On an (Interacting) Field Theories With Tensorial Momentum

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Abstract

The construction of field theories with space-time symmetries, including tensorial charges (i.e. of M-theory type), initiated in hep-th/9907011, is extended to include interaction. For SO(2,2) gravity in a tensorial space-time, with space-time symmetry consisting of Lorentz generators and "translations", represented by second-rank antisymmetric tensor, the cubic interaction terms are constructed by requirement of maintaining the gauge invariance property of theory. This interaction is essentially unique.

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1 Introduction

Space-time algebras with tensorial momentum appeared in study of branes theory, see [1] for a review. The most general such algebra is that of M-theory, where anticommutator of supercharges includes all possible tensors:

$$\begin{aligned} \{\bar{Q}, Q\} &= \Gamma^i P_i + \Gamma^{ij} Z_{ij} + \Gamma^{ijklm} Z_{ijklm}, \\ i, j, ... &= 0, 1, 2, ..10. \end{aligned} \tag{1}$$

The natural approach from the point of view of modern field theory is to try to construct the field theories, invariant w.r.t such (super)-algebras. Such an approach was initiated in [2]. There we didn't address the problem in whole generality, but make the following simplifications. First, as noted in [3], algebra (1) can be rewritten in a manifestly SO(2,10) invariant form:

$$\begin{aligned} \{\bar{Q}, Q\} &= \Gamma^{\mu\nu} P_{\mu\nu} + \Gamma^{\mu\nu\lambda\rho\sigma\delta} Z^{+}_{\mu\nu\lambda\rho\sigma\delta}, \\ \mu\nu, ... &= 0', 0, 1, ...10 \end{aligned} \tag{2}$$

As seen, energy-momentum vector disappear from the r.h.s, namely it joins with 11D second rank tensor into 12D second rank tensor. Then we took the bosonic part of (2) (with Lorentz generators), with non-zero second-rank tensor only, and considered that algebras at different lower dimensions. So, the algebra we shall consider consists of the following set of generators:

$$\begin{gathered} M_{\mu\nu}, P_{\mu\nu} \\ \mu, \nu = 0', 0, 1, ...q \end{gathered} \tag{3}$$

where $M_{\mu\nu}$ are SO(2,q) Lorentz generators, $P_{\mu\nu}$ are Abelian "translations". The first step in construction of (interacting) field theories is construction of the unitary irreducible representations of algebra (1) on the language of relativistic fields equations. Such an equations are constructed in [2], [4]. The usual principles of construction of such a representations are applicable for more general cases than usual Poincare algebra, particularly in a given case. The simplest representation is in the space of scalar function on space, dual to momenta in (1), selected by Klein-Gordon type equations. Back in momentum representation they are:

$$(TrP^2 - 2m_1^2)\Phi(P_{\mu\nu}) = 0, \tag{4}$$

$$(TrP^4 - 2m_2^4)\Phi(P_{\mu\nu}) = 0, \tag{5}$$

$$...$$

In a similar way the equations of reduced spin 1/2 can be constructed. The novelty appears for spin 1 equation, which can be expected to incorporate some generalization of notion of gauge invariance. It appears that statement of gauge symmetry for the theories considered have to be generalized into the form

$$\delta_1 S_1 + \delta_2 S_2 + ... + \delta_n S_n = 0 \tag{6}$$

where S_1, S_2 ... are actions for equations, corresponding to Eqs. (4), (5), ... for spin 1, and δ_1, δ_2, ... are corresponding variations. In this report we shall consider the spin 2 case in dimension 2+2. The exact equations for that case are the following. The first level action S_1 is

$$\begin{aligned} S_1 = \int dx(&-\partial_{\nu\kappa} h_{\alpha\beta}\partial_{\nu\kappa} h_{\alpha\beta} + 4\partial_{\beta\kappa} h_{\alpha\beta}\partial_{\nu\kappa} h_{\alpha\nu} - \partial_{\beta\kappa} h_{\alpha\beta}\partial_{\nu\alpha} h_{\kappa\nu} \\ &+4\partial_{\alpha\beta} h_{\alpha\kappa}\partial_{\beta\kappa} h + \partial_{\beta\kappa} h\partial_{\beta\kappa} h) \end{aligned} \tag{7}$$

with $h = h^{\mu}_{\mu}$. This expression is unique among those of second order over derivatives and over field $h_{\mu\nu}$, which goes into the quadratic part of the General Relativity Lagrangian after reduction, i.e on the mass shell of higher level equations (see [2]).

Second level action S_2 is

$$\begin{aligned} S_2 = \int dx(&-4\partial_{\alpha\beta} h_{\beta\kappa}\partial_{\kappa\lambda}\partial_{\lambda\mu}\partial_{\mu\nu} h_{\alpha\nu} - 2\partial_{\alpha\beta} h_{\beta\mu}\partial_{\kappa\lambda}\partial_{\kappa\lambda}\partial_{\mu\nu} h_{\alpha\nu} \\ &-s(\tfrac{1}{2}\partial_{\kappa\lambda}\partial_{\lambda\mu} h_{\alpha\beta}\partial_{\mu\nu}\partial_{\nu\kappa} h_{\alpha\beta} - \tfrac{1}{4}\partial_{\kappa\lambda}\partial_{\kappa\lambda} h_{\alpha\beta}\partial_{\mu\nu}\partial_{\mu\nu} h_{\alpha\beta})) \end{aligned} \tag{8}$$

where s is an arbitrary real parameter, which can be fixed from the requirement that 2+2 theory should be a part of 2+4 theory, as shown in [4]. This theory has the following gauge invariance

$$\delta_1 S_1 + \delta_2 S_2 = 0 \tag{9}$$

with variations, in momentum representation:

$$\delta_1 h_{\mu\nu} = (p^2\xi + \xi p^2 - sp\xi p)_{\mu\nu}, \tag{10}$$

$$\delta_2 h_{\mu\nu} = \xi_{\mu\nu} \tag{11}$$

where gauge transformation parameter $\xi_{\mu\nu}$ is symmetric tensor.

Our main aim in this report is construction of first non-trivial interaction term for the theory (7), (8). The basis of construction will be the principle of maintaining the (deformed) gauge transformation (9). In other words, we shall use Noether procedure, generalized to the case of gauge invariance with few Lagrangians (9). The construction of field theories in space-times with coordinates, corresponding to (some) of tensorial charges, was first, at best of our knowledge, addressed in [2]. The particle models in such spaces (with all tensor coordinates activated) are constructed in a number of papers, see [5], [6]. In the beautiful paper [7] a free field equations are constructed in the space of second-rank symmetric tensor coordinates for OSp type algebras, which (equations) describe the whole tower of higher spin fields, and analysis of properties of these equations is given. Differences and similarities with present approach are, first, that we are constructing Lagrangians and actions, while in [7] the equations of motion are suggested, although, the number of equations exceeds the number of field, which is counterpart of necessity of few Lagrangians for one field in our approach. Second, we are constructing a field theories for one irrep of tensorial Poincare, while [7] contains, as mentioned, the whole tower of higher spins. See also references in [7] for an earlier ideas on field theories in a tensorial space-time.

2 Cubic Interaction Terms For 2+2 Gravity

So, we have to write down all possible "next order" over $h_{\mu\nu}$ terms, which are first non-trivial, cubic, interaction terms in both actions (7), (8) and in transformations (10), (11), with arbitrary coefficients, then calculate the variations and require the fulfilment of gauge invariance equation (9) in corresponding order. This procedure will give a set of linear equations on coefficients of different terms in next order actions and variations. That system will be a strongly overdetermined, so one can't guarantee the existence of solution. Indeed, calculations for the simplest possible case of Chern-Simons Lagrangian with linear over derivatives variations were unsuccessful [8]. So, we find encouraging the fact, that in this case the non-trivial solution exists. Moreover, as we shall see below, in some sense that solution is unique. Now we shall describe the derivation of solution and present an exact formulae. First, to the action S_1 the terms can be added of the form $hhh\partial\partial$ where derivative are implied to be acting on fields, and indexes should be contracted in some way. There are many ways of constructing such terms, but integrating by parts leads to the 19 independent terms. Some terms can be excluded due to the finite dimensionality - 4, of range, run by indexes, so that antisymmetrization over 5 of them is identically zero. Direct check gives exactly one relation between mentioned 19 terms, so that we have precisely 18 independent 3-rd order terms to be added to S_1. Next, corresponding terms should be added to variation (10), namely of the type $\xi h\partial\partial$, where, again, derivatives are implied to be acting on a field h or parameter ξ, and indexes are contracted in appropriate way. There are 67 possible terms in the first variation δ_1, correspondingly another 67 unknown parameters are introduced. Turning to the second Lagrangian, the similar considerations lead to the 148 coefficients of the terms of type $hhhh\partial\partial\partial\partial$, added to second Lagrangian, and three terms of type $h\xi$ added to second variation, i.e. the variation (11). There are relations among mentioned 148 terms, similar to that in S_1, coming from the finiteness of range run by indexes. Direct calculation with the use of "Mathematica" shows existence of 48 such relations. So, we have to substitute variations into actions and check the relation (9) up to the order $\xi hh\partial\partial\partial$. Direct calculation gives the 9-parametric solution for the overdetermined system of linear equations (9). One of these parameters corresponds to solution with zero variation of second action, and variation of first one is proportional to its equation of motion. Three parameters are appearing as redefinition of parameter $\xi \to \xi h$ in free actions' gauge-invariance statement (9) with (7), (8), (10), (11). Two parameters are also appearing from free actions gauge-invariance statement through redefinition of field $h \to hh$. All these solutions are in a reasonable sense trivial. Among remaining three parameters two have a property that in corresponding solutions all coefficients of third order terms in S_1 are zero, although in S_2 third-order terms are truly non-zero, so these solutions are "half non-trivial". Finally, one parameter requires fully non-trivial third-order interaction terms in both actions. That solution is presented below. We present only next order terms w.r.t. the (7), (8), (10), (11).

$$
\begin{aligned}
& S_1^{(3)} = g \int (2h_{\kappa\nu}\partial_{\nu\pi}h_{\xi\kappa}\partial_{\pi\alpha}h_{\alpha\xi} + 2h_{\alpha\xi}\partial_{\nu\pi}h_{\xi\kappa}\partial_{\pi\alpha}h_{\kappa\nu} \\
& +h_{\kappa\nu}\partial_{\xi\kappa}h_{\alpha\xi}\partial_{\pi\alpha}h_{\nu\pi} + h_{\pi\nu}\partial_{\xi\kappa}h_{\alpha\xi}\partial_{\pi\alpha}h_{\nu\kappa} \\
& -h_{\alpha\xi}\partial_{\nu\pi}h_{\xi\kappa}\partial_{\pi\nu}h_{\kappa\xi} - 2h_{\kappa\nu}\partial_{\xi\pi}h_{\alpha\xi}\partial_{\pi\alpha}h_{\nu\kappa} - h_{\pi\nu}\partial_{\xi\kappa}h_{\alpha\alpha}\partial_{\pi\xi}h_{\nu\kappa} \\
& -h_{\pi\nu}\partial_{\xi\kappa}h_{\kappa\nu}\partial_{\pi\xi}h_{\alpha\alpha} + h_{\xi\kappa}\partial_{\nu\pi}h_{\alpha\alpha}\partial_{\pi\nu}h_{\xi\kappa})
\end{aligned}
\tag{12}
$$

$$
\delta_1^{(1)} h_{\alpha\beta} = g(\partial_{\pi\kappa}\partial_{\kappa\nu}\xi_{\alpha\pi}h_{\nu\beta} + \partial_{\kappa\nu}\partial_{\nu\beta}\xi_{\kappa\pi}h_{\alpha\pi} - s\partial_{\alpha\pi}\partial_{\kappa\nu}\xi_{\pi\kappa}h_{\nu\beta}) + (\alpha \Leftrightarrow \beta)
\tag{13}
$$

$$
\begin{aligned}
S_2^{(3)} = g \int dx (&24\partial_{o\alpha}\partial_{\nu o}h_{\alpha\beta}\partial_{\mu\nu}h_{\beta\kappa}\partial_{\lambda\mu}h_{\kappa\lambda} + 12\partial_{o\alpha}\partial_{\nu o}h_{\alpha\beta}\partial_{\mu\nu}h_{\kappa\lambda}\partial_{\lambda\mu}h_{\beta\kappa} \\
&-12\partial_{o\alpha}\partial_{\nu o}h_{\kappa\lambda}\partial_{\lambda\mu}h_{\alpha\beta}\partial_{\mu\nu}h_{\beta\kappa} + 24\partial_{\kappa\lambda}\partial_{\mu\nu}h_{\lambda\mu}\partial_{\nu o}h_{\beta\kappa}\partial_{o\alpha}h_{\alpha\beta} - 24\partial_{\kappa\lambda}\partial_{\nu o}h_{\alpha\beta}\partial_{o\alpha}h_{\beta\kappa}\partial_{\mu\nu}h_{\lambda\mu} \\
&+24\partial_{\mu\nu}\partial_{\nu o}h_{\alpha\beta}\partial_{o\alpha}h_{\beta\kappa}\partial_{\kappa\lambda}h_{\lambda\mu} + 8h_{\alpha\beta}\partial_{o\alpha}\partial_{\nu o}h_{\mu\nu}\partial_{\kappa\lambda}\partial_{\lambda\mu}h_{\beta\kappa} + 16\partial_{\kappa\lambda}\partial_{\lambda\mu}h_{\mu\nu}\partial_{\nu o}h_{\beta\kappa}\partial_{o\alpha}h_{\alpha\beta} \\
&+16\partial_{\kappa\lambda}\partial_{\lambda\mu}h_{\mu\nu}\partial_{\nu o}h_{\alpha\beta}\partial_{o\alpha}h_{\beta\kappa} + 8\partial_{\kappa\lambda}\partial_{\nu o}h_{\mu\nu}\partial_{\lambda\mu}h_{\beta\kappa}\partial_{o\alpha}h_{\alpha\beta} - 8\partial_{\kappa\lambda}\partial_{\nu o}h_{\mu\nu}\partial_{\lambda\mu}h_{\alpha\beta}\partial_{o\alpha}h_{\beta\kappa} \\
&+24\partial_{\kappa\lambda}\partial_{o\alpha}h_{\beta\kappa}\partial_{\nu o}h_{\alpha\beta}\partial_{\lambda\mu}h_{\mu\nu} - 12\partial_{\mu\alpha}\partial_{\nu o}h_{\alpha\beta}\partial_{o\nu}h_{\beta\kappa}\partial_{\lambda\mu}h_{\kappa\lambda} + 6\partial_{\mu\alpha}\partial_{\nu o}h_{\alpha\beta}\partial_{o\nu}h_{\kappa\lambda}\partial_{\lambda\mu}h_{\beta\kappa} \\
&+6\partial_{\lambda\mu}\partial_{\nu o}h_{\alpha\beta}\partial_{\mu\alpha}h_{\kappa\lambda}\partial_{o\nu}h_{\beta\kappa} + 4h_{\beta\kappa}\partial_{o\nu}\partial_{\nu o}h_{\alpha\beta}\partial_{\mu\alpha}\partial_{\kappa\lambda}h_{\lambda\mu} - 4\partial_{\mu\alpha}\partial_{\nu o}h_{\alpha\beta}\partial_{o\nu}h_{\beta\kappa}\partial_{\kappa\lambda}h_{\lambda\mu} \\
&+8\partial_{\kappa\lambda}\partial_{\nu o}h_{\alpha\beta}\partial_{\mu\alpha}h_{\lambda\mu}\partial_{o\nu}h_{\beta\kappa} + (2+2s)h_{\alpha\beta}\partial_{o\lambda}\partial_{\lambda\mu}h_{\kappa\alpha}\partial_{\mu\nu}\partial_{\nu o}h_{\beta\kappa} \\
&+(14+2s)\partial_{\lambda\mu}\partial_{\mu\nu}h_{\alpha\beta}\partial_{o\lambda}h_{\kappa\alpha}\partial_{\nu o}h_{\beta\kappa} \\
&-8h_{\alpha\beta}\partial_{o\kappa}\partial_{\lambda\mu}h_{\alpha\beta}\partial_{\mu\nu}\partial_{\nu o}h_{\kappa\lambda} + 8\partial_{o\kappa}\partial_{\nu o}h_{\alpha\beta}\partial_{\lambda\mu}h_{\kappa\lambda}\partial_{\mu\nu}h_{\beta\alpha} - 8h_{\beta\kappa}\partial_{\lambda\alpha}\partial_{\kappa\lambda}h_{\alpha\beta}\partial_{o\mu}\partial_{\nu o}h_{\mu\nu} \\
&+8\partial_{\kappa\lambda}\partial_{\nu o}h_{\alpha\beta}\partial_{\lambda\alpha}h_{\beta\kappa}\partial_{o\mu}h_{\mu\nu} + 32\partial_{\lambda\alpha}\partial_{\nu o}h_{\alpha\beta}\partial_{\kappa\lambda}h_{\beta\kappa}\partial_{o\mu}h_{\mu\nu} - 8\partial_{\mu\nu}\partial_{\nu o}h_{\alpha\alpha}\partial_{\lambda\mu}h_{\beta\kappa}\partial_{o\lambda}h_{\beta\kappa} \\
&+4\partial_{\lambda\mu}\partial_{\nu o}h_{\alpha\alpha}\partial_{\mu\nu}h_{\beta\kappa}\partial_{o\lambda}h_{\beta\kappa} - (1+s)h_{\kappa\alpha}\partial_{\lambda\mu}\partial_{\mu\lambda}h_{\alpha\beta}\partial_{o\nu}\partial_{\nu o}h_{\beta\kappa} \\
&-(\tfrac{5}{2}+s)\partial_{\lambda\mu}\partial_{\mu\lambda}h_{\alpha\beta}\partial_{\nu o}h_{\beta\kappa}\partial_{o\nu}h_{\kappa\alpha} \\
&+4h_{\alpha\beta}\partial_{\nu o}\partial_{o\nu}h_{\beta\alpha}\partial_{\mu\kappa}\partial_{\lambda\mu}h_{\kappa\lambda} + 6\partial_{\lambda\mu}\partial_{\mu\kappa}h_{\kappa\lambda}\partial_{\nu o}h_{\alpha\beta}\partial_{o\nu}h_{\beta\alpha} + 8\partial_{\mu\kappa}\partial_{\nu o}h_{\alpha\beta}\partial_{\lambda\mu}h_{\beta\alpha}\partial_{o\nu}h_{\kappa\lambda} \\
&+2\partial_{o\nu}\partial_{\nu o}h_{\kappa\kappa}\partial_{\lambda\mu}h_{\alpha\beta}\partial_{\mu\lambda}h_{\beta\alpha})
\end{aligned}
\tag{14}
$$

$$
\delta_2^{(1)} h_{\alpha\beta} = g(\xi_{\alpha\kappa}h_{\kappa\beta} + \xi_{\beta\kappa}h_{\kappa\alpha}) \tag{15}
$$

where g is an interaction constant. There are many possible forms of rewriting this solution, by using identities between different third-order terms in second action, presented form seems to be most compact.

3 Conclusion

Having symmetry variations in non-zero over field order, one usually obtains a non-Abelian algebra of symmetries. However, in our case we have an obstacle in calculating an algebra of gauge transformations. In usual case symmetry statement (6) include one term, and commutator of symmetries is again a symmetry. When number of terms is more than one, as is in our case, it is easy to understand, that commutator of symmetries is no longer a symmetry. We are not aware on a procedure (which should exist, nevertheless) of obtaining third symmetry from two existing symmetries of type (6) with more than one action. It is worth to mention that one other property, namely, the fact that symmetry of equations of motion follows from that of action, is maintained in this generalized case. That can be simply proved by differentiating (6).

There are few directions of development of the present results. Next order terms can be derived, in principle, in the same way, but will require too much calculations, so we need an understanding of symmetry formulae, e.g. whether some geometry exists behind these symmetries. One can try to construct the interaction terms for few spin 1 fields, i.e. generalize the Yang-Mills theory. These calculations also are sufficiently complicated. The quantization of these models is another problem. Approach developed in [7], particularly the notion of positive and negative modes, seems to be applicable in above case, also. The existence of actions in the present approach perhaps should provide a possibility of generalization of path integral quantization.

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