On Deriving Space–Time From Quantum Observables and States

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Dedicated to Rudolf Haag on the occasion of his eightieth birthday

Abstract

We prove that, under suitable assumptions, operationally motivated quantum data completely determine a space–time in which the quantum systems can be interpreted as evolving. At the same time, the dynamics of the quantum system is also determined. To minimize technical complications, this is done in the example of three-dimensional Minkowski space.

1 Introduction

The problem of determining from which physical observations one can derive the properties of the space–time in which the observer is located is as old as the theory of general relativity itself, and has received many sorts of answers. As this is not the place to give either a review of these or a discussion of their relative advantages and disadvantages, we shall content ourselves with referring the interested reader to the papers [26, 2, 3, 6, 7] and the further references to be found there. Some of these approaches use only classical objects and data as input, others use a mixture of classical and quantum data, while yet others begin with purely quantum data. In this paper we propose a novel approach to the problem which is based upon recent advances in algebraic quantum field theory.

Over a quarter century ago, groundbreaking work [9] revealed a connection between the Poincaré group and the modular objects which Tomita-Takesaki theory [34, 12, 30] associates with the vacuum state on Minkowski space and certain local algebras of observables. This has led to many unexpected applications of modular theory to quantum field theory — see [11] for a recent review. One line of research [13, 33, 14, 15, 18] growing out of Bisognano and Wichmann's work has drawn attention to the group $\mathcal{J}$ generated by the modular involutions J_W associated to the algebras of observables $\mathcal{A}(W)$ localized in the wedge–shaped regions W in Minkowski space. That program (which has been generalized to curved space–times) will also not be reviewed here. But it is essential to understand that the modular involutions J_W are uniquely determined by the state, which models the preparation of the quantum system, and the algebras $\mathcal{A}(W)$, whose self-adjoint elements model the observables of the system. In other words, the group $\mathcal{J}$ is determined, at least in principle, by operational quantum data. To eliminate the tacit reference to Minkowski space in the local algebras, and to strengthen the purely operational nature of the initial data, we shall consider a state ω on a collection $\{\mathcal{A}_i\}_{i\in I}$ of algebras $\mathcal{A}_i$ indexed by "laboratories" $i \in I$.[1] The algebra $\mathcal{A}_i$ will be thought of as being generated by all the observables measurable in the laboratory $i \in I$. One then has a collection of modular involutions J_i acting on some Hilbert space, and they generate a group $\mathcal{J}$, which has been called the modular symmetry group.

We therefore have an abstract group $\mathcal{J}$ generated by involutions. This is precisely the starting point of the program of absolute geometry, see *e.g.* [1, 4, 5]. From such a group and a suitable set of axioms to be satisfied by the generators of that group, absolute geometers derive various metric spaces such as Minkowski spaces and Euclidean spaces upon which the abstract group acts as the isometry group of the metric space. Different sets of axioms on the group yield different metric spaces. This affords us with the possibility of deriving a space–time from the group $\mathcal{J}$, *i.e.*, the operational data $(\omega, \{\mathcal{A}_i\}_{i\in I})$ would determine the space–time in which the quantum systems could naturally be considered to be evolving. Different sets of algebraic relations in $\mathcal{J}$ would lead to different space–times.

It is the purpose of this paper to indicate how this could be possible. To minimize technical complications which would detract from the point of principle we wish to make, we shall illustrate this program using the example of three-dimensional Minkowski space. We emphasize that we are establishing a conceptual point of principle — we are not proposing a concrete operational procedure to determine space–time.

In Section 2 we shall present the absolute geometry relevant for our immediate purposes. Although Minkowski space is one of the cases the absolute geometers have already treated, their tacit geometric starting point is different from ours, and so we are obliged to provide another chain of arguments to derive three-dimensional Minkowski space from the group $\mathcal{J}$. The quantum data which then "determine" three-dimensional Minkowski space will be discussed in Section 3. It will be shown that under purely algebraic conditions on the group $\mathcal{J}$, there exists an identification of laboratories $i \in I$ with subregions W_i of $\mathbb{R}^3$ such that (1) $\mathcal{J}$ contains a representation of the Poincaré group on $\mathbb{R}^3$ and (2) the collection $\{\mathcal{A}(W_i)\}_{i\in I}$ is a Poincaré covariant and local net of von Neumann algebras on $\mathbb{R}^3$ satisfying Haag duality. Some further results are proven in Section 3, as well. We make some final comments in Section 4.

2 Absolute Geometry and Three-Dimensional Minkowski Space

We first provide an overview of our reasoning in this section. Three-dimensional Minkowski space is an affine space whose plane at infinity is a hyperbolic projective–metric plane [22]. In [5], Bachmann, Pejas, Wolff, and Baur (BPWB) consider an abstract group $\mathfrak{G}$ which is generated by an invariant system $\mathcal{G}$ of generators in which each of the generators is involutory and which satisfies a certain set of axioms. From this they construct a hyperbolic

[1] More detailed definitions will be given in Section 3.

projective–metric plane in which the given group $\mathfrak{G}$ is isomorphic to a subgroup of the group of congruent transformations (motions) of the projective–metric plane. By interpreting the elements of $\mathcal{G}$ as line reflections in a hyperbolic plane, BPWB show that the hyperbolic projective–metric plane can be generated by these line reflections in such a way that these line reflections form a subgroup of the group of motions of the projective–metric plane.

Coxeter shows in [24] that every motion of the hyperbolic plane is generated by a suitable product of orthogonal line reflections, where an orthogonal line reflection is defined as a harmonic homology with center exterior point and axis the given ordinary line and where the center and axis are a pole-polar pair. In the following we show that Coxeter's and BPWB's notions of motions coincide in the hyperbolic projective–metric plane and that the reflections can be viewed as reflections about exterior points.

Then we embed our projective–metric plane into a three-dimensional projective space. By singling out our original plane as the plane at infinity, we obtain an affine space whose plane at infinity is a hyperbolic projective–metric plane, which is the well-known characterization of three-dimensional Minkowski space. We show that the motions of our original plane induce motions in the affine space and, by a suitable identification, we show that any motion in this Minkowski space can be generated by reflections about spacelike lines. Thus, to construct a three-dimensional Minkowski space, one can start with a generating set $\mathcal{G}$ of reflections about spacelike lines, equivalently, reflections about exterior points in the hyperbolic projective–metric plane at infinity. This equivalence is important for our argument. We therefore obtain a three-dimensional affine space with the Minkowski metric, which is constructed out of a group generated by a set of isometries.

The approach in this paper differs from the method used both in [37] for two-dimensional Minkowski space and in [32] for four-dimensional Minkowski space. In these papers one begins by constructing the affine space. In the two-dimensional case [37], the elements of the generating set $\mathcal{G}$ are identified with line reflections in an affine plane, while in the four-dimensional case [32], the elements of the generating set $\mathcal{G}$ are identified with reflections about hyperplanes in an affine space. Thus, in each of these papers, the generating set $\mathcal{G}$ is identified with a set of symmetries. A map of affine subspaces is then obtained using the definition of orthogonality given by commuting generators. This map induces a hyperbolic polarity in the hyperplane at infinity, thereby yielding the Minkowski metric.

In our approach, we take the dual view, beginning with points instead of lines and constructing the affine space out of the plane at infinity. The definition of orthogonality induced by the commutation relations of the generators in the hyperplane at infinity is used to obtain the polarity and then the hyperplane at infinity is embedded in an affine space to get Minkowski space. This argument is necessitated here not only because of the different dimensionality of the space, but also because, for reasons made clear in Section 3, our generating involutions must ultimately have the geometric interpretation of reflections about spacelike lines. Despite the new elements in our approach, much of our argument in this section consists of appropriate re-interpretations and modifications of work already in the literature.

2.1 Construction of Π

As the ideas and results of the absolute geometers are not widely known, particularly among theoretical and mathematical physicists, we shall give here the definitions, axioms and main results we shall need and also provide a sketch of some of the pertinent arguments. For detailed proofs, the reader is referred to [5] or to [4].

One begins with a group $\mathfrak{G}$ generated by an invariant system $\mathcal{G}$ of involution elements. The elements of $\mathcal{G}$ will be denoted by lowercase Latin letters. Those involutory elements of $\mathfrak{G}$ which can be represented as a product ab, where $a, b \in \mathcal{G}$, will be denoted by uppercase Latin letters. If $\xi, \eta \in \mathfrak{G}$ and $\xi\eta$ is an involution, we shall write $\xi|\eta$. The notation $\xi, \eta|\varphi, \psi$ means $\xi|\varphi$ and $\xi|\psi$ and $\eta|\varphi$ and $\eta|\psi$.

The axioms we shall use to derive three-dimensional Minkowski space from $(\mathfrak{G}, \mathcal{G})$ are:

Axioms

A.1: For every P, Q there exists a g with $P, Q\ |g.$

A.2: If $P, Q\ |g, h$ then $P = Q$ or $g = h.$

A.3: If $a, b, c\ |P$ then $abc = d \in \mathcal{G}.$

A.4: If $a, b, c\ |g$ then $abc = d \in \mathcal{G}.$

A.5: There exist g, h, j such that $g\ |h$ but $j \nmid g, h, gh.$

A.6: For each P and g with $P \nmid g$ there exist exactly two distinct elements $h_1, h_2 \in \mathcal{G}$ such that $h_1, h_2 \mid P$ and $g, h_i \nmid R, c$ for any R, c and $i = 1, 2$.

We shall call a pair $(\mathfrak{G}, \mathcal{G})$ consisting of a group $\mathfrak{G}$ and an invariant system $\mathcal{G}$ of generators of the group $\mathfrak{G}$ satisfying the axioms above a group of motions.

In [5] the elements of $\mathcal{G}$ are interpreted as secant or ordinary lines in a hyperbolic plane. In our approach, we view the elements of $\mathcal{G}$ initially as exterior points in a hyperbolic plane. After embedding our hyperbolic projective–metric plane into an affine space, we will be able to identify the elements of $\mathcal{G}$ with spacelike lines and their corresponding reflections in a three-dimensional Minkowski space. After realizing that statements about the geometry of the plane at infinity correspond to statements about the geometry of the whole space where all lines and all planes are considered through a point, we see that the axioms are also statements about spacelike lines — the elements of $\mathcal{G}$ — and timelike lines — the elements P of $\mathfrak{G}$ — through any point in Minkowski 3-space.

These algebraic axioms have a geometric interpretation in the group plane $(\mathfrak{G}, \mathcal{G})$, which we now indicate. The elements of $\mathcal{G}$ are called lines of the group plane, and those involutory group elements which can be represented as the product of two elements of $\mathcal{G}$ are called points of the group plane. Two lines g and h of the group plane are said to be perpendicular if $g \mid h$. Thus, the points are those elements of the group which can be written as the product of two perpendicular lines. A point P is incident with a line g in the group plane if $P \mid g$. Two lines g, h are said to be parallel if $g, h \nmid P, c$, for all P, c, in other words, if they have neither a common perpendicular line nor a common point. Thus, if $P \neq Q$, then by A.1 and A.2, the points P and Q in the group plane are joined by a unique line. If $P \nmid g$ then A.6 says that there are precisely two lines through P parallel to $g.$

Lemma 2.1 *For each $\alpha \in \mathfrak{G}$, the mappings $\sigma_a : g \longmapsto g^\alpha \equiv \alpha g \alpha^{-1}$ and $\sigma_\alpha : P \longmapsto P^\alpha \equiv \alpha P \alpha^{-1}$ are one-to-one mappings of the set of lines and the set of points, each onto itself in the group plane.*

Proof. Let $\alpha \in \mathfrak{G},$ and consider the mapping $\gamma \longmapsto \gamma^\alpha \equiv \alpha\gamma\alpha^{-1}$ of $\mathfrak{G}$ onto itself. It is easily seen that this mapping is bijective. Since $\mathcal{G}$ is an invariant system ($a^b \in \mathcal{G}$ for every $a \in \mathcal{G}, b \in \mathfrak{G}$) $\mathcal{G}$ will be mapped onto itself, and if P is a point, so that $P = gh$ with $g|h,$ then $P^\alpha = g^\alpha h^\alpha$ and $g^\alpha|h^\alpha$, so that P^α is also a point. Thus, $g \longmapsto g^\alpha$, $P \longmapsto P^\alpha$ are one-to-one mappings of the set of lines and the set of points, each onto itself in the group plane. $\square$

Definition 2.1 *A one-to-one mapping σ of the set of points and the set of lines each onto itself is called an* orthogonal collineation *if it preserves incidence and orthogonality.*

Since the "$|$" relation is preserved under the above mappings, orthogonal collineations also preserve incidence and orthogonality as defined above.

Corollary 2.1 *The mappings $\sigma_\alpha : g \longmapsto g^\alpha$ and $\sigma_\alpha : P \longmapsto P^\alpha$ are orthogonal collineations of the group plane and are called motions of the group plane induced by α.*

In particular, if α is a line a, we have a reflection about the line a in the group plane, and if α is a point A, we have a point reflection about A in the group plane.

If to every $\alpha \in \mathfrak{G}$ one assigns the motion of the group plane induced by α, one obtains a homomorphism of $\mathfrak{G}$ onto the group of motions of the group plane. Bachmann shows in [4] that this homomorphism is in fact an isomorphism, so that points and lines in the group plane may be identified with their respective reflections. Thus, $\mathfrak{G}$ is seen to be the group of orthogonal collineations of $\mathfrak{G}$ generated by $\mathcal{G}$.

Definition 2.2 *Planes which are representable as an isomorphic image, with respect to incidence and orthogonality, of the group plane of a group of motions $(\mathfrak{G}, \mathcal{G})$, are called metric planes.*

In [5], BPWB show how one can embed a metric plane into a projective–metric plane by constructing an ideal plane using pencils of lines. We shall outline how this is done.

Definition 2.3 *Three lines are said to lie in a pencil if their product is a line; i.e., a, b, c lie in a pencil if*

$$abc = d \in \mathcal{G} \quad . \tag{1}$$

Definition 2.4 *Given two lines a, b with $a \neq b$, the set of lines c satisfying equation (1) is called a pencil of lines and is denoted by $G(ab)$, since it depends only on the product ab.*

Note that the relation (1) is symmetric, *i.e.* it is independent of the order in which the three lines are taken: since $cba = (abc)^{-1}$ is a line, the invariance of $\mathcal{G}$ implies that $cab = (abc)^c$ is a line and that every motion of the group plane takes triples of lines lying in a pencil into triples in a pencil. The invariance of $\mathcal{G}$ also shows that (1) holds whenever at least two of the three lines coincide.

Using axioms implied by A.1 – A.6, BPWB then show that there are three distinct classes of pencils.

(1) If $a, b|V$ then $G(ab) = \{c : c|V\}$. In this case, $G(ab)$ is called a pencil of lines with center V and is denoted by $G(V)$.

(2) If $a, b\ |c$ then $G(ab) = \{d : d|c\}$. In this case, $G(ab)$ is called a pencil of lines with axis c and is denoted by $G(c)$.

(3) By A.6, there exist parallel lines a, b. Thus, in this case $G(ab) = \{c : c \parallel a, b$ where $a \parallel b\}$, which we denote by $G(ab)_\infty$.

An ideal projective plane Π is constructed in the following manner. An ideal point is any pencil of lines $G(ab)$ of the metric plane. The pencils $G(P)$ correspond in a one-to-one way to the points of the metric plane. An ideal line is a certain set of ideal points. There are three types:

(1) A proper ideal line $g(a)$ is the set of ideal points which have in common a line a of the metric plane.

(2) The set of pencils $G(x)$ with $x\,|P$ for a fixed point P of the metric plane, which we denote by $g(P)$.

(3) Sets of ideal points which can be transformed by a halfrotation[2] about a fixed point P of the metric plane into a proper ideal line; these we denote by $g(ab)_\infty$.

The polarity is defined by mapping $G(C) \longmapsto g(C)$ and $g(C) \longmapsto G(C)$; $G(ab)_\infty \longmapsto g(ab)_\infty$ and $g(ab)_\infty \longmapsto G(ab)_\infty$; and $G(c) \longmapsto g(c)$ and $g(c) \longmapsto G(c)$. In [4], Bachmann shows that the resulting ideal plane is a hyperbolic projective plane in which the theorem of Pappus and the Fano axiom both hold, *i.e.* a hyperbolic projective–metric plane.

In this model, the ideal points of the form $G(P)$ are the interior points of the hyperbolic projective–metric plane; thus the points of the metric plane correspond in a one–to–one manner with the interior points of the hyperbolic projective–metric plane. The ideal points $G(x)$, for $x \in \mathcal{G}$ are the exterior points of the hyperbolic projective–metric plane.

Proposition 2.1 *Each $x \in \mathcal{G}$ corresponds in a one-to-one manner with the exterior points of the hyperbolic projective–metric plane.*

Proof. Since each line d of the metric plane is incident with at least three points (Theorem 5 in [5]) and a point is of the form ab with $a|b$, it follows that each $x \in \mathcal{G}$ is the axis of a pencil. From the uniqueness of perpendiculars (Theorem 4 [5]), each $x \in \mathcal{G}$ corresponds in a one-to-one manner with the pencils $G(x)$. Hence, each $x \in \mathcal{G}$ corresponds in a one-to-one manner with the exterior points of the hyperbolic projective–metric plane. $\square$

Thus, one may view the axioms as referring to the interior and exterior points of a hyperbolic projective–metric plane. The ideal points of the form $G(ab)_\infty$, where $a \parallel b$, are the points on the absolute, *i.e.*, the points at infinity in the hyperbolic projective–metric plane.

We turn to the ideal lines. A proper ideal line $g(a)$ is a set of ideal points which have in common a line a of the metric plane.

Proposition 2.2 *A proper ideal line $g(a)$ is a secant line of the form*
$g(a) = \{G(P), G(x), G(bc)_\infty : x, P|a$ *and* $abc \in \mathcal{G}$ *where* $b \parallel c\}$.

Proof. By Theorem 23 of [5], any two pencils of lines of the metric plane have at most one line in common. By A.6, each line belongs to at least two pencils of parallels and by Theorem 13 of [5] and A.6 again, each line $g \in \mathcal{G}$ belongs to precisely two such pencils. Thus, a proper ideal line contains two points on the absolute, interior points, and exterior points. Hence, a proper ideal line is a secant line. A secant line is the set
$g(c) = \{G(P), G(x), G(ab)_\infty : x, P\,|c$ and $abc \in \mathcal{G}$ where $a \parallel b\}$. $\square$

Corollary 2.2 *The ideal line which consists of pencils $G(x)$ with $x\,|P$ for a fixed point P of the metric plane consists only of exterior points, i.e., it is an exterior line. Therefore,* $g(P) = \{G(x) : x|P\}$.

[2]see p. 161 in [5]

The final type of ideal line is a tangent line. It contains only one point $G(ab)_\infty$ on the absolute. Denoting this line by $g(ab)_\infty$, we have $g(ab)_\infty = \{G(ab)_\infty\} \cup \{G(x) : x \in \mathcal{G}$ and $abx \in \mathcal{G}\}$ where $a \parallel b$. Recalling that each $x \in \mathcal{G}$ corresponds to an exterior point in the hyperbolic projective–metric plane, we see that a tangent line consists of one point on the absolute and every other point is an exterior point.

We also note that under the above identifications, each secant line $g(c)$ corresponds to a unique "exterior point" $G(c)$, $G(c) \notin g(c)$, since one only considers those $x, P \,|c$ such that $xc \neq 1_{\mathfrak{G}}$ and $Pc \neq 1_{\mathfrak{G}}$. Each exterior line corresponds to a unique interior point P and each tangent line corresponds to a unique point on the absolute.

Proposition 2.3 *The map* Φ *given by*

(i) $\Phi(G(c)) = g(c)$, $\Phi(g(c)) = G(c)$

(ii) $\Phi(G(P)) = g(P)$, $\Phi(g(P)) = G(P)$

(iii) $\Phi(G(ab)_\infty) = g(ab)_\infty$, $\Phi(g(ab)_\infty) = G(ab)_\infty$

is a polarity.

Proof. Let $\mathcal{P}$ be the set of all points of Π and $\mathcal{L}$ the set of all lines of Π. From the remarks above it follows that Φ is a well-defined one-to-one point–to–line mapping of $\mathcal{P}$ onto $\mathcal{L}$ and a well-defined one-to-one line–to–point mapping of $\mathcal{L}$ onto $\mathcal{P}$.

Next, it is shown that Φ is a correlation and for this it suffices to show that Φ preserves incidence. Let $g(c) = \{G(P), G(x), G(ab)_\infty : x, P|c$ and $abc \in \mathcal{G}$ where $a \parallel b\}$ be a secant line. Let $G(A), G(B), G(d), G(ef)_\infty \in g(c)$, where $G(ef)_\infty = \{x \in \mathcal{G} : xef \in \mathcal{G}$ and $e \parallel f\}$. Then $A, B, d|c$ and $cab \in \mathcal{G}$. Now $\Phi(G(A)) = g(A) = \{G(x) : x|A\}$, $\Phi(G(B)) = g(B) = \{G(x) : x|B\}$, $\Phi(G(d)) = g(d)$, and $\Phi(G(ef)_\infty) = g(ef)_\infty = \{G(ef)_\infty\} \cup \{G(x) : efx \in \mathcal{G}\}$. Thus, it follows that $\Phi(g(c)) = G(c) \in g(A) \cap g(B) \cap g(d) \cap g(ef)_\infty$, so that $\Phi(g(c)) \in \Phi(G(A)), \Phi(G(B)), \Phi(G(d)), \Phi(G(ef)_\infty)$ and Φ preserves incidence on a secant line.

Consider an exterior line $g(P) = \{G(x) : x|P\}$, and let $G(a), G(b) \in g(P)$. Then $a, b|P$ and it follows that $G(P) \in g(a) \cap g(b)$, *i.e.* $\Phi(g(P)) \in \Phi(G(a)) \cap \Phi(G(b))$ and Φ preserves incidence on an exterior line.

Finally, let $g(ab)_\infty = \{G(ab)_\infty\} \cup \{G(x) : abx \in \mathcal{G}$ where $a \parallel b\}$ be a tangent line. Clearly, since $\Phi(G(ab)_\infty) = g(ab)_\infty$, one has $G(ab)_\infty \in g(ab)_\infty$. Now suppose that $G(d) \in g(ab)_\infty$. Then $abd \in \mathcal{G}$ and $\Phi(G(d)) = g(d) = \{G(A), G(x), G(ef)_\infty : A, x|d$ and $def \in \mathcal{G}$ where $e \parallel f\}$. Thus, $G(d) \in G(ab)_\infty \cap G(ef)_\infty$ and $G(ef)_\infty \in g(d)$, which implies that $G(d) \in g(ef)_\infty$ and $\Phi(g(ef)_\infty) \in \Phi(G(d))$. Hence, Φ preserves incidence and is a correlation. $\square$

It follows from the above observations that Φ transforms the points $G(Y)$ on a line $g(b)$ into the lines $\Phi(G(Y))$ through the point $\Phi(g(b))$. Thus, Φ is a projective correlation. Since Φ^2 is the identity map, then Φ is a polarity. Moreover, since $\Phi(g(ab)_\infty) = G(ab)_\infty$ with $G(ab)_\infty \in g(ab)_\infty$, then Φ is a hyperbolic polarity.

Proposition 2.4 *The definition of orthogonality given by the polarity coincides with and is induced by the definition of orthogonality in the group plane.*

Proof. Declaring a perpendicularity with respect to the polarity defined above, one has, on the one hand, $g(c) \perp g(a)$ if and only if $\Phi(g(c)) = G(c) \in g(a)$ and, on the other, $\Phi(g(a)) = G(a) \in g(c)$ if and only if $a|c$. Similarly, one has $g(c) \perp g(P)$ if and only if $\Phi(g(c)) = G(c) \in g(P)$ if and only if $c|P$. In addition, $g(P) \perp g(Q)$ if and only if $\Phi(g(P)) = G(P) \in g(Q)$ if and only if $P|Q$. In fact, this is excluded by Theorem 2.1(c) in [5], in conformity with hyperbolic geometry, since two interior points cannot be conjugate under the hyperbolic polarity.

Further, one sees that $G(C) \perp G(p)$ if and only if $G(C) \in \Phi(G(p)) = g(p)$ and $G(p) \in \Phi(G(C)) = g(C)$ if and only if $P|c$. Finally, one also has $G(c) \perp G(x)$ if and only if $G(c) \in \Phi(G(x)) = g(x)$ and $G(x) \in g(c) = \Phi(G(c))$ if and only if $x|c$. $\square$

We also see that if instead of interpreting our original generators as ordinary lines in a hyperbolic plane, we interpret them as exterior points, then we can construct a hyperbolic projective–metric plane in which the theorem of Pappus and Fano's axiom both hold and which is generated by the exterior points of the hyperbolic projective–metric plane.

With the identifications and the geometric objects defined above, we show in the next subsection that the motions of the hyperbolic projective–metric plane above can be generated by reflections about exterior points, *i.e.* any transformation in the hyperbolic plane which leaves the absolute invariant can be generated by a suitable product of reflections about exterior points.

2.2 Reflections About Exterior Points

In [24] Coxeter showed that any congruent transformation of the hyperbolic plane is a collineation which preserves the absolute and that any such transformation is a product of reflections about ordinary lines in the hyperbolic plane, where a line reflection about a line m is a harmonic homology with center M and axis m, where M and m are a pole–polar pair and M is an exterior point. A point reflection is defined similarly: a harmonic homology with center M and axis m, where M and m are a pole-polar pair, M is an interior point, and m is an exterior line. Note that in both cases, M and m are nonincident. We recall a series of definitions for the convenience of the reader.

Definition 2.5 *A collineation is a one-to-one map of the set of points onto the set of points and a one-to-one map of the set of lines onto the set of lines that preserves the incidence relation.*

Definition 2.6 *A perspective collineation is a collineation which leaves a line pointwise fixed — called its axis — and a point line-wise fixed — called its center.*

Definition 2.7 *A homology is a perspective collineation with center a point B and axis a line b where B is not incident with b.*

Definition 2.8 *A harmonic homology with center B and axis b, where B is not incident with b, is a homology which relates each point A in the plane to its harmonic conjugate with respect to the two points B and $(b, [A, B])$, where $[A, B]$ is the line joining A and B and $(b, [A, B])$ is the point of intersection of b and $[A, B]$.*

Definition 2.9 *A complete quadrangle is a figure consisting of four points (the vertices), no three of which are collinear, and of the six lines joining pairs of these points. If l is one of these lines, called a side, then it lies on two of the vertices, and the line joining the other two vertices is called the opposite side to l. The intersection of two opposite sides is called a diagonal point.*

Definition 2.10 *A point D is the harmonic conjugate of a point C with respect to points A and B if A and B are two vertices of a complete quadrangle, C is the diagonal point on the line joining A and B, and D is the point where the line joining the other two diagonal points cuts $[A, B]$. One denotes this relationship by $H(AB, CD)$.*

In keeping with the notation employed at the end of § 2.1, let $G(b)$ be an exterior point and $g(b)$ its pole.

Lemma 2.2 *The map* $\Psi_b : \left\{ \begin{array}{ccccccc} G(A) & \longmapsto & G(A)^b & \textit{and} & G(d) & \longmapsto & G(d)^b \\ g(A) & \longmapsto & g(A)^b & & g(d) & \longmapsto & g(d)^b \\ G(cd)_\infty & \longmapsto & G(cd)^b_\infty & & g(cd)_\infty & \longmapsto & g(cd)^b_\infty \end{array} \right\}$

is a collineation.

Proof. This follows from the earlier observation that the motions of the group plane map pencils onto pencils preserving the "$|$" relation. $\square$

Lemma 2.3 Ψ_b *is a perspective collineation and, hence, a homology.*

Proof. Recall that $g(b) = \{G(A), G(x), G(cd)_\infty : x, A \mid b$ and where b lies in the pencil $G(cd)_\infty\}$. For any $G(A)$ and $G(x)$ in $g(b)$ one has $A^b = A$ and $x^b = x$, since $A, x \mid b$ and if $G(A), G(x) \notin g(b)$ then $A, x \nmid b$ and $A^b \neq A$, $x^b \neq x$, and $A^b, x^b \nmid b$. Thus, $G(A)^b, G(x)^b \notin g(b)$.

Recall also that $G(cd)_\infty = \{f \mid fcd \in \mathcal{G}$, where c and d have neither a common point nor a common perpendicular $\}$. Now $g(b)$ is a secant line, so that it contains two such distinct points, $G(mn)_\infty$ and $G(pq)_\infty$, say, on the absolute. Since the motions of the group plane map pencils onto pencils preserving the "$|$" relation it follows that if $c, d \in G(mn)_\infty$ then $c^b, d^b \in G(mn)_\infty$ and hence, $G(mn)^b_\infty = G(mn)_\infty$ and $G(pq)^b_\infty = G(pq)_\infty$. Moreover, if $G(rs)_\infty \notin g(b)$, then it follows that $G(rs)^b_\infty \notin g(b)$. Thus, Ψ_b leaves $g(b)$ pointwise invariant.

Let $g(d), g(Q)$, and $g(rs)_\infty$ be a secant line, exterior line, and tangent line, respectively, containing $G(b)$. For $G(e) \in g(d)$ one has $e \mid d$ and $e^b \mid d^b = d$, since $b \mid d$, thus $G(e)^b \in g(d)$. For $G(A) \in g(d)$, $A^b \mid d^b = d$, so $G(A)^b \in g(d)$. Similarly, it follows that if $G(ef)_\infty \in g(d)$ then $G(ef)^b_\infty \in g(d)$, and $g(d)^b = g(d)$. One easily sees that $g(Q)^b = g(Q)$ and $g(rs)^b_\infty = g(rs)_\infty$. Thus, Ψ_b leaves every line through b invariant and Ψ_b is a perspective collineation for each $b \in \mathcal{G}$. $\square$

Proposition 2.5 Ψ_b *is a harmonic homology.*

Proof. Since A^b is again a point in the original group plane and since d^b is again a line in the original group plane, it follows that, for each $b \in \mathcal{G}$, Ψ_b maps interior points to interior points, exterior points to exterior points, points on the absolute to points on the absolute, secant lines to secant lines, exterior lines to exterior lines, and tangent lines to tangent lines. Moreover, since $(\xi^b)^b = \xi$ for any $\xi \in \mathfrak{G}$, Ψ_b is involutory for each $b \in \mathcal{G}$. But in a projective plane in which the theorem of Pappus holds, the only collineations which are involutory are harmonic homologies [21]. Thus Ψ_b is a harmonic homology for each $b \in \mathcal{G}$. $\square$

Proposition 2.6 *Point reflections about interior points are generated by reflections about exterior points.*

Proof. Arguing in a similar manner, one sees that for each interior point $G(A)$, Ψ_A is a harmonic homology with center $G(A)$ and axis $g(A)$, where $g(A)$ is the polar of $G(A)$, $G(A) \notin g(A)$, and where Ψ_A is defined analogously to Ψ_b. Thus, each Ψ_A is a point reflection, and since A is the product of two exterior points, one sees that point reflections about interior points are generated by reflections about exterior points. $\square$

Proposition 2.7 *The reflection of an interior point about a secant line coincides with the reflection of the same interior point about an exterior point. Moreover, since any motion of the hyperbolic plane is a product of line reflections about secant lines, any motion of the hyperbolic plane is generated by reflections about exterior points.*

Proof. Consider a line reflection in the hyperbolic plane, *i.e.* the harmonic homology with axis $g(b)$ and center $G(b)$. Let $G(A)$ be an interior point and $g(d)$ a line through $G(A)$ meeting $g(b)$. Since $G(b) \in g(d)$, one has $b|d$ and $g(d)$ is orthogonal to $g(b)$. Let $G(E)$ be the point where $g(b)$ meets $g(d)$. Since $G(E) \in g(b)$, then $E|b$ and $Eb = f$ for some $f \in \mathcal{G}$. It follows that the reflection of $G(A)$ about $g(b)$ is the same as the reflection of $G(A)$ about $G(E)$. Since $b|d$ and $E|d$, then $bd = C$ and one has $E, C|b, d$ with $b \neq d$. Thus, by A.2, $E = C = bd$. Hence, $A^E = A^{db}$ and $A|d$ as $G(A) \in g(d) = A^b$. $\square$

Since the motions of the projective–metric plane are precisely those collineations which leave the absolute invariant, we have the following result.

Theorem 2.1 *Reflections of exterior points about exterior points and about exterior lines are also motions of the projective–metric plane. Hence, the* Ψ_b*'s for* $b \in \mathcal{G}$ *acting on exterior points and exterior lines are motions of the hyperbolic projective–metric plane.*

We also point out that the proof that each Ψ_b is an involutory homology also shows that the Fano axiom holds, since in a projective plane in which the Fano axiom does not hold no homology can be an involution [8].

2.3 Embedding a Hyperbolic Projective–Metric Plane Into a Projective 3-Space

We embed our hyperbolic projective–metric plane into a three-dimensional projective space, finally obtaining an affine space whose plane at infinity is isomorphic to our original projective–metric plane. Any projective plane Π in which the theorem of Pappus holds can be represented as the projective coordinate plane over a field $\mathcal{K}$. (The theorem of Pappus guarantees the commutativity of $\mathcal{K}$.) Then by considering quadruples of elements of $\mathcal{K}$, one can define a projective space $P_3(\mathcal{K})$ in which the coordinate plane corresponding to Π is included. Since the Fano axiom holds, the corresponding coordinate field $\mathcal{K}$ is not of characteristic 2 [8]. In fact, A.6 entails that $\mathcal{K}$ is a Euclidean field. By singling out the coordinate plane corresponding to Π as the plane at infinity, one obtains an affine space whose plane at infinity is a hyperbolic projective–metric plane, *i.e.* three-dimensional Minkowski space.

To say that a plane Π is a projective coordinate plane over a field $\mathcal{K}$ means that each point of Π is a triple of numbers (x_0, x_1, x_2), not all zero, together with all multiples $(\lambda x_0, \lambda x_1, \lambda x_2)$, $\lambda \neq 0$. Similarly, each line of Π is a triple of numbers $[u_0, u_1, u_2]$, not all zero, together with all multiples $[\lambda u_0, \lambda u_1, \lambda u_2]$, $\lambda \neq 0$. In $P_3(\mathcal{K})$ all the quadruples of numbers with the last entry zero correspond to Π. One can now obtain an affine space $\mathcal{A}$ by defining the points of $\mathcal{A}$ to be those of $P_3(\mathcal{K}) - \Pi$, *i.e.* those points whose last entry is nonzero; a line l of $\mathcal{A}$ to be a line l' in $P_3(\mathcal{K}) - \Pi$ minus the intersection point of the line l' with Π; and by defining a point P in $\mathcal{A}$ to be incident with a line l of $\mathcal{A}$ if and only if P is incident with the corresponding l'. Planes of $\mathcal{A}$ are obtained in a similar way [25].

Thus, each point in Π represents the set of all lines in $\mathcal{A}$ parallel to a given line, where lines and planes are said to be parallel if their first three coordinates are the same, and each line in Π represents the set of all planes parallel to a given plane. Since parallel objects can be considered to intersect at infinity, we call Π the plane at infinity.

2.3.1 Exterior Point Reflections Generate Motions in an Affine Space

In [22], Coxeter shows that three-dimensional Minkowski space is an affine space whose plane at infinity is a hyperbolic projective–metric plane. He also classifies the lines and planes of the affine space according to their sections by the plane at infinity:

Line or Plane	Section at Infinity
Timelike line	Interior point
Lightlike line	Point on the absolute
Spacelike line	Exterior point
Characteristic plane	Tangent line
Minkowski plane	Secant line
Spacelike plane	Exterior line

He shows that if one starts with an affine space and introduces a hyperbolic polarity in the plane at infinity of the affine space, then the polarity induces a Minkowskian metric in the whole space. With this hyperbolic polarity one considers as perpendicular any line and plane or any plane and plane whose elements at infinity correspond under this polarity. Two lines are said to be perpendicular if they intersect and their elements at infinity correspond under the polarity.

The proof that the group of motions of three-dimensional Minkowski space is generated by the "reflections about spacelike lines" defined above is the final step of our argumentation in this section.

Theorem 2.2 *Exterior point reflections generate any motion in the affine space. Moreover, since exterior points correspond to spacelike lines, any motion in Minkowski 3-space is generated by reflections about spacelike lines.*

Proof. Since any motion in Minkowski space can be generated by a suitable product of plane reflections, it suffices to show that reflections about exterior points generate plane reflections.

Let α be any Minkowski plane or spacelike plane (note that reflections about characteristic planes and lightlike lines do not exist since they are self-perpendicular or see [21]). Let P be any point in Minkowski space. Let l be the line through P parallel to α. Let α_∞ denote the section of α at infinity. Applying the polarity to α_∞, one obtains a point $g_\infty \perp \alpha_\infty$. Let g be a line through P whose section at infinity is g_∞, so that g is a line through P orthogonal to α. Since each line in the plane at infinity contains at least 3 points, there exists a line l in α which is orthogonal to g as $g_\infty \perp \alpha_\infty$. Now let m be a line through P not in α which intersects l. It follows that the reflection of P about α is the same as reflecting m about l and taking the intersection of the image of m under the reflection with g. By the construction of the affine space and the definition of orthogonality in the affine space, it follows that l and m must act as their sections at infinity act. Since any point reflection in the hyperbolic projective–metric plane can be generated by reflections about exterior points, it follows that the reflection of P about α is generated by reflections of P about spacelike lines. $\square$

From $(\mathfrak{G}, \mathcal{G})$ we have therefore constructed a model of three-dimensional Minkowski space in which each element of $\mathcal{G}$ is identified as a spacelike line (and every spacelike line in the Minkowski space is such an element) and on which each element of $\mathcal{G}$ acts adjointly as the reflection about the spacelike line. Such reflections generate the proper Poincaré group $\mathcal{P}_+$ on three-dimensional Minkowski space. The group $\mathfrak{G}$ is therefore isomorphic to $\mathcal{P}_+$, and the adjoint action of the identity component of $\mathfrak{G}$ upon $\mathcal{G}$ is transitive.

3 From States and Observables to Space–Time

An operationally motivated and mathematically powerful approach to quantum field theory is algebraic quantum field theory (AQFT) (cf. [29]). The initial data in AQFT are a collection $\{\mathcal{A}(\mathcal{O})\}$ of unital C^*-algebras indexed by a suitable set of open subregions $\mathcal{O}$ of the space–time of interest, with $\mathcal{A}(\mathcal{O})$ understood as being generated by all the observables measurable in the spacetime region $\mathcal{O}$, and a state ω on these algebras, understood as representing the preparation of the quantum system under investigation. For the reasons mentioned in the Introduction, we shall replace the index set of subregions of a specified space–time with some abstract set I, which for our purposes may be viewed as indexing possible laboratories. Hence, $\mathcal{A}_i$ is interpreted as the algebra generated by all observables measurable in the "laboratory" $i \in I$. It is understood that the description of the laboratory would include not only "spatial" but also "temporal" specifications. These specifications would be made with respect to suitable measuring devices, which themselves do not presuppose a particular space–time. There may be some structure on the index set I. For example, it makes sense to represent the fact that laboratory i is contained in the laboratory j with $i < j$. Then one would certainly have the relation $\mathcal{A}_i \subset \mathcal{A}_j$, *i.e.* $\mathcal{A}_i$ is a subalgebra of $\mathcal{A}_j$.[3] Hence, if $(I, \leq)$ is a partially ordered set, then one may expect that the property of isotony holds. We would therefore be working with two partially ordered sets, $(I, \leq)$ and $(\{\mathcal{A}_i\}_{i\in I}, \subseteq)$, and we require that the assignment $i \mapsto \mathcal{A}_i$ be an order-preserving bijection (*i.e.* it is an isomorphism in the structure class of partially ordered sets). Any such assignment which is not an isomorphism in this sense would involve some kind of redundancy in the description. To different laboratories should correspond different algebras.[4]

If $\{\mathcal{A}_i\}_{i\in I}$ is a net, then the inductive limit $\mathcal{A}$ of $\{\mathcal{A}_i\}_{i\in I}$ exists and may be used as a reference algebra. However, even if $\{\mathcal{A}_i\}_{i\in I}$ is not a net, it is still possible [28] to naturally

[3]This property of the net $\{\mathcal{A}_i\}_{i\in I}$ is called isotony in the AQFT literature.

[4]This truism need not hold in certain space–times such as anti-de Sitter space–time [16, 20], where there exist closed timelike curves.

embed the algebras $\mathcal{A}_i$ in a C^*-algebra $\mathcal{A}$ in such a way that the inclusion relations are preserved. In the following we need not distinguish these two cases and refer, somewhat loosely, to any collection $\{\mathcal{A}_i\}_{i\in I}$ of algebras, as specified, as a net. Any state on $\mathcal{A}$ restricts to a state on $\mathcal{A}_i$, for each $i \in I$. For that reason, we shall speak of a state on $\mathcal{A}$ as being a state on the net $\{\mathcal{A}_i\}_{i\in I}$.

Given a state ω on the algebra $\mathcal{A}$, one can consider the corresponding GNS representation $(\mathcal{H}_\omega, \pi_\omega, \Omega)$ and the von Neumann algebras $\mathcal{R}_i \equiv \pi_\omega(\mathcal{A}_i)''$, $i \in I$. We shall assume that the representation space $\mathcal{H}_\omega$ is separable. We extend the assumption of nonredundancy of indexing to the net $\{\mathcal{R}_i\}_{i\in I}$, *i.e.* we assume that also the map $i \mapsto \mathcal{R}_i$ is an order-preserving bijection.[5] If the GNS vector Ω is cyclic and separating for each algebra $\mathcal{R}_i$, $i \in I$, then from the modular theory of Tomita-Takesaki [34, 12], we are presented with a collection $\{J_i\}_{i\in I}$ of modular involutions (and a collection $\{\Delta_i\}_{i\in I}$ of modular operators), directly derivable from the state and the algebras. This collection $\{J_i\}_{i\in I}$ of operators on $\mathcal{H}_\omega$ generates a group $\mathcal{J}$. Note that $J\Omega = \Omega$ for all $J \in \mathcal{J}$. In the following we shall denote the adjoint action of J_i upon the elements of the net $\{\mathcal{R}_i\}_{i\in I}$ by $\mathrm{ad}J_i$, *i.e.* $\mathrm{ad}J_i(\mathcal{R}_j) \equiv J_i\mathcal{R}_jJ_i = \{J_iAJ_i : A \in \mathcal{R}_j\}$. Note that if $\mathcal{R}_1 \subset \mathcal{R}_2$, then one necessarily has $\mathrm{ad}J_i(\mathcal{R}_1) \subset \mathrm{ad}J_i(\mathcal{R}_2)$, in other words the map $\mathrm{ad}J_i$ is order-preserving.

The Condition of Geometric Modular Action (CGMA) was first introduced in [13] and has received a great deal of development since then — see, *e.g.*, [10, 15, 14, 17, 11]. In the present abstract setting, the CGMA is the condition that each map $\mathrm{ad}J_i$ leaves the set $\{\mathcal{R}_i\}_{i\in I}$ invariant, *i.e.* $\mathrm{ad}J_i$ is a net automorphism, for each $i \in I$. By the uniqueness of the modular objects, it follows that $\{J_i\}_{i\in I}$ is an invariant generating set of involutions for the group $\mathcal{J}$, as required for the purposes of absolute geometry. Furthermore, we note that the CGMA implies that, for each $i \in I$, there exists an order-preserving bijection τ_i on I such that $\mathrm{ad}J_i(\mathcal{R}_j) = \mathcal{R}_{\tau_i(j)}$ and $J_iJ_jJ_i = J_{\tau_i(j)}$, $i,j \in I$ [15]. The group generated by the involutions τ_i, $i \in I$, is denoted by $\mathcal{T}$ and forms a subgroup of the permutation group on the index set I. The set $\{\tau_i\}_{i\in I}$ is also an invariant generating set of involutions for the group $\mathcal{T}$. Thus, the pair $(\mathcal{T}, \{\tau_i\}_{i\in I})$ also provides a candidate for an absolute geometric treatment. In fact, as shown in [15], the group $\mathcal{J}$ is a central extension of $\mathcal{T}$ by a subgroup $\mathcal{Z}$ of the center of $\mathcal{J}$. So, in general, one should possibly consider the pair $(\mathcal{T}, \{\tau_i\}_{i\in I})$ as the initial data for the considerations of the previous section. However, in the case we are examining, the center of $\mathcal{J}$ turns out to be trivial, so that $\mathcal{J}$ and $\mathcal{T}$ are isomorphic. Hence, we shall avoid some technical complications and impose the axioms A.1 – A.6 on the pair $(\mathcal{J}, \{J_i\}_{i\in I})$. And to avoid certain degeneracies, we shall assume all algebras $\mathcal{A}_i$ to be nonabelian.

For the convenience of the reader, we summarize our standing assumptions.

Standing Assumptions For the net $\{\mathcal{A}_i\}_{i\in I}$ of nonabelian C^*-algebras and the state ω on $\mathcal{A}$ we assume

(i) $i \mapsto \mathcal{R}_i$ is an order-preserving bijection;
(ii) Ω is cyclic and separating for each algebra $\mathcal{R}_i$, $i \in I$;
(iii) each $\mathrm{ad}J_i$ leaves the set $\{\mathcal{R}_i\}_{i\in I}$ invariant.

Already these assumptions restrict significantly the class of admissible groups $\mathcal{T}$ and $\mathcal{J}$ [15]. In general, it may be necessary to pass to a suitable subcollection of $\{\mathcal{R}_i\}_{i\in I}$ in order for the Standing Assumptions to be satisfied [15] — see the final section for a brief

[5] This is automatically the case if the algebras $\mathcal{A}_i$ are W^*-algebras and ω induces a faithful representation of $\cup_{i\in I}\mathcal{A}_i$.

discussion of this point. Note also that the Standing Assumptions imply

$$\mathcal{R}_{\tau_i(i)} = J_i \mathcal{R}_i J_i = \mathcal{R}_i{}' \quad , \tag{2}$$

for all $i \in I$. Hence, the surjective map $i \mapsto J_i$ is two-to-one, since $J_i = J_{\tau_i(i)}$.

Consider three-dimensional Minkowski space with the standard metric

$$g = \mathrm{diag}(1,-1,-1) \equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

given in proper coordinates. The isometry group of this space is the Poincaré group $\mathcal{P}$ and the family $\mathcal{W}$ of wedges is obtained by applying the elements of $\mathcal{P}$ to a single wedge-shaped region of the form

$$W_R \equiv \{x \in \mathbb{R}^3 : x_1 > |x_0|\} \quad ,$$

i.e. $\mathcal{W} = \{\lambda W_R : \lambda \in \mathcal{P}\}$, where $\lambda W_R = \{\lambda(x) : x \in W_R\}$. We remark that, in fact, one has $\mathcal{W} = \{\lambda W_R : \lambda \in \mathcal{P}_+^\uparrow\}$, where $\mathcal{P}_+^\uparrow$ is the identity component of the Poincaré group. Each wedge $W = \lambda W_R$ determines a spacelike line called the edge E_W of the wedge: $E_W = \lambda E_R$, where

$$E_R = \{x \in \mathbb{R}^3 : x_1 = 0 = x_0\} \quad .$$

Note that W and its causal complement[6] W' share the same edge, *i.e.* $E_W = E_{W'}$. Moreover, the equality $E_{W_1} = E_{W_2}$ entails that either $W_1 = W_2$ or $W_1 = W_2{}'$. Conversely, each spacelike line $l \subset \mathbb{R}^3$ determines a pair of wedges W_1, W_2 with $W_1 = W_2{}'$ — in fact, the causal complement of l consists of two connected components, each a wedge, each the causal complement of the other, and each having l as its edge.

Now assume that ω and $\{\mathcal{A}_i\}_{i\in I}$ satisfy the Standing Assumptions, and that the pair $(\mathcal{J}, \{J_i\}_{i\in I})$ fulfills Axioms A.1 – A.6 in Section 2.1. With $\mathfrak{G} = \mathcal{J}$ and $\mathcal{G} = \{J_i\}_{i\in I}$, the results of Section 2 entail that there exists a realization of three-dimensional Minkowski space $\mathbb{R}^3$ in which each J_i corresponds uniquely to a spacelike line l_i and on which each J_i acts adjointly as the reflection about l_i. Hence, to each J_i corresponds a pair W_i, W_i' of complementary wedges whose common edge is l_i (recall that in this construction l_i is, in fact, equal to J_i). One of these spacelike lines is the set $l_{i_0} \equiv E_R = \{(0,0,x) : x \in \mathbb{R}\}$. Define $\chi(i_0) \equiv W_R$. From the results in Section 2, the adjoint action of the identity component (isomorphic to $\mathcal{P}_+^\uparrow$) of the group $\mathcal{J}$ upon $\{J_i\}_{i\in I}$ is transitive. In light of (2), this entails that the adjoint action of $\mathcal{J}$ upon $\{\mathcal{R}_i\}_{i\in I}$ is also transitive. Hence, for every $i \in I$ there exists a $g_i \in \mathcal{J}$ such that

$$\mathcal{R}_i = g_i \mathcal{R}_{i_0} g_i{}^{-1} = \mathcal{R}_{\tau_{g_i}(i_0)} \quad . \tag{3}$$

By the Standing Assumptions, this implies $i = \tau_{g_i}(i_0)$, for every $i \in I$. Of course, for fixed $i \in I$ the group element g_i is not unique — it is determined only up to an element of the subgroup of $\mathcal{J}$ which leaves the algebra $\mathcal{R}_{i_0}$ fixed, *i.e.* the commutator subgroup of J_{i_0}, which in our construction is also the subgroup of $\mathcal{P}_+$ leaving the line l_{i_0} fixed. g_i itself can be expressed as a product of a finite number of elements in $\{J_i\}_{i\in I}$, and g_i acts adjointly upon our model of Minkowski 3-space as the product of the corresponding reflections.

Let $g\chi(i_0) \equiv \{gPg^{-1} : P \in \chi(i_0)\}$ denote the image under $g \in \mathcal{J}$ of the wedge $\chi(i_0)$; $g\chi(i_0)$ is itself a wedge. For each $i \in I$ and a particular choice of g_i as above, define

[6] The causal complement of a set $S \subset \mathbb{R}^3$ is the interior of the set of all points in $\mathbb{R}^3$ which are spacelike separated from every point in S.

$\chi(i) = \chi(\tau_{g_i}(i_0))$ to be $g_i\chi(i_0)$. Note that $g_i\chi(i_0)$ is independent of the choice of $g_i \in \mathcal{J}$ satisfying equation (3), since any element of $\mathcal{P}_+$ leaving l_{i_0} fixed also leaves $\chi(i_0)$ fixed. One then has

$$J_i\chi(j) = J_i g_j \chi(i_0) = \chi(\tau_{J_i g_j}(i_0)) = \chi(\tau_i(\tau_{g_j}(i_0))) = \chi(\tau_i(j)) \quad . \tag{4}$$

Since the results of Section 2 entail that $\mathcal{J}$ is isomorphic to the proper Poincaré group $\mathcal{P}_+$, $\mathcal{J}$ contains an (anti-)unitary representation $U(\mathcal{P}_+)$ of $\mathcal{P}_+$; indeed, from the results of Section 2, one actually has $\mathcal{J} = U(\mathcal{P}_+)$. We define the algebra $\mathcal{R}(\chi(i))$ corresponding to the wedge $\chi(i)$ to be $\mathcal{R}_i$. Using (4), one then finds that

$$J_i\mathcal{R}(\chi(j))J_i = J_i\mathcal{R}_j J_i = \mathcal{R}_{\tau_i(j)} = \mathcal{R}(\chi(\tau_i(j))) = \mathcal{R}(J_i\chi(j)) \quad ,$$

for every $i, j \in I$. This implies that the net $\{\mathcal{R}(\chi(i))\}_{i\in I}$ is covariant under the representation $U(\mathcal{P}_+)$. In addition, since by construction $J_i\chi(i) = \chi(i)'$ (J_i is the reflection about the line $l_i = J_i$ and the edge of the wedge $\chi(i)$ is $g_i J_{i_0} {g_i}^{-1} = J_i$), it follows that one has

$$\mathcal{R}(\chi(i))' = \mathcal{R}_i{}' = J_i\mathcal{R}_i J_i = J_i\mathcal{R}(\chi(i))J_i = \mathcal{R}(J_i\chi(i)) = \mathcal{R}(\chi(i)') \quad .$$

This is the property known as Haag duality in AQFT.

We have therefore proven the following theorem.

Theorem 3.1 *Let ω and $\{\mathcal{A}_i\}_{i\in I}$ satisfy the Standing Assumptions, and let the pair $(\mathcal{J}, \{J_i\}_{i\in I})$ fulfill Axioms A.1 – A.6 in Section 2.1. Then there exists a bijection $\chi : I \to \mathcal{W}$ such that $J_i\chi(j) = \chi(\tau_i(j))$, for all $i, j \in I$. The group $\mathcal{J}$ forms an (anti-)unitary representation of the proper Poincaré group $\mathcal{P}_+$, wherein each J_i represents the reflection about the spacelike line in $\mathbb{R}^3$ which is the edge of the wedge $\chi(i)$. Moreover, the bijection χ can be chosen so that with $\mathcal{R}(\chi(i)) \equiv \mathcal{R}_i$, the collection $\{\mathcal{R}(\chi(i))\}_{i\in I}$ forms a collection of von Neumann algebras satisfying Haag duality which is covariant under this representation of $\mathcal{P}_+$.*

We have at present no proof that the bijection χ can be selected in such a manner that the resulting set $\{\mathcal{R}(\chi(i))\}_{i\in I}$ satisfies isotony. This is because $\{\mathcal{R}(\chi(i))\}_{i\in I}$ is isotonous if and only if the map $\chi : I \to \mathcal{W}$ is order-preserving, and we do not know how to assure this. If the order structure on I expresses the ordering of the laboratories discussed above, then unless such a choice of χ can be made, the conceptual problem at hand has not yet been satisfactorily solved. We feel it likely that the group $\mathcal{J}$ can only satisfy all the assumptions A1 – A6 if the order structure on I is consistent with the order structure on $\{\chi(i)\}_{i\in I}$, but a proof to this effect would require a development of modular theory in a direction which has hardly been considered in the literature. In anticipation of such a future theory, we can at least prove the following theorem.

Theorem 3.2 *Let ω and $\{\mathcal{A}_i\}_{i\in I}$ satisfy the Standing Assumptions, and let the pair $(\mathcal{J}, \{J_i\}_{i\in I})$ fulfill Axioms A.1 – A.6 in Section 2.1. If the bijection $\chi : I \to \mathcal{W}$ from Theorem 3.1 is order-preserving, then $\{\mathcal{R}(\chi(i))\}_{i\in I}$ is a net of von Neumann algebras satisfying locality. Furthermore, the CGMA is satisfied by the pair $(\omega, \{\mathcal{R}(\chi(i))\}_{i\in I})$.*

Proof. In light of the assumed isotony of $\{\mathcal{R}(\chi(i))\}_{i\in I}$ and the Haag duality from Theorem 3.1, it follows that if $\chi(j) \subset \chi(i)'$, then $\mathcal{R}(\chi(j)) \subset \mathcal{R}(\chi(i)') = \mathcal{R}(\chi(i))'$, *i.e.* the net is local. The fact that the CGMA is satisfied by $(\omega, \{\mathcal{R}(\chi(i))\}_{i\in I})$ is a trivial consequence of the construction and isotony. $\square$

Once one has such a net of wedge algebras, it is then standard [9] to generate a maximal local net $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{S}}$ which also contains algebras of observables localized in compact regions and is Poincaré covariant under $U(\mathcal{P}_+)$. For the sake of completeness, we mention that there exist examples of quantum fields, *e.g.* the free scalar Bose field on three-dimensional Minkowski space in the vacuum state, which fulfill the hypotheses of Theorems 3.1 and 3.2 [35].

An immediate consequence of these theorems is that the modular symmetry group $\mathcal{J}$ must also contain a strongly continuous unitary representation of the time translation subgroup of $\mathcal{P}_+$, which is usually interpreted as describing the dynamics of the covariant quantum system. Hence, also the dynamics of the quantum system is determined by ω and $\{\mathcal{A}_i\}_{i\in I}$, under the stated conditions.

In [15] a purely algebraic stability condition called the Modular Stability Condition was identified for reasons we shall not explain here. The Modular Stability Condition requires that the modular unitaries Δ_j^{it} associated with $(\Omega, \mathcal{R}_j)$ in Tomita–Takesaki theory are contained in $\mathcal{J}$, for each $j \in I$ and $t \in \mathbb{R}$.

Corollary 3.1 *If, in addition to the hypotheses of Theorem 3.2, the pair $(\omega, \{\mathcal{A}_i\}_{i\in I})$ fulfills the Modular Stability Condition, then the modular unitaries ${\Delta_{\chi(j)}}^{it}$ associated with $(\Omega, \mathcal{R}(\chi(j)))$ are contained in $U(\mathcal{P}_+^\uparrow) \subset \mathcal{J}$, for each $j \in I$ and $t \in \mathbb{R}$, and represent the boost group leaving the wedge $\chi(j)$ invariant.*[7]

Proof. In Section 5.2 of [15] are given sufficient conditions for the adjoint action of the modular unitaries Δ_j^{it} upon the net constructed above to be implemented by the specified Poincaré transformations. All of those conditions are satisfied here with the possible exception that the adjoint action of all modular unitaries may not act transitively upon the net and that the conditions (ii) and (iii) of the CGMA in [15] may not hold. However, in [19, 27] it is shown that the former assumption may be dropped. Furthermore, the role of conditions (ii) and (iii) in [15] was to assure that the adjoint action of modular objects was implemented by Poincaré transformations — but this is assured here by the construction above and the Modular Stability Condition. Hence, here these assumptions may also be dropped. $\square$

4 Conclusion and Outlook

We have shown that it is possible to derive a space–time from the operationally motivated quantum data of a state modelling the preparation of the quantum system and a net of algebras of the observables of the quantum system which is indexed in some suitable manner, *e.g.* by the laboratories in which the observables were measured. This has been done for the simplest nontrivial case — three-dimensional Minkowski space. A similar derivation of four-dimensional Minkowski space has been made in [35], though surely not in the optimal manner. Although it is not yet clear which class of space–times could be attainable through this approach, it is likely that one could at least be able to derive in this manner all space–times with a sufficiently large isometry group.

[7]This property is called modular covariance in the AQFT literature.

We mentioned above the likelihood that a given net $\{\mathcal{A}_i\}_{i\in I}$ will not satisfy the Standing Assumptions and that it may be necessary to pass to a subnet. This is because experience has shown that only the modular involutions associated with algebras localized in certain types of regions will have a suitable adjoint action upon the net - cf. [15] for a discussion of this matter. However, a subnet of the original net may well satisfy the Standing Assumptions. From this point of view, the results of [9] assert that in any finite component quantum field theory on Minkowski space satisfying the Wightman axioms to which can be locally associated a net $\{\mathcal{R}(\mathcal{O})\}_{\mathcal{O}\in\mathcal{S}}$ of local von Neumann algebras, there always exists a subnet $\{\mathcal{R}(W)\}_{W\in\mathcal{W}}$ which satisfies our Standing Assumptions. Hence, given a net $\{\mathcal{A}_i\}_{i\in I}$, one would proceed to a subnet satisfying the Standing Assumptions and suitable absolute geometric axioms, construct the space–time, make a suitable identification between the elements of the subnet and algebras associated with special regions of the space–time, then attempt to use the inclusion relations in the original net $\{\mathcal{A}_i\}_{i\in I}$ to identify the remaining algebras in $\{\mathcal{A}_i\}_{i\in I}$ with algebras associated with suitable regions in the derived space–time.

In Theorem 3.1 the crucial hypothesis that Axioms A.1 – A.6 are satisfied is imposed upon the auxiliary (and non-operational) object $\mathcal{J}$. And in order to obtain Theorem 3.2 it was necessary for us to posit that χ was order-preserving. It would be desirable to determine conditions upon the net $\{\mathcal{A}_i\}_{i\in I}$ directly which would, by some suitable extension of the current state of modular theory, imply that the group $\mathcal{J}$ fulfills the said axioms and that χ is order-preserving. This will involve making progress in a field of mathematics which is yet in its infancy. The modular theory of Tomita–Takesaki was initially formulated and developed for a state on a single algebra. Only relatively recently, particularly motivated by questions in AQFT, have researchers considered a state on a pair of algebras and studied relations among the modular objects implied by relations between the algebras. A notable extension of the theory to more than two algebras can be found in the recent papers [36, 31] (though the essential nature of the insights there were for pairs of algebras, as well). To attain a theorem of the type we would like to see some day, it will be necessary to develop the modular theory of a state and a net of algebras. Such a theory would have many applications besides the one we envision.

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