Toward an Infinite-component Field Theory with a Double Symmetry: Interaction of Fields

L.M.Slad*

Skobeltsyn Institute of Nuclear Physics,
Moscow State University, Moscow 119899

Abstract

We complete the first stage of constructing a theory of fields not investigated before; these fields transform according to Lorentz group representations decomposable into an infinite direct sum of finite-dimensional irreducible representations. We consider only those theories that initially have a double symmetry: relativistic invariance and the invariance under the transformations of a secondary symmetry generated by the polar or the axial four-vector representation of the orthochronous Lorentz group. The high symmetry of the theory results in an infinite degeneracy of the particle mass spectrum with respect to spin. To eliminate this degeneracy, we postulate a spontaneous secondary-symmetry breaking and then solve the problems on the existence and the structure of nontrivial interaction Lagrangians.

1. Introduction

We recently began constructing a theory of fields that transform according to representations of the proper Lorentz group $L_+^\uparrow$ that are decomposable into an infinite direct sum of finite-dimensional irreducible representations (we call such fields the ISFIR-class fields) [1]. This construction is based on a group theory approach formulated in [2] and called a double symmetry. It aims to study a possible of effectively describing hadrons in terms of the monolocal infinite-component fields.

A special feature of the general relativistically invariant theory of the free ISFIR-class fields is an infinite number of arbitrary constants in its Lagrangians of the form

$$\mathcal{L}_0 = \frac{i}{2}[(\Psi, \Gamma^\mu \partial_\mu \Psi) - (\partial_\mu \Psi, \Gamma^\mu \Psi)] - (\Psi, R\Psi) \tag{1}$$

and in their corresponding Gelfand–Yaglom equations [3], [4]. Such constants correspond to each pair of irreducible $L_+^\uparrow$-group representations such that the matrix elements of the four-vector operator Γ^μ are nonzero. Because of the infinite arbitrariness with respect to constants, nobody analyzed the infinite-component ISFIR-class field theory or spoke about its physical properties until [1]. The elemination of this arbitrariness was the main problem posed in [1] and solved there by extracting that part of the general ISFIR-class field theory having relativistic invariance (the primary symmetry) that is additionally invariant under global secondary-symmetry transformations of the form

$$\Psi(x) \to \Psi'(x) = \exp[-iD^\mu \theta_\mu]\Psi(x), \tag{2}$$

*E-mail: slad@theory.sinp.msu.ru

where D^μ are matrix operators and the parameters θ_μ are the components of the polar or the axial four-vector of the orthochronous Lorentz group $L^\uparrow$.

The notion of the secondary symmetry introduced in [2] generalizes the long-known symmetries like the symmetry of the Gell-Mann–Levy σ-model [5] and supersymmetry (see, e.g., [6]). By definition, the secondary-symmetry transformations do not break the primary symmetry, and their parameters belong to the space of a certain representation of the primary-symmetry group. In the σ-model, the parameters of secondary-symmetry transformations are the pseudoscalars with respect to the orthochronous Lorentz group, whereas supersymmetry is generated by the bispinor representation of the proper Lorentz group. The secondary symmetry used in [1], whose transformations are given by relation (2), is a natural member of this series of the topical symmetries of field theories. Transformations (2) connect sectors belonging to the spaces of different irreducible $L_+^\uparrow$-group representations because of which the problem of eliminating the infinite arbitrariness in the constants of the relativistically invariant theory of the ISFIR-class fields can be completely solved.

In each of the variants of the free infinite-component field theory with the double symmetry that were found and described in [1], we know, first, a representation of the proper Lorentz group, according to which the field transforms, second, the four-vector operator D^μ from secondary-symmetry transformations (2) (up to a common numerical factor), and, third, the four-vector operator Γ^μ and therefore the kinetic term of Lagrangian (1) (up to a common numerical factor). It is also known that the operator R defining the mass term of Lagrangian (1) is a multiple of the identity operator.

In complete accordance with the known Coleman–Mandula theorem [7] relating to consequences of the Lorentz-group extension, the mass spectrum corresponding to any of the variants of the infinite-component field theory with the double symmetry is infinitely degenerate with respect to spin. This requires introducing a spontaneous secondary-symmetry breaking that leaves the primary symmetry (the relativistic invariance) of the theory unchanged. To be specific, we suppose that the scalar (with respect to the group $L^\uparrow$) components of one or several bosonic infinite-component ISFIR-class fields have nonzero vacuum expectation values. A spontaneous secondary-symmetry breaking can introduce only one correction into the free-field theory with the double symmetry described in [1], namely, it can change the operators R, i.e., change the mass terms in the Lagrangians of form (1). To find those variants of the theory for which such nonzero changes are possible and to obtain new expressions for the operators R, we must solve the problem of the existence and the structure of nontrivial interaction Lagrangians for the infinite-component fields with the initial double symmetry.

In this paper, we confine ourselves to finding only those Lagrangians knowing which completes the construction of the free fermionic ISFIR-class field theories and brings us right up to the beginning of the study of a number of physical properties of such fields: their mass spectra and the characteristics of electromagnetic interaction. Constructing the free and the self-interacting bosonic ISFIR-class field theories should be completed only in the case where describing baryons in terms of the considered infinite-component fields proves successful.

We now present the subject matter of our paper in more concrete formulations.

We analyze the structure of three-particle interaction Lagrangians of the form[1]

$$\mathcal{L}_{\rm int} = \sum_{i,\tau,l,m} \overline{\psi}(x) Q_i^{\tau lm} \psi(x) \varphi^i_{\tau lm}(x) \equiv \sum_{i,\tau,l,m} (\psi(x), Q_i^{\tau lm} \varphi^i_{\tau lm}(x) \psi(x)), \qquad (3)$$

[1]All the necessary information about the notions used and notation relating to the Lorentz group can be found in [1].

where $\psi(x)$ is a fermion field, $\varphi^i_{\tau lm}(x)$ is the bosonic-field component characterized by the irreducible $L^{\uparrow}_{+}$-group representation $\tau = (l_0, l_1)$, by the spin l, and by its third-axis projection m, and $Q_i^{\tau lm} \equiv Q_i^{(l_0,l_1)lm}$ are matrix operators. The index i indicates that according to conditions 1 and 3A in [1], two types of bosonic field that differ by the transformation properties under space reflection are included in Lagrangian (3).

We consider only those Lagrangians (3) for which the bosonic fields have the scalar or the pseudoscalar (with respect to the group $L^{\uparrow}$) components (these fields are respectively denoted by $\varphi^{+}(x)$ and $\varphi^{-}(x)$, and the index i is assigned the values $+$ and $-$). Therefore, at this stage of constructing the theory, only two representations, S^1 given by (I.88) and S^B given by (I.89)[2], are selected from the countable set of the proper Lorentz group representations with integer spin that was found in [1].

We impose the requirement on the interaction Lagrangians, as well as on the free Lagrangians in [1], that they initially have the double symmetry: relativistic invariance and the invariance under secondary-symmetry transformations (2) or (I.94) generated by the polar or the axial four-vector representation of the orthochronous Lorentz group.

If the scalar (with respect to the group $L^{\uparrow}$) bosonic-field component $\varphi^{+}_{(0,1)00}(x)$ acquires a nonzero vacuum expectation value λ as a result of a spontaneous secondary-symmetry breaking, then the sum of Lagrangians (I.12) and (3) results in fermionic-field Lagrangian (1) with the operator R of the form

$$R = \kappa E + \lambda Q_{+}^{(0,1)00}, \tag{4}$$

where E is the identity operator. The operator $Q_{+}^{(0,1)00}$ depends on the ratio of the normalization constants for the four-vector operators D^{μ} in the secondary-symmetry transformations of the bosonic and the fermionic fields, this ratio plays the role of a parameter. At the stage of comparing the properties of the infinite-component fields and the properties of hadrons, we consider a sum of Lagrangians of type (3) that contain several bosonic fields that differ, in particular, in the mentioned normalization constants. In this case, formula (4) contains several terms of the type $\lambda Q_{+}^{(0,1)00}$.

The subject matter is further presented in the following order. First, we write formulas involving the four-vector operators; these formulas are repeatedly used to solve the problems in the paper. Next, we successively consider the question about the existense of the nontrivial fermion-boson interaction Lagrangians and the matrix elements of the operators $Q_{\pm}^{(0,1)00}$ for the cases where the bosonic fields are described by the respective $L^{\uparrow}_{+}$-group representations S^1 and S^B. In conclusion, we give an example of a fermionic infinite-component ISFIR-class field with interesting mass spectrum properties described by the constructed double-symmetry theory. We note that the proofs of the formulas given in this paper are based on evident algebraic operations and, as a rule, require lengthy calculations.

2. Some relations involving four-vector operators

We take two four-vector operators V^{μ} and W^{μ} defined by the respective quantities $v_{\tau'\tau}$ and $w_{\tau'\tau}$. From the analogues of formulas (I.17)–(I.23), as well as from relations (I.7)–(I.10), we obtain

$$\sum_{l',m'} g_{\mu\nu} V^{\mu}_{\tau''l''m'',\tau'l'm'} W^{\nu}_{\tau'l'm',\tau lm} = b_{\tau\tau'} v_{\tau\tau'} w_{\tau'\tau} \delta_{\tau''\tau} \delta_{l''l} \delta_{m''m}, \tag{5}$$

[2]Here and in what folows, the references to the formulas in [1] are given in the form (I.N), where N is the number of the corresponding formula.

where

$$b_{\tau\tau'} = \begin{cases} 2(l_1 - l_0 - 1)(l_1 + l_0 + 1), & \text{if } \tau' = (l_0 + 1, l_1), \\ 2(l_1 - l_0 + 1)(l_1 + l_0 - 1), & \text{if } \tau' = (l_0 - 1, l_1), \\ 2(l_0 - l_1 - 1)(l_1 + l_0 + 1), & \text{if } \tau' = (l_0, l_1 + 1), \\ 2(l_0 - l_1 + 1)(l_1 + l_0 - 1), & \text{if } \tau' = (l_0, l_1 - 1), \end{cases} \tag{6}$$

for $\tau = (l_0, l_1)$. There is no summation over the repeated index τ' in the left side of relation (5).

In the free-field theory with the double symmetry, the operators D^μ in transformations (2) satisfy the relation

$$D^\mu D_\mu = HE, \tag{7}$$

where H is a numerical factor. This relation is a consequence (I.26) and is easily verified by direct calculations.

Using formulas (5) and (6) and the values of the quantities $d_{\tau'\tau}$ chracterizing the four-vector operator D^μ and written in Corollaries 1–4 in [1], we have

$$H = \begin{cases} 4d_0^2 & \text{for Corollaries 1 and 3 (point 1),} \\ -4d_0^2 & \text{for Corollary 2 (point 2),} \\ -8d_0^2 & \text{for Corollary 2 (point 3),} \\ 8d_0^2 & \text{for Corollary 3 (point 2),} \end{cases} \tag{8}$$

where $d_0 = g_0 c_0$.

We now introduce the four-vector operator Ω^μ described by the quantities $\omega_{\tau'\tau}$ and constructed from the given operators V^μ and W^μ as

$$\Omega^\mu = V^\nu W^\mu V_\nu.$$

Using the analogues of formulas (I.17)–(I.23) and relations (I.7)–(I.10), we obtain

$$\begin{gathered} \frac{1}{2}\omega(l_0 + 1, l_1; l_0, l_1) = \\ = (l_1 - l_0 + 1)(l_1 + l_0 - 1)v(l_0 + 1, l_1; l_0, l_1)w(l_0, l_1; l_0 - 1, l_1)v(l_0 - 1, l_1; l_0, l_1) \\ -v(l_0 + 1, l_1; l_0, l_1)w(l_0, l_1; l_0 + 1, l_1)v(l_0 + 1, l_1; l_0, l_1) \\ +(l_1 - l_0 - 2)(l_1 + l_0 + 2)v(l_0 + 1, l_1; l_0 + 2, l_1)w(l_0 + 2, l_1; l_0 + 1, l_1)v(l_0 + 1, l_1; l_0, l_1) \\ +(l_1 + l_0 - 1)v(l_0 + 1, l_1; l_0, l_1)w(l_0, l_1; l_0, l_1 - 1)v(l_0, l_1 - 1; l_0, l_1) \\ -(l_1 - l_0 - 2)(l_1 + l_0 - 1)v(l_0 + 1, l_1; l_0 + 1, l_1 - 1)w(l_0 + 1, l_1 - 1; l_0, l_1 - 1)v(l_0, l_1 - 1; l_0, l_1) \\ -(l_1 - l_0 + 1)v(l_0 + 1, l_1; l_0, l_1)w(l_0, l_1; l_0, l_1 + 1)v(l_0, l_1 + 1; l_0, l_1) \\ -(l_1 - l_0 + 1)(l_1 + l_0 + 2)v(l_0 + 1, l_1; l_0 + 1, l_1 + 1)w(l_0 + 1, l_1 + 1; l_0, l_1 + 1)v(l_0, l_1 + 1; l_0, l_1) \\ +(l_1 - l_0 - 2)v(l_0 + 1, l_1; l_0 + 1, l_1 - 1)w(l_0 + 1, l_1 - 1; l_0 + 1, l_1)v(l_0 + 1, l_1; l_0, l_1) \\ -(l_1 + l_0 + 2)v(l_0 + 1, l_1; l_0 + 1, l_1 + 1)w(l_0 + 1, l_1 + 1; l_0 + 1, l_1)v(l_0 + 1, l_1; l_0, l_1). \end{gathered} \tag{9}$$

The expression for $\frac{1}{2}\omega(l_0, l_1; l_0+1, l_1)$ is obtained from (9) by replacing $v_{\tau'\tau}$, $w_{\tau'\tau}$, and $\omega_{\tau'\tau}$ with $v_{\tau\tau'}$, $w_{\tau\tau'}$ and $\omega_{\tau\tau'}$, and the formulas for $\frac{1}{2}\omega(l_0, l_1+1; l_0, l_1)$ and $\omega(l_0, l_1; l_0, l_1+1)$ are obtained from the respective formulas for $\frac{1}{2}\omega(l_0 + 1, l_1; l_0, l_1)$ and $\frac{1}{2}\omega(l_0, l_1; l_0 + 1, l_1)$ by replacing $l_0 \leftrightarrow l_1$.

If the four-vector operator D^μ from secondary-symmetry transformations (2) corresponds to the variants of the free-field theory described in Corollary 2 (point 3) or in Corollary 3 (point 2) in [1], then using formula (9), we obtain

$$D^\nu D^\mu D_\nu = 0. \tag{10}$$

For the same operators D^μ, the relation

$$\varepsilon_{\mu\nu\rho\sigma} D^\mu D^\nu D^\rho D^\sigma = 0 \tag{11}$$

also holds.

It is evident that interaction Lagrangian (3) can be invariant under transformations (2) or (I.94), if and only if their generated secondary-symmetry group is common to both the fermionic and the bosonic fields. All the secondary-symmetry groups obtained in [1] can be referred to one of the following three types.

The first type includes the Abelian groups, for which

$$[D^\mu, D^\nu] = 0. \tag{12}$$

This equality holds for the operators D^μ corresponding to Corollaries 1, 2 (point 2), and 3 (point 1) in [1]. For instance, formulas (I.75) and (I.76) relating to Corollaries 1 and 3 (point 1) imply that $D^\mu = g_0\Gamma^\mu$, and relation (I.26) then leads to equality (12). If the operators D^μ are described by formulas (I.79)–(I.82) in Corollary 2 (point 2), then we can verify the relations that are equivalent to equality (12) and are obtained from Eqs. (I.32)–(I.44) by replacing $c_{\tau'\tau} \to d_{\tau'\tau}$.

The second type of the secondary-symmetry groups corresponds to Corollaries 2 (point 3) and 3 (point 2) in [1]. For this type, the antisymmetric operators $G^{\mu\nu}$ defined by

$$G^{\mu\nu} = [D^\mu, D^\nu] \tag{13}$$

are nonzero. The equalities

$$[G^{\mu\nu}, G^{\rho\sigma}] = 0, \tag{14}$$

$$\{D^\mu, G^{\nu\rho}\} = 0 \tag{15}$$

hold. Using relation (15), we can show that the secondary-symmetry groups of the second type are infinite.

The third type of the secondary-symmetry groups corresponds to Corollaries 2 (point 1) and 4 in [1]. For this type, the inequality $[G^{\mu\nu}, G^{\rho\sigma}] \neq 0$ holds.

Until now, we used the same notion for a number of quantities defined in the spaces of both the fermionic and the bosonic fields. In what follows, we include the additional indices F and B in the notation of such quantities: $D^{F\mu}$, H^F, and $G^{F\mu\nu}$ and $D^{B\mu}$, H^B, and $G^{B\mu\nu}$.

We now list the variants of interaction Lagrangians (3) to be considered in accordance with the group properties of the fields involved.

Variants 1 and 1A. Each of the bosonic fields transforms according to the $L_+^\uparrow$-group representation S^1 given by (I.88). The fermionic field trasforms according to any of the representations S^{k_1} given by (I.72). The secondary-symmetry transformations are generated by the polar and the axial four-vector representations of the group $L^\uparrow$ in the respective variants 1 and 1A. Such bosonic-field transformations have form (2) and (I.94) in the respective variants 1 and 1A. The operators $D^{B\mu}$ are described by formulas (I.74) and (I.76), where $k_1 = 1$. The secondary-symmetry transformations for the fermionic field have form (2). The operators $D^{F\mu}$ involved are described by formulas (I.73)–(I.76) and (I.79)–(I.82) in the respective variants 1 and 1A.

Variant 2. Each of the bosonic and the fermionic fields transforms according to the respective $L_+^\uparrow$-group representation S^B given by (I.89) and S^F given by (I.83). The secondary-symmetry transformations for the bosonic and the fermionic fields have the respective forms (I.94) and (2), and the parameter θ_μ in formulas (I.94) and (2) is the axial four-vector of the group $L^\uparrow$. The operators $D^{B\mu}$ and $D^{F\mu}$ are described by the respective formulas (I.90)–(I.93) and (I.84)–(I.87).

The secondary-symmetry groups for variants 1 and 1A of the interaction Lagrangian refer to the first type and for variant 2 do to the second type.

In what follows, we present detailed calculations relating to the structure of the interaction Lagrangians only for variants 1 and 2; for variant 1A, we indicate the changes that must be introduced in variant 1.

3. The representation S^1 for the bosonic field and the fermion-boson interaction Lagrangian

We let $Q_\pm$ denote the scalar (with respect to the group $L_+^\uparrow$) operator $Q_\pm^{(0,1)00}$ in Lagrangian (3). Let the fermionic field $\psi(x)$ belong to the representation space S of the orthochronous Lorentz group, and let its Lorentz transformation have form (I.3). Then the fulfillment of the relation

$$S^{-1}(g)Q_\pm S(g) = Q_\pm \tag{16}$$

is the necessary condition for the relativistic invariance of Lagrangian (3). Here and in what follows, either the upper or the lower signs are taken simultaneously.

From relation (16), we find that for any irreducible $L_+^\uparrow$-qroup representation (l_0, l_1) belonging to the representation S, the equaities

$$Q_\pm \xi_{(l_0,l_1)lm} = q^\pm(l_0, l_1)\xi_{(l_0,l_1)lm}, \tag{17}$$

$$q^\pm(-l_0, l_1) = \pm q^\pm(l_0, l_1), \tag{18}$$

where the quantities $q(|l_0|, l_1)$ are arbitrary, must hold.

It is easy to prove that the fulfillment of relation (16) and equalities (21) given below and expressing the operators $Q_\pm^{\tau lm}$ in Lagrangian (3) in terms of the operators $Q_\pm$ is the enough condition for the relativistic invariance of Lagrangian (3).

Taking equality (I.27) into account, we can easily see that secondary-symmetry transformations (2) generated by the polar four-vector of the group $L^\uparrow$ leave Lagrangian (3) unchanged if the operators $Q^{\tau lm}$ satisfy the system of equations

$$[D^{F\mu}, Q_\pm^{\tau lm}] = \sum_{\tau',l',m'} Q_\pm^{\tau' l' m'} D^{B\mu}_{\tau' l' m', \tau lm}, \tag{19}$$

where $\tau, \tau' \in S$ and $D^{B\mu}_{\tau' l' m', \tau lm}$ is the matrix element of the operator $D^{B\mu}$.

Equations (19) with formulas (5) and (6) give a possibility to express the operators $Q_\pm^{\tau_1 lm}$ and $Q_\pm^{\tau_0 lm}$ in terms of each other for any preassigned τ_1 and τ_0 belonging to the representation S, and this is possible in an infinite number of ways. The main question is therefore about the equivalence of all of such expressions or, what is the same, about the consistency of system of equations (19).

The following statement holds.

Proposition 1. *Let the transformation properties of the fields in Lagrangian* (3) *coincide with those listed in variant* 1. *Then system of equations* (19) *is equivalent to the equations*

$$D^{F\mu} Q_{\pm} D^F_{\mu} = (H^F - H^B/2) Q_{\pm} \tag{20}$$

in the operator $Q_{\pm}$ *and the system of independent equalities*

$$Q_{\pm}^{(0,n+1)lm} = (n+1) \left(\frac{2}{H^B}\right)^n \times$$

$$\times [D^F_{\nu_n}, \ldots, [D^F_{\nu_2}, [D^F_{\nu_1}, Q_{\pm}]] \ldots] \left(D^{B\nu_1} D^{B\nu_2} \ldots D^{B\nu_n}\right)_{(0,1)00,(0,n+1)lm}, \tag{21}$$

where $n \geq 1$ *and the quantities* H^F *and* H^B *and given by respective relations* (7) *and* (8).

Proof. We first prove that Eqs. (20) and equalities (21) are the consequence of system of equations (19).

We multiply both sides of relation (19) by the operator $D^{F\nu}$ first from the left an then from the right. We subtract the second of the obtained equations from the first one and eiminate the operator $D^{F\nu}$ in the right side of the resulting equation using initial Eq. (19). We have

$$[D^{F\nu}, [D^{F\mu}, Q_{\pm}^{\tau lm}]] = \sum_{\tau', l', m'} Q_{\pm}^{\tau' l' m'} \left(D^{B\nu} D^{B\mu}\right)_{\tau' l' m', \tau lm}. \tag{22}$$

We multiply both sides of this equation by $g_{\mu\nu}$ and sum over the indices μ and ν. Taking relation (7) into account and setting $\tau = (0,1)$, $l = 0$, and $m = 0$, we obtain Eqs. (20).

We now take Eq. (19) with $\tau = (0,j)$ and $j \geq 1$. We multiply both sides by $g_{\mu\nu} D^{B\nu}_{(0,j)lm,(0,j+1)l''m''}$ and sum over the indices μ, ν, l, and m. Using relatios (5), (6), (I.74), (I.76), and (8), we obtain

$$Q_{\pm}^{(0,j+1)l''m''} = \frac{j+1}{j} \cdot \frac{2}{H^B} \sum_{lm} [D^F_{\nu}, Q_{\pm}^{(0,j)lm}] D^{B\nu}_{(0,j)lm,(0,j+1)l''m''},$$

whence equalities (21) immediately follow.

We now prove that all equations of system (19) are fulfilled if equalities (21) and Eqs. (20) are fulfilled.

We first verify that the relation

$$\sum_{lm} \left(D^{B\nu_1} D^{B\nu_2} \ldots D^{B\nu_n}\right)_{(0,1)00,(0,n+1)lm} \left(D^{B\mu_n} \ldots D^{B\mu_2} D^{B\mu_1}\right)_{(0,n+1)lm,(0,1)00} =$$

$$= s_0(n) \sum g^{\nu_1\mu_{j_1}} g^{\nu_2\mu_{j_2}} \ldots g^{\nu_n\mu_{j_n}} + s_1(n) \sum g^{\nu_{i_1}\nu_{i_2}} g^{\mu_{j_1}\mu_{j_2}} g^{\nu_{i_3}\mu_{j_3}} \ldots g^{\nu_{i_n}\mu_{j_n}} + \ldots \tag{23}$$

holds. The first sum in the right side of this equality is taken over all permutations of the indices $\mu_1, \mu_2, \ldots, \mu_n$. The second sum is taken over all combinations of two indices ν_{i_1} and ν_{i_2} extracted from the collection $\nu_1, \nu_2, \ldots, \nu_n$, over all combinations of two indices μ_{j_1} and μ_{j_2} extracted from the collection $\mu_1, \mu_2, \ldots, \mu_n$, and over the permutations of all remaining indices μ_j with some fixed order of the remaining indices ν_i and so on.

Let $a^i_{\nu_i}$ and $b^j_{\mu_j}$, where $i, j = 1, \ldots, n$, be the covariant components of arbitrary four-vectors. We multiply the left side of formula (23) by $a^1_{\nu_1} \ldots a^n_{\nu_n} b^n_{\mu_n} \ldots b^1_{\mu_1}$ and take the sum over $\nu_1, \ldots, \nu_n, \mu_n, \ldots, \mu_1$. The obtained expression $\mathcal{A}$ is invariant under the proper Lorentz group transformations. According to the first main theorem of the invariant theory [8], the expression $\mathcal{A}$ is a function of the standard basic invariants for the group $L_+^{\uparrow}$ that are the scalar product of two four-vectors $(p^1 p^2) \equiv g^{\mu\nu} p^1_{\mu} p^2_{\nu}$ and the component determinant of four four-vectors $[p^1 p^2 p^3 p^4] \equiv \varepsilon^{\mu\nu\rho\sigma} p^1_{\mu} p^2_{\nu} p^3_{\rho} p^4_{\sigma}$. Because different components of the operator $D^{B\mu}$ commute with each other in the considered variant of Lagrangian (3), the expression $\mathcal{A}$ remains unchanged under a permutation of any two vectors $a^i_{\nu_i}$ and $a^{i'}_{\nu_{i'}}$ or any two vectors $b^j_{\mu_j}$ and $b^{j'}_{\mu_{j'}}$. Consequently, the expression $\mathcal{A}$ cannot contain the component determinants. Taking the arbitrariness of the auxiliary four-vectors $a^i_{\nu_i}$ and $b^j_{\mu_j}$ into account, we thus verify the validity of relation (23).

We mutipy both sides of equality (23) by $g_{\nu_1\nu_2}$ and sum over the indices ν_1 and ν_2. As a result, the left side of the obtained equality vanishes according to relation (5). We thus have

$$s_1(n) = -s_0(n)/n. \tag{24}$$

We multiply both sides of equality (23) by $g_{\nu_n\mu_n}$ and sum over indices ν_n and μ_n. Using formulas (5), (6), (I.74), (I.76), and (8), we obtain

$$(n+3)s_0(n) + (n-1)s_1(n) = \frac{n+1}{n} \cdot \frac{H^B}{2} s_0(n-1).$$

Using relation (24) and the equality $s_0(1) = H^B/4$, we thus obtain

$$s_0(n) = \frac{1}{(n+1)!}\left(\frac{H^B}{2}\right)^n. \tag{25}$$

We now multiply both sides of equality (21) by $(D^{B\mu_n}\dots D^{B\mu_2}D^{B\mu_1})_{(0,n+1)lm,(0,1)00}$ and use relations (23)–(25), commutativity (12) of different components of the operator $D^{F\mu}$, formula (7), and condition (20), which the operators $Q_\pm$ satisfy. We have

$$\begin{aligned}
&[D^{F\mu_n}, \dots, [D^{F\mu_2}, [D^{F\mu_1}, Q_\pm]] \dots] \\
&-\frac{H^B}{2n}\sum g^{\mu_i\mu_j}[D^{F\mu_n}, \dots, [D^{F\mu_2}, [D^{F\mu_1}, Q_\pm]] \dots]_{\mu_i\mu_j} + \dots = \\
&= \sum_{l,m} Q_\pm^{(0,n+1)lm}\left(D^{B\mu_n}\dots D^{B\mu_2}D^{B\mu_1}\right)_{(0,n+1)lm,(0,1)00}.
\end{aligned} \tag{26}$$

Here and in what follows, $[D^{F\mu_n}, \dots, [D^{F\mu_1}, Q_\pm] \dots]_{\mu_{i_1}\cdots\mu_{i_k}}$ is the $(n-k)$-fold commutator obtained from the n-fold commutator $[D^{F\mu_n}, \dots, [D^{F\mu_1}, Q_\pm] \dots]$ by deleting the operators $D^{F\mu_{i_1}}, \dots, D^{F\mu_{i_k}}$. The first sum in the left side of equality (26) is taken over all combinations of two indices μ_i and μ_j extracted from the collection $\mu_1, \mu_2, \dots, \mu_n$, and so on.

We also need one more formula obtained from relations (5), (6), (I.74), (I.76), and (8), namely,

$$\begin{aligned}
&(D^{B\nu_n}\dots D^{B\nu_2}D^{B\nu_1})_{(0,n+1)l'm',(0,1)00}(D^B_{\nu_1}D^B_{\nu_2}\dots D^B_{\nu_n})_{(0,1)00,(0,n+1)lm} = \\
&= \frac{1}{n+1}\left(\frac{H^B}{2}\right)^n \delta_{ll'}\delta_{mm'}.
\end{aligned} \tag{27}$$

Successively using equalities (21), (26), (7), (12), (27), and again (21), we now obtain the chain of equalities

$$\begin{aligned}
&[D^{F\mu}, Q_\pm^{(0,n+1)lm}] = \\
&= (n+1)\left(\frac{2}{H^B}\right)^n [D^{F\mu}, [D^F_{\nu_n}, \dots, [D^F_{\nu_1}, Q_\pm] \dots]] \left(D^{B\nu_1}\dots D^{B\nu_n}\right)_{(0,1)00,(0,n+1)lm} = \\
&= \left\{(n+1)\left(\frac{2}{H^B}\right)^n \sum_{l'm'} Q_\pm^{(0,n+2)l'm'}\left(D^{B\mu}D^B_{\nu_n}\dots D^B_{\nu_1}\right)_{(0,n+2)l'm',(0,1)00}\right.
\end{aligned}$$

$$+\left(\frac{2}{H^B}\right)^{n-1}\sum_{i=1}^{n}\delta^{\mu}_{\nu_i}[D^F_{\nu_n},\ldots,[D^F_{\nu_1},Q_{\pm}]\ldots]_{\nu_i}\Bigg\}\left(D^{B\nu_1}\ldots D^{B\nu_n}\right)_{(0,1)00,(0,n+1)lm}=$$
$$=\sum_{l',m'}\left(Q_{\pm}^{(0,n+2)l'm'}D^{B\mu}_{(0,n+2)l'm',(0,n+1)lm}+Q_{\pm}^{(0,n)l'm'}D^{B\mu}_{(0,n)l'm',(0,n+1)lm}\right),$$

which ultimately reproduces the equalities of system (19) corresponding to variant 1 of Lagrangian (3). We note that those terms in formulas (23) and (26) that are not written explicity and a part of the terms in the sum before the dots contribute zero to the final result of the chain of equalities written above. This is due to the appearing of expressions $g_{\nu_i\nu_j}(D^{B\nu_1}\ldots D^{B\nu_n})_{(0,1)00,(0,n+1)lm}$ whose factors reduce to the form $\sum_{l'm'} g_{\mu\nu}D^{B\mu}_{(0,1)00,(0,2)l'm'}\ D^{B\nu}_{(0,2)l'm',(0,3)l''m''}$ because of the commutativity of different components of the operator $D^{B\mu}$ given by (12). The vanishing of the last sum is ensured by relation (5). The proof of Proposition 1 is thus completed.

We write relation (20) as a system of equations in the quantities $q(l_0,l_1)$ defined by formula (17). Using formulas (5)–(8) and (I.73)–(I.76), we have

$$(k_1-l_0-1)(k_1+l_0)q^{\pm}(l_0+1,l_1)+(k_1-l_0)(k_1+l_0-1)q^{\pm}(l_0-1,l_1)$$
$$-(k_1-l_1-1)(k_1+l_1)q^{\pm}(l_0,l_1+1)-(k_1-l_1)(k_1+l_1-1)q^{\pm}(l_0,l_1-1)=$$
$$=z(l_1-l_0)(l_1+l_0)q^{\pm}(l_0,l_1), \tag{28}$$

where $z=2-H^B/H^F$ for all irreducible $L_+^{\uparrow}$-group representations (l_0,l_1) belonging to representation S^{k_1} given by (I.72) (k_1 is a half-integer, $k_1\geq 3/2$). In the variant under consideration, we have $H^B>0$ and $H^F>0$; therefore, $z\in(-\infty,2)$.

It is easy to see that nontrivial solutions of system of equations (18) and (28) exist and the general solution contains $k_1-1/2$ arbitrary constants, for which we can take the quantities $q(l_0,k_1)$ with $1/2\leq l_0\leq k_1-1$. Directly using Eqs. (18) and (28) as recursive relations can be preferable to using the explicit form of the quantities $q(l_0,l_1)$. Even in the simplest case, where $k_1=3/2$, the quantities $q(l_0,l_1)$ are expressed in terms of higher transcendental functions,

$$q^{\pm}\left(\frac{1}{2},\frac{3}{2}\right)=q_0^{\pm},\quad q^{\pm}\left(\frac{1}{2},l_1\right)=\frac{2q_0^{\pm}}{l_1^2-1/4}\left[C^2_{l_1-3/2}\left(\frac{z}{2}\right)\mp C^2_{l_1-5/2}\left(\frac{z}{2}\right)\right],\quad l_1\geq\frac{5}{2}, \tag{29}$$

where $q_0^{\pm}$ are arbitrary constants and $C_n^{\nu}(x)$ is the Gegenbauer polynomial [9]. Expressing the Gegenbauer polynomial in terms of the hypergeometric series $F(a,b;c;y)$, we can represent formulas (29) as

$$q^{\pm}\left(\frac{1}{2},2N-\frac{1}{2}\right)=q_0^{\pm}\frac{(-1)^{N+1}}{2N-1}\left[F\left(1-N,N+1;\frac{1}{2};\frac{z^2}{4}\right)\right.$$
$$\left.\pm(N-1)zF\left(2-N,N+1;\frac{3}{2};\frac{z^2}{4}\right)\right], \tag{30}$$
$$q^{\pm}\left(\frac{1}{2},2N+\frac{1}{2}\right)=q_0^{\pm}\frac{(-1)^{N}}{2N+1}\left[\pm F\left(1-N,N+1;\frac{1}{2};\frac{z^2}{4}\right)\right.$$
$$\left.-(N+1)zF\left(1-N,N+2;\frac{3}{2};\frac{z^2}{4}\right)\right], \tag{31}$$

where $N\geq 1$.

When passing from variant 1 of Lagrangian (3) to variant 1A, we replace the indices $\pm$ with the indices $\mp$ in the right side of Eqs. (19) and, if n is odd, also in the left side of Eqs. (21) and in the right side of relations (26). In the chain of equalities, we need to make such a replacement in the third and fourth links and, if n is odd, also in the second link. The system of equations for the quantities $q^{\pm}(l_0, l_1)$ are identical in variants 1 and 1A. According to relation (10), we have $H^B > 0$ and $H^F < 0$ in variant 1A; therefore, $z \in (2, +\infty)$.

In variants 1 and 1A, each representation S^{k_1} (k_1 is a half-integer, $k_1 \geq 3/2$) for the fermionic field is thus assigned nontrivial Lagrangian (3) containing $k_1 - 1/2$ arbitrary constants. The matrix elements of the operators $Q_{\pm}$ are found from Eqs. (18) and (28). The operators $Q_{\pm}^{(0,n+1)lm}$ are expressed in terms of the operators $Q_{\pm}$ using equalities of type (21).

4. The representation S^B for the bosonic field and the fermion-boson interaction Lagrangian

Proposition 2. *Let the transformation properties of the fields in Lagrangian* (3) *coincide with the ones listed for variant* 2. *Then a nontrivial Lagrangian* (3) *does not exist.*

Proof. The invariance of Lagrangian (3) under the secondary-symmetry transformations is provided by the fulfillment of conditions (19), in whose right side it is necessary to replace the indices $\pm$ with $\mp$. These conditions imply, first, Eqs. (20) for the operators $Q_{\pm}$ and, second, the relations

$$[G^{F\mu\nu}, Q_{\pm}^{\tau lm}] = \sum_{\tau' l' m'} Q_{\pm}^{\tau' l' m'} G^{B\mu\nu}_{\tau' l' m', \tau lm}. \tag{32}$$

From relations (32), we can obtain the analogue of Eqs. (22), in which the four-vector operators are replaced by the operators $G^{\mu\nu}$ and $G^{\rho\sigma}$. We multiply the obtained analogue of Eqs. (22) first by $g_{\mu\rho}g_{\nu\sigma}$ and then by $\varepsilon_{\mu\nu\rho\sigma}$. In both cases, we then take the sum over the indices μ, ν, ρ, and σ and take formulas (13), (7), (10), and (11) into account. Then two further pairs of equations,

$$G^{F\mu\nu} Q_{\pm} G^F_{\mu\nu} = [(H^B)^2 - 2(H^F)^2] Q_{\pm}, \tag{33}$$

$$\varepsilon_{\mu\nu\rho\sigma} G^{F\mu\nu} Q_{\pm} G^{F\rho\sigma} = 0, \tag{34}$$

are added to Eqs. (20) for the operator $Q_{\pm}$.

Using formulas (5), (6), (8)–(10), and (I.84)–(I.87), Eqs. (20), (33), and (34) are reduced to a system of equations for the quantities $q^{\pm}(l_0, l_1)$ in which $(l_0, l_1) \in S^F$, and the representation S^F is given by formula (I.83). If $l_0 + l_1$ is an even number, we have

$$\begin{aligned}(l_1 - l_0 - 1)q^{\pm}(l_0 + 1, l_1) + (l_1 - l_0 + 1)q^{\pm}(l_0 - 1, l_1) + (l_1 - l_0 + 1)q^{\pm}(l_0, l_1 + 1)\\ +(l_1 - l_0 - 1)q^{\pm}(l_0, l_1 - 1) = 2(2 - y)(l_1 - l_0)q^{\pm}(l_0, l_1),\end{aligned} \tag{35}$$

$$\begin{aligned}\frac{l_1 + l_0 - 2}{l_1 + l_0 - 1} q^{\pm}(l_0 - 1, l_1 - 1) + \frac{l_1 + l_0 + 2}{l_1 + l_0 + 1} q^{\pm}(l_0 + 1, l_1 + 1)\\ + \frac{2}{(l_1 + l_0 - 1)(l_1 + l_0 + 1)} q^{\pm}(l_0, l_1) = (2 - y^2) q^{\pm}(l_0, l_1),\end{aligned} \tag{36}$$

$$\begin{aligned}(l_1 - l_0 + 1)q^{\pm}(l_0 - 1, l_1 + 1) + (l_1 - l_0 - 1)q^{\pm}(l_0 + 1, l_1 - 1) =\\ = (2 - y^2)(l_1 - l_0) q^{\pm}(l_0, l_1);\end{aligned} \tag{37}$$

if $l_0 + l_1$ is an odd number, then

$$(l_1 + l_0 + 1)q^{\pm}(l_0 + 1, l_1) + (l_1 + l_0 - 1)q^{\pm}(l_0 - 1, l_1) + (l_1 + l_0 + 1)q^{\pm}(l_0, l_1 + 1)$$
$$+(l_1 + l_0 - 1)q^{\pm}(l_0, l_1 - 1) = 2(2 - y)(l_1 + l_0)q^{\pm}(l_0, l_1), \quad (38)$$
$$(l_1 + l_0 - 1)q^{\pm}(l_0 - 1, l_1 - 1) + (l_1 + l_0 + 1)q^{\pm}(l_0 + 1, l_1 + 1) =$$
$$= (2 - y^2)(l_1 + l_0)q^{\pm}(l_0, l_1), \quad (39)$$
$$\frac{l_1 - l_0 + 2}{l_1 - l_0 + 1}q^{\pm}(l_0 - 1, l_1 + 1) + \frac{l_1 - l_0 - 2}{l_1 - l_0 - 1}q^{\pm}(l_0 + 1, l_1 - 1)$$
$$+\frac{2}{(l_1 - l_0 - 1)(l_1 - l_0 + 1)}q^{\pm}(l_0, l_1) = (2 - y^2)q^{\pm}(l_0, l_1), \quad (40)$$

where $y = H^B/H^F$. In variant 2, the inequalities $H^B > 0$, $H^F < 0$, and $y < 0$ hold according to relation (10).

It is easy to verify that the system of Eqs. (18) and (35)–(40) has a nontrivial solution, if and only if $y = 2$ and this solution is

$$q^{\pm}(l_0, l_1) = \begin{cases} (-1)^{l_1 - 1}(l_1 + l_0)q_0^{\pm} & \text{for even } \ l_1 + l_0, \\ \pm(-1)^{l_1 - 1}(l_1 + l_0)q_0^{\pm} & \text{for odd } \ l_1 + l_0, \end{cases} \quad (41)$$

where $q_0^{\pm}$ are arbitrary constants.

Therefore, nontrivial solutions for the system of Eqs. (18) and (35)–(40) in the quantities $q^{\pm}(l_0, l_1)$ do not exist in the domain of admissible values of the parameter y ($y \in (-\infty, 0)$), i.e., there are no nonzero operators $Q_{\pm}$ in variant 2 of Lagrangian (3). It follows from the conditions of type (19) that any operator $Q_{\pm}^{\tau lm}$ ($\tau \in S^B$) is linearly and homogeneously expressed in terms of the operators $Q_{\pm}$. Consequently, all the operators $Q_{\pm}^{\tau lm}$ in the considered variant are zero, and interaction Lagrangian (3) is trivial.

5. An example of the mass spectrum in the infinite-component field theory with the double symmetry

Announcing the completion of the first stage of the construction of the theory of the infinite-component ISFIR-class fields, we demonstrate some properties of the mass spectrum of the fermionic states in such a theory with one example without details of proofs.

Let a free fermionic field theory be assigned the description given in Corollary 2 (point 2) in [1] with $k_1 = 3/2$, and let the mass term of its Lagrangian arises from interaction Lagrangian (3) in variant 1A as a result of a spontaneous secondary-symmetry breaking.

The state vector ψ_M of a particle of mass M in its rest frame must satisfy a relativistically invariant equation (of Gelfand–Yaglom type (I.1)) taking the form

$$(M\Gamma^0 - R)\psi_M = 0 \quad (42)$$

and the normalization condition

$$|\overline{\psi_M}\Gamma^0\psi_M| < +\infty \quad (43)$$

if M is a discrete point of the mass spectrum or

$$\overline{\psi_{M'}}\Gamma^0\psi_M = a_0\delta(M - M') \quad (44)$$

(a_0 is some number) if M and M' belong to the continuous part of the mass spectrum.

We find the operator Γ^0 in Eq. (42) using relations (I.18)–(I.23), (I.79), and (I.80). We define the operator R by formula (4), where we set $\kappa = 0$ and use relation (17) for the operator $Q_+^{(0,1)00}$.

Because the operator Γ^0 is diagonal with respect to the spin index l and the spin-projection index m, the vector states ψ_M are assigned certain values of spin and its projection. We let J denote the spin of the particle in its rest frame. Because, according to condition 4 in [1], the operators Γ^0 is Hermitian and its matrix elements in the case considered are real, the state vectors ψ_M satisfying Eq. (42) possess the certain spatial parity $P\psi_M = \pm\psi_M$, i.e., $(\psi_M)_{(-1/2,l_1)Jm} = \pm(\psi_M)_{(1/2,l_1)Jm}$. Taking the abovementioned into account, we write relation (42) as a system of equations in the components $\psi(l_1) \equiv (\psi_M)_{(1/2,l_1)Jm}$ of the state vector with positive parity. We have

$$\left(l_1 - \frac{1}{2}\right)\sqrt{(l_1+J+1)(l_1-J)}\psi(l_1+1) + \left[(-1)^{l_1+\frac{1}{2}}2l_1\left(J+\frac{1}{2}\right)\right.$$

$$\left.+\frac{2\lambda}{Mc_0}\left(l_1^2 - \frac{1}{4}\right)q^+(\frac{1}{2},l_1)\right]\psi(l_1) + \left(l_1+\frac{1}{2}\right)\sqrt{(l_1+J)(l_1-J-1)}\psi(l_1-1) = 0, \quad (45)$$

where $J \geq l_1 - 1$ and $l_1 \geq 3/2$. We calculate the quantities $q^+(1/2, l_1)$ using recursive relations (18) and (28) with $z > 2$.

Using the system of equation (18) and (28), we find the asymptotic behavior as $l_1 \to +\infty$,

$$q^+(\frac{1}{2},l_1) = r_0\frac{u^{l_1-\frac{1}{2}}}{l_1-\frac{1}{2}}\left(1+\mathcal{O}\left(\frac{1}{l_1}\right)\right), \quad (46)$$

where $u = (z+\sqrt{z^2-4})/2$ and r_0 is a constant depend on the parameter z.

Using Eqs. (45), we next find the asymptotic behavior of the quantity $\psi(l_1)$ as $l_1 \to +\infty$. We obtain

$$\psi(l_1) = r_1\left(-\frac{2\lambda r_0}{Mc_0}\right)^{l_1-\frac{1}{2}}\frac{u^{[\frac{1}{2}(l_1-\frac{1}{2})(l_1-\frac{3}{2})]}}{(l_1-\frac{3}{2})!}\left(1+\mathcal{O}\left(\frac{1}{l_1}\right)\right)$$

$$+r_2\left(-\frac{Mc_0}{2\lambda r_0}\right)^{l_1-\frac{1}{2}}\frac{(l_1-\frac{3}{2})!}{u^{[\frac{1}{2}(l_1-\frac{1}{2})(l_1-\frac{3}{2})]}}\left(1+\mathcal{O}\left(\frac{1}{l_1}\right)\right), \quad (47)$$

where r_1 and r_2 is constants dependent on the quantities Mc_0/λ and z as on parameters.

If the constant r_1 does not vanish for some value of the quantity M, then the corresponding solution of system of equations (45) in the quantities $\psi(l_1)$ evidently satisfies neither normalization condition (43) nor condition (44). If $r_1 = 0$, then the solution of system (45) satisfies normalization condition (43), i.e., this solution is the state vector of the particle, and the corresponding quantity M is its mass. Therefore, if the mass spectrum in the considered variant of the theory is not empty for some set of z-parameter values from the interval $z \in (2, +\infty)$, then the spectrum is discrete for all values of z from this set.

The normalized solutions of system of equations (45) can hardly be expressed in terms of some known special functions. We therefore use numerical methods that allow finding any number of the least values of the mass M and of the initial components $\psi(l_1)$ of the corresponding state vectors of the particles. We choose the value and the sign of the parameter $\lambda q_0^+/c_0$ such that the least-enegy state has the positive spatial parity and its mass is equal to unity. For $z = 3.0$, the least masses M and the corresponding spin and parity J^P of the particles then have the values presented in Table 1.

Table 1

J^P	$\frac{1}{2}^+$	$\frac{1}{2}^-$	$\frac{3}{2}^+$	$\frac{3}{2}^-$	$\frac{5}{2}^+$	$\frac{5}{2}^-$	$\frac{7}{2}^+$	$\frac{7}{2}^-$	$\frac{9}{2}^+$	$\frac{9}{2}^-$	$\frac{11}{2}^+$	$\frac{11}{2}^-$
M	1.00	1.99	1.59	3.44	2.95	6.71	5.97	14.0	12.7	30.4	28.1	68.0
	4.16	9.04	7.68	17.5	15.5	36.5	33.1	79.3	73.2	178.	166.	406.
	20.2	46.1	40.6	95.5	86.6	207.	191.	464.	433.			
	107.	251.	227.	544.	501.							
	597.											

This example allows making the following conclusion related to the constructed theory of the infinite-component ISFIR-class fields. There exist relativistically invariant equations of Gelfand–Yaglom type (I.1) such that, first, their mass spectrum has no continuous part, and, second, each value of spin and parity is assigned a countable set of masses that is not bounded from above, and, third, the minimum value of mass for the given spin grows infinitely with the spin. These conclusions qualitatively correspond to the picture that is provided by the parton model of hadrons together with the concept of bags for quarks and gluons. A more detailed analysis of the mass spectra in the constucted infinite-component field theory with the double symmetry and their relation to the real baryon spectra is the subject of our subsequent paper.

Acknowledgments. The author is grateful to A.U. Klimyk, A.A. Komar, and V.I. Fushchich for useful discussions of the problems considered in this paper.

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