mu A^a_\nu - D_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$. For the covariant derivative to commute with the gauge transformation,

$$\nabla_\mu(\Psi) \mapsto \exp(i\alpha^a\mathbf{t}^a)\nabla_\mu\Psi, \tag{83}$$

we must require that, for infinitesimal α^a,

$$A^a_\mu \mathbf{t}^a \mapsto A^a_\mu \mathbf{t}^a + \frac{1}{g}\left(D_\mu \alpha^a\right)\mathbf{t}^a + i\alpha^a A^b_\mu f^{abc}\mathbf{t}^c \tag{84}$$

under a local gauge transformation. The full Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\gamma^\mu \nabla_\mu - M)\Psi - \mathcal{L}_A \tag{85}$$

is then seen to be gauge invariant if, and only if, we constrain the gauge parameters to zero-divisor values $\mathbb{P}^0_+$,

$$\alpha^a : \mathbb{P}^{1,3} \longrightarrow \mathbb{P}^0_+ \cong \mathbb{R}, \tag{86}$$

so that the change of $A^a_\mu \mathbf{t}^a$ in (84) is $\mathbb{P}^0_+$-valued, and therefore the mass term in (82) is gauge invariant. Hence, none of the gauge symmetry present in standard non-abelian gauge theory is lost.

This shows that it is merely the spacetime formulation of standard quantum field theory that causes the conflict between the gauge principle and Pauli-Villars regularisation. It is an open question as to what extent one can exploit this symmetry, e.g., obtain Ward identities, in pseudo-complex quantum field theory. The investigation of such questions requires a careful analysis of non-trivial interaction effects due to the existence of zero-divisors. It should be interesting to address these questions in future research.

VIII. CONCLUSION

The pseudo-complexified Poincaré group encodes the kinematics of a relativistic particle with submaximal acceleration. Its representation theory reveals that a pseudo-complex quantum particle gives rise to a doublet of real particles, with different real masses but equal spin, if described by a pseudo-complex field theory. Exactly one of these particles is identified as a Weyl ghost that acts as a Pauli-Villars regulator of the other, proper particle. Hence, in pseudo-complex quantum field theory, particles always carry their regulators around.

The abstract results from the representation theory are confirmed by the explict construction of the theory for a scalar field, which is governed by a pseudo-complexified Klein-Gordon equation. The pseudo-complex denominator of the corresponding scalar propagator generates a double infinity of poles due to the existence of zero divisors. An analysis of its projection to a spacetime field confirms that this is equivalent to a Pauli-Villars regularisation.

Remarkably, the pseudo-complex structure of the full theory, which is broken by the spacetime projection, resurfaces as the geometrical structure of the phase space for the regularised spacetime field. We find that the Pauli-Villars regularisation of a real spacetime theory induces a pseudo-complex field theory, and vice versa.

This equivalence between *(i)* maximal acceleration kinematics, *(ii)* pseudo-complex quantum field theory, and *(iii)* Pauli-Villars regularisation, rigorously establishes a conjecture of Nesterenko, Feoli, Lambiase and Scarpetta [9] on the regularising property of the maximal acceleration principle in scalar quantum field theory. The extension to spinor and vector fields is straightforward. Pseudo-complex gauge theory features the standard gauge symmetry, although it projects to a Pauli-Villars regularised theory on spacetime.

An in-depth analysis of interacting pseudo-complex quantum field theory remains to be done in future work, where particular attention should be paid to non-trivial effects due to the existence of zero divisors in $\mathbb{P}$.

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