A NOTE ON NONCOMMUTATIVE AND FALSE NONCOMMUTATIVE SPACES

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ABSTRACT. We show that the algebra of functions on noncommutative space allows two different representations. One is describing the genuine noncommutative space, while another one can be rewritten in commutative form by a redefinition of generators.

Noncommutative geometry (for a recent review see e.g. [1]) plays an important rôle in both string theory, since it provides a tool for description of brane dynamics. There is a wide activity and literature in this field, so we refer reader to the appropriate reviews, e.g. [2] and references therein/thereon.

The flat noncommutative space is given by coordinates x^μ, $\mu = 1, \ldots, D$, satisfying the noncommutativity relations,

$$[x^\mu, x^\nu] = \mathrm{i}\theta^{\mu\nu}, \tag{1}$$

where for simplicity we assume that the matrix formed of $\theta^{\mu\nu}$ is nondegenerate (therefore the space-time dimensionality is even), the generalisation to a degenerate $\theta^{\mu\nu}$ being also possible. These coordinate can be viewed as generators of algebra of operators acting on a Hilbert space $\mathcal{H}$. The properties of the representation of algebra (1) are less discussed in the physicist's literature, so we hope that this note may serve at least for the pedagogical purposes. We are going to show that the above algebra can both generate an irreducible representation which is the input of a genuine noncommutative space or it be incorporated in a larger algebra, in the last case the noncommutativity is fake and can be eliminated by a simple (but nonlocal) shift of coordinates.

In addition to x^μ satisfying eq. (1), consider also the translation operators p_μ, having canonical commutators with x^μ,

$$[p_\mu, x^\nu] = -\mathrm{i}\delta^\nu_\mu. \tag{2}$$

One can observe, that the quantity $\pi_\mu = p_\mu + \theta^{-1}_{\mu\nu} x^\nu$, where $\theta^{-1}_{\mu\nu}$ is the inverse matrix to $\theta_{\mu\nu}$, commute with all the x^μ,

$$[\pi_\mu, x^\nu] = [p_\mu, x^\nu] + \theta^{-1}_{\mu\alpha}[x^\alpha, x^\nu] = 0 \tag{3}$$

From now, there are (at least) two possibilities: depending on if the representation of algebra generated by x's alone is irreducible or not on the Hilbert space $\mathcal{H}$. In the first case, one immediately has (by the virtue of the Schur's lemma), that $\pi_\mu = 0$. Thus, one has the translation operators p_μ expressed in terms of x^μ, the only independent operators,

$$p_\mu = -\mathrm{i}\theta^{-1}_{\mu\nu} x^\nu, \tag{4}$$

which leads to the commutators,

$$[p_\mu, p_\nu] = -\mathrm{i}\theta^{-1}_{\mu\nu}. \tag{5}$$

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This is the standard D-dimensional noncommutative plane. In this case one can pass through the Lorentz transformation to the coordinates z^m and $\bar{z}_{\bar{m}}$, satisfying,

$$[\bar{z}_m, z_n] = -\mathrm{i}\theta^{(m)}\delta_{mn}, \tag{6}$$

while the matrix $\theta^{\mu\nu}$ is brought to the block diagonal form with $D/2$ antisymmetric (2×2)-blocks, $m = 1, \dots, D/2$,

$$\begin{pmatrix} 0 & \theta_{(m)} \\ -\theta_{(m)} & 0 \end{pmatrix}$$

There no sum over m in eq. (6) is assumed.

This is represented as the *standard Heisenberg algebra in $D/2$ dimensions* with planckian constant depending on the direction.

Consider now a different situation. Let this turn x^μ not to generate an irreducible representation by themselves, but only together with p_μ.

In this case, in spite of commuting with all x^μ, the quantity π_μ, should not vanish unless it commutes also with all p_μ. The last happens when the commutators of p_μ satisfy eq. (5). In this case, all $\pi_\mu = 0$, and one returns back to the previous situation. Consider the case when p_μ has a generic c-number commutator,

$$[p_\mu, p_\nu] = -\mathrm{i}B_{\mu\nu}, \tag{7}$$

where $B_{\mu\nu}$ is an antisymmetric matrix.

It is not difficult to see, that the criterium for the existence of a nontrivial π_μ is,

$$r \equiv \mathrm{rank}(B_{\mu\nu} - \theta^{-1}_{\mu\nu}) > 0, \tag{8}$$

r giving the number of independent operators p_μ, the maximal case, when p_μ are all independent, being when $\det(B - \theta^{-1}) \neq 0$. In this case, one has an usual *D-dimensional Heisenberg algebra* generated by p_μ and q^μ (compare with $D/2$-dimensional one in the previous case). One can again pass to the canonical variables, P_μ and X^μ which have the standard commutators,

$$[P_\mu, X^\nu] = -\mathrm{i}\delta^\nu_\mu, \qquad [P, P] = [X, X] = 0, \tag{9}$$

by a shift,

$$x^\mu \to X^\mu = x^\mu + \xi^{\mu\alpha}p_\alpha, \tag{10}$$

$$p_\mu \to P_\mu = p_\mu + \zeta_{\mu\alpha}x^\alpha, \tag{11}$$

where $\xi^{\mu\nu}$ and $\zeta_{\mu\nu}$ satisfy the following equations,

$$\theta^{\mu\nu} - 2\xi^{[\mu\nu]} - \xi^{\mu\alpha}B_{\alpha\beta}\xi^{\nu\beta} = 0, \tag{12}$$

$$-B_{\mu\nu} + 2\zeta_{[\mu\nu]} + \zeta_{\mu\alpha}\theta^{\alpha\beta}\zeta_{\nu\beta} = 0, \tag{13}$$

$$\xi^{\mu\alpha}B_{\alpha\nu} + \zeta_{\mu\alpha}\theta^{\alpha\nu} + \zeta_{\mu\alpha}\xi^{\nu\alpha} = 0, \tag{14}$$

"[...]" denotes the antisymmetric part of the matrix. In what follows we will not try to analyse the solutions to eqs. (12-14), but only point that the existence of solution follows from the Darboux theorem. In the case when $B_{\mu\nu} = 0$, the above equations greatly simplify and one has,

$$X^\mu = x^\mu + \frac{1}{2}\theta^{\mu\nu}p_\nu, \tag{15}$$

$$P_\mu = p_\mu. \tag{16}$$

As a result, we have a *commutative* space generated by X^μ common with vector algebra on it generated by P_μ. However, from the point of view of the Quantum Field Theory the shift $x \to X$ and $p \to P$ produce a nonlocal field redefinitions containing an infinite number of derivatives.

Finally, let us note that the last situation is common (up to interchange $p \leftrightarrow x$) in usual mechanics. In the presence of a constant electromagnetic field the translation operators p_μ have a nontrivial commutator, like in eq. (7), where $B_{\mu\nu}$ is the field strength. This, however, does not lead to any noncommutativity of the space-time.

A more appropriate example is given by a model with higher derivative terms considered in [3], where it was shown that a noncommutativity removable by transformation similar to (15) arises.

Discussions. Let us summarise the results so far obtained. We considered the algebra of operators x^μ and p_μ satisfying commutator relations (1,2) and (7). We required this algebra to have irreducible representation on the Hilbert space $\mathcal{H}$. We obtained that in the case when $B = \theta^{-1}$ this is representation of the $D/2$-dimensional Heisenberg algebra and this is “genuine” noncommutative space, while in the case when $\det(B - \theta) \neq 0$ it is one of D-dimensional Heisenberg algebra and it is reduced, although by nonlocal transformation, to the usual commutative space.

Using the equivalence of Heisenberg algebra representations in different dimensions [4], which falls in general context of background independence of noncommutative theories [5], one may conclude that gauge models based on these representations are also equivalent. We think that this equivalence should be provided by some analog of the Seiberg–Witten map [2] when the noncommutativity is treated dynamically as in [6, 7, 8].

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