Observations on noncommuting coordinates and on fields depending on them

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Abstract

The original ideas about noncommuting coordinates are recalled. The connection between U(1) gauge fields defined on noncommuting coordinates and fluid mechanics is explained.

The idea that configuration-space coordinates may not commute

$$[x^i, x^j] = i\theta^{ij} \tag{1}$$

where θ^{ij} is a constant, anti-symmetric two-index object, has arisen recently from string theory, but in fact it has a longer history. Like many interesting quantum-mechanical ideas, it was first suggested by Heisenberg, in the late 1930s, who reasoned that coordinate noncommutativity would entail a coordinate uncertainty and would ameliorate short-distance singularities, which beset quantum fields. He told his idea to Peierls, who eventually made use of it when analyzing electronic systems in an external magnetic field, so strong that projection to the lowest Landau level is justified. But this phenomenological realization of Heisenberg's idea did not address issues in fundamental science, so Peierls told Pauli about it, who in turn told Oppenheimer, who asked his student Snyder to work it out and this led to the first published paper on the subject [1].

The coordinate noncommutativity in the lowest Landau level is very similar to today's string-theory origins of noncommutativity – both rely on the presence of a strong background field. Also, thus far, it is the only physically realized example of noncommuting coordinates, so let me describe it in a little detail [2]. We consider the

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motion of a charged (e) and massive (m) particle in a constant magnetic field (B) pointing along the z direction. All interesting physics is in the x–y plane. The Lagrangian for this planar motion is

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{e}{c}(\dot{x}A_x + \dot{y}A_y) - V(x,y) \tag{2}$$

where the vector potential $\boldsymbol{A}$ can be chosen as $(0, xB)$ and $V(x,y)$ describes additional interactions ("impurities"). In the absence of V, the quantum spectrum consists of the well-known Landau levels $|N, d\rangle$, where N indexes the level's energy eigenvalue, and d describes the infinite degeneracy of each level. The separation between levels is $O(B/m)$, so that in the strong magnetic field limit only the lowest Landau level $|0, d\rangle$ is relevant. But observe that the large B limit corresponds to small m, so projection on the lowest Landau level is also achieved by setting m to zero in (2). In that limit the Lagrangian (2), in the chosen gauge, becomes

$$L_{\ell L\ell} = \frac{e}{c}Bx\dot{y} - V(x,y) \ . \tag{3}$$

This is of the form $p\dot{q} - H(p,q)$, and immediately identifies $\frac{e}{c}Bx$ and y as canonical conjugates, leading in the usual way to the commutator

$$[x,y] = -i\frac{\hbar c}{eB} \ . \tag{4}$$

[The "Peierls substitution" consists of determining the effect of the impurity by computing the eigenvalues of $V(x,y)$, where x and y are noncommuting.]

For another perspective, consider a calculation of the lowest Landau level matrix elements of the $[x,y]$ commutator.

$$\langle 0, d | xy - yx | 0, d' \rangle = M(d, d') - M^*(d', d) \tag{5}$$

where

$$M(d, d') = \langle 0, d | xy | 0, d' \rangle \ . \tag{6}$$

We evaluate (6) by inserting intermediate states in product xy:

$$M(d, d') = \sum_s \langle 0, d | x | s \rangle \langle s | y | 0, d' \rangle \ . \tag{7}$$

If the sum is over *all* the degenerate Landau levels, then one finds that (5) vanishes: x and y do commute! But if one pretends that the world is restricted to the lowest Landau level and includes only that level (with its degeneracy) in the intermediate state sum

$$M_{\ell L\ell} = \sum_{d''} \langle 0, d | x | 0, d'' \rangle \langle 0, d'' | y | 0, d' \rangle \tag{8}$$

one finds that in this truncated state space, eq. (7) becomes consistent with (4):

$$\langle 0, d | [x,y] | 0, d' \rangle = -i\frac{\hbar c}{eB} \langle 0, d | 0, d' \rangle \ . \tag{9}$$

Let me now return to the general and abstract problem of noncommuting coordinates. When confronting the noncommutativity postulate (1), it is natural to ask which (infinitesimal) coordinate transformations

$$\delta x^i = -f^i(x) \tag{10}$$

leave (1) unchanged. The answer is that the (infinitesimal) transformation vector function $f^i(x)$ must be determined by a scalar $f(x)$ through the expression [3]

$$f^i(x) = \theta^{ij}\partial_j f(x)\ . \tag{11}$$

Since $\partial_i f^i(x) = 0$, these are recognized as volume-preserving transformations. [They do not exhaust all volume preserving transformations, except in two dimensions. In dimensions greater two, (11) defines a subgroup of volume-preserving transforms that also leave θ^{ij} invariant.]

The volume-preserving transformations form the link between noncommuting coordinates and fluid mechanics. Since the theory of fluid mechanics is not widely known outside the circle of fluid mechanicians, let me put down some relevant facts [4]. There are two, physically equivalent descriptions of fluid motion: One is the Lagrange formulation, wherein the fluid elements are labeled, first by a discrete index n: $\boldsymbol{X}_n(t)$ is the position as a function of time of the nth fluid element. Then one passes to a continuous labeling variable $n \to \boldsymbol{x} : \boldsymbol{X}_n(t) \to \boldsymbol{X}(t, \boldsymbol{x})$, and $\boldsymbol{x}$ may be taken to be the position of the fluid element at initial time $\boldsymbol{X}(0, \boldsymbol{x}) = \boldsymbol{x}$. This is a comoving description. Because labels can be arbitrarily rearranged, without affecting physical content, the continuum description is invariant against volume-preserving transformations of $\boldsymbol{x}$, and in particular, it is invariant against the specific volume-preserving transformations (11), provided the fluid coordinate $\boldsymbol{X}$ transforms as a scalar:

$$\delta_f \boldsymbol{X} = f^i(\boldsymbol{x})\frac{\partial}{\partial x^i}\boldsymbol{X} = \theta^{ij}\partial_i \boldsymbol{X}\partial_j f\ . \tag{12}$$

The common invariance of Lagrange fluids and of noncommuting coordinates is a strong hint of a connection between the two.

Formula (12) will take a very suggestive form when we rewrite it in terms of a bracket defined for functions of $\boldsymbol{x}$ by

$$\big\{\mathcal{O}_1(\boldsymbol{x}), \mathcal{O}_2(\boldsymbol{x})\big\} = \theta^{ij}\partial_i\mathcal{O}_1(\boldsymbol{x})\partial_j\mathcal{O}_2(\boldsymbol{x})\ . \tag{13}$$

Note that with this bracket we have

$$\big\{x^i, x^j\big\} = \theta^{ij}\ . \tag{14}$$

So we can think of bracket relations as classical precursors of commutators for a noncommutative field theory – the latter obtained from the former by replacing brackets by $-i$ times commutators, à la Dirac. More specifically, we shall see that the noncommuting field theory that emerges from the Lagrange fluid is a noncommuting U(1) gauge theory.

This happens when the following steps are taken. We define the evolving portion of $\boldsymbol{X}$ by

$$X^i(t,\boldsymbol{x}) = x^i + \theta^{ij}\widehat{A}_j(t,\boldsymbol{x}) \ . \tag{15}$$

(It is assumed that θ^{ij} has an inverse.) Then (12) is equivalent to the suggestive expression

$$\delta_f \widehat{A}_i = \partial_i f + \{\widehat{A}_i, f\} \ . \tag{16}$$

When the bracket is replaced by $(-i)$ times the commutator, this is precisely the gauge transformation for a noncommuting U(1) gauge potential $\widehat{A}_i$. Moreover, the gauge field $\widehat{F}_{ij}$ emerges from the bracket of two Lagrange coordinates

$$\{X^i, X^j\} = \theta^{ij} + \theta^{im}\theta^{jn}\widehat{F}_{mn} \tag{17}$$

$$\widehat{F}_{mn} = \partial_m\widehat{A}_n - \partial_n\widehat{A}_m + \{\widehat{A}_m, \widehat{A}_n\} \ . \tag{18}$$

Again (18) is recognized from the analogous formula in noncommuting gauge theory.

What can one learn from the parallelism of the formalism for a Lagrange fluid and a noncommuting gauge field? One result that has been obtained addresses the question of what is a gauge field's covariant response to a coordinate transformation. This question can be put already for commuting, non-Abelian gauge fields, where conventionally the response is given in terms of a Lie derivative L_f:

$$\delta_f x^\mu = -f^\mu(x) \tag{19}$$

$$\delta_f A_\mu = L_f A_\mu \equiv f^\alpha \partial_\alpha A_\mu + \partial_\mu f^\alpha A_\alpha \ . \tag{20}$$

But this implies

$$\delta_f F_{\mu\nu} = L_f F_{\mu\nu} \equiv f^\alpha\partial_\alpha F_{\mu\nu} + \partial_\mu f^\alpha F_{\alpha\nu} + \partial_\nu f^\alpha F_{\mu\alpha} \tag{21}$$

which is not covariant since the derivative in the first term on the right is not the covariant one. The cure in this, commuting, situation has been given some time ago [5]: Observe that (20) may be equivalently presented as

$$\begin{aligned} \delta_f A_\mu = L_f A_\mu &= f^\alpha\big(\partial_\alpha A_\mu - \partial_\mu A_\alpha - i[A_\alpha, A_\mu]\big) \\ &\qquad + f^\alpha\partial_\mu A_\alpha - i[A_\mu, f^\alpha A_\alpha] + \partial_\mu f^\alpha A_\alpha \\ &= f^\alpha F_{\alpha\mu} + D_\mu(f^\alpha A_\alpha) \ . \end{aligned} \tag{22}$$

Thus, if the coordinate transformation generated by f^α is supplemented by a gauge transformation generated by $-f^\alpha A_\alpha$, the result is a gauge covariant coordinate transformation

$$\delta'_f A_\mu = f^\alpha F_{\alpha\mu} \tag{23}$$

and the modified response of $F_{\mu\nu}$ involves the gauge-covariant Lie derivative L'_f:

$$\delta'_f F_{\mu\nu} = L'_f F_{\mu\nu} \equiv f^\alpha D_\alpha F_{\mu\nu} + \partial_\mu f^\alpha F_{\alpha\nu} + \partial_\nu f^\alpha F_{\mu\alpha} \ . \tag{24}$$

In the noncommuting situation, loss of covariance in the ordinary Lie derivative is even greater, because in general the coordinate transformation functions f^α do not commute with the fields $A_\mu, F_{\mu\nu}$; moreover, multiplication of x-dependent quantities is not a covariant operation. All these issues can be addressed and resolved by considering them in the fluid mechanical context, at least, for linear and volume-preserving diffeomorphisms. The analysis is technical and I refer you to the published papers [3, 6]. The final result for the covariant coordinate transformation on the noncommuting gauge potential $\widehat{A}_\mu$, generated by $f^\alpha(X)$, is

$$\delta'_f \widehat{A}_\mu = \tfrac{1}{2}\{f^\alpha(X), \widehat{F}_{\alpha\mu}\} + \text{reordering terms.} \tag{25}$$

Note that the generating function $f^\alpha(X)$ enters the anticommutator with covariant argument X. f^α is restricted to be either linear or volume-preserving; in the latter case there are reordering terms, whose form is explicitly determined by the fluid mechanical antecedent.

Next, I shall discuss the Seiberg-Witten map [7], which can be made very transparent by the fluid analogy. The Seiberg-Witten map replaces the noncommuting vector potential $\widehat{A}_\mu$ by a nonlocal function of a commuting potential a_μ and of θ; i.e., the former is viewed as a function of the latter. The relationship between the two follows from the requirement of stability against gauge transformations: a noncommuting gauge transformation of the noncommuting gauge potential should be equivalent to a commuting gauge transformation on the commuting vector potential on which the noncommuting potential depends. Formally:

$$\widehat{A}_\mu(a + \mathrm{d}\lambda) = \widehat{A}_\mu^{G(a,\lambda)}(a) \ . \tag{26}$$

Here λ is the Abelian gauge transformation function that transforms the Abelian, commuting gauge potential a_μ; $G(a, \lambda)$ is the noncommuting gauge function that transforms the noncommuting gauge potential $\widehat{A}_\mu$. G depends on a_μ and λ, and one can show that it is a noncommuting 1-cocycle [8].

Moreover, when the action and the equations of motion of the noncommuting theory are transformed into commuting variables, the dynamical content is preserved: the physics described by noncommuting variables is equivalently described by the commuting variables, albeit in a complicated, nonlocal fashion.

The Seiberg-Witten map is intrinsically interesting in the unexpected equivalence that it establishes. Moreover, it is practically useful for the following reason. It is difficult to extract gauge invariant content from a noncommuting gauge theory because quantities constructed locally from $\widehat{F}_{\mu\nu}$ are not gauge invariant; to achieve gauge invariance, one must integrate over space-time. Yet for physical analysis one wants local quantities: profiles of propagating waves, etc. Such local quantities can be extracted in a gauge invariant manner from the physically equivalent, Seiberg-Witten mapped commutative gauge theory [9].

Let me now use the fluid analogy to obtain an explicit formula for the Seiberg-Witten map; actually, we shall present the inverse map, expressing commuting fields in terms of noncommuting ones. For our development we must refer to a second, alternative formulation of fluid mechanics, the so-called Euler formulation. This is not a comoving description, rather the experimenter observes the fluid density ρ and velocity $\boldsymbol{v}$ at given point in space-time $(t, \boldsymbol{r})$. The current is $\rho\boldsymbol{v}$ and satisfies with ρ a continuity equation

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0 \ . \tag{27}$$

The theory is completed by positing an "Euler equation" for $\partial\boldsymbol{v}/\partial t$, but we shall not need this here.

Of interest to us is the relation between the Lagrange description and the Euler description. This is given by the formulas

$$\rho(t,\boldsymbol{r}) = \int \mathrm{d}x\, \delta\big(\boldsymbol{X}(t,\boldsymbol{x}) - \boldsymbol{r}\big) \tag{28a}$$

$$\rho(t,\boldsymbol{r})\boldsymbol{v}(t,\boldsymbol{r}) \equiv \boldsymbol{j}(t,\boldsymbol{r}) = \int \mathrm{d}x\, \frac{\partial}{\partial t}\boldsymbol{X}(t,\boldsymbol{x})\delta\big(\boldsymbol{X}(t,\boldsymbol{x}) - \boldsymbol{r}\big) \ . \tag{28b}$$

(The integration and the δ-function carry the dimensionality of space.) Observe that the continuity equation (27) follows from the definitions (28), which can be summarized by

$$j^\mu(t,\boldsymbol{r}) = \int \mathrm{d}r\, \frac{\partial}{\partial t}X^\mu\delta(\boldsymbol{X} - \boldsymbol{r}) \tag{29}$$

$$X^0 = t$$

$$\partial_\mu j^\mu = 0 \ . \tag{30}$$

The (inverse) Seiberg-Witten map, for the case of two spatial dimensions, can be extracted from (29), (30) [3]. (The argument can be generalized to arbitrary dimensions, but there it is more complicated [3].) Observe that the right side of (29) depends on $\widehat{\boldsymbol{A}}$ through $\boldsymbol{X}$ [see (15)]. It is easy to check that the integral (29) is invariant under the transformations (12); equivalently viewed as a function of $\widehat{\boldsymbol{A}}$, it is gauge invariant [see (16)]. Owing to the conservation of j^μ [see (30)], its dual $\varepsilon_{\alpha\beta\mu}j^\mu$ satisfies a conventional, commuting Bianchi identity, and therefore can be written as the curl of an Abelian vector potential a_α, apart from proportionality and additive constants:

$$\begin{aligned} \partial_\alpha a_\beta - \partial_\beta a_\alpha + \text{constant} &\propto \varepsilon_{\alpha\beta\mu}\int \mathrm{d}x\, \frac{\partial}{\partial t}X^\mu\delta(\boldsymbol{X} - \boldsymbol{r}) \\ \partial_i a_j - \partial_j a_i + \text{constant} &\propto \varepsilon_{ij}\int \mathrm{d}x\, \delta(\boldsymbol{X} - \boldsymbol{r}) = \varepsilon_{ij}\rho \ . \end{aligned} \tag{31}$$

This is the (inverse) Seiberg-Witten map, relating the $\boldsymbol{a}$ to $\widehat{\boldsymbol{A}}$.

Thus far operator noncommutativity has not been taken into account. To do so, we must provide an ordering for the δ-function depending on the operator $X^i = x^i + \theta^{ij}\widehat{A}_j$. This we do with the Weyl prescription by Fourier transforming. The final operator version of equation (31), restricted to the two-dimensional spatial components, reads

$$\int \mathrm{d}r\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}(\partial_i a_j - \partial_j a_i) = -\varepsilon^{ij}\Big[\int \mathrm{d}x\, e^{i\boldsymbol{k}\cdot\boldsymbol{X}} - (2\pi)^2\delta(\boldsymbol{k})\Big] \tag{32}$$

where the x integral over an operator ($\boldsymbol{X}$) dependent integrand is interpreted as a trace. Here the additive and proportionality constants are determined by requiring agreement for weak noncommuting fields.

Formula (32) has previously appeared in a direct analysis of the Seiberg-Witten relation [10]. Now we recognize it as the (quantized) expression relating Lagrange and Euler formulations for fluid mechanics.

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