

Perturbative Equivalent Theorem in q -Deformed Dynamics

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Abstract

Corresponding to two ways of realizing the q -deformed Heisenberg algebra by the undeformed variables there are two q -perturbative Hamiltonians with the additional momentum-dependent interactions, one originates from the perturbative expansion of the potential, the other originates from that of the kinetic energy term. At the level of operators, these two q -perturbative Hamiltonians are different. In order to establish a reliable foundation of the perturbative calculations in q -deformed dynamics, except examples of the harmonic-oscillator and the Morse potential demonstrated before, the general q -perturbative equivalent theorem is demonstrated, which states that for any regular potential which is singularity free the expectation values of two q -perturbative Hamiltonians in the eigenstates of the undeformed Hamiltonian are equivalent. For the q -deformed “free” particle case, the perturbative Hamiltonian originated from the kinetic energy term still keeps its general expression, but it does not lead to energy shift.

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The ordinary quantum mechanics, which is based on the Heisenberg commutation relation, has obtained every successes from the space scale 10^{-8} cm to 10^{-18} cm. There is a possibility that the Heisenberg commutation relation at short distances, say, much smaller than 10^{-18} cm, may need generalizing. In search for such possibility the q -deformed Heisenberg algebra is a candidate. In literature different frameworks of q -deformed quantum mechanics were established [1–18]. The framework of the q -deformed Heisenberg algebra developed in Refs. [2, 4] shows clear physical content: its relation to the corresponding q -deformed boson commutation relations and the limiting process of the q -deformed harmonic oscillator to the undeformed one are clear. In this framework the new features of q -deformed quantum mechanics are explored. The q -deformed uncertainty relation shows essential deviation from the Heisenberg one [15, 17]: the ordinary minimal uncertainty relation is undercut. A non-perturbative feature of the q -deformed Schrödinger equation is that the energy spectrum exhibits an exponential structure [3, 4, 16], which qualitatively explains the pattern of quark and lepton masses [16]. The perturbative expansion of the q -deformed Hamiltonian possesses a complex structure, which amounts to some additional momentum-dependent interaction [2, 4, 16].

Recent studies of the perturbative aspects of the q -deformed Schrödinger equation in the above framework explored interesting characteristics. Corresponding to two ways of realizing the q -deformed Heisenberg algebra by the undeformed variables which are related by the canonical transformation there are two q -perturbative Hamiltonians, one originates from the perturbative expansion of the potential, the other originates from that of the kinetic energy term. At the level of operators, these two q -perturbative Hamiltonians are different. Studies in the harmonic-oscillator potential and the Morse potential showed that [18] expectation values of these two q -perturbative Hamiltonians in the eigenstates of the undeformed Hamiltonian are equivalent. In the example of the harmonic-oscillator potential, two q -perturbative Hamiltonians only differ by terms $a^m a^{\dagger n}$ with $m \neq n$, thus lead to the same energy shifts $\Delta E_n^{(q)} = -\frac{f^2 \omega}{48} (4n^3 + 6n^2 + 20n + 9)$, where a and a^\dagger are the annihilation operator and the frequency of the harmonic-oscillator.

In order to establish a reliable foundation of the perturbative calculations in q -deformed dynamics one should clarify: whether such equivalence explored in these two examples

holds for the general case ? In this letter we demonstrate that for any regular potential which is singularity free the expectation values of these two q -perturbative Hamiltonians in the eigenstates of the undeformed Hamiltonian are equal. This is summarized as the perturbative equivalent theorem in q -deformed dynamics. The equivalent theorem means that at the level of operators, these two perturbative Hamiltonians are different, however, they differ only by a quantity whose expectation value in the undeformed stationary states vanishes. As a self-consistent check we consider the q -deformed “free” particle. In this case the perturbative Hamiltonian originated from the potential vanishes, but the other one originated from the kinetic energy term still keeps its general expression. The calculation confirms that in this case the energy shift is zero.

In the following, we first review the necessary background of q -deformed quantum mechanics. In terms of q -deformed phase space variables — the position operator \mathbf{X} and the momentum operator \mathbf{P} , the following q -deformed Heisenberg algebra has been developed [2, 4]:

$$q^{1/2}XP - q^{-1/2}PX = iU, \quad UX = q^{-1}XU, \quad UP = qPU, \quad (1)$$

where \mathbf{X} and \mathbf{P} are hermitian and \mathbf{U} is unitary: $\mathbf{X}^\dagger = \mathbf{X}$, $\mathbf{P}^\dagger = \mathbf{P}$, $\mathbf{U}^\dagger = \mathbf{U}^{-1}$. Compared to the Heisenberg algebra the operator \mathbf{U} is a new member, called the scaling operator. The necessity of introducing the operator \mathbf{U} is as follows.

The simultaneous hermitian of \mathbf{X} and \mathbf{P} is a delicate point in q -deformed dynamics. The definition of the algebra (1) is based on the definition of the hermitian momentum operator \mathbf{P} . However, if \mathbf{X} is assumed to be a hermitian operator in a Hilbert space, the usual quantization rule $\mathbf{P} \rightarrow -i\partial_X$ does not yield a hermitian momentum operator. A hermitian momentum operator \mathbf{P} is related to ∂_X and \mathbf{X} in a nonlinear way by introducing a scaling operator \mathbf{U} [4]

$$U^{-1} \equiv q^{1/2}[1 + (q - 1)X\partial_X], \quad \bar{\partial}_X \equiv -q^{-1/2}U\partial_X, \quad P \equiv -\frac{i}{2}(\partial_X - \bar{\partial}_X), \quad (2)$$

where $\bar{\partial}_X$ is the conjugate of ∂_X . The operator \mathbf{U} is introduced in the definition of the hermitian momentum, thus it closely relates to properties of dynamics and plays an essential role in q -deformed quantum mechanics. The nontrivial properties of \mathbf{U} imply that

the algebra (1) has a richer structure than the Heisenberg commutation relation. In (1) the parameter q is a fixed real number. It is important to make distinctions for different realizations of the q -algebra by different ranges of q values [19–21]. Following Refs. [2, 4] we only consider the case $q > 1$ in this paper. The reason is that such choice of the parameter q leads to consistent dynamics. In the limit $q \rightarrow 1^+$ the scaling operator U reduces to the unit operator, thus the algebra (1) reduces to the Heisenberg commutation relation.

Such defined hermitian momentum P leads to q -deformation effects, which exhibit in the dynamical equation. Eq. (2) shows that the momentum P depends non-linearly on X and ∂_X . Thus the q -deformed Schrödinger equation is difficult to treat. In this letter we demonstrate that there is a reliable foundation for its perturbative calculation.

The q -deformed phase space variables X , P and the scaling operator U can be realized in terms of two pairs of the undeformed variables [4].

(I) The variables \hat{x} , \hat{p} of the ordinary quantum mechanics, where \hat{x} , \hat{p} satisfy: $[\hat{x}, \hat{p}] = i$, $\hat{x} = \hat{x}^\dagger$, $\hat{p} = \hat{p}^\dagger$. The variables X , P and the scaling operator U are related to \hat{x} , \hat{p} by:

$$X = \frac{[\hat{z} + \frac{1}{2}]}{\hat{z} + \frac{1}{2}} \hat{x}, \quad P = \hat{p}, \quad U = q^{\hat{z}}, \quad \hat{z} = -\frac{i}{2}(\hat{x}\hat{p} + \hat{p}\hat{x}) \quad (3)$$

where $[A]$ is the q -deformation of A , defined by $[A] = (q^A - q^{-A})/(q - q^{-1})$. It is easy to check that X , P and U satisfy (1).

(II) The variables \tilde{x} and \tilde{p} of an undeformed algebra, which are obtained by a canonical transformation of \hat{x} and \hat{p} :

$$\tilde{x} = \hat{x}F^{-1}(\hat{z}), \quad \tilde{p} = F(\hat{z})\hat{p}, \quad (4)$$

where

$$F^{-1}(\hat{z}) = \frac{[\hat{z} - \frac{1}{2}]}{\hat{z} - \frac{1}{2}}. \quad (5)$$

Such defined variables \tilde{x} and \tilde{p} also satisfy the undeformed algebra: $[\tilde{x}, \tilde{p}] = i$, and $\tilde{x} = \tilde{x}^\dagger$, $\tilde{p} = \tilde{p}^\dagger$. Thus $\tilde{p} = -i\partial_{\tilde{x}}$. The q -deformed variables X , P and the scaling operator U are related to \tilde{x} and \tilde{p} as follows:

$$X = \tilde{x}, \quad P = F^{-1}(\tilde{z})\tilde{p}, \quad U = q^{\tilde{z}}, \quad \tilde{z} = -\frac{i}{2}(\tilde{x}\tilde{p} + \tilde{p}\tilde{x}), \quad (6)$$

where $F^{-1}(\tilde{z})$ defined by Eq. (5) for the variables (\tilde{x}, \tilde{p}) . From Eqs. (4)–(6) it follows that such defined \mathbf{X} , \mathbf{P} and \mathbf{U} also satisfy (1), and Eq. (6) is equivalent to Eq. (3).

The q -deformed phase space (\mathbf{X}, \mathbf{P}) governed by the q -algebra (1) is a q -deformation of the phase space (\tilde{x}, \tilde{p}) of the ordinary quantum mechanics, thus all machinery of the ordinary quantum mechanics can be applied to the q -deformed quantum mechanics. It means that dynamical equations of the quantum system are the same for the undeformed phase space variables (\tilde{x}, \tilde{p}) and for the q -deformed phase space variables (\mathbf{X}, \mathbf{P}) , that is, the q -deformed Hamiltonian with potential $V(\mathbf{X})$ is

$$H(\mathbf{X}, \mathbf{P}) = \frac{1}{2\mu} \mathbf{P}^2 + V(\mathbf{X}). \quad (7)$$

In the (\tilde{x}, \tilde{p}) system \mathbf{X} is a non-linear function of (\tilde{x}, \tilde{p}) . From (3) it follows that \mathbf{X} can be represented as

$$\mathbf{X} = i(q - q^{-1})^{-1} (q^{(\hat{z}+1/2)} - q^{-(\hat{z}+1/2)}) \hat{p}^{-1}. \quad (8)$$

Using (8) it is convenient to discuss the perturbative expansion of \mathbf{X} . In view of every success of the ordinary quantum mechanics the effects of q -deformation must be extremely small. So we can let $q = e^f = 1 + f$, with $0 < f \ll 1$. To the order f^2 of the perturbative expansion, \mathbf{X} reduces to

$$\mathbf{X} = \hat{x} + f^2 g(\hat{x}, \hat{p}), \quad g(\hat{x}, \hat{p}) = -\frac{1}{6} (1 + \hat{x} \hat{p} \hat{x} \hat{p}) \hat{x}. \quad (9)$$

For any regular potential $V(\mathbf{X})$, which is singularity free, to the order f^2 , such potentials can be expressed by the undeformed variables (\tilde{x}, \tilde{p}) as

$$V(\mathbf{X}) = V(\hat{x}) + \hat{H}_I^{(q)}(\hat{x}, \hat{p}), \quad (10)$$

with the perturbation

$$\hat{H}_I^{(q)}(\hat{x}, \hat{p}) = f^2 \sum_{k=1}^{\infty} \frac{V^{(k)}(0)}{k!} \left(\sum_{i=0}^{k-1} \hat{x}^{(k-1)-i} g(\hat{x}, \hat{p}) \hat{x}^i \right), \quad (11)$$

where $V^{(k)}(0)$ is the k -th derivative of $V(\hat{x})$ at $\hat{x} = 0$. In (11) the ordering between the non-commutative quantities \tilde{x} and $g(\hat{x}, \hat{p})$ is carefully considered. Substituting for $g(\hat{x}, \hat{p})$ and summing over i and k , the above result can be expressed as

$$\hat{H}_I^{(q)}(\hat{x}, \hat{p}) = \frac{f^2}{6} \{ \hat{x}^3 V'(\hat{x}) \partial_{\hat{x}}^2 + [\hat{x}^3 V''(\hat{x}) + 3\hat{x}^2 V'(\hat{x})] \partial_{\hat{x}} + \frac{1}{3} \hat{x}^3 V'''(\hat{x}) + \frac{3}{2} \hat{x}^2 V''(\hat{x}) \}. \quad (12)$$

In the (\tilde{x}, \tilde{p}) system P is a non-linear function of (\tilde{x}, \tilde{p}) . Using (6), to the order f^2 , the perturbative expansions of the momentum P and the kinetic energy $P^2/(2\mu)$ read

$$P = \tilde{p} + f^2 h(\tilde{x}, \tilde{p}), \quad h(\tilde{x}, \tilde{p}) = -\frac{1}{6}(1 + \tilde{p}\tilde{x}\tilde{p})\tilde{p}, \quad (13)$$

$$\frac{1}{2\mu}P^2 = \frac{1}{2\mu}\tilde{p}^2 + \tilde{H}_I^{(q)}(\tilde{x}, \tilde{p}), \quad (14)$$

with

$$\begin{aligned} \tilde{H}_I^{(q)}(\tilde{x}, \tilde{p}) &= \frac{1}{2\mu}f^2[\tilde{p}h(\tilde{x}, \tilde{p}) + h(\tilde{x}, \tilde{p})\tilde{p}] \\ &= -\frac{1}{12\mu}f^2[2\tilde{x}^2\partial_{\tilde{x}}^4 + 8\tilde{x}\partial_{\tilde{x}}^3 + 3\partial_{\tilde{x}}^2] \end{aligned} \quad (15)$$

Eqs. (14) and (15) show that in the (\tilde{x}, \tilde{p}) system the perturbative contribution comes from the kinetic-energy term, which is different from Eq. (11), where in the (\tilde{x}, \tilde{p}) system the perturbative contribution comes from the potential.

From Eqs. (7),(9), (10), and (12) - (15), it follows that the perturbative expansion of the q -deformed Hamiltonian $H(X, P)$ can be written down in the (\hat{x}, \hat{p}) system or in the (\tilde{x}, \tilde{p}) system. In the (\hat{x}, \hat{p}) system

$$H(X(\hat{x}, \hat{p}), P(\hat{x}, \hat{p})) = H_{\text{un}}(\hat{x}, \hat{p}) + \hat{H}_I^{(q)}(\hat{x}, \hat{p}). \quad (16)$$

In the (\tilde{x}, \tilde{p}) system

$$H(X(\tilde{x}, \tilde{p}), P(\tilde{x}, \tilde{p})) = H_{\text{un}}(\tilde{x}, \tilde{p}) + \tilde{H}_I^{(q)}(\tilde{x}, \tilde{p}). \quad (17)$$

In the above

$$H_{\text{un}}(\xi, \kappa) = \frac{1}{2\mu}\kappa^2 + V(\xi) \quad (18)$$

is the corresponding undeformed Hamiltonian in the (ξ, κ) system, where (ξ, κ) represents (\hat{x}, \hat{p}) or (\tilde{x}, \tilde{p}) .

The above two perturbative Hamiltonian $\hat{H}_I^{(q)}(\hat{x}, \hat{p})$ and $\tilde{H}_I^{(q)}(\tilde{x}, \tilde{p})$ originate, separately, from the perturbative expansions of the potential and the kinetic energy. At the level of operator they are different. It is interesting to note that their contributions to the perturbative shifts of energy spectrum for the undeformed Hamiltonian in the (\hat{x}, \hat{p}) system and the (\tilde{x}, \tilde{p}) system are the same. This is summarized in the following theorem.

Perturbative Equivalent Theorem: For any regular potential which is singularity free the expectation values of two q-perturbative Hamiltonians $\hat{H}_I^{(q)}(\hat{x}, \hat{p})$ and $\tilde{H}_I^{(q)}(\tilde{x}, \tilde{p})$ defined, separately, by Eqs. (12) and (15), in the eigenstates of the undeformed Hamiltonian are equivalent.

Suppose that the Schrödinger equation for the undeformed system is solved in the configuration space ξ_0 , i.e., the manifold of the spectrum ξ_0 of ξ :

$$H_{\text{un}}(\xi, \kappa)\psi_n^{(0)}(\xi_0) = E_n^{(\text{un})}\psi_n^{(0)}(\xi_0). \quad (19)$$

The structure of $\psi_n^{(0)}(\hat{x}_0)$ in the configuration space \hat{x}_0 and the structure of $\psi_n^{(0)}(\tilde{x}_0)$ in the configuration space \tilde{x}_0 are the same.

The expectation values of $\hat{H}_I^{(q)}$ and $\tilde{H}_I^{(q)}$ in the undeformed stationary states $\psi_n^{(0)}$ are

$$\begin{aligned} \Delta \hat{E}_n^{(q)} &= \int d\hat{x}_0 \psi_n^{(0)*}(\hat{x}_0) \hat{H}_I^{(q)}(\hat{x}_0, -i\partial_{\hat{x}_0}) \psi_n^{(0)}(\hat{x}_0) \\ \Delta \tilde{E}_n^{(q)} &= \int d\tilde{x}_0 \psi_n^{(0)*}(\tilde{x}_0) \tilde{H}_I^{(q)}(\tilde{x}_0, -i\partial_{\tilde{x}_0}) \psi_n^{(0)}(\tilde{x}_0), \end{aligned} \quad (20)$$

From Eqs. (12) and (15), using the Schrödinger equation and integrating Eq. (20) by parts, it follows that Eq. (20) can be rewritten as

$$\Delta \hat{E}_n^{(q)} = \frac{f^2}{6} \int_{-\infty}^{\infty} d\hat{x}_0 \psi_n^{(0)*}(\hat{x}_0) \left\{ V(\hat{x}_0) \left[1 - 4\mu \hat{x}_0^2 (V(\hat{x}_0) - E_n^{(\text{un})}) \right] - \frac{2}{3} \mu E_n^{(\text{un})} \hat{x}_0^3 V'(\hat{x}_0) \right\} \psi_n^{(0)}(\hat{x}_0). \quad (21)$$

$$\Delta \tilde{E}_n^{(q)} = \frac{f^2}{6} \int_{-\infty}^{\infty} d\tilde{x}_0 \psi_n^{(0)*}(\tilde{x}_0) \left(V(\tilde{x}_0) - E_n^{(\text{un})} \right) \left[1 - 4\mu \tilde{x}_0^2 (V(\tilde{x}_0) - E_n^{(\text{un})}) \right] \psi_n^{(0)}(\tilde{x}_0). \quad (22)$$

Using the Schrödinger equation again, because the structure of $\psi_n^{(0)}(\hat{x}_0)$ in the configuration space \hat{x}_0 and the structure of $\psi_n^{(0)}(\tilde{x}_0)$ in the configuration space \tilde{x}_0 are the same, the difference of $\Delta \hat{E}_n^{(q)}$ and $\Delta \tilde{E}_n^{(q)}$ is given by

$$\Delta \hat{E}_n^{(q)} - \Delta \tilde{E}_n^{(q)} = \frac{f^2}{6} E_n^{(\text{un})} \int_{-\infty}^{\infty} dx \psi_n^{(0)*}(x) \left[1 - 2x^2 \partial_x^2 - \frac{2}{3} x^3 \mu V'(x) \right] \psi_n^{(0)}(x). \quad (23)$$

In the undeformed stationary states $|\psi^{(0)}\rangle$ we have

$$i \frac{d}{dt} \langle \psi^{(0)} | \hat{x}^m \hat{p}^n | \psi^{(0)} \rangle = \langle \psi^{(0)} | \left[\hat{x}^m \hat{p}^n, \frac{1}{2\mu} \hat{p}^2 + V(\hat{x}) \right] | \psi^{(0)} \rangle = 0. \quad (24)$$

From Eq. (24) for the cases of $m=3$, $n=1$ and $m=2$, $n=0$ it follows that

$$\langle \psi^{(0)} | \hat{x}^3 V' - \frac{3}{\mu} \hat{x}^2 \hat{p}^2 + \frac{3i}{\mu} \hat{x} \hat{p} | \psi^{(0)} \rangle = 0, \quad \langle \psi^{(0)} | 1 + 2i \hat{x} \hat{p} | \psi^{(0)} \rangle = 0.$$

Putting these two equations in Eq. (23), it shows $\Delta \tilde{E}_n^{(q)} - \Delta \tilde{E}_n^{(q)} = 0$.

As a consistent check of the equivalent theorem we consider the q -deformed “free” particle described by the Hamiltonian $H(X, P) = \frac{1}{2\mu}P^2$. In this case $\tilde{H}_I^{(q)}(\hat{x}, \hat{p}) = 0$, but $\tilde{H}_I^{(q)}(\tilde{x}, \tilde{p})$ is still described by Eq. (15). The question is whether in this case $\Delta \tilde{E}_n^{(q)} = 0$? In the eigenstate $|\psi_p^{(0)}\rangle$ of the undeformed free Hamiltonian $H_{\text{un}}(\tilde{x}, \tilde{p}) = \frac{1}{2\mu}\tilde{p}^2$, from Eq. (24) for the cases of $m = n = 3$ and $m = n = 2$ it follows that

$$\langle \psi_p^{(0)} | i\tilde{x}^2\tilde{p}^4 + \tilde{x}\tilde{p}^3 | \psi_p^{(0)} \rangle = 0, \quad \langle \psi_p^{(0)} | 2i\tilde{x}\tilde{p}^3 + \tilde{p}^2 | \psi_p^{(0)} \rangle = 0.$$

Putting these results in Eq. (15), we obtain $\Delta \tilde{E}_n^{(q)} = \langle \psi_p^{(0)} | \tilde{H}_I^{(q)}(\tilde{x}, \tilde{p}) | \psi_p^{(0)} \rangle = 0$.

It should be pointed out that if q -deformed quantum mechanics is a realistic physical theory, its effects mainly manifest at very short distances much smaller than 10^{-18} cm; its correction to the ordinary quantum mechanics must be extremely small in the energy range of nowadays experiments, which means that the parameter q must be very close to one. So the perturbative investigation of q -deformed dynamics is meaningful, which shows the clear indication of q -deformed modifications to the ordinary quantum mechanics, and in some interesting cases, for example in the q -squeezed state [17], may provide some evidence about such effects to nowadays experiments.

The equivalent theorem establishes a reliable foundation for the perturbative calculations in q -deformed dynamics. Base on the equivalent theorem we can use any one of two q -perturbative Hamiltonians to calculate the energy shifts. For systems with complicated potentials, it is convenient to calculate the q -perturbative shifts of the energy spectrum in the (\tilde{x}, \tilde{p}) system.

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