

# The Seiberg-Witten prepotential and the Euler class of the reduced moduli space of instantons <sup>\*</sup>

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## Abstract

The  $n$ -instanton contribution to the Seiberg-Witten prepotential of  $\mathcal{N} = 2$  supersymmetric  $d = 4$  Yang Mills theory is represented as the integral of the exponential of an equivariantly exact form. Integrating out an overall scale and a  $U(1)$  angle the integral is rewritten as  $(4n - 3)$  fold product of a closed two form. This two form is, formally, a representative of the Euler class of the Instanton moduli space viewed as a principal  $U(1)$  bundle, because its pullback under bundle projection is the exterior derivative of an angular one-form.

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# 1 Introduction

The conjecture of Seiberg and Witten [1] of the exact formula for the prepotential of  $\mathcal{N}=2$ ,  $d=4$  supersymmetric Yang-Mills theories has been verified in microscopical instanton calculations for contributions of topological charge one [9] - [11] and two [5]. The propose of the present communication is to report on our findings concerning the representations of the general  $n$ -instanton coefficient of the Seiberg - Witten prepotential in algebraic - geometric terms.

We do not achieve the determination of the coefficients, but we hope that our contribution will be a step into this direction. This letter communication contains results, which will be published in more extended form elsewhere [6].

We will constrain ourselves to a discussion of the simplest possible setting for the appearance of an  $\mathcal{N}=2$  prepotential  $\mathcal{F}$ . That is, we choose  $SU(2)$  as underlying gauge group (spontaneously broken to  $U(1)$ ) and restrict the field content to an  $\mathcal{N}=2$  vector multiplet, denoted in the following by  $\Psi$ <sup>1</sup>. The SW proposal for  $\mathcal{F}(\Psi)$  as a function of  $\Psi$  amounts under this circumstances to the ansatz for  $\mathcal{F}$  as inverse of the modular elliptic function  $\mathcal{J}$  [2] with the following representation in terms of a power series expansion

$$\mathcal{F}(\Psi) = \frac{i}{2\pi} \log \Psi^2 \log \frac{\Psi^2}{\Lambda^2} - \frac{i}{\pi} \sum_{n=1}^{\infty} \mathcal{F}_n \left( \frac{\Lambda}{\Psi} \right)^{4n} \Psi^2, \quad (1)$$

where  $\Lambda$  denotes a dynamical scale enforced by ultraviolet divergencies. The first (logarithmic) term on the r.h.s. of eq. (1) comprises the perturbative contributions which are the classical prepotential  $\propto \Psi^2$  and the one-loop contribution  $\propto \Psi^2 \log \Psi^2$ . A non-renormalization theorem [3] implies that there are no other perturbative (higher loop) contributions to  $\mathcal{F}(\Psi)$ . The second term in the r.h.s. of eq. (1) with numerical coefficients  $\mathcal{F}_n$  is of nonperturbative origin due to instanton configurations. The perturbative and non-perturbative pieces add up to the inverse elliptic modular function reflecting therewith a dynamically realized (S-) duality. For the arguments leading to the ansatz, (1), which might be called macroscopic - including in particular duality and a certain minimality assumption (which might be coined as the assumption of minimal analytical consistency [4]) - we refer to [1].

The plan of the paper is as follows. In Sec. 2 we recall part of the pioneering work by Dorey, Khoze and Mattis (DKM) [5], [7], [8], leading to the determination of the one- and

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<sup>1</sup> $\Psi$  is assumed to be composed of reduced  $U(1)$ -degrees of freedom due to symmetry breaking. In  $\Psi$  is incorporated the breaking parameter  $u$ :  $\Psi = v + \dots$ .

two-instanton coefficients [5] and also providing a suggestive ansatz for the measure in the  $n$ -instanton moduli space[8]. It will be noted that the DKM measure has to be supplemented by the specification of a domain of integration, since the space of integration proposed by DKM containing redundant degrees of freedom is not orientable. In Sec. 3 we show that  $\mathcal{F}_n$  can be represented as integral of the top form of the exponential of an equivariantly exact form. Sec. 4 contains our main result, the representation of  $\mathcal{F}_n$  as integral of a  $(4n-3)$  fold product of a closed two-form. The two form is formally a representative of the Euler class associated with the ADHM moduli space viewed as a (principal)  $U(1)$  bundle. We conclude the paper in section 5 with some comments.

## 2 The DKM results and the necessity of $SO(n)$ gauge fixing

The coefficients  $\mathcal{F}_n$  are given in terms of certain “reduced” matrix elements, that is, expressions in which the parameters corresponding to overall translation invariance are eliminated together with their supersymmetry partners. These matrix elements are represented by integrals over the supersymmetrized moduli space of  $n$ -instanton configurations - the Atiyah, Drinfeld, Manin (ADHM) [14] parameters -

$$\mathcal{F}_n = \int d\mu^{(n)} e^{-S^{(n)}}. \quad (2)$$

$d\mu^{(n)}$  denotes here a particular measure on the  $\mathcal{N}=2$  supersymmetrised ADHM moduli space and  $S^{(n)}$  stands for an effective  $\mathcal{N}=2$  action. We will quote the results of DKM concerning  $d\mu^{(n)}$  and  $S^{(n)}$  referring for detailed explanations to the articles [5], [7] and [8].

The supersymmetrized ADHM moduli space may be embedded into a larger space whose bosonic coordinates are the quaternionic entries <sup>2</sup> of a  $(n+1) \times n$  matrix,

$$a_{\alpha\dot{\beta}} = \begin{pmatrix} w_1 & \dots & w_n \\ a'_{11} & \dots & a'_{1n} \\ & \dots & \\ a'_{n1} & \dots & a'_{nn} \end{pmatrix}_{\alpha\dot{\beta}}, \quad a'_{ij} = a'_{ji}, \quad \alpha, \dot{\beta} = 1, 2 \quad (3)$$

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<sup>2</sup>Our conventions concerning quaternions are the following: an arbitrary quaternion  $x_{\alpha\dot{\beta}}$  can be decomposed in a basis of unit quaternions  $x_{\alpha\dot{\beta}} = \sum_{\mu=1}^4 x_{\mu} (e^{\mu})_{\alpha\dot{\beta}}$ ,  $x_{\mu}$  real,  $\{e^{\mu}\} = -i\sigma^{\mu}$ ,  $\mu = 1, 2, 3$ ;  $\sigma^4 = \mathbf{I}_2$ ;  $(\hat{x})^{\beta\alpha} = \epsilon^{\dot{\beta}\dot{\gamma}} \epsilon^{\alpha\delta} x_{\delta\dot{\gamma}}$ ;  $\epsilon^{12} = \epsilon^{1\bar{2}} = 1$ . The notions of imaginary ( $\Im m$ ) and real ( $\Re e$ ) parts, which we will use further on, should be understood with respect to this conjugation.

and fermionic partners encoded into spinor valued  $(n+1) \times n$  matrices

$$\begin{aligned} \mathcal{M}_\alpha &= \begin{pmatrix} \mu_1 & \cdots & \mu_n \\ \mathcal{M}'_{11} & \cdots & \mathcal{M}'_{1n} \\ & \cdots & \\ \mathcal{M}'_{n1} & \cdots & \mathcal{M}'_{nn} \end{pmatrix}_\alpha, \quad \mathcal{N}_\alpha = \begin{pmatrix} \nu_1 & \cdots & \nu_n \\ \mathcal{N}'_{11} & \cdots & \mathcal{N}'_{1n} \\ & \cdots & \\ \mathcal{N}'_{n1} & \cdots & \mathcal{N}'_{nn} \end{pmatrix}_\alpha, \quad \alpha = 1, 2 \\ \mathcal{M}'_{ij} &= \mathcal{M}'_{ji}, \quad \mathcal{N}'_{ij} = \mathcal{N}'_{ji}. \end{aligned} \quad (4)$$

The action  $S^{(n)}$  and the measure  $d\mu^{(n)}$  have been found by DKM to be of the form

$$\pi^{-2} S^{(n)} = \frac{8n}{g^2} + 16|v|^2 \sum_{k=1}^n |w_k|^2 - 8tr_n \bar{\mathbf{L}} \mathbf{L}^{-1} (\Lambda + \Lambda_f) + 4\sqrt{2} \sum_{k=1}^n \bar{\mu}_k \bar{\nu}_k, \quad (5)$$

$$\begin{aligned} d\mu^{(n)} &= \frac{C_n}{Vol(O(n))} \prod_{i=1}^n d^4 w_i d^2 \mu_i d^2 \nu_i \prod_{i \leq j} d^4 a'_{ij} d^2 \mathcal{M}'_{ij} d^2 \mathcal{N}'_{ij} \prod_{i < j} d(\mathcal{A}_{tot})_{ij} \\ &\quad \times \prod_{i < j} \delta \left( (\mathbf{L} \cdot \mathcal{A}_{tot} - \Lambda - \Lambda_f)_{ij} \right) \delta^{(3)} \left( \frac{1}{2} \left( (\bar{a}a)_{ij} - (\bar{a}a)_{ji} \right) \right) \\ &\quad \times \delta^{(2)} \left( (\bar{a}\mathcal{M})_{ij} - (\bar{a}\mathcal{M})_{ji} \right) \delta^{(2)} \left( (\bar{a}\mathcal{N})_{ij} - (\bar{a}\mathcal{N})_{ji} \right). \end{aligned} \quad (6)$$

In (5)  $\Lambda$  and  $\Lambda_f$  are the  $n \times n$  antisymmetric matrices

$$\Lambda_{kl} = \bar{w}_k v w_l - \bar{w}_l v w_k; \quad (7)$$

$$\Lambda_f = \frac{1}{2\sqrt{2}} (\bar{\mathcal{M}}\mathcal{N} - \bar{\mathcal{N}}\mathcal{M}), \quad (8)$$

and  $\mathbf{L}$  is a real operator acting on any skew symmetric  $n \times n$  matrix  $\mathbf{X}$  by

$$\mathbf{L} \cdot \mathbf{X} = \frac{1}{2} \{X, W\} + \bar{a}' [a', X] - [\bar{a}', X] a', \quad (9)$$

where  $W$  is the real valued symmetric  $n \times n$  matrix

$$W_{kl} = \bar{w}_k w_l + \bar{w}_l w_k. \quad (10)$$

The measure  $d\mu^{(n)}$  is according to DKM [8] uniquely determined by the requirements of supersymmetry and the cluster property. The factor  $1/Vol(O(n))$  takes care of the fact that redundant  $O(n)$  degrees of freedom have not been discarded from the integral. The normalization constant  $C_n$  is fixed through the cluster condition and the normalization of the one-instanton coefficient. The last three products of  $\delta$  - functions on the r.h.s. of (6) enforce the supersymmetrized ADHM constraints. It can be shown that  $\mathbf{L}$  is invertible for  $\bar{a}'$  and  $\mathbf{u}$ 's satisfying the ADHM constraints. The auxiliary variables  $\mathcal{A}_{tot}$  can be integrated out leaving behind a factor  $1/\det \mathbf{L}$ .

The DKM representation, eq. (2), of  $\mathcal{F}_n$  together with the specifications in eq.'s (5), (6) has to be supplemented by an  $O(n)$  gauge fixing condition since the larger space including redundant  $O(n)$  degrees of freedom turns out to be non-orientable. To see the necessity of such a gauge fixing procedure we choose a gauge and verify a posteriori that the restriction is unavoidable to obtain a non-vanishing result for  $\mathcal{F}_n$ . We may consider for this purpose any representation built from the variables  $a$  and  $\bar{a}$  and impose on this gauge fixing conditions. To concretize the ideas let us consider the real symmetric matrix

$$Y_{ik} = \Re(\bar{a}_{ij}a_{jk})$$

which transforms under the adjoint representation of  $O(n)$ .  $Y$  can be brought through an  $O(n)$  transformation into diagonal form with the diagonal elements arranged in increasing order of their absolute values<sup>3</sup>. There will not appear in this procedure a non-trivial Faddeev-Popov determinant because of supersymmetry. (The assertion will become evident from the deductions in the next section where we identify fermions with one-forms.)

The gauge fixing condition degenerates at places where two or more than two eigenvalues of  $Y$  coincide. There part of the  $O(n)$  group is restored. In the generic case of two coinciding eigenvalues an  $O(2)$  sub-group is revived. It means that we hit a Gribov horizon [15]. Points at the two sides of the horizon of codimension one are related by a permutation (considered as an element of the group  $O(n)$ ). This implies that the corresponding volume elements have opposite orientations what confirms the above claim<sup>4</sup>.

### 3 Simplifications

To evaluate for general  $n$  the integral (2) with the measure (6) and the induced action, eq. (5), appears to be a formidable task. The evaluation of  $\mathcal{F}_2$ , achieved by DKM, [5], already seems, at least at first sight, to be miraculous. We propose in the following two subsections simplifications of the integrals, which we hope, will give some insight into the algebraic-geometric nature of the problem. The results of these two subsections have also been derived by Bellisai, Bruzzo, Fucito, Tanzini and Travaglini [12], [13], who use an approach technically different from our.

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<sup>3</sup>This condition does not fix completely the gauge since some discrete transformations are still possible. Dropping the factor  $1/Vol(O(n))$  from (6) one has to take care of the latter leftover redundancies.

<sup>4</sup>DKM avoid in [5] the horizon problem by integrating the *modulus* of the integrand and the measure over the larger space.

### 3.1 Fermions as differential forms

It is a well known fact that fermion zero modes in selfdual Yang-Mills backgrounds are in correspondence with the fluctuation modes of the vector fields which makes it appearing natural to identify part or all of the cotangent space of the ADHM moduli with the Grassmann valued fermion zero modes. For the case of the  $\mathcal{N}=2$  supersymmetry the correspondence of fermion modes and cotangent space turns out to be one-to-one.

To start with we combine the Grassmannian spinor valued matrices  $\mathcal{M}, \mathcal{N}$  into a quaternion valued matrix denoted by  $\mathcal{P}$ . Let  $A^{\dot{\alpha}}, B^{\dot{\alpha}}$  be some two component ( $\dot{\alpha} = 1, 2$ ) c-number spinors satisfying

$$\begin{aligned} A \cdot B &\equiv A^{\dot{\alpha}} B_{\dot{\alpha}} \equiv A^{\dot{\alpha}} B^{\dot{\beta}} \epsilon_{\dot{\alpha}\dot{\beta}} = 1 \\ B_{\dot{\alpha}} &= \epsilon_{\dot{\alpha}\dot{\beta}} B^{\dot{\beta}}, \epsilon_{12} = -\epsilon_{21} = 1, \end{aligned}$$

and define

$$(\mathcal{P})_{\alpha\dot{\alpha}} = \mathcal{M}_{\alpha} B_{\dot{\alpha}} - \mathcal{N}_{\alpha} A_{\dot{\alpha}} \quad (11)$$

with the inverse relation

$$\mathcal{M}_{\alpha} = (\mathcal{P})_{\alpha\dot{\alpha}} A^{\dot{\alpha}}; \quad \mathcal{N}_{\alpha} = (\mathcal{P})_{\alpha\dot{\alpha}} B^{\dot{\alpha}}. \quad (12)$$

ADHM matrix labels are here suppressed. The fermionic constraints read in terms of the new variables as

$$\bar{a}\mathcal{P} - (\bar{a}\mathcal{P})^T = 0, \quad (13)$$

and the fermionic  $\delta$ -functions appearing in the measure (6) read as

$$\prod_{i < j} \delta^{(4)} \left( (\bar{a}\mathcal{P})_{ij} - (\bar{a}\mathcal{P})_{ji} \right). \quad (14)$$

It is easily seen that the imaginary parts of the fermionic constraints (13) are automatically fulfilled if one substitutes for  $\mathcal{P}$  any  $O(n)$ -covariant derivative of  $\mathbf{a}$ ,

$$\begin{aligned} \mathcal{P} &= (d + X) a \equiv \mathcal{D}a; \\ X \cdot a &\equiv X \cdot \begin{pmatrix} w \\ a' \end{pmatrix} = \begin{pmatrix} -w \cdot X \\ [X, a'] \end{pmatrix} \end{aligned} \quad (15)$$

with  $X$  being any real skewsymmetric  $n \times n$  matrix of one-forms. Indeed, one has the identities:

$$\begin{aligned} \Im m \left( \bar{a} \mathcal{D}a - (\bar{a} \mathcal{D}a)^T \right) &= d \Im m (\bar{a}a), \\ \Im m \left( \bar{a} (X \cdot a) - (\bar{a} (X \cdot a))^T \right) &= [X, \Im m (\bar{a}a)], \end{aligned} \quad (16)$$

so that both, the ordinary exterior differential as well as the  $O(n)$  connection part, lead to vanishing contributions inserted into the imaginary part of eq. (13) as long as  $\mathbf{a}$  satisfies the ADHM constraints. For an arbitrary  $O(n)$  Lie algebra valued one form  $\mathbf{X}$  holds

$$\Re \left( \bar{a} (X \cdot a) - (\bar{a} (X \cdot a))^T \right) = -\mathbf{L} \cdot X \quad (17)$$

with  $\mathbf{L}$  as introduced in eq. (9). This allows to choose a connection s.t. also the remaining real part of (13) vanishes which is found to be given by

$$X = \mathbf{L}^{-1} \cdot \Re \left( \bar{a} da - (\bar{a} da)^T \right). \quad (18)$$

We assume now that, the bosonic ADHM moduli  $\mathbf{a}$  satisfy the ADHM constraints and the gauge fixing condition

$$Y_{ik} \equiv \Re (\bar{a} a)_{ik} = 0; \text{ for } i \neq k. \quad (19)$$

The fermionic variables are determined by eq.'s (15) and (18) and satisfy therefore the constraints (13). The Jacobian factors arising from the integration of the bosonic  $\mathbf{a}$ -functions in eq. (6) and the imaginary projections of the fermionic  $\mathbf{a}$ -functions, eq. (14), cancel each other as a consequence of the relations (16). From the real part of the fermionic  $\mathbf{a}$ -functions survives the  $SO(n)$  Haar measure (multiplied by  $\mathbf{L}$ )

$$\int_{g \in SO(n)} \prod_{i < j} (\mathbf{L} (g^{-1} dg))_{ij} = Vol(SO(n)) \det \mathbf{L}.$$

The factor  $\det \mathbf{L}$  from the last integration drops out together with the  $1/\det \mathbf{L}$  factor of the integration of the auxiliary variables  $\mathbf{A}_{tot}$  in (6). Viewing the induced action  $S^{(n)}$  as a mixed differential form

$$S^{(n)} = S^{(n)}(a, \mathcal{D}a)$$

we are all in all lead to rewrite eq. (2) as

$$\mathcal{F}_n \simeq C_n \frac{Vol(SO(n))}{Vol(O(n))} \int_{\mathcal{M}_n} e^{-S^{(n)}(a, \mathcal{D}a)} \quad (20)$$

with  $\mathcal{M}_n$  denoting the  $n$ -instanton moduli space. The exponential under the integral has to be expanded s.t. the top form on  $\mathcal{M}_n$  is reached.

### 3.2 $S^{(n)}$ as an equivariantly exact form

The integrand on the r.h.s. of eq. (20) is also invariant under a  $U(1)$  symmetry (besides its  $O(n)$  invariance), the remainder of the original  $SU(2)$  gauge group. We want to introduce

an equivariant calculus with respect to this  $U(1)$  symmetry, (for a detailed account of the equivariant differential calculus one may consult chapter 7 of [16]).

Let  $\mathbf{M}$  denote a manifold with a continuous group  $\mathbf{G}$  acting on it. Vectors  $\mathbf{X}$  of the Lie algebra  $\mathfrak{g}$  of  $\mathbf{G}$  are mapped to vector fields  $L_X$  on  $\mathbf{M}$ . With  $i_X$  we denote the nilpotent operation of contraction of the vector field  $L_X$  with differential forms on  $\mathbf{M}$ . Following Cartan [17] one introduces the “equivariant” external differential

$$d_X = d - i_X. \quad (21)$$

$d_X$  is nilpotent on  $\mathbf{G}$ -invariant differential forms since one has

$$d_X^2 = - (d \circ i_X + i_X \circ d) = \mathcal{L}_X \quad (22)$$

with  $L_X$  being identified according to Cartan’s homotopy formula with the Lie derivative along the vector field  $L_X$ . The extension of the formalism to the setting of a vector bundle over  $\mathbf{M}$  is straightforward.

Let  $\mathcal{D} = d + A$  denote a covariant derivative acting on sections of such a bundle (in the case under consideration a bundle associated to  $O(n)$ ). The equivariant operation  $\mathcal{D}_X$  is defined in analogy to eq. (21) by

$$\mathcal{D}_X = \mathcal{D} - i_X. \quad (23)$$

This gives rise to the notion of equivariant curvature

$$\mathcal{F}_X (\cdot) = (\mathcal{D}_X^2 + \mathcal{L}_X) (\cdot) \quad (24)$$

satisfying the equivariant Bianchi identity

$$\mathcal{D}_X \mathcal{F}_X = 0. \quad (25)$$

The  $U(1)$  symmetry in question is represented infinitesimally on the bosonic ADHM parameters  $w, a$  by

$$\delta w \sim vw, \quad \delta \bar{w} \sim -\bar{w}v, \quad \delta a' = 0, \quad (26)$$

$v$  being the breaking parameter introduced above. Let  $L_v$  denote the vector field on the moduli space corresponding to the transformation laws (26). The contraction operation associated to this vector field is given by

$$i_v \cdot dw = vw, \quad i_v \cdot d\bar{w} = -\bar{w}v, \quad i_v \cdot da' = 0. \quad (27)$$



A simple calculation reveals that the induced action  $S^{(n)}$  can be represented as equivariant external exterior derivative of an  $U(1)$  invariant one form denoted by  $\omega$

$$S^{(n)} = \mathcal{D}_v \cdot \omega \equiv d_v \omega \quad (28)$$

$$\begin{aligned} \mathcal{D}_v &= d + X - i_v, \\ d_v &= d - i_v, \\ \omega &= - \sum_{i=1}^n 2\Re e(\bar{w}_i \bar{v} \mathcal{D} w_i) \end{aligned} \quad (29)$$

The two alternative representations of  $S^{(n)}$ <sup>5</sup> in (28) are a consequence of the fact that the one-form  $\omega$  is  $O(n)$  invariant.

So,  $\mathcal{F}_n$  can be represented as the integral of an exponential of an equivariantly exact form (superseding eq. (20) )

$$\mathcal{F}_n \simeq \int_{\mathcal{M}_n} e^{-d_v \omega}. \quad (30)$$

To demonstrate the naturalness of the equivariant calculus in the present context (apart from the concise appearance of the action (28), (29)) we quote the antichiral supersymmetry transformations induced from  $\mathbf{N}=2$  field theory to the ADHM supermoduli (as shown in [7])<sup>6</sup>:

$$\begin{aligned} \delta a_{\alpha\dot{\beta}} &= (\mathcal{D}_v a)_{\alpha\dot{\beta}} \quad (\equiv \mathcal{D} a_{\alpha\dot{\beta}}); \\ \delta (\mathcal{D}_v a)_{\alpha\dot{\beta}} &= \mathcal{D}_v^2 a_{\alpha\dot{\beta}} = \mathcal{F}_v a_{\alpha\dot{\beta}} \equiv \begin{pmatrix} -\omega_{\alpha\dot{\beta}} \mathcal{F}_v \\ [\mathcal{F}_v, a'_{\alpha\dot{\beta}}] \end{pmatrix}. \end{aligned} \quad (31)$$

The closure of this representation of supersymmetry transformations - modulo  $O(n)$  transformations - is a consequence of the equivariant Bianchi identity, eq (25).

## 4 $\mathcal{F}_n$ as a formal intersection number

Standard localization theory for exact equivariant forms [16], [18] might suggest that the integral (30) can be localized at the set of critical points of the vector field  $L_v$ , that is , at  $w_1 = \dots w_n = 0$ . This turns out not to be the case, as the residuum in question at the

<sup>5</sup> $S^{(n)}$  in eq. (28) differs by an irrelevant overall normalization factor from the DKM action, eq. (5).

<sup>6</sup>we quote here a special supersymmetry transformation. The general transformations are of the form  $Q_{\alpha}^i a_{\alpha\dot{\beta}} = \epsilon_{\alpha\dot{\beta}}^i (\mathcal{D} a)_{\alpha}^i$ ,  $i = 1, 2$  etc.

locus  $w = 0$  vanishes. It should also be noted that the standard theory, tailored for compact manifolds without boundaries, does not obviously apply to our problem since there are at least three potential obstacles

- (i) The variables  $w_i$  and  $a'_{ij}$  reach out to infinity;
- (ii) We cannot ignore a Gribov horizon, as noted above, which supplies one type of boundary.
- (iii) The other type of boundary, the Donaldson - Uhlenbeck boundary [19], [20], appears at places where the rank condition (see [14]) is violated.

To deal with item (i) we introduce a scaling variable by setting  $a = R^{1/2}\hat{a}$ , i.e.

$$w = R^{1/2}\hat{w}, \quad a'_{ij} = R^{1/2}\hat{a}'_{ij} \quad (32)$$

s.t. holds <sup>7</sup>

$$\sum_{i=1} |\hat{w}_i|^2 + \sum_{i,j=1}^n |\hat{a}'_{ij}|^2 = 1 \quad (33)$$

$S^{(n)}$  reads in terms of the variables  $R, \hat{a}$  as

$$S^{(n)} = dR\hat{\omega}^{(n)} + Rd_v\hat{\omega}^{(n)} \quad (34)$$

with the notation  $\hat{\omega} \equiv \omega(\hat{a}, d\hat{a})$ . The scaling variable  $R$  can be integrated straightforwardly,

$$\mathcal{F}_n \simeq \int_{\mathcal{M}_n} e^{-dR\hat{\omega} - Rd_v\hat{\omega}} = - \int_{\hat{\mathcal{M}}_n} \int_0^\infty dR \hat{\omega} e^{-Rd_v\hat{\omega}} = \int_{\hat{\mathcal{M}}_n} \frac{\hat{\omega}}{d_v\hat{\omega}}. \quad (35)$$

( $\hat{\mathcal{M}}_n$  is the manifold of the rescaled moduli  $\hat{a}$  )

Using the notation  $\rho = \hat{\omega}/i_v \cdot \hat{\omega}$  we rewrite the previous equation in the form

$$\mathcal{F}_n \simeq \int_{\hat{\mathcal{M}}_n} \rho (d\rho)^{4n-3}. \quad (36)$$

To verify the equality of (35) and (36) one has to expand the denominator in the integrand of (35) and to take into account the identities

$$\frac{\hat{\omega}}{i_v \cdot \hat{\omega}} \left( \frac{d\hat{\omega}}{i_v \cdot \hat{\omega}} \right)^k = \frac{\hat{\omega}}{i_v \cdot \hat{\omega}} \left( d \frac{\hat{\omega}}{i_v \cdot \hat{\omega}} \right)^k \equiv \rho (d\rho)^k \quad (37)$$

for any nonnegative integer  $k$ . The most important properties of the forms  $\rho, d\rho$  are contained in the equations

$$i_v \cdot \rho = 1; \quad (38)$$

$$i_v \cdot d\rho = 0, \quad (39)$$

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<sup>7</sup>The appearance of the quadratic form on the l.h.s. of eq. (33) is irrelevant as long as it is non-degenerate.

the latter equality being a consequence of the relations

$$\rho = \frac{\hat{\omega}}{i_v \cdot \hat{\omega}}, \quad \mathcal{L}_v \hat{\omega} \equiv (i_v \circ d + d \circ i_v) \hat{\omega} = 0. \quad (40)$$

One may view  $\hat{\mathcal{M}}_n$  as a  $S^1$  bundle vis-à-vis the action of the  $U(1)$  symmetry. Eq. (39) means that  $\rho$  contains with coefficient  $1/2|v|$  the differential of an angle parameterizing the  $U(1)$  group orbits.  $2|v|\rho$  is, with other words, an angular one-form of the  $S^1$ -bundle and  $2|v|d\rho$  would have to be identified with the Euler class of that bundle (see e.g. [21]) were there no boundaries. The differential of the  $U(1)$  angle, call it  $\varphi$ , only shows up in  $\rho$  but not, according to eq. (39), in  $d\rho$ . We may therefore substitute  $\rho$  in the integrand on the r.h.s. of eq. (36) by  $2|v|d\varphi$  (other parts of  $\rho$  lead to vanishing contributions). Integrating out  $\varphi$  we arrive at

$$\mathcal{F}_n \simeq \int_{\hat{\mathcal{M}}'_n} (d\rho)^{4n-3}, \quad (41)$$

where  $\hat{\mathcal{M}}'_n$  denotes the base of the  $S^1$ -bundle of which  $\hat{\mathcal{M}}_n$  is the total bundle space.  $\mathcal{F}_n$  as represented in eq. (41), appears as an intersection number, i.e. the integral of the  $(4n-3)$ -fold product of  $d\rho$ .

### Remarks:

1) The concrete coordinatization of the  $U(1)$  group requires the choice of a concrete, complex scalar  $U(1)$  representation. It seems advisable to choose an  $O(n)$  invariant combination of variables forming such a representation. (Only in this case the  $U(1)$  angle will not be changed along  $O(n)$  orbits and, hence, can be naturally defined in the space  $\hat{\mathcal{M}}_n$ .) One may, for example, combine the quaternionic components of

$$w = (w_1, \dots, w_n) \quad (42)$$

$$w_i = \sum_{\mu=0}^3 e_\mu \cdot w_i^\mu, \quad (43)$$

into complex eigenvectors  $\psi_i, \chi_i$  of the  $U(1)$  transformations

$$\begin{aligned} \psi_i &= w_i^0 + iw_i^3; \\ \chi_i &= w_i^2 + iw_i^1 \end{aligned}$$

and build with them  $O(n)$ -invariant variables

$$\begin{aligned} Z &= \sum_{i=1}^n \psi_i^2 \\ Z' &= \sum_{i=1}^n \chi_i^2. \end{aligned}$$

The real and imaginary part of  $Z$  ( $\mathbb{Z}$ ) are of the type of doublets we are looking for. Suppose we parameterize the  $\mathbb{Z}$ - plane with a radial and an angular variable. The latter may be identified with the above variable  $\varphi$ . One is now confronted with three kinds of boundaries, the Gribov horizon, the Donaldson-Uhlenbeck (DU) boundary and the submanifold given by  $Z=0$ , where the chosen coordinatization of the  $U(1)$  symmetry breaks down. The Gribov horizon does not give a contribution because there at least one  $O(2)$  subgroup of  $O(n)$  is restored as symmetry and this is not covered by the locally  $O(n)$  invariant combinations of differential forms showing up in  $S^{(n)}$ . The vanishing of the contributions at the DU boundary is the consequence of simple estimates. We conclude that the integral (41) is localized at  $Z=0$

$$\mathcal{F}_n \simeq \int_{\mathcal{M}'_n \cap \{Z=0\}} \rho (d\rho)^{4n-4}. \quad (44)$$

We can iterate this procedure choosing as new  $U(1)$  variable, e.g., the argument of the complex variable  $Z$  and afterwards possibly other combinations of variables. Unfortunately we have not been able to execute the recursion efficiently enough to reach a determination of  $\mathcal{F}_n$  for general  $n$ .

2) We want to emphasize that the representation (41) for  $\mathcal{F}_n$  is by no means unique because of the quasi-cohomological character of the problem. To illustrate the point we show that the result for  $\mathcal{F}_n$  is to a large extent independent of the choice of an  $O(n)$ - connection.

Let  $\mathbf{L}(\alpha, \beta)$  denote the operator

$$\mathbf{L}(\alpha, \beta) (\cdot) = \alpha \{W, \cdot\} + \beta (\bar{a}' [a', \cdot] - [\bar{a}', \cdot] a') \quad (45)$$

with  $\alpha$  and  $\beta$  being positive real numbers.  $\mathbf{L}(\alpha, \beta)$  is invertible on  $\hat{\mathcal{M}}_n$ . We may consider the modified equivariant connection

$$\begin{aligned} \mathcal{D}_v^{(\alpha, \beta)} &= d + A^{(\alpha, \beta)} - i_v, \\ A^{(\alpha, \beta)} &= \frac{1}{\mathbf{L}(\alpha, \beta)} \Re e \left\{ \alpha \bar{w} dw + \beta \bar{a}' da' - (\cdots)^T \right\} \end{aligned}$$

and the modified action

$$S^{(n)(\alpha, \beta)} = -d_v \cdot \Re e \left( \bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w \right) \quad (46)$$

and

$$\mathcal{F}_n^{(\alpha, \beta)} \simeq \int_{\mathcal{M}_n} e^{-S^{(n)(\alpha, \beta)}}. \quad (47)$$

We want to show that  $\mathcal{F}_n^{(\alpha, \beta)} = \mathcal{F}_n$ . In fact we have

$$\begin{aligned} \mathcal{F}_n - \mathcal{F}_n^{(\alpha, \beta)} &= \int_{\mathcal{M}_n} \left( e^{-S^{(n)}} - e^{-S^{(n)(\alpha, \beta)}} \right) = \\ &= \int_{\mathcal{M}_n} \left( \frac{\Re e (\bar{w} \bar{v} \mathcal{D}_v w)}{d_v \cdot \Re e (\bar{w} \bar{v} \mathcal{D}_v w)} - \frac{\Re e (\bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w)}{d_v \cdot \Re e (\bar{w} \bar{v} \mathcal{D}_v^{(\alpha, \beta)} w)} \right). \end{aligned} \quad (48)$$

It is easily seen that the relevant topform (t.f.) part of the integrand is exact. One concludes from arguments as used above in remark (i) that there are no boundary contributions to the integral of this exact form and hence  $\mathcal{F}_n = \mathcal{F}_n^{(\alpha, \beta)}$ .

## 5 Concluding remarks

Matone [?] has pointed out an interesting analogy between the recursive determination of the Weil - Peterson volume of the moduli space  $\mathcal{M}_{0,n}$  of punctured Riemann spheres and the calculation of the  $\mathbf{N} = 2$  prepotential. There are several facets to this analogy of the which we only quote the one which might be called microscopic. It has been shown by Zograf [23] that the Weil - Peterson volume of  $\mathcal{M}_{0,n}$

$$Vol_{WP}(\mathcal{M}_{0,n}) = \frac{1}{(n-3)!} \int_{\mathcal{M}_{0,n}} \omega_{WP}^{n-3} \quad (49)$$

with  $\omega_{WP}$  denoting the (closed) Weil - Peterson two form, may be determined through a two - term recursion relation (the analogue in the instanton case is a three term relation)

$$v_n = \sum_{j=1}^{n-3} \frac{j(n-j-2)}{n-1} \binom{n-4}{j-1} \binom{n}{j+1} v_{j+2} v_{n-j}; \quad n \geq 4, \quad v_3 = 1, \quad (50)$$

$$v_n = \frac{(n-3)!}{\pi^{2(n-3)}} Vol_{WP}(\mathcal{M}_{0,n}).$$

The origin of the two-term structure of (50) is easy to understand within Zograf's approach to the problem. He is executing the integral of one of the  $n-3$  two forms in (49). The homology cycles separates the punctures on  $\mathbb{C}$  into two groups leading to a factorization into the product of the volumes of two spheres, each with a smaller number of punctures, as a result of the localization of the two-dimensional integration on the Deligne, Knudsen, Mumford boundary [24] of  $\mathcal{M}_{0,n}$ . The missing link for a perfect analogy between the two problems relies on the fact that we are not able to localize the integrals over the instanton moduli space somewhere.

A crucial point for any further progress in this matter will be - to our opinion - that one finds a (so far missing) idea as to which geometrical fact the three-term recursion could be attached. (In the case of punctured spheres it is the effectiveness of a lasso as a tool for catching whatever one wants to catch in two dimensions.) For that purpose a detailed knowledge of second cohomology classes  $H^2$  of the space  $\hat{\mathcal{M}}'_n$  would be of prime importance. The complete analogy between punctured Riemann spheres and the SW prepotential in four dimensions,

enthusiastically announced in [22] has still to be found. We hope that our representation (41) of the coefficients  $\mathcal{F}_n$  as formal intersection numbers may be a useful step into this direction.

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