Diff-invariant Kinetic Terms in Arbitrary Dimensions

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We study the physical content of quadratic diff-invariant Lagrangians in arbitrary dimensions by using covariant symplectic techniques. This paper extends previous results in dimension four. We discuss the difference between the even and odd dimensional cases.

I. INTRODUCTION

In recent years an interesting program devoted to the systematic study of deformations of gauge theories has been developed [1]. The purpose of this line of research is to study quantum field theories "continuously connected" to other, usually simpler, ones. In a typical situation one considers a free theory, such as Maxwell electromagnetism, takes one or more copies of the Lagrangian describing it, and studies the consistent introduction of interaction terms [1]. By doing this it has been shown [2] that Yang-Mills theories are the only consistent way to introduce interactions in the Maxwell Lagrangian. However, this does not prove that these are the only consistent interacting theories for 1-form fields because for this to be true one must first know for sure that the only possible starting point is the Maxwell action [3].

This example suggests the problem of classifying all the free theories with the scope of using them as a way to obtain interesting interacting models. With this idea in mind, and with the purpose of studying the possibility of finding suitable diff-invariant kinetic terms that could be used to build perturbative treatable gravitational actions we studied in [4] all the possible diff-invariant kinetic terms in four space-time dimensions. We found out that these theories do not describe any local degrees of freedom and, hence, their consistent deformations cannot describe any gravitational theory with local degrees of freedom. Anyway, having a full classification of these kinetic terms one can consider to study all their possible deformations as a means to classify the topological theories in four dimensions of Schwarz type (see [5] and references therein)[1] (or, at least, a big subset of them). Here we extend the analysis of [4] to arbitrary dimensions. Even and odd dimensional spacetimes show some

diferences worth of study; in particular in odd dimensions the kinetic term can have "diagonal" parts (as those appearing in the Chern-Simons Lagrangian) that are not present in the even dimensional situation; we discuss this, and related issues here. The paper is organized as follows. After this introduction we study in section II the physical content of purely quadratic, diff-invariant Lagrangians in even dimensions completing the results of [4]. We will do it by resorting to covariant symplectic methods as in the quoted paper. Section III will discuss the odd dimensional case with special emphasis on the differences with the even dimensional one. We end the paper in section IV with our conclusions and directions for future work.

II. DIFF-INVARIANT KINETIC TERMS: THE EVEN DIMENSIONAL CASE

We start by writing the most general diff-invariant local quadratic action in an even-dimensional differentiable manifold. Diff invariance demands the absence of background structures in the Lagrangian, i.e. all the objects that appear must be taken as dynamical. This precludes the appearance of background metrics or connections (other than the ones defined directly by the differential structure). The condition that the action be quadratic also constraints its form in a twofold way because the only derivative operator that we can use is the exterior differential and the only fields that can appear are differential forms, acted upon by the exterior differential. If one considers the inclusion of other types of tensor fields[2] one would be forced to use covariant derivatives that would give rise to non-quadratic terms in addition to the quadratic ones (or include these terms as total divergencies that would drop in the absence of boundaries).

Let us consider a $D+1$-dimensional differentiable manifold $\mathcal{M} = \mathbb{R} \times \Sigma_D$ where Σ_D is a compact, orientable,

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[1] Topological theories of the Witten type involve background structures, such as metrics, and are outside the scope of this paper.

[2] One can trivially add terms quadratic in tensor densities *without any derivatives* that do not describe any physical degrees of freedom.

D-dimensional manifold without boundary ($D = 2N-1$, $N \in \mathbb{N}$). As we stated before we will work with differential forms $\{A_n\}_{n=0}^{D+1}$, $A_n \in \Omega_n^{\mathrm{range}_n}(\mathcal{M})$. Here A_n denotes a set of n-forms labelled by an internal index (taking values from 1 to range_n) that we do not make explicit; in practice we will take A_n as a column vector with transpose denoted as A_n^{t}. The most general action under the conditions expressed above is

$$S_{2N}[A] = \int_{\mathcal{M}} \left\{ \sum_{n=0}^{N-1} d\,\overset{1}{A}{}_n^{\mathrm{t}} \wedge \overset{1}{A}{}_{2N-n-1} + \sum_{n=0}^{N-1} A_n^{\mathrm{t}} \wedge \Theta_n A_{2N-n} + \frac{1}{2} A_N^{\mathrm{t}} \wedge \Theta_N A_N + \theta^{\mathrm{t}} A_{2N} \right\}. \tag{1}$$

We discuss here the meaning and structure of the different terms in (1). The first term is the only one that involves derivatives. In the following we will make a distinction between two types of fields that we denote $\overset{1}{A}$ and $\overset{2}{A}$; $\overset{2}{A}$ are those acted upon by derivatives (either directly or after integration by parts) whereas no derivatives act on the $\overset{1}{A}$ fields. Notice that as we have the freedom to integrate by parts we can extend the sum in the first term only to $N-1$. Another interesting feature of this first term is that, as $n \neq 2N-n-1$ for all values of $n, N \in \mathbb{N}$, the two fields that appear in it must be necessarily different so, as shown in appendix A, it is actually possible to avoid the introduction of a "coupling matrix" and take $\overset{1}{A}{}_n$ and $\overset{2}{A}{}_{2N-n-1}$ with the same number of internal components. This allows us to write it in the following convenient form

$$d\,\overset{1}{A}{}_n^{\mathrm{t}} \wedge \overset{1}{A}{}_{2N-n-1} = \left[\, d\,\overset{1}{A}{}_n^{\mathrm{t}} \;\; d\,\overset{2}{A}{}_n^{\mathrm{t}} \,\right] \wedge \overset{1}{\Pi} \begin{bmatrix} \overset{1}{A}{}_{2N-n-1} \\[4pt] \overset{2}{A}{}_{2N-n-1} \end{bmatrix}$$

with $\overset{1}{\Pi} := \begin{bmatrix} \mathbb{I}_n & 0 \\ 0 & 0 \end{bmatrix}$ ($\mathbb{I}_n$ is the $n \times n$ identity matrix). The second term involves both type 1 and type 2 fields and it does not contain any derivatives. We have used the following compact notation

$$A_n^{\mathrm{t}} \wedge \Theta_n A_{2N-n} = \left[\, \overset{1}{A}{}_n^{\mathrm{t}} \;\; \overset{2}{A}{}_n^{\mathrm{t}} \,\right] \wedge \begin{bmatrix} \Theta_n^{11} & \Theta_n^{12} \\[4pt] \Theta_n^{21} & \Theta_n^{22} \end{bmatrix} \begin{bmatrix} \overset{1}{A}{}_{2N-n} \\[4pt] \overset{2}{A}{}_{2N-n} \end{bmatrix}.$$

Here we must allow the appearance of arbitrary, real, coupling matrices Θ_n^{11}, Θ_n^{12}, Θ_n^{21}, and Θ_n^{22}. These matrices are not square, in general, because the dimensions of A_n and A_{2N-n} do not necessarily match. We are also writing type-1 and type-2 fields in a single object

$$A_n = \begin{bmatrix} \overset{1}{A}{}_n \\[4pt] \overset{2}{A}{}_n \end{bmatrix}.$$

The third term in (1) involves only the A_N field so the matrix Θ_N is symmetric or antisymmetric for even and odd values of N respectively; for convenience we introduce a $1/2$ factor in front of it. Finally the last, linear term, involves a 2N-form A_{2N} and a constant vector θ.

The field equations obtained from (1) are

$$\Theta_0^{\mathrm{t}} A_0 + \theta = 0, \tag{2a}$$
$$\overset{1}{\Pi} dA_n + \Theta_{n+1}^{\mathrm{t}} A_{n+1} = 0, \qquad 0 \leq n < N-1 \tag{2b}$$
$$(-1)^N \overset{1}{\Pi} dA_{N-1} + \Theta_N A_N = 0, \tag{2c}$$
$$(-1)^n \overset{1}{\Pi} dA_n + \Theta_{2N-n-1} A_{n+1} = 0, \qquad N \leq n < 2N. \tag{2d}$$

Equation (2c) will be referred to as the "central equation" because it plays a special role in the resolution of the equations (2).

The process of solving these equations is tedious but straightforward. At every step, i.e. for every equation, we get a certain consistency condition, a partial solution for one of the fields, and a complete solution for another. The only obvious exceptions are the first and last equations. For (2a) we do not get a complete solution for any of the A_n whereas for (2d) the solution process terminates and we get no partial solutions.

A. Equation (2a)

Let us take a set of linearly independent elements of the kernel of Θ_0, $\rho_{c_o}^{|0|}$ labelled by c_o ($\Theta_0 \rho_{c_o}^{|0|} = 0$). We immediately see that the constant vector θ must satisfy the consistency condition

$$\rho_{c_o}^{|0|\mathrm{t}} \theta = 0. \tag{3}$$

When this condition holds we can solve the linear set of equations (2a) to get

$$A_0(x) = -\Theta_0^{-1\mathrm{t}} \theta + \lambda_{\alpha_0}^{|0|} A_0^{\alpha_0}(x) \tag{4}$$

where $A_0^{\alpha_0}(x)$ are *arbitrary* 0-forms at this stage, $\lambda_{\alpha_0}^{|0|}$, labelled by α_0, belong to the kernel of Θ_0^{t} ($\Theta_0^{\mathrm{t}} \lambda_{\alpha_0}^{|0|} = 0$), and $\Theta_0^{-1\mathrm{t}} \theta$ denotes a particular solution to the inhomogeneous equations (2a)[3]. At this point we have a consistency condition (3) and a partial solution for $A_0(x)$ that we use in the next step.

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[3] Upper and lower indices appearing in pairs are summed over. In order to avoid confusion we use vertical bars in the labels of ρ

B. Equations (2b)

If $N > 1$ Eqs. (2b) are a set of $N-1$ equations that are all solved by following essentially the same procedure as before though there are some minor differences depending on the type of differential forms involved. Let us consider first the equation for $n = 0$,

$$\overset{1}{\Pi} dA_0 + \Theta_1^{\text{t}} A_1 = 0 \,.$$

Taking a set of linearly independent elements of the kernel of Θ_1, $\rho_{c_1}^{|1|}$ labelled by c_1 ($\Theta_1 \rho_{c_1}^{|1|} = 0$) we obtain the consistency condition

$$\rho_{c_1}^{|1|\text{t}} \overset{1}{\Pi} \lambda_{\alpha_0}^{|0|} dA_0^{\alpha_0}(x) := \mathcal{M}_{c_1 \alpha_0}^{|0|} dA_0^{\alpha_0}(x) = 0 \qquad (5)$$

where we have defined a new matrix $\mathcal{M}_{c_1 \alpha_0}^{|0|}$. Notice that the presence of $\overset{1}{\Pi}$ in its definition means that (5) is a condition on type-1 fields only. This condition can be solved in the following way. We expand

$$A_0^{\alpha_0}(x) = [r_{p_0}^{|0|}]^{\alpha_0} A_0^{p_0}(x) + [v_{q_0}^{|0|}]^{\alpha_0} A_0^{q_0}(x)$$

where $\mathcal{M}_{c_1 \alpha_0}^{|0|} [r_{p_0}^{|0|}]^{\alpha_0} = 0$ and the $[v_{q_0}^{|0|}]^{\alpha_0}$ together with $[r_{p_0}^{|0|}]^{\alpha_0}$ define a basis of the real vector space on which $\mathcal{M}_{c_1 \alpha_0}^{|0|}$ is defined. Here p_0, q_0 are supposed to take different sets of values (so that $A^{p_0}(x)$ and $A^{q_0}(x)$ are independent objects) and label the vectors in the basis; α_0 is an explicit vector index. Plugging the previous decomposition in (5) gives

$$\mathcal{M}_{c_1 \alpha_0}^{|0|} [v_{q_0}^{|0|}]^{\alpha_0} dA_0^{q_0}(x) = 0 \,, \qquad (6)$$

and taking into account that $\mathcal{M}_{c_1 \alpha_0}^{|0|} [v_{q_0}^{|0|}]^{\alpha_0}$ is a basis of the image of $\mathcal{M}_{c_1 \alpha_0}^{|0|}$ we conclude that $dA_0^{q_0}(x) = 0$ and, hence

$$A_0^{q_0}(x) = \mathbf{a}_{|0|}^{q_0 i_0} A_{i_0}^{|0|}(x) \,, \qquad (7)$$

where $\{A_{i_0}^{|0|}(x)\}_{i_0=0}^{\dim H^0(\mathcal{M})}$ is a basis of the 0th-de Rham cohomology group[4] of $\mathcal{M}$ and $\mathbf{a}_{|0|}^{q_0 i_0} \in \mathbb{R}$. With this information we can complete now the partial solution for $A_0(x)$ that we obtained in the previous step

$$A_0(x) = \qquad (8)$$
$$- \Theta_0^{-1\text{t}} \theta + \lambda_{\alpha_0}^{|0|} \{ [r_{p_0}^{|0|}]^{\alpha_0} A_0^{p_0}(x) + [v_{q_0}^{|0|}]^{\alpha_0} \mathbf{a}_{|0|}^{q_0 i_0} A_{i_0}^{|0|}(x) \} \,.$$

We can also get in this step, by using (8), a partial solution for A_1 in the form

$$A_1(x) = -\Theta_1^{-1\text{t}} \overset{1}{\Pi} \lambda_{\alpha_0}^{|0|} [r_{p_0}^{|0|}]^{\alpha_0} dA_0^{p_0}(x) + \lambda_{\alpha_1}^{|1|} A_1^{\alpha_1}(x) \,. \qquad (9)$$

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and λ that indicate to which of the Θ matrices they refer to; this indices are *never* summed over. Θ_0^{-1} can be thought of as a pseudoinverse of Θ_0.

[4] For a short review on the de-Rham cohomology groups in this context we refer the reader to [4].

The remaining equations in (2b) are solved in exactly the same way. The only difference with the case just discussed appears in the solutions of conditions analogous to (6) for p-forms with $p > 0$. If we have, for example,

$$\mathcal{M}_{c_2 \alpha_1}^{|1|} [v_{q_1}^{|1|}]^{\alpha_1} dA_1^{q_1}(x) = 0$$

with $\mathcal{M}_{c_2 \alpha_1}^{|1|} := \rho_{c_2}^{|2|\text{t}} \overset{1}{\Pi} \lambda_{\alpha_1}^{|1|}$ we expand

$$A_1^{\alpha_1}(x) = [r_{p_1}^{|1|}]^{\alpha_1} A_1^{p_1}(x) + [v_{q_1}^{|1|}]^{\alpha_1} A_1^{q_1}(x)$$

where $\mathcal{M}_{c_2 \alpha_1}^{|1|} [r_{p_1}^{|1|}]^{\alpha_1} = 0$ and the $[v_{q_1}^{|1|}]^{\alpha_1}$ are introduced to complete a basis of the real vector space on which $\mathcal{M}_{c_2 \alpha_1}^{|1|}$ is defined. By doing this we get the equation $dA_1^{q_1}(x) = 0$ with solutions given by

$$A_1^{q_1}(x) = \mathbf{a}_{|1|}^{q_1 i_1} A_{i_1}^{|1|}(x) + d\varpi_0^{q_1}(x). \qquad (10)$$

Here $\{A_{i_1}^{|1|}(x)\}_{i_1=0}^{\dim H^1(\mathcal{M})}$ is a basis of the 1st-de Rham cohomology group of $\mathcal{M}$, $\mathbf{a}_{|1|}^{q_1 i_1} \in \mathbb{R}$, and $\varpi_0^{q_1}(x)$ are arbitrary 0-forms. The only difference with the case discussed above is the appearance of $\varpi_0^{q_1}(x)$. Similar objects appear for the remaining equations. We end this subsection by giving the solutions obtained from a generic equation (labelled by n) in this set: the complete solution for A_n, $(0 < n < N-1)$

$$A_n(x) = -\Theta_n^{-1\text{t}} \overset{1}{\Pi} \lambda_{\alpha_{n-1}}^{|n-1|} [r_{p_{n-1}}^{|n-1|}]^{\alpha_{n-1}} dA_{n-1}^{p_{n-1}}(x) +$$
$$+ \lambda_{\alpha_n}^{|n|} \Big\{ [r_{p_n}^{|n|}]^{\alpha_n} A_n^{p_n}(x) + \qquad (11)$$
$$+ [v_{q_n}^{|n|}]^{\alpha_n} \Big[\mathbf{a}_{|n|}^{q_n i_n} A_{i_n}^{|n|}(x) + d\varpi_{n-1}^{q_n}(x) \Big] \Big\}$$

and a partial solution for A_{n+1} $(0 \leq n < N-1)$

$$A_{n+1}(x) = \qquad (12)$$
$$- \Theta_{n+1}^{-1\text{t}} \overset{1}{\Pi} \lambda_{\alpha_n}^{|n|} [r_{p_n}^{|n|}]^{\alpha_n} dA_n^{p_n}(x) + \lambda_{\alpha_{n+1}}^{|n+1|} A_{n+1}^{\alpha_{n+1}}(x) \,.$$

C. Central equation (2c)

The matrix Θ_N that appears in this equation is either symmetric or antisymmetric and hence the kernel of Θ_N and Θ_N^{t} coincide. Here we get the complete solution for $A_{N-1}(x)$

$$A_{N-1}(x) = -\Theta_{N-1}^{-1\text{t}} \overset{1}{\Pi} \lambda_{\alpha_{N-2}}^{|N-2|} [r_{p_{N-2}}^{|N-2|}]^{\alpha_{N-2}} dA_{N-2}^{p_{N-2}}(x)$$
$$+ \lambda_{\alpha_{N-1}}^{|N-1|} \Big\{ [r_{p_{N-1}}^{|N-1|}]^{\alpha_{N-1}} A_{N-1}^{p_{N-1}}(x) \qquad (13)$$
$$+ [v_{q_{N-1}}^{|N-1|}]^{\alpha_{N-1}} \Big[\mathbf{a}_{|N-1|}^{q_{N-1} i_{N-1}} A_{i_{N-1}}^{|N-1|}(x) + d\varpi_{N-2}^{q_{N-1}}(x) \Big] \Big\}$$

and a partial solution for $A_N(x)$

$$A_N(x) = (-1)^{N+1} \Theta_N^{-1} \overset{1}{\Pi} \lambda_{\alpha_{N-1}}^{|N-1|} [r_{p_{N-1}}^{|N-1|}]^{\alpha_{N-1}} dA_{N-1}^{p_{N-1}}(x)$$
$$+ \rho_{c_N}^{|N|} A_N^{c_N}(x) \,. \qquad (14)$$

D. Equations (2d)

At this point the procedure that we use to solve the equations must be clear and, in fact, the rationale to give some details in this and the following sections is that of introducing in a systematic way the notation that we are using. Let us consider first the equation ($n = N$)

$$(-1)^N \overset{1}{\Pi} dA_N + \Theta_{N-1} A_{N+1} = 0.$$

We have now the consistency condition given by

$$\lambda^{|N-1|\mathrm{t}}_{\alpha_{N-1}} \overset{1}{\Pi} \rho^{|N|}_{c_N} dA^{c_N}_N(x) := \mathcal{N}^{|N|}_{\alpha_{N-1} c_N} dA^{c_N}_N(x) = 0.$$

where $\mathcal{N}^{|k|}_{\alpha_{k-1} c_k}$ satisfies $\mathcal{N}^{|k|}_{\alpha_{k-1} c_k} = \mathcal{M}^{|k-1|}_{c_k \alpha_{k-1}}$ for every k. Expanding

$$A^{c_N}_N(x) = [l^{|N|}_{p_N}]^{c_N} A^{p_N}_N(x) + [w^{|N|}_{q_N}]^{c_N} A^{q_N}_N(x)$$

with $\mathcal{N}^{|N|}_{\alpha_{N-1} c_N}[l^{|N|}_{p_N}]^{c_N} = 0$ and $[w^{|N|}_{q_N}]^{\alpha_N}$ introduced to complete a basis of the vector space where $\mathcal{N}^{|N|}_{\alpha_{N-1} c_N}$ is defined. The complete solution for $A_N(x)$ is

$$A_N(x) = (-1)^{N+1} \Theta^{-1}_N \overset{1}{\Pi} \lambda^{|N-1|}_{\alpha_{N-1}} [r^{|N-1|}_{p_{N-1}}]^{\alpha_{N-1}} dA^{p_{N-1}}_{N-1}(x)$$
$$+ \rho^{|N|}_{c_N} \Big\{ [l^{|N|}_{p_N}]^{c_N} A^{p_N}_N(x) \tag{15}$$
$$+ [w^{|N|}_{q_N}]^{c_N} \Big[\mathbf{a}^{q_N i_N}_{|N|} A^{|N|}_{i_N}(x) + d\varpi^{q_N}_{N-1}(x) \Big] \Big\}$$

and we get the folowing partial solution for $A_{N+1}(x)$

$$A_{N+1}(x) = (-1)^{N+1} \Theta^{-1}_{N-1} \overset{1}{\Pi} \rho^{|N|}_{c_N} [l^{|N|}_{p_N}]^{c_N} dA^{p_N}_N(x)$$
$$+ \rho^{|N-1|}_{c_{N-1}} A^{c_{N-1}}_{N+1}(x). \tag{16}$$

From a generic equation in this set ($N < n < 2N - 1$) we get a complete solution for $A_n(x)$

$$A_n(x) =$$
$$(-1)^n \Theta^{-1}_{2N-n} \overset{1}{\Pi} \rho^{|2N-n+1|}_{c_{2N-n+1}} [l^{|2N-n+1|}_{p_{n-1}}]^{c_{2N-n+1}} dA^{p_{n-1}}_{n-1}(x)$$
$$+ \rho^{|2N-n|}_{c_{2N-n}} \Big\{ [l^{|2N-n|}_{p_n}]^{c_{2N-n}} A^{p_n}_n(x) \tag{17}$$
$$+ [w^{|2N-n|}_{q_n}]^{c_{2N-n}} \Big[\mathbf{a}^{q_n i_n}_{|n|} A^{|n|}_{i_n}(x) + d\varpi^{q_n}_{n-1}(x) \Big] \Big\}$$

and a partial solution for $A_{n+1}(x)$

$$A_{n+1}(x) =$$
$$(-1)^{n+1} \Theta^{-1}_{2N-n-1} \overset{1}{\Pi} \rho^{|2N-n|}_{c_{2N-n}} [l^{|2N-n|}_{p_n}]^{c_{2N-n}} dA^{p_n}_n(x)$$
$$+ \rho^{|2N-n-1|}_{c_{2N-n-1}} A^{c_{2N-n-1}}_{n+1}(x), \tag{18}$$

where the objects appearing in these equations are defined in analogy with the ones introduced at the beginning of this subsection. The solutions for all the equations (2d), are given by (17) except the first one ($n = N$) which is given by (15) and the last one that is given by

$$A_{2N}(x) = \tag{19}$$
$$\Theta^{-1}_0 \overset{1}{\Pi} \rho^{|1|}_{c_1} [l^{|1|}_{p_{2N-1}}]^{c_1} dA^{p_{2N-1}}_{2N-1}(x) + \rho^{|0|}_{c_0} A^{c_0}_{2N}(x).$$

As we see the solutions are parametrized by three different types of objects: Arbitrary m-forms $A^{p_n}_n(x)$, ($n = 0, \ldots, 2N-1$), and $A^{c_0}_{2N}(x)$; arbitrary m-forms $\varpi^{q_{n+1}}_n(x)$, ($n = 0, \ldots, 2N - 2$), and a set of real numbers $\mathbf{a}^{q_n i_n}_{|n|}$ ($n = 0, \ldots, 2N-1$).

If $N = 1$ the solutions are given by (8), (15), and (19) with $N = 1$ in these last two equations.

E. Symplectic structure

Once we have obtained the solutions to the field equations we must compute the symplectic structure on the solution space. This will allow us to identify the physical degrees of freedom and the gauge symmetries of the lagrangians introduced above. To this end we must substitute the solutions to the field equations into the symplectic structure obtained from the action

$$\Omega = \int_\Sigma \sum_{n=0}^{N-1} d\!\!\!I \, \overset{\blacksquare}{A}{}^{\mathrm{t}}_n \wedge d\!\!\!I \, \overset{\blacksquare}{A}{}_{2N-n-1} \tag{20}$$

After a series of cancellations between the different terms we find the following expression for the symplectic structure in the space of fields

$$\Omega = \sum_{n=0}^{N-1} \left[[v^{|n|}_{q_n}]^{\alpha_n} \mathcal{N}^{|n+1|}_{\alpha_n c_{n+1}} [w^{|n+1|}_{q_{2N-n-1}}]^{c_{n+1}} \int_\Sigma A^{|n|}_{i_n} \wedge A^{|2N-n-1|}_{i_{2N-n-1}} \right] d\!\!\!I \mathbf{a}^{q_n i_n}_{|n|} \wedge d\!\!\!I \mathbf{a}^{q_{2N-n-1} i_{2N-n-1}}_{|2N-n-1|} \tag{21}$$

As we can see the only degrees of freedom are described by $\mathbf{a}^{q_n i_n}_{|n|}$; all the remaining objects in the solutions

obtained above represent gauge transformations. This means that all these models (in even dimensional space-times) are topological theories of the Schwarz type because they do not depend on any metric but only on the topology of the manifold $\mathcal{M}$ (or, equivalently Σ). Some comments on the structure of (21) are in order now. First it is straightforward to prove that the factors in front of $d\mathbf{a}_{|n|}^{q_n i_n} \wedge d\mathbf{a}_{|2N-n-1|}^{q_{2N-n-1} i_{2N-n-1}}$ are, in fact, non singular. This is so because $\mathcal{N}_{\alpha_n c_{n+1}}^{|n+1|}$ is non singular in the vector spaces spanned by $[w_{q_{2N-n-1}}^{|n+1|}]^{c_{n+1}}$ and $[v_{q_n}^{|n|}]^{\alpha_n}$ and the integral is non-singular as a consequence of Poincaré duality. Second it is possible to prove, by Poincaré's lemma that the de-Rham cohomology groups of $\mathcal{M}$ and Σ coincide ($H^k(\mathcal{M}) = H^k(\Sigma)$). Finally we want to point out an interesting feature of (21) wich is the fact that every term in the sum depends on two consecutive matrices Θ_n and Θ_{n+1} and there are no "diagonal terms" (involving $\mathbf{a}_{|n|}^{q_n i_n}$'s with the same index $|n|$).

III. DIFF-INVARIANT KINETIC TERMS: THE ODD DIMENSIONAL CASE.

Let us consider now the following action in $2N+1$ dimensions

$$S_{2N+1}[A] = \int_{\mathcal{M}} \left\{ \sum_{n=0}^{N-1} d\overset{\mathbb{1}}{A}_n^{\text{t}} \wedge \overset{\mathbb{1}}{A}_{2N-n} + \frac{1}{2} d\overset{\mathbb{1}}{A}_N \wedge \Xi_N \overset{\mathbb{1}}{A}_N + \sum_{n=0}^{N} A_n^{\text{t}} \wedge \Theta_n A_{2N-n+1} + \theta^{\text{t}} A_{2N+1} \right\} \tag{22}$$

where the notation that we use is analogous to the one introduced in the previous section; in particular we work now with sets of differential forms A_n (with an internal index that we do not make explicit) of order $n = 0, \ldots, 2N+1$. We see that we have now the possibility of having a diagonal derivative term (that we will refer to as the Ξ-term) $\frac{1}{2} d\overset{\mathbb{1}}{A}_N \wedge \Xi_N \overset{\mathbb{1}}{A}_N$. At variance with the previous case we cannot eliminate the coupling matrix Ξ_N by non-singular field redefinitions. With the conventions that we are using we can take Ξ_N as a non-singular matrix; if N is odd we can choose it to be symmetric whereas for even N it is antisymmetric. If Ξ_N is symmetric and non-singular we can diagonalize Ξ_N and write it as a diagonal matrix of 1 and -1 entries. If it is antisymmetric it is always possible to write it as the matrix

$$\begin{bmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{bmatrix}.$$

This structure will determine, in part, the types of internal symmetries of the actions derived from (22) by means of consistent deformations. The structure of the remaining terms of this action is analogous to the ones considered in the previous section. The field equations derived from (22) are

$$\Theta_0^{\text{t}} A_0 + \theta = 0, \tag{23a}$$

$$\overset{1}{\Pi} dA_n + \Theta_{n+1}^{\text{t}} A_{n+1} = 0, \qquad 0 \le n \le N-1 \tag{23b}$$

$$(-1)^{N+1} \Xi_N \overset{1}{\Pi} dA_N + \Theta_N A_{N+1} = 0, \tag{23c}$$

$$(-1)^{n+1} \overset{1}{\Pi} dA_n + \Theta_{2N-n} A_{n+1} = 0, \quad N+1 \le n \le 2N. \tag{23d}$$

The structure of equations (23a-23b) is exactly the same as in the previous case; we write their solutions here for completeness. From (23a) we get the partial solution

$$A_0(x) = -\Theta_0^{-1\text{t}}\theta + \lambda_{\alpha_0}^{|0|} A_0^{\alpha_0}(x) \tag{24}$$

with $\rho_{c_o}^{|0|\text{t}}\theta = 0$; and proceeding as before we finally get

$$A_0(x) =$$
$$-\Theta_0^{-1\text{t}}\theta + \lambda_{\alpha_0}^{|0|} \left\{ [r_{p_0}^{|0|}]^{\alpha_0} A_0^{p_0}(x) + [v_{q_0}^{|0|}]^{\alpha_0} \mathbf{a}_{|0|}^{q_0 i_0} A_{i_0}^{|0|}(x) \right\}.$$

Equations (23b) give the complete solution for A_n ($n = 1, \ldots, N-1$)

$$A_n(x) = -\Theta_n^{-1\text{t}} \overset{1}{\Pi} \lambda_{\alpha_{n-1}}^{|n-1|} [r_{p_{n-1}}^{|n-1|}]^{\alpha_{n-1}} dA_{n-1}^{p_{n-1}}(x) +$$
$$+ \lambda_{\alpha_n}^{|n|} \left\{ [r_{p_n}^{|n|}]^{\alpha_n} A_n^{p_n}(x) + \right. \tag{25}$$
$$\left. + [v_{q_n}^{|n|}]^{\alpha_n} \left[\mathbf{a}_{|n|}^{q_n i_n} A_{i_n}^{|n|}(x) + d\varpi_{n-1}^{q_n}(x) \right] \right\}$$

and a partial solution for A_{n+1} ($n = 0, \ldots, N-1$)

$$A_{n+1}(x) = \tag{26}$$
$$-\Theta_{n+1}^{-1\text{t}} \overset{1}{\Pi} \lambda_{\alpha_n}^{|n|} [r_{p_n}^{|n|}]^{\alpha_n} dA_n^{p_n}(x) + \lambda_{\alpha_{n+1}}^{|n+1|} A_{n+1}^{\alpha_{n+1}}(x)$$

with all the algebraic objects appearing in these expressions defined as in the even case.

A. Central equation (23c)

The first consistency condition that we get from the central equation is

$$\lambda_{\alpha_N}^{|N|\text{t}} \Xi_N \overset{1}{\Pi} \lambda_{\alpha_N'}^{|N|} dA_N^{\alpha_N'}(x) := \mathbb{M}_{\alpha_N \alpha_N'} dA_N^{\alpha_N'}(x) = 0. \tag{27}$$

Expanding

$$A_n^{\alpha_N}(x) = [r_{p_N}^{|N|}]^{\alpha_N} A_N^{p_N}(x) + [v_{q_N}^{|N|}]^{\alpha_N} A_N^{q_N}(x)$$

where $\mathbb{M}_{\alpha_N \alpha_N'}[r_{p_N}^{|N|}]^{\alpha_N'} = 0$ and $[v_{q_N}^{|N|}]^{\alpha_N}$ are introduced, as in previous instances, to complete a basis of the relevant vector space. The complete solution for $A_N(x)$ is

$$A_N(x) = -\Theta_n^{-1\mathrm{t}} \overset{1}{\Pi} \lambda_{\alpha_{N-1}}^{|N-1|}[r_{p_{N-1}}^{|N-1|}]^{\alpha_{N-1}} dA_{N-1}^{p_{N-1}}(x) +$$

$$+ \lambda_{\alpha_N}^{|N|}\left\{ [r_{p_N}^{|N|}]^{\alpha_N} A_N^{p_N}(x) + \right. \tag{28}$$

$$\left. + [v_{q_N}^{|N|}]^{\alpha_N}\left[\mathbf{a}_{|N|}^{q_N i_N} A_{i_N}^{|N|}(x) + d\varpi_{N-1}^{q_N}(x) \right] \right\}$$

and we get the following partial solution for $A_{N+1}(x)$

$$A_{N+1}(x) = \tag{29}$$

$$(-1)^N \Theta_N^{-1} \Xi_N \overset{1}{\Pi} \lambda_{\alpha_N}^{|N|}[r_{p_N}^{|N|}]^{\alpha_N} dA_N^{p_N}(x) + \rho_{c_N}^{|N|} A_{N+1}^{c_N}(x).$$

B. Equations (23d)

Let us consider the equation corresponding to $n = N + 1$

$$(-1)^N \overset{1}{\Pi} dA_{N+1} + \Theta_{N-1} A_{N+2} = 0.$$

As before we get the consistency condition

$$\lambda_{\alpha_{N-1}}^{|N-1|} \overset{1}{\Pi} \rho_{c_N}^{|N|} dA_{N+1}^{c_N}(x) := \mathcal{N}_{\alpha_{N-1}c_N}^{|N|} dA_{N+1}^{c_N}(x) = 0.$$

Expanding

$$A_{N+1}^{c_N}(x) = [l_{p_{N+1}}^{|N|}]^{c_N} A_{N+1}^{p_{N+1}}(x) + [w_{q_{N+1}}^{|N|}]^{c_N} A_{N+1}^{q_{N+1}}(x)$$

with $\mathcal{N}_{\alpha_{N-1}c_N}^{|N|}[l_{p_{N+1}}^{|N|}]^{c_N} = 0$ and $[w_{q_{N+1}}^{|N|}]^{c_N}$ introduced to complete a basis of the relevant vector space. The complete solution for A_{N+1} is

$$A_{N+1}(x) = (-1)^N \Theta_N^{-1} \Xi_N \overset{1}{\Pi} \lambda_{\alpha_N}^{|N|}[r_{p_N}^{|N|}]^{\alpha_N} dA_N^{p_N}(x)$$

$$+ \rho_{c_N}^{|N|}\left\{ [l_{p_{N+1}}^{|N|}]^{c_N} A_{N+1}^{p_{N+1}}(x) \right. \tag{30}$$

$$\left. + [w_{q_{N+1}}^{|N|}]^{c_N}\left[\mathbf{a}_{|N+1|}^{q_{N+1} i_{N+1}} A_{i_{N+1}}^{|N+1|}(x) + d\varpi_N^{q_{N+1}}(x) \right] \right\}$$

and we get the following partial solution for A_{N+2}

$$A_{N+2}(x) = (-1)^{N+1} \Theta_{N-1}^{-1} \overset{1}{\Pi} \rho_{c_N}^{|N|}[l_{p_{N+1}}^{|N|}]^{c_N} dA_{N+1}^{p_{N+1}}(x)$$

$$+ \rho_{c_{N-1}}^{|N-1|} A_{N+2}^{c_{N-1}}(x). \tag{31}$$

For a generic equation in this set ($N + 1 < n < 2N$) we get a complete solution for $A_n(x)$

$$A_n(x) =$$

$$(-1)^{n+1}\Theta_{2N-n+1}^{-1}\overset{1}{\Pi}\rho_{c_{2N-n+2}}^{|2N-n+2|}[l_{p_{n-1}}^{|2N-n+2|}]^{c_{2N-n+1}} dA_{n-1}^{p_{n-1}}(x)$$

$$+ \rho_{c_{2N-n+1}}^{|2N-n+1|}\left\{ [l_{p_n}^{|2N-n+1|}]^{c_{2N-n+1}} A_n^{p_n}(x) \right. \tag{32}$$

$$\left. + [w_{q_n}^{|2N-n+1|}]^{c_{2N-n+1}}\left[\mathbf{a}_{|n|}^{q_n i_n} A_{i_n}^{|n|}(x) + d\varpi_{n-1}^{q_n}(x) \right] \right\}$$

and a partial solution for $A_{n+1}(x)$

$$A_{n+1}(x) =$$

$$(-1)^n \Theta_{2N-n}^{-1} \overset{1}{\Pi} \rho_{c_{2N-n+1}}^{|2N-n+1|}[l_{p_n}^{|2N-n+1|}]^{c_{2N-n+1}} dA_n^{p_n}(x)$$

$$+ \rho_{c_{2N-n}}^{|2N-n|} A_{n+1}^{c_{2N-n}}(x), \tag{33}$$

where the objects appearing in these equations are defined in analogy with the ones previously introduced in the paper. The solutions for all the equations (23), are given by (32) except the first one ($n = N + 1$) which is given by (30) and the last one that is given by

$$A_{2N+1}(x) =$$

$$\Theta_0^{-1} \overset{1}{\Pi} \rho_{c_1}^{|1|}[l_{p_{2N}}^{|1|}]^{c_1} dA_{2N}^{p_{2N}}(x) + \rho_{c_0}^{|0|} A_{2N+1}^{c_0}(x). \tag{34}$$

The solutions are parametrized by three different types of objects: Arbitrary m-forms $A_n^{p_n}(x)$, ($n = 0, \ldots, 2N$), and $A_{2N+1}^{c_0}(x)$; arbitrary m-forms $\varpi_n^{q_{n+1}}(x)$, ($n = 0, \ldots, 2N - 1$), and a set of real numbers $\mathbf{a}_{|n|}^{q_n i_n}$ ($n = 0, \ldots, 2N$). If $N = 1$ the solutions are

$$A_0(x) = -\Theta_0^{-1\mathrm{t}}\theta$$

$$+ \lambda_{\alpha_0}^{|0|}\left\{ [r_{p_0}^{|0|}]^{\alpha_0} A_0^{p_0}(x) + [v_{q_0}^{|0|}]^{\alpha_0} \mathbf{a}_{|0|}^{q_0 i_0} A_{i_0}^{|0|}(x) \right\}$$

$$A_1(x) = -\Theta_1^{-1\mathrm{t}} \overset{1}{\Pi} \lambda_{\alpha_0}^{|0|}[r_{p_0}^{|0|}]^{\alpha_0} dA_0^{p_0}(x)$$

$$+ \lambda_{\alpha_1}^{|1|}\left\{ [r_{p_1}^{|1|}]^{\alpha_1} A_1^{p_1}(x) + [v_{q_1}^{|1|}]^{\alpha_1}\left[\mathbf{a}_{|1|}^{q_1 i_1} A_{i_1}^{|1|}(x) + d\varpi_0^{q_1}(x) \right] \right\}$$

$$A_2(x) = -\Theta_1^{-1\mathrm{t}}\Xi_1 \overset{1}{\Pi} \lambda_{\alpha_1}^{|1|}[r_{p_1}^{|1|}]^{\alpha_1} dA_1^{p_1}(x)$$

$$+ \rho_{c_1}^{|1|}\left\{ [l_{p_2}^{|1|}]^{c_1} A_2^{p_2}(x) + [w_{q_2}^{|1|}]^{c_1}\left[\mathbf{a}_{|2|}^{q_2 i_2} A_{i_2}^{|2|}(x) + d\varpi_1^{q_2}(x) \right] \right\}$$

$$A_3(x) = \Theta_0^{-1} \overset{1}{\Pi} \rho_{c_1}^{|1|}[l_{p_2}^{|1|}]^{c_1} dA_2^{p_2}(x) + \rho_{c_0}^{|0|} A_3^{c_0}(x).$$

C. Symplectic structure

The symplectic structure in this case is obtained by substituting the solutions to the field equations in

$$\Omega = \int_\Sigma \left[\sum_{n=0}^{N-1} d\!\!\!I\, \overset{\mathrm{t}}{A}_n \wedge d\!\!\!I\, A_{2N-n} + d\!\!\!I\, \overset{\mathrm{t}}{A}_N \wedge \Xi_N d\!\!\!I\, A_N \right] \tag{35}$$

to get

$$\Omega = \sum_{n=0}^{N-1} \left[[v_{q_n}^{|n|}]^{\alpha_n} \mathcal{N}_{\alpha_n c_{n+1}}^{|n+1|} [w_{q_{2N-n}}^{|n+1|}]^{c_{n+1}} \int_{\Sigma} A_{i_n}^{|n|} \wedge A_{i_{2N-n}}^{|2N-n|} \right] \mathbb{d}\mathbb{a}_{|n|}^{q_n i_n} \mathbb{\wedge}\mathbb{d}\mathbb{a}_{|2N-n|}^{q_{2N-n} i_{2N-n}} \tag{36}$$

$$+ \left[[v_{q_N}^{|N|}]^{\alpha_N} \mathbb{M}_{\alpha_N \alpha'_N} [v_{q'_N}^{|N|}]^{\alpha'_N} \int_{\Sigma} A_{i_N}^{|N|} \wedge A_{i'_N}^{|N|} \right] \mathbb{d}\mathbb{a}_{|N|}^{q_N i_N} \mathbb{\wedge}\mathbb{d}\mathbb{a}_{|N|}^{q'_N i'_N} \, .$$

The most interesting feature of Ω is the appearence of diagonal $\mathbb{d}\mathbb{a} \wedge \mathbb{d}\mathbb{a}$ terms that depend on both Ξ_N and Θ_N (through λ_N).

IV. CONCLUDING REMARKS

We have studied in this paper the most general quadratic diff-invariant theories in arbitrary dimensions; in particular we have found that they always describe topological (non-local) degrees of freedom. This result is easy to understand in the covariant symplectic framework that we are using here to study the dynamical content and the symmetries of the actions (1) and (22) because the symplectic structure Ω on the solution space given by eqs. (21) and (36) is *independent* of the choice of the hypersurface Σ. As the functions that appear in the solutions to the field equations are completely arbitrary the only way to avoid a hypersurface dependence of Ω is if they do not appear in the symplectic structure. This can be checked by the explicit computation of the restriction of (20) and (35) to the corresponding solution spaces.

In the actions that we have considered the fields that appear (or rather, some combinations of them determined by the coupling matrices that we introduce) can be thought of as Lagrange multipliers imposing constraints on the exterior differentials of other fields. The fact that the coupling matrices that we introduce are completely general makes it difficult to disentangle this complicated set of constraints. In fact, if instead of the covariant symplectic techniques that we use here one resorts to the more familiar Dirac constraint analysis, the problem becomes very hard to solve. By finding out the most general solutions of the field equations and the symplectic structure we can write down the gauge symmetries of all these actions and identify their degrees of freedom in a straightforward way.

As we can see by looking at the symplectic 2-forms that we find in the paper the matrices that multiply the $\mathbb{d}\mathbb{a} \wedge \mathbb{d}\mathbb{a}$ terms depend on pairs of consecutive Θ matrices and for odd dimensions also on Ξ and Θ_N.

An interesting and open problem is to study the consistent deformations of the actions given in this paper (some work in this direction has been carried out for the abelian Chern-Simons Lagrangian in [6]). They would provide coupled topological theories of the BF type in even dimensions or coupled Chern-Simons and BF theories in odd dimensions. One would expect that some modes that are decoupled in the free models that we consider here are actually coupled in their deformations as it happens, in a slightly different context, in the BFYM Lagrangians discussed in [7]. For every Lagrangian of the type that we are considering in this paper it is easy to find another one involving only derivative terms with the same dynamical content (changing, if necessary the internal dimensions of some of the fields involved); however, we do not know if the consistent deformations of Lagrangians equivalent in this sense will be equivalent.

Appendix A: Structure of the derivative term in even dimensions

We show here that, with the exception of the Chern-Simons-like terms that appear in the odd-dimensional case (coupling A_N with dA_N) it is always possible to avoid introducing coupling matrices in the derivative terms by using linear field redefinitions.

If $\Delta \in \mathcal{M}_{M \times N}(\mathbb{R})$ and we have $dA_m^{\mathrm{t}} \wedge \Delta A_n$, we can introduce bases for $\mathbb{R}^N$ and $\mathbb{R}^M$ as $\mathcal{B}_n = \{v_1, \ldots, v_r, \rho_1, \ldots, \rho_{N-r}\}$, $\mathcal{B}_m = \{w_1, \ldots, w_r, \lambda_1, \ldots, \lambda_{M-r}\}$ where $r = \mathrm{rank}(\Delta)$, $\Delta \rho_k = 0$ for $k = 1, \ldots, N-r$, $\lambda_j^{\dagger} \Delta = 0$ for $j = 1, \ldots, M-r$. We have now

$$dA_m^{\mathrm{t}} \wedge \Delta A_n = dA_m^{\mathrm{t}} (\mathcal{B}_m^{-1})^{\mathrm{t}} \mathcal{B}_m^{\mathrm{t}} \Delta \mathcal{B}_n \mathcal{B}_n^{-1} A_n \quad ,$$

where $\mathcal{B}_m^{\mathrm{t}} \Delta \mathcal{B}_n$ has the following block form

$$\begin{bmatrix} w_a^{\mathrm{t}} \Delta v_b & 0 \\ 0 & 0 \end{bmatrix} ,$$

$w_a^{\mathrm{t}} \Delta v_b \in \mathcal{M}_{r \times r}(\mathbb{R})$ and is regular so that by independent linear redefinitions in A_m and A_n it can be taken to be the identity. By using the convention that fields that do no couple to derivatives are "type 2" we see that the derivative terms can be taken as in (1) with all generality and in particular, that the number of internal components in A_n and A_{2N-n+1} (in the even case) and in A_n and A_{2N-n} (in the odd one) are the same.

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