Skyrmions in the Quantum Hall effect and noncommutative solitons

V. Pasquier
CEA/Saclay, Service de Physique Théorique
F-91191 Gif-sur-Yvette Cedex, FRANCE

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Abstract

It has been recently shown that solitons are fundamental classical solutions of non-commutative field theories. We reconsider this issue from the standpoint of the Hall effect and identify some solutions with known solutions in the integer Hall effect with no Zeeman coupling.

1 Introduction

This letter is made of two parts.

In the first part we show that the wave functions of two particles interacting by a rotationally invariant potential in their Lowest Landau Levels is independent of the potential. This is well known [1] when the two particles have equal and opposite charge and remains true when the charges are not equal. The interest of this result is that the bound states formed by the two particles are the relevant excitations of the fractional quantum Hall effect (FQHE). In this case the two particles obey opposite statistics and the bound states are the so called Composite fermions of the FQHE. The universality of their wave function is the seed for the rigidity of the many-body wave functions of the FQHE [15] as explained in [11]. When the two charges are equal and opposite, the bound states form neutral fermionic dipoles [11, 12]. The field theory of these neutral dipoles is related to the Fermi liquid observed at $\nu = 1/2$ [16]. We briefly recall why the theory of these dipoles is expected to be noncommutative (see also [13, 14]).

In the second part we study the condensation of bosonic dipoles made of two fermions using the techniques of [11]. These bosonic dipoles are useful to describe the quantum Hall effect with spin or the double layer systems. This allows us to establish a link between the skyrmion of the Hall effect [3, 6] and solitons recently introduced in the context of non commutative string theory [7, 8, 9, 10]. The main physical difference lies in the extensiveness of the skyrmions made out of a macroscopic number of dipoles whereas the noncommutative solitons [10] coincide with their microscopic limit.

2 Dipoles and displacement group

In this section we consider the motion of two particles with charges of opposite sign but not necessarily equal in magnitude which interact attractively in a strong magnetic field. If V is a translation and rotation invariant potential and P_0 projects each particle in its Lowest Landau Level, the eigenstates of the matrix $P_0 V P_0$ acting in the Hilbert space of the two particles do not depend on V and can be organized into representations of a deformation of the displacement algebra.

Consider a charge $q > 0$ particle in a magnetic field. We use the symmetric gauge and take a units of length such that the gauge field is given by: $A_x = y, \; A_y = -x$. In terms of the coordinates $z = x + iy, \; \overline{z} = x - iy$ one can define two sets of mutually commuting oscillators:

$$a = \partial_{\overline{z}} + qz/2 \quad , \quad a^+ = -\partial_z + q\overline{z}/2$$

$$b = \partial_z + q\bar{z}/2 \quad , \quad b^+ = -\partial_{\bar{z}} + qz/2$$

The first set of oscillators define the free Hamiltonian $H = a^+a$ the eigenvalues of which correspond to the Landau levels (LL) $E_n = nq$ with $n \geq 0$. The second set of oscillators b, b^+ are the guiding center coordinates. Together with angular the angular momentum $L = (b^+b - a^+a)/q$ they generate a deformation of the displacement algebra:

$$[b, b^+] = q, \;\; [L, b^+] = b^+, \;\; [L, b] = -b \tag{1}$$

and they commute with H so that this algebra acts within each LL. The representations are characterized by the eigenvalues $n \geq 0$ of $b^+b/q - L$ which index the LL's and the states $|n, l\rangle$ within a representation are labeled by the angular momentum eigenvalues $l \geq -n$. The wave functions in the Lowest LL $(n = 0)$ are:

$$\langle z|l\rangle = (q^{1/2}z)^l/(2\pi l!)^{1/2} e^{-qz\bar{z}/2} \tag{2}$$

It is helpful to visualize them as thin shells of radius $\sqrt{l/q}$ occupying an area π/q. The orbital degeneracy of a charge q is thus $N_\phi = A/2\pi l_q^2$ where A is the total area and $l_q = (2q)^{-1/2}$ defines the magnetic length. It is also useful to introduce coherent states $|z\rangle$ defined by $\langle l|z\rangle = \langle z|l\rangle^*$ which are the most localized states in the Lowest Landau level (LLL).

Consider now two particles with positive charge q_1, q_2 interacting by a translation and rotation invariant potential V within their respective LLL ($n_1 = n_2 = 0$). If we label $1, 2$ the oscillators associated to each particle, the operators $B = b_1 + b_2, \;\; B^+ = b_1^+ + b_2^+$ and $L = L_1 + L_2$ act within the LLL and commute with the interaction V. They obey the relations (1) with the parameter $q = q_1 + q_2$. Inside the LLL the states $|n, l\rangle$ are labeled by the eigenvalues of $B^+B/q - L$ $n \leq 0$ and the angular momentum eigenvalues $l \geq -n$. As a result the eigenstates of P_0VP_0 where P_0 projects each particle in its LLL do not depend on V and their eigenvalues E_n are labeled by n.

We are interested in the case where the first particle has a positive charge q_1 and the second one has a negative charge $-q_2$, $q_1 > q_2$. In this case, a_2^+ and b_2^+ are an annihilation operator and it follows that the states associated to the second particle in the n^{th} LL have their angular momentum $l_2 \leq n_2$. When the two particles 1 and 2 interact the operators B, B^+, L obey the relations (1) with the parameter $q^* = q_1 - q_2$ and the states in the LLL $(n_1 = n_2 = 0)$ $|n, l\rangle$ have $n \geq 0$ and $l \geq -n$. They are the eigenstates of P_0VP_0 and in the case where V is attractive they form bound states which can be interpreted as charge q^* particles in the n^{th} LL. For example, the wave functions of the states with zero angular momentum ($l = 0$) are given by:

$$\phi_n(z_1, \bar{z}_2) = L_n(q^* z_1\bar{z}_2) e^{q_2 z_1\bar{z}_2 - q_1 z_1\bar{z}_1/2 - q_2 z_2\bar{z}_2/2} \tag{3}$$

where the L_n are the Laguerre polynomials. If the two charges are equal and opposite, B and B^+ can be diagonalized simultaneously and play the role of a generalized momentum. The corresponding plane waves are then:

$$\phi_{p,\bar{p}}(z_1, \bar{z}_2) = e^{q(z_1\bar{z}_2 - z_1\bar{z}_1/2 - z_2\bar{z}_2/2) + i(\bar{p}z_1 + p\bar{z}_2)} \tag{4}$$

This dynamical generation of higher Landau level wave functions or plane waves inside the LLL plays an important role in the FQHE. The bound states obtained in this way are called composite particles and the field theory of these particles is non-commutative as we show now (see also [11, 12]). Let us consider the case where the two charges are equal in magnitude $q_1 = q_2$. Then the bound states form neutral dipoles and the non-commutativity originates from their extensiveness [11, 12, 14]. The projection into the LLL transforms $f(\vec{x})$ into a matrix :

$$\hat{f}_{l,l'} = \langle l|f|l'\rangle$$

where the matrix elements are taken in between LLL orbitals. The action of this matrix on the state $|l\rangle$ being given by $\sum_{l'} f_{l,l'}|l'\rangle$. The function $f(z, \bar{z})$ is called the P-symbol for the matrix $\hat{f}$

[17]. If we use the coherent state basis to represent the LLL orbitals one can rewrite the matrix $\hat{f}$ as:

$$\hat{f} = \int \; |z\rangle f_p(z,\overline{z})\langle z| \; dz d\overline{z}.$$

and because the LLL basis is not complete, the matrices associated to different functions not commute [11, 12].

Another way to associate a function to a matrix consists in bracketing the matrix between coherent states:

$$f_q(z,\overline{z}) \quad = \quad \langle z|\hat{f}|z\rangle \tag{5}$$

It is called the Q-symbol and will be useful when we consider fields in the next section. In particular one has:

$$e^{i(\bar{p}z+p\overline{z})} = \langle z|e^{i\frac{\bar{p}b^+}{q}}e^{i\frac{pb}{q}}|z\rangle \tag{6}$$

The Q-symbol induces a noncommutative product on functions which we denote by $*$: $f_q * g_q = (\hat{f}\hat{g})_q$. In the following, the Q-symbol is understood whenever we consider a *-product.

In the dipole case there are 4 type of operators to consider according to how they intertwine the particles 1 and 2. They can be represented in the matrix form:

$$\begin{pmatrix} \hat{f}_{11} & \hat{f}_{12} \\ \hat{f}_{21} & \hat{f}_{22} \end{pmatrix} \tag{7}$$

where each matrix element is a N_ϕ^2 matrix. We can also consider the Q-symbol which is a two by two matrix with functions of $z, \overline{z}$ as matrix elements.

3 Skyrmions and non-commutative solitons

The simplest model for dipoles occurs for a system of N_e electrons interacting repulsively at $\nu = 1$: $N_e = N_\phi$. The electrons carry a spin $1/2$ which is not coupled to the magnetic field in the limit of zero Zeeman coupling. At this filling the ground state is a ferromagnet with total spin $N_e/2$ where $N_e = N_\phi$ is the total number of electrons in the system.

The addition or removal of a single electron dramatically reduces the spin as seen numerically [3] and experimentally [4]. This behavior may be explained using the so called skyrmion spin textures [6, 5] in a non linear σ model approach. These textures carry a topological number equal to the electric charge. In this section we review these solutions and reinterpret them as non-commutative solitons.

Let us see heuristically how this problem is related to the dipoles and why we expect them to condense in presence of an electric charge. The picture we have in mind is a ferromagnetic ground state where all spins are aligned in the minus direction. Each time we flip a spin a spin one dipole is formed made by the spin up electron and the hole created in the spin down electron sea. If we remove an electron from the ground state, a condensate will form if it is energetically favorable to flip a large number of spins to lower the energy. In the dipole language it means that a large number of dipoles agglomerate around the charge. To see that this occurs consider one dipole in presence of a positive charge equal in magnitude to the charges of the dipole. It is equivalent to consider two positive charges denoted $1, 2$ in presence of a negative charge denoted 3. The charges with the same sign repel each other through the interaction V while the charges with opposite sign attract each other through $-V$. In the strong field limit we can neglect the masses of the particle and by the canonical quantization procedure the coordinate $R_{13} = R$ becomes conjugated to $R_{23} = P^\perp$, $[P_x^\perp, R_y] = -[P_y^\perp, R_x] = 1$. The Hamiltonian which describes the three charges is the sum of their interactions: $H = V(R - P^\perp) - V(R) - V(P^\perp)$. For a reasonable repulsive potential a bound state forms with the three particles aligned and the negative charge in the middle of two positive charges (see [2] for a more complete treatment). The angular momentum is proportional to the area since $L = R.P^\perp$. Since a single dipole binds to the charge we expect that several (bosonic) dipoles condense in the presence of the charge and that the condensate carries an angular momentum L proportional to its mean square radius.

To make the discussion simplest we consider the case of a hard core potential $V = \delta^{(2)}(\vec{x})$ and the electrons live on a sphere of area $(2L+1)\pi$ thread by $2L$ quantum fluxes where L is half integer. The LLL is $2L+1$ degenerate and in the azimuthal gauge $a_\phi = 1 - \cos(\theta)$ the LLL orbitals take the form:

$$\langle \hat{r} | l \rangle = \begin{pmatrix} 2L \\ l+L \end{pmatrix}^{1/2} u^{L+l} v^{L-l} \tag{8}$$

$-L \leq l \leq L$ where $u = \sin(\theta/2) e^{i\phi}$ and $v = \cos(\theta/2)$.

The zero energy wave functions are homogeneous polynomial of degree $2L$ in the coordinates u_i, v_i of the electrons and vanish when two particles are at the same position. When there are exactly $2L+1$ electrons on the sphere the Pauli principle forces it to be symmetric under the spin permutations and the ground state is thus ferromagnetic. If we remove 1 electron from the system, Fertig et al. [5] proposed a wave function satisfying these constraints :

$$\Phi = \prod_{i=1}^{N_e} (u_i | \downarrow\rangle + v_i | \uparrow\rangle) \prod_{i<j} (u_i v_j - u_j v_i) \tag{9}$$

The effect of the first factor is to expel the spin down electrons from the first orbital $(l=-L)$ and the spin up electron from the last orbital $(l=L)$ so that the spin is up at the north pole and down at the south pole. This wave function being invariant under simultaneous space and spin rotation, it follows that the spin takes a hedgehog shape around the sphere. Although it is not a spin eigenstate, it has its maximal weight for the spin and angular momentum $S=L=0$. It fits our qualitative picture where on a sphere of area $(2L+1)\pi$, L dipoles bind to the charge to form a bound state of spin and the angular momentum L.

To relate this discussion to the noncommutative field theory it is useful to introduce the fermionic operators which create a spin up or down electron in the l^{th} orbital: $c^+_{l\uparrow}, c^+_{l\downarrow}$. The products $c^+_{l\uparrow} c_{l'\downarrow}$ create a spin one dipole made of charge $q=1$ particle in the l orbital and a charge $q=-1$ hole in the l' orbital. It is possible to express the relevant observables in terms of a fundamental field which creates the dipoles. For this we define the matrix fields $\sigma_{ij}, \bar{\sigma}_{kl}$ which obey canonical commutation relations:

$$\begin{aligned} [\sigma_{ij}, \sigma_{kl}] &= [\bar{\sigma}_{ij}, \bar{\sigma}_{kl}] = 0 \\ [\sigma_{ij}, \bar{\sigma}_{kl}] &= \delta_{il}\delta_{jk} \end{aligned} \tag{10}$$

Next we identify fields in the two representations by requiring that they obey the same commutation relations:

$$A = \begin{pmatrix} c^+_{i\downarrow} c_{j\downarrow} & c^+_{i\uparrow} c_{l\downarrow} \\ c^+_{k\downarrow} c_{j\uparrow} & c^+_{k\uparrow} c_{l\uparrow} \end{pmatrix} = \begin{pmatrix} (\bar{\sigma}\sigma)_{ij} & (\bar{\sigma}.\sqrt{1-\sigma\bar{\sigma}})_{il} \\ (\sqrt{1-\sigma\bar{\sigma}}.\sigma)_{kj} & (1-\sigma\bar{\sigma})_{kl} \end{pmatrix} \tag{11}$$

Indeed, the matrix elements of the left-hand side matrix obey the commutation relations of a $U(2N)$ Lie algebra and the right-hand side matrix can be viewed as a generalized Holstein-Primakov representation for these generators [11]. If we go to the Q-representation, the equality (11) translates into an identification between fields:

$$\begin{pmatrix} \phi^+_\downarrow \phi_\downarrow & \phi^+_\uparrow \phi_\downarrow \\ \phi^+_\downarrow \phi_\uparrow & \phi^+_\uparrow \phi_\uparrow \end{pmatrix} (\hat{r}) = \begin{pmatrix} \bar{\sigma} * \sigma & \bar{\sigma} * \sqrt{1 - \sigma * \bar{\sigma}} \\ \sqrt{1 - \sigma * \bar{\sigma}} * \sigma & 1 - \sigma * \bar{\sigma} \end{pmatrix} (\hat{r}) \tag{12}$$

where the fields $\phi^+_{\uparrow,\downarrow}(\hat{r}) = \sum_l \langle \hat{r} | l \rangle c^+_{l\uparrow,\downarrow}$ create an electron up or down at a given position. The diagonal matrix elements are the density of up and down electrons. In the case of a δ potential the energy is the integral of the total density squared which in term of $\sigma(\hat{r})$ is given by:

$$E = \int (\sigma * \bar{\sigma} - \bar{\sigma} * \sigma - 1)^2 d^2\hat{r} \tag{13}$$

We then minimize the energy for a classical field and we thus replace σ_{ij} by a c-number in this expression. The minimum is clearly for a constant density:

$$[\sigma, \bar{\sigma}] = \frac{1}{2L+1} \tag{14}$$

A solution valid in the north hemisphere is $\sigma_{l-1,l} = \sqrt{\frac{l+L}{2L+1}}$. The expression which follows for A is then:

$$A = \frac{1}{2L+1} \begin{pmatrix} L+l & L_+ \\ L_- & L-l \end{pmatrix} \tag{15}$$

where $\vec{L}$ is the angular momentum operator. In the south hemisphere we must use the gauge $\tilde{a}_\phi = 1 + \cos(\theta)$. The LLL basis $|\tilde{l}\rangle$ takes the form (8) where $\tilde{u} = \cos(\theta/2)e^{-i\phi}$ and $\tilde{v} = \sin(\theta/2)$. In the intermediate region the two basis are related by a gauge transformation $e^{2iL\phi}\langle\hat{r}|\tilde{l}\rangle = \langle\hat{r}|l\rangle$ with $l = -\tilde{l}$. In the basis $|\tilde{l}\rangle$ the expression (15) of A must be conjugated by σ_x. Thus we use the same expression $\tilde{\sigma}_{ll'} = \sigma_{ll'}$ in the right hand side of (11) to represent $\sigma_x A \sigma_x$ in the south hemisphere. Clearly we cannot use a single matrix σ to represent A globally since the trace of the left-hand side of (11) must be equal to the number of electrons $2L$ and the trace of the right-hand side is equal to $2L+1$ for any finite size matrix σ.

The Q-symbol expression of A at scales large compared to the magnetic length is obtained by substituting in (12) $\sigma_q(\hat{r}) \simeq \sin(\theta/2)e^{i\phi}$. It has the characteristic hedgehog shape of the spin texture.

Recently, non-commutative solitons were introduced in relation with strings [7]. The solution presented here has many similarities with the one presented in [10] following [8, 9]. In particular the field A is a projector $A^2 = A$. There is however a major physical difference: on the plane, the equivalent of the skyrmion [5] has a parameter which controls its size. The solution A of [10] coincides with the 'zero size soliton' as can easily be seen by interpreting its Q-symbol as a spin texture. In the case of the δ potential, the energy of the skyrmions is independent of their size but as soon as some nonlocal repulsion is introduced large skyrmions are energetically favored.

One of the aim of this letter was to establish a link between an experimentally observed phenomena in the integer Hall effect and classical solutions of noncommutative field theories. In the context of the fractional Hall effect the dipoles that have been considered in [11, 12] are fermions and the field σ is thus fermionic. In this case nonlinear classical solutions, if they exist, may be related to the striped phases [18] (see [20] for a proposal in this direction) or pair condensation [19] predicted for higher Landau levels.

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