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Exact amplitudes in four dimensional non-critical string theories

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The large N expansion of $\mathcal{N}=2$ supersymmetric Yang-Mills theory with gauge group $\mathrm{SU}(N)$ has recently been shown to break down at singularities on the moduli space. We conjecture that by taking $N\to\infty$ and approaching the singularities in a correlated way, all the observables of the theory have a finite universal limit yielding amplitudes in string theories dual to field theories describing the light degrees of freedom. We explicitly calculate the amplitudes corresponding to the Seiberg-Witten period integrals for an A_{n-1} series of multicritical points as well as for other critical points exhibiting a scaling reminiscent of the $c=1$ matrix model. Our results extend the matrix model approach to non-critical strings in less than one dimension to non-critical strings in four dimensions.

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1 Introduction

Deriving exact results in string theory, or more modestly results valid to all orders of perturbation theory, is a challenge that can be rarely taken up. The first real success in this direction was achieved in 1989 [1] by using the relationship between the large N expansion of matrix integrals and a discretized version of the sum over world sheets [2, 3]. A crucial property is that the large N expansion of a matrix theory can break down at some critical value g_c of the 't Hooft coupling g. For example, in simple integrals over hermitian matrices [4], the partition function has an asymptotic expansion of the form

$$Z(g) = \sum_{h\geq 0} N^{2-2h}\, Z_h(g), \tag{1}$$

where the coefficients $Z_h(g)$ are given themselves by a convergent series in g. The radius of convergence of this series, which turns out to be independent of h, gives the critical coupling g_c. For g near g_c, the terms with a high power of g, or equivalently the Feynman diagrams with a very large number of vertices, dominate. In the dual double-line representation of the graphs, such Feynman diagrams correspond to discretized oriented Riemann surfaces with a very large number of faces, and thus give a good approximation to the smooth world sheets of an oriented closed string theory. It is then possible to take a double scaling limit $N \to \infty$ and $g \to g_c$ from which a genuine continuum string theory emerges [1]. The double scaled observables, which are identified with amplitudes in non-critical string theories, can be explicitly studied in various cases corresponding to a world sheet theory of central charge $c \leq 1$ (critical strings propagating in less than two dimensions). For reviews and references on these developments, the reader may consult [5].

In spite of its impressive achievements, the above approach suffers from two main drawbacks that have never been satisfactorily solved. The first disappointment is that the double scaling limits do not provide a non-perturbative definition of unitary theories. Technically, one can trace the problem to the fact that in the unitary cases the critical coupling g_c either is negative (and thus the original matrix integral is not convergent at $g = g_c$), or does not correspond to a true saddle point. All the information is encoded in non-linear differential equations (like the Painlevé equation) which yield recursion relations determining the perturbative expansions to all orders, but can accomodate various non-perturbative extensions. In the case of non-unitary models, the perturbative series are Borel summable and thus a natural ansatz for a non-perturbative definition can be given, but this is not possible for the most interesting unitary cases. A second major difficulty is that the method can be used only on tractable matrix integrals, which narrows drastically the possible applications.

Critical strings propagating in $D+1$ dimensions, or equivalently non-critical string theories in D dimensions, are a priori related to D dimensional matrix path integrals near critical points where the large N expansion breaks down, of which no example were known for $D>1$ in the 90s. Only strings propagating in less than two space-time dimensions (the so-called $c=1$ barrier) were thus studied at that time.

This situation has changed very recently with a work of the author on the large N limit of four dimensional $\mathcal{N}=2$ supersymmetric gauge theories [6]. The matrix integrals one considers in this context are supersymmetrized versions of the gauge theory path integrals that were originally discussed by 't Hooft [2]. Such path integrals need to be renormalized and the 't Hooft coupling g transmutes into a unit of mass and is not a free parameter. The "parameters" that can be adjusted to critical values are in these cases Higgs vaccuum expectation values (moduli). One of the main result of [6] was to show explicitly that the large N limit does break down at critical points (more commonly called singularities) on the moduli space. The aim of the present work is to argue that double scaling limits can be defined near those critical points, and to extract some exact string amplitudes from the double scaled Seiberg-Witten period integrals. Since the gauge theory path integrals are non-perturbatively defined for all moduli, we always obtain a full non-perturbative prescription for the amplitudes, in sharp contrast to the $c\leq 1$ cases.

Our main results were originally guessed by studying simple two dimensional toy models [7] whose main virtue is that they are tractable both in supersymmetric and non-supersymmetric examples. A supersymmetric version of the models, closely related to $\mathcal{N}=2$ super Yang-Mills [8], turned out to be particularly instructive [9]. The works [7, 9] themselves built on early attempts to study double scaling limits for $D>1$ in simple $\mathrm{O}(N)$ vector models [11].

We have organized the paper as follows. In Section 2 we start by giving an elementary discussion of the simplest critical point. Then we state the general conjecture and discuss its physical content in some details. We have tried to give a comprehensive qualitative presentation of our main points, without entering into any detailed calculation. In Section 3 we define the scalings near various multicritical points and show that the Seiberg-Witten period integrals have a finite limit. We give explicit formulas for the resulting integrals, which correspond to exact amplitudes in four dimensional non-critical string theories. Section 4 is a short conclusion.

2 A simple example and the general conjecture

2.1 The simplest critical point

Let us start with the final formula of [6],

$$\frac{i\pi z_1}{N} = 2\sqrt{1-1/r^2} - 2\ln\Bigl(1+\sqrt{1-1/r^2}\Bigr) - 2\ln r - \frac{2\sqrt{1-1/r^2}\,\ln 2}{N} + \frac{\pi^2/12-(\ln 2)^2}{N^2r^2\sqrt{1-1/r^2}} + \mathcal{O}(1/N^3). \tag{2}$$

It gives the large N expansion of a Seiberg-Witten period integral z_1 in pure $\mathcal{N}=2$, $\mathrm{SU}(N)$ super Yang-Mills theory, as a function of a dimensionless modulus

$$r = v/\Lambda\,, \tag{3}$$

where v is the global mass scale setting the Higgs eigenvalues distribution, and Λ the dynamically generated scale of the theory. At $r=r_{\mathrm{c}}=1$ there is a singularity where a dyon is becoming massless. The mass of the dyon is given by

$$M_{\mathrm{BPS}} = \sqrt{2}\,|v\,z_1|. \tag{4}$$

At leading $N=\infty$ order, the formula (2) incorrectly predicts $M_{\mathrm{BPS}}\propto (r-r_{\mathrm{c}})^{3/2}$ as $r\rightarrow r_{\mathrm{c}}$. The correct answer is known from electric/magnetic duality [12] to be $M_{\mathrm{BPS}}\propto r-r_{\mathrm{c}}$. Equation (2) also explicitly shows that higher order corrections are infinite at $r=r_{\mathrm{c}}$. These infrared divergences were discussed at length in [6].

Let us now introduce the deviation to the critical point

$$\delta = r - r_{\mathrm{c}} \tag{5}$$

and consider the quantity $z_1/\sqrt{\delta}$ in the limit $\delta\rightarrow 0$. By keeping only the dominant contribution at each order in $1/N$, we get

$$\frac{i\pi z_1}{\sqrt{\delta}} = -\frac{2}{3}\,N\delta - 2\ln 2 + \frac{\pi^2/12-(\ln 2)^2}{N\delta} + \mathcal{O}(1/N^2). \tag{6}$$

In the double scaling limit

$$N\rightarrow\infty\,,\quad \delta\rightarrow 0\,,\quad N\delta = \mathrm{constant}\,, \tag{7}$$

we see that the first three terms in the expansion of $z_1/\sqrt{\delta}$ remain finite. Introducing the dimensionless coupling constant

$$\kappa^{-1} = N\delta \tag{8}$$

and the rescaled amplitude

$$\mathcal{A}_1 = \frac{i\pi z_1}{\sqrt{\delta}} \tag{9}$$

we have

$$\mathcal{A}_1 = -\frac{2}{3\kappa} - 2\ln 2 + \left(\pi^2/12 - (\ln 2)^2\right)\kappa + \cdots \tag{10}$$

We want to interpret (10) as giving the first terms in the perturbative expansion of a string theory amplitude $\mathcal{A}_1$. The full amplitude is given by the full Seiberg-Witten period $i\pi z_1/\sqrt{\delta}$ in the scaling limit (7), which indeed yields a finite result as we will show in Section 3. But before we embark upon those calculations, we are going to discuss the general ideas at the root of the identification of $\mathcal{A}_1$ and its generalizations with string theory amplitudes.

2.2 The conjecture

The main claims of this paper can be summarized as follows. Near singularities on the moduli space of $\mathcal{N}=2$ supersymmetric Yang-Mills theory where the large N expansion breaks down [6], it is possible to define double scaling limits, similar to (7), with the following properties:

i) after a suitable rescaling of the space-time variables, corresponding to a low energy limit, all the correlation functions of the gauge theory have a finite limit. Most of the degrees of freedom of the original gauge theory decouple in the scaling limit, and we are left with a double scaled field theory describing the interacting light degrees of freedom only.

ii) the double scaling limits define $D=4$ continuum unitary non-critical string theories dual to the double scaled field theory.

The ideas underlying the conjecture are largely independent of supersymmetry, as the study of two dimensional models clearly shows [7], and similar statements could be made in the more general context of four dimensional gauge theories with Higgs fields. However, concrete examples are limited so far to $\mathcal{N}=2$ supersymmetric theories.

2.3 Discussion

2.3.1 The double scaled gauge theory

The ability to define the double scaled gauge theory near the critical points relies on the fact that the infrared divergences responsible for the breakdown of the $1/N$

expansion [6] are specific enough so that they can be compensated for by taking $N \to \infty$ in a suitable way. We will prove in Section 3 that this is possible for the low energy observables for which Seiberg and Witten have provided exact formulas [12]. The conjecture is then that all the other correlators of the gauge theory will also have a finite limit. It is actually very important to realize that the double scaling limit is always a low energy limit. This is not surprising, since the terms that survive are the most divergent at each order in $1/N$, and those divergences are infrared effects as stressed in [6]. This is also the simplest way to understand the universality of the limit. For example, the amplitude $\mathcal{A}_1$ discussed in Section 2.1 was defined by equation (9) to be the Seiberg-Witten period z_1 rescaled by a factor $1/\sqrt{\delta}$. Since z_1 gives the physical BPS mass (4), we see that the limit involves a rescaling of the mass scales. The precise formulation of the conjecture is thus that the correlators of the gauge theory have a finite limit when expressed in terms of rescaled space-time variables

$$x' = \sqrt{\delta}\, x. \tag{11}$$

For more general critical points, the scaling involves different positive powers of δ, as explained in Section 3, but the idea remains the same.

The fact that the double scaling limit is a low energy limit shows that the study of the Seiberg-Witten periods, which determine the leading terms of the effective action in a derivative expansion, goes a long way toward proving that all the other correlators must have a well-defined limit as well. The reason is that all the non-zero correlators at large distance are determined by this effective action. However, there are two important subtleties that must be taken into account. The first one is that in general, the standard local abelian effective action is not valid near the critical points, because we can have light dyons which are not local with respect to each other [13]. This argument does not apply to the simplest critical point discussed in Section 2.1, for which the low energy theory is a simple abelian gauge theory coupled to a single charged hypermultiplet. In that case, it may be tempting to incorrectly identify the double scaled gauge theory with this simple abelian theory. A way to understand that this cannot possibly be the right answer is to note that an abelian gauge theory is not a well-defined quantum theory on all scales, whereas the double scaled theory, which is obtained by a consistent limiting procedure from a non-abelian gauge theory, must be a well-defined quantum theory on all scales. What we must get is a modified version of an abelian gauge theory with a well-defined UV fixed point. Interestingly, it is actually possible to get different UV regularizations for a given critical point, as we will see at the end of Section 3.

The above discussion may seem a little bit abstract, but all this physics can be studied very concretely in the context of two dimensional non-linear σ models [9]. Typically, the low energy CFT is then an element of the minimal series. The double scaled theory can be the associated Landau-Ginzburg quantum field theory, but also a more general field theory describing the same physics in the IR but different in the UV [9].

2.3.2 The continuum string theory

The most difficult part of the conjecture is to understand that a continuum string theory is defined by the double scaling limit. Even in the $D \leq 1$ cases studied in [1, 5], it is not possible to give a rigorous proof, and our four dimensional examples come with several additional subtleties.

The standard heuristic argument is based on the analysis of the $1/N$ expansion of the partition function (1), or more generally of any amplitude $\mathcal{A}$. By taking into account the contributions from Feynman diagrams only (which obviously are the only one that survive in the large N limit in the case of simple zero dimensional integrals), a suitably normalized amplitude can be expanded,

$$\mathcal{A} = \sum_{h\geq 0} N^{2-2h} A_h(g)\,, \tag{12}$$

with

$$A_h(g) = \sum_{n\geq 0} A_{h,n}\, g^{2n}. \tag{13}$$

The asymptotic estimate of the coefficients $A_{h,n}$ is

$$A_{h,n} \underset{n\rightarrow\infty}{\propto} a_h\, n^{\gamma_h - 3} g_c^{-2n} \tag{14}$$

for some a_h (for a detailed account of this property and related topics, see [10]), which shows that the series for $A_h(g)$ has a finite radius of convergence g_c (independent of h) and

$$A_h(g) \underset{g\rightarrow g_c}{\propto} \frac{a_h}{(1-(g/g_c)^2)^{\gamma_h - 2}}\,. \tag{15}$$

When $g \rightarrow g_c$, the terms at very large n dominate in the sum (13). They are associated with discretized Riemann surfaces of area $A = na^2$, where a sets the length scales on the discretized world sheet. From (14) and (13) we see that the contribution of a surface of area A is proportional to $\exp(-\lambda A)$, with a renormalized world sheet cosmological constant λ given by

$$\lambda = \frac{2\ln(g_c/g)}{a^2} = \frac{\delta_g}{a^2}\,. \tag{16}$$

The continuum limit corresponds to letting the world sheet UV cutoff $1/a$ goes to infinity while $\delta_g \to 0$, keeping fixed λ. A scaling limit for which contributions from all genera survive can be defined because the exponent γ_h in (14) satisfies $2-\gamma_h = (2-\gamma_{\rm str})(1-h)$ for some h-independent string susceptibility $\gamma_{\rm str}$. By taking

$$N \to \infty\,, \quad \delta_g \to 0\,, \quad N\delta_g^{1-\gamma_{\rm str}/2} = \text{constant}, \tag{17}$$

the amplitude (12) reduces to an amplitude for an oriented closed string theory,

$$\mathcal{A} = \sum_{h\geq 0} A_h\, \kappa^{2h-2}. \tag{18}$$

The scaling (17) has the interesting consequence that the classical string coupling $1/N$ is renormalized to a dimensionfull coupling

$$g_{\rm str} = \frac{1}{Na^{2-\gamma_{\rm str}}}, \tag{19}$$

of world sheet dimension $2-\gamma_{\rm str}$. The dimensionless coupling κ in terms of which the string amplitudes are expanded is

$$\kappa = g_{\rm str}\lambda^{\gamma_{\rm str}-1/2}. \tag{20}$$

In these simple non-critical string theories, a variation of the string coupling $g_{\rm str}$ can thus be exactly compensated for by a variation of a world sheet coupling λ.

In our four dimensional examples, several of the above statements need to be refined, but we are going to argue that the general ideas remain valid. The first obvious difference is that the 't Hooft coupling g is no longer a free parameter. Ordinary perturbation theory is defined by renormalizing the space-time theory at some scale μ, and the 't Hooft coupling is then fixed and equal to

$$\frac{1}{g^2(\mu/\Lambda)} = \frac{1}{4\pi^2}\ln\frac{\mu}{\Lambda}. \tag{21}$$

Any choice $\mu > \Lambda$ is consistent and yields a real coupling. For simplicity, let us consider a case where we vary only one Higgs vev parameter v corresponding to the most relevant deformation from the critical point. The gauge theory depends on the dimensionless ratio r defined by (3). The expansion (13) is then more precisely written

$$A_h(r) = \sum_{n\geq 0} A_{h,n}(\mu/v)\, g^{2n}(\mu/\Lambda). \tag{22}$$

Note that the Λ-dependence only appears through the coupling g in perturbation theory. We thus have a different expansion for each different choices of μ. However, the critical properties of the series, such as (14) or (15), cannot depend of the choice of μ, because the physical amplitude $\mathcal{A}$ is μ-independent. This implies that the A_h themselves are μ-independent, because the β function for the 't Hooft coupling does not depend on N in our theories. In other words, the RG equations insure that the $n \to \infty$ asymptotics of the coefficients $A_{h,n}(\mu/v)$ are independent of μ. We can be more precise: if the radius of convergence of (22) for the special choice $\mu = v$ is $g_{\rm c}$ ($g_{\rm c}$ is actually infinite in all the cases we have studied), then the radius of convergence of (22) for an arbitrary $\mu > \Lambda$ is

$$g_{\rm c}^2(\mu/v) = \frac{g_{\rm c}^2}{1 + \dfrac{g_{\rm c}^2}{4\pi^2} \,|\ln\dfrac{\mu}{v}\,|}\cdot \tag{23}$$

This is demonstrated by noting that the series at general μ is obtained by substituting

$$g^2 = \frac{g^2(\mu/\Lambda)}{1 + \dfrac{g^2(\mu/\Lambda)}{4\pi^2}\,\ln\dfrac{v}{\mu}} \tag{24}$$

in the series for $\mu = v$. This implies that $g_{\rm c}^2(\mu/v) \geq g_{\rm c}^2/(1 + (g_{\rm c}^2/4\pi^2)|\ln(\mu/v)|)$, a strict inequality being possible mathematically. However, since the μ-independent quantities A_h diverge for $r = r_{\rm c} = v_{\rm c}/\Lambda$ such that $g^2(r_{\rm c}) = g_{\rm c}^2$, we must have $g^2(\mu/\Lambda) = g_{\rm c}^2(\mu/v_{\rm c})$ for all μ, which proves (23). When $r \to r_{\rm c}$, $g_{\rm c}^2 - g^2 \propto r - r_{\rm c} = \delta$, and the μ-independent world sheet cosmological constant is given by $\lambda = \delta/a^2$. The continuum limit is then $\delta \to 0$ at fixed λ and the double scaling limit is as in (17) with δ_g replaced by δ.

Let us note that the simplifying assumptions used in the above discussion, in particular the fact that the β function is N-independent, are by no means necessary conditions for double scaling limits to exist. If the β function is N-dependent, as occurs in less supersymmetric situations and in particular in the two dimensional models studied in [7], then double scaling limits must involve a mixing between different orders in $1/N$, and in particular the naive scaling (17) is corrected by logarithms. Even in our $\mathcal{N} = 2$ examples, this complication can occur, because simple scaling laws like (15) can be corrected by logarithmic terms, as we will see in Section 3.

2.3.3 A theory of open strings

Our preceding discussion focused exclusively on the contributions coming from Feynman diagrams. However, one of the main results of [6] was to show that, in $\mathcal{N} = 2$ super Yang-Mills, there is another type of contribution that plays an equally important rôle, the fractional instantons. These fractional instantons were interpreted as coming from the quantum disintegration of the large instantons that are present semi-classically. Remarkably, these fractional instantons generate a series in $1/N$, which fits perfectly in a string picture, provided one introduces open strings in addition to closed strings. We do not have a graphical representation for these open strings, analogous to the double line representation of Feynman diagrams for the closed strings. However, our conjecture is that the double scaling limit corresponds to a continuum limit for both open and closed strings. This is particularly significant, because the Seiberg-Witten periods that we will study in Section 3 have only a trivial one-loop contribution from the Feynman diagrams.

The presence of open strings might be interpreted by considering that the dual field theories are not confining. However, it is not clear whether our open strings can live in the bulk of space-time, or are actually stuck on some D-brane. The latter possibility seems more likely, because the open strings in $\mathcal{N}=2$ super Yang-Mills [6] could be attached to the inflating branes that make the supergravity background well-defined through the enhançon mechanism [15].

2.3.4 Non-perturbative string theory

An important feature of our double scaling limits is that they always provide a non-perturbative definition of the string theories, thus bypassing a fundamental limitation of the $c \leq 1$ matrix models. The way this non-perturbative definition is obtained is in some sense conservative, since it is based on the use of gauge theory path integrals. Note however that our string theories are not dual to gauge theories, but to other types of field theories. Our point is that the gauge theory path integrals *in the vicinity of critical points* automatically generate string theories.

Good simplified models for the $c \leq 1$ matrix models are the vector models, for which one integrates over N-vectors instead of $N \times N$-matrices. Double scaling limits can be defined in that context, and they generate continuum theories of polymers [16, 11]. For the same reasons as in the matrix models, the double scaling limits only give a perturbative definition in the cases that are not Borel summable. We have argued that for the matrix integrals, this problem is solved by considering gauge theories integrals. The similar idea for vector models is to consider non-linear σ models integrals. In bosonic cases, one can then explicitly generate non-Borel summable partition functions that are nevertheless non-perturbatively defined by the original vector integrals [17].

Another related interesting feature of our conjecture is that it makes a direct link between the existence of CFT in four dimensions and the fact that 4D field theories can be dual to string theories. The modern starting point for the field theory/string theory duality is indeed a correspondence between a conformal gauge theory and a string theory [18], but this is an a priori surprising feature because the string picture most naturally emerges in confining theories. The fact that a string description of CFT is possible comes from the rather non-trivial observation that the varying string tension is set by the Liouville coordinate [19]. In our approach, the string theory is directly generated by the Feynman diagrams of a gauge theory at some critical point, without refering to flux tubes or confinement.

One may wonder whether a correspondence between conformal *gauge* theories and string theories could be derived within our approach. It is well-known that such critical points can appear on the moduli space of various $\mathcal{N}=2$ supersymmetric gauge theories. For example, the scale invariant $\mathrm{SU}(N)$ theory with $N_{\mathrm{f}}=2N$ quark hypermultiplets in the fundamenetal representation, and various related cousins, are discussed in [20]. In those cases, it is very natural to obtain a theory with both open and closed strings, because the large N limit is also a large N_{f} limit. This suggests that the discussion of the present paper could be generalized. As for the $\mathcal{N}=4$ theory, which is not believed to contain open strings, an explicit construction along the lines of [20] does not seem to have appeared. It would be extremely interesting to investigate this particular theory in more details.

3 Exact amplitudes

Let us now compute explicitly the double scaled Seiberg-Witten period integrals z for a variety of critical points. The basic formula for z, as reviewed in [6], is [12, 14]

$$z_\alpha = \frac{1}{2i\pi}\oint_\alpha \frac{x\,dp}{y}\,, \tag{25}$$

where α is a cycle in the integer homology of the genus $N-1$ hyperelliptic curve

$$\mathcal{C}:\ y^2 = q(x)^2 = p(x)^2 - 1/r^{2N} = \prod_{i=1}^{N}(x-\phi_i)^2 - 1/r^{2N}. \tag{26}$$

The ϕ_is are related to the Higgs vevs moduli and satisfy $\sum_{i=1}^N \phi_i = 0$. We choose the contours α to be the vanishing cycles at the critical point under study. If those cycles have a non-zero intersection form, the low energy theory is an interacting CFT. As explained in [6], from the point of view of the large N limit, it is useful to distinguish between two classes of singularities depending on whether they occur when a classical root of q melts with the enhançon (first class) or when two disconnected pieces of the enhançon collide with each other (second class). The low energy space-time CFT does not depend upon the singularity being first class or second class, but the double scaled theory, the scaling, and the world sheet theory do depend on the class. In all the examples we have studied, the string susceptibility turns out to be

$$\gamma_{\rm str} = 0, \tag{27}$$

but the simple scaling (17) can be corrected by logarithms for the first class, in a way reminiscent of the $c = 1$ matrix model [21]. In the following, we discuss an A_{n-1} series of second class singularities generalizing the example A_1 of Section 2.1, and then we present the case of the simplest first class singularity.

3.1 An A_{n-1} series of multicritical points

We can get a second class A_{n-1} critical point by choosing N to be a multiple of n and the polynomial p entering in (26) to be

$$p(x) = \Bigl(x^n + \sum_{k=2}^{n-1} u_{n-k} x^{n-k} + 1\Bigr)^{N/n}. \tag{28}$$

The scaling limit is universal and does not depend of the precise way the second class critical point is embedded in the original gauge theory, so the special choice (28) is not restrictive. The parameters u_k, $1 \le k \le n-2$, and

$$u_0 = 1 - 1/r^n = \delta\,, \tag{29}$$

correspond to $\mathcal{N} = 2$ preserving relevant deformations around the critical point at $u_k = 0$. Their space-time anomalous dimensions can be calculated by noting that the curve (26) takes the form

$$y^2 \approx \frac{2N}{n}\Bigl(x^n + \sum_{k=2}^{n} u_{n-k} x^{n-k}\Bigr) \tag{30}$$

in the vicinity of the singularity and by using the fact that the Seiberg-Witten periods are of dimension one. The result is

$$[u_k] = \frac{2(n-k)}{n+2}\,, \qquad 0 \le k \le n-2. \tag{31}$$

The most relevant operator couples to u_0. This series of critical points corresponds to the M_n^0 SCFTs discussed in [22].

To understand how the double scaling limit works, let us write the polynomial p as

$$p(x) = \exp\left[\frac{N}{n}\,\ln\Big(1 + \sum_{k=1}^{n-2} u_k x^k + x^n\Big)\right] \tag{32}$$

and change the variable to

$$w = N^{1/n}\,x. \tag{33}$$

If we take

$$N \to \infty\,, \quad u_k \to 0\,, \quad t_k = N^{1-k/n}\,u_k = \text{constant}\,, \tag{34}$$

then p has a finite limit $\exp(t(w)/n)$ with

$$t(w) = \sum_{k=1}^{n-2} t_k w^k + w^n. \tag{35}$$

If, in addition to (34), we take

$$N \to \infty\,, \quad \delta \to 0\,, \quad t_0 \stackrel{\text{def}}{=} N\delta = \text{constant}, \tag{36}$$

then the rescaled amplitude $N^{1/n} z_\alpha$ will also have a finite limit,

$$N^{1/n}\,z_\alpha \longrightarrow \frac{1}{2i\pi n}\oint_\alpha \frac{wt'(w)}{\sqrt{1-e^{-2(t(w)+t_0)/n}}}\,dw. \tag{37}$$

The renormalized amplitude, which is finite in the continuum limit, is defined to be

$$\mathcal{A}_\alpha = \frac{i\pi z_\alpha}{\delta^{1/n}}. \tag{38}$$

The scalings (34) and (36) relate the *world-sheet* dimensions (not to be confused with the *space-time* dimensions (31)) of the renormalized parameters t_k, $\Delta_k = (1-k/n)\Delta_0$. By identifying the most relevant parameter t_0 with the cosmological constant, we obtain

$$\Delta_k = 2 - \frac{2k}{n}. \tag{39}$$

This formula suggests that the t_ks couple to bulk world sheet operators $\mathcal{O}_k$ of dimension $2k/n$. Similarly, the scaling (38) shows that $\mathcal{A}_\alpha$ is of dimension $2/n$.

It is convenient to introduce the new variables and polynomial

$$\tau_k = (2/n)^{1-k/n}\,t_k\,, \quad u = (2/n)^{1/n}w\,, \quad T(u) = \sum_{k=0}^{n-2}\tau_k u^k + u^n, \tag{40}$$

in terms of which

$$\mathcal{A}_\alpha = \frac{1}{4\tau_0^{1/n}}\oint_\alpha \frac{uT'(u)}{\sqrt{1-e^{-T(u)}}}\,du. \tag{41}$$

The contour α encircles any two roots of the polynomial T. Equation (41) gives our basic formula for the string theory amplitudes corresponding to the A_n multicritical points. Let us look in more details at the special case where only the most relevant operator is turned on. It is then natural to introduce the coupling $\kappa = 1/\tau_0$, to rescale $u \to \kappa^{-1/n}u$, and to consider the contours α_{jk} encircling the roots $u_j = \exp(i\pi(1+2j)/n)$ and u_k. A straightforward calculation then yields

$$\mathcal{A}_{\alpha_{jk}} = \frac{n}{4\kappa}\oint_{\alpha_{jk}} \frac{u^n\,du}{\sqrt{1-e^{-(1+u^n)/\kappa}}} = e^{i\pi(j+k+1)/n}\sin(\pi(j-k)/n)\,\mathcal{I}_n(\kappa) \tag{42}$$

where

$$\mathcal{I}_n(\kappa) = \frac{n}{(n+1)\kappa} + \int_0^{1/\kappa}\Big(\frac{1}{\sqrt{1-e^{-x}}}-1\Big)(1-\kappa x)^{1/n}dx. \tag{43}$$

The asymptotic expansion of $\mathcal{I}_n(\kappa)$ can be obtained by noting that when $x\sim 1/\kappa$, $1/\sqrt{1-\exp(-x)}-1$ is exponentially small. We thus have

$$\mathcal{I}_n(\kappa) = \frac{n}{(n+1)\kappa} + \sum_{k=0}^{K}\frac{\Gamma(k-1/n)}{\Gamma(-1/n)\Gamma(k+1)} I_k\,\kappa^k + \mathcal{O}(\kappa^{K+1}) \tag{44}$$

with

$$I_k = \int_0^\infty\Big(\frac{1}{\sqrt{1-e^{-x}}}-1\Big)x^k\,dx = \frac{(-1)^{k+1}}{k+1}\int_0^1\frac{\big(\ln(1-t)\big)^{k+1}}{2t^{3/2}}\,dt. \tag{45}$$

The first integrals I_k can be calculated by expanding the logarithm in powers of t,

$$I_0 = 2\ln 2\,,\quad I_1 = \frac{\pi^2}{6}-2(\ln 2)^2\,,\quad I_2 = \frac{8(\ln 2)^2}{3}-\frac{2\pi^2\ln 2}{3}+4\zeta(3)\,,\ \text{etc}\ldots \tag{46}$$

In particular, we recover the expansion (10).

3.2 Other scalings

We are now going to study the simplest first class singularity. More general cases, including mixed examples where first class and second class singularities collide, can be studied along the same lines. Let us choose

$$p(x) = (x+1/N)^{N-1}(x-1+1/N). \tag{47}$$

The first class singularity occurs at

$$r = r_{\rm c} = \frac{N}{(N-1)^{1-1/N}} \tag{48}$$

when the two positive real roots x_1 and $x_- > x_1$ of $p(x)+1/r^N$, that exist for $r>r_{\rm c}$, coincide (see Section 4.1 of [6]).

3.2.1 Elementary analysis

The large N expansion of the Seiberg-Witten period

$$z = \frac{1}{i\pi}\int_{x_1}^{x_-} \frac{xp'(x)}{\sqrt{p(x)^2 - 1/r^{2N}}} \tag{49}$$

was evaluated up to terms of order $1/N^3$ in [6], equation (59),

$$\begin{aligned} \frac{i\pi z}{N} &= -\frac{r-1}{r} + \ln r + \frac{1}{N}\left(-\ln r + \frac{r-1}{r}\ln 2 + \frac{(r-1)\ln(r-1)}{r}\right) \\ &\quad + \frac{1}{N^2 r}\left(\frac{1}{2}\left(\ln(r-1)\right)^2 + (\ln 2)\ln(r-1) + \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{24}\right) + \mathcal{O}(1/N^3). \end{aligned} \tag{50}$$

An important qualitative difference with the expansion at the second class critical points (as in (2) for example) is that there are logarithmic singularities when $r \to 1$, in addition to the simple power-like divergences. It is then impossible to find a simple scaling of the form (17). However, let us consider the renormalized amplitude

$$\mathcal{A} = i\pi N z \tag{51}$$

and the modified scaling

$$N \to \infty\,, \quad \delta = r - 1 \to 0\,, \quad N\delta - \ln N = \text{constant} = 1/\kappa'. \tag{52}$$

Then the leading term in (50) gives $1/(2\kappa'^2)$ plus divergent $(\ln N)/\kappa'$ and $(\ln N)^2$ terms. The first divergence is actually exactly canceled by taking into account the $1/N$ term in (50). All the divergences at order κ'^0, which come from all three terms in (50), turn out to cancel likewise, yielding the finite expansion

$$\mathcal{A} = \frac{1}{2\kappa'^2} - \frac{\ln(e\kappa'/2)}{\kappa'} + \frac{(\ln\kappa')^2}{2} - (\ln 2)\ln\kappa' + \frac{(\ln 2)^2}{2} - \frac{\pi^2}{24} + \cdots \tag{53}$$

We will show below that these cancellations actually work to all orders and beyond. The scaling (51) then shows that $\mathcal{A}$ has world sheet dimension two. The expansion (53) can be put in a more familiar form by introducing a new coupling κ defined by

$$1/\kappa' = 1/\kappa + \ln\kappa \tag{54}$$

and in terms of which

$$\mathcal{A} = \frac{1}{2\kappa^2} + \frac{\ln 2 - 1}{\kappa} + \frac{1}{2}(\ln 2)^2 - \frac{\pi^2}{24} + \mathcal{O}(\kappa). \tag{55}$$

3.2.2 The full amplitude

By changing the variable to $z = N(1/(x+1/N)-1)$ in (49) we get

$$\mathcal{A} = N\int_{z_-}^{z_1} dz\, \frac{1-z}{z(1+z/N)^2}\frac{[1-1/N-z/N^2][1+z/(N(1-z))]}{\sqrt{1-N^2(1+z/N)^{2N}/(z^2 r^{2N})}}\,. \tag{56}$$

A complication with respect to the cases studied in Section 3.1 stems from the fact that $\mathcal{A}$ comes with a factor of N in front of the integral. To show that the scaling (52) yields a finite limit, we thus have to check that the leading term is actually zero. The finite string amplitude is then extracted by taking into account subleading terms in the integrand and in z_- and z_1. Up to terms that trivially go to zero, we obtain

$$\mathcal{A} = \mathcal{A}_a + \mathcal{A}_b \tag{57}$$

with

$$\mathcal{A}_a = \int_{Z_-}^{Z_1} dz\, \frac{2z^2-1}{z}\frac{1}{\sqrt{1-e^{2(z-1/\kappa)}/(z\kappa)^2}}\,, \tag{58}$$

$$\mathcal{A}_b = N\int_{\hat{Z}_-}^{\hat{Z}_1} dz\, \frac{1-z}{z}\frac{1}{\sqrt{1-(1-z^2/N)e^{2(z-1/\kappa)}/(z\kappa)^2}}\,. \tag{59}$$

The boundaries of the integration region for $\mathcal{A}_a$, Z_- and $Z_1 > Z_-$, are the two positive solutions of

$$f(z) = ze^{-z} = f(1/\kappa) \stackrel{\text{def}}{=} \rho. \tag{60}$$

It is useful to introduce the inverses of the function f,

$$x = f(z) \iff \begin{cases} z = W_-(x) \text{ for } z\in[0,1] \\ z = W_1(x) \text{ for } z\in[1,+\infty[\,. \end{cases} \tag{61}$$

One then has

$$Z_- = W_-(\rho)\,,\qquad Z_1 = W_1(\rho) = 1/\kappa\,. \tag{62}$$

On the other hand, $\hat{Z}_-$ and $\hat{Z}_1 > \hat{Z}_-$ are solutions of

$$f(z) = \Big(1-\frac{z^2}{2N}\Big)\rho\,, \tag{63}$$

and in particular

$$f(\hat{Z}_-) = \Big(1-\frac{Z_-^2}{2N}\Big)\rho + \mathcal{O}(1/N^2)\,,\quad f(\hat{Z}_1) = \Big(1-\frac{Z_1^2}{2N}\Big)\rho + \mathcal{O}(1/N^2)\,. \tag{64}$$

Let us now change the variable in the integral (59) from z to $x = f(z)$. We get

$$\mathcal{A}_b^< = N \int_{f(\hat{Z}_-)}^{1/e} \frac{dx}{x} \frac{1}{\sqrt{1-(1-W_-^2(x)/N)\rho^2/x^2}}, \tag{65}$$

$$\mathcal{A}_b^> = N \int_{1/e}^{f(\hat{Z}_1)} \frac{dx}{x} \frac{1}{\sqrt{1-(1-W_1^2(x)/N)\rho^2/x^2}}. \tag{66}$$

The large N limits of these two integrals are similar. For example, to study (65), we substitute $x \to f(\hat{Z}_-) + \rho(x-1)$, and expand the integrand at large N by carefully taking into account (64). The result is

$$\mathcal{A}_b^< = -\frac{e\rho Z_-^2}{2\sqrt{1-e^2\rho^2}} + N\int_1^{\frac{1}{e\rho}} \frac{dx}{\sqrt{x^2-1}} + \frac{1}{2}\int_1^{\frac{1}{e\rho}} \frac{dx}{(x^2-1)^{3/2}} \left(xZ_-^2 - W_-^2(\rho x)\right) \cdot \tag{67}$$

The formula for $\mathcal{A}_b^>$ is the same except for a global sign change and the replacement of W_- and Z_- by W_1 and Z_1 respectively. The term proportional to N in thus finally

$$N\left(\int_1^{\frac{1}{e\rho}} + \int_{\frac{1}{e\rho}}^1\right) \frac{dx}{\sqrt{x^2-1}} = 0 \tag{68}$$

which shows that the amplitude indeed has a finite limit in the scaling (52),

$$\mathcal{A}_b = \frac{e\rho\left(Z_1^2 - Z_-^2\right)}{2\sqrt{1-e^2\rho^2}} + \frac{1}{2}\int_1^{\frac{1}{e\rho}} \frac{dx}{(x^2-1)^{3/2}} \left(W_1^2(\rho x) - W_-^2(\rho x) - x(Z_1^2 - Z_-^2)\right) \cdot \tag{69}$$

In the original variables, we finally end up with

$$\begin{aligned}\mathcal{A} = {} & \frac{e\rho\left(Z_1^2 - Z_-^2\right)}{2\sqrt{1-e^2\rho^2}} + \int_{Z_-}^{Z_1} dz\, \frac{2z^2-1}{z} \frac{1}{\sqrt{1-e^{2(z-1/\kappa)}/(z\kappa)^2}} \\ & + \frac{\rho^2}{2}\int_{Z_-}^1 dz\, \frac{z-1}{z^2} \frac{\left(z - Z_-^2 e^{-z}/\rho\right)e^{2z}}{(1-\rho^2 e^{2z}/z^2)^{3/2}} + \frac{\rho^2}{2}\int_1^{Z_1} dz\, \frac{z-1}{z^2} \frac{\left(z - Z_1^2 e^{-z}/\rho\right)e^{2z}}{(1-\rho^2 e^{2z}/z^2)^{3/2}} \cdot \end{aligned} \tag{70}$$

4 Conclusion

We have generalized the matrix model approach to non-critical string theories in less than one dimension [3, 1] to non-critical string theories in four dimensions. The main differences between the two cases is that the latter is always non-perturbatively defined and contains open strings in addition to closed strings. There are many tantalizing questions that we have not addressed. In particular, a standard continuum construction of the world sheet theory is lacking. It would be extremely interesting to be able to recover some of our results in this more familiar context.

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