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Functional approach to $2+1$ dimensional gravity coupled to particles [1]

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Abstract

The quantum gravity problem of $\mathcal{N}$ point particles interacting with the gravitational field in $2+1$ dimensions is approached working out the phase-space functional integral. The maximally slicing gauge is adopted for a non compact open universe with the topology of the plane. The conjugate momenta to the gravitational field are related to a class of meromorphic quadratic differentials. The boundary term for the non compact space is worked out in detail. In the extraction of the physical degrees of freedom functional determinants related to the puncture formulation of string theory occur and cancel out in the final reduction. Finally the ordering problem in the definition of the functional integral is discussed.

1 Introduction

In this paper we examine the functional approach to a problem of quantum gravity i.e. the quantum treatment of $\mathcal{N}$ particles interacting with the gravitational field in $2+1$ dimensions. The analogous problem in absence of matter has been dealt with by Carlip [1]. It is well known that in absence of matter gravity in $2+1$ dimensions acquires a non trivial dynamics only on closed universes and here the physical degrees of freedom are encoded in the moduli of the space sections. The hamiltonian treatment of such a problem is found in [2, 3, 4]; for genus 1 i.e. torus topology the classical hamiltonian is explicitly known and its quantum transcription, choosing a proper ordering, gives rise to the Maass laplacian, thoroughly studied by mathematicians [5]. For higher genus no explicit expression is known even for the classical hamiltonian.

In [1] Carlip starts from the general form of the phase space functional integral and through a process of gauge fixing reduces it to a functional phase space integral on the physical degrees of freedom i.e. the moduli and their conjugate momenta. The final result is what one would have naively obtained by writing down the simple minded phase space integral using the reduced action. This is not obtained through particular tricks but simply interpreting the functional δ functions in such a way as to preserve invariance under diffeomorphisms.

In presence of particles the problem acquires a highly non trivial dynamics also on open spaces. For the open space with the topology of the plane the maximally slicing gauge can be adopted, leading to notable simplifications [6, 7, 8]. The two particle case can be solved exactly both at the classical and quantum level while in presence of three or more particles the hamiltonian even if perfectly defined [8, 9] cannot be written in explicit form.

In the following we deal with the functional formulation of quantum $2+1$ dimensional gravity coupled to particles in an open universe with the topology of the plane. There are two noteworthy differences with respect to the problem dealt with in [1]. a) We are in presence of a Riemann surface with punctures at the location of the particles. b) The boundary terms play an essential role in the dynamics of the problem.

With regard to point (a) the main difference is that the transverse traceless part of the space cotangent to the space of the spacial metrics is described by a class of meromorphic quadratic differentials which turn out to be parametrized by the canonical momenta of the particles.

With regard to point (b) we treat the non compact case as suggested in [10] i.e. to consider first the problem with a fixed boundary and then to take the limit when the boundary goes to infinity. In such a limiting process the values at the boundary of the fields have to be chosen as to provide a regular non trivial dynamics.

As always the central problem with the functional integration in presence of gauge symmetries, like quantum gravity, is to extract the relevant degrees of freedom from the gauge degrees of freedom. This can be performed by introducing gauge fixings and evaluating the ensuing Faddeev- Popov determinants, as is usually done, or by applying the so called geometric approach [12, 13, 14] in which the gauge volume is simply factorized. Both procedures will be applied here and as expected they show up to be equivalent. The geometric approach has the aesthetic advantage to extract directly the result without introducing two gauge fixings whose explicit form at the end is completely irrelevant. In the extraction of the relevant degrees of freedom several non trivial functional determinants are produced; this are analogous to the determinants occurring in the puncture formulation of string theory [16]. Despite the complexity of the intermediate steps all such determinants except one cancel out exactly. The last remaining determinant can be reduced to 1 by a simple and natural choice of the canonical variables.

The final expression of the functional integral is the reduced phase space integral in which only the particle positions and momenta occur. Unfortunately, as it happens also in the case examined in [1], such a functional expression tells us little about the ordering problem which is related to the final definition of the functional integral. We shall discuss shortly this problem in the last section.

The paper is organized as follows: In Section 2 after recalling the structure of the classical action with boundary terms included, we write down the phase space functional integral. Integration over the lapse and shift functions provides the hamiltonian and diffeomorphism constraints. In Section 3 we deal with the integration over the space metric and the momenta conjugate to the space metric. Here the geometric approach is most clear: one describes the general metric as the result of the application of a space diffeomorphism to the metric given in the conformal gauge, where the conformal factor is allowed singularities at the punctures which are the particle positions. It is useful to distinguish the diffeomorphisms which do not move the punctures from those which move them.

Then one considers the variations of the metric which are square integrable in the De Witt metric. An important point is that in order to ensure square integrability of such variations we find that the diffeomorphisms which describe the motion of the punctures have to be correlated with the motion of the singularities of the conformal factor. Then one examines the space cotangent to such variations i.e. the space of the conjugate momenta to the metric. It turns out that the transverse traceless part of such a space is described by the meromorphic quadratic differentials having only simple poles on the punctures. In addition square integrability, in the De Witt metric, of such conjugate momenta imply a sum rule on the residues; such a restriction was already found in the classical treatment along a different path by Deser [17] as a necessary restriction on the conjugate momenta if one wants to avoid singularities in the space metric. In [8, 19] it arose from the consistency of the asymptotic behavior of the conformal factor with the hamiltonian constraint. Here it appears as a necessary restriction on the square integrability of the conjugate momenta and their variations.

In Section 4 we deal with the gauge fixings; the main role is played by the maximally slicing gauge fixing; it is in fact such a fixing which allows the solution of the diffeomorphism constraint in terms of the quadratic differentials and reduces the hamiltonian constraint to an equation equivalent to the Liouville equation. At the end of this section we briefly describe the geometric approach to the space diffeomorphisms which as announced is equivalent to the Faddeev-Popov procedure. In Section 5 we give a detailed derivation of the effective hamiltonian for the non compact space as the limit of the boundary term, when the boundary goes to infinity. As already mentioned the procedure is the one suggested in [10] which in our case can be taken to the end explicitly. Finally we give a discussion of the ordering problem in relation to other approaches to the quantization of $2+1$ dimensional gravity.

2 The action and the gauge fixings

The gravitational part of the action in $D+1$ dimensions, in hamiltonian form is given by [10, 11]

$$\begin{aligned} S^{Grav} = & \int_{\mathcal{M}} dt d^D x \left[\pi^{ij}\dot{h}_{ij} - N^i H_i^{Grav} - N H^{Grav}\right] + \\ & + 2\int dt \int_{B_t} d^{(D-1)}x \sqrt{\rho} N \left(K_{B_t} + \frac{\eta}{\cosh\eta}\mathcal{D}_\alpha v^\alpha\right) - 2\int dt \int_{B_t} d^{(D-1)}x \, r_\alpha \pi^{\alpha\beta}_{(\rho)} N_\beta \end{aligned} \tag{1}$$

where h_{ij} is the D-dimensional metric of the constant time slices Σ_t, $\sinh\eta = n_\mu u^\mu$ with n^μ the future-pointing unit normal to the time slices Σ_t and u^μ the outward-pointing unit normal to the space like boundary B, $B_t = \Sigma_t \cap B$, $\sqrt{\rho}$ is the volume form induced by the space metric on B_t, K_{B_t} is the extrinsic curvature of B_t as a surface embedded in Σ_t, $v_\alpha = \frac{1}{\cosh\eta}(n_\alpha - \sinh\eta\, u_\alpha)$ and r_α is the versor normal to B_t in Σ_t. The subscript ρ in $\pi^{\alpha\beta}_{(\rho)}$ is a reminder that it has to be considered as a tensor density with respect to the measure $\sqrt{\rho}$. $\mathcal{D}_\alpha$ is the covariant derivative induced by the metric on the space-like boundary B. For the ADM metric [21] we use the notation

$$ds^2 = -N^2 dt^2 + h_{ij}(dx^i + N^i dt)(dx^j + N^j dt). \tag{2}$$

Moreover we have

$$H_i^{Grav} = -2\sqrt{h} D_j \frac{\pi^j{}_i}{\sqrt{h}} \tag{3}$$

$$H^{Grav} = \frac{1}{\sqrt{h}} h_{ij} h_{kl} (\pi^{ik}\pi^{jl} - \pi^{ij}\pi^{kl}) - \sqrt{h} R, \tag{4}$$

where D is the covariant derivative induced by the metric h_{ij} on the surfaces Σ_t and R is the intrinsic curvature of these surfaces.

The matter part of the action is given by

$$S_m = \int dt \sum_n P_{ni} \dot{q}_n^i + N^i(q_n) P_{ni} - N(q_n) \sqrt{P_{ni} P_{nj} h^{ij}(q_n) + m_n^2}. \tag{5}$$

In the following we shall denote

$$H_i = H_i^{Grav} - \sum_n P_{ni} \delta^2(x - q_n) \tag{6}$$

and

$$H_0 = H = H^{Grav} + \sum_n \sqrt{P_{ni} P_{nj} h^{ij}(q_n) + m_n^2}\ \delta^2(x - q_n) \tag{7}$$

and $S = S^{Grav} + S_m$.

As discussed in detail in [10, 11] S is the classical action when the fields are kept constant on the boundary B_t i.e. the variation of S provides the correct equations of motion when such a variation is performed by keeping constant the $2+1$ dimensional metric $g_{\mu\nu}$ on the boundary B_t. As we shall be interested in an open universe we must take at the end the limit when the boundary B_t goes to infinity. For doing that we shall need to give the asymptotic behavior of the fields at the boundary $|z| = r_0$ for $r_0 \to \infty$. This limit process will be dealt with in Section 5 following the procedure described in [10].

We write the phase space functional integral as

$$Z = \int \prod_{n=1}^{\mathcal{N}} D[P_n] D[\pi^{ij}] D[h_{ij}] D[N^i] D[N] e^{iS}. \tag{8}$$

The metric h_{ij} is defined on the punctured plane $R^2 \setminus \{q_1, \dots q_{\mathcal{N}}\}$, where $q_1, \dots q_{\mathcal{N}}$ are the particle positions. Thus integration on the metric h_{ij} implicitly contains the integration on the particle positions. We shall derive the explicit form of such a dependence in Section 3. The functional integral (8) is ill defined due to the invariance of the action under space-time diffeomorphisms and as well known one has to introduce $D+1=3$ gauge fixings, which we shall denote by $\delta(\chi)\prod_{i=1}^{2}\delta(\chi^i)$. In presence of such gauge fixings the functional integral takes the form, known as Faddeev formula [34]

$$Z=\int\prod_{n=1}^{\mathcal{N}}D[P_n]D[\pi^{ij}]D[h_{ij}]D[N^i]D[N]\delta(\chi)\prod_{i=1}^{2}\delta(\chi^i)|\mathrm{Det}\{\chi^\mu,H_\nu\}|e^{iS}. \tag{9}$$

$|\mathrm{Det}\{\chi^\mu,H_\nu\}|$ is the jacobian which assures the invariance under diffeomorphisms i.e. the Faddeev-Popov determinant. We now integrate over the Lagrange multipliers N^i and N obtaining apart for a multiplicative constant

$$Z=\int\prod_{n=1}^{\mathcal{N}}D[P_n]D[\pi^{ij}]D[h_{ij}]\delta(\chi)\delta(\chi^1)\delta(\chi^2)|\mathrm{Det}\{\chi^\mu,H_\nu\}|\delta(\frac{H_i}{\sqrt{h}})\delta(\frac{H}{\sqrt{h}})e^{iS}. \tag{10}$$

The integration over $D[N]$ gives rise to $\delta(\frac{H}{\sqrt{h}})$ due to the following reason (see also [1]). We recall that N is a scalar on the hypersurface Σ_t (time-slice) and H is a scalar density on the same hypersurface. Then the diffeomorphism invariant functional extension of the formula $\delta(x)=\frac{1}{2\pi}\int e^{ipx}dp$ for a scalar s is

$$\delta(s)=\text{const.}\int D[N]e^{i\int Ns\sqrt{h}d^Dx}$$

and thus

$$\int D[N]e^{i\int HNd^Dx}=\text{const.}\ \delta(\frac{H}{\sqrt{h}})$$

and similarly for the densities H_i.

3 Integration over $D[h_{ij}]$ and $D[\pi^{ij}]$

In order to perform the integration in $D[h_{ij}]$ and $D[\pi^{ij}]$ we have to study the space of the metrics over the punctured plane and its cotangent space [15, 16]. The general parameterization of these metrics is

$$h_{ij}=F^*(e^{2\sigma}\delta_{ij}) \tag{11}$$

being F a 2-dimensional diffeomorphism. In fact for the punctured 2-dimensional plane the only Teichmüller parameters are the positions of the punctures [15] and thus all the metric through a diffeomorphism can be brought to the conformal type. $e^{2\sigma}$ will be a smooth conformal factor on the punctured plane. Finite geodesic distance among the punctures allows singularities

$$[(z-z^c_n)(\bar{z}-\bar{z}^c_n)]^{-\mu_n} \qquad \text{with} \qquad \mu_n<1, \tag{12}$$

where we have employed the complex coordinate $z=x+iy$. We denote with z^c_n the position of the particles in the conformal gauge $z^c_n=F(z_n)$.

As usual in the ADM approach [21] we have to fix the boundary conditions on the fields; we shall assume on a large circle of radius R the space metric diffeomorphic to const $\times$ $(z\bar{z})^{-\mu_0}\delta_{ij}$ with $0<\mu_0<1$. In the variational problem we have to keep the metric at the boundary fixed or better the variation of the metric $g_{\mu\nu}$ has to be such as not to vary the metric induced on the boundary [29]. So we can write $e^{2\sigma}=e^{2\sigma_R}e^{2\sigma_S}$, where σ_R is a regular conformal factor, and σ_S is given by

$$2\sigma_S=\sum_n -\mu_n\rho(z-z^c_n)\ln|z-z^c_n|^2-\mu_0\;\rho(\frac{1}{z})\ln(z\bar{z}), \tag{13}$$

with $\rho(z)$ a smooth function having support inside a circle of radius 1.

For the functional integration in $D[h_{ij}]$ as done in [1, 12, 13] we assume the measure induced by the diffeomorphism invariant distance provided by the De Witt metric

$$(\delta h_{ij},\delta h_{ij})=\int\sqrt{h}\;\delta h_{ij}G^{iji'j'}\delta h_{i'j'}d^2x \tag{14}$$

with

$$2G^{iji'j'}=(h^{ii'}h^{jj'}+h^{ij'}h^{ji'}-\frac{2}{D}h^{ij}h^{i'j'})+C\;h^{ij}h^{i'j'}, \tag{15}$$

that is we set

$$1=\int D[\delta h_{ij}]e^{-(\delta h_{ij},\delta h_{ij})}. \tag{16}$$

In order to have a positive definite metric we need $C>0$; it is however well known that the integration on the Weyl deformations of the metric makes C to disappear from the final result [18, 20].

If we want the measure $D[h_{ij}]$ to be well defined we must admit only variations of the metric with finite norm $(\delta h_{ij}, \delta h_{ij}) < \infty$. This condition imposes the behavior at infinity

$$|\delta h_{ij}| \simeq (z\bar{z})^{-\frac{\mu_0+1}{2}-\epsilon}, \qquad \epsilon > 0. \tag{17}$$

Similarly the behavior at the punctures z_n^c must be

$$|\delta h_{zz}| \simeq (\zeta\bar{\zeta})^{-\frac{\mu_n+1}{2}+\epsilon} \quad \text{with} \quad \zeta = z - z_n^c. \tag{18}$$

We write now the variation of the metric. This is given by

$$\delta h_{ij} = \mathcal{L}_\eta h_{ij} + F^*((\delta e^{2\sigma})\delta_{ij}) \tag{19}$$

with η a vector field and $\mathcal{L}_\eta$ the related Lie derivative. For F equal to the identity we have

$$\delta h_{ij} = \mathcal{L}_\eta(e^{2\sigma}\delta_{ij}) + \delta e^{2\sigma}\delta_{ij}. \tag{20}$$

Due to the invariance of the De Witt metric under diffeomorphisms the above reasoning and bounds extend also to the case $F \neq I$. In fact

$$\mathcal{L}_\eta F^*(e^{2\sigma}\delta_{ij}) = F^*\mathcal{L}_\xi(e^{2\sigma}\delta_{ij}) \tag{21}$$

with $\xi = F_*\eta$. Our purpose will be to change over from the integration on h_{ij}, to the integration on the diffeomorphisms and the conformal factor. Due to the presence of the gauge fixings the integration on the diffeomorphisms explores only the infinitesimal ones i.e. the tangent space described by vector fields ξ. With regard to the term $\mathcal{L}_\xi h_{ij}$ it will be instrumental to decompose the field ξ which generates the infinitesimal diffeomorphism as the sum

$$\xi = \xi^0 + \sum_{k=1}^{\mathcal{N}} \alpha_k \xi^k + \sum_{k=1}^{\mathcal{N}} \bar{\alpha}_k \bar{\xi}^k \tag{22}$$

with

$$\bar{\xi}^{k\bar{z}}(z) = \overline{\xi^{kz}(z)}, \quad \xi^{kz}(z_n^c) = \bar{\xi}^{k\bar{z}}(z_n^c) = \delta_{kn}, \quad \xi^{k\bar{z}}(z) = 0, \quad \bar{\xi}^{kz}(z) = 0 \tag{23}$$

and the field ξ^0 vanishes at the punctures. The variation of the metric due to infinitesimal diffeomorphisms becomes

$$\delta h_{ij} = (P\xi^0)_{ij} + \sum_{k=1}^{\mathcal{N}} \alpha_k (P\xi^k)_{ij} + \sum_{k=1}^{\mathcal{N}} \bar{\alpha}_k (P\bar{\xi}^k)_{ij} + h_{ij} D_l \xi^l \tag{24}$$

with

$$(P\xi)_{ij} = D_i\xi_j + D_j\xi_i - D_l\xi^l h_{ij}. \tag{25}$$

We shall come back to eq.(24) after eq.(45). Eq.(25) in the conformal metric and complex coordinates takes the form [12]

$$(P\xi)_{zz} = 2e^{2\sigma}\frac{\partial}{\partial z}(e^{-2\sigma}\xi_z) = e^{2\sigma}\frac{\partial}{\partial z}\xi^{\bar z}, \tag{26}$$

$$(P\xi)_{\bar z\bar z} = 2e^{2\sigma}\frac{\partial}{\partial \bar z}(e^{-2\sigma}\xi_{\bar z}) = e^{2\sigma}\frac{\partial}{\partial \bar z}\xi^{z}, \tag{27}$$

$$(P\xi)_{\bar z z} = (P\xi)_{z\bar z} = 0. \tag{28}$$

The adjoint P^+ of the operator P acting on the space of ξ^0 equipped with the diffeomorphism invariant metric

$$(\xi,\xi) = \int \sqrt{h}\ \xi^i\xi^j h_{ij}\frac{i}{2}dz\wedge d\bar z. \tag{29}$$

for $\delta h \in \mathcal{D}(P^+)$ is given by

$$(P^+\delta h)_z = -4e^{-2\sigma}\frac{\partial}{\partial \bar z}\delta h_{zz} \tag{30}$$

and

$$(P^+\delta h)_{\bar z} = -4e^{-2\sigma}\frac{\partial}{\partial z}\delta h_{\bar z\bar z}. \tag{31}$$

The contribution to δh_{ij} due to the variation of the conformal factor is

$$(\delta e^{2\sigma})\delta_{ij} = \delta_{ij}e^{2\sigma}\delta(2\sigma_R) + \delta_{ij}e^{2\sigma}\delta(2\sigma_S) \tag{32}$$

where

$$\delta(2\sigma_S) = \sum_n \mu_n\left(\frac{\delta z_n^c}{z-z_n^c} + \frac{\delta\bar z_n^c}{\bar z-\bar z_n^c}\right)\rho(z-z_n^c) + \sum_n \mu_n \ln|z-z_n^c|^2\left(\frac{\partial\rho}{\partial z}\delta z_n^c + \frac{\partial\rho}{\partial\bar z}\delta\bar z_n^c\right). \tag{33}$$

The variation of $\delta\mu_n$ would give rise to $\delta\sigma_S$ which are square integrable and thus such variation can be reabsorbed in $\delta\sigma_R$; in the following the hamiltonian constraint will fix $\mu_n = \frac{m_n}{4\pi}$.

We shall see now that the imposition that δh_{ij} be square integrable in the De Witt metric imposes a relation among the δz_n^c and the ξ^k.

The variation $\delta h_{zz} = (P\xi)_{zz} = e^{2\sigma}\frac{\partial}{\partial z}\xi^{\bar z}$ and $\delta h_{z\bar z} = \delta_{ij}e^{2\sigma}\delta(2\sigma_R)$ are always square integrable for regular $\xi^{\bar z}$ and $\delta(2\sigma_R)$, square integrable in their respective norms. In fact in order to have a finite norm, ξ must behave at infinity as

$$|\xi^z| \simeq (z\bar z)^{\mu_0-\frac{1}{2}-\epsilon} \tag{34}$$

and so we have

$$\delta h_{zz} = e^{2\sigma}\frac{\partial}{\partial z}\xi^{\bar z} \simeq (z\bar z)^{-1-\epsilon} \tag{35}$$

satisfying condition (17) at infinity and

$$(\delta h_{zz}, \delta h_{zz}) = \int \sqrt{h}\ e^{2\sigma}\frac{\partial}{\partial z}\xi^{\bar z} h^{z\bar z}h^{z\bar z}e^{2\sigma}\frac{\partial}{\partial \bar z}\xi^{z}\frac{i}{2}dz\wedge d\bar z = \int\sqrt{h}\ \text{regular}\ \frac{i}{2}dz\wedge d\bar z \tag{36}$$

is finite because of the local finiteness of the area and the behavior (35) at ∞. The analysis for $\delta(2\sigma_R)$ is even easier because

$$(h_{ij}\delta(2\sigma_R), h_{ij}\delta(2\sigma_R)) = \frac{C}{2}\int\sqrt{h}\ h_{ij}\delta(2\sigma_R)h_{lk}\delta(2\sigma_R)h^{ij}h^{lk}\ \frac{i}{2}dz\wedge d\bar z =$$

$$= 2C\int\sqrt{h}\ (\delta(2\sigma_R))^2\ \frac{i}{2}dz\wedge d\bar z = 2C(\delta(2\sigma_R), \delta(2\sigma_R)), \tag{37}$$

and the finiteness of the norm of $\delta(2\sigma_R)$ implies directly the finiteness of the norm of $h_{ij}\delta(2\sigma_R)$.

The problem arises with the contributions $\delta(2\sigma_S) \simeq \mu_n(\frac{\delta z_n^c}{z-z_n^c} + \frac{\delta \bar z_n^c}{\bar z-\bar z_n^c})$ and $D_l\xi^l \simeq \partial_l\xi^l - \mu_n\left(\frac{\xi^z}{z-z_n^c} + \frac{\xi^{\bar z}}{\bar z-\bar z_n^c}\right)$ which are not separately square integrable as

$$\int\sqrt{h}\frac{\mu_n^2}{(z-z_n^c)(\bar z-\bar z_n^c)}\ \frac{i}{2}dz\wedge d\bar z \tag{38}$$

is always divergent at the singularity $z = z_n^c$. Thus in order to have an integrable δh_{ij} we need

$$\alpha_n = \delta z_n^c, \qquad \bar\alpha_n = \delta\bar z_n^c. \tag{39}$$

Our next job is to compute the functional jacobian in the transition from the integration variable h_{ij} to the $\sigma_R, z_n^c, \bar z_n^c, \xi^0$. This is achieved with standards methods going over to the tangent space [12]

$$1 = \int D[\delta h_{ij}]e^{-(\delta h_{ij}, \delta h_{ij})} =$$

$$= J_h \int \prod_{k=1}^{\mathcal{N}} dz_k^c d\bar{z}_k^c D[\delta\sigma_R] D[\xi^0] e^{-(\delta h_{ij}, \delta h_{ij})}. \tag{40}$$

The decomposition (22) seems to introduce an arbitrariness in the further developments, as conditions (23) are very far from fixing the ξ^k completely, but we shall see that all our results will be independent on the choice of such ξ^k respecting condition (23).

Now

$$(\delta h_{ij}, \delta h_{ij}) = (P\xi^0 + \alpha_k P\xi^k + \bar{\alpha}_k P\bar{\xi}^k, P\xi^0 + \alpha_k P\xi^k + \bar{\alpha}_k P\bar{\xi}^k) + 2C(\delta\sigma_R, \delta\sigma_R). \tag{41}$$

In order to proceed further we characterize the orthogonal complement to $P\xi^0$. This is given by the solutions of

$$(P^+\delta h)_z = -4e^{-2\sigma} \frac{\partial \delta h_{zz}}{\partial \bar{z}} = 0 \tag{42}$$

for $\delta h_{zz} \in D(P^+)$. From expression (26, 27) of P we see that in order that $\delta h_{zz} \in D(P^+)$ it is necessary that $\delta h_{zz} = O(\zeta^\alpha)$ at the singularities with $\alpha \geq -1$. On the other hand (42) tells us that δh_{zz} has to be analytic on the punctured plane i.e. meromorphic on the plane while $\alpha \geq -1$ excludes double poles and thus $\delta h \perp P\xi^0$ is of the form

$$\delta h_{zz} = \sum_{k=1}^{\mathcal{N}} \frac{\lambda_k}{z - z_k^c}; \qquad \delta h_{\bar{z}\bar{z}} = \sum_{k=1}^{\mathcal{N}} \frac{\bar{\lambda}_k}{\bar{z} - \bar{z}_k^c}. \tag{43}$$

Square integrability at infinity imposes $\sum_{k=1}^{\mathcal{N}} \lambda_k = 0$ and thus a complete basis of square integrable holomorphic quadratic differentials is given by

$$Q_{kzz} = \frac{1}{z - z_k^c} - \frac{1}{z - z_1^c}, \qquad Q_{k\bar{z}\bar{z}} = 0, \qquad (k = 2, \dots \mathcal{N}). \tag{44}$$

Taking the complex conjugate we have also

$$\bar{Q}_{kzz} = 0, \qquad \bar{Q}_{k\bar{z}\bar{z}} = \frac{1}{\bar{z} - \bar{z}_k^c} - \frac{1}{\bar{z} - \bar{z}_1^c} \qquad (k = 2, \dots \mathcal{N}). \tag{45}$$

We come back to eq.(24). We saw that the orthogonal complement to $P(\xi^0)$ in the space of the traceless square integrable δh_{ij} is $\mathcal{N} - 1$ dimensional. Thus if we leave in eq.(24) the α_k arbitrary we introduce an overcounting. This is due to the fact that for $\alpha_1 = \alpha_2 = \cdots = \alpha_{\mathcal{N}}$,

$P\left(\sum_{k=1}^{\mathcal{N}}(\alpha_k\xi^k+\bar\alpha_k\bar\xi^k)\right)$ is orthogonal to all the meromorphic quadratic differentials Q_k and $\bar Q_k$ and as such it belongs to the closure of the space $P(\xi^0)$. Several choices, all physically equivalent, can be done; we shall choose $\alpha_1=0$ which as we shall see shortly describes the dynamics in the relative coordinates $z'^c_n=z^c_n-z^c_1$. Now we decompose $P\bar\xi^k$ into the two mutually orthogonal contributions

$$(P\bar\xi^k)_{zz}=(P\bar\xi^{0k})_{zz}+\sum_{m=2}^{\mathcal{N}}\beta^k_m Q_{mzz}, \tag{46}$$

with

$$\beta^k_m=\sum_{n=2}^{\mathcal{N}}(Q_m,\bar Q_n)^{-1}(\bar Q_n,P\bar\xi^k) \tag{47}$$

where $(Q_k,\bar Q_m)^{-1}$ denotes the inverse of the $(\mathcal{N}-1)\times(\mathcal{N}-1)$ matrix $(\bar Q_m,Q_n)$ i.e. $\sum_{m=2}^{\mathcal{N}}(Q_k,\bar Q_m)^{-1}(\bar Q_m,Q_n)=\delta_{kn}$. In the Appendix it is shown that for a proper choice of the ξ^k, always satisfying condition (23), such square integrable ξ^{0k} exists. With the round brackets as always we understand the invariant scalar product according to the metric h_{ij}. E.g.

$$(\bar Q_m,Q_n)=\int\sqrt{h}\bar Q_{mij}\;h^{ii'}\;h^{jj'}\;Q_{ni'j'}\frac{i}{2}dz\wedge d\bar z=\int\sqrt{h}\bar Q_{m\bar z\bar z}\;h^{\bar zz}\;h^{\bar zz}\;Q_{nzz}\frac{i}{2}dz\wedge d\bar z. \tag{48}$$

As a consequence $(Q_m,Q_n)=(\bar Q_m,\bar Q_n)=0$.

Thus performing the shift $\xi^0\to\xi^0+\sum_{k=2}^{\mathcal{N}}(\delta z'^c_k\xi^{0k}+\delta\bar z'^c_k\bar\xi^{0k})$, in the integration over the space of the fields ξ^0 we have

$$1=J_h\int\prod_{k=2}^{\mathcal{N}}dz'^c_k d\bar z'^c_k D[\delta\sigma_R]D[\xi^0]e^{-2C(\delta\sigma_R,\delta\sigma_R)}e^{-(P\xi^0,P\xi^0)}e^{-2\delta z'^c_k\delta\bar z'^c_l(P\xi^k,Q_n)(Q_n,\bar Q_s)^{-1}(\bar Q_s,P\bar\xi^l)} \tag{49}$$

and thus apart from irrelevant numerical factors we have

$$J_h=\mathrm{Det}^*(P^+P)^{\frac12}\;\det(\bar Q_k,Q_l)^{-1}\det(Q_s,P\xi^l)\det(P\bar\xi^k,\bar Q_n) \tag{50}$$

where $\mathrm{Det}^*(P^+P)$ is the determinant of the operator P^+P, restricted to the space of the ξ^0 [16] and the functional measure on the space of the ξ^0 is defined as

$$1=\int D[\xi^0]e^{-(\xi^0,\xi^0)}, \tag{51}$$

and that on the space of $\delta\sigma_R$

$$1 = \int D[\delta\sigma_R] e^{-2C(\delta\sigma_R,\delta\sigma_R)}. \tag{52}$$

Thus

$$D[h_{ij}] = D[F^0] \prod_{k=2}^{\mathcal{N}} dz'^c_k d\bar{z}'^c_k \ D[\sigma_R] \mathrm{Det}^*(P^+P)^{\frac{1}{2}} \ \det(Q_l, \bar{Q}_k)^{-1} \ \det(Q_s, \mu_l) \ \det(\bar{\mu}_l, \bar{Q}_s) \tag{53}$$

with μ_l the Beltrami differential

$$\mu_l = P\xi^l. \tag{54}$$

Now we pass to the integration on the π^{ij}, that is on the space cotangent to the space of punctured metrics. We give a useful parameterization of π^{ij} in term of which to perform the integration, provided we compute the functional determinant related to the change of parameterization. The measure $D[\pi^{ij}]$ is defined by

$$1 = \int D[\pi^{ij}] e^{-(\pi^{ij},\pi^{ij})} \tag{55}$$

where

$$(\pi^{ij}, \pi^{ij}) = \int \frac{1}{\sqrt{h}} \pi^{ij} G_{ijmn} \pi^{mn} \frac{i}{2} dz \wedge d\bar{z}. \tag{56}$$

As in the case of the integration over the variation of the metric δh_{ij}, we must impose the square integrability of π^{ij} in the norm (56) in order to have a well defined functional measure. Moreover we want to have a well defined action, so δh_{ij} and π^{ij} must satisfy the following condition

$$\int \pi^{ij} \delta h_{ij} \frac{i}{2} dz \wedge d\bar{z} < \infty. \tag{57}$$

Taking into account the restrictions previously imposed on δh_{ij} at the singularities and at infinity we obtain

$$\left(\frac{\pi^{ij}}{\sqrt{h}}\right) \simeq (z\bar{z})^{\frac{3}{2}\mu_0 - \frac{1}{2} - \epsilon} \quad \text{at } \infty \tag{58}$$

as at infinity

$$\delta h_{ij} \simeq (z\bar{z})^{-\frac{\mu_0+1}{2} - \epsilon} \tag{59}$$

and on each singularity

$$\left(\frac{\pi^{ij}}{\sqrt{h}}\right) \simeq (\zeta\bar{\zeta})^{\frac{3}{2}\mu_n-\frac{1}{2}+\epsilon} \tag{60}$$

as the behavior of δh_{ij} is

$$\delta h_{ij} \simeq (\zeta\bar{\zeta})^{-\frac{\mu_n+1}{2}+\epsilon}. \tag{61}$$

It is easy to verify that these behaviors assure also the square summability of π^{ij}.

We give now the orthogonal decomposition of π^{ij} obeying to the restrictions described above. We set

$$\pi^{ij} = \frac{\pi}{2}h^{ij} + \pi^{Tij} \tag{62}$$

where $\pi = \pi^{ij}h_{ij}$ and π^{Tij} is a traceless tensor. Furthermore we decompose π^{Tij} in a transverse part and a remainder, i.e. we write

$$\frac{\pi^{Tij}}{\sqrt{h}} = \frac{\pi^{TTij}}{\sqrt{h}} + (PY^0)^{ij}, \tag{63}$$

where by definition π^{TT} belongs to the orthogonal (traceless) complement to $(PY^0)^{ij}$, being Y^0 the square integrable vector fields vanishing at the punctures. The previously defined π^{TT} are solutions (square integrable) of the equation

$$\left(P^+\frac{\pi^{TT}}{\sqrt{h}}\right)_j = 0 \tag{64}$$

on the punctured plane. In the conformal metric we can rewrite eq.(64) as

$$\left(P^+\frac{\pi^{TT}}{\sqrt{h}}\right)_z = -4e^{-2\sigma}\frac{\partial}{\partial\bar{z}}\frac{\pi^{TT}_{zz}}{\sqrt{h}} = 0, \tag{65}$$

i.e. $\dfrac{\pi^{TT}_{\ \ zz}}{\sqrt{h}}$ is a meromorphic function. On the other hand square integrability forbids poles of order higher than the first at the punctures z_n^c and the behavior $1/z$ at ∞. Thus in terms of the basis (44,45) we can write

$$\frac{\pi^{TT}_{\ \ zz}}{\sqrt{h}} = -\frac{1}{4\pi}\sum_{k=2}^{\mathcal{N}} t_k Q_{kzz}, \qquad \frac{\pi^{TT}_{\ \ \bar{z}\bar{z}}}{\sqrt{h}} = -\frac{1}{4\pi}\sum_{k=2}^{\mathcal{N}} \bar{t}_k \bar{Q}_{k\bar{z}\bar{z}}. \tag{66}$$

We come now to the integration measure $D[\pi^{ij}]$; we want to express it in terms of the variables, $Y^0, t_k, \bar{t}_k, \frac{\pi}{\sqrt{h}}$. The Jacobian relative to this change of parameterization is given by

$$1 = J_\pi \int D\left[\frac{\pi}{\sqrt{h}}\right] \prod_{k=2}^{\mathcal{N}} dt_k d\bar{t}_k \; D[Y^0] \; e^{-(\pi^{ij},\pi^{ij})}. \tag{67}$$

But

$$(\pi^{ij}, \pi^{ij}) = 2C \int \sqrt{h} \left(\frac{\pi}{\sqrt{h}}\right)^2 \frac{i}{2} dz \wedge d\bar{z} +$$
$$+(PY^0, PY^0) + \frac{1}{16\pi^2} \sum_{k,l} \bar{t}_k t_l \int 2\sqrt{h} \; \bar{Q}_{k\bar{z}\bar{z}} \; h^{\bar{z}z} h^{\bar{z}z} \; Q_{lzz} \frac{i}{2} dz \wedge d\bar{z} \tag{68}$$

from which

$$J_\pi = \mathrm{Det}^*(P^+P)^{\frac{1}{2}} \; \det(\bar{Q}_k, Q_l), \tag{69}$$

having used the normalization

$$1 = \int D\left[\frac{\pi}{\sqrt{h}}\right] e^{-2C(\frac{\pi}{\sqrt{h}}, \frac{\pi}{\sqrt{h}})}. \tag{70}$$

Putting together J_h and J_π we have

$$J_h \times J_\pi = \mathrm{Det}^*(P^+P)^{\frac{1}{2}} \; \det(\bar{Q}_l, Q_k)^{-1} \; \det(Q_s, \mu_n) \det(\bar{\mu}_n, \bar{Q}_s) \; \mathrm{Det}^*(P^+P)^{\frac{1}{2}} \det(\bar{Q}_l, Q_k) =$$
$$= \mathrm{Det}^*(P^+P) \; \det(Q_s, \mu_n) \det(\bar{\mu}_n, \bar{Q}_s). \tag{71}$$

The phase space functional integral now reads

$$Z = \int \prod_{n=1}^{\mathcal{N}} D[P_n] \prod_{n=2}^{\mathcal{N}} D[t_n] D[\bar{t}_n] D[\frac{\pi}{\sqrt{h}}] D[Y^0] D[z'^c_n] D[\bar{z}'^c_n] D[\sigma_R] D[F^0] \; \mathrm{Det}^*(P^+P) \times$$
$$\det(Q_s, \mu_n) \det(\bar{\mu}_n, \bar{Q}_s) \delta(\chi) \delta(\chi^1) \delta(\chi^2) |\mathrm{Det}\{\chi^\mu, H_\nu\}| \delta(\frac{H_i}{\sqrt{h}}) \delta(\frac{H}{\sqrt{h}}) e^{iS}. \tag{72}$$

We notice that with our choice of ξ^k eq.(23) we have $(Q_s, \mu_n) = -4\pi\delta_{sn}$ and thus the last two determinants give the numerical constant $(-4\pi)^{2(N-1)}$.

4 Constraints and gauge fixings

As already mentioned in the introduction we shall adopt the maximally slicing gauge $\delta(\chi)$ with $\chi = \frac{\pi}{\sqrt{h}} = \frac{\pi^{ij}h_{ij}}{\sqrt{h}}$. This will be the only relevant gauge fixing as the other two $\delta(\chi^1)$, $\delta(\chi^2)$ have the simple role of locking the space diffeomorphisms. The results are completely independent of the explicit form of χ^1 and χ^2; one could also adopt the viewpoint of [12, 13, 14] i.e. not to put any gauge fixing for the diffeomorphisms and simply factorize the infinite volume of the integration on the diffeomorphisms outside the functional integral, as we shall discuss at the end of this section.

We first consider the constraint

$$\frac{H_i}{\sqrt{h}} = -2D_j\frac{\pi^j{}_i}{\sqrt{h}} - \sum_{n=1}^{\mathcal{N}}\frac{P_{ni}}{\sqrt{h}}\delta^2(z-z_n). \tag{73}$$

This can be written in terms of the $\dfrac{\pi_{zz}}{\sqrt{h}}$ in the conformal metric as

$$-4e^{-2\sigma}\partial_{\bar z}(\frac{\pi_{zz}}{\sqrt{h}}) =$$

$$\left((P^+P)Y^0\right)_z + \sum_{n=2}^{\mathcal{N}}\ \left(\delta^2(z-z^c_n)-\delta^2(z-z^c_1)\right)\ \frac{t_n}{\sqrt{h}} = F_*\left(\sum_{n=1}^{\mathcal{N}}\frac{P_n}{\sqrt{h}}\delta^2(z-z_n)\right)_z. \tag{74}$$

Integration over Y^0 gives $Y^0=0$ and produces the diffeomorphism invariant determinant $[\mathrm{Det}^*(P^+P)]^{-1}$ which will cancel the one appearing in eq.(72), while integration over P_n produces

$$t_n = (F_*P_n)_z \qquad (n=2\ldots\mathcal{N}), \qquad \sum_{n=2}^{\mathcal{N}} t_n = -\left(F_*P_1\right)_z \tag{75}$$

$$\bar t_n = (F_*P_n)_{\bar z} \qquad (n=2\ldots\mathcal{N}), \qquad \sum_{n=2}^{\mathcal{N}} \bar t_n = -\left(F_*P_1\right)_{\bar z}. \tag{76}$$

The phase space path integral now reads

$$Z = \int\prod_{n=2}^{\mathcal{N}} D[t_n]D[\bar t_n]D[z'^c_n]D[\bar z'^c_n]D[\sigma_R]\ D[F^0]\ \delta(\chi^1)\delta(\chi^2)|\mathrm{Det}\{\chi^\mu, H_\nu\}|\delta(\frac{H}{\sqrt{h}})e^{iS}. \tag{77}$$

As noticed in [1] the gauge fixing $\frac{\pi}{\sqrt{h}} = 0$ has zero Poisson bracket with H_i, being H_i the generators of the diffeomorphisms and thus

$$\mathrm{Det}\{\chi^\mu, H_\nu\} = \mathrm{Det}\{\chi^i, H_j\}\mathrm{Det}\{\chi, H\}. \tag{78}$$

We notice

$$\int D[F^0]|\mathrm{Det}\{\chi^i, H_j\}|\delta(\chi^1)\delta(\chi^2) = \text{const.} \tag{79}$$

The functional integral becomes

$$Z = \int \prod_{n=2}^{\mathcal{N}} D[t_n]D[\bar{t}_n]D[z'^c_n]D[\bar{z}'^c_n]D[\sigma_R]\ |\mathrm{Det}\{\chi, H\}|\delta(\frac{H}{\sqrt{h}})e^{iS}, \tag{80}$$

which is explicitly independent of the choice of the gauge fixings χ^1, χ^2. We give now the explicit form of the hamiltonian constraint

$$0 = \frac{H}{\sqrt{h}} = \frac{1}{h}\pi^{ab}\pi_{ab} - R + \sum_n \frac{1}{\sqrt{h}}\delta^2(z - z_n)\sqrt{m_n^2 + P_{ni}h^{ij}P_{nj}} =$$

$$= F^*\left(e^{-2\sigma}\left(e^{-2\sigma}2\pi^z{}_{\bar{z}}\pi^{\bar{z}}{}_z + \Delta_0(2\sigma) + \sum_n \delta^2(z - z^c_n)\sqrt{m_n^2 + 2e^{-2\sigma(z^c_n)}t_n\bar{t}_n}\right)\right), \tag{81}$$

where we took into account that $\pi = 0$. Eq.(81) is satisfied by the solution of the equation [7]

$$\Delta_0 2\sigma = -2\pi^z_{\bar{z}}\pi^{\bar{z}}_z e^{-2\sigma} - \sum_n m_n\delta^2(z - z^c_n) \tag{82}$$

or equivalently by the solution of the Liouville equation

$$\Delta_0 2\tilde{\sigma} = -e^{-2\tilde{\sigma}} - \sum_n (m_n - 4\pi)\delta^2(z - z^c_n) - 4\pi\sum_A \delta^2(z - z^c_A) \tag{83}$$

being z^c_A the position of the apparent singularities i.e. those values for which

$$\sum_{n=2}^{\mathcal{N}} \frac{t_n}{z^c_A - z^c_n} - \frac{\sum_{n=2}^{\mathcal{N}} t_n}{z^c_A - z^c_1} = 0, \tag{84}$$

and $e^{-2\tilde{\sigma}} = e^{-2\sigma}2\pi^z{}_{\bar{z}}\pi^{\bar{z}}{}_z$. As proved in [22, 23, 24] the solution of eq.(82) is unique for $e^{2\sigma}$ behaving at infinity like $(z\bar{z})^{-\mu}$ with $\mu < 1$, $0 < \frac{m_n}{4\pi}$ and $\sum_n \frac{m_n}{4\pi} < \mu$. Substituting the behaviors (12) in eq.(82) we see that the μ_i are fixed to the constant particle masses $\mu_n = \frac{m_n}{4\pi}$ and the requirement of $2+1$ dimensional gravity $0 < \mu_n < 1,\ 0 < \mu < 1$ satisfy Picard bound. We recall that $0 < \mu_n < 1$ states that the n-th particle mass is positive and that the conical deficit at q_n^c is less than 2π; $0 < \mu < 1$ states that the total energy is positive and that the total conical defect is less than 2π, while $\sum_n \mu_n < \mu$ states that the total energy exceed the sum of the rest masses [7, 8].

Next we have to compute $\text{Det}\{\chi, H\} = \text{Det}\{\frac{\pi}{\sqrt{h}}, H\}$ when the two conditions $\pi = 0, H = 0$ are satisfied. Such a calculation can be performed by using the relations $\{A(x), \pi^{ij}(y)\} = \frac{\delta A(x)}{\delta h_{ij}(y)}$ and $\{A(x), h_{ij}(y)\} = -\frac{\delta A(x)}{\delta \pi^{ij}(y)}$. The functional derivative with respect to h_{ij} can be computed with the help of the standard formula [29]

$$h^{ab}(x)\frac{\delta R_{ab}(x)}{\delta h_{ij}(y)} = (D^i D^j - h^{ij} D^a D_a)\delta^2(x-y) \tag{85}$$

obtaining for such a determinant

$$\text{Det}\{\chi, H\} = \text{Det}\left[\left(\frac{1}{h}\pi^{ab}\pi_{ab} - D^2\right)\delta^2(x-y)\right] \tag{86}$$

being $D^2 = D^a D_a$ the scalar Laplace-Beltrami operator.

We show now that the determinant (86) cancels with the one arising from $\delta(\frac{H}{\sqrt{h}})$. In fact from eq.(81) when we integrate $\delta(\frac{H}{\sqrt{h}})$ in $D[\sigma_R]$ we obtain as a result

$$\left[\text{Det}F^*(\ e^{-2\sigma}(-e^{-2\sigma}2\pi^z{}_{\bar{z}}\pi^{\bar{z}}{}_z + \Delta_0)\delta^2(x-y)\ )\right]^{-1} \tag{87}$$

where we took into account the constraint $\frac{H}{\sqrt{h}} = 0$ and the vanishing of the functional derivative of the source term, always on $\frac{H}{\sqrt{h}} = 0$, due to the behavior $e^{-2\sigma} \simeq \text{const.}(\zeta\bar{\zeta})^{\mu_n}$ at the particle singularities [7].

The action S after solving the constraints becomes

$$\int dt \left[\sum_{n=1}^{\mathcal{N}} P_{ni}\ \dot{q}_n^i + \int \pi^{ij}\dot{h}_{ij}\frac{i}{2}dz \wedge d\bar{z} - H_B\right] \tag{88}$$

where by $-H_B$ we have denoted the boundary term in eq.(1). With regard to the kinetic terms we notice that

$$\dot{q}_n^c - \xi(q_n^c) = F_*\dot{q}_n \tag{89}$$

if $\xi(z)$ generates the diffeomorphism F and recalling eq.(76) we have

$$\sum_{n=1}^{\mathcal{N}} P_{ni}\dot{q}^i_n = \sum_{n=1}^{\mathcal{N}} \left(F_* P_n\right)_z \left(F_* \dot{q}_n\right)^z + \left(F_* P_n\right)_{\bar{z}} \left(F_* \dot{q}_n\right)^{\bar{z}}$$

$$= \sum_{n=2}^{\mathcal{N}} t_n \left(\dot{z}^c_n - \dot{z}^c_1 - \xi^z(q^c_n) + \xi^z(q^c_1)\right) + c.c. \tag{90}$$

Then we notice that

$$\int \pi^{ij}\dot{h}_{ij}\frac{i}{2}dz\wedge d\bar{z} = \int F^*(\pi)^{ij}\frac{d}{dt}(F^*(\delta_{ij}e^{2\sigma}))\frac{i}{2}dz\wedge d\bar{z} =$$

$$= \int F^*\left(-\frac{\sqrt{h}}{4\pi}\sum_{k=2}^{\mathcal{N}}(Q_k t_k + \bar{Q}_k\bar{t}_k) + \sqrt{h}\ P(Y^0)\right)^{ij} [F^*(\delta_{ij}\frac{d}{dt}e^{2\sigma}) + F^*(\mathcal{L}_\xi(e^{2\sigma}\delta_{ij}))]\frac{i}{2}dz\wedge d\bar{z} =$$

$$= \int \left(-\frac{\sqrt{h}}{4\pi}\sum_{k=2}^{\mathcal{N}}(Q_k t_k + \bar{Q}_k\bar{t}_k)\right)^{ij} P(\xi)_{ij}\ \frac{i}{2}dz\wedge d\bar{z} \tag{91}$$

where we took into account that

$$\frac{d}{dt}F(t)^*[A(t)] = F(t)^*[\dot{A}(t)] + F(t)^*\mathcal{L}_\xi[A(t)]. \tag{92}$$

Explicit evaluation of eq.(91) gives

$$-\frac{1}{\pi}\int\sum_{n=2}^{\mathcal{N}} Q_{nzz}t_n\frac{\partial}{\partial\bar{z}}\xi^z\frac{i}{2}dz\wedge d\bar{z} + c.c. = \sum_{n=2}^{\mathcal{N}} t_n(\xi^z(q^c_n) - \xi^z(q^c_1)) + c.c. \tag{93}$$

Then summing eq.(90) to eq.(91) we obtain

$$\sum_{n=1}^{\mathcal{N}} P_{ni}\dot{q}^i_n + \int \pi^{ij}\dot{h}_{ij}\frac{i}{2}dz\wedge d\bar{z} = \sum_{n=2}^{\mathcal{N}} t_n \dot{z'}^c_n + \bar{t}_n\dot{\bar{z}}'^c_n \tag{94}$$

with ${z'}^c_n = z^c_n - z^c_1$.

Thus we have reached the functional integral

$$Z = \int\prod_{n=2}^{\mathcal{N}} D[{z'}^c_n]D[{\bar{z}'}^c_n]D[t_n]D[\bar{t}_n]\ e^{i\int(\sum_{n=2}^{\mathcal{N}}(t_n\dot{z'}^c_n + \bar{t}_n\dot{\bar{z}}'^c_n - H_B)dt}, \tag{95}$$

i.e. all functional determinants cancel out. The main point in achieving such a result is the remark in [1] that the expression

$$\int D[N^l]\ e^{-i\int N^l H_l d^2z} \tag{96}$$

if we want to respect invariance under diffeomorphisms has to be understood as $\delta(\frac{H_i}{\sqrt{h}})$ and similarly for N and H. The role of the two gauge fixings $\delta(\chi^1)$ and $\delta(\chi^2)$ is simply to lock the space diffeomorphisms; their explicit form does not intervene in the final result. One could approach with some advantage the functional integral along the so called geometric procedure [12, 13, 14]. Here one start with the functional integral without introducing the gauge fixings χ^1 and χ^2

$$Z = \int \prod_{n=1}^{\mathcal{N}} D[P_n]D[\pi^{ij}]D[h_{ij}]D[N^i]D[N]\delta(\chi)|\mathrm{Det}\{\chi, H\}|e^{iS}, \tag{97}$$

with a scalar χ. We proceed exactly as before reaching instead of eq.(80)

$$\int D[F^0]\ \times\ A \tag{98}$$

where A is the diffeomorphism invariant quantity

$$\int \prod_{n=2}^{\mathcal{N}} D[{z'}^c_n]D[{\bar{z}'}^c_n]D[t_n]D[\bar{t}_n]D[\sigma_R]\ |\mathrm{Det}\{\chi, H\}|\delta(\frac{H}{\sqrt{h}})e^{iS}. \tag{99}$$

Factorising away the infinite gauge volume $\int D[F^0]$ we obtain again eq.(95). This procedure clarifies the fact that in eq.(80) no trace is left of χ^1 and χ^2.

5 The boundary terms

We come now to the boundary term

$$-H_B = 2\int_{B_t} d^{(D-1)}x\,\sqrt{\rho}N\left(K_{B_t} + \frac{\eta}{\cosh\eta}\mathcal{D}_\alpha v^\alpha\right) - 2\int_{B_t} d^{(D-1)}x\, r_\alpha \pi^{\alpha\beta}_{(\rho)} N_\beta \tag{100}$$

which plays the role of the hamiltonian. Such a term was already computed in [8] and found to be given by

$$H_B = \ln s^2 \tag{101}$$

being $\ln s^2$ the constant term in the asymptotic behavior of 2σ i.e.

$$2\sigma = -\mu \ln z\bar{z} + \ln s^2 + O(\frac{1}{z}) + O(\frac{1}{\bar{z}}) + O((z\bar{z})^{\mu-1}) = -\mu \ln z\bar{z} + \ln s^2 + O((z\bar{z})^{-\alpha}), \tag{102}$$

where $\alpha = \min(1/2, 1-\mu)$.

We recall also that the same hamiltonian H_B was derived in [8] directly from the equations of motion obtained by solving the constraints in the maximally slicing gauge $K = 0$.

For completeness we want to discuss again such a term in some detail. Our treatment as the one in [1] is based on the ADM formalism, with the difference that instead of having a compact space, our space (the plane) is non compact.

In [10] is described the procedure for computing the boundary term for non compact spaces. Such a procedure amounts to compute the boundary term appearing in (100) on a compact region and then letting the boundary go to infinity. In order to do so one has to supply the fields N, N^z and σ for $|z| = r_0$ and the behavior of such boundary conditions for $r_0 \to \infty$. Such asymptotic behaviors define the ADM frame at infinity. As a rotating frame has $N^z \simeq z$ we shall impose a N^z behaving at infinity like r^α with $\alpha < 1$. It is immediately seen [8] that under such a condition the only surviving boundary term in H_B with N constant on the boundary is

$$-2N(r^0) \int_{Bt} dx \sqrt{\rho} K_{Bt} \tag{103}$$

whose exact expression in terms of the conformal factor is [8]

$$-2N(r_0) \int_{Bt} d\theta (r_0 \partial_r \sigma(r_0, \theta) + 1). \tag{104}$$

Thus one should proceed as follows: one solves eq.(82) for $|z| \leq r_0$ with the boundary condition

$$2\sigma(z) = -\mu_0 \ln r_0^2 + c_0 \quad \text{for} \quad |z| = r_0. \tag{105}$$

We shall denote such solution as σ_V. Then the boundary term on the non compact plane is given by the limit for $r_0 \to \infty$ of

$$2N(r_0) \int d\theta(-r_0 \partial_r \sigma_V(r_0, \theta) - 1). \tag{106}$$

In general as pointed out in [10] some divergence may originate in this limiting procedure and such a divergence can be eliminated by taking the difference between the boundary expression and the same expression computed on some background space time. Such a procedure can be applied to our case but, as we shall see, the divergent term in our case is a number independent of the dynamical variables and as such irrelevant in the hamiltonian. Thus even if such a subtraction can be performed it is not necessary for our purposes. Clearly solving eq.(82) on the finite region $|z| \leq r_0$ with the boundary condition (105) is far more difficult that solving eq.(82) on the whole plane with a given asymptotic behavior $2\sigma = -\mu \ln z\bar{z} + O(1)$ as in the first case the problem cannot be reduced to an ordinary linear fuchsian equation. However one expects that for large r_0 one can replace in the calculation of the boundary term eq.(106) σ_V with σ with a proper μ, as for large r_0 eq.(102) becomes more and more circularly symmetric. To prove this assertion let us consider the solution of eq.(82) $2\sigma(z, \mu)$ with μ defined by

$$-\mu \ln r_0^2 + \ln s^2({z'}^c_n, {\bar{z}'}^c_n, t_n, \bar{t}_n, \mu) = -\mu_0 \ln r_0^2 + c_0. \tag{107}$$

The difference $\eta = 2\sigma_V - 2\sigma$ satisfies

$$\Delta_0 \eta(z) = -2\pi^{\bar{z}}_z \pi^z_{\bar{z}} e^{-2\sigma(z)}(e^{-\eta(z)} - 1) = -e^{-2\tilde{\sigma}}(e^{-\eta(z)} - 1) \tag{108}$$

where for $|z| = r_0$, $\eta(z) = O(r_0^{-2\alpha})$ according to eq.(102). Picard [22] examined eq.(108) on a bounded domain proving that the function η assumes its maxima and minima on the boundary, in our case on the circle of radius r_0. The proof is based on the positivity of $e^{-2\tilde{\sigma}}$ and the result holds also when $e^{-2\tilde{\sigma}}$ possesses locally integrable singularities [22], as it happens in our case. Thus we can conclude that $\eta(z) = O(r_0^{-2\alpha})$ on the whole disk of radius $|z| = r_0$. Integrating now eq.(108) on the disk $|z| \leq r_0$ we have

$$r_0 \int [\partial_r \sigma_V(r_0) - \partial_r \sigma(r_0, \theta)] d\theta = \int_{|z| \leq r_0} \left(-2\pi^{\bar{z}}_z \pi^z_{\bar{z}} e^{-2\sigma(z)}(e^{-\eta(z)} - 1)\right) d^2 z = O(r_0^{-2\alpha}), \tag{109}$$

as

$$\int -2\pi^{\bar{z}}_{z}\pi^{z}_{\bar{z}}e^{-2\sigma}d^2z \tag{110}$$

extended to the whole plane is convergent due to the asymptotic behavior of 2σ. Then using eq.(107) we have

$$H_B = 4\pi N(r_0)[\mu - 1 + O(r_0^{-2\alpha})]. \tag{111}$$

Using the fact that $\ln s^2$ is a real analytic function of μ [9] and the implicit function theorem [33] we obtain solving eq.(107) for large r_0

$$\mu - \mu_0 = \frac{\ln s^2(z'^c_n, \bar{z}'^c_n, t_n, \bar{t}_n, \mu_0) - c_0}{\ln r_0^2}\left(1 + O(\frac{1}{\ln r_0^2})\right). \tag{112}$$

Thus

$$H_B = 4\pi N(r_0)\left[\frac{\ln s^2(z'^c_n, \bar{z}'^c_n, t_n, \bar{t}_n, \mu_0) - c_0}{\ln r_0^2}\left(1 + O(\frac{1}{\ln r_0^2})\right) + \mu_0 - 1 + O(r_0^{-2\alpha})\right] \tag{113}$$

and we have to take the limit of H_B for $r_0 \to \infty$. We recall that $N(r_0)$ describes the asymptotic time and it is immediately seen that the only choice of $N(r_0)$ which gives rise in the limit $r_0 \to \infty$, to a dynamics which is neither singular nor frozen is

$$N(r_0) = c_1 \ln r_0^2 + c_2 \tag{114}$$

producing in the limit $r_0 \to \infty$, with the conventional normalization adopted in [8] $c_1 = \frac{1}{4\pi}$, the hamiltonian

$$H_B = \ln s^2(z'^c_n, \bar{z}'^c_n, t_n, \bar{t}_n, \mu_0) + \text{const.} \tag{115}$$

where the divergent numerical constant $(-c_0 + (\mu_0 - 1)\ln r_0^2)$, is irrelevant to the hamiltonian. Subtracting to the boundary terms the contribution due to the background we would have obtained

$$H_B = \ln s^2(z'^c_n, \bar{z}'^c_n, t_n, \bar{t}_n, \mu_0) - c_0 \tag{116}$$

with no divergent term. The non trivial part in H_B is contained in $\ln s^2$ which is a function of the canonical variables z'^c_n and t_n. We notice that the behavior (114) is the asymptotic behavior of N obtained from the solution of the classical equation of motion in the $K = 0$

gauge [8]. Thus we reproduced for the functional integral the same expression which would have been derived from the reduced particle dynamics i.e.

$$Z = \int \prod_{n=2}^{\mathcal{N}} D[{q'}^c_n] D[t_n] \; e^{i \int (\sum_{n=2}^{\mathcal{N}} (t_n \dot{z}'^c_n + \bar{t}_n \dot{\bar{z}}'^c_n) - \ln s^2 ({z'}^c_n, \bar{z}'^c_n, t_n, \bar{t}_n, \mu_0)) dt}. \tag{117}$$

In principle one would have expected measure terms i.e. quantum corrections to such naive translation, but as we saw, detailed treatment starting from formula (8) shows that all the intervening measure terms (determinants) cancel out exactly.

Obviously the fact that we have reached expressions (95,117) for the functional integral tells us little about the ordering problem. In [8] we gave a detailed quantum treatment of the two particle problem. The choice performed in [8] was dictated by naturalness and aesthetic reasons reaching the logarithm of the Laplace-Beltrami operator on a cone. As discussed in [8] this is very similar to the quantum treatment of a test particle on a cone given in [35]. But there is no a priori reason for that choice. A standard choice for the functional integral is the mid point rule [31] which is equivalent to the Weyl ordering at the operator level. In our case

$$H_B = \ln \left[(q\bar{q})^{\mu_0} P\bar{P} \right] = \ln \left[(q_1^2 + q_2^2)^{\mu_0} (P_1^2 + P_2^2) \right] \tag{118}$$

and the Weyl ordering gives rise simply to the operator

$$\mu_0 \ln(\hat{q}_1^2 + \hat{q}_2^2) + \ln(\hat{P}_1^2 + \hat{P}_2^2). \tag{119}$$

This is the choice examined by Ciafaloni and Munier in [32] in the context of high energy behavior of Yang-Mills field theory. The time evolution operator induced by any hamiltonian can be described by a functional integral provided the classical hamiltonian is defined through a special ordering procedure [36]. It appears that the logarithm of the Laplace-Beltrami operator can be obtained only through a rather complicated ordering process and the same can be said for the functional translation of the Maass laplacian adopted in [3, 4]. For more than two particles the hamiltonian becomes very complicated and here up to now no guiding principle has emerged for addressing the ordering problem. It is not clear on the other hand whether the functional or the operator approach should be the guiding principles in quantum gravity. Other issues common to the problem treated in [1] are the relations to the proper time quantization [37, 38] and to the causal quantization of gravity [39]. E.g. if N is integrated only on positive values one does not obtain the strict constraint $\delta(H/\sqrt{h})$ but only a kind of smeared form of it and one does not expect equivalence with the functional integral (117).

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Appendix

Here we show that for a proper choice of the ξ^k, always respecting eq.(23), a square integrable ξ^{0k} exists satisfying eq.(46). Such an equation is solved by

$$\bar{\xi}^{0k\bar{z}}(z) = \bar{\xi}^{k\bar{z}}(z) - 4\Delta_0^{-1}\left(\partial_{\bar{z}}(e^{-2\sigma}\sum_{k=2}^{\mathcal{N}}\beta_m^k Q_{mzz})\right)(z) + c_k \tag{120}$$

where $\Delta_0^{-1} = \frac{1}{4\pi}\ln(z-z')(\bar{z}-\bar{z}')$. It is easily seen that

$$\int \partial_{\bar{z}}(e^{-2\sigma}\sum_{k=2}^{\mathcal{N}}\beta_m^k Q_{mzz})\frac{i}{2}dz\wedge d\bar{z} = 0 \tag{121}$$

and thus

$$-4\Delta_0^{-1}\left(\partial_{\bar{z}}(e^{-2\sigma}\sum_{k=2}^{\mathcal{N}}\beta_m^k Q_{mzz})\right)(z) \tag{122}$$

goes to a constant at infinity. Taking the scalar product of eq.(46) with $\bar{Q}_k$ we have

$$\bar{\xi}^{0k\bar{z}}(z_n) - \bar{\xi}^{0k\bar{z}}(z_1) = 0 \tag{123}$$

and thus using the freedom on the c_k we can have $\xi^{0k\bar{z}}(z_n) = 0$ for all z_n and k. Now by properly choosing $\xi^{k\bar{z}}$ outside a circle of radius R containing all singularities, always respecting conditions (23), we obtain $\bar{\xi}^{0k\bar{z}} = 0$ outside such a circle.

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