q-Integration on Quantum Spaces

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Abstract

In this article we present explicit formulae for q-integration on quantum spaces which could be of particular importance in physics, i.e. q-deformed Minkowski space and q-deformed Euclidean space in 3 or 4 dimensions. Furthermore, our formulae can be regarded as a generalization of Jackson's q-integral to 3 and 4 dimensions.

1 Introduction

One might say the ideas of differential calculus are as old as physical science itself. Since its invention by J. Newton and G.W. Leibniz there hasn't been a necessity for an essential change. Although this can be seen as a great success one cannot ignore the fact that up to now physicists haven't been able to present a unified description of nature by using this traditional tool, i.e., a theory which does not break down at any possible space-time distances.

Quantum spaces, however, which are defined as co-module algebras of quantum groups and which can be interpreted as deformations of ordinary co-ordinate algebras [1] could provide a proper framework for developing a new kind of non-commutative analysis [2], [3]. For our purposes it is sufficient to consider a quantum space as an algebra $\mathcal{A}_q$ of formal power series in the non-commuting co-ordinates $X_1, X_2, \ldots, X_n$

$$\mathcal{A}_q = \mathbb{C}\left[[X_1, \ldots X_n]\right]/\mathcal{I} \tag{1}$$

where $\mathcal{I}$ denotes the ideal generated by the relations of the non-commuting co-ordinates.

The algebra $\mathcal{A}_q$ satisfies the Poincaré-Birkhoff-Witt property, i.e., the dimension of the subspace of homogenous polynomials should be the same as for commuting co-ordinates. This property is the deeper reason why the monomials of normal ordering $X_1 X_2 \ldots X_n$ constitute a basis of $\mathcal{A}_q$. In particular, we can establish a vector space

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isomorphism between $\mathcal{A}_q$ and the commutative algebra $\mathcal{A}$ generated by ordinary coordinates $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} \mathcal{W} &: \mathcal{A} \longrightarrow \mathcal{A}_q, \\ \mathcal{W}\left(x_1^{i_1} \ldots x_n^{i_n}\right) &= X_1^{i_1} \ldots X_n^{i_n}. \end{aligned} \tag{2}$$

This vector space isomorphism can be extended to an algebra isomorphism introducing a non-commutative product in $\mathcal{A}$, the so-called $\star$-product [4], [5]. This product is defined by the relation

$$\mathcal{W}\left(f \star g\right) = \mathcal{W}\left(f\right) \cdot \mathcal{W}\left(g\right) \tag{3}$$

where f and g are formal power series in $\mathcal{A}$. In [6] we have calculated the $\star$-product for quantum spaces which could be of particular importance in physics, i.e., q-deformed Minkowski space and q-deformed Euclidean space in three or four dimensions.

Additionally, for each of these quantum spaces exists a symmetry algebra [7], [8] and a covariant differential calculus [9], which can provide an action upon the quantum spaces under consideration. If $\mathcal{H}$ is an algebra which acts on $\mathcal{A}$ covariantly we are also able to introduce an action upon the corresponding commutative algebra by means of the relation

$$\mathcal{W}\left(h \triangleright f\right) := h \triangleright \mathcal{W}\left(f\right), \quad h \in \mathcal{H},\, f \in \mathcal{A}\,,. \tag{4}$$

In our previous work [10] we have presented explicit formulae for such representations.

In the following it is our aim to derive some sort of q-integration on commutative algebras which are, via the algebra isomorphism $\mathcal{W}$, related to q-deformed Minkowski space or q-deformed Euclidean space in three and four dimensions. In some sense our considerations can be thought of as a generalization of the celebrated Jackson-integral [11] to higher dimensions. To this end we will start with introducing new elements which are formally inverse to the partial derivatives of the given covariant differential calculi. Consequently, such an extension of the algebra of partial derivatives will lead to additional commutation relations. Finally, the representations in [10] will aid us in identifying this new elements with particular solutions of some q-difference equations.

In this way, we can interpret our results as a method to discretise classical integrals of more than one dimension. It is worth noting that from a mathematical point of view it is not quite clear for which functions apart from polynomials our integral expressions will converge. However, because of some physical arguments, we will cover later on, it is reasonable to assume that functions which do not lead to finite integrals should be of no relevance in physics.

Furthermore, it is necessary to distinguish between left and right integrals. For both types formulae for integration by parts can be derived. It is also possible to define volume integrals which are invariant under translations or the action of symmetry generators if surface terms are neglected.

2 q-deformed Euclidean Space in three Dimensions

The formulae for the $\star$-product in [6] allow us to introduce inverse objects on the co-ordinate algebra of the quantum spaces under consideration. Furthermore, the commutation relations the new elements have to be subject to can be read off from these formulae quite easily. As the partial derivatives obey the same commutation relations as the non-commutative co-ordinates [12], namely

$$\partial^3\partial^+ = q^2\partial^+\partial^3, \quad \partial^-\partial^3 = q^2\partial^3\partial^-, \quad \partial^-\partial^+ = \partial^+\partial^- + \lambda\left(\partial^3\right)^2 \tag{5}$$

where $\lambda = q - q^{-1}$ and $q > 1$, we can also apply this method in order to introduce elements $\left(\partial^A\right)^{-1}$, $A = \pm, 3$, with

$$\begin{aligned} \partial^+\left(\partial^+\right)^{-1} &= \left(\partial^+\right)^{-1}\partial^+ = 1, \\ \partial^3\left(\partial^3\right)^{-1} &= \left(\partial^3\right)^{-1}\partial^3 = 1, \\ \partial^-\left(\partial^-\right)^{-1} &= \left(\partial^-\right)^{-1}\partial^- = 1. \end{aligned} \tag{6}$$

Then the remaining commutation relations take the form

$$\begin{aligned} \left(\partial^3\right)^{-1}\partial^+ &= q^{-2}\partial^+\left(\partial^3\right)^{-1}, \quad \partial^3\left(\partial^+\right)^{-1} = q^{-2}\left(\partial^+\right)^{-1}\partial^3, \\ \left(\partial^-\right)^{-1}\partial^3 &= q^{-2}\partial^3\left(\partial^-\right)^{-1}, \quad \partial^-\left(\partial^3\right)^{-1} = q^{-2}\left(\partial^3\right)^{-1}\partial^-, \\ \left(\partial^-\right)^{-1}\partial^+ &= \partial^+\left(\partial^-\right)^{-1} - q^{-4}\lambda\left(\partial^3\right)^2\left(\partial^-\right)^{-2}, \\ \partial^-\left(\partial^+\right)^{-1} &= \left(\partial^+\right)^{-1}\partial^- - q^{-4}\lambda\left(\partial^+\right)^{-2}\left(\partial^3\right)^2. \end{aligned} \tag{7}$$

In addition, the inverse elements $\left(\partial^A\right)^{-1}$, $A = \pm, 3$, have to be subject to the following identities:

$$\begin{aligned} \left(\partial^3\right)^{-1}\left(\partial^+\right)^{-1} &= q^2\left(\partial^+\right)^{-1}\left(\partial^3\right)^{-1}, \quad \left(\partial^-\right)^{-1}\left(\partial^3\right)^{-1} = q^2\left(\partial^3\right)^{-1}\left(\partial^-\right)^{-1}, \\ \left(\partial^-\right)^{-1}\left(\partial^+\right)^{-1} &= \sum_{i=0}^{\infty}\lambda^i\,[[i]]_{q^4}!\left(\begin{bmatrix} -1 \\ i \end{bmatrix}_{q^4}\right)^2\left(\partial^+\right)^{-(i+1)}\left(\partial^3\right)^{2i}\left(\partial^-\right)^{-(i+1)}. \end{aligned} \tag{8}$$

For the commutation relations with the symmetry generators we find the expressions

$$\begin{aligned} L^+\left(\partial^+\right)^{-1} &= \left(\partial^+\right)^{-1}L^+, \\ L^+\left(\partial^3\right)^{-1} &= \left(\partial^3\right)^{-1}L^+ + q^{-1}\partial^+\left(\partial^3\right)^{-2}\tau^{-\frac{1}{2}}, \\ L^+\left(\partial^-\right)^{-1} &= \left(\partial^-\right)^{-1}L^+ + q^{-1}\partial^3\left(\partial^-\right)^{-2}\tau^{-\frac{1}{2}}, \end{aligned} \tag{9}$$

$$\begin{aligned} [0.1in]L^-\left(\partial^-\right)^{-1} &= \left(\partial^-\right)^{-1}L^-, \\ L^-\left(\partial^3\right)^{-1} &= \left(\partial^3\right)^{-1}L^- - q^{-3}\left(\partial^3\right)^{-2}\partial^-\tau^{-\frac{1}{2}}, \\ L^-\left(\partial^+\right)^{-1} &= \left(\partial^+\right)^{-1}L^- - q^{-4}\left(\partial^+\right)^{-2}\partial^3\tau^{-\frac{1}{2}}, \end{aligned} \tag{10}$$

$$
\begin{aligned}
[0.1in]\tau^{-\frac{1}{2}}\left(\partial^{+}\right)^{-1} &= q^{-2}\left(\partial^{+}\right)^{-1}\tau^{-\frac{1}{2}}, \qquad (11)\\
\tau^{-\frac{1}{2}}\left(\partial^{-}\right)^{-1} &= q^{2}\left(\partial^{-}\right)^{-1}\tau^{-\frac{1}{2}},\\
\tau^{-\frac{1}{2}}\left(\partial^{3}\right)^{-1} &= \left(\partial^{3}\right)^{-1}\tau^{-\frac{1}{2}},\\
[0.1in]\Lambda^{-\frac{1}{2}}\left(\partial^{A}\right)^{-1} &= q^{2}\left(\partial^{A}\right)^{-1}\Lambda^{\frac{1}{2}}, \quad A=\pm,3. \qquad (12)
\end{aligned}
$$

Applying the substitutions

$$
\partial^{A}\rightarrow\hat{\partial}^{A}, \quad \left(\partial^{A}\right)^{-1}\rightarrow\left(\hat{\partial}^{A}\right)^{-1}, \quad A=\pm,3 \qquad (13)
$$

and

$$
\partial^{A}\rightarrow P^{A}, \quad \left(\partial^{A}\right)^{-1}\rightarrow\left(P^{A}\right)^{-1}, \quad A=\pm,3 \qquad (14)
$$

to all expressions presented so far we get the corresponding relations of the second differential calculus (generated by the conjugated partial derivatives $\hat{\partial}^{A}$) and that of the algebra of hermitean momentum generators P^{A}, respectively [12].

In [10] it was shown that according to

$$
\partial^{A}F=\left(\left(\partial_{(i=0)}^{A}\right)+\left(\partial_{(i>0)}^{A}\right)\right)F \qquad (15)
$$

the representations of our partial derivatives can be divided up into a classical part and corrections vanishing in the undeformed limit $q\rightarrow 1$. Thus, seeking a solution to the equation $\partial^{A}F=f$, for given f it is reasonable to consider the following expression:

$$
\begin{aligned}
F &= \left(\partial^{A}\right)^{-1}f=\frac{1}{\left(\partial_{(i=0)}^{A}\right)+\left(\partial_{(i>0)}^{A}\right)}f \qquad (16)\\
&= \frac{1}{\left(\partial_{(i=0)}^{A}\right)\left(1+\left(\partial_{(i=0)}^{A}\right)^{-1}\left(\partial_{(i>0)}^{A}\right)\right)}f\\
&= \frac{1}{\left(1+\left(\partial_{(i=0)}^{A}\right)^{-1}\left(\partial_{(i>0)}^{A}\right)\right)}\cdot\frac{1}{\left(\partial_{(i=0)}^{A}\right)}f\\
&= \sum_{k=0}^{\infty}(-1)^{k}\left[\left(\partial_{(i=0)}^{A}\right)^{-1}\left(\partial_{(i>0)}^{A}\right)\right]^{k}\left(\partial_{(i=0)}^{A}\right)^{-1}f.
\end{aligned}
$$

With this formula at hand the new elements $\left(\partial^{A}\right)^{-1}$, $A=\pm,3,$ can be represented on the commutative algebra as

$$
\begin{aligned}
\left(\partial^{-}\right)_{L}^{-1}f &= -q\left(D_{q^{4}}^{+}\right)^{-1}f, \qquad (17)\\
\left(\partial^{3}\right)_{L}^{-1}f &= \left(D_{q^{2}}^{3}\right)^{-1}f\left(q^{-2}x^{+}\right),
\end{aligned}
$$

$$
\begin{aligned}
\left(\partial^{+}\right)_L^{-1} f &= -q^{-1}\sum_{k=0}^{\infty}(-\lambda)^k q^{2k(k+1)}\left[\left(D_{q^4}^{-}\right)^{-1} x^{+}\left(D_{q^2}^{3}\right)^2\right]^k \\
&\quad\times\left(D_{q^4}^{-}\right)^{-1} f\left(q^{-2(k+1)}x^3\right)
\end{aligned}
$$

where the symbols $D_{q^a}^A$ and $\left(D_{q^a}^A\right)^{-1}$, $A=\pm,3,$ denote Jackson derivatives and Jackson integrals, respectively. Their explicit form can be found in appendix A. It is worth noting that these expressions have to be considered as definite integrals with their integration limits determining the limits of the appearing Jackson integrals. We have to emphasize that in general the identity $\left(\partial^A\right)^{-1}\partial^A f = f$ does not hold any longer, as $\left(\partial^A\right)^{-1} F$ is only one possible solution to the equation $\partial^A F = f$. However, this problem should not arise, if we restrict attention to continuous functions which fulfill certain boundary conditions.

Analogously, in the case of the second differential calculus the inverse elements $\left(\hat{\partial}^A\right)^{-1}$, $A=\pm,3,$ can now be written as

$$
\begin{aligned}
\left(\hat{\partial}^{+}\right)_L^{-1} f &= -q^{-1}\left(D_{q^4}^{-}\right)^{-1} f\left(q^2x^3, q^4x^-\right), \qquad (18)\\
\left(\hat{\partial}^{3}\right)_L^{-1} f &= \sum_{k=0}^{\infty}\lambda^k\left(q\lambda_+\right)^k\left[\left(D_{q^2}^{3}\right)^{-1} x^3 D_{q^4}^{+} D_{q^4}^{-}\right]^k \\
&\quad\times\left(D_{q^2}^{3}\right)^{-1} f\left(q^2x^+, q^2x^3, q^4x^-\right), \\
\left(\hat{\partial}^{-}\right)_L^{-1} f &= -\sum_{k=0}^{\infty}\lambda^k q^k\left[H^{-1}\left(\left(D_{q^4}^{+}\right)^{-1} x^{-}\left(D_{q^2}^{3}\right)^2\right) - q^{-2}\lambda\lambda_+\left(x^3\right) D_{q^4}^{+} D_{q^4}^{-k}\right]^k \\
&\quad\times H^{-1}\left(D_{q^4}^{+}\right)^{-1} f\left(q^4x^+, q^{-2k}x^3, q^{-4k}x^-\right)
\end{aligned}
$$

where $\lambda_+ = q+q^{-1}$ and H^{-1} denotes the inverse of the operator H introduced in [10]. Their explicit form can be found in appendix B.

Finally, formula (16) can also be applied to the algebra of momentum generators P^A, yielding

$$
\begin{aligned}
-\frac{i}{[2]_{q^3}}\left(P^{+}\right)_L^{-1} f &= -q^{-2}\sum_{k=0}^{\infty}\left(-q^2\lambda\right)^k\left[\left(H^{+}\right)^{-1}\left(D_{q^4}^{-}\right)^{-1} x^{+}\left(D_{q^2}^{3}\right)^2\right]^k \qquad (19)\\
&\quad\times\left(H^{+}\right)^{-1}\left(D_{q^4}^{-}\right)^{-1} f, \\
-\frac{i}{[2]_{q^3}}\left(P^{3}\right)_L^{-1} f &= -q^{-1}\sum_{k=0}^{\infty}\left(\lambda\lambda_+\right)^k q^{-k(k+4)}\left[\left(H^{3}\right)^{-1}\left(D_{q^2}^{3}\right)^{-1} x^3 D_{q^4}^{+} D_{q^{-4}}^{-}\right]^k \\
&\quad\times\left(H^{3}\right)^{-1}\left(D_{q^2}^{3}\right)^{-1} f\left(q^{-2k}x^+, q^{-2k}x^3\right),
\end{aligned}
$$

$$
\begin{aligned}
-\frac{i}{[2]_{q^3}}\left(P^-\right)_L^{-1} f \quad = \quad & -q^4 \sum_{k=0}^{\infty}\left(q^2\lambda\right)^k \\
& \times\left[\left(H^-\right)^{-1}\left(\left(D_{q^4}^+\right)^{-1} x^-\left(D_{q^2}^3\right)^2 - q^{-2}\lambda\lambda_+\left(x^3\right)^2 D_{q^4}^+ D_{q^4}^-\right)\right]^k \\
& \times\left(H^-\right)^{-1}\left(D_{q^4}^+\right)^{-1} f\left(q^4 x^+, q^{-2k} x^3, q^{-4k} x^-\right).
\end{aligned}
$$

Again, the operators $\left(H^A\right)^{-1}$, $A=\pm,3,$ denote the inverse of the operators H^A, $A=\pm,3,$ introduced in [10]. (Their explicit form is written out in appendix B.)

As indicated by an index L all of the above integral operators have been derived from left derivatives. Depending on the fact whether the expression for the integral under consideration refers to left or right derivatives we will call it from now on a left or right integral and denote this by an additional index L or R. From the relation between left and right derivatives presented in [10] we can at once deduce simple rules for transforming left and right integrals into each other, namely

$$
\begin{aligned}
\left(\partial^+\right)_R^{-1} f \quad & \overset{+}{\longleftrightarrow}\overset{-}{} \quad -q^6\left(\hat{\partial}^-\right)_L^{-1} f, \\
\left(\partial^3\right)_R^{-1} f \quad & \overset{+}{\longleftrightarrow}\overset{-}{} \quad -q^6\left(\hat{\partial}^3\right)_L^{-1} f, \\
\left(\partial^-\right)_R^{-1} f \quad & \overset{+}{\longleftrightarrow}\overset{-}{} \quad -q^6\left(\hat{\partial}^+\right)_L^{-1} f.
\end{aligned}
\tag{20}
$$

And in the same manner we also get

$$
\begin{aligned}
\left(\hat{\partial}^+\right)_R^{-1} f \quad & \overset{+}{\longleftrightarrow}\overset{-}{} \quad -q^{-6}\left(\partial^-\right)_L^{-1} f, \\
\left(\hat{\partial}^3\right)_R^{-1} f \quad & \overset{+}{\longleftrightarrow}\overset{-}{} \quad -q^{-6}\left(\partial^3\right)_L^{-1} f, \\
\left(\hat{\partial}^-\right)_R^{-1} f \quad & \overset{+}{\longleftrightarrow}\overset{-}{} \quad -q^{-6}\left(\partial^+\right)_L^{-1} f.
\end{aligned}
\tag{21}
$$

In the case of the algebra of momentum generators we can finally write

$$
\begin{aligned}
\left(P^+\right)_R^{-1} f \quad & \overset{+}{\longleftrightarrow}\overset{-}{} \quad -\left(P^-\right)_L^{-1} f, \\
\left(P^3\right)_R^{-1} f \quad & \overset{+}{\longleftrightarrow}\overset{-}{} \quad -\left(P^3\right)_L^{-1} f, \\
\left(P^-\right)_R^{-1} f \quad & \overset{+}{\longleftrightarrow}\overset{-}{} \quad -\left(P^+\right)_L^{-1} f
\end{aligned}
\tag{22}
$$

where the symbol $\overset{+}{\longleftrightarrow}\overset{-}{}$ denotes that one can make a transition between the two expressions by applying the substitutions [1]

$$
x^+ \longleftrightarrow x^-, \qquad \left(D_{q^a}^+\right)^{\pm 1} \longleftrightarrow \left(D_{q^a}^-\right)^{\pm 1}. \tag{23}
$$

[1] Let us note that the operators H^{-1}, $\left(H^A\right)^{-1}$, $A=\pm,3,$ have to be affected by this substitution, too.

For both types of integrals we can write down formulae for integration by parts. In the case of left integrals they read

$$
\begin{aligned}
(\partial^-)^{-1}_L (\partial^-_L f) \star g \Big|^b_{x^+=a} &= f \star g \Big|^b_{x^+=a} - (\partial^-)^{-1}_L \left(\Lambda^{1/2}\tau^{-1/2} f\right) \star \partial^-_L g \Big|^b_{x^+=a}, \qquad (24)\\
(\partial^3)^{-1}_L (\partial^3_L f) \star g \Big|^b_{x^3=a} &= f \star g \Big|^b_{x^3=a} - (\partial^3)^{-1}_L \left(\Lambda^{1/2} f\right) \star \partial^3_L g \Big|^b_{x^3=a} \\
&\quad - \lambda\lambda_+ (\partial^3)^{-1}_L \left(\Lambda^{1/2} L^+ f\right) \star \partial^-_L g \Big|^b_{x^3=a}, \\
(\partial^+)^{-1}_L (\partial^+_L f) \star g \Big|^b_{x^-=a} &= f \star g \Big|^b_{x^-=a} - (\partial^+)^{-1}_L \left(\Lambda^{1/2}\tau^{1/2} f\right) \star \partial^+_L g \Big|^b_{x^-=a} \\
&\quad - q\lambda\lambda_+ (\partial^+)^{-1}_L \left(\Lambda^{1/2}\tau^{1/2} L^+ f\right) \star \partial^3_L g \Big|^b_{x^-=a} \\
&\quad - q^2\lambda^2\lambda_+ (\partial^+)^{-1}_L \left(\Lambda^{1/2}\tau^{1/2} (L^+)^2 f\right) \star \partial^-_L g \Big|^b_{x^-=a}
\end{aligned}
$$

and for the second covariant differential calculus we get accordingly

$$
\begin{aligned}
\left(\hat\partial^+\right)^{-1}_L \left(\hat\partial^+_L f\right) \star g \Big|^b_{x^-=a} &= f \star g \Big|^b_{x^-=a} - \left(\hat\partial^+\right)^{-1}_L \left(\Lambda^{-1/2}\tau^{-1/2} f\right) \star \hat\partial^+_L g \Big|^b_{x^-=a}, \qquad (25)\\
\left(\hat\partial^3\right)^{-1}_L \left(\hat\partial^3_L f\right) \star g \Big|^b_{x^3=a} &= f \star g \Big|^b_{x^3=a} - \left(\hat\partial^3\right)^{-1}_L \left(\Lambda^{-1/2} f\right) \star \hat\partial^3_L g \Big|^b_{x^3=a} \\
&\quad - \lambda\lambda_+ \left(\hat\partial^3\right)^{-1}_L \left(\Lambda^{-1/2} L^- f\right) \star \hat\partial^+_L g \Big|^b_{x^3=a}, \\
\left(\hat\partial^-\right)^{-1}_L \left(\hat\partial^-_L f\right) \star g \Big|^b_{x^+=a} &= f \star g \Big|^b_{x^+=a} - \left(\hat\partial^-\right)^{-1}_L \left(\Lambda^{-1/2}\tau^{1/2} f\right) \star \hat\partial^-_L g \Big|^b_{x^+=a} \\
&\quad - q^{-1}\lambda\lambda_+ \left(\hat\partial^-\right)^{-1}_L \left(\Lambda^{-1/2}\tau^{1/2} L^- f\right) \star \hat\partial^3_L g \Big|^b_{x^+=a} \\
&\quad - q^{-2}\lambda^2\lambda_+ \left(\hat\partial^-\right)^{-1}_L \left(\Lambda^{-1/2}\tau^{1/2} (L^-)^2 f\right) \star \hat\partial^+_L g \Big|^b_{x^+=a}.
\end{aligned}
$$

In the case of right integrals the rules for integration by parts take the form

$$
\begin{aligned}
(\partial^-)^{-1}_R f \star \partial^-_R g \Big|^b_{x^+=a} &= f \star g \Big|^b_{x^+=a} - (\partial^-)^{-1}_R (\partial^-_R f) \star \left(\Lambda^{-1/2}\tau^{1/2} g\right) \Big|^b_{x^+=a}, \qquad (26)\\
(\partial^3)^{-1}_R f \star \partial^3_R g \Big|^b_{x^3=a} &= f \star g \Big|^b_{x^3=a} - (\partial^3)^{-1}_R (\partial^3_R f) \star \left(\Lambda^{-1/2} g\right) \Big|^b_{x^3=a} \\
&\quad + \lambda\lambda_+ (\partial^3)^{-1}_R (\partial^-_R f) \star \left(\Lambda^{-1/2}\tau^{1/2} L^+ g\right) \Big|^b_{x^3=a}, \\
(\partial^+)^{-1}_R f \star \partial^+_R g \Big|^b_{x^-=a} &= f \star g \Big|^b_{x^-=a} - (\partial^+)^{-1}_R (\partial^+_R f) \star \left(\Lambda^{-1/2}\tau^{-1/2} g\right) \Big|^b_{x^-=a} \\
&\quad + q^{-1}\lambda\lambda_+ (\partial^+)^{-1}_R (\partial^3_R f) \star \left(\Lambda^{-1/2} L^+ g\right) \Big|^b_{x^-=a} \\
&\quad - \lambda^2\lambda_+ (\partial^+)^{-1}_R (\partial^-_R f) \star \left(\Lambda^{-1/2}\tau^{1/2} (L^+)^2 g\right) \Big|^b_{x^-=a}
\end{aligned}
$$

and for the second differential calculus we have again the formulae

$$
\left(\hat\partial^+\right)^{-1}_R f \star \hat\partial^+_R g \Big|^b_{x^-=a} = f \star g \Big|^b_{x^-=a} - \left(\hat\partial^+\right)^{-1}_R \left(\hat\partial^+_R f\right) \star \left(\Lambda^{1/2}\tau^{1/2} g\right) \Big|^b_{x^-=a}, \qquad (27)
$$

$$
\begin{aligned}
\left(\hat{\partial}^3\right)_R^{-1} f \star \hat{\partial}^3_R g \Big|_{x^3=a}^{b} &= f \star g \Big|_{x^3=a}^{b} - \left(\hat{\partial}^3\right)_R^{-1} \left(\hat{\partial}^3_R f\right) \star \left(\Lambda^{1/2} g\right) \Big|_{x^3=a}^{b} \\
&\quad + \lambda\lambda_+ \left(\hat{\partial}^3\right)_R^{-1} \left(\hat{\partial}^+_R f\right) \star \left(\Lambda^{1/2}\tau^{1/2} L^- g\right) \Big|_{x^3=a}^{b}, \\
\left(\hat{\partial}^-\right)_R^{-1} f \star \hat{\partial}^-_R g \Big|_{x^+=a}^{b} &= f \star g \Big|_{x^+=a}^{b} - \left(\hat{\partial}^-\right)_R^{-1} \left(\hat{\partial}^-_R f\right) \star \left(\Lambda^{1/2}\tau^{-1/2} g\right) \Big|_{x^+=a}^{b} \\
&\quad + q\lambda\lambda_+ \left(\hat{\partial}^-\right)_R^{-1} \left(\hat{\partial}^3_R f\right) \star \left(\Lambda^{1/2} L^- g\right) \Big|_{x^+=a}^{b} \\
&\quad - \lambda^2\lambda_+ \left(\hat{\partial}^-\right)_R^{-1} \left(\hat{\partial}^+_R f\right) \star \left(\Lambda^{1/2}\tau^{1/2} \left(L^-\right)^2 g\right) \Big|_{x^+=a}^{b} .
\end{aligned}
$$

Let us now make contact with expressions of the form $(\partial^+)^{-1}(\partial^3)^{-1}(\partial^-)^{-1}$ which can be interpreted as some sort of 3-dimensional volume integrals. As the operators $(\partial^A)^{-1}$, $A = \pm, 3,$ do not mutually commute, it should be clear that in general the result of our volume integration depends on the order in which the operators $(\partial^A)^{-1}$, $A = \pm, 3$ are arranged. However, if we consequently drop surface terms, the considered volume integrals will only change their normalisation, as the following calculation illustrates:

$$
\begin{aligned}
(\partial^-)^{-1}(\partial^3)^{-1}(\partial^+)^{-1} f &= q^2 (\partial^-)^{-1}(\partial^+)^{-1}(\partial^3)^{-1} f \\
&= \sum_{i=0}^{\infty} q^{2(i+2)} \lambda^i \, [[i]]_{q^4}! \left(\begin{bmatrix} -1 \\ i \end{bmatrix}_{q^4} \right)^2 (\partial^+)^{-(i+1)} (\partial^3)^{2i-1} (\partial^-)^{-(i+1)} f \\
&= q^4 (\partial^+)^{-1}(\partial^3)^{-1}(\partial^-)^{-1} f + S.T.
\end{aligned} \tag{28}
$$

where $S.T.$ stands for neglected surface terms. For the first and second equality we have used here the identities of (8). Finally, the list containing all possible relations between the different volume integrals is given by

$$
\begin{aligned}
(\partial^+)^{-1}(\partial^3)^{-1}(\partial^-)^{-1} f &= q^{-4} (\partial^-)^{-1}(\partial^3)^{-1}(\partial^+)^{-1} f = \\
q^{-2} (\partial^3)^{-1}(\partial^-)^{-1}(\partial^+)^{-1} f &= q^{-2} (\partial^3)^{-1}(\partial^+)^{-1}(\partial^-)^{-1} f = \\
q^{-2} (\partial^-)^{-1}(\partial^+)^{-1}(\partial^3)^{-1} f &= q^{-2} (\partial^+)^{-1}(\partial^-)^{-1}(\partial^3)^{-1} f .
\end{aligned} \tag{29}
$$

Let us note that the same identities also hold for the operators $\left(\hat{\partial}^A\right)^{-1}$ and $(P^A)^{-1}$, $A = \pm, 3$.

Our next goal is to provide formulae which enable an explicit calculation of the new volume integrals. To this end we have to insert the above representations (17), (18) and (19) for $(\partial^A)^{-1}$, $\left(\hat{\partial}^A\right)^{-1}$ and $(P^A)^{-1}$, respectively, into expressions (29) defining our volume integrals. A detailed analysis of the resulting formulae shows that all contributions depending on λ^k, $k \geq 1$, lead to surface terms and can, in turn, be neglected. Due to this fact we end up with the identities

$$
(\partial^+)_L^{-1} (\partial^3)_L^{-1} (\partial^-)_L^{-1} f = \tag{30}
$$

$$= q^{-4}\left(D^-_{q^4}\right)^{-1}\left(D^3_{q^2}\right)^{-1}\left(D^+_{q^4}\right)^{-1} f\left(q^{-2}x^+, q^{-2}x^3\right),$$

$$[0.1in]\left(\hat{\partial}^-\right)^{-1}_L\left(\hat{\partial}^3\right)^{-1}_L\left(\hat{\partial}^+\right)^{-1}_L f =$$

$$= -q^3 H^{-1}\left(D^+_{q^{-4}}\right)^{-1}\left(D^3_{q^{-2}}\right)^{-1}\left(D^-_{q^{-4}}\right)^{-1} f\left(q^2x^+, q^2x^3, q^4x^-\right),$$

$$[0.1in]\left(-\frac{i}{[2]_{q^3}}\right)^3\left(P^-\right)^{-1}_L\left(P^3\right)^{-1}_L\left(P^+\right)^{-1}_L f =$$

$$= q\left(H^-\right)^{-1}\left(D^+_{q^4}\right)^{-1}\left(H^3\right)^{-1}\left(D^3_{q^2}\right)^{-1}\left(H^+\right)^{-1}\left(D^-_{q^4}\right)^{-1} f\left(q^4x^+\right).$$

As our volume integrals are built up from operators which are inverse to partial derivatives or momentum generators their translation invariance is obvious if surface terms are dropped. Likewise, we can easily prove rotation invariance by applying the adjoint action of the symmetry generators L^+, L^- and $\tau^{-1/2}$ to our volume integrals [13], [14]. For example, we have

$$\begin{aligned} L^+ \triangleright \left(\left(\partial^+\right)^{-1}\left(\partial^3\right)^{-1}\left(\partial^-\right)^{-1}\right) &= \left(\partial^+\right)^{-1} L^+ \triangleright \left(\left(\partial^3\right)^{-1}\left(\partial^-\right)^{-1}\right) \\ &= \left(\partial^+\right)^{-1}\left(\partial^3\right)^{-1} L^+ \triangleright \left(\partial^-\right)^{-1} + q^{-1}\left(\partial^+\right)^{-1}\partial^+\left(\partial^3\right)^{-2}\tau^{-1/2} \triangleright \left(\partial^-\right)^{-1} \\ &= \left(\partial^+\right)^{-1}\left(\partial^3\right)^{-1}\left(\partial^-\right)^{-1} L^+ \triangleright 1 + q\left(\partial^3\right)^{-2}\left(\partial^-\right)^{-1} \\ &= 0 + S.T. \end{aligned} \tag{31}$$

Repeating this calculation for L^- and $\tau^{-1/2}$ yields the same result and therefore shows rotation invariance of the volume integrals (29), if again surface terms are dropped.

3 q-deformed Euclidean Space in four Dimensions

The 4-dimensional Euclidean space can be treated in very much the same way as the 3-dimensional one. Again, we start with the commutation relations [15]

$$\begin{array}{ll} \partial^1\partial^2 = q\partial^2\partial^1, & \partial^1\partial^3 = q\partial^3\partial^1, \\ \partial^2\partial^4 = q\partial^4\partial^2, & \partial^3\partial^4 = q\partial^4\partial^3, \\ \partial^2\partial^3 = \partial^3\partial^2, & \partial^4\partial^1 = \partial^1\partial^4 + \lambda\partial^2\partial^3 \end{array} \tag{32}$$

where $\lambda = q - q^{-1}$ with $q > 1$, and introduce inverse elements $\left(\partial^i\right)^{-1}$, $i = 1, \ldots, 4$, by

$$\partial^i\left(\partial^i\right)^{-1} = \left(\partial^i\right)^{-1}\partial^i = 1. \tag{33}$$

Now, the remaining commutation relations become

$$\begin{aligned} \left(\partial^2\right)^{-1}\partial^1 &= q\partial^1\left(\partial^2\right)^{-1}, \quad \partial^2\left(\partial^1\right)^{-1} = q\left(\partial^1\right)^{-1}\partial^2, \\ \left(\partial^3\right)^{-1}\partial^1 &= q\partial^1\left(\partial^3\right)^{-1}, \quad \partial^3\left(\partial^1\right)^{-1} = q\left(\partial^1\right)^{-1}\partial^3, \end{aligned} \tag{34}$$

$$
\begin{aligned}
\left(\partial^4\right)^{-1}\partial^2 &= q\partial^2\left(\partial^4\right)^{-1}, \quad \partial^4\left(\partial^2\right)^{-1} = q\left(\partial^2\right)^{-1}\partial^4, \\
\left(\partial^4\right)^{-1}\partial^3 &= q\partial^3\left(\partial^4\right)^{-1}, \quad \partial^4\left(\partial^3\right)^{-1} = q\left(\partial^3\right)^{-1}\partial^4, \\
\left(\partial^3\right)^{-1}\partial^2 &= \partial^2\left(\partial^3\right)^{-1}, \quad \partial^3\left(\partial^2\right)^{-1} = \left(\partial^2\right)^{-1}\partial^3, \\
\left(\partial^4\right)^{-1}\partial^1 &= \partial^1\left(\partial^4\right)^{-1} - q^2\lambda\partial^2\partial^3\left(\partial^4\right)^{-2}, \\
\partial^4\left(\partial^1\right)^{-1} &= \left(\partial^1\right)^{-1}\partial^4 - q^2\lambda\left(\partial^1\right)^{-2}\partial^2\partial^3.
\end{aligned}
$$

In addition, we obtain further identities of the form

$$
\begin{aligned}
\left(\partial^2\right)^{-1}\left(\partial^1\right)^{-1} &= q^{-1}\left(\partial^1\right)^{-1}\left(\partial^2\right)^{-1}, \quad \left(\partial^3\right)^{-1}\left(\partial^1\right)^{-1} = q^{-1}\left(\partial^1\right)^{-1}\left(\partial^3\right)^{-1}, \qquad (35) \\
\left(\partial^4\right)^{-1}\left(\partial^2\right)^{-1} &= q^{-1}\left(\partial^2\right)^{-1}\left(\partial^4\right)^{-1}, \quad \left(\partial^4\right)^{-1}\left(\partial^3\right)^{-1} = q^{-1}\left(\partial^3\right)^{-1}\left(\partial^4\right)^{-1}, \\
\left(\partial^3\right)^{-1}\left(\partial^2\right)^{-1} &= \left(\partial^2\right)^{-1}\left(\partial^3\right)^{-1}, \\
\left(\partial^4\right)^{-1}\left(\partial^1\right)^{-1} &= \sum_{i=0}^{\infty}\lambda^i\,[[i]]_{q^{-2}}!\left(\begin{bmatrix} -1 \\ i \end{bmatrix}_{q^{-2}}\right)^2\left(\partial^1\right)^{-(i+1)}\left(\partial^2\right)^i\left(\partial^3\right)^i\left(\partial^4\right)^{-(i+1)}.
\end{aligned}
$$

Next, the commutation relations with the symmetry generators read

$$
\begin{aligned}
L_1^+\left(\partial^1\right)^{-1} &= q^{-1}\left(\partial^1\right)^{-1}L_1^+ + q^{-1}\left(\partial^1\right)^{-2}\partial^2, && (36) \\
L_1^+\left(\partial^2\right)^{-1} &= q\left(\partial^2\right)^{-1}L_1^+, \\
L_1^+\left(\partial^3\right)^{-1} &= q^{-1}\left(\partial^3\right)^{-1}L_1^+ - q^{-1}\left(\partial^3\right)^{-2}\partial^4, \\
L_1^+\left(\partial^4\right)^{-1} &= q\left(\partial^4\right)^{-1}L_1^+, \\
[0.16in]L_2^+\left(\partial^1\right)^{-1} &= q^{-1}\left(\partial^1\right)^{-1}L_2^+ + q^{-1}\left(\partial^1\right)^{-2}\partial^3, && (37) \\
L_2^+\left(\partial^2\right)^{-1} &= q^{-1}\left(\partial^2\right)^{-1}L_2^+ - q^{-1}\left(\partial^2\right)^{-2}\partial^4, \\
L_2^+\left(\partial^3\right)^{-1} &= q\left(\partial^3\right)^{-1}L_2^+, \\
L_2^+\left(\partial^4\right)^{-1} &= q\left(\partial^4\right)^{-1}L_2^+, \\
[0.4cm]L_1^-\left(\partial^1\right)^{-1} &= q^{-1}\left(\partial^1\right)^{-1}L_1^-, && (38) \\
L_1^-\left(\partial^2\right)^{-1} &= q\left(\partial^2\right)^{-1}L_1^- + q^3\partial^1\left(\partial^2\right)^{-2}, \\
L_1^-\left(\partial^3\right)^{-1} &= q^{-1}\left(\partial^3\right)^{-1}L_1^-, \\
L_1^-\left(\partial^4\right)^{-1} &= q\left(\partial^4\right)^{-1}L_1^- - q^3\partial^3\left(\partial^4\right)^{-2}, \\
[0.4cm]L_2^-\left(\partial^1\right)^{-1} &= q^{-1}\left(\partial^1\right)^{-1}L_2^-, && (39) \\
L_2^-\left(\partial^2\right)^{-1} &= q^{-1}\left(\partial^2\right)^{-1}L_2^-, \\
L_2^-\left(\partial^3\right)^{-1} &= q\left(\partial^3\right)^{-1}L_2^- + q^3\partial^1\left(\partial^3\right)^{-2}, \\
L_2^-\left(\partial^4\right)^{-1} &= q\left(\partial^4\right)^{-1}L_2^- - q^3\partial^2\left(\partial^4\right)^{-2}, \\
[0.4cm]K_1\left(\partial^1\right)^{-1} &= q\left(\partial^1\right)^{-1}K_1, && (40)
\end{aligned}
$$

$$
\begin{aligned}
K_1\left(\partial^2\right)^{-1} &= q^{-1}\left(\partial^2\right)^{-1} K_1, \\
K_1\left(\partial^3\right)^{-1} &= q\left(\partial^3\right)^{-1} K_1, \\
K_1\left(\partial^4\right)^{-1} &= q^{-1}\left(\partial^4\right)^{-1} K_1, \\
[0.4cm] K_2\left(\partial^1\right)^{-1} &= q\left(\partial^1\right)^{-1} K_2, \\
K_2\left(\partial^2\right)^{-1} &= q\left(\partial^2\right)^{-1} K_2, \\
K_2\left(\partial^3\right)^{-1} &= q^{-1}\left(\partial^3\right)^{-1} K_2, \\
K_2\left(\partial^4\right)^{-1} &= q^{-1}\left(\partial^4\right)^{-1} K_2, \\
[0.4cm] \Lambda\left(\partial^i\right)^{-1} &= q^2\left(\partial^i\right)^{-1}\Lambda, \qquad i = 1, \ldots, 4.
\end{aligned} \tag{41, 42}
$$

With the substitutions

$$
\partial^i \rightarrow \hat{\partial}^i, \quad \left(\partial^i\right)^{-1} \rightarrow \left(\hat{\partial}^i\right)^{-1}, \quad i = 1, \ldots, 4 \tag{43}
$$

and

$$
\partial^i \rightarrow P^i, \quad \left(\partial^i\right)^{-1} \rightarrow \left(P^i\right)^{-1}, \quad i = 1, \ldots, 4 \tag{44}
$$

we get the corresponding relations for the second differential calculus and the algebra of momentum generators, respectively.

Now, the same reasoning we have already applied to the 3-dimensional Euclidean space leads immediately to the representations

$$
\begin{aligned}
\left(\partial^1\right)_L^{-1} f &= q\left(D_{q^2}^4\right)^{-1} f\left(q^{-1}x^2, q^{-1}x^3\right), \\
\left(\partial^2\right)_L^{-1} f &= \left(D_{q^2}^3\right)^{-1} f\left(q^{-1}x^1, q^{-2}x^4\right) \\
&\quad + q^{-1}\sum_{k=1}^{\infty}\lambda^k q^{k^2}\left[\left(D_{q^2}^3\right)^{-1} x^2 D_{q^{-2}}^1 D_{q^{-2}}^4\right]^k \\
&\quad \times\left(D_{q^2}^3\right)^{-1} f\left(q^{-1}x^1, q^k x^2, q^k x^3, q^{-2}x^4\right), \\
\left(\partial^3\right)_L^{-1} f &= \left(D_{q^2}^2\right)^{-1} f\left(q^{-1}x^1, q^{-2}x^4\right) \\
&\quad + q^{-1}\sum_{k=1}^{\infty}\lambda^k q^{k^2}\left[\left(D_{q^2}^2\right)^{-1} x^3 D_{q^{-2}}^1 D_{q^{-2}}^4\right]^k \\
&\quad \times\left(D_{q^2}^2\right)^{-1} f\left(q^{-1}x^1, q^k x^2, q^k x^3, q^{-2}x^4\right), \\
\left(\partial^4\right)_L^{-1} f &= q^{-1}\sum_{k=0}^{\infty}\lambda^k q^k \\
&\quad \times\left\{N^{-1}\left(D_{q^2}^1\right)^{-1}\left[q^{-1}x^4 D_{q^{-2}}^2 D_{q^{-2}}^3 + \lambda x^2 x^3\left(D_{q^{-2}}^1\right)^2 D_{q^{-2}}^4\right]\right\}^k \\
&\quad \times N^{-1}\left(D_{q^2}^1\right)^{-1} f\left(q^{2k}x^1, q^k x^2, q^k x^3, q^{-2}x^4\right).
\end{aligned} \tag{45}
$$

Similarly, we have for the second differential calculus

$$
\begin{aligned}
\left(\hat{\partial}^1\right)^{-1}_L f &= q\sum_{k=0}^{\infty}(-\lambda)^k\, q^{-k(k+1)}\left[\left(D^4_{q^{-2}}\right)^{-1} x^1 D^2_{q^{-2}} D^3_{q^{-2}}\right]^k \\
&\quad\times\left(D^4_{q^{-2}}\right)^{-1} f\left(q^{k+1}x^2, q^{k+1}x^3\right), \\
\left(\hat{\partial}^2\right)^{-1}_L f &= \left(D^3_{q^{-2}}\right)^{-1} f\left(qx^1\right), \\
\left(\hat{\partial}^3\right)^{-1}_L f &= \left(D^2_{q^{-2}}\right)^{-1} f\left(qx^1\right), \\
\left(\hat{\partial}^4\right)^{-1}_L f &= q^{-1}\left(D^1_{q^{-2}}\right)^{-1} f.
\end{aligned}
\tag{46}
$$

Finally, we obtain for the inverse momentum generators the expressions

$$
\begin{aligned}
-\frac{i}{[2]_{q^2}}\left(P^1\right)^{-1}_L f &= \sum_{k=0}^{\infty}(-q\lambda)^k\left[\left(N^1\right)^{-1}\left(D^4_{q^2}\right)^{-1} x^1 D^2_{q^{-2}} D^3_{q^{-2}}\right]^k \\
&\quad\times\left(N^1\right)^{-1}\left(D^4_{q^2}\right)^{-1} f, \\
-\frac{i}{[2]_{q^2}}\left(P^2\right)^{-1}_L f &= q^{-1}\left(N^2\right)^{-1}\left(D^3_{q^2}\right)^{-1} f \\
&\quad+ q^{-1}\sum_{k=1}^{\infty}\lambda^k q^{\frac{1}{2}k(3k-5)}\left[\left(N^2\right)^{-1}\left(D^3_{q^2}\right)^{-1} x^2 D^1_{q^2} D^4_{q^2}\right]^k \\
&\quad\times\left(N^2\right)^{-1}\left(D^3_{q^2}\right)^{-1} f\left(q^{-k}x^1, q^k x^2, q^k x^3\right), \\
-\frac{i}{[2]_{q^2}}\left(P^3\right)^{-1}_L f &= q^{-1}\left(N^3\right)^{-1}\left(D^2_{q^2}\right)^{-1} f \\
&\quad+ q^{-1}\sum_{k=1}^{\infty}\lambda^k q^{\frac{1}{2}k(3k-5)}\left[\left(N^3\right)^{-1}\left(D^2_{q^2}\right)^{-1} x^3 D^1_{q^2} D^4_{q^2}\right]^k \\
&\quad\times\left(N^3\right)^{-1}\left(D^2_{q^2}\right)^{-1} f\left(q^{-k}x^1, q^k x^2, q^k x^3\right), \\
-\frac{i}{[2]_{q^2}}\left(P^4\right)^{-1}_L f &= q^{-2}\sum_{k=0}^{\infty}\left(q^{-4}\lambda\right)^k \\
&\quad\times\left\{\left(N^4\right)^{-1}\left(D^1_{q^2}\right)^{-1}\left[q^{-1}x^4 D^2_{q^{-2}} D^3_{q^{-2}} + \lambda x^2 x^3\left(D^1_{q^{-2}}\right)^2 D^4_{q^{-2}}\right]\right\}^k \\
&\quad\times\left(N^4\right)^{-1}\left(D^1_{q^2}\right)^{-1} f\left(q^{2k}x^1, q^k x^2, q^k x^3, q^{-2}x^4\right)
\end{aligned}
\tag{47}
$$

where N^{-1} and $\left(N^i\right)^{-1}$, $i = 1, \ldots, 4,$ denote the inverse of the operators N and N^i, $1, \ldots, 4,$ as introduced previously in [10]. Their explicit form can be looked up in appendix B. One should also recall that the above expressions have to be considered as definite integrals with their integration limits determining the limits of the appearing Jackson integrals $\left(D^i_{q^a}\right)^{-1}$, $i = 1, \ldots, 4$.

Again we can establish a correspondence between left and right integrals

$$\begin{aligned} \left(\partial^i\right)^{-1}_R f &\overset{j}{\longleftrightarrow}{}^{j'} && -q^4 \left(\hat{\partial}^{i'}\right)^{-1}_L f, \\ \left(\hat{\partial}^i\right)^{-1}_R f &\overset{j}{\longleftrightarrow}{}^{j'} && -q^{-4} \left(\partial^{i'}\right)^{-1}_L f, \\ \left(P^i\right)^{-1}_R f &\overset{j}{\longleftrightarrow}{}^{j'} && -\left(P^{i'}\right)^{-1}_L f \end{aligned} \tag{48}$$

where $i = 1, \ldots, 4$ and $i' = 5 - i$. The symbol $\overset{j}{\longleftrightarrow}{}^{j'}$ now denotes that one can make a transition between the two expressions by applying the substitutions

$$x^i \rightarrow x^{i'}, \qquad \hat{\sigma}_i \rightarrow \hat{\sigma}_{i'}, \qquad \hat{\sigma}_i \equiv x^i \frac{\partial}{\partial x^i} \tag{49}$$

and

$$D^i_{q^a} \rightarrow D^{i'}_{q^a}, \qquad \left(D^i_{q^a}\right)^{-1} \rightarrow \left(D^{i'}_{q^a}\right)^{-1}. \tag{50}$$

It is important to realize that the expressions for N^{-1} and $\left(N^i\right)^{-1}$, $i = 1, \ldots, 4$, have to be affected by the above substitutions,too. A simple example shall clarify the meaning of this point:

$$\left(D^4_{q^{-2}}\right)^{-1} D^2_{q^{-2}} q^{(\hat{\sigma}_4 - 2)\hat{\sigma}_3} f\left(q^2 x^2\right) \overset{j}{\longleftrightarrow}{}^{j'} \left(D^1_{q^{-2}}\right)^{-1} D^3_{q^{-2}} q^{(\hat{\sigma}_1 - 2)\hat{\sigma}_2} f\left(q^2 x^3\right). \tag{51}$$

The formulae for integration by parts are now given by

$$\begin{aligned} \left(\partial^1\right)^{-1}_L \left(\partial^1_L f\right) \star g\Big|^b_{x^4=a} &= f \star g\big|^b_{x^4=a} - \left(\partial^1\right)^{-1}_L \left(\Lambda^{1/2} K_1^{1/2} K_2^{1/2} f\right) \star \partial^1_L g\Big|^b_{x^4=a}, \\ \left(\partial^2\right)^{-1}_L \left(\partial^2_L f\right) \star g\Big|^b_{x^3=a} &= f \star g\big|^b_{x^3=a} - \left(\partial^2\right)^{-1}_L \left(\Lambda^{1/2} K_1^{-1/2} K_2^{1/2} f\right) \star \partial^2_L g\Big|^b_{x^3=a} \\ &\quad - q\lambda \left(\partial^2\right)^{-1}_L \left(\Lambda^{1/2} K_1^{1/2} K_2^{1/2} L_1^+ f\right) \star \partial^1_L g\Big|^b_{x^3=a}, \\ \left(\partial^3\right)^{-1}_L \left(\partial^3_L f\right) \star g\Big|^b_{x^2=a} &= f \star g\big|^b_{x^2=a} - \left(\partial^3\right)^{-1}_L \left(\Lambda^{1/2} K_1^{1/2} K_2^{-1/2} f\right) \star \partial^3_L g\Big|^b_{x^3=a} \\ &\quad - q\lambda \left(\partial^3\right)^{-1}_L \left(\Lambda^{1/2} K_1^{1/2} K_2^{1/2} L_2^+ f\right) \star \partial^1_L g\Big|^b_{x^2=a}, \\ \left(\partial^4\right)^{-1}_L \left(\partial^4_L f\right) \star g\Big|^b_{x^1=a} &= f \star g\big|^b_{x^1=a} - \left(\partial^4\right)^{-1}_L \left(\Lambda^{1/2} K_1^{-1/2} K_2^{-1/2} f\right) \star \partial^4_L g\Big|^b_{x^1=a} \\ &\quad + q\lambda \left(\partial^4\right)^{-1}_L \left(\Lambda^{1/2} K_1^{1/2} K_2^{-1/2} L_1^+ f\right) \star \partial^3_L g\Big|^b_{x^1=a} \\ &\quad + q\lambda \left(\partial^4\right)^{-1}_L \left(\Lambda^{1/2} K_1^{-1/2} K_2^{1/2} L_2^+ f\right) \star \partial^2_L g\Big|^b_{x^1=a} \\ &\quad + q^2\lambda^2 \left(\partial^4\right)^{-1}_L \left(\Lambda^{1/2} K_1^{1/2} K_2^{1/2} L_1^+ L_2^+ f\right) \star \partial^1_L g\Big|^b_{x^1=a}. \end{aligned} \tag{52}$$

For the second covariant differential calculus one can compute likewise

$$
\begin{aligned}
\left(\hat{\partial}^1\right)^{-1}_L \left(\hat{\partial}^1_L f\right) \star g\Big|^b_{x^4=a} &= f \star g|^{\,b}_{x^4=a} - \left(\hat{\partial}^1\right)^{-1}_L \left(\Lambda^{-1/2} K_1^{-1/2} K_2^{-1/2} f\right) \star \hat{\partial}^1_L g\Big|^{\,b}_{x^4=a} \\
&\quad + q^{-1}\lambda \left(\hat{\partial}^1\right)^{-1}_L \left(\Lambda^{-1/2} K_1^{1/2} K_2^{-1/2} L_1^- f\right) \star \hat{\partial}^2_L g\Big|^b_{x^4=a} \\
&\quad + q^{-1}\lambda \left(\hat{\partial}^1\right)^{-1}_L \left(\Lambda^{-1/2} K_1^{-1/2} K_2^{1/2} L_2^- f\right) \star \hat{\partial}^3_L g\Big|^b_{x^4=a} \\
&\quad + q^{-2}\lambda^2 \left(\hat{\partial}^1\right)^{-1}_L \left(\Lambda^{-1/2} K_1^{1/2} K_2^{-1/2} L_1^- L_2^- f\right) \star \partial^4_L g\Big|^b_{x^4=a}, \\
\left(\hat{\partial}^2\right)^{-1}_L \left(\hat{\partial}^2_L f\right) \star g\Big|^b_{x^3=a} &= f \star g|^{\,b}_{x^3=a} - \left(\hat{\partial}^2\right)^{-1}_L \left(\Lambda^{-1/2} K_1^{1/2} K_2^{-1/2} f\right) \star \hat{\partial}^2_L g\Big|^b_{x^3=a} \\
&\quad - q^{-1}\lambda \left(\hat{\partial}^2\right)^{-1}_L \left(\Lambda^{-1/2} K_1^{1/2} K_2^{1/2} L_2^- f\right) \star \hat{\partial}^4_L g\Big|^b_{x^3=a}, \\
\left(\hat{\partial}^3\right)^{-1}_L \left(\hat{\partial}^3_L f\right) \star g\Big|^b_{x^2=a} &= f \star g|^{\,b}_{x^2=a} - \left(\hat{\partial}^3\right)^{-1}_L \left(\Lambda^{-1/2} K_1^{-1/2} K_2^{1/2} f\right) \star \hat{\partial}^3_L g\Big|^b_{x^2=a} \\
&\quad - q^{-1}\lambda \left(\hat{\partial}^3\right)^{-1}_L \left(\Lambda^{-1/2} K_1^{1/2} K_2^{1/2} L_1^- f\right) \star \hat{\partial}^4_L g\Big|^b_{x^2=a}, \\
\left(\hat{\partial}^4\right)^{-1}_L \left(\hat{\partial}^4_L f\right) \star g\Big|^b_{x^1=a} &= f \star g|^{\,b}_{x^1=a} - \left(\hat{\partial}^4\right)^{-1}_L \left(\Lambda^{-1/2} K_1^{1/2} K_2^{1/2} f\right) \star \hat{\partial}^4_L g\Big|^{\,b}_{x^1=a}.
\end{aligned}
\tag{53}
$$

For right integrals, however, we have

$$
\begin{aligned}
\left(\partial^1\right)^{-1}_R f \star \partial^1_R g\Big|^{\,b}_{x^4=a} &= f \star g|^b_{x^4=a} - \left(\partial^1\right)^{-1}_R \left(\partial^1_R f\right) \star \left(\Lambda^{-1/2} K_1^{-1/2} K_2^{-1/2} g\right)\Big|^b_{x^4=a} \\
\left(\partial^2\right)^{-1}_R f \star \partial^2_R g\Big|^{\,b}_{x^3=a} &= f \star g|^b_{x^3=a} - \left(\partial^2\right)^{-1}_R \left(\partial^2_R f\right) \star \left(\Lambda^{-1/2} K_1^{1/2} K_2^{-1/2} g\right)\Big|^b_{x^3=a} \\
&\quad + \lambda \left(\partial^2\right)^{-1}_R \left(\partial^1_R f\right) \star \left(\Lambda^{-1/2} K_1^{1/2} K_2^{-1/2} L_1^+ g\right)\Big|^b_{x^3=a}, \\
\left(\partial^3\right)^{-1}_R f \star \partial^3_R g\Big|^{\,b}_{x^2=a} &= f \star g|^b_{x^2=a} - \left(\partial^3\right)^{-1}_R \left(\partial^3_R f\right) \star \left(\Lambda^{-1/2} K_1^{-1/2} K_2^{1/2} g\right)\Big|^b_{x^2=a} \\
&\quad + \lambda \left(\partial^3\right)^{-1}_R \left(\partial^1_R f\right) \star \left(\Lambda^{-1/2} K_1^{-1/2} K_2^{1/2} L_2^+ g\right)\Big|^b_{x^2=a}, \\
\left(\partial^4\right)^{-1}_R f \star \partial^4_R g\Big|^{\,b}_{x^1=a} &= f \star g|^b_{x^1=a} - \left(\partial^4\right)^{-1}_R \left(\partial^4_R f\right) \star \left(\Lambda^{-1/2} K_1^{1/2} K_2^{1/2} g\right)\Big|^b_{x^1=a} \\
&\quad - \lambda \left(\partial^4\right)^{-1}_R \left(\partial^3_R f\right) \star \left(\Lambda^{-1/2} K_1^{1/2} K_2^{1/2} L_1^+ g\right)\Big|^b_{x^1=a} \\
&\quad - \lambda \left(\partial^4\right)^{-1}_R \left(\partial^2_R f\right) \star \left(\Lambda^{-1/2} K_1^{1/2} K_2^{1/2} L_2^+ g\right)\Big|^b_{x^1=a} \\
&\quad + \lambda^2 \left(\partial^4\right)^{-1}_R \left(\partial^1_R f\right) \star \left(\Lambda^{-1/2} K_1^{1/2} K_2^{1/2} L_1^+ L_2^+ g\right)\Big|^{\,b}_{x^1=a}
\end{aligned}
\tag{54}
$$

and for the second covariant differential calculus

$$
\left(\hat{\partial}^1\right)^{-1}_R f \star \hat{\partial}^1_R g\Big|^b_{x^4=a} = f \star g|^{\,b}_{x^4=a} - \left(\hat{\partial}^1\right)^{-1}_R \left(\hat{\partial}^1_R f\right) \star \left(\Lambda^{1/2} K_1^{1/2} K_2^{1/2} g\right)\Big|^b_{x^4=a} \tag{55}
$$

$$
\begin{aligned}
&- \lambda \left(\hat{\partial}^1\right)_R^{-1} \left(\hat{\partial}^2_R f\right) \star \left(\Lambda^{1/2} K_1^{1/2} K_2^{1/2} L_1^- g\right)\Big|_{x^4=a}^{b} \\
&- \lambda \left(\hat{\partial}^1\right)_R^{-1} \left(\hat{\partial}^3_R f\right) \star \left(\Lambda^{1/2} K_1^{1/2} K_2^{1/2} L_2^- g\right)\Big|_{x^4=a}^{b} \\
&+ \lambda^2 \left(\hat{\partial}^1\right)_R^{-1} \left(\hat{\partial}^4_R f\right) \star \left(\Lambda^{1/2} K_1^{1/2} K_2^{1/2} L_1^- L_1^- g\right)\Big|_{x^4=a}^{b},
\end{aligned}
$$

$$
\begin{aligned}
\left(\hat{\partial}^2\right)_R^{-1} f \star \hat{\partial}^2_R g\Big|_{x^3=a}^{b} &= f \star g\big|_{x^3=a}^{b} - \left(\hat{\partial}^2\right)_R^{-1} \left(\hat{\partial}^2_R f\right) \star \left(\Lambda^{1/2} K_1^{-1/2} K_2^{1/2} g\right)\Big|_{x^3=a}^{b} \\
&+ \lambda \left(\hat{\partial}^2\right)_R^{-1} \left(\hat{\partial}^4_R f\right) \star \left(\Lambda^{1/2} K_1^{-1/2} K_2^{1/2} L_2^- g\right)\Big|_{x^3=a}^{b}, \\
\left(\hat{\partial}^3\right)_R^{-1} f \star \hat{\partial}^3_R g\Big|_{x^2=a}^{b} &= f \star g\big|_{x^2=a}^{b} - \left(\hat{\partial}^3\right)_R^{-1} \left(\hat{\partial}^3_R f\right) \star \left(\Lambda^{1/2} K_1^{-1/2} K_2^{-1/2} g\right)\Big|_{x^2=a}^{b} \\
&+ \lambda \left(\hat{\partial}^3\right)_R^{-1} \left(\hat{\partial}^4_R f\right) \star \left(\Lambda^{1/2} K_1^{1/2} K_2^{-1/2} L_1^- g\right)\Big|_{x^2=a}^{b}, \\
\left(\hat{\partial}^4\right)_R^{-1} f \star \hat{\partial}^4_R g\Big|_{x^1=a}^{b} &= f \star g\big|_{x^1=a}^{b} - \left(\hat{\partial}^4\right)_R^{-1} \left(\hat{\partial}^4_R f\right) \star \left(\Lambda^{1/2} K_1^{-1/2} K_2^{-1/2} g\right)\Big|_{x^1=a}^{b}.
\end{aligned}
$$

Next we turn to expressions of the form $(\partial^1)^{-1}(\partial^2)^{-1}(\partial^3)^{-1}(\partial^4)^{-1}$ which can again be regarded as a q-deformed version of four dimensional volume integrals. If surface terms are neglected, we can state the identities

$$
\begin{aligned}
(\partial^1)^{-1}(\partial^2)^{-1}(\partial^3)^{-1}(\partial^4)^{-1} &= (\partial^1)^{-1}(\partial^3)^{-1}(\partial^2)^{-1}(\partial^4)^{-1} = \\
q\,(\partial^1)^{-1}(\partial^2)^{-1}(\partial^4)^{-1}(\partial^3)^{-1} &= q\,(\partial^1)^{-1}(\partial^3)^{-1}(\partial^4)^{-1}(\partial^2)^{-1} = \\
q\,(\partial^2)^{-1}(\partial^1)^{-1}(\partial^3)^{-1}(\partial^4)^{-1} &= q\,(\partial^3)^{-1}(\partial^1)^{-1}(\partial^2)^{-1}(\partial^4)^{-1} = \\
q^3\,(\partial^2)^{-1}(\partial^4)^{-1}(\partial^3)^{-1}(\partial^1)^{-1} &= q^3\,(\partial^3)^{-1}(\partial^4)^{-1}(\partial^2)^{-1}(\partial^1)^{-1} = \\
q^3\,(\partial^4)^{-1}(\partial^2)^{-1}(\partial^1)^{-1}(\partial^3)^{-1} &= q^3\,(\partial^4)^{-1}(\partial^3)^{-1}(\partial^1)^{-1}(\partial^2)^{-1} = \\
q^4\,(\partial^4)^{-1}(\partial^2)^{-1}(\partial^3)^{-1}(\partial^1)^{-1} &= q^4\,(\partial^4)^{-1}(\partial^3)^{-1}(\partial^2)^{-1}(\partial^1)^{-1} = \\
&= q^2 \times \text{remaining combinations.}
\end{aligned} \tag{56}
$$

Let us note that the elements $\left(\hat{\partial}^i\right)^{-1}$ and $(P^i)^{-1}$, $i = 1, \ldots, 4,$ obey the same relations. Since the results of the various volume integrals in (56) differ by a normalisation factor only, we can restrict attention to one of the above expressions. Hence, it is sufficient to consider the following explicit formulae:

$$
\begin{aligned}
&(\partial^4)_L^{-1}(\partial^3)_L^{-1}(\partial^2)_L^{-1}(\partial^1)_L^{-1} f = \\
&\quad = q^{-4} N^{-1} \left(D^1_{q^2}\right)^{-1}\left(D^2_{q^2}\right)^{-1}\left(D^3_{q^2}\right)^{-1}\left(D^4_{q^2}\right)^{-1} \\
&\qquad \times f(q^{-2}x^1, q^{-1}x^2, q^{-1}x^3, q^{-4}x^4),
\end{aligned} \tag{57}
$$

$$
\begin{aligned}
&\left(\hat{\partial}^1\right)^{-1}_L \left(\hat{\partial}^2\right)^{-1}_L \left(\hat{\partial}^3\right)^{-1} \left(\hat{\partial}^4\right)^{-1} f = \\
&= q^4 \left(D^4_{q^{-2}}\right)^{-1} \left(D^3_{q^{-2}}\right)^{-1} \left(D^2_{q^{-2}}\right)^{-1} \left(D^1_{q^{-2}}\right)^{-1} f(q^2x^1, qx^2, qx^3), \\
&\left(-\frac{i}{[2]_{q^2}}\right)^4 (P^4)^{-1}_L (P^3)^{-1}_L (P^2)^{-1}_L (P^1)^{-1}_L f = \\
&= q^{-4} (N^4)^{-1} \left(D^1_{q^2}\right)^{-1} (N^3)^{-1} \left(D^2_{q^2}\right)^{-1} \\
&\quad \times (N^2)^{-1} \left(D^3_{q^2}\right)^{-1} (N^1)^{-1} \left(D^4_{q^2}\right)^{-1} f.
\end{aligned}
$$

With the same reasoning applied to q-deformed Euclidean space in three dimensions we can immediately varify rotation and translation invariance of our four dimensional volume integrals.

4 q-deformed Minkowski-Space

In principle all considerations of the previous two sections pertain equally to q-deformed Minkowski space [12] apart from the fact that the results now obey a more involved structure. The partial derivatives of q-deformed Minkowski space satisfy the relations

$$
\begin{aligned}
\partial^0 \partial^- &= \partial^- \partial^0, \quad \partial^0 \partial^+ = \partial^+ \partial^0, \quad \partial^0 \tilde{\partial}^3 = \tilde{\partial}^3 \partial^0, \\
\tilde{\partial}^3 \partial^+ &= q^2 \partial^+ \tilde{\partial}^3, \quad \partial^- \tilde{\partial}^3 = q^2 \tilde{\partial}^3 \partial^-, \\
\partial^- \partial^+ &= \partial^+ \partial^- + \lambda \left(\tilde{\partial}^3 \tilde{\partial}^3 + \partial^0 \tilde{\partial}^3\right),
\end{aligned} \tag{58}
$$

with $\lambda = q - q^{-1}$ and $q > 1$. As usual, the inverse elements $(\partial^\mu)^{-1}$, $\mu = \pm, 0, \tilde{3}$, are defined by

$$
(\partial^\mu)^{-1} \partial^\mu = \partial^\mu (\partial^\mu)^{-1} = 1 \tag{59}
$$

and the remaining commutation relations involving partial derivatives are given by

$$
\begin{aligned}
\left(\tilde{\partial}^3\right)^{-1} \partial^+ &= q^{-2} \partial^+ \left(\tilde{\partial}^3\right)^{-1}, \quad \tilde{\partial}^3 (\partial^+)^{-1} = q^{-2} (\partial^+)^{-1} \tilde{\partial}^3, \\
(\partial^-)^{-1} \tilde{\partial}^3 &= q^{-2} \tilde{\partial}^3 (\partial^-)^{-1}, \quad \partial^- \left(\tilde{\partial}^3\right)^{-1} = q^{-2} \left(\tilde{\partial}^3\right)^{-1} \partial^-, \\
(\partial^-)^{-1} \partial^+ &= \partial^+ (\partial^-)^{-1} - q^{-2} \lambda \partial^0 \tilde{\partial}^3 (\partial^-)^{-2} - q^{-4} \lambda \left(\tilde{\partial}^3\right)^2 (\partial^-)^{-2}, \\
\partial^- (\partial^+)^{-1} &= (\partial^+)^{-1} \partial^- - q^{-2} \lambda (\partial^+)^{-2} \partial^0 \tilde{\partial}^3 - q^{-4} \lambda (\partial^+)^{-2} \left(\tilde{\partial}^3\right)^2, \\
(\partial^0)^{-1} \partial^A &= \partial^A (\partial^0)^{-1}, \quad \partial^0 (\partial^A)^{-1} = (\partial^A)^{-1} \partial^0, \quad A = \pm, \tilde{3}.
\end{aligned} \tag{60}
$$

Finally, there are the commutation relations

$$
(\partial^0)^{-1} (\partial^\mu)^{-1} = (\partial^\mu)^{-1} (\partial^0)^{-1}, \quad \mu = \pm, \tilde{3}, \tag{61}
$$

$$\left(\partial^{-}\right)^{-1}\left(\tilde{\partial}^{3}\right)^{-1} = q^{2}\left(\tilde{\partial}^{3}\right)^{-1}\left(\partial^{-}\right)^{-1}, \quad \left(\tilde{\partial}^{3}\right)^{-1}\left(\partial^{+}\right)^{-1} = q^{2}\left(\partial^{+}\right)^{-1}\left(\tilde{\partial}^{3}\right)^{-1},$$

$$\begin{aligned}\left(\partial^{-}\right)^{-1}\left(\partial^{+}\right)^{-1} &= q^{2}\sum_{i=0}^{\infty}\left(\frac{\lambda}{\lambda_{+}}\right)^{i}[[i]]_{q^{2}}!\left(\left[\begin{array}{c}-1\\ i\end{array}\right]_{q^{2}}\right)^{2}\sum_{j+k=i}\left(-q^{6}\right)q^{i(i+2k)}\left[\begin{array}{c}i\\ k\end{array}\right]_{q^{2}}\\ &\quad\times\sum_{p=0}^{k}\left(q^{4j}\lambda_{+}\right)^{p}\left(\partial^{+}\right)^{p-(i+1)}\left(\tilde{\partial}^{3}\right)^{2j}S_{k,p}\left(\partial^{0},\tilde{\partial}^{3}\right)\left(\partial^{-}\right)^{p-(i+1)}\end{aligned}$$

where the symbols $S_{k,p}$ stand for polynomials of degree $2(k-p)$, with their explicit form presented in appendix A.

Next, we come to the commutation relations with the Lorentz generators [16], [17], [18] which become

$$\begin{aligned}T^{+}\left(\partial^{0}\right)^{-1} &= \left(\partial^{0}\right)^{-1}T^{+}, \\ T^{+}\left(\tilde{\partial}^{3}\right)^{-1} &= \left(\tilde{\partial}^{3}\right)^{-1}T^{+} - q^{1/2}\lambda_{+}^{1/2}\left(\tilde{\partial}^{3}\right)^{-2}\partial^{+}, \\ T^{+}\left(\partial^{+}\right)^{-1} &= q^{2}\left(\partial^{+}\right)^{-1}T^{+}, \\ T^{+}\left(\partial^{-}\right)^{-1} &= q^{-2}\left(\partial^{-}\right)^{-1}T^{+} - q^{-1/2}\lambda_{+}^{1/2}\left(\partial^{-}\right)^{-2}\left(\tilde{\partial}^{3}+q^{-2}\partial^{0}\right),\end{aligned} \tag{62}$$

$$\begin{aligned}[0.16in]T^{-}\left(\partial^{0}\right)^{-1} &= \left(\partial^{0}\right)^{-1}T^{-}, \\ T^{-}\left(\tilde{\partial}^{3}\right)^{-1} &= \left(\tilde{\partial}^{3}\right)^{-1}T^{-} - q^{-1/2}\lambda_{+}^{1/2}\left(\tilde{\partial}^{3}\right)^{-2}\partial^{-}, \\ T^{-}\left(\partial^{-}\right)^{-1} &= q^{-2}\left(\partial^{-}\right)^{-1}T^{-}, \\ T^{-}\left(\partial^{+}\right)^{-1} &= q^{2}\left(\partial^{+}\right)^{-1}T^{-} - q^{1/2}\lambda_{+}^{1/2}\left(\partial^{+}\right)^{-2}\left(\tilde{\partial}^{3}+q^{2}\partial^{0}\right),\end{aligned} \tag{63}$$

$$\begin{aligned}[0.16in]\tau^{3}\left(\partial^{0}\right)^{-1} &= \left(\partial^{0}\right)^{-1}\tau^{3}, \quad \tau^{3}\left(\tilde{\partial}^{3}\right)^{-1} = \left(\tilde{\partial}^{3}\right)^{-1}\tau^{3}, \\ \tau^{3}\left(\partial^{+}\right)^{-1} &= q^{4}\left(\partial^{+}\right)^{-1}\tau^{3}, \quad \tau^{3}\left(\partial^{-}\right)^{-1} = q^{-4}\left(\partial^{-}\right)^{-1}\tau^{3},\end{aligned} \tag{64}$$

$$\begin{aligned}[0.16in]T^{2}\left(\tilde{\partial}^{3}\right)^{-1} &= q\left(\tilde{\partial}^{3}\right)^{-1}T^{2}, \quad T^{2}\left(\partial^{+}\right)^{-1} = q^{-1}\left(\partial^{+}\right)^{-1}T^{2}, \\ T^{2}\left(\partial^{-}\right)^{-1} &= q\left(\partial^{-}\right)^{-1}T^{2} - q^{-3/2}\lambda_{+}^{-1/2}\tilde{\partial}^{3}\left(\partial^{-}\right)^{-2}\tau^{1}, \\ T^{2}\left(\partial^{3}\right)^{-1} &= \left(T^{2}\triangleright\left(\partial^{3}\right)^{-1}\right)\tau^{1} + \left(\sigma^{2}\triangleright\left(\partial^{3}\right)^{-1}\right)T^{2},\end{aligned} \tag{65}$$

$$\begin{aligned}[0.16in]S^{1}\left(\tilde{\partial}^{3}\right)^{-1} &= q^{-1}\left(\tilde{\partial}^{3}\right)^{-1}S^{1}, \quad S^{1}\left(\partial^{-}\right)^{-1} = q^{-1}\left(\partial^{-}\right)^{-1}S^{1}, \\ S^{1}\left(\partial^{+}\right)^{-1} &= q\left(\partial^{+}\right)^{-1}S^{1} + q^{-1/2}\lambda_{+}^{-1/2}\left(\partial^{+}\right)^{-2}\tilde{\partial}^{3}\sigma^{2}, \\ S^{1}\left(\partial^{3}\right)^{-1} &= \left(S^{1}\triangleright\left(\partial^{3}\right)^{-1}\right)\sigma^{2} + \left(\tau^{1}\triangleright\left(\partial^{3}\right)^{-1}\right)S^{1},\end{aligned} \tag{66}$$

$$\begin{aligned}[0.16in]\tau^{1}\left(\tilde{\partial}^{3}\right)^{-1} &= q^{-1}\left(\tilde{\partial}^{3}\right)^{-1}\tau^{1}, \quad \tau^{1}\left(\partial^{-}\right)^{-1} = q\left(\partial^{-}\right)^{-1}\tau^{1}, \\ \tau^{1}\left(\partial^{+}\right)^{-1} &= q^{-1}\left(\partial^{+}\right)^{-1}\tau^{1} + q^{3/2}\lambda_{+}^{-1/2}\lambda^{2}\tilde{\partial}^{3}\left(\partial^{+}\right)^{-2}T^{2},\end{aligned} \tag{67}$$

$$
\begin{aligned}
\tau^1 \left(\partial^3\right)^{-1} &= \left(\tau^1 \triangleright \left(\partial^3\right)^{-1}\right) \tau^1 + \lambda^2 \left(S^1 \triangleright \left(\partial^3\right)^{-1}\right) T^2, \\
[0.16in]\sigma^2 \left(\tilde{\partial}^3\right)^{-1} &= q \left(\tilde{\partial}^3\right)^{-1} \sigma^2, \quad \sigma^2 \left(\partial^+\right)^{-1} = q \left(\partial^+\right)^{-1} \sigma^2, \qquad (68) \\
\sigma^2 \left(\partial^-\right)^{-1} &= q^{-1} \left(\partial^-\right)^{-1} \sigma^2 - q^{1/2} \lambda_+^{-1/2} \lambda^2 \left(\partial^-\right)^{-2} \tilde{\partial}^3 S^1, \\
\sigma^2 \left(\partial^3\right)^{-1} &= \left(\sigma^2 \triangleright \left(\partial^3\right)^{-1}\right) \sigma^2 + \lambda^2 \left(T^2 \triangleright \left(\partial^3\right)^{-1}\right) S^1
\end{aligned}
$$

wher

$$
\begin{aligned}
T^2 \triangleright \left(\partial^3\right)^{-1} &= q^{-3/2} \sum_{k=0}^{\infty} \left(-q^2 \frac{\lambda^2}{\lambda_+^2}\right)^k \left(K_{1,q^2}\right)_{-1}^{(k,k+1)} \qquad (69) \\
&\times \sum_{0 \leq i+j \leq k} \lambda_+^{j-1/2} \binom{k}{i} \left(\partial^+\right)^{j+1} \left(a_+ \left(q^{2j} \tilde{\partial}^3\right)\right)^i \\
&\times S_{k-i,j}\left(\partial^0, \tilde{\partial}^3\right) \left(\partial^0 + \frac{2q^{2j+1}}{[2]_q} \tilde{\partial}^3\right)^{-2(k+1)} \left(\partial^-\right)^j, \\
[0.1in]S^1 \triangleright \left(\partial^3\right)^{-1} &= -q^{-3/2} \sum_{k=0}^{\infty} \left(-q^{-2} \frac{\lambda^2}{\lambda_+^2}\right)^k \left(K_{1,q^{-2}}\right)_{-1}^{(k,k+1)} \qquad (70) \\
&\times \sum_{0 \leq i+j \leq k} \lambda_+^{j-1/2} \binom{k}{i} \left(\partial^+\right)^{j} \left(a_+ \left(q^{2j} \tilde{\partial}^3\right)\right)^i \\
&\times S_{k-i,j}\left(\partial^0, \tilde{\partial}^3\right) \left(\partial^0 + \frac{2q^{2j+1}}{[2]_q} \tilde{\partial}^3\right)^{-2(k+1)} \left(\partial^-\right)^{j+1}, \\
[0.1in]\tau^1 \triangleright \left(\partial^3\right)^{-1} &= q \sum_{k=0}^{\infty} \left(-q^2 \frac{\lambda^2}{\lambda_+^2}\right)^k \left(K_{1,q^2}\right)_{-1}^{(k,k)} \qquad (71) \\
&\times \sum_{0 \leq i+j \leq k} \lambda_+^{j} \binom{k}{i} \left(\partial^+\right)^{j} \left(a_+ \left(q^{2j} \tilde{\partial}^3\right)\right)^i \\
&\times S_{k-i,j}\left(\partial^0, \tilde{\partial}^3\right) \left(\partial^0 + \frac{2q^{2j+1}}{[2]_q} \tilde{\partial}^3\right)^{-2k-1} \left(\partial^-\right)^j, \\
[0.1in]\sigma^2 \triangleright \left(\partial^3\right)^{-1} &= q^{-1} \sum_{k=0}^{\infty} \left(-q^{-2} \frac{\lambda^2}{\lambda_+^2}\right)^k \left(K_{1,q^{-2}}\right)_{-1}^{(k,k)} \qquad (72) \\
&\times \sum_{0 \leq i+j \leq k} \lambda_+^{j} \binom{k}{i} \left(\partial^+\right)^{j} \left(a_- \left(q^{2j} \tilde{\partial}^3\right)\right)^i \\
&S_{k-i,j}\left(\partial^0, \tilde{\partial}^3\right) \left(\partial^0 + \frac{2q^{2j-1}}{[2]_q} \tilde{\partial}^3\right)^{-2k-1} \left(\partial^-\right)^j.
\end{aligned}
$$

Note that these expressions have been formulated by using the abbreviations and conventions listed in appendix A. Applying the substitutions

$$\partial^\mu \rightarrow \hat{\partial}^\mu, \qquad (\partial^\mu)^{-1} \rightarrow \left(\hat{\partial}^\mu\right)^{-1} \tag{73}$$

and

$$\partial^\mu \rightarrow P^\mu, \qquad (\partial^\mu)^{-1} \rightarrow (P^\mu)^{-1} \tag{74}$$

to the formulae (58)-(72) again yields the corresponding expressions for the second covariant differential calculus and for the algebra of momentum generators, respectively.

As in the Euclidean case the representations of partial derivatives of q-deformed Minkowski space [10] can again be split into two parts

$$\partial^\mu F = \left(\left(\partial^\mu_{(i=0)}\right) + \left(\partial^\mu_{(i>0)}\right)\right) F, \quad \mu = \pm, 0, \tilde{3}. \tag{75}$$

Now, we can follow the same lines as in the previous sections. Thus, solutions to the difference equations

$$\partial^\mu F = f, \quad \mu = \pm, 0, \tilde{3}, \tag{76}$$

are given by

$$\begin{aligned} F &= (\partial^\mu)^{-1} f \\ &= \sum_{k=0}^{\infty} (-1)^k \left[\left(\partial^\mu_{(i=0)}\right)^{-1} \left(\partial^\mu_{(i>0)}\right)\right]^k \left(\partial^\mu_{(i=0)}\right)^{-1} f. \end{aligned} \tag{77}$$

For the differential calculus with partial derivatives ∂^μ, $\mu = \pm, 0, \tilde{3}$, the operators needed for computing (77) can in explicit terms be written as

$$\begin{aligned} \left(\tilde{\partial}^3_{(i=0)}\right)^{-1}_L f &= \left(D^3_{q_+ q^{-2}}\right)^{-1} f, \qquad (78) \\ \left(\tilde{\partial}^3_{(i>0)}\right)_L f &= -q\lambda^2\lambda_+^{-1} x^+ \tilde{x}^3 D^+_{q^2} \left(D^3_{2,q^{-1}}\right)^{1,1} f \\ &\quad - \lambda\left\{q^{-1} x^- D^-_{q^{-2}} \left(D^3_{q^{-2}} f\right)(q_+ x^3) + \tilde{x}^3 D^+_{q^2} D^-_{q^{-2}} f\left(q^{-2}\tilde{x}^3, q_- x^3\right)\right\} \\ &\quad + \sum_{l=1}^{\infty} \alpha^l_- \sum_{0\leq i+j\leq l} \left\{-q^2\lambda \left(M^-\right)^l_{i,j}(\underline{x}) \left(\tilde{T}^3_1\right)^l_j f + \left(M^+\right)^l_{i,j}(\underline{x}) \left(\tilde{T}^3_2\right)^l_j f\right\}\Big|_{x^0 \rightarrow x^3 - \tilde{x}^3}, \\ [0.16in] \left(\partial^+_{(i=0)}\right)^{-1}_L f &= -q^{-1} \left(D^-_{q^{-2}}\right)^{-1} f\left(q^2\tilde{x}^3, q_+ x^3\right), \qquad (79) \\ \left(\partial^+_{(i>0)}\right)_L f &= -\frac{\lambda}{\lambda_+} x^+ \left(D^3_{2,q^{-1}}\right)^{1,1} f \\ &\quad + \sum_{l=1}^{\infty} \alpha^l_- \sum_{0\leq i+j\leq l} \left\{-q\left(M^-\right)^l_{i,j}(\underline{x}) \left(T^+_1\right)^l_j f - \frac{\lambda}{\lambda_+}\left(M^+\right)^l_{i,j}(\underline{x}) \left(T^+_2\right)^l_j f\right\}\Big|_{x^0 \rightarrow x^3 - \tilde{x}^3}, \\ [0.16in] \left(\partial^0_{(i=0)}\right)^{-1}_L f &= \left(\tilde{D}^3_{q^{-2}}\right)^{-1} f\left(q^2 x^+, q_+ x^3, q^2 x^-\right), \qquad (80) \end{aligned}$$

$$
\begin{aligned}
\left(\partial^0_{(i>0)}\right)_L f \;&=\; -\frac{\lambda}{\lambda_+}x^+ D^+_{q^{-2}} D^3_{q_+q^{-2}} f + \\
&- \lambda\left(x^3 - q^{-1}\lambda_+^{-1}\tilde{x}^3\right) D^+_{q^{-2}} D^-_{q^{-2}} f\left(q^{-2}\tilde{x}^3, q_- x^3\right) \\
&+ q^{-1}\lambda_+^{-1}\lambda^2 x^+ x^- D^+_{q^{-2}} D^-_{q^{-2}}\left(D^3_{q^{-2}} f\right) f\left(q_+ x^3\right) \\
&+ q^{-1}\lambda_+^{-1}\lambda^2 x^+\left(x^3 - q^{-1}\lambda_+^{-1}\tilde{x}^3\right) D^+_{q^{-2}}\left(D^3_{2,q^{-1}}\right)^{1,1} f \\
&+ \sum_{l=1}^{\infty}\alpha_-^l \sum_{0\leq i+j\leq l}\left\{\left(M^-\right)^l_{i,j}(\underline{x})\;\left(T^0_1\right)^l_j f - \frac{\lambda}{\lambda_+}\left(M^+\right)^l_{i,j}(\underline{x})\left(T^0_2\right)^l_j f\right\}\Big|_{x^0\to x^3-\tilde{x}^3} \\
&+ \frac{\lambda}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_-^{l+m}\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{0\leq u\leq i+j}\left(M^{-+}_{q^{-1}}\right)^{l,m}_{i,j,u}(\underline{x})\;\left(T^0_3\right)^{l,m}_u f\Big|_{x^0\to x^3-\tilde{x}^3} \\
&+ q^{-1}\frac{\beta_-}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_-^{l+m+1}\sum_{i=0}^{l+1}\sum_{j=0}^{m}\sum_{0\leq u\leq i+j}\left(M^{-+}_{q^{-1}}\right)^{l+1,m}_{i,j,u}(\underline{x})\;\left(T^0_4\right)^{l,m}_u f\Big|_{x^0\to x^3-\tilde{x}^3} \\
&- q^{-1}\lambda_+^{-1}\sum_{0\leq l+m<\infty}\alpha_-^{l+m+1}\sum_{i=0}^{l}\sum_{j=0}^{m+1}\sum_{0\leq u\leq i+j}\left(M^{-+}_{q^{-1}}\right)^{l,m+1}_{i,j,u}(\underline{x})\;\left(T^0_5\right)^{l,m}_u f\Big|_{x^0\to x^3-\tilde{x}^3},
\end{aligned}
$$

$$
[0.16in]\left(\partial^-_{(i=0)}\right)^{-1}_L f \;=\; -q\left(D^+_{q^{-2}}\right)^{-1} f, \tag{81}
$$

$$
\begin{aligned}
\left(\partial^-_{(i>0)}\right)_L f \;&=\; \\
&\lambda\sum_{l=0}^{\infty}\alpha_-^l\sum_{0\leq i+j\leq l}\left\{\left(M^-\right)^l_{i,j}(\underline{x})\left(T^-_1\right)^l_j f + \left(M^+\right)^l_{i,j}(\underline{x})\left(T^-_2\right)^l_j f\right\}\Big|_{x^0\to x^3-\tilde{x}^3} \\
&+ q^2\lambda\sum_{l=0}^{\infty}\alpha_-^{l+1}\sum_{0\leq i+j\leq l+1}\left(M^+\right)^{l+1}_{i,j}(\underline{x})\;\left(T^-_3\right)^l_j f\Big|_{x^0\to x^3-\tilde{x}^3} \\
&+ \frac{\lambda}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_-^{l+m}\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{0\leq u\leq i+j}\Big\{q^{-1}\left(M^{-+}_q\right)^{l,m}_{i,j,u}(\underline{x})\;\left(T^-_4\right)^{l,m}_u f \\
&\qquad + \lambda\left(M^{-+}_{q^{-1}}\right)^{l,m}_{i,j,u}(\underline{x})\left(T^-_5\right)^{l,m}_u f\Big\}\Big|_{x^0\to x^3-\tilde{x}^3} \\
&+ \beta_-\frac{\lambda}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_-^{l+m+1}\sum_{i=0}^{l+1}\sum_{j=0}^{m}\sum_{0\leq u\leq i+j}\left(M^{-+}_{q^{-1}}\right)^{l+1,m}_{i,j,u}(\underline{x})\;\left(T^-_6\right)^{l,m}_u f\Big|_{x^0\to x^3-\tilde{x}^3} \\
&+ \frac{\lambda}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_-^{l+m+1}\sum_{i=0}^{l}\sum_{j=0}^{m+1}\sum_{0\leq u\leq i+j}\Big\{\left(M^{-+}_{q^{-1}}\right)^{l,m+1}_{i,j,u}(\underline{x})\left(T^-_7\right)^{l,m}_u f \\
&\qquad + q\left(M^{-+}_{q^{-1}}\right)^{l,m+1}_{i,j,u}(\underline{x})\left(T^-_8\right)^{l,m}_u f\Big\}\Big|_{x^0\to x^3-\tilde{x}^3}
\end{aligned} \tag{82}
$$

where

$$\left(\tilde{T}_1^3\right)_j^l f = \left[\left(\tilde{O}_1^3\right)_l f\Big|_{x^3\to y_+}\right]\left(q^{2(j-1)}\tilde{x}^3\right), \tag{83}$$

$$\left(\tilde{T}_2^3\right)_j^l f = \left[\left(\tilde{O}_2^3\right)_l f\Big|_{x^3\to x^0+\tilde{x}^3} - q^{-1}\lambda\left(\tilde{O}_3^3\right)_l f\Big|_{x^3\to y_+}\right]\left(q^{2j}\tilde{x}^3\right),$$

$$[.16in]\left(\tilde{T}_1^+\right)_j^l f = \left[\left(O_1^+\right)_l f\big|_{x^3\to y_+}\right]\left(q^{2(j-1)}\tilde{x}^3\right), \tag{84}$$

$$\left(T_2^+\right)_j^l f = \left[\left(O_2^+\right)_l f\big|_{x^3\to x^0+\tilde{x}^3}\right]\left(q^{2j}\tilde{x}^3\right),$$

$$[.16in]\left(T_1^0\right)_j^l f = \left[\left(O_1^0\right)_l f\big|_{x^3\to y_-}\right]\left(q^{2j}\tilde{x}^3\right), \tag{85}$$

$$\left(T_2^0\right)_j^l f = \left[\left(O_2^0\right)_l f\big|_{x^3\to x^0+\tilde{x}^3} - q^{-1}\lambda\left(O_3^0\right)_l f\big|_{x^3\to y_+}\right]\left(q^{2j}\tilde{x}^3\right),$$

$$\left(T_3^0\right)_u^{l,m} f = \left[q^{-2}\left(Q_1^0\right)_{l,m} f\big|_{x^3\to x^0+\tilde{x}^3} - \left(Q_2^0\right)_{l,m} f\Big|_{x^3\to y_+}\right]\left(q^{2u}\tilde{x}^3\right),$$

$$\left(T_4^0\right)_u^{l,m} f = \left[\left(Q_3^0\right)_{l,m} f\big|_{x^3\to x^0+\tilde{x}^3} - q^{-1}\lambda\left(Q_4^0\right)_{l,m} f\Big|_{x^3\to y_+}\right]\left(q^{2u}\tilde{x}^3\right),$$

$$\left(T_5^0\right)_u^{l,m} f = \left[\left(Q_5^0\right)_{l,m} f\big|_{x^3\to x^0+\tilde{x}^3}\right]\left(q^{2u}\tilde{x}^3\right),$$

$$[.16in]\left(T_1^-\right)_j^l f = \left[\left(O_1^-\right)_l f\big|_{x^3\to y_-}\right]\left(q^{2j}\tilde{x}^3\right), \tag{86}$$

$$\left(T_2^-\right)_j^l f = \left[\left(O_2^-\right)_l f\big|_{x^3\to x^0+\tilde{x}^3} + \left(O_3^-\right)_l f\big|_{x^3\to y_+}\right]\left(q^{2j}\tilde{x}^3\right),$$

$$\left(T_3^-\right)_j^l f = \left[\left(O_4^-\right)_l f\big|_{x^3\to x^0+\tilde{x}^3}\right]\left(q^{2j}\tilde{x}^3\right),$$

$$\left(T_4^-\right)_u^{l,m} f = \left[\left(Q_1^-\right)_{l,m} f\big|_{x^3\to x^0+\tilde{x}^3} - \lambda\left(Q_2^-\right)_{l,m} f\Big|_{x^3\to y_+}\right]\left(q^{2(u+1)}\tilde{x}^3\right),$$

$$\left(T_5^-\right)_u^{l,m} f = \left[q^{-2}\left(Q_3^-\right)_{l,m} f\big|_{x^3\to x^0+\tilde{x}^3} - \left(Q_4^-\right)_{l,m} f\big|_{x^3\to y_+}\right]\left(q^{2u}\tilde{x}^3\right),$$

$$\left(T_6^-\right)_u^{l,m} f = \left[\left(Q_5^-\right)_{l,m} f\big|_{x^3\to x^0+\tilde{x}^3} - q^{-1}\lambda\left(Q_6^-\right)_{l,m} f\Big|_{x^3\to y_+}\right]\left(q^{2u}\tilde{x}^3\right),$$

$$\left(T_7^-\right)_u^{l,m} f = \left[\left(Q_7^-\right)_{l,m} f\big|_{x^3\to x^0+\tilde{x}^3}\right]\left(q^{2(u+1)}\tilde{x}^3\right),$$

$$\left(T_8^-\right)_u^{l,m} f = \left[\left(Q_8^-\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3}\right]\left(q^{2u}\tilde{x}^3\right).$$

The operators $(O_i^\mu)_l$ and $(Q_i^\mu)_{l,m}$, $\mu = \pm, \tilde{3}, 0,$ as well as the polynomials $(M^\pm)_{i,j}^l\,(\underline{x})$ and $\left(M_{q^\pm}^{-+}\right)_{i,j,u}^{l,m}(\underline{x})$ have already been introduced in [10]. Their explicit form is once again listed in appendix A. In addition, we have used the abbreviations

$$\alpha_\pm = -q^{\pm 2}\frac{\lambda^2}{\lambda_+^2}, \qquad q_\pm = 1 \pm \frac{\lambda}{\lambda_+}\frac{\tilde{x}^3}{x^3}, \tag{87}$$

$$\beta_\pm = \left(q^\pm + \lambda_+\right), \qquad y_\pm = x^0 + \frac{2}{\lambda_+}q^{\pm 1}\tilde{x}^3.$$

$$\left(\hat{\tilde{\partial}}^3{}_{(i=0)}\right)^{-1}_L f = \left(D^3_{q-q^2}\right)^{-1} f\left(q^{-2}x^+\right), \tag{88}$$

$$\left(\hat{\tilde{\partial}}^3{}_{(i>0)}\right)_L f = \sum_{l=1}^{\infty}\alpha_+^l \sum_{0\leq i+j\leq l} \left(M^-\right)^l_{i,j}(\underline{x}) \left(\hat{\tilde{T}}^3\right)^l_j f\Big|_{x^0\to x^3-\tilde{x}^3},$$

$$[0.16in]\left(\hat{\partial}^-_{(i=0)}\right)^{-1}_L f = -q\left(D^+_{q^2}\right)^{-1} f, \tag{89}$$

$$\left(\hat{\partial}^-_{(i>0)}\right)_L f = \tag{90}$$

$$\frac{\lambda}{\lambda_+}\sum_{l=0}^{\infty}\alpha_+^l \sum_{0\leq i+j\leq l}\left\{\left(M^+\right)^l_{i,j}(\underline{x})\left(\hat{T}^-_1\right)^l_j f + q^{-1}\left(M^-\right)^l_{i,j}(\underline{x})\left(\hat{T}^-_2\right)^l_j f\right\}\Big|_{x^0\to x^3-\tilde{x}^3},$$

$$[0.16in]\left(\hat{\partial}^0_{(i=0)}\right)^{-1}_L f = \left(\tilde{D}^3_{q^2}\right)^{-1} f\left(q_- x^3\right), \tag{91}$$

$$\begin{aligned}
\left(\hat{\partial}^0_{(i>0)}\right)_L f =& -q\frac{\lambda}{\lambda_+}\left\{qx^- D^-_{q^{-2}}\left(D^3_{q^{-2}} f\right)\left(q^2\tilde{x}^3, q^2 q_+ x^3\right) + qx^+ D^+_{q^2} D^3_{q^2 q_-} f\right\} \\
&- q\frac{\lambda}{\lambda_+}\left\{\tilde{x}^3 D^+_{q^2} D^-_{q^2} f\left(q^2 q_- x^3\right) + q^2\lambda x^+ \tilde{x}^3 D^+_{q^2}\left(\tilde{D}^3_{q^2} D^3_{q^2} f\right)\left(q_+ x^3\right)\right\} \\
&+ \sum_{l=1}^{\infty}\alpha_+^l \sum_{0\leq i+j\leq l}\left\{\left(M^+\right)^l_{i,j}(\underline{x})\left(\hat{T}^0_1\right)^l_j f + q\frac{\lambda}{\lambda_+}\left(M^-\right)^l_{i,j}(\underline{x})\left(\hat{T}^0_2\right)^l_j f\right\}\Big|_{x^0\to x^3-\tilde{x}^3} \\
&- q^2\frac{\lambda}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_+^{l+m}\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{0\leq u\leq i+j}\left(M^{+-}_{q^{-1}}\right)^{l,m}_{i,j,u}(\underline{x})\left(\hat{T}^0_3\right)^{l,m}_u f\Big|_{x^0\to x^3-\tilde{x}^3} \\
&+ \frac{\beta_+}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_+^{l+m+1}\sum_{i=0}^{l+1}\sum_{j=0}^{m}\sum_{0\leq u\leq i+j}\left(M^{+-}_{q^{-1}}\right)^{l+1,m}_{i,j,u}(\underline{x})\left(\hat{T}^0_4\right)^{l,m}_u f\Big|_{x^0\to x^3-\tilde{x}^3} \\
&+ \lambda_+^{-1}\sum_{0\leq l+m<\infty}\alpha_+^{l+m+1}\sum_{i=0}^{l}\sum_{j=0}^{m+1}\sum_{0\leq u\leq i+j}\left(M^{+-}_{q^{-1}}\right)^{l,m+1}_{i,j,u}(\underline{x})\left(\hat{T}^0_5\right)^{l,m}_u f\Big|_{x^0\to x^3-\tilde{x}^3},
\end{aligned}$$

$$[0.16in]\left(\hat{\partial}^+_{(i=0)}\right)^{-1}_L f = -q^{-1}\left(D^-_{q^2}\right)^{-1} f\left(q^{-2}q_+ x^3\right), \tag{92}$$

$$\begin{aligned}
\left(\hat{\partial}^+_{(i>0)}\right)_L f =& -q\lambda x^+\left(\tilde{D}^3_{q^2} D^3_{q^2} f\right)^{-1} f\left(q_+ x^3\right) \\
&- q\sum_{l=1}^{\infty}\alpha_+^l\sum_{0\leq i+j\leq l}\left\{\left(M^-\right)^l_{i,j}(\underline{x})\left(\hat{T}^+_1\right)^{l,m}_u f + \lambda\left(M^+\right)^l_{i,j}(\underline{x})\left(\hat{T}^+_2\right)^{l,m}_u f\right\}\Big|_{x^0\to x^3-\tilde{x}^3}
\end{aligned}$$

$$
\begin{aligned}
&- q\frac{\lambda}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_+^{l+m}\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{0\leq u\leq i+j}\left(M^{+-}_{q^{-1}}\right)^{l,m}_{i,j,u}(\underline{x})\left(\hat{T}_3^{+}\right)^{l,m}_{u} f\Big|_{x^0\to x^3-\tilde{x}^3}\\
&- \frac{\lambda}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_+^{l+m+1}\sum_{i=0}^{l}\sum_{j=0}^{m+1}\sum_{0\leq u\leq i+j}\left(M^{+-}_{q^{-1}}\right)^{l,m+1}_{i,j,u}(\underline{x})\left(\hat{T}_4^{+}\right)^{l,m}_{u} f\Big|_{x^0\to x^3-\tilde{x}^3}
\end{aligned}
$$

where

$$\left(\hat{\tilde{T}}^3\right)^l_j f = \left[\left(\hat{\tilde{O}}^3\right)_l f\Big|_{x^3\to x^0+\tilde{x}^3}\right](q^{2j}\tilde{x}^3), \tag{93}$$

$$[0.16in]\left(\hat{T}_1^-\right)^l_j f = \left[\left(\hat{O}_1^-\right)_l f\Big|_{x^3\to x^0+\tilde{x}^3}\right]\left(q^{2(j+1)}\tilde{x}^3\right), \tag{94}$$

$$\left(\hat{T}_2^-\right)^l_j f = \left[\left(\hat{O}_2^-\right)_l f\Big|_{x^3\to x^0+\tilde{x}^3}\right](q^{2j}\tilde{x}^3),$$

$$[0.16in]\left(\hat{T}_1^0\right)^l_j f = \left[\left(\hat{O}_1^0\right)_l f\Big|_{x^3\to y_+} - q^2\frac{\lambda}{\lambda_+}\left(\hat{O}_2^0\right)_l f\Big|_{x^3\to q^2 y_+}\right](q^{2j}\tilde{x}^3), \tag{95}$$

$$\left(\hat{T}_2^0\right)^l_j f = \left[\left(\hat{O}_3^0\right)_l f\Big|_{x^3\to x^0+\tilde{x}^3} + \left(\hat{O}_4^0\right)_l f\Big|_{x^3\to q^2 y_-}\right](q^{2j}\tilde{x}^3),$$

$$\left(\hat{T}_3^0\right)^{l,m}_u f = \left[\left(\hat{Q}_1^0\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3}\right](q^{2u}\tilde{x}^3),$$

$$\left(\hat{T}_4^0\right)^{l,m}_u f = \left[\left(\hat{Q}_2^0\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3}\right](q^{2u}\tilde{x}^3),$$

$$\left(\hat{T}_5^0\right)^{l,m}_u f = \left[\left(\hat{Q}_3^0\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3}\right](q^{2u}\tilde{x}^3),$$

$$[0.16in]\left(\hat{T}_1^+\right)^l_j f = \left[\left(\hat{O}_1^+\right)_l f\Big|_{x^3\to q^2 y_-}\right](q^{2j}\tilde{x}^3), \tag{96}$$

$$\left(\hat{T}_2^+\right)^l_j f = \left[\left(\hat{O}_2^+\right)_l f\Big|_{x^3\to y_+}\right](q^{2j}\tilde{x}^3),$$

$$\left(\hat{T}_3^+\right)^{l,m}_u f = \left[\left(\hat{Q}_1^+\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3}\right](q^{2u}\tilde{x}^3),$$

$$\left(\hat{T}_4^+\right)^{l,m}_u f = \left[\left(\hat{Q}_2^+\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3}\right](q^{2u}\tilde{x}^3).$$

And likewise for the algebra of momentum generators we have

$$
\begin{aligned}
-\frac{i}{[2]_{q^2}}\left(\tilde{P}^3_{(i=0)}\right)^{-1}_L f &= \left(\tilde{J}^3\right)^{-1}\left(D^3_{q^2}\right)^{-1} f,\\
i\,[2]_{q^2}\left(\tilde{P}^3_{(i>0)}\right)_L f &= -q^{-3}\lambda x^- D^-_{q^{-2}}\left(D^3_{q^{-2}} f\right)(q_+ x^3)\\
&\quad - q^{-1}\lambda\left\{q^{-1}\tilde{x}^3 D^+_{q^2} D^-_{q^{-2}} f\left(q^{-2}\tilde{x}^3, q_- x^3\right) + \frac{\lambda}{\lambda_+}x^+\tilde{x}^3 D^+_{q^2}\left(D^3_{2,q^{-1}}\right)^{1,1} f\right\}
\end{aligned}
\tag{97}
$$

$$
\begin{aligned}
&+\sum_{l=1}^{\infty}\alpha_0^l\sum_{0\leq i+j\leq l}\left\{\left(M^+\right)^l_{i,j}(\underline{x})\left(T_1^{P_3}\right)^l_j f+\left(M^-\right)^l_{i,j}(\underline{x})\left(T_2^{P_3}\right)^l_j f\right\}\Big|_{x^0\to x^3-\tilde{x}^3},
\end{aligned}
$$

$$
[0.16in]-\frac{i}{[2]_{q^2}}\left(P^+_{(i=0)}\right)^{-1}_L f=-q^{-1}\left(J^+\right)^{-1}\left(D^-_{q^2}\right)^{-1}f, \tag{98}
$$

$$
\begin{aligned}
i\,[2]_{q^2}\left(P^+_{(i>0)}\right)_L f={}&-\lambda x^+\left\{q^{-2}\lambda_+^{-1}\left(D^3_{2,q^{-1}}\right)^{1,1}f+q^3\left(\tilde{D}^3_{q^2}D^3_{q^2}f\right)\left(q_+x^3\right)\right\}\\
&-\sum_{l=1}^{\infty}\alpha_0^l\sum_{0\leq i+j\leq l}\left\{q\left(M^-\right)^l_{i,j}(\underline{x})\left(T_1^{P_+}\right)^l_j f+\lambda\left(M^+\right)^l_{i,j}(\underline{x})\left(T_2^{P_+}\right)^l_j f\right\}\Big|_{x^0\to x^3-\tilde{x}^3}\\
&-q\frac{\lambda}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_0^{l+m}\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{0\leq u\leq i+j}\left(M^{+-}_{q^{-1}}\right)^{l,m}_{i,j,u}(\underline{x})\left(T_3^{P_+}\right)^{l,m}_u f\Big|_{x^0\to x^3-\tilde{x}^3}\\
&-\frac{\lambda}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_0^{l+m+1}\sum_{i=0}^{l}\sum_{j=0}^{m+1}\sum_{0\leq u\leq i+j}\left(M^{+-}_{q^{-1}}\right)^{l,m+1}_{i,j,u}(\underline{x})\left(T_4^{P_+}\right)^{l,m}_u f\Big|_{x^0\to x^3-\tilde{x}^3},
\end{aligned}
$$

$$
[0.16in]-\frac{i}{[2]_{q^2}}\left(P^0_{(i=0)}\right)^{-1}_L f=\left(J^0\right)^{-1}\left(\tilde{D}^3_{q^2}\right)^{-1}f, \tag{99}
$$

$$
\begin{aligned}
i\,[2]_{q^2}\left(P^0_{(i>0)}\right)_L f={}&-q^{-2}\lambda x^3D^+_{q^{-2}}D^-_{q^{-2}}f\left(q^{-2}\tilde{x}^3,q_-x^3\right)\\
&-\frac{\lambda}{\lambda_+}x^+D^+_{q^2}\left(D^3_{q_+q^{-2}}f\left(q^{-2}x^+\right)+q^4D_{q_-q^2}f\right)\\
&-q^4\frac{\lambda}{\lambda_+}x^-D^-_{q^{-2}}\left(D^3_{q^{-2}}f\right)\left(q^2\tilde{x}^3,q_+q^2x^3\right)\\
&-q^3\frac{\lambda}{\lambda_+}\tilde{x}^3D^+_{q^2}D^-_{q^2}\left(f\left(q_-q^2x^3\right)-f\left(q^{-2}x^+,q^{-2}\tilde{x}^3,q_-x^3,q^{-2}x^-\right)\right)\\
&+q^{-3}\lambda_+^{-1}\lambda^2x^+x^-D^+_{q^{-2}}D^-_{q^{-2}}\left(D^3_{q^{-2}}f\right)\left(q_+x^3\right)\\
&+q^{-3}\lambda_+^{-1}\lambda^2x^+\left(x^3-q^{-1}\lambda_+^{-1}\tilde{x}^3\right)D^+_{q^{-2}}\left(D^3_{2,q^{-1}}\right)^{1,1}f\\
&-q^5\lambda_+^{-1}\lambda^2x^+\tilde{x}^3D^+_{q^2}\left(\tilde{D}^3_{q^2}D^3_{q^2}f\right)\left(q_+x^3\right)\\
&+\sum_{l=1}^{\infty}\alpha_0^l\sum_{0\leq i+j\leq l}\left\{\left(M^-\right)^l_{i,j}(\underline{x})\left(T_1^{P_0}\right)^l_j f+\left(M^+\right)^l_{i,j}(\underline{x})\left(T_2^{P_0}\right)^l_j f\right\}\Big|_{x^0\to x^3-\tilde{x}^3}\\
&-\frac{\lambda}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_0^{l+m}\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{0\leq u\leq i+j}\left(M^{-+}_{q^{-1}}\right)^{l,m}_{i,j,u}(\underline{x})\left(T_3^{P_0}\right)^{l,m}_u f\Big|_{x^0\to x^3-\tilde{x}^3}\\
&+\sum_{0\leq l+m<\infty}\alpha_0^{l+m+1}\sum_{i=0}^{l+1}\sum_{j=0}^{m}\sum_{0\leq u\leq i+j}\left(M^{-+}_{q^{-1}}\right)^{l+1,m}_{i,j,u}(\underline{x})\left(T_4^{P_0}\right)^{l,m}_u f\Big|_{x^0\to x^3-\tilde{x}^3}\\
&-\sum_{0\leq l+m<\infty}\alpha_0^{l+m+1}\sum_{i=0}^{l}\sum_{j=0}^{m+1}\sum_{0\leq u\leq i+j}\left(M^{-+}_{q^{-1}}\right)^{l,m+1}_{i,j,u}(\underline{x})\left(T_5^{P_0}\right)^{l,m}_u f\Big|_{x^0\to x^3-\tilde{x}^3},
\end{aligned}
$$

$$[0.16in] -\frac{i}{[2]_{q^2}}\left(P^-_{(i=0)}\right)^{-1}_{L} f = -q\left(J^-\right)^{-1}\left(D^+_{q^2}\right)^{-1} f, \tag{100}$$

$$i\,[2]_{q^2}\left(P^-_{(i>0)}\right)_L f = \tag{101}$$

$$\lambda\sum_{l=0}^{\infty}\alpha_0^l\sum_{0\leq i+j\leq l}\left\{\left(M^-\right)^l_{i,j}(\underline{x})\left(T_1^{P_-}\right)^l_j f+\left(M^+\right)^l_{i,j}(\underline{x})\left(T_2^{P_-}\right)^l_j f\right\}\Big|_{x^0\to x^3-\tilde{x}^3}$$

$$+\,q^2\lambda\sum_{l=0}^{\infty}\alpha_0^{l+1}\sum_{0\leq i+j\leq l+1}\left(M^+\right)^{l+1}_{i,j}(\underline{x})\left(T_3^{P_-}\right)^l_j f\Big|_{x^0\to x^3-\tilde{x}^3}$$

$$+\,\frac{\lambda}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_0^{l+m}\sum_{i=0}^{l}\sum_{j=0}^{m}\sum_{0\leq u\leq i+j}\left\{q^{-1}\left(M_q^{-+}\right)^{l,m}_{i,j,u}(\underline{x})\left(T_4^{P_-}\right)^{l,m}_u f\right.$$

$$\left.+\,\lambda\left(M_{q^{-1}}^{-+}\right)^{l,m}_{i,j,u}(\underline{x})\left(T_5^{P_-}\right)^{l,m}_u f\right\}\Big|_{x^0\to x^3-\tilde{x}^3}$$

$$+\,\beta_-\frac{\lambda}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_0^{l+m+1}\sum_{i=0}^{l+1}\sum_{j=0}^{m}\sum_{0\leq u\leq i+j}\left(M_{q^{-1}}^{-+}\right)^{l+1,m}_{i,j,u}(\underline{x})\left(T_6^{P_-}\right)^{l,m}_u f\Big|_{x^0\to x^3-\tilde{x}^3}$$

$$+\,\frac{\lambda}{\lambda_+}\sum_{0\leq l+m<\infty}\alpha_0^{l+m+1}\sum_{i=0}^{l}\sum_{j=0}^{m+1}\sum_{0\leq u\leq i+j}\left\{\left(M_q^{-+}\right)^{l,m+1}_{i,j,u}(\underline{x})\left(T_7^{P_-}\right)^{l,m}_u f\right.$$

$$\left.+\,q\left(M_{q^{-1}}^{-+}\right)^{l,m+1}_{i,j,u}(\underline{x})\left(T_8^{P_-}\right)^{l,m}_u f\right\}\Big|_{x^0\to x^3-\tilde{x}^3}$$

where

$$\left(T_1^{P_3}\right)^l_j f = \left[\left(\tilde{O}_1^{P_3}\right)_l f\Big|_{x^3\to x^0+\tilde{x}^3} - q^{-1}\lambda\left(\tilde{O}_2^{P_3}\right)_L f\Big|_{x^3\to y_+}\right](q^{2j}\tilde{x}^3), \tag{102}$$

$$\left(T_2^{P_3}\right)^l_j f = \left[\left(\tilde{O}_3^{P_3}\right)_l f\Big|_{x^3\to x^0+\tilde{x}^3} - q^2\lambda\left(\tilde{O}_4^{P_3}\right)_L f\Big|_{x^3\to y_-}\right](q^{2j}\tilde{x}^3),$$

$$[0.16in]\left(T_1^{P_+}\right)^l_j f = \left[\left(O_1^{P_+}\right)_l f\Big|_{x^3\to y_-}\right](q^{2j}\tilde{x}^3), \tag{103}$$

$$\left(T_2^{P_+}\right)^l_j f = \left[\lambda_+^{-1}\left(O_2^{P_+}\right)_l f\Big|_{x^3\to x^0+\tilde{x}^3} + q\left(O_3^{P_+}\right)_L f\Big|_{x^3\to y_+}\right](q^{2j}\tilde{x}^3),$$

$$\left(T_3^{P_+}\right)^{l,m}_u f = \left[\left(Q_1^{P_+}\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3}\right](q^{2u}\tilde{x}^3),$$

$$\left(T_4^{P_+}\right)^{l,m}_u f = \left[\left(Q_2^{P_+}\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3}\right](q^{2u}\tilde{x}^3),$$

$$[0.16in]\left(T_1^{P_0}\right)^l_j f = \left[\left(O_1^{P_0}\right)_l f\Big|_{x^3\to y_-} - q\frac{\lambda}{\lambda_+}\left(O_2^{P_0}\right)_l f\Big|_{x^3\to x^0+\tilde{x}^3}\right](q^{2j}\tilde{x}^3), \tag{104}$$

$$\left(T_2^{P_0}\right)^l_j f = \left[\left(O_3^{P_0}\right)_l f\Big|_{x^3\to y_+} - \frac{\lambda}{\lambda_+}\left(O_4^{P_0}\right)_L f\Big|_{x^3\to x^0+\tilde{x}^3}\right](q^{2j}\tilde{x}^3),$$

$$
\begin{aligned}
\left(T_3^{P_0}\right)_u^{l,m} f &= \left[\left(Q_1^{P_0}\right)_{l,m} f\Big|_{x^3\to y_+} - \left(Q_2^{P_0}\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3}\right]\left(q^{2u}\tilde{x}^3\right),\\
\left(T_4^{P_0}\right)_u^{l,m} f &= \left[\left(Q_3^{P_0}\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3} - q^{-1}\beta_-\frac{\lambda}{\lambda_+}\left(Q_4^{P_0}\right)_{l,m} f\Big|_{x^3\to y_+}\right]\left(q^{2u}\tilde{x}^3\right),\\
\left(T_5^{P_0}\right)_u^{l,m} f &= \left[\left(Q_5^{P_0}\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3}\right]\left(q^{2u}\tilde{x}^3\right),\\
[0.16in]\left(T_1^{P_-}\right)_j^{l} f &= \left[q^{-1}\lambda_+^{-1}\left(O_1^{P_-}\right)_l f\Big|_{x^3\to x^0+\tilde{x}^3} + \left(O_2^{P_-}\right)_l f\Big|_{x^3\to y_-}\right]\left(q^{2j}\tilde{x}^3\right), \qquad (105)\\
\left(T_2^{P_-}\right)_u^{l} f &= \left[\left(O_3^{P_-}\right)_l f\Big|_{x^3\to x^0+\tilde{x}^3} + \left(O_4^{P_-}\right)_l f\Big|_{x^3\to y_+}\right.\\
&\quad \left. + \left(O_5^{P_-}\right) f\Big|_{x^3\to x^0+q^2\tilde{x}^3}\right]\left(q^{2j}\tilde{x}^3\right),\\
\left(T_3^{P_-}\right)_j^{l} f &= \left[\left(O_6^{P_-}\right)_l f\Big|_{x^3\to x^0+\tilde{x}^3}\right]\left(q^{2j}\tilde{x}^3\right),\\
\left(T_4^{P_-}\right)_u^{l,m} f &= \left[\left(Q_1^{P_-}\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3} - \lambda\left(Q_2^{P_-}\right)_{l,m} f\Big|_{x^3\to y_+}\right]\left(q^{2(u+1)}\tilde{x}^3\right),\\
\left(T_5^{P_-}\right)_u^{l,m} f &= \left[q^{-2}\left(Q_3^{P_-}\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3} - \left(Q_4^{P_-}\right)_{l,m} f\Big|_{x^3\to y_+}\right]\left(q^{2u}\tilde{x}^3\right),\\
\left(T_6^{P_-}\right)_u^{l,m} f &= \left[\left(Q_5^{P_-}\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3} - q^{-1}\lambda\left(Q_6^{P_-}\right)_{l,m} f\Big|_{x^3\to y_+}\right]\left(q^{2u}\tilde{x}^3\right),\\
\left(T_7^{P_-}\right)_u^{l,m} f &= \left[\left(Q_7^{P_-}\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3}\right]\left(q^{2(u+1)}\tilde{x}^3\right),\\
\left(T_8^{P_-}\right)_u^{l,m} f &= \left[\left(Q_8^{P_-}\right)_{l,m} f\Big|_{x^3\to x^0+\tilde{x}^3}\right]\left(q^{2u}\tilde{x}^3\right).
\end{aligned}
$$

The explicit form of the operators $\left(\hat{O}_i^{\mu}\right)_l$, $\left(\hat{Q}_i^{\mu}\right)_{l,m}$ and $\left(O_i^{P_\mu}\right)_l$, $\left(Q_i^{P_\mu}\right)_{l,m}$ can be found in appendix A, whereas the scaling operators $(J^\mu)^{-1}$ are defined in appendix B. For the purpose of abbreviation we have additionally $\alpha_0 = -\lambda^2/\lambda_+^2$ introduced.

Again, left and right integrals are related to each other by the following transitions:

$$
\begin{aligned}
\left(\partial^0\right)_R^{-1} f &\overset{+}{\longleftrightarrow}{}^{-} -q^{-4}\left(\hat{\partial}^0\right)_L^{-1} f,\\
\left(\tilde{\partial}^3\right)_R^{-1} f &\overset{+}{\longleftrightarrow}{}^{-} -q^{-4}\left(\hat{\tilde{\partial}}^3\right)_L^{-1} f,\\
\left(\partial^+\right)_R^{-1} f &\overset{+}{\longleftrightarrow}{}^{-} -q^{-4}\left(\hat{\partial}^-\right)_L^{-1} f,\\
\left(\partial^-\right)_R^{-1} f &\overset{+}{\longleftrightarrow}{}^{-} -q^{-4}\left(\hat{\partial}^+\right)_L^{-1} f.
\end{aligned}
\qquad (106)
$$

Analogously, for the second differential calculus we have

$$
\left(\hat{\partial}^0\right)_R^{-1} f \overset{+}{\longleftrightarrow}{}^{-} -q^{4}\left(\partial^0\right)_L^{-1} f, \qquad (107)
$$

$$
\begin{aligned}
\left(\hat{\tilde{\partial}}^3\right)_R^{-1} f &\overset{+}{\longleftrightarrow}{}^{-} -q^4\left(\tilde{\partial}^3\right)_L^{-1} f, \\
\left(\hat{\partial}^+\right)_R^{-1} f &\overset{+}{\longleftrightarrow}{}^{-} -q^4\left(\partial^-\right)_L^{-1} f, \\
\left(\hat{\partial}^-\right)_R^{-1} f &\overset{+}{\longleftrightarrow}{}^{-} -q^4\left(\partial^+\right)_L^{-1} f.
\end{aligned}
$$

It remains to treat the case of the algebra of momentum generators for which we find

$$
\begin{aligned}
\left(P^0\right)_R^{-1} f &\overset{+}{\longleftrightarrow}{}^{-} -\left(P^0\right)_L^{-1} f, \\
\left(\tilde{P}^3\right)_R^{-1} f &\overset{+}{\longleftrightarrow}{}^{-} -\left(\tilde{P}^3\right)_L^{-1} f, \\
\left(P^+\right)_R^{-1} f &\overset{+}{\longleftrightarrow}{}^{-} -\left(P^-\right)_L^{-1} f, \\
\left(P^-\right)_R^{-1} f &\overset{+}{\longleftrightarrow}{}^{-} -\left(P^+\right)_L^{-1} f.
\end{aligned}
\tag{108}
$$

The rules for integration by parts in the case of left integrals are given by

$$
\begin{aligned}
\left(\tilde{\partial}^3\right)_L^{-1}\left(\tilde{\partial}_L^3 f\right)\star g\Big|_{x^3=a}^{b} &= f\star g|_{x^3=a}^{b} - \left(\tilde{\partial}^3\right)_L^{-1}\left(\Lambda^{1/2}\tau^1 f\right)\star\tilde{\partial}_L^3 g\Big|_{x^3=a}^{b} \\
&\quad + q^{1/2}\lambda_+^{1/2}\lambda\left(\tilde{\partial}^3\right)_L^{-1}\left(\Lambda^{1/2}\left(\tau^3\right)^{-1/2}S^1 f\right)\star\partial_L^+ g\big|_{x^3=a}^{b}, \\
\left(\partial^+\right)_L^{-1}\left(\partial_L^+ f\right)\star g\Big|_{x^-=a}^{b} &= f\star g|_{x^-=a}^{b} - \left(\partial^+\right)_L^{-1}\left(\Lambda^{1/2}\left(\tau^3\right)^{-1/2}\sigma^2 f\right)\star\partial_L^+ g\big|_{x^-=a}^{b} \\
&\quad + q^{3/2}\lambda_+^{-1/2}\lambda\left(\partial^+\right)_L^{-1}\left(\Lambda^{1/2}T^2 f\right)\star\tilde{\partial}_L^3 g\Big|_{x^-=a}^{b}, \\
\left(\partial^-\right)_L^{-1}\left(\partial_L^- f\right)\star g\Big|_{x^+=a}^{b} &= f\star g|_{x^+=a}^{b} - \left(\partial^-\right)_L^{-1}\left(\Lambda^{1/2}\left(\tau^3\right)^{1/2}\tau^1 f\right)\star\partial_L^- g\big|_{x^+=a}^{b} \\
&\quad + q^{-1/2}\lambda_+^{1/2}\lambda\left(\partial^-\right)_L^{-1}\left(\Lambda^{1/2}S^1 f\right)\star\partial_L^0 g\big|_{x^+=a}^{b} \\
&\quad + \lambda^2\left(\partial^-\right)_L^{-1}\left(\Lambda^{1/2}\left(\tau^3\right)^{-1/2}T^-S^1 f\right)\star\partial_L^+ g\big|_{x^+=a}^{b} \\
&\quad - q^{-1/2}\lambda_+^{-1/2}\lambda\left(\partial^-\right)_L^{-1}\left(\Lambda^{1/2}\left(\tau^1T^- - q^{-1}S^1\right)f\right)\star\tilde{\partial}_L^3 g\Big|_{x^+=a}^{b}, \\
\left(\partial^0\right)_L^{-1}\left(\partial_L^0 f\right)\star g\Big|_{\tilde{x}^3=a}^{b} &= f\star g|_{\tilde{x}^3=a}^{b} - \left(\partial^0\right)_L^{-1}\left(\Lambda^{1/2}\sigma^2 f\right)\star\partial_L^0 g\big|_{\tilde{x}^3=a}^{b} \\
&\quad + q^{1/2}\lambda_+^{-1/2}\lambda\left(\partial^0\right)_L^{-1}\left(\Lambda^{1/2}T^2\left(\tau^3\right)^{1/2} f\right)\star\partial_L^- g\big|_{\tilde{x}^3=a}^{b} \\
&\quad - q^{-1/2}\lambda_+^{-1/2}\lambda\left(\partial^0\right)_L^{-1}\left(\Lambda^{1/2}\left(\tau^3\right)^{-1/2}\left(T^-\sigma^2 + qS^1\right)f\right)\star\partial_L^+ g\big|_{\tilde{x}^3=a}^{b} \\
&\quad + \lambda_+^{-1}\left(\partial^0\right)_L^{-1}\left(\Lambda^{1/2}\left(\lambda_-^2T^-T^2 + q\left(\tau^1-\sigma^2\right)\right)f\right)\star\tilde{\partial}_L^3 g\Big|_{\tilde{x}^3=a}^{b}.
\end{aligned}
\tag{109}
$$

Similarly in the case of the second differential calculus we get

$$
\left(\hat{\tilde{\partial}}^3\right)_L^{-1}\left(\hat{\tilde{\partial}}_L^3 f\right)\star g\Big|_{x^3=a}^{b} = f\star g|_{x^3=a}^{b}
\tag{110}
$$

$$
\begin{aligned}
&- \left(\hat{\tilde{\partial}}^3\right)_L^{-1} \left(\Lambda^{-1/2} \left(\tau^3\right)^{-1/2} \sigma^2 f\right) \star \hat{\tilde{\partial}}^3_L g \Big|_{x^3=a}^{b} \\
&+ q^{3/2}\lambda_+^{1/2}\lambda \left(\hat{\tilde{\partial}}^3\right)_L^{-1} \left(\Lambda^{-1/2} T^2 f\right) \star \hat{\partial}^-_L g \Big|_{x^3=a}^{b},
\end{aligned}
$$

$$
\begin{aligned}
\left(\hat{\partial}^-\right)_L^{-1} \left(\hat{\partial}^-_L f\right) \star g \Big|_{x^+=a}^{b} &= f \star g\big|_{x^+=a}^{b} - \left(\hat{\partial}^-\right)_L^{-1} \left(\Lambda^{-1/2}\tau^1 f\right) \star \hat{\partial}^-_L g \Big|_{x^+=a}^{b} \\
&+ q^{1/2}\lambda_+^{-1/2}\lambda \left(\hat{\partial}^-\right)_L^{-1} \left(\Lambda^{-1/2} \left(\tau^3\right)^{-1/2} S^1 f\right) \star \hat{\tilde{\partial}}^3_L g \Big|_{x^+=a}^{b},
\end{aligned}
$$

$$
\begin{aligned}
\left(\hat{\partial}^+\right)_L^{-1} \left(\hat{\partial}^+_L f\right) \star g \Big|_{x^-=a}^{b} &= f \star g\big|_{x^-=a}^{b} - \left(\hat{\partial}^+\right)_L^{-1} \left(\Lambda^{-1/2}\sigma^2 f\right) \star \hat{\partial}^+_L g \Big|_{x^-=a}^{b} \\
&+ q^{1/2}\lambda_+^{1/2}\lambda \left(\hat{\partial}^+\right)_L^{-1} \left(\Lambda^{-1/2} T^2 \left(\tau^3\right)^{1/2} f\right) \star \hat{\partial}^0_L g \Big|_{x^-=a}^{b} \\
&+ q^{1/2}\lambda_+^{-1/2}\lambda \left(\hat{\partial}^+\right)_L^{-1} \left(\Lambda^{-1/2} \left(\tau^3\right)^{-1/2} \left(T^+\sigma^2 + q\tau^3 T^2\right) f\right) \star \hat{\tilde{\partial}}^3_L g \Big|_{x^-=a}^{b} \\
&- q^2\lambda^2 \left(\hat{\partial}^+\right)_L^{-1} \left(\Lambda^{-1/2} T^2 T^+ f\right) \star \hat{\partial}^-_L g \Big|_{x^-=a}^{b},
\end{aligned}
$$

$$
\begin{aligned}
\left(\hat{\partial}^0\right)_L^{-1} \left(\hat{\partial}^0_L f\right) \star g \Big|_{\tilde{x}^3=a}^{b} &= f \star g\big|_{\tilde{x}^3=a}^{b} - \left(\hat{\partial}^0\right)_L^{-1} \left(\Lambda^{-1/2} \left(\tau^3\right)^{1/2} \tau^1 f\right) \star \hat{\partial}^0_L g \Big|_{\tilde{x}^3=a}^{b} \\
&+ q^{-1/2}\lambda_+^{-1/2}\lambda \left(\hat{\partial}^0\right)_L^{-1} \left(\Lambda^{-1/2} S^1 f\right) \star \hat{\partial}^+_L g \Big|_{\tilde{x}^3=a}^{b} \\
&+ q^{1/2}\lambda_+^{-1/2}\lambda \left(\hat{\partial}^0\right)_L^{-1} \left(\Lambda^{-1/2} \left(qT^+\tau^1 - T^2\right) f\right) \star \hat{\partial}^-_L g \Big|_{\tilde{x}^3=a}^{b} \\
&- \lambda_+^{-1} \left(\hat{\partial}^0_L\right)^{-1} \left(\Lambda^{-1/2} \left(\tau^3\right)^{-1/2} \left(\lambda^2 T^+ S^1 + q^{-1}\left(\tau^3\tau^1 - \sigma^2\right)\right) f\right) \star \hat{\tilde{\partial}}^3_L g \Big|_{\tilde{x}^3=a}^{b}.
\end{aligned}
$$

In the case of right integrals the rules for integration by parts take the form

$$
\begin{aligned}
\left(\tilde{\partial}^3\right)_R^{-1} f \star \left(\tilde{\partial}^3_R g\right) \Big|_{x^3=a}^{b} &= f \star g\big|_{x^3=a}^{b} - \left(\tilde{\partial}^3\right)_R^{-1} \left(\tilde{\partial}^3_R f\right) \star \left(\Lambda^{-1/2}\sigma^2 g\right) \Big|_{x^3=a}^{b} \\
&- q^{1/2}\lambda_+^{1/2}\lambda \left(\tilde{\partial}^3\right)_R^{-1} \left(\partial^+_R f\right) \star \left(\Lambda^{-1/2} S^1 g\right) \Big|_{x^3=a}^{b},
\end{aligned} \tag{111}
$$

$$
\begin{aligned}
\left(\partial^+\right)_R^{-1} f \star \left(\partial^+_R g\right)\big|_{x^-=a}^{b} &= f \star g\big|_{x^-=a}^{b} - \left(\partial^+\right)_R^{-1} \left(\partial^+_R f\right) \star \left(\Lambda^{-1/2} \left(\tau^3\right)^{1/2} \tau^1 g\right) \Big|_{x^-=a}^{b} \\
&- q^{3/2}\lambda_+^{-1/2}\lambda \left(\partial^+\right)_R^{-1} \left(\tilde{\partial}^3_R f\right) \star \left(\Lambda^{-1/2} \left(\tau^3\right)^{1/2} T^2 g\right) \Big|_{x^-=a}^{b},
\end{aligned}
$$

$$
\begin{aligned}
\left(\partial^-\right)_R^{-1} f \star \left(\partial^-_R g\right)\big|_{x^+=a}^{b} &= f \star g\big|_{x^+=a}^{b} - \left(\partial^-\right)_R^{-1} \left(\partial^-_R f\right) \star \left(\Lambda^{-1/2} \left(\tau^3\right)^{-1/2} \sigma^2 g\right) \Big|_{x^+=a}^{b} \\
&- q^{3/2}\lambda_+^{1/2}\lambda \left(\partial^-\right)_R^{-1} \left(\partial^0_R f\right) \star \left(\Lambda^{-1/2} \left(\tau^3\right)^{-1/2} S^1 g\right) \Big|_{x^+=a}^{b} \\
&- q^2\lambda^2 \left(\partial^-\right)_R^{-1} \left(\partial^+_R f\right) \star \left(\Lambda^{-1/2} \left(\tau^3\right)^{-1/2} T^- S^1 g\right) \Big|_{x^+=a}^{b} \\
&- q^{1/2}\lambda_+^{-1/2}\lambda \left(\partial^-\right)_R^{-1} \left(\tilde{\partial}^3_R f\right) \star \left(\Lambda^{-1/2} \left(\tau^3\right)^{-1/2} \left(S^1 - qT^-\sigma^2\right) g\right) \Big|_{x^+=a}^{b},
\end{aligned}
$$

$$
\left(\partial^0\right)_R^{-1} f \star \left(\partial^0_R g\right)\big|_{\tilde{x}^3=a}^{b} = f \star g\big|_{\tilde{x}^3=a}^{b} - \left(\partial^0\right)_R^{-1} \left(\partial^0_R f\right) \star \left(\Lambda^{-1/2}\tau^1 g\right) \Big|_{\tilde{x}^3=a}^{b}
$$

$$
\begin{aligned}
&- q^{1/2}\lambda_+^{-1/2}\lambda \left(\partial^0\right)_R^{-1} \left(\partial_R^- f\right) \star \left(\Lambda^{-1/2} T^2 g\right)\Big|_{\tilde{x}^3=a}^{b} \\
&- q^{1/2}\lambda_+^{-1/2}\lambda \left(\partial^0\right)_R^{-1} \left(\partial_R^+ f\right) \star \left(\Lambda^{-1/2}\left(qS^1 - \tau^1 T^-\right) g\right)\Big|_{\tilde{x}^3=a}^{b} \\
&+ \lambda_+^{-1} \left(\partial^0\right)_R^{-1} \left(\tilde{\partial}_R^3 f\right) \star \Lambda^{-1/2}\left(\lambda^2 T^2 T^- + q\left(\sigma^2 - \tau^1\right)\right) g\Big|_{\tilde{x}^3=a}^{b}
\end{aligned}
$$

and analogously for the second differential calculus we have

$$
\begin{aligned}
\left(\hat{\tilde{\partial}}^3\right)_R^{-1} f \star \left(\hat{\tilde{\partial}}_R^3 g\right)\Big|_{x^3=a}^{b} &= f \star g|_{x^3=a}^{b} \\
&- \left(\hat{\tilde{\partial}}^3\right)_R^{-1} \left(\hat{\tilde{\partial}}_R^3 g\right) \star \left(\Lambda^{1/2}\left(\tau^3\right)^{1/2} \tau^1 g\right)\Big|_{x^3=a}^{b} \\
&- q^{3/2}\lambda_+^{1/2}\lambda \left(\hat{\tilde{\partial}}^3\right)_R^{-1} \left(\hat{\partial}_R^- f\right) \star \left(\Lambda^{1/2}\left(\tau^3\right)^{1/2} T^2 g\right)\Big|_{x^3=a}^{b}, \\
\left(\hat{\partial}^-\right)_R^{-1} f \star \left(\hat{\partial}_R^- g\right)\Big|_{x^+=a}^{b} &= f \star g|_{x^+=a}^{b} - \left(\hat{\partial}^-\right)_R^{-1} \left(\hat{\partial}_R^- f\right) \star \left(\Lambda^{1/2} \sigma^2 g\right)\Big|_{x^+=a}^{b} \\
&- q^{1/2}\lambda_+^{-1/2}\lambda \left(\hat{\partial}^-\right)_R^{-1} \left(\hat{\tilde{\partial}}_R^3 f\right) \star \left(\Lambda^{1/2} S^1 g\right)\Big|_{x^+=a}^{b}, \\
\left(\hat{\partial}^+\right)_R^{-1} f \star \left(\hat{\partial}_R^+ g\right)\Big|_{x^-=a}^{b} &= f \star g|_{x^-=a}^{b} - \left(\hat{\partial}^+\right)_R^{-1} \left(\hat{\partial}_R^+ f\right) \star \left(\Lambda^{1/2} \tau^1 g\right)\Big|_{x^-=a}^{b} \\
&- q^{1/2}\lambda_+^{1/2}\lambda \left(\hat{\partial}^+\right)_R^{-1} \left(\hat{\tilde{\partial}}_R^3 f\right) \star \left(\Lambda^{1/2}\left(\tau^1 T^+ + q^{-1}\tau^3 T^2\right) g\right)\Big|_{x^-=a}^{b} \\
&- \lambda^2 \left(\hat{\partial}^+\right)_R^{-1} \left(\hat{\partial}_R^- f\right) \star \left(\Lambda^{1/2} T^2 T^+ g\right)\Big|_{x^-=a}^{b}, \\
\left(\hat{\partial}^0\right)_R^{-1} f \star \left(\hat{\partial}_R^0 g\right)\Big|_{\tilde{x}^3=a}^{b} &= f \star g|_{\tilde{x}^3=a}^{b} - \left(\hat{\partial}^0\right)_R^{-1} \left(\hat{\partial}_R^0 f\right) \star \left(\Lambda^{1/2}\left(\tau^3\right)^{1/2} \sigma^2 g\right)\Big|_{\tilde{x}^3=a}^{b} \\
&- q^{3/2}\lambda_+^{-1/2}\lambda \left(\hat{\partial}^0\right)_R^{-1} \left(\hat{\partial}_R^+ f\right) \star \left(\Lambda^{1/2}\left(\tau^3\right)^{-1/2} S^1 g\right)\Big|_{\tilde{x}^3=a}^{b} \\
&+ q^{-1/2}\lambda_+^{-1/2}\lambda \left(\hat{\partial}^0\right)_R^{-1} \left(\hat{\partial}_R^- f\right) \star \left(\Lambda^{1/2}\left(\tau^3\right)^{-1/2}\left(q\tau^3 T^2 - \sigma^2 T^+\right) g\right)\Big|_{\tilde{x}^3=a}^{b} \\
&- q^{-1}\lambda_+^{-1}\lambda \left(\hat{\partial}^0\right)_R^{-1} \left(\hat{\tilde{\partial}}_R^3 f\right) \\
&\star \left(\Lambda^{1/2}\left(\tau^3\right)^{1/2}\left(q^{-1}\lambda^2 S^1 T^+ + \left(\tau^3\right)^{-1}\sigma^2 - \tau^2\right) g\right)\Big|_{\tilde{x}^3=a}^{b}.
\end{aligned}
\tag{112}
$$

As $\left(\partial^0\right)^{-1}$ commutes with the elements $\left(\partial^\mu\right)^{-1}$, $\mu = \pm, \tilde{3}$ one can at once realize that the different possibilities for a volume integration are related to each other by the following identities, if surface terms are neglected :

$$
\begin{aligned}
\left(\partial^+\right)^{-1}\left(\tilde{\partial}^3\right)^{-1}\left(\partial^-\right)^{-1}\left(\partial^0\right)^{-1} f &= q^{-4}\left(\partial^-\right)^{-1}\left(\tilde{\partial}^3\right)^{-1}\left(\partial^+\right)^{-1}\left(\partial^0\right)^{-1} f \\
= q^{-2}\left(\tilde{\partial}^3\right)^{-1}\left(\partial^-\right)^{-1}\left(\partial^+\right)^{-1}\left(\partial^0\right)^{-1} f &= q^{-2}\left(\tilde{\partial}^3\right)^{-1}\left(\partial^+\right)^{-1}\left(\partial^-\right)^{-1}\left(\partial^0\right)^{-1} f
\end{aligned}
\tag{113}
$$

$$= q^{-2}\left(\partial^{-}\right)^{-1}\left(\partial^{+}\right)^{-1}\left(\tilde{\partial}^{3}\right)^{-1}\left(\partial^{0}\right)^{-1} f \;=\; q^{-2}\left(\partial^{+}\right)^{-1}\left(\partial^{-}\right)^{-1}\left(\tilde{\partial}^{3}\right)^{-1}\left(\partial^{0}\right)^{-1} f.$$

These relations follow from the same reasonings we have already applied to the Euclidean cases and carry over directly to volume integrals with the operators $\left(\hat{\partial}^{\mu}\right)^{-1}$ or $(P^{\mu})^{-1}$, $\mu = \pm, \tilde{3}, 0.$ For calculating volume integrals over the whole Minkowski-space one can use the expressions

$$\begin{aligned}
&\left(\partial^{-}\right)^{-1}\left(\partial^{0}\right)^{-1}\left(\partial^{+}\right)^{-1}\left(\tilde{\partial}^{3}\right)^{-1} f = \\
&\quad = q^{2}\left(D^{+}_{q^{-2}}\right)\left(\tilde{D}^{3}_{q^{-2}}\right)^{-1}\left(D^{-}_{q^{-2}}\right) \\
&\qquad \times\left(\left(D^{3}_{q+q^{-2}}\right) f\right)\left(q^{2}x^{+}, q^{2}\tilde{x}^{3}, \left(1+2\frac{\lambda}{\lambda_{+}}\frac{\tilde{x}^{3}}{x^{3}}, q^{2}x^{-}\right)\right), \\
&\left(\hat{\partial}^{0}\right)^{-1}\left(\hat{\partial}^{+}\right)^{-1}\left(\hat{\partial}^{-}\right)^{-1}\left(\hat{\tilde{\partial}}^{3}\right)^{-1} f = \\
&\quad = \left(\tilde{D}^{3}_{q^{2}}\right)\left(D^{-}_{q^{2}}\right)^{-1}\left(D^{+}_{q^{2}}\right)\left(\left(D^{3}_{q-q^{2}}\right) f\right)\left(q^{-2}x^{+}, q^{-2}x^{3}\right), \\
&\left(-\frac{i}{[2]_{q^{2}}}\right)^{4}\left(P^{-}\right)^{-1}\left(P^{0}\right)^{-1}\left(P^{+}\right)^{-1}\left(\tilde{P}^{3}\right)^{-1} f = \\
&\quad = \left(J^{-}\right)^{-1}\left(D^{+}_{q^{2}}\right)\left(J^{0}\right)^{-1}\left(\tilde{D}^{3}_{q^{2}}\right)^{-1} \\
&\qquad \times\left(J^{+}\right)^{-1}\left(D^{-}_{q^{2}}\right)\left(\tilde{J}^{3}\right)^{-1}\left(D^{3}_{q^{2}}\right)^{-1} f
\end{aligned} \tag{114}$$

where again surface terms have been neglected. Translation and Lorentz invariance of these volume integrals can be shown by the same method already applied to the Euclidean cases.

5 Remarks

In the past several attempts have been made to introduce the notion of integration on quantum spaces [19],[20],[21],[22],[23]. However, our way of introducing integration is much more in the spirit of [24], where integration has also been considered as an inverse of q-differentiation. While in [24] the algebraic point of view dominates, our integration formulae are directly derived by inversion of q-difference operators. Thus, this sort of integrals represent nothing else than a generalization of Jackson's celebrated q-integral to higher dimensions.

The expressions for our integrals are affected by the non-commutative structure of the underlying quantum space in two ways. First of all, integration can always be reduced to a process of summation. Additionally, there is a correction term for each order of λ vanishing in the undeformed limit $(q = 1)$. This new kind of terms result from the finite boundaries of integration and are responsible for the fact that integral operators referring to different directions do not commute.

Let us end with a few comments on the question of integrability. It can easily be seen that polynomials lead to finite integrals, thus, they are always integrable. Furthermore, it can be shown by numerical simulations or a detailed analysis of our formulae that for functions with large characteristic wave length the correction terms in our integral formulae become very small. If we assume λ to be some sort of a fundamental length in space-time, from a physical point of view, only such functions should be considered whose characteristic wave length is not essentially smaller than λ. For these functions the assumption seems to be reasonable that the corresponding correction terms constitute a convergent series.

A Notation

1. The *q-number* is defined by [13]

$$[[c]]_{q^a} \equiv \frac{1-q^{ac}}{1-q^a}, \qquad a, c \in \mathbb{C}. \tag{115}$$

For $m \in \mathbb{N}$, we can introduce the *q-factorial* by setting

$$[[m]]_{q^a}! \equiv [[1]]_{q^a}\,[[2]]_{q^a} \ldots [[m]]_{q^a}\,, \qquad [[0]]_{q^a}! \equiv 1. \tag{116}$$

There is also a q-analogue of the usual binomial coefficients, the so-called *q-binomial coefficients* defined by the formula

$$\begin{bmatrix} \alpha \\ m \end{bmatrix}_{q^a} \equiv \frac{[[\alpha]]_{q^a}\,[[\alpha-1]]_{q^a} \ldots [[\alpha-m+1]]_{q^a}}{[[m]]_{q^a}!} \tag{117}$$

where $\alpha \in \mathbb{C}$, $m \in \mathbb{N}$.

2. Note that in functions only such variables are explicitly displayed which are effected by a scaling. For example,

$$f\left(q^2x^+\right) \qquad \text{instead of} \qquad f\left(q^2x^+, x^3, x^-\right). \tag{118}$$

3. The *Jackson derivative* referring to the coordinate x^A is defined by

$$D^A_{q^a} f := \frac{f\left(x^A\right) - f\left(q^a x^A\right)}{\left(1-q^a\right) x^A} \tag{119}$$

where f may depend on all other coordinates as well. Higher Jackson derivatives are obtained by applying the above operator $D^A_{q^a}$ several times:

$$\left(D^A_{q^a}\right)^i f := \underbrace{D^A_{q^a} D^A_{q^a} \ldots D^A_{q^a}}_{i \text{ times}} f. \tag{120}$$

4. For $a > 0$, $q > 1$ and $x^A > 0$, the definition of the *Jackson integral* is

$$\begin{aligned}
\left(D_{q^a}^A\right)^{-1} f\Big|_0^{x^A} &= -(1-q^a)\sum_{k=1}^{\infty}\left(q^{-ak}x^A\right) f\left(q^{-ak}x^A\right), \\
\left(D_{q^a}^A\right)^{-1} f\Big|_{x^A}^{\infty} &= -(1-q^a)\sum_{k=0}^{\infty}\left(q^{ak}x^A\right) f\left(q^{ak}x^A\right), \\
\left(D_{q^{-a}}^A\right)^{-1} f\Big|_0^{x^A} &= (1-q^{-a})\sum_{k=0}^{\infty}\left(q^{-ak}x^A\right) f\left(q^{-ak}x^A\right), \\
\left(D_{q^{-a}}^A\right)^{-1} f\Big|_{x^A}^{\infty} &= (1-q^{-a})\sum_{k=1}^{\infty}\left(q^{ak}x^A\right) f\left(q^{ak}x^A\right).
\end{aligned} \tag{121}$$

For $a > 0$, $q > 1$ and $x^A < 0$, we set

$$\begin{aligned}
\left(D_{q^a}^A\right)^{-1} f\Big|_{x^A}^{0} &= (1-q^a)\sum_{k=1}^{\infty}\left(q^{-ak}x^A\right) f\left(q^{-ak}x^A\right), \\
\left(D_{q^a}^A\right)^{-1} f\Big|_{-\infty}^{x^A} &= (1-q^a)\sum_{k=0}^{\infty}\left(q^{ak}x^A\right) f\left(q^{ak}x^A\right), \\
\left(D_{q^{-a}}^A\right)^{-1} f\Big|_{x^A}^{0} &= -(1-q^{-a})\sum_{k=0}^{\infty}\left(q^{-ak}x^A\right) f\left(q^{-ak}x^A\right), \\
\left(D_{q^{-a}}^A\right)^{-1} f\Big|_{-\infty}^{x^A} &= -(1-q^{-a})\sum_{k=0}^{\infty}\left(q^{ak}x^A\right) f\left(q^{ak}x^A\right).
\end{aligned} \tag{122}$$

Note that the formulae (121) and (122) also yield expressions for the q-integrals over any other interval [13].

5. Arguments enclosed in brackets refer to the first object on their left. For example, we have

$$D_{q^2}^{+} f\left(q^2 x^+\right) = D_{q^2}^{+}\left(f\left(q^2 x^+\right)\right) \tag{123}$$

or

$$D_{q^2}^{+}\left[D_{q^2}^{+} f + D_{q^2}^{-} f\right]\left(q^2 x^+\right) = D_{q^2}^{+}\left(\left[D_{q^2}^{+} f + D_{q^2}^{-} f\right]\left(q^2 x^+\right)\right). \tag{124}$$

However, the symbol $|_{x' \to x}$ applies to the whole expression on its left side reaching up to the next opening bracket or $\pm$ sign.

6. Calculations for q-deformed Minkowski space show that it is reasonable to give the following repeatedly appearing polynomials a name of their own:

$$S_{i,j}\left(x^0, \tilde{x}^3\right) \equiv \begin{cases} \sum\limits_{p_1=0}^{j}\sum\limits_{p_2=0}^{p_1}\cdots\sum\limits_{p_{i-j}=0}^{p_{i-j-1}}\prod\limits_{l=0}^{i-j} a_-\left(x^0, q^{2p_l}\tilde{x}^3\right), & \text{if } 0 \leq j < i, \\ 1, & \text{if } j = i, \end{cases}$$

$$
\begin{aligned}
a_{\pm}\left(x^0,\tilde{x}^3\right) &\equiv q^{\pm 1}\tilde{x}^3\left(q^{\pm 1}\tilde{x}^3+\lambda_+ x^0\right), \qquad (125)\\
\left(M^{\pm}\right)^{k}_{i,j}\left(\underline{x}\right) &\equiv \left(M^{\pm}\right)^{k}_{i,j}\left(x^0,x^+,\tilde{x}^3,x^-\right)\\
&= \binom{k}{i}\lambda_+^j\left(a_{\pm}\left(q^{2j}\tilde{x}^3\right)\right)^i\left(x^+x^-\right)^j S_{k-j,j}\left(x^0,\tilde{x}^3\right),\\
\left(M_{\alpha}^{-+}\right)^{k,l}_{i,j,u}\left(\underline{x}\right) &\equiv \left(M_{\alpha}^{-+}\right)^{k,l}_{i,j,u}\left(x^0,x^+,\tilde{x}^3,x^-\right)\\
&= \binom{k}{i}\binom{l}{j}\lambda_+^u\left(a_-\left(\alpha q^{2u+1}\tilde{x}^3\right)^{k-i}\right)\left(a_+\left(\alpha q^{2u+1}\tilde{x}^3\right)\right)^{l-j}\\
&\quad\times\left(x^+x^-\right)^u S_{i+j,u}\left(x^0,\tilde{x}^3\right),\\
\left(M_{\alpha}^{+-}\right)^{k,l}_{i,j,u}\left(\underline{x}\right) &\equiv \left(M_{\alpha}^{-+}\right)^{k,l}_{i,j,u}\left(\underline{x}\right).
\end{aligned}
$$

7. In [10] we have introduced the quantities $\left(K_{a_1,\ldots,a_l}\right)_{\alpha}^{(k_1,\ldots,k_l)}$, $k_i\in\mathbb{N}$, $a_i,\alpha\in\mathbb{R}$. We also refer to [10] for a review of their explicit calculation. These quantities can be used to define new differentiation operators by setting

$$
\left(D_{a_1,\ldots,a_l}\right)^{(k_1,\ldots,k_l)}x^n=\begin{cases}\left(K_{a_1,\ldots,a_l}\right)_n^{(k_1,\ldots,k_l)}x^{n-k_1-\ldots-k_l},\\ 0,\qquad \text{if } n<k_1+\ldots+k_l,\end{cases} \qquad (126)
$$

Especially, in the case of q-deformed Minkowski space we need the following operators:

$$
\begin{aligned}
\left(D^3_{1,q}\right)^{k,l} &\equiv \left(D_{1,q^2}\right)^{(k,l)}, \qquad (127)\\
\left(D^3_{2,q}\right)^{k,l} &\equiv \left(D_{y_-/x^3,q^2y_-/x^3}\right)^{(k,l)},\\
\left(D^3_{3,q}\right)^{k,l}_{i,j} &\equiv \left(D_{y_+/x^3,q^2y_+/x^3,y_-/x^3,q^2y_-/x^3}\right)^{(k,l,i,j)},\\
\left(D^3_{4,q}\right)^{k,l}_{i,j} &\equiv \left(D_{y_+/y_-,q^2y_+/y_-,1,q^2}\right)^{(k,l,i,j)}.
\end{aligned}
$$

Notice that these operators should act upon the coordinate x^3 only.

8. I

$$
\begin{aligned}
\left(\tilde{O}^3_1\right)_k f &= \tilde{x}^3 D^+_{q^2}\left(D^3_{1,q^{-1}}\right)^{k,k}D^-_{q^{-2}}f, \qquad (128)\\
\left(\tilde{O}^3_2\right)_k f &= \left(D^3_{2,q^{-1}}\right)^{k,k+1}f-q\lambda_+^{-1}\lambda^2x^+\tilde{x}^3D^+_{q^2}\left(D^3_{2,q^{-1}}\right)^{k+1,k+1}f,\\
\left(\tilde{O}^3_3\right)_k f &= x^-\left(D^3_{1,q^{-1}}\right)^{k,k+1}D^-_{q^{-2}}f,\\
[0.16in]\left(O^+_1\right)_k f &= D^-_{q^{-2}}\left(D^3_{1,q^{-1}}\right)^{k,k}f, \qquad (129)\\
\left(O^+_2\right)_k f &= x^+\left(D^3_{2,q^{-1}}\right)^{k+1,k+1}f,\\
[0.16in]\left(O^0_1\right)_k f &= \tilde{D}^3_{q^{-2}}\left(D^3_{1,q^{-1}}\right)^{k,k}f\left(q^{-2}x^+,q^{-2}x^-\right) \qquad (130)
\end{aligned}
$$

$$
\begin{aligned}
& - \lambda \left(x^0 + q^{-1}\lambda_+^{-1}\tilde{x}^3\right) D_{q^{-2}}^{+} \left(D_{1,q^{-1}}^3\right)^{k,k} D_{q^{-2}}^{-} f\left(q^{-2}\tilde{x}^3\right), \\
\left(O_2^0\right)_k f \;\; = \;\; & x^+ D_{q^{-2}}^{+} \left(D_{2,q^{-1}}^3\right)^{k,k+1} f \\
& + q^{-1}\lambda x^+ \left(x^0 + q\lambda_+^{-1}\tilde{x}^3\right) D_{q^{-2}}^{+} \left(D_{2,q^{-1}}^3\right)^{k+1,k+1} f, \\
\left(O_3^0\right)_k f \;\; = \;\; & x^+ x^- D_{q^{-2}}^{+} \left(D_{1,q^{-1}}^3\right)^{k,k+1} D_{q^{-2}}^{-} f,
\end{aligned}
$$

$$
\begin{aligned}
[0.16in] \left(Q_1^0\right)_{k,l} f \;\; = \;\; & \left(x^0 - q\lambda\tilde{x}^3\right) \left(D_{3,q^{-1}}^3\right)_{l,l+1}^{k+1,k} f\left(q^{-2}x^+\right), \\
\left(Q_2^0\right)_{k,l} f \;\; = \;\; & x^- \left(D_{4,q^{-1}}^3\right)_{l,l}^{k+1,k} D_{q^{-2}} f\left(q^{-2}x^+\right) \\
& + q^{-3}\lambda \left(x^0 - q\lambda\tilde{x}^3\right) x^- \left(D_{4,q^{-1}}^3\right)_{l,l+1}^{k+1,k} D_{q^{-2}}^{-} f\left(q^{-2}x^+\right), \\
\left(Q_3^0\right)_{k,l} f \;\; = \;\; & \left(D_{3,q^{-1}}^3\right)_{l,l+1}^{k+1,k+1} f\left(q^{-2}x^+\right), \\
\left(Q_4^0\right)_{k,l} f \;\; = \;\; & x^- \left(D_{4,q^{-1}}^3\right)_{l,l+1}^{k+1,k+1} D_{q^{-2}}^{-} f\left(q^{-2}x^+\right), \\
\left(Q_5^0\right)_{k,l} f \;\; = \;\; & \left(D_{3,q^{-1}}^3\right)_{l+1,l+1}^{k+1,k} f\left(q^{-2}x^+\right),
\end{aligned}
\tag{131}
$$

$$
\begin{aligned}
[0.16in] \left(O_1^-\right)_k f \;\; = \;\; & x^3 D_{q^2}^{+} \tilde{D}_{q^{-2}}^3 \left(D_{1,q^{-1}}^3\right)^{k,k} f\left(q^{-2}x^+, q^{-2}x^-\right) \\
& - q^2\lambda\tilde{x}^3 \left(x^0 + q^{-1}\lambda_+^{-1}\tilde{x}^3\right) \\
& \times \left(D_{q^2}^{+}\right)^2 \left(D_{1,q^{-1}}^3\right)^{k,k} D_{q^{-2}}^{-} f\left(q^{-2}x^+, q^{-2}\tilde{x}^3\right) \\
& - q^{-1}\frac{\lambda}{\lambda_+} \left(\tilde{x}^3\right)^2 \left(D_{q^2}^{+}\right)^2 \left(D_{1,q^{-1}}^3\right)^{k,k} D_{q^{-2}}^{-} f\left(q^{-2}\tilde{x}^3\right), \\
\left(O_2^-\right)_k f \;\; = \;\; & x^3 D_{q^2}^{+} \left(D_{2,q^{-1}}^3\right)^{k,k+1} f\left(q^{-2}x^+\right) \\
& + q^{-1}\lambda_+^{-1}\tilde{x}^3 D_{q^2}^{+} \left(D_{2,q^{-1}}^3\right)^{k,k+1} f \\
& - q^3\lambda_+^{-1}\lambda^2 x^+ \tilde{x}^3 \left(x^0 + q\lambda_+^{-1}\tilde{x}^3\right) \\
& \times \left(D_{q^2}^{+}\right)^2 \left(D_{2,q^{-1}}^3\right)^{k+1,k+1} f\left(q^{-2}x^+\right) \\
& - q^2\frac{\lambda^2}{\lambda_+^2} x^+ \left(\tilde{x}^3\right)^2 \left(D_{q^2}^{+}\right)^2 \left(D_{2,q^{-1}}^3\right)^{k+1,k+1} f, \\
\left(O_3^-\right)_k f \;\; = \;\; & q^{-1} x^- \tilde{D}_{q^2}^3 \left(D_{1,q^{-1}}^3\right)^{k,k+1} f\left(q^{-2}x^+, q^{-2}x^-\right) \\
& - q^{-1}\lambda \left(x^0 + \tilde{x}^3\right) x^- D_{q^2}^{+} \left(D_{1,q^{-1}}^3\right)^{k,k+1} D_{q^{-2}}^{-} f\left(q^{-2}x^+\right) \\
& - q\lambda x^- D_{q^2}^{+} \left(D_{1,q^{-1}}^3\right)^{k,k} D_{q^{-2}}^{-} f\left(q^{-2}x^+\right) \\
& - q^{-2}\frac{\lambda}{\lambda_+}\tilde{x}^3 x^- D_{q^2}^{+} \left(D_{1,q^{-1}}^3\right)^{k,k+1} D_{q^{-2}}^{-} f,
\end{aligned}
\tag{132}
$$

$$
\begin{aligned}
\left(O_4^-\right)_k f &= D_{q^2}^+\left(D_{2,q^{-1}}^3\right)^{k+1,k+1} f\left(q^{-2}x^+\right), \\
[0.16in]\left(Q_1^-\right)_{k,l} f &= \left(q^{-1}+\lambda_+\right) x^-\left(D_{3,q^{-1}}^3\right)_{l,l+1}^{k,k+1} f\left(q^{-2}x^+\right) \qquad (133)\\
&+ q^{-2}\lambda\left(x^0-q\lambda\tilde{x}^3\right) x^-\left(D_{3,q^{-1}}^3\right)_{l,l+1}^{k+1,k+1} f\left(q^{-2}x^+\right), \\
\left(Q_2^-\right)_{k,l} f &= \left(x^-\right)^2\left(D_{4,q^{-1}}^3\right)_{l,l}^{k+1,k+1} D_{q^{-2}}^- f\left(q^{-2}x^+\right) \\
&+ q^{-1}\left(q^{-1}+\lambda_+\right)\left(x^-\right)^2\left(D_{4,q^{-1}}^3\right)_{l,l+1}^{k,k+1} D_{q^{-2}}^- f\left(q^{-2}x^+\right) \\
&+ q^{-3}\lambda\left(x^0-q\lambda\tilde{x}^3\right)\left(x^-\right)^2\left(D_{4,q^{-1}}^3\right)_{l,l+1}^{k+1,k+1} f\left(q^{-2}x^+\right), \\
\left(Q_3^-\right)_{k,l} f &= \left(x^0-q\lambda\tilde{x}^3\right)\tilde{x}^3 D_{q^2}^+\left(D_{3,q^{-1}}^3\right)_{l,l+1}^{k+1,k} f\left(q^{-2}x^+\right), \\
\left(Q_4^-\right)_{k,l} f &= \tilde{x}^3 x^- D_{q^2}^+\left(D_{4,q^{-1}}^3\right)_{l,l}^{k+1,k} f\left(q^{-2}x^+\right) \\
&+ q^{-3}\lambda\left(x^0-q\lambda\,\tilde{x}^3\right)\tilde{x}^3 x^- D_{q^2}^+\left(D_{4,q^{-1}}^3\right)_{l,l+1}^{k+1,k} D_{q^{-2}}^- f\left(q^{-2}x^+\right), \\
\left(Q_5^-\right)_{k,l} f &= \tilde{x}^3 D_{q^2}^+\left(D_{3,q^{-1}}^3\right)_{l,l+1}^{k+1,k+1} f\left(q^{-2}x^+\right), \\
\left(Q_6^-\right)_{k,l} f &= \tilde{x}^3 x^- D_{q^2}^+\left(D_{4,q^{-1}}^3\right)_{l,l+1}^{k+1,k+1} D_{q^{-2}}^- f\left(q^{-2}x^+\right), \\
\left(Q_7^-\right)_{k,l} f &= x^-\left(D_{3,q^{-1}}^3\right)_{l+1,l+1}^{k+1,k+1} f\left(q^{-2}x^+\right), \\
\left(Q_8^-\right)_{k,l} f &= \tilde{x}^3 D_{q^2}^+\left(D_{3,q^{-1}}^3\right)_{l+1,l}^{k+1,k} f\left(q^{-2}x^+\right).
\end{aligned}
$$

Ad [illegible] need the operators

$$
\begin{aligned}
\left(\hat{\tilde{O}}^3\right)_k f &= \left(D_{2,q}^3\right)^{k,k+1} f\left(q^2x^+\right), \qquad (134)\\
[0.16in]\left(\hat{O}_1^-\right)_k f &= x^-\left(D_{2,q}^3\right)^{k+1,k+1} f\left(q^2x^+\right), \qquad (135)\\
\left(\hat{O}_2^-\right)_k f &= \tilde{x}^3 D_{q^2}^+\left(D_{2,q}^3\right)^{k,k+1} f\left(q^2x^+\right), \\
[0.16in]\left(\hat{O}_1^0\right)_k f &= \tilde{D}_{q^2}^3\left(D_{1,q}^3\right)^{k,k} f \qquad (136)\\
&- q^3\lambda_+^{-1}\lambda^2 x^+\tilde{x}^3 D_{q^2}^+\tilde{D}_{q^2}^3\left(D_{1,q}^3\right)^{k,k+1} f, \\
\left(\hat{O}_2^0\right)_k f &= x^-\left(D_{1,q^{-1}}^3\right)^{k,k+1} D_{q^2}^- f\left(q^2\tilde{x}^3\right), \\
\left(\hat{O}_3^0\right)_k f &= qx^+ D_{q^2}^+\left(D_{2,q}^3\right)^{k,k+1} f, \\
\left(\hat{O}_4^0\right)_k f &= \tilde{x}^3 D_{q^2}^+\left(D_{1,q^{-1}}^3\right)^{k,k} D_{q^2}^- f,
\end{aligned}
$$

$$
\begin{aligned}
[0.16in]\left(\hat{Q}_1^0\right)_{k,l} f &= \left(x^0 + q^{-1}\lambda\tilde{x}^3\right)\left(D_{3,q}^3\right)_{l,l+1}^{k+1,k} f \\
&\quad + q\lambda_+^{-1}\lambda\left(q+\lambda_+\right)x^+\tilde{x}^3 D_{q^2}^+\left(D_{3,q}^3\right)_{l,l+1}^{k,k+1} f \\
&\quad - q^3\lambda_+^{-1}\lambda^2 x^+\tilde{x}^3\left(x^0 + q^{-1}\lambda\tilde{x}^3\right) D_{q^2}^+\left(D_{3,q}^3\right)_{l,l+1}^{k+1,k+1} f, \qquad (137)\\
\left(\hat{Q}_2^0\right)_{k,l} f &= \left(D_{3,q}^3\right)_{l,l+1}^{k+1,k+1} f, \\
\left(\hat{Q}_3^0\right)_{k,l} f &= q^{-1}\left(D_{3,q}^3\right)_{l+1,l+1}^{k+1,k} f - q^2\lambda_+^{-1}\lambda^2 x^+\tilde{x}^3 D_{q^2}^+\left(D_{3,q}^3\right)_{l+1,l+1}^{k+1,k+1} f, \\
[0.16in]\left(\hat{O}_1^+\right)_k f &= \left(D_{1,q^{-1}}^3\right)^{k,k} D_{q^2}^- f, \qquad (138)\\
\left(\hat{O}_2^+\right)_k f &= x^+\tilde{D}_{q^2}^3\left(D_{1,q}^3\right)^{k,k+1} f, \\
[0.16in]\left(\hat{Q}_1^+\right)_{k,l} f &= \left(q+\lambda_+\right)x^+\left(D_{3,q}^3\right)_{l,l+1}^{k,k+1} f \\
&\quad - q^2\lambda x^+\left(x^0 + q^{-1}\lambda\tilde{x}^3\right)\left(D_{3,q}^3\right)_{l,l+1}^{k+1,k+1} f, \qquad (139)\\
\left(\hat{Q}_2^+\right)_{k,l} f &= x^+\left(D_{3,q}^3\right)_{l+1,l+1}^{k+1,k+1} f.
\end{aligned}
$$

Finally, the representations of $(P^\mu)^{-1}$, $\mu = \pm, 0, \tilde{3}$, have been formulated by using the following abbreviations:

$$
\begin{aligned}
\left(\tilde{O}_1^{p_3}\right)_k f &= q^{-2(k+1)}\left(\tilde{O}_2^3\right)_k f, \qquad (140)\\
\left(\tilde{O}_2^{p_3}\right)_k f &= q^{-2(k+1)}\left(\tilde{O}_3^3\right)_k f, \\
\left(\tilde{O}_3^{p_3}\right)_k f &= q^{2(k+1)}\left(\hat{\tilde{O}}^3\right)_k f, \\
\left(\tilde{O}_4^{p_3}\right)_k f &= q^{-2(k+1)}\left(\left(\tilde{O}_1^3\right)_k f\right)\left(q^{-2}\tilde{x}^3\right), \\
[0.16in]\left(O_1^{p+}\right)_k f &= q^{-2(k+1)}\left(\left(O_1^+\right)_k f\right)\left(q^{-2}\tilde{x}^3\right) \\
&\quad + q^{2(k+1)}\left(\left(\hat{O}_1^+\right) f\right)\left(q^2\tilde{x}^3\right), \qquad (141)\\
\left(O_2^{p+}\right)_k f &= q^{-2(k+1)}\left(O_2^+\right)_k f, \\
\left(O_3^{p+}\right)_k f &= q^{2(k+1)}\left(\hat{O}_2^+\right)_k f, \\
[0.16in]\left(Q_1^{p+}\right)_{k,l} f &= q^{2(k+l+1)}\left(\hat{Q}_1^+\right)_{k,l} f, \qquad (142)\\
\left(Q_2^{p+}\right)_{k,l} f &= q^{2(k+l+2)}\left(\hat{Q}_2^+\right)_{k,l} f, \\
[0.16in]\left(O_1^{p_0}\right)_k f &= q^{-2(k+1)}\left(O_1^0\right)_k f - q\lambda_+^{-1}\lambda q^{2(k+1)}\left(\left(\hat{O}_4^0\right) f\right)\left(q^2 x^3\right), \qquad (143)\\
\left(O_2^{p_0}\right)_k f &= q^{2(k+1)}\left(\hat{O}_3^0\right)_k f, \\
\left(O_3^{p_0}\right)_k f &= q^{2(k+1)}\left(\hat{O}_1^0\right)_k f - \lambda_+^{-1}\lambda q^{2(k+2)}\left(\left(\hat{O}_2^0\right) f\right)\left(q^2 x^3\right)
\end{aligned}
$$

$$
\begin{aligned}
& && + q^{-1}\lambda^2\lambda_+^{-1}q^{-2(k+1)}\left(O_3^0\right)_k f, \\
\left(O_4^{p0}\right)_k f &=& q^{-2(k+1)}\left(O_2^0\right)_k f, \\
[0.16in]\left(Q_1^{p0}\right)_{k,l} f &=& q^{-2(k+l+1)}\left(Q_2^0\right)_{k,l} f, && (144) \\
\left(Q_2^{p0}\right)_{k,l} f &=& q^{-2(k+l+2)}\left(Q_1^0\right)_{k,l} f - q^{2(k+l+2)}\left(\hat{Q}_1^0\right)_{l,k} f, \\
\left(Q_3^{p0}\right)_{k,l} f &=& \left(1+q^{-1}\lambda_+^{-1}\right)q^{-2(k+l+2)}\left(Q_3^0\right)_{k,l} f + \lambda_+^{-1}q^{2(k+l+2)}\left(\hat{Q}_3^0\right)_{l,k} f, \\
\left(Q_4^{p0}\right)_{k,l} f &=& q^{-2(k+l+2)}\left(Q_4^0\right)_{k,l} f, \\
\left(Q_5^{p0}\right)_{k,l} f &=& q\lambda_+^{-1}q^{-2(k+l+2)}\left(Q_5^0\right)_{k,l} f - \left(1+q\lambda_+^{-1}\right)q^{2(k+l+2)}\left(\hat{Q}_2^0\right)_{l,k} f, \\
[0.16in]\left(O_1^{p-}\right)_k f &=& q^{2(k+1)}\left(\hat{O}_2^-\right)_k f, && (145) \\
\left(O_2^{p-}\right)_k f &=& q^{-2(k+1)}\left(O_1^-\right)_k f, \\
\left(O_3^{p-}\right)_k f &=& q^{-2(k+1)}\left(O_2^-\right)_k f, \\
\left(O_4^{p-}\right)_k f &=& q^{-2(k+1)}\left(O_3^-\right)_k f, \\
\left(O_5^{p-}\right)_k f &=& \lambda_+^{-1}q^{2(k+1)}\left(\left(\hat{O}_1^-\right)_k f\right)\left(q^2\tilde{x}^3\right), \\
\left(O_6^{p-}\right)_k f &=& q^{-2(k+2)}\left(O_4^-\right)_k f, \\
[0.16in]\left(Q_i^{p-}\right)_{k,l} f &=& q^{-2(k+l+1)}\left(Q_i^-\right)_{k,l} f, \qquad i=1,\ldots,4, && (146) \\
\left(Q_i^{p-}\right)_{k,l} f &=& q^{-2(k+l+2)}\left(Q_i^-\right)_{k,l} f, \qquad i=5,\ldots,8.
\end{aligned}
$$

B Scaling Operators

For the representations of the covariant differential calculus of three dimensional Euclidean space we need the following operators:

$$
\begin{aligned}
Hf &=& qf - q^{-1}[2]_{q^2} f\left(q^{-4}x^-\right) + q^{-2}[2]_q f\left(q^{-2}x^3, q^{-4}x^-\right), && (147) \\
H^{-1}f &=& q^{-1}\sum_{k=0}^{\infty}\sum_{l=0}^{k}(-1)^l q^{-2k-l}\binom{k}{i}\left([2]_{q^2}\right)^{k-l}\left([2]_q\right)^l f\left(q^{-2l}x^3, q^{-4k}x^-\right), \\
[0.16in]H^+f &=& q^2 f\left(q^2x^3\right) + f\left(q^{-2}x^+, q^{-4}x^-\right), && (148) \\
\left(H^+\right)^{-1}f &=& q^{-2}\sum_{k=0}^{\infty}(-1)^k q^{-2k} f\left(q^{-4k-2}x^3, q^{-4k}x^-\right), \\
[0.16in]H^3 f &=& q^2 f\left(q^2x^+\right) + q^{-2}f\left(q^{-2}x^+, q^{-2}x^3, q^{-4}x^-\right), && (149) \\
\left(H^3\right)^{-1}f &=& q^{-2}\sum_{k=0}^{\infty}(-1)^k q^{-4k} f\left(q^{-4k-2}x^+, q^{-2k}x^3, q^{-4k}x^-\right), \\
[0.16in]H^- f &=& q^2\left(f + f\left(q^4x^+\right)\right) - [2]_{q^2} f\left(q^{-4}x^-\right) + q^{-1}[2]_q f\left(q^{-2}x^3, q^{-4}x^-\right), && (150)
\end{aligned}
$$

$$\left(H^{-}\right)^{-1} f = q^{-2}\sum_{i=0}^{\infty}\sum_{j+k+l=i}(-1)^{j+l}\frac{i!}{j!k!l!}q^{-2k-3l}\left([2]_{q^2}\right)^k\left([2]_q\right)^l \times f\left(q^{-4(i+k+l+1)}x^{+}, q^{-2l}x^3, q^{-4(k+l)}x^{-}\right)$$

where $[2]_{q^a} = q^a + q^{-a}$, $a \in \mathbb{C}$. It is worth noting that this fomulae are only valid for $q > 1$.

Furthermore, we introduce the functions

$$W_{[a,b]}(x) \equiv \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(y)\, e^{ixy} dy = \begin{cases} 1, & \text{if } a \leq x \leq b, \\ 0, & \text{otherwise} \end{cases} \tag{151}$$

where

$$F(y) = \begin{cases} \frac{i}{y\sqrt{2\pi}}\left(e^{-iby} - e^{-iay}\right), & \text{if } y \neq 0, \\ \frac{b-a}{\sqrt{2\pi}}, & \text{if } y = 0, \end{cases} . \tag{152}$$

With these functions at hand we can also define

$$\begin{aligned} W_{]0,1]}(x) &\equiv W_{[-1,1]}(x) - W_{[-1,0]}(x), \\ W_{[-1,0[}(x) &\equiv W_{[-1,1]}(x) - W_{[0,1]}(x), \\ W_{\overline{[0,1]}}(x) &\equiv 1 - W_{[0,1]}(x), \\ W_0(x) &\equiv W_{[-1,0]}(x) + W_{[0,1]}(x) - W_{[-1,1]}(x). \end{aligned} \tag{153}$$

Using the operators [6]

$$\hat{\sigma}_i \equiv x^i \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, 4, \tag{154}$$

and

$$\hat{L}^{i_1,i_4}_{i_2,i_3} f \equiv f(q^{i_1}x^1, q^{i_2}x^2, q^{i_3}x^3, q^{i_4}x^4), \quad i_1, i_2, i_3, i_4 = 1, \ldots, 4, \tag{155}$$

the scaling operators needed for q-deformed Euclidean space in four dimensions take the form

$$\begin{aligned} Nf &= f\left(q^{-2}x^4\right) + q\lambda\left(x^2 D^2_{q^2} + x^3 D^3_{q^2}\right) f + q^{-2}\lambda^2 x^4 D^4_{q^{-2}} f, \\ N^{-1} f &= \frac{1}{2}q^2 W_0\left(\hat{S}^0_1\right) f\left(q^2 x^4\right) \\ &\quad + q^2\sum_{k=0}^{\infty}\left(-q^2\right)^k\sum_{l_1+l_2+l_3=k}\frac{k!}{l_1!l_2!l_3!}\left(-q^{-1}\lambda_+\right)^{l_3} W_{]0,1]}\left(\hat{S}^0_1\right)\hat{L}^{0,2(k+1)}_{2l_1,2l_2} f \\ &\quad + \sum_{k=0}^{\infty}\left(-q^{-2}\right)^k\sum_{l=0}^{\infty}\binom{-k-1}{l}\left(-q^{-1}\lambda_+\right)^l W_{\overline{[0,1]}}\left(\hat{S}^0_1\right) \\ &\quad\quad \times \left\{ 2^{-(k+l+1)} W_0\left(\hat{S}^0_2\right)\hat{L}^{0,-2k}_{-(k+l+1),-(k+l+1)} f \right. \end{aligned} \tag{156}$$

$$
\begin{aligned}
& + \sum_{u=0}^{\infty} \binom{-k-l-1}{u} \left[W_{]0,1]} \left(\hat{S}_2^0 \right) \hat{L}^{0,-2k}_{-2(k+l+u+1),2u} f \right. \\
& \left. + W_{]0,1]} \left(\hat{S}_3^0 \right) \hat{L}^{0,-2k}_{2u,-2(k+l+u+1)} f \right] \Big\}, \\
[0.16in] N^1 f &= q^{-3} f\left(qx^2, qx^3\right) + q^3 f\left(q^{-1}x^2, q^{-1}x^3, q^{-2}x^4\right), \qquad (157) \\
\left(N^1\right)^{-1} f &= \frac{1}{2} q^3 W_0 \left(\hat{S}^1\right) f\left(q^{-1}x^2, q^{-1}x^3\right) \\
& + q^3 \sum_{k=0}^{\infty} \left(-q^6\right)^k W_{]0,1]} \left(\hat{S}^1\right) \hat{L}^{0,-2k}_{-(2k+1),-2(k+1)} f \\
& + q^{-3} \sum_{k=0}^{\infty} \left(-q^{-6}\right)^k W_{\overline{[0,1]}} \left(\hat{S}^1\right) \hat{L}^{0,2(k+1)}_{2k+1,2k+1} f, \\
[0.16in] N^2 f &= q^{-3} f\left(qx^1, q^2x^4\right) + q^3 f\left(q^{-1}x^1, q^{-2}x^3\right), \qquad (158) \\
\left(N^2\right)^{-1} f &= \frac{1}{2} q^3 W_0 \left(\hat{S}^2\right) f\left(q^{-1}x^2, q^{-2}x^4\right) \\
& + q^3 \sum_{k=0}^{\infty} \left(-q^6\right)^k W_{]0,1]} \left(\hat{S}^2\right) \hat{L}^{-(2k+1),-(2k+1)}_{0,-2k} f \\
& + q^{-3} \sum_{k=0}^{\infty} \left(-q^{-6}\right)^k W_{\overline{[0,1]}} \left(\hat{S}^2\right) \hat{L}^{2k+1,2k}_{0,2(k+1)} f \\
[0.16in] N^3 f &= q^{-3} f\left(qx^1, q^2x^4\right) + q^3 f\left(q^{-1}x^1, q^{-2}x^2\right), \qquad (159) \\
\left(N^3\right)^{-1} f &= \frac{1}{2} q^3 W_0 \left(\hat{S}^3\right) f\left(q^{-1}x^2, q^{-2}x^4\right) \\
& + q^3 \sum_{k=0}^{\infty} \left(-q^6\right)^k W_{]0,1]} \left(\hat{S}^3\right) \hat{L}^{-2(k+1),-2(k+1)}_{-2k,0} f \\
& + q^{-3} \sum_{k=0}^{\infty} \left(-q^{-6}\right)^k W_{\overline{[0,1]}} \left(\hat{S}^3\right) \hat{L}^{2k+1,2k}_{2(k+1),0} f, \\
[0.16in] N^4 f &= q^{-3} f\left(q^{-2}x^4\right) + q^3 f\left(q^{-2}x^1, q^{-2}x^4\right) \qquad (160) \\
& + q^{-2}\lambda \left(x^2 D^2_{q^2} + x^3 D^3_{q^2} \right) f + q^{-5}\lambda^2 x^4 D^4_{q^{-2}} f, \\
\left(N^4\right)^{-1} f &= \frac{1}{2} q^5 W_0 \left(\hat{S}^4_1\right) f\left(q^2x^4\right) \\
& + q^5 \sum_{k=0}^{\infty} \left(-q^5\right)^k \sum_{l_1+\ldots+l_4=k} \frac{k!}{l_1!\ldots l_4!} \left(-q^{-4}\lambda_+\right)^{l_1} q^{3(l_2-l_3-l_4)} \\
& \quad \times W_{]0,1]} \left(\hat{S}^4_1\right) \hat{L}^{-2l_2,2(k-m_2+1)}_{2l_3,2l_4} f \\
& + \frac{1}{2} q^{-3} \sum_{k=0}^{\infty} \left(-2q^{-8}\right)^k W_{\overline{[0,1]}} \left(\hat{S}^4_1\right) W_0 \left(\hat{S}^4_2\right) \hat{L}^{2(k+1),2}_{0,0} f
\end{aligned}
$$

$$
\begin{aligned}
&+ q^{-3}\sum_{k=0}^{\infty}\left(-q^{-8}\right)^{k}\sum_{l=0}^{\infty}\binom{-k-1}{l}q^{-6l}\sum_{m_1+m_2+m_3=l}\frac{l!}{m_1!m_2!m_3!}\left(-q^{-1}\lambda_+\right)^{m_1}\\
&\quad\times W_{\overline{[0,1]}}\left(\hat{S}_1^4\right)W_{]0,1]}\left(\hat{S}_2^4\right)\hat{L}^{2(k+l+1),2(l+1)}_{2m_2,2m_3}f\\
&+ q^{3}\sum_{k=0}^{\infty}\left(-q^{-2}\right)^{k}\sum_{l=0}^{\infty}\binom{-k-1}{l}q^{6l}\sum_{u=0}^{\infty}\binom{-k-l-1}{u}\left(-q^{-1}\lambda_+\right)^{u}\\
&\quad\times W_{\overline{[0,1]}}\left(\hat{S}_1^4\right)W_{\overline{[0,1]}}\left(\hat{S}_2^4\right)\\
&\quad\times\Big\{2^{-(k+l+u+1)}W_0\left(\hat{S}_3^4\right)\hat{L}^{-2l,-2(k+l)}_{-(k+l+u+1),-(k+l+u+1)}f\\
&\quad+\sum_{v=0}^{\infty}\binom{-k-l-u-1}{v}\Big[W_{]0,1]}\left(\hat{S}_3^4\right)\hat{L}^{-2l,-2(k+l)}_{2v,-2(k+l+u+v+1)}\\
&\quad+W_{\overline{[0,1]}}\left(\hat{S}_3^4\right)\hat{L}^{-2l,-2(k+l)}_{-2(k+l+u+v+1),2v}f\Big]\Big\}
\end{aligned}
$$

where

$$
\begin{aligned}
\hat{S}_1^0 &= q^{-2}-q^{2\hat{\sigma}_4}\left(q^{2\hat{\sigma}_2}+q^{2\hat{\sigma}_3}-q^{-1}\lambda_+\right), && (161)\\
\hat{S}_2^0 &= 1-q^{2(\hat{\sigma}_3-\hat{\sigma}_2)},\\
[0.16in]\hat{S}^1 &= q^{-6}-q^{-2(\hat{\sigma}_2+\hat{\sigma}_3+\hat{\sigma}_4)}, && (162)\\
[0.16in]\hat{S}^2 &= q^{-6}-q^{-2(\hat{\sigma}_1+\hat{\sigma}_3+\hat{\sigma}_4)}, && (163)\\
[0.16in]\hat{S}^3 &= q^{-6}-q^{-2(\hat{\sigma}_1+\hat{\sigma}_2+\hat{\sigma}_4)}, && (164)\\
[0.16in]\hat{S}_1^4 &= q^{-5}-q^{3-2\hat{\sigma}_1}-q^{-3+2\hat{\sigma}_4}\left(q^{2\hat{\sigma}_2}+q^{2\hat{\sigma}_3}-q^{-1}\lambda_+\right), && (165)\\
\hat{S}_2^4 &= q^{3}-q^{-3+2(\hat{\sigma}_1+\hat{\sigma}_4)}\left(q^{2\hat{\sigma}_2}+q^{2\hat{\sigma}_3}-q^{-1}\lambda_+\right),\\
\hat{S}_3^4 &= 1-q^{2(\hat{\sigma}_2-\hat{\sigma}_3)}.
\end{aligned}
$$

Again we have to emphasize that these expressions hold for $q>1$ only.

Finally, in the case of q-deformed Minkowski space we need the following operators to perform our integrals:

$$
\begin{aligned}
J^{-}f &= f\left(q^{-2}x^{+}\right)+q^{2}f, && (166)\\
\left(J^{-}\right)^{-1} &= q^{-2}\sum_{k=0}^{\infty}\left(-q^{-2}\right)^{k}f\left(q^{-2k}x^{+}\right),\\
[0.16in]J^{+}f &= f\left(q^{-2}\tilde{x}^{3},q_-x^{3},q^{-2}x^{-}\right)+q^{2}f\left(q_-q^{2}x^{3}\right), && (167)\\
\left(J^{+}\right)^{-1}f &= q^{-2}\sum_{k=0}^{\infty}\left(-q^{-2}\right)^{k}f\left(q_-^{-1}q^{-2(k+1)}x^{3},q^{-2}\tilde{x}^{3},q^{-2}x^{-}\right),\\
[0.16in]J_k^0 f &= \frac{\left(\tilde{x}^{3}\partial_3\right)^{k}}{k!}\left(q^{2}+(-1)^{k}q^{-2\left(\hat{\sigma}_++\tilde{\hat{\sigma}}_3+\hat{\sigma}_-\right)}\right), && (168)
\end{aligned}
$$

$$
\begin{aligned}
\left(J_0^0\right)^{-1} f &= q^{-2} \sum_{k=0}^{\infty} \left(-q^{-2}\right)^k f\left(q^{-2k} x^+, q^{-2k} \tilde{x}^3, q^{-2k} x^-\right), \\
\left(J^0\right)^{-1} f &= \left(J_0^0\right)^{-1} f - \sum_{k=1}^{\infty} \left(\frac{\lambda}{\lambda_+}\right)^k \left(J_0^0\right)^{-1} J_k^0 \left(J_0^0\right)^{-1} f \\
&\quad - \sum_{k=0}^{\infty} \left(\frac{\lambda}{\lambda_+}\right)^{k+1} \sum_{l=1}^{k} (-1)^l \sum_{0 \leq m_1 + \ldots + m_l \leq k-l} \left(\prod_{i=1}^{l} \left(J_0^0\right)^{-1} J_{m_i+1}^0\right) \\
&\quad \times \left(J_0^0\right)^{-1} J_{k+1-l-m_1-\ldots-m_l}^0 \left(J_0^0\right)^{-1} f, \\
[.16in] \tilde{J}_0^3 f &= f\left(q^{-2} x^3\right) + q^2 f\left(q^2 x^+\right), \qquad (169) \\
\tilde{J}_k^3 f &= \left(\tilde{x}^3\right)^k \left[\left(D_{1,q^2}^3\right)^{(k,1)} f\left(q^{-2} x^3\right) + (-1)^k \left(D_{1,q^{-2}}\right)^{(k,1)} f\left(q^2 x^+, q^2 x^3\right)\right], \\
\left(\tilde{J}_0^3\right)^{-1} f &= q^{-2} \sum_{k=0}^{\infty} \left(-q^{-2}\right)^k f\left(q^{-2(k+1)} x^+, q^{-2k} x^3\right), \\
\left(\tilde{J}^3\right)^{-1} f &= \left(\tilde{J}_0^3\right)^{-1} f - \sum_{k=1}^{\infty} \left(\frac{\lambda}{\lambda_+}\right)^k \left(\tilde{J}_0^3\right)^{-1} \left(D_{q^2}^3\right)^{-1} \tilde{J}_k^3 \left(\tilde{J}_0^3\right)^{-1} f \\
&\quad - \sum_{k=1}^{\infty} \left(\frac{\lambda}{\lambda_+}\right)^{k+1} \sum_{l=1}^{k} (-1)^l \sum_{0 \leq m_1 + \ldots + m_l \leq k-l} \left(\tilde{J}_0^3\right)^{-1} \left(D_{q^2}^3\right)^{-1} \\
&\quad \times \left(\prod_{i=1}^{l} \tilde{J}_{m_i+1}^3 \left(\tilde{J}_0^3\right)^{-1} \left(D_{q^2}^3\right)^{-1}\right) \tilde{J}_{k+1-l-m_1-\ldots-m_l}^3 \left(\tilde{J}_0^3\right)^{-1} f
\end{aligned}
$$

where again $q > 1$. Let us also note that the product $\prod_{i=1}^{n} \hat{a}_i$ of non-commuting operators $\hat{a}_i$ is given by

$$
\prod_{i=1}^{n} \hat{a}^i \equiv \hat{a}^1 \cdot \ldots \cdot \hat{a}^n. \qquad (170)
$$

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