QUANTUM MECHANICAL MIRROR SYMMETRY, D BRANES, AND B FIELDS

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ABSTRACT. We construct quantum mechanical models which mimic many features of string theory. We use these models to gain improved descriptions of B fields and gerbes. We examine the analogs of T duality, D branes, and mirror symmetry in these models and derive quantum mechanical analogs of standard phenomena, such as the noncommutative geometry induced by a B field.

1. INTRODUCTION

In this note we attempt to gain insight into mirror symmetry from a quantum mechanical perspective. Replacing the superconformal field theories of superstring theory with quantum mechanics leads to constructions which are extremely elementary (and which can be expressed in terms easily accessible to a mathematician with no familiarity with physics). Of course, such elementary constructions do not lead to such deep predictions as the enumeration of rational curves. Nonetheless, our quantum mechanical analogs appear, in many cases, to provide low energy approximations to string theoretic phenomena. These approximations can then be applied to gain new understanding of some poorly understood features of string theory, most notably B fields and gerbes.

We begin with an examination of supersymmetric quantum mechanics with 4 real supercharges on a Calabi Yau manifold, X. We recall here, for the reader completely unfamiliar with mirror symmetry, some of mirror symmetry's coarse features. (See for example [4]). One associates a superconformal field theory to an n complex dimensional Calabi Yau manifold X. This field theory generates two filtered differential complexes, which might be called the A' and B' complexes. The A' complex can be identified with the de Rham complex on X, with a deformed ring structure. Let $T_X^{1,0}$ denotes the holomorphic tangent bundle of X. The B' complex can be identified with the Dolbeault complex with the ring structure obtained by identifying $H^{n-p,q}(X)$ with $H^q(W, \bigwedge^p T_X^{1,0})$. Here the ring structure on $H^q(W, \bigwedge^p T_X^{1,0})$ is given by exterior product of forms and tangent vectors. If two manifolds are mirror, there is an isomorphism of the associated field theories which exchanges these two complexes. In particular, when two Calabi Yau manifolds M and W of complex dimension n form a mirror pair then:

(1) $h^{p,q}(M) = h^{n-p,q}(W).$
(2) There is a natural local correspondence between the complex moduli of M and the Kahler moduli of W.

(3) The Hodge filtration on M is mirror to a filtration on W which converges in the large Kahler limit to a filtration by degree.

(4) The Dolbeault cohomology rings on M and W are isomorphic with respect to the quantum deformed ring structure of $H^{\cdot,\cdot}(M)$, and the ring structure on W obtained via the isomorphism $H^{n-p,q}(W) \simeq H^q(W, \bigwedge^p T_W^{1,0})$.

We construct two filtered differential complexes on X, which we will call the A and B complexes. These play the role of the A' and B' complexes above. In particular, they are simply de Rham and Dolbeault complexes. The A complex has the Hodge filtration as usual. The filtration on the Dolbeault complex, seems to be new, however, and has interesting features closely parallel to those appearing in mirror symmetry. In particular, features (1),(2), and (3) above have their analogs in our construction. The deformations of ring structure that we find seem, however, to lack the interesting features of those given by string theory. As an amusing consequence of our examination of these complexes, we find a slight strengthening of the Hodge theorem for compact Kahler manifolds.

A quantum mechanical analog of mirror symmetry should have spectra of Hamiltonians preserved under mirror symmetry. The simple example of an elliptic curve immediately shows that this fails for the supersymmetric quantum mechanics discussed in the first half of this paper. In order to remedy this, we construct a quantum mechanical analog of T duality and show formally how to enlarge both our Hamiltonian and the space of sections on which it acts so that, at least for tori, T duality preserves the spectrum. We warn the more mathematical reader that the construction of these models (sections 7- 10) is carried out in a formal fashion, ignoring questions of what types of convergence to impose on various infinite series.

The introduction of an analog of T duality immediately leads to analogs of D branes and to a very simple explicit realization of B fields, gerbes, and n-gerbes. The effects of the B field in our quantum mechanics include the automatic introduction of a Moyal product and a modification of the metric to the effective metric seen by open strings in the presence of a B field ([14]). The realization of a gerbe provided by our model distinguishes between gerbes appropriate for closed strings and those suitable for both open and closed strings. The obstruction to a closed string gerbe being lifted to an open string gerbe is given by the cohomology class of the field strength of the B field. The extension of this gerbe construction to n gerbes, $n > 1$, does not appear to be suitable for modelling the effects of the Ramond-Ramond fields. In subsequent work, we will offer a different method for including these fields in a quantum mechanical model.

2. The complexes

Consider a supersymmetric quantum mechanics with 4 real supercharges. Thus we have 4 self adjoint operators

$$Q_1, Q_2, Q_3, Q_4,$$

densely defined on some Hilbert space. The operators satisfy the algebra

$$(1) \qquad\qquad Q_i Q_j + Q_j Q_i = 2\delta_{ij} H,$$

for some Hamiltonian H.

We may realize this algebra by fixing our Hilbert space to be the L_2 differential forms on a complete Kahler manifold X and setting

$$Q_1 = \bar{\partial} + \bar{\partial}^*, iQ_2 = \bar{\partial} - \bar{\partial}^*, Q_3 = \partial + \partial^*, iQ_4 = \partial - \partial^*.$$

The Kahler identities guarantee that (1) is satisfied with $2H = \Delta$. We could also, of course, realize the algebra in other ways, for example, by setting

$$Q_1' = d + d*, iQ_2' = d - d^*, Q_3' = d_c + d_c^*, iQ_4' = d_c - d_c^*.$$

With this data, we may construct our complexes. The $A-$ complex is simply the de Rham complex, $(A^{\cdot}, d, F^{\cdot})$, equipped with the Hodge filtration:

$$F^p A^j = \oplus_{k=0}^{j-p} A^{p+k, j-p-k},$$

for $j \geq p$, and zero for $p > j$. The Hodge filtration induces the decomposition $d = \bar{\partial} + \partial$ and leads to the associated graded complex $(\oplus_p A^{p, \cdot}, \bar{\partial})$. Let $H_A^{\cdot}$ denote the cohomology of the A complex, and let $H_A^{\cdot, \cdot}$ denote the cohomology of the associated gradeds. Then we have the usual Hodge decomposition of the cohomology,

$$H_A^k = \oplus_{p+q=k} H_A^{p,q}.$$

(Here the cohomology is to be interpreted as the reduced L_2 cohomology if X is not compact.)

The B complex is a filtered Dolbeault complex which we denote $(B^{\cdot}, \bar{\partial}, K^{\cdot})$. Here

$$B^p := \oplus_{j=0}^p A^{n-p+j, j}.$$

The filtration $K^{\cdot}$ depends on both the metric and the complex structure. We will define it in terms of differential operators analogous to $\bar{\partial}$ and ∂ in the A complex.

Choose a unit complex number b and define

$$P := (\bar{\partial} + b\partial^*),$$

and

$$P' := (\bar{\partial} - b\partial^*).$$

Then P and P' sum to $2\bar{\partial}$, square to zero, and anticommute. In Kahler normal coordinates centered at a point a

$$P = \sum_j (e(d\bar{z}_j) - be(dz_j)^*)\frac{\partial}{\partial \bar{z}_j} + O(a),$$

where $e(X)$ denotes exterior multiplication by X, e^* denotes the adjoint operation, and $O(a)$ denotes terms vanishing to first order at a. Similarly,

$$P' = \sum_j (e(d\bar{z}_j) + be(dz_j)^*)\frac{\partial}{\partial \bar{z}_j} + O(a).$$

At the origin of a Kahler normal coordinate system centered at a, we define

$$u^j = e(d\bar{z}_j) - be(dz_j)^*, v^j = e(d\bar{z}_j) + be(dz_j)^*,$$

and for a multiindex I, set $u^I = u^{i_1} \cdots u^{i_{|I|}}$, and similarly define v^I.

The operators P and P', naturally defined on a Kahler manifold with a choice of unit complex number b, lead us to define summands of $B^\cdot$ by

$$B^{p,q} = \{f \in K^{p+q} : f = \sum_{|I|=p, |J|=q} f_{IJ} u^J v^I dz_1 \wedge \cdots \wedge dz_n)\}.$$

Then we have the decomposition,

$$B^j = \oplus_{p+q=j} B^{p,q}.$$

We now define the filtration $K^\cdot$ analogously to the Hodge filtration. For $j \geq p$,

$$K^p B^j = \oplus_{k=0}^{j-p} B^{p+k, j-p-k},$$

and zero otherwise.

The Kahler identities allow us to extend the usual proof of the Hodge decomposition to this bigraded complex, giving

$$H_B^k = \oplus_{p+q=k} H_B^{p,q},$$

where $H_B^\cdot$ denotes the cohomology of the B complex and $H_B^{p,q}$ is the cohomology of the graded complex (interpreted as reduced L_2 cohomology if X is not compact).

3. IDENTIFYING $H_B^{p,q}$

Observe that a harmonic representative of an element of $H_B^{p,q}$ is also a harmonic representative of an element of the de Rham cohomology $H_A^\cdot$, but need not be of fixed degree with respect to the (usual) degree of the A complex. So, do the spaces $H_B^{p,q}$ lead to new invariants of Kahler manifolds? We will see below.

Standard arguments show that the complex $(B^{p,\cdot}, P)$ is a flabby resolution of the kernel of P on $B^{p,0}$. Computing in local coordinates, we see that this latter sheaf is isomorphic to Ω^{n-p}, the holomorphic $n-p$ forms. Hence, we find that

$$(2) \qquad\qquad H_B^{p,q}(X) \simeq H_A^{n-p,q}(X).$$

Note, however, that the harmonic embedding of $H_B^{p,q}(X)$ in $H_A(X)$ does *not* lie in $H_A^{n-p,q}(X)$. So, the actual embedding may hold new data for us. In order to analyze this embedding, we let L denote exterior multiplication by the Kahler form and L^* its adjoint. Observe that the Kahler identities

$$\partial^* = [\bar{\partial}, -iL^*],$$

and

$$[\partial^*, -iL^*] = 0,$$

imply

$$e^{ibL^*} \bar{\partial} e^{-ibL^*} = P.$$

Thus the associated graded complex of the B complex is *conjugate* to the original B complex. This is a rather strange result whose analogous statement for the A complex would be that the (global) Dolbeault complex is isomorphic (not even just quasi-isomorphic) to the de Rham complex. Assuming that there is a strong (mirror) correspondence between the A and B complex, we are led to discover the following strong version of the Hodge decomposition theorem.

Theorem 3.1. *Let X be a compact Kahler manifold. There exists a bounded pseuodifferential operator h of order zero so that*

$$e^{-h}de^h = \bar{\partial}.$$

In particular, the de Rham complex (of global sections) is isomorphic to the Dolbeault complex via the map from de Rham to Dolbeault:

$$f \to e^{-h}f.$$

Proof: The proof is trivial once one suspects the existence of such an h. h may be chosen (nonuniquely) to be

$$h := -d^*\partial\Delta^{-1}.$$

Here we follow the convention that Δ^{-1} vanishes on the harmonic forms, has range perpendicular to the harmonic forms, and satisfies $(dd^* + d^*d)\Delta^{-1} = Id$ on the orthogonal complement to the harmonics. ■

Observe that for the B complex we conjugate by a local operator to obtain an isomorphism between the $\bar{\partial}$ and P complexes but could also construct a global operator of a form analogous to h above. Clearly, however, there is no local operator conjugating d to $\bar{\partial}$.

4. Variation of structure

Now specialize to the case where X has a metrically trivial canonical bundle. We will call such an X Calabi-Yau (or CY). This is a slight abuse of notation as one usually requires the vanishing of $H_A^{p,0}$ for $p < n$. Let ω denote a generator for $H^{n,0}$.

We list some correspondences between forms (up to scalars) in the A and B complexes given by relating operators and various harmonic generators of one dimensional spaces $H_A^{p,q}$ and $H_B^{p,q}$.

A complex	B complex
1	ω
ω	e^{ibL}
$\bar{\omega}$	e^{-ibL}
$dvol$	$\bar{\omega}$
$d + d^*$	$\bar{\partial} + \bar{\partial}^*$
$\bar{\partial} + \bar{\partial}^*$	$d + bd^*$.

If M is finite volume, the correspondence between $dvol$ and $\bar{\omega}$ suggests that the bounded linear functional on the A complex obtained by integrating a form over M should be paired with the functional on the B complex obtained by taking the L_2 inner product with ω.

For a CY manifold,

$$H_A^{n-p,q} \simeq H^{-p,q},$$

where $H^{-p,q}$ denotes cohomology with coefficients in the p-th exterior power of the holomorphic tangent bundle. The isomorphism is given by sending $f_{IJ}d\bar{z}^I\theta^J$ to $f := f_{IJ}e(d\bar{z}^I)\theta^J\omega$, the section θ^J of $\bigwedge^p T_X$ acts by interior multiplication. In kahler normal coordinates, we write the latter as $f_{IJ}d\bar{z}^I e^*(dz^J)\omega$. Now an elementary computation shows that the map

$$f \to f_{IJ}u^I v^J\omega,$$

gives the isomorphism (on harmonic representatives) between $H_A^{n-p,q}(X)$ and $H_B^{p,q}(X)$. We can also write the image of f under this map as

$$F_L : f \to f_{IJ} e^{ibL^*} e(d\bar{z}^J) e^{-ibL^*} e^{ibL} e^*(dz^I) e^{-ibL} \omega.$$

As we noted in the introduction, mirror symmetry is supposed to exchange metric data with complex structure data. Hence the analog of the variation of Hodge structure on the A complex, which is, of course, obtained by varying the complex structure would be the variation of $H_B^{p,q}$ as we vary the metric. Holding the complex structure fixed, this corresponds to varying the Kahler form. So, a reasonable analog of variation of Hodge structure is studying the infinitesimal variation of F_L as L varies. This is easiest to do for $H_B^{0,q}$ and $H_B^{p,0}$, for in those cases harmonic representatives in $H^{-p,0}$ and $H^{0,q}$ are independent of the Kahler class. For $f \in H^{-p,0}$ we get

$$F_L(f) = f_I e^{ibL} \theta^I e^{-ibL} \omega = f_I e^{ibL} \theta^I \omega.$$

Hence, the first variation of $F_L(f)$ is

$$ibL_1 F_L(f),$$

where L_1 denotes exterior multiplication by the first variation of the kahler form. Similarly, higher variations are easily computed in terms of further wedge products.

Next let us consider the large Kahler structure limit of the filtration $\bar{K}^{\cdot}$. For $l \geq p$, we have

$$\bar{K}^p B^l = \{f \in B^l : f = \sum_{|I|=p} u^I f_I, f_I \in B^{l-p}\}.$$

The above decomposition, $f = \sum_{|I|=p} u^I f_I$, need not be uniquely defined. For simplicity, we consider a 1 parameter variation of Kahler structure; i.e., we simply considering scaling, replacing

$$L \to TL, T \to \infty.$$

This replaces L^* by $T^{-1}L^*$. Since we will be varying the metric, we choose local Kahler normal coordinates $\{z^i\}$ with respect to $T = 1$. Then we rewrite $\bar{K}^p B^l$ as

$$\bar{K}^p B^l = \{f \in B^l : f = \sum_{|I|=p} e^{ibL^*/T} e(d\bar{z}^I) e^{-ibL^*/T} f_I, f_I \in B^{l-p}\}.$$

In particular, taking a limit as $T \to \infty$, we obtain a limiting filtration

$$\bar{K}_\infty^p B^l = \{f \in B^l : f = \sum_{|I|=p} e(d\bar{z}^I) f_I, f_I \in B^{l-p}\}.$$

This filtration reduces on B^l to the filtration by degree:

$$\bar{K}_\infty^p B^l = \{f \in B^l : \text{degree} f \geq n - l + 2p\},$$

where degree is in the usual sense of degree of a differential form.

Thus, as with mirror symmetry, we obtain a filtration which is a deformation of the degree filtration.

We remark that in the case of $h_A^{1,1} = 1$, the freedom to choose the phase b in our construction effectively makes our construction dependent on the

complexified Kahler cone rather than the real Kahler cone. This gives b a role similar to that of a B field. This similarity recurs in Section 10.

5. PRODUCTS

Our analogy with mirror symmetry weakens considerably when we turn to product structures. Our constructions provide a natural deformation of the ring structure of the B complex, and not the A complex as with mirror symmetry.

We consider ring structures to be determined, in part, by the differentials of our complexes; we require the differentials to satisfy a graded Leibniz rule. Thus, before we introduce the filtration $K^{\cdot}$ and the differentials P, P', we have two obvious ring structures on the complex $(B^{\cdot}, \partial)$. (We continue to restrict to the case of CY manifolds). The first ring structure is the usual one given by the wedge product. The second is induced, as previously mentioned, by writing forms as $g = g_{IJ} d\bar{z}^J \theta^I \omega$ and $h = h_{LK} d\bar{z}^K \theta^L \omega$, taking their product to be

$$g_{IJ} h_{LK} (-1)^{|I||K|} d\bar{z}^{J \cup K} \theta^{I \cup L} \omega.$$

The operators P and P' are also derivations with respect to the second of these ring structures. Moreover, with this structure, the map $F_L : H^q(M, \bigwedge^p T_M^{1,0}) \to H_B^{p,q}$ is a ring isomorphism. Thus our complexes do not seem to introduce any natural deformations of ring structure.

We can use F_L to impose a deformed ring structure if we consider the B complex equipped with the usual wedge product. Consider the string of isomorphisms

$$H_A^{n-p,q} \simeq H^q(M, \bigwedge^p T_M^{1,0}) \overset{F_L}{\simeq} H_B^{p,q}.$$

For the A complex it is more natural to express a form f as $f = \hat{f}_{KJ} d\bar{z}^J dz^K$ rather than $f = f_{IJ} e(d\bar{z}^J) e^*(dz^I) \omega$. Here (in Kahler normal coordinates as always), $f_{KJ} = \pm \hat{f}_{K^c J}$. Then we may write

$$F_L(f) = f_{IJ} e^{ibL^*} e(d\bar{z}^J) e^{-ibL^*} e^{ibL} e^*(dz^I) e^{-ibL} \omega$$

$$= \hat{f}_{KJ} e^{ibL^*} e(d\bar{z}^J) e^{-ibL^*} dz^K e^{ibL},$$

Set

$$\hat{F}_L(f) = \hat{f}_{IK} e^{ibL^*} e(d\bar{z}^I) e^{-ibL^*} dz^K.$$

Then imposing a ring structure on the A complex by making $\hat{F}_L$ a ring isomorphism with respect to the wedge product on the image complex defines a deformation of the product structure on the A complex, which in the large L limit obviously converges to the usual ring structure. This product does not appear to be analogous to the deformation of the ring structure obtained in mirror symmetry, but we see that we have deformed the usual product in such a way that as the metric grows (and therefore L^* shrinks) we return to the usual product structure. We are viewing forms as operators here. Under this identification, the "vacuum vector" i.e., the "unit" degree zero vector in the exterior algebra passes from 1 to e^{ibL}. This is also the natural vacuum vector if we define a grading of our complex using $d + d_c^*$ instead of d.

An alternate approach to constructing a deformed product from this structure is to consider analogs of the Yukawa coupling:

$$(u_1, u_2, u_3) := \int_M \omega \wedge \nabla_{u_1} \nabla_{u_2} \nabla_{u_3} \omega,$$

where ∇ denotes the Gauss Manin connection. If we consider ω to be a generator for $H_A^{n,0}$, we have seen that the analog in the B complex is

$$e^{ibL},$$

variations of which are simply given by wedge products with appropriate forms. This approach seems to return the undeformed wedge product.

6. Dropping the Calabi Yau Hypothesis

If we drop the assumption that our manifold has trivial canonical bundle, the preceding construction becomes modified in an obvious fashion. The construction of the A and B complexes is unchanged, but we lose the identification of $H_B^{p,q}$ with $H^{-p,q}$ and consequently any ring structure associated to this identification. Nonetheless, we do not lose the basic isomorphism,

$$H_A^{n-p,q} \simeq H_B^{p,q}.$$

The actual map realizing this isomorphism on harmonic representatives must be re-expressed in the obvious fashion as

$$F_L(f_{IJ} d\bar{z}^J dz^I) = f_{IJ} e^{ibL^*} d\bar{z}^J e^{-ibL^*} e^{ibL} dz^I.$$

7. Quantum Mechanical T Duality

Let us recall from a geometric perspective, the rules for T-duality worked out for WZW sigma models in Buscher [1] and [2]. Suppose that we are considering a manifold M which is a Riemannian oriented circle bundle over a base manifold B. In particular, we assume that the associated circle action is an isometry. Let U_a be a coordinate cover of B and set $V_a = \pi^{-1}(U_a)$, where $\pi : M \to B$ is the projection. Choose coordinates (x_a^i, x_a^0) for V_a, where x_a^0 is a choice of angular coordinate for the fiber and x_a^i are coordinates for the base, U_a. Buscher derives the following transformation rules for the metric, g, the B field b, and the dilaton ϕ. $T-$ duality replaces these quantities with $\hat{g}, \hat{b}, \hat{\phi}$, where

$$\hat{g}_{00} = 1/g^{00},$$

$$\hat{g}_{0i} = b_{0i}/g^{00},$$

$$\hat{g}_{ij} = g_{ij} - (g_{0i}g_{0j} - b_{0i}b_{0j})/g_{00},$$

$$\hat{b}_{0i} = -\hat{b}_{i0} = g_{0i}/g_{00},$$

$$\hat{b}_{ij} = b_{ij} + (g_{0i}b_{0j} - b_{0i}g_{0j})/g_{00},$$

and

$$\hat{\phi} = \phi - \frac{1}{2}log(g_{00}).$$

Let us understand these transformations more geometrically. First we assume that $b = 0$. The metric g can be rewritten as

$$\hat{g}_{ij} dx^i \otimes dx^j + g_{00}(dx^0 + A_i dx^i)^{\otimes 2},$$

where $A_i = g_{0i}/g_{00}$, is independent of x^0 since we the circle action is an isometry. Thus the horizontal lift of the vector $\frac{\partial}{\partial x^i}$ is the vector

$$H_i = \frac{\partial}{\partial x^i} - A_i \frac{\partial}{\partial x^0}.$$

Observe that

$$< H_i, H_j > = \hat{g}_{ij}.$$

Because the H_i are the horizontal lifts of the vectors $\frac{\partial}{\partial x^i}$ on U_a, they transform as these vectors do under a change of coordinate neighborhoods. Hence, we see that the metric $\hat{g}$ is realized (as is well known) on the trivial bundle $B \times S^1$. We would like to understand the role of the B field better. At first approximation it is supposed to define a potential for the torsion tensor, T defined by

$$(3) \qquad\qquad \hat{\nabla}_{\frac{\partial}{\partial x^a}} \frac{\partial}{\partial x^b} - \hat{\nabla}_{\frac{\partial}{\partial x^b}} \frac{\partial}{\partial x^a} = T^c_{ab} \frac{\partial}{\partial x^b}.$$

The T dual connection $\hat{\nabla}$ is assumed to be metric compatible. The torsion is related to the B field by

$$(db)_{abc} = T_{abc}.$$

Here we lower indices with the metric $\hat{g}$. Observe

$$\nabla_{H_i} H_j - \nabla_{H_j} H_i = (db)_{ij0} \frac{\partial}{\partial x^0} = T^0_{ij} \hat{g}_{00} \frac{\partial}{\partial x^0} = T^0_{ij} |dx^0 + A_i dx^i|^2 \frac{\partial}{\partial x^0}.$$

The similarity of this expression to (3) suggests the possibilty of intepreting T duality literally as an exchange of $dx^0 + A_i dx^i$ and $g^{00} \frac{\partial}{\partial x^0}$; i.e., an exchange of a vector with its metric dual covector. In order to make sense of such an exchange, we need to represent cotangent vectors in some fashion akin to differential operators. A natural guess for a representation is to define a representation on some subspace of partial differential operators defined by

$$du^k f(u) \frac{\partial}{\partial u^j} = -\delta^k_i f(u)/2\pi,$$

and extend as a ($C^\infty(M)$ linear) derivation. We will denote this operator

$$ad(u^k).$$

We have not resolved the question of which subspaces of operators to consider. In particular, we will in some cases consider formal power series of operators without addressing in what sense we require convergence. As will become clear in later sections, different physical models will require different subspaces.

If we want to consider quantum mechanical theories which are invariant under T duality, then we need to include the T dual of the Laplace operator. This should take the form,

$$g_{ab} ad(x^a) ad(x^b).$$

Thus the dual of the Laplacian is essentially the metric.

Let us consider this construction on a circle of length $2\pi L$. Write $g = L^2 du^2$, with u a 2π periodic coordinate. The formal eigenvectors of $L^2 ad(u)^2$ are then of the form

$$F(u)e^{2\pi p \frac{\partial}{\partial u}},$$

with eigenvector $4L^2 p^2$. We may interpret this latter operator as translation by $2\pi p$ followed by multiplication by $F(u)$. This interpretation requires viewing translation by p as the endpoint of a 1 parameter family of translations. We see that we have a continuous spectrum for this dual Laplacian. We have, however, forgotten to impose a condition. We were intially considering functions which were 2π periodic. In particular, they were annihilated by the operator

$$e^{2\pi \frac{\partial}{\partial u}} - 1.$$

The T dual condition, then, is to restrict to the kernel of

(4) $$e^{iad(u)} - 1.$$

Imposing this condition then restricts us to (the span of) those eigen-operators, $e^{2\pi p \frac{\partial}{\partial u}}$, with p an integer, and we get a spectrum which is fixed under T duality and includes an analog of winding modes. In (9.2) we will give a geometric interpretation of the condition (4) in terms of D branes.

Next, the guess for the role of the B field, based on our T duality computations is that it enters in a manner similar to a connection term, modifying our self dual Laplacian to something like

$$-\frac{1}{\sqrt[2]{g}}(\frac{\partial}{\partial x^i} + b_{ik}ad(x^k)/i)g^{ij}\sqrt[2]{g}(\frac{\partial}{\partial x^j} + b_{jk}ad(x^k)/i) + g_{ij}ad(x^i)ad(x^j).$$

Here we have placed the B field on the Laplacian side, as seems natural for a map from $TM :\to T^*M$. If instead we collect the B terms on the metric side we get (for say constant g^{ij}, b_{ik}, and g)

$$-g^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} + 2ig^{ij}\frac{\partial}{\partial x^i}b_{jk}ad(x^k) + (g_{ij} - bg^{-1}b_{ij})ad(x^i)ad(x^j).$$

Let $G_{ij} := g_{ij} - bg^{-1}b_{ij}$, and following ([14]) (up to a constant) set $\theta^{pi} = ig^{pq}b_{qn}G^{ni}$. Then our operator reduces to

$$-G^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j} + G_{ij}(dx^i + \theta^{pi}\frac{\partial}{\partial x_p})(dx^j + \theta^{qj}\frac{\partial}{\partial x_q}).$$

In particular, we see that we may interpret this as defining a new metric G. This is the effective metric seen by open strings in the presence of a B field ([14]).

We next observe that the Moyal product associated to a constant B field naturally arises from our formalism. (See [14] and references therein). We initially considered functions and scalar valued forms, but were led by considerations of T duality to enlarge our coefficients to include partial differential operators. We can recover our original space of sections as the intersection of the kernels of $ad(x^k)$ for all k. In the context of open strings, as we shall

explain in a subsequent section, this subspace is analogous to Neumann boundary conditions. As the B field has the effect of deforming

$$(5) \qquad g_{ij}ad(x^i)ad(x^j) \to G_{ij}(dx^i + \theta^{pi}\frac{\partial}{\partial x_p})(dx^j + \theta^{qj}\frac{\partial}{\partial x_q}),$$

it is natural to replace the above analog of the Neumann condition by the condition that we consider sections in the kernel of $(dx^i + \theta^{pi}\frac{\partial}{\partial x_p})$.

Formally, we may write a section in the kernel of the deformed $ad(x^i)'s$ as

$$\hat{f} := f(x^j + \theta^{jk}\frac{\partial}{\partial x^k}),$$

interpreted as a formal power series in the symbol $\frac{\partial}{\partial x^p}$. Such a section is uniquely determined by the zero order term, $f(x)$, in the power series expansion and may be viewed as a "flat" lift of the function f to a formal partial differential operator. See [5], Chapter 5, for similar constructions in the context of deformation quantization. The product as differential operators, $\hat{f} \circ \hat{h}$ of two such flat lifts is again a flat lift (by the Leibniz rule) whose zero order term is given by $f * h$, where $*$ denotes the Moyal product with respect to θ^{ij}:

$$f * h = \sum_{k=0}^{\infty} \sum_{|I|=|J|=k} \theta^{IJ} f_I h_J / k!.$$

Here $\theta^{IJ} := \theta^{i_1 j_1} \cdots \theta^{i_{|I|} j_{|I|}}$, and f_I, for $|I| = k$ and denotes $\frac{\partial^k}{\partial x^{i_1} \cdots \partial x^{i_k}} f$.

Thus, we are naturally led to a Moyal product induced by the B field.

8. TORI

In this section, as an elementary exercise, we consider the equality of the spectrum of the operators $\Delta + g_{ij}ad(x^i)ad(x^j)$ on a pair of T dual tori E_1 and E_2. The first torus, E_1 has area A, B field equal to 0, and complex structure defined in the usual way by the parameter $\tau = \tau_1 + i\tau_2$. The T dual torus has area τ_2, B field given by $\tau_1 ds \wedge dt$, and complex structure parameter $\tau = iA$. Thus

$$E_1 := C/\lambda Z + \lambda(\tau_1 + i\tau_2)Z.$$

We parametrize E_1 by

$$(x(s,t), y(s,t)) = (\lambda s + \lambda\tau_1 t, \lambda\tau_2 t).$$

Set $A = \lambda^2 \tau_2$, the area of T. Then

$$dx^2 + dy^2 = \frac{A}{\tau_2}(ds + \tau_1 dt)^2 + A\tau_2 dt^2,$$

and

$$\Delta = -\frac{\tau_2}{A}\frac{\partial^2}{\partial s^2} - (A\tau_2)^{-1}(\frac{\partial}{\partial t} - \tau_1\frac{\partial}{\partial s})^2.$$

According to our definition of the spectrum for the "metric", we find the spectrum of $\Delta + dx^2 + dy^2$ is given by

$$\frac{A}{\tau_2}(m + \tau_1 n)^2 + A\tau_2 n^2 + \frac{\tau_2}{A}p^2 + (A\tau_2)^{-1}(q - \tau_1 p)^2,$$

with m, n, p, q integer.

Now a T duality transformation changes the metric to

$$\frac{\tau_2}{A}ds^2 + A\tau_2 dt^2,$$

and the Laplacian becomes

$$-\frac{A}{\tau_2}\frac{\partial^2}{\partial s^2} - (A\tau_2)^{-1}\frac{\partial^2}{\partial t^2}.$$

According to our earlier discussion, the duality transformation should also introduce a B field $\tau_1 ds \wedge dt$, which enters by deforming $dx^2 + dy^2 + \Delta$ to

$$\frac{\tau_2}{A}ds^2 + A\tau_2 dt^2 + -\frac{A}{\tau_2}(\frac{\partial}{\partial s} + \tau_1 dt/i)^2 - (A\tau_2)^{-1}(\frac{\partial}{\partial t} - \tau_1 ds/i)^2.$$

This new operator has spectrum

$$\frac{\tau_2}{A}p^2 + A\tau_2 n^2 + \frac{A}{\tau_2}(m + \tau_1 n)^2 + (A\tau_2)^{-1}(q - \tau_1 p)^2,$$

with integer m, n, p, q. The spectrum is thus invariant under T duality.

9. D branes

9.1. **Dictionary.** In this subsection we construct a dictionary relating operators we have defined to standard super conformal field theory constructions. This will allow us to interpret some of our results crudely in terms of D branes and permit the reader to extend these interpretations in many directions. Let $X^i(t, s)$ denote, as usual, the pull back to the string world sheet of a local coordinate on M. We use coordinates s and t on the string, satisfying s is constant on the boundary of the string. The translation of stringy constructions into our model begins with the correspondence

$$X_t^i \to g^{ij}\nabla_{\frac{\partial}{\partial x^j}},$$

$$X_s^i \to ad(x^i).$$

Hence,

$$\partial X^k \to g^{kj}\nabla_{\frac{\partial}{\partial x^j}} + iad(x^k).$$

Following the notation of [9], we let $Q^{\pm}$ and J denote the left moving supercurrents and the $U(1)$ current respectively. Then we associate

$$Q^+ \to i(e(dz^j) + e^*(d\bar{z}^j))(\nabla_{\frac{\partial}{\partial z^j}} - ig_{j\bar{k}}ad(\bar{z}^k)) \sim \partial - \bar{\partial}^*,$$

$$Q^- \to i(e(d\bar{z}^j) + e^*(dz^j))(\nabla_{\frac{\partial}{\partial \bar{z}^j}} - ig_{\bar{j}k}ad(z^k)) \sim \bar{\partial} - \partial^*,$$

$$\bar{Q}^+ \to i(e(dz^j) - e^*(d\bar{z}^j))(\nabla_{\frac{\partial}{\partial z^j}} + ig_{j\bar{k}}ad(\bar{z}^k)) \sim \partial + \bar{\partial}^*,$$

$$\bar{Q}^- \to -i(e(d\bar{z}^j) - e^*(dz^j))(\nabla_{\frac{\partial}{\partial \bar{z}^j}} + ig_{\bar{j}k}ad(z^k)) \sim \bar{\partial} + \partial^*,$$

$$J \to -ig_{j\bar{k}}(-e(dz^j)e(d\bar{z}^k) + e(dz^k)^*e(d\bar{z}^j)^*) + 2q - 2p,$$

$$\bar{J} \to ig_{j\bar{k}}(-e(dz^j)e(d\bar{z}^k) + e(dz^k)^*e(d\bar{z}^j)^*) + 2p - 2q.$$

9.2. **Boundary conditions.** In the absence of D branes, the usual boundary condition on an open string is the Neumann boundary condition :

$$X_s^i = 0,$$

on the string boundary for all i. According to our dictionary, this corresponds to the condition that

$$ad(x^i) = 0.$$

Thus the space of scalar valued differential forms is the subspace of sections corresponding to standard boundary conditions. Pursuing the analogy, then a p brane boundary condition corresponds to (locally) forcing the condition on our sections that they are annihilated by $\nabla_{\frac{\partial}{\partial x^i}}$ for $n-p$ directions $p < i \leq n$. The sections are allowed to have coefficients in partial differential operators which involve only these same $n-p$ "normal" derivatives. Thus, this subspace should correspond to D branes given by $x^i = constant_i$, for $p < i \leq n$. (See [11][p.265-267]). We interpret the normal derivatives as determining the position of the D brane. In the example of the circle considered in Section 7, we saw that the eigenstates for the metric component of the Hamiltonian were of the form $e^{2\pi p \frac{\partial}{\partial u}}$. This operator is naturally interpreted as translation by $2\pi p$. Hence, if we view this as determining the relative positions of the D branes, we find we have 1 brane located at each point of $2\pi Z$ on the real line if we impose (4). This is the standard representation for a single brane on a circle. If we do not impose (4), then the continuous spectrum could be interpreted as a family of branes filling space.

The construction of D branes as in [10], in terms of an equality

$$(6) \qquad\qquad\qquad \partial X^m = R_n^m \bar{\partial} X^n,$$

for some orthogonal matrix R generalizes this interpretation of D branes as choices of subspaces (or possibly quotient spaces) of partial differential operators and leads to analogs of A and B branes in our model. We will not pursue this here.

The notion of D branes being given as kernels of operators does not seem nicely self dual. Once we include kernels, one should expect to see also cokernels. The arrival of complexes does not seem far behind. It seems likely that our construction will lead to relations between D branes and holonomic D-modules and that this will clarify the role of complexes. We have not yet explored this possibility.

9.3. **D branes in the presence of a Wilson line.** We next consider an extension of our construction to include the presence of a background vector bundle. Thus we consider partial differential operators acting on sections of a flat C^n bundle E. Our partial differential operator coefficients are now $end(E)$ valued. Neumann boundary conditions still are represented by considering the subspace given by the kernel of the $ad(x^i)'s$. Suppose we now T dualize on a circle factor, in the x^1 direction. Assume for simplicty that we are in flat space. Then the condition that $ad(x^1)$ annihilate our partial differential operator (henceforth pdo) is replaced by the condition

that $ad(\nabla_{\frac{\partial}{\partial x^1}})$ annihilate it. Suppose that we can choose a frame so that

$$\nabla_{\frac{\partial}{\partial x^1}} = \frac{\partial}{\partial x^1} + \Lambda,$$

for some constant diagonal n by n matrix Λ with eigenvalues λ_j. (We are considering a situation like that discussed in [11][p.263-267].) The pdos we are allowed then are of the form

$$f_k(x)(\frac{\partial}{\partial x^1})^k,$$

where f_k is annihilated by $ad(\nabla_{\frac{\partial}{\partial x^1}})$.

Expressing f_k as a matrix with respect to a basis in which Λ is diagonal, we find that the diagonal entries f_k^{ii} are constant in x^1. The off diagonal entries f_k^{ij} are of the form an x^1 independent function multiplied by $e^{-x^1(\lambda_i - \lambda_j)}$. For generic Λ this is not periodic; therefore f_k is forced to be diagonal, except when two or more eigenvalues coincide (mod $2\pi i Z$) at which point there is a gauge enhancement and the f_k take values in a larger algebra. In order to localize the branes as in [11][p.263-267], we must consider again the T dual of the periodicity condition,

$$e^{2\pi \frac{\partial}{\partial u}} - 1.$$

We rewrite this as

$$e^{2\pi(\nabla_{\frac{\partial}{\partial u}} - \Lambda)} - 1 = 0.$$

Dualizing, this becomes

$$e^{iad(u)} - e^{2\pi\Lambda} = 0.$$

Restricting to eigenstates of the metric again, we see that we may write i^{th} diagonal entry of our allowed pdos as

$$F_p^{ii} e^{2\pi(p+\lambda_i)\frac{\partial}{\partial x^1}}.$$

As in the preceding section, we may interpret this as a brane located at position λ_i on the circle. Hence, this model reproduces the expected result that, in the presence of a flat C^n bundle, the T dual (along a circle) of the Neumann boundary condition is n D branes, whose relative position is determined by the monodromy of the connection. This description fails when two or more eigenvalues coincide, at which point there is an enlargement of the space of sections.

10. T duality and Mirror symmetry

In this section, I consider the natural question: can we map the A complex to the B complex via T duality? There are at least two methods for doing this. The first, which we shall not explore here, requires assuming a torus fibration of our manifold (as in [15]) on which we may T dualize. For this to succeed, however, we are required to modify our operators somewhat, for example, replacing $\bar{\partial}$ by $Q^- - \bar{Q}^-$ and ∂ by $Q^+ + \bar{Q}^+$. This approach closely tracks the usual SCFT analysis. The method which we shall instead explore here is to introduce a complex version of T duality which may be viewed as a restriction of T duality to objects associated to the holomorphic tangent bundle and its dual. This approach leads to constructions which are weakly reminiscent of A branes and B branes.

So, we consider an antilinear T duality replacing $\frac{\partial}{\partial z^i}$ by $ig_{\bar{i}j}ad(z^j)$, $e(dz^i)$ by $e^*(dz^i)$, leaving the conjugate variable unchanged, and extend by making $T^2 = 1$. Now consider our A and B complexes, with their coboundary operators. How can we use complex T duality to pass from one to the other? Start with

$$d = e(d\bar{z}^i)\frac{\partial}{\partial \bar{z}^i} + e(dz^i)\frac{\partial}{\partial z^i}.$$

If we T dualize following the above recipe, this becomes

$$d_T := e(d\bar{z}^i)\frac{\partial}{\partial \bar{z}^i} + ie(dz^i)^* g_{\bar{i}j}ad(z^j).$$

Now our coboundary operator for the B complex has the form (in nice coordinates)

$$P = (e(d\bar{z}^i) - be(dz^i)^*)\frac{\partial}{\partial \bar{z}^i}.$$

How can we connect these? We alter our "boundary conditions". Initially we considered scalar valued forms for our A and B complexes, but in order to define T duality, we were forced to consider forms with coefficients in partial differential operators. Our original space is recovered by imposing the analog of Neumann boundary conditions: $ad(x^i) = 0$. In general, we should expect to impose $dim_R(M)$ conditions to get a complex "as large as" the original one. (Presumably there is a holonomic condition that is the correct generalization). The usual complex satisfies $ad(z^i) = 0$ and $ad(\bar{z}^i) = 0$, for all i. Let's preserve this as much as possible. So, we assume for both complexes that $ad(\bar{z}^i) = 0$. In order for T duality to transform d to P, we must deform the first condition, $ad(\bar{z}^i) = 0$, to $ad(z^j) - ibg^{j\bar{k}}\frac{\partial}{\partial \bar{z}^k} = 0$. On this restricted subspace, $d_T = P$. Note the similarity between the modification of the condition that $ad(z^i) = 0$ and the deformation of the Neumann boundary condition induced by a B field in (5). This parallel inclines us to interpret $-ib\omega$ as a B field, except that here we are only using it to deform $ad(z^i)$ and not $ad(\bar{z}^i)$.

Thus we consider sections of the form $f(z, \frac{\partial}{\partial z^j} + ig_{j\bar{k}}\bar{z}^k/b)$, again interpreted as formal power series. These are determined uniquely by $f(z, \frac{\partial}{\partial z})$, a collection of holomorphic sections of powers of the holomorphic tangent bundle. On the other hand, if we now work backwards, the conditions that $ad(\bar{z}^i) = 0$ and $ad(z^j) - ibg^{j\bar{k}}\frac{\partial}{\partial \bar{z}^k} = 0$ are T dual to the condition (for the A complex) that $ad(\bar{z}^i) = 0$ and $\frac{\partial}{\partial z^i} - ib\frac{\partial}{\partial \bar{z}^i} = 0$. This latter condition is, of course, not coordinate independent. Because we have a sum of a complex linear and a complex anti-linear map, it is not well defined independent of a choice of complex basis. Its definition requires a choice of underlying real structure for the holomorphic tangent space $T^{1,0} = T_R^{1,0} \otimes C$, where (at a point) $T_R^{1,0}$ is a subspace spanned by the $\frac{\partial}{\partial x^i}$ in some local coordinate system with $z^i = x^i + iy^i$.

Assuming the existence of such a structure, the A complex, for say $b = 1$ consists of sections of the form

$$f(y, \frac{\partial}{\partial z}).$$

As b varies, these families rotate in an obvious fashion. Thus, the sections are determined by their values on Lagrangian submanifolds determined by this real structure and the choice of b.

11. Global Structure of the B field

11.1. Global structures associated to B fields. We have seen how a B field can alter differential operators and lead to simple noncommutative geometries. Now we will examine the global structures associated to B fields.

We adopt the notation $A \diamond ad(x^j)$ to denote the operator which sends $(\frac{\partial}{\partial x^j})^k$ to $\sum_{p=0}^{k-1} (\frac{\partial}{\partial x^j})^p A (\frac{\partial}{\partial x^j})^{k-p-1}$. In particular, for pdos of degree greater than one, $A \diamond ad(x^j)$ does not denote $ad(x^j)$ followed by multiplication on the left by A. This definition is natural if we wish to associate to a 1 form $b_j dx^j$ a derivation on the space of pdos which is independent of coordinate choice. On the other hand, this operator now distinguishes between $\frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}$ and $\frac{\partial}{\partial x^j}\frac{\partial}{\partial x^i}$. In order to remedy this, we must consider the space of differential operators with their ordering retained; i.e. we do not quotient out by the relations which hold for operators acting on the smooth functions. In fact, all of the constructions in this section may be defined on spaces of pdos of degree 1, in which case we may ignore these ordering difficulties. Hence, for the remainder of this section, we restrict to that case for simplicity. The extension to spaces of higher order pdos is immediate.

We recall that the B field has the effect of replacing the derivative $\nabla_{\frac{\partial}{\partial x^i}}$ by $\nabla_{\frac{\partial}{\partial x^i}} - b_{ij} ad(x^j)$. If b were constant, this shift could be interpreted as conjugation

$$\nabla_{\frac{\partial}{\partial x^i}} \to e^{(x^i b_{ij}) \diamond ad(x^j)} \nabla_{\frac{\partial}{\partial x^i}} e^{-(x^i b_{ij}) \diamond ad(x^j)}.$$

This conjugation corresponds to replacing a partial differential operator $a_I \partial^I$ by $e^{b_j ad(x^j)} a_I \partial^I$, where $b_j = x^i b_{ij}$, in the example at hand.

If we assume our pdos act on sections of a $U(1)$ bundle, then under a change of frame, the pdos are modified by $e^{u_j \diamond ad(x^j)}$, where $u_j dx^j = g^{-1} dg$ for some S^1 valued function g. This transforms $\frac{\partial}{\partial x^i}$ to $\frac{\partial}{\partial x^i} + u_i$, u_i acting by the adjoint action. Here we are using the equality

$$ad(u_i) = u_{i,j} \diamond ad(x^j).$$

Note that conjugation by the exponential of an arbitrary 1 form,

$$\nabla_{\frac{\partial}{\partial x^i}} \to e^{\lambda_j \diamond ad(x^j)} \nabla_{\frac{\partial}{\partial x^i}} e^{-\lambda_j \diamond ad(x^j)},$$

will modify $\nabla_{\frac{\partial}{\partial x^i}}$ by (in geodesic coordinates) $-\lambda_{j,i} \diamond dx^j$, with $\lambda_{j,i} dx^i \otimes dx^j$ not necessarily a 2 form; i.e., $\lambda_{i,j}$ need not be skew in its indices. In order for this to be a 2 form, it would be necessary that $\lambda^i \frac{\partial}{\partial x^i}$ be a Killing vector. On the other hand, as noted above,

$$ad(\lambda_i) - \lambda_{i,j} ad(x^j) = 0.$$

Hence, we see that we can interpet this conjugation as transforming

$$B \to B + d\lambda$$

if we simultaneously transform

$$\nabla_i \to \nabla_i - \lambda_i.$$

Thus, up to a sign convention, the connection transforms as expected [11][(8.7.7) and (8.78)] under a change of gauge for the B field.

We see that we can interpret b as a component of a connection form for a covariant derivative on a space of pdos with allowed transition functions of the form $e^{w_j \diamond ad(x^j)}$. It is easy to pass from this perspective to the Deligne cohomology class of a gerbe. Fix an open cover $\{U_\alpha\}$ of M. Consider collections p_α of partial differential operators on each U_α such that on $U_\alpha \cap U_\beta$, we have

$$p_\alpha = e^{w_j^{\alpha\beta} \diamond ad(x^j)} p_\beta,$$

for some collection of 1 forms $w_j^{\alpha\beta} dx^j$, satisfying the cocycle condition

$$w_j^{\alpha\beta} dx^j + w_j^{\beta\mu} dx^j + w_j^{\mu\alpha} dx^j = d\lambda^{\alpha\beta\mu}.$$

Here we should either choose

$$d\lambda^{\alpha\beta\mu} = 0,$$

or

$$d\lambda^{\alpha\beta\mu} = g_{\alpha\beta\mu}^{-1} dg_{\alpha\beta\mu},$$

for some S^1 valued cocycle $g_{\alpha\beta\mu}$. The first choice is the standard cocycle condition for transition functions of the above form. Note, however, that forcing w to satisfy $\delta w = 0$ (with δ the Cech coboundary) implies that the field strength

$$H := dB$$

is cohomologically trivial.

We may realize the less restrictive condition $\delta w = d\lambda$ for nonzero $d\lambda$ if we modify our construction to consider connections not on pdos but on gauge equivalence classes of pdos. The action of $ad(\lambda_i)$ is trivial on gauge equivalence classes, and therefore the background gauge field can be dropped from the theory. One does not expect coupling of the B field to a connection in a closed string theory; hence, the gauge equivalence class representation should correspond to a closed string theory, while the representation with $\delta w = 0$ should correspond to an open string theory, where one expects a coupling to a gauge field. In our model then, we see a cohomological obstruction to lifting a B field background from closed to open string theory. The field strength H must be cohomologically trivial for a closed string B field to lift to an open string theory.

Let us consider what data we have now assembled in order to view the B field as a connection on the space of pdos (or gauge equivalence classes of pdos). On each open set U_α we have a 2 form b^α, on the intersection of two open sets $U_\alpha \cap U_\beta$, we have a 1 form $w^{\alpha\beta}$, and on each triple intersection, $U_\alpha \cap U_\beta \cap U_\mu$, we have an S^1 valued function $g_{\alpha\beta\mu}$. Let δ denote the Cech coboundary operator. Then this collection satisfies

$$\delta b = dw, \ \delta w = g^{-1} dg, \ \text{and } \delta g = 0.$$

Thus, our B field, as numerous authors have predicted (see for example [6] and [9]), defines a Deligne cohomology class

$$(g, w, b) \in H^2(M, D^2),$$

in the notation of [3].

As $e^{b_j \diamond ad(x^j)} \partial_i = \partial_i + b_i,$ we may interpret this change of gauge as a change of connection. We note that it is the shift that follows from tensoring a bundle with a line bundle with connection $\nabla = d + b.$ Hence, we may interpret the transition functions associated to our B field construction as corresponding to a line bundle with connection on the intersection of each coordinate patch, as in Hitchin's picture of gerbes (see [7]).

We compute the curvature of the connection

$$\nabla_{\frac{\partial}{\partial x^i}} - b_{ij} ad(x^j).$$

Write $\nabla_{\frac{\partial}{\partial x^i}} = \frac{\partial}{\partial x^i} + A_i$ in a local frame. Then the curvature is given by

$$dx^i \wedge dx^k [\frac{\partial}{\partial x^i} + A_i - b_{ij} ad(x^j), \frac{\partial}{\partial x^k} + A_k - b_{kp} ad(x^p)] =$$

$$dx^i \wedge dx^k [A_{k,i} - A_{i,k} - (b_{kp,i} + b_{pi,k}) ad(x^p)] =$$

$$dx^i \wedge dx^k [A_{k,i} - A_{i,k} + b_{ik,p} ad(x^p) - db_{ikp} ad(x^p)].$$

Using

$$b_{ik,p} ad(x^p) = ad(b_{ik}),$$

and setting

$$\tilde{F} = dA + b,$$

we see that the curvature of our connection can be written

$$\tilde{F} - i_p(db) ad(x^p).$$

Here i_p denotes interior multiplication by $\frac{\partial}{\partial x^p}$. Note that here $(dA + b)_{ik}$ acts on a space of partial differential operators by the adjoint action. Observe that the combination $\tilde{F} = dA + b$ is invariant under both the gauge symmetries of the underlying connection and the connection on the space of pdos.

11.2. **Higher order gerbes.** The construction of B fields as connections on spaces of pdos can now be iterated. For example, we get a three form connection if we consider connections on the B field connections.

A general B field connection locally has the form

$$\nabla_i = \partial_i + \lambda_i - b_{ik} \diamond ad(x^k).$$

We define a new ad operator ad_2 which acts on these connections by

$$ad_2(x^i)(\nabla_k) = -\delta^i_k.$$

Then the arbitrary change of λ_i and b_{ik} is given by

$$e^{(P_j + L_{jk} \diamond ad(x^k)) \diamond_2 ad_2(x^j)},$$

with L a 2 form and $\diamond_2$ the obvious analog of $\diamond$. We will view such transformations as a change of frame on our space of B field connections.

A connection on the space of B field connections has the local form

$$\nabla_i^B = \partial_i + \lambda_i + b_{ik} \diamond ad(x^k) - W_{ij} \diamond_2 ad_2(x^j) - c_{ijk} \diamond ad(x^j) \diamond_2 ad(x^k).$$

Here λ_i and $b_{ik} \diamond ad(x^k)$ act by the adjoint action. A change of frame $e^{(P_j + L_{jk} \diamond ad(x^k)) \diamond_2 ad_2(x^j)}$ now induces a change of gauge

$$\nabla_i^B \to \partial_i + \lambda_i + b_{ik} \diamond ad(x^k) - W_{ij} \diamond_2 ad_2(x^j) - c_{ijk} \diamond ad(x^j) \diamond_2 ad_2(x^k)$$

$$- (P_{j,i} + L_{jk,i} \diamond ad(x^k)) \diamond_2 ad_2(x^j) =$$

$$\partial_i + \lambda_i - P_i + (b-L)_{ik} \diamond ad(x^k) - (W - dP)_{ij} \diamond_2 ad_2(x^j) - (c + dL)_{ijk} \diamond ad(x^j) \diamond_2 ad_2(x^k)$$

$$+ L_{ij,k} \diamond ad(x^k) \diamond_2 ad_2(x^j).$$

Recall that $L_{ij,k} \diamond ad(x^k) = ad(L_{ij})$. Hence we can rewrite the connection in the new gauge as

$$\partial_i + \lambda_i - P_i + (b - L)_{ik} \diamond ad(x^k) - (W - dP - L)_{ij} \diamond_2 ad_2(x^j)$$

$$- (c + dL)_{ijk} \diamond ad(x^j) \diamond_2 ad_2(x^k).$$

Thus we have a gauge transformation which transforms the three form c to $c + dL$ and the two form $b - W$ to $b - W + dP$, while sending λ_i to $\lambda_i - P_i$ and $(b + W)/2$ to $(b + W)/2 - L$. This appears to have a somewhat different field content from what we expect for type *IIA* supergravity, where three form potentials such as c arise. (See [12]Chapter 12). In subsequent work, we shall explore an alternate construction of a quantum mechanical model for Ramond-Ramond fields.

We may now consider various choices of representation: connections or gauge equivalence classes of connection, gauge equivalence classes of B field connection, etc. One can iterate such constructions ad nauseum.

11.3. Non-abelian gauge groups.

It is natural to ask what happens if we allow our connections to take values in a nonabelian group. So, we consider a "gauge transformation" $\nabla_{\frac{\partial}{\partial x^k}} \to \nabla_{\frac{\partial}{\partial x^k}} + \lambda_k$. Here λ_k is endomorphism valued.

$$ad(\nabla_{\frac{\partial}{\partial x^i}} - b_{ij} ad(x^j)) \nabla_{\frac{\partial}{\partial x^k}} = F_{ik} + b_{ik},$$

but $ad(\nabla_{\frac{\partial}{\partial x^i}} - b_{ij} ad(x^j))(\nabla_{\frac{\partial}{\partial x^k}} + \lambda_k) = F_{ik} + \lambda_{k,i} + b_{ik}$. Hence, as before, we must modify $ad(\nabla_{\frac{\partial}{\partial x^i}} - b_{ij} ad(x^j))$. We replace $\nabla_{\frac{\partial}{\partial x^i}}$ by $\nabla_{\frac{\partial}{\partial x^i}} - \lambda_i$. We expect that we must replace b by $b + d\lambda$, but now there is some choice as to which connection we use to compute $d\lambda$, d or $d + \lambda$. If we choose the average $d_{\lambda/2} := d + \lambda/2$, then we get

$$ad(\nabla_{\frac{\partial}{\partial x^i}} + \lambda_i - (b + d_{\lambda/2}\lambda)_{ij} ad(x^j))(\nabla_{\frac{\partial}{\partial x^k}} + \lambda_k) = F_{ik} + b_{ik},$$

as desired.

This extension of gerbes does not look suitable for string theory. In particular, it is no longer clear how to decouple the B field from the connection for closed strings. If we assume, however, that the B field takes values in the center, C, of the gauge group then we may pass to a closed string theory if we consider $C-$ gauge equivalence classes of pdos.

12. Acknowledgements

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