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Supermatrix models for M-theory based on $\mathfrak{osp}(1|32,\mathbb{R})$

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Abstract

Taking seriously the hypothesis that the full symmetry algebra of M-theory is $\mathfrak{osp}(1|32,\mathbb{R})$, we derive the supersymmetry transformations for all fields that appear in 11- and 12-dimensional realizations and give the associated SUSY algebras. We study the background-independent $\mathfrak{osp}(1|32,\mathbb{R})$ cubic matrix model action expressed in terms of representations of the Lorentz groups $SO(10,2)$ and $SO(10,1)$. We explore further the 11-dimensional case and compute an effective action for the BFSS-like degrees of freedom. We find the usual BFSS action with additional terms incorporating couplings to transverse 5-branes, as well as a mass-term and an infinite tower of higher-order interactions.

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1 Introduction

M-theory [1] should eventually provide a unifying framework for non-perturbative string theory. While there is lot of compelling evidence for this underlying M-theory, it is still a rather elusive theory, lacking a satisfactory intrinsic formulation. It is probably the matrix model by Banks, Fischler, Shenker and Susskind (BFSS) [2] which still comes closest to this goal. In the absence of a microscopic description, quite some information can be obtained by simply looking at the eleven-dimensional superalgebra [3] whose central charges correspond to the extended objects, i.e. membranes and five-branes present in M-theory. Relations with the hidden symmetries of eleven-dimensional supergravity [4] and its compactifications and associated BPS configurations (see e.g. [5, 6] and references therein) underlined further the importance of the algebraic aspects. It has been conjectured [7] that the large superalgebra $\mathfrak{osp}(1|32)$ may play an important and maybe unifying rôle in M and F theory [8].

The hidden symmetries of the $11D$ supergravity action points to a non-linearly realized Lorentzian Kac-Moody algebra $\mathfrak{e}_{11}$, whose supersymmetric extension contains $\mathfrak{osp}(1|32)$ as a finite-dimensional subalgebra. It would be interesting to investigate further the relationships between those two aspects of the symmetries underlying M-theory.

In this paper, we have chosen to explore further the possible unifying rôle of $\mathfrak{osp}(1|32)$ and study its implications for matrix models. One of our main motivations is to investigate the dynamics of extended objects such as membranes and five-branes, when they are treated on the same footing as the "elementary" degrees of freedom. In order to see eleven and twelve-dimensional structures emerge, we have to embed the $SO(10,2)$ Lorentz algebra and the $SO(10,1)$ Poincaré algebra into the large $\mathfrak{osp}(1|32)$ superalgebra. This will yield certain deformations and extensions of these algebras which nicely include new symmetry generators related to the charges of the extended objects appearing in the eleven and twelve-dimensional theories. The supersymmetry transformations of the associated fields also appear naturally.

Besides these algebraic aspects, we are interested in the dynamics arising from matrix models derived from such algebras. Following ideas initially advocated by Smolin [9], we start with matrices $M \in \mathfrak{osp}(1|32)$ as basic dynamical objects, write down a very simple action for them and then decompose the result according to the different representations of the eleven and twelve-dimensional algebras. In the eleven-dimensional case, we expect this action to contain the scalars X_i of the BFSS matrix model and the associated fermions together with five-branes. In ten dimensions, cubic supermatrix models have already been studied by Azuma, Iso, Kawai and Ohwashi [10] (more details can be found in Azuma's master thesis [11]) in an attempt to compare it with the IIB matrix model of Ishibashi, Kawai, Kitazawa and Tsuchiya [12].

To test the relevance of our model, we try to exhibit its relations with the BFSS matrix model. For this purpose, we perform a boost to the infinite momentum frame (IMF), thus reducing the explicit symmetry of the action to $SO(9)$. Then, we integrate out conjugate momenta and auxiliary fields and calculate an effective action for the scalars X_i, the associated fermions, and higher form fields. What we obtain in the end is the BFSS matrix model with additional terms. In particular, our effective action explicitly contains couplings to 5-brane degrees of freedom, which are thus naturally incorporated in our model as fully dynamical entities. Moreover, we also get additional interactions and masslike terms. This should not be too surprising since we started with a larger theory. The interaction terms we obtain are somewhat similar to the higher-dimensional operators one expects when integrating out (massive) fields in quantum field theory. This can be viewed as an extension of the BFSS theory describing M-theoretical physics in certain non-Minkowskian backgrounds.

The outline of this paper is the following: in the next section we begin by recalling the form of the $\mathfrak{osp}(1|32)$ algebra and the decomposition of its matrices. In section 3 and 4, we study the embedding of the twelve-, resp. eleven-dimensional superalgebras into $\mathfrak{osp}(1|32)$, and obtain the corresponding algebraic structure including the extended objects described by a six- resp. five-form. We establish the supersymmetry transformations of the fields, and write down a cubic matrix model which yields an action for the various twelve- resp. eleven-dimensional fields. Finally, in section 5, we study further the eleven-dimensional matrix model, compute an effective action and do the comparison with the BFSS model.

2 The $\mathfrak{osp}(1|32,\mathbb{R})$ superalgebra

We first recall some definitions and properties of the unifying superalgebra $\mathfrak{osp}(1|32,\mathbb{R})$ which will be useful in the following chapters. The superalgebra is defined by the following three equations:

$$\begin{aligned}
[Z_{AB}, Z_{CD}] &= \Omega_{AD} Z_{CB} + \Omega_{AC} Z_{DB} + \Omega_{BD} Z_{CA} + \Omega_{BC} Z_{DA} \ , \\
[Z_{AB}, Q_C] &= \Omega_{AC} Q_B + \Omega_{BC} Q_A \ , \\
\{Q_A, Q_B\} &= Z_{AB} \ ,
\end{aligned} \tag{1}$$

where Ω_{AB} is the antisymmetric matrix defining the $\mathfrak{sp}(32,\mathbb{R})$ symplectic Lie algebra. Let us now give an equivalent description of elements of $\mathfrak{osp}(1|32,\mathbb{R})$. Following Cornwell [13], we call $\mathbb{R}B_L$ the real Grassmann algebra with L generators, and $\mathbb{R}B_{L0}$ and $\mathbb{R}B_{L1}$ its even and odd subspace respectively. Similarly, we define a $(p|q)$ supermatrix to be even (degree 0) if it can be written as:

$$M = \begin{pmatrix} A & B \\ F & D \end{pmatrix} \ .$$

where A and D are $p \times p$, resp. $q \times q$ matrices with entries in $\mathbb{R}B_{L0}$, while B and F are $p \times q$ (resp. $q \times p$) matrices, with entries in $\mathbb{R}B_{L1}$. On the other hand, odd supermatrices (degree 1) are characterized by 4 blocks with the opposite parities.

We define the supertranspose of a supermatrix M as[1]:

$$M^{ST} = \begin{pmatrix} A^\top & (-1)^{deg(M)} F^\top \\ -(-1)^{deg(M)} B^\top & D^\top \end{pmatrix} \ .$$

If one chooses the orthosymplectic metric to be the following 33×33 matrix:

$$G = \begin{pmatrix} 0 & -1\!\!1_{16} & 0 \\ 1\!\!1_{16} & 0 & 0 \\ 0 & 0 & i \end{pmatrix} \ ,$$

(where the i is chosen for later convenience to yield a hermitian action), we can define the $\mathfrak{osp}(1|32,\mathbb{R})$ superalgebra as the algebra of $(32|1)$ supermatrices M satisfying the equation:

$$M^{ST} \cdot G + (-1)^{deg(Z)} G \cdot M = 0 \ .$$

[1]We warn the reader that this is not the same convention as in [11].

From this defining relation, it is easy to see that an even orthosymplectic matrix should be of the form:

$$M = \begin{pmatrix} A & B & \Phi_1 \\ F & -A^\top & \Phi_2 \\ -i\Phi_2^\top & i\Phi_1^\top & 0 \end{pmatrix} = \begin{pmatrix} m & \Psi \\ -i\Psi^\top C & 0 \end{pmatrix} , \tag{2}$$

where A,B and F are 16×16 matrices with entries in $\mathbb{R}B_{L0}$ and $\Psi = (\Phi_1, \Phi_2)^\top$ is a 32-components Majorana spinors with entries in $\mathbb{R}B_{L1}$. Furthermore, $B = B^\top$, $F = F^\top$ so that $m \in \mathfrak{sp}(32, \mathbb{R})$ and C is the following 32×32 matrix:

$$C = \begin{pmatrix} 0 & -\mathbb{1}_{16} \\ \mathbb{1}_{16} & 0 \end{pmatrix} , \tag{3}$$

and will turn out to act as the charge conjugation matrix later on.

Such a matrix in the Lie superalgebra $\mathfrak{osp}(1|32, \mathbb{R})$ can also be regarded as a linear combination of the generators thereof, which we decompose in a bosonic and a fermionic part as:

$$H = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \chi \\ -i\chi^\top C & 0 \end{pmatrix} = h^{AB} Z_{AB} + \chi^A Q_A \tag{4}$$

where Z_{AB} and Q_A are the same as in (1). An orthosymplectic transformation will then act as:

$$\delta_H^{(1)} = [H, \bullet] = h^{AB}[Z_{AB}, \bullet] + \chi^A [Q_A, \bullet] = \delta_h^{(1)} + \delta_\chi^{(1)} . \tag{5}$$

This notation allows us to compute the commutation relations of two orthosymplectic transformations characterized by $H = (h, \chi)$ and $E = (e, \epsilon)$. Recalling that for Majorana fermions $\chi^\top C \epsilon = \epsilon^\top C \chi$, we can extract from $[\delta_H^{(1)}, \delta_E^{(1)}]$ the commutation relation of two symplectic transformations:

$$[\delta_h^{(1)}, \delta_e^{(1)}]_A{}^B = \begin{pmatrix} [h, e]_A{}^B & 0 \\ 0 & 0 \end{pmatrix} , \tag{6}$$

the commutation relation between a symplectic transformation and a supersymmetry:

$$[\delta_h^{(1)}, \delta_\chi^{(1)}]_A{}^B = \begin{pmatrix} 0 & h_A{}^D \chi_D \\ i(\chi^\top C)^D h_D{}^B & 0 \end{pmatrix} , \tag{7}$$

and the commutator of two supersymmetries:

$$[\delta_\epsilon^{(1)}, \delta_\chi^{(1)}]_A{}^B = \begin{pmatrix} -i(\chi_A (\epsilon^\top C)^B - \epsilon_A (\chi^\top C)^B) & 0 \\ 0 & 0 \end{pmatrix} . \tag{8}$$

3 The 12-dimensional case

In order to be embedded into $\mathfrak{osp}(1|32, \mathbb{R})$, a Lorentz algebra must have a fermionic representation of 32 real components at most. The biggest number of dimensions in which this is the case is 12, where Dirac matrices are 64×64. As this dimension is even, there exists a Weyl representation of 32 complex components. We need furthermore a Majorana condition to make them real. This depends of course on the signature of space-time and is possible only for signatures $(10, 2)$, $(6, 6)$ and $(2, 10)$, when (s,t) are such that $s-t=0 \mod 8$. Let us concentrate in this paper on the most physical case (possibly relevant for F-theory) where the number of timelike dimensions is 2. However, since we choose to concentrate on the next section's M-theoretical case, we will not push this analysis too far and will thus restrict ourselves to the computation of the algebra and the cubic action.

To express the $\mathfrak{osp}(1|32,\mathbb{R})$ superalgebra in terms of 12-dimensional objects, we have to embed the $SO(10,2)$ Dirac matrices into $\mathfrak{sp}(32,\mathbb{R})$ and replace the fundamental representation of $\mathfrak{sp}(32,\mathbb{R})$ by $SO(10,2)$ Majorana-Weyl spinors. A convenient choice of 64×64 Gamma matrices is the following:

$$\Gamma^0=\begin{pmatrix}0 & -\mathbb{I}_{32}\\ \mathbb{I}_{32} & 0\end{pmatrix},\ \Gamma^{11}=\begin{pmatrix}0 & \tilde{\Gamma}^0\\ \tilde{\Gamma}^0 & 0\end{pmatrix},\ \Gamma^i=\begin{pmatrix}0 & \tilde{\Gamma}^i\\ \tilde{\Gamma}^i & 0\end{pmatrix}\ \ \forall\, i=1,\ldots,10, \tag{9}$$

where $\tilde{\Gamma}^0$ is the 32×32 symplectic form:

$$\tilde{\Gamma}^0=\begin{pmatrix}0 & -\mathbb{I}_{16}\\ \mathbb{I}_{16} & 0\end{pmatrix}$$

which, with the $\tilde{\Gamma}^i$'s and $\tilde{\Gamma}^{10}$, builds a Majorana representation of the $10+1$-dimensional Clifford algebra $\{\tilde{\Gamma}^\mu,\tilde{\Gamma}^\nu\}=2\eta^{\mu\nu}\mathbb{I}_{32}$ for the mostly + metric . Of course, $\tilde{\Gamma}^{10}=\tilde{\Gamma}^0\tilde{\Gamma}^1\ldots\tilde{\Gamma}^9$. This choice has $(\Gamma^0)^2=(\Gamma^{11})^2=-\mathbb{I}_{64}$, while $(\Gamma^i)^2=\mathbb{I}_{64}$, $\forall i=1\ldots 10$, and gives a representation of $\{\Gamma^M,\Gamma^N\}=2\eta^{MN}\mathbb{I}_{64}$ for a metric of the type $(-,+,\ldots,+,-)$. As we have chosen all Γ's to be real, this allows to take $B=\mathbb{I}$ in $\Psi^*=B\Psi$, which implies that the charge conjugation matrix $C=\Gamma^0\Gamma^{11}$, i.e.

$$C=\begin{pmatrix}-\tilde{\Gamma}^0 & 0\\ 0 & \tilde{\Gamma}^0\end{pmatrix}\ .$$

This will then automatically satisfy:

$$C\Gamma^M C^{-1}=(\Gamma^M)^\top\ ,\ C\Gamma^{MN}C^{-1}=-(\Gamma^{MN})^\top \tag{10}$$

and more generally:

$$C\Gamma^{M_1\ldots M_n}C^{-1}=(-1)^{n(n-1)/2}(\Gamma^{M_1\ldots M_n})^\top\ . \tag{11}$$

The chirality matrix for this choice will be:

$$\Gamma_*=\Gamma^0\ldots\Gamma^{11}=\begin{pmatrix}-\mathbb{I}_{32} & 0\\ 0 & \mathbb{I}_{32}\end{pmatrix}\ .$$

We will identify the fundamental representation of $\mathfrak{sp}(32,\mathbb{R})$ with positive chirality Majorana-Weyl spinors of $SO(10,2)$, i.e. those satisfying: $\mathcal{P}_+\Psi=\Psi$, for:

$$\mathcal{P}_+=\frac{1}{2}(1+\Gamma_*)=\begin{pmatrix}0 & 0\\ 0 & \mathbb{I}_{32}\end{pmatrix}\ .$$

Decomposing the 64 real components of the positive chirality spinor Ψ into $32+32$ or $16+16+16+16$, we can write: $\Psi^\top=(0,\Phi^\top)=(0,0,\Phi_1^\top,\Phi_2^\top)$. Because $\overline{\Psi}=\Psi^\dagger\Gamma^0\Gamma^{11}=\Psi^\top C$, this choice for the charge conjugation matrix C is convenient since it will act as C in equation (3) (though with a slight abuse of notation), and thus:

$$(0,0,-i\Phi_2^\top,i\Phi_1^\top)=(0,-i\Phi^\top\tilde{\Gamma}^0)=-i\Psi^\top C=-i\overline{\Psi}.$$

3.1 Embedding of $SO(10,2)$ in $OSp(1|32,\mathbb{R})$

We would now like to study how the Lie superalgebra of $OSp(1|32,\mathbb{R})$ can be expressed in terms of generators of the Super-Lorentz algebra in 10+2 dimensions with additional symmetry generators. In other words, if we separate the $\mathfrak{sp}(32,\mathbb{R})$ transformations h into a part sitting in the Lorentz algebra and a residual $\mathfrak{sp}(32,\mathbb{R})$ part, we can give an explicit description of this enhanced super-Poincaré algebra (1) where we promote the former central charges to new generators of the enhanced superalgebra.

To do so, we need to expand a symplectic matrix in irreducible tensors of $SO(10,2)$. This can be done as follows:

$$h_A{}^B = \frac{1}{2!}(\mathcal{P}_+\Gamma^{MN})_A{}^B h_{MN} + \frac{1}{6!}(\mathcal{P}_+\Gamma^{M_1\ldots M_6})_A{}^B h^+_{M_1\ldots M_6} \tag{12}$$

where the $+$ on $h_{M_1\ldots M_6}$ recalls its self-duality, and the components of h in the decomposition in irreducible tensors of $SO(10,2)$ are given by $h_{MN} = -\frac{1}{32}Tr_{\mathfrak{sp}(32,\mathbb{R})}(h\Gamma_{MN})$ and $h^+_{M_1\ldots M_6} = -\frac{1}{32}Tr_{\mathfrak{sp}(32,\mathbb{R})}(h\Gamma_{M_1\ldots M_6})$. Indeed, a real symplectic 32×32 matrix satisfies $m\tilde{\Gamma}^0 = -\tilde{\Gamma}^0 m^\top$, and C acts like $\tilde{\Gamma}^0$ on $\mathcal{P}_+\Gamma^{M_1\ldots M_n}$. Furthermore, (11) indicates that:

$$C(1+\Gamma_*)\Gamma^{M_1\ldots M_n} = (-1)^{n(n-1)/2}((1+(-1)^n\Gamma_*)\Gamma^{M_1\ldots M_n})^T C\ . \tag{13}$$

Thus, $\mathcal{P}_+\Gamma^{M_1\ldots M_n}$ is symplectic iff n is even and $(-1)^{n(n-1)/2} = -1$. For $0 \le n \le 6$, this is only the case if $n=2$ or 6. As a matter of fact, the numbers of independent components match since: $12\cdot 11/2 + 1/2\cdot 12!/(6!)^2 = 528 = 16\cdot 33$.

The symplectic transformation δ_h may then be decomposed into irreducible 12-dimensional tensors of symmetry generators, namely the $\mathfrak{so}(10,2)$ Lorentz algebra generator J^{MN} and a new 6-form symmetry generator $J^{M_1\ldots M_6}$. To calculate the commutation relations of this enhanced Lorentz algebra, we will choose the following representation of the symmetry generators:

$$J^{MN} = \frac{1}{2!}\,\mathcal{P}_+\,\Gamma^{MN}\,,\qquad J^{M_1\ldots M_6} = \frac{1}{6!}\,\mathcal{P}_+\,\Gamma^{M_1\ldots M_6}\ .$$

so that a symplectic transformation will be given in this base by:

$$h = h_{MN}\,J^{MN} + h_{M_1\ldots M_6}\,J^{M_1\ldots M_6}\ .$$

We will now turn to computing the superalgebra induced by the above bosonic generators and the supercharges for $D = 10+2$. The bosonic commutators may readily be computed using:

$$[\Gamma_{M_1\ldots M_k},\Gamma_{N_1\ldots N_l}] = \begin{cases} \displaystyle\sum_{j=0}^{\lfloor(min(k,l)-1)/2\rfloor}(-1)^{k-j-1}\,2\cdot(2j+1)!\binom{k}{2j+1}\binom{l}{2j+1}\times \\ \qquad \times\eta_{M_1N_1}\ldots\eta_{M_{2j+1}N_{2j+1}}\Gamma_{M_{2j+2}N_{2j+2}\ldots M_kN_l} \text{ if } k\cdot l \text{ is even and,} \\ \displaystyle\sum_{j=0}^{(min(k,l)-1)/2}(-1)^j\,2\cdot(2j)!\binom{k}{2j}\binom{l}{2j}\times \\ \qquad \times\eta_{M_1N_1}\ldots\eta_{M_{2j}N_{2j}}\Gamma_{M_{2j+1}N_{2j+1}\ldots M_kN_l} \qquad \text{if } k\cdot l \text{ is odd.} \end{cases} \tag{14}$$

On the other hand, for the commutation relations involving fermionic generators, we proceed as follows. We expand equation (7) of the preceding chapter in irreducible tensors of $SO(10,2)$:

$$[\delta_\chi, \delta_h] \;=\; -\frac{1}{2!}\,\chi^A\, h_{MN}(\mathcal{P}_+\,\Gamma^{MN})^B{}_A Q_B - \frac{1}{6!}\,\chi^A\, h_{M_1\ldots M_6}(\mathcal{P}_+\,\Gamma^{M_1\ldots M_6})^B{}_A Q_B\;,$$

which is also given by:

$$[\delta_\chi, \delta_h] \;=\; \chi^A\, h_{MN}[Q_A, J^{MN}] + \chi^A\, h_{M_1\ldots M_6}[Q_A, J^{M_1\ldots M_6}]\;. \tag{15}$$

Comparing terms pairwise, we see that the supercharges transform as:

$$[J^{MN}, Q_A] \;=\; \frac{1}{2!}\,(\mathcal{P}_+\,\Gamma^{MN})^B{}_A Q_B\;, \qquad [J^{M_1\ldots M_6}, Q_A] \;=\; \frac{1}{6!}\,(\mathcal{P}_+\,J^{M_1\ldots M_6})^B{}_A Q_B\;.$$

Finally, in order to obtain the anti-commutator of two supercharges, we expand the RHS of (8) in the bosonic generators J^{MN} and $J^{M_1\ldots M_6}$:

$$-\chi^A\epsilon_B\{Q_A, Q^B\} \;\equiv\; [\delta_\chi, \delta_\epsilon] \;=\; \frac{i}{16}(\chi^\top C\Gamma_{MN}\epsilon)J^{MN} + \frac{i}{16}(\chi^\top C\Gamma_{M_1\ldots M_6}\epsilon)J^{M_1\ldots M_6}\;, \tag{16}$$

and match the first and the last term of the equation.

Summarizing the results of this section, we get the following 12-dimensional realization of the superalgebra $\mathfrak{osp}(1|32,\mathbb{R})$[2]:

$$\begin{aligned}
[J^{MN}, J^{OP}] \;&=\; -4\eta^{[M[O}J^{N]P]} \\
[J^{MN}, J^{M_1\ldots M_6}] \;&=\; -12\,\eta^{[M[M_1}\,J^{N]M_2\ldots M_6]} \\
[J^{N_1\ldots N_6}, J^{M_1\ldots M_6}] \;&=\; -4!\,6!\,\eta^{[N_1[M_1}\,\eta^{N_2\,M_2}\,\eta^{N_3\,M_3}\,\eta^{N_4\,M_4}\,\eta^{N_5\,M_5}\,J^{N_6]M_6]} \\
&\quad+\;2\cdot 6^2\,\eta^{[N_1[M_1}\,\varepsilon^{N_2\ldots N_6]M_2\ldots M_6]}{}_{AB}\,J^{AB} \\
&\quad+\;4\left(\frac{6!}{4!}\right)^3\,\eta^{[N_1[M_1}\,\eta^{N_2\,M_2}\,\eta^{N_3\,M_3}\,J^{N_4\ldots N_6]M_4\ldots M_6]}
\end{aligned} \tag{17}$$

$$\begin{aligned}
[J^{MN}, Q_A] \;&=\; \frac{1}{2}\,(\mathcal{P}_+\,\Gamma^{MN})^B{}_A Q_B \\
[J^{M_1\ldots M_6}, Q_A] \;&=\; \frac{1}{6!}\,(\mathcal{P}_+\,\Gamma^{M_1\ldots M_6})^B{}_A Q_B \\
\{Q_A, Q^B\} \;&=\; -\frac{i}{16}(C\Gamma_{MN})_A{}^B J^{MN} - \frac{i}{16}(C\Gamma_{M_1\ldots M_6})_A{}^B J^{M_1\ldots M_6}\;,
\end{aligned}$$

where antisymmetrization brackets on the RHS are meant to match the anti-symmetry of indices on the LHS.

[2] Notice that the second term appearing on the right handside of the third commutator is in fact proportional to $\Gamma^{M_1\ldots M_{10}}$, which, in turn, can be reexpressed as $\Gamma^{M_1\ldots M_{10}} = (1/2)\varepsilon^{AB\,M_1\ldots M_{10}}\,\Gamma_{AB}\Gamma_*$. Indeed, in $10+2$ dimensions, we always have:

$$\Gamma^{M_1\ldots M_k} = \frac{1}{(12-k)!}\varepsilon^{M_1\ldots M_k M_{k+1}\ldots M_{12}}\,\Gamma_{M_{k+1}\ldots M_{12}}\Gamma_*$$

3.2 Supersymmetry transformations of 12D matrix fields

In the following, we will construct a dynamical matrix model based on the symmetry group $\mathfrak{osp}(1|32,\mathbb{R})$ using elements in the adjoint representation of this superalgebra, i.e. matrices in this superalgebra. We can write such a matrix as:

$$M = \begin{pmatrix} m & \Psi \\ -i\Psi^\top C & 0 \end{pmatrix} , \tag{18}$$

where m is in the adjoint representation of $\mathfrak{sp}(32,\mathbb{R})$ and Ψ is in the fundamental. Since M belongs to the adjoint representation, a SUSY will act on it in the following way:

$$\delta^{(1)}_\chi M_A{}^B = \chi^D [Q_D, M]_A{}^B = \begin{pmatrix} -i(\chi_A(\Psi^\top C)^B - \Psi_A(\chi^\top C)^B) & -m_A{}^D \chi_D \\ -i(\chi^\top C)^D m_D{}^B & 0 \end{pmatrix} \tag{19}$$

In our particular 12D setting, m gives rise to a 2-form field C (with $SO(10,2)$ indices, not to be confused with the charge conjugation matrix with $\mathfrak{sp}(32,\mathbb{R})$ indices) and a self-dual 6-form field Z^+, as follows:

$$m_A{}^B = \frac{1}{2!}(\mathcal{P}_+\Gamma^{MN})_A{}^B C_{MN} + \frac{1}{6!}(\mathcal{P}_+\Gamma^{M_1\ldots M_6})_A{}^B Z^+_{M_1\ldots M_6} \, . \tag{20}$$

We can extract the supersymmetry transformations of C, Z^+ and Ψ from (19) and we obtain:

$$\begin{aligned} \delta^{(1)}_\chi C_{MN} &= \frac{i}{16}\overline{\chi}\,\Gamma_{MN}\Psi \, , \\ \delta^{(1)}_\chi Z^+_{M_1\ldots M_6} &= \frac{i}{16}\overline{\chi}\,\Gamma_{M_1\ldots M_6}\Psi \, , \\ \delta^{(1)}_\chi \Psi &= -\frac{1}{2}\Gamma^{MN}\chi\, C_{MN} - \frac{1}{6!}\Gamma^{M_1\ldots M_6}\chi\, Z^+_{M_1\ldots M_6} \, . \end{aligned} \tag{21}$$

These formulæ allow us to compute the effect of two successive supersymmetry transformations using (11) and (14):

$$\begin{aligned} [\delta^{(1)}_\chi, \delta^{(1)}_\epsilon]\Psi &= \frac{i}{16}\Big\{(\overline{\epsilon}\Psi)\,\chi - (\overline{\chi}\,\Psi)\,\epsilon\Big\} \, , \\ [\delta^{(1)}_\chi, \delta^{(1)}_\epsilon] C_{MN} &= \frac{i}{4}\overline{\chi}\left\{\Gamma_{[M}{}^P C_{N]P} + \frac{1}{5!}\Gamma_{[M}{}^{M_1\ldots M_5} Z^+_{N]M_1\ldots M_5}\right\}\mathcal{P}_+\,\epsilon \, , \\ [\delta^{(1)}_\chi, \delta^{(1)}_\epsilon] Z^+_{M_1\ldots M_6} &= \overline{\chi}\left\{\frac{3i}{4}\Gamma_{[M_1\ldots M_5}{}^N C_{M_6]N} + \frac{3i}{2}\Gamma_{[M_1}{}^N Z^+_{M_2\ldots M_6]N} - \right. \\ &\quad \left. - \frac{5i}{12}\Gamma_{[M_1 M_2 M_3}{}^{N_1 N_2 N_3} Z^+_{M_4 M_5 M_6]N_1 N_2 N_3}\right\}\mathcal{P}_+\,\epsilon \, , \end{aligned} \tag{22}$$

where we used the self-duality[3] of Z^+. At this stage, we can mention that the above results are in perfect agreement with the adjoint representation of $[\delta^{(1)}_\chi, \delta^{(1)}_\epsilon]$ (viz. (8)) on the matrix fields.

[3] Z^+ satisfies $Z^+_{M_1\ldots M_6} = \frac{1}{6!}\varepsilon_{M_1\ldots M_6}{}^{N_1\ldots N_6} Z^+_{N_1\ldots N_6}$

3.3 $\mathfrak{sp}(32,\mathbb{R})$ transformations of the fields and their commutation relation with supersymmetries

To see under which transformations an $\mathfrak{osp}(1|32,\mathbb{R})$-based matrix model should be invariant, one should look at the full transformation properties including the bosonic $\mathfrak{sp}(32,\mathbb{R})$ transformations. In close analogy with equation (19), we have the following full transformation law of M:

$$\delta_H^{(1)} M_A{}^B = \left[\begin{pmatrix} h & \chi \\ -i\overline{\chi} & 0 \end{pmatrix}, \begin{pmatrix} m & \Psi \\ -i\overline{\Psi} & 0 \end{pmatrix}\right]_A{}^B , \tag{23}$$

implying the following transformation rules:

$$\delta_H^{(1)} m_A{}^B = [h,m]_A{}^B - i(\chi_A\overline{\Psi}^B - \Psi_A\overline{\chi}^B) , \tag{24}$$

$$\delta_H^{(1)} \Psi_A = h_A{}^C \Psi_C - m_A{}^C \chi_C . \tag{25}$$

We then want to extract from the first of the above equations the full transformation properties of C_{MN} and $Z^+_{M_1\dots M_6}$. From (17) and (22) or directly using (14) and the cyclicity of the trace, the bosonic transformations are:

$$\begin{aligned} \delta_h^{(1)} C_{MN} &= 4h^P{}_{[N} C_{M]P} + \frac{4}{5!} h^{N_1\dots N_5}{}_{[N} Z^+_{M]N_1\dots N_5} , \\ \delta_h^{(1)} Z^+_{M_1\dots M_6} &= 12 h_{[M_1\dots M_5}{}^P C_{M_6]P} - 24\, h^N{}_{[M_1} Z^+_{M_2\dots M_6]N} - \\ &\quad + \frac{20}{3} h^{N_1N_2N_3}{}_{[M_1M_2M_3} Z^+_{M_4M_5M_6]N_1N_2N_3} , \end{aligned} \tag{26}$$

while the fermionic part is as in (21). If one uses (26) to compute the commutator of a supersymmetry and an $\mathfrak{sp}(32,\mathbb{R})$ transformation, the results will look very complicated. On the other hand, the commutator of two symmetry transformations may be cast in a compact form using the graded Jacobi identity of the $\mathfrak{osp}(1|32,\mathbb{R})$ superalgebra, which comes into the game since matrix fields are in the adjoint representations of $\mathfrak{osp}(1|32,\mathbb{R})$.

Such a commutator acting on the fermionic field Ψ yields:

$$\begin{aligned} [\delta_\chi^{(1)}, \delta_h^{(1)}]\, \Psi &= -hm\chi + [h,m]\chi = -mh\chi = \\ &= -\frac{1}{2!}(\mathcal{P}_+\Gamma^{MN} h\chi) C_{MN} - \frac{1}{6!}(\mathcal{P}_+\Gamma^{M_1\dots M_6} h\chi) Z^+_{M_1\dots M_6} . \end{aligned} \tag{27}$$

The same transformation on m leads to:

$$[\delta_\chi^{(1)}, \delta_h^{(1)}]\, m_A{}^B = i\left(\Psi_A(\chi^\top h^\top C)^B - (h\chi)_A(\Psi^\top C)^B\right) , \tag{28}$$

which in components reads:

$$[\delta_\chi^{(1)}, \delta_h^{(1)}]\, C_{MN} = \frac{i}{16}\, \chi^\top C h \Gamma_{MN} \Psi , \tag{29}$$

$$[\delta_\chi^{(1)}, \delta_h^{(1)}]\, Z^+_{M_1\dots M_6} = \frac{i}{16}\, \chi^\top C h \Gamma_{M_1\dots M_6} \Psi . \tag{30}$$

In eqns. (27), (29) and (30), one could write h in components as in (12) and use:

$$\Gamma_{M_1\dots M_k}\Gamma_{N_1\dots N_l} = \sum_{j=0}^{min(k,l)} (-1)^{k-j-1}\, 2\cdot j! \binom{k}{j}\binom{l}{j} \eta_{M_1N_1}\dots\eta_{M_jN_j}\Gamma_{M_{j+1}N_{j+1}\dots M_kN_l} \tag{31}$$

to develop the products of Gamma matrices in irreducible tensors of $SO(10,2)$ and obtain a more explicit result. The final expression for (27) and (30) will contain Gamma matrices with an even number of indices ranging from 0 to 12, while in (29) the number of indices will stop at 8. Since we won't use this result as such in the following, we won't give it here explicitly.

3.4 A note on translational invariance and kinematical supersymmetries

At this point, we want to make a comment on so-called kinematical supersymmetries that have been discussed in the literature on matrix models ([12], [10]). Indeed, commutation relations of dynamical supersymmetries do not close to give space-time translations, i.e. they do not shift the target space-time fields X^M by a constant vector.

However, as was pointed out in [12] and [10], if one introduces so-called kinematical supersymmetry transformations, their commutator with dynamical supersymmetries yields the expected translations by a constant vector. By kinematical supersymmetries, one simply means translations of fermions by a constant Grassmannian odd parameter. In our case, this assumes the form:

$$\begin{aligned} \delta^{(2)}_\xi C_{MN} \;=\; \delta^{(2)}_\xi Z^+_{M_1\dots M_6} \;=\; 0\;, \qquad\qquad & \delta^{(2)}_\xi \Psi \;=\; \xi\;, \\ \Longrightarrow [\delta^{(2)}_\xi, \delta^{(2)}_\zeta]M \;=\; 0 & \end{aligned} \tag{32}$$

Since there is no vector field to be interpreted as space-time coordinates in this 12-dimensional setting, it is interesting to look at the interplay between dynamical and kinematical supersymmetries (which we denote respectively by $\delta^{(1)}$ and $\delta^{(2)}$) when acting on higher-rank tensors. In our case:

$$[\delta^{(1)}_\chi, \delta^{(2)}_\xi]C_{MN} \;=\; -\frac{i}{16}(\chi^\top C\Gamma_{MN}\xi)\;, \qquad [\delta^{(1)}_\chi, \delta^{(2)}_\xi]Z^+_{M_1\dots M_6} = -\frac{i}{16}(\chi^\top C\Gamma_{M_1\dots M_6}\xi)\;. \tag{33}$$

Thus, $[\delta^{(1)}_\chi, \delta^{(2)}_\xi]$ applied to p–forms closes to translations by a constant p-form, generalizing the vector case mentioned above.

For fermions, we have as expected:

$$[\delta^{(1)}_\chi, \delta^{(2)}_\xi]\Psi = 0\;. \tag{34}$$

It is however more natural to consider dynamical and kinematical symmetries to be independent. We would thus expect them to commute. With this in mind, we suggest a generalised version of the translational symmetries introduced in (32):

$$\delta^{(2)}_K\Psi = \xi, \quad \delta^{(2)}_K C_{MN} = k_{MN}, \quad \delta^{(2)}_K Z^+_{M_1\dots M_6} = k^+_{M_1\dots M_6}\;. \tag{35}$$

It is then natural that the matrix

$$K = \begin{pmatrix} k & \xi \\ -i\xi^\top C & 0 \end{pmatrix} \tag{36}$$

should transform in the adjoint of $\mathfrak{osp}(1|32,\mathbb{R})$, which means that:

$$\delta_H^{(1)} k_A{}^B = [h,k]_A{}^B - i(\chi_A(\xi^\top C)^B - \xi_A(\chi^\top C)^B) \tag{37}$$

$$\delta_H^{(1)} \xi_A = h_A{}^C \xi_C - k_A{}^C \chi_C . \tag{38}$$

We can now compute the general commutation relations between translational symmetries $M \to M + K$ and $\mathfrak{osp}(1|32,\mathbb{R})$ transformations and conclude that these operations actually commute:

$$[\delta_H^{(1)}, \delta_K^{(2)}]M = 0 \ . \tag{39}$$

3.5 12-dimensional action for supersymmetric cubic matrix model

We will now build the simplest gauge- and translational-invariant $\mathfrak{osp}(1|32,\mathbb{R})$ supermatrix model with $U(N)$ gauge group. For this purpose, we promote each entry of the matrix M to a hermitian matrix in the Lie algebra of $\mathfrak{u}(N)$ for some value of N. We choose the generators $\{t^a\}_{a=1,\ldots,N^2}$ of $\mathfrak{u}(N)$ so that: $[t^a, t^b] = i f^{abc} t^c$ and $Tr_{\mathfrak{u}(N)}(t^a \cdot t^b) = \delta^{ab}$.

In order to preserve both orthosymplectic and gauge invariance of the model, it suffices to write its action as a supertrace over $\mathfrak{osp}(1|32,\mathbb{R})$ and a trace over $\mathfrak{u}(N)$ of a polynomial of $\mathfrak{osp}(1|32,\mathbb{R})\otimes\mathfrak{u}(N)$ matrices. Following [9], we consider the simplest model containing interactions, namely: $STr_{\mathfrak{osp}(1|32,\mathbb{R})} Tr_{\mathfrak{u}(N)}(M[M,M]_{\mathfrak{u}(N)})$. For hermiticity's sake one has to multiply such an action by a factor of i. We also introduce a coupling constant g^2. This cubic action takes the following form:

$$\begin{aligned} I = \ & \frac{i}{g^2} STr_{\mathfrak{osp}(1|32,\mathbb{R})} Tr_{\mathfrak{u}(N)}(M[M,M]_{\mathfrak{u}(N)}) \ = \ -\frac{1}{g^2} f^{abc} STr_{\mathfrak{osp}(1|32,\mathbb{R})}(M^a M^b M^c) \ = \\ = \ & -\frac{1}{g^2} f^{abc} \left(Tr_{\mathfrak{sp}(32,\mathbb{R})}(m^a m^b m^c) + 3i\Psi^{a\top} C m^b \Psi^c \right) \end{aligned} \tag{40}$$

which we can now express in terms of 12-dimensional representations, where the symplectic matrix m is given by (20).

Let us give a short overview of the steps involved in the computation of (40). It amounts to performing traces of triple products of m^a's over $\mathfrak{sp}(32,\mathbb{R})$, i.e. traces of products of Dirac matrices. We proceed by decomposing such products into their irreps using (31). The only contributions surviving the trace are those proportional to the unit matrix. Thus, the only terms left in (40) will be those containing traces over triple products of 2-forms, over products of a 2-form and two 6-forms, and over triple products of 6-forms, while terms proportional to products of two 2-forms and a 6-form will yield zero contributions.

The two terms involving Z^+'s (to wit CZ^+Z^+ and $Z^+Z^+Z^+$) require some care, since $\Gamma^{A_1\ldots A_{12}}$ is proportional to $\Gamma_\star$ in $12D$, and hence $Tr(\mathcal{P}^+\Gamma^{A_1\ldots A_{12}}) \propto Tr(\Gamma_\star^2) \neq 0$. Since double products of six-indices Gamma matrices decompose into $\mathbb{I}$ and Gamma matrices with 2, 4 up to 12 indices, their trace with Γ^{MN} will keep terms with 2, 10 or 12 indices (the last two containing Levi-Civita tensors) while their trace with $\Gamma^{M_1\ldots M_6}$ will only keep those terms with 6, 8, 10 and 12 indices.

Finally, putting everything together, exploiting the self-duality of Z^+ and rewriting cubic products of fields contracted by f^{abc} as a trace over $\mathfrak{u}(N)$, we get:

$$I = \frac{32i}{g^2} Tr_{\mathfrak{u}(N)} \Bigg(C_M{}^N [C_N{}^O, C_O{}^M]_{\mathfrak{u}(N)} - \frac{1}{20} C_A{}^B [Z_B^{+M_1 \dots M_5}, Z^{+}_{M_1 \dots M_5}{}^A]_{\mathfrak{u}(N)} + \\ + \frac{61}{2(3!)^3} Z^{+}_{ABC}{}^{DEF} [Z^{+}_{DEF}{}^{GHI}, Z^{+}_{GHI}{}^{ABC}]_{\mathfrak{u}(N)} + \\ + \frac{3i}{64} \Psi^\top C \mathcal{P}_+ \Gamma^{MN} [C_{MN}, \Psi]_{\mathfrak{u}(N)} + \frac{3i}{32 \cdot 6!} \Psi^\top C \mathcal{P}_+ \Gamma^{M_1 \dots M_6} [Z^{+}_{M_1 \dots M_6}, \Psi]_{\mathfrak{u}(N)} \Bigg)$$

where we have chosen: $\varepsilon^{0 \dots 11} = \varepsilon_{0 \dots 11} = +1$, since the metric contains two time-like indices. Similarly, one can decompose invariant terms such as $STr_{\mathfrak{osp}(1|32,\mathbb{R})} Tr_{\mathfrak{u}(N)}(M^2)$ and $STr_{\mathfrak{osp}(1|32,\mathbb{R})} Tr_{\mathfrak{u}(N)}([M,M]_{\mathfrak{u}(N)} [M,M]_{\mathfrak{u}(N)})$, etc. While it might be interesting to investigate further the $12D$ physics obtained from such models and compare it to F-theory dynamics, we will not do so here. We will instead move to a detailed study of the better known $11D$ case, possibly relevant for M-theory.

4 Study of the 11D M-theory case

We now want to study the $11D$ matrix model more thoroughly. Similarly to the 12 dimensional case, we embed the $SO(10,1)$ Clifford algebra into $\mathfrak{sp}(32,\mathbb{R})$ and replace the fundamental representation of $\mathfrak{sp}(32,\mathbb{R})$ by $SO(10,1)$ Majorana spinors. A convenient choice of 32×32 Gamma matrices are the $\tilde{\Gamma}$'s we used in the $12D$ case. We choose them as follows:

$$\tilde{\Gamma}^0 = \begin{pmatrix} 0 & -\mathbb{1}_{16} \\ \mathbb{1}_{16} & 0 \end{pmatrix}, \ \tilde{\Gamma}^{10} = \begin{pmatrix} 0 & \mathbb{1}_{16} \\ \mathbb{1}_{16} & 0 \end{pmatrix}, \ \tilde{\Gamma}^i = \begin{pmatrix} \gamma^i & 0 \\ 0 & -\gamma^i \end{pmatrix} \ \ \forall i = 1, \dots, 9, \tag{41}$$

where the γ^i's build a Majorana representation of the Clifford algebra of $SO(9)$, $\{\gamma^i, \gamma^j\} = 2\delta^{ij} \mathbb{1}_{16}$. As before, we have $\tilde{\Gamma}^{10} = \tilde{\Gamma}^0 \tilde{\Gamma}^1 \dots \tilde{\Gamma}^9$ provided $\gamma^1 \dots \gamma^9 = \mathbb{1}_{16}$, since we can define γ^9 to be $\gamma^9 = \gamma^1 \dots \gamma^8$. This choice has $(\tilde{\Gamma}^0)^2 = -\mathbb{1}_{32}$, while $(\tilde{\Gamma}^M)^2 = \mathbb{1}_{32}$, $\forall M = 1 \dots 10$ and gives a representation of $\{\tilde{\Gamma}^M, \tilde{\Gamma}^N\} = 2\eta^{MN} \mathbb{1}_{32}$ for the choice $(-,+,\dots,+)$ of the metric. As we have again chosen all $\tilde{\Gamma}$'s to be real, this allows to take $\tilde{B} = \mathbb{1}$ in $\Psi^* = \tilde{B}\Psi$, which implies that the charge conjugation matrix is $\tilde{C} = \tilde{\Gamma}^0$. Moreover, we have the following transposition rules for the $\tilde{\Gamma}$ matrices:

$$\tilde{C} \tilde{\Gamma}^{M_1 \dots M_n} \tilde{C}^{-1} = (-1)^{n(n+1)/2} (\tilde{\Gamma}^{M_1 \dots M_n})^\top \tag{42}$$

We will identify the fundamental representation of $\mathfrak{sp}(32,\mathbb{R})$ with a 32-component Majorana spinor of $SO(10,1)$. Splitting the 32 real components of the Ψ into $16+16$ as in: $\Psi^\top = (\Phi_1^\top, \Phi_2^\top)$, we can use the following identity:

$$(-i\Phi_2^\top, i\Phi_1^\top) = -i\Psi^\top \tilde{\Gamma}^0 = -i\Psi^\top \tilde{C} = -i\overline{\Psi}$$

to write orthosymplectic matrices again as in (2).

4.1 Embedding of the 11D Super-Poincaré algebra in $\mathfrak{osp}(1|32,\mathbb{R})$

In $11D$, we can also express the $\mathfrak{sp}(32,\mathbb{R})$ transformations in terms of translations, Lorentz transformations and new 5-form symmetries, by defining:

$$h = h_M P^M + h_{MN} J^{MN} + h_{M_1\dots M_5} J^{M_1\dots M_5} \,. \tag{43}$$

With the help of (14), we can compute this enhanced Super-Poincaré algebra as in dimension 12, using the following explicit representation of the generators:

$$P^M = \tilde\Gamma^M \,, \quad J^{MN} = \frac{1}{2}\tilde\Gamma^{MN} \,, \quad J^{M_1\dots M_5} = \frac{1}{5!}\tilde\Gamma^{M_1\dots M_5} \tag{44}$$

In order to express everything in terms of the above generators, we need to dualize forms using the formula: $\frac{1}{(11-k)!}\varepsilon^{M_1\dots M_{11}}\tilde\Gamma_{M_{k+1}\dots M_{11}} = -\tilde\Gamma^{M_1\dots M_k}$. This leads to the following superalgebra:

$$\begin{aligned}
[P^M, P^N] &= 4J^{MN} \\
[P^M, J^{OP}] &= 2\eta^{M[O}P^{P]} \\
[J^{MN}, J^{OP}] &= -4\eta^{[M[O}J^{N]P]} \\
[P^M, J^{M_1\dots M_5}] &= -\frac{2}{5!}\varepsilon^{MM_1\dots M_5}{}_{N_1\dots N_5}J^{N_1\dots N_5} \\
[J^{MN}, J^{M_1\dots M_5}] &= -10\,\eta^{[M[M_1}\,J^{N]M_2\dots M_5]} \\
[J^{M_1\dots M_5}, J^{N_1\dots N_5}] &= -\frac{2}{(5!)^2}\varepsilon^{M_1\dots M_5 N_1\dots N_5}{}_A P^A + \frac{1}{(3!)^2}\eta^{[M_1[N_1}\,\eta^{M_2\,N_2}\varepsilon^{M_3\dots M_5]N_3\dots N_5]}{}_{O_1\dots O_5}J^{O_1\dots O_5} + \\
&+ \frac{1}{3!}\eta^{[M_1[N_1}\,\eta^{M_2\,N_2}\,\eta^{M_3\,N_3}\,\eta^{M_4\,N_4}\,J^{M_5]N_5]} \\
[P^M, Q_A] &= (\tilde\Gamma^M)^B{}_A Q_B \\
[J^{MN}, Q_A] &= \frac{1}{2}(\tilde\Gamma^{MN})^B{}_A Q_B \\
[J^{M_1\dots M_5}, Q_A] &= \frac{1}{5!}(\tilde\Gamma^{M_1\dots M_5})^B{}_A Q_B \\
\{Q_A, Q^B\} &= \frac{i}{16}(\tilde C\tilde\Gamma_M)_A{}^B P^M - \frac{i}{16}(\tilde C\tilde\Gamma_{MN})_A{}^B J^{MN} + \frac{i}{16}(\tilde C\tilde\Gamma_{M_1\dots M_5})_A{}^B J^{M_1\dots M_5}\,.
\end{aligned} \tag{45}$$

Note that this algebra is the dimensional reduction from $12D$ to $11D$ of (17). In particular, the first three lines build the $\mathfrak{so}(10,2)$ Lie algebra, but appear in this new 11-dimensional context as the Lie algebra of symmetries of AdS_{11} space (it is of course also the conformal algebra in 9+1 dimensions). We may wonder whether this superalgebra is a minimal supersymmetric extension of the AdS_{11} Lie algebra or not. If we try to construct an algebra without the five-form symmetry generators, the graded Jacobi identity forbids the appearence of a five-form central charge on the RHS of the $\{Q_A, Q^B\}$ anticommutator. The number of independent components in this last line of the superalgebra will thus be bigger on the LHS than on the RHS. This is not strictly forbidden, but it has implications on the representation theory of the superalgebra. The absence of central charges will for example forbid the existence of shortened representations with a non-minimal eigenvalue of the quadratic Casimir operator $C = -1/4P_M P^M + J_{MN}J^{MN}$ ("spin") of the AdS_{11} symmetry group (see [14]). More generally, in

11D, either all objects in the RHS of the last line are central charges (this case corresponds simply to the $11D$ Super-Poincaré algebra) or they should all be symmetry generators. Thus, although it is not strictly-speaking the minimal supersymmetric extension of the AdS_{11} Lie algebra, it is certainly the most natural one. That's why some authors [7] call $\mathfrak{osp}(1|32,\mathbb{R})$ the super-AdS algebra in $11D$. Here, we will stick to the more neutral $\mathfrak{osp}(1|32,\mathbb{R})$ terminology. Furthermore, $\mathfrak{osp}(1|32,\mathbb{R})$ is also the maximal $\mathcal{N}=1$ extension of the AdS_{11} algebra. In principle, one could consider even bigger superalgebras, but we will not investigate them in this article.

It is also worth remarking that similar algebras have been studied in [15] where they are called topological extensions of the supersymmetry algebras for supermembranes and super-5-branes.

4.2 The supersymmetry properties of the 11D matrix fields

Let us now look at the action of supersymmetries on the fields of an $\mathfrak{osp}(1|32,\mathbb{R})$ eleven-dimensional matrix model. We expand once again the bosonic part of our former matrix M on the irrep of $SO(10,1)$ in terms of 32-dimensional Γ matrices:

$$m = X_M\,\widetilde{\Gamma}^M + \frac{1}{2!}\,C_{MN}\,\widetilde{\Gamma}^{MN} + \frac{1}{5!}\,Z_{M_1\dots M_5}\,\widetilde{\Gamma}^{M_1\dots M_5}\,,$$

where the vector, the 2- and 5-form are given by:

$$X_M = \frac{1}{32}\,Tr_{\mathfrak{sp}(32,\mathbb{R})}(m\,\widetilde{\Gamma}_M)\,,\; C_{MN} = -\frac{1}{32}\,Tr_{\mathfrak{sp}(32,\mathbb{R})}(m\,\widetilde{\Gamma}_{MN})\,,\; Z_{M_1\dots M_5} = \frac{1}{32}\,Tr_{\mathfrak{sp}(32,\mathbb{R})}(m\,\widetilde{\Gamma}_{M_1\dots M_5})\,.$$

Let us give the whole $\delta_H^{(1)}$ transformation acting on the fields (using the cyclic property of the trace, for instance: $Tr([h,m]\widetilde{\Gamma}^M) = Tr(h[m,\widetilde{\Gamma}^M])$):

$$\begin{aligned}
\delta_H^{(1)}\,X^M &= 2\left(h^{MQ}\,X_Q + h^Q\,C_Q{}^M - \frac{1}{(5!)^2}\,\varepsilon^{MM_1\dots M_5}{}_{N_1\dots N_5}h^{N_1\dots N_5}\,Z_{M_1\dots M_5}\right) - \frac{i}{16}\,\chi^\top\widetilde{\Gamma}^0\widetilde{\Gamma}^M\Psi\,,\\
\delta_H^{(1)}\,C^{MN} &= -4\left(h^{[M}X^{N]} - h^{[M}{}_Q C^{N]Q} + \frac{1}{4!}\,h_{M_1\dots M_4}{}^{[M}\,Z^{N]M_1\dots M_4}\right) + \frac{i}{16}\,\chi^\top\widetilde{\Gamma}^0\widetilde{\Gamma}^{MN}\Psi\,,\\
\delta_H^{(1)}\,Z^{M_1\dots M_5} &= 2\left(\frac{1}{5!}\varepsilon^{M_1\dots M_5}{}_{N_1\dots N_5 Q}\,h^{N_1\dots N_5}\,X^Q + 5\,h_Q{}^{[M_1\dots M_4}\,C^{M_5]Q} - 5\,h_Q{}^{[M_1}\,Z^{M_2\dots M_5]Q} +\right.\\
&\quad\left. + \frac{1}{5!}\,\varepsilon^{M_1\dots M_5}{}_{ON_1\dots N_5}\,h^O\,Z^{N_1\dots N_5} - \frac{1}{3\cdot 4!}h^{O_1\dots O_5}\,\varepsilon_{O_1\dots O_5 N_1 N_2 N_3}{}^{[M_1 M_2 M_3}\,Z^{M_4 M_5]N_1 N_2 N_3}\right) -\\
&\quad - \frac{i}{16}\,\chi^\top\widetilde{\Gamma}^0\widetilde{\Gamma}^{M_1\dots M_5}\Psi\,,\\
\delta_H^{(1)}\,\Psi &= \left(h_M\,\widetilde{\Gamma}^M + h_{MN}\,\widetilde{\Gamma}^{MN} + h_{M_1\dots M_5}\,\widetilde{\Gamma}^{M_1\dots M_5}\right)\Psi -\\
&\quad - \widetilde{\Gamma}^M\chi X_M - \frac{1}{2}\widetilde{\Gamma}^{MN}\chi C_{MN} - \frac{1}{5!}\widetilde{\Gamma}^{M_1\dots M_5}\chi Z_{M_1\dots M_5}\,,
\end{aligned}$$

where the part between parentheses describes the symplectic transformations, while the rest represents the supersymmetry variations. Note that we used $\frac{1}{(11-k)!}\varepsilon^{M_1\dots M_{11}}\tilde{\Gamma}_{M_{k+1}\dots M_{11}} = -\tilde{\Gamma}^{M_1\dots M_k}$ in $\delta_H^{(1)}\,Z^{M_1\dots M_5}$ to dualize the Gamma matrices when needed.

4.3 11-dimensional action for a supersymmetric matrix model

As in the $12D$ case, we will now consider a specific model, invariant under $U(N)$ gauge and $\mathfrak{osp}(1|32,\mathbb{R})$ transformations. The simplest such model containing interactions and "propagators" is a cubic action along with a quadratic term. Hence, we choose:

$$I = STr_{\mathfrak{osp}(1|32,\mathbb{R})\otimes\mathfrak{u}(N)}\left(-\mu M^2 + \frac{i}{g^2}M[M,M]_{\mathfrak{u}(N)}\right). \tag{46}$$

Contrary to a purely cubic model, one loses invariance under $M\to M+K$ for a constant diagonal matrix K, which contains the space-time translations of the BFSS model. In contrast with the BFSS theory, our model doesn't exhibit the symmetries of flat $11D$ Minkowski space-time, so we don't really expect this sort of invariance. However, the symmetries generated by P^M remain unbroken, as well as all other $\mathfrak{osp}(1|32,\mathbb{R})$ transformations. Indeed, the related bosonic part of the algebra (45) contains the symmetries of AdS_{11} as a subalgebra, and as was pointed out in [16] and [17], massive matrix models with a tachyonic mass-term for the coordinate X's fields appear in attempts to describe gravity in de Sitter spaces (an alternative approach can be found in [18]). Note that we take the opposite sign for the quadratic term of (46), this choice being motivated by the belief that AdS vacua are more stable than dS ones, so that the potential energy for physical bosonic fields should be positive definite in our setting.

The computation of the 11-dimensional action for this supermatrix model is analogous to the one performed in 12 dimensions. We remind the reader that each entry of the matrix M now becomes a hermitian matrix in the Lie algebra of $\mathfrak{u}(N)$ for some large value of N whose generators are defined as in the $12D$ case.

After performing in (46) the traces on products of Gamma matrices, it comes out that the terms of the form XXX, XXZ, XCC, CCZ and XCZ have vanishing trace (since products of Gamma matrices related to these terms have decomposition in irreducible tensors that do not contain a term proportional to $\mathbb{1}_{32}$) so that only terms of the form XXC, XZZ, CZZ, CCC, ZZZ will remain from the cubic bosonic terms. As for terms containing fermions and the mass terms, they are trivial to compute. Using (31) and the usual duality relation for Gamma matrices in $11D$, one finally obtains the following result:

$$\begin{aligned}
I = & -32\mu\, Tr_{\mathfrak{u}(N)}\left\{X_M X^M - \frac{1}{2!}C_{MN}C^{MN} + \frac{1}{5!}Z_{M_1\ldots M_5}Z^{M_1\ldots M_5} + \frac{i}{16}\overline{\Psi}\Psi\right\} + \\
& + \frac{32i}{g^2}Tr_{\mathfrak{u}(N)}\Bigg(3\,C_{NM}\,[X^M,X^N]_{\mathfrak{u}(N)} - \varepsilon^{M_1\ldots M_{11}}\left\{\frac{3}{(5!)^2}Z_{M_1\ldots M_5}\,[X_{M_6},Z_{M_7\ldots M_{11}}]_{\mathfrak{u}(N)} - \right. \\
& \left. - \frac{2^3\,5^2}{(5!)^3}Z_{M_1\,M_2\,M_3}{}^{AB}\,[Z_{AB\,M_4\,M_5\,M_6},Z_{M_7\ldots M_{11}}]_{\mathfrak{u}(N)}\right\} + \frac{3}{4!}C_{MN}\,[Z_{A_1\ldots A_4}{}^N, Z^{A_1\ldots A_4\,M}]_{\mathfrak{u}(N)} + \\
& + C_{MN}\,[C^N{}_O,C^{OM}]_{\mathfrak{u}(N)} + \frac{3i}{32}\left\{\overline{\Psi}\,\widetilde{\Gamma}^M[X_M,\Psi]_{\mathfrak{u}(N)} + \frac{1}{2!}\overline{\Psi}\,\widetilde{\Gamma}^{MN}\,[C_{MN},\Psi]_{\mathfrak{u}(N)} + \right. \\
& \left. + \frac{1}{5!}\overline{\Psi}\,\widetilde{\Gamma}^{M_1\ldots M_5}\,[Z_{M_1\ldots M_5},\Psi]_{\mathfrak{u}(N)}\right\}\Bigg).
\end{aligned} \tag{47}$$

5 Dynamics of the $11D$ supermatrix model and its relation to BFFS theory

Now, we will try to see to what extent our model may describe at least part of the dynamics of M-theory. Since the physics of the BFSS matrix model and its relationships to 11D supergravity and superstring theory are relatively well understood, if our model is to be relevant to M-theory, we expect it to be related to BFSS theory at least in some regime. To see such a relationship, we should reduce our model to one of its ten-dimensional sectors and turn it into a matrix quantum mechanics.

5.1 Compactification and T-duality of the $11D$ supermatrix action

If we want to link (47) to BFSS, which is basically a quantum mechanical supersymmetric matrix model, we should reduce the eleven-dimensional target-space spanned by the X^M's to 10 dimensions, and, at the same time, let a "time" parameter t appear. At this stage, the world-volume of the theory is reduced to one point. We start by decompactifying it along two directions, following the standard procedure outlined in [19]. Namely, we compactify the target-space coordinates X_0 and X_{10} on circles of respective radii $R_0 = R$ and $R_{10} = \omega R$. We introduce the rescaled field $X'_{10} \equiv X_{10}/\omega$ which has the same $2\pi R$ periodicity as X_0. We can then perform T-dualities on X_0 and X'_{10} to circles of dual radii $\widehat{R} \equiv l_{11}^2/R$ (parametrized by τ and y), where l_{11} is some scale, typically the 11-dimensional Planck length. The fields of our theory, for simplicity denoted here by Y, now depend on the world-sheet coordinates τ and y as follows:

$$Y(\tau, y) \;=\; \sum_{m,n} Y_{mn}\, e^{i(m\tau+ny)/\widehat{R}} \;. \tag{48}$$

As a consequence, we now need to average the action over τ and y with the measure $d\tau dy/(2\pi\widehat{R})^2$. Finally, one should identify under T-duality:

$$X_0 \sim 2\pi l_{11}^2 \Big(i\partial_\tau - A_\tau(\tau,y)\Big) \triangleq i\widehat{\mathcal{D}}_\tau \;, \quad X_{10} \equiv \omega X'_{10} \sim 2\pi\omega l_{11}^2 \Big(i\partial_y - A_y(\tau,y)\Big) \triangleq i\omega\widehat{\mathcal{D}}_y \;, \tag{49}$$

where A_τ and A_y are the connections on the $U(N)$ gauge bundle over the world-sheet. For notational convenience, we rewrite $\phi \triangleq C_{0\,10}$, $F_{\tau y} \triangleq -i\,[\widehat{\mathcal{D}}_\tau, \widehat{\mathcal{D}}_y]$ and $\widetilde{\Gamma}_* \triangleq \widetilde{\Gamma}_{10}$ and encode the possible values of the indices in the following notation:

$$\begin{aligned} A,\ B \;&=\; 0,\ldots,10 \;, \qquad i,\ j,\ k \;=\; 1,\ldots,9 \;, \\ \alpha \;&=\; 1,\ldots,10 \;, \qquad \beta \;=\; 0,\ldots,9 \;. \end{aligned}$$

Then, the compactified version of (47) reads:

$$
\begin{aligned}
I_c = \frac{32i}{g^2} \int \frac{d\tau dy}{(2\pi \widehat{R})^2} Tr_{\mathrm{u}(N)} \bigg(& - 6\, C_{i0}\, i[\widehat{\mathcal{D}}_\tau, X_i] \,+\, 6\omega\, C_{i10}\, i[\widehat{\mathcal{D}}_y, X_i] \,+\, \frac{3}{32}\, \overline{\Psi}\, \widetilde{\Gamma}_0\, [\widehat{\mathcal{D}}_\tau, \Psi] - \\
& - \frac{3\omega}{32}\, \overline{\Psi}\, \widetilde{\Gamma}_*\, [\widehat{\mathcal{D}}_y, \Psi] \,-\, \frac{3}{(5!)^2}\, \varepsilon_{\alpha_1 \cdots \alpha_{10} 0}\, Z_{\alpha_1 \cdots \alpha_5}\, i[\widehat{\mathcal{D}}_\tau, Z_{\alpha_6 \cdots \alpha_{10}}] \,+\, \frac{3\omega}{(5!)^2}\, \varepsilon^{\beta_1 \cdots \beta_{10} 10}\, Z_{\beta_1 \cdots \beta_5}\, i[\widehat{\mathcal{D}}_y, Z_{\beta_6 \cdots \beta_{10}}] + \\
& + 6i\omega\, \phi\, F_{\tau y} \,+\, 3\, C_{ij}\, [X_j, X_i] \,+\, \frac{3}{(5!)^2}\, \varepsilon^{A_1 \cdots A_{10}}{}_j\, Z_{A_1 \cdots A_5}\, [X_j, Z_{A_6 \cdots A_{10}}] - \\
& - \frac{2^3 5^2}{(5!)^3}\, \varepsilon^{A_1 \cdots A_{11}}\, Z_{A_1 A_2 A_3}{}^{B_1 B_2}\, [Z_{B_1 B_2 A_4 A_5 A_6}, Z_{A_7 \cdots A_{11}}] \,+\, \frac{3}{4!} \Big\{ C_{ij}\, [Z_{j\, A_1 \cdots A_4}, Z_i{}^{A_1 \cdots A_4}] - \\
& - \, 2\, C_{i0}\, [Z_{0\, \alpha_1 \cdots \alpha_4}, Z_{i\, \alpha_1 \cdots \alpha_4}] \,+\, 2\, C_{i10}\, [Z_{10\, \beta_1 \cdots \beta_4}, Z_i{}^{\beta_1 \cdots \beta_4}] \,-\, 2\, \phi\, [Z_{10\, i_1 \cdots i_4}, Z_{0\, i_1 \cdots i_4}] \Big\} + \\
& + C_{ij}\, [C_{jk}, C_{ki}] \,+\, 3\, C_{i0}\, [C_{k0}, C_{ki}] \,-\, 3\, C_{i10}\, [C_{k10}, C_{ki}] \,+\, 6\, \phi\, [C_{k10}, C_{k0}] + \\
& + \frac{3i}{32} \Big\{ \overline{\Psi}\, \widetilde{\Gamma}_i\, [X_i, \Psi] \,+\, \frac{1}{2!}\, \overline{\Psi}\, \widetilde{\Gamma}_{ij} [C_{ij}, \Psi] \,-\, \overline{\Psi}\, \widetilde{\Gamma}_i \widetilde{\Gamma}_0\, [C_{i0}, \Psi] \,+\, \overline{\Psi}\, \widetilde{\Gamma}_i \widetilde{\Gamma}_*\, [C_{i10}, \Psi] \,-\, \overline{\Psi}\, \widetilde{\Gamma}_0 \widetilde{\Gamma}_*\, [\phi, \Psi] + \\
& + \frac{1}{5!}\, \overline{\Psi}\, \widetilde{\Gamma}^{A_1 \cdots A_5}\, [Z_{A_1 \cdots A_5}, \Psi] \Big\} \,+\, i\mu g^2 \Big(\widehat{\mathcal{D}}_\tau \widehat{\mathcal{D}}_\tau \,-\, \omega^2\, \widehat{\mathcal{D}}_y \widehat{\mathcal{D}}_y \,+\, X_i X_i \,+\, \frac{i}{16} \overline{\Psi} \Psi \,+\, \phi^2 - \\
& - \frac{1}{2!}\, C_{ij} C_{ij} \,+\, C_{i0} C_{i0} \,-\, C_{i10} C_{i10} \,+\, \frac{1}{5!} Z_{A_1 \cdots A_5} Z^{A_1 \cdots A_5} \Big) \bigg) \,.
\end{aligned} \tag{50}
$$

Repeated indices are contracted, and when they appear alternately up and down, Minkowskian signature applies, whereas Euclidian signature is in force when both are down.

5.2 Ten-dimensional limits and IMF

Since the BFSS matrix model is conjectured to describe M-theory in the infinite momentum frame, we shall investigate our model in this particular limit. For this purpose, let's define the light-cone coordinates $t_+ \equiv (\tau + y)/\sqrt{2}$ and $t_- \equiv (\tau - y)/\sqrt{2}$ and perform a boost in the y direction. In the limit where the boost parameter u is large, the boost acts as $(t_+, t_-) \stackrel{\sim}{\longrightarrow} (ut_+, u^{-1} t_-)$, or as $(\tau, y) \stackrel{\sim}{\longrightarrow} \sqrt{2}(ut_+, ut_+)$ on the original coordinates. In particular, when $u \to \infty$, the t_- dependence disappears from the action and we can perform the trivial t_- integration. The dynamics is now solely described by the parameter $t \equiv \sqrt{2} u t_+$, which is decompactified through this procedure. In particular, both $\widehat{\mathcal{D}}_\tau$ and $\widehat{\mathcal{D}}_y$ are mapped into $\widehat{\mathcal{D}}_t$.

So far, the ratio of the compactification radii ω is left undetermined and it parametrizes a continuous family of frames. It affects the kinetic terms as:

$$
\begin{aligned}
I_c = \frac{32i}{g^2} \lim_{u \to \infty} \int_{-\pi \widehat{R} u}^{\pi \widehat{R} u} \frac{dt}{2\sqrt{2}\pi \widehat{R} u} Tr_{\mathrm{u}(N)} \bigg(& - 6 \left(C_{i0} - \omega\, C_{i10} \right) i[\widehat{\mathcal{D}}_t, X_i] \,+\, \frac{3}{32}\, \overline{\Psi} \left(\widetilde{\Gamma}_0 - \omega\, \widetilde{\Gamma}_* \right) [\widehat{\mathcal{D}}_t, \Psi] - \\
& - \frac{3}{(5!)^2}\, \varepsilon_{\alpha_1 \cdots \alpha_{10} 0}\, Z_{\alpha_1 \cdots \alpha_5}\, i[\widehat{\mathcal{D}}_t, Z_{\alpha_6 \cdots \alpha_{10}}] \,+\, \frac{3\omega}{(5!)^2}\, \varepsilon^{\beta_1 \cdots \beta_{10} 10}\, Z_{\beta_1 \cdots \beta_5}\, i[\widehat{\mathcal{D}}_t, Z_{\beta_6 \cdots \beta_{10}}] \,+\, \ldots \bigg)
\end{aligned} \tag{51}
$$

In order to have a non-trivial action, as in the BFSS case, we must take the limit $u \to \infty$ together with $N \to \infty$ in such a way that $N/(\widehat{R}u) \to \infty$. In the following, we will write $\overline{R} \equiv \widehat{R}u$, implicitly take the limit $(\overline{R}, N) \to \infty$ and let t run from $-\infty$ to ∞.

In the usual IMF limit, one starts from an uncompactified X_0. In our notation, this corresponds to $R \to \infty$, i.e. to the particular choice $\omega = R_{10}/R \to 0$. So, in the IMF limit, all terms proportional to ω drop out of (51). In the following chapters, we will restrict ourselves to this case, since we are especially interested in the physics of our model in the infinite momentum frame.

5.3 Dualization of the mass term

Let us comment on the meaning of the $\widehat{\mathcal{D}}_t^2$ term arising from the T-dualization of the mass term $Tr((X_0)^2)$, which naively breaks gauge invariance. To understand how it works, we should recall that the trace is defined by the following sum:

$$Tr_{\mathfrak{u}(N)}(-\widehat{\mathcal{D}}_t^2) \; = \; -\sum_a \langle u_a(t)|\widehat{\mathcal{D}}_t^2|u_a(t)\rangle \; = \; \sum_a \|i\widehat{\mathcal{D}}_t|u_a(t)\rangle\|^2 \quad . \tag{52}$$

for a set of basis elements $\{|u_a(t)\rangle\}_a$ of $\mathfrak{u}(N)$, which might have some t-dependence or not. If the $|u_a(t)\rangle$ are covariantly constant, the expression (52) is obviously zero. Choosing the $|u_a(t)\rangle$ to be covariantly constant seems to be the only coherent possibility. Such a covariantly constant basis is:

$$|u_a(t)\rangle \; \triangleq \; e^{-i\int_{t_0}^t A_0(t')\,dt'}|u_a\rangle \quad ,$$

(where the $|u_a\rangle$'s form a constant basis, for instance, the generators of $\mathfrak{u}(N)$ in the adjoint representation). Now, t lives on a circle and the function $\exp i\int_{t_0}^t A_0(t')\,dt'$ is well-defined only if the zero-mode $A_0^{(0)} = 2\pi n$, $n \in \mathbb{Z}$. But we can always set $A_0^{(0)}$ to zero, since it doesn't affect the behaviour of the system, as it amounts to a mere constant shift in "energy". With this choice, we can integrate $\widehat{\mathcal{D}}_t$ by part without worrying about the trace.

5.4 Decomposition of the 5-forms

In (49), the only fields to be dynamical are the X_i, the $Z_{\alpha_1\cdots\alpha_5}$ and the Ψ. The remaining ones are either the conjugate *momentum*-like fields when they multiply derivatives of dynamical fields, or *constraint*-like when they only appear algebraically.

Thus, the C_{i0} and $\overline{\Psi}$ have a straightforward interpretation as *momenta* conjugate respectively to the X_i and to Ψ. For the 5-form fields $Z_{A_1\cdots A_5}$ however, the matter is a bit more subtle, due to the presence of the $11D$ ε tensor in the kinetic term for the 5-form fields. Actually, the real degrees of freedom contained in $Z_{A_1\cdots A_5}$ decompose as follows, when going down from $(10+1)$ to 9 dimensions:

$$\Omega^5(\mathcal{M}_{10,1},\mathbb{R}) \quad \longrightarrow \quad 3\times\Omega^4(\mathcal{M}_9,\mathbb{R})\oplus\Omega^3(\mathcal{M}_9,\mathbb{R}) \quad . \tag{53}$$

To be more specific (as in our previous convention, $i_k = 1,\ldots,9$ are purely spacelike indices in $9D$), the 3-form fields on the RHS of (53) are $Z_{i_1i_2i_30,10} \triangleq B_{i_1i_2i_3}$, while the 4-form fields are $Z_{i_1i_2i_3i_410} \triangleq Z_{i_1i_2i_3i_4}$, $Z_{i_1i_2i_3i_40} \triangleq H_{i_1i_2i_3i_4}$ and[4] $\Pi^{i_1\cdots i_4} \triangleq 1/5!\,\varepsilon^{j_1\cdots j_5 i_1\cdots i_4 0,10}Z_{j_1\cdots j_5}$; these conventions allow us to cast the

[4]Using

$$\varepsilon^{j_1\cdots j_N i_{N+1}\cdots i_9 0,10}\,\varepsilon_{k_1\cdots k_N i_{N+1}\cdots i_9 0,10} \; = \; -(9-N)!\sum_\pi \sigma(\pi)\prod_{n=1}^N \delta^{j_n}_{k_{\pi(n)}} \quad ,$$

where π is any permutation of $(1,2,..,N)$ and $\sigma(\pi)$ is the signature thereof, this relation can be inverted: $Z_{i_1\cdots i_5} = \frac{1}{4!}\,\varepsilon_{i_1\cdots i_5 j_6\cdots j_9}\,\Pi^{j_6\cdots j_9}$, kinetic term for the 5-form fields into the expression $6/4!\,\Pi^{i_1\cdots i_4}\,[\widehat{\mathcal{D}}_t, Z_{i_1\cdots i_4}]$, while B and H turn out to be *constraint*-like fields, the whole topic being summarized in Table 1.

dynamical var.	*number of real comp.*	*conjugate momenta*	*constraint-like*	*number of real comp.*
X_i	9	C_{i0}	C_{ij}	36
			C_{i10}	9
			ϕ	1
$Z_{i_1\cdots i_4}$	126	$\Pi_{i_1\cdots i_4}$	$H_{i_1\cdots i_4}$	126
			$B_{i_1 i_2 i_3}$	84
Ψ	32	$\overline{\Psi}$		

Table 1: *Momentum*-like and *constraint*-like auxiliary fields

We see that longitudinal 5-brane degrees of freedom are described by the 4-form $Z_{i_1\cdots i_4}$, while transverse 5-brane fields $Z_{i_1\cdots i_5}$ appear in the definition of the conjugate momenta. As they are dual to one another, we could also have exchanged their respective rôles. Both choices describe the same physics. We can thus interpret these degrees of freedom as transverse 5-branes, completing the BFSS theory, which already contains longitudinal 5-branes as bound states of D0-branes.

Choosing the $\varepsilon_{i_1\cdots i_9}$ tensor in 9 spatial dimensions to be:

$$\varepsilon_{i_1\cdots i_9} \triangleq \varepsilon_{i_1\cdots i_9}{}^{0,10} = -\varepsilon_{i_1\cdots i_9 0,10} \quad ,$$

we can express the action I_c in terms of the degrees of freedom appearing in Table 1 (note that from now on all indices will be down, the signature for squared expressions is Euclidean and we write $\mathcal{D}_t$ instead of $\widehat{\mathcal{D}}_t$):

$$\begin{aligned}
I_c \;=\; & \frac{8\sqrt{2}i}{\pi g^2 \overline{R}} \int dt\, Tr_{\mathrm{u}(N)} \Bigg(-6i\, C_{i0}\, [\mathcal{D}_t, X_i] \;-\; \frac{i}{4}\, \Pi_{i_1\cdots i_4}\, [\mathcal{D}_t, Z_{i_1\cdots i_4}] \;+\; \frac{3}{32}\, \overline{\Psi}\, \widetilde{\Gamma}_0\, [\mathcal{D}_t, \Psi] \;+\; 3\, C_{ij}\, [X_j, X_i] - \\
& + \left(\Pi_{i_1 i_2 i_3 j}\, [X_j, B_{i_1 i_2 i_3}] \;-\; \frac{1}{4\cdot 4!}\, \varepsilon_{i_1\cdots i_8\, j} Z_{i_1\cdots i_4}\, [X_j, H_{i_5\cdots i_8}] \right) \;+\; \frac{1}{3!\cdot 4!}\, W(Z,\Pi,H,B) + \\
& + \frac{1}{2} \Bigg\{ C_{ij}\, K_{ij}(Z,\Pi,H,B) \;-\; 2\, C_{i0} \left(\frac{1}{4\cdot 4!} \varepsilon_{i\, j_1\cdots j_4 k_1\cdots k_4} [H_{j_1\cdots j_4}, \Pi_{k_1\cdots k_4}] \;+\; [Z_{i\, j_1 j_2 j_3}, B_{j_1 j_2 j_3}] \right) + \\
& + \; 2\, C_{i10} \left(\frac{1}{4\cdot 4!} \varepsilon_{i\, j_1\cdots j_4 k_1\cdots k_4} [Z_{j_1\cdots j_4}, \Pi_{k_1\cdots k_4}] \;-\; [H_{i\, j_1 j_2 j_3}, B_{j_1 j_2 j_3}] \right) \;-\; \frac{1}{2}\, \phi\, [Z_{i_1\cdots i_4}, H_{i_1\cdots i_4}] \Bigg\} + \\
& + C_{ij}\, [C_{jk}, C_{ki}] \;+\; 3\, C_{i0}\, [C_{k0}, C_{ki}] \;-\; 3\, C_{i10}\, [C_{k10}, C_{ki}] \;+\; 6\, \phi\, [C_{k10}, C_{k0}] + \\
& + \frac{3i}{32} \Bigg\{ \overline{\Psi}\, \widetilde{\Gamma}_i\, [X_i, \Psi] \;+\; \frac{1}{2!}\, \overline{\Psi}\, \widetilde{\Gamma}_{ij} [C_{ij}, \Psi] \;-\; \overline{\Psi}\, \widetilde{\Gamma}_i \widetilde{\Gamma}_0\, [C_{i0}, \Psi] \;+\; \overline{\Psi}\, \widetilde{\Gamma}_i \widetilde{\Gamma}_*\, [C_{i10}, \Psi] - \\
& - \; \overline{\Psi}\, \widetilde{\Gamma}_0 \widetilde{\Gamma}_*\, [\phi, \Psi] \;+\; \frac{1}{4!}\, \overline{\Psi}\, \widetilde{\Gamma}_{i_1\cdots i_4} \widetilde{\Gamma}_* [Z_{i_1\cdots i_4}, \Psi] \;+\; \frac{1}{4!}\, \overline{\Psi}\, \widetilde{\Gamma}_{i_1\cdots i_4} \widetilde{\Gamma}_0 \widetilde{\Gamma}_* [\Pi_{i_1\cdots i_4}, \Psi] + \\
& - \; \frac{1}{4!}\, \overline{\Psi}\, \widetilde{\Gamma}_{i_1\cdots i_4} \widetilde{\Gamma}_0 [H_{i_1\cdots i_4}, \Psi] \;-\; \frac{1}{3!}\, \overline{\Psi}\, \widetilde{\Gamma}_{i_1 i_2 i_3} \widetilde{\Gamma}_0 \widetilde{\Gamma}_* [B_{i_1 i_2 i_3}, \Psi] \Bigg\} \;+\; \mu g^2 i \Bigg\{ (X_i)^2 \;+\; \frac{i}{16}\, \overline{\Psi}\Psi \;+\; \phi^2 - \\
& - \frac{1}{2!}\, (C_{ij})^2 \;+\; (C_{i0})^2 \;-\; (C_{i10})^2 \;+\; \frac{1}{4!} \Big((Z_{i_1\cdots i_4})^2 \;+\; (\Pi_{i_1\cdots i_4})^2 \;-\; (H_{i_1\cdots i_4})^2 \;-\; 4\, (B_{i_1 i_2 i_3})^2 \Big) \Bigg\} \Bigg) \; .
\end{aligned} \tag{54}$$

We have redefined the two following lengthy expressions in a compact way to cut short: first the term coupling the various 5-form components to the C_{ij}:

$$K_{ij}(Z,\Pi,H,B) \triangleq [Z_{j\,k_1k_2k_3}, Z_{i\,k_1k_2k_3}] + [\Pi_{j\,k_1k_2k_3}, \Pi_{i\,k_1k_2k_3}] - 3[B_{j\,k_1k_2}, B_{i\,k_1k_2}] - [H_{j\,k_1k_2k_3}, H_{i\,k_1k_2k_3}]\,,$$

and second, the trilinear couplings amongst the 5-form components:

$$\begin{aligned} W(Z,\Pi,H,B) \quad \triangleq \quad & \varepsilon_{i_1\cdots i_9}\Big\{ B_{i_1i_2\,j}\ (2\,[\Pi_{j\,i_3i_4i_5}, \Pi_{i_6\cdots i_9}] - [Z_{j\,i_3i_4i_5}, Z_{i_6\cdots i_9}] - [H_{j\,i_3i_4i_5}, H_{i_6\cdots i_9}]) + \\ & + \frac{2}{3}\,B_{i_1i_2i_3}\ ([B_{i_4i_5i_6}, B_{i_7i_8i_9}] + [Z_{i_4i_5i_6\,j}, Z_{j\,i_7i_8i_9}] - [H_{i_4i_5i_6\,j}, H_{j\,i_7i_8i_9}])\Big\} \\ & + (3!)^2\,\Pi_{i_1i_2j_1j_2}\,[Z_{j_1j_2k_1k_2}, H_{k_1k_2i_1i_2}] \quad . \end{aligned}$$

5.5 Computation of the effective action

We now intend to study the effective dynamics of the X_i and Ψ fields, in order to compare it to the physics of D0-branes as it is described by the BFSS matrix model. For this purpose, we start by integrating out the 2-form momentum-like and constraint-like fields, which will yield an action containing the BFSS matrix model as its leading term with, in addition, an infinite series of couplings between the fields. Similarly, one would like to integrate out the Z-type momenta and constraints Π, H and B, to get an effective action for the 5-brane (described by Z_{ijkl}) coupled to the D0-branes. We will however not do so in the present paper, but leave this for further investigation.

To simplify our expressions, we set:[5]

$$\beta \triangleq \mu g^2 \quad , \qquad\qquad \gamma \triangleq \frac{8\sqrt{2}}{\pi g^2 \overline{R}} \quad ,$$

and write (54) as (after taking the trace over $\mathfrak{u}(N)$):

$$I_c = \gamma \int dt\ \Big(\beta(\mathbf{C}_i^a)^{\intercal}(\mathcal{J}_{ij}^{ab} + \Delta_{ij}^{ab})\mathbf{C}_j^b + \mathbf{C}_i^a\cdot\mathbf{F}_i^a + \mathcal{L}_C + \mathcal{L}_\phi + \widehat{\mathcal{L}}\Big) \quad . \tag{55}$$

For convenience, we have resorted to a very compact notation, where:

$$\mathbf{C}_i^a \triangleq \begin{pmatrix} C_{i0}^a \\ C_{i10}^a \end{pmatrix} , \qquad \mathcal{J}_{ij}^{ab} \triangleq \begin{pmatrix} -\delta^{ab}\delta_{ij} & 0 \\ 0 & \delta^{ab}\delta_{ij} \end{pmatrix} , \qquad \Delta_{ij}^{ab} \triangleq \frac{3f^{abc}}{\beta}\begin{pmatrix} C_{ij}^c & \phi^c\delta_{ij} \\ -\phi^c\delta_{ij} & -C_{ij}^c \end{pmatrix} \quad ,$$

and where the components of the vector $\mathbf{F}_i^a = \begin{pmatrix} F_i^a \\ G_i^a \end{pmatrix}$, are given by the following expressions:

$$\begin{aligned} F_i \quad &\triangleq \quad 6\,[\mathcal{D}_t, X_i] - \frac{i}{4\cdot 4!}\varepsilon_{i\,j_1\cdots j_4k_1\cdots k_4}\,[H_{j_1\cdots j_4}, \Pi_{k_1\cdots k_4}] - i\,[Z_{i\,j_1j_2j_3}, B_{j_1j_2j_3}] - \frac{3}{32}\{\overline{\Psi}, \widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\} \quad , \\ G_i \quad &\triangleq \quad \frac{i}{4\cdot 4!}\varepsilon_{i\,j_1\cdots j_4k_1\cdots k_4}[Z_{j_1\cdots j_4}, \Pi_{k_1\cdots k_4}] - i\,[H_{i\,j_1j_2j_3}, B_{j_1j_2j_3}] + \frac{3}{32}\{\overline{\Psi}, \widetilde{\Gamma}_i\widetilde{\Gamma}_*\Psi\} \quad . \end{aligned}$$

[5]If we consider X and hence C, Z and Ψ to have the engineering dimension of a length, then so has β, while γ has dimension (length)$^{-4}$.

Note that we have written $if^{abc}\overline{\Psi}^b\widetilde{\Gamma}_{\ldots}\Psi^c$ as $\{\overline{\Psi},\widetilde{\Gamma}_{\ldots}\Psi\}^a$ with a slight abuse of notation. The remaining terms in the action (55) depending on C_{ij} and ϕ are contained in

$$\begin{aligned} \mathcal{L}_C &\triangleq \frac{\beta}{2}(C^a_{ij})^2 + E^a_{ij}C^a_{ij} - f^{abc}C^a_{ij}C^b_{jk}C^c_{ki} \quad , \\ \mathcal{L}_\phi &\triangleq -\beta(\phi^a)^2 + J^a\phi^a \quad , \end{aligned}$$

with the following definitions

$$\begin{aligned} E_{ij} &\triangleq \frac{i}{2}K_{ij} + 3i\,[X_i,X_j] + \frac{3}{64}\{\overline{\Psi},\widetilde{\Gamma}_{ij}\Psi\} \quad , \\ J &\triangleq \frac{-i}{4}[Z_{i_1\cdots i_4},H_{i_1\cdots i_4}] - \frac{3}{32}\{\overline{\Psi},\widetilde{\Gamma}_0\widetilde{\Gamma}_*\Psi\} \quad , \end{aligned}$$

and finally $\widehat{\mathcal{L}}$ is the part of I_c in (54) independent of C_{ij}, C_{i10}, C_{i0} and ϕ. in other words the part containing only dynamical fields (fermions Ψ and coordinates X_i) as well as all fields related to the 5-brane (the dynamical ones: Z and Π, as well as the constrained ones: B and H).

Now, (55) is obviously bilinear in the $\mathbf{C}^a_i$ (note that Δ^{ab}_{ij} *is* symmetric, since C_{ij} is actually antisymmetric in i and j). So one may safely integrate them out, after performing a Wick rotation such as

$$t \to \tau = it \quad , \qquad\qquad C_{i10} \to \overline{C}_{i10} = \pm iC_{i10} \quad .$$

The indeterminacy in the choice of the direction in which to perform the Wick rotation will turn out to be irrelevant after the integration of C_{i10} (indeed, this $\pm$ sign appears in each factor of ϕ and each factor of G, which always come in pairs).

We then get the Euclidean version of (55):

$$I_{\mathrm{E}} = \gamma\int d\tau\,\Big(\beta(\overline{\mathbf{C}}^a_i)^\intercal(\mathbb{I}^{ab}_{ij}+\overline{\Delta}^{ab}_{ij})\overline{\mathbf{C}}^b_j + (\overline{\mathbf{C}}^a_i)^\intercal\overline{\mathbf{F}}^a_i - \mathcal{L}_C - \mathcal{L}_\phi - \widehat{\mathcal{L}}\Big) \quad ,$$

where the new rotated fields assume the following form:

$$\overline{\mathbf{C}}^a_i \triangleq \begin{pmatrix} C^a_{i0} \\ \overline{C}^a_{i10} \end{pmatrix} , \qquad \overline{\mathbf{F}}^a_i \triangleq \begin{pmatrix} -F^a_i \\ \pm iG^a_i \end{pmatrix} ,$$

$$\mathbb{I}^{ab}_{ij} \triangleq \begin{pmatrix} \delta^{ab}\delta_{ij} & 0 \\ 0 & \delta^{ab}\delta_{ij} \end{pmatrix} , \qquad \overline{\Delta}^{ab}_{ij} \triangleq \frac{3f^{abc}}{\beta}\begin{pmatrix} -C^c_{ij} & \pm i\phi^c\delta_{ij} \\ \mp i\phi^c\delta_{ij} & C^c_{ij} \end{pmatrix} .$$

The gaussian integration is straightforward, and yields, after exponentiation of the non trivial part of the determinant:

$$\begin{aligned} &\int D\overline{C}_{i10}\,DC_{i0}\,\exp\Big\{-I_{\mathrm{E}}\Big\} \\ &\quad\propto \exp\Big\{-\frac{1}{2}Tr\Big(\ln(\mathbb{I}^{ab}_{ij}+\overline{\Delta}^{ab}_{ij})\Big) - \gamma\int d\tau\,\Big(-\frac{1}{4\beta}(\overline{\mathbf{F}}^a_i)^\intercal(\mathbb{I}^{ab}_{ij}+\overline{\Delta}^{ab}_{ij})^{-1}\overline{\mathbf{F}}^b_j - \mathcal{L}_C - \mathcal{L}_\phi - \widehat{\mathcal{L}}\Big)\Big\} \quad . \end{aligned}$$

The term quadratic in $\mathbf{F}$ is obviously tree-level, whereas the first one is a 1-loop correction to the effective action. The 1-loop "behaviour" is encoded in the divergence associated with the trace of an operator, since

$$Tr\widehat{O} = \int d\tau\,O^i{}_i(\tau)\langle\tau|\tau\rangle = \Lambda\int d\tau\,O^i{}_i(\tau) \quad , \tag{56}$$

where the integration in Fourier space is divergent, and has been replaced by the cutoff Λ. Transforming back to real Minkowskian time t, we obtain the following effective action

$$I_{\text{eff}} = \gamma \int dt \left(\widehat{\mathcal{L}} + \mathcal{L}_C + \mathcal{L}_\phi + \frac{1}{4\beta} (\overline{\mathbf{F}}_i^a)^\intercal (\mathbb{I}_{ij}^{ab} + \overline{\Delta}_{ij}^{ab})^{-1} \overline{\mathbf{F}}_j^b - \frac{\Lambda}{2\gamma} \big(\ln(\mathbb{I} + \overline{\Delta}(t)) \big)_{ii}^{aa} \right) . \tag{57}$$

5.6 Analysis of the different contributions to the effective action

The natural scale of (57) is β, which is proportional to the mass parameter μ. We therefore expand (57) in powers of $1/\beta$, which amounts to expanding (57) in powers of $\overline{\Delta}$. Now, this procedure must be regarded as a formal expansion, since we don't want to set β to a particular value. However, this formal expansion in $1/\beta$ actually conceals a true expansion in $[X_i, X_j]$, which should be small to minimize the potential energy, as will become clear later on.

First of all, let us consider the expansion of the tree-level term up to $\mathcal{O}(1/\beta^3)$. The first order term is given by:

$$\frac{1}{\beta} \int dt\, (\overline{\mathbf{F}}_i^a)^\intercal \overline{\mathbf{F}}_i^a = \frac{1}{\beta} \int dt\, Tr \Big((F_i)^2 - (G_i)^2 \Big) .$$

Since F_i contains $[\mathcal{D}_t, X_i]$ and $\{\overline{\Psi}, \Psi\}$, while G_i contains only $\{\overline{\Psi}, \Psi\}$ (ignoring Z-type contributions), this term will generate a kinetic term for the X^i's as well as trilinear and quartic interactions.

The second-order term is:

$$\frac{1}{\beta} \int dt\, (\overline{\mathbf{F}}_i^a)^\intercal \overline{\Delta}_{ij}^{ab} \overline{\mathbf{F}}_j^b = \frac{3i}{\beta^2} \int dt\, Tr \Big(C_{ij} \Big\{ [F_i, F_j] - [G_i, G_j] \Big\} - 2\,\phi\, [F_i, G_i] \Big).$$

All vertices generated by this term contain either one C, with 2 to 4 X or Ψ, or one ϕ, with 3 or 4 X or Ψ.

Finally, the third-order contibution is as follows:

$$\begin{aligned} \frac{1}{\beta} \int dt\ \ (\overline{\mathbf{F}}_i^a)^\intercal (\overline{\Delta}^2)_{ij}^{ab} \overline{\mathbf{F}}_j^b = -\frac{3^2}{\beta^3} \int dt\ \ Tr \Big(& [F_i, C_{ij}][C_{jk}, F_k] - [G_i, C_{ij}][C_{jk}, G_k] + \\ & + [F_i, \phi][\phi, F_i] - [G_i, \phi][\phi, G_i] + 2\,[G_i, C_{ij}][\phi, F_j] - 2\,[F_i, C_{ij}][\phi, G_j] \Big) , \end{aligned}$$

producing vertices with 2 ϕ's or 2 C's, together with 2 to 4 X or Ψ, as well as vertices with 1 ϕ or 1 C, with 3 to 4 X or Ψ.

Next we turn to the 1-loop term, where we expand the logarithm up to $\mathcal{O}(1/\beta^3)$. Because of the total antisymmetry of both f^{abc} and C_{ij}, one has $Tr\overline{\Delta} = 0$, so that the first term cancels. Now, keeping in mind that

$$f^{abc} f^{bad} = -C_2(\mathfrak{ad}) \delta^{cd} \qquad \text{and} \qquad f^{amn} f^{bno} f^{com} = \frac{1}{2} C_2(\mathfrak{ad}) f^{abc} \quad ,$$

$C_2(\mathfrak{ad})$ referring to the quadratic Casimir operator in the adjoint representation of the Lie algebra, one readily finds:

(i). $Tr\overline{\Delta}^2 = \left(\frac{3}{\beta}\right)^2 2iC_2(\mathfrak{ad})\Lambda \int dt\, Tr\Big((C_{ij})^2 - 9(\phi)^2 \Big)$, (ii). $Tr\overline{\Delta}^3 \;=\; -\left(\frac{3}{\beta}\right)^3 C_2(\mathfrak{ad})\Lambda \int dt\, Tr\Big(C_{ij}[C_{jk}, C_{ki}]\Big)\,.$

In other words, the 1-loop correction (i) renormalizes the mass terms for C_{ij} and ϕ in $\widetilde{I}_c$ as follows:

- Mass renormalization for C_{ij}: $\frac{1}{2}\gamma\beta \longrightarrow \frac{1}{2}\gamma\beta\left(1+\frac{3^2}{\gamma\beta^3}C_2(\mathfrak{ad})\Lambda\right)$
- Mass renormalization for ϕ: $\gamma\beta \longrightarrow \gamma\beta\left(1+\frac{3^4}{2\gamma\beta^3}C_2(\mathfrak{ad})\Lambda\right)$

Whereas the 1-loop correction (ii) renormalizes the trilinear coupling between the C_{ij} in I_c:

- Renormalization of the $C_{ij}[C_{jk}, C_{ki}]$ coupling: $\gamma \longrightarrow \gamma\left(1-\frac{3^2}{2\gamma\beta^3}C_2(\mathfrak{ad})\Lambda\right)$

Up to $Tr\overline{\Delta}^3$, the 1-loop corrections actually only renormalize terms already present in I_c from the start. This is not the case for the higher order subsequent 1-loop corrections: there is an infinite number of such corrections, each one diverging like Λ. A full quantization of (57) is obviously a formidable task, which we will not attempt in the present paper. A sensible regularization of the divergent contributions should take into account the symmetries of the classical action, which are not explicit anymore after performing T-dualities and the IMF limit. However, since our model is quantum-mechanical, we believe it to be finite even if we haven't come up with a fully quantized formulation.

Summing up the different contributions computed in this section, one gets the following 1-loop effective action up to $\mathcal{O}(1/\beta^3)$:

$$\begin{aligned}
\frac{1}{\gamma}\quad I_{\text{eff}} \;&=\; \int dt\,\left(\mathcal{L}_C + \mathcal{L}_\phi + \widehat{\mathcal{L}}\right) + \frac{\gamma}{4\beta}\int dt\; Tr\left(F_i^2 - G_i^2\right) - \\
&- \frac{3i\gamma}{4\beta^2}\int dt\, Tr\bigg(C_{ij}\Big([F_i,F_j]-[G_i,G_j]\Big) - 2\,\phi\,[F_i,G_i]\bigg) + \frac{\gamma\lambda}{2\beta^2}\int dt\, Tr\bigg(C_{ij}^2 - 9\phi^2\bigg) - \\
&- \frac{9\gamma}{4\beta^3}\int dt\, Tr\bigg([F_i,C_{ij}][C_{jk},F_k] \,-\, [G_i,C_{ij}][C_{jk},G_k] \,+\, [F_i,\phi][\phi,F_i] \,-\, [G_i,\phi][\phi,G_i] \,+ \\
&+ 2\,[G_i,C_{ij}][\phi,F_j] \,-2\,[F_i,C_{ij}][\phi,G_j]\bigg) - \frac{i\lambda\gamma}{2\beta^3}\int dt\, Tr\bigg(\,C_{ij}\,[C_{jk},C_{ki}]\bigg) \,+\, \mathcal{O}(1/\beta^4) \quad . \qquad (58)
\end{aligned}$$

where λ is proportional to the cutoff Λ:

$$\lambda \;\triangleq\; \frac{9\,C_2(\mathfrak{ad})\Lambda}{\gamma} \quad .$$

Note that the $\mathcal{O}(1/\beta^4)$ terms that we haven't written contain at least three powers of C_{ij} or ϕ.

5.7 Iterative solution of the constraint equations

The 1-loop corrected action (58) still contains the constraint fields C_{ij} and ϕ, which should in principle be integrated out in order to get the final form of the effective action. Since I_{eff} contains arbitrarily high powers of C_{ij} and ϕ, we cannot perform a full path integration. We can however solve the equations for C_{ij} and ϕ perturbatively in $1/\beta$. This allows to replace these fields in (58) with the solution to their equations of motion. Thus, in contrast with the preceeding subsection, here we remain at tree-level.

The equation of motion for C_{ij} may be computed from (58), and reads:

$$
\begin{aligned}
&C_{ij} + \frac{1}{\beta}\Big(E_{ij} + 3i[C_{jk}, C_{ki}]\Big) + \frac{1}{\beta^3}\left(\frac{3}{4}i\Big\{[G_i, G_j] - [F_i, F_j]\Big\} + \lambda C_{ij}\right) + \\
&\quad + \frac{1}{\beta^4}\frac{9}{2}\left(\Big\{[F_{[i}, [C_{j]k}, F_k]] - [G_{[i}, [C_{j]k}, G_k]] + [G_{[i}, [\phi, F_{j]}]] - [F_{[i}, [\phi, G_{j]}]]\Big\} + \right. \\
&\quad \left. - \frac{i\lambda}{3}[C_{jk}, C_{ki}]\right) + \mathcal{O}(1/\beta^5) = 0 \,, \qquad (59)
\end{aligned}
$$

while the equation of motion for ϕ is:

$$
\begin{aligned}
&\phi - \frac{1}{2\beta}J - \frac{3}{\beta^3}\left(\frac{i}{4}[F_i, G_i] - 3\lambda\phi\right) + \frac{3^2}{4\beta^4}\Big([F_i, [F_i, \phi]] - \\
&\quad - [G_i, [G_i, \phi]] + [F_i, [C_{ij}, G_j]] - [G_i, [C_{ij}, F_j]]\Big) + \mathcal{O}(1/\beta^5) = 0 \,. \qquad (60)
\end{aligned}
$$

By solving the coupled equations of motion (59) and (60) recursively, one gets C_{ij} and ϕ up to $\mathcal{O}(1/\beta^5)$. We can safely stop at $\mathcal{O}(1/\beta^5)$, because the terms contributing to that order in (59) and (60) are, on the one hand, $\beta^{-1}\Lambda(\delta/\delta C_{ij})Tr\overline{\Delta}^4$ and $\beta^{-1}\Lambda(\delta/\delta\phi)Tr\overline{\Delta}^4$, whose lowest order is $\mathcal{O}(1/\beta^8)$, and on the other hand $\beta^{-2}(\delta/\delta C_{ij})\mathbf{F}^{\intercal}\overline{\Delta}^3\mathbf{F}$ and $\beta^{-2}(\delta/\delta\phi)\mathbf{F}^{\intercal}\overline{\Delta}^3\mathbf{F}$, whose lowest order is $\mathcal{O}(1/\beta^7)$, so that the eom don't get any corrections from contributions of $\mathcal{O}(1/\beta^4)$ coming from I_{eff}.

Subsequently, the $1/\beta$ expansion for C_{ij} reads

$$
\begin{aligned}
C_{ij} &= -\frac{1}{\beta}E_{ij} + \frac{3i}{\beta^3}\left([E_{ik}, E_{kj}] + \frac{1}{4}[F_i, F_j] - \frac{1}{4}[G_i, G_j]\right) + \frac{\lambda}{\beta^4}E_{ij} + \\
&+ \frac{9}{\beta^5}\Big(-2[E_{[ik}, [E_{kl}, E_{lj]}]] + \frac{1}{2}[E_{[ik}, [F_k, F_{j]}]] - \frac{1}{2}[E_{[ik}, [G_k, G_{j]}]] + \frac{1}{2}[[E_{[ik}, F_k], F_{j]}] - \\
&- \frac{1}{2}[[E_{[ik}, G_k], G_{j]}] + \frac{1}{4}[G_{[i}, [F_{j]}, J]] - \frac{1}{4}[F_{[i}, [G_{j]}, J]]\Big) + \mathcal{O}(1/\beta^6) \quad , \qquad (61)
\end{aligned}
$$

and the expansion for ϕ:

$$
\begin{aligned}
\phi &= \frac{1}{2\beta}J + \frac{3i}{4\beta^3}[F_i, G_i] - \left(\frac{3}{2}\right)^2\frac{\lambda}{\beta^4}J - \qquad (62) \\
&- \frac{9}{8\beta^5}\Big([F_i, [F_i, J]] - [G_i, [G_i, J]] - 2[[F_i, E_{ij}], G_j] + 2[[G_i, E_{ij}], F_j]\Big) + \mathcal{O}(1/\beta^6) \,.
\end{aligned}
$$

Now, plugging the result for C_{ij} and ϕ into I_{eff}, one arrives at the "perturbative" effective action, which we have written up to and including $\mathcal{O}(1/\beta^5)$, since the highest order ($\mathcal{O}(1/\beta^3)$) we calculated in I_{eff} is quadratic in C and ϕ[6], and since the $\mathcal{O}(1/\beta^4)$-terms in (58) only generate $\mathcal{O}(1/\beta^7)$ - terms.

[6]note that their expansion starts at $\mathcal{O}(1/\beta)$

This effective action takes the following form:

$$\begin{aligned}
\frac{1}{\gamma}\, I_{\text{eff}} \;&=\; \int dt \Bigg(\widehat{\mathcal{L}} + \frac{1}{4\beta}\, Tr\left(F_i^2 - G_i^2 + J^2 - 2\,(E_{ij})^2\right) + \\
&+ \frac{i}{\beta^3}\, Tr\Big(- E_{ij}[E_{jk}, E_{ki}] \,+\, \frac{3}{4} E_{ij} \Big\{ [F_i, F_j] - [G_i, G_j] \Big\} + \frac{3}{4} J [F_i, G_i] \Big) \Bigg) + \\
&+ \frac{\lambda}{2\beta^4}\, Tr\Big((E_{ij})^2 - \frac{9}{4} J^2 \Big) + \frac{9}{2\beta^5}\, Tr\Big(([E_{ik}, E_{kj}])^2 + \frac{1}{16} ([F_i, F_j] - [G_i, G_j])^2 - \\
&+ \frac{1}{2} [E_{ik}, E_{kj}] \big([F_i, F_j] - [G_i, G_j] \big) - \frac{1}{8} ([F_i, G_i])^2 - \frac{1}{2} \Big\{ ([F_i, E_{ij}])^2 - ([G_i, E_{ij}])^2 \Big\} + \\
&+ \frac{1}{4} \Big\{ ([F_i, J])^2 - ([G_i, J])^2 \Big\} - \frac{1}{2} [G_i, E_{ij}][J, F_j] + \frac{1}{2} [F_i, E_{ij}][J, G_j] \Big) \Bigg) + \mathcal{O}(1/\beta^6)\,.
\end{aligned}$$

At that point, we can replace the aliases E, F, G and J by their expression in terms of the fundamental fields X, Ψ, Z, Π, B and H. The result of this lengthy computation (already to order $1/\beta$) is presented in the Appendix. Here, we will only display the somewhat simpler result obtained by ignoring all 5-form induced fields. Furthermore, we will remove the parameter β from the action, since it was only useful as a reminder of the order of calculation in the perturbative approach. To do so, we absorb a factor of $1/\beta$ in every field, as well as in $\mathcal{D}_t$ (so that the measure of integration scales with β). Thus, β only appears in the prefactor in front of the action, at the 4^{th} power. This is similar to the case of Yang-Mills theory, where one can choose either to have a factor of the coupling constant in the covariant derivatives or have it as a prefactor in front of the action. To be more precise, we set:

$$\Theta \;=\; \frac{1}{4\sqrt{6}\beta}\, \Psi\,, \quad \widetilde{X}_i \;=\; \frac{1}{\beta}\, X_i\,, \quad \widetilde{A}_0 \;=\; \frac{1}{\beta}\, A_0\,, \quad G \;=\; 9\beta^4\, \gamma\,, \quad \tilde{t} \;=\; \beta t\;,$$

and similarily for the Z sector: $(Z, \Pi, H, B) \to (Z/\beta, \Pi/\beta, H/\beta, B/\beta)$.

With this redefinition, it becomes clear that our developpment is really an expansion in higher commutators and not in β. It makes thus sense to limit it to the lowest orders since the commutators should remain small to minimize the potential energy. To get a clearer picture of the final result, we will put all the 5-form-induced fields (Z, Π, H, B) to zero. For convenience we will still write $\widetilde{X}$ as X

and $\tilde{t}$ as t in the final result, which reads:

$$
\begin{aligned}
I(X,\Theta) =& \frac{1}{G}\int dt\, Tr_{\mathfrak{u}(N)}\Bigg(([\mathcal{D}_t, X_i])^2 + \frac{1}{2}([X_i, X_j])^2 + i\overline{\Theta}\tilde{\Gamma}_0[\mathcal{D}_t,\Theta] - \overline{\Theta}\tilde{\Gamma}_i[X_i,\Theta] - \\
& -\frac{1}{9}(X_i)^2 - \frac{2i}{3}\overline{\Theta}\Theta - 3[\mathcal{D}_t, X_i]\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\} - \frac{3i}{2}[X_i,X_j]\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\} + \\
& +\frac{9}{4}(\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\})^2 - \frac{9}{4}(\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\})^2 + \frac{9}{4}(\{\overline{\Theta},\tilde{\Gamma}_0\tilde{\Gamma}_*\Theta\})^2 - \frac{9}{8}(\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\})^2 + \\
& +3[X_i,X_j][[X_j,X_k],[X_k,X_i]] - 9[X_i,X_j][[\mathcal{D}_t,X_i],[\mathcal{D}_t,X_j]] - \\
& -\frac{3^3 i}{2}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[[X_j,X_k],[X_k,X_i]] + \frac{3^4}{2^2}[X_i,X_j][\{\overline{\Theta},\tilde{\Gamma}_{jk}\Theta\},\{\overline{\Theta},\tilde{\Gamma}_{ki}\Theta\}] - \\
& -\frac{3^4 i}{2^3}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[\{\overline{\Theta},\tilde{\Gamma}_{jk}\Theta\},\{\overline{\Theta},\tilde{\Gamma}_{ki}\Theta\}] + \frac{3^3 i}{2}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[[\mathcal{D}_t,X_i],[\mathcal{D}_t,X_j]] + \\
& +3^3[X_i,X_j][[\mathcal{D}_t,X_i],\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_0\Theta\}] - \frac{3^4 i}{2^2}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[[\mathcal{D}_t,X_i],\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_0\Theta\}] - \\
& -\frac{3^4}{2^2}[X_i,X_j][\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\},\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_0\Theta\}] + \frac{3^5 i}{2^3}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\},\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_0\Theta\}] + \\
& +\frac{3^4}{2^2}[X_i,X_j][\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\},\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_*\Theta\}] - \frac{3^5 i}{2^3}\{\overline{\Theta},\tilde{\Gamma}_{ij}\Theta\}[\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\},\{\overline{\Theta},\tilde{\Gamma}_j\tilde{\Gamma}_*\Theta\}] - \\
& -\frac{3^4 i}{2}\{\overline{\Theta},\tilde{\Gamma}_0\tilde{\Gamma}_*\Theta\}[[\mathcal{D}_t,X_i],\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\}] +, \frac{3^5 i}{2}\{\overline{\Theta},\tilde{\Gamma}_0\tilde{\Gamma}_*\Theta\}[\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_0\Theta\},\{\overline{\Theta},\tilde{\Gamma}_i\tilde{\Gamma}_*\Theta\}]\Bigg) + \\
& + \text{eighth-order interactions.}
\end{aligned}
$$

We see that the first four terms in this action correspond to the BFSS matrix model, but with a doubled number of fermions. So, in order to maintain half of the original supersymmetries (i.e. $\mathcal{N}=1$ in $10D$), one could project out half of the original fermions with $\mathcal{P}_- \xrightarrow{\text{IMF}} (1+\tilde{\Gamma}_*)/2$. Finally, in addition to the BFSS-like terms, we have mass terms and an infinite tower of interactions possibly containing information about the behaviour of brane dynamics in the non-perturbative sector.

6 Discussion

After a general description of $\mathfrak{osp}(1|32)$ and its adjoint representation, we have studied its expression as a symmetry algebra in $12D$. We have described the resulting transformations of matrix fields and their commutation relations. Finally, we have proposed a matrix theory action possessing this symmetry in $12D$. We have then repeated this analysis in the 11-dimensional case, where $\mathfrak{osp}(1|32)$ is a sort of super-AdS algebra. Compactification and T-dualization of two coordinates produced a one-parameter family of singular limiting procedures that shrink the world-sheet along a world-line. We have then identified one of them as the usual IMF limit, which gave rise to a non-compact dynamical evolution parameter that has allowed us to distinguish dynamical from auxiliary fields. Integrating out the latter and solving some constraints recursively, we have obtained a matrix model with a highly non-trivial dynamics, which is similar to the BFSS matrix model when both X^2 and multiple commutators are small. The restriction to a low-energy sector where both X^2 and $[X,X]$ are small seems to correspond to a space-time with weakly interacting (small $[X, X]$) D-particles that are nevertheless not far apart (small X^2). The stable classical solutions correspond to vanishing matrices, i.e. to D-particles stacked at the origin, which diplays some common features with matrix models in pp-wave backgrounds (see for instance [20, 21, 22]).

Since the promotion of the membrane charges in the $11D$ super-Poincaré algebra to symmetry generators implied the non-commutativity of the P's, and thus the AdS_{11} symmetry, the membranes are responsible for some background curvature of the space-time. Indeed, since the C_{MN} don't appear as dynamical degrees of freedom, their rôle is to produce the precise tower of higher-order interactions necessary to enforce such a global symmetry on the space-time dynamically generated by the X_i's. The presence of mass terms is thus no surprise since they were also conjectured to appear in matrix models aimed at describing gravity in deSitter spaces, albeit with a tachyonic sign reflecting the unusual causal structure of deSitter space ([16, 17]). One might also wonder whether the higher interaction terms we get are somehow related to the high energy corrections to BFSS one would obtain from the non-abelian Dirac-Born-Infeld action. Another question one could address is what kind of corrections a term of the form $STr_{\mathfrak{osp}(1|32)\otimes\mathfrak{u}(n)}([M, M][M, M])$ would induce.

It would also be interesting to investigate the dynamics of the 5-branes degrees of freedom more thoroughly by computing the effective action for Z (from I_{eff} of the Appendix) and give a definite proposal for the physics of 5-branes in M-theory. Note that there is some controversy about the ability of the BFFS model to describe transverse 5-branes (see e.g. [23, 24] and references therein for details). Our model would provide an interesting extension of the BFSS theory by introducing in a very natural way transverse 5-branes (through the fields dual to Z_{ijkl}) in addition to the D0-branes bound states describing longitudinal 5-branes, which are already present in BFSS theory.

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8 Appendix

We give here the complete effective action at order $1/\beta$.

$$
\begin{aligned}
I_{\mathrm{eff}} \;=\;& \frac{1}{G}\int dt\, Tr_{\mathfrak{u}(N)}\Bigg(-\beta\Big\{(X_i)^2+\frac{i}{16}\,\overline{\Psi}\Psi+\frac{1}{4!}\Big((Z_{i_1\cdots i_4})^2+(\Pi_{i_1\cdots i_4})^2-(H_{i_1\cdots i_4})^2-4\,(B_{i_1i_2i_3})^2\Big)\Big\}+\\
+\;& \Big\{\frac{1}{4}\Pi_{i_1\cdots i_4}[\mathcal{D}_t,Z_{i_1\cdots i_4}]+\frac{3i}{32}\overline{\Psi}\widetilde{\Gamma}_0[\mathcal{D}_t,\Psi]+i\Pi_{i_1i_2i_3\,j}[X_j,B_{i_1i_2i_3}]-\frac{i}{4\cdot 4!}\varepsilon_{i_1\cdots i_8j}Z_{i_1\cdots i_4}[X_j,H_{i_5\cdots i_8}]+\\
+\;& \frac{i}{3!\cdot 4!}\varepsilon_{i_1\cdots i_9}\Bigg(B_{i_1i_2\,j}\Big(2\,[\Pi_{j\,i_3i_4i_5},\Pi_{i_6\cdots i_9}]+[Z_{j\,i_3i_4i_5},Z_{i_6\cdots i_9}]-[H_{j\,i_3i_4i_5},H_{i_6\cdots i_9}]\Big)+\\
+\;& \frac{2}{3}\,B_{i_1i_2i_3}\Big([B_{i_4i_5i_6},B_{i_7i_8i_9}]+[Z_{i_4i_5i_6\,j},Z_{j\,i_7i_8i_9}]-[H_{i_4i_5i_6\,j},H_{j\,i_7i_8i_9}]\Big)\Bigg)+\\
+\;& \frac{i}{4}\,\Pi_{i_1i_2j_1j_2}\,[Z_{j_1j_2k_1k_2},H_{k_1k_2i_1i_2}]-\frac{3}{32}\Bigg(\overline{\Psi}\,\widetilde{\Gamma}_i\,[X_i,\Psi]+\frac{1}{4!}\,\overline{\Psi}\,\Big(\widetilde{\Gamma}_{i_1\cdots i_4}\widetilde{\Gamma}_*[Z_{i_1\cdots i_4},\Psi]+\\
+\;& \widetilde{\Gamma}_{i_1\cdots i_4}\widetilde{\Gamma}_0\widetilde{\Gamma}_*[\Pi_{i_1\cdots i_4},\Psi]-\widetilde{\Gamma}_{i_1\cdots i_4}\widetilde{\Gamma}_0[H_{i_1\cdots i_4},\Psi]-4\,\widetilde{\Gamma}_{i_1i_2i_3}\widetilde{\Gamma}_0\widetilde{\Gamma}_*[B_{i_1i_2i_3},\Psi]\Big)\Bigg)\Big\}+\\
+\;\frac{1}{4\beta}\;& \Big\{36\,([\mathcal{D}_t,X_i])^2-\frac{i}{8}\varepsilon_{ij_1\cdots j_8}[\mathcal{D}_t,X_i][H_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}]-12i\,[\mathcal{D}_t,X_i][Z_{ij_1\cdots j_3},B_{j_1\cdots j_3}]-\\
-\;& \frac{9}{8}[\mathcal{D}_t,X_i]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\}-\frac{1}{16}[H_{i_1\cdots i_4},\Pi_{j_1\cdots j_4}]\Big([H_{i_1\cdots i_4},\Pi_{j_1\cdots j_4}]-16[H_{i_1i_2i_3j_4},\Pi_{j_1j_2j_3i_4}]+\\
+\;& 36[H_{i_1i_2j_3j_4},\Pi_{j_1j_2i_3i_4}]-16[H_{i_1j_2j_3j_4},\Pi_{j_1i_2i_3i_4}]+[H_{j_1j_2j_3j_4},\Pi_{i_1i_2i_3i_4}]\Big)-\\
-\;& \frac{1}{2\cdot 4!}\varepsilon_{ij_1\cdots j_8}[H_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}][Z_{ik_1\cdots k_3},B_{k_1\cdots k_3}]+\frac{i}{2^9}\varepsilon_{ij_1\cdots j_8}[H_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\}-\\
-\;& ([Z_{ij_1\cdots j_3},B_{j_1\cdots j_3}])^2+\frac{3i}{16}[Z_{ij_1\cdots j_3},B_{j_1\cdots j_3}]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\}+\frac{9}{2^{10}}(\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_0\Psi\})^2+\\
+\;& \frac{1}{16}\,[Z_{i_1\cdots i_4},\Pi_{j_1\cdots j_4}]\Big([Z_{i_1\cdots i_4},\Pi_{j_1\cdots j_4}]-16[Z_{i_1i_2i_3j_4},\Pi_{j_1j_2j_3i_4}]+36[Z_{i_1i_2j_3j_4},\Pi_{j_1j_2i_3i_4}]-\\
-\;& 16[Z_{i_1j_2j_3j_4},\Pi_{j_1i_2i_3i_4}]+[Z_{j_1j_2j_3j_4},\Pi_{i_1i_2i_3i_4}]\Big)-\frac{1}{2\cdot 4!}\varepsilon_{ij_1\cdots j_8}[Z_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}][H_{ik_1\cdots k_3},B_{k_1\cdots k_3}]-\\
-\;& \frac{i}{2^9}\varepsilon_{ij_1\cdots j_8}[Z_{j_1\cdots j_4},\Pi_{j_5\cdots j_8}]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_*\Psi\}+([H_{ij_1\cdots j_3},B_{j_1\cdots j_3}])^2+\frac{3i}{16}[H_{ij_1\cdots j_3},B_{j_1\cdots j_3}]\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_*\Psi\}-\\
-\;& \frac{9}{2^{10}}(\{\overline{\Psi},\widetilde{\Gamma}_i\widetilde{\Gamma}_*\Psi\})^2-\frac{1}{16}([Z_{i_1\cdots i_4},H_{i_4\cdots i_4}])^2+\frac{3i}{2^6}[Z_{i_1\cdots i_4},H_{i_4\cdots i_4}]\{\overline{\Psi},\widetilde{\Gamma}_0\widetilde{\Gamma}_*\Psi\}+\frac{9}{2}([B_{ik_1k_2},B_{jk_1k_2}])^2+\\
+\;& \frac{1}{2}([Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}])^2+[Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}][\Pi_{il_1l_2l_3},\Pi_{jl_1l_2l_3}]+\frac{1}{2}([\Pi_{ik_1k_2k_3},\Pi_{jk_1k_2k_3}])^2-\\
-\;& 3[Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}][B_{il_1l_2},B_{jl_1l_2}]-[Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}][H_{il_1l_2l_3},H_{jl_1l_2l_3}]+\frac{9}{2^{10}}(\{\overline{\Psi},\widetilde{\Gamma}_0\widetilde{\Gamma}_*\Psi\})^2-\\
-\;& 3\,[\Pi_{ik_1k_2k_3},\Pi_{jk_1k_2k_3}][B_{il_1l_2},B_{jl_1l_2}]-[\Pi_{ik_1k_2k_3},\Pi_{jk_1k_2k_3}][H_{il_1l_2l_3},H_{jl_1l_2l_3}]+\\
+\;& 3\,[B_{ik_1k_2},B_{jk_1k_2}][H_{il_1l_2l_3},H_{jl_1l_2l_3}]+\frac{1}{2}([H_{ik_1k_2k_3},H_{jk_1k_2k_3}])^2-6\,[Z_{ik_1k_2k_3},Z_{jk_1k_2k_3}][X_i,X_j]+
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{3i}{32}[Z_{ik_1k_2k_3}, Z_{jk_1k_2k_3}]\{\overline{\Psi}, \widetilde{\Gamma}_{ij}\Psi\} - 6[\Pi_{ik_1k_2k_3}, \Pi_{jk_1k_2k_3}][X_i, X_j] + \frac{3i}{32}[\Pi_{ik_1k_2k_3}, \Pi_{jk_1k_2k_3}]\{\overline{\Psi}, \widetilde{\Gamma}_{ij}\Psi\} + \\
&+ 18\,[B_{ik_1k_2}, B_{jk_1k_2}][X_i, X_j] - \frac{9i}{32}[B_{ik_1k_2}, B_{jk_1k_2}]\{\overline{\Psi}, \widetilde{\Gamma}_{ij}\Psi\} + 6\,[H_{ik_1k_2k_3}, H_{jk_1k_2k_3}][X_i, X_j] - \\
&- \frac{3i}{32}[H_{ik_1k_2k_3}, H_{jk_1k_2k_3}]\{\overline{\Psi}, \widetilde{\Gamma}_{ij}\Psi\} + 18\,([X_i, X_j])^2 - \frac{9i}{16}[X_i, X_j]\{\overline{\Psi}, \widetilde{\Gamma}_{ij}\Psi\} - \frac{9}{2^{11}}(\{\overline{\Psi}, \widetilde{\Gamma}_{ij}\Psi\})^2 \Bigg\} + \\
&+ \mathcal{O}(1/\beta^3)\Bigg) .
\end{aligned}
$$

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