Macroconstraints from Microsymmetries

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Abstract

The dynamics governing the evolution of a many body system is constrained by a nonabelian local symmetry. We obtain explicit forms of the global macroscopic condition assuring that at the microscopic level the evolution respects the overall symmetry constraint.

Let us consider a multiparticle quantum system with the local interaction invariant with respect to the internal symmetry group G. We shall call this underlying symmetry of microscopic interactions a <u>microsymmetry</u> of the system. The system transforms under a given representation of the symmetry group. We shall call this property a <u>macrosymmetry</u> of the system. Let us try to answer a question: is macrosymmetry preserved during a time evolution of the system? This problem is treated with more details by J. Rafelski and the author[1].

The system consists of particles belonging to multiplets of the symmetry group. One denotes $f_{(\zeta)}^{(\alpha_i,\nu_i)}(\Gamma,\vec{r},t)$ a distribution function of the particle, transforming under α_i representation of the symmetry group, with quantum numbers of ν_i member of the multiplet. The variables $(\Gamma,\vec{r})$ denotes a set of the phase - space variables such as $(\vec{p},\vec{r})$ and t is time. A subscript ζ denotes other quantum numbers characterizing different multiplets of the same representation α.

The number of particles of the specie $\{\alpha,\nu_\alpha\}$ is:

$$N_{\nu_\alpha;(\zeta)}^{(\alpha)}(t) = \int dV d\Gamma f_{(\zeta)}^{(\alpha,\nu_\alpha)}(\Gamma,\vec{r},t). \tag{1}$$

Let us consider the corresponding state vector in particle number representation: $\left| N_{\nu_{\alpha_1}}^{(\alpha_1)},\ldots,N_{\nu_{\alpha_n}}^{(\alpha_n)} \right\rangle$. All other variables, related to phase-space properties of the system are suppressed here. This vector describes symmetry properties of our systems and transforms as a direct product representation of the symmetry group G. This representation is of the form:

$$\alpha_1^{N^{(\alpha_1)}} \otimes \alpha_2^{N^{(\alpha_2)}} \otimes \cdots \otimes \alpha_n^{N^{(\alpha_n)}}. \tag{2}$$

A multiplicity $N^{(\alpha_j)}$ of the representation α_j in this product is equal to a number of particles which transform under this representation:

$$N^{(\alpha_j)} = \sum_j \left(\sum_{\zeta_j} N_{\nu_{\alpha_j};(\zeta_j)}^{(\alpha_j)} \right) = \sum_j N_{\nu_{\alpha_j}}^{(\alpha_j)}. \tag{3}$$

The representation given by Eq. (2) can be decomposed into direct sum of irreducible representations Λ_k. Corresponding states are denoted as $|\Lambda_k,\lambda_{\Lambda_k};\mathcal{N}\rangle$ where λ_{Λ_k} is an index numbering members of the representation Λ and $\mathcal{N}$ is a total number of particles

$$\mathcal{N} = \sum_k N_{\nu_{\alpha_k}}^{(\alpha_k)}. \tag{4}$$

Each physical state can be decomposed into irreducible representation base states $|\Lambda_k,\lambda_{\Lambda_k};\mathcal{N};\xi_{\Lambda_k}\rangle$. Variables ξ_Λ are degeneracy parameters required for the full description of a state in the "symmetry space". Let us consider a projection operator $\mathcal{P}^\Lambda$ on the subspace spanned by all states transforming under representation Λ.

$$\mathcal{P}^\Lambda \left| N_{\nu_{\alpha_1}}^{(\alpha_1)},\ldots,N_{\nu_{\alpha_n}}^{(\alpha_n)} \right\rangle = \sum_{\xi_\Lambda}^{\oplus} |\Lambda,\lambda_\Lambda;\xi_\Lambda\rangle\, \mathcal{C}^{\Lambda,\lambda_\Lambda}_{\{N_{\nu_{\alpha_1}}^{(\alpha_1)},\ldots,N_{\nu_{\alpha_n}}^{(\alpha_n)}\}}(\xi_\Lambda). \tag{5}$$

This operator has the generic form (see e.g. [2]):

$$\mathcal{P}^\Lambda = d(\Lambda) \int_G d\mu(g) \bar{\chi}^{(\Lambda)}(g) U(g). \tag{6}$$

Here $d(\Lambda)$ is the dimension of the representation Λ, $\chi^{(\Lambda)}$ is the character of the representation Λ, $d\mu(g)$ is the invariant Haar measure on the group, and $U(g)$ is an operator transforming a state under consideration. We will use the matrix representation:

$$
\begin{aligned}
& U(g)\left|N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right\rangle \\
= & \sum_{\nu_1^{(1)},\ldots,\nu_n^{(N_{\nu_n})}} D^{(\alpha_1)}_{\nu_1^{(1)}\nu_1}\cdots D^{(\alpha_1)}_{\nu_1^{(N_{\nu_1})}\nu_1}\cdots D^{(\alpha_n)}_{\nu_n^{(1)}\nu_n} \\
& \cdots D^{(\alpha_n)}_{\nu_n^{(N_{\nu_n})}\nu_n}\left|N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right\rangle. \qquad (7)
\end{aligned}
$$

$D^{(\alpha_n)}_{\nu,\nu'}$ is a matrix elements of the group element g corresponding to the representation α. Notation convention in Eq. (7) arises since there are $N^{(\alpha_j)}_{\nu_{\alpha_j}}$ states transforming under representation α_j and having quantum numbers of the ν_{α_j}-th member of a given multiplet.

The probability

$$
\overline{P^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}}
$$

that $N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}$ particles transforming under the symmetry group representations $\alpha_1,\ldots,\alpha_n$ combine into $\mathcal{N}$ particle state transforming under representation Λ of the symmetry group is given by

$$
\begin{aligned}
& \left\langle N^{(\alpha_1)}_{\nu_{\alpha_1}},\cdots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right|\mathcal{P}^{\Lambda}\left|N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right\rangle \\
& \qquad = \sum_{\xi_\Lambda}|\mathcal{C}^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}(\xi_\Lambda)|^2. \qquad (8)
\end{aligned}
$$

Left hand side of this equation can be calculated directly from Eqs.(6) and (7). One gets finally

$$
\begin{aligned}
& \overline{P^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}} \\
= \; & \mathcal{A}^{\{\mathcal{N}\}}d(\Lambda)\int_G d\mu(g)\bar{\chi}^{(\Lambda)}(g)[D^{(\alpha_1)}_{\nu_1\nu_1}]^{N^{(\alpha_1)}_{\nu_{\alpha_1}}} \\
& \cdots[D^{(\alpha_n)}_{\nu_n\nu_n}]^{N^{(\alpha_n)}_{\nu_{\alpha_n}}}. \qquad (9)
\end{aligned}
$$

where $\mathcal{A}^{\{\mathcal{N}\}}$ is a permutation normalization factor. For particles of the kind $\{\alpha,\zeta\}$ we included in Eq. (9) the permutation factor:

$$
\mathcal{A}^{\alpha}_{(\zeta)} = \frac{\mathcal{N}^{(\alpha)}_{(\zeta)}!}{\prod\limits_{\nu_\alpha}\mathcal{N}^{(\alpha)}_{\nu_\alpha;(\zeta)}!}. \qquad (10)
$$

The permutation factor $\mathcal{A}^{\{\mathcal{N}\}}$ is a product of all "partial" factors

$$
\mathcal{A}^{\{\mathcal{N}\}} = \prod_j\prod_{\zeta_j}\mathcal{A}^{\alpha_j}_{(\zeta_j)}. \qquad (11)
$$

Because of macrosymmetry all weights in Eq. (9) are constant in time. It provides subsidiary constraints on distribution functions $f^{(\alpha_i,\nu_i)}$. These conditions assure that in a dynamical evolution the symmetry of the system is preserved.

We now convert the global constraint into a time evolution condition and consider:

$$
\frac{d}{dt}\overline{P^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}} = 0. \qquad (12)
$$

Introducing here the result of Eq. (9) one obtains:

$$
\begin{aligned}
0 = & \frac{d\,\mathcal{A}^{\{\mathcal{N}\}}}{dt}d(\Lambda)\times \\
& \int_G d\mu(g)\bar{\chi}^{(\Lambda)}(g)[D^{(\alpha_1)}_{\nu_1\nu_1}]^{N^{(\alpha_1)}_{\nu_{\alpha_1}}}\cdots[D^{(\alpha_n)}_{\nu_n\nu_n}]^{N^{(\alpha_n)}_{\nu_{\alpha_n}}} \\
& +\sum_{j=1}^{n}\sum_{\nu_{\alpha_j}}\frac{d\,N^{(\alpha_j)}_{\nu_{\alpha_j}}}{dt}\mathcal{A}^{\{\mathcal{N}\}}d(\Lambda) \\
& \times\int_G d\mu(g)\bar{\chi}^{(\Lambda)}(g)[D^{(\alpha_1)}_{\nu_1\nu_1}]^{N^{(\alpha_1)}_{\nu_{\alpha_1}}} \\
& \cdots[D^{(\alpha_n)}_{\nu_n\nu_n}]^{N^{(\alpha_n)}_{\nu_{\alpha_n}}}\log[D^{(\alpha_j)}_{\nu_j\nu_j}]. \qquad (13)
\end{aligned}
$$

All integrals which appear in Eq. (9) and Eq. (13) can be expressed explicitly in an analytic form for any compact symmetry group.

To write an expression for the time derivative of the normalization factor $\mathcal{A}^{\{\mathcal{N}\}}$ we perform analytic continuation from integer to continuous values of variables $N^{(\alpha_n)}_{\nu_{\alpha_n}}$. Thus we replace all factorials by the Γ–function of corresponding arguments. We encounter here also the digamma function ψ [3]:

$$
\psi(x) = \frac{d\log\Gamma(x)}{d\,x}. \qquad (14)
$$

This allows to write:

$$
\begin{aligned}
& \frac{d\,\mathcal{A}^{\{\mathcal{N}\}}}{dt} \\
& = \mathcal{A}^{\{\mathcal{N}\}}\sum_j\sum_{\zeta_j}\left[\frac{d\,\mathcal{N}^{(\alpha_j)}_{(\zeta_j)}}{dt}\psi(\mathcal{N}^{(\alpha_j)}_{(\zeta_j)}+1)\right. \\
& \left. -\sum_{\nu_{\alpha_j}}\frac{d\mathcal{N}^{(\alpha_j)}_{\nu_{\alpha_j};(\zeta_j)}}{dt}\psi(\mathcal{N}^{(\alpha_j)}_{\nu_\alpha;(\zeta_j)}+1)\right]. \qquad (15)
\end{aligned}
$$

The time derivatives $d\mathcal{N}^{(\alpha)}_{\nu_\alpha;(\zeta)}/dt$ are obtained from the integrated kinetic equation fulfilled by a set of distribution functions $f^{(\alpha_i,\nu_i)}_{(\zeta)}(\Gamma,\vec{r},t)$. The case of the generalized Vlasov - Boltzmann kinetic equations was considered in [1].

These subsidiary conditions fulfilled by the microscopic kinetic equations are the necessary conditions for preserving the internal symmetry on the macroscopic level. Rates of change $d\mathcal{N}^{(\alpha)}_{\nu_\alpha;(\zeta)}/dt$ are related to "macrocurrents", which are counterparts of "microcurrents" related directly to a symmetry on a microscopic level via the Noether theorem. This can be considered as a set of conditions on macrocurrents to provide consistency with the overall symmetry of the system. These conditions leads to nontrivial results only for nonabelian symmetry groups. In the abelian case all charges are additive ones. The charge conservation on the microscopic level is equivalent to the global charge conservation of the multiparticle system. This is not the case for nonabelian symmetries, where nonabelian charges can combine to different representations of the symmetry group.

New constraints on kinetic equations lead to decreasing number of available states for the system during its time evolution. One can expect that such a system when approaching its equilibrium would produce less entropy. This should give some observable effects, e.g. for particle production processes in heavy-ion collision.

It should be noted that these new constraints on evolution equations are purely quantum effect. In the case of classical systems a concept of representations of the symmetry group is not applicable.

Acknowledgments

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