Descendants of the Chiral Anomaly

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Dirac Medalist Meeting, Trieste, Italy, November 2000

MIT-CTP#3051

Abstract

Chern-Simons terms are well-known descendants of chiral anomalies, when the latter are presented as total derivatives. Here I explain that also Chern-Simons terms, when defined on a 3-manifold, may be expressed as total derivatives.

The axial anomaly, that is, the departure from transversality of the correlation function for fermion vector, vector, and axial vector currents, involves ${}^*FF$, an expression constructed from the gauge fields to which the fermions couple. Specifically, in the Abelian case one encounters

$$ {}^*F^{\mu\nu} F_{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = -4\boldsymbol{E}\cdot\boldsymbol{B} \tag{1} $$

where $F_{\mu\nu}$ is the covariant electromagnetic tensor

$$ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu \tag{2a} $$

while $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields

$$ E^i = F^{io} \ , \quad B^i = -\tfrac{1}{2}\varepsilon^{ijk} F_{jk} \ . \tag{2b} $$

The non-Abelian generalization reads

$$ {}^*F^{\mu\nu a} F^a_{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta} F^a_{\mu\nu} F^a_{\alpha\beta} \tag{3} $$

where $F^a_{\mu\nu}$ is the Yang-Mills gauge field strength

$$ F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu \tag{4} $$

and a labels the components of the gauge group, whose structure constants are f^{abc}.

The quantity ${}^*FF$ is topologically interesting. Its integral over 4-space is quantized, and measures the topological class (labeled by integers) to which the vector potential A belongs. Consequently, the integral of ${}^*FF$ is a topological invariant and we expect that, as befits a topological invariant, it should be possible to present ${}^*FF$ as a total derivative, so that its 4-volume integral becomes converted by Gauss' law into a surface integral, sensitive only to long distance, global properties of the gauge fields. That a total derivative form for ${}^*FF$ indeed holds is seen when $F_{\mu\nu}$ is expressed in terms of potentials. In the Abelian case, we use (2a) and find immediately

$$\tfrac{1}{2}{}^*F^{\mu\nu}F_{\mu\nu} = \partial_\mu\big(\varepsilon^{\mu\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma\big) \ . \tag{5}$$

For non-Abelian fields, (4) establishes the result we desire:

$$\tfrac{1}{2}{}^*F^{\mu\nu a}F^a_{\mu\nu} = \partial_\mu\varepsilon^{\mu\alpha\beta\gamma}\big(A^a_\alpha\partial_\beta A^a_\gamma + \tfrac{1}{3}f^{abc}A^a_\alpha A^b_\beta A^c_\gamma\big) \ . \tag{6}$$

The quantities whose divergence gives ${}^*FF$ are called Chern-Simons terms. By suppressing one dimension they become naturally defined on a 3-dimensional manifold (they are 3-forms), and we are thus led to consider the Chern-Simons terms in their own right [1]:

$$\mathrm{CS}(A) = \varepsilon^{ijk}A_i\partial_j A_k \qquad \text{(Abelian)} \tag{7}$$

$$\mathrm{CS}(A) = \varepsilon^{ijk}\big(A^a_i\partial_j A^a_k + \tfrac{1}{3}f^{abc}A^a_i A^b_j A^c_k\big) \qquad \text{(non-Abelian).} \tag{8}$$

The 3-dimensional integral of these quantities is again topologically interesting. When the non-Abelian Chern-Simons term is evaluated on a pure gauge, non-Abelian vector potential

$$A_i = g^{-1}\partial_i g \tag{9}$$

the 3-dimensional volume integral of $\mathrm{CS}(g^{-1}\partial g)$ measures the topological class (labeled by integers) to which the group element g belongs. The integral in the Abelian case – the case of electrodynamics – is called the magnetic helicity $\int \mathrm{d}^3r\, \boldsymbol{A}\cdot\boldsymbol{B}$, $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, and measures the linkage of magnetic flux lines. An analogous quantity arises in fluid mechanics, with the local fluid velocity $\boldsymbol{v}$ replacing $\boldsymbol{A}$, and the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v}$ replacing $\boldsymbol{B}$. Then the integral $\int \mathrm{d}^3r\, \boldsymbol{v}\cdot\boldsymbol{\omega}$ is called kinetic helicity [2].

I shall not review here the many uses to which the Chern-Simons terms, Abelian and non-Abelian, introduced in [1], have been put. The applications range from the mathematical characterizations of knots to the physical descriptions of electrons in the quantum Hall effect [3], vivid evidence for the deep significance of the Chern-Simons structure and of its antecedent, the chiral anomaly.

Instead, I pose the following question: Can one write the Chern-Simons term as a total derivative, so that (as befits a topological quantity) the spatial volume integral becomes a surface integral? An argument that this should be possible is the following: The Chern-Simons term is a 3-form on 3-space, hence it is maximal and its exterior derivative vanishes because there are no 4-forms on 3-space. This establishes that on 3-space the Chern-Simons term is closed, so one can expect that it is also exact, at least locally, that is, it can be written as a total derivative. Of course such a representation for the Chern-Simons term requires expressing the potentials in terms of "prepotentials", since the formulas (7), (8) in terms of potentials show no evidence of derivative structure. [Recall that the total derivative formulas (5), (6) for the axial anomaly also require using potentials to express F.]

There is a physical, practical reason for wanting the Abelian Chern-Simons term to be a total derivative. It is known in fluid mechanics that there exists an obstruction to constructing a Lagrangian for Euler's fluid equations, and this obstruction is just the kinetic helicity $\int \mathrm{d}^3 r\, \boldsymbol{v} \cdot \boldsymbol{\omega}$, that is, the volume integral of the Abelian Chern-Simons term, constructed from the velocity 3-vector $\boldsymbol{v}$. This obstruction is removed when the integrand is a total derivative, because then the kinetic helicity volume integral is converted to a surface integral by Gauss' theorem. When the integral obtains contributions only from a surface, the obstruction disappears from the 3-volume, where the fluid equation acts [4].

It is easy to show that the Abelian Chern-Simons term can be presented as a total derivative. We use the Clebsch parameterization for a 3-vector [5]:

$$\boldsymbol{A} = \boldsymbol{\nabla}\theta + \alpha\boldsymbol{\nabla}\beta \ . \tag{10}$$

This nineteenth century parameterization of a 3-vector $\boldsymbol{A}$ in terms of the prepotentials (θ, α, β) is an alternative to the usual transverse/longitudinal parameterization. In modern language it is a statement of Darboux's theorem that the 1-form $A_i\,\mathrm{d}r^i$ can be written as $\mathrm{d}\theta + \alpha\,\mathrm{d}\beta$ [6]. With this parameterization for $\boldsymbol{A}$, one sees that the Abelian Chern-Simons term indeed is a total derivative:

$$\begin{aligned} \mathrm{CS}(A) &= \varepsilon^{ijk} A_i \partial_j A_k \\ &= \varepsilon^{ijk} \partial_i\theta \partial_j\alpha \partial_k\beta \\ &= \partial_i\big(\varepsilon^{ijk}\theta\partial_j\alpha\partial_k\beta\big) \ . \end{aligned} \tag{11}$$

When the Clebsch parameterization is employed for $\boldsymbol{v}$ in the fluid dynamical context, the situation is analogous to the force law in electrodynamics. While the Lorentz equation is written in terms of field strengths, a Lagrangian formulation needs potentials from which the field strengths are reconstructed. Similarly, Euler's equation involves the velocity vector $\boldsymbol{v}$, but in a Lagrangian for this equation the velocity must be parameterized in terms of the prepotentials θ, α, and β.

In a natural generalization of the above, we ask whether a non-Abelian vector potential can also be parameterized in such a way that the non-Abelian Chern-Simons term (8) becomes a total derivative. We have answered this question affirmatively and we have found appropriate prepotentials that do the job [4, 7, 8].

In order to describe our non-Abelian construction, we first revisit the Abelian problem. As we have stated, the solution in the Abelian case is immediately provided by the Clebsch parameterization (10). However, finding the non-Abelian generalization requires an indirect construction, which we first present for the Abelian case.

Although in the Abelian case we are concerned with U(1) potentials, we begin by considering a bigger group SU(2), which contains our group of interest U(1). Let g be a group element of SU(2) and construct a pure-gauge SU(2) gauge potential

$$\mathcal{A} = g^{-1}\,\mathrm{d}g \ . \tag{12}$$

We know that $\mathrm{tr}(g^{-1}\,\mathrm{d}g)^3$ is a total derivative [1]; indeed, its 3-volume integral measures the topological winding number of g and therefore can be expressed as a surface integral, as befits a topological quantity. The separate SU(2) component potentials $\mathcal{A}^a$ can be projected from (12) as

$$\mathcal{A}^a = i\,\mathrm{tr}\,\sigma^a g^{-1}\,\mathrm{d}g \ , \quad \mathcal{A} = \mathcal{A}^a\sigma^a/2i \tag{13}$$

and

$$\begin{aligned} -\tfrac{2}{3}\,\mathrm{tr}(g^{-1}\,\mathrm{d}g)^3 &= \frac{1}{3!}\varepsilon^{abc}\mathcal{A}^a\mathcal{A}^b\mathcal{A}^c \\ &= \mathcal{A}^1\mathcal{A}^2\mathcal{A}^3 \ . \end{aligned} \tag{14}$$

Moreover, since $\mathcal{A}^a$ is a pure gauge, it satisfies

$$\mathrm{d}\mathcal{A}^a = -\tfrac{1}{2}\varepsilon^{abc}\mathcal{A}^b\mathcal{A}^c \ . \tag{15}$$

Next define an Abelian vector potential A by projecting one component of $g^{-1}\,\mathrm{d}g$

$$A = i\,\mathrm{tr}\,\sigma^3 g^{-1}\,\mathrm{d}g = \mathcal{A}^3 \ . \tag{16}$$

Note that A is *not* an Abelian pure gauge $\boldsymbol{\nabla}\times\boldsymbol{A} = \boldsymbol{B} \neq 0$. It now follows from (15) that

$$\begin{aligned} \boldsymbol{A}\cdot\boldsymbol{B}\,\mathrm{d}^3r = A\,\mathrm{d}A = \mathcal{A}^3\,\mathrm{d}\mathcal{A}^3 &= -\mathcal{A}^1\mathcal{A}^2\mathcal{A}^3 \\ &= \tfrac{2}{3}\,\mathrm{tr}(g^{-1}\,\mathrm{d}g)^3 \ . \end{aligned} \tag{17}$$

The last equality is a consequence of (14) and shows that the Abelian Chern-Simons term is proportional to the winding number density of the non-Abelian group element, and therefore is a total derivative. Note that the projected formula (16) involves three arbitrary functions – the three parameter functions of the SU(2) group – which is the correct number needed to represent an Abelian vector potential in 3-space.

It is instructive to see how this works explicitly. The most general SU(2) group element reads $\exp(\sigma^a\omega^a/2i)$. The three functions ω^a are presented as $\widehat{\omega}^a\omega$, where $\widehat{\omega}^a$ is a unit SU(2) 3-vector and ω is the magnitude of ω^a. The unit vector may be parameterized as

$$\widehat{\omega}^a = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta) \tag{18a}$$

where Θ and Φ are functions on 3-space, as is ω. A simple calculation shows that

$$g^{-1}\,\mathrm{d}g = \frac{\sigma^a}{2i}\big(\widehat{\omega}^a\,\mathrm{d}\omega + \sin\omega\,\mathrm{d}\widehat{\omega}^a - (1-\cos\omega)\varepsilon^{abc}\,\widehat{\omega}^b\,\mathrm{d}\widehat{\omega}^c\big) \tag{18b}$$

$$
\begin{aligned}
A &= \operatorname{tr} i\sigma^3 g^{-1}\,\mathrm{d}g \\
&= \cos\Theta\,\mathrm{d}\omega - \sin\omega\sin\Theta\,\mathrm{d}\Theta - (1-\cos\omega)\sin^2\Theta\,\mathrm{d}\Phi && (19)\\
A\,\mathrm{d}A &= -2\,\mathrm{d}(\omega - \sin\omega)\,\mathrm{d}(\cos\Theta)\,\mathrm{d}\Phi = \mathrm{d}\Omega && (20)\\
\Omega &= -2\Phi\,\mathrm{d}(\omega-\sin\omega)\,\mathrm{d}(\cos\Theta)\ . && (21)
\end{aligned}
$$

The last two equations show that our SU(2)-projected, U(1) potential possesses a total-derivative Chern-Simons term. Once we have in hand a parameterization for A such that $A\,\mathrm{d}A$ is a total derivative, it is easy to find the Clebsch parameterization for A. In the above,

$$A = \mathrm{d}(-2\Phi) + 2\big(1 - (\sin^2\tfrac{\omega}{2})\sin^2\Theta\big)\,\mathrm{d}\big(\Phi + \tan^{-1}[(\tan\tfrac{\omega}{2})\cos\Theta]\big)\ . \tag{22}$$

The projected formula (19), (22) for A, contains three arbitrary functions ω, Θ, and Φ; this offers sufficient generality to parameterize an arbitrary 3-vector $\boldsymbol{A}$. Moreover, in spite of the total derivative expression for $A\,\mathrm{d}A$, its spatial integral need not vanish. In our example, the functions ω, Θ, and Φ in general depend on $\boldsymbol{r}$; however, if we take ω to be a function only of $r = |\boldsymbol{r}|$, and identify Θ and Φ with the polar and azimuthal angles θ and φ of $\boldsymbol{r}$, then

$$
\begin{aligned}
\int A\,\mathrm{d}A &= 4\pi\int_0^\infty \mathrm{d}r\,\frac{\mathrm{d}}{\mathrm{d}r}(\omega-\sin\omega)\\
&= 4\pi(\omega-\sin\omega)\Big|_{r=0}^{r=\infty}\ . && (23)
\end{aligned}
$$

Thus if $\omega(0) = 0$ and $\omega(\infty) = \pi N$, N an integer, the integral is nonvanishing, giving $4\pi^2 N$; the contribution comes entirely from the bounding surface at infinity [7].

With this preparation, I can now describe the non-Abelian construction [8]. We are addressing the following mathematical problem: We wish to parameterize a non-Abelian vector potential A^a belonging to a group H, so that the non-Abelian Chern-Simons term (8) is a total derivative. Since we are in three dimensions, the vector potential has $3\times(\dim H)$ components, so our parameterization should have that many arbitrary functions.

The solution to our mathematical problem is to choose a large group G (compact, semi-simple) that contains H as a subgroup. The generators of H are called I^m $[m = 1,\ldots,(\dim H)]$ while those of G not in H are called S^A $[A = 1,\ldots,(\dim G) - (\dim H)]$. We further demand that G/H is a symmetric space; that is, the structure of the Lie algebra is

$$[I^m, I^n] = f^{mno} I^o \tag{24a}$$

$$[I^m, S^A] = h^{mAB} S^B \tag{24b}$$

$$[S^A, S^B] \propto h^{mAB} I^m\ . \tag{24c}$$

Here f^{mno} are the structure constants of H. Eq. (24b) shows that the S^A provide a representation for I^m and, according to (24c), their commutator closes on I^m. The normalization of the H-generators is fixed by $\operatorname{tr} I^m I^n = -N\delta^{mn}$. With g, a generic group element of G, giving rise to a pure gauge potential $\mathcal{A} = g^{-1}\,\mathrm{d}g$ in G, we define the H-vector potential A by projecting with generators belonging to H:

$$A = \frac{1}{N}\operatorname{tr} I^m g^{-1}\,\mathrm{d}g\ . \tag{25}$$

We see that the Abelian [U(1)] construction presented in (12)–(16) follows the above pattern: $\mathrm{SU}(2) = G \supset H = \mathrm{U}(1)$; $I^m = \sigma^3/2i$, $S^A = \sigma^2/2i, \sigma^3/2i$. Moreover, a chain of equations analogous to (14)–(17) shows that the H Chern-Simons term is proportional to $\mathrm{tr}(g^{-1}\,\mathrm{d}g)^3$, which is a total derivative [3, 7]:

$$\mathrm{CS}(A \in H) = \frac{1}{48\pi^2 N}\,\mathrm{tr}(g^{-1}\,\mathrm{d}g)^3 \ . \tag{26}$$

Two comments elaborate on our result. It may be useful to choose for H a direct product $H_1 \otimes H_2 \subset G$, where it has already been established that the Chern-Simons term of H_2 is a total derivative, and one wants to prove the same for the H_1 Chern-Simons term. The result (26) implies that

$$\mathrm{CS}(A \in H_1) + \mathrm{CS}(A \in H_2) = \frac{1}{48\pi^2 N}\,\mathrm{tr}(g^{-1}\,\mathrm{d}g)^3 \ . \tag{27}$$

Since the right side is known to be a total derivative, and the second term on the left side is also a total derivative by hypothesis, Eq. (27) implies the desired result that $\mathrm{CS}(A \in H_1)$ is a total derivative. Furthermore, since the total derivative property of $\mathrm{tr}(g^{-1}\,\mathrm{d}g)^3$ is not explicitly evident, our "total derivative" construction for a non-Abelian Chern-Simons term may in fact result in an expression of the form $a\,\mathrm{d}a$, where a is an Abelian potential. At this stage one can appeal to known properties of an Abelian Chern-Simons term to cast $a\,\mathrm{d}a$ into total derivative form, for example, by employing a Clebsch parameterization for a. In other words, our construction may be more accurately described as an "Abelianization" of a non-Abelian Chern-Simons term.

To illustrate explicitly the workings of this construction, I present now the parameterization for an SU(2) potential $A_i = A_i^a \sigma^a/2i$, which contains $3 \times 3 = 9$ functions in three dimensions. For G we take O(5), while H is chosen as $\mathrm{O}(3) \otimes \mathrm{O}(2) \approx \mathrm{SU}(2) \otimes \mathrm{U}(1)$, and we already know that an Abelian [U(1)] Chern-Simons term is a total derivative. We employ a 4-dimensional representation for O(5) and take the $\mathrm{O}(2) \approx \mathrm{U}(1)$ generator to be I^0:

$$I^0 = \frac{1}{2i}\begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix} \tag{28a}$$

while the $\mathrm{O}(3) \approx \mathrm{SU}(2)$ generators are I^m, $m = 1, 2, 3$:

$$I^m = \frac{1}{2i}\begin{pmatrix} \sigma^m & 0 \\ 0 & \sigma^m \end{pmatrix} \ . \tag{28b}$$

Finally, the complementary generators of O(5), which do not belong to H, are S^A and $\tilde{S}^A$, $A = 1, 2, 3$:

$$S^A = \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & 0 \\ \sigma^A & 0 \end{pmatrix} \ , \quad \tilde{S}^A = \frac{1}{i\sqrt{2}}\begin{pmatrix} 0 & \sigma^A \\ 0 & 0 \end{pmatrix} \ . \tag{28c}$$

There are a total of ten generators, which is the dimension of O(5), and one verifies that their Lie algebra is as in (24).

Next we construct a generic O(5) group element g, which is a 4×4 matrix. The construction begins by choosing a special O(5) matrix M, depending on six functions, a generic O(3) matrix h with three functions, and a generic O(2) matrix k involving a single function, for a total of ten functions,

$$g = Mhk \tag{29}$$

where M is given by

$$M = \frac{1}{\sqrt{1+\boldsymbol{\omega}\cdot\boldsymbol{\omega}^* - \frac{1}{4}(\boldsymbol{\omega}\times\boldsymbol{\omega}^*)}} \begin{pmatrix} 1-\frac{i}{2}(\boldsymbol{\omega}\times\boldsymbol{\omega}^*)\cdot\boldsymbol{\sigma} & -\boldsymbol{\omega}\cdot\boldsymbol{\sigma} \\ \boldsymbol{\omega}^*\cdot\boldsymbol{\sigma} & 1+\frac{i}{2}(\boldsymbol{\omega}\times\boldsymbol{\omega}^*)\cdot\boldsymbol{\sigma} \end{pmatrix} . \tag{30}$$

Here $\boldsymbol{\omega}$ is a complex 3-vector, involving six arbitrary functions. The SU(2) connection is now taken as in (25)

$$A^m = -\operatorname{tr}(I^m g^{-1}\,\mathrm{d}g) \tag{31a}$$

and with (29) this becomes

$$A = h^{-1}\tilde{A}h + h^{-1}\,\mathrm{d}h \tag{31b}$$

$$\tilde{A} = -\operatorname{tr}(I^m M^{-1}\,\mathrm{d}M) . \tag{31c}$$

We see that k disappears from the formula for A, which is an SU(2) gauge-transform (with h) of the connection $\tilde{A}$ that is constructed just from M. It is evident that A depends on the required nine parameters: three in h and six in M.

[Interestingly, the parameterization (31) of the SU(2) connection possess a structure analogous to the Clebsch parameterization of an Abelian vector. Both present their connection as a gauge transformation of another, "core" connection: θ in the Abelian formula $\boldsymbol{\nabla}\theta + \alpha\boldsymbol{\nabla}\beta$, and h in (31b).]

The Chern-Simons term (31b) of A in (31a) relates to that of (31c) by a gauge transformation:

$$\mathrm{CS}(A) = \mathrm{CS}(\tilde{A}) + \mathrm{d}\operatorname{tr}\Big(-\frac{1}{8\pi^2}h^{-1}\,\mathrm{d}h\,\tilde{A}\Big) + \frac{1}{48\pi^2}\operatorname{tr}(h^{-1}\,\mathrm{d}h)^3 . \tag{32}$$

The last two terms on the right describe the response of a Chern-Simons term to a gauge transformation; the next-to-last is manifestly a total derivative, as is the last – in a "hidden" fashion. Finally,

$$\mathrm{CS}(\tilde{A}) = \frac{1}{16\pi^2}a\,\mathrm{d}a \tag{33}$$

where

$$a = \frac{\boldsymbol{\omega}\cdot\mathrm{d}\boldsymbol{\omega}^* - \boldsymbol{\omega}^*\cdot\mathrm{d}\omega}{1+\boldsymbol{\omega}\cdot\boldsymbol{\omega}^* - \frac{1}{4}(\boldsymbol{\omega}\times\boldsymbol{\omega}^*)^2} \tag{34}$$

We remark that a can now be parameterized in the Clebsch manner, so that $a\,\mathrm{d}a$ appears as a total derivative, completing our construction.

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