Comments on Noncommutative ADHM Construction

Yu Tian
School of Physics, Peking University, Beijing 100871, China
E-mail: phytian@yahoo.com

Chuan-Jie Zhu
Institute of Mathematics, Henan University, Kaifeng 475001
and
Institute for Theoretical Physics, Chinese Academy of Sciences
P. O. Box 2735, Beijing 100080
E-mail: zhucj@itp.ac.cn

Abstract

We extend the method of matrix partition to obtain explicitly the gauge field for noncommutative ADHM construction in some general cases. As an application of this method we apply it to the $U(2)$ 2-instanton and get explicit result for the gauge fields in the coincident instanton limit. We also easily apply it to the noncommutative 't Hooft instantons in the appendix.

1 Introduction

In a previous paper [1], we studied the $U(1)$ and $U(2)$ instanton solutions on general noncommutative $\mathbf{R^4}$ [2], where for the $U(1)$ 1-instanton, 2-instanton and $U(2)$ 1-instanton cases we obtain explicitly the gauge field by ADHM construction [3].

In (noncommutative) ADHM construction, there are two steps to obtain the instanton configuration (the gauge field) explicitly. The first step is to solve the ADHM equation to get the matrix Δ (we follow closely the notations of [1]). The second step is to construct the matrix U from Δ, or in other words, to find all the zero modes of Δ. The gauge field can then be constructed from U. The difficulty of the first step has been well-known for a long time. But it was only recently realized that the second step is also a difficult problem in the noncommutative case. [4] discussed the $U(1)$ $\theta_2 > 0$ case, and our previous paper [1] resolved the problem in some of the $U(1)$ and $U(2)$ cases (see also [5, 6, 7]), but for the general cases there were no systematic methods to obtain the matrix U yet. It would be a formidable task to find the matrix U for higher N and k.

Keeping this problem in mind, we will try to develop a method to accomplish the second step systematically. In fact there is a general method in the commutative case [9]. What we will do is to extend this method to the noncommutative case. By using this method we have rechecked all the results obtained in our previous paper [1]. As a new application of this method, we will demonstrate how to deal with some of the $U(2)$ 2-instanton cases. As far as we know there is no general solution of noncommutative $U(2)$ 2-instanton ADHM equation. So we will use this method to study a special case in the coincident instanton limit.

This paper is organized as follows: in section 2 we recall briefly the noncommutative $\mathbf{R^4}$ and set our notations and then the ADHM construction. In section 3 we present the modified matrix partition method and in section 4 we solve a special case of $U(2)$ 2-instanton. Finally we mention that our method can be used to give a more general class of $U(2)$ multi-instantons.

2 $\mathrm{R^4_{NC}}$, the (anti-)self-dual equations and ADHM construction

2.1 $\mathrm{R^4_{NC}}$ and the (anti-)self-dual equations

First let us recall briefly the noncommutative $\mathbf{R^4}$ and set our notations[1]. For a general noncommutative $\mathbf{R^4}$ we mean a space with (operator) coordinates x^m, $m = 1, \cdots, 4$, which satisfy the following relations:

$$[x^m, x^n] = i\theta^{mn}, \tag{1}$$

where θ^{mn} are real constants. If we assume the standard (Euclidean) metric for the noncommutative $\mathbf{R^4}$, we can use the orthogonal transformation with

[1]For general reviews on noncommutative geometry and field theory, see, for example, [4, 10, 11, 12].

positive determinant to change θ^{mn} into the following standard form:

$$(\theta^{mn}) = \begin{pmatrix} 0 & \theta^{12} & 0 & 0 \\ -\theta^{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta^{34} \\ 0 & 0 & -\theta^{34} & 0 \end{pmatrix}, \qquad (2)$$

with $\theta^{12} > 0$ and $\theta^{12} + \theta^{34} \geq 0$. If the original θ^{mn} is non-degenerate, θ^{34} in this standard form is non-vanishing. Otherwise the considered noncommutative $\mathbf{R^4}$ can be identified with a direct product $\mathbf{R^2_{\mathrm{NC}}} \times \mathbf{R^2}$. One can refer [8] for this degenerate case, which we will not dwell on in this paper. By using this form of θ^{mn}, the only non-vanishing commutators are as follows:

$$[x^1, x^2] = i\theta^{12}, \qquad [x^3, x^4] = i\theta^{34}, \qquad (3)$$

and the other twos obtained by using the anti-symmetric property of commutators. Introducing complex coordinates:

$$\begin{array}{ll} z_1 = x^2 + ix^1, & \bar{z}_1 = x^2 - ix^1, \\ z_2 = x^4 + ix^3, & \bar{z}_2 = x^4 - ix^3, \end{array} \qquad (4)$$

the non-vanishing commutation relations are

$$[\bar{z}_1, z_1] = 2\theta^{12} \equiv \theta_1, \quad [\bar{z}_2, z_2] = 2\theta^{34} \equiv \theta_2. \qquad (5)$$

By a noncommutative gauge field A_m we mean an operator valued field. The (anti-hermitian) field strength F_{mn} is defined similarly as in the commutative case:

$$F_{mn} = \hat{\partial}_{[m}A_{n]} + A_{[m}A_{n]} \equiv \hat{\partial}_m A_n - \hat{\partial}_n A_m + [A_m, A_n], \qquad (6)$$

where the derivative operator $\hat{\partial}_m$ is defined as follows:

$$\hat{\partial}_m f \equiv -i\theta_{mn}[x^n, f], \qquad (7)$$

where θ_{mn} is the inverse of θ^{mn}. For our standard form (2) of θ^{mn} we have

$$\hat{\partial}_1 f = \frac{i}{\theta^{12}}[x^2, f], \qquad \hat{\partial}_2 f = -\frac{i}{\theta^{12}}[x^1, f], \qquad (8)$$

which can be expressed by the complex coordinates (4) as follows:

$$\partial_1 f \equiv \hat{\partial}_{z_1} f = \frac{1}{\theta_1}[\bar{z}_1, f], \qquad \bar{\partial}_1 f \equiv \hat{\partial}_{\bar{z}_1} f = -\frac{1}{\theta_1}[z_1, f], \qquad (9)$$

and similar relations for $x^{3,4}$ and $z_2, \bar{z}_2$.

For a general metric g_{mn} the instanton equations are

$$F_{mn} = \pm \frac{\epsilon^{pqrs}}{2\sqrt{g}} g_{mp} g_{nq} F_{rs}, \qquad (10)$$

and the solutions are known as self-dual (SD, for "+" sign) and anti-self-dual (ASD, for "−" sign) instantons. Here ϵ^{pqrs} is the totally anti-symmetric tensor ($\epsilon^{1234} = 1$ etc.) and g is the metric. We will take the standard metric $g_{mn} = \delta_{mn}$ and take the noncommutative parameters $\theta_{1,2}$ as free parameters. We also note that the notions of self-dual and anti-self-dual are interchanged by a parity transformation. A parity transformation also changes the sign of θ^{mn}. In the following discussion we will consider only the ASD instantons. So we should not restrict θ_2 to be positive.

2.2 ADHM construction for ordinary gauge theory

For ordinary gauge theory all the (ASD) instanton solutions are obtained by ADHM (Atiyah-Drinfeld-Hitchin-Manin) construction [3]. In this construction we introduce the following ingredients (for $U(N)$ gauge theory with instanton number k):

- complex vector spaces V and W of dimensions k and N,
- $k \times k$ matrix $B_{1,2}$, $k \times N$ matrix I and $N \times k$ matrix J,
- the following quantities:

$$\begin{aligned} \mu_r &= [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + I\, I^\dagger - J^\dagger J, && (11) \\ \mu_c &= [B_1, B_2] + I\, J. && (12) \end{aligned}$$

The claim of ADHM is as follows:

- Given $B_{1,2}$, I and J such that $\mu_r = \mu_c = 0$, an ASD gauge field can be constructed;
- All ASD gauge fields can be obtained in this way.

It is convenient to introduce a quaternionic notation for the 4-dimensional Euclidean space-time indices:

$$x \equiv x^n \sigma_n, \qquad \bar{x} \equiv x^n \bar{\sigma}_n, \tag{13}$$

where $\sigma_n = (i\vec{\tau}, 1)$ and τ^c, $c = 1, 2, 3$ are the three Pauli matrices, and the conjugate matrices $\bar{\sigma}_n = \sigma_n^\dagger = (-i\vec{\tau}, 1)$. In terms of the complex coordinates (4) we have

$$(x_{\alpha\dot{\alpha}}) = \begin{pmatrix} z_2 & z_1 \\ -\bar{z}_1 & \bar{z}_2 \end{pmatrix}, \qquad (\bar{x}^{\dot{\alpha}\alpha}) = \begin{pmatrix} \bar{z}_2 & -z_1 \\ \bar{z}_1 & z_2 \end{pmatrix}. \tag{14}$$

Then the basic object in the ADHM construction is the $(N + 2k) \times 2k$ matrix Δ which is linear in the space-time coordinates:

$$\Delta = a + b\bar{x}, \tag{15}$$

where the constant matrices

$$a = \begin{pmatrix} I^\dagger & J \\ B_2^\dagger & -B_1 \\ B_1^\dagger & B_2 \end{pmatrix}, \quad b = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{16}$$

Consider the conjugate operator of Δ:

$$\Delta^\dagger = a^\dagger + x b^\dagger = \begin{pmatrix} I & B_2 + z_2 & B_1 + z_1 \\ J^\dagger & -B_1^\dagger - \bar{z}_1 & B_2^\dagger + \bar{z}_2 \end{pmatrix}. \tag{17}$$

It is easy to check that the ADHM equations (11) and (12) are equivalent to the so-called factorization condition:

$$\Delta^\dagger \Delta = \begin{pmatrix} f^{-1} & 0 \\ 0 & f^{-1} \end{pmatrix}, \tag{18}$$

where $f(x)$ is a $k \times k$ hermitian matrix. From the above condition we can construct a hermitian projection operator P as follows:[2]

$$\begin{aligned} P &= \Delta f \Delta^\dagger, \\ P^2 &= \Delta f f^{-1} f \Delta^\dagger = P. \end{aligned} \tag{20}$$

Obviously, the null-space of $\Delta^\dagger(x)$ is of N dimension for generic x. The basis vectors for this null-space can be assembled into an $(N+2k) \times N$ matrix $U(x)$:

$$\Delta^\dagger U = 0, \tag{21}$$

which can be chosen to satisfy the following orthonormal condition:

$$U^\dagger U = 1. \tag{22}$$

The above orthonormal condition guarantees that $UU^\dagger$ is also a hermitian projection operator. Now it can be proved that the completeness relation [6]

$$P + UU^\dagger = 1 \tag{23}$$

holds if U contains the whole null-space of $\Delta^\dagger$. In other words, this completeness relation requires that U consists of all the zero modes of $\Delta^\dagger$.[3]

The (anti-hermitian) gauge potential is constructed from U by the following formula:

$$A_m = U^\dagger \partial_m U. \tag{24}$$

Substituting this expression into (6), we get the following field strength:

$$\begin{aligned} F_{mn} &= \partial_{[m}(U^\dagger \partial_{n]} U) + (U^\dagger \partial_{[m} U)(U^\dagger \partial_{n]} U) = \partial_{[m} U^\dagger (1 - UU^\dagger) \partial_{n]} U \\ &= \partial_{[m} U^\dagger \Delta f \Delta^\dagger \partial_{n]} U = U^\dagger \partial_{[m} \Delta f \partial_{n]} \Delta^\dagger U = U^\dagger b \bar{\sigma}_{[m} \sigma_{n]} f b^\dagger U \\ &= 2i \bar{\eta}^c_{mn} U^\dagger b (\tau^c f) b^\dagger U. \end{aligned} \tag{25}$$

Here $\bar{\eta}^a_{ij}$ is the standard 't Hooft η-symbol, which is anti-self-dual:

$$\frac{1}{2} \epsilon_{ijkl} \bar{\eta}^a_{kl} = -\bar{\eta}^a_{ij}. \tag{26}$$

2.3 Noncommutative ADHM construction

The above construction has been extended to noncommutative gauge theory [2]. We recall this construction briefly here. By introducing the same data as

[2]We use the following abbreviation for expressions with f:

$$\Delta f \Delta^\dagger \equiv \Delta \begin{pmatrix} f & 0 \\ 0 & f \end{pmatrix} \Delta^\dagger = \Delta (f \otimes 1_2) \Delta^\dagger. \tag{19}$$

[3]The proof is sketched as follows: the two projection operators P and $UU^\dagger$ are orthogonal to each other, and so $1 - P - UU^\dagger$ is also a hermitian projection operator. Now this can always be written as the form $VV^\dagger$. Here Δ and f are both of maximum rank and $PVV^\dagger = 0$, then V must consist of some zero modes of $\Delta^\dagger$ other than those in U. This conclusion is in conflict with the assumption that U contains all the zero modes of $\Delta^\dagger$. Although the notion of maximum rank is ambiguous in the infinite-dimensional case as we will encounter in noncommutative ADHM construction, it can be defined as follows so that the above proof is also true in the infinite-dimensional case. We say that an $\infty \times \infty$ matrix X is of maximum rank if it has no zero modes, i.e., $X\Phi \neq 0$ if $\Phi \neq 0$.

above but considering the z_i's as noncommutative we see that the factorization condition (18) still gives $\mu_c = 0$, but μ_r no longer vanishes. It is easy to check that the following relation holds:

$$\mu_r = \zeta \equiv \theta_1 + \theta_2. \tag{27}$$

In this case the two ADHM equations (11) and (12) can be combined into one [7]:

$$\tau^{c\dot{\alpha}}{}_{\dot{\beta}}(\bar{a}^{\dot{\beta}} a_{\dot{\alpha}})_{ij} = \delta_{ij}\delta^{c3}\zeta. \tag{28}$$

As studied mathematically by various people (see, for example, the lectures by H. Nakajima [13]), the moduli space of the noncommutative instantons is better behaved than their commutative counterpart. In the noncommutative case the operator $\Delta^\dagger\Delta$ always has maximum rank (see [10]).

Though there is no much difference between the noncommutative ADHM construction and the commutative one, we should study the noncommutative case in more detail. In order to study the instanton solutions precisely, we use a Fock space representation as follows ($n_1, n_2 \geq 0$):

$$\begin{aligned} z_1|n_1, n_2\rangle &= \sqrt{\theta_1}\sqrt{n_1+1}|n_1+1, n_2\rangle, & (29)\\ \bar{z}_1|n_1, n_2\rangle &= \sqrt{\theta_1}\sqrt{n_1}|n_1-1, n_2\rangle, & (30)\end{aligned}$$

by using the commutation relation (5). Similar expressions for z_2 and $\bar{z}_2$ also apply (but paying a little attention to the sign of θ_2 which is not restricted to be positive). In this representation the z_i's are infinite-dimensional matrices, and so are the operator Δ, $\Delta^\dagger$ etc. Because of infinite dimensions are involved we can not determine the dimension of null-space of $\Delta^\dagger$ straightforwardly from the difference of the numbers of its rows and columns. But it turns out that $\Delta^\dagger$ also has infinite number of zero modes, and they can be arranged into an $(N+2k)\times N$ matrix with entries from the (noncommutative) algebra generated by the coordinates, which resembles the commutative case.

3 The modified matrix partition method

In the commutative case, there is a standard method [9], which we call the 'matrix partition method' in ADHM construction, to obtain the matrix U from Δ as follows. First we introduce the following decomposition of the matrices U and Δ:

$$U = \begin{pmatrix} V_{N\times N} \\ U'_{2k\times N}\end{pmatrix}, \qquad \Delta = \begin{pmatrix} K_{N\times 2k} \\ \Delta'_{2k\times 2k}\end{pmatrix}. \tag{31}$$

So we have

$$K = \begin{pmatrix} I^\dagger & J \end{pmatrix}, \quad \Delta' = \begin{pmatrix} B_2^\dagger + \bar{z}_2 & -B_1 - z_1 \\ B_1^\dagger + \bar{z}_1 & B_2 + z_2 \end{pmatrix}. \tag{32}$$

Substituting (31) into the completeness relation (23), we obtain

$$\begin{pmatrix} VV^\dagger & VU'^\dagger \\ U'V^\dagger & U'U'^\dagger \end{pmatrix} = \begin{pmatrix} 1 - KfK^\dagger & -Kf\Delta'^\dagger \\ -\Delta' f K^\dagger & 1 - \Delta' f \Delta'^\dagger \end{pmatrix}. \tag{33}$$

Now we can choose

$$V = V^\dagger = (1 - KfK^\dagger)^{1/2} \tag{34}$$

as a solution of the (1,1) element of the matrix equation (33). Then by the (2,1) element of (33) the matrix U' can be expressed as

$$U' = -\Delta' f K^\dagger V^{-1}. \tag{35}$$

The choice (34) of V is known as the 'singular gauge', and any other choices of V which solve (33) are related to (34) by a gauge transformation.

Corresponding to the singularity arising in the commutative case, the choice (34) also brings problems if we extend the matrix partition method to the non-commutative case. In the following subsections we will show that slight modifications of this method can remedy these problems.

In the noncommutative case, the solutions of the (1,1) element of (33) are not always related to each other by gauge transformations. This can be seen as follows. If V is a solution of the (1,1) element of (33) and S is an operator such that $SS^\dagger = 1$, then VS is also a solution:

$$(VS)(VS)^\dagger = VSS^\dagger V^\dagger = VV^\dagger, \tag{36}$$

while we can have $S^\dagger S \neq 1$. This S induces a new solution US of the completeness relation (23). However, US does not satisfy the orthonormal condition (22):

$$(US)^\dagger (US) = S^\dagger U^\dagger US \neq 1, \tag{37}$$

which is not a problem in the commutative case: any solution of (23) must satisfy (22). The above freedom of V can be used to resolve the problems of the matrix partition method in noncommutative ADHM construction, while we must consider the completeness relation (23) and the orthonormal condition (22) as two independent equations.

The two cases, $\theta_2 > 0$ and $\theta_2 < 0$, have an essential difference: for $\theta_2 > 0$ the matrix Δ' has zero modes and $\Delta'^\dagger$ is of maximum rank, while for $\theta_2 < 0$ the matrix Δ' is of maximum rank and $\Delta'^\dagger$ has zero modes. As we can see in the following subsections, the two cases must be considered separately just because of this difference.

3.1 $\theta_2 > 0$ case

The immediate consequence of that Δ' has zero modes in this case is that the matrix V in (34) is not invertible and we we can not use (35) to determine U'. Here we will try to directly project out the zero modes of $VV^\dagger$ first.

Let Π_0 be the projector whose range is the null-space of $(1 - KfK^\dagger)$. We can still use (34) and let

$$U' = -\Delta' f K^\dagger (1 - KfK^\dagger)_\Pi^{-1/2}, \tag{38}$$

where the inversion of $(1 - KfK^\dagger)$ is restricted on $\Pi \equiv 1 - \Pi_0$. Noting that

$$\begin{aligned} \Delta'^\dagger \Delta' f K^\dagger \Pi_0 &= \Delta'^\dagger \Delta' (K^\dagger K + \Delta'^\dagger \Delta')^{-1} K^\dagger \Pi_0 \\ &= [1 - K^\dagger K (K^\dagger K + \Delta'^\dagger \Delta')^{-1}] K^\dagger \Pi_0 \\ &= K^\dagger (1 - KfK^\dagger) \Pi_0 = 0 \end{aligned} \tag{39}$$

and $\Delta'^\dagger$ is of maximum rank by (32), we have

$$\Delta' f K^\dagger \Pi_0 = 0. \tag{40}$$

The (2,1) element of the matrix equation (33) is now satisfied:

$$\begin{aligned} U'V^\dagger &= -\Delta' f K^\dagger (1-KfK^\dagger)_\Pi^{-1/2}(1-KfK^\dagger)^{1/2} \\ &= -\Delta' f K^\dagger (1-\Pi_0) = -\Delta' f K^\dagger. \end{aligned} \tag{41}$$

Moreover we can easily check that the (2,2) element of (33) holds:

$$\begin{aligned} \Delta'^\dagger U' U'^\dagger &= \Delta'^\dagger \Delta' f K^\dagger (1-KfK^\dagger)_\Pi^{-1} K f \Delta'^\dagger \\ &= K^\dagger (1-KfK^\dagger)(1-KfK^\dagger)_\Pi^{-1} K f \Delta'^\dagger \\ &= K^\dagger (1-\Pi_0) K f \Delta'^\dagger = K^\dagger K f \Delta'^\dagger \end{aligned} \tag{42}$$

and

$$\begin{aligned} \Delta'^\dagger (1-\Delta' f \Delta'^\dagger) &= [1-\Delta'^\dagger \Delta' (K^\dagger K + \Delta'^\dagger \Delta')^{-1}]\Delta'^\dagger \\ &= K^\dagger K f \Delta'^\dagger, \end{aligned} \tag{43}$$

so

$$U' U'^\dagger = 1 - \Delta' f \Delta'^\dagger. \tag{44}$$

Even though (34) together with (38) satisfy equation (33), they do not satisfy the orthonormal condition (22):

$$\begin{aligned} U^\dagger U &= V^\dagger V + U'^\dagger U' \\ &= 1 - KfK^\dagger + (1-KfK^\dagger)_\Pi^{-1/2} K f \Delta'^\dagger \Delta' f K^\dagger (1-KfK^\dagger)_\Pi^{-1/2} \\ &= 1 - KfK^\dagger \\ &\quad + (1-KfK^\dagger)_\Pi^{-1/2} K f (f^{-1} - K^\dagger K) f K^\dagger (1-KfK^\dagger)_\Pi^{-1/2} \\ &= 1 - KfK^\dagger \\ &\quad + (1-KfK^\dagger)_\Pi^{-1/2} (KfK^\dagger - KfK^\dagger KfK^\dagger)(1-KfK^\dagger)_\Pi^{-1/2} \\ &= 1 - KfK^\dagger + \Pi KfK^\dagger \Pi \\ &= 1 - \Pi_0 KfK^\dagger - KfK^\dagger \Pi_0 + \Pi_0 = \Pi \neq 1. \end{aligned} \tag{45}$$

The solution to this problem is to introduce a shift operator s as follows:

$$s s^\dagger = 1, \quad s^\dagger s = \Pi, \tag{46}$$

and we set

$$V = (1-KfK^\dagger)^{1/2} s^\dagger, \tag{47}$$

$$U' = -\Delta' f K^\dagger (1-KfK^\dagger)^{-1/2} s^\dagger. \tag{48}$$

The operator $s^\dagger$ removes the zero modes of the operator $(1-KfK^\dagger)$ and makes its inversion well-defined. Then we can easily see that the completeness relation (33) still holds and the orthonormal condition is now satisfied:

$$\begin{aligned} U^\dagger U &= s(1-KfK^\dagger)s^\dagger \\ &\quad + s(1-KfK^\dagger)^{-1/2} K f \Delta'^\dagger \Delta' f K^\dagger (1-KfK^\dagger)^{-1/2} s^\dagger \\ &= s \Pi s^\dagger = 1. \end{aligned} \tag{49}$$

So (47) and (48) constitute the required matrix U.

Now we determine the null-space of $(1-KfK^\dagger)$. As

$$\begin{aligned} (1-KfK^\dagger)K &= K[1-(K^\dagger K + \Delta'^\dagger \Delta')^{-1} K^\dagger K] \\ &= K f \Delta'^\dagger \Delta', \end{aligned} \tag{50}$$

we can first determine the null-space of Δ'. It is easy to see from (32) that the zero modes of Δ' must take the form:

$$\Phi = \begin{pmatrix} \Upsilon \\ 0 \end{pmatrix} \tag{51}$$

because the second column of Δ' is of maximum rank. By (50), we introduce an operator

$$\Psi \equiv K\Phi = I^\dagger \Upsilon, \tag{52}$$

which belongs to the null-space of $(1 - KfK^\dagger)$:

$$(1 - KfK^\dagger)\Psi = Kf\Delta'^\dagger \Delta' \Phi = 0. \tag{53}$$

Then a hermitian operator G can be constructed as follows:

$$G = \Psi^\dagger \Psi = \Upsilon^\dagger I I^\dagger \Upsilon = \Upsilon^\dagger [II^\dagger + \sum_{\alpha=1,2} (B_\alpha + z_\alpha)(B_\alpha^\dagger + \bar{z}_\alpha)]\Upsilon = \Upsilon^\dagger f^{-1} \Upsilon, \tag{54}$$

which is positive definite because f^{-1} has no zero mode. Here we have used the fact

$$(B_\alpha^\dagger + \bar{z}_\alpha)\Upsilon = 0, \quad \alpha = 1, 2. \tag{55}$$

If Ψ contains all the zero modes of $(1 - KfK^\dagger)$, it is easy to show that

$$\Pi_0 = \Psi G^{-1} \Psi^\dagger. \tag{56}$$

We will prove this completeness of Ψ as follows. From (40) we have

$$fI\Pi_0 \subset \Upsilon, \tag{57}$$
$$J^\dagger \Pi_0 = 0, \tag{58}$$

where the notation '$\subset$' now means 'included in the space spanned by the column vectors of', and (58) gives

$$(1 - KfK^\dagger)\Pi_0 = (1 - I^\dagger f I)\Pi_0 = 0, \tag{59}$$

which leads to

$$\Pi_0 = I^\dagger f I \Pi_0 \subset I^\dagger \Upsilon. \tag{60}$$

So (51) does span the whole null-space of $(1 - KfK^\dagger)$.

For $[B_1, B_2] \neq 0$, we can not give any more explicit results. But when $[B_1, B_2] = 0$, we have [4]:

$$\Upsilon = e^{-\sum_\alpha \theta_\alpha^{-1} B_\alpha^\dagger z_\alpha} |0, 0\rangle. \tag{61}$$

In general, the condition $[B_1, B_2] = 0$ can not be satisfied. When $N = 1$ it can be proved [13] that we can always take $J = 0$, so the above condition holds. Moreover, it also holds in the $U(2)$ 1-instanton case, which we had considered in [1]. A special $U(2)$ 2-instanton case also satisfies this condition. We will give the explicit solution in the next section.

3.2 $\theta_2 < 0$ case

In this case the problem is that $\Delta'^\dagger$ has zero modes. Here Δ' is of maximum rank as we can see from (32), so is $(1 - KfK^\dagger)$ for the reason stated in the previous subsection. But if we use (34) and (35) as the solution of (33), equations (42) (where $\Pi_0 = 0$) and (43) do not give (44).

Suppose that the null-space of $\Delta'^\dagger$ is of k' dimension and Γ is the matrix composed of all the orthonormal zero modes of $\Delta'^\dagger$:

$$\Gamma^\dagger\Gamma = 1_{k'\times k'}, \qquad \Delta'^\dagger\Gamma = 0, \tag{62}$$

we have

$$1 - \Delta' f \Delta'^\dagger - U'U'^\dagger \subset \Gamma. \tag{63}$$

Noting that a matrix $\tilde{U}$ composed of (34) and (35) satisfies

$$\tilde{U}^\dagger\tilde{U} = 1, \qquad \Delta^\dagger\tilde{U} = 0, \tag{64}$$

the only thing we must do is to add some extra zero modes to $\tilde{U}$ so as to make it satisfy the completeness relation. In what follows, we will prove that these extra zero modes can be assembled into a matrix X of the form

$$X = \begin{pmatrix} 0 \\ \Gamma \end{pmatrix}. \tag{65}$$

Firstly, X belongs to the null-space of $\Delta^\dagger$:

$$\Delta^\dagger X = \Delta'^\dagger\Gamma = 0. \tag{66}$$

Secondly, it is not difficult to see that a non-vanishing (1,1) element of X will break down the (1,1) element of equation (33). Finally, any vectors belong to the space spanned by Γ appearing in the (2,1) element of X will violate (63). So we conclude that the combination of $\tilde{U}$ and X gives all the zero modes of $\Delta^\dagger$.

To make the above conclusion more explicit, we introduce the following shift operator:

$$SS^\dagger = 1, \quad S^\dagger S = 1 - \sum_{n=0}^{k'-1} |n,0\rangle\langle n,0| \tag{67}$$

and a $k' \times N$ matrix

$$Y = \begin{pmatrix} \langle 0,0| & 0 & \cdots & 0 \\ \langle 1,0| & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \langle k'-1,0| & 0 & \cdots & 0 \end{pmatrix}. \tag{68}$$

An $N \times N$ matrix operator $\mathcal{S}$ is defined as

$$\mathcal{S} = \mathrm{diag}(S, 1, \cdots, 1). \tag{69}$$

Then the required matrix U can be expressed as follows:

$$\begin{aligned} V &= (1 - KfK^\dagger)^{1/2}\mathcal{S}, &(70)\\ U' &= \Gamma Y - \Delta' f K^\dagger (1 - KfK^\dagger)^{-1/2}\mathcal{S}, &(71) \end{aligned}$$

which can be straightforwardly checked to satisfy both conditions (22) and (23).

Again for $[B_1, B_2] = 0$ we can explicitly give

$$\Gamma = \begin{pmatrix} e^{\theta_2^{-1}B_2\bar{z}_2 - \theta_1^{-1}B_1^\dagger z_1}|0,0\rangle \\ 0 \end{pmatrix} W^{-1/2}, \tag{72}$$

where the normalization factor W is:

$$W = \langle 0,0| e^{\theta_2^{-1}B_2^\dagger z_2 - \theta_1^{-1}B_1\bar{z}_1} e^{\theta_2^{-1}B_2\bar{z}_2 - \theta_1^{-1}B_1^\dagger z_1}|0,0\rangle, \tag{73}$$

and so we have $k' = k$.

4 A special $U(2)$ 2-instanton solution

To explicitly give the moduli space of a $U(N)$ multi-instanton is a very difficult problem [14]. In the commutative case, [15] solved this problem when the instanton number $k = 2$. For the noncommutative case a similar solution has not appeared yet. However, it is much easier to find some special solutions of $U(N)$ multi-instanton. To illustrate our method, we construct one of noncommutative $U(2)$ 2-instanton in the limit of coincident instantons.

In a special case, these ADHM data (15) can be written as follows:

$$\Delta = \begin{pmatrix} 0 & \sqrt{2\zeta+\rho^2} & 0 & 0 \\ 0 & 0 & \rho & 0 \\ \bar{z}_2 & 0 & -z_1 & -\sqrt{\zeta+\rho^2} \\ 0 & \bar{z}_2 & 0 & -z_1 \\ \bar{z}_1 & 0 & z_2 & 0 \\ \sqrt{\zeta+\rho^2} & \bar{z}_1 & 0 & z_2 \end{pmatrix}, \tag{74}$$

$$\Delta^\dagger = \begin{pmatrix} 0 & 0 & z_2 & 0 & z_1 & \sqrt{\zeta+\rho^2} \\ \sqrt{2\zeta+\rho^2} & 0 & 0 & z_2 & 0 & z_1 \\ 0 & \rho & -\bar{z}_1 & 0 & \bar{z}_2 & 0 \\ 0 & 0 & -\sqrt{\zeta+\rho^2} & -\bar{z}_1 & 0 & \bar{z}_2 \end{pmatrix}, \tag{75}$$

where ρ is the scale of the instanton size. In other words, we have

$$I = \begin{pmatrix} 0 & 0 \\ \sqrt{2\zeta+\rho^2} & 0 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & 0 \\ \rho & 0 \end{pmatrix}, \tag{76}$$

$$B_1 = \begin{pmatrix} 0 & \sqrt{\zeta+\rho^2} \\ 0 & 0 \end{pmatrix}, \qquad B_2 = 0. \tag{77}$$

These ADHM data lead to

$$f = \begin{pmatrix} (Z_1'' + Z_2' + \zeta + \rho^2)(Z_\rho''\,')^{-1} & -\sqrt{\zeta+\rho^2}(Z_\rho''\,')^{-1}\bar{z}_1 \\ -\sqrt{\zeta+\rho^2}z_1(Z_\rho''\,')^{-1} & (Z_1' + Z_2' - \theta_1 + \rho^2)(Z_\rho'\,')^{-1} \end{pmatrix}. \tag{78}$$

In order to make the above expressions more similar to that of the $U(1)$ 2-instanton discussed in [1], we have carefully chosen our notations as follows:

$$Z_1 \equiv z_1\bar{z}_1, \qquad Z_2 \equiv z_2\bar{z}_2, \tag{79}$$

$$Z_1' \equiv Z_1 + \theta_1, \quad Z_1'' \equiv Z_1 + 2\theta_1, \quad \text{etc.} \tag{80}$$
$$Z_2' \equiv Z_2 + \theta_2, \quad Z_2'' \equiv Z_2 + 2\theta_2, \quad \text{etc.} \tag{81}$$

$$Z \equiv (Z_1 + Z_2)^2 - \theta_1 Z_1 + (\theta_2 + \rho^2) Z_2, \tag{82}$$
$$Z'\,' \equiv (Z_1' + Z_2')^2 - \theta_1 Z_1' + (\theta_2 + \rho^2) Z_2', \quad \text{etc.} \tag{83}$$

$$Z_\rho \equiv Z + \rho^2 (Z_1 + Z_2 + \zeta + \rho^2), \tag{84}$$
$$Z_\rho'\,' \equiv Z'\,' + \rho^2 (Z_1' + Z_2' + \zeta + \rho^2), \quad \text{etc.} \tag{85}$$

It is not difficult to obtain

$$1 - KfK^\dagger = 1 - \begin{pmatrix} (2\zeta + \rho^2) f_{22} & 0 \\ 0 & \rho^2 f_{11} \end{pmatrix} = \begin{pmatrix} Z/Z_\rho'\,' & 0 \\ 0 & Z''\,'/Z_\rho''\,' \end{pmatrix}, \tag{86}$$

$$-(\Delta' f K^\dagger)_{*1} = \sqrt{2\zeta + \rho^2} \begin{pmatrix} \sqrt{\zeta + \rho^2}\, \bar{z}_2 \bar{z}_1 \\ -\bar{z}_2 (Z_1 + Z_2 + \theta_2 + \rho^2) \\ \sqrt{\zeta + \rho^2}\, \bar{z}_1 \bar{z}_1 \\ -\bar{z}_1 (Z_1 + Z_2 - \theta_1) \end{pmatrix} / Z_\rho'\,', \tag{87}$$

$$-(\Delta' f K^\dagger)_{*2} = \rho \begin{pmatrix} z_1 (Z_1'' + Z_2') \\ -\sqrt{\zeta + \rho^2}\, z_1 z_1 \\ -z_2 (Z_1'' + Z_2' + \zeta + \rho^2) \\ \sqrt{\zeta + \rho^2}\, z_2 z_1 \end{pmatrix} / Z_\rho''\,', \tag{88}$$

where the subscripts '*1' and '*2' mean the first column and the second column of the matrix $\Delta' f K^\dagger$ respectively. Now we solve the two cases $\theta_2 > 0$ and $\theta_2 < 0$ in sequence.

4.1 The $\theta_2 > 0$ case

Because $[B_1, B_2] = 0$ for this special solution, we have

$$\Psi = I^\dagger e^{-\sum_\alpha \theta_\alpha^{-1} B_\alpha^\dagger z_\alpha} |0,0\rangle = \sqrt{2\zeta + \rho^2} \begin{pmatrix} -\sqrt{\frac{\zeta + \rho^2}{\theta_1}} |1,0\rangle & |0,0\rangle \\ 0 & 0 \end{pmatrix}, \tag{89}$$

so

$$\Pi_0 = \Psi (\Psi^\dagger \Psi)^{-1} \Psi^\dagger = \begin{pmatrix} |0,0\rangle\langle 0,0| + |1,0\rangle\langle 1,0| & 0 \\ 0 & 0 \end{pmatrix}. \tag{90}$$

In fact, a simple comparison between (56) in [1] and the above (82) makes it clear that here Z also annihilates two states: $|0,0\rangle$ and $|1,0\rangle$, so (90) can be obtained directly.

By (46) we introduce a shift operator

$$s = \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, \tag{91}$$

$$t t^\dagger = 1, \quad t^\dagger t = 1 - |0,0\rangle\langle 0,0| - |1,0\rangle\langle 1,0|. \tag{92}$$

The required U is given as follows:

$$V = \begin{pmatrix} (Z/Z'_\rho{}')^{1/2}t^\dagger & 0 \\ 0 & (Z''{}'/Z''_\rho{}')^{1/2} \end{pmatrix}, \tag{93}$$

$$U'_{*1} = \sqrt{2\zeta+\rho^2}\begin{pmatrix} \sqrt{\zeta+\rho^2}\bar{z}_2\bar{z}_1 \\ -\bar{z}_2(Z_1+Z_2+\theta_2+\rho^2) \\ \sqrt{\zeta+\rho^2}\bar{z}_1\bar{z}_1 \\ -\bar{z}_1(Z_1+Z_2-\theta_1) \end{pmatrix}(ZZ'_\rho{}')^{-1/2}t^\dagger, \tag{94}$$

$$U'_{*2} = \rho\begin{pmatrix} z_1(Z''_1+Z'_2) \\ -\sqrt{\zeta+\rho^2}z_1z_1 \\ -z_2(Z''_1+Z'_2+\zeta+\rho^2) \\ \sqrt{\zeta+\rho^2}z_2z_1 \end{pmatrix}(Z''{}'Z''_\rho{}')^{-1/2}. \tag{95}$$

4.2 The $\theta_2 < 0$ case

Unlike the $U(1)$ 2-instanton case in [1], the (2,2) element of the matrix f is well-defined on the vacuum $|0,0\rangle$:

$$f_{22}|0,0\rangle = (2\theta_1+\theta_2+\rho^2)^{-1}|0,0\rangle. \tag{96}$$

Note that this f_{22} tends to that of the $U(1)$ 2-instanton case when ρ tends to zero, as we have expected.

Now we have

$$e^{\theta_2^{-1}B_2\bar{z}_2-\theta_1^{-1}B_1^\dagger z_1}|0,0\rangle = \begin{pmatrix} |0,0\rangle & 0 \\ -\sqrt{\frac{\zeta+\rho^2}{\theta_1}}|1,0\rangle & |0,0\rangle \end{pmatrix}, \tag{97}$$

and so

$$\Gamma = \begin{pmatrix} \sqrt{\frac{\theta_1}{2\theta_1+\theta_2+\rho^2}}|0,0\rangle & 0 \\ -\sqrt{\frac{\zeta+\rho^2}{2\theta_1+\theta_2+\rho^2}}|1,0\rangle & |0,0\rangle \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \tag{98}$$

Then the required U is given as follows:

$$V = \begin{pmatrix} (Z/Z'_\rho{}')^{1/2}t & 0 \\ 0 & (Z''{}'/Z''_\rho{}')^{1/2} \end{pmatrix}, \tag{99}$$

$$\begin{aligned} U'_{*1} &= \begin{pmatrix} \sqrt{\frac{\theta_1}{2\theta_1+\theta_2+\rho^2}}|0,0\rangle \\ -\sqrt{\frac{\zeta+\rho^2}{2\theta_1+\theta_2+\rho^2}}|1,0\rangle \\ 0 \\ 0 \end{pmatrix}\langle 0,0| + \begin{pmatrix} 0 \\ |0,0\rangle \\ 0 \\ 0 \end{pmatrix}\langle 1,0| \\ &\quad + \sqrt{2\zeta+\rho^2}\begin{pmatrix} \sqrt{\zeta+\rho^2}\bar{z}_2\bar{z}_1 \\ -\bar{z}_2(Z_1+Z_2+\theta_2+\rho^2) \\ \sqrt{\zeta+\rho^2}\bar{z}_1\bar{z}_1 \\ -\bar{z}_1(Z_1+Z_2-\theta_1) \end{pmatrix}(ZZ'_\rho{}')^{-1/2}t \end{aligned} \tag{100}$$

and U'_{*2} is the same as given in (95).

5 Discussion

To end this paper, we will explain here how our method can be used to deal with an interesting class of $U(2)$ multi-instantons. This case is quite similar to the elongated $U(1)$ multi-instanton.

We will take the following ADHM data:

$$I = \sqrt{k\zeta + \rho^2} \begin{pmatrix} e_k & 0 \end{pmatrix}, \quad J = \rho \begin{pmatrix} 0 \\ e_1^\dagger \end{pmatrix}, \tag{101}$$

$$B_1 = \sum_{i=1}^{k-1} \sqrt{i\zeta + \rho^2} e_i e_{i+1}^\dagger, \quad B_2 = 0, \tag{102}$$

where

$$e_i^\dagger = (\overset{1}{0}, \cdots, \overset{i-1}{0}, \overset{i}{1}, \overset{i+1}{0}, \cdots, \overset{k}{0}). \tag{103}$$

One can easily check that the above data do form a $U(2)$ k-instanton solution, which includes the $U(2)$ 2-instanton solution in the previous section as a special case. It is the $U(2)$ analog of the elongated $U(1)$ k-instanton [16], and we may call it the 'elongated $U(2)$ k-instanton'.

Because the above data always satisfy the condition $[B_1, B_2] = 0$, it is easy to obtain

$$\Pi_0 = \begin{pmatrix} \sum_{n=0}^{k-1} |n,0\rangle\langle n,0| & 0 \\ 0 & 0 \end{pmatrix}, \tag{104}$$

and to calculate Γ by (72). Then there is no difficulty to work out the instanton configuration.

Acknowledgments

Chuan-Jie Zhu would like to thank Prof. Zhu-Jun Zheng and the hospitality at the Institute of Mathematics, Henan University.

A ADHM Construction for Noncommutative 't Hooft Instantons

In this appendix we apply our method to the construction of noncommutative 't Hooft instantons [17, 18]. We will only discuss the ASD 't Hooft instantons on ASD $\mathbf{R}^{\mathbf{4}}_{\mathrm{NC}}$ (i.e., $\zeta = \theta_1 + \theta_2 = 0$ while $\theta_{1,2} \neq 0$).

Take the following ADHM data [19, 20]:

$$a = \begin{pmatrix} \rho \otimes 1_2 \\ -a^n \otimes \bar{\sigma}_n \end{pmatrix}, \quad \rho = (\rho_1, \cdots, \rho_k), \quad a^n = \mathrm{diag}(a_1^n, \cdots, a_k^n), \tag{105}$$

where ρ_i and a_i^n are constants parameterizing the scale and the position of the ith instanton respectively. By direct calculation one finds

$$f = r^{-2} - r^{-2}\rho^\dagger \phi^{-1} \rho r^{-2}, \qquad r^2 \equiv r^n r^n \tag{106}$$

where

$$r^n = x^n - a^n \equiv \mathrm{diag}(r_1^n, \cdots, r_k^n), \quad \phi = 1 + \rho_i^2 r_i^{-2}. \tag{107}$$

Summation of repeated indices is assumed here for both n and i. It is straightforward to obtain

$$1 - KfK^\dagger = \phi^{-1} \otimes 1_2, \tag{108}$$

$$-\Delta' f K^\dagger = -\Delta' r^{-2} \rho^\dagger \phi^{-1}, \tag{109}$$

where a tensor product with 1_2 is omitted in the second equation.

From the ADHM data (105) we see that the matrices B_1 and B_2 take very simple diagonal form. So we have also $[B_1, B_2] = 0$ and the extra zero modes from (72) are just k coherent states respectively shifted by the complex components of $a_1^n, \cdots, a_k^n$. Denote these (orthonormal) coherent states as $|a_i\rangle$, we will have

$$\Gamma = \begin{pmatrix} \mathrm{diag}(|a_1\rangle, \cdots, |a_k\rangle) \\ 0_k \end{pmatrix}. \tag{110}$$

Then the required U is again obtained by (67-71) where $k' = k$:

$$U = \begin{pmatrix} 0 \\ \Gamma \end{pmatrix} Y + \begin{pmatrix} 1_2 \\ -\Delta' r^{-2} \rho^\dagger \end{pmatrix} \phi^{-1/2} \mathcal{S}. \tag{111}$$

References

[1] Y. Tian and C.-J. Zhu, *Instantons on General Noncommutative R^4*, hep-th/0205110.

[2] N. Nekrasov and A. Schwarz, *Instantons on Noncommutative R^4, and (2,0) Superconformal Six Dimensional Theory*, Commun. Math. Phys. 198 (1998) 689, hep-th/9802068.

[3] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, *Construction of Instantons*, Phys. Lett. A65 (1978) 185.

[4] N. Nekrasov, *Trieste Lectures on Solitons in Noncommutative Gauge Theories*, hep-th/0011095.

[5] K. Furuuchi, *Instantons on Noncommutative R^4 and Projection Operators*, Prog. Theor. Phys. 103 (2000) 1043, hep-th/9912047.

[6] K. Kim, H. Lee and H. S. Yang, *Comments on Instantons on Noncommutative R^4*, hep-th/0003093.

[7] C.-S. Chu, V. V. Khoze and G. Travaglini, *Notes on Noncommutative Instantons*, Nucl.Phys. B621 (2002) 101, hep-th/0108007.

[8] K.-Y. Kim, B.-H. Lee and H. S. Yang, Phys. Lett. B523, 357 (2001), hep-th/0109121.

[9] V. G. Drinfeld and Y. I. Manin, Commun. Math. Phys. 63, 177 (1978),
H. Osborn, Nucl. Phys. B159, 497 (1979).
V. V. Khoze, M. P. Mattis and M. J. Slater, Nucl. Phys. B **536** (1998) 69, hep-th/9804009.

[10] A. Konechny and A. Schwarz, *Introduction to M(atrix) Theory and Noncommutative Geometry*, hep-th/0012145; *Introduction to M(atrix) Theory and Noncommutative Geometry, Part II*, hep-th/0107251.

[11] M. R. Douglas and N. A. Nekrasov, *Noncommutative Field Theory*, Rev. Mod. Phys. 73 (2002) 977, hep-th/0106048.

[12] R. J. Szabo, *Quantum Field Theory on Noncommutative Spaces*, hep-th/0109162.

[13] H. Nakajima, *Heisenberg Algebra and Hilbert Schemes of Points on Projective Surfaces*, alg-geom/9507012 and *Lectures on Hilbert Schemes of Points on Surfaces*.

[14] N. H. Christ, N. K. Stanton, and E. J. Weinberg, Phys. Rev. D18, 2013 (1978).

[15] N. B. Pomeroy, *The U(N) ADHM Two-Instanton*, hep-th/0203184.

[16] T. Ishikawa, S. Kuroki and A. Sako, *Elongated U(1) Instantons on Noncommutative* R^4, J. High Energy Phys. **0112** (2001) 000, hep-th/0109111.

[17] D. H. Correa, G. S. Lozano, E. F. Moreno and F. A. Schaposnik, *Comments on the U(2) noncommutative instanton*, Phys. Lett. B **515** (2001) 206, hep-th/0105085.

[18] O. Lechtenfeld and A. D. Popov, JHEP 03, 040 (2002), hep-th/0109209.

[19] E. Corrigan, D. B. Fairlie, S. Templeton and P. Goddard, *A Green's function for the general self-dual gauge field*, Nucl. Phys. B **140** (1978) 31.

[20] H. Osborn, *Solutions of the Dirac equation for general instanton solutions*, Nucl. Phys. B **140** (1978) 45.