A note on quantization of matrix models

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Abstract

The issue of non-perturbative background independent quantization of matrix models is addressed. The analysis is carried out by considering a simple matrix model which is a matrix extension of ordinary mechanics reduced to 0 dimension. It is shown that this model has an ordinary mechanical system evolving in time as a classical solution. But in this treatment the action principle admits a natural modification which results in algebraic relations describing quantum theory. The origin of quantization is similar to that in Adler's generalized quantum dynamics. The problem with extension of this formalism to many degrees of freedom is solved by packing all the degrees of freedom into a single matrix. The possibility to apply this scheme to field theory and to various matrix models is discussed.

1 Introduction

Matrix models were proposed as a non-perturbative definition of string theory some time ago. There are several versions of matrix models. Generally they are defined as reductions of certain Super Yang Mills theories to one dimension (BFSS matrix model [1]) or to zero dimension (IKKT matrix model [2]).

One of the most interesting properties of these models is the dynamical origin of spacetime. For example the action of IKKT model

$$S = -Tr(\frac{1}{4}[A_\mu, A_\nu][A^\mu, A^\nu] + fermions) \qquad (1)$$

does not contain any a priori spacetime structure at all. The later arises from classical solutions of the model (1) and it is associate to the distribution of the eigenvalues of the matrices A_μ [3]. Spacetime is said to be "generated dynamically" in these models.

Besides spacetime structure matrix models also give rise to various objects propagating in spacetime such as D-instantons, D-strings, D0-branes, membranes etc., depending on

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a particular model chosen, and also to local interactions between those objects. This provides some evidence for the conjecture that matrix models can be a constructive non-perturbative definition of string theory.

Because the definition (1) of a matrix model do not include any background spacetime structure one may hope that this model could provide a background independent definition of string theory so that different string theories in various background spacetimes would be different solutions of a single theory.

Classicaly the theory indeed looks background independent[1]. However the way it is generally quantized is the following. First one picks up a classical solution to the matrix model, $A_\mu = X_\mu$, representing a particular spacetime structure with some background fields in it. Then one considers small perturbations around the classical solution

$$A_\mu = X_\mu + \tilde{A}_\mu \tag{2}$$

representing the objects moving in the background spacetime and interactions between them. Then one quantizes $\tilde{A}_\mu$. It is clear that the resulting theory is no longer background independent.

The key question which is addressed in this paper is whether we can define a non-perturbative background independent quantization of matrix models. The definition of non-prturbative quantization of a theory is generally straightforward. One has to represent all the observables and all the symmetries of the theory by operators on a certain Hilbert space. However when we try to apply this to a matrix model a certain puzzle emerges. Basic observables, matrices A_μ, which are to be represented by hermitian operators are already hermitian operators on a certain Hilbert space. And the basic symmetries of the theory, U such that $S[U^{-1}A_\mu U] = S[A_\mu]$, which are to be represented by unitary transformations are already unitary operators in the same Hilbert space. The straightforward application of the quantization procedure then would lead us to something like "operator-valued operators" which would be simply operators because any two Hilbert spaces are isomorphic to each other. Therefore quantization doesn't seem to change the shape of the theory.

A natural question then arises: Can matrix models produce quantum theory without quantization in the usual sense? If so, how can it be realized?

One proposal for this was made by Smolin in a recent paper [5] which is that matrix model can be interpreted as a non-local hidden variables theory, the eigenvalues being quantum observables and the entries being hidden variables. Quantum mechanics for the eigenvalues is then reproduced in the ordinary statistical mechanical description of the model.

In this paper we will study the possibility that in matrix model framework where there is no a priory spacetime structure classical and quantum theory can be brought much closer

[1] The action (1) contains a fixed flat Minkowskian metrics $\eta^{\mu\nu}$ contracting lower indices, though. However in [3] was considered a possibility that this model can also in principle describe curved spacetime. The metric $\eta^{\mu\nu}$ is to be understood in this case as a metric in tangent space and the frame field forming the metrics of the manifold originates from matrices. All the background information encoded in $\eta^{\mu\nu}$ is the dimension and the signature. Here it is worth mentioning also cubic matrix models [4] which do not depend on any background metric structure at all.

to each other than they are in the presence of a background spacetime. The later would imply a possible relation between origin of space-time and quantization. There are some indications on existence of such a relation also pointed out in [5]. Many of notions on the basis of which the distinction between classical and quantum theory is generally made rely on the existence of classical spacetime. For example classical system is recognized as having a definite smooth trajectory, something what quantum system doesn't have. But the very notion of smooth trajectory can not be defined without using the the classical notion of spacetime and its differential structure. Also quantum theory is characterized by its generic non-locality. But the notion of locality also relies on classical spacetime. Given all the above it is very plausible that matrix models with the generalized notion of spasetime that they give rise to can provide framework which is general for classical and quantum theory. All the distinction s between them should be quantitative such as spectra of physical observables.

In the present paper the problem of quantization of matrix models is studied on a simple example which is a matrix extension of mechanical system. In section 2 the model is defined. It is shown that this matrix model reproduces ordinary mechanics. In section 3 a natural modification of this model is proposed. The equations of motion of this modified model describe quantum mechanics. Two alternative derivations of basic equations of quantum mechanics are presented each giving the same result. In section 4 other possible modifications of the model are discussed. They provide some generalizations of quantum mechanics such as that with minimum length uncertainty relation. In section 5 the scheme is generalized to systems with many degrees of freedom. This is done by a specific way of encoding the degrees of freedom which is very natural for matrix models. In section 6 a possibility to generalize the model to field theory as well as to various matrix models related to string theory and loop quantum gravity is discussed.

2 Matrix model for a mechanical system with one degree of freedom

In this section we will define a matrix model which reproduces the ordinary mechanics of a system with one degree of freedom x described by the following action S_L in the Lagrangian form.

$$S_L[x] = \int dt \Big\{ \frac{1}{2}\Big(\frac{dx}{dt}\Big)^2 - V(x) \Big\} \tag{3}$$

Here V is an arbitrary function of x which represents the self-interaction of x. The equation of motion can be obtained by requiring that the variation of the action (3) with respect to x vanish.

$$\frac{\delta S_L}{\delta X(t)} = 0. \tag{4}$$

This results in the standard Newtonian equation of motion:

$$\frac{d^2x}{dt^2} + \partial_x V(x) = 0. \tag{5}$$

All the constructions of this section can be equally done for both the Lagrangian action principle (3,4) and the Hamiltonian action principle

$$S_H[x,p] = \int dt \Big\{ p\frac{dx}{dt} - \frac{p^2}{2} - V(x) \Big\}, \tag{6}$$

which is equivalent to (3) in the sense that if minimize this action with respect to the momentum p,

$$\frac{\delta S_H[x,p]}{\delta p} = 0 \tag{7}$$

and substitute the solution to the equation (7) $p = p(x)$ into the action (6) we will find that

$$S_H[x, p(x)] = S_L[x], \tag{8}$$

and therefore the action (3) and the action (6) result in the same equation for x. Below we will consider the Lagrangian form of the matrix action. The generalization to the Hamiltonian form is straightforward.

The basic entries of the action (3) are the configuration coordinate x and the time derivative $\frac{d}{dt}$ and also the integration operation $\int dt$. All this operations can be thought of as realizations of a multiplication operator, a derivative operator, and a trace operation respectively on a certain space of functions $\psi(t)$.

$$\begin{aligned} x(t) \times \psi(t) \quad &\leftrightarrow \quad X\psi \\ \frac{d}{dt}\psi(t) \quad &\leftrightarrow \quad D\psi \\ \int dt x(t) \quad &\leftrightarrow \quad TrX. \end{aligned} \tag{9}$$

We do not specify the class of functions $\psi(t)$ nor do we endow the space of them with any Hilbert space structure. So all that matters is the algebraic relations between those operators.

Given the correspondence (9) the action (3) can be rewritten in the following operator form:

$$S_L[x] = S_{op}[d/dt, x(t)], \tag{10}$$

where

$$S_{op}[D, X] = Tr\Big\{ \frac{1}{2}[D, X]^2 - V(X) \Big\}, \tag{11}$$

where $[A, B] = AB - BA$ is the operator commutator.

Now one can loosen the requirement of the correspondence (9) and consider D and X to be unknown operator valued variables. Of course the physical meaning of the operators X and D as the position of the particle and the evolution generator respectively has to be retained. All we get rid of is the relation between them as between a smooth function of a certain parameter and a derivative operator with respect to this parameter.

Another way to define the action (11) which is more similar to the usual way how matrix models are introduced is to consider the action for a Higgs field in 0+1 dimension

$$S = \int dx^0 Tr[(D_0 x)^2 - V(x)]. \tag{12}$$

Here x is the $SU(N)$ Higgs field and $D_0 x = \partial_0 x + [A_0, x]$ is the $SU(N)$ covariant derivative along x^0 direction. Now by reduction of this model down to 0 dimension and taking the limit $N \to \infty$ we obtain (11).

Now the statement is that the matrix model defined by the action (11) has the mechanical system defined by the action (3) as its solution.

To see this consider the variation of the action (11) with respect to D and X:

$$\delta S_{op}[D, X] = Tr\Big\{\Big[- [D, [D, X]] - \partial_X V(X)\Big]\delta X\Big\} + Tr\Big\{[D, \delta X[D, X]]\Big\}$$
$$+Tr\Big\{[X, [D, X]]\delta D\Big\} + Tr\Big\{[X, \delta D[D, X]]\Big\}. \tag{13}$$

The second and the fourth terms in the variation would be equal to zero if we were allowed to do cyclic permutations in the trace of a product of several operators. The later is true for finite dimensional operators but this property generally does not extend to infinite dimensional ones. Here we will simply require that the variations δX and δD are such that

$$Tr\Big\{[D, \delta X[D, X]]\Big\} = 0 \tag{14}$$

and

$$Tr\Big\{[X, \delta D[D, X]]\Big\} = 0. \tag{15}$$

The condition (14) have a simple classical interpretation. According to classical correspondence (9)we can rewrite (14) as

$$0 = Tr\Big\{[D, \delta X[D, X]]\Big\} = \int_{t_1}^{t_2} dt \frac{d}{dt}\Big(\delta x(t) \frac{dx(t)}{dt}\Big) = \delta x(t) \frac{dx(t)}{dt}\Big|_{t=t_1}^{t=t_2} \tag{16}$$

The later holds when

$$\delta x(t_1) = \delta x(t_2) = 0. \tag{17}$$

Thus, the condition (14) means that we vary the action with respect to a function keeping it fixed at the endpoints. This is how it is usually done in the Euler-Lagrange variational principle. The condition (15) has no classical analog and therefore it is difficult to visualize.

As usual the variational principle is

$$\delta S_{op}[D, X] = 0. \tag{18}$$

Given the conditions (14) and (15) and that the variations δX and δD are independent we have the following equations of motion

$$[D, [D, X]] + \partial_X V(X) = 0, \tag{19}$$

$$[X,[D,X]] = 0. \tag{20}$$

Classically eq. (19) coincides with the newtonian equation of motion. Eq. (20) has no classical analogs.

From now on we will look for solutions of operator equations such as (19,20) in a representation where X is a multiplication operator. Thus we assume that the operator algebra can be realized on a space of functions $\psi(X)$. This means that any operator from our algebra can be represented by the following series (finite or infinite)

$$O = \sum_{i,j=0}^{\infty} o_{ij} X^i \left(\frac{d}{dX}\right)^j = \sum_{j=0}^{\infty} o_j(X) \left(\frac{d}{dX}\right)^j. \tag{21}$$

We can now use this form for D and substitute it into eq.(20). This will result in the following solution for D

$$D = d_1(X)\frac{d}{dX} + d_0(X), \tag{22}$$

where $d_1(X)$ and $d_2(X)$ are arbitrary functions of X. Now we can introduce a new variable t such that

$$\frac{dx(t)}{dt} = d_1(x(t)). \tag{23}$$

Then the solution to the equation (20) can be rewritten as

$$X = x(t)\times, \quad D = \frac{d}{dt} + d_0(t). \tag{24}$$

Here t is an arbitrary parameter, which means that the relations (9) were derived up to reparameterization. But if we recall that physically by D we mean the evolution generator in a background time this freedom is fixed. Thus, all the relations (9) are recovered and if we substitute them to the initial action (11) we recover the classical action (3). Therefore the matrix model based on the action (11) describes classical dynamics of the system defined by the action (3).

3 Quantization without quantization

Now one can show that by a minor modification of the action (11) the equations of motion (19,20) can be turned into those of quantum mechanics. The equations of motion in this framework are derived from variation of the action not only with respect to X but also with respect to D. Therefore the action (11) admits a nontrivial modification by adding an extra term to the action which depends purely on D. In this section we consider the simplest possible such term which is linear in D. The issue of uniqueness of this modification and the results of other possible modifications will be discussed in the next section.

By adding a linear in D term to the action (11) we obtain

$$S_{op}[D,X] = Tr\Big\{\frac{1}{2}[D,X]^2 - V(X) - iD\Big\}. \tag{25}$$

Here the coefficient in front of D is taken to be imaginary to provide hermiticity of the extra term. Being interpreted as a derivative operator D has to be anti-hermitian. Because we don't have any Hilbert space structure yet the hermiticity relations are to be understood as formal *-relations.

By variation of the action (25) we obtain the following modified equations of motion

$$[D,[D,X]]+\partial_X V(X)=0, \tag{26}$$

$$[[D,X],X]=iI, \tag{27}$$

where I is the identity operator. The equations (19) and (26) are the same, eqs. 20) and (27) differ by them of iI in the r.h.s. We can now solve eq. (27) for D in the form (21). The result satisfying anti-hermiticity condition have the form

$$D=\frac{i}{2}\frac{d^2}{dX^2}-d_1(X)\frac{d}{dX}-\frac{1}{2}\partial_X(d_1(X))-id_0(X). \tag{28}$$

Here d_0 and d_1 are arbitrary real functions of x. To find them we should substitute D in the form (28) to eq.(26), which will result in the following equation

$$V(X)=d_0(X)+\frac{1}{2}d_1(X)^2+c. \tag{29}$$

Here c is an arbitrary constant. Equation (29) can be solved with respect to d_0 and substituted to (28). The resulting expression for D

$$D=\frac{i}{2}\frac{d^2}{dX^2}-d_1(X)\frac{d}{dX}-\frac{1}{2}\partial_X(d_1(X))-\frac{i}{2}d_1(X)^2-iV(X)+ic \tag{30}$$

depends on an arbitrary function $d_1(X)$ and is not therefore completely determined. Here we can recall that the equations (26,27) are invariant with respect to unitary transformations

$$X\to UXU^{-1},\quad D\to UDU^{-1}. \tag{31}$$

Now consider the following unitary transformation

$$U=\exp i\int\limits_{X_0}^{X} d_1(X)dX. \tag{32}$$

It is obvious that X is not changed by this transformation

$$X'=UXU^{-1}=X. \tag{33}$$

By direct computation one can check that

$$D'=UDU^{-1}=\frac{i}{2}\frac{d^2}{dX^2}-iV(X)+ic. \tag{34}$$

Thus the dependence on $d_1(X)$ can be removed from the final expression for D by a unitary transformation and the result is precisely the relation between the evolution generator and, the hamiltonian of ordinary quantum mechanics. It may be interpreted as the Schoedinger equation when represented on states or the Heisinberg equation when applied to operators. The only remaining freedom is the constant c indicating the arbitrariness in defining the ground state energy of the system.

The origin of quantum mechanical relationships between operators become more explicit if we consider the hamiltonian version of the action (11)

$$S_{op,h}[D,X,P] = Tr\Big\{P[D,X] - \frac{1}{2}P^2 - V(X)\Big\}, \tag{35}$$

which results in the following equivalent to (19,20) equations of motion

$$[D,X] - P = 0, \tag{36}$$

$$[D,P] + \partial_X V(X) = 0, \tag{37}$$

$$[X,P] = 0. \tag{38}$$

Now if we again add an extra term of iD to it

$$S_{op,q}[D,X,P] = Tr\Big\{P[D,X] - \frac{1}{2}P^2 - V(X) + iD\Big\}, \tag{39}$$

the modified equations of motion take the form

$$[D,X] - P = 0, \tag{40}$$

$$[D,P] + \partial_X V(X) = 0, \tag{41}$$

$$[P,X] = iI. \tag{42}$$

Eqs. (40,41) are those of ordinary hamiltonian mechanics which hold both in classical and quantum regime. Eq.(42) is the usual quantum mechanical commutation relation. This is where the quantization comes from. The origin of quantum mechanical commutation relation (42) in this model is analogous to that in generalized quantum mechanics proposed by Adler [6]. There a matrix extension of ordinary mechanics of the form (12) was considered. Then the invariance with respect to unitary transformations was interpreted as a gauge symmetry. The commutation relation of the form (42) was then induced by a term linear in the corresponding gauge field in the action. The relation between Adler's formulation of quantum mechanics and BFSS matrix model was also studied by Minic [7].

Now from (40,41,42) one can derive the equation analogous to (34). First, from the equations (41) and (40) and also (42) it follows that

$$[\frac{P^2}{2} + V(X), D] = 0, \tag{43}$$

which implies that

$$\frac{P^2}{2} + V(X) = f(D). \tag{44}$$

By taking the commutator of the last equation with X and using the equations (42) and (40) we find that $f(D) = iD + c$, where c is an arbitrary constant playing the same role as that in (34), and therefore

$$\frac{P^2}{2} + V(X) + c = iD. \tag{45}$$

This equation coincides with (34) given that the momentum is represented by derivative operator $P = i\frac{d}{dX}$. Together with the equation (42) it forms the compete set of equations of quantum mechanics of the system under consideration, the equations (40) and (41) can be derived from them.

4 Deformations of higher order. Modified uncertainty principle and Schrodinger equation

One can note that the addition of a term of iD to the action (39) is not the only possible modification of the action (35) preserving the classical part of it. In principle we can add to the action (35) an arbitrary function of D, $f(D)$. The function of D is to be understood as a series in powers of D.

$$f(D) = f_0 + f_1 D + f_2 D^2 + ... \tag{46}$$

As to whether this series can be understood as an asymptotic expansion we can note that D has a dimension of inverse time. Therefore for dimension to mach we will need to introduce a constant of the dimension of time. The expansion will be in powers of τD where τ is a certain fixed time parameter. Therefore the expansion (46) is asymptotic if the time scale of phenomena considered is much larger than τ. This is satisfied if τ is Plank time, the only known fundamental time scale in nature. Therefore it makes sense to consider the modification of the action (35) to the next (second) order in D:

$$S_{op,q}[D, X, P] = Tr\Big\{P[D, X] - \frac{1}{2}P^2 - V(X) + iD + \tau D^2\Big\}. \tag{47}$$

For simplicity we consider the system which is classically a harmonic oscillator, i.e. $V(X) = \omega^2 X^2$. Then the equations of motion obtained by variation of the action (47) read

$$[D, X] - P = 0, \tag{48}$$

$$[D, P] + \omega^2 X = 0, \tag{49}$$

$$[P, X] = iI + 2\tau D. \tag{50}$$

One can check by direct computation that the equation (43) holds also in this case and because $V(X)$ is quadratic this follows from equations (48) and (49) only. Thus, as before we have the following relation

$$\frac{P^2}{2} + \omega^2 X^2 = f(D). \tag{51}$$

Finally, $f(D)$ can be found by commuting the last relation with X or P using the equations (48,49,50). We find that $f(D) = iD + \tau D^2$. Therefore instead of the evolution equation (45) we will have

$$\frac{P^2}{2} + \omega^2 X^2 = iD + \tau D^2. \tag{52}$$

This is a quadratic equation for D and the solution is

$$iD = \frac{1 \pm \sqrt{1 - 4\tau\left[\frac{P^2}{2} + \omega^2 X^2\right]}}{2\tau} \tag{53}$$

Now we can take the solution which has the right limit as $\tau \to 0$ and expand it in powers of τ. We will have

$$iD = \frac{P^2}{2} + \omega^2 X^2 - \tau\left[\frac{P^2}{2} + \omega^2 X^2\right]^2 + ... \tag{54}$$

This is what the evolution generator looks like to the first order in τ.

The commutation relation between the position and the momentum operators is also modified. By substituting (54) into (50) we find to the first order in τ

$$[P, X] = i\left[I + 2\tau\left(\frac{P^2}{2} + \omega^2 X^2\right)\right]. \tag{55}$$

Such a commutation relation is a natural generalization of the basic commutation relations of quantum mechanics which may underlie an uncertainty principle with minimum length and minimum momenta or, in the case of free particle ($\omega = 0$), minimum length only. Such possibility was considered by Kempf *et al* [8], [9]. For (55) to be minimum length uncertainty relation it is necessary that τ be positive. For negative τ (55) doesn't have sensible interpretation.

The modification of the action considered in this section is possible only if our model don't have to be general covariant. If we consider a possibility to quantize reparameterization invariant model then the modification of the action (11) shouldn't violate this invariance. As Tr is interpreted as an integral over dt and D is interpreted as the time derivative the only possible reparameterization invariant modification of the action (11) is linear in D. Also we can notice that the correction to the ordinary quantum mechanical commutation relation in (55) is proportional to the hamiltonian. But in reparameterization invariant system hamiltonian vanishes. Thus, for general covariant systems the modification of the action (11) resulting in ordinary quantum mechanics is unique.

5 Generalization to the systems with many degrees of freedom

The generalization of all the above to systems with N degrees of freedom is not straightforward. If we simply took one copy of the action (39) per one degree or freedom and considered a sum of them adding also some terms representing interaction between these degrees of freedom then instead of having one copy of the equation (42) per each degree of freedom we would have a single equation

$$\sum_{i=1}^{N}[P_i, X_i] = iN, \tag{56}$$

for the whole system. This is not enough to recover the the complete set of quantum mechanical equations of the system considered. The same problem also arose in [6]. A solution to this problem based on thermodynamical considerations has been given by Adler and Millard [10]. It is based on the fact that the net non-commutativity for all degrees of freedom given by (56) is distributed uniformly between different degrees of freedom in thermodynamical approximation. This is analogous to how each degree of freedom carry a $(1/2)kT$ portion of energy at equilibrium.

Here we will give another solution to this problem which is very natural in the context of matrix models and which is exact, i.e. valid also apart from thermodynamical equilibrium. It is based on a specific way of encoding of the degrees of freedom of the theory which is extensively used in various matrix models [1, 2, 4].

Let our system have N degrees of freedom. They can be represented by N infinite dimensional matrices commuting with each other $X_1, X_2, ..., X_N$, $[X_I, X_J] = 0$. Then we can construct the following operator

$$\mathcal{X} = \begin{pmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & X_N \end{pmatrix} = diag(X_1, X_2, .., X_N). \tag{57}$$

This is an infinite dimensional matrix divided into N infinite dimensional blocks located at diagonal. It acts on extended space which is direct sum of N copies of the space where a matrix representing a single degree of freedom act.

The matrix (57) will be treated as a whole i.e. as a single matrix-valued degree of freedom. Therefore we need to introduce some tools which would allow us to identify the degrees of freedom of our interest within this matrix. The natural additional structure we have here is that of linear space $\mathbb{R}^N$ the dimensions of which are associate with the degrees of freedom of the system. One can construct linear transformations on this space. They are $N \times N$ matrices whose entries are arbitrary c-numbers. Among these matrices we can find projectors and by applying them extract the degrees of freedom of our interest. Invertible matrices form a group $GL(N)$. In the case of identical non-interacting particles this is a symmetry of the theory. However, in general case this symmetry is broken by interaction.

Now one needs to define the time evolution generator D acting on the extended space. In the present paper we will consider the situation in which time is unique for all degrees of freedom. This is the way how ordinary mechanics is usually described. However keeping in mind the possibility to apply this scheme to background independent matrix models one may notice that this framework admits a natural generalization for multi-fingered time.

Because time is unique for all degrees of freedom D must be a c-number in $\mathbb{R}^N$, i.e. it must commute with all matrices from $GL(N)$,

$$\mathcal{D} = D \otimes I_{N\times N} = \begin{pmatrix} D & 0 & \cdots & 0 \\ 0 & D & \cdots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & \cdots & 0 & D \end{pmatrix}. \tag{58}$$

This is where the information about the number of degrees of freedom of the system is encoded. The system has N degrees of freedom if there is a unitary transformation by which the evolution generator $\mathcal{D}$ can be brought in the form (58) with N identical blocks D on the diagonal. Transformations from $GL(N)$ keep the form (58) unchanged. Now these transformations can be used to bring an arbitrary matrix $\mathcal{X}$ to block-diagonal form (57). Given the usual relation between coordinate and momentum $\mathcal{P} = [\mathcal{D}, \mathcal{X}]$ and the form (58) of the operator $\mathcal{D}$ canonical coordinate and momentum can be block-diagonalized simultaneously. So the canonical momentum can be also represented in the form $\mathcal{P} = diag(P_1, P_2, .., P_N)$.

Thus, the generalization of the equation (39) for a system with N degrees of freedom has the following form.

$$S_{op,q}[D, \mathcal{X}, \mathcal{P}] = Tr\Big\{\mathcal{P}[D, \mathcal{X}] - \frac{1}{2}\mathcal{P}^2 - \mathcal{V}[\mathcal{X}] + iD\Big\}, \tag{59}$$

where the interaction is introduced via

$$\mathcal{V}[\mathcal{X}] = \sum_{m=1}^{\infty} \mathcal{X}V_{m,1}\mathcal{X}V_{m,2}...\mathcal{X}V_{m,m}. \tag{60}$$

Here $V_{i,j}$ are fixed $N\times N$ matrices whose entries are c-numbers. A model can be specified by a particular choice of the matrices $V_{i,j}$. Specifically, if all the $V_{i,j}$ are diagonal in the same basis where $\mathcal{X}$ and $\mathcal{P}$ are block diagonal they introduce self interaction of the degrees of freedom X_I. Interaction between X_I and X_J, $I \neq J$, can be introduced via non-diagonal elements of $V_{i,j}$. Note that $\mathcal{V}[\mathcal{X}]$ is a matrix of potential functions, not a single potential function. This is a generalization of ordinary mechanical system in which the forces may be non-conservative. In the present paper we are interested in reproducing ordinary mechanical systems, i.e. those with conservative forces induced by a single overall potential $V(X_1, X_2, ..., X_N)$. So we will restrict ourselves to the situation when the matrices $V_{i,j}$ are such that $\mathcal{V}[\mathcal{X}] = V(X_1, X_2, ..., X_N) \otimes I_{N\times N}$.

The action (59) results in the following equations of motion

$$[\mathcal{D}, \mathcal{X}] - \mathcal{P} = 0, \tag{61}$$

$$[\mathcal{D}, \mathcal{P}] + \partial_{\mathcal{X}}\mathcal{V}(\mathcal{X}) = 0, \tag{62}$$

$$[\mathcal{P}, \mathcal{X}] = iI. \tag{63}$$

which completely coincide to the equations (40,41,42) of the system with one degree of freedom. As we mentioned above the fact that this system describes N degrees of freedom follows from the assumption that the evolution generator $\mathcal{D}$ can be cast in the block-diagonal form (58) by a unitary transformation.

And there is still a question: is the set of equations (61,62,63) sufficient to completely describe quantum theory of the system? Note that the operator commutation relations are encoded in the equation (63) which also can be rewritten as

$$[P_J, X_J] = iI, \tag{64}$$

while in canonical quantization the following set of commutation relations is imposed

$$[P_J, X_K] = iI\delta_{JK}. \tag{65}$$

So the commutation relations (65) with $J \neq K$ are missed in the equation (65). And the question is whether they can be recovered somehow from the equations (61,62,63).

We will address this question by considering a system with two degrees of freedom. The generalization of this scheme to arbitrary number of degrees of freedom is straightforward but require longer calculations. Let us start with Lagrangian equivalent of eqs.(61,62,63)

$$[\mathcal{D}, [\mathcal{D}, \mathcal{X}]] + \partial_{\mathcal{X}} \mathcal{V}(\mathcal{X}) = 0, \tag{66}$$

$$[[\mathcal{D}, \mathcal{X}], \mathcal{X}] = iI. \tag{67}$$

By taking into account that $\mathcal{D} = D \otimes I_{2\times 2}$ and $\mathcal{V} = V \otimes I_{2\times 2}$ and taking $\mathcal{X}$ in block-diagonal form (57) for $N = 2$ the equations (66,67) can be rewritten as the following set of equations for D

$$\begin{aligned} &[D, [D, X_1]] + \partial_{X_1} V(X_1, X_2) = 0 \\ &[D, [D, X_2]] + \partial_{X_2} V(X_1, X_2) = 0 \end{aligned} \tag{68}$$

$$\begin{aligned} &[[D, X_1], X_1] = iI \\ &[[D, X_2], X_2] = iI \end{aligned} \tag{69}$$

To solve these equations for D we will use a pattern similar to (21). Generalized to the case of two degrees of freedom it will have the following form

$$D = \sum_{i,j=0}^{\infty} d_{ij}(X_1, X_2) \left(\frac{\partial}{\partial X_1}\right)^i \left(\frac{\partial}{\partial X_2}\right)^j. \tag{70}$$

The coefficients d_{ij} are to be found from eqs. (68,69). By substituting (70) into (69) we find (for convenience we introduced the following notations for nonzero coefficients $f = d_{11}$, $g_1 = d_{10}$, $g_2 = d_{01}$, $h = d_{00}$)

$$iD = -\frac{1}{2}\frac{\partial^2}{\partial X_1^2} - \frac{1}{2}\frac{\partial^2}{\partial X_1^2} + f\frac{\partial^2}{\partial X_1 X_2} + g_1\frac{\partial}{\partial X_1} + g_2\frac{\partial}{\partial X_2} + h. \tag{71}$$

From (71) one can derive expressions for momenta

$$P_1 = i[D, X_1] = -\frac{\partial}{\partial X_1} + f\frac{\partial}{\partial X_2} + g_1$$

$$P_2 = i[D, X_2] = -\frac{\partial}{\partial X_2} + f\frac{\partial}{\partial X_1} + g_2. \tag{72}$$

It is easy to see that in this special case the commutation relations (65) with $J \neq K$ which do not enter the equations of motion explicitly are determined by the function f in (71). And the question is now whether f can be determined by eqs. (20). To recover ordinary quantum mechanics we need to prove that $f = 0$.

Eqs. (20) after substituting (71) and (72) into them can be solved by considering the resulting expression as a polynomial in derivatives and requiring that the coefficient in front of each term vanish. The terms with second order derivatives vanish if

$$\partial_{X_2} f + f\partial_{X_1} f = 0 \quad \text{and} \quad \partial_{X_1} f + f\partial_{X_2} f = 0, \tag{73}$$

which immediately implies that $f = const$. The later also simplifies the condition of vanishing of the first order terms, which is now

$$\partial_{X_1} g_2 - f\partial_{X_2} g_2 + f\partial_{X_1} g_1 - \partial_{X_2} g_1 = 0. \tag{74}$$

Now one can try to remove the linear in derivatives terms from the expression (71) for D by a unitary transformation of the type (31) with $U = \exp(F)$. For this F has to satisfy the following equations

$$-\partial_{X_1} F + f\partial_{X_2} F + g_1 = 0$$

$$-\partial_{X_2} F + f\partial_{X_1} F + g_2 = 0. \tag{75}$$

This system of equations doesn't have a solution for arbitrary g_1 and g_2. They have to satisfy a certain condition. By taking derivatives of the equations (75) and combining them we find that solvability condition for the system (75) exactly coincide with (74) and is therefore satisfied automatically. Thus, there always exist a unitary transformation by which the evolution generator can be reduced to the form

$$iD = -\frac{1}{2}\frac{\partial^2}{\partial X_1^2} - \frac{1}{2}\frac{\partial^2}{\partial X_1^2} + f\frac{\partial^2}{\partial X_1 X_2} + h, \tag{76}$$

where f is a constant. Finally, by substituting D in this form into (20) we find the following equations for h.

$$\partial_{X_1} h - f\partial_{X_2} h = \partial_{X_1} V$$

$$\partial_{X_2} h - f\partial_{X_1} h = \partial_{X_2} V. \tag{77}$$

Again we can derive the solvability condition for this system of equations which is

$$f(\partial_{X_2}^2 - \partial_{X_1}^2)V = 0. \tag{78}$$

For arbitrary V there is only one solution to this equation, $f = 0$, which as it was mentioned above results in the standard quantum mechanical commutation relations. Then the solution for (77) is $h = V + c$ and the resulting expression for the evolution generator is

$$iD = -\frac{1}{2}\frac{\partial^2}{\partial X_1^2} - \frac{1}{2}\frac{\partial^2}{\partial X_2^2} + V + c. \tag{79}$$

Thus, although the equations of motion derived from the action principle (59) do not include all the equations of quantum mechanics explicitly, they do unambiguously describe the standard quantum mechanics.

There is however an exception. The equation (78) may have also solutions with $f \neq 0$. This happens when

$$(\partial_{X_2}^2 - \partial_{X_1}^2)V = 0, \tag{80}$$

i.e. when the degrees of freedom are identical and the interaction between them is linear. This is however a very special case and it's difficult to judge on this basis whether the solutions with $f \neq 0$ lead to any new physics.

6 Discussion

In this section we will discuss the possibility to apply the above scheme to systems with infinite number of degrees of freedom, i.e. to second quantized theories.

Field theory

It is straightforward to write down an action analogous to (11) for a scalar field. It will have the form of early reduced models [11]

$$S = Tr\Big\{[D_\mu, \Phi]^2 - V(\Phi)\Big\}, \tag{81}$$

where Tr stays for $\int d^4x$ and D_μ for $\frac{\partial}{\partial x_\mu}$. Contrary to [11] D_μ are now not fixed and are treated as independent variables. It is easy to derive the equations of motion for Φ and D_μ by varying (81) and see that the ordinary classical scalar field theory solves those equations.

One can consider a modification of the action (81) analogous to (39) by adding a term to S which depends only on D_μ. There are many possible expressions that could be made of D_μ. Here we will mention only those which are generally-covariant.

There is no covariant term linear in D_μ that could be added to (81). There is a covariant first order derivative operator acting on spinors which is the Dirac operator $\not{D} = \gamma^\mu D_\mu$. However $\not{D}$ is traceless and therefore doesn't result in a nontrivial contribution to (81). Thus, the only way to get a covariant term linear in D_μ is to introduce a fixed background operator and combine it with D_μ in such a way that the resulting expression would be covariant. We consider an expression depending on D_0 only. In this case the equations for D_i are the same as for classical system. We can solve them in the same way thereby introducing a three dimensional space. We can use differential structure of this space for further construction. The expression linear in D_0 can be made covariant by multiplying it by a three dimensional density. To be non-dynamical this density should be made completely of coordinates. The only such density known is the three dimensional delta function $\delta^3(x_i - x_i')$. Thus, the expression to be added to (81) is

$$S_q = S + iTr \sum_{x_i'} D_0 \delta^3(x_i - x_i'), \tag{82}$$

where sum is taken over the continuum of the points of the three dimensional manifold.

The matrix model (82) although very singular and ill-defined does have the standard quantum field theory as its solution. The question whether this solution is unique is difficult to answer. The most problematic point is to check whether the equation analogous to Eq. (65) for $J \neq K$ holds for this model. Now it has the following involving a continuous parameter form

$$[[D_0, \Phi], e^{x^i D_i} \Phi e^{-x^i D_i}] = \delta^3(x_i). \tag{83}$$

To see whether it is a consequence of the equations of motion of (82) we should solve the later by using a pattern analogous to (70) which will now involve variational derivatives. The problem is that the series in this pattern can not be made finite as it was done in finite dimensional case. Therefore it is a difficult technical problem to check whether this model has solutions other than the standard quantum field theory.

We can consider higher order in D_μ corrections to the action (81) not involving metrics. The most obvious one is

$$S' = S + \epsilon^{\mu\nu\alpha\beta} Tr D_\mu D_\nu D_\alpha D_\beta. \tag{84}$$

This correction is identically zero in four dimensional space however it is non-zero in odd dimension and can be interpreted as Chern-Simons term.

Another possibility involves the Dirac operator

$$S' = S + Tr \not{D}^2. \tag{85}$$

Such term is used in the spectral action principle in non-commutative geometry [12, 13] and contains in particular Einstein-Hilbert action.

This are all the possible modifications of the action (81) not involving metrics.

Various matrix models

If we try to apply the result of this paper for example directly to IKKT model(1) we encounter the same problem as with mechanical system with many degrees of freedom. The equation of motion for A_0 has the form

$$\sum_j [E_j, A_j] = iI. \tag{86}$$

The presence of a sum over j in this equation means that the quantum commutation relations are not completely determined.

The solution to this problem would be the packing of all the matrices A_j into a single matrix. The resulting matrix model wouldn't even have a preexisting dimension. The IKKT model then could be derived from this model by a symmetry breaking.

Such models already exist in literature. These are cubic matrix models [4] based on the following action principle

$$S = \epsilon^{\mu\nu\alpha} Tr A_\mu A_\nu A_\alpha. \tag{87}$$

This action is explicitly background independent and may be relevant not only to string theory but also to loop quantum gravity given the relation between loop quantum gravity and topological field theory [14]. As this action doesn't include any background metric no equations of the type (86) can appear as the sum in (86) is the contraction of indices by a metric. Therefore there will be no ambiguity in definition of quantum commutation relations derived from the action

$$S' = S + iTrA_0. \tag{88}$$

To make the action (88) covariant we can rewrite it in the following form

$$S' = S + i\epsilon^{\mu\nu\alpha} Tr\theta_{\mu\nu} A_\alpha, \tag{89}$$

where all θ's are fixed. It is interesting to note that exactly the action (89) was considered as giving rise to the space non-commutativity in non-commutative Chern-Simons theory also playing important role in the description of quantum Hall effect [15]. This may imply a possible relation between space-time non-commutativity in non-commutative geometry and quantum mechanical non-commutativity. This issue will be discussed elsewhere.

A comment on diffeomorphism invariance

The mechanical model considered in this paper was not diffeomorphism invariant. It is worth making some comments on how to apply this formalism to a diffeomorphism invariant theory. As we mentioned above extra terms in the action resulting in quantization do not violate diffeomorphism invariance while almost any other extra term would violate it. It is suggestive that if we require diffeomorphism invariance we reproduce the known physics. Now the question is how to cast the ordinary diffeomorphism invariance of a field theory in the language of matrix models.

When we consider an action of a field theory we may notice that there are generally two kinds of entries in it. First, it contains fields. This may be fields describing geometry of the manifold such as metrics and connections as well as matter fields. All the fields can be evaluated as functions of coordinates on the manifold x^μ. The other type of entries of the classical action are derivative operators ∂_μ. We generally assume that these entries form the following Poisson algebra with respect to commutator:

$$[x^\mu, x^\nu] = 0, \tag{90}$$

$$[\partial_\mu, \partial_\nu] = 0, \tag{91}$$

$$[\partial_\mu, x^\nu] = \delta^\nu_\mu. \tag{92}$$

It is natural to identify the action of diffeomorphism group with the coordinate transformation, $x^{\mu\prime} = x^{\mu\prime}(x^\mu)$. Below for simplicity we will consider small transformations,

$$x^{\mu\prime} = x^\mu + f^\mu(x^\mu), \tag{93}$$

where $f^\mu \ll x^\mu$. By analogy with hamiltonian mechanics we can think of x^μ as canonical coordinates parameterizing configuration space and of ∂_μ as canonical momenta, with (90,91,92) being the Poisson bracket relations. For any transformation of configuration space of the form (93) there exists a canonical transformation of the phase space with the generating function

$$D = f^\mu(x^\mu)\partial_\mu \tag{94}$$

defined by

$$x^{\mu\prime} = x^\mu + [D, x^\mu] = x^\mu + f^\mu(x^\mu), \tag{95}$$

$$\partial'_\mu = \partial_\mu + [D, \partial_\mu] = \partial_\mu + \frac{\partial f^\nu}{\partial x^\mu}\partial_\nu. \tag{96}$$

One can say that under the action of the diffeomorphism group the algebraic elements x^μ and p_μ transform according to (95,96). This action preserves the relations (90,91,92). If we represent x^μ and p_μ by hermitian operators on certain Hilbert space H diffeomorphisms (95,96) will be represented by unitary transformations in this space. Therefore the diffeomorphism group is a subgroup of the group of all unitary transformations in H.

However the diffeomorphism transformation introduced above are not those playing the crucial role in General Relativity. Here one has to recall that there are two kinds of diffeomorphisms: active and passive. Any field theory is trivially invariant with respect to passive diffeomorphisms which act simultaneously on coordinates and fields. On the other hand only general relativistic theory is invariant with respect to active diffeomorphisms which act on fields only. For detailed discussion on the distinction between active and passive diffeomorphisms see [16].

In order to distinguish active and passive diffeomorphisms here one needs to see how they act on the fields. For simplicity let us take all the fields in the theory to be scalars $\phi_n(x^\mu)$, $n = 1, ..., N$. If ϕ_n is a smooth function of x^μ then given the transformation (95) for x^μ one can see that $\phi_n(x^\mu)$ transforms with respect to diffeomorphisms as follows

$$\phi_n{}'(x^\mu) = \phi_n(x^\mu) + [D, \phi_n(x^\mu)] = \phi_n(x^\mu) + \frac{\partial \phi_n(x^\mu)}{\partial x^\nu} f^\nu(x^\mu). \tag{97}$$

This transformation law together with (96) is exactly the way fields and derivatives should transform with respect to passive diffeomorphisms. Under active diffeomorphisms fields should transform according to (97) while derivative operators should remain unchanged.

Since the action principle doesn't depend on coordinates explicitly it is convenient to rewrite the relations (90,91,92) in terms of fields and derivatives operators. They have the following form

$$[\phi_n, \phi_m] = 0, \tag{98}$$

$$[\partial_\mu, \partial_\nu] = 0, \tag{99}$$

$$[[\partial_\mu, \phi_n]\phi_m] = 0. \tag{100}$$

It is interesting to note that unlike the relations (90,91,92) the relations (98,99,100) are invariant not only with respect to passive but also with respect to active diffeomorphisms, which can be checked by direct computation. Thus, for a field theory containing scalar fields only it is easy to include the active diffeomorphisms in this framework. They are a subgroup of the full group of unitary transformations of the matrix model with respect to which derivative operators are c-numbers.

The situation becomes more difficult if we consider a matrix model based on a gauge theory. Gauge fields have the following transformation low with respect to diffeomorphisms

$$A'_\mu = A_\mu + f^\mu F_{\mu\nu} \neq A_\mu[D, A_\mu], \tag{101}$$

where $F_{\mu\nu}$ is a curvature tensor of the connection A_μ. So if we consider a gauge field A_μ as a matrix or equivalently consider the full covariant derivative $D_\mu = \partial_\mu + A_\mu$ as a matrix then the action of the diffeomorphism group is not a unitary transformation in the space of such matrices. The situation can be fixed by multiplying the covariant derivative by a frame field e_a^μ where the index a refers to the tangent space. Given that e_a^μ transforms as a vector the matrices G_a defined by

$$G_a = e_a^\mu D_\mu \tag{102}$$

have the following transformation law with respect to diffeomorphisms

$$G'_a = G_a + [f^\mu D_\mu, G_a] = G_a + [D, G_a]. \tag{103}$$

So the active diffeomorphisms are unitary transformations in the space of matrices G_a. But one can easily see that the passive diffeomorphisms have exactly the same action on G_a which is the property of not only generally relativistic theories . Therefore the question of the meaning of diffeomorphism invariance in gauge theory matrix models is very subtle.

Finnaly, it is worth making a comment on how the relations of the type (98,99,100) are understood in this framework. If the algebra $\mathcal{A} = \{\phi_n, \partial\}$ is an algebra of fields and derivative operators on a certain classical differential manifold then the relations (98,99,100) are satisfied. We can invert the statement and say that if the relations (98,99,100) are satisfied than the algebra $\mathcal{A}$ admits a representation as a classical manifold. Possible deformations of the relations (98,99,100) are to be understood as generalizations of the notion of manifold. In particular deformations of the relation (98) can be understood in the framework of Non-commutative Geometry [13]. Thus, matrix models can be understood as a framework in which the relations (98,99,100) (or part of them) are not fixed in the beginning. Instead these relations are determined by dynamics, i.e. derived from the equations of motion of the theory.

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