An Algorithm for the Microscopic Evaluation of the Coefficients of the Seiberg-Witten Prepotential

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Abstract

A procedure, allowing to calculate the coefficients of the SW prepotential in the framework of the instanton calculus is presented. As a demonstration explicit calculations for 2, 3 and 4− instanton contributions are carried out.

To the memory of Lochlainn O'Raifeartaigh

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1 Introduction

One of the challenges put forward by the Seiberg-Witten proposal [1] for an exact expression of the prepotential in $N = 2$ extended $d = 4$ Super Yang-Mills theory is its verification through microscopic instanton calculations. Dorey, Khoze and Mattis made the first steps into that direction determining the 1 and 2-instanton contribution to the prepotential [2]. (The 1-instanton contribution was already calculated before in references [3] and [4].) These authors also derived a representation of the integration measure of instanton moduli in general [5]. After this work further progress was hampered by the fact that beyond instanton number 3 the ADHM (Atiyah, Drinfeld, Hitchin, Manin [6]) moduli space is not known explicitly. In [7], [8] we derived a standard algebraic-geometric form, to be reproduced below, representing the expansion coefficients of the prepotential as integral of the exponential of an equivariantly exact mixed differential form. This can be reduced in the k-instanton sector to an integral of a (4k-3)-fold wedge product of a closed differential 2-form, which is the formal representative of the Euler class of the moduli space viewed as U(1) principal bundle. So we were led to an algebraic-geometric interpretation of the coefficients. But we failed in our main objective, which was to calculate via localization technique concrete numbers. The reason for our failure was clarified and cured in a recent work by Hollowood [9], where it is argued that one should, to start with, resolve the short distance singularities of the moduli space[1].

An elegant method to achieve the resolution consists in deforming the instanton configurations into a non-commutative domain [10]. The parameter measuring the non-commutativity, a new length scale, acts as an ultraviolet regulator, while the integrals in question are - as formally shown below- insensitive to the deformation. The integrals can now be localized at the surface of critical points of the abelean vector field which goes with the remaining torus symmetry after spontaneous breakdown of the non-abelian gauge symmetry of the N=2 Yang-Mills theory[2].

The critical surface happens to coincide with the the manifold of non-commutative U(1) instantons (imbedded as submanifold into the ADHM moduli space of the non-abelian gauge group). The remaining integral along the critical surface has according to the recipes of the equivariant localization technique [11], [12] the appearance of a sum of characteristic classes of the unitary bundle connected with the ADHM construction. Their evaluation is for general

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[1]R.F. thanks Francesco Fucito for a discussion of this point.

[2]Localization without regularization renders a vanishing residuum at the corresponding critical surface.

instanton number still a highly cumbersome task.

A conceivable way to overcome the difficulties relies on the deformation invariance of the characteristic classes. So one may follow backwards a gradient flow along a Gl(k) orbit [13] (chapter 6), [14], which relates the relevant matrices of the ADHM construction to diagonal matrices. We will not follow this approach here since we were reminded meanwhile by Nekrasov [15] of a technically simpler method[3]. The idea, already used some time ago in a macroscopical context [16], [17], is to modify the vector field underlying the localization method by adding pieces corresponding to space rotations. The critical set of the modified vector field consists of discrete points and the evaluation of the integrals is reduced to a manageable task: to find the eigenvalues of the vector field action on the tangent space of the respective critical point. The purpose of this paper is to show that the program can be implemented on the microscopic level following the conceptual lines of our previous papers [7], [8]. Similar results have been announced in [15]. We collect in the following section 2 some material concerning the instanton calculus. Section 3 is devoted to a description of the algorithm allowing to determine (in principle) the contributions from arbitrary instanton sectors. We will work out for the purpose of illustration the 2 ,3 and 4-instanton coefficient of the pure N=2 vector theory (i.e. without matter multipletts) with SU(2) as underlying gauge group in section 4. We end this section presenting a formula for the determinant of the deformed vector field action on the neighborhood of an arbitrary critical point in the general case of $SU(N)$ as gauge group.

2   Instanton calculus

We quote the ADHM data for the construction of $SU(2)$ instantons in a form which is generalizable to $SU(N)$, $N \geq 2$. (The restriction to $SU(2)$ is a matter of convenience; all our results can easily be extended to $SU(N)$). These data consist of complex matrices, two $k \times k$ matrices B_1, B_2, a $k \times 2$ matrix I and a $2 \times k$ matrix J, fulfilling a certain rank condition (cf. [6], [18]) and obeying the relations

$$[B_1, B_2] + IJ = 0; \tag{2.1}$$

$$\left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + II^\dagger - J^\dagger J = 0, \tag{2.2}$$

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[3]We still believe that it is worthwhile to pursue attempts either to evaluate the integrals without regularization or to handle the U(1)-instanton manifolds. It may lead to new mathematical insights.

where "†" denotes hermitean conjugation. The data are redundant in the sense that sets of matrices related by $U(k)$ transformations,

$$\mathcal{A} \equiv (B_1, B_2, I, J) \leftrightarrow \mathcal{A}',$$

$$B_i' = gB_ig^{-1}, \;\; I' = gI, \;\; J' = Jg^{-1};$$

$$i = 1, 2; \;\; g \in U(k) \tag{2.3}$$

give rise to the same self dual Yang-Mills configuration of topological charge k. Factoring out the redundancy from the above data (e.g. by imposing gauge fixing conditions) one is lead to the instanton moduli space as the set of gauge equivalence classes of solutions of Eq.'s (2.1), (2.2). This space is not smooth. It contains boundary points where the previously mentioned rank condition is violated (which corresponds to Yang-Mills configurations with topological charge density concentrated in part at single points). A resolution of singularities is achieved [14] through a modification of Eq. (2.2):

$$\left[B_1, B_1^\dagger\right] + \left[B_2, B_2^\dagger\right] + II^\dagger - J^\dagger J = \zeta\, \mathbf{1}_{k\times k}, \tag{2.4}$$

where ζ denotes a positive real number. Eq. (2.4) is in fact the starting point for the construction of solutions of the Yang-Mills selfduality equation on a non-commutative space-time [10] with ζ setting the scale of the non-commutativity. The appearance of the non-commutative space will be of no relevance for our purposes. Going from Eq. (2.2) to Eq. (2.4) we will content ourselves with an argument, to be given below, that the numbers we want to calculate do not depend on ζ.

Dealing with a supersymmetric theory in quasi-classical approximation one has to take into account fermionic degrees of freedom which are here the Weyl zero modes of positive chirality in the selfdual Yang-Mills background. It turns out, [7], [8], that a neat realization of the fermions is supplied on the level of moduli through their identification with a basis of the cotangent bundle of these moduli. To adapt the cotangent space in a $U(k)$-invariant way we introduce some notation. Let $\mathbf{L}$ be a selfadjoint operator acting on the space of anti-hermitean $k \times k$ matrices M by

$$\mathbf{L} \cdot M = \left\{II^\dagger + J^\dagger J, M\right\} + \sum_{l=1,2} \left(\left[B_l, \left[B_l^\dagger, M\right]\right] + \left[B_l^\dagger, [B_l, M]\right]\right). \tag{2.5}$$

It can be shown that $\mathbf{L}$ is invertible (in fact positive) as long as the above mentioned rank condition is satisfied (under the supposition of Eq.'s (2.1), (2.2). The invertibility is guaranteed per se if one relies on Eq.'s (2.1), (2.4)). Let $\mathbf{X}$ be the matrix valued differential one-form

$$\mathbf{X} = \sum_{i=1,2} \left[B_i^\dagger, dB_i\right] + J^\dagger dJ - dII^\dagger - h.c. \tag{2.6}$$

3

We introduce a $U(k)$-covariantized exterior derivative on the ADHM matrices $\mathcal{A} \equiv (B_1, B_2, I, J)$ setting

$$\mathcal{D}\mathcal{A} \equiv d\mathcal{A} + Y \cdot \mathcal{A}, \tag{2.7}$$

with $Y = \mathbf{L}^{-1}\mathbf{X}$ and $Y \cdot \mathcal{A} \equiv ([Y, B_1], [Y, B_2], YI, -JY)$. $\mathcal{D}\mathcal{A}$ satisfies by construction (cf. [7], [8]), the fermionic part of the supersymmetrized ADHM conditions (2.1) and (2.2) (or (2.4) resp.) thus giving rise to Weyl zero modes. It is worth to mention the geometric meaning of the $U(k)$ connection field Y. The flat euclidean space of data $\mathcal{A}$ (without Eq.'s (2.1), (2.2) ((2.4)) can be supplied with a Kähler metric. The Kähler property is preserved by the restriction imposed by Eq.'s (2.1), (2.2) ((2.4)) provided one projects in the tangent bundle of the moduli manifold onto the lifted (with respect to the $U(k)$ action) horizontal subspace [19], [20]. To come to terms with the latter notion let us assume that the moduli space is realized through the imposition of a gauge fixing condition, e.g. by demanding that, say, the hermitean part of B_1 is diagonal. The tangent space of the larger manifold (without gauge fixing) may be decomposed into the subspace spanned by the infinitesimal $U(k)$ generators, the so called "vertical" subspace and its orthogonal[4] complement, the "horizontal" subspace, the latter being isomorphic to the tangent space of the gauge fixed moduli manifold $\mathcal{M}_k$. Given a tangent vector T on $\mathcal{M}_k$ one finds a unique vector V in the vertical subspace s.t. $(V+T)$, called "the horizontal lift of T", is contained in the horizontal space. All that extends naturally to the corresponding cotangent bundles. The covariant derivative introduced in Eq. (2.7) serves just this purpose: it lifts the ordinary exterior derivative on $\mathcal{M}_k$ to the horizontal subspace of the larger cotangent bundle. A natural metric $\tilde{g}$ on the horizontal subspace induced from the flat one is given by

$$\tilde{g}\left(d\mathcal{A}^\dagger, d\mathcal{A}\right) = g\left(\mathcal{D}\mathcal{A}^\dagger, \mathcal{D}\mathcal{A}\right), \tag{2.8}$$

g being the flat metric

$$g\left(d\mathcal{A}^\dagger, d\mathcal{A}\right) = tr\left(\sum_{l=1,2} dB_l^\dagger dB_l + dI^\dagger dI + dJ dJ^\dagger\right). \tag{2.9}$$

One of the results of [7], [8] is that the coefficient $\mathcal{F}_k$ of the $\mathcal{N} = 2$ prepotential

$$\mathcal{F}(\Psi) = \frac{i}{2\pi}\Psi^2 \log \frac{2\Psi^2}{e^3\Lambda^2} - \frac{i}{\pi}\sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{\Psi}\right)^{4k} \Psi^2, \tag{2.10}$$

quoted here as a function of the $\mathcal{N} = 2$ chiral vector superfield Ψ can be represented as an integral over the reduced moduli space $\mathcal{M}'_k$ - that is the space from which the coordinates of

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[4]orthogonal with respect to the induced metric.

the instanton center and its fermionic counterparts have been eliminated,

$$\mathcal{F}_k \simeq \int_{\mathcal{M}'_k} e^{-d_x \omega}, \tag{2.11}$$

where ω is here the differential one-form

$$\omega = \Re e \left(\mathcal{D} I \bar{v} I^\dagger + J^\dagger v \mathcal{D} J \right), \tag{2.12}$$

v denotes the Higgs field (the scalar component of the above mentioned superfield Ψ) vacuum expectation value breaking the gauge group $SU(2)$ down to $U(1)$. We will chose v to be of the form

$$v = \begin{pmatrix} ia & 0 \\ 0 & -ia \end{pmatrix}, \tag{2.13}$$

where a is real. d_x in Eq. (2.11) stands for an equivariant differential

$$d_x \equiv d + i_x, \tag{2.14}$$

d being the ordinary exterior derivative and i_x meaning contraction with the $U(1)$ vector field (denoted below by x) going along with the infinitesimal transformation

$$\delta B_i \sim 0; \quad \delta I \sim Iv; \quad \delta J \sim -vJ. \tag{2.15}$$

It is worthwhile to note, that the one-form ω is dual to the vector field (2.15) with respect to the metric $\tilde{g}$:

$$\omega = \Re e \, \tilde{g} \left(x^\dagger, d\mathcal{A} \right). \tag{2.16}$$

The coefficient $\mathcal{F}_k$ may be deformed into

$$\mathcal{F}_k(t) \equiv \int_{\mathcal{M}'_k} e^{-\frac{1}{t} d_x \omega} \tag{2.17}$$

and we obtain

$$\frac{d}{dt}\mathcal{F}_k(t) = -\frac{1}{t^2} \int_{\mathcal{M}'_k} d_x \left(\omega e^{-\frac{1}{t} d_x \omega} \right) = = -\frac{1}{t^2} \int_{\mathcal{M}'_k} d \left(\omega e^{-\frac{1}{t} d_x \omega} \right). \tag{2.18}$$

For the equality (2.18) use has been made of the equivariance of the one-form ω,

$$d_x^2 \omega = (d \circ i_x + i_x \circ d) \, \omega = \mathcal{L}_x \omega = 0, \tag{2.19}$$

where $\mathcal{L}_x$ is the Lie-derivative of the above introduced $U(1)$ vector field. To understand the second equality in (2.18) one has to note that the the integral over $\mathcal{M}'_k$ has to be taken with the top form from $\exp -(\frac{1}{t} d_x \omega)$ inserted, which can be reached (being a top form) only by d, not by

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i_x. The t-dependence of $\mathcal{F}_k(t)$ hinges on whether the total derivative integral in (2.18) picks up boundary terms. Boundary terms are apparently present in the unregularized version of $\mathcal{M}'_k$ defined by Eq.'s (2.1), (2.2) (see [7], [8]), but are not present, as has been convincingly argued by Hollowood [9], after ζ-regularization[5]. $\mathcal{F}_k$ does nevertheless not depend on ζ. Indeed, the integral (2.18) over manifolds $\mathcal{M}'_k$ associated to different parameters ζ are related by a simple rescaling which can be absorbed into a change of the parameter t and so does not, according to Hollowood, alter $\mathcal{F}_k$.

The result

$$\left. \frac{d\mathcal{F}_k}{dt} \right|_{\zeta \neq 0} = 0 \qquad (2.20)$$

suggests a saddle point evaluation of the integral over $\mathcal{M}'_k$, which should render an exact result. The saddle points are determined by $i_x\omega$,

$$
\begin{aligned}
i_x\omega = \tilde{g}\left(x^\dagger, x\right) = tr\Bigg(&\sum_{l=1,2} \left[\mathbf{L}^{-1}\Lambda, B_l^\dagger\right]\left[\mathbf{L}^{-1}\Lambda, B_l\right] \\
&+ \left(-vI^\dagger - I^\dagger\mathbf{L}^{-1}\Lambda\right)\left(Iv + \mathbf{L}^{-1}\Lambda I\right) \\
&+ \left(J^\dagger v + \mathbf{L}^{-1}\Lambda J^\dagger\right)\left(-vJ - J\mathbf{L}^{-1}\Lambda\right)\Bigg).
\end{aligned}
\qquad (2.21)
$$

The r.h.s. of Eq. (2.21) is a sum of squares, each of which should vanish for itself at a saddle point. In this way we are lead to the saddle point equalities

$$
\begin{aligned}
\left[\mathbf{L}^{-1}\Lambda, B_l\right] &= 0; \\
Iv + \mathbf{L}^{-1}\Lambda I &= 0; \\
-vJ - J\mathbf{L}^{-1}\Lambda &= 0.
\end{aligned}
\qquad (2.22)
$$

One should note that the $\mathbf{L}^{-1}\Lambda$ appearing in these equations are emerging from the $U(k)$ connections of the $U(1)$ vector field due to the horizontal lift of the latter. The interpretation of Eq.'s (2.22) is obvious: The saddle points are identical with the places at which the lifted vector field vanishes and can be shown, [9], to coincide with the union of direct products of two $U(1)$ instanton configurations with topological charges k_1 and k_2 satisfying the condition $k_1 + k_2 = k$. Using the general localization formula (see [12], p. 216 ff.) we can reduce the $8k-4$ dimensional integral (2.11) over the moduli space $\mathcal{M}'_k$ to a sum of $4k_1 + 4k_2 - 4 = 4k - 4$ dimensional integrals over the spaces $\mathcal{M}'_{k_1,k_2} \equiv \mathcal{M}_{k_1}(U(1)) \times \mathcal{M}_{k_2}(U(1))\backslash\mathbf{C}^2$ (the common

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[5]One may worry that the non-compactness of $\mathcal{M}'_k$ may create problems. This is not the case . Estimates involving the dilute gas approximation for instantons show that the integrands decay fast enough into the non-compact directions to avoid trouble.

center of two $U(1)$ instantons is eliminated)

$$\int_{\mathcal{M}'_k} e^{-d_x\omega} = \sum_{\substack{k_1,k_2 \\ k_1+k_2=1}} \int_{\mathcal{M}_{k_1,k_2}} \frac{1}{\det^{1/2}(\mathcal{L}_\mathcal{N} + \mathcal{R}_\mathcal{N})}, \tag{2.23}$$

where $\mathcal{L}_\mathcal{N}$ is the action of the vector field on the subspaces of horizontal spaces orthogonal to $\mathcal{M}_{k_1,k_2}$ (these subspaces constitute the so called normal bundle) and $\mathcal{R}_\mathcal{N}$, a two form along $\mathcal{M}_{k_1,k_2}$ and a linear operator acting on these orthogonal subspaces, is the curvature of the normal bundle. Eq. (2.23) shows that each SW coefficient $\mathcal{F}_k$ is equal to a certain polynomial of the characteristic classes of the normal bundle. The formula (2.23) providing a drastic simplification of the initial problem, still does not allow us to perform computations beyond 2-instantons[6].

3   The modified vector field

We have seen in section 2, that the main building block of the superinstanton action is the vector field x (see (2.11), (2.12), (2.16)). The difficulty we have encountered was a too large zero set of this vector field. We describe a natural deformation of this vector field whose zero set is only a discrete finite set. Such a vectorfield has been for good use in mathematics (cf. [14], and references therein[7]). We will heavily rely on some of the results of Nakajima in the following. Consider two independent rotations of space-time, first on the x_1, x_2 plane with the rotation angle ϵ_1 and the second one on the plane x_3, x_4 with rotation angle ϵ_2. It is convenient to introduce complex coordinates $z_1 = x_1 + ix_2$, $z_2 = x_3 + ix_4$ in (euclidean) space-time. The group element specified by the parameters (t_1, t_2) $(t_l \equiv \exp i\epsilon_l,\ l = 1, 2)$ acts on (z_1, z_2) as $(z_1, z_2) \rightarrow (t_1 z_1, t_2 z_2)$. The respective action on the ADHM data is given by:

$$B_l \rightarrow t_l B_l; \quad I \rightarrow I; \quad J \rightarrow t_1 t_2 J. \tag{3.1}$$

The unbroken $U(1)$ subgroup of the gauge group acts as:

$$B_l \rightarrow B_l; \quad I \rightarrow I t_v; \quad J \rightarrow t_v^{-1} J, \tag{3.2}$$

where $t_v = \exp ia\sigma_3$ (our vector field x is just the generator of the transformations (3.2)). It is evident that the transformations (3.2), (3.1) act properly also on the moduli space $\mathcal{M}_k$, because they commute with $U(k)$. Let us combine the transformations (3.2), (3.1)

$$B_l \rightarrow t_l B_l; \quad I \rightarrow I t_v; \quad J \rightarrow t_1 t_2 t_v^{-1} J \tag{3.3}$$

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[6]The 2-instanton calculation goes smoothly and gives the desired result.

[7]We became aware of the relevance of Nakajima's work through [15].

and consider its generating vector field - to be denoted $\tilde{x}$ - in the construction of a modified superinstanton action as the main building block substituting the vector field x. The components of the vector field $\tilde{x}$ are determined through the infinitesimal form of the transformations (3.3)

$$\delta B_l \sim \epsilon_l B_l; \quad \delta I \sim Iv; \quad \delta J \sim -\epsilon v J, \tag{3.4}$$

where $\epsilon = \epsilon_1 + \epsilon_2$. To find the deformed one-form $\tilde{\omega}$ we substitute in Eq. (2.16) x by $\tilde{x}$. As a result we get

$$\tilde{\omega} = \Re e\, tr \left(-i \sum_{l=1,2} \epsilon_l \mathcal{D}B_l B_l^\dagger - \mathcal{D}IvI^\dagger + J^\dagger \left(v - i\epsilon \right) \mathcal{D}J \right). \tag{3.5}$$

For the bosonic part of the deformed superinstanton action $d_{\tilde{x}}\tilde{\omega}$ we obtain

$$i_{\tilde{x}}\tilde{\omega} = \tilde{g}\left(\tilde{x}^\dagger, \tilde{x} \right) = tr \left(\sum_{l=1,2} \left(-i\epsilon_l B_l^\dagger + \left[\mathbf{L}^{-1}\tilde{\Lambda}, B_l^\dagger \right] \right) \left(i\epsilon_l B_l + \left[\mathbf{L}^{-1}\tilde{\Lambda}, B_l \right] \right) \right.$$
$$+ \left(-vI^\dagger - I^\dagger \mathbf{L}^{-1}\tilde{\Lambda} \right) \left(Iv + \mathbf{L}^{-1}\tilde{\Lambda}I \right)$$
$$\left. + \left(J^\dagger \left(v - i\epsilon \right) + \mathbf{L}^{-1}\tilde{\Lambda}J^\dagger \right) \left(\left(-v + i\epsilon \right) J - J\mathbf{L}^{-1}\tilde{\Lambda} \right) \right), \tag{3.6}$$

where

$$\tilde{\Lambda} = i_{\tilde{x}}\mathbf{X} = \sum_{l=1,2} i\epsilon_l \left[B_l^\dagger, B_l \right] + J^\dagger \left(-v + i\epsilon \right) J - IvI^\dagger - h.c. \,. \tag{3.7}$$

The deformed super-instanton action has an explicit dependence on the instanton center. Hence we are not allowed to restrict the moduli space of instantons imposing the condition $tr B_1 = tr B_2 = 0$ which means that the center of instantons has zero coordinates. We define the deformed version of the Eq. (2.11) as

$$Z_k \left(a, \epsilon_1, \epsilon_2 \right) \equiv \int_{\mathcal{M}_k} e^{-d_{\tilde{x}}\tilde{\omega}}, \tag{3.8}$$

where integration is over the entire moduli space $\mathcal{M}_k$. The evaluation of this integral by means of localization technic is much simpler, because on the right hand side of the localization formula (2.23) we will have a finite sum over the zero locus set of vector field $\tilde{x}$ and therefore the curvature term of the normal bundle $\mathcal{R}_\mathcal{N}$ is absent. A zero locus of the vector field $\tilde{x}$ (equivalently a fixed point of the combined action (3.3)) on the moduli space) is defined by the conditions:

$$t_l B_l = g^{-1} B_l g; \quad I t_v = g^{-1} I; \quad t_1 t_2 t_v^{-1} J = J g. \tag{3.9}$$

Using the freedom of $U(k)$ transformations we will assume that the group element g out of $U(k)$ is diagonal

$$g = \begin{pmatrix} e^{i\Phi_1} & \cdots & 0 \\ & \cdots & \\ 0 & \cdots & e^{i\Phi_k} \end{pmatrix}. \tag{3.10}$$

The fixed point condition (3.9) implies:

$$\delta B_{l,ij} \equiv (\Phi_{ij} + \epsilon_l)\, B_{l,ij} = 0;$$
$$\delta I_{i\lambda} \equiv (\Phi_i + a_\lambda)\, I_{i\lambda} = 0;$$
$$\delta J_{\lambda i} \equiv (-\Phi_i - a_\lambda + \epsilon)\, J_{\lambda i} = 0;$$
$$i,j = 1, 2, \cdots, k; \quad l,\lambda = 1, 2; \quad \Phi_{ij} \equiv \Phi_i - \Phi_j. \tag{3.11}$$

The geometrical meaning of these equations (as before for x) is the criticality of the horizontally lifted vector field $\tilde{x}'$ whose action in the neighborhood of a critical point is given by equation

$$\delta_{\tilde{x}'} B_{l,ij} \equiv (\Phi_{ij} + \epsilon_l)\, B_{l,ij} \ ;$$
$$\delta_{\tilde{x}'} I_{i\lambda} \equiv (\Phi_i + a_\lambda)\, I_{i\lambda} \ ;$$
$$\delta_{\tilde{x}'} J_{\lambda i} \equiv (-\Phi_i - a_\lambda + \epsilon)\, J_{\lambda i} \ . \tag{3.12}$$

The classification of fixed points of the torus action (3.1) in the case of $U(1)$ (non-commutative) instantons can be found in [14] chapter 5.2. Fortunately, to adapt this to our case of $SU(2)$ (or $SU(N)$ in general) instantons and the combined action (3.3) only some minor modifications are needed.

There is one-to-one correspondence between the fixed points of $\tilde{x}'$ and ordered pairs of Young diagrams (Y_1, Y_2) with k_1 and k_2 boxes resp. , s.t. $k_1 + k_2 = k$. Denote the number of boxes in the rows of the Young diagram Y_l by $\nu_{l,1} \geq \nu_{l,2} \geq \cdots \geq 0$ and the number of boxes in columns[8] by $\nu'_{l,1} \geq \nu'_{l,2} \geq \cdots \geq 0$. Then we distribute the phases $\Phi_1, \Phi_2, \cdots, \Phi_{k_1}$ in the Young diagrams Y_1 and $\Phi_{k_1+1}, \Phi_{k_1+2}, \cdots, \Phi_k$ in Y_2 subsequently filling row after row beginning from the upper left corner boxes. To the phase Φ_m distributed in a box of Y_l in the i-th row and j-th column we assign the value

$$\Phi_m = -a_l - (i-1)\epsilon_2 - (j-1)\epsilon_1, \tag{3.13}$$

where $a_1 = -a_2 = a$. The matrix element $B_{1,mn}$ ($B_{2,mn}$) is nonzero if and only if Φ_m and Φ_n are neighbors in a row (column). The only nonzero matrix elements of I are $I_{1,1}$ and $I_{k_1+1,2}$. J vanishes identically. These are some special points belonging to the manifold $\mathcal{M}_{k_1,k_2}$ introduced earlier. It is easy to see that the conditions (3.11) are fulfilled. Of course, to determine the actual values of nonzero matrix elements one should impose the ADHM equations (2.1), (2.4) but this is a more delicate problem. Below we will solve this problem for instanton charges $k = 2, 3, 4$.

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[8]Our convention is to numerate rows beginning from the top to the bottom and columns from left to right.

The next step is the construction of tangent spaces passing through the fixed points and to calculate the determinants of $\tilde{x}'$-action on this spaces. To do this, one needs to find all solutions of the linearized ADHM equations (the so called fermionic ADHM equations)

$$[\delta B_1, B_2] + [B_1, \delta B_2] + \delta I J + I \delta J = 0;$$
$$\sum_{l=1,2} \left[\delta B_l, B_l^\dagger\right] + \delta I I^\dagger - J^\dagger \delta J = 0 \tag{3.14}$$

around the fixed points. Denote the fixed point ADHM data corresponding to the Young diagrams (Y_1, Y_2) as $\mathcal{A}_{Y_1,Y_2}$. The function $Z_k(a, \epsilon_1, \epsilon_2)$ can be completely determined in terms of above data:

$$Z_k(a, \epsilon_1, \epsilon_2) = \sum_{Y_1,Y_2} \frac{1}{\det \mathcal{L}_{\tilde{x}'}} \Bigg|_{\mathcal{A}_{Y_1,Y_2}}. \tag{3.15}$$

The reason of appearance of $1/\det$ instead of $1/\sqrt{\det}$ as in Eq. (2.23) is due to our convention to consider complex tangent spaces and linear operators acting on them instead of their real forms.

Though the deformed superinstanton action $d_{\tilde{x}'}\tilde{\omega}$ tends to the initial one as $\epsilon_1, \epsilon_2 \to 0$, it is not true for the $Z_k(a, \epsilon_1, \epsilon_2)$ which is highly singular at this limit. In his recent paper N.Nekrasov [15] brings some field theoretical arguments to come to the remarkable conclusion that the generating function

$$Z(q, a, \epsilon_1, \epsilon_2) \equiv 1 + \sum_{k=1}^{\infty} Z_k(a, \epsilon_1, \epsilon_2) q^k \tag{3.16}$$

can be represented as

$$Z(q, a, \epsilon_1, \epsilon_2) = \exp -\frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}(q, a, \epsilon_1, \epsilon_2), \tag{3.17}$$

where $\mathcal{F}$ is regular at $\epsilon_1, \epsilon_2 = 0$ and that the SW coefficients $\mathcal{F}_k$ are nothing else than the coefficients of the Taylor expansion of $\mathcal{F}(q, a, \epsilon_1, \epsilon_2)$:

$$\mathcal{F}(q, a, \epsilon_1, \epsilon_2) = \sum_{k=1}^{\infty} \mathcal{F}_k(a, \epsilon_1, \epsilon_2) q^k \tag{3.18}$$

at $\epsilon_1, \epsilon_2 = 0$. More precisely[9]

$$\mathcal{F}_k a^{2-4k} = 2^{2k-2} \mathcal{F}_k(a, \epsilon_1, \epsilon_2). \tag{3.19}$$

We hope to return to this point in a future publication and to present an intrinsic explanation of the singularity structure (3.17) using directly the definition of $Z(q, a, \epsilon_1, \epsilon_2)$.

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[9]The factor 2^{2k-2} is needed to have the normalization adopted in [2].

4 Low charge instanton calculations

In this section we carry out explicit calculations for instanton charges up to four. For simplicity we rescale ADHM data and set $\zeta = 1$.

- **One-instantons**

This case is almost trivial. We have two pairs of Young diagrams[10]:
a) $Y_1 = \{\nu_{1,1} = 1\}$, $Y_2 = \{\emptyset\}$;
b) $Y_1 = \{\emptyset\}$, $Y_2 = \{\nu_{1,1} = 1\}$. In the case a) $B_1 = B_2 = 0$, $I = \begin{pmatrix} 1 & 0 \end{pmatrix}$, $\Phi_1 = -a_1$. The tangent space vectors are given by

$$\delta B_l = (\delta B_{l,11}); \quad \delta I = \begin{pmatrix} 0 & \delta I_{12} \end{pmatrix}; \quad \delta J = \begin{pmatrix} \delta J_{11} \\ 0 \end{pmatrix}. \tag{4.1}$$

The dimension of the tangent space is $2 \times 4 = 8$ as it should. Taking into account (3.12) one easily calculates the determinant

$$\det \mathcal{L}_{\tilde{x}'} = \epsilon_1 \epsilon_2 a_{21} (a_{12} + \epsilon), \tag{4.2}$$

where $a_{\lambda\mu} \equiv a_\lambda - a_\mu; \lambda, \mu = 1, 2$. There is no need to carry out calculation for the case b) because for interchanged Y_1 and Y_2 one obtains the same determinant with interchanged a_1 and a_2. Note also that simultaneous transposition of both Young diagrams gives rise to a determinant with interchanged $\epsilon_1 \leftrightarrow \epsilon_2$. Below we will explicitly mention only one pair of such symmetry related diagrams, but of course we will take all of them into account in final expressions of Z_k Eq. (3.15). Thus for the 1-instantons

$$Z_1 (a, \epsilon_1, \epsilon_2) = (\epsilon_1 \epsilon_2 a_{21} (a_{12} + \epsilon))^{-1} + (\epsilon_1 \epsilon_2 a_{12} (a_{21} + \epsilon))^{-1}. \tag{4.3}$$

From Eq. (3.16)-(3.18)

$$\mathcal{F}_1 (a, \epsilon_1, \epsilon_2) = -\epsilon_1 \epsilon_2 Z_1 = \frac{2}{a_{12}^2 - \epsilon^2}. \tag{4.4}$$

Taking the limit $\epsilon_{1,2} \to 0$ one obtains

$$\mathcal{F}_1 = 1/2.$$

- **2-instantons**

---

[10]We always note only nonzero ν's.

There are two basic cases.

a) $Y_1 = \{\nu_{1,1} = 2)\}$, $Y_2 = \{\emptyset\}$; $\Phi_1 = -a_1$, $\Phi_2 = -a_1 - \epsilon_1$.

The fixed point ADHM data:

$$B_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad B_2 = 0; \quad I = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.5}$$

The tangent space:

$$\delta B_1 = \begin{pmatrix} \delta B_{1,11} & \delta B_{1,12} \\ 0 & \delta B_{1,11} \end{pmatrix}; \quad \delta B_2 = \begin{pmatrix} \delta B_{2,11} & 0 \\ \delta B_{2,21} & \delta B_{2,11} \end{pmatrix};$$

$$\delta I = \begin{pmatrix} 0 & \delta I_{12} \\ 0 & \delta I_{22} \end{pmatrix}; \qquad \delta J = \begin{pmatrix} 0 & 0 \\ \delta J_{21} & \delta J_{22} \end{pmatrix}. \tag{4.6}$$

The determinant:

$$\det \mathcal{L}_{\tilde{x}'} = 2\epsilon_1^2 \epsilon_2 \epsilon_{21} a_{21} (a_{21} - \epsilon_1)(a_{12} + \epsilon)(a_{12} + 2\epsilon_1 + \epsilon_2). \tag{4.7}$$

b) $Y_1 = \{\nu_{1,1} = 1)\}$, $Y_2 = \{\nu_{2,1} = 1\}$; $\Phi_1 = -a_1$, $\Phi_2 = -a_2$.

Fixed point:

$$B_1 = B_2 = 0; \quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.8}$$

Tangent space:

$$\delta B_1 = \begin{pmatrix} \delta B_{1,11} & \delta B_{1,12} \\ \delta B_{1,21} & \delta B_{1,22} \end{pmatrix}; \quad \delta B_2 = \begin{pmatrix} \delta B_{2,11} & \delta B_{2,12} \\ \delta B_{2,21} & \delta B_{2,22} \end{pmatrix};$$

$$\delta I = 0; \quad \delta J = 0. \tag{4.9}$$

Determinant:

$$\det \mathcal{L}_{\tilde{x}'} = \epsilon_1^2 \epsilon_2^2 \left(a_{12}^2 - \epsilon_1^2\right)\left(a_{12}^2 - \epsilon_2^2\right). \tag{4.10}$$

Using above data one first calculates Z_2 and then

$$\mathcal{F}_2(a, \epsilon_1, \epsilon_2) = -\epsilon_1 \epsilon_2 \left(Z_2 - \frac{1}{2} Z_1^2\right).$$

The final result reads:

$$\mathcal{F}_2(a, \epsilon_1, \epsilon_2) = \frac{20a^2 + 7\epsilon_1^2 + 16\epsilon_1\epsilon_2 + 7\epsilon_2^2}{\left(4a^2 - \epsilon^2\right)^2 \left(4a^2 - (2\epsilon_1 + \epsilon_2)^2\right)\left(4a^2 - (\epsilon_1 + 2\epsilon_2)^2\right)}. \tag{4.11}$$

Now let us take the limit $\epsilon_{1,2} \to 0$:

$$\mathcal{F}_2 = \frac{5}{2^4}.$$

12

- **3-instantons**

There are three basic cases.

a) $Y_1 = \{\nu_{1,1} = 3\}$, $Y_2 = \{\emptyset\}$; $\Phi_1 = -a_1$, $\Phi_2 = -a_1 - \epsilon_1$, $\Phi_3 = -a_1 - 2\epsilon_1$.

Fixed point:

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}; \quad B_2 = 0; \quad I = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.12}$$

Tangent space:

$$\delta B_1 = \begin{pmatrix} \delta B_{1,11} & \delta B_{1,12} & \delta B_{1,13} \\ 0 & \delta B_{1,11} & \frac{1}{\sqrt{2}}\delta B_{1,12} \\ 0 & 0 & \delta B_{1,11} \end{pmatrix}; \quad \delta B_2 = \begin{pmatrix} \delta B_{2,11} & 0 & 0 \\ \delta B_{2,21} & \delta B_{2,11} & 0 \\ \delta B_{2,31} & \frac{1}{\sqrt{2}}\delta B_{2,21} & \delta B_{2,11} \end{pmatrix};$$

$$\delta I = \begin{pmatrix} 0 & \delta I_{12} \\ 0 & \delta I_{22} \\ 0 & \delta I_{32} \end{pmatrix}; \quad \delta J = \begin{pmatrix} 0 & 0 & 0 \\ \delta J_{21} & \delta J_{22} & \delta J_{23} \end{pmatrix}. \tag{4.13}$$

The determinant:

$$\det \mathcal{L}_{\tilde{x}'} = 6\epsilon_1^3 \epsilon_2 \epsilon_{21} \left(\epsilon_2 - 2\epsilon_1\right) a_{21} \left(a_{21} - \epsilon_1\right) \left(a_{21} - 2\epsilon_1\right)$$
$$\times \left(a_{12} + \epsilon\right)\left(a_{12} + 2\epsilon_1 + \epsilon_2\right)\left(a_{12} + 3\epsilon_1 + \epsilon_2\right). \tag{4.14}$$

b) $Y_1 = \{\nu_{1,1} = 2, \nu_{1,2} = 1\}$, $Y_2 = \{\emptyset\}$; $\Phi_1 = -a_1$, $\Phi_2 = -a_1 - \epsilon_1$, $\Phi_3 = -a_1 - \epsilon_1 - \epsilon_2$.

Fixed point:

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad I = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.15}$$

Tangent space:

$$\delta B_1 = \begin{pmatrix} \delta B_{1,11} & 0 & 0 \\ 0 & \delta B_{1,22} & \delta B_{2,22} - \delta B_{2,11} \\ 0 & \delta B_{1,32} & 2\delta B_{1,11} - \delta B_{1,22} \end{pmatrix};$$

$$\delta B_2 = \begin{pmatrix} \delta B_{2,11} & 0 & 0 \\ 0 & \delta B_{2,22} & \delta B_{2,23} \\ 0 & \delta B_{1,11} - \delta B_{1,22} & 2\delta B_{211} - \delta B_{2,22} \end{pmatrix};$$

$$\delta I = \begin{pmatrix} 0 & \delta I_{12} \\ 0 & \delta I_{22} \\ 0 & \delta I_{32} \end{pmatrix}; \quad \delta J = \begin{pmatrix} 0 & 0 & 0 \\ \delta J_{21} & \delta J_{22} & \delta J_{23} \end{pmatrix}. \tag{4.16}$$

Determinant:

$$\det \mathcal{L}_{\tilde{x}'} = \epsilon_1^2 \epsilon_2^2 \left(2\epsilon_1 - \epsilon_2\right) \left(2\epsilon_2 - \epsilon_1\right) a_{21} \left(a_{21} - \epsilon_1\right) \left(a_{21} - \epsilon_2\right)$$
$$\times \left(a_{12} + \epsilon\right) \left(a_{12} + 2\epsilon_1 + \epsilon_2\right) \left(a_{12} + 2\epsilon_2 + \epsilon_1\right). \tag{4.17}$$

c) $Y_1 = \{\nu_{1,1} = 1, \nu_{1,2} = 1\}$, $Y_2 = \{\nu_{2,1} = 1\}$; $\Phi_1 = -a_1$, $\Phi_2 = -a_1 - \epsilon_2$, $\Phi_3 = -a_2$.

Fixed point:

$$B_1 = 0; \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad I = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.18}$$

Tangent space:

$$\delta B_1 = \begin{pmatrix} \delta B_{1,11} & 0 & 0 \\ \delta B_{1,21} & \delta B_{1,11} & \delta B_{1,23} \\ \delta B_{1,31} & \delta B_{1,32} & \delta B_{1,33} \end{pmatrix};$$

$$\delta B_2 = \begin{pmatrix} \delta B_{2,11} & \delta B_{2,12} & \delta B_{2,13} \\ 0 & \delta B_{2,11} & \delta B_{2,23} \\ 0 & \delta B_{2,32} & \delta B_{2,33} \end{pmatrix};$$

$$\delta I = \begin{pmatrix} 0 & \delta B_{2,23} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad \delta J = \begin{pmatrix} 0 & 0 & 0 \\ -\delta B_{1,32} & 0 & 0 \end{pmatrix}. \tag{4.19}$$

Determinant:

$$\det \mathcal{L}_{\tilde{x}'} = 2\epsilon_1^2 \epsilon_2^3 \epsilon_{12} a_{21} \left(a_{21} + \epsilon_1 - \epsilon_2\right) \left(a_{12} + \epsilon_1\right) \left(a_{21} + \epsilon_2\right)$$
$$\times \left(a_{12} + \epsilon\right) \left(a_{12} + 2\epsilon_2\right). \tag{4.20}$$

Using these data we have calculated Z_3 and then

$$\mathcal{F}_3\left(a, \epsilon_1, \epsilon_2\right) = -\epsilon_1 \epsilon_2 \left(Z_3 + \frac{1}{3} Z_1^3 - Z_1 Z_2\right).$$

Here is the final result:

$$\mathcal{F}_3\left(a, \epsilon_1, \epsilon_2\right) = \tag{4.21}$$

$$\frac{16 \left(144a^4 + 29\epsilon_1^4 + 154\epsilon_1^3\epsilon_2 + 258\epsilon_1^2\epsilon_2^2 + 154\epsilon_1\epsilon_2^3 + 29\epsilon_2^4 + 8a^2 \left(29\epsilon_1^2 + 71\epsilon_1\epsilon_2 + 29\epsilon_2^2\right)\right)}{3 \left(4a^2 - \epsilon^2\right)^3 \left(4a^2 - \left(2\epsilon_1 + \epsilon_2\right)^2\right) \left(4a^2 - \left(3\epsilon_1 + \epsilon_2\right)^2\right) \left(4a^2 - \left(\epsilon_1 + 2\epsilon_2\right)^2\right) \left(4a^2 - \left(\epsilon_1 + 3\epsilon_2\right)^2\right)}.$$

Now let us take the limit $\epsilon_{1,2} \to 0$:

$$\mathcal{F}_3 = \frac{3}{4}.$$

- **4-instantons**

Now we need to investigate 7 different cases.

a) $Y_1 = \{\nu_{1,1} = 4\}$, $Y_2 = \{\emptyset\}$; $\Phi_1 = -a_1$, $\Phi_2 = -a_1 - \epsilon_1$, $\Phi_3 = -a_1 - 2\epsilon_1$, $\Phi_4 = -a_1 - 3\epsilon_1$.

Fixed point:

$$
B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{3} & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad B_2 = 0; \quad I = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.
\tag{4.22}
$$

Tangent space:

$$
\delta B_1 = \begin{pmatrix} \delta B_{1,11} & \delta B_{1,12} & \delta B_{1,13} & \delta B_{1,14} \\ 0 & \delta B_{1,11} & \sqrt{\frac{2}{3}}\delta B_{1,12} & \frac{1}{\sqrt{3}}\delta B_{1,13} \\ 0 & 0 & \delta B_{1,11} & \frac{1}{\sqrt{3}}\delta B_{1,12} \\ 0 & 0 & 0 & \delta B_{1,11} \end{pmatrix};
$$

$$
\delta B_2 = \begin{pmatrix} \delta B_{2,11} & 0 & 0 & 0 \\ \delta B_{2,21} & \delta B_{211} & 0 & 0 \\ \delta B_{2,31} & \sqrt{\frac{2}{3}}\delta B_{2,21} & \delta B_{2,11} & 0 \\ \delta B_{2,41} & \frac{1}{\sqrt{3}}\delta B_{2,31} & \frac{1}{\sqrt{3}}\delta B_{2,21} & \delta B_{2,11} \end{pmatrix};
$$

$$
\delta I = \begin{pmatrix} 0 & \delta I_{12} \\ 0 & \delta I_{22} \\ 0 & \delta I_{32} \\ 0 & \delta I_{42} \end{pmatrix}; \quad \delta J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta J_{21} & \delta J_{22} & \delta J_{23} & \delta J_{24} \end{pmatrix}.
\tag{4.23}
$$

The determinant:

$$
\det \mathcal{L}_{\tilde{x}'} = 24\epsilon_1^4 \prod_{j=0}^{3} (\epsilon_2 - j\epsilon_1)(a_{21} - j\epsilon_1)(a_{12} + \epsilon + j\epsilon_1).
\tag{4.24}
$$

b) $Y_1 = \{\nu_{1,1} = 3, \nu_{1,2} = 1\}$, $Y_2 = \{\emptyset\}$; $\Phi_1 = -a_1$, $\Phi_2 = -a_1 - \epsilon_1$, $\Phi_3 = -a_1 - 2\epsilon_1$, $\Phi_4 = -a_1 - \epsilon_2$.

Fixed point:

$$
B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}; \quad I = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.
\tag{4.25}
$$

Tangent space:

$$
\delta B_1 = \begin{pmatrix} \delta B_{1,11} & \delta B_{1,12} & 0 & 0 \\ 0 & \delta B_{1,22} & \sqrt{2}\delta B_{1,12} & \sqrt{2}(\delta B_{2,22} - \delta B_{2,11}) \\ 0 & 0 & \delta B_{1,22} & \delta B_{1,34} \\ 0 & 0 & \delta_{1,43} & 3\delta B_{1,11} - 2\delta B_{1,22} \end{pmatrix};
$$

15

$$\delta B_2 = \begin{pmatrix} \delta B_{2,11} & 0 & 0 & 0 \\ \delta B_{1,34} & \delta B_{2,22} & 0 & 0 \\ 0 & \sqrt{2}\delta B_{1,34} & \delta B_{2,22} & \delta B_{2,34} \\ 0 & \sqrt{2}\left(\delta B_{1,11} - \delta B_{1,22}\right) & -\delta B_{1,12} & 3\delta B_{2,11} - 2\delta B_{2,22} \end{pmatrix};$$

$$\delta I = \begin{pmatrix} 0 & \delta I_{12} \\ 0 & \delta I_{22} \\ 0 & \delta I_{32} \\ 0 & \delta I_{42} \end{pmatrix}; \quad \delta J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta J_{21} & \delta J_{22} & \delta J_{23} & \delta J_{24} \end{pmatrix}. \tag{4.26}$$

The determinant:

$$\det \mathcal{L}_{\tilde{x}'} = 4\epsilon_1^3 \epsilon_2^2 \epsilon_{21}^2 \left(3\epsilon_1 - \epsilon_2\right) a_{21} \left(a_{21} - \epsilon_1\right) \left(a_{21} - 2\epsilon_1\right) \left(a_{21} - \epsilon_2\right) \left(a_{12} + \epsilon\right)$$
$$\times \left(a_{12} + 2\epsilon_1 + \epsilon_2\right) \left(a_{12} + 3\epsilon_1 + \epsilon_2\right) \left(a_{12} + \epsilon_1 + 2\epsilon_2\right). \tag{4.27}$$

c) $Y_1 = \{\nu_{1,1} = 2, \nu_{1,2} = 2\}$, $Y_2 = \{\emptyset\}$; $\Phi_1 = -a_1$, $\Phi_2 = -a_1 - \epsilon_1$, $\Phi_3 = -a_1 - \epsilon_2$, $\Phi_4 = -a_1 - \epsilon$.
Fixed point:

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}} & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix}; \quad I = \begin{pmatrix} 2 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.28}$$

Tangent space:

$$\delta B_1 = \begin{pmatrix} \delta B_{1,11} & \delta B_{1,12} & 0 & 0 \\ 0 & \delta B_{1,11} & 0 & 0 \\ \delta B_{1,31} & \delta B_{1,32} & \delta B_{1,11} & \sqrt{3}\delta B_{1,12} \\ 0 & \sqrt{3}\delta B_{1,31} & 0 & \delta B_{1,11} \end{pmatrix};$$

$$\delta B_2 = \begin{pmatrix} \delta B_{2,11} & 0 & \delta B_{2,13} & 0 \\ \delta B_{2,21} & \delta B_{2,11} & \delta B_{2,23} & \sqrt{3}\delta B_{2,13} \\ 0 & 0 & \delta B_{2,11} & 0 \\ 0 & 0 & \sqrt{3}\delta B_{2,21} & \delta B_{2,11} \end{pmatrix};$$

$$\delta I = \begin{pmatrix} 0 & \delta I_{12} \\ 0 & \delta I_{22} \\ 0 & \delta I_{32} \\ 0 & \delta I_{42} \end{pmatrix}; \quad \delta J = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \delta J_{21} & \delta J_{22} & \delta J_{23} & \delta J_{24} \end{pmatrix}. \tag{4.29}$$

The determinant:

$$\det \mathcal{L}_{\tilde{x}'} = 4\epsilon_1^2 \epsilon_2^2 \epsilon_{12}^2 \left(2\epsilon_1 - \epsilon_2\right) \left(2\epsilon_2 - \epsilon_1\right) a_{21} \left(a_{21} - \epsilon_1\right) \left(a_{21} - \epsilon_2\right) \left(a_{12} + \epsilon\right)^2$$
$$\times \left(a_{12} + 2\epsilon_1 + \epsilon_2\right) \left(a_{12} + 2\epsilon_2 + \epsilon_1\right) \left(a_{12} + 2\epsilon\right). \tag{4.30}$$

d) $Y_1 = \{\nu_{1,1} = 1\}$, $Y_2 = \{\nu_{2,1} = 3\}$; $\Phi_1 = -a_1$, $\Phi_2 = -a_2$, $\Phi_3 = -a_2 - \epsilon_1$, $\Phi_4 = -a_2 - 2\epsilon_1$.

Fixed point:

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} ; \quad B_2 = 0; \quad I = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.31}$$

Tangent space:

$$\delta B_1 = \begin{pmatrix} \delta B_{1,11} & 0 & 0 & \delta B_{1,14} \\ \delta B_{1,21} & \delta B_{1,22} & \delta B_{1,23} & \delta B_{1,24} \\ \delta B_{1,31} & 0 & \delta B_{1,22} & \frac{1}{\sqrt{2}}\delta B_{1,23} \\ \delta B_{1,41} & 0 & 0 & \delta B_{1,22} \end{pmatrix} ;$$

$$\delta B_2 = \begin{pmatrix} \delta B_{2,11} & \delta B_{2,12} & \delta B_{2,13} & \delta B_{2,14} \\ 0 & \delta B_{2,22} & 0 & 0 \\ 0 & \delta B_{2,32} & \delta B_{2,22} & 0 \\ \delta B_{2,41} & \delta B_{2,42} & \frac{1}{\sqrt{2}}\delta B_{2,32} & \delta B_{2,22} \end{pmatrix} ;$$

$$\delta I = \begin{pmatrix} 0 & 0 \\ \sqrt{2}\delta B_{1,31} & 0 \\ \delta B_{1,41} & 0 \\ 0 & 0 \end{pmatrix} ; \quad \delta J = \begin{pmatrix} 0 & \sqrt{2}\delta B_{2,13} & \delta B_{2,14} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.32}$$

The determinant:

$$\det \mathcal{L}_{\tilde{x}'} = 6\epsilon_1^4 \epsilon_2^2 \epsilon_{21} \left(\epsilon_2 - 2\epsilon_1\right) a_{12} \left(a_{21} + 3\epsilon_1\right) \left(a_{12}^2 - \epsilon_1^2\right) \left(a_{21} + \epsilon_2\right)$$
$$\times \left(a_{21} + \epsilon\right) \left(a_{21} + 2\epsilon_1 + \epsilon_2\right) \left(a_{12} - 2\epsilon_1 + \epsilon_2\right). \tag{4.33}$$

e) $Y_1 = \{\nu_{1,1} = 1\}$, $Y_2 = \{\nu_{2,1} = 2, \nu_{2,2} = 1\}$; $\Phi_1 = -a_1$, $\Phi_2 = -a_2$, $\Phi_3 = -a_2 - \epsilon_1$, $\Phi_4 = -a_2 - \epsilon_2$.

Fixed point:

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} ; \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} ; \quad I = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{3} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.34}$$

Tangent space:

$$\delta B_1 = \begin{pmatrix} \delta B_{1,11} & 0 & \delta B_{1,13} & \delta B_{1,14} \\ 0 & \delta B_{1,22} & 0 & 0 \\ \delta B_{1,31} & 0 & \delta B_{1,33} & \delta B_{2,33} - \delta B_{2,22} \\ \delta B_{1,41} & 0 & \delta B_{1,43} & 2\delta B_{1,22} - \delta B_{1,33} \end{pmatrix} ;$$

17

$$\delta B_2 \;=\; \begin{pmatrix} \delta B_{2,11} & 0 & \delta B_{2,13} & \delta B_{2,14} \\ 0 & \delta B_{2,22} & 0 & 0 \\ \delta B_{2,31} & 0 & \delta B_{2,33} & \delta B_{2,34} \\ \delta B_{2,41} & 0 & \delta B_{1,22} - \delta B_{1,33} & 2\delta B_{2,22} - \delta B_{2,33} \end{pmatrix};$$

$$\delta I \;=\; \begin{pmatrix} 0 & 0 \\ \delta B_{1,31} + \delta B_{2,41} & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}; \quad \delta J = \begin{pmatrix} 0 & \delta B_{2,13} - \delta B_{1,14} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (4.35)$$

The determinant:

$$\det \mathcal{L}_{\tilde{x}'} = \epsilon_1^3 \epsilon_2^3 \left(2\epsilon_1 - \epsilon_2\right)\left(2\epsilon_2 - \epsilon_1\right) a_{12}^2 \left(a_{21} + 2\epsilon_1\right)\left(a_{21} + 2\epsilon_2\right)\left(a_{21} + \epsilon\right)^2 \left(a_{12}^2 - \epsilon_{12}^2\right). \qquad (4.36)$$

f) $Y_1 = \{\nu_{1,1} = 2\}$, $Y_2 = \{\nu_{2,1} = 2\}$; $\Phi_1 = -a_1$, $\Phi_2 = -a_1 - \epsilon_1$, $\Phi_3 = -a_2$, $\Phi_4 = -a_2 - \epsilon_1$.

Fixed point:

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad B_2 = 0; \quad I = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}. \qquad (4.37)$$

Tangent space:

$$\delta B_1 \;=\; \begin{pmatrix} \delta B_{1,11} & \delta B_{1,12} & \delta B_{1,13} & \delta B_{1,14} \\ 0 & \delta B_{1,11} & 0 & \delta B_{1,13} \\ \delta B_{1,31} & \delta B_{1,32} & \delta B_{1,33} & \delta B_{1,34} \\ 0 & \delta B_{1,31} & 0 & \delta B_{1,33} \end{pmatrix};$$

$$\delta B_2 \;=\; \begin{pmatrix} \delta B_{2,11} & 0 & \delta B_{2,13} & 0 \\ \delta B_{2,21} & \delta B_{2,11} & \delta B_{2,23} & \delta B_{2,13} \\ \delta B_{2,31} & 0 & \delta B_{2,33} & 0 \\ \delta B_{2,41} & \delta B_{2,31} & \delta B_{2,43} & \delta B_{2,33} \end{pmatrix};$$

$$\delta I \;=\; 0; \quad \delta J = 0. \qquad (4.38)$$

The determinant:

$$\det \mathcal{L}_{\tilde{x}'} = 4\epsilon_1^4 \epsilon_2^2 \epsilon_{12}^2 \left(a_{12}^2 - \epsilon_1^2\right)\left(a_{12}^2 - 4\epsilon_1^2\right)\left(a_{12}^2 - \epsilon_2^2\right)\left(a_{12}^2 - \epsilon_{12}^2\right). \qquad (4.39)$$

g) $Y_1 = \{\nu_{1,1} = 2\}$, $Y_2 = \{\nu_{2,1} = 1, \nu_{2,2} = 1\}$; $\Phi_1 = -a_1$, $\Phi_2 = -a_1 - \epsilon_1$, $\Phi_3 = -a_2$, $\Phi_4 = -a_2 - \epsilon_2$.

Fixed point:

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}; \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}; \quad I = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}. \qquad (4.40)$$

Tangent space:

$$\delta B_1 = \begin{pmatrix} \delta B_{1,11} & \delta B_{1,12} & \delta B_{1,13} & \delta B_{1,14} \\ 0 & \delta B_{1,11} & \delta B_{1,23} & 0 \\ 0 & 0 & \delta B_{1,33} & 0 \\ 0 & \delta B_{1,42} & \delta B_{1,43} & \delta B_{1,33} \end{pmatrix};$$

$$\delta B_2 = \begin{pmatrix} \delta B_{2,11} & 0 & 0 & 0 \\ \delta B_{2,21} & \delta B_{2,11} & 0 & \delta B_{2,24} \\ \delta B_{2,31} & \delta B_{2,32} & \delta B_{2,33} & \delta B_{2,34} \\ \delta B_{2,41} & 0 & 0 & \delta B_{2,33} \end{pmatrix};$$

$$\delta I = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}\delta B_{1,23} \\ 0 & 0 \\ \frac{1}{\sqrt{2}}\delta B_{2,41} & 0 \\ 0 & 0 \end{pmatrix}; \quad \delta J = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{2}}\delta B_{1,14} & 0 \\ \frac{1}{\sqrt{2}}\delta B_{2,32} & 0 & 0 & 0 \end{pmatrix}; \quad (4.41)$$

The determinant:

$$\det \mathcal{L}_{\tilde{x}'} = 4\epsilon_1^3\epsilon_2^3\epsilon_{12}^2 a_{12}^2 \left(a_{12}^2 - \epsilon^2\right)\left(a_{12} - \epsilon_1\right)\left(a_{12} + \epsilon_2\right)\left(a_{12} + 2\epsilon_1 - \epsilon_2\right)\left(a_{21} + 2\epsilon_2 - \epsilon_1\right). \quad (4.42)$$

The expression for

$$\mathcal{F}_4\left(a, \epsilon_1, \epsilon_2\right) = \epsilon_1\epsilon_2\left(Z_4 - Z_1 Z_3 - \frac{1}{2}Z_2^2 + Z_1^2 Z_2 - \frac{1}{4}Z_1^4\right)$$

is very lengthy to present here. We only note here that it is indeed regular at $\epsilon_{1,2} \to 0$. Here are the first nontrivial terms of its expansion:

$$\mathcal{F}_4\left(a, \epsilon_1, \epsilon_2\right) = \frac{1469}{2^{15}a^{14}} + \frac{18445}{2^{15}a^{16}}\left(\epsilon_1^2 + \epsilon_2^2\right) + \frac{15151}{2^{14}a^{16}}\epsilon_1\epsilon_2 + \cdots, \quad (4.43)$$

Thus

$$\mathcal{F}_4 = \frac{1469}{2^9},$$

which is as it should be.

Finally we quote a general formula for the determinant of the vector field action on the tangent space of a generic critical point at arbitrary instanton number and gauge group $SU(N)$. To obtain this formula we closely follow the line of arguments presented in [14], Section 5.2, where the characters of the torus action around fixed points are calculated. For our purposes we need to calculate the character of the representation of the group

$$U(1)^{N-1} \times U(1)^2 \quad (4.44)$$

(the first factor is the Cartan subgroup of the group $SU(N)$ and the second is the 2-torus acting in space-time and on ADHM data) in the tangent space at the fixed point specified by

19

the Young diagrams $Y_1, \ldots, Y_N$. The result reads (cf. with the formula in proposition 5.8 page 67 of Nakajima's book [14]):

$$\chi\left(a_l, \epsilon_1, \epsilon_2\right) = \sum_{\lambda,\mu=1}^{N} T_{a_\mu} T_{a_\lambda}^{-1} \left(\sum_{s\in Y_\lambda} T_1^{-h_\lambda(s)} T_2^{1+v_\mu(s)} + \sum_{s'\in Y_\mu} T_1^{1+h_\mu(s')} T_2^{-v_\lambda(s')} \right), \qquad (4.45)$$

where $h_\lambda(s) = \nu_{\lambda,i} - j$, $v_\lambda(s) = \nu'_{\lambda,j} - i$ if the box s is located on the i-th row and the j-th column of a Young diagram. It is assumed that $\nu_{\lambda,i}$, $\nu'_{\lambda,j}$ are defined for arbitrary positive integers i, j. For $i > \nu'_{\lambda,1}$ and $j > \nu_{\lambda,1}$ by definition they are identically zero. In (4.45) $T_{a_\lambda} \equiv \exp i a_\lambda$, $T_l \equiv \exp i\epsilon_l$ are elements of respective $U(1)$ factors of the group (4.44) taken in the fundamental representations. A term of the form $T_{a_\mu} T_{a_\lambda}^{-1} T_1^m T_2^n$ in Eq. (4.45) indicates that the tangent space includes a (complex) one dimensional invariant subspace of our deformed vector field $\tilde{x}'$ action with eigenvalue $a_\mu - a_\nu + m\epsilon_1 + n\epsilon_2$. Multiplying all these eigenvalues for the determinant of $\tilde{x}'$ action we find[11]

$$\det \mathcal{L}_{\tilde{x}'}\big|_{(Y_1,\cdots,Y_N)} = \prod_{\lambda,\mu=1}^{N} \left(\prod_{s\in Y_\lambda} \left(a_{\mu\lambda} - \epsilon_1 h_\lambda(s) + \epsilon_2 \left(1 + v_\mu(s)\right)\right) \right.$$

$$\left. \times \prod_{s'\in Y_\mu} \left(a_{\mu\lambda} + \epsilon_1 \left(1 + h_\mu(s')\right) - \epsilon_2 v_\lambda(s')\right) \right). \qquad (4.46)$$

The repercussions of this formula will be discussed in a subsequent publication.

5   Acknowledgements

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[11]A more complicated formula for the (inverse) determinant is presented also in [15] Eq. (3.20) which has a discrepancy as compared to our formula (4.46). We have checked that in all cases explicitly described in this section the formula (4.46) gives correct results.

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