Cohomology and Bessel Functions Theory

M.Mekhfi *†

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Abstract

By studying cohomological quantum mechanics on the punctured plane,we were led to identify (reduced) Bessel functions with homotopic loops living on the plane.This identification led us to correspondence rules between exponentials and Bessel functions.The use of these rules makes us retrieve known but also new formulas in Bessel functions theory.

1 Cohomology and topological actions

A recall of few basic notions of cohomology together with associated topological actions and topological quantum mechanical systems is necessary in order to understand the relationships of these backgrounds to the theory of Bessel functions.Let M be a compact manifold with local coordinates x^{μ} and let $H^{r}(M)$ be the r^{th} de Rham cohomology group.Let $c_{1,}$.........c_{k} be elements of the homology group $H_{r}(M)$ with k the r^{th} Betti number and of the same class $[c_{i}]=[c_{j}]$.Then for any set of numbers $b_{1},......b_{k}$ a corollary of the de Rham's theorem states that there exist a closed r-form ω such that

$$\int_{c_{i}}\omega=b_{i}\qquad 1\leq i\leq k \tag{1}$$

We may include in this expression the trivial period $b_{i}=0$ which corresponds to ω being a closed and exact form.The numbers in 1 are the periods of closed r-forms over cycles c_{i}.The main features of such actions is that there are defined on the product $\mathrm{H}_{r}(\mathrm{M})\times\mathrm{H}^{r}(\mathrm{M})$ and are therefore topological actions,that is invariant under any infinitesimal deformations which keeps the cycle within its homologyclass $H^{r}(M)$

$$\delta x^{\mu}=\epsilon^{\mu}$$

*Mailing address:Laboratoire des sciences de la matière condensée, Département de physique Université Es-senia 31100 Oran ALGERIE

†mekhfi@myrealbox.com

Topological (cohomological) quantum mechanical systems we are interested in are those described by the topological actions 1 .The application to Bessel functions comes from considerations of the first non trivial action with period $b_1$on the punctured plane .The topological ingredients are loops and 1-forms .

A very practical way to gauge fix these actions is to use BRST symmetry.To the deformation symmetry at hand we associate the BRST symmetry with generator s

$$\begin{aligned} sx^\mu &= \psi^\mu \\ s\psi^\mu &= 0 \\ s\bar{\psi}^\mu &= \lambda^\mu \\ s\lambda^\mu &= 0 \end{aligned}$$

To gauge fix the symmetry such that covariance is maintained and to get an action quadratic in velocities Baulieu and Singer proposed the following gauge function $\dot{x}^\mu + \frac{\partial V}{\partial x^\mu} + \cdots$ where $\dot{x}^\mu = \frac{dx^\mu}{dt}$ and where we omit the Christoffel symbol term as these are known matters and not relevant to what follows.The important ingredient here is the prepotential V .It is a priori an arbitrary function of the coordinates x^μ but to correctly define topological invariants one has to define the prepotential properly.First write the gauge fixed action

$$S_{GF} = \int_{c\in H_1} \omega + \int dt \;\; s\bar{\psi}^\mu (g_{\mu\nu}\dot{x}^\nu + \frac{\partial V}{\partial x^\mu} - \frac{1}{2} g_{\mu\nu}\lambda^\mu + \cdots \tag{2}$$

After integrating out the auxiliary field λ we select out the bosonic linear term of interest.

$$\int dt\, \dot{x}^\mu \frac{\partial V}{\partial x^\mu} = \int dV$$

The second step is to cancel out the classical action.This cancellation is necessary for the resulting gauge fixed action to possess a secondary (dual) BRST symmetry $\bar{s}$ which in turn allows us to write the hamiltonian in the form $H \propto \{Q,\ \bar{Q}\}$ with Q and $\bar{Q}$ both nilpotent.The proper definition of the prepotential is therefore given by the equation

$$\int_{c\in H_1} \omega = \int_{c\in H_1} dV \tag{3}$$

In other word one should not simply look for a prepotential on the basis that the integral of it numerically cancels the classical action but shoud first define a period (here b_1).We hereafter will restrict to the punctured plane $R^2/(0)$and take it as our target manifold as it is simple with a non trivial homology and specially because it has a direct application to the theory of

Bessel functions on concern in this paper.We thus have $H^1(R^2/(0), R) \cong R$ that is $\omega^{`s}$are one forms labelled by real numbers and the cycles c_i are homotopic loops encircling the whole as $H_1 \cong \Pi \cong Z$ where Π is the homotopy group or the fundamental group and Z is the set of integers.The topological action associated with the punctured plane is ($\lambda \in R$ and $x^1 + ix^2 = r\exp(i\theta)$)

$$S_{cl} = \lambda \int_c d\theta$$

The complete solution to the equation 3 for the punctured plane is

$$V = \lambda(\theta + \Phi(\theta)) \tag{4}$$

Where the function $\Phi(\theta)$ is any function but periodic .The simplest case without $\Phi(\theta)$ has been selected by Baulieu and Rabinovici [1].Such truncated solution neither lead to a complete description of the invariants on the punctured plane nor to the theory of Bessel functions .Let us remember that the topological invariants on the punctured plane are W and $\Pi(m)$ $m \in Z$ which are respectively the winding number operator and the fundamental group operator (m indexing the group elements).They are defined as follows[1]

$$\begin{aligned} W \quad & \mid \quad n\rangle = n \mid n\rangle \\ \Pi(m) \quad & \mid \quad n\rangle = \mid n+m\rangle \end{aligned}$$

Where the state $\mid n\rangle$ describes the "quantum" state of the loop encircling the whole n times.The topological invariant which came out of the analysis of reference [1] is

$$W + \lambda\Pi(0)$$

where $\Pi(0) = 1$ is the group identity .The other group elements $\Pi(m)$ $m \neq 0$ are missing.We have completed their analysis [2] [3] by adding the missing part $\Phi(\theta)$ in the complete solution 4 . The topological invariant turned out to be in this case

$$W + \lambda \sum_{m \in Z} \Pi(m)$$

Where all the invariants are represented.The presence of the periodic function Φ in V is primordial and is at the heart of the connection to Bessel function.

[1]We may use the representation$\mid \theta \rangle$ instead of $\mid n \rangle$.In this case we have the actions

$$\begin{aligned} W \quad & \mid \quad \theta\rangle = -i\partial_\theta \mid \theta \rangle \\ \Pi(m) \quad & \mid \quad \theta \rangle = \exp im\theta \mid \theta \rangle \end{aligned}$$

2 Bessel functions are realization of homotopic loops

The hamiltonian associated with the general action 2 and adapted to the punctured plane is

$$H = \frac{1}{2}p^2 + \frac{1}{2r^2}(\frac{\partial V}{\partial \theta})^2 - \frac{1}{2}[\bar{\psi}_i, \psi_j]\frac{\partial^2 V}{\partial x_i \partial x_j} = \frac{1}{2}\{Q,\ \bar{Q}\} \tag{5}$$

Where $P_i = -i\frac{\partial}{\partial x_i}$ and $\bar{\psi}_i = \frac{\partial}{\partial \psi_i}$ are canonical momenta for the coordinates x_i and the ghost fields ψ_i respectively and where the generators Q and $\bar{Q}$ are given by

$$\begin{aligned} Q &= \psi_i(p_i + i\frac{\partial V}{\partial x_i}) \\ \bar{Q} &= \bar{\psi}_i(p_i - i\frac{\partial V}{\partial x_i}) \end{aligned}$$

Let us note that these generators are nilpotent ,that is $Q^2 = \bar{Q}^2 = 0$ this is because there are built out of the anticommuting variables $\psi(\bar{\psi})$.In fact one can identify the first term in Q (p_i term)and $\bar{Q}$ respectively with the exterior derivative d ($d^2 = 0$) and its adjoint $d^\dagger(d^{\dagger 2} = 0)$ and the generator Q itself with the twisted or deformed exterior derivative $d_\alpha = \exp(\alpha V)\ d$ $\exp(-\alpha V)$ first introduced by E.Witten [4] and which served to investigate Morse theory in a very novel and deep way.Let us rewrite Q in the form of exponential as d_α by using the relation $p_i + i\frac{\partial V}{\partial x_i} = \exp(V)\, p_i \exp(-V)$ .Doing this we get

$$\begin{aligned} Q &= \exp(V)\, Q_0 \exp(-V) \\ \bar{Q} &= \exp(-V)\, Q_0 \exp(V) \end{aligned}$$

Where Q_0 is identified with the exterior derivative d .

Bessel functions will enter in play when we consider the eigenvalue problem associated to the Hamiltonian.The more elegant way to study the above hamiltonian is to use the superwavefunction formalism [2].Let $\Phi(x, \psi)$denotes the superwavefunction

$$\Phi(x, \psi) = \phi + \psi_i A^i + \frac{1}{2} \in_{ij} \psi_i \psi_j B$$

Where the four states $\phi, A^i$and B are functions of the coordinates x_i only.We will only retain ghost free solutions i.e. $\phi^{‘s}$ .These are eigenstates of the hamiltonian (The ghost solutions will probably give no direct informations on Bessel functioms !!)

$$\begin{aligned} H\phi &= \phi \\ H &= -\frac{\partial^2}{\partial x_i \partial x^i} + \frac{1}{2r^2}[(\frac{\partial V}{\partial \theta})^2 - \frac{\partial^2 V}{\partial^2 \theta}] \end{aligned} \quad (6)$$

Let us look for a solution of the form

$$F(r)\mathrm{f}(\theta)$$

Putting this into the ϕ equation 6 and separating the variables,we get $(W = (\frac{\partial \Phi}{\partial \theta})^2 - \frac{\partial^2 \Phi}{\partial^2 \theta})$

$$\begin{aligned} 0 &= \ddot{\mathrm{f}}(\theta) + (\zeta^2 - \lambda^2 - W)\mathrm{f}(\theta) \\ 0 &= \ddot{F}(r) + \frac{1}{r}\dot{F} + (-\frac{\zeta^2}{r^2} + 2E)F \end{aligned}$$

Where ζ is the separation parameter.The r- component equation is the differential equation defining Bessel functions $J_\zeta(\sqrt{2}Er)$,while the θ-component equation is of Sturn-Liouville type.This equation has been analyzed in [2] and put on the simplified form

$$\ddot{u} + (\zeta^2 - \lambda^2 - W\,)\,u = 0$$

Where the solutions are here periodic.There is a set of theorems of eigenvalues and boundaries problems which state that the above equation independently of the potential W provided it is continuous,has real eigenvalues and are all ordered as $0\langle\zeta_1^2\langle\cdots\zeta_p^2$ and $\lim_{p\to\infty}\zeta_p = \infty$ and moreover that the eigenfunctions $u_{\zeta_p}(\theta)$ are orthonormal with weight 1,that is $\int u^*_{\zeta_q}(\theta)u_{\zeta_p}(\theta)$ $d\theta = \delta_{qp}$.(These are generalization of $\exp[i\sqrt{\zeta^2 - \lambda^2}\theta]$) when $W = 0$).

We are now ready to write down expectation values for topological invariants.On dimensional grounds ,one may select the following candidates,together with their hermitian conjugates .[2]

$$\begin{aligned} \{Q, \epsilon^{ij} x_i \bar{\psi}_j\} &= -i\partial_\theta + i\partial_\theta V \\ \{Q, x_i \bar{\psi}_j\} &= r\partial_r \end{aligned}$$

In writing these expressions we dropped out ghost terms as they give vanishing actions on the ϕ eigenfunctions of zero ghost numbers we are considering.The expectation value of the θ-component topological invariant is (we only include the θ-component of the wave function since the operator only depends on θ)

[2]The topological invariant $-i\partial_\theta + i\partial_\theta V = -i\partial_\theta + \lambda \sum_{m\in Z} \phi_m \exp(im\theta)$ can be written in an equivalent way in the $\mid n\rangle$ states as $W + \lambda \sum_{m\in Z} \Pi(m)$ for specific values of the $\phi_m^{`s}$.

$$
\begin{aligned}
\langle\{Q,\epsilon^{ij}x_i\bar{\psi}_j\}\rangle_{E,\zeta_p} &= \frac{\int u^*_{\zeta p}(-i\partial_\theta + i\partial_\theta V)u_{\zeta_p}}{\int u^*_{\zeta p}u_{\zeta_p}d\theta} = \langle -i\partial_\theta + i\partial_\theta V\rangle^{\zeta_p} \\
&= \sqrt{\zeta_p^2-\lambda^2} + \text{higher order in } V \\
&= \zeta_p + \cdots
\end{aligned}
$$

This is an effective winding number which is ζ_p corrected by the interaction V.

To define the second topological invariant one needs to interpret Bessel functions as describing homotopic loops .To this end we have to restrict ourselves to non interacting case i.e. $\lambda = 0; W = 0; \zeta_p = n$ .In this case the wavefunctions are simple

$$
\begin{aligned}
u_n(\theta) &= \exp in\theta \\
\frac{J_n(z)}{z^n};\ z &= \sqrt{2}Er
\end{aligned}
$$

If usually one associates $\exp in\theta$ to the homotopic n-loop the corresponding Bessel function does not .Bessel functions scale as $J_n(z) \propto z^n$ when z goes to zero and so could not describe a loop winding around the puncture (origin) which is physically present as it is trapped by the puncture .We therefore consider that positively oriented loops ($n \succ 0$) are described rather by the reduced Bessel function $\frac{J_\zeta(z)}{z^n}$ as $\lim_{z\to 0}\frac{J_\zeta(z)}{z^n} \neq 0$. Negatively oriented loops ($n \prec 0$) on the other hand are descibed by $z^nJ_n(z)$.

Now that Bessel functions are supposed to realize homotopic loops in the z variable .It is natural to look for the operator in the z variable which is analogous to the winding number operator W

The operator that plays the role of the winding number operator; W_r has the form

$$
W_r = -\frac{1}{2}(z\frac{d}{dz} + (\frac{d}{zdz})^{-1})
$$

and it acts on the wavefunctions as follows

$$
\begin{aligned}
W_r\ \frac{J_n(z)}{z^n} &= n\ \frac{J_n(z)}{z^n} \\
W_r\ (z^nJ_n(z)) &= -n\ z^nJ_n(z)
\end{aligned}
$$

The proof is as follows.

$$
W_r\,\frac{J_n(z)}{z^n} = -\frac{1}{2}(z\frac{d}{dz} + (\frac{d}{zdz})^{-1})\ \frac{J_n(z)}{z^n}
$$

$$
\begin{aligned}
&= -\frac{1}{2}z^2\frac{d}{zdz}\frac{J_n(z)}{z^n}+\frac{1}{2}\frac{J_{n-1}(z)}{z^{n-1}} \\
&= \frac{1}{2}z^2\frac{J_{n+1}(z)}{z^{n+1}}+\frac{1}{2}\frac{J_{n-1}(z)}{z^{n-1}} \\
&= \frac{1}{2}\frac{(J_{n+1}(z)+J_{n-1}(z))}{z^{n-1}} \\
&= n\frac{J_n(z)}{z^n}
\end{aligned}
$$

Where in the fourth line we apply the recursion formula $J_{n+1}(z) + J_{n-1}(z) = 2n\ J_n(z)$ .There is another operator which applies on reduced Bessel functions in a special way.It is

$$
\begin{aligned}
d_m &= \frac{d}{zdz}\cdots\frac{d}{zdz} = (\frac{d}{zdz})^m \ ; m \in Z \\
d_{-|m|}d_{|m|} &= d_{|m|}d_{-|m|} = 1 \\
(-1)^m d_m\frac{J_n(z)}{z^n} &= \frac{J_{n+m}(z)}{z^{n+m}} \ ; m \in N \\
d_m\ (z^n J_n(z)) &= z^{n-m}J_{n-m}(z);\ m \in N
\end{aligned}
$$

We thus have two operators acting on reduced Bessel functions W_r and d_m and these are the analogous (r-components) of $W_\theta = -id_\theta$ and $\Pi_\theta(m) = \exp im\theta$

3 Exp$^{\mathbf{S}}$ vs Bessel$^{\mathbf{‘S}}$:The correspondence Rules

The important remark we make toward unraveling new properties of Bessel functions (one in this paper) is that ($\zeta = n$ The separation parameter) is common a parameter to $\exp in\theta$ and to $\frac{J_n(z)}{z^n}$, both describing the same object :The $n-$loop.,Hence we suggest and prove that :Certain specific relationships ammong exponentials will be transported to reduced Bessel functions via the following correspondence rules CR$^{‘s}$

These CR$^{‘s}$ are collected in the following tableau.(The notation E$^{im\theta}$

means that $\exp in\theta$ acts as an operator)

θ-components	r -components
Function	Function
$\exp in\theta$; $-\pi \leq \theta \leq \pi$	$\frac{J_n(z)}{z^n}$
$1; (n = 0)$	$J_0(z)$
Homotopic group elements	Homotopic group elements
$\Pi_\theta\,(m) = \mathrm{E}^{im\theta}$; $m \in Z$	$\Pi_r(m) = (-1)^m (\frac{d}{zdz})^m$; $m \in Z$
$1; (m = 0)$	$1 \equiv d_{-\lvert m\rvert} d_{\lvert m\rvert} = d_{\lvert m\rvert} d_{-\lvert m\rvert}$
Winding number	Winding number
$W_\theta = -i\frac{d}{d\theta}$	$W_r = -\frac{1}{2}(z\frac{d}{dz} + (\frac{d}{zdz})^{-1})$

(7)

Few remarks are in order at this stage .First note that the topological invariant $z\partial_z$ gets a meaning in the new framework.That is we have $r\partial_r = -2W_r + \Pi_r(-1)$ .It is a combination of the winding number W_r and the homotopy group element $\Pi_r(-1)$ .Second, note the global restriction on the angle $-\pi \leq \theta \leq \pi$.Relationships valid within truncated values of the angle or in another interval other that the above will not map to corresponding relationships in Bessel functions.To understand the globality of the restriction let us consider the identity

$$-i\theta = \sum_{m \in Z/0} (-1)^m \frac{\exp im\theta}{m} \; ; -\pi \prec \theta \prec \pi$$

This formula serves to define the interval of validity of the the angle θ as the right hand side only involve transportable quantities that is $\exp im\theta$ and the accompanying constants $(-1)^m$; $\frac{1}{m}$ which are invariant under transport .If on the other hand we shift the angle to $\theta \to \theta + \pi$,we get a "bad" non tranportable formula

$$-i\theta = \sum_{m \in Z/0} \frac{\exp im\theta}{m} - i\pi \; ; 0 \prec \theta \prec 2\pi$$

as the interval of definition is no longer $-\pi \prec \theta \prec \pi$

Finally we may note that the periodicity of the exponential is not transported by CR^{s}as globality fixes the angle to be in a fixed interval.

Applying the CR^{s}we get a series of interesting correspondences listed in the following tableau.

θ-components	r -components
$\mathrm{E}^{im\theta}\exp in\theta=\exp i(n+m)\theta$	$(-1)^m(\frac{d}{zdz})^m\frac{J_n(z)}{z^n}=$
	$\frac{J_{n+m}(z)}{z^{n+m}}$
$\mathrm{E}^{im\theta}\mathrm{E}^{in\theta}=\mathrm{E}^{i(m+n)\theta}$	$(-1)^m(\frac{d}{zdz})^m(-1)^n(\frac{d}{zdz})^n=$
	$(-1)^{m+n}(\frac{d}{zdz})^{m+n}$
$1\exp in\theta=\exp in\theta$	$(\frac{d^2}{dz^2}+(2n+1)\frac{d}{zdz}+1)\frac{J_n(z)}{z^n}=0$
$\left[-i\frac{d}{d\theta},E^{im\theta}\right]=mE^{im\theta}$	$\left[W_z,(-1)^m(\frac{d}{zdz})^m\right]=$
	$m(-1)^m(\frac{d}{zdz})^m$
$(i\theta)^n$; c number	$\frac{d^n}{d\lambda^n}\frac{J_\lambda(z)}{z^\lambda}\mid_{\lambda=0}$
$i\theta$; operator	$\sum_{m\in Z/0}\frac{(-1)^m}{m}(\frac{d}{zdz})^m$
$\exp i\lambda\theta$	$\frac{J_\lambda(z)}{z^\lambda}$
$\mathrm{E}^{i\lambda\theta}$	$(\frac{d}{zdz})^\lambda$
$-i\frac{d}{d\theta}\exp i\lambda\theta=\lambda\exp i\lambda\theta$	$z(J_{\lambda+1}+J_{\lambda-1})=2\lambda\ J_\lambda(z)$
$-i\theta=\sum_{m\in Z/0}(-1)^m\frac{\exp im\theta}{m}$	$\sum_{m\in Z/0}\frac{(-1)^m}{m}\frac{J_m(z)}{z^m}=-\frac{d}{d\lambda}\frac{J_\lambda(z)}{z^\lambda}\mid_{\lambda=0}$
$;-\pi\prec\theta\prec\pi$	$=J_0(z)\ln z-\frac{\pi}{2}N_0(z)$
The Unification Formula for Bessel Functions of Different Orders	
$\exp i(n+\lambda)\theta=$	$\frac{J_{n+\lambda}(z)}{z^{n+\lambda}}=$
$\exp(-\lambda\sum_{m\in Z/0}(-1)^m\frac{\mathrm{E}^{im\theta}}{m})\exp in\theta$	$\exp(-\lambda\sum_{m\in Z/0}\frac{1}{m}(\frac{d}{zdz})^m)\frac{J_n(z)}{z^n}$
etc	etc
$\vdots$	$\vdots$

Let us prove some correspondences to illustrate the CR^s.We take as examples the one leading to the Bessel differential equation and that leading to the recursion formula.The remaining correspondences present no difficulties to be proved .Start from the trivial identity

$$1\exp in\theta=\exp in\theta$$

where 1 is the unit operator .CR$^{\text{`}s}$,according to the tableau 7 ,dictate the following expression in terms of Bessel functions

$$\begin{aligned}
1\exp in\theta &= \exp in\theta\\
&\Downarrow\\
\frac{d}{zdz}(\frac{d}{zdz})^{-1}\frac{J_n(z)}{z^n} &= \frac{J_n(z)}{z^n}\\
&\Downarrow\\
\left[\frac{d}{zdz}(\frac{d}{zdz})^{-1}+\frac{d}{zdz}(z\frac{d}{dz})-\frac{d}{zdz}(z\frac{d}{dz})\right]\frac{J_n(z)}{z^n} &= \frac{J_n(z)}{z^n}\\
\left[\frac{d}{zdz}(z\frac{d}{dz}+(\frac{d}{zdz})^{-1})-\frac{d^2}{dz^2}-\frac{1}{z}\frac{d}{zdz}-1\right]\frac{J_n(z)}{z^n} &= 0
\end{aligned}$$

$$
\begin{aligned}
\left[\frac{d^2}{dz^2}+\frac{1}{z}\frac{d}{zdz}+2\frac{d}{zdz}W_z+1\right]\frac{J_n(z)}{z^n} &= 0\\
\left[\frac{d^2}{dz^2}+(2n+1)\frac{d}{zdz}+1\right]\frac{J_n(z)}{z^n} &= 0
\end{aligned}
$$

In the second line we added and substracted the quantity $\frac{d}{zdz}(z\frac{d}{dz})$ and grouped terms such as to show the winding operator W_z which acts on $\frac{J_n(z)}{z^n}$by singling out the index n.The result is the Bessel differential equation .The proof of the second example is as follows

$$
\begin{aligned}
-i\frac{d}{d\theta}\exp i\lambda\theta &= \lambda\exp i\lambda\theta\\
&\Downarrow\\
W_z\frac{J_n(z)}{z^n} &= \lambda\frac{J_n(z)}{z^n}\\
-\frac{1}{2}(z\frac{\partial}{dz}+(\frac{\partial}{zdz})^{-1})\frac{J_n(z)}{z^n} &= \lambda\frac{J_n(z)}{z^n}\\
z^2\frac{J_{\lambda+1}(z)}{z^{\lambda+1}}+\frac{J_{\lambda-1}(z)}{z^{\lambda-1}} &= 2\lambda\frac{J_\lambda(z)}{z^\lambda}\\
z(J_{\lambda+1}(z)+J_{\lambda-1}(z)) &= 2\lambda J_\lambda(z)
\end{aligned}
$$

The last formula is a well known recursion formula .

4 A new property of Bessel functions

The CR$^{‘s}$gave us among various known recursions formulas or differential equations a very specific and new formula

$$
\frac{J_{n+\lambda}(z)}{z^{n+\lambda}}=\exp(-\lambda\sum_{m\in Z/(0)}\frac{1}{m}(\frac{d}{zdz})^m)\frac{J_n(z)}{z^n} \tag{8}
$$

This formula has been tested using different methods.It has been shown to be true by direct analytical computations in . [5] and by mapping the integer order Bessel function differential equation to the real order one in a paper yet to be submitted .On the other hand such mapping has been successfully applied to Neumman$^{‘s}$and Hankel functions as well with a formula similar to the above and when applied to Polynomials such as Hermite and Laguerre it gives deformed versions of these polynomials which are generalization of the older one with certain shared properties .

5 Outlooks

In studying the topological action b_1 on the punctured plane we pull out all the topological invariants realized in the r as well as the θ variables

.As a consequence this leads us to CR$^{`s}$ between exponentials and Bessel functions.In this specific case CR$^{`s}$ could have led us to discover the well known formula $\frac{J_n}{z^n} \propto \int_{unit-circle}(2\tau)^{-n}\exp(\tau-\frac{z^2}{4\tau})\,\frac{d\tau}{\tau}$ which summarize these rules (the exponential in the integrand has an essential singularity at the origin,hence the punctured plane is singled out ,the map $unit-circle \rightarrow \tau^{-n}$ maps the circle to the n-loop and finally any deformation (*homotopy*) of the integration path is allowed as this is a property of the complex plane).The above analysis is however more promising if applied to more general actions b_i on manifolds of richer cohomology $\mathrm{H}_r(\mathrm{M})\times\mathrm{H}^r(\mathrm{M})$ .In this more general case new topological invariants will show up together with the functions they act on and new CR$^{`s}$ between these new functions.

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