Induced commutative Chern-Simons term from noncommutativity in planar systems

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Abstract

We test the consistency of the use of a noncommutative theory description for charged particles in a strong magnetic field, by deriving the induced Chern-Simons (CS) term for an external Abelian gauge field in $2+1$ dimensions. In this description, the system is modeled by a noncommutative matter field coupled to a $U(1)$ noncommutative gauge field, related to the original, commutative one, by a Seiberg-Witten transformation. We show that an Abelian CS term for the commutative gauge field is indeed induced, and moreover that it matches the result of previous commutative field theory calculations.

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1 Introduction

The relevance of noncommutative field theories for the description of 2+1 dimensional systems in a strong external magnetic field, notably the Quantum Hall effect, hardly needs to be stressed. Indeed, it has been shown some time ago that noncommutative field theories in $2+1$ dimensions may be a useful tool for the approximate description of some planar condensed matter models, when the kinetic term may be neglected in comparison with the coupling between the matter current and the external field. When this is the case, the two spatial coordinates become conjugate to each other, so that the system may be thought of as defined on a 'quantum phase space': a noncommutative quantum field theory [1, 2, 3, 4, 5, 6, 7, 8, 9]. In such an effective description, the noncommutativity of space is characterized by the antisymmetric ('Poisson') tensor θ^{ij}:

$$\theta^{ij} \;=\; \theta\,\epsilon^{ij} \quad , \quad \theta \;=\; -(eB)^{-1} \quad i,j \;=\; 1,2. \tag{1}$$

Hence, all the dependence on the external magnetic field is traded for the noncommutativity, so that if B manifests itself through, for example, parity or time-reversal symmetry breaking effects, this has to appear as a pure consequence of the noncommutativity of the effective description.

A particularly interesting opportunity to check the noncommutative description arises when one considers a commutative theory corresponding to charged particles in the presence of a strong magnetic field and of an abelian gauge field $\mathcal{A}_\mu$ (unrelated to B) which plays the role of a 'source' or external probe. It is well known that, in a purely commutative description, one may apply perturbation theory to obtain the induced CS term for $\mathcal{A}_\mu$. Of course this is only possible in the case in which the relation between the external magnetic field (B) and the electronic density (ρ) is such that there is an integer number of filled Landau levels, i.e., the filling fraction is $\nu = \frac{\rho}{eB}2\pi = n$ with n integer (in units in which $\hbar = c = 1$). In this evaluation, the constant magnetic field is taken into account exactly, since the full propagator in the presence of the magnetic field and at a finite density is used in the perturbative series [10]. The emergence of an induced CS term may be seen, in this picture, to be a parity breaking effect due to the magnetic field. In the more general case, i.e., when a perturbative calculation is not correct, it can be argued, based on symmetry reasons, that there should be a Chern-Simons term in the effective action for the external probe. Equivalently, due to the presence of the external magnetic field there should be a transverse response to a static electric field. In two spatial dimensions, such a response can be obtained only if there is a Chern-Simons term. Moreover, the coefficient should be determined by the Hall conductance which can be shown to be $\sigma_{xy} = \frac{\rho c e}{B}$ in a translational invariant system [11].

Our aim, in this paper, is to show that the same object, namely, the commutative induced CS term, may also be derived in the corresponding effective noncommutative framework. This result will look, at first sight, paradoxical, since a perturbative evaluation of the effective action in the noncommutative model yields no induced CS term for the *noncommutative* gauge field.

The organization of this paper is as follows: in section 2, we derive the noncommutative theory as an effective description of the original commutative model, in section 3 we present a calculation of the induced CS term in the noncommutative theory context, and in section 4 we present our conclusions.

2 The noncommutative effective theory

To begin with, let us assume that the commutative system is defined by a generating functional $\mathcal{Z}_B[\mathcal{A}_\mu]$, where $\mathcal{A}_\mu$ denotes an external source

$$\mathcal{Z}_B[\mathcal{A}_\mu] \;=\; \int \mathcal{D}\bar{\psi}\,\mathcal{D}\psi \,\exp\left\{i\,S[\bar{\psi},\psi;B,\mathcal{A}_\mu]\right\} \;, \tag{2}$$

and we have made explicit the fact that the action S depends on the magnetic field B. The generating functional has been written as a path integral over a charged field, which will typically be a spinless fermionic field. The corresponding action for this (non-relativistic, Grassmann) field is

$$S[\bar{\psi},\psi;B,\mathcal{A}] \;=\; \int d^3x \left[\bar{\psi} i D_0 \psi - \frac{1}{2m}\overline{D_i\psi}(D_i\psi)\right] , \tag{3}$$

where the covariant derivatives are defined by

$$D_0 \;=\; \partial_0 - ie(a_0 + \mathcal{A}_0) \qquad D_i \;=\; \partial_i - ie(A_i + \mathcal{A}_i) \;, \tag{4}$$

where a_0 is a constant, which plays the role of a chemical potential, fixing the total charge Q of the system, and A_i $(i = 1, 2)$ is the vector potential of the constant magnetic field, namely: $\epsilon^{ij}\partial_i A_j = B$. In order to understand the emergence of an effective noncommutative description, we shall first set $\mathcal{A}_\mu = 0$, and then reintroduce it (as well as possible interactions) afterwards, so that the generating functional we will first consider is

$$\mathcal{Z}_B \;\equiv\; \mathcal{Z}_B[0] \;=\; \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \;\exp\left\{iS[\bar{\psi},\psi;B,0]\right\} \;. \tag{5}$$

We then use the complex combinations $D = D_1 + iD_2$ and $\overline{D} = D_1 - iD_2$, plus an integration by parts, to write

$$S[\bar{\psi}, \psi; B, 0] = \int d^3x \left[\bar{\psi} \left(i\partial_t + ea_0 - \frac{eB}{2m} \right) \psi + \frac{1}{2m} \bar{\psi} \overline{D} D \psi \right], \tag{6}$$

where we have used the elementary relation $[D_1, D_2] = -ieB$. The noncommutative theory will be obtained by performing an approximation which relies upon the smallness of the scale defined by the mass in comparison with the one defined by the magnetic field B. Moreover, in the noncommutative theory, the matter fields will verify first order equations. In order to take these facts into account, it is convenient to use an equivalent action $\widetilde{S}$, obtained by introducing two auxiliary complex fields λ and $\bar{\lambda}$:

$$\widetilde{S}[\bar{\psi}, \psi, \bar{\lambda}, \lambda; B] = \int d^3x \left[\bar{\psi} \left(i\partial_t + ea_0 - \frac{eB}{2m} \right) \psi + \bar{\lambda} D \psi + \bar{\psi} \bar{D} \lambda - \frac{m}{2} \bar{\lambda} \lambda \right], \tag{7}$$

which, of course, reproduces S when $\bar{\lambda}$ and λ are integrated out.

Assuming the charge density of the system to be fixed to some value ρ, we may write $ea_0 = -\frac{2\pi}{m}\rho$, thus we see that the ea_0 term scales like m^{-1}. In fact, all the terms in the first line of (7) have the same dependence in m, since when one uses an expansion of the time dependence of the fields in terms of the Landau level energies

$$E_n = \omega_c \left(n + \frac{1}{2} \right) \quad , \quad \omega_c = \frac{|eB|}{m} \quad , \tag{8}$$

the time derivative also picks up a factor of m^{-1}, regardless of the level considered. The noncommutative description arises when the $m \to 0$ limit is taken. This requires the vanishing of the term which is quadratic in the Grassmann fields. Using the explicit values of the energies, we see that this implies a constraint between the density and the magnetic field:

$$\rho = |eB| \frac{n}{2\pi} \quad n \in \mathbb{N} \, . \tag{9}$$

In this limit, the term quadratic in the auxiliary field vanishes, and the equations of motion are first order: $D\psi = 0$, i.e., the fields are in the lowest Landau level (LLL). This means that only the $n = 1$ case actually appears in (9), when m is strictly zero. This reduction to the LLL implies that, if there are more terms in the action (like interactions) one has to replace the standard product by the Moyal product: the theory becomes noncommutative [9]. This is true, in particular, for the terms that correspond to the coupling to the external field $\mathcal{A}_\mu$. We start from that standpoint in the following section.

3 Induced Chern-Simons term

The existence of an effective noncommutative description when the fields are projected to the LLL amounts to:

$$\mathcal{Z}_B[\mathcal{A}_\mu] \simeq \widehat{\mathcal{Z}}_\theta[\widehat{\mathcal{A}}^\theta_\mu] \quad (m \to 0) \tag{10}$$

where $\widehat{\mathcal{Z}}_\theta[\widehat{\mathcal{A}}^\theta_\mu]$ is the generating functional for the noncommutative theory

$$\widehat{\mathcal{Z}}_\theta[\widehat{\mathcal{A}}^\theta_\mu] = \int \mathcal{D}\widehat{\bar{\psi}}\mathcal{D}\widehat{\psi} \, \exp\{iS^\theta[\widehat{\bar{\psi}}, \widehat{\psi}; 0, \widehat{\mathcal{A}}^\theta_\mu]\} \, . \tag{11}$$

Independently of the details of the particular model considered, the noncommutative action will be assumed to have the same structure as the commutative one, except for the fact that there will be no coupling to an external (noncommutative) magnetic field (B is entirely absorbed in θ). As we mentioned in section 1, when there is an integer number of filled Landau levels, a local CS term in $\mathcal{A}_\mu$ appears as the leading term in a derivative expansion of the commutative generating functional, namely,

$$\mathcal{Z}_B[\mathcal{A}_\mu] \simeq \exp\{i\kappa(B) \, S_{CS}[\mathcal{A}_\mu]\} \, , \tag{12}$$

where S_{CS} denotes the CS action,

$$S_{CS}[A] = \frac{1}{2} \int d^3x \, \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \tag{13}$$

while κ is the corresponding coefficient, determined by the number of filled Landau levels n, through the relation:

$$\kappa(B) = \frac{n}{2\pi} = \frac{\rho e}{B} \, . \tag{14}$$

Our aim is to show that the same object can be derived in the noncommutative theory framework.

We shall provide an explicit calculation. To that end, we will assume that the commutative theory action S is given by:

$$S \;=\; \int d^3x \left[\bar{\psi} i D_0 \psi - \frac{1}{2m} (\overline{D_i \psi})(D_i \psi) \right] , \tag{15}$$

where $D_j \;=\; \partial_j + A_j^B + A_j$, with A_j^B denoting the part of the gauge field that corresponds to the magnetic field B, i.e., $\epsilon^{jk} \partial_j A_k \;=\; ieB$, and $D_0 \;=\; \partial_0 + i\mu + A_0$, where μ is the chemical potential, to be fixed latter, in terms of the total charge of the system. To simplify the comparison with the noncommutative theory, we shall use the convention that A_μ is anti-Hermitian, and moreover that the charge e has been absorbed into the gauge field definition. The noncommutative action S^θ is, on the other hand, explicitly given by

$$S^\theta \;=\; \int d^3x \left[i\overline{\widehat{\psi}} \star \widehat{D}_0 \widehat{\psi} \;-\; \frac{1}{2m} (\overline{\widehat{D}_i \widehat{\psi}}) \star \widehat{D}_i \widehat{\psi} \right] \tag{16}$$

where $\star$ denotes the noncommutative Moyal product:

$$f(x) \star g(x) \;=\; \exp \left(-\frac{i}{2} \theta \epsilon^{ij} \frac{\partial}{\partial \zeta^i} \frac{\partial}{\partial \xi^j} \right) f(x+\zeta) g(x+\xi) \bigg|_{\zeta=0, \xi=0} , \tag{17}$$

$\widehat{D}_j \widehat{\psi} \;=\; \partial_j \widehat{\psi} + \widehat{\psi} \star \widehat{A}_j$, and $\widehat{D}_0 \;=\; \partial_0 \widehat{\psi} + \widehat{\psi} \star (\widehat{A}_0 + i\mu)$ (we have chosen the antifundamental representation). After some standard manipulations, we may rewrite S^θ as:

$$S^\theta \;=\; S_{\rm free} \,+\, S_{\rm int} \tag{18}$$

where

$$\begin{aligned} S_{\rm free} \;&=\; \int d^3x \left[i\overline{\widehat{\psi}} \star (\partial_0 + i\mu) \widehat{\psi} - \frac{1}{2m} \partial_i \overline{\widehat{\psi}} \star \partial_i \widehat{\psi} \right] \\ S_{\rm int} \;&=\; \int d^3x \left\{ i\widehat{\psi} \star \widehat{A}_0 \star \overline{\widehat{\psi}} - \frac{1}{2m} \left[\widehat{\psi} \star \widehat{A}_i \star \partial_i \overline{\widehat{\psi}} - \partial_i \widehat{\psi} \star \widehat{A}_i \star \overline{\widehat{\psi}} \right.\right. \\ &\quad \left.\left. - \widehat{\psi} \star \widehat{A}_i \star \widehat{A}_i \star \overline{\widehat{\psi}} \right] \right\} . \end{aligned} \tag{19}$$

The induced CS term will be obtained, as usual, from the fermionic effective action $S_{\rm eff}$,

$$\widehat{\mathcal{Z}}^\theta[\widehat{A}_\mu^\theta] \;=\; \int \mathcal{D}\overline{\widehat{\psi}} \mathcal{D}\widehat{\psi} \exp \left[i(S_{\rm free} + S_{\rm int}) \right] = \exp \left(i S_{\rm eff}[\widehat{A}] \right) , \tag{20}$$

with:

$$
\begin{aligned}
S_{\text{eff}} &= \operatorname{Tr}\left\{ \ln\left[(k_0 - \frac{\vec{k}^2}{2m} - \mu)\delta(k_1 - k_2) \right.\right. \\
&\quad + i \int dp \left(\tilde{A}_0(p) + \frac{(\vec{k}_1 + \vec{k}_2)\vec{\tilde{A}}(p)}{2m} \right) e^{-i/2\, k\theta p} \delta(k_1 + p - k_2) \\
&\quad \left.\left. + \int \frac{dp\, dq}{2m} \vec{\tilde{A}}(p)\vec{\tilde{A}}(q) e^{-i/2\xi} \delta(k_1 + p + q - k_2) \right] \right\}
\end{aligned} \tag{21}
$$

and $\widehat{A}_\mu(x) = \int d^3p \widetilde{A}_\mu(p)\, e^{ip\cdot x}$. We are using the following notation:

$$
\begin{aligned}
\xi &= k_1\theta p + k_1\theta q - k_1\theta k_2 + p\theta q - p\theta k_2 - q\theta k_2 \\
p\theta q &= p_i\theta^{ij} q_j = \theta p_i \epsilon^{ij} q_j .
\end{aligned}
$$

Expanding S_{eff} up to second order in A_μ, we find:

$$
S_{\text{eff}} \simeq S_{\text{eff}}^{(0)} + S_{\text{eff}}^{(1)} + S_{\text{eff}}^{(2)} \tag{22}
$$

where

$$
S_{\text{eff}}^{(0)} = \operatorname{Tr}\left[\ln\left(\Delta^{-1}(k_1, \mu)\delta(k_1 - k_2) \right) \right] \tag{23}
$$

$$
S_{\text{eff}}^{(1)} = \operatorname{Tr}\left[\Delta(k_1, \mu)\delta(k_1 - k_2) T_1(k_2, k_3) \right] \tag{24}
$$

$$
\begin{aligned}
S_{\text{eff}}^{(2)} &= \operatorname{Tr}\left[\Delta(k_1, \mu)\delta(k_1 - k_2) T_2(k_2, k_3) - \frac{1}{2}\Delta(k_1, \mu)\delta(k_1 - k_2) \right. \\
&\quad \left. T_1(k_2, k_3)\Delta(k_3, \mu)\delta(k_3 - k_4) T_1(k_4, k_5) \right]
\end{aligned} \tag{25}
$$

and we defined

$$
\begin{aligned}
\Delta(k, \mu) &= \left(k_0 - \frac{\vec{k}^2}{2m} - \mu \right)^{-1} \\
T_1(k_1, k_2) &= i \int d^d p \left(\tilde{A}_0(p) + \frac{(\vec{k}_1 + \vec{k}_2)\vec{\tilde{A}}(p)}{2m} \right) \delta(p + k_1 - k_2) e^{-i/2\, k_1\theta p} \\
T_2(k_1, k_2) &= \int \frac{dp\, dq}{2m} \vec{\tilde{A}}(p)\vec{\tilde{A}}(q) e^{-i/2\xi} \delta(p + q + k_1 - k_2) .
\end{aligned}
$$

By analogy with the commutative field theory calculation, one may be inclined to think that the CS term comes from the term of second order in

A_μ:

$$
\begin{aligned}
S^{(2)}_{\rm eff} &= \int d^3p d^3k \Delta(k,\mu) \left\{ \frac{\vec{\tilde{A}}(-p)\vec{\tilde{A}}(p)}{2m} + \frac{1}{2}\Delta(k+p,\mu) \left[\tilde{A}_0(p) + \frac{2\vec{k}+\vec{p}}{2m}\vec{\tilde{A}}(p) \right] \right. \\
& \left. \left[\tilde{A}_0(-p) + \frac{2\vec{k}+\vec{p}}{2m}\vec{\tilde{A}}(-p) \right] \exp(i/2\; p\theta p) \right\} \; ; \qquad (26)
\end{aligned}
$$

however, it is straightforward to check that the transverse conductivity coming from that term vanishes. This result has been derived under the assumption $[\partial_i, \partial_j]A_\mu = 0$, namely, 'trivial' A_μ configurations. At the level of the calculation, this amounts to using the relation: $\exp(ip\theta p)\widetilde{A}_\mu(p) = \widetilde{A}_\mu(p)$ in Fourier space. Relaxing that condition, however, does not help to induce a CS term, in spite of the fact that it might produce parity breaking effects for some special, singular configurations (i.e., vortex like configurations) of the gauge field.

The resolution of this apparent paradox lies in the fact that the relation between the commutative and noncommutative generating functionals is less direct than it seems, since the commutative one depends on A while the noncommutative functional has $\widehat{A}^\theta_\mu$ as its argument. The relation between these two gauge fields is determined by the requisite that the mapping should preserve the respective orbits of each theory. It is given by the Seiberg-Witten map [12], which provides a relation of the type:

$$
\widehat{A}^\theta_\mu \;=\; \widehat{A}^\theta_\mu(A,\theta) \qquad (27)
$$

where the dependence is highly nonlinear, and in general it may be found only by applying some sort of approximation scheme. Thus, our claim is that, in order to recover the induced CS term of the commutative theory, one should use the SW relation to bring the effective action back to a functional of A_μ, the commutative gauge field. Since the CS term is quadratic in A_μ, the use of a first order expansion in θ is sufficient, since a θ expansion of the Seiberg-Witten map yields [12, 13]:

$$
\widehat{A}_\mu \;=\; A_\mu - \frac{\theta^{\rho\sigma}}{2} A_\rho \{\partial_\sigma A_\mu + F_{\sigma\mu}\} \,+\, \mathcal{O}(\theta^2) \; . \qquad (28)
$$

With this in mind, we may now calculate the term $S_q[A]$, of second order in the *commutative* gauge field A, coming from the effective action, obtaining:

$$
S_q[A] \;=\; S_{\rm even} + S_{\rm odd} \qquad (29)
$$

with $S_{\rm even}$ given by $S_{\rm eff}^{(2)}$ as in eq. (26), and

$$S_{\rm odd}[A] \;=\; -i\frac{\theta^{\rho\sigma}}{2}\int d^3x\int\frac{d^3k}{(2\pi)^3}\Delta(k,\mu)\;A_\rho(x)[(\partial_\sigma A_0+F_{\sigma\,0}) + \frac{2\vec{k}}{2m}(\partial_\sigma\vec{A}+\vec{F_\sigma})]\;. \tag{30}$$

where we have assumed trivial configurations for the field $A_\mu(x)$. Finally, we may write the explicit form of the $S_{\rm odd}$ term:

$$S_{\rm odd} \;=\; -iI(\mu)\,\frac{\theta^{\rho\sigma}}{2}\int d^3x\,A_\rho(\partial_\sigma A_0+F_{\sigma\,0}) \tag{31}$$

where

$$I(\mu) \;=\; \int\frac{d^3k}{(2\pi)^3}\,\Delta(k,\mu) \;=\; -\,\frac{i}{2\pi}\,m\mu\;. \tag{32}$$

Using this result, the relation $\theta = -(eB)^{-1}$, and rewriting the gauge field dependent terms in covariant form, we arrive at the expression:

$$S_{\rm odd} \;=\; \frac{m\mu}{2\pi eB}\int d^3x\,\epsilon^{\alpha\beta\gamma}\,A_\alpha\partial_\beta A_\gamma\;, \tag{33}$$

which is a Chern-Simons action, with a coefficient

$$\kappa \;=\; \frac{\rho e}{B} \tag{34}$$

where we used the fact that $\rho = \frac{m\mu}{2\pi}$, and we have restored the e factors that were absorbed into the gauge field definition at the begining of this section. This results coincides with the one obtained from the commutative field theory calculation [10]. We remark that, as in the commutative case, the Chern-Simons coefficient will not be renormalized by higher order terms in the perturbative expansion in $\tilde{A}$. Moreover, higher order terms in the S-W relation will not contribute to this term since they involve higher order powers in θ.

Regarding the general situation, i.e., the case of a general planar field theory corresponding to charged particles, we note that the crucial property which is required is the existence of a finite density (i.e., a finite chemical potential). The existence of a finite chemical potential implies that in the evaluation of the fermionic determinant, the effective action will always have a term proportional to μ and to the integral of $\widehat{A}_0$. The latter, when written in terms of A_μ, is a commutative CS term.

4 Conclusions

We have shown that, for a system of non-relativistic fermions in a magnetic field, the induced CS term for an external gauge field may be re-derived using the effective noncommutative description corresponding to the original commutative theory. We insist that the object we calculate is *not* the noncommutative induced CS term, but rather we show how the usual, commutative CS term is indeed captured by the approximations made to introduce the noncommutative description. The corresponding coefficients for the CS terms agree, and moreover in both the commutative and noncommutative calculations there are constraints between the density and the magnetic field. In the former, the reason is that perturbation theory would be ill-defined for a system with partially filled Landau levels (degeneracy), while for the latter the constraint appears in the very derivation of the noncommutative description (9).

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