Hierarchy of Dirac, Pauli and Klein-Gordon conserved operators in Taub-NUT background

Ion I. Cotăescu [*]

West University of Timişoara,

V. Pârvan Ave. 4, RO-1900 Timişoara, Romania

Mihai Visinescu [†]

Department of Theoretical Physics,

National Institute for Physics and Nuclear Engineering,

P.O.Box M.G.-6, Magurele, Bucharest, Romania

April 25, 2020

Abstract

The algebra of conserved observables of the $SO(4,1)$ gauge-invariant theory of the Dirac fermions in the external field of the Kaluza-Klein monopole is investigated. It is shown that the Dirac conserved operators have physical parts associated with Pauli operators that are also conserved in the sense of the Klein-Gordon theory. In this way one gets simpler methods of analyzing the properties of the conserved Dirac operators and their main algebraic structures including the representations of dynamical algebras governing the Dirac quantum modes.

Pacs 04.62.+v

[*]E-mail: cota@physics.uvt.ro

[†]E-mail: mvisin@theor1.theory.nipne.ro

1 Introduction

The relativistic quantum mechanics, seen as the one-particle restriction of the Lagrangian quantum field theory on curved spacetimes, give rise to interesting mathematical problems concerning the properties of the physical observables. It is known that one of the largest algebras of conserved operators is produced by the Euclidean Taub-NUT geometry since beside usual isometries this has a hidden symmetry of the Kepler type [1, 2]. This is related to the existence of Stäckel-Killing tensors connected with the components of an analogue to the Runge-Lenz vector of the Kepler type problem for which, in addition, can be expressed in terms of four Killing-Yano tensors [2, 3, 4].

   The theory of the Dirac equation in the Kaluza-Klein monopole field was studied in the mid eighties [5]. An attempt to take into account the Runge-Lenz vector of this problem was done in [6]. We have continued this study showing that the Dirac equation is analytically solvable [7] and determining the energy eigenspinors of the central modes. Moreover, we derived all the conserved observables of this theory, including those associated with the hidden symmetries of the Taub-NUT geometry. Thus we obtained the Runge-Lenz vector-operator of the Dirac theory, pointing out its specific properties [8]. The consequences of the existence of this operator were studied in [9] showing that the dynamical algebras of the Dirac theory corresponding to different spectral domains are the same as in the scalar case [2] but involving other irreducible representations. Thus for the discrete energy spectra we obtained two irreducible representations of the $o(4)$ algebra describing distinguish quantum modes for each energy level [9].

   This new phenomenon encourage us to continue the mathematical study of the whole algebra of conserved observables of the Dirac theory in Taub-NUT background. In our opinion, the operators related to the manifest or hidden symmetries of the Taub-NUT geometry are of a special interest since they reflect the effects of the geometry on the behavior of different quantum systems, with integer or half-integer spin. However, in the Dirac case there are several complicated operators whose manipulation can be sometime extremely difficult. We hope that a general study of their action on the Dirac spinors could lead to simpler calculation methods.

   The present article is devoted to this problem. Our goal is to separate the active parts of the conserved Dirac operators, called here physical parts, which determine the effects on the Dirac energy eigenspinors. Obviously these are projections obtained with the help of the projection operator on the whole space of physical spinors. We show that these are of a specific diagonal (even) or off-diagonal (odd) forms depending on Pauli operators that are also

conserved in the sense of the Klein-Gordon theory. In this way we derive simpler calculation rules and identify associations among conserved Dirac and Pauli operators as those currently used in theories involving monopoles [10, 11, 12] or new other we write down here.

We start in the second section with a brief review of some previous results [7, 8, 9] we need. In the next section we study the algebra of the conserved Dirac operators and introduce a new type of even operators which help us to define the projection operator that separates the physical parts. Furthermore, we point out that the diagonal (even) physical parts of the Dirac observables can be associated with well-defined conserved Pauli operators obeying the same algebraic relations. In Section 4 we discuss the physical parts of the main conserved Dirac operators [7, 8, 9] and we identify their associated Pauli conserved operators. Here after presenting the simplest conserved Pauli operators we derive those corresponding to isometries or hidden symmetries including the generators of the dynamical algebras. In this way we show that the results of [9] hold also in the case of continuous energy spectra, for $so(3,1)$ or $e(3)$ dynamical algebras. The conclusions and comments are presented in the last section and in a short Appendix some formulas involving an important Pauli operator studied in [11, 12] are given.

We work in natural units with $\hbar = c = 1$.

2 Preliminaries

The background of the gauge-invariant five-dimensional theory of the Dirac fermions in the external field of the Kaluza-Klein monopole [13] is the the Taub-NUT space with the time coordinate trivially added. It is convenient to consider the static chart of Cartesian coordinates x^μ, ($\mu, \nu, ... = 0, 1, 2, 3, 5$), with the line element

$$ ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - \frac{1}{V} dl^2 - V(dx^5 + A_i dx^i)^2, \qquad (1) $$

where $dl^2 = (d\vec{x})^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ is the usual Euclidean three-dimensional line element in Cartesian physical space coordinates x^i ($i, j, ... = 1, 2, 3$). The other coordinates are the time, $x^0 = t$, and the Cartesian Kaluza-Klein extra-coordinate, x^5. In (1) the function $1/V(r) = 1 + \mu/r$ depends on $r = |\vec{x}|$ and the real parameter μ while A_i are the potentials of the Dirac monopole.

This background has the isometry group $G_s = SO(3) \otimes U(1)_5 \otimes T_t(1)$ formed by the rotations of the Cartesian space coordinates and x^5 and t translations. The $U(1)_5$ symmetry is important since this eliminates the so

3

called NUT singularity if x^5 has the period $4\pi\mu$. The Killing vectors $k_{(i)}$ ($i = 1, 2, 3$) and $k_{(5)}$ are directly connected with the conserved operators which appear in the scalar case. They can be expressed in terms of momentum operators $P_i = -i(\partial_i - A_i\partial_5)$ and $P_5 = -i\partial_5$ [2]. The last one, for negative mass models, can be interpreted as the "relative electric charge" and it is always conserved. Moreover, the Taub-NUT geometry possesses four Killing-Yano tensors, $f_{\mu\nu}^{(i)}$ ($i = 1, 2, 3$) and $f_{\mu\nu}^Y$, of valence 2, related to the hidden symmetries of the Taub-NUT geometry reflected by the existence of the non-trivial Stäckel-Killing tensors $k_{(i)}^{\mu\nu}$ [2, 4, 7, 14].

In this Kaluza-Klein geometry there is a pentad gauge fixing [15] where the massless Dirac field, ψ, satisfies a simple gauge-covariant Dirac equation, $\slashed{D}\psi = 0$, where $\slashed{D} = i\gamma^0\partial_t - \slashed{D}_s$ [16, 5, 7, 9]. In the standard representation of the Dirac matrices (with diagonal γ^0 [17]) the Hamiltonian operator [7, 9]

$$H = \gamma^0\,\slashed{D}_s = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix} \tag{2}$$

has manifest supersymmetry. It is expressed in terms of Pauli operators,

$$\alpha \quad = \sqrt{V}\,\pi = \sqrt{V}\left(\sigma_P - \frac{iP_5}{V}\right), \tag{3}$$

$$\alpha^* \quad = V\pi^*\frac{1}{\sqrt{V}} = V\left(\sigma_P + \frac{iP_5}{V}\right)\frac{1}{\sqrt{V}}, \tag{4}$$

where $\sigma_P = \vec{\sigma}\cdot\vec{P}$ involves the Pauli matrices, σ_i. These operators give the space part of the Klein-Gordon operator as [7, 9],

$$\Delta = \alpha^*\alpha = V\pi^*\pi = V\vec{P}^2 + \frac{1}{V}P_5{}^2. \tag{5}$$

We specify that the star superscript is a mere notation that does not represent the Hermitian conjugation because here we use a non-unitary representation of the algebra of Dirac operators. Of course, this is equivalent with the unitary representation where all of these operators are self-adjoint [7].

Since P_5 commutes with all the other conserved observables, we settle its eigenvalue, $\hat{q}$, such that $q \equiv -\mu\hat{q} = 0, \pm 1/2, \pm 1, ...$ [1, 2]. We denote by $\mathfrak{S}$ the space of usual and generalized energy eigenspinors of the form $U_E = (u_E, E^{-1}\alpha u_E)^T$ which solve the eigenvalue problem $HU_E = EU_E$ [7]. In [7, 9] we showed that u_E is a solution of the static Klein-Gordon equation, $\Delta u_E = E^2 u_E$, that may be square integrable with respect to the specific relativistic scalar product of the Dirac theory [7] or behave as tempered distributions. The Klein-Gordon equation is analytically solvable producing

4

continuous energy spectra, $E \geq |\hat{q}|$, for any real μ and discrete energy levels, E_n with $n > |q| > 0$, only for $\mu < 0$. These are included in the domain $(0, \hat{q})$ such that $\lim_{n \to \infty} E_n = |\hat{q}|$. Hence, there are no zero modes and the operator Δ can be considered invertible on the space of of the spinors u_E. Therefore, we can conclude that the Dirac equation produces the *same* energy spectra as the Klein-Gordon one. Moreover, since there are no zero modes the Hamiltonian operator (2) is also invertible. The meaning of this operation will be discussed in the next section.

3 The algebra of conserved Dirac operators

Our aim is to study here the form and the action of the *conserved* operators of the Dirac theory which, by definition, are the Dirac operators that *commute* with the Hamiltonian (2). We denote by $\mathbf{D}$ the algebra of these operators. We say that a Pauli operator $\hat{X}$ acting on the space of the two-component Pauli spinors u_E is *conserved* if it commutes with Δ. We denote by $\mathbf{P}$ the algebra of the conserved Pauli operators, including the conserved observables of the Klein-Gordon theory, called here *orbital* operators. In this section we denote systematically by capitals the operators of $\mathbf{D}$ and by hated ones the operators of $\mathbf{P}$ without to use special notations for the identity operators of these algebras.

In general, the Pauli blocks, $\hat{X}^{(ab)}$ ($a, b = 1, 2$), of any conserved Dirac operator

$$X = \left(\begin{array}{cc} \hat{X}^{(11)} & \hat{X}^{(12)} \\ \hat{X}^{(21)} & \hat{X}^{(22)} \end{array} \right) \in \mathbf{D} \tag{6}$$

satisfy the conditions

$$\hat{X}^{(22)} \alpha = \alpha \hat{X}^{(11)}, \quad \alpha^* \hat{X}^{(22)} = \hat{X}^{(11)} \alpha^*, \tag{7}$$

$$\hat{X}^{(12)} \alpha = \alpha^* \hat{X}^{(21)}, \quad \alpha \hat{X}^{(12)} = \hat{X}^{(21)} \alpha^* \tag{8}$$

which are equivalent with $[X, H] = 0$. Hereby it results that

$$\hat{X}^{(21)} = \alpha \hat{X}^{(12)} \alpha \Delta^{-1} \tag{9}$$

and

$$[\hat{X}^{(11)}, \Delta] = [\hat{X}^{(12)} \alpha, \Delta] = [\alpha^* \hat{X}^{(21)}, \Delta] = 0 \tag{10}$$

which means that $\hat{X}^{(11)}, \hat{X}^{(12)} \alpha, \alpha^* \hat{X}^{(21)} \in \mathbf{P}$.

We observe that possible solutions of Eqs.(7) and (8) are the diagonal operators

$$\mathcal{D}(\hat{X}) = \left(\begin{array}{cc} \hat{X} & 0 \\ 0 & \alpha \hat{X} \Delta^{-1} \alpha^* \end{array} \right) \tag{11}$$

where $\hat{X} \in \mathbf{P}$. Particularly, for $\hat{X} = 1$ we obtain the projection operator

$$I = \mathcal{D}(1) = \begin{pmatrix} 1 & 0 \\ 0 & \alpha\Delta^{-1}\alpha^* \end{pmatrix} \tag{12}$$

on the space $\mathfrak{S}$. This split the algebra $\mathbf{D} = \mathbf{D}_0 \oplus \mathbf{D}_1$ in two subspaces of the projections $XI \in \mathbf{D}_0$ and $X(1-I) \in \mathbf{D}_1$ of all $X \in \mathbf{D}$. According to Eqs.(7) and (8) we find that the projections of two arbitrary operators $X, Y \in \mathbf{D}$ satisfy $(XI)(YI) = (XY)I$ and $[X(1-I)](YI) = 0$ which lead to the conclusion that $\mathbf{D}_0$ is a subalgebra while $\mathbf{D}_1$ is even an ideal in $\mathbf{D}$. Obviously, I is the identity operator of $\mathbf{D}_0$. On the other hand, in [7] we introduced the $\mathcal{Q}$-operators defined as

$$\mathcal{Q}(\hat{X}) = \left\{ H, \begin{pmatrix} \hat{X} & 0 \\ 0 & 0 \end{pmatrix} \right\} = \begin{pmatrix} 0 & \hat{X}\alpha^* \\ \alpha\hat{X} & 0 \end{pmatrix}, \tag{13}$$

where $\hat{X}$ may be any Pauli operator. However, if $\hat{X} \in \mathbf{P}$ then $\mathcal{Q}(\hat{X}) \in \mathbf{D}_0$ since $[\mathcal{Q}(\hat{X}), H] = 0$ and $\mathcal{Q}(\hat{X})I = \mathcal{Q}(\hat{X})$. If $\hat{X} = 1$ we obtain just the Hamiltonian operator $H = \mathcal{Q}(1) \in \mathbf{D}_0$. Consequently, the inverse of H with respect to I can be represented as $H^{-1} = \mathcal{Q}(\Delta^{-1})$. The mappings $\mathcal{D} : \mathbf{P} \to \mathbf{D}_0$ and $\mathcal{Q} : \mathbf{P} \to \mathbf{D}_0$ are linear and have the following algebraic properties

$$\mathcal{D}(\hat{X})\mathcal{D}(\hat{Y}) = \mathcal{D}(\hat{X}\hat{Y}), \tag{14}$$
$$\mathcal{Q}(\hat{X})\mathcal{Q}(\hat{Y}) = \mathcal{D}(\hat{X}\hat{Y}\Delta), \tag{15}$$
$$\mathcal{D}(\hat{X})\mathcal{Q}(\hat{Y}) = \mathcal{Q}(\hat{X})\mathcal{D}(\hat{Y}) = \mathcal{Q}(\hat{X}\hat{Y}), \tag{16}$$

for any $\hat{X}, \hat{Y} \in \mathbf{P}$. Moreover, the relations

$$[\gamma^0, \mathcal{D}(\hat{X})] = 0, \quad \{\gamma^0, \mathcal{Q}(\hat{X})\} = 0 \tag{17}$$

indicate that, according to the usual terminology [17], $\mathcal{D}$ and $\gamma^0\mathcal{D}$ are *even* Dirac operators while $\mathcal{Q}$ and $\gamma^0\mathcal{Q}$ are *odd* ones. We note that there are many other odd or even operators which do not have such forms.

In general, since I is the projection operator on the space of the Dirac energy eigenspinors $\mathfrak{S}$, we say that the projection IXI of any Dirac operator X, conserved or not, represents the *physical part* of X. We can convince ourselves that if $X \in \mathbf{D}$ then

$$IXI \equiv XI = \mathcal{D}(\hat{X}^{(11)}) + \mathcal{Q}(\hat{X}^{(12)}\alpha\Delta^{-1}), \tag{18}$$

which means that all the operators from $\mathbf{D}_0$ can be written as $\mathcal{D}$ or $\mathcal{Q}$-operators. Thus the action of X reduces to that of the Pauli operators

6

involved in (18) allowing us to rewrite the problems of the Dirac theory in terms of Pauli operators [8, 9]. Indeed, it is easy to show that the action of any operator $X \in \mathbf{D}$ on $U_E \in \mathfrak{G}$ is

$$XU_E = XIU_E = \begin{pmatrix} \hat{\mathcal{P}}_E(X)\, u_E \\ E^{-1}\alpha\hat{\mathcal{P}}_E(X)\, u_E \end{pmatrix}, \qquad (19)$$

where, by definition,

$$\hat{\mathcal{P}}_E(X) = \hat{X}^{(11)} + E^{-1}\hat{X}^{(12)}\alpha \qquad (20)$$

is the *conserved* Pauli operator *associated* to X. Since the mapping $\hat{\mathcal{P}}_E : \mathbf{D} \to \mathbf{P}$ is linear and satisfies $\hat{\mathcal{P}}_E(X) = \hat{\mathcal{P}}_E(XI)$ it results that $\mathrm{Ker}\,\hat{\mathcal{P}}_E = \mathbf{D}_1$. In other respects, Eqs.(7) and (8) lead to the important property

$$\hat{\mathcal{P}}_E(XY) = \hat{\mathcal{P}}_E(X)\hat{\mathcal{P}}_E(Y)\,, \quad \forall X, Y \in \mathbf{D}\,. \qquad (21)$$

which guarantees that $\hat{\mathcal{P}}_E$ preserves the algebraic relations, mapping any algebra or superalgebra of $\mathbf{D}_0$ into an *isomorphic* algebra or superalgebra of $\mathbf{P}$, with the same commutation and anticommutation rules.

4 Conserved observables

In what follows we focus on the physical parts of the main Dirac conserved observables pointing out the technical advantages of using $\mathcal{D}$ and $\mathcal{Q}$-operators that help us to identify the associated Pauli operators defined by (20). The even physical parts, $\mathcal{D}$, are associated to Pauli operators independent on E which are, therefore, well-defined physical observables. For this reason it is useful to briefly review the most important conserved Pauli operators and then turn to the physical parts of the Dirac ones.

4.1 Conserved orbital and Pauli operators

In general, the Pauli operators are 2×2 matrix differential operators acting on two-component Pauli spinors. There are many non conserved operators which do not commute with Δ as, for example, $\alpha,\ \alpha^*,\ \pi,\ \pi^*,\ \sigma_r = \vec{\sigma}\cdot\vec{x}/r$ or the operator $\lambda = \vec{\sigma}\cdot(\vec{x}\times\vec{P}) + 1$ proposed in [11] and discussed in [12]. Some algebraic properties of these operators are given in Appendix.

By definition, the conserved operators of $\mathbf{P}$ commute with Δ which is the static part of the Klein-Gordon operator. As mentioned, these can be the usual conserved orbital operators of the scalar fields or more complicated ones involving, in addition, the Pauli matrices which also commute with Δ.

The main conserved orbital operators are the basis generators of the natural representation of the group G_s carried by the space of scalar fields. These generators are defined up to the factor $-i$ as the Killing vector fields corresponding to isometries or the operators given by the Killing tensors associated to the hidden symmetries. The $U_5(1)$ generator is P_5 and the Killing vectors $k^\mu_{(i)}$ give the $SO(3)$ generators which are the components L_i of the orbital angular momentum operator [7, 9]

$$\vec{L} = \vec{x} \times \vec{P} - \mu \frac{\vec{x}}{r} P_5 \,. \tag{22}$$

These commute with Δ and satisfy the canonical commutation relations among themselves and with the components of all the other vector operators (e.g. coordinates, momenta, etc.). On the other hand, the specific Killing tensors $k^{\mu\nu}_{(i)}$ of the Taub-NUT geometry allow one to define the Runge-Lenz operator for scalar particles [2]

$$\vec{K} = \frac{1}{2}(\vec{P} \times \vec{L} - \vec{L} \times \vec{P}) - \frac{\mu}{2}\frac{\vec{x}}{r}\Delta + \mu\frac{\vec{x}}{r}P_5{}^2 \,, \tag{23}$$

which commute with Δ and its components satisfy the commutation relations

$$[L_i,\, K_j] = i\varepsilon_{ijk} K_k \,, \quad [K_i,\, K_j] = i\varepsilon_{ijk} L_k F^2 \,, \tag{24}$$

where $F^2 = P_5{}^2 - \Delta$. For given values of E and $\hat{q}$ this operator can be rescaled in order to recover the dynamical algebras corresponding to different spectral domains of the Kepler-type problems [2]. The new operators

$$R_i = \begin{cases} F^{-1}K_i & \text{for} \quad \mu < 0 \quad \text{and} \quad E < |\hat{q}| \\ K_i & \text{for} \quad \text{any } \mu \quad \text{and} \quad E = |\hat{q}| \\ \pm iF^{-1}K_i & \text{for} \quad \text{any } \mu \quad \text{and} \quad E > |\hat{q}| \end{cases} \tag{25}$$

and L_i ($i = 1, 2, 3$) generate either a representation of the $o(4)$ algebra for the discrete energy spectrum in the domain $0 < E < |\hat{q}|$ or a representation of the $o(3,1)$ algebra for continuous spectrum in the domain $E > |\hat{q}|$. A special case is that of the dynamical algebra $e(3)$ which corresponds only to the ground energy of the continuous spectrum, $E = |\hat{q}|$.

The operators of $\mathbf{P}$ involving Pauli matrices can be vector operators as the total angular momentum,

$$\vec{J} = \vec{L} + \frac{\vec{\sigma}}{2} \,, \tag{26}$$

or scalar operators of the form $\sigma_L = \vec{\sigma} \cdot \vec{L}$, $\sigma_K = \vec{\sigma} \cdot \vec{K}$ or $\sigma_R = \vec{\sigma} \cdot \vec{R}$, involved in superalgebras as

$$\{\sigma_K,\, \sigma_L + 1\} = 0 \,. \tag{27}$$

Other conserved Pauli operators with more complicated structure have to be derived in association with the physical parts of the conserved Dirac observables.

4.2 Associated Dirac and Pauli operators

We have seen that the physical parts of the conserved Dirac observables can have diagonal or off-diagonal terms among them only the diagonal ones can be correctly associated to conserved Pauli operators independent on E. However, the off-diagonal operators can be transformed at any time in diagonal ones using the multiplication with H or H^{-1}. For example, H itself which is off-diagonal is related to the diagonal operators $H^2 = \mathcal{D}(\Delta)$ or I. Thus each conserved Dirac operator can be brought in a diagonal form associated with an operator from $\mathbf{P}$.

Let us start with the generators of the representations of the group G_s carried by the space of the Dirac spinors. The $U(1)_5$ generator remains the former operator P_5 but the $SO(3)$ generators get the usual spin terms, $S_i = \frac{1}{2}\mathrm{diag}(\sigma_i, \sigma_i)$ of the total angular momentum whose components, $\mathcal{J}_i = L_i + S_i$, commute with H even if neither L_i nor S_i do not have this property [7, 9]. However, the effect on the spinors of $\mathbf{S}$ is due only to the physical parts which read

$$\mathcal{J}_i I = \mathcal{D}(J_i) = \mathcal{D}(L_i) + \frac{1}{2}\mathcal{D}(\sigma_i) \tag{28}$$

where both the orbital and the spin terms are *separately* conserved since L_i and σ_i commute with Δ. Obviously, in this case the associated Pauli operators are just J_i defined by (26).

The simplest conserved off-diagonal operators are the so called Dirac-type operators generated by the first three Killing-Yano tensors, $f^{(i)}$. We have shown [7, 8] that these can be written simply in the form $Q_i = \mathcal{Q}(\sigma_i)$ which explains why their algebraic properties are close to those of the Pauli matrices. Now, we can prove that the diagonal operators $H^{-1}Q_i = \mathcal{D}(\sigma_i)$ form a representation of the algebra of Pauli matrices with values in $\mathbf{D}_0$ since

$$H^{-1}Q_i H^{-1}Q_j = \delta_{ij}I + i\varepsilon_{ijk}H^{-1}Q_k \,. \tag{29}$$

The corresponding Dirac-type operator of the last Killing-Yano tensor, f^Y, calculated according to the general rule of [18] has been obtained in [8]. This has the form

$$Q^Y = -\mathcal{Q}(\sigma_r) + \frac{2i}{\mu\sqrt{V}}\begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}. \tag{30}$$

Using the identities presented in Appendix one finds the equivalent forms reported in [8] and verify that Q^Y commutes with H and P_5 and anticommutes with $\not{D}_s$ and γ^0. Moreover, after a little calculation, we obtain the remarkable identity

$$\mu P_5 \left[Q^Y + \mathcal{Q}(\sigma_r) \right] = \{H, \Lambda\} \tag{31}$$

involving the operator $\Lambda = \operatorname{diag}(\lambda, \lambda)$ that is a particular version of the Biedenharn operator [19]. This is not conserved but $\Lambda^2 = \vec{\mathcal{J}}^2 - \mu^2 P_5{}^2 + 1/4$ has this property. Furthermore, we observe that, according to (A.1) and (A.3), the physical part of Q^Y can be put in the form

$$Q^Y I = \mathcal{Q} \left(-\sigma_r + \frac{2i}{\mu} \lambda \pi \Delta^{-1} \right) = \mathcal{Q}(\sigma^Y \Delta^{-1}) \,, \tag{32}$$

where

$$\sigma^Y = \frac{2}{\mu} \left[\sigma_K + (\sigma_L + 1) P_5 \right] \tag{33}$$

is a new conserved Pauli operator associated to $HQ^Y = HQ^Y I = \mathcal{D}(\sigma^Y)$.

4.3 The Runge-Lenz operator and dynamical algebras

These results allow us to calculate directly the physical parts of the Runge-Lenz operator of the Dirac theory (related to the Killing tensor $\vec{k}^{\mu\nu}$), following the same procedure as in [8]. We start with the equivalent definition of the physical parts of the auxiliary operators [8]

$$\mathcal{N}_i I = \frac{\mu}{4} \left\{ HQ^Y , H^{-1} Q_i \right\} - \mathcal{J}_i P_5 I \,, \tag{34}$$

which can be written as

$$\mathcal{N}_i I = \mathcal{D} \left(\frac{\mu}{4} \left\{ \sigma^Y , \sigma_i \right\} - J_i P_5 \right) = \mathcal{D}(\hat{N}_i) \,, \tag{35}$$

in terms of their associated conserved Pauli operators,

$$\hat{N}_i = K_i + \frac{\sigma_i}{2} P_5 \,. \tag{36}$$

Furthermore, we define the physical parts of the components of the conserved Runge-Lenz operator [8, 9]

$$\mathcal{K}_i I = \mathcal{N}_i I + \frac{1}{2} (\mathcal{F} - P_5) H^{-1} Q_i \tag{37}$$

10

where $\mathcal{F}^2 = P_5{}^2 - H^2$. Since $\mathcal{F}^2 I = \mathcal{D}(F^2)$, we can express $\mathcal{K}_i I = \mathcal{D}(\hat{K}_i)$, now the associated conserved Pauli operators being,

$$\hat{K}_i = K_i + \frac{\sigma_i}{2} F. \tag{38}$$

All these associations help us to understand the significance of the isomorphism among the algebra of the Dirac operators [8, 9],

$$[\mathcal{J}_i, \mathcal{K}_j] = i\varepsilon_{ijk}\mathcal{K}_k, \quad [\mathcal{K}_i, \mathcal{K}_j] = i\varepsilon_{ijk}\mathcal{J}_k\mathcal{F}^2, \tag{39}$$

that of the Pauli operators,

$$\left[J_i, \hat{K}_j\right] = i\varepsilon_{ijk}\hat{K}_k, \quad \left[\hat{K}_i, \hat{K}_j\right] = i\varepsilon_{ijk}J_k F^2, \tag{40}$$

and (24).

Re-scaling (37) as in the case of the orbital operators (25), but using $\mathcal{F}$ instead of $\mathcal{E}$, one obtains the even operators $\mathcal{R}_i \in \mathbf{D}$ [9] having simple physical parts, $\mathcal{R}_i I = \mathcal{D}(\hat{R}_i)$, associated with the conserved Pauli operators

$$\hat{R}_i = \begin{cases} R_i + \frac{\sigma_i}{2} & \text{for} \quad E \neq \hat{q} \\ K_i & \text{for} \quad E = \hat{q} \end{cases}. \tag{41}$$

We specify that the orbital and spin terms of $\mathcal{N}_i I$, $\mathcal{K}_i I$ and $\mathcal{R}_i I$ (for $E \neq \hat{q}$) are also separately conserved, as in the case of the angular momentum, since K_i and F commute with Δ.

The representations of the dynamical algebras $o(4)$ or $o(3,1)$ that govern the Dirac modes for $E \neq \hat{q}$ are generated by $\mathcal{J}_i$ and $\mathcal{R}_i$ [9] whose physical parts, $\mathcal{J}_i I$ and $\mathcal{R}_i I$, have the same spin terms, $\mathcal{D}(\sigma_i)/2$. Therefore, each of these representations is the direct product between the irreducible representation of scalar modes and a spin half two-dimensional (fundamental) representation of the dynamical algebra [9]. When $E = \hat{q}$ then $\mathcal{F}$ and F vanish such that the representation of the subalgebra $so(3) \subset e(3)$ remains generated by the operators (28) while the operators $\mathcal{K}_i I$ lose their spin terms becoming the translation generators of $e(3)$. All these representations arising from direct products are reducible. We note that this phenomenon is new since in the scalar (Klein-Gordon) case the representations of the dynamical algebras of the Kepler-type problems are irreducible [2]. However, these results could be easily obtained analyzing the *equivalent* representations generated by the associated Pauli operators J_i and $\hat{R}_i$ as we did already in [9] for the discrete energy spectrum. We recall that therein we introduced the new conserved operator $\mathcal{C} = 2\vec{\mathcal{J}} \cdot \vec{\mathcal{R}} - 1/2$ in order to distinguish between the irreducible representations of the $o(4)$ dynamical algebra. Now we see that $\mathcal{C} I = \mathcal{D}(\sigma_R + \sigma_L + 1)$ where, according to (27), we have $\{\sigma_R, \sigma_L + 1\} = 0$.

5 Conclusions

The first conclusion is that our approach allows one to associate the conserved Dirac operators of diagonal (even) form to conserved Pauli operators independent on E. Thus for each type of symmetry we have conserved operators at three levels: Dirac, Pauli and orbital (of the Klein-Gordon theory). The following table resumes this hierarchy (K is an abbreviation for Killing, K-Y for Killing-Yano; * denotes entries which involve issues too complex to be abbreviated in the table and some comments are given below).

geometric object	nature	symmetry	Dirac operator	Pauli operator	Klein-Gordon operator
$f_{\mu\nu}^{(i)}$	K-Y tensor	*	$H^{-1}Q_i$	σ_i	-
$f_{\mu\nu}^{Y}$	K-Y tensor	*	HQ^Y	σ^Y	-
$k_{(5)}^{\mu}$	K vector	$U(1)_5$	P_5	P_5	P_5
$k_{(i)}^{\mu}$	K vector	$SO(3)$	$\mathcal{J}_i$	J_i	L_i
$k_{(i)}^{\mu\nu}$	K tensor	hidden	$\mathcal{K}_i, \mathcal{R}_i$	$\hat{K}_i, \hat{R}_i$	K_i, R_i

However, there are many other even or odd conserved Dirac operators (e.g., $\mathcal{D}(\sigma_K^2)$, $\mathcal{Q}(L_i)$, $\mathcal{Q}(\sigma^Y)$, etc.) which can be constructed with the help of the conserved Pauli or orbital ones. This large collection of conserved observables is in fact a rich algebra freely generated by those related to the manifest or hidden symmetries of the Taub-NUT geometry.

In $N = 1$ supersymmetric quantum models with standard supersymmetry there is a single supercharge Q that closes $Q^2 = H$ on the Hamiltonian. In many of these models, and that is the case of the Taub-NUT manifold, one can find additional or hidden, non standard supercharges involving Killing-Yano tensors. The Killing-Yano tensors $f_{\mu\nu}^{(i)}$ ($i = 1, 2, 3$) give a vector representation of $SO(3)$ and their existence is connected with the complex structures of the hyper-Kähler Taub-NUT space. The forth Killing-Yano tensor $f_{\mu\nu}^{Y}$ is a singlet and exists by virtue of the metric being type D. All four Killing-Yano tensors are invariant under the action of $U(1)_5$ which physically represents the relative electric charge of two monopoles.

For spin-$\frac{1}{2}$ particles, the Killing-Yano tensors are essential in construction of Dirac-type operators and evaluation of the spin contributions to the conserved quantities from the scalar case. The antisymmetric feature of these operators make them the natural object used in description of the Dirac

fermion in a curved spacetime. On the other hand, the fact that the Stäckel-Killing tensors involved in the Runge-Lenz vector (23) can be expressed as symmetrized products of Killing-Yano tensors seems to be useless for scalar particles described by Schrödinger or Klein-Gordon equations. Therefore the existence of a certain square root of the Stäckel-Killing tensors becomes relevant only in the presence of fermions.

In other respects, we can eliminate many difficulties due to the spin terms of the Dirac theory if we restrict ourselves only to the physical parts XI of the operators $X \in \mathbf{D}$ giving up the projections $X(1 - I) \in \mathbf{D_1}$ which can give rise sometime to very complicated calculations. Moreover, we get the advantage of reducing the algebraic operations among the physical parts from $\mathbf{D_0}$ to calculations involving only the associated Pauli operators from $\mathbf{P}$. For example, if instead of $[\vec{Q}, Q^Y]$, we calculate only its physical part, $[\vec{Q}, Q^Y]I = [H^{-1}\vec{Q}, HQ^Y] = \mathcal{D}([\vec{\sigma}, \sigma^Y])$, we avoid a tedious algebra easily obtaining the interesting identity

$$\frac{\mu}{4}[\vec{Q}, Q^Y I] = i(\vec{\mathcal{K}} + \vec{\mathcal{J}}P_5) \times (H^{-1}\vec{Q}) + (F + P_5)H^{-1}\vec{Q} \qquad (42)$$

which shows that this commutator does not produce new conserved observables.

Finally we note that our method based on the separation of the physical parts expressed in terms of $\mathcal{D}$ and $\mathcal{Q}$-operators could be used in any problem where the Hamiltonian is invertible and has manifest supersymmetry.

Appendix A: The operator λ

The operator

$$\lambda = \vec{\sigma} \cdot (\vec{x} \times \vec{P}) + 1 = \sigma_L + 1 + \mu\sigma_r P_5 \qquad (A.1)$$

has the properties

$$\{\sigma_r, \lambda\} = 0, \quad [\sigma_r, \sigma_P] = \frac{2i}{r}\lambda \qquad (A.2)$$

and

$$\sigma_P \lambda = -\lambda \sigma_P = \frac{1}{2}\vec{\sigma} \cdot (\vec{P} \times \vec{L} - \vec{L} \times \vec{P}) - \frac{i\mu}{r}\lambda P_5 \qquad (A.3)$$

which lead to

$$\{\alpha^*, \lambda\} = \frac{2i}{\sqrt{V}}\lambda P_5, \quad \{\alpha, \lambda\} = -\frac{2i}{\sqrt{V}}\lambda P_5. \qquad (A.4)$$

13

References

[1] G. W.. Gibbons and N. S. Manton, *Nucl. Phys.* **B274** (1986) 183.

[2] G. W. Gibbons and P. J. Ruback, *Phys. Lett.* **B188** (1987)226; *Commun. Math. Phys.* **115** (1988) 267; L. Gy. Feher and P. A. Horváthy, *Phys. Lett.* **B183** (1987) 182; B. Cordani, Gy. Feher and P. A. Horváthy, *Phys. Lett.* **B201** (1988) 481.

[3] J. W. van Holten, *Phys. Lett.* **B342** (1995) 47.

[4] D. Vaman and M. Visinescu, *Phys. Rev.* **D57** (1998) 3790; D. Vaman and M. Visinescu, *Fortschr. Phys.* **47** (1999) 493.

[5] Z. F. Ezawa and A. Iwazaki, *Phys. Lett.* **B138** (1984) 81; M. Kobayashi and A. Sugamoto, *Progr. Theor. Phys.* **72** (1984) 122; A. Bais and P. Batenberg, *Nucl. Phys.* **B245** (1984) 469.

[6] A. Comtet and P. A. Horváthy, *Phys. Lett.* **B349** (1995) 49.

[7] I. I. Cotăescu and M. Visinescu, `hep-th/0008181`; *Int. J. Mod. Phys. A* **16** (2001) 1743.

[8] I. I. Cotăescu and M. Visinescu, `hep-th/0101163`; *Phys. Lett.* **B502** (2001) 229.

[9] I. I. Cotăescu and M. Visinescu, `hep-th/0102083`; *Class. Quantum Grav.* to appear.

[10] E. D'Hoker and L. Vinet, *Phys. Lett.* **B137** (1984) 72; F. Bloore and P. A. Horváthy, *J. Math. Phys.* **33** (1992) 1869.

[11] F. De Jonge, A. J. Macfarlane, K. Peters and J. - W. van Holten, *Phys. Lett.* **B359** (1995) 114.

[12] P. A. Horváthy, A. J. Macfarlane and J.- W. van Holten *Phys. Lett.* **B486** (2000) 346.

[13] D. J. Gross and M. J. Perry, *Nucl. Phys.* **B226** (1983) 29; R. D. Sorkin, *Phys. Rev. Lett.* **51** (1983) 87.

[14] I. I. Cotăescu and M. Visinescu, *Mod. Phys. Lett. A***15** (2000) 145.

[15] H. Boutaleb - Joutei and A. Chakrabarti, *Phys. Rev.* **D21** (1979) 2280.

[16] A. Macias and H. Dehnen, *Class. Quantum Grav.* **8** (1991) 203.

[17] B. Thaller, *The Dirac Equation* (Springer Verlag, Berlin Heidelberg, 1992).

[18] B. Carter and R. G. McLenaghan, *Phys. Rev.* **D19** (1979) 1093.

[19] L. C. Biedenharn, *Phys. Rev.* **162** (1962) 845; L. C. Biedenharn and N. V. V. Swamy, *ibid* **B5** (1964) 1353;