Covariant Thermodynamics of Quantum Systems: Passivity, Semipassivity, and the Unruh Effect

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Abstract

According to the Second Law of Thermodynamics, cycles applied to thermodynamic equilibrium states cannot perform any work (passivity property of thermodynamic equilibrium states). In the presence of matter this can hold only in the rest frame of the matter, as moving matter drives, e.g., windmills and turbines. If, however, a homogeneous and stationary state has the property that no cycle can perform more work than an ideal windmill, then it can be shown that there is some inertial frame where the state is a thermodynamic equilibrium state. This provides a covariant characterization of thermodynamic equilibrium states.

In the absence of matter, cycles should perform work only when driven by nonstationary inertial forces caused by the observer's motion. If a (pure) state of a relativistic quantum field theory behaves this way, it satisfies the spectrum condition and exhibits the Unruh effect.

1   Introduction

In [34], Pusz and Woronowicz analyzed thermodynamic equilibrium states in a general quantum theoretical setting. They established that the condition of complete passivity, which can be derived from the first principles of thermodynamics, is, at nonzero temperature, equivalent to the Kubo-Martin-Schwinger (KMS-) condition, a widely used generalization of the Gibbs characterization of thermodynamic equilibrium states that is appropriate also for systems with infinitely many degrees of freedom [23, 28, 21, 20].

Complete passivity is a strengthening of the principle that cycles, when applied to thermodynamic equilibrium states, cannot perform any work, which is the passivity property of thermodynamic equilibrium states. In the

presence of matter, however, a system can exhibit passivity in one distinguished frame of reference at most, as moving matter can drive cycles (e.g., windmills and turbines). So the question arises what a covariant version of the (complete) passivity condition could look like. This problem is of particular interest if the state under consideration is stationary and homogeneous, as its invariance properties alone do not distinguish any frame of reference in this case. If a thermodynamic system is covariant under a representation of the spacetime translation group, it possesses at least one stationary and homogeneous nonequilibrium state [31], and thermal equilibrium states in quantum field theories exhibit themselves as nonequilibrium states to moving observers [31, 33].

In this article, the Pusz-Wornowicz analysis will be generalized to a version that can be used in each inertial frame of reference. It is well known that the power of a windmill or a turbine depends on the third power of the velocity of the current driving the device. It will be shown in Sect. 3 that if a cycle applied to a stationary and homogeneous state ω cannot perform more work than such a device, then there is a frame of reference where the considered state is a thermodynamic equilibrium state in the sense that it satisfies the KMS-condition or is a ground state of the Hamiltonian, which corresponds to the case of zero temperature.

The first covariant characterization of thermal states in quantum field theory was recently given by Bros and Buchholz [9]. Their criterion is a relativistic KMS-condition, and they expected that this condition could eventually be derived from the assumption that the work a cycle can extract from the system within a given spacetime region is bounded by a constant that depends on the size of the spacetime region. This conjecture motivated the analysis to follow and is partially confirmed by its results.

In Sect. 4 it is briefly discussed how the results of Sect. 3 can be applied to describe the chemical potential.

In Sect. 5, the notion of a vacuum state is discussed. Due to the absence of matter and energy flows, only nonstationary inertial forces caused by an acceleration of the observer should drive cycles when applied to a vacuum state. It is shown that the spectrum condition holds for each pure state behaving this way.

In Sect. 6, it is shown that if a state of a local quantum field theory behaves this way, then it exhibits the Unruh effect. The Unruh effect first has been established by Unruh [37] for the free field and, independently and simultaneously, by Bisognano and Wichmann [3, 4] for finite-component Wightman fields. A recent derivation for massive particle states in algebraic quantum field theory is due to Mund [30], and a couple of uniqueness results concerning the Unruh effect in this setting can be found in [6, 25] and the references quoted there.

Results similar to those of Sects. 5 and 6 have recently been presented in [12, 13] for quantum fields on de Sitter and Anti-de Sitter spacetimes,

2

respectively: assuming that a given state exhibits the KMS-condition to all uniformly accelerated observers, covariance under a representation of the spacetime's symmetry group was established, and the corresponding Hawking temperatures were computed.

Sect. 7 summarizes the results.

2 Thermodynamic equilibrium and passivity

In the quantum statistical mechanics of general quantum systems [7, 8], each quantum system is characterized by its algebra $\mathcal{M}$ of observables, which we here assume to be a von Neumann algebra[1]. Each state of the system is described by a linear functional ω on $\mathcal{M}$ that associates with each $A \in \mathcal{M}$ the corresponding expectation value $\omega(A) \in \mathbf{C}$ such that $\omega(A^*A) \geq 0$ for all $A \in \mathcal{M}$ and $\omega(1) = 1$.

In what follows, one state ω will be our object of investigation, and it will be assumed that this state is induced by a cyclic vector Ω of $\mathcal{M}$ (i.e., $\omega(A) = \langle \Omega, A\Omega \rangle$, and $\overline{\mathcal{M}\Omega} = \mathcal{H}$). The existence of Ω can be assumed without loss of generality by using the GNS-representation of ω, as all properties of $\mathcal{M}$ assumed below are inherited by this representation.

A selfadjoint operator H will be considered as the free Hamiltonian of the system: for each $A \in \mathcal{M}$ and each time $t \in \mathbf{R}$, it is assumed that $A_t := e^{itH}Ae^{-itH}$ lies in $\mathcal{M}$. We consider the case that ω is stationary with respect to this time evolution, i.e., that $\omega(A) = \omega(A_t)$ for all $A \in \mathcal{M}$ and all $t \in \mathbf{R}$. In this case, Ω is an eigenvector of H, and subtracting the corresponding eigenvalue from H, one can choose H such that $H\Omega = 0$.

If $e^{-\beta H}$ is trace class for a $\beta \geq 0$, then ω is a thermodynamic equilibrium state if it is a Gibbs state at the inverse temperature β. In this case, the two-point function $z \mapsto (\operatorname{tr}(e^{-\beta H}))^{-1}\operatorname{tr}(e^{-\beta H}e^{izH}Be^{-izH}A) =: F(z)$ is analytic in the open strip $S_\beta := \{z \in \mathbf{C} : -\beta < \operatorname{Im} z < 0\}$ and continuous on $\overline{S}_\beta$, and it satisfies

$$F(t) = \omega(B_t A) \quad \text{and} \quad F(t - i\beta) = \omega(AB_t) \quad \text{for all } t \in \mathbf{R}, \quad A, B \in \mathcal{M}. \tag{1}$$

In general, ω is called a *KMS-state (at the inverse temperature β)* of the dynamics generated by H if there exists a continuous function $F : \overline{S}_\beta \to \mathbf{C}$ that is analytic in S_β and satisfies the boundary condition (1). While the KMS-condition is equivalent to the Gibbs condition for finite volume

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[1]A von Neumann algebra is a linear space $\mathcal{M}$ of bounded operators in $\mathcal{H}$ that contains the adjoint of each of its elements and the operator product of any two of its elements, and that coincides with its *bicommutant* $\mathcal{M}''$; here $\mathcal{M}'$ denotes the *commutant* of $\mathcal{M}$, i.e., the (von Neumann) algebra of all bounded linear operators that commute with all elements of $\mathcal{M}$, and $\mathcal{M}'' := (\mathcal{M}')'$. One could choose $\mathcal{M}$ within a larger class of algebras as well, namely, the C*-algebras with a unit element, but to save notation, we confine ourselves to von Neumann algebras.

systems, it remains a meaningful condition in the general case (of, e.g., an infinitely extended system) that $e^{-\beta H}$ is not trace class, as it merely refers to the two-point function of ω. For this reason, the KMS-condition is used to characterize thermodynamic equilibrium states in such cases ([21], cf. also [20]). KMS-states at infinite temperature are traces, i.e., states satisfying $\omega(AB) = \omega(BA)$ for all $A, B \in \mathcal{M}$, and ω may be considered a "KMS-state at zero temperature" if $H \geq 0$, i.e., if ω is a *ground state* of H.

Physically, a thermodynamic equilibrium state can be characterized by its reaction to a cyclic reversible change of the external conditions, a *cycle*. For the time being, we consider as a cycle any perturbation of H by (time-dependent) self-adjoint elements $h(t)$ of $\mathcal{M}$ that depend on $t \in \mathbf{R}$ in a norm-continuously differentiable fashion and vanish for $t \notin [0, T]$ for some $T > 0$. The duration t_h of the cycle h is the smallest $T > 0$ satisfying this condition.

Dyson's perturbation series yields the perturbed unitary time evolution $(U_h(t))_{t \geq 0}$ that solves the equation

$$\frac{\mathrm{d}}{\mathrm{d}t} U_h(t) = -i e^{itH} h(t) e^{-itH} U_h(t), \qquad t \geq 0, \tag{2}$$

with the initial condition $U_h(0) = 1$. If Ω is the system's state vector at $t = 0$, this time evolution determines the state vector $\psi(t) := e^{-itH} U_h(t)\Omega$ for $t > 0$; formally, this is expressed by the Schrödinger equation $i\frac{\mathrm{d}}{\mathrm{d}t}\psi(t) = (H + h(t))\psi(t)$. The expectation value of the rate at which the process h adds energy to the system at the time t is $\langle \psi(t), \dot{h}(t)\psi(t) \rangle$, and the expectation value of the energy added to the system by the time t_h is

$$L_h := \int_0^{t_h} \langle \psi(t), \dot{h}(t)\psi(t) \rangle \, dt.$$

As h is a reversible process, there is no heat extracted from or added to the system, so by the First Law of Thermodynamics, L_h is the work required for the cycle h; equivalently, $-L_h$ is the work performed by h. The Second Law of Thermodynamics requires that $-L_h \leq 0$ for all cycles h if ω is a thermodynamic equilibrium state.

ω is called a *passive* state if $-L_h \leq 0$ for all cycles h. Let $U(\mathcal{M})$ denote the group of unitary elements of $\mathcal{M}$ and $U_1(\mathcal{M})$ the norm-connected component of $U(\mathcal{M})$ that contains the unit operator. By Thm. 2.1 in [34], ω exhibits passivity if and only if one has

$$-\langle W\Omega, [H, W]\Omega \rangle = -\langle W\Omega, HW\Omega \rangle \leq 0 \tag{3}$$

for all $W \in U_1(\mathcal{M})$ with $[H, W] \in \mathcal{M}$.[2] For the typical finite-temperature case that ω is known not to be a trace, a glance at the proof of this theorem

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[2]The expression $\langle Hx, Wy \rangle - \langle x, WHy \rangle$ is defined for all x, y in the domain of H. $[H, W] \in \mathcal{M}$ means that the sesquilinear form defined this way is bounded and that the

in [34] shows that passivity holds if and only if Ineq. (3) holds for *all* unitary elements W of $\mathcal{M}$ with $[H, W] \in \mathcal{M}$.

The unitaries W can be interpreted as propagators $U_h(t_h)$: if h is a cycle with $[H, h(t)] \in \mathcal{M}$ for all $t \in \mathbf{R}$, then $L_h = \langle U_h(t_h)\Omega, H U_h(t_h)\Omega \rangle$. In the form of Ineq. (3), the passivity condition does no longer depend on the technical condition that $h(t)$ depends on t in a norm continuously differentiable fashion; it applies to any cycle that provides some appropriate unitary propagator $W \in \mathcal{M}$ with $[H, W] \in \mathcal{M}$.

While a mixture of passive states is passive, a mixture of states at different temperatures does not have any well-defined temperature and, hence, cannot be a thermodynamic equilibrium state according to the Zeroth Law. It follows that the class of passive states is considerably larger than that of thermodynamic equilibrium states, and that a stronger assumption is needed to single out the thermodynamic equilibrium states.

As one way to strengthen the notion of passivity accordingly, the condition of complete passivity was used in [34].[3] It can be physically motivated as follows. The Zeroth Law of thermodynamics requires that if a system in thermodynamic equilibrium is coupled to an identical system in thermodynamic equilibrium and at the same temperature, the combined system should be in thermodynamic equilibrium as well, and this should also hold if more than two identical copies of the system are coupled. So if ω is a thermodynamic equilibrium state and if $\mathcal{M}^{\otimes N}$ is the Nth tensorial power of $\mathcal{M}$, the product state defined by

$$\mathcal{M}^{\otimes N} \ni A_1 \otimes A_2 \otimes \cdots \otimes A_N \mapsto \omega(A_1)\omega(A_2)\ldots\omega(A_n)$$

should be passive as well. ω is then called *completely passive*. As shown in [34], a state is completely passive if and only if it is a KMS-state or a ground state.

3   Moving systems and semipassivity

A more general version of (complete) passivity can be used to characterize thermodynamic equilibrium states in a covariant fashion, i.e., by a criterion that does not only hold in the system's rest frame. This issue is of interest if the state ω is stationary and homogeneous: if ω is stationary in no inertial frame, there is no thermodynamic equilibrium to characterize, and if it is stationary in a unique inertial frame, this frame is already distinguished by

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associated bounded operator is an element of $\mathcal{M}$; if commutators involving one unbounded selfadjoint operator are referred to as elements of $\mathcal{M}$ in what follows, this is to be read this way. Note that the vector $A\Omega$ is in the domain of H for all $A \in \mathcal{M}$ with $[H, A] \in \mathcal{M}$ (cf. Prop. 3.2.55 in [7] and p. 280 in [34]).

[3]As an alternative, Pusz and Woronowicz strengthen the passivity assumption by assuming a cluster property in addition, which characterizes pure thermodynamic phases. It is straightforward to check that our results to follow can be modified accordingly.

this fact, and it suffices to apply the results of Pusz and Woronowicz. But stationarity with respect to two distinct time evolutions implies invariance of ω under translations in at least one spatial direction.

In what follows, we consider the case that there are $s \geq 1$ spatial directions with this property. The above generator H of the time translations and s other self-adjoint operators $P_1, \ldots, P_s$, which we consider as the generators of the spatial translations and which we collect in the vector operator $\mathbf{P}$, will be assumed to generate a unitary representation V of the $(1+s)$-dimensional translation group $(\mathbf{R}^{1+s}, +)$ such that $V(x)AV(x)^* \in \mathcal{M}$ for all $A \in \mathcal{M}$ and all $x \in \mathbf{R}^{1+s}$. ω is assumed to be invariant under V, i.e., $\omega(V(x)AV(x)^*) = \omega(A)$ for all $A \in \mathcal{M}$ and all $x \in \mathbf{R}^{1+s}$, so Ω is an eigenvector of $\mathbf{P}$, and again, we can (without loss) choose $\mathbf{P}$ such that $P_1\Omega = \cdots = P_s\Omega = 0$.

Suppose that ω is passive with respect to the Hamiltonian H. If the system is not at rest, but moves at a velocity $\mathbf{u}$, then the time evolution is not generated by H, but by $H + \mathbf{u}\mathbf{P}$.[4] If $W \in U_1(\mathcal{M})$ satisfies $[H, W] \in \mathcal{M}$ and $[\mathbf{P}, W] \in \mathcal{M}$, then the work performed by the corresponding cycle is

$$- L = -\langle W\Omega, (H + \mathbf{u}\mathbf{P}) W\Omega\rangle$$
$$\leq -\langle W\Omega, \mathbf{u}\mathbf{P}\, W\Omega\rangle,$$

as ω is passive with respect to H. Defining $|\mathbf{P}| := \sqrt{P_1^2 + \cdots + P_s^2}$, one finds $-\mathbf{u}\mathbf{P} \leq |\mathbf{u}|\,|\mathbf{P}|$, so

$$- L \leq |\mathbf{u}|\langle W\Omega, |\mathbf{P}|\, W\Omega\rangle. \tag{4}$$

Now suppose that ω is not necessarily passive with respect to H. We call ω *semipassive* if the work a cycle can perform is bounded as in Ineq. (4), i.e., if there is a constant $\mathcal{E} \geq 0$ such that

$$- \langle W\Omega, H\, W\Omega\rangle \leq \mathcal{E}\langle W\Omega, |\mathbf{P}|\, W\Omega\rangle \tag{5}$$

for all $W \in U_1(\mathcal{M})$ with $[H, W] \in \mathcal{M}$ and $[\mathbf{P}, W] \in \mathcal{M}$. The constant $\mathcal{E}$ will be referred to as an *efficiency bound* of ω. Generalizing also the notion of complete passivity, ω will be called a *completely semipassive* state if all its finite tensorial powers are semipassive with respect to one fixed efficiency bound $\mathcal{E}$.

By the above considerations, a state is completely semipassive in all inertial frames if it is completely passive in some inertial frame. We will prove now that if, conversely, a state is completely semipassive in a given inertial frame, then there exists an inertial frame where it is completely passive.

We proceed in two steps by distinguishing the cases that ω is faithful, i.e., that given any $A \in \mathcal{M}$, $\omega(A^*A) = 0$ implies $A = 0$, and that ω is not faithful.

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[4]In a relativistic theory, this generator must be multiplied by the time dilation factor $\gamma = (1 - \mathbf{u}^2/c^2)^{-1/2}$; see below.

Proposition 3.1 *Let ω be completely semipassive with efficiency bound $\mathcal{E}$. If ω is faithful, then there exists a $\mathbf{u} \in \mathbf{R}^s$ with $|\mathbf{u}| \leq \mathcal{E}$ such that*

  (i) $H + \mathbf{u}\mathbf{P} = 0$, *or*

  (ii) ω *is a KMS-state at finite $\beta \geq 0$ with respect to $H + \mathbf{u}\mathbf{P}$.*

Proof. Using the reasoning that lead to Ineq. (3.8) in [34], one can derive from semipassivity that the useful inequality

$$\langle A\Omega, (H + \mathcal{E}|\mathbf{P}|)e^{-H^2 - |\mathbf{P}|^2} A\Omega \rangle + \langle A^*\Omega, (H + \mathcal{E}|\mathbf{P}|)e^{-H^2 - |\mathbf{P}|^2} A^*\Omega \rangle \geq 0 \quad (6)$$

holds for *all* $A \in \mathcal{M}$. The factor $e^{-H^2 - |\mathbf{P}|^2}$ is a mollifier that avoids domain problems.

As ω is faithful, Ω is separating, so Tomita-Takesaki theory (cf. App. A) defines the modular operator Δ and the infinitesimal generator $K = \ln(\Delta)$ of the modular automorphism group. As the dynamics generated by K satisfies the KMS-condition and is, up to a scalar multiplication of K, the only such dynamics, we compare H and $\mathbf{P}$ with K.

As H and $\mathbf{P}$ generate one-parameter unitary groups that act as automorphisms on $\mathcal{M}$ and leave Ω fixed, they strongly commute with K (cf. App. A). One obtains from Ineq. (6) (see [34] for details) that

$$-H(1 - \Delta)e^{-H^2 - |\mathbf{P}|^2} \leq \mathcal{E}|\mathbf{P}|(1 + \Delta)e^{-H^2 - |\mathbf{P}|^2}.$$

By this inequality, the joint spectrum[5] $\sigma_{H,\mathbf{P},K}$ of H, $\mathbf{P}$, and K is a subset of the set

$$\sigma^{(\mathcal{E})} := \{(\eta, \mathbf{k}, \kappa) \in \mathbf{R}^{s+2} : -\eta(1 - e^{\kappa}) \leq \mathcal{E}|\mathbf{k}|(1 + e^{\kappa})\},$$

which contains the entire $\kappa = 0$-plane and all $(\eta, \mathbf{k}, \kappa) \in \mathbf{R}^{s+2}$ with $\kappa > 0$ and

$$\eta \leq \mathcal{E}|\mathbf{k}|\frac{e^{\kappa} + 1}{e^{\kappa} - 1}, \quad (7)$$

and all $(\eta, \mathbf{k}, \kappa) \in \mathbf{R}^{s+2}$ with $\kappa < 0$ and

$$\eta \geq -\mathcal{E}|\mathbf{k}|\frac{1 + e^{\kappa}}{1 - e^{\kappa}}. \quad (8)$$

Using complete semipassivity and the identities $JHJ = -H$, $J\mathbf{P}J = -\mathbf{P}$, and $JKJ = -K$, (cf. App. A) one can argue like in [34] to show that $\sigma_{H,\mathbf{P},K}$ is a subset of a subgroup $\tilde{\sigma}_{H,\mathbf{P},K}$ of $(\mathbf{R}^{s+2}, +)$ that is a subset of $\sigma^{(\mathcal{E})}$, so the above estimates imply that it must be a subset of an at most (s+1)-dimensional subspace of $\mathbf{R}^{s+2}$. The smallest such subspace X must be a subset of $\sigma^{(\mathcal{E})}$ as well. Namely, if $(\eta, \mathbf{k}, \kappa) \in \sigma^{(\mathcal{E})}$, then $(\lambda\eta, \lambda\mathbf{k}, \lambda\kappa) \in \sigma^{(\mathcal{E})}$ for all $\lambda \in [0, 1]$, so $\bigcup_{\lambda \in [0,1]} \lambda\tilde{\sigma}_{H,\mathbf{P},K} \subset \sigma^{(\mathcal{E})}$. As X is the closure of the left-hand side, and as $\sigma^{(\mathcal{E})}$ is a closed set, it follows that $X \subset \sigma^{(\mathcal{E})}$, as stated.

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[5]See App. B for some remarks on joint spectra.

Alternative (i) states that H is a linear function of $\mathbf{P}$. In particular, this holds if X contains the κ-axis: by Lemma B.2, the joint spectrum of H and $\mathbf{P}$ is the closure of the image of $\sigma_{H,\mathbf{P},K}$ under the orthogonal projection π_κ along the κ-axis onto the η-$\mathbf{k}$-plane. As $\sigma_{H,\mathbf{P},K} \subset X$ and as $\pi_\kappa(X)$ is closed, it follows that $\sigma_{H,\mathbf{P}} \subset \pi_\kappa(X)$. Since X contains the κ-axis, it now follows that a point $(\eta, \mathbf{k})$ can be in $\sigma_{H,\mathbf{P}}$ only if $\{(\eta, \mathbf{k})\} \times \mathbf{R} \subset X$. It follows that Ineqs. (7) and (8) hold for all $\kappa > 0$ and all $\kappa < 0$, respectively, and one finds $-\mathcal{E}|\mathbf{P}| \leq H \leq \mathcal{E}|\mathbf{P}|$. But as the joint spectrum of H and $\mathbf{P}$ is a subspace of $\pi_\kappa(X)$, this inequality, together with Lemma B.1, entails that H is a linear function of $\mathbf{P}$, as stated.

It remains to prove Alternative (ii) for the case that H is not a linear function of $\mathbf{P}$. By what we just proved, X does not contain the κ-axis in this case, so K is a linear function of H and $\mathbf{P}$ (cf. Lemma B.1), i.e., there are $\beta \in \mathbf{R}$ and $\mathbf{v} \in \mathbf{R}^s$ such that

$$K = -\beta H + \mathbf{v}\mathbf{P}. \tag{9}$$

The vector $\mathbf{v}$ is unique up to a component that is perpendicular to the smallest linear subspace Y of $\mathbf{R}^s$ containing the joint spectrum of the components of $\mathbf{P}$, so $\mathbf{v}$ can and will be chosen in Y.

If $\mathbf{v}\mathbf{P} = 0$, then $K = -\beta H$, and Ineq. (7) reads

$$-\frac{\kappa}{\beta} \leq \mathcal{E}|\mathbf{k}|\frac{e^\kappa + 1}{e^\kappa - 1}.$$

for all $\kappa > 0$ and all $\mathbf{k} \in Y$, so $\beta > 0$, which yields Alternative (ii).

In the remaining case, $\mathbf{v}\mathbf{P} \neq 0$, and since $\mathbf{v} \in Y$, the unit vector $\mathbf{e_v} := |\mathbf{v}|^{-1}\mathbf{v}$ is in Y.

If $\beta \neq 0$, then Eq. (9) and the assumption that H is not a function of $\mathbf{P}$ entail $K \neq 0$, so for each $\kappa > 0$ and each $\lambda > 0$, one has $(\eta(\lambda, \kappa), \lambda\mathbf{e_v}, \kappa) \in X$, where

$$\eta(\lambda, \kappa) := -\frac{1}{\beta}(\kappa + \lambda\mathbf{e_v}\mathbf{v}) = -\frac{1}{\beta}(\kappa + \lambda|\mathbf{v}|).$$

Since $X \subset \sigma^{(\mathcal{E})}$, Ineq. (7) yields

$$-\frac{1}{\beta}(\kappa + \lambda|\mathbf{v}|) \leq \lambda\mathcal{E}\frac{e^\kappa + 1}{e^\kappa - 1}$$

for all $\kappa, \lambda > 0$, so $\beta > 0$ and $|\frac{\mathbf{v}}{\beta}| \leq \mathcal{E}$, and putting $\mathbf{u} := \mathbf{v}\beta$, one obtains Alternative (ii).

We can now complete the proof by showing that in the remaining case that $\beta = 0$, one arrives at $K = 0$, so that ω is a trace, i.e., a KMS-state at infinite temperature. As H is not a linear function of $\mathbf{P}$, while $\beta = 0$ implies $K = \mathbf{v}\mathbf{P}$, H cannot be a linear function of K and $\mathbf{P}$, so X must contain the η-axis. But if K did not equal zero, there would exist a $\mathbf{k} \in \mathbf{R}^s$ such that $\mathbf{v}\mathbf{k} > 0$ and $(\eta, \mathbf{k}, \mathbf{v}\mathbf{k}) \in X$ for all $\eta \in \mathbf{R}$, so Ineq. (7) would imply

8

$\eta \leq \mathcal{E}|\mathbf{k}|\frac{e^{\mathbf{vk}}+1}{e^{\mathbf{vk}}-1}$ for all $\eta \in \mathbf{R}$. As $\mathbf{vk} > 0$ by assumption, this is impossible, so $K = 0$, as stated. $\qquad\square$

The next proposition considers the case that ω is not faithful.

Proposition 3.2 *Let ω be completely semipassive with efficiency bound $\mathcal{E}$. If ω is not faithful, then there exists a $\mathbf{u} \in \mathbf{R}^s$ with $|\mathbf{u}| \leq \mathcal{E}$ such that $H + \mathbf{uP} \geq 0$.*

Proof. As ω is cyclic with respect to $\mathcal{M}$, it is separating with respect to $\mathcal{M}'$. As the projection operator $E_{\mathbf{h}}$ onto the closed subspace $\mathbf{h} := \overline{\mathcal{M}'\Omega}$ is easily seen to be an element of $\mathcal{M}$, the algebra $E_{\mathbf{h}}\mathcal{M}E_{\mathbf{h}} := \{E_{\mathbf{h}}ME_{\mathbf{h}} : M \in \mathcal{M}\}$ is a von Neumann subalgebra of $\mathcal{M}$, and with respect to the von Neumann algebra

$$\mathcal{N} := \{E_{\mathbf{h}}M|_{\mathbf{h}} : M \in \mathcal{M}\}$$

of operators in the Hilbert space $\mathbf{h}$, Ω is both cyclic and separating. It is also straightforward to check that the representation V maps $\mathbf{h}$ and $\mathbf{h}^\perp$ onto themselves, that it strongly commutes with $E_{\mathbf{h}}$ and, hence, implements automorphisms of $\mathcal{N}$.

Now let Δ be the modular operator of $\mathcal{N}$ and Ω, and define the positive operator $\tilde{\Delta} := \Delta E_{\mathbf{h}}$. One checks (cf. App. A) that $\tilde{\Delta}$ strongly commutes with H, so one can consider the joint spectrum $\sigma_{H,\mathbf{P},\tilde{\Delta}}$ of H, $\mathbf{P}$, and $\tilde{\Delta}$.

If $A \in \mathcal{M}$, then $B := AE_{\mathbf{h}}$ lies in $\mathcal{M}$ as well, and inserting B into Ineq. (6) yields, after the procedure followed earlier,

$$-H(1 - \tilde{\Delta})e^{-H^2 - |\mathbf{P}|^2} \leq \mathcal{E}|\mathbf{P}|(1 + \tilde{\Delta})e^{-H^2 - |\mathbf{P}|^2}.$$

It follows that $\sigma_{H,\mathbf{P},\tilde{\Delta}}$ is a subset of the set

$$\sigma^{(\mathcal{E})} := \{(\eta, \mathbf{k}, \delta) \in \mathbf{R} \times \mathbf{R}^s \times \mathbf{R}^{\geq 0} : -\eta(1 - \delta) \leq \mathcal{E}|\mathbf{k}|(1 + \delta)\}.$$

The points in $\sigma^{(\mathcal{E})}$ of the form $(\eta, \mathbf{k}, 0)$ satisfy the estimate

$$-\eta \leq \mathcal{E}|\mathbf{k}|. \tag{10}$$

The spectrum $\sigma_{H,\mathbf{P},\tilde{\Delta}}$ contains at least one such point: as Ω is not separating with respect to $\mathcal{M}$, while being separating with respect to $\mathcal{N}$ by construction, one has $\mathcal{M} \neq \mathcal{N}$ and $\mathbf{h} \neq \mathcal{H}$, and as $\tilde{\Delta}$ annihilates all elements of $\mathbf{h}^\perp \neq \{0\}$, the elements of $\mathbf{h}^\perp$ are zero eigenvectors of $\tilde{\Delta}$. Meanwhile, the representation V maps the subspace $\mathbf{h}^\perp$ onto itself, so it follows that $H|_{\mathbf{h}^\perp}$ and $\mathbf{P}|_{\mathbf{h}^\perp}$ are self-adjoint operators in $\mathbf{h}^\perp$, whose spectral projections are restrictions of the corresponding spectral projections of H and $\mathbf{P}$, respectively. But this implies that $\sigma_{H,\mathbf{P},\tilde{\Delta}}$ contains some point of the form $(\eta, \mathbf{k}, 0)$.

Next we prove that all points in $\sigma_{H,\mathbf{P},\tilde{\Delta}}$ must satisfy Ineq. (10) (even though not all of them are of the form $(\eta, \mathbf{k}, 0)$).

To show that the opposite case cannot occur, choose $(\eta, \mathbf{k}, 0) \in \sigma_{H,\mathbf{P},\tilde{\Delta}}$, and let $(\eta', \mathbf{k}', \delta')$ be any point in $\sigma_{H,\mathbf{P},\tilde{\Delta}}$ that violates Ineq. (10). As ω is completely semipassive, it follows that

$$(\eta + n\eta', \mathbf{k} + n\mathbf{k}', 0 \cdot n\delta') \in \sigma^{(\mathcal{E})} \qquad \text{for all } n \in \mathbf{N},$$

so

$$-(\eta + n\eta') \leq \mathcal{E}|\mathbf{k} + n\mathbf{k}'| \qquad \text{for all } n \in \mathbf{N}.$$

Choosing n sufficiently large, one now arrives at a contradiction with the assumption that $-\eta' > \mathcal{E}|\mathbf{k}'|$. It follows that Ineq. (10) must hold for all elements of $\sigma_{H,\mathbf{P},\tilde{\Delta}}$, as stated.

If one now applies Lemma B.2, one finds Ineq. (10) for all $(\eta, \mathbf{k}) \in \sigma_{H,\mathbf{P}}$. But by complete semipassivity, the corresponding estimate should hold for all tensorial powers of ω, which, as above, implies that $\sigma_{H,\mathbf{P}}$ is a subset of a sub-semigroup of $\mathbf{R}^{1+s}$ whose elements satisfy Ineq. (10). But such a semigroup must be a subset of a half space whose elements satisfy Ineq. (10) as well, so it follows from Lemma B.2 that there exists a $\mathbf{u} \in \mathbf{R}^s$ such that $|\mathbf{u}| \leq \mathcal{E}$ and such that the operator $H + \mathbf{u}\mathbf{P}$ is positive, which is the statement.

$\blacksquare$

Summing up our results, one now obtains the following theorem:

Theorem 3.3 *The state ω is completely semipassive with efficiency bound $\mathcal{E}$ if and only if there exists a $\mathbf{u} \in \mathbf{R}^s$ with $|\mathbf{u}| \leq \mathcal{E}$ such that with respect to $H + \mathbf{u}\mathbf{P}$, ω is a ground state or a KMS-state at a finite inverse temperature $\beta \geq 0$.*

In a relativistic theory, the generator of the time evolution of the system moving at velocity $\mathbf{u} < c$ is not $H + \mathbf{u}\mathbf{P}$, but $\gamma(H + \mathbf{u}\mathbf{P})$, where $\gamma = (1 - |\mathbf{u}|^2/c^2)^{-\frac{1}{2}} \equiv (1 - |\mathbf{u}|^2)^{-\frac{1}{2}}$. Theorem 3.3 still holds without any modification, but the inverse temperature of the system is not the parameter β found there, but β/γ.

4   Semipassivity and the chemical potential

Above, we have considered the operators $\mathbf{P}$ as the generators of the spatial translations. But that $\mathbf{P}$ plays this concrete role, has not been used in the proofs, so other applications can easily be thought of. A diffusion of particles or charges from or into infinite reservoirs can take place in a stationary fashion (cf. also [35, 18, 2, 20, 22]). In this case, there should be a vector $(N_1, \ldots, N_n) =: N$ of self-adjoint operators that can modify the generator of the time evolution accordingly. Again, we assume that $N_1\Omega = \cdots = N_n\Omega = 0$.

Such diffusion processes can make cycles perform work (cf. also [32]), and the condition of semipassivity with respect to N means that any work performed by a cycle can be performed by these effects only. The state ω is semipassive with respect to N if there exists a nonnegative constant $\mathcal{E}_N$ such that

$$-\langle W\Omega, HW\Omega \rangle \leq \langle W\Omega, \mathcal{E}_N|N|W\Omega \rangle$$

for all $W \in U_1(\mathcal{M})$ with $[H,W] \in \mathcal{M}$ and $[N,W] \in \mathcal{M}$. Mimicking the proofs of Props. 3.1 and 3.2, one directly obtains the following corollary.

Corollary 4.1 *With respect to N, the state ω is completely semipassive with efficiency bound $\mathcal{E}_N$ if and only if there is a vector $\mu := (\mu_1, \ldots, \mu_n)$ such that with respect to $H + \mu N$, ω is either a ground state or a KMS-state at a finite inverse temperature $\beta \geq 0$.*

The vector μ collects the chemical potentials associated with the different particles or charges.

5 Passivity and vacuum states

A cycle should perform work only if there is either some flow of matter or if the cycle is driven by a nonstationary inertial force due to the observer's motion. Since matter is completely absent in vacuum, each vacuum state should be passive in every uniformly accelerated frame (whose acceleration may be zero).

In this section, we show that if a pure state ω behaves this way, then it is invariant under spacetime translations, and the four-momentum spectrum is contained in the forward lightcone (spectrum condition); these are the familiar defining properties of a vacuum state. For the case that, in addition, $\mathcal{M}$ arises from a relativistic quantum field theory and the vacuum state exhibits passivity in each uniformly accelerating frame, it will be shown in the next section that in the eyes of each uniformly accelerating observer, ω is a KMS-state at a nonzero and finite positive temperature proportional to the acceleration, which is the Unruh effect.

As above, let ω be a state of a von Neumann algebra $\mathcal{M}$, let ω be induced by a cyclic vector Ω as above, and assume that there is a strongly continuous unitary representation V of $(\mathbf{R}^{1+s}, +)$ with generators H and $\mathbf{P}$ and with the property that $V(x)\mathcal{M}V(x)^* = \mathcal{M}$ for all $x \in \mathbf{R}^{1+s}$.

If ω is a vacuum state, then the above considerations suggest that it should, in particular, be passive with respect to each Hamiltonian of the form $\gamma(H + \mathbf{v}\mathbf{P})$ with $|\mathbf{v}| < c = 1$, and $\gamma = (1 - \mathbf{v}^2/c^2)^{-1/2}$. As passivity implies stationarity, it follows that ω should be invariant under all spacetime translations; again, we can assume without loss that $H\Omega = P_1\Omega = \cdots = P_s\Omega = 0$.

If $V(x) \in \mathcal{M}$ for all $x \in \mathbf{R}^{1+s}$, which holds, in particular, if ω is a pure state, as $\mathcal{M} = \mathcal{B}(\mathcal{H})$ in this case, one can prove the following:

Proposition 5.1 *Let the state ω exhibit passivity with respect to $\gamma(H + \mathbf{v}\mathbf{P})$ for each $\mathbf{v} \in \mathbf{R}^s$ with $|\mathbf{v}| < 1$, and suppose that $V(x) \in \mathcal{M}$ for all $x \in \mathbf{R}^{1+s}$. Then the joint spectrum of H and $\mathbf{P}$ is contained in the cone*

$$V_+ := \{(\eta, \mathbf{k}) \in \mathbf{R}^{1+s} : \eta \geq 0, \eta^2 - \mathbf{k}^2 \geq 0\},$$

i.e., the spectrum condition holds.

Proof. As ω is passive with respect to $K := \gamma(H + \mathbf{v}\mathbf{P})$ for each $\mathbf{v} \in \mathbf{R}^s$ with $|\mathbf{v}| < 1$, it follows (see Ineq. (3.8) in [34]) that

$$\langle A\Omega, Ke^{-K^2} A\Omega \rangle + \langle A^*\Omega, Ke^{-K^2} A^*\Omega \rangle \quad \text{for all } A \in \mathcal{M}. \qquad (11)$$

Note that Ke^{-K^2} is a bounded operator.

As $V(x) \in \mathcal{M}$ for all $x \in \mathbf{R}^{1+s}$, the spectral projection $E := E_K(\{0\})$ of K associated with $\{0\}$ is an element of $\mathcal{M}$ for every $\mathbf{v} \in \mathbf{R}^s$, and $A := (1-E)BE \in \mathcal{M}$ for all $B \in \mathcal{M}$. Inserting A into Ineq. (11), and taking into account that $K\Omega = 0$, one finds

$$\langle B\Omega, Ke^{-K^2}(1-E)B\Omega \rangle + \underbrace{\langle EB^*(1-E)\Omega, Ke^{-K^2}B^*(1-E)\Omega \rangle}_{=0} \geq 0 \quad (12)$$

(note that $E\Omega = \Omega$ by construction), so $Ke^{-K^2}(1-E)$ is a positive (bounded) operator. Since the function $x \mapsto xe^{-x^2}$ preserves signs, it follows that $K(1-E)$ is a positive operator as well. But on the other hand, $KE = 0$, so K is positive. This immediately implies the spectrum condition; note that the forward light cone is an intersection of half spaces.

$\square$

If, conversely, ω is known to be spacetime translation invariant and to satisfy the spectrum condition, it can be shown that the unitary operators $V(x)$, $x \in \mathbf{R}^{1+s}$, are elements of $\mathcal{M}$ ([1], see also Thm. III.3.2.4 in [20]), and ω can be decomposed into pure states that are invariant under spacetime translations and satisfy the spectrum condition as well (see, e.g., Sect. III.3.2 in [20]).

6 Passivity and the Unruh effect

Let $\mathcal{M}$, ω, V and Ω be as above. We now need some basic structures of local quantum fields, which associate von Neumann algebras $\mathcal{M}(\mathcal{O})$ of local observables with all bounded open spacetime regions $\mathcal{O} \subset \mathbf{R}^{1+s}$ in such a way that the following conditions are satisfied:

(A) Isotony. If $\mathcal{O}$ and P are bounded open regions in $\mathbf{R}^{1+s}$ such that $\mathcal{O} \subset P$, then $\mathcal{M}(\mathcal{O}) \subset \mathcal{M}(P)$.

(B) Locality. If $\mathcal{O}$ and P are spacelike separated bounded open regions in $\mathbf{R}^{1+s}$ and if $A \in \mathcal{M}(\mathcal{O})$ and $B \in \mathcal{M}(P)$, then $AB = BA$.

(C) Spacetime Translation Covariance. The representation V of $(\mathbf{R}^{1+s}, +)$ satisfies

$$V(x)\mathcal{M}(\mathcal{O})V(x)^* = \mathcal{M}(\mathcal{O} + x)$$

for all bounded open sets $\mathcal{O} \subset \mathbf{R}^{1+s}$ and for all $x \in \mathbf{R}^{1+s}$.

(D) Spectrum Condition. The joint spectrum of the generators of V is contained in the closed forward light cone.

$\mathcal{M}$ is assumed to be the smallest von Neumann algebra that contains all local algebras $\mathcal{M}(\mathcal{O})$ associated with bounded open regions.

The trajectory of a (pointlike) observer who is uniformly accelerated in the 1-direction with acceleration a can be translated to the curve

$$t \mapsto \frac{c^2}{a}\left(\sinh\frac{at}{c}, \cosh\frac{at}{c}, 0, \ldots, 0\right), \qquad \tau \in \mathbf{R},$$

where $t \in \mathbf{R}$ denotes the accelerated observer's eigentime. The wedge $W_1 := \{x \in \mathbf{R}^{1+s} : x_1 > |x_0|\}$, which is referred to as the *Rindler wedge*, is the region of all spacetime points the accelerated observer can communicate with using causal signals. Therefore, the elements of the algebra $\mathcal{M}(W_1)$ are precisely those observables the uniformly accelerated observer can measure. The images of W_1 under Poincaré transformations are referred to as *wedges*.

We assume that some uniformly accelerated observer exists:

(E) There is a self-adjoint operator K_1 generating, within W_1, the free dynamics of the uniformly accelerating observer, i.e.,

$$e^{i\tau K_1}\mathcal{M}(\mathcal{O})e^{-i\tau K_1} = \mathcal{M}(\Lambda_1(\tfrac{a}{c}\tau)\mathcal{O})$$

for all $\tau \in \mathbf{R}$ and all bounded open sets $\mathcal{O} \subset W_1$. K_1 strongly commutes with $P_2, \ldots, P_s$, and $K_1\Omega = 0$.

Here, $\Lambda_1(\tfrac{a}{c}\tau)$ denotes the Lorentz boost by $\tfrac{a}{c}\tau$ in the 1-direction. $\mathcal{M}$ is not yet assumed to be covariant under a full representation of the Poincaré group, although the assumption that K_1 strongly commutes with $P_2, \ldots, P_s$ is already a part of this condition.

Proposition 6.1 *With the above assumptions, assume ω to exhibit passivity with respect to the dynamics generated by K_1. Then ω is a KMS-state of $\mathcal{M}(W_1)$ with respect to K_1 at the Unruh temperature*

$$T_U = \frac{\hbar a}{2\pi ck}.$$

13

Proof. As a consequence of the spectrum condition, Ω is cyclic not only with respect to $\mathcal{M}$, but even with respect to $\mathcal{M}(W_1)$ and $\mathcal{M}(-W_1)$ (cf. [11], p. 279). By Prop. 2.2 in [14], it also follows from the spectrum condition and locality that the space of vectors that are invariant under translations in the 2-direction is 1-dimensional (and, thus, spanned by Ω), and as Ω is cyclic with respect to $\mathcal{M}(W_1)$, the state $\omega|_{\mathcal{M}(W_1)}$ weakly clusters (in the sense of [34]) with respect to the translations in the 2-direction, (note that W_1 is invariant under these translations). As the translations in the 2-direction strongly commute with K_1, one can apply Thm. 1.3 in [34] to conclude that ω must be a KMS-state or a ground state with respect to the dynamics generated by K_1. In particular, ω exhibits complete passivity.

As Ω is cyclic with respect to $\mathcal{M}(-W_1)$ and as $-W_1$ is spacelike with respect to W_1, locality implies that Ω is separating with respect to $\mathcal{M}(W_1)$, so $\omega|_{\mathcal{M}(W_1)}$ is faithful. It follows that $\omega|_{\mathcal{M}(W_1)}$ is a KMS-state at some inverse temperature $0 \leq \beta < \infty$ (use, e.g., Prop. 3.1 above for $\mathcal{E} = 0$). As the action of the modular unitary operators is nontrivial, their generator differs from zero, so $\omega|_{\mathcal{M}(W_1)}$ cannot be a trace, and one even has $\beta > 0$. The modular group is generated by the operator $\beta\frac{\hbar a}{c}K_1$.

It is now easy to compute β and the *Unruh temperature* $T_U := (k\beta)^{-1}$. It follows from Thm. II.9 in [5][6] that the spectrum condition entails

$$\exp\left(it\beta\frac{\hbar a}{c}K_1\right) V(x) \exp\left(-it\beta\frac{\hbar a}{c}K_1\right) = V(\Lambda_1(-2\pi t)x)$$

for all $x \in \mathbf{R}^{1+s}$, so it is evident that $\beta\frac{\hbar a}{c} = 2\pi$, whence the stated formula for the Unruh temperature follows.

$\square$

Assuming the statement of this proposition in all Lorentz frames, group cohomological arguments imply that V and the operators K_W associated with all wedges W, generate a representation of the proper Poincaré group $\mathcal{P}_+$ [10], and this representation is even a representation of the restricted Poincaré group $\mathcal{P}_+^\uparrow$ [16]. The modular conjugation of the Rindler wedge implements a P₁CT-symmetry, i.e., a spatial reflection of the 1-component, a time reflection, and a charge conjugation [16]. This fact was found to imply the spin-statistics for massive (para-) bosonic and (para-) fermionic particles [16, 24], and as these proofs did not use any spinor calculus, it was possible to generalize them to conformal quantum field theories [17], to massive particles with braid group statistics in 1+2 dimensions such as anyons [27, 29], and to special quantum field theories on (sufficiently symmetric) curved spacetimes [19].

---

[6]See also [15] for a considerably shorter proof.

7 Conclusion

The behaviour of cycles can be used to characterize thermodynamic equilibrium states in a covariant fashion. Cycles cannot extract any energy from a system in thermodynamic equilibrium by performing exterior work, i.e., thermodynamic equilibrium states exhibit passivity. It follows that if a thermodynamic equilibrium state is observed from a uniformly moving frame of reference, it ceases to be a thermodynamic equilibrium state, as cycles can perform work there. The amount of work a cycle can perform when applied to a moving thermodynamic equilibrium state is bounded by the amount of work an ideal windmill or turbine could perform; this property is called *semipassivity*, and the factor $\mathcal{E} \geq 0$ characterizing the bound is called an *efficiency bound*. The Zeroth Law justifies a strengthening of passivity and semipassivity called *complete passivity* and *complete semipassivity*, respectively.

For the description of homogeneous states, the condition of complete semipassivity turns out to be the appropriate generalization of complete passivity to moving frames of reference. If it holds, an inertial frame can be found where the system is in thermodynamic equilibrium. Semipassivity can also be used to measure the violation of passivity due to stationary diffusion processes and to define the corresponding chemical potentials.

When applied to states without matter, cycles should not perform any work unless there are nonstationary inertial forces to drive them. Each pure state behaving this way satisfies the spectrum condition, and in the general setting of local quantum field theory in Minkowski spacetime, such a state appears as a thermodynamic equilibrium state at the Unruh temperature $\frac{\hbar a}{2\pi ck}$ to each observer who is uniformly accelerated with acceleration a.

Acknowledgements

Appendix

A Some Tomita-Takesaki theory

The modular theory founded by Tomita and Takesaki [36] plays an important role in quantum field theory and quantum statistical mechanics (cf.,

e.g., [20, 6]). It is used in the above proofs, so for the convenience of the reader, some relevant facts and notation of Tomita-Takesaki theory are summarized here in a nutshell.

As above, let a state ω of $\mathcal{M}$ be induced by the cyclic vector Ω. Suppose that ω is faithful, then Ω is also *separating* with respect to $\mathcal{M}$, i.e., given an $A \in \mathcal{M}$, $A\Omega = 0$ implies $A = 0$. This implies that the map

$$A\Omega \mapsto A^*\Omega, \qquad A \in \mathcal{M},$$

defines an antilinear operator on the dense domain $\mathcal{M}\Omega$. This operator is closable, and its closure S can, like a complex number, be polar-decomposed, $S = J\Delta^{1/2}$, into a positive operator $\Delta^{1/2}$ (its "modulus") and an antiunitary operator J (its "phase"). Δ and J, like S, leave Ω fixed by construction, and J is a conjugation, i.e., $J^2 = 1$. By a theorem of Tomita and Takesaki [36], one has

$$\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M} \qquad \text{for all } t \in \mathbf{R};$$
$$J\mathcal{M}J = \mathcal{M}'.$$

With respect to the dynamics $A \mapsto \Delta^{it}A\Delta^{-it} =: A_t$, $A \in \mathcal{M}$, $t \in \mathbf{R}$, the state ω satisfies the KMS-condition:

$$\langle \Omega, AB_t\Omega \rangle = \langle A^*\Omega, \Delta^{it}B\Delta^{-it}\Omega \rangle = \langle J\Delta^{1/2}A\Omega, J\Delta^{1/2}\Delta^{it}B^*\Delta^{-it}\Omega \rangle$$
$$= \langle \Delta^{1/2}\Delta^{it}B^*\Delta^{-it}, \Delta^{1/2}A\Omega \rangle = \langle \Omega, \Delta^{it}B\Delta^{-it}\Delta A\Omega \rangle$$
$$= \langle \Delta\Omega, \Delta^{it}B\Delta^{-it}\Delta A\Omega \rangle = \langle \Omega, B_{t-i}A\Omega \rangle$$

(cf. also Lemma 8.1.10 (p. 351) in [26]), and for each faithful state ω of a von Neumann algebra there is only one strongly continuous automorphism group of $\mathcal{M}$ with this property [36]. The positive operator Δ is referred to as the *modular operator of $\mathcal{M}$ and Ω*, the group of the automorphisms $A \mapsto A_t$, $t \in \mathbf{R}$, of $\mathcal{M}$ is called the *modular (automorphism) group*, and the conjugation J is the *modular conjugation* of $\mathcal{M}$ and Ω.

If U is a unitary operator in $\mathcal{H}$ such that $U\mathcal{M}U^* = \mathcal{M}$ and $U\Omega = \Omega$, then one has, for all $A \in \mathcal{M}$:

$$USU^*A\Omega = US(U^*AU)\Omega = U(U^*AU)^*\Omega = A^*\Omega = SA\Omega,$$

and this suffices to conclude that $USU^* = S$. The fact that

$$J\Delta^{1/2} = S = USU^* = UJ\Delta^{1/2}U^* = UJU^*U\Delta^{1/2}U^*,$$

together with the uniqueness of the polar decomposition of a closed linear or antilinear operator, implies that $UJU^* = J$, that $U\Delta^{1/2}U^* = \Delta^{1/2}$, and that $U\Delta^{it}U^* = \Delta^{it}$. It follows that each self-adjoint operator G in $\mathcal{H}$ with $G\Omega = 0$ and with $e^{itG}\mathcal{M}e^{-itG} = \mathcal{M}$ for all $t \in \mathbf{R}$, strongly commutes with K and J. This implies $e^{itG} = Je^{itG}J$ for all $t \in \mathbf{R}$, and it is not difficult to conclude that $iG = J(iG)J = -iJGJ$, so one arrives at the relation $JGJ = -G$, which is used for $G = H$, $G = K$, and $G = \mathbf{P}$ in the text.

B Some remarks on joint spectra

For the reader's convenience, we recall some basic facts on joint spectra of strongly commuting self-adjoint operators. We work with three operators for notational convenience; the generalization to n operators is straightforward.

Let A, B, and C be self-adjoint operators that commute strongly, i.e., whose spectral projections commute and, hence, define a product spectral measure. The *joint spectrum* $\sigma_{A,B,C}$ of A, B, and C is the support of the product measure $E_{A,B,C}$ of the spectral measures E_A, E_B, and E_C. It is a closed set by construction. A point $(x,y,z) \in \mathbf{R}^3$ is in $\sigma_{A,B,C}$ if and only if for all $\varepsilon > 0$, one has

$$E_A([x-\varepsilon, x+\varepsilon])E_B([y-\varepsilon, y+\varepsilon])E_C([z-\varepsilon, z+\varepsilon]) \neq 0.$$

The following lemmas are used in the above proofs.

Lemma B.1 *Let X be a two-dimensional subspace of $\mathbf{R}^3$ that does not contain the z-axis. If $\sigma_{A,B,C} \subset X$, then C is a linear function of the operators A and B.*

Proof. As X does not contain the z-axis, there is a linear function $f : \mathbf{R}^2 \to \mathbf{R}$ such that $X = \{(x, y, f(x,y)) : (x,y) \in \mathbf{R}^2\}$, and if $I \subset \mathbf{R}$ is any Borel set, one checks that

$$
\begin{aligned}
E_C(I) &= E_A(\mathbf{R})E_B(\mathbf{R})E_C(I) \\
&= E_{A,B,C}(\mathbf{R}^2 \times I) = E_{A,B,C}((\mathbf{R}^2 \times I) \cap X) \\
&= E_{A,B,C}((f^{-1}(I) \times \mathbf{R}) \cap X) = E_{A,B,C}((f^{-1}(I) \times \mathbf{R}) \\
&= E_{A,B}(f^{-1}(I))E_C(\mathbf{R}) = E_{A,B}(f^{-1}(I)).
\end{aligned}
$$

∎

Lemma B.2 *Let π_z denote the orthogonal projection along the z-axis onto the x-y-plane. Then*

$$\sigma_{A,B} = \overline{\pi_z(\sigma_{A,B,C})}.$$

Proof. If $(x,y,z) \in \sigma_{A,B,C}$, then for each $\varepsilon > 0$, one has

$$
\begin{aligned}
F_\varepsilon &:= E_A([x-\varepsilon, x+\varepsilon])E_B([y-\varepsilon, y+\varepsilon]) \\
&= F_\varepsilon E_C(\mathbf{R}) \geq F_\varepsilon E_C([z-\varepsilon, z+\varepsilon]) \neq 0,
\end{aligned}
$$

so $(x,y) \in \sigma_{A,B}$, proving $\pi_z(\sigma_{A,B,C}) \subset \sigma_{A,B}$. As $\sigma_{A,B}$ is closed, it follows that $\overline{\pi_z(\sigma_{A,B,C})} \subset \sigma_{A,B}$.

If, conversely, $(x,y) \in \sigma_{A,B}$, then one has $F_\varepsilon \neq 0$ for all $\varepsilon > 0$, so if the set

$$M_\varepsilon := \pi_z^{-1}([x-\varepsilon, x+\varepsilon] \times [y-\varepsilon, y+\varepsilon])$$

had empty intersection with $\sigma_{A,B,C}$, the product of F_ε with all spectral projections of C would equal zero, and in particular, $F_\varepsilon = F_\varepsilon E_C(\mathbf{R}) = 0$, which is in conclict with $(x,y) \in \sigma_{A,B}$.

We conclude that each open neighbourhood of (x,y) contains a point in $\pi_z(\sigma_{A,B,C})$, so $\sigma_{A,B} \subset \overline{\pi_z(\sigma_{A,B,C})}$, and the proof is complete.

$\square$

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