Particle description of zero energy vacuum.
I. Virtual particles

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Abstract

First the "frame problem" is sketched: the motion of an isolated particle obeys a simple law in galilean frames, but how does the galilean character of the frame manifest itself at the place of the particle? A description of vacuum as a system of virtual particles will help to answer this question. For future application to such a description, the notion of global particle is defined and studied. To this end, a systematic use of the Fourier transformation on the Poincaré group is needed. The state of a system of n free particles is represented by a statistical operator W, which defines an operator-valued measure on $\widehat{\boldsymbol{P}}^n$ ($\widehat{\boldsymbol{P}}$ is the dual of the Poincaré group). The inverse Fourier-Stieltjes transform of that measure is called the characteristic function of the system; it is a function on $\boldsymbol{P}^n$. The main notion is that of global characteristic function: it is the restriction of the characteristic function to the diagonal subgroup of $\boldsymbol{P}^n$; it represents the state of the system, considered as a single particle. The main properties of characteristic functions, and particularly of global characteristic functions, are studied. A mathematical Appendix defines two functional spaces involved.

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1 INTRODUCTION

In this paper and in the companion article(1), we present a description of the vacuum as a system of virtual particles. During the last half-century or so, virtual particles have been stowaways in theoretical physics; we intend to show that they deserve to have a better status. It will turn out that their state can be described quantum-mechanically, at the price of a natural extension of the usual density matrix formalism. The extended formalism will then be used to give an explicit description of the vacuum in terms of virtual particles. It will be expedient to use a mathematical tool which may be not so familiar to physicists, namely harmonic analysis on the Poincaré group; but as far as physics is concerned, we shall be elementary.

2 THE FRAME PROBLEM

In classical mechanics (non-relativistic or relativistic) the motion of an isolated particle has simple kinematical properties summarized by the law of inertia, which holds true only in galilean frames. In quantum mechanics it takes the form of the law of conservation of momentum for an isolated particle, again restricted to galilean frames. But neither classical mechanics, nor non-relativistic quantum mechanics give any sensible solution to the *frame problem: how does the galilean character of the frame manifest itself physically at the place of the particle?* In textbooks and treatises it is systematically avoided, and students are taught to refrain from asking about it.

On the other hand, relativistic quantum mechanics gives an essentially different description of the inertial motion, which fundamentally solves the frame problem. Already in the Dirac equation there are clues to that, namely the *Zitterbewegung* and the fact that the only eigenvalues of the velocity operator are $\pm c$. As stressed with particular strength by Weinberg(2), however, any relativistic quantum theory is equivalent to a quantum field theory [3].

Now relativistic quantum field theory does not recognize permanent particles: it works from start to finish with the creation and annihilation of particles. This should not be forgotten even when dealing with inertial motion. Frenkel, who made this remark long ago(3), described the inertial motion of an electron as a sequence of processes of the following type: annihilation of the electron with the positron of a virtual pair, and transition of the virtual electron to a real state.

A similar description was given by Thirring in his textbook(4). It may be worth noting that Rosenfeld severely criticized this approach(5): he insisted that virtual pairs should be considered as nothing more than "easily remembered semi-quantitative features of the formalism". While his remark rightly warns us against a too literal interpretation of the Frenkel-Thirring process, we shall take the essential physical idea of this process as an incentive (among others) to build a rigorous description of vacuum as a system of virtual particles.

2.1 Vacuum Virtual Particles as a Possible Solution of the Frame Problem

Our central assumption is that vacuum is a system of virtual particles. This will bring us to consider the inertial motion as a guided motion: a process of permanent interaction with the virtual particles of vacuum. The relativistic invariance of vacuum implies, as will be seen in (II), that the virtual

[3] More precisely, Weinberg states that if one lets aside string and similar theories, quantum field theory is the only way to reconcile the principles of quantum mechanics (including the cluster decomposition property) with those of relativity.

particles are homogeneously distributed. Accordingly, the momentum of a free particle (i.e. a particle interacting only with the vacuum distribution) is conserved as it proceeds. Such is the physical idea of the solution of the frame problem.

Now what about non-galilean frames? In such a frame, the motion of a particle is again a process of interaction with the virtual particles of vacuum. But now momentum is no longer conserved, hence the distribution of the vacuum particles is no longer homogeneous. We shall say that vacuum is deformed, and that the accelerated motion of a particle is due to its interaction with the deformed vacuum.

2.2 The Semantic Incompleteness of Non-vacuum Theories

Maybe this kind of ideas could help to deal with physical problems at the boundary between gravitation and quantum mechanics. Ignoring this possibility for the time being, let us merely remark that our reformulation cures theoretical physics from an illness common to classical mechanics and non-relativistic quantum mechanics: in so far as they cannot solve the frame problem, their descriptions of motion and of gravitational field bear a strongly formal character. Physics indeed tries to explain the phenomena which it considers by interactions between material objects. But in theories which cannot take into account the existence of vacuum, there is no physical interaction between the reference frame and the moving particle. Hence, even though the difference between galilean and non-galilean frames is exactly described (above all in general relativity), it lacks *Anschaulichkeit.* (Or to tell it in Feynman's terms(6)(7), it hardly allows visualization). Of course this does not lead to any logical incompleteness; but these theories show a deficit of intelligibility. With growing habit physicists no longer notice the deficit, which is sometimes better felt by students; still one has to admit that non-vacuum theories suffer from a semantic incompleteness.

2.3 Resonances as Virtual Particles

Another reason to take more seriously the concept of virtual particles can be found in the phenomenology of strong interactions. It was found indeed very early, even before the advent of quarks, that resonances are not only peculiarities of interactions between stable particles, but also full-fledged particles. But what about their status: are they real or virtual particles? A virtual particle is one which has too short a lifetime to be detectable "directly", i.e., by its interaction with a macroscopic apparatus. It can be detected only through the mediation of its interaction with some real particles. This definition is usually accepted, most often implicitly. It implies immediately that resonances are virtual particles. As an almost trivial corollary, there should be no shame in speaking about virtual particles, not only as ways of describing a formalism, but also as physical objects. Hence the quantum-mechanical description of the state of a resonance(8) should be considered as a matter of course; that this is not the case presently is one more clue to the unreasonable status of virtual particles in current physics.

3 THE FOURIER TRANSFORMATION

In the following we shall proceed up to a characterization of the state of virtual particles. A virtual particle is an object present in the intermediate stage of a process. In the case of a formation reaction, the initial (real) particles transform into the virtual particle of interest; hence we shall define the state of the virtual particle as the global state of the initial particles (or equivalently, of the final particles). By global state of a system of (real) particles, we understand(9) the state of that system considered as a single particle; of course this is a natural generalization of the notion of centre-of-mass motion. We shall also say that the global state is the state of the global particle of the system.

3.1 Why the Fourier Transformation?

In order to define the state of the global particle of a given system of particles, we shall have to introduce Fourier transformation on the invariance group.

This transformation will turn out to be a natural tool of quantum relativistic kinematics. By the usual Fourier transformation, a function is represented as a combination of exponentials. Why exponentials, and not another family of functions? Because the function $e^{i\omega t}$ defines a unitary representation of the translation group. In the frequency analysis of a signal, there is indeed an implicit reference to the signal passing through a linear filter, invariant by the translation group(10).

In a physical theory invariant by a group G larger than the translation group, one should therefore expect to meet the Fourier transformation on G. The Fourier transformation on a group has been long known to mathematicians(11)(12). Up to now, however, physicists hardly used the resources of this mathematical theory. In the following we shall show that it is a natural and necessary tool of quantum relativistic kinematics.

First some basic notions of Fourier transformation will be recalled. Then the characteristic function(8)(9) of a one-particle state will be defined: it is a function on the Poincaré group P, which turns out to be the inverse Fourier-Stieltjes transform of the operator-valued measure on the dual $\widehat{P}$, defined by the statistical operator (usually called "density matrix") of the state.

The characteristic function of an n particle state will be defined in an analogous manner: it is a function on P^n. The restriction of this function to the diagonal subgroup of P^n is the global characteristic function (9) of the state. Its physical meaning is simple and important: it characterizes the global state of the system, i.e. the state of the n particles, considered as a single particle. This notion is an elementary quantum mechanical one; particle physics uses it implicitly, especially in the study of resonances in production reactions, but it does not seem to have ever been explicitly defined.

An Appendix shows that the two types of characteristic functions which have been introduced belong respectively to two mathematical objects - the Fourier algebra and the Fourier-Stieltjes algebra - which play an important role in harmonic analysis.

3.2 The Fourier Transform of a Function

The mathematical properties stated in this article are meant to be applied to the case where G is P, the inhomogeneous $SL(2,C)$, or universal covering group of the Poincaré group, or else a direct power of this group. They will be stated mostly under a rather general form; the groups P and P^n, as well as their extensions to discrete and internal symmetries, belong to the class of groups which will be considered. (For brevity's sake, P will sometimes be called, somewhat improperly, the "Poincaré group").

Let G be a separable unimodular type I Lie group(13). The invariant measure on G (Haar measure) is defined up to a factor; this factor will be choosed once for all, and the so fixed Haar measure will be denoted by dg.

The set of the equivalence classes of irreducible continuous unitary representations of G is called the dual of G, denoted by $\widehat{G}$. (In the following, we shall say "representation" instead of "unitary continuous representation"). In each equivalence class we shall choose a certain representation, hence $\widehat{G}$ will be considered as a set of representations. A representation will be labeled, in the case $m^2 > 0$, by $\chi = (m, s, \epsilon)$, ϵ being the sign of energy; this χ will be called the signature of the representation. In the representation $\chi \in \widehat{G}$, the element $g \in G$ is represented by the unitary operator $U_{\chi,g}$; the space of this representation is denoted by $\mathcal{H}_\chi$.

The Plancherel measure on $\widehat{G}$ will be denoted by $d\chi$; its normalization is fixed as soon as the normalization of the Haar measure is chosen.

The definition of an operator field on $\widehat{G}$ amounts to giving, for almost all $\chi \in \widehat{G}$, an operator A_χ on the space $\mathcal{H}_\chi$. Let φ be a complex valued function on G. If φ is integrable, the Fourier transform of φ is the operator field on $\widehat{G}$ defined by

$$\mathcal{T}(\varphi)_\chi = \int_G \varphi_g U_{\chi,g} dg \tag{1}$$

(For reasons to appear later, the names of variables will be written as indices).

If the function $\chi \to Tr[\mathcal{T}(\varphi)_\chi U_{\chi,g}]$ is integrable on $\widehat{G}$ for any $g \in G$, equation (1) can be inverted:

$$\varphi_g = \int_{\widehat{G}} Tr[\mathcal{T}(\varphi)_\chi U_{\chi,g^{-1}}] d\chi \tag{2}$$

3.3 The Inverse Fourier-Stieltjes Transform of an Operator-valued Measure

Mathematicians usually start with a function on a group and define its Fourier transform, which is an operator field on the dual; the inverse Fourier transformation allows one to come back to the function. But for the problems of interest here the starting point is an object on the dual. The statistical operator which describes the state of a particle is indeed related to an irreducible representation of the invariance group, and therefore to a point of the dual. We shall also introduce states which have a mass spectrum (the global states); here we shall have an operator field on the dual. The notion which includes these two different situations is an operator-valued measure on the dual, which has an inverse Fourier-Stieltjes transform (9)(14). Let us now proceed to define these notions.

Let μ be a positive scalar measure on $\widehat{G}$, and let F be an operator field defined on the support of μ: to any $\chi \in supp(\mu)$ there corresponds an operator F_χ on $\mathcal{H}_\chi$. The couple $\mathcal{M} = (\mu, F)$ defines an operator-valued measure on $\widehat{G}$. If the operator F_χ is positive and trace class μ-almost everywhere, and if the function $\chi \to Tr(F_\chi)$ is μ-integrable, one says that the operator-valued measure $\mathcal{M}$ is *trace class.*

If $\mathcal{M} = (\mu, F)$ is a trace class operator-valued measure on G , its inverse Fourier-Stieltjes transform φ is defined by a Stieltjes integral:

$$\varphi_g = \int_{\widehat{G}} Tr[F_\chi U_{\chi,g^{-1}}] d\mu(\chi) \tag{3}$$

If the measure μ is absolutely continuous with respect to the Plancherel measure, one can assume without restriction that it is equal to the Plancherel measure; the operator field F then appears as the *spectral density* of the operator-valued measure $\mathcal{M}$ with respect to the Plancherel measure (otherwise stated, as its Radon-Nikodym derivative). If μ is equal to the Plancherel measure, definition (3) becomes

$$\varphi_g = \int_{\widehat{G}} Tr[F_\chi U_{\chi,g^{-1}}]d\chi \tag{4}$$

and we get back formula (2).

We shall need also another particular case: the case when μ is a Dirac measure at the point $\chi \in \widehat{G}$, whereas the field F reduces to a single positive trace class operator W on $\mathcal{H}_\chi$. The operator-valued measure thus defined will be called the Dirac measure W at the point χ. Its inverse Fourier-Stieltjes transform is given by

$$\varphi_g = Tr[WU_{\chi,g^{-1}}] \tag{5}$$

3.4 Examples

$SU(2)$

For a compact group such as $SU(2)$, the dual is discrete; in Eq. (2), the integration reduces to a summation. The natural normalisation of dg is defined by putting the total measure of the group equal to 1; as a result, the Plancherel measure of a representation is equal to its dimension.

For $SU(2)$, the Haar measure is defined by

$$\int_G f_g dg = (8\pi^2)^{-1} \int_0^{2\pi} d\alpha \int_0^{\pi} sin\beta d\beta \int_0^{2\pi} d\gamma f(\alpha,\beta,\gamma)$$

where α, β, γ are the Euler angles.

The dual can be labelled by the set of the possible values of the angular momentum:

$$\widehat{G} = \{0, 1/2, 1, 3/2, ...\}$$

The Plancherel measure consists in masses $2j+1$ at the points j:

$$p_j = 2j+1 \tag{6}$$

For any $j \in \widehat{G}$, the space $\mathcal{H}_j$ has dimension $2j+1$ and the operator $U_{j,g}$ is defined by the matrix

$$< m|U_{j,g}|m' >= D^j_{mm'}(g)$$

where the $D^j_{mm'}$ are the well known matrix elements of the j representation of $SU(2)$. The Fourier transformation sends the function f on G onto the operator field defined thus: to any j corresponds the operator

$$\mathcal{T}(f)_j = \int_G f_g U_{j,g} dg\ . \tag{7}$$

The Fourier inversion formula expresses the function f in terms of the operator field $\mathcal{T}(f)$:

$$f_g = \sum_j p_j Tr[\mathcal{T}(f)_j U_{j,g^{-1}}] \tag{8}$$

It can be obtained directly by using the orthogonality relations of the matrix elements of the representations of $SU(2)$. The form of equation (8) makes obvious the fact that it is a particular case of equation (2).

But we shall need, rather than the inverse Fourier transform (8), the inverse Fourier-Stieltjes transform of an operator-valued measure, especially that of the Dirac measure W at the point j. This transform is a particular case of Eq. (5):

$$f_g = Tr[WU_{j,g^{-1}}] \tag{9}$$

Poincaré group P

The above defined group P is locally compact, but not compact: its total Haar measure is infinite. Hence the dual is not discrete. In physics one uses most often the part of the dual which corresponds to the irreducible representations with positive squared mass and energy. Let us remark at once that the representations which differ from the ones just mentioned only by having a negative energy are as simple, they can be obtained for instance by taking the complex conjugate of the operator for positive energy. These two classes of representations are relevant for the study of vacuum (see the companion paper (II)); hence their products (which contain some representations with $m^2 < 0$) are relevant also. The following considerations hold for all the mentioned classes of representations. However, we shall always take as examples the well known representations with $m^2 > 0$ and positive energy.

In order to define the Haar measure and to fix its normalization, let us write with Nghiêm(15)(16) an element of $SL(2,C)$ under the canonical form

$$A = UH_k$$

In this equation U denotes an element of $SU(2)$; H_k is the unique hermitian positive matrix which sends the vector k onto the vector $(1,0,0,0)$; finally, k is the image of the vector $(1,0,0,0)$ by A^{-1}. We can now define the Haar measure on $SL(2,C)$: it is given by

$$d^6A = d^3Ud^3\mathbf{k}/k_0 \ ,$$

where d^3U is the Haar measure on $SU(2)$. Hence the Haar measure on P is given by

$$dg = d^4ad^6A \tag{10}$$

This normalization of the Haar measure fixes that of the Plancherel measure. In particular, the latter reads on the representations with positive m^2 thus (s denotes the spin value):

$$d\chi = [2(2\pi)^4]^{-1}(2s+1)m^2dm^2 \tag{11}$$

The Fourier transform of a function on P is the operator field on $\widehat{P}$ defined by equation (1). The Fourier inversion formula is given by equation (2). The extension to the case $G = P^n$ is straightforward.

Practically, to compute the operator $\mathcal{T}(\varphi)_\chi$ and to evaluate the trace in equation (2) one uses an improper basis in the space $\mathcal{H}_\chi$. See Ref. 9, and for more details Nghiêm's papers(15)(16).

Finally, the inverse Fourier-Stieltjes transform of the operator-valued Dirac measure W at the point χ is given by equation (5).

4 CHARACTERISTIC FUNCTIONS

4.1 One Particle

Let us first consider a spin j, i.e. a particle of spin j, of which we consider only the spin degrees of freedom. Its state is defined by a statistical operator W on the space $\mathcal{H}_j$ (whose dimension is $2j+1$) of the representation j of $SU(2)$. To this state one may also associate the Dirac operator-valued measure W at the point j.

We define the *characteristic function* of the state as the inverse Fourier-Stieltjes transform of the latter operator-valued measure. This transform is the function φ given by equation (9). As W is trace class, the function is defined everywhere on the group; the trace of W is equal to the value of the characteristic function at the neutral element of the group:

$$TrW = \varphi_e \tag{12}$$

The positivity of the operator W is equivalent to the fact that φ is a positive-definite function(13)(17). It should be noticed that the characteristic function is not the inverse Fourier transform of the operator field defined by putting W at the point j, zero elsewhere; the latter is given by Eq. (8), not Eq. (9); equation (8) defines the same function, but multiplied by $(2j+1)$. The choice made extends to a non-compact group.

Let us now consider a particle transforming according to the representation χ of the group $G = P$. Its state is represented by a statistical operator W on the space $\mathcal{H}_\chi$. The associated operator-valued measure is the Dirac measure W at the point χ. Again, we define the characteristic function of the state as the inverse Fourier-Stieltjes transform of the latter operator-valued measure. It is given by equation (5). As W is trace class, this function is defined everywhere on G.

As for a spin, the positivity of W is equivalent to the fact that φ is a positive-definite function. The trace of W is again given by equation (12), where e denotes the neutral element of G.

4.2 Several Particles

Let us now consider the case of n spins $j_1, ..., j_n$. The state of the system is represented by an operator W on the space

$$\mathcal{H}_{j_1 \dots j_n} = \mathcal{H}_{j_1} \otimes \dots \otimes \mathcal{H}_{j_n} \ .$$

This case is analogous to that of a single spin: one should only replace the group $SU(2)$ by $SU(2)^n$, a group whose elements read $(g_1 ... g_n)$. The elements of the dual of $SU(2)^n$ read $(j_1, ..., j_n)$; the operator $U_{j_1 \dots j_n, g_1 \dots g_n}$ is the tensor product

$$U_{j_1 \dots j_n, g_1 \dots g_n} = U_{j_1, g_1} \otimes \dots \otimes U_{j_n, g_n} \ .$$

The characteristic function of the state W reads

$$\varphi_{g_1 \dots g_n} = Tr\{WU_{j_1 \dots j_n, g_1^{-1} \dots g_n^{-1}}\} \tag{13}$$

As in the case $n = 1$, this equation defines the inverse Fourier-Stieltjes transform of an operator-valued measure, namely the Dirac measure W at the point $(j_1, ..., j_n)$ of the dual.

Let us consider now n particles, corresponding to the irreducible representations $\chi_1, ..., \chi_n$ of the group P. The state of the system is represented by an operator W on the space

$$\mathcal{H}_{\chi_1 ... \chi_n} = \mathcal{H}_{\chi_1} \otimes ... \otimes \mathcal{H}_{\chi_n} .$$

This case is analogous to that of a single particle: one should only replace the group P by P^n, a group whose elements read $(g_1 ... g_n)$. The elements of the dual of P^n read $(\chi_1, ..., \chi_n)$; the operator $U_{\chi_1 ... \chi_n, g_1 ... g_n}$ is the tensor product

$$U_{\chi_1 ... \chi_n, g_1 ... g_n} = U_{\chi_1, g_1} \otimes ... \otimes U_{\chi_n, g_n} . \tag{14}$$

The characteristic function of the state W reads

$$\varphi_{g_1 ... g_n} = Tr\{W U_{\chi_1 ... \chi_n, g_1^{-1} ... g_n^{-1}}\} \tag{15}$$

As in the case $n = 1$, this equation defines the inverse Fourier-Stieltjes transform of an operator-valued measure, namely the Dirac measure W at the point $(\chi_1, ..., \chi_n)$ of the dual.

5 PROPERTIES OF THE CHARACTERISTIC FUNCTIONS

The properties and the physical meaning of characteristic functions have been studied in detail in Ref. 9. Let us recall here some essential results.

5.1 Meaning of the Characteristic Function

As shown by equation (5), the value of the characteristic function φ of a state at the element g of the group is the expectation value in this state of the operator $U_{\chi, g^{-1}}$. This definition is analogous to that of the characteristic function of a random variable X: its value at t is the expectation value of e^{itX}. Furthermore in the case $G = P$, the restriction of φ to the translation subgroup, $a \rightarrow \varphi(a, e)$, turns out to be the characteristic function (in the usual sense of probability theory) of the energy-momentum vector. If indeed we write an element of P under the form (a, A) with $a \in T$ (the translation subgroup) and $A \in SL(2, C)$, the operator $U(a, e)$ can be written $exp(ia_\mu P^\mu)$. (Both the neutral elements of P and of $SL(2, C)$ will be denoted by e). Hence the expectation values of the components of the energy-momentum vector are given by

$$< P^\mu >= [i/\varphi_e] \partial \varphi_{a,e} / \partial a_\mu |_{a=0} . \tag{16}$$

(We did not assume that the statistical operator is normalized; we have used equation (12).

More generally, the expectation values of the dynamical variables of the system can be obtained by applying to the characteristic function suitable differential operators, which correspond to elements of the universal enveloping algebra of the Lie algebra of the Poincaré group.

Let us define the partial Fourier transform(16) (on the translation subgroup) of the characteristic function φ:

$$\overline{\varphi}_{k,A} = \int e^{ika} \varphi_{a,A} d^4 a .$$

As $\varphi(a,e)$ is the characteristic function (in the probabilistic sense) of the energy-momentum vector, the restriction of $\overline{\varphi}$ defined by putting $A=e$ is the (unnormalized) probability density of the energy-momentum vector. Let us now consider the restriction of $\overline{\varphi}$ defined by giving to k a fixed value and restraining A to the little group of k, i.e. to the subgroup of the elements of $SL(2,C)$ which leave k fixed: $Ak=k$. It can be shown(9), thanks to Nghiêm's results(16), that this restriction gives the spin characteristic function (cf. Section 2.3) of the particle. Explicitly, the spin characteristic function is given by

$$\omega_{k,g} = \overline{\varphi}_{k,H_k^{-1}gH_k} \tag{17}$$

where H_k has been defined in Section 2.3, and g is an element of $SU(2)$.

5.2 Transformation Law

Let us consider a one particle state defined by its statistical operator W and by its characteristic function φ; let χ be the irreducible representation associated to the particle. The element γ of the invariance group transforms this state into another one, whose statistical operator is

$$W' = U_{\chi,\gamma} W U_{\chi,\gamma^{-1}} \quad .$$

Equation (5) then gives the new characteristic function φ' :

$$\varphi_g^{'} = Tr[U_{\chi,\gamma} W U_{\chi,\gamma^{-1}} U_{\chi,g^{-1}}] = Tr[W U_{\chi,\gamma^{-1}} U_{\chi,g^{-1}} U_{\chi,\gamma}] \quad ,$$

and finally

$$\varphi_g^{'} = \varphi_{\gamma^{-1} g \gamma} \quad .$$

This transformation can be generalized to the case of an n particle state. If there is no interaction between the particles, we might consider P^n as the invariance group, because in this case the dynamical variables relative to each single particle are separately conserved. However, the interesting cases are those where there is an interaction somewhere; for instance, the state might be an asymptotic ingoing or outgoing state. The invariance group is then the diagonal subgroup G_d of G, defined by

$$g_1 = \ldots = g_n \quad .$$

The transformation law reads

$$\varphi_{g_1 \ldots g_n}^{'} = \varphi_{\gamma^{-1} g_1 \gamma \ldots \gamma^{-1} g_n \gamma} \quad . \tag{18}$$

5.3 Inclusive Characteristic Functions

Let us note finally that the characteristic function of a system of particles also allows one to express the inclusive characteristic functions. An inclusive state of a system of n particles is obtained by neglecting some of them, for instance those numbered by $p+1, \ldots, n$. One has to take the partial trace of the n particle statistical operator with respect to the degrees of freedom corresponding to the $(n-p)$ neglected particles. This operation becomes very simple with characteristic functions:

$$\varphi_{g_1 \ldots g_p}^{(p)} = \varphi_{g_1 \ldots g_p e \ldots e}^{(n)} \quad . \tag{19}$$

One puts equal to e (the neutral element of the group) the variables corresponding to neglected particles.

6 GLOBAL CHARACTERISTIC FUNCTIONS

6.1 The State of a Global Particle

Let us consider a system of n particles described by irreducible representations of the invariance group. What are the properties of this system, if we consider it as a single particle? (In the following, this particle will be called the *global particle*). This problem is often approached, although not quite explicitly. For instance, one may study the motion of the center of mass. One may also consider the total angular momentum of the system, or else its effective mass (i.e. the mass of the global system); the latter case occurs in the experimental study of resonances in production processes.

To get the state of the global particle, one has to take the partial trace(18) of the statistical operator of the n particle system over all degrees of freedom, except those which correspond to the global motion. This operation, however, is not always easy to carry out. Let us show that it is very simple if one uses characteristic functions.

Let φ be the characteristic function of the system: it is a function on P^n, defined by

$$\varphi_{g_1 \ldots g_n} = Tr\{W[U_{\chi_1, g_1^{-1}} \otimes \ldots \otimes U_{\chi_n, g_n^{-1}}]\} \tag{20}$$

Let us now consider the restriction φ_{glob} of φ to the diagonal subgroup. Clearly, this is the characteristic function of the global particle (or, as we shall call it, the *global characteristic function*). Indeed, the restriction to the diagonal subgroup means that one considers only the combined motions of the system. Besides, definition (20) implies that the expectation values of the dynamical variables $P_\mu, M_{\mu\nu}$, which correspond to elements of the Lie algebra of the group, are obtained by adding the expectation values of the n single particle variables. (See Eq. (16) for the variable P_μ).

To study the global characteristic function in more detail, let us consider the operator field which is its Fourier transform:

$$\rho_\chi = \int_G \varphi_{glob\ g} U_{\chi,g} dg \quad . \tag{21}$$

This field is the spectral density of an operator-valued measure. Let K be a Borel part of G: the operator-valued measure assigns to it the operator

$$W(K) = \int_K^{\oplus} \rho_\chi d\chi \tag{22}$$

on the space

$$\mathcal{H}(K) = \int_K^{\oplus} \mathcal{H}_\chi d\chi \quad . \tag{23}$$

The operator W(K) will be called a *conditional statistical operator* (it is conditioned by K). It is not normalized.

The operator-valued measure just defined is another representation of the state of the global particle. The mathematical object which has an immediate physical meaning is not the *density of statistical operator* ρ_χ, but rather the operator $W(K)$. The latter is dimensionless, as are the probabilities computed from it, whereas ρ_χ has the dimension M^{-4} because $d\chi$ has the dimension M^4, see equation (11). On the dual another measure than the Plancherel measure might be used, for instance dm^2; in such a case the density of statistical operator would be modified, but $W(K)$

would not be. From this point of view, the expression (22) of the statistical operator is analogous to that of the power radiated by a source in the frequency range $[\nu_1, \nu_2]$:

$$W([\nu_1, \nu_2]) = \int_{\nu_1}^{\nu_2} (dW/d\nu) d\nu \quad .$$

This equation expresses the fact that the radiated power is an additive function of the frequency interval. If instead of the frequencies one uses the wavelengths, the spectral density of power becomes $dW/d\lambda = (\nu^2/c)(dW/d\nu)$, but the radiated power is of course not changed.

On the other hand, the dynamical variables of the global particle are represented by operator fields of the form

$$Q : \chi \to Q_\chi \quad .$$

The expectation value of the dynamical variable Q is given in terms of the statistical operator (22) by

$$< Q >_K = \int_K Tr[Q_\chi \rho_\chi] d\chi \quad . \tag{24}$$

If we use another measure on the dual, the operator field Q will not be modified. This is due to the fact that Q_χ depends on the χ (the mass and spin, say) of the global particle, whereas W has the additivity property characteristic of a measure: if K_1 and K_2 are two disjoint Borel parts of $\widehat{G}$, one has

$$W(K_1 \cup K_2) = W(K_1) \oplus W(K_2) \quad .$$

Q_χ is analogous to physical quantities like the refraction index or the absorption coefficient, which are not densities of additive functions of the frequency interval, and which are therefore independent of the measure used on the frequency axis.

The representation of a state by an operator-valued measure on the dual of the invariance group is not a familiar one; it is, however, unavoidable for a system like the global particle, which has the same dynamical variables as an elementary system(19), but is described by a reducible representation of the invariance group.

Indeed, the definition of the global particle implies that its algebra of observables is isomorphic to that of an elementary system. Now the center of the universal enveloping algebra of the Lie algebra of the Poincaré group is generated by the elements $P_\mu P^\mu$ and $W_\mu W^\mu$. For an elementary system, the representation of the group is irreducible and the operators which represent $P_\mu P^\mu$ and $W_\mu W^\mu$ are multiples of the identity operator. For a global particle the representation is no longer irreducible, it is only multiplicity free; but the operators which represent the elements of the center of the universal enveloping algebra still commute with all the dynamical variables (in other terms, they still belong to the center of the algebra of observables). Hence any state of a global particle is an incoherent superposition of states with fixed values of χ. This is indeed precisely what is expressed by equation (22). Besides, if we take for K the whole dual we get an unconditional statistical operator:

$$W = \int_{\widehat{G}}^{\oplus} \rho_\chi d\chi \quad . \tag{25}$$

In the just mentioned formalism of the algebra of observables, all observables are represented by operators on the same Hilbert space, which could be here only the unconditional space

$$\mathcal{H} = \int_{\widehat{G}}^{\oplus} \mathcal{H}_\chi d\chi \quad . \tag{26}$$

Such a representation is useful in certain cases, but here it would oblige us to assign to dynamical variables spectral densities, which are devoid of direct physical meaning.

Let us point out at last that the formalism just introduced fits in well with the methods of experimental analysis of global particles, which have been used for instance in the study of strong processes. In such a case the measurements are conditioned by the choice of an interval of effective mass, and often also by the choice of a value for the total angular momentum of the system. This was especially conspicuous, for instance, in an experiment by Baton and Laurens(20): they studied dipions $\pi^-\pi_0$ in the mass region of the ρ meson; their statistics was high enough to allow them conditioning their measurements by different mass intervals, whose width was most often smaller than that of the ρ meson. But for any value of the mass width chosen, the measured dynamical variables are always functions of the mass of the global particle, whereas the elements of density matrix which can be measured depend additively of the mass interval.

For those reasons the dynamical variables must be represented by operator fields, and the state must be represented by an operator-valued measure on the dual which allows conditional statistical operators such as (22) to be defined.

6.2 Conservation of the Global Particle

The value of the characteristic function of a particle (simple particle or global particle) at the element g^{-1} of the invariance group is the expectation value of the operator of the representation of the invariance group. For a simple particle this operator is $U_{\chi,g}$; for a global particle it is given by the tensor product of the n irreducible representations, i.e. by Eq. (14) (where one must put every g_i equal to g), or equivalently by an operator field $\chi \to U_{\chi,g}$. Now the operators of the representation of the invariance group are conserved quantities, as are those which represent the elements of the Lie algebra (the former are the exponentials of the latter). Hence the expectation values of those operators are also conserved quantities. This implies that in any tranformation process, the global characteristic function of the system is conserved. We shall use in article II a particular case of this: if n real particles transform into a virtual particle, the characteristic function of the virtual particle is equal to the global characteristic function of the initial n particle system.

For a process described by an S matrix, one can give a formal proof of the conservation of the global characteristic function. Let W_{in} and W_{out} be respectively the ingoing and outgoing (normalized) statistical operators. Let $g --> U_g$ be the tensor product of the n irreducible representations corresponding to the n initial particles. The S matrix commutes with the U_g . The initial global characteristic function reads

$$\varphi(in)_g = Tr(W_{in}U_{g^{-1}})$$

The final characteristic function reads

$$\begin{aligned} \varphi(out)_g &= Tr(W_{out}U_{g^{-1}}) \\ &= Tr(SW_{in}S^{-1}U_{g^{-1}}) \\ &= Tr(W_{in}S^{-1}U_{g^{-1}}S) \\ &= Tr(W_{in}U_{g^{-1}}) \qquad = \varphi(in)_g \quad . \end{aligned}$$

6.3 Some Details about the Meaning of Global Characteristic Functions

In Ref. 9 it has been checked by elementary calculations (with use of the explicit forms of the matrix elements of the representations of $SU(2)$ and P, respectively) that the restrictions of the characteristic functions to the diagonal subgroups describe indeed the global state. These computations will not repeated here, but we shall now recall their main results. They show how the restriction of the characteristic function to the diagonal subgroup is reflected on the statistical operator, on the other side of the Fourier transformation.

Example 1: $SU(2)$

Let us consider first the case of two spins j_1 and j_2. The complete state of the system is described by a statistical operator W which can be written either in the basis of individual states $|m_1m_2>$, or in the basis of global states $|JM>$. To go from one basis to the other one one uses the Clebsch-Gordan coefficients $<m_1m_2|JM>$. In the basis $|JM>$, the operator W is not diagonal in general. To pass from W to W_{glob}, we must eliminate the elements which are non-diagonal with respect to J :

$$W_{glob} = \oplus W_J \tag{27}$$

with

$$<M|W_J|M'> = <JM|W|JM'> \tag{28}$$

Otherwise stated, the operator W_glob is obtained from W by diagonal truncation with respect to J : if we call Π_J the projector onto the subspace defined by the value J for the total angular momentum, one has

$$W_J = \Pi_J W \Pi_J \tag{29}$$

If one computes the characteristic function f corresponding to the operator (27):

$$f_g = \sum_J Tr[W_J U_J(g^{-1})] \quad , \tag{30}$$

one can see that it is equal to the diagonal restriction of the complete characteristic function.

Let us now consider the case of more than two spins. We have to do with a degenerate case: the common eigenspaces of J^2 and J_z are now more than one-dimensional. We shall denote the vectors of the collective basis by $|JMa>$, where a stands for one or several degeneracy parameters. The vectors of the individual basis will be denoted by $|m_1...m_n>$, or $|(m_i)>$ for short. The equations analogous to (27)-(28) read:

$$W_{glob} = \oplus W_J \tag{31}$$

with

$$<M|W_J|M'> = \sum_a <JMa|W|JM'a> \quad . \tag{32}$$

Here going from W to W_J is more complicate: equation (29) is replaced by

$$W_J = Tr_a[\Pi_J W \Pi_J] \quad . \tag{33}$$

We have taken the partial trace with respect to the degeneracy parameters a.

Here again, one can check explicitly that the characteristic function corresponding to the operator (31) is equal to the diagonal restriction of the complete characteristic function.

Example 2: Poincaré group

The main difference with the $SU(2)$ case is due to the fact that the dual is no longer discrete. In the definition of the global statistical operator W_{glob}, the direct sum of Eq. (27) is replaced by a direct integral(8)(13):

$$W_{glob} = \int_{\widehat{G}}^{\oplus} \rho_\chi d\chi \quad . \tag{34}$$

To make explicit the definition of ρ_χ, we shall use continuous pseudobases in the spaces $\mathcal{H}_\chi$. We shall use the results and the notations of Joos(21). Let us take the case $n = 2$. A one particle state will be denoted by $|k>$, where k stands for both the energy-momentum and the spin or helicity index. Similarly, a two particle state will be denoted by $|k_1 k_2>$. As we have almost always a degenerate case (the only exception is the case where at least one of the spins is zero), a global state will always be denoted by $|\chi k \eta>$, where η stands for one or several degeneracy parameters. The Clebsch-Gordan coefficients (whose explicit expressions are given by Joos) will be denoted by $<k_1 k_2|\chi k \eta>$. The expression of an element of the global pseudobasis is

$$|\chi k \eta> = \sum_{sp} \int d^3k_1 [2w_1]^{-1} d^3k_2 [2w_2]^{-1} |k_1 k_2 >< k_1 k_2|\chi k \eta>$$

where one has put

$$w_i = (\mathbf{k}_i^2 + m_i^2)^{1/2} \quad ;$$

here $\mathbf{k}$ stands for the space part of k; $\sum_s p$ denotes summation over the spin indices contained in $|k_1 k_2>$.

Let us call, as in Eq. (24), ρ the operator field "density of global statistical operator". The (improper) matrix elements of ρ_χ can be expressed in terms of those of the complete statistical operator W:

$$< k|\rho_\chi|k' > = \sum_\eta < \chi k \eta|W|\chi k' \eta > \quad . \tag{35}$$

As shown by equation (21), the characteristic function corresponding to the operator-valued measure whose density is ρ_χ is the inverse Fourier transform of the field ρ. It can be checked explicitly that the latter is equal to the diagonal restriction of the complete characteristic function.

The cases $n > 2$ differ from the case $n = 2$ only by some complications; η becomes a continuous parameter. The Clebsch-Gordan coefficients for any n have been computed by Klink and Smith(22)(23).

Let us point out that all the results stated about the global state and the global characteristic function follow from the sole assumption of relativistic invariance. The properties of these objects are elementary kinematical properties. Their meaning did not appear clearly in Ref. 8 and Ref. 9, where they were mixed with dynamical considerations.

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APPENDIX: FUNCTIONAL SPACES

We have introduced two types of characteristic functions. The functions which describe one or several particles are the inverse Fourier-Stieltjes transforms of Dirac measures on the dual of a group (the invariance group, or a direct power of it); the functions which describe global particles are the inverse Fourier transforms of operator fields on the dual of the invariance group. Let us show that these two types of functions belong respectively to two types of functional spaces. These spaces, which play an important role in harmonic analysis, have been introduced by Eymard(24) in 1964. (For a recent review, see Eymard(25)).

The Fourier-Stieltjes algebra $B(G)$ of a locally compact group G is the vector space of the functions over G defined by: $g \to < x|U_g|\eta >$, where U is a representation (not necessarily irreducible) of the group. This space is also a Banach algebra, for a product and a norm which need not be specified here.

The Fourier algebra $A(G)$ of a locally compact group G is the closure in $B(G)$ of the intersection of $B(G)$ with the space $L(G)$ of the continuous functions with compact support on G. All the functions of $A(G)$ tend to zero at infinity (Ref. 22, proposition (3.7)).

These definitions may look somewhat abstract; they will become more familiar by specifiying the relations of $A(G)$ and $B(G)$ with the operator fields and the operator-valued measures on the dual.

Let $\mathcal{L}_1(G)$ be the vector space of the trace class operator fields on $\widehat{G}$, i.e. of the operator fields $\chi \to A_\chi$ such that the function $\chi \to Tr(|A_\chi|)$ exists and is integrable. The relation of almost everywhere equality (for the Plancherel measure) is an equivalence relation, and the space of classes modulo this relation is denoted by $L_1(G)$. We shall say for simplicity's sake that $L_1(G)$ is the space of integrable fields on G.

It follows immediately from the definition that an integrable field has an inverse Fourier transform:

$$\mathcal{T}^{-1}(A)_g = \int_G Tr[A_\chi U_{\chi, g-1}] d\chi \quad .$$

Let $\mathcal{M} = (\mu, F)$ be a trace class operator-valued measure on $\widehat{G}$ (see Section IIb), and let $M(G)$ be the space of all trace class operator-valued measures. We know that $\mathcal{M}$ has an inverse Fourier-Stieltjes transform.

We can now characterize $A(G)$ and $B(G)$ by using Fourier transforms. The Fourier algebra $A(G)$ is the image by the inverse Fourier transformation of the space $L_1(G)$. The Fourier-Stieltjes algebra $B(G)$ is the image by the inverse Fourier-Stieltjes transformation of the space $M(G)$.

It follows then from Section 4 that the characteristic function of an n particle state ($n \geq 1$) belongs to the Fourier-Stieltjes algebra of the n^{th} direct power of the invariance group.

According to Eq. (21) the global characteristic functions are the inverse Fourier transforms of elements of $L_1(G)$. Hence they belong to the Fourier algebra of the invariance group.

The difference between the two cases is due to the fact that the characteristic function of an n particle state corresponds to fixed values of the masses; on the contrary, a global particle has a continuous mass spectrum.

We can see thus that the Fourier and Fourier-Stieltjes algebras might have been tailor-made for quantum mechanics.

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