Integrable Ladder t-J Model with Staggered Shift of the Spectral Parameter

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Abstract

The generalization of the Yang-Baxter equations (YBE) in the presence of $\mathbb{Z}_2$ grading along both chain and time directions is presented and an integrable model of $t-J$ type with staggered disposition along a chain of shifts of the spectral parameter is constructed. The Hamiltonian of the model is computed in fermionic formulation. It involves three neighbour site interactions and therefore can be considered as a zig-zag ladder model. The Algebraic Bethe Ansatz technique is applied and the eigenstates, along with eigenvalues of the transfer matrix of the model are found. In the thermodynamic limit, the lowest energy of the model is formed by the quarter filling of the states by fermions instead of usual half filling.

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1 Introduction

The interest to ladder type models was raised in a beginning of 90-s (see for a review [1]) in connection with high temperature superconductivity problems in metal oxides. It is believed that quasi-one dimensional multi-ladder chains of strongly interacting electrons reflects the most important aspects of two dimensional systems and also can reveal some properties of the week coupling between conducting planes.

Recently there has been considerable interest in the construction of integrable ladder type models motivated by the desire to use powerful technique of Algebraic Bethe Ansatz (ABA) [2, 3] in the exact investigations of the variety of physical phases of the models.

In the articles [5] integrable ladder models were constructed by extension of symmetry algebra, in [6] by defining first the ground state and then formulating a model, which has it. The higher conservation laws of an integrable models, which contains next to nearest neighbour interactions, were used in construction of ladder models in [7] by developing the approach of the article [8]. There are also some other type attempts in this area [9, 10].

Usually integrable models are homogeneous along a chain, namely, the spectral u and model parameters are the same in the product of R-matrices along a chain. It is obvious, that if one considers arbitrary shifts of the spectral parameters by some z_i in the monodromy matrix we still will have an integrable model. But in order to have a local Hamiltonian we need to consider shifts with the fixed periodicity n, which will cause the interaction of spins (or electrons) within an amount of n neighbours leading to n-ladder model. This type of model was first considered in [11], developing the ideas of the articles [12] (see also [13]), then in a chain of articles [14, 15, 16].

In the article [4] we have proposed an inhomogeneous model based on XXZ spin-chain, where inhomogeneouty appeared not only in the staggered shifts of the spectral parameter, but also in change of structure of R-matrices in a product along a chain. Namely, two monodromy matrices of chains $M_s, s = 0, 1$ were considered along a time direction, where the R-matrices in the product have an alternating disposition of the anisotropy parameter $\pm\Delta$ of the XXZ model. Besides that, and contrary to case considered in [13, 14, 15, 16], the spectral parameter of the second line has an opposite sign. Due to the double space translational invariance the Hamiltonian of the model contains interaction between three neighbours sites of the chain and therefore represents a zig-zag type ladder model. At the free fermionic point $\Delta = 0$ the model becomes a model of two noninteracting fermions, hopping separately in the odd and even sites of the chain.

In this article we are extending our approach to $spin - 1$ case and considering the t-J model [19, 21, 22, 23, 24, 25, 26]. Following [4] we are writing two Yang-Baxter equations (YBE) (see equations 4 and 5 below) for each step of two alternating R-matrices in the chain, but as it appeared in the solution, in this isotropic in the spin space model two intertwining R-matrices are coinciding. As a result we are being left with the alternating shift of the spectral parameter and the change of sign of the spectral parameter in second line. As we will see, though this gives us the same Bethe Equations (BE) for the spectral

parameters of the excitations as for the model derived in [14, 15, 16], their energy and the energy of ground state are different. It appears that in the thermodynamic limit the lowest energy of the model is reached by the quarter filling of the states by fermions, instead of the usual half filling.

In Section 3 we formulate the model and find a local Hamiltonian, which has a zig-zag ladder form. It consists of two chains with $t-J$ type Hamiltonians on each of them, the hopping term of electrons from one chain to the other and two type of interaction terms between chains. The first interaction term has the form of spin-spin interaction, where one spin is composed from two fermions on the same site of the chain, while the other spin is composed from two different fermions on the neighbour sites of the other chain of the ladder. The second interaction term has a topological form of interacting spins and is written for the triangles consisting of the zig-zag rungs.

In Section 4 we apply ABA in order to find the eigenvalues and the eigenstates of the model. At the end of Section we find the ground state energy of the model and the spectrum of excitations in thermodynamic limit.

2 The Yang-Baxter Equations and its solution.

Following [4] let us define the monodromy operator of the model as a product of the two chain monodromy operators $M_0(u)$ and $M_1(u)$

$$M(u) = M_0(u)M_1(u), \tag{1}$$

where

$$\begin{aligned} M_1(u) &= \bar{R}^{\iota_2}_{01}(u){R_{02}}^{\iota_1\iota_2}(u)\bar{R}^{\iota_2}_{03}(u).... \\ M_0(u) &= R_{01}(u)\bar{R}^{\iota_1}_{02}(u)R_{03}(u).... \end{aligned} \tag{2}$$

and we have two operations ι_1 and ι_2, defined with property ${\iota_1}^2 = {\iota_2}^2 = 1$.

The transfer matrix of the model will be defined as a trace over auxiliary spaces 0 and $0'$

$$\tau(u) = Tr_0 Tr_{0'} M(u) = \tau_0(u)\tau_1(u). \tag{3}$$

As it was shown in the article [4] the commutativity of transfer matrices $\tau(u), \tau(v)$ for arbitrary values of the spectral parameters u and v can be ensured by the following set of Yang-Baxter Equations

$$\check{R}_{12}(u,v)\check{\bar{R}}^{\iota_2}_{23}(u)\check{R}_{12}(v) = \check{R}^{\iota_2}_{23}(v)\check{\bar{R}}_{12}(u)\check{\bar{R}}_{23}(u,v), \tag{4}$$

and

$$\check{\bar{R}}_{12}(u,v)\check{R}^{\iota_1\iota_2}_{23}(u)\check{\bar{R}}^{\iota_1}_{12}(v) = \check{\bar{R}}^{\iota_1\iota_2}_{23}(v)\check{R}^{\iota_1}_{12}(u)\check{R}_{23}(u,v), \tag{5}$$

where we have used the braid (check) formalism for the convenience.

It is also convenient to use the fermionic operator formalism for R-matrices and the YBE developed in [18, 19, 20, 17].

For the $t-J$ model under consideration [19, 21, 22, 23, 24, 25, 26] we have to realize the spin 1 algebra in the fermionic approach. A minimum of two sorts of fermions is needed in order to express three basic states $\mid +\rangle, \mid 0\rangle, \mid -\rangle$ of the $spin-1$ particle with the z component of the spin equal to 1, 0, -1 correspondingly.

Now let's define ${c_\sigma}^+$, c_σ, where $\sigma = \uparrow\downarrow$, as a creation-annihilation operators of fermions with the up and down spins respectively, together with their Fock space $\mid 0\rangle, \mid \sigma\rangle$.

The states with definite third projection of the algebra $SU(2)$ can be realized through fermionic Fock space as follows

$$\mid -\rangle \equiv \mid 0, \downarrow\rangle, \mid +\rangle \equiv \mid \uparrow, 0\rangle, \mid 0\rangle \equiv \mid 0, 0\rangle, \tag{6}$$

numerated as $\mid 1\rangle, \mid 2\rangle, \mid 3\rangle$ respectively.

As it is obvious from the formulas 6, we have constructed a graded space with the following parities for the basic vectors

$$p(\mid +\rangle) = p(\mid -\rangle) = 1, \qquad p(\mid 0\rangle) = 0. \tag{7}$$

In order to proceed further and write the fermionic R-matrix we should calculate the Hubbard operator $X_m^n = \mid m\rangle\langle n \mid; \qquad m, n = 1, 2, 3$

$$\begin{aligned} X_m^k &= \begin{pmatrix} |-\rangle\langle -| & |-\rangle\langle +| & |-\rangle\langle 0| \\ |+\rangle\langle -| & |+\rangle\langle +| & |+\rangle\langle 0| \\ |0\rangle\langle -| & |0\rangle\langle +| & |0\rangle\langle 0| \end{pmatrix} \\ &= \begin{pmatrix} (1-n_\uparrow)n_\downarrow & c_\downarrow^+ c_\uparrow & (1-n_\uparrow)c_\downarrow^+ \\ c_\uparrow^+ c_\downarrow & n_\uparrow(1-n_\downarrow) & c_\uparrow^+(1-n_\downarrow) \\ (1-n_\uparrow)c_\downarrow & c_\uparrow(1-n_\downarrow) & (1-n_\uparrow)(1-n_\downarrow) \end{pmatrix}, \end{aligned} \tag{8}$$

The trace of this operator is

$$\Delta = X_m^m = 1 - n_\uparrow n_\downarrow, \tag{9}$$

which is an identity operator on the space of states,where the double occupancy of the sites by fermions is excluded.

Following [19] let's write down the fermionic R-operator for the t-J model ($spin$ 1 [21, 22, 23] model).

$$\check{R}_{i,j}(u) = a(u) I_{i,j} + b(u)\Pi_{i,j} = a(u) I_{i,j} + b(u) \sum_{m,n=1}^{N} (-1)^{p(m)} {X_i}_n^m {X_j}_m^n, \tag{10}$$

where $\Pi_{i,j}$ is the graded permutation operator of the spaces V_i and V_J.

Now by putting the R-matrix form (10) into the YBE (4) and (5) and after some calculations one can find 12 equations which require the operations $\tilde{}$ and transformation ι_2 to be

$$\begin{array}{rcl} \tilde{a}(u,v) = a(u,v), & & \tilde{b}(u,v) = b(u,v), \\ \dfrac{a^{\iota_2}(u)}{b^{\iota_2}(u)} & = & \dfrac{a(u)}{b(u)}. \end{array} \tag{11}$$

The conditions (11) are reducing the 12 equations to following two equations

$$\begin{array}{rcl} a(u,v)[\bar{a}(u)b(v) - \bar{b}(u)a(v)] + b(u,v)\bar{a}(u)a(v) & = & 0 \\ a(u,v)[a^{\iota_1}(u)\bar{b}^{\iota_1}(v) - b^{\iota_1}(u)\bar{a}^{\iota_1}(v)] + b(u,v)a^{\iota_1}(u)\bar{a}^{\iota_1}(v) & = & 0 \end{array} \tag{12}$$

the consistency condition of which can be found easily as follows

$$\frac{b(v)}{a(v)} - \frac{\bar{b}^{\iota_1}(v)}{\bar{a}^{\iota_1}(v)} = \frac{\bar{b}(u)}{\bar{a}(u)} - \frac{b^{\iota_1}(u)}{a^{\iota_1}(u)} = constant = \theta. \tag{13}$$

Here θ is the new parameter of our model.

Then the solution of (12) for the intertwinner parameters $a(u,v)$ and $b(u,v)$ will be

$$\frac{b(u,v)}{a(u,v)} = \frac{\bar{b}(u)}{\bar{a}(u)} - \frac{b(v)}{a(v)} \tag{14}$$

We should now define the ι_1 operation. It is easy to see from the expression (13) for $v = u$ that ι_1 operation can be consistently defined as follows

$$\frac{b^{\iota_1}(u)}{a^{\iota_1}(u)} = -\frac{b(u)}{a(u)}, \quad \frac{\bar{b}^{\iota_1}u}{\bar{a}^{\iota_1}(u)} = -\frac{\bar{b}(u)}{\bar{a}(u)}. \tag{15}$$

It is clear from the formulas (12-14) that the ratio $\frac{b(u)}{a(u)}$ can be taken as a spectral parameter $\frac{b(u)}{a(u)} = u$, $\frac{\bar{b}(u)}{\bar{a}(u)} = \bar{u} = \theta - u$.

Finally, after appropriate normalization of $a(u)$ and $b(u)$ in order to have $a(u)+b(u) = 1$, one finds the following solution of YBE

$$\begin{array}{rcl} a(u) & = & \dfrac{1}{1+u}, \quad \bar{a}(u) = \dfrac{1}{1+\theta-u}, \quad a(\bar{u},v) = \dfrac{1}{1+\theta-u-v}, \\ b(u) & = & \dfrac{u}{1+u}, \quad \bar{b}(u) = \dfrac{\theta-u}{1+\theta-u}. \quad b(\bar{u},v) = \dfrac{\theta-u-v}{1+\theta-u-v}. \end{array} \tag{16}$$

According to standard prescription of the ABA technique the logarithmic derivative of the transfer matrix at some point defines the Hamiltonian of the model

$$H = -\frac{\partial \ln \tau(u)}{\partial u}|_{u=0}. \tag{17}$$

As it is known, in order Hamiltonian to be local, it is necessary to have a value u_0, such that

$$\check{R}_{i,j}(u_0) = I_{i,j}. \tag{18}$$

Analyzing the solutions of our YBE (16), one can see from (10) that, at the point $u_0 = 0$, only $R_{i,j}(0) = I_{i,j}$ and $\bar{R}_{i,j}(0) = I_{i,j}$. As calculations shows the Hamiltonian is nevertheless local, but it contains interaction between four neighbour points.

Technically, in order to calculate the logarithmic derivative (17), one should put the expression of $\check{R}_{i,j}$ operators around $u_0 = 0$ up to linear terms

$$\begin{aligned} R_{i,j} &= I_{i,j} + uH_{i,j}, \\ \bar{R}_{i,j}(u) &= R_{i,j}(\theta) - uH_{i,j}, \\ R^{\iota_1}_{i,j}(u) &= I_{i,j} - uH_{i,j}, \\ \bar{R}^{\iota_1}_{i,j}(u) &= R_{i,j}(-\theta) + uH_{i,j}, \end{aligned} \tag{19}$$

with

$$H_{i,j} = \sum_{m,n} (-1)^{p(m)} X_{i}^{\ m}_{n} X_{j}^{\ n}_{m} \tag{20}$$

into the expression (1) of the Monodromy Matrix

$$M(u) = \bar{R}^{\iota_1}_{01}(u) R_{12}(u) \bar{R}^{\iota_1}_{23} R_{34}(u) .... R^{\iota_1\iota_2}_{12}(u) \bar{R}^{\iota_2}_{23}(u) R^{\iota_1\iota_2}_{34}(u) \bar{R}^{\iota_2}_{45} .... \tag{21}$$

As a result, after some algebraic calculations, we will obtain the following Hamiltonian for the present staggered t-J model

$$\begin{aligned} H &= \theta\Delta \sum_{i=1}^{N} \Bigg\{ \sum_{\sigma=\uparrow\downarrow} \left(2 - \frac{n_{i-1}}{2} - \frac{n_{i-2}}{2}\right) (c^{+}_{i,\sigma} c_{i+1,\sigma} - c^{+}_{i+1,\sigma} c_{i,\sigma}) \\ &+ \sum_{\sigma=\uparrow\downarrow} \left[\left(1 - \frac{n_{i+1}}{2} + (-1)^i \theta\right) c^{+}_{i+2,\sigma} c_{i,\sigma} - \left(1 - \frac{n_{i+1}}{2} - (-1)^i \theta\right) c^{+}_{i,\sigma} c_{i+2,\sigma} \right] \\ &+ 2 \Big[(\vec{S}_{i+2} + \vec{S}_{i-1})(\vec{S}_{i,i+1} - \vec{S}_{i+1,i}) + \vec{S}_{i+1}(\vec{S}_{i+2,i} - \vec{S}_{i,i+2}) \\ &+ \theta(-1)^i \left(\vec{S}_{i-1}\vec{S}_{i+1} - \frac{1}{4} n_{i-1} n_{i+1} + \frac{n_{i-1} + n_{i+1}}{2} \right) - i\epsilon^{abc} S^a_i S^b_{i+1} S^c_{i+2} \Big] \Bigg\} \Delta, \end{aligned} \tag{22}$$

where $\Delta = \prod_{i=1}^{N} \Delta_i = \prod_{i=1}^{N} (1 - n_{i\uparrow} n_{i\downarrow})$ is the projector which excludes the double occupancy by electrons at any site i.

The spin operators $\vec{S}_i$ and $\vec{S}_{i,j}$ are defined as follows

$$\begin{aligned} \vec{S}_i &= \frac{1}{2} \Psi^{+}_{i,\beta} \vec{\sigma}^{\beta}_{\alpha} \Psi^{\alpha}_{i} = \frac{1}{2} \Delta_i c^{+}_{i,\alpha} \vec{\sigma}^{\alpha}_{\beta} c^{\beta}_{i} \Delta_i \\ \vec{S}_{i,j} &= \frac{1}{2} \Psi^{+}_{i,\beta} \vec{\sigma}^{\beta}_{\alpha} \Psi^{\alpha}_{j} = \frac{1}{2} \Delta_i c^{+}_{i,\alpha} \vec{\sigma}^{\alpha}_{\beta} c^{\beta}_{j} \Delta_j, \end{aligned} \tag{23}$$

where

$$\Psi^1 = (1-n_\uparrow)c_\downarrow, \Psi^2 = c_\uparrow(1-n_\downarrow), \tag{24}$$

and $\vec{\sigma}$ are Pauli matrices.

3 Algebraic Bethe Ansatz for the staggered t-J model.

In this section we will apply the technique of ABA [3, 25, 26] to the present model and find the eigenvalues and eigenstates of the Hamiltonian (22).

For this purpose let's introduce the L operators as follows

$$(L_{i,j})^{k'}_{k} = \langle k \mid R_{i,j} \mid k' \rangle \tag{25}$$

which is a matrix in the horizontal auxiliary space and an operator in the quantum space. In matrix form it looks like

$$L_{i,j} = \begin{pmatrix} a(u) - b(u)(1-n_\uparrow)n_\downarrow & -b(u)c_\uparrow c^+_\downarrow & b(u)(1-n_\uparrow)c^+_\downarrow \\ -b(u)c^+_\uparrow c_\downarrow & a(u) - b(u)n_\uparrow(1-n_\downarrow) & b(u)c^+_\uparrow(1-n_\downarrow) \\ b(u)(1-n_\uparrow)c_\downarrow & b(u)c_\uparrow(1-n_\downarrow) & a(u) + b(u)(1-n_\uparrow)(1-n_\downarrow) \end{pmatrix} \tag{26}$$

The monodromy matrix $M^k_{k'}(u)$, which defined by matrix elements of the monodromy operators (1,2) in the auxiliary space can be expressed as a product of $L_{i,j}$ matrices as follows

$$\begin{aligned} M_0(u)^k_{k'} &= \langle k \mid M_0(u) \mid k' \rangle = (-1)^{p(k)p(k')}(\bar{L}^{\iota_1}_{01})^k_{k_1}(L_{02})^{k_1}_{k_2}....(L_{0N})^{k_{N-1}}_{k'}, \\ M_1(u)^k_{k'} &= \langle k \mid M_1(u) \mid k' \rangle = (-1)^{p(k)p(k')}(L^{\iota_1\iota_2}_{01})^k_{k_1}(\bar{L}^{\iota_2}_{02})^{k_1}_{k_2}....(\bar{L}^{\iota_2}_{0N})^{k_{N-1}}_{k'}. \end{aligned} \tag{27}$$

Following the notations of the article [26], one can write

$$M_s(u)^k_{k'} = \begin{pmatrix} A_{s,11}(u) & A_{s,12}(u) & B_{s,1}(u) \\ A_{s,12}(u) & A_{s,22}(u) & B_{s,2}(u) \\ C_{s,1}(u) & C_{s,2}(u) & D_s(u) \end{pmatrix} \quad s = 0, 1 \tag{28}$$

where $A_{s,ab}$, $B_{s,a}$, $C_{s,a}$, D_s; $(a, b = 1, 2)$ act on the quantum space.

Then, as a super-trace of the Monodromy matrix (28) the transfer matrix (3) will have the form

$$\tau_s(u) = -A_{s,11}(u) - A_{s,22}(u) + D_s(u), \quad s = 0, 1. \tag{29}$$

This matrix elements of the Monodromy matrix are obeying the algebraic relations

$$
\begin{aligned}
&(-1)^{p(k'')(p(m')+p(m''))}\check{R}^{km}_{k'm'}(u,v)M^{m'}_{1,m''}(u)M^{k'}_{0,k''}(v)\\
&\qquad=(-1)^{p(k')(p(m)+p(m'))}M^{m}_{1,m'}(v)M^{k}_{0,k'}(u)\check{R}^{k'm'}_{k''m''}(u,v),\\
&(-1)^{p(k'')(p(m')+p(m''))}\check{R}^{km}_{k'm'}(u,v)M^{m'}_{0,m''}(u)M^{k'}_{1,k''}(v)\\
&\qquad=(-1)^{p(k')(p(m)+p(m'))}M^{m}_{0,m'}(v)M^{k}_{1,k'}(u)\check{R}^{k'm'}_{k''m''}(u,v),
\end{aligned}
\tag{30}
$$

which are the consequence of YBE (4-5). In getting (30) we have used the properties (11).

Consider now the empty fermionic state as a test "vacuum"

$$
\mid\Omega\rangle_s=\mid 0,0,...,0\rangle_s=\mid 0\rangle_{1s}\mid 0\rangle_{2s}...\mid 0\rangle_{Ns},\quad s=0,1,
\tag{31}
$$

and let's check that $\mid\Omega\rangle$ indeed is a eigenstate of transfer matrix (29)

$$
\tau_s(u)\mid\Omega\rangle_s=\nu^{(0)}_s\mid\Omega\rangle_{1-s}
\tag{32}
$$

From (27) $\tau_s(u)$ is a product of $L_{i,j}$ matrices. Hence, in order to check (32) we should first calculate $L_{0k}\mid 0\rangle_{s,k}$. It appears that

$$
\bar{L}^{\iota_2}_{0k}\mid 0\rangle_{s,k}=\begin{pmatrix}\bar{b}(u)^{\iota_2} & 0 & \bar{b}^{\iota_2}(u)c^+_{k\downarrow}\\ 0 & \bar{b}^{\iota_2}(u) & \bar{b}^{\iota_2}(u)c^+_{k\uparrow}\\ 0 & 0 & \bar{a}^{\iota_2}(u)+\bar{b}^{\iota_2}(u)\end{pmatrix}\mid\Omega\rangle_k\quad s=0,1.
\tag{33}
$$

We see that $L_{0k}\mid 0\rangle_k$ is a upper-triangular matrix. Therefore the action of the product of L_{0k} in the formula (27) on vacuum $\mid\Omega\rangle_k$ as a matrix will also have upper triangular form

$$
\begin{aligned}
M_1(u)^k_{k'}\mid\Omega\rangle_1 &= \begin{pmatrix}[b^{\iota_1\iota_2}(u)\bar{b}^{\iota_2}(u)]^{\frac{N}{2}} & 0 & B_{1,1}(u)\\ 0 & [b^{\iota_1\iota_2}(u)\bar{b}^{\iota_2}(u)]^{\frac{N}{2}} & B_{1,2}(u)\\ 0 & 0 & 1\end{pmatrix}\mid\Omega\rangle_1,\\
M_0(u)^k_{k'}\mid\Omega\rangle_0 &= \begin{pmatrix}[\bar{b}^{\iota_1}(u)b(u)]^{\frac{N}{2}} & 0 & B_{0,1}(u)\\ 0 & [\bar{b}^{\iota_1}(u)b(u)]^{\frac{N}{2}} & B_{0,2}(u)\\ 0 & 0 & 1\end{pmatrix}\mid\Omega\rangle_0,
\end{aligned}
\tag{34}
$$

where we have used that $a(u)+b(u)=1$.

We see that the $B_{s,1}(u)$ and $B_{s,2}(u),(s=0,1)$ operators create one particle states while $C_{s,1}(u)$, $C_{s,2}(u)$ operators annihilate them

$$
C_{s,a}(u)\mid\Omega\rangle_s=0,\quad s=0,1;\ \ a=1,2
\tag{35}
$$

We see from the expression (34) that

$$
\nu^{(0)}_s(u)=1-2[b^{\iota_1}(u)\bar{b}(u)]^{\frac{N}{2}},
\tag{36}
$$

where $b^{\iota_1}(u)$ and $\bar{b}(u)$ defined by the equations (16) and N is the length of the chain.

This observation leads us to the following Ansatz for the eigenstates of $\tau(v)$

$$\mid v_1, v_2, ...v_n \mid F\rangle_0 = F^{a_n...a_1} B_{0,a_1}(v_1) B_{1,a_2}(v_2)...B_{0,a_n}(v_n) \mid \Omega\rangle_0, \quad a_i = 1, 2; \tag{37}$$

is a n particle state. The $F^{a_n...a_1}$ is a function of spectral parameters v_j to be specified later.

The action of the transfer matrix (29) on the states (37) is determined by the relations (34) and the intertwining properties of the $A_{s,ab}(u)$, $D_s(u)$, $B_{s,a}(u)$ operators defined from the YBE (30). The components of the intertwining relations, which we need for the construction of the ABA are

$$\begin{aligned} D_1(u)B_{0,a}(v) &= \frac{1}{b(u,v)} B_{1,a}(v)D_0(u) - \frac{a(v,u)}{b(v,u)} B_{1,a}(u)D_0(v), \\ A_{1,ba}(u)B_{0,c}(v) &= \frac{r_{bc}^{b'c'}(u,v)}{b(u,v)} B_{1,c'}(v)A_{0,b'a}(u) + \frac{a(u,v)}{b(u,v)} B_{1,b}(u)A_{0,ca}(v), \\ B_{1,a}(u)B_{0,b}(v) &= r_{ab}^{b'a'}(u,v) B_{1,a'}B_{0,b'}(u), \end{aligned} \tag{38}$$

where

$$r_{bc}^{b'c'}(v) = -a(v)\delta_b^{c'}\delta_c^{b'} + b(v)\delta_b^{b'}\delta_c^{c'} = -a(v)I_{bc}^{c'b'} - b(v)\Pi_{bc}^{(1)b'c'}. \tag{39}$$

Here $\Pi_{bc}^{(1),b'c'}$ is a graded permutation operator for $p(1) = p(2) = 1$, one can check that it fulfills the following YBE

$$r(\lambda-\mu)_{a_3c_3}^{a_2c_2} r(\lambda)_{c_2d_2}^{a_1b_1} r(\mu)_{a_2c_2}^{d_2b_2} = r(\mu)_{a_2c_2}^{a_1c_1} r(\lambda)_{a_3b_3}^{c_2d_2} r(\lambda-\mu)_{d_2b_2}^{c_1b_1}. \tag{40}$$

Now by use of (38), we can obtain, that the diagonal elements of the monodromy matrix act on the states (37) as follows

$$\begin{aligned} D_1(u) \mid v_1, ..., v_n \mid F\rangle_1 &= \prod_{j=1}^{n} \frac{1}{a(v_j,u)} \mid v_1, ..., v_n \mid F\rangle_0 + \\ &+ \sum_{k=1}^{n} (\tilde{\Lambda}_k)_{a_1...a_n}^{b_1...b_n} B_{1,b_k}(u) \prod_{j=1, j\neq k}^{n} B_{b_j}(v_j) \mid \Omega\rangle_0, \\ [A_{1,11}(u) + A_{1,22}(u)] \mid v_1, ..., v_n \mid F\rangle_1 &= \\ = -\prod_{i=1}^{n} \frac{1}{b(u,v_i)} [b^{t_1t_2}(u)\bar{b}^{t_1}(u)]^{\frac{N}{2}} \tau_{a_1...a_n}^{(1)a'_1...a'_n}(u) F^{a_n...a_1} \prod_{i=1}^{n} B_{a'_i}(v_i) \mid \Omega\rangle_0 + \\ + \sum_{k=1}^{n} (\Lambda_k)_{a_1...a_n}^{b_1...b_n} F^{a_n...a_1} B_{1,b_k}(u) \prod_{i=1, j\neq k}^{n} B_{b_j}(v_j) \mid \Omega\rangle_0, \end{aligned} \tag{41}$$

where

$$\begin{aligned}\tau^{(1)a'_1...a'_n}_{a_1...a_n}(u) &= -r^{b_1a'_1}_{ca_1}(u,v_1)...r^{ca'_n}_{b_{n-1}a_n}(u,v_n)\\ &= str[l_n(u,v_n)l(n-1)(u,v_{n-1})...l_1(u,v_1)],\end{aligned} \tag{42}$$

and

$$[l_k(u,v_k)]^{b_k}_{b_{k-1}} = r^{b_ka'_k}_{b_{k-1}a_k}(u,v_k). \tag{43}$$

As it follows from (43), l_k is a 2×2 matrix, the elements of which are the operators

$$l_k(u) = \begin{pmatrix} l^1_{k,1} & l^2_{k,1} \\ l^1_{k,2} & l^2_{k,2}\end{pmatrix} = \begin{pmatrix} b(u)I - a(u)e^1_1 & -a(u)e^2_1 \\ -a(u)e^1_2 & b(u)I - a(u)e^2_2\end{pmatrix}. \tag{44}$$

where e^b_a are quantum operators in the n-th space with matrix representation $(e^b_a)^\alpha_\beta = \delta^\alpha_a\delta^b_\beta$.

It is obvious, that the eigenvalue condition

$$(D_{s,1}(u) - A_{s,11}(u) - A_{s,22}(u)) \mid v_1,...v_n \mid F\rangle_s = \nu_s(u,v_1,...,v_n) \mid v_1,...,v_n \mid F\rangle_{1-s} \tag{45}$$

will be fulfilled if

i) we impose the cancellation of unwanted terms in (41)

$$[(\tilde{\Lambda}_k)^{b_1...b_n}_{a_1...a_n} - (\Lambda_k)^{b_1...b_n}_{a_1...a_n}]F^{a_n...a_1} = 0 \tag{46}$$

called Bethe equations (BE), and

ii) we solve the eigenvalue problem for the small transfer matrix (42)

$$\tau^{(1)a'_1...a'_n}_{a_1...a_n}(u;v_1,...,v_n)F^{a_1...a_n} = \nu^{(1)}(u,v_i)F^{a'_1...a'_n}, \tag{47}$$

then we have the following expression for eigenvalues

$$\begin{aligned}\nu_1(u;v_1,...,v_n) &= \prod_{i=1}^n \frac{1}{b(v_i,u)} + [b^{t_1t_2}(u)\bar{b}^{t_2}(u)]^{\frac{N}{2}}\prod_{j=1}^n\frac{1}{b(u,v_j)}\nu^{(1)}(u,v_i),\\ \nu_0(u;v_1,...,v_n) &= \prod_{i=1}^n \frac{1}{b(v_i,\bar{u})} + [b^{t_2}(\bar{u})\bar{b}^{t_2}(\bar{u})]^{\frac{N}{2}}\prod_{j=1}^n\frac{1}{b(\bar{u},v_j)}\nu^{(1)}(\bar{u},v_i).\end{aligned} \tag{48}$$

For the solution of second equation *ii*) we should make the ABA for a small auxiliary problem of chain, with length n (number of particles) and "nested" transfer matrix $\tau^{(1)a'_1...a'_n}_{a_1...a_n}(u;v_1,...,v_n)$. This is why all this procedure is called Nested Algebraic Bethe Ansatz (NABA). In the article [26], it was demonstrated how to calculate $\Lambda^{b_1...b_n}_{a_1...a_n}$ and $\tilde{\Lambda}^{b_1...b_n}_{a_1...a_n}$ and to reduce the condition of cancellation of the unwanted terms for the ordinary $t-J$ model to some equation. It is not necessary to repeat the same calculation here

since it differs very little from the carried one. The only difference is appearing in the term

$$(A_{1,11}+A_{1,22}) \mid \Omega\rangle = [b^{t_1t_2}(u)\bar{b}^{t_2}(u)]^{\frac{N}{2}} \mid \Omega\rangle, \tag{49}$$

therefore we obtain the following conditions

$$\tau^{(1)b'_1\dots b'_n}_{b_1\dots b_n}(v_k \mid v_1,...,v_n)F^{b_n\dots b_1} = [b^{t_1t_2}(v_k)\bar{b}^{t_2}(v_k)]^{-\frac{N}{2}} \prod_{i=1,i\neq k}^{n} \frac{b(v_kv_i)}{b(v_iv_k)} F^{b'_n\dots b'_1}, \tag{50}$$

as a Bethe equations.

In the next step of the $NABA$ we should find the eigenvalues and eigenstates of $\tau^{(1)}(u)$. It is clear from the equations (40) that we have another small integrable model with the R matrix $r^{a'b'}_{ab}(u)$ defined by the formula (39) and the corresponding transfer matrix $\tau^{(1)}(u)$.

Therefore we should apply a non ordinary ABA to this problem. The YBE for the problem is

$$r^{a'b'}_{ab}(u,v)\hat{M}^{(1)a''}_{a'}(u)\hat{M}^{(1)b''}_{b'}(v) = \hat{M}^{(1)b'}_{b}(v)\hat{M}^{(1)a'}_{a}(u)r^{a''b''}_{a'b'}(u,v) \tag{51}$$

where $M^{(1)a'}_{a}$ is the corresponding (nested) Monodromy matrix.

Now if we define

$$M^{(1)}(u) = \begin{pmatrix} A^{(1)}(u) & B^{(1)}(u) \\ C^{(1)}(u) & D^{(1)}(u) \end{pmatrix}, \quad \tau^{(1)}(u) = -A^{(1)}(u) - D^{(1)}(u), \tag{52}$$

then by use of the formula (39) and YBE (51), we find

$$\begin{aligned} D^{(1)}(u)B^{(1)}(v) &= \frac{1}{b(u,v)}B^{(1)}(v)D^{(1)}(u) + \frac{a(v,u)}{b(v,u)}B^{(1)}(u)D^{(1)}(v), \\ A^{(1)}(u)B^{(1)}(v) &= \frac{a(u,v)}{b(u,v)}B^{(1)}(u)A^{(1)}(v) + \frac{1}{b(v,u)}B^{(1)}(v)A^{(1)}(u), \\ B^{(1)}(u)B^{(1)}(v) &= B^{(1)}(v)B^{(1)}(u). \end{aligned} \tag{53}$$

Let's take as reference state

$$\begin{aligned} \mid 0\rangle^{(1)}_k &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ \mid \Omega\rangle^{(1)} &= \mid 0\rangle^{(1)}_1 \dots \mid 0\rangle^{(1)}_n = \bigotimes_{k=1}^{n} \mid 0\rangle^{(1)}_k. \end{aligned} \tag{54}$$

The action of the nested monodromy matrix $M^{(1)}(u)$ on the reference state $\mid \Omega\rangle^{(1)}$ is described by the action $l_k(u)$ on $\mid 0\rangle^{(1)}_k$, which we can find from (44). So we obtain

$$\begin{aligned} A^{(1)}(u) \mid \Omega\rangle^{(1)} = \prod_{i=1}^{n}[b(u,v_i) - a(u,v_i)] \mid \Omega\rangle^{(1)} = \prod_{i=1}^{n}\frac{b(u,v_i)}{b(v_i,u)} \mid \Omega\rangle^{(1)}, \\ D^{(1)}(u) \mid \Omega\rangle^{(1)} = \prod_{i=1}^{n} b(u,v_j) \mid \Omega\rangle^{(1)}. \end{aligned} \tag{55}$$

For the eigenstates of $\tau^{(1)}(v)$, we are going to do the following Ansatz

$$\mid v_1^{(1)},\ldots,v_m^{(1)}\rangle = B^{(1)}(v_1^{(1)})B^{(1)}(v_2^{(1)})\ldots B^{(1)}(v_m^{(1)}) \mid \Omega\rangle^{(1)}. \tag{56}$$

The action of $\tau^{(1)}(u)$ on the states (56) is the same as the action of the diagonal elements of (52) on that states. By use of (38) we will obtain

$$\begin{aligned}
D^{(1)}(u) \mid v_1^{(1)},\ldots,v_m^{(1)}\rangle &= \prod_{j=1}^{n} b(u,v_j) \mid v_1^{(1)},\ldots,v_m^{(1)}\rangle \\
&+ \sum_{k=1}^{m} \Lambda_k^{(1)} B^{(1)}(u) \prod_{i=1,i\neq k}^{m} B^{(1)}(v_i) \mid \Omega\rangle^{(1)}, \\
A^{(1)}(u) \mid v_1^{(1)},\ldots,v_m^{(1)}\rangle &= \prod_{i=1}^{m} \frac{1}{b(v_i^{(1)},u)} \prod_{j=1}^{n} \frac{b(u,v_j)}{b(v_j,u)} \mid v_1^{(1)},\ldots,v_m^{(1)}\rangle \\
&+ \sum_{k=1}^{m} \tilde{\Lambda}_k^{(1)} B^{(1)}(u) \prod_{j=1,j\neq k}^{m} B^{(1)}(v_j) \mid \Omega\rangle^{(1)}.
\end{aligned} \tag{57}$$

From the expression (57) we can easily write the eigenvalues of $\tau^{(1)}(u)$

$$\begin{aligned}
\tau^{(1)} \mid v_1^{(1)},\ldots,v_m^{(1)}\rangle = &- \left[\prod_{i=1}^{m} \frac{1}{b(v_i^{(1)},u)} \prod_{j=1}^{n} \frac{b(u,v_j)}{b(v_j,u)} + \right. \\
&+ \left. \prod_{i=1}^{m} \frac{1}{b(u,v_j^{(1)})} \prod_{j=1}^{n} b(u,v_j)\right] \mid v_1^{(1)},\ldots,v_m^{(1)}\rangle.
\end{aligned} \tag{58}$$

One can get simply the first set of Bethe equations by comparing (58) with the formula (50). Inputing $u = v_k$ in (50) we obtain

$$[b^{\iota_1}(v_k)\bar{b}(v_k)]^{\frac{N}{2}} = \prod_{i=1}^{m} b(v_i^{(1)},v_k), \quad k = 1,2,\ldots,n. \tag{59}$$

The second set of Bethe equations, which are the conditions of cancellation of the unwanted terms $\Lambda_k^{(1)}$ and $\tilde{\Lambda}_k^{(1)}$ are similar to the corresponding equations of the standard XXX model and can be found easily as

$$\prod_{j=1}^{n} b(v_j,v_k^{(1)}) = \prod_{i\neq k} \frac{b(v_k^{(1)},v_i^{(1)})}{b(v_i^{(1)},v_k^{(1)})}, \quad k = 1,2,\ldots,m. \tag{60}$$

This is exactly the same equation as found in [26].

Finally we find

$$\begin{aligned}
\nu_1(u,\{v_i\}) &= \prod_{i=1}^{n}\frac{1}{b(v_i,u)} - [b^{t_1}(u)\bar{b}(u)]^{\frac{N}{2}}\prod_{j=1}^{n}\frac{1}{b(u,v_j)}\left[\prod_{i=1}^{m}\frac{1}{b(v_i^{(1)},u)}\prod_{j=1}^{n}\frac{b(u,v_j)}{b(v_j,u)}\right.\\
&+ \left.\prod_{i=1}^{m}\frac{1}{b(u,v_i^{(1)})}\prod_{j=1}^{n}b(u,v_j)\right],\\
\nu_0(u,\{v_i\}) &= \prod_{i=1}^{n}\frac{1}{b(v_i,\bar{u})} - [\bar{b}^{t_1}(u)b(u)]^{\frac{N}{2}}\prod_{j=1}^{n}\frac{1}{b(\bar{u},v_j)}\left[\prod_{i=1}^{m}\frac{1}{b(v_i^{(1)},\bar{u})}\prod_{j=1}^{n}\frac{b(\bar{u},v_j)}{b(v_j,\bar{u})}\right.\\
&+ \left.\prod_{i=1}^{m}\frac{1}{b(\bar{u},v_i^{(1)})}\prod_{j=1}^{n}b(\bar{u},v_j)\right] \qquad (61)
\end{aligned}$$

as the n particle state eigenvalues of transfer matrices $\tau_1(u)$ and $\tau_0(u)$ respectively.

But the transfer matrix of our staggered model is the product of $\tau_0(u)$ and $\tau_1(u)$, therefore the eigenvalues $\nu(u,\{v_i\})$ of $\tau(u)$ are

$$\nu(u,\{v_i\}) = \nu_0(u,\{v_i\})\nu_1(u,\{v_i\}) \qquad (62)$$

with the Bethe equations (59) and (60) unchanged.

Let us now to calculate the energy of excitations over the test "vacuum" $\mid\Omega\rangle$, called bare energy, which will be dressed in a real ground state due to interactions with particles in a filled Dirac sea. The bare energy is a logarithmic derivative of eigenvalues (61) and (62) at the point $u=0$. The simple calculation gives the energy and the momentum of n-particle state $\mid v_1,...v_n \mid F\rangle$ as it follows

$$\begin{aligned}
E_0(\{v_j\}) &= -\sum_{j=1}^{n}\left\{\frac{1}{v_j^2+1/4} - \frac{1}{(v_j-\theta)^2+1/4}\right\},\\
iP(\{v_j\}) &= \sum_{j}^{n}\left\{\log\frac{v_j+1/2}{v_j-1/2} + \log\frac{v_j-\theta+1/2}{v_j-\theta-1/2}\right\}, \qquad (63)
\end{aligned}$$

where we have redefined the spectral parameters as $v_j \to v_j - 1/2$.

The solution of the BE (59) and (60) is usually obtained in the thermodynamic limit ($N,n,m\to\infty$, with the fixed ratio $\frac{n}{N},\frac{m}{N}$). In this case instead of a discrete set of spectral parameters v_j one introduces the distribution of continuous density $\rho(v)$ of rapidities. The ground state is defined by filling up the Dirac sea(s) of negative energies by the electrons. It was argued in the article [25] that the ground state of the $t-J$ model is defined by the string solutions of length two, which are filling of all states with negative energy. The lowest energy value can be reached by maximal filling of negative energy states, which corresponds to $\frac{n}{N}=1$ and with zero magnetization, corresponding to $m=\frac{n}{2}$. In our

model it is clear from the expression of the energy (63) that only the spectral parameters of the interval

$$-\infty < u < \frac{\theta}{2} \tag{64}$$

have to be filled in order to form a ground state. But since this is exactly equal to half of lattice sites N, we will have a ground state corresponding to quarter filling of all states.

After introducing the densities, the BE becomes an integral equation of the form [25]

$$\pi\rho(v) + \int_Q^\infty du \frac{\rho(u)}{(v-u)^2+1} = \frac{1}{2}\left\{\frac{1}{v^2+1/4} + \frac{1}{(v-\theta)^2+1/4}\right\}\ , \tag{65}$$

where Q defines the rapidity of the Fermi level. In case of quarter filling it is equal to zero.

The energy of ground state is defined by the equation

$$\begin{aligned} E_0 &= -2N\int_Q^\infty \left\{\frac{1}{v^2+1} - \frac{1}{(v-\theta)^2+1}\right\} \\ &= -2N(\rho(0)-\rho(\theta)). \end{aligned} \tag{66}$$

At usual half filling $Q = -\infty$, and as it follows from the expression (66), $E = 0$.

It seams to us, that this model provides an interesting possibility to analyze by means of exact integrability the physics of systems with the quarter filled ground state.

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