Rotations and e, ν Propagators, Part III

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Abstract

In Parts I and II we showed that e,ν propagators can be derived from rotation invariant projection operators, thereby providing examples of how quantities with spacetime symmetry can be obtained by constraining rotationally symmetric objects. One constraint is the restriction of the basis; only two kinds of bases were considered, one for the electron and one for the neutrino. In this part, we find that, of a wide range of bases each consistent with the constraint process, only the two kinds of bases considered in Parts I and II give spacetime symmetric propagators. We interpret the result geometrically. The spinor representation is unfaithful in four dimensional Euclidean space which explains why spin 1/2 wave functions have four, not two, components. Then we show how a basis relates to two planes in four dimensional Euclidean space. A pair of planes spanning two or three dimensions does not allow spacetime symmetry. Spacetime symmetry requires two planes that span four dimensions.

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1   Introduction

We begin with rotations and vectors in four dimensional Euclidean space E^4. With a two dimensional representation we should be able to represent vectors with two component quantities, the 2-vectors. But we can't. In four dimensional Euclidean space the two dimensional representation is unfaithful, one matrix represents two different rotations. We need two 2-vectors to describe the motion of one vector in E^4. The geometry is discussed in Sec. 2 more fully but basically one 2-vector moves under rotations in one plane while the second 2-vector moves with rotations in a second plane. This may be why electron wave functions, which to begin with here are acted on only by rotations, have four components.

In Sec. 3 we consider bases whose upper 2-vectors are eigenvectors of rotations in one plane and the lower 2-vectors are eigenvectors in a second plane. In Sec. 2 the two planes are complementary, i.e. the two span all four dimensions, while in Sec. 3 we consider any two planes in the four dimensional Euclidean space. Eigenvectors of one plane are related to the eigenvectors of the second plane by a U(1)$\times$SU(2) transformation, in general.

In Sec. 4 the four projection operators for the general eigenvector basis are molded to yield two spacetime projection operators by the process developed in Parts I and II, [1] and [2]. The requirements of spacetime symmetry determine the U(1)$\times$SU(2) transformation relating the two rotation planes of the upper and lower 2-vectors. We find that only when the two planes for the basis span all four dimensions does the basis yield spacetime symmetric operators. These bases are just the ones considered in Parts I and II that turned out electron and neutrino propagators. Thus the process of constraining rotation-based quantities to make spacetime symmetric objects with a two dimensional representation can give only electron or neutrino propagators. Furthermore, from our viewpoint, having spacetime symmetry is equivalent to choosing complementary planes for rotating the upper and lower 2-vectors in the wave function.

2   The Implied Geometry of Internal Spin Space

Why pairs of 2-vectors? Each electron or neutrino wave function has four components arranged in a pair of 2-vectors (a chiral representation). Only the free neutrino is discussed in Part II, so agreement with the standard electroweak model would require that only two of the four components participate in interactions.

The explanation depends on the fact that the 2×2 matrix representations of rotations are faithful in three dimensional Euclidean space E^3 but unfaithful in E^4. For each of the two inequivalent representations of rotations in E^3, there is a one-to-one correspondence between matrices and rotations but in E^4 the correspondence is one-to-two. As we now show, to properly represent a vector quantity χ in E^4, one needs two 2-vectors so that the effects on χ of the two rotations can be distinguished. See Fig. 1 and 2.

E^3. A rotation r in E^3 can be represented by a 2×2 matrix R, I(1), in either of two in-

Figure 1: Vectors in planes 1 and 2 rotate. In four dimensional Euclidean space E^4, the rotations r_1 and r_2 are independent whenever planes 1 and 2 intersect only at the origin O. In such cases, the 2×2 matrices R_1 and R_2 that represent the rotations must commute. Hence the matrices R_1 are the same matrices as the R_2s. Since one matrix represents two rotations, the representation is unfaithful.

equivalent matrix representations, also known as inequivalent 'spinor' representations. Each representation is faithful, there is exactly one matrix R for each rotation r. Let x be a 2-vector representing some vector quantity χ. Then applying the rotation r with arbitrary rotation angles θ gives the evolution of the quantity, i.e. $r\chi$. Matrix multiplication, Rx, gives the 2-vector that corresponds to $r\chi$. Hence the evolution of the vector χ is well represented by one 2-vector x.

Figure 2: The vector χ reacts differently to rotations r_1 and r_2. But when planes 1 and 2 intersect only at the origin, the same matrix R represents both rotations. To keep track of what is happening to χ in two planes, we need two 2-vectors, thereby making a four component quantity ψ. The upper 2-vector in ψ represents χ rotating by r_1 and the lower represents rotation by r_2. In Sec. 4 we show that spacetime can result only when the two planes span E^4.

E^4. In four dimensional Euclidean space E^4 things are different, see Fig. 1. Given any 2-dimensional plane, plane 1, that contains the origin, there is a second plane, plane 2, that intersects 1 only at the origin. A rotation r_1 about the origin in plane 1 is independent of any rotation r_2 about the origin in plane 2, hence order is unimportant and $r_1 r_2 - r_2 r_1 = 0$. This implies that any matrix R_1 that represents r_1 must commute with the matrix R_2 that represents r_2. For R_1 and R_2 this in turn means the unit 3-vectors n^k in $R = \exp\left(i n^k \sigma^k \theta / 2\right)$ are equal, $n_1^k = n_2^k$. Thus R_1 and R_2 are the same matrix except with possibly different rotation angles θ_1 and θ_2. This one-matrix-for-two-rotations association makes the 2×2 matrix representation 'unfaithful' in E^4.

When a 2-vector x evolves by multiplying it by a rotation matrix R with rotation angle θ, all we know is that a rotation r_1 through an angle θ_1 and an independent rotation r_2 through an angle θ_2 has taken place, where

$$\theta = \theta_1 + \theta_2 \tag{1}$$

But the evolution of a vector quantity χ will in general depend on whether it is transformed via r_1 or r_2, see Fig. 2. Hence to keep track of the evolution of χ we need two 2-vectors, one for its evolution with rotations in plane 1, and one 2-vector for rotations in plane 2.

The spin 1/2 wave functions ψ are just combinations of the two 2-vectors that represent vectors like χ in four dimensional Euclidean space E^4. Rotation of the upper 2-vector in ψ can take place in plane 1, while the rotations of the lower 2-vector take place in plane 2.

We can now answer the question posed at the start of the section. Each ψ must have four components because the 2×2 matrix representations of rotations in E^4 are unfaithful and it takes two 2-vectors to represent one vector.

3   An Eigenvector Basis with Independent Planes

Up to this point in Parts I and II, the basis pairs have contained matched upper and lower 2-vectors, 'matched' because they are eigenvectors u^+ or u^- of the same rotation matrix R, I(1) and I(3). We use eigenvectors because they give a phase factor when acted on by the rotation matrix and we use the phase factors in the definition of time and space. But, as just discussed, the rotation plane may not be the same for the upper and lower 2-vectors even when the same rotation matrix R is applied to the upper and lower 2-vectors. What happens when the rotation matrices, and therefore the rotation planes, for the upper and lower 2-vectors are allowed to differ?

Eigenvector Pairs. First we consider two general pairs. Let α, β, a, and b be four 2-vectors arranged in two pairs,

$$u_a^\alpha = \begin{pmatrix} e^{w_1/2}\alpha \\ e^{w_2/2}a \end{pmatrix} \qquad u_b^\beta = \begin{pmatrix} e^{w_3/2}\beta \\ e^{w_4/2}b \end{pmatrix}, \tag{2}$$

where we absorb phase factors in the 2-vectors thereby making the ws real and we absorb the normalization factors in the ws so that the 2-vectors are normalized to unity,

$$\alpha^\dagger \alpha = 1 \qquad \beta^\dagger \beta = 1 \qquad a^\dagger a = 1 \qquad b^\dagger b = 1. \tag{3}$$

The two pairs (2) are orthogonal when they are 'eigenvector pairs,' i.e. α and β are eigenvectors of a rotation matrix R and a and b are eigenvectors of a possibly different rotation matrix R_a. To show this, note that since eigenvectors are orthogonal, I(7), we have

$$\alpha^\dagger \beta = 0 \qquad a^\dagger b = 0 \tag{4}$$

and (2) then implies that u_a^α, and u_b^β are orthogonal,

$$u_a^{\alpha\dagger} u_b^\beta = 0. \tag{5}$$

The inverse is not true. If the pairs (2) are orthogonal as in (5) then it does not follow that $\{\alpha, \beta\}$ and $\{a, b\}$ are sets of eigenvectors and orthogonal.

Hence we can associate planes with the upper and lower 2-vectors of eigenvector pairs. α and β are eigenvectors for rotations in one plane and a and b are eigenvectors for rotations in a second plane. We may just as well take α and β be the eigenvectors u^+ and u^- for the rotation matrix R in I(1) and I(3). We have

$$\alpha = u^+ \quad \beta = u^-. \tag{6}$$

U(1)×SU(2) Theorem. One orthonormal set of eigenvectors is related to a second orthonormal set of eigenvectors by the action of a rotation matrix like R, I(1), and a phase factor, i.e. a U(1)×SU(2) transformation. This is a well known assertion which is obtained in Appendix A for completeness. By (42) we get

$$\begin{pmatrix} a \\ b \end{pmatrix} = T \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = e^{i\chi}[\tau^4 \cos(\kappa/2) + i N^k \tau^k \sin(\kappa/2)] \begin{pmatrix} u^+ \\ u^- \end{pmatrix} = e^{i\chi} e^{i N^k \tau^k \kappa/2} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}, \tag{7}$$

where T is a phase factor times a rotation matrix, χ and κ are real, N^k is a unit 3-vector and the 2×2 matrices τ are the same as the σ matrices in I(2). (We write a spin matrix as a τ when it is applied to a pair of 2-vectors and we write the spin matrix as a σ when it is applied to the two components of one 2-vector.)

The group U(1)×SU(2) is the familiar gauge group for the electroweak interaction. It is important that a gauge group realize some special property of the system described. We have just found that U(1)×SU(2) relates the eigenvectors of rotations in the upper 2-vector plane with those of the lower 2-vector plane. Put another way, the two pairs (2) are orthogonal for any values of χ, N^k, and κ in T, (7).

Basis. A basis contains four pairs, so we take two sets of eigenvector pairs like (2). We have

$$u_a^+ = \begin{pmatrix} e^{d_1/2}u^+ \\ e^{d_2/2}a \end{pmatrix} \quad u_b^- = \begin{pmatrix} e^{d_3/2}u^- \\ e^{d_4/2}b \end{pmatrix} \quad v_a^+ = \begin{pmatrix} e^{i\alpha_1}e^{d_5/2}u^+ \\ e^{i\alpha_2}e^{d_6/2}a \end{pmatrix} \quad v_b^- = \begin{pmatrix} e^{i\alpha_3}e^{d_7/2}u^- \\ e^{i\alpha_4}e^{d_8/2}b \end{pmatrix}, \tag{8}$$

where $\{u^+, u^-\}$ and $\{a, b\}$ are orthonormal sets of eigenvectors just as before and the phases α_i are arbitrary.

We assume the basis pairs are orthogonal. From $u_a^{+\dagger} v_a^+ = u_b^{-\dagger} v_b^- = 0$, we get

$$\alpha_1 - \alpha_2 = \pi + 2n\pi \quad \alpha_3 - \alpha_4 = \pi + 2m\pi \quad d_1 - d_2 = -(d_5 - d_6) \quad d_3 - d_4 = -(d_7 - d_8), \tag{9}$$

where n and m are integers. By orthogonality, (9), we can rewrite the basis (8) as

$$u_a^+ = e^{w_1/2} \begin{pmatrix} e^{w_2/2}u^+ \\ e^{-w_2/2}a \end{pmatrix} \quad u_b^- = e^{w_3/2} \begin{pmatrix} e^{-w_4/2}u^- \\ e^{w_4/2}b \end{pmatrix}$$

$$v_a^+ = e^{i\alpha}e^{w_5/2} \begin{pmatrix} e^{-w_2/2}u^+ \\ -e^{w_2/2}a \end{pmatrix} \quad v_b^- = e^{i\delta}e^{w_6/2} \begin{pmatrix} -e^{w_4/2}u^- \\ e^{-w_4/2}b \end{pmatrix}, \tag{10}$$

where the ws are related to the ds, $w_1 = (d_1 + d_2)/2$, $w_2 = (d_1 - d_2)/2$, etc.

As with previous bases we rotate both upper and lower 2-vectors so that the basis acquires a common phase. We get

$$R_a^+ u_a^+ = e^{+i\theta/2} u_a^+ \quad R_b^- u_b^- = e^{+i\theta/2} u_b^- \quad R_a^+ v_a^+ = e^{+i\theta/2} v_a^+ \quad R_b^- v_b^- = e^{+i\theta/2} v_b^-, \tag{11}$$

where R_a^+ means applying R to the upper 2-vector to generate the eigenvalue $\exp(i\theta/2)$ and R_a to the lower 2-vector to generate the eigenvalue $\exp(i\theta/2)$, R_b^- means applying R^{-1} to the upper 2-vector to generate the eigenvalue $\exp(i\theta/2)$ and $R_b = R_a^{-1}$ to the lower 2-vector to generate the eigenvalue $\exp(i\theta/2)$.

The rotated basis pairs (11) are not yet normalized to the same value. For example, we get

$$(R_a^+ u_a^+)^\dagger R_a^+ u_a^+ = u_a^{+\dagger} u_a^+ = 2e^{w_1} \cosh w_2, \tag{12}$$

with similar expressions for the other normalization values. Normalization values are constrained by spacetime symmetry, as we find in the next section.

4   Projection Operators and Space-time

Any ordered set of four complex numbers can be written in terms of the basis pairs (11). Choose one such set and call it ψ. We get

$$\psi = \alpha R_a^+ u_a^+ + \beta R_b^- u_b^- + \gamma R_a^+ v_a^+ + \delta R_b^- v_b^- = e^{i\theta/2}(\alpha u_a^+ + \beta u_b^- + \gamma v_a^+ + \delta v_b^-), \tag{13}$$

where $\alpha = (R_a^+ u_a^+)^\dagger \psi / (2e^{w_1} \cosh w_2)$, etc.

We now follow the procedure that gave projection operators I(22) and II(6) for the uv basis in Sec. 4 of Part I and the fg basis in Sec. 3 in Part II. We consider the basis-specifying parameters to be the ws in (10) and the n^k of u^+ and u^-. This implies that a and b are known in terms of n^k via the U(1)$\times$SU(2) transformation (7). Hence when the as-yet-unknown delta function parameters p^μ are determined, they will be functions of w and n^k, just as in Parts I and II.

Projection Operators. One can verify that the resulting projection operator for $R_a^+ u_a^+$ is

$$K(2, 1, R_a^+ u_a^+)\gamma^4 \equiv \int \frac{d^3 p'}{(2\pi)^3} \frac{1}{2e^{w_1{}'} \cosh w_2{}'} e^{i\theta_2'/2} u_a^{+}{}' u_a^{+}{}'^\dagger e^{-i\theta_1'/2}, \tag{14}$$

which gives

$$\int d^3 x_1 K(2, 1, R_a^+ u_a^+)\gamma^4 \psi(1) = e^{i\theta_2/2}(\alpha u_a^+) = \psi(2)_{\beta=\gamma=\delta=0}, \tag{15}$$

where $\psi(1)$ is ψ with $\theta = \theta_1$ and $\theta_i'/2 = -\Delta_i' + p^{k'} x_i^k$. The projection operators for the other basis pairs are similar.

Time, Surface Integral. Following the steps leading to I(28) that give space-time symmetry in all parts except for the matrix part, we can write the action of the projection operator in near-space-time form. We get

$$\psi(2)_{\beta=\gamma=\delta=0} = i \int \frac{d^4 x_1 d^4 p'}{(2\pi)^4} e^{-ip'_\mu x_2^\mu} \frac{(\pm m) u_a^{+\prime} u_a^{+\prime\dagger} \gamma^4}{p^{4\prime 2} - (m e^{w_1\prime} \cosh w_2\prime)^2 + i\epsilon} e^{ip'_\nu x_1^\nu} N_\sigma \gamma^\sigma \psi(1), \qquad (16)$$

where we introduce a fourth component of momentum and a fourth component of displacement, i.e. energy and time,

$$p^{4\prime} \equiv \pm m e^{w_1\prime} \cosh w_2\prime a \qquad a\Delta_2' = p^{4\prime} t_2 \qquad x_2^4 = t_2, \qquad (17)$$

a is real, m is a positive constant, and the p^k are as-yet-unknown functions of w_i and n^k. In (16), the integral over x^μ is a surface integral over the surface S which in the frame for (15) is just $x^4 = 0$. In the same frame $N^\mu = \{0, 0, 0, 1\}$.

Matrices. At this point in the process, we look to combine projection operators so that the matrices have a space-time invariant form. One can show that none of the four projection operators is space-time symmetric on its own. Hence, we require that two projection operators combine to make a space-time invariant matrix. In particular, we assume that

$$m(ii^\dagger + jj^\dagger)\gamma^4 = +(E\gamma^4 - p^k \gamma^k + M \cdot 1)$$

$$m(kk^\dagger + ll^\dagger)\gamma^4 = -(E'\gamma^4 - p^{k\prime}\gamma^k + M' \cdot 1), \qquad i, j, k, l \in \{u_a^+, u_b^-, v_a^+, v_b^-\}, \qquad (18)$$

where we can use the $\theta = 0$ basis because the phase factors would cancel anyway. The parameters m, M, and M' are positive or zero. The choice of signs on the right in (18) is made for convenience. We assume that $\{p^k, E\}$ and $\{p^{k\prime}, E'\}$ transform as 4-vectors in order to have energy-momentum p_μ connect the invariant matrix $p_\mu \gamma^\mu$ with the invariant phase $p_\mu x^\mu$ in (16).

One purpose of developing a characterization of spacetime objects is to provide a format for investigating alternatives. Such considerations have their place elsewhere. Hence ways other than (18) to make an invariant matrix are to be considered elsewhere.

There are just three ways to select two sets of two pairs $\{i, j\}$ and $\{k, l\}$ from a set of four. Hence we consider three cases.

Case 1: $i = u_a^+$, $j = u_b^-$, $k = v_a^+$, and $l = v_b^-$. By (7), (10), and the ij part of (18), we get

$$a = u^+ \qquad b = u^-, \qquad (19)$$

$$me^{w_1} = me^{w_3} = M \qquad w_2 = w_4$$

$$E = M \cosh w_2 \qquad p^k = M \sinh(w_2) n^k \qquad (\text{Case 1}).$$

By (19), the expressions (10) for v_a^+ and v_b^- simplify. Then the kl part of (18) gives

$$me^{w_5} = me^{w_6} = M' \qquad E' = -M' \cosh w_2 \qquad p^{k\prime} = M' \sinh(w_2) n^k. \qquad (20)$$

Comparing (19) and (20), we see that the masses M and M' may differ. This is a normalization problem that can be fixed by making the sum of the projection matrices proportional to unity. By (18), one can show that $M = M'$ if and only if

$$u_a^+ u_a^{+\dagger} + u_b^- u_b^{-\dagger} + v_a^+ v_a^{+\dagger} + v_b^- v_b^{-\dagger} \propto 1, \tag{21}$$

where 1 stands for the unit 4×4 matrix.

The normalization values for u_a^+ and u_b^- must be the same to make a spacetime invariant, i.e. by (12) and (19) we have $e^{w_1}\cosh w_2 = e^{w_3}\cosh w_4$, and, by (20), the normalization values for v_a^+ and v_b^- must be equal. But the normalizations for all four are the same only for equal particle and antiparticle masses $M = M'$.

By (19), (20), and (21), the eigenvector basis (10) reduces to

$$u_a^+ = \sqrt{\frac{M}{m}}\begin{pmatrix} e^{w_2/2}u^+ \\ e^{-w_2/2}u^+ \end{pmatrix} \qquad u_b^- = \sqrt{\frac{M}{m}}\begin{pmatrix} e^{-w_2/2}u^- \\ e^{w_2/2}u^- \end{pmatrix}$$

$$v_a^+ = e^{i\alpha}\sqrt{\frac{M}{m}}\begin{pmatrix} e^{-w_2/2}u^+ \\ -e^{w_2/2}u^+ \end{pmatrix} \qquad v_b^- = e^{i\delta}\sqrt{\frac{M}{m}}\begin{pmatrix} -e^{w_2/2}u^- \\ e^{-w_2/2}u^- \end{pmatrix}, \tag{22}$$

Thus the bases found for Case 1 are all essentially the same as the uv basis of Part I, I(9). We recover the uv basis when we select

$$m = M \qquad w_2 = w \qquad \alpha = \delta = 0 \qquad (\text{Case } 1). \tag{23}$$

By Part I, we conclude that Case 1 gives propagators that can describe a free electron and a free positron.

Case 2: $i = u_a^+$, $j = v_a^+$, $k = u_b^-$, and $l = v_b^-$. By (7), (10), and the ij expression in (18), we get

$$a = e^{i\nu}u^- \qquad b = e^{i\zeta}u^+ \qquad (\text{Case } 2) \tag{24}$$

$$M = 0 \qquad w_1 = w_5 \qquad E = me^{w_1}\cosh w_2 \qquad p^k = me^{w_1}\cosh(w_2)n^k,$$

where ν and ζ are arbitrary phases. By (24), the kl expression in (18) gives

$$M' = 0 \quad w_3 = w_6 \quad E' = -me^{w_3}\cosh w_4 \quad p^k = me^{w_3}\cosh(w_4)n^k. \tag{25}$$

By (24) and (25) we see that spacetime invariance allows particle and antiparticle to have different energy-momenta when they are in one of the states of this basis. When we require the sum of the projection operators to be a multiple of the unit matrix, (21), we have

$$E = -E' \quad p^k = p^{k'} \quad e^{w_3}\cosh(w_4) = e^{w_1}\cosh(w_2). \tag{26}$$

For simplicity, we take $w_3 = w_1$ and $w_4 = w_2$.

By (24), (25), and (26), the eigenvector basis (10) reduces to

$$u_a^+ = e^{w_1/2} \begin{pmatrix} e^{w_2/2}u^+ \\ e^{i\nu}e^{-w_2/2}u^- \end{pmatrix} \qquad u_b^- = e^{w_1/2} \begin{pmatrix} e^{-w_2/2}u^- \\ e^{i\zeta}e^{w_2/2}u^+ \end{pmatrix}$$

$$v_a^+ = e^{i\alpha}e^{w_1/2} \begin{pmatrix} e^{-w_2/2}u^+ \\ -e^{i\nu}e^{w_2/2}u^- \end{pmatrix} \qquad v_b^- = e^{i\delta}e^{w_1/2} \begin{pmatrix} -e^{w_2/2}u^- \\ e^{i\zeta}e^{-w_2/2}u^+ \end{pmatrix}, \qquad (27)$$

The bases that have spacetime symmetry under Case 2 are equivalent to the fg basis II(1). To duplicate those particular pairs we choose

$$w_1 = 0 \qquad w_2 = w \qquad \alpha = \delta = \nu = \zeta = 0. \quad \text{(Case 2)} \qquad (28)$$

The normalization parameter w_1 can be absorbed in m and α, δ, ν, and ζ are arbitrary phases. Hence, by Part II, Case 2 gives bases that lead to the neutrino and antineutrino propagators II(17) and II(21).

Case 3: $i = u_a^+$, $j = v_b^-$, $k = u_b^-$, and $l = v_a^+$. By (7), (10), and the ij part of (18), we get

$$a = u^+ \qquad b = -u^-, \qquad \text{(Case 3)} \qquad (29)$$

$$me^{w_1} = me^{w_6} = M \qquad w_2 = -w_4$$

$$E = M \cosh w_2 \qquad p^k = M \sinh(w_2)n^k.$$

By (29), the expressions (10) for u_b^- and v_a^+ simplify. Then the kl version of (18) yields

$$me^{w_3} = M' \qquad w_3 = w_5 \qquad E' = -M' \cosh w_2 \qquad p^{k'} = M' \sinh(w_2)n^k, \qquad (30)$$

Requiring that the sum of the projection matrices be proportional to unity, (21), gives

$$E = -E' \qquad p^k = p^{k'} \qquad M = M'. \qquad (31)$$

By (29), (30), and (31), the eigenvector basis (10) reduces to

$$u_a^+ = \sqrt{\frac{M}{m}} \begin{pmatrix} e^{w_2/2}u^+ \\ e^{-w_2/2}u^+ \end{pmatrix} \qquad u_b^- = \sqrt{\frac{M}{m}} \begin{pmatrix} e^{w_2/2}u^- \\ -e^{-w_2/2}u^- \end{pmatrix}$$

$$v_a^+ = e^{i\alpha}\sqrt{\frac{M}{m}} \begin{pmatrix} e^{-w_2/2}u^+ \\ -e^{w_2/2}u^+ \end{pmatrix} \qquad v_b^- = e^{i\delta}\sqrt{\frac{M}{m}} \begin{pmatrix} -e^{-w_2/2}u^- \\ -e^{w_2/2}u^- \end{pmatrix}, \qquad (32)$$

The basis (32) is essentially the same as the uv basis in Part I, I(9). To get the uv basis within a couple of signs, $u_b^- \to -v_-^-$ and $v_b^- \to -u_-^-$, put

$$m = M \qquad w_2 = w \qquad \alpha = \delta = 0. \quad \text{(Case 3)} \qquad (33)$$

Thus, by Part I, Case 3 yields a basis that gives a free electron and a free positron propagator.

Spacetime invariance, i.e. the requirement (18), has returned only eigenvector bases that give electron or neutrino-like propagators. It follows that the only space-time invariant spin 1/2 propagators are electron-like or neutrino-like, within the limits of the assumptions applied in the process developed in this series of papers.

Furthermore spacetime symmetry results only when the two planes of an eigenvector basis are complementary in E^4. Hence we have found a geometric property that signals the ability to construct spacetime.

A   U(1)×SU(2) Theorem

Let u^+ and u^- be orthonormal 2-vectors. Let a and b be a second set of orthonormal 2-vectors, as in (3) and (4). As shown in Problem 2, it follows that u^+ and u^- are eigenvectors for rotations in one plane, plane 1, and a and b are eigenvectors in some plane, plane 2.

Theorem: One orthonormal 2-vector set $\{a, b\}$ can be found from another orthonormal set $\{u^+, u^-\}$ by multiplication with a rotation matrix like R, I(1), and a phase factor, i.e. by the action of a U(1)×SU(2) transformation. The proof follows.

Since it takes two linearly independent 2-vectors to span the space of 2-vectors, a and b can be expressed in terms of u^+ and u^-. Let T be the matrix of coefficients relating a and b to u^+ and u^-.

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} \begin{pmatrix} u^+ \\ u^- \end{pmatrix}. \tag{34}$$

We need to show that T has the form of a phase factor times a matrix like R, I(1).

Since $a^\dagger a = 1$, $b^\dagger b = 1$, and $a^\dagger b = 0$, we get

$$| T_{11} |^2 + | T_{12} |^2 = 1 \qquad | T_{21} |^2 + | T_{22} |^2 = 1 \qquad T_{11}^* T_{21} = -T_{12}^* T_{22} \tag{35}$$

Write $(a^\dagger b)^* (a^\dagger b) = 0$ in terms of the T_{ij}s. We have

$$| T_{11} |^2 | T_{21} |^2 + T_{11}^* T_{21} T_{12} T_{22}^* + T_{11} T_{21}^* T_{12}^* T_{22} + | T_{12} |^2 | T_{22} |^2 = 0. \tag{36}$$

Now use (35) to eliminate T_{11} and T_{11}^* from the first three terms in (36). Note that it is not necessary to divide by any T_{ij} components. By simplifying the result we get

$$| T_{21} |^2 = | T_{12} |^2. \tag{37}$$

By (35) and (37), we get

$$| T_{11} |^2 = | T_{22} |^2. \tag{38}$$

These equations force T to take the following form in terms of real valued phases α, β, γ, δ, and η. We get

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = \begin{pmatrix} \exp i\alpha \cos \eta & \exp i\beta \sin \eta \\ -\exp i\gamma \sin \eta & \exp i\delta \cos \eta \end{pmatrix}, \tag{39}$$

where the new parameters here on the right have no relation to the other uses of the same symbols elsewhere in this paper.

By the rightmost equation in (35), i.e. $a^\dagger b = 0$, we find that $\beta + \gamma = \alpha + \delta + 2n\pi$. Since $\alpha = (\alpha + \delta)/2 + (\alpha - \delta)/2$, $\delta = (\alpha + \delta)/2 - (\alpha - \delta)/2$, $\beta = (\alpha + \delta)/2 + (\beta - \gamma)/2 + n\pi$, $\gamma = (\alpha + \delta)/2 - (\beta - \gamma)/2 = n\pi$, we have

$$\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix} = e^{i(\alpha + \delta)/2} \begin{pmatrix} \exp[i(\alpha - \delta)/2] \cos \eta & (-1)^n \exp[i(\beta - \gamma)/2] \sin \eta \\ (-1)^{n+1} \exp[-i(\beta - \gamma)/2] \sin \eta & \exp[-i(\alpha - \delta)/2] \cos \eta \end{pmatrix}. \tag{40}$$

We see that T is a phase factor times a unimodular unitary matrix; the set of such matrices forms a 2×2 representation of SU(2), which is equivalent to a representation of rotations in E^3 or E^4.

To put the SU(2) matrix in (40) in the same form as R, see I(1), we introduce new variables κ and N^k,

$$\cos\left(\kappa/2\right) = \cos\left[(\alpha - \delta)/2\right]\cos\eta \qquad N^3\sin\left(\kappa/2\right) = \sin\left[(\alpha - \delta)/2\right]\cos\eta \qquad (41)$$

$$N^1\sin\left(\kappa/2\right) = (-1)^n\sin\left[(\beta - \gamma)/2\right]\sin\eta \qquad N^2\sin\left(\kappa/2\right) = (-1)^n\cos\left[(\beta - \gamma)/2\right]\sin\eta.$$

One can show that N^k is a unit 3-vector, i.e. $\sum {N^k}^2 = 1$, for $\kappa \neq 0$. In terms of the new variables, we get

$$\begin{pmatrix} a \\ b \end{pmatrix} = T\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = e^{i\chi}[\tau^4\cos(\kappa/2) + iN^k\tau^k\sin(\kappa/2)]\begin{pmatrix} u^+ \\ u^- \end{pmatrix} = e^{i\chi}e^{iN^k\tau^k\kappa/2}\begin{pmatrix} u^+ \\ u^- \end{pmatrix}, \qquad (42)$$

where $\chi = (\alpha + \delta)/2$. The 2×2 matrices τ are the same as the σ matrices, I(2),

$$\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \tau^4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \qquad (43)$$

We write a spin matrix as a τ when it is applied to a pair of 2-vectors and we write the spin matrix as a σ when it is applied to the two components of a 2-vector.

Comparing (42) and I(1) completes the proof.

B   Problems

1. Consider the pair $u_+^+ = \begin{pmatrix} e^{w/2}u^+ \\ e^{-w/2}u^+ \end{pmatrix}$ of the uu basis, I(9). Show that u_+^+ does not represent any vector quantity χ in E^4. Find two pairs X_1 and X_2 that do represent vector quantities χ_1 and χ_2 and whose sum is u_+^+, i.e. $X_1 + X_2 = u_+^+$. Assume that the upper eigenvector u^+ (a 2-vector) is an eigenvector for rotations about the origin in plane 1 and the lower eigenvector u^+ is an eigenvector for rotations about the origin in plane 2. Also assume that plane 1 and plane 2 span E^4.

[Hint: An eigenvector represents a vector perpendicular to the plane of rotation. A vector in the plane of rotation is represented by a sum of the eigenvectors for rotations in the plane.]

2. Prove the following theorem. Theorem: two orthonormal 2-vectors, say δ and ϵ, are the eigenvectors of some rotation. (One way is identify the parameters in u^+ in I(3) in terms of the components of one 2-vector and use orthonormality to show the other must have the form of u^- in I(3). Then construct the matrix R and show $R\delta = \exp\left(i\theta/2\right)\delta$ and that $R\epsilon = \exp\left(-i\theta/2\right)\epsilon$, thereby completing the proof.)

3. Consider two coordinate planes 12 and 34, where 1234 indicates rectangular coordinates. If we decide the positive rotation direction for plane 12 then there are two ways to orient the rotations in plane 34. If one orientation gives a positive phase $+\theta/2$ for a rotation in the plane 34 then the other orientation gives a negative phase $-\theta/2$. Thus the occurance of two orientations for rotations in plane 34 means the lower 2-vector can have either the same phase as or the opposite phase of the upper 2-vector.

This suggests that we investigate both combinations of phases, $++$ and $+-$, for upper and lower 2-vectors. The rotated basis pairs for an electron in Part I and a neutrino in Part II have upper and lower 2-vectors with like phase $\theta/2$. Carry through the processes from rotated basis to spacetime invariant electron and neutrino propagators with the upper 2-vectors having phase $\theta/2$ and the lower 2-vectors having the opposite phase $-\theta/2$.

4. Show that 2×2 matrices cannot provide a nontrivial representation of all rotations about the origin in Euclidean spaces of five or more dimensions. One way to show this is to consider a 2×2 matrix representation in a four dimensional subspace and then try to extend that representation to rotations in unrepresented planes. (For an application of 2×2 matrix representations of rotations in 4d subspaces of 16d, see [3].)

References

[1] Shurtleff, R., hep-th/0007232.

[2] Shurtleff, R., hep-th/0007245.

[3] Shurtleff, R., quant-ph/9907012.