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Chern-Simons in the Seiberg-Witten map for non-commutative Abelian gauge theories in $4D$

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Abstract

A cohomological BRST characterization of the Seiberg-Witten (SW) map is given. We prove that the coefficients of the SW map can be identified with elements of the cohomology of the BRST operator modulo a total derivative. As an example, it will be illustrated how the first coefficients of the SW map can be written in terms of the Chern-Simons three form. This suggests a deep topological and geometrical origin of the SW map. The existence of the map for both Abelian and non-Abelian case is discussed. By using a recursive argument and the associativity of the $\star$-product, we shall be able to prove that the Wess-Zumino consistency condition for non-commutative BRST transformations is fulfilled. The recipe of obtaining an explicit solution by use of the homotopy operator is briefly reviewed in the Abelian case.

Key words: Seiberg-Witten map, non-commutative gauge theories, Chern-Simons.

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1 Introduction

The construction of non-commutative gauge theories via the Seiberg-Witten (SW) map [1] has been recently considered by many authors [2, 3, 4, 5, 6]. The main issue is whether it is possible to deform [4, 5] the structure equations of ordinary gauge theories into the non-commutative counterpart.

The commutative ghost and gauge fields c, a_i take values in the Lie algebra $\mathfrak{G}$ and transform as

$$sc = ic^2 \tag{1a}$$
$$sa_i = \partial_i c + i[c, a_i]\,, \tag{1b}$$

where s is the BRST differential. Analogously, C, A_i denote the corresponding non-commutative fields, which are formal power series in θ whose coefficients are local polynomials in c and a_i. They are required to obey the conditions [5]

$$sC = iC \star C \tag{2a}$$
$$sA_i = \partial_i C + i[C \stackrel{\star}{,} A_i]\,. \tag{2b}$$

The fields C, A_i reduce to c, a_i at $\theta = 0$

$$C = c + O(\theta)\,, \tag{3a}$$
$$A_i = a_i + O(\theta)\,. \tag{3b}$$

In addition, C is taken to be linear in c. Moreover, it has been underlined [4] that C, A_i cannot be Lie-algebra valued if they have to fulfill eqs.(2). They are in fact elements of the enveloping algebra $Env(\mathfrak{G})$ [4].

For the star operation we adopt the Weyl-Moyal product

$$\begin{aligned} f \star g &= \exp\left(\frac{i}{2}\theta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right) f(x)g(y)\Big|_{y\to x} \\ &= f(x)g(x) + \frac{i}{2}\theta^{ij}\partial_i f(x)\partial_j g(x) + \ldots \end{aligned} \tag{4}$$

and

$$\begin{aligned} [A \stackrel{\star}{,} B] &= A \star B - B \star A \\ &= i\theta^{ij}\partial_i A \partial_j B + \ldots\,. \end{aligned} \tag{5}$$

where θ^{ij} is an anti-symmetric constant matrix.

The solutions of eqs.(2) turn out to be affected by several ambiguities [2, 4, 5], which can be conveniently analysed from a cohomological point of view. Indeed, as shown in [5], a nilpotent differential Δ is naturally generated when one tries to solve eqs.(2) order by order in the θ expansion. In the Abelian case Δ reduces to the ordinary Abelian BRST differential. Thus, the problem of finding an explicit solution of eqs.(2) is equivalent to the construction of a suitable homotopy operator for Δ, provided that the corresponding Wess-Zumino consistency condition is fulfilled. This is the approach taken in ref.[5]. Recently, the homotopy operator relevant for the Freedman-Townsend Abelian non-commutative model has been obtained in [6].

The aim of the present work is to pursue the investigation of the cohomological properties of the SW map. We first focus on the Abelian case. Here we point out that the most general solution of eqs.(2) can be characterized in terms of elements of the cohomology of s modulo d, $H(s|d)$. The aforementioned ambiguities show up as elements of the local cohomology of s, $H(s)$, and can be interpreted as field redefinitions, according to [2, 4, 5].

It is worth reminding that the elements of the cohomology of s modulo d can be written in terms of topological quantities like the Chern-Simons three-form and its generalizations. As a consequence, we shall prove that the lowest order coefficients of the expansion of C and A_i can be expressed indeed in terms of the Chern-Simons form, up to a total derivative. This observation leads us to suggest that the coefficients of the SW map possess a deep topological origin.

We also check that the Wess-Zumino consistency condition of ref. [5] associated to eqs.(2) is satisfied due to the associativity of the $\star$-product, both for the Abelian and the non-Abelian case. This is a necessary condition for the existence of the SW map. It becomes a sufficient condition if the homotopy operator for Δ is obtained. This point will be illustrated in detail for the Abelian case by providing an explicit formula for the homotopy.

The paper is organized as follows. In Sect. 2 we present the most general solution for the non-commutative transformations of the ghost and gauge field, and we shall check that the lowest order coefficients of the SW map can be cast in the form of the Chern-Simons. Sect. 3 is devoted to the recursive proof of the fulfillment of the Wess-Zumino consistency condition and of the existence of the SW map in the Abelian case.

2 General solution of non-commutative Abelian gauge transformations

In the Abelian case the BRST differential s is

$$sa_i = \partial_i c\,, \qquad sc = 0\,. \tag{6}$$

Besides the θ-expansion, in this case there is an additional grading [3] which is useful in order to solve the equations

$$sC = iC \star C\,, \tag{7a}$$
$$sA_i = \partial_i C + i[C \stackrel{\star}{,} A_i]\,. \tag{7b}$$

This grading consists in counting the number of gauge fields a_j present in a given expression. This amounts to expand both C and A_i as

$$C = c + \sum_{n=1}^{\infty} C^{(n)}(a_j, c, \theta) \,, \tag{8a}$$

$$A_i = a_i + \sum_{n=2}^{\infty} A_i^{(n)}(a_j, \theta) \,, \tag{8b}$$

where $C^{(n)}$, $A_i^{(n)}$ are assumed to be of order n in a_j. Each coefficient $C^{(n)}$, $A_i^{(n)}$ can be further expanded in powers of θ. As it can be seen from eq.(6), the BRST differential s has degree -1 with respect to the degree induced by counting the fields a_j.

Let us begin with eq.(7a). Expanding in powers of a_j one gets

$$\begin{aligned}
&sc = 0 \,, \\
&sC^{(1)} = \frac{i}{2}\{c \stackrel{\star}{,} c\} \,, \\
&sC^{(2)} = i\{c \stackrel{\star}{,} C^{(1)}\} \,, \\
&sC^{(3)} = i\{c \stackrel{\star}{,} C^{(2)}\} + \frac{i}{2}\{C^{(1)} \stackrel{\star}{,} C^{(1)}\} \,, \\
&\ldots
\end{aligned} \tag{9}$$

A particular solution of these equations has been found in [4] and takes the form

$$C^{(1)} = \frac{1}{2}\theta^{ij} a_j \partial_i c \,, \tag{10a}$$

$$C^{(2)} = \frac{1}{6}\theta^{kl}\theta^{ij} a_l (\partial_k (a_j \partial_i c) - f_{jk}\partial_i c) \,, \tag{10b}$$

where the expansion in θ has been truncated at the first non trivial orders and f_{jk} stands for the Abelian commutative field strength

$$f_{jk} \equiv \partial_j a_k - \partial_k a_j \,. \tag{11}$$

It should be observed that these expressions provide only a particular solution to eq.(7a). Other solutions have been found in [3, 4] and differ by elements which are BRST invariant. In fact, from the eqs.(9) it follows that the coefficients $C^{(n)}$ are always defined modulo terms which are BRST invariant. Moreover, as remarked in [4, 7], these extra terms should be taken into account in order to obtain a consistent quantum version of the non-commutative theories.

It is the purpose of the next subsection to give a precise cohomological interpretation of the general solution of eq.(7a).

2.1 The general solution for C

Let us work out the most general solution of eq.(7a). This equation can be solved if and only if there exists a Λ such that

$$iC \star C = \frac{i}{2}\{C \overset{\star}{,} C\} = s\Lambda\,, \tag{12}$$

where Λ is a local formal power series in a_j, θ and linear in c. If Λ exists[4], C must satisfy the homogeneous equation

$$s(\Lambda - C) = 0\,, \tag{13}$$

which in turn implies

$$C = \Lambda + \xi\,, \qquad \xi \in H(s)\,. \tag{14}$$

In the above equation ξ stands for a representative of $H(s)$, the local cohomology of s. Of course, $H(s)$ contains an infinite number of elements, as for instance

$$\theta^{ij} c f_{ij}\,, \quad \theta^{ij}\theta^{mn} c f_{ij} f_{mn}\,, \quad \theta^{ij}\theta^{mn}\theta^{pq} c f_{ij} f_{mn} f_{pq}\,, \quad \ldots \tag{15}$$

We can now proceed to characterize Λ. For that purpose we recall a few properties of the $\star$ product. From the Taylor expansion

$$\begin{aligned} f(x) \star g(x) &= \exp\left(\frac{i}{2}\theta^{ij}\frac{\partial}{\partial x^i}\frac{\partial}{\partial y^j}\right) f(x)g(y)\Big|_{y\to x} \\ &= f(x)g(x) + \sum_{n=1}^{\infty}\frac{1}{n!}\left(\frac{i}{2}\right)^n \theta^{i_1 j_1}\ldots\theta^{i_n j_n}\partial_{i_1}\ldots\partial_{i_n} f(x)\partial_{j_1}\ldots\partial_{j_n} g(x) \end{aligned} \tag{16}$$

one can easily prove the following relation

$$\begin{aligned} [f(x) \overset{\star}{,} g(x)] \;=\; & \sum_{n=1}^{\infty\frac{1}{n!}} \left(\frac{i}{2}\right)^n \theta^{i_1 j_1}\ldots\theta^{i_n j_n} \\ & \times (1-(-1)^n)\partial_{i_1}\ldots\partial_{i_n} f(x)\partial_{j_1}\ldots\partial_{j_n} g(x) \end{aligned} \tag{17}$$

where $[f, g]$ is an anti-commutator if both f and g are fermionic and a commutator otherwise.

The above expression can be written as a total derivative

$$[f(x) \overset{\star}{,} g(x)] = \partial_i j^i \tag{18}$$

for a suitable j^i, thanks to eq.(17) and the anti-symmetry of θ.

[4]We refer the reader to Sect. 3 for a proof of the existence of Λ.

Therefore, eq.(12) can be recast in the form

$$\frac{i}{2}\{C \stackrel{\star}{,} C\} = \partial_i \omega^i \tag{19}$$

where ω^i has ghost number two. By using the above equation eq.(12) reads

$$s\Lambda = \partial_i \omega^i\,, \tag{20}$$

implying that Λ can be characterized as an element of $H(s|d)$, the cohomology of s modulo d, in the space of the formal power series in a_j, c and θ. Notice that this is an all-order statement with respect to the expansion in powers of θ.

The requirement of linearity of Λ (and of C) in c is guaranteed since Λ is of ghost number one and c is the only field with positive ghost number and there are no fields with negative ghost number.

In summary, the most general solution of eq.(7a) can be written as

$$C = c + \Xi + \xi\,, \qquad \Xi \in H(s|d)\,, \qquad \xi \in H(s) \tag{21}$$

where the first term corresponds to the initial condition of eq.(3a), Ξ is a representative of the cohomology of s modulo d and ξ is a representative of the local cohomology of s.

In the following, we shall check that the coefficients of the solutions for $C^{(1)}$ and $C^{(2)}$ given in eqs.(10) belong in fact to $H(s|d)$. In particular, we will prove that $C^{(2)}$ is nothing but the Chern-Simons three form, up to a total derivative.

2.2 Analysis of the coefficients $C^{(1)}$ and $C^{(2)}$

In this subsection we verify that the coefficients $C^{(1)}$ and $C^{(2)}$ in eqs.(10) belong to $H(s|d)$.

Let us consider first $C^{(1)}$. Its variation under the BRST operator s gives

$$sC^{(1)} = \frac{1}{2}\theta^{ij}\partial_j c \partial_i c = \partial_j \left(\frac{1}{2}\theta^{ij} c \partial_i c\right)\,. \tag{22}$$

The latter equality follows from the anti-symmetry of θ^{ij}. By direct inspection one concludes that $C^{(1)}$ cannot be written as a trivial term in the sense of the cohomology of s modulo d. Thus $C^{(1)}$ identifies a representative of $H(s|d)$.

Let us now look at $C^{(2)}$. In this case it is not difficult to verify that the expression (10b) can be rewritten as a Chern-Simons term plus a total derivative:

$$\begin{aligned} C^{(2)} \;&=\; \frac{1}{6}\partial_k(\theta^{kl}\theta^{ij} a_l a_j \partial_i c) \\ &\quad - \frac{1}{12}\theta^{kl}\theta^{ij}(a_j f_{kl} + a_k f_{lj} + a_l f_{jk})\partial_i c\,. \end{aligned} \tag{23}$$

The Chern-Simons term is put in evidence in the last line. Let us now check that $C^{(2)}$ belongs to $H(s|d)$. For that purpose it is sufficient to look at the Chern-Simons. Thus

$$\begin{aligned} s\left(\theta^{kl}\theta^{ij}(a_j f_{kl}+a_k f_{lj}+a_l f_{jk})\partial_i c\right) &= \theta^{ij}\theta^{kl}\left(\partial_j c f_{kl}+\partial_k c f_{lj}+\partial_l c f_{jk}\right) \\ &= \theta^{ij}\theta^{kl}\left(\partial_j c f_{kl}+\partial_k c f_{lj}+\partial_l c f_{jk}\right. \\ &\quad \left.+c(\partial_j f_{kl}+\partial_k f_{lj}+\partial_l f_{jk})\right)\partial_i c\,, \end{aligned} \tag{24}$$

where the term in the last line is zero due to the Bianchi identity. Therefore

$$\begin{aligned} s\left(\theta^{kl}\theta^{ij}(a_j f_{kl}+a_k f_{lj}+a_l f_{jk})\partial_i c\right) &= \partial_j(\theta^{ij}\theta^{kl}c f_{kl}\partial_i c) \\ &+ \partial_k(\theta^{ij}\theta^{kl}c f_{lj}\partial_i c)+\partial_l(\theta^{ij}\theta^{kl}c f_{jk}\partial_i c) \\ &- \theta^{ij}\theta^{kl}\left(c f_{kl}\partial_j\partial_i c+c f_{lj}\partial_k\partial_i c+c f_{jk}\partial_l\partial_i c\right)\,. \end{aligned} \tag{25}$$

Upon reshuffling of the indices one can see that the terms in the last line of the above equation sum up to zero due to the anti-symmetry of the θ. Finally

$$\begin{aligned} & s\left(\theta^{kl}\theta^{ij}(a_j f_{kl}+a_k f_{lj}+a_l f_{jk})\partial_i c\right)= \\ &\qquad \partial_j(\theta^{ij}\theta^{kl}c f_{kl}\partial_i c)+\partial_k(\theta^{ij}\theta^{kl}c f_{lj}\partial_i c)+\partial_l(\theta^{ij}\theta^{kl}c f_{jk}\partial_i c)\,. \end{aligned} \tag{26}$$

Thus the BRST variation of $C^{(2)}$ is a total derivative. Moreover, the presence of the Chern-Simons term ensures that $C^{(2)}$ is a non-trivial element of the cohomology of s modulo d [8]. Its appearance is a consequence of the general results established in the previous section.

2.3 The gauge field A_i

The analysis of the non-commutative Abelian gauge field A_i can be performed along the same lines of C. Of course, a solution of

$$sA_i = \partial_i C + i[C \stackrel{\star}{,} A_i] \tag{27}$$

exists if and only if there is a suitable Ω_i with ghost number zero such that

$$\partial_i C + i[C \stackrel{\star}{,} A_i] = s\Omega_i\,. \tag{28}$$

Ω_i cannot depend on c since it has ghost number zero. If eq.(28) is verified, from eq.(27) we get

$$s(A_i - \Omega_i) = 0\,. \tag{29}$$

Therefore

$$A_i = \Omega_i + \sigma_i\,, \qquad \sigma_i \in H(s) \tag{30}$$

where σ_i is an element of the local cohomology of s in the space of the formal power series in a_j and θ with a Lorentz vector index.

Moreover from eq.(18) we conclude that

$$i[C \stackrel{\star}{,} A_i] = \partial_k \Omega_i^k \tag{31}$$

for a suitable Ω_i^k. From eq.(28) we get

$$s\Omega_i = \partial_k(\Omega_i^k + \delta_i^k C) \,, \tag{32}$$

which means that also Ω_i can be characterized in terms of the cohomology of s modulo d.

The most general solution for the non-commutative gauge field A_i can thus be written as

$$A_i = a_i + \Sigma_i + \sigma_i \,, \quad \Sigma_i \in H(s|d) \,, \quad \sigma_i \in H(s) \tag{33}$$

where Σ_i is an element of $H(s|d)$.
In order to check that the coefficients of the expansion of A_i in eq.(8b) can be identified with elements of $H(s|d)$ let us discuss $A_i^{(2)}$, whose expression has been found in [4]. Up to the first order in θ it reads

$$A_i^{(2)} = -\frac{1}{2}\theta^{kl} a_k(\partial_l a_i + f_{li}) \,. \tag{34}$$

It is interesting to observe that also in this case it is possible to put in evidence the Chern-Simons term. Indeed, up to a total derivative, expression (34) can be written as

$$A_i^{(2)} \;=\; -\frac{1}{2}\partial_l(\theta^{kl} a_k a_i) + \frac{1}{4}\theta^{kl} a_i f_{lk} - \frac{1}{2}\theta^{kl} a_k f_{li} \,. \tag{35}$$

Anti-symmetry of θ^{kl} and of f_{li} allows us to cast the last two terms in the form of a Chern-Simons, namely

$$A_i^{(2)} \;=\; -\frac{1}{2}\partial_l(\theta^{kl} a_k a_i) + \frac{1}{4}\theta^{kl}(a_i f_{lk} + a_l f_{ki} + a_k f_{il}) \,, \tag{36}$$

ensuring that $A_i^{(2)}$ is a non-trivial element of $H(s|d)$. In much the same way one can easily prove that the BRST variation of the coefficient $A_i^{(3)}$ found in [3] yields a total derivative and belongs to $H(s|d)$.

Concerning the matter fields, their inclusion can be discussed along the same line. For instance, the first coefficients for the non-commutative matter fields can be found in [4], and can be easily proven to be characterized in terms of $H(s|d)$ and $H(s)$.

3 Non-commutative Wess-Zumino consistency condition

Although in the Abelian case the expansion in powers of a_i proves to be very useful in discussing the cohomological properties of the coefficients of C and A_i, the expansion in powers of θ is better suited to discuss the existence of the SW map.

In this section we provide a complete proof of the existence of the SW map in the Abelian case for solutions that can be Taylor-expanded in powers of θ. The proof will be constructed recursively in the θ expansion. The main idea of the proof has been given in [9], the key ingredient being the associativity of the $\star$-product.

We will focus on the proof of the existence of C. The non-commutative gauge field A_i can be treated similarly. In the course of the proof the nilpotent operator Δ introduced in [5] will naturally arise.

We wish to prove that the equation

$$sC = iC \star C \tag{37}$$

can be solved by a formal power series

$$C = \sum_{n=0}^{\infty} C^n(a_j, c, \theta) \tag{38}$$

with the condition $C^0 = c$. C^n is now a local polynomial in (a_j, c, θ) of order n in θ and is linear in c.

Also, it will be useful to introduce the following notation. For arbitrary local functions $f(x), g(x)$, we expand the product $f \star g$ in powers of θ and define

$$f(x) \star g(x) \equiv \sum_{q=0}^{\infty} f(x) \overset{q}{\star} g(x) \,. \tag{39}$$

where $f(x) \overset{q}{\star} g(x)$ denotes the term of order q in the θ expansion of the $\star$-product.

The proof of the existence will be given by induction in the powers of θ. At the order zero in θ eq.(37) is verified since $C^0 = c$. Let us now assume that eq.(37) is verified up to order $n-1$, so that

$$sC^m = i \sum_{p+q+r=m} C^p \overset{q}{\star} C^r \,, \qquad m = 0, \dots, n-1 \,. \tag{40}$$

We have to prove that eq.(37) can be verified at the order n by a suitable choice of C^n. At the n-th order we get

$$\begin{aligned} sC^n &= i \sum_{p+q+r=n} C^p \overset{q}{\star} C^r \\ &= iC^0C^n + iC^nC^0 + i \sum_{\substack{p+q+r=n \\ p\neq n,\, r\neq n}} C^p \overset{q}{\star} C^r \,, \end{aligned} \tag{41}$$

which can be rewritten as

$$\begin{aligned} sC^n - i\{c, C^n\} &= Z^n \\ &\equiv i \sum_{\substack{p+q+r=n \\ p\neq n,\, r\neq n}} C^p \overset{q}{\star} C^r \,. \end{aligned} \tag{42}$$

Following [5] it is useful to introduce the nilpotent operator Δ defined as

$$\Delta(X) = sX - i[c, X\} \,, \qquad \Delta^2 = 0 \,, \tag{43}$$

where $[c, X\}$ is a commutator if X is bosonic and an anti-commutator if X is fermionic. Eq.(42) takes now the form

$$\Delta C^n = Z^n \,. \tag{44}$$

If a solution C^n of eq.(43) exists, then Z^n must satisfy the Wess-Zumino consistency condition

$$\Delta Z^n = 0 \,, \tag{45}$$

due to the nilpotency of Δ. We will show that eq.(45) holds true due to the associativity of the $\star$-product and the recursive assumption in eq.(40).

Some remarks are here in order. In the Abelian case the first two terms in eq.(41) cancel out and the operator Δ reduces to the usual BRST operator s. This is no more true in the non-Abelian case. However, since the proof of the Wess-Zumino consistency condition in eq.(45) can be carried out along the same lines for both the Abelian and the non-Abelian case, we choose to work with the full operator Δ.

Let us compute ΔZ^n. We get

$$\begin{aligned} \Delta Z^n &= sZ^n - i[c, Z^n] \\ &= i \sum_{\substack{p+q+r=n \\ p\neq n,\, r\neq n}} (sC^p) \overset{q}{\star} C^r - i \sum_{\substack{p+q+r=n \\ p\neq n,\, r\neq n}} C^p \overset{q}{\star} (sC^r) \\ &\quad + \sum_{\substack{p+q+r=n \\ p\neq n\,, r\neq n}} c\, C^p \overset{q}{\star} C^r - \sum_{\substack{p+q+r=n \\ p\neq n\,, r\neq n}} C^p \overset{q}{\star} C^r c \,. \end{aligned} \tag{46}$$

Taking into account eq.(40), we can rewrite the BRST variations in the above equation in terms of $\star$-product as follows:

$$
\begin{aligned}
\Delta Z^n &= - \sum_{\substack{p+q+r+s+t=n \\ p+q+r\neq n,\, t\neq n}} \left(C^p \overset{q}{\star} C^r\right) \overset{s}{\star} C^t \\
&\quad + \sum_{\substack{p+q+r+s+t=n \\ p\neq n,\, r+s+t\neq n}} C^p \overset{q}{\star} \left(C^r \overset{s}{\star} C^t\right) \\
&\quad + \sum_{\substack{p+q+r=n \\ p\neq n,\, r\neq n}} c\, C^p \overset{q}{\star} C^r - \sum_{\substack{p+q+r=n \\ p\neq n,\, r\neq n}} C^p \overset{q}{\star} C^r c\,.
\end{aligned}
\tag{47}
$$

We now recast each term appearing in the r.h.s. of the above equation in a form convenient to put in evidence the cancellations occurring among the various terms:

$$
\begin{aligned}
- \sum_{\substack{p+q+r+s+t=n \\ p+q+r\neq n,\, t\neq n}} \left(C^p \overset{q}{\star} C^r\right) \overset{s}{\star} C^t &= - \sum_{p+q+r+s+t=n} \left(C^p \overset{q}{\star} C^r\right) \overset{s}{\star} C^t \\
&\quad + \sum_{p+q+r=n} \left(C^p \overset{q}{\star} C^r\right) \, c + c\, c\, C^n\,, \\
\sum_{\substack{p+q+r+s+t=n \\ p\neq n,\, r+s+t\neq n}} C^p \overset{q}{\star} \left(C^r \overset{s}{\star} C^t\right) &= \sum_{p+q+r+s+t=n} C^p \overset{q}{\star} \left(C^r \overset{s}{\star} C^t\right) \\
&\quad - c \sum_{p+q+r=n} C^p \overset{q}{\star} C^r - C^n c\, c\,, \\
\sum_{\substack{p+q+r=n \\ p\neq n,\, r\neq n}} c\, C^p \overset{q}{\star} C^r &= \sum_{p+q+r=n} c\, C^p \overset{q}{\star} C^r - c\, C^n c - c\, c\, C^n\,, \\
- \sum_{\substack{p+q+r=n \\ p\neq n,\, r\neq n}} C^p \overset{q}{\star} C^r c &= - \sum_{p+q+r=n} C^p \overset{q}{\star} C^r c + c\, C^n c + C^n c\, c\,.
\end{aligned}
\tag{48}
$$

Finally, summing up all the terms we are left with

$$
\begin{aligned}
\Delta Z^{(n)} &= - \sum_{p+q+r+s+t=n} \left(C^p \overset{q}{\star} C^r\right) \overset{s}{\star} C^t + \sum_{p+q+r+s+t=n} C^p \overset{q}{\star} \left(C^r \overset{s}{\star} C^t\right) \\
&= 0\,,
\end{aligned}
\tag{49}
$$

where the associativity of the $\star$-product has been used. Therefore Z^n obeys the Wess-Zumino consistency condition eq.(45) thanks to the associativity of the $\star$-product and to the recursive assumption in eq.(40).

We stress that this result holds for both Abelian and non-Abelian gauge groups.

To conclude that eq.(44) can be solved it remains to prove that the cohomology of the operator Δ is empty in the sector with ghost number two. In particular, if one is able to find out a homotopy operator K for Δ

$$\{K, \Delta\} = \mathcal{I}\,, \tag{50}$$

where $\mathcal{I}$ in the r.h.s. of eq.(50) stands for the identity operator when acting on formal power series depending on c and zero otherwise, then an explicit solution for C^n is provided by

$$C^n = \Delta K Z^n\,. \tag{51}$$

Eq.(51) follows from the relation

$$Z^n = \mathcal{I} Z^n = \{K, \Delta\} Z^n = \Delta K Z^n \tag{52}$$

which holds true due to the Wess-Zumino consistency condition in eq.(45).

For the non-Abelian case such a homotopy operator K has been proposed in [5]. For the Abelian case the homotopy operator takes a very simple form [10]

$$K = \int_0^1 dt \left(a_i \lambda_t \frac{\partial}{\partial(\partial_i c)} + \partial_{(i} a_{j)} \lambda_t \frac{\partial}{\partial(\partial_i \partial_j c)} + \partial_{(i} \partial_j a_{k)} \lambda_t \frac{\partial}{\partial(\partial_i \partial_j \partial_k c)} + \dots \right) \tag{53}$$

where $(i, \dots, k)$ denotes total symmetrization and the operator λ_t acts as

$$\begin{aligned} &\lambda_t X(c, f_{ij}, \partial^n f_{ij}, \partial^n c, a_i, \partial_{i_1} \dots \partial_{i_{n-2}} \partial_{(i_{n-1}} a_{i_n)}) \\ &\qquad = X(c, f_{ij}, \partial^n f_{ij}, t\partial^n c, t a_i, t\partial_{i_1} \dots \partial_{i_{n-2}} \partial_{(i_{n-1}} a_{i_n)})\,, \end{aligned} \tag{54}$$

where X is an arbitrary local formal power series. The operator λ_t leaves unchanged the undifferentiated ghost c and the field strength f_{ij} and its derivatives $\partial^n f_{ij}$. The homotopy K allows us to obtain representatives of the higher order coefficients in a systematic way. In the case of the Abelian Freedman-Townsend model the analogue of the homotopy operator in eq.(53) can be found in [6]. The proof outlined for the case of C can be paraphrased along the same lines for the case of A_i.

4 Conclusions

In this work we have provided a cohomological interpretation of the solution of the Seiberg-Witten (SW) map for Abelian gauge theories in terms of $H(s|d)$, the cohomology of the Abelian BRST differential s modulo d, with the addition of elements of $H(s)$, the local cohomology of s. This feature might help in clarifying the geometrical and topological nature of the SW map. The examples of the first coefficients for the non-commutative fields A_i and C have been analysed, and shown to be written in terms of the Chern-Simons three form.

It is worth mentioning here that, due to the presence of the θ's, suitable generalizations of the Chern-Simons are to be expected for the higher order coefficients. Consider for instance the term of order three in a_i

$$\theta^{km}\theta^{pq}\left(a_{[i}f_{km]}f_{pq} + a_{[p}f_{km]}f_{qi} + a_{[q}f_{km]}f_{ip}\right) \tag{55}$$

where $[ikm]$ means complete antisymmetrization in the Lorentz indices i, k, m. By using the Bianchi identity, it is almost immediate to verify that the BRST variation of the term (55) gives a total derivative. On the other hand expression (55) turns out to be BRST non trivial. We see thus that the presence of the θ's allows for the existence of new elements of the cohomology of s modulo d, which would be automatically vanishing if the only antisymmetric quantity at our disposal would be the Levi-Civita tensor ϵ^{ijkl}.

The full characterization of $H(s|d)$ in the presence of the θ's is therefore of primary importance in order to work out the most general solution of the SW map. This is an important point which deserves further investigation [11].

We have also discussed the Wess-Zumino consistency condition on general grounds, proving that it can be fulfilled order by order in θ for both Abelian and non-Abelian case, due to the associativity of the $\star$-product. This condition plays a key role in the proof of the existence of the SW map. It is indeed a necessary condition. It becomes a sufficient condition provided one is able to show that the cohomology of the coboundary operator Δ is empty in the sector of ghost number two and one, which would ensure the existence of C and A_i. The homotopy operator for the differential Δ in the Abelian case has been used in order to find an explicit solution for the SW map. This concludes the proof of the existence of the map in this case.

As a final comment, we emphasize that the cohomological analysis carried out here concerns only classical aspects. Moreover, the knowledge of the most general solution of the SW map should be the natural starting point for a possible consistent quantization of the non-commutative theories. Indeed, there are indications that the intrinsic ambiguities of the SW map show up at quantum level even if they are not included in the starting classical action [4, 7].

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