More about generalized maximally superintegrable systems of Winternitz type

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Abstract

Recently proposed procedure of constructing maximally superintegrable systems of Winternitz type is further developed and illustrated by an example of a system admitting an explicit construction of angle variables and additional integrals of motion.

A possible application of the method to Liouville system is briefly presented.

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Recently we have outlined [1] general procedure leading to superintegrable models generalizing Winternitz system [1]–[6]. We have shown that for completely separated hamiltonians the potentials corresponding to superintegrable models have the form $u(x) = \beta^2(\phi(x) - x)^2$ with $\phi(x)$ having property $\phi(\phi(x)) = x$.

The system consisting of the set of the independent degrees of freedom, each governed by the natural hamiltonian, with the potential energy of the above form and the mass μ_k is maximally superintegrable iff all ratios $\frac{\beta_k}{\sqrt{\mu_k}} / \frac{\beta_l}{\sqrt{\mu_l}}$ are rational.

In the present letter we give further examples of our method. In particular, we show how it can be applied to more general separable theories like Liouville systems.

The key role in our approach is played by the function $\phi(x)$ obeying $\phi \circ \phi = \mathrm{id}$. It is not difficult to classify such ϕ's.

Assume that ϕ has the following properties:

i) $\phi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and onto

ii) $\phi \circ \phi = \mathrm{id}$

Then ϕ is one-to-one, $\phi^{-1} = \phi$, and ϕ is either strictly increasing or strictly decreasing. In the former case $\phi(x) = x$ while in the latter

$$\phi(x) = F(c - F^{-1}(x)) \tag{1}$$

where $F:\mathbb{R}\to\mathbb{R}$ is continuously differentiable one-to-one mapping of $\mathbb{R}$ onto $\mathbb{R}$.

To show this let us first assume that $\phi(x)$ is strictly increasing. Then $\phi(x)\geq x$ implies $x=\phi(\phi(x))\geq\phi(x)$ so that $\phi(x)=x$ and the same applies if $\phi(x)\leq x$.

Assume now that ϕ is strictly decreasing. Then $\phi(x)=x$ has exactly one solution, $x=a$. Put $c=2a$ and define

$$F(x)=\begin{cases} x, & x\in(-\infty,a\rangle \\ \phi(c-x), & x\in(a,\infty) \end{cases}$$

It is then easy to check that $F:\mathbb{R}\to\mathbb{R}$ is one-to-one continuously differentiable and onto and eq.(1) holds. Note that the representation (1) is by far nonunique.

Having a general solution (1) at our disposal we can construct a wide class of superintegrable systems following prescription given in [1]. The relevant completely separated hamiltonians read

$$\begin{gathered} H=\sum_{k=1}^{N}\left(\frac{p_k^2}{2\mu_k}+U_k(x_k)\right)\equiv\sum_{k=1}^{N}H_k(x_k,p_k) \\ \text{where} \\ U_k(x)\equiv\beta_k^2(\phi_k(x)-x)^2,\quad \beta_k^2=\frac{\alpha^2m_k^2\mu_k^2}{8} \end{gathered} \tag{2}$$

and m_k are arbitrary integers.

To see how it works let us take simple F giving rise to a nontrivial potential:

$$F^{-1}(x)=\varrho x^3 \tag{3}$$

where ϱ is a constant of dimension of inverse length squared. Then with c being another constant of dimension of length we get:

$$\phi(x)=\sqrt[3]{\frac{c-\varrho x^3}{\varrho}} \tag{4}$$

and ($\sigma_k\equiv c_k/\varrho_k$)

$$H=\sum_{k=1}^{N}\left(\frac{p_k^2}{2\mu_k}+\frac{\alpha^2\mu_k^2m_k^2}{8}\left(\sqrt[3]{\sigma_k-x_k^3}-x_k\right)^2\right) \tag{5}$$

The general considerations of [1] give at once the action variables while for obtaining the angle variables an explicit integration is necessary which, in our example, is not easy. Once the angle variables are known it is straightforward (in principle, at least) to find the additional integrals of motion (see, for example [7, 8]). However, in general it appears that the relevant integrations cannot be performed explicitly. Therefore, our construction gives a very wide class of superintegrable systems with rather complicated, in general, additional integrals of motion. It means in particular that the freedom in choice of the variables which separate the Hamilton-Jacobi equation appears only on the level of general canonical transformations (the additional integrals are not quadratic in momenta).

To show more explicitly how that works consider another simple choice:

$$F(x) = a \sinh\left(\frac{x}{a}\right) \tag{6}$$

Then $\phi(x)$ reads

$$\phi(x) = a(\sinh\left(\frac{c}{a}\right)\sqrt{1+\frac{x^2}{a^2}} - \cosh\left(\frac{c}{a}\right)\frac{x}{a}) \tag{7}$$

and the relevant model is ($\sigma_k \equiv \frac{c_k}{a_k}$)

$$\begin{aligned} H \quad = \quad & \sum_{k=1}^{N}\left(\frac{p_k^2}{2\mu_k} + \beta_k^2 a_k^2 (2(1+\cosh\sigma_k)\cosh\sigma_k \frac{x_k^2}{a_k^2} - \right. \\ & \left. - 2\sinh\sigma_k(1+\cosh\sigma_k)\frac{x_k}{a_k}\sqrt{1+\frac{x_k^2}{a_k^2}} + \sinh^2\sigma_k)\right) \end{aligned} \tag{8}$$

The advantage of this model is that one can find the angle variables by elementary integration.

In terms of action variables our hamiltonian reads [1]

$$H = \sum_{k=1}^{N} E_k = \alpha \sum_{k=1}^{N} m_k J_k \tag{9}$$

therefore,

$$\frac{\partial E_k}{\partial J_k} = \alpha m_k. \tag{10}$$

The angle variables are given by

$$\phi_k = \alpha\mu_k m_k \int^{x_k} \frac{dx_k}{\sqrt{2\mu_k(E_k - U_k)}} \tag{11}$$

and the lower integration limit may be chosen arbitrarily. In our case the integration can be performed explicitly. The result reads:

$$\phi_k \quad = \quad \arcsin\left(\frac{2\beta_k a_k \cosh\left(\frac{\sigma_k}{2}\right)}{\sqrt{E_k}}\left(\frac{x_k}{a_k}\cosh\left(\frac{\sigma_k}{2}\right) + \right.\right.$$

$$
\begin{aligned}
&-\sqrt{1+\frac{x_k^2}{a_k^2}}\sinh\left(\frac{\sigma_k}{2}\right)\Bigg)\Bigg) + \\
&+\tanh\left(\frac{\sigma_k}{2}\right)\arcsin\left(\frac{2\beta_k a_k\cosh\left(\frac{\sigma_k}{2}\right)}{\sqrt{E_k+4\beta_k^2 a_k^2\cosh^2\left(\frac{\sigma_k}{2}\right)}}\cdot\right. \\
&\left.\cdot\left(\sqrt{1+\frac{x_k^2}{a_k^2}}\cosh\left(\frac{\sigma_k}{2}\right)-\frac{x_k}{a_k}\sinh\left(\frac{\sigma_k}{2}\right)\right)\right)
\end{aligned}
\tag{12}
$$

The angle ϕ_k is well-defined. Indeed, the first term on the right hand side changes by 2π while the second comes back to the initial value when x_k attains again initial value (for that reason $\tanh\left(\frac{\sigma_k}{2}\right)$ can be also arbitrary).

In order to construct the additional integral of motion let us consider the simplest case $N = 2$, $m_1 = m_2 = 1$, $\mu_1 = \mu_2 = \mu$, ,$a_1 = a_2 = a$, $\sigma_1 = \sigma_2 = \sigma$ and write (12) as

$$
\phi_k = \psi_k + \tanh\left(\frac{\sigma}{2}\right)\chi_k, \quad k = 1,2 \tag{13}
$$

Under the above assumptions $\dot{\phi}_1 = \dot{\phi}_2$, so $\phi_1 - \phi_2$ is an integral of motion. Morover, any periodic function of $\phi_1 - \phi_2$ is well defined on phase space so that, for example $\sigma(I_1, I_2)\sin(\phi_1 - \phi_2)$ can be taken as an additional integral of motion (for arbitrary $\sigma(I_1, I_2)$).

However, there is still some technical obstacle. Due to the properties of ϕ_k mentioned above any periodic function of ϕ_k's can be expressed in terms x_k's and p_k's; in general the resulting expression will be very complicated and not expressible in terms of elementary functions. The exceptional case would be if the periodic functions of ϕ_k are expressible in terms of trigonometric functions of both ψ_k and χ_k. This is possible if $\tanh\left(\frac{\sigma}{2}\right)$ is rational. In particular we assume that $\tanh\left(\frac{\sigma}{2}\right) = \frac{1}{2}$, i.e. $\sigma = \ln 3$. **Let us stress again that this condition is of technical nature and has nothing to do with the superintegrability condition.**

Now, under our assumptions

$$
\sin 2(\phi_1 - \phi_2) = \sin(2(\psi_1 - \psi_2) + (\chi_1 - \chi_2)) \tag{14}
$$

is an integral of motion. It is polynomially expressible in terms of

$\sin\psi_k$, $\cos\psi_k$, $\sin\chi_k$, $\cos\chi_k$. In turn,

$$\begin{aligned}
\sin\psi_k &= \frac{4\beta a}{\sqrt{3H_k}}\left(\frac{2x_k}{\sqrt{3}a} - \frac{1}{\sqrt{3}}\sqrt{1+\frac{x_k^2}{a^2}}\right) \\
\sin\chi_k &= \frac{4\beta a}{\sqrt{3H_k+16\beta^2a^2}}\left(\frac{2}{\sqrt{3}}\sqrt{1+\frac{x_k^2}{a^2}} - \frac{1}{\sqrt{3}}\frac{x_k}{a}\right) \\
\cos\psi_k &= \frac{p_k}{\sqrt{2\mu H_k}} \\
\cos\chi_k &= \sqrt{\frac{p_k^2+\frac{32}{3}\beta^2a^2\mu}{2\mu H_k}}
\end{aligned} \tag{15}$$

The sign ambiguity for cosines of ψ_k's is properly taken into account by replacing $\sqrt{E_k - U_k(x)}$ by $p_k/\sqrt{2\mu}$. Also the form of $\cos\chi_k$ shows that χ_k is well defined over Liouville torus.

Let us note that even in this simplest case the additional integral cannot be chosen as a polynomial function of momenta.

The original Winternitz model was defined on semiaxis. It is not difficult to generalize our discussion to this case. The counterpart of eq.(1) reads now

$$\phi(x) = F\left(\frac{c}{F^{-1}(x)}\right) \tag{16}$$

where $F:\mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuously differentiable isomorphism. Indeed, $l = \ln x$ is a smooth inverible mapping from $\mathbb{R}_+$ to $\mathbb{R}$; therefore, if $\phi:\mathbb{R}\rightarrow\mathbb{R}$ and $\phi\circ\phi = \mathrm{id}$ then $\tilde{\phi} = l^{-1}\circ\phi\circ l : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ obeys also $\tilde{\phi}\circ\tilde{\phi} = \mathrm{id}$. Taking $F(x) \sim x^\lambda$ one obtains Winternitz model. Again a variety of models can be constructed by selecting various F. For example,

$$F(x) = x\left(1+\frac{x}{a}\right) \tag{17}$$

gives

$$\phi(x) = \frac{c\left(\sqrt{1+\frac{4x}{a}}+1\right)}{2x}\left(1+\frac{c\left(\sqrt{1+\frac{4x}{a}}+1\right)}{2ax}\right) \tag{18}$$

We obtain a deformation of Winternitz model which is recovered in $a\rightarrow\infty$ limit.

Our method is by far not restricted to the hamiltonians of the form given by eq.(2). Consider the Liouville system (see [9] for example)

$$H = \frac{\sum_{k=1}^{N} \left(\frac{p_k^2}{2\mu_k} + V_k(x_k) \right)}{\sum_{k=1}^{N} c_k(x_k)} \tag{19}$$

It is separable because the H-J equation $H(x,p) = E$ separates into the set of equations

$$\frac{p_k^2}{2\mu_k} + (V_k(x_k) - Ec_k(x_k)) = \epsilon_k, \quad \sum_{k=1}^{N} \epsilon_k = 0 \tag{20}$$

corresponding to totally separable hamiltonian

$$\tilde{H}(x,p,E) = \sum_{k=1}^{N} \left(\frac{p_k^2}{2\mu_k} + V_k(x_k) - Ec_k(x_k) \right) \tag{21}$$

The total energy E enters now as a parameter of effective potentials

$$U_k \equiv V_k(x) - Ec_k(x) \tag{22}$$

We see that our method can be applied if we can select potentials which depend linearly on some parameter E. One straghtforward choice is to take $V_k(x_k) = c_k(x_k) = \beta_k^2(x_k - \phi_k(x_k))^2$ as in eq.(2); also in the original Winternitz model one can separate the potential into two pieces in arbitrary way. To see that less obvious possibilities exist consider the model given by eq.(6). The potential in eq.(8) can be rewritten as

$$U(x) = (V(x) - Ec(x) + U_0)\gamma(E) \tag{23}$$

where the following identifications have been made

$$\begin{aligned}
V(x) &\equiv 2\beta^2 x^2 && (24)\\
c(x) &\equiv \frac{x}{a}\sqrt{1 + \frac{x^2}{a^2}} && (25)\\
E &= 2\beta^2 a^2 \tanh\left(\frac{c}{a}\right) && (26)\\
U_0 &= -\frac{1}{2}\sqrt{4\beta^2 a^4 - E^2} + \beta a^2 && (27)\\
\gamma(E) &= \frac{1}{\sqrt{1 - \frac{E^2}{2\beta^2 a^2}}} \left(\frac{1}{\sqrt{1 - \frac{E^2}{2\beta^2 a^2}}} + 1 \right) && (28)
\end{aligned}$$

Consider now the Liouville model given by

$$H = \frac{\sum_{k=1}^{N}\left(\frac{p_k^2}{2\mu_k}+2\beta_k^2x_k^2\right)}{1+\sum_{k=1}^{N}\frac{x_k}{a_k}\sqrt{1+\frac{x_k^2}{a_k^2}}}, \qquad \begin{aligned}\beta_k^2 &= \frac{\alpha^2\mu_k m_k^2}{8}, \\ \beta_1^2a_1^2 &= \ldots = \beta_N^2a_N^2\end{aligned} \tag{29}$$

where m_k are arbitrary integers. For $|E| < \min_k(2\beta_k^2a_k^2)$ the motion is bounded with respect to all variables x_k. Morover, one can choose $c_k = a_k\mathrm{arctanh}\left(\frac{E}{2\beta_k^2a_k^2}\right)$ to find that our model is superintegrable.

In fact, the equivalent hamiltonian (21) is given by eq.(24–28) with the replacement $\beta_k^2 \to \beta_k^2/\gamma(E)$; note that the function $\gamma(E)$ is universal, i.e. does not depend on k.

As a special example consider the case $m_k = 1$; then $\beta_1 = \ldots = \beta_N = \beta$, $a_1 = \ldots a_N = a$ and we obtain the deformation of isotropic oscillator which is still superintegrable.

More general discussion of Liouville and Staeckel systems will be given elsewhere.

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