

# D-BRANES AND $\mathcal{N} = 1$ SUPERSYMMETRY

MICHAEL R. DOUGLAS

ABSTRACT. We discuss the recent proposal that BPS D-branes in Calabi-Yau compactification of type II string theory are  $\Pi$ -stable objects in the derived category of coherent sheaves.

To appear in the proceedings of Strings 2001, Mumbai, India.

## 1. INTRODUCTION

Superstring theory has made enormous progress over the last decade. We now think of it as part of a larger framework, string/M theory, which has given us some nonperturbative understanding and ability to address questions of relatively direct physical interest, such as the qualitative dynamics of strongly coupled gauge theory and the origin of black hole entropy.

Less progress has been made on the primary question from the 80's, namely to make contact with particle physics phenomenology. Nevertheless, I believe this will be the central problem for the theory over the coming decade, partly because our improved understanding does bear on the difficulties encountered then, and even more (one hopes) because of the prospect of entirely new physics to be discovered at LHC, Fermilab and elsewhere. Many physicists believe that supersymmetry will be found at the energies to be probed there, in which case the central problem will be to understand backgrounds of string/M theory with  $\mathcal{N} = 1$  supersymmetry at low energies.

Any systematic approach to these questions requires techniques in which all, or at least some large class of backgrounds, can be studied. One might be lucky and find that a particularly simple background turns out to show interesting parallels with observed physics, but without the larger picture, one will not know how much significance to ascribe to this. Furthermore, our experience with duality suggests that it can often be easier to understand and get exact results for all backgrounds of a theory than for any given background; the Seiberg-Witten solution of  $\mathcal{N} = 2$  Yang-Mills theory provides a compelling example.

Looking back over the results from string/M theory duality, it seems fair to say that a good understanding was achieved of four dimensional backgrounds with  $\mathcal{N} = 2$  supersymmetry, and much less was achieved for  $\mathcal{N} = 1$  supersymmetry. Although we are far from a complete classification of backgrounds with  $\mathcal{N} = 2$  supersymmetry, the basic test of

this statement is that when one considers a particular  $\mathcal{N} = 2$  construction in enough depth, one is always able to find dualities or connections to other constructions and link it into a larger picture.

The broadest and most useful standard construction of  $\mathcal{N} = 2$  backgrounds appears to be the compactification of type II superstring theory on three (complex) dimensional Calabi-Yau manifolds. This provides a geometric picture and one can take advantage of the large body of work constructing and classifying these manifolds. Furthermore, by using mirror symmetry, one can compute the basic observable of  $\mathcal{N} = 2$  supergravity, the prepotential governing the dynamics of vector multiplets and the central charges of BPS states, including all world-sheet instanton corrections. This provides exact results in an *a priori* highly stringy and nongeometric regime. Further dualities such as type II-heterotic allow reinterpreting these instanton corrections as space-time instanton corrections [26], and all of the interesting physics of  $\mathcal{N} = 2$  compactification discovered so far can be realized in this framework.

Because the structure of  $\mathcal{N} = 2$  supergravity fits so well with the geometry of these manifolds [39], it is even tempting to conjecture that all  $\mathcal{N} = 2$  compactifications can be realized (in some dual picture) as type II on Calabi-Yau (perhaps with simple generalizations such as adding discrete torsion). Since the Calabi-Yau manifolds themselves are not classified, it is hard to evaluate this conjecture at present, but one might imagine finding some general argument which given an  $\mathcal{N} = 2$  compactification produces the appropriate Calabi-Yau. If this turned out to be true, we would have a clear sense in which all  $\mathcal{N} = 2$  compactifications were classifiable.

$\mathcal{N} = 1$  compactification is more difficult to study for many reasons. These models (especially the phenomenologically interesting ones) can break supersymmetry spontaneously; if not, the supersymmetric vacua are generically isolated. Thus, the strategy of matching moduli spaces of dual pairs which works well for  $\mathcal{N} = 2$  will not work in all cases.

Experience with  $\mathcal{N} = 1$  suggests however that there are many models with moduli spaces, and that a rich picture of dualities should already exist for these. In a larger picture, we might find additional relations between this subset of theories and those with isolated or no supersymmetric vacua, bringing these cases within reach.

Thus we should look for some standard construction of  $\mathcal{N} = 1$  compactifications of string theory, which ideally would be geometric and share the good properties of type II on Calabi-Yau. The traditional answer to this question is to compactify the heterotic string on a Calabi-Yau threefold. This depends on the additional choice of a vector bundle with structure group  $G \subset E_8 \times E_8$  or  $G \subset Spin(32)/\mathbb{Z}_2$  and a gauge field solving the hermitian Yang-Mills equations [21]. This additional choice is geometric and involves both topological parameters (the Chern or K

theory class) and holomorphic parameters, leading to an interesting field content, superpotential and symmetry breaking structure.

One general problem with using this as our standard construction of  $\mathcal{N} = 1$  compactifications is that in general these results obtain both world-sheet and space-time instanton corrections, and there is no clear limit which keeps a controllable subset of the stringy corrections. Even the classical and large volume limit would be interesting to have under control, but the larger problem is that no clear picture of the set of such bundles is to be found in the physics literature, which relies almost entirely on *ad hoc* constructions. (The construction of [19] is not *ad hoc* and is a definite step forward, but it does not describe all bundles.)

One can try enlisting mathematical help for this problem, but the specific question we just asked, what are all the bundles satisfying certain topological constraints (the anomaly matching conditions), is generally considered difficult. On the other hand, the mathematicians do know a great deal about this type of problem, it is just not in the predigested form one might like.

Two other candidate standard constructions have emerged in more recent times, F theory compactified on Calabi-Yau fourfolds [41], and type I theory on Calabi-Yau threefolds [34]. Both take advantage of the equivalence between branes and gauge field configurations which is the hallmark of the Dirichlet brane and which was already exemplified in Witten's work on small instantons [43]: the moduli space of a configuration of Dirichlet 5-branes and 9-branes in flat space, is equivalent to the moduli space of instantons, and furthermore their world-volume gauge theory contains precisely the ADHM construction of the moduli space and the instantons themselves, making a powerful but rather abstruse mathematical construction quite concrete and usable, as has been demonstrated in numerous recent works, such as [10].

This relation suggests that type I on Calabi-Yau could serve as a better standard construction, and that the power of the Dirichlet brane to turn complicated mathematics into physics might allow us to proceed much farther in this direction.

Over the last two years, this hope has been realized in the related but somewhat simpler problem of Dirichlet branes in type II compactification on Calabi-Yau threefolds. In the classical (zero string coupling or sphere and disk world-sheet) limit, the type II and type I problems are very closely related: one can obtain type I models and type II orientifolds from type II with branes by a second step of quotient by a world-sheet orientation reversing symmetry.

Our work so far has concentrated on the classical type II problem, both from the desire to focus on the essentials of the problem, but also because this liberates us from the tadpole/anomaly matching constraints (one can consider D-branes which sit at points in Minkowski space, which at the classical level is essentially the same problem as

space-filling branes) and allows putting the problem in a larger context. In particular, it turns out to be very useful to start with a finite set of generating D-branes, which individually would not have satisfied tadpole cancellation, and build up the spectrum as bound states of arbitrary numbers of these.

On the other hand, we work with finite  $\alpha'$ , so the problem retains highly nontrivial instanton corrections (sphere and disk world-sheet instantons). Thus if we could find a construction which described all brane configurations in this limit, we would have made a good start on the problem of finding a standard construction of  $\mathcal{N} = 1$  models.

In [12], building on [4, 7, 11, 14, 15], we have found a simple characterization of all the BPS branes on a Calabi-Yau, which works in the stringy regime and predicts the dependence of the spectrum on the CY moduli (i.e. lines of marginal stability):

**Claim 1.1.** *A (B-type) BPS brane on the CY manifold  $M$  is a  $\Pi$ -stable object in the derived category of coherent sheaves on  $M$ .*

This proposal relies on some rather unfamiliar mathematics, which in this context first appeared in Kontsevich's homological mirror symmetry proposal [28], and we will not be able to explain it in detail here. What we will try to do is explain the basic ideas, and why this mathematics is actually very pertinent both for the physics of branes and for the general problem of studying  $\mathcal{N} = 1$  supersymmetric gauge theories. After this general introduction, we will give an example of a Gepner model boundary state (constructed by Recknagel and Schomerus [35]) which turns out to be one of the “nonclassical” objects of the derived category.

## 2. THE MATHEMATICS OF $\mathcal{N} = 1$ SUPERSYMMETRY

Consider a BPS brane in a  $d = 3 + 1$ ,  $\mathcal{N} = 2$  compactification which extends in the Minkowski dimensions. Its world-volume theory will be an  $\mathcal{N} = 1$  gauge theory, which given our assumptions will have a  $\prod U(n_i)$  gauge group, chiral matter in bifundamentals, an action which can be written as a single trace, and can be treated classically. Its moduli space of supersymmetric vacua will be a moduli space of BPS brane configurations.

Study of explicit stringy constructions such as boundary states in orbifolds and Gepner models suggests a generalization of this idea. Suppose we take a finite number of generating branes  $B_i$  and write the general world-volume theory with  $N_i$  copies of the brane  $B_i$ . If all of these branes preserve the same  $\mathcal{N} = 1$ , everything we just said will still be true, and we can find brane configurations with RR charge  $\sum_i N_i [B_i]$  (we use the notation  $[B]$  for the charge or K theory class of the brane  $B$ ) by finding supersymmetric vacua of the combined theory with unbroken  $U(1)$  gauge symmetry (more unbroken symmetry means

we do not have a single bound state). Since it is possible to match the RR charge (or K theory class) of any brane configuration using a finite set of generating branes, and there is no obvious reason that this cannot realize all BPS configurations with this charge, this suggests that we try to use this as our general construction. One can gain further simplicity by taking “rigid” branes as the  $B_i$  (i.e. with no moduli space); all chiral matter will then appear as open strings between branes, so the bound state moduli space will be explicitly constructed by the gauge theory.

These assumptions all hold for orbifold models, but more generally (e.g. in Gepner models), one needs to use generating branes which do not all preserve the same  $\mathcal{N} = 1$ . The  $\mathcal{N} = 2$  superalgebra contains a continuous family of  $\mathcal{N} = 1$  subalgebras parameterized by a  $U(1)$ ; a BPS brane with central charge  $Z$  will leave unbroken the  $\mathcal{N} = 1$  subalgebra  $Q = e^{i\pi\varphi}\bar{Q}$  with  $\pi\varphi$  equal to the phase of  $Z$ . Unless the phase for each  $B_i$  is the same, these combinations will break supersymmetry.

However, at least in cases where the collection of branes actually does decay to a BPS bound state, this might be expected to be a spontaneous breaking of the  $\mathcal{N} = 1$  supersymmetry of the final state. We will pursue this assumption, eventually finding its justification in string theory.

We now recall the structure of a general  $d = 4$ ,  $\mathcal{N} = 1$  theory which is relevant for the problem of finding supersymmetric vacua. This is the gauge group  $G$ , the spectrum of chiral multiplets  $\phi_i$ , the superpotential  $W$ , and finally the D-flatness conditions, which almost follow from the previous data but require the specification of a real Fayet-Iliopoulos parameter for each  $U(1)$  factor in  $G$ . In terms of this data, the problem of finding supersymmetric vacua splits into a holomorphic part, finding the solutions of F-flatness  $\partial W / \partial \phi_i = 0$  up to complex gauge equivalence, and then within each gauge equivalence class finding a solution of the D-flatness conditions.

The first important point we want to make is that this paradigm can serve as a general approach to all problems of finding BPS D-branes. Let us consider what at first looks like the opposite limit to the one we consider, the large volume limit, in which the brane world-volume does not have a finite number of fields but is actually a higher dimensional gauge theory. In this limit, for B-type branes, the problem is to solve the hermitian Yang-Mills equations,

$$(2.1) \quad F_{ij} dz^i dz^j = 0$$

$$(2.2) \quad F_{i\bar{j}} dz^i d\bar{z}^j \wedge \omega^{d-1} = c\omega^d$$

where  $\omega$  is the Kähler form and  $d$  is the complex dimension of the space.

As is well known, these are hard equations to solve explicitly, and it is better to proceed as follows: first, find a holomorphic bundle  $E$ , which can be considered as a solution of (2.1); second, appeal to the

theorems of Donaldson and Uhlenbeck-Yau [9], which state necessary and sufficient conditions (which we quote below) for (2.2) to admit a solution.

This two-step procedure is precisely an infinite dimensional analog of the two-step procedure for finding supersymmetric vacua: the equation (2.1) is precisely  $\partial W / \partial \phi_i = 0$  where  $W$  can be taken to be the holomorphic Chern-Simons action, while (2.2) is precisely the D-flatness condition partner to local complex gauge transformations on the bundle  $E$ .

This raises the possibility that a unified procedure can cover both the large volume limit and the approach discussed above of finding bound states of generating branes. A further indication that this should be true is the “decoupling statement” of BDLR [4]. This states that the F flatness part of the problem is independent of half of the Calabi-Yau moduli, depending only on complex structure moduli for B-type branes, and only on (stringy) Kähler moduli for A-type branes. In some sense this is implicit in the statement that the superpotential is computable in topologically twisted open string theory (e.g. see [42, 3]), but in BDLR it was realized that this implies that the holomorphic structure of B brane world-volume theories, even in the stringy regime, should be exactly computable at large volume.

### 3. F FLATNESS

The decoupling statement predicts that there should be some very direct equivalence between F-flat configurations of supersymmetric gauge theories describing bound states of branes, for example the quiver theories of [16], and holomorphic bundles (or coherent sheaves [22]) on the corresponding resolved space. Such an equivalence was first noticed (in the physics literature) in [15] which studied the  $\mathbb{C}^3/\mathbb{Z}_3$  orbifold in depth, and it turns out to have a long mathematical history. Indeed, the usual  $\mathbb{C}^3/\mathbb{Z}_3$  quiver theory precisely contains the standard mathematical representation of the moduli space of coherent sheaves on  $\mathbb{CP}^2$  (which is the relevant part of the resolved space), as given by a theorem of Beilinson [2] which actually slightly predates the ADHM construction. This theorem has already been generalized to any  $\mathbb{C}^3/\Gamma$  orbifold in mathematical work on the generalized McKay correspondence [36] and turns out to provide a very useful part of the story for compact Calabi-Yau manifolds as well, at least for the cases with a Gepner model realization, since the Gepner model can be thought of as a Landau-Ginzburg orbifold  $\mathbb{C}^5/\Gamma$  and the same technology applied [7, 20, 29, 40].

Although similar in spirit to constructions of sheaves which have been used by physicists, the relations given in these theorems are better, primarily because they are one-to-one: if a given topological class

of sheaf can be constructed, all of the sheaves in this class can be constructed, and each sheaf corresponds to a unique configuration (up to complex gauge equivalence) of the gauge theory. Thus these theorems validate the decoupling statement and provide a faithful translation of the problem of finding bundles into gauge theory terms, just like the original small instanton construction.

It turns out, however, that there is a deeper subtlety in the relation between gauge theory configurations and sheaves: not all sheaves can be constructed as bound states of branes; one needs antibranes as well.

This becomes apparent on considering the large volume interpretation of the fractional branes. For  $\mathbb{C}^3/\mathbb{Z}_3$  this was first found by Diaconescu and Gomis [8], and since duplicated by the many other considerations we just described:

$$(3.1) \quad B_1 \cong \mathcal{O}_{\mathbb{P}^2}(-1)$$

$$(3.2) \quad B_2 \cong \bar{\Omega}_{\mathbb{P}^2}(1)$$

$$(3.3) \quad B_3 \cong \mathcal{O}_{\mathbb{P}^2}.$$

The topological (K theory) class of a D-brane in this problem is specified by three integers which could be regarded as D0, D2 and D4-brane charge, or just as well expressed in the basis provided by the charges of these three fractional branes  $[B_i]$  (in fact all charges are integral in the latter basis). Thus the minimal condition for a brane to be a bound state of fractional branes is that its charges be nonnegative in the fractional brane basis. It is easy to find counterexamples, though, starting with the D2-brane:

$$(3.4) \quad [\mathcal{O}_\Sigma] = [\mathcal{O}_{\mathbb{P}^2}] - [\mathcal{O}_{\mathbb{P}^2}(-1)].$$

Although this directly contradicts a naive form of the decoupling statement, the physical resolution of this point is just to allow bound states of fractional branes and their antibranes.

There is a far more general argument that one cannot avoid talking about bound states of branes and antibranes in these problems. The essential point is that the question of what is a “brane” and what is an “antibrane” actually depends on Kähler moduli. Except in the special case of the literal antibrane  $\bar{B}$  to a brane  $B$  (i.e. its orientation reversal), the only principled way to make such a distinction is to ask whether two objects preserve the same  $\mathcal{N} = 1$  supersymmetry or not. As we discussed, this will be true if their BPS central charges have the same phase.

However, in  $\mathcal{N} = 2$  supersymmetry, the BPS central charges depend on the vector multiplet moduli (here the stringy Kähler moduli) in a rather complicated way, determined by the  $\mathcal{N} = 2$  prepotential. By looking at examples, one quickly finds that two branes with different RR charge (K theory class) can have aligned BPS central charge at one point in moduli space, and anti-aligned at another, continuously

interpolating along the path in between. In particular, while in the large volume limit the central charge is dominated by the brane of highest dimension and thus all branes made from coherent sheaves preserve the same  $\mathcal{N} = 1$  supersymmetry,<sup>1</sup> elsewhere in Kähler moduli space this will no longer be true. The  $\mathbb{C}^3/\mathbb{Z}_3$  results we just quoted are the simplest example: at large volume  $B_1$  and  $B_3$  are “branes” while  $B_2$  is an “antibrane,” while at the orbifold point  $\mathbb{Z}_3$  symmetry guarantees that all three have the same central charge and are “branes.”

In large volume terms, it is not natural to restrict attention only to bundles or to coherent sheaves; one must consider a larger class of objects. This is rather more apparent in the A brane picture, in which continuous variations of BPS central charge can be made in a purely geometric way. Thus any framework which makes mirror symmetry manifest must have some way to treat this problem.

In fact, there is such a framework, the homological mirror symmetry proposal of Kontsevich, which is going to be the primary new example in our discussion of abstruse mathematics turning into physics. This proposal was loosely inspired by Witten’s discussion of topological open string theory [42], which identified allowed boundary conditions in the A and B twisted string theories as (respectively) isotopy classes of lagrangian submanifolds and holomorphic bundles. A mathematical theory of the A objects had been developed by Fukaya, which had all of the structure required to match up with B branes as coherent sheaves, but did not. Rather, Kontsevich proposed that B branes had to be identified with objects in the derived category of coherent sheaves.

Again without going into technical details (which are spelled out to some extent in [12] and in more detail in [1, 6]), the basic idea of the derived category is somewhat analogous to K theory, which is now a generally accepted element in the discussion of D-branes, in that it allows discussing arbitrary combinations of branes and antibranes in a precise way. Where the derived category goes far beyond K theory is that it keeps track of all massless fermionic open strings between a pair of D-branes. This depends on much more than the topological class of the branes – for example, a pair of D0-branes will come with extra massless strings only if they are located at the same point.

If we know the massless fermions between a pair of branes, and we assume  $\mathcal{N} = 1$  supersymmetry (which may be spontaneously broken), we effectively know all of the vector and chiral multiplets in the world-volume theory which are not lifted by superpotential-induced mass terms. In a precise sense, this is all of the holomorphic information about the branes which does not depend on the stringy Kähler moduli of the CY, and does not depend on which branes are “branes” and which are “antibranes.”

---

<sup>1</sup>This assumes that the volume in string units is much larger than any of the RR charges of the branes.



The appearance of the derived category can also be motivated quite simply from considerations of topological open string theory: one just generalizes the definitions to allow  $Q_{BRST}$  to contain non-trivial dependence on the Chan-Paton factors, and then imposes an equivalence relation which identifies configurations which are related by adding cancelling brane-antibrane pairs. The difference with K theory is that one only considers a brane-antibrane pair to cancel if all open strings to them entirely cancel out of the  $Q$ -cohomology, a condition which requires the brane and antibrane to be identical as holomorphic objects.

As is usual in topological string theory, a theory which is topological on the world-sheet can describe non-topological information in space-time. Topologically twisted type II strings on CY generically contain holomorphic information; the  $\mathcal{N} = 2$  prepotential for the closed string, and the  $\mathcal{N} = 1$  superpotential for the open string. An essential difference between closed and open strings for the present problem is that the existence of boundary conditions (D-branes) for the open string problem depends on non-topological data, the Kähler moduli, which drop out of the B twisted theory. Thus it is useful to distinguish “topological” (one might also say “holomorphic”) D-branes, whose existence does not depend on Kähler moduli, from “physical” D-branes whose existence does depend on these moduli.

It is important to note that the formalism of the derived category is general; one can take any “category” (obeying certain axioms) and form the corresponding derived category. In particular, one can define categories of configurations of  $\mathcal{N} = 1$  supersymmetric gauge theories, which correspond to branes as discussed above, and then form the corresponding derived category. This will lead to a structure which can describe all bound states both of the original generating branes  $B_i$  and all of their antibranes  $\bar{B}_i$ .

There is a lot of mathematical evidence by now that the derived category is the correct framework for this discussion. For example, if one follows a closed loop in Kähler moduli space, one obtains a monodromy on the RR charges of B branes; in fact explicit candidate transformations on the entire topological brane spectrum have been proposed [23, 37], which act naturally on the derived category. Furthermore, the simplest statement of Beilinson’s theorem and the other theorems we mentioned, is that the the derived category of F-flat configurations of a quiver gauge theory is equivalent to the derived category of coherent sheaves on the corresponding space. This class of objects seems to be large enough to describe all the mathematical and physical phenomena discovered so far (at least for BPS branes; non-BPS branes are not so well understood yet) and combined with the explicit constructions of [12, 1, 6] we seem to have adequate physical confirmation of Kontsevich’s proposal.

Besides describing all branes in principle, these ideas tell us that we can find concrete ways to describe all branes in practice – if we can find the appropriate quiver gauge theory, its derived category will be a usable description of all of the branes. These theories are known for orbifolds, and quite a lot has already been worked out for Gepner models which describe compact CY's [7, 20, 29, 40]. Perhaps the main problem in getting an exact description along these lines is to work out an exact superpotential for Gepner model boundary states.

An intriguing conjecture, supported by the known examples, is that these superpotentials always take a form such that the equations  $\partial W / \partial \phi_i = 0$  are the conditions for an operator  $D$  constructed from the fields  $\phi$  to square to zero [13]. A heuristic argument for this is that this is also true of the equation (2.1).

#### 4. D-FLATNESS

Having made what we believe is a correct statement for the set of “all solutions of F-flatness conditions” or topological B branes, we can now go on to discuss the D-flatness conditions. A direct approach to this problem would be to find the stringy corrections to the hermitian Yang-Mills equation, and then either solve the resulting equation or find the necessary and sufficient conditions for its solution. Although there is a better approximation to the “correct” equation available (the MMMS equation [30]), this includes only powerlike  $\alpha'$  and not world-sheet instanton corrections, and it seems likely that including the latter in a direct approach would be complicated.

As we discussed, these equations are already too hard to solve in the large volume limit, and what is more useful is to know the necessary and sufficient conditions under which they will have a solution. According to the DUY theorems, a holomorphic bundle on a Kähler manifold  $M$  will correspond to a solution of hermitian Yang-Mills (which will be unique) if and only if it is  $\mu$ -stable. To define this condition, we first define the *slope*  $\mu(E)$  of a bundle  $E$  to be

$$(4.1) \quad \mu(E) = \frac{1}{\text{rk } E} \int c_1(E) \wedge \omega^{n-1},$$

where  $\text{rk } E$  is the rank of  $E$ ,  $\omega$  is the Kähler class of  $M$ , and  $n$  is the complex dimension of  $M$ .

A bundle  $E$  is then  $\mu$ -stable iff, for all subbundles  $E'$ , one has

$$(4.2) \quad \mu(E') < \mu(E).$$

Although the condition is a little complicated to use in practice (one must be able to work with lists of subbundles, either explicitly or implicitly), it is both the simplest general mathematical condition which has been found in work on this subject, and it is physically meaningful: if one varies the Kähler moduli in a way which causes a degeneration

$\mu(E') = \mu(E)$ , one can show that one reaches a line of marginal stability on which the D-brane associated to  $E$  will decay into products including  $E'$ . Thus the mathematical idea of “subbundle” or “sub-object” corresponds directly to a physical idea of “subbrane,” which suggests that we should be able to generalize this condition to any point in stringy Kähler moduli space.

Such a generalization, called  $\Pi$ -stability, was proposed in [14] and then in a more general form in [12]. The quantity which plays the role of the slope turns out to be the phase of the BPS central charge, which is computed from the periods  $\Pi$  of the mirror CY. Important inputs into this proposal were work of Sharpe on  $\mu$ -stability and D-flatness [38], work of Joyce on A branes [24] and of Kachru and McGreevy relating this to D-flatness [25], and finally work of King [27] which (as applied in [14]) gives the necessary and sufficient conditions for D-flatness conditions in  $\mathcal{N} = 1$  gauge theory to admit solutions. Somewhat surprisingly, these last conditions were not previously known in the physics literature (except for the special case of zero FI terms).

All of these conditions are interesting, but space prohibits their discussion here. In any case, there is a general argument from world-sheet conformal field theory [12] which leads to a general stability condition which reduces to each of the ones cited above in the appropriate limit, which we now describe.

Consider the  $(2, 2)$  superconformal field theory associated to the CY, with two B-type boundary conditions. These are discussed in [32]; the important point for us is that they can be regarded as Dirichlet boundary conditions on the boson representing the  $U(1)$  of the  $(2, 2)$  algebra, with the position just being the phase of the BPS central charge. Two branes which preserve different space-time  $\mathcal{N} = 1$ 's preserve the world-sheet  $\mathcal{N} = 2$ , but spontaneously break space-time supersymmetry, because the world-sheet boundary conditions eliminate the zero mode of the spectral flow operator. Nevertheless, one can relate space-time bosons to fermions, by using the spectral flow operator associated to either of the original BPS boundary states. The results differ by a phase, so this is not a symmetry, but one can still identify a partner bosonic operator to each massless fermion (thus each state in the topological open string theory). Furthermore, one can compute its mass by these considerations; it is still

$$(4.3) \quad m^2 = \frac{1}{2}(Q - 1)$$

where  $Q$  is the  $U(1)$  charge.

This is the standard formula; what has changed is that  $Q$  need not take integer eigenvalues anymore, and bose-fermi masses are not degenerate. If the two Dirichlet boundary conditions on the boson are  $\varphi_1$

and  $\varphi_2$ , the  $U(1)$  charge (which is winding number) will take values

$$(4.4) \quad Q = n + \varphi_2 - \varphi_1; \quad n \in \mathbb{Z}.$$

Suppose we now consider a given pair of boundary conditions and open string, and vary the Kähler moduli. The only effect on this state (since it is chiral under the world-sheet  $\mathcal{N} = 2$ ) will be to vary the positions  $\varphi_1$  and  $\varphi_2$ , and thus the  $U(1)$  charge  $Q$ , according to (4.4).

This provides a rule which determines the mass squared of every boson in a chiral multiplet, everywhere in Kähler moduli space, if we know it at one point. For B branes, the  $U(1)$  charge  $Q$  is also the rank of an associated differential form (for example,  $H^1(M, A^* \otimes B)$  corresponds to a charge 1 chiral primary and massless matter), so this can be computed from geometry or from the quiver constructions.

This result is the key to understanding bound state formation and decay: if a boson goes tachyonic, condensing it can form a bound state, while if it goes massive, a previously existing bound state can go unstable. The first conclusion is probably uncontroversial; the second conclusion can be proven by showing that if at some point in Kähler moduli space two branes  $A$  and  $B$  formed a bound state  $C$  by tachyon condensation; then if at some other point the string  $A \rightarrow B$  becomes massive, some other chiral operator  $B \rightarrow C$  or  $C \rightarrow \bar{A}$  will have its  $U(1)$  charge drop below zero. However this contradicts the axioms of unitary CFT, a contradiction which can only be resolved by the decay of the heaviest of the branes involved.

All of these considerations can be summarized in the following rules. We need a definition from the formalism of the derived category, the “distinguished triangle.” Certain triples of branes are “distinguished,” in physical terms because tachyon condensation between a pair of them can produce the third one as a bound state. This is denoted by the following diagram:

$$(4.5) \quad \dots \longrightarrow C[-1] \xrightarrow{\psi} A \xrightarrow{\rho} B \xrightarrow{\phi} C \longrightarrow A[1] \longrightarrow B[1] \longrightarrow \dots$$

The arrows denote open strings (called morphisms in the categorical terminology), while the bracketed notation  $A[1]$  indicates an “image brane” as explained in [12]; the odd “images” are antibranes.

There are various ways to read this diagram:  $A$  and  $C$  can form the bound state  $B$  by condensing  $\psi$ ;  $\bar{A}$  and  $B$  can form  $C$  by condensing  $\rho$ ; and so forth. In many works (e.g. [22]), it has been noted that bound state formation can be described by exact sequences; for example  $A+C$  binding to make  $B$  is

$$(4.6) \quad 0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Every such exact sequence leads to the corresponding distinguished triangle in the derived category, describing various related brane-antibrane

processes. Furthermore, every pair of objects (branes) and every morphism between them can be completed to a triangle, so there are many more possible bound states in this framework.

To find lines of marginal stability or predict bound state formation, one works with these triangles, and keeps track of the  $U(1)$  charge or “grading” of each of the three open strings involved. By the definitions, these will always sum to 1. One then enforces the basic rule that no chiral field can have negative  $U(1)$  charge. Thus, if all three objects in the triangle actually exist as physical branes (are stable) at a given point in Kähler moduli space, all three strings must have  $U(1)$  charges between 0 and 1. We refer to this as a “stable triangle.” Conversely, if two stable objects are related by an open string with  $Q > 1$ , the third object in the triangle must not be stable.

Suppose we have all this information and we then follow some path in Kähler moduli space. The  $U(1)$  charges of the open strings will vary, and when they reach 0 or 1, decays are possible. More specifically, if some  $U(1)$  charge for a stable triangle becomes 0, one can show that the others must be 0 and 1 (this follows from the relation  $Z_A + Z_C = Z_B$  between the central charges). As we cross this line, the brane between the 0’s will decay (it will always be the heaviest of the three). On the other hand, suppose we start with two stable objects with a massive open string  $Q > 1$  which becomes massless. As we cross this line, the third object (their bound state) will go stable.

There are many consistency conditions which these rules must satisfy. For example, in the second process, one requires that the new bound state is not destabilized by some other object. Furthermore there are relations required to make this rule unambiguous; for example it cannot be that an object both enters into bound state formation and decays on the same line. Some but not all of these conditions have been proven from the formalism at present. Furthermore, many nontrivial examples seem to make sense, and as we mentioned one can rederive the known stability conditions in appropriate limits, so this rule appears to be a good candidate for the necessary and sufficient condition replacing the D-flatness conditions on a general stringy Calabi-Yau.

## 5. AN EXAMPLE

The simplest example of all of this is the formation of a  $Dp-2$ -brane by tachyon condensation between a  $Dp$ -brane and an anti- $Dp$ -brane carrying flux, as discussed by many authors (in the CY context, by [33]). This is described by the exact sequence

$$(5.1) \quad 0 \longrightarrow \mathcal{O}(-1) \xrightarrow{0} \mathcal{O} \xrightarrow{1/2} \mathcal{O}_\Sigma \longrightarrow 0.$$

Here  $\mathcal{O}$  is the  $Dp$ -brane with trivial gauge bundle wrapping the entire CY or some compact cycle in it.  $\mathcal{O}(-1)$  is a brane carrying  $-1$  unit of flux, and  $\mathcal{O}_\Sigma$  is the resulting  $Dp-2$ -brane.

We have indicated the  $U(1)$  charges of the maps involved, computed in the large volume limit, by the numbers above the arrows. The  $\xrightarrow{0}$  indicates a standard brane-antibrane tachyon with  $m^2 = -1/2$ , while the  $\xrightarrow{1/2}$  indicates a  $Dp$ - $D(p-2)$  tachyon with  $m^2 = -1/4$  (by the usual large volume rules); the charges are then deduced using (4.3). If we complete this to a triangle as in (4.5), the third map will also have charge  $1/2$ .

Now, to see what happens as we decrease the volume of the CY, we need to know how the BPS central charges of these branes vary. This can be computed from mirror symmetry, but in examples studied so far it turns out that the qualitative behavior we are interested in is already predicted by the results with world-sheet instanton corrections left out. In this case, we have (for a cycle of dimension  $n$ )

$$(5.2) \quad Z(\mathcal{O}) = \frac{1}{n!}(B - iV)^n;$$

$$(5.3) \quad Z(\mathcal{O}(-1)) = \frac{1}{n!}(-1 + B - iV)^n;$$

$$(5.4) \quad Z(\mathcal{O}_\Sigma) = Z(\mathcal{O}) - Z(\mathcal{O}(-1)).$$

As we decrease  $V$ ,  $\varphi(\mathcal{O})$  will increase, and  $\varphi(\mathcal{O}(-1))$  will decrease, so the charge 0 will increase, while the  $1/2$  charges can be checked to decrease. Eventually the first charge reaches 1, and the brane  $\mathcal{O}_\Sigma$  will decay into these two constituents (assuming it didn't decay into something else first).<sup>2</sup> Thus this brane does not exist in the small volume region. For reasons we will not get into here, it is natural to identify this marginal stability line with a “phase boundary.”

This conclusion would presumably follow from any correct treatment of the bound state. However, an amusing and nontrivial prediction of the present formalism is that if one continues to smaller volume, another bound state forms. This is because the map  $\mathcal{O}(-1) \rightarrow \mathcal{O}$  will have a Serre dual, which on a Calabi-Yau will be an element of  $H^n(\mathcal{O}, \mathcal{O}(-1))$ . (This is the dual under the natural pairing  $(\alpha, \beta) = \int \bar{\Omega}^{(n)} \wedge \text{Tr} \alpha \wedge \beta$ .) This fits into another triangle

$$(5.5) \quad \dots \rightarrow \mathcal{O} \xrightarrow{n} \mathcal{O}(-1) \rightarrow X \rightarrow \mathcal{O}[1] \rightarrow \dots$$

where the object  $X$  is not a coherent sheaf, and is not stable at large volume.

Now, the same considerations which made the  $U(1)$  charge of  $\xrightarrow{0}$  increase with decreasing volume, will make the  $U(1)$  charge of  $\xrightarrow{n}$  decrease. Eventually it will cross 1 (at some smaller volume than the previous decay) and  $X$  will become a new stable brane.

In the Gepner model of the quintic, one can check that this happens before one reaches the Gepner point. In fact,  $X$  turns out to be one of the states constructed by Recknagel and Schomerus, the “mysterious”

---

<sup>2</sup>The same argument predicts that a higher degree  $Dp - 2$ -brane will also decay [18], resolving a paradox encountered in [31].

state discussed in [11, 5] whose central charge would have vanished on a path leading back to large volume. Thus we have an example of a brane which does not correspond to a coherent sheaf yet which has been proven to exist in the stringy regime, which serves as a nontrivial confirmation of these ideas.

## REFERENCES

- [1] P. S. Aspinwall and A. Lawrence, “Derived Categories and Zero-Brane Stability,” hep-th/0104147.
- [2] A. A. Beilinson, “Coherent sheaves on  $\mathbb{P}^n$  and problems of linear algebra”, *Funct. Anal. Appl.* **12** (1978) 214–216.
- [3] M. Bershadsky, S. Cecotti, H. Ooguri, and C. Vafa, “Kodaira-Spencer Theory of Gravity and Exact Results for Quantum String Amplitudes,” *Comm. Math. Phys.* **165** (1994) 311–428; hep-th/9309140.
- [4] I. Brunner, M. R. Douglas, A. Lawrence, and C. Römelsberger, “D-branes on the quintic,” hep-th/9906200.
- [5] F. Denef, “Supergravity flows and D-brane stability,” *JHEP* **0008** (2000) 050, hep-th/0005049.
- [6] D.-E. Diaconescu, “Enhanced D-Brane Categories From String Field Theory,” hep-th/0104200.
- [7] D.-E. Diaconescu and M. R. Douglas, “D-branes on Stringy Calabi-Yau Manifolds,” hep-th/0006224.
- [8] D.-E. Diaconescu and J. Gomis, “Fractional branes and boundary states in orbifold theories,” hep-th/9906242.
- [9] S. K. Donaldson and P. B. Kronheimer, *The Geometry of Four-Manifolds*, Oxford Univ. Press, 1990.
- [10] N. Dorey, T. J. Hollowood, V. V. Khoze, M. P. Mattis and S. Vandoren, “Multi-Instanton Calculus and the AdS/CFT Correspondence in  $N=4$  Superconformal Field Theory,” *Nucl. Phys.* **B552** (1999) 88–168; hep-th/9901128.
- [11] M. R. Douglas, “Topics in D-geometry”, *Class. Quant. Grav.* **17** (2000) 1057, hep-th/9910170.
- [12] M. R. Douglas, “D-branes, Categories, and  $\mathcal{N} = 1$  Supersymmetry,” to appear in *J. Math. Phys.*; hep-th/0011017.
- [13] M. R. Douglas, to appear.
- [14] M. R. Douglas, B. Fiol, and C. Römelsberger, “Stability and BPS branes”, hep-th/0002037.
- [15] M. R. Douglas, B. Fiol, and C. Römelsberger, “The spectrum of BPS branes on a noncompact Calabi-Yau”, hep-th/0003263.
- [16] M. R. Douglas, B. R. Greene and D. R. Morrison, “Orbifold resolution by D-branes,” *Nucl. Phys.* **B506**, 84 (1997) hep-th/9704151.
- [17] M. R. Douglas and G. Moore, “D-branes, Quivers, and ALE Instantons,” hep-th/9603167.
- [18] M. R. Douglas and E. Witten, to appear.
- [19] R. Friedman, J. W. Morgan and E. Witten, “Vector Bundles over Elliptic Fibrations,” alg-geom/9709029.
- [20] S. Govindarajan and T. Jayaraman, “D-branes, Exceptional Sheaves and Quivers on Calabi-Yau manifolds: From Mukai to McKay,” hep-th/0010196.
- [21] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, vol. 2, Cambridge Univ. Press, 1987.
- [22] J. Harvey and G. Moore, “On the algebras of BPS states,” *Comm. Math. Phys.* **197** (1998) 489–519, hep-th/9609017.

- [23] R. P. Horja, “Derived Category Automorphisms from Mirror Symmetry,” math.AG/0103231.
- [24] D. Joyce, “On counting special Lagrangian homology 3-spheres,” hep-th/9907013.
- [25] S. Kachru and J. McGreevy, “Supersymmetric three-cycles and (super)symmetry breaking,” hep-th/9908135.
- [26] S. Kachru and C. Vafa, “Exact Results for N=2 Compactifications of Heterotic Strings,” Nucl.Phys. B450 (1995) 69-89; hep-th/9505105.
- [27] A. D. King, “Moduli of Representations of Finite Dimensional Algebras”, Quart. J. Math. Oxford (2), 45 (1994), 515-530.
- [28] M. Kontsevich, “Homological algebra of mirror symmetry”, alg-geom/9411018.
- [29] P. Mayr, “Phases of Supersymmetric D-branes on Kaehler Manifolds and the McKay correspondence,” hep-th/0010223.
- [30] M. Marino, R. Minasian, G. Moore and A. Strominger, “Nonlinear Instantons from Supersymmetric p-Branes,” JHEP 0001 (2000) 005; hep-th/9911206.
- [31] J. Maldacena, A. Strominger and E. Witten, “Black Hole Entropy in M-Theory,” JHEP 9712 (1997) 002; hep-th/9711053.
- [32] H. Ooguri, Y. Oz and Z. Yin, “D-Branes on Calabi-Yau Spaces and Their Mirrors,” Nucl.Phys. B477 (1996) 407-430, hep-th/9606112.
- [33] Y. Oz, T. Pantev and D. Waldram, “Brane-Antibrane Systems on Calabi-Yau Spaces,” hep-th/0009112.
- [34] G. Pradisi and A. Sagnotti, “Open String Orbifolds,” Phys. Lett. B216 (1989) 59.
- [35] A. Recknagel and V. Schomerus, “D-branes in Gepner models”, Nucl. Phys. **B531** (1998) 185, hep-th/9712186.
- [36] M. Reid, “La correspondance de McKay”, Séminaire Bourbaki (novembre 1999), no. 867, math.AG/9911165.
- [37] P. Seidel and R. Thomas, “Braid group actions on derived categories of sheaves,” math.AG/0001043.
- [38] E. Sharpe, “Kaehler cone substructure,” Adv. Theor. Math. Phys. **2**, 1441 (1999), hep-th/9810064.
- [39] A. Strominger, Comm. Math. Phys. 133 (1990) 163.
- [40] A. Tomasiello, “D-branes on Calabi-Yau manifolds and helices,” hep-th/0010217.
- [41] C. Vafa, “Evidence for F-Theory,” Nucl.Phys. B469 (1996) 403-418; hep-th/9602022.
- [42] E. Witten, “Chern-Simons Gauge Theory as a String Theory,” hep-th/9207094.
- [43] E. Witten, “Small Instantons in String Theory,” Nucl. Phys. B460 (1996) 541-559; hep-th/9511030.

(Michael R. Douglas) DEPT. OF PHYSICS AND ASTRONOMY, RUTGERS UNIVERSITY, PISCATAWAY, NJ 08855 USA *and* I.H.E.S., LE BOIS-MARIE, BURES-SUR-YVETTE 91440 FRANCE

*Email address:* mrd@physics.rutgers.edu