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Light-Cone Gauge for N=2 Strings *

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Abstract

Covariant quantization of self-dual strings in 2+2 flat dimensions reduces them to their zero modes, a consequence of extended world-sheet supersymmetry. We demonstrate how to arrive at the same result more directly by employing a 'double' light-cone gauge. An unconventional feature of this gauge is the removal of anticommuting degrees of freedom by commuting symmetries and vice versa. The reducibility of the N=4 string and its equivalence with the N=2 string become apparent.

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1. N=2 and N=4 strings in superconformal gauge. String theories with more than one world-sheet supersymmetry 'suffer' from the absence of higher dimensions [1, 2]. Indeed, covariant quantization and naive BRST ghost-counting for the $N{=}2$ and $N{=}4$ superconformal algebras of constraints yield critical dimensions of four and minus eight(!), respectively. However, as was realized by Siegel [3], the $N{=}4$ constraints are reducible, and the $N{=}4$ string turns out to be the same as the $N{=}2$ string. Yet, in contrast to the $N{=}2$ formulation, the $N{=}4$ description is manifestly Lorentz covariant. Since the required signature of the four-dimensional target is $(++--)$, by 'Lorentz group' we mean $SO(2,2) \simeq SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R})'$. These global symmetries are not to be confused with the local R symmetry of $N{=}4$ supersymmetry, denoted by $SL(2,\mathbb{R})''$.

In this letter we shall employ (Majorana) spinor notation for all Lorentz and internal indices. We distinguish the different groups by using

$$\alpha \ \leftrightarrow \ SL(2,\mathbb{R}) \quad , \qquad \dot\alpha \ \leftrightarrow \ SL(2,\mathbb{R})' \quad , \qquad \ddot\alpha \ \leftrightarrow \ SL(2,\mathbb{R})'' \quad , \qquad \text{where} \quad \alpha \in \{0,1\} \quad . \tag{1}$$

In particular, fundamental spinors are taken to be real.[1] In the NSR formulation, both $N{=}2$ and $N{=}4$ strings are parametrized by the four coordinates $X^{\alpha\dot\alpha}$ plus four anticommuting NSR fields $\psi^{\ddot\alpha\dot\alpha}$. From the world-sheet point of view, the former are scalars while the latter form two-component Majorana spinors, $\psi^{\ddot\alpha\dot\alpha} = (\psi_+^{\ddot\alpha\dot\alpha}, \psi_-^{\ddot\alpha\dot\alpha})$ in a Weyl basis. With '$\pm$' we generally indicate light-cone components of world-sheet tensors, i.e. $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$. Sharing a world-sheet supersymmetry multiplet with $X^{\alpha\dot\alpha}$, the anticommuting coordinates $\psi^{\ddot\alpha\dot\alpha}$ should also carry an undotted $SL(2,\mathbb{R})$ index. However, its value is coupled to that of the $SL(2,\mathbb{R})''$ index and we suppress it. The action defines a (target-space) light-cone pairing,

$$(X^{0\dot0}, X^{1\dot1}) \quad , \quad (\psi^{\ddot0\dot0}, \psi^{\ddot1\dot1}) \qquad \text{and} \qquad (X^{0\dot1}, X^{1\dot0}) \quad , \quad (\psi^{\ddot0\dot1}, \psi^{\ddot1\dot0}) \quad , \tag{2}$$

which decomposes the variables into two $N{=}1$ light-cone sets.

Starting from the formulation with auxiliary world-sheet $N{=}2$ or $N{=}4$ supergravity [4, 5], we advance to the superconformal gauge. In the critical dimension, all supergravity remnants then disappear thanks to super Weyl invariance. The residual (superconformal) freedom in this gauge does not prevent quantization but implies that physical states are subject to constraints and gauge identifications. The $N{=}2$ constraints $(T, G^{\ddot0 1}, G^{\ddot1 0}, J^{\ddot0\ddot1})$ represent a non-degenerate subset of the (reducible) $N{=}4$ constraints $(T, G^{\ddot\alpha\alpha}, J^{(\ddot\alpha\ddot\beta)})$, which entails the selection of a one-parameter subgroup of the $SL(2,\mathbb{R})''$ R-symmetry group generated by the spin-one constraints $J^{(\ddot\alpha\ddot\beta)}$. Because we work with real spinors it is preferable to choose a noncompact subgroup $GL(1,\mathbb{R}) \subset SL(2,\mathbb{R})''$. As the R-symmetry index of $\psi^{\ddot\alpha\dot\alpha}$ is tied to the space-time $SL(2,\mathbb{R})$ index, the choice of $J^{\ddot0\ddot1}$ incidentally also breaks the Lorentz group,

$$SO(2,2) \ \simeq \ SL(2,\mathbb{R}) \otimes SL(2,\mathbb{R})' \quad \longrightarrow \quad GL(1,\mathbb{R}) \otimes SL(2,\mathbb{R})' \quad . \tag{3}$$

Being non-degenerate, each (commuting or anticommuting) $N{=}2$ constraint essentially removes one timelike and one spacelike degree of freedom (of matching statistics). Hence, we expect $(T, G^{\ddot0 1}, G^{\ddot1 0}, J^{\ddot0\ddot1})$ to eliminate *all* string coordinates $X^{\alpha\dot\alpha}$ and their partners $\psi^{\ddot\alpha\dot\alpha}$ in the $2{+}2$ dimensional space-time. Indeed, this expectation is confirmed by direct analysis [6] as well as by amplitude computations which reveal that the $N{=}2$ string is just a point particle encoding the dynamics of self-dual Yang-Mills and gravity [7], at least at tree-level.[2]

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[1] Hence, v^α and $v^{\dot\alpha}$ are not related by complex conjugation, as in $3{+}1$ dimensions.

[2] For a review see [8, 9]. Quantization is detailed in [10]. The loop structure is subject of [11, 12].

Yet, it is unclear how to reproduce this result by further gauge-fixing to a light-cone gauge. As in other string theories, conformal reparametrizations can trivialize one light-cone coordinate, e.g. $\partial X^{0\dot{0}} = p^{0\dot{0}}$, and solving $T{=}0$ fixes a second one, e.g. $\partial X^{1\dot{1}}$. The remaining commuting gauge transformations, $GL(1,\mathbb{R})$ generated by $J^{\ddot{0}\ddot{1}}$, only affect ψ but not X, and so cannot eliminate the pair $(X^{0\dot{1}}, X^{1\dot{0}})$. It seems that one is still left with $1{+}1$ dimensional 'transverse' string excitations. In the following, we resolve this contradiction by showing that the light-cone gauge can still be used to get rid of *all* string excitations, provided we permit *commuting* transformations to gauge-fix *anticommuting* degrees of freedom and vice versa.

2. Residual N=2 superconformal symmetry. As is well known, the superconformal gauge possesses residual gauge freedom in the form of $N{=}2$ superconformal transformations. In light-cone world-sheet coordinates, these read

$$\begin{aligned}
\delta X^{0\dot{\alpha}} &= (\xi^+\partial_+ + \xi^-\partial_-)X^{0\dot{\alpha}} - \mathrm{i}\epsilon_1^+\psi_+^{\ddot{0}\dot{\alpha}} - \mathrm{i}\epsilon_1^-\psi_-^{\ddot{0}\dot{\alpha}} \quad , && (4)\\
\delta X^{1\dot{\alpha}} &= (\xi^+\partial_+ + \xi^-\partial_-)X^{1\dot{\alpha}} + \mathrm{i}\epsilon_0^+\psi_+^{\ddot{1}\dot{\alpha}} + \mathrm{i}\epsilon_0^-\psi_-^{\ddot{1}\dot{\alpha}} \quad , && (5)\\
\delta\psi_\pm^{\ddot{0}\dot{\alpha}} &= \xi^\pm\partial_\pm\psi_\pm^{\ddot{0}\dot{\alpha}} - \epsilon_0^\pm\partial_\pm X^{0\dot{\alpha}} - \lambda^\pm\psi_\pm^{\ddot{0}\dot{\alpha}} \quad , && (6)\\
\delta\psi_\pm^{\ddot{1}\dot{\alpha}} &= \xi^\pm\partial_\pm\psi_\pm^{\ddot{1}\dot{\alpha}} + \epsilon_1^\pm\partial_\pm X^{1\dot{\alpha}} + \lambda^\pm\psi_\pm^{\ddot{1}\dot{\alpha}} \quad , && (7)
\end{aligned}$$

where the commuting parameter functions (ξ^+,λ^+) and (ξ^-,λ^-) depend only on $\tau{+}\sigma$ and $\tau{-}\sigma$, respectively, while the anticommuting parameter functions $(\epsilon_0^+,\epsilon_1^+)$ and $(\epsilon_0^-,\epsilon_1^-)$ do likewise.

The transformations parametrized by $(\xi^+,\epsilon_0^+,\epsilon_1^+,\lambda^+)$ are generated by the left-moving $N{=}2$ currents $(T, G^{\ddot{1}0}, G^{\ddot{0}1}, J^{\ddot{0}\ddot{1}})$, in that order. The other half (carrying a '$-$' superscript) goes with a right-moving copy of those currents. We shall focus on the left-movers and drop all world-sheet indices. Then the $N{=}2$ generators read

$$\begin{aligned}
T &= \partial X^{0\dot{0}}\partial X^{1\dot{1}} - \partial X^{0\dot{1}}\partial X^{1\dot{0}} + \mathrm{i}\psi^{\ddot{0}\dot{0}}\partial\psi^{\ddot{1}\dot{1}} - \mathrm{i}\psi^{\ddot{0}\dot{1}}\partial\psi^{\ddot{1}\dot{0}} - \mathrm{i}\psi^{\ddot{1}\dot{0}}\partial\psi^{\ddot{0}\dot{1}} + \mathrm{i}\psi^{\ddot{1}\dot{1}}\partial\psi^{\ddot{0}\dot{0}} \quad , && (8)\\
G^{\ddot{0}1} &= \psi^{\ddot{0}\dot{0}}\partial X^{1\dot{1}} - \psi^{\ddot{0}\dot{1}}\partial X^{1\dot{0}} \quad , && (9)\\
G^{\ddot{1}0} &= \psi^{\ddot{1}\dot{0}}\partial X^{0\dot{1}} - \psi^{\ddot{1}\dot{1}}\partial X^{0\dot{0}} \quad , && (10)\\
J^{\ddot{0}\ddot{1}} &= \psi^{\ddot{0}\dot{0}}\psi^{\ddot{1}\dot{1}} - \psi^{\ddot{0}\dot{1}}\psi^{\ddot{1}\dot{0}} \quad . && (11)
\end{aligned}$$

As usual, T is associated with conformal coordinate transformations while $J^{\ddot{0}\ddot{1}}$ holomorphically rescales the NSR fields only. Note that the $GL(1,\mathbb{R})$ current is antihermitian. More interesting is the action of the anticommuting generators, depicted in full detail as

$$\begin{array}{ccccc}
X^{0\dot{0}} & \xrightarrow{G^{\ddot{0}1}} & \psi^{\ddot{0}\dot{0}} & \xrightarrow{\cdot\, G^{\ddot{1}0}} & \partial X^{0\dot{0}} \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\
X^{1\dot{1}} & \xrightarrow{G^{\ddot{1}0}\cdot} & \psi^{\ddot{1}\dot{1}} & \xrightarrow{G^{\ddot{0}1}} & \partial X^{1\dot{1}}
\end{array}
\quad \text{and} \quad
\begin{array}{ccccc}
X^{0\dot{1}} & \xrightarrow{\cdot\, G^{\ddot{0}1}} & \psi^{\ddot{0}\dot{1}} & \xrightarrow{G^{\ddot{1}0}} & \partial X^{0\dot{1}} \\
\uparrow\downarrow & & \uparrow\downarrow & & \uparrow\downarrow \\
X^{1\dot{0}} & \xrightarrow{G^{\ddot{1}0}} & \psi^{\ddot{1}\dot{0}} & \xrightarrow{G^{\ddot{0}1}\cdot} & \partial X^{1\dot{0}}
\end{array} \qquad (12)$$

where the vertical arrows relate (target-space) light-cone conjugate[3] variables.

[3] Two variables are light-cone conjugate when their part of the (target-space) metric is $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. It implies that the momentum and annihilation parts of one variable are canonically conjugate to the position and creation parts of the other.

3. Gauge fixing part one – the conventional part. As is well-known, the residual-invariance generators play a double role because their vanishing has to be imposed as a constraint on the theory. In the bosonic string, for example, one employs T first to gauge-fix a light-cone coordinate, say $\partial X^+ = p^+$, and then a second time to solve $T{=}0$ for the light-cone conjugate coordinate ∂X^-, in effect getting rid of $1{+}1$ dimensions. This mechanism generalizes to the $N{=}1$ supersymmetric case, where an anticommuting generator G allows one to transform ψ^+ to zero and obtain ψ^- as a function of the transversal coordinates by solving $G{=}0$.

Let us do the same for the light-cone pair $(X^{0\dot0}, X^{1\dot1})$ plus $(\psi^{\ddot0\dot0}, \psi^{\ddot1\dot1})$ by making use of the generators T and $G^{\ddot1 0}$. By additional dots in the left part of (12) we indicate the function of the anticommuting generator. In this way we arrive at

$$\partial X^{0\dot0} \;=\; p^{0\dot0} \quad , \qquad \partial X^{1\dot1} \;=\; \tfrac{1}{p^{0\dot0}}(\partial X^{0\dot1}\partial X^{1\dot0} + \mathrm{i}\psi^{\ddot0\dot1}\partial\psi^{\ddot1\dot0} + \mathrm{i}\psi^{\ddot1\dot0}\partial\psi^{\ddot0\dot1}) \tag{13}$$

$$\psi^{\ddot0\dot0} \;=\; 0 \quad , \qquad \psi^{\ddot1\dot1} \;=\; \tfrac{1}{p^{0\dot0}}\,\psi^{\ddot1\dot0}\partial X^{0\dot1} \quad , \tag{14}$$

and the remaining $N{=}2$ generators simplify to

$$G^{\ddot0 1} \;=\; -\psi^{\ddot0\dot1}\partial X^{1\dot0} \qquad \text{and} \qquad J^{\ddot0\ddot1} \;=\; -\psi^{\ddot0\dot1}\psi^{\ddot1\dot0} \quad . \tag{15}$$

Although this reasoning is purely classical, it can be incorporated in the quantum theory by reading (13)–(15) as operator statements and replacing $p^{0\dot0}$ by the c-number $k^{0\dot0}$.

4. Chiral bosonization. In order to get rid of the remaining pair $(X^{0\dot1}, X^{1\dot0})$ plus $(\psi^{\ddot0\dot1}, \psi^{\ddot1\dot0})$ we must attempt to employ the *commuting* generator $J^{\ddot0\ddot1}$ to eliminate the *anticommuting* ψ's and then kill the X's with $G^{\ddot0 1}$. On the classical level this makes no sense because ordinary and Grassmann numbers are not related in any way. Upon quantization, however, this distinction blurs: fermionic creation and annihilation operators may be represented by finite matrices (with commuting entries), simply leading to multi-component wave functions. In two dimensions, one even has a direct relation between commuting and anticommuting fields by means of bosonization. Following this idea, we describe the $(\psi^{\ddot0\dot1}, \psi^{\ddot1\dot0})$ system by a chiral boson ϕ (normal-ordering implied),

$$\psi^{\ddot0\dot1} \;=\; \mathrm{e}^{-\phi} \qquad \text{and} \qquad \psi^{\ddot1\dot0} \;=\; \mathrm{e}^{+\phi} \quad , \qquad \text{so that} \qquad \psi^{\ddot1\dot0}\psi^{\ddot0\dot1} \;=\; \mathrm{i}\,\partial\phi \quad . \tag{16}$$

Note that ϕ is not an angle but a scale; there are no winding modes. Let us remark on the statistics. For more than one pair of anticommuting fields the above equations have to be supplemented by suitable cocycle (a.k.a. Jordan-Wigner or Klein) factors which ensure that the different pairs mutually anticommute. Since in the present case, however, only a single pair of NSR fields is left, its statistics is irrelevant and we can fully describe it by the chiral commuting ϕ.

Invoking the decomposition

$$\phi \;=\; \phi_< + q + (\tau{+}\sigma)p + \phi_> \tag{17}$$

into negative-, zero-, and positive-frequency parts,[4] the exponential operators read

$$\mathrm{e}^{n\phi} \;=\; \mathrm{e}^{n\phi_<}\,\mathrm{e}^{nq+n(\tau+\sigma)p}\,\mathrm{e}^{n\phi_>} \;=\; \mathrm{e}^{n\phi_<}\,\mathrm{e}^{\frac{n}{2}q}\,\mathrm{e}^{n(\tau+\sigma)p}\,\mathrm{e}^{\frac{n}{2}q}\,\mathrm{e}^{n\phi_>} \quad . \tag{18}$$

[4]Positive frequency means positive Fourier modes, i.e. the annihilation part. Zero modes obey $[q,p]=\mathrm{i}$.

The ϕ Fock space is generated by ϕ creation operators (contained in $\phi_<$) acting on a momentum eigenstate $|a\rangle$, defined by

$$p\,|a\rangle \;=\; a\,|a\rangle \tag{19}$$

and created from the vacuum state via

$$|a\rangle \;=\; \mathrm{e}^{\mathrm{i}aq}\,|0\rangle \;=\; \mathrm{e}^{\mathrm{i}a\phi}(\tau{\to}{-\infty})\,|0\rangle \quad . \tag{20}$$

On such a state, exponential operators act as

$$\mathrm{e}^{n\phi}\,|a\rangle \;=\; \mathrm{e}^{n(a-\mathrm{i}\frac{n}{2})(\tau+\sigma)}\;\mathrm{e}^{n\phi_<}\,|a{-}n\mathrm{i}\rangle \quad . \tag{21}$$

From this we learn two things. First, $\mathrm{i}p$ is nothing but fermion number because the NSR fields $\mathrm{e}^{\pm\phi}$ shift the eigenvalue by ±1 unit. [5] Second, a is tied to the monodromy for the NSR fields,

$$\mathrm{e}^{\pm\phi}(\sigma{+}2\pi)\,|a\rangle \;=\; -\,\mathrm{e}^{\pm2\pi a}\,\mathrm{e}^{\pm\phi}(\sigma)\,|a\rangle \quad . \tag{22}$$

It should be noted that, due to the reality of the NSR fields, the monodromy group is $\mathbb{R}_+$ and not $U(1)$. The Fourier modes are

$$\psi^{\ddot{0}\dot{1}}_{m+\frac12+\mathrm{i}a} \qquad \text{and} \qquad \psi^{\ddot{1}\dot{0}}_{m+\frac12-\mathrm{i}a} \tag{23}$$

with $m\in\mathbb{Z}$, so all monodromy sectors are NS-like. Due to the spectral flow isometry, which acts as

$$\psi \;\longmapsto\; \mathrm{e}^{-\mathrm{i}aq}\,\psi\,\mathrm{e}^{\mathrm{i}aq} \;=\; \mathrm{e}^{\pm(\tau+\sigma)a}\,\psi \quad , \tag{24}$$

all monodromy sectors are equivalent. In the following, we choose $a{=}0$.

5. Gauge fixing part two – the unconventional part. How is the gauge-fixing accomplished in this framework? As for example in the Gupta-Bleuler method, we shall impose the remaining constraints

$$G^{\ddot{0}1} \;=\; -\mathrm{e}^{-\phi}\,\partial X^{1\dot{0}} \qquad \text{and} \qquad J^{\ddot{0}\ddot{1}} \;=\; \mathrm{i}\,\partial\phi \tag{25}$$

not as operator equations but rather demand that their positive- and zero-frequency parts annihilate the physical states,[6]

$$G^{\ddot{0}1}_{\geq}(\sigma)\,|\mathrm{phys}\rangle \;=\; 0 \qquad \text{and} \qquad J^{\ddot{0}\ddot{1}}_{\geq}(\sigma)\,|\mathrm{phys}\rangle \;=\; \mathrm{i}a\,|\mathrm{phys}\rangle \quad . \tag{26}$$

If a constraint is not self-conjugate its conjugate will create a gauge invariance which allows us to further restrict the physical states. Our state space consists of the zero modes and excitations of the free fields ϕ, $X^{\ddot{0}\dot{1}}$, and $X^{1\dot{0}}$. In other words, its basis is generated by acting with their creation operators on the momentum eigenstates $|a,k^{\ddot{0}\dot{1}},k^{1\dot{0}}\rangle$.

We first attend to the commuting generator, $J^{\ddot{0}\ddot{1}}=\mathrm{i}\partial\phi$. Classically, it effects local translations of ϕ. On quantum states, we demand

$$\partial\phi_>\,|\mathrm{phys}\rangle \;=\; 0 \quad , \tag{27}$$

which removes all ϕ creation operators from $|\mathrm{phys}\rangle$. The ϕ content of $|\mathrm{phys}\rangle$ is thus reduced to its zero mode, whose value we chose to be $a{=}0$.

[5] Due to the indefinite target-space metric there is no conflict with the anti-hermiticity of $\mathrm{i}p$.

[6] except for the zero mode $J^{\ddot{0}\ddot{1}}_0=\mathrm{i}p$ as seen above. $G^{\ddot{0}1}$ has no zero modes.

Finally, we expose the action of the anticommuting generator, $G^{\ddot{0}1} = -\mathrm{e}^{-\phi}\partial X^{1\dot{0}}$. Since

$$\mathrm{e}^{-\phi}\,|0\rangle \;=\; \mathrm{e}^{-\frac{\mathrm{i}}{2}(\tau+\sigma)}\,\mathrm{e}^{-\phi_<}\,|\mathrm{i}\rangle \tag{28}$$

the r.h.s. contains only negative-frequency (and no zero-mode) parts. Therefore, $G^{\ddot{0}1}_{\geq}$ on physical states will involve all positive modes of $\partial X^{1\dot{0}}$ but no others. Consequently, the requirement

$$(\partial X^{1\dot{0}}\,\mathrm{e}^{-\phi})_{\geq}\,|\mathrm{phys}\rangle \;=\; 0 \tag{29}$$

eliminates all $X^{0\dot{\mathrm{i}}}$ creation operators from $|\mathrm{phys}\rangle$. By conjugation, the associated gauge symmetry allows us to gauge away all $X^{1\dot{0}}$ creation operators as well. We are now left with

$$|\mathrm{phys}\rangle \;=\; |0, k^{0\dot{\mathrm{i}}}, k^{1\dot{0}}\rangle \quad . \tag{30}$$

Finally, we may include the momenta $k^{0\dot{0}}$ and $k^{1\dot{\mathrm{i}}}$ back into the parametrization of the physical states. Then, the '$1\dot{1}$' coordinates from (13) and (14) may be expressed as[7]

$$\partial X^{1\dot{\mathrm{i}}} \;=\; \tfrac{1}{k^{0\dot{0}}}(\partial X^{0\dot{\mathrm{i}}}\partial X^{1\dot{0}} + a^2 - \partial\phi\partial\phi) \qquad \text{and} \qquad \psi^{\ddot{\mathrm{i}}\dot{\mathrm{i}}} \;=\; \tfrac{1}{k^{0\dot{0}}}\,\partial X^{0\dot{\mathrm{i}}}\,\mathrm{e}^{+\phi} \quad . \tag{31}$$

On the states (30) only the non-positive modes contribute. The zero mode of $\partial X^{1\dot{\mathrm{i}}}$ yields the mass-shell condition $k^{0\dot{0}}k^{1\dot{\mathrm{i}}} - k^{0\dot{\mathrm{i}}}k^{1\dot{0}} = 0$.

Summarizing, the 'double' light-cone gauge reduces the $N{=}2$ string degrees of freedom to the zero modes of its bosonic coordinates and puts them on mass-shell. As in the covariant treatment, a massless free boson makes up the entire spectrum of states.

We finally remark that the elimination mechanism works differently for bosons and fermions. It is crucial that the bosons come in light-cone pairs, i.e. they support an indefinite space-time metric, while each (light-cone conjugate) fermion pair gives rise (via bosonization) to a single self-conjugate boson, so that these chiral bosons support only a Euclidean metric. For this reason it is impossible[8] to employ a commuting gauge invariance to gauge-fix only half of a fermionic pair and eliminate the other half through the constraint, as is done for the X's. After both gauge-fixing and imposing the constraints, however, the counting for bosons and fermions again coincides.

6. The N=4 case. To streamline the equations for the $N{=}4$ string, we only display the left-moving degrees of freedom and drop the world-sheet $(\pm)$ indices. The right-moving part behaves completely analogously. The $N{=}4$ superconformal transformations

$$\delta X^{0\dot{\alpha}} \;=\; \xi\,\partial X^{0\dot{\alpha}} \;+\; \mathrm{i}\epsilon_1^{\ddot{0}}\psi^{\ddot{\mathrm{i}}\dot{\alpha}} \;-\; \mathrm{i}\epsilon_1^{\ddot{\mathrm{i}}}\psi^{\ddot{0}\dot{\alpha}} \quad , \tag{32}$$

$$\delta X^{1\dot{\alpha}} \;=\; \xi\,\partial X^{1\dot{\alpha}} \;+\; \mathrm{i}\epsilon_0^{\ddot{0}}\psi^{\ddot{\mathrm{i}}\dot{\alpha}} \;-\; \mathrm{i}\epsilon_0^{\ddot{\mathrm{i}}}\psi^{\ddot{0}\dot{\alpha}} \quad , \tag{33}$$

$$\delta\psi^{\ddot{0}\dot{\alpha}} \;=\; \xi\,\partial\psi^{\ddot{0}\dot{\alpha}} \;+\; \epsilon_0^{\ddot{0}}\partial X^{0\dot{\alpha}} \;+\; \epsilon_1^{\ddot{0}}\partial X^{1\dot{\alpha}} \;+\; \lambda^{\ddot{0}\ddot{0}}\psi^{\ddot{\mathrm{i}}\dot{\alpha}} \;-\; \lambda^{\ddot{0}\ddot{\mathrm{i}}}\psi^{\ddot{0}\dot{\alpha}} \quad , \tag{34}$$

$$\delta\psi^{\ddot{\mathrm{i}}\dot{\alpha}} \;=\; \xi\,\partial\psi^{\ddot{\mathrm{i}}\dot{\alpha}} \;+\; \epsilon_0^{\ddot{\mathrm{i}}}\partial X^{0\dot{\alpha}} \;+\; \epsilon_1^{\ddot{\mathrm{i}}}\partial X^{1\dot{\alpha}} \;+\; \lambda^{\ddot{\mathrm{i}}\ddot{0}}\psi^{\ddot{\mathrm{i}}\dot{\alpha}} \;-\; \lambda^{\ddot{\mathrm{i}}\ddot{\mathrm{i}}}\psi^{\ddot{0}\dot{\alpha}} \tag{35}$$

involve the four commuting left-moving parameters $(\xi, \lambda^{(\ddot{\alpha}\ddot{\beta})})$ and four anticommuting left-moving parameters $(\epsilon_\alpha^{\ddot{\alpha}})$. Clearly, the relation with the $N{=}2$ parameters is

$$\xi = \xi \quad , \qquad \lambda = \lambda^{\ddot{0}\ddot{\mathrm{i}}} = \lambda^{\ddot{\mathrm{i}}\ddot{0}} \quad , \qquad \epsilon_0 = \epsilon_0^{\ddot{0}} \quad , \qquad \epsilon_1 = \epsilon_1^{\ddot{\mathrm{i}}} \quad . \tag{36}$$

[7] The classical constraint $T{=}0$ gets modified to $T{=}a^2$. This cancels the contribution of the NSR momenta to L_0.
[8] It is possible only for the zero modes.

The $N{=}4$ constraints which generate the above transformations consist of the $N{=}2$ set (8)–(11) enlarged by

$$
\begin{aligned}
\epsilon_0^{\ddot{1}} :\ G^{\ddot{0}0} &= \psi^{\ddot{0}\dot{0}} \partial X^{0\dot{1}} - \psi^{\ddot{0}\dot{1}} \partial X^{0\dot{0}} \quad , && (37)\\
\epsilon_1^{\ddot{0}} :\ G^{\ddot{1}1} &= \psi^{\ddot{1}\dot{0}} \partial X^{1\dot{1}} - \psi^{\ddot{1}\dot{1}} \partial X^{1\dot{0}} \quad , && (38)\\
\lambda^{\ddot{1}\ddot{1}} :\ J^{\ddot{0}\ddot{0}} &= 2\, \psi^{\ddot{0}\dot{0}} \psi^{\ddot{0}\dot{1}} \quad , && (39)\\
\lambda^{\ddot{0}\ddot{0}} :\ J^{\ddot{1}\ddot{1}} &= 2\, \psi^{\ddot{1}\dot{0}} \psi^{\ddot{1}\dot{1}} \quad . && (40)
\end{aligned}
$$

Let us investigate if the non-negative-frequency parts of the additional four constraints imply any further restrictions on the physical states of the $N{=}2$ string. On the gauge slice (13) and (14) the above generators reduce to

$$
\begin{aligned}
G^{\ddot{0}0} &= -\mathrm{e}^{-\phi}\, k^{0\dot{0}} \quad , && (41)\\
G^{\ddot{1}1} &= -\mathrm{e}^{+\phi}\, (\partial\phi\partial\phi - a^2)/k^{0\dot{0}} \quad , && (42)\\
J^{\ddot{0}\ddot{0}} &= 0 \quad , && (43)\\
J^{\ddot{1}\ddot{1}} &= 2\, :\mathrm{e}^{+\phi}\mathrm{e}^{+\phi}:\, \partial X^{0\dot{1}}/k^{0\dot{0}} \;=\; 0 \quad . && (44)
\end{aligned}
$$

On the physical states $|0, k^{\alpha\dot{\alpha}}\rangle$, only negative-frequency modes will be created, and $G^{\ddot{\alpha}\beta}$ do not contain zero modes. Therefore, indeed

$$
G^{\ddot{0}0}_{\geq}\, |0, k^{\alpha\dot{\alpha}}\rangle \;=\; 0 \;=\; G^{\ddot{1}1}_{\geq}\, |0, k^{\alpha\dot{\alpha}}\rangle \quad . \qquad (45)
$$

Hence, the additional $N{=}4$ constraints are obsolete on the physical states of the $N{=}2$ string, as expected.

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