Non-commuting coordinates in vortex dynamics and in the Hall effect related to "exotic" Galilean symmetry

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Abstract

Vortex dynamics in a thin superfluid ^{4}He film as well as in a type II superconductor is described by the classical counterpart of the model advocated by Peierls, and used for deriving the ground states of the Fractional Quantum Hall Effect. The model has non-commuting coordinates, and is obtained by reduction from a particle associated with the "exotic" extension of the planar Galilei group.

1 Vortex dynamics and the Peierls substitution

(Quantum) Mechanics with non-commuting coordinates[1, 2],

$$\{x, y\} = \theta, \tag{1}$$

has become the focus of recent research. Such a relation may appear rather puzzling. Below we argue, however, that it is inherent in a number of physical instances, and could indeed have been recognized many years ago.

Our first example of non-commuting coordinates is provided by the effective dynamics of point-like flux lines in a thin film of superfluid ^{4}He [3, 4]. For the sake of simplicity, we restrict ourselves to two vortices of identical vorticity. The center-of-vorticity coordinates are constants of the motion. For the relative coordinates $x = x_1 - x_2$ and $y = y_1 - y_2$, respectively, the equations of motion become[3, 4, 5]

$$(\rho L \kappa)\, \dot{x} = \partial_y H, \qquad (\rho L \kappa)\, \dot{y} = -\partial_x H, \tag{2}$$

where ρ and L are the density and the thickness of the film, respectively; κ is the (quantized) vorticity. The Hamiltonian reads

$$H = -\frac{\rho L \kappa^2}{4\pi} \ln r. \tag{3}$$

Eq. (2) is plainly a Hamiltonian system,

$$\dot{\xi} = \{\xi, H\}, \qquad \xi = (x, y), \tag{4}$$

with the Poisson bracket associated with the symplectic structure of the plane, $\theta^{-1} dx \wedge dy$, $\theta \equiv (\rho L \kappa)^{-1}$. Vortex dynamics in a superfluid helium film provides us therefore with non-commuting coordinates, since (1) holds. Let us emphasize that the symplectic plane should be viewed as the classical *phase space*.

The classical motions are determined at once: owing to energy conservation and consistently with the conservation of the vorticity, the motions are uniform rotations with angular velocity is $-(\kappa/2\pi)r^{-2}$.

The equations (4) can be derived[4] from the Euler equations of an incompressible fluid, $\dot{\vec{v}} + \vec{v} \cdot \vec{\nabla}\vec{v} = -\vec{\nabla}p$, $\vec{\nabla} \cdot \vec{v} = 0$, where p is the pression. Defining the *vorticity field* as $\omega = \vec{\nabla} \times \vec{v}$ and taking the curl of the Euler equations yields

$$\dot{\omega} + \vec{v} \cdot \vec{\nabla}\omega = 0. \tag{5}$$

These equations are Hamiltonian, $\dot{\omega} = \{\omega, H\}$, with

$$\{F, G\} = \int \omega \left\{ \frac{\delta F}{\delta \omega}, \frac{\delta G}{\delta \omega} \right\} dxdy \quad \text{and} \quad H = \int \tfrac{1}{2}L\rho\vec{v}^2 dxdy, \tag{6}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket on the symplectic plane. By incompressibility, $\vec{v} = \vec{\nabla} \times \psi$, so that $H = -(L\rho/2) \int \omega\psi dxdy$. Then, assuming that the vorticity is supported by pointlike objects, $\omega = \sum_i \kappa\delta(\vec{x} - \vec{x}_i)$, eqns. (3) and (4) follow.

Vortices in a thin superconducting film behave in a similar manner[6]. Here one starts with the magnetohydrodynamic equations $\dot{v}_i - \epsilon_{ij}v_j\vec{\nabla} \times \vec{v} = (e/m)(E_i + \epsilon_{ij}v_jB)$ of a charged incompressible fluid. The vorticity, which reads now rather $\omega = \vec{\nabla} \times \vec{v} + eB/m$, satisfies the same equation, Eq. (5), as in the superfluid case. Then the generalized London equation requires

$$\vec{\nabla} \times \vec{v} + eB/m = (h/2e) \sum_i \delta(\vec{x} - \vec{x}_i) \tag{7}$$

where $h/2e$ is the flux quantum. The vorticity is hence supported again by pointlike objects, and one recovers the same model as for a neutral superfluid. The Hamiltonian, H, is the sum of the interaction potentials[6].

Yet another example is provided by a extreme type II bulk superconductor[6]. The mean distance of the quantized flux lines is much larger than their core size, so they can be viewed as vortex filaments in an incompressible and frictionless fluid. Using the London equation it is shown[6] that the displacements of the vortices from their fixed positions satisfy, to first order, the same pairs of equations above (2), i. e. (4). The Hamiltonian is the sum of more involved non-local expressions[6].

Another peculiarity is the absence of a mass term in the hamiltonian (3). The analogy with the motion of *massless* particles in a magnetic field, noticed before[5], can be further amplified. The model can indeed also obtained by considering the massless limit of an ordinary, charged particle in the plane subject to an electromagnetic field[7]. The $m \to 0$ limit yields in fact the first-order Lagrangian without mass term,

$$L = \frac{eB}{2}(x\dot{y} - y\dot{x}) - eV, \tag{8}$$

whose associated Hamiltonian system is (4) with $\theta^{-1} = eB$ and $H = eV$.

Quantizing the classical system (4) yields the so-called he "Peierls substitution". Seventy years ago Peierls[8] argued in fact that in a strong magnetic field B and weak potential V the electrons remain in the lowest Landau level, and the energy is simply $E_n = eB/2m + \epsilon_n$, where the ϵ_n are the eigenvalues of the operator $\hat{V}(\hat{x}, \hat{y})$, obtained from the potential alone, but such that $\hat{x}$ and $\hat{y}$ are canonically conjugate,

$$[\hat{x}, \hat{y}] = \frac{i}{eB}. \tag{9}$$

In more recent times, the "Peierls substitution" has re-emerged in the theory of the Quantum Hall Effect[9, 10, 11]. It is argued[9] that the "$\frac{1}{3}$", or more generally, the fractional effect arises from "condensation into a collective ground state which represents a novel state of matter"[9]. This latter consists of a multi-electron system represented by Laughlin's quasiparticles[11], all of which lie in the lowest Landau level and obey hence the Peierls dynamics.

The clue of the relation between the fractional Hall effect and superfluid vortices is that Laughlin's quasiparticles correspond to the vortex solutions in the effective Landau-Ginzburg field theory of the QHE[12]. The simple hamiltonian system (4) lies hence at the heart of this deep relation.

2   Exotic particles and the Hall effect

A slightly different derivation[13] of the system (4) starts with a particle associated with the "exotic" two-parameter central extension of the planar Galilei group [14]. Let us describe this point in some detail.

According to geometric quantization[15, 16], elementary particles correspond to coadjoint orbits of their fundamental symmetry groups, endowed with their canonical symplectic structures. These latter are labeled in turn by the cohomology classes. In any dimension $d \geq 3$, the Galilei group has one-dimensional cohomology labeled by a real parameter m, identified with the mass. The *planar* Galilei group admits however, a second cohomology label, k [14]. The orbits are still $\mathbf{R}^4$, but the corresponding "exotic" symplectic structure is, rather,

$$d\vec{p} \wedge d\vec{x} + \frac{\theta}{2} \epsilon_{ij} dp_i \wedge dp_j, \qquad \theta \equiv \frac{k}{m^2}, \tag{1}$$

so that the Poisson bracket reads

$$\{f, g\} = \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial x_i} \frac{\partial f}{\partial p_i} \right) + \theta \left(\frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_2} - \frac{\partial g}{\partial x_1} \frac{\partial f}{\partial x_2} \right). \tag{2}$$

The spatial coordinates are therefore again non-commuting, $\{x, y\} = \theta$ cf. (1).

Hamilton's equations associated with (2) and with the standard free Hamiltonian $H_0 = \vec{p}^2/2m$ describe the usual free motions; the "exotic" structure only appears in the conserved quantities. While the energy, H_0, and the momentum, $\vec{p}$, have the conventional form, the angular momentum and Galilean boosts get indeed new contributions,

$$\begin{aligned} j &= \vec{x} \times \vec{p} + \tfrac{1}{2}\theta\, \vec{p}^2, \\ g_i &= mx_i - p_i t + m\theta\, \epsilon_{ij} p_j, \end{aligned} \tag{3}$$

The commutation relations are those of the "exotic" [meaning two-fold centrally extended] Galilei group which are the usual ones except for the boosts which satisfy rather

$$\{g_1, g_2\} = -m^2\theta = k. \tag{4}$$

Conversely, positing the commutation relations (1), augmented with the standard Heisenberg relations $\{x_i, p_j\} = \delta_{ij}$ and $\{p_i, p_j\} = 0$, yields the unique symplectic form (1), showing that exotic Galilean symmetry and non-commutative quantum mechanics are indeed equivalent [13].

Let us mention that the Hamiltonian structure presented here for a free exotic particle is consistent with the "exotic" Lagrangian

$$L_0 = \vec{p} \cdot \dot{\vec{x}} - \frac{\vec{p}^2}{2m} + \frac{\theta}{2}\vec{p} \times \dot{\vec{p}}. \tag{5}$$

The "exotic" extension plays hence little rôle for a free particle, explaining (perhap) why it has only attracted little attention until recently. The situation changes dramatically, though, when coupling to an abelian gauge field is considered. Minimal coupling is achieved using Souriau's prescription [15], which simply means adding the electromagnetic 2-form to the symplectic form of the system. In terms of the Lagrangian, the minimally coupled expression reads

$$L = \int (\vec{p} - e\vec{A}) \cdot \dot{\vec{x}} - \frac{\vec{p}^2}{2m} + eV + \frac{\theta}{2}\vec{p} \times \dot{\vec{p}}. \tag{6}$$

In the associated Euler-Lagrange equations,

$$\begin{aligned} m^*\dot{x}_i &= p_i - em\theta\epsilon_{ij}E_j, \\ \dot{p}_i &= eE_i + eB\epsilon_{ij}\dot{x}_j, \end{aligned} \tag{7}$$

the extension parameters combine with the magnetic field into and *effective mass*,

$$m^* = m\left(1 - e\theta B\right). \tag{8}$$

For non-vanishing effective mass, $m^* \neq 0$, the motions are roughly similar to that of an ordinary particle in a planar electromagnetic field [7].

When the effective mass, vanishes, $m^* = 0$ i. e. for

$$eB = \frac{1}{\theta}, \tag{9}$$

however, the system becomes singular, and the consistency of the equations of motion (7) can only be maintained when *all particles move according to the Hall law*

$$\dot{Q}_i = \varepsilon_{ij}\frac{E_j}{B}. \tag{10}$$

where $E_i = -\partial_i V$, and the $Q_i = x_i - E_i/B^2$, are suitable coordinates [13].

Put in another way, symplectic[13, 15] (alias Hamiltonian[17]) reduction reduces the dimension of the phase space from 4 to 2, yielding namely the classical model (2) with H given by the potential eV and with magnetic field $eB = \rho\delta\kappa$.

Geometrically, the 4-dimensional phase space reduces to a two-dimensional one, namely to (4). Similarly, the minimally coupled Lagrangian (6) reduces to the simple first-order Lagrangian (8).

Note that our minimal coupling prescription is different from the one proposed in Ref. 1 using the Poisson structure. This latter only allows a constant magnetic field, and yields conslusions similar to but still different from ours.

3 Quantization

Then the Peierls substitution is recovered by quantizing the reduced model [7, 13], conveniently carried out in the Bargmann-Fock framework. Setting $z = \theta^{-1/2}(x + iy)$, and choosing the holomorphic polarization, the the wave functions are $f(z)e^{-|z|^2/4}$ with $f(z)$ analytic; the fundamental (creation and annihilation) operators $\widehat{z} = z\cdot$, $\widehat{\bar{z}} = 2\partial_z$ satisfy $[\widehat{\bar{z}}, \widehat{z}] = 2$.

Finding the spectrum requires quantizing the Hamiltonian $H = eV$. The point is that the answer depends crucially on the chosen quantization scheme. Let us restrict ourselves to the radial case $V = U(r^2)$. Then the classical system is symmetric w. r. t. rotations. The conserved angular momentum for the planar system (4) is in particular

$$J = \frac{1}{2\theta}r^2 = \tfrac{1}{2}z\bar{z}. \tag{1}$$

The system has therefore fractional angular momentum, $j_n = \alpha_0 + n$, $n = 0, 1, \ldots$, as it is readily derived[5] using the representation theory of the symplectic group sp(1).

Let us now turn to the quantization of the radial potential $U(r^2) = U(2\theta J)$. One of the schemes[2, 5] says that the spectrum of is simply $U(2\theta j_n)$. This scheme ignores, however, the problem of operator ordering. According to another proposal[10, 7], $\widehat{H}$ is obtained by *anti-normal ordering*. In terms of the complex coordinates z and $\bar{z}$, this amounts to "putting all the $\widehat{\bar{z}}$ to the left and all the $\widehat{z}$ to the right".

Recently[13] we argued that this prescription requires further modification, and proposed to use instead Bergman quantization[18], which identifies the quantum operator associated to $H(z, \bar{z})$ as

$$\widehat{H}\psi(z) = \int \exp\left[\tfrac{1}{2}\bar{\zeta}(z - \zeta)\right](H - \partial_z\partial_{\bar{z}}H)\,\psi(\zeta)d\zeta d\bar{\zeta}. \tag{2}$$

Although we have not yet evaluated this for the vortex Hamiltonian (3), it is obvious that it will be different from the simple formula above. When V is a polynomial, the prescription (2) simplifies to first subtracting a correction term, $V \to \widetilde{V} = V - \partial_z\partial_{\bar{z}}V$, and then anti-normal order $\widetilde{V}$ [18].

The model discussed here has been used in the context of the Fractional Quantum Hall Effect, namely to calculate the "1/3 effect"[10, 11] using anti-normal ordering. There the LLL electrons were supposed to be in pairwise Coulomb interaction. For two quasiparticles, this means $V = r^{-1}$. Bergmann quantization yields[18] instead that the operator $\widehat{V}$ is multiplication with

$$\frac{1}{r}\operatorname{erf}(\sqrt{r/\theta}) + (\theta\pi)^{-1/2}e^{-r^2/\theta}, \tag{3}$$

which only differs from the usual expression in the region $r < \theta$. The consequences of this short-distance modifications have not yet been drawn.

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