# Particle Weights and their Disintegration I

Martin Porrmann
II. Institut für Theoretische Physik, Universität Hamburg
Luruper Chaussee 149, D-22761 Hamburg, Germany
e-mail: martin.porrmann@desy.de

November 6, 2002

**Abstract**

The notion of Wigner particles is attached to irreducible unitary representations of the Poincaré group, characterized by parameters $m$ and $s$ of mass and spin, respectively. However, the Lorentz symmetry is broken in theories with long-range interactions, rendering this approach inapplicable (infraparticle problem). A unified treatment of both particles and infraparticles via the concept of particle weights can be given within the framework of local quantum physics. They arise as temporal limits of physical states in the vacuum sector and describe the asymptotic particle content. In this paper their definition and characteristic properties, already presented in [9] and [14], are worked out in detail. The existence of the temporal limits is established by use of suitably defined seminorms which are also essential in proving the characteristic features of particle weights.

## 1 Introduction

Physical phenomena in high energy physics are analyzed in terms of particles arising as asymptotic configurations of elementary localized entities in scattering experiments. These particles are characterized by certain intrinsic properties expressed by quantum numbers; their consistent and complete theoretical explanation is a goal of quantum field theory. The usual theoretical description of particles goes back to Wigner's classical analysis of irreducible unitary representations of the Poincaré group [32]. He gives a complete classification of these representations labelled by two parameters $m$ and $s$. It is standard to assume that states of an elementary particle pertain to one of these, interpreting the corresponding parameters as the intrinsic mass and spin, respectively, of the particle. However, this approach is not universally applicable, for there are quantum field theoretic models allowing for particles of zero rest mass, where states of particles coupled to these cannot be described in terms of eigenstates of the mass operator. An example is quantum electrodynamics, where charged particles are inevitably accompanied by soft photons. It is an open question, known as the infraparticle problem [29], how mass and spin are to be described in this situation. Moreover, standard collision theory does not work either. The present paper describes an alternative approach to the concept of particles within the framework of local quantum physics [15, 14]. The fundamental ideas and results are due to Buchholz and have first been presented in [6]. Concepts and statements may be familiar from [9] (cf. also [14, Section VI.2.2]); the more detailed elaboration below is the result of joint work of Detlev Buchholz and the author.

The particle concept to be set forth is motivated by the experimental situation encountered in high energy physics, where certain entities show up as *particles*, being traced by specific measuring devices called *detectors*. The invariant characteristic of these physical systems in different circumstances is their being localized in the interaction with a detector. In 1967 Araki and Haag [1] presented an analysis of the particle content of scattering states for *massive* models, investigating the asymptotics of physical states by means of certain operators to be interpreted as counters. These are assumed to be well localized and, at the same time, to annihilate the vacuum in order to be insensitive to it. Due to the Reeh-Schlieder-Theorem, the second requirement is incompatible with strict locality [28] (cf. also [14, Theorem 5.3.2]). The appropriate localization concept for detectors turns out to be *almost locality* (or *quasilocality of infinite order* [1]) meaning that their dislocalized part (outside a finite region of radius $r$) falls off more rapidly than any power of $r$. Within this setting, Araki and Haag arrived at a decomposition of scattering states in terms of energy-momentum eigenstates. They got to the following asymptotic relation which holds true for quasilocal operators satisfying $C\Omega = C^*\Omega = 0$, $\Omega$ the vacuum state, and for arbitrary vectors $\Phi$ and certain specific vectors $\Psi$ representing outgoing particle configurations [1, Theorem 4]:

$$\lim_{t\to\infty} \langle \Phi | t^3 C(h,t) | \Psi \rangle = \sum_{i,j}{}' \int d^3p \, \Gamma_{ij}(\boldsymbol{p}) \langle \Phi | a_j^{\dagger\text{out}}(\boldsymbol{p}) \, a_i^{\text{out}}(\boldsymbol{p}) | \Psi \rangle \, h(\boldsymbol{v}_i). \tag{1.1}$$

Here $\Gamma_{ij}(\boldsymbol{p}) \doteq 8\pi^3 \langle \boldsymbol{p}\, j | C(0) | \boldsymbol{p}\, i \rangle$, and $h$ denotes a function of the velocity that is defined by $\boldsymbol{v}_i \doteq (\boldsymbol{p}^2 + m_i^2)^{-1/2}\boldsymbol{p}$. The indices $i$ and $j$ in the above formula denote the particle types, characterized by intrinsic quantum numbers including spin, and summation runs over pairs of particles with equal mass: $m_i = m_j$. The structure of the right-hand side of this equation is based on the *a priori* knowledge of the particle content of the massive theory.

Our final goal is to develop a result similar to (1.1) in a model-independent framework without excluding massless states. To this end a considerably smaller class of operators is treated as representatives of particle detectors. It is based on the observation that, to produce a signal, a minimal energy is needed which depends on the characteristics of the detector. Henceforth, the relevant operators are required to annihilate all physical states with bounded energy below a specific threshold. Again, on account of the Reeh-Schlieder-Theorem, this feature is incompatible with strict locality. The construction of the algebra of counters thus starts with operators transferring energy-momenta appropriate to enforce insensitivity below a certain energy bound. Among these the subset of almost local ones is selected to localize the states. Upon complementing these requirements by an assumption of smoothness with respect to Poincaré transformations, the actual definition of detectors includes the possibility to perform measurements after localization. Passing to the limit of asymptotic times in investigating physical states of bounded energy by means of these particle counters, one arrives at linear functionals $\sigma$ on the algebra of detectors which are continuous with respect to suitable topologies and exhibit properties of singly localized systems to be interpreted as particles. They give rise to certain specific sesquilinear forms on the space of localizing operators, the *particle weights*. Having thus established a result corresponding to the left-hand side of (1.1), we are faced with the problem of decomposing the asymptotic functional $\sigma$ in terms of energy-momentum eigenstates as suggested by the right-hand side. In the corresponding reformulation of (1.1) all expressions occurring in the integrand apart from $\Gamma_{ij}$ are absorbed into measures $\mu_{i,j}$, so that the asymptotic functional $\sigma$ is represented as a mixture of linear forms on the algebra of detectors with Dirac kets representing improper momentum eigenstates:

$$\sigma(C) = \sum_{i,j}{}' \int d\mu_{i,j}(\boldsymbol{p}) \, \langle \boldsymbol{p}\, j | C(0) | \boldsymbol{p}\, i \rangle. \tag{1.2}$$

A result of this kind can indeed be established in terms of the corresponding sesquilinear forms (particle weights). Here the assumption of smoothness with respect to space-time translations imposed on the localizing operators turns out to be vital to render the concept of particle weights stable in the course of disintegration.

The structure of this article is as follows: Section 2 develops the concept of detectors and investigates suitable topologies. A basic ingredient is the interplay between locality and the spectrum condition. In Section 3 the resulting continuous functionals in the dual space are analyzed. Then, on physical grounds, a certain subclass is distinguished, arising as asymptotic limits of functionals constructed from physical states of bounded energy. These limits are to be interpreted as representing asymptotic particle configurations. A characteristic of the limiting procedure is its ability to directly reproduce charged systems, in contrast to the LSZ-theory where charge-carrying unobservable operators are necessary. The representations induced by these asymptotic functionals are highly reducible. Their disintegration in terms of irreducible representations corresponding to *pure* particle weights will be presented in a second paper, where a formula analogous to equation (1.2) will be a central result. To facilitate reading, proofs have been deferred to Sections 4 and 5. The Conclusions (Section 6) put this approach into proper place and list topics of further study.

# 2 Localizing Operators and Spectral Seminorms

## 2.1 The Algebra of Detectors

We start this section by giving the exact definitions underlying the concept of detectors to be used in the sequel. As mentioned in the Introduction, a detector should annihilate states of bounded energy below a certain threshold. This feature is implemented by the following definition.

**Definition 2.1** (Vacuum Annihilation Property)**.** An operator $A \in \mathfrak{A}$ has the *vacuum annihilation property*, if, in the sense of operator-valued distributions, the mapping

$$\mathbb{R}^{s+1} \ni x \mapsto \alpha_x(A) \doteq U(x) A U(x)^* \in \mathfrak{A} \tag{2.1}$$

has a Fourier transform with compact support $\Gamma$ contained in the complement of the forward light cone $\overline{V}_+$. The collection of all vacuum annihilation operators is a subspace $\mathfrak{A}_{ann}$ of $\mathfrak{A}$.

The support of the Fourier transform of (2.1) is precisely the energy-momentum transfer of $A$, and the energy threshold for the annihilation of states depends on the distance $d(\Gamma, \overline{V}_+)$ between $\Gamma$ and $\overline{V}_+$ and the position of $\Gamma$.

A second property of detectors should be their localization in spacetime. Strict locality being incompatible with the vacuum annihilation property, we confine ourselves to almost locality.

**Definition 2.2** (Almost Locality)**.** Let $\mathscr{O}_r \doteq \{(x^0, \boldsymbol{x}) \in \mathbb{R}^{s+1} : |x^0| + |\boldsymbol{x}| < r\}$, $r > 0$, denote the double cone (standard diamond) with basis $\boldsymbol{O}_r \doteq \{\boldsymbol{x} \in \mathbb{R}^s : |\boldsymbol{x}| < r\}$. An operator $A \in \mathfrak{A}$ is called *almost local*, if there exists a net $\{A_r \in \mathfrak{A}(\mathscr{O}_r) : r > 0\}$ of local operators such that

$$\lim_{r \to \infty} r^k \|A - A_r\| = 0 \tag{2.2}$$

for any $k \in \mathbb{N}_0$. The set of almost local operators is a $^*$-subalgebra $\mathfrak{A}_{\mathscr{S}}$ of $\mathfrak{A}$.

The spaces $\mathfrak{A}_{ann}$ and $\mathfrak{A}_{\mathscr{S}}$ are both invariant under Poincaré transformations $(\Lambda, x) \in \mathsf{P}_+^\uparrow$; the operator $\alpha_{(\Lambda,x)}(A)$ transfers energy-momentum in $\Lambda\Gamma$ which again belongs to the complement of $\overline{V}_+$ like the support $\Gamma$ pertaining to $A$ itself.

The construction of a subalgebra $\mathfrak{C}$ in $\mathfrak{A}$ containing those self-adjoint operators to be interpreted as representatives of particle detectors is now accomplished in three steps (Definitions 2.3–2.5), where the above characteristics are supplemented by requiring smoothness with respect to the Poincaré group.

**Definition 2.3.** The almost local vacuum annihilation operators $L_0$ with the accessory property that the mapping

$$\mathsf{P}_+^\uparrow \ni (\Lambda, x) \mapsto \alpha_{(\Lambda,x)}(L_0) \in \mathfrak{A} \tag{2.3}$$

is infinitely often differentiable with respect to the initial topology of the Poincaré group $\mathsf{P}_+^\uparrow$ and the norm topology of $\mathfrak{A}$ constitute a subspace of $\mathfrak{A}_{\mathscr{S}} \cap \mathfrak{A}_{ann}$. The additional requirement that all partial derivatives of any order be again almost local distinguishes a *vector space* $\mathfrak{L}_0 \subset \mathfrak{A}$.

*Remark.* (i) Partial derivatives of any order of (2.3) correspond to vacuum annihilation operators inheriting their energy-momentum transfer from $L_0$. But they need not necessarily be almost local. By virtue of the last condition, the space $\mathfrak{L}_0$ is stable not only under Poincaré transformations but also under differentiation.

(ii) A huge number of elements of $\mathfrak{L}_0$ can be constructed by regularizing almost local operators with respect to rapidly decreasing functions on the Poincaré group furnished with the Haar measure $\mu$. (Note, that the semi-direct product Lie group $\mathsf{P}_+^\uparrow = \mathsf{L}_+^\uparrow \ltimes \mathbb{R}^{s+1}$ is unimodular [23, Proposition II.29 and Corollary], since $\mathsf{L}_+^\uparrow$ is a simple thus semisimple Lie group [16, Proposition I.1.6]). For $A \in \mathfrak{A}_{\mathscr{S}}$ the operator

$$A(F) \doteq \int d\mu(\Lambda,x)\, F(\Lambda,x)\, \alpha_{(\Lambda,x)}(A) \tag{2.4}$$

belongs to $\mathfrak{L}_0$ and transfers energy-momentum in the compact set $\Gamma \subseteq \complement\overline{V}_+$, if the infinitely often differentiable function $F$ is rapidly decreasing on the subgroup $\mathbb{R}^{s+1}$ and compactly supported on $\mathsf{L}_+^\uparrow$ (i. e., $F \in \mathscr{S}_0(\mathsf{P}_+^\uparrow) = \mathscr{S}_0(\mathsf{L}_+^\uparrow \ltimes \mathbb{R}^{s+1})$, cf. [4]) and has the additional property that the Fourier transforms of the partial functions $F_\Lambda(\,.\,) \doteq F(\Lambda,\,.\,)$ have common support in $\Gamma$ for any $\Lambda \in \mathsf{L}_+^\uparrow$.

The following definition specifies a left ideal $\mathfrak{L}$ of the algebra $\mathfrak{A}$.

**Definition 2.4.** Let $\mathfrak{L}$ denote the linear span of all operators $L \in \mathfrak{A}$ of the form $L = A\,L_0$ where $A \in \mathfrak{A}$ and $L_0 \in \mathfrak{L}_0$; i. e.,

$$\mathfrak{L} \doteq \mathfrak{A}\,\mathfrak{L}_0 = \operatorname{span}\{A\,L_0 : A \in \mathfrak{A}, L_0 \in \mathfrak{L}_0\}.$$

Then $\mathfrak{L}$ is a left ideal of $\mathfrak{A}$, called the *left ideal of localizing operators*.

By its very construction, $L \in \mathfrak{L}$ annihilates the vacuum and all states of bounded energy below a certain threshold which is determined by the minimal threshold associated with a vacuum annihilation operator occurring in any of the representations $L = \sum_{i=1}^N A_i\,L_i$, $A_i \in \mathfrak{A}$, $L_i \in \mathfrak{L}_0$. The elements of $\mathfrak{L}$ are used to define that algebra of operators of which the self-adjoint elements are to be interpreted as representatives of particle detectors.

**Definition 2.5.** Let $\mathfrak{C}$ denote the linear span of all operators $C \in \mathfrak{A}$ which can be represented in the form $C = L_1{}^* L_2$ with $L_1$, $L_2 \in \mathfrak{L}$; i. e.,

$$\mathfrak{C} \doteq \mathfrak{L}^*\,\mathfrak{L} = \operatorname{span}\{L_1{}^* L_2 : L_1, L_2 \in \mathfrak{L}\}.$$

Then $\mathfrak{C}$ is a $^*$-subalgebra of $\mathfrak{A}$, called the *algebra of detectors*.

This algebra is smaller than that used by Araki and Haag [1]. It is neither closed in the uniform topology of $\mathfrak{A}$ nor does it contain a unit. Note that $\mathfrak{L}$ as well as $\mathfrak{C}$ are stable under Poincaré transformations.

## 2.2 Spectral Seminorms on the Algebra of Detectors

Triggering a detector $C \in \mathfrak{C}$ requires a minimal energy $\varepsilon$ to be deposited. Take a state $\omega$ of bounded energy $E$, then we expect to encounter a finite number of localization centers, their number being equal to or less than $E/\varepsilon$. According to this heuristic picture, placing the counter $C$ at every point $\boldsymbol{x}$ in space $\mathbb{R}^s$ at given time $t$ and adding up the corresponding expectation values $\omega(\alpha_{(t,\boldsymbol{x})}(C))$ should result in finiteness of the integral

$$\int_{\mathbb{R}^s} d^s x \, \big|\omega\big(\alpha_{(t,\boldsymbol{x})}(C)\big)\big| < \infty. \tag{2.5}$$

As a matter of fact, the elements of $\mathfrak{C}$ turn out to have this property as can be shown by using results of Buchholz [7]. The essential ingredient here is the interplay between spatial localization and energy bounds, hinting at the importance of phase-space properties of quantum field theory.

**Proposition 2.6.** *Suppose that $\Delta \subseteq \mathbb{R}^{s+1}$ is a bounded Borel set.*
*(i) Let $L \in \mathfrak{L}$ be arbitrary, then the net $\Big\{E(\Delta) \int_K d^s x \, \alpha_{\boldsymbol{x}}(L^* L) \, E(\Delta) : K \subseteq \mathbb{R}^s \text{compact}\Big\}$ of operator-valued Bochner integrals converges $\sigma$-strongly for $K \nearrow \mathbb{R}^s$, its limit being the $\sigma$-weak integral $\int_{\mathbb{R}^s} d^s x \, E(\Delta)\alpha_{\boldsymbol{x}}(L^* L)E(\Delta)$.*
*(ii) Let $C$ be an arbitrary detector in $\mathfrak{C}$, then the net of operator-valued Bochner integrals $\Big\{E(\Delta) \int_K d^s x \, \alpha_{\boldsymbol{x}}(C) \, E(\Delta) : K \subseteq \mathbb{R}^s \text{compact}\Big\}$ is $\sigma$-strongly convergent in the limit $K \nearrow \mathbb{R}^s$, and its limit is the $\sigma$-weak integral $\int_{\mathbb{R}^s} d^s x \, E(\Delta)\alpha_{\boldsymbol{x}}(C)E(\Delta)$. Furthermore,*

$$\sup\Big\{\int_{\mathbb{R}^s} d^s x \, \big|\phi\big(E(\Delta)\alpha_{\boldsymbol{x}}(C)E(\Delta)\big)\big| : \phi \in \mathscr{B}(\mathscr{H})_{*,1}\Big\} < \infty. \tag{2.6}$$

Relation (2.6) constitutes the sharpened version of (2.5) which was based on heuristic considerations.

The preceding result suggests the introduction of topologies on the left ideal $\mathfrak{L}$ and on the $^*$-algebra $\mathfrak{C}$, respectively, using specific seminorms indexed by bounded Borel subsets $\Delta$ of $\mathbb{R}^{s+1}$. By their very construction, these are especially well-adapted to the problems at hand.

**Definition 2.7.** (a) The left ideal $\mathfrak{L}$ is equipped with a family of seminorms $q_\Delta$ via

$$q_\Delta(L) \doteq \Big\| \int_{\mathbb{R}^s} d^s x \, E(\Delta)\alpha_{\boldsymbol{x}}(L^* L)E(\Delta) \Big\|^{1/2}, \quad L \in \mathfrak{L}. \tag{2.7a}$$

(b) The $^*$-algebra $\mathfrak{C}$ is furnished with seminorms $p_\Delta$ by assigning

$$p_\Delta(C) \doteq \sup\Big\{\int_{\mathbb{R}^s} d^s x \, \big|\phi\big(E(\Delta)\alpha_{\boldsymbol{x}}(C)E(\Delta)\big)\big| : \phi \in \mathscr{B}(\mathscr{H})_{*,1}\Big\}, \quad C \in \mathfrak{C}. \tag{2.7b}$$

(c) These families of seminorms give $\mathfrak{L}$ and $\mathfrak{C}$ the structure of locally convex Hausdorff spaces denoted $(\mathfrak{L}, \mathfrak{T}_q)$ and $(\mathfrak{C}, \mathfrak{T}_p)$, respectively.

An alternative expression for $q_\Delta$ can be given in terms of the positive cone $\mathscr{B}(\mathscr{H})_*^+$ in the pre-dual $\mathscr{B}(\mathscr{H})_*$:

$$q_\Delta(L)^2 = \sup\Big\{\int_{\mathbb{R}^s} d^s x \, \omega\big(E(\Delta)\alpha_{\boldsymbol{x}}(L^* L)E(\Delta)\big) : \omega \in \mathscr{B}(\mathscr{H})_{*,1}^+\Big\}. \tag{2.7c}$$

The seminorm properties of $q_\Delta$ and $p_\Delta$ are easily checked. That the corresponding families separate elements of $\mathfrak{L}$ and $\mathfrak{C}$, respectively, is established as follows: From the very definition of the seminorms $q_\Delta$ and $p_\Delta$ we infer that the conditions $q_\Delta(L) = 0$ and $p_\Delta(C) = 0$, $L \in \mathfrak{L}$, $C \in \mathfrak{C}$, by rendering the integrands in (2.7c) and (2.7b) identically zero, imply $LE(\Delta) = 0$ and $E(\Delta)CE(\Delta) = 0$ for any bounded Borel set $\Delta$, since $\mathscr{B}(\mathscr{H})_{*,1}^+$ as well as $\mathscr{B}(\mathscr{H})_{*,1}$ are separating sets of functionals for $\mathscr{B}(\mathscr{H})$. Now, states of bounded energy constitute a dense subspace of $\mathscr{H}$ so that in conclusion $L = 0$ and $C = 0$.

## 2.3 Characteristics of the Spectral Seminorms

The subsequent investigations very much depend on special properties of these seminorms. Here we present only the most important ones, deferring their proof as well as the formulation of a couple of important Lemmas to Section 4. Furthermore, we are, in the present context, not aiming at utmost generality of statements. A more elaborate discussion can be found in [27]. $\Delta$, with or without sub- or superscripts, generically denotes bounded subsets of the energy-momentum space $\mathbb{R}^{s+1}$. The first result concerns the net structure of the family of seminorms.

**Proposition 2.8.** *The families of seminorms $q_\Delta$ and $p_\Delta$ on $\mathfrak{L}$ and $\mathfrak{C}$, respectively, constitute nets with respect to the inclusion relation. For any $\Delta$ and $\Delta'$ we have*

$$\begin{aligned} \Delta \subseteq \Delta' \quad &\Rightarrow \quad q_\Delta(L) \leqslant q_{\Delta'}(L), \quad L \in \mathfrak{L}, \\ \Delta \subseteq \Delta' \quad &\Rightarrow \quad p_\Delta(C) \leqslant p_{\Delta'}(C), \quad C \in \mathfrak{C}. \end{aligned}$$

Of particular importance in connection with the characteristics of particle weights are the invariance of the seminorms with respect to spacetime translations as well as special properties of continuity and differentiability with respect to the Poincaré group.

**Proposition 2.9.** *Let $x \in \mathbb{R}^{s+1}$ be arbitrary, then*

*(i)* $q_\Delta\big(\alpha_x(L)\big) = q_\Delta(L)\,, \quad L \in \mathfrak{L}\,;$

*(ii)* $p_\Delta\big(\alpha_x(C)\big) = p_\Delta(C)\,, \quad C \in \mathfrak{C}\,.$

It is assumed that the automorphism group of Poincaré transformations acts strongly continuous on the $C^*$-algebra $\mathfrak{A}$, meaning that the mapping $(\Lambda, x) \mapsto \alpha_{(\Lambda,x)}(A)$ for given $A \in \mathfrak{A}$ is continuous with respect to the initial topology of $\mathsf{P}_+^\uparrow$ and the uniform topology of $\mathfrak{A}$. This characteristic is preserved in passing to the subspaces $\mathfrak{L}$ and $\mathfrak{C}$ with their respective locally convex topologies. For operators in $\mathfrak{L}_0$ even infinite differentiability is seen to hold with respect to $(\mathfrak{L}_0, \mathfrak{T}_q)$. The seminorm topologies not being finer than the norm topology, this result is not a corollary of strong continuity of the automorphism group.

**Proposition 2.10.** *(i) Given $L \in \mathfrak{L}$ and $C \in \mathfrak{C}$, the mappings*

$$\begin{aligned} \mathsf{P}_+^\uparrow \ni (\Lambda, x) &\mapsto \alpha_{(\Lambda,x)}(L) \in \mathfrak{L}, \\ \mathsf{P}_+^\uparrow \ni (\Lambda, x) &\mapsto \alpha_{(\Lambda,x)}(C) \in \mathfrak{C} \end{aligned}$$

*are continuous in the locally convex spaces $(\mathfrak{L}, \mathfrak{T}_q)$ and $(\mathfrak{C}, \mathfrak{T}_p)$.*

*(ii) Given $L_0 \in \mathfrak{L}_0$, the mapping*

$$\mathsf{P}_+^\uparrow \ni (\Lambda, x) \mapsto \alpha_{(\Lambda,x)}(L_0) \in \mathfrak{L}_0$$

*is infinitely often differentiable in the locally convex space $(\mathfrak{L}_0, \mathfrak{T}_q)$. Furthermore, its partial derivatives coincide with those arising from the presupposed differentiability of this mapping with respect to the uniform topology (cf. Definition 2.3).*

# 3 Particle Weights as Asymptotic Plane Waves

We now turn to the investigation of the topological dual space of $(\mathfrak{C}, \mathfrak{T}_p)$.

**Definition 3.1.** (a) The linear functionals on $\mathfrak{C}$, which are continuous with respect to the seminorm $p_\Delta$, constitute a normed vector space $\mathfrak{C}_\Delta^*$ via

$$\|\varsigma\|_\Delta \doteq \sup\big\{|\varsigma(C)| : C \in \mathfrak{C}, p_\Delta(C) \leqslant 1\big\}, \quad \varsigma \in \mathfrak{C}_\Delta{}^*.$$

(b) The topological dual of the locally convex space $(\mathfrak{C}, \mathfrak{T}_p)$ is denoted $\mathfrak{C}^*$.

Due to the net property of the family of seminorms $p_\Delta$ (Proposition 2.8), a linear functional on $\mathfrak{C}$ belongs to the topological dual $\mathfrak{C}^*$ if and only if it is continuous with respect to a specific one of these seminorms [21, Proposition 1.2.8]. Hence $\mathfrak{C}^*$ is the union of all the spaces $\mathfrak{C}_\Delta^*$.

### 3.1 General Properties

Before proceeding to extract certain elements from $\mathfrak{C}^*$ to be interpreted as representing asymptotic mixtures of particle-like entities, we are first going to collect a number of properties common to *all* functionals from the topological dual of $\mathfrak{C}$. First of all, continuity as established in Proposition 2.10 directly carries over.

**Proposition 3.2.** *Continuous linear functionals $\varsigma \in \mathfrak{C}^*$ have the following properties.*
*(i) The mapping $(\Lambda, x) \mapsto \varsigma\big(L_1{}^* \alpha_{(\Lambda,x)}(L_2)\big)$ is continuous for given $L_1$, $L_2 \in \mathfrak{L}$.*

*(ii) The mapping $(\Lambda, x) \mapsto \varsigma\big(\alpha_{(\Lambda,x)}(C)\big)$ is continuous for given $C \in \mathfrak{C}$.*

Every *positive* functional $\varsigma$ on the $^*$-algebra $\mathfrak{C} = \mathfrak{L}^* \mathfrak{L}$ defines a non-negative sesquilinear form on $\mathfrak{L}$ through

$$\langle\, . \,|\, . \,\rangle_\varsigma : \mathfrak{L} \times \mathfrak{L} \to \mathbb{C} \quad (L_1, L_2) \mapsto \langle L_1 | L_2 \rangle_\varsigma \doteq \varsigma(L_1{}^* L_2), \tag{3.1a}$$

and thus induces a seminorm $q_\varsigma$ on $\mathfrak{L}$ via

$$q_\varsigma : \mathfrak{L} \to \mathbb{R}_+ \quad L \mapsto q_\varsigma(L) \doteq \langle L | L \rangle_\varsigma^{1/2}, \tag{3.1b}$$

respectively a norm $\|\, . \,\|_\varsigma$ on the quotient space $\mathfrak{L}/\mathfrak{N}_\varsigma$, where $\mathfrak{N}_\varsigma$ denotes the null space of $q_\varsigma$. Using square brackets to designate the cosets in $\mathfrak{L}/\mathfrak{N}_\varsigma$, we immediately get the following result on differentiability since, by the supposed continuity of $\varsigma$, the seminorm $q_\varsigma$ is continuous with respect to at least one of the seminorms $q_\Delta$.

**Proposition 3.3.** *Let $\varsigma$ be a continuous positive functional on the $^*$-algebra $\mathfrak{C}$, in short $\varsigma \in \mathfrak{C}^{*+}$. Then, given $L_0 \in \mathfrak{L}_0$, the mapping*

$$\mathsf{P}_+^\uparrow \ni (\Lambda, x) \mapsto \big[\alpha_{(\Lambda,x)}(L_0)\big]_\varsigma \in (\mathfrak{L}_0/\mathfrak{N}_\varsigma, \|\, . \,\|_\varsigma)$$

*is infinitely often differentiable with respect to the norm $\|\, . \,\|_\varsigma$.*

The following Cluster Property of positive functionals in $\mathfrak{C}^*$ is important in characterizing particle weights as being singly localized whereas the Spectral Property restricts their possible energy-momenta.

**Proposition 3.4** (Cluster Property)**.** *Let $L_i$ and $L_i'$ be elements of $\mathfrak{L}_0$ and let $A_i \in \mathfrak{A}$, $i = 1$, $2$, be almost local operators, then the function*

$$\mathbb{R}^s \ni \boldsymbol{x} \mapsto \varsigma\big((L_1{}^* A_1 L_1')\alpha_{\boldsymbol{x}}(L_2{}^* A_2 L_2')\big) \in \mathbb{C} \tag{3.2}$$

*is an element of $L^1(\mathbb{R}^s, d^s x)$ for any $\varsigma \in \mathfrak{C}^{*+}$ and satisfies*

$$\int_{\mathbb{R}^s} d^s x \, \big|\varsigma\big((L_1{}^* A_1 L_1')\alpha_{\boldsymbol{x}}(L_2{}^* A_2 L_2')\big)\big| \leqslant \|\varsigma\|_\Delta M_\Delta \tag{3.3}$$

*for any bounded Borel set $\Delta$ for which $\varsigma$ belongs to $\mathfrak{C}_\Delta^*$; $M_\Delta$ is a constant depending on $\Delta$ and the operators involved.*

The Spectral Property of functionals $\varsigma \in \mathfrak{C}^*$ expressed in the subsequent proposition will prove to be of importance in defining the energy-momentum of particle weights.

**Proposition 3.5** (Spectral Property)**.** *Let $L_1$, $L_2 \in \mathfrak{L}$ and $\varsigma \in \mathfrak{C}^*$. Then the support of the Fourier transform of the distribution*

$$\mathbb{R}^{s+1} \ni x \mapsto \varsigma\big(L_1{}^* \alpha_x(L_2)\big) \in \mathbb{C}$$

*is contained in a shifted light cone $\overline{V}_+ - q$ for some $q \in \overline{V}_+$. More specifically, $q$ is determined by the condition $\Delta \subseteq q - \overline{V}_+$, where $\varsigma \in \mathfrak{C}_\Delta^*$.*

## 3.2 Asymptotic Functionals

Now we turn to functionals in $\mathfrak{C}^*$ arising as temporal limits of physical states with bounded energy. The development of such a state $\omega \in \mathscr{S}(\Delta)$, $\Delta$ a bounded Borel set, can be explored by considering the following integral

$$\int_{\mathbb{R}^s} d^s v \, h(\boldsymbol{v}) \, \omega\big(\alpha_{(\tau,\tau\boldsymbol{v})}(C)\big), \tag{3.4}$$

where $h$ denotes a bounded measurable function on the unit ball of $\mathbb{R}^s$, $\boldsymbol{v}$ representing velocity. Apart from $h$, (3.4) coincides with the integral (2.5) encountered in the heuristic considerations of Section 2. The investigations of that place (cf. Proposition 2.6) imply that (3.4) takes on a finite value for any counter $C \in \mathfrak{C}$ at given time $\tau$.

The physical interpretation is as follows: Consider a function $h$ of bounded support $\mathsf{V} \subseteq \mathbb{R}^s \setminus \{0\}$ in velocity space, then the integral (3.4) corresponds to summing up the expectation values of measurements of $C$ in the state $\omega$ at time $\tau$, their locus comprising the bounded section $\tau \cdot \mathsf{V}$ of configuration space. The distance of this portion from the origin together with its total extension increases with time. More exactly, the measurements take place in a cone with apex at $\boldsymbol{0}$, its orientation being determined by the support of $h$. For different times $\tau$ the counter $C$ is set up in specific parts of that cone, the extension of which grows as $|\tau|^s$ (compensating for the quantum mechanical spreading of wave packets), while their distance from the origin increases proportionally to $|\tau|$. If in the limit of large (positive or negative) times the physical state $\omega$ has evolved into a configuration containing a particle (incoming or outgoing) travelling with velocity $\boldsymbol{v}_0 \in \mathsf{V}$, then for a counter $C_0$ sensitive for that specific particle the above experimental setup is expected to asymptotically yield a constant signal. Mathematically, this corresponds to the existence of limits of the above integral at asymptotic times, evaluated for the counter $C_0$ and a function $h_0$ with support around $\boldsymbol{v}_0$. Thus, the problem has to be settled in which (topological) sense such limits can be established. To tackle this assignment we turn to a slightly modified version of (3.4) involving a certain time average for technical reasons.

**Definition 3.6.** Let $\Delta$ be a bounded Borel subset of $\mathbb{R}^{s+1}$, let $\omega \in \mathscr{S}(\Delta)$ denote a physical state of bounded energy and let $\boldsymbol{v} \mapsto h(\boldsymbol{v})$ be a bounded measurable function on the unit ball of $\mathbb{R}^s$. Furthermore suppose that $t \mapsto T(t)$ is a continuous real-valued function, approaching $+\infty$ or $-\infty$ for asymptotic positive or negative times, respectively, not as fast as $|t|$. Then we define a net $\{\rho_{h,t} : t \in \mathbb{R}\}$ of linear functionals on $\mathfrak{C}$ by

$$\rho_{h,t}(C) \doteq T(t)^{-1} \int_t^{t+T(t)} d\tau \int_{\mathbb{R}^s} d^s x \, h(\tau^{-1}\boldsymbol{x}) \, \omega\big(\alpha_{(\tau,\boldsymbol{x})}(C)\big), \quad C \in \mathfrak{C}. \tag{3.5}$$

The operators $U(\tau)$ implementing time translations commute with $E(\Delta)$, a fact that allows (3.5) to be re-written for the physical state $\omega(\,.\,) = \omega\big(E(\Delta)\,.\,E(\Delta)\big)$ as

$$\rho_{h,t}(C) = T(t)^{-1} \int_t^{t+T(t)} d\tau \int_{\mathbb{R}^s} d^s x \, h(\tau^{-1}\boldsymbol{x}) \, \omega\big(U(\tau)E(\Delta)\alpha_{\boldsymbol{x}}(C)E(\Delta)U(\tau)^*\big). \tag{3.6}$$

Since all the functionals $\omega\big(U(\tau)\,.\,U(\tau)^*\big)$, $\tau \in \mathbb{R}$, belong to $\mathscr{B}(\mathscr{H})_{*,1}$ the left-hand side of (3.6) can be estimated by

$$\begin{aligned} \big|\rho_{h,t}(C)\big| &\leqslant \sup_{\tau \in I_t} \Big| \int_{\mathbb{R}^s} d^s x \, h(\tau^{-1}\boldsymbol{x}) \, \omega\big(U(\tau)E(\Delta)\alpha_{\boldsymbol{x}}(C)E(\Delta)U(\tau)^*\big) \Big| \\ &\leqslant \|h\|_\infty \sup_{\phi \in \mathscr{B}(\mathscr{H})_{*,1}} \int_{\mathbb{R}^s} d^s x \big|\phi\big(E(\Delta)\alpha_{\boldsymbol{x}}(C)E(\Delta)\big)\big| = \|h\|_\infty p_\Delta(C), \end{aligned} \tag{3.7}$$

where $I_t$ denotes the interval of $\tau$-integration at time $t$. This inequality implies that all the functionals $\rho_{h,t}$ are continuous with respect to $p_\Delta$, i. e., $\rho_{h,t} \in \mathfrak{C}_\Delta^*$. Moreover, (3.7) is uniform in $t$ so that the net $\{\rho_{h,t} : t \in \mathbb{R}\}$ even turns out to be an equicontinuous subset of $\mathfrak{C}^*$. The Theorem of Alaoğlu-Bourbaki [20, Theorem 8.5.2] then tells us that this net is relatively compact with respect to the weak topology, leading to the following fundamental result.

**Theorem 3.7** (Existence of Limits)**.** *Under the assumptions of Definition 3.6 the net* $\{\rho_{h,t} : t \in \mathbb{R}\} \subseteq \mathfrak{C}_\Delta^*$ *possesses weak limit points in* $\mathfrak{C}^*$ *at asymptotic times. This means that there exist functionals* $\sigma_{h,\omega}^{(+)}$ *and* $\sigma_{h,\omega}^{(-)}$ *on* $\mathfrak{C}$ *together with corresponding subnets* $\{\rho_{h,t_\iota} : \iota \in J\}$ *and* $\{\rho_{h,t_\kappa} : \kappa \in K\}$*, where* $\lim_\iota t_\iota = +\infty$ *and* $\lim_\kappa t_\kappa = -\infty$*, such that for arbitrary* $C \in \mathfrak{C}$

$$\rho_{h,t_\iota}(C) \underset{\iota}{\longrightarrow} \sigma_{h,\omega}^{(+)}(C), \tag{3.8a}$$

$$\rho_{h,t_\kappa}(C) \underset{\kappa}{\longrightarrow} \sigma_{h,\omega}^{(-)}(C). \tag{3.8b}$$

The heuristic picture laid open above suggests that in theories with a complete particle interpretation the net of functionals actually converges, but the proof is still lacking. One has to assure that in the limit of large times multiple scattering does no longer withhold the measurement results $\rho_{h,t}(C)$ from growing stable. Another problem is the possible vanishing of the limit functionals on all of $\mathfrak{C}$, a phenomenon we anticipate to encounter in theories without a particle interpretation (e. g. the generalized free field). They both seem to be connected with the problem of asymptotic completeness of quantum field theoretic models and reasonable criteria to restrict their phase space structure. The denomination of asymptotic functionals by $\sigma$ reflects their *singular* nature: the values that the functionals $\rho_{h,t}$ return for finite times $t$ when applied to the identity operator $\mathbb{1}$ (which is not contained in $\mathfrak{C}$) are divergent as $|t|^s$ at asymptotic times.

The convergence problem as yet only partially solved in the sense of Theorem 3.7, one can nevertheless establish a number of distinctive properties of the limit functionals. An immediate first consequence of the above construction is the following proposition.

**Proposition 3.8** (Positivity and Continuity of Limits)**.** *Suppose that* $\Delta$ *is a bounded Borel subset of* $\mathbb{R}^{s+1}$, $\omega \in \mathscr{S}(\Delta)$ *a physical state of bounded energy and* $h \in L^\infty(\mathbb{R}^s, d^s x)$ *a non-negative function. Then the limit functionals* $\sigma$ *of the net* $\{\rho_{h,t} : t \in \mathbb{R}\}$ *are positive elements of* $\mathfrak{C}_\Delta^*$*:*

$$\big|\sigma(C)\big| \leqslant \|h\|_\infty \, p_\Delta(C), \quad C \in \mathfrak{C}; \tag{3.9a}$$

$$0 \leqslant \sigma(C), \quad C \in \mathfrak{C}^+. \tag{3.9b}$$

The proofs of Propositions 3.9 and 3.10 require a special asymptotic behaviour of the function $h \in L^\infty(\mathbb{R}^s, d^s v)$: it has to be continuous, approximating a constant value in the limit $|\boldsymbol{v}| \to \infty$, i. e., $h - M_h \in C_0(\mathbb{R}^s)$ for a suitable constant $M_h$; these functions constitute a subspace of $C(\mathbb{R}^s)$ that will be denoted $C_{0,c}(\mathbb{R}^s)$.

**Proposition 3.9** (Translation Invariance)**.** *Let* $\Delta \subseteq \mathbb{R}^{s+1}$ *be a bounded Borel set, let* $\omega \in \mathscr{S}(\Delta)$ *and* $h \in C_{0,c}(\mathbb{R}^s)$*. Then the limit functionals* $\sigma$ *of the net* $\{\rho_{h,t} : t \in \mathbb{R}\}$ *are invariant under spacetime translations, i. e.,* $\sigma(\alpha_x(C)) = \sigma(C)$ *for any* $C \in \mathfrak{C}$ *and any* $x \in \mathbb{R}^{s+1}$.

The last property of those special elements $\sigma \in \mathfrak{C}_\Delta^{*+}$ arising as limits of nets of functionals $\{\rho_{h,t_\iota} : \iota \in J\}$ complements the Cluster Property 3.4. It asserts, given certain specific operators $C \in \mathfrak{C}$, the existence of a *lower* bound for the integral of $\boldsymbol{x} \mapsto \sigma\big(C^* \alpha_{\boldsymbol{x}}(C)\big)$.

**Proposition 3.10** (Existence of Lower Bounds)**.** *Let* $C \in \mathfrak{C}$ *be a counter which has the property that the function* $\boldsymbol{x} \mapsto p_\Delta\big(C^* \alpha_{\boldsymbol{x}}(C)\big)$ *is integrable (cf. Lemma 5.3). Let furthermore* $\sigma \in \mathfrak{C}_\Delta^{*+}$ *be the limit of a net of functionals* $\{\rho_{h,t_\iota} : \iota \in J\}$*, where the velocity function* $h$ *is non-negative and belongs to* $C_{0,c}(\mathbb{R}^s)$*. Then*

$$\big|\sigma(C)\big|^2 \leqslant \|h\|_\infty \int_{\mathbb{R}^s} d^s x \, \sigma\big(C^* \alpha_{\boldsymbol{x}}(C)\big). \tag{3.10}$$

### 3.3 Particle Weights

The features of limit functionals collected thus far suggest their interpretation as representing mixtures of particle-like quantities with sharp energy-momentum: being translationally invariant according to Proposition 3.9, they appear as plane waves, i. e., energy-momentum eigenstates; on the other hand, they are singly localized at all times by Proposition 3.4, a feature to be expected for a particle-like system. In addition, Proposition 3.5 determines the energy-momentum spectrum. Systems of this kind shall be summarized under the concept of *particle weights*, a term reflecting the connection to the notion of *weights* or *extended positive functionals* in the theory of $C^*$-algebras. Their definition goes back to Dixmier [11, Section I.4.2] (cf. also [25, Section 5.1] and [24]) and designates functionals on the positive cone $\mathfrak{A}^+$ of a $C^*$-algebra $\mathfrak{A}$ that can attain infinite values, a property they share with the singular functionals of Theorem 3.7. Their domain $\mathfrak{C}$ does not comprise the unit $\mathbf{1}$ of the quasi-local algebra, since the defining approximation yields the value $\sigma(\mathbf{1}) = +\infty$.

As mentioned on page 7, every positive functional $\sigma$ on $\mathfrak{C} = \mathfrak{L}^* \mathfrak{L}$ defines a non-negative sesquilinear form $\langle\, . \mid . \,\rangle_\sigma$ on $\mathfrak{L} \times \mathfrak{L}$ that induces a seminorm $q_\sigma$ on $\mathfrak{L}$ and a norm $\|\, . \,\|_\sigma$ on the corresponding quotient $\mathfrak{L}/\mathfrak{N}_\sigma$ with respect to the null space $\mathfrak{N}_\sigma$ of $q_\sigma$. Taking advantage of these constructions, we shall depart from functionals and proceed to sesquilinear forms, which are intimately connected with representations of $\mathfrak{A}$. The following definition reformulates the results on limit functionals within this setting.

**Definition 3.11.** A particle weight is a non-trivial, non-negative sesquilinear form on $\mathfrak{L}$, denoted $\langle\, . \mid . \,\rangle$, which induces a seminorm $q_w$ with null space $\mathfrak{N}_w$ on the ideal $\mathfrak{L}$ and a norm $\|\, . \,\|_w$ on the quotient $\mathfrak{L}/\mathfrak{N}_w$ complying with the following conditions:

(i) for any $L_1, L_2 \in \mathfrak{L}$ and $A \in \mathfrak{A}$ one has $\langle L_1 | A L_2 \rangle = \langle A^* L_1 | L_2 \rangle$;

(ii) for given $L \in \mathfrak{L}$ the following mapping is continuous with respect to $q_w$:

$$\mathsf{P}_+^\uparrow \ni (\Lambda, x) \mapsto \alpha_{(\Lambda,x)}(L) \in \mathfrak{L};$$

(iii) given $L_0 \in \mathfrak{L}_0$, the mapping

$$\mathsf{P}_+^\uparrow \ni (\Lambda, x) \mapsto \left[\alpha_{(\Lambda,x)}(L_0)\right]_w \in (\mathfrak{L}_0/\mathfrak{N}_w, \|\, . \,\|_w)$$

is infinitely often differentiable with respect to the norm $\|\, . \,\|_w$ (cf. Proposition 3.3).

(iv) $\langle\, . \mid . \,\rangle$ is invariant with respect to spacetime translations $x \in \mathbb{R}^{s+1}$, i. e.,

$$\big\langle \alpha_x(L_1) \big| \alpha_x(L_2) \big\rangle = \langle L_1 | L_2 \rangle, \quad L_1, L_2 \in \mathfrak{L},$$

and the $(s+1)$-dimensional Fourier transforms of the distributions $x \mapsto \big\langle L_1 \big| \alpha_x(L_2) \big\rangle$ have support in a shifted forward light cone $\overline{V}_+ - q$ with $q \in \overline{V}_+$.

Note that we did not use any of the $q_\Delta$-topologies on $\mathfrak{L}$ in formulating this definition, for properties given in terms of these will in general get lost in the disintegration to be expounded in a second article. Instead, the relevant topologies are those induced on $\mathfrak{L}$ by the sesquilinear form itself. The constituent properties are preserved under the operations of addition and of multiplication by positive numbers. Therefore, the totality of particle weights, supplemented by the trivial form, proves to be a positive (proper convex) cone, denoted $\boldsymbol{W}$, in the linear space of all sesquilinear forms on $\mathfrak{L}$ (cf. [26, 2]). It is by no means self-evident how to pass from a sesquilinear form of the above type to a positive linear functional on $\mathfrak{C}$. It requires additional restrictive assumptions on the structure of $\mathfrak{C}$ to render the definition of the associated functional unambiguous.

A completely equivalent characterization of particle weights can be given in terms of representations $(\pi_w, \mathscr{H}_w)$ of the quasi-local algebra $\mathfrak{A}$, obtained by a GNS-construction (cf. [24, Theorem 3.2] and [25, Proposition 5.1.3]).

**Theorem 3.12.** *(I) To any particle weight* $\langle\,.\,|\,.\,\rangle$ *there corresponds a non-zero, non-degenerate representation* $(\pi_w, \mathscr{H}_w)$ *of the quasi-local* $C^*$*-algebra* $\mathfrak{A}$ *with the following properties:*

*(i) there exists a linear mapping* $|\,.\,\rangle$ *from* $\mathfrak{L}$ *onto a dense subspace of* $\mathscr{H}_w$*:*

$$|\,.\,\rangle : \mathfrak{L} \to \mathscr{H}_w \qquad L \mapsto |L\rangle,$$

*such that the representation* $\pi_w$ *is given by*

$$\pi_w(A)|L\rangle = |AL\rangle, \quad A \in \mathfrak{A}, \quad L \in \mathfrak{L};$$

*(ii) the following mapping is continuous for given* $L \in \mathfrak{L}$*:*

$$\mathsf{P}_+^\uparrow \ni (\Lambda, x) \mapsto \big|\alpha_{(\Lambda,x)}(L)\big\rangle \in \mathscr{H}_w; \tag{3.11}$$

*(iii) for* $L_0 \in \mathfrak{L}_0$ *the mapping* (3.11) *is infinitely often differentiable;*

*(iv) there exists a strongly continuous unitary representation* $x \mapsto U_w(x)$ *of spacetime translations* $x \in \mathbb{R}^{s+1}$ *on* $\mathscr{H}_w$ *defined by*

$$U_w(x)|L\rangle \doteq \big|\alpha_x(L)\big\rangle, \quad L \in \mathfrak{L},$$

*with spectrum in a displaced forward light cone* $\overline{V}_+ - q$, $q \in \overline{V}_+$.

*(II) Any representation* $(\pi_w, \mathscr{H}_w)$ *which has the above characteristics defines a particle weight through the scalar product on* $\mathscr{H}_w$.

*Remark.* The unitaries $U_w(x)$ implement the group $\big\{\alpha_x : x \in \mathbb{R}^{s+1}\big\}$ of automorphisms of $\mathfrak{A}$ in $(\mathscr{H}_w, \pi_w)$ via

$$U_w(x)\pi_w(A)U_w(x)^* = \pi_w\big(\alpha_x(A)\big), \quad A \in \mathfrak{A}, \quad x \in \mathbb{R}^{s+1}. \tag{3.12}$$

The energy-momentum transfer of operators in $\mathfrak{L}$ determines the spectral subspace to which the corresponding vectors in $\mathscr{H}_w$ pertain. Moreover, particle weights enjoy a Cluster Property parallel to that established in Proposition 3.4 for positive functionals in $\mathfrak{C}_\Delta^*$.

**Proposition 3.13** (Spectral Subspaces)**.** *Let* $L \in \mathfrak{L}$ *have energy-momentum transfer in the Borel subset* $\Delta'$ *of* $\mathbb{R}^{s+1}$*. Then, in the representation* $(\pi_w, \mathscr{H}_w)$ *corresponding to the particle weight* $\langle\,.\,|\,.\,\rangle$*, the vector* $|L\rangle$ *belongs to the spectral subspace pertaining to* $\Delta'$ *with respect to the generator of the intrinsic unitary representation* $x \mapsto U_w(x)$ *of spacetime translations:*

$$|L\rangle = E_w(\Delta')|L\rangle. \tag{3.13}$$

**Proposition 3.14** (Cluster Property for Particle Weights)**.** *Let* $L_i$ *and* $L_i'$ *be elements of* $\mathfrak{L}_0$ *with energy-momentum transfer in* $\Gamma_i$ *respectively* $\Gamma_i'$*, and let* $A_i \in \mathfrak{A}$*,* $i = 1, 2$*, be almost local. Then, for the particle weight* $\langle\,.\,|\,.\,\rangle$ *with GNS-representation* $(\pi_w, \mathscr{H}_w)$*,*

$$\mathbb{R}^s \ni \boldsymbol{x} \mapsto \big\langle L_1{}^* A_1 L_1' \big| \alpha_{\boldsymbol{x}}(L_2{}^* A_2 L_2')\big\rangle = \big\langle L_1{}^* A_1 L_1' \big| U_w(\boldsymbol{x}) \big| L_2{}^* A_2 L_2'\big\rangle \in \mathbb{C}$$

*is a function in* $L^1\big(\mathbb{R}^s, d^s x\big)$.

At this point a brief comment on the notation chosen seems appropriate (cf. [9]). We deliberately utilize the typographical token $|\,.\,\rangle$ introduced by Dirac [10, § 23] for ket vectors describing improper momentum eigenstates $|\boldsymbol{p}\rangle$, $\boldsymbol{p} \in \mathbb{R}^s$. These act as distributions on the space of momentum wave functions with values in the physical Hilbert space $\mathscr{H}$, thereby presupposing a superposition principle to hold without limitations. This assumption collapses in an infraparticle situation as described in the Introduction. In contrast to this, the *pure* particle weights, that will appear in connection with elementary physical systems, are seen to be associated with sharp momentum and yet capable of describing infraparticles. Here the operators $L \in \mathfrak{L}$ take on the role of the previously mentioned momentum space wave functions in that they localize the particle weight in order to produce a normalizable vector $|L\rangle$ in the Hilbert space $\mathscr{H}_w$. This in turn substantiates the terminology introduced in Definition 2.4. Describing elementary physical systems, pure particle weights should give rise to irreducible representations of the quasi-local algebra, which motivates the subsequent definition. It is supplemented by a notion of boundedness which is in particular shared by functionals $\sigma$ arising from the construction expounded in Subsection 3.2 (cf. Lemma 3.16).

**Definition 3.15.** A particle weight is said to be
(a) *pure*, if the corresponding representation $(\pi_w, \mathscr{H}_w)$ is irreducible;
(b) $\Delta$-*bounded*, if to any bounded Borel subset $\Delta'$ of $\mathbb{R}^{s+1}$ there exists another such set $\overline{\Delta} \supseteq \Delta + \Delta'$ so that the GNS-representation $(\pi_w, \mathscr{H}_w)$ of the particle weight and the defining representation are connected by the following inequality, valid for any $A \in \mathfrak{A}$,

$$\|E_w(\Delta')\pi_w(A)E_w(\Delta')\| \leqslant c \cdot \|E(\overline{\Delta})AE(\overline{\Delta})\| \tag{3.14}$$

with a suitable positive constant $c$ that is independent of the Borel sets involved. Obviously, $\Delta$ ought to be a bounded Borel set as well.

**Lemma 3.16.** *Any positive asymptotic functional $\sigma \in \mathfrak{C}_\Delta^*$, constructed according to Theorem 3.7 under the assumptions of Proposition 3.8, gives rise to a $\Delta$-bounded particle weight $\langle\,.\,|\,.\,\rangle_\sigma$.*

The above concepts are important in connection with the disintegration theory (pure particle weights) and with the local normality of representations ($\Delta$-boundedness). Both will be the topic of the forthcoming second article.

# 4 Proofs for Section 2

In a first step to prove the validity of (2.5) for detectors in $\mathfrak{C}$, we consider detectors built from special operators $L_0 \in \mathfrak{L}_0$ having energy-momentum transfer in *convex* sets.

**Proposition 4.1.** *Let $E(\,.\,)$ be the spectral resolution corresponding to the unitary representation $x \mapsto U(x)$ of spacetime translations and let $L_0 \in \mathfrak{L}_0$ have energy-momentum transfer in the compact and convex subset $\Gamma$ of $\complement \overline{V}_+$. Then the net of operator-valued Bochner integrals $\left\{E(\Delta)\int_K d^s x\, \alpha_{\boldsymbol{x}}(L_0{}^* L_0)\, E(\Delta) : K \subseteq \mathbb{R}^s \text{compact}\right\}$ is $\sigma$-strongly convergent in $\mathscr{B}(\mathscr{H})$ as $K \nearrow \mathbb{R}^s$ for any bounded Borel set $\Delta \subseteq \mathbb{R}^{s+1}$. Its limit is the $\sigma$-weak integral of the $\mathscr{B}(\mathscr{H})^+$-valued mapping $\boldsymbol{x} \mapsto E(\Delta)\alpha_{\boldsymbol{x}}(L_0{}^* L_0)E(\Delta)$ and satisfies the following estimate for suitable $N(\Delta,\Gamma) \in \mathbb{N}$ depending on $\Delta$ and $\Gamma$:*

$$\left\| \int_{\mathbb{R}^s} d^s x\, E(\Delta)\alpha_{\boldsymbol{x}}(L_0{}^* L_0)E(\Delta) \right\| \leqslant N(\Delta,\Gamma) \int_{\mathbb{R}^s} d^s x\, \big\| \big[\alpha_{\boldsymbol{x}}(L_0), L_0{}^*\big] \big\|. \tag{4.1}$$

*Remark.* The integral on the right-hand side of (4.1) is finite due to almost locality of $L_0$: Let $A$ and $B$ be almost local with approximating nets $\{A_r \in \mathfrak{A}(\mathscr{O}_r) : r > 0\}$ and $\{B_r \in \mathfrak{A}(\mathscr{O}_r) : r > 0\}$, respectively. Since $\mathscr{O}_r$ and $\mathscr{O}_r + 2\boldsymbol{x}$ are spacelike separated for $r \leqslant |\boldsymbol{x}|$, $\boldsymbol{x} \in \mathbb{R}^s \setminus \{\boldsymbol{0}\}$, one has

$$\big\| \big[\alpha_{2\boldsymbol{x}}(A), B\big] \big\| \leqslant 2\left(\|A - A_{|\boldsymbol{x}|}\|\,\|B\| + \|A - A_{|\boldsymbol{x}|}\|\,\|B - B_{|\boldsymbol{x}|}\| + \|A\|\,\|B - B_{|\boldsymbol{x}|}\|\right), \tag{4.2}$$

implying integrability of the left-hand side over all of $\mathbb{R}^s$.

*Proof.* $\Delta$ being a bounded Borel set with closure $\overline{\Delta}$, there exists, due to compactness and convexity of $\Gamma$, a number $n \in \mathbb{N}$ such that $(\overline{\Delta} + \Gamma_n) \cap \overline{V}_+ = \emptyset$ is satisfied, where $\Gamma_n$ denotes the $n$-fold sum of the compactum $\Gamma$. The spectrum condition then entails $E(\overline{\Delta + \Gamma_n}) = 0$. As a consequence, for arbitrary $x_1, \dots, x_n \in \mathbb{R}^s$, the product $\prod_{i=1}^{n} \alpha_{x_i}(L_0)$ annihilates any state with energy-momentum in $\Delta$ so that $E(\Delta)$ turns out to be a projection with range in the intersection of the kernels of $n$-fold products of the above kind. An application of [7, Lemma 2.2] then yields the estimate

$$\left\| E(\Delta) \int_K d^s x \, \alpha_x(L_0{}^* L_0) \, E(\Delta) \right\| \leqslant (n-1) \int_{\mathbb{R}^s} d^s x \, \left\| \left[ \alpha_x(L_0), L_0{}^* \right] \right\| \tag{4.3}$$

for any compact $K$, where the right-hand side is finite due to almost locality of $L_0$. The positive operators on the left-hand side thus constitute an increasing net that is bounded and converges $\sigma$-strongly to its least upper bound in $\mathscr{B}(\mathscr{H})^+$ [5, Lemma 2.4.19].

The $\sigma$-weak topology on $\mathscr{B}(\mathscr{H})$ is induced by the positive normal functionals $\psi$ in the cone $\mathscr{B}(\mathscr{H})_*^+$. Thus, integrability of $x \mapsto E(\Delta)\alpha_x(L_0{}^*L_0)E(\Delta)$ with respect to the $\sigma$-weak topology is implied by integrability of all of the corresponding functions $x \mapsto \left|\psi\big(E(\Delta)\alpha_x(L_0{}^*L_0)E(\Delta)\big)\right| = \psi\big(E(\Delta)\alpha_x(L_0{}^*L_0)E(\Delta)\big)$. Given any compact subset $K$ of $\mathbb{R}^s$, we have, by (4.3),

$$\begin{aligned}\int_K d^s x \, \left|\psi\big(E(\Delta)\alpha_x(L_0{}^*L_0)E(\Delta)\big)\right| &= \psi\Big(\int_K d^s x \, E(\Delta)\alpha_x(L_0{}^*L_0)E(\Delta)\Big) \\ &\leqslant \|\psi\| \left\| \int_K d^s x \, E(\Delta)\alpha_x(L_0{}^*L_0)E(\Delta) \right\| \leqslant \|\psi\| \, (n-1) \int_{\mathbb{R}^s} d^s x \, \left\| \left[\alpha_x(L_0), L_0{}^*\right] \right\|\end{aligned}$$

so that, as a consequence of the Monotone Convergence Theorem [12, II.2.7], the functions $x \mapsto \left|\psi\big(E(\Delta)\alpha_x(L_0{}^*L_0)E(\Delta)\big)\right|$ indeed turn out to be integrable. The integral of $x \mapsto E(\Delta)\alpha_x(L_0{}^*L_0)E(\Delta)$ over $\mathbb{R}^s$ with respect to the $\sigma$-weak topology thus exists [12, II.6.2] and, by application of Lebesgue's Dominated Convergence Theorem [12, II.5.6], is seen to be the $\sigma$-weak limit of the net of compactly supported integrals. It thus coincides with the $\sigma$-strong limit established above and, according to (4.3), satisfies (4.1) with $N(\Delta, \Gamma) \doteq n - 1$. □

The special result of Proposition 4.1 for $L_0 \in \mathfrak{L}_0$ can now easily be generalized to arbitrary $L \in \mathfrak{L}$ and $C \in \mathfrak{C}$ as laid down in Proposition 2.6.

*Proposition 2.6.* (i) By partition of unity applied to the Fourier transform of the mapping $x \mapsto U(x) L_0 U(x)^*$, any $L_0 \in \mathfrak{L}_0$ is seen to be representable as a finite sum of operators $L_i \in \mathfrak{L}_0$ transferring energy-momentum in compact and *convex* subsets $\Gamma_i$ of $\complement \overline{V}_+$. Accordingly, any $L \in \mathfrak{L}$ can be written as $L = \sum_{j=1}^{m} A_j L_j$ with $A_j \in \mathfrak{A}$ and $L_j \in \mathfrak{L}_0$ of this special kind. The relation

$$L^* L \leqslant 2^{m-1} \Big( \sup_{1 \leqslant j \leqslant m} \|A_j\|^2 \Big) \sum_{j=1}^{m} L_j{}^* L_j,$$

implies for any compact $K \subseteq \mathbb{R}^s$

$$\left\| E(\Delta) \int_K d^s x \, \alpha_x(L^* L) \, E(\Delta) \right\| \leqslant 2^{m-1} \Big( \sup_{1 \leqslant j \leqslant m} \|A_j\|^2 \Big) \sum_{j=1}^{m} \left\| E(\Delta) \int_K d^s x \, \alpha_x(L_j{}^* L_j) \, E(\Delta) \right\|,$$

and the statement now follows by use of the arguments given in the proof of Proposition 4.1 which apply to the vacuum annihilation operators $L_j$.

(ii) By polarization, an arbitrary element $C_0 = L_1{}^* L_2 \in \mathfrak{C}$ with $L_1, L_2 \in \mathfrak{L}$ can be written as a linear combination of four positive operators of the form $L^* L$, $L \in \mathfrak{L}$. So the problem of $\sigma$-strong convergence of the net of Bochner integrals constructed with $C_0$ reduces to the case already established in the first part. Polar decomposition of the normal functional $\varphi$ [30, Theorem III.4.2(i), Proposition III.4.6], yields a partial isometry $V \in \mathscr{B}(\mathscr{H})$ and a *positive* normal functional $|\phi|$ subject to $\||\phi|\| = \|\phi\|$ such that $\phi(\,.\,) = |\phi|(\,.\,V)$. This entails for arbitrary $\boldsymbol{x} \in \mathbb{R}^s$ and $\lambda > 0$ the estimate

$$
\begin{aligned}
2\left|\phi\big(E(\Delta)\alpha_{\boldsymbol{x}}(C_0)E(\Delta)\big)\right| &= 2\left||\phi|\big(E(\Delta)\alpha_{\boldsymbol{x}}(L_1{}^*L_2)E(\Delta)V\big)\right| \\
&\leqslant 2\sqrt{|\phi|\big(E(\Delta)\alpha_{\boldsymbol{x}}(L_1{}^*L_1)E(\Delta)\big)}\sqrt{|\phi|\big(V^*E(\Delta)\alpha_{\boldsymbol{x}}(L_2{}^*L_2)E(\Delta)V\big)} \\
&\leqslant \lambda^{-1}|\phi|\big(E(\Delta)\alpha_{\boldsymbol{x}}(L_1{}^*L_1)E(\Delta)\big) + \lambda|\phi|\big(V^*E(\Delta)\alpha_{\boldsymbol{x}}(L_2{}^*L_2)E(\Delta)V\big).
\end{aligned}
$$

Both sides of this inequality are integrable over $\mathbb{R}^s$, since from part (i) we infer

$$
\begin{aligned}
&2\int_{\mathbb{R}^s} d^sx\,\left|\phi\big(E(\Delta)\alpha_{\boldsymbol{x}}(C_0)E(\Delta)\big)\right| \\
&\qquad \leqslant \lambda^{-1}\|\phi\|\left\|\int_{\mathbb{R}^s} d^sx\,E(\Delta)\alpha_{\boldsymbol{x}}(L_1{}^*L_1)E(\Delta)\right\| + \lambda\|\phi\|\left\|\int_{\mathbb{R}^s} d^sx\,E(\Delta)\alpha_{\boldsymbol{x}}(L_2{}^*L_2)E(\Delta)\right\|,
\end{aligned}
$$

by noting that normal functionals and $\sigma$-weak integrals commute due to [12, Proposition II.5.7 adapted to integrals in locally convex spaces]. Taking the infimum with respect to $\lambda$, one arrives at

$$
\begin{aligned}
&\int_{\mathbb{R}^s} d^sx\,\left|\phi\big(E(\Delta)\alpha_{\boldsymbol{x}}(C_0)E(\Delta)\big)\right| \\
&\qquad \leqslant \|\phi\|\left\|\int_{\mathbb{R}^s} d^sx\,E(\Delta)\alpha_{\boldsymbol{x}}(L_1{}^*L_1)E(\Delta)\right\|^{1/2}\left\|\int_{\mathbb{R}^s} d^sx\,E(\Delta)\alpha_{\boldsymbol{x}}(L_2{}^*L_2)E(\Delta)\right\|^{1/2}.
\end{aligned} \tag{4.4}
$$

This relation being valid for any normal functional $\phi \in \mathscr{B}(\mathscr{H})_*$, the asserted existence of the $\sigma$-weak integral is established for $C_0 \in \mathfrak{C}$, which obviously coincides with the $\sigma$-strong limit of the corresponding net. As a by-product of (4.4) we have

$$
\begin{aligned}
&\sup\left\{\int_{\mathbb{R}^s} d^sx\,\left|\phi\big(E(\Delta)\alpha_{\boldsymbol{x}}(C_0)E(\Delta)\big)\right| : \phi \in \mathscr{B}(\mathscr{H})_{*,1}\right\} \\
&\qquad \leqslant \left\|\int_{\mathbb{R}^s} d^sx\,E(\Delta)\alpha_{\boldsymbol{x}}(L_1{}^*L_1)E(\Delta)\right\|^{1/2}\left\|\int_{\mathbb{R}^s} d^sx\,E(\Delta)\alpha_{\boldsymbol{x}}(L_2{}^*L_2)E(\Delta)\right\|^{1/2}.
\end{aligned} \tag{4.5}
$$

These results are readily extended to establish part (ii) in the general case, since any $C \in \mathfrak{C}$ is a linear combination of operators of the form of $C_0$. $\square$

What follows are proofs as well as statements of seminorm properties.

*Proposition 2.8.* Let $L \in \mathfrak{L}$ and let $\Delta \subseteq \Delta'$, then

$$
0 \leqslant \int_{\mathbb{R}^s} d^sx\,E(\Delta)\alpha_{\boldsymbol{x}}(L^*L)E(\Delta) \leqslant \int_{\mathbb{R}^s} d^sx\,E(\Delta')\alpha_{\boldsymbol{x}}(L^*L)E(\Delta'),
$$

which by (2.7a) implies $q_\Delta(L)^2 \leqslant q_{\Delta'}(L)^2$. In case of the $p_\Delta$-topologies, note that for any Borel set $\Delta$ the functional $\phi^{E(\Delta)}$, defined through $\phi^{E(\Delta)}(\,.\,) \doteq \phi(E(\Delta)\,.\,E(\Delta))$, belongs to $\mathscr{B}(\mathscr{H})_{*,1}$ if $\phi$ does. From this we infer, since moreover $E(\Delta) = E(\Delta)E(\Delta') = E(\Delta')E(\Delta)$ is implied by $\Delta \subseteq \Delta'$, that

$$
\begin{aligned}
&\left\{\int_{\mathbb{R}^s} d^sx\,\left|\phi\big(E(\Delta)\alpha_{\boldsymbol{x}}(C)E(\Delta)\big)\right| : \phi \in \mathscr{B}(\mathscr{H})_{*,1}\right\} \\
&\qquad\qquad \subseteq \left\{\int_{\mathbb{R}^s} d^sx\,\left|\phi\big(E(\Delta')\alpha_{\boldsymbol{x}}(C)E(\Delta')\big)\right| : \phi \in \mathscr{B}(\mathscr{H})_{*,1}\right\}
\end{aligned}
$$

for any $C \in \mathfrak{C}$ and thus, by (2.7b), that $p_\Delta(C) \leqslant p_{\Delta'}(C)$. $\square$

The statements of the following Lemmas can easily be established by use of the integral representations for the seminorms.

**Lemma 4.2.** *(i) For $L \in \mathfrak{L}$ with $LE(\Delta) = 0$, one has $q_\Delta(L) = 0$.*
*(ii) If $C \in \mathfrak{C}$ satisfies $E(\Delta)CE(\Delta) = 0$, then $p_\Delta(C) = 0$.*

**Lemma 4.3.** *(i) For any $L \in \mathfrak{L}$ and $A \in \mathfrak{A}$*

$$q_\Delta(AL) \leqslant \|A\| \, q_\Delta(L). \tag{4.6}$$

*(ii) Let $L_i$, $i = 1, 2$, be operators in $\mathfrak{L}$ having energy-momentum transfer in $\Gamma_i \subseteq \mathbb{R}^{s+1}$, and let $\Delta_i$ denote Borel subsets of $\mathbb{R}^{s+1}$ containing $\Delta + \Gamma_i$, respectively. Then, for any $A \in \mathfrak{A}$,*

$$p_\Delta(L_1{}^* A L_2) \leqslant \|E(\Delta_1) A E(\Delta_2)\| \, q_\Delta(L_1) \, q_\Delta(L_2). \tag{4.7}$$

*Proof.* (i) For $L \in \mathfrak{L}$ and $A \in \mathfrak{A}$ the relation $L^* A^* A L \leqslant \|A\|^2 L^* L$ implies

$$\int_{\mathbb{R}^s} d^s x \, \omega\big(E(\Delta)\alpha_{\boldsymbol{x}}(L^* A^* A L)E(\Delta)\big) \leqslant \|A\|^2 \int_{\mathbb{R}^s} d^s x \, \omega\big(E(\Delta)\alpha_{\boldsymbol{x}}(L^* L)E(\Delta)\big)$$

for any $\omega \in \mathscr{B}(\mathscr{H})_{*,1}^+$ and thus, by virtue of (2.7c),

$$\begin{aligned} q_\Delta(AL)^2 &= \sup\Big\{ \int_{\mathbb{R}^s} d^s x \, \omega\big(E(\Delta)\alpha_{\boldsymbol{x}}(L^* A^* A L)E(\Delta)\big) : \omega \in \mathscr{B}(\mathscr{H})_{*,1}^+ \Big\} \\ &\leqslant \|A\|^2 \sup\Big\{ \int_{\mathbb{R}^s} d^s x \, \omega\big(E(\Delta)\alpha_{\boldsymbol{x}}(L^* L)E(\Delta)\big) : \omega \in \mathscr{B}(\mathscr{H})_{*,1}^+ \Big\} = \|A\|^2 q_\Delta(L)^2. \end{aligned}$$

(ii) Let $\phi$ be a normal functional on $\mathscr{B}(\mathscr{H})$ with $\|\phi\| \leqslant 1$. By polar decomposition, there exist a partial isometry $V$ and a positive normal functional $|\phi|$ with $\||\phi|\| \leqslant 1$ such that $\phi(\,.\,) = |\phi|(\,.\,V)$. Then for any $\boldsymbol{x} \in \mathbb{R}^s$

$$\begin{aligned} \big|\phi\big(E(\Delta)\alpha_{\boldsymbol{x}}(L_1{}^* A L_2)E(\Delta)\big)\big| &= |\phi|\big(E(\Delta)\alpha_{\boldsymbol{x}}(L_1{}^*)E(\Delta_1)\alpha_{\boldsymbol{x}}(A)E(\Delta_2)\alpha_{\boldsymbol{x}}(L_2)E(\Delta)V\big) \\ &\leqslant \|E(\Delta_1)\alpha_{\boldsymbol{x}}(A)E(\Delta_2)\| \sqrt{|\phi|\big(E(\Delta)\alpha_{\boldsymbol{x}}(L_1{}^* L_1)E(\Delta)\big)} \sqrt{|\phi|\big(V^* E(\Delta)\alpha_{\boldsymbol{x}}(L_2{}^* L_2)E(\Delta)V\big)}, \end{aligned}$$

and the method used in the proof of Proposition 2.6 yields, in analogy to (4.5),

$$\begin{aligned} p_\Delta(L_1{}^* A L_2) &= \sup\Big\{ \int_{\mathbb{R}^s} d^s x \, \big|\phi\big(E(\Delta)\alpha_{\boldsymbol{x}}(L_1{}^* A L_2)E(\Delta)\big)\big| : \phi \in \mathscr{B}(\mathscr{H})_{*,1} \Big\} \\ &\leqslant \|E(\Delta_1) A E(\Delta_2)\| \, q_\Delta(L_1) \, q_\Delta(L_2), \end{aligned}$$

where use was made of (2.7b). □

An immediate consequence of the second part of this Lemma is

**Corollary 4.4.** *The sesquilinear mapping on the topological product of the locally convex space $(\mathfrak{L}, \mathfrak{T}_q)$ with itself, defined by*

$$\mathfrak{L} \times \mathfrak{L} \ni (L_1, L_2) \mapsto L_1{}^* L_2 \in \mathfrak{C},$$

*is continuous with regard to the respective locally convex topologies.*

**Lemma 4.5.** *(i) For any operator $L \in \mathfrak{L}$ there holds the relation*

$$p_\Delta(L^* L) = q_\Delta(L)^2.$$

*(ii) Let $C$ be an element of $\mathfrak{C}$, then*

$$p_\Delta(C^*) = p_\Delta(C).$$

*Proof.* (i) By (2.7c) and (2.7b), we have for $L \in \mathfrak{L}$

$$q_\Delta(L)^2 = \sup\Big\{\int_{\mathbb{R}^s} d^s x\, \omega\big(E(\Delta)\alpha_{\boldsymbol{x}}(L^*L)E(\Delta)\big) : \omega \in \mathscr{B}(\mathscr{H})^+_{*,1}\Big\}$$
$$\leqslant \sup\Big\{\int_{\mathbb{R}^s} d^s x\, \big|\phi\big(E(\Delta)\alpha_{\boldsymbol{x}}(L^*L)E(\Delta)\big)\big| : \phi \in \mathscr{B}(\mathscr{H})_{*,1}\Big\} = p_\Delta(L^*L),$$

whereas the reverse inequality is a consequence of Lemma 4.3.

(ii) Note that $\mathscr{B}(\mathscr{H})_{*,1}$ is invariant under the operation of taking adjoints defined by $\psi \mapsto \psi^*$ with $\psi^*(A) \doteq \overline{\psi(A^*)}$, $A \in \mathscr{B}(\mathscr{H})$, for any linear functional $\psi$ on $\mathscr{B}(\mathscr{H})$. Thus

$$p_\Delta(C^*) = \sup\Big\{\int_{\mathbb{R}^s} d^s x\, \big|\phi\big(E(\Delta)\alpha_{\boldsymbol{x}}(C^*)E(\Delta)\big)\big| : \phi \in \mathscr{B}(\mathscr{H})_{*,1}\Big\}$$
$$= \sup\Big\{\int_{\mathbb{R}^s} d^s x\, \big|\phi^*\big(E(\Delta)\alpha_{\boldsymbol{x}}(C)E(\Delta)\big)\big| : \phi \in \mathscr{B}(\mathscr{H})_{*,1}\Big\} = p_\Delta(C)$$

for any $C \in \mathfrak{C}$. □

The above results are used in the proofs of Propositions 2.9 and 2.10.

*Proposition 2.9.* $\mathscr{B}(\mathscr{H})_{*,1}$ as well as its intersection $\mathscr{B}(\mathscr{H})^+_{*,1}$ with the positive cone $\mathscr{B}(\mathscr{H})^+_*$ are invariant under the mapping $\psi \mapsto \psi^U$ defined on $\mathscr{B}(\mathscr{H})^*$ for any unitary operator $U \in \mathscr{B}(\mathscr{H})$ by $\psi^U(\ .\ ) \doteq \psi(U\ .\ U^*)$.

(i) Now, $\alpha_x(L^*L) = U_t\alpha_{\boldsymbol{x}}(L^*L){U_t}^*$ for any $x = (t, \boldsymbol{x}) \in \mathbb{R}^{s+1}$. This implies

$$\omega\big(E(\Delta)\alpha_{\boldsymbol{y}}\big(\alpha_x(L^*L)\big)E(\Delta)\big) = \omega\big(U_tE(\Delta)\alpha_{\boldsymbol{x}+\boldsymbol{y}}(L^*L)E(\Delta){U_t}^*\big)$$

for any $\boldsymbol{y} \in \mathbb{R}^s$ and any $\omega \in \mathscr{B}(\mathscr{H})_*$. Henceforth

$$\int_{\mathbb{R}^s} d^s y\, \omega\big(E(\Delta)\alpha_{\boldsymbol{y}}\big(\alpha_x(L^*L)\big)E(\Delta)\big) = \int_{\mathbb{R}^s} d^s y\, \omega\big(U_tE(\Delta)\alpha_{\boldsymbol{x}+\boldsymbol{y}}(L^*L)E(\Delta){U_t}^*\big)$$
$$= \int_{\mathbb{R}^s} d^s y\, \omega\big(U_tE(\Delta)\alpha_{\boldsymbol{y}}(L^*L)E(\Delta){U_t}^*\big)$$

so that, by virtue of (2.7c), for any $L \in \mathfrak{L}$

$$q_\Delta\big(\alpha_x(L)\big)^2 = \sup\Big\{\int_{\mathbb{R}^s} d^s y\, \omega\big(E(\Delta)\alpha_{\boldsymbol{y}}\big(\alpha_x(L^*L)\big)E(\Delta)\big) : \omega \in \mathscr{B}(\mathscr{H})^+_{*,1}\Big\}$$
$$= \sup\Big\{\int_{\mathbb{R}^s} d^s y\, \omega\big(U_tE(\Delta)\alpha_{\boldsymbol{y}}(L^*L)E(\Delta){U_t}^*\big) : \omega \in \mathscr{B}(\mathscr{H})^+_{*,1}\Big\}$$
$$= \sup\Big\{\int_{\mathbb{R}^s} d^s y\, \omega\big(E(\Delta)\alpha_{\boldsymbol{y}}(L^*L)E(\Delta)\big) : \omega \in \mathscr{B}(\mathscr{H})^+_{*,1}\Big\} = q_\Delta(L)^2.$$

(ii) The same argument applies to $p_\Delta$, so that

$$p_\Delta\big(\alpha_x(C)\big) = \sup\Big\{\int_{\mathbb{R}^s} d^s y\, \big|\phi\big(E(\Delta)\alpha_{\boldsymbol{y}}\big(\alpha_x(C)\big)E(\Delta)\big)\big| : \phi \in \mathscr{B}(\mathscr{H})_{*,1}\Big\}$$
$$= \sup\Big\{\int_{\mathbb{R}^s} d^s y\, \big|\phi\big(U_tE(\Delta)\alpha_{\boldsymbol{y}}(C)E(\Delta){U_t}^*\big)\big| : \phi \in \mathscr{B}(\mathscr{H})_{*,1}\Big\}$$
$$= \sup\Big\{\int_{\mathbb{R}^s} d^s y\, \big|\phi\big(E(\Delta)\alpha_{\boldsymbol{y}}(C)E(\Delta)\big)\big| : \phi \in \mathscr{B}(\mathscr{H})_{*,1}\Big\} = p_\Delta(C)$$

for $C \in \mathfrak{C}$. □

*Proposition 2.10.* (i) Continuity of the mapping $(\Lambda,x) \mapsto \alpha_{(\Lambda,x)}(L)$ in the locally convex space $(\mathfrak{L},\mathfrak{T}_q)$ means continuity on all of $\mathsf{P}_+^\uparrow$ with respect to each of the seminorms $q_\Delta$, and this in turn is established by demonstrating continuity in $(\mathbf{1},0)$ for arbitrary localizing operators $L$, since $\mathfrak{L}$ is invariant under Poincaré transformations.

Let the Borel subset $\Delta$ of $\mathbb{R}^{s+1}$ be arbitrary but fixed and consider $L' \in \mathfrak{L}_0$ having energy-momentum transfer $\Gamma$ which, under sufficiently small Poincaré transformations, stays within a compact and convex subset $\widehat{\Gamma}$ of $\complement \overline{V}_+$. Then relation (4.1) in Proposition 4.1 tells us that

$$q_\Delta\big(\alpha_{(\Lambda,x)}(L') - L'\big)^2 \leqslant N(\Delta,\widehat{\Gamma}) \int_{\mathbb{R}^s} d^s y \, \big\| \big[\alpha_{\boldsymbol{y}}\big(\alpha_{(\Lambda,x)}(L') - L'\big), \big(\alpha_{(\Lambda,x)}(L') - L'\big)^*\big] \big\|. \tag{4.8}$$

Exploiting almost locality of the operators on the right-hand side to control the norm of the large radius part of the function $\boldsymbol{y} \mapsto \big\| \big[\alpha_{\boldsymbol{y}}\big(\alpha_{(\Lambda,x)}(L') - L'\big), \big(\alpha_{(\Lambda,x)}(L') - L'\big)^*\big] \big\|$, one can establish the existence of an integrable majorizing function, independent of $(\Lambda,x)$ in a small neighbourhood of $(\mathbf{1},0)$. Moreover, this integrand vanishes pointwise in the limit $(\Lambda,x) \to (\mathbf{1},0)$ due to strong continuity of the automorphism group. Therefore, by Lebesgue's Dominated Convergence Theorem, we infer that for any sequence of transformations approaching $(\mathbf{1},0)$

$$\lim_{n\to\infty} q_\Delta\big(\alpha_{(\Lambda_n,x_n)}(L') - L'\big) = 0.$$

Since $\mathsf{P}_+^\uparrow$ as a topological space satisfies the first axiom of countability, this suffices to establish continuity of the mapping $(\Lambda,x) \mapsto \alpha_{(\Lambda,x)}(L')$ in $(\mathbf{1},0)$ with respect to the $q_\Delta$-topology. An arbitrary operator $L \in \mathfrak{L}$ can be represented as $L = \sum_{i=1}^N A_i L_i'$, where $L_i' \in \mathfrak{L}_0$ comply with the above assumptions on $L'$ and the operators $A_i$ belong to the quasi-local algebra $\mathfrak{A}$ for any $i = 1, \ldots, N$. According to Lemma 4.3, one can then derive an estimate for $q_\Delta\big(\alpha_{(\Lambda,x)}(L) - L\big)$ in terms of $q_\Delta\big(\alpha_{(\Lambda,x)}(L_i') - L_i'\big)$ and $\big\|\alpha_{(\Lambda,x)}(A_i) - A_i\big\|$, so that the mapping $(\Lambda,x) \mapsto \alpha_{(\Lambda,x)}(L)$ is seen to be continuous in $(\mathbf{1},0)$ with respect to $q_\Delta$ for arbitrary $L \in \mathfrak{L}$.

Continuity of the mapping $(\Lambda,x) \mapsto \alpha_{(\Lambda,x)}(C)$ in the locally convex space $(\mathfrak{C},\mathfrak{T}_p)$ is equivalent to its being continuous with respect to all seminorms $p_\Delta$. This problem again reduces to the one already solved above, if one takes into account the shape of general elements of $\mathfrak{C}$ according to Definition 2.5 in connection with Corollary 4.4.

(ii) Let $L_0 \in \mathfrak{L}_0$ be given and consider the parametrization $\boldsymbol{h} \mapsto (\Lambda_{\boldsymbol{h}}, x_{\boldsymbol{h}})$ of an open neighbourhood of $(\Lambda_{\boldsymbol{0}}, x_{\boldsymbol{0}})$. With respect to this, the derivative at $(\Lambda_{\boldsymbol{0}}, x_{\boldsymbol{0}})$ of the mapping $(\Lambda,x) \mapsto \alpha_{(\Lambda,x)}(L_0)$ yields an approximation that differs from the actual change with $\boldsymbol{h}$ by the residual term

$$R(\boldsymbol{h}) = \alpha_{(\Lambda_{\boldsymbol{h}},x_{\boldsymbol{h}})}(L_0) - \alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}(L_0) - \sum_{i,j} h_j C_{ij}(\boldsymbol{0})\, \alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}\big(\delta^i(L_0)\big). \tag{4.9a}$$

Here $\boldsymbol{h} \mapsto C_{ij}(\boldsymbol{h})$ are analytic functions and $\delta^i(L_0)$ denote the partial derivatives associated with $L_0$ at $(\mathbf{1},0) \in \mathsf{P}_+^\uparrow$. Due to the presupposed differentiability with respect to the uniform topology this residual term satisfies

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} |\boldsymbol{h}|^{-1} \big\|R(\boldsymbol{h})\big\| = 0, \tag{4.9b}$$

a relation which now has to be shown to stay true with the norm replaced by any of the seminorms $q_\Delta$. According to the Mean Value Theorem in a version generalized to vector-valued differentiable mappings on manifolds, for small $\boldsymbol{h}$

$$\alpha_{(\Lambda_{\boldsymbol{h}},x_{\boldsymbol{h}})}(L_0) - \alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}(L_0) = \int_0^1 d\vartheta \sum_{i,j} h_j C_{ij}(\vartheta\boldsymbol{h})\, \alpha_{(\Lambda_{\vartheta\boldsymbol{h}},x_{\vartheta\boldsymbol{h}})}\big(\delta^i(L_0)\big),$$

where the integral is to be understood with respect to the norm topology of $\mathfrak{A}$. Thus (4.9a) can be re-written as

$$R(\boldsymbol{h}) = \sum_{i,j} h_j \int_0^1 d\vartheta \left( C_{ij}(\vartheta\boldsymbol{h})\, \alpha_{(\Lambda_{\vartheta\boldsymbol{h}},x_{\vartheta\boldsymbol{h}})}\big(\delta^i(L_0)\big) - C_{ij}(\boldsymbol{0})\, \alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}\big(\delta^i(L_0)\big) \right). \tag{4.9c}$$

As a consequence of the first statement, the integrand on the right-hand side is continuous with respect to all seminorms $q_\Delta$, so that the integral exists in the completion of the locally convex space $(\mathfrak{L}, \mathfrak{T}_q)$. By application of [12, II.6.2 and 5.4], this leads to

$$|\boldsymbol{h}|^{-1} q_\Delta\big(R(\boldsymbol{h})\big) \leqslant \sum_{i,j} \max_{0 \leqslant \vartheta \leqslant 1} q_\Delta\left( C_{ij}(\vartheta\boldsymbol{h})\, \alpha_{(\Lambda_{\vartheta\boldsymbol{h}},x_{\vartheta\boldsymbol{h}})}\big(\delta^i(L_0)\big) - C_{ij}(\boldsymbol{0})\, \alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}\big(\delta^i(L_0)\big) \right),$$

where evidently the right-hand side vanishes for $\boldsymbol{h} \to \boldsymbol{0}$. This establishes differentiability of the mapping $(\Lambda, x) \mapsto \alpha_{(\Lambda,x)}(L_0)$ for $L_0 \in \mathfrak{L}_0$ with respect to $(\mathfrak{L}, \mathfrak{T}_q)$ and, by (4.9a), coincidence of the derivatives with those presupposed by the definition of $\mathfrak{L}_0$. □

Based on the continuity result of Proposition 2.10, it is possible to construct new elements of (the completions of) the locally convex spaces by way of integration.

**Lemma 4.6.** *Let the function $F \in L^1\big(\mathsf{P}_+^\uparrow, d\mu(\Lambda,x)\big)$ have compact support $\mathsf{S}$.*
*(i) To any $L_0 \in \mathfrak{L}_0$ another operator in $\mathfrak{L}_0$ is associated by*

$$\alpha_F(L_0) \doteq \int d\mu(\Lambda,x)\, F(\Lambda,x)\, \alpha_{(\Lambda,x)}(L_0). \tag{4.10}$$

*(ii) If $L \in \mathfrak{L}$ and $C \in \mathfrak{C}$, then*

$$\alpha_F(L) \doteq \int d\mu(\Lambda,x)\, F(\Lambda,x)\, \alpha_{(\Lambda,x)}(L), \tag{4.11a}$$

$$\alpha_F(C) \doteq \int d\mu(\Lambda,x)\, F(\Lambda,x)\, \alpha_{(\Lambda,x)}(C) \tag{4.11b}$$

*exist as integrals in the completions of $(\mathfrak{L}, \mathfrak{T}_q)$ and $(\mathfrak{C}, \mathfrak{T}_p)$, respectively, and satisfy*

$$q_\Delta\big(\alpha_F(L)\big) \leqslant \|F\|_1 \sup_{(\Lambda,x)\in\mathsf{S}} q_\Delta\big(\alpha_{(\Lambda,x)}(L)\big), \tag{4.12a}$$

$$p_\Delta\big(\alpha_F(C)\big) \leqslant \|F\|_1 \sup_{(\Lambda,x)\in\mathsf{S}} p_\Delta\big(\alpha_{(\Lambda,x)}(C)\big). \tag{4.12b}$$

*Proof.* (i) By assumption, $(\Lambda,x) \mapsto |F(\Lambda,x)|\,\|\alpha_{(\Lambda,x)}(L_0)\| = |F(\Lambda,x)|\,\|L_0\|$ is an integrable majorizing function for the integrand of (4.10), so $\alpha_F(L_0)$ exists as a Bochner integral in $\mathfrak{A}$. The same holds true with $L_0$ replaced by approximating local operators $L_{0,r} \in \mathfrak{A}(\mathscr{O}_r)$. Due to compactness of $\mathsf{S}$, these integrals belong to the local algebras $\mathfrak{A}\big(\mathscr{O}_{r(\mathsf{S})}\big)$ pertaining to standard diamonds in $\mathbb{R}^{s+1}$ with an $s$-dimensional basis of radius $r(\mathsf{S}) \doteq a(\mathsf{S})r + b(\mathsf{S})$, $a(\mathsf{S})$ and $b(\mathsf{S})$ suitable positive constants. Now,

$$\alpha_F(L_0) - \alpha_F(L_{0,r}) = \int_{\mathsf{S}} d\mu(\Lambda,x)\, F(\Lambda,x) \big(\alpha_{(\Lambda,x)}(L_0) - \alpha_{(\Lambda,x)}(L_{0,r})\big)$$

can be estimated for any $k \in \mathbb{N}$ ($\mu(\mathsf{S})$ is the measure of the compact set $\mathsf{S}$) by

$$0 \leqslant r(\mathsf{S})^k \big\|\alpha_F(L_0) - \alpha_F(L_{0,r})\big\| \leqslant \mu(\mathsf{S}) \|F\|_1 \big(a(\mathsf{S})r + b(\mathsf{S})\big)^k \|L_0 - L_{0,r}\|.$$

Due to almost locality of $L_0$, the right-hand side vanishes in the limit of large $r$ so that $\alpha_F(L_0)$ itself is almost local with approximating net $\left\{\alpha_F(L_{0,r}) \in \mathfrak{A}\big(\mathscr{O}_{r(\mathsf{S})}\big) : r > 0\right\}$.

Let $\Gamma \subseteq \complement \overline{V}_+$ denote the energy-momentum transfer of the vacuum annihilation operator $L_0$, then, by the Fubini Theorem [12, II.16.3], for any $g \in L^1(\mathbb{R}^{s+1}, d^{s+1}y)$

$$\int_{\mathbb{R}^{s+1}} d^{s+1}y\, g(y)\, \alpha_y\big(\alpha_F(L_0)\big) = \int_{\mathsf{S}} d\mu(\Lambda,x)\, F(\Lambda,x) \int_{\mathbb{R}^{s+1}} d^{s+1}y\, g(y)\, \alpha_y\big(\alpha_{(\Lambda,x)}(L_0)\big).$$

If the support of the Fourier transform $\tilde{g}$ of $g$ satisfies $\operatorname{supp}\tilde{g} \subseteq \bigcap_{(\Lambda,x)\in\mathsf{S}} \complement(\Lambda\Gamma)$, the inner integrals on the right-hand side vanish for any $(\Lambda,x) \in \mathsf{S}$ so that

$$\int_{\mathbb{R}^{s+1}} d^{s+1}y\, g(y)\, \alpha_y\big(\alpha_F(L_0)\big) = 0.$$

The energy-momentum transfer of $\alpha_F(L_0)$ is thus contained in $\bigcup_{(\Lambda,x)\in\mathsf{S}} \Lambda\Gamma$, a compact subset of $\complement\overline{V}_+$, and $\alpha_F(L_0)$ turns out to be a vacuum annihilation operator.

Finally, infinite differentiability of the mapping $(\Lambda,x) \mapsto \alpha_{(\Lambda,x)}\big(\alpha_F(L_0)\big)$ with respect to the uniform topology has to be established. Using the notation introduced in the proof of the second part of Proposition 2.10, we get the counterparts of equations (4.9a) and (4.9c) with $\alpha_{(\Lambda,x)}(L_0)$ (likewise infinitely often differentiable) in place of $L_0$:

$$\begin{aligned}
&R^{(\Lambda,x)}(\boldsymbol{h})\\
&\quad= \alpha_{(\Lambda_{\boldsymbol{h}},x_{\boldsymbol{h}})}\big(\alpha_{(\Lambda,x)}(L_0)\big) - \alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}\big(\alpha_{(\Lambda,x)}(L_0)\big) - \sum_{i,j} h_j C_{ij}(\boldsymbol{0})\, \alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}\big(\delta^i\big(\alpha_{(\Lambda,x)}(L_0)\big)\big)\\
&\quad= \sum_{i,j} h_j \int_0^1 d\vartheta\, \Big(C_{ij}(\vartheta\boldsymbol{h})\, \alpha_{(\Lambda_{\vartheta\boldsymbol{h}},x_{\vartheta\boldsymbol{h}})}\big(\delta^i\big(\alpha_{(\Lambda,x)}(L_0)\big)\big) - C_{ij}(\boldsymbol{0})\, \alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}\big(\delta^i\big(\alpha_{(\Lambda,x)}(L_0)\big)\big)\Big).
\end{aligned} \tag{4.13a}$$

Upon multiplication by the compactly supported function $F$, integration of all parts in this sequence of equations (which are continuous in $(\Lambda,x)$) yields, :

$$\begin{aligned}
&\int d\mu(\Lambda,x)\, F(\Lambda,x)\, R^{(\Lambda,x)}(\boldsymbol{h})\\
&= \alpha_{(\Lambda_{\boldsymbol{h}},x_{\boldsymbol{h}})}\big(\alpha_F(L_0)\big) - \alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}\big(\alpha_F(L_0)\big)\\
&\qquad\qquad - \sum_{i,j} h_j C_{ij}(\boldsymbol{0}) \int d\mu(\Lambda,x)\, F(\Lambda,x)\, \alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}\big(\delta^i\big(\alpha_{(\Lambda,x)}(L_0)\big)\big)\\
&= \sum_{i,j} h_j \int d\mu(\Lambda,x) \int_0^1 d\vartheta\, F(\Lambda,x) \cdot\\
&\qquad \cdot \Big(C_{ij}(\vartheta\boldsymbol{h})\, \alpha_{(\Lambda_{\vartheta\boldsymbol{h}},x_{\vartheta\boldsymbol{h}})}\big(\delta^i\big(\alpha_{(\Lambda,x)}(L_0)\big)\big) - C_{ij}(\boldsymbol{0})\, \alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}\big(\delta^i\big(\alpha_{(\Lambda,x)}(L_0)\big)\big)\Big).
\end{aligned} \tag{4.13b}$$

The second equation suggests to consider its last term as the derivative at $(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})$ of the mapping in question and the left-hand side as the residual term in the chosen parametrization. This interpretation is correct if the norm of the left-hand side multiplied by $|\boldsymbol{h}|^{-1}$ vanishes in the limit $\boldsymbol{h} \to \boldsymbol{0}$. This is true since, by the third part of (4.13b),

$$\begin{aligned}
&|\boldsymbol{h}|^{-1} \Big\| \int d\mu(\Lambda,x)\, F(\Lambda,x)\, R^{(\Lambda,x)}(\boldsymbol{h}) \Big\|\\
&\leqslant \sum_{i,j} \int d\mu(\Lambda,x) \int_0^1 d\vartheta\, |F(\Lambda,x)| \cdot\\
&\qquad \cdot \Big\| C_{ij}(\vartheta\boldsymbol{h})\, \alpha_{(\Lambda_{\vartheta\boldsymbol{h}},x_{\vartheta\boldsymbol{h}})}\big(\delta^i\big(\alpha_{(\Lambda,x)}(L_0)\big)\big) - C_{ij}(\boldsymbol{0})\, \alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}\big(\delta^i\big(\alpha_{(\Lambda,x)}(L_0)\big)\big) \Big\|,
\end{aligned}$$

and the right-hand side is easily seen to tend to $0$ as $\boldsymbol{h} \to \boldsymbol{0}$, by an application of Lebesgue's Dominated Convergence Theorem [12, II.5.6] upon noting the pointwise vanishing of the integrand in this limit. Now,

$$\delta^i\big(\alpha_{(\Lambda,x)}(L_0)\big) = \sum_k D_{ik}(\Lambda,x)\alpha_{(\Lambda,x)}\big(\delta^k(L_0)\big)$$

with analytic functions $D_{ik}$. Thus, the derivative of $(\Lambda,x) \mapsto \alpha_{(\Lambda,x)}\big(\alpha_F(L_0)\big)$ at $(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})$ in the $j$-th direction resulting from the second part of (4.13b) can be written

$$\begin{aligned}\sum_i C_{ij}(\boldsymbol{0}) \int d\mu(\Lambda,x)\, F(\Lambda,x)\, \alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}\Big(\delta^i\big(\alpha_{(\Lambda,x)}(L_0)\big)\Big) \\ = \sum_{ik} C_{ij}(\boldsymbol{0}) D_{ik}(\Lambda,x)\alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}\Big(\int d\mu(\Lambda,x)\, F(\Lambda,x)\, \alpha_{(\Lambda,x)}\big(\delta^k(L_0)\big)\Big) \\ = \sum_{ik} C_{ij}(\boldsymbol{0}) D_{ik}(\Lambda,x)\alpha_{(\Lambda_{\boldsymbol{0}},x_{\boldsymbol{0}})}\Big(\alpha_F\big(\delta^k(L_0)\big)\Big).\end{aligned}$$

The operators $\delta^k(L_0)$ belong to $\mathfrak{L}_0$. So, as a result of the above reasoning, $\alpha_F\big(\delta^k(L_0)\big)$ is an almost local vacuum annihilation operator which in addition is differentiable. Thus, derivatives of $(\Lambda,x) \mapsto \alpha_{(\Lambda,x)}\big(\alpha_F(L_0)\big)$ of arbitrary order exist and belong to $\mathfrak{L}_0$.

(ii) By Proposition 2.10, the mappings $(\Lambda,x) \mapsto \alpha_{(\Lambda,x)}(L)$ and $(\Lambda,x) \mapsto \alpha_{(\Lambda,x)}(C)$ are continuous with respect to the uniform topology and all the $q_\Delta$- and $p_\Delta$-topologies, hence bounded on the compact support of $\boldsymbol{F}$. This implies their measurability together with the fact that their product with the integrable function $\boldsymbol{F}$ is majorized in each case by a multiple of $|F|$. As a consequence, the integrals $\alpha_F(L)$ and $\alpha_F(C)$ exist in the completions of the locally convex spaces $(\mathfrak{L},\mathfrak{T}_q)$ and $(\mathfrak{C},\mathfrak{T}_p)$, respectively, and (4.12) is an immediate upshot of [12, II.6.2 and 5.4]. □

There exists a version of the second part of this Lemma for Lebesgue-integrable functions on $\mathbb{R}^{s+1}$.

**Lemma 4.7.** *Let $L \in \mathfrak{L}$ and let $g \in L^1\big(\mathbb{R}^{s+1}, d^{s+1}x\big)$. Then*

$$\alpha_g(L) \doteq \int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x)\, \alpha_x(L) \tag{4.14}$$

*is an operator in the completion of $(\mathfrak{L},\mathfrak{T}_q)$, satisfying*

$$q_\Delta\big(\alpha_g(L)\big) \leqslant \|g\|_1\, q_\Delta(L). \tag{4.15}$$

*The energy-momentum transfer of $\alpha_g(L)$ is contained in $\operatorname{supp}\tilde{g}$.*

*Proof.* By translation invariance of the norm $\|\,.\,\|$ and of the seminorms $q_\Delta$ as established in Proposition 2.9, the (measurable) integrand on the right-hand side of (4.14) is majorized by the functions $x \mapsto |g(x)|\,\|L\|$ and $x \mapsto |g(x)|\, q_\Delta(L)$ for any bounded Borel set $\Delta$. These are Lebesgue-integrable and, therefore, $\alpha_g(L)$ exists as a unique element of the completion of $(\mathfrak{L},\mathfrak{T}_q)$, satisfying the claimed estimates (4.15).

Consider an arbitrary function $h \in L^1\big(\mathbb{R}^{s+1}, d^{s+1}x\big)$. According to Fubini's Theorem [12, II.16.3] in combination with translation invariance of Lebesgue measure,

$$\begin{aligned}\int_{\mathbb{R}^{s+1}} d^{s+1}y\, h(y)\, \alpha_y\big(\alpha_g(L)\big) &= \int_{\mathbb{R}^{s+1}} d^{s+1}y\, h(y) \int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x)\, \alpha_{x+y}(L) \\ &= \int_{\mathbb{R}^{s+1}} d^{s+1}x \Big(\int_{\mathbb{R}^{s+1}} d^{s+1}y\, h(y)\, g(x-y)\Big) \alpha_x(L),\end{aligned}$$

where the term in parentheses on the right-hand side is the convolution product $h*g$. Its Fourier transform is $\widetilde{h*g}(p) = (2\pi)^{(s+1)/2}\tilde{h}(p)\tilde{g}(p)$ [18, Theorem VI.(21.41)] so that it vanishes if $\tilde{h}$ and $\tilde{g}$ have disjoint supports. Therefore, $\operatorname{supp}\tilde{h}\cap\operatorname{supp}\tilde{g} = \emptyset$ entails

$$\int_{\mathbb{R}^{s+1}} d^{s+1}y\, h(y)\,\alpha_y\big(\alpha_g(L)\big) = 0,$$

demonstrating that the Fourier transform of $y \mapsto \alpha_y\big(\alpha_g(L)\big)$ has support in $\operatorname{supp}\tilde{g}$ which henceforth contains the energy-momentum transfer of $\alpha_g(L)$. $\square$

Finally, it is possible to establish a property of rapid decay with respect to the seminorms $q_\Delta$ for commutators of *almost local* elements of $\mathfrak{L}$.

**Lemma 4.8.** *Let $L_1$ and $L_2$ belong to $\mathfrak{L}_0$ and let $A_1$, $A_2 \in \mathfrak{A}$ be almost local. Then*

$$\boldsymbol{x} \mapsto q_\Delta\big(\big[\alpha_{\boldsymbol{x}}(A_1L_1), A_2L_2\big]\big)$$

*is a function that decreases faster than any power of $|\boldsymbol{x}|^{-1}$ when $|\boldsymbol{x}| \to \infty$.*

*Proof.* Given an approximating net $\big\{A_r \in \mathfrak{A}(\mathscr{O}_r) : r > 0\big\}$ for an almost local operator $A$, this can be used to construct a second one, $\big\{A'_r \in \mathfrak{A}(\mathscr{O}_r) : r > 0\big\}$, with $\|A'_r\| \leqslant \|A\|$ and $\|A - A'_r\| \leqslant 2\|A - A'_r\|$. Nets with this additional property allow for an improved version of (4.2) to be used later:

$$\big\|\big[\alpha_{2\boldsymbol{x}}(A), B\big]\big\| \leqslant 2\left(\|A - A_{|\boldsymbol{x}|}\|\,\|B\| + \|A\|\,\|B - B_{|\boldsymbol{x}|}\|\right), \quad \boldsymbol{x} \in \mathbb{R}^s \setminus \{\boldsymbol{0}\}. \tag{4.16}$$

First, we consider the special case of two elements $L_a$ and $L_b$ in $\mathfrak{L}_0$ having energy-momentum transfer in compact and *convex* subsets $\Gamma_a$ and $\Gamma_b$ of $\complement\overline{V}_+$, respectively, such that $\Gamma_{a,b} \doteq (\Gamma_a + \Gamma_b) - \Gamma_a$ and $\Gamma_{b,a}$, defined accordingly, both belong to the complement of $\overline{V}_+$, too. According to Lemmas 4.5 and 4.3,

$$\begin{aligned} q_\Delta\big(\big[\alpha_{\boldsymbol{x}}(L_a), L_b\big]\big)^2 &= p_\Delta\big(\big[\alpha_{\boldsymbol{x}}(L_a), L_b\big]^*\big[\alpha_{\boldsymbol{x}}(L_a), L_b\big]\big) \\ &\leqslant q_\Delta(L_b)\, q_\Delta\big(\alpha_{\boldsymbol{x}}(L_a)^*\big[\alpha_{\boldsymbol{x}}(L_a), L_b\big]\big) + q_\Delta(L_a)\, q_\Delta\big({L_b}^*\big[\alpha_{\boldsymbol{x}}(L_a), L_b\big]\big), \end{aligned} \tag{4.17}$$

and we are left with the task to investigate $\boldsymbol{x} \mapsto q_\Delta\big(\alpha_{\boldsymbol{x}}(L_a)^*\big[\alpha_{\boldsymbol{x}}(L_a), L_b\big]\big)$ as well as $\boldsymbol{x} \mapsto q_\Delta\big({L_b}^*\big[\alpha_{\boldsymbol{x}}(L_a), L_b\big]\big)$ in the limit of large $|\boldsymbol{x}|$. The arguments of both terms belong to $\mathfrak{L}_0$ with energy-momentum transfer in the compact and convex sets $\Gamma_{a,b}$ and $\Gamma_{b,a}$. Thus, relation (4.1) of Proposition 4.1 together with (2.7a) yields for the second term

$$\begin{aligned} &|\boldsymbol{x}|^{2k} q_\Delta\big({L_b}^*\big[\alpha_{\boldsymbol{x}}(L_a), L_b\big]\big)^2 \\ &\qquad \leqslant N(\Delta, \Gamma_{b,a}) \int_{\mathbb{R}^s} d^s y\, |\boldsymbol{x}|^{2k} \Big\|\Big[\alpha_y\big({L_b}^*\big[\alpha_{\boldsymbol{x}}(L_a), L_b\big]\big), \big({L_b}^*\big[\alpha_{\boldsymbol{x}}(L_a), L_b\big]\big)^*\Big]\Big\|. \end{aligned} \tag{4.18}$$

Let $\big\{L_{a,r} \in \mathfrak{A}(\mathscr{O}_r) : r > 0\big\}$ and $\big\{L_{b,r} \in \mathfrak{A}(\mathscr{O}_r) : r > 0\big\}$ be approximating nets for $L_a$ and $L_b$, respectively, with $\|L_{a,r}\| \leqslant \|L_a\|$ and $\|L_{b,r}\| \leqslant \|L_b\|$; the operators ${L_{b,r}}^*\big[\alpha_{\boldsymbol{x}}(L_{a,r}), L_{b,r}\big] \in \mathfrak{A}\big(\mathscr{O}_{r+|\boldsymbol{x}|}\big)$ then constitute the large radius part of approximating nets for the almost local operators ${L_b}^*\big[\alpha_{\boldsymbol{x}}(L_a), L_b\big]$, $\boldsymbol{x} \in \mathbb{R}^s$, so that for suitable $C_l > 0$, $l \in \mathbb{N}$,

$$\big\|{L_b}^*\big[\alpha_{\boldsymbol{x}}(L_a), L_b\big] - {L_{b,r}}^*\big[\alpha_{\boldsymbol{x}}(L_{a,r}), L_{b,r}\big]\big\| \leqslant C_l\, r^{-l}. \tag{4.19}$$

Therefore, approximating nets $\big\{L(a,b;\boldsymbol{x})_r \in \mathfrak{A}(\mathscr{O}_r) : r > 0\big\}$, $\boldsymbol{x} \in \mathbb{R}^s$, exist which satisfy $\|L(a,b;\boldsymbol{x})_r\| \leqslant \big\|{L_b}^*\big[\alpha_{\boldsymbol{x}}(L_a), L_b\big]\big\|$ and $\big\|{L_b}^*\big[\alpha_{\boldsymbol{x}}(L_a), L_b\big] - L(a,b;\boldsymbol{x})_{r+|\boldsymbol{x}|}\big\| \leqslant 2C_l\, r^{-l}$, due to the introductory remark. This implies, according to (4.16), that the integrand of (4.18) is bounded by

$$
\begin{aligned}
&|\boldsymbol{x}|^{2k}\left\|\left[\alpha_{\boldsymbol{y}}\left(L_b{}^*\left[\alpha_{\boldsymbol{x}}(L_a),L_b\right]\right),\left(L_b{}^*\left[\alpha_{\boldsymbol{x}}(L_a),L_b\right]\right)^*\right]\right\| \\
&\qquad \leqslant |\boldsymbol{x}|^{2k}4\left\|L_b{}^*\left[\alpha_{\boldsymbol{x}}(L_a),L_b\right]\right\|\left\|L_b{}^*\left[\alpha_{\boldsymbol{x}}(L_a),L_b\right]-L(a,b;\boldsymbol{x})_{2^{-1}|\boldsymbol{y}|}\right\| \\
&\qquad \leqslant \begin{cases} 8\,|\boldsymbol{x}|^{2k}\|L_b\|^2\left\|\left[\alpha_{\boldsymbol{x}}(L_a),L_b\right]\right\|^2 & ,\ |\boldsymbol{y}|\leqslant 2(|\boldsymbol{x}|+1), \\ 8\,\|L_b\||\boldsymbol{x}|^{2k}\left\|\left[\alpha_{\boldsymbol{x}}(L_a),L_b\right]\right\|C_l(2^{-1}|\boldsymbol{y}|-|\boldsymbol{x}|)^{-l} & ,\ |\boldsymbol{y}|>2(|\boldsymbol{x}|+1). \end{cases}
\end{aligned} \tag{4.20}
$$

Having these estimates at hand, integration with respect to $\boldsymbol{y}$ of the right-hand side yields in both cases for $l \geqslant s+2$ polynomials of degree $s$ in $|\boldsymbol{x}|$ so that, due to the decay properties of the function $\boldsymbol{x} \mapsto \left\|\left[\alpha_{\boldsymbol{x}}(L_a),L_b\right]\right\|$, there exists a uniform bound

$$
|\boldsymbol{x}|^k q_\Delta\left(L_b{}^*\left[\alpha_{\boldsymbol{x}}(L_a),L_b\right]\right)^2 \leqslant M, \quad \boldsymbol{x}\in\mathbb{R}^s. \tag{4.21}
$$

The same reasoning applies to the term $q_\Delta\left(\alpha_{\boldsymbol{x}}(L_a)^*\left[\alpha_{\boldsymbol{x}}(L_a),L_b\right]\right)$, thus, by virtue of (4.17), establishing the asserted rapid decrease of the mapping $\boldsymbol{x}\mapsto q_\Delta\left(\left[\alpha_{\boldsymbol{x}}(L_a),L_b\right]\right)$.

For arbitrary almost local elements $A_1$, $A_2 \in \mathfrak{A}$ and $L_1$, $L_2 \in \mathfrak{L}_0$ one has, by use of Lemma 4.3,

$$
\begin{aligned}
&q_\Delta\left(\left[\alpha_{\boldsymbol{x}}(A_1L_1),A_2L_2\right]\right) \\
&\leqslant \|A_1\|\left\|\left[\alpha_{\boldsymbol{x}}(L_1),A_2\right]\right\|q_\Delta(L_2)+\|A_1\|\|A_2\|\,q_\Delta\left(\left[\alpha_{\boldsymbol{x}}(L_1),L_2\right]\right) \\
&\qquad\qquad +\left\|\left[\alpha_{\boldsymbol{x}}(A_1),A_2\right]\right\|\|L_2\|\,q_\Delta(L_1)+\|A_2\|\left\|\left[\alpha_{\boldsymbol{x}}(A_1),L_2\right]\right\|q_\Delta(L_1),
\end{aligned}
$$

and rapid decay is an immediate consequence of almost locality for all terms but the second one on the right-hand side of this inequality. Using suitable decompositions of $L_1$ and $L_2$ in terms of elements of $\mathfrak{L}_0$ complying pairwise with the special properties exploited in the previous paragraph, the remaining problem of decrease of the function $\boldsymbol{x}\mapsto q_\Delta\left(\left[\alpha_{\boldsymbol{x}}(L_1),L_2\right]\right)$ reduces to the case that has already been considered above. □

# 5 Proofs for Section 3

The following results are concerned with integrability properties of functionals in $\mathfrak{C}^*$. Lemma 5.1 is an immediate consequence of Lemmas 4.6 and 4.7, whereas Lemma 5.2 prepares the proof of a kind of Cluster Property for *positive* functionals in $\mathfrak{C}^*$, formulated in Proposition 3.4.

**Lemma 5.1.** *Let $\varsigma\in\mathfrak{C}^*$, $L_1$, $L_2\in\mathfrak{L}$ and $C\in\mathfrak{C}$.*
*(i) Let $F\in L^1\left(\mathsf{P}_+^\uparrow,d\mu(\Lambda,x)\right)$ have compact support $\mathsf{S}$, then*

$$
\varsigma\left(L_1{}^*\alpha_F(L_2)\right)=\int d\mu(\Lambda,x)\,F(\Lambda,x)\,\varsigma\left(L_1{}^*\alpha_{(\Lambda,x)}(L_2)\right), \tag{5.1a}
$$

$$
\varsigma\left(\alpha_F(C)\right)=\int d\mu(\Lambda,x)\,F(\Lambda,x)\,\varsigma\left(\alpha_{(\Lambda,x)}(C)\right), \tag{5.1b}
$$

*and there hold the estimates*

$$
\left|\varsigma\left(L_1{}^*\alpha_F(L_2)\right)\right|\leqslant\|F\|_1\|\varsigma\|_\Delta\,q_\Delta(L_1)\sup_{(\Lambda,x)\in\mathsf{S}}q_\Delta\left(\alpha_{(\Lambda,x)}(L_2)\right), \tag{5.2a}
$$

$$
\left|\varsigma\left(\alpha_F(C)\right)\right|\leqslant\|F\|_1\|\varsigma\|_\Delta\sup_{(\Lambda,x)\in\mathsf{S}}p_\Delta\left(\alpha_{(\Lambda,x)}(C)\right) \tag{5.2b}
$$

*for any $\Delta$ such that $\varsigma\in\mathfrak{C}_\Delta{}^*$.*

*(ii) For any function* $g \in L^1(\mathbb{R}^{s+1}, d^{s+1}x)$

$$\varsigma(L_1{}^*\alpha_g(L_2)) = \int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x)\, \varsigma(L_1{}^*\alpha_x(L_2)), \tag{5.3}$$

*and a bound is given by*

$$\big|\varsigma(L_1{}^*\alpha_g(L_2))\big| \leqslant \|g\|_1 \|\varsigma\|_\Delta\, q_\Delta(L_1)\, q_\Delta(L_2) \tag{5.4}$$

*for any* $\Delta$ *with* $\varsigma \in \mathfrak{C}_\Delta^*$.

**Lemma 5.2.** *Let* $L' \in \mathfrak{L}$ *and let* $L \in \mathfrak{L}$ *have energy-momentum transfer in the compact set* $\Gamma \subset \complement V_+$. *If* $\varsigma \in \mathfrak{C}^{*+}$ *is a positive functional belonging to* $\mathfrak{C}_\Delta^*$ *and* $\Delta'$ *denotes any bounded Borel set containing* $\Delta + \Gamma$, *then*

$$\int_{\mathbb{R}^s} d^s x\, \varsigma(L^* \alpha_{\boldsymbol{x}}(L'^* L')L) \leqslant \|\varsigma\|_\Delta\, q_\Delta(L)^2 q_{\Delta'}(L')^2. \tag{5.5}$$

*Proof.* Let $\boldsymbol{K}$ be an arbitrary compact subset of $\mathbb{R}^s$. Then

$$\int_{\boldsymbol{K}} d^s x\, L^* \alpha_{\boldsymbol{x}}(L'^* L')L = L^*\Big(\int_{\boldsymbol{K}} d^s x\, \alpha_{\boldsymbol{x}}(L'^* L')\Big)L$$

belongs to the algebra of counters and exists furthermore as an integral in the completion of $\mathfrak{C}$ with respect to the $p_\Delta$-seminorms. Therefore, the continuous functional $\varsigma$ can be interchanged with the integral [12, Proposition II.5.7] to give

$$\int_{\boldsymbol{K}} d^s x\, \varsigma(L^* \alpha_{\boldsymbol{x}}(L'^* L')L) = \varsigma\Big(L^* \int_{\boldsymbol{K}} d^s x\, \alpha_{\boldsymbol{x}}(L'^* L')\, L\Big).$$

Making use of the positivity of $\varsigma$, an application of Lemma 4.3 leads to the estimate

$$\begin{aligned} 0 \leqslant \int_{\boldsymbol{K}} d^s x\, \varsigma(L^* \alpha_{\boldsymbol{x}}(L'^* L')L) &\leqslant \|\varsigma\|_\Delta\, p_\Delta\Big(L^* \int_{\boldsymbol{K}} d^s x\, \alpha_{\boldsymbol{x}}(L'^* L')\, L\Big) \\ &\leqslant \|\varsigma\|_\Delta\, q_\Delta(L)^2 \Big\| E(\Delta') \int_{\boldsymbol{K}} d^s x\, \alpha_{\boldsymbol{x}}(L'^* L')\, E(\Delta')\Big\|, \end{aligned}$$

which survives in the limit $\boldsymbol{K} \nearrow \mathbb{R}^s$. Since the right-hand side stays finite in this procedure the function $\boldsymbol{x} \mapsto \varsigma(L^* \alpha_{\boldsymbol{x}}(L'^* L')L)$ is integrable as a consequence of the Monotone Convergence Theorem; its integral over $\mathbb{R}^s$ satisfies the asserted estimate due to equation (2.7a). □

Next comes the proof of the Cluster Property for positive functionals in $\mathfrak{C}^*$.

*Proposition 3.4.* Commuting $A_1 L_1'$ and $\alpha_{\boldsymbol{x}}(L_2{}^* A_2)$ in the argument of (3.2), we get the estimate

$$\begin{aligned} &\big|\varsigma\big((L_1{}^* A_1 L_1')\alpha_{\boldsymbol{x}}(L_2{}^* A_2 L_2')\big)\big| \\ &\qquad \leqslant \big|\varsigma\big(L_1{}^*\big[A_1 L_1', \alpha_{\boldsymbol{x}}(L_2{}^* A_2)\big]\alpha_{\boldsymbol{x}}(L_2')\big)\big| + \big|\varsigma\big(L_1{}^* \alpha_{\boldsymbol{x}}(L_2{}^* A_2) A_1 L_1' \alpha_{\boldsymbol{x}}(L_2')\big)\big|. \end{aligned} \tag{5.6}$$

Making use of Lemma 4.3 and Proposition 2.9, the first term on the right-hand side turns out to be integrable over $\mathbb{R}^s$, due to almost locality of the operators encompassed by the commutator:

$$\begin{aligned} &\int_{\mathbb{R}^s} d^s x\, \big|\varsigma\big(L_1{}^*\big[A_1 L_1', \alpha_{\boldsymbol{x}}(L_2{}^* A_2)\big]\alpha_{\boldsymbol{x}}(L_2')\big)\big| \\ &\qquad \leqslant \|\varsigma\|_\Delta\, q_\Delta(L_1)\, q_\Delta(L_2') \int_{\mathbb{R}^s} d^s x\, \big\|\big[A_1 L_1', \alpha_{\boldsymbol{x}}(L_2{}^* A_2)\big]\big\|. \end{aligned} \tag{5.7}$$

By positivity of $\varsigma$, application of the Cauchy-Schwarz inequality yields the following bound for the second term of (5.6):

$$\varsigma\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2{}^*A_2A_2{}^*L_2)L_1\big)^{1/2}\varsigma\big(\alpha_{\boldsymbol{x}}(L_2'^*)L_1'^*A_1{}^*A_1L_1'\alpha_{\boldsymbol{x}}(L_2')\big)^{1/2} \\ = 2^{-1}\Big(\varsigma\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2{}^*A_2A_2{}^*L_2)L_1\big) + \varsigma\big(\alpha_{\boldsymbol{x}}(L_2'^*)L_1'^*A_1{}^*A_1L_1'\alpha_{\boldsymbol{x}}(L_2')\big)\Big). \tag{5.8}$$

Integration of the first term on the right-hand side is possible, according to Lemma 5.2:

$$\int_{\mathbb{R}^s} d^sx\,\varsigma\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2{}^*A_2A_2{}^*L_2)L_1\big) \leqslant \|\varsigma\|_\Delta\, q_\Delta(L_1)^2 q_{\Delta_1}(A_2{}^*L_2)^2, \tag{5.9}$$

where $\Delta_1$ is any bounded Borel set containing the sum of $\Delta$ and the energy-momentum transfer $\Gamma_1$ of $L_1$. Upon commuting $\alpha_{\boldsymbol{x}}(L_2'^*)$ and $\alpha_{\boldsymbol{x}}(L_2')$ to the interior in the second term of (5.8), it turns out to be bounded by (cf. Lemma 4.3 and Proposition 2.9)

$$\big|\varsigma\big(\big[\alpha_{\boldsymbol{x}}(L_2'^*),L_1'^*\big]A_1{}^*A_1L_1'\alpha_{\boldsymbol{x}}(L_2')\big)\big| + \big|\varsigma\big(L_1'^*\alpha_{\boldsymbol{x}}(L_2'^*)A_1{}^*A_1\big[L_1',\alpha_{\boldsymbol{x}}(L_2')\big]\big)\big| \\ + \big|\varsigma\big(L_1'^*\alpha_{\boldsymbol{x}}(L_2'^*)A_1{}^*A_1\alpha_{\boldsymbol{x}}(L_2')L_1'\big)\big| \\ \leqslant \|A_1\|^2\Big(\|\varsigma\|_\Delta\big(\|L_1'\|\,q_\Delta(L_2') + \|L_2'\|\,q_\Delta(L_1')\big)q_\Delta\big(\big[L_1',\alpha_{\boldsymbol{x}}(L_2')\big]\big) + \varsigma\big(L_1'^*\alpha_{\boldsymbol{x}}(L_2'^*L_2')L_1'\big)\Big), \tag{5.10}$$

where again use is made of the positivity of $\varsigma$. Lemma 4.8 on rapid decay of commutators of almost local operators with respect to the $q_\Delta$-seminorm and Lemma 5.2 show integrability of the right-hand side of (5.10) over $\mathbb{R}^s$, where in view of (5.5) the integral is bounded by a term proportional to $\|\varsigma\|_\Delta$. Combining this result with (5.7) and (5.9) establishes the assertion. □

The Cluster Property has been proved under the fairly general assumption of almost locality of the operators involved. It also holds, if the mapping $\boldsymbol{x} \mapsto p_\Delta\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2)\big)$, $\boldsymbol{x} \in \mathbb{R}^s$, is integrable for given $L_1$, $L_2 \in \mathfrak{L}$ and the continuous functional $\varsigma$ belongs to $\mathfrak{C}_\Delta{}^*$. Another consequence of this combination of properties concerns weakly convergent nets $\{\varsigma_\iota : \iota \in J\}$ of functionals from bounded subsets of $\mathfrak{C}_\Delta{}^*$: a kind of Dominated Convergence Theorem.

**Lemma 5.3.** *Let $L_1$, $L_2 \in \mathfrak{L}$ be such that $\boldsymbol{x} \mapsto p_\Delta\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2)\big)$ is integrable and consider the weakly convergent net $\{\varsigma_\iota : \iota \in J\}$ in a bounded subset of $\mathfrak{C}_\Delta{}^*$ with limit $\varsigma$. Then*

$$\int_{\mathbb{R}^s} d^sx\,\varsigma\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2)\big) = \lim_\iota \int_{\mathbb{R}^s} d^sx\,\varsigma_\iota\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2)\big). \tag{5.11}$$

*Proof.* By assumption of integrability of $\boldsymbol{x} \mapsto p_\Delta\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2)\big)$, there exists a compact set $\boldsymbol{K}$ to any $\varepsilon > 0$ such that

$$\int_{\complement \boldsymbol{K}} d^sx\,p_\Delta\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2)\big) < \varepsilon. \tag{5.12}$$

Moreover, by Proposition 2.10 and Corollary 4.4, $\boldsymbol{x} \mapsto L_1{}^*\alpha_{\boldsymbol{x}}(L_2)$ is a continuous mapping on $\mathbb{R}^s$ with respect to the $p_\Delta$-topology, hence uniformly continuous on $\boldsymbol{K}$. This means that there exists $\delta > 0$ such that $\boldsymbol{x}$, $\boldsymbol{x}' \in \boldsymbol{K}$ and $|\boldsymbol{x}-\boldsymbol{x}'| < \delta$ imply

$$p_\Delta\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2) - L_1{}^*\alpha_{\boldsymbol{x}'}(L_2)\big) < \varepsilon. \tag{5.13}$$

By compactness of $\boldsymbol{K}$, we can find finitely many elements $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N \in \boldsymbol{K}$ so that the $\delta$-balls around these points cover all of $\boldsymbol{K}$, and, since $\varsigma$ is the weak limit of the net $\{\varsigma_\iota : \iota \in J\}$, there exists $\iota_0 \in J$ with the property that $\iota \succ \iota_0$ entails

$$\big|\varsigma\big(L_1{}^*\alpha_{\boldsymbol{x}_i}(L_2)\big) - \varsigma_\iota\big(L_1{}^*\alpha_{\boldsymbol{x}_i}(L_2)\big)\big| < \varepsilon \tag{5.14}$$

for any $i = 1, \ldots, N$. Now, selecting for $\boldsymbol{x} \in \boldsymbol{K}$ an appropriate $\boldsymbol{x}_k$ in a distance less than $\delta$ and making use of (5.13) and (5.14), one has for any $\boldsymbol{x} \in \boldsymbol{K}$ and $\iota \succ \iota_0$

$$\begin{aligned}&\left|\varsigma\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2)\big) - \varsigma_\iota\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2)\big)\right| \\ &\qquad \leqslant \|\varsigma\|_\Delta\,\varepsilon + \left|\varsigma\big(L_1{}^*\alpha_{\boldsymbol{x}_k}(L_2)\big) - \varsigma_\iota\big(L_1{}^*\alpha_{\boldsymbol{x}_k}(L_2)\big)\right| + \|\varsigma_\iota\|_\Delta\,\varepsilon \leqslant \varepsilon\,(1 + \|\varsigma\|_\Delta + \|\varsigma_\iota\|_\Delta).\end{aligned}$$

For these indices we thus arrive at the estimate

$$\begin{aligned}&\left|\int_{\mathbb{R}^s} d^s x\,\Big(\varsigma\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2)\big) - \varsigma_\iota\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2)\big)\Big)\right| \\ &\leqslant \left|\int_{\boldsymbol{K}} d^s x\,\Big(\varsigma\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2)\big) - \varsigma_\iota\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2)\big)\Big)\right| \\ &\qquad + \left|\int_{\complement\boldsymbol{K}} d^s x\,\Big(\varsigma\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2)\big) - \varsigma_\iota\big(L_1{}^*\alpha_{\boldsymbol{x}}(L_2)\big)\Big)\right| \\ &\qquad\qquad \leqslant \varepsilon\,(1 + \|\varsigma\|_\Delta + \|\varsigma_\iota\|_\Delta)\int_{\boldsymbol{K}} d^s x + (\|\varsigma\|_\Delta + \|\varsigma_\iota\|_\Delta)\,\varepsilon,\end{aligned}$$

where use is made of (5.12). Since the index $\iota_0$ can be determined appropriately for arbitrarily small $\varepsilon$, this inequality proves the possibility to interchange integration and the limit with respect to $\iota$ as asserted in (5.11). □

The proof of the Spectral Property of functionals in $\mathfrak{C}^*$ relies on the Lemmas established above.

*Proposition 3.5.* According to Lemma 4.7, for $g \in L^1(\mathbb{R}^{s+1}, d^{s+1}x)$ the operator

$$\alpha_g(L_2) = \int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x)\,\alpha_x(L_2)$$

lies in the completion of $(\mathfrak{L}, \mathfrak{T}_q)$ with energy-momentum transfer contained in $\operatorname{supp}\tilde{g}$. If $\varsigma$ belongs to $\mathfrak{C}_\Delta^*$, we infer from $\operatorname{supp}\tilde{g} \subseteq \complement(\overline{V}_+ - \Delta)$ that $\alpha_g(L_2)E(\Delta) = 0$ and henceforth, by Lemma 4.2, $q_\Delta\big(\alpha_g(L_2)\big) = 0$. Lemma 5.1 then yields

$$\left|\int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x)\,\varsigma\big(L_1{}^*\alpha_x(L_2)\big)\right| = \left|\varsigma\big(L_1{}^*\alpha_g(L_2)\big)\right| \leqslant \|\varsigma\|_\Delta\, q_\Delta(L_1)\, q_\Delta\big(\alpha_g(L_2)\big)$$

which, according to the preceding considerations, entails

$$\int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x)\,\varsigma\big(L_1{}^*\alpha_x(L_2)\big) = 0. \tag{5.15}$$

Now, let $g' \in L^1(\mathbb{R}^{s+1}, d^{s+1}x)$ have $\operatorname{supp}\tilde{g'} \subseteq \complement(\overline{V}_+ - q)$ with $\Delta$ lying in $q - \overline{V}_+$, then $\operatorname{supp}\tilde{g'} \subseteq \complement(\overline{V}_+ - \Delta)$ and (5.15) is satisfied, thus proving the assertion. □

Recall that the following two proofs require the function $h$ to belong to $C_{0,c}(\mathbb{R}^s)$, the space of continuous functions in $C(\mathbb{R}^s)$ which approximate a constant value in the limit $|\boldsymbol{v}| \to \infty$.

*Proposition 3.9.* Due to translation invariance of Lebesgue measure, one has for any finite time $t$ and any given $x = (x^0, \boldsymbol{x}) \in \mathbb{R}^{s+1}$

$$\rho_{h,t}\big(\alpha_{(x^0,\boldsymbol{x})}(C)\big) = T(t)^{-1}\int_{t+x^0}^{t+x^0+T(t)} d\tau \int_{\mathbb{R}^s} d^s y\, h\big((\tau - x^0)^{-1}(\boldsymbol{y} - \boldsymbol{x})\big)\,\omega\big(\alpha_{(\tau,\boldsymbol{y})}(C)\big).$$

Accordingly, $\left|\rho_{h,t}(C)-\rho_{h,t}\left(\alpha_{(x^0,\boldsymbol{x})}(C)\right)\right|$ can be split into a sum of three integrals to be estimated separately:

$$\begin{aligned}\left|T(t)^{-1}\int_{t}^{t+x^0} d\tau \int_{\mathbb{R}^s} d^s y\, h(\tau^{-1}\boldsymbol{y})\,\omega\left(\alpha_{(\tau,\boldsymbol{y})}(C)\right)\right| &\leqslant |T(t)|^{-1}|x^0|\,\|h\|_\infty\, p_\Delta(C),\\ \left|T(t)^{-1}\int_{t+x^0+T(t)}^{t+T(t)} d\tau \int_{\mathbb{R}^s} d^s y\, h(\tau^{-1}\boldsymbol{y})\,\omega\left(\alpha_{(\tau,\boldsymbol{y})}(C)\right)\right| &\leqslant |T(t)|^{-1}|x^0|\,\|h\|_\infty\, p_\Delta(C).\end{aligned}$$

Both $\rho_{h,t}(C)$ and $\rho_{h,t}\left(\alpha_{(x^0,\boldsymbol{x})}(C)\right)$ contribute to the third one

$$\begin{aligned}&\left|T(t)^{-1}\int_{t+x^0}^{t+x^0+T(t)} d\tau \int_{\mathbb{R}^s} d^s y\,\left[h(\tau^{-1}\boldsymbol{y})-h\left((\tau-x^0)^{-1}(\boldsymbol{y}-\boldsymbol{x})\right)\right]\omega\left(\alpha_{(\tau,\boldsymbol{y})}(C)\right)\right|\\ &\qquad\qquad\leqslant \sup_{\tau\in I_{t,x^0}}\sup_{\boldsymbol{y}\in\mathbb{R}^s}\left|h(\tau^{-1}\boldsymbol{y})-h\left((\tau-x^0)^{-1}(\boldsymbol{y}-\boldsymbol{x})\right)\right| p_\Delta(C),\end{aligned}$$

where we used the abbreviation $I_{t,x^0}$ for the interval of $\tau$-integration. Note that for $|t|$ large enough division by $\tau$ and $\tau-x^0$ presents no problem. We finally arrive at the following estimate, setting $\boldsymbol{z}_\tau \doteq \boldsymbol{z}+(\tau-x^0)^{-1}(x^0\boldsymbol{z}-\boldsymbol{x})$,

$$\left|\rho_{h,t}(C)-\rho_{h,t}\left(\alpha_{(x^0,\boldsymbol{x})}(C)\right)\right| \leqslant \left(2\,|T(t)|^{-1}|x^0|\,\|h\|_\infty + \sup_{\tau\in I_{t,x^0}}\sup_{\boldsymbol{z}\in\mathbb{R}^s}\left|h(\boldsymbol{z})-h(\boldsymbol{z}_\tau)\right|\right) p_\Delta(C). \tag{5.16}$$

Since, by assumption, $h$ approaches a constant value for $|\boldsymbol{z}|\to\infty$, there exists to $\varepsilon>0$ a compact ball $K$ in $\mathbb{R}^s$ so that $\boldsymbol{z}\in\complement K$ implies $|h(\boldsymbol{z})-h(\boldsymbol{z}_\tau)|<\varepsilon$ for large $|\tau|$. On the other hand, the net $\{\boldsymbol{z}_\tau : \tau\in\mathbb{R}\}$ approximates $\boldsymbol{z}$ uniformly on compact subsets of $\mathbb{R}^s$ in the limit $|\tau|\to\infty$; as a consequence of continuity of $h$, i. e., uniform continuity on compacta, $|h(\boldsymbol{z})-h(\boldsymbol{z}_\tau)|<\varepsilon$ also holds for $\boldsymbol{z}\in K$ in this limit. Thus, for large $|\tau|$ the term $\sup_{\boldsymbol{z}\in\mathbb{R}^s}|h(\boldsymbol{z})-h(\boldsymbol{z}_\tau)|$ falls below any given positive bound so that the right-hand side of (5.16) is seen to vanish with $|t|\to\infty$, since then $|T(t)|$ exceeds any positive value. The same holds true for the limit of the left-hand-side, $|\sigma(C)-\sigma(\alpha_x(C))|$, which establishes the assertion. □

Last in this sequence of proofs comes that of the existence of lower bounds.

*Proposition 3.10.* Consider the functional $\rho_{h,t}$ defined via $\omega\in\mathscr{S}(\Delta)$ at finite time $t$. Application of the Cauchy-Schwarz inequality with respect to the inner product of square-integrable functions $f$ and $g$,

$$(f,g)_t \doteq T(t)^{-1}\int_t^{t+T(t)} d\tau\,\overline{f(\tau)}\,g(\tau),$$

to the absolute square of $\rho_{h,t}$ yields the estimate

$$\left|\rho_{h,t}(C)\right|^2 \leqslant T(t)^{-1}\int_t^{t+T(t)} d\tau\left|\int_{\mathbb{R}^s} d^s x\, h(\tau^{-1}\boldsymbol{x})\,\omega\left(\alpha_\tau\left(\alpha_{\boldsymbol{x}}(C)\right)\right)\right|^2. \tag{5.17}$$

Now, let $K\subset\mathbb{R}^s$ be compact; due to the positivity of the state $\omega\in\mathscr{S}(\Delta)$, [5, Proposition 2.3.11(b)] and Fubini's Theorem [12, II.16.3] lead to the following estimate for arbitrary $\tau$:

$$\begin{aligned}&\left|\omega\left(\alpha_\tau\left(\int_K d^s x\, h(\tau^{-1}\boldsymbol{x})\,\alpha_{\boldsymbol{x}}(C)\right)\right)\right|^2\\ &\qquad\leqslant \omega\left(\alpha_\tau\left(\int_K d^s x\int_K d^s y\, h(\tau^{-1}\boldsymbol{y})\,h(\tau^{-1}\boldsymbol{x})\,\alpha_{\boldsymbol{y}}(C^*)\,\alpha_{\boldsymbol{x}}(C)\right)\right).\end{aligned} \tag{5.18}$$

Commuting $\omega \circ \alpha_\tau$ and the integrals and passing to the limit $K \nearrow \mathbb{R}^s$, one arrives, on account of the assumed integrability of the mapping $\boldsymbol{x} \mapsto p_\Delta\big(C^* \alpha_{\boldsymbol{x}}(C)\big)$, at

$$\begin{aligned}
&\Big| \int_{\mathbb{R}^s} d^s x \, h(\tau^{-1}\boldsymbol{x}) \, \omega\big(\alpha_\tau\big(\alpha_{\boldsymbol{x}}(C)\big)\big) \Big|^2 \\
&\leqslant \int_{\mathbb{R}^s} d^s x \int_{\mathbb{R}^s} d^s y \, h(\tau^{-1}\boldsymbol{y}) \, h(\tau^{-1}\boldsymbol{x}) \, \omega\big(\alpha_\tau\big(\alpha_{\boldsymbol{y}}(C^*)\alpha_{\boldsymbol{x}}(C)\big)\big) \leqslant \|h\|_\infty^2 \int_{\mathbb{R}^s} d^s x \, p_\Delta\big(C^* \alpha_{\boldsymbol{x}}(C)\big).
\end{aligned} \tag{5.19}$$

Combination of (5.17) and (5.19) finally yields

$$\big|\rho_{h,t}(C)\big|^2 \leqslant T(t)^{-1} \int_t^{t+T(t)} d\tau \int_{\mathbb{R}^s} d^s x \int_{\mathbb{R}^s} d^s y \, h(\tau^{-1}\boldsymbol{y}) \, h(\tau^{-1}\boldsymbol{x}) \, \omega\big(\alpha_\tau\big(\alpha_{\boldsymbol{y}}(C^*)\alpha_{\boldsymbol{x}}(C)\big)\big). \tag{5.20}$$

We want to replace the term $h(\tau^{-1}\boldsymbol{x})$ by the norm $\|h\|_\infty$ and, to do so, define the function $h_+ \doteq (\|h\|_\infty h - h^2)^{1/2}$ which is a non-negative element of $C_{0,c}(\mathbb{R}^s)$ as is $h$ itself. Then for any $\boldsymbol{z}, \boldsymbol{z}' \in \mathbb{R}^s$ there holds the equation

$$\|h\|_\infty h(\boldsymbol{z}) = h(\boldsymbol{z})\, h(\boldsymbol{z}') + h_+(\boldsymbol{z})\, h_+(\boldsymbol{z}') + h_+(\boldsymbol{z})\big(h_+(\boldsymbol{z}) - h_+(\boldsymbol{z}')\big) + h(\boldsymbol{z})\big(h(\boldsymbol{z}) - h(\boldsymbol{z}')\big). \tag{5.21}$$

Next, for an arbitrary function $g \in C_{0,c}(\mathbb{R}^s)$ the following inequality can be based on an application of Fubini's Theorem and the reasoning of (3.7):

$$\begin{aligned}
&\Big| T(t)^{-1} \int_t^{t+T(t)} d\tau \int_{\mathbb{R}^s} d^s x \int_{\mathbb{R}^s} d^s y \, g(\tau^{-1}\boldsymbol{y}) \big(g(\tau^{-1}\boldsymbol{y}) - g(\tau^{-1}\boldsymbol{x})\big) \, \omega\big(\alpha_\tau\big(\alpha_{\boldsymbol{y}}(C^*)\alpha_{\boldsymbol{x}}(C)\big)\big) \Big| \\
&\quad = \Big| \int_{\mathbb{R}^s} d^s x \, T(t)^{-1} \int_t^{t+T(t)} d\tau \int_{\mathbb{R}^s} d^s z \, \tau^s g(\boldsymbol{z}) \big(g(\boldsymbol{z}) - g(\boldsymbol{z}_\tau(\boldsymbol{x}))\big) \, \omega\big(\alpha_{(\tau,\tau\boldsymbol{z})}\big(C^* \alpha_{\boldsymbol{x}}(C)\big)\big) \Big| \\
&\qquad\qquad \leqslant \|g\|_\infty \int_{\mathbb{R}^s} d^s x \, \sup_{\tau \in I_t} \sup_{\boldsymbol{z} \in \mathbb{R}^s} \big| g(\boldsymbol{z}) - g\big(\boldsymbol{z}_\tau(\boldsymbol{x})\big) \big| \, p_\Delta\big(C^* \alpha_{\boldsymbol{x}}(C)\big).
\end{aligned} \tag{5.22}$$

Here we use the coordinate transformation $\boldsymbol{x} \rightsquigarrow \boldsymbol{x} + \boldsymbol{y}$ followed by the transformation $\boldsymbol{y} \rightsquigarrow \boldsymbol{z} \doteq \tau^{-1}\boldsymbol{y}$ and introduce the abbreviations $\boldsymbol{z}_\tau(\boldsymbol{x}) \doteq \tau^{-1}\boldsymbol{x} + \boldsymbol{z}$ as well as $I_t$ for the interval of $\tau$-integration. Similar to the proof of Proposition 3.9, the expression $\sup_{\tau \in I_t} \sup_{\boldsymbol{z} \in \mathbb{R}^s} \big| g(\boldsymbol{z}) - g\big(\boldsymbol{z}_\tau(\boldsymbol{x})\big) \big|$ is seen to vanish for all $\boldsymbol{x} \in \mathbb{R}^s$ in the limit of large $|t|$ so that by Lebesgue's Dominated Convergence Theorem the left-hand side of (5.22) converges to 0. This reasoning in particular applies to the functions $h$ as well as $h_+$ and thus to the third and fourth term on the right of equation (5.21). On the other hand, substitution of $h$ by $h_+$ in the integral (5.20) likewise gives a non-negative result for all times $t$. Combining all this information and specializing to a subnet $\{t_\iota : \iota \in J\}$ approximating $+\infty$ or $-\infty$, one arrives at the following asymptotic version of (5.20):

$$\begin{aligned}
&\lim_\iota \big|\rho_{h,t_\iota}(C)\big|^2 \\
&\quad \leqslant \lim_\iota \|h\|_\infty T(t_\iota)^{-1} \int_{t_\iota}^{t_\iota+T(t_\iota)} d\tau \int_{\mathbb{R}^s} d^s x \int_{\mathbb{R}^s} d^s y \, h(\tau^{-1}\boldsymbol{y}) \, \omega\big(\alpha_\tau\big(\alpha_{\boldsymbol{y}}(C^*)\alpha_{\boldsymbol{x}}(C)\big)\big) \\
&\qquad = \|h\|_\infty \lim_\iota \int_{\mathbb{R}^s} d^s x \, \rho_{h,t_\iota}\big(C^* \alpha_{\boldsymbol{x}}(C)\big).
\end{aligned}$$

By Lemma 5.3, this result extends to the limit functional $\sigma$, thus yielding (3.10). □

The equivalent characterization of particle weights in Theorem 3.12 is immediate apart from an application of Stone's Theorem.

*Theorem 3.12.* Part (I): The various properties stated in the Theorem are readily established, once the GNS-construction has been carried out. The existence of a strongly continuous unitary representation of spacetime translations in $(\pi_w, \mathscr{H}_w)$ is a direct consequence of translation invariance of the particle weight $\langle\,.\,|\,.\,\rangle$ and its continuity under Poincaré transformations with respect to $q_w$. Stone's Theorem (cf. [3, Chapter 6, § 2] and [17, Theorem VIII.(33.8)]) connects the spectrum of the generator $P_w = (P_w^\mu)$ of the unitary representation with the support of the Fourier transform of $x \mapsto \langle L_1 | \alpha_x(L_2) \rangle$, by virtue of the relation

$$\begin{aligned}\int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x) \langle L_1 | \alpha_x(L_2) \rangle &= \int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x) \langle L_1 | U_w(x) | L_2 \rangle \\ &= (2\pi)^{(s+1)/2} \langle L_1 | \tilde{g}(P_w) | L_2 \rangle\end{aligned} \tag{5.23}$$

which holds for any $L_1$, $L_2 \in \mathfrak{L}$ and any $g \in L^1(\mathbb{R}^{s+1}, d^{s+1}x)$. To clarify this fact, note, that the projection-valued measure $E_w(\,.\,)$ corresponding to $P_w$ is regular, which means that for each Borel set $\Delta'$ the associated projection $E_w(\Delta')$ is the strong limit of the net $\{E_w(\Gamma') : \Gamma' \subseteq \Delta' \text{ compact}\}$. For each compact $\Gamma \subseteq \complement(\overline{V}_+ - q)$ consider an infinitely often differentiable function $\tilde{g}_\Gamma$ with support in the set $\complement(\overline{V}_+ - q)$ enveloping the characteristic function for $\Gamma$ [19, Satz 7.7]: $0 \leqslant \chi_\Gamma \leqslant \tilde{g}_\Gamma$. According to the assumption of Definition 3.11, the left-hand side of (5.23) vanishes for any $g_\Gamma$ of the above kind, and this means that all the bounded operators $\tilde{g}_\Gamma(P_w)$ equal $0$ not only on the dense subspace spanned by vectors $|L\rangle$, $L \in \mathfrak{L}$, but on all of $\mathscr{H}_w$. Due to the fact that $\tilde{g}_\Gamma$ majorizes $\chi_\Gamma$, this in turn implies $\chi_\Gamma(P_w) = E_w(\Gamma) = 0$ and thus, by arbitrariness of $\Gamma \subseteq \complement(\overline{V}_+ - q)$ in connection with regularity, the desired relation $E_w\big(\complement(\overline{V}_+ - q)\big) = 0$.

Part (II): The reversion of the above arguments in order to establish that the scalar product on $\mathscr{H}_w$ possesses the characteristics of a particle weight is self-evident. □

The following analogue of Lemmas 4.6 and 4.7 in terms of the $q_w$-topology is of importance not only for the results of Section 3, but also plays an important role in the constructions that underlie the theory of disintegration.

**Lemma 5.4.** *Let $L \in \mathfrak{L}$ and let $\langle\,.\,|\,.\,\rangle$ be a particle weight.*
*(i) Let $F \in L^1\big(\mathsf{P}_+^\uparrow, d\mu(\Lambda, x)\big)$ have compact support $\mathsf{S}$, then the Bochner integral*

$$\alpha_F(L) = \int d\mu(\Lambda, x)\, F(\Lambda, x)\, \alpha_{(\Lambda,x)}(L) \tag{5.24a}$$

*lies in the completion of $\mathfrak{L}$ with respect to the locally convex topology induced on it by the initial norm $\|\,.\,\|$ and the $q_w$-seminorm defined by the particle weight. Moreover, $|\alpha_F(L)\rangle$ is a vector in the corresponding Hilbert space $\mathscr{H}_w$ that can be written*

$$|\alpha_F(L)\rangle = \int d\mu(\Lambda, x)\, F(\Lambda, x)\, |\alpha_{(\Lambda,x)}(L)\rangle, \tag{5.24b}$$

*its norm being bounded by $\|F\|_1 \sup_{(\Lambda,x)\in\mathsf{S}} \big\| |\alpha_{(\Lambda,x)}(L)\rangle \big\|$.*
*(ii) For any function $g \in L^1\big(\mathbb{R}^{s+1}, d^{s+1}x\big)$ the Bochner integral*

$$\alpha_g(L) = \int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x)\, \alpha_x(L) \tag{5.25a}$$

*likewise lies in the completion of $\mathfrak{L}$ with respect to the locally convex topology mentioned above. $|\alpha_g(L)\rangle$ is a vector in the Hilbert space $\mathscr{H}_w$ subject to the relation*

$$|\alpha_g(L)\rangle = \int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x)\, |\alpha_x(L)\rangle = (2\pi)^{(s+1)/2} \tilde{g}(P_w) |L\rangle \tag{5.25b}$$

*with norm bounded by $\|g\|_1 \big\| |L\rangle \big\|$. Here $P_w = (P_w^\mu)$ denotes the generator of the unitary representation of spacetime translations in $(\pi_w, \mathscr{H}_w)$.*

*Proof.* (i) The seminorm $q_w$ induced on $\mathfrak{L}$ by the particle weight is continuous with respect to Poincaré transformations so that the integrand of (5.24a) can be estimated by the Lebesgue-integrable function $(\Lambda,x) \mapsto |F(\Lambda,x)| \cdot \sup_{(\Lambda,x)\in S} q_w(\alpha_{(\Lambda,x)}(L))$. Therefore, the integral in question indeed exists in the completion of the locally convex space $\mathfrak{L}$ not only with respect to the norm topology but also with respect to the seminorm $q_w$. Now, $\||L'\rangle\|$ coincides with $q_w(L')$ for any $L' \in \mathfrak{L}$, a relation which extends to the respective completions [22, Chapter One, § 5 4.(4)] thus resulting in (5.24b) and the given bound for this integral.

(ii) Invariance of $\langle\, .\, |\, .\, \rangle$ with respect to spacetime translations implies translation invariance of the seminorm $q_w$. Therefore the integrand of (5.25a) is majorized by the Lebesgue-integrable function $x \mapsto |g(x)|\, q_w(L)$ so that the respective integral exists in the completion of $\mathfrak{L}$. The first equation of (5.25b) and its norm bound arise from the arguments that were already applied in the first part, whereas the second one is a consequence of Stone's Theorem (cf. (5.23)). □

With this result, we are in the position to prove Proposition 3.13 on spectral subspaces of $\mathscr{H}_w$.

*Proposition 3.13.* The energy-momentum transfer of $A \in \mathfrak{A}$ can be described by the support of the Fourier transform of $x \mapsto \alpha_x(A)$ considered as an operator-valued distribution. Thus, by assumption, for any Lebesgue-integrable function $g$ satisfying $\operatorname{supp}\tilde{g} \cap \Delta' = \emptyset$ we have $\alpha_g(L) = 0$ and, by virtue of Lemma 5.4,

$$\int_{\mathbb{R}^{s+1}} d^{s+1}x\, g(x)\, \big|\alpha_x(L)\big\rangle = \big|\alpha_g(L)\big\rangle = 0. \tag{5.26}$$

Upon insertion of (5.26) into the formulation (5.23) of Stone's Theorem, the reasoning applied in the proof of the first part of Theorem 3.12 yields the assertion. □

The proofs of Proposition 3.14 and Lemma 3.16 require considerably more work than the last ones.

*Proposition 3.14.* To establish this result we follow the strategy of the proof of Proposition 3.4. Applied to the problem at hand, expressed in terms of $(\pi_w, \mathscr{H}_w)$, this yields for any $\boldsymbol{x} \in \mathbb{R}^s$

$$\begin{aligned} &\big|\big\langle L_1{}^*A_1L_1'\big|U_w(\boldsymbol{x})\big|L_2{}^*A_2L_2'\big\rangle\big| \\ &\quad \leqslant \big|\big\langle L_1'\big|\pi_w\big(\big[A_1{}^*L_1, \alpha_{\boldsymbol{x}}(L_2{}^*A_2)\big]\big)U_w(\boldsymbol{x})\big|L_2'\big\rangle\big| + \big|\big\langle L_1'\big|\pi_w\big(\alpha_{\boldsymbol{x}}(L_2{}^*A_2)A_1{}^*L_1\big)U_w(\boldsymbol{x})\big|L_2'\big\rangle\big|. \end{aligned} \tag{5.27}$$

The first term on the right-hand side is majorized by the product of norms of its constituents $\big\|\big[A_1{}^*L_1, \alpha_{\boldsymbol{x}}(L_2{}^*A_2)\big]\big\|\, \||L_1'\rangle\|\, \||L_2'\rangle\|$ as the particle weight is translation invariant and the representation is continuous. The operators involved are almost local without exception, so the norm of the commutator decreases rapidly, rendering this term integrable. The second term requires a closer inspection:

$$\begin{aligned} &\big|\big\langle L_1'\big|\pi_w\big(\alpha_{\boldsymbol{x}}(L_2{}^*A_2)A_1{}^*L_1\big)U_w(\boldsymbol{x})\big|L_2'\big\rangle\big| \\ &\qquad \leqslant \big\|\pi_w\big(\alpha_{\boldsymbol{x}}(A_2{}^*L_2)\big)|L_1'\rangle\big\|\, \big\|\pi_w\big(A_1{}^*L_1\big)U_w(\boldsymbol{x})|L_2'\rangle\big\| \\ &\qquad\qquad \leqslant 2^{-1}\Big(\big\|\pi_w\big(\alpha_{\boldsymbol{x}}(A_2{}^*L_2)\big)|L_1'\rangle\big\|^2 + \big\|\pi_w\big(\alpha_{(-\boldsymbol{x})}(A_1{}^*L_1)\big)|L_2'\rangle\big\|^2\Big). \end{aligned} \tag{5.28}$$

Now, $\pi_w(A')$ has the same energy-momentum transfer with respect to the unitary representation $x \mapsto U_w(x)$ as the operator $A' \in \mathfrak{A}$ has regarding the underlying positive energy representation, and, according to Proposition 3.13, $|L_i'\rangle = E_w(\Gamma_i')|L_i'\rangle$, $i = 1, 2$. Since the spectrum of $x \mapsto U_w(x)$ is restricted to a displaced forward light cone, all of the arguments given in the proofs of Propositions 4.1 and 2.6 also apply to the representation $(\pi_w, \mathscr{H}_w)$ so that, e. g., the integral

$$\int_{\mathbb{R}^s} d^s x\, E_w(\Gamma_1')\pi_w\big(\alpha_{\boldsymbol{x}}(L_2{}^* A_2 A_2{}^* L_2)\big)E_w(\Gamma_1')$$

is seen to exist in the $\sigma$-weak-topology on $\mathscr{B}(\mathscr{H}_w)$. Thus

$$\begin{aligned}\int_{\mathbb{R}^s} d^s x\, \big\|\pi_w\big(\alpha_{\boldsymbol{x}}(A_2{}^* L_2)\big)E_w(\Gamma_1')|L_1'\rangle\big\|^2 \\ = \int_{\mathbb{R}^s} d^s x\, \big\langle L_1'\big|E_w(\Gamma_1')\pi_w\big(\alpha_{\boldsymbol{x}}(L_2{}^* A_2 A_2{}^* L_2)\big)E_w(\Gamma_1')\big|L_1'\big\rangle < \infty.\end{aligned} \tag{5.29}$$

The same holds true for the other term on the right-hand side of (5.28), so its left-hand side is seen to be an integrable function of $\boldsymbol{x}$, too. Altogether, we have thus established the Cluster Property for particle weights. □

*Lemma 3.16.* Let $(\pi_\sigma, \mathscr{H}_\sigma)$ denote the GNS-representation corresponding to the functional $\sigma \in \mathfrak{C}_\Delta{}^*$ and let $E_\sigma(\,.\,)$ be the spectral measure associated with the generator $P_\sigma = (P_\sigma^\mu)$ of the intrinsic representation of spacetime translations. For the time being, suppose that $\Delta'$ is an *open* bounded Borel set in $\mathbb{R}^{s+1}$. Let furthermore $L$ be an arbitrary element of $\mathfrak{L}$ and $A \in \mathfrak{A}$. We are interested in an estimate of the term $\langle L|E_\sigma(\Delta')\pi_\sigma(A)E_\sigma(\Delta')|L\rangle_\sigma$. The spectral measure is regular, so $E_\sigma(\Delta')$ is the strong limit of the net $\{E_\sigma(\Gamma) : \Gamma \subset \Delta' \text{ compact}\}$. As $\Delta'$ is assumed to be open, there exists for each compact subset $\Gamma$ of $\Delta'$ an infinitely often differentiable function $\tilde{g}_\Gamma$ with $\operatorname{supp}\tilde{g}_\Gamma \subset \Delta'$ that fits between the corresponding characteristic functions [19, Satz 7.7]: $\chi_\Gamma \leqslant \tilde{g}_\Gamma \leqslant \chi_{\Delta'}$. Thus

$$0 \leqslant \big(E_\sigma(\Delta') - \tilde{g}_\Gamma(P_\sigma)\big)^2 \leqslant \big(E_\sigma(\Delta') - E_\sigma(\Gamma)\big)^2,$$

from which we infer that for arbitrary $L' \in \mathfrak{L}$

$$0 \leqslant \big\|\big(E_\sigma(\Delta') - \tilde{g}_\Gamma(P_\sigma)\big)|L'\rangle\big\|^2 \leqslant \big\|\big(E_\sigma(\Delta') - E_\sigma(\Gamma)\big)|L'\rangle\big\|^2 \xrightarrow[\Gamma\nearrow\Delta']{} 0. \tag{5.30}$$

By density of all vectors $|L'\rangle$ in $\mathscr{H}_\sigma$, this means that $E_\sigma(\Delta')$ is the strong limit of the net $\big\{\tilde{g}_\Gamma(P_\sigma) : \Gamma \subset \Delta' thickspace \text{compact}\big\}$ for $\Gamma \nearrow \Delta'$ and

$$\langle L|E_\sigma(\Delta')\pi_\sigma(A)E_\sigma(\Delta')|L\rangle_\sigma = \lim_{\Gamma\nearrow\Delta'} \langle L|\tilde{g}_\Gamma(P_\sigma)\pi_\sigma(A)\tilde{g}_\Gamma(P_\sigma)|L\rangle. \tag{5.31}$$

The Fourier transform $\tilde{g}_\Gamma$ of the rapidly decreasing function $g_\Gamma$ belongs to $L^1\big(\mathbb{R}^{s+1}, d^{s+1}x\big)$, so Lemma 5.4 yields for the right-hand side of (5.31)

$$\begin{aligned}\langle L|\tilde{g}_\Gamma(P_\sigma)\pi_\sigma(A)\tilde{g}_\Gamma(P_\sigma)|L\rangle = (2\pi)^{-(s+1)}\langle \alpha_{g_\Gamma}(L)|\pi_\sigma(A)|\alpha_{g_\Gamma}(L)\rangle_\sigma \\ = (2\pi)^{-(s+1)}\sigma\big(\alpha_{g_\Gamma}(L)^* A \alpha_{g_\Gamma}(L)\big),\end{aligned} \tag{5.32}$$

where in the last equation we use the continuous extension of $\sigma$ to the argument at hand (cf. Lemmas 4.7 and 4.3 in connection with Corollary 4.4). The approximating functionals $\rho_{h,t}$ for $\sigma$ in the form (3.6) with a non-negative function $h \in L^\infty(\mathbb{R}^s, d^s x)$ satisfy the following estimate of their integrand by an application of [5, Proposition 2.3.11]:

$$\begin{aligned}&\big|h(\tau^{-1}\boldsymbol{x})\,\omega\big(U(\tau)E(\Delta)\alpha_{\boldsymbol{x}}(\alpha_{g_\Gamma}(L)^* A \alpha_{g_\Gamma}(L))E(\Delta)U(\tau)^*\big)\big| \\ &\quad = h(\tau^{-1}\boldsymbol{x})\,\big|\omega\big(U(\tau)E(\Delta)\alpha_{\boldsymbol{x}}(\alpha_{g_\Gamma}(L)^*)E(\overline{\Delta})\alpha_{\boldsymbol{x}}(A)E(\overline{\Delta})\alpha_{\boldsymbol{x}}(\alpha_{g_\Gamma}(L))E(\Delta)U(\tau)^*\big)\big| \\ &\qquad \leqslant \|E(\overline{\Delta})AE(\overline{\Delta})\|\; h(\tau^{-1}\boldsymbol{x})\,\omega\big(U(\tau)E(\Delta)\alpha_{\boldsymbol{x}}(\alpha_{g_\Gamma}(L)^*\alpha_{g_\Gamma}(L))E(\Delta)U(\tau)^*\big).\end{aligned}$$

The spectral projections $E(\overline{\Delta})$ pertaining to the bounded, open Borel set $\overline{\Delta} = \Delta + \Delta'$ can be inserted here, since, according to Lemma 4.7, the energy-momentum transfer of $\alpha_{g_\Gamma}(L)$ is contained in $\Delta'$ by construction. An immediate consequence is

$$\left|\rho_{h,t}\left(\alpha_{g_\Gamma}(L)^* A \alpha_{g_\Gamma}(L)\right)\right| \leqslant \|E(\overline{\Delta})AE(\overline{\Delta})\| \, \rho_{h,t}\left(\alpha_{g_\Gamma}(L)^* \alpha_{g_\Gamma}(L)\right),$$

which extends to the limit functional $\sigma$

$$\left|\sigma\left(\alpha_{g_\Gamma}(L)^* A \alpha_{g_\Gamma}(L)\right)\right| \leqslant \|E(\overline{\Delta})AE(\overline{\Delta})\| \, \sigma\left(\alpha_{g_\Gamma}(L)^* \alpha_{g_\Gamma}(L)\right). \tag{5.33}$$

Insertion of this result into (5.32) and passing to the limit $\Gamma \nearrow \Delta'$ in (5.31) yields

$$\left|\langle L|E_\sigma(\Delta')\pi_\sigma(A)E_\sigma(\Delta')|L\rangle_\sigma\right| \leqslant \|E(\overline{\Delta})AE(\overline{\Delta})\| \, \langle L|E_\sigma(\Delta')|L\rangle_\sigma \leqslant \|E(\overline{\Delta})AE(\overline{\Delta})\| \, \langle L|L\rangle_\sigma. \tag{5.34}$$

Taking the supremum with respect to all $L \in \mathfrak{L}$ with $\||L\rangle_\sigma\| \leqslant 1$ (these constitute a dense subset of the unit ball in $\mathscr{H}_\sigma$), we get, through an application of [31, Satz 4.4],

$$\|E_\sigma(\Delta')\pi_\sigma(A)E_\sigma(\Delta')\| \leqslant 2 \cdot \|E(\overline{\Delta})AE(\overline{\Delta})\|. \tag{5.35}$$

This establishes the defining condition (3.14) for $\Delta$-boundedness with $c = 2$ in the case of an *open* bounded Borel set $\Delta'$. But this is not an essential restriction, since an arbitrary bounded Borel set $\Delta'$ is contained in the open set $\Delta'_\eta$, $\eta > 0$, consisting of all those points $p \in \mathbb{R}^{s+1}$ with $\inf_{p' \in \Delta'} |p - p'| < \eta$. Since $\Delta'_\eta$ is likewise a bounded Borel set, we get

$$\|E_\sigma(\Delta')\pi_\sigma(A)E_\sigma(\Delta')\| \leqslant \|E_\sigma(\Delta'_\eta)\pi_\sigma(A)E_\sigma(\Delta'_\eta)\| \leqslant 2 \cdot \|E(\overline{\Delta}_\eta)AE(\overline{\Delta}_\eta)\| \tag{5.36}$$

as an immediate consequence of (5.35) with $\overline{\Delta}_\eta \doteq \Delta + \Delta'_\eta$. This covers the general case and thereby proves $\Delta$-boundedness for positive asymptotic functionals $\sigma \in \mathfrak{C}_\Delta^*$. □

# 6 Conclusions

The present article is based on the general point of view that the concept of *particles* is asymptotic in nature and simultaneously has to be founded by making appropriate use of locality. This reflects the conviction that the long-standing problem of *asymptotic completeness* of quantum field theory, i. e., the question if a quantum field theoretic model can be interpreted completely in terms of particles, has to be tackled by the aid of further restrictions on the general structure, which essentially are of a local character. The question is which local structure of a theory is appropriate in order that it governs scattering processes in such a way that asymptotically the physical states appear to clot in terms of certain entities named particles. The compactness and nuclearity conditions discussed in [8] and the references therein are examples of this kind of approach. It is not claimed that they already give a complete answer, but that they indicate the right direction.

Asymptotic functionals on a certain algebra of detectors have been constructed that give rise to particle weights which are to be interpreted as mixtures of particle states. Their disintegration (presented in a forthcoming paper) constitutes the basis for the definition of mass and spin even in the case of charged states (cf. [9]). The technical problems arising in the course of these investigations, e. g., that of convergence in connection with Theorem 3.7 and those to be encountered when establishing the disintegration theory, might be solvable with additional information at hand that could be provided by the investigation of concrete models. Quantum electrodynamics is an example [13]. So far only *single-particle* weights have been considered. Another field of future research should be the inspection of coincidence arrangements of detectors as in [1]. In this respect, too, the analysis of concrete models might prove helpful.

## Acknowledgements

The results presented above have been worked out by Detlev Buchholz and the author on the basis of ideas of Buchholz. I should like to thank him for this opportunity of collaboration. Bernd Kuckert has given valuable advice in editing the final version of the manuscript, his help is gratefully acknowledged. Also acknowledged is financial support by Deutsche Forschungsgemeinschaft which I obtained from the Graduiertenkolleg "Theoretische Elementarteilchenphysik" at the II. Institut für Theoretische Physik of the University of Hamburg.

# References

[1] ARAKI, H., AND HAAG, R. Collision cross sections in terms of local observables. *Commun. Math. Phys. 4* (1967), 77–91.

[2] ASIMOW, L., AND ELLIS, A. *Convexity Theory and its Applications in Functional Analysis*. Academic Press, Inc., London, New York, 1980.

[3] BARUT, A. O., AND RĄCZKA, R. *Theory of Group Representations and Applications*, 2nd rev. ed. Państwowe Wydawnictwo Naukowe–Polish Scientific Publishers, Warszawa, 1980.

[4] BAUMGÄRTEL, H., AND WOLLENBERG, M. *Causal Nets of Operator Algebras*. Akademie-Verlag, Berlin, 1992.

[5] BRATTELI, O., AND ROBINSON, D. W. *Operator Algebras and Quantum Statistical Mechanics 1*, 2nd ed. Springer-Verlag, New York, Berlin, Heidelberg, 1987.

[6] BUCHHOLZ, D. On particles, infraparticles, and the problem of asymptotic completeness. In *VIIIth International Congress on Mathematical Physics,* Marseille 1986 (Singapore, 1987), M. Mebkhout and R. Sénéor, Eds., World Scientific, pp. 381–389.

[7] BUCHHOLZ, D. Harmonic analysis of local operators. *Commun. Math. Phys. 129* (1990), 631–641.

[8] BUCHHOLZ, D., AND PORRMANN, M. How small is the phase space in quantum field theory? *Ann. Inst. Henri Poincaré - Physique théorique 52* (1990), 237–257.

[9] BUCHHOLZ, D., PORRMANN, M., AND STEIN, U. Dirac versus Wigner: Towards a universal particle concept in local quantum field theory. *Phys. Lett. B267* (1991), 377–381.

[10] DIRAC, P. A. M. *The Principles of Quantum Mechanics*, 4th ed. At The Clarendon Press, Oxford, 1958.

[11] DIXMIER, J. *Von Neumann Algebras*. North-Holland Publishing Co., Amsterdam, New York, Oxford, 1981.

[12] FELL, J. M. G., AND DORAN, R. S. *Representations of *-Algebras, Locally Compact Groups, and Banach *-Algebraic Bundles – Volume 1*. Academic Press, Inc., San Diego, London, 1988.

[13] FREDENHAGEN, K., AND FREUND, J. Work in progress.

[14] HAAG, R. *Local Quantum Physics*, 2nd rev. and enl. ed. Springer-Verlag, Berlin, Heidelberg, New York, 1996.

[15] HAAG, R., AND KASTLER, D. An algebraic approach to quantum field theory. *J. Math. Phys. 5* (1964), 848–861.

[16] HELGASON, S. *Groups and Geometric Analysis*. Academic Press, Inc., Orlando, London, 1984.

[17] HEWITT, E., AND ROSS, K. A. *Abstract Harmonic Analysis II*. Springer-Verlag, Berlin, Heidelberg, New York, 1970.

[18] HEWITT, E., AND STROMBERG, K. *Real and Abstract Analysis*, 2nd ed. Springer-Verlag, Berlin, Heidelberg, New York, 1969.

[19] JANTSCHER, L. *Distributionen*. Walter de Gruyter, Berlin, New York, 1971.

[20] JARCHOW, H. *Locally Convex Spaces*. B. G. Teubner, Stuttgart, 1981.

[21] KADISON, R. V., AND RINGROSE, J. R. *Fundamentals of the Theory of Operator Algebras – Volume I*. Academic Press, Inc., New York, London, 1983.

[22] KÖTHE, G. *Topological Vector Spaces I*, 2nd rev. ed. Springer-Verlag, Berlin, Heidelberg, New York, 1983.

[23] NACHBIN, L. *The Haar Integral*. D. Van Nostrand Company, Inc., Princeton, New Jersey, Toronto, New York, London, 1965.

[24] PEDERSEN, G. K. Measure theory for $C^*$ algebras. *Math. Scand. 19* (1966), 131–145.

[25] PEDERSEN, G. K. *$C^*$-Algebras and their Automorphism Groups*. Academic Press, Inc., London, New York, San Francisco, 1979.

[26] PERESSINI, A. L. *Ordered Topological Vector Spaces*. Harper & Row Publishers, New York, Evanston, London, 1967.

[27] PORRMANN, M. *The Concept of Particle Weights in Local Quantum Field Theory*. PhD thesis, Universität Göttingen, January 2000. hep-th/0005057.

[28] REEH, H., AND SCHLIEDER, S. Bemerkungen zur Unitäräquivalenz von Lorentz-invarianten Feldern. *Nuovo Cimento 22* (1961), 1051–1068.

[29] SCHROER, B. Infrateilchen in der Quantenfeldtheorie. *Fortschr. Phys. 11* (1963), 1–32.

[30] TAKESAKI, M. *Theory of Operator Algebras I*. Springer-Verlag, New York, Heidelberg, Berlin, 1979.

[31] WEIDMANN, J. *Lineare Operatoren in Hilberträumen*. B. G. Teubner, Stuttgart, 1976.

[32] WIGNER, E. P. On unitary representations of the inhomogeneous Lorentz group. *Ann. Math. 40* (1939), 149–204.