Generalized WDVV equations for B_r and C_r pure N=2 Super-Yang-Mills theory

L.K. Hoevenaars, R. Martini

Abstract

A proof that the prepotential for pure N=2 Super-Yang-Mills theory associated with Lie algebras B_r and C_r satisfies the generalized WDVV (Witten-Dijkgraaf-Verlinde-Verlinde) system was given by Marshakov, Mironov and Morozov. Among other things, they use an associative algebra of holomorphic differentials. Later Ito and Yang used a different approach to try to accomplish the same result, but they encountered objects of which it is unclear whether they form structure constants of an associative algebra. We show by explicit calculation that these objects are none other than the structure constants of the algebra of holomorphic differentials.

1 Introduction

In 1994, Seiberg and Witten [?] solved the low energy behaviour of pure N=2 Super-Yang-Mills theory by giving the solution of the prepotential $\mathcal{F}$. The essential ingredients in their construction are a family of Riemann surfaces Σ, a meromorphic differential λ_{SW} on it and the definition of the prepotential in terms of period integrals of λ_{SW}

$$a_i = \int_{A_i} \lambda_{SW} \qquad \frac{\partial \mathcal{F}}{\partial a_i} = \int_{B_i} \lambda_{SW} \tag{1.1}$$

where A_i and B_i belong to a subset of the canonical cycles on the surface Σ and the a_i are a subset of the moduli parameters of the family of surfaces. These formulas define the prepotential $\mathcal{F}\left(a_1, ..., a_r\right)$ implicitly; here r denotes the rank of the gauge group under consideration.

A link between the prepotential and the Witten-Dijkgraaf-Verlinde-Verlinde equations [?],[?] was first suggested in [?]. Since then an extensive literature on the subject was formed. It was found that the perturbative piece of the prepotential $\mathcal{F}(a_1, ..., a_r)$ for pure N=2 SYM theory satisfies the generalized WDVV equations [?],[?],[?]

$$\mathcal{F}_i \mathcal{F}_k^{-1} \mathcal{F}_j = \mathcal{F}_j \mathcal{F}_k^{-1} \mathcal{F}_i \qquad \forall i, j, k = 1, ..., r \tag{1.2}$$

where the $\mathcal{F}_i$ are matrices given by $\left(\mathcal{F}_i\right)_{jk} = \frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k}$.
Moreover, it was shown that the full prepotential for simple Lie algebras of type A,B,C,D [?] and type E [?] and F [?] satisfies this generalized WDVV system[1]. The approach used by Ito and Yang in [?] differs from the other two, due to the type of associative algebra that is being used: they use the Landau-Ginzburg chiral ring while the others use an algebra of holomorphic differentials. For the A,D,E cases this difference in approach is negligible since the two different types of algebras are isomorphic. For the Lie algebras of B,C type this is not the case and this leads to some problems. The present article deals with these problems and shows that the proper algebra to use is the one

[1] The generalized WDVV equations for G_2 are trivial

suggested in [?]. A survey of these matters, as well as the results of the present paper can be found in the internal publication [?].

This paper is outlined as follows: in the first section we will review Ito and Yang's method for the A,D,E Lie algebras. In the second section their approach to B,C Lie algebras is discussed. Finally in section three we show that Ito and Yang's construction naturally leads to the algebra of holomorphic differentials used in [?].

2 A review of the simply laced case

In this section, we will describe the proof in [?] that the prepotential of 4-dimensional pure $N = 2$ SYM theory with Lie algebra of simply laced (ADE) type satisfies the generalized WDVV system. The Seiberg-Witten data [?], [?], [?] consists of:

- a family of Riemann surfaces Σ of genus g given by

$$z + \frac{\mu}{z} = W(x, u_1, \ldots, u_r) \tag{2.1}$$

  where W is a particular function which is polynomial for Lie algebras A_r, rational for D_r and contains some radicals for the exceptional algebras. Furthermore μ is a scaling parameter which plays a role in the physical theory, r is the rank of the Lie algebra and the u_i its Casimirs, which play the role of moduli parameters for the family of surfaces.

- a meromorphic differential λ_{SW} on the surfaces, called the Seiberg-Witten differential. It is defined by

$$\lambda_{SW} = x\frac{dz}{z} \tag{2.2}$$

  and has the property that $\frac{\partial \lambda_{SW}}{\partial u_i}$ is a holomorphic differential modulo exact forms.

- a particular subset $\{A_i, B_i\}$ containing $2r$ of a set of $2g$ canonical homology cycles.

From this data one can implicitly construct the prepotential by first defining

$$a_i := \oint_{A_i} \lambda_{SW} \qquad \mathcal{F}_j := \oint_{B_j} \lambda_{SW} \tag{2.3}$$

and noting that the period matrix $\frac{\partial \mathcal{F}_j}{\partial a_i}$ is symmetric. This implies that $\mathcal{F}_j$ can be thought of as a gradient, which leads to the following

Definition 1 *The prepotential is a function $\mathcal{F}(a_1, \ldots, a_r)$ such that*

$$\mathcal{F}_j = \frac{\partial \mathcal{F}}{\partial a_j} \tag{2.4}$$

We also define a system of third order nonlinear partial differential equations:

Definition 2 *Let $f : \mathbb{C}^r \to \mathbb{C}$, then the generalized WDVV system [**?**], [**?**] for f is*

$$f_i K^{-1} f_j = f_j K^{-1} f_i \quad \forall i, j \in \{1, ..., r\} \tag{2.5}$$

where the f_i are matrices with entries

$$(f_i)_{jk} = \frac{\partial^3 f(a_1, ..., a_r)}{\partial a_i \partial a_j \partial a_k} \tag{2.6}$$

and $K = \sum_{l=1}^{r} \alpha_l f_l$ is an invertible linear combination of them.

The main result of this section is

Proposition 3 *The prepotential satisfies the generalized WDVV system*

Proof. We will prove that the prepotential satisfies this system by showing existence of the following:

1. an 'invertible metric' K
2. an associative algebra with structure constants C_{ij}^k
3. a relation between the third order derivatives of $\mathcal{F}$ and the structure constants:

$$(\mathcal{F}_i)_{jk} = C_{ij}^l K_{kl} \tag{2.7}$$

Lemma 4 *If conditions $1-3$ are met, then F satisfies the generalized WDVV system.*

Proof. *Associativity of the algebra can be expressed through an identity on the structure constants*

$$C_i C_j = C_j C_i \tag{2.8}$$

which due to 3 also reads

$$\mathcal{F}_i K^{-1} \mathcal{F}_j K^{-1} = \mathcal{F}_j K^{-1} \mathcal{F}_i K^{-1} \tag{2.9}$$

and multiplying by K from the right gives the desired result. ■

The rest of the proof deals with a discussion of the conditions 1-3. It is well-known [**?**] that the right hand side of (2.1) equals the Landau-Ginzburg superpotential associated with the corresponding Lie algebra. Using this connection, we can define the primary fields $\phi_i(u) := -\frac{\partial W}{\partial u_i}$ and

Definition 5 *The chiral ring is an associative algebra defined by*

$$\phi_i(u)\phi_j(u) = C_{ij}^k(u)\phi_k(u) \qquad mod\left(\frac{\partial W}{\partial x}\right) \tag{2.10}$$

Instead of using the u_i as coordinates on the part of the moduli space we're interested in, we want to use the a_i. For the chiral ring this implies that in the new coordinates

$$\begin{aligned}
(-\frac{\partial W}{\partial a_i})(-\frac{\partial W}{\partial a_j}) &= \frac{\partial u_x}{\partial a_i}\frac{\partial u_y}{\partial a_j} C_{xy}^z(u) \frac{\partial a_k}{\partial u_z}(-\frac{\partial W}{\partial a_k}) \qquad \mathrm{mod}(\frac{\partial W}{\partial x}) \qquad \Longrightarrow \\
\phi_i(a)\phi_j(a) &= C_{ij}^k(a)\phi_k(a) \qquad \mathrm{mod}(\frac{\partial W}{\partial x})
\end{aligned} \tag{2.11}$$

which again is an associative algebra, but with different structure constants $C_{ij}^k(a) \neq C_{ij}^k(u)$. This is the algebra we will use in the rest of the proof.

For the relation (2.7) we turn to another aspect of Landau-Ginzburg theory: the Picard-Fuchs equations (see e.g [**?**] and references therein). These form a coupled set of first order partial differential equations which express how the integrals of holomorphic differentials over homology cycles of a Riemann surface in a family depend on the moduli.

Definition 6 *Flat coordinates of the Landau-Ginzburg theory are a set of coordinates* $\{t_i\}$ *on moduli space such that*

$$\frac{\partial^2 W}{\partial t_i \partial t_j} = \frac{\partial Q_{ij}}{\partial x} \tag{2.12}$$

where Q_{ij} *is given by*

$$\phi_i(t)\phi_j(t) = C_{ij}^k(t)\phi_k(t) + Q_{ij}\frac{\partial W}{\partial x} \tag{2.13}$$

The fact that such coordinates indeed exist will not be discussed here. In terms of these coordinates the following set of Picard-Fuchs equations hold [**?**]

$$\frac{\partial}{\partial t_i}\left[\oint_\Gamma \frac{\partial \lambda sw}{\partial t_j}\right] = C_{ij}^k(t)\frac{\partial}{\partial t_k}\left[\oint_\Gamma \frac{\partial \lambda sw}{\partial t_r}\right] \tag{2.14}$$

for any cycle $\Gamma \in \{A_i, B_i\}$. These equations were derived by making use of the chiral ring, expressed in the flat coordinates, and therefore the structure constants $C_{ij}^k(t)$ appear. Making a change of coordinates to the a_i and using the fact that the a_i satisfy (2.14) one finds

$$\frac{\partial}{\partial a_i}\left[\oint_\Gamma \frac{\partial \lambda sw}{\partial a_j}\right] - C_{ij}^l(a)\frac{\partial}{\partial a_l}\left[\oint_\Gamma \frac{\partial \lambda sw}{\partial a_m}\right]\frac{\partial a_m}{\partial t_r} \tag{2.15}$$

Taking $\Gamma = B_k$ we get

$$\mathcal{F}_{ijk} = C_{ij}^l(a)K_{kl} \tag{2.16}$$

which is the intended relation (2.7). The only thing that is left to do, is to prove that $K_{kl} = \frac{\partial a_m}{\partial t_r}\mathcal{F}_{mkl}$ is invertible. This will not be discussed in the present paper. ∎

In conclusion, the most important ingredients in the proof are the chiral ring and the Picard-Fuchs equations. In the following sections we will show that in the case of B_r, C_r Lie algebras, the Picard-Fuchs equations can still play an important role, but the chiral ring should be replaced by the algebra of holomorphic differentials considered by the authors of [**?**]. These algebras are isomorphic to the chiral rings in the ADE cases, but not for Lie algebras B_r, C_r.

3 Ito & Yang's approach to B_r and C_r

In this section, we discuss the attempt made in [**?**] to generalize the contents of the previous section to the Lie algebras B_r, C_r. We will discuss only B_r since the situation for C_r is completely analogous. The Riemann surfaces are given by

$$z + \frac{\mu}{z} = \frac{W_{BC}(x, u_i)}{x} \tag{3.1}$$

where W_{BC} is the Landau-Ginzburg superpotential associated with the theory of type BC. From the superpotential we again construct the chiral ring in flat coordinates where

$$\phi_i(t) := -\frac{\partial W_{BC}}{\partial t_i} \qquad \phi_i(t)\phi_j(t) = C_{ij}^k(t)\phi_k(t) + Q_{ij}\left[\frac{\partial W_{BC}}{\partial x}\right] \tag{3.2}$$

However, the fact that the right-hand side of (3.1) does not equal the superpotential is reflected by the Picard-Fuchs equations, which no longer relate the third order derivatives of $\mathcal{F}$ with the structure constants $C_{ij}^k(a)$. Instead, they read

$$\mathcal{F}_{ijk} = \tilde{C}_{ij}^l(a)K_{kl} \tag{3.3}$$

where $K_{kl} = \frac{\partial a_m}{\partial t_r}\mathcal{F}_{mkl}$ and

$$\tilde{C}_{ij}^k(t) = C_{ij}^k(t) + D_{ij}^l \sum_{n=1}^{r} \frac{2nt_n}{2r-1}\tilde{C}_{nl}^k(t). \tag{3.4}$$

The D_{ij}^l are defined by

$$Q_{ij} = xD_{ij}^l\phi_l \tag{3.5}$$

and we switched from $\tilde{C}_{ij}^k(a)$ to $\tilde{C}_{ij}^k(t)$ in order to compare these with the structure constants $C_{ij}^k(t)$. At this point, it is unknown[2] whether the $\tilde{C}_{ij}^k(t)$ (and therefore the $\tilde{C}_{ij}^k(a)$) are structure constants of an associative algebra. This issue will be resolved in the next section.

4 The identification of the structure constants

The method of proof that is being used in [?] for the B_r, C_r case also involves an associative algebra. However, theirs is an algebra of holomorphic differentials which is isomorphic to

$$\phi_i(t)\phi_j(t) = \gamma_{ij}^k(t)\phi_k(t) \qquad \mathrm{mod}(x\frac{\partial W_{BC}}{\partial x} - W_{BC}). \tag{4.1}$$

In the rest of this section we will show that

Theorem 7 $\boxed{\tilde{C}_{ij}^k(t) = \gamma_{ij}^k(t)}$

Proof. Starting from the multiplication structure in (3.2) we find (see (3.5))

$$\phi_i(t)\phi_j(t) = \sum_{k=1}^{r} C_{ij}^k(t)\phi_k(t) + \sum_{k=1}^{r} D_{ij}^k\phi_k(t)x\partial_x W_{BC} \tag{4.2}$$

we will rewrite it in such a way that it becomes of the form

$$\phi_i(t)\phi_j(t) = \sum_{k=1}^{r} \tilde{C}_{ij}^k(t)\phi_k(t) + P_{ij}\left[x\partial_x W_{BC} - W_{BC}\right] \tag{4.3}$$

[2] Except for rank 3 and 4, for which explicit calculations of $\tilde{C}_{ij}^k(t)$ were made in [?]

As a first step, we use (3.4):

$$
\begin{aligned}
\phi_i\phi_j &= \left[C_i \cdot \overrightarrow{\phi} + D_i \cdot \overrightarrow{\phi}\, x\partial_x W_{BC}\right]_j \\
&= \left[\left(\widetilde{C}_i - D_i \cdot \sum_{n=1}^{r} \frac{2nt^n}{2r-1}\widetilde{C}_n\right) \cdot \overrightarrow{\phi} + D_i \cdot \overrightarrow{\phi}\, x\partial_x W_{BC}\right]_j \\
&= \left[\widetilde{C}_i \cdot \overrightarrow{\phi} - D_i \cdot \sum_{n=1}^{r} \frac{2nt^n}{2r-1}\widetilde{C}_n \cdot \overrightarrow{\phi} + D_i \cdot \overrightarrow{\phi}\, x\partial_x W_{BC}\right]_j
\end{aligned} \tag{4.4}
$$

The notation $\overrightarrow{\phi}$ stands for the vector with components ϕ_k and we used a matrix notation for the structure constants.
The proof becomes somewhat technical, so let us first give a general outline of it. The strategy will be to get rid of the second term of (4.4) by cancelling it with part of the third term, since we want an algebra in which the first term gives the structure constants. For this cancelling we'll use equation (3.4) in combination with the following relation which expresses the fact that W_{BC} is a graded function

$$
x\frac{\partial W_{BC}}{\partial x} + \sum_{n=1}^{r} 2nt_n \frac{\partial W_{BC}}{\partial t_n} = 2rW_{BC} \tag{4.5}
$$

Cancelling is possible at the expense of introducing yet another term which then has to be canceled etcetera. This recursive process does come to an end however, and by performing it we automatically calculate modulo $x\partial_x W_{BC} - W_{BC}$ instead of $x\partial_x W_{BC}$.

We rewrite (4.4) by splitting up the third term and rewriting one part of it using (4.5):

$$
\begin{aligned}
\left[D_i \cdot \overrightarrow{\phi}\, x\partial_x W_{BC}\right]_j &= \left[-\frac{1}{2r-1} D_i \cdot \overrightarrow{\phi}\, x\partial_x W_{BC} + \left(1 + \frac{1}{2r-1}\right) D_i \cdot \overrightarrow{\phi}\, x\partial_x W_{BC}\right]_j \\
&= \left[-\frac{D_i}{2r-1} \cdot \overrightarrow{\phi}\left(2rW_{BC} - \sum_{n=1}^{r} 2nt_n\phi_n\right) + \frac{2rD_i}{2r-1} \cdot \overrightarrow{\phi}\, x\partial_x W_{BC}\right]_j
\end{aligned} \tag{4.6}
$$

Now we use (4.2) to work out the product $\phi_k\phi_n$ and the result is:

$$
\begin{aligned}
\phi_i\phi_j &= \left[\widetilde{C}_i \cdot \overrightarrow{\phi} - \frac{D_i}{2r-1} \cdot \sum_{n=1}^{r} 2nt_n \left(\widetilde{C}_n \cdot \overrightarrow{\phi} - C_n \cdot \overrightarrow{\phi}\right) - \frac{D_i}{2r-1} \cdot \sum_{n=1}^{r} 2nt_n D_n \cdot \overrightarrow{\phi}\, x\partial_x W_{BC}\right]_j \\
&+ \frac{2rD_i}{2r-1} \cdot \left[x\partial_x W_{BC} - W_{BC}\right]_j
\end{aligned} \tag{4.7}
$$

We now use (3.4) again to rewrite the second term in the first line:

$$
\begin{aligned}
\phi_i\phi_j &= \left[\widetilde{C}_i \cdot \overrightarrow{\phi} + \frac{D_i}{2r-1} \cdot \sum_{n=1}^{r} 2nt_n \left(-D_n \cdot \sum_{m=1}^{r} \frac{2mt^m}{2r-1}\widetilde{C}_m \cdot \overrightarrow{\phi} + D_n \cdot \overrightarrow{\phi}\, x\partial_x W_{BC}\right)\right]_j \\
&\quad + \frac{2rD_i}{2r-1} \left[x\partial_x W_{BC} - W_{BC}\right]_j
\end{aligned} \tag{4.8}
$$

Note that by cancelling the one term, we automatically calculate modulo $x\partial_x W_{BC} - W_{BC}$. The expression between brackets in the first line seems to spoil our achievement but it doesn't: until now we rewrote

$$
\left[-D_i \cdot \sum_{n=1}^{r} \frac{2nt_n}{2r-1}\widetilde{C}_n \cdot \overrightarrow{\phi} + D_i \cdot \overrightarrow{\phi}\, x\partial_x W_{BC}\right]_j \tag{4.9}
$$

and we can now rewrite using the same procedure

$$\left[-D_n \cdot \sum_{m=1}^{r} \frac{2mt^m}{2r-1} \widetilde{C}_m \cdot \overrightarrow{\phi} + D_n \cdot \overrightarrow{\phi} x \partial_x W_{BC}\right]_j \tag{4.10}$$

This is a recursive process. If it stops at some point, then we get a multiplication structure

$$\phi_i \phi_j = \sum_{k=1}^{r} \widetilde{C}_{ij}^k \phi_k + P_{ij}\left(x \partial_x W_{BC} - W_{BC}\right) \tag{4.11}$$

for some polynomial P_{ij} and the theorem is proven. To see that the process indeed stops, we refer to the lemma below. ■

We note that the recursive process adds a matrix D_i each time, and these matrices have a pleasant property:

Lemma 8 *The matrices D_i are nilpotent.*

Proof. One finds [?] that the degree of Q_{ij} is

$$\deg\left(Q_{ij}\right) = 2r + 1 - 2\left(i + j\right) \tag{4.12}$$

Dividing this by x, we get an object of degree $2r - 2\left(i + j\right)$. The D_{ij}^k are defined through (3.5) and if we can show that for $j \geq k$ we can't divide $\frac{Q_{ij}}{x}$ by ϕ_k, we have shown that D_i is nilpotent since it is strictly upper triangular. Since

$$\deg\left(\phi_k\right) = 2r - 2k \tag{4.13}$$

we find that indeed for $j \geq k$ the degree of ϕ_k is bigger than the degree of $\frac{Q_{ij}}{x}$ and since they are both polynomials we can't divide the two. This finishes the proof of the lemma. ■

5 Conclusions and outlook

In this letter we have shown that the unknown quantities $\tilde{C}_{ij}^k$ of [?] are none other than the structure constants of the algebra of holomorphic differentials introduced in [?]. Therefore this is the algebra that should be used, and not the Landau-Ginzburg chiral ring. However, the connection with Landau-Ginzburg can still be very useful since the Picard-Fuchs equations may serve as an alternative to the residue formulas considered in [?].