The Proof that the Standard Transformations of E and B and the Maxwell Equations with E and B are not Relativistically Correct

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In this paper it is exactly proved that the standard transformations of the three-dimensional (3D) vectors of the electric and magnetic fields **E** and **B** *are not* relativistically correct transformations. Thence the 3D vectors **E** and **B** *are not* well-defined quantities in the 4D spacetime and, contrary to the general belief, the usual Maxwell equations with the 3D **E** and **B** *are not* in agreement with the special relativity. The 4-vectors E^a and B^a, as well-defined 4D quantities, are introduced instead of ill-defined 3D **E** and **B**. The proof is given in the tensor and the Clifford algebra formalisms.

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It is generally accepted by physics community that there is an agreement between the classical electromagnetism and the special relativity (SR). The standard transformations of the three-dimensional (3D) vectors of the electric and magnetic fields, **E** and **B** respectively, are considered to be the Lorentz transformations (LT) of these vectors, see, e.g., [1], [2] Sec. 11.10, or [3] par.24. The usual Maxwell equations (ME) with the three-vectors (3-vectors) **E** and **B** are assumed to be physically equivalent to the field equations (FE) expressed in terms of the electromagnetic field tensor F^{ab}. In this paper it will be exactly proved that the above mentioned standard transformations of **E** and **B** *are not* relativistically correct transformations in the 4D spacetime and consequently that the usual ME with **E** and **B** and the FE with F^{ab} *are not* physically equivalent. The whole consideration will be mainly presented in the tensor formalism (TF), since it is better known, and only briefly in the Clifford algebra formalism (CAF). It will be shown that in the 4D spacetime the well-defined 4D quantities, the 4-vectors of the electric and magnetic fields E^a and B^a in the TF (as in [4,5]), or, e.g., the 1-vectors E and B in the CAF (as in [6]), have to be introduced instead of ill-defined 3-vectors **E** and **B**.

Let us start with some general definitions. The electromagnetic field tensor F^{ab} is defined without reference frames, i.e., it is an abstract tensor, a geometric quantity; Latin indices a,b,c, are to be read according to the abstract index notation, as in [7] and [4,5]. When some reference frame (a physical object) is introduced and the system of coordinates (a mathematical object) is adopted in it, then F^{ab} can be written as a coordinate-based-geometric quantity (CBGQ) containing components and a basis. The system of coordinates with the Einstein synchronization of clocks and Cartesian spatial coordinates (it will be called the Einstein system of coordinates (ESC)) is almost always chosen in the usual treatments. (In my approach to SR that uses 4D quantities defined without reference frames, [4,5] and [8] in the TF, and [6] in the CAF, any permissible system of coordinates, not necessary the ESC, can be used on an equal footing.) When F^{ab} is written as a CBGQ (with the ESC) it becomes $F^{ab} = F^{\mu\nu}\gamma_\mu \otimes \gamma_\nu$, where Greek indices μ, ν in $F^{\mu\nu}$ run from 0 to 3 and they denote the components of the geometric object F^{ab} in some system of coordinates, here the ESC, γ_μ are the basis 4-vectors (not the components) forming the standard basis $\{\gamma_\mu\}$ and $'\otimes'$ denotes the tensor product of the basis 4-vectors. In the TF I shall often denote the unit 4-vector in the time direction γ_0 as t^b as well. Then in some reference frame (with the ESC, i.e., with the standard basis $\{\gamma_\mu\}$) t^b can be also written as a CBGQ, $t^b = t^\mu \gamma_\mu$, where t^μ is a set of components of the unit 4-vector in the time direction ($t^\mu = (1, 0, 0, 0)$). Almost always in the standard covariant approaches to SR one considers only the components of the geometric quantities taken in the ESC, and thus not the whole tensor. However the components are coordinate quantities and they do not contain the whole information about the physical quantity.

In the standard treatments one defines the sets of components of the electric and magnetic fields as

$$\begin{aligned} E^\mu &= F^{\mu\nu} t_\nu, \quad B^\mu = (1/2)\varepsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma} t_\nu = (F^*)^{\mu\nu} t_\nu, \\ E^0 &= 0, E^i = F^{i0}; \quad B^0 = 0, B^i = (1/2)\varepsilon^{0ikl} F_{lk}, \end{aligned} \tag{1}$$

where $(F^*)^{\alpha\beta} = (1/2)\varepsilon^{\alpha\beta\gamma\delta} F_{\gamma\delta}$ is the dual tensor (the components). (The signature of the Minkowski tensor is -2, $\varepsilon^{0123} = 1$, and $c = 1$). *The temporal components of* E^μ *and* B^μ *are zero.* Note that we can select a particular - but otherwise arbitrary - inertial frame of reference (IFR) S as the frame in which the relations (1) hold. t^μ can be interpreted as the 4-velocity (the components in the ESC) of the observers that are at rest in S. In the standard treatments the 3-vectors **E** and **B**, as *geometric quantities in the 3D space*, are constructed from the spatial components E^i and B^i from (1) and *the unit 3-vectors* **i**, **j**, **k**, e.g., $\mathbf{E} = F^{10}\mathbf{i} + F^{20}\mathbf{j} + F^{30}\mathbf{k}$. These results are quoted in numerous textbooks and papers treating relativistic electrodynamics in the TF, see, e.g., [2,3]. Actually in the usual covariant approaches, e.g., [2,3], one forgets about E^0 and B^0 components and simply makes the identification of six independent components of $F^{\mu\nu}$ with three components E^i, $E^i = F^{i0}$, and three components B^i, $B^i = (1/2)\varepsilon^{ikl}F_{lk}$. Since in SR we work with the 4D spacetime the mapping between the components of $F^{\mu\nu}$ and the components of the 3D vectors $\mathbf{E}$ and $\mathbf{B}$ is mathematically better founded by the relations (1) than by their simple identification. Therefore we proceed the consideration using (1). Note that the whole procedure is made in an IFR with the ESC. In another system of coordinates that is different than the ESC, e.g., differing in the chosen synchronization (as it is the 'r' synchronization considered in [4]), the identification of E^i with F^{i0}, as in (1) (and also for B^i), is impossible and meaningless. Further the components E^i and B^i are determined in the 4D spacetime in the standard basis $\{\gamma_\mu\}$. Thence when forming the geometric quantities the components would need to be multiplied with the unit 4-vectors γ_i and not with the unit 3-vectors.

Let us now apply the LT to the components given in (1). Under the passive LT the sets of components E^μ and B^μ from (1) transform to E'^μ and B'^μ in the relatively moving IFR S'

$$\begin{aligned} E'^\mu &= F'^{\mu\nu}v'_\nu, \quad B'^\mu = (1/2)\varepsilon^{\mu\nu\lambda\sigma}F'_{\lambda\sigma}v'_\nu = (F^*)'^{\mu\nu}v'_\nu, \\ E'^\mu &= \left(-\beta\gamma E^1, \gamma E^1, E^2, E^3\right), \ B'^\mu = \left(-\beta\gamma B^1, \gamma B^1, B^2, B^3\right), \end{aligned} \tag{2}$$

where $v'_\nu = (\gamma, \beta\gamma, 0, 0)$. The unit 4-vector (the components) t^μ in the time direction in S transforms upon the LT into the unit 4-vector v'^ν, the 4-velocity of the moving observers, that contains not only the temporal component but also $\neq 0$ spatial component. Thence, the LT transform the set of components (1) into (2). Note that *E'^μ and B'^μ do have the temporal components as well.* Further the components E^μ (B^μ) in S transform upon the LT again to the components E'^μ (B'^μ) in S'; there is no mixing of components. Actually this is the way in which every well-defined 4-vector (the components) transforms upon the LT. A geometric quantity, an abstract tensor E^a, can be represented by CBGQs in S and S' (both with the ESC) as $E^\mu\gamma_\mu$ and $E'^\mu\gamma'_\mu$, where E^μ and E'^μ are given by the relations (1) and (2) respectively. *All the primed quantities (components and the basis) are obtained from the corresponding unprimed quantities through the LT.* Of course it must hold that

$$E^a = E^\mu\gamma_\mu = E'^\mu\gamma'_\mu, \tag{3}$$

since the components E^μ transform by the LT and the basis γ_μ transforms by the inverse LT thus leaving the whole CBGQ invariant upon the passive LT. The invariance of some 4D CBGQ upon the passive LT is the crucial requirement that must be satisfied by any well-defined 4D quantity. It reflects the fact that such mathematical, invariant, geometric 4D quantity represents *the same physical object* for relatively moving observers. The use of CBGQs enables us to have clearly and correctly defined the concept of sameness of a physical system for different observers. The importance of this concept in SR was first pointed out in [9,10]. However they also worked with components in the ESC (the covariant quantities) and not with geometric quantities (the invariant quantities). It should be noted that in all other standard treatments, e.g., [1-3], the importance of such concept is completely overlooked what caused many difficulties in understanding SR. It can be easily checked by the direct inspection that (3) holds when E^μ and E'^μ are given by (1) and (2). (The same holds for B^a.)

In contrast to the above consideration in all usual treatments, e.g., [2,3] and [11] eqs. (3.5) and (3.24), in S' one again simply makes the identification of six independent components of $F'^{\mu\nu}$ with three components E'^i, $E'^i = F'^{i0}$, and three components B'^i, $B'^i = (1/2)\varepsilon^{ikl}F'_{lk}$. This means that standard treatments assume that under the passive LT the set of components $t^\mu = (1, 0, 0, 0)$ from S transform to $t'^\nu = (1, 0, 0, 0)$ (t'^ν are the components of the unit 4-vector in the time direction in S' and in the ESC), and consequently that E^μ and B^μ from (1) transform to $E'^\mu_{st.}$ and $B'^\mu_{st.}$ in S',

$$\begin{aligned} E'^\mu_{st.} &= F'^{\mu\nu}t'_\nu, B'^\mu_{st.} = (F^*)'^{\mu\nu}t'_\nu; \ E'^\mu_{st.} = \left(0, E^1, \gamma E^2 - \gamma\beta B^3, \gamma E^3 + \gamma\beta B^2\right), \\ B'^\mu_{st.} &= \left(0, B^1, \gamma B^2 + \gamma\beta E^3, \gamma B^3 - \gamma\beta E^2\right), \end{aligned} \tag{4}$$

where the subscript - st. is for - standard. *The temporal components of $E'^\mu_{st.}$ and $B'^\mu_{st.}$ in S' are again zero as are the temporal components of E^μ and B^μ in S.* This fact clearly shows that *the transformations given by the relation (4) are not the LT of some well-defined 4D quantities; the LT cannot transform a 4-vector for which the temporal component is zero in one frame S to the 4-vector with the same property in relatively moving frame S'.* Also *the LT cannot transform the unit 4-vector in the time direction in one frame S to the unit 4-vector in the time direction in another relatively moving frame S'.* Obviously $E'^\mu_{st.}$ and $B'^\mu_{st.}$ are completely different quantities than E'^μ and B'^μ (2) that are obtained by the LT. We can easily check that

$$E'^\mu_{st.}\gamma'_\mu \neq E^\mu\gamma_\mu, \quad B'^\mu_{st.}\gamma'_\mu \neq B^\mu\gamma_\mu. \tag{5}$$

This means that, e.g., $E^\mu\gamma_\mu$ and $E'^\mu_{st.}\gamma'_\mu$ *are not the same quantity for observers in S and S'.* As far as relativity is concerned the quantities, e.g., $E^\mu\gamma_\mu$ and $E'^\mu_{st.}\gamma'_\mu$, are not related to one another. Their identification is the typical case of mistaken identity. The fact that they are measured by two observers does not mean that relativity has something to do with the problem. The reason is that observers S and S' are not looking at the same physical object but at two different objects. *Every observer makes measurement on its own object and such measurements are not related by the LT.* Thus the transformations (4) are not the LT and $E_{st.}^{\prime\mu}$ and $B_{st.}^{\prime\mu}$, in contrast to $E^{\prime\mu}$ and $B^{\prime\mu}$, are not well-defined 4D quantities.

From *the relativistically incorrect transformations* (4) one simply derives the transformations of the spatial components $E_{st.}^{\prime i}$ and $B_{st.}^{\prime i}$. As can be seen from (4) *the transformations of* $E_{st.}^{\prime i}$ *and* $B_{st.}^{\prime i}$ *are exactly the standard transformations of components of the 3-vectors* $\mathbf{E}$ *and* $\mathbf{B}$ that are obtained by Einstein in [1] and subsequently quoted in almost every textbook and paper on relativistic electrodynamics. Then in the same way as in S the 3-vectors $\mathbf{E}'$ and $\mathbf{B}'$ as *geometric quantities in the 3D space* are constructed in S' from the spatial components $E_{st.}^{\prime i}$ and $B_{st.}^{\prime i}$ and *the unit 3-vectors* $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$, e.g., $\mathbf{E}'=F^{\prime 10}\mathbf{i}'+F^{\prime 20}\mathbf{j}'+F^{\prime 30}\mathbf{k}'$. Both the transformations (4) and the transformations for $E_{st.}^{\prime i}$ and $B_{st.}^{\prime i}$ (i.e., for $\mathbf{E}'$ and $\mathbf{B}'$) are typical examples of the "apparent" transformations (AT) that are first discussed in [9] and [10]. The AT of the spatial distances (the Lorentz contraction) and the temporal distances (the dilatation of time) are elaborated in detail in [4] and [8] (see also [12]), and in [4] I have discussed the AT of $\mathbf{E}$ and $\mathbf{B}$. It is explicitly shown in [8] that the true agreement with experiments that test SR exists only when the theory deals with well-defined 4D quantities, i.e., the quantities that are invariant upon the passive LT.

In all previous treatments of SR, e.g., [1-3] [11], the transformations for $E_{st.}^{\prime i}$ and $B_{st.}^{\prime i}$ are considered to be the LT of the 3D electric and magnetic fields. However our analysis shows that the transformations for $E_{st.}^{\prime i}$ and $B_{st.}^{\prime i}$ are derived from *the relativistically incorrect transformations* (4) and that the 3-vectors $\mathbf{E}'$ and $\mathbf{B}'$ are formed by an incorrect procedure in 4D spacetime, i.e., by multiplying these *relativistically incorrect components* with *the unit 3-vectors.* All this together exactly proves that the standard transformations for $\mathbf{E}'$ and $\mathbf{B}'$ have absolutely nothing to do with the LT, and that the quantities $E_{st.}^{\prime i}$ and $B_{st.}^{\prime i}$, i.e., the 3-vectors $\mathbf{E}$ and $\mathbf{B}$ are not well-defined 4D quantities. Consequently *the usual ME with 3D* $\mathbf{E}$ *and* $\mathbf{B}$ *are not in agreement with SR and they are not physically equivalent with relativistically correct FE with* F^{ab} (see also [4]).

The relation (1) reveals that we can always select a particular - but otherwise arbitrary - IFR S in which the temporal components of E^{μ} and B^{μ} are zero. Then in that frame the usual ME for the spatial components E^{i} and B^{i} (of E^{μ} and B^{μ}) will be fulfilled. As a consequence the usual ME can explain all experiments that are performed in one reference frame. However as shown above the temporal components of $E^{\prime\mu}$ and $B^{\prime\mu}$ are not zero; (2) is relativistically correct, but it is not the case with (4). This means that the usual ME cannot be used for the explanation of any experiment that test SR, i.e., in which relatively moving observers have to compare their data *obtained by measurements on the same physical object.*

The relations (2) imply that in the ESC the well-defined 4D electric and magnetic fields will be CBGQs, the 4-vectors, $E^{\mu}\gamma_{\mu}=F^{\mu\nu}v_{\nu}\gamma_{\mu}$ and $B^{\mu}\gamma_{\mu}=(F^{*})^{\mu\nu}v_{\nu}\gamma_{\mu}$ respectively. In an arbitrary chosen IFR S v_{ν} can be taken to be in the time direction, i.e., $v_{\nu}=t_{\nu}$, whence one finds (1) in S and (2) in any relatively moving IFR S'. (The components in the ESC, e.g., $E^{\mu}=F^{\mu\nu}v_{\nu}$, and the covariant formulation of electrodynamics with them, are considered in [13], [12] and [14]). In order to have the electric and magnetic 4-vectors defined without reference frames, i.e., *independent of the chosen reference frame and of the chosen system of coordinates in it*, we employ the abstract tensors and write $E^{a}=F^{ab}v_{b}$ and $B^{a}=-(1/2)\varepsilon^{abcd}v_{b}F_{cd}$, see [4,5] and [7]. The velocity v_{b} and all other quantities entering into these relations are defined without reference frames. v_{b} characterizes some general observer. Thus *the relations for* E^{a} *and* B^{a} *hold for any observer.* When some reference frame is chosen with the ESC in it and when v_{b} is specified to be in the time direction in that frame, i.e., $v_{b}=t_{b}$, then all results of the classical electromagnetism are recovered in that frame. However, in contrast to the description of the electromagnetism with the 3D $\mathbf{E}$ and $\mathbf{B}$, the description with E^{a} and B^{a} is correct not only in that frame but in all other relatively moving frames and it holds for any permissible choice of coordinates. In [4] I have also presented the form of the LT that is independent of the chosen coordinates. Furthermore I have developed three equivalent but independent, consistent and complete formulations of electrodynamics with abstract tensors, with F^{ab}, E^{a} and B^{a}, and with their complex combination. (Note that the above relations for E^{a} and B^{a} are not the physical definitions of E^{a} and B^{a} but they simply connect the independent and complete formulations with F^{ab} and E^{a}, B^{a}. The physical definitions of E^{a} and B^{a} are given in terms of the Lorentz force expressed with E^{a} and B^{a} and Newton's second law, as in [4] in the TF and in the first paper in [6] in the CAF.)

The same situation as in TF happens in the standard CAF, e.g., [15,16]. The ESC, i.e., the standard basis $\{\gamma_{\mu}\}$, is chosen from the outset and the relations for E and B are written explicitly using the basis 1-vector in the time direction, γ_{0}. In some IFR S, $E=F\cdot\gamma_{0}=F^{k0}\gamma_{k}$, and $B=-\gamma_{5}(F\wedge\gamma_{0})=(1/2)\varepsilon^{0ikl}F_{lk}\gamma_{i}$, where γ_{5} is the pseudoscalar for the frame $\{\gamma_{\mu}\}$, '$\cdot$' and '$\wedge$' denote the inner and outer products of the basis 1-vectors. (This form for E and B is equivalent to the forms given in the standard CAF, e.g., [15,16].) Then it is wrongly assumed that under the active LT (expressed in CAF by rotors; see [15,16], [6]) the new, i.e., the Lorentz transformed, E' and B' are $E_{st.}^{\prime}=F'\cdot\gamma_{0}$ and $B_{st.}^{\prime}=-\gamma_{5}(F'\wedge\gamma_{0})$. However when E is transformed by the active LT then $R(F\cdot\gamma_{0})\widetilde{R}$ *is not* equal to $(RF\widetilde{R})\cdot\gamma_{0}=E_{st.}^{\prime}$ (for the explicit form of the rotor R see, e.g., [16] ch.6 and [6]). Really the components of $E'_{st.}$ ($B'_{st.}$) are the same as in the AT (4), while the components of $RE\widetilde{R}$ ($RB\widetilde{R}$) are the same as in the correct LT (2). Finally the standard transformations for the 3D **E** and **B** are derived from the AT for $E'_{st.}$ and $B'_{st.}$, see [15] Sec. 18 and [16] Ch. 7. In [6] I have presented the relativistically correct form for E and B by replacing γ_0 with v, the velocity of some general observer, which is also defined, as in TF, without reference frames. The CAF from [6] enabled me to develop *four* equivalent but independent, consistent and complete formulations of electrodynamics with field bivector F, with 1-vectors E and B , with complex 1-vector Ψ and, what is specific for the CAF, with a *real* Clifford multivector Ψ. Furthermore all relevant quantities for the electromagnetism, the stress-energy 1-vector $T(v)$, the energy density U (scalar), the Poynting 1-vector S, the angular momentum density M (bivector) and the Lorentz force K (1-vector) are directly derived from the FE and all of them are defined without reference frames.

The whole consideration explicitly shows that the 3D quantities **E** and **B**, their transformations and the equations with them are ill-defined in the 4D spacetime. More generally, the 3D quantities do not have an independent physical reality in the 4D spacetime. Thence the relativistically correct physics must be formulated with 4D quantities that are defined without reference frames, or by the 4D CBGQs (e.g., as in [4,5], [8] in the TF and [6] in the CAF). The principle of relativity is automatically included in such theory with well-defined 4D quantities, while in the standard approach to SR [1] it is postulated outside the mathematical formulation of the theory. The comparison with experiments from [8] and [6] reveals that the true agreement with experiments that test SR can be achieved only when such well-defined 4D quantities are considered.

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