Improved Epstein–Glaser renormalization II. Lorentz invariant framework

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Abstract

The Epstein–Glaser type T-subtraction introduced by one of the authors in a previous paper is extended to the Lorentz invariant framework. The advantage of using our subtraction instead of Epstein and Glaser's standard W-subtraction method is especially important when working in Minkowski space, as then the counterterms necessary to keep Lorentz invariance are simplified. We show how T-renormalization of primitive diagrams in the Lorentz invariant framework directly relates to causal Riesz distributions. A covariant subtraction rule in momentum space is found, sharply improving upon the BPHZL method for massless theories.

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1 Introduction

This paper is a continuation of [1] by the second named author, in which an extension of functionals, v.gr. (convolutions of) powers of Feynman propagators, to distributions on configuration space, according to the spirit of Epstein–Glaser [2], was presented.

By modifying the use of, and relaxing the conditions on, the infrared regulators w introduced by Epstein and Glaser, very useful results on the distributions at the crossroads of mathematics and quantum field theory have been obtained; the relationship of this improved Epstein–Glaser subtraction, dubbed "T-renormalization", with Hadamard regularization, the minimal subtraction scheme in analytical regularization, and differential renormalization, has been treated there at length. Hereinafter that paper [1] will be denoted by I.

The discussion in I took place in the Euclidean framework introduced by Stora [3] in the realm of the Epstein–Glaser construction. We tackle in this paper the problem of going to the "physical world" with its symmetry group of transformations, namely the Minkowski space and the Lorentz group respectively.

The Lorentz covariance properties of extensions for powers of propagators are deeply related to the S-matrix covariance. Prima facie, Epstein–Glaser procedures are not covariant. A proof of existence of covariant time ordered products was first given in [2], working on momentum space. About ten years later, the problem was translated into a group cohomological question on configuration space by Popineau and Stora in [4] —another work which has remained at the status of preprint. The latter analysis is available in textbook form [5]. In relatively recent (also apparently unpublished) preprints [6, 7], explicit computations for "counterterms" reestablishing Lorentz invariance of the extensions have been performed.

A perfectly covariant method for the extension of distributions employed in quantum field theory exists, although it is rarely used: the "analytical regularization" method [8] of Bollini and Giambiagi.[1] It leads to quantum versions of the Riesz distributions of classical field theory [9–11].

In this paper we extend T-renormalization of primitive diagrams to the Lorentz-invariant framework, and show that it generalizes directly the causal or quantum Riesz distributions.

The proof that all difficulties in renormalization theory can be overcome through exclusive use of the T operation is work in progress; the third paper of the series [12] —from now on denoted III— deals with renormalization of nonprimitive diagrams and the Hopf algebra of Feynman graphs [13–15].

We do not suppose the reader to be familiar with causal Riesz distributions. Thus we turn to them in Section 3, after recalling the main properties of our "natural" extension method in Section 2. The covariance properties of T-renormalization are then addressed in Section 4. To fix ideas, we consider in Section 5 the basic primitively divergent graphs in massless φ^4_4 theory, and demonstrate by way of example the link to analytical regularization and causal Riesz distributions. Section 6 shows how T-renormalization improves upon the BPHZL formalism. In Section 7 we deal with the massive theory. We conclude with a brief discussion and outlook.

[1] the 'coinventors' of dimensional regularization: Il Nuovo Cimento **XIIB** (1972) 20.

2 Main features of T-renormalization

Let us begin by fixing some conventions. The scalar functionals we shall be dealing with (coming from the Wick theorem) are to be extended to the main diagonal of $\mathbb{R}^{4(n+1)}$. The latter can be regarded as the origin in $\mathbb{R}^{4n}$ thanks to translation invariance, allowing to set $x_{n+1} = 0$, for instance.

For fixed $j \in \{1, \dots, n\}$, to each fourcoordinate x_j (a point in $\mathbb{R}^4$) let $\alpha^j = (\alpha_0^j, \alpha_1^j, \alpha_2^j, \alpha_3^j)$ be a quadri-index, where α_μ^j is a nonnegative integer for each j and each Lorentz index μ. One has, according to Schwartz's notation [16], $\alpha^j! = \alpha_0^j!\alpha_1^j!\alpha_2^j!\alpha_3^j!$ and $|\alpha^j| = \alpha_0^j + \alpha_1^j + \alpha_2^j + \alpha_3^j$, as well as

$$x_j^{\alpha^j} := \prod_{\mu=0}^{3} (x_j^\mu)^{\alpha_\mu^j}, \qquad \partial_{\alpha^j} := \left(\frac{\partial}{\partial x_j}\right)^{\alpha^j} = \prod_{\mu=0}^{3} \left(\frac{\partial}{\partial x_j^\mu}\right)^{\alpha_\mu^j} =: \prod_{\mu=0}^{3} (\partial_\mu^j)^{\alpha_\mu^j}.$$

We also use a multiquadri-index $\alpha = (\alpha_\mu^j)$ notation: $\alpha! = \alpha^1!\alpha^2! \dots \alpha^n!$ and $|\alpha| = |\alpha^1| + |\alpha^2| + \dots + |\alpha^n|$, as well as

$$x^\alpha := \prod_{j=1}^{n} x_j^{\alpha^j}, \qquad \partial_\alpha := \prod_{j=1}^{n} \partial_{\alpha^j}.$$

We assume that the reader is familiar with the basic concepts of distribution theory. In this paper operators (Fourier transforms, actions of the Lorentz group, subtractions, ...) are defined on distributions always by transposition (or adjoint mapping), and we denote them by the *same* letter denoting the original action on test functions.

Consider scalar functionals $f(x_j)$ defined on $M_4^{n+1} \setminus D_{n+1}$, where $D_{n+1} := \{x_1 = \dots = x_{n+1}\}$ is the full diagonal; thus we assume that the diagram is primitive or that Bogoliubov's [17] disentangling operation $\bar{R}$ corresponding to all subgraphs (see III) has been performed. Those will be considered as functionals of the difference variables in $\mathbb{R}^{4n} \setminus \{0\}$. A tempered distribution $\tilde{f} \in \mathcal{S}'(\mathbb{R}^{4n})$ is an *extension* or *renormalization* of f if

$$\tilde{f}[\phi] \equiv \langle \tilde{f}, \phi \rangle = \int_{\mathbb{R}^{4n}} f(x)\phi(x)\, d^{4n}x$$

holds whenever ϕ belongs to $\mathcal{S}(\mathbb{R}^{4n} \setminus \{0\})$. In QFT one considers a generalized homogeneity degree, the *scaling degree* [18]. The scaling degree σ of a scalar distribution f at the origin of $\mathbb{R}^d$ is defined to be

$$\sigma(f) = \inf\{\, s : \lim_{\lambda \to 0} \lambda^s f(\lambda x) = 0 \,\} \qquad \text{for } f \in \mathcal{S}'(\mathbb{R}^{4n}),$$

where the limit is taken in the sense of distributions. Essentially this means that $f(x) = O(|x|^{-\sigma(f)})$ as $x \to 0$ in the Cesàro or distributional average sense [19].

Let then $\sigma(f) = a$, with a an integer, and $k = a - 4n \geq 0$. Then, $f \notin L^1_{\mathrm{loc}}(\mathbb{R}^{4n})$. The simplest way to get an extension of f would appear to be standard Taylor series surgery: throw away the k-jet $j_0^k\phi$ of ϕ at the origin, and define $\tilde{f}$ by transposition:

$$\langle \tilde{f}, \phi \rangle = \langle R_0^k f, \phi \rangle := \langle f, R_0^k \phi \rangle,$$

where $R_0^k\phi := \phi - j_0^k\phi$ is the Taylor remainder. Using Lagrange's integral formula for R_0^k, and exchanging integrations, one appears to obtain an explicit integral formula for $R_0^k f$:

$$R_0^k f(x) \equiv T_1 f(x) := (-)^{k+1}(k+1) \sum_{|\beta|=k+1} \partial_\beta \left[\frac{x^\beta}{\beta!} \int_0^1 dt\, \frac{(1-t)^k}{t^{k+4n+1}} f\left(\frac{x}{t}\right)\right]. \tag{1}$$

The trouble with (1) is that the remainder $R_0^k\phi$ is not a test function and therefore, unless the infrared behaviour of f is good, we can end up with an undefined integral. Actually, for the needs of theories with only massive fields, formula (1) is largely sufficient. However in theories with massless particles, f is typically an homogeneous function with an algebraic singularity, the infrared behaviour is pretty bad, and $-4n$ is also the critical degree. A way to avoid the problem is to *weight* the Taylor subtraction. Epstein and Glaser [2] introduced infrared regulators w with the properties $w(0) = 1$ and $w^{(\alpha)}(0) = 0$ for $0 < |\alpha| \leq k$, as well as projector maps $\phi \mapsto W_w\phi$ on $\mathcal{S}(\mathbb{R}^d)$ given by

$$W_w\phi(x) := \phi(x) - w(x) j_0^k \phi(x). \tag{2}$$

There is a considerable amount of overkill in (2). We argued in I that one can, and should, weight only the *last* term of the Taylor expansion. This leads to the definition used in this paper, at variance with Epstein and Glaser's:

$$T_w\phi(x) := \phi(x) - j_0^{k-1}(\phi)(x) - w(x) \sum_{|\alpha|=k} \frac{x^\alpha}{\alpha!}\, \phi^{(\alpha)}(0). \tag{3}$$

Just $w(0) = 1$ is now required in principle for the weight function. T_w is also a projector. To obtain an integral formula for it, start from

$$T_w\phi = (1-w) R_0^{k-1}\phi + w R_0^k \phi.$$

By transposition, using (1), we derive

$$\begin{aligned} T_w f(x) = (-)^k k \sum_{|\alpha|=k} \partial_\alpha &\left[\frac{x^\alpha}{\alpha!} \int_0^1 dt\, \frac{(1-t)^{k-1}}{t^{k+d}} f\left(\frac{x}{t}\right)\left(1 - w\left(\frac{x}{t}\right)\right)\right] \\ &+ (-)^{k+1}(k+1) \sum_{|\beta|=k+1} \partial_\beta \left[\frac{x^\beta}{\beta!} \int_0^1 dt\, \frac{(1-t)^k}{t^{k+d+1}} f\left(\frac{x}{t}\right) w\left(\frac{x}{t}\right)\right]. \end{aligned} \tag{4}$$

Consider the functional variation of the renormalized amplitudes with respect to w. One has

$$\left\langle \frac{\delta}{\delta w} T_w f, \psi \right\rangle := \frac{d}{d\lambda} T_{w+\lambda\psi}\, f \,\Big|_{\lambda=0}.$$

Equation (3) yields:

$$\frac{\delta}{\delta w} T_w f[\cdot] = (-)^{k+1} \sum_{|\alpha|=k} f[x^\alpha \cdot]\, \frac{\partial_\alpha \delta}{\alpha!}\,, \tag{5}$$

independently of w. The combination $\frac{(-)^{|\alpha|}}{\alpha!}\partial_\alpha\delta$ is rebaptized δ_α. A central fact of renormalization theory, under its distributional guise, is that there is no unique way to construct the renormalized amplitudes, the inherent ambiguity being represented by the undetermined coefficients of the δ and its derivatives, describing how the chosen extension acts on the (finite codimension) space of test functions which do *not* vanish to some order in a neighborhood of 0. There is, however, a more natural way —in which the ambiguity is reduced to terms in the higher-order derivatives of δ, as seen in (5), exclusively. This is guaranteed by our choice of T_w.

A word on the space of infrared regulators w is in order. In I it was shown that, for the extension of homogeneous f of the kind found in massless field models, any element of the space $\mathcal{K}'$ (dual of the Grossmann–Loupias–Stein function space) of distributions rapidly decreasing in the Cesàro sense [20], taking the value 1 at zero, qualifies as a weight "function". The space $\mathcal{K}'$ is a kind of distributional analogue of the Schwartz space $\mathcal{S}$. Elements of $\mathcal{K}'$ have moments of all orders. In particular, exponential functions e^{iqx} do qualify; this was realized by Prange, at the heuristical level [21]. See the consequences in Section 6. The usefulness of $\mathcal{K}'$ has appeared by now in many different contexts [22]. The regulator $w(\mu x) = H(1-\mu|x|)$ of I, with H the Heaviside function, will be mainly used here. Call T_μ the corresponding renormalization.

The results just summarized go a long way to justify the conjecture (made by Connes and independently by Estrada) that Hadamard's finite part theory is in principle enough to deal with quantum field theory divergences. At least in the Euclidean context. When going from there to the physical signature for the spacetime, both the ultraviolet and the infrared problems immediately turn nastier, in a tangled sort of way. For the first, since the singular support of the Feynman propagator lies on the whole light-cone, it would appear that we have to worry about singularities supported on the entire cone, and not just at the origin. For the second, it is easy to see that if we approach infinity in directions parallel to the light cone, the Feynman propagator decays as $1/|x|$ and not "naïvely" anymore as $1/|x|^2$.

Both kinds of trouble are a bit less ferocious than they seem. There are techniques for dealing with the worsened infrared problem, conjuring at need combinations of diagrams [23]. Microlocal analysis [24, Sec. 8.2] can be invoked [25] to argue that the ultraviolet troubles remain concentrated at the origin, just as in the Euclidean case. In the next section, we show the same for the square and the cube of the Feynman propagator (i.e., the basic four-point and two-point divergent graphs in the φ^4_4 model), by a direct calculation.

Whereas T_1 of (1) ostensibly preserves the Lorentz covariance properties of f, the operator T_w of (4) in general does not. Our main task is to fix this problem.

3 Causal Riesz distributions

Riesz's method consists in generalizing to the Lorentz-invariant context the well known holomorphic family of distributions on $\mathbb{R}$,

$$\Phi^\lambda(x) := \frac{x_+^{\lambda-1}}{\Gamma(\lambda)}$$

for complex λ, which have the properties

$$\Phi^\lambda * \Phi^\mu = \Phi^{\lambda+\mu}; \quad \frac{d\Phi^\lambda}{dx} = \Phi^{\lambda-1}; \quad \Phi^0(x) = \delta(x),$$

where $*$ denotes convolution of distributions. In fact, Riesz only dealt with the (advanced and) retarded propagators. He was able to show that the holomorphic family of distributions G^λ_{ret} defined on $\mathbb{R}^4$ as follows:

$$G^\lambda_{\text{ret}}(x) = C_\lambda\, x_+^{2(\lambda-2)},$$

with x_+^2 equal to $t^2 - |\vec{x}|^2 = t^2 - r^2$ on the forward light cone and to 0 anywhere else, and

$$C_\lambda = \frac{1}{2^{2\lambda-1}\pi\,\Gamma(\lambda)\Gamma(\lambda-1)},$$

fulfils

$$G^\lambda_{\text{ret}} * G^\mu_{\text{ret}} = G^{\lambda+\mu}_{\text{ret}}; \quad \Box G^\lambda_{\text{ret}} = G^{\lambda-1}_{\text{ret}}; \quad G^0_{\text{ret}} = \delta.$$

In particular, $x_+^{2(\lambda-2)}$ is at once seen to have (generally double) poles with residues concentrated at the origin as the only singularities. The G^λ_{ret} constitute a set of convolution inverse powers of the d'Alembertian, verifying $\Box^\lambda G^\lambda_{\text{ret}} = \delta(x)$.

In quantum field theory, as stressed in [8], the relevant set of inverse powers is related to the Feynman propagator

$$D_F(x) = \frac{-i}{4\pi^2(t^2 - r^2 - i\epsilon)}.$$

We therefore focus on $(t^2 - r^2 \pm i\epsilon)^{\alpha-2}$, with poles at $\alpha = 0, -1, \dots$, with the aim of studying the renormalization by analytic regularization of the functionals $(t^2-r^2\pm i\epsilon)^{-2}$, $(t^2-r^2\pm i\epsilon)^{-3}$ and so on, ill-defined as distributions. For a start, we need to compute the residues at the poles.

It is instructive and convenient for the purpose to look at the Euclidean $\mathbb{R}^4$ first, in a somewhat unconventional manner. Consider $\rho^2 := |x|^2 = t^2 + r^2$. The singularities of ρ^α are well known (see I): simple poles at $\alpha = -4 - 2k$, for $k = 0, 1, \dots$, with residues

$$\operatorname*{Res}_{\alpha=-4-2k} \rho^\alpha = \frac{\Omega_4\, \Delta^k \delta}{2^k k!\, 4.6.8\dots(2+2k)}, \tag{6}$$

the denominator in the case $k = 0$ being 1; here $\Omega_4 = 2\pi^2$ is the area of the sphere in dimension 4, and Δ the Laplacian in dimension 4. It is possible, and usually done, to define a holomorphic family of distributions that encodes the pole structure of ρ^α in the same way that Φ^λ encodes that of x_+^λ. However, we consider instead the following *meromorphic* family:

$$G^\alpha_{\text{eucl}}(x) = C_\alpha\, \rho^{2(\alpha-2)},$$

with

$$C_\alpha = \frac{e^{-i\pi\alpha}\Gamma(2-\alpha)}{4^\alpha \pi^2 \Gamma(\alpha)}.$$

Notice that $G^1_{\rm eucl}$ is the Green function for the Laplace equation on $\mathbb{R}^4$; that $\Delta G^\alpha_{\rm eucl} = G^{\alpha-1}_{\rm eucl}$, which is quickly seen from $\Delta\rho^\mu = \mu(\mu+2)\rho^{\mu-2}$; and that $G^{-m}_{\rm eucl}(x) = \Delta^m\delta(x)$; the latter of course is just another way of writing (6).

It is also true that $G^\alpha_{\rm eucl} * G^\beta_{\rm eucl} = G^{\alpha+\beta}_{\rm eucl}$. For that, define the Fourier transforms on test functions by

$$F[\phi](p) \equiv \hat\phi(p) := \int \frac{d^4x}{(2\pi)^2}\, e^{-ipx}\phi(x), \quad F^{-1}[\phi](p) \equiv \check\phi(p) := \int \frac{d^4x}{(2\pi)^2}\, e^{ipx}\phi(x),$$

and on distributions by transposition. Then $Ff\,Fg = (2\pi)^{-2}F(f*g)$ for convolvable distributions. It turns out [26, Thm. 5.9] that

$$\hat G^\alpha_{\rm eucl}(p) = (2\pi)^{-2}e^{-i\pi\alpha}|p|^{-2\alpha}, \tag{7}$$

a most interesting duality. From this the convolution identity follows.

Now we obtain the poles of $(x^2 \pm i\epsilon)^\alpha$ from the poles of $\rho^{2\alpha}$. We follow Gelfand and Shilov [27, Ch. III, Secs 2.3, 2.4] in this. Consider the quadratic forms $g_\pm(x) := \pm i\rho^2$. Then $g^\alpha_\pm = e^{\pm i\pi\alpha/2}\rho^{2\alpha}$. Rewrite equation (6) as

$$\operatorname*{Res}_{\alpha=-2-l} g^\alpha_\pm = \frac{-\pi^2\,\Delta^l_{g_\pm}\delta}{4^l l!(l+1)!\sqrt{(\mp i)^4\det g_\pm}},$$

with $\Delta_{g_\pm}$ the Laplacian canonically associated to $g_\pm$, which is $\mp i\Delta$ on this occasion. To be precise, if $\tilde g^{ij}$ is the inverse matrix of the quadratic form g, then

$$\Delta_g := \sum_{i,j}\tilde g^{ij}\,\partial_i\,\partial_j.$$

For the forms $g(x) = t^2 - r^2 \pm i\epsilon(t^2+r^2)$, we find then by analytic continuation:

$$\operatorname*{Res}_{\alpha=-2-l}(x^2 \pm i\epsilon)^\alpha = \frac{\pm i\pi^2\,\Box^l\delta}{4^l l!(l+1)!}.$$

(This analytic continuation is not to be confused with the one involved in the definition of the G^α for α complex.)

The information on the singularity structure of $(x^2 \pm i\epsilon)^\alpha$ —and of its Fourier transform— can now be codified in *causal Riesz distributions* $G^\alpha_\pm$. To wit, we define

$$G^\alpha_\pm(x) := \frac{\mp ie^{\mp i\pi\alpha}\Gamma(2-\alpha)}{4^\alpha\pi^2\Gamma(\alpha)}(t^2 - r^2 \pm i\epsilon)^{\alpha-2}, \tag{8}$$

and, sure enough,

$$G^1_-(x) = D_F(x); \qquad G^{-l}_\pm(x) = \Box^l\delta(x)$$

for $l \geq 0$. Also, $\Box G^\alpha_\pm(x) = G^{\alpha-1}_\pm(x)$, just as for the ordinary Riesz distributions. This is clear from

$$\Box(t^2 - r^2 \pm i\epsilon)^{\alpha-2} = 4(\alpha-1)(\alpha-2)(t^2 - r^2 \pm i\epsilon)^{\alpha-3}$$

valid for $1 < \Re\alpha < 2$, and then analytically extended. It follows that $\Box^l f = G_\pm^{-l} * f$ for appropriately convolvable f; and $\Box^\alpha$ for complex α can be defined by $\Box^\alpha f = G_\pm^{-\alpha} * f$.

We can perform the (covariant, if one wishes) Fourier transforms by the same method of analytic prolongation from the Fourier transforms of the $r^{2\alpha}$. The result is

$$\hat{G}_\pm^\alpha(p) = (2\pi)^{-2} e^{\mp i\pi\alpha}(p^2 \mp i\epsilon)^{-\alpha}, \tag{9}$$

where $p^2 = E^2 - |\vec{p}|^2$. For instance, $\hat{G}_-^0(p) = 1/4\pi^2$, $\hat{G}_-^1(p) = \hat{D}_F(p) = \frac{-1}{4\pi^2(p^2+i\epsilon)}$, as expected [17]. There is still an interesting duality at work here. Moreover,

$$G_\pm^\alpha * G_\pm^\beta = G_\pm^{\alpha+\beta}.$$

In summary, thanks to (rigorous) "Wick rotation", the structure of the causal Riesz distributions $G_\pm^\alpha$ is remarkably simpler than the structure of the retarded Riesz distributions $G_{\rm ret}^\alpha$. It largely parallels the positive signature case, vindicating Schwinger's contention on the "Euclidean" character of quantum field theory [28].

We turn finally to the renormalization of the functionals $(t^2 - r^2 - i\epsilon)^{-l}$ with $l \geq 2$ from G_-^α. We may define the extension $[(t^2 - r^2 - i\epsilon)^{-2}]_{\rm AR}$, as a distribution, to be the second term on the right hand side of the expansion

$$(t^2 - r^2 - i\epsilon)^{\kappa-2} =: \frac{-i\pi^2\delta(x)}{\kappa} + [(t^2 - r^2 - i\epsilon)^{-2}]_{\rm AR} + O(\kappa).$$

That is to say,

$$[(t^2 - r^2 - i\epsilon)^{-2}]_{\rm AR} = \lim_{\kappa\to 0} \frac{d}{d\kappa}[\kappa(t^2 - r^2 - i\epsilon)^{\kappa-2}].$$

Analogously,

$$(t^2 - r^2 - i\epsilon)^{\kappa-3} =: \frac{-i\pi^2\Box\delta(x)}{8\kappa} + [(t^2 - r^2 - i\epsilon)^{-3}]_{\rm AR} + O(\kappa), \tag{10}$$

as $\kappa \to 0$ defines $[(t^2 - r^2 - i\epsilon)^{-3}]_{\rm AR}$, and so on.

4 Lorentz covariance of the T-renormalization

The action of an element Λ of the Lorentz group on $\mathbb{R}^{4n}$ is given by the tensorial representation

$$\Lambda^{\otimes n} x := (\Lambda x_1, \ldots, \Lambda x_n),$$

to be denoted Λ as well, according to custom. The action of the Lorentz group on functionals is defined by

$$\langle \Lambda f(x), \phi(x)\rangle \equiv \langle f(\Lambda x), \phi(x)\rangle := \langle f(x), \Lambda\phi(x)\rangle, \quad \text{with} \quad \Lambda\phi(x) := \phi(\Lambda^{-1}x).$$

It follows that $\langle \Lambda f(x), \Lambda^{-1}\phi(x)\rangle = \langle f(x), \phi(x)\rangle$.

A Lorentz invariant functional fulfils

$$f(\Lambda x) = f(x). \tag{11}$$

(More generally, in the nonscalar case, f would have tensorial and/or spinorial character and one would have a covariant transformation

$$f(\Lambda x) = [D(\Lambda) f](x)$$

with $D(\Lambda)$ a finite dimensional representation of $SL(2,\mathbb{C})$ —making no notational distinction between belonging to the Lorentz group and to its cover— acting on functionals in the obvious way.)

Derivatives will transform according to the (tensor powers of the) contragredient representation: one has

$$x^\alpha\,\partial_\alpha(\Lambda\phi) = x^\alpha\,\partial_\alpha(\phi\circ\Lambda^{-1}) = x^\alpha[\Lambda^{-1}]^\beta{}_\alpha(\partial_\beta\phi)\circ\Lambda^{-1} = (\Lambda x)^\beta(\partial_\beta\phi)\circ\Lambda^{-1}.$$

In particular,

$$x^\alpha\,\partial_\alpha(\Lambda\phi)(0) = [\Lambda^{-1}x]^\beta\,\partial_\beta\phi(0), \tag{12}$$

that is to say $R_0^k\Lambda = \Lambda R_0^k$, and

$$\delta_\alpha(\Lambda x) = [\Lambda^{-1}x]^\beta_\alpha\,\delta_\beta(x). \tag{13}$$

Suppose that f is Lorentz invariant and a particular extension $T_w f$ to the whole of $\mathbb{R}^{4n}$ has been constructed, according to our scheme. *All* the extensions of f are given by

$$T_w f + \sum_{|\alpha|\le k} a^\alpha\,\delta_\alpha, \tag{14}$$

with $\binom{4n+k}{k}$ coefficients a^α. Our goal is to show that a Lorentz invariant extension $T_w^{\mathrm{cov}} f$ can be obtained within the class of T-extensions, advocated in this series of papers. Namely,

$$T_w^{\mathrm{cov}} f(\Lambda x) = T_w^{\mathrm{cov}} f(x),$$

with

$$T_w^{\mathrm{cov}} f = T_w f + \sum_{|\alpha|=k} a^\alpha\,\delta_\alpha, \tag{15}$$

for at most $\binom{4n-1+k}{k}$ coefficients a^α; so that the ambiguity (14) in all the smaller orders drops out.

By a theorem of Gårding and Lions [29], the difference between two covariant extensions must be of the form $P(\Box)\delta$, where $P(\Box)$ is a polynomial in $\Box$; in our case, a monomial.

The proof is by direct computation; since $\Lambda f = f$, we get

$$\begin{aligned}
\langle \Lambda(T_w f) - T_w f, \phi\rangle &= \langle T_w f, \Lambda\phi\rangle - \langle T_w f, \phi\rangle = \langle f, T_w\Lambda\phi\rangle - \langle f, T_w\phi\rangle \\
&= \langle f, (1-w)R_0^{k-1}\Lambda\phi + wR_0^k\Lambda\phi\rangle - \langle f, (1-w)R_0^{k-1}\phi + wR_0^k\phi\rangle \\
&\overset{(12)}{=} \langle f, (1-w)\Lambda R_0^{k-1}\phi + w\Lambda R_0^k\phi\rangle - \langle f, (1-w)R_0^{k-1}\phi + wR_0^k\phi\rangle \\
&= \langle \Lambda f, (1-\Lambda^{-1}w)R_0^{k-1}\phi + \Lambda^{-1}wR_0^k\phi\rangle - \langle f, (1-w)R_0^{k-1}\phi + wR_0^k\phi\rangle \\
&\overset{(11)}{=} \langle f, (1-\Lambda^{-1}w)R_0^{k-1}\phi + \Lambda^{-1}wR_0^k\phi\rangle - \langle f, (1-w)R_0^{k-1}\phi + wR_0^k\phi\rangle \\
&= \langle f, (w-\Lambda^{-1}w)(R_0^{k-1}\phi - R_0^k\phi)\rangle \\
&= \sum_{|\alpha|=k}\langle f, (w-\Lambda^{-1}w)x^\alpha\rangle\frac{\partial_\alpha\phi(0)}{\alpha!}.
\end{aligned}$$

The integral $\langle f, (w - \Lambda^{-1}w)x^\alpha\rangle$ exists under the hypothesis we have made.

This shows that

$$\Lambda(T_w f) - T_w f = \sum_{|\alpha|=k} b^\alpha(\Lambda)\delta_\alpha,$$

with coefficients

$$b^\alpha(\Lambda) = (-)^k\langle f, (w - \Lambda^{-1}w)x^\alpha\rangle,$$

with $|\alpha| = k$. One also has

$$\sum_{|\alpha|=k} b^\alpha(\Lambda)\delta_\alpha = k\sum_{|\alpha|=k}\partial^\alpha\left[\frac{x^\alpha}{\alpha!}\int_0^1 dt\,\frac{(1-t)^{k-1}}{t^{k+4n}} f\Big(\frac{x}{t}\Big)\Big(w\Big(\frac{x}{t}\Big) - w\Big(\frac{\Lambda x}{t}\Big)\Big)\right]$$
$$+ (k+1)\sum_{|\beta|=k+1}\partial^\beta\left[\frac{x^\beta}{\beta!}\int_0^1 dt\,\frac{(1-t)^k}{t^{k+4n+1}} f\Big(\frac{x}{t}\Big)\Big(w\Big(\frac{\Lambda x}{t}\Big) - w\Big(\frac{x}{t}\Big)\Big)\right].$$

The rest of the proof just follows the steps of the cohomological argument in [4]: applying two Lorentz transformations, on use of (13) —and omitting indices— one obtains

$$b(\Lambda_1\Lambda_2) = \Lambda_2^{-1}b(\Lambda_1) + b(\Lambda_2), \tag{16}$$

where Λ_2^{-1} denotes the tensor antirepresentation. A solution for this equation is given by

$$b(\Lambda) = (1 - \Lambda^{-1})a \tag{17}$$

with $a \in \mathbb{R}^{4k}$ independent of Λ. Actually (16) is a group 1-cocycle equation, for $SL(2,\mathbb{C})$, with values in the space carrying the contragredient representation, and because of the vanishing of the the first cohomology group $H^1(SL(2,\mathbb{C}); \mathbb{R}^{4k})$ [30] its only solutions are of the trivial form (17).

Now, we conclude that, if a satisfies (17), then, in view of (13), formula (15) gives indeed a Lorentz invariant renormalization of f.

For logarithmic divergences, $b = 0$, and, as $[\Lambda^{-1}]^{\otimes 0} = 1$, one can take a arbitrary (this is a priori obvious in view of the Lorentz invariance of δ). The choice $a = 0$ commends itself.

For higher order divergences, in principle one solves (17) for a and plugs the (in general non unique) solution in (15). However, following a suggestion in [25], a wiser course can be devised. For simplicity, we take $n = 1$ from now on. Then we can assume that f depends only on x^2 [31]. It will be seen that only the symmetric part of the Lorentz content of a^α counts. Consider $\sigma(f) = 6$, i.e. $k = 2$, a quadratic divergence. We revert to a Lorentz quadri-index notation: $|\alpha| = 2,\ \alpha \leftrightarrow (\mu_1\mu_2)$. It is found [6] that the totally symmetric part

$$a^{(\mu_1\mu_2)} = -\tfrac{1}{4}\langle f, \big(x^{\mu_1}x^{\mu_2}x^\rho\partial_\rho - x^2x^{(\mu_1}\partial^{\mu_2)}\big)w\rangle,$$

is a possible choice for a; this choice is canonical in that $a^\mu_\mu = 0$. Integrating by parts the previous expression, on account of $\partial_\mu f = 2x_\mu f'$, we obtain

$$a^{(\mu_1\mu_2)} = \langle f, \big(x^{\mu_1}x^{\mu_2} - \tfrac{1}{4}x^2g^{\mu_1\mu_2}\big)w\rangle.$$

In (15) with use of (3) we see cancellation of the first term on the right hand side of this equation, and finally the canonical expression

$$\langle T_w^{\rm cov} f(x), \phi(x)\rangle = \langle f(x), \phi(x) - \phi(0) - w(x)\frac{\Box\phi(0)}{8}x^2\rangle,$$

emerges for the Lorentz-invariant distribution extending a quadratically divergent Lorentz-invariant functional f. This formula supplants the case $k = 2$ of (3) in practice.

More generally, from the formulae in [6], we can derive, for $\sigma(f) = 2m + 4$ or $2m + 5$:

$$\langle T_w^{\rm cov} f(x), \phi(x)\rangle = \left\langle f(x), \phi(x) - \phi(0) - \frac{\Box\phi(0)}{8}x^2 - \cdots - w(x)\xi_m x^{2m}\right\rangle, \tag{18}$$

where

$$\xi_m := \frac{2(2m-1)!!}{(2m+2)!!(2m)!}.$$

Concentrate now in $T_\mu^{\rm cov}$. In Section 4.2 of I we proved that the Euclidean $\langle[\rho^{-4-2m}]_{\rm AR}, \phi(x)\rangle$ is given by

$$\left\langle \rho^{-4-2m}, \phi(x) - \phi(0) - \frac{\Delta\phi(0)}{8}\rho^2 - \cdots - H(1-\rho)\xi_m\, \Delta^m\phi(0)\rho^{2m}\right\rangle.$$

This expression Wick-rotates into $\langle[(x^2 \pm i\epsilon)^{-2-m}]_{\rm AR}, \phi(x)\rangle$, which therefore is given by

$$\left\langle (x^2 \pm i\epsilon)^{-2-m}, \phi(x) - \phi(0) - \frac{\Box\phi(0)}{8}x^2 - \cdots - H(1-\rho)\xi_m\Box^m\phi(0)(x^2)^m\right\rangle.$$

The conclusion is that $[(x^2 \pm i\epsilon)^{-2-m}]_{\rm AR} = T_{\mu=1}^{\rm cov}(x^2 \pm i\epsilon)^{-2-m}$: for negative powers of the Feynman propagator, Bollini and Giambiagi's analytic regularization and our canonical covariant renormalization using the *improved* subtraction $T_{\mu=1}$ give one and the same result.

This coincidence is extended to T_μ for all values of μ by introduction of a 't Hooft factor in the definition of $[(x^2 \pm i\epsilon)^{-2-m}]_{\rm AR}$. The procedure will be clear from the examples in the next section.

5 Computing examples

The singularities of the powers of D_F are concentrated at the origin, so that the improved method of Epstein and Glaser is directly applicable here. Consider first the T_μ-renormalization of $(t^2 - r^2 - i\epsilon)^{-2}$, corresponding to the "fish" diagram in the φ_4^4 model. We use the notations $[f]_{\rm R} := [f]_{{\rm R},\mu} := T_\mu^{\rm cov} f$. From (4) we obtain, in full analogy with the Euclidean case (see Sec. 3 of I):

$$[(x^2 - i\epsilon)^{-2}]_{{\rm R},\mu} = \tfrac{1}{2}\partial_\nu\left[x^\nu \frac{\log\mu^2(x^2 - i\epsilon)}{(x^2 - i\epsilon)^2}\right].$$

This is the very same result coming from analytical regularization: just check

$$\mu^{2\kappa}(x^2 - i\epsilon)^{\kappa-2} = \frac{\mu^{2\kappa}}{2\kappa}\partial_\nu[x^\nu(x^2 - i\epsilon)^{\kappa-2}],$$

and expand in κ the right hand side. In conclusion:

$$[(t^2 - r^2 - i\epsilon)^{-2}]_{\mathrm{R},\mu} = [(t^2 - r^2 - i\epsilon)^{-2}]_{\mathrm{AR},\mu},$$

or $[(D_F)^2]_{\mathrm{R}} = [(D_F)^2]_{\mathrm{AR}}$, with this generalized definition of $[\cdot]_{\mathrm{AR}}$.

Consider now the "sunset" diagram in the same model. Have another look at Sec. 3 of I; one obtains

$$\left[(x^2 - i\epsilon)^{-3}\right]_{\mathrm{R},\mu} = 3 \sum_{|\beta|=3} \partial_\beta \left[\frac{x^\beta}{\beta!} \frac{\log(\mu^2(x^2 - i\epsilon))}{(x^2 - i\epsilon)^3}\right] - \frac{3i\pi^2}{8} \Box\delta(x).$$

Analogously, one checks by a longer but straightforward calculation:

$$\mu^{2\kappa}(x^2 - i\epsilon)^{\kappa-3} = \frac{3\mu^{2\kappa}}{2\kappa(1 - 3\kappa + 2\kappa^2)} \sum_{|\beta|=3} \partial_\beta \Big(\frac{x^\beta (x^2 - i\epsilon)^{\kappa-3}}{\beta!}\Big).$$

It is clear from (10) that

$$3\,\partial_\beta \Big(\frac{x^\beta (x^2 - i\epsilon)^{\kappa-3}}{\beta!}\Big) = -\frac{i\pi^2}{4} \Box\delta(x),$$

from which

$$\mu^{2\kappa}(x^2 - i\epsilon)^{\kappa-3} = \frac{-i\pi^2 \Box\delta(x)}{8\kappa} + \left[(x^2 - i\epsilon)^{-3}\right]_{\mathrm{R},\mu} + O(\kappa).$$

Therefore $[(D_F)^3]_{\mathrm{R}} = [(D_F)^3]_{\mathrm{AR}}$.

For higher order powers of G_F similar arguments show that

$$[(D_F)^l]_{\mathrm{R}} = [(D_F)^l]_{\mathrm{AR}}$$

is true generally for $l \geq 2$.

6 BPHZ interlude

It is well known that for zero-mass models, the basic BPHZ scheme runs into trouble. This is due to the failure of $\partial^\mu \hat{f}(0)$ to exist for $|\mu| = k$, on account of the infrared problem. Now, one can try subtraction at some suitable external momentum $q \neq 0$, providing a mass scale. It is patent that this last subtraction will introduce in the Minkowskian context a noncovariance. This prompted Lowenstein and Zimmermann to introduce their "soft mass insertions" [32]; but that BPHZL method is quite awkward in practice.

A far simpler solution to the problem is now available to us. It comes from the observation in I that the BPHZ method is ancillary to Epstein and Glaser's: from the definitions

$$\langle F[R_0^k f], F^{-1}[\phi]\rangle = \langle F[f], F^{-1}[R_0^k \phi]\rangle. \tag{19}$$

An expression such as $F[f]$ is *not* a priori meaningless: it is a well defined functional on the linear subspace of Schwartz functions ϕ whose first moments $\int p^\alpha \phi(p)\, d^d p$ up to order

$k+1$ happen to vanish: this is the Fourier counterpart of the space of distributions on configuration space acting on Schwartz test functions vanishing up to order $k+1$ at the origin.

Now, one has

$$(x^\mu \phi)\check{}(p) = (-i)^{|\mu|}\, \partial^\mu \check{\phi}(p),$$

where μ denotes a multiindex; so that, in particular,

$$(x^\mu)\check{}(p) = (-i)^{|\mu|}(2\pi)^{d/2}\, \partial^\mu \delta(p).$$

Also,

$$\partial_\mu \phi(0) = (-i)^{|\mu|}(2\pi)^{-d/2}\langle p^\mu, \check{\phi}\rangle.$$

From this, with an integration by parts in the right hand side of (19), we conclude

$$\langle F[R_0^k f], F^{-1}[\phi]\rangle = \langle R_0^k F[f], F^{-1}[\phi]\rangle;$$

that is to say, F and R_0^k *commute*. Thus the BPHZ subtraction rule in momentum space is equivalent to (1).

Now use (18) instead of (1), with employment of $w(x) = \exp(-iqx)$, with $q \neq 0$, which, as discussed earlier, is a perfectly good infrared regulator. From that follows the simple, obviously covariant, rule:

$$T_q^{\text{cov}} f(p) = f(p) - \frac{\Box f(0) p^2}{8} - \cdots - \xi_m \Box^m f(q)\,(p^2)^m.$$

for a Feynman amplitude f in momentum space. Note $\Box^m T_q^{\text{cov}} f(q) = 0$. The difference between two of these recipes is, as it should be, a Lorentz-invariant polynomial in p, of degree the divergence index.

7 The massive case

We set out to obtain the massive propagator D_F^m by perturbation of the massless theory propagator. We found this method to work in the Euclidean context (see Section 6 of I), if *and only if* T-renormalization of the resulting series of divergent convolution integrals is used.

Here the miracle recurs. To find D_F^m, recall that δ is the unit for convolution, think of D_F as $[\Box\delta]^{*-1}$ and try to compute the convolution inverse of $[(\Box + m^2)\delta]^{*-1} = [\Box\delta]^{*-1} * [\delta + m^2[\Box\delta]^{*-1}]^{*-1}$ by the time-honoured geometric series trick:

$$D_F^m = [\Box\delta]^{*-1} * \left[\delta + \sum_{l=1}^{\infty}(-)^l m^{2l}[\Box\delta]^{*-l}\right] = D_F + \sum_{l=1}^{\infty}(-)^l m^{2l} G_-^{l+1}. \tag{20}$$

Now, all the G_-^l for $l \geq 2$ are in need of renormalization! The shortest way to an explicit expression is analytic regularization:

$$m^{2(\kappa-1)} G_-^\kappa =: m^{2(l-1)} \frac{\operatorname{Res}_{\kappa=l} G_-^\kappa}{\kappa - l} + [m^{2(l-1)} G_-^l]_{\text{AR}} + O(\kappa);$$

experience with this kind of expressions in Section 3 led us to include the mass factor in the renormalization.

Therefore $[m^{2(l-1)}G_-^l]_{\mathrm{R}}$ for $l \geq 2$ is computed by

$$\lim_{\kappa\to l}\frac{\partial}{\partial\kappa}[m^{2(\kappa-1)}(\kappa-l)G_-^\kappa], \tag{21}$$

with the G_-^κ given by (8). We leave to the reader the computation of the residues.

Before exhibiting $[m^{2(l-1)}G_-^l]_{\mathrm{R}}$, note that we are renormalizing (infrared) divergent convolution integrals like

$$D_F * D_F(x) \propto \int d^4z\,\frac{1}{((x-z)^2-i\epsilon)(z^2-i\epsilon)}.$$

Also note that, in view of the duality (9), read in terms of configuration variables, we will in fact be computing the Fourier transforms of the $(x^2-i\epsilon)^l$; which means that we remain in the realm of our canonical covariant renormalization procedure, with m substituted for μ (that is reflected in the notation $[\dots]_{\mathrm{R}}$ instead of $[\dots]_{\mathrm{AR}}$). This is also true in the Euclidean case, on account of (7) —as was remarked in subsection 5.3 of I.

A most important observation [8] is that actually $\square^l[G_-^l]_{\mathrm{R}} = \delta$, so that now $\square^\alpha$ is defined for all α; the iterated d'Alembertians of the corresponding residues vanish.

Let $H_0 = 0$ and let H_l be the sum of the l first terms of the harmonic series. Working from (21) we find

$$[G_-^l]_R(x) = \frac{(-)^{l-1}i(x^2)^{l-2}}{\pi^2 4^l(l-2)!(l-1)!}\left[\log\frac{m^2(x^2-i\epsilon)}{4} + 2\gamma - H_{l-1} - H_{l-2} - i\pi\right],$$

where γ is the Euler–Mascheroni constant. Now the previous formal series becomes a convergent series, to wit:

$$D_F^m(x) = \sum_{l=1}^{\infty}(-)^l m^{2(l-1)}[G_-^l]_R = \frac{imK_1(m\sqrt{-x^2+i\epsilon})}{4\pi^2\sqrt{-x^2+i\epsilon}}. \tag{22}$$

Here the known expansion of the (modified) Bessel function K_1 [33] has been used.

Some more remarks are in order: the calculational trick (20) works also for retarded propagators, say, but there it is easier in that the terms of the series need not be renormalized; it is well known that the result involves other types of Bessel functions. Thus the previous computation of D_F^m is mind-boggling in more than one way. First of all, it is unclear why the formal series (20) would transmute into a convergent one; and second, the slightest deviation from our canonical procedure (for instance, using dimensional regularization) is sure to produce a wrong result. An attempt at explanation of the first prodigy was made in [34]; there the same phenomena were noted in the Euclidean framework. The parallelism between Euclidean and causal propagators is of course kept, and (22) can be regarded again as the outcome of a Wick rotation.

Powers of D_F^m are renormalizable in our standard way, by use of (1), now applicable, and automatically Lorentz-covariance preserving. The scaling degree of the (powers of) propagators is the same in the massive and in the massless cases. One routinely finds, for instance,

$$[(D_F^m)^2]_{\mathrm{R}}(x) = -\frac{m^2}{32\pi^4}\partial_\mu x^\mu \left(\frac{K_1^2(m\sqrt{-x^2+i\epsilon}) - K_0(m\sqrt{-x^2+i\epsilon})K_2(m\sqrt{-x^2+i\epsilon})}{-x^2+i\epsilon}\right).$$

8 Outlook

The "missing link" between the Epstein–Glaser subtraction method and the literature on prolongation of distributions found in I has here been extended to the Minkowskian context. Before rendering in the language of T-renormalization the full complexity of the construction of time-ordered products, and the main result of perturbative renormalization theory, one needs to handle the combinatorial aspects of diagrams with subdivergences. This we do in the next paper of the series III, using a variant of the Connes–Kreimer Hopf algebraic paradigm.

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