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All minimal supergravity extensions of 2d dilaton theories+)

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Abstract

The formulation of 2d-dilaton theories, like spherically reduced Einstein gravity, is greatly facilitated in a formulation as a first order theory with nonvanishing bosonic torsion. This is especially also true at the quantum level. The interpretation of superextensions as graded Poisson sigma models is found to cover generically all possible 2d supergravities. Superfields and thus superfluous auxiliary fields are avoided altogether. The procedure shows that generalizations of bosonic 2d models are highly ambiguous.

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Despite the fact that so far no tangible direct evidence for supersymmetry has been discovered in nature, supersymmetry [1] managed to retain continual interest within the aim to arrive at a fundamental 'theory of everything' ever since its discovery: first in supergravity [2] in $d = 4$, then in generalizations to higher dimensions of higher N [3], and finally incorporated as a low energy limit of superstrings [4] or of even more fundamental theories [5] in 11 dimensions.

Even before the advent of strings and superstrings the importance of studies in $1 + 1$ 'spacetime' had been emphasized [6] in connection with the study of possible superspace formulations. To the best of our knowledge, however, to this day there have been only few attempts to generalize the supergravity formulation of (trivial) Einstein-gravity in $d = 2$ to the consideration of two-dimensional $(1, 1)$ supermanifolds for which the condition of vanishing (bosonic) torsion is removed [7, 8]. Only attempts to formulate theories with higher powers of curvature (at vanishing torsion) seem to exist [9]. There seem to be only very few exact solutions of supergravity in $d = 4$ as well [10].

Especially at times when the number of arguments in favour of the existence of an, as yet undiscovered, fundamental theory increases [3] it may seem appropriate to also exploit — if possible — *all* generalizations of the two-dimensional stringy world sheet. Actually, such an undertaking can be (and indeed is) successful, as suggested by the recent much improved insight, attained for all (non-supersymmetric) two-dimensional diffeomorphism invariant theories, including dilaton theory, and also comprising torsion besides curvature [11] in the most general manner. In the absence of matter-fields (non-geometrical degrees of freedom) all these models are integrable at the classical level and admit the analysis of all global solutions [12]. Even many aspects of quantization of any such theory now seem to be well understood [13]. By contrast, in the presence of matter and if singularities like black holes (BH) occur in such models, integrable solutions are known only for very few cases. These include interactions with fermions of one chirality [14] and, if scalar fields are present, only the dilaton black hole [15], "chiral" scalars [16] and models which have asymptotical Rindler behaviour [17]. Therefore, a supersymmetric extension of the matterless case suggests that the solvability may carry over, in general. Then, at least part of "matter" could be represented by superpartners of the geometric bosonic field variables. At the quantum level for bosonic gravity in two dimensions the path integral formalism has proved to be invaluable to exactly integrate geometry and to treat matter in a consistent perturbation theory [18]. Also this exact geometrical integration should carry over to the supergravity case.

Within the realm of bosonic two dimensional gravity theories, including those with nonvanishing torsion, the reformulation as a first order action [19, 20] with auxiliary fields ϕ and X^a ($Y = X^a X_a/2$)

$$L^{\mathrm{FOG}} = \int_{\mathcal{M}} \phi d\omega + X_a De^a + \epsilon v(\phi, Y) \tag{1}$$

has led to new insights. Indeed L^{FOG} for a potential v quadratic in torsion

$$v^{\mathrm{dil}}(\phi, Y) = YZ(\phi) + V(\phi). \tag{2}$$

is exatly equivalent [12],[18] to a generalized dilaton theory

$$L^{\mathrm{dil}} = \int d^2x \sqrt{-g} \left[\frac{\widetilde{R}}{2}\phi - \frac{1}{2} Z(\phi)(\partial^n \phi)(\partial_n \phi) + V(\phi) \right]. \tag{3}$$

where $\tilde{R}$ represents the torsion free curvature. A special case of such dilaton theories is spherically reduced Einstein gravity (SRG) in D dimensions

$$
\begin{aligned}
Z_{SRG} &= -\frac{D-3}{D-2}\,\phi^{-1} \\
V_{SRG} &= -\lambda^2\,\phi^{\frac{D-4}{D-2}}
\end{aligned}
\tag{4}
$$

with the Schwarzschild BH solution. Also the so-called dilaton BH [15] ($D \to \infty$ in (4)) and a "Poincaré gauge" [21] theory quadratic in curvature and torsion [11] and simpler models with $Z = 0$, like the Jackiw-Teitelboim gravity ($v = -\Lambda\phi$) [22] are covered by (3) and thus also by (1).
Our present approach to obtain the minimal supergravity extensions of generic models of type (1) is based upon the concept of the Poisson-Sigma models (PSM) [19, 23, 24].

Collecting zero form and one-form fields, respectively, within (1) as

$$
(X^i) := (\phi, X^a), \qquad (A_i) = (dx^m A_{mi}(x)) := (\omega, e_a), \tag{5}
$$

and after a partial integration, the action (1) may be rewritten as

$$
L^{\text{PSM}} = \int_{\mathcal{M}} dX^i \wedge A_i + \frac{1}{2}\mathcal{P}^{ij} A_j \wedge A_i, \tag{6}
$$

where the matrix $\mathcal{P}^{ij}$ may be read off by direct comparison. The basic observation in this framework is that this matrix defines a Poisson bracket on the space spanned by target space coordinates X^i of a Sigma Model. In the present context the related bracket $\{X^i, X^j\} := \mathcal{P}^{ij}$ has the form

$$
\{X^a, \phi\} = X^b \epsilon_b{}^a, \tag{7}
$$

$$
\{X^a, X^b\} = v(\phi, Y)\epsilon^{ab}\,. \tag{8}
$$

This bracket may be verified to obey the Jacobi identity $\{\{X^a, X^b\}, X^c\} + \text{cycl. perm.} = 0$. Eq. (7) shows that ϕ is the generator of Lorentz transformations (with respect to that bracket) on the target space $\mathbb{R}^3$. Minimal supersymmetric extensions are obtained [25, 26] and [27] by adding additional anticommuting target space coordinates χ^α and corresponding Rarita-Schwinger 1-form fields ψ_α in (5), $X^I = (X^i, \chi^\alpha)$ and $A_I = (A_i, \psi_\alpha))$.

Thus the PSM action (6) generalizes to

$$
L^{\text{gPSM}} = \int_{\mathcal{M}} dX^I A_I + \frac{1}{2}\mathcal{P}^{IJ} A_J A_I\,. \tag{9}
$$

Both χ^α and ψ_α denote Majorana fields, when, as in what follows, $N = 1$ supergravity is considered. The graded Poisson tensor $\mathcal{P}^{IJ} = (-1)^{IJ+1}\mathcal{P}^{JI}$ is again assumed to fulfil a "graded" Jacobi identity

$$
J^{IJK} = \mathcal{P}^{IL}\overrightarrow{\partial}_L \mathcal{P}^{JK} + \text{gcycl}(IJK) = 0\,. \tag{10}
$$

Together with (10) the e.o.m-s with right derivatives $\vec{\partial}_I = \frac{\vec{\partial}}{\partial X^I}$

$$
dX^I + \mathcal{P}^{IJ} A_J = 0, \tag{11}
$$

$$
dA_I + \frac{1}{2}(\overrightarrow{\partial}_I \mathcal{P}^{JK}) A_K A_J = 0 \tag{12}
$$

provide the on-shell symmetries of the action (9)

$$\delta X^I = \mathcal{P}^{IJ}\epsilon_J , \qquad \delta A_I = -d\epsilon_I - (\overrightarrow{\partial}_I \mathcal{P}^{JK})\epsilon_K A_J \, , \tag{13}$$

which depend on infinitesimal local parameters $\epsilon_I = (\epsilon_\phi, \epsilon_a, \epsilon_\alpha)$. The mixed components $\mathcal{P}^{\alpha\phi}$ are constructed by analogy to $\mathcal{P}^{a\phi}$ in (7) with the appropriate generator $(-\gamma_5/2)$ of Lorentz transformations in 2d spinor space. Then $d\epsilon_\alpha$ in the second set of eq. (13) acquires an additional term casting it into the covariant $(D\epsilon)_\alpha$, with covariant derivative D appropriate for a supergravity transformation.
As the Poisson tensor $\mathcal{P}^{IJ}$ is not of full rank, Casimir functions $\mathcal{C}(Y, \phi, \chi^2)$ exist which are the solutions of $\{X^I, \mathcal{C}\} = \mathcal{P}^{IJ}\partial_J \mathcal{C} = 0$ and thus correspond to conserved quantities $d\mathcal{C} = dX^I \partial_I \mathcal{C} = 0$ when (11) is used. The bosonic $\mathcal{C}$ in supergravity is of the form

$$\mathcal{C} = c + \frac{1}{2}\chi^2 c_2 \, , \tag{14}$$

where c and c_2 are functions of ϕ and Y only. However, also fermionic Casimir functions may occur (see below). In the pure bosonic case ($\chi = \psi = 0$) and for the potential (2) the differential equation for $\mathcal{C}$ allows an analytic solution. For instance, c for SRG simply coincides (up to a factor) with the ADM mass of the Schwarzschild BH. It is interesting, though, that such a conservation law continues to exist also in interactions with additional matter contributions [28], i.e. beyond the range of validity of the PSM concept.
The determination of all possible minimal supergravities [29, 7] now reduces to finding the solutions of the Jacobi identities (10). In the general ansatz for $\mathcal{P}^{IJ}$

$$\mathcal{P}^{ab} = V\,\epsilon^{ab} \, , \tag{15}$$

$$\mathcal{P}^{b\phi} = X^a\,\epsilon_a{}^b \, , \tag{16}$$

$$\mathcal{P}^{\alpha\phi} = -\frac{1}{2}\chi^\beta(\gamma_5)_\beta{}^\alpha \, , \tag{17}$$

$$\mathcal{P}^{\alpha b} = \chi^\beta(F^b)_\beta{}^\alpha \, , \tag{18}$$

$$\mathcal{P}^{\alpha\beta} = v^{\alpha\beta} + \frac{\chi^2}{2}\,v_2^{\alpha\beta} \, , \tag{19}$$

the function

$$V = v(\phi, Y) + \frac{\chi^2}{2}\,v_2(\phi, Y) \tag{20}$$

contains the original bosonic potential v. As explained above, eqs. (16) and (17) are fixed by Lorentz invariance. Each one of the (symmetric) spinor-tensors $v^{\alpha\beta}$ and $v_2^{\alpha,\beta}$ in (19) can be further expanded into three scalar functions of Y and ϕ, multiplying the symmetric matrices $(\gamma_5)^{\alpha\beta}, X^a(\gamma_a)^{\alpha\beta}, X^a(\gamma_5\gamma_a)^{\alpha\beta}$. The quantity $(F^b)_\beta{}^\alpha$ is easily seen to depend on another set of eight scalar functions. Thus the task to solve the Jacobi identities (10), which are differential equations, at first sight seems to be quite formidable.
Fortunately, in the course of our extensive analysis [29, 7] it turned out that by starting from the solutions of the Casimir functions, obeying equations like the one sketched after eq. (14), the problem, relating the many unknown functions above to the original bosonic potential v, may be reduced to the solution of *algebraic* equations.

We have classified the different cases according to the rank of $\mathcal{P}^{IJ}$, when the fermionic degrees of freedom are included. The bosonic sub-space is odd-dimensional which produced one Casimir function c. For full "fermionic rank", i.e. when the rank of $v^{\alpha\beta}$ in (19 is two, the single Casimir function (14) appears and the general solution still depends on five arbitrary (bosonic) functions of ϕ and Y beside v.
If the fermionic rank is reduced by one, beside the bosonic Casimir function (14) a fermionic one exists. It is of the generic form

$$\mathcal{C}^{(\pm)} = \chi^{\pm} \left| \frac{X^{--}}{X^{++}} \right|^{\pm 1/4} c_{(\pm)}(\phi, X) \tag{21}$$

and owes its Lorentz invariance to the abelian boost transformation $\exp(\pm\beta)$ of the light cone coordinates $X^{\pm\pm}$, related to X^a and $\exp(\pm\beta/2)$ of the chiral spinor components $\chi^{\pm}$. Then the general solution of the gPSM algebra contains four arbitrary functions beside v.
For rank zero of the fermionic extension, i.e. rank three as in the pure bosonic case, in $\mathcal{P}^{IJ}$ beside (14) both fermionic Casimir functions (21) are conserved and three functions remain arbitrary for a given bosonic potential v.
This arbitrariness can be understood as well by studying reparametrizations of the target space, spanned by the X^I in the gPSM. Those reparametrizations may generate new models. Therefore, they can be useful to create a more general gPSM from a simpler one, although this approach is difficult to handle if v in (20) is assumed to be the given starting point. However, within the present context the subset of those reparametrizations may be analysed which leaves the bosonic theory unchanged. Again the same number of arbitrary functions emerges for the different cases described in the paragraphs above.
A generic property of the fermionic extensions obtained in our analysis was the appearance of "obstructions". The first type of those consisted in singular functions of the bosonic variables ϕ and Y, multiplying the fermionic parts of a supergravity action, when no such singularities were present in the bosonic part. But even in the absence of such additional singularities, the relation of the original potential to some prepotential dictated by the corresponding supergravity theory, either led to a restriction of the range of ϕ and/or Y as given by the original bosonic one, or even altogether prevented any extension of the latter. Remarkably, a known 2d supergravity model like the one of Howe [6] which originally had been constructed with the full machinery of the superfield technique, escapes such obstructions. There, in our language, the PSM potential $v = -2\lambda^2\phi^3$ permits an expansion in terms of the prepotential $u(\phi)$ through $v = -du^2/d\phi$. An example where obstructions seem to be inevitable is the KV-model [11] with quadratic bosonic torsion.
The hope that a link could be found between the possibility of reducing the arbitariness of extensions referred to above, and the absence of such obstructions, did not materialize. We could give several counter examples, including different singular and nonsingular extensions of SRG.
Another very important point concerns the "triviality", proved earlier by one of us [27]. It was based upon the observation that locally a formulation of the dynamics in terms of Darboux coordinates allows to elevate the infinitesimal transformations (13) to finite ones. Then the latter may be used to gauge the fermionic fields to zero. Providing now the explicit form of those Darboux coor-

dinates in the explicit solution of a generic model we also give additional support to the original argument of [27]. However, the appearance of the obstructions and the ensuing singular factors in the transition to the Darboux coordinates may introduce a new aspect. When those new singularities appear at isolated points without restriction of the range for the original bosonic field variable, they may be interpreted and discarded much like coordinate singularities. Another way to circumvent this problem in the presence of restrictions to the range and thus to retain triviality is to allow a continuation of our (real) theory to complex variables. This triviality disappears anyhow, when interactions with additional matter fields are introduced, obeying the same symmetry as given by the gPSM-theory. An example for this has been proposed already in ref. [26].
In order to eliminate the arbitrariness of superdilaton extensions the only viable argument seems to consist in starting from a supergravity theory in higher dimensions (e.g. $D = 4$) and to reduce it (spherically or toroidally) to a $D = 2$ effective theory. However, the Killing spinors needed in that case must be Dirac spinors, requiring the generalization of the work [29, 7] described here to (at least) $N = 2$, where, however, the same technique of gPSM-s can be applied.

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