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Free q-Deformed Relativistic Wave Equations by Representation Theory

Christian Blohmann

Ludwig-Maximilians-Universität München, Sektion Physik
Lehrstuhl Prof. Wess, Theresienstr. 37, D-80333 München

Max-Planck-Institut für Physik, Föhringer Ring 6, D-80805 München

Abstract

In a representation theoretic approach a free q-relativistic wave equation must be such, that the space of solutions is an irreducible representation of the q-Poincaré algebra. It is shown how this requirement uniquely determines the q-wave equations. As examples, the q-Dirac equation (including q-gamma matrices which satisfy a q-Clifford algebra), the q-Weyl equations, and the q-Maxwell equations are computed explicitly.

1 Introduction

Free elementary particles can be identified with irreducible representations of the Poincaré group [1] or, equivalently, the Poincaré algebra. These representations are realized by Wigner spinors, that is, on-shell wave functions with spin indices carrying representations of the little algebras. If we want to describe interactions where energy and momentum can be transfered from one particle onto another, we need to leave the mass shell. And we need a way to describe several particle types and their coupling in one common formalism.

This can be done by introducing Lorentz spinor wave functions. That is, tensor products of the algebra of functions on spacetime with a finite vector space containing the spin degrees of freedom, the whole space carrying a tensor representation of the Lorentz algebra. The additional mathematical structure we need in order to couple two wave functions is provided by the multiplication within the algebra of space functions. However, these Lorentz spinor representations are not irreducible. Therefore, only an irreducible subrepresentation can be the space of physical states. This subrepresentation is conveniently described as kernel of a linear operator $\mathbb{A}$, that is, we demand all physical states ψ to satisfy the wave equation $\mathbb{A}\psi = 0$.

This line of thought relies on the sole assumption that the Poincaré algebra describes the basic symmetry of spacetime. In this work we replace the Poincaré algebra by its q-deformation [2], which describes the basic symmetry of q-deformed spacetime. Then we construct q-wave equations proceeding in exactly the same way as in the undeformed case.

Various other methods to construct q-deformed wave equations have already been proposed, based on q-Clifford algebras [3], q-deformed co-spinors [4], or differential calculi on quantum spaces [5–7], leading to mutually different results. While each approach may be justified in its own right, the situation as a whole is unsatisfactory since it should be possible to determine the wave equations *uniquely* as in the undeformed case [8] without needing any additional mathematical structure besides the q-Poincaré algebra and the basic apparatus of quantum mechanics.

Throughout this article, it is assumed that q is a real number $q > 1$. We will frequently use the abbreviations $\lambda = q - q^{-1}$ and $[2] = q + q^{-1}$. The lower case Greek letters μ, ν, σ, τ denote 4-vector indices running through $\{0, -, +, 3\}$. The upper case Roman letters A, B, C denote 3-vector indices running through $\{-1, 0, +1\} = \{-, 3, +\}$. A very short introduction to the q-Poincaré algebra is given in Appendix A. Some more mathematical background information for this article has been compiled in [9].

2 q-Spinor Wave Functions

2.1 General q-Wave Equations

We seek linear wave equations $\mathbb{A}\psi = 0$, where $\mathbb{A}$ is a linear operator. For $\ker \mathbb{A}$ to be a subrepresentation, the operator must satisfy

$$\mathbb{A}\psi = 0 \quad \Rightarrow \quad \mathbb{A}h\psi = 0 \tag{1}$$

for all q-Poincaré transformations h. Depending on the particle type under consideration we might include charge and parity transformations. $\mathbb{A}$ is not unique since the wave equations for $\mathbb{A}$ and $\mathbb{A}'$ must be considered equivalent as long as their solutions are the same, $\ker(\mathbb{A}) = \ker(\mathbb{A}')$.

Ideally, $\mathbb{A}$ is a projection operator, $\mathbb{A} = \mathbb{P}$, with $\mathbb{P}^2 = \mathbb{P}$, $\mathbb{P}^* = \mathbb{P}$. Condition (1) is then equivalent to

$$[\mathbb{P}, h] = 0 \tag{2}$$

for all q-Poincaré transformations h. Whether the wave equation is written with a projection is a matter of convenience. The Dirac equation is commonly written with such a projection which is determined uniquely (up to complement) by condition (2). For the Maxwell equations a projection can be found but yields a second order differential equation. For this reason, the Maxwell equations are commonly described by a more general operator $\mathbb{A}$, which leads to a first order equation.

2.2 q-Lorentz Spinors

We define a general, single particle q-Lorentz spinor wave function as element of the tensor product $\mathcal{S} \otimes \mathcal{X}$ of a finite vector space $\mathcal{S} = \mathbb{C}^n$ holding the spin degrees of freedom and the space of q-Minkowski space functions $\mathcal{X} = \mathbb{R}_q^{1,3}$ (App. A).

Let $\{e_i\}$ be a basis of $\mathcal{S}$ transforming under a q-Lorentz transformation $h \in \mathcal{H} = \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ as $h \triangleright e_j = e_i\, \rho(h)^i{}_j$, where $\rho : \mathcal{H} \to \mathrm{End}(\mathcal{S})$ is the representation map. Any spinor $\psi \in \mathcal{S} \otimes \mathcal{X}$ can be written as

$$\psi = e_j \otimes \psi^j \,, \tag{3}$$

where j is summed over and the ψ^j are elements of $\mathcal{X}$. The action of $h \in \mathcal{H}$ on a spinor is

$$h\psi = (h_{(1)} \triangleright e_j) \otimes (h_{(2)} \triangleright \psi^j) = e_i \otimes \rho(h_{(1)})^i{}_j (h_{(2)} \triangleright \psi^j) \,. \tag{4}$$

This tells us that, if we want to work directly with the $\mathcal{X}$-valued components ψ^i, the action of h is

$$h\psi^i = \rho(h_{(1)})^i{}_j (h_{(2)} \triangleright \psi^j) \,. \tag{5}$$

Do not confuse the total action $h\psi^i$ with the action of h on each component of ψ^i denoted by $h \triangleright \psi^i$. The transformation of ψ^i can easily be generalized to the case where $\mathcal{S}$ carries a tensor representation of two finite representations, that is, we have spinors with two or more indices

$$h\psi^{ij} = \rho(h_{(1)})^i{}_{i'} \rho'(h_{(2)})^j{}_{j'} (h_{(3)} \triangleright \psi^{i'j'}) \,, \tag{6}$$

where ρ and ρ' are the representation maps of the first and second index, respectively.

Furthermore, we get spinors from the action of tensor operators. Let T^i be a upper left ρ-tensor operator, $\mathrm{ad_L} h \triangleright T^i \equiv h_{(1)} T^i S(h_{(2)}) = \rho(Sh)^i{}_j T^j$, and $\psi = e_j \otimes \psi^j$ a ρ'-spinor. Any operator T^i can be written as $T^i = \sum_k A_k^i \otimes B_k^i \in \mathrm{End}(\mathcal{S}) \otimes \mathrm{End}(\mathcal{X})$ such that the action of T^i becomes

$$T^i \psi = e_j \otimes \sum_k \rho(A_k^i)^j{}_{j'} B_k^i \triangleright \psi^{j'} = e_j \otimes (T^i \psi^j) =: e_j \otimes \phi^{ij} \,. \tag{7}$$

How does this new array of wave functions $\phi^{ij} = T^i \psi^j$ transform under q-Lorentz transformations? Letting h act from the left, we find

$$h\phi^{ij} = \rho(h_{(1)})^j{}_{j'} (h_{(2)} \triangleright \phi^{ij'}) \,, \tag{8}$$

that is, h acts only on the index of the wave functions ψ^j. However, if we transform ϕ^{ij} by transforming ψ^j inside, we find

$$\begin{aligned} T^i(h\psi^j) &= (T^i h)\psi^j = h_{(2)}[\mathrm{ad}_{\mathrm{L}} S^{-1}(h_{(1)}) \triangleright T^i]\psi^j \\ &= \rho(h_{(1)})^i{}_{i'} h_{(2)} T^{i'}\psi^j = \rho(h_{(1)})^i{}_{i'} h_{(2)} \phi^{i'j} \\ &= \rho(h_{(1)})^i{}_{i'} \rho'(h_{(2)})^j{}_{j'} (h_{(3)} \triangleright \phi^{i'j'}) \,. \end{aligned} \tag{9}$$

In other words, if ψ^j is transformed $\phi^{ij} = T^i\psi^j$ transforms as a $\rho \otimes \rho'$-spinor. Note, that for the last calculation the order in the tensor product $\mathcal{S} \otimes \mathcal{X}$ is essential. It would not have worked out as nicely if we had constructed the spinor space as $\mathcal{X} \otimes \mathcal{S}$. Chief examples of this construction would be the gauge term $P^\mu\phi$ of the vector potential A^μ, or the derivatives of the vector potential $P^\mu A^\nu$ used to construct the electromagnetic field strength tensor $F^{\mu\nu}$.

2.3 q-Derivatives

We have not yet said how the momenta P^μ act on q-Lorentz spinors. One might be tempted to assume that they act on the wave function part only, that is, as $1 \otimes P^\mu$ on the tensor product. However, this is not possible because, unless the spin $\mathcal{S} = \mathbb{C}$ is trivial, $1 \otimes P^\mu$ is no 4-vector operator and thus cannot represent 4-momentum. We can turn $1 \otimes P_\mu$ into a 4-vector operator, though, by twisting $1 \otimes P_\mu$ with a universal $\mathcal{R}$-matrix $\mathcal{R} = \mathcal{R}_{[1]} \otimes \mathcal{R}_{[2]}$ of the q-Lorentz algebra,

$$P^\mu := \mathcal{R}^{-1}(1 \otimes P^\mu)\mathcal{R} = (L_+^\Lambda)^\mu{}_\nu \otimes P^\nu \,, \tag{10}$$

with the L-matrix $(L_+^\Lambda)^\mu{}_\nu := \mathcal{R}_{[1]}\Lambda(\mathcal{R}_{[2]})^\mu{}_\nu$. Out of the two universal $\mathcal{R}$-matrices of the q-Lorentz algebra we opt for the antireal $\mathcal{R}_{\mathrm{I}}$, $\mathcal{R}_{\mathrm{I}}^{*\otimes *} = \mathcal{R}_{\mathrm{I}}^{-1}$, because only then the twisting is compatible with the $*$-structure. The momenta act on a ρ-spinor as

$$P^\mu\psi^i = \rho\big((L_{\mathrm{I}+}^\Lambda)^\mu{}_\nu\big)^i{}_j \,(P^\nu \triangleright \psi^j) \,, \tag{11}$$

where the L-matrix has been calculated in [9]. The action of P^μ on each component of ψ^j can be viewed as derivation within the algebra of q-Minkowski space functions $\mathcal{X}$,

$$\partial^\mu := 1 \otimes \mathrm{i}P^\mu \,. \tag{12}$$

Now we can interpret an operator linear in the momenta as q-differential operator. If $C_\mu = C_\mu \otimes 1$ are operators that act on the spinor indices only,

$$\mathrm{i}\, C_\mu P^\mu = C_\mu\, \rho\big((L_{\mathrm{I}+}^\Lambda)^\mu{}_\nu\big)\partial^\nu = \tilde{C}_\nu \partial^\nu \,, \tag{13}$$

where

$$\tilde{C}_\nu := C_\mu\, \rho\big((L_{\mathrm{I}+}^\Lambda)^\mu{}_\nu\big) \tag{14}$$

such that $\tilde{C}_\nu$ still acts on the spinor index only, while ∂^ν acts componentwise, so the two operators commute $[\tilde{C}_\mu, \partial^\nu] = 0$. We will calculate the transformation

$C_\mu \to \tilde{C}_\mu$ for particular representations below. Finally, we remark that for the mass Casimir we have $P_\mu P^\mu = \mathcal{R}^{-1}(1 \otimes P_\mu P^\mu)\mathcal{R} = 1 \otimes P_\mu P^\mu$, hence, $P_\mu P^\mu = -\partial_\mu \partial^\mu$. This means, that mass irreducibility for a spinor is the same as mass irreducibility for each component of the spinor.

2.4 Conjugate Spinors

One of the effects of using Lorentz spinors is that the underlying representations can no longer be unitary, since there are no unitary finite representations of the non-compact Lorentz algebra — in the q-deformed as well as in undeformed case. However, we can introduce non-degenerate but indefinite bilinear forms playing the role of the scalar product. With respect to these pseudo scalar products the spinors carry $*$-representations, that is, the $*$-operation on the algebra side is the same as the pseudo adjoint on the operator side.

The problem of non-unitarity arises from the finiteness of the spin part $\mathcal{S}$ within the space of spinor wave functions $\mathcal{S} \otimes \mathcal{X}$, so we can assume that the wave function part $\mathcal{X}$ does carry a $*$-representation. It is then sufficient to redefine the scalar product on $\mathcal{S}$ only. Consider a $D^{(j,0)}$-representation of $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ with orthonormal basis $\{e_n\}$ and the canonical scalar product $\langle e_m | e_n \rangle = \delta_{mn}$. We want to define a pseudo scalar product by

$$(e_m|e_n) := A_{mn} \quad \text{such that} \quad (e_m|(g \otimes h) \triangleright e_n) = ((g \otimes h)^* \triangleright e_m|e_n) \tag{15}$$

for any $g \otimes h \in \mathcal{U}_q(\mathrm{sl}_2) \otimes \mathcal{U}_q(\mathrm{sl}_2) \cong \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$. For a pseudo scalar product we must suppose A_{mn} to be a non-degenerate, hermitian, but not necessarily positive definite matrix. Inserting the definition of the pseudo scalar product, the pseudo-unitarity condition (15) reads

$$\begin{aligned} (e_m|(g \otimes h) \triangleright e_n) &= (e_m|\, e_{n'} \rho^j(g)^{n'}{}_n\, \varepsilon(h)) = A_{mn'} \rho^j(g)^{n'}{}_n\, \varepsilon(h) \\ &\overset{!}{=} ((g \otimes h)^* \triangleright e_m|\, e_n) = (e_{m'} \varepsilon(g^*) \rho^j(h^*)^{m'}{}_m|\, e_n) \\ &= A_{m'n}\, \overline{\varepsilon(g^*) \rho^j(h^*)^{m'}{}_m} = A_{m'n}\, \varepsilon(g) \rho^j(h)^m{}_{m'} \,, \end{aligned} \tag{16}$$

where we have used that $(g \otimes h)^* = \mathcal{R}_{21}(h^* \otimes g^*)\mathcal{R}_{21}^{-1}$ [10] and $\varepsilon(\mathcal{R}_{[1]})\mathcal{R}_{[2]} = 1$. Traditionally, the scalar product is not described by a matrix A_{mn} but by introducing a conjugate spinor basis $\{\bar{e}_n\}$ demanding

$$(e_m|e_n) = \langle \bar{e}_m | e_n \rangle \quad \Rightarrow \quad \bar{e}_m = e_{m'} A_{m'm} \,. \tag{17}$$

Using (16) the conjugate basis turns out to transform as

$$\begin{aligned} (g \otimes h) \triangleright \bar{e}_n &= e_{m'} \rho^j(g)^{m'}{}_m \varepsilon(h)\, A_{mn} = e_{m'} A_{m'n'} \varepsilon(g) \rho^j(h)^{n'}{}_n \\ &= \bar{e}_{n'} \varepsilon(g) \rho^j(h)^{n'}{}_n \,, \end{aligned} \tag{18}$$

that is, $\bar{e}_n$ must transform according to a $D^{(0,j)}$-representation. $D^{(j,0)}$ and $D^{(0,j)}$ being inequivalent representations, the conjugate basis $\bar{e}_n$ cannot be expressed as a linear combination of the original basis vectors e_n. In order to allow for a conjugate spinor basis we must consider a representation which contains both, $D^{(j,0)}$ and $D^{(0,j)}$, and thus at least their direct sum $D^{(j,0)} \oplus D^{(0,j)}$ as subrepresentation.

So far, it seems that everything is almost trivially analogous to the undeformed case. It is not. If we consider irreducible representations of mixed chirality, $D^{(i,j)}$, we find that the appearance of the $\mathcal{R}$-matrix in $(g \otimes h)^*$ makes it impossible to define conjugate spinors. It only works for $D^{(j,0)}$, because $\rho^0 = \varepsilon$ and $\varepsilon(\mathcal{R}_{[1]})\mathcal{R}_{[2]} = 1$. Fortunately, we do have conjugate spinors for the most interesting cases: Dirac spinors ($D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$) and the Maxwell tensor ($D^{(1,0)} \oplus D^{(0,1)}$). For these cases everything is analogous to the undeformed case.

Let us consider a $D^{(j,0)} \oplus D^{(0,j)}$ representation with basis $\{e^{\mathrm{L}}_n\}$ for the left chiral subrepresentation $D^{(j,0)}$ and the basis $\{e^{\mathrm{R}}_n\}$ for $D^{(0,j)}$. We define the conjugate basis by $\overline{e^{\mathrm{L}}_n} := e^{\mathrm{R}}_n$ and $\overline{e^{\mathrm{R}}_n} = e^{\mathrm{L}}_n$. Let us call $\mathcal{P}$ the parity operator that exchanges the left and right chiral part. Its matrix representation with respect to the basis $\{e^{\mathrm{L}}_n, e^{\mathrm{R}}_n\}$ is

$$\mathcal{P}_{mn} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{19}$$

where 1 is the $(2j+1)$-dimensional unit matrix. This is the matrix that represents our new pseudo scalar product as a bilinear form. The pseudo Hermitian conjugate of some operator A can now be written as

$$j(A) := \mathcal{P} A^{\dagger} \mathcal{P}, \tag{20}$$

which is an involution because $\mathcal{P}^{\dagger} = \mathcal{P}^{-1}$ and an algebra anti-homomorphism because $\mathcal{P} = \mathcal{P}^{-1}$.

We apply this result to the whole space of spinor wave functions $\mathcal{S} \otimes \mathcal{X}$. Let us assume that the scalar product of two wave functions $f, g \in \mathcal{X}$ can be written (at least formally) as some sort of integral $\langle f|g\rangle = \int f^* g$. The pseudo scalar product of two $D^{(j,0)} \oplus D^{(0,j)}$ spinors ψ, ϕ becomes

$$\begin{aligned}(\psi|\phi) = (e_m \otimes \psi^m | e_n \otimes \phi^n) &= (e_m|e_n)\langle \psi^m|\phi^n\rangle \\ &= \int (\psi^m)^* \mathcal{P}_{mn} \phi^n = \int \bar{\psi}^n \phi^n ,\end{aligned} \tag{21}$$

with the conjugate spinor wave function defined as

$$\bar{\psi}^n := (\psi^m)^* \mathcal{P}_{mn} . \tag{22}$$

In summary, we have convinced ourselves that in the case of $D^{(j,0)} \oplus D^{(0,j)}$ representations the conjugation of spinors, of spinor wave functions, and of operators works exactly as in the undeformed case.

3 The q-Dirac Equation

3.1 The q-Dirac Equation in the Rest Frame

In this section we consider q-Dirac spinors $\psi = e_i \otimes \psi^i$ with the spin part transforming according to a $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ representation [9]. We want to write the q-Dirac equation as expression which involves momenta only to first order, corresponding to a first order differential equation

$$\mathbb{P}\psi := \frac{1}{2m}(m + \gamma_\mu P^\mu)\psi = 0\,, \tag{23}$$

with the γ_μ being some operators acting on ψ^i. We can already say that γ_μ must be a left 4-vector operator. If it were not, $\gamma_\mu P^\mu$ would not be scalar and, hence, would not commute with the q-Lorentz transformations as required in Eq. (2).

We consider here a massive q-Dirac spinor representation, so there is a rest frame [11], that is, a set of states ψ^i which the momenta act upon as $P^0\psi^i = m\psi^i$, $P^A\psi^i = 0$. We start the search for a projection $\mathbb{P}$ that reduces the q-Dirac representation by computing how it acts on the rest frame, where we have

$$\mathbb{P}_0 = \tfrac{1}{2}(1 + \gamma_0)\,, \tag{24}$$

the zero indicating that this is a projection within the rest frame only. We assume that we can realize the operator γ_0 as 4×4-matrix that acts on the spin degrees of freedom only. This is not unreasonable, for if γ_μ is a set of matrices that form a 4-vector operator in the $D^{(\frac{1}{2},0)} \oplus D^{(0,\frac{1}{2})}$ representation then $\gamma_\mu \otimes 1$ will also be a 4-vector operator in the representation of spinor wave functions. So let us assume we can write $\mathbb{P}_0 = \mathbb{P}_0 \otimes 1$ in block form as $\mathbb{P}_0 = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where A, B, C, D are 2×2-matrices.

The restriction of condition (2) to the rest frame means that $\mathbb{P}_0$ must commute with the little algebra. The little algebra for the massive case is the $\mathcal{U}_q(\mathrm{su}_2)$ subalgebra of rotations [11]. A rotation $l \in \mathcal{U}_q(\mathrm{su}_2)$ is represented by

$$\rho(l) = \begin{pmatrix} \rho^{\frac{1}{2}}(l) & 0 \\ 0 & \rho^{\frac{1}{2}}(l) \end{pmatrix}. \tag{25}$$

Since the $\rho^{\frac{1}{2}}$-representations of the rotations generate all 2×2-matrices (the q-Pauli matrices are a basis), $\mathbb{P}_0$ will only commute with all rotations if A, B, C, D are numbers, that is, complex multiples of the unit matrix.

Furthermore, $\mathbb{P}_0$ has to be a projection operator, $\mathbb{P}_0^2 = \mathbb{P}_0$, $\mathbb{P}_0^\dagger = \mathbb{P}_0$, and, as in the undeformed case, we require it to commute with the parity operator, $[\mathbb{P}_0, \mathcal{P}] = 0$. Together these conditions fix $\mathbb{P}_0$ and hence γ_0 uniquely to

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{26}$$

the same as in the undeformed case.

3.2 The q-Gamma Matrices and the q-Clifford Algebra

If γ_0 is to be a 4-vector operator, we have to define the other gamma matrices as in Eq. (83) by

$$\begin{aligned}
\gamma_- &= \mathrm{ad}_\mathrm{L}(-q^{-\frac{1}{2}}\lambda^{-1}[2]^{\frac{1}{2}}\,c) \triangleright \gamma_0 \\
\gamma_+ &= \mathrm{ad}_\mathrm{L}(q^{\frac{1}{2}}\lambda^{-1}[2]^{\frac{1}{2}}\,b) \triangleright \gamma_0 \\
\gamma_3 &= \mathrm{ad}_\mathrm{L}(\lambda^{-1}\,(d-a)) \triangleright \gamma_0\,,
\end{aligned} \tag{27}$$

where the adjoint action is understood with respect to the q-Dirac representation. In order to compute this explicitly, we have to calculate the representations of the boosts (77) first.

$$\rho(a) = \begin{pmatrix} \rho^{\frac{1}{2}}(K^{\frac{1}{2}}) & 0 \\ 0 & \rho^{\frac{1}{2}}(K^{-\frac{1}{2}}) \end{pmatrix}, \qquad \rho(b) = \begin{pmatrix} 0 & 0 \\ 0 & q^{-\frac{1}{2}}\lambda\rho^{\frac{1}{2}}(K^{-\frac{1}{2}}E) \end{pmatrix} \tag{28a}$$

$$\rho(c) = \begin{pmatrix} -q^{\frac{1}{2}}\lambda\rho^{\frac{1}{2}}(FK^{\frac{1}{2}}) & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho(d) = \begin{pmatrix} \rho^{\frac{1}{2}}(K^{-\frac{1}{2}}) & 0 \\ 0 & \rho^{\frac{1}{2}}(K^{\frac{1}{2}}) \end{pmatrix} \tag{28b}$$

This gives us for example

$$\begin{aligned}
\gamma_+ &= \mathrm{ad}_\mathrm{L}(q^{\frac{1}{2}}\lambda^{-1}[2]^{\frac{1}{2}}\,b) \triangleright \gamma_0 = q^{\frac{1}{2}}\lambda^{-1}[2]^{\frac{1}{2}}[\rho(b)\gamma_0\rho(a) - q\rho(a)\gamma_0\rho(b)] \\
&= [2]^{\frac{1}{2}}\left[\begin{pmatrix} 0 & 0 \\ \rho^{\frac{1}{2}}(K^{-\frac{1}{2}}EK^{\frac{1}{2}}) & 0 \end{pmatrix} - q\begin{pmatrix} 0 & \rho^{\frac{1}{2}}(E) \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 0 & q\,\sigma_+ \\ -q^{-1}\sigma_+ & 0 \end{pmatrix},
\end{aligned} \tag{29}$$

where σ_+ is one of the q-Pauli matrices defined as q-Clebsch-Gordan coefficients or, equivalently, as spin-$\frac{1}{2}$ representation of the angular momentum generators, $\sigma_A = [2]\rho^{\frac{1}{2}}(J_A)$ [9]. The analogous calculations for γ_- and γ_+ yield

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_A = \begin{pmatrix} 0 & q\,\sigma_A \\ -q^{-1}\sigma_A & 0 \end{pmatrix}, \tag{30}$$

where A runs as usual through $\{-,+,3\}$.

This result can be easily generalized to higher spin. All we have to do for a massive $D^{(j,0)} \oplus D^{(0,j)}$-spinor is to replace $\rho^{\frac{1}{2}}$ with ρ^j. The result is higher dimensional γ-matrices

$$\gamma_0^{(j)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_A^{(j)} = [2]\begin{pmatrix} 0 & q\,\rho^j(J_A) \\ -q^{-1}\rho^j(J_A) & 0 \end{pmatrix}. \tag{31}$$

If we want to write the q-Dirac equation as q-differential equation, we need to calculate $\tilde{\gamma}_\mu$ by formula (14). For the q-Pauli matrices we get

$$\sigma_A\,\rho^{(\frac{1}{2},0)}\big((L^\Lambda_{\mathrm{I}+})^A{}_B\big) = q^2\tilde{\sigma}_B\,, \qquad \sigma_A\,\rho^{(0,\frac{1}{2})}\big((L^\Lambda_{\mathrm{I}+})^A{}_B\big) = q^{-2}\tilde{\sigma}_B\,, \tag{32}$$

where we have used a variant of the q-Pauli matrices, defined as $\tilde{\sigma}_A := -[2]\rho^{\frac{1}{2}}(SJ_A)$. Explicitly, these are

$$\tilde{\sigma}_- = [2]^{\frac{1}{2}} \begin{pmatrix} 0 & q^{\frac{1}{2}} \\ 0 & 0 \end{pmatrix}, \qquad \tilde{\sigma}_+ = [2]^{\frac{1}{2}} \begin{pmatrix} 0 & 0 \\ -q^{-\frac{1}{2}} & 0 \end{pmatrix}, \qquad \tilde{\sigma}_3 = \begin{pmatrix} -q^{-1} & 0 \\ 0 & q \end{pmatrix} \tag{33}$$

with respect to the $\{-,+\}$ basis. For the transformed q-gamma matrices we obtain

$$\tilde{\gamma}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \tilde{\gamma}_A = \begin{pmatrix} 0 & q^{-1}\tilde{\sigma}_A \\ -q\tilde{\sigma}_A & 0 \end{pmatrix}, \tag{34}$$

and the q-Dirac equation written as q-differential equation becomes

$$(m - \mathrm{i}\tilde{\gamma}_\mu \partial^\mu)\psi = 0\,. \tag{35}$$

After lengthy calculations we find that the gamma matrices satisfy the relations

$$\tilde{\gamma}_\sigma \tilde{\gamma}_\tau = \eta_{\tau\sigma} + \tilde{\gamma}_\mu \tilde{\gamma}_\nu {\mathbb{P}_{\mathrm{A}}}^{\nu\mu}{}_{\tau\sigma} \quad \Leftrightarrow \quad \tilde{\gamma}_\mu \tilde{\gamma}_\nu {\mathbb{P}_{\mathrm{S}}}^{\nu\mu}{}_{\sigma\tau} = \eta_{\sigma\tau}\,, \tag{36}$$

where $\mathbb{P}_{\mathrm{A}}$ is the antisymmetrizer and $\mathbb{P}_{\mathrm{S}} = 1 - \mathbb{P}_{\mathrm{A}}$ is the symmetrizer of the Clebsch-Gordan series (78). This is the q-deformation of the Clifford algebra relations, from which now follows that the square of q-Dirac operator is indeed the mass Casimir,

$$(\tilde{\gamma}_\mu \partial^\mu)^2 = \partial_\mu \partial^\mu = -P_\mu P^\mu\,. \tag{37}$$

As in the undeformed case we conclude that a solution ψ to the q-Dirac equation satisfies automatically the mass shell condition $P_\mu P^\mu \psi = m^2\psi$, and that $\mathbb{P} = \frac{1}{2m}(m + \gamma_\mu P^\mu)$ really is a projection operator.

One could have started directly from relations (36) trying to find matrices that satisfy them [3]. This approach has a number of disadvantages: a) It is computationally much more cumbersome than boosting γ_0. b) The result is not unique, that is, we would get many solutions to the q-Clifford algebra not knowing which representations they belong to. c) Having determined a solution $\tilde{\gamma}_\mu$, the covariance of the q-Dirac equation remains unclear as $\tilde{\gamma}_\mu$ cannot be a 4-vector operator.

3.3 The Zero Mass Limit and the q-Weyl Equations

The zero mass limit of the q-Dirac equation is formally

$$\mathbb{A}\psi := \gamma_\mu P^\mu \psi = 0\,, \tag{38}$$

where $\mathbb{A}$ is no longer a projection operator. The wave equation is now decoupled into two independent equations for a left handed $D^{(\frac{1}{2},0)}$-spinor ψ_{L} and a right handed $D^{(0,\frac{1}{2})}$-spinor ψ_{R},

$$\sigma_A P^A \psi_{\mathrm{L}} = q^{-1} P^0 \psi_{\mathrm{L}}\,, \qquad \sigma_A P^A \psi_{\mathrm{R}} = -q P^0 \psi_{\mathrm{R}}\,, \tag{39}$$

the q-Weyl equations for massless left and right handed spin-$\frac{1}{2}$ particles. Written as q-differential equation they become

$$\tilde{\sigma}_A \partial^A \psi_{\mathrm{L}} = -q \partial^0 \psi_{\mathrm{L}} , \qquad \tilde{\sigma}_A \partial^A \psi_{\mathrm{R}} = q^{-1} \partial^0 \psi_{\mathrm{R}} . \tag{40}$$

The operator $\mathbb{A}$ inherits property (1) from $\mathbb{P}$, so $\mathbb{A}\psi = 0$ is a viable wave equation. On the massless momentum eigenspace [11] where $(P_0, P_-, P_+, P_3) = (k, 0, 0, k)$, $\mathbb{A}$ acts as $\mathbb{A}_0 = k\left(\begin{smallmatrix} 0 & 1-q\sigma_3 \\ 1+q^{-1}\sigma_3 & 0 \end{smallmatrix}\right)$. The kernel of this operator is 2-dimensional, the solution states corresponding to helicity $\pm\frac{1}{2}$.

If we generalize these considerations to higher spin $D^{(j,0)} \otimes D^{(0,j)}$ Dirac type spinors, we find that the corresponding operator $\mathbb{A}$ has zero kernel, so the space of solutions is trivial. This applies in particular to q-Maxwell spinors. Therefore, we need a different approach to find the q-Maxwell equations — in complete analogy to the undeformed case [8].

4 The q-Maxwell Equations

4.1 The q-Maxwell Equations in the Momentum Eigenspaces

In this section we consider massless $D^{(1,0)} \oplus D^{(0,1)}$ spinors. According to the Clebsch-Gordan series (78) this type of spinor is equivalent to an antisymmetric tensor $F^{\mu\nu}$ with two 4-vector indices. These are the types of spinors commonly used to describe the electromagnetic field.

We start our calculations in the massless momentum eigenspace with momentum eigenvalues $(P_0, P_-, P_+, P_3) = (k, 0, 0, k)$ for some real parameter k. It has been shown in [11] that this eigenspace is invariant under the little algebra generated by the group-like generator of q-rotations around the z-axis K and

$$N_- := q^{\frac{1}{2}} [2]^{\frac{1}{2}} ac , \qquad N_+ := q^{\frac{1}{2}} [2]^{\frac{1}{2}} bd , \qquad N_3 := 1 + [2] bc . \tag{41}$$

The irreducible $*$-representations of this algebra are one-dimensional, given by $K = \kappa = N_3$ and $N_\pm = 0$ for real κ. Within the momentum eigenspace the little algebra acts only on the spinor index, here, by the $D^{(1,0)} \oplus D^{(0,1)}$ matrix representation

$$\begin{gathered} K = \begin{pmatrix} \rho^1(K) & 0 \\ 0 & \rho^1(K) \end{pmatrix} , \quad N_- = -q[2] \begin{pmatrix} \rho^1(J_-) & 0 \\ 0 & 0 \end{pmatrix} \\ N_+ = -q^{-1}[2] \begin{pmatrix} 0 & 0 \\ 0 & \rho^1(J_+) \end{pmatrix} , \quad N_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{gathered} \tag{42}$$

where ρ^1 is the vector representation of $\mathcal{U}_q(\mathrm{su}_2)$. We seek an operator $\mathbb{P}_0 = \mathbb{P}_0 \otimes 1$ that projects onto an irreducible subrepresentation of the momentum eigenspace.

As before, we write it in block form as $\mathbb{P}_0 = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)$, where A, B, C, D are 3×3-matrices. We must have $\mathbb{P}_0^\dagger = \mathbb{P}_0$, so A and D must be Hermitian matrices and $C = B^\dagger$. Within an irreducible representation of the little algebra we have $N_\pm = 0$, so we must demand $N_\pm \mathbb{P}_0 = 0$. This leads to the conditions

$$\rho^1(J_-)\, A = 0\,, \qquad \rho^1(J_+)\, D = 0\,, \qquad \rho^1(J_-)\, B = 0\,, \qquad \rho^1(J_+)\, B^\dagger = 0\,. \tag{43}$$

To satisfy these conditions A, B, and D must be of the form

$$A = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 0 & \beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta \end{pmatrix}, \tag{44}$$

for α, δ real and β complex. Furthermore, $\mathbb{P}_0$ must project onto an eigenvector of K. From this it follows that $\beta = 0$ and either $\alpha = 1$, $\delta = 0$ or $\alpha = 0$, $\delta = 1$. To summarize, there are two possible projections

$$\mathbb{P}_\mathrm{L} = \begin{pmatrix} 1 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix}, \qquad \mathbb{P}_\mathrm{R} = \begin{pmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & 1 \end{pmatrix} \tag{45}$$

projecting each on a irreducible one-dimensional representation of the little algebra. The image of $\mathbb{P}_\mathrm{L}$ is part of the left handed $D^{(1,0)}$ component while $\mathbb{P}_\mathrm{R}$ projects onto the right handed $D^{(0,1)}$ component of the spinor. Physically, this corresponds to left and right handed circular waves. We want to allow for parity transformations exchanging the left and right handed parts, so we need both parts $\mathbb{P}_0 = \mathbb{P}_\mathrm{L} + \mathbb{P}_\mathrm{R}$. With the parity transformation included, the two dimensional space which $\mathbb{P}_0$ projects onto is irreducible.

4.2 Computing the q-Maxwell Equations

We want to write the q-Maxwell equations in the form of a first order differential equation

$$\mathbb{A}\psi := C_\mu P^\mu\, \psi = 0\,, \tag{46}$$

hoping that again the operators C_μ can be chosen to act on the spinor index only, $C_\mu = C_\mu \otimes 1$. Recall from the last section, that as long as we do not include parity transformations, we must have two independent equations for the right and the left handed part of the spinor, ψ_L carrying a $D^{(1,0)}$ representation and ψ_R carrying a $D^{(0,1)}$ representation

$$\mathbb{A}_\mathrm{L}\psi_\mathrm{L} = 0\,, \qquad \mathbb{A}_\mathrm{R}\psi_\mathrm{R} = 0\,. \tag{47}$$

For condition (1) it would be sufficient (but not necessary) if $\mathbb{A}_\mathrm{L}$, $\mathbb{A}_\mathrm{R}$ were scalar operators. Let us try to choose $\mathbb{A}_\mathrm{L} = C^\mathrm{L}_\mu P^\mu$ and $\mathbb{A}_\mathrm{R} = C^\mathrm{R}_\mu P^\mu$ to be scalars with respect to rotations. For this to be possible C_0^{L}, C_0^{R} must be scalars with respect to rotations while C_A^{L}, C_A^{R} must transform as 3-vectors. The only scalar operators within the D^1-representation of $\mathcal{U}_q(\mathrm{su}_2)$ are multiples of the unit matrix, while every 3-vector operator is proportional to $\rho^1(J_A)$. Hence, up to an overall constant factor our wave equations would be written as

$$\big(P^0 + \alpha_{\mathrm{L}}\, \rho^1(J_A) P^A\big)\psi_{\mathrm{L}} = 0\,, \qquad \big(P^0 + \alpha_{\mathrm{R}}\, \rho^1(J_A) P^A\big)\psi_{\mathrm{R}} = 0\,, \tag{48}$$

where α_{L}, α_{R} are constants. We determine these constants by considering the wave equations in the momentum eigenspace,

$$\big(1 + \alpha_{\mathrm{L}}\, \rho^1(J_3)\big)\psi_{\mathrm{L}} = 0\,, \qquad \big(1 + \alpha_{\mathrm{R}}\, \rho^1(J_3)\big)\psi_{\mathrm{R}} = 0\,. \tag{49}$$

The space of solutions of each of these equations must equal the image of the projections $\mathbb{P}_{\mathrm{L}}$ and $\mathbb{P}_{\mathrm{R}}$, respectively. This requirement fixes the constants to $\alpha_{\mathrm{L}} = q^{-1}$ and $\alpha_{\mathrm{R}} = -q$.

Although this determines our candidate for the q-Maxwell equations, condition (1) has yet to be checked for the boosts. Let ψ_0 be an element of the momentum eigenspace where $(P_0, P_-, P_+, P_3) = (k, 0, 0, k) =: (p_\mu)$. Using the commutation relations between boosts and momentum generators we find [9]

$$P_\mu(a\psi_0) = q^{-1} p_\mu(a\psi_0)\,, \qquad P_\mu(b\psi_0) = q^{-1} p_\mu(b\psi_0) \tag{50a}$$
$$P_\mu(c\psi_0) = q p_\mu(c\psi_0)\,, \qquad P_\mu(d\psi_0) = q p_\mu(d\psi_0)\,. \tag{50b}$$

By induction it follows, that for any monomial in the boosts, $h = a^i b^j c^k d^l$, we have $P_\mu(h\psi_0) = q^{k+l-i-j} p_\mu(h\psi_0)$. Thus, for $\psi := h\psi_0$, the wave equation (46) takes the form

$$(C_0 - C_3)\psi = 0\,. \tag{51}$$

Looking separately at the left and right handed part of $\psi = \psi_{\mathrm{L}} + \psi_{\mathrm{R}}$ this equation writes out

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & q^{-2} & 0 \\ 0 & 0 & q^{-1}[2] \end{pmatrix} \begin{pmatrix} \psi_{\mathrm{L}}^- \\ \psi_{\mathrm{L}}^3 \\ \psi_{\mathrm{L}}^+ \end{pmatrix} = 0\,, \qquad \begin{pmatrix} q[2] & 0 & 0 \\ 0 & q^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_{\mathrm{R}}^- \\ \psi_{\mathrm{R}}^3 \\ \psi_{\mathrm{R}}^+ \end{pmatrix} = 0\,, \tag{52}$$

which is equivalent to $\psi_{\mathrm{L}}^3 = \psi_{\mathrm{L}}^+ = 0$ and $\psi_{\mathrm{R}}^- = \psi_{\mathrm{R}}^3 = 0$. If we now have a solution of Eq. (51), that is, a spinor ψ whose only non-vanishing components are ψ_{L}^- and ψ_{R}^+, could it happen that by boosting it gets other non-vanishing components, thus turning a solution into a non-solution? The answer to this question is no. We exemplify this, applying formula (5) for the action of the boost generator c

on a left handed spinor,

$$\begin{aligned}
c\,\psi_{\rm L}^A &= \rho^{(1,0)}(c_{(1)})^A{}_{A'}\big(c_{(2)} \triangleright \psi_{\rm L}^{A'}\big) \\
&= \rho^{(1,0)}(c)^A{}_{A'}\big(a \triangleright \psi_{\rm L}^{A'}\big) + \rho^{(1,0)}(d)^A{}_{A'}\big(c \triangleright \psi_{\rm L}^{A'}\big) \\
&= -q^{\frac12}\lambda\rho^1(FK^{\frac12})^A{}_{A'}\big(a \triangleright \psi_{\rm L}^{A'}\big) + \rho^1(K^{-\frac12})^A{}_{A'}\big(c \triangleright \psi_{\rm L}^{A'}\big) \\
&= -q^{\frac12}\lambda[2]^{\frac12}\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}\begin{pmatrix}a \triangleright \psi_{\rm L}^-\\ a \triangleright \psi_{\rm L}^3\\ a \triangleright \psi_{\rm L}^+\end{pmatrix} + \begin{pmatrix}q&0&0\\0&1&0\\0&0&q^{-1}\end{pmatrix}\begin{pmatrix}c \triangleright \psi_{\rm L}^-\\ c \triangleright \psi_{\rm L}^3\\ c \triangleright \psi_{\rm L}^+\end{pmatrix} \\
&= \begin{pmatrix}-q^{\frac12}\lambda[2]^{\frac12} a \triangleright \psi_{\rm L}^3 + q\,c \triangleright \psi_{\rm L}^-\\ -q^{\frac12}\lambda[2]^{\frac12} a \triangleright \psi_{\rm L}^+ + c \triangleright \psi_{\rm L}^3\\ q^{-1} c \triangleright \psi_{\rm L}^+\end{pmatrix},
\end{aligned} \tag{53}$$

which clearly shows that, if $\psi_{\rm L}^3$ and $\psi_{\rm L}^+$ vanish, so do $c\psi_{\rm L}^3$ and $c\psi_{\rm L}^+$. Similar calculations can be done for the other boost generators and right handed spinors.

By induction we conclude, that if ψ_0 is a solution of Eq. (51) and $h = a^i b^j c^k d^l$ is a monomial in the boosts, the spinor $\psi = h\psi_0$ will be a solution, as well. The algebra of all boosts, $SU_q(2)^{\rm op}$, is generated as linear space by the monomials, thus, $h\psi_0$ is a solution for any boost $h \in SU_q(2)^{\rm op}$. Since furthermore every q-Lorentz transformation can be written as a sum of products of rotations and boost, $h\psi_0$ is a solution for *any* q-Lorentz transformation h. We assume that the space of solutions, $\ker \mathbb{A}$, is an irreducible representation. This means in particular that the q-Lorentz algebra acts transitively on $\ker \mathbb{A}$, so any solution can be written as $h\psi_0$. Hence, the wave equations

$$\rho^1(J_A)P^A\psi_{\rm L} = -qP_0\psi_{\rm L}\,, \qquad \rho^1(J_A)P^A\psi_{\rm R} = q^{-1}P_0\psi_{\rm R} \tag{54}$$

do indeed satisfy property (1).

Now we want to write these equations as q-differential equations $\tilde C_\mu \partial^\mu \psi = 0$, where $\tilde C_\mu$ is defined in Eq. (14). After lengthy calculations we get

$$\begin{aligned}
\rho^1(J_{A'})^B{}_{C'}\,\rho^{(1,0)}\big((L^\Lambda_{\rm I+})^{A'}{}_A\big)^{C'}{}_C &= -q^2\varepsilon_C{}^B{}_A \\
\rho^1(J_{A'})^B{}_{C'}\,\rho^{(0,1)}\big((L^\Lambda_{\rm I+})^{A'}{}_A\big)^{C'}{}_C &= -q^{-2}\varepsilon_C{}^B{}_A\,,
\end{aligned} \tag{55}$$

so the wave equations (54) can be written as

$$\vec\partial \times \vec\psi_{\rm L} = {\rm i}q^{-1}\partial_0\vec\psi_{\rm L}\,, \qquad \vec\partial \times \vec\psi_{\rm R} = -{\rm i}q\,\partial_0\vec\psi_{\rm R}\,, \tag{56}$$

where $\vec\psi_{\rm R} = (\psi_{\rm R}^A)$, $\vec\psi_{\rm L} = (\psi_{\rm L}^A)$ and where the cross product is defined in Eq. (75). A spinor $\vec\psi_{\rm L}$ which is a solution to this equation must also satisfy the mass zero condition. Using identities (76) for the cross product, commutation relations (79) of the derivations, $\vec\partial \times \vec\partial = -{\rm i}\lambda\partial_0\vec\partial$, and the wave equation (56), we rewrite the mass zero condition as

$$
\begin{aligned}
0 &= \partial_\mu \partial^\mu \vec{\psi}_{\mathrm{L}} = (\partial_0^2 - \vec{\partial}\cdot\vec{\partial})\vec{\psi}_{\mathrm{L}} \\
&= \partial_0^2 \vec{\psi}_{\mathrm{L}} - (\vec{\partial}\times\vec{\partial})\times\vec{\psi}_{\mathrm{L}} + \vec{\partial}\times(\vec{\partial}\times\vec{\psi}_{\mathrm{L}}) - \vec{\partial}(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}) \\
&= \partial_0^2 \vec{\psi}_{\mathrm{L}} + \mathrm{i}\lambda\partial_0(\vec{\partial}\times\vec{\psi}_{\mathrm{L}}) + \vec{\partial}\times(\mathrm{i}q^{-1}\partial_0\vec{\psi}_{\mathrm{L}}) - \vec{\partial}(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}) \\
&= \partial_0^2 \vec{\psi}_{\mathrm{L}} - q^{-1}\lambda\partial_0^2\vec{\psi}_{\mathrm{L}} - q^{-2}\partial_0^2\vec{\psi}_{\mathrm{L}} - \vec{\partial}(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}) \\
&= -\vec{\partial}(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}})\,.
\end{aligned} \tag{57}
$$

Contracting the wave equation with $\vec{\partial}$

$$
\vec{\partial}\cdot(\vec{\partial}\times\vec{\psi}_{\mathrm{L}}) = (\vec{\partial}\times\vec{\partial})\cdot\vec{\psi}_{\mathrm{L}} = -\mathrm{i}\lambda\partial_0(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}) = \mathrm{i}q^{-1}\partial_0(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}})\,, \tag{58}
$$

we see that $\partial_0(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}) = 0$ if $\vec{\psi}_{\mathrm{L}}$ is to satisfy the wave equation. Together with Eq. (57) this means that the mass zero condition is equivalent to $\partial_\mu(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}) = 0$, that is, $\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}$ must be a constant number. In a momentum eigenspace we have $\partial_0(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}}) = k(\vec{\partial}\cdot\vec{\psi}_{\mathrm{L}})$, so this constant number must be zero. The same reasoning applies to the right handed spinor $\vec{\psi}_{\mathrm{R}}$.

We conclude that the wave equations (56) together with the mass zero condition $\partial_\mu\partial^\mu\psi = 0$ are equivalent to

$$
\vec{\partial}\times\vec{\psi}_{\mathrm{L}} = \mathrm{i}q^{-1}\partial_0\vec{\psi}_{\mathrm{L}}\,, \qquad \vec{\partial}\cdot\vec{\psi}_{\mathrm{L}} = 0 \tag{59}
$$

$$
\vec{\partial}\times\vec{\psi}_{\mathrm{R}} = -\mathrm{i}q\,\partial_0\vec{\psi}_{\mathrm{R}}\,, \qquad \vec{\partial}\cdot\vec{\psi}_{\mathrm{R}} = 0\,, \tag{60}
$$

which we will call the q-Maxwell equations.

4.3 The q-Electromagnetic Field

Finally, we write the q-Maxwell equations in a more familiar form, that is, in terms of the q-deformed electric and magnetic fields. In the undeformed case the electric and magnetic fields can — up to constant factors — be characterized within the $D^{(1,0)}\oplus D^{(0,1)}$ representation as eigenstates of the parity operator (19). The electric field should transform like a polar vector $\mathcal{P}\vec{E} = -\vec{E}$, while the magnetic field must be an axial vector $\mathcal{P}\vec{B} = \vec{B}$. Recall, that the parity operator $\mathcal{P}$ acts on q-spinors by exchanging the left and the right handed parts $\mathcal{P}\psi_{\mathrm{L}} = \psi_{\mathrm{R}}$, $\mathcal{P}\psi_{\mathrm{R}} = \psi_{\mathrm{L}}$. This fixes the fields

$$
\vec{E} = \mathrm{i}(\vec{\psi}_{\mathrm{R}} - \vec{\psi}_{\mathrm{L}})\,, \qquad \vec{B} = \vec{\psi}_{\mathrm{R}} + \vec{\psi}_{\mathrm{L}} \tag{61}
$$

up to constant factors which have been chosen to give the right undeformed limit. Spinor conjugation of the fields is now the same as ordinary conjugation $\bar{E}^A = (E^A)^*$, $\bar{B}^A = (B^A)^*$. In terms of these fields, the q-Maxwell equations (59) take the form

$$
\vec{\partial}\times\vec{E} = \tfrac{1}{2}[2]\,\partial_0\vec{B} - \tfrac{1}{2}\mathrm{i}\lambda\,\partial_0\vec{E}\,, \qquad \vec{\partial}\cdot\vec{E} = 0 \tag{62}
$$

$$
\vec{\partial}\times\vec{B} = -\tfrac{1}{2}[2]\,\partial_0\vec{E} - \tfrac{1}{2}\mathrm{i}\lambda\,\partial_0\vec{B}\,, \qquad \vec{\partial}\cdot\vec{B} = 0\,. \tag{63}
$$

We can also express the q-Maxwell equations in terms of a field strength tensor $F^{\mu\nu}$. According to the Clebsch-Gordan series (78) we can embed, say, a $D^{(1,0)}$ representation in to a $D^{(\frac{1}{2},\frac{1}{2})} \otimes D^{(\frac{1}{2},\frac{1}{2})}$ representation. Explicitly, a basis $\{e_C\}$ of the former is mapped to a basis $\{e'_\mu \otimes e'_\nu\}$ of the latter by [9]

$$e_C \mapsto e'_A \otimes e'_B \, \varepsilon^{AB}{}_C + q e'_0 \otimes e'_C - q^{-1} e'_C \otimes e'_0 \,. \tag{64}$$

Accordingly, we map a left $\mathbf{3}$-vector

$$\psi_\mathrm{L} = e_C \otimes \psi_\mathrm{L}^C \mapsto (e'_\mu \otimes e'_\nu) \otimes F_\mathrm{L}^{\mu\nu} \,, \tag{65}$$

where

$$F_\mathrm{L}^{\mu\nu} := \begin{pmatrix} F_\mathrm{L}^{00} & F_\mathrm{L}^{0N} \\ F_\mathrm{L}^{M0} & F_\mathrm{L}^{MN} \end{pmatrix} = \begin{pmatrix} 0 & q\psi_\mathrm{L}^N \\ -q^{-1}\psi_\mathrm{L}^M & \varepsilon^{MN}{}_C \, \psi_\mathrm{L}^C \end{pmatrix} , \tag{66}$$

and where M, N run through $\{-,+,3\}$. In the same manner we obtain for the right handed part

$$F_\mathrm{R}^{\mu\nu} := \begin{pmatrix} 0 & -q^{-1}\psi_\mathrm{R}^N \\ q\psi_\mathrm{R}^M & \varepsilon^{MN}{}_C \, \psi_\mathrm{R}^C \end{pmatrix} . \tag{67}$$

In terms of these matrices the q-Maxwell equations (59) take the form $\partial_\nu F_\mathrm{L}^{\mu\nu} = 0$ and $\partial_\nu F_\mathrm{R}^{\mu\nu} = 0$. By construction, we have $\mathbb{P}^{\mu\nu}_{(1,0)\sigma\tau} F_\mathrm{L}^{\sigma\tau} = F_\mathrm{L}^{\mu\nu}$ and $\mathbb{P}^{\mu\nu}_{(0,1)\sigma\tau} F_\mathrm{R}^{\sigma\tau} = F_\mathrm{R}^{\mu\nu}$. This suggests to introduce the field strength tensor and its dual

$$F^{\mu\nu} := \mathrm{i}(F_\mathrm{L}^{\mu\nu} + F_\mathrm{R}^{\mu\nu}) \,, \qquad \tilde{F}^{\mu\nu} := \mathrm{i}(F_\mathrm{L}^{\mu\nu} - F_\mathrm{R}^{\mu\nu}) \,, \tag{68}$$

for which we have

$$F^{\mu\nu} = \mathbb{P}_\mathrm{A}{}^{\mu\nu}{}_{\sigma\tau} F^{\sigma\tau} \,, \qquad \tilde{F}^{\mu\nu} = \varepsilon^{\mu\nu}{}_{\sigma\tau} F^{\sigma\tau} \,, \tag{69}$$

where the q-epsilon tensor is commonly defined as $\varepsilon^{\mu\nu}{}_{\sigma\tau} = \mathbb{P}^{\mu\nu}_{(1,0)\sigma\tau} - \mathbb{P}^{\mu\nu}_{(0,1)\sigma\tau}$. In terms of the electric and the magnetic field this is

$$\begin{aligned} F^{\mu\nu} &:= \begin{pmatrix} 0 & -\frac{1}{2}[2]E^N + \frac{1}{2}\mathrm{i}\lambda B^N \\ \frac{1}{2}[2]E^M + \frac{1}{2}\mathrm{i}\lambda B^M & \mathrm{i}\varepsilon^{MN}{}_C \, B^C \end{pmatrix} \\ \tilde{F}^{\mu\nu} &:= \begin{pmatrix} 0 & \frac{1}{2}[2]\mathrm{i}B^N - \frac{1}{2}\lambda E^N \\ -\frac{1}{2}[2]\mathrm{i}B^M - \frac{1}{2}\lambda E^M & -\varepsilon^{MN}{}_C \, E^C \end{pmatrix} . \end{aligned} \tag{70}$$

Finally, the q-Maxwell equations take the form

$$\partial_\nu F^{\mu\nu} = 0 \,, \qquad \partial_\nu \tilde{F}^{\mu\nu} = 0 \,, \tag{71}$$

in complete analogy to the undeformed case.

A Useful Formulas

Let E, F, K, and K^{-1} be the generators of $\mathcal{U}_q(\mathrm{su}_2)$. The set of generators $\{J_A\} = \{J_-, J_3, J_+\}$ of $\mathcal{U}_q(\mathrm{su}_2)$ defined as

$$\begin{aligned} J_- &:= q[2]^{-\frac{1}{2}} KF \\ J_3 &:= [2]^{-1}(q^{-1}EF - qFE) \\ J_+ &:= -[2]^{-\frac{1}{2}} E \end{aligned} \tag{72}$$

is the left 3-vector operator of angular momentum. The center of $\mathcal{U}_q(\mathrm{su}_2)$ is generated by

$$W := K - \lambda J_3 = K - \lambda [2]^{-1}(q^{-1}EF - qFE)\,, \tag{73}$$

the Casimir operator of angular momentum. W is related to J_A by

$$W^2 - 1 = \lambda^2(J_3^2 - q^{-1}J_-J_+ - qJ_+J_-) = \lambda^2 J_A J_B g^{AB}\,,$$

thus defining the 3-metric g^{AB}, by which we raise 3-vector indices $X^A = g^{AB}X_B$. There is also an ε-tensor

$$\begin{aligned} &\varepsilon^{-3}{}_- = q^{-1} && \varepsilon^{3-}{}_- = -q && && \text{(74a)} \\ &\varepsilon^{-+}{}_3 = 1 && \varepsilon^{+-}{}_3 = -1 && \varepsilon^{33}{}_3 = -\lambda && \text{(74b)} \\ &\varepsilon^{3+}{}_+ = q^{-1} && \varepsilon^{+3}{}_+ = -q\,, && && \text{(74c)} \end{aligned}$$

so we can define a scalar and a vector product by

$$\vec{X} \cdot \vec{Y} := g^{AB} X_A Y_B\,, \qquad (\vec{X} \times \vec{Y})_C := \mathrm{i}\, X_A Y_B \varepsilon^{AB}{}_C\,, \tag{75}$$

for which we have the useful identities

$$\begin{gathered} \vec{X} \cdot (\vec{Y} \times \vec{Z}) = (\vec{X} \times \vec{Y}) \cdot \vec{Z} \\ (\vec{X} \times \vec{Y}) \times \vec{Z} - (\vec{X} \cdot \vec{Y})\vec{Z} = \vec{X} \times (\vec{Y} \times \vec{Z}) - \vec{X}(\vec{Y} \cdot \vec{Z})\,. \end{gathered} \tag{76}$$

Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be the matrix of generators of $SU_q(2)^{\mathrm{op}}$, the opposite algebra of the quantum group $SU_q(2)$. The Hopf-$*$ algebra generated by the Hopf-$*$ subalgebras $\mathcal{U}_q(\mathrm{su}_2)$ and $SU_q(2)^{\mathrm{op}}$ with cross commutation relations

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} E &= \begin{pmatrix} qEa - q^{\frac{3}{2}}b & q^{-1}Eb \\ qEc + q^{\frac{3}{2}}Ka - q^{\frac{3}{2}}d & q^{-1}Ed + q^{-\frac{1}{2}}Kb \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} F &= \begin{pmatrix} qFa + q^{-\frac{1}{2}}c & qFb - q^{-\frac{1}{2}}K^{-1}a + q^{-\frac{1}{2}}d \\ q^{-1}Fc & q^{-1}Fd - q^{-\frac{5}{2}}K^{-1}c \end{pmatrix} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} K &= K \begin{pmatrix} a & q^{-2}b \\ q^2c & d \end{pmatrix}, \end{aligned}$$

which is the Drinfeld double of $\mathcal{U}_q(\mathrm{su}_2)$ and $SU_q(2)^{\mathrm{op}}$, is the ***q*-Lorentz algebra** $\mathcal{H} = \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ [12].

Other forms of the q-Lorentz algebra can be found in the literature [13–15], which are essentially equivalent [9, 16]. Very useful for the representation theory is the form where $\mathcal{U}_q(\mathrm{sl}_2(\mathbb{C})) \cong \mathcal{U}_q(\mathrm{sl}_2) \otimes \mathcal{U}_q(\mathrm{sl}_2)$ as algebra. This isomorphism is defined on rotations $l \in \mathcal{U}_q(\mathrm{su}_2)$ by the coproduct $l \mapsto \Delta(l) \in \mathcal{U}_q(\mathrm{sl}_2) \otimes \mathcal{U}_q(\mathrm{sl}_2)$ and for the generators of boosts as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} K^{\frac{1}{2}} \otimes K^{-\frac{1}{2}} & q^{-\frac{1}{2}}\lambda K^{\frac{1}{2}} \otimes K^{-\frac{1}{2}}E \\ -q^{\frac{1}{2}}\lambda F K^{\frac{1}{2}} \otimes K^{-\frac{1}{2}} & K^{-\frac{1}{2}} \otimes K^{\frac{1}{2}} - \lambda^2 F K^{\frac{1}{2}} \otimes K^{-\frac{1}{2}}E \end{pmatrix}. \tag{77}$$

On this form of the q-Lorentz algebra the representation maps of the irreducible representations are $\rho^{j_1} \otimes \rho^{j_2}$, where ρ^j is the spin-j representation of $\mathcal{U}_q(\mathrm{sl}_2)$. For the 4-vector representation we have the usual Clebsch-Gordan series

$$D^{(\frac{1}{2},\frac{1}{2})} \otimes D^{(\frac{1}{2},\frac{1}{2})} \cong D^{(0,0)} \oplus D^{(1,0)} \oplus D^{(0,1)} \oplus D^{(1,1)}. \tag{78}$$

The projection matrices on the according subspaces are denoted by $\mathbb{P}_{(0,0)}$, $\mathbb{P}_{(1,0)}$, $\mathbb{P}_{(0,1)}$, $\mathbb{P}_{(1,1)}$, the antisymmetrizer by $\mathbb{P}_{\mathrm{A}} := \mathbb{P}_{(1,0)} + \mathbb{P}_{(0,1)}$, and the symmetrizer by $\mathbb{P}_{\mathrm{S}} := \mathbb{P}_{(0,0)} + \mathbb{P}_{(1,1)} = 1 - \mathbb{P}_{\mathrm{A}}$. With respect to the 4-vector basis

$[2]^2(\mathbb{P}_{\mathrm{A}})^{ab}{}_{cd} =$

	$C0$	$0D$	CD
$A0$	$2\delta^A_C$	$-[4][2]^{-1}\delta^A_D$	$\lambda\varepsilon_C{}^A{}_D$
$0B$	$-[4][2]^{-1}\delta^B_C$	$2\delta^B_D$	$\lambda\varepsilon_C{}^B{}_D$
AB	$-\lambda\varepsilon^{AB}{}_C$	$-\lambda\varepsilon^{AB}{}_D$	$2\varepsilon^{AB}{}_X\varepsilon_C{}^X{}_D$

where A, B, C, D run through $\{-, +, 3\}$.

The $*$-algebra generated by P_0, P_-, P_+, P_3 with commutation relations

$$P_0 P_A = P_A P_0\,, \qquad P_A P_B\, \varepsilon^{AB}{}_C = -\lambda P_0 P_C\,, \tag{79}$$

and $*$-structure $P_0^* = P_0$, $P_-^* = -q^{-1}P_+$, $P_+^* = -qP_-$, $P_3^* = P_3$ is the ***q*-Minkowski space algebra** $\mathcal{X} = \mathbb{R}^{1,3}_q$. The center of $\mathbb{R}^{1,3}_q$ is generated by

$$m^2 := P_\mu P_\nu \eta^{\mu\nu} = P_0^2 + q^{-1}P_-P_+ + qP_+P_- - P_3^2\,, \tag{80}$$

the mass Casimir, thus defining the 4-metric $\eta^{\mu\nu}$. It is related to the 3-metric by $\eta^{AB} = -g^{AB}$ for $A, B \in \{-, +, 3\}$. The generators P_μ carry a 4-vector representation of $\mathcal{H}$ which in this particular basis is denoted by $h \triangleright P_\nu = P_\mu \Lambda(h)^\nu{}_\mu$. It turns $\mathcal{X}$ into a $\mathcal{H}$-module $*$-algebra.

The **q-Poincaré algebra** $\mathcal{A}$ is the $*$-algebra generated by the q-Lorentz algebra $\mathcal{H} = \mathcal{U}_q(\mathrm{sl}_2(\mathbb{C}))$ and the q-Minkowski algebra $\mathcal{X} = \mathbb{R}_q^{1,3}$ with cross commutation relations

$$h\, P_\nu = P_\mu \Lambda(h_{(1)})^\mu{}_\nu\, h_{(2)}\,, \tag{81}$$

for all $h \in \mathcal{H}$. In other words, $\mathcal{A}$ is the Hopf semidirect product $\mathcal{A} = \mathcal{X} \rtimes \mathcal{H}$.

The left Hopf adjoint action of $\mathcal{H}$ on $\mathcal{A}$ is defined as

$$\mathrm{ad}_\mathrm{L} h \triangleright a := h_{(1)} a S(h_{(2)})\,. \tag{82}$$

The commutation relations (81) are precisely such that the left Hopf adjoint action equals the 4-vector action $\mathrm{ad}_\mathrm{L} h \triangleright P_\nu = P_\mu \Lambda(h)^\mu{}_\nu$. Any set of operators with this property will be called a 4-vector operator. In particular we have

$$\begin{aligned} P_- &= \mathrm{ad}_\mathrm{L}(-q^{-\frac{1}{2}}\lambda^{-1}[2]^{\frac{1}{2}}\, c) \triangleright P_0 \\ P_+ &= \mathrm{ad}_\mathrm{L}(q^{\frac{1}{2}}\lambda^{-1}[2]^{\frac{1}{2}}\, b) \triangleright P_0 \\ P_3 &= \mathrm{ad}_\mathrm{L}(\lambda^{-1}\,(d-a)) \triangleright P_0\,, \end{aligned} \tag{83}$$

so, if we know the zero component of a 4-vector operator, we can easily compute the other components.

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