Dual description of $U(1)$ charged fields in $(2+1)$ dimensions

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Abstract

We show that the functional bosonization procedure can be generalized in such a way that, to any field theory with a conserved Abelian charge in $2+1$ dimensions, there corresponds a dual Abelian gauge field theory. The properties of this mapping and of the dual theory are discussed in detail, presenting different explicit examples. In particular, the meaning and effect of the coefficient of the Chern-Simons term in the dual action is interpreted in terms of the spin and statistics connection.

1 Introduction

Recently, the functional bosonization of fermionic systems in $(2+1)$ dimensions has been a subject of intense research, with various attempts to extend to higher dimensions some of the non perturbative techniques and results known for the two-dimensional case [1]. These efforts have led to a set of interesting and promising results [2]-[7], which turned out to be relevant, for instance, to the understanding of the universal behavior of the Hall conductance in interacting electron systems [10]. An important point to be remarked is that, in this approach, the bosonization of a charged fermionic system in $(2+1)$ dimensions is achieved through the introduction of an Abelian vector gauge field A_μ [2]-[8].

More precisely, as in the two-dimensional case, the $U(1)$ fermionic current $J_\mu = \bar{\psi}\gamma_\mu\psi$ is mapped into a topologically conserved current j^T_μ [2]-[7]:

$$J_\mu = \bar{\psi}\gamma_\mu\psi \;\longrightarrow\; j^T_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda \; . \tag{1.1}$$

Correspondingly, the free massive Dirac action

$$S_F[\bar{\psi},\psi] = \int d^3x \, \bar{\psi} \left(\not{\partial} + m \right) \psi \; , \tag{1.2}$$

is transformed into a bosonic, gauge invariant action $S_B[A]$

$$S_F[\bar{\psi},\psi] \longrightarrow S_B[A] \; , \tag{1.3}$$

whose exact form is, in general, unknown, since its evaluation requires the calculation of a fermionic determinant in $(2+1)$ dimensions. It is possible, however, to carry out the bosonization procedure in such a way that it is consistent with a perturbative expansion of the fermionic determinant, by a decoupling transformation of the fermionic fields in the functional integral [9].

However, just on general grounds, we may say that S_B consists of a Chern-Simons action (the leading term), plus an infinite series of terms depending on the curvature $\widetilde{F}_\mu = \frac{1}{2}\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$, namely

$$S_B[A] \;=\; i\,\frac{1}{\eta}\, S_{CS}[A] \;+\; R[\widetilde{F}] \; , \tag{1.4}$$

where $S_{CS}[A]$ stands for the Chern-Simons action [1]

$$S_{CS}[A] \;=\; \frac{1}{2}\int d^3x \, \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \; , \tag{1.5}$$

and $R[\widetilde{F}]$ denotes the remainder, higher order contributions in the curvature, including those which are non local and non quadratic in $\widetilde{F}$ [2]-[7]. It is worth mentioning that, recently, the current bosonization rule (1.1) has been proven to hold even in the presence of interactions, giving to eq.(1.1) a

[1]We adopt Euclidean conventions.

universal character [7]. This means that the bosonization rule for a fermionic system whose action contains an interaction part $I[J^\mu]$, depending only on the current J^μ, generalizes to:

$$S_F[,\bar{\psi},\psi] + I[J^\mu] \leftrightarrow S_B[A] + I[\epsilon^{\mu\nu\rho}\partial_\nu A_\rho] \tag{1.6}$$

where $S_B[A]$ is the same functional that bosonizes the *free* fermionic action $S_F[\psi]$. This relationship has to be understood as an equality between correlation functions of currents, as obtained from the action corresponding to each description, namely

$$\langle J_{\mu_1}...J_{\mu_n}\rangle_{(S_F+I[J])} = \langle j^T_{\mu_1}...j^T_{\mu_n}\rangle_{(S_B+I[j^T])} \; . \tag{1.7}$$

The parameter η in the leading term of eq.(1.3) is the coefficient of the induced Chern-Simons action. It turns out that the determination of this coefficient is plagued by finite ambiguities, related to the choice of the regularization procedure. In other words, as usual in any field theory computation, a set of physical conditions has to be imposed on the system in order to fix all the ambiguities. We shall, for the time being, leave this parameter unspecified, coming back to it later on, when discussing the relationship of η with the statistics of the excitations present in the bosonized version of the theory.

In the low energy regime, only the Chern-Simons term survives in the right hand side of eq.(1.3), and this yields a simple, closed expression for the bosonized action:

$$\lim_{m\to\infty} S_B[A] \;=\; i\,\frac{1}{\eta}\,S_{CS}[A] \; . \tag{1.8}$$

The aim of this work is to generalize this construction, originally used for the bosonization of fermionic actions, to the case of an arbitrary field theory model with a conserved global $U(1)$ charge. We shall see that, regardless of the spin (and statistics) of the fields in the original action, any three-dimensional model with an Abelian conserved charge can be mapped into a dual Abelian gauge theory. We shall refer to this mapping as *duality* rather than *bosonization*, in view of the more general meaning of its defining characteristics. As the most distinctive feature of this mapping, we mention that the conserved Noether current J_μ corresponding to the $U(1)$ global symmetry will, again, have a topological current j^T_μ as its dual partner in the Abelian gauge theory:

$$J_\mu \longrightarrow \; j^T_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda \; . \tag{1.9}$$

The topological current j^T_μ becomes then identified with $\tilde{F}_\mu$, the *topological local*, *gauge invariant*, and *identically conserved* current that can be written in terms of A_μ in the dual gauge invariant theory. Moreover, the gauge invariance of the bosonized theory will be shown to be related to the renormalizability of the functional integrals leading to the dual theory.

Another property of the bosonization procedure, which will be extended to the general case, shall be that the dual gauge invariant action will also contain a Chern-Simons term, whenever there is a breaking of parity in the original theory.
It is worth noting that the possibility of describing three-dimensional $U(1)$ charged models in terms of gauge fields can bring us new perspectives in order to understand in a deeper way the non perturbative dynamics of planar systems. Needless to say, much is already known about the geometry and the topology of gauge theories.

The present work is organized as follows: in section 2 we briefly review the bosonization of the massive Dirac field, fixing our conventions, and extracting from that case the essential properties of the procedure. This procedure is then generalized to an arbitrary field with a $U(1)$ charge in section 3. In section 4 a few examples are dealt with in detail. Section 5 is devoted to the physical interpretation of the coefficient of the induced Chern-Simons in the dual gauge invariant action form. Finally, in section 6, we present our conclusions.

2 Massive Dirac field

Let us begin by defining $\mathcal{Z}[s_\mu]$, the generating functional of current correlation functions for a massive Dirac field in the presence of an external source s_μ, in its 'fermionic representation':

$$\mathcal{Z}[s_\mu] \;=\; \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\{-S_F[\bar{\psi},\psi;s_\mu]\} \;, \tag{2.10}$$

where

$$S_F[\bar{\psi},\psi;s_\mu] \;=\; \int d^3x \, \bar{\psi}(x)[\not{\partial} + m + i \not{s}(x)]\psi(x) \;, \tag{2.11}$$

and we adopted the Euclidean spacetime conventions:

$$g_{\mu\nu} \;=\; \delta_{\mu\nu} \quad (\gamma_\mu)^\dagger \;=\; \gamma_\mu \quad \{\gamma_\mu,\gamma_\nu\} \;=\; 2\,\delta_{\mu\nu} \;. \tag{2.12}$$

Current correlation functions are averages of the conserved current operator $J_\mu(x) = \bar{\psi}(x)\gamma_\mu\psi(x)$

$$\langle J_{\mu_1}(x_1)\ldots J_{\mu_n}(x_n)\rangle_s \;=\; \frac{1}{\mathcal{Z}[s_\mu]}\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, J_{\mu_1}(x_1)\ldots J_{\mu_n}(x_n)$$

$$\times \exp\{-\int d^3x\bar{\psi}[\not{\partial} + m + i \not{s}]\psi\} \;, \tag{2.13}$$

and may, of course, be obtained by functional differentiation,

$$\langle J_{\mu_1}(x_1)\ldots J_{\mu_n}(x_n)\rangle_s \;=\; \frac{1}{\mathcal{Z}[s_\mu]} i\frac{\delta}{\delta s_{\mu_1}(x_1)}\ldots i\frac{\delta}{\delta s_{\mu_n}(x_n)}\mathcal{Z}[s_\mu]$$

$$= \; i\frac{\delta}{\delta s_{\mu_1}(x_1)}\ldots i\frac{\delta}{\delta s_{\mu_n}(x_n)}\mathcal{W}[s_\mu] \;, \tag{2.14}$$

with $\mathcal{W}[s_\mu] \;=\; \ln \mathcal{Z}[s_\mu]$. Correlation functions for vanishing s_μ shall be denoted as in (2.13), except for the omission of the subindex 's' in the average symbol.

The first step in the bosonization procedure consists of performing the non-anomalous change of variables:

$$\psi(x) \;\to\; e^{i\alpha(x)}\psi(x) \qquad \bar\psi(x) \;\to\; e^{-i\alpha(x)}\bar\psi(x) \tag{2.15}$$

for the fermionic fields in the functional integral. This leads to an equivalent expression for $\mathcal{Z}[s_\mu]$,

$$\mathcal{Z}[s_\mu] \;=\; \int \mathcal{D}\bar\psi\mathcal{D}\psi \,\exp\left\{-S_F[\bar\psi,\psi;s_\mu+\partial_\mu\alpha]\right\} \;, \tag{2.16}$$

or,

$$\mathcal{Z}[s_\mu] \;=\; \mathcal{Z}[s_\mu+\partial_\mu\alpha] \;. \tag{2.17}$$

As the left hand side of (2.17) is independent of α, so must be the right hand side. Then, we obtain an equivalent form for $\mathcal{Z}[s_\mu]$ by integrating over α and, if we want to keep also source-independent factors, dividing by the corresponding volume factor $\mathcal{N}_1$:

$$\mathcal{Z}[s_\mu] \;=\; \frac{1}{\mathcal{N}_1}\int \mathcal{D}\alpha \; \mathcal{Z}[s_\mu+\partial_\mu\alpha]$$

$$= \; \frac{1}{\mathcal{N}_1}\int \mathcal{D}\alpha\mathcal{D}\bar\psi\mathcal{D}\psi \,\exp\left\{-S_F[\bar\psi,\psi;s_\mu+\partial_\mu\alpha]\right\} \;, \tag{2.18}$$

where $\mathcal{N}_1 = \int \mathcal{D}\alpha$. This is of course a divergent factor, similar to the gauge group volume element which is factorized in the application of the Faddeev-Popov procedure to an Abelian gauge invariant theory. We assume that it has been regulated, for example, by introducing an Euclidean cutoff Λ, and by using a finite volume spatial region. We shall not need its explicit form, however, and will keep the same notation $\mathcal{N}_1$, in the understanding that this object has been regularized. Then we go on to an expression where the integration over the scalar field α is transformed into one over a 'pure gauge' vector field b_μ, such that $b_\mu = \partial_\mu \alpha$. The integration over this vector field is thus constrained to satisfy a null-curvature condition, namely

$$\mathcal{Z}[s_\mu] \;=\; \frac{\mathcal{N}_2}{\mathcal{N}_1} \int \mathcal{D}b_\mu \delta[\tilde{f}_\mu(b)] \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left\{-S_F[\bar{\psi},\psi; s_\mu + b_\mu]\right\} \;, \tag{2.19}$$

where $\tilde{f}_\mu \;=\; \epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda$. In (2.19), the new constant factor $\mathcal{N}_2$ appears as a consequence of the fact that the δ function of $\tilde{f}_\mu(b)$ differs from the 'pure gauge' condition for b_μ by a constant, ill-defined factor. One easily sees that:

$$\mathcal{N}_2 \;=\; \det(\epsilon_{\mu\lambda\nu}\partial_\lambda) \;. \tag{2.20}$$

This factor is actually zero, because of the zero eigenvalue corresponding to the longitudinal mode of the operator $\epsilon_{\mu\lambda\nu}\partial_\lambda$, and it is formally canceled by an analogous factor in the denominator, which comes from the δ function. A possible way to cure this would be to represent the determinant as a functional integral over vector Grassmann fields,

$$\mathcal{N}_2 \;=\; \int \mathcal{D}\bar{c}_\mu \mathcal{D}c_\mu \; \exp\{\int d^3x \bar{c}_\mu \epsilon_{\mu\lambda\nu}\partial_\lambda c_\nu\} \tag{2.21}$$

and to fix the gauge (through the Faddeev-Popov procedure) for this gauge invariant functional integral. We shall not detail this, however, since we will not need the explicit form of $\mathcal{N}_2$. We shall keep the notation '$\mathcal{N}_2$' to denote this (gauge fixed) object.

One then shifts the b_μ field: $b_\mu \to b_\mu - s_\mu$ to decouple it from the source s_μ in the fermionic action, and exponentiates the δ function by using a Lagrange multiplier field A_μ, to obtain

$$\mathcal{Z}[s_\mu] \;=\; \frac{\mathcal{N}_2}{\mathcal{N}_1} \int \mathcal{D}A_\mu \mathcal{D}b_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left\{-S_F[\bar{\psi},\psi; b_\mu] + i\int d^3x A_\mu \tilde{f}_\mu(b-s)\right\} \;. \tag{2.22}$$

Equation (2.22) then leads to the bosonic representation

$$\mathcal{Z}[s_\mu] \;=\; \int \mathcal{D}A_\mu \, e^{-S_B[A] - i\int d^3x s_\mu \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda} \tag{2.23}$$

where the 'bosonized action' $S_B[A]$ is defined by

$$e^{-S_B[A]} \;=\; \frac{\mathcal{N}_2}{\mathcal{N}_1} \int \mathcal{D}b_\mu \, \det[\not{\partial} + m + i \not{b}] \, e^{i\int d^3x b_\mu \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda} \;. \tag{2.24}$$

The role of the factors $\mathcal{N}_{1,2}$ is evident in (2.24): their presence adds a constant to the bosonized action. To understand their effect, one can consider the $s_\mu = 0$ case, which leads to

$$\mathcal{Z}[0] \;=\; \int \mathcal{D}A_\mu \, e^{-S_B[A]}$$

$$= \frac{\mathcal{N}_2}{\mathcal{N}_1} \int \mathcal{D}A_\mu \int \mathcal{D}b_\mu \, \det[\not\partial + m + i \not b] \, e^{i\int d^3x b_\mu \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda} \,, \tag{2.25}$$

or, integrating over A_μ,

$$\mathcal{Z}[0] \;=\; \frac{\mathcal{N}_2}{\mathcal{N}_1} \int \mathcal{D}b_\mu \, \delta[\tilde{f}(b)] \, \det[\not\partial + m + i \not b] \,,$$

$$= \frac{1}{\mathcal{N}_1} \int \mathcal{D}\alpha \, \det[\not\partial + m + i \not\partial\alpha] \;=\; \frac{1}{\mathcal{N}_1} \int \mathcal{D}\alpha \, \det[\not\partial + m]$$

$$= \det[\not\partial + m] \,, \tag{2.26}$$

where we used the gauge invariance of the fermionic determinant. This shows that the factors are required if one is interested in evaluating quantities that are interesting even in the absence of sources, like the vacuum energy in a gravitational background, or in a finite volume. In what follows, however, we shall disregard the constant factors, since we will be mostly interested in flat spacetime correlation functions, where those factors cancel out.

To find the explicit form of the bosonized action, one needs of course to evaluate the fermionic determinant *and* of the integral over the auxiliary field b_μ, which yields an expression with the structure of (1.4). To make sense of expression (2.24) when the fermionic determinant is evaluated beyond the quadratic approximation, one should say something about the renormalizability of the functional integral over the auxiliary field b_μ. We note that this integral could be regarded as corresponding to the generating functional of complete Green's functions for a dynamical vector gauge field b_μ, equipped with a gauge invariant 'action':

$$S_b[b_\mu] \;=\; -\mathcal{W}[b_\mu] \,, \tag{2.27}$$

and with a 'source' $s^b_\mu = i\epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$, namely,

$$e^{-S_B[A]} \;=\; \int [\mathcal{D}b_\mu] \, e^{-S_b[b_\mu] + \int d^3x s^b_\mu b_\mu} \,, \tag{2.28}$$

where $[\mathcal{D}b_\mu]$ denotes the measure including gauge fixing. We assume the Landau gauge has been chosen, for the sake of simplicity.

The UV properties of this integral are not explicit, due to the non-local and non-polynomial character of S_b. However, by using a functional expansion of $\mathcal{W}$, we see that

$$S_b[b_\mu] \;=\; -\frac{1}{2} \int d^3x_1 d^3x_2 \mathcal{W}^{(2)}_{\mu_1\mu_2}(x_1,x_2) b_{\mu_1}(x_1) b_{\mu_2}(x_2)$$

$$- \frac{1}{3!} \int d^3x_1 d^3x_2 d^3x_3 \mathcal{W}^{(3)}_{\mu_1\mu_2\mu_3}(x_1,x_2,x_3) b_{\mu_1}(x_1) b_{\mu_2}(x_2) b_{\mu_3}(x_3) \;+\; \ldots \tag{2.29}$$

where we adopted the notation:

$$\mathcal{W}^{(n)}_{\mu_1\cdots\mu_n}(x_1,\cdots,x_n) \;=\; \frac{\delta^n \mathcal{W}[b_\mu]}{\delta b_{\mu_1}(x_1)\cdots\delta b_{\mu_n}(x_n)}\Big|_{b_\mu=0} \;. \tag{2.30}$$

The first step to understand the convergence properties of the functional integral over b_μ, is to find out the large momentum behaviour of the propagator, which is obviously determined by $\mathcal{W}^{(2)}$. Regardless of the details of the theory, the Fourier transform of the function $\mathcal{W}^{(2)}$ will grow linearly with k, for large values of the momentum k, as a dimensional analysis shows: gauge invariance (together with the Landau gauge choice) of the fermionic determinant implies that $\mathcal{W}^{(2)}$ is transverse, and its coefficient has to be a scalar function of k_μ, which for large values of $|k|$ behave like $\sim |k|$. Then the b_μ field propagator will behave like $\sim 1/k$ in the same regime. We note that, in the absence of gauge invariance for the fermionic determinant, there is no reason to exclude worse behaviours for the propagator.

The b_μ propagator will connect to generalized vertices corresponding to all the $\mathcal{W}^{(n)}$'s. These vertices are momentum dependent, and actually, they behave as a negative power of k for large k. For an arbitrary n, we have the behaviour $\mathcal{W}^{(n)} \sim k^{3-n}$, with n even and greater than 4. Thus the power counting of a 1PI diagram G corresponding to the functional integral over b_μ will have a superficial degree of divergence $\omega(G)$ given by

$$\omega(G) \;=\; 3L - I + \sum_{n\geq 4} k_n (3-n) \;, \tag{2.31}$$

where L is the number of independent loops, I the number of internal b_μ lines, and k_n the number of vertices of n legs in the diagram G. Using the standard relation $L = I - V + 1$, where V is the total number of vertices (of any kind), we see that

$$\omega(G) \;=\; 3 + 2I - 3V + \sum_{n\geq 4} k_n (3-n) \;. \tag{2.32}$$

On the other hand, any proper diagram verifies the 'topological' relation:

$$\sum_{n\geq 4} n k_n \;=\; 2I + E \;, \tag{2.33}$$

where E denotes the number of external b_μ field lines. When (2.33) is inserted into (2.32), it yields

$$\omega(G) \;=\; 3 - E \;, \tag{2.34}$$

which evidently corresponds to a renormalizable theory, since only the two-point function *might* require a subtraction. Gauge invariance, combined with the linear degree of divergence, imply that the counterterm could only be a Chern-Simons like term. This, however, is not renormalized beyond one loop [11].

We conclude from this discussion that the bosonized action exists in perturbation theory, to all orders (in a loop expansion). It should be noted that, when the fermionic determinant is evaluated in a derivative expansion, the previous power counting is spoiled. This is, however, just an artifact of the approximation.

To conclude this section, we enumerate some essential features of the bosonization algorithm, to take them into account in the generalization we shall develop in the next section:

1. The starting point should be a model with a *global* $U(1)$ invariance. A local invariance is not good, since then the original action is not changed by (the analog of) the coordinate transformation (2.15).

2. The generating functional for the current correlation functions, $\mathcal{Z}[s_\mu]$, should verify (2.17). This allows for the decoupling of the source s_μ and the b_μ field through a shift of the latter. Of course, the way to prove that this equation is verified will, in general, depend on the particular model considered, and it may, as in the fermionic case, require a change of variables for the charged fields.

3. As a matter of principle, we should require the bosonized action to exist, and to avoid defining it through a non-renormalizable functional integral. We have seen that, if the property (2.17) holds, that integral is finite. Moreover, there is no need to assume that $S_b = -\mathcal{W}[b_\mu]$ proceeds from a fermionic matter field; just gauge invariance of the result is required.

3 The general framework

As mentioned at the the end of the preceeding section, the extension of the procedure originally developed for the Dirac field to an arbitrary model with a global Abelian charge is straightforward, as long as property (2.17) holds. The reason is that then all the steps leading to the bosonic representation (2.23) hold true, with the only change that the expression for $\mathcal{Z}[s_\mu]$ as an integral over charged fields is not written explicitly. One sees that

$$\mathcal{Z}[s_\mu] \;=\; \int \mathcal{D}A_\mu \, e^{-S_B[A]-i\int d^3x s_\mu \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda} \tag{3.35}$$

where now the 'bosonized action' $S_B[A]$ is defined by

$$e^{-S_B[A]} \;=\; \frac{\mathcal{N}_2}{\mathcal{N}_1} \int \mathcal{D}b_\mu \, \mathcal{Z}[b_\mu] \, e^{i \int d^3x b_\mu \epsilon_{\mu\nu\lambda} \partial_\nu A_\lambda} \,, \tag{3.36}$$

an expression of which (2.24) is a particular case, where $\mathcal{Z}[s_\mu]$ is a fermionic determinant. Different field theories coupled to an external source s_μ will give rise to an identical expression, although the representation of $\mathcal{Z}[s_\mu]$ as a functional integral over the original matter field will of course be different for each model. By expanding (2.17) in powers of α and integrating by parts, one can see that it is equivalent to the (infinite) set of Ward identities:

$$\langle \partial \cdot J(x_1) \cdots \partial \cdot J(x_n) \, J_{\nu_1}(y_1) \cdots J_{\nu_k}(y_k) \rangle_s \;=\; 0 \quad , \forall n = 1, 2, \ldots, k = 0, 1, \ldots \tag{3.37}$$

This of course also implies the corresponding identities for $s_\mu = 0$, but one cannot obtain the $s_\mu \neq 0$ identities from the $s_\mu = 0$ ones. If one just requires that the $s_\mu = 0$ identities hold, and the source s_μ is only an artifact to derive $s_\mu = 0$ Green's functions, then there is no lost of generality in assuming (2.17) (this will of course not affect the $s_\mu = 0$ averages). That property is tantamount to requiring that $\mathcal{Z}[s_\mu]$ is invariant under 'gauge transformations' of the source. This can be guaranteed if the coupling of the source to the matter fields is gauge invariant, since then the matter fields can be gauge transformed in order to compensate for the gauge transformation of the 'gauge field' s_μ. In what follows, we shall then require current conservation even in the presence of a non-vanishing source s_μ, as this implies conservation when $s_\mu \to 0$ as well. More importantly, the gauge invariance is necessary to assure the finiteness of the b_μ functional integral. In order to work out the general setup, now in terms of the matter fields, let us consider a pair $\phi, \phi^\dagger$ of complex $U(1)$-charged fields (scalars, vectors, spinors, ...), and let

$$S[\phi^\dagger, \phi] \;=\; \int d^3x \, \mathcal{L}(\phi^\dagger, \phi; \partial\phi^\dagger, \partial\phi) \;\; , \tag{3.38}$$

be the corresponding classical action. As already remarked, $S[\phi^\dagger, \phi]$ will be assumed to be invariant under the global $U(1)$ transformations

$$\begin{aligned} \phi \;&\to\; e^{i\alpha}\phi \,, \\ \phi^\dagger \;&\to\; e^{-i\alpha}\phi^\dagger \,, \end{aligned} \tag{3.39}$$

yielding a conserved $U(1)$ Noether current J_μ, *i.e.*,

$$\partial_\mu J_\mu = 0 \tag{3.40}$$

when the equations of motion hold. Now we require the conservation of the current even when the source does not vanish. As we said above, the coupling to the source has to be then gauge invariant, and this is assured by replacing normal derivatives by covariant derivatives with respect to the source. Note that, in general, the conserved current will then depend on the source s_μ. With this in mind, we introduce the generating functional $\mathcal{Z}[s_\mu]$

$$\mathcal{Z}[s_\mu] \;=\; \int D\phi D\phi^\dagger e^{-\int d^3x \left(\mathcal{L}[\phi^\dagger,\phi;(D\phi)^\dagger,D\phi]\,\right)} \;, \tag{3.41}$$

where $D_\mu\phi = (\partial_\mu + is_\mu)\phi$. Following the steps already applied in section 2, one performs the local change of variables in (3.41)

$$\begin{aligned} \phi(x) \;&\rightarrow\; e^{i\alpha(x)}\phi(x) \;, \\ \phi^\dagger(x) \;&\rightarrow\; e^{-i\alpha(x)}\phi^\dagger(x) \;. \end{aligned} \tag{3.42}$$

which immediately leads to the desired property

$$\mathcal{Z}[s_\mu] \;=\; \mathcal{Z}[s_\mu + \partial_\mu\alpha] \;. \tag{3.43}$$

Using now the following relation

$$\begin{aligned} \mathcal{L}\left(\phi,\phi^\dagger,(\partial_\mu + is_\mu)\phi,(\partial_\mu - is_\mu)\phi^\dagger\right) \;&=\; \\ \mathcal{L}\left(\phi,\phi^\dagger,\partial_\mu\phi,\partial_\mu\phi^\dagger\right) &+ s_\mu J^\mu \;\;, \end{aligned} \tag{3.44}$$

where $J_\mu(\phi,\phi^\dagger)$ is the Noether current associated to the global $U(1)$ invariance of the action $S(\phi)$, for the generating functional $\mathcal{Z}[s_\mu]$ we obtain

$$\mathcal{Z}[s_\mu] \;=\; \int D\phi D\phi^\dagger e^{-\int d^3x \left(\mathcal{L}[\phi^\dagger,\phi;(\partial\phi)^\dagger,\partial\phi] + s_\mu J^\mu\,\right)} \;. \tag{3.45}$$

The meaning of the equation (3.44) is that coupling the $U(1)$ current J^μ to an external source s_μ, is equivalent to replace the space-time derivative ∂_μ with the covariant derivative $(\partial_\mu + is_\mu)$, according to the principle of the minimal coupling. Although the relation (3.44) is self-evident for actions which are linear in the space-time derivatives of the fields, as for instance in the case of the usual fermionic action, it is worth underlining that eq.(3.44) can be in fact satisfied also in more general cases, the recipe here being that of using a first-order formalism by means of the introduction of suitable auxiliary fields, as it will be shown in detail in the next section in the case of the complex scalar fields.
Using now eq.(3.42), we get

$$\mathcal{Z}[s_\mu] = \int D\phi D\phi^\dagger D\eta_\mu DA_\mu$$

$$\exp\left\{-\int d^3x[\mathcal{L}(\phi,\phi^\dagger,(\partial_\mu+i\eta_\mu-is_\mu)\phi,(\partial_\mu-i\eta_\mu+is_\mu)\phi^\dagger] \right.$$
$$\left. +i\ A_\mu\epsilon^{\mu\nu\rho}\partial_\nu\eta_\rho]\right\}\ , \tag{3.46}$$

so that, performing the change of variables

$$\eta_\mu \to \eta_\mu - s_\mu\ , \tag{3.47}$$

the external source s_μ decouples from the fields $\phi, \phi^\dagger$, *i.e.*

$$\mathcal{Z}[s_\mu] = \int D\phi D\phi^\dagger D\eta_\mu DA_\mu \exp\left\{-\int d^3x[\mathcal{L}(\phi,\phi^\dagger,(\partial_\mu+i\eta_\mu)\phi,(\partial_\mu-i\eta_\mu)\phi^\dagger]\right.$$
$$\left. +i\ A_\mu\epsilon^{\mu\nu\rho}\partial_\nu(\eta_\rho-s_\rho)]\right\}\ , \tag{3.48}$$

Introducing thus the effective action $S_{\rm eff}(\eta)=\int d^3x\mathcal{L}_{\rm eff}(\eta)$ through the equation

$$e^{-S_{\rm eff}[\eta_\mu]} = \int D\phi D\phi^\dagger e^{-\int d^3x\mathcal{L}\left(\phi,\phi^\dagger,(\partial_\mu+i\eta_\mu)\phi,(\partial_\mu-i\eta_\mu)\phi^\dagger\right)}\ , \tag{3.49}$$

we have

$$\mathcal{Z}(s) = \int D\eta_\mu DA_\mu e^{-\int d^3x(\mathcal{L}_{\rm eff}(\eta)\ -i\ A_\mu\epsilon^{\mu\nu\rho}\partial_\nu\eta_\rho+iA_\mu\epsilon^{\mu\nu\rho}\partial_\nu s_\rho)}\ .$$

Finally, defining the so called dual action $S_{\rm dual}(A)$

$$e^{-S_{\rm dual}[A_\mu]} = \int D\eta_\mu e^{-\int d^3x(\mathcal{L}_{\rm eff}(\eta)\ -\ iA_\mu\epsilon^{\mu\nu\rho}\partial_\nu\eta_\rho)}\ , \tag{3.50}$$

we obtain the *dual representation* of the generating functional $\mathcal{Z}[s_\mu]$, namely

$$\mathcal{Z}[s_\mu] = \int DA_\mu e^{-\left(S_{\rm dual}[A_\mu]+i\int d^3x s_\mu\epsilon^{\mu\nu\rho}\partial_\nu A_\rho\right)}\ . \tag{3.51}$$

It should be observed that the dual field A_μ can be actually interpreted as a genuine gauge field, as the dual action $S_{\rm dual}(A)$ in eq.(3.50) is manifestly gauge invariant,

$$S_{\rm dual}[A_\mu]\ =\ S_{\rm dual}[A_\mu+\partial_\mu\omega]\ . \tag{3.52}$$

Expression (3.51) provides a representation of the correlation functions in terms of a $(2+1)$ gauge field theory. In particular, from the coupling of the external source s_μ in the expression (3.51), it becomes apparent that, as announced, the $U(1)$ Noether current J_μ is mapped into a dual topological current

$$J_\mu(\phi,\phi^\dagger) \longrightarrow j^T_\mu(A) = \epsilon^{\mu\nu\rho}\partial_\nu A_\rho\ . \tag{3.53}$$

Correspondingly, for the classical action we have

$$S[\phi] = \int d^3x\mathcal{L}\left(\phi,\phi^\dagger,\partial_\mu\phi,\partial_\mu\phi^\dagger\right) \longrightarrow S_{\rm dual}[A_\mu]\ . \tag{3.54}$$

Equations (3.53) , (3.54) represent our duality mapping and have to be understood in terms of an equality among the correlations functions,

$$\begin{aligned} & \langle J_{\mu_1}(x_1)J_{\mu_2}(x_2)\ \ldots\ J_{\mu_n}(x_n)\rangle_{S[\phi]} \\ = & \left\langle j^T_{\mu_1}(x_1)j^T_{\mu_2}(x_2)\ \ldots\ j^T_{\mu_n}(x_n)\right\rangle_{S_{\rm dual}[A_\mu]} \end{aligned} \tag{3.55}$$

obtained by differentiating the generating functionals (3.41) and (3.51) with respect to the external source s_μ. Equation (3.55) states that the Green's functions of $U(1)$ Noether currents have a dual representation in terms of topological currents built up with a gauge field.

As a final comment we remark that the universal character of the bosonization rules for fermionic systems (eq.(1.6)), are generalized to the present dual mapping. This means that, adding to the initial action $S[\phi]$ a current interaction term $I[J^\mu]$, the final dual result is found to be

$$S[\phi]+I[J^\mu]\leftrightarrow S_{\rm dual}[A]+I[\epsilon^{\mu\nu\rho}\partial_\nu A_\rho]\ . \tag{3.56}$$

Indeed, this can be easily seen by representing the interaction term $I[J^\mu]$ in the following Fourier functional integral form

$$e^{-I[J^\mu]}=\int Da_\mu e^{-(\int d^3x a_\mu J^\mu+\tilde I[a_\mu])} \tag{3.57}$$

for a suitable action $\tilde I[a_\mu]$. Therefore, for the generating functional we get

$$\begin{aligned} \mathcal{Z}[s_\mu] &= \int D\phi D\phi^\dagger e^{-(S[\phi]+I[J^\mu]+\int d^3x s_\mu J^\mu)} \\ &= \int D\phi D\phi^\dagger Da_\mu e^{-(S[\phi]+\tilde I[a_\mu]+\int d^3x(s_\mu+a_\mu)J^\mu)}\ . \end{aligned} \tag{3.58}$$

Performing now the path integration over the variables ϕ, $\phi^\dagger$ using the dual representation (3.51), it follows

$$\mathcal{Z}[s_\mu]=\int DA_\mu Da_\mu e^{-(S_{\rm dual}[A_\mu]+\tilde I[a_\mu]+\int d^3x(s^\mu+a^\mu)j^T_\mu)}\ , \tag{3.59}$$

which, upon integration over a_μ and use of eq.(3.57), yields the final dual representation

$$\mathcal{Z}[s_\mu]=\int DA_\mu e^{-(S_{\rm dual}[A_\mu]+I[j^T_\mu]+\int d^3x s^\mu j^T_\mu)}\ , \tag{3.60}$$

which implies, in particular, eq.(3.56).

4 Examples

For a better understanding of the previous results, let us work out in detail some examples, beginning by the simplest case of the dual mapping applied to a complex scalar field.

4.1 Complex scalar field

The model is described by the $U(1)$ invariant action

$$S(\varphi,\varphi^\dagger) = \int d^3x \left(\partial_\mu \varphi \partial^\mu \varphi^\dagger + m^2 \varphi \varphi^\dagger \ + V(|\varphi|) \right) \ , \tag{4.61}$$

$\varphi,\ \varphi^\dagger$ being complex scalar fields and $V(|\varphi|)$ stands for the scalar potential. As already underlined, it is convenient to switch to the equivalent first order formalism by introducing suitable auxiliary fields. In the present case this task amounts to rewriting the action (4.61) in the following form:

$$S(\varphi,b) = \int d^3x \left(b^\dagger_\mu \partial^\mu \varphi + b_\mu \partial^\mu \varphi^\dagger - b^\dagger_\mu b^\mu + m^2 \varphi \varphi^\dagger \ + V(|\varphi|) \right) \ . \tag{4.62}$$

Expression (4.62) is easily seen to be completely equivalent to (4.61) upon elimination of the two auxiliary fields $b_\mu,\ b^\dagger_\mu$ through the equations of motion

$$\begin{aligned} \frac{\delta S}{\delta b^\mu} &= \partial^\mu \varphi^\dagger - b^\dagger_\mu \ , \\ \frac{\delta S}{\delta b^\dagger_\mu} &= \partial^\mu \varphi - b^\mu \ . \end{aligned} \tag{4.63}$$

The Noether current J^μ corresponding to the $U(1)$ global invariance

$$\begin{aligned} \varphi \ &\rightarrow \ e^{i\alpha} \varphi \ , \qquad & b_\mu \rightarrow e^{-i\alpha} b_\mu \\ \varphi^\dagger \ &\rightarrow \ e^{-i\alpha} \varphi^\dagger \ , \qquad & b^\dagger_\mu \rightarrow e^{i\alpha} {b_\mu}^\dagger \ , \end{aligned} \tag{4.64}$$

is found to be

$$\begin{aligned} J^\mu &= i \left(b^{\dagger\mu} \ \varphi - b^\mu \varphi^\dagger \right) \ , \\ \partial_\mu J^\mu &= 0 \ + \ \text{eqs. of motion} \ . \end{aligned} \tag{4.65}$$

For the generating functional $\mathcal{Z}(s)$

$$\mathcal{Z}(s) = \mathcal{N} \int D\varphi D\varphi^\dagger Db_\mu Db^\dagger_\mu e^{-\left(S(\varphi,b) \ + \int d^3x s_\mu J^\mu\right)} \ , \tag{4.66}$$

we get

$$\mathcal{Z} \ = \ \mathcal{N} \int D\varphi D\varphi^\dagger Db_\mu Db^\dagger_\mu D\eta_\mu DA_\mu$$

$$e^{-\left(S(\varphi,(\partial+i\eta)\varphi,b)+\int d^3x(s_\mu J^\mu+ A_\mu\epsilon^{\mu\nu\rho}\partial_\nu\eta_\rho)\right)} . \tag{4.67}$$

Moreover, according to eq.(3.44) , the following relationship is easily proven to hold

$$S(\varphi,(\partial+i\eta)\varphi,b)+\int d^3x s_\mu J^\mu = S(\varphi,(\partial+i(\eta+s)\varphi,b) . \tag{4.68}$$

Repeating now the same steps of the previous section, for the final form of the generating functional $\mathcal{Z}(s)$ we obtain

$$\mathcal{Z}(s)=\mathcal{N}\int DA_\mu e^{-\left(S_{\rm dual}(A)+\int d^3x s_\mu\epsilon^{\mu\nu\rho}\partial_\nu A_\rho\right)} , \tag{4.69}$$

with the gauge invariant dual action $S_{\rm dual}(A)$ being defined by

$$e^{-S_{\rm dual}(A)} = \int D\varphi D\varphi^\dagger Db_\mu Db^\dagger_\mu D\eta_\mu e^{-\left(S(\varphi,(\partial+i\eta)\varphi,b)+\int d^3x A_\mu\epsilon^{\mu\nu\rho}\partial_\nu\eta_\rho\right)} . \tag{4.70}$$

We see therefore that, as announced, the Noether current (4.65) is mapped into the topological current $J^\mu_T(A)=\epsilon^{\mu\nu\rho}\partial_\nu A_\rho$. Equation (4.69) defines the dual representation of a charged scalar field in terms of a $(2+1)$ gauge theory. It is also worth mentioning that, in the present case, some of the one-loop contributions to the dual gauge invariant action $S_{\rm dual}(A)$ have been computed [13]. In particular, from the expression of the vacuum polarization [13], it follows that there is no induced Chern-Simons term in $S_{\rm dual}(A)$. As expected, this result is due to the absence of parity breaking terms in the starting action (4.61).

4.2 Complex vector field

As a second example, let us discuss the case of the complex vector field whose classical action reads

$$S(B,B^\dagger)=\int d^3x \left(\epsilon^{\mu\nu\rho}B^\dagger_\mu\partial_\nu B_\rho-\mu B^\dagger_\mu B^\mu\right) . \tag{4.71}$$

This model is known as the complex self-dual model [14], and it has been proven to be equivalent to the (complex) Maxwell-Chern-Simons model. It describes a massive excitation of mass μ and spin $s=\mu/|\mu|$. It should also be observed that the action (4.71) is not parity invariant, due to the presence of the mass term $\mu B^\dagger_\mu B^\mu$.
Obviously, expression (4.71) is left invariant by the global $U(1)$ transformations

$$
\begin{aligned}
B_\mu &\to e^{i\alpha}B_\mu \\
B^\dagger_\mu &\to e^{-i\alpha}B^\dagger_\mu \ ,
\end{aligned} \tag{4.72}
$$

which yield the following Noether current J^μ

$$
\begin{aligned}
J^\mu(B,B^\dagger) &= i\epsilon^{\mu\nu\rho}B^\dagger_\nu B_\rho \ , \\
\partial_\mu J^\mu &= 0 \ + \ \text{eqs. of motion} \ .
\end{aligned} \tag{4.73}
$$

As usual, let us start with the generating functional $\mathcal{Z}(s)$

$$
\mathcal{Z}(s) = \mathcal{N}\int DBDB^\dagger e^{-\left(S(B,B^\dagger) \, + \, \int d^3x s_\mu J^\mu\right)} \ . \tag{4.74}
$$

Repeating the same steps as before, it can be easily shown that the dual form for the generating functional (4.74) is found to be

$$
\mathcal{Z}(s) = \mathcal{N}\int DA_\mu e^{-\left(S_{\mathrm{dual}}(A)+\int d^3x s_\mu \epsilon^{\mu\nu\rho}\partial_\nu A_\rho\right)} \ , \tag{4.75}
$$

with

$$
e^{-S_{\mathrm{dual}}(A)} = \int DB_\mu DB^\dagger_\mu D\eta_\mu e^{-\left(S(B,(\partial+i\eta)B)+\int d^3x A_\mu \epsilon^{\mu\nu\rho}\partial_\nu \eta_\rho\right)} \ , \tag{4.76}
$$

being the corresponding gauge invariant dual action. We see thus that, once again, the $U(1)$ Noether current J^μ is mapped into the topological current $J^\mu_T(A)$

$$
J^\mu(B,B^\dagger) = i\epsilon^{\mu\nu\rho}B^\dagger_\nu B_\rho \ \longrightarrow \ J^\mu_T(A) = \epsilon^{\mu\nu\rho}\partial_\nu A_\rho \ . \tag{4.77}
$$

Concerning now the form of the dual gauge invariant action $S_{\mathrm{dual}}(A)$, the presence of an induced Chern-Simons term can be easily confirmed, due to the parity breaking mass term in the starting action (4.71). This statement follows also by simple inspection of the Feynman rules needed for the computation of the effective dual action $S_{\mathrm{dual}}(A)$. Indeed, from the expression (4.76) we see that to each interaction vertex of the type $BB^\dagger\eta$ there is an associated factor $i\epsilon_{\mu\nu\rho}$, namely

$$
\left[(BB^\dagger\eta) - \text{vertex}\right] \ \longrightarrow \mathrm{i}\epsilon_{\mu\nu\rho} \ , \tag{4.78}
$$

Moreover, for the propagator $\left\langle B^\dagger_\mu B_\nu\right\rangle$ we have

$$
\left\langle B^\dagger_\mu B_\nu\right\rangle = \mathcal{G}^{\mathrm{even}}_{\mu\nu}(p) \ + \ \mathcal{G}^{\mathrm{odd}}_{\mu\nu}(p) \ , \tag{4.79}
$$

$$
\mathcal{G}^{\mathrm{even}}_{\mu\nu}(p) = \frac{\mu}{p^2-\mu^2}\left(g_{\mu\nu} - \frac{p_\mu p_\nu}{\mu^2}\right) \tag{4.80}
$$

$$
\mathcal{G}^{\mathrm{odd}}_{\mu\nu}(p) = \frac{i\epsilon_{\mu\nu\lambda}p^\lambda}{p^2-\mu^2} \ , \tag{4.81}
$$

$\mathcal{G}^{\rm even}_{\mu\nu}(p)$, $\mathcal{G}^{\rm odd}_{\mu\nu}(p)$ denoting the even and the odd contributions, respectively. Looking then at the one-loop contribution for the vacuum polarization diagram, it is easily realized that the combination of vertices and propagators will always lead to an odd parity contribution, coming from expressions of the type

$$\epsilon^{\mu\alpha\beta}\epsilon^{\nu\sigma\tau}\int\frac{d^3k}{(2\pi)^3}\mathcal{G}^{\rm odd}_{\alpha\tau}(p-k)\mathcal{G}^{\rm even}_{\beta\sigma}(k)\ . \tag{4.82}$$

These contributions, properly renormalized, will result in the existence of a Chern-Simons term in the dual effective action $S_{\rm dual}(A)$.

Let us end this section by spending a few words on the interesting case in which the parity violating mass term is absent in the initial action. We start therefore with a topological action, corresponding to a complex pure Chern-Simons term, *i.e.*

$$S_{\rm top}=\int d^3x\epsilon^{\mu\nu\rho}B^\dagger_\mu\partial_\nu B_\rho\ . \tag{4.83}$$

Taking into account that the $U(1)$ Noether current (4.73) gets unmodified, we end up with a functional generator whose dual expression is provided by the same formula (4.75), with $S_{\rm dual}(A)$ given by

$$e^{-S_{\rm dual}(A)}\ =\int DB_\mu DB^\dagger_\mu D\eta_\mu e^{-\left(S_{\rm top}(B,(\partial+i\eta)B)+\int d^3xA_\mu\epsilon^{\mu\nu\rho}\partial_\nu\eta_\rho\right)}\ . \tag{4.84}$$

The only difference with the previous massive case relies on the form of the propagator, which now reads

$$\left\langle B^\dagger_\mu B_\nu\right\rangle_{\rm top}=\frac{i\epsilon_{\mu\nu\lambda}p^\lambda}{p^2}\ . \tag{4.85}$$

We could expect thus that, in spite of the presence of the $\epsilon_{\mu\nu\rho}$ tensor in the starting action (4.83), no induced Chern-Simons term should be generated in the final effective dual action $S_{\rm dual}(A)$. This should follow form the observation that a generic Feynman diagram obtained from the action $S_{\rm top}\,(B,(\partial+i\eta)B)$ will contain as many vertices as propagators, so that there will be no ϵ tensor left at the end. However, as in the massless fermionic case [12], the use of a gauge invariant regularization (*i.e.* Pauli-Villars) will introduce a parity even term, which no longer protects the generation of a possible induced Chern-Simons term in the effective action.

4.3 Fractional statistics model

We may consider as a starting point a theory of Dirac fermions coupled to a statistical gauge field a_μ, with a partition function

$$\mathcal{Z}[s_\mu]\ =\int\mathcal{D}a_\mu\mathcal{D}\bar\psi\mathcal{D}\psi\ \exp\left\{-\int d^3x\bar\psi[\partial\!\!\!/+i\ a\!\!\!/+i\ s\!\!\!/+m]\psi\right.$$

$$+i\theta S_{CS}[a_\mu]\} \ . \tag{4.86}$$

It should be noted that the statistical field a_μ is not related to A_μ, the field appearing in the dual representation, but is introduced just as a trick to represent the statistical interaction. This representation is of course simpler than the one that would result from using 'anyon field operators' and no statistical gauge field. As the Dirac field describes fermions, we see that the total statistical phase under particle interchange is $e^{i\pi\nu}$, where:

$$\nu \ = \ 1 \ + \ \frac{1}{2\pi\theta} \ . \tag{4.87}$$

Let us now see what is the dual theory that corresponds to this model, by a direct application of the procedure described in Sect. 3. We first note that the partition function is, indeed, invariant under gauge transformations of s_μ, and that a_μ must be integrated, since it is really part of the original matter (anyonic) field.

Then, integrating over a_μ, we obtain

$$\begin{aligned} \mathcal{Z}[s_\mu] \ &= \ \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \ \exp\Big\{-\int d^3x \bar{\psi}[\not{\partial} + i \not{s} + m]\psi \ + \\ &+ \frac{i}{\theta} \int d^3x d^3y J_\mu(x) \mathcal{K}_{\mu\nu}(x-y) J_\nu(y)\Big\} \end{aligned} \tag{4.88}$$

where $J_\mu = \bar{\psi}\gamma_\mu\psi$ is the fermionic current and $\mathcal{K}_{\mu\nu}$ is defined by

$$\epsilon_{\mu\lambda\rho}\partial_\lambda^x K_{\rho\nu}(x-y) \ = \ \delta_{\mu\nu}\delta^{(3)}(x-y) \ . \tag{4.89}$$

We are left thus with the bosonization of a massive Dirac fermion in the presence of an interaction depending only on the current $J_\mu(x)$. Therefore, making use of the universality properties of the bosonization rules for the fermionic current given in eq.(1.6) (see also ref.[7]), the above expression (4.88) is bosonized into

$$S_B[A] \ = \ i\,\frac{1}{\eta}\,S_{CS}[A] \ + \ R[\tilde{F}] + \frac{i}{\theta}\int d^3x d^3y j_\mu^T(x)\mathcal{K}_{\mu\nu}(x-y)j_\nu^T(y) \ , \tag{4.90}$$

with $j_\mu^T = \epsilon_{\mu\nu\lambda}\partial_\nu A_\lambda$ being the topological current. Moreover, from eq. (4.89), we easily obtain

$$S_B[A] \ = \ i\,(\frac{1}{\eta} + \frac{1}{\theta})\,S_{CS}[A] \ + \ R[\tilde{F}] \ . \tag{4.91}$$

We are now ready to discuss the physical interpretation of the results so far obtained. This will be the task of the next section.

5 Physical interpretation

Let us now deal with the dual gauge invariant action for the field A_μ, with the aim of interpreting some of its properties.

We shall see that the general structure of the dual action is strongly related to the statistical properties of its excitations.

The bosonized action will certainly display nontrivial excitations corresponding to field configurations with finite energy, labeled by an integer n, corresponding to the quantization of the 'magnetic' flux [16]. Indeed, this quantization is a consequence of the presence of the R-term in eq.(4.91), which implies that a finite energy field configuration A_μ must be a (regular) pure gauge at spatial infinity. As a matter of fact, the energy density is entirely due to the R-term, whose leading low energy contribution is found to be the Maxwell term. These configurations correspond typically to localized vortices.

If we now consider a configuration in the dual theory where two well separated vortices are adiabatically interchanged, the final phase factor relative to the case where the vortices are not interchanged should be dominated by the Chern-Simons term in eq.(4.91), which can be re-expressed as

$$i(\frac{1}{\eta}+\frac{1}{\theta})\int d^3x d^3y j^T_\mu(x)\mathcal{K}_{\mu\nu}(x-y)j^T_\nu(y)\ . \tag{5.92}$$

In addition, taking into account that $\partial_\mu j^T_\mu(x)=0$, we see that the term (5.92) corresponds to the well-known form of the 'statistical interaction' between particle currents (see, for example, [15]). This means that under an interchange of the positions of two vortices in the dual gauge theory, the wave function will pick up a phase factor $e^{i\pi\nu}$, where

$$\nu\ =\frac{1}{2\pi}(\frac{1}{\eta}+\frac{1}{\theta}). \tag{5.93}$$

At this point, in order to go further in the analysis of the statistics of these excitations, we have to specify the value of the induced Chern-Simons coefficient η. As already underlined in the introduction, the computation of this coefficient is plagued by finite regularization ambiguities, which have to be fixed by imposing suitable physical requirements on the theory. In what follows, we shall adopt a determination of η compatible with the requirement of invariance of the three dimensional fermionic determinant under large gauge transformations, when the euclidean time coordinate is compactified to a circle S^1[12].

In particular, for the free massive Dirac fermion, this condition leads to the value

$$\eta = \frac{1}{2\pi} . \tag{5.94}$$

We underline here that with this value of η, the statistical factor in eq.(5.93) coincides precisely with that of eq.(4.87).

It is also interesting to understand this result from the spin and statistics theorem point of view. If one assumes that the density of particles (defined in terms of the time component of the bosonized current) is localized, and that point-like densities are avoided by the introduction of smeared-out densities (that tend to point-like ones at the end of the calculation), one can show [15] that the spin of each excitation becomes

$$S = \frac{1}{2}\nu . \tag{5.95}$$

This is the well known result that the spin and statistics theorem holds when the interaction is mediated by a Chern-Simons gauge field. However, note that here we see an interesting new relation: the statistics (and spin) of the matter field is encoded in the coefficient of the induced Chern-Simons term. Of course, the same analysis holds also in the case where the initial fractional statistical action is that of a spinless scalar complex field, in the presence of the statistical field a. In this case the statistical factor ν contains only the contribution coming from the θ term, due to the absence of the induced Chern-Simons term in the bosonic polarization tensor.

6 Conclusions

We have shown that the bosonization program in three dimensions can be generalized to any Abelian field theory system displaying a global U(1) invariance. In particular, the $U(1)$ Noether current J_μ is mapped into the topological dual current j^T_μ, this mapping being exact and universal.

We would like to observe that, in the case where a second order field theory is considered, the above mentioned properties of the dual mapping have been derived by recasting the theory in a first order formalism. Moreover, the current correlation functions corresponding to these two formalisms are known to differ by contact terms, i.e. by terms proportional to the derivatives of the delta function. However, these terms are not relevant to the calculation of correlation functions at different points, implying thus the vanishing of the contact terms. Also as a by-product of the use of a first-order formulation, we note that the decoupling transformations of [9] can be trivially extended to the case of arbitrary matter fields, by replacing the Dirac operator with the appropriate one. This would produce the formal relation between interacting (Ψ) and free (χ) matter fields in the functional integral, which generalizes the one for the Dirac case:

$$
\begin{aligned}
\Psi(x) &= [1 + e(\not{\partial} + m)^{-1} \not{A}]^{-\frac{1}{2}} \chi(x) \\
\Psi^{\dagger}(x) &= \chi^{\dagger}(x) [1 + e \not{A}(\not{\partial} + m)^{-1}]^{-\frac{1}{2}} .
\end{aligned}
\tag{6.96}
$$

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Concerning the spectrum of the dual theory, we have been able to relate the induced Chern-Simons coefficient with the statistics of vortex like excitations, which are expected to be relevant for the understanding of the physical properties of the dual theory [16]. In particular, in the case of a dual theory corresponding to spin one-half massive Dirac fermions, these excitations turn out to obey Fermi Dirac statistics, provided one fixes the ambiguities of the induced Chern-Simons coefficient by requiring large gauge invariance of the fermionic determinant, when the time coordinate is compactified to a circle S^1. We also note that the vortex flux quantization necessary for the validity of this invariance argument, is strongly related to the presence of terms in the dual action which depend on the curvature, as for instance the Maxwell term, and which allow for finite energy solutions.

Finally, it is interesting to make a parallel with the SU(2) skyrmion quantization. In this case, Witten [17] has discussed the possible spin and statistics of the skyrmions, introducing a compactified time coordinate on the circle S^1. The $SU(2)$ Skyrme configurations defined on $S^1 \times S^3$ can be separated into two distinct homotopy classes, yielding two inequivalent quantizations, corresponding either to fermions or to bosons. These two possibilities can be implemented by the introduction of a topological Wess-Zumino-Witten (WZW) action, which allows to properly distinguish the two different statistics. In this sense, the Chern-Simons term of the dual theory can be interpreted as playing a role analogous to that of the WZW term in the Skyrme model. Both terms, being of first order in the time derivative, survive the process where two topological excitations are adiabatically interchanged.

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