On the evaluation of the evolution operator $Z_{\rm Reg}(R_2, R_1)$ in the Diakonov–Petrov approach to the Wilson loop

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Abstract

We evaluate the evolution operator $Z_{\rm Reg}(R_2, R_1)$ introduced by Diakonov and Petrov for the definition of the Wilson loop in terms of a path integral over gauge degrees of freedom. We use the procedure suggested by Diakonov and Petrov (Phys. Lett. B224 (1989) 131) and show that the evolution operator vanishes.

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Path integral for the evolution operator $Z_{\rm Reg}(R_2, R_1)$

In Ref.[1] for the representation of the Wilson loop in terms of the path integral over gauge degrees of freedom Diakonov and Petrov used the functional $Z(R_2, R_1)$ defined by (see Eq.(8) of Ref.[1])

$$Z(R_2,R_1) = \int\limits_{R_1}^{R_2} DR(t)\,\exp\Bigg(iT\int\limits_{t_1}^{t_2}{\rm Tr}\,(iR\,\dot{R}\,\tau_3)\Bigg), \tag{1}$$

where $\dot{R} = dR/dt$ and $T = 1/2, 1, 3/2, \ldots$ is the colour isospin quantum number. According to Diakonov and Petrov $Z(R_2, R_1)$ should be regularized by the analogy to an axial–symmetric top. The regularized expression of $Z(R_2, R_1)$ has been determined in Eq.(9) of Ref.[1] and reads

$$Z_{\rm Reg}(R_2,R_1) = \int\limits_{R_1}^{R_2} DR(t)\,\exp\Bigg(i\int\limits_{t_1}^{t_2}\Big[\frac{1}{2}\,I_{\perp}\,(\Omega^2_1 + \Omega^2_2) + \frac{1}{2}\,I_{\parallel}\,\Omega^2_3 + T\,\Omega_3\Big]\Bigg), \tag{2}$$

where $\Omega_a = i\,{\rm Tr}(R\,\dot{R}\,\tau_a)$ are angular velocities of the top, τ_a are Pauli matrices $a = 1,2,3$, $I_{\perp}$ and $I_{\parallel}$ are the moments of inertia of the top which should be taken to zero. According to

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the prescription of Ref.[1] one should take first the limit $I_{\parallel} \to 0$ and then $I_{\perp} \to 0$. For the confirmation of the result, given in Eq.(13) of Ref.[1],

$$Z_{\rm Reg}(R_2, R_1) = (2T+1)\, D^T_{TT}(R_2 R_1^{\dagger}) = (2T+1)\, D^T_{-T-T}(R_1 R_2^{\dagger}), \tag{3}$$

where $D^T(U)$ is a Wigner rotational matrix in the representation T, Diakonov and Petrov suggested to evaluate the evolution operator (2) explicitly via the discretization of the path integral over R. The discretized form of the path integral Eq.(2) is given by Eq.(14) of Ref.[1] and reads

$$\begin{aligned} Z_{\rm Reg}(R_{N+1}, R_0) &= \lim_{\substack{N \to \infty \\ \delta \to 0}} \mathcal{N} \int \prod^{N}_{n=1} dR_n \\ &\times \exp\left[\sum^{N}_{n=0}\left(- i\,\frac{I_{\perp}}{2\delta}\Big[({\rm Tr}\,V_n\tau_1)^2 + ({\rm Tr}\,V_n\tau_2)^2\Big] - i\,\frac{I_{\parallel}}{2\delta}\,({\rm Tr}\,V_n\tau_3)^2 - T\,({\rm Tr}\,V_n\tau_3)\right)\right], \end{aligned} \tag{4}$$

where $R_n = R(s_n)$ with $s_n = t_1 + n\,\delta$ and $V_n = R_n R^{\dagger}_{n+1}$ are the relative orientations of the top at neighbouring points [1]. The normalization factor $\mathcal{N}$ is determined by

$$\mathcal{N} = \left(\frac{I_{\perp}}{2\pi i\delta}\sqrt{\frac{I_{\parallel}}{2\pi i\delta}}\right)^{N+1}. \tag{5}$$

(see Eq.(19) of Ref.[1]). Following the prescription of Ref.[1] one should take the limits $\delta \to 0$ and $I_{\parallel}, I_{\perp} \to 0$ but keeping the ratios I_i/δ, where ($i = \parallel, \perp$), much greater than unity, $I_i/\delta \gg 1$.

The main point of the evaluation of the path integral is to show that the evolution operator $Z_{\rm Reg}(R_2, R_1)$ given by the path integral (2) reduces to the representation in the form of *a sum over possible intermediate states*, i.e. eigenfunctions of the axial–symmetric top [1]

$$Z_{\rm Reg}(R_2, R_1) = \sum^{\infty}_{J=0}\sum^{J}_{m=-J}(2J+1)\, D^J_{mm}(R_2 R_1^{\dagger})\, e^{-i(t_2 - t_1)\, E_{Jm}}, \tag{6}$$

(see Eq.(12) of Ref.[1]), where E_{Jm} are the eigenvalues of the Hamiltonian of the axial–symmetric top

$$E_{Jm} = \frac{J(J+1) - m^2}{2I_{\perp}} + \frac{(m-T)^2}{2I_{\parallel}} \tag{7}$$

(see Eq.(11) of Ref.[1]).

According to Diakonov's and Petrov's statement the integral has a saddle–point at $V_n \simeq 1$. For the calculation of the integral around the saddle–point Diakonov and Petrov suggested the following procedure. Let us denote the exponent of Eq.(4) as

$$f[V_n] = - i\,\frac{I_{\perp}}{2\delta}\Big[({\rm Tr}\,V_n\tau_1)^2 + ({\rm Tr}\,V_n\tau_2)^2\Big] - i\,\frac{I_{\parallel}}{2\delta}\,({\rm Tr}\,V_n\tau_3)^2 - T\,({\rm Tr}\,V_n\tau_3) \tag{8}$$

and represent the exponential in the following form

$$e^{f[V_n]} = \sum^{\infty}_{J=0}\sum^{J}_{p=-J}\sum^{J}_{q=-J}(2J+1)\lambda^J_{pq} D^J_{pq}(V_n). \tag{9}$$

The coefficients λ^J_{pq} are given by

$$\lambda^J_{pq} = \int dU_n\, D^J_{qp}(U^\dagger_n)\, e^{f[U_n]}. \tag{10}$$

Substituting Eq.(10) in Eq.(9) we get the identity

$$e^{f[V_n]} = \sum^{\infty}_{J=0}\sum^{J}_{p=-J}\sum^{J}_{q=-J}(2J+1)\, D^J_{pq}(V_n)\int dU_n\, D^J_{qp}(U^\dagger_n)\, e^{f[U_n]}. \tag{11}$$

Let us show that Eq.(11) is the identity. For this aim we have to use the relation

$$\sum^{\infty}_{J=0}\sum^{J}_{p=-J}\sum^{J}_{q=-J}(2J+1)\, D^J_{pq}(V_n)\, D^J_{qp}(U^\dagger_n) = \sum^{\infty}_{J=0}(2J+1)\,\chi_J[V_nU^\dagger_n]. \tag{12}$$

By using Eq.(12) the r.h.s. of Eq.(11) reads

$$\int dU_n\, e^{f[U_n]}\sum^{\infty}_{J=0}(2J+1)\,\chi_J[V_nU^\dagger_n] = \int dU_n\, e^{f[U_n]}\,\delta(V_nU^\dagger_n) = e^{f[V_n]}, \tag{13}$$

where $\delta(V_nU^\dagger_n)$ is a δ–function defined by

$$\sum^{\infty}_{J=0}(2J+1)\,\chi_J[V_nU^\dagger_n] = \delta(V_nU^\dagger_n). \tag{14}$$

The important consequence of these steps is that dU_n as well as dV_n is a standard Haar measure normalized to unity

$$\int dU_n = \int dV_n = 1. \tag{15}$$

This point alters crucially the results of Ref.[1].

Inserting the expansion Eq.(11) in the r.h.s. of Eq.(4) we obtain

$$\begin{aligned}
Z_{\rm Reg}(R_{N+1},R_0) = &\lim_{\substack{N\to\infty\\ \delta\to 0}}\ \mathcal{N}\int\ldots\int dR_1\,dR_2\ldots dR_{N-1}\,dR_N\\
\times &\sum^{\infty}_{J_0=0}\sum^{J_0}_{p_0=-J_0}\sum^{J_0}_{q_0=-J_0}(2J_0+1)\, D^{J_0}_{p_0q_0}(R_0R^\dagger_1)\int dU_0\, D^{J_0}_{q_0p_0}(U^\dagger_0)\, e^{f[U_0]}\\
\times &\sum^{\infty}_{J_1=0}\sum^{J_1}_{p_1=-J_1}\sum^{J_1}_{q_1=-J_1}(2J_1+1)\, D^{J_1}_{p_1q_1}(R_1R^\dagger_2)\int dU_1\, D^{J_1}_{q_1p_1}(U^\dagger_1)\, e^{f[U_1]}\\
\times &\sum^{\infty}_{J_2=0}\sum^{J_2}_{p_2=-J_2}\sum^{J_2}_{q_2=-J_2}(2J_2+1)\, D^{J_2}_{p_2q_2}(R_2R^\dagger_3)\int dU_2\, D^{J_2}_{q_2p_2}(U^\dagger_2)\, e^{f[U_2]}\\
\times &\ldots\\
\times &\sum^{\infty}_{J_N=0}\sum^{J_N}_{p_N=-J_N}\sum^{J_N}_{q_N=-J_N}(2J_N+1)\, D^{J_N}_{p_Nq_N}(R_NR^\dagger_{N+1})\int dU_N\, D^{J_N}_{q_Np_N}(U^\dagger_N)\, e^{f[U_N]}
\end{aligned} \tag{16}$$

Integrating over $R_n\,(n = 1, 2, \ldots, N)$ and using the orthogonality relation for the group elements we arrive at the expression

$$Z_{\rm Reg}(R_{N+1}, R_0) = \lim_{\substack{N\to\infty\\ \delta\to 0}} \sum^{\infty}_{J=0}\sum^{J}_{p=-J}\sum^{J}_{q=-J}(2J+1)\, D^J_{pq}(R_0R^{\dagger}_{N+1})\,\mathcal{N}\int dU_0\, D^J_{qp}(U^{\dagger}_0)\, e^{f[U_0]}$$
$$\times\int dU_1\, D^J_{qp}(U^{\dagger}_1)\, e^{f[U_1]}\int dU_2\, D^J_{qp}(U^{\dagger}_2)\, e^{f[U_2]}\ldots\int dU_N\, D^J_{qp}(U^{\dagger}_N)\, e^{f[U_N]} =$$
$$= \lim_{\substack{N\to\infty\\ \delta\to 0}} \sum^{\infty}_{J=0}\sum^{J}_{p=-J}\sum^{J}_{q=-J}(2J+1)\, D^J_{pq}(R_0R^{\dagger}_{N+1})\,[Z^J_{qp}]^{N+1}, \tag{17}$$

where Z^J_{qp} is defined by

$$Z^J_{qp} = \frac{I_{\perp}}{2\pi i\delta}\sqrt{\frac{I_{\parallel}}{2\pi i\delta}}\int dU\, D^J_{qp}(U^{\dagger})\, e^{f[U]}. \tag{18}$$

Recall that dU is the Haar measure normalized to unity Eq.(15).

For the subsequent evaluation of the integral over U we follow Diakonov and Petrov and use

$$U = e^{i\,\frac{1}{2}\,\vec{\omega}\,\cdot\,\vec{\tau}} \tag{19}$$

for the fundamental representation and

$$D^J_{qp}(U^{\dagger}) = \Big(e^{-\,i\,\vec{\omega}\,\cdot\,\vec{T}}\Big)_{qp} \tag{20}$$

for $J \neq 1/2$. In the parameterization (19) the Haar measure dU reads

$$dU = \frac{d\omega_1 d\omega_2 d\omega_3}{16\pi^2}\left(\frac{2}{\omega}\,\sin\frac{\omega}{2}\right)^2, \tag{21}$$

where $\omega = \sqrt{\omega^2_1+\omega^2_2+\omega^2_3}$. According to the Diakonov and Petrov point of view the integral over U calculated in the limit $I_{\parallel}/\delta, I_{\perp}/\delta \to \infty$ has a saddle point at $U \simeq 1$ [1]. Expanding the integrand around the saddle–point, keeping only quadric terms and **neglecting the contribution of the terms coming from the Haar measure**, we get

$$Z^J_{qp} = \frac{I_{\perp}}{2\pi i\delta}\sqrt{\frac{I_{\parallel}}{2\pi i\delta}}\int^{\infty}_{-\infty} d\omega_1\int^{\infty}_{-\infty} d\omega_2\int^{\infty}_{-\infty} d\omega_3\,\exp\left\{i\frac{I_{\perp}}{2\delta}\,(\omega^2_1+\omega^2_2) + i\frac{I_{\parallel}}{2\delta}\,\omega^2_3\right\}$$
$$\times\left[\delta_{qp} - \frac{1}{2}\,[\omega^2_1(T^2_1)_{qp} + \omega^2_2(T^2_2)_{qp}] - \frac{1}{2}\,\omega^2_3\,((T_3+T)^2)_{qp}\right]. \tag{22}$$

Integrating over $\omega_a\,(a = 1, 2, 3)$ we arrive at the expression

$$Z^J_{qp} = \delta_{qp} - i\,\delta\left[\frac{(T^2_1+T^2_2)_{qp}}{2I_{\perp}} + \frac{((T_3+T)^2)_{qp}}{2I_{\parallel}}\right] =$$
$$= \delta_{qp}\left\{1 - i\,\delta\left[\frac{(J(J+1)-p^2)}{2I_{\perp}} + \frac{(p+T)^2}{2I_{\parallel}}\right]\right\}. \tag{23}$$

[1]Below we do not pay attention to the factor $1/16\pi^2$ that has to be included in the normalization factor $\mathcal{N}$ in the form $(16\pi^2)^{N+1}$.

This agrees with the result obtained by Diakonov and Petrov (see Eq.(18) of Ref.[1])

Substituting Eq.(23) in Eq.(17) we obtain the evolution operator $Z_{\rm Reg}(R_0R^{\dagger}_{N+1})$ defined by

$$\begin{aligned}
Z_{\rm Reg}(R_{N+1},R_0) &= \lim_{\substack{N\to\infty\\ \delta\to 0}}\sum^{\infty}_{J=0}\sum^{J}_{p=-J}\sum^{J}_{q=-J}(2J+1)\,D^J_{pq}(R_0R^{\dagger}_{N+1})\,[Z^J_{qp}]^{N+1} = \\
&= \lim_{\substack{N\to\infty\\ \delta\to 0}}\sum^{\infty}_{J=0}\sum^{J}_{p=-J}(2J+1)\,D^J_{pp}(R_0R^{\dagger}_{N+1})\Big\{1 - i\,\delta\Big[\frac{(J(J+1)-p^2)}{2I_{\perp}} + \frac{(p+T)^2}{2I_{\parallel}}\Big]\Big\}^{N+1} = \\
&= \lim_{N\to\infty}\sum^{\infty}_{J=0}\sum^{J}_{p=-J}(2J+1)\,D^J_{pp}(R_0R^{\dagger}_{N+1})\Big\{1 - i\,\frac{t_2-t_1}{N+1}\Big[\frac{(J(J+1)-p^2)}{2I_{\perp}} + \frac{(p+T)^2}{2I_{\parallel}}\Big]\Big\}^{N+1}, \quad (24)
\end{aligned}$$

where we have used the definition of δ: $\delta = (t_2-t_1)/(N+1)$ [1].

Taking the limit $N\to\infty$ we get

$$\begin{aligned}
Z_{\rm Reg}(R_{\infty},R_0) &= \sum^{\infty}_{J=0}\sum^{J}_{p=-J}(2J+1)\,D^J_{pp}(R_0R^{\dagger}_{\infty}) \\
&\times\ \exp\Big\{-i(t_2-t_1)\Big[\frac{(J(J+1)-p^2)}{2I_{\perp}} + \frac{(p+T)^2}{2I_{\parallel}}\Big]\Big\}. \quad (25)
\end{aligned}$$

Replacing $R_0\to R_1$ and $R^{\dagger}_{\infty}\to R^{\dagger}_2$ we arrive at the expression

$$\begin{aligned}
Z_{\rm Reg}(R_2,R_1) &= \sum^{\infty}_{J=0}\sum^{J}_{p=-J}(2J+1)\,D^J_{pp}(R_1R^{\dagger}_2) \\
&\times\ \exp\Big\{-i\,(t_2-t_1)\Big[\frac{(J(J+1)-p^2)}{2I_{\perp}} + \frac{(p+T)^2}{2I_{\parallel}}\Big]\Big\}. \quad (26)
\end{aligned}$$

This expression coincides fully with the result obtained by Diakonov and Petrov (see Eq.(22) of Ref.[1]) and reproduces the expansion of the evolution operator (6) (see Eq.(12) of Ref.[1]).

Now taking the limits $I_{\parallel}\to 0$ and $I_{\perp}\to 0$ we have to keep the term $-p = J = T$ [1] and obtain

$$Z_{\rm Reg}(R_2,R_1) = (2T+1)\,D^T_{-T-T}(R_1R^{\dagger}_2)\,\exp\Big[-i(t_2-t_1)\,\frac{T}{2I_{\perp}}\Big]. \quad (27)$$

In the limit $I_{\perp}\to 0$ due to this strongly oscillating factor the r.h.s. of Eq.(27) vanishes. This point has been discussed in detail in Refs.[2,3]. Such a vanishing of the evolution operator confirms the statement in Refs.[2,3] that the path integral representation of the Wilson loop by Diakonov and Petrov is erroneous.

We would like to accentuate that following Diakonov's and Petrov's evaluation of the integral over U we have not taken into account the contribution of the Haar measure. From the Haar measure (21) we should get an additional contribution

$$dU = \frac{d\omega_1 d\omega_2 d\omega_3}{16\pi^2}\Big(\frac{2}{\omega}\,\sin\frac{\omega}{2}\Big)^2 = \frac{d\omega_1 d\omega_2 d\omega_3}{16\pi^2}\Big(1 - \frac{1}{12}\,(\omega^2_1+\omega^2_2+\omega^2_3)\Big). \quad (28)$$

This changes the value Z^J_{qp} in Eq.(23) as follows

$$Z^J_{qp} = \delta_{qp}\Big\{1 - i\,\delta\,\frac{1}{12}\Big(\frac{2}{I_{\perp}} + \frac{1}{I_{\parallel}}\Big) - i\,\delta\Big[\frac{(J(J+1)-p^2)}{2I_{\perp}} + \frac{(p+T)^2}{2I_{\parallel}}\Big]\Big\}. \quad (29)$$

However, it is not the complete set of contributions of order $O(\delta/I_\perp)$ and $O(\delta/I_\parallel)$ to Z^J_{qp}. In order to take into account all of them we have to expand too the exponential $\exp\, f[U]$ keeping the terms of order $\omega_1^4 I_\perp/\delta$, $\omega_2^4 I_\perp/\delta$, $\omega_3^4 I_\parallel/\delta$ and so on. The corresponding expansion of the exponential $\exp\, f[U]$ reads

$$\begin{aligned}\exp\, f[U] \;&=\; \exp\left\{i\frac{I_\perp}{2\delta}\,(\omega_1^2+\omega_2^2)+i\frac{I_\parallel}{2\delta}\,\omega_3^2\right\}\\ &\times\left[1-i\frac{I_\perp}{24\delta}(\omega_1^2+\omega_2^2)^2-i\frac{I_\perp+I_\parallel}{24\delta}(\omega_1^2+\omega_2^2)\,\omega_3^2-i\frac{I_\parallel}{24\delta}\omega_3^4+\ldots\right],\end{aligned}\tag{30}$$

where ellipses denote the terms that have been taken into account in (22).

The contribution of the terms in Eq.(30) changes Z^J_{qp} (29) as follows

$$Z^J_{qp}=\delta_{qp}\left\{1+i\,\delta\,\frac{1}{8}\left(\frac{2}{I_\perp}+\frac{1}{I_\parallel}\right)-i\,\delta\left[\frac{(J(J+1)-p^2)}{2I_\perp}+\frac{(p+T)^2}{2I_\parallel}\right]\right\}.\tag{31}$$

This describes the total contribution of the terms of order $O(\delta/I_\perp)$ and $O(\delta/I_\parallel)$. Due to Eq.(31) the evolution operator reads

$$\begin{aligned}&Z_{\rm Reg}(R_2,R_1)=\exp\left\{i\,(t_2-t_1)\,\frac{1}{8}\left(\frac{2}{I_\perp}+\frac{1}{I_\parallel}\right)\right\}\\ &\times\sum_J^\infty\sum_{p=-J}^{J}(2J+1)\,D^J_{pp}(R_1R_2^\dagger)\,\exp\left\{-\,i\,(t_2-t_1)\left[\frac{(J(J+1)-p^2)}{2I_\perp}+\frac{(p+T)^2}{2I_\parallel}\right]\right\}.\end{aligned}\tag{32}$$

Hence, the evaluation of the path integral (2) with the correct account for all contributions of order $O(\delta/I_\perp)$ and $O(\delta/I_\parallel)$ around the saddle–point, including the contributions of the Haar measure and the terms of order $\omega_1^4 I_\perp/\delta$, $\omega_2^4 I_\perp/\delta$, $\omega_3^4 I_\parallel/\delta$ and so on, leads to a result that differs fully from the expansion (6) derived from the quantum mechanical consideration of $Z_{\rm Reg}(R_2,R_1)$ in terms of eigenfunctions of the axial–symmetric top. This means that the path integral (2) representing the evolution operator $Z_{\rm Reg}(R_2,R_1)$ has no relation to the axial–symmetric top and predicts a completely different energy spectrum than that given by Eq.(7) for the quantum axial–symmetric top. In the limit $I_\parallel\to 0$ and $I_\perp\to 0$ the evolution operator vanishes by virtue of the strongly oscillating factors.

Thus, the only well defined magnitude of the evolution operator is zero. This confirms fully the results obtained in Refs.[2,3] that the evolution operator $Z_{\rm Reg}(R_2,R_1)$ vanishes and the path integral representation of the Wilson loop suggested by Diakonov and Petrov in terms of the evolution operator $Z_{\rm Reg}(R_2,R_1)$ is erroneous. All of these statements are completely applicable to the results discussed by Diakonov and Petrov in their recent manuscript hep–lat/0008004 [4].

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