

Use of Nilpotent weights in Logarithmic Conformal Field Theories *

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Abstract

We show that logarithmic conformal field theories may be derived using nilpotent scale transformation. Using such nilpotent weights we derive properties of LCFT's, such as two and three point correlation functions solely from symmetry arguments. Singular vectors and the Kac determinant may also be obtained using these nilpotent variables, hence the structure of the four point functions can also be derived. This leads to non homogeneous hypergeometric functions. We can construct "superfields" using a nilpotent variable. Using this construct we show that the superfield of conformal weight zero, composed of the identity and the pseudo identity is related to a superfield of conformal dimension two, which comprises of energy momentum tensor and its logarithmic partner. This device also allows us to derive the operator product expansion for logarithmic operators. Finally we consider LCFT's near a boundary.

Keywords: Field theory, Conformal, Logarithmic.

1 Introduction

Logarithmic conformal field theories (LCFT) were first introduced by Gurarie [1] in the context of $c = -2$ conformal field theory (CFT). The difference between an LCFT and a CFT [2], lies in the appearance of logarithmic as well powers in the singular behavior of the correlation functions. In an LCFT, degenerate groups of operators may exist which all have the same conformal weight. They form a Jordan cell under the action of L_0 . In the simplest case a pair of operators exist which transform according to

$$\begin{aligned}\phi(\lambda z) &= \lambda^{-\Delta} \phi(z), \\ \psi(\lambda z) &= \lambda^{-\Delta} [\psi(z) - \phi(z) \ln \lambda].\end{aligned}\tag{1}$$

The correlation function of this pair were derived in [3, 4, 5]. Now using nilpotent variables [6]

$$\begin{aligned}\theta_i^2 &= 0, \\ \theta_i \theta_j &= \theta_j \theta_i,\end{aligned}\tag{2}$$

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and postulating that the conformal weights may have a nilpotent part

$$\Phi(\lambda z, \theta) = \lambda^{-(\Delta+\theta)} \Phi(z, \theta), \quad (3)$$

and the construct $\Phi(z, \theta) = \phi(z) + \theta\psi(z)$, we arrive at equation (1). In order to construct bigger Jordan cells it is sufficient to expand our construct, using $\theta^n = 0$ and

$$\Phi(z, \theta) = \phi_0(z) + \phi_1(z)\theta + \phi_2(z)\theta^2 + \dots + \phi_{n-1}(z)\theta^{n-1}. \quad (4)$$

We will however restrict ourselves to rank two cell, and generalization to higher order cells is straight forward.

2 Two and three point correlation functions

Now consider the following two point function

$$G(z_1, z_2, \theta_1, \theta_2) = \langle \Phi_1(z_1, \theta_1) \Phi_2(z_2, \theta_2) \rangle, \quad (5)$$

invariance under translation and rotation force dependence of $G(z_1, z_2, \theta_1, \theta_2)$ to be on $z_1 - z_2$. Under scale transformation we have

$$G(\lambda(z_1 - z_2), \theta_1, \theta_2) = \lambda^{-(\Delta_1+\theta_1)} \lambda^{-(\Delta_2+\theta_2)} G((z_1 - z_2), \theta_1, \theta_2), \quad (6)$$

and under special conformal transformation $z \rightarrow \frac{z}{1+bz}$ we find that $\Delta_1 = \Delta_2 =: \Delta$ and constant term in the expansion of $G(z_1, z_2, \theta_1, \theta_2)$ vanishes. Then

$$\langle \Phi(z_1, \theta_1) \Phi(z_2, \theta_2) \rangle = \frac{1}{(z_1 - z_2)^{2\Delta+(\theta_1+\theta_2)}} (a_1(\theta_1 + \theta_2) + a_{12}\theta_1\theta_2). \quad (7)$$

Expanding both sides in terms of θ_1 and θ_2 leads to all possible correlation functions including ϕ and ψ

$$\begin{aligned} \langle \phi(z') \phi(z) \rangle &= 0, \\ \langle \psi(z') \psi(z) \rangle &= \frac{1}{(z' - z)^{2\Delta}} (-2a_1 \ln(z' - z) + a_{12}), \\ \langle \psi(z') \phi(z) \rangle &= \frac{a_1}{(z' - z)^{2\Delta}}. \end{aligned} \quad (8)$$

which is consistent with previous works [3].

Let us next consider the three point function

$$G(z_1, z_2, z_3, \theta_1, \theta_2, \theta_3) = \langle \Phi_1(z_1, \theta_1) \Phi_2(z_2, \theta_2) \Phi_3(z_3, \theta_3) \rangle. \quad (9)$$

Again it is clear that if one obtains this correlation function, all the correlators such as $\langle \phi_1 \phi_2 \phi_3 \rangle$, $\langle \phi_1 \phi_2 \psi_3 \rangle$, $\langle \psi_1 \psi_2 \psi_3 \rangle$ can be calculated readily by expanding this correlation function in terms of θ_1 , θ_2 and θ_3 . Also note that these fields may belong to different Jordan cells. The procedure of finding this correlator is just the same as the one we did for the two point function. Like ordinary CFTs, the three point function is obtained up to some constants. Of course, in our case it is found up to a function of θ_i 's, *i.e.*

$$G(z_1, z_2, z_3, \theta_1, \theta_2, \theta_3) = f(\theta_1, \theta_2, \theta_3) z_{12}^{-a_{12}} z_{23}^{-a_{23}} z_{31}^{-a_{31}}, \quad (10)$$

where $z_{ij} = (z_i - z_j)$ and $a_{ij} = \Delta_i + \Delta_j - \Delta_k + (\theta_i + \theta_j - \theta_k)$. There are some constraints on $f(\theta_1, \theta_2, \theta_3)$, but further reduction requires specification the rank of Jordan cell. As we have taken it to be 2, we have

$$f(\theta_1, \theta_2, \theta_3) = \sum_{i=1}^3 C_i \theta_i + \sum_{1 \leq i < j \leq 3} C_{ij} \theta_i \theta_j + C_{123} \theta_1 \theta_2 \theta_3. \quad (11)$$

While symmetry considerations do not rule out a constant term on the right hand side of equation (11), but a consistent OPE forces this constant to vanish [7, 8]. This form together with the equations (10) and (11) leads to correlation functions already obtained in the literature. This is also consistent with the observation that in all the known LCFT's so far the three point function of the first field in the Jordan cell vanishes.

In a similar fashion one can derive the form of the four point functions. But before this is done, we need to address the question of singular vectors in an LCFT.

3 Hilbert Space

Considering the infinitesimal transformation consistent with equation (3) we have

$$\delta \Phi = -\epsilon \partial \Phi - (\Delta + \theta) \Phi \partial \epsilon. \quad (12)$$

This defines the action of the generators of the Virasoro algebra on the primary fields and points to the existence of a highest weight vector with nilpotent eigenvalue

$$\begin{aligned} L_0 |\Delta + \theta\rangle &= (\Delta + \theta) |\Delta + \theta\rangle, \\ L_n |\Delta + \theta\rangle &= 0, \quad n \geq 1. \end{aligned} \quad (13)$$

Nilpotent state $|\Delta + \theta\rangle$ can be considered as

$$\begin{aligned} |\Delta + \theta\rangle &= \Phi(0, \theta) |0\rangle = [\phi(0) + \theta \psi(0)] |0\rangle, \\ &= |\Delta, 0\rangle + \theta |\Delta, 1\rangle. \end{aligned} \quad (14)$$

It can be easily seen that the law written in equation (14), leads to the well known equations

$$\begin{aligned} L_0 |\Delta, 0\rangle &= \Delta |\Delta, 0\rangle, \\ L_0 |\Delta, 1\rangle &= \Delta |\Delta, 1\rangle + |\Delta, 0\rangle. \end{aligned} \quad (15)$$

Now we define an out state in an LCFT

$$\begin{aligned} \langle \Delta + \theta | &= \langle 0 | \Phi^\dagger(0, \theta), \\ &= \langle 0 | [\phi(0) + \theta \psi(0)]^\dagger, \end{aligned} \quad (16)$$

where dagger means adjoint of fields, just the same as CFT. So

$$\langle \Delta + \theta | = \lim_{\bar{z} \rightarrow \infty} \langle 0 | \left(\phi(\bar{z}) \bar{z}^{2\Delta} + \bar{\theta} [\psi(\bar{z}) + \ln \bar{z}^2 \phi(\bar{z})] \bar{z}^{2\Delta} \right), \quad (17)$$

which together 'in' state defined in equation (14) and using form of the two point correlation functions in equation (8) leads to

$$\langle \Delta + \theta | \Delta + \theta \rangle = \theta + \bar{\theta} + d \bar{\theta} \theta, \quad (18)$$

where \vec{n} is a_{12} if we normalize a_1 to one. By expanding left hand side of the last equation, we find

$$\langle \Delta, 0 | \Delta, 0 \rangle = 0, \quad \langle \Delta, 0 | \Delta, 1 \rangle = \langle \Delta, 1 | \Delta, 0 \rangle = 1, \quad \langle \Delta, 1 | \Delta, 1 \rangle = d. \quad (19)$$

In addition to these highest weight states, there are descendants which can be obtained by applying $L_{-\vec{n}}$'s on the highest weight vectors

$$|\Delta + n_1 + n_2 + \dots + n_k + \theta\rangle = L_{-n_1} L_{-n_2} \dots L_{-n_k} |\Delta + \theta\rangle. \quad (20)$$

Most general state of Hilbert space at level n can be constructed as follows

$$\begin{aligned} |\chi_{\Delta,c}^{(n)}(\theta)\rangle &= \sum_{|\vec{n}|=n} b^{\vec{n}} L_{-\vec{n}} |\Delta + \theta\rangle, \\ &= \sum_{\{n_1+n_2+\dots+n_k=n\}} b^{(n_1, n_2, \dots, n_k)} L_{-n_1} L_{-n_2} \dots L_{-n_k} |\Delta + \theta\rangle. \end{aligned} \quad (21)$$

As an application of the above definitions we compute the character formula

$$\chi_{\Delta}(\theta, \bar{\theta}) = \sum_N \langle N + \Delta + \theta | \eta^{L_0 - \frac{c}{24}} | N + \Delta + \theta \rangle, \quad (22)$$

which by equation (20) simplifies to

$$\chi_{\Delta}(\theta, \bar{\theta}) = \eta^{\Delta + \theta - \frac{c}{24}} \sum_N \eta^N p(N, \theta) \langle \Delta + \theta | \Delta + \theta \rangle. \quad (23)$$

Writing $p(N, \theta) = p_0(N) + \theta p_1(N)$ we obtain four characters

$$\begin{aligned} \chi_{\Delta}^{(\phi, \phi)} &= 0, \\ \chi_{\Delta}^{(\phi, \psi)} &= \chi_{\Delta}^{(\psi, \phi)} = \eta^{\Delta - \frac{c}{24}} \sum_N \eta^N p_0(N), \\ \chi_{\Delta}^{(\psi, \psi)} &= \eta^{\Delta - \frac{c}{24}} \sum_N \eta^N [p_1(N) + (d + \ln \eta) p_0(N)]. \end{aligned} \quad (24)$$

Appearance of logarithms in character formula have been discussed in [9, 10].

4 Singular Vectors in LCFT

We define a singular vector at level n by $|\chi_{\Delta,c}^{(n)}\rangle_S$ as a vector that is orthogonal to all vectors in its level

$$\langle \chi_{\Delta',c}^{(n)} | \chi_{\Delta,c}^{(n)} \rangle_S = 0. \quad (25)$$

As a result it has a zero norm and is orthogonal to any vector at higher levels. According to equation (20), last condition is equivalent to

$$\langle \Delta' + \theta | L_{\vec{n}'} | \chi_{\Delta,c}^{(n)} \rangle_S = 0, \quad \forall \vec{n}' : |\vec{n}'| = n. \quad (26)$$

The number of $L_{\vec{n}'}$'s is $p(n)$, the number of partitions of the integer n . So equation (26) is equivalent to $p(n)$ equation with $p(n)$ unknown coefficients. By putting equation (21) in the last equation we have

$$\langle \Delta' + \theta | L_{\vec{n}'} | \chi_{\Delta,c}^{(n)} \rangle_S = \sum_{|\vec{n}|=n} b^{\vec{n}} \langle \Delta' + \theta | L_{\vec{n}'} L_{-\vec{n}} | \Delta + \theta \rangle = 0, \quad \forall \vec{n}' : |\vec{n}'| = n, \quad (27)$$

Since $L_k|\Delta + \theta\rangle = 0$ (for $k \geq 1$) and using Virasoro algebra we find

$$L_{\vec{n}'} L_{-\vec{n}} |\Delta + \theta\rangle = \sum_{m=0} \alpha_{\vec{n}', \vec{n}}^m(c) L_0^m |\Delta + \theta\rangle, \quad (28)$$

where coefficients $\alpha_{\vec{n}', \vec{n}}^m(c)$ are numbers or constants dependent on central charge c . By putting this expression in equation (27)

$$\sum_{|\vec{n}|=n} b^{\vec{n}} \left(\sum_{m=0} \alpha_{\vec{n}', \vec{n}}^m(c) (\Delta + \theta)^m \right) \langle \Delta' + \theta | \Delta + \theta \rangle = 0, \quad \forall \vec{n}' : |\vec{n}'| = n, \quad (29)$$

where $(\Delta + \theta)^m$ is eigenvalue of $L_0^m |\Delta + \theta\rangle$. Non zero solutions for $b^{\vec{n}}$'s leads to Kac determinant in LCFT

$$\det \left(\sum_{m=0} \alpha_{\vec{n}', \vec{n}}^m(c) (\Delta + \theta)^m \right) = 0 \quad \forall |\vec{n}'| = n, |\vec{n}| = n. \quad (30)$$

Singular vectors exist for those values of Δ and c that Kac determinant is zero just the same as CFT.

In the following we determine the null vectors at level 2 for a Jordan cell of rank 2. We thus have

$$|\chi_{\Delta, c}^{(2)}(\theta)\rangle_S = \left(b^{(1,1)} L_{-1}^2 + b^{(2)} L_{-2} \right) |\Delta + \theta\rangle. \quad (31)$$

Equation (29) is equivalent to

$$\begin{pmatrix} 4(2\Delta^2 + \Delta) + 4(4\Delta + 1)\theta & 6(\Delta + \theta) \\ 6(\Delta + \theta) & 4(\Delta + \theta) + \frac{c}{2} \end{pmatrix} \begin{pmatrix} b^{(1,1)} \\ b^{(2)} \end{pmatrix} = 0, \quad (32)$$

which leads to

$$\Delta \left[8\Delta^2 - 5\Delta + c(\Delta + \frac{1}{2}) \right] + \left[24\Delta^2 - 10\Delta + c(2\Delta + \frac{1}{2}) \right] \theta = 0. \quad (33)$$

So Kac determinant vanishes for $(\Delta, c) = (0, 0), (-\frac{5}{4}, 25), (\frac{1}{4}, 1)$. Because of homogeneity of the equation (32) at least one of the coefficients is arbitrary. Therefore we write

$$b^{(1,1)} = 3, \quad b^{(2)} = -[4(\Delta + \theta) + 2], \quad (34)$$

and since for a Jordan cell of rank 2, $|\chi_{\Delta, c}^{(2)}(\theta)\rangle_S = |\chi_{\Delta, c}^{(2)}(0)\rangle_S + \theta |\chi_{\Delta, c}^{(2)}(1)\rangle_S$ we find

$$|\chi_{\Delta, c}^{(2)}(1)\rangle_S = \left[3L_{-1}^2 - 2(2\Delta + 1)L_{-2} \right] |\Delta, 1\rangle - 4L_{-2} |\Delta, 0\rangle, \quad (35)$$

for $\Delta = \frac{1}{4}, -\frac{5}{4}$ as a logarithmic singular vector. $\Delta = 0$ does not lead to a logarithmic singular vector because $|\Delta, 1\rangle$ does not appear in $|\chi_{\Delta, c}^{(2)}(1)\rangle_S$. By the same technique logarithmic singular vectors can be obtained at higher levels which is consistent with [11].

5 Roots of Kac determinant in LCFT

In the previous section we saw that in an LCFT singular vectors exist for those values of (Δ, c) which Kac determinant vanishes just the same as CFT. If we compare equations (28)

and (30) with their counterparts in CFT, we see that Δ in CFT has been replaced by $\Delta + \theta$. One thus concludes that in an LCFT and at level n Kac determinant has the form

$$\det_n(c, \Delta + \theta) = \prod_{r,s=1; 1 \leq rs \leq n}^n (\Delta + \theta - \Delta_{r,s}(c))^{p(n-rs)}, \quad (36)$$

where $\Delta_{r,s}(c)$ is

$$\Delta_{r,s}(c) = \frac{1}{96} \left[(r+s)\sqrt{1-c} + (r-s)\sqrt{25-c} \right]^2 - \frac{1-c}{24}. \quad (37)$$

and $p(n-rs)$ is the number of partitions of the integer $n-rs$. In an ordinary CFT when the Kac determinant vanishes we have a singular vector. In our case and for a rank 2 Jordan cell the condition of vanishing $\det_n(c, \Delta + \theta)$ are:

(i) If $p(n-rs) \geq 2$ for some r and s , Kac determinant vanishes for all values of Δ that satisfy in $\Delta = \Delta_{r,s}(c)$.

(ii) If $p(n-rs) = 1$ for some pairs of $(r, s) = (r_1, s_1), (r_2, s_2), \dots$ we can have vanishing determinant if at least $\Delta = \Delta_{r_i, s_i}(c) = \Delta_{r_j, s_j}(c)$. In this case unlike (i) we are limited to special values for Δ and c which last condition is held. As an example we consider Kac determinant at level 2. Since

$$p(2-rs) = \begin{cases} 1 & r=1, s=1 \\ 1 & r=1, s=2 \text{ or } r=2, s=1 \end{cases}, \quad (38)$$

all of them are cases of (ii). So

$$\begin{aligned} \Delta_{1,2} &= \Delta_{2,1} \Rightarrow \begin{cases} c = 1, \Delta = \frac{1}{4} \\ c = 25, \Delta = -\frac{5}{4}, \end{cases} \\ \Delta_{1,2} &= \Delta_{1,1} \Rightarrow c = 0, \Delta = 0. \end{aligned} \quad (39)$$

This approach can be extended easily to higher levels and Jordan cells of bigger rank. These results are consistent with those of [11].

6 Four point functions

To obtain further information about the theory with which we are concerned, such as surface critical exponents, OPE structure, monodromy group and etc. one should compute four point correlation functions. In the language we have developed so far, the four point correlation functions depend on four θ 's in addition to the coordinates of points

$$\begin{aligned} G(z_1, z_2, z_3, z_4, \theta_1, \theta_2, \theta_3, \theta_4) &= \langle \Phi_1(z_1, \theta_1) \dots \Phi_4(z_4, \theta_4) \rangle \\ &= f(\eta, \theta_1, \theta_2, \theta_3, \theta_4) \prod_{1 \leq i < j \leq 4} z_{ij}^{\mu_{ij}}. \end{aligned} \quad (40)$$

where

$$\mu_{ij} = \frac{1}{3} \sum_{k=1}^4 (\Delta_k + \theta_k) - (\Delta_i + \theta_i) - (\Delta_j + \theta_j), \quad \eta = \frac{z_{41} z_{23}}{z_{43} z_{21}}. \quad (41)$$

This form is invariant under all conformal transformations. Although there is no other restrictions on θ due to symmetry considerations, but because of OPE structure, the four-point function $\langle \phi \phi \phi \phi \rangle$ should vanish [7, 8], that is, the term independent of θ_i 's in θ is

zero. Thus in addition to the differential equations which should be satisfied by \mathbf{G} , one must impose the condition $\langle \phi\phi\phi\phi \rangle = 0$ on the solution derived.

If there is a singular vector in the theory, a differential equation can be derived for $f(\eta, \theta_1, \theta_2, \theta_3, \theta_4)$. Let us consider a theory which contains a singular vector at level two. As seen in previous section the singular vector in such a theory is

$$\chi^{(2)}(z_4, \theta_4) = [3L_{-1}^2 - (2(2\Delta_4 + 1) + 4\theta_4)L_{-2}] \Phi_4(z_4, \theta_4). \quad (42)$$

As this vector is orthogonal to all the other operators in the Verma module

$$\langle \Phi_1 \Phi_2 \Phi_3 \chi^{(2)} \rangle = 0, \quad (43)$$

one immediately is led to the differential equation

$$\left[3\partial_{z_4}^2 - (2(2\Delta_4 + 1) + 4\theta_4) \sum_{i=1}^3 \frac{\Delta_i + \theta_i}{(z_i - z_4)^2} - \frac{\partial_{z_i}}{z_i - z_4} \right] \langle \Phi_1 \Phi_2 \Phi_3 \Phi_4 \rangle = 0. \quad (44)$$

By sending points to $z_1 = 0$, $z_2 = 1$, $z_3 \rightarrow \infty$ and $z_4 = \eta$, we find

$$\begin{aligned} \partial_\eta^2 f + \left[\frac{2\mu_{14}}{\eta} - \frac{2\mu_{24}}{1-\eta} - \alpha \frac{2\eta-1}{\eta(1-\eta)} \right] \partial_\eta f + \left[\frac{\mu_{14}(\mu_{14}-1)}{\eta^2} + \frac{\mu_{24}(\mu_{24}-1)}{(1-\eta)^2} \right. \\ \left. - \frac{2\mu_{14}\mu_{24}}{\eta(1-\eta)} - \frac{\alpha(\Delta_1 + \theta_1 - \mu_{14})}{\eta^2} - \frac{\alpha(\Delta_2 + \theta_2 - \mu_{24})}{(1-\eta)^2} + \frac{\alpha\mu_{12}}{\eta(1-\eta)} \right] f = 0, \end{aligned} \quad (45)$$

where $\alpha = \frac{1}{3} [2(2\Delta_4 + 1) + 4\theta_4]$. Renormalizing using

$$H(\eta, \theta_1, \theta_2, \theta_3, \theta_4) = \eta^{-\beta_1 + \mu_{14}} (1-\eta)^{-\beta_2 + \mu_{24}} f(\eta, \theta_1, \theta_2, \theta_3, \theta_4), \quad (46)$$

we find that β_i satisfy

$$\beta_i(\beta_i - 1) + \alpha(\beta_i - \Delta_i - \theta_i) = 0, \quad i = 1, 2 \quad (47)$$

and H satisfies the hypergeometric equation

$$\eta(1-\eta) \frac{d^2 H}{d\eta^2} + [c - (a+b+1)\eta] \frac{dH}{d\eta} - abH = 0, \quad (48)$$

where

$$\begin{aligned} ab &= (\beta_1 + \beta_2)(\beta_1 + \beta_2 + 2\alpha - 1) + \alpha(\Delta_4 - \Delta_3 + \theta_4 - \theta_3), \\ a + b + 1 &= 2(\beta_1 + \beta_2 + \alpha), \\ c &= 2\beta_1 + \alpha. \end{aligned} \quad (49)$$

We can now write down the solution of equation (48) in terms of the hypergeometric series

$$H(a, b, c; \eta) = K(\theta_1, \theta_2, \theta_3, \theta_4) h(a, b, c; \eta), \quad (50)$$

where

$$h(a, b, c; \eta) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} \eta^n, \quad (51)$$

with $(x)_n = x(x+1)\dots(x+n-1)$, $(x)_0 = 1$. Also

$$K(\theta_1, \theta_2, \theta_3, \theta_4) = \sum_{i=1}^4 k_i \theta_i + \sum_{1 \leq i < j \leq 4} k_{ij} \theta_i \theta_j + \sum_{1 \leq i < j < k \leq 4} k_{ijk} \theta_i \theta_j \theta_k + k_{1234} \theta_1 \theta_2 \theta_3 \theta_4. \quad (52)$$

Note that the coefficients in equation (51) contain nilpotent terms. Therefore equation (51) actually describes more than one solution. This expansion results in 16 functions. This is natural because as seen in the case of two and three point functions, inside the correlator (40) there exist sixteen distinct correlation functions. Of course, if all the fields belong to the same Jordan cell, only four of them may be independent and the rest are related by crossing symmetry. Also, one of them which only contains ϕ fields, vanishes and hence only three independent functions remain. The form of these functions may be obtained by expanding equation (50) and collecting powers of θ_i 's. Note that expanding equation (48) leads to sixteen differential equations. The general form of these equations is given in appendix. As an example we solve equation (48) for the special case of $\Delta_1 = \Delta_2 = \Delta_3 = \Delta_4 = \frac{1}{4}$. In this case one of the solutions of equation (47) is

$$\begin{aligned}\beta_1 &= \frac{1}{2} + \theta_1 - \frac{1}{3}\theta_4 + \frac{2}{3}\theta_1\theta_4, \\ \beta_2 &= \frac{1}{2} + \theta_2 - \frac{1}{3}\theta_4 + \frac{2}{3}\theta_2\theta_4.\end{aligned}\tag{53}$$

We then find from equation (49)

$$\begin{aligned}a &= 2 + \theta_1 + \theta_2 + \theta_3 + \theta_4 + \frac{2}{3}(\theta_1 + \theta_2 + \theta_3)\theta_4, \\ b &= 1 + \theta_1 + \theta_2 - \theta_3 + \frac{1}{3}\theta_4 + \frac{2}{3}(\theta_1 + \theta_2 - \theta_3)\theta_4, \\ c &= 2 + 2\theta_1 + \frac{2}{3}\theta_4 + \frac{4}{3}\theta_1\theta_4.\end{aligned}\tag{54}$$

Finally from equations (40), (46) and (50) we get expressions for the various four point functions. For example

$$\langle \psi(z_1)\phi(z_2)\phi(z_3)\phi(z_4) \rangle = k_1 \eta^{\frac{2}{3}}(1-\eta)^{\frac{2}{3}} h_0(\eta) \prod_{1 \leq i < j \leq 4} z_{ij}^{-\frac{1}{6}}.\tag{55}$$

Where $h_0(\eta)$ is given in the appendix. The other four point functions having 2, 3 or 4 ψ 's can be calculated in the same way. In these functions, depending on how many ψ 's are present in the correlators, different functions appear on the right hand side. If there is no ψ in the correlator, the correlator is zero, if there is one $\psi(z_i)$, only $H_i(=k_i h_0(\eta))$ appears, if there are two ψ 's, H_i and H_{ij} appear, and so on. The situation is just the same for the two or three point functions. For example the two point function $\langle \phi(z)\psi(0) \rangle$ is written in terms of the function $z^{-2\Delta}$ and in the correlator $\langle \psi(z)\psi(0) \rangle$ there exist both functions $z^{-2\Delta}$ and $z^{-2\Delta} \ln z$.

7 Energy momentum tensor

Two central operators in a CFT are the energy momentum tensor, T with conformal weight $\Delta=2$ and the identity operator, I with conformal weight $\Delta=0$. However, T is a secondary field of the identity, because $L_{-2}I = T$.

In an LCFT degenerate operators exist which form a Jordan cell under conformal transformation. This holds true for the identity as well. The existence of a logarithmic identity operator has been discussed by a number of authors [1, 12, 13].

Now consider the identity operator I and its logarithmic partner ω . According to equation (1) this pair transforms as

$$\begin{aligned}\Omega(\lambda z) &= \Omega(z), \\ \omega(\lambda z) &= \omega(z) - \Omega(z) \ln \lambda.\end{aligned}\tag{56}$$

So according to our convention, we define a primary field $\Phi_0(z, \theta)$

$$\Phi_0(z, \theta) = \Omega(z) + \theta\omega(z), \quad (57)$$

with conformal weight θ . Under scaling, $\Phi_0(z, \theta)$ transforms according to equation (3). Thus we have

$$\begin{aligned} L_0\Omega(z) &= 0, \\ L_0\omega(z) &= \Omega(z), \end{aligned} \quad (58)$$

where was first observed in $c = -2$ theory by Gurarie [1].

Here we wish to find the field $T(z, \theta)$ with conformal weight $2 + \theta$ which is a secondary of $\Phi_0(z, \theta)$ in the sense

$$L_{-2}\Phi_0(z, \theta) = T(z, \theta). \quad (59)$$

By writing $T(z, \theta) = T_0(z) + \theta t(z)$ and since $L_0T(z, \theta) = (2 + \theta)T(z, \theta)$ we have

$$\begin{aligned} L_0T_0(z) &= 2T_0(z), \\ L_0t(z) &= 2t(z) + T_0(z). \end{aligned} \quad (60)$$

This points to the existence of an extra energy momentum tensor [12, 13, 14]. By applying L_2 on both sides of equation (59) we have

$$\begin{aligned} L_2T_0(z) &= \frac{c}{2}\Omega(z), \\ L_2t(z) &= \frac{c}{2}\omega(z) + 4\Omega(z). \end{aligned} \quad (61)$$

The first of this pair exists in an ordinary CFT, so $T_0(z)$ leads to the Virasoro algebra, while $t(z)$ must leads to a new algebra [14]. We now attempt at finding the OPE of the $T_0(z')$ with $\psi(z)$ and extra energy momentum tensor, $t(z)$. Because of OPE's invariance under scaling and according to our convention it is sufficient to change conformal weight of each field to $\Delta + \theta$. Consider the following OPE

$$T_0(z')\Phi(z, \theta) = \frac{\Delta + \theta}{(z' - z)^2}\Phi(z, \theta) + \frac{\partial_z\Phi(z, \theta)}{z' - z} + \dots \quad (62)$$

This relation leads to the familiar OPE for $T_0(z')\phi(z)$ and a new OPE

$$T_0(z')\psi(z) = \frac{\phi(z) + \Delta\psi(z)}{(z' - z)^2} + \frac{\partial_z\psi(z)}{z' - z} + \dots \quad (63)$$

Also

$$T_0(z')T(z, \theta) = \frac{\frac{c(\theta)}{2}\Phi_0(z, \theta)}{(z' - z)^4} + \frac{2 + \theta}{(z' - z)^2}T(z, \theta) + \frac{\partial_zT(z, \theta)}{z' - z} + \dots, \quad (64)$$

where $c(\theta) = c_1 + \theta c_2$. Again we obtain two OPE, one of them is $T_0(z')T_0(z)$ which is known from CFT and the other is

$$T_0(z')t(z) = \frac{\frac{c_1}{2}\omega(z) + \frac{c_2}{2}\Omega(z)}{(z' - z)^4} + \frac{T_0(z) + 2t(z)}{(z' - z)^2} + \frac{\partial_zt(z)}{z' - z} + \dots \quad (65)$$

The emergence of an extra energy momentum tensor and central charge have been noticed by Gurarie and Ludwig [14], although our approach is very different. Recently it has been argued [15] that in theories with zero central charge ($c = 0$) $\mathbf{1}$, the logarithmic partner of T_0 is not a descendant of any other field. So these theories can have non degenerate

vacua. Such theories will not fit into the framework presented here. Theories with non zero central charge ($c \neq 0$) behave very differently. In these theories there exist a logarithmic partner for T_0 only if the vacua is degenerate and so \mathbb{I} is a descendant field [15].

It is worth noting that equations (56) imply that $\langle \Omega \rangle$ vanishes whereas $\langle \omega \rangle = 1$. This immediately results in the vanishing of $\langle T_0 T_0 \rangle$, even though the central charge may not vanish.

8 Boundary

Let us now consider the problem of LCFT near a boundary. As shown in [16] in an ordinary CFT, if the real axis is taken to be the boundary, with certain boundary condition that $T = \bar{T}$ on the real axis, the differential equation satisfied by n-point function near a boundary are the same as the differential equations satisfied by 2n-point function in the bulk. This trick may be used in order to derive correlations of an LCFT near a boundary [10, 17]. Here we rederive the same results using the nilpotent formalism. Again we consider an LCFT with a rank 2 Jordan cell. First we find the one point functions of this theory. By applying L_0, L_{+1} on the correlators, one obtains

$$\begin{aligned} (\partial_z + \partial_{\bar{z}}) \langle \Phi(z, \bar{z}, \theta) \rangle &= 0, \\ (z\partial_z + \bar{z}\partial_{\bar{z}} + 2(\Delta + \theta)) \langle \Phi(z, \bar{z}, \theta) \rangle &= 0, \\ (z^2\partial_z + \bar{z}^2\partial_{\bar{z}} + 2z(\Delta + \theta) + 2\bar{z}(\Delta + \theta)) \langle \Phi(z, \bar{z}, \theta) \rangle &= 0. \end{aligned} \quad (66)$$

In these equations, we have assumed that Φ is a scalar field so that $\Delta = \bar{\Delta}$. The first equation states $\langle \Phi(z, \bar{z}, \theta) \rangle$ is a function of $z - \bar{z}$ and the solution to the second equation is

$$\langle \Phi(y, \theta) \rangle = \frac{f(\theta)}{y^{2(\Delta + \theta)}}, \quad (67)$$

where $y = z - \bar{z}$. The third line of equation (66) is automatically satisfied by this solution. Expanding $f(\theta)$ as $a + b\theta$ one finds

$$\langle \Phi(y, \theta) \rangle = \frac{a}{y^{2\Delta}} + \frac{\theta}{y^{2\Delta}}(b - 2a \ln y). \quad (68)$$

As the field $\Phi(y, \theta)$ is decomposed to $\phi(y) + \theta\psi(y)$ one can read the one-point functions $\langle \phi(y) \rangle$ and $\langle \psi(y) \rangle$ from the equation (68)

$$\begin{aligned} \langle \phi(y) \rangle &= \frac{a}{y^{2\Delta}}, \\ \langle \psi(y) \rangle &= \frac{1}{y^{2\Delta}}(b - 2a \ln y). \end{aligned} \quad (69)$$

To go further, one can investigate the two-point function

$$G(z_1, \bar{z}_1, z_2, \bar{z}_2, \theta_1, \theta_2) = \langle \Phi(z_1, \bar{z}_1, \theta_1) \Phi(z_2, \bar{z}_2, \theta_2) \rangle, \quad (70)$$

in the same theory. Invariance under the action of L_{-1} implies

$$(\partial_{z_1} + \partial_{\bar{z}_1} + \partial_{z_2} + \partial_{\bar{z}_2})G = 0. \quad (71)$$

The most general solution of this equation is $G = G(y_1, y_2, x_1, x_2, \theta_1, \theta_2)$, where $y_1 = z_1 - \bar{z}_1$, $y_2 = z_2 - \bar{z}_2$, $x = x_2 - x_1$ and $x_i = z_i + \bar{z}_i$. By invariance under the action of L_0 we should have

$$\left[y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} + x \frac{\partial}{\partial x} + 2(\Delta + \theta_1) + 2(\Delta + \theta_2) \right] G = 0, \quad (72)$$

which implies

$$G = \frac{1}{x^{4\Delta+2\theta_1+2\theta_2}} f(\alpha_1, \alpha_2, \theta_1, \theta_2), \quad (73)$$

where $\alpha_1 = \frac{y_1}{x}$ and $\alpha_2 = \frac{y_2}{x}$. Now consider the action of L_1 on G

$$\begin{aligned} (x_1 + x_2) \left[y_1 \frac{\partial}{\partial y_1} + y_2 \frac{\partial}{\partial y_2} + x \frac{\partial}{\partial x} + 2(\Delta + \theta_1) + 2(\Delta + \theta_2) \right] G \\ + \left[xy_1 \frac{\partial}{\partial y_1} - xy_2 \frac{\partial}{\partial y_2} + (y_1^2 - y_2^2) \frac{\partial}{\partial x} + 2x(\theta_1 - \theta_2) \right] G = 0. \end{aligned} \quad (74)$$

The first bracket is zero because of equation (72). Substituting the solution (73) in equation (74) the function f satisfies

$$\left(\alpha_1 + \frac{\alpha_1}{\alpha_1^2 - \alpha_2^2} \right) \frac{\partial f}{\partial \alpha_1} + \left(\alpha_2 + \frac{\alpha_2}{\alpha_2^2 - \alpha_1^2} \right) \frac{\partial f}{\partial \alpha_2} + 2 \left(2\Delta + \theta_1 + \theta_2 + \frac{\theta_1 - \theta_2}{\alpha_1^2 - \alpha_2^2} \right) f = 0. \quad (75)$$

The most general solution of above is

$$f(\alpha_1, \alpha_2, \theta_1, \theta_2) = \frac{1}{(\alpha_1 \alpha_2)^{2\Delta+\theta_1+\theta_2}} \left(\frac{\alpha_2}{\alpha_1} \right)^{\theta_1-\theta_2} g \left(\frac{1 + \alpha_1^2 + \alpha_2^2}{\alpha_1 \alpha_2}, \theta_1, \theta_2 \right), \quad (76)$$

where g is an arbitrary function. So the two point function G is found up to an unknown function

$$\langle \Phi(z_1, \bar{z}_1, \theta_1) \Phi(z_2, \bar{z}_2, \theta_2) \rangle = \frac{1}{(y_1 y_2)^{2\Delta+\theta_1+\theta_2}} \left(\frac{y_1}{y_2} \right)^{\theta_1-\theta_2} g \left(\frac{x^2 + y_1^2 + y_2^2}{y_1 y_2}, \theta_1, \theta_2 \right), \quad (77)$$

which is the same as the solution obtained in [17].

9 Extension of nilpotent variable to a four component superfield

In section 1 we introduced a two component superfield $\Phi(z, \theta) = \phi(z) + \theta\psi(z)$ which its components forms a Jordan cell under conformal transformations. In this section we extend our superfield to a four component one, exploiting a grassman variable η

$$\Phi(z, \eta) = \phi(z) + \bar{\alpha}(z)\eta + \bar{\eta}\alpha(z) + \bar{\eta}\eta\psi(z). \quad (78)$$

Here a fermionic field $\alpha(z)$ with the same conformal dimension as $\phi(z)$ has been added to the multiplet, and the nilpotent variable η has been interpreted as $\bar{\eta}\eta$. Note that both α and $\bar{\alpha}$ live in the holomorphic section of the theory. Now we observe that $\Phi(z, \eta)$ has the following transformation law under scaling

$$\Phi(\lambda z, \eta) = \lambda^{-(\Delta+\bar{\eta}\eta)} \Phi(z, \eta). \quad (79)$$

To find out what this scaling law means, one should expand both sides of equation (79) in terms of η and $\bar{\eta}$. Doing this and comparing the two sides of equation (79), it is found that $\phi(z)$ and $\psi(z)$ transform as equation (1) and α and $\bar{\alpha}$ are ordinary fields of dimension Δ . The appearance of such fields has been proposed by Kausch [18], within the $c = -2$ theory.

In previous sections we saw that using this structure one can derive most of the properties of LCFTs. Let us first consider the identity operator. In a two component superfield, in

addition to the ordinary unit operator $\mathbb{1}$, with the property $\Omega S = S$ for any field S , and its logarithmic partner, which we denoted by $\bar{\mathbb{1}}$, there exist two other fields with zero conformal dimension for the multiplet to complete

$$\Phi_0(z, \eta) = \Omega + \bar{\xi}(z)\eta + \bar{\eta}\xi(z) + \bar{\eta}\eta\omega(z), \quad (80)$$

with the property that $\langle \Phi_0(z, \eta) \rangle = \bar{\eta}\eta$. Note that LCFTs have the curious property that $\langle \Omega \rangle = 0$. The existence of these fields and their OPEs have been discussed by Kausch [18]. Kausch takes the ghost action of

$$S = \frac{1}{\pi} \int d^2z (a\bar{\partial}b + \bar{a}\partial\bar{b}), \quad (81)$$

with $c = -2$. Instead of the degrees of freedom a and b , he takes two fields χ^α on equal footing, where $\alpha = 1, 2$. Taking the Laurent expansion

$$\chi^\alpha = \sum_n \chi_n^\alpha z^{-n-1}, \quad (82)$$

he observes that χ_0^α plays the role of a ladder operator for the multiplet defined in equation (80)

$$\begin{aligned} \chi_0^\alpha \Omega &= 0, \\ \chi_0^\alpha \omega &= \xi^\alpha, \\ \chi_0^\alpha \xi^\beta &= d^{\alpha\beta} \Omega, \end{aligned} \quad (83)$$

where $d^{\alpha\beta}$ is a totally antisymmetric matrix. Note that over the vacuum multiplet we have

$$\chi_0^\alpha \chi_0^\beta = d^{\alpha\beta} L_0. \quad (84)$$

An interesting question is whether it is possible to write the OPEs of the fields in the multiplet in terms of our fields $\Phi_0(z, \eta)$. The OPE of Φ_0 with itself has to give back Φ_0 to lowest order

$$\Phi_0(z_1, \eta_1) \Phi_0(z_2, \eta_2) \sim (z_1 - z_2)^{\bar{\eta}_1 \eta_2 + \bar{\eta}_2 \eta_1} \Phi_0(z_1, \eta_3), \quad (85)$$

where $\eta_3 = \eta_1 + \eta_2$. To see why this OPE has been proposed, one can look at the behaviour of both sides of equation (85) under scaling transformation. Under the transformation $z \rightarrow \lambda z$ the LHS of equation (85) transforms as $LHS \rightarrow \lambda^{-(\bar{\eta}_1 \eta_1 + \bar{\eta}_2 \eta_2)} LHS$, (See the transformation law (79)) and the RHS of equation (85) transforms as $RHS \rightarrow \lambda^{\bar{\eta}_1 \eta_2 + \bar{\eta}_2 \eta_1 - \bar{\eta}_3 \eta_3} RHS$, which are the same. So the OPE proposed here seems reasonable.

The OPEs of the primary fields involved can then be extracted by expanding in powers of η . Taking $z_2 = 0$ and renaming $z_1 = z$ the OPEs of these fields, aside from trivial OPEs of Ω , are

$$\begin{aligned} \bar{\xi}(z)\xi(0) &= 2i(\omega + \Omega \log z), \\ \xi(z)\omega(0) &= -\xi \log z, \\ \omega(z)\omega(0) &= -\log z(2\omega + \Omega \log z). \end{aligned} \quad (86)$$

These results are consistent with those of [18]. An obvious generalization of (85) leads to

$$\Phi_a(z_1, \eta_1) \Phi_b(z_2, \eta_2) \sim (z_1 - z_2)^{\Delta_c - \Delta_b - \Delta_a + \bar{\eta}_1 \eta_2 + \bar{\eta}_2 \eta_1} C_{abc} \Phi_c(z_1, \eta_3). \quad (87)$$

for any three primary fields with arbitrary conformal dimensions. An immediate implication of these equations is that the OPE of two Φ fields will be the first components of the

multiplets such as an Ω or another Φ , thus the expectation values of an arbitrary string $\langle \phi_1 \phi_2 \cdots \phi_n \rangle$ will vanish in any LCFT [8].

A consequence of existence of Φ_0 is the existence of the energy momentum multiplet

$$T(z, \eta) = L_{-2}\Phi_0(z, \eta). \quad (88)$$

We thus observe that we have three partners for the energy momentum tensor $T(z, \eta) = T_0(z) + \bar{\eta}\zeta(z) + \bar{\zeta}(z)\eta + \bar{\eta}\eta t(z)$, as discussed by Gurarie and Ludwig [14]. In their paper, they have considered some specific theories and have suggested some OPEs for different fields of energy momentum tensor multiplet. However, getting some insight from the OPEs we have written so far, we propose

$$T(z, \eta_1)T(0, \eta_2) = z^{\bar{\eta}_1\eta_2 + \bar{\eta}_2\eta_1} \left[\frac{\frac{c(\eta_3)}{2}\Phi_0(\eta_3)}{z^4} + \frac{d(\eta_3)\chi(\eta_3)}{z^3} + \frac{e(\eta_3)T(\eta_3)}{z^2} + \frac{f(\eta_3)\partial_z T(\eta_3)}{z} \right]. \quad (89)$$

There are some points in this OPE which should be clarified. First of all, in contrast with the OPE of ordinary energy momentum tensor, there exists a $1/z^3$ term. In the ordinary OPE, this term vanishes because $L_{-1}\Omega = 0$. Such a reasoning can not be extended to the case of other fields of the energy momentum multiplet, that is, one can not assume that $L_{-1}\omega$, $L_{-1}\xi$ and $L_{-1}\bar{\xi}$ all vanish. In equation (89) we have denoted $L_{-1}\Phi_0(\eta)$ by $\chi(\eta) = \bar{\eta}\sigma + \bar{\sigma}\eta + \bar{\eta}\eta J$.

The other point is that the constants appearing on the RHS depend only on η_3 . This seems a reasonable assumption. The constants such as $e(\eta)$ have only two components, that is we can express them as $e(\eta) = c_1 + c_2\bar{\eta}\eta$, since we wish to avoid non scalar constants in our theory. Note that c_1 corresponds to the usual central charge in the ordinary theories. It will be more clear when one looks at the OPE of two T_0 's which is given below. The constant $e(\eta)$ is taken to be $e(\eta) = 2 + \bar{\eta}\eta$ for consistency with known conformal dimension of $T_0(z)$ and with the fact that $t(z)$ is the logarithmic partner of $T_0(z)$. Also $f(\eta)$ is taken to be unit, in order to obtain the familiar action of T_0 on the members of the multiplet. By now, there is no restrictions on $d(\eta)$ and we will take it to be $d_1 + \bar{\eta}\eta d_2$, however as shown below, the constant d_2 plays no role in the OPE.

Expanding both sides of equation (89), we find the explicit form of the OPEs

$$\begin{aligned} T_0(z)T_0(0) &= \frac{c_1\Omega}{z^4} + \frac{2T_0}{z^2} + \frac{\partial_z T_0}{z}, \\ T_0(z)t(0) &= \frac{\frac{c_2}{2}\Omega + \frac{c_1}{2}\omega}{z^4} + \frac{d_1 J}{z^3} + \frac{2t + T_0}{z^2} + \frac{\partial_z t}{z}, \\ t(z)t(0) &= -\frac{1}{2} \frac{\log z ((c_1 \log z + 2c_2)\Omega + 2c_1\omega)}{z^4} - \frac{2d_1 \log z J}{z^3} \\ &\quad - \frac{2 \log z (2t + T_0 \log z) + 2 \log z T_0}{z^2} - \frac{\partial_z (\log z (2t + T_0 \log z))}{z}, \\ T_0(z)\zeta(0) &= \frac{1}{2} \frac{c_1\xi}{z^4} + \frac{d_1\sigma}{z^3} + \frac{2\zeta}{z^2} + \frac{\partial_z \zeta}{z}, \\ \zeta(z)t(0) &= -\frac{1}{2} \frac{c_1\xi \log z}{z^4} - \frac{d_1 \log z \sigma}{z^3} - \frac{2\zeta \log z}{z^2} - \frac{\partial_z (\zeta \log z)}{z}, \\ \bar{\zeta}(z)\zeta(0) &= -\frac{1}{2} \frac{c_1(\omega + \Omega \log z) + c_2\Omega}{z^4} - \frac{d_1 J}{z^3} - \frac{2(t + T_0 \log z) + T_0}{z^2} \\ &\quad - \frac{\partial_z (t + T_0 \log z)}{z}. \end{aligned} \quad (90)$$

We observe that these expressions are not the same as those found by [14]. Aside from the presence of the $1/z^3$ terms, the most important difference is that we have the logarithmic

partner of unity operator in our OPEs as well as unity, itself. The existence of 'pseudo-unity' when there is a logarithmic partner for the energy momentum tensor is obligatory. In fact one can easily show that $L_2 t(z)$ is the logarithmic partner of unity. The presence of pseudo-unity causes some problems with the OPEs derived by [14], because now the expectation value of $\mathbb{1}$ is zero. This leads to

$$\langle T_0(z_1)T_0(z_2) \rangle = 0 \quad (91)$$

regardless whether c_1 is zero or not. Instead we have

$$\langle T_0(z_1)t(z_2) \rangle = \frac{c_1}{2(z_1 - z_2)^4} \quad (92)$$

which is just the one written by [14] with $\mathbb{1}$ being $c_1/2$ of our theory. So it is observed that in such theories, there is no need to set $c_1 = 0$. However as suggested by [14], two central charges appear as is seen in equation (89). Since this structure is reminiscent of supersymmetry, most authors have set $c_1 = 0$. Setting c_1 to zero, we do find a much simpler OPE. However, two obvious differences with supersymmetry may be observed. First, $\mathbb{1}$ does not transform like the supercurrent of SCFT. Second, the role of $t(z)$ is not clear. We therefore believe that connection with supersymmetry, if any, has to appear at a deeper level. Thus identifying $\mathbb{1}$ with the similar object in [14] may not be right.

In this section we saw that the idea of grassman variables in LCFT can be extended to include fermionic fields in the theory, and this naturally leads to a current algebra involving the energy momentum tensor, its logarithmic partner and two fermionic currents. Despite the superficial resemblance to supersymmetry such as super multiplets etc., there is need of further clarification if there is any supersymmetry within the theory.

10 AdS/LCFT correspondence and correlation functions

Much work has been done in the last few years based on the AdS/CFT correspondence with the aim of understanding conformal field theories [19]. Within this framework, the correlation functions of operators on the boundary of Anti de Sitter space are determined in terms of appropriate bulk propagators. While the form of the two and three point functions within CFT are fixed by conformal invariance, it is interesting to find actions in the bulk which result in the desired boundary green functions. In particular it is interesting to discover which actions give rise to logarithmic conformal field theories which leads to AdS/LCFT correspondence. Two points should be clarified, first what is meant by ordinary AdS/CFT correspondence and second what is an LCFT, and how does it fit into the correspondence.

The conjecture states that a correspondence between theories defined on AdS_{d+1} and CFT_d can be found. Suppose that a classical theory is defined on the AdS_{d+1} via the action $S[\Phi]$. On the boundary of this space the field is constrained to take certain boundary value $\Phi|_{\partial \text{AdS}} = \Phi_0$. With this constraint, one can calculate the partition function

$$Z[\Phi_0] = e^{-S_{\text{Cl}}[\Phi]}|_{\Phi|_{\partial \text{AdS}} = \Phi_0}. \quad (93)$$

On the other hand in the CFT_d space, there exist operators like $\mathbb{1}$ which belong to some conformal tower. Now the correspondence states that the partition function calculated in AdS is the generating function of the theory in CFT with Φ_0 being the source, that is

$$Z[\Phi_0] = \langle e^{\int \mathcal{O} \Phi_0} \rangle. \quad (94)$$

How do logarithmic conformal field theories fit into this picture? The bulk actions defined on AdS_d which give rise to logarithmic operators on the boundary where first discussed

in [20, 21] and have consequently been discussed by a number of authors [22, 23]. More recently a connection with world sheet supersymmetry has been discussed in [25]. In this section the correspondence is explained explicitly and the two point correlation functions of different fields of CFT are derived [24].

To begin, one should propose an action on AdS. As we will have an operator like

$$O(z, \eta) = A(z) + \bar{\zeta}(z)\eta + \bar{\eta}\zeta(z) + \bar{\eta}\eta B(z), \quad (95)$$

with scaling law

$$O(\lambda z, \eta) = \lambda^{-(\Delta + \bar{\eta}\eta)} O(z, \eta). \quad (96)$$

in LCFT part of the theory, there should be a corresponding field in AdS, $\Phi(x, \eta)$, which can be expanded as

$$\Phi(x, \eta) = C(x) + \bar{\eta}\beta(x) + \bar{\beta}(x)\eta + \bar{\eta}\eta D(x), \quad (97)$$

where \mathbf{x} is $(d+1)$ dimensional with components x^0, \dots, x^d . Of course $d=2$ is the case we are most interested in, any how the result can be applied to any dimension and so we will consider the general case in our calculations. Let us then consider the action

$$S = -\frac{1}{2} \int d^{d+1}x \int d\bar{\eta}d\eta [(\nabla\Phi(x, \eta)) \cdot (\nabla\Phi(x, -\eta)) + m^2(\eta)\Phi(x, \eta)\Phi(x, -\eta)]. \quad (98)$$

This action seems to be the simplest non trivial action for the field $\Phi(x, \eta)$. To write it explicitly in terms of the four components of the the field, one should expand equation (98) in powers of $\bar{\eta}$ and η using equation (97). Integrating over $\bar{\eta}$ and η one finds

$$S = -\frac{1}{2} \int d^{d+1}x [2(\nabla C) \cdot (\nabla D) + 2m^2_1 CD + m^2_2 C^2 + 2(\nabla \bar{\beta}) \cdot (\nabla \beta) + 2m^2_1 \bar{\beta}\beta]. \quad (99)$$

To derive expression (99) we have assumed $m^2(\eta) = m^2_1 + m^2_2 \bar{\eta}\eta$. Note that the bosonic part of this action is the same as the one proposed by [20, 21] with $m^2_1 = \Delta(\Delta - d)$ and $m^2_2 = 2\Delta - d$. In our theory with proper scaling of the fields, one can recover these relations. The equation of motion for the field Φ is

$$(\nabla^2 - m^2(\eta))\Phi(x, \eta) = 0. \quad (100)$$

The Dirichlet Green function for this system satisfies the equation

$$(\nabla^2 - m^2(\eta))G(x, y, \eta) = \delta(x, y), \quad (101)$$

together with the boundary condition

$$G(x, y, \eta)|_{x \in \partial_{AdS}} = 0. \quad (102)$$

With this Green function, the Dirichlet problem for Φ in AdS can be solved readily. However, near the boundary of AdS, that is $x^d \simeq 0$, the metric diverges so the problem should be studied more carefully. One can first solve the problem for the boundary at $x^d = \varepsilon$ and then let ε tend to zero. With properly redefined scaled fields at the boundary one can avoid the singularities in the theory. So we first take the boundary at $x^d = \varepsilon$ and find the Green function [26]

$$G(x, y, \eta)|_{y^d = \varepsilon} = -a(\eta)\varepsilon^{\Delta + \bar{\eta}\eta - d} \left(\frac{x^d}{(x^d)^2 + |\mathbf{x} - \mathbf{y}|^2} \right)^{\Delta + \bar{\eta}\eta}, \quad (103)$$

where the bold face letters are \mathbf{d} -dimensional and live on the boundary. The field in the bulk is related to the boundary fields by

$$\Phi(x, \eta) = 2a(\eta)(\Delta + \bar{\eta}\eta)\varepsilon^{\Delta+\bar{\eta}\eta-d} \int_{y^d=\varepsilon} d^d y \Phi(\mathbf{y}, \varepsilon, \eta) \left(\frac{x^d}{(x^d)^2 + |\mathbf{x} - \mathbf{y}|^2} \right)^{\Delta+\bar{\eta}\eta}, \quad (104)$$

with $a = \frac{\Gamma(\Delta+\bar{\eta}\eta)}{2\pi^{d/2}\Gamma(\alpha+1)}$ and $\alpha = \Delta + \bar{\eta}\eta - d/2$. To compute Gamma functions in whose argument appears $\bar{\eta}\eta$, one should make a Taylor expansion for the function, that is

$$\Gamma(a + \bar{\eta}\eta) = \Gamma(a) + \bar{\eta}\eta\Gamma'(a). \quad (105)$$

There is no higher terms in this Taylor expansion because $(\bar{\eta}\eta)^2 = 0$. Now defining $\Phi_b(\mathbf{x}, \eta) = \lim_{\varepsilon \rightarrow 0} (\Delta + \bar{\eta}\eta)\varepsilon^{\Delta+\bar{\eta}\eta-d} \Phi(\mathbf{x}, \varepsilon, \eta)$, we have

$$\Phi(x, \eta) = \int d^d \mathbf{y} \left(\frac{x^d}{(x^d)^2 + |\mathbf{x} - \mathbf{y}|^2} \right)^{\Delta+\bar{\eta}\eta} \Phi_b(\mathbf{y}, \eta). \quad (106)$$

Using the solution derived, one should compute the classical action. First note that the action can be written as (using the equation of motion and integrating by parts)

$$S_{cl.} = \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \varepsilon^{1-d} \int d\bar{\eta} d\eta \int d^d \mathbf{y} \left[\Phi(\mathbf{y}, \varepsilon, \eta) \frac{\partial \Phi(\mathbf{y}, \varepsilon, -\eta)}{\partial x^d} \right], \quad (107)$$

putting the solution (106) into equation (107), the classical action becomes

$$S_{cl.}(\Phi_b) = \frac{1}{2} \int d\bar{\eta} d\eta \int d^d \mathbf{x} d^d \mathbf{y} \frac{a(\eta) \Phi_b(\mathbf{x}, \eta) \Phi_b(\mathbf{y}, -\eta)}{|\mathbf{x} - \mathbf{y}|^{2\Delta+2\bar{\eta}\eta}}. \quad (108)$$

The next step is to derive correlation functions of the operator fields on the boundary by using AdS/CFT correspondence. In our language the operator \mathbf{O} has an η dependence, in addition to its usual coordinate dependence. It lives in the LCFT space and can be expanded as

$$\mathbf{O}(\mathbf{x}, \eta) = A(\mathbf{x}) + \bar{\eta}\zeta(\mathbf{x}) + \bar{\zeta}(\mathbf{x})\eta + \bar{\eta}\eta B(\mathbf{x}), \quad (109)$$

so the AdS/LCFT correspondence becomes

$$\left\langle \exp \left(\int d\bar{\eta} d\eta \int d^d \mathbf{x} \mathbf{O}(\mathbf{x}, \eta) \Phi_b(\mathbf{x}, \eta) \right) \right\rangle = e^{S_{cl}(\Phi_b)}. \quad (110)$$

Expanding both sides of this equation in powers of Φ_b and integrating over η 's, the two point function of different components of $\mathbf{O}(\mathbf{x}, \eta)$ can be found

$$\begin{aligned} \langle A(\mathbf{x}) A(\mathbf{y}) \rangle &= 0, \\ \langle A(\mathbf{x}) B(\mathbf{y}) \rangle &= \frac{a_1}{(\mathbf{x} - \mathbf{y})^{2\Delta}}, \\ \langle B(\mathbf{x}) B(\mathbf{y}) \rangle &= \frac{1}{(\mathbf{x} - \mathbf{y})^{2\Delta}} (a_2 - 2a_1 \log(\mathbf{x} - \mathbf{y})), \\ \langle \bar{\zeta}(\mathbf{x}) \zeta(\mathbf{y}) \rangle &= \frac{-a_1}{(\mathbf{x} - \mathbf{y})^{2\Delta}}, \end{aligned} \quad (111)$$

with all other correlation functions being zero. These correlation functions can be obtained in another way. Knowing the behaviour of the fields under conformal transformations, the form of two-point functions are determined. The scaling law is given by equation (96). Using this scaling law, most of the correlation functions derived here are fulfilled. However

it does not lead to vanishing correlation functions of $\langle A(\mathbf{x})\zeta(\mathbf{y}) \rangle$ and $\langle B(\mathbf{x})\zeta(\mathbf{y}) \rangle$. These correlation functions are found to be

$$\begin{aligned}\langle A(\mathbf{x})\zeta(\mathbf{y}) \rangle &= \frac{b_1}{(\mathbf{x} - \mathbf{y})^{2\Delta}}, \\ \langle B(\mathbf{x})\zeta(\mathbf{y}) \rangle &= \frac{1}{(\mathbf{x} - \mathbf{y})^{2\Delta}} (b_2 - 2b_1 \log(\mathbf{x} - \mathbf{y})).\end{aligned}\quad (112)$$

Of course, assuming $b_1 = b_2 = 0$ one finds the forms derived above. However, the vanishing value of such correlators comes from some other properties of the theory. What forces these constants to vanish is the fact that the total fermion number is odd¹. One way of seeing this is to look at the OPE as given in [7]. The OPE of two \mathbf{Q} -fields has been proposed to be

$$O(z)O(0) \sim z^{\bar{\eta}_1\eta_2 + \bar{\eta}_2\eta_1} \frac{\Phi_0(\eta_3)}{z^{2(\Delta + \bar{\eta}_3\eta_3)}}, \quad (113)$$

where $\eta_3 = \eta_1 + \eta_2$ and Φ_0 is the identity multiplet

$$\Phi_0(\eta) = \Omega + \bar{\eta}\xi + \bar{\xi}\eta + \bar{\eta}\eta\omega, \quad (114)$$

with the property

$$\langle \Phi_0(\eta) \rangle = \bar{\eta}\eta. \quad (115)$$

Note that the ordinary identity operator is Ω which has the unusual property that $\langle \Omega \rangle = 0$, but its logarithmic partner, ω , has nonvanishing norm. Calculating the expectational value of both sides of equation (113), the correlation functions of different fields inside \mathbf{Q} are found which leads to vanishing correlators $\langle A(\mathbf{x})\zeta(\mathbf{y}) \rangle$ and $\langle B(\mathbf{x})\zeta(\mathbf{y}) \rangle$, just the same result as derived by ADS/LCFT correspondence.

11 BRST symmetry of the theory

The existence of some ghost fields in the action of the theory considered so far, is reminiscent of BRST symmetry. A few remarks clarifying the word "ghost" may be in order here. The action defined by equation (99) has two types of fields in it. Assume \mathbf{C} and \mathbf{D} are scalar fields and \mathbf{B} and $\bar{\mathbf{B}}$ and fermionic fields in the superfield structure. We have called the fermionic fields "ghosts" because they are scalar fermions and also because they participate in the BRST symmetry as we shall see below.

As the action is quadratic, the partition function can be calculated explicitly

$$Z = \int \mathbf{D}C(x) \mathbf{D}D(x) \mathbf{D}\bar{\mathbf{B}}(x) \mathbf{D}\mathbf{B}(x) e^{-S[C, D, \bar{\mathbf{B}}, \mathbf{B}]}. \quad (116)$$

The bosonic and fermionic parts of the action are decoupled and integration over each of them can be performed independently. For bosonic part one has

$$Z_b = \int \prod_p dC_p \int \prod_p dD_p \exp \left\{ -\frac{1}{2} \begin{pmatrix} C_p & D_p \end{pmatrix} G(p) \begin{pmatrix} C_p \\ D_p \end{pmatrix} \right\}, \quad (117)$$

where a Fourier transform has been done and

$$G(p) = \begin{pmatrix} p^2 + m^2_1 & 0 \\ m^2_2 & p^2 + m^2_1 \end{pmatrix}. \quad (118)$$

¹this observation is due to M. Flohr

These integrals are simple Gaussian ones. So, apart from some unimportant numbers, this partition function becomes

$$Z_b = \prod_p \left[\det \begin{pmatrix} p^2 + m^2_1 & 0 \\ m^2_2 & p^2 + m^2_1 \end{pmatrix} \right]^{-1/2} = \prod_p (p^2 + m^2_1)^{-1}. \quad (119)$$

For the fermionic part the the same steps can be done and the result is

$$Z_f = \prod_p (p^2 + m^2_1). \quad (120)$$

Now it is easily seen that the total partition function is merely a number, independent of the parameters of the theory and this is the signature of BRST symmetry. Before proceeding, it is worth mentioning that this symmetry will be induced onto LCFT part of the correspondence and the correlation functions in that space will also be invariant under proper transformations.

In BRST transformation, the fermionic and bosonic fields should transform into each other. In our case this can be done using η and $\bar{\eta}$. Also one needs an infinitesimal anticommuting parameters. Now let ϵ_1 and ϵ_2 be infinitesimal anticommuting parameters and consider the following infinitesimal transformation of the the field Φ

$$\delta\Phi(\mathbf{x}, \eta) = (\bar{\epsilon}\eta + \bar{\eta}\epsilon)\Phi(\mathbf{x}, \eta). \quad (121)$$

It can be easily seen that this transformation leaves the action invariant, because in the action the only terms which exist are in the form of $\Phi(\eta)\Phi(-\eta)$ and under such a transformation this term will become

$$\delta(\Phi(\eta)\Phi(-\eta)) = \Phi(\eta)(-\bar{\epsilon}\eta - \bar{\eta}\epsilon)\Phi(-\eta) + (\bar{\epsilon}\eta + \bar{\eta}\epsilon)\Phi(\eta)\Phi(-\eta), \quad (122)$$

which is identically zero. We can therefore interpret this transformation as the action of two charges Q and \bar{Q} . The explicit action of Q for each component of Φ is

$$\begin{aligned} QC &= 0, \\ Q\beta &= 0, \\ Q\bar{\beta} &= C \\ QD &= -\beta. \end{aligned} \quad (123)$$

As expected the bosonic and fermionic fields are transformed into each other, and the square of Q vanishes. The action of \bar{Q} is similar except that it vanishes on $\bar{\beta}$ and not on β .

To see how this symmetry is induced onto the LCFT part of the theory one should first find the proper transformation. Going back to equation (110) and using the symmetry obtained for the classical action one finds

$$\exp(S_{Cl}[\Phi]) = \left\langle \exp \left(\int O(\Phi + \delta\Phi) \right) \right\rangle. \quad (124)$$

As the transformation of Φ is $(\bar{\epsilon}\eta + \bar{\eta}\epsilon)\Phi$ the integrand on the right hand side of equation (124) is just $O\Phi + (\bar{\epsilon}\eta + \bar{\eta}\epsilon)O\Phi$ which can be regarded as $(O + \delta O)\Phi$ with $\delta O = (\bar{\epsilon}\eta + \bar{\eta}\epsilon)O$. So equation (124) can be rewritten as

$$\left\langle \exp \left(\int O\Phi \right) \right\rangle = \left\langle \exp \left(\int (O + \delta O)\Phi \right) \right\rangle. \quad (125)$$

This shows that the correlation functions of the \mathbf{O} field are invariant under the BRST transformation, that is $\delta\langle O_1 O_2 \cdots O_n \rangle = 0$ if the BRST transformation is taken to be

$$\delta O = (\bar{\epsilon}\eta + \bar{\eta}\epsilon)O. \quad (126)$$

Again one can rewrite this transformation in terms of the components of \mathbf{O} and the result is just the same as equation (123). This invariance can be tested using two point correlation functions derived in previous section. These correlation functions are easily found to be invariant under the transformation law (126), as an example

$$Q\langle B\bar{\zeta} \rangle = \langle (QB)\bar{\zeta} \rangle + \langle B(Q\bar{\zeta}) \rangle = \langle \bar{\zeta}\zeta \rangle - \langle BA \rangle = 0. \quad (127)$$

12 Appendix: Hypergeometric functions

In this appendix we show that equation (48) can be considered as 16 differential equations. However one of them is trivial because it vanishes due to OPE constraints [7, 8]. According to equation (49) \mathbf{a} , \mathbf{b} , \mathbf{c} and so \mathbf{H} are functions of \mathbf{U} 's. We write them in a general form

$$\begin{aligned} H &= \sum_{i=1}^4 H_i \theta_i + \sum_{1 \leq i < j \leq 4} H_{ij} \theta_i \theta_j + \sum_{1 \leq i < j < k \leq 4} H_{ijk} \theta_i \theta_j \theta_k + H_{1234} \theta_1 \theta_2 \theta_3 \theta_4, \\ a &= a_0 + \sum_{i=1}^4 a_i \theta_i + \sum_{1 \leq i < j \leq 4} a_{ij} \theta_i \theta_j + \sum_{1 \leq i < j < k \leq 4} a_{ijk} \theta_i \theta_j \theta_k + a_{1234} \theta_1 \theta_2 \theta_3 \theta_4, \end{aligned} \quad (128)$$

and in a similar way for \mathbf{b} and \mathbf{c} . Now by substitution of them in equation (48) we obtain 15 differential equations

$$\begin{aligned} DH_i &= 0, \\ DH_{ij} &= \{-[c_i - (a_i + b_i)\eta] \frac{dH_j}{d\eta} + (a_0 b_i + a_i b_0) H_j + i \longleftrightarrow j\}, \\ DH_{ijk} &= [-[c_k - (a_k + b_k)\eta] \frac{dH_{ij}}{d\eta} + (a_0 b_k + a_k b_0) H_{ij} \\ &\quad - [c_{ij} - (a_{ij} + b_{ij})\eta] \frac{dH_k}{d\eta} + (a_0 b_{ij} + a_i b_j + a_{ij} b_0) H_k + \text{cyclic terms}], \\ DH_{1234} &= [-[c_l - (a_l + b_l)\eta] \frac{dH_{ijk}}{d\eta} + (a_0 b_l + a_l b_0) H_{ijk} \\ &\quad - [c_{ij} - (a_{ij} + b_{ij})\eta] \frac{dH_{kl}}{d\eta} + (a_0 b_{ij} + a_i b_j + a_{ij} b_0) H_{kl} \\ &\quad - [c_{ijk} - (a_{ijk} + b_{ijk})\eta] \frac{dH_l}{d\eta} + (a_0 b_{ijk} + a_{ijk} b_0 + a_k b_{ij} + a_{ij} b_k) H_l \\ &\quad + \text{cyclic terms}], \end{aligned} \quad (129)$$

where

$$D := \eta(1 - \eta) \frac{d^2}{d\eta^2} + [c_0 - (a_0 + b_0 + 1)\eta] \frac{d}{d\eta} - a_0 b_0. \quad (130)$$

Let us now obtain from equation (51), first few terms of 16 functions that appear in solutions of differential equations, given above for the special case of $\Delta_i = \frac{1}{4}$

$$\begin{aligned} h_0 &= F(2, 1, 2, \eta) = 1 + \eta + \eta^2 + \eta^3 + \cdots \\ h_1 &= \frac{1}{2}\eta + \frac{2}{3}\eta^2 + \frac{3}{4}\eta^3 + \cdots, \quad h_2 = \frac{3}{2}\eta + \frac{7}{3}\eta^2 + \frac{35}{12}\eta^3 + \cdots \end{aligned}$$

$$\begin{aligned}
h_3 &= -\frac{1}{2}\eta - \frac{2}{3}\eta^2 - \frac{3}{4}\eta^3 + \dots, & h_4 &= \frac{1}{2}\eta + \frac{7}{9}\eta^2 + \frac{35}{36}\eta^3 + \dots \\
h_{12} &= -\frac{1}{2}\eta - \frac{1}{18}\eta^2 + \frac{29}{72}\eta^3 + \dots, & h_{13} &= \frac{1}{2}\eta + \frac{4}{9}\eta^2 + \frac{3}{8}\eta^3 + \dots \\
h_{23} &= -\frac{2}{3}\eta^2 - \frac{5}{4}\eta^3 + \dots, & h_{14} &= \frac{1}{3}\eta + \frac{2}{3}\eta^2 + \frac{11}{12}\eta^3 \\
h_{24} &= \frac{7}{6}\eta + \frac{70}{27}\eta^2 + \frac{281}{72}\eta^3 + \dots, & h_{34} &= -\frac{1}{2}\eta - \frac{49}{54}\eta^2 - \frac{259}{216}\eta^3 + \dots \\
h_{123} &= \frac{7}{9}\eta^2 + \frac{7}{6}\eta^3 + \dots, & h_{124} &= -\frac{2}{3}\eta - \frac{41}{162}\eta^2 + \frac{317}{648}\eta^3 + \dots \\
h_{134} &= \frac{2}{3}\eta + \frac{121}{162}\eta^2 + \frac{455}{648}\eta^3 + \dots, & h_{234} &= -\frac{32}{27}\eta^2 - \frac{46}{18}\eta^3 + \dots \\
h_{1234} &= \frac{137}{81}\eta^2 + \frac{17}{6}\eta^3 + \dots.
\end{aligned} \tag{131}$$

where $Dh_0 = 0$.

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