Conformal Gaussian Approximation

Aníbal Iucci[a] and Carlos M. Naón[a]

April 25, 2020

Abstract

We present an alternative way to determine the unknown parameter associated to a gaussian approximation in a generic two-dimensional model. Instead of the standard variational approach, we propose a procedure based on a quantitative prediction of conformal invariance, valid for systems in the scaling regime, away from criticality. We illustrate our idea by considering, as an example, the sine-Gordon model. Our method gives a good approximation for the soliton mass as function of β.

Pacs: 11.10.Lm
11.10.Kk

[a] Instituto de Física La Plata, Departamento de Física, Facultad de Ciencias Exactas, Universidad Nacional de La Plata. CC 67, 1900 La Plata, Argentina.
emails: iucci@fisica.unlp.edu.ar, naon@fisica.unlp.edu.ar

The so called "self consistent harmonic approximation" (SCHA) is a non-perturbative technique that has been extensively employed in Statistical Mechanics [1][2] and Condensed Matter physics [3][4][5][6] applications. Roughly speaking it amounts to replacing an exact action $S_{\rm true}$ by a trial action $S_{\rm trial}$ that makes the problem tractable. Usually $S_{\rm trial}$ is just a quadratic action that depends on certain unknown parameter Ω that must be determined through some criterion such as the minimization of the free energy of the system. This approximation is intimately related the the "gaussian effective potential" (GEP) [7][8] in Quantum Field Theories (QFT's), a variational approximation to the effective potential which uses a gaussian wave functional depending on some mass parameter as the trial ground state. It also relies on a minimization principle often called "principle of minimal sensitivity" [9] to determine the additional parameter. The main purpose of this Letter is to point out that in two-dimensional problems (1+1 QFT's) there is an alternative way to obtain the quantity Ω. Since, as we shall see, this method is based on Conformal Field Theory (CFT) [10], we call it "conformal gaussian approximation" (CGA). In view of the interesting physics that is currently under investigation in low-dimensional systems (organic conductors [11], charge transfer salts [12], quantum wires [13], edge states in a two-dimensional (2D) electron system in the fractional quantum Hall regime [14], Carbon Nanotubes (CNT) [15], Luttinger-like systems in 2D high temperature superconductors [16]) the dimensional restriction in the validity of our approach does not diminish its practical interest.

We shall begin by depicting the main features of the standard SCHA. One starts from a partition function

$$\mathcal{Z}_{\rm true} = \int \mathcal{D}\mu e^{-S_{\rm true}} \tag{1}$$

where $\mathcal{D}\mu$ is a generic integration measure. An elementary manipulation leads to

$$\mathcal{Z}_{\rm true} = \frac{\int \mathcal{D}\mu e^{-(S_{\rm true}-S_{\rm trial})}\, e^{-S_{\rm trial}}}{\int \mathcal{D}\mu e^{-S_{\rm trial}}} \int \mathcal{D}\mu e^{-S_{\rm trial}} = \mathcal{Z}_{\rm trial} \left\langle e^{-(S_{\rm true}-S_{\rm trial})} \right\rangle_{\rm trial} \tag{2}$$

for any trial action $S_{\rm trial}$. Now, by means of the property

$$\left\langle e^{-f} \right\rangle \geq e^{-\langle f \rangle}, \tag{3}$$

for f real, and taking natural logarithm in equation (2), we obtain Feynman's inequality [17]

$$\ln \mathcal{Z}_{\rm true} \geq \ln \mathcal{Z}_{\rm trial} - \langle S_{\rm true} - S_{\rm trial} \rangle_{\rm trial} . \tag{4}$$

In general, $S_{\rm trial}$ depends on some parameters, which are fixed by minimizing the right hand side of the last equation.

From now on, in order to illustrate the procedure, we shall consider scalar 2D models, with action

$$S_{\rm true} = \int \frac{d^2p}{(2\pi)^2}\, \varphi(p) \frac{F(p)}{2} \varphi(-p) + \int d^2x \;\; V(\varphi) \tag{5}$$

where $\varphi(p)$ is a scalar field and $F(p)$ is usually of the form $F(p) \sim p^2$. For simplicity, in this formula we have written the kinetic term in Fourier space but we kept the interaction term in coordinate space.

As the trial action one proposes a quadratic one,

$$S_{\rm trial} = \int \frac{d^2p}{(2\pi)^2} \left[\varphi(p) \frac{F(p)}{2} \varphi(-p) + \frac{\Omega^2}{2} \varphi(p) \varphi(-p) \right] , \tag{6}$$

where Ω is the trial parameter. To proceed with the minimization, we first write the partition function for the trial action in the form

$$\mathcal{Z}_{\rm trial} = \exp \left[-\frac{\mathcal{V}}{2} I_0(\Omega) \right] \tag{7}$$

where $\mathcal{V}$ is the volume (infinite) of the whole space $\int d^2x$, and we have defined

$$I_0(\Omega) = \int \frac{d^2p}{(2\pi)^2} \ln[F(p) + \Omega^2]. \tag{8}$$

$$I_n(\Omega) = \int \frac{d^2p}{(2\pi)^2} \frac{1}{[F(p) + \Omega^2]^n} \tag{9}$$

with the formal properties

$$\frac{dI_0(\Omega)}{d\Omega} = 2\Omega I_1(\Omega) \tag{10}$$

$$\frac{dI_n(\Omega)}{d\Omega} = -2\Omega n I_{n+1}(\Omega). \tag{11}$$

To go further we must specify the potential $V(\varphi)$. Let us specialize the discussion to the sine-Gordon model (SGM), taking

$$V(\varphi) = \frac{\alpha}{\beta^2} \left[1 - \cos(\beta\varphi) \right] , \tag{12}$$

and $F(p) = p^2$.

As it stands the theory is divergent. To take into account the divergences we can implement Coleman's normal order prescription [18] with respect to any given mass-dimensional constant ρ from the beginning [19], by replacing in the lagrangian $V(\varphi)$ by its normal ordered form

$$\mathcal{N}_\rho[V(\varphi)] = \frac{\alpha}{\beta^2}\left[1 - \cos(\beta\varphi)e^{\frac{1}{2}\beta^2 I_1(\rho)}\right] \tag{13}$$

where $\mathcal{N}_\rho[...]$ means the normal ordering form with respect to ρ.

It is now straightforward to compute $\langle S_{\rm true} - S_{\rm trial}\rangle$, by following, for instance, the steps explained in ref. [20]. The result is

$$\langle S_{\rm true} - S_{\rm trial}\rangle_{\rm trial} = \mathcal{V}\left[\frac{\alpha}{\beta^2}\left(1 - e^{-\frac{1}{2}\beta^2[I_1(\Omega)-I_1(\rho)]}\right) - \frac{\Omega^2}{2}I_1(\Omega)\right]. \tag{14}$$

Now inserting equations (7) and (14) in equation (4), and extremizing the r.h.s. with respect to Ω, we finally obtain

$$\Omega^2 - \alpha e^{-\beta^2/2(I_1(\Omega)-I_1(\rho))} = 0. \tag{15}$$

This gap equation allows to extract a finite answer for Ω, depending on the mass parameter ρ (the difference $I_1(\Omega) - I_1(\rho)$ is finite). Note that the value of ρ is completely arbitrary, if one chooses it to be equal to the trial mass Ω, the solution to the equation is

$$\Omega^2 = \alpha. \tag{16}$$

The same result would be obtained if instead of $\rho = \Omega$ one had taken $\rho = \sqrt{\alpha}$.

Let us now present an alternative route to determine Ω. To this end we will exploit a quantitative prediction of conformal invariance for 2D systems in the scaling regime, away from the critical point. Indeed, starting from the so called 'c-theorem' [21] Cardy [22] showed that the value of the conformal anomaly c, which characterizes the model at the critical point, and the second moment of the energy-density correlator in the scaling regime of the non-critical theory are related by

$$\int d^2x\,|x|^2\;\langle\varepsilon(x)\varepsilon(0)\rangle = \frac{c}{3\,\pi\,t^2\,(2-\Delta_\varepsilon)^2}, \tag{17}$$

where ε is the energy-density operator, Δ_ε is its scaling dimension and $t \propto (T - T_c)$ is the coupling constant of the interaction term that takes the system away from criticality. The validity of this formula has been explicitly verified for several models [22] [23] [24] [25]. For the SGM, the energy density operator is given by the cosine term, its conformal dimension is $\Delta_\varepsilon = \beta^2/4\pi$, t is the coupling constant α/β^2 and the associated free bosonic CFT has $c = 1$.

Now we claim that Ω can be determined in a completely different, not variational way, by enforcing the validity of the above conformal identity for the trial action. In other words, we will demand that the following equation holds:

$$\frac{\alpha^2}{\beta^4}\int d^2x\,|x|^2\;\langle\cos\beta\varphi(x)\;\cos\beta\varphi(0)\rangle_{\rm trial} = \frac{1}{3\,\pi\;(2-\frac{\beta^2}{4\pi})^2}, \tag{18}$$

which is to be viewed as an equation for the mass parameter Ω. Of course, if one is interested in comparing the answer given by this formula with the SCHA result, when evaluating the left hand side of (18) one must adopt a regularizing prescription equivalent to the normal ordering implemented in the SCHA calculation. A careful computation leads to the following gap equation:

$$(\frac{\Omega}{\rho})^{2(2-u)} = (\frac{\alpha}{\rho^2})^2\,\frac{3}{32}\,\frac{2-u}{u^2} \tag{19}$$

where we have defined the variable $u = \beta^2/4\pi$ $(0 \leq u < 2)$ and ρ is the normal ordering parameter, as before. We see that, as in the standard SCHA equation (15), one has different answers for different choices of ρ, but in this case, the results obtained for the values $\sqrt{\alpha}$ and Ω are different. In any case one gets a non trivial dependence of Ω on β^2 in contrast with the SCHA. This is interesting if one recalls the physical meaning of mass gaps in the context of the SGM. Indeed, as it is well-known, Dashen, Hasslacher and Neveu (DHN) [26] have computed by semiclassical techniques the mass spectrum for the SGM. It consists of a soliton (associated to the fermion of the Thirring model) with mass

$$M_{sol} = \frac{2-u}{\pi\,u}\,\sqrt{\alpha}, \tag{20}$$

and a sequence of doublet bound states with masses

$$M_N = \frac{2(2-u)}{\pi\,u}\,\sin\left[N\frac{\pi\,u}{2(2-u)}\right]\,\sqrt{\alpha}, \tag{21}$$

with $N = 1, 2, ... < (2-u)/u$. (From this last condition it is easy to see that in order to have N bound states one must have $u < 2/(N+1)$. As a consequence there is no bound state for $u > 1$). Therefore the masses in the

SGM spectrum also depend on u as the CGA mass of equation (19). Thus, in this respect our proposal seems to be able to improve the standard gaussian prediction for the SGM, at least qualitatively. In order to perform a more specific and quantitative discussion let us compare equations (19) and (20) as functions of u. We set $\rho = \sqrt{\alpha}$, which corresponds to the prescription employed by DHN when deriving (20) and (21). The result is shown in Fig. 1 where one can observe a general qualitative analogy between both curves. In particular, for $0.7 \leq u \leq 1$ ($u = 1$ corresponds to the free fermion point of the Thirring model and to the Luther-Emery point in the backscattering model [27]) our CGA prediction is in full agreement with the values of the soliton mass as computed by DHN. We want to stress that for $u = 1$ we get $\Omega/\sqrt{\alpha} = \sqrt{3/32} \approx 0.30$ whereas the exact value given by (20) is $1/\pi \approx 0.31$ (standard SCHA yields, of course, $\Omega/\sqrt{\alpha} = 1$).

To conclude, we have reconsidered the well-known SCHA method in which a comparatively complex action is replaced by a simpler quadratic system depending on a mass parameter Ω which is usually determined through a variational calculation. Taking into account the (1+1)-dimensional case, we have proposed an alternative way for evaluating Ω. Our technique (CGA) is based on a consequence of Zamolodchikov's [21] c-theorem first derived by Cardy [22]. We have illustrated the proposal by considering the sine-Gordon (SGM) model. We showed that for this model CGA gives a quite good prediction for the behavior of the soliton mass as function of β^2 (see equations (19) and (20) and Fig. 1. It would be interesting to test our approach in other models such as the continuum version of the tricritical Ising model, which is described by the second model of the unitary minimal series [28] [29] with central charge $c = 7/10$.

Acknowledgements

This work was supported by the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET) and Universidad Nacional de La Plata (UNLP), Argentina.

References

[1] Y. Saito, Z. Phys. B **32**, 75, (1978).

[2] M. P. A. Fisher and W. Zwerger, Phys. Rev. B **32**, 6190, (1985).

[3] A. O. Gogolin, Phys. Rev. Lett. **71**, 2995, (1993).

[4] R. Egger, H. Grabert, Phys. Rev. Lett. **75**, 3505, (1995).

[5] B.W. Xu, Y.M. Zhang, J. Phys. A **29**, 7349, (1996).

[6] A. Iucci, K. Li, C. Naón, Nucl. Phys. B **601**, 607, (2001).

[7] P. M. Stevenson, Phys. Rev. D **30**, 1712, (1984); **32**, 1389, (1985).

[8] R. Ingermanson, Nucl. Phys. B **266**, 620 (1986).

[9] P. M. Stevenson, Phys. Rev. D **23**, 2916 (1981).
S. K. Kauffman and S. M. Perez, J. Phys. A **17**, 2027 (1984)

[10] D. Boyanovsky, C. M. Naón *An introduction to conformal invariance in quantum field theory and statistical mechanics* Riv. Nuovo Cim. **13** N2 1 (1990).
P. Di Francesco, P. Mathieu, D. Sénéchal *Conformal Field Theory* (New York: Springer-Verlag) (1999).

[11] D. Jerome, H. Schulz, Adv. Phys. **31**, 299 (1982).

[12] A.J. Epstein et al., Phys. Rev. Lett. **47**, 741 (1981).

[13] A. R. Goni et al., Phys. Rev. Lett. **67**, 3298 (1991).

[14] X. G. Wen, Phys. Rev. Lett. **64** 2206 (1990), Phys. Rev. B **41** 12838 (1990), Int. Journal Mod. Phys. B **6** 1711 (1992).

[15] M. Bockrath, D. H. Cobden, J. Lu, A. G. Rinzler, R. E. Smalley, L. Balents, P. L. Mc-Euen, Nature **397** 598 (1999).
Z. Yao, H. W. J. Postma, L. Balents, C. Dekker, Nature **402** 273 (1999).

[16] D. Orgad et al., Phys. Rev. Lett. **86**, 4362, (2001).

[17] R. P. Feynman, "Statistical Mechanics - A set of Lectures", Frontiers in Physics, W.A.Benjamin Inc., (1972).

[18] S. Coleman, Phys. Rev. D **11** 2088 (1975).

[19] C. Naón, Phys. Rev. D **31**, 2035 (1985).
Wen-Fa Lu and Chul Koo Kim, hep-th/0104205.

[20] K. Li and C. Naón, J. Phys. A **31** 7929 (1998).

[21] A. B. Zamolodchikov, JETP Lett. **43** 730, (1986); translated from Pis'ma Zh. Eksp. Teor. Fiz. **43** 565.

[22] J. L. Cardy, Phys. Rev. Lett. **60** 2709, (1988)

[23] J. L. Cardy, J. Phys. A **21**, L797, (1988)

[24] C. M. Naón, J. Phys. A **22**, 2877, (1989)

[25] C. Naón and M. Salvay, hep-th/0104148.

[26] R. Dashen, B. Hasslacher and A. Neveu, Phys. Rev. D **11**, 3424, (1975).
See also R. Rajaraman, Phys. Rep. **21C**, 227, (1975).

[27] A. H. Luther, V. J. Emery, Phys. Rev. Lett. **33**, 589, (1974).

[28] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. B **241**, 333, (1984).

[29] D. Friedan, Z. Qiu and S. Shenker, Phys. Rev. Lett. **52**, 1575, (1984).

Figure caption

Figure 1: Masses in units of $\sqrt{\alpha}$ as functions of u. The dashed line is $M_{sol}/\sqrt{\alpha}$, whereas the filled line represents $\Omega/\sqrt{\alpha}$ as given by CGA.