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Fedosov Deformation Quantization as a BRST Theory.

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The relationship is established between the Fedosov deformation quantization of a general symplectic manifold and the BFV-BRST quantization of constrained dynamical systems. The original symplectic manifold $\mathcal{M}$ is presented as a second class constrained surface in the fibre bundle $\mathcal{T}^*_\rho\mathcal{M}$ which is a certain modification of a usual cotangent bundle equipped with a natural symplectic structure. The second class system is converted into the first class one by continuation of the constraints into the extended manifold, being a direct sum of $\mathcal{T}^*_\rho\mathcal{M}$ and the tangent bundle $T\mathcal{M}$. This extended manifold is equipped with a nontrivial Poisson bracket which naturally involves two basic ingredients of Fedosov geometry: the symplectic structure and the symplectic connection. The constructed first class constrained theory, being equivalent to the original symplectic manifold, is quantized through the BFV-BRST procedure. The existence theorem is proven for the quantum BRST charge and the quantum BRST invariant observables. The adjoint action of the quantum BRST charge is identified with the Abelian Fedosov connection while any observable, being proven to be a unique BRST invariant continuation for the values defined in the original symplectic manifold, is identified with the Fedosov flat section of the Weyl bundle. The Fedosov fibrewise star multiplication is thus recognized as a conventional product of the quantum BRST invariant observables.

1. Introduction

Different trends are recognized among the approaches to quantization of systems whose classical mechanics is based on the Poisson bracket. In physics, the quantization strategy evolves, in a sense, in the opposite direction to the main stream developing in mathematics. From the physical viewpoint, the phase manifold is usually treated as a constraint surface in a flat manifold or in a manifold whose geometric structure is rather simpler than that of constraint surface. And the efforts are not directed to reduce the dynamics on the curved shell before quantization. The matter is that the physical models should usually possess an explicit relativistic covariance and space-time locality, whereas the reduction to the constraint surface usually breaks both. So, the main trend in physics is to quantize the system as it originally occurs, i.e. with constraints. The reduction is achieved in quantum theory by means of restrictions imposed to the class of admissible observables and states. The most sophisticated quantization scheme developed in this direction is the BFV method [1] (for review see [2]) based on the idea of the BRST symmetry. The method allows, in principle, to quantize any first class constraint theory, with the exception of the special case of the so-called infinitely reducible constraints. As to the second class constrained theories, various methods are known to adopt the BFV-BRST approach for the case. In this paper we turn to the idea to convert the second class theory into the first class extending the phase manifold by extra degrees of freedom which are going to be eventually gauged out by the introduced gauge symmetry related to the effective first class constraints. A number of the general conversion schemes is known today [3, 4, 5, 6]. The conversion ideas are widely applied in practical physical problems concerning quantization of the second class constrained systems.

The mathematical insight into the quantization problem always starts with the reduced Poisson manifold where the constraints, if they could originally present, have already been resolved. The general concept of the deformation quantization was introduced in Ref. [11, 12]. The existence of the star product on the general symplectic manifold was proven in Ref. [13] where the default was ascertained for the cohomological obstructions to the deformation of the associative multiplication.

Independently, Fedosov suggested the explicit construction of the star product on any symplectic manifold [14] (see also the subsequent book [15]). Now the general statement regarding the existence of the star product for the most general Poisson manifold is established by Kontsevich [16]. Recently the Kontsevich quantization formula was also supplied with an interesting physical explanation [17].

However, in the case of symplectic manifolds the Fedosov construction of the star product seems to be most useful in applications. The advantage is in the explicit description of the algebra of quantum observables. In the Fedosov approach the quantum observable algebra is the space of the flat sections of the Weyl algebra bundle over the symplectic manifold, with the multiplication being the fibrewise Weyl product. The Fedosov star-product allows a generalisation to the case of super-Poison bracket [18].

Mention that the deformation quantization structures are coming now to the gauge field theory not only as a tool of quantizing but rather as the means of constructing new classical models, e.g. gauge theories on noncommutative spaces [7] and higher-spin interactions [8]. The recent developments have also revealed a deep relationship between the strings and the Yang-Mills theory on the noncommutative spaces [9, 10].

The BRST approach to the quantization of the systems with the geometrically nontrivial phase space was initiated by Batalin and Fradkin who suggested to present the original symplectic manifold as a second class constraint surface embedded into the linear symplectic space [19][1]

From the viewpoint of the BFV method, the question of deformation quantization of general symplectic manifold was considered in Ref. [20] where one could actually observe (although it was not explicitly mentioned about in the paper) that the generating equations for the Abelian conversion [6], being applied to the embedding of the second class constraints of the Ref.[19, 21], naturally involve the characteristic structures of the Fedosov geometry: the symmetric symplectic connection and the curvature.

However, according to our knowledge, the relationship has not been established yet between the second class constraint approach of [19, 20, 21] and the Fedosov construction.

In this paper we show that the Fedosov quantization scheme can be completely derived from the BFV-BRST quantization of the constrained dynamical systems.

First the symplectic manifold $\mathcal{M}$ is extended to the fibre bundle $\mathcal{T}^*_\rho\mathcal{M}$, being a certain modification of the usual cotangent bundle, which still carries the canonical symplectic structure. The original

[1]The global geometric properties of this embedding were not in the focus of the original papers [19, 20, 21]. In the present paper we suggest a slightly different constrained embedding of the symplectic manifold with an explicit account for the global geometry, although our basic goal is beyond the geometry of the embedding itself.

manifold $\mathcal{M}$ is identified to the second class constrained surface in $\mathcal{T}^*_\rho\mathcal{M}$. This allows to view the Poisson bracket on the base manifold as the Dirac bracket associated to the second class constraints. Further, the second class constraints are converted into the first class ones in spirit of the Abelian conversion procedure [6]. In the case at hand, we choose the conversion variables to be the coordinates on the fibres of the tangent bundle over the symplectic manifold. The phase space of the converted system, in distinction to the direct application of the conventional conversion scheme [6] exploited in the Ref. [20], is equipped with a natural nonlinear symplectic structure. This symplectic structure involves the initial symplectic form and a symmetric symplectic connection. Remarkably, these structures are known as those determining the so called Fedosov geometry [22]. In its turn the Jacobi identity for the Poisson bracket, being defined in this extended manifold, encodes all the respective compatibility conditions for the Fedosov manifold. So, the embedding and converting procedure make the relationship transparent between the constrained Hamiltonian dynamics and Fedosov's geometry.

We quantize the resulting gauge invariant system, being globally equivalent to the original symplectic one, according to the standard BFV quantization prescription. As the extended phase space of the BFV quantization is a geometrically nontrivial symplectic manifold, it is a problem to quantize it directly. Fortunately, to proceed with the BFV scheme when the constraints have the special structure as in the case in hand, one needs to define only the quantization of some subalgebra of functions. Namely, we consider subalgebra $\mathfrak{A}$ of functions at most linear in the momenta which is closed w.r.t. the associative multiplication and the Poisson bracket. This subalgebra contains all the BRST observables, BRST charge Ω and the ghost charge. Unlike the entire algebra of functions on the extended phase space, the construction of the star-multiplication in $\mathfrak{A}$ is evident in this case. At the quantum level we arrive at the quantum BRST charge Ω satisfying the quantum nilpotency condition $[\Omega, \Omega]_\star = 0$. The algebra of quantum observables is thus the zero-ghost-number cohomology of $Ad\ \Omega$. This algebra, being viewed as a vector space, is isomorphic to the algebra of classical observables. The noncommutative product from the algebra of quantum BRST observables is carried over to the space of functions on the symplectic manifold giving a deformation quantization.

This approach allows us to identify all the basic structures of Fedosov's method as those of the BRST theory. In particular, the auxiliary variables y^i (which appear in Ref [14] as the generators of Weyl algebra) turns out to be the conversion variables, the basic one-forms dx^i on the symplectic manifold should be identified with the ghost variables associated to the converted constraints. Further, the Fedosov flat connection D is the adjoint action of the quantum BRST charge Ω; the flat sections of the Weyl bundle is thus nothing but the BRST cohomology. Under this identification, the Fedosov quantization statements regarding the existence of the Abelian connection, lift of the functions from the symplectic manifold to the flat sections of the Weyl bundle can be recognized as the standard existence theorems of the BRST theory.

2. Representation of a general symplectic manifold as a constrained Hamiltonian dynamics.

In this section we first represent a general symplectic manifold $\mathcal{M}$, $\dim(\mathcal{M}) = N$ as a second class constraint surface embedded into the fibre bundle $\mathcal{T}^*_\rho\mathcal{M}$, $\dim(\mathcal{T}^*_\rho\mathcal{M}) = 2N$ being equipped with globally defined canonical symplectic structure. Next we develop the procedure to convert the second class constraints into the first class ones extending the manifold $\mathcal{T}^*_\rho\mathcal{M}$ to the direct sum $\mathcal{T}^*_\rho\mathcal{M} \oplus T\mathcal{M}$ which possesses a nontrivial Poisson structure. This structure generates, in a sense, all the structure relations of the symplectic geometry of the original symplectic manifold $\mathcal{M}$. The extra degrees of freedom introduced with this embedding are effectively gauged out due to the gauge symmetry related to the effective first class constraints. And finally we construct the classical BRST embedding for the effective first class system, which serves as a starting point for the BFV-BRST quantization of the symplectic manifold, that is done in the next section.

2.1. **Second class constraint formulation of the symplectic structure.** Let $\mathcal{M}$ be the symplectic manifold with symplectic form ω. Denote by $\{\,\cdot\,,\,\cdot\,\}_\mathcal{M}$ the respective Poisson bracket on $\mathcal{M}$. Let x^i be a local coordinate system on $\mathcal{M}$. In the local coordinates the symplectic form and the Poisson bracket read as

$$\omega = \omega_{ij}(x)dx^i \wedge dx^j\,, \quad d\omega = 0, \tag{2.1}$$

$$\{a(x)\,, b(x)\,\}_\mathcal{M} = \omega^{ij}(x)\frac{\partial a(x)}{\partial x^i}\frac{\partial b(x)}{\partial x^j}\,, \quad \omega^{ij}\omega_{jk} = \delta^i_k. \tag{2.2}$$

Let Γ be a symmetric symplectic connection on $\mathcal{M}$, which always exists (for details of the geometry based on this connection see [22]). In the local coordinates x^i we have

$$\frac{\partial}{\partial x^i}\omega_{jk} - \Gamma^l_{ij}\omega_{lk} - \Gamma^l_{ik}\omega_{jl} = 0\,, \qquad \nabla_i(\frac{\partial}{\partial x^j}) = \Gamma^k_{ij}\frac{\partial}{\partial x^k}\,. \tag{2.3}$$

Introduce a curvature tensor $R^k_{l;ij}$ of Γ by $R^k_{l;ij}\dfrac{\partial}{\partial x^k} = [\nabla_i, \nabla_j]\dfrac{\partial}{\partial x^l}$. In the local coordinates it reads

$$R^k_{l;ij} = \partial_i\Gamma^k_{jl} + \Gamma^n_{jl}\Gamma^k_{in} - \partial_j\Gamma^k_{il} - \Gamma^n_{il}\Gamma^k_{jn}\,. \tag{2.4}$$

In the symplectic geometry it is convenient to use the coefficients Γ_{ijk} defined by $\Gamma_{ijk} = \omega_{in}\Gamma^n_{jk}$ and $R_{kl;ij} = \omega_{kn}R^k_{l;ij}$. The curvature tensor $R_{kl;ij}$ satisfies corresponding Bianchi identities:

$$\nabla_m R_{kl;ij} + \ldots = 0 \tag{2.5}$$

The following properties are known of the symmetric symplectic connection (see e.g. [22]): Γ_{ijk} is total symmetric in each Darboux coordinate system and the curvature tensor has the symmetry property

$$R_{kl;ij} = R_{lk;ij}\,. \tag{2.6}$$

This fact could be immediately seen by choosing a coordinate system where ω_{ij} are constant.

Consider an open covering of $\mathcal{M}$. In each domain U_α the symplectic form can be represented as

$$\omega = d\rho^\alpha\,, \qquad \omega_{ij} = \partial_i\rho^\alpha_j - \partial_j\rho^\alpha_i\,, \tag{2.7}$$

where $\rho^\alpha = \rho^\alpha_i dx^i$ is the symplectic potential in U_α. In the overlapping $U_{\alpha\beta} = U_\alpha \cap U_\beta$ we have

$$\rho^\alpha - \rho^\beta = \phi^{\alpha\,\beta}\,, \qquad d\phi^{\alpha\beta} = 0\,. \tag{2.8}$$

The transition 1-forms $\phi^{\alpha\beta}$ obviously satisfies

$$\phi^{\alpha\beta} + \phi^{\beta\alpha} = 0\,, \qquad \phi^{\alpha\beta} + \phi^{\beta\gamma} + \phi^{\gamma\alpha} = 0\,. \tag{2.9}$$

in the overlappings $U_\alpha \cap U_\beta$ and $U_\alpha \cap U_\beta \cap U_\gamma$ respectively.

Given an atlas U_α and the symplectic potential ρ^α defined in each domain U_α one can construct an affine bundle $\mathcal{T}^*_\rho\mathcal{M}$ over $\mathcal{M}$. Namely, for each domain U_α with the local coordinates x^i_α (index α indicates that x^i_α are the coordinates on U_α) choose the fibre to be R^N ($N = \dim(\mathcal{M})$) with the coordinates p^α_i. In the overlapping $U_\alpha \cap U_\beta$ we prescribe the following transition law

$$p^\alpha_i = p^\beta_j \frac{\partial x^j_\beta}{\partial x^i_\alpha} + \phi^{\alpha\beta}_i\,. \tag{2.10}$$

(summation over the repeating indices $\alpha, \beta, \ldots$ is not implied). Here the coefficients $\phi^{\alpha\beta}_i$ of the 1-form $\phi^{\alpha\beta}$ are introduced by $\phi^{\alpha\beta} = \phi^{\alpha\beta}_i dx^i_\alpha$. It is easy to check that the transition law (2.10) satisfies standard conditions in the overlapping of two and three domains and thus it determines $\mathcal{T}^*_\rho\mathcal{M}$ as a bundle.

The difference between usual cotangent bundle $T^*\mathcal{M}$ and $\mathcal{T}^*_\rho\mathcal{M}$ is that the structure group of the former is $GL(N, R)$ while that of the later is a group of affine transformations of R^N.

As the transformation law (2.10) of the variables p_i differs from that of the coordinates on the fibres of the standard cotangent bundle by a closed 1-form only, then $\mathcal{T}^*_\rho\mathcal{M}$ is also equipped with the canonical symplectic form $dp_i \wedge dx^i$. In particular, the corresponding Poisson bracket has also the canonical form

$$\{f, g\} = \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial x^i}\,. \tag{2.11}$$

An important feature of this construction is that the surface $\mathcal{L}$ defined by the equations

$$\theta_i(x, p) \equiv \rho_i - p_i = 0\,, \tag{2.12}$$

is a submanifold in $\mathcal{T}^*_\rho\mathcal{M}$. Indeed, these equations can be considered as those determining the smooth section of $\mathcal{T}^*_\rho\mathcal{M}$. Moreover, considered as a manifold, $\mathcal{L}$ is isomorphic to the original manifold $\mathcal{M}$. Indeed, $\mathcal{L}$ is a section of the bundle $\mathcal{T}^*_\rho\mathcal{M}$ and $\mathcal{M}$ is a base of $\mathcal{T}^*_\rho\mathcal{M}$; the projection $\pi : \mathcal{T}^*_\rho\mathcal{M} \to \mathcal{M}$ to $\mathcal{L} \subset \mathcal{T}^*_\rho\mathcal{M}$ obviously establishes an isomorphism between $\mathcal{M}$ and $\mathcal{L}$. Note also that quantities θ_i transforms as coefficients of a 1-form.

From the viewpoint of the Hamiltonian constrained dynamics, θ_i are the second class constraints, as their Poisson brackets in $\mathcal{T}^*_\rho\mathcal{M}$ form an invertible matrix

$$\{\theta_i, \theta_j\} = \omega_{ij} \tag{2.13}$$

The Dirac bracket in $\mathcal{T}^*_\rho\mathcal{M}$, being built of the constraints (2.12),

$$\{f(x,p), g(x,p)\}_D \equiv \{f(x,p), g(x,p)\} - \{f(x,p), \theta_i\}\,\omega^{ij}\,\{\theta_j, g(x,p)\} \tag{2.14}$$

can be considered as Poisson bracket defined on the constraint surface $\mathcal{L}$. As the Dirac bracket is nondegenerate on the constraint surface $\mathcal{L}$, the latter is a symplectic manifold. One can see that

$\mathcal{L}$ is isomorphic to $\mathcal{M}$ when each one is considered as a symplectic manifold. Indeed, any function $f(x,p)$ on $\mathcal{T}^*_\rho\mathcal{M}$ can be reduced on $\mathcal{L}$ to the function $f_0(x) = f(x,p)|_{p_i=\rho_i(x)}$, while the function $f_0(x)$ can be understood as defined on the original manifold $\mathcal{M}$. The Dirac bracket (2.14) between any functions $f(x,p)$, $g(x,p)$ coincides on the constraint surface $\mathcal{L}$ determined by constraints (2.12) to the Poisson bracket between their projections to $\mathcal{M}$:

$$\begin{aligned} &\{f(x,p), g(x,p)\}_D|_{p_i\,=\,\rho_i(x)} = \{f_0(x), g_0(x)\}_{\mathcal{M}}\,, \\ &f_0(x) = f(x,p)|_{p_i\,=\,\rho_i(x)}\,, \qquad g_0(x) = g(x,p)|_{p_i\,=\,\rho_i(x)}\,. \end{aligned} \tag{2.15}$$

This obvious fact provides the equivalence of the constrained dynamics in $\mathcal{T}^*_\rho\mathcal{M}$ and the Hamiltonian one in $\mathcal{M}$. The quantization problem for the symplectic manifold $\mathcal{M}$ is thereby equivalent to the quantization of the second class constrained theory in $\mathcal{T}^*_\rho\mathcal{M}$.

2.2. **Conversion to the first class.** In this section we suggest a procedure to convert second class constraints (2.12) into the first class ones. The procedure explicitly accounts the geometry of the original manifold $\mathcal{M}$, and makes transparent the relationship between the BRST and Fedosov's constructions.

Now we further enlarge the phase space. Namely we embed $\mathcal{T}^*_\rho\mathcal{M}$ into the $\mathcal{T}^*_\rho\mathcal{M} \oplus T\mathcal{M}$. Let y^i be the natural coordinates on the fibres of tangent bundle $T\mathcal{M}$. In order to equip the extended phase space $\mathcal{T}^*_\rho\mathcal{M} \oplus T\mathcal{M}$ with the Poisson bracket one has to engage an additional structure, a symplectic connection.

In view of the properties (2.3), (2.4) and (2.5) one can equip $\mathcal{T}^*_\rho\mathcal{M} \oplus T\mathcal{M}$ with a symplectic structure. Indeed, let the bracket operation $\{\cdot\,,\cdot\}$ on $\mathcal{T}^*_\rho\mathcal{M} \oplus T\mathcal{M}$ be given by

$$\begin{array}{rclrcl} \{x^i, p_j\} &=& \delta^i_j\,, & \{y^i, y^j\} &=& \omega^{ij}\,, \\ \{p_i, y^j\} &=& \Gamma^j_{il}y^l\,, & \{p_i, p_j\} &=& \frac{1}{2}R_{mn;\,ij}y^m y^n\,, \\ \{x^i, x^j\} &=& 0\,, & \{x^i, y^j\} &=& 0\,. \end{array} \tag{2.16}$$

Then (2.16) is a Poisson bracket provided Γ is a symmetric symplectic connection.

Considering Jacobi identities for the Poisson brackets (2.16) between y^i, y^j, p_k; $\quad y^i, p_j, p_k$ and p_i, p_j, p_k one arrives at (2.3),(2.4) and (2.5) respectively. In this sense, the Poisson bracket (2.16) might be viewed as a generating structure for the Fedosov geometry which is based just on the relations (2.3), (2.4) and (2.5).

In what follows instead of smooth functions $\mathcal{C}^\infty(\mathcal{T}^*_\rho\mathcal{M} \oplus T\mathcal{M})$ on $\mathcal{T}^*_\rho\mathcal{M} \oplus T\mathcal{M}$, we will consider formal power series in y^i with coefficients in $\mathcal{C}^\infty(\mathcal{T}^*_\rho\mathcal{M})$. Moreover, we restrict the coefficients to be polynomials in p_i. The reason for this is that y^i serve as "conversion variables" and one has to allow formal power series in y. As p_i play a role of momenta, it is a usual technical restriction in physics to allow only polynomials in p_i. Thus speaking about "functions" on $\mathcal{T}^*_\rho\mathcal{M} \oplus T\mathcal{M}$ we mean sections of the appropriate vector bundle over $\mathcal{M}$. The Poisson bracket (2.16) is well defined in the algebra of these "functions".

There is a simple formula which clarifies the geometrical meaning of this Poisson bracket:

$$\{p_i, f(x,y)\} = -\nabla_i f(x,y)\,, \tag{2.17}$$

where the function $f(x,y)$ (formal power series in y) is understood in the r.h.s. of (2.17) as an inhomogeneous symmetric tensor field on $\mathcal{M}$, i.e.

$$f(x,y)=\sum_{k=0}^{\infty} f_{i_1\dots i_k}(x)\,y^{i_1}\dots y^{i_k}\,,\quad \{f(x,y),p_j\}=\sum_{k=0}^{\infty}\big(\nabla_j f_{i_1\dots i_k}(x)\big)\,y^{i_1}\dots y^{i_k}\,. \tag{2.18}$$

The goal of the conversion procedure is to continue the second class constraints $\theta_i(x,p)$ (2.12), being the functions on $\mathcal{T}^*_\rho\mathcal{M}$, into $\mathcal{T}^*_\rho\mathcal{M}\oplus T\mathcal{M}$ $\theta_i\to \mathrm{T}_i(x,p,y)$, $\mathrm{T}_i|_{y=0}=\theta_i$ in such a way that T_i have to be the first class in the extended manifold. Thus we look for the functions T_i such that

$$\{\mathrm{T}_i,\mathrm{T}_j\}=0\,,\quad \mathrm{T}_i|_{y=0}=\theta_i\,. \tag{2.19}$$

We also prescribe T_i to transform as the coefficients of the 1-form under the change of coordinates on $\mathcal{M}$, as the original constraints θ_i (2.12) have the same transformation property. Existence of the Abelian conversion is established by the following[2]

Proposition 2.1. *Equation* (2.19) *has a solution.*

Proof. Let us look for the solution to the equation (2.19) in the form of the explicit power series expansion in the variables y^i.

$$\mathrm{T}_i=\sum_{r=0}^{\infty}\tau_i^r \tag{2.20}$$

It turns out that it is sufficient to consider functions τ_i^r, $r\geq 1$ which do not depend on the momenta p_i:

$$\tau_i^0(x)=\theta_i\,,\quad \tau_i^r=\tau^r_{i\,j_1\dots j_r}(x)y^{j_1}\dots y^{j_r}\,. \tag{2.21}$$

Functions $\tau^r_{i\,j_1\dots j_r}(x)$ can be considered as the coefficients of the tensor field on $\mathcal{M}$ that is symmetric w.r.t. all indices except the first one. In the zeroth and first order we respectively have

$$\omega_{ij}+\tau^1_{i\,l}\omega^{l\,k}\tau^1_{jk}=0\,,\qquad \nabla_{[i}\tau^1_{j]k}+2\tau^2_{[ilk}\omega^{lm}\tau^1_{j]m}=0\,, \tag{2.22}$$

with $[i,j]$ standing for antisymmetrisation in i,j. There is a particular solution to these equations:

$$\tau^1_{i\,k}=-\omega_{ik}\,,\qquad \tau^2_{ijk}=0\,. \tag{2.23}$$

Taking $\tau^1_{ij}=-\omega_{ij}$ one can in fact consider more general solutions for τ^2_{ijk}. In this case, the second equation of (2.22) implies that $\Gamma_{ijk}+\tau^2_{ijk}$ are the coefficients of a symmetric symplectic connection on $\mathcal{M}$. This arbitrariness in the solution of (2.22) can be absorbed by the redefinition of the symmetric symplectic connection entering the Poisson bracket (2.16).

The ambiguity in τ^1_{il} might be able to reflect additional geometrical structures on $\mathcal{M}$. As we will see below, standard Fedosov's construction of the star-product on $\mathcal{M}$ corresponds to the "minimal" solution (2.23). However we consider here a general solution to (2.22).

Taking any fixed solution to Eq. (2.22) for τ_i^1,τ_i^2 one sees that in the r-th ($r\geq 2$) order in y Eq. (2.19) implies

$$\left\{\tau^1_{[i},\tau^{r+1}_{j]}\right\}+B^r_{ij}=0\,, \tag{2.24}$$

[2]The local proof for the respective existence theorem is known [6]. However we give here a proof with a due regard to the global geometry which is based on the Poisson bracket (2.16).

where the quantities B^r_{ij} are given by

$$(2.25)\qquad \begin{aligned} B^2_{ij} &= \nabla_{[i}\tau^2_{j]} + \{\tau^2_i, \tau^2_j\} + \frac{1}{2}R_{mn;ij}y^m y^n\,,\\ B^r_{ij} &= \nabla_{[i}\tau^r_{j]} + \sum_{t=0}^{r-2}\left\{\tau^{r-t}_i, \tau^{t+2}_j\right\}\,, \qquad r \geq 3\,. \end{aligned}$$

Now relations (2.24) are to be considered as the equations determining τ^{r+1}_i. We need the following

Lemma 2.1. *Let the quantity $A_{ij}(x,y)$ be such that $A_{ij} + A_{ji} = 0$ and $\{\tau^1_i, A_{jk}\} + \mathrm{cycle}(i,j,k) = 0$ then there exist C_i such that*

$$(2.26)\qquad A_{ij} = \left\{\tau^1_{[i}, C_{j]}\right\}\,.$$

The statement is an obvious generalisation of the standard Poincare Lemma. In the case where $\tau^1_{i\,k} = -\omega_{ik}$, it is precisely the Poincare Lemma.

It follows from the lemma that equation (2.24) has a solution iff B_{ij} satisfies

$$(2.27)\qquad \{\tau^1_i, B_{jk}\} + \mathrm{cycle}(i,j,k) = 0\,.$$

To show that it takes place let us introduce the partial sum

$$(2.28)\qquad \mathrm{T}^s_i = \sum_{t=0}^{s}\tau^t_i$$

and consider expression

$$(2.29)\qquad \{\mathrm{T}^s_i, \mathrm{T}^s_j\} = \sum_{t=1}^{(s-1)}(\left\{\tau^1_{[i}, \tau^{t+1}_{j]}\right\} + B^t_{ij}) + B^s_{ij} + \dots\,.$$

where $\dots$ denote terms of order higher than s in y^i. Assume that Eqs. (2.24) hold for $1 \leq r \leq s-1$. Excluding the contribution of order $s-1$ from the Jacobi identity

$$(2.30)\qquad \{\mathrm{T}^s_i, \{\mathrm{T}^s_j, \mathrm{T}^s_k\}\} + cycle(i,j,k) = 0\,,$$

one arrives at

$$(2.31)\qquad \{\tau^1_i, B^s_{jk}\} + cycle(i,j,k) = 0\,.$$

It follows from the lemma that for $r = s$ equation (2.24) considered as that on τ^{r+1} admits a solution. The induction implies that Eq. (2.19) also admits solution, at least locally.

To show that solution exists globally we construct the particular solution to the equation (2.24) for $r = s$:

$$(2.32)\qquad \tau^{s+1}_i = -\frac{2}{s+2}B^s_{ij}(K^{-1})^j_m y^m\,, \qquad K^j_i = \tau^1_{il}\omega^{lj}\,.$$

This solution satisfies the condition

$$(2.33)\qquad \tau^{s+1}_i(K^{-1})^i_j y^j = 0\,,$$

which does not depend on the choice of the local coordinates on $\mathcal{M}$. Given a fixed first term τ^1_i, Eq. (2.33) can be considered as the condition on the solution to the equation (2.19). It is easy to see that solution to (2.19) is unique provided the condition (2.33) is imposed. Indeed, general solution to the Eq. (2.24) is given by

$$(2.34)\qquad \overline{\tau}^{s+1}_i = \tau^{s+1}_i + \{\tau^1_i, C^s\}\,.$$

where τ_i^{s+1} is the particular solution (2.32) and $C^s = C^s(x,y)$ is an arbitrary function. One can check that condition (2.33) implies that second term in (2.34) vanishes. Choosing τ_i to satisfy (2.33) in each domain U_α one gets the global solution to Eq. (2.19). □

Thus we have arrived at the first class constrained system, with the constraints being (2.20). An observable of the first class constrained system is a function $A(x,y,p)$ satisfying

$$\{\mathrm{T}_i, A\} = V_i^j \mathrm{T}_j \,. \tag{2.35}$$

for some functions $V_j^i(x,y,p)$. Observables $A(x,y,p)$ and $B(x,y,p)$ are said equivalent iff their difference is proportional to the constraints, i.e.

$$A - B = V^i \mathrm{T}_i \tag{2.36}$$

for some functions $V^i(x,y,p)$. In each equivalence class of the observables there is a unique representative which does not depend on the momenta p_i. Indeed, let $A(x,y,p)$ be an observable. Since it is a polynomial in p it can be rewritten as

$$A = a(x,y) + a_1^i(x,y)\mathrm{T}_i + \dots \,, \tag{2.37}$$

where $\dots$ stands for higher (but finite) orders in T. It follows from (2.35) that $a(x,y)$ satisfies

$$\{\mathrm{T}_i, a(x,y)\} = 0\,. \tag{2.38}$$

Now we are going to show that the Poisson algebra of the inequivalent observables is isomorphic to the algebra of functions on $\mathcal{M}$.

Proposition 2.2. *Eq.* (2.38) *has a unique solution $a(x,y)$ satisfying $a(x,0) = a_0(x)$ for any given function $a_0 \in \mathcal{C}^\infty(\mathcal{M})$*

A proof is a direct analogue of that of Proposition (2.1).

Given two solutions $a(x,y)$ and $b(x,y)$ of Eq. (2.38) corresponding to the boundary conditions $a(x,0) = a_0(x)$ and $b(x,0) = b_0(x)$ one can check that

$$\{a,b\}\,|_{y=0} = \{a_0, b_0\}_{\mathcal{M}} \,. \tag{2.39}$$

Thus the isomorphism is obviously seen between Poisson algebra of observables of the first class theory and the Poisson algebra of functions of symplectic manifold $\mathcal{M}$. This shows that the constructed first class constrained system is equivalent to the original unconstrained system on $\mathcal{M}$.

2.3. **An extended Poisson bracket and the BRST charge.** According to the BFV quantization prescription we have to extend the phase space introducing Grassmann odd ghost variable $\mathcal{C}^i$ to each constraint T_i and the ghost momenta $\mathcal{P}_i$ canonically conjugated to $\mathcal{C}^i$.

We wish the ghost variables $\mathcal{C}^i$ and $\mathcal{P}_i$ to transform under the change of local coordinate system on the base $\mathcal{M}$ as the components of the vector field and 1-form respectively. Thus in the intersection $U_\alpha \cap U_\beta$ one has

$$\mathcal{C}^i_\alpha = \mathcal{C}^j_\beta \frac{\partial x^i_\alpha}{\partial x^j_\beta}\,, \qquad \mathcal{P}^\alpha_i = \mathcal{P}^\beta_j \frac{\partial x^j_\beta}{\partial x^i_\alpha}\,. \tag{2.40}$$

Further define the following Poisson brackets on the extended phase space:

$$\{\mathcal{C}^i, \mathcal{P}_j\} = \delta^i_j\,, \quad \{\mathcal{C}^i, \mathcal{C}^j\} = 0\,, \quad \{\mathcal{P}_i, \mathcal{P}_j\} = 0\,, \tag{2.41}$$

the brackets between ghosts and other variables vanish and the brackets among x, y, p keep their form (2.16). If the momenta p_i were still transformed according to (2.10) the Poisson bracket relations would not be invariantly defined. In order to make them invariant we modify the transformation properties of the momenta p_i: in the overlapping $U_\alpha \cap U_\beta$ of coordinate neighborhoods the transition law (2.10) is modified by ghost contribution as follows

$$p^\alpha_i = p^\beta_j \frac{\partial x^j_\beta}{\partial x^i_\alpha} + \phi^{\alpha\beta}_i + \mathcal{P}^\beta_l \mathcal{C}^k_\beta \frac{\partial x^j_\alpha}{\partial x^k_\beta} \frac{\partial^2 x^l_\beta}{\partial x^i_\alpha \partial x^j_\alpha}\,. \tag{2.42}$$

One can easily check that the Poisson brackets (2.41) preserve their form under the change of coordinates on $\mathcal{M}$ and the corresponding change of other variables. Thus the extended Poisson bracket is globally defined.

Let us explain the geometry of the extended phase space constructed above. Let $\overline{\rho}$ be the pullback of the symplectic potential ρ from the base $\mathcal{M}$ to the odd tangent bundle $\Pi T\mathcal{M}$ over $\mathcal{M}$ (we view ghost variables $\mathcal{C}^i$ as natural coordinates on the fibres of $\Pi T\mathcal{M}$ over $\mathcal{M}$). It was shown in section **2.1** that given a (locally defined) 1-form ρ_α in each coordinate neighborhood U_α of manifold $\mathcal{M}$ one can construct a modified cotangent bundle $\mathcal{T}^*_\rho\mathcal{M}$ over $\mathcal{M}$. If in addition 1-form is such that $d\rho_\alpha = d\rho_\beta$ in the intersection $U_\alpha \cap U_\beta$ the modified cotangent bundle is equipped with the canonical symplectic structure. Applying this construction to the $\Pi T\mathcal{M}$ with the (locally defined) 1-form $\overline{\rho}$ one arrives at the affine bundle $\mathcal{T}^*_{\overline{\rho}}(\Pi T\mathcal{M})$.

Now it is easy to see that the Poisson bracket (2.41) is nothing but the canonical Poisson bracket on the modified cotangent bundle $\mathcal{T}^*_{\overline{\rho}}(\Pi T\mathcal{M})$. While the variable p_i, being canonically conjugated to the variable x^i, has the transition law (2.42).

Finally, the whole extended phase space $\mathcal{E}$ of the BFV formulation of the converted system is the vector bundle

$$\mathcal{E} = \mathcal{T}^*_{\overline{\rho}}(\Pi T\mathcal{M}) \oplus T\mathcal{M}, \tag{2.43}$$

with y^i being the natural coordinates on the fibres of $T\mathcal{M}$ (here $\mathcal{T}^*_{\overline{\rho}}(\Pi T\mathcal{M})$ is considered as the vector bundle over $\mathcal{M}$ and $\oplus$ denotes the direct sum of vector bundles.) It goes without saying that "functions" on $\mathcal{E}$ are formal power series in y^i. In what follows we denote the algebra of "functions" on the extended phase space by $\mathcal{F}(\mathcal{E})$.

According to the BFV quantization procedure we prescribe ghost degrees to each the variable

$$\mathrm{gh}(\mathcal{C}^i) = 1\,, \qquad \mathrm{gh}(\mathcal{P}_i) = -1\,, \qquad \mathrm{gh}(x^i) = \mathrm{gh}(y^i) = \mathrm{gh}(p_i) = 0\,. \tag{2.44}$$

Thus the Poisson bracket carries vanishing ghost number. A ghost charge $\mathbf{G}$ can be realized as

$$\mathbf{G} = \mathcal{C}^i \mathcal{P}_i\,. \tag{2.45}$$

Indeed

$$\{\mathbf{G}, \mathcal{C}^i\} = \mathcal{C}^i\,, \quad \{\mathbf{G}, \mathcal{P}_i\} = -\mathcal{P}_i\,, \quad \{\mathbf{G}, x^i\} = \{\mathbf{G}, y^i\} = \{\mathbf{G}, p_i\} = 0\,. \tag{2.46}$$

The BRST charge of the converted system is given by

$$\Omega = \mathcal{C}^i \mathrm{T}_i \,. \tag{2.47}$$

It satisfies the nilpotency condition

$$\{\Omega, \Omega\} = 0 \,. \tag{2.48}$$

w.r.t. the extended Poisson bracket. The BRST charge Ω carries unit ghost number:

$$\{\mathbf{G}, \Omega\} = \Omega \,. \tag{2.49}$$

Relations (2.48) and (2.49) are known as the BRST algebra. A BRST observable is a function A satisfying

$$\{\Omega, A\} = 0 \,, \qquad \mathrm{gh}(A) = 0 \,. \tag{2.50}$$

A BRST observable of the form $\{\Omega, B\}$ is called trivial. The algebra of the inequivalent observables (i.e. quotient of all observables modulo trivial ones) is thus the zero-ghost-number cohomology of the classical BRST operator $\{\Omega, \,\cdot\,\}$. The Poisson bracket on the extended phase space obviously determines the Poisson bracket in the BRST cohomology. Thus the space of inequivalent BRST observables is a Poisson algebra.

Proposition 2.3. *The Poisson algebra of inequivalent observables of the BFV theory with the BRST charge* (2.47) *and ghost charge* (2.45) *is isomorphic with the Poisson algebra of functions on the symplectic manifold* $\mathcal{M}$*.*

Proof. Let $\mathfrak{A}_0 \subset \mathcal{F}(\mathcal{E})$ be the algebra of functions depending on $x, y, \mathcal{C}$ only. Any function from $\mathfrak{A}_0$ is a pullback of some function on $\Pi T\mathcal{M} \oplus T\mathcal{M}$ to the entire extended phase space $\mathcal{E}$ (2.43). Let also $A = A(x, y, p, \mathcal{C}, \mathcal{P})$ be a BRST observable. Then one can check that there exist functions $a \in \mathfrak{A}_0$ and $\Psi \in \mathcal{F}(\mathcal{E})$ such that

$$A = a + \{\Omega, \Psi\} \,, \qquad \mathrm{gh}(a) = 0 \,, \quad \{a, \Omega\} = 0 \,. \tag{2.51}$$

As a matter of fact a can not depend on $\mathcal{C}^i$ as it has zero ghost number. Finally, it follows from Proposition **2.2** that the Poisson algebra of function on $\mathcal{M}$ is isomorphic with the Poisson algebra of zero-ghost-number BRST invariant functions from $\mathfrak{A}_0$. □

Thus at the classical level the initial Hamiltonian dynamics on $\mathcal{M}$ is equivalently represented as the BFV theory.

3. Quantization and quantum observables

In this section we find a Poisson subalgebra $\mathfrak{A}$ in the algebra of functions on the extended phase space which contains all the physical observables and the generators of the BRST algebra. Thus instead of quantizing the entire extended phase space it is sufficient to quantize just Poisson subalgebra $\mathfrak{A}$. This subalgebra can be easily quantized that results in a quantum BRST formulation of the effective first class constrained theory. The quantum BRST observables of the constructed system are isomorphic to the space of functions on the symplectic manifold $\mathcal{M}$. This isomorphism carries the star-multiplication from the algebra of quantum observables to the algebra of functions on $\mathcal{M}$, giving thus a deformation quantization of $\mathcal{M}$. It turns out that the star multiplication of the quantum BRST observables is the fibrewise multiplication of the Fedosov flat sections of the Weyl algebra bundle over $\mathcal{M}$. Finally we interpret all the basic objects of the Fedosov deformation quantization as those of the BRST theory.

3.1. **Quantization of the extended phase space.** Consider a Poisson subalgebra $\mathfrak{A} \subset \mathcal{F}(\mathcal{E})$ which is generated by subalgebra $\mathfrak{A}_0$ (subalgebra of functions depending on $x, y, \mathcal{C}$ only) and the elements

$$\mathbf{P} = \mathcal{C}^i\theta_i \equiv \mathcal{C}^i(\rho_i(x) - p_i)\,, \quad \mathbf{G} = \mathcal{C}^i\mathcal{P}_i\,, \qquad \mathbf{P},\, \mathbf{G} \in \mathcal{F}(\mathcal{E})\,. \tag{3.1}$$

The reason for considering $\mathfrak{A}$ is that $\mathfrak{A}$ is a minimal Poisson subalgebra of $\mathcal{F}(\mathcal{E})$ which contains, at least classically, all the BRST observables and both BRST and ghost charges (recall that $\mathbf{G}$ is precisely a ghost charge while BRST charge Ω can be represented in the form $\Omega = \mathbf{P} + \overline{\Omega}$, with $\overline{\Omega}$ being some element of $\mathfrak{A}_0$). A general homogeneous element a of $\mathfrak{A}$ has the form

$$a = \mathbf{P}^m\mathbf{G}^n a(x, y, \mathcal{C})\,, \quad m = 0, 1\,, \quad n = 0, 1, \ldots N\,, \quad N = \dim(\mathcal{M})\,, \quad a \in \mathfrak{A}_0\,, \tag{3.2}$$

Note that algebra $\mathfrak{A}$ is not free, it can be considered as the quotient of the free algebra generated (as a supercommutative algebra) by $\mathfrak{A}_0$ and the elements $\mathbf{P}, \mathbf{G}$ modulo the relations

$$\begin{gathered} \mathbf{P}^m\mathcal{C}^{i_1}, \ldots, \mathcal{C}^{i_{N-k-m+1}}\mathbf{G}^k = 0\,, \\ N = \dim(\mathcal{M})\,, \qquad k = 0, 1, \ldots, N+1\,, \quad m = 0, 1.\,, \quad N - k - m + 1 \geq 0\,. \end{gathered} \tag{3.3}$$

The definition of $\mathfrak{A}$ is in fact invariant in the sense that it is independent of the choice of the coordinates on $\mathcal{M}$. The basic Poisson bracket relations in $\mathfrak{A}$ read as

$$\begin{array}{rclrcl} \{\mathbf{P}, \mathbf{P}\} & = & R + \omega\,, & \{\mathbf{G}, \mathbf{P}\} & = & \mathbf{P}\,, \\ \{\mathbf{P}, a\} & = & \nabla a\,, & \{\mathbf{G}, \mathbf{G}\} & = & 0\,, \\ \{\mathbf{G}, a\} & = & \mathcal{C}^i \dfrac{\partial a}{\partial \mathcal{C}^i}\,, & \{a, b\} & = & \omega^{ij}\dfrac{\partial a}{\partial y^i}\dfrac{\partial b}{\partial y^j}\,, \end{array} \tag{3.4}$$

where $a(x, y, \mathcal{C})$ and $b(x, y, \mathcal{C})$ are arbitrary elements of $\mathfrak{A}_0$, $\nabla = \mathcal{C}^i\nabla_i$ is a covariant differential in $\mathfrak{A}_0$ and

$$R = \frac{1}{2}R_{kl;ij}\mathcal{C}^i\mathcal{C}^j\, y^k y^l\,, \quad \omega = \omega_{ij}\mathcal{C}^i\mathcal{C}^j\,, \qquad R, \omega \in \mathfrak{A}_0\,, \tag{3.5}$$

is the curvature of the covariant differential ∇ and the symplectic form respectively. It is easy to see that $\mathfrak{A}$ is closed w.r.t. the Poisson bracket and thus it is a Poisson algebra.

There is almost obvious star product which realizes deformation quantization of $\mathfrak{A}$ as a Poisson algebra. The explicit construction of the star product in $\mathfrak{A}$ is presented in Appendix **A**. In order to proceed with the BRST quantization of our system we will actually engage the star multiplication in $\mathfrak{A}_0 \subset \mathfrak{A}$ given by the Weyl star-product

$$(a \star b)(x, y, \mathcal{C}) = exp(-\frac{i\hbar}{2}\omega^{ij}\frac{\partial}{\partial y_1^i}\frac{\partial}{\partial y_2^j})a(x, y_1, \mathcal{C})b(x, y_2, \mathcal{C})|_{y_1 = y_2 = y}\,, \tag{3.6}$$

and the following commutation relations in $\mathfrak{A}$:

$$\begin{array}{rclrcl} [\mathbf{P}, a]_\star & = & -i\hbar\nabla a\,, & [\mathbf{P}, f(\mathbf{G})]_\star & = & i\hbar\mathbf{P}\dfrac{\partial}{\partial \mathbf{G}}F\,, \\ [\mathbf{P}, \mathbf{P}]_\star & = & -i\hbar(R + \Omega)\,, & [\mathbf{G}, a]_\star & = & -i\hbar\mathcal{C}^i\dfrac{\partial}{\partial \mathcal{C}^i}a\,, \end{array} \tag{3.7}$$

for any element $a \in \mathfrak{A}_0$ and a function $f(\mathbf{G})$ depending on $\mathbf{G}$ only. In what follows $\mathfrak{A}$ and $\mathfrak{A}_0$ considered as the associative algebras with respect to star-multiplication will be denoted by $\mathfrak{A}^q$ and $\mathfrak{A}_0^q$ respectively.

3.2. **The quantum BRST charge.** At the classical level, all the physical observables and generators of the BRST algebra (ghost charge $\mathbf{G}$ and the BRST charge Ω) belong to $\mathfrak{A}$. Thus to perform the BRST quantization of the first class constrained system one may restrict himself by the quantum counterpart $\mathfrak{A}^q$ of the Poisson algebra $\mathfrak{A}$.

Consider relations of the BRST algebra [3]

$$[\hat{\Omega},\hat{\Omega}]_\star \equiv 2\hat{\Omega} * \hat{\Omega} = 0\,, \qquad [\hat{\mathbf{G}},\hat{\Omega}]_\star = \hat{\Omega}\,, \qquad \hat{\Omega},\hat{\mathbf{G}} \in \mathfrak{A}^q\,. \tag{3.8}$$

The first equation implies the nilpotency of the adjoint action D of $\hat{\Omega}$ defined by

$$DA = \frac{i}{\hbar}[\hat{\Omega},A]_\star\,, \qquad a \in \mathfrak{A}^q\,, \tag{3.9}$$

Note that D preserves subalgebra $\mathfrak{A}_0^q \subset \mathfrak{A}^q$ and therefore D can be considered as an odd nilpotent differential in $\mathfrak{A}_0^q$.

Show the existence of the quantum BRST charge satisfying (3.8) whose classical limit coincides with classical BRST charge Ω from previous section. Instead of finding $\hbar$-corrections to the classical BRST charge it is convenient to construct $\hat{\Omega}$ at the quantum level from the very beginning. In order to formulate boundary conditions to be imposed on $\hat{\Omega}$ and for the technical convenience we introduce a useful degree [20, 14]. Namely, we prescribe the following degrees to the variables

$$\begin{aligned} \deg(x^i) = \deg(\mathcal{C}^j) = 0\,, \qquad \deg(p_i) = \deg(\mathcal{P}_i) = 2\,,\\ \deg(y^i) = 1\,, \qquad \deg(\hbar) = 2\,. \end{aligned} \tag{3.10}$$

The star-commutator in $\mathfrak{A}^q$ apparently preserves the degree.

Let us expand $\hat{\Omega}$ into the sum of homogeneous components

$$\hat{\Omega} = \sum_{r=0}^{\infty} \Omega^r\,, \qquad \deg(\Omega^r) = r\,. \tag{3.11}$$

Given a classical BRST charge Ω (2.47) which starts as $\Omega = \mathcal{C}^i\rho_i - \mathcal{C}^i p_i + \mathcal{C}^i\tau^1_{ij}y^j + \mathcal{C}^i\tau^2_{ijl}y^jy^l + \dots$ one can formulate boundary condition on the solution of (3.8) as follows

$$\Omega^0 = \mathcal{C}^i\rho_i\,, \qquad \Omega^1 = \mathcal{C}^i\tau^1_{ij}y^j\,. \qquad \Omega^2 = -\mathcal{C}^i p_i + \mathcal{C}^i\tau^2_{ijl}y^jy^l\,. \tag{3.12}$$

Proposition 3.1. *Equations* (3.8) *considered as those for* $\hat{\Omega}$ *has a solution satisfying boundary condition* (3.12)*.*

Proof. Eq. (3.8) evidently holds in the lowest order w.r.t. degree (3.10) provided respective classical BRST charge Ω satisfies $\{\Omega,\Omega\} = 0$. At the classical level the higher order terms in expansion of Ω w.r.t. y do not depend on the momenta $p_i, \mathcal{P}_i$. Thus these terms belong to $\mathfrak{A}_0$. It is useful to assume that the same occurs at the quantum level:

$$\Omega^r \in \mathfrak{A}_0^q \qquad r \geq 3\,. \tag{3.13}$$

[3]Here we have introduced a separate notation $\hat{\mathbf{G}}$ for the quantum ghost charge because it can differ from the classical ghost charge $\mathbf{G} = \mathcal{C}^i\mathcal{P}_i$ by an imaginary constant $\frac{i\hbar N}{2}$. This constant, of course, can not contribute to the commutation relations.

In the $r+2$-th ($r \geq 2$) degree Eq. (3.8) implies

$$\delta\Omega^{r+1} + B^r = 0\,, \tag{3.14}$$

where the quantity B^r is defined by Ω^t, $t \leq r$:

$$B^r = \frac{i}{2\hbar}\sum_{t=0}^{r-2}[\Omega^{t+2}, \Omega^{r-t}]_\star\,, \qquad \deg(B^r) = r\,, \tag{3.15}$$

and $\delta\,:\,\mathfrak{A}_0^q \to \mathfrak{A}_0^q$ stands for

$$\delta a = \frac{i}{\hbar}[\Omega^1, a]_\star = \mathcal{C}^i \tau^1_{ij}\omega^{jl}\frac{\partial}{\partial y^l}a\,, \qquad a \in \mathfrak{A}_0^q\,. \tag{3.16}$$

Note that δ is obviously nilpotent in $\mathfrak{A}_0^q$. It follows from the nilpotency of δ that the compatibility condition for the Eq. (3.14) is $\delta B^r = 0$. In fact it is a sufficient condition for Eq. (3.14) to admit a solution. Indeed, the cohomology of the differential δ is trivial when evaluated on functions at least linear in $\mathcal{C}$. To show it, we construct the "contracting homotopy " δ^{-1}. Namely, let δ^{-1} be defined by its action on a homogeneous element

$$a_{pq} = a_{i_1,\ldots,i_p\,;\,j_1,\ldots,j_q}(x)\, y^{i_1}\ldots y^{i_p}\mathcal{C}^{j_1}\ldots\mathcal{C}^{j_q}\,, \tag{3.17}$$

by

$$\begin{aligned} \delta^{-1}a_{pq} &= \frac{1}{p+q}y^i(K^{-1})_i^j\frac{\partial}{\partial\mathcal{C}^j}a_{pq}\,, \qquad p+q \neq 0 \\ \delta^{-1}a_{00} &= 0\,, \end{aligned} \tag{3.18}$$

where $(K^{-1})_i^j$ is inverse to $K_j^i = \tau^1_{il}\omega^{lj}$. For any $a = a(x, y, \mathcal{C})$ we have

$$a|_{y=\mathcal{C}=0} + \delta\delta^{-1}a + \delta^{-1}\delta a = a\,. \tag{3.19}$$

Since B^r is quadratic in $\mathcal{C}$ then the 3-rd term vanishes and $\delta B^r = 0$ implies $B^r = \delta\delta^{-1}B^r$ which in turn implies that Eq. (3.14) admits a solution.

Let us show that the necessary condition $\delta B^r = 0$ is fulfilled. To this end assume Ω^r to satisfy (3.14) for $r \leq s$. Thus the Jacobi identity

$$[\sum_{t=0}^{s}\Omega^t, [\sum_{t=0}^{s}\Omega^t, \sum_{t=0}^{s}\Omega^t]_\star]_\star = 0 \tag{3.20}$$

implies in the $s+3$-th degree that $\delta B^s = 0$.

The particular solution to (3.14) for $r = s$ evidently reads as

$$\Omega^{s+1} = -\delta^{-1}B^s \tag{3.21}$$

Iteratively applying this procedure one can construct a solution to Eq. (3.8) at least locally. To show that Eq. (3.8) admit a global solution we note that operators δ as well as δ^{-1} are defined in a coordinate independent way. It implies that particular solution (3.21) does not depend on the choice of the local coordinate system and thus it is a global solution.

The quantum BRST charge constructed above obviously satisfies

$$\delta^{-1}\Omega^r = 0\,, \qquad r \geq 3\,, \tag{3.22}$$

which can be considered as an additional condition on the solution to the Eq. (3.8). One can actually show that solution to the Eq. (3.8) is unique provided condition (3.22) is imposed on $\hat{\Omega}$.

Thus we have shown how to construct quantum BRST charge associated to the first class constraints T_i. $\square$

The operator δ which is extensively used in the proof plays crucial role in the BRST formalism. In the case of the first class constrained system, the counterpart of δ is known as the Koszule-Tate differential associated to the constraint surface [24, 23] while in the Lagrangian BV quantization [25] the respective counterpart of δ is the Koszule-Tate differential associated to the stationary surface.

3.3. **An algebra of the quantum BRST observables and the star-multiplication.** Observables in the BFV quantization are recognized as zero-ghost-number values closed w.r.t. to adjoint action D of BRST charge $\hat{\Omega}$ modulo exact ones; $\hat{a}$ is an observable iff

$$D\hat{a} \equiv \frac{i}{\hbar}[\hat{\Omega},\hat{a}]_\star = 0\,, \qquad [\hat{\mathbf{G}},\hat{a}]_\star = 0\,. \tag{3.23}$$

Two observables are said equivalent iff their difference is D-exact. The space of inequivalent observables is thus the zero-ghost-number cohomology of D.

Initially, the classical observables are the functions on the symplectic manifold $\mathcal{M}$. Now $\mathcal{M}$ is embedded into the extended phase space $\mathcal{E}$ (2.43). According to the BFV prescription the quantum extension of the initial observable a is an operator (symbol in our case) $\hat{a}$ of the quantum converted system that is the solution to the Eqs. (3.23) subjected to the boundary condition

$$\hat{a}|_{y=0} = a_0(x)\,. \tag{3.24}$$

Let us consider the algebra of quantum BRST observables in $\mathfrak{A}^q$. First study observables in $\mathfrak{A}^q_0 \subset \mathfrak{A}^q$.

Proposition 3.2. *Equations* (3.23) *have a unique solution belonging to* $\mathfrak{A}^q_0$ *for each initial observable* $a_0 = a_0(x)$.

Proof. Consider an expansion of $\hat{a}$ in the homogeneous components

$$\hat{a} = \sum_{r=0}^{\infty} a^r\,, \qquad \deg(a^r) = r\,. \tag{3.25}$$

The boundary condition (3.24) implies $a^0 = a_0(x)$. Eq. (3.23) obviously holds in the first degree. In the higher degrees we have

$$\delta a^{r+1} + B^r = 0\,, \tag{3.26}$$

where B^r is given by

$$B^r = \frac{i}{\hbar}\sum_{t=0}^{r-2}[\Omega^{t+2}, a^{r-t}]_\star\,, \qquad \deg(B^r) = r\,, \tag{3.27}$$

and Ω^t are terms of the expansion (3.11) of $\hat{\Omega}$ w.r.t. degree. Similarly to the proof of the Proposition **3.1** the necessary and sufficient condition for (3.26) to admit solution is $\delta B^r = 0$. To show that the condition holds indeed assume that Eqs. (3.26) hold for all $r \leq s$. Then consider the identity

$$[\hat{\Omega},[\hat{\Omega},\hat{a}]_\star]_\star = 0\,. \tag{3.28}$$

Excluding contribution of degree $s+3$ we arrive at

$$\delta B^s = 0\,. \tag{3.29}$$

The particular solution to the Eq. (3.26) for $r = s$ is

$$a^{s+1} = -\delta^{-1} B^s\,. \tag{3.30}$$

Iteratively applying this procedure one arrives at the particular solution to the Eq. (3.23) satisfying boundary condition $\hat{a}|_{y=0} = a_0(x)$.

Finally let us show the uniqueness. Taking into account that a^{s+1} belongs to $\mathfrak{A}^q_0$ and $\mathrm{gh}(a^{s+1}) = 0$ we conclude that a^{s+1} does not depend on $\mathcal{C}$. Thus the general solution to the equation (3.26) is given by

$$a^{s+1} = -\delta^{-1} B^s + C^{s+1}(x,\hbar)\,. \tag{3.31}$$

It is easy to see that the boundary condition requires $C^{s+1}(x,\hbar) = 0$ (recall that $(\delta^{-1}B^n)|_{y=0} = 0$.) □

Since equation (3.23) is linear it has a unique solution $\hat{a}$ satisfying the boundary condition $\hat{a}_{y=0} = a_0(x,\hbar)$ even if the initial observable a_0 was allowed to depend formally on $\hbar$. It follows from the Proposition **3.2** that the space of inequivalent quantum observables coincides with the space of classical observables (functions on $\mathcal{M}$) tensored by formal power series in $\hbar$. In other words, the space of zero-ghost-number cohomology of D evaluated in $\mathfrak{A}^q_0$ is isomorphic with $\mathcal{C}^\infty(\mathcal{M}) \otimes [[\hbar]]$, where $\mathcal{C}^\infty(\mathcal{M})$ is the algebra of functions on $\mathcal{M}$ and $[[\hbar]]$ denotes the space of formal power series in $\hbar$. In fact even a stronger statement holds

Theorem 3.1. *The space of inequivalent quantum BRST observables, i.e. the zero-ghost-number cohomology evaluated in $\mathfrak{A}^q$, is isomorphic to $\mathcal{C}^\infty(\mathcal{M}) \otimes [[\hbar]]$.*

A proof follows from observation that each zero-ghost-number cohomology class from $\mathfrak{A}^q$ has a representative in $\mathfrak{A}^q_0$. Further, Proposition **3.2** implies that the representative is unique and provides us with the explicit isomorphism between $\mathcal{C}^\infty(\mathcal{M}) \otimes [[\hbar]]$ and the space of inequivalent quantum observables.

Since the BRST differential $D = \frac{i}{\hbar}[\hat{\Omega},\cdot\,]_\star$ is an inner derivation in $\mathfrak{A}^q$ then the star multiplication in $\mathfrak{A}^q$ determines the star multiplication in the space of quantum BRST cohomology. Making use of isomorphism from **3.1** one can equip $\mathcal{C}^\infty(\mathcal{M}) \otimes [[\hbar]]$ with the associative multiplication, determining thereby a star product on $\mathcal{M}$. As the algebra of quantum BRST observables is a deformation of the Poisson algebra of classical ones, which in turn is isomorphic with the Poisson algebra $\mathcal{C}^\infty(\mathcal{M})$, the star-product on $\mathcal{M}$ satisfies standard correspondence conditions

$$\hat{a} \star \hat{b}|_{y=\hbar=0} = a_0 b_0\,, \qquad \frac{i}{\hbar}[\hat{a},\hat{b}]_\star|_{y=\hbar=0} = \{a_0, b_0\}_{\mathcal{M}}\,. \tag{3.32}$$

Here $\hat{a}$ and $\hat{b}$ are the symbols obtained by means of Proposition **3.2** starting from $a_0, b_0 \in \mathcal{C}^\infty(\mathcal{M})$.

3.4. **BFV-Fedosov correspondence.** To establish the correspondence with the Fedosov construction of the star product we note that the quantum algebra $\mathfrak{A}^q_0$ consisting of functions[4] of $x, y, \mathcal{C}$ is precisely the algebra of sections of the Weyl algebra bundle from [14] provided one identifies ghosts $\mathcal{C}^i$ with the basis 1-forms dx^i.

[4]Recall that in the BRST approach functions of auxiliary variables y are formal power series in y

Let us consider the quantum BRST charge $\hat{\Omega}$ corresponding to the boundary condition (2.23). An adjoint action of $\hat{\Omega}$ on $\mathfrak{A}^q_0$

$$Da \equiv \frac{i}{\hbar}[\hat{\Omega}, a]_\star = (\mathcal{C}_i \nabla_i - \mathcal{C}^i \frac{\partial}{\partial y^i})a + \frac{i}{\hbar}[\sum_{t=3}^{\infty} \Omega^t, a]_\star\,, \quad a \in \mathfrak{A}^q_0\,, \tag{3.33}$$

is precisely the Fedosov connection in the Weyl algebra bundle. Indeed, in the Fedosov-like notations

$$\delta = \mathcal{C}^i \frac{\partial}{\partial y^i}\,, \qquad \partial = \mathcal{C}^i \nabla_i\,, \qquad r = \sum_{t=3}^{\infty} \Omega^t \in \mathfrak{A}_0\,, \tag{3.34}$$

the star-commutator with quantum BRST charge $\hat{\Omega}$ can be rewritten as

$$Da \equiv \frac{i}{\hbar}[\Omega, a]_\star = (\partial - \delta)a + \frac{i}{\hbar}[r, a]_\star\,, \tag{3.35}$$

that makes transparent identification of the $\frac{i}{\hbar}[\hat{\Omega}, \cdot\,]_\star$ and the Fedosov connection in the Weyl algebra bundle. In particular, the zero-curvature condition is precisely the BFV quantum master equation $[\hat{\Omega}, \hat{\Omega}]_\star = 0$.

It follows from Theorem **3.1** that each equivalence class of the quantum BRST observables has a unique representative in $\mathfrak{A}^q_0$. Thus the inequivalent quantum BRST observables are the flat sections of the Weyl algebra bundle. In its turn the star multiplication (3.6) of quantum BRST observables (considered as the BRST invariant functions from $\mathfrak{A}^q_0$) is nothing but the Weyl product of the Fedosov flat sections of Weyl algebra bundle.

There is a certain distinction between BRST and Fedosov quantization. Unlike the Fedosov Abelian connection, the adjoint action of the BRST charge can be realized as an inner derivation of the associative algebra $\mathfrak{A}^q$. In particular, in the BRST approach the covariant differential D is strictly flat.

4. Conclusion

Summarize the results of this paper. First we construct a global embedding of a general symplectic manifold $\mathcal{M}$ into the modified cotangent bundle $\mathcal{T}^*_\rho\mathcal{M}$ as a second class constrained surface. Then we have elaborated globally defined procedure which converts the second class constrained system into the first class one that allows to construct the BRST description for the Hamiltonian dynamics in the original symplectic manifold. We have explicitly established the structure of the classical BRST cohomology in this theory and perform a straightforward quantum deformation of the classical Poisson algebra which contains all the observables and the BRST algebra generators. As all the values on the original symplectic manifold are identified with the observables of the BRST theory, we have thus quantized the general symplectic manifold. Finally, we establish a detailed relationship between the quantum BFV-BRST theory of the symplectic manifolds and Fedosov's deformation quantization.

The construction of the BRST embedding of the second class constrained theory, being done by means of the cohomological technique, allows to recognize the conversion procedure as some sort of deformation of the classical Poisson algebra of the second class system. This deformation has an essential distinction from that one which is usually studied in relation to switching on the interactions in classical gauge theories [26], although the cohomological technique is quite similar. As soon as *classical* deformation has been performed, the problem of the *quantum* deformation becomes transparent in the theory. Thus in the BRST approach a part of the deformation quantization problem is transformed, in a sense, in the problem of another deformation, classical in essence, while the quantization itself is almost obvious in the classically deformed system.

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Appendix A. Star-multiplication in $\mathfrak{A}$

Here we present an explicit form of the star product in the Poisson algebra $\mathfrak{A}$. Recall, that $\mathfrak{A}$ is a Poisson subalgebra of $\mathcal{F}(\mathcal{E})$ generated by $\mathfrak{A}_0$ (Poisson algebra of functions depending of $x, y, \mathcal{C}$ only) and the elements $\mathbf{P} = \mathcal{C}^i\theta_i$, $\mathbf{G} = \mathcal{C}^i\mathcal{P}_i$. The star product in $\mathfrak{A}_0$ could be defined by the Weyl multiplication (3.6). As for general elements of $\mathfrak{A}$, let us consider first $\mathbf{P}$-independent ones. For the general $\mathbf{P}$-independent elements $A(x, y, \mathcal{C}, \mathbf{G})$ and $B(x, y, \mathcal{C}, \mathbf{G})$ we postulate

$$\begin{aligned} A(x,y,\mathcal{C},\mathbf{G}) \star B(x,y,\mathcal{C},\mathbf{G}) = \big[&exp(-i\hbar(\mathbf{G}\frac{\partial}{\partial \mathbf{G}_1}\frac{\partial}{\partial \mathbf{G}_2} + \mathcal{C}^i\frac{\partial}{\partial \mathcal{C}^i_2}\frac{\partial}{\partial \mathbf{G}_1})) \\ &A(x,y,\mathcal{C}_1,\mathbf{G}_1) \star B(x,y,\mathcal{C}_2,\mathbf{G}_2)\big]\Big|_{\mathcal{C}_1=\mathcal{C}_2=\mathcal{C},\, \mathbf{G}_1=\mathbf{G}_2=\mathbf{G},} \end{aligned} \tag{A.1}$$

where the $\star$ in the r.h.s. is the Weyl multiplication (3.6) acting on y only. Finally, taking into account commutation relations (3.7) for the $\mathbf{P}$-dependent elements one can choose

$$\begin{aligned} (\mathbf{P}A) \star (B) &= \mathbf{P}(A \star B)\,, \\ (A) \star (\mathbf{P}B) &= (-1)^{\mathrm{p}(A)}\big[\mathbf{P}(A \star B) - i\hbar\mathbf{P}((\frac{\partial}{\partial \mathbf{G}}A) \star B) \\ &\quad + i\hbar((\nabla A) \star B) - (i\hbar)^2(\nabla\frac{\partial}{\partial \mathbf{G}}A) \star B\big]\,, \\ (\mathbf{P}A) \star (\mathbf{P}B) &= (-1)^{\mathrm{p}(A)}\big[\tfrac{i\hbar}{2}(R+\omega) \star A \star B + \tfrac{(i\hbar)^2}{2}(R+\omega) \star (\frac{\partial}{\partial \mathbf{G}}A) \star B \\ &\quad + i\hbar\mathbf{P}((\nabla A) \star B) - (i\hbar)^2\mathbf{P}((\nabla\frac{\partial}{\partial \mathbf{G}}A) \star B)\big]\,, \end{aligned} \tag{A.2}$$

where A and B are general $\mathbf{P}$-independent elements from $\mathfrak{A}$ and the star-product in the right hand sides of Eqs. (A.2) is that given by (A.1). The associativity of multiplication (A.1) and (A.2) can be verified directly.

This star multiplication can be thought of as that of $\mathbf{P}, x, y, \mathcal{C}, \mathbf{G}$-symbols which is also a Weyl symbol w.r.t. y-variables. If one considered the star-product in $\mathfrak{A}$ as that on functions explicitly depending on $p, \mathcal{P}$ then it would correspond to the $p, x, y, \mathcal{C}, \mathcal{P}$-symbol.

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