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On Fractional Instanton Numbers in Six Dimensional Heterotic $E_8 \times E_8$ Orbifolds

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Abstract

We derive the precise relation between level matching condition and fractional instanton numbers in six dimensional, abelian and supersymmetric orbifolds of $E_8 \times E_8$ heterotic string theory. The fractional part of the two E_8 instanton numbers is explicitly calculated in terms of the gauge twist. This relation is then used to show that the classification of these orbifolds can be given in terms of flat bundles away from the orbifold singularities under the only constraint that the sum of the fractional parts of the gauge instanton numbers match the fractional part of the gravitational instanton number locally at every fixed point. This directly carries over to M-theory on $S^1/\mathbb{Z}_2$.

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1 Introduction and Summary

Since the $E_8 \times E_8$ heterotic string theory has been related to M-theory on $S^1/\mathbb{Z}_2$ for the first time [1, 2] (see also [3]) some effort has been made to describe heterotic orbifolds in this setting [4, 5, 6, 7]. However, in the phenomenologically interesting case of compactification to four dimensions our understanding is still extremely limited (see [8]) and even in the much simpler case of six noncompact dimensions only some models have been treated successfully, whereas a large class of models still lacks a fully satisfying description [6].

In the present paper we focus on symmetric, abelian and perturbative orbifolds of the $E_8 \times E_8$ heterotic string in six dimensions preserving 8 supersymmetries as described in [9, 10] (without discrete torsion).

At first, we consider a single fixed point in an arbitrary model located at the origin of $\mathbb{C}^2$. Associated to it is a generator (r, γ) of $\mathbb{Z}_N$ consisting of a rotation r acting like $\exp(2\pi i \Phi_i)$ on the complex coordinate Z^i and a gauge shift γ acting like $\exp(2\pi i \beta_1^I H_I^1) \exp(2\pi i \beta_2^I H_I^2)$ where the $H_I^{1,2}$ are the eight generators of the Cartan subalgebra in the adjoint representation of $E_8^{(1,2)}$. The H_I will be normalized such that the E_8 lattice Λ_8 is given by the vectors $p^I = (n_1, \ldots, n_8)$ and $p^I = (\frac{1}{2} + n_1, \ldots, \frac{1}{2} + n_8)$ with $n_i \in \mathbb{Z}$ and $\sum_i n_i \in 2\mathbb{Z}$. This implies that $q_{1,2}^I = N\beta_{1,2}^I$ is a lattice vector.

Consistency of the model requires the so called level matching condition

$$\Phi^2 = \beta_1^2 + \beta_2^2 \mod \frac{2}{N} \tag{1.1}$$

to be satisfied [10, 15]. It has long been suspected [11, 14][2], that this condition corresponds to the well known relation[3]

$$\operatorname{tr} R^2 = \frac{1}{30} \operatorname{tr} F_1^2 + \frac{1}{30} \operatorname{tr} F_2^2 \tag{1.2}$$

required by the Green-Schwarz mechanism [16] ($\mathrm{d}H = \operatorname{tr} R^2 - 1/30 \operatorname{tr} F^2$) and the perturbativity of the orbifold ($H = 0$). However, for example by inspecting the different $\mathbb{Z}_3$ orbifolds by trial and error, naive application leads to inconsistencies.

Our main result, to be shown in section 2, is that for an E_8 bundle that corresponds to a shrunken instanton on a $\mathbb{Z}_N$ orbifold singularity the fractional part of the instanton

[2] In case of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ string theory such issues have been analyzed in [12, 13]. See also [14].

[3] All traces for the gauge group will be in the adjoint representation.

number is given by[4]

$$I = -\frac{1}{60}\frac{1}{8\pi^2}\int_U \operatorname{tr} F^2 = \frac{N}{2}\beta^2 \pmod 1 \tag{1.3}$$

where β is defined as above. The rotation r is given by $\Phi = (1/N, -1/N)$ and U is a small neighbourhood surrounding the shrunken instanton. Using this result (1.1) translates into the requirement that locally the sum of the fractional parts of the gauge instanton numbers match the fractional part of the gravitational instanton number (which is $1/N$):

$$-\frac{1}{2}\frac{1}{8\pi^2}\int_U \operatorname{tr} R^2 = -\frac{1}{60}\frac{1}{8\pi^2}\int_U \operatorname{tr} F_1^2 - \frac{1}{60}\frac{1}{8\pi^2}\int_U \operatorname{tr} F_2^2 \pmod 1 \tag{1.4}$$

Even though this might look trivial from a weak coupling perspective, in light of M-theory on $S^1/\mathbb{Z}_2$ (1.3) fixes the fractional instanton numbers on each $\mathbb{Z}_2$ fixed point seperately. Therefore the distribution of the integer part on the two $\mathbb{Z}_2$ fixed points is not directly given by (1.3) and has to be investigated by different methods [4, 5, 6, 7].

Turning to global aspects of the orbifold we define it in the usual way [10] (for an introduction see [18] or [19], chapter 16) and explicitly allow for quantized Wilson lines. Abstractly the orbifold is $O = \mathbb{C}^2/S$ where S, the space group, is generated by affine transformations $D : x \mapsto rx + h$. To include the gauge degrees of freedom we augment D by a gauge transformation γ_D and require that the map $\gamma : S \to \mathrm{E}_8 \times \mathrm{E}_8$ is a group homomorphism. Furthermore we require that all γ_D commute with each other. This restricts the map γ quite severely and especially implies that each γ_D generates a cyclic subgroup of E_8. Given a fixed point $Dx_0 = x_0$ of some D with $Dx = rx + h$ the twist is generated by (r, γ_D) which has to fulfill (1.1).

To see that γ defines a flat gauge bundle consider $\mathbb{C}^2 - F$ where F is the set of all fixed points of S. Since, by supersymmetry, F consists of isolated fixed points only the fundamental group π_1 of $\mathbb{C}^2 - F$ is zero and $\mathbb{C}^2 - F$ is the universal covering space of $(\mathbb{C}^2 - F)/S = O - F$, that is, the orbifold with the fixed points taken out. This implies (see, for example [17], Chapter III) that π_1 of $O - F$ is isomorphic to S and the map γ provides a homomorphism of $\pi_1(O - F)$ to the gauge group. This is nothing but the data of a flat gauge bundle[5] on $O - F$ (up to gauge transformations).

[4]In analogy to the $\mathrm{Spin}32/\mathbb{Z}_2$ case this relation has already been used in [14]. See the footnote in section 2.3 on how the E_8 instanton number is related to that of $\mathrm{Spin}(16)$.

[5]Since on a flat bundle parallel translation around closed paths (with a fixed staring point) is invariant under continous deformations of the path, every class in π_1 corresponds to precisely one group element. Since concatenation of two pathes corresponds to group multiplication this is a homomorphism from π_1 to the gauge group.

In conclusion, we have shown that the orbifolds in our class correspond to all possible flat $E_8 \times E_8$ bundles on the orbifold $\mathbb{C}^2/S$ with the fixed points taken out under the only restriction that the sum of the fractional parts of the gauge instanton numbers (computed from the flat bundle data via (1.1)) match the fractional part of the gravitational one locally for every fixed point. Since the fractional instanton numbers are computed seperately for every E_8, this classification fully applies to M-theory on $S^1/\mathbb{Z}_2$.

Moreover, (1.3) allows to calculate the fractional part of an E_8 instanton number for any instanton sitting on a (possibly blown up) orbifold singularity as long as the gauge bundle approaches a flat bundle away from the singularity.

2 Fractional Instanton Numbers

This section is devoted to the derivation of equation (1.3). We consider the situation of an E_8 instanton localized in the interior of a real four-manifold[6] M with boundary L, that is we treat the gauge bundle as being flat on L. We start by showing that the fractional part of the instanton number depends only on the isomorphism class of the bundle on L. After that we construct one explicit element of that class corresponding to the given data and finally compute the fractional instanton number of that element[7].

The first two steps are mostly standard (see for example [20, 21]) whereas the third one requires a bit more technology (see especially [22] chapter III and [23]).

2.1 Basic Facts

The instanton number is defined as (a discussion of the normalization can be found in [26])

$$I = -\frac{1}{60}\frac{1}{8\pi^2}\int_M \operatorname{tr} F^2 \tag{2.1}$$

with the trace in the adjoint representation. Since E_8 is semi-simple the first nontrivial homotopy group is $\pi_3(E_8) = \mathbb{Z}$. This implies, due to the presence of the boundary, that the bundle is trivial on M. This can be seen as follows: a possible obstruction in constructing a section of a principal E_8-bundle on M (and thereby showing triviality of the bundle) is

[6]To be precise, we require M to be orientable, compact and connected with a connected boundary.

[7]More mathematically speaking in step two we construct a bundle map from the Hopf fibration $S^3 \longrightarrow S^2$ to $E_8 \longrightarrow E_8/T^8$ with T^8 a maximal torus of E_8 and in step three we calculate the image of the fundamental cycle of S^3 under that map by a spectral sequence.

given by an nonzero element of $H^4(M,\mathbb{Z})$. Since M is an orientable and compact manifold with boundary, we have by duality $H^4(M,\mathbb{Z}) \simeq H_0(M,\partial M,\mathbb{Z}) = 0$. (see for example [21] part III, or [27] for a nice introduction) (2.1) now reads

$$I = -\frac{1}{30}\frac{1}{8\pi^2}\int_L \mathrm{tr}\left(AF - \frac{1}{3}A^3\right) = \frac{1}{3\cdot 60}\frac{1}{8\pi^2}\int_L \mathrm{tr}\, A^3 \tag{2.2}$$

To make contact to the situation studied in the introduction we have to take $L = S^3/\mathbb{Z}_N$ (a lens space) with $\pi_1(L) = \mathbb{Z}_N$. By the same reasoning as in the introduction, S^3 is the covering space of L and (r,γ) specifies a flat bundle on L. Moreover, by pulling back the bundle via the covering map $\pi : S^3 \to S^3/\mathbb{Z}_N$ we get a bundle on S^3 on which A can be gauge transformed to $A' = 0$ because $\pi_1(S^3) = 0$. Denoting the gauge transformation by $g : S^3 \to \mathrm{E}_8$ we get $A = g^{-1}A'g + g^{-1}\mathrm{d}g = g^{-1}\mathrm{d}g$. Plugging that into (2.2) we get $I = I_S/N$ with I_S defined by

$$I_S = \frac{1}{3\cdot 60}\frac{1}{8\pi^2}\int_{S^3} \mathrm{tr}(g^{-1}\mathrm{d}g)^3 \tag{2.3}$$

Of course g represents an element of $\pi_3(\mathrm{E}_8)$ which is identified with $\mathbb{Z}$ by (2.3) (see again [26]). This especially shows that I is a multiple of $1/N$.

To show that I_S changes by a multiple of N when the bundle on L is gauge transformed we consider two flat connections A_1 and A_2 on L, both corresponding to the same generator (r,γ). Since both bundles are isomorphic we have $A_2 = h^{-1}A_1h + h^{-1}\mathrm{d}h$ for some $h : L \to \mathrm{E}_8$. Now h can be lifted to S^3 giving $h_S = h \circ \pi$ and we get $A_2 = g_2^{-1}\mathrm{d}g_2 = h_S^{-1}g_1^{-1}(\mathrm{d}g_1)h_S + h_S^{-1}\mathrm{d}h_S = (g_1h_S)^{-1}\mathrm{d}(g_1h_S)$. But since the pointwise product $g\cdot g'$ for two elements of π_3 of a Lie group G corresponds to the group addition of the elements $g_2 = g_1 + h_S$.

To proceed, we note that, since π_3 is the first nontrivial homotopy class of E_8, by the Hurewicz isomorphism $\pi_3(\mathrm{E}_8)$ is isomorphic to $H_3(\mathrm{E}_8,\mathbb{Z})$ and further by the universal coefficient theorem to $H^3(\mathrm{E}_8,\mathbb{Z})$ because, by the same argument we have $H^1 = H^2 = H_1 = H_2 = 0$. Therefore the instanton number I_S is given by the pullback via g of the generator ω of $H^3(\mathrm{E}_8,\mathbb{Z})$ evaluated on the fundamental cycle C_S of S^3:

$$I_S = g^*\omega(C_S) \tag{2.4}$$

Since $\pi : S^3 \to L$ is N to one we have $\pi_*C_S = NC_L$ with C_L the fundamental cycle of L. This immediately yields $h_S^*\omega(C_S) = \pi^*h^*\omega(C_S) = h^*\omega(\pi_*C_S) = Nh^*\omega(C_L) \in N\mathbb{Z}$.

2.2 Construction of the Bundle

We now construct a bundle on S^3 simply by constructing the map g, which we require to obey $g(rx) = \gamma g(x)$. This bundle will obviously be the pullback of a bundle on L that fulfills our requirements.

To construct g (and for the final step) we will need some basic facts about lens spaces (see [22], § 18). As above we define L by the fibration $\mathbb{Z}_N \longrightarrow S^3 \xrightarrow{\pi} L$ where S^3 is identified with the unit sphere in $\mathbb{C}^2$ and the generator of $\mathbb{Z}_N$ acts on S^3 like

$$e^{2\pi i/N} : (Z^1, Z^2) \mapsto (e^{2\pi i/N} Z^1, e^{2\pi i/N} Z^2) \tag{2.5}$$

This action is of course compatible with the $\mathrm{U}(1)$ action on S^3 (where $\mathrm{U}(1)$ is identified with the unit circle in $\mathbb{C}$)

$$(Z^1, Z^2) \mapsto (\lambda Z^1, \lambda Z^2) \qquad \lambda \in S^1 \subset \mathbb{C} \tag{2.6}$$

and we get the Hopf-fibration $S^1 \longrightarrow S^3 \xrightarrow{\pi_S} \mathbb{C}P^1 \simeq S^2$. Now since $\mathbb{Z}_N \subset \mathrm{U}(1)$ the $\mathrm{U}(1)$ action descends to an action on L and we have

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^3 & \xrightarrow{\pi_S} & S^2 \\ & & \pi \downarrow & & \mathrm{id} \downarrow \\ S^1 & \longrightarrow & L & \xrightarrow{\pi_L} & S^2 \end{array} \tag{2.7}$$

In this diagram π is a bundle map which is N to one on the standard fibre.

This can be made more explicit by identifying S^2 with a cylinder $I \times S^1$ where at both ends of the interval S^1 is identified to a point. We parametrize $I \times S^1$ by (ϱ, Ψ) where $\varrho \in \left[0, \frac{\pi}{2}\right], \Psi \in [0, 2\pi[$. To write down the bundle explicitly we divide S^2 into upper (D^2_+) and lower (D^2_-) hemisphere. Using $(Z^1, Z^2) = (\cos\varrho\, e^{i\varphi_1}, \sin\varrho\, e^{i\varphi_2})$ we write

$$\begin{array}{llll} D^2_+ : & \varrho \geq \frac{\pi}{4} & \varphi_2 = \varphi_+ & \varphi_1 = \varphi_+ - \Psi \\ D^2_- : & \varrho \leq \frac{\pi}{4} & \varphi_1 = \varphi_- & \varphi_2 = \varphi_- + \Psi \end{array} \tag{2.8}$$

Therefore $\lambda = e^{i\varphi} \in \mathrm{U}(1)$ acts like $\varphi_- \mapsto \varphi_- + \varphi,\ \varphi_+ \mapsto \varphi_+ + \varphi$ and the fibre S^1 is parametrized by $\varphi_-, \varphi_+ \in [0, 2\pi[$.

Since on the equator $\varphi_+ = \varphi_- + \Psi$ the sphere S^3 as a bundle ist made of two trivial S^1 bundles on the hemispheres clutched together by the generator of $\pi_1(S^1)$. Analogously, by (2.7), the clutching function of L is $N \in \pi_1(S^1)$.

To simplify the construction of g we insert a cylinder $C = I \times S^1$ parametrized by (x, Ψ), $x \in [0,1], \Psi \in [0, 2\pi[$ between the hemispheres by attatching D^2_+ at $x = 1$ and D^2_- at $x = 0$. This is extended to the bundle by parametrizing the fibre as φ_+ as on D^2_+. Therefore the bundle is now clutched nontrivially at $x = 0$ and trivially at $x = 1$.

Finally we define $g(\varrho, \Psi, \varphi_+) = g_+(\varrho, \Psi, \varphi_+) = e^{iqH\varphi_+}$ on D^2_+ and $g(\varrho, \Psi, \varphi_-) = g_-(\varrho, \Psi, \varphi_-) = e^{iqH\varphi_-}$ on D^2_-. This yields

$$\begin{aligned} g(x=0, \Psi, \varphi_+) &= g_-(\varrho = \tfrac{\pi}{4}, \Psi, \varphi_+ - \Psi) &&= e^{iqH(\varphi_+ - \Psi)} \\ g(x=1, \Psi, \varphi_+) &= g_+(\varrho = \tfrac{\pi}{4}, \Psi, \varphi_+) &&= e^{iqH\varphi_+} \end{aligned} \tag{2.9}$$

on the cylinder C and g can be extended to $g(x, \Psi, \varphi_+) = e^{iqH\varphi_+}$ at $\Psi = 0, 2\pi$.

g at $\varphi_+ = 0$ is now defined on the boundary of the square $0 \leq x \leq 1,\ 0 \leq \Psi \leq 2\pi$ (the identification of $\Psi = 0$ with $\Psi = 2\pi$ plays no role in the consideration) and the only nontrivial step in the construction is to extend this definition to the interior of the square. Topologically this is the same as to extend a map $g : S^1 \to \mathrm{U}(1) \subset \mathrm{E}_8$ with $S^1 = \partial D^2$ to the whole of D^2. Therefore we consider the fibration $\mathrm{U}(1) \longrightarrow \mathrm{E}_8 \longrightarrow \mathrm{E}_8/\mathrm{U}(1)$ where $\mathrm{U}(1)$ is the subgroup of E_8 generated from $e^{iqH\varphi}$. The relative homotopy sequence of this fibration contains the following part

$$\ldots \longrightarrow \pi_2(\mathrm{E}_8) = 0 \longrightarrow \pi_2(\mathrm{E}_8, \mathrm{U}(1)) \xrightarrow{\partial} \pi_1(\mathrm{U}(1)) \longrightarrow \pi_1(\mathrm{E}_8) = 0 \longrightarrow \ldots \tag{2.10}$$

Since the sequence is exact the map ∂ is an isomorphism and g can be extended to $\varphi = 0$ in C. Finally we define g on the whole of C by $g(x, \Psi, \varphi_+) = e^{iqH\varphi_+} g(x, \Psi, \varphi_+ = 0)$.

2.3 Calculation of the Instanton Number

As the map g constructed in the last section actually fulfills $g(e^{i\varphi} Z^i) = e^{iqH\varphi} g(Z^i)$ it is a bundle map from S^3 to E_8 which especially provides a homomorhism from the (cohomology) spectral sequence of the former to that of the latter[8].

[8] It can be easily seen that we could restrict ourselves to a map to $\mathrm{Spin}(16)$ at this point: by acting with the Weyl group we can map g into the $\mathrm{Spin}(16)$ sublattice of E_8. Since $\pi_1(\mathrm{Spin}(16)) = 0$ all steps of the last section apply as before and we are left with a pure $\mathrm{Spin}(16)$ bundle. This then allows to compute the instanton number with the formulas given in [13, 14]. However, we will proceed in a different way, since our calculation gives the instanton number (including the integer part) for gauge bundles satisfying $g(e^{i\varphi} Z^i) = e^{iqH\varphi} g(Z^i)$.

We start by writing down the standard example of the spectral sequence of the Hopf fibration. The E_2 term is given by $E_2^{p,q} = H^p(S^2) \otimes H^q(S^1)$, explicitly:

$$E_2 = \begin{array}{c|ccc} 1 & \mathbb{Z}/a_S & 0 & \mathbb{Z}/a_S b \\ 0 & \mathbb{Z}/1 & 0 & \mathbb{Z}/b \\ \hline & 0 & 1 & 2 \end{array} \tag{2.11}$$

where p, q label columns and rows and the diagonal arrow denotes the map d_2. By H/h we denote the group H generated by h. As the sequence stops at E_3 we have $E_3 \simeq H^*(S^3) = (\mathbb{Z}/1, 0, 0, \mathbb{Z}/c_S)$ with $c_S = a_S b$. This implies that d_2 is an isomorphism from $E_2^{0,1}$ to $E_2^{2,0}$.

We now turn to the spectral sequence of the fibration $T^8 \longrightarrow \mathrm{E}_8 \longrightarrow \mathrm{E}_8/T^8$ where T^8 is a maximal torus of E_8 containing the $\mathrm{U}(1)$ generated from the elements $e^{iqH\varphi}$. First we need to verify that the base is simply connected. This is clear from the homotopy sequence

$$\ldots \longrightarrow \pi_1(\mathrm{E}_8) = 0 \longrightarrow \pi_1(\mathrm{E}_8/T^8) \longrightarrow \pi_0(T^8) = \{*\} \longrightarrow \cdots \tag{2.12}$$

(π_0 is not a group here and consists only of the (arbitrary) base point $*$). Furthermore, as shown by Morse theoretic methods in [23], the base is torsion free and $H^{2n+1}(\mathrm{E}_8/T^8) = 0$ for $n \in \mathbb{Z}$. With this information the E_2 term is[9]

$$E_2 = \begin{array}{c|ccccc} 3 & 56\mathbb{Z} & & & & \\ 2 & 28\mathbb{Z} & 0 & & & \\ 1 & 8\mathbb{Z} & 0 & 64\mathbb{Z} & & \\ 0 & \mathbb{Z} & 0 & 8\mathbb{Z} & 0 & 35\mathbb{Z} \\ \hline & 0 & 1 & 2 & 3 & 4 \end{array} \tag{2.13}$$

This can be seen as follows: Firstly, we note that $H^*(T^8) = (\mathbb{Z}, 8\mathbb{Z}, 28\mathbb{Z}, 56\mathbb{Z}, \ldots)$. Secondly, since $H^*(\mathrm{E}_8) = (\mathbb{Z}/1, 0, 0, \mathbb{Z}/\omega, \text{higher groups})$ by Hurewicz E_3 must be trivial for degree 1 and 2. Therefore, again $\mathrm{d}_2 : E_2^{0,1} \to E_2^{2,0}$ is an isomorphism. This implies, since by the Künneth formula $H^*(T^8) = (H^*(S^1))^8$ (with respect to $\otimes$), that all maps d_2 from the first to the third column are invertible.

Moreover, since d_3 maps everything to zero all elements of $H^*(\mathrm{E}_8)$ up to degree three are given by E_3 up to degree three. However, because d_2 from the first to the third column is invertible, the only nonvanishing element up to degree three of E_3 must be $E_3^{2,1} = \mathbb{Z}/\omega$. This implies $E_2^{4,0} = 35\mathbb{Z}$.

[9] Addition and multiplication of groups are written with respect to the operations $\oplus$ and $\otimes$ on abelian groups, i.e. $2\mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{Z}^2 = \mathbb{Z} \otimes \mathbb{Z}$.

This can independently be verified by calculating $H^*(\mathrm{E}_8/T^8)$ as described in [23]: the dimension of $H^{2n}(\mathrm{E}_8/T^8)$ is given by the number of elements of the Weyl group which change the sign of precisely q of the positive roots. This can be computed easily, since, for a given Weyl reflection σ, q is given by the length of σ (see for example [24] section 10.3), i.e. the (minimal) number of simple Weyl reflections σ can be composed of $\sigma = \sigma_{\alpha_1} \cdot \ldots \cdot \sigma_{\alpha_q}$. For $q = 1$ there are 8 simple roots so $H^2(\mathrm{E}_8/T^8) = 8\mathbb{Z}$. For $q = 2$ there are $8 \cdot 7/2 + 7 = 35$ combinations, because simple Weyl reflections of two simple roots which are not connected by a line in the Dynkin diagram commute. So $H^4(\mathrm{E}_8/T^8) = 35\mathbb{Z}$.

Explicitly we denote the generators of $H^1(T^8)$ by $a^i, i = 1, \ldots, 8$ and those of $H^2(\mathrm{E}_8/T^8)$ by $b^i = \mathrm{d}_2 a^i$. As $E_2^{0,2} = H^2(T^8)$ is generated by the 28 elements $a^i a^j$ with $i < j$ their image under d_2 in $E_2^{2,1}$ is given by the elements $\mathrm{d}_2(a^i a^j) = b^i a^j - a^i b^j = -a^i b^j + a^j b^i$.

To calculate ω we write all elements in terms of an euclidian basis A_I of the real homology $H_1(T^8, \mathbb{R})$ and its dual a^I in $H^1(T^8, \mathbb{R})$. We choose the basis such that it is compatible with the lattice Λ_8 given in the introduction. Then ω can be written as $\omega = w_{IJ} a^I b^J$ where w_{IJ} is a symmetric matrix, because the image of d_2 consists of all elements $w_{IJ} a^I b^J$ with antisymmetric w_{IJ}. Under an element T of the Weyl group w is transformed to $T^{-1} w T$ (T is an orthogonal matrix). As w must be invariant unter the Weyl group, which acts irreducible on a vector, by Schur's lemma, $\omega = w \delta_{IJ} a^I b^J$.

We now turn to the calculation of the instanton number. By construction g maps the fundamental cycle A_S of the S^1 fiber of the Hopf fibration to $q^I A_I$. This implies

$$g^* a^I(A_S) = a^I(g_* A_S) = a^I(q^J A_J) = q^I \tag{2.14}$$

and therefore $g^* a^I = q^I a_S$. Finally we have

$$\begin{aligned} I_S = g^*\omega(C_S) &= w\delta_{IJ}\, (g^* a^I\, g^* b^J)(C_S) = w\delta_{IJ}\, (g^* a^I\, g^*(\mathrm{d}_2 a^J))(C_S) \\ &= w\delta_{IJ}\, (g^* a^I\, \mathrm{d}_2(g^* a^J))(C_S) = w\delta_{IJ}\, (q^I a_S\, q^J \mathrm{d}_2 a_S)(C_S) \\ &= w\delta_{IJ}\, q^I q^J\, (a_S b)(C_S) = w q^2 \end{aligned} \tag{2.15}$$

To normalize this equation we compare to the standard embedding $I_S = 1$, $q^I = (1, 1, 0, \ldots)$ resulting in $w = 1/2$. Therefore we have

$$I = \frac{I_S}{N} = \frac{1}{2N} q^2 = \frac{N}{2} \beta^2 \pmod 1 \tag{2.16}$$

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