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NONASSOCIATIVE STAR PRODUCT DEFORMATIONS FOR D–BRANE WORLDVOLUMES IN CURVED BACKGROUNDS

Lorenzo Cornalba[1] and **Ricardo Schiappa**[2]

1 – Laboratoire de Physique Théorique
École Normale Supérieure
F-75231 Paris Cedex 05, FRANCE

2 – Department of Physics
Harvard University
Cambridge, MA 02138, U.S.A.

Abstract

We investigate the deformation of D–brane world–volumes in curved backgrounds. We calculate the leading corrections to the boundary conformal field theory involving the background fields, and in particular we study the correlation functions of the resulting system. This allows us to obtain the world–volume deformation, identifying the open string metric and the noncommutative deformation parameter. The picture that unfolds is the following: when the gauge invariant combination $\omega = B + F$ is constant one obtains the standard Moyal deformation of the brane world–volume. Similarly, when $d\omega = 0$ one obtains the noncommutative Kontsevich deformation, physically corresponding to a curved brane in a flat background. When the background is curved, $H = d\omega \neq 0$, we find that the relevant algebraic structure is still based on the Kontsevich expansion, which now defines a nonassociative star product with an A_∞ homotopy associative algebraic structure. We then recover, within this formalism, some known results of Matrix theory in curved backgrounds. In particular, we show how the effective action obtained in this framework describes, as expected, the dielectric effect of D–branes. The polarized branes are interpreted as a soliton, associated to the condensation of the brane gauge field.

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[1]E-mail: cornalba@lpt.ens.fr
[2]E-mail: ricardo@lorentz.harvard.edu

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1 Introduction and Summary

Noncommutative quantum field theoretic limits of string theory have received considerable attention in the recent literature, and have been studied in a variety of papers (see, *e.g.*, [1, 2, 3, 4, 5, 6] and references therein). The attention is focused on a specific scaling limit, where the effects of large magnetic backgrounds are translated into Moyal noncommutative deformations of the D–brane world–volume algebra of functions. The open string physics is therefore captured within a quantum field theory (which is renormalizable, despite appearances [7, 8]). A common point to most previous investigations is that the background (sigma model) fields are taken to be constant and that, as a consequence, the target space is flat. One may then ask the natural question of what happens if the background is curved, *i.e.*, if the background fields are no longer constant? This question received some attention in a couple of recent papers [9, 10, 11, 12], but there is no general answer to it (other papers of interest with some relation to this subject are, *e.g.*, [13, 14, 15, 16]). Our goal in this work is to address this problem in the context of a simple model with weakly curved backgrounds, which can be on one side connected to the known flat background framework, and on the other hand can be related to formal results of brane physics in WZW models, which can be analyzed exactly with conformal field theory techniques [10, 11, 17].

More concretely, the aim of this paper is to understand how the presence of a non–trivial background field affects the world–volume deformation of a D–brane. It is known that, in the presence of a constant background B–field, the physics can be exactly described either by a sigma model approach [18, 19, 20, 21, 22, 23, 24, 25], or alternatively, by translating the background B–field into a noncommutative Moyal deformation of the brane world–volume algebra of functions [3, 5]. The constant field situation represents a particular choice of background and one can ask what happens in more complicated situations. One thing to keep in mind is that (as for the Born–Infeld action [26, 27, 28]) the gauge covariant combination to consider is not B alone, but $B+F$, which we shall denote by $\omega \equiv B+F$ in the following. One may then consider three cases of increasing complexity: the case of constant ω, the case where $d\omega = 0$ but ω is *not* constant, and the most general case where $d\omega \neq 0$ and we have NS–NS three form flux (as $d\omega = dB + dF = H$) and a curved background.

The analysis leads to the following complete picture. The first case, corresponding to constant ω, has been extensively studied in the literature where one obtains a noncommutative Moyal deformation of the brane world–volume [3, 4, 5, 6]. The physics is by now very well understood, corresponding to a flat brane embedded in a flat background space. The second case, when ω is not constant but $d\omega = 0$, has also been studied in the literature, though to a much less extent. This gives the so–called Cattaneo–Felder model of [29]. One therefore obtains the natural extension of the Moyal deformation to the case of varying symplectic form, corresponding to the noncommutative Kontsevich star product deformation of the brane world–volume algebra of functions [30]. This situation corresponds to the embedding of a curved brane in a flat background space. These configurations have also been studied from the point of view of BPS membranes in Matrix theory, where the varying F–field physically corresponds to a varying density of zero–branes over a curved membrane [31, 32]. Finally, the general case where $d\omega \neq 0$ is the main subject of this paper. One no longer has a symplectic form and apparently no obvious definition of a star product —which usually comes from a given Poisson structure on the world–volume of the D–brane. In this general situation, we will find that the world–volume algebra of functions is deformed to an algebra which is not only noncommutative, but also nonassociative. One interesting point we shall uncover is that this nonassociative star product can still be defined using Kontsevich's formula [30]. Therefore, the nonassociativity can be traced, thanks to Kontsevich's formality formulae, to the Schouten–Nijenhuis bracket of ω^{-1} with itself, which is proportional to the NS–NS field strength $d\omega = H$ [30, 29]. These nonassociative algebras have the structure of an A_∞ homotopy associative algebra (see, *e.g.*, [33, 34, 35]) which have previously received some attention in the string field theory literature since they are the natural algebras that appear in general open–closed string field theories [36, 37].

Our approach in this paper will rely on a perturbative calculation of n–point functions on the disk, using the background field method applied to open string theory [18, 19, 22, 23]. The background fields are expanded in Taylor series, and the derivative terms that appear are treated as new interactions, which we treat in a perturbative expansion. This allows us to obtain the open string parameters, metric G and deformation θ, generalizing the results in [23, 3, 5]. It also allows us to identify the star product deformations, as described in the previous paragraph. We begin, in section 2, by describing the specific closed string backgrounds which we shall consider in this paper. These will be the class of parallelizable manifolds, exact background solutions for closed string theory [20]. Then, in section 3, we shall describe in detail the perturbation theory on the disk for open strings in these curved backgrounds, *i.e.*, we will study the new interaction vertices due to the curvature terms. In particular, we present the general methods that we then use in section 4 for the calculation of n–point functions on the disk, with particular emphasis on the conformal properties of these disk correlators. These correlators also yield the open string parameters and the nonassociative Kontsevich star product. Section 5 includes a brief resume of the different situations and the different world–volume deformations and star products, which can be read directly by the reader who wishes to skip the calculations in the preceding sections. It also describes in some detail the concept of a nonassociative star product deformation, which could be a topic of great interest for future research.

Most of the previous treatment is done in a particular $\alpha' \to 0$ scaling limit [5], where the closed string metric, g, scales to zero. In section 6 we move away from this limit and compute corrections to the previous results which explicitly depend on the closed string metric. These calculations yield the formulas relating open and closed string parameters. It is interesting to observe that the final answer is a simple generalization of the flat background results of [23, 3, 5]. In section 7 we make contact with previous results and in particular we describe, within our formalism, the dielectric effect of D–branes [38] in these curved backgrounds. Indeed, these solutions describing polarization of lower dimensional branes, obtained first in [38] and then further studied in different situations involving D–branes and fundamental strings in R–R or NS–NS backgrounds by, *e.g.*, [39, 40, 41, 42, 43, 44, 45], is now reinterpreted, dually, as an instability of the space filling brane, which condenses to a lower dimensional brane. This is accomplished by first studying the relation between the partition function —the correlators we computed in the earlier sections— and the effective action. Once this connection is made (using boundary string field theory arguments), we obtain the usual matrix action in the presence of an H–field, and we can then use the previous results on the subject.

Finally, we discuss in the concluding sections how further studies of these nonassociative geometries could lead to a proper definition of Matrix theory [46, 47, 48, 49, 50] in a general curved background. These nonassociative geometries could provide the proper framework to generalize the arguments in [51, 52, 53] and the weak field calculations of

[54, 55] in order to build the Matrix theory action in a general curved target space.

2 Open Strings in Parallelizable Backgrounds

The physics of a string propagating in a curved background is conveniently described in terms of a nonlinear sigma model. In the presence of a background metric $g_{ab}(x)$ and NS–NS 2–form field $B_{ab}(x)$ the action which governs the motion of the string is given by [18, 19, 20, 22, 23],

$$S = \frac{1}{4\pi\alpha'}\int_\Sigma g_{ab}(X)\, dX^a \wedge *dX^b + \frac{i}{4\pi\alpha'}\int_\Sigma B_{ab}(X)\, dX^a \wedge dX^b, \tag{1}$$

where Σ is the string world–volume. Moreover, when considering open strings one can include boundary interactions on $\partial\Sigma$. In the sequel, we will mainly focus on the coupling to the $U(1)$ gauge field $A_a(x)$, given by

$$S_B = i\oint_{\partial\Sigma} A_a(X)\, dX^a.$$

In this paper we will consider only the physics at weak string coupling, and we will consequently assume Σ to have the topology of a disk. Other background fields (such as the dilaton) will not play a role in our subsequent analysis. We shall mainly address maximal branes, though our results are completely general. Also, from now on, we will work in units such that

$$2\pi\alpha' = 1.$$

The action (1) is written in a generic coordinate system x^a in spacetime. On the other hand, in order to use (1) to compute correlators in perturbation theory, it is natural to follow the standard techniques of the background field method and use coordinates x^a which are *Riemann normal coordinates at the origin* —*i.e.* defined using geodesic paths in target space which start at $x^a = 0$ [18, 20]. We recall that the main advantage of this choice is that the Taylor series expansion of any tensor around $x^a = 0$ is explicitly given in terms of covariant tensors evaluated at the origin. In particular one has, up to quadratic order in the coordinates,

$$g_{ab}(x) = g_{ab} - \frac{1}{3} R_{acbd} x^c x^d + \cdots . \tag{2}$$

Let us now consider the expansion of the NS–NS 2–form field, by first recalling that we have some gauge freedom in the definition of $B_{ab}(x)$. In fact, the transformations $B \to B + d\Lambda$, $A \to A - \Lambda$ leave the total action $S + S_B$ invariant, and we can use this freedom to impose the following (radial) gauge[1]

$$x^a B_{ab}(x) = x^a B_{ab}(0) .$$

One can explicitly solve the above equation in terms of the NS–NS three–form field strength

$$H = dB,$$

and obtain

$$B_{ab}(x) = B_{ab} + x^c \int_0^1 s^2 H_{abc}(sx)\, ds.$$

Therefore, the normal coordinate expansion for the field B_{ab} is explicitly given by

$$B_{ab}(x) = B_{ab} + \frac{1}{3} H_{abc} x^c + \frac{1}{4} \nabla_d H_{abc} x^c x^d + \cdots . \tag{3}$$

Using the expressions (2) and (3), one can expand (1) about the classical constant background $\partial X^a = 0$ and obtain

$$S = S_0 + S_1 + \cdots , \tag{4}$$

where S_n contains $n+2$ powers of the coordinate fields X^a and where, in particular,

$$\begin{aligned} S_0 &= \frac{1}{2} g_{ab} \int_\Sigma dX^a \wedge *dX^b + \frac{i}{2} B_{ab} \int_\Sigma dX^a \wedge dX^b, \\ S_1 &= \frac{i}{6} H_{abc} \int_\Sigma X^a dX^b \wedge dX^c. \end{aligned} \tag{5}$$

[1]Given a generic field $B_{ab}(x)$, we can consider the gauge transformation parameter $\Lambda_a(x)$ given by $\Lambda_a(x) = x^b \int_0^1 s B_{ab}(sx)\, ds$. It is then a simple computation to see that the combination $\partial_a \Lambda_b - \partial_b \Lambda_a$ equals $-B_{ab}(x) + x^c \int_0^1 s^2 H_{abc}(sx)\, ds$.

In this paper, we will be primarily interested in the effects of the term S_1, which describes a small curved deviation from the flat closed string background. Let us elaborate more on this point. To leading order in α', the beta function equations which describe consistent closed string backgrounds read [18, 20]:

$$R_{ab} = \frac{1}{4} H_{acd} H_b{}^{cd} \ , \qquad\qquad \nabla^a H_{abc} = 0. \tag{6}$$

If we work to first order in H, one may then neglect the presence of curvature coming from the metric and only consider the effects of H coming from (5). We can actually make these arguments more systematic if we consider a general class of conjectured solutions to the beta function equations, called parallelizable manifolds [20]. These configurations are characterized by the following properties. First of all, the tensor H_{abc} is covariantly constant,

$$\nabla_a H_{bcd} = 0.$$

Moreover, if we consider the generalized connection $\Gamma + \frac{1}{2}H$, then the corresponding curvature tensor,

$$\mathcal{R}_{abcd} = R_{abcd} + \frac{1}{2}\nabla_a H_{bcd} - \frac{1}{2}\nabla_b H_{acd} + \frac{1}{4} H_{ade} H_{bc}{}^e - \frac{1}{4} H_{ace} H_{bd}{}^e \ ,$$

must vanish. Using the fact that $R_{a[bcd]} = 0$, one can easily show that the field H_{abc} must satisfy a Jacobi identity, in the sense that

$$H_{abe} H_{cd}{}^e + \mathrm{cyclic}_{abc} = 0.$$

These facts then imply

$$R_{abcd} = \frac{1}{4} H_{abe} H_{cd}{}^e \ ,$$

and therefore (6). Moreover, at a more fundamental level, it was explicitly shown that when the target is parallelizable, the string sigma model is ultra–violet finite to two loops, with vanishing beta functions [20]. It was moreover suggested that this holds true to higher orders for the superstring, and one thus has a consistent solution of closed string theory [20].

In the parallelizable situation the expansion (4) drastically simplifies. In the sequel we shall only need the explicit forms of S_0 and S_1 given above. On the other hand, in order to extend the results of this paper to higher order in H, one needs the expressions of S_n for $n \geq 2$. We include, for completeness, the first of these terms explicitly given by:

$$S_2 = -\frac{1}{24} H_{abe} H_{cd}{}^{e} \int_{\Sigma} X^a X^c dX^b \wedge *dX^d.$$

3 Perturbation Theory

In the last section we have reviewed the general form of the sigma model action which describes open string dynamics in curved backgrounds. From now on we shall only consider backgrounds which are weakly curved. More precisely we will work, for the rest of the paper, to *leading* order in the background field H, and consequently we shall focus our analysis on the action $S_0 + S_1 + S_B$. If we denote with $F = dA$ the $U(1)$ field strength, and with ω the symplectic structure

$$\omega_{ab}(x) = B_{ab} + F_{ab}(x)\,, \tag{7}$$

then the relevant action is given by

$$\frac{1}{2} g_{ab} \int_{\Sigma} dX^a \wedge *dX^b + i \int_{\Sigma} \omega + \frac{i}{6} H_{abc} \int_{\Sigma} X^a dX^b \wedge dX^c. \tag{8}$$

Before we start the detailed discussion of the perturbation theory for the action (8), and in order to set the stage and motivate the subsequent results, let us begin by recalling some known facts which are valid in the flat space limit of $H_{abc} = 0$. On one side, the conventional approach to open string physics starts by considering the simple free action $\frac{1}{2} g_{ab} \int_{\Sigma} dX^a \wedge *dX^b$, or even the full free action S_0. One then analyzes the physics of boundary interactions by considering the coupling S_B to the $U(1)$ gauge field, A, and one treats (following, for example, the approaches in [19, 22, 23]) the interactions perturbatively in $F = dA$. In this scheme the basic interaction vertex with n external legs involves $n-2$ derivatives of F, and the perturbation theory quickly becomes unmanageable as soon as one considers rapidly varying gauge fields. It was noted, on the other hand, in [29] that, if one considers the simple topological action $i \int_{\Sigma} \omega$ (that is, one looks at (8) in the limit g_{ab}, $H_{abc} \to 0$), then the resulting path integral drastically simplifies. In fact, if one considers the n–point function of n generic functions $f_1(x), \ldots, f_n(x)$, placed cyclically on the boundary $\partial\Sigma$ of the string world–volume, one obtains the simple result (independently of the moduli of the insertion points) [29, 5]:

$$\langle f_1 \cdots f_n \rangle = \int V(\omega)\, dx\; (f_1 \star \cdots \star f_n)\,. \tag{9}$$

In the above, $\star$ is the *associative* Kontsevich star product[2] with respect to the Poisson structure $\alpha = \omega^{-1}$,

$$f \star g = f \cdot g + \frac{i}{2}\alpha^{ab}\partial_a f\, \partial_b g + \cdots, \tag{10}$$

and $V(\omega) = \sqrt{\det \omega}\,(1 + \cdots)$ is a volume form[3] such that $\int V(\omega)\; dx$ acts as a trace for the product $\star$. The basic point we would like to stress is that the product (10) contains derivatives of α (and therefore of F) to all orders, and is therefore valid for arbitrary gauge field configurations. This means that the perturbation theory in A_a becomes tractable to all orders when $g_{ab} \to 0$, and is conveniently described in terms of the algebraic operation $\star$.

We shall see in this paper that, when one introduces the perturbation S_1 but still considers the limit $g_{ab} \to 0$, then one can still re–sum the perturbation theory to all orders in A_a. We will see that the relevant algebraic structure is still given by a Kontsevich product of the general form (10), but now with ω replaced in a natural way by the gauge invariant combination:

$$\widetilde{\omega}_{ab}(x) = \omega_{ab}(x) + \frac{1}{3}H_{abc}x^c = B_{ab}(x) + F_{ab}(x)\,,$$

and with α replaced by $\widetilde{\alpha} = \widetilde{\omega}^{-1}$. In order to clearly distinguish the two cases, we shall denote this second product (relative to $\widetilde{\alpha}$) with $\bullet$, given by the usual Kontsevich expansion,

$$f \bullet g = f \cdot g + \frac{i}{2}\widetilde{\alpha}^{ab}\partial_a f\, \partial_b g + \cdots.$$

The two–form $\widetilde{\omega}$ is not closed and correspondingly the product $\bullet$ is now *nonassociative.* We will discuss later how the nonassociativity is controlled by the field strength $H = d\widetilde{\omega}$.

[2] The terms hidden behind the dots $\cdots$ in (10) are given by explicit diagrammatic expressions, as explained in [30], valid for any bi–vector field $\alpha^{ab}(x)$ in terms of the functions f, g, the tensor α^{ab} and their derivatives. If α^{-1} is closed, then the corresponding product is associative.

[3] For more details on $V(\omega)$ we refer the reader to [56].

The n–point functions are again given by an equation similar to (9), with $\star$ replaced by $\bullet$. On the other hand, expressions like $f_1 \bullet \cdots \bullet f_n$ are ambiguous, due to the nonassociativity of the product, and one needs to insert parenthesis to precisely define their meaning. This can be done in various ways, and this fact is reflected in the dependence of n–point functions on the $n-3$ *conformal* moduli of the insertion points on the boundary $\partial\Sigma$. The n–point functions will then be interpolations, parameterized by $n-3$ moduli, between the various possible positions of the parenthesis in the expression $f_1 \bullet \cdots \bullet f_n$.

From now until section 4.5 we will concentrate on the simplest case of $F = 0$ or

$$\omega_{ab}(x) = B_{ab}.$$

We thus neglect the boundary interaction S_B and concentrate on the action $S_0 + S_1$. The generalization to the case (7) will be comparatively simple (as for the $d\widetilde{\omega} = 0$ case) and is left to section 4.5, which also summarizes the results in the general context. We now turn to a systematic discussion of the perturbation theory for the action $S_0 + S_1$.

3.1 The Free Theory

Let us first recall some facts about the unperturbed action S_0. Since S_0 is invariant under translations $X^a \to X^a + c^a$, the field X^a can be split into a constant zero mode x^a and a fluctuating quantum field ζ^a,

$$X^a = x^a + \zeta^a. \tag{11}$$

Path integrals with the free action S_0 are then explicitly given by a path integral over the quantum field ζ^a and an ordinary integral over the zero–mode x^a as [19]:

$$\int [dX]\ e^{-S_0(X)} \to \int dx \int [d\zeta]\ e^{-S_0(\zeta)}.$$

The integral in $[d\zeta]$ is gaussian and is determined once one obtains the two–point function for the fluctuating field ζ. From now on, and unless otherwise specified, we will parameterize the disk Σ with the complex upper–half plane $\mathbf{H}^+$. As discussed in [22, 3, 5], the two–point function can be more conveniently written if one introduces the open string metric G_{ab} and noncommutativity tensor θ^{ab} as given by

$$\frac{1}{G} + \theta = \frac{1}{g+B}.$$

It then has the general form,

$$\langle \zeta^a (z) \zeta^b (w) \rangle = \frac{i}{\pi} \theta^{ab} \mathcal{A} (z, w) - \frac{1}{\pi} G^{ab} \mathcal{B} (z, w) + \frac{1}{2\pi} g^{ab} \mathcal{C} (z, w) , \tag{12}$$

where

$$\begin{aligned} \mathcal{A} (z, w) &= \frac{1}{2i} \ln \left(\frac{\overline{w} - z}{\overline{z} - w} \right) , \\ \mathcal{B} (z, w) &= \ln \left| z - \overline{w} \right| , \\ \mathcal{C} (z, w) &= \ln \left| \frac{z - w}{z - \overline{w}} \right| . \end{aligned}$$

In the sequel, we shall only need to consider the propagator (12) when one point (say w) is placed at the boundary $\partial\Sigma$ of the string world–sheet. In this case $w = \overline{w}$ and $\mathcal{C} (z, w) = 0$. Also, in the case $w = \overline{w}$, the coefficients $\mathcal{A} (z, w)$ and $\mathcal{B} (z, w)$ have a simple geometrical interpretation. $\mathcal{A}$ measures the angle between the line z–w and the vertical line passing through w, and $\mathcal{B}$ gives the logarithm of the distance between z and w.

We now consider the limit $g_{ab} \to 0$. In this limit, the effective open string metric G_{ab} becomes large and therefore the term in (12) proportional to G^{ab} becomes irrelevant. Also, one has in this limit, that $\theta = B^{-1} = \alpha(x)$. In this case the propagator (12) reduces to

$$\langle \zeta^a (z) \zeta^b (w) \rangle = \frac{i}{\pi} \theta^{ab} \mathcal{A} (z, w) ,$$

and the computation of path integrals becomes simple. As we discussed in the previous subsection, if one considers n functions $f_1, \ldots, f_n$, positioned at ordered points $\tau_1 < \cdots < \tau_n$ on the boundary $\partial\Sigma$ of the string world–sheet, then the path integral

$$\int [dX] \; e^{-S_0(X)} \, f_1 \left(X \left(\tau_1 \right) \right) \cdots f_n \left(X \left(\tau_n \right) \right) \tag{13}$$

can be evaluated [3, 5] with the result,

$$\int V (B) \; dx \; \left(f_1 \star \cdots \star f_n \right) .$$

Since $\omega(x) = B$ is constant, the product $\star$ is the usual Moyal star product and $V(B) = \sqrt{\det B}$.

A word on notation. From now on we will omit the explicit reference to the volume form in the integrals. We shall therefore use the following short–hand notation:

$$\int V(\omega)\, dx \cdots \to \int \cdots .$$

3.2 The Interaction

Let us now consider the effects of the perturbation S_1. Corresponding to the split (11), the effect of S_1 is to introduce two bulk graphs:

$$\begin{aligned} \mathcal{V} &= -\frac{i}{6} H_{abc} x^c \int_\Sigma d\zeta^a \wedge d\zeta^b , && (14) \\ \mathcal{W} &= -\frac{i}{6} H_{abc} \int_\Sigma \zeta^a d\zeta^b \wedge d\zeta^c . && (15) \end{aligned}$$

We will then consider the following path integral

$$\begin{aligned} &\int [dX]\; e^{-S_0(X)-S_1(X)}\, f_1(X(\tau_1)) \cdots f_n(X(\tau_n)) \simeq \\ &\int [dX]\; e^{-S_0(X)}\, [1 + \mathcal{V} + \mathcal{W}]\;\, f_1(X(\tau_1)) \cdots f_n(X(\tau_n)) . \end{aligned} \qquad (16)$$

In order to analyze the effects of $\mathcal{V}$ and $\mathcal{W}$, let us first introduce some notation and discuss some useful simple results. Consider a generic point $z \in \mathbf{H}^+$, and consider the path integral:

$$\int [dX]\; e^{-S_0(X)}\, \zeta^a(z)\;\, f_1(X(\tau_1)) \cdots f_n(X(\tau_n)) .$$

If we introduce the short–hand notation,

$$\mathcal{A}(z, \tau_i) = \mathcal{A}_i ,$$

for the angle between the line z–τ_i and the vertical through τ_i, then the result of the above path integral is simply given by

$$\sum_{i=1}^{n} \frac{i}{\pi} \mathcal{A}_i \, \theta^{a\widetilde{a}} \int f_1 \star \cdots \star \partial_{\widetilde{a}} f_i \star \cdots \star f_n.$$

The above result is easy to understand once one considers the expansion of the functions $f_i\,(X) = f_i\,(x + \zeta)$ in Taylor series in powers of ζ. The contraction of the field $\zeta^a\,(z)$ with a field $\zeta^{\widetilde{a}}\,(\tau_i)$ coming from the Taylor expansion of the function f_i gives a factor of $\frac{i}{\pi}\mathcal{A}_i \, \theta^{a\widetilde{a}}$. We are then left with a path integral of the form (13), where the function f_i has been replaced with its derivative $\partial_{\widetilde{a}} f_i$. More generally, when a free field $\zeta^a\,(z)$ is contracted with one of the boundary functions it acts as a differentiation:

$$\frac{i}{\pi} \mathcal{A} \, \theta^{a\widetilde{a}} \partial_{\widetilde{a}} \; . \tag{17}$$

With this result, we can now consider the effects of the perturbation vertices $\mathcal{V}$ and $\mathcal{W}$ in the path integral (16). Let us start with the analysis of $\mathcal{V}$. Choose any two indices $i < j$ and consider the term where the two ζ's in $\mathcal{V}$ differentiate the two functions f_i and f_j (in the sense just described above). If ζ^a differentiates f_i and ζ^b differentiates f_j one then gets

$$\frac{i}{6\pi^2} H_{abc} \theta^{a\widetilde{a}} \theta^{b\widetilde{b}} \left(\int_\Sigma d\mathcal{A}_i \wedge d\mathcal{A}_j \right) \int x^c \left(\cdots \star \partial_{\widetilde{a}} f_i \star \cdots \star \partial_{\widetilde{b}} f_j \star \cdots \right).$$

The integral over Σ can be evaluated by noting that the upper–half plane $\mathbf{H}^+$ corresponds to the simplex $-\frac{\pi}{2} < \mathcal{A}_i < \mathcal{A}_j < \frac{\pi}{2}$ in the $\mathcal{A}_i$–$\mathcal{A}_j$ plane. Therefore the integral $\int_\Sigma d\mathcal{A}_i \wedge d\mathcal{A}_j$ is equal to $\frac{1}{2}\pi^2$. Moreover, if we instead let ζ^a differentiate f_j and ζ^b differentiate f_i we obtain, using the antisymmetry of H_{abc}, the same result as above. Summing the two contributions, and summing over all possible pairs $i < j$, one then obtains

$$\sum_{i<j} V_{ij},$$

where, for $i < j$, we have defined

$$V_{ij} = \frac{i}{6} H_{abc} \theta^{a\widetilde{a}} \theta^{b\widetilde{b}} \int x^c \star \left(f_1 \star \cdots \star \partial_{\widetilde{a}} f_i \star \cdots \star \partial_{\widetilde{b}} f_j \star \cdots \star f_n \right).$$

In the above equation we have used the fact that (for the Moyal product) $\int f \cdot g = \int f \star g$, in order to rewrite everything in terms of $\star$ products, including the multiplication by the coordinate function x^c. To conclude the analysis of the effect of the two–vertex $\mathcal{V}$, one must also consider the term coming from the contraction of the two ζ's in $\mathcal{V}$ among themselves. This term will require some care, since we must regularize the contraction of two fields at coincident points. On the other hand, the general structure of the contribution can be obtained with little effort by recalling that the two indices a and b in (14) are contracted with the antisymmetric tensor H_{abc}. This implies that the contribution in question must have the form

$$V = \mathcal{N}\, H_{abc}\theta^{bc} \int x^a \star (f_1 \star \cdots \star f_n)\,,$$

where $\mathcal{N}$ is an unknown constant which will later be determined to be 1/3.

We now move to the analysis of the contributions coming from the three–graph $\mathcal{W}$. First, given three indices $i < j < k$, let us define

$$W_{ijk} = -\frac{1}{12} H_{abc}\theta^{a\widetilde{a}}\theta^{b\widetilde{b}}\theta^{c\widetilde{c}} \int f_1 \star \cdots \star \partial_{\widetilde{a}} f_i \star \cdots \star \partial_{\widetilde{b}} f_j \star \cdots \star \partial_{\widetilde{c}} f_k \star \cdots \star f_n.$$

It is then easy to check, using the general result (17), that the contribution from the three–vertex which comes from the contraction of the fields ζ's in $\mathcal{W}$ with the functions f_i, f_j, f_k is given by

$$\sum_{i<j<k} S\,(\tau_i, \tau_j, \tau_k)\, W_{ijk},$$

where the function S is

$$\begin{aligned} S\,(\tau_i, \tau_j, \tau_k) &= \frac{2}{\pi^3}\left[\int_\Sigma \mathcal{A}_i d\mathcal{A}_j \wedge d\mathcal{A}_k \pm \text{permutation}_{ijk}\right] \\ &= \frac{4}{\pi^3}\left[\int_\Sigma \mathcal{A}_i d\mathcal{A}_j \wedge d\mathcal{A}_k + \text{cyclic}_{ijk}\right]. \end{aligned}$$

Other combinations, which involve contractions of the ζ's amongst themselves, yield a vanishing contribution to the result of the three–vertex $\mathcal{W}$.

Let us analyze the function S in more detail. As we saw, it depends on three ordered points $\tau_i < \tau_j < \tau_k$ on the boundary $\partial\Sigma$. On the other hand, since it is written explicitly in terms of integrals of angle functions $\mathcal{A}$, it is actually invariant under translations $\tau \to \tau + c$ and scalings $\tau \to \lambda\tau$, *i.e.*, under the subgroup of the modular group $SL\,(2, \mathbf{R})$ which leaves invariant the point at infinity. Therefore, by sending τ_i to 0 and τ_k to 1, it becomes clear that S actually only depends on a single parameter ranging between 0 and 1. Explicitly, one has:

$$S(\tau_i, \tau_j, \tau_k) = S\left(\frac{\tau_{ji}}{\tau_{ki}}\right),$$

where, from now on, we use the notation

$$\tau_{ji} = \tau_j - \tau_i.$$

As we explicitly show in the appendix, the function $S(x)$ can be computed exactly. It is a monotonically decreasing function defined on $[0, 1]$ ranging from 1 to -1. It satisfies $S(1-x) = -S(x)$ and is explicitly given by

$$S(x) = 1 - 2L(x).$$

The function $L(x)$ is the so called normalized Rogers dilogarithm [57], defined in terms of the usual dilogarithm $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$ as:

$$L(x) = \frac{6}{\pi^2}\left[\mathrm{Li}_2(x) + \frac{1}{2}\ln(x)\ln(1-x)\right].$$

We then conclude that the contribution coming from the three–vertex $\mathcal{W}$ is given by

$$\sum_{i<j<k} S\left(\frac{\tau_{ji}}{\tau_{ki}}\right) W_{ijk}.$$

4 Computation of n–point Functions

We now use the general results derived in the previous section in order to analyze the conformal properties of the n–point functions (16). Let us recall that we are still working in the simple case of constant symplectic structure $\omega_{ab}(x) = B_{ab}$, so that $\widetilde{\omega}_{ab} = B_{ab} + \frac{1}{3}H_{abc}x^c$. The generalization to arbitrary symplectic structure $\omega_{ab}(x) = B_{ab} + F_{ab}(x)$ is left to section 4.5.

In order to simplify the expressions in this section, we introduce the following short–hand notation:

$$
\begin{aligned}
K^{abc} &= \theta^{a\tilde{a}}\theta^{b\tilde{b}}\theta^{c\tilde{c}}H_{\tilde{a}\tilde{b}\tilde{c}} \\
y_a &= B_{ab}x^b.
\end{aligned}
$$

4.1 2–point Function

We shall first analyze the two–point function in some detail, since the manipulations for the higher point functions will be similar. One considers two functions f_1 and f_2, placed at points τ_1 and τ_2 on the real line, with $\tau_1 < \tau_2$. A simple computation, using the general results of the previous section, shows that the two–point function is explicitly given by:

$$
\int f_1 \star f_2 + \int \left(-\frac{i}{6} K^{abc}\, y_c \star \partial_a f_1 \star \partial_b f_2 + \mathcal{N} B_{bc} K^{abc} y_a \star f_1 \star f_2 \right). \tag{18}
$$

The above expression does *not* depend on the explicit values of τ_1, τ_2, but depends only on the order of the points τ_i on the real line. On the other hand, since two points on the boundary of a disk have no (conformal) moduli, the two–point function must be a *symmetric* bilinear of f_1, f_2. The first term in (18) is clearly symmetric. Let us then concentrate on the second term, by rewriting it with f_1 and f_2 interchanged. This gives, after a small rearranging,

$$
\int \left(\frac{i}{6} K^{abc}\, \partial_a f_1 \star y_c \star \partial_b f_2 + \mathcal{N} B_{bc} K^{abc} f_1 \star y_a \star f_2 \right). \tag{19}
$$

Using that $K^{abc}\, \partial_a f_1 \star y_c = K^{abc}\, y_c \star \partial_a f_1$, and differentiating by parts, we see that the difference between (19) and the second term of (18) reads

$$
\int \left(\frac{i}{3} B_{bc} K^{abc}\, \partial_a f_1 \star f_2 - \mathcal{N} B_{bc} K^{abc}\, [y_a, f_1] \star f_2 \right).
$$

The above is then vanishing if one has

$$
\mathcal{N} = \frac{1}{3}.
$$

With this value the two–point function is conformally invariant and is a simple symmetric bilinear of the functions f_1, f_2. Let us denote the n–point function by P_n. A little computation shows, using the identity $\int f \star g = \int fg$, that the two–point function (18) is given by the explicitly symmetric expression

$$P_2(f_1,f_2) = \int f_1 f_2 \left(1 + \frac{1}{3} H_{abc} x^a \theta^{bc}\right). \tag{20}$$

4.2 3–point Function

Let $\tau_1 < \tau_2 < \tau_3$ be there *ordered* points on the real line, and let us consider the three–point function of three functions $f_1,\ f_2,\ f_3$. One now has a contribution from the three–vertex $\mathcal{W}$, but it vanishes since

$$K^{abc} \int \partial_a f_1 \star \partial_b f_2 \star \partial_c f_3 = 0.$$

The only contribution then comes from the two–vertex $\mathcal{V}$, and it is given explicitly by (using the value 1/3 for $\mathcal{N}$)

$$\begin{aligned} P_3(f_1,f_2,f_3) = &\int f_1 \star f_2 \star f_3 + \frac{1}{3} B_{bc} K^{abc} \int y_a \star f_1 \star f_2 \star f_3 \\ -\frac{i}{6} K^{abc} &\int y_c \star (\partial_a f_1 \star \partial_b f_2 \star f_3 + \partial_a f_1 \star f_2 \star \partial_b f_3 + f_1 \star \partial_a f_2 \star \partial_b f_3). \end{aligned} \tag{21}$$

As for the two–point function, the above expression does not depend on the explicit values of the τ_i's, but only on their order on the real line. On the other hand, since three points on the disk have no moduli (as in the two–point function case), the above expression should actually be invariant under cyclic permutations of the three functions, and in particular under the replacement $f_1,\ f_2,\ f_3 \to f_2,\ f_3,\ f_1$. One must then show that (21) is equal to:

$$\begin{aligned} &\int f_2 \star f_3 \star f_1 + \frac{1}{3} B_{bc} K^{abc} \int f_1 \star y_a \star f_2 \star f_3 \\ &-\frac{i}{6} K^{abc} \int (-\partial_a f_1 \star y_c \star \partial_b f_2 \star f_3 - \partial_a f_1 \star y_c \star f_2 \star \partial_b f_3 + f_1 \star y_c \star \partial_a f_2 \star \partial_b f_3). \end{aligned}$$

In the second line of the above expression, we are free to move the function y_c all the way to the left, since we are contracting with the totally antisymmetric object K^{abc}. Given this fact, it is simple to show that the above expression is identical to (21), thus proving that also the three–point function is invariant under conformal transformations, and is therefore a cyclic trilinear of its inputs. Note that the *same* value for $\mathcal{N}$ makes both P_2 and P_3 invariant.

4.3 4–point Function

Let us now consider the four–point function. As usual we choose four ordered points $\tau_1 < \cdots < \tau_4$ on the real line and four functions $f_1, \ldots, f_4$. Following the general results in the previous sections, the result of the path integral (16) breaks into three parts. First we have the unperturbed result, given by

$$\int f_1 \star \cdots \star f_4.$$

The above is independent of the positions of the τ's, and is conformally (actually topologically) invariant by itself, since it is a cyclic multilinear function of the f's. Second, we have the term coming from the two–vertex $\mathcal{V}$, given by (in the notation of section 3.2)

$$V(f_1, \ldots, f_4) = V + \sum_{i<j} V_{ij}. \tag{22}$$

Finally we have, for the first time, a non–vanishing contributions to the path integral coming from the three–vertex $\mathcal{W}$, which is given explicitly by

$$\frac{1}{12} K^{abc} \int \left[S(\tau_1, \tau_2, \tau_3)\, \partial_a f_1 \star \partial_b f_2 \star \partial_c f_3 \star f_4 + \cdots \right], \tag{23}$$

where $\cdots$ stands for three more terms which are weighted with the corresponding factor $S(\tau_i, \tau_j, \tau_k)$, and with the derivatives ∂_a, ∂_b and ∂_c acting on all possible groups of three functions —as explained in section 3.2. Note that all the terms in (23) are actually the same after integration by parts (for example $K^{abc} \int \partial_a f_1 \star \partial_b f_2 \star f_3 \star \partial_c f_4 = -K^{abc} \int \partial_a f_1 \star \partial_b f_2 \star \partial_c f_3 \star f_4$, and so on), so that the above equation can be rewritten as

$$\kappa(\tau_i)\, \frac{1}{12} K^{abc} \int f_1 \star \partial_a f_2 \star \partial_b f_3 \star \partial_c f_4, \tag{24}$$

where the coefficient κ is given by

$$\kappa(\tau_i) = -S(\tau_1, \tau_2, \tau_3) + S(\tau_1, \tau_2, \tau_4) - S(\tau_1, \tau_3, \tau_4) + S(\tau_2, \tau_3, \tau_4).$$

Let us now discuss the conformal invariance of the above four–point function. Start by considering a general $SL(2, \mathbf{R})$ transformation which *preserves* the order of the points τ_1, $\ldots$, τ_4, on the real line. In this case the term (22) is invariant by itself, since it depends only on the order of the insertion points and not their specific positions. It must then be true that (24) is also invariant, and this will be the case if the coefficient $\kappa(\tau_i)$ itself is unchanged under the $SL(2,\mathbf{R})$ transformation. We first recall that four points on the real line have a unique invariant module m, with $0 < m < 1$, which can be taken to be the position of point 2 once one maps τ_1, τ_3, τ_4 to 0, 1, $+\infty$. Using the standard notation $\tau_{ij} = \tau_i - \tau_j$, the module m can also be invariantly described by the cross–ratio

$$m = \frac{\tau_{43}\tau_{21}}{\tau_{42}\tau_{31}}.$$

Let us now rewrite κ in terms of Rogers dilogarithms (see the appendix)

$$\frac{1}{2}\kappa = L\left(\frac{\tau_{21}}{\tau_{31}}\right) - L\left(\frac{\tau_{21}}{\tau_{41}}\right) + L\left(\frac{\tau_{31}}{\tau_{41}}\right) - L\left(\frac{\tau_{32}}{\tau_{42}}\right).$$

If we use the general identity (49), from the appendix, with $x = \frac{\tau_{21}}{\tau_{31}}$ and $y = \frac{\tau_{31}}{\tau_{41}}$, one quickly discovers that

$$\kappa(\tau_i) = 2L(m),$$

thus showing that the expression (23) is conformally invariant, for an *order–preserving* $SL(2,\mathbf{R})$ transformation.

One now needs to show that the full four–point function is invariant under order–changing conformal transformations. We will actually be done once we have considered the following special case. Start with the following configuration of points $\tau_1 = 0$, $\tau_2 = m$, $\tau_3 = 1$ and $\tau_4 = +\infty$. The K–dependent part of the four–point function is given by

$$V(f_1, f_2, f_3, f_4) + \frac{1}{6}L(m)\, K^{abc} \int f_1 \star \partial_a f_2 \star \partial_b f_3 \star \partial_c f_4.$$

Let us now move the point τ_4 from $+\infty$ to $-\infty$. In this case, the path integral gives

$$\begin{aligned} & V(f_4, f_1, f_2, f_4) + \frac{1}{6}L(1-m)\, K^{abc} \int f_4 \star \partial_a f_1 \star \partial_b f_2 \star \partial_c f_3 \\ = \; & V(f_4, f_1, f_2, f_4) + \frac{1}{6}\left(L(m) - 1\right) K^{abc} \int f_1 \star \partial_a f_2 \star \partial_b f_3 \star \partial_c f_4. \end{aligned}$$

One then needs to prove that

$$V(f_4, f_1, f_2, f_4) - V(f_1, f_2, f_3, f_4) = \frac{1}{6}K^{abc} \int f_1 \star \partial_a f_2 \star \partial_b f_3 \star \partial_c f_4.$$

We see that the dependence on the modulus m has dropped out and this must be the case since the LHS of the above equation depends only on the order of the points, not on their positions. The above equation is a special case of a more general formula which we shall prove in the next section, where we consider the conformal invariance of n–point functions.

4.4 General n–point Functions

We finally turn our analysis to the n–point functions by considering the path integral with n functions $f_1, \ldots, f_n$ inserted on the real line in points $\tau_1 < \cdots < \tau_n$ which are ordered from the left to the right. The unperturbed result is just,

$$\int f_1 \star \cdots \star f_n,$$

which is invariant under all diffeomorphisms of the disk. The H_{abc} dependent terms in the path integral divide as always in an expression coming from two–vertex,

$$V\left(f_1, \ldots, f_n\right) = V + \sum_{i<j} V_{ij} \ , \tag{25}$$

and a part coming from the three–vertex $\sum_{i<j<k} S_{ijk} W_{ijk}$, where we compactly write $S_{ijk} = S\left(\tau_i, \tau_j, \tau_k\right)$. We have defined the symbols W_{ijk} and S_{ijk} for $1 \leq i < j < k \leq n$, but one can extend the definition to all indices i, j, k by demanding that both W_{ijk} and S_{ijk} be totally antisymmetric tensors. Then the last contribution to the path integral is just

$$\frac{1}{6} \sum_{i,j,k} S_{ijk} W_{ijk}. \tag{26}$$

The terms W_{ijk} are not linearly independent since one can show, differentiating by parts, that

$$\sum_k W_{ijk} = 0.$$

This implies that[4] the number of independent coefficients W_{ijk} is $\binom{n-1}{3}$, and that there is a totally antisymmetric tensor, W_{ijkl}, such that $W_{ijk} = \sum_l W_{ijkl}$. Concretely, one can

[4]These facts follow from the following (trivial) cohomology computation. Let C^k be the space of totally

choose

$$W_{ijkl} = \frac{1}{n} \left(W_{ijk} - W_{ijl} + W_{ikl} - W_{jkl} \right). \tag{27}$$

Therefore equation (26) can be written as

$$\frac{1}{6} \sum_{i,j,k,l} S_{ijk} W_{ijkl} = \sum_{i<j<k<l} W_{ijkl} \left(S_{ijk} - S_{ijl} + S_{ikl} - S_{jkl} \right)$$

As we have already seen in section 4.3 (see also the appendix), the properties of the Rogers dilogarithmic function imply that, for $i < j < k < l$,

$$S_{ijk} - S_{ijl} + S_{ikl} - S_{jkl} = -2L \left(\frac{\tau_{lk}\tau_{ji}}{\tau_{lj}\tau_{ki}} \right).$$

Therefore the final result,

$$-2 \sum_{i<j<k<l} W_{ijkl} L \left(\frac{\tau_{lk}\tau_{ji}}{\tau_{lj}\tau_{ki}} \right), \tag{28}$$

is written as a function only of the cross–ratios and is therefore conformally invariant.

As in the case of the four–point function, the above reasoning is valid as long as the conformal transformation preserves the order of the points on the real line. In order to complete the proof of conformal invariance one must also consider the behavior of the full path integral as we pass one point from $+\infty$ to $-\infty$. Let us then consider the simple setup with $\tau_1, \ldots, \tau_{n-1}$ at fixed positions, and $\tau_n \to +\infty$. Then the sum (26) breaks into two parts:

$$\sum_{i<j<k<n} S_{ijk} W_{ijk} + \sum_{i<j<n} S_{ijn} W_{ijn} = \sum_{i<j<k<n} S_{ijk} W_{ijk} + \sum_{i<j<n} W_{ijn},$$

where we have used the fact that $S_{ijn} = S(0) = 1$. Let us now "move τ_n across infinity", so that $\tau_n \to -\infty$. The first term in the above expression is invariant, since it does not contain the point n, and the function f_n in W_{ijk} is not differentiated. The only change is in the term with W_{ijn}. As we move the point τ_n from $+\infty$ to $-\infty$, the coefficients

antisymmetric tensors with k indices, and let $\delta_{k+1} : C^{k+1} \to C^k$ be defined by $\delta T_{i_1 \cdots i_k} = \sum_j T_{i_1 \cdots i_k j}$. Then $\delta_{k+1}\delta_k = 0$. It is easy to show that the corresponding cohomology is trivial (see Equation 27). Therefore if $\delta_3 B = 0$, it must be that $B = \delta_4 \cdots$. Moreover, one has that $\dim \ker \delta_{k+1} = \dim C^{k+1} - \dim Im \delta_k = \dim C^{k+1} - \dim \ker \delta_k$, so that $\dim \ker \delta_3 = \dim C^3 - \dim C^2 + \dim C^1 - \dim C^0$.

W_{ijn} are multiplied not with $S(0) = 1$ but with $S(1) = -1$, so that the total expression changes by

$$2 \sum_{i<j<n} W_{ijn}.$$

The above term is purely topological, *i.e.*, it does not depend on the explicit position of the points $\tau_1, \ldots, \tau_{n-1}$, and it must be canceled by the variation of expression (25) as we change the ordering of the functions. More precisely, one must have that:

$$V(f_n, f_1, \ldots, f_{n-1}) - V(f_1, \ldots, f_n) = 2 \sum_{i<j<n} W_{ijn}.$$

To prove the above statement let us denote with $\widetilde{V}$ and $\widetilde{V}_{ij}$ the quantities corresponding to V and V_{ij}, with the functions $f_1, \ldots, f_n$ permuted to $f_n, f_1, \ldots, f_{n-1}$, so that $V(f_n, f_1, \ldots, f_{n-1}) = \widetilde{V} + \sum_{i<j} \widetilde{V}_{ij}$. It is easy to show that,

$$\begin{aligned}
\widetilde{V}_{ij} &= 2W_{i-1,j-1,n} + V_{i-1,j-1}\ , && (1 < i < j) \\
\widetilde{V}_{1j} &= -V_{j-1,n}\ . && (1 < j)
\end{aligned}$$

Also, since $\widetilde{V} - V = \frac{i}{3} B_{bc} K^{abc} \int f_1 \star \cdots \star f_{n-1} \star \partial_a f_n$ one can show that

$$\widetilde{V} - V = 2 \sum_{j<n} V_{jn}.$$

Putting everything together, one finally obtains

$$\begin{aligned}
& V(f_n, f_1, \ldots, f_{n-1}) - V(f_1, \ldots, f_n) = \\
= \ & \widetilde{V} - V + \sum_{1<i<j} \widetilde{V}_{ij} + \sum_{1<j} \widetilde{V}_{1j} - \sum_{i<j<n} V_{ij} - \sum_{j<n} V_{jn} \\
= \ & 2 \sum_{i<j<n} W_{ijn},
\end{aligned}$$

as was to be shown.

4.5 Including the Boundary Interaction S_B

In this section we are going to extend the results of the previous section by including the effects of the boundary interaction S_B in the computation of the n–point functions (16). We have not checked with path integral computations all the details of what follows, but the extension is quite natural. We will leave for future work a detailed path integral analysis of the results of this section.

It is natural in this context to change notation and to represent, as usual (see, *e.g.*, [56]), functions as operators and $\star$ products with operator multiplication. Finally, integrals $\int$ will be denoted by traces Tr. Therefore, we shall shift notation for functions as follows

$$x^a \to X^a \ , \qquad\qquad f_i \to F_i \ ,$$

and for traces as

$$\int V\left(\omega\right)\, dx \to \mathrm{Tr} \ .$$

One then has the simple correspondences:

$$\begin{aligned} \theta^{a\widetilde{a}}\partial_{\widetilde{a}} f &\to -i[X^a, F], \\ \theta^{ab} &\to -i[X^a, X^b]. \end{aligned}$$

This allows us to rewrite the expressions for V, V_{ij} and W_{ijk} in operator notation as

$$\begin{aligned} V &= -\frac{2i}{3} H_{abc} \mathrm{Tr}\left(X^a X^b X^c F_1 \cdots F_n\right), \\ V_{ij} &= -\frac{i}{6} H_{abc} \mathrm{Tr}\left(X^c F_1 \cdots [X^a, F_i] \cdots [X^b, F_j] \cdots F_n\right), \end{aligned}$$

and,

$$W_{ijk} = -\frac{i}{12} H_{abc} \mathrm{Tr}\left(F_1 \cdots [X^a, F_i] \cdots [X^b, F_j] \cdots [X^c, F_k] \cdots F_n\right).$$

We now consider the general case of $\omega_{ab}\left(x\right) = B_{ab} + F_{ab}\left(x\right)$. The expressions above are still well–defined and are the natural generalizations of the $\omega_{ab}\left(x\right) = B_{ab}$ expressions previously derived. On the other hand, for general ω, we have that:

$$\sum_k W_{ijk} = W_{ij},$$

where, for $i < j$,

$$\begin{aligned} W_{ij} &= \frac{i}{24} H_{abc} \mathrm{Tr}\left(F_1 \cdots \left[\left[X^a, X^b\right], F_i\right] \cdots [X^c, F_j] \cdots F_n\right) \\ &\quad - \frac{i}{24} H_{abc} \mathrm{Tr}\left(F_1 \cdots [X^a, F_i] \cdots \left[\left[X^b, X^c\right], F_j\right] \cdots F_n\right). \end{aligned}$$

Note that, when $[X^a, X^b] = i\theta^{ab}$ is constant, W_{ij} vanishes. In order to get a conformally invariant expression, one is the forced to replace

$$W_{ijk} \to \mathbf{W}_{ijk} = W_{ijk} - \frac{1}{n}\left(W_{ij} - W_{ik} + W_{jk}\right).$$

It is then clear that $\sum_k \mathbf{W}_{ijk} = 0$, so that the expression (using the notation of the previous sections)

$$\mathbf{W} = \sum_{i<j<k} S_{ijk} \mathbf{W}_{ijk}$$

is invariant under conformal transformations which do not change the order of the insertion points on the real line. In the case analyzed in the previous section, the term above (coming from the three–vertex) was supplemented with the term coming from the two–vertex,

$$V\left(F_1, \ldots, F_n\right) = V + \sum_{i<j} V_{ij}.$$

We recall that the above expression is important in the case when τ_n "goes around ∞". In particular, when $[X^a, X^b]$ is constant, we have that $V\left(F_n, F_1, \ldots, F_{n-1}\right) - V\left(F_1, \ldots, F_n\right) = 2\sum_{i<j<n} \mathbf{W}_{ijn}$, so that the full n–point function is conformally invariant. Again, for general $[X^a, X^b]$, we must add to $V\left(F_1, \ldots, F_n\right)$ terms which vanish for constant $[X^a, X^b]$. The simplest way to find the correct result is the following. First let us introduce a bit of notation. As in section 4.4, given any expression $\cdots$, we will denote with $\widetilde{\cdots}$ the same expression, with the functions $F_1, \ldots, F_n$ cyclically permuted to $F_n, F_1, \ldots, F_{n-1}$. In particular, for $1 < i < j < k$, one has that $\widetilde{\mathbf{W}}_{ijk} = \mathbf{W}_{i-1,j-1,k-1}$ and that $\widetilde{\mathbf{W}}_{1ij} = \mathbf{W}_{i-1,j-1,n}$. Let us then consider the following expression:

$$v\left(F_1,\ldots,F_n\right)=\frac{2}{n}\sum_{i<j<k}\left(i+j+k\right)\mathbf{W}_{ijk}.$$

A small computation shows that

$$\begin{aligned}\widetilde{v} &= \frac{2}{n}\sum_{i<j<k<n}\left(i+j+k+3\right)\mathbf{W}_{ijk}+\frac{2}{n}\sum_{i<j<n}\left(i+j+3\right)\mathbf{W}_{ijn}\\ &= v-2\sum_{i<j<n}\mathbf{W}_{ijn},\end{aligned}$$

where we have used the fact that $\sum_{i<j<k}\mathbf{W}_{ijk}=0$, which follows simply from $\sum_k\mathbf{W}_{ijk}=0$. One can then consider the combination:

$$V\left(F_1,\ldots,F_n\right)+v\left(F_1,\ldots,F_n\right). \tag{29}$$

The previous discussion implies that expression (29), in the case of constant $[X^a,X^b]=i\theta^{ab}$, is a *cyclic* function in the arguments $F_1,\ldots,F_n$. In general, though, the above need not be cyclic. We can nonetheless construct the correct generalization, $\mathbf{V}$ of $V\left(F_1,\ldots,F_n\right)$, by cyclically symmetrizing. In particular, if we define

$$\mathbf{V}=\frac{1}{n}\left[V\left(F_1,\ldots,F_n\right)+v\left(F_1,\ldots,F_n\right)+\mathrm{cyclic}_{1\cdots n}\right]-v\left(F_1,\ldots,F_n\right),$$

then this satisfies

$$\widetilde{\mathbf{V}}-\mathbf{V}=2\sum_{i<j<n}\mathbf{W}_{ijn}$$

and, following the same arguments as in section 4.4, we have restored conformal invariance. Therefore the final result for the n–point function is given by:

$$\mathbf{V}+\sum_{i<j<k}S_{ijk}\mathbf{W}_{ijk}.$$

5 Nonassociative Deformations of Worldvolumes

We are now in a position to show the importance of the Kontsevich product $\bullet$ in the above construction. In this section we shall first discuss in some detail Kontsevich products defined starting from various different bi–vector fields (in section 5.1), and then see how one can reinterpret, in this framework, the results of the last section (in 5.2 and 5.3).

Let us start the discussion by considering the simplest case when $\omega = B + F$ is constant. We are then considering the standard Moyal product deformation of the brane world–volume, which is described in [3, 5]. Physically, it corresponds to the embedding of a flat brane in a flat background space. The relevant product is the Moyal star product, given by the formula

$$(f \star g)\ (x) = e^{\frac{i}{2}\theta^{ij}\partial_i^x\partial_j^y}\ f(x)g(y)|_{x=y}. \tag{30}$$

The open string parameters can be written in terms of the closed string parameters with the formulas (43), where $\theta^{ab} = -i[x^a, x^b]_\star$. In the zero slope limit [5], correlators are computed according to:

$$\left\langle \prod_{i=1}^{n} f_i\left(X(\tau_i)\right)\right\rangle = \int \sqrt{\det\omega}\ d^{p+1}x\ f_1 \star \cdots \star f_n.$$

Now let us consider the case when $\omega\left(x\right)$ is no longer constant, but $d\omega = 0$. Then, ω still defines a symplectic structure on the brane world–volume. Physically, this corresponds to embeddings of a curved brane in a flat background space, as can be most easily seen from the Matrix theory point of view (for example this is described, in the context of holomorphic curves in flat space, in [31, 32]). Recall, in fact, that the F field represents the zero–brane density on a two–brane, such that

$$N = \frac{1}{2\pi}\int_S F$$

is the total number of zero–branes. For static solutions F is proportional to the area element, and is therefore no longer constant with respect to the Euclidean coordinates of the flat background. The zero–brane density varies along the two–brane, which in turn effectively amounts to building a curved $M2$–brane in the flat space background.

From the σ–model point of view, the case of $d\omega = 0$ is very similar to the constant one (after all, all symplectic structures are locally related by a coordinate change), but now the Moyal star product is replaced by Kontsevich's formula [30], as shown in detail in [29]. Then the star product is (we denote the noncommutative parameter by $\alpha^{ab}\left(x\right)$ in here),

$$
\begin{aligned}
f \star g &= fg + \frac{i}{2}\alpha^{ab}\partial_a f \partial_b g - \frac{1}{8}\alpha^{ac}\alpha^{bd}\partial_a\partial_b f \partial_c\partial_d g \\
&\quad - \frac{1}{12}\alpha^{ad}\partial_d\alpha^{bc}\left(\partial_a\partial_b f\partial_c g - \partial_b f \partial_a\partial_c g\right) + \mathcal{O}(\alpha^3),
\end{aligned} \tag{31}
$$

while open string parameters are still given by the same formulas. Finally, correlators are computed in the $\alpha' \to 0$ limit as follows,

$$
\left\langle \prod_{i=1}^{n} f_i\left(X(\tau_i)\right) \right\rangle = \int V(\omega) d^{p+1}x \ \left(f_1 \star \cdots \star f_n\right).
$$

The situation we analyze in detail in this paper is when $H = d\omega \neq 0$. The target is then no longer flat and one is thus embedding a curved brane in a curved background. At first sight it seems that, since one no longer has a symplectic manifold (and therefore a Poisson structure), one can no longer identify the correct algebraic structure —if any— which controls the deformation in this case. The result we have obtained is that this is not the case. As we will explain at length in this section, the Kontsevich formula is still relevant in the description of the physics. Indeed, we find that the deformation is *still* given by the Kontsevich star product expansion, as written in coordinates (we shall now denote the star product by $\bullet$ and the inverse two–form by $\widetilde{\alpha}$ in order to distinguish the two cases),

$$
\begin{aligned}
f \bullet g &= fg + \frac{i}{2}\widetilde{\alpha}^{ab}\partial_a f \partial_b g - \frac{1}{8}\widetilde{\alpha}^{ac}\widetilde{\alpha}^{bd}\partial_a\partial_b f \partial_c\partial_d g \\
&\quad - \frac{1}{12}\widetilde{\alpha}^{ad}\partial_d\widetilde{\alpha}^{bc}\left(\partial_a\partial_b f\partial_c g - \partial_b f \partial_a\partial_c g\right) + \mathcal{O}(\widetilde{\alpha}^3).
\end{aligned} \tag{32}
$$

The difference now is that the star product is no longer associative. Therefore, when in curved backgrounds, the brane world–volume is deformed not only through a noncommutative parameter ($\widetilde{\alpha} = \widetilde{\omega}^{-1}$), but also through a nonassociative parameter which —as we shall see— is essentially $H = d\widetilde{\omega}$.

Again, open string parameters are given by the same formulas as in the Moyal case (as will be later shown in section 6). As we have seen in the previous section, correlators in the topological limit $g_{ab} \to 0$ require a detailed analysis. The results can again be written, as we shall show in this section, in terms of $\bullet$ and the general formula will still be

$$\left\langle \prod_{i=1}^{n} f_i\left(X(\tau_i)\right)\right\rangle \sim \int V(\widetilde{\omega})d^{p+1}x \ \ (f_1 \bullet \cdots \bullet f_n)\,.$$

On the other hand, due to the non–associativity of $\bullet$, one has to define precisely what one means by the RHS of the above equation, which now depends explicitly on the moduli of the insertion points τ_i.

5.1 Nonassociative Star Products

We shall now study the properties of the nonassociative Kontsevich star product $\bullet$. In the last part of this section we will work in the $g_{ab} \to 0$ limit, so that $\alpha = \omega^{-1}$, $\widetilde{\alpha} = \widetilde{\omega}^{-1}$. Also, unless explicitly needed, we shall drop the tildes.

Let us start by considering the associativity properties of the Kontsevich expansion (31) or (32). To this end one needs to compute, given three generic functions f, g, h, the difference $(f \star g) \star h - f \star (g \star h)$. Using the expansions (31), (32), it is not difficult to show that:

$$(f \star g) \star h - f \star (g \star h) = \frac{1}{6}\left(\alpha^{i\ell}\partial_\ell \alpha^{jk} + \alpha^{j\ell}\partial_\ell \alpha^{ki} + \alpha^{k\ell}\partial_\ell \alpha^{ij}\right)(\partial_i f \partial_j g \partial_k h) + \mathcal{O}(\alpha^3). \quad (33)$$

If α is a Poisson structure, *i.e.*, satisfies

$$\alpha^{i\ell}\partial_\ell \alpha^{jk} + \alpha^{j\ell}\partial_\ell \alpha^{ki} + \alpha^{k\ell}\partial_\ell \alpha^{ij} = 0,$$

then the associated product is associative (in fact to all orders). Note that, when α is invertible, the above equation is equivalent to $d\left(\alpha^{-1}\right) = d\omega = 0$, so that the expansion (31) defines an associative product.

Now consider the product (32). In this case one has both a noncommutative deformation, with parameter $\widetilde{\alpha}$, and a nonassociative deformation with parameter $H = d\widetilde{\omega}$. To better understand this point let us re–write expression (33) in terms of the 3–form field H. Indeed, using that $\partial_k \widetilde{\alpha}^{ij} = \widetilde{\alpha}^{ia}\widetilde{\alpha}^{jb}\partial_k \widetilde{\omega}_{ab}$, one can rewrite (33) as

$$(f \bullet g) \bullet h - f \bullet (g \bullet h) = \frac{1}{6}\,\widetilde{\alpha}^{ia}\widetilde{\alpha}^{jb}\widetilde{\alpha}^{kc}\,H_{abc}\,\partial_i f\,\partial_j g\,\partial_k h + \cdots. \quad (34)$$

Precisely because we have $H \neq 0$, the star product (32) is *not* associative.

We then have two products, $\star$ and $\bullet$, given by the Kontsevich expansion in terms of $\omega_{ab}(x)$ and $\widetilde{\omega}_{ab}(x) = \omega_{ab}(x) + \frac{1}{3}H_{abc}x^c$, respectively. We wish to explicitly relate the product $\bullet$ to the associative product $\star$. First, it is clear that:

$$\widetilde{\alpha}^{ij} = \alpha^{ij} + \frac{1}{3}\alpha^{ai}\alpha^{bj}H_{abc}x^c + \cdots.$$

Therefore one has $f \bullet g = f \star g + \frac{i}{6}\alpha^{ai}\alpha^{bj}H_{abc}x^c\partial_i f\partial_j g + \cdots$. Recalling that $[x^a, f] = i\alpha^{ai}\partial_i f + \cdots$, it is not hard to show:

$$f \bullet g = f \star g - \frac{i}{12}H_{abc}\left\{x^c, [x^a, f]_\star \star [x^b, g]_\star\right\}_\star.$$

One can check the correctness of the above formula by expanding equation (32) to order α^2. It is moreover convenient, as in section 4.5, to move to operator notation for the associative product $\star$. Therefore the above expression for the product $\bullet$ can be compactly written as

$$F \bullet G = FG - \frac{i}{12}H_{abc}\left\{X^c, [X^a, F][X^b, G]\right\}. \tag{35}$$

It is then simple to show that:

$$(F \bullet G) \bullet H - F \bullet (G \bullet H) = -\frac{i}{6}\; H_{abc}\; [X^a, F][X^b, G][X^c, H],$$

which is the generalization of (34) to all orders in α.

Let us now take the functions f, g and h to be the local coordinate functions x^i in $\mathbf{R}^n$. By direct use of the nonassociative Kontsevich formula (32) one obtains,

$$x^i \bullet x^j = x^i x^j + \frac{i}{2}\widetilde{\alpha}^{ij}(x), \tag{36}$$

and from (34),

$$\left(x^i \bullet x^j\right) \bullet x^k - x^i \bullet \left(x^j \bullet x^k\right) = \frac{1}{6}\,\widetilde{\alpha}^{ia}\widetilde{\alpha}^{jb}\widetilde{\alpha}^{kc}\,H_{abc}. \tag{37}$$

Calculating the star bracket commutator (making use of (36)) one obtains the noncommutative algebra,

$$[x^i, x^j]_\bullet = i\widetilde{\alpha}^{ij}(x),$$

which is a very similar result to the standard Kontsevich deformation. On the other hand, in order to compute the Jacobi expression, one uses (37) to obtain:

$$[x^i, [x^j, x^k]_\bullet]_\bullet + [x^j, [x^k, x^i]_\bullet]_\bullet + [x^k, [x^i, x^j]_\bullet]_\bullet = -\widetilde{\alpha}^{ia}\widetilde{\alpha}^{jb}\widetilde{\alpha}^{kc} H_{abc},$$

which is a violation of the Jacobi identity.

5.2 Operator Product Expansions and Factorization

In the previous subsection we have reviewed the basic properties of the nonassociative Kontsevich product $\bullet$. We may now use the general results of section 4.5 in order to show the relevance of the algebraic operation $\bullet$ in the computation of n–point functions. Let us then first summarize the results of section 4. We have constructed n–point functions $P_n[F_1, \ldots, F_n]$ which depend uniquely on the $n-3$ conformal moduli of the insertion points τ_i of the functions F_i. In particular, the one–point function P_1, which we shall call P in the sequel, is a generalization of the trace, Tr, and is given by:

$$P_1[F] = P[F] = \mathrm{Tr}(F) - \frac{2i}{3} H_{abc} \mathrm{Tr}\left(X^a X^b X^c F\right).$$

Now consider a general n–point function for functions F_i at points τ_i. Let us scale the insertion points $\tau_i \to \varepsilon\, \tau_i$ for $\varepsilon \to 0$. On one hand the result of the n–point function does not change, since it is invariant under $SL(2, \mathbf{R})$ transformations. On the other hand we can use an OPE argument to conclude that there must exist a function, $O_n[F_1, \ldots, F_n](\tau_i)$, such that

$$P_n[F_1, \ldots, F_n](\tau_i) = P[O_n[F_1, \ldots, F_n](\tau_i)]. \tag{38}$$

The operations $O_n[F_1, \ldots, F_n](\tau_i)$ are —informally— untraced versions of the P_n's, and are invariant under the subgroup of $SL(2, \mathbf{R})$ which leaves the point at ∞ invariant, *i.e.*, translations and rescalings. They will depend on $n-2$ moduli. In particular one can now see the relevance of the $\bullet$ product, which is nothing but the operation O_2. More precisely, one can check that

$$\begin{aligned} O_1[F](\tau) &= F, \\ O_2[F, G](\tau_1, \tau_2) &= F \bullet G, \end{aligned}$$

where, for the second expression, it is simple to use its explicit expansion, (35), and insert it in (38) in order to check that one does get the right result, $P_2[F,G]$, as derived in section 4.5. With a little more work, and using the results on the n–point functions of section 4 and the facts on the $\bullet$ product of the previous subsection, one can likewise obtain,

$$\begin{aligned} O_3\left[F,G,H\right](\tau_i) &= L\left(1-m\right)\left(F\bullet G\right)\bullet H + L\left(m\right) F\bullet\left(G\bullet H\right), \\ m &= \frac{\tau_{21}}{\tau_{31}}. \end{aligned}$$

In particular O_3, which depends on a single modulus, is explicitly written in terms of the product $\bullet$ and interpolates between the two possible positionings of the parenthesis. More generally, the operations O_n will depend on $n-2$ moduli and will interpolate between the various possible ways of taking products of n functions with the $\bullet$ product.

OPE arguments can be used, in the general case, to compute n–point functions at the boundary of the moduli space of the insertion points τ_i. Again consider an n–point function with functions F_i at points τ_i. Let a subset of the points —say $\tau_1, \ldots, \tau_m$— converge to zero via a common rescaling $\tau_i \to \varepsilon\tau_i$, $i = 1, \ldots, m$. Then one can use an OPE argument to show that (the indices i and j indicate the two sets $1, \ldots, m$, and $m+1, \ldots, n$, respectively)

$$\lim_{\varepsilon\to 0} P_n\left[F_1,\ldots,F_n\right](\varepsilon\tau_i,\tau_j) = P_{n-m+1}\left[O_m\left[F_1,\ldots,F_m\right](\tau_i), F_{m+1},\ldots,F_n\right](0,\tau_j). \quad (39)$$

For example, one can show that:

$$P\left[(F\bullet G)\bullet H\right] = P\left[F\bullet(G\bullet H)\right].$$

This follows from applying (39) and recalling the fact that the three–point function $P_3\left[F,G,H\right](\tau_1,\tau_2,\tau_3)$ is independent of the moduli. If one considers the two limits $\tau_2 \to \tau_1$ and $\tau_2 \to \tau_3$, and uses factorization with $O_2 \sim \bullet$, one quickly arrives at the above result.

5.3 The Homotopy Associative Algebraic Structure

We have seen that the operations O_n define, as a function of the modular parameters, a structure which extends that of an associative algebra. In fact, the failure of the product

$\bullet \sim O_2$ to be associative is measured by O_3, which now interpolates (thanks to the modular parameter $0 < m < 1$) between the two possible "placements" of the parenthesis. In a very crude sense, the nonassociativity at each order is controlled by higher order terms. These type of structures have appeared in the literature on string theory, starting from the use in string field theory of the Batalin–Vilkovisky formalism to quantize gauge theories which do not close off-shell [34, 35, 36, 37]. These are the A_∞ homotopy associative algebras, where the failure of the associativity property is controlled by a third order term, and similarly at higher orders [33].

Let us formalize these concepts a bit further, and show that the structure $C^\infty(M)$ and $O_n[F_1, \ldots, F_n](\tau)$ is actually that of an A_∞ space[5]. The idea of an A_∞ space is the same as that of an A_∞ algebra, only the definition of homotopy is changed (one uses a map instead of a differential). So we first follow the original work of Stasheff [33] and recall the definition of homotopy associativity. The intuitive notion is the following. A space X and a multiplication $m : X \times X \to X$ is a homotopy associative space if the maps $m(\mathbf{1} \otimes m)$ and $m(m \otimes \mathbf{1})$ are homotopic as maps $X \times X \times X \to X$. If we are given three functions, F_1, F_2 and F_3, with the nonassociative product $\bullet$ there are two distinct ways to insert parenthesis in the natural application $C^\infty(M) \times C^\infty(M) \times C^\infty(M) \to C^\infty(M)$, *i.e.*, the standard options $(F_1 \bullet F_2) \bullet F_3$ and $F_1 \bullet (F_2 \bullet F_3)$. But a quick reminder of the previous section also tells us that there is a homotopy, $O_3[F_1, F_2, F_3](m) : [0, 1] \times C^\infty(M)^{\times 3} \to C^\infty(M)$, between these two seemingly distinct ways to associate brackets under the $\bullet$ product.

In order to realize that there are stronger conditions of "associativity modulo homotopy" than the previous one, let us proceed by analyzing the situation with four functions, $F_1, \ldots, F_4$. There are now five distinct ways to insert parenthesis for the nonassociative product,

$$\begin{array}{ccc} & (F_1 \bullet F_2) \bullet (F_3 \bullet F_4), & \\ ((F_1 \bullet F_2) \bullet F_3) \bullet F_4, & & F_1 \bullet (F_2 \bullet (F_3 \bullet F_4)), \\ (F_1 \bullet (F_2 \bullet F_3)) \bullet F_4, & & F_1 \bullet ((F_2 \bullet F_3) \bullet F_4), \end{array}$$

which can actually be pictorially written at the vertices of a pentagon. The point is now that while the O_3 homotopy naturally yields homotopies that run between the vertices, it is not necessarily true that one can extend the homotopy to the interior of the pentagon. If

[5]In here we take M to be the brane world–volume.

one can *not* extend the homotopy to this situation, the algebraic structure of the product is denoted A_3. If, on the other hand, one *can* extend the homotopy to the interior of the whole pentagon, the algebraic structure is denoted A_4. As we go further along this way one is led to consider higher poliedra, and if one can always extend homotopies to the interior of these poliedra, then the algebraic structure is A_∞ homotopy associative [33].

Let us illustrate these concepts by explictly writing down the $O_4[F_1,\ldots,F_4](x,y)$ operation. One can compute it to be

$$\begin{aligned}
O_4[F_1,\ldots,F_4](x,y) &= L\left[\left(1-\frac{x}{y}\right)\left(1-\frac{1-y}{1-x}\right)\right](F_1\bullet F_2)\bullet(F_3\bullet F_4)\\
&\quad +L\left[\left(1-\frac{x}{y}\right)\left(\frac{1-y}{1-x}\right)\right]((F_1\bullet F_2)\bullet F_3)\bullet F_4\\
&\quad +L\left[\frac{x}{y}\left(1-\frac{1-y}{1-x}\right)\right]F_1\bullet(F_2\bullet(F_3\bullet F_4))\\
&\quad +L\left[\frac{x}{y}\left(1-y\right)\right](F_1\bullet(F_2\bullet F_3))\bullet F_4\\
&\quad +L\left[x\left(\frac{1-y}{1-x}\right)\right]F_1\bullet((F_2\bullet F_3)\bullet F_4),\\
x &= \frac{\tau_{21}}{\tau_{41}}\,, \qquad y=\frac{\tau_{31}}{\tau_{41}},
\end{aligned}$$

where $0<x<y<1$. At first sight one would say that $\{x,y\}$ take values in a triangle. However, a glance at the expression above also tells us that while one of the vertices of this triangle, $\{x,y\}=\{0,1\}$, is perfectly regular, the other two, $\{x,y\}=\{0,0\}$ and $\{x,y\}=\{1,1\}$, are actually singular. Each of these singular points can actually be resolved into two distinct limits, once we scale x and y in the two possible different ways. For instance, the limit where $\{x,y\}\to\{0,0\}$ can be approached with both x and y scaling as $\epsilon\to 0$, with $\frac{x}{y}\to 1$ or with x scaling as ϵ^2 and y as ϵ, and $\frac{x}{y}\to 0$. A similar situation occurs for the limit where $\{x,y\}\to\{1,1\}$. So, the resolution of the singular vertices of the triangle actually produces the expected pentagon. Once this is realized, it is simple to see that O_4 plays the role of the A_4 homotopy,

$$O_4[F_1,\ldots,F_4](x,y):\mathcal{P}\times C^\infty(M)^{\times 4}\to C^\infty(M),$$

where $\mathcal{P}$ is the pentagon spanned by x and y. Observe that, as explained in the previous section, $P[O_4]=O_4$. It is then very natural to conjecture that general n–point functions will thus produce the necessary homotopies in order that $(C^{\infty}(M), \bullet, \{O_n\}_{n=2}^{\infty})$ is an A_∞ homotopy associative algebra (and where $P_n = P[O_n]$).

The homotopies $O_n[F_1, \ldots, F_n](\tau)$ also induce the necessary homotopies to create an L_∞ commutator homotopy Lie algebra, and to create homotopy differential operators (which may contain non–trivial topological information for the tensor bundles of M). Indeed, the commutator algebra, $[x^i, x^j]_\bullet$, is an L_∞ homotopy Lie algebra: using the basic homotopy, $O_3[F, G, H](m)$, one can define a "composite" homotopy between zero and $-\frac{i}{6} H_{abc}[X^a, F][X^b, G][X^c, H]$, the term that violates Jacobi's identity in the $\bullet$ commutator algebra. With this homotopy, Jacobi's identity will be satisfied up to homotopy, and one thus obtains an L_∞ homotopy Lie algebra. In order to build a differential structure (and thus, gauge theory) one still needs a covariant derivative in the sense that $\nabla(F \bullet G) = \nabla F \bullet G + F \bullet \nabla G$. While this may not seem as a viable course of action, one can use *homotopy* to impose the Leibnitz rule: the derivative operation $\nabla_X F \sim [X, F]$ will satisfy the Leibnitz rule *up to* homotopy. Again using the basic homotopy one can define a composite homotopy between $[X, F \bullet G]$ and $[X, F] \bullet G + F \bullet [X, G]$, so that the commutator $[X, F]$ becomes a homotopy derivative for the $\bullet$ product.

6 Corrections Involving the Metric Tensor

Up to now we have studied the topological limit $g_{ab} \to 0$ in great detail. In these last sections we shall discuss corrections to the above results, when one includes a non–vanishing closed string metric g_{ab} in the calculations. This will allow us to identify the open string effective parameters —metric G^{ab} and noncommutative parameter θ^{ab}— in terms of the closed string parameters g_{ab} and B_{ab}.

Recall that the two–point function of the fluctuating field ζ is

$$\langle \zeta^a(z) \zeta^b(\tau) \rangle = \frac{i}{\pi} \theta^{ab} \mathcal{A}(z, \tau) - \frac{1}{\pi} G^{ab} \mathcal{B}(z, \tau),$$

where

$$\begin{aligned} \mathcal{A}(z, \tau) &= \frac{1}{2i} \ln\left(\frac{\tau - z}{\overline{z} - \tau}\right), \\ \mathcal{B}(z, \tau) &= \ln|z - \tau|, \end{aligned}$$

and where we have placed the point τ at the world–sheet boundary $\partial\Sigma$. In the previous sections we have worked only with the term in $\mathcal{A}(z,\tau)$. Here, we will evaluate the two–point function $P_2(x^a,x^b)$ including the contribution arising from the term in $\mathcal{B}(z,\tau)$. The only diagrammatic contribution still comes from the two–graph $\mathcal{V}$, (14), so that one can easily compute the relevant Feynman diagrams. First, observe that the contribution of the $\mathcal{B}(z,\tau)$ term to the volume form $V(\omega)$ is proportional to $\propto H_{abc}G^{ab}$ and therefore vanishes due to the antisymmetry of H_{abc}. Thus, the volume form is unchanged.

Schematically, the propagator looks like $\mathcal{A}-\mathcal{B}$. In the previous sections we computed the correction to the two–point function going like $\mathcal{A}^2$, with the result:

$$\frac{i}{3\pi}\theta^{ia}\theta^{jb}H_{abc}x^c\mathcal{A}\left(\tau_1,\tau_2\right),$$

therefore yielding a correction to the noncommutative θ parameter as,

$$\theta^{ij}\to\theta^{ij}+\frac{1}{3}\theta^{ia}\theta^{jb}H_{abc}x^c. \tag{40}$$

To this result we now add the correction to the two–point function which goes as $\mathcal{B}^2$,

$$-\frac{i}{3\pi}G^{ia}G^{jb}H_{abc}x^c\mathcal{A}\left(\tau_1,\tau_2\right).$$

These two results produce the full correction to the noncommutative parameter,

$$\theta^{ij}\to\theta^{ij}-\frac{1}{3}G^{ia}G^{jb}H_{abc}x^c+\frac{1}{3}\theta^{ia}\theta^{jb}H_{abc}x^c. \tag{41}$$

Finally, there are also mixed corrections going as $\mathcal{AB}$ (and also the "symmetric" $\mathcal{BA}$). They are,

$$-\frac{1}{3\pi}\theta^{ia}G^{jb}H_{abc}x^c\mathcal{B}\left(\tau_1,\tau_2\right),$$

plus the symmetric contribution in i and j. These contributions yield the correction to the effective open string metric,

$$G^{ij}\to G^{ij}-\frac{1}{3}\theta^{ia}G^{jb}H_{abc}x^c+\frac{1}{3}G^{ia}\theta^{jb}H_{abc}x^c. \tag{42}$$

All these results can be nicely combined with the flat spaces formulas which connect closed string and open string parameters [3, 5], to yield new but identical formulas in this curved background scenario.

6.1 Open String Parameters

Let us first recall how, in the flat case, one relates open and closed string parameters [3, 5]:

$$
\begin{aligned}
\frac{1}{G} &= \frac{1}{g+\omega}\, g\, \frac{1}{g-\omega}\ , \\
\theta &= -\frac{1}{g+\omega}\, \omega\, \frac{1}{g-\omega}\ .
\end{aligned} \tag{43}
$$

In here, $\omega = B + F$ is constant, G_{ij} is the metric effectively seen by the open strings and θ^{ij} is the noncommutativity parameter on the brane world–volume (we never consider noncommutativity along the time direction), $[x^i, x^j] = i\theta^{ij}$.

Let us now consider the above formulae (43) with ω replaced with the curved background expression,

$$\widetilde{\omega}_{ab}(x) = \omega_{ab} + \frac{1}{3} H_{abc} x^c,$$

and let us expand (43) to first order in H. Denoting by $\widetilde{G}$ and $\widetilde{\theta}$ the "curved" open string parameters, one obtains:

$$
\begin{aligned}
\frac{1}{\widetilde{G}} + \widetilde{\theta} &= \frac{1}{g+\omega+\frac{1}{3}Hx} \\
&\simeq \frac{1}{g+\omega} - \frac{1}{g+\omega}\, \frac{1}{3}Hx\, \frac{1}{g+\omega} \\
&= \left(\frac{1}{G} - \frac{1}{G}\, \frac{1}{3}Hx\ \theta - \theta\ \frac{1}{3}Hx\ \frac{1}{G}\right) + \left(\theta - \frac{1}{G}\ \frac{1}{3}Hx\ \frac{1}{G} - \theta\ \frac{1}{3}Hx\ \theta\right).
\end{aligned}
$$

It is clear that we have just obtained the previous results (42) and (41). Therefore, formulas (43) are still valid in the curved background situation, but now the fields are taken to be varying fields rather than constant fields. In other words, formulas (43) are still valid for weakly varying non–closed gauge invariant two–form $\widetilde{\omega}$.

In particular, in the zero slope $\alpha' \to 0$ limit of [5], the effective open string parameters are given by

$$\frac{1}{\widetilde{G}} = -\frac{1}{\widetilde{\omega}}\, g\, \frac{1}{\widetilde{\omega}}\ , \qquad \widetilde{\theta} = \frac{1}{\widetilde{\omega}}\ . \tag{44}$$

7 Tachyons and Matrix Models in Curved Backgrounds

In this section we analyze in some detail the zero–point function, *i.e.*, the partition function, and connect the discussion of this paper, in this simple case, to some known results in Matrix models.

Let us start by considering the Born–Infeld action in the presence of a weak background field H. We shall be brief, since the arguments which follow are very well known. One starts by expanding the determinant in order to obtain

$$S = \int \sqrt{\det\left(\delta_{ab} + \frac{1}{3}H_{abc}x^c + F_{ab}\right)} \simeq \int \left(1 + \frac{1}{4}F^2 + \frac{1}{6}H_{abc}x^cF_{ab}\right).$$

We can then use the canonical correspondences $F_{ab} \to i[X^a, X^b]$ and $\int \to \mathrm{Tr}$ in order to rewrite the RHS above as

$$\begin{aligned} S &\simeq \mathrm{Tr}\left(1 - \frac{1}{4}g_{ac}g_{bd}[X^a, X^b][X^c, X^d] + \frac{i}{3}H_{abc}X^aX^bX^c\right) \\ &\simeq \mathrm{Tr}\left(1 + \frac{i}{3}H_{abc}X^aX^bX^c\right) + \mathcal{O}(g^2). \end{aligned} \tag{45}$$

Note that the above action has been found by [10, 11] in the context of studies of branes in WZW models at large level k —that is, at small $H_{abc} \sim k^{-1/2}$— and by Myers in [38] in the context of studies of polarization of lower–dimensional branes in the presence of R–R background fields. In the sequel we will just look at the terms in the above equation which are independent of g_{ab}, and concentrate on the linear terms in H_{abc}. A special case of the results of this paper is the zero–point function, or partition function,

$$Z = P[1] = \mathrm{Tr}\left(1 - \frac{2i}{3}H_{abc}X^aX^bX^c\right). \tag{46}$$

At first sight, there is an incompatibility between equations (45) and (46), since one expects that $Z \sim S$. We have, on the other hand, a difference

$$S - Z \simeq H_{abc}\mathrm{Tr}\left(X^aX^bX^c\right). \tag{47}$$

Recall though, see *e.g.* [58, 59, 60], that the partition function and the action need not be equal and are expected to differ by a renormalization group beta function contribution. More precisely,

$$S = (1 + \beta \frac{\partial}{\partial g}) Z.$$

We will show that the difference (47) is nothing but this extra term, thus resolving the apparent contradiction.

Recall that the coefficient $-2i/3$ in equation (46) was fixed in section 4 in order to obtain conformally invariant results. This corresponded to an ill defined vacuum graph (which contributes to the volume form) which is linearly divergent. In this paper we have chosen a *specific* regularization scheme which preserves the conformal invariance of the results. This scheme, however, does *not* correspond to the usual minimal subtraction scheme, as we will show in a moment, and contributes a finite part to the tachyon beta function, thus explaining the difference (47).

Let us be more specific. Let us consider the more general boundary interaction S_B, including the tachyon field,

$$\int d\tau \left[\frac{1}{2\pi} T_B\left(X\right) + iA_a\left(X\right)\dot{X}^a \right].$$

In the above, T_B is the *bare* tachyon field, given by $T_B = T + \Delta T$, where ΔT are the tachyon counterterms. Now let us consider the vacuum graph in question and let us regularize it following, *e.g.*, the prescription of [60]. The graph then contributes (including the tachyon counterterm):

$$-\Delta T - \frac{i}{6} H_{abc} x^c \int d\tau \, \langle \zeta^a \dot{\zeta}^b \rangle. \tag{48}$$

Working on the disk and regularizing [60] the result, one finds that:

$$\int d\tau \, \langle \zeta^a \dot{\zeta}^b \rangle = 2i\theta^{ab} \frac{e^{-2\varepsilon}}{1 - e^{-2\varepsilon}} = i\theta^{ab} \left[\frac{1}{\varepsilon} - 1 + \mathcal{O}\left(\varepsilon\right) \right].$$

Therefore one obtains that equation (48) yields a result of

$$-\Delta T - \frac{i}{3} H_{abc} X^a X^b X^c \left[\frac{1}{\varepsilon} - 1 \right],$$

where we have used that $\theta^{ab} = -i[X^a, X^b]$. The usual prescription is the one of minimal subtraction, *i.e.*, the counterterm just cancels the pole leaving a finite result of $\frac{i}{3} H_{abc} X^a X^b X^c$. We choose, on the other hand, a different renormalization prescription dictated by conformal invariance, which gives as a total contribution $-\frac{2i}{3}H_{abc}X^aX^bX^c$. This implies that the tachyon counterterm must be

$$\Delta T = -i\left(\frac{1}{3\varepsilon} - 1\right) H_{abc}X^aX^bX^c,$$

and that the corresponding beta function is

$$\beta_T = -T - H_{abc}X^aX^bX^c.$$

Following the methods of [58, 59, 60], this contribution to the beta function implies that the total action, including the tachyon potential, is given by

$$\mathrm{Tr}\left[\left(1 + T + \frac{i}{3}H_{abc}X^aX^bX^c\right)e^{-T}\right],$$

which is extremized at $T = -\frac{i}{3}H_{abc}X^aX^bX^c$. The value of the action at the extremum is then

$$\mathrm{Tr}\left(1 + \frac{i}{3}H_{abc}X^aX^bX^c\right),$$

thus showing that the difference between the partition function and the action is compensated by a condensation of the tachyon.

7.1 11–Dimensional Language and the 3–form Field

In order to be complete, one still needs to relate the previous Matrix theory action, which is written in the 10–dimensional type IIA language, to the full M–theory 11–dimensional language. Using the 11-dimensional light–cone notation and the previous operator form of the action, we have actually built a Matrix theory action in a weakly curved background. One can use the ideas from [49, 50], and their application to curved backgrounds [61], to make precise the relation between the Matrix theory and the D–brane Born–Infeld actions. We shall now briefly look at these issues, with a particular attention to the 11–dimensional 3–form field.

Let us start by considering M–theory with a background metric g_{IJ}, in a frame with a compact coordinate X^- of size R, which is light–like in the flat space limit $g_{IJ} \to \eta_{IJ}$. This theory can be described as the limit of a family of space–like compactified theories

[49, 50]. Define the theory $\widetilde{M}$ with background metric $\widetilde{g}_{IJ}$ in a frame with a space–like compact direction X^{10} of size $\widetilde{R}$. The DLCQ limit of the original theory, M, can be found by boosting the theory $\widetilde{M}$ in the X^{10} direction with boost parameter,

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \sqrt{1+\frac{1}{2}\left(\frac{R}{\widetilde{R}}\right)^2},$$

and then taking the limit $\widetilde{R} \to 0$. The 3–form field $\widetilde{A}_{IJK}$ in the theory $\widetilde{M}$ is related to that of the original theory M by the same Lorentz transformation as above. Moreover, the theory $\widetilde{M}$ on a small space–like circle of radius $\widetilde{R}$ is equivalent to type IIA string theory with background form fields given by:

$$\widetilde{A}_{IJK} d\widetilde{X}^I \wedge d\widetilde{X}^J \wedge d\widetilde{X}^K = C^{D2}_{\mu\nu\rho} dX^\mu \wedge dX^\nu \wedge dX^\rho + B_{\mu\nu} dX^\mu \wedge dX^\nu \wedge dX^{10}.$$

The configurations of interest in this paper carry no $D0$ or $D2$–brane charge. One will thus be left with a $B_{\mu\nu}$ background form field which will give rise to the following M–theory background 3–form field (here we take $\alpha \equiv \gamma(1-\beta)$):

$$A_{+-i} = 0 \ , \quad A_{ijk} = 0 \ , \quad A_{+ij} = \frac{1}{\sqrt{2}\alpha} B_{ij} \ , \quad A_{-ij} = -\frac{\alpha}{\sqrt{2}} B_{ij} \ ,$$

where one should recall that we have only space–space B–field turned on.

It is interesting to observe that, at the M–theory level, the nonassociative Kontsevich deformation obtained is associated to the A_{+ij} and A_{-ij} components of the 11–dimensional field A_{IJK} (see also [62] for the standard Moyal deformation). One is therefore led to speculate that, from a purely M–theoretic point of view, one should be able to construct deformations associated to 3–index tensor structures, which should reduce to Kontsevich type of deformations for configurations considered in this paper. This is an interesting venue to explore, as it may also aid in understanding brane world–volume deformations associated to non–zero varying R–R fields (and R–R field strengths). One thing one can say is that 3–index tensor structures will probably be naturally associated to 3–component products or 3–brackets in the sense that one can write, given some (constant) tensor C^{ijk},

$$\{f,g,h\}(x) = e^{\frac{i}{2}C^{ijk}\partial^x_i \partial^y_j \partial^z_k} f(x)g(y)h(z)|_{x=y=z}.$$

Structures involving 3–brackets have been previously discussed in the context of covariant Matrix theory actions [63]. It would be interesting to further explore these ideas.

8 Future Perspectives

In this paper we have shown how to use open string perturbation theory in order to describe brane physics in weakly curved backgrounds. The method described allows one to translate the properties of the curved background —which traditionally show up as sigma model couplings— into a given deformation of the algebra of functions on the brane world–volume, which depends in general on the specific closed string background considered. In particular, the presence of an NS–NS field strength H induces a nonassociative deformation of the algebra of functions.

Our choice of background is of the type $R + \frac{1}{4}H^2 = 0$, and it would be interesting to further develop the disk perturbation theory in order to investigate the properties of the star product deformation to higher order. It would also be interesting to study tachyon condensation in such a background. Given that one can compute correlators using star product prescriptions (as thoroughly explained in this work), one can then use standard boundary string field techniques in order to compute the minima of the tachyon potential in this background.

Another interesting point is to further study the $\bullet$ product. Defining gauge theory on these "nonassociative manifolds" is not straightforward. Also, given the discussion about Matrix theory in curved backgrounds, it seems clear that understanding the geometry of these "nonassociative manifolds", much like there is a geometrical understanding of Kontsevich's noncommutative manifolds [64, 65, 66], would be needed in order to fully understand Matrix theory in any given background. More pragmatically, we were able to map functions to matrices because we wrote everything in terms of the $\star$ product, which is associative. A question that immediately rises is whether there is a "matrix" formulation of the theory which can be written exclusively in terms of the $\bullet$ product. Answering this question could be of great interest for the goal of defining Matrix theory in general curved backgrounds. This definitely requires a full understanding of the role of homotopy associative algebras. We hope to address some of these questions in the near future.

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A Dilogarithm Identities

Let us recall that the standard dilogarithmic function is defined by the following expression:

$$\mathrm{Li}_2(x) = -\int_1^x \frac{\ln(1-s)}{s} ds = \sum_{n=1}^{\infty} \frac{1}{n^2} x^n.$$

In particular $\mathrm{Li}_2(0) = 0$ and $\mathrm{Li}_2(1) = \zeta(2) = \frac{1}{6}\pi^2$. A related function, more useful for our purposes, is the Rogers normalized dilogarithm [57] $L(x)$ defined for $0 \leq x \leq 1$ by

$$L(x) = \frac{6}{\pi^2}\left[\mathrm{Li}_2(x) + \frac{1}{2}\ln(x)\ln(1-x)\right].$$

The Rogers dilogarithm is monotonically increasing on $[0,1]$ and, at the end points of the interval, is given by

$$L(0) = 0 \ , \qquad L(1) = 1.$$

The function L satisfies a fundamental property, which is crucial in our computations, namely, given $x, y \in [0,1]$ the following holds:

$$L(x) + L(y) = L(xy) + L\left(\frac{x(1-y)}{1-xy}\right) + L\left(\frac{y(1-x)}{1-xy}\right). \tag{49}$$

A special important case of the above equation is Euler's identity,

$$L(x) + L(1-x) = 1.$$

B Computation of the Function $S(x)$

Let us start the computation of $S(x)$ by analyzing an auxiliary function, which we shall denote by $s(x)$, and which will be defined for all $x \in \mathbf{R}$. Let us consider three points 0, 1 and x on the real line and let us denote, given a point p in the upper–half plane, with α, β and γ the angles formed by the lines p–0, p–1 and p–x with the vertical line. A little plane geometry shows that:

$$\tan\gamma = (1-x)\tan\alpha + x\tan\beta. \tag{50}$$

The function s will then be given by:

$$s(x) = \frac{4}{\pi^3} \int_\Sigma \gamma d\alpha \wedge d\beta.$$

From the geometric construction it is clear that

$$s(1-x) = -s(x).$$

We also recall that the upper–half plane Σ corresponds to the simplex $-\frac{\pi}{2} \leq \alpha \leq \beta \leq \frac{\pi}{2}$ in the α–β plane. This fact can be used to compute special values of s,

$$\begin{aligned} s(+\infty) &= -s(-\infty) = 1 \ , \\ s(1) &= -s(0) = \frac{1}{3} \ , \end{aligned} \tag{51}$$

and $s(1/2) = 0$. It is clear that, from the definition of S, one has for $x \in [0,1]$,

$$S(x) = s\left(\frac{1}{x}\right) - s(x) + s\left(\frac{-x}{1-x}\right). \tag{52}$$

Consider the derivative $\frac{d}{dx} s(x)$. From (50) we deduce that:

$$\frac{d}{dx}\gamma = \frac{\tan\beta - \tan\alpha}{1 + ((1-x)\tan\alpha + x\tan\beta)^2} \ ,$$

so that,

$$\begin{aligned} \frac{d}{dx}s &= \int_\Sigma \frac{d}{dx}\gamma d\alpha \wedge d\beta = \\ &= \int_{-\infty<z<w<\infty} \frac{dzdw}{(1+z^2)(1+w^2)} \frac{w-z}{1+((1-x)z+xw)^2} \ , \end{aligned}$$

where we have defined $z = \tan\alpha$, $w = \tan\beta$. The above integral can be evaluated with the result:

$$\frac{d}{dx}s = -\frac{1}{\pi^2}\left[\frac{\ln(x)^2}{1-x} + \frac{\ln(1-x)^2}{x}\right].$$

The above equation can be easily integrated by noting that $\frac{d}{dx}\mathrm{Li}_2(x) = -\frac{\ln x}{1-x}$. Using the boundary values (51) one obtains the following expression for S,

$$
\begin{aligned}
s &= -\frac{1}{3}+\frac{4}{\pi^2}\left[\mathrm{Li}_2\left(x\right)+\frac{1}{2}\ln\left(-x\right)\ln\left(1-x\right)\right] \ , && (x<0)\\
s &= \frac{2}{\pi^2}\left[-\mathrm{Li}_2\left(1-x\right)+\mathrm{Li}_2\left(x\right)\right] \ , && (0<x<1)\\
s &= \frac{1}{3}-\frac{4}{\pi^2}\left[\mathrm{Li}_2\left(1-x\right)+\frac{1}{2}\ln\left(x\right)\ln\left(x-1\right)\right] \ . && (x>0)
\end{aligned}
$$

We finally use equation (52) and the fact that $\mathrm{Li}_2\left(1-1/x\right)=-\mathrm{Li}_2\left(1-x\right)-\frac{1}{2}\left(\ln\left(x\right)\right)^2$ to show that

$$
\begin{aligned}
S\left(x\right) &= \frac{6}{\pi^3}\left(-\mathrm{Li}_2\left(x\right)+\mathrm{Li}_2\left(1-x\right)\right)\\
&= -L\left(x\right)+L\left(1-x\right)\\
&= 1-2L\left(x\right).
\end{aligned}
$$

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