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The no-ghost theorem for string theory in curved backgrounds with a flat timelike direction

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ABSTRACT

It is well-known that the standard no-ghost theorem can be extended straightforwardly to the general $c = 26$ CFT on $\mathbb{R}^{d-1,1} \times K$, where $2 \leq d \leq 26$ and K is a compact unitary CFT of appropriate central charge. We prove the no-ghost theorem for $d = 1$, *i.e.*, when only the timelike direction is flat. This is done using the technique of Frenkel, Garland and Zuckerman.

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1 Introduction

As is well-known, string theory generally contains negative norm states (ghosts) from timelike oscillators. However, they do not appear as physical states. This is the famous *no-ghost theorem* [1]-[11].[1]

When the background spacetime is curved, things are not clear though. First of all, nonstationary situations are complicated even in field theory. There is no well-defined ground state with respect to a time translation and the "particle interpretation" becomes ambiguous. Moreover, even for static or stationary backgrounds, it is not currently possible to show the no-ghost theorem from the knowledge of the algebra alone. In fact, Refs. [16] and Ref. [17] found ghosts for strings on three-dimensional anti-de Sitter space (AdS_3) and three-dimensional black holes, respectively. There are references which have proposed resolutions to the ghost problem for AdS_3 [18, 19]. However, the issue is not settled yet and proof for the other backgrounds is still lacking.

Viewing the situation, we would like to ask the converse question: how general can the no-ghost theorem definitely apply? This is the theme of this paper.

In order to answer to the question, let us look at the known proofs more carefully. There are two approaches. The first approach uses the "old covariant quantization" (OCQ). Some proofs in this approach use the "DDF operators" to explicitly construct the observable Hilbert space to show the theorem [1, 12]. This proof assumes flat spacetime and cannot be extended to the more general cases. Goddard and Thorn's proof [2, 6] is similar to this traditional one, but is formulated without explicit reference to the DDF operators. Since it only requires the existence of a flat light-cone vector, the proof can be extended easily to $2 \leq d \leq 26$.

There is another approach using the BRST quantization. These work more generally. Many proofs can be easily extended to the general $c = 26$ CFT on $\mathbb{R}^{d-1,1} \times K$, where $2 \leq d \leq 26$ and K corresponds to a compact unitary CFT of central charge $c_K = 26 - d$ [3, 9, 10, 11, 20].

On the other hand, the theorem has not been studied in detail when $0 < d < 2$. The purpose of this paper is to show an explicit proof for $d = 1$.

We show the no-ghost theorem using the technique of Frenkel, Garland and Zuckerman (FGZ) [9]. Their proof is different from the others. For

[1]For the NSR string, see Refs. [12]-[15].

example, the standard BRST quantization assumes $d \geq 2$ in order to prove the "vanishing theorem," *i.e.*, the BRST cohomology is trivial except at the zero ghost number. However, FGZ's proof of the vanishing theorem essentially does not require this as we will see later. Establishing the no-ghost theorem for $d = 1$ is then straightforward by calculating the "index" and "signature" of the cohomology groups.

Thus, in a sense our result is obvious a priori, from Refs. [9, 14, 15]. But it does not seem to be known well, so it is worth working out this point explicitly in detail.

The organization of the present paper is as follows. First, in the next section, we set up our notations and briefly review the BRST quantization of string theory. In Section 3, we will see a standard proof of the vanishing theorem and review why the standard no-ghost theorem cannot be extended to $d < 2$. Then in Section 4, we prove the vanishing theorem due to FGZ, following Refs. [9, 14, 15]. We use this result in Section 5 to prove the no-ghost theorem for $d = 1$.

For the other attempts of the $d = 1$ proof, see Section 6 (iv). Among them, Ref. [11] considers the same kind of CFT as ours. However, the proof relies heavily on the proof of the flat $d = 26$ string, and thus the proof is somewhat roundabout and is not transparent as ours. Moreover, our proof admits the extention to more general curved backgrounds [see Section 6 (v)].

2 BRST Quantization

In this section, we briefly review the BRST quantization of string theory [3, 20, 21]. We make the following assumptions:

(i). Our world-sheet theory consists of d free bosons X^μ $(\mu = 0, \cdots, d-1)$ with signature $(1, d-1)$ and a compact unitary CFT K of central charge $c_K = 26 - d$. Although we focus on the $d = 1$ case below, the extension to $1 \leq d \leq 26$ is straightforward [Section 6 (i)].

(ii). We assume that K is unitary and that its spectrum is discrete and bounded below. Thus, all states in K lie in highest weight representations. The weight of highest weight states should have $h^K > 0$ from the Kac determinant; therefore, the eigenvalue of L_0^K is always non-negative.

(iii). The momentum of states is $k^\mu \neq 0$. [See Section 6 (iii) for the exceptional case $k^\mu = 0$.]

Then, the total L_m of the theory is given by $L_m = L_m^X + L_m^g + L_m^K$, where

$$L_m^X = \frac{1}{2} \sum_{n=-\infty}^{\infty} :\alpha^\mu_{m-n} \alpha_{\mu,n}:, \tag{1}$$

$$L_m^g = \sum_{n=-\infty}^{\infty} (m-n) :b_{m+n} c_{-n}: - \delta_m. \tag{2}$$

Here,

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m+n}\, \eta^{\mu\nu}, \qquad \{b_m, c_n\} = \delta_{m+n}, \tag{3}$$

and $\delta_m = \delta_{m,0}$. With the d-dimensional momentum k^μ, $\alpha^\mu_0 = \sqrt{2\alpha'} k^\mu$.

The ghost number operator $\hat{N}^g$ counts the number of c minus the number of b excitations: [2]

$$\hat{N}^g = \sum_{m=1}^{\infty} (c_{-m} b_m - b_{-m} c_m) = \sum_{m=1}^{\infty} (N_m^c - N_m^b). \tag{4}$$

We will call the total Hilbert space $\mathcal{H}_{total}$. Recall that the physical state conditions are

$$Q|\mathrm{phys}\rangle = 0, \qquad b_0|\mathrm{phys}\rangle = 0. \tag{5}$$

These conditions imply

$$0 = \{Q, b_0\}|\mathrm{phys}\rangle = L_0|\mathrm{phys}\rangle. \tag{6}$$

Thus, we define the following subspaces of $\mathcal{H}_{total}$:

$$\mathcal{H} = \{\phi \in \mathcal{H}_{total} : b_0\phi = 0\}, \tag{7a}$$

$$\hat{\mathcal{H}} = \mathcal{H}^{L_0} = \{\phi \in \mathcal{H}_{total} : b_0\phi = L_0\phi = 0\}. \tag{7b}$$

We will consider the cohomology on $\hat{\mathcal{H}}$ since Q takes $\hat{\mathcal{H}}$ into itself from $\{Q, b_0\} = L_0$ and $[Q, L_0] = 0$. The subspace $\mathcal{H}$ will be useful in our proof of the vanishing theorem (Section 4).

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[2] The ghost zero modes will not matter to our discussion. $\hat{N}^g$ is related to the standard ghost number operator N^g as $N^g = \hat{N}^g + c_0 b_0 - \frac{1}{2}$. Note that the operator $\hat{N}^g$ is also normalized so that $\hat{N}^g|\downarrow\rangle = 0$.

The Hilbert space $\hat{\mathcal{H}}$ is classified according to mass eigenvalues. $\hat{\mathcal{H}}$ at a particular mass level will be often written as $\hat{\mathcal{H}}(k^2)$. For a state $|\phi\rangle \in \hat{\mathcal{H}}(k^2)$,

$$L_0|\phi\rangle = (\alpha' k^2 + L_0^{int})|\phi\rangle = 0, \tag{8}$$

where L_0^{int} counts the level number. One can further take an eigenstate of $\hat{N}^g$ since $[L_0^{int}, \hat{N}^g] = 0$. $\hat{\mathcal{H}}$ is decomposed by the eigenvalues of $\hat{N}^g$ as

$$\hat{\mathcal{H}} = \bigoplus_{n\in\mathbf{Z}} \hat{\mathcal{H}}^n. \tag{9}$$

We define the raising operators as $\alpha^\mu_{-m}, b_{-m}, c_{-m}, x^\mu$ and c_0. The ground state in $\hat{\mathcal{H}}(k^2)$ is given by

$$|0;k\rangle \otimes |h^K\rangle = e^{ik\cdot x}|0;\downarrow\rangle \otimes |h^K\rangle, \tag{10}$$

where $|0;\downarrow\rangle$ is the vacuum state annihilated by all lowering operators and $|h^K\rangle$ is a highest weight state in K. Then, $\hat{\mathcal{H}}(k^2)$ is written as

$$\hat{\mathcal{H}}(k^2) = \left(\mathcal{F}(\alpha^\mu_{-m}, b_{-m}, c_{-m}; k) \otimes \mathcal{H}_K\right)^{L_0}. \tag{11}$$

Here, $*^{L_0}$ denotes the L_0-invariant subspace: $F^{L_0} = F \cap \mathrm{Ker} L_0$. A state in $\mathcal{H}_K$ is constructed by Verma modules of K. The Fock space $\mathcal{F}(\alpha^\mu_{-m}, b_{-m}, c_{-m}; k)$ is spanned by all states of the form

$$|N;k\rangle = \prod_{\mu=0}^{d-1}\prod_{m=1}^{\infty} \frac{(\alpha^\mu_{-m})^{N^\mu_m}}{\sqrt{m^{N^\mu_m} N^\mu_m!}} \prod_{m=1}^{\infty}(b_{-m})^{N^b_m} \prod_{m=1}^{\infty}(c_{-m})^{N^c_m}|0;k\rangle, \tag{12}$$

where $N^\mu_m = \frac{1}{m}\alpha^\mu_{-m}\alpha_{\mu,m}$ are the number operators for α^μ_m. In terms of the number operators,

$$L_0^{int} = \sum_{m=1}^{\infty} m\left(N^b_m + N^c_m + \sum_{\mu=0}^{d-1} N^\mu_m\right) + L_0^K - 1. \tag{13}$$

The BRST operator

$$Q = \sum_{m=-\infty}^{\infty}(L^X_{-m} + L^K_{-m})c_m - \frac{1}{2}\sum_{m,n=-\infty}^{\infty}(m-n) : c_{-m}c_{-n}b_{m+n} : - c_0 \tag{14}$$

can be decomposed in terms of ghost zero modes as follows:

$$Q = \hat{Q} + c_0 L_0 + b_0 M, \tag{15}$$

where $M = -2\sum_{m=1}^{\infty} mc_{-m}c_m$ and $\hat{Q}$ is the collection of the terms in Q without b_0 or c_0. Using the above decomposition (15), for a state $|\phi\rangle \in \hat{\mathcal{H}}$,

$$Q|\phi\rangle = \hat{Q}|\phi\rangle. \tag{16}$$

Therefore, the physical state condition reduces to

$$\hat{Q}|\phi\rangle = 0. \tag{17}$$

Also, $\hat{Q}^2 = 0$ on $\hat{\mathcal{H}}$ from Eq. (16). Thus, $\hat{Q} : \hat{\mathcal{H}}^n \to \hat{\mathcal{H}}^{n+1}$ defines a BRST complex, which is called the *relative* BRST complex. So, we can define $\hat{\mathcal{H}}_c, \hat{\mathcal{H}}_e \subset \hat{\mathcal{H}}$ by

$$\hat{Q}\hat{\mathcal{H}}_c = 0, \qquad \hat{\mathcal{H}}_e = \hat{Q}\hat{\mathcal{H}}, \tag{18}$$

and define the relative BRST cohomology of Q by

$$\hat{\mathcal{H}}_{obs} = \hat{\mathcal{H}}_c/\hat{\mathcal{H}}_e. \tag{19}$$

In terms of the cohomology group, $\hat{\mathcal{H}}_{obs}(k^2) = \oplus_{n\in\mathbf{Z}} H^n(\hat{\mathcal{H}}(k^2), \hat{Q}(k))$.

Now, the inner product in $\mathcal{H}_{total}$ is given by

$$\langle 0, I; k|c_0|0, I'; k'\rangle = (2\pi)^d\delta^d(k - k')\delta_{I,I'}, \tag{20}$$

where I labels the states of the compact CFT K. We take the basis I to be orthonormal. Then, the inner product $\langle||\rangle$ in $\hat{\mathcal{H}}$ is defined by $\langle|\rangle$ as follows:

$$\langle 0, I; k|c_0|0, I'; k'\rangle = 2\pi\delta(k^2 - k'^2)\langle 0, I; k||0, I'; k'\rangle. \tag{21}$$

The inner products of the other states follow from the algebra with the hermiticity property, $b_m^\dagger = b_{-m}$, $c_m^\dagger = c_{-m}$ and $(\alpha_m^\mu)^\dagger = \alpha_{-m}^\mu$.[3]

3 The Vanishing Theorem and Standard Proofs

In order to prove the no-ghost theorem, it is useful to show the following theorem:

Theorem 3.1 (The Vanishing Theorem)**.** *The $\hat{Q}$-cohomology can be non-zero only at $\hat{N}^g = 0$, i.e., $H^n(\hat{\mathcal{H}}, \hat{Q}) = 0$ for $n \neq 0$.*

[3]We will write $\langle\cdots||\cdots\rangle$ as $\langle\cdots|\cdots\rangle$ below.

To prove this, we use the notion of *filtration*. We first explain the method and then give an example of filtration used in [3, 21]. The filtration is part of the reason why $d \geq 2$ in standard proofs.

A filtration is a procedure to break up $\hat{Q}$ according to a quantum number N_f (filtration degree):

$$\hat{Q} = Q_0 + Q_1 + \cdots + Q_N, \tag{22}$$

where

$$[N_f, Q_m] = mQ_m. \tag{23}$$

We also require

$$[N_f, \hat{N}^g] = [N_f, L_0] = 0. \tag{24}$$

Then, $\hat{\mathcal{H}}$ breaks up according to the filtration degree $N_f(=q)$ as well as the ghost number $\hat{N}^g(=n)$:

$$\hat{\mathcal{H}} = \bigoplus_{q,n \in \mathbf{Z}} \hat{\mathcal{H}}^{n;q}. \tag{25}$$

If $\hat{\mathcal{H}}^q$ can be nonzero only for a finite range of degrees, the filtration is called *bounded*.

The nilpotency of $\hat{Q}^2$ implies

$$\sum_{\substack{m,n \\ m+n=l}} Q_m Q_n = 0, \qquad l = 0, \ldots, 2N \tag{26}$$

since they have different values of N_f. In particular,

$$Q_0^2 = 0. \tag{27}$$

The point is that we can first study the cohomology of $Q_0 : \hat{\mathcal{H}}^{n;q} \to \hat{\mathcal{H}}^{n+1;q}$. This is easier since Q_0 is often simpler than $\hat{Q}$. Knowing the cohomology of Q_0 then tells us about the cohomology of $\hat{Q}$. In fact, one can show the following lemma:

Lemma 3.1. *If the Q_0-cohomology is trivial, so is the $\hat{Q}$-cohomology.*

Proof. [4] Let ϕ be a state of ghost number $\hat{N}^g = n$ and $\hat{Q}$-invariant ($\phi \in \hat{\mathcal{H}}_c^n$). Assuming that the filtration is bounded, we write

$$\phi = \phi_k + \phi_{k+1} + \cdots + \phi_p, \tag{28}$$

[4] The following argument is due to Ref. [21].

where $\phi_q \in \hat{\mathcal{H}}^{n;q}$. Then,

$$\hat{Q}\phi = (Q_0\phi_k) + (Q_0\phi_{k+1} + Q_1\phi_k) + \cdots + (Q_N\phi_p). \tag{29}$$

Each parenthesis vanishes separately since they carry different N_f. So, $Q_0\phi_k = 0$. The Q_0-cohomology is trivial by assumption, thus $\phi_k = Q_0\chi_k$. But then $\phi - \hat{Q}\chi_k$, which is cohomologous to ϕ, has no $N_f = k$ piece. By induction, we can eliminate all ϕ_q, so $\phi = \hat{Q}(\chi_k + \ldots + \chi_p)$; ϕ is actually $\hat{Q}$-exact. □

Moreover, one can show that the Q_0-cohomology is isomorphic to that of $\hat{Q}$ if the Q_0-cohomology is nontrivial for at most one filtration degree [20, 21]. We do not present the proof because our derivation does not need this. In the language of a spectral sequence [22], the first term and the limit term of the sequence are

$$E_1 \cong \bigoplus_q H(\hat{\mathcal{H}}^q, d_0), \qquad E_\infty \cong H(\hat{\mathcal{H}}, \hat{Q}). \tag{30}$$

The above results state that the sequence collapses after the first term:

$$E_1 \cong E_\infty. \tag{31}$$

Then, a standard proof proceeds to show that states in the nontrivial degree are in fact light-cone spectra, and thus there is no ghost in the $\hat{Q}$-cohomology [20].

Now, we have to find an appropriate filtration and show that the Q_0-cohomology is trivial if $\hat{N}^g \neq 0$. This completes the proof of the vanishing theorem. The standard proof of the theorem uses the following filtration [3, 21]: [5]

$$N_f^{(KO)} = \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m}\alpha_{-m}^- \alpha_m^+ + \hat{N}^g. \tag{32}$$

The degree $N_f^{(KO)}$ counts the number of α^+ minus the number of α^- excitations. So, this filtration assumes two flat directions. The degree zero part of $\hat{Q}$ is

$$Q_0^{(KO)} = -\sqrt{2\alpha'}k^+ \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} c_m \alpha_{-m}^-. \tag{33}$$

[5] The $\hat{N}^g$ piece is not really necessary. We include this to make the filtration degree non-negative.

The operator $Q_0^{(KO)}$ is nilpotent since α_m^- commute and c_m anticommute. Obviously, we cannot use α_m^0 in place of α_m^- since α_m^0 do not commute. Thus, we have to take a different approach for $d=1$.

4 The Vanishing Theorem (FGZ)

Since we want to show the no-ghost theorem for $d=1$, we cannot use $N_f^{(KO)}$ as our filtration degree. Fortunately, there is a different proof of the vanishing theorem [9, 14, 15], which uses a different filtration. Their filtration is unique in that Q_0 can actually be written as a sum of two differentials, d' and d''. This effectively reduces the problem to a "$c=1$" CFT, which contains the timelike part and the b ghost part. Then, a Künneth formula relates the theorem to the whole complex. This is the reason why the proof does not require $d\geq 2$. In addition, in this approach the matter Virasoro generators themselves play a role similar to that of the light-cone oscillators in Kato-Ogawa's approach. In this section, we prove the theorem using the technique of Refs. [9, 14, 15], but for more mathematically rigorous discussion, consult the original references.

Proof of the vanishing theorem for $d=1$. FGZ's filtration is originally given for the $d=26$ flat spacetime as

$$N_f^{(FGZ)} = -L_0^{X(d=26)} + \sum_{m=1}^{\infty} m(N_m^c - N_m^b). \tag{34}$$

The filtration itself does not require $d\geq 2$; this filtration can be naturally used for $d=1$, replacing $L_0^{X(d=26)}$ with L_0^X. Then, the modified filtration assigns the following degrees to the operators:

$$\mathrm{fdeg}(c_m) = |m|, \qquad \mathrm{fdeg}(b_m) = -|m|, \tag{35a}$$
$$\mathrm{fdeg}(L_m^X) = m, \qquad \mathrm{fdeg}(L_m^K) = 0. \tag{35b}$$

The operator $N_f^{(FGZ)}$ satisfies conditions (24) and the degree of each term in $\hat{Q}$ is non-negative. Because the eigenvalue of L_0^{int} is bounded below from Eq. (13), the total number of oscillators for a given mass level is bounded. Thus, the degree for the states is bounded for each mass level. Note that the unitarity of the compact CFT K is essential for the filtration to be bounded.

The degree zero part of $\hat{Q}$ is given by

$$
\begin{aligned}
Q_0^{(FGZ)} &= d' + d'', &\text{(36a)}\\
d' &= \sum_{m>0} c_m L_{-m}^X + \sum_{m,n>0} \frac{1}{2}(m-n) b_{-m-n} c_m c_n, &\text{(36b)}\\
d'' &= -\sum_{m,n>0} \frac{1}{2}(m-n) c_{-m} c_{-n} b_{m+n}. &\text{(36c)}
\end{aligned}
$$

We break $\hat{\mathcal{H}}$ as follows:

$$
\begin{aligned}
\hat{\mathcal{H}} &= \left(\mathcal{F}(\alpha_{-m}^0, b_{-m}, c_{-m}; k^0) \otimes \mathcal{H}_K\right)^{L_0} &\text{(37a)}\\
&= \left(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0) \otimes \mathcal{F}(c_{-m}) \otimes \mathcal{H}_K\right)^{L_0}. &\text{(37b)}
\end{aligned}
$$

The Hilbert spaces $\hat{\mathcal{H}}$, $\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)$ and $\mathcal{F}(c_{-m})$ are decomposed according to the ghost number $\hat{N}^g = n$:

$$
\hat{\mathcal{H}}^n = \left(\left(\bigoplus_{\substack{n=N^c-N^b\\ N^c,N^b\geq 0}} \mathcal{F}^{-N^b}(\alpha_{-m}^0, b_{-m}; k^0) \otimes \mathcal{F}^{N^c}(c_{-m})\right) \otimes \mathcal{H}_K\right)^{L_0}. \tag{38}
$$

From Eqs. (36), the differentials act as follows:

$$
\begin{aligned}
&Q_0^{(FGZ)} : \mathcal{H}^n \to \mathcal{H}^{n+1}, &\text{(39a)}\\
&d' : \mathcal{F}^n(\alpha_{-m}^0, b_{-m}; k^0) \to \mathcal{F}^{n+1}(\alpha_{-m}^0, b_{-m}; k^0), &\text{(39b)}\\
&d'' : \mathcal{F}^n(c_{-m}) \to \mathcal{F}^{n+1}(c_{-m}), &\text{(39c)}
\end{aligned}
$$

and $d'^2 = d''^2 = 0$. Thus, $\mathcal{F}^n(\alpha_{-m}^0, b_{-m}; k^0)$ and $\mathcal{F}^n(c_{-m})$ are complexes with differentials d' and d''. Note that $Q_0^{(FGZ)}$ is the differential for $\mathcal{H}^n$ as well as for $\hat{\mathcal{H}}^n$.

Then, the Künneth formula (Appendix A) relates the cohomology group of $\mathcal{H}$ to those of $\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)$ and $\mathcal{F}(c_{-m})$:

$$
H^n(\mathcal{H}) = \left(\bigoplus_{\substack{n=N^c-N^b\\ N^c,N^b\geq 0}} H^{-N^b}\left(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)\right) \otimes H^{N^c}\left(\mathcal{F}(c_{-m})\right)\right) \otimes \mathcal{H}_K. \tag{40}
$$

Later we will prove the following lemma:

Lemma 4.1. *$H^{-N^b}\left(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)\right) = 0$ if $N^b > 0$ and $(k^0)^2 > 0$.*

Then, Eq. (40) reduces to

$$H^n(\mathcal{H})^{L_0} = \left(\bigoplus_{n=N^c} H^0\left(\mathcal{F}(\alpha^0_{-m}, b_{-m}; k^0)\right) \otimes H^{N^c}\left(\mathcal{F}(c_{-m})\right) \otimes \mathcal{H}_K \right)^{L_0}, \tag{41}$$

which leads to $H^n(\mathcal{H})^{L_0} = 0$ for $n < 0$ because $N^c \geq 0$. The cohomology group $H^n(\mathcal{H})^{L_0}$ is not exactly what we want. However, Lian and Zuckerman have shown that

$$H^n(\hat{\mathcal{H}}) \cong H^n(\mathcal{H})^{L_0}. \tag{42}$$

See pages 325-326 of Ref. [14]. Thus,

$$H^n(\hat{\mathcal{H}}, Q_0^{(FGZ)}) = H^n(\hat{\mathcal{H}}, \hat{Q}) = 0 \quad \text{if } n < 0. \tag{43}$$

We will later prove the Poincaré duality theorem, $H^n(\hat{\mathcal{H}}, \hat{Q}) = H^{-n}(\hat{\mathcal{H}}, \hat{Q})$ (Theorem 5.2). Therefore,

$$H^n(\hat{\mathcal{H}}, \hat{Q}) = 0 \quad \text{if } n \neq 0. \tag{44}$$

This is the vanishing theorem for $d = 1$. □

Now we will show Lemma 4.1. The proof is twofold: the first is to map the $c = 1$ Fock space $\mathcal{F}(\alpha^0_{-m}; k^0)$ to a Verma module, and the second is to show the lemma using the Verma module.

Let $\mathcal{V}(c, h)$ be a Verma module with highest weight h and central charge c. Then, we first show the isomorphism

$$\mathcal{F}(\alpha^0_{-m}; k^0) \cong \mathcal{V}(1, h^X) \quad \text{if } (k^0)^2 > 0. \tag{45}$$

Here, $h^X = -\alpha'(k^0)^2$. This is plausible from the defining formula of L^X_m,

$$L^X_{-m} = \sqrt{2\alpha'} k_0 \alpha^0_{-m} + \cdots, \tag{46}$$

where $+\cdots$ denotes terms with more than one oscillators. The actual proof is rather similar to an argument in [4, 23].

Proof of Eq. (45). The number of states of the Fock space $\mathcal{F}(\alpha^0_{-m}; k^0)$ and that of the Verma module $\mathcal{V}(1, h^X)$ are the same for a given level N. Thus, the Verma module[6] furnishes a basis of the Fock space if all the states in a highest weight representation,

$$|h^X, \{\lambda\}\rangle = L^X_{-\lambda_1} L^X_{-\lambda_2} \ldots L^X_{-\lambda_M} |h^X\rangle, \tag{47}$$

[6] With a slight abuse of terminology, we use the word "Verma module" even if $|h^X, \{\lambda\}\rangle$ are not all independent.

are linearly independent, where $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_M$. This can be shown using the Kac determinant.

Consider the matrix of inner products for the states at level N:

$$\mathcal{M}^N_{\{\lambda\},\{\lambda'\}}(c, h^X) = \langle h^X, \{\lambda\} | h^X, \{\lambda'\}\rangle, \qquad \sum_i \lambda_i = N. \tag{48}$$

The Kac determinant is then given by

$$\det[\mathcal{M}^N(c, h^X)] = K_N \prod_{1 \leq rs \leq N} (h^X - h_{r,s})^{P(N-rs)}, \tag{49}$$

where K_N is a positive constant and the multiplicity of the roots, $P(N-rs)$, is the partition of $N-rs$. The zeros of the Kac determinant are at

$$h_{r,s} = \frac{c-1}{24} + \frac{1}{4}(r\alpha_+ + s\alpha_-)^2, \tag{50}$$

where

$$\alpha_\pm = \frac{1}{\sqrt{24}}(\sqrt{1-c} \pm \sqrt{25-c}). \tag{51}$$

For $c = 1$, $\alpha_\pm = \pm 1$ so that $h_{r,s} = (r-s)^2/4 \geq 0$. Thus, the states $|h^X, \{\lambda\}\rangle$ are linearly independent if $h^X < 0$. □

Let us check what spectrum actually appears in $\hat{\mathcal{H}}$. Using assumption (ii) of Section 2, Eqs. (8) and (13), we get $h^X \leq 1$ for a state in $\hat{\mathcal{H}}$. Also, $h^X \neq 0$ from assumption (iii).[7] Thus, we need to consider the Fock spaces $\mathcal{F}(\alpha^0_{-m}; k^0)$ with $h^X \leq 1$ ($h^X \neq 0$). Those with $h^X < 0$ are expressed by Verma modules from Eq. (45). On the other hand, those with $0 < h^X \leq 1$ are not. However, there is only the ground state in this region as the states in $\hat{\mathcal{H}}$. This state has $\hat{N}^g = 0$, so the state does not affect the vanishing theorem.

The isomorphism (45) is essential for proving the vanishing theorem. In the language of FGZ, what we have shown is that $\mathcal{F}(\alpha^0_{-m}; k^0)$ is an "$\mathcal{L}_-$-free module," which is a prime assumption of the vanishing theorem (Theorem 1.12 of [9]). The proof of Lemma 4.1 is now straightforward using Eq. (45) and an argument given in [15]:

Proof of Lemma 4.1. Using Eq. (45), a state $|\phi\rangle \in \mathcal{F}(\alpha^0_{-m}, b_{-m}; k^0)$ can be written as

$$|\phi\rangle = b_{-i_1} \dots b_{-i_L} L^X_{-\lambda_1} \dots L^X_{-\lambda_M} |h^X\rangle, \tag{52}$$

[7] The Verma module $\mathcal{V}(1,0)$ fails to furnish the basis of $\mathcal{F}(\alpha^0_{-m}; 0)$ at the first level because $L^X_{-1}|h^X = 0\rangle = 0$ for $d = 1$.

where $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_M$ and $0 < i_1 < i_2 < \cdots < i_L$. Note that the states in $\mathcal{F}(\alpha^0_{-m}, b_{-m}; k^0)$ all have nonpositive ghost number: $\hat{N}^g|\phi\rangle = -L|\phi\rangle$.

We define a new filtration degree $N_f^{(FK)}$ as

$$N_f^{(FK)}|\phi\rangle = -(L+M)|\phi\rangle, \tag{53}$$

which corresponds to

$$\mathrm{fdeg}(L^X_{-m}) = \mathrm{fdeg}(b_{-m}) = -1 \quad \text{for } m > 0. \tag{54}$$

The algebra then determines $\mathrm{fdeg}(c_m) = 1$ (for $m > 0$) from the assignment. The operator $N_f^{(FK)}$ satisfies conditions (24) and the degree of each term in d' is non-negative. The degree zero part of d' is given by

$$d'_0 = \sum_{m>0} c_m L^X_{-m}. \tag{55}$$

Since we want a bounded filtration, break up $\mathcal{F}(\alpha^0_{-m}, b_{-m}; k^0)$ according to L_0 eigenvalue l_0 :

$$\mathcal{F}(\alpha^0_{-m}, b_{-m}; k^0) = \bigoplus_{l_0} \mathcal{F}(\alpha^0_{-m}, b_{-m}; k^0)^{l_0}, \tag{56}$$

where

$$\mathcal{F}(\alpha^0_{-m}, b_{-m}; k^0)^{l_0} = \mathcal{F}(\alpha^0_{-m}, b_{-m}; k^0) \cap \mathrm{Ker}(L_0 - l_0). \tag{57}$$

Note that $\mathcal{F}(\alpha^0_{-m}, b_{-m}; k^0)^{l_0}$ is finite dimensional since $|\phi\rangle \in \mathcal{F}(\alpha^0_{-m}, b_{-m}; k^0)^{l_0}$ satisfies

$$\sum_{k=1}^{L} i_k + \sum_{k=1}^{M} \lambda_k + h^X = l_0. \tag{58}$$

Thus, the above filtration is bounded for each $\mathcal{F}(\alpha^0_{-m}, b_{-m}; k^0)^{l_0}$.

We first consider the d'_0-cohomology on $\mathcal{F}(\alpha^0_{-m}, b_{-m}; k^0)^{l_0}$ for each l_0. Define an operator Γ such as

$$\Gamma|\phi\rangle = \sum_{l=1}^{M} b_{-\lambda_l} \left(b_{-i_1} \ldots b_{-i_L}\right) L^X_{-\lambda_1} \ldots \widehat{L^X_{-\lambda_l}} \ldots L^X_{-\lambda_M} |h^X\rangle, \tag{59}$$

where $\widehat{L^X_{-\lambda_l}}$ means that the term is missing (When $M = 0$, $\Gamma|\phi\rangle \overset{\mathrm{def}}{=} 0$). Then, it is straightforward to show that

$$\{d'_0, \Gamma\}|\phi\rangle = (L+M)|\phi\rangle. \tag{60}$$

The operator Γ is called a *homotopy operator* for d_0'. Its significance is that the d_0'-cohomology is trivial except for $L+M=0$. If $|\phi\rangle$ is closed, then

$$|\phi\rangle = \frac{\{d_0', \Gamma\}}{L+M}|\phi\rangle = \frac{1}{L+M} d_0' \Gamma |\phi\rangle. \tag{61}$$

Thus, a closed state $|\phi\rangle$ is actually an exact state if $L+M \neq 0$. Therefore, the d_0'-cohomology is trivial if $\hat{N}^g < 0$ since $\hat{N}^g = -L$. And now, again using Lemma 3.1, the d'-cohomology $H^n(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)^{l_0})$ is trivial if $n < 0$.

Because $[d', L_0] = 0$, we can define

$$H^n(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0))^{l_0} = H^n(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)) \cap \mathrm{Ker}(L_0 - l_0). \tag{62}$$

Furthermore, the isomorphism

$$H^n(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0))^{l_0} \cong H^n(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)^{l_0}) \tag{63}$$

can be established. Consequently, $H^n(\mathcal{F}(\alpha_{-m}^0, b_{-m}; k^0)) = 0$ if $n < 0$. □

5 The No-Ghost Theorem

Having shown the vanishing theorem, it is straightforward to show the no-ghost theorem:

Theorem 5.1 (The No-Ghost Theorem)**.** *$\hat{\mathcal{H}}_{obs}$ is a positive definite space when $1 \leq d \leq 26$.*

The calculation below is essentially the same as the one in Refs. [9, 10, 15], but we repeat it here for completeness.

In order to prove the theorem, the notion of *signature* is useful. For a vector space V with an inner product, we can choose a basis e_a such that

$$\langle e_a | e_b \rangle = \delta_{ab} C_a, \tag{64}$$

where $C_a \in \{0, \pm 1\}$. Then, the signature of V is defined as

$$\mathrm{sign}(V) = \sum_a C_a, \tag{65}$$

which is independent of the choice of e_a. Note that if $\mathrm{sign}(V) = \dim(V)$, all the C_a are 1, so V has positive definite norm.

So, the statement of the no-ghost theorem is equivalent to [8]

$$\mathrm{sign}(V_i^{obs}) = \dim(V_i^{obs}). \tag{66}$$

This can be replaced as a more useful form

$$\sum_i e^{-\lambda\alpha' m_i^2} \mathrm{sign}(V_i^{obs}) = \sum_i e^{-\lambda\alpha' m_i^2} \dim(V_i^{obs}), \tag{67}$$

where λ is a constant. Equation (66) can be retrieved from Eq. (67) by expanding in powers of λ. We write Eq. (67) as

$$\mathrm{tr}_{obs}\, e^{-\lambda L_0^{int}} C = \mathrm{tr}_{obs}\, e^{-\lambda L_0^{int}}, \tag{68}$$

where C is an operator which gives eigenvalues C_a.

Equation (68) is not easy to calculate; however, the following relation is straightforward to prove:

$$\mathrm{tr}\, e^{-\lambda L_0^{int}} C = \mathrm{tr}\, e^{-\lambda L_0^{int}} (-)^{\hat{N}^g}. \tag{69a}$$

Here, the trace is taken over V_i and we take a basis which diagonalizes $(-)^{\hat{N}^g}$. Thus, we can prove Eq. (68) by showing Eq. (69a) and

$$\begin{aligned} \mathrm{tr}\, e^{-\lambda L_0^{int}} (-)^{\hat{N}^g} &= \mathrm{tr}_{obs}\, e^{-\lambda L_0^{int}}, & \text{(69b)} \\ \mathrm{tr}\, e^{-\lambda L_0^{int}} C &= \mathrm{tr}_{obs}\, e^{-\lambda L_0^{int}} C. & \text{(69c)} \end{aligned}$$

Thus, the trace weighted by $(-)^{\hat{N}^g}$ is an *index*.

Proof of Eq. (69b). At each mass level, states φ_m in V_i are classified into two kinds of representations: BRST singlets $\phi_{\tilde{a}} \in V_i^{obs}$ and BRST doublets (χ_a, ψ_a), where $\chi_a = \hat{Q}\psi_a$. The ghost number of χ_a is the ghost number of ψ_a plus 1. Therefore, $(-)^{\hat{N}^g}$ causes these pairs of states to cancel in the index and only the singlets contribute:

$$\begin{aligned} \mathrm{tr}\, e^{-\lambda L_0^{int}} (-)^{\hat{N}^g} &= \mathrm{tr}_{obs}\, e^{-\lambda L_0^{int}} (-)^{\hat{N}^g} & \text{(70)} \\ &= \mathrm{tr}_{obs}\, e^{-\lambda L_0^{int}}. & \text{(71)} \end{aligned}$$

We have used the vanishing theorem on the last line. □

[8] In this section, we also write $V_i^{obs} = \hat{\mathcal{H}}_{obs}(k^2)$ and $V_i = \hat{\mathcal{H}}(k^2)$, where the subscript i labels different mass levels.

Proof of Eq. (69c). At a given mass level, the matrix of inner products among $|\varphi_m\rangle$ takes the form

$$\langle\varphi_m|\varphi_n\rangle = \left(\begin{array}{c} \langle\chi_a| \\ \langle\psi_a| \\ \langle\phi_{\tilde{a}}| \end{array}\right) (|\chi_b\rangle, |\psi_b\rangle, |\phi_{\tilde{b}}\rangle) = \left(\begin{array}{ccc} 0 & M & 0 \\ M^\dagger & A & B \\ 0 & B^\dagger & D \end{array}\right). \tag{72}$$

We have used $\hat{Q}^\dagger = \hat{Q}$, $\langle\chi|\chi\rangle = \langle\chi|\hat{Q}|\psi\rangle = 0$ and $\langle\chi|\phi\rangle = \langle\psi|\hat{Q}|\phi\rangle = 0$. If M were degenerate, there would be a state χ_a which is orthogonal to all states in V_i. Thus, the matrix M should be nondegenerate. (Similarly, the matrix D should be nondegenerate as well.) So, a change of basis

$$\begin{aligned} |\chi'_a\rangle &= |\chi_a\rangle, \\ |\psi'_a\rangle &= |\psi_a\rangle - \frac{1}{2}(M^{-\dagger}A)_{ba}|\chi_b\rangle, \\ |\phi'_{\tilde{a}}\rangle &= |\phi_{\tilde{a}}\rangle - (M^{-\dagger}B)_{b\tilde{a}}|\chi_b\rangle, \end{aligned} \tag{73}$$

sets $A = B = 0$. Finally, going to a basis,

$$\begin{aligned} |\chi''_a\rangle &= \frac{1}{\sqrt{2}}(|\chi'_a\rangle + M^{-1}_{ba}|\psi'_b\rangle), \\ |\psi''_a\rangle &= \frac{1}{\sqrt{2}}(|\chi'_a\rangle - M^{-1}_{ba}|\psi'_b\rangle), \\ |\phi''_{\tilde{a}}\rangle &= |\phi'_{\tilde{a}}\rangle, \end{aligned} \tag{74}$$

the inner product $\langle\varphi''_m|\varphi''_n\rangle$ becomes block-diagonal:

$$\langle\varphi''_m|\varphi''_n\rangle = \left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & D \end{array}\right). \tag{75}$$

Therefore, BRST doublets again make no net contribution:

$$\mathrm{tr}\, e^{-\lambda L_0^{int}} C = \mathrm{tr}_{obs}\, e^{-\lambda L_0^{int}} C. \tag{76}$$

This proves Eq. (69c). □

One can indeed check that M and D are nondegenerate. The inner product in V_i is written as the product of inner products in $\mathcal{F}(\alpha^0_{-m}; k^0)$, ghost sector and $\mathcal{H}_K$. The inner product in $\mathcal{F}(\alpha^0_{-m}; k^0)$ is easily seen to be diagonal and nondegenerate. For the ghost sector, the inner product becomes diagonal and nondegenerate as well by taking the basis $p_m = (b_m + c_m)/\sqrt{2}$ and $m_m = (b_m - c_m)/\sqrt{2}$, where

$$\{p_m, p_n\} = \delta_{m+n}, \qquad \{p_m, m_n\} = 0, \qquad \{m_m, m_n\} = -\delta_{m+n}. \tag{77}$$

$\mathcal{H}_K$ is assumed to have a positive-definite inner product. Therefore, the matrix $\langle \varphi_m | \varphi_n \rangle$ is nondegenerate. Consequently, the matrices M and D are also nondegenerate.

The inner product is nonvanishing only between the states with opposite ghost numbers. Since D is nondegenerate, BRST singlets of opposite ghost number must pair up. We have therefore established the Poincaré duality theorem as well:

Theorem 5.2 (Poincaré Duality)**.** $H^{\hat{N}^g}(\hat{\mathcal{H}}, \hat{Q}) = H^{-\hat{N}^g}(\hat{\mathcal{H}}, \hat{Q})$.

Proof of Eq. (69a). We prove Eq. (69a) by explicitly calculating the both sides.

In order to calculate the left-hand side of Eq. (69a), take an orthonormal basis of definite N^p_m, N^m_m [the basis (77)], N^0_m and an orthonormal basis of $\mathcal{H}_K$. Then, $C = (-)^{N^m_m + N^0_m}$. Similarly, for the right-hand side, take an orthonormal basis of definite N^b_m, N^c_m, N^0_m and an orthonormal basis of $\mathcal{H}_K$.

From these relations, the left-hand side of Eq. (69a) becomes

$$\begin{aligned}
&\mathrm{tr}\, e^{-\lambda L_0^{int}} C \\
&\quad = e^{\lambda} \prod_{m=1}^{\infty} \left(\sum_{N^p_m=0}^{1} e^{-\lambda m N^p_m} \right) \left(\sum_{N^m_m=0}^{1} e^{-\lambda m N^m_m} (-)^{N^m_m} \right) \\
&\qquad\qquad \times \left(\sum_{N^0_m=0}^{\infty} e^{-\lambda m N^0_m} (-)^{N^0_m} \right) \mathrm{tr}_{\mathcal{H}_K} e^{-\lambda L_0^K} \\
&\quad = e^{\lambda} \prod_m (1 + e^{-\lambda m})(1 - e^{-\lambda m})(1 + e^{-\lambda m})^{-1}\, \mathrm{tr}_{\mathcal{H}_K} e^{-\lambda L_0^K} \\
&\quad = e^{\lambda} \prod_m (1 - e^{-\lambda m})\, \mathrm{tr}_{\mathcal{H}_K} e^{-\lambda L_0^K}.
\end{aligned} \tag{78}$$

The right-hand side becomes

$$
\begin{aligned}
&\mathrm{tr}\, e^{-\lambda L_0^{int}}(-)^{\hat{N}^g} \\
&\quad = e^{\lambda}\prod_{m=1}^{\infty}\left(\sum_{N_m^b=0}^{1} e^{-\lambda m N_m^b}(-)^{N_m^b}\right)\left(\sum_{N_m^c=0}^{1} e^{-\lambda m N_m^c}(-)^{N_m^c}\right) \\
&\qquad\qquad \times \left(\sum_{N_m^0=0}^{\infty} e^{-\lambda m N_m^0}\right) \mathrm{tr}_{\mathcal{H}_K}\, e^{-\lambda L_0^K} \\
&\quad = e^{\lambda}\prod_{m}(1-e^{-\lambda m})(1-e^{-\lambda m})(1-e^{-\lambda m})^{-1}\,\mathrm{tr}_{\mathcal{H}_K}\, e^{-\lambda L_0^K} \\
&\quad = e^{\lambda}\prod_{m}(1-e^{-\lambda m})\,\mathrm{tr}_{\mathcal{H}_K}\, e^{-\lambda L_0^K}. \qquad (79)
\end{aligned}
$$

This proves Eq. (69a). □

6 Discussion

(i). The extension of the vanishing theorem to $d \geq 1$ is straightforward. Write $\hat{\mathcal{H}}$ such that

$$\hat{\mathcal{H}} = \left(\mathcal{F}(\alpha_{-m}^0, b_{-m}, c_{-m}; k^0) \otimes \mathcal{H}_s\right)^{L_0}, \qquad (80)$$

where $\mathcal{H}_s = \mathcal{F}(\alpha_{-m}^i; k^i) \otimes \mathcal{H}_K$. The superscript i runs from 1 to $d-1$. Similarly, break up L_m. In particular,

$$L_0 = L_0^{(0)} + L_0^g + L_0^{(s)}, \qquad (81a)$$

where

$$
\begin{aligned}
L_0^{(0)} + L_0^g &= h^{(0)} + \sum_{m=1}^{\infty} m(N_m^0 + N_m^b + N_m^c) - 1, && (81b) \\
L_0^{(s)} &= h^{(s)} + \sum_{i=1}^{d-1}\sum_{m=1}^{\infty} m N_m^i + L_0^K, && (81c) \\
h^{(0)} &= -\alpha'(k^0)^2, \qquad h^{(s)} = \sum_{i=1}^{d-1}\alpha'(k^i)^2. && (81d)
\end{aligned}
$$

Just like $\mathcal{H}_K$, the spectrum of $\mathcal{H}_s$ is bounded below and $L_0^{(s)} \geq 0$ for $(k^i)^2 \geq 0$.[9] Thus, our derivation applies to $d > 1$ essentially with no modification; simply make the following replacements:

$$\mathcal{H}_K \to \mathcal{H}_s, L_0^X \to L_0^{(0)}, L_0^K \to L_0^{(s)}, h^X \to h^{(0)}. \tag{82}$$

(However, use the only momentum independent piece of $L_0^{(s)}$ in calculating the index and the signature.)

(ii). The standard proofs of the no-ghost theorem do not only show the theorem, but also show that the BRST cohomology is isomorphic to the light-cone spectra. Since we do not have light-cone directions in general, we do not show this. In other words, we do not construct physical states explicitly.

(iii). Our proof does not apply at the exceptional value of momentum $k^\mu = 0$ because the vanishing theorem fails (See footnote 7). Even in the flat $d = 26$ case, the exceptional case needs a separate treatment [8, 9, 21]. For the flat case, the relative cohomology is nonzero at three ghost numbers and is represented by

$$\alpha_{-1}^\mu|0; k^\mu = 0\rangle, \quad b_{-1}|0; k^\mu = 0\rangle, \quad \text{and} \quad c_{-1}|0; k^\mu = 0\rangle. \tag{83}$$

Thus, there are negative norm states. However, the physical interpretation of these infrared states is unclear [8, 24].

(iv). The original no-ghost theorem by Goddard and Thorn [2] can be applied to $d = 1$ via a slight modification. The $d = 1$ Hilbert space $\mathcal{H}$ can be decomposed as

$$\mathcal{H}_{h^X,h^K} = \left\{ Polynomial(\alpha_{-m}^0, L_{-m}^K)|h^X, h^K\rangle \right\}. \tag{84}$$

Goddard and Thorn's proof applies to any invariant subspace of the $d = 26$ Hilbert space. The subspace $\mathcal{H}_{h^X,h^K}$ is invariant under the action of Virasoro generators. Moreover, any state of K can be constructed by a free-field representation since $h^K > 0$. Reference [11] uses these facts to show the theorem for $d = 1$. Incidentally, Thorn [4] also used OCQ and proved the no-ghost theorem for $1 \leq d \leq 25$. The proof does not assume the compact CFT, and there is no known way to give such a theory consistent interactions at loop levels [25].

[9] In fact, FGZ have shown that an infinite sum of Verma modules with $h > 0$ furnish a basis of the Fock space $\mathcal{F}(\alpha_{-m}^i; k^i)$. Thus, not only $\mathcal{H}_K$, but the whole $\mathcal{H}_s$ must be written by Verma modules with $h > 0$.

(v). Finally, as is clear from our proof, the vanishing theorem itself does not require even $d=1$, and the extention to more general backgrounds is possible. In particular,

Theorem 6.1. *$H^n(\hat{\mathcal{H}}, \hat{Q}) = 0$ for $n \neq 0$ if $\mathcal{H}$ can be decomposed as*

$$\mathcal{H}_{h,h'} = \mathcal{V}(c=1, h<0) \otimes \mathcal{V}(c'=25, h'>0) \otimes \mathcal{F}(b_{-m}, c_{-m}). \tag{85}$$

Here, $\mathcal{V}(c=1, h<0)$ necessarily corresponds to a nonunitary CFT Hilbert space, and $\mathcal{V}(c'=25, h'>0)$ corresponds to a unitary CFT Hilbert space. Of course, we cannot prove the no-ghost theorem in our current technology. However, the no-ghost theorem implies the vanishing theorem; so, the proof of the vanishing theorem provides a consistency check or a circumstantial evidence of the no-ghost theorem for more general backgrounds.[10]

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A The Künneth Formula

To simplify the notation, denote the complexes appeared in Section 4 as follows:

$$(\mathcal{F}(\alpha^0_{-m}, b_{-m}, c_{-m}; k^0), Q_0^{(FGZ)}) \;\to\; (\mathcal{F}, Q), \tag{86a}$$

$$(\mathcal{F}(\alpha^0_{-m}, b_{-m}; k^0), d') \;\to\; (\mathcal{F}_1, d'), \tag{86b}$$

$$(\mathcal{F}(c_{-m}), d'') \;\to\; (\mathcal{F}_2, d''). \tag{86c}$$

[10] work in progress

Let $\{\omega_i^{-b}\}$ and $\{\eta_j^{n+b}\}$ be bases of $H^{-b}(\mathcal{F}_1)$ and $H^{n+b}(\mathcal{F}_2)$ respectively. Then, $\phi^n = \omega_i^{-b}\eta_j^{n+b}$ is a closed state in $\mathcal{F}\,(=\mathcal{F}_1\otimes\mathcal{F}_2)$. We show that this is not an exact state. If it were exact, it would be written as

$$\phi^n = \omega_i^{-b}\eta_j^{n+b} = Q(\alpha^{-b-1}\beta^{n+b} + \gamma^{-b}\delta^{n+b-1}) \tag{87}$$

for some α^{-b-1}, β^{n+b}, γ^{-b}, and δ^{n+b-1}. Executing the differential, we get

$$\begin{aligned} \omega_i^{-b}\eta_j^{n+b} &= (d'\alpha^{-b-1})\beta^{n+b} + \alpha^{-b-1}(-)^{b+1}(d''\beta^{n+b}) \\ &\quad + (d'\gamma^{-b})\delta^{n+b-1} + \gamma^{-b}(-)^{b}(d''\delta^{n+b-1}). \end{aligned} \tag{88}$$

Comparing the left-hand side with the right-hand side, we get $\alpha^{-b-1} = \delta^{n+b-1} = 0$; thus, $\phi^n = 0$ contradicting our assumption. Thus, ϕ^n is an element of $H^n(\mathcal{F})$. Conversely, any element of $H^n(\mathcal{F})$ can be decomposed into a sum of a product of the elements of $H^{-b}(\mathcal{F}_1)$ and $H^{n+b}(\mathcal{F}_2)$. Thus, we obtain

$$H^n(\mathcal{F}) = \bigoplus_{\substack{n=c-b \\ c,b\geq 0}} H^{-b}(\mathcal{F}_1)\otimes H^{c}(\mathcal{F}_2). \tag{89}$$

Here, the restriction of the values b and c comes from the fact $\mathcal{F}_1^n = \mathcal{F}_2^{-n} = 0$ for $n > 0$.

This is the Künneth formula (for the "torsion-free" case [22].) Our discussion here is close to the one of Ref. [26] for the de Rham cohomology.

B Some Useful Commutators

In this appendix, we collect some useful commutators:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n},$$

$$[L_m, \alpha_n^\nu] = -n\alpha_{m+n}^\nu, \qquad [L_m, b_n] = (m-n)b_{m+n},$$

$$[L_m, c_n] = (-2m-n)c_{m+n},$$

$$[Q, L_m] = 0, \qquad [Q, \alpha_m^\nu] = -\sum_{n=-\infty}^{\infty} mc_n\alpha_{m-n}^\nu,$$

$$\{Q, b_m\} = L_m, \qquad \{Q, c_m\} = -\sum_{n=-\infty}^{\infty} nc_{-n}c_{m+n},$$

$$[N^g, b_m] = -b_m, \qquad [N^g, c_m] = c_m,$$

$$[N^g, L_m] = 0, \qquad [N^g, Q] = Q.$$

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