Quantum mechanics on noncommutative plane and sphere from constrained systems.

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Abstract

It is shown that quantum mechanics on noncommutative (NC) spaces can be obtained by canonical quantization of some underlying constrained systems. Noncommutative geometry arises after taking into account the second class constraints presented in the models. It leads, in particular, to a possibility of quantization in terms of the initial NC variables. For a two-dimensional plane we present two Lagrangian actions, one of which admits addition of an arbitrary potential. Quantization leads to quantum mechanics with ordinary product replaced by the Moyal product. For a three-dimensional case we present Lagrangian formulations for a particle on NC sphere as well as for a particle on commutative sphere with a magnetic monopole at the center, the latter is shown to be equivalent to the model of usual rotor. There are several natural possibilities to choose physical variables, which lead either to commutative or to NC brackets for space variables. In the NC representation all information on the space variable dynamics is encoded in the NC geometry. Potential of special form can be added, which leads to an example of quantum mechanics on the NC sphere.

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1 Introduction and symmary.

Recently quantum mechanics on noncommutative spaces (NQM) have received a considerable discussion [1-7]. For two-dimensional

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plane it can be characterized by the following brackets ($\epsilon_{ab} = -\epsilon_{ba}$, $a, b = 1, 2,\ \epsilon_{12} = 1$)

$$\{x_a, x_b\} = \theta\epsilon_{ab}, \qquad \{x_a, p_b\} = \delta_{ab}, \qquad \{p_a, p_b\} = 0, \tag{1}$$

and by the Hamiltonian $H = \frac{p^2}{2m} + V(x)$ with some potential $V(x)$. To make this situation tractable, the prescription is to consider new variables

$$\tilde{x}_a = x_a + \frac{\theta}{2}\epsilon_{ab}p_b, \qquad \tilde{p}_a = p_a, \tag{2}$$

which obey the canonical brackets and thus can be quantized in the standard way. It leads to the Schrödinger equation

$$E\Psi(\tilde{x}) = \frac{1}{2m}\tilde{p}^2\Psi(\tilde{x}) + V(\tilde{x}_a - \frac{\theta}{2}\epsilon_{ab}\tilde{p}_b)\Psi(\tilde{x}), \tag{3}$$

where the last term can be rewritten [8, 9, 3] through the Moyal product

$$V(\tilde{x}_a - \frac{\theta}{2}\epsilon_{ab}\tilde{p}_b)\Psi(\tilde{x}) = V(\tilde{x}) \star \Psi(\tilde{x}). \tag{4}$$

Thus one obtains quantum mechanics in terms of the commutative variables $\tilde{x}, \tilde{p}$, but with the ordinary product replaced by the star product.

Let us recall that in some cases appearance of the noncommutative geometry [10] has a natural interpretation as resulting from the canonical quantization of some underlying constrained system. In particular, this interpretation is possible for the lowest level Landau problem [11, 9] and for the open string in a B-field background [12-14]. Regarding the NC quantum mechanics on the plane, one possibility was proposed in [1], starting from higher derivative mechanical action. It leads to the NC particle with extra physical degrees of freedom. In this relation it is natural to ask whether a similar interpretation is possible for NC quantum mechanics of the scalar particle (1)-(4), as well as for the corresponding generalization on three dimensional sphere. Here we demonstrate that it is actually the case. Our starting point will be some mechanical system (without higher derivatives) formulated in an appropriately extended configuration space. Nonphysical character of the corresponding extra degrees of freedom is supplied by second class constraints. The noncommutative geometry (1) arises after introduction the Dirac bracket corresponding to these constraints, while the prescription (2) becomes, in fact, the standard necessary step for the canonical quantization of a system with second class constraints [15, 16].

The work is organised as follows. In Sect. 2 we discuss models which lead to quantum mechanics on the NC plane. The Lagrangian action, which is appropriate for at most quadratic potential, looks as follows

$$S = \int d\tau \left[-\frac{m}{2}\dot{v}^2 + 2(\dot{v} - \dot{x})\theta^{-1}v - \frac{2}{m\bar{\theta}^2}v^2 - U(x) \right], \tag{5}$$

where $x_a(\tau),\ v_a(\tau)$ are the configuration space variables and $\theta_{ab} = \bar{\theta}\epsilon_{ab}$. The variables v_a are subject to the second class constraints and can be omitted from the consideration after the Dirac bracket is introduced. The physical sector consist of x_a and the conjugated momentum p_a. The Dirac bracket for x_a turns out to be nontrivial, with the noncommutativity parameter being $\theta = \bar{\theta}\det^{-1}(1 - \frac{m\bar{\theta}^2}{4}\partial\partial U)$. The parameter (and rank of matrix of the constraint brackets) depend on the potential. It explains appearance of two phases [4-6] of the resulting NQM: for a critical value of the parameter, the model involves first class constraints instead of the second class ones. For the case $\partial_a\partial_b U = const$ one can easily find the canonical variables, see Eq.(17). Quantization leads to NQM (3) with the potential $V = -\frac{m\bar{\theta}^2}{8}\partial_a U \partial_a U + U$. For an arbitrary potential $U(x)$, the noncommutativity parameter θ depends on x_a and one is faced with the problem of diagonalization of the brackets, Eq.(15) below. Surprisingly enough, the problem can be resolved if one starts from the action, which is obtained from (5) omitting the first term

$$S = \int d\tau \left[2(\dot{v} - \dot{x})\theta^{-1}v - \frac{2}{m\bar{\theta}^2}v^2 - V(x) \right]. \tag{6}$$

It can be considered as the action of ordinary particle (with position x_a) written in the first order form, with the "Chern-Simons term" for v added: $\dot{v}\theta^{-1}v$. The action is similar to the one discussed by Lukierski at all [1], but does not involve of higher derivatives. As a concequence, there are no "internal" oscillator modes in the physical sector. Below we show that this action leads to NQM of the scalar particle (1)-(4) with the same potential V.

In Sect. 3 we demonstrate that the same procedure works for noncommutative sphere in three dimensions [19-21]. We propose the following action

$$S = \int d\tau \left[-\epsilon_{ijk}\dot{v}_i v_j x_k - v^2 + \phi(x_i^2 - 1) - V(v^2) \right], \tag{7}$$

where $\epsilon_{ijk} = \epsilon_{[ijk]}$, $\epsilon_{123} = 1$. The variables v_i play the same role as in the previous case (5). The variables x_i are restricted to lie on the sphere $x^2 = 1$. The kinematic constraint is included into the action by using of the Lagrangian multiplier ϕ. Dynamics is governed by second order differential equations, which is supplied by the presence of the term v^2. We restrict ourselves to $SO(3)$-invariant potential $V(v^2)$. The combination $x_i v_i$ is not included into the potential since it would lead to deformation of the constraint system algebra as compared with the free case $V = 0$ (see below). In the Hamiltonian formalism the essential constraints of the model are

$$G_i \equiv p_i + \epsilon_{ijk} v_j x_k = 0, \quad \pi_i = 0, \quad x_i^2 - 1 = 0, \quad x_i v_i = 0, \tag{8}$$

where $(x,\ \pi)$ and $(v,\ p)$ form canonical pairs (in these notations commutative relations appear in the standard form, see Eq.(41)). The constraints form the second class system. The corresponding Dirac bracket is constructed and brackets for the phase space variables are presented in $SO(3)$ covariant form. Using the constraints $G_i = 0$ one can represent one of the variables $(x,\ v,\ p)$ through the remaining ones, which leads either to commutative or to NC brackets for space variables. The representations are discussed in Sect. 4. $(x,\ p)$-representation is characterized by NC space geometry and trivial dynamics for the corresponding space variables. In $(v,\ p)$-representation the geometry can be made commutative by transition to the canonical variables, the dynamics of which is governed then by nonlinear equations. In Sect. 5 we present and discuss slight modification of the action (7) which describes a particle on the commutative sphere with a monopole at the center. In particular, we show canonical equivalence of this model and model of the rotor. In the end of the section some possible generalizations of the action (7) are discussed.

2 Particle on the noncommutative plane.

Starting from the action (5), one finds in the Hamiltonian formalism the primary constraints

$$G_a \equiv p_a + 2\theta^{-1}_{ab} v_b = 0, \tag{9}$$

and the Hamiltonian

$$H = -\frac{1}{2m}\pi^2 + \frac{2}{m}\pi\theta^{-1}v + U(x) + \lambda(p + \theta^{-1}y). \tag{10}$$

Here $p,\ \pi$ are conjugated momenta for $x,\ v$ and λ is the Lagrangian multiplier for the constraint. Further analysis gives the secondary constraints

$$\dot{G}_a = 0 \Longrightarrow T_a \equiv \pi_a - 2\theta^{-1}_{ab} v_b + \frac{m}{2}\theta_{ab}\partial_b U = 0, \tag{11}$$

as well as equations for determining the Lagrangian multipliers

$$F\lambda = -\frac{2}{m}(\pi - \theta^{-1}v), \qquad F_{ab} \equiv \delta_{ab} - \frac{m\bar{\theta}^2}{4}\partial_a\partial_b U. \tag{12}$$

Next step depends on the rank of the matrix F. If $\det F = 0$, the model involves first class constraints (see also Eq.(13)), which explains appearance of two phases [4-6] of the resulting NQM. Let us consider the nondegenerated case $\det F \neq 0$. Then the constraints form the second class system

$$\{G_a, G_b\} = 0, \qquad \{T_a, T_b\} = -4\theta^{-1}_{ab}, \qquad \{G_a, T_b\} = 2F_{ac}\theta^{-1}_{cb}. \tag{13}$$

Introducing the Dirac bracket

$$\begin{aligned} \{A, B\}_D = \{A, B\} - \{A, G\}\theta\triangle^{-1}\{G, B\} - \\ \{A, G\}\frac{1}{2}F^{-1}\theta\{T, B\} - \{A, T\}\frac{1}{2}\theta F^{-1}\{G, B\}, \end{aligned} \tag{14}$$

the variables v, π can be omitted from consideration, while for the remaining physical variables $x,\ p$ one obtains from Eq.(14) the following brackets:

$$\{x_a, x_b\} = \triangle^{-1}\theta_{ab}, \quad \{x_a, p_b\} = F^{-1}_{ab}, \quad \{p_a, p_b\} = 0. \tag{15}$$

The noncommutativity parameter depends on the potential through the quantity

$$\triangle \equiv \det(1 - \frac{m\bar{\theta}^2}{4}\partial\partial U). \tag{16}$$

Let us restrict ourselves to the case $\partial_a \partial_b U = const$. To quantize the system one needs to find the canonical variables [16], which in this case turn out to be

$$\tilde{x}_a = F_{ab} x_b + \frac{1}{2}\theta_{ab} p_b, \quad \tilde{p}_a = p_a. \tag{17}$$

They obey the standard brackets $\{\tilde{x}_a, \tilde{x}_b\} = 0$, $\{\tilde{x}_a, \tilde{p}_b\} = \delta_{ab}$, $\{\tilde{p}_a, \tilde{p}_b\} = 0$. The Hamiltonian in terms of the canonical variables is (the term F can be equally included into the kinetic part of the Hamiltonian [26])

$$H_{ph} = \frac{1}{2m}\tilde{p}^2 - \frac{m\bar{\theta}^2}{8}\partial_a U \partial_a U|_{x(\tilde{x},\tilde{p})} + U[F^{-1}(\tilde{x} - \frac{1}{2}\theta\tilde{p})], \tag{18}$$

where the term with derivatives of the potential comes from Eq.(11). The resulting system can be quantized now in the standard way. Note that the underlying potential U and the final one turn out to be different for this model. For example, starting from the harmonic oscillator $U = \frac{k}{2}|x|^2$, one obtains the NQM which corresponds to oscillator with renormalized rigidity $\tilde{k} = (1 - \frac{m\bar{\theta}^2 k}{4})^{-1} k$, namely

$$V = [-\frac{m\bar{\theta}^2}{8}\partial_a U \partial_a U + U]|_{x(\tilde{x},\tilde{p})} = \frac{\tilde{k}}{2}|\tilde{x} - \frac{1}{2}\theta\tilde{p}|^2. \tag{19}$$

Note also that in absence of the potential ($U = 0$) the model (5) describes the free NC particle which is characterised by the equations of motion $\dot{x}_a = \frac{1}{m}p_a$, $\dot{p}_a = 0$ and by the relations (1).

Let us return to the case of an arbitrary potential. As it was mentioned, the complicated brackets (15) arise due to the fact that the secondary constraints (11) involve derivative of the potential. While existence of the canonical variables is guaranteed by the known theorems [16], it is problematic to find a solution in the manifest form. One possibility to avoid the problem is to construct action which will create the primary constraints only. Since $U(x)$ does not contain the time derivative, it can not give contribution into the primary constraints. An appropriate action is[1]

$$S = \int d\tau \left[2(\dot{v} - \dot{x})\theta^{-1} v - \frac{2}{m\bar{\theta}^2}v^2 - V(x) \right], \tag{20}$$

where x_a, v_a are the configuration space variables. Configuration space dynamics is governed by the second order equations which is

[1] An equivalent form of the action can be obtained by the shift: $x \to x' = x - v$.

supplied by the term v^2. Following the Dirac procedure one obtains the primary second class constraints

$$G_a \equiv p_a + 2\theta^{-1}_{ab}v_b = 0, \qquad T_a = \pi_a - 2\theta^{-1}_{ab}v_b = 0, \tag{21}$$

and the Hamiltonian

$$H = \frac{2}{m\bar{\theta}^2}v^2 + V(x) + \lambda_1 G + \lambda_2 T. \tag{22}$$

Remaining analysis is similar to the previous case. Introducing the Dirac bracket (14) (where $F = 1$ now) , the variables v, π can be omitted, while for x, p one has the brackets (1). Defining the canonical variables $\tilde{x}_a = x_a + \frac{1}{2}\theta_{ab}p_b$, $\tilde{p}_a = p_a$, one obtains the physical Hamiltonian $H = \frac{2}{m}\tilde{p}^2 + V(\tilde{x} - \frac{1}{2}\theta\tilde{p})$, thus reproducing the NQM (3), (4) for the case of an arbitrary potential.

We have demonstrated that quantum mechanics on NC plane can be considered as resulting from direct canonical quantization of the underlying constrained systems (5), (6). It implies, that instead of the star product (3), (4), one can equally use now other possibilities to quantize the system. In particular, the conversion scheme [17] or the embedding formalism [18] can be applied. For example, it is not difficult to rewrite the formulation (20)- (22) as a first class constrained system. Namely, let us keep G-constraint only and define the deformed Hamiltonian as

$$\tilde{H} = \frac{2}{m\bar{\theta}^2}v^2 + V[x - \frac{1}{2}\theta(\pi - 2\theta^{-1}v)] + \lambda G. \tag{23}$$

Since $\{G, \tilde{H}\} = 0$, it is equivalent formulation of the problem (22), the latter is reproduced in the gauge $T = 0$. Now one can quantize all the variables canonically, while the first class constraint $G = 0$ can be imposed as restriction on the wave function. It implies quantization in terms of the initial NC variables. Another possibility is to consider the gauges different from $T = 0$. For example, one can take $\pi = 0$, which can lead to simplification of the eigenvalue problem (3).

3 Particle on the noncommutative sphere.

Here we show that dynamics on the NC sphere can be described in a similar fashion, starting from the action (7). From manifest form of the action it follows that velocities do not enter into expressions for definition of conjugated momentum in the Hamiltonian formulation. On the first stage of the Dirac procedure one finds the primary constraints

$$G_i \equiv p_i + \epsilon_{ijk} v_j x_k = 0, \quad T_i \equiv \pi_i = 0, \quad p_\phi = 0, \tag{24}$$

where p_i are conjugated momentum for v_i while π_i corresponds to x_i. The Hamiltonian is

$$H = v^2 - \phi(x_i^2 - 1) + V(v^2) + \lambda_i G_i + \bar{\lambda}_i T_i + \lambda p_\phi, \tag{25}$$

where λ are the Lagrangian multipliers for the corresponding constraints. The constraints obey the following Poisson bracket algebra

$$\{G_i, G_j\} = 2\epsilon_{ijk} x_k, \quad \{G_i, T_j\} = -\epsilon_{ijk} v_k, \quad \{T_i, T_j\} = 0. \tag{26}$$

Matrix composed from the brackets admits two null-vectors $w_1 = (0, v_j),\ w_2 = (v_i, -2x_j)$, so the system $(G,\ T)$ involve two first class constraints at this stage: $v_i T_i$ and $v_i G_i - 2x_j T_j$. From (24) one has the consequences $v_i p_i = 0,\ x_i p_i = 0$. At the second stage of the Dirac procedure there appear the equations

$$\begin{aligned} \dot{p}_\phi = 0 \implies & \ x_i^2 - 1 = 0, \\ \dot{G}_i = 0 \implies -2(1+V')v_i + 2\epsilon_{ijk}\lambda_j x_k - & \ \epsilon_{ijk}\bar{\lambda}_j v_k = 0, \\ \dot{T}_i = 0 \implies & \ 2\phi x_i - \epsilon_{ijk}\lambda_j v_k = 0, \end{aligned} \tag{27}$$

where $V' \equiv \frac{\partial V}{\partial v^2}$. From these equations one extracts three secondary constraints

$$S \equiv x_i^2 - 1 = 0, \quad \bar{S} \equiv x_i v_i = 0, \quad \Phi \equiv \phi + \frac{1}{2} v_i^2 (1 + V') = 0, \tag{28}$$

while the remaining equations involve the Lagrangian multipliers. They will be resolved in the manifestly $SO(3)$-covariant form below. On the next step there arise equations for the Lagrangian multipliers only

$$\begin{aligned} \dot{\Phi} = 0 \implies & \ \lambda + \{\Phi, G_j\}\lambda_j + \{\Phi, T_j\}\bar{\lambda}_j = 0, \\ \dot{S} = 0 \implies x_i \bar{\lambda}_i = 0, \qquad \dot{\bar{S}} = 0 \implies & \ v_i \bar{\lambda}_i + x_j \lambda_j = 0, \end{aligned} \tag{29}$$

which finishes the Dirac procedure for revealing the constraints. To determine the Lagrangian multipliers one has now equations (27), (29). Their consequences are $x_i\lambda_i = x_i\bar{\lambda}_i = v_i\lambda_i = v_i\bar{\lambda}_i = 0$. Using these equations, one resolves Eqs.(27), (29) as

$$\lambda_i = (1+V')p_i, \qquad \bar{\lambda}_i = \lambda = 0. \tag{30}$$

The Hamiltonian equations of motion for the model can be obtained with the help of Eqs.(25), (30). They will be discussed in the next section. Since all the multipliers have been determined, the constraints (24), (28) form the second class system and thus can be taken into account by transition to the Dirac bracket. After introduction of the Dirac bracket corresponding to the pair $p_\phi = 0,\ \Phi = 0$, the variables $\phi,\ p_\phi$ can be omitted from consideration. The Dirac brackets for the remaining variables coincide with the Poison one. To find the Dirac bracket which corresponds to the remaining eight constraints one needs to invert 8×8 matrix composed from Poisson brackets of these constraints. To simplify the problem, we prefer to do this in two steps: first, let us construct an intermediate Dirac bracket which corresponds to the constraints $G_a = 0,\ T_a = 0,\ a = 1, 2$, and then bracket which corresponds to the remaining constraints $G_3,\ T_3,\ S,\ \bar{S}$. Consistency of this procedure is guaranteed by the known theorems [16]. On the first step one has the Poisson brackets

$$\{G_a, G_b\} = 2\epsilon_{ab}x_3, \quad \{G_a, T_b\} = -\epsilon_{ab}v_3, \quad \{T_a, T_b\} = 0. \tag{31}$$

Then the intermediate Dirac bracket is

$$\begin{aligned}\{A, B\}_{D1} = \{A, B\} - \{A, G_a\}\frac{\epsilon_{ab}}{v_3}\{T_b, B\} - \\ \{A, T_a\}\frac{\epsilon_{ab}}{v_3}\{G_b, B\} - \{A, T_a\}\frac{2x_3}{v_3^2}\epsilon_{ab}\{T_b, B\}.\end{aligned} \tag{32}$$

Now one can use the equations $G_a = 0,\ T_a = 0$ in any expression. As a consequence, the remaining constraints can be taken in the form

$$\begin{aligned} G_3 \equiv x_i p_i = 0, \qquad T_3 \equiv \pi_3 = 0, \\ S \equiv x_i^2 - 1 = 0, \qquad \bar{S} \equiv \frac{1}{x_3}(v_3 + J_3) = 0, \end{aligned} \tag{33}$$

and obey the $D1$-algebra

$$\{G_3, S\}_{D1} = -\frac{4x_3}{J_3}, \quad \{G_3, \bar{S}\}_{D1} = -2\left(1 + \frac{p_a^2}{J_3^2}\right),$$

$$\{T_3, S\}_{D1} = 0, \quad \{T_3, \bar{S}\}_{D1} = -\frac{J_3^2 - p_a^2}{x_3^2 J_3}, \quad \{S, \bar{S}\}_{D1} = \frac{4x_3 p_3}{J_3^2}, \tag{34}$$

where J_i are the rotation generators: $J_i \equiv \epsilon_{ijk} x_j p_k$. The corresponding matrix can be easily inverted, and the final expression for the Dirac bracket is

$$\begin{aligned}
\{A, B\}_D = \{A, B\} - \{A, G_3\}\frac{x_3^2 p_3}{J_3^2 - p_a^2}\{T_3, B\} - \\
\{A, G_3\}\frac{J_3}{4x_3}\{S, B\} + \{A, T_3\}\frac{x_3^2 p_3}{J_3^2 - p_a^2}\{G_3, B\} + \\
\{A, T_3\}\frac{x_3(J_3^2 + p_a^2)}{2(J_3^2 - p_a^2)}\{S, B\} - \{A, T_3\}\frac{x_3^2 J_3}{J_3^2 - p_a^2}\{\bar{S}, B\} + \\
\{A, S\}\frac{J_3}{4x_3}\{G_3, B\} - \{A, S\}\frac{x_3(J_3^2 + p_a^2)}{2(J_3^2 - p_a^2)}\{T_3, B\} + \\
\{A, \bar{S}\}\frac{x_3^2 J_3}{J_3^2 - p_a^2}\{T_3, B\},
\end{aligned} \tag{35}$$

where all the brackets on the r.h.s. are $D1$-brackets. Note that the complete constraint system (24), (28) is $SO(3)$-covariant. Consequently, one expects that the final expressions for the brackets can be rewritten in $SO(3)$-covariant form also. It is actually the case. For example, from Eq.(35) one obtains for the variables x_i

$$\begin{aligned}
\{x_1, x_2\}_D = -\frac{x_3}{J_3^2}\left[2 + \frac{x_3^2(2p_i^2 + J_3^2)}{J_3^2 - p_a^2}\right], \\
\{x_a, x_3\}_D = -\frac{1}{J_3 p_i^2}\left[\frac{2x_3 p_3}{J_3}\epsilon_{ac}p_c + 2p_a + (J_3^2 + 2p_a^2)\epsilon_{ac}x_c\right].
\end{aligned} \tag{36}$$

Using the equalities

$$\begin{aligned}
J^2 \equiv J_i^2 = p_i^2 = v_i^2, \qquad J_3^2 - p_a^2 = -x_3^2 p_i^2, \\
x_3 p_3 p_a - J_3 \epsilon_{ab} p_b + p_b^2 x_a = 0,
\end{aligned} \tag{37}$$

which are true on the constraint surface (24), (28), the equations (36) can be presented in $SO(3)$-covariant form $\{x_i, x_j\}_D = \frac{1}{J^2}\epsilon_{ijk}x_k$, $i, j = 1, 2, 3$. Other brackets can be computed from Eq.(35) in a similar fashion. After tedious calculations one obtains the following result

$$\{x_i, x_j\} = \frac{1}{J^2}\epsilon_{ijk}x_k, \qquad \{x_i, p_j\} = \frac{1}{J^2}J_i x_j,$$

$$\{p_i, p_j\} = -\frac{1}{2}\epsilon_{ijk}x_k, \tag{38}$$

$$\{v_i, v_j\} = -\frac{1}{2}\epsilon_{ijk}x_k, \qquad \{v_i, x_j\} = -\frac{1}{J^2}x_i p_j,$$
$$\{v_i, p_j\} = \frac{1}{2}(\delta_{ij} + x_i x_j). \tag{39}$$

Since $\{x_i, J^2\} = 0$, the operator J^2 can be included into redefinition of x_i: $\tilde{x}_i \equiv J^2 x_i$, then $\tilde{x}_i$ obeys $SU(2)$ algebra $\{\tilde{x}_i, \tilde{x}_j\} = \epsilon_{ijk}\tilde{x}_k$, and is constrained to lie on the fuzzy sphere $\tilde{x}^2 = (J^2)^2$. Quantum realization and irreducible representations of such a kind of an algebraic structure were considered, in particular, in [20, 21]. One notes that the algebra obtained (38) has much simpler structure as compared with the one proposed in [20] from algebraic considerations.

Since the second class constraints were taken into account, one can now use them in any expression. In particular, from Eqs.(24), (28) it follows that as the physical sector variables one can choose either (x_i, p_i) or (v_i, p_i), or (x_i, v_i). Relation between these representations is given by the first equation from (24), which can be written in one of the following forms[2]

$$p_i = -\epsilon_{ijk}v_j x_k, \qquad v_i = -\epsilon_{ijk}x_j p_k, \qquad x_i = \frac{1}{v^2}\epsilon_{ijk}v_j p_k. \tag{40}$$

Let us point out that for any given choice, the remaining nonphysical variable looks formally as the rotation generator in the corresponding representation. The equations (40) relate different representations of the particle dynamics on NC sphere which are discussed in the next section.

4 Three representations for the particle dynamics on noncommutative sphere.

To discuss classical dynamics of the particle on NC sphere it will be sufficient to consider the free case $V = 0$. In what follows, we will preserve SO(3) covariance which implies that two of the constraints are not resolved in the manifest form. Note also that the variables

[2]The representations for NC plane can be obtained in a similar fashion starting from Eq.(21), and are not interesting due to linear character of the constraints.

$\phi,\ p_\phi$ are trivially constrained $\phi = 0,\ p_\phi = 0$ and thus are omitted from consideration.

Noncommutative $(x_i,\ p_i)$**-representation.** Taking $x,\ p$ as the basic variables, their algebra is

$$\{x_i, x_j\} = \frac{1}{J^2}\epsilon_{ijk}x_k, \quad \{x_i, p_j\} = \frac{1}{J^2}J_i x_j, \quad \{p_i, p_j\} = -\frac{1}{2}\epsilon_{ijk}x_k. \tag{41}$$

Equations of motion follow from (25), (30)

$$\dot{x}_i = 0, \qquad \dot{p}_i = \epsilon_{ijk}x_j p_k, \tag{42}$$

and are accompanied by two constraints

$$x_i^2 - 1 = 0, \qquad x_i p_i = 0. \tag{43}$$

The physical Hamiltonian has the form $H_{ph} = p^2$. One notes that $\{x_i, J^2\} = 0$, so J^2 can be absorbed into redefinition of $x_i : \tilde{x}_i \equiv J^2 x_i$. The algebra acquires then the form

$$\{\tilde{x}_i, \tilde{x}_j\} = \epsilon_{ijk}\tilde{x}_k, \quad \{\tilde{x}_i, p_j\} = \epsilon_{ijk}p_k, \quad \{p_i, p_j\} = -\frac{1}{2J^2}\epsilon_{ijk}\tilde{x}_k. \tag{44}$$

Commutative $(v_i,\ p_i)$**-representation.** In this case the bracket algebra is

$$\{v_i, v_j\} = -\frac{1}{2p^2}v_{[i}p_{j]}, \qquad \{v_i, p_j\} = \frac{1}{2}(\delta_{ij} - \frac{v_i v_j + p_i p_j}{v^2}),$$
$$\{p_i, p_j\} = -\frac{1}{2p^2}v_{[i}p_{j]}. \tag{45}$$

Dynamics turns out to be nontrivial for both variables

$$\dot{v}_i = p_i, \quad \dot{p}_i = -v_i, \quad v_i^2 = p_i^2, \quad v_i p_i = 0, \tag{46}$$

which implies $\ddot{v} + v = 0$ for the configuration space variable. Note that the representation turns out to be symmetric under the change $v \to -p, \ \ p \to v$.

Let us compare these two representations of the particle dynamics on NC sphere. Since the NC geometry has been obtained by using the Dirac bracket, there exists transformation to new variables, in terms of which the bracket acquires the canonical form [16]. The corresponding theorem states that constraints of a theory become a part of the new variables after this transformation. Being applied to the case under consideration, it means that the new variables will have the following structure:

$$(x_i,\ \pi_i,\ v_i,\ p_i) \Longrightarrow (\tilde{x}_i = x_i - \frac{\epsilon_{ijk}v_jp_k}{v^2},\ \pi_i,\ \tilde{v}_3,\ \tilde{p}_3,\ \tilde{v}_a,\ \tilde{p}_a), \quad (47)$$

where $\tilde{v}_a,\ \tilde{p}_a$ are the physical variables with the canonical brackets, in particular: $\{\tilde{v}_a, \tilde{v}_b\} = 0$. From the expression (24), (47) one notes that the theorem naturally selects $(v,\ p)$-representation for transition to the canonical brackets, which is the reason for the name: "commutative representation". From Eq.(46) it follows that the canonical coordinates have nontrivial equations of motion (see also the next section). In contrast, in the NC $(x,\ p)$-representation the configuration space dynamics (42) turns out to be trivial. Thus, the NC description implies that all information on the dynamics is encoded in NC geometry. Similar situation was observed for $SO(n)$ nonlinear sigma-model in [22] and for the Green-Schwarz superstring in the covariant gauge in [23].

$(x_i,\ v_i)$**-representation** coincides with (x_i, p_i)-representation.

5 Particle on commutative sphere with a magnetic monopole at the center and the Rotor.

In this section we show that slight modification of the action (7) gives description for a particle with a monopole at the center of the sphere [24]. It will be demonstrated also that this model is equivalent to the model of usual rotor.

Let us consider the action (7) with the variables x and v interchanged in the first term

$$S = \int d\tau \left[-\epsilon_{ijk}\dot{x}_ix_jv_k - v^2 + \phi(x_i^2 - 1) - V(v^2) \right]. \quad (48)$$

Canonical momentum for x_i is denoted through p_i while π_i corresponds to the variable v_i (the notations are opposite to the ones adopted for the model (7)). In these notations analysis of the model turns out to be similar to the previous case, so we present the final results only. The essential constraints of the theory are

$$\begin{aligned} G_i \equiv p_i + \epsilon_{ijk}x_jv_k = 0, \quad T_i \equiv \pi_i = 0, \\ S \equiv x_i^2 - 1 = 0, \quad \bar{S} \equiv x_iv_i = 0, \end{aligned} \quad (49)$$

and can be taken into account by transition to the Dirac bracket. After that, dynamics of the model can be presented in one of the following three forms.

(x_i, v_i)**-representation.** In terms of these variables the bracket algebra is

$$\{x_i, x_j\} = 0, \quad \{x_i, v_j\} = \epsilon_{ijk}x_k, \quad \{v_i, v_j\} = \epsilon_{ijk}v_k, \tag{50}$$

while their dynamics is governed by the equations (free case)

$$\dot{x}_i = -2\epsilon_{ijk}x_j v_k, \quad \dot{v}_i = 0, \quad x_i^2 - 1 = 0, \quad x_i v_i = 0. \tag{51}$$

For the physical Hamiltonian one has the expression (remember that v is noncommutative variable)

$$H_{ph} = v^2 + V(v^2). \tag{52}$$

The algebra obtained (50) corresponds to the particle on commutative sphere with a monopole at the center (note the relations (37))[24].

$(x_i,\ p_i)$**-representation.** In this case one has the brackets

$$\{x_i, x_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij} - x_i x_j, \quad \{p_i, p_j\} = -(x_i p_j - x_j p_i). \tag{53}$$

Equations of motion turn out to be nontrivial for both variables

$$\dot{x}_i = 2p_i, \quad \dot{p}_i = -2p^2 x_i, \quad x_i^2 - 1 = 0, \quad x_i p_i = 0. \tag{54}$$

which implies $\ddot{x}_i + 4p^2 x_i = 0$ for the configuration space variables. The equations (53), (54) correspond to model of the rotor and can be equally obtained from the action

$$S = \int d\tau \left[\frac{1}{2}\dot{x}^2 + \phi(x^2 - 1)\right]. \tag{55}$$

Thus we have demonstrated canonical equivalence of the models (50), (51) and (53), (54). They correspond to different choices of physical variables in the underlying action (48). Equivalently, they are related by the change of variables $v_i = \epsilon_{ijk}x_j p_k$.

Resolving the remaining constraints from Eq.(54), it is not difficult to find the canonical variables of the model $x_a = \tilde{x}_a,\ p_a = \tilde{p}_a - \frac{(\tilde{x}\tilde{p})}{1+\tilde{x}^2}\tilde{x}_a,\ a = 1, 2$, which obey $\{\tilde{x}_a, \tilde{x}_b\} = \{\tilde{p}_a, \tilde{p}_b\} = 0,\ \{\tilde{x}_a, \tilde{p}_b\} = \delta_{ab}$. Dynamics is governed by the nonlinear equations $\dot{\tilde{x}}_a = \tilde{p}_a +$

$\frac{(\tilde{x}\tilde{p})}{1-\tilde{x}^2}\tilde{x}_a$, $\dot{\tilde{p}}_a = 0$. Let us point out that the relation established between the particle with a monopole and the rotor allows one to construct NC quantum mechanics corresponding to the geometry given in Eq.(50), with the nontrivial potential (52), following the same procedure as in Sect. 2. Ssince the bracket kernel of (53) is degenerated, see Eq.(54), the star product constructed using all six variables turns out to be nonassociative [25]. This subject will be discussed elsewhere.

$(v_i,\ p_i)$**-representation.** For these variables the algebra is ($J_i \equiv \epsilon_{ijk}v_jp_k$)

$$\{v_i, v_j\} = \epsilon_{ijk}v_k, \quad \{v_i, p_j\} = -\frac{1}{p^2}(v_iJ_j - v_jJ_i), \quad \{p_i, p_j\} = -\epsilon_{ijk}v_k, \tag{56}$$

while the equations of motion are similar to (x, v)-representation

$$\dot{v}_i = 0, \quad \dot{p}_i = 2\epsilon_{ijk}v_jp_k, \quad v_i^2 = p_i^2, \quad p_iv_i = 0. \tag{57}$$

Comparing this representation with $(x,\ p)$-representation one observes the same property as for NC sphere: transition from commutative description (53) to NC description (56) implies trivial dynamics for space variables in the latter representation.

Thus we have presented the Lagrangian formulations for a particle on the noncommutative sphere (7) as well as for a particle on the commutative sphere with a monopole at the center (48), the latter is shown to be canonically equivalent to the model of rotor. In both cases the desired algebraic structure (41), (50) arises as the Dirac bracket corresponding to the second class constraints presented in the model. After introduction of the Dirac bracket, the constraints can be used to represent part of variables through the remaining ones. There exist several ($SO(3)$ covariant) possibilities to choose the basic variables, which leads to different representations for the two models. In both cases there is the "commutative representation" which is appropriate for determining the canonical variables starting from the known constraint system. Using relation between NC and commutative representations one is able to construct quantum mechanics which corresponds to the NC representation.

In conclusion, let us comment on possible generalizations of the model (7). One possibility is to consider immersion of the model into a locally invariant system. Let us omit the term $\phi(x^2 - 1)$ in the action (1). Then the formulation involves one first class constraint which corresponds to the local symmetry $\delta x_i = \gamma v_i$. Thus, one is able now to consider different gauges of the model ($x^2 - 1 = 0$ and $v^2 - 1 = 0$ are equally admissible now). We suggest that it can give unified description of the three models considered in this work. Other possibility may be NC quantum mechanics on three-dimensional plane. To construct it, one needs to modify the action (7) in such a way that only primary constraints of the type (24) are generated and form the second class system. These problems will be considered elsewhere.

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