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Exact braneworld cosmology induced from bulk black holes

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Abstract

We use a new, exact approach in calculating the energy density measured by an observer living on a brane embedded in a charged black hole spacetime. We find that the bulk Weyl tensor gives rise to non-linear terms in the energy density and pressure in the FRW equations for the brane. Remarkably, these take exactly the same form as the "unconventional" terms found in the cosmology of branes embedded in pure AdS, with extra matter living on the brane. Black hole driven cosmologies have the benefit that there is no ambiguity in splitting the braneword energy momentum into tension and additional matter. We propose a new, enlarged relationship between the two descriptions of braneworld cosmology. We also study the exact thermodynamics of the field theory and present a generalised Cardy-Verlinde formula in this set up.

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1 Introduction

Recently there has been much interest in a holographic description of braneworld cosmology. Such work was inspired by Savonije and Verlinde [?, ?] who considered a braneworld in a Schwarzschild Anti-de Sitter (AdS) background. They demonstrated that the standard cosmology driven by the energy density/pressure of an $(n-1)$ dimensional conformal field theory (CFT), has a dual description – it can be regarded as being a braneworld cosmology driven by the bulk black hole, when the brane is near the AdS boundary. All of the details of the cosmology can be captured by the parameters defining the background black hole spacetime. Furthermore, from the field theory point of view, one can derive equations which relate the thermodynamic variables of the CFT. As the brane evolves in the black hole background it was noticed that these equations coincide with the cosmological evolution equations of the braneworld at the point at which the brane crosses the black hole horizon [?, ?].

The notion of holography – that all of the physics of a bulk spacetime is encoded in a lower dimensional field theory – is realised in a very concise manner by the AdS/CFT correspondence [?, ?, ?]. In this, the gravitational physics in an $\boldsymbol{n}$ dimensional Anti-de Sitter bulk spacetime is captured by an $(n-1)$ dimensional field theory which lives on the boundary of AdS. Witten later introduced finite temperature into the bulk and boundary theories [?], which initiated the study of the dual field theory for the Schwarzschild-AdS spacetime. The study of bulk black holes and their dual theories continued to be developed for a wide class of spacetimes. What will interest us here is a brane embedded in a background of bulk Reissner-Nordström AdS black holes (It was demonstrated in [?], that in this case the dual field theory is coupled to a background global current). We choose to study the RNAdS background as it exhibits the full generality of many of the results discussed in earlier studies of braneworld holography [?, ?, ?, ?, ?, ?, ?, ?, ?]. RNAdS has also been used to exhibit other interesting features of the AdS/CFT correspondence which cannot be observed in the case of non-zero temperature alone. Firstly, because the dual field theory has a much richer structure [?, ?], and secondly, because we have more control on interpreting the field theory effects caused by the different parameters of the black hole spacetime [?].

In braneworld holography, we normally think of the bulk gravity theory as being dual to a CFT on the brane itself. Paradoxically, the CFT has an ultra violet cut off, so it is actually a broken CFT [?, ?]. In most studies of dual theories on the brane, the precise nature of the dual theory is unknown. We regard it as an abstract field theory, some of whose properties we can derive. We should also note that early studies of branes [?] had fine tuned tensions that ensured the cosmological constant on the brane vanished (critical branes). If we avoid this fine tuning we are able to induce a non-zero cosmological constant on the brane [?] (non-critical branes). In particular, recent observations indicating that our universe has a small positive cosmological constant [?, ?] suggest it is important to consider de Sitter braneworlds. These inflationary braneworlds are naturally induced by quantum effects of a field theory on the brane [?, ?, ?].

When the bulk spacetime has a non-vanishing Weyl tensor[1], Shiromizu et al [?] pointed out that the "electric" part of this tensor appears in the Einstein's equations on the brane. This contribution is thought of as being "holographic" in that it can also be interpreted as coming from a dual theory on the brane. By considering a braneworld observer, we can derive properties of the dual theory and get a better understanding of the Weyl tensor contribution. However, in all the previous literature, it has been assumed that the brane is at large radial distance from the centre of AdS space, i.e. near the boundary. This allows us to assume that the energy density of the braneworld universe is small, and the true holographic description of an $(n-1)$ dimensional braneworld in an n dimensional bulk can be understood. However, these results are all approximations in the sense that for a general brane evolution it is not necessary for the brane to remain close to the boundary. In this paper, it is our objective to undertake a new study of the braneworld in which we calculate the energy of the braneworld field theory exactly, regardless of the brane's position in the bulk. We also allow the brane tension to be arbitrary, thereby including both critical and non-critical branes.

We find that this exact approach introduces quadratic terms in the Friedmann Robertson Walker (FRW) equations of the braneworld universe. Remarkably these turn out to have exactly the same form as the quadratic terms found in the cosmology of branes embedded in a pure AdS background, when extra matter is placed on the brane [?, ?]. This suggests a new, enlarged relationship between the two descriptions of braneworld cosmology, that includes even the "unconventional" quadratic terms in the FRW equation. In the linear approximation we recover the standard dual descriptions observed by Savonije and Verlinde [?, ?]. Furthermore, unlike in [?, ?] there is no ambiguity in splitting the braneworld energy momentum tensor into tension and additional matter. This is because we do not have any additional matter, only tension and a bulk Weyl tensor contribution that we have interpreted on the brane.

The rest of the paper is organised as follows: in section 2 we derive the equations of motion for the brane embedded in RNAdS and interpret them as FRW equations for the brane universe. In section 3 we derive the energy density measured by a braneworld observer exactly. We see how the FRW equations contain the same quadratic terms found in "unconventional' brane cosmology and suggest a new enlarged duality. In section 4 we examine the thermodynamics of the field theory on the brane and present a generalised (local) Cardy-Verlinde formula that includes the exact non-linear terms. We then demonstrate how the correspondences between the braneworld field theory and the brane's evolution in the bulk continue to hold in this case. Finally, section 5 contains some concluding remarks.

[1]This can occur naturally for a hot critical braneworld due to the emission of radiation into the bulk [?].

2 Equations of Motion

We will consider an $(n-1)$ dimensional brane of tension σ sandwiched in between two n dimensional black holes. In order to show the generality of our work we are allowing the black holes to be charged [2], although our brane will be uncharged. Since this means that lines of flux must not converge to or diverge from the brane, we must have black holes of equal but opposite charge. In this case, the flux lines will pass through the brane since one black hole will act as a source for the charge whilst the other acts as a sink. It should be noted that although we do not have $\mathbb{Z}_2$ symmetry across the brane for the electromagnetic field, the geometry **is** $\mathbb{Z}_2$ symmetric.

We denote our two spacetimes by $\mathcal{M}^+$ and $\mathcal{M}^-$ for the positively and negatively charged black holes respectively. Their boundaries, $\partial\mathcal{M}^+$ and $\partial\mathcal{M}^-$, both coincide with the brane. Our braneworld scenario is then described by the following action:

$$\begin{aligned} S &= \frac{1}{16\pi G_n}\int_{\mathcal{M}^+ + \mathcal{M}^-} d^n x\sqrt{g}\left(R - 2\Lambda_n - F^2\right) + \frac{1}{8\pi G_n}\int_{\partial\mathcal{M}^+ + \partial\mathcal{M}^-} d^{n-1}x\sqrt{h}K \\ &\quad + \frac{1}{4\pi G_n}\int_{\partial\mathcal{M}^+ + \partial\mathcal{M}^-} d^{n-1}x\sqrt{h}F^{ab}n_a A_b + \sigma\int_{brane} d^{n-1}x\sqrt{h}, \end{aligned} \tag{1}$$

where g_{ab} is the bulk metric and h_{ab} is the induced metric on the brane. K is the trace of the extrinsic curvature of the brane, and n_a is the unit normal to the brane pointing from $\mathcal{M}^+$ to $\mathcal{M}^-$. Notice the presence of the Hawking-Ross term in the action (1) which is necessary for black holes with a fixed charge [?].

The bulk equations of motion which result from this action are given by

$$R_{ab} - \frac{1}{2}Rg_{ab} = -\Lambda_n g_{ab} + 2F_{ac}F_b{}^c - \frac{1}{2}g_{ab}F^2 \tag{2}$$

$$\partial_a\left(\sqrt{g}F^{ab}\right) = 0 \tag{3}$$

These admit the following 2 parameter family of electrically charged black hole solutions for the bulk metric

$$ds_n{}^2 = -h(Z)dt^2 + \frac{dZ^2}{h(Z)} + Z^2 d\Omega_{n-2}, \tag{4}$$

in which

$$h(Z) = k_n{}^2 Z^2 + 1 - \frac{c}{Z^{n-3}} + \frac{q^2}{Z^{2n-6}}, \tag{5}$$

and the electromagnetic field strength

$$F = dA \quad \text{where} \quad A = \left(-\frac{1}{\kappa}\frac{q}{Z^{n-3}} + \Phi\right)dt \quad \text{and} \quad \kappa = \sqrt{\frac{2(n-3)}{n-2}}. \tag{6}$$

Note that $d\Omega_{n-2}$ is the metric on a unit $(n-2)$ sphere. k_n is related to the bulk cosmological constant by $\Lambda_n = -\frac{1}{2}(n-1)(n-2)k_n^2$, whereas c and q are constants

[2] We note that for branes moving in a Schwarzschild AdS bulk, there is evidence that gravity is localised on the brane [?]. We suspect that this still holds for a Reissner-Nordström AdS bulk.

of integration. If q is set to zero in this solution, we regain the AdS-Schwarzschild solution in which c introduces a black hole mass. The presence of q introduces black hole charge for which Φ is an electrostatic potential difference. In this general metric, $h(Z)$ has two zeros, the larger of which, Z_H, represents the event horizon of the black hole.

Here, charge is a localised quantity. It can be evaluated from a surface integral on any closed shell wrapping the black hole (Gauss' Law). In $\mathcal{M}^{\pm}$ the total charge is

$$Q = \pm \frac{(n-2)\kappa\Omega_{n-2}}{8\pi G_n} q, \tag{7}$$

where Ω_{n-2} is the volume of a unit $(n-2)$ sphere. Note also that the ADM mass [?] of each black hole is given by

$$M = \frac{(n-2)\Omega_{n-2}c}{16\pi G_n}. \tag{8}$$

Let us now consider the dynamics of our brane embedded in this background of charged black holes. We use τ to parametrise the brane so that it is given by the section $(\mathbf{x}^\mu, t(\tau), Z(\tau))$ of the bulk metric. The Israel equations for the jump in extrinsic curvature across the brane give the brane's equations of motion. One might suspect that the presence of the Hawking-Ross term in the action will affect the form of these equations. However, since the charge on the black holes is fixed, the flux across the brane does not vary and the Israel equations take their usual form

$$2K_{ab} - 2Kh_{ab} = -8\pi G_n \sigma h_{ab}, \tag{9}$$

where

$$K_{ab} = h_a^c h_b^d \nabla_{(c} n_{d)} \quad \text{and} \quad n_a = (\mathbf{0}, -\dot{Z}, \dot{t}). \tag{10}$$

Here we use an overdot to denote differentiation with respect to τ. Note that n_a is the **unit** normal, so we have the condition

$$-h(Z)\dot{t}^2 + \frac{\dot{Z}^2}{h(Z)} = -1 \tag{11}$$

The resulting equations of motion for the brane are:

$$\dot{Z}^2 = aZ^2 - 1 + \frac{c}{Z^{n-3}} - \frac{q^2}{Z^{2n-6}} \tag{12a}$$

$$\ddot{Z} = aZ - \left(\frac{n-3}{2}\right)\frac{c}{Z^{n-2}} + (n-3)\frac{q^2}{Z^{2n-5}} \tag{12b}$$

$$\dot{t} = \frac{\sigma_n Z}{h(Z)} \tag{12c}$$

where $a = \sigma_n^2 - k_n^2$ and $\sigma_n = \frac{4\pi G_n \sigma}{n-2}$. This analysis has been presented in more detail, at least for uncharged black holes, in [?, ?].

We shall now examine the cosmology of our braneworld in this background. The induced metric is given by the following

$$ds^2_{n-1} = -d\tau^2 + Z(\tau)^2 d\Omega_{n-2}. \tag{13}$$

We see that we may interpret $Z(\tau)$ as corresponding to the scale factor in an expanding/contracting universe and that equations (12a) and (12b) should be regarded as giving rise to the Friedmann equations of our braneworld. Indeed, if we define the Hubble parameter as $H = \frac{\dot{Z}}{Z}$, we arrive at the following equations for the cosmological evolution of the brane

$$H^2 = a - \frac{1}{Z^2} + \frac{c}{Z^{n-1}} - \frac{q^2}{Z^{2n-4}} \tag{14a}$$

$$\dot{H} = \frac{1}{Z^2} - \left(\frac{n-1}{2}\right)\frac{c}{Z^{n-1}} + (n-2)\frac{q^2}{Z^{2n-4}}. \tag{14b}$$

Let us examine these equations in more detail. Equation (14a) contains the cosmological constant term a. For $a = 0$ we have a critical wall with vanishing cosmological constant. For $a > 0/a < 0$ we have super/subcritical walls that correspond to asymptotically de Sitter/anti-de Sitter spacetimes. Note that for subcritical and critical walls, Z has a maximum and minimum value. For supercritical walls, we have two possibilities: either Z is bounded above and below or it is only bounded below and may stretch out to infinity.

However, our real interest in equations (14a) and (14b), lies in understanding the c and q^2 terms. As discussed in [?, ?], we can interpret the contribution of the c term to the cosmological evolution in two different ways. On the one hand, the evolution is driven, in part, by the masses of the bulk black holes, as is evident in equations (14a) and (14b). On the other hand, we can ignore the bulk and describe the evolution as being driven by the energy density and pressure of radiation in a dual field theory living on the brane. At least for critical walls, it was showed by Biswas and Mukherji [?] that we could interpret the q^2 terms as corresponding to stiff matter in the dual theory. These discussions were motivated by "holography" in the sense that the duality related an n dimensional bulk theory to an $(n-1)$ dimensional field theory on the brane. In the spirit of AdS/CFT, the brane was close to the AdS boundary. In the next section we will not make that assumption. We will nevertheless discover an interesting alternative picture, although it will be "unconventional", at least in terms of cosmology!

3 Energy on the brane

Consider an observer living on the brane. He measures time using the braneworld coordinate, τ, rather than the bulk time coordinate, t. This of course affects his measurement of the energy density. In previous literature, the energy on the brane has been obtained by scaling the energy of the bulk accordingly. For example, in [?], a detailed calculation using Euclidean gravity techniques yielded a bulk energy,

$E_{bulk} \approx 2M\left(\frac{k_n^2}{\sigma_n^2}\right)$, where M is the ADM mass of the black holes. This was then scaled using $\dot{t}$ to give the energy on the brane. In the limit of large $Z(\tau)$, $\dot{t} \approx \frac{\sigma_n}{k_n^2 Z}$ so that a braneworld observer measures the energy to be $E \approx \frac{2M}{\sigma_n Z}$. This analysis enabled us to have the dual picture described in the previous section, even for non-critical branes.

While the results obtained from these methods are interesting, they are approximations. Indeed, the Euclidean analysis of [?] is only valid in certain limits, imposed because we require time translational symmetry to Wick rotate to Euclidean signature. We should note that the deviation from the expected bulk energy, $2M$, is not so startling as we might originally think. The limits of the analysis impose that as the brane approaches the AdS boundary, $\sigma_n \to k_n$. Therefore, if the brane actually strikes the AdS boundary, it has to be critical and we recover what we might have naïvely expected: the bulk energy is given by the sum of the ADM masses.

In this paper, we will bypass all of these approximations and limitations by ignoring the bulk energy and calculating the energy on the brane directly. We will use the techniques of [?] to evaluate the gravitational energy using τ as our chosen time coordinate. Happily we will not need to Wick rotate to Euclidean signature, enabling us to exactly calculate the energy density on the brane, even at smaller values of $Z(\tau)$.

We begin by focusing on the contribution from the positively charged black hole spacetime, $\mathcal{M}^+$ and its boundary, $\partial\mathcal{M}^+$. This boundary of course coincides with the brane. Consider the timelike vector field defined on $\partial\mathcal{M}^+$

$$\tau^a = (\mathbf{0}, \dot{t}, \dot{Z}). \tag{15}$$

This maps the boundary/brane onto itself, and satisfies $\tau^a \nabla_a \tau = 1$. In principle we can extend the definition of τ^a into the bulk, stating only that it approaches the form given by equation (15) as it nears the brane. We now introduce a family of spacelike surfaces, Σ_τ, labelled by τ that are always normal to τ^a. This family provide a slicing of the spacetime, $\mathcal{M}^+$ and each slice meets the brane orthogonally. As usual we decompose τ^a into the lapse function and shift vector, $\tau^a = Nr^a + N^a$, where r^a is the unit normal to Σ_τ. However, when we lie on the brane, τ^a **is** the unit normal to Σ_τ, because there we have the condition (11). Therefore, on $\partial\mathcal{M}^+$, the lapse function, $N = 1$ and the shift vector, $N^a = 0$. Before we consider whether or not we need to subtract off a background energy, let us first state that the relevant part of the action at this stage of our analysis is the following:

$$I^+ = \frac{1}{16\pi G_n}\int_{\mathcal{M}^+} R - 2\Lambda_n - F^2 + \frac{1}{8\pi G_n}\int_{\partial\mathcal{M}^+} K + \frac{1}{4\pi G_n}\int_{\partial\mathcal{M}^+} F_{ab} n^a A^b. \tag{16}$$

As stated earlier, we do not include any contribution from $\mathcal{M}^-$ or $\partial\mathcal{M}^-$, nor do we include the term involving the brane tension. This is because we want to calculate the gravitational Hamiltonian, without the extra contribution of a source. The brane tension has already been included in the analysis as a cosmological constant term, and it would be wrong to double count.

Given the slicing Σ_τ, the Hamiltonian that we derive from I^+ is given by

$$\begin{aligned} H^+ &= \frac{1}{8\pi G_n}\int_{\Sigma_\tau} N\mathcal{H} + N^a\mathcal{H}_a - 2NA_\tau\nabla_a E^a \\ &\quad -\frac{1}{8\pi G_n}\int_{S_\tau} N\Theta + N^a p_{ab}n^b - 2NA_\tau n_a E^a + 2NF^{ab}n_a A_b \end{aligned} \tag{17}$$

where $\mathcal{H}$ and $\mathcal{H}_a$ are the Hamiltonian and the momentum constraints respectively. p^{ab} is the canonical momentum conjugate to the induced metric on Σ_τ and E^a is the momentum conjugate to A_a. The surface S_τ is the intersection of Σ_τ and the brane, while Θ is the trace of the extrinsic curvature of S_τ in Σ_τ.

Note that the momentum $E^a = F^{a\tau}$. In particular, $E^\tau = 0$ and we regard A_τ as an ignorable coordinate. We will now evaluate this Hamiltonian for the RNAdS spacetime described by equations (4), (5) and (6). Each of the constraints vanish because this is a solution to the equations of motion.

$$\mathcal{H} = \mathcal{H}_a = \nabla_a E^a = 0. \tag{18}$$

The last constraint is of course Gauss' Law. When evaluated on the surface S_τ, the potential, $A = \left(-\frac{1}{\kappa}\frac{q}{Z(\tau)^{n-3}} + \Phi\right)\dot{t}\, d\tau$. The important thing here is that it only has components in the τ direction. This ensures that the last two terms in the Hamiltonian cancel one another. Since $N = 1$ and $N^a = 0$ on S_τ, it only remains to evaluate the extrinsic curvature Θ. If γ_{ab} is the induced metric on S_τ, it is easy to show that

$$\Theta = \Theta_{ab}\gamma^{ab} = K_{ab}\gamma^{ab} = (n-2)\frac{h(Z)\dot{t}}{Z}. \tag{19}$$

The energy is then evaluated as

$$\mathcal{E} = -\frac{1}{8\pi G_n}\int_{S_\tau}(n-2)\frac{h(Z)\dot{t}}{Z}. \tag{20}$$

We will now address the issue of background energy. This is usually necessary to cancel divergences in the Hamiltonian. In our case, the brane cuts off the spacetime. If the brane does not stretch to the AdS boundary there will not be any divergences that need to be cancelled. However it is important to define a zero energy solution. In this work we will choose pure AdS space. This is because the FRW equations for a brane embedded in pure AdS space would include all but the holographic terms that appear in equations (14a) and (14b). These are the terms we are trying to interpret with this analysis.

We will denote the background spacetime by $\mathcal{M}_0$. We have chosen this to be pure AdS space cut off at a surface $\partial\mathcal{M}_0$ whose geometry is the same as our brane. As is described in [?] this means we have the bulk metric given by

$$ds_n{}^2 = -h_{AdS}(Z)dT^2 + \frac{dZ^2}{h_{AdS}(Z)} + Z^2 d\Omega_{n-2}, \tag{21}$$

in which

$$h_{AdS}(Z) = {k_n}^2 Z^2 + 1. \tag{22}$$

There is of course no electromagnetic field. The surface $\partial\mathcal{M}_0$ is described by the section $(\mathbf{x}^\mu, T(\tau), Z(\tau))$ of the bulk spacetime. In order that this surface has the same geometry as our brane we impose the condition

$$-h_{AdS}(Z)\dot{T}^2 + \frac{\dot{Z}^2}{h_{AdS}(Z)} = -1 \tag{23}$$

which is analogous to the condition given in equation (11).

We now repeat the above evaluation of the Hamiltonian for the background spacetime. This gives the following value for the background energy

$$\mathcal{E}_0 = -\frac{1}{8\pi G_n}\int_{S_\tau}(n-2)\frac{h_{AdS}(Z)\dot{T}}{Z}. \tag{24}$$

Making use of equations (12a), (12c) and (23) we find that the energy of $\mathcal{M}^+$ above the background $\mathcal{M}_0$ is given by

$$E_+ = \mathcal{E} - \mathcal{E}_0 = \frac{(n-2)}{8\pi G_n}\int_{S_\tau}\sqrt{\sigma_n^2 - \frac{\Delta h}{Z^2}} - \sigma_n \tag{25}$$

where

$$\Delta h = h(Z) - h_{AdS}(Z) = -\frac{c}{Z^{n-3}} + \frac{q^2}{Z^{2n-6}}. \tag{26}$$

In this relation Δh is negative everywhere outside of the black hole horizon and so it is clear that E_+ is positive. We now turn our attention to the contribution to the energy from $\mathcal{M}^-$. Since the derivation of E_+ saw the cancellation of the last two terms in the Hamiltonian (17) we note that the result is purely geometrical. Even though $\mathcal{M}^+$ and $\mathcal{M}^-$ have opposite charge, they have the same geometry and so $E_+ = E_-$. We deduce then that the total energy

$$E = E_+ + E_- = \frac{(n-2)}{4\pi G_n}\int_{S_\tau}\sqrt{\sigma_n^2 - \frac{\Delta h}{Z^2}} - \sigma_n \tag{27}$$

Since the spatial volume of the braneworld $V = \int_{S_\tau} = \Omega_{n-2}Z^{n-2}$, we arrive at the following expression for the energy density measured by an observer living on the brane

$$\rho = \frac{(n-2)\sigma_n}{4\pi G_n}\left(\sqrt{1 - \frac{\Delta h}{\sigma_n^2 Z^2}} - 1\right) \tag{28}$$

where we have pulled out a factor of σ_n. If we insert this expression back into the first Friedmann equation (14a) we find that

$$H^2 = a - \frac{1}{Z^2} + \frac{8\pi G_n\sigma_n}{n-2}\rho + \left(\frac{4\pi G_n}{n-2}\right)^2\rho^2 \tag{29}$$

Although we will leave a more detailed analysis of the pressure, p, until section 4, we can make use of the Conservation of Energy equation

$$\dot{\rho} = -(n-2)H(\rho + p) \tag{30}$$

to derive the second Friedmann equation

$$\dot{H} = \frac{1}{Z^2} - 4\pi G_n \sigma_n (\rho + p) - (n-2)\left(\frac{4\pi G_n}{n-2}\right)^2 \rho(\rho + p) \tag{31}$$

These are clearly not the standard Friedmann equations for an $(n-1)$ dimensional universe with energy density ρ and pressure p. However, we should not expect them to be. We have not made any approximations in arriving at these results so it is possible that we would see non-linear terms. What is exciting is that the quadratic terms we see here have exactly the same form as the unconventional terms that were originally noticed by [?, ?] in the study of brane cosmologies. In that case, one places extra matter on the brane to discover this unconventional cosmology. We have no extra matter on the brane but by including a bulk black hole, we get exactly the same type of cosmology. Clearly there is an alternative description.

We also note that in [?, ?], the energy momentum tensor on the brane is split between tension and additional matter in an arbitrary way. In our analysis the tension is the **only** explicit source of energy momentum on the brane so there is no split required. With this in mind we are able to interpret each term in the FRW equations more confidently, in particular, the cosmological constant term. Furthermore, we have not yet made any assumptions on the form of the braneworld Newton's constant.

Finally, we see that for small ρ and p, we can neglect the ρ^2 and ρp terms and recover the standard Friedmann equations for an $(n-1)$ dimensional universe

$$H^2 = a - \frac{1}{Z^2} + \frac{16\pi G_{n-1}}{(n-2)(n-3)}\rho \tag{32}$$

$$\dot{H} = \frac{1}{Z^2} - \frac{8\pi G_{n-1}}{(n-3)}(\rho + p) \tag{33}$$

Here we have taken the $(n-1)$ dimensional Newton's Constant to be

$$G_{n-1} = \frac{(n-3)}{2} G_n \sigma_n \tag{34}$$

as is suggested by [?, ?, ?, ?, ?, ?]. We see, then, how the relationships noticed in previous studies of braneworld holography are just an approximation of the relationship described here.

4 Thermodynamics on the brane

In [?] it was noticed that there was a remarkable relationship between the thermodynamics of a field theory on the brane and its gravitational dynamics in the AdS bulk. That work was of course based on the assumption that the brane reached close to the AdS boundary. Our work is not limited by these approximations and we shall henceforth attempt to generalise those results to this exact setting.

We shall assume that our field theory on the brane is in thermodynamic equilibrium. We would therefore expect it to satisfy the first law of thermodynamics

$$T\mathrm{d}S = \mathrm{d}E - \Phi\mathrm{d}Q + p\mathrm{d}V. \tag{35}$$

Notice the presence of the chemical potential Φ, which is conjugate to the R charge, Q of the field theory. This is typical of theories that are dual to RNAdS in the bulk [?].

In reality, this first law arises from the contributions of both of our black holes, $\mathcal{M}^+$ and $\mathcal{M}^-$. Written in terms of the field theory quantities which are derived from each of these black holes, the first law should actually take the extended form

$$T_+\mathrm{d}S_+ + T_-\mathrm{d}S_- = \mathrm{d}E_+ + \mathrm{d}E_- - \Phi_+\mathrm{d}Q_+ - \Phi_-\mathrm{d}Q_- + (p_+ + p_-)\mathrm{d}V. \tag{36}$$

We noted in section 2 that $\mathcal{M}^+$ and $\mathcal{M}^-$ have the same geometry. Field theory quantities that only see this geometry, as opposed to the difference in charge, will therefore be the same whether they are derived from $\mathcal{M}^+$ or $\mathcal{M}^-$. We deduce then that we should derive a single temperature, T, entropy, S, energy, E and pressure, p where

$$\begin{aligned}
T &= T_+ &= T_- \\
S &= 2S_+ &= 2S_- \\
E &= 2E_+ &= 2E_- \\
p &= 2p_+ &= 2p_-.
\end{aligned}$$

However we need to be more careful in discussing the chemical potential and the charge because these are aware of the different sign in the charges in $\mathcal{M}^+$ and $\mathcal{M}^-$. Since the charges are equal and opposite we define

$$\begin{aligned}
\Phi &= \Phi_+ &= -\Phi_- \\
Q &= 2Q_+ &= -2Q_-
\end{aligned}$$

Given these relations, we do indeed recover a simplified first law of the form (35).

To proceed further we will have to assume that the entropy and charge of our field theory is given exactly by the entropy and charge of the black holes. Taking into account the doubling up from the two black holes we find that

$$S = \frac{\Omega_{n-2}Z_H^{n-2}}{2G_n} \quad \text{and} \quad Q = \frac{(n-2)\kappa\Omega_{n-2}}{4\pi G_n}q. \tag{37}$$

Recall that the energy E is given by

$$E = \frac{(n-2)\sigma_n\Omega_{n-2}Z^{n-2}}{4\pi G_n}\left(\xi(Z) - 1\right). \tag{38}$$

where we have taken out a factor of σ_n in equation (27) and used the simplifying notation

$$\xi(Z) = \sqrt{1 - \frac{\Delta h}{\sigma_n^2 Z^2}}. \tag{39}$$

The remaining thermodynamic variables, p, T and Φ, can now be obtained by making use of the first law (35).

$$\begin{aligned}
p = -\left(\frac{\partial E}{\partial V}\right)_{S,Q} &= -\rho + \frac{1}{8\pi G_n \sigma_n \xi(Z)}\left[\frac{(n-1)c}{Z^{n-1}} - \frac{2(n-2)q^2}{Z^{2n-4}}\right], &(40)\\
T = \left(\frac{\partial E}{\partial S}\right)_{Q,V} &= \frac{1}{\xi(Z)}\frac{T_{BH}}{\sigma_n Z}, &(41)\\
\Phi = \left(\frac{\partial E}{\partial Q}\right)_{S,V} &= \frac{1}{\kappa\sigma_n \xi(Z) Z}\left(\frac{q}{Z_H^{n-3}} - \frac{q}{Z^{n-3}}\right), &(42)
\end{aligned}$$

where $T_{BH} = \frac{h'(Z_H)}{4\pi}$ is the temperature of the black holes. Given the value of p that we have derived above, it is a useful check of the consistency of these results, to observe that equations (31) and (14b) are indeed the same. Notice also that the chemical potential, Φ, vanishes at the horizon.

In earlier studies of such braneworlds, the nonlinear terms in p were not taken into account due to various approximations which were made during the calculations. It is therefore far from trivial to state that any further results connecting the bulk spacetime physics and the field theory thermodynamics should continue to hold in our scenario. We therefore proceed to consider these connections between the FRW equations and the field theory variables, including all such non-linear terms, to observe how the connection is affected in their presence.

Our first task is to rewrite the first law of thermodynamics in terms of densities and we find that

$$T\mathrm{d}s = \mathrm{d}\rho - \Phi \mathrm{d}\rho_Q - \gamma\, \mathrm{d}\left(\frac{1}{Z^2}\right), \tag{43}$$

where

$$\gamma = \frac{(n-2)Z^2}{2}(\rho + p - \Phi\rho_Q - Ts). \tag{44}$$

and the densities $s = \frac{S}{V}$ and $\rho_Q = \frac{Q}{V}$. This equation defines γ which is the variation of ρ with respect to the spatial curvature $1/Z^2$. It therefore represents the geometrical Casimir part of the energy density. Using the values we have derived for our field theory thermodynamic variables, we find that

$$\gamma = \frac{n-2}{8\pi G_n \sigma_n \xi(Z)}\frac{Z_H^{n-3}}{Z^{n-3}}. \tag{45}$$

We are now ready to present a generalised (local) Cardy-Verlinde formula for the entropy density, s, in terms of γ and the thermodynamic variables

$$\begin{aligned} s &= \frac{4\pi\sigma_n}{(n-2)k_n}\left[\left(1+\frac{4\pi G_n\rho}{(n-2)\sigma_n}\right)^2\gamma\left(\rho-\frac{1}{2}\Phi\rho_Q-\frac{\gamma}{Z^2}\right)\right. \\ &\qquad \left. -\frac{2\pi G_n}{(n-2)\sigma_n}\rho^2\gamma\left(1+\frac{4\pi G_n\rho}{(n-2)\sigma_n}\right)\right]^{1/2}. \end{aligned} \tag{46}$$

In the limit of small ρ, we can ignore the quadratic terms and this reduces to the formula found in [?, ?, ?, ?]

$$s = \frac{4\pi\sigma_n}{(n-2)k_n}\sqrt{\gamma\left(\rho-\frac{1}{2}\Phi\rho_Q-\frac{\gamma}{Z^2}\right)} \tag{47}$$

In [?], it was noticed that, when a critical brane $(\sigma_n = k_n)$ crosses the black hole horizon, the Hubble Parameter is given by $H^2 = k_n^2$. At this instance, the entropy density can be written as

$$s = \frac{H}{2G_n k_n} = \frac{(n-3)H}{4G_{n-1}} \tag{48}$$

where we have made use of equation (34) when $\sigma_n = k_n$. Notice that this corresponds to the Hubble entropy described in [?]. The key observation is that, at the horizon, equation (47) then reduces to the linear form of the first Friedmann equation (32). Confining ourselves to the case of critical branes, this result generalises to the non-linear case when we evaluate equation (46) at the horizon, since we then recover the exact form of the Friedmann equation (29).

For non-critical branes $(\sigma_n \neq k_n)$ at the horizon, the connection between entropy density and the Hubble parameter is not a linear one. We instead find that $H^2 = \sigma_n^2 = k_n^2 + a$, therefore using equation (37), the entropy density on non-critical branes should be rewritten as

$$s = \frac{\sqrt{H^2-a}}{2G_n k_n} = \left(\frac{\sigma_n}{k_n}\right)\frac{(n-3)\sqrt{H^2-a}}{4G_{n-1}} \tag{49}$$

where we have once again made use of equation (34). Unlike in the critical case, it is not clear how we should interpret this entropy formula. Nevertheless we note that when equations (47) and (46) are evaluated at the horizon they reproduce the linear (32) and non-linear (29) Friedmann equations respectively.

We also present the generalised form of the temperature at the horizon crossing

$$T = -\frac{k_n\dot{H}}{2\pi\sigma_n\sqrt{H^2-a}}\left(1+\frac{4\pi G_n\rho}{(n-2)\sigma_n}\right)^{-1}. \tag{50}$$

Notice that this reduces to the formula quoted in [?] when we ignore quadratic terms and set $\sigma_n = k_n$ for the critical brane. Using this generalised form, we see that if we evaluate equation (44) at the horizon, we reproduce the second Friedmann equation (31), even in the exact, non-linear case.

5 Discussion

We have noticed a remarkable new relationship for braneworld cosmology that includes the unconventional terms first spotted in [?, ?]. Consider the approach of [?, ?] and embed a brane in pure AdS space with additional matter placed on the brane. The braneworld observer sees a different cosmology to what we might expect. The Friedmann equations include quadratic terms in the energy density and pressure of the additional matter. In our work, we see that we obtain exactly the same unconventional terms when one considers a brane embedded in between two black holes, with no extra matter being placed on the brane. In previous studies of branes in black hole backgrounds, the cosmological evolution has been described in two different ways. As above, it may be regarded as being driven by the bulk black hole or by a field theory living on the brane. However, these results assume that the brane is close to the AdS boundary. We have used techniques that do not require us to make such approximations in our calculations. It seems that the "holographic" relationships of [?] are just an approximation of the "unconventional" relationship we have noticed.

Another appealing feature of our analysis is that it is free of ambiguities. In [?, ?] it is not clear why we should split the brane energy momentum tensor into tension and additional matter in the way that we do. In our approach there is no additional matter, only tension, so there is no ambiguous split. We find that our exact interpretation of the bulk Weyl tensor suggests that the FRW equations should indeed take the form given in [?, ?]. However, by comparing our values for ρ and p we also see that we have a highly non-trivial equation of state. It only takes a recognisable form as the brane approaches the AdS boundary. This is expected because by allowing the brane to probe deep into the bulk we are inserting a significant cut-off in the braneworld field theory. The approach of [?, ?] has the benefit that it allows for an arbitrary linear equation of state.

Given our exact work, we have also been able to present a generalised Cardy-Verlinde formula for the field theory on the brane. We should note that we made various assumptions (e.g. entropy on brane is entropy of bulk) that perhaps require further justification – Our results are as given but it would be worth taking a closer look at how a braneworld observer should really see entropy in this exact setting. Nevertheless, we find that just as in [?], the Friedmann equations are reproduced when we evaluate the relevant thermodynamic formulas at the horizon, including all non-linear terms. We should note that we do not know how we should interpret the expression for the entropy (49) when there is a cosmological constant induced on the brane.

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