Fock Vacuum Instability and Causality

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Abstract

The vacuum diagram is calculated at second order for theories with self-interacting massless fields in the framework of finite causal perturbation theory. It is pointed out that the infrared behaviour of the vacuum diagram leads to unstable Fock vacua for QCD or massless QED, but not for quantum gravity. Therefore a radical rearrangement of the physical system must take place for such theories. Conversely, stability of the Fock vacuum for massless interacting fields is another hint at the possibility that quantum gravity should be treated as an effective theory.

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1 Introduction

Soon after the beginnings of Quantum Field Theory (QFT), it turned out that the perturbative expansion of several quantities like the electron self-energy are plagued by divergences of different kind. These divergences in QFT can be classified into four types, namely infrared, ultraviolet (UV), infinite volume and particle number divergences. A large amount of work has been done to overcome these problems, but even after the construction of a renomalized perturbation expansions based on the Lagrangean formalism for interacting fields or the scattering matrix (S-matrix), little is known until today about the convergence properties of the perturbation expansion.

The first attempt to go beyond the widely used Lagrangean framework can be traced back to W. Heisenberg in 1943 [1], where he came to the conclusion that the basic observable is the S-matrix, which describes actually the quantities that can be measured in an experiment. Heisenberg even proposed to construct a theory directly in terms of the elements of the S-matrix, eliminating the notion of fields and the hypothesis of adiabatic switching of the interaction. However, the complete banishing of local quantities from the theory turned out to be too radical.

A very natural and rigorous approach to axiomatic perturbation theory which has been applied successfully to all relevant interactions of the Standard model [2, 3, 4] was provided by H. Epstein and V. Glaser in 1971 [5, 6, 7]. Their method, often called finite causal perturbation theory or FCPT for short in this paper, avoids UV-divergences from the start by defining mathematically correct time-ordered products for distributions. The resulting distributions are smeared out by test functions, avoiding provisionally volume and infrared divergences at once. In FCPT, the ansatz for the S-matrix as a power series in the coupling constant is crucial, namely S is considered as a sum of smeared operator-valued distributions

$$S(g) = \mathbf{1} + \sum_{n=1}^{\infty} \frac{1}{n!} \int d^4x_1 \dots d^4x_n \, T_n(x_1, \dots, x_n) \, g(x_1) \cdot \dots \cdot g(x_n) \quad , \tag{1}$$

where the Schwartz test function $g \in \mathcal{S}(\mathbf{R^4})$ switches the interaction and provides the infrared cutoff. The basic formulation of causality in FCPT, which had already been used by Bogoliubov *et al.* [8], is

$$S(g_1 + g_2) = S(g_2)S(g_1) \quad , \tag{2}$$

if the support of $g_2(x)$ is later than $\operatorname{supp} g_1$ in some Lorentz frame. We denote this by $\operatorname{supp} g_2 > \operatorname{supp} g_1$. The condition allows the construction of the n-point distributions T_n as a well-defined 'renormalized' time-ordered product expressed in terms of Wick monomials of free fields $:\mathcal{O}(x_1, \dots, x_n):$

$$T_n(x_1, \dots, x_n) = \sum_{\mathcal{O}} :\mathcal{O}(x_1, \dots, x_n): \, t_n^{\mathcal{O}}(x_1 - x_n, \dots, x_{n-1} - x_n) \, . \tag{3}$$

Possible ambiguities in the decomposition of T_n according to (3) will play no role for us in the sequel. The t_n are C-number distributions. T_n is constructed inductively from the first order $T_1(x)$, which describes the interaction among the quantum fields, and from the lower orders T_j, $j = 2, \dots, n-1$ by means of Poincaré covariance and causality. The inductive construction of the n-point distributions T_n can be considered as the main technical drawback of the theory, since all lower orders $T_1, ...T_{n-1}$ must be calculated first in order to construct T_n. But a naive definition of T_n like

$$T_n(x_1, ..., x_n) =$$

$$\sum_{\pi} \Theta(x^0_{\pi_1} - x^0_{\pi_2}) \cdot ... \cdot \Theta(x^0_{\pi_{n-1}} - x^0_{\pi_n}) T_1(x_{\pi_1}) \cdot ... \cdot T_1(x_{\pi_n}) \tag{4}$$

where the sum runs over all $n!$ permutations, is not well-defined, since it contains the product of Heaviside distributions with other singular distributions. This error leads to the well-known UV divergences in the calculation of Feynman diagrams. We illustrate this fact by the following very simple example in one dimension: It is of course meaningless to multiply a Heaviside distribution $\Theta(t)$ with a δ-distribution $\delta(t)$ or even its derivative $\delta'(t)$. The Fourier transforms are

$$\hat{\Theta}(\omega) = -\frac{i}{\sqrt{2\pi}}\frac{1}{\omega - i0} \quad , \quad \hat{\delta'}(\omega) = \frac{i\omega}{\sqrt{2\pi}} \quad . \tag{5}$$

The ill-defined product $(\Theta\delta')(t)$ goes over into a non-existing convolution by Fourier transform

$$(\Theta\delta')(t) \to \frac{1}{2\pi}(\hat{\Theta} * \delta')(\omega) = \frac{1}{(2\pi)^2}\int_{-\infty}^{+\infty} d\omega' \, \frac{\omega - \omega'}{\omega' - i0} \tag{6}$$

which is definitely 'UV-divergent'.

$T_2(x,y)$ contains typically tree (one contraction), loop (two contractions) and vacuum graph (three contractions) contributions. Due to the presence of normal ordering, tadpole diagrams do not show up. In this paper, we will concentrate on the vacuum graph, which has been considered already in [9] in a preliminary way. Since this graph contains no external fields, we can directly identify $T_2^{vac}(x_1, x_2) = t_2^{vac}(x_1 - x_2)$. Usually, vacuum bubbles are neglected by simply absorbing them into a phase of the S-matrix. In FCPT, this is not necessary, since their contribution to the S-matrix vanishes in the adiabatic limit for theories like ordinary QED with a massive electron. But this is not always the case for theories with self-interacting massless fields like QCD.

2 Preliminary remarks

The first order interaction $T_1(x) \sim i : \mathcal{L}_{int} :$ is usually motivated from a classical Lagrangean interaction density. For QED, $T_1(x)$ is given by means of the free electron and photon field

$$T_1(x) = ie : \overline{\Psi}(x)\gamma^\mu\Psi(x) : A_\mu(x) \quad . \tag{7}$$

For the purpose of this paper, we outline briefly the main steps in the construction of $T_2(x,y)$ from a given first order interaction. In FCPT, one constructs first the operator valued distributions

$$R_2'(x,y) = -T_1(y)T_1(x) \quad , \tag{8}$$
$$A_2'(x,y) = -T_1(x)T_1(y) \quad , \quad \text{and} \tag{9}$$
$$D_2(x,y) = R_2'(x,y) - A_2'(x,y) = [T_1(x), T_1(y)] \quad . \tag{10}$$

A meaningful theory ensures causality conditions (in a distributive sense) like (2), which implies also

$$[T_1(x), T_1(y)] = 0 \quad \text{for} \quad (x-y)^2 < 0 \tag{11}$$

such that

$$\operatorname{supp} D_2(x,y) \subseteq \{(x,y) \in \mathbf{R}^8 | (x-y)^2 \geq 0\} = \overline{V}_2 \quad . \tag{12}$$

This property of D_2 must be checked first before one may start the inductive construction of the S-matrix. From $D_2(x,y)$ we construct the Lorentz covariant retarded distribution $R_2(x,y)$, which coincides with $D_2(x,y)$ in the sense that $(R_2, \varphi) = (D_2, \varphi)$ for all test functions $\varphi \in \mathcal{S}(\mathbf{R}^8)$ with (compact) $\operatorname{supp}\varphi \subset \{(x,y) \in \mathbf{R}^8 | (x^0 - y^0) > 0\}$ and $(R_2, \varphi) = 0$ for all φ with $\operatorname{supp}\varphi \subset \{(x,y) \in$

$\mathbf{R}^8|(x^0 - y^0) < 0\}$. Such a distribution can always be found, but it is only defined up to local operator-valued distributions with support on the complete diagonal $\Delta_2 = \{(x,y) \in \mathbf{R}^8 | x = y\}$. Symmetry and renormalizability considerations further restrict these ambiguous terms. Obviously, $T_2(x,y) = R_2(x,y) - R_2'(x,y)$ is then a time-ordered product, since by construction

$$T_2(x,y) = \begin{cases} T_1(x)T_1(y) & : \quad x^0 - y^0 > 0 \\ T_1(y)T_1(x) & : \quad x^0 - y^0 < 0 \end{cases} \quad . \tag{13}$$

We pursue the strategy described above for the calculation of the vacuum diagram described by the C-number distribution $t_2^{vac}(x-y)$.

3 Calculation of the vacuum diagram

We calculate the vacuum diagram for the example of pure QCD, where the first order self-interaction is described by the help of $SU(3)$ structure constants f_{abc}

$$\begin{aligned} T_1(x) &= igf_{abc} : A_{\mu a}(x) A_{\nu b}(x) \partial^\nu A_c^\mu(x) : \\ &\quad -igf_{abc} : A_{\mu a}(x) u_b(x) \partial^\mu \tilde{u}_c(x) : \quad . \end{aligned} \tag{14}$$

Here, $A_{\mu a}$ are the free gauge potentials, satisfying the commutation relations

$$[A_a^{\mu(\mp)}(x), A_b^{\nu(\pm)}(y)] = i\delta_{ab} g^{\mu\nu} D_0^\pm(x-y) \quad , \tag{15}$$

where $A^{(\pm)}$ are the emission and absorption parts of A and $D_0^\pm$ the zero-mass Pauli-Jordan distributions defined by

$$D_0^\pm(x) = \pm \frac{i}{(2\pi)^3} \int d^4p \, \Theta(\pm p^0) \delta(p^2) e^{-ipx} \quad . \tag{16}$$

The free ghosts are Fermi fields and satisfy the anticommutation relations

$$\{u_a^{(\mp)}(x), \tilde{u}_b^{(\pm)}(y)\} = -i\delta_{ab} D_0^\pm(x-y) \quad , \tag{17}$$

and all other anticommutators vanish. The quartic gluon interaction is missing in T_1 since it is enforced by gauge invariance as a local counterterm at second order in FCPT [2]. For the sake of convenience, we will use g for the QCD coupling constant as well as for test functions, since a mix up is impossible.

As described in the previous section, we calculate first the vacuum expectation value

$$R_2'^{vac}(x,y) = r_2'^{vac}(x-y) = - < \Omega | T_1(y) T_1(x) | \Omega > \quad . \tag{18}$$

The calculation is straightforward, therefore we present the result which can be decomposed in a purely gluonic part and a ghost part:

$$r_2'^{vac,gl}(x-y) = -3ig^2 N(N^2-1) D_0^-(x-y) \partial_\nu^x D_0^-(x-y) \partial_y^\nu D_0^-(x-y) \quad , \tag{19}$$

$$r_2'^{vac,gh}(x-y) = +ig^2 N(N^2-1) D_0^-(x-y) \partial_\nu^x D_0^-(x-y) \partial_y^\nu D_0^-(x-y) \quad , \tag{20}$$

where we have used that the fully antisymmetric structure constants satisfy $f_{abc} f_{abc} = N(N^2-1)$ for $SU(N)$.

It is difficult to give a meaning to the product of three singular objects in x-space, but $r_2'^{vac}$ is of course well-defined via the impulse space, where the products in x-space go over into convolutions. The C-number distribution we consider now is therefore (omitting temporarily some prefactors)

$$r_2'^{vac}(x) = -D_0^-(x)\partial_\mu D_0^-(x)\partial^\mu D_0^-(x) = -\frac{1}{6}\partial_\mu\partial^\mu (D_0^-(x))^3 \quad , \tag{21}$$

$$\hat{r}_2'^{vac}(p) = \frac{p^2}{6(2\pi)^4}(\hat{D}_0^- * \hat{D}_0^- * \hat{D}_0^-)(p) \quad . \tag{22}$$

The preliminary result

$$(\hat{D}_0^- * \hat{D}_0^-)(p) = -\frac{1}{8\pi}\Theta(p^2)\Theta(-p^0) \tag{23}$$

can be readily obtained. Then we have

$$\hat{r}_2'^{vac}(p) = \frac{ip^2}{24(2\pi)^6}\int d^4q\,\delta((p-q)^2)\Theta(q^0-p^0)\Theta(q^2)\Theta(-q^0) \quad . \tag{24}$$

If p is spacelike, i.e. $p^2 < 0$, then we can choose a special Lorentz frame where $p = (0, \vec{p})$, and it follows that $\hat{r}_2'^{vac}(p) = 0$ due to the term $\Theta(q^0)\Theta(-q^0)$ in (24). For timelike p we choose a Lorentz frame where $p = (p^0, \vec{0})$. The four-dimensional integral can be restricted to three dimensions because of the δ-distribution in $(p-q)^2$. Taking the Heaviside distributions in (24) into account,

$$(p-q)^2 = p_0^2 - 2p^0q^0 + q_0^2 - \vec{q}^{\,2} = 0 \tag{25}$$

implies $q^0 = p^0 + |\vec{q}|$ and the integral goes over into

$$\begin{aligned}
\hat{r}_2'^{vac}(p^0, \vec{0}) &= \frac{ip_0^2}{24(2\pi)^6}\int \frac{d^3q}{2|\vec{q}|}\Theta(p_0^2 + 2p^0|\vec{q}|)\Theta(-|\vec{q}| - p^0) \\
&= \frac{ip_0^2}{24(2\pi)^5}\Theta(-p^0)\int_0^{-p^0/2} d|\vec{q}||\vec{q}| \\
&= \frac{ip_0^2}{192(2\pi)^5}\Theta(-p^0)p_0^2 \quad .
\end{aligned} \tag{26}$$

The final result is by Lorentz covariance

$$\hat{r}_2'^{vac}(p) = \frac{i}{192(2\pi)^5}\Theta(-p^0)\Theta(p^2)(p^2)^2 \quad . \tag{27}$$

From symmetry considerations we have automatically

$$\hat{a}_2'^{vac}(p) = \hat{r}_2'^{vac}(-p) \tag{28}$$

and therefore

$$\hat{d}_2^{vac}(p) = -\frac{i}{192(2\pi)^5}\mathrm{sgn}(p^0)\Theta(p^2)(p^2)^2 \quad . \tag{29}$$

The calculation of the retarded part r_2^{vac} of d_2^{vac} is demonstrated for an analogous case in [10] in full detail. The result turns out to be

$$\hat{r}_2^{vac}(p) = \frac{1}{192(2\pi)^6}(p^2)^2\log\frac{-(p^2 + ip^0 0)}{M^2} \quad , \tag{30}$$

and the final result for the vacuum diagram is

$$\hat{t}_2^{vac}(p) = -\frac{ig^2N(N^2-1)}{96(2\pi)^6}(p^2)^2\log\frac{-(p^2+i0)}{M^2} \quad . \tag{31}$$

Scale invariance of the causal distribution is destroyed by the time-ordering procedure. The parameter M^2 remains undefined. Changing the value of M^2 changes t_2^{vac} only by local terms in x-space, which are not relevant for the following discussion.

4 Volume divergence

A problem arises when we calculate the vacuum transition amplitude at second order

$$\begin{aligned} <\Omega|S_2(\Omega)|\Omega> &= \frac{1}{2}\int d^4x d^4y\, g(x)t_2^{vac}(x-y)g(y) \\ &= \frac{1}{2}(2\pi)^2\int d^4p\,\hat{g}(-p)\hat{t}_2^{vac}(p)\hat{g}(p) \end{aligned} \tag{32}$$

in the adiabatic limit $g \to 1$. Since $1 \notin \mathcal{S}(\mathbf{R}^4)$, we have to specify the limiting process. Usually, one chooses a fixed test function $g_0 \in \mathcal{S}(\mathbf{R}^4)$ with $g_0(0) = 1$ and performs a scaling limit, i.e. for $\varepsilon \to 0$ we have $g_\varepsilon(x) = g_0(\varepsilon x) \to 1$. This implies

$$\begin{aligned} <\Omega|S_2(\Omega)|\Omega> &= \frac{1}{2}\int d^4x d^4y\, g_0(\varepsilon x)t_2^{vac}(x-y)g_0(\varepsilon y) \\ &= \frac{1}{2\varepsilon^8}\int d^4x d^4y\, g_0(x)t_2^{vac}((x-y)/\varepsilon)g_0(y) \\ &= \frac{(2\pi)^2}{2\varepsilon^4}\int d^4p\,\hat{g}_0(-p)\hat{t}_2^{vac}(\varepsilon p)\hat{g}_0(p) \quad . \end{aligned} \tag{33}$$

This expression diverges in the adiabatic limit, since

$$\frac{1}{\varepsilon^4}\int d^4p\,\hat{g}_0(-p)(\varepsilon p)^4\log\frac{-(\varepsilon^2p^2+i0)}{M^2}\hat{g}_0(p) \to \tag{34}$$

$$\int d^4p\,\hat{g}_0(-p)p^4\log\frac{-(p^2+i0)}{M^2}\hat{g}_0(p) + \log(\varepsilon^2)\int d^4p\,\hat{g}_0(p)p^4\hat{g}_0(-p) \tag{35}$$

due to the logarithmic term in ε.

5 Conclusion

The Fock vacuum is unstable against the interaction of the massless gluon fields. This is a consequence of the factor p^4 in the vacuum diagram, which expresses simply its power-counting degree $\omega = 4$. The instability has nothing to do with the non-abelian character of QCD, since the same phenomenon also occurs in massless QED. There, t_2^{vac} has exactly the same form as in QCD up to a negative prefactor

$$\hat{t}_2^{vac} = \frac{ie^2}{24(2\pi)^6}p^4\log\frac{-(p^2+i0)}{M'^2} \quad . \tag{36}$$

It is tempting to look for a cancellation of the QCD vacuum divergence through massless fermions, but in view of the present properties of QCD (confinement), this is probably not even desirable.

It is interesting to observe that in quantum gravity, the situation is remedied by the non-renormalizable coupling that leads to a factor p^6 in $\hat{t}_2$ which cancels the logarithmic divergence. The same is true for the Euler effective action for photon-photon scattering

$$T_1(x) = \frac{2i\alpha^2}{45m_e^4}[(\vec{E}^2 - \vec{B}^2)^2 + 7(\vec{E}\vec{B})^2] \quad . \tag{37}$$

One might therefore argue for QCD that the volume divergence encountered in eq. (35) must be compensated by the interaction, which signals a radical rearrangement of the system as in statistical mechanics. Such a rearrangement is well-known from (1+1)-dimensional QFT models, especially for the Schwinger model [11, 12, 13].

Our result shows also that it is really a fair assumption that quantum gravity should be treated as an effective theory [14], which incorporates the 'true' physical vacuum. The price one has to pay for perturbative vacuum stability is non-renormalizability, since we do not know the underlying theory of quantum gravity.

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