Nonrenormalization theorems for $N = 2$ Super Yang-Mills

V.E.R. Lemes[(a)], N. Maggiore[(b)], M.S. Sarandy[(c)], S.P. Sorella[(a)], A. Tanzini[(d)] and O.S. Ventura[(a)]

[(a)] *UERJ, Universidade do Estado do Rio de Janeiro, Rua São Francisco Xavier 524, 20550-013 Maracanã, Rio de Janeiro, Brazil.*

[(b)] *Dipartimento di Fisica, Universitá di Genova, Via Dodecaneso 33 - I-16146 Genova, Italy - and INFN, Sezione di Genova.*

[(c)] *Centro Brasileiro de Pesquisas Físicas (CBPF), Rua Dr. Xavier Sigaud 150 - 22290-180 - Rio de Janeiro - RJ - Brazil.*

[(d)] *Dipartimento di Fisica, Universitá di Roma "Tor Vergata ", Via della Ricerca Scientifica 1, 00173 Roma, Italy.*

It is known that supersymmetry affects in a deep way the ultraviolet behaviour of quantum field theories leading, in the case of massless four-dimensional $N = 2$ and $N = 4$ super Yang-Mills (SYM), to a set of remarkable nonrenormalization theorems for the corresponding β_g-functions [1].

In the case of $N = 2$, β_g receives only one-loop contributions, while in $N = 4$ it vanishes to all orders of perturbation theory. The proof of these theorems is reviewed within the algebraic BRST approach [2,3].

The quantum properties of $N = 2$ and $N = 4$ SYM can be derived by making use of the twisting procedure, which allows to replace the spinor indices of supersymmetry $(\alpha, \dot{\alpha})$ with Lorentz vector indices. The physical content of the theory is left unchanged, for the twist is a linear change of variables, and the twisted version is perturbatively indistinguishable from the original one.

The proofs of the nonrenormalization theorems rely on the key observation that the action of both $N = 2$ and $N = 4$ SYM is obtained from the local gauge invariant polynomial $\mathrm{Tr}\phi^2$, ϕ being the scalar field of the vector $N = 2$ multiplet. This polynomial enjoys the important property of having vanishing anomalous dimension, from which the nonrenormalization theorems for $N = 2, 4$ follow [2].

The global symmetry group of $N = 2$ in four dimensional flat euclidean space-time is $SU(2)_L \times SU(2)_R \times SU(2)_I \times U(1)_{\mathcal{R}}$, where $SU(2)_L \times SU(2)_R$ is the rotation group and $SU(2)_I$ and $U(1)_{\mathcal{R}}$ are the symmetry groups corresponding to the internal $SU(2)$-transformations and to the $\mathcal{R}$-symmetry. The twisting procedure consists of replacing the rotation group by $SU(2)'_L \times SU(2)_R$, where $SU(2)'_L$ is the diagonal sum of $SU(2)_L$ and $SU(2)_I$, allowing to identify the internal indices with the spinor indices. The fields of the $N = 2$ vector multiplet in the WZ gauge are given by $(A_\mu, \psi^i_\alpha, \overline{\psi}^i_{\dot{\alpha}}, \phi, \overline{\phi})$, where $\psi^i_\alpha, \overline{\psi}^i_{\dot{\alpha}}$ are Weyl spinors with $i = 1, 2$ being the internal index of the fundamental representation of $SU(2)_I$, and $\phi, \overline{\phi}$ are complex scalars; all fields belonging to the adjoint representation of the gauge group. Under the twisted group, these fields decompose as [2] $A_\mu \to A_\mu, \quad (\phi, \overline{\phi}) \to (\phi, \overline{\phi}), \quad \psi^i_\alpha \to (\eta,\ \chi_{\mu\nu}), \quad \overline{\psi}^i_{\dot{\alpha}} \to \psi_\mu$.

The fields $(\psi_\mu, \chi_{\mu\nu}, \eta)$ anticommute due to their spinor nature, and $\chi_{\mu\nu}$ is a self-dual tensor field. The action of $N=2$ SYM in terms of the twisted variables is [2]

$$S^{N=2} = \frac{1}{g^2}\mathrm{Tr}\int d^4x \left(\frac{1}{2}F^+_{\mu\nu}F^{+\mu\nu} + \frac{1}{2}\overline{\phi}\left\{\psi^\mu, \psi_\mu\right\} - \chi^{\mu\nu}(D_\mu\psi_\nu - D_\nu\psi_\mu)^+ \right.$$
$$\left. +\eta D_\mu\psi^\mu - \frac{1}{2}\overline{\phi}D_\mu D^\mu\phi - \frac{1}{2}\phi\left\{\chi^{\mu\nu}, \chi_{\mu\nu}\right\} - \frac{1}{8}\left[\phi, \eta\right]\eta - \frac{1}{32}\left[\phi, \overline{\phi}\right]\left[\phi, \overline{\phi}\right] \right),$$

where g is the coupling constant and $F^+_{\mu\nu} = F_{\mu\nu} + \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$.

Concerning the supersymmetry generators $(\delta^\alpha_i, \overline{\delta}^i_{\dot\alpha})$ of the $N=2$ superalgebra, the twisting procedure gives rise to the twisted generators: a scalar δ, a vector δ_μ and a self-dual tensor $\delta_{\mu\nu}$, which leave the action invariant. Only the generators δ and δ_μ are relevant for the nonrenormalization theorems [2].

The theory is quantized by collecting all the generators (s, δ, δ_μ) into an extended operator $Q \equiv s + \omega\delta + \varepsilon^\mu\delta_\mu$, s being the BRST operator of the gauge transformations and ω and ε^μ are global ghosts.

¿From the Batalin-Vilkovisky procedure, the complete gauge-fixed action is $\Sigma = S^{N=2} + S_{\mathrm{gf}} + S_{\mathrm{ext}}$, where S_{gf} is the gauge-fixing term in the Landau gauge and S_{ext} contains the coupling of the non-linear transformations $Q\Phi_i$ to antifields Φ^*_i [2]. The complete action Σ satisfies the Slavnov-Taylor identity

$$\mathcal{S}(\Sigma) \equiv \mathrm{Tr}\int d^4x \left(\frac{\delta\Sigma}{\delta\Phi^*_i}\frac{\delta\Sigma}{\delta\Phi_i} + b\frac{\delta\Sigma}{\delta\bar{c}} + \omega\varepsilon^\mu\partial_\mu\bar{c}\frac{\delta\Sigma}{\delta b} \right) = \omega\varepsilon^\mu\Delta^{\mathrm{cl}}_\mu,$$

where $\bar{c}$ and b are respectively the antighost and the Lagrange multiplier, and Δ^{cl}_μ is an integrated local polynomial which is linear in the quantum fields and hence not affected by the quantum corrections [2].

¿From the Slavnov-Taylor identity it follows that the linearized operator $\mathcal{S}_\Sigma$

$$\mathcal{S}_\Sigma = \mathrm{Tr}\int d^4x \left(\frac{\delta\Sigma}{\delta\Phi^*_i}\frac{\delta}{\delta\Phi_i} + \frac{\delta\Sigma}{\delta\Phi_i}\frac{\delta}{\delta\Phi^*_i} + b\frac{\delta}{\delta\bar{c}} + \omega\varepsilon^\mu\partial_\mu\bar{c}\frac{\delta}{\delta b} \right)$$

is nilpotent modulo a total space-time derivative, namely $\mathcal{S}_\Sigma\mathcal{S}_\Sigma = \omega\varepsilon^\mu\partial_\mu$.

Introducing the operator $\mathcal{W}_\mu$

$$\mathcal{W}_\mu \equiv \frac{1}{\omega}\left[\frac{\partial}{\partial\varepsilon^\mu}, \mathcal{S}_\Sigma\right], \qquad \{\mathcal{W}_\mu, \mathcal{S}_\Sigma\} = \partial_\mu, \qquad \{\mathcal{W}_\mu, \mathcal{W}_\nu\} = 0,$$

it follows that the complete action Σ is obtained by repeated applications of $\mathcal{W}_\mu$ on $\mathrm{Tr}\phi^2$, according to the formula [2]

$$\frac{\partial\Sigma}{\partial g} = \frac{1}{3g^3}\epsilon^{\mu\nu\rho\sigma}\mathcal{W}_\mu\mathcal{W}_\nu\mathcal{W}_\rho\mathcal{W}_\sigma\int d^4x\ \mathrm{Tr}\phi^2 \ + \ \mathcal{S}_\Sigma\Xi^{-1},$$

for some irrelevant trivial cocycle $\mathcal{S}_\Sigma\Xi^{-1}$. This equation is of fundamental importance for the nonrenormalization theorems. It states that the bulk of the theory is the composite operator $\mathrm{Tr}\phi^2$, which contains all the information on the ultraviolet behaviour. This is a consequence of the fact that the operator $\mathrm{Tr}\phi^2$ obeys a Callan-Symanzik equation with vanishing anomalous dimension [2]. It follows thus that the β_g-function obeys the differential equation [2]

$$g\frac{\partial\beta_g}{\partial g} - 3\beta_g = 0 \qquad \Rightarrow \qquad \beta_g = k\,g^3, \quad (k\ \text{constant}).$$

This equation expresses the celebrated nonrenormalization theorem of $N = 2$ SYM, stating that, to all orders of perturbation theory, the β_g-function has only one-loop contributions.

The above procedure has been successfully extended to the presence of matter, with the hypermultiplets belonging to a generic representation of the gauge group. Also in that case, a complete twisted formulation of the theory, namely action, fields and symmetries, has been given.

A direct one-loop computation of k yields [4]

$$k = -\frac{1}{8\pi^2}\left(C_1 - HC_2\right) \ ,$$

where C_1 and C_2 are respectively the Casimir invariants of the representations of gauge and matter $N = 2$ multiplets, and H is the number of matter multiplets. The $N = 4$ is recovered as a particular case of $N = 2$, with matter in the adjoint representation of the gauge group and $H = 1$. It remains true that the fully quantized twisted action of $N = 4$ is related to the invariant polynomial $\mathrm{Tr}\phi^2$ [3]. This implies that the beta function of $N = 4$ can be at most of one-loop order. Therefore, from $H = 1$ and $C_1 = C_2$, it vanishes to all orders of perturbation theory.

BIBLIOGRAPHY.

[1] P. West, "Introduction to Supersymmetry and Supergravity", Singapore, *Singapore: World Scientific* (1990).

[2] A. Blasi, V.E.R. Lemes, N. Maggiore, S.P.Sorella, A. Tanzini, O.S. Ventura and L.C.Q. Vilar, JHEP 0005 (2000) 419 (hep-th/0004048).

[3] V.E.R. Lemes, M.S. Sarandy, S.P. Sorella, A. Tanzini and O.S. Ventura, (hep-th/0011001).

[4] S. Weinberg, "The Quantum Theory of Fields, Vol.III Supersymmetry", Cambridge Univ. Press, 2000.