A gauge invariant formulation for the $SU(N)$ non-linear σ-model in $2+1$ dimensions

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Abstract

We derive a local, gauge invariant action for the $SU(N)$ non-linear σ-model in $2+1$ dimensions. In this setting, the model is defined in terms of a self-interacting pseudo-vector field θ_μ, with values in the Lie algebra of the group $SU(N)$. Thanks to a non-trivially realized gauge invariance, the model has the correct number of physical degrees of freedom: only one polarization of θ_μ, like in the case of the familiar Yang-Mills theory in $2+1$ dimensions. Moreover, since θ_μ is a pseudo-vector, the physical content corresponds to one massless *pseudo-scalar* field in the Lie algebra of $SU(N)$, as in the standard representation of the model. We show that the dynamics of the physical polarization corresponds to that of the $SU(N)$ non-linear σ-model in the standard representation, and also construct the corresponding BRST invariant gauge-fixed action.

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1 Introduction

The non-linear σ-model [1] is a very important tool for the description of the effective, low-energy dynamics of systems with a broken continuous (global) symmetry [2]. Many of its interesting and distinctive features stem from the fact that the symmetry group is realized in a non-linear way, as this endows the theory with a rich structure of interactions. Indeed, it has an infinite number of interaction vertices, when defined in terms of field variables which are themselves group coordinates. Nonetheless, this holds true in spite of the model having a 'universality': its properties are completely determined when the symmetry group and the spacetime dimension are known.

Of course, the same non-linearity is also responsible for the fact that, except for the $1+1$ dimensional case, the theory becomes non-renormalizable from the point of view of the usual loop expansion [2]. However, even in more than two spacetime dimensions, the model still has a reasonable predictive power, if properly understood as an effective theory [3]. This approach has been successfully applied to chiral perturbation theory [4], as a convenient effective model for QCD. Note, however, that in $2+1$ dimensions, the non-linear σ-model is renormalizable if a large-N expansion is used [5], instead of the standard loopwise perturbation theory.

The non-linearity may usually be tackled by resorting to an auxiliary, 'Lagrange multiplier' field, which enforces a constraint on the (otherwise free) field variables. The typical example of this is, perhaps, the $O(N)$ non-linear σ-model, where an auxiliary field imposes a constant-modulus constraint on an N-component scalar field $\vec{\phi} = (\phi_1, \ldots, \phi_N)$, which is a vector field in internal space. An important by-product of this construction is that the auxiliary field is a $O(N)$ singlet, hence, the large-N expansion is easier to formulate after one 'integrates out' the ϕ field, leaving an action for the Lagrange multiplier.

Indeed, the procedure of 'linearizing' an action, by the introduction of auxiliary fields, and afterwards integrating the original fields out to obtain an effective theory for the auxiliary fields, has frequently proven to be very useful. This is particularly true when the auxiliary field has some convenient symmetry or transformation properties [6]. In particular, it allows one to obtain an effective theory where the symmetry properties are inherited from the ones of the Lagrange multiplier in the linearized theory.

In this paper, we introduce a gauge invariant, non-trivially realized Abelian quantum field theory model in $2+1$ dimensions, which is derived by the procedure of integrating out the original variables, in order to obtain an effective theory for the auxiliary field. Since our starting point shall be a representation of the non linear σ-model where the Lagrange multiplier has a local gauge symmetry, that feature will be preserved in the resulting action. The realization of the Abelian gauge symmetry is non trivial, because the commutator of two gauge transformations is zero only on-shell, i.e., on the configurations that satisfy the equations of motion. Equivalently, the commutator between two 'true' gauge transformations yields a trivial, 'equation of motion' gauge transformation [7, 8].

The structure of this paper is as follows: in section 2 we derive the action for model, showing that it is indeed defined by a gauge invariant action. Then we consider the realization and structure of the gauge and global symmetries in section 3, leaving for section 4 the quantum treatment of the model. Section 5 contains our conclusions.

2 The model

We shall begin by reviewing the main features of the polynomial representation for the $SU(N)$ non-linear σ-model in $2+1$ dimensions, as presented in [9, 10]. This formulation may be defined in terms of a gauge invariant Euclidean action S_{inv}, which determines the dynamics of two fields L_μ (vector) and θ_μ (pseudo-vector) in the Lie algebra of $SU(N)$:

$$S_{inv}[L,\theta] \;=\; \int d^3x \, \mathcal{L}_{inv}(L,\theta) \tag{1}$$

with

$$\mathcal{L}_{inv}(L,\theta) \;=\; \frac{1}{2} g^2 L_\mu \cdot L_\mu + ig\,\theta_\mu \cdot \tilde{F}_\mu(L) \tag{2}$$

where g is a constant with the dimensions of a mass (it is in fact the exact analog of f_π in the $3+1$ dimensional case), and $\tilde{F}_\mu(L)$ denotes the dual of the non Abelian field strength tensor for the vector field L_μ, namely,

$$\tilde{F}_\mu(L) \;=\; \frac{1}{2}\epsilon_{\mu\nu\lambda} F_{\nu\lambda}(L) \quad , \quad F_{\mu\nu}(L) \;=\; \partial_\mu L_\nu - \partial_\nu L_\mu \,+\, g^{\frac{1}{2}}[L_\mu, L_\nu]\;. \tag{3}$$

Being L_μ an element in the Lie algebra, with the convention that $L_\mu = -L_\mu^\dagger$, it can be written as

$$\begin{aligned} L_\mu(x) \;&=\; L_\mu^a(x)\lambda_a \;\;,\;\; \lambda_a^\dagger \;=\; -\lambda_a \;, \\ \mathrm{tr}(\lambda_a\lambda_b) \;&=\; -\delta_{ab} \;\;,\;\; [\lambda_a,\lambda_b] \;=\; f_{abc}\,\lambda_c \end{aligned} \tag{4}$$

where f_{abc} is real and completely antisymmetric. Group indices will be indistinctly written as subscripts or superscripts; no meaning should be assigned to the difference. In (2), we also used the notation: $U \cdot V \equiv U_a V_a$, and

$(U \times V)_a = f_{abc} U_b V_c$ for any two elements U, V in the algebra. Also, both L and θ have the mass dimensions of $g^{1/2}$.

The 'inv' subscript in the action has been introduced in order to emphasize the fact that it is, indeed, invariant under the (local) gauge transformations:

$$\delta_\omega L_\mu \;=\; 0 \qquad \delta_\omega \theta_\mu \;=\; D_\mu \omega \;, \tag{5}$$

where the covariant derivative is compatible with the parallel transport defined by L, namely,

$$D_\mu \omega = \partial_\mu \omega + g^{\frac{1}{2}} [L_\mu, \omega] \;, \tag{6}$$

or in components:

$$(D_\mu \omega)^a = \partial_\mu \omega^a + g^{\frac{1}{2}} \, f_{abc} \, L^b_\mu \, \omega_c \;. \tag{7}$$

It must be noted that this gauge symmetry is valid *of-shell*, namely, it holds true regardless of whether the fields verify the equations of motion or not. Besides, equation (5) tells us that L is a gauge-invariant object, and this implies that the commutator of two gauge transformations vanishes:

$$[\delta_\eta \, , \, \delta_\omega] \;=\; 0 \;. \tag{8}$$

Here δ_ω and δ_η denote the operators that perform a gauge transformation on a given functional (eventually a function) of the fields. Namely, if I is a functional of L and θ,

$$\delta_\omega I[L, \theta] \;=\; \int d^3x \, \delta_\omega \theta^a_\mu(x) \, \frac{\delta I[L, \theta]}{\delta \theta^a_\mu(x)} \;, \tag{9}$$

where $\delta_\omega \theta^a_\mu$ is defined as in (5). This of course means that the gauge group is *Abelian*, in spite of the non-Abelian looking transformation rule for θ.

Had we wanted to work with this representation, we should have considered fixing the gauge as the next step. Rather than doing that, we shall move on to derive an 'effective theory' for θ_μ, an auxiliary field which transforms as a vector field in the adjoint representation. To that end, we define the effective action $S_{inv}[\theta]$ by the following expression:

$$\int [\mathcal{D}\theta] \; e^{-S_{inv}[\theta]} \;=\; \int \mathcal{D}\theta \, \mathcal{D}L \; e^{-S_{inv}[L,\theta]} \tag{10}$$

where $[\mathcal{D}\theta]$ denotes the integration measure for θ in the effective theory (the brackets denote possible group factors). Of course, the integration over θ_μ is ill-defined, since the theory is gauge invariant. There is, however, no obstruction to the integration of the L-field, since θ_μ is, in that case, regarded as a background field. We shall, of course, have to deal with the gauge-fixing for $S_{inv}[\theta]$ afterwards.

The integral over L_μ in (10) is a Gaussian, and its evaluation yields the result:

$$S_{inv}[\theta] \;=\; \int d^3x\, \mathcal{L}_{inv}(\theta) \;\;,\;\; \mathcal{L}_{inv}(\theta) \;=\; \frac{1}{2}\tilde{f}^a_\mu G^{ab}_{\mu\nu}(\theta)\tilde{f}^b_\nu \tag{11}$$

where $\tilde{f}$ is the dual of the *Abelian* field [1] strength: $\tilde{f}^a_\mu \equiv \epsilon_{\mu\nu\lambda}\partial_\nu\theta^a_\lambda$, and

$$G^{ab}_{\mu\nu} \;=\; [M^{-1}]^{ab}_{\mu\nu} \;\;,\;\; M^{ab}_{\mu\nu} \;=\; \delta_{\mu\nu}\delta^{ab} + ig^{-\frac{1}{2}}\epsilon_{\mu\lambda\nu}f^{acb}\theta^c_\lambda \;. \tag{12}$$

The fact that G is the inverse of M must be understood in the sense that the relations:

$$G^{ac}_{\mu\lambda}M^{cb}_{\lambda\nu} \;=\; \delta_{\mu\nu}\delta^{ab} \tag{13}$$

are valid. Fortunately, the explicit for of G is not required for most of our presentation. Note, however, that one may easily obtain an approximate expression for it by performing an expansion in powers of the (dimensionless) object $\theta g^{-\frac{1}{2}}$. There arises also from the Gaussian integral a factor which modifies the θ-field integration measure,

$$[\mathcal{D}\theta] \;=\; \mathcal{D}\theta\,[\det(M)]^{-\frac{1}{2}} \tag{14}$$

A question that immediately presents itself at this point is what has happened to the gauge invariance; indeed, the gauge invariance in the polynomial representation, equation (5), involves L_μ in its definition, and L_μ is precisely the field that has been eliminated from the action.

Of course, a standard Maxwell-like gauge transformation will not do, since, although $\tilde{f}_\mu$ is invariant under the Abelian gauge transformations of the Maxwell theory, G, that depends on θ_μ, is not. Indeed, looking for example at the explicit form of the action (11), with G expanded up to terms of order $\frac{\theta^2}{g}$, we see that:

$$S_{inv}[\theta] \;=\; \int d^3x\, \Big[\frac{1}{2}\tilde{f}_\mu(\theta)\cdot\tilde{f}_\mu(\theta) \;-\; \frac{i}{2}g^{-\frac{1}{2}}\epsilon_{\mu\nu\lambda}\,\theta_\mu\cdot\tilde{f}_\nu(\theta)\times\tilde{f}_\lambda(\theta)$$

$$-\frac{1}{2g}(\theta_\mu\cdot\tilde{f}_\mu\theta_\nu\cdot\tilde{f}_\nu \;-\; \theta_\mu\cdot\tilde{f}_\nu\,\theta_\mu\cdot\tilde{f}_\nu \;+\; \tilde{f}_\mu\cdot\tilde{f}_\mu\,\theta_\nu\cdot\theta_\nu - \;\tilde{f}_\mu\cdot\tilde{f}_\nu\,\theta_\mu\cdot\theta_\nu)\Big] \;, \tag{15}$$

where only the term in the first line is invariant under Abelian gauge transformations. In spite of this, we do expect a gauge invariance to exist for $S_{inv}[\theta]$, since we know there are two unphysical components (for each value

[1] We adopt the convention that a lowercase f_μ refers to the dual of the *Abelian*field strength, while the uppercase one is reserved for the dual non Abelian one.

of a) in θ_μ, which do appear in the free propagator. This propagator will of course be determined by the free action

$$S^{(0)}_{inv}[\theta] \;=\; \int d^3x \, \frac{1}{2} \tilde{f}^a_\mu(\theta) \tilde{f}^a_\mu(\theta) \;=\; \int d^3x \, \frac{1}{4} f^a_{\mu\nu}(\theta) f^a_{\mu\nu}(\theta) \tag{16}$$

after adding a gauge fixing term.

It is then reasonable to assume that the gauge transformations for θ should be of the form

$$\delta_\omega \theta_\mu \;=\; \partial_\mu \omega + g^{\frac{1}{2}} [L_\mu(\theta), \omega] \tag{17}$$

where $L_\mu(\theta)$ is a *dependent* field which plays the role of a connection, and should of course be defined in terms of θ.

A possible hint to find the explicit form of $L_\mu(\theta)$ comes from the fact that performing the Gaussian integration is tantamount to 'replacing the integrated field by their values at the extreme of the exponent'. Denoting by $\hat{L}_\mu(\theta)$ the expression that maximizes the exponent, we see that it is given by:

$$\hat{L}^a_\mu \;=\; -ig^{-1} \, G^{ab}_{\mu\nu}(\theta) \tilde{f}^b_\nu \; . \tag{18}$$

Thus we shall adopt the ansatz $L_\mu(\theta) \equiv \hat{L}_\mu(\theta)$, the consistency of which we will verify now: to see whether the transformation (17) is a (gauge) symmetry of the action (11) or not, we first evaluate the first variation of $S_{inv}[\theta]$ under a general, not necessarily gauge, infinitesimal variation of θ.

After some elementary algebra, we obtain:

$$\delta S_{inv}[\theta] \;=\; \int d^3x \, \delta\theta^a_\mu \, \Big\{ \epsilon_{\mu\nu\lambda} \, \partial_\nu [G^{ab}_{\lambda\rho}(\theta) \tilde{f}^b_\rho(\theta)]$$

$$- \frac{i}{2} g^{-\frac{1}{2}} \, \epsilon_{\mu\nu\lambda} \, f_{abc} \, G^{bd}_{\nu\alpha}(\theta) \, \tilde{f}_\alpha]^d \, G^{ce}_{\lambda\beta}(\theta) \, \tilde{f}^e_\beta \Big\} \tag{19}$$

where we used the symmetry property $G^{ab}_{\mu\nu} = G^{ba}_{\nu\mu}$, and the relation

$$\delta G^{ab}_{\mu\nu} \;=\; -ig^{-\frac{1}{2}} \, G^{ac}_{\mu\lambda}(\theta) \epsilon_{\lambda\rho\sigma} \, f^{cde} \, \delta\theta^d_\rho \, G^{eb}_{\sigma\nu}(\theta) \; , \tag{20}$$

both of them consequences of the fact that $G = M^{-1}$. Recalling the definition of $L_\mu(\theta)$, we may also write (19) as:

$$\delta S_{inv}[\theta] \;=\; ig \int d^3x \, \delta\theta^a_\mu \, \tilde{F}^a_\mu(L(\theta)) \tag{21}$$

where

$$\tilde{F}^a_\mu(L(\theta)) \;=\; \frac{1}{2} \epsilon_{\mu\nu\lambda} F^a_{\nu\lambda}(L(\theta)) \; ,$$

$$F^a_{\mu\nu}(L(\theta)) \;=\; \partial_\mu L^a_\nu(\theta) - \partial_\nu L^a_\mu(\theta) + g^{\frac{1}{2}} f^{abc} L^b_\mu(\theta) L^c_\nu(\theta) \;. \qquad (22)$$

Using now the explicit form for $\delta\theta_\mu$ that corresponds to a gauge variation, equation (17), we see that

$$\delta S_{inv}[\theta] \;=\; -ig \int d^3x \; \omega^a(x) [D_\mu \tilde{F}_\mu]^a(L) \;=\; 0 \;, \qquad (23)$$

as a consequence of the Bianchi identity, which is of course true regardless of L being an independent field or not. We shall henceforth omit writing the dependence of L on θ explicitly, since L shall always be assumed to be a dependent field. A small technical point (absent in the real time formulation) is that the relation (18) includes complex factors: an i multiplying G, but also G itself has both real an imaginary parts. That should be hardly surprising, since the action itself is not purely real, as it happens with Euclidean actions including Chern-Simons terms (and with other topological objects in different numbers of dimensions). Thus the relation (18), to have non-trivial solutions, require the continuation of the fields to complex values. Of course, the gauge invariant action in Minkowski spacetime, S^M_{inv}, is real,

$$S^M_{inv} \;=\; \int d^3x \, \frac{1}{2} \tilde{f}^\mu_a G^{ab}_{\mu\nu}(\theta) \tilde{f}^\nu_b \qquad (24)$$

where $\tilde{f}^\mu_a = \epsilon^{\mu\nu\lambda}\partial_\nu \theta^a_\lambda$ and $G^{ab}_{\mu\nu}(\theta)$ is determined by the equations:

$$G^{ac}_{\mu\rho}(\theta)\, M^{\rho\nu}_{cb}(\theta) \;=\; \delta^\nu_\mu \delta^a_b \;\;,\;\; M^{\mu\nu}_{ab} \;=\; g^{\mu\nu}\delta_{ab} + g^{-\frac{1}{2}} \epsilon^{\mu\lambda\nu} f^{acb} \theta^c_\lambda \;. \qquad (25)$$

Thus we have verified the consistency of the definition of the covariant derivative with the gauge invariance of the action. Note, however, that there is an important difference with the polynomial formulation, in that the gauge transformations for θ involve L, which is itself a function on θ. Thus L will, in general, change under a gauge transformation in this formulation. In particular, this implies that finite gauge transformations will be different to infinitesimal ones. This is a consequence in fact of the algebra of gauge transformations being open, as it will be discussed in the next section.

Also, expression (21) tells us that the classical equations of motion deriving from $S_{inv}[\theta]$ are:

$$F_{\mu\nu}(L) \;=\; 0 \;. \qquad (26)$$

i.e., the Maurer-Cartan equations for L, which obviously have a gauge invariant set of solutions.

Regarding the integration measure $[\mathcal{D}\theta]$, it is straightforward to verify that the gauge variation of $[\mathcal{D}\theta]$ is zero. We conclude that the action (11) is indeed gauge invariant. The gauge invariance is not of the Yang-Mills type, but rather involves as a connection a vector field L_μ which is a composite field, defined in terms of θ_μ and its derivatives. As we shall see in the next section, the gauge group is indeed Abelian, but the albegra of gauge transformations is not closed off-shell.

It may seem surprising at first sight that the only 'content' of the classical equations of motion is that the Maurer-Cartan equations for a field are satisfied, since we still need the dynamics for the true degrees of freedom. Of course, such a dynamics is also present in this description: L is a pure gauge field, i.e., $L_\mu = U^\dagger \partial_\mu U$ with $U(x) \in SU(N)$, and besides (see (42) below) $\partial_\mu \cdot L_\mu = 0$. These two equations are the equivalent to the classical equations of motion for the non-linear σ-model.

3 Symmetries

The actual form of the gauge transformations, as acting on the field θ_μ, has been obtained by the procedure of borrowing the (known) form of the corresponding transformations from the polynomial version, and afterwards replacing the field L_μ by its value at the extreme (a function of θ). This yields, for a transformation parametrized by the function $\omega(x)$, the variation:

$$\delta_\omega \theta_\mu(x) \; = \; D_\mu^L \omega(x) \tag{27}$$

where

$$D_\mu^L \omega \; = \; \partial_\mu \omega + g^{\frac{1}{2}} [L_\mu, \omega] \; , \tag{28}$$

with:

$$L_\mu^a \; = \; -i g^{-1} \, G_{\mu\nu}^{ab}(\theta) \, \tilde{f}_\nu^b(\theta) \; . \tag{29}$$

In spite of the presence of a covariant derivative, the transformations do not correspond to a non-Abelian Yang-Mills theory. Indeed, it should be noted that the transformations (27) involve the covariant derivative, defined in terms of a composite field which plays the role of a connection. However, they are not of the strictly Abelian type either, since the transformation law for θ does not correspond to that case.

We shall now see that what happens is that the transformations are, indeed, Abelian, but only *on-shell*, i.e., on the equations of motion. To be specific, consider the commutator of two gauge transformations, corresponding to the gauge functions ω and η. We find that the result may be written, after some algebraic manipulations, as follows:

$$[\delta_\eta, \delta_\omega] \theta_\mu^a \; = \; \Sigma_{\mu\nu}^{ab}(\theta) \, \frac{\delta S_{inv}[\theta]}{\delta \theta_\nu^b} \tag{30}$$

where we introduced the object:

$$\Sigma^{ab}_{\mu\nu}(\theta) \;=\; -\frac{1}{g}\,\eta^h\,\omega^c\,(f^{aec}f^{dbh} - f^{aeh}f^{dbc})G^{ed}_{\mu\nu}(\theta)\;. \tag{31}$$

It is important to realize that $\Sigma^{ab}_{\mu\nu}$ is antisymmetric, namely,

$$\Sigma^{ab}_{\mu\nu} \;=\; -\Sigma^{ba}_{\nu\mu}\;, \tag{32}$$

since this means that the right hand side of (30) is a trivial gauge transformation [8]. Indeed, for a given action $S[\theta]$, a transformation of the kind

$$\delta\theta^a_\mu \;=\; \Lambda^{ab}_{\mu\nu}(\theta)\frac{\delta S[\theta]}{\delta\theta^b_\nu} \tag{33}$$

with an arbitrary antisymmetric function $\Lambda^{ab}_{\mu\nu} = -\Lambda^{ba}_{\nu\mu}$, is a symmetry of $S[\theta]$, regardless of the form of $S[\theta]$. It can also be shown [8], that the commutator between a non-trivial gauge transformation and a trivial one yields a trivial gauge transformation. Thus, we see that the physically relevant gauge group is Abelian, and isomorphic to $U(1)^{(N^2-1)}$ (for $SU(N)$), although realized in a non-trivial way, since the 'trivial' part of the gauge transformations cannot be easily eliminated within the present formulation of the model.

A related property is that the composite field L_μ, which is gauge invariant in the polynomial transformation, is now also gauge-invariant but only on-shell:

$$\delta_\omega L^a_\mu \;=\; -ig^{-\frac{1}{2}}\,G^{ab}_{\mu\nu}(\theta)f^{bcd}\tilde{F}^c_\nu(L)\omega_d\;, \tag{34}$$

i.e., it vanishes when $\tilde{F}_\mu(L) = 0$.

The question that immediately presents itself is what are the conditions a gauge invariant functional must verify. This is of course important, since gauge invariant functionals are naturally associated to physical observables. Besides, in the functional integral approach to a quantum gauge field theory, the condition a gauge invariant functional must satisfy is an important part of the formulation.

So, assuming $I[\theta]$ to be a gauge invariant functional of θ, it must verify the condition:

$$\delta_\omega\, I[\theta] \;=\; 0\;, \tag{35}$$

where

$$\delta_\omega \;=\; \int d^3x\,\delta_\omega\theta^a_\mu(x)\,\frac{\delta}{\delta\theta^a_\mu(x)}\;. \tag{36}$$

However, if such a gauge invariant functional exists, one immediately gets a consistency condition by applying two successive gauge transformations on I and subtracting them, namely:

$$\delta_\omega I[\theta] = 0 \;\;\Rightarrow\;\; [\delta_\eta\,,\,\delta_\omega]\,I[\theta] \;=\; 0\;. \tag{37}$$

On the other hand, we may of course evaluate the commutator of two gauge transformations; after some algebra, we find:

$$[\delta_\eta \,,\, \delta_\omega] \;=\; \int d^3x \, \Sigma^{ab}_{\mu\nu}(\theta) \, \frac{\delta S_{inv}}{\delta\theta^a_\mu(x)} \frac{\delta}{\delta\theta^b_\nu(x)} \;. \tag{38}$$

Thus, for non-trivial gauge invariant functional I to exist, since Σ depends on the arbitrary functions η and ω, we have to impose the additional condition:

$$F_{\mu\nu}(L) \;=\; 0 \;. \tag{39}$$

This is nothing new from the classical point of view, but it makes a difference for the quantum theory, where all the configurations matter, and not just the extrema of the action. This seems to lead us to the inclusion of (39) as a constraint, what is not what we want. Fortunately, there are ways out of this [8], that does not require the introduction of extra constraints (wich might even reduce the number of degrees of freedom.

Regarding the global symmetries, we know that L_μ is a conserved current, associated to a global symmetry of the non-linear σ-model. To see that L_μ is conserved in this formalism is a bit tricky. One possible way to prove that is to use the property that the composite field L_μ as given by (29) may also be written, after some algebra, as:

$$L_\mu \;=\; -ig^{-1} \, \epsilon_{\mu\nu\lambda} D_\nu \theta_\lambda \;, \tag{40}$$

where we used the property:

$$G^{ab}_{\mu\nu}(\theta) \;=\; \delta^{ab}_{\mu\nu} \;-\; ig^{-\frac{1}{2}} \epsilon_{\mu\lambda\sigma} f^{acd} \theta^c_\lambda G^{db}_{\sigma\nu}(\theta) \;. \tag{41}$$

Then it follows that

$$\partial_\mu L_\mu \;=\; D_\mu L_\mu \;=\; -ig^{-1} \, \epsilon_{\mu\nu\lambda} D_\mu D_\nu \theta_\lambda \;=\; -ig^{-\frac{1}{2}} [\tilde{F}_\mu(L), \theta_\mu] \tag{42}$$

which vanishes on shell, and implies the conservation of L_μ. The conserved charge is of course given by the space integral of L_0. It is instructive to consider the particular case of a static point-like static charge of color a and strength q located at $\mathbf{x} = \mathbf{x}_0$. This corresponds to a charge density

$$L^a_0(x) \;=\; -iq\delta(\mathbf{x} - \mathbf{x}_0), \quad L_j(x) \;=\; 0 \;. \tag{43}$$

Inserting this into the relation (29) yields

$$\tilde{f}^a_\mu \;=\; q \, \delta_{\mu 0} \, \delta(\mathbf{x} - \mathbf{x}_0) \tag{44}$$

i.e., it corresponds to a point like magnetic flux sitting on the same point. The conserved charge is then equal to the total magnetic flux (for that color).

4 Quantum theory

We shall consider here the quantum theory corresponding to this gauge invariant model, from the path integral approach. The natural object to consider is then of course the generating functional for θ-field correlation functions. The ill-defined (gauge invariant) partition function shall be given by the expression:

$$\mathcal{Z}_{inv}[J] \;=\; \int [\mathcal{D}\theta] \;\exp\left\{-S_{inv}[\theta] + \int d^3x J_\mu \cdot \theta_\mu\right\} \;. \tag{45}$$

The generating functional (45), being gauge invariant, requires the introduction of a gauge-fixing term and its companion ghost action to be well-defined. However, a standard Faddeev-Popov approach to the definition of the gauge-fixed action will not do, since the resulting action is neither BRST invariant, nor the transformation becomes nilpotent. The difficulty lies, of course, in the fact that the algebra of the gauge transformations is 'open', namely, it closes only when the equations of motion are satisfied. However, a modified action, which generally involves quartic ghost terms may be constructed, such that the action is invariant under an extended BRST transformation [7, 8]. By an application of such method to this case, we obtain the gauge-fixed action S:

$$S[\theta_\mu; b, \bar{c}, c] \;=\; S_{inv}[\theta] \,+\, S_{gf}[b,\theta] \,+\, S_{gh}[\bar{c}, c; \theta] \tag{46}$$

where We shall adopt the covariant gauge-fixing term:

$$S_{gf}[\theta] \;=\; \int d^3x \,(-\frac{1}{2\lambda} b^2 \,+\, b \cdot \partial_\mu \theta_\mu) \tag{47}$$

and the corresponding ghost action becomes

$$S_{gh}[\bar{c}, c; \theta] \;=\; \int d^3x \left[\partial_\mu \bar{c} \cdot D^L_\mu c \,+\, \frac{1}{2g} (\partial_\mu \bar{c} \times c)^a G^{ab}_{\mu\nu}(\theta) (\partial_\nu \bar{c} \times c)^b\right] \;. \tag{48}$$

The existence of a quartic term in the ghosts makes it evident that the BRST transformations are not of the standard form. Indeed, we find that the precise form for the transformations is:

$$\delta\theta^a_\mu \;=\; \xi\,(D_\mu c)^a \,+\, \xi \frac{i}{g} f^{abe} G^{bd}_{\mu\nu} (\partial_\nu \bar{c} \times c)^d c^e$$

$$\delta c \,=\, 0 \;, \quad \delta\bar{c} \,=\, i\xi b \;, \quad \delta b \,=\, 0 \;. \tag{49}$$

They leave the action S invariant, and the transformation is besides nilpotent.

The generating functional for the gauge-fixed action is then defined as follows:

$$\mathcal{Z}[J; j, \bar{\eta}, \eta] \;=\; \int [\mathcal{D}\theta]\, \mathcal{D}b\, \mathcal{D}\bar{c}\, \mathcal{D}c$$
$$\times \, \exp\left\{ -S[\theta; b, \bar{c}, c] + \int d^3x (J_\mu \cdot \theta_\mu + j \cdot b \,+\, \bar{\eta} \cdot c + \bar{c} \cdot \eta) \right\} \,. \tag{50}$$

It should be noted that, in all of the above equations, the covariant derivative is defined in terms of the dependent field L, which is a function of θ.

This may be thought of as the main result of this letter, namely, there exists a gauge invariant description for the non-linear σ-model in $2+1$ dimensions; that description is built in terms of θ, a pseudo-vector field in the algebra of the group. The gauge algebra is however open, what makes the BRST quantization less immediate than for the Yang-Mills case (although the algebra is Abelian on-shell). The resulting gauge fixed action contains terms quartic in the ghosts, and is invariant under a global BRST symmetry. This BRST symmetry may be applied to, for example, the derivation of Ward identities that will restrict the form of the counterterms.

Regarding the quantum corrections, it should be noted that there is another (equivalent) possibility to tackle the problem of open gauge algebras, through the introduction of auxiliary field. Their function is to render the on-shell symmetry into an off-shell one, where the Faddeev-Popov trick may be applied. The upshot of this procedure here, leads one to the 'polynomial formulation' Lagrangian of (2), whose renormalization properties have been considered in [9].

5 Conclusions

We have shown that the $SU(N)$ non-linear σ-model in $2+1$ dimensions may indeed be described by a gauge invariant action $S_{inv}[\theta]$, for a single pseudo-vector field θ. That action has a gauge invariance which involves a composite field L (a function of θ) that plays a role similar to a connection. This, however, is so only when one considers infinitesimal gauge transformations. Finite gauge transformations, and the composition of two gauge transformations show that the gauge algebra is open. The resulting classical theory shows no difference with the standard formulation of the non-linear σ-model, since the classical trajectories are the only important part of the action, and there the algebra closes.

For the quantum theory, however, the situation is more complicated, as the BRST quantization requires the introduction of a term which is quartic in the ghosts. However, the corresponding global BRST symmetry exists, and may indeed be used as a starting point in the construction of the quantum effective action. We also note that this open algebra formulation is also equivalent to the polynomial formulation, where the algebra is closed and Abelian.

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References

[1] S. Weinberg, Phys. Rev. **166**, 1568 (1968).

[2] See, for example:
section **14** of: J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, Clarendon Press, Oxford (2002).

[3] S. Weinberg, Phys. Lett. B**91**, 51 (1980).

[4] See for example:
A. Pich, Rept. Prog. Phys. **58**, 563 (1995), for a modern review.

[5] I. Y. Arefeva, Annals Phys. **117**, 393 (1979).

[6] See, for example, Chapter **17** of:
R. J. Rivers, *Path Integral Methods in Quantum Field Theory*, Cambridge University Press, Cambridge (1988).

[7] For an early reference, see for example:
B. de Wit and J. W. van Holten, Phys. Lett. B **79**, 389 (1978).

[8] For a review, see for example:
M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press, Princeton, NJ (1992).

[9] C. D. Fosco and T. Matsuyama, Int. J. Mod. Phys. A **10** (1995) 1655.

[10] C. D. Fosco and T. Matsuyama, Prog. Theor. Phys. **93** (1995) 441.