Towards a new quantization of Dirac's monopole

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There are several mathematical and physical reasons why Dirac's quantization must hold. How far one can go without it remains an open problem. The present work outlines a few steps in this direction.

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In his pioneer work [1] Dirac showed that the existence of just one magnetic monopole would explain electric charge quantization. Nowadays this is known as *Dirac's quantization rule*. In sixties Lipkin *et al* [2] showed that in the presence of the magnetic monopole the Jacobi identity for the translational group failed. Later Jackiw [3] found that the Dirac's quantization restores the associativity of the finite translations, but the infinitesimal generators remain "nonassociative" and the Jacobi identity fails.

There exist various arguments based on the quantum mechanics, theory of representations, topology and differential geometry in behalf of the Dirac's rule [3–8]. All of them are related to the necessary conditions of saving the situation and the problem of finding the sufficient conditions of existing of magnetic point like monopole is still open. It is therefore of interest to construct self consistent nonasociative quantum mechanics with an arbitrary magnetic monopole charge or maybe involving the distinct quantization rules related to the magnetic monopole. The present work outlines a few steps in this direction and the problems we have found.

Let us consider a point particle with electric charge e and mass m moving in the field of a magnetic monopole of charge q. The non-relativistic classical Hamiltonian may be written as follows [9]

$$H = \frac{1}{2mr^2}(\mathbf{p}\cdot\mathbf{r})^2 + \frac{1}{2mr^2}(\mathbf{J}^2 - \mu^2)$$

where

$$\mathbf{J} = \mathbf{r}\times(\mathbf{p} - e\mathbf{A}) - \mu\frac{\mathbf{r}}{r}, \quad \mu = eq \tag{1}$$

is the conserved total angular momentum and we set $\hbar = c = 1$. The last term in the formula (1) usually is interpreted as the contribution of the electromagnetic field [9–12], which carries an angular momentum

$$\mathbf{L}_{em} = \frac{1}{4\pi}\int \mathbf{r}\times(\mathbf{E}\times\mathbf{B})d^3r = -\mu\frac{\mathbf{r}}{r}$$

Also this is known as *Poincaré magnetic angular momentum* [13]. A magnetic field of the Dirac's monopole is

$$\mathbf{B} = q\frac{\mathbf{r}}{r^3} \tag{2}$$

and any choice of the vector potential $\mathbf{A}$ being compatible with (2) must have singularities, the so-called *Dirac's string*.

At the quantum level the operator

$$\mathbf{J} = \mathbf{r}\times(-i\nabla - e\mathbf{A}) - \mu\frac{\mathbf{r}}{r} \tag{3}$$

representing the angular momentum $\mathbf{J}$ has the same properties as a standard angular momentum and obeys the following commutation relations

$$[H,\mathbf{J}^2] = 0, \quad [H,J_i] = 0, \quad [\mathbf{J}^2, J_i] = 0 \tag{4}$$

$$[J_i, J_j] = i\epsilon_{ijk}J_k, \tag{5}$$

Choosing the vector potential as

$$\mathbf{A} = -q\frac{1+\cos\theta}{r\sin\theta}\,\hat{\mathbf{e}}_\varphi.$$

we find

$$J_\pm = e^{\pm i\varphi}\left(\pm\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi} - \mu\frac{1+\cos\theta}{\sin\theta}\right), \tag{6}$$

$$J_0 = -i\frac{\partial}{\partial\varphi} + \mu, \tag{7}$$

$$\mathbf{J}^2 = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) - \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} - {} - i\frac{2\mu}{1-\cos\theta}\frac{\partial}{\partial\varphi} + \mu^2\frac{1+\cos\theta}{1-\cos\theta} + \mu^2 \tag{8}$$

where $J_\pm = J_x \pm iJ_y$ are the raising and the lowering operators for $J_0 = J_z$ satisfying the standard commutations relations

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0.$$

It is known that for the Dirac's monopole problem the Schrödinger equation written in the spherical coordinates admits the separation of variables and the eigenfunctions of the angular part of the Hamiltonian are [5]

$$Y_{jn\mu}(\theta,\varphi) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{i(j-\mu-n)\varphi}P_{jn\mu}(\cos\theta), \tag{9}$$

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with n as a non-negative integer and

$$\begin{aligned} &P_{jn\mu}(u) = \frac{1}{\sqrt{h_{jn\mu}}}(1-u)^{\alpha/2}(1+u)^{\beta/2}P_n^{(\alpha,\beta)}(u), \\ &h_{jn\mu} = \frac{2^{2(j-n)+1}}{2j+1}\frac{\Gamma(j+\mu+1)\Gamma(j-\mu+1)}{n!\Gamma(2j-n+1)}, \\ &\alpha = j+\mu-n, \quad \beta = j-\mu-n, \end{aligned}$$

here Γ is the Gamma function, $j(j+1)$ and $j-n$ are the eigenvalues of $\mathbf{J}^2$ and J_z respectively, and $P_n^{(\alpha,\beta)}$ are the Jacobi polynomials.

Using the rotational invariance of the system and restricting by the unitary representations of the group SO(3) one can obtain the Dirac's quantization rule $2\mu =$ integer. This results in the following: *the only way to avoid Dirac's rule is to consider a nonunitary representation of the rotation group.*

Further we will try to describe the charge-monopole system as a free particle with an arbitrary spin, the so-called *anyons*, and relate this with the nonunitary representations of the rotation group. The idea to describe Dirac's monopole as a particle with a spin is not new [9, 11, 14]. However, in the known models anyon's statistics is based on the SO(2,1)-symmetry [14, 15]. and it is not clear how anyons may be appeared in a non-relativistic theory and its relation with the group SO(3).

Following [14] we consider the Lagrangian describing a non-relativistic charged particle in the field of a magnetic monopole

$$L = \frac{m}{2}\dot{\mathbf{r}}^2 + e\mathbf{A}\cdot\dot{\mathbf{r}}$$

where $\mathbf{A}$ is the vector potential determining the magnetic field (2). In the limit $m \to 0$, the action is given by the integral of the one-form $\theta = eA_i dx^i$ and we have

$$d\theta = -\frac{1}{2\mathbf{S}^2}\varepsilon_{ijk}S_i dS_j \wedge dS_k, \quad \mathbf{S} = -\mu\frac{\mathbf{r}}{r} \tag{10}$$

The two-form $d\theta$ is closed and nondegenerate, and is interpreted as a symplectic form corresponding to the symplectic potential θ defined on the unit two-sphere. It can be used to introduce Poisson brackets on S^2 as follows

$$\{S_i, S_j\} = \varepsilon_{ijk}S_k, \tag{11}$$

and thus the angular momentum of the electromagnetic field $\mathbf{L}_{em}$ can be considered as a *classical spin.*

Developing this idea, one can give the alternative description of the charge-monopole system as a free particle of the fixed spin with translational and spin degrees of freedom. The corresponding constrained Hamiltonian is given by

$$H = \frac{1}{2m}\Big(\mathbf{p} - \frac{1}{r^2}(\mathbf{r}\times\mathbf{S})\Big)^2 + \lambda(\mathbf{S}\cdot\mathbf{r} + \mu r) \tag{12}$$

where the second term describes the spin-orbit interaction. On the constraint surface $\mathbf{S}\cdot\mathbf{r} + \mu r \approx 0$ the dynamics generated by the Hamiltonian (12) is exactly the same as the dynamics of the initial charge-monopole system [14].

The above considerations is based on the conventional approach to the spin and therefore involves implicitly the Dirac's quantization rule. The case of μ being not necessarily integer or half integer implies making use of the nonunitary representations of the group SO(3).

Let the set of $\{S_\pm, S_0\}$ and $\{L_\pm, L_0\}$ form the algebra $so(2,1)$ and $so(3)$ respectively

$$[S_+, S_-] = -2S_0, \quad [S_\pm, S_0] = \mp S_\pm \tag{13}$$

$$[L_+, L_-] = 2L_0, \quad [L_\pm, L_0] = \mp L_\pm. \tag{14}$$

We introduce a direct sum $so(3) \oplus so(2,1)$ as follows

$$J_0 = L_0 + S_0, \; J_+ = L_+ - S_+, \; J_- = L_- + S_-. \tag{15}$$

The computation shows that the operator $\mathbf{J}$ obeys the commutation relations of the algebra $so(3)$

$$[J_0, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_0$$

and identifying J_i's as

$$J_1 = L_1 + iS_2, \; J_2 = L_2 + iS_1, \; J_3 = J_0 \tag{16}$$

we find

$$[J_i, J_j] = i\varepsilon_{ijk}J_k.$$

Notice that the group $\mathrm{SO(3)}\otimes\mathrm{SO(2,1)}$ has previously appeared in Dirac monopole theory, as a dynamical invariance group [16].

We construct the representation by using the generators of the group $SO(3)$ with the standard action on the state $|l, \tilde{m}\rangle$ given by

$$\begin{aligned} &\mathbf{L}^2|l,\tilde{m}\rangle = l(l+1)|l,\tilde{m}\rangle, \\ &L_0|l,\tilde{m}\rangle = \tilde{m}|l,\tilde{m}\rangle, \\ &L_+|l,\tilde{m}\rangle = \sqrt{(l-\tilde{m})(l+\tilde{m}+1)}|l,\tilde{m}+1\rangle, \\ &L_-|l,\tilde{m}\rangle = \sqrt{(l+\tilde{m})(l-\tilde{m}+1)}|l,\tilde{m}-1\rangle, \end{aligned}$$

and the unitary infinite dimensional representation of the group $SU(1,1)$ relating to the generators S_i's.

For the representation bounded above we have [15]

$$\begin{aligned} &\mathbf{S}^2|\lambda,n\rangle = \lambda(\lambda-1)|\lambda,n\rangle, \\ &S_0|\lambda,n\rangle = -(\lambda+n)|\lambda,n\rangle, \\ &S_+|\lambda,n\rangle = -\sqrt{(2\lambda+n-1)n}|\lambda,n-1\rangle, \\ &S_-|\lambda,n\rangle = -\sqrt{(2\lambda+n)(n+1)}|\lambda,n+1\rangle, \end{aligned}$$

where $n = 0, 1, 2, \ldots, \infty$ and λ is an arbitrary parameter, and for the highest-weight state $|\lambda, 0\rangle$ one obtains

$$S_0|\lambda,0\rangle = -\lambda|\lambda,0\rangle, \quad S_+|\lambda,0\rangle = 0.$$

Thus λ is the eigenvalue of J_0 for the state $|\lambda, 0\rangle$ and characterizes the representation. Recently the infinite dimensional representations of the group SO(2,1) have been used for the description of a theory of anyons where the spin of the one-particle states is taken to be $s = 1 - \lambda$ [15] (see also [17, 18]).

Starting with the highest-weight state $|j, j\rangle = |l, l\rangle \otimes |\lambda, 0\rangle$ corresponding to the eigenvalue $l - \lambda$ of the operator J_0 and applying the lowering operator J_- one can construct all states as follows

$$J_-|j, m\rangle = \sqrt{(j+m)(j-m+1)}|j, m-1\rangle, \quad (17)$$
$$J_+|j, m\rangle = \sqrt{(j-m)(j+m+1)}|j, m-1\rangle, \quad (18)$$
$$m = j, j-1, \ldots, -\infty$$

It is easy to see that the arised representation is nonunitary infinite dimensional and bounded above by $m \leq j$, instead of $-j \leq m \leq j$ well known for the unitary finite-dimensional representations of the rotation group.

The analysis of the nonunitary representation shows that the allowed values of j, μ are $\mu^2 \leq j(j+1)$ and $j - \mu$ = integer, which is an alternative choice to the Dirac rule [19].

In our model the charge-monopole system is interpreted as a free anyon with translational and spin degrees of freedom. The close approach has been applied in [20, 21] for description of a fractional spin in $(2+1)-$ dimensions.

Recently Martínez-y-Romero *et al* [22, 23] have used a nonunitary representation for a Dirac particle in a Coulomb-like field and Davis and Ghandour [24] have showed that nonunitary transformations are not so strange in quantum mechanics. The physical consequences of using nonunitary representations in Dirac's monopole problem are not clear till now and the work is in progress.

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