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A Family of Quasi-solvable Quantum Many-body Systems

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Abstract

We construct a family of quasi-solvable quantum many-body systems by an algebraic method. The models contain up to two-body interactions and have permutation symmetry. We classify these models under the consideration of invariance property. It turns out that this family includes the rational, hyperbolic (trigonometric) and elliptic Inozemtsev models as the particular cases.

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I. INTRODUCTION

New findings of solvable or integrable models have stimulated the development of new and wide research directions and ideas in both physics and mathematics. The discovery of *quasi-solvability* in quantum mechanics [1] is a typical example. By quasi-solvability we mean that a part of the spectra can be solved, at least, algebraically[1]. One of the most successful approach to construct a quasi-solvable model is the algebraic method introduced by Turbiner in 1988 [2], in which a family of quasi-solvable one-body models was constructed by the $\mathfrak{sl}(2)$ generators on a polynomial space. This family was later completely classified under the consideration of the $GL(2,\mathbb{R})$ invariance of the models [3, 4]. Recently, this family have been paid much attention to in the context of $\mathcal{N}$-fold supersymmetry [5–15]. Several attempts were made to construct quasi-solvable many-body models by naive extension to higher-rank algebras. Especially, construction of two-body problems by the rank 2 algebras were extensively investigated [4, 16–21]. These approaches however leaded to Schrödinger operators in curved space in general and could hardly apply to more than two-body problems.

In 1995, a significant progress was made in Ref. [22], where the exact solvability of the rational and trigonometric A type Calogero-Sutherland (CS) models [23–25] for any finite number of particles were shown by a similar algebraic method. The key ingredient is the introduction of the elementary symmetric polynomials which reflect the permutation symmetry of the original models. The algebra for the M-body system is $\mathfrak{sl}(M+1)$. This idea was further applied to show the exact solvability of rational and trigonometric A and BC type CS models and their supersymmetric generalizations [26], and to show the quasi-exact solvability of various deformed CS models [27, 28]. Therefore, one can say the approach starting from Ref. [22] is, up to now, the most successful in investigating the quasi-solvable quantum many-body problems. However, one has not yet known all the models that can be obtained by this approach. In other words, we have not obtained the classification of these $\mathfrak{sl}(M+1)$ M-body models like that of the $\mathfrak{sl}(2)$ one-body models. Recently, this classification problem was partly accessed in Ref. [29] though, as was stressed by the authors themselves, the results depend on the specific ansatz and thus are incomplete. In this paper, we will show the complete classification of the quantum many-body systems with *up to two-body interactions* which can be constructed by the $\mathfrak{sl}(M+1)$ method.

II. CONSTRUCTION OF THE MODELS

Consider an M-body quantum Hamiltonian,

$$H_{\mathcal{N}} = -\frac{1}{2}\sum_{i=1}^{M}\frac{\partial^2}{\partial q_i^2} + V(q_1,\cdots,q_M), \tag{1}$$

which possesses permutation symmetry, that is,

$$V(\ldots,q_i,\ldots,q_j,\ldots) = V(\ldots,q_j,\ldots,q_i,\ldots), \tag{2}$$

[1] The term *quasi-exact solvability* has been widely used in this meaning. However, we keep it to express the case where the state vectors corresponding to the solvable spectra are normalizable. Importance of this distinction is explained in Ref. [12, 15]

for $\forall\, i \neq j$. To algebraize the Hamiltonian (1), we will proceed the following three steps. At first, we make a *gauge* transformation on the Hamiltonian (1):

$$\tilde{H}_{\mathcal{N}} = e^{\mathcal{W}(q)} H_{\mathcal{N}} e^{-\mathcal{W}(q)}. \tag{3}$$

The function $\mathcal{W}(q)$ is to be determined later and plays the role of the superpotential when the system Eq. (1) is supersymmetric. As in Eq. (3), we will hereafter attach the tilde to both operators and vector spaces to indicate that they are gauge-transformed from the original ones. In the next, we change the variables q_i to h_i by a function h of a single variable; $h_i = h(q_i)$. Note that the way of the change of the variables preserves the permutation symmetry. The third step is the introduction of the elementary symmetric polynomials of h_i defined by,

$$\sigma_k(h) = \sum_{i_1 < \cdots < i_k} h_{i_1} \cdots h_{i_k} \quad (k = 1, \ldots, M), \tag{4}$$

from which we further change the variables to σ_i. Then, we give the components of the $\mathcal{N}$-fold supercharges in terms of the above variables σ_i as follows,

$$\tilde{P}_{\mathcal{N}}^{\{i\}} = \frac{\partial^{\mathcal{N}}}{\partial \sigma_{i_1} \cdots \partial \sigma_{i_{\mathcal{N}}}} \quad (1 \leq i_1 \leq \cdots \leq i_{\mathcal{N}} \leq M), \tag{5}$$

where $\{i\}$ is an abbreviation of the set $\{i_1, \ldots, i_{\mathcal{N}}\}$. Using these $\mathcal{N}$-fold supercharges, we define the vector space $\tilde{\mathcal{V}}_{\mathcal{N}} \equiv \bigcap_{\{i\}} \ker \tilde{P}_{\mathcal{N}}^{\{i\}}$, which now becomes,

$$\tilde{\mathcal{V}}_{\mathcal{N}} = \text{span}\ \left\{ \sigma_1^{n_1} \cdots \sigma_M^{n_M} : 0 \leq \textstyle\sum_{i=1}^{M} n_i \leq \mathcal{N} - 1 \right\}. \tag{6}$$

For given M and $\mathcal{N}$, the dimension of the vector space (6) amounts to,

$$\dim \tilde{\mathcal{V}}_{\mathcal{N}} = \sum_{n=0}^{\mathcal{N}-1} \frac{(n + M - 1)!}{n!\,(M-1)!} = \frac{(\mathcal{N} + M - 1)!}{(\mathcal{N} - 1)!\, M!}. \tag{7}$$

We will construct the system (3) to be quasi-solvable so that the solvable subspace are given by just Eq. (6). This can be achieved by imposing the following quasi-solvability condition [12, 13],

$$\tilde{P}_{\mathcal{N}}^{\{i\}} \tilde{H}_{\mathcal{N}} \tilde{\mathcal{V}}_{\mathcal{N}} = 0 \quad \text{for} \quad \forall \{i\}. \tag{8}$$

The general solution of Eq. (8) can be obtained in completely the same way as shown in Ref. [13]. As in the case of the one-body models, it is sufficient to find out up to the second-order differential operators as the solutions for $\tilde{H}_{\mathcal{N}}$ since we are constructing a Schrödinger operator in the original variables q_i. It turns out that the general solution which contains up to the second derivatives takes the following form,

$$\tilde{H}_{\mathcal{N}} = -\sum_{\kappa,\lambda,\mu,\nu=0}^{M} A_{\kappa\lambda,\mu\nu} E_{\kappa\lambda} E_{\mu\nu} + \sum_{\kappa,\lambda=0}^{M} B_{\kappa\lambda} E_{\kappa\lambda} - C, \tag{9}$$

where $A_{\kappa\lambda,\mu\nu}$, $B_{\kappa\lambda}$, C are arbitrary constants, and $E_{\kappa\lambda}$ are the first-order differential operators which constitute the Lie algebra $\mathfrak{sl}(M+1)$:

$$E_{0i} = \frac{\partial}{\partial\sigma_i}, \quad E_{ij} = \sigma_i \frac{\partial}{\partial\sigma_j}, \tag{10a}$$

$$E_{i0} = \sigma_i E_{00} = \sigma_i \left(\mathcal{N} - 1 - \sum_{k=1}^{M} \sigma_k \frac{\partial}{\partial\sigma_k} \right). \tag{10b}$$

If we explicitly express the general solution (9) in terms of σ_i, we obtain the following expression,

$$\begin{aligned} \tilde{H}_{\mathcal{N}} &= -\sum_{k,l=1}^{M} \left[\boldsymbol{A}_0(\sigma)\sigma_k\sigma_l - \boldsymbol{A}_k(\sigma)\sigma_l + \boldsymbol{A}_{kl}(\sigma) \right] \frac{\partial^2}{\partial\sigma_k\partial\sigma_l} \\ &\quad + \sum_{k=1}^{M} \left[\boldsymbol{B}_0(\sigma)\sigma_k - \boldsymbol{B}_k(\sigma) \right] \frac{\partial}{\partial\sigma_k} - \boldsymbol{C}(\sigma), \end{aligned} \tag{11}$$

where $\boldsymbol{A}_\kappa$, $\boldsymbol{A}_{kl}$, $\boldsymbol{B}_\kappa$ and $\boldsymbol{C}$ are second-order polynomials of several variables.

One of the most difficult problems that one would come across on the algebraic approach to the quasi-solvable many-body systems is to solve the canonical form condition:

$$H_{\mathcal{N}} = e^{-\mathcal{W}(q)} \tilde{H}_{\mathcal{N}} e^{\mathcal{W}(q)} = -\frac{1}{2} \sum_{i=1}^{M} \frac{\partial^2}{\partial q_i^2} + V(q). \tag{12}$$

If the Hamiltonian (11) is gauge-transformed back to the original one, it is in general not the canonical form of the Schrödinger operator like Eq. (1) and one can hardly solve for arbitrary M the conditions under which a gauge-transform of Eq. (11) could be cast into the Schrödinger form. This difficulty can, however, be partly overcome by the following observation, which is the key ingredient in this paper. Suppose we can solve the canonical condition for $M = 2$ and obtain a quasi-solvable two-body Hamiltonian which would have the following form,

$$H_{\mathcal{N}} = -\frac{1}{2} \sum_{i=1}^{2} \frac{\partial^2}{\partial q_i^2} + \sum_{i=1}^{2} V_1(q_i) + V_2(q_1, q_2). \tag{13}$$

The Hamiltonian (13) and the corresponding solvable wave functions are, due to the way of the construction, invariant under the exchange of the variables q_1 and q_2. This permutation symmetry ensures that the simple extension of the two-body Hamiltonian (13) to the M-body one,

$$H_{\mathcal{N}} = -\frac{1}{2} \sum_{i=1}^{M} \frac{\partial^2}{\partial q_i^2} + \sum_{i=1}^{M} V_1(q_i) + \sum_{i<j}^{M} V_2(q_i, q_j), \tag{14}$$

is also quasi-solvable for $M > 2$. Conversely, if we can solve the condition for an M, we would obtain a quasi-solvable Hamiltonian which might contain up to M-body interactions. Then, we can get a quasi-solvable M-body model with up to the two-body interactions if we turn off all the coupling constants of the interactions except for the one- and two-body ones. The resultant model should be, when we put $M = 2$, one of the models (13) constructed by the $\mathfrak{sl}(3)$. Therefore, as far as up to two-body interactions are concerned, it is necessary and sufficient to solve the $M = 2$ case by virtue of the permutation symmetric construction. We have found that we can actually solve the canonical form condition for $M = 2$ and that $\tilde{H}_{\mathcal{N}}$ must have the following expression in terms of the variables h_i,

$$
\begin{aligned}
\tilde{H}_{\mathcal{N}}(h) \;=\; & -\sum_{i=1}^{M} P(h_i)\frac{\partial^2}{\partial h_i^2} \\
& -\sum_{i=1}^{M}\left[Q(h_i) - \frac{\mathcal{N}-2+c}{2}P'(h_i)\right]\frac{\partial}{\partial h_i} \\
& -2c\sum_{i\neq j}^{M}\frac{P(h_i)}{h_i - h_j}\frac{\partial}{\partial h_i} - \boldsymbol{C}\big(\sigma(h)\big),
\end{aligned} \tag{15}
$$

where $\boldsymbol{C}$ is given by,

$$
\begin{aligned}
\boldsymbol{C}\big(\sigma(h)\big) \;=\; & \frac{(\mathcal{N}-1)(\mathcal{N}-2+2c)}{12}\sum_{i=1}^{M} P''(h_i) \\
& -\frac{\mathcal{N}-1}{2}\sum_{i=1}^{M} Q'(h_i) - \frac{\mathcal{N}-1}{2}c\sum_{i\neq j}^{M}\frac{P'(h_i)}{h_i - h_j} + R.
\end{aligned} \tag{16}
$$

The P and Q in Eqs. (15) and (16) are a fourth- and a second-order polynomial, respectively:

$$
P(h) \;=\; a_4 h^4 + a_3 h^3 + a_2 h^2 + a_1 h + a_0, \tag{17a}
$$

$$
Q(h) \;=\; b_2 h^2 + b_1 h + b_0. \tag{17b}
$$

Thus, there are 10 parameters a_n, b_n, c, R, which characterize the quasi-solvable Hamiltonian (15). The function $h(q)$, which determine the change of variables, is given by the solution of the differential equation,

$$
h'(q)^2 = 2P\big(h(q)\big). \tag{18}
$$

One may notice that the resultant Eqs. (15)–(18) have resemblance to those of the one-body quasi-solvable models constructed by $\mathfrak{sl}(2)$ generators [2–4], or equivalently, the Type A $\mathcal{N}$-fold supersymmetric models [13, 14]. Indeed, we can easily see that the above results reduce to the one-body $\mathfrak{sl}(2)$ quasi-solvable and Type A $\mathcal{N}$-fold supersymmetric models if we set $M = 1$ and $c = 0$. This is consistent with the fact that in the case of $M = 1$ the above procedure is essentially equivalent to that in the $\mathfrak{sl}(2)$ construction of Type A $\mathcal{N}$-fold supersymmetry [13]. The new parameter c is the coupling constant of the two-body interaction, as is indicated in Eqs. (15) and (16), and does not appear in the one-body models. Under the above conditions (15)–(18) satisfied, the original Hamiltonian becomes the following Schrödinger type,

$$
H_{\mathcal{N}} = -\frac{1}{2}\sum_{i=1}^{M}\frac{\partial^2}{\partial q_i^2} + \frac{1}{2}\sum_{i=1}^{M}\left[\left(\frac{\partial \mathcal{W}(q)}{\partial q_i}\right)^2 - \frac{\partial^2 \mathcal{W}(q)}{\partial q_i^2}\right] - \boldsymbol{C}\big(\sigma(h)\big), \tag{19}
$$

and the *superpotential* $\mathcal{W}(q)$ is given by,

$$\mathcal{W}(q) = -\sum_{i=1}^{M}\int dh_i \frac{Q(h_i)}{2P(h_i)} + \frac{\mathcal{N}-1+c}{2}\sum_{i=1}^{M}\ln|h_i'| - c\sum_{i<j}^{M}\ln|h_i - h_j|\,. \tag{20}$$

It is evident by the construction that the solvable wave functions $\psi(q)$ of the Hamiltonian (19) take the following form,

$$\psi(q) = \tilde{\psi}(q)\, e^{-\mathcal{W}(q)}, \quad \tilde{\psi}(q) \in \tilde{\mathcal{V}}_{\mathcal{N}}. \tag{21}$$

The Hamiltonian (19) with Eqs. (16) and (20) is the most general quasi-solvable many-body systems with two-body interactions which can be constructed by the $\mathfrak{sl}(M+1)$ generators (10).

Before investigating what kind of particular models emerges from the general Hamiltonian (19), we will refer to an interesting feature of the result. If the algebraic Hamiltonian (9) does not contain any *raising* operator E_{i0}, it preserves the vector space $\tilde{\mathcal{V}}_{\mathcal{N}}$ for arbitrary $\mathcal{N}$ and becomes a solvable model [20, 21]. In this case, it turns out that $\boldsymbol{C}(\sigma) = C$, one of the constants involved in Eq. (9), and thus the original Hamiltonian (19) becomes supersymmetric. A system is always quasi-solvable if it is supersymmetric, since the ground state is always solvable. From the above result, we can conclude that a system is always supersymmetric if it is solvable and all its states have the form (21).

III. CLASSIFICATION OF THE MODELS

It was shown that the one-body $\mathfrak{sl}(2)$ quasi-solvable models can be classified using the shape invariance of the Hamiltonian under the action of $GL(2,\mathbb{R})$ of linear fractional transformations [3, 4]. We can see that the many-body Hamiltonian (15) also has the same shape invariance property. The linear fractional transformation of h_i is introduced by,

$$h_i \mapsto \hat{h}_i = \frac{\alpha h_i + \beta}{\gamma h_i + \delta} \quad (\Delta \equiv \alpha\delta - \beta\gamma \neq 0). \tag{22}$$

Then, it turns out that the Hamiltonian (15) is shape invariant under the following transformation induced by Eq. (22),

$$\tilde{H}_{\mathcal{N}}(h) \mapsto \widehat{\tilde{H}}_{\mathcal{N}}(h) = \prod_{i=1}^{M}(\gamma h_i + \delta)^{\mathcal{N}-1}\,\tilde{H}_{\mathcal{N}}(\hat{h})\prod_{i=1}^{M}(\gamma h_i + \delta)^{-(\mathcal{N}-1)}, \tag{23}$$

where the polynnomials $P(h)$ and $Q(h)$ in the $\tilde{H}_{\mathcal{N}}(h)$ are transformed according to,

$$P(h) \mapsto \hat{P}(h) = \Delta^{-2}(\gamma h + \delta)^4 P(\hat{h}), \tag{24a}$$
$$Q(h) \mapsto \hat{Q}(h) = \Delta^{-1}(\gamma h + \delta)^2 Q(\hat{h}). \tag{24b}$$

For a given $P(h)$, the function $h(q)$ is determined by Eq. (18) and a particular model is obtained by substituting this $h(q)$ for Eq. (19) and (20). Under the transformation (24b) of $GL(2,\mathbb{R})$, every real quartic polynomial $P(h)$ is equivalent to one of the following eight forms:

$$
\begin{aligned}
&1).\ \frac{1}{2}, \quad 2).\ 2h, \quad 3).\ 2\nu h^2, \quad 4).\ 2\nu(h^2-1),\\
&5).\ 2\nu(h^2+1), \quad 6).\ \frac{\nu}{2}(h^2+1)^2,\\
&7).\ 2h^3-\frac{g_2}{2}h-\frac{g_3}{2}, \quad 8).\ \frac{\nu}{2}(h^2+1)\big((1-k^2)h^2+1\big).
\end{aligned}
$$

where ν, k, g_2 and g_3 are all real numbers satisfying $\nu\neq 0$, $0<k<1$ and $g_2^3-27g_3^2\neq 0$. Thus, the quasi-solvable models (19) can be classified by the above eight cases.

Case 1). $h(q)=q$:
This leads to the rational A type Inozemtsev model [30–32]. The main difference between quantum and classical case is that the quantum quasi-solvability holds only for *quantized* values of the parameter, say, for integer $\mathcal{N}$, while the classical integrability holds for continuous values. This is one of the common features that the quantum quasi-solvable models share.

Case 2). $h(q)=q^2$:
This leads to the rational BC type Inozemtsev model. The quasi-exactly solvable model reported in Ref. [28] is just this case.

Case 3). $h(q)=e^{2\sqrt{\nu}q}$:
This leads to the hyperbolic ($\nu>0$) and trigonometric ($\nu<0$) A type Inozemtsev model.

Case 4). $h(q)=\cosh 2\sqrt{\nu}q$:
This leads to the hyperbolic ($\nu>0$) and trigonometric ($\nu<0$) BC type Inozemtsev model. The quasi-solvabilty of the special cases of the above four were recently shown in Ref. [33] by an ansatz method. In the solvable cases, All the above four include the corresponding CS models.

Case 5). $h(q)=\sinh 2\sqrt{\nu}q$:
This leads to neither the Inozemtsev nor the Olshanetsky-Perelomov type [34] even in the solvable case. The paper Ref. [29] covers most of the above five models.

Case 6). $h(q)=\tan\sqrt{\nu}q$:
This is a genuine quasi-solvable model which does not have any solvable cases.

Case 7). $h(q)=\wp(q;g_2,g_3)$:
This case includes the elliptic BC type Inozemtsev model and the twisted CS models [35–37]. The elliptic model in Ref. [38] may be also included in this case.

Case 8). $h(q)=\mathrm{sn}\,(\nu q|k)/\,\mathrm{cn}\,(\sqrt{\nu}q|k)$:
This is also a genuine quasi-solvable model. We have not appreciated whether the models 6 and 8 include any known model or not.

More details on the results presented here and further development will be reported in the near future.

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