

Continuum limits of Berenstein–Maldacena–Nastase matrix theory: Where is the (nonabelian) gauge group?

Corneliu Sochichiu*

High Energy and Elementary Particle
Physics Division, University of Crete,
P.O. Box 2208, GR-710 03 Heraclion,
Crete, GREECE

and

Laboratori Nazionali di Frascati,
INFN, Via E. Fermi 40,
C.P. 13, 00044 Frascati
Italy

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Abstract

We discuss the continuum limits of Berenstein–Maldacena–Nastase matrix model. The special attention is paid to limits that give rise to Poisson bracket gauge field theories with gauge groups $U(n)$ on the ordinary two sphere. The gauge group and the space depending on the degeneracy of the classical solution about which the model is considered. We compare these limits as well as different solutions in the framework of the same limit model. We show that these models fail to be equivalent in the continuum limit, i.e. the continuum limit does not commute with dualities of the matrix theory.

1 Introduction

The correspondence between gauge fields and strings is a long standing problem [1]. To establish such an equivalence a good understanding of both gauge and string theories is needed, inclusively on the nonperturbative level. Some progress in this direction was reached recent years. (For a comprehensive review see [2].)

On the other hand, hints for connection between nonperturbative string dynamics and matrix models which are dimensionally reduced Yang–Mills theories where given by matrix theories [3, 4], describing branes.

*On leave from: *Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, 141980 Dubna, Moscow Reg., RUSSIA* and *Institutul de Fizică Aplicată AS, str. Academiei, nr. 5, Chişinău, MD2028 MOLDOVA*; e-mail: `sochichi@lnf.infn.it`

In recent papers [5]–[7], the correspondence between Yang–Mills theory and string theory in pp-wave background [8, 9], was considered. The pp-wave background is a curved space corresponding to a plane parallel gravitational wave. In such a space the string theory preserves many of its nice features, in particular it is treatable and can be quantized [10].

The DLCQ compactification which in the flat space gives the BFSS model [3], in the case of pp-wave leads to a modified matrix theory for zero-brane. The modification with respect to the BFSS model results in the addition of mass terms for all matrix fields and of Chern–Simons terms to some of them. The result of this modification is that the stationary vacua of the model are given by fuzzy sphere or by a set of fuzzy spheres. The perturbation theory around such classical vacua was analyzed [11] together with the continuum limit (see also [16]). Also, in [11], it was shown that BMN matrix theory arises as the result of the world sheet quantization of membranes in pp-wave background described by a Poisson bracket model. Like BFSS matrix theory the BMN matrix theory also possesses a number of supersymmetric solutions [12]–[15].

Here we review the fact that in the scaling limit proposed by BMN one in fact recovers the above Poisson bracket action of the commutative spherical brane. Depending on what solution of the matrix model is chosen, one can get in the above limit field models with different local gauge groups. In earlier papers [17]–[21] we analyzed the equivalence relations arising in the $N \rightarrow \infty$ limit of “flat” matrix theories. In that cases, $N \rightarrow \infty$ limit yielded noncommutative gauge models in different dimensions and/or having different gauge groups. Now, the situation is slightly different. Since the $N \rightarrow \infty$ limit of BMN matrix theory is combined also with the commutative limit, the scaling prescription leads to a commutative, although exotic, field model on an ordinary sphere or set of spheres. The equivalence of these different limits in this case is not clear *a priori*. To find such an equivalence, if it exists, one has to identify the solutions in the limit model, which correspond to different sets of spheres and analyze the model in the vicinity of such solutions. We do this and find that, in fact, the models are not equivalent. One can recover, starting with the $U(1)$ model, the spectrum corresponding to the Abelian subalgebra only of nonabelian models.

The plan of the paper is as follows. In the next section we review the classical solutions and continuum limit of the BMN matrix model. After that, we consider the Poisson gauge model with the gauge group $U(1)$ and find in this model solutions which are commutative analogs of $U(n)$ backgrounds of matrix theory. We find that it is only maximal abelian subgroup $U(1)^n$ of $U(n)$ which is manifest while the remaining part is hidden in the large gauge transformations of the original irreducible limit. The world sheet quantization makes these modes to arise explicitly.

2 Classical solutions and continuum limit

The BMN matrix model appears as the DLCQ quantization of zero brane in the pp-wave background,

$$ds^2 = -4dx^+dx^- - \left[\left(\frac{\mu}{3} \right)^2 x_\alpha^2 + \left(\frac{\mu}{6} \right)^2 x_\mu^2 \right] (dx^+)^2 + dx_i^2, \quad (1)$$

were the early Greek indices run $\alpha, \beta = 1, 2, 3$, late ones $\mu, \nu = 4, \dots, 9$, while the Latin ones span both of these sets, i.e. $i = 1, 2, \dots, 9$, $x_i = (x_\alpha, x_\mu)$. In this approach, the sector of M-theory corresponding to the light cone momentum $2p^+ = -p_- = N/R$ (R is the DLCQ radius) is described by the following matrix action [6],

$$S = \int dt \operatorname{tr} \left[\frac{1}{2(2R)} (D_0 \phi^i)^2 + \frac{(2R)}{4} [\phi^i, \phi^j]^2 - \frac{1}{2(2R)} \left(\left(\frac{\mu}{3} \right)^2 \phi_\alpha^2 + \left(\frac{\mu}{6} \right)^2 \phi_\mu^2 \right) - \frac{\mu}{3} i \epsilon_{\alpha\beta\gamma} \phi_\alpha \phi_\beta \phi_\gamma \right] + \text{fermions}, \quad (2)$$

where ϕ_i are $N \times N$ hermitian matrices and “fermions” denotes the fermionic part of the action which is not written explicitly, since it is not important for our further analysis. In (2) the indices run according to the same convention as in eq. (1).

One can see, that there are no stable nontrivial vacua involving only fields ϕ_μ (cfy. Ref. [16]), while one can build nontrivial vacuum solutions out of ϕ_α . In what follows, we will consider the model about such configurations.

The ϕ_α dependent part of the action can be rewritten in the following form:

$$S_0 = \int dt \operatorname{tr} \left[\frac{1}{2(2R)} (D_0 \phi_\alpha)^2 + \frac{(2R)}{4} \left([\phi_\alpha, \phi_\beta] - i \frac{\mu}{6R} \epsilon_{\alpha\beta\gamma} \phi_\gamma \right)^2 \right]. \quad (3)$$

As it is not difficult to see from the form (3) of the action, the vacua of this sector of the model are given by matrices satisfying $su(2)$ algebra,

$$[\phi_\alpha, \phi_\beta] = i \frac{\mu}{6R} \epsilon_{\alpha\beta\gamma} \phi_\gamma. \quad (4)$$

The matrices ϕ_α , satisfying vacuum condition (4) can be split into blocks corresponding to irreducible representations R_λ of $su(2)$ of spins j_λ , having the total dimensionality,

$$\sum_\lambda (2j_\lambda + 1) = N. \quad (5)$$

The cases of interest for us are when the solution is represented by n -times degenerate irreducible representation of the spin j satisfying $2j = N/n - 1$. In particular, the simplest case is for $n = 1$ when one has a simple irreducible representation with $2j + 1 = N$. Although there are other interesting cases, in what follows we concentrate mainly on the above ones.

So, let us consider a solution $\phi_\alpha \equiv Y_\alpha$, which is n times irreducible representation. An arbitrary Hermitian $n \times n$ matrix can be uniquely expanded in terms of $n \times n$ Hermitian matrices whose entries of symmetrised traceless polynomials of Y_α . This polynomials are noncommutative analogues of spherical functions and treating them as such one has a map from the space of operators on N dimensional space to the space of $n \times n$ matrix valued functions on a sphere of the radius,

$$Y_\alpha^2 = \left(\frac{\mu}{6R} \right)^2 j(j+1). \quad (6)$$

These functions are subject to the star product on fuzzy sphere whose exact form we will not need¹.

¹For the details referring the fuzzy sphere star product we send readers to [16, 11].

Thus, an arbitrary matrix configuration can be considered as a perturbation of the background solution,

$$\phi_\alpha = Y_\alpha + A_\alpha, \quad (7a)$$

$$\phi_\mu = \phi_\mu, \quad (7b)$$

$$\dots, \quad (7c)$$

where A_α and ϕ_μ are now fields on the fuzzy sphere and dots stay for fermions. In this parametrization the action (3) is essentially one of Yang–Mills–Higgs model on a fuzzy sphere.

Having in mind this map one can switch between different solutions and re-expand as in (7) to obtain equivalence maps between models with different gauge groups living on fuzzy spheres of different radii which are related as $rn = \text{const}$ (see [17]–[21]).

According to BMN prescription, as N goes to infinity the radius of the sphere remains finite,

$$Y_\alpha^2 \rightarrow r_0^2 = \left(\frac{\mu p^+}{6n} \right)^2, \quad N \rightarrow \infty, \quad (2R) \sim \frac{N}{p^+}, \quad (8)$$

while the background becomes commutative,

$$[Y_\alpha, Y_\beta] = i \frac{n}{N} r_0 \epsilon_{\alpha\beta\gamma} Y_\gamma \rightarrow 0. \quad (9)$$

Ploughing this into action one should be careful with divergent factors of $(2R)$. The contribution to the action will be given by the leading term in the expansion of commutators,

$$[f, g] \approx i \frac{n}{N} \{f, g\} + O(N^{-2}), \quad (10)$$

where $\{, \}$ is the Poisson bracket on the sphere which is given by,

$$\{f, g\} = \epsilon_{\alpha\beta\gamma} Y_\alpha \partial_\beta f \partial_\gamma g. \quad (11)$$

Also the trace is replaced by integration over the sphere according to,

$$\frac{4\pi n}{N} \text{tr} \rightarrow \int d\Omega. \quad (12)$$

Then, the action becomes,

$$S_{N \rightarrow \infty} = \int dt d\Omega \text{tr}_{u(n)} \left(\frac{1}{2} (D_0 \phi_i)^2 + \frac{1}{4g^2} (\{\phi_\alpha, \phi_\beta\} - r \epsilon_{\alpha\beta\gamma} \phi_\gamma)^2 \right. \\ \left. + \frac{1}{4g^2} \{\phi_\mu, \phi_\nu\}^2 + \frac{1}{2} \left(\frac{r}{2g} \right)^2 \phi_\mu^2 \right) + \text{fermions}, \quad (13)$$

where $\phi_\alpha = Y_\alpha + A_\alpha$, $g^2 = (p^+)^2$. Integration is performed over time and a sphere whose radius is $r = (\mu p^+)/6n$, all fields are n dimensional Hermitian matrices subject to $u(n)$ trace.

The model possesses a Poisson bracket gauge symmetry,

$$\phi_i \rightarrow \phi_i + \{\phi_i, u\}, \quad (14)$$

where u is an arbitrary hermitian $n \times n$ matrix valued function.

3 Commutative “dualities”

In the previous section we found that the $N \rightarrow \infty$ limit of the BMN matrix theory is sensitive to the background around which we are considering it. For any finite N and finite noncommutativity they are just different parameterizations of the same matrix model and, therefore, these models are all equivalent. This may not hold true as N goes to infinity and the noncommutativity vanishes. Let us try to check, however, at which extend this equivalence is still present in the limiting model (13).

In order to do this consider the model (13) for $n = 1$ and $r = r_0$ which is obtained from the $N \rightarrow \infty$ limit of the irreducible algebra. The gauge symmetry here is just $U(1)$ Poisson bracket gauge symmetry. Let us find static vacuum solutions of this model most close to one producing the $U(n)$ model.

The static vacua satisfy an equation analogous to (4) that the commutator is replaced with the Poisson bracket,

$$\{\phi_\alpha, \phi_\beta\} = r_0 \epsilon_{\alpha\beta\gamma} \phi_\gamma. \quad (15)$$

ϕ_α are functions on (ordinary) sphere of radius r_0 . Since ϕ_α^2 P.b.-commutes with all ϕ_α the solution is, in fact a map of two spheres: $S^2 \rightarrow S^2$. Nontrivial solutions are given, therefore, by the homotopically nontrivial maps. Since $\pi_2(S^2) = \mathbb{Z}$ it is natural to identify the homotopy class of the solution with the rank n of the gauge group $U(n)$. Indeed, zero and one class solutions correspond to $\phi_\alpha = 0$ and $\phi_\alpha = Y_\alpha$, respectively.

Let us find higher classes. In spherical coordinates,

$$Y_1 = r_0 \sin \theta \cos \varphi, \quad (16a)$$

$$Y_2 = r_0 \sin \theta \sin \varphi, \quad (16b)$$

$$Y_3 = r_0 \cos \theta, \quad (16c)$$

the Poisson bracket is given by,

$$\{f, g\} = \frac{1}{\sin \theta} (\partial_\theta f \partial_\varphi g - \partial_\theta g \partial_\varphi f). \quad (17)$$

On the other hand, the simplest way to get a map of n -th homotopically class is to wrap along φ ,

$$\phi_1 \equiv Y_1^{(n)} = r \sin \theta \cos n\varphi, \quad (18a)$$

$$\phi_2 \equiv Y_2^{(n)} = r \sin \theta \sin n\varphi, \quad (18b)$$

$$\phi_3 \equiv Y_3^{(n)} = Y_3 = \cos \theta. \quad (18c)$$

Fortunately, we are lucky enough and the map as it appears in (18) satisfies the vacuum condition (15) if the radius r is chosen to be $r = r_0/n$. This relation is encouraging since it is exactly the relation of the radii of the spheres on which $U(n)$ models live in the $N \rightarrow \infty$ limit (cfy. (8)).

To proceed further we have to consider the functions (18) as new “coordinates” by which we should substitute the old ones. Since $Y^{(1)}$ are wrapping n times about $Y^{(n)}$ a generic function of $Y^{(1)}$ becomes a n -fold ambiguous function of $Y^{(n)}$. Locally, any function of $Y^{(1)}$ will become a set of n functions of $Y^{(n)}$,

$$\phi(Y^{(1)}) \mapsto \phi^a(Y^{(n)}), \quad a = 1, \dots, n, \quad (19)$$

one for each sheet. (In general functions are mapped to sections of a nontrivial n dimensional fibre bundle.)

Unfortunately, this is not in a total accordance with our expectations, since in order to get $U(n)$ gauge group the fields should map to $n \times n$ dimensional matrices rather than to n component fields. In fact, the fields in new coordinates represent the diagonal part of the expected matrices. Indeed, the gauge transformation (14) splits in n $U(1)$ parts (one $U(1)$ for each sheet) which is an indication that the gauge group is $U(1)^n$.

Summarizing, it appears that the maximum we can get in the limiting model is to map the $U(1)$ model to a model where $U(n)$ is truncated down to $U(1)^n$.

4 World volume quantization and restoration of the whole $U(n)$ group

Let us try to recover the remaining non-diagonal part of the desired $U(n)$ symmetry group.

In fact, the symmetry can be restored upon the worldvolume quantization. The idea can be illustrated by the following example. Consider a particle moving in a space consisting of n sheets (branes). The position of the particle is given by its (continuous) coordinate x and the number of the sheet $a = 1, \dots, n$. Classically, the particle can move smoothly along x and jump through indices a . Suppose the observer does not care about the sheet numbers. So far there is no nonabelian symmetry in the system.

Now, consider the above model as being quantum. Since the particle can be found on different branes, the wave function of the particle is a n dimensional vector $\psi_a(x)$. There are n^2 Hermitian operators describing the jumps of the particle from a to b , which commute with x and p and which generate a $U(n)$ symmetry group.

Let us return now to our model. The gauge symmetry (14) is in fact only the infinitesimal version of the whole gauge invariance. Eq. (14) can be integrated to Hamiltonian flows to yield the finite gauge transformations. For example, a rotation by $\Delta\varphi$ along the Y_3 axis can be formally written as,

$$\phi(\varphi) \mapsto \phi(\varphi + \Delta\varphi) = e^{\{\Delta\varphi Y_3, \phi\}}(\varphi), \quad (20)$$

where we symbolically denoted the exponentiated Poisson bracket,

$$e^{\{A, B\}} = B + \frac{1}{1!}\{A, B\} + \frac{1}{2!}\{A, \{A, B\}\} + \dots \quad (21)$$

Thus the “large” rotations of φ by, say, $2\pi k$ where $k < n$ is an integer, result in cyclic jumping over k sheets. $Y^{(n)}$ are invariant under such transformations since they are degenerate along the sheet numbers. The total number of independent large rotations is precisely n^2 (including identical rotations). Thus the nonabelian structure is hidden in large gauge transformations!

Consider now the world volume quantization. It results in the replacement of the Poisson bracket algebra,

$$\{Y_\alpha^{(n)}, Y_\beta^{(n)}\} = r\epsilon_{\alpha\beta\gamma} Y_\gamma^{(n)}, \quad (22)$$

by an operator one,

$$[Y_\alpha^{(n)}, Y_\beta^{(n)}] = i\hbar r \epsilon_{\alpha\beta\gamma} Y_\gamma^{(n)}, \quad (23)$$

in such a way that it forms an irreducible representation modulo the action of the sheet jump operators. It is n -tuple degenerate one and this degeneracy is governed by $U(n)$ gauge group. Now arbitrary operator about the background $Y^{(n)}$ is represented by n -dimensional matrix valued noncommutative function on the fuzzy sphere (23).

Let us note that the gauge group was restored at the moment when we replaced the “classical” sheet number label by an operator.

5 Discussion

In this note we considered the properties of $N \rightarrow \infty$ limit of the BMN matrix theory.

Field theory models describing the fluctuations of the matrix theory in $N \rightarrow \infty$ limit depend on the classical background around which the fluctuations are measured. Different backgrounds lead to models in different spaces or having different gauge groups. In BMN matrix theory there is a class of classical backgrounds given by fields satisfying $su(2)$ algebra. In the case when such a background is given by a n times degenerate irreducible representation of spin $j \sim N/n$, the continuum limit of the matrix model is described by a Poisson gauge model with group $U(n)$. Since at finite N the models with different gauge groups appear as different parameterizations of the same matrix theory there is an equivalence between them. As N goes to infinity, however, the models become nonequivalent. Thus, we are able to map the Poisson gauge model with group $U(1)$ to at most the abelian sector of the $U(n)$ model. Quantization of the worldsheet seems to restore the whole gauge group.

The string interpretation of the above properties of the limit is as follows. Continuum limit of the BMN matrix theory corresponds to zero slope or infinite tension limit of the string theory. So, the transition from a one-brane configuration, which corresponds to gauge group $U(1)$, to a multi-brane configuration, corresponding to group $U(n)$, passes through intermediary configurations with concentric branes of different radii. Infinite tension strings can not stretch between branes of different radii, but rather can begin and end on the same brane. The last gives modes in the abelian sector of the gauge theory, while nonabelian modes can not appear even when spherical branes become of the same radius.

Here we considered only very specific backgrounds given by irreducible representations or product of identical irreducible representations. Another interesting class is given by background with products of *different* irreducible representations. In the case of these solutions one can expect to have a Poisson gauge model with spontaneously broken gauge group $U(n) \rightarrow U(n_1) \times \cdots \times U(n_k)$, $n_1 + \cdots + n_k = n$, where n_i is the multiplicity of the i -th irreducible factor. These can serve as subjects for a future study.

Unfortunately, there is no known direct relation between BMN matrix theory and BMN super-Yang-Mills/PP (SYM/PP) string correspondence [5], except that the respective matrix theory was designed to describe the brane dynamics on the pp-wave. It would be interesting to know such a relation in case it exists, but so far the SYM/PP correspondence is formulated in terms of perturbative

string states and extending the analysis beyond the perturbative limits is a good challenge.

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