Joint Description of Periodic SL(2,R) WZNW Model and Its Coset Theories

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Abstract

Liouville, $SL(2,R)/U(1)$ and $SL(2,R)/R_+$ coset structures are completely described by gauge invariant Hamiltonian reduction of the $SL(2,R)$ WZNW theory.

1 Introduction

Wess-Zumino-Novikov-Witten (WZNW) models are fascinating two-dimensional conformal field theories with reach symmetry and dynamical structures. Their cosets form important classes of integrable theories. An outstanding example is the $SL(2,R)$ WZNW model which is related to the Liouville theory and other cosets with interesting ('black hole') space-time properties. Using Hamiltonian reduction one can show what happens under this reduction with the $SL(2,R)$ WZNW fields, the symplectic structure, the Kac-Moody currents, the Sugawara energy-momentum tensor, and most importantly how the general $SL(2,R)$ solution reduces to those of the gauged theories. Free-field parametrisation allows canonical quantisation, and we present as a typical quantum result a causal quantum group commutator.

The talk is based on a series of papers [1]-[5].

2 SL(2,R) WZNW theory

WZNW models are invariant under left (chiral) and right (anti-chiral) multiplications of the WZNW field $g(z,\bar{z})$. These Kac-Moody symmetries provide integrability of the theory and give its general solution as a product of chiral and anti-chiral fields, $g(z)$ respectively $\bar{g}(\bar{z})$, which for periodic boundary conditions in σ have monodromies $g(z+2\pi) = g(z)M$ and $\bar{g}(\bar{z}-2\pi) = M^{-1}\bar{g}(\bar{z})$ with $M \in SL(2,R)$. $z = \tau + \sigma$, $\bar{z} = \tau - \sigma$ are light cone coordinates. The conformal symmetry is generated by the traceless Sugawara energy-momentum tensor.

The basic (anti-)chiral Poisson brackets follow by inverting the corresponding symplectic form of the $SL(2,R)$ theory. But piecing together the chiral and anti-chiral results surprisingly simple causal non-equal time Poisson brackets follow for the $SL(2,R)$ WZNW fields [5]

$$\{\, g_{ab}(z,\bar{z}),\, g_{cd}(y,\bar{y})\,\} = \frac{\gamma^2}{4}\,\Theta\,[2g_{ad}(z,\bar{y})\,g_{cb}(y,\bar{z}) - g_{ab}(z,\bar{z})\,g_{cd}(y,\bar{y})], \tag{1}$$

where $\Theta = \epsilon(z-y) + \epsilon(\bar{z}-\bar{y})$ is the causal factor. The stair-step function $\epsilon(z) = 2n+1$ for $2\pi n < z < 2\pi(n+1)$ ensures that causality. This Poisson bracket was derived for hyperbolic monodromy in the 'fundamental' interval $z - y \in (-2\pi, 2\pi)$ where $\epsilon(z-y) = sign\,(z-y)$, and it holds therefore also on the line. Eq.(1) can be generalised even outside this domain [2].

From (1) one can derive, e.g., the canonical Poisson brackets, the Kac-Moody algebra and Poisson bracket relations of the energy-momentum tensor with itself or any other field.

The causal Poisson brackets encode the full WZNW dynamics.

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3 Coset theories

Gauging the SL(2,R) WZNW theory with respect to the three different types of one-dimensional subgroups $h = e^{\alpha t} \in SL(2,R)$

$$e^{\alpha t_0} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}, \quad e^{\alpha t_2} = \begin{pmatrix} e^{\alpha} & 0 \\ 0 & e^{-\alpha} \end{pmatrix}, \quad e^{\alpha t_+} = \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}, \tag{2}$$

and considering axial respectively vector gauging $g \mapsto hgh$ and $g \mapsto hgh^{-1}$ one finds six integrable [3] coset theories. The subgroups are called compact ($t = t_0$), non-compact ($t = t_2$) and nilpotent ($t = t_+$), where the t_n are elements of the $sl(2,R)$ algebra given by the Pauli matrices $t_0 = -i\sigma_2$, $t_1 = \sigma_1$, $t_2 = \sigma_3$, and $t_+ = t_0 + t_1$ with $t_+^2 = 0$ is a nilpotent element. The cosets will be described in terms of gauge invariant components of the WZNW field. Taking into account the condition $\det g = 1$ and parametrising the WZNW field by

$$g = cI + v^n\, t_n = \begin{pmatrix} c - v_2 & -v_1 - v_0 \\ -v_1 + v_0 & c + v_2 \end{pmatrix}, \qquad \text{with} \qquad c^2 + v^n v_n = 1, \tag{3}$$

for each of the coset models only a couple of these field components will be gauge invariant.

It is worth to consider both Lagrangean and Hamiltonian reduction.

3.1 Lagrangean reduction by gauging

The considered $SL(2,R)$ WZNW theory on the cylinder can be gauged in the standard manner and it yields for the compact axial respectively vector cases the gauge invariant Lagrangeans with the euclidian target space geometries of a cigar (the 'euclidian black hole') and a trumpet

$$\mathcal{L}_G^{(1)}| = \frac{1}{\gamma^2}\,\frac{\partial_z v_1 \partial_{\bar z} v_1 + \partial_z v_2 \partial_{\bar z} v_2}{1 + v_1^2 + v_2^2}, \qquad \mathcal{L}_G^{(2)}| = \frac{1}{\gamma^2}\,\frac{\partial_z c \partial_{\bar z} c + \partial_z v_0 \partial_{\bar z} v_0}{c^2 + v_0^2 - 1}. \tag{4}$$

Whereas the target space of $\mathcal{L}_G^{(1)}|$ is R^2, for $\mathcal{L}_G^{(2)}|$ the unit disk $c^2 + v_0^2 < 1$ is missing and this Lagrangean is singular at the disk boundary, but both coset theories are mutually related [5].

For the non-compact cases we obtain two equivalent minkowskian ('black hole') actions

$$\mathcal{L}_G^{(3)}| = \frac{1}{\gamma^2}\,\frac{\partial_z v_1 \partial_{\bar z} v_1 - \partial_z v_0 \partial_{\bar z} v_0}{1 + v_1^2 - v_0^2}, \qquad \mathcal{L}_G^{(4)}| = \frac{1}{\gamma^2}\,\frac{\partial_z v_2 \partial_{\bar z} v_2 - \partial_z c \partial_{\bar z} c}{1 + v_2^2 - c^2}, \tag{5}$$

which are analytically related to (4). The target space is R^2 and $\mathcal{L}_G^{(3)}|$, e.g., has two singularity lines $v_0 = \pm\sqrt{1+v_1^2}$. There are so three different regular domains in the target space

$$v_0 > \sqrt{1+v_1^2}, \qquad -\sqrt{1+v_1^2} < v_0 < \sqrt{1+v_1^2}, \qquad v_0 < -\sqrt{1+v_1^2}, \tag{6}$$

and this coset theory has to be investigated in each of them separately.

Finally, for the nilpotent gaugings only two identical gauged Lagrangeans arise for the field $V = g_{12}(z,\bar z)$ whereas the other gauge invariant components v_2 or c simply disappear

$$\mathcal{L}_G^{(5)}| = \mathcal{L}_G^{(6)}| = \frac{1}{\gamma^2}\,\frac{\partial_z V \partial_{\bar z} V}{V^2}. \tag{7}$$

$V = 0$ is a singularity of the Lagrangian, but for the regular parametrisation $V = \pm e^{\gamma\phi}$ we get as a result free-field theories only

$$\mathcal{L}_G^{(5)}| = \mathcal{L}_G^{(6)}| = \partial_z \phi \partial_{\bar z} \phi. \tag{8}$$

Note that the Liouville theory does not arise by this standard gauging.

3.2 Hamiltonian reduction by constraints

Hamiltonian reduction is an alternative but more flexible method to construct and investigate coset theories. Here the constrained Kac-Moody currents $J_0 = 0 = \bar{J}_0$, $J_2 = 0 = \bar{J}_2$ and $J_+ = 0 = \bar{J}_+$ provide the cosets (4), (5) and (7) respectively. But both, the axial and the vector gauged Lagrangeans arise by one and the same constraints [4, 5]. Although these systems are described by different components of the WZNW field they live on the same constrained surface, and are therefore mutually related with each other. It is important to mention that the two first current constraints are of second class and the nilpotent gauging is of first class.

Imposing to the $SL(2,R)$ WZNW theory the alternative nilpotent constraints $J_+ = \rho,\ \bar{J}_+ = \bar{\rho}$ with non-vanishing constants ρ and $\bar{\rho}$, and write the only gauge invariant field component as $g_{12}(z,\bar{z}) = \psi(z)\bar{\psi}(\bar{z}) + \chi(z)\bar{\chi}(\bar{z})$, then $\psi(z) = g_{11}(z),\ \chi(z) = g_{12}(z)$ etc., satisfy constant Wronskians $\psi(z)\chi'(z) - \psi'(z)\chi(z) = \rho\gamma^2$ etc.. The following identification

$$e^{-\gamma\varphi(z,\bar{z})} = \psi(z)\bar{\psi}(\bar{z}) + \chi(z)\bar{\chi}(\bar{z}) \tag{9}$$

leads us to the Liouville equation with the 'cosmological' constant given by $\mu = -\rho\bar{\rho}\gamma^3$

$$\partial_{z\bar{z}}\,\varphi + \mu e^{2\gamma\varphi} = 0. \tag{10}$$

The equation (9) obviously also provides the general solution of the Liouville equation.

Hamiltonian reduction is in fact a method for integrating coset theories [4].

4 Reduction of Poisson brackets

Since the nilpotent constraints are of first class, for the gauge invariant Liouville exponential $e^{-\gamma\varphi} = g_{12}(z,\bar{z})$ the reduced non-equal time Poisson bracket can be read off directly from the relation (1) without any further calculations. For the (anti-)chiral fields $\psi(z) = g_{11}(z), \bar{\psi}(\bar{z}) = \bar{g}_{11}(\bar{z}),\ \chi(z) = g_{12}(z),\ \bar{\chi}(\bar{z}) = \bar{g}_{12}(\bar{z})$ the classical form of the exchange algebra results, which quantum mechanically become the celebrated Gervais-Neveu quantum group relations.

It might be worth to note that the gauge invariant nilpotent reduction of the Sugawara energy-momentum tensor immediately generates the Liouville form with the standard classical 'improvement' term included. But in this case there do not exist coset currents.

The situation is different if we reduce the $SL(2,R)$ WZNW theory by the second class constraints. Using the isomorphism between $SL(2,R)$ and $SU(1,1)$ there is a natural complex structure given, e.g. for the euclidian case (4), by the complex coordinates $u = v_1 + iv_2$ and $x = c + iv_0$ which are related by $|x|^2 - |u|^2 = 1$. $u(z,\bar{z})$ and $x(z,\bar{z})$ are the physical fields of the axial respectively vector gauged cosets. These gauge invariant fields can be expressed similarly as in the Liouville case (9) by $u(z,\bar{z}) = \psi(z)\bar{\psi}(\bar{z}) + \chi(z)\bar{\chi}(\bar{z})$, but now in terms of the complex fields $\psi(z) = g_{11}(z) + ig_{12}(z)$, $\chi(z) = g_{21}(z) + ig_{22}(z)$ etc., and with the non-constant Wronskians $\psi(z)\chi'(z) - \psi'(z)\chi(z) = 2W(z)$ and the anti-chiral one. Here $W(z) = J_1(z) + iJ_2(z)$ is the parafermionic coset current [3, 4]. The algebra of the coset fields is given by Dirac brackets [5], and there are causal relations for each coset

$$\begin{aligned}
\{\,u(z,\bar{z}),\,u(y,\bar{y})\,\}_D &= \gamma^2\ \Theta\,[u(z,\bar{y})\,u(y,\bar{z}) - u(z,\bar{z})\,u(y,\bar{y})],\\
\{\,u(z,\bar{z}),\,u^*(y,\bar{y})\,\}_D &= \gamma^2\ \Theta\,x(z,\bar{y})\,x^*(y,\bar{z}),\\
\{\,x(z,\bar{z}),\,x(y,\bar{y})\,\}_D &= \gamma^2\ \Theta\,[x(z,\bar{y})\,x(y,\bar{z}) - x(z,\bar{z})\,x(y,\bar{y})],\\
\{\,x(z,\bar{z}),\,x^*(y,\bar{y})\,\}_D &= \gamma^2\ \Theta\,u(z,\bar{y})\,u^*(y,\bar{z}),
\end{aligned} \tag{11}$$

and non-causal connections which with the notation $2E = \epsilon(z-y)$, $2\bar{E} = \epsilon(\bar{z}-\bar{y})$ are

$$\begin{aligned}
\{\,u(z,\bar{z}),\,x(y,\bar{y})\,\}_D &= \gamma^2\ \Theta\,x(z,\bar{y})\,u(y,\bar{z}) - \gamma^2\,E\,u(z,\bar{z})\,x(y,\bar{y}),\\
\{\,u(z,\bar{z}),\,x^*(y,\bar{y})\,\}_D &= \gamma^2\,\Theta\,u(z,\bar{y})\,x^*(y,\bar{z}) - \gamma^2\,\bar{E}\,u(z,\bar{z})\,x^*(y,\bar{y}).
\end{aligned} \tag{12}$$

As expected the axial and vector gauged theories form a coupled algebra.

5 Canonical quantisation

The canonical quantisation of the cosets can be performed in the same way as it has been done for the Liouville theory [1]. Here one uses the general solution of the coset as a canonical transformation between the non-linear coset fields and free fields. The quantisation will be defined by replacing the Poisson brackets of the canonical free fields by commutators. Non-linear expressions in the free fields will be normal ordered. But calculations with normal ordered operators usually yield anomalous contributions. Such anomalies can be avoided by quantum mechanically deforming the composite operators of the cosets. The deformations are determined by requiring the classical symmetry transformations, and locality, to be valid as commutator relations. As a result, we show the non-equal time commutator for the Liouville exponential $u(z,\bar{z}) = e^{-\gamma\varphi(z,\bar{z})}$ which is written here for convenience as a Moyal bracket [2]

$$\begin{aligned}&\{\check{u}(z,\bar{z}),\, \check{u}(y,\bar{y})\}_* = \frac{1}{\hbar}\sin(\hbar\gamma^2/4)\,[\epsilon(z-y)+\epsilon(\bar{z}-\bar{y})]\times \\ &\left[\check{u}(z,\bar{y}) * \check{u}(y,\bar{z}) + \check{u}(y,\bar{z}) * \check{u}(z,\bar{y}) - \frac{\check{u}(z,\bar{z}) * \check{u}(y,\bar{y}) + \check{u}(y,\bar{y}) * \check{u}(z,\bar{z})}{2\cos(\hbar\gamma^2/4)}\right].\end{aligned} \tag{13}$$

Its expansion in $\hbar$ reproduces the Poisson bracket. We can define by (13) other operators and their commutators. The operator of $e^{-2\gamma\varphi(z,\bar{z})}$ simply follows by differentiation for equal time.

Further results will be discussed in the lecture notes [6].

6 Final remarks

There is a complete classical understanding of the whole set of $SL(2,R)$ theories. Quantum mechanically the Liouville theory is best worked out, but still incomplete. The zero mode structure and the Hilbert space require further intensive study. Coset currents only exist for the non-nilpotent gauged $SL(2,R)$ theories as parafermions. They generate quantum mechanically a dilaton which might render the classically non-dynamical metric dynamical.

Besides being interesting in its own right the $SL(2,R)$ WZNW model and its cosets also appear in many applications, in particular in string calculations. An exact and complete quantum mechanically treatment of the $SL(2,R)$ family would be helpful to understand, e.g., the AdS_3/CFT correspondence, which is an intensively discussed contemporary problem.

Minor knowledge exists for the quantum mechanical $SL(2,R)$ WZNW model.

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