κ-Deformations of $D = 3$ Conformal Versus Deformations of $D = 4$ AdS Symmetries

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Abstract

We describe the classical $o(3,2)$ r-matrices as generating the quantum deformations of either D=3 conformal algebra with mass-like deformation parameters or D=4 AdS algebra with dimensionless deformation parameters. We describe the quantization of classical $o(3,2)$ r-matrices via Drinfeld twist method which locates the deformation in the coalgebra sector. Further we obtain the quantum $o(3,2)$ algebra in a convenient Hopf algebra form by considering suitable deformation maps from classical to deformed $o(3,2)$ algebra basis. It appears that if we pass from κ-deformed D =3 conformal algebra basis to the deformed D=4 AdS generators basis the role of dimensionfull parameter is taken over by the AdS radius R. We provide also the bilinear $o(3,2)$ Casimir which we express using the deformed D=3 conformal basis.

1 Introduction

Let us recall that the nonstandard quantum deformation of $sl(2;R) \simeq o(2,1)$ has been interpreted in [1] as the κ-deformation of D=1 conformal algebra. Moreover, it appears that if we consider the solutions of classical Yang-Baxter equations with

support in Borel subalgebra of $o(D, 2)$ (for $D > 1$), after quantization we obtain the nonstandard quantum deformations[1] of D-dimensional conformal algebra with mass-like deformation parameter. The $D = 2$ case of $o(2, 2)$ due to the algebra isomorphism $o(2, 2) \simeq o(2, 1) \oplus o(2, 1)$ can be reduced to D=1 case (see e.g. [6]). The simplest new case is given by D=3 conformal algebra $o(3, 2)$[2].

The quantum deformation of $o(3, 2)$ which can be interpreted as κ-deformation of D=3 conformal algebra has been first given by Herranz [8]. His result was obtained by checking the Hopf algebra relations, with coproduct sector defining the classical $o(3, 2)$ r-matrix by its lowest order term. Our approach is different: firstly we consider the general formula for classical $o(3, 2)$ r-matrices providing κ-deformations, then we perform the quantization using the Drinfeld twist technique [9]–[11]. In this way we obtain the quantum $o(3, 2)$ Hopf algebra in classical Lie algebra basis and achieve better control over the structure of this quantum Hopf algebra. Further we perform the nonlinear deformation map which provides for D=3 conformal algebra the deformed algebraic relations, but leads to more convenient form of coalgebra relations. However one can also interpret the quantum algebra $o(3, 2)$ as a quantum deformation of D=4 AdS algebra, with dimensionfull parameter R describing the AdS radius. If we reexpress the considered here κ-deformations of D=3 conformal algebra as describing the deformations of D=4 AdS symmetry we obtain rather surprising result that former dimensionfull "κ-parameters" should be considered as dimensionless. Further we shall address the question how to obtain deformed $o(3, 2)$ Casimirs (for simplicity we shall consider only bilinear case) which we can interpret in D=3 conformal or D=4 AdS basis.

The plan of our paper is the following:

In Sect. 2 we shall classify in "mathematical" Cartan-Weyl basis the solutions of CYBE describing the three-parameter family of classical r-matrices with support in $B_+ \wedge B_+$ where B_+ denotes the Borel algebra of $o(3, 2)$. Using inner automorphisms one can consider the classes of mathematically equivalent r-matrices. We shall show that up to this equivalence the considered three-parameter family is the continuous one parameter set of (mathematically) nonequivalent classical r-matrices. Subsequently we impose the $o(3, 2)$ reality conditions for the generators and choose respectively conditions for the deformation parameters permitting to define the real $o(3, 2)$ quantum Hopf algebras.

We shall calculate the Drinfeld twist and obtain explicite formulae providing the coproducts in classical $o(3, 2)$ basis (see Appendix). Further we perform the nonlinear transformation (deformation map) of $o(3, 2)$ basis which provides the quantum

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[1]We remind here that standard quantum deformation is given by Drinfeld-Jimbo deformation scheme [2, 3], which we shall call q-deformation. The q-deformation of $o(D, 2)$ implies that the deformation parameter q is dimensionless (for D=4 see [4,5]).

[2]The case $D = 4$, i.e. nonstandard quantum deformations of D=4 conformal algebra were considered recently by the same authors in [7].

deformation of $o(3.2)$ algebra in a convenient (Hopf algebraic) form[3]. The new multiplication and coproducts are given by explicite formulae. Finally, as a byproduct of our method the bilinear Casimir in deformed quantum $o(3,2)$ algebra basis is given.

In Sect. 3 we shall interpret the result of Sect. 2 in physical D=3 conformal and D=4 AdS bases. It appears that the same mathematical deformation parameters have different meaning in these two frameworks: for D=3 conformal algebra they have the dimension of the inverse of mass and describe κ-deformation, and for D=4 AdS algebra they are dimensionless.

The paper is concluded by Sect. 4 with an outlook. In this last Section we provide the formula for universal quantum R-matrix.

2 Classical $o(3,2)$ r-matrices and twist quantization of $U(o(3,2))$ in mathematical basis

i) Classical $o(3,2)$ r-matrices in Cartan-Weyl basis

The simple complex Lie algebra $B_2 = o(5) \simeq sp(4) = C_2$ has two simple roots α_1, α_2 with length squares one and two, and symmetrized Cartan matrix ($i = 1, 2$)

$$\alpha_{ij} = (\alpha_i, \alpha_j) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}. \tag{2.1}$$

The Cartan-Chevaley basis for $B_2 \simeq C_2$

$$[h_i, e_{\pm j}] = \pm \alpha_{ij}\, e_{\pm j}, \qquad\qquad [h_i, h_j] = 0,$$

$$[e_{+i}, e_{-j}] = \delta_{ij}\, h_i, \tag{2.2}$$

should be enlarged by the generators $e_{\pm 3}, e_{\pm 4}$ corresponding to nonsimple roots $\alpha_3 = \alpha_1 + \alpha_2$ and $\alpha_4 = \alpha_1 + \alpha_3$

$$e_{\pm 3} = \pm[e_{\pm 1}, e_{\pm 2}], \qquad e_{\pm 4} = \pm[e_{\pm 1}, e_{\pm 3}], \tag{2.3}$$

which together with the Chevalley basis (2.2) describe, due to Serre relations, the whole 10-generator Lie algebra $B_2 \simeq C_2$. It can be checked that this algebra is invariant under the following complex rescaling of generators:

$$h'_l = h_i, \qquad e'_{\pm 1} = e_{\pm 1}, \qquad e'_{\pm l} = \lambda^{\pm 1} e_{\pm l}, \qquad l = 2, 3, 4, \tag{2.4}$$

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[3]For the special choice of deformation parameter our formulae are similar to the ones given by Herranz [8] but not identical. We find also some problems with the reality condition for deformed D=3 conformal algebra proposed in [8].

which for the real form $o(3,2)$ (λ real) will be interpreted physically in Sect. 3.

In order to obtain the real form $o(3,2)$ of $o(5,C)$ we shall use the Hermitian conjugation map[4] which leaves invariant the Borel subalgebra $B^+ = (h_i, e_A)$ ($A = 1,\ldots 4$). The $o(3,2)$ generators satisfy the following reality conditions

$$h_i^+ = -h\,, \qquad\qquad e_{\pm i}^+ = \lambda_i\, e_{\pm i}\,, \qquad\qquad (2.5)$$

where $\lambda_i = \pm 1$ [3, 1] that implies $e_{\pm 3}^+ = -\lambda_1\lambda_2 e_{\pm 3}$ and $e_{\pm 4}^+ = -\lambda_2 e_{\pm 4}$.

Smooth triangular quantum deformations of $o(5,C)$ are described infinitesimally by classical r-matrices, satisfying classical Yang-Baxter equation. We have obtained the following set of three-parameter classical r-matrices with support in $B^+ \wedge B^+$.

$$r(\alpha, \xi, \rho) = \alpha[(2h_1 + h_2) \wedge e_4 + 2e_1 \wedge e_3] + \xi\, h_2 \wedge e_2 + \rho\, e_2 \wedge e_4\,. \qquad (2.6)$$

The invariance of (2.6) under rescaling (2.4) implies that

$$\alpha' = \lambda^{-1}\alpha\,, \qquad \xi' = \lambda^{-1}\xi\,, \qquad \rho' = \lambda^{-2}\rho\,. \qquad (2.7)$$

We shall show in Sect.3 that the scaling properties (2.7) determine D=3 mass dimensions and imply the interpretation of the deformation (2.6) as representing D=3 κ-deformations. Due to the most general two-parameter scaling automorphisms of $o(5,C)$ algebra[5]

$$\begin{aligned}
&e_1 \Rightarrow \alpha^{\frac{1}{2}}\xi^{-\frac{1}{2}}e_1, \quad e_{-1} \Rightarrow \alpha^{-\frac{1}{2}}\xi^{\frac{1}{2}}e_{-1}, \quad e_2 \Rightarrow \xi e_2, \quad e_{-2} \Rightarrow \xi^{-1}e_{-2}\,, \\
&e_3 \Rightarrow \alpha^{\frac{1}{2}}\xi^{\frac{1}{2}}e_3, \quad e_{-3} \Rightarrow \alpha^{-\frac{1}{2}}\xi^{-\frac{1}{2}}e_{-3}, \quad e_4 \Rightarrow \alpha e_4, \quad e_{-4} \Rightarrow \alpha^{-1}e_{-4}\,.
\end{aligned} \qquad (2.8)$$

two out of three parameters α, ξ, ρ can be fixed in a particular way.

If we choose in (2.5) $\lambda_i = -1$ all 10 generators describing $o(3,2)$ are anti-Hermitian. One gets in formula (2.6) the Hermitian classical $o(3,2)$ r-matrices if the parameters α, ξ, ρ are real; the purely imaginary choice of of α, ξ, ρ implies that the classical r-matrices are anti-Hermitian. Further we shall employ the second choice, which will provide after twist quantization the real quantum $o(3,2)$ Hopf algebras.

Below we shall assume that $\rho = 0$[6], i.e. the substitution (2.8) can transform the parameters $\alpha = 1$, $\xi = 1$ in (2.6) and attach to them arbitrary nonzero values. The example of κ-Poincaré algebra [2]–[14] shows that such deformations being equivalent mathematically are not equivalent physically, so further we shall keep α and ξ arbitrary.

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[4]Such a map is defined e.g. for defining matrix representation of $o(5,C)$ as well as for the differential realization describing the infinitesimal D=5 rotations.

[5]The number of parameters for such scale automorphisms is equal to the rank of the Lie algebra

[6]The third term in (2.6), proportional to ρ, introduces Reshetikhin twisting factor on Abelian subalgebra ($[e_2, e_3] = 0$). Such terms in classical r-matrix generate "soft deformations".

Finally we observe that the matrix (2.6) for $\xi = \rho = 0$ was presented in [1], and the choice corresponding to $\alpha = \xi = \frac{1}{2}$, $\rho = 0$ was obtained by Herranz [8].

ii) **Twisting Elements and their Parametrizations**

Twist deformations of enveloping algebra $U(g)$ are defined by the twisting elements $\Phi = \sum f_{(1)} \otimes f_{(2)} \in U(g) \otimes U(g)$ that satisfy the twist equations [9]:

$$(\Phi)_{12}\,(\Delta \otimes \mathrm{id})\;\Phi = (\;\Phi)_{23}\,(\mathrm{id} \otimes \Delta)\;\Phi, \qquad (\epsilon \otimes \mathrm{id})\;\Phi = (\mathrm{id} \otimes \epsilon)\;\Phi = 1. \quad (2.9)$$

The quantized algebra $U_\Phi(g)$ has the same commutation relations as the classical enveloping algebra $U(g)$, the deformation is present only in the new coproducts that can be obtained in the form

$$\Delta_\Phi = \;\Phi\Delta\;\Phi^{-1}, \qquad\qquad (2.10)$$

and in the antipode formulae. Recently the effective methods to find explicit solutions to (2.9) were elaborated [10, 11] for classical simple Lie algebras. In particular the deformed carrier space algorithm was discovered [15] and chains of twists for orthogonal algebras were constructed in [16], where the technique of finding the factorized solution with carrier in $B^+(o(M))$ is given.

Applying it to $U(o(3,2))$ one can check that the two-parameter twisting element for arbitrary parameters α and ξ in (2.6) is described by the product of canonical twist element (corresponding to $\xi = \rho = 0$) and deformed Jordanian twist (corresponding to $\alpha = \rho = 0$). One gets the formula:

$$\mathcal{F}(\alpha, \xi) = \exp\frac{1}{2}\left(h_2 \otimes \sigma_2(\alpha, \xi)\right) \exp\left(\alpha e_1 \otimes e_3\, e^{-\frac{1}{2}\sigma_4(\alpha)}\right) \exp\frac{1}{2}\left(h_4 \otimes \sigma_4(\alpha)\right), \quad (2.11)$$

Notice that here the formal power series σ_2, σ_4 are now functions of α and ξ

$$\begin{aligned}
\sigma_4(\alpha) &= \ln(1 + \alpha e_4), & (2.12)\\
\sigma_2(\alpha, \xi) &= \ln\left(1 - \xi e_2 + \frac{1}{2}\alpha\,\xi\,e_3^2\,e^{-\sigma_4(\alpha)}\right).
\end{aligned}$$

iii) **Deformed twisted $o(3,2)$ coproducts**

The universal enveloping algebra $U(o(3,2))$ is not modified by twisting procedure and the whole deformation is located in the coalgebra sector. The costructure and correspondingly the tensor product rules are defined by the new coproducts according to the formula (2.10). We present the explicit form of these coproducts in the Appendix.

It can be shown that for imaginary parameters α and ξ the coproduct map is real, i.e. $\Delta(a^+) = (\Delta(a))^+$, where $a \in U(o(3,2))$ and $(a \otimes b)^+ = a^+ \otimes b^+$.

iv) **The nonlinear deformation map and quantum $o(3,2)$ algebra in non-classical basis**

We see from the formulae in the Appendix that the generators $(h_i, e_A,) \subset B^+$ have considerably simpler coproducts than the generators $e_{-A} \in B^-$. According to

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the duality defined in $U_{\mathcal{F}}$ by the twisting element $\mathcal{F}$ (see the deformed carrier space approach in [16]) the appropriate generators for the carrier subalgebra are just the tensor multipliers in $\ln(\mathcal{F})$. In our case these are $h_2, h_4, \sigma_2, \sigma_2, e_1$ and $e_3 e^{-\sigma_4}$. The remaining four elements must be correlated with the obtained new positive root generators $(E_1, \ldots E_4)$ to conserve the relations $[E_p, E_{-p}] = H_p$. Finally we get the following deformed basis:

$$
\begin{aligned}
&H_2 = h_2\,, &&H_4 = h_4\,, \\
&E_4 = \alpha^{-1} \ln(1 + \alpha\, e_4)\,, \\
&E_2 = -\xi^{-1} \ln\left[1 - \xi\, e_2 + \frac{1}{2}\alpha\xi\, e_3^2\, (1 + \alpha\, e_4)^{-1}\right]\,, \\
&E_3 = e_3 \cdot (1 + \alpha\, e_4)^{-1}\,, \\
&E_1 = e_1\,, &&E_{-1} = e_{-1} + \alpha\, \xi^{-1}\, e_1\,, \\
&E_{-2} = e_{-2} + \frac{1}{4}\xi\, h_2^2 - \frac{1}{2}\alpha\, e_1^2\,, \\
&E_{-3} = e_{-3} - \frac{1}{2}\alpha\, (h_2 + h_4)e_1\,, \\
&E_{-4} = e_{-4} + \alpha\xi^{-1}\, e_{-2} + \frac{1}{4}\alpha\, h_2^2 - \frac{1}{4}\alpha\, h_4^2\,. &&(2.13)
\end{aligned}
$$

It is easy to check that for imaginary parameters α and ξ the deformed generators (2.14) remain anti-Hermitian.

The set of generators (2.13) is chosen in such a way that it provides in the particular limit $(\alpha \to 0,\ \xi \to 0;\ \alpha\xi^{-1} \to 0)$ the classical generators of $o(3,2)$ and leads to simplified structure of coproducts[7]. Under rescaling (2.4) and (2.7) the deformed generators (2.13) transform in the same way as the corresponding undeformed classical generators

$$
H_i' = H_i\,, \qquad E_{\pm 1}' = E_{\pm 1}\,, \qquad E_{\pm l}' = \lambda^{\pm 1}\, E_{\pm l}\,, \qquad (l = 2, 3, 4)\,, \qquad (2.14)
$$

i.e. they have the same mass dimensions as the undeformed generators.

The coproducts of the generators (2.13) are the following:

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[7]If we wish to define te generators having finite limit in the limit $\alpha \to 0,\ \xi \to 0$ for any order then one has to redefine the generators E_{-1} and E_{-4} in (2.13)

$$
\begin{aligned}
\Delta_F(H_4) &= \begin{pmatrix} H_4 \otimes e^{-\alpha E_4} + 1 \otimes H_4 - \\ -\alpha\, E_1 \otimes E_3\, e^{-\frac{1}{2}(\alpha E_4 - \xi E_2)} - \frac{1}{2}\alpha\xi\, H_2 \otimes E_3^2 e^{\xi E_2} \end{pmatrix}, \\
\Delta_F(H_2) &= H_2 \otimes e^{\xi E_2} + 1 \otimes H_2, \\
\Delta_F(E_4) &= E_4 \otimes 1 + 1 \otimes E_4, \qquad\qquad \Delta_F(E_2) = E_2 \otimes 1 + 1 \otimes E_2, \\
\Delta_F(E_3) &= E_3 \otimes e^{\frac{1}{2}(-\xi E_2 - \alpha E_4)} + 1 \otimes E_3, \\
\Delta_F(E_1) &= E_1 \otimes e^{-\frac{1}{2}(\alpha E_4 - \xi E_2)} + 1 \otimes E_1 - \frac{1}{2}\xi H_2 \otimes E_3 e^{\xi E_2}, \\
\Delta_F(E_{-1}) &= E_{-1} \otimes e^{\frac{1}{2}(-\xi E_2 - \alpha E_4)} + 1 \otimes E_{-1}, \\
\Delta_F(E_{-2}) &= E_{-2} \otimes e^{\xi E_2} + 1 \otimes E_{-2}, \\
\Delta_F(E_{-3}) &= \begin{pmatrix} E_{-3} \otimes e^{-\frac{1}{2}(\alpha E_4 - \xi E_2)} + 1 \otimes E_{-3} - \\ -\frac{1}{2}\xi H_2 \otimes E_{-1} e^{-\alpha E_4} + \alpha\left(\frac{1}{4}\xi H_2 - E_{-2}\right) \otimes E_3 e^{\xi E_2} \end{pmatrix}, \\
\Delta_F(E_{-4}) &= \begin{pmatrix} E_{-4} \otimes e^{-\alpha E_4} + 1 \otimes E_{-4} + \alpha E_1 \otimes E_{-1} e^{-\frac{1}{2}(\alpha E_4 - \xi E_2)} \\ -\alpha E_{-3} \otimes E_3 e^{-\frac{1}{2}(\alpha E_4 - \xi E_2)} + \frac{1}{2}\alpha\xi H_2 \otimes E_{-1} E_3 e^{\xi E_2} \\ +\left(-\xi^{-1}\alpha E_{-2} + \frac{1}{4}\alpha H_2\right) \otimes \left(e^{-\alpha E_4} - e^{\xi E_2} - \frac{1}{2}\alpha\xi E_3^2 e^{\xi E_2}\right) \end{pmatrix}.
\end{aligned}
$$
$$(2.15)$$

The set of generators (2.13) satisfy the following set of deformed commutators (we write down only nontrivial commutators):

$$
\begin{aligned}
[H_2, E_1] &= -E_1, & [H_2, E_{-1}] &= -2\frac{\alpha}{\xi}E_1 + E_{-1}, \\
[H_2, E_2] &= -\frac{2}{\xi}\left(1 - e^{\xi E_2}\right), & [H_2, E_{-2}] &= -2E_{-2} - \frac{1}{2}\xi H_2^2, \\
[H_2, E_3] &= E_3, & [H_2, E_{-3}] &= -E_{-3}, \\
[H_2, E_{-4}] &= -2\frac{\alpha}{\xi}E_{-2} + \frac{1}{2}\alpha H_2^2 - \frac{\alpha^2}{\xi}E_1^2, \\
[H_4, E_1] &= E_1, & [H_4, E_{-1}] &= 2\frac{\alpha}{\xi}E_1 - E_{-1}, \\
[H_4, E_2] &= -\alpha E_3^2 e^{\xi E_2}, & [H_4, E_{-2}] &= -\alpha E_1^2, \\
[H_4, E_3] &= 2E_3\left(e^{-\alpha E_4} - \frac{1}{2}\right), \\
[H_4, E_{-3}] &= -E_{-3} - \alpha(H_2 + H_4)E_1, \\
[H_4, E_4] &= \frac{2}{\alpha}\left(1 - e^{-\alpha E_4}\right), \\
[H_4, E_{-4}] &= -\frac{\alpha}{2}H_4^2 + \frac{\alpha^2}{\xi}E_1^2 + 2(\frac{\alpha}{\xi}E_{-2} - E_{-4}), \\
[E_1, E_2] &= E_3 e^{\xi E_2}, & [E_1, E_{-1}] &= \frac{1}{2}\left(H_4 - H_2\right), \\
[E_1, E_3] &= \frac{1}{\alpha}\left(1 - e^{-\alpha E_4}\right), & [E_1, E_{-2}] &= \frac{\xi}{2}\left(H_2 + \frac{1}{2}\right)E_1, \\
[E_1, E_{-3}] &= -(E_{-2} - \frac{\xi}{4}H_2^2 + \frac{1}{2}\alpha E_1^2), & [E_1, E_{-4}] &= -E_{-3},
\end{aligned}
$$

$$
\begin{aligned}
\left[E_2, E_{-1}\right] &= -\tfrac{\alpha}{\xi}E_3\,, & \left[E_2, E_{-2}\right] &= -H_2\,, \\
\left[E_2, E_{-3}\right] &= -(\tfrac{\alpha}{2}E_3 - E_{-1})e^{\xi E_2} + \tfrac{\alpha}{\xi}E_1\,, \\
\left[E_2, E_{-4}\right] &= \tfrac{\alpha}{\xi}H_2 - (\alpha E_{-1} - \tfrac{1}{4}\alpha^2 E_3)\,E_3\,e^{-\alpha E_4} - \tfrac{\alpha}{2\xi}(e^{-\alpha E_4} - e^{\xi E_2})\,, \\
\left[E_3, E_{-1}\right] &= \tfrac{1}{\xi}\left(e^{-\xi E_2 - \alpha E_4} - 1\right) + \tfrac{\alpha}{2}E_3^2\,, \\
\left[E_3, E_{-2}\right] &= E_1 + \xi\tfrac{1}{2}(\tfrac{1}{2} - H_2)\,E_3\,, & \left[E_3, E_{-3}\right] &= \tfrac{1}{2}\left(H_2 + H_4\right)\,, \\
\left[E_3, E_{-4}\right] &= E_{-1}e^{-\alpha E_4} - \tfrac{\alpha}{2}(H_2 + H_4)\,E_3\,, & \left[E_4, E_{-1}\right] &= -E_3\,, \\
\left[E_4, E_{-3}\right] &= E_1\,, & \left[E_4, E_{-4}\right] &= H_4\,, \\
\left[E_{-1}, E_{-2}\right] &= -E_{-3} - \tfrac{1}{2}\alpha E_1 + \tfrac{1}{2}\xi(\tfrac{1}{2} - H_2)\,E_{-1}\,, & \left[E_{-1}, E_{-3}\right] &= -E_{-4}\,, \\
\left[E_{-1}, E_{-4}\right] &= -2\tfrac{\alpha}{\xi}E_{-3} - \tfrac{\alpha}{2}(H_2 + H_4)\,E_{-1}\,, \\
\left[E_{-2}, E_{-3}\right] &= -\xi\tfrac{1}{2}(H_2 + \tfrac{1}{2})\,E_{-3}\,, \\
\left[E_{-2}, E_{-4}\right] &= \alpha E_{-3}E_1 + \tfrac{\alpha}{2}(-E_{-2} + \tfrac{\xi}{4}H_2^2 - \tfrac{1}{2}\alpha E_1^2)\,, \\
\left[E_{-3}, E_{-4}\right] &= \alpha H_2\,E_{-3} - \alpha E_{-4}E_1\,,
\end{aligned}
$$
$$(2.16)$$

It can be shown that the rhs of relations (2.16) are anti-Hermitian, in consistency with the anti-Hermiticity of generators (2.13).

In order to describe completely the Hopf algebra structure of deformed $o(3,2)$ we present the formulae for antipodes:

$$
\begin{aligned}
S_F(H_4) &= \left(-H_4 - \alpha\,E_1 E_3 + \tfrac{1}{2}\alpha\xi\,H_2 E_3^2\right)e^{\alpha E_4}, \\
S_F(H_2) &= -H_2 e^{-\xi E_2}, \\
S_F(E_4) &= -E_4, \\
S_F(E_2) &= -E_2, \\
S_F(E_3) &= -E_3 e^{\frac{1}{2}(\alpha E_4 + \xi E_2)}, \\
S_F(E_{-1}) &= -E_{-1} e^{\frac{1}{2}(\alpha E_4 + \xi E_2)}, \\
S_F(E_1) &= \left(-E_1 + \tfrac{1}{2}\xi H_2 E_3\right) e^{\frac{1}{2}(\alpha E_4 - \xi E_2)}, \\
S_F(E_{-2}) &= -E_{-2} e^{-\xi E_2}, \\
S_F(E_{-3}) &= \left(\begin{array}{c} -E_{-3} - \alpha\left(E_{-2} - \tfrac{1}{4}\xi H_2\right)E_3 \\ +\tfrac{1}{2}\xi H_2\left(E_{-1} - \alpha E_3\right)e^{(\alpha E_4 + \xi E_2)} \end{array}\right) e^{\frac{1}{2}(\alpha E_4 - \xi E_2)}, \\
S_F(E_{-4}) &= \left(\begin{array}{c} -E_{-4} + \alpha E_1 E_{-1} - \alpha E_{-3}E_3 - \tfrac{1}{2}\alpha\xi H_2 E_3 E_{-1} \\ +\left(\xi^{-1}\alpha E_{-2} - \tfrac{1}{4}\alpha H_2\right)\left(1 - e^{(-\xi E_2 - \alpha E_4)} - \tfrac{1}{2}\alpha\xi E_3^2\right) \end{array}\right) e^{\alpha E_4}.
\end{aligned}
$$
$$(2.17)$$

The counits remain classical.

v) Deformed Bilinear Casimir

The classical Casimir operator of $o(5)$ has the form ($r, s = 1, 2$; α_{rs} is the symmetric Cartan matrix).

$$
C_2 = \alpha_{rs}\,h_r\,h_s + \frac{1}{2}(e_A e_{-A} + e_{-A}e_A) = h_2^2 + h_4^2 - 2h_4 - h_2 + e_A e_{-A}\,. \qquad (2.18)
$$

In order to apply the formula (2.18) to o(3,2) we should impose the proper reality conditions (see (2.5)) If we introduce the inverse deformation map (i.e. inverse the relations (2.13)) the Casimir operator (2.18) can be expressed in terms of deformed $o(3, 2)$ generators (2.13) as follows.

$$
\begin{aligned}
C \;=\; & 2H_1^2 + H_2^2 + 2H_1\,H_2 + 4H_1 + 3H_1 + 2\Big\{E_1\,E_{-1} - \alpha\xi^{-1}E_1^2 \\
& + \Big[\frac{1}{\xi}(e^{-\xi E_2} - 1) - \frac{1}{2}\alpha\,E_3^2\,e^{-2\alpha E_4}[2 - e^{\alpha E_4}]^{-1}\Big]\Big(E_{-2} - \frac{1}{4}\xi\,H_2^2 + \frac{1}{2}\alpha E_1^2\Big) \\
& + E_3 e^{\alpha E_4}\,(E_{-3} + \alpha H_3 E_1) + \frac{1}{\alpha}\Big(e^{\alpha E_4} - 1\Big) \\
& \cdot \Big(E_{-4} - \alpha\,\xi^{-1}\Big(E_{-2} + \frac{1}{2}\alpha\,E_1^2\Big) + \frac{1}{4}\alpha\,H_4^2\Big)\Big\}.
\end{aligned}
\tag{2.19}
$$

3 Physical Bases: D=3 Conformal and D=4 AdS

i) **D=3 conformal basis**

Let us introduce purely imaginary $o(3,2)$ generators $M_{\mu\nu} = -M_{\mu\nu}^+$ ($\mu,\nu = 0, 1, 2, 3, 4$) by the relation

$$
[M_{\mu\nu}, M_{\rho\tau}] = g_{\mu\tau}\,M_{\nu\rho} - g_{\nu\tau}M_{\mu\rho} + g_{\nu\rho}M_{\mu\tau} - g_{\mu\rho}M_{\nu\tau},
\tag{3.1}
$$

where $o(3,2)$ metric has the form $g_{\mu\nu} = diag(-++-+)$.

The D=3 Lorentz generators are

$$
L_1 = M_{10}, \qquad L_2 = M_{02}, \qquad J = M_{21}.
\tag{3.2}
$$

The threemomenta P_r, K_r ($r = 0, 1, 2$) and dilatation generator D are given by the formulae

$$
P_r = \frac{1}{\sqrt{2}}(M_{3r} - M_{4r}), \qquad K_r = \frac{1}{\sqrt{2}}(M_{3r} + M_{4r}), \qquad D = M_{34}.
\tag{3.3}
$$

The relation between CW basis $(h_i, e_{\pm A})$ for $o(3,2)$ and the generators $M_{\mu\nu}$ is the following

$$
\begin{aligned}
& M_{10} = h_1, \qquad M_{34} = h_3, \qquad M_{02} = \tfrac{1}{\sqrt{2}}(e_1 + e_{-1}), \\
& M_{32} = \tfrac{1}{\sqrt{2}}(e_3 + e_{-3}), \quad M_{12} = -\tfrac{1}{\sqrt{2}}(e_1 - e_{-1}), \quad M_{24} = \tfrac{1}{\sqrt{2}}(e_3 - e_{-3}), \\
& M_{04} = -\tfrac{1}{2}(e_2 + e_{-2} + e_4 + e_{-4}), \qquad M_{14} = -\tfrac{1}{2}(e_2 - e_{-2} - e_4 + e_{-4}), \\
& M_{30} = -\tfrac{1}{2}(e_2 - e_{-2} + e_4 - e_{-4}), \\
& M_{31} = -\tfrac{1}{2}(e_2 + e_{-2} - e_4 - e_{-4}).
\end{aligned}
\tag{3.4}
$$

The algebra (3.1) takes the form of D=3 conformal algebra ($r, s, u, v = 0, 1, 2$; $g_{rs} = diag(-1, 1, 1)$):

$$[M_{rs}, M_{uv}] = g_{rv} M_{su} - g_{ru} M_{sv} + g_{su} M_{rv} - g_{sv} M_{ru} \,,$$
$$[M_{rs}, P_u] = g_{su} P_r - g_{ru} P_s \,,$$
$$[M_{rs}, K_u] = g_{ru} K_s - g_{su} K_r \,,$$
$$[M_{rs}, D] = 0 \,, \qquad\qquad\qquad\qquad\qquad\qquad (3.5)$$
$$[D, P_r] = -P_r \,, \qquad\qquad [P_r, P_s] = 0 \,,$$
$$[D, K_r] = K_r \,, \qquad\qquad [K_r, K_s] = 0 \,,$$
$$[K_r, P_s] = g_{rs} D + M_{rs} \,.$$

The general r-matrix (2.5) is given in D=3 conformal basis (3.2–3.3) by the formula

$$r(\alpha, \xi, \rho) = \alpha[(D + L_1) \wedge P_- + \sqrt{2}(L_2 - J) \wedge P_2] - \xi(D - L_1) \wedge P_+ + \rho P_+ \wedge P_- \quad (3.6)$$

where $P_\pm = \frac{1}{\sqrt{2}}(P_1 \pm P_0)$. One can show that the invariance of r-matrix (3.6) under the scale transformation

$$P'_r = \lambda^{-1} P_r \,, \qquad K'_r = \lambda K_r \,, \qquad M'_{rs} = M_{rs} \,, \qquad D' = D \,, \qquad (3.7)$$

implies that the deformation parameters α, ξ, ρ are dimensionfull in accordance with (2.7). We see therefore that the classical r-matrix (2.6) describes the generalized κ-deformation of D=3 conformal algebra. Introducing the fundamental mass parameter κ one can write

$$\alpha = \frac{1}{2\kappa} \,, \qquad \xi = \frac{\gamma}{2\kappa} \,, \qquad \rho = \frac{\delta}{\kappa^2} \,, \qquad\qquad (3.8)$$

The arbitrary choices of the dimensionless parameters γ, δ correspond to generalized κ-deformation of D=3 conformal algebra.

If we put $\gamma = 1$, i.e. $\alpha = \xi = \frac{1}{2\kappa}$ and $\rho = 0$ one gets in our D=3 conformal basis (3.2–4) the r-matrix proposed by Herranz [8]

$$r^H(\frac{1}{2}, \frac{1}{2}, 0) = L_1 \wedge P_1 - D \wedge P_0 + (L_2 - J) \wedge P_2 \,. \qquad (3.9)$$

In order to rewrite the results of Sect. 2 in the framework of D=3 conformal algebra we should invert the relations (3.4). One gets (we put $K_\pm = \frac{1}{\sqrt{2}}(K_1 \pm K_0)$)

$$\begin{aligned}
h_1 &= H_{12} = L_1 \,, & h_3 &= D \,, \\
e_1 &= \tfrac{1}{\sqrt{2}}(L_2 + J) \,, & e_{-1} &= \tfrac{1}{\sqrt{2}}(L_2 - J) \,, \\
e_2 &= P_+ \,, & e_{-2} &= K_- \,, \qquad\qquad (3.10) \\
e_3 &= P_2 & e_{-3} &= K_2 \,, \\
e_4 &= P_- \,, & e_{-4} &= K_+ \,,
\end{aligned}$$

We assume further that the deformed nonclassical D=3 conformal generators $\hat{J}, \hat{L}_r, \widehat{D}, \hat{P}_l, \widehat{K}_l$ are related with the deformed generators $H_2, H_4, E_{\pm A}$ (A=1...4) by

the same relations (3.4) and (3.10). The deformation map (2.13)) in terms of D=3 conformal generators looks as follows

$$
\begin{aligned}
\widehat{J} &= \frac{1}{2}[(2 - \alpha\xi^{-1})J - \alpha\xi^{-1}L_2]\,, & \widehat{L}_1 &= L_1\,, \\
\widehat{L}_2 &= \frac{1}{2}[(2 + \alpha\xi^{-1})L_2 + \alpha\xi^{-1}J]\,, & \widehat{D} &= D\,, \\
\widehat{P}_0 &= -\frac{1}{\sqrt{2}}\Big\{-\xi^{-1}\ln[1 + \xi\,P_+ + \tfrac{1}{2}\alpha\xi P_2(1 - \alpha\,P_-)^{-1}] + \alpha^{-1}\ln(1 + \alpha\,P_-)\Big\}\,, \\
\widehat{P}_1 &= \frac{1}{\sqrt{2}}\Big\{\xi^{-1}\ln[1 + \xi\,P_+ + \tfrac{1}{2}\alpha\xi P_2(1 - \alpha\,P_-)^{-1}] + \alpha^{-1}\ln(1 + \alpha\,P_-)\Big\}\,, \\
\widehat{P}_2 &= P_2[1 + \alpha\,P_-]^{-1}\,, \\
\widehat{K}_0 &= K_0 + \frac{1}{\sqrt{2}}\Big[-\alpha D L_1 + \tfrac{1}{4}\xi(D - L_1)^2 - \alpha\,\xi^{-1}\,K_- - \frac{\alpha}{4}(L_2 + J)^2\Big]\,, \\
\widehat{K}_1 &= K_1 + \frac{1}{\sqrt{2}}\Big[-\alpha D L_1 - \tfrac{1}{4}\xi(D - L_1)^2 - \alpha\,\xi^{-1}\,K_- + \frac{\alpha}{4}(L_2 + J)^2\Big]\,, \\
\widehat{K}_2 &= K_2 - \frac{\alpha}{\sqrt{2}}D(L_2 + J)\,, & (3.11)
\end{aligned}
$$

Applying the formulae (3.10) to deformed generators we can express the relations (2.15–17) in terms of deformed D=3 conformal generators (3.11)

ii) **D=4 AdS basis**

Ten generators of $o(3,2)$ satisfying the relations (3.1) describe $D = 4\,AdS$ algebra defined by the commutation relations

for D=4 Lorentz generators $(M_i, N_i; i = 1, 2, 3)$

$$[M_i, M_j] = \epsilon_{ijk}\,M_k\,, \qquad [N_i, N_j] = -\epsilon_{ijk}\,M_k\,, \qquad [M_i, N_i] = \epsilon_{ijk}\,N_k\,, \qquad (3.12)$$

and for the extension by the four curved AdS translations $\mathcal{P}_\mu = (\mathcal{P}_i, \mathcal{P}_0)$

$$
\begin{array}{ll}
[\mathcal{P}_i, \mathcal{P}_j] = -\frac{1}{R^2}\,\epsilon_{ijk}\,M_k\,, & [\mathcal{P}_0, \mathcal{P}_i] = \frac{1}{R^2}\,N_i\,, \\
[M_i, \mathcal{P}_0] = 0\,, & [M_i, \mathcal{P}_j] = \epsilon_{ijk}\,\mathcal{P}_k\,, \\
[N_i, \mathcal{P}_0] = \mathcal{P}_i\,, & [N_i, \mathcal{P}_j] = \delta_{ij}\,\mathcal{P}_0\,,
\end{array}
\qquad (3.13)
$$

where R describes dimensionfull (inverse mass dimension) AdS radius.

The D=4 AdS generators $(M_i, N_i, \mathcal{P}_\mu)$ can be expressed in terms of "mathematical" generators $h_2, h_4, e_{\pm A}$ as follows

$$
\begin{array}{ll}
M_1 = \frac{1}{\sqrt{2}}(e_{-1} - e_1)\,, & M_2 = \frac{1}{\sqrt{2}}(e_{-3} - e_3)\,, \\
M_3 = \frac{1}{2}(e_{-4} - e_2 + e_2 - e_4)\,, & \\
N_1 = -\frac{1}{2}(e_{-2} + e_4 + e_2 + e_4)\,, & N_2 = \frac{1}{2}(h_4 - h_2)\,, \\
N_3 = \frac{1}{\sqrt{2}}(e_{-1} + e_1)\,, & \\
\mathcal{P}_0 = \frac{1}{2R}(e_{-2} + e_{-4} - e_2 - e_4)\,, & \mathcal{P}_1 = \frac{1}{2R}(h_4 + h_2)\,, \\
\mathcal{P}_2 = -\frac{1}{2R}(e_{-4} - e_{-2} - e_2 + e_4)\,, & \mathcal{P}_3 = \frac{1}{2R}(e_{-3} + e_3)\,,
\end{array}
\qquad (3.14)
$$

10

The inverse formulae to the relations (3.14) permit to express the classical r-matrix (2.6) in D=4 AdS basis:

$$
\begin{aligned}
r(\alpha, \xi, \rho; R) & = \frac{1}{2} N_2 \wedge \left[(\xi - \alpha) N_1 - (\alpha + \xi) M_3\right] + \alpha M_2 \wedge (N_3 - M_1) \\
& \quad - \frac{1}{2} \rho M_3 \wedge N_1 + \frac{R}{2} \Big\{ N_2 \wedge \left[(\xi - \alpha) \mathcal{P}_0 - (\alpha + \xi) \mathcal{P}_2\right] \\
& \quad + \mathcal{P}_1 \wedge \left[(\xi - \alpha) M_3 - (\alpha + \xi) N_1\right] - \rho M_3 \wedge \mathcal{P}_0 \\
& \quad + \rho N_1 \wedge \mathcal{P}_2 - 2 \alpha \mathcal{P}_3 \wedge (N_3 - M_1) \Big\} \\
& \quad + \frac{R^2}{2} \Big\{ \mathcal{P}_1 \wedge \left[(\xi - \alpha) \mathcal{P}_2 - (\alpha + \xi) \mathcal{P}_0\right] - \rho \mathcal{P}_0 \wedge \mathcal{P}_2 \Big\}. \quad (3.15)
\end{aligned}
$$

We had already seen in the formulae (3.14) that in order to obtain the physical rescaling of AdS generators ($P_\mu \longrightarrow \lambda^{-1} P_\mu$, $M_i \longrightarrow M_i$, $N_i \longrightarrow N_i$) it is sufficient to consider the AdS radius R as the dimensionfull parameter ($R \longrightarrow \lambda R$), i.e. we need not transform the generators e_{+A}, e_{-A}, h_r under the scaling. Subsequently, from the formulae (2.6) as well as (3.15) it follows that the deformation parameters α, ξ, ρ in D=4 AdS basis (3.14) remain dimensionless.

Using further the relations (3.14) one can express the twist factor (2.11) as well as the twisted coproduct formulae from Appendix in terms of D=4 AdS generators. The choice of deformation map introducing nonclassical basis suitably adjusted to D=4 AdS algebra is under consideration.

4 Outlook

This paper contributes to the studies of nonstandard quantum deformations of D=4 space-time algebras which are described infinitesimally by the classical r-matrices satisfying CYBE. Our approach is based on Drinfeld twist technique, which provides the complete set of Hopf algebra relations and gives the universal $\mathcal{R}$-matrix via the formula $\mathcal{R} = \mathcal{F}^{21} \mathcal{F}^{-1}$. For example performing the twist quantization of the classical r-matrix (2.6) with $\rho = 0$ we obtain the universal $\mathcal{R}$-matrix in the deformed $o(3, 2)$-basis which looks as follows (see also [16])

$$
\begin{aligned}
\mathcal{R} & = \exp\left(E_2 \otimes H_2\right) \exp\left(-\alpha E_1 e^{-\frac{1}{2} E_4} \otimes E_2\right) \exp\left(E_4 \otimes H_4\right) \\
& \quad \times \exp\left(-H_4 \otimes E_4\right) \exp\left(\alpha E_2 \otimes E_1 e^{-\frac{1}{2} E_4}\right) \exp\left(-H_2 \otimes E_2\right). \quad (4.1)
\end{aligned}
$$

Further, using the formulae (3.14) and (2.13) one can express the formula (4.1) in terms of D=3 conformal or D=4 AdS generators. Finally let us recall that the twist quantization of space-time symmetries in classical Lie algebra basis does not modify the irreducible representations of space-time algebra (one-particle sectors), but provides deformed tensor product representations, i.e. non-Fock formula for n-particle

states. Such a modification of tensoring procedure of irreducible representations can be interpreted as the introduction of particle interactions in algebraic way (see e.g. [17, 18]). Such interpretation of our scheme follows from the fact that the Hamiltonian H is the time component of momentum vector and belongs to the set of symmetry generators with deformed coproducts. If we interpret the coproduct $\Delta(H)$ as describing two-particle energy operator, we obtain nonstandard formula for nonsymmetric two-particle interaction energy which is not invariant under the classical particle exchange transformation. It is an interesting task to find the suitable definition of deformed statistics and its applications in physical models with interaction terms appearing due to quantum deformations.

Acknowledgments

Appendix

Substituting twist function (2.11) in the formula (2.10) we obtain the following coproducts for the generators $h_2, h_4 = 2h_1 + h_2, e_{\pm A}$ ($A = 1, 2, 3, 4$) describing $o(3, 2)$ Lie algebra:

$$
\begin{aligned}
\Delta_{\mathcal{F}}(h_4) &= \begin{array}{c} h_4 \otimes e^{-\sigma_4(\alpha)} + 1 \otimes h_4 \\ -2\alpha e_1 \otimes e_3 e^{-\frac{3}{2}\sigma_4(\alpha) - \frac{1}{2}\sigma_2(\alpha, \xi)} - \frac{\alpha\xi}{2} h_2 \otimes e_3^2 e^{-2\sigma_4(\alpha) - \sigma_2(\alpha, \xi)}, \end{array} \\
\Delta_{\mathcal{F}}(h_2) &= h_2 \otimes e^{-\sigma_2(\alpha, \xi)} + 1 \otimes h_2, \\
\Delta_{\mathcal{F}}(e_4) &= e_4 \otimes e^{\sigma_4(\alpha)} + 1 \otimes e_4, \\
\Delta_{\mathcal{F}}(e_1) &= e_1 \otimes e^{-\frac{1}{2}\sigma_4(\alpha) - \frac{1}{2}\sigma_2(\alpha, \xi)} + 1 \otimes e_1 - \frac{1}{2}\xi h_2 \otimes e_3 e^{-\sigma_4(\alpha) - \sigma_2(\alpha, \xi)}, \\
\Delta_{\mathcal{F}}(e_3) &= e_3 \otimes e^{\frac{1}{2}\sigma_4(\alpha) + \frac{1}{2}\sigma_2(\alpha, \xi)} + e^{\sigma_4(\alpha)} \otimes e_3, \\
\Delta_{\mathcal{F}}(e_2) &= \begin{array}{c} e_2 \otimes e^{\sigma_2(\alpha, \xi)} + 1 \otimes e_2 \\ +\xi e_3 \otimes e_3 e^{-\frac{1}{2}\sigma_4(\alpha) + \frac{1}{2}\sigma_2(\alpha, \xi)} + \frac{1}{2}\xi^2 e_4 \otimes e_3^2 e^{-\sigma_{1+2}}. \end{array}
\end{aligned}
$$

$$
\Delta_{\mathcal{F}}(e_{-1}) = e_{-1} \otimes e^{-\frac{1}{2}\sigma_4(\alpha) + \frac{1}{2}\sigma_2(\alpha, \xi)} + 1 \otimes e_{-1}
$$
$$
+\alpha e_1 \otimes \xi^{-1} \left(e^{\sigma_2(\alpha, \xi)} - 1 \right) e^{-\frac{1}{2}\sigma_4(\alpha) - \frac{1}{2}\sigma_2(\alpha, \xi)} - \frac{1}{2}\alpha h_2 \otimes e_3 e^{-\sigma_2(\alpha, \xi)} e^{-\sigma_4(\alpha)};
$$

$$\Delta_{\mathcal{F}}\left(e_{-2}\right) = e_{-2} \otimes e^{-\sigma_2(\sigma,\xi)} + 1 \otimes e_{-2} + \tfrac{1}{2}\alpha h_2 e_1 \otimes e_3\, e^{-\frac{3}{2}(\sigma_4(\alpha)+\sigma_2(\alpha,\xi))}$$

$$-\tfrac{1}{2}\xi h_2 \otimes \left(\begin{array}{c} \left(1 - e^{-\sigma_2(\alpha,\xi)}\right) - \tfrac{1}{2}\left(1 - e^{-\sigma_4(\alpha)}\right) \\ +h_2 + \tfrac{1}{2}\alpha\xi e_3^2 e^{-2\sigma_4(\alpha)} e^{-\sigma_2(\alpha,\xi)} - \alpha e_3 e_1 e^{-\sigma_4(\alpha)} \end{array}\right) e^{-\sigma_2(\alpha,\xi)}$$

$$-\tfrac{1}{8}h_2^2 \otimes \left(\begin{array}{c} 2\left(e^{-\sigma_2(\alpha,\xi)} - 1\right) \\ -\alpha\xi e_3^2 e^{-2\sigma_4(\alpha)} e^{-\sigma_2(\alpha,\xi)} \end{array}\right) e^{-\sigma_2(\alpha,\xi)}$$

$$+\alpha e_1 \otimes e_1 e^{-\frac{1}{2}(\sigma_4(\alpha)+\sigma_2(\alpha,\xi))} - \tfrac{1}{2}\alpha e_1^2 \otimes \left(1 - e^{-\sigma_4(\alpha)}\right) e^{-\sigma_2(\alpha,\xi)};$$

$$\Delta_{\mathcal{F}}\left(e_{-3}\right) = e_{-3} \otimes e^{-\frac{1}{2}\sigma_4(\alpha)-\frac{1}{2}\sigma_2(\alpha,\xi)} + 1 \otimes e_{-3}$$

$$+\tfrac{1}{2}\xi h_2 \otimes \left(-e_{-1} + \alpha h_3 e_3 e^{-\sigma_4} - \tfrac{1}{2}\alpha^2 e_3^2 e_1 e^{-2\sigma_4(\alpha)}\right) e^{-\sigma_2(\alpha,\xi)}$$

$$+\tfrac{1}{2}\alpha\xi h_2 \otimes e_3 \left(-e^{-\sigma_2(\alpha,\xi)} + \tfrac{1}{2}\alpha\xi e_3^2 e^{-2\sigma_4(\alpha)-\sigma_2(\alpha,\xi)} + \tfrac{1}{2}\right) e^{-(\sigma_2(\alpha,\xi)+\sigma_4(\alpha))} +$$

$$\tfrac{1}{4}\alpha\xi h_2^2 \otimes e_3 \left(e^{-\sigma_2(\alpha,\xi)} - \tfrac{1}{2}\alpha\xi e_3^2 e^{-2\sigma_4(\alpha)-\sigma_2(\alpha,\xi)} - 1\right) e^{-(\sigma_2(\alpha,\xi)+\sigma_4(\alpha))}$$

$$-\tfrac{1}{2}\alpha h_2 e_1 \otimes \left(-e^{-\sigma_2(\alpha,\xi)} + \tfrac{3}{2}\alpha\xi e_3^2 e^{-2\sigma_4(\alpha)} e^{-\sigma_2(\alpha,\xi)} + 1\right) e^{-\frac{1}{2}(\sigma_2(\alpha,\xi)+\sigma_4(\alpha))}$$

$$+\tfrac{1}{2}\alpha h_4 \otimes e_1 e^{-\sigma_4(\alpha)} + \tfrac{1}{4}\alpha\xi h_4 h_2 \otimes e_3 e^{-2\sigma_4(\alpha)} e^{-\sigma_2(\alpha,\xi)} - \alpha e_{-2} \otimes e_3 e^{-\sigma_4(\alpha)} e^{-\sigma_2(\alpha,\xi)}$$

$$+\alpha e_1 \otimes h_3 e^{-\frac{1}{2}(\sigma_2(\alpha,\xi)+\sigma_4(\alpha))} - \tfrac{1}{2}\alpha h_4 e_1 \otimes \left(e^{-\frac{1}{2}\sigma_4(\alpha)} - e^{-\frac{3}{2}\sigma_4(\alpha)}\right) e^{-\frac{1}{2}\sigma_2(\alpha,\xi)}$$

$$-\alpha^2 e_1 \otimes e_3 e_1 e^{-\frac{3}{2}\sigma_4(\alpha)-\frac{1}{2}\sigma_2(\alpha,\xi)} + \alpha^2 e_1^2 \otimes e_3 \left(\tfrac{1}{2} - e^{-\sigma_4(\alpha)}\right) e^{-\sigma_4(\alpha)} e^{-\sigma_2(\alpha,\xi)};$$

$$\Delta_{\mathcal{F}}\left(e_{-4}\right) = e_{-4} \otimes e^{-\sigma_4(\alpha)} + 1 \otimes e_{-4}$$

$$\tfrac{1}{2}\,h_2 \otimes \left[\begin{array}{c} +\left(\alpha\xi e_3 e_{-1} + \tfrac{1}{2}\alpha\xi e_2\right) e^{-\sigma_4(\alpha)-\sigma_2(\alpha,\xi)} \\ -\tfrac{1}{2}\alpha^2\xi e_3^2 \left(e^{\sigma_2(\alpha,\xi)} - 1\right) e^{-2\sigma_4(\alpha)-2\sigma_2(\alpha,\xi)} \\ +\alpha^2\xi e_3^2 \left(e^{-3\sigma_4(\alpha)-\sigma_2(\alpha,\xi)} - \tfrac{1}{4}\alpha\xi e_3^2 e^{-4\sigma_4(\alpha)-2\sigma_2(\alpha,\xi)}\right) \end{array}\right] +$$

$$+\tfrac{1}{8}\alpha^2\xi h_2^2 \otimes e_3^2 \left(e^{\sigma_2(\alpha,\xi)} - 1 + \tfrac{1}{2}\alpha\xi e_3^2 e^{-2\sigma_4(\alpha)}\right) e^{-2\sigma_4(\alpha)} e^{-2\sigma_2(\alpha,\xi)} +$$

$$-\alpha e_{-3} \otimes e_3 e^{-\frac{3}{2}\sigma_4(\alpha)} e^{-\frac{1}{2}\sigma_2(\alpha,\xi)} + \alpha e_1 \otimes e_{-1} e^{-\frac{1}{2}\sigma_{1+2}} e^{-\frac{1}{2}\sigma_2(\alpha,\xi)}$$

$$+\tfrac{1}{2}\alpha^2 h_2 e_1 \otimes e_3 \left(e^{\sigma_2(\alpha,\xi)} - 1 + \alpha\xi e_3^2 e^{-2\sigma_4(\alpha)}\right) e^{-\frac{3}{2}\sigma_{1+2}} e^{-\frac{3}{2}\sigma_2(\alpha,\xi)}$$

$$+\alpha^2 e_1^2 \otimes \left(-\tfrac{1}{2}e_2 e^{-\sigma_{1+2}} e^{-\sigma_2(\alpha,\xi)} + \alpha e_3^2 e^{-3\sigma_4(\alpha)-\sigma_2(\alpha,\xi)}\right)$$

$$+\tfrac{1}{2}\alpha^2 e_{-2} \otimes e_3^2 e^{-2\sigma_4(\alpha)} e^{-\sigma_2(\alpha,\xi)}$$

$$-\alpha^2 \left(\tfrac{1}{2}h_4 e_1 + e_1\right) \otimes e_3 \left(2e^{-\sigma_4(\alpha)} - 1\right) e^{-\frac{3}{2}\sigma_4(\alpha)} e^{-\frac{1}{2}\sigma_2(\alpha,\xi)}$$

$$+\tfrac{1}{2}h_4 \otimes h_4 e^{-\sigma_4(\alpha)} + \tfrac{1}{2}h_4 \otimes \left(e^{-\sigma_4(\alpha)} - e^{-2\sigma_4(\alpha)}\right)$$

$$-\alpha^2 e_1 \otimes h_4 e_3 e^{-\frac{3}{2}\sigma_4(\alpha)-\frac{1}{2}\sigma_2(\alpha,\xi)}$$

$$-\tfrac{1}{4}\alpha^2\xi(h_4 h_2 \otimes e_3^2 e^{-3\sigma_4(\alpha)-\sigma_2(\alpha,\xi)} + h_2 \otimes h_4 e_3^2 e^{-2+\sigma_4(\alpha)-\sigma_2(\alpha,\xi)}).$$

where $h_3 = h_1 + h_2 = \tfrac{1}{2}(h_2 + h_4)$.

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