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Noncommutative Instantons via Dressing and Splitting Approaches

Zalán Horváth[1], Olaf Lechtenfeld[2] and Martin Wolf[2,3]

[1] *Department of Theoretical Physics*
Eötvös University
1117, Budapest, Hungary

[2] *Institut für Theoretische Physik*
Universität Hannover
30167, Hannover, Germany

[3] *Institut für Theoretische Physik*
Technische Universität Dresden
01062, Dresden, Germany

Email: zalanh@ludens.elte.hu, lechtenf, wolf@itp.uni-hannover.de

Abstract

Almost all known instanton solutions in noncommutative Yang-Mills theory have been obtained in the modified ADHM scheme. In this paper we employ two alternative methods for the construction of the self-dual $U(2)$ BPST instanton on a noncommutative Euclidean four-dimensional space with self-dual noncommutativity tensor. Firstly, we use the method of dressing transformations, an iterative procedure for generating solutions from a given seed solution, and thereby generalize Belavin's and Zakharov's work to the noncommutative setup. Secondly, we relate the dressing approach with Ward's splitting method based on the twistor construction and rederive the solution in this context. It seems feasible to produce nonsingular noncommutative multi-instantons with these techniques.

1 Introduction

The idea of a noncommutative space-time is more than fifty years old [1]. It offers a mild way to introduce nonlocality into field theories without loosing too much control. Motivated by string theory [2] the investigation of non-Abelian gauge theories defined on noncommutative space-times took center stage during the last couple of years. It is well known that the dynamics of non-Abelian gauge fields involves nonperturbative field configurations, like instantons and monopoles, in an essential way. Hence, before attempting to quantize a gauge theory it is mandatory to study its classical solutions and to characterize their moduli spaces.

Nekrasov and Schwarz gave first examples of noncommutative instantons [3]. They modified the Atiyah-Drinfeld-Hitchin-Manin (ADHM) method [4] and showed that (anti-self-dual) noncommutativity resolves the singularities of the instanton moduli space. Moreover, they discovered that the noncommutative Euclidean four-space $\mathbb{R}^4_\theta$ permits nonsingular Abelian instanton solutions, which do not survive in the commutative limit. Since then, a lot of work has been devoted to this subject (see e. g. [5]-[31] and references therein). For reviews on this matter see [32]-[36].

Most authors used the modified ADHM equations for the construction of instantons on $\mathbb{R}^4_\theta$. In the present paper we advocate two complementing methods termed the dressing approach and the twistor approach. These proved to be successful in the commutative case: The twistor approach underlies the ADHM scheme [4] and can produce generic n-instantons via a sequence of Atiyah-Ward ansätze [37, 38]. In particular, the 't Hooft n-instanton solution can be obtained in a very explicit way. The dressing approach [39] (see also [40]-[42]) produces this solution equally well [44].

In the noncommutative extension of $\mathbb{R}^4$ the classical fields are best represented as (Lie-algebra valued) operators in an auxiliary two-oscillator Fock space. The authors of [22] employed the twistor approach to construct $U(2)$ 't Hooft n-instantons on $\mathbb{R}^4_\theta$. However, they were led to a gauge field which violates self-duality on an n-dimensional subspace of the two-oscillator Fock space. This deficiency originated from the singular gauge choice pertinent to the 't Hooft solution and was repaired by a suitable Murray-von Neumann transformation after a specific projection of the gauge potential. The proper noncommutative 't Hooft multi-instanton field strength was written down explicitly but its associated gauge potential could be given only implicitly. In order to get around these difficulties, Correa et al. [16] suggested to use the Belavin-Polyakov-Schwarz-Tyupkin (BPST) [45] ansatz for constructing the noncommutative $U(2)$ one-instanton. In contrast to [22] they did obtain an explicit expression for the self-dual gauge potential but the reality of the gauge potential and field strength was lost.

In this paper we concentrate on the one-instanton case for self-dual $\mathbb{R}^4_\theta$. Instead of trying to generalize the BPST ansatz [45] we choose the method of dressing transformations and generalize the approach of Belavin and Zakharov [46] to the noncommutative case. This eventually results in explicit expressions for a real gauge potential with self-dual field strength for the noncommutative $U(2)$ BPST instanton. Exploiting the connection between the dressing method and Ward's splitting method we rederive the same configuration by generalizing Crane's construction [47]. In fact, Crane's ansatz for the transition matrix substitutes the Atiyah-Ward ansatz and leads directly to the nonsingular instanton configuration without the necessity of a singular gauge transformation. Its generalization may pave the way to nonsingular noncommutative multi-instantons.

The organization of the paper is as follows: First, we briefly discuss Yang-Mills theory on commutative and noncommutative Euclidean four-dimensional space by introducing the basic notions and definitions. We then present the dressing method and illustrate it by constructing the noncommutative $U(2)$ BPST instanton solution for a self-dual noncommutativity tensor. In the last section we outline the connection between this method and the twistor approach and again provide the noncommutative BPST instanton solution. An appendix briefly reviews the geometry of the commutative twistor space.

2 Yang-Mills theory on commutative $\mathbb{R}^4$ and noncommutative $\mathbb{R}^4_\theta$

Commutative Yang-Mills theory. We consider the Euclidean four-dimensional space $\mathbb{R}^4$ with the canonical metric $\delta_{\mu\nu}$. Furthermore we specify to a principal bundle of the form $P=\mathbb{R}^4\times U(2)$ with a connection $A=A_\mu\,\mathrm{d}x^\mu$ and the Yang-Mills curvature $F=\mathrm{d}\,A+A\wedge A$. In components the latter equation reads $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]$. Here, ∂_μ denotes the partial derivative with respect to x^μ, and Greek indices always run from 1 to 4.

The self-dual Yang-Mills (SDYM) equations take the form

$$F\;=\;*F\qquad\Longleftrightarrow\qquad F_{\mu\nu}\;=\;\tfrac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\ , \tag{2.1}$$

where '$*$' denotes the Hodge star and $\epsilon_{\mu\nu\rho\sigma}$ is the Levi-Civita symbol on $\mathbb{R}^4$ with $\epsilon_{1234}=1$. Solutions of (2.1) having finite Yang-Mills action are called instantons. Their action is given by

$$S\;=\;-\frac{1}{g^2_{\mathrm{YM}}}\int\mathrm{tr}\,F\wedge *F\;=\;\frac{8\pi^2}{g^2_{\mathrm{YM}}}Q\ , \tag{2.2}$$

where Q is the topological charge

$$Q\;=\;-\frac{1}{8\pi^2}\int\mathrm{tr}\,F\wedge F\ . \tag{2.3}$$

Here, 'tr' is the trace over the $\mathfrak{u}(2)$ gauge algebra and g_{YM} the Yang-Mills coupling hidden in the definition of the Lie algebra components of A and F.

Introducing complex coordinates[1]

$$z^1=x^1+\mathrm{i}x^2,\quad \bar z^1=x^1-\mathrm{i}x^2,\quad z^2=x^3-\mathrm{i}x^4,\quad \bar z^2=x^3+\mathrm{i}x^4 \tag{2.4}$$

and defining

$$A_{z^1}\;=\;\tfrac{1}{2}(A_1-\mathrm{i}A_2)\qquad\text{and}\qquad A_{z^2}\;=\;\tfrac{1}{2}(A_3+\mathrm{i}A_4) \tag{2.5a}$$

as well as

$$A_{\bar z^1}\;=\;\tfrac{1}{2}(A_1+\mathrm{i}A_2)\qquad\text{and}\qquad A_{\bar z^2}\;=\;\tfrac{1}{2}(A_3-\mathrm{i}A_4)\ , \tag{2.5b}$$

we can rewrite the SDYM equations (2.1) in the form

$$[D_{z^1},D_{z^2}]=0\ ,\quad [D_{\bar z^1},D_{\bar z^2}]=0\ ,\quad [D_{z^1},D_{\bar z^1}]+[D_{z^2},D_{\bar z^2}]=0\ , \tag{2.6}$$

with $D_{z^a}=\partial_{z^a}+A_{z^a}$ and $D_{\bar z^a}=\partial_{\bar z^a}+A_{\bar z^a}$, respectively for $a=1,2$.

[1] This particular choice and its geometrical meaning will be clarified in section four and in the appendix.

Noncommutative Yang-Mills theory. A noncommutative extension of $\mathbb{R}^4$ is defined via deforming the ring of functions on it. More precisely, the pointwise product between functions gets replaced with the Moyal star product which is defined by

$$(f \star g)(x) \;:=\; f(x)\, \exp\left\{\tfrac{\mathrm{i}}{2}\overleftarrow{\partial_\mu}\theta^{\mu\nu}\overrightarrow{\partial_\nu}\right\}\, g(x)\ , \tag{2.7}$$

where $f, g \in C^\infty(\mathbb{R}^4, \mathbb{C})$ and $\theta^{\mu\nu}$ is chosen to be a constant antisymmetric tensor. Equation (2.7) implies that

$$[x^\mu, x^\nu]_\star \;:=\; x^\mu \star x^\nu - x^\nu \star x^\mu \;=\; \mathrm{i}\theta^{\mu\nu}\ . \tag{2.8}$$

In this paper we restrict $\theta^{\mu\nu}$ to be self-dual with

$$\theta^{12} = -\theta^{21} = \theta^{34} = -\theta^{43} =: \theta > 0 \tag{2.9}$$

and all other components being identically zero. The action (2.2) and the SDYM equations (2.1) are formally unchanged, but the ordinary product needs to be replaced by the star product, e. g.

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \star A_\nu - A_\nu \star A_\mu\ . \tag{2.10}$$

It is well known that the nonlocality of the star product can be cumbersome for explicit calculations. It is therefore convenient to pass to the operator formalism via the Weyl correspondence given by

$$\tilde{f}(k) \;\mapsto\; \hat{f}(\hat{x}) \;=\; \frac{1}{(2\pi)^4}\int \mathrm{d}^4\mathrm{k}\ \tilde{f}(k)\, \mathrm{e}^{\mathrm{i}k\hat{x}}\ , \tag{2.11a}$$

$$\hat{f}(\hat{x}) \;\mapsto\; \tilde{f}(k) \;=\; (2\pi\theta)^2\ \mathrm{Tr}\big\{\mathrm{e}^{-\mathrm{i}k\hat{x}}\hat{f}(\hat{x})\big\}\ , \tag{2.11b}$$

where 'Tr' denotes the trace over the operator representation of the noncommutative algebra and $\tilde{f}(k)$ is the Fourier transform of $f(x)$, i. e.

$$f(x) \;\mapsto\; \tilde{f}(k) \;=\; \int \mathrm{d}^4\mathrm{x}\ f(x)\, \mathrm{e}^{-\mathrm{i}kx}\ . \tag{2.12}$$

In these equations kx is shorthand notation for $k_\mu x^\mu$. Important relations are

$$f \star g \;\mapsto\; \hat{f}\,\hat{g} \qquad \text{and} \qquad \int \mathrm{d}^4\mathrm{x}\ f \;=\; (2\pi\theta)^2\, \mathrm{Tr}\,\hat{f}\ , \tag{2.13}$$

whenever both sides of the latter equation exist. Thus, we obtain operator-valued coordinates $\hat{x}^\mu$ satisfying the commutation relations $[\hat{x}^\mu, \hat{x}^\nu] = \mathrm{i}\theta^{\mu\nu}$ which define the noncommutative Euclidean four-dimensional space denoted by $\mathbb{R}^4_\theta$.

In complex coordinates (2.4) our choice of $\theta^{\mu\nu}$ (2.9) implies

$$[\hat{z}^1, \hat{\bar{z}}^1] \;=\; 2\theta \qquad \text{and} \qquad [\hat{z}^2, \hat{\bar{z}}^2] \;=\; -2\theta \tag{2.14}$$

with all other commutators being equal to zero. Coordinate derivatives turn into inner derivations of the operator algebra, i. e.

$$\hat{\partial}_{z^1}\hat{f} \;=\; -\frac{1}{2\theta}\,\mathrm{ad}(\hat{\bar{z}}^1)(\hat{f}) \qquad \text{and} \qquad \hat{\partial}_{\bar{z}^1}\hat{f} \;=\; \frac{1}{2\theta}\,\mathrm{ad}(\hat{z}^1)(\hat{f}) \tag{2.15a}$$

as well as

$$\hat{\partial}_{z^2}\hat{f} \;=\; \frac{1}{2\theta}\,\mathrm{ad}(\hat{\bar{z}}^2)(\hat{f}) \qquad \text{and} \qquad \hat{\partial}_{\bar{z}^2}\hat{f} \;=\; -\frac{1}{2\theta}\,\mathrm{ad}(\hat{z}^2)(\hat{f})\ . \tag{2.15b}$$

Since the commutation relations (2.14) identify our operator algebra with a pair of Heisenberg algebras it can obviously be represented on the two-oscillator Fock space $\mathcal{H} = \bigoplus_{n_1,n_2} \mathbb{C}\,|n_1,n_2\rangle$ with

$$|n_1,n_2\rangle \;=\; \frac{1}{\sqrt{n_1!n_2!(2\theta)^{n_1+n_2}}}\,(\hat{\bar{z}}^1)^{n_1}(\hat{z}^2)^{n_2}|0,0\rangle\ . \tag{2.16}$$

Hence, coordinates and fields on $\mathbb{R}^4_\theta$ both correspond to operators acting on $\mathcal{H}$. Sitting in the adjoint representation of $\mathfrak{u}(2)$ the components $\hat{A}_\mu$ and $\hat{F}_{\mu\nu}$ act from the left in the space $\mathcal{H}\otimes\mathbb{C}^2 \cong \mathcal{H}\oplus\mathcal{H}$, which carries a fundamental representation of the group $U(2)$. In the operator formulation the action (2.2) reads

$$S \;=\; -\tfrac{1}{2}\Big(\frac{2\pi\theta}{g_{\mathrm{YM}}}\Big)^2\,\mathrm{Tr}\big\{\mathrm{tr}\,\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\big\}\ , \tag{2.17}$$

and the SDYM equations (2.6) retain their familiar form

$$\hat{F}_{z^1z^2}=0\ ,\quad \hat{F}_{\bar{z}^1\bar{z}^2}=0\ ,\quad \hat{F}_{z^1\bar{z}^1}+\hat{F}_{z^2\bar{z}^2}=0\ . \tag{2.18}$$

In order to streamline our notation we will from now on omit the hats over the operators.

3 Instantons from the dressing approach

Linear system. The key observation is that the SDYM equations (2.18) can be obtained as the integrability conditions of the linear system (already in the noncommutative setup)

$$(D_{\bar{z}^1}-\lambda D_{z^2})\psi \;=\; 0\ ,\qquad (D_{\bar{z}^2}+\lambda D_{z^1})\psi \;=\; 0\qquad\text{and}\qquad \partial_{\bar{\lambda}}\psi \;=\; 0, \tag{3.1}$$

where $\lambda\in\mathbb{C}\cup\{\infty\}\cong\mathbb{CP}^1$ is called the spectral parameter and the $2{\times}2$ matrix ψ depends on $(x^1,x^2,x^3,x^4,\lambda)$ (or equivalently on $(z^1,\bar{z}^1,z^2,\bar{z}^2,\lambda)$). Hence, it lives on a space $\mathcal{P}=\mathbb{R}^4_\theta\times\mathbb{CP}^1$ known as the twistor space. Since this system is overdetermined there are integrability conditions which turn out to be exactly the SDYM equations (2.18).

Following Belavin and Zakharov [46] we impose a reality condition on ψ:

$$[\psi(x,-\bar{\lambda}^{-1})]^\dagger \;=\; \psi^{-1}(x,\lambda)\ , \tag{3.2}$$

with $x\in\mathbb{R}^4_\theta$. This restriction ensures that the gauge potential A_μ is anti-Hermitian, i. e. $A_\mu^\dagger=-A_\mu$, as is required by the gauge group $U(2)$.

The linear system (3.1) can be rewritten as

$$\psi(\partial_{\bar{z}^1}-\lambda\partial_{z^2})\psi^\dagger \;=\; A_{\bar{z}^1}-\lambda A_{z^2}\ , \tag{3.3a}$$
$$\psi(\partial_{\bar{z}^2}+\lambda\partial_{z^1})\psi^\dagger \;=\; A_{\bar{z}^2}+\lambda A_{z^1}\ , \tag{3.3b}$$

whereby $\psi^\dagger$ is shorthand notation for $[\psi(x,-\bar{\lambda}^{-1})]^\dagger$. The task is to find the auxiliary field ψ, since the gauge potential and, hence, the curvature then follow immediately from the equations above.

Dressing method. In the commutative setup Belavin and Zakharov constructed [46] solutions to the SDYM equations (2.1) by using the method of dressing transformations, which is a recursive procedure for generating solutions from a given seed solution. After having derived the one-instanton BPST solution they used it as the seed solution to give a recursion relation for the construction of the n-instanton configuration. The extension of the dressing method to the noncommutative case is readily accomplished (cf. [48], [49] and [50]). We now recall this extension and generalize the Belavin-Zakharov ansatz to the noncommutative realm.

Let ψ be a given solution of the linear system (3.3). To generate a new solution out of ψ we multiply the so-called dressing factor χ on the left, i. e. we consider

$$\tilde{\psi}(z^1,\bar{z}^1,z^2,\bar{z}^2,\lambda) \;=\; \chi(z^1,\bar{z}^1,z^2,\bar{z}^2,\lambda)\,\psi(z^1,\bar{z}^1,z^2,\bar{z}^2,\lambda)\;. \tag{3.4}$$

The dressing factor χ is assumed to be a global meromorphic operator-valued[2] function in the spectral parameter $\lambda\in\mathbb{CP}^1$, which implies the expansion

$$\chi \;=\; \lambda R_{-1}+R_0+\sum_{i=1}^{r}\frac{R_i}{(\mu_i\lambda+\nu_i)} \tag{3.5}$$

with μ_i, $\nu_i\in\mathbb{C}$ and the coefficients R_{-1}, R_0 and R_i depending on $\{z^a\}$ and $\{\bar{z}^a\}$ but not on the spectral parameter λ. We have restricted ourselves to first-order poles, since one can generate higher-order poles by a successive multiplication of such expressions. Thus, χ has finitely many poles which are located at $\lambda_\infty=\infty$ and $\lambda_i=-\nu_i/\mu_i$ with $\mu_i\neq 0$ for $i=1,\dots,r$.

Since $\tilde{\psi}$ is supposed to be a new solution we can write

$$\tilde{\psi}(\partial_{\bar{z}^1}-\lambda\partial_{z^2})\tilde{\psi}^\dagger \;=\; \tilde{A}_{\bar{z}^1}-\lambda\tilde{A}_{z^2}\;, \tag{3.6a}$$
$$\tilde{\psi}(\partial_{\bar{z}^2}+\lambda\partial_{z^1})\tilde{\psi}^\dagger \;=\; \tilde{A}_{\bar{z}^2}+\lambda\tilde{A}_{z^1}\;. \tag{3.6b}$$

A short calculation shows that

$$\chi(D_{\bar{z}^1}-\lambda D_{z^2})\chi^\dagger \;=\; \tilde{A}_{\bar{z}^1}-\lambda\tilde{A}_{z^2}\;, \tag{3.7a}$$
$$\chi(D_{\bar{z}^2}+\lambda D_{z^1})\chi^\dagger \;=\; \tilde{A}_{\bar{z}^2}+\lambda\tilde{A}_{z^1}\;, \tag{3.7b}$$

where $D_\mu=\partial_\mu+A_\mu$ is the covariant derivative in the background of the old gauge potential A_μ determined through ψ. Since the gauge potential $\tilde{A}_\mu$ is λ-independent the left hand side of (3.7) is at most linear in λ. The ansatz (3.5) for χ, however, contains finitely many poles in the spectral parameter. Therefore, all proper residues must vanish. This requirement yields differential equations for the coefficients R_{-1}, R_0 and R_i for $i=1,\dots,r$. After solving these equations one obtains a new solution $\tilde{A}_\mu$ of the noncommutative SDYM equations. Finally, this procedure may be iterated to get new solutions from old ones.

Ansatz and noncommutative BPST instanton. Let us obtain a one-instanton solution by way of dressing. The trivial solution of (3.3) is $\psi=1$ and $A_\mu=0$. We take this solution as the seed solution and choose the dressing factor of the form [46][3]

$$\psi(x,\lambda) \;=\; G\Big(1+2H+\lambda S^\dagger+\frac{1}{\lambda}S\Big)\;, \tag{3.8}$$

where all λ dependence is made explicit and G and H are taken to be Hermitian diagonal $2{\times}2$ matrix functions of $\{z^a,\bar{z}^a\}$, i. e.

$$G \;=:\; \mathrm{diag}(g_-,g_+) \qquad\text{and}\qquad H \;=:\; \mathrm{diag}(h_-,h_+)\;. \tag{3.9}$$

[2]When we say operator-valued function we imply a $\{z^a\}$ and $\{\bar{z}^a\}$ dependence.

[3]Here we have renamed the dressing factor χ and called it ψ in order to be conform with the literature.

From equation (3.8) we obtain

$$[\psi(x,-\bar\lambda^{-1})]^\dagger \;=\; \Big(1+2H-\lambda S^\dagger-\frac{1}{\lambda}S\Big)\,G\;. \tag{3.10}$$

In order to simplify further calculations we put some restrictions on H and G, namely we require that

$$[G,H] \;=\; [G,S] \;=\; [H,S] \;=\; 0\;. \tag{3.11}$$

Then the reality condition (3.2) implies

$$S^2 \;=\; 0 \qquad\text{and}\qquad (S^\dagger)^2 \;=\; 0 \tag{3.12a}$$

as well as

$$G^2(1{+}2H)^2 \;=\; 1+G^2\{S,S^\dagger\}\;. \tag{3.12b}$$

The λ dependence of the differential equations (3.3) leads to

$$G\big\{S^\dagger\partial_{z^a}S^\dagger\big\}G \;=\; 0 \qquad\text{for}\quad a=1,2 \tag{3.13a}$$

as well as

$$G\big\{(1{+}2H)\partial_{z^a}S^\dagger-S^\dagger\partial_{z^a}(1{+}2H)+\epsilon_{ab}S^\dagger\partial_{\bar z^b}S^\dagger\big\}G \;=\; 0 \qquad\text{for}\quad a=1,2\;, \tag{3.13b}$$

where the nilpotency of $S^\dagger$ has been used. We shall find matrix functions H and S for which the brackets above will vanish by themselves.

In order to construct a 2×2 matrix S satisfying (3.12a) and (3.13a) (which do not involve H), we consider the two vectors

$$T_1 \;=\; \begin{pmatrix} v_1 \\ -v_2\end{pmatrix} \qquad\text{and}\qquad T_2 \;=\; \begin{pmatrix} v_2^\dagger \\ v_1^\dagger\end{pmatrix} \tag{3.14}$$

built from functions v_1 and v_2 and introduce some real function $K=K(r^2)$ which depends only on the combination

$$r^2 \;:=\; \bar z^1z^1+\bar z^2z^2\;. \tag{3.15}$$

With these ingredients we parametrize $S^\dagger$ as follows,

$$S^\dagger \;=\; T_1\frac{1}{K}T_2^\dagger \;=\; \begin{pmatrix} v_1\frac{1}{K}v_2 & v_1\frac{1}{K}v_1 \\ -v_2\frac{1}{K}v_2 & -v_2\frac{1}{K}v_1\end{pmatrix}\;, \tag{3.16}$$

and expect conditions on v_1, v_2, and K. Due to the nilpotency property (3.12a) of S and $S^\dagger$ we find $[v_1,v_2]=0$ and hence, $T_1^\dagger T_2=T_2^\dagger T_1=0$. The condition (3.13a) then tells us that v_1 and v_2 are anti-holomorphic functions depending on $\bar z^1$ and $\bar z^2$ only. A simple choice is $v_1=\bar z^1$ and $v_2=\bar z^2$, which specifies

$$S^\dagger \;=\; \begin{pmatrix} \bar z^1\frac{1}{K}\bar z^2 & \bar z^1\frac{1}{K}\bar z^1 \\ -\bar z^2\frac{1}{K}\bar z^2 & -\bar z^2\frac{1}{K}\bar z^1\end{pmatrix}\;. \tag{3.17}$$

The commutation property (3.11) of G and H with this form of S can be achieved by choosing $g_\pm$ and $h_\pm$ in (3.9) to depend only on r^2 and by relating them via

$$g_\pm(r^2) \;=\; g(r^2\pm2\theta) \qquad\text{and}\qquad h_\pm(r^2) \;=\; h(r^2\pm2\theta) \tag{3.18}$$

to yet unknown scalar functions g and h. It is useful to define the notation $f_\pm(r^2) := f(r^2 \pm 2\theta)$ for arbitrary functions f.

Next we address equations (3.13b). Remembering that partial derivatives are given by the inner derivations (2.15a) and (2.15b) we reformulate:

$$G\big\{(1{+}2H)\,\bar z^1 S^\dagger - S^\dagger \bar z^1 (1{+}2H) + S^\dagger z^2 S^\dagger\big\}G \;=\; 0\;, \tag{3.19a}$$

$$G\big\{(1{+}2H)\,\bar z^2 S^\dagger - S^\dagger \bar z^2 (1{+}2H) - S^\dagger z^1 S^\dagger\big\}G \;=\; 0\;. \tag{3.19b}$$

Demanding that the brackets vanish gives us two relations for the functions h and K. We claim that these are fulfilled for

$$h(r^2) \;=\; -\tfrac12\Big(1+\frac{K(r^2)}{K(r^2)-2C}\Big) \qquad \text{and} \qquad K(r^2+2\theta) \;=\; K(r^2)+2\theta \tag{3.20}$$

with $C \in \mathbb{R}$ being some (almost) arbitrary constant. The proof of this assertion is rather straightforward and we therefore refrain from presenting it here. The general solution to the functional equation in (3.20) is

$$K(r^2) \;=\; r^2 + \tilde K(r^2) \tag{3.21}$$

with an arbitrary 2θ-periodic function $\tilde K$. By imposing a smooth $\theta \to 0$ limit we force $\tilde K$ to be constant, i. e.

$$K(r^2) \;=\; r^2 + 2\Lambda^2 \tag{3.22}$$

with some new real constant $\Lambda^2 \geq 0$.[4] We are going to see that Λ corresponds to the size of the instanton. The corresponding h reads

$$h(r^2) \;=\; -\frac{r^2+2\Lambda^2-C}{r^2+2\Lambda^2-2C}\;. \tag{3.23}$$

A particularly simple choice is $C = 0$ which results in $h = -1$. More generally, we should only exclude the values $C = \Lambda^2 + n\theta$ for $n \in \mathbb{Z}_+$, where H becomes singular.

It remains to impose the condition (3.12b) which serves to determine the function g. With the help of

$$\{S, S^\dagger\} \;=\; \begin{pmatrix} \frac{(r^2)_-^2}{K_-^2} & 0 \\ 0 & \frac{(r^2)_+^2}{K_+^2} \end{pmatrix} \tag{3.24}$$

equation (3.12b) reduces to

$$g^2(1{+}2h)^2 \;=\; 1 + g^2\frac{(r^2)^2}{K^2} \tag{3.25}$$

which yields

$$g(r^2) \;=\; \pm\frac{1}{2\Lambda}\,\frac{r^2+2\Lambda^2-2C}{\sqrt{r^2+\Lambda^2}}\;. \tag{3.26}$$

During this computation we tacitly assumed that all quantities are well behaved such that we could legally perform all necessary operations. However, this need not always be the case. As a prime example, $(r^2)_-^{-1} \equiv (r^2{-}2\theta)^{-1}$ is ill behaved when acting on the ground state $|0,0\rangle$ of $\mathcal{H}$.

[4]The positivity of Λ^2 will follow from the hermiticity of g.

Luckily, the potentially dangerous K_-^{-1} is regulated by Λ^2. Only the limit $\Lambda^2 \to 0$ is apparently singular. The results above may be recombined to compose the final expression for ψ:

$$\psi \;=\; \frac{1}{2\Lambda}\begin{pmatrix} \frac{1}{\sqrt{r^2+\Lambda^2-2\theta}} & 0 \\ 0 & \frac{1}{\sqrt{r^2+\Lambda^2+2\theta}} \end{pmatrix} \begin{pmatrix} r^2+2\Lambda^2-2\theta-\lambda\bar{z}^1\bar{z}^2-\frac{z^1z^2}{\lambda} & -\lambda(\bar{z}^1)^2+\frac{(z^2)^2}{\lambda} \\ \lambda(\bar{z}^2)^2-\frac{(z^1)^2}{\lambda} & r^2+2\Lambda^2+2\theta+\lambda\bar{z}^1\bar{z}^2+\frac{z^1z^2}{\lambda} \end{pmatrix}\;, \tag{3.27}$$

where we have chosen the negative sign in (3.26). The θ dependence of this noncommutative solution is very simple. It is also quite remarkable that it differs from the commutative solution (as given by Belavin and Zakharov [46]) only in a few spots. Moreover, the final expression is independent of the parameter C, in accordance with the latter's interpretation as a regulator of intermediate singularities in the course of the calculation.

Connection and curvature. We are now in the position to construct the gauge potential using our solution (3.27). By looking at the terms of (3.3) which are linear in λ we can compute A_{z^1} and A_{z^2}. The calculation is kind of tedious but straightforward. We therefore omit the explicit derivation and give only the results

$$A_{z^1} \;=\; \begin{pmatrix} -\frac{\bar{z}^1}{2\theta}\left(\sqrt{\frac{r^2+\Lambda^2-2\theta}{r^2+\Lambda^2}}-1\right) & 0 \\ -\bar{z}^2\frac{1}{\sqrt{r^2+\Lambda^2}\sqrt{r^2+\Lambda^2-2\theta}} & -\frac{\bar{z}^1}{2\theta}\left(\sqrt{\frac{r^2+\Lambda^2+4\theta}{r^2+\Lambda^2+2\theta}}-1\right) \end{pmatrix} \tag{3.28a}$$

and

$$A_{z^2} \;=\; \begin{pmatrix} \left(\sqrt{\frac{r^2+\Lambda^2-2\theta}{r^2+\Lambda^2}}-1\right)\frac{\bar{z}^2}{2\theta} & -\frac{1}{\sqrt{r^2+\Lambda^2}\sqrt{r^2+\Lambda^2-2\theta}}\bar{z}^1 \\ 0 & \left(\sqrt{\frac{r^2+\Lambda^2+4\theta}{r^2+\Lambda^2+2\theta}}-1\right)\frac{\bar{z}^2}{2\theta} \end{pmatrix}\;, \tag{3.28b}$$

which in the commutative limit $\theta \to 0$ coincide with the Belavin-Zakharov solution [46]. The remaining components are $A_{\bar{z}^a} = -A_{z^a}^\dagger$ for $a = 1, 2$.

Furthermore, we compute the nonvanishing components of the Yang-Mills curvature using $F_{z^a\bar{z}^b} = \partial_{z^a}A_{\bar{z}^b} - \partial_{\bar{z}^b}A_{z^a} + [A_{z^a}, A_{\bar{z}^b}]$ for $a, b = 1, 2$. Again, this task is lengthy but not difficult. Ultimately we find the self-dual configuration

$$F_{z^1\bar{z}^1} \;=\; -F_{z^2\bar{z}^2} \;=\; \begin{pmatrix} -\frac{\Lambda^2}{(r^2+\Lambda^2)(r^2+\Lambda^2-2\theta)} & 0 \\ 0 & \frac{\Lambda^2}{(r^2+\Lambda^2-2\theta)(r^2+\Lambda^2+2\theta)} \end{pmatrix} \tag{3.29a}$$

and

$$F_{z^1\bar{z}^2} \;=\; F_{z^2\bar{z}^1}^\dagger \;=\; \begin{pmatrix} 0 & 0 \\ \frac{2\Lambda^2}{(r^2+\Lambda^2)\sqrt{r^2+\Lambda^2-2\theta}\sqrt{r^2+\Lambda^2+2\theta}} & 0 \end{pmatrix}. \tag{3.29b}$$

Comparing this solution, constructed by the dressing method, with the one obtained via the ADHM approach [14] we recognize complete agreement if we identify Λ with the size of the instanton. The zero-size limit produces a pure-gauge configuration as it should for self-dual $\theta^{\mu\nu}$.

Recall that the calculation of the topological charge produces a surface integral at infinity, where the noncommutativity goes to zero.[5] Since our solution coincides with the standard BPST instanton in the commutative limit, we conclude that it has topological charge $Q = 1$. This result was also obtained by Furuuchi [14] in a direct evaluation of Q from the expressions (3.28).

[5] However, when $U(1)$ is gauged this is not true [12].

4 Instantons from the splitting approach

In the appendix we give a brief review of the geometric picture of the linear system (3.3) in the commutative case in terms of holomorphic vector bundles over the twistor space $\mathcal{P} = \mathbb{R}^4 \times \mathbb{CP}^1$ for the space $\mathbb{R}^4$ [37, 38]. This eventually results in the Ward correspondence. For the noncommutative extension, the notion of vector bundles does not exist anymore and is replaced by the notion of projective modules (see e. g. [32], [34] and [35]). Still, the assertion of Ward's theorem remains the same on noncommutative $\mathbb{R}^4_\theta \times \mathbb{CP}^1$ in the following sense. Let $U_\pm$ be the two canonical coordinate patches covering $\mathbb{CP}^1$. Taking an operator-valued holomorphic matrix f_{+-} restricted to $U_+ \cap U_- \subset \mathbb{CP}^1$, one may try to split $f_{+-} = \psi_+^{-1}\psi_-$ into ψ_+ and ψ_- having no singularities in the spectral parameter λ on U_+ and U_-, respectively. If successful then one can use the equations

$$A_{\bar{z}^a} \;=\; \psi_+ \partial_{\bar{z}^a} \psi_+^{-1}\big|_{\lambda=0} \qquad \text{for} \quad a = 1, 2 \tag{4.1}$$

to find a self-dual gauge potential.[6] This defines a parametric Riemann-Hilbert problem. For a more detailed discussion we refer to [7], [9] and [22].

The linear system (3.3) is equivalent to the equations[6]

$$\begin{aligned} \psi_+(\partial_{\bar{z}^1} - \lambda\partial_{z^2})\psi_+^{-1} \;&=\; \psi_-(\partial_{\bar{z}^1} - \lambda\partial_{z^2})\psi_-^{-1} \;=\; A_{\bar{z}^1} - \lambda A_{z^2} \;, && (4.2a)\\ \psi_+(\partial_{\bar{z}^2} + \lambda\partial_{z^1})\psi_+^{-1} \;&=\; \psi_-(\partial_{\bar{z}^2} + \lambda\partial_{z^1})\psi_-^{-1} \;=\; A_{\bar{z}^2} + \lambda A_{z^1} \;, && (4.2b) \end{aligned}$$

since using the solution (3.27) for ψ one can construct a matrix Υ which depends meromorphically on

$$w^1 \;=\; z^1 - \lambda\bar{z}^2 \;, \qquad w^2 \;=\; z^2 + \lambda\bar{z}^1 \;, \qquad \text{and} \quad w^3 \;=\; \lambda \tag{4.3}$$

and acts by right multiplication so that

$$\psi_+ \;=\; \psi\,\Upsilon \;. \tag{4.4}$$

We then define ψ_- by $\psi_-(\lambda) := [\psi_+^{-1}(-1/\bar{\lambda})]^\dagger$. In the commutative case this was shown by Crane [47]. We shall now generalize his construction to the noncommutative setup.

One-instanton solution. For notational simplicity let us rewrite the solution (3.27) for ψ as

$$\psi \;=\; X + \lambda Y^\dagger + \frac{1}{\lambda} Y \;, \tag{4.5}$$

i. e. we abbreviate the known expressions by $X = G(1{+}2H)$ and $Y = GS = GT_2\frac{1}{K}T_1^\dagger$, where the matrices G, H, T_i and K are given in the previous section. Moreover, we expand ψ_+ and Υ on $U_+ \cap U_-$ as

$$\psi_+(x,\lambda) \;=\; \sum_{n\in\mathbb{Z}_+} \psi_{+n}(x)\,\lambda^n \qquad \text{and} \qquad \Upsilon(w^1,w^2,\lambda) \;=\; \sum_{n\in\mathbb{Z}} \Upsilon_n(w^1,w^2)\,\lambda^n \;, \tag{4.6}$$

since $w^3 = \lambda$. Comparing coefficients of λ^n in $\psi_+ = \psi\,\Upsilon$ then yields the equations

$$\begin{aligned} \psi_{+n} \;&=\; X\Upsilon_n + Y^\dagger\Upsilon_{n-1} + Y\Upsilon_{n+1} \qquad \text{for} \quad n \geq 0 \;, && (4.7a)\\ 0 \;&=\; X\Upsilon_n + Y^\dagger\Upsilon_{n-1} + Y\Upsilon_{n+1} \qquad \text{for} \quad n < 0 \;, && (4.7b) \end{aligned}$$

[6] These equations are derived and discussed in the appendix.

while the holomorphicity conditions $\partial_{\bar{w}^a}\Upsilon = 0$ imply the recursion relations

$$\partial_{\bar{z}^1}\Upsilon_n \;=\; \partial_{z^2}\Upsilon_{n-1} \qquad \text{and} \qquad \partial_{\bar{z}^2}\Upsilon_n \;=\; -\partial_{z^1}\Upsilon_{n-1}\ . \tag{4.8}$$

Bearing in mind the commutative limit [47] we would like to truncate to

$$\psi_+(\lambda) \;=\; \psi_{+0}+\psi_{+1}\lambda \qquad \text{and} \qquad \Upsilon(\lambda) \;=\; \Upsilon_{-1}\lambda^{-1}+\Upsilon_0+\Upsilon_1\lambda\ . \tag{4.9}$$

We claim that it is indeed consistent to require $\Upsilon_{n<-1}=\Upsilon_{n>1}=0$ and $\psi_{+n<0}=\psi_{+n>1}=0$, which reduces the infinite set (4.7a) and (4.7b) to

$$0 \;=\; Y\Upsilon_{-1}\ , \tag{4.10a}$$
$$0 \;=\; X\Upsilon_{-1}+Y\Upsilon_0\ , \tag{4.10b}$$
$$\psi_{+0} \;=\; X\Upsilon_0+Y^\dagger\Upsilon_{-1}+Y\Upsilon_1\ , \tag{4.10c}$$
$$\psi_{+1} \;=\; X\Upsilon_1+Y^\dagger\Upsilon_0\ , \tag{4.10d}$$
$$0 \;=\; Y^\dagger\Upsilon_1\ . \tag{4.10e}$$

The truncated recursion relations (4.8) imply that

$$\partial_{\bar{z}^a}\Upsilon_{-1} \;=\; 0 \;=\; \partial_{z^a}\partial_{z^b}\partial_{z^c}\Upsilon_{-1}\ , \tag{4.11a}$$
$$\partial_{\bar{z}^a}\partial_{\bar{z}^b}\Upsilon_0 \;=\; 0 \;=\; \partial_{z^a}\partial_{z^b}\Upsilon_0 \qquad \text{and} \qquad \Box\,\Upsilon_0 \;=\; 0\ , \tag{4.11b}$$
$$\partial_{\bar{z}^a}\partial_{\bar{z}^b}\partial_{\bar{z}^c}\Upsilon_1 \;=\; 0 \;=\; \partial_{z^a}\Upsilon_1\ , \tag{4.11c}$$

so that the 2×2 matrices Υ_0 and $\Upsilon_\pm$ are quadratic functions of $\{z^a\}$ and $\{\bar{z}^a\}$. Demanding invariance under reflection on the origin and additionally imposing $\Upsilon(\lambda)=[\Upsilon(-1/\bar{\lambda})]^\dagger$ the functional dependence takes the form

$$\Upsilon_{-1} \;=\; -\tau_-(z^1)^2+\tau_+(z^2)^2-\tau_3\,z^1z^2\ , \tag{4.12a}$$
$$\Upsilon_0 \;=\; -\tau_3\,(z^1\bar{z}^1-z^2\bar{z}^2)+2\tau_-z^1\bar{z}^2+2\tau_+z^2\bar{z}^1-\tau_4\ , \tag{4.12b}$$
$$\Upsilon_1 \;=\; \tau_+(\bar{z}^1)^2-\tau_-(\bar{z}^2)^2+\tau_3\,\bar{z}^1\bar{z}^2\ , \tag{4.12c}$$

with constant matrices $\tau_-=\tau_+^\dagger$ as well as τ_3 and τ_4 (the latter two being Hermitian).

Equations (4.10a) and (4.10e) with $\Upsilon_{-1}^\dagger=-\Upsilon_1$ are clearly solved by putting

$$\Upsilon_{-1} \;=\; -\tfrac{1}{2\Lambda}T_2T_1^\dagger \qquad \text{and} \qquad \Upsilon_1 \;=\; \tfrac{1}{2\Lambda}T_1T_2^\dagger\ , \tag{4.13}$$

where a convenient normalization has been chosen. This fixes $\tau_\pm=\frac{1}{2\Lambda}(\sigma_1\pm\mathrm{i}\sigma_2)$ and $\tau_3=\frac{1}{2\Lambda}\sigma_3$, where σ_i denotes the Pauli matrices. The remaining condition (4.10b) then determines the matrix $\tau_4=\mathbf{1}$. The equations (4.10c) and (4.10d) finally serve to compute ψ_+. Expressed in terms of $\{w^a\}$ coordinates, we thus arrive at

$$\Upsilon(w^1,w^2,\lambda) \;=\; \frac{1}{2\Lambda^2}\begin{pmatrix} -2\Lambda^2-\frac{w^1w^2}{\lambda} & \frac{(w^2)^2}{\lambda} \\ -\frac{(w^1)^2}{\lambda} & -2\Lambda^2+\frac{w^1w^2}{\lambda}\end{pmatrix}\ . \tag{4.14}$$

The matrix ψ_+ is then given by

$$\begin{aligned}\psi_+(x,\lambda) \;&=\; \psi(x,\lambda)\,\Upsilon(w^1,w^2,\lambda)\\ &=\; -\frac{1}{\Lambda}\begin{pmatrix}\frac{1}{\sqrt{r^2+\Lambda^2-2\theta}} & 0\\ 0 & \frac{1}{\sqrt{r^2+\Lambda^2+2\theta}}\end{pmatrix}\begin{pmatrix}\bar{z}^1z^1+\Lambda^2-\lambda\bar{z}^1\bar{z}^2 & -\bar{z}^1z^2-\lambda(\bar{z}^1)^2\\ -z^1\bar{z}^2+\lambda(\bar{z}^2)^2 & \bar{z}^2z^2+\Lambda^2+\lambda\bar{z}^1\bar{z}^2\end{pmatrix}\end{aligned} \tag{4.15}$$

which coincides with Crane's solution [47] in the commutative limit. The matrix f_{+-} then becomes

$$f_{+-}(w^a,\lambda) \;=\; \psi_+^{-1}(w^a,\lambda)\,\psi_-(w^a,\lambda) \;=\; \Upsilon^{-2}(w^a,\lambda) \;=\; \frac{1}{\Lambda^2}\begin{pmatrix} \Lambda^2-\frac{w^1w^2}{\lambda} & \frac{(w^2)^2}{\lambda} \\ -\frac{(w^1)^2}{\lambda} & \Lambda^2+\frac{w^1w^2}{\lambda} \end{pmatrix}\,. \tag{4.16}$$

In order to construct the gauge potential we need the inverse of ψ_+ at $\lambda=0$ which is

$$\psi_+^{-1}\big|_{\lambda=0} \;=\; -\frac{1}{\Lambda}\begin{pmatrix} \frac{1}{\sqrt{r^2+\Lambda^2-2\theta}} & 0 \\ 0 & \frac{1}{\sqrt{r^2+\Lambda^2+2\theta}} \end{pmatrix}\begin{pmatrix} \bar z^2 z^2+\Lambda^2-2\theta & \bar z^1 z^2 \\ z^1\bar z^2 & \bar z^1 z^1+\Lambda^2+2\theta \end{pmatrix}\,. \tag{4.17}$$

With these ingredients we can use equation (4.1) to reconstruct the gauge potential and hence, the curvature. What we find in this way is, of course, identical to the result of the previous section, namely (3.28) and (3.29).

5 Concluding remarks

In this paper we have extended the self-dual BPST instanton solution to Yang-Mills theory defined on a noncommutative Euclidean space. We have chosen the noncommutative deformation matrix $\theta^{\mu\nu}$ to be self-dual as well, because self-dual instantons on an anti-self-dual background cannot be captured with our noncommutative extension of the Belavin-Zakharov ansatz. Our calculations demonstrate that noncommutativity with a self-dual $\theta^{\mu\nu}$ causes no difficulties in constructing solutions. Potential singularities like $(r^2-2\theta)^{-1}$, as occurring for the noncommutative 't Hooft instanton [16, 22], are regulated in our case by the instanton size. Moreover, in the framework of the dressing and splitting approaches described in this paper we were able to solve the reality problem of the gauge field which was encountered by the authors of [16] in generalizing the BPST ansatz [45].

It is tempting to recycle the constructed one-instanton solution as the seed solution in the dressing method, in order to generate multi-instantons. Perhaps a combination of the dressing and splitting method will do the job. One may hope that computing the two-instanton[7] configuration in terms of a matrix $\Upsilon_{n=2}$ will point towards a recursive procedure for the construction of n-instantons. However, further work in this direction needs to be done.

The splitting and dressing approaches presented here have recently been lifted from gauge field theory to string field theory [51, 52]. Since $10d$ superstrings in Berkovits' nonpolynomial formulation [53] as well as $4d$ self-dual strings à la Berkovits and Siegel [54] are classically integrable [55], their field equations can be linearized and classical backgrounds can be constructed using these methods. Moreover, string field theory may be viewed as an infinite-dimensional noncommutative field theory, so that the techniques of the present paper are directly applicable. A program in this direction has been initiated [51, 52].

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[7] Using the dressing method Belavin and Zakharov found the two-instanton solution in the commutative case [46].

A Geometry of the twistor space for $\mathbb{R}^4$

To motivate the linear system (3.3) and to understand the geometry behind it, this appendix analyzes its commutative analog. It has a geometrical interpretation in terms of holomorphic bundles over the twistor space $\mathcal{P} = \mathbb{R}^4 \times \mathbb{CP}^1$ for the space $\mathbb{R}^4$ [37, 38].

It is well known that the two-sphere $S^2 \cong \mathbb{CP}^1$ can be covered by two coordinate patches $U_\pm$ and hence, also the whole twistor space $\mathcal{P}$:

$$\mathcal{P} \;=\; \mathcal{U}_+ \cup \mathcal{U}_- \;, \qquad \mathcal{U}_+ \;=\; \mathbb{R}^4 \times U_+ \;, \qquad \mathcal{U}_- \;=\; \mathbb{R}^4 \times U_- \;, \tag{A.1}$$

with $U_+ = \mathbb{CP}^1 \setminus \{\infty\}$ and $U_- = \mathbb{CP}^1 \setminus \{0\}$. The complexified tangent space $\mathbb{C}^4 \cong \mathbb{R}^4 \otimes \mathbb{C}$ of $\mathbb{R}^4$ can be decomposed into two subspaces with respect to a chosen constant almost complex structure $J = (J^\mu_{\ \nu})$ on $\mathbb{R}^4$, i. e. $\mathbb{C}^4 \cong \mathcal{V} \oplus \bar{\mathcal{V}}$ with[8]

$$\mathcal{V} \;=\; \{V \in \mathbb{C}^4 \,|\, J^\mu_{\ \nu} V^\nu = \mathrm{i} V^\mu\} \qquad \text{and} \qquad \bar{\mathcal{V}} \;=\; \{V \in \mathbb{C}^4 \,|\, J^\mu_{\ \nu} V^\nu = -\mathrm{i} V^\mu\} \;. \tag{A.2}$$

For instance, on $\bar{\mathcal{V}}$ we may take the basis

$$(\bar{V}^\mu_1) \;=\; \left(\tfrac{1}{2}, \tfrac{\mathrm{i}}{2}, -\tfrac{1}{2}\lambda, -\tfrac{\mathrm{i}}{2}\lambda\right) \qquad \text{and} \qquad (\bar{V}^\mu_2) \;=\; \left(\tfrac{1}{2}\lambda, -\tfrac{\mathrm{i}}{2}\lambda, \tfrac{1}{2}, -\tfrac{\mathrm{i}}{2}\right) \;, \tag{A.3}$$

where λ is the local holomorphic coordinate on the two-sphere $S^2 \cong SO(4)/U(2)$ which parametrizes J, i. e. $\lambda = (\xi^1 + \mathrm{i}\xi^2)/(1+\xi^3)$ with $\xi^a \xi^a = 1$. Therefore, we can introduce on $\mathcal{U}_+$ (and similarly on $\mathcal{U}_-$) the anti-holomorphic vector fields $\bar{V}_a = \bar{V}^\mu_a \partial_\mu$ $(a = 1,2,3)$[8]

$$\bar{V}_1 \;=\; \partial_{\bar{z}^1} - \lambda \partial_{z^2} \;, \qquad \bar{V}_2 \;=\; \partial_{\bar{z}^2} + \lambda \partial_{z^2} \qquad \text{and} \qquad \bar{V}_3 \;=\; \partial_{\bar{\lambda}} \;, \tag{A.4}$$

where we have chosen the standard complex structure on $S^2 \cong \mathbb{CP}^1$. This means that the coordinates (2.4) are the canonical complex coordinates on $\mathbb{R}^4 \cong \mathbb{C}^2$ corresponding to $\lambda = 0$. Hence, the appropriate coordinates on the twistor space $\mathcal{P}$ are

$$w^1 = z^1 - \lambda \bar{z}^2 \;, \qquad w^2 = z^2 + \lambda \bar{z}^1 \qquad \text{and} \qquad w^3 = \lambda \qquad \text{on} \quad \mathcal{U}_+ \;, \tag{A.5a}$$

$$\tilde{w}^1 = \tilde{\lambda} z^1 - \bar{z}^2 \;, \qquad \tilde{w}^2 = \tilde{\lambda} z^2 + \bar{z}^1 \qquad \text{and} \qquad \tilde{w}^3 = \tilde{\lambda} \qquad \text{on} \quad \mathcal{U}_- \;, \tag{A.5b}$$

which are related on the intersection $\mathcal{U}_+ \cap \mathcal{U}_- \cong \mathbb{R}^4 \times \mathbb{C}^*$ via

$$w^1 = \frac{\tilde{w}^1}{\tilde{w}^3} \;, \qquad w^2 = \frac{\tilde{w}^2}{\tilde{w}^3} \qquad \text{and} \qquad w^3 = \frac{1}{\tilde{w}^3} \;. \tag{A.6}$$

Moreover, we can construct a local basis of 1-forms $\bar{\theta}^a(\bar{V}_b) = \delta^a_{\ b}$ on $\mathcal{U}_+$ which read

$$\bar{\theta}^1 \;=\; \gamma\,(\mathrm{d}\,\bar{z}^1 - \bar{\lambda}\,\mathrm{d}\,z^2) \;, \qquad \bar{\theta}^2 \;=\; \gamma\,(\mathrm{d}\,\bar{z}^2 + \bar{\lambda}\,\mathrm{d}\,z^1) \qquad \text{and} \qquad \bar{\theta}^3 \;=\; \mathrm{d}\,\bar{\lambda} \;, \tag{A.7}$$

with $\gamma = (1 + \lambda\bar{\lambda})^{-1}$.

On the principal bundle $P = \mathbb{R}^4 \times U(2)$ over $\mathbb{R}^4$ the connection 1-form A determines the connection $D = \mathrm{d} + A$ on P. Furthermore, let $\rho : U(2) \to GL(2,\mathbb{C})$ be the fundamental representation of $U(2)$. Then, as usual, we associate to P the complex vector bundle $E = P \times_\rho \mathbb{C}^2$. Using the projection $\pi : \mathcal{P} \to \mathbb{R}^4$ we can pull back E to a bundle $\pi^* E$ over $\mathcal{P}$. By definition, the pulled-back connection 1-form $\pi^* A$ on $\pi^* E$ is flat along the fibers $\mathbb{CP}^1$ and, hence, the pulled-back connection

[8] For a detailed description see e. g. [56].

π^*D is nothing but $\pi^*D|_{\mathcal{U}_+} = D + \mathrm{d}\,\lambda\,\partial_\lambda + \mathrm{d}\,\bar\lambda\,\partial_{\bar\lambda}$. Writing $\pi^*A =: B + \bar B$ with $\bar B =: \bar B_a \bar\theta^a$ we discover the components

$$\bar B_1 \;=\; A_{\bar z^1} - \lambda A_{z^2} \;, \qquad \bar B_2 \;=\; A_{\bar z^2} + \lambda A_{z^1} \qquad \text{and} \qquad \bar B_3 \;=\; 0 \;, \tag{A.8}$$

whereby on the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$ we have $\bar B_a = \lambda \tilde{\bar B}_a$. The pulled-back connection π^*D is then given in terms of B by $\pi^*D = \partial_B + \bar\partial_{\bar B}$ with $\bar\partial_{\bar B} = \bar\partial + \bar B = \bar\theta^a(\bar V_a + \bar B_a)$.

Let us consider the equations $\bar\partial_{\bar B} s = 0$ for a local smooth section s of π^*E. By definition the local solutions of these equations are just the local holomorphic sections of π^*E. The bundle $E' := \pi^*E \to \mathcal{P}$ is then termed holomorphic iff these equations are compatible in the sense of $\bar\partial^2_{\bar B} = 0$. Writing down $\bar\partial_{\bar B} s = 0$ explicitly e. g. on $\mathcal{U}_+$ one realizes that $s_+ := s|_{\mathcal{U}_+}$ does not dependent on $\bar\lambda$. The compatibility equations $\bar\partial^2_{\bar B} = 0$ coincide with the SDYM equations (2.6). Therefore, we have local solutions $s_\pm$ on $\mathcal{U}_\pm$ with $s_+ = s_-$ on the intersection $\mathcal{U}_+ \cap \mathcal{U}_-$. Note that we can always decompose $s_\pm = \psi_\pm \eta_\pm$, where $\psi_\pm$ lie in the complexified gauge group $U(2) \otimes \mathbb{C} \cong GL(2, \mathbb{C})$, are nonsingular on $\mathcal{U}_\pm$, and satisfy $\bar\partial_{\bar B} \psi_\pm = 0$ on $\mathcal{U}_\pm$. The vector functions $\eta_\pm \in \mathbb{C}^2$ are holomorphic on $\mathcal{U}_\pm$, i. e. they are only functions of $\{w^a\}$ and $\{\tilde w^a\}$, respectively. We therefore have

$$\psi_+(\partial_{\bar z^1} - \lambda\partial_{z^2})\psi_+^{-1} \;=\; \psi_-(\partial_{\bar z^1} - \lambda\partial_{z^2})\psi_-^{-1} \;=\; A_{\bar z^1} - \lambda A_{z^2} \;, \tag{A.9a}$$

$$\psi_+(\partial_{\bar z^2} + \lambda\partial_{z^1})\psi_+^{-1} \;=\; \psi_-(\partial_{\bar z^2} + \lambda\partial_{z^1})\psi_-^{-1} \;=\; A_{\bar z^2} + \lambda A_{z^1} \;, \tag{A.9b}$$

$$\partial_{\bar\lambda}\psi_+ \;=\; \partial_{\bar\lambda}\psi_- \;=\; 0 \tag{A.9c}$$

on $\mathcal{U}_+ \cap \mathcal{U}_-$, and consequently

$$A_{\bar z^a} \;=\; \psi_+ \partial_{\bar z^a} \psi_+^{-1}\big|_{\lambda=0} \qquad \text{for} \quad a = 1, 2 \;. \tag{A.10}$$

Furthermore, the vector functions $\eta_\pm$ are related via

$$\eta_+ \;=\; f_{+-}\,\eta_- \qquad \text{with} \qquad f_{+-} \;=\; \psi_+^{-1}\,\psi_- \qquad \text{on} \quad \mathcal{U}_+ \cap \mathcal{U}_- \;, \tag{A.11}$$

which implies the holomorphicity of f_{+-}.

In summary, we have described a one-to-one correspondence between gauge equivalence classes of self-dual connection 1-forms A on a complex vector bundle E over $\mathbb{R}^4$ and equivalence classes of holomorphic vector bundles E' over the twistor space $\mathcal{P}$ trivial on $\mathbb{CP}^1 \hookrightarrow \mathcal{P}$. A local gauge transformation of the gauge potential A is reflected by $\psi_\pm \mapsto \psi_\pm^g := g^{-1}\psi_\pm$ and, hence, leaves the transition function f_{+-} invariant. On the other hand, the gauge potential A is inert under a transformation $\psi_\pm \mapsto \psi_\pm^{h_\pm} := \psi_\pm h_\pm^{-1}$, where $h_\pm$ live in the complexified gauge group and are regular holomorphic on $\mathcal{U}_\pm$, respectively. This is known as the twistor correspondence or the Euclidean version of Ward's theorem [37, 38].

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