Observable Algebra

Merab Gogberashvili
Andronikashvili Institute of Physics
6 Tamarashvili Str.,Tbilisi 380077, Georgia
E-mail: gogber@hotmail.com

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Abstract

A physical applicability of normed split-algebras, such as hyperbolic numbers, split-quaternions and split-octonions is considered. We argue that the observable geometry can be described by the algebra of split-octonions, which is naturally equipped by zero divisors (the elements of split-algebras corresponding to zero norm). In such a picture physical phenomena are described by the ordinary elements of chosen algebra, while the zero divisors give raise the coordinatization and two fundamental constants, namely velocity of light and Planck constant. It turns to be possible that uncertainty principle appears from the condition of positively defined norm, and has the same geometrical meaning as the existence of the maximal value of speed. The property of non-associativity of octonions could correspond to the appearance of fundamental probabilities in physics. Grassmann elements and a non-commutativity of space coordinates, which are widely used in various physical theories, appear naturally in our approach.

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1 Introduction

Real physical phenomena always expose themselves experimentally as a set of measured numbers. The theory of representation of algebra serves as a tool, which gives a possibility to interpret abstract mathematical quantities from the point of view of the observable reality. The geometry of space-time being the main characteristic of the reality can be also described in the language of some algebras and symmetries. Usually in physics the geometry is thought to be objective without any connection to the way it gets observed. In other words, it is assumed the existence of some massive objects, with respect to which physical events can be numerated. However, geometry cannot be introduced independently. Results of any observation depend not only upon the phenomena we study but also on the way we make the observation and on the algebra one uses for a coordinatization.

Let us start from the assumption that physical quantities must correspond to elements of some algebra, so they can be summed and multiplied. Since all observable quantities we know are real it is possible to restrict our self with a real algebra. To have a transition from a manifold consisted of elements being the results of measurements to geometry one must be able to introduce a distance between two objects. Introduction of a distance always means some comparison of two objects using one of them as an etalon. Thus we need algebra with a unit element.

In the algebraic language all above requirements mean that to describe the geometry of the real world we need the normed composition algebra with unit elements over the real numbers. Besides of the usual algebra of real numbers according to the Hurwitz theorem there are three extraordinary and unique algebras satisfied the required conditions, namely the algebra of complex numbers, the algebra of quaternions and the octonions algebra [1].

Essential for all these algebras is the existence of the real unit element e_0 and different number of adjoined hyper-complex units e_n. For the case of complex number $n = 1$, for quaternions $n = 3$ and for octonions $n = 7$. Square of unit element e_0 is always positive and square of hyper-complex units e_n can be positive or negative as well

$$e_n^2 = \pm e_0 \ . \tag{1}$$

In applications of division algebras mainly the negative sign in (1) is used. In this case their norms are positively defined.

We note that in real world any comparison of two objects can be done only by exchange of some signals between them. The new physical requirement on the algebra we would like to apply is that it must contain special elements corresponding to unit signals, which can be used for distance measurements. One-way signals can correspond to any kind of number. The important is to have the real norm, which is actually the multiplication of direct and backward signals, being the result of one measurement. To have a possibility to introduction some dynamics we wont to choose positive sign in (1), or

$$e_n^2 = -e_0 \ . \tag{2}$$

This leads to so-called split algebras with several negative terms in definition of there norms. Adjoining to real unit element (which could be connected with time) any invertible hyper-complex units with negative norms could be understood as an appearance of dynamics. Critical elements of split-algebras, so-called zero divisors, corresponding to zero norms can be used as unit signals to measure distances between physical objects (which are described by usual elements of the algebra).

We believe that dimension of proper algebra and properties of its unit elements also can be extracted from the physical considerations. We know that physical quantities are divided into three main classes, scalars, vectors and spinors. We require having a unique way to describe these objects in appropriate algebra.

Scalar quantities have magnitude only, and do not involve directions. The complete specification of a scalar quantity requires a unit of the same kind and a number counting how many times the unit is contained in the quantity. Scalar quantities are manipulated by applying the rules of ordinary algebra.

Vast range of physical phenomena finds their most natural description as vector quantities, having magnitude and direction. A vector quantity requires for its specification a real number, which gives the magnitude to the quantity in terms of the unit (as for scalars) and an assignment of direction. By revising the process of addition from standard vector algebra it was found that any vector could be decomposed into components

$$\mathbf{a} = a_n e_n \ , \tag{3}$$

using unit orthogonal basis vectors e_n. The ordinary idea of a product in scalar algebra cannot apply to vectors because of their directional properties. Since vectors have their origin in physical problems, definitions of the products of vectors historically was obtained with the way in which such products occur in physical applications. It was introduced two types of multiplication of vectors - the scalar product

$$(\mathbf{ab}) = ab\cos\theta \ , \tag{4}$$

and the vector product

$$[\mathbf{ab}] = ab\sin\theta \ . \tag{5}$$

These formulae result the following properties of unit basis vectors

$$(e_n e_m) = 0 \ , \qquad e_n e_m = -e_m e_n \ , \qquad (e_n e_n) = e_n^2 = 1 \ , \qquad [e_n e_m] = \varepsilon_{nmk} e_k \ , \tag{6}$$

where ε_{nmk} is fully anti-symmetric tensor. Or vice versa, we can say that postulated algebra (6) for basis elements one receive the observed multiplication laws (4) and (5) of physical vectors.

Since non-zero are only vector product of different basis vectors and scalar product of a basis vector on itself, we don't need to write different kind of brackets in expressions of their multiplications.

Because of ambiguity of choosing of direction of basis vectors one must introduce conjugate elements e_n^*, which differ from e_n by the sign $e_n^* = -e_n$, what can be understood as a reflection of the vector ort.

The algebra of the vectors basis elements (6) is exactly the same we have for basis units of split algebras (with the positive sign (2)), which we had choose from some general requirements.

Next restriction for appropriate algebra can be imposed from the properties of spinors, another important class of observed physical quantities. Spinors also can be characterized by some magnitude and direction. In difference with vectors, spinors changes the sign after full rotation. Using multi-dimensional picture this feature can be understood as when we transform ordinary space coordinates spinors rotate also in some extra space. It is well known that there exists an equivalence of vectors and spinors in eight dimensions [2], what renders the 8-dimensional theory (connected with octonions) a very special case. In eight dimensions a vector and a left and right spinor all have an equal number of components (namely, eight) and their invariant forms all look the same. If one considers the vectors rotation as the primary rotation that induces the transformations of two kinds of spinors, it is equivalent if one starts from spinor rotation, which induce corresponding rotation of vector and second kind of spinor. This property is referred in the literature as "principle of triality". The physical result of this principle can be the observed fact that interaction in particle physics exhibits by the vertex containing left spinor, right spinor and vector [3].

To resume our general consideration we conclude that the proper algebra needed to describe the geometry of real world must be the 8-dimensional algebra of split-octonions over the real numbers.

In the next section the properties of zero divisors, which appear in split-algebras is considered. The following three sections is devoted to geometrical applications of hyper-numbers, split-quaternions and split-octonions respectively.

2 Zero Divisors

From the different combinations of basis elements of split-algebra special objects, called zero divisors can be constructed [1, 4]. We assume that these objects (we shall call critical elements of the algebra) as the unit signals for characterization of the physical events could be used. Sommerfeld specially notes physical fitness of zero divisors of split algebras in his book [5].

The first objects we wont to study are projective operators with the property

$$D^2 = D \ , \tag{7}$$

for any non-zero D. In division algebras only projective operator is the identity $1 = e_o$.

Most general form of projective operator in split-algebra is

$$D = 1/2 + a_n e_n \ , \tag{8}$$

with $a_n a_n = 1/4$, where n runs over the number of the orthogonal unit elements e_n. Commutating projective operator can be constructed by decomposition of the identity element

$$1 = D_n^+ + D_n^- \ , \tag{9}$$

with a particular choice

$$D_n^+ = \frac{1}{2}(1 + e_n) \ , \qquad D_n^- = \frac{1}{2}(1 - e_n) \ . \tag{10}$$

The elements D_n^+ and D_n^- are commutating projective operators, since

$$D_n^+ D_n^- = D_n^- D_n^+ = 0 \ . \tag{11}$$

Another type of zero divisors in split-algebras are Grassmann numbers defined as the set of anti-commuting numbers $G_1, G_2, ..., G_n$ with the properties

$$G_n^2 = 0 \ , \qquad G_n G_m = 0 \ . \tag{12}$$

Since hyper-complex basis units satisfying the relations:

$$e_n e_m = 2\eta_{nm} \ , \tag{13}$$

the Grassmann numbers can be constructed by coupling of two basis elements (except of unity) with the opposite choice of signs in (1). For example, Grassmann numbers are the sums

$$G^{\pm} = \frac{1}{2}(e_1 \pm e_2) \tag{14}$$

of the two basis elements e_1 and e_2 with the properties

$$e_1^2 = 1 \ , \qquad e_2^2 = -1 \ , \qquad e_1^* = -e_1 \ , \qquad e_2^* = -e_2 \ . \tag{15}$$

The Grassmann numbers not contains the unit element 1 as one of the terms (structure with the unit element forms projective operator) and it can be represented as

$$G = e_1 D \ , \tag{16}$$

where D is projective operator and e_1 is the vector-like basis element of algebra with the positive square $e_1^2 = 1$.

It can be checked that Grassmann numbers $G^{\pm}$ and projective operators $D^{\pm}$ obey the following algebra:

$$\begin{aligned} D^{\pm}D^{\mp} &= 0 \ , \qquad & D^{\pm}D^{\pm} &= D^{\pm} \ , \\ G^{\pm}G^{\pm} &= 0 \ , \qquad & G^{\pm}G^{\mp} &= D^{\mp} \ , \\ D^{\pm}G^{\pm} &= 0 \ , \qquad & D^{\pm}G^{\mp} &= G^{\mp} \ , \\ G^{\pm}D^{\mp} &= 0 \ , \qquad & G^{\pm}D^{\pm} &= G^{\pm} \ . \end{aligned} \tag{17}$$

From this relations we see that separately quantities G^+ and G^- are Grassmann numbers, but they not commute with each other (in difference with the projective operators D^+ and D^-) and thus don't obey to full Gtassmann algebra (12). Instead quantities G^+ and G^- are the elements of so-called algebra of Fermi operators with the anticommutator

$$G^+G^- + G^-G^+ = 1 \ . \tag{18}$$

Algebra of Fermi operators is some syntheses of Grassmann and Clifford algebras.

Mutually commuting zero divisors of algebra corresponds to simultaneously measurable signals and can serve as the geometrical basis for the physical events, which correspond to usual elements of the algebra.

3 Hyper-Numbers

There are certain important types of physical phenomenon, which are intrinsically 2-dimensional in nature. Many physical functions occur naturally in pairs and was expressed by 2-dimensional real numbers that satisfies a different type of algebra [6]. Most common 2-dimensional algebra is the algebra of complex numbers, however, there is no special reason to prefer one algebra to the other [7].

Historically, the motivation to introduce complex numbers was mathematical rather than physical. This numbers made sense of the solution of algebraic equations, of the convergence of series, of formulae for trigonometric functions, differential equations, and many other things. This was quite unlike the initial motivation for using real numbers, which come about as an idealization of the kind of quantity that directly arose from physical measurements.

Essential for the complex numbers

$$z = x + iy \ , \tag{19}$$

where x and y are some real numbers, is existence of the unit element and one imaginary element i, with the property $i^2 = -1$. Later it was found that if we introduce conjugation of complex numbers

$$z^* = x - iy \ , \tag{20}$$

there algebra is division. A regular complex number can be represented geometrically by the amplitude and by the polar angle

$$\rho^2 = zz^* = x^2 + y^2 \ , \qquad \theta = \arctan \frac{y}{x} \ . \tag{21}$$

The amplitude ρ is multiplicative and the polar angle θ is additive upon the multiplication of complex numbers.

Complex number had found many uses in physics, but these were considered as a "mathematical tricks", like the employment of complex numbers in 2-dimensional hydrodynamics, electrical circuit theory, or the theory of vibrations.

In quantum mechanics complex numbers enter at the foundation of the theory from probability amplitudes and superposition principle. To pass from a quantum information link, which does not respect the rules of ordinary classical causality (as in the case of Einstein-Podolsky-Rosen phenomenon) to a classical information link (which necessarily propagates causally into the future) and to obtain real number corresponding to probability, one must multiply the complex amplitude by its complex conjugate. In fact, the complex conjured amplitude may be thought of as applying to the quantum-information link in the reverse direction in time [8].

Up to now it is admitted without justification that two-valuedness of physical quantities naturally appearing in quantum mechanics is to be described in terms of complex numbers. Complex numbers achieve a particular representation of quantum mechanics in term of which the fundamental equations take their simplest form. Other choices for the representation of the two-valuedness also possible, but would give to the Schrodinger equation a more complicated form (involving additional terms), although its physical meaning would be unchanged [9].

The main reason why complex numbers are popular is Euler's formula. As it was mentioned by Feynman [10], "the most remarkable formula in mathematics is:

$$e^{i\theta} = \cos\theta + i \sin\theta \ . \tag{22}$$

This is our jewel. We may relate the geometry to the algebra by representing complex numbers in a plane

$$x + iy = \rho e^{i\theta} \ . \tag{23}$$

This is the unification of algebra and geometry."

Using (23) the rule for complex multiplication looks almost obvious as a consequence of the behavior of rotations in plane. We know that a rotation of α-angle around the z-axis, can be represented by

$$e^{i\alpha}(x + iy) = \rho e^{i(\theta+\alpha)} \ . \qquad (24)$$

Another possible 2-dimensional normed algebra - algebra of hyper-numbers

$$z = t + hx \qquad (25)$$

has also a long history but is rarely used in physics [11]. In (25) the quantities t, x are the real numbers and hyper-unit h has the properties similar to ordinary unit vector

$$h^2 = 1 \ . \qquad (26)$$

Conjugation of h means space-reflection when h changes its direction on opposite $h^* = -h$. In this case in difference with ordinary complex numbers norm of hyper-number is full invariant of rotation with reflection.

Hyper-numbers constitute a commutative ring, but not a field, since norm of a nonzero hyper-number

$$N = zz^* = t^2 - x^2 \ , \qquad (27)$$

can be zero.

Hyper-number does not have a inverse when its norm is zero, i. e., when $x = \pm t$, or, alternatively, when $z = t(1 \pm h)$. These two lines whose elements have no inverses, play the same role as the point $z = 0$ does for the complexes, and provide the essential property of the light cone, which makes the hyper-numbers relevant for representation of relativistic coordinate transformations. Lorentz boosts can be succinctly expressed as

$$(t + hx) = (t' + hx')e^{h\theta} \ , \qquad (28)$$

where $\tanh\theta = v$.

Any hyper-number (25) can be expressed in terms of light cone coordinates

$$z = t + hx = (t + x)D^+ + (t - x)D^- \ , \qquad (29)$$

where we introduced projective operators

$$D^\pm = \frac{1}{2}(1 \pm h) \qquad (30)$$

corresponding to the critical signals.

So we can say that hyper-numbers can be used to describe the dynamic in 1-dimensional space. Unit element of algebra corresponding to positive term in the expression of the norm (27) is connected with time. Orthogonal to time hyper-element giving the negative term in (27) can be consider as describing one space dimension. Unit of algebra needed to measure distances is connected with the critical signals and is universal for all physical quantities, which described by usual elements of the algebra. Thus it is just the hyper-unit that seems to be responsible for the signature of space-time metric and the velocity of light c is related with it.

4 Quaternions

William Hamilton's discovery of quaternions in 1843 was the first event in history when the concept of 2-dimensional numbers was successfully generalized. Many authors focused on the adoption of quaternions in physics [7, 12].

General element of quaternion algebra can be written by using of the only two basis elements i and j in the form

$$q = a + bi + (c + di)j \ , \tag{31}$$

where a, b, c and d are some real numbers. The third basis element (ij) can be described by conjugation of the first two.

Quaternion reverse to (31), or conjugated quaternion q^* can be received using the properties of basis units under the conjugation (reflection)

$$i^* = -i \ , \qquad j^* = -j \ , \qquad (ij)^* = -(ij) \ , \tag{32}$$

Physically observable quantity is the norm of quaternion $N = qq^*$ corresponding to the multiplication of the direct and reverse signals.

When the basis elements i and j are imaginary $i^2 = j^2 = -1$ as ordinary complex unit we have Hamilton's quaternion with the positively defined norm

$$N = qq^* = q^*q = a^2 + b^2 + c^2 + d^2 \ . \tag{33}$$

In this case the third orthogonal basis unit (ij) has the analogous to i and j properties.

For the positive squares $i^2 = j^2 = 1$ we have the algebra of split-quaternions and the unit elements i, j have the properties of ordinary unit vectors. Norm of split-quaternions

$$N = qq^* = q^*q = a^2 - b^2 - c^2 + d^2 \ , \tag{34}$$

has $(2 + 2)$ signature and in general is not positive defined.

We see that the third unit element (ij) of split-quaternions is the vector product of unit vectors i and j and thus is pseudo-vector. It is important to realize that a pseudo-vector is not a geometrical object in the usual sense. In particular, it should not be considered as a real physical arrow in space. The norm of quaternion is defined geometrically without reference to the arbitrary system of coordinates used, but in determining of direction of (ij) the notion of right-handedness is needed. Vector multiplication of two vectors is ordinary vector only if we fix a right-handed Cartesian coordinate system and use right-handed rule. But coordinate system can be defined only if we have the second quaternion for comparison. In our approach we cannot do this, there is no independent geometry and coordinate systems as itself, we ca define distances only by critical signals. The pseudo-vector (ij) differs from i and j and behaves like pure imaginary object, since

$$(ij)^* = -(ij) \ , \qquad (ij)(ij) = -i^2j^2 = -1 \ . \tag{35}$$

In this fashion one can justify the origins of complex numbers without introducing them ad-hoc. The imaginary unit is (ij) and the number $x + (ij)y$ has the properties of ordinary complex numbers. In this sense we can say that split-quaternions are more rich structure in compare to Hamilton's quaternios and complex and hyper-numbers are there particular case.

In the algebra of split quaternions can be introduced two classes (totally four) projective operators

$$D_i^{\pm} = \frac{1}{2}(1 \pm i) \ , \qquad D_j^{\pm} = \frac{1}{2}(1 \pm j) \ , \tag{36}$$

which don't commute with each other. Commutated projective operators with the properties

$$[D^+ D^-] = 0 \ , \qquad D^+ D^+ = D^+ \ , \qquad D^- D^- = D^- \ , \tag{37}$$

can be only two of them $D_i^{\pm}$, or $D_j^{\pm}$. Operators D^+ and D^- differs from the each other by reflecting of basis element and thus correspond to the direct and reverse critical signals along the one from the two real directions.

In algebra we have also two classes of Grassmann numbers

$$G_i^{\pm} = \frac{1}{2}(1 \pm i)j \ , \qquad G_j^{\pm} = \frac{1}{2}(1 \pm j)i \ . \tag{38}$$

From the properties of quaternion zero divisors

$$\begin{aligned} D_i^{\pm} G_i^{\pm} = G_i^{\pm} \ , \qquad & D_j^{\pm} G_j^{\pm} = G_j^{\pm} \ , \\ D_i^{\pm} G_i^{\mp} = 0 \ , \qquad & D_j^{\pm} G_j^{\mp} = 0 \ , \end{aligned} \tag{39}$$

we see that Grassmann numbers (38) have pair wise commute relations with the projective operators (36)

$$[D_i^{\pm} G_i^{\mp}] = 0 \ , \qquad [D_j^{\pm} G_j^{\mp}] = 0 \ . \tag{40}$$

Using commutating zero divisors any quaternion can be written in the form

$$q = a + ib + (c + id)j = D^+[a + b + (c + d)G^+] + D^-[a - b + (c - d)G^-] \ , \tag{41}$$

where $D^{\pm}$ and $G^{\pm}$ are projective operators and Grassmann elements, which belong to the one of the classes from (36) and (38) labeled by i, or j.

The quaternion algebra is associative and therefore matrices can represent them. We get the simplest non-trivial representation of the split-quaternion basis units if we choose the real Pauli matrices accompanied by the unit matrix. Note that for the real matrix representation of Hamilton's quaternions one needs 4-dimensional matrices.

The independent unit elements of split quaternions i and j have the following matrix representation

$$i = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ , \qquad j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ .$$

The third unit element of quaternion algebra (ij) is formed by multiplication of i and j and has the representation

$$ij = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \ .$$

It is easy to noticed that in difference with complex case three real Pauli matrices have different properties, since their square gives the unit matrix

$$(1) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

with the different signs

$$i^2 = j^2 = (1) \ , \qquad (ij)^2 = -(1) \ . \tag{42}$$

Conjugation of unit vectors i and j means changing of signs of matrices i, j and (ij) and

$$ii^* = jj^* = -(1) \ , \qquad (ij)(ij)^* = (1) \ . \tag{43}$$

Matrix representation of independent projective operators and Grassmann elements from (36) and (38) labeled by i are

$$D_i^+ = \frac{1}{2}(1+i) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \ , \qquad D_i^- = \frac{1}{2}(1-i) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \ ,$$

$$G_i^+ = \frac{1}{2}(j+ij) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \ , \qquad G_i^- = \frac{1}{2}(j-ij) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \ .$$

It is easy to find also matrix representation of zero divisors labeled by j.

Decomposition of the quaternion (41) now can be written in the form

$$q = \begin{pmatrix} (a+b) & (c+d) \\ (c-d) & (a-b) \end{pmatrix} \ .$$

Norm of the quaternion (34) is now the determinant of this matrix

$$detq = (a^2 - b^2) - (c^2 - d^2) \ . \tag{44}$$

If we treat unit element of algebra as a time then split-quaternions could be used for describing of rotation in 2-dimensional space, similar, as hyper-complex numbers are useful to study dynamic in (1+1)-space. We have now two rotations in $t-i$ and $t-j$ planes described by $D^{\pm}$, which introduces fundamental constant c.

But what is geometrical meaning of Grassmanns numbers and fourth orthogonal element in quaternion algebra? Similar elements was absent in algebra of hyper-numbers containing only one fundamental constant c.

It can be shown, that pseudo-vector (ij) connected with rotation in $(i-j)$ plane. Indeed operators

$$R^{\pm} = \frac{1}{\sqrt{2}}[1 \pm (ij)] \ , \tag{45}$$

with the property $R^+R^- = 1$, represent rotation on the 90^0 in $(i-j)$ plane and can transform i to j

$$R^+ i R^- = -j \ , \qquad R^+ j R^- = i \ . \tag{46}$$

So extra pseudo-direction (ij) is similar to angular momentum and can be connected with the spin.

There is the symmetry with choosing of independent projective operator $D_i^\pm$ or $D_j^\pm$ for decomposition of a quaternion (41). This means that we have two independent space directions i and j where we can in principle exchange signals. However, we can simultaneously measure the signals only from one direction, since operators $D_i^\pm$ and $D_j^\pm$ not commute. This property is equivalent to non-commutativity of space coordinates [13].

The constant c (which is universal since connected with critical signals $D^\pm$) for the convenience can be extracted explicitly from the unit element of any usual quaternion

$$q = ct + xi + yj + \lambda(ij) \ , \tag{47}$$

where λ is some quantity with the dimension of length.

Grassmann numbers correspond to critical rotations in $(i - ij)$ and $(j - ij)$ planes, similar as projective operators does in $(1-i)$ and $(1-j)$ planes. Existence of maximal velocity following from the requirement to have positive norm for the components of quaternion (47) means

$$\frac{\Delta x}{c\Delta t} < 1 \ , \qquad \frac{\Delta y}{c\Delta t} < 1 \ . \tag{48}$$

Now we can write similar extra relations

$$\frac{\Delta x}{\Delta\lambda} < 1 \ , \qquad \frac{\Delta y}{\Delta\lambda} < 1 \ , \tag{49}$$

connected with the Grassmann numbers $G^\pm$. So there must exists second fundamental constant (which can be extracted from λ) characterizing this critical property of algebra. It is natural to identify it with the Plank's constant $\hbar$ and write λ in the form

$$\lambda = \frac{\hbar}{P} \ , \tag{50}$$

where quantity P has the dimension of the momentum. Inserting (50) into (49) we found that probably uncertainty principle of quantum mechanics follows from the condition of the positively defined of the quaternion norm and has the geometrical meaning similar as existence of the maximal velocity.

To resume split quaternion (47) with the norm

$$N = c^2t^2 + \frac{\hbar^2}{P^2} - x^2 - y^2 \ , \tag{51}$$

could be used to describe dynamics of particle with spin in 2-dimensional space. Two fundamental constants c and $\hbar$ have the geometrical origin and correspond to two kinds of critical signals in (2+2)-dimensions with one time coordinate. The Lorentz factor

$$\gamma = \sqrt{1 - \frac{v^2}{c^2} + \frac{\hbar^2}{c^2}\frac{F^2}{P^2}} \ , \tag{52}$$

corresponding to general bust in the space (51) except of velocity v^2 contains extra positive term, where F is some kind of force. So dispersion relation in (2+2)-space (51) will have form similar to the double-special relativity models [16].

5 Octonions

Despite the fascination of octonions for over a century (in 1844-1845 by Graves and Cayley) it is fair to say that they still await universal acceptance. However, this is not to say that there have not been various attempts to find appropriate uses for them in physics (see, for example [4, 12, 14, 15]).

As we had seen hyper-numbers are useful to describe dynamics in 1-dimensional space, wile split-quaternion - in 2-dimensional space. According to the Hurwitz theorem there is yet only one composite division algebra, the octonion algebra [1]. We wont to show that split-octonions can describe dynamics in usually 3-dimensional space.

For constructing of the octonion algebra with a general element

$$O = a_0 e_0 + a_n e_n \ , \qquad n = 1, 2, ..., 7 \tag{53}$$

where e_0 is the unit element and a_0, a_n are the real numbers, multiplication law of its eight basis elements e_0, e_n usually is given. For the case of ordinary octonions

$$e_0^2 = e_0 \ , \qquad e_n^2 = -e_0 \ , \qquad e_0^* = e_0 \ , \qquad e_n^* = -e_n \ , \tag{54}$$

with the positively defined norm

$$N = OO^* = O^*O = a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + a_5^2 + a_6^2 + a_7^2 \ , \tag{55}$$

multiplication table can be written in the form

$$(e_n e_m) = -\delta_{nm} e_0 + \epsilon_{nmk} \epsilon_k \ , \tag{56}$$

where δ_{nm} is the Kronecker symbol and ϵ_{nmk} is the fully anti-symmetric tensor with the postulated value $\epsilon_{nmk} = +1$ for the following values of indices

$$nmk = 123, 145, 176, 246, 257, 347, 365 \ .$$

This definition of structure constants used by Cayley in his original paper but is not unique. There are 16 different possibilities of definition of ϵ_{nmk} leading to the equivalent algebras. Sometimes for visualization of multiplication of octonion basis units except of tables the geometrical picture of Fano plane (a little gadget with 7 points and 7 lines) is used [3]. Unfortunately both of these methods are almost unenlightening. We wont to consider more obvious picture using properties of usual vectors products.

For the case of quaternions we had shown that not all basis elements of the algebra are independent. We easily recover full algebra from two basis elements without considering of the multiplication tables or graphics. The same can be done with the split-octonions. Now beside of unit element 1 we have three fundamental basis elements i, j and l with the properties of ordinary unit vectors

$$i^2 = j^2 = l^2 = 1 \ . \tag{57}$$

Other four basis elements of octonion algebra $(ij), (il), (jl), ((ij)l)$ can be constructed by vector multiplications of i, j and l. All brackets here denote vector product and they show only the order of multiplication. Here we have some ambiguity of placing brackets for the eights element $((ij)l)$. However, we can choose any convenient form of placing of brackets, all the other possibilities are defined by ordinary laws of triple vector multiplication. As it was mentioned in the introduction for the orthogonal basis vectors we don't need to use different kind of brackets to denote there scalar and vector multiplication, since the product of the different basis vectors is always vector product, and the product of two same basis vectors is always scalar product.

Using the properties of standard scalar and vector products of orthogonal vectors i, j and l we find the properties of the other four, not fundamental basis elements of octonion algebra

$$\begin{aligned} (i(jl)) = -((ij)l) = -((li)j) \ , \\ ij = -ji \ , \qquad il = -li \ , \qquad jl = -lj \ , \\ (ij)^2 = (il)^2 = (jl)^2 = -1 \ , \qquad ((ij)l)^2 = 1 \ , \\ (ij)(ij)^* = (il)(il)^* = (jl)(jl)^* = 1 \ , \qquad ((ij)l)((ij)l)^* = -1 \ . \end{aligned} \tag{58}$$

Conjugation as for the quaternion case means reflection of the basis vectors, or changing of the signs of basis elements except of unit element. These obvious properties of the scalar and vector products of three fundamental basis vectors are hidden if we write abstract octonion algebra (56).

In the algebra of split-octonions there is possible to introduced the four classes of projective operators

$$D_i^\pm = \frac{1}{2}(1 \pm i) \ , \qquad D_j^\pm = \frac{1}{2}(1 \pm j) \ , \qquad D_l^\pm = \frac{1}{2}(1 \pm l) \ , \qquad D_{(ij)l}^\pm = \frac{1}{2}(1 \pm (ij)l) \ , \tag{59}$$

corresponding to the critical unit signal along the i, j, l and $(ij)l$ directions. This four classes of projective operators don't commute with each other. Independent projective operators can be only one class from them, i.e. any pair $D^\pm$ from (59) with the same label.

In the algebra we have also the four classes of Grasman numbers

$$\begin{aligned} G_{i1}^\pm = \frac{1}{2}(1 \pm i)j \ , \qquad G_{i2}^\pm = \frac{1}{2}(1 \pm i)l \ , \qquad G_{i3}^\pm = \frac{1}{2}(1 \pm i)(jl) \ , \\ G_{j1}^\pm = \frac{1}{2}(1 \pm j)i \ , \qquad G_{j2}^\pm = \frac{1}{2}(1 \pm j)l \ , \qquad G_{j3}^\pm = \frac{1}{2}(1 \pm j)(il) \ , \\ G_{l1}^\pm = \frac{1}{2}(1 \pm l)i \ , \qquad G_{l2}^\pm = \frac{1}{2}(1 \pm l)j \ , \qquad G_{l3}^\pm = \frac{1}{2}(1 \pm l)(ij) \ , \\ G_{(ij)l1}^\pm = \frac{1}{2}(1 \pm (ij)l)i \ , \quad G_{(ij)l2}^\pm = \frac{1}{2}(1 \pm (ij)l)j \ , \quad G_{(ij)l3}^\pm = \frac{1}{2}(1 \pm (ij)l)l \ . \end{aligned} \tag{60}$$

Mutually commute only the numbers G_1, G_2 and G_3 from each class with the same sign. So we have 8 independent real Grassmann algebras with the elements $G_{n1}^\pm, G_{n2}^\pm$ and $G_{n3}^\pm$, where n runs over i, j, l and $((ij)l)$.

In difference with quaternions, octonions cannot be represented by matrices with the usual multiplication lows [1]. The reason is non-associativity leading to different rules for left and right multiplications.

Using commuting projective operators and Grassmann numbers any split octonion

$$O = A + Bi + Cj + D(ij) + El + F(il) + G(jl) + H(i(jl)) \ , \tag{61}$$

where A, B, C, D, E, F, G and H are some real numbers, could be written in the form

$$\begin{aligned} O = D^+ \left[(A+B) + (C+D)G_1^+ + (E+F)G_2^+ + (G+H)G_3^+\right] + \\ + D^- \left[(A-B) + (C-D)G_1^- + (E-F)G_2^- + (G-H)G_3^-\right] \ . \end{aligned} \tag{62}$$

Because of symmetry between i, j and l using (59) and (60) we can find 3 different representation of (62).

Similar to quaternion case here we also have two fundamental constants c and $\hbar$ corresponding to the two types of critical signals $D^\pm$ and $G^\pm$ and any octonion can be written in the form

$$O = ct + xi + yj + zl + \frac{\hbar}{P_z}(ij) + \frac{\hbar}{P_y}(il) + \frac{\hbar}{P_x}(jl) + \omega(i(jl)) \ , \tag{63}$$

where ω is some quantity with the dimension of length. Here we had used similar to (50) relations to extract Planck's constants form the components of (61) and thus introduced the quantities P_x, P_y and P_z with the dimensions of momentums.

Two terms in formula (62) with different signs of zero divisors correspond to direct and backward signals in the direction of one of the axis i, j, or l. However, as for the quaternions different class of projective operators D_i, D_j and D_l are not commute, what is the analog of non-commutivity of coordinates in the models [13].

Specific for the representation (62) also is the existence of three zero divisors forming full Grassmann algebras with three elements for any direction. So for octonions we have four type of commuting critical signals. One is rotation in $t-x$ planes and corresponds to the constant c. Two are different rotations in $x-P$ planes and gives uncertainty principles similar to (49) containing $\hbar$ and the last one is rotation in $P-\omega$ plane.

Requiring to have the positively defined norm for octonion (63), in addition to (48) and (49), we can write the extra relations

$$\frac{\Delta\omega}{c\Delta t} < 1 \ , \qquad \frac{\Delta\omega\Delta P}{\hbar} < 1 \ . \tag{64}$$

If we extract two fundamental constants c and $\hbar$ from ω have the quantity with the dimension of energy

$$E = \frac{c\hbar}{\omega} \ . \tag{65}$$

Then norm of the octonion (63)

$$N = OO^* = c^2t^2 + \frac{\hbar^2}{P_x^2} + \frac{\hbar^2}{P_y^2} + \frac{\hbar^2}{P_z^2} - x^2 - y^2 - z^2 - \frac{c^2\hbar^2}{E^2} \ , \tag{66}$$

will be similar to the distance in some kind of phase space.

In spite that in the split-octonion algebra there are four real basis vectors i, j, l and $((ij)l)$ we can consider to have only the three space dimensions, since we have only the three independent rotation operators in $(i-j)$, $(i-l)$ and $(j-l)$ planes respectively

$$R^{\pm}_{ij} = \frac{1}{\sqrt{2}}[1 \pm (ij)] \ , \qquad R^{\pm}_{il} = \frac{1}{\sqrt{2}}[1 \pm (il)] \ , \qquad R^{\pm}_{jl} = \frac{1}{\sqrt{2}}[1 \pm (jl)] \ , \qquad (67)$$

with the property $R^+R^- = 1$. Another fact is that in formulae (59) and (60) there is no symmetry between zero divisors along the directions i, j, l and along $((ij)l)$. The reason is non-associativity of octonions resulting not unique answer of the product of $D^{\pm}_{(ij)l}$ with $G^{\pm}_{(ij)l}$. It means that commuting this operators or not is depends on the order of multiplication of basis elements. As the result we can't write decomposition of a octonion (62) using $D^{\pm}_{(ij)l}$ and $G^{\pm}_{(ij)l}$ in the unique way.

The basis element $((ij)l)$ all time appears when we study rotations of i, j and l. For example, the rotation of basis element i around itself (in $(j-l)$ plane) on the 90^0 by the operator (67) gives the vector $((jl)i)$. After the rotation on π we receive $-i$ and only after the full rotation on 2π we come back to i, since

$$R^+_{jl} i R^-_{jl} = -i(jl) = (ij)l \ , \qquad R^+_{jl}((ij)l)R^-_{jl} = -i \ . \qquad (68)$$

Similar situation is for the rotations around j and l.

So fourth vector direction $((ij)l)$ corresponds to the multi-valuence of the components of octonion and non-associativity, which corresponds of non-visibility of fourth direction, introduces fundamental uncertainty in physics.

Non-associativity of octonions follows from the property of triple vector products and appears when we consider multiplication of three different basis elements of octonion necessarily including the eighth basis element $((ij)l)$ (in our representation) as one of the term. For example, we have

$$j\,(l\ ((ij)l)) = j\,(-(ij)) = i \ , \qquad (69)$$
$$(jl)\ ((ij)l) = (jl)((jl)i) = -i \ . \qquad (70)$$

Non-associativity of octonions is difficult to understand only studing the properties of basis elements. For example, if we introduce so-called open multiplication laws for the basis elements [15], i.e. we don't place the brackets, and formally using only the anti-commute properties of the basis elements we found

$$ijl = lij \ . \qquad (71)$$

But anti-commuting comes from the property of vector products and its impossible to use separately. If we put the brackets from the properties of vector products we shall receive the another result

$$((ij)l) = -(l(ij)) \ , \qquad (72)$$

since this expression is the vector product of the vectors l and (ij).

Non-associativity follows from the fact that the basis elements of octonions $(ij), (il)$ and (jl) are pseudo-vectors and have properties of imaginary unit, i.e. there square (scalar product) is negative. Even in three dimensions ordinary vector multiplication is non-associative. For example, if **a** and **b** are two real 3-vectors, there triple vector product is not associative

$$[[\mathbf{aa}]\mathbf{b}] = 0 \ , \qquad [\mathbf{a}[\mathbf{ab}]] = -\mathbf{b} \ . \tag{73}$$

In 3-space triple vector product

$$[\mathbf{a}[\mathbf{bc}]] = (\mathbf{ac})\mathbf{b} - (\mathbf{ab})\mathbf{c} \tag{74}$$

of three orthogonal vectors (describing by the determinant of three-on-three matrix) is zero. Considering vector products in more than three dimensions there possible to construct extra vectors, normal to all three (using matrix with the more than three columns and rows). Then triple vector product of three orthogonal vectors is not zero and we shall automatically receive their non-associative properties. In particular, when we have product of three basis element, which contains the eights basis unit $((ij)l)$, for some order of multiplication of the basis elements there necessarily arises scalar product (square) of two pseudo-vectors (ij), (il), or (jl) giving the opposite sign in the result.

Octonions have a weak form of associativity called alternativity, what in the language of basis elements means that in the expressions when only two fundamental basis elements exists the placing of brackets is arbitrary, for example

$$(ii)l = i(il) \ , \qquad (li)i = l(ii) \ . \tag{75}$$

This property, following from multidimensional generalization of (74), allows us to simplify expressions containing more than three octonionic unit elements. Unit vectors, which after commutating appears to be the neighbors, can be take out from the brackets and using scalar product to identify there square with the unit of algebra.

At the end we want to note, that in general relations (48), (49) and (64) are not to be satisfied separately. Positively definition of the norm (66) gives only one relation containing all components of octonion. The Lorentz factor

$$\gamma = \sqrt{1 - \frac{v^2}{c^2} - \hbar^2 \left(\frac{A^2}{E^2} - \frac{1}{c^2}\frac{F^2}{P^2} \right)} \ , \tag{76}$$

corresponding to general bust in (4+4)-space. Here the quantities A and F have the dimensions of work and force respectively. The formula (76) contains the extra terms, which change the 4-dimensional dispersion relation similar as it is for double-special relativity theory [16].

6 Conclusion

A physical applicability of normed split-algebras, such as hyperbolic numbers, split-quaternions and split-octonions was considered. We argue that the observable geometry can be described by the algebra of split-octonions, which is naturally equipped by zero divisors (the elements of split-algebras corresponding to zero norm). In such a picture physical phenomena are described by the ordinary elements of chosen algebra, while the zero divisors give raise the coordinatization and two fundamental constants, namely velocity of light and Planck constant. It turns to be possible that uncertainty principle appears from the condition of positively defined norm, and has the same geometrical meaning as the existence of the maximal value of speed. The property of non-associativity of octonions could correspond to the appearance of fundamental probabilities in physics. Grassmann elements and a non-commutativity of space coordinates, which are widely used in various physical theories, appear naturally in our approach.

We do not yet introduce any physical fields or equations. We had only shown that the proper algebra could describe geometry and introduce some necessary characteristics of future particle physics models.

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