Quantum field theory of a free massless (pseudo)scalar field in 1+1–dimensional space–time as a test for the massless Thirring model

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Abstract

We analyse different approaches to the description of the quantum field theory of a free massless (pseudo)scalar field defined in 1+1–dimensional space–time which describes the bosonized version of the massless Thirring model. These are (i) axiomatic quantum field theory, (ii) current algebra and (iii) path–integral. We show that the quantum field theory of a free massless (pseudo)scalar field defined on the class of the Schwartz test functions $\mathcal{S}_0(\mathbb{R}^2)$ connects all these approaches. This quantum field theory is well–defined within the framework of Wightman's axioms and Wightman's positive definiteness condition. The physical meaning of the definition of Wightman's observables on the class of test functions from $\mathcal{S}_0(\mathbb{R}^2)$ instead of $\mathcal{S}(\mathbb{R}^2)$, as required by Wightman's axioms, is the irrelevance of the collective zero–mode related to the collective motion of the "center of mass" of the free massless (pseudo)scalar field, which can be deleted from the intermediate states of correlation functions (Eur. Phys. J. C **24**; 653 (2002)). In such a theory the continuous symmetry, induced by shifts of the massless (pseudo)scalar field, is spontaneously broken and there is a non–vanishing spontaneous magnetization. The obtained results are discussed in connection with Coleman's theorem asserting the absence of Goldstone bosons and spontaneously broken continuous symmetry in quantum field theories defined in 1+1–dimensional space–time.

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1 Introduction

In 1964 Wightman has delivered his seminal lectures [1], where he has formulated his point of view concerning the quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$, which can be described by the Lagrangian

$$\mathcal{L}(x) = \frac{1}{2}\,\partial_{\mu}\vartheta(x)\partial^{\mu}\vartheta(x), \tag{1.1}$$

invariant under the continuous Abelian symmetry group induced by field shifts

$$\vartheta(x) \to \vartheta\,'(x) = \vartheta(x) + \alpha \tag{1.2}$$

with the parameter $\alpha \in \mathbb{R}^{\,1}$.

For the definition of the quantum field theory, described by the Lagrangian (1.1), Wightman introduced observables defined by

$$\vartheta(h) = \int d^2x\, h(x)\, \vartheta(x), \tag{1.3}$$

where $h(x)$ are test functions from the Schwartz class $\mathcal{S}(\mathbb{R}^{\,2})$ or $\mathcal{S}_0(\mathbb{R}^{\,2}) = \{h(x) \in \mathcal{S}(\mathbb{R}^{\,2}); \tilde{h}(0) = 0\}$ [1]. The function $\tilde{h}(k)$ is the Fourier transform of $h(x)$ defined by

$$\tilde{h}(k) = \int d^2x\, h(x)\, e^{+ik\,\cdot\,x} \quad , \quad h(x) \;=\; \int \frac{d^2k}{(2\pi)^2}\,\tilde{h}(k)\, e^{-ik\,\cdot\,x}. \tag{1.4}$$

In terms of Wightman's observable $\vartheta(h)$ one can define a quantum state $|h\rangle$

$$|h\rangle = \vartheta(h)|\Psi_0\rangle \tag{1.5}$$

with the norm $||h||$ given by

$$\begin{aligned} ||h||^2 \;&=\; \langle h|h\rangle = \langle\Psi_0|\vartheta(h)\vartheta(h)|\Psi_0\rangle = \iint d^2x d^2y\, h^*(x)\,\langle\Psi_0|\vartheta(x)\vartheta(y)|\Psi_0\rangle\, h(y) = \\ &=\; \iint d^2x d^2y\, h^*(x)\, D^{(+)}(x-y;\mu)\, h(y). \end{aligned} \tag{1.6}$$

Here $|\Psi_0\rangle$ is a vacuum state and $D^{(+)}(x-y;\mu)$ is the Wightman function defined by [1]

$$\begin{aligned} &D^{(+)}(x-y;\mu) = \langle\Psi_0|\vartheta(x)\vartheta(y)|\Psi_0\rangle = \\ &= \frac{1}{2\pi}\int^{\infty}_{-\infty}\frac{dk^1}{2k^0}\, e^{-\,i\,k\,\cdot\,(x\,-\,y)} = -\frac{1}{4\pi}\ell n[-\mu^2(x-y^2 + i\,0\cdot\varepsilon(x^0-y^0)], \end{aligned} \tag{1.7}$$

where $\varepsilon(x^0-y^0)$ is the sign function, $(x-y)^2 = (x^0-y^0)^2 - (x^1-y^1)^2$, $k\cdot(x-y) = k^0(x^0-y^0) - k^1(x^1-y^1)$, $k^0 = |k^1|$ is the energy of a free massless (pseudo)scalar quantum with momentum k^1 and μ is the infrared cut–off reflecting the infrared divergence of the Wightman function (1.7).

Wightman's analysis of a quantum field theory, described by the Lagrangian (1.1), can be summarized as "… there is no such mathematical object as a free field with mass zero in two–dimensional space–time unless one of the usual assumptions is abandoned."

[1]. The usual assumptions are Wightman's axioms and Wightman's positive definiteness condition [1–4].

The main problem promoting Wightman to make such a strong assertion was the infrared divergence of the Wightman function (1.7). Due to this infrared divergence the Wightman function $D^{(+)}(x-y;\mu)$ does not satisfy Wightman's positive definiteness condition [1–3]

$$||h||^2 = \iint d^2x d^2y h^*(x) D^{(+)}(x-y;\mu) h(y) \geq 0 \tag{1.8}$$

on the Schwartz test functions $h(x)$ from $\mathcal{S}(\mathbb{R}^2)$ [1–3]. This can be easily seen in the momentum representation. Substituting (1.4) and (1.7) in (1.8) we get

$$||h||^2 = \frac{1}{2\pi}\int^{\infty}_{-\infty}\frac{dk^1}{2k^0}\,|\tilde{h}(k^0,k^1)|^2. \tag{1.9}$$

In the light–cone variables $k_+ = k^0 + k^1$, $k_- = k^0 - k^1$ and $d^2k = \frac{1}{2}dk_+dk_-$ the r.h.s. of (1.9) reads [5]

$$||h||^2 = \frac{1}{2\pi}\int^{\infty}_{0}\frac{dk_+}{k_+}\,|\tilde{h}(k_+,0)|^2, \tag{1.10}$$

where we have used the Fourier transform of the Wightman function $D^{(+)}(x-y;\mu)$ equal to [5]

$$\begin{aligned} F^{(+)}(k) &= \int d^2(x-y)\, e^{+ik\,\cdot\,(x-y)}\, D^{(+)}(x-y;\mu) = 2\pi\,\theta(k^0)\,\delta(k^2) = \\ &= 2\pi\,\frac{\theta(k_+)}{k_+}\,\delta(k_-) + 2\pi\,\frac{\theta(k_-)}{k_-}\,\delta(k_+) \end{aligned} \tag{1.11}$$

and symmetric test functions $\tilde{h}(k_+,k_-) = \tilde{h}(k_-,k_+)$ for simplicity.

If the test functions $h(x)$ belong to the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ with $\tilde{h}(0) \neq 0$, the integral over k^1 in (1.9) has a logarithmic divergence. Since this integral is related to a norm of a quantum state, it cannot be infinite. Therefore, the integral over k^1 should be regularized. The former can be carried out within the theory of generalized functions [26]. The regularized expression is equal to

$$\begin{aligned} ||h||^2 &= \iint d^2x d^2y\, h^*(x)\, D^{(+)}(x-y;\mu)\, h(y)_R = \\ &= \frac{1}{2\pi}\int^{\infty}_{-\infty}\frac{dk^1}{2k^0}\,(|\tilde{h}(k)|^2 - |\tilde{h}(0)|^2\,) = \frac{1}{2\pi}\int^{\infty}_{0}\frac{dk_+}{k_+}(|\tilde{h}(k_+,0)|^2 - |\tilde{h}(0,0)|^2\,). \end{aligned} \tag{1.12}$$

The test functions of the Schwartz class are fast decreasing for $k^1 \to \pm\infty$ and correspondingly for $k_\pm \to \infty$. This implies that the momentum integrals of the regularized expression (1.12) are negative definite. This yields

$$||h||^2 = \iint d^2x d^2y\, h^*(x)\, D^{(+)}(x-y;\mu)\, h(y)_R < 0. \tag{1.13}$$

Hence, Wightman's positive definiteness condition (1.8) is violated for quantum states in the quantum field theory of a free massless (pseudo)scalar field described in terms of Wightman's observables defined on test functions $h(x)$ from the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ with $\tilde{h}(0) \neq 0$.

According to Wightman "one way to make the positive definiteness of" (1.8) "irrelevant is to restrict the class of test functions" from $\mathcal{S}(\mathbb{R}^2)$ to $\mathcal{S}_0(\mathbb{R}^2) = \{h(x) \in \mathcal{S}(\mathbb{R}^2); \tilde{h}(0) = 0\}$ [1].

In this connection the question could be asked: "Why can quantities like test functions, which do not enter to the Lagrangian of a free massless (pseudo)scalar field and, correspondingly, do not affect the dynamics of this system, play such a crucial role for the existence of a quantum field theory of a free massless (pseudo)scalar field.

In our recent paper [5] we have made an attempt to understand Wightman's observables $\vartheta(h)$, defined by (1.3). We consider the Fourier transform $\tilde{h}(k)$ of the test function $h(x)$ as an *apparatus* function related to the *resolving power* of the device, which the observer uses for the detection of quanta of a free massless (pseudo)scalar field. Of course, such an interpretation is also rather questionable. This is due to the neutrality and decoupling of the quanta of the free massless (pseudo)scalar field $\vartheta(x)$ from everything. Nevertheless, if in spite of this we assume a measurability of these quanta, Wightman's statement, concerning the possibility to define a quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$ on the class of test functions from $\mathcal{S}_0(\mathbb{R}^2) = \{h(x) \in \mathcal{S}(\mathbb{R}^2); \tilde{h}(0) = 0\}$, can be interpreted as impossibility to detect quanta with zero energy and momentum $k^0 = k^1 = 0$. We call them below *zero–mode quanta*.

In our article [6] we have found that zero–mode quanta are related to the collective shift of the field $\vartheta(x)$ describing the motion of the "center of mass". We have shown that the removal of the collective zero–mode allows to formulate the quantum field theory of the free massless (pseudo)scalar field without infrared divergences [6]. For a free quantum system the exclusion of the collective zero–mode does not affect the evolution of the system caused by a relative motion in it [6]. The possibility to remove the collective zero–mode has been realized within the path–integral approach to the description of a quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$ in terms of the generating functional of Green functions $Z[J]$ of the free massless (pseudo)scalar field $\vartheta(x)$ [6], where $J(x)$ is an external source of the ϑ–field. The collective zero–mode of the free massless (pseudo)scalar field has been removed by the constraint [6]

$$\int d^2x\, J(x) = \tilde{J}(0) = 0. \tag{1.14}$$

The removal of the collective zero–mode from the intermediate states of correlation functions, described by the generating functional of Green functions $Z[J]$, makes it reasonable to delete this mode from Wightman's observables. For this aim the test functions should obey the constraint $\tilde{h}(0) = 0$. This is fulfilled for the test functions $h(x)$ from $\mathcal{S}_0(\mathbb{R}^2)$. Hence, defining Wightman's observables on the test functions from $\mathcal{S}_0(\mathbb{R}^2) = \{h(x) \in \mathcal{S}(\mathbb{R}^2; \tilde{h}(0) = 0\}$ one makes them insensitive to the collective zero–mode of the "center of mass" of the ϑ–field described by the Lagrangian (1.1). In other words, the collective zero–mode of a free massless (pseudo)scalar field cannot be measured by Wightman's observables.

The interpretation of Wightman's observables suggested in [5] allows to bridge standard [7,8] and axiomatic quantum field theory [1–4]. Four years after Wightman's lectures

[1], where he declared that the free massless (pseudo)scalar field $\vartheta(x)$ in 1+1–dimensional space–time does not exist from the point of view of axiomatic quantum field theory, Callan, Dashen and Sharp have published their seminal paper [9] entitled "Solvable Two–Dimensional Field Theory Based on Currents". In this paper the authors wrote: "Our discussion is not directed toward formal mathematical questions regarding the model. The results are of interest, instead, because they provide a tractable model of a theory based on the currents and the energy–momentum tensor, and because they allow one to see very readily (a) why the Thirring model is solvable and (b) why it has trivial physical consequences."

The last statement concerning the triviality of the massless Thirring model pointed out first by Wightman [1] and then by Callan, Dashen and Sharp [9] we have revised recently in [10]. We have shown that the massless Thirring model possesses a non–trivial non–perturbative phase of spontaneously broken chiral symmetry. The wave function of the non–perturbative vacuum in the chirally broken phase is of the BCS–type [10,11] with fermions acquiring a dynamical mass. Therefore, the massless Thirring model is by no means trivial and enriched by non–perturbative phenomena [10].

In spite of Wightman's declaration Callan, Dashen and Sharp analysed the massless Thirring model, [12] defined by the Lagrangian [1]

$$\mathcal{L}_{\rm Th}(x) = \bar{\psi}(x)i\gamma^{\mu}\partial_{\mu}\psi(x) - \frac{1}{2}\,g\,\bar{\psi}(x)\gamma^{\mu}\psi(x)\bar{\psi}(x)\gamma_{\mu}\psi(x), \tag{1.15}$$

and have expressed the energy–momentum tensor $\theta_{\mu\nu}$ of the massless Thirring model

$$\theta_{\mu\nu} = \frac{1}{2c}[j_{\mu}(x)j_{\nu}(x) + j_{\nu}(x)j_{\mu}(x) - g_{\mu\nu}j_{\alpha}(x)j^{\alpha}(x)], \tag{1.16}$$

where $j_{\mu}(x) = \bar{\psi}(x)\gamma_{\mu}\psi(x)$ and c is the Schwinger term [13], in terms of the massless scalar field $\varphi(x)$

$$\theta_{\mu\nu} = \frac{1}{2}[\partial_{\mu}\varphi(x)\partial_{\nu}\varphi(x) + \partial_{\nu}\varphi(x)\partial_{\mu}\varphi(x) - g_{\mu\nu}\partial_{\alpha}\varphi(x)\partial^{\alpha}\varphi(x)] \tag{1.17}$$

with the bosonization rules

$$\begin{aligned}\frac{1}{\sqrt{c}}\,j_0(x^0,x^1) &= \Pi(x^0,x^1),\\ \frac{1}{\sqrt{c}}\,j_1(x^0,x^1) &= \frac{\partial\varphi(x^0,x^1)}{\partial x^1}\end{aligned} \tag{1.18}$$

solving the current algebra of the massless Thirring model [9] defined by

$$\begin{aligned}&[j_0(x^0,x^1), j_0(x^0,y^1)] = 0,\\ &[j_1(x^0,x^1), j_1(x^0,y^1)] = 0,\\ &[j_0(x^0,x^1), j_1(x^0,y^1)] = i\,c\,\frac{\partial}{\partial x^1}\delta(x^1 - y^1),\end{aligned} \tag{1.19}$$

[1]Here g is a dimensionless coupling constant that can be both positive and negative as well. The field $\psi(x)$ is a spinor field with two components $\psi_1(x)$ and $\psi_2(x)$. The γ–matrices are defined in terms of the well–known 2×2 Pauli matrices σ_1, σ_2 and σ_3: $\gamma^0 = \sigma_1$, $\gamma^1 = -i\sigma_2$ and $\gamma^5 = \gamma^0\gamma^1 = \sigma_3$ [10].

Using a relation, analogous to that suggested by Morchio, Pierotti and Strocchi [14]

$$\frac{\partial\varphi(x)}{\partial x^\mu} = \varepsilon_{\mu\nu}\frac{\partial\vartheta(x)}{\partial x_\nu}, \tag{1.20}$$

where $\varepsilon^{\mu\nu}$ is the anti–symmetric tensor defined by $\varepsilon^{01} = -\varepsilon^{10} = 1$, one can reduce the bosonization rules (1.18) to a form agreeing with those suggested in [10]

$$\begin{aligned}\frac{1}{\sqrt{c}}\,j_0(x^0,x^1) &= \frac{\partial\vartheta(x^0,x^1)}{\partial x^1},\\ \frac{1}{\sqrt{c}}\,j_1(x^0,x^1) &= \frac{\partial\vartheta(x^0,x^1)}{\partial x^0} = \Pi(x).\end{aligned} \tag{1.21}$$

The Schwinger term c can be expressed in terms of the coupling constant of the massless Thirring model as follows [10]

$$c = \frac{1}{\pi}\Big(1 - e^{-2\pi/g}\Big). \tag{1.22}$$

This result has been obtained for the chirally broken phase of the massless Thirring model [10].

Due to the canonical commutation relation

$$[\Pi(x^0,x^1),\varphi(x^0,y^1)] = -[\vartheta(x^0,x^1),\Pi(x^0,y^1)] = -i\,\delta(x^1-y^1) \tag{1.23}$$

the spatial derivatives of the φ and ϑ fields reproduce fully the Schwinger equal–time commutation relation at the level of the canonical quantum field theory of a free massless (pseudo)scalar field

$$\begin{aligned}[j_0(x^0,x^1),j_1(x^0,y^1)] &= \Big[\Pi(x^0,x^1),\frac{\partial\varphi(x^0,y^1)}{\partial y^1}\Big] = \Big[\frac{\partial\vartheta(x^0,x^1)}{\partial x^1},\Pi(x^0,y^1)\Big] =\\ &= i\,c\,\frac{\partial}{\partial x^1}\delta(x^1-y^1).\end{aligned} \tag{1.24}$$

Without reference to a certain class of test functions, on which Wightman's positive definiteness condition should be positive and finite, Callan, Dashen and Sharp comment concerning the energy–momentum tensor (1.17): "That is, it is the energy–momentum tensor of a free massless scalar field. At this point, one could introduce the Fock representation for the scalar field, annihilation and creation operators, etc., and verify in detail that the energy and momentum operators have the expected properties, but there is little to be gained by going over these well–known details."

We would like to emphasize that one of the main open problems of axiomatic quantum field theory is the construction of a Fock space of all *observable* states of a free massless (pseudo)scalar field in 1+1–dimensional space–time. However, the stress of this problem can be relaxed if one takes into account that such states can be never detected due to their sterility and decoupling from everything.

The equivalence between the massless Thirring model and the quantum field theory of the free massless (pseudo)scalar field, obtained by Callan, Dashen and Sharp at the level of current algebra, testifies a one–to–one correspondence between these two theories and the irrefutable fact that the non–existence of the free massless (pseudo)scalar field in 1+1–dimensional space–time entails the non–existence of the massless Thirring model with self–coupled fermion fields.

Starting with Klaiber [15] the problem of the solution of the massless Thirring model was understood as the possibility to evaluate any correlation function. In his seminal paper [15] Klaiber suggested a solution of the massless Thirring model in terms of arbitrary correlation functions. In our recent paper [16] we have given a detail analysis of Klaiber's operator formalism and Klaiber's solution of the massless Thirring model. We have displayed weak and strong sides of Klaiber's results. We have also shown that the infrared cut–off μ, appearing in the correlation functions in Klaiber's approach, can be replaced by the ultra–violet cut–off Λ by means of a non–perturbative renormalization of the wave functions of massless Thirring fermion fields. This evidences that the massless Thirring model does not suffer from the problem of infrared divergences. Thereby, in the bosonized version, described by the quantum field theory of the free massless (pseudo)scalar field, such a problem should also not exist when it is treated well.

Klaiber's understanding of the solution of the massless Thirring model was then realized within the path–integral approach [16–18] (see also [16]) supplemented by the analysis of chiral Jacobians induced by local chiral rotations [16,19–24]. The path–integral approach is a nice tool for the evaluation of any correlation function in the massless Thirring model in terms of degrees of freedom of a free massless fermion field and two free massless scalar and pseudoscalar fields [16–18]. The final expressions for correlation functions do not depend on the infrared cut–off and contain only the ultra–violet cut–off Λ. The dependence of correlation functions on the ultra–violet cut–off Λ can be removed and the cut–off Λ can be replaced by a finite arbitrary scale M by means of a non–perturbative renormalization of the wave functions of massless Thirring fermion fields, described by the renormalization constant Z_2 [16].

It is important to emphasize that for the solution of the massless Thirring model within the path–integral approach Wightman's positive definiteness condition [1–3] and test functions do not concern.

In order to reconcile all of these approaches (i) the axiomatic quantum field theory based on Wightman's axioms and Wightman's positive definiteness condition, (ii) the current algebra, using the equivalence of the massless Thirring model and the Sugawara model, where currents are dynamical variables, and (iii) the path–integral we see only one way to assume that for the axiomatic quantum field theoretic description of the free massless (pseudo)scalar field theory we should use only test functions from $\mathcal{S}_0(\mathbb{R}^2)$ as has been pointed out by Wightman [1]. The physical meaning of such a constraint is the suppression of the collective zero–mode of the free massless (pseudo)scalar field $\vartheta(x)$ in the definition of Wightman's observables $\vartheta(h)$. Such a suppression agrees well with our conclusion that the collective zero–mode of a free massless (pseudo)scalar field does not affect the dynamics of relative motions of the system. Therefore, the definition of Wightman's observables on the test functions from $\mathcal{S}_0(\mathbb{R}^2) = \{h(x) \in \mathcal{S}(\mathbb{R}^2); \tilde{h}(0) = 0\}$ instead of $\mathcal{S}(\mathbb{R}^2)$ is well–motivated and does not contradict Wightman's axioms and Wightman's positive definiteness condition [1].

As has been shown in [6] such a quantum field theory is enriched by non–perturbative phenomena. Indeed, it possesses a non–trivial phase of spontaneously broken continuous symmetry (1.2) characterized by non–vanishing spontaneous magnetization.

In this context let us discuss Wightman's observables $\vartheta(h)$ defined on the test functions $h(x) \in \mathcal{S}(\mathbb{R}^2)$ and $\mathcal{S}_0(\mathbb{R}^2)$. The generator $Q(x^0)$ responsible for shifts of the ϑ–field (1.2) is defined by

$$Q(x^0) = \int^{\infty}_{-\infty} dx^1\, j_0(x^0, x^1) = \int^{\infty}_{-\infty} dx^1\, \Pi(x^0, x^1). \tag{1.25}$$

One can show [5] that under the symmetry transformation (1.2) Wightman's observable (1.3) is changed by

$$e^{+i\alpha\, Q(x^0)}\, \vartheta(h)\, e^{-i\alpha\, Q(x^0)} = \vartheta(h) + \alpha \int d^2x\, h(x). \tag{1.26}$$

This yields the variation of Wightman's observable

$$\delta\vartheta(h) = \alpha \int d^2x\, h(x). \tag{1.27}$$

Thus, for the general case of test functions $h(x) \in \mathcal{S}(\mathbb{R}^2)$ Wightman's observable $\vartheta(h)$ is not invariant under the field–shifts (1.2).

It is important to emphasize that $\delta\vartheta(h)$ given by (1.27) is not an operator–valued quantity. Therefore, the vacuum expectation value coincides with the quantity itself

$$\langle\Psi_0|\delta\vartheta(h)|\Psi_0\rangle = \delta\vartheta(h) = \alpha \int d^2x\, h(x). \tag{1.28}$$

Hence, the variation of Wightman's observable $\delta\vartheta(h)$ contains neither the information about spontaneous breaking of continuous symmetry nor Goldstone bosons.

In order to make this more obvious let us narrow the class of the test functions from $\mathcal{S}(\mathbb{R}^2)$ to $\mathcal{S}_0(\mathbb{R}^2)$. In this case Wightman's observable $\vartheta(h)$ becomes invariant under shifts (1.2) and the variation of Wightman's observable $\delta\vartheta(h)$ is identically zero, $\delta\vartheta(h) = 0$. However, this does not give new information about Goldstone bosons and a spontaneously broken continuous symmetry in addition to that we have got on the class of the test functions from $\mathcal{S}(\mathbb{R}^2)$.

In this connection Coleman's theorem [25] asserting the absence of Goldstone bosons in 1+1–dimensional space-time seems to be doubtful. Since Coleman has interpreted this theorem too strong:"... in two dimensions there is no spontaneous breakdown of continuous symmetries ..." [27], we would like to turn to the analysis of Coleman's theorem in this paper.

The paper is organized as follows. In Section 2 we give a cursory outline of Wightman's axioms and Wightman's positive definiteness condition. In Section 3 we consider a free massless (pseudo)scalar field theory free of infrared divergences and defined on the test functions from $\mathcal{S}_0(\mathbb{R}^2)$. In Section 4 we discuss a canonical quantum field theory of a massless self–coupled (pseudo)scalar field with current conservation, satisfying Wightman's axioms and Wightman's positive definiteness condition. In this quantum field theory continuous symmetry is spontaneously broken and Goldstone bosons are quanta of a massless (pseudo)scalar field. In Section 5 we analyse Coleman's proof and his theorem. In the Conclusion we summarize the results.

2 Wightman's axioms and Wightman's positive definiteness condition

According to Wightman [1–4] [2] any quantum field theory should satisfy the following set of axioms:

- **W1** (Covariance). There is a continuous unitary representation of the imhomogeneous Lorentz group $g \to U(g)$ on the Hilbert space $\mathcal{H}$ of quantum theory states. The generators $H = (P^0, P^1)$ of the translation subgroup have spectrum in the forward cone $(p^0)^2 - (p^1)^2 \ge 0$, $p^0 \ge 0$. There is a vector $|\Psi_0\rangle \in \mathcal{H}$ (the vacuum) invariant under the operators $U(g)$.

- **W2** (Observables). There are field operators $\{\vartheta(h) : h(x) \in \mathcal{S}(\mathbb{R}^2)\}$ densely defined on $\mathcal{H}$. The vector $|\Psi_0\rangle$ is in the domain of any polynomial in the $\vartheta(h)$'s, and the subspace $\mathcal{H}'$ spanned algebraically by the vectors $\{\vartheta(h_1)\ldots\vartheta(h_n)|\Psi_0\rangle; n \ge 0, h_i \in \mathcal{S}(\mathbb{R}^2)\}$ is dense in $\mathcal{H}$. The field $\vartheta(h)$ is covariant under the action of the Lorentz group on $\mathcal{H}$, and depends linearly on h. In particular, $U^\dagger(g)\vartheta(h)U(g) = \vartheta(h_g)$.

- **W3** (Locality). If the supports of $h(x)$ and $h'(x)$ are space–like separated, then $[\vartheta(h), \vartheta(h')] = 0$ on $\mathcal{H}'$.

- **W4** (Vacuum). The vacuum vector $|\Psi_0\rangle$ is the unique vector (up to scalar multiples) in $\mathcal{H}$ which is invariant under time translations.

These axioms should be supplemented by Wightman's positive definiteness condition which reads [1–3]

$$\|\Psi\|^2 = \Big\|\alpha_0|\Psi_0\rangle + \alpha_1\int d^2x_1\, h(x_1)\,\vartheta(x_1)|\Psi_0\rangle + \frac{\alpha_2}{2!}\iint d^2x_1 d^2x_2\, h(x_1)h(x_2)\,\vartheta(x_1)\,\vartheta(x_2)|\Psi_0\rangle + \ldots\Big\|^2 \ge 0 \quad (2.1)$$

for all $\alpha_i \in \mathbb{R}^1\,(i = 0, 1, \ldots)$ and the test functions $h(x)$ from the the Schwartz class $\mathcal{S}(\mathbb{R}^2)$, $h(x) \in \mathcal{S}(\mathbb{R}^2)$ [1–3], and $|\Psi_0\rangle$ is a vacuum wave function.

The wave function $|\Psi\rangle$ is a linear superposition of all quantum states $|\Psi_n\rangle$

$$|\Psi_n\rangle = \frac{1}{\sqrt{n!}}\int\ldots\int d^2x_1\ldots d^2x_n\, h(x_1)\ldots h(x_n)\,\vartheta(x_1)\ldots\vartheta(x_n)\,|\Psi_0\rangle, \quad (2.2)$$

which are vectors in the Hilbert space $\mathcal{H}$ [1–4]. In terms of the two–point Wightman function the relation (2.1) reads

$$\iint d^2x d^2y\, h^*(x)\, D^{(+)}(x-y)\, h(y) \ge 0, \quad (2.3)$$

which is so called Wightman's positive definiteness condition [1–3].

[2] We cite these axioms from the textbook by Glimm and Jaffe [4].

3 A free massless (pseudo)scalar field theory without infrared divergences. Path–integral approach

The quantum field theory of the free massless (pseudo)scalar field $\vartheta(x)$ without infrared divergences has been developed in Ref.[6] within the path–integral approach. The removal of infrared divergences is caused by the constraint (1.14) on external sources $J(x)$ of the ϑ–field. We have shown that such a theory can be fully determined by the generating functional of Green functions

$$\begin{aligned} Z[J] &= \Big\langle\Psi_0\Big|\mathrm{T}\Big(e^{i\int d^2x\,\vartheta(x)J(x)}\Big)\Big|\Psi_0\Big\rangle = \\ &= \int \mathcal{D}\vartheta\, e^{\displaystyle i\int d^2x\,\Big[\frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \vartheta(x)J(x)\Big]}, \end{aligned} \tag{3.1}$$

where T is a time–ordering operator.

In terms of $Z[J]$ an arbitrary correlation function of the ϑ–field can be defined as follows

$$\begin{aligned} &G(x_1,\ldots,x_n;y_1,\ldots,y_p) = \langle\Psi_0|F(\vartheta(x_1),\ldots,\vartheta(x_n);\vartheta(y_1),\ldots,\vartheta(y_p))|\Psi_0\rangle = \\ &= F\Big(-i\frac{\delta}{\delta J(x_1)},\ldots,-i\frac{\delta}{\delta J(x_n)};-i\frac{\delta}{\delta J(y_1)},\ldots,-i\frac{\delta}{\delta J(y_p)}\Big)Z[J]\Big|_{J\,=\,0}. \end{aligned} \tag{3.2}$$

Relative to the massless Thirring model one encounters the problem of the evaluation of correlation functions of the following kind

$$\begin{aligned} &G(x_1,\ldots,x_n;y_1,\ldots,y_p) = \Big\langle\Psi_0\Big|\mathrm{T}\Big(\prod^{n}_{j=1}e^{+i\beta\vartheta(x_j)}\prod^{p}_{k=1}e^{-i\beta\vartheta(y_k)}\Big)\Big|\Psi_0\Big\rangle = \\ &= \exp\Big\{-i\beta\sum^{n}_{j=1}\frac{\delta}{\delta J(x_j)} + i\beta\sum^{p}_{k=1}\frac{\delta}{\delta J(y_k)}\Big\}Z[J]\Big|_{J\,=\,0}. \end{aligned} \tag{3.3}$$

Since the path–integral over the ϑ–field (3.1) is Gaussian, it can be evaluated explicitly. The result reads

$$Z[J] = \exp\Big\{i\,\frac{1}{2}\int d^2x\,d^2y\,J(x)\,\Delta(x-y;M)\,J(y)\Big\}, \tag{3.4}$$

where $\Delta(x-y;M)$, the causal two–point Green function, obeys the equation

$$\Box\,\Delta(x-y;M) = \delta^{(2)}(x-y) \tag{3.5}$$

and relates to the Wightman functions as

$$\Delta(x;M) = i\,\theta(+x^0)\,D^{(+)}(x;M) + i\,\theta(-x^0)\,D^{(-)}(x;M), \tag{3.6}$$

where M is a finite scale.

Due to the constraint $\tilde{J}(0) = 0$ (1.14) the collective zero–mode of the ϑ–field can be deleted from the intermediate states defining vacuum expectation values (3.2), therefore the measurement of this configuration in terms of Wightman's observables $\vartheta(h)$ (1.3), defined on the class of test functions from $\mathcal{S}(\mathbb{R}^2)$, has no physical meaning.

In order to show that in the quantum field theory of a free massless (pseudo)scalar field without infrared divergences the continuous symmetry, caused by the field shifts (1.2), is spontaneously broken we suggest to consider the massless (pseudo)scalar field $\vartheta(x)$ coupled to an external "magnetic" field $h_\lambda(x)$ [28], where $h_\lambda(x)$ is a sequence of Schwartz functions from $\mathcal{S}_0(\mathbb{R}^2)$ with vanishing norm at $\lambda \to \infty$. The Lagrangian (1.1) should be changed as follows

$$\mathcal{L}(x;h_\lambda) = \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + h_\lambda(x)\,\vartheta(x). \tag{3.7}$$

The Lagrangian (3.7) defines the action of a massless (pseudo)scalar field $\vartheta(x)$ coupled to the "magnetic"field $h_\lambda(x)$

$$S[\vartheta,h_\lambda] = \int d^2x\,\mathcal{L}(x;h_\lambda) = \frac{1}{2}\int d^2x\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \int d^2x\,h_\lambda(x)\,\vartheta(x). \tag{3.8}$$

Since the "magnetic"field $h_\lambda(x)$ belongs to the Schwartz class $\mathcal{S}_0(\mathbb{R}^2)$ obeying the constraint

$$\int d^2x\,h_\lambda(x) = \tilde{h}_\lambda(0) = 0, \tag{3.9}$$

the action $S[\vartheta,h_\lambda]$ is invariant under the symmetry transformation (1.2).

Making a field–shift (1.2) we get

$$\begin{aligned} S[\vartheta,h_\lambda] \to S'[\vartheta,h_\lambda] &= \frac{1}{2}\int d^2x\,\partial_\mu\vartheta'(x)\partial^\mu\vartheta'(x) + \int d^2x\,h_\lambda(x)\,\vartheta'(x) = \\ &= S[\vartheta,h_\lambda] + \alpha\int d^2x\,h_\lambda(x). \end{aligned} \tag{3.10}$$

Due to the constraint (3.9) the r.h.s. of (3.10) is equal to $S[\vartheta,h_\lambda]$. This confirms the invariance of the action under the symmetry transformations (1.2).

The generating functional of Green functions reads now

$$\begin{aligned} Z[J;h_\lambda] &= \Big\langle\Psi_0\Big|\mathrm{T}\Big(e^{i\int d^2x\,\vartheta(x)(h_\lambda(x)+J(x))}\Big)\Big|\Psi_0\Big\rangle = \\ &= \int\mathcal{D}\vartheta\,e^{i\int d^2x\,\left[\frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \vartheta(x)(h_\lambda(x)+J(x))\right]} = \\ &= \exp\Big\{i\,\frac{1}{2}\int d^2x\,d^2y\,(h_\lambda(x)+J(x))\,\Delta(x-y;M)\,(h_\lambda(y)+J(y))\Big\}. \end{aligned} \tag{3.11}$$

We remind that due the constraints $\tilde{J}(0) = \tilde{h}_\lambda(0) = 0$ the generating functional of Green functions $Z[J;h_\lambda]$ is invariant under field–shifts (1.2).

According to Itzykson and Drouffe [28] the magnetization $\mathcal{M}(h_\lambda)$ can be defined by [6] [3]

$$\begin{aligned} \mathcal{M}(h_\lambda) &= \langle\Psi_0|\cos\vartheta(h_\lambda)|\Psi_0\rangle = \exp\Big\{-\frac{1}{2}\int d^2x\,d^2y\,h^*_\lambda(x)\,D^{(+)}(x-y;M)\,h_\lambda(y)\Big\} = \\ &= \exp\Big\{-\frac{1}{4\pi}\int^\infty_0\frac{dk_+}{k_+}\,|\tilde{h}_\lambda(k_+,0)|^2\Big\}. \end{aligned} \tag{3.12}$$

[3]For simplicity we consider symmetric functions $\tilde{h}_\lambda(k_+,k_-) = \tilde{h}_\lambda(k_-,k_+)$.

Since the operator $\cos\vartheta(h_\lambda)$ is not time–ordered, the vacuum expectation value (3.12) is defined in terms of the Wightman function $D^{(+)}(x-y;M)$ (1.7), with the replacement of the infrared cut–off μ by the finite scale M [6], but not the causal Green function $\Delta(x-y;M)$ of (3.6).

We would like to emphasize that the exponent in the r.h.s. of (3.12) has the form of Wightman's positive definiteness condition (1.10). Due to the fast decreasing of the functions $\tilde{h}_\lambda(k_+,0)$ for $k_+\to\infty$ and the constraint $\tilde{h}_\lambda(0,0)=0$, the integral over k_+ is convergent and positive definite.

If we switch off the "magnetic" field taking the limit $h_\lambda\to 0$, this can be done adiabatically defining $h_\lambda(x)=e^{-\varepsilon\,\lambda}\,h(x)$ for $\lambda\to\infty$ with ε, a positive, infinitesimally small parameter, we get

$$\mathcal{M}=\lim_{\lambda\to\infty}\mathcal{M}(h_\lambda)=1. \tag{3.13}$$

This agrees with our results obtained in [6]. The quantity $\mathcal{M}$ is the spontaneous magnetization. Since the spontaneous magnetization does not vanish, $\mathcal{M}=1$, the continuous symmetry, caused by the field–shifts (1.2), is spontaneously broken. This confirms our statement concerning the existence of the chirally broken phase in the massless Thirring model [10].

Thus, we have shown that in the quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$, defined on test functions $h(x)$ from $\mathcal{S}_0(\mathbb{R}^2)$ and external sources $J(x)$, obeying the constraint $\tilde{J}(0)=0$, there is a non–perturbative phase, characterized by a non–vanishing spontaneous magnetization $\mathcal{M}=1$, testifying the existence of a spontaneously broken continuous symmetry (1.2).

4 Canonical quantum field theory of a massless self–coupled (pseudo)scalar field

For the most general version of a canonical quantum field theory, which we consider as a candidate for a test of Coleman's theorem, we assume a quantum field theory of a massless self–coupled (pseudo)scalar field $\vartheta(x)$ with current conservation $\partial_\mu j^\mu(x)=0$ satisfying Wightman's axioms **W1** – **W4** and Wightman's positive definiteness condition on the test functions $h(x)$ from $\mathcal{S}(\mathbb{R}^2)$ [5]. Nevertheless, below we will deal only with test functions from $\mathcal{S}_0(\mathbb{R}^2)=\{h(x)\in\mathcal{S}(\mathbb{R}^2);\tilde{h}(0)=0\}$. The most useful tool for the analysis of this theory is the Källen–Lehmann representation [29].

For the proof of his theorem Coleman has taken the quantum field theory of a massless (pseudo)scalar field $\vartheta(x)$ with a conserved current $\partial^\mu j_\mu(x)=0$ and considered the Fourier transforms $F^{(+)}(k)$, $F^{(+)}_\mu(k)$ and $F^{(+)}_{\mu\nu}(k)$ of the two–point functions defined by

$$\begin{aligned}
F^{(+)}(k) &= \int d^2x\,e^{i\,k\,\cdot\,x}\,D^{(+)}(x)=\int d^2x\,e^{i\,k\,\cdot\,x}\,\langle\Psi_0|\vartheta(x)\vartheta(0)|\Psi_0\rangle,\\
F^{(+)}_\mu(k) &= i\int d^2x\,e^{i\,k\,\cdot\,x}\,D^{(+)}_\mu(x)=i\int d^2x\,e^{i\,k\,\cdot\,x}\,\langle\Psi_0|j_\mu(x)\vartheta(0)|\Psi_0\rangle,\\
F^{(+)}_{\mu\nu}(k) &= \int d^2x\,e^{i\,k\,\cdot\,x}\,D^{(+)}_{\mu\nu}(x)=\int d^2x\,e^{i\,k\,\cdot\,x}\,\langle\Psi_0|j_\mu(x)j_\nu(0)|\Psi_0\rangle,
\end{aligned}\tag{4.1}$$

where $D^{(+)}(x)$, $D^{(+)}_{\mu}(x)$ and $D^{(+)}_{\mu\nu}(x)$ are the Wightman function (1.7), current–field and current–current correlation functions calculated with respect to the vacuum state $|\Psi_0\rangle$, invariant under space and time translations (see Wightman's axioms).

Since only the Fourier transform $F^{(+)}_{\mu}(k)$ can test Goldstone bosons we turn to the consideration of this function only. Inserting a complete set of intermediate states, the Fourier transform $F^{(+)}_{\mu}(k)$ can be transcribed into the form

$$F^{(+)}_{\mu}(k) = i\sum_{n}\int d^2x\, e^{+ik\,\cdot\,x}\,\langle\Psi_0|j_{\mu}(x)|n\rangle\langle n|\vartheta(0)|\Psi_0\rangle, \qquad (4.2)$$

Due to the invariance of the vacuum state $|\Psi_0\rangle$ under space and time translations and Lorentz covariance $\langle\Psi_0|j_{\mu}(x)|\Psi_0\rangle = 0$, we have

$$F^{(+)}_{\mu}(k) \;=\; i\sum_{n\neq\Psi_0}\int d^2x\, e^{+ik\,\cdot\,x}\,\langle\Psi_0|j_{\mu}(x)|n\rangle\langle n|\vartheta(0)|\Psi_0\rangle. \qquad (4.3)$$

Using again the invariance of the vacuum state $|\Psi_0\rangle$ under space and time translations we obtain [29]

$$F^{(+)}_{\mu}(k) \;=\; i(2\pi)^2\sum_{n\neq\Psi_0}\delta^{(2)}(k-p_n)\langle\Psi_0|j_{\mu}(0)|n\rangle\langle n|\vartheta(0)|\Psi_0\rangle \qquad (4.4)$$

The r.h.s. can be rewritten in the form of the Källen–Lehmann representation in terms of the spectral function $\rho(m^2)$ which is defined by [29]

$$\begin{aligned} F^{(+)}_{\mu}(k) &= i\,(2\pi)^2\sum_{n\neq\Psi_0}\delta^{(2)}(k-p_n)\langle\Psi_0|j_{\mu}(0)|n\rangle\langle n|\vartheta(0)|\Psi_0\rangle = \\ &= -\varepsilon_{\mu\nu}\,k^{\nu}\,\varepsilon(k^1)\,\theta(k^0)\int^{\infty}_{0}\delta(k^2-m^2)\,\rho(m^2)dm^2. \end{aligned} \qquad (4.5)$$

We notice that for massless states $-\varepsilon_{\mu\nu}\,k^{\nu}\,\varepsilon(k^1) = k_{\mu}$.

This is the most general form of a *tempered* distribution in the quantum field theory of a massless self–coupled (pseudo)scalar field $\vartheta(x)$ with current conservation $\partial^{\mu}j_{\mu}(x) = 0$ in 1+1–dimensional space–time satisfying Wightman's axioms **W1** – **W4** and Wightman's positive definiteness condition on the test functions $h(x)$ from $\mathcal{S}(\mathbb{R}^2)$.

Let us isolate the contribution of the state with $m^2 = 0$ to $F^{(+)}_{\mu}(k)$. Setting the spectral function $\rho(m^2)$ equal to

$$\rho(m^2) \;=\; \sigma\,\delta(m^2) + \rho\,'(m^2), \qquad (4.6)$$

we obtain the Fourier transform $F^{(+)}_{\mu}(k)$ in the form

$$F^{(+)}_{\mu}(k) \;=\; \sigma k_{\mu}\theta(k^0)\delta(k^2) - \varepsilon_{\mu\nu}\,k^{\nu}\,\varepsilon(k^1)\,\theta(k^0)\int^{\infty}_{M^2}\delta(k^2-m^2)\rho\,'(m^2)dm^2, \qquad (4.7)$$

where the spectral function $\rho\,'(m^2)$ contains only the contributions of the states with $m^2 > 0$ and the scale M^2 separates the state with $m^2 = 0$ from the states with $m^2 > 0$.

The original of the Fourier transform given by (4.5) is given by

$$iD^{(+)}_{\mu}(x) = -i\varepsilon_{\mu\nu}\frac{\partial}{\partial x_{\nu}}\int^{\infty}_{0}\frac{dm^2}{8\pi^2}\,\rho(m^2)\int^{\varphi_0}_{-\varphi_0}d\varphi\, e^{-m\sqrt{-x^2 + i0\cdot\varepsilon(x^0)}\,\cosh\varphi}, \tag{4.8}$$

where φ_0 is defined by [16]

$$\varphi_0 = \frac{1}{2}\,\ell n\Big(\frac{x^0 + x^1 - i0}{x^0 - x^1 - i0}\Big). \tag{4.9}$$

No we transcribe the r.h.s. of $iD^{(+)}_{\mu}(x)$ as follows

$$\begin{aligned}
&iD^{(+)}_{\mu}(x) = -i\varepsilon_{\mu\nu}\frac{\partial\varphi_0}{\partial x_{\nu}}\Big[\frac{1}{4\pi^2}\int^{\infty}_{0}dm^2\,\rho(m^2)\Big]\\
&-i\varepsilon_{\mu\nu}\frac{\partial}{\partial x_{\nu}}\frac{1}{8\pi^2}\int^{\infty}_{0}dm^2\,\rho(m^2)\int^{\varphi_0}_{-\varphi_0}d\varphi\,\Big(e^{-m\sqrt{-x^2 + i0\cdot\varepsilon(x^0)}\,\cosh\varphi} - 1\Big) =\\
&= \Big[\frac{1}{2\pi}\int^{\infty}_{0}dm^2\,\rho(m^2)\Big]\frac{i}{2\pi}\,\frac{x_{\mu}}{-x^2 + i0\cdot\varepsilon(x^0)}\\
&-i\varepsilon_{\mu\nu}\frac{\partial}{\partial x_{\nu}}\frac{1}{8\pi^2}\int^{\infty}_{0}dm^2\,\rho(m^2)\int^{\varphi_0}_{-\varphi_0}d\varphi\,\Big(e^{-m\sqrt{-x^2 + i0\cdot\varepsilon(x^0)}\,\cosh\varphi} - 1\Big) =\\
&= \int^{\infty}_{0}dm^2\,\rho(m^2)\int\frac{d^2q}{(2\pi)^2}\,q_{\mu}\,\theta(q^0)\,\delta(q^2)\,e^{-iq\cdot x}\\
&-i\varepsilon_{\mu\nu}\frac{\partial}{\partial x_{\nu}}\frac{1}{8\pi^2}\int^{\infty}_{0}dm^2\,\rho(m^2)\int^{\varphi_0}_{-\varphi_0}d\varphi\,\Big(e^{-m\sqrt{-x^2 + i0\cdot\varepsilon(x^0)}\,\cosh\varphi} - 1\Big).
\end{aligned} \tag{4.10}$$

This defines $iD^{(+)}_{\mu}(x)$ in the following general form

$$\begin{aligned}
&iD^{(+)}_{\mu}(x) = \int^{\infty}_{0}dm^2\,\rho(m^2)\int\frac{d^2q}{(2\pi)^2}\,q_{\mu}\,\theta(q^0)\,\delta(q^2)\,e^{-iq\cdot x}\\
&-i\varepsilon_{\mu\nu}\frac{\partial}{\partial x_{\nu}}\frac{1}{8\pi^2}\int^{\infty}_{0}dm^2\,\rho(m^2)\int^{\varphi_0}_{-\varphi_0}d\varphi\,\Big(e^{-m\sqrt{-x^2 + i0\cdot\varepsilon(x^0)}\,\cosh\varphi} - 1\Big).
\end{aligned} \tag{4.11}$$

The first term describes the contribution of the state with $m^2 = 0$, whereas the second one contains the contributions of all states with $m^2 > 0$. Since the contribution of the state with $m^2 = 0$ is defined by the expression $F^{(+)}_{\mu}(k; m^2 = 0) = \sigma\,k_{\mu}\,\theta(k^0)\,\delta(k^2)$ [5,25], we get sum rules for the spectral function $\rho(m^2)$, which read

$$\int^{\infty}_{0}dm^2\,\rho(m^2) = \sigma. \tag{4.12}$$

In order to investigate further properties of the spectral function $\rho(m^2)$ we suggest to consider the vacuum expectation value $\langle\Psi_0|[j_{\mu}(x),\vartheta(0)]|\Psi_0\rangle$. Following the standard procedure expounded above we get

$$\begin{aligned}
&\langle\Psi_0|[j_{\mu}(x),\vartheta(0)]|\Psi_0\rangle =\\
&= i\int^{\infty}_{0}dm^2\,\rho(m^2)\int\frac{d^2k}{(2\pi)^2}\,\varepsilon_{\mu\nu}\,k^{\nu}\,\varepsilon(k^1)\,\theta(k^0)\,\delta(k^2 - m^2)\,(e^{-ik\cdot x} + e^{+ik\cdot x}).
\end{aligned} \tag{4.13}$$

The vacuum expectation value of the equal–time commutation relation for the time–component of the current $j_0(0,x^1)$ and the field $\vartheta(0)$ reads

$$\langle\Psi_0|[j_0(0,x^1),\vartheta(0)]|\Psi_0\rangle = -\frac{1}{2\pi}\int^\infty_0 dm^2\,\rho(m^2)\,i\,\delta(x^1)$$
$$-i\int^\infty_0 dm^2\,\rho(m^2)\int^\infty_0\frac{dk^1}{2\pi^2}\Big(\frac{k^1}{\sqrt{(k^1)^2+m^2}}-1\Big)\,\cos(k^1x^1). \qquad (4.14)$$

The state with $m^2=0$ does not contribute to the second term. This term is defined by the spectral function $\rho\,'(m^2)$ only

$$\langle\Psi_0|[j_0(0,x^1),\vartheta(0)]|\Psi_0\rangle = -\frac{1}{2\pi}\int^\infty_0 dm^2\,\rho(m^2)\,i\,\delta(x^1)$$
$$-i\int^\infty_{M^2} dm^2\,\rho\,'(m^2)\int^\infty_0\frac{dk^1}{2\pi^2}\Big(\frac{k^1}{\sqrt{(k^1)^2+m^2}}-1\Big)\,\cos(k^1x^1). \qquad (4.15)$$

Let us show that in the canonical quantum field theory the second term in (4.15) should vanish.

For this aim it is sufficient to prove that in the quantum field theory of a massless (pseudo)scalar field $\vartheta(x)$ the canonical conjugate momentum $\Pi(x)$ coincides with the time component of the current $j_\mu(x)$, i.e. $\Pi(x) = j_0(x)$. This results in the l.h.s. of (4.15) equal to $-i\,\delta(x^1)$.

The fact that $j_0(x)$ is equal to the conjugate momentum $\Pi(x)$ of the field $\vartheta(x)$ can be easily illustrated in terms of the Lagrangian. The general Lagrangian of the massless self–coupled (pseudo)scalar field $\vartheta(x)$, invariant under the field–shifts $\vartheta(x)\to\vartheta\,'(x)=\vartheta(x)+\alpha$, should depend only on $\partial_\mu\vartheta(x)$ and can be written as

$$\mathcal{L}(x) = \mathcal{L}[\partial_\mu\vartheta(x)]. \qquad (4.16)$$

In the Lagrange approach the current $j_\mu(x)$ is defined by

$$j_\mu(x) = \frac{\delta\mathcal{L}[\partial_\mu\vartheta(x)]}{\delta\partial^\mu\vartheta(x)}. \qquad (4.17)$$

This testifies the coincidence of $j_0(x)$ with the conjugate momentum $\Pi(x)$, which is the derivative of the Lagrangian with respect to the time–derivative of the ϑ–field, $\dot{\vartheta}(x)$. We get

$$\Pi(x) = \frac{\delta\mathcal{L}[\partial_\mu\vartheta(x)]}{\delta\dot{\vartheta}(x)} = j_0(x). \qquad (4.18)$$

Using the canonical equal–time commutation relation

$$[j_0(0,x^1),\vartheta(0)] = [\Pi(0,x^1),\vartheta(0)] = -i\delta(x^1) \qquad (4.19)$$

we derive from (4.15) the sum rules

$$\int^\infty_0 dm^2\,\rho(m^2) = 2\pi. \qquad (4.20)$$

Comparing (4.20) with (4.12) we get

$$\sigma = 2\pi. \tag{4.21}$$

This rules out Coleman's result asserting $\sigma = 0$.

As the second term in the r.h.s. of (4.15) can be never proportional to $\delta(x^1)$ it should be zero. This yields

$$\rho\,'(m^2) \equiv 0. \tag{4.22}$$

Hence, the spectral function $\rho(m^2)$ is equal to

$$\rho(m^2) = \sigma\,\delta(m^2) = 2\pi\,\delta(m^2). \tag{4.23}$$

This means that in the case of current conservation $\partial^\mu j_\mu(x) = 0$ the Fourier transform $F^{(+)}_\mu(k)$ is defined by the contribution of the state with $m^2 = 0$ only. This confirms that the expression $F^{(+)}_\mu(k) = \sigma\,k_\mu\,\theta(k^0)\,\delta(k^2)$, postulated by Coleman [25], is general for canonical quantum field theories with conserved current $\partial^\mu j_\mu(x) = 0$ but rules out Coleman's result $\sigma = 0$ [5]. In non–canonical quantum field theories the expression $F^{(+)}_\mu(k) = \sigma\,k_\mu\,\theta(k^0)\,\delta(k^2)$, postulated by Coleman, is not general and should be rewritten in the form (4.7) with $\rho\,'(m^2) \neq 0$. Hence, Coleman's result can only be understood by the fact that he removed the canonical massless field $\vartheta(x)$ from the consideration.

Multiplying (4.15) by $i\,\alpha$ and integrating over x^1 we obtain $\langle\Psi_0|\delta\vartheta(0)|\Psi_0\rangle$, which reads

$$\langle\Psi_0|\delta\vartheta(0)|\Psi_0\rangle = i\,\alpha\int^{\infty}_{-\infty} dx^1\,\langle\Psi_0|[j_0(0,x^1),\vartheta(0)]|\Psi_0\rangle = \frac{\alpha}{2\pi}\int^{\infty}_{0} dm^2\,\rho(m^2) = \alpha, \tag{4.24}$$

where we have taken into account the discussion above and the expression for the spectral function $\rho(m^2)$ given by (4.23).

5 Triviality of Coleman's proof of his theorem

In this section we would like to show that Coleman's theorem is a trivial consequence of the exclusions of massless one–particle states and has no relation to the suppression of spontaneous symmetry breakdown.

For the proof of his theorem Coleman treated the Cauchy–Schwarz inequality

$$\int\frac{d^2k}{2\pi}\,|\tilde{h}_\lambda(k)|^2\,F^{(+)}(k)\int\frac{d^2k}{2\pi}\,|\tilde{h}_\lambda(k)|^2\,F^{(+)}_{00}(k) \ge \Big[\int\frac{d^2k}{2\pi}\,|\tilde{h}_\lambda(k)|^2\,F^{(+)}_{0}(k)\Big]^2, \tag{5.1}$$

defined on the test functions $h_\lambda(x)$ from $\mathcal{S}(\mathbb{R}^{\,1})\otimes\mathcal{S}_0(\mathbb{R}^{\,1})$:

$$h_\lambda(x) = \frac{1}{\lambda}\,f\Big(\frac{x_+}{\lambda}\Big)\,g(x_-) + \frac{1}{\lambda}\,f\Big(\frac{x_-}{\lambda}\Big)\,g(x_+) \tag{5.2}$$

the Fourier transform of which is given by

$$\tilde{h}_\lambda(k_+,k_-) = \tilde{f}(\lambda k_-)\,\tilde{g}(k_+) + \tilde{f}(\lambda k_+)\,\tilde{g}(k_-) \tag{5.3}$$

with $f(x_\pm) \in \mathcal{S}(\mathbb{R}^1)$ and $g(x_\pm) \in \mathcal{S}_0(\mathbb{R}^1)$.

In order to make Coleman's exclusions more transparent we suggest to rewrite the inequality (5.1) in the equivalent form

$$\int d^2xd^2y\, h^*(x)\, D^{(+)}(x-y)\, h(y) \int d^2xd^2y\, h^*(x)\, D^{(+)}_{00}(x-y)\, h(y)$$
$$\geq \Big[\int d^2xd^2y\, h^*(x)\, iD^{(+)}_{0}(x-y)\, h(y)\Big]^2. \tag{5.4}$$

In terms of vacuum expectation values it reads

$$\Big[\int d^2xd^2y\, h^*(x)\, \langle\Psi_0|\vartheta(x)\vartheta(y)|\Psi_0\rangle\, h(y)\Big]\Big[\int d^2xd^2y\, h^*(x)\, \langle\Psi_0|j_0(x)j_0(y)|\Psi_0\rangle\, h(y)\Big]$$
$$\geq \Big[\int d^2xd^2y\, h^*(x)\, i\langle\Psi_0|j_0(x)\vartheta(y)|\Psi_0\rangle\, h(y)\Big]^2. \tag{5.5}$$

Inserting a complete set of intermediate states, the eigenstates $|n\rangle$ of the full Hamiltonian, we get

$$\sum_n \int d^2xd^2y\, h^*(x)\, \langle\Psi_0|\vartheta(x)|n\rangle\langle n|\vartheta(y)|\Psi_0\rangle\, h(y)$$
$$\times \sum_n \int d^2xd^2y\, h^*(x)\, \langle\Psi_0|j_0(x)|n\rangle\langle n|j_0(y)|\Psi_0\rangle\, h(y)$$
$$\geq \Big[\sum_n \int d^2xd^2y\, h^*(x)\, i\langle\Psi_0|j_0(x)|n\rangle\langle n|\vartheta(y)|\Psi_0\rangle\, h(y)\Big]^2. \tag{5.6}$$

Using the invariance of the vacuum state under space and time translations and Lorentz covariance we get

$$\sum_n |\tilde{h}(p_n)|^2|\langle n|\vartheta(0)|\Psi_0\rangle|^2 \sum_n |\tilde{h}(p_n)|^2|\langle n|j_0(0)|\Psi_0\rangle|^2$$
$$\geq \Big[\sum_n |\tilde{h}(p_n)|^2 i\langle\Psi_0|j_0(0)|n\rangle\langle n|\vartheta(0)|\Psi_0\rangle\Big]^2. \tag{5.7}$$

Now it is convenient to introduce the following notations

$$\sum_n |\tilde{h}(p_n)|^2\, |\langle n|\vartheta(0)|\Psi_0\rangle|^2 = \langle\Psi_0|\vartheta(0)|\Psi_0\rangle^2|\tilde{h}(0)|^2$$
$$+ \sum_{n\neq\Psi_0, p_n^2=0} |\tilde{h}(p_n)|^2|\langle n|\vartheta(0)|\Psi_0\rangle|^2 + \sum_{n\neq\Psi_0, p_n^2\neq 0} |\tilde{h}(p_n)|^2|\langle n|\vartheta(0)|\Psi_0\rangle|^2 =$$
$$= a_0^2 + a_1^2 + a_2^2,$$
$$\sum_n |\tilde{h}(p_n)|^2\, |\langle n|j_0(0)|\Psi_0\rangle|^2 =$$
$$= \sum_{n\neq\Psi_0, p_n^2=0} |\tilde{h}(p_n)|^2|\langle n|j_0(0)|\Psi_0\rangle|^2 + \sum_{n\neq\Psi_0, p_n^2\neq 0} |\tilde{h}(p_n)|^2|\langle n|j_0(0)|\Psi_0\rangle|^2 =$$
$$= b_1^2 + b_2^2,$$

$$
\begin{aligned}
&\sum_n |\tilde{h}(p_n)|^2\, i\, \langle\Psi_0|j_0(0)|n\rangle\langle n|\vartheta(0)|\Psi_0\rangle = \\
&= \sum_{n\neq\Psi_0, p_n^2=0} |\tilde{h}(p_n)|^2\, i\, \langle\Psi_0|j_0(0)|n\rangle\langle n|\vartheta(0)|\Psi_0\rangle \\
&+ \sum_{n\neq\Psi_0, p_n^2\neq 0} |\tilde{h}(p_n)|^2\, i\, \langle\Psi_0|j_0(0)|n\rangle\langle n|\vartheta(0)|\Psi_0\rangle = a_1 b_1 + a_2 b_2,
\end{aligned}
\tag{5.8}
$$

where the indices $i = 0, 1, 2$ correspond to the contributions of the vacuum state, the massless state with $p_n^2 = 0$ and the states with $p_n^2 \neq 0$, respectively.

In terms of the vectors $\vec{a} = (a_0, a_1, a_2)$ and $\vec{b} = (0, b_1, b_2)$ the inequality (5.7) reads

$$
(a_0^2 + a_1^2 + a_2^2)(b_1^2 + b_2^2) \geq (a_1 b_1 + a_2 b_2)^2. \tag{5.9}
$$

This is the Cauchy–Schwarz inequality for $\vec{a}$ and $\vec{b}$.

The inequality (5.9) is still correct if an arbitrary number of intermediate states is removed from the sums $\sum_n$. In his proof Coleman has used this fact carrying out the following steps:

- (i) $a_1 = 0$: Due to the extension of test functions from $\mathcal{S}_0(\mathbb{R}^2)$ to $\mathcal{S}(\mathbb{R}^2)$ zero–mass contributions to the Wightman functions are excluded from the very beginning.
- (ii) $a_0 = 0$: By the special choice of the test function $g(0) = 0$.
- (iii) $b_1 = 0$: By claiming that "Because $F_{00}^{(+)}(k)$ is a positive distribution, the second integral is monotone decreasing" for λ going to infinity. Coleman refers with "the second integral" to the expression $\int d^2k\, |\tilde{h}_\lambda(k)|^2 F_{00}(k)$.
- (iv) $a_2 = b_2 = 0$: By the limit $\lambda \to \infty$ removing all intermediate states with non–zero squared invariant masses, $p_n^2 \neq 0$.

Thus, in the limit $\lambda \to \infty$ the Cauchy–Schwarz inequality (5.9) reads

$$
0 \cdot 0 \geq (0 \cdot 0)^2. \tag{5.10}
$$

Hence, Coleman removed step by step all intermediate states. Since no eigenstates of the Hamiltonian are left, there are no contributions to the r.h.s. of (5.1) and (5.10) and consequently no massless bosons. But this does not say anything what happens if the zero–mass modes are not removed.

The trivial conclusion of these steps is: *If one excludes zero–mass modes from the spectrum of the full Hamiltonian, i.e. from the intermediate states, they are not present in the theory.* In fact, since we have shown that in a canonical quantum field theory the only contributions to $F_0^{(+)}(k)$ come from zero–mass modes, an exclusion of these modes leads to $F_0^{(+)}(k) = 0$.

Thus, we argue that if one understands Coleman's paper as a proof of the absence of Goldstone bosons in two dimensions than this conclusion is wrong. Goldstone bosons in Coleman's paper are excluded by Wightman's positive definiteness condition, which he requests for test functions from $\mathcal{S}(\mathbb{R}^2)$.

6 Conclusion

We argue that the inconsistencies of Coleman's theorem are shortly in the following:

- Coleman's theorem contradicts canonical quantum field theory of a massless self–coupled (pseudo)scalar field $\vartheta(x)$ with current conservation $\partial^\mu j_\mu(x) = 0$ in which the parameter $\sigma = 2\pi$ but not $\sigma = 0$. As has been shown in [5] such a quantum field theory satisfies Wightman's axioms and Wightman's positive definiteness condition on the test functions $h(x)$ from $\mathcal{S}(\mathbb{R}^2)$.

- Coleman's theorem testifies the obvious and trivial assertion: "If one removes a canonical massless (pseudo)scalar field from the theory, this field does not appear in the further consideration."

Then, accepting Coleman's theorem as the proof of the absence of the quantum field theory of a free massless (pseudo)scalar field, one can be confused by the following consequence of this theorem demanding the absence of the quantum field theory of Thirring fermion fields. In fact, since the massless Thirring model bosonizes to the quantum field theory of a free massless (pseudo)scalar field, any suppression of this quantum field theory would lead to the suppression of the massless Thirring model. But in this case how do we have to understand the results obtained within current algebra and path–integral approach?

Therefore, the only way to reconcile different approaches to the description of the quantum field theory of a free massless (pseudo)scalar field defined in 1+1–dimensional space–time: (i) axiomatic, based on Wightman's axioms and Wightman's positive definiteness condition, (ii) current algebra and (iii) path–integral, is to use the test functions from $\mathcal{S}_0(\mathbb{R}^2)$ [1]. In fact, in vacuum expectation values defined in current algebra and path–integral approach the contribution of the zero–mode collective configuration of a free massless (pseudo)scalar field can be removed without influence on the evolution of relative motion of the system. Therefore, from a physical point of view the definition of Wightman's observables on the class of test functions from $\mathcal{S}_0(\mathbb{R}^2)$, suppressing a measurement of the collective zero–mode, describing a shift of a free massless (pseudo)scalar field, is well–motivated [1].

The canonical quantum field theory of the free massless (pseudo)scalar field $\vartheta(x)$ without infrared divergences, formulated in [6], is well defined on the class of the test functions from $\mathcal{S}_0(\mathbb{R}^2)$. This quantum field theory solves the problem of infrared divergences of the Wightman functions and describes the bosonized version of the massless Thirring model with fermion fields quantized in the chirally broken phase. The chirally broken phase of the massless Thirring model is characterized by a fermion condensate [10], the non–zero value of which is caused by the non–vanishing spontaneous magnetization, $\mathcal{M} = 1$, in the quantum field theory of a free massless (pseudo)scalar field without infrared divergences [6].

Recently [30] we have shown that the boson field representation for the massless Thirring fermion fields, suggested by Morchio, Pierotti and Strocchi [14,31], agrees fully with the existence of the chirally broken phase in the massless Thirring model and the fermion condensation. Moreover, such a representation satisfies the constant of motion for the massless Thirring model which we have found in [10].

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