

## A note on theta dependence

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The dependence on the topological  $\theta$  angle term in quantum field theory is usually discussed in the context of instanton calculus. There the observables are  $2\pi$  periodic, analytic functions of  $\theta$ . However, in strongly coupled theories, the semi-classical instanton approximation can break down due to infrared divergences. Instances are indeed known where analyticity in  $\theta$  can be lost, while the  $2\pi$  periodicity is preserved. In this short note we exhibit a simple two dimensional example where the  $2\pi$  periodicity is lost. The observables remain periodic under the transformation  $\theta \mapsto \theta + 2k\pi$  for some  $k > 2$ . We also briefly discuss the case of four dimensional  $\mathcal{N} = 2$  supersymmetric gauge theories.

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A topological  $\theta$  angle term  $L_\theta$  can be added to the lagrangian in various quantum field theories. The most important and well-known example is the case of four dimensional gauge theories, for which

$$L_\theta^{4D} = \frac{\theta}{32\pi^2} \epsilon^{\mu\nu\rho\kappa} \text{tr} F_{\mu\nu} F_{\rho\kappa}. \quad (1)$$

Other examples include two dimensional gauge theories, for which

$$L_\theta^{2D} = \frac{\theta}{4\pi} \epsilon^{\mu\nu} F_{\mu\nu}, \quad (2)$$

or two dimensional non-linear  $\sigma$  models with target space  $\mathcal{M}$ , for which

$$L_\theta^\sigma = \frac{\theta}{4\pi} \epsilon^{\mu\nu} B_{ij}(\phi) \partial_\mu \phi^i \partial_\nu \phi^j, \quad (3)$$

where  $B \in H^2(\mathcal{M}, \mathbb{Z})$  has integer periods (for Kähler  $\sigma$  models like the  $\mathbb{C}P^N$  model in which we will be interested in,  $B$  is the suitably normalized Kähler form). All the  $\theta$  angle terms can be written locally as total derivatives, and are normalized in such a way that with classical boundary conditions at infinity (determined by the requirement of finite classical action),

$$\int L_\theta / \theta \in \mathbb{Z}. \quad (4)$$

These two properties have two important consequences in *weakly coupled* field theories. The fact that  $L_\theta$  is a total derivative implies that there is no  $\theta$  dependence in perturbation theory. The fact that  $\int L_\theta / \theta$  is quantized implies that topologically non-trivial classical field configurations, called instantons, that can contribute to the path integral in a semiclassical approximation, may induce a  $2\pi$  periodic and smooth  $\theta$  dependence. More precisely, a  $k$  instanton or  $k$  anti-instanton contribution,  $k \in \mathbb{Z}$ , is proportional to

$$e^{-8\pi^2 k/g^2} e^{\pm i k \theta}, \quad (5)$$

where  $g$  is the conventionally normalized coupling constant. In theories where dimensional transmutation takes place, the running coupling  $g$  is replaced by a scale  $|\Lambda|$ . It is convenient to introduce a complexified scale

$$\Lambda = |\Lambda| e^{i\theta/\beta}, \quad (6)$$

where  $\beta$  is the coefficient of the one-loop  $\beta$  function. Contributions from  $k$  instantons and  $k$  anti-instantons in the one-loop approximation are then respectively proportional to  $\Lambda^{k\beta}$  and  $\Lambda^{k\beta}$ .

Understanding the  $\theta$  dependence in *strongly coupled* field theories is a much more difficult problem. Boundary conditions at infinity, or equivalently the structure of the vacuum, should be determined by using the (generally unknown) quantum effective action, not the classical action, and quantization laws such as (4) can be invalidated. Moreover, subtle infrared effects can induce a  $\theta$  dependence in Feynman diagrams. The conclusion is that analyticity or  $2\pi$  periodicity are not ensured a priori. A typical example where the dependence in  $\theta$  is not consistent with instanton calculus is the two dimensional quantum electrodynamics, or equivalently the purely bosonic  $\mathbb{CP}^N$  non-linear  $\sigma$  model whose effective action is QED $_{\sigma}$  [1]. There it is well-known that the term (2) induces a constant background electric field  $E = e^2\theta/2\pi$ , where  $e$  is the electric charge [2]. The phenomenon of pair creation, which is energetically favoured for  $|E| > e^2/2$ , ensures  $2\pi$  periodicity in  $\theta$ , but analyticity at  $\theta = \pm\pi$  is lost. Discussions of similar effects in various contexts can be found for example in [3].

An interesting generalization of the standard QED $_{\sigma}$  discussed above is the  $\mathcal{N}=2$  supersymmetric QED $_{\sigma}$  with a twisted superpotential  $W$ , or equivalently the  $\mathcal{N}=2$  supersymmetric  $\mathbb{CP}^N$  model with twisted masses  $m_i$  [4, 5, 6]. There, in addition to the gauge field  $E = F^{01}$ , one has a Dirac fermion  $\lambda$  and a complex scalar  $\sigma$ . All these fields belong to the same supersymmetry multiplet and are packed up in a single (twisted) chiral superfield

$$\Sigma = \sigma - 2i(\theta_- \bar{\lambda}_+ + \bar{\theta}_+ \lambda_-) + 2\bar{\theta}_+ \theta_- (D - iE). \quad (7)$$

The real auxiliary field  $D$  appears on the same footing as the gauge field  $E$ , consistently with the fact that gauge fields do not propagate in two dimensions. The equation for the background electric field is in this case  $E = -2e^2 \text{Im } W'(\sigma)$ , which shows that in the supersymmetric vacua for which  $W'(\sigma) = 0$ , there is actually no background electric field, whatever  $\theta$  may be. The phenomenon of pair creation never occurs, since the supersymmetric vacua have zero vacuum energy and are thus stable. What is left then of the  $2\pi$  periodicity in  $\theta$ ? For the  $\mathbb{CP}^N$  model, the twisted superpotential is [7, 5, 6]

$$W(\sigma) = \frac{\sigma}{4\pi} \ln \prod_{i=1}^{N+1} \frac{\sigma + m_i}{e\Lambda} + \frac{1}{4\pi} \sum_{i=1}^{N+1} m_i \ln \frac{\sigma + m_i}{e\Lambda}, \quad (8)$$

where  $\Lambda$  is given by (6) with  $\beta = N+1$ . The field  $\sigma$  is chosen so that  $\sum_{i=1}^{N+1} m_i = 0$ . The vacuum equation  $W'(\langle\sigma\rangle) = 0$  reduces to

$$\prod_{i=1}^{N+1} (\langle\sigma\rangle + m_i) = \Lambda^{N+1}. \quad (9)$$

This equation has  $N+1$  solutions that correspond generically to  $N+1$  physically inequivalent vacua  $|i\rangle$ ,  $1 \leq i \leq N+1$ . The question we want to address is then the following: can we trust instanton calculus, and in particular the quantization condition (4), in those vacua?

An interesting property of the theory described by (8), shared more generally by asymptotically free non-linear  $\sigma$  models with mass terms [8], is that any of the vacua can be made arbitrarily weakly coupled by suitably choosing the mass parameters. If  $|m_i - m_k| \gg |\Lambda|$  for all  $i \neq k$ , then the vacuum  $|k\rangle$  characterized by  $\langle k|\sigma|k\rangle \simeq -m_k$  is weakly coupled because the coordinate fields on  $\mathbb{C}P^N$  have large masses  $|m_i - m_k|$ .<sup>1</sup> The exact formula (9) then predicts that  $\langle k|\sigma|k\rangle$  is given by an instanton series of the form

$$\langle k|\sigma|k\rangle = -m_k + \sum_{j=1}^{\infty} c_j^{(k)} \Lambda^{j(N+1)}, \quad (10)$$

where the  $j$ -instanton contribution  $c_j^{(k)}$  is a calculable function of the masses, for example  $c_1^{(k)} = 1/\prod_{j \neq k} (m_j - m_k)$ . Obviously, in the regime  $|m_i - m_k| \gg |\Lambda|$ , instanton calculus is valid: the expectation value  $\langle k|\sigma|k\rangle$  is a smooth,  $2\pi$  periodic function of  $\theta$ ,

$$\langle k|\sigma|k\rangle(\theta + 2\pi) = \langle k|\sigma|k\rangle(\theta), \quad (11)$$

as are the other correlators in the  $k$ -th vacuum.

When the masses  $|m_i - m_k|$  decrease, the coupling grows, and we enter a regime where the semiclassical instanton calculus is no longer reliable. In our model, there is actually a sharp transition between an instanton dominated semiclassical regime and a purely strongly coupled regime. Mathematically, the transition occurs because series in  $\Lambda^{N+1}$  like (10) have a finite radius of convergence  $R(m_i)$ . When  $|\Lambda|^{N+1} > R(m_i)$ , all the deductions based on a naïve discussion of instanton series can be invalidated.<sup>2</sup> To be more specific, let us consider the case  $N=1$ ,  $m_1 = -m_2 = m$ . For  $|m| > |\Lambda|$ , the expectation values are given by convergent instanton series deduced from (9),

$$\langle 1|\sigma|1\rangle = -\langle 2|\sigma|2\rangle = -m \sum_{j=0}^{\infty} \frac{(-1)^{j+1} \Gamma(j-1/2)}{2\sqrt{\pi} j!} \left(\frac{\Lambda}{m}\right)^{2j}. \quad (12)$$

However, for  $|m| < |\Lambda|$ , one must use the analytic continuation to (12), given by

$$\langle 1|\sigma|1\rangle = -\langle 2|\sigma|2\rangle = -\sqrt{m^2 + \Lambda^2}. \quad (13)$$

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<sup>1</sup>The masses  $m_i$  play the same rôle as Higgs vevs in four dimensional gauge theories, and the coordinate fields are like W bosons [8].

<sup>2</sup>For an analysis of the large  $N$  limit of the analytic continuations, see [9, 10].

A very important property of the analytic continuations is that they have branch cuts. Equation (11) is no longer valid and is replaced by

$$\langle \sigma \rangle_1(\theta + 2\pi) = \langle \sigma \rangle_2(\theta). \quad (14)$$

The statement for arbitrary  $N$  is that the vacua are permuted for  $\theta \rightarrow \theta + 2\pi$  at strong coupling. Since for a generic choice of the masses  $m_i$  the  $N+1$  vacua are physically inequivalent, we see that in general *the physics is not  $2\pi$  periodic in  $\theta$* .<sup>3</sup> Let us note that if we sum up over all the vacua, as would be the case in a calculation of the path integral in finite volume, then the  $2\pi$  periodicity is restored. However, in infinite volume, cluster decomposition implies that we should restrict ourselves to a given vacuum. In that case, though the path integral is not  $2\pi$  periodic, we still have

$$\langle \sigma \rangle_k((\theta + 2\pi(N+1))) = \langle \sigma \rangle_k(\theta). \quad (15)$$

This is consistent with a modified quantization law

$$(N+1) \int L_\theta / \theta \in \mathbb{Z} \quad (16)$$

that is to be compared with the classical quantization law (4). Equation (16) would correspond to so-called “fractional instantons,” field configurations with fractional topological charge  $1/(N+1)$ . The fractional instanton picture remains elusive, however, because we are not able to exhibit the corresponding field configurations.<sup>4</sup>

A precise way to state the result obtained above is to say that, at strong coupling, the transformation  $\theta \rightarrow \theta + 2\pi$  corresponds to a non-trivial monodromy of the vacua, and that, interestingly, *this monodromy is not a symmetry transformation*. Another context where similar phenomena take place is four dimensional  $N=2$  gauge theories. The analogy with the  $CP^N$  model discussed above was emphasized in [6], and fractional instanton contributions were computed in the large  $N$  limit in [9]. The formulas of [9] clearly show that some observables transform in a complicated way under  $\theta \rightarrow \theta + 2\pi$ . The transformation  $\theta \rightarrow \theta + 2\pi$  is actually implemented in  $N=2$  gauge theories by a non-trivial duality transformation, which is a symmetry of the low energy effective action. Unlike the monodromy transformation of our two dimensional model, it could be a symmetry of the whole theory. To check this explicitly is however a highly non-trivial issue.

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<sup>3</sup>In the case  $N=1$ , the vacua [1] and [2] are physically equivalent due to a special  $\mathbb{Z}_2$  symmetry of the theory, but for  $N \geq 2$  this does not occur.

<sup>4</sup>They cannot be smooth configurations of the microscopic fields, but might be naturally described in terms of composite fields that enter in the quantum effective action.

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