On T-Duality in Brane Gas Cosmology

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In the context of homogeneous and isotropic superstring cosmology, the T-duality symmetry of string theory has been used to argue that for a background space-time described by dilaton gravity with strings as matter sources, the cosmological evolution of the Universe will be nonsingular. In this Letter we discuss how T-duality extends to brane gas cosmology, an approximation in which the background space-time is again described by dilaton gravity with a gas of branes as a matter source. We conclude that the arguments for nonsingular cosmological evolution remain valid.

I. INTRODUCTION

In [1] it was suggested that due to a new string theory-specific symmetry called T-duality, string theory has the potential to resolve the initial singularity problem of Standard Big Bang cosmology, a singularity which also plagues scalar field-driven inflationary cosmology [2, 3].

The framework of [1] was based on an approximation in which the mathematical background space-time is described by the equations of dilaton gravity (see [4, 5]), with the matter source consisting of a gas of strings. The background spatial sections were assumed to be toroidal such as to admit one-cycles. Thus, the degrees of freedom of the string gas consist of winding modes in addition to the momentum modes and the oscillatory modes. Then, both momentum and winding numbers take on discrete values, and the energy spectrum of the theory is invariant under inversion of the radii of the torus, i.e. $R \to \alpha'/R$, where $\alpha'^{1/2}$ is the string length l_s. The mass of a state with momentum and winding numbers n and ω, respectively, in a compact space of radius R, is the same as that of a state with momentum and winding numbers ω and n, respectively, in the space of radius α'/R. This symmetry was used to argue [1] that as the radii of the torus decrease to very low values, no physical singularities will occur. Firstly, under the assumption of thermal equilibrium, the temperature of a string gas at radius R will be equal to the temperature of the string gas at radius α'/R. Secondly, any process computed for strings on a space with radius R, is identical to a dual process computed for strings on a space with radius α'/R. Therefore there exists a 'minimal' radius $\alpha'^{1/2}$ in the sense that physics on length scales below this radius can equally well be described by physics on length scales larger than that.

Since the work of [1] our knowledge of string theory has evolved in important ways. In particular, it has been realized [6] that string theory must contain degrees of freedom other than the perturbative string degrees of freedom used in [1]. These new degrees of freedom are Dp-branes of various dimensionalities (depending on which string theory one is considering). Since the T-duality symmetry was used in an essential way (see e.g. [7]) to arrive at the existence of Dp-branes (p-branes for short in the following), it is clear that T-duality symmetry will extend to a cosmological scenario including p-branes. However, since a T-duality transformation changes the dimensionality of branes, it is useful to explicitly verify that the arguments of [1] for a nonsingular cosmological evolution carry over when the gas of perturbative string modes is generalized to a gas of branes. A model for superstring cosmology in which the background space-time is described (as in [4, 5]) by dilaton gravity, and the matter source is a gas of branes, has recently been studied under the name of "brane gas cosmology" [8, 9] (see also [10, 11] for extensions to backgrounds which are not toroidal, and [12] for an extension to an anisotropic background).

In this Letter, we establish the explicit action of T-duality in the context of brane gas cosmology on a toroidal background. For a solution of the background geometry appropriate for cosmological considerations in which the radii of the torus are decreasing from large to small values as we go back in time, we must consider T-dualizing in all spatial dimensions. We demonstrate that the mass spectrum of branes remains invariant under this action. Thus, if the background dynamics is adiabatic, then the temperature of the brane gas will be invariant under the change $R \to \alpha'/R$, i.e.

$$T(R) = T\left(\frac{\alpha'}{R}\right), \tag{1}$$

thus demonstrating that superstring cosmology can avoid the temperature singularity problem of standard and inflationary cosmology. Similarly, our results can be used to show that the arguments for the existence of a minimal physical length given in [1] extend to brane gas cosmology.

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The outline of this Letter is as follows. In the following section we give a brief review of brane gas cosmology. In Section III we (partially re-)derive the energy, the momentum, and the pressure for p-branes. Next, we review the action of T-duality on winding states of p-branes. The main section of this Letter is Section V in which we show that the mass spectrum of a p-brane gas of superstring theory is invariant under T-duality. Section VI contains a discussion of some implications of the result, and conclusions. A word on notation: the string coupling constant is denoted by g, and, since the original string cosmology of [1] was formulated in terms of weakly coupled string theory, we will assume weak string coupling, i.e. $g \ll 1$.

II. REVIEW OF BRANE GAS COSMOLOGY

As already mentioned in the Introduction, the framework of brane gas cosmology consists of a homogeneous and isotropic background of dilaton gravity coupled to a gas of p-branes as a matter source. We are living in the bulk [1].

The initial conditions in the early Universe are 'conservative' and 'democratic'; conservative in the sense that they are close to the initial conditions assumed to hold in standard big bang cosmology (i.e. a hot dense gas of matter), democratic in the sense that all 9 spatial dimensions of critical string theory are considered on an equal basis [2]. Thus, matter is taken to be a gas of p-branes of all allowed values of p in thermal equilibrium. In particular, all modes of the branes are excited, including the winding modes.

The background space-time is taken to be $\mathcal{R} \times T^9$ where T^9 denotes a nine-torus. The key feature of T^9 which is used in the analysis is the fact that it admits one-cycles which makes it possible for closed strings to have conserved winding numbers [3]. It is also assumed that the initial radius in each toroidal direction is the same, and comparable to the self-dual radius $\alpha'^{1/2}$. Initially, all directions are expanding isotropically with $R > \alpha'^{1/2}$ (the extension to anisotropic initial conditions has recently been considered in [12]).

As shown in this Letter, as a consequence of T-duality symmetry, brane gas cosmology provides a background evolution without cosmological singularities. The scenario also provides a possible dynamical explanation for why only three spatial dimensions can become large. Winding modes (and thus T-duality) play a crucial role in the argument. Let us first focus on the winding modes of fundamental strings [1].

The winding and anti-winding modes ω and $\bar{\omega}$ of the strings are initially in thermal equilibrium with the other states in the string gas. Thermal equilibrium is maintained by the process

$$\omega + \bar{\omega} \rightleftarrows \text{loops, radiation.} \tag{2}$$

When strings cross each other, they can intercommute such that a winding and an anti-winding mode annihilate, producing fundamental string loops or radiation without winding number. This process is analogous to infinite cosmic strings intersecting and producing cosmic string loops and radiation during their interaction (see e.g. [14, 15] for reviews of cosmic string dynamics).

As the spatial sections continue to expand, matter degrees of freedom will gradually fall out of equilibrium. In the context of string gas cosmology with $R > \alpha'^{1/2}$, the winding strings are the heaviest objects and will hence fall out of equilibrium first. Since the energy of a winding mode is proportional to R, Newtonian intuition would imply that the presence of winding modes would prevent further expansion. This is contrary to what would be obtained by using the Einstein equations. However, the equations of dilaton gravity yield a similar result to what is obtained from Newtonian intuition [4]: the presence of winding modes (with negative pressure) acts as a confining potential for the scale factor.

As long as winding modes are in thermal equilibrium, the total energy can be minimized by transferring it to momentum or oscillatory modes (of the fundamental string). Thereby, the number of winding modes decreases, and the expansion can go on. However, if the winding modes fall out of equilibrium, such that there is a large number of them left, the expansion is slowed down and eventually stopped. If we now try to make d of the original 9 spatial dimensions much larger than the string scale, then an obstruction is encountered if $d > 3$: in this case the probability for crossing and therefore for equilibrating according to the process (2) is zero. On the other hand, in a three-dimensional subspace of the nine-torus, two strings will generically meet. Therefore, the winding modes can annihilate, thermal equilibrium can be maintained, and, since the decay modes of the winding strings have positive pressure, the expansion can go

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[1] In this sense, brane gas cosmology is completely different in ideology than brane world scenarios in which it is assumed (in general without any dynamical explanation) that we live on a specific brane embedded in a warped bulk space-time. From the point of view of heterotic M-theory [13], our considerations should be viewed as applying to the 10 dimensional orbifold space-time on which we live.

[2] For consistency, critical superstring theories need a 10 dimensional target space-time which is in apparent contradiction with the observed four. Usually it is assumed that six dimensions are compactified from the outset due to some unknown physics. However, following the usual approach in cosmology it seems more natural that initially all nine spatial dimensions were compact and small, and that three of them have grown large by a dynamical *decompactification* process. The scenario originally proposed in [1] offers such a dynamical decompactification mechanism.

[3] Recently, the scenario of [1, 8] was generalized [10, 11] to spatial backgrounds such as Calabi-Yau manifolds which admit 2-cycles but no 1-cycles.

on [4]. As a result, three dimensions of the torus will grow large while the other six stay small (of size $R \sim \alpha'^{1/2}$). Observationally there is no obstruction against our three space dimensions being large and compact as long as their radius is bigger than the Hubble radius today.

It is not hard to include p-branes into the above scenario [8]. Now the initial state is a hot, dense gas of all branes allowed in a particular theory. In particular, brane winding modes are excited, in addition to modes corresponding to fluctuations of the brane. Since the winding modes play the most important role in the dynamical decompactification mechanism of [1, 8], we will focus our attention on these modes. The analogous classical counting argument as given above for strings yields the result that p-brane winding modes can interact in at most $2p+1$ spatial dimensions. Since for weak string coupling and for spatial sizes larger than the self-dual radius the mass of a p-brane (with a fixed winding number in all of its p spatial dimensions) increases as p increases, p-branes will fall out of equilibrium earlier the larger p is. Thus, e.g. in a scenario with 2-branes, these will fall out of equilibrium before the fundamental strings and allow five spatial dimensions to start to grow [8]. Within these five spatial dimensions, the fundamental string winding modes will then allow only a three-dimensional subspace to become macroscopic. Thus, a hierarchy of internal dimensions is generated dynamically.

III. ENERGY, MOMENTUM, AND PRESSURE OF P-BRANES

This section is devoted to the derivation of physical quantities describing the brane gas which determine the cosmological evolution of the background space-time.

Starting from the Dirac-Born-Infeld action, we obtain expressions for the energy and the momentum of a p-brane in $D = d + 1$ dimensional space-time. We show that there is no momentum flowing along the p tangential directions. From the energy–momentum tensor one can also define a pressure, and hence an equation of state, for the whole brane gas. Even though some of the results in this section are already known, we find it useful to give a self-consistent overview.

Let $\sigma = (\sigma^0, \sigma^i), i = 1, \cdots, p$, denote some intrinsic coordinates on the worldsheet of a p-brane. Its position (or embedding) in D-dimensional space time is described by $x^\mu = X^\mu(\sigma)$, where $\mu = 0, \cdots, d$, and the induced metric is $\gamma_{ab} = \eta_{\mu\nu} X^\mu_{,a} X^\nu_{,b}$, where $a, b = 0, \cdots, p$.

The Dirac-Born-Infeld action is (in the string frame)

$$S_p = -T_p \int d^{p+1}\sigma e^{-\phi} \sqrt{-\gamma} \,, \qquad (3)$$

where T_p denotes the tension (charge) of a p-brane, $\gamma = \det(\gamma_{ab})$, and ϕ is the dilaton of the compactified theory. For our adiabatic considerations, we assume that it is constant (taking its asymptotic value), and absorb it into a physical tension

$$\tau_p = e^{-\phi} T_p = \frac{T_p}{g} = \frac{1}{(2\pi)^p} \frac{1}{g \alpha'^{(p+1)/2}} \,, \qquad (4)$$

where in the final step we have used the expression for T_p (see e.g. [7]). Note that for any p-brane τ_p goes like $1/g$, and that hence, in the weak string coupling regime which we are considering, the branes are heavy. The action (3) can be written as an integral over D-dimensional space-time

$$S_p = \int d^D x \left(-\tau_p \int d^{p+1}\sigma \delta^{(D)}(x^\mu - X^\mu(\sigma)) \sqrt{-\gamma} \right) . \qquad (5)$$

As the integration domain is a torus, both integrals are finite.

Varying the action (5) with respect to the background metric and comparing with the usual definition of the space-time energy-momentum tensor, one obtains[5]

$$\begin{aligned} &T^{\alpha\beta}(x^\mu) = \qquad (6) \\ &- \tau_p \int d^{p+1}\sigma \delta^{(D)}(x^\mu - X^\mu(\sigma)) \sqrt{-\gamma} \gamma^{ab} X^\alpha_{,a} X^\beta_{,b} \,. \end{aligned}$$

The DBI action (3) is invariant under $p+1$ reparametrizations $\sigma \to \tilde{\sigma}(\sigma)$, and we can use this freedom to choose

$$\gamma_{00} = -\sqrt{-\gamma} \,, \qquad \gamma_{0i} = 0 \,. \qquad (7)$$

Notice that $\det(\gamma_{ij}) = -\sqrt{-\gamma}$ and $\gamma^{ik}\gamma_{kj} = \delta^i_j$. By choosing this gauge, we do not specify a particular embedding which will be convenient later when treating a brane gas, where the branes have arbitrary orientations. Furthermore, it is consistent to set $X^0 = \sigma^0$.

To calculate the energy E_p of a p-brane, one observes that in the gauge (7)

$$\begin{aligned} &\gamma^{ab} X^0_{,a} X^0_{,b} = \gamma^{00} = -\frac{1}{\sqrt{-\gamma}} \\ &\Rightarrow \quad T^{00}(x^\mu) = \tau_p \int d^{p+1}\sigma \delta^{(D)}(x^\mu - X^\mu(\sigma)) \,. \qquad (8) \end{aligned}$$

Writing $x^\mu = (t, x^n), n = 1, \cdots, d$, and splitting the delta-function into a product, the integral over σ^0 can be

[4] Since quantum mechanically, the thickness of the strings is given by the string length [16], it is important for the brane gas scenario that the initial size of the spatial sections was string scale. Otherwise, it would always be the total dimensionality of space which would be relevant in the classical counting argument of [1], and there could be no expansion in any direction.

[5] In curved backgrounds the energy-momentum tensor gets multiplied by $\sqrt{-g}$.

carried out. The energy density of a p-brane in d spatial dimensions is

$$\rho_p \equiv T^{00}(t, x^n) = \tau_p \int d^p\sigma \delta^{(d)}(x^n - X^n(t, \sigma^i)), \quad (9)$$

and its total energy is

$$E_p = \int d^d x \rho_p = \tau_p \int d^p\sigma = \tau_p Vol_p\,. \quad (10)$$

The volume of a p-brane in its rest frame, Vol_p, is finite as the brane is wrapped around a torus. Eq. (10) provides a formula for the lowest mass state, $M_p = E_p$, which will be used in section V. As expected intuitively, the minimal mass is equal to the tension times the volume of a brane.

To calculate the space-time momentum P_p^n of a p-brane, one first evaluates

$$\gamma^{ab} X^0_{,a} X^n_{,b} = -\frac{X^n_{,0}}{\sqrt{-\gamma}}$$
$$\Rightarrow \quad T^{0n}(x^\mu) = \tau_p \int d^{p+1}\sigma \delta^{(D)}(x^\mu - X^\mu(\sigma)) X^n_{,0} \quad (11)$$

Proceeding similarly as before, the total momentum of a p-brane is found to be

$$P_p^n = \tau_p \int d^p\sigma \dot{X}^n(t, \sigma^i)\,, \quad (12)$$

where the dot denotes the derivative w.r.t. time t. The gauge conditions (7) can be written as $0 = \gamma_{0i} = \dot{X}^m X_{m,i}$, where the sum over $m = 1, \cdots, d$ is the ordinary Euclidean scalar product. This is equivalent to saying that the (spatial) velocity vector $\dot{X}$ is perpendicular onto each of the tangential vectors $X^m_{,i}$. Therefore, only the transverse momentum is observable [6]. Assuming that the brane is a pointlike classical object w.r.t. the transverse directions, this momentum is not quantized despite of the compactness of space. In particular, the question whether there might exist a T-duality correspondence between transverse momentum modes and winding modes does not arise. Moreover, we neglect the possibility of open strings travelling on the brane which would in fact lead to a non-zero tangential momentum.[7] Hence, in what follows, we focus on the zero modes of p-branes.

Finally, the pressure $\mathcal{P}_p$ of a p-brane is given by averaging over the trace T^m_m. First, notice that

$$\begin{aligned}\gamma^{ab} X^m_{,a} X_{m,b} &= \gamma^{00} X^m_{,0} X_{m,0} + \gamma^{11}\gamma_{11} + \cdots + \gamma^{pp}\gamma_{pp} \\ &\quad + 2\gamma^{12}\gamma_{21} + \cdots + 2\gamma^{1p}\gamma_{p1} \\ &\quad + 2\gamma^{23}\gamma_{32} + \cdots + 2\gamma^{2p}\gamma_{p2} + \cdots \\ &\quad + 2\gamma^{p-1,p}\gamma_{p,p-1} \\ &= -\frac{1}{\sqrt{-\gamma}} X^m_{,0} X_{m,0} + p\,. \end{aligned} \quad (13)$$

In the first step we have used that the products of the embedding functions can be expressed in terms of the induced metric, e.g. $\gamma_{11} = X^m_{,1} X_{m,1}$, and in the second step the fact that $\gamma^{ik}\gamma_{kj} = \delta^i_j$. Inserting this into Eq. (6), eliminating the remaining $\sqrt{-\gamma}$ by $-\sqrt{-\gamma} = \gamma_{00} = -1 + X^m_{,0} X_{m,0}$, and integrating out the σ^0 dependence, one finds

$$\begin{aligned} &T^m_m(t, x^n) = \\ &\tau_p \int d^p\sigma \delta^{(d)}(x^n - X^n(t, \sigma^i))[(p+1)\dot{X}^m \dot{X}_m - p]\,. \end{aligned} \quad (14)$$

The quantity $\dot{X}^m \dot{X}_m$ is the squared velocity of a point on a brane parameterized by (t, σ^i). We define the mean squared velocity of the branes in the gas by averaging over all σ^i, i.e. $v^2(t) \equiv \langle \dot{X}^m \dot{X}_m \rangle$. In the averaged trace, $\langle T^m_m \rangle$, the velocity term can be taken out of the integral. Comparing with Eq. (9), one obtains the equation of state of a p-brane gas

$$\mathcal{P}_p \equiv \frac{1}{d}\langle T^m_m \rangle = \left[\frac{p+1}{d} v^2 - \frac{p}{d}\right] \rho_p\,. \quad (15)$$

In the relativistic limit ($v^2 \to 1$) the branes behave like ordinary relativistic particles: $\mathcal{P}_p = \frac{1}{d}\rho_p$, whereas in the non-relativistic limit ($v^2 \to 0$) $\mathcal{P}_p = -\frac{p}{d}\rho_p$. For domain walls this result was obtained in [18].

The pressure $\mathcal{P}_p$ and the energy density ρ_p are the source terms in the Einstein equations for the brane gas [4, 8].

IV. WINDING STATES AND T-DUALITY

We briefly review some of the properties of T-duality that are needed subsequently.

Consider a nine-torus T^9 with radii $(R_1, \cdots, R_9)$. Under a T-duality transformation in n-direction

$$R_n \to R'_n = \frac{\alpha'}{R_n}\,, \quad (16)$$

and all other radii stay invariant. T-duality also acts on the dilaton (which is constant in our case), and hence on the string coupling constant, as

$$g \to g' = \frac{\alpha'^{1/2}}{R_n} g\,. \quad (17)$$

Note, however, that the fundamental string length $l_s \equiv \alpha'^{1/2}$ is an invariant. The transformation law (17) follows

[6] Note the analogy with topological defects in field theory, where also only the transverse momentum of the defects - here taken to be straight - is observable.

[7] A relation between brane winding modes and open string momentum modes, quantized as n_m/R_m, was shown in [17].

from the requirement that the gravitational constant in the effective theory remains invariant under T-duality. In general, T-duality changes also the background geometry. However, a Minkowski background (as we are using here) is invariant.

For a p-brane on T^9, a particular winding state is described by a vector $\omega = (\omega_1, \cdots, \omega_9)$. There are $9!/[p!(9-p)!]$ such vectors corresponding to all possible winding configurations. For illustration take a 2-brane on a three-torus: it can wrap around the (12), (13), (23) directions, and hence there are $3!/2! = 3$ vectors $\omega = (\omega_1, \omega_2, 0), \omega = (\omega_1, 0, \omega_3), \omega = (0, \omega_2, \omega_3)$.

Whereas T-duality preserves the nature of a fundamental string, it turns a p-brane into a different object. To see this consider a brane with p single windings $\omega = (1, \cdots, 1, 0, \cdots, 0)$ which represents a p-dimensional hypersurface on which open strings end. Along the brane the open string ends are subject to Neumann boundary conditions. These become Dirichlet boundary conditions on the T-dual coordinate R'_n (if n denotes a tangential direction), i.e. for each string endpoint R'_n is fixed. Thus a T-duality in a tangential direction turns a p-brane into a (p-1)-brane. Similarly, a T-duality in an orthogonal direction turns it into a (p+1)-brane (see e.g. [7] for more details).

Next, consider a T-duality transformation in a direction in which the p-brane has multiple windings $\omega_n > 1$. One obtains a number ω_n of (p-1)-branes which are equally spaced along this direction. As an example take a 1-brane with winding $\omega_1 = 2$ on a circle with radius R_1. This configuration is equivalent to a 1-brane with *single* winding on a circle with radius $2R_1$. T-dualizing in 1-direction gives a single 0-brane on a circle of radius $\alpha'/2R_1$ which is equivalent to two 0-branes on a circle of radius α'/R_1 (see e.g. [19]). Since applying a T-duality transformation twice in the same direction yields the original state (up to a sign in the RR field), also the inverse is true: a number ω_n of (p-1)-branes correspond to a single p-brane with winding ω_n.

So far we have discussed T-duality transformations in a single direction. For applications to isotropic brane gas cosmology we need to consider T-dualizing in all nine spatial directions. Given a gas of branes, $\mathcal{B}$, on a nine-torus with radii $(R_1, \cdots, R_9)$ consisting of a large number of branes of all types admitted by a particular string theory, we want to find the corresponding gas $\mathcal{B}^*$ on the dual torus $\mathcal{T}^*$ with radii $(R'_1, \cdots, R'_9)$. To that end one performs a T-duality transformation in each of the nine spatial directions. From what we have discussed so far, it is now easy to see that a p-brane in a winding state $\omega = (\omega_1, \cdots, \omega_p, 0, \cdots, 0)$ is mapped into a number $\omega_1 \cdots \omega_p$ of (9-p)-branes, each of which is in a state $\omega^* = (0, \cdots, 0, 1, \ldots, 1)$. The (9-p)-brane wraps in the (9-p) directions orthogonal to the original p-brane. It is clear that the above considerations hold for any winding configuration.

After these preparatory steps, we now turn to the main part of this Letter.

V. MASS SPECTRA AND T-DUALITY

A. Masses of p-branes with single winding

In this section we show that each mass state in a brane gas $\mathcal{B}$ has a corresponding state with equal mass in the brane gas $\mathcal{B}^*$. Based on type IIA superstring theory we take $\mathcal{B}$ to consist of 0, 2, 4, 6 and 8 branes. Then, by the discussion in the preceding section, the brane gas $\mathcal{B}^*$ contains 9, 7, 5, 3, 1 branes which are the states of type IIB as we have carried out an odd number (nine) of T-duality transformations. Notice that this follows from the T-duality symmetry for fundamental strings, not from T-duality arguments applied to the above brane gases which we actually want to show. Our demonstration is done by carrying out explicitly nine T-duality transformations on a mass state in $\mathcal{B}$, and showing that there is a corresponding and equal mass state in $\mathcal{B}^*$. In this sense the two brane gases are T-dual.

Suppose that the branes in $\mathcal{B}$ are wrapped around some of the cycles of a nine-torus with radii $(R_1, \cdots, R_9)$. Then, the volume Vol_p of a p-brane in Eq. (10) is simply the product of the p circumferences, and the minimal masses $M_p = E_p$ (in the string frame) are

$$M_0 = \frac{1}{g\alpha'^{1/2}} , \tag{18}$$

$$M_2 = (2\pi)^2 R_9 R_8 \tau_2 = \frac{R_9 R_8}{g\alpha'^{3/2}} , \tag{19}$$

$$M_4 = (2\pi)^4 R_9 R_8 R_7 R_6 \tau_4 = \frac{R_9 R_8 R_7 R_6}{g\alpha'^{5/2}} , \tag{20}$$

$$M_6 = (2\pi)^6 R_9 \cdots R_4 \tau_6 = \frac{R_9 \cdots R_4}{g\alpha'^{7/2}} , \tag{21}$$

$$M_8 = (2\pi)^8 R_9 \cdots R_2 \tau_8 = \frac{R_9 \cdots R_2}{g\alpha'^{9/2}} , \tag{22}$$

(see also [7]) where in the second step we have used expression (4) for the tension of a p-brane. For notational convenience we have fixed a particular winding configuration. The argument is generalized for arbitrary winding configurations and winding numbers at the end of this section. If, as we have assumed, $R_n > \alpha'^{1/2}$, then the heaviest object in the theory is the 8-brane.

The DBI action (3) is invariant under T-duality. Hence, all formulae derived from it (energy, mass and pressure) are valid in both the original brane gas $\mathcal{B}$ and in the dual brane gas $\mathcal{B}^*$. Thus, the mass spectrum of the $\mathcal{B}^*$ brane gas is

$$M_9^* = (2\pi)^9 R'_9 \cdots R'_1 \tau_9^* = \frac{R'_9 \cdots R'_1}{g^*\alpha'^{10/2}} , \tag{23}$$

$$M_7^* = (2\pi)^7 R'_7 \cdots R'_1 \tau_7^* = \frac{R'_7 \cdots R'_1}{g^*\alpha'^{8/2}} , \tag{24}$$

$$M_5^* = (2\pi)^5 R'_5 \cdots R'_1 \tau_5^* = \frac{R'_5 \cdots R'_1}{g^*\alpha'^{6/2}} , \tag{25}$$

$$M_3^* = (2\pi)^3 R'_3 R'_2 R'_1 \tau_3^* = \frac{R'_3 R'_2 R'_1}{g^*\alpha'^{4/2}} , \tag{26}$$

$$M_1^* = 2\pi R_1' \tau_1^* = \frac{R_1'}{g^* \alpha'} \,. \tag{27}$$

Since $R_i' < \alpha'^{1/2}$, the heaviest brane of the dual gas $\mathcal{B}^*$ is now the 1-brane. The coupling constant in $\mathcal{B}^*$ is given by

$$g^* = \frac{\alpha'^{9/2}}{R_9 \cdots R_1} g \,. \tag{28}$$

Note that if the radii $(R_1, \cdots, R_9)$ of the initial nine-torus are bigger than the self-dual radius $\alpha'^{1/2}$, then $g^* < g$, and thus the assumption of a small string coupling constant is safe.

Given the two mass spectra, one can easily verify that each state in the brane gas $\mathcal{B}$ has a corresponding state with equal mass in the dual brane gas $\mathcal{B}^*$:

$$M_{9-p}^* = M_p \,. \tag{29}$$

This establishes explicitly that the T-duality of the string gas used in [1] extends to the brane gas cosmology of [8].

As an explicit example, consider a 2-brane wrapped around the 8 and 9 directions. Its mass is (19)

$$M_2 = \frac{R_8 R_9}{g \alpha'^{3/2}} \,. \tag{30}$$

If we replace the string coupling constant g by the dual string coupling constant g^* via (28), and the radii R_8 and R_9 by the dual radii R_8' and R_9' via (16), one obtains

$$M_2 = \frac{R_1' \cdots R_7'}{g^* \alpha'^{8/2}} = M_7^* \,. \tag{31}$$

In the above example, we have specified a particular winding configuration for simplicity, but clearly the argument holds as well in the general case, where a p-brane wraps around some directions $n_1 \cdots n_p$:

$$M_p = (2\pi)^p R_{n_1} \cdots R_{n_p} \tau_p = \frac{R_{n_1} \cdots R_{n_p}}{g \alpha'^{(p+1)/2}} \,. \tag{32}$$

Via the same steps as in the above example, it follows that

$$\begin{aligned} M_{9-p}^* &= (2\pi)^{9-p} R_{m_1}' \cdots R_{m_{9-p}}' \tau_{9-p}^* = \frac{R_{m_1}' \cdots R_{m_{9-p}}'}{g^* \alpha'^{(10-p)/2}} \\ &= M_p \,, \end{aligned} \tag{33}$$

where $\{m_1, \cdots, m_{9-p}\} \neq \{n_1, \cdots, n_p\}$.

B. Multiple windings

Consider now a p-brane with multiple windings $\omega = (\omega_1, \cdots, \omega_p, 0, \cdots, 0)$. Its mass is

$$M_p(\omega) = \omega_1 \cdots \omega_p M_p \,. \tag{34}$$

In the $\mathcal{B}^*$ brane gas this corresponds to a number $\omega_1 \cdots \omega_p$ of (9-p)-branes each with winding $\omega^* = (0, \cdots, 0, 1, \cdots, 1)$ and mass M_{9-p}^*. Since $M_{9-p}^* = M_p$, the total mass of this 'multi-brane' state is equal to the mass of the original brane, namely

$$(\omega_1 \cdots \omega_p) M_{9-p}^* = M_p(\omega) \,, \tag{35}$$

which establishes the correspondence of $\mathcal{B}$ and $\mathcal{B}^*$ in the case of multiple windings.

One should also add fundamental strings to the brane gas $\mathcal{B}$. Since their mass squared is

$$M^2 = \left(\frac{n_n}{R_n}\right)^2 + \left(\frac{\omega_n R_n}{\alpha'}\right)^2 \,, \tag{36}$$

is its clear that every fundamental string state in $\mathcal{B}$ has a corresponding state in $\mathcal{B}^*$ when $n_n \leftrightarrow \omega_n$.

VI. COSMOLOGICAL IMPLICATIONS AND DISCUSSION

We have demonstrated explicitly how T-duality acts on a brane gas in a toroidal cosmological background, and have in particular shown that the mass spectrum of the theory is invariant under T-duality. Thus, the arguments of [1] which led to the conclusion that cosmological singularities can be avoided in string cosmology extend to brane gas cosmology.

Whereas T-duality does not change the nature of fundamental strings, but simply interchanges winding and momentum numbers, it changes the nature of branes: after T-dualizing in all d spatial dimensions, a p-brane becomes a $(d-p)$-brane which, however, was shown to have the same mass as the 'original' brane.

In [8] it was shown that the dynamical decompactification mechanism proposed in [1] remains valid if, in addition to fundamental strings, the degrees of freedom of type IIA superstring theory are enclosed. We briefly comment on the decompactification mechanism in the presence of a type IIB brane gas on a nine-torus. As before, we assume a hot, dense initial state where, in particular, the brane winding modes are excited and in thermal equilibrium with the other degrees of freedom. All directions of the torus are roughly of string scale size, l_s, and start to expand isotropically. For the 9-,7, and 5-brane winding modes there is no dimensional obstruction to continuously meet and to remain in equilibrium, thereby transferring their energy to less costly momentum or oscillatory modes: these degrees of freedom do not constrain the number of expanding dimensions. However, the 3-branes allow only seven dimensions to grow further, and out of these, three dimensions can become large due to the 1-branes and the fundamental strings. As far as the 'intercommutation' and equilibration process is concerned, the 1-branes and the strings play the same role, but since the winding modes of the former are heavier ($\frac{R}{g\alpha'} \gg \frac{R}{\alpha'}$ at weak coupling), they disappear earlier.

We have focused our attention on how T-duality acts on brane winding modes. However, since in a hot and dense initial state we expect all degrees of freedom of a brane to be excited, we should also include transverse fluctuations (oscillatory modes) in our considerations concerning T-duality. To our knowledge, the quantization of such modes is, however, not yet understood, and we leave this point for future studies.

Another interesting issue is to investigate how the present picture of brane gas cosmology gets modified when gauge fields on the branes are included. These correspond to $U(N)$ Chan-Paton factors at the open string ends. In this case, a T-duality in a transverse direction yields a number N of parallel (p-1)-branes at different positions [20].

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