Kustaanheimo-Stiefel transformation and static zero modes of Dirac operator.

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Abstract

By exploiting the relation between static zero modes of massless Dirac operator and Kustaanheimo-Stiefel (Hopf) bundle sections, a general zero modes Ansatz which depends on an arbitrary real vector-function on R^3 is constructed.

1. The problem of zero modes of the two, three and four-dimensional Dirac operator in external gauge field is of interest in quantum mechanics as well as quantum field theory. The zero modes affect strongly the ground state of spin 1/2 charged particle in two-dimensional magnetic field, the stability or the collapse of Coulomb systems with magnetic field and the behaviour of the Fermi determinant in quantum field theory.

It is common knowledge [1] the crucial role of degeneracy of zero modes in some physical applications. The explicit examples of such a degeneracy have been constructed in [2]. But in [3] were proposed two new examples of zero modes in three dimensions.

In the present paper we deal with the three-dimensional static massless Dirac (Pauli) equation. A general zero modes Ansatz incorporating the previously known solutions as special cases is proposed by using the quaternion form of expressions for planar $SO(3)$ rotations in the vector parameterization [4] as well as sections of the Kustaanheimo-Stiefel (Hopf) bundle $\dot{R}^4 \to \dot{R}^3$ ($S^3 \to S^2$) [5], [6].

2. For this purpose the linear biquaternion formulation of Dirac equation [7] is used. In our approach all variables are biquaternions (quaternions over the complex number field). Arbitrary biquaternion is written as $q = q_0 e_0 + q_1 e_1 + q_2 e_2 + q_3 e_3 = q_0 + \mathbf{q}$, where e_0, e_1, e_2 and e_3 are basis elements. The simplest realization of the basis is $I, -i\sigma_1, -i\sigma_2$ and $-i\sigma_3$, where I is the identity 2×2 matrix, σ_l are Pauli matrices. Symbols q^* and $\overline{q}$ denote the complex and quaternion conjugations, respectively: $q^* = q_0^* + \mathbf{q}^*$, $\overline{q} = q_0 - \mathbf{q}$. Scalar part of biquaternion is $(q)_S = q_0$, vector one is $(q)_V = \mathbf{q}$. The product of two biquaternions $a = a_0 + \mathbf{a}$ and $b = b_0 + \mathbf{b}$ is a biquaternion $c = ab$, where $(c)_S = c_0 = a_0 b_0 - (\mathbf{ab})$, $(c)_V = \mathbf{c} = a_0\mathbf{b} + b_0\mathbf{a} + [\mathbf{ab}]$; $(\mathbf{ab}) = \delta_{lm} a_l b_m$ and $[\mathbf{ab}] = \varepsilon_{klm} a_l b_m$, $k, l, m = 1, 2, 3$.

There exists another useful biquaternion basis:

$$\Pi_{1(2)} = 1/2(e_0 \pm ie_3), \quad S_1 = ie_1, \quad S_2 = ie_2.$$

Projective quaternions $\Pi_{1(2)}$ (divisor of zero) and quaternions $S_{1(2)}$ possess the following properties:

$$\Pi^2_{1(2)} = \Pi_{1(2)}, \quad \Pi_1\Pi_2 = \Pi_2\Pi_1 = 0, \quad \Pi_1 + \Pi_2 = 1, \quad \overline{\Pi}_1 = \Pi_1^* = \Pi_2.$$

$$\Pi_1 S_{1(2)} = S_{1(2)}\Pi_2, \quad S^2_{1(2)} = 1, \quad S^*_{1(2)} = \overline{S}_{1(2)} = -S_{1(2)}.$$

Instead of e_1, e_2 and e_3 one can choose, certainly, three arbitrary orthogonal unit vectors $\mathbf{n}_1, \mathbf{n}_2$ and $\mathbf{n}_3$, where $(\mathbf{n}_l \mathbf{n}_m) = \delta_{lm}$. Then the biquaternion form of Dirac equation [7] is

$$(\nabla_{(4)} + ieA)\Psi\Pi_1 S_1 + (\overline{\nabla}_{(4)} + ie\overline{A})\Psi\Pi_2 S_1 - m\Psi = 0, \tag{1}$$

where $\Psi = 2(\varphi_1 + S_1\varphi_2)\Pi_1 + 2(\xi_1 S_1 + \xi_2)\Pi_2$, $A = iA_0 + \mathbf{A}$, $A^* = -\overline{A}$, $\nabla_{(4)} = i\partial_t - \nabla$, $\nabla^*_{(4)} = -\overline{\nabla}_{(4)}$.

For the massless static case the equation (1) can be divided in two equations

$$(\nabla - ie\mathbf{A})\Psi\Pi_1 = 0, \quad (\nabla - ie\mathbf{A})\Psi\Pi_2 = 0, \tag{2}$$

where $\Psi\Pi_1$ and $\Psi\Pi_2$ are realized by two ideals of biquaternion algebra. The equations (2) are equivalent to each other. For this reason it is enough to solve, for instance, the first equation for biquaternion $\Psi\Pi_1$.

Let us take the basis as $e_0, \mathbf{n}_1, \mathbf{n}_2$ and $\mathbf{n}_3$. Then $\Pi_1 = 1/2(e_0 + i\mathbf{n}_3)$, $S_1 = i\mathbf{n}_1$. The first biquaternion equation (2) may be written now in terms of real quaternion. Indeed, taking into account that $i\Pi_1 = -\mathbf{n}_3\Pi_1$ we have

$$\begin{aligned} \Psi\Pi_1 &= 2(\varphi_1 + S_1\varphi_2)\Pi_1 = 2(\mathrm{Re}\varphi_1 + i\mathrm{Im}\varphi_1 - \mathrm{Im}\varphi_2\mathbf{n}_1 + i\mathrm{Re}\varphi_2\mathbf{n}_1)\Pi_1 = \\ &= 2(\mathrm{Re}\varphi_1 - \mathrm{Im}\varphi_2\,\mathbf{n}_1 + \mathrm{Re}\varphi_2\,\mathbf{n}_2 - \mathrm{Im}\,\varphi_1\mathbf{n}_3)\Pi_1 = 2U\Pi_1. \end{aligned} \tag{3}$$

Here U is real quaternion. Substituting (3) into (2) we obtain the same equation for both the real and imaginary parts of biquaternion:

$$\nabla\{U\}\mathbf{n}_3 - \mathbf{A}U = 0, \tag{4}$$

The quaternion derivative operator ∇ acts on the quaternion being situated within the curly brackets only. From (4) we get

$$\begin{aligned} &\Big(\frac{\nabla\{U\}\mathbf{n}_3\overline{U}}{U\overline{U}}\Big)_V = \mathbf{A}, \\ &\Big(\frac{\nabla\{U\}\mathbf{n}_3\overline{U}}{U\overline{U}}\Big)_S = 0. \end{aligned} \tag{5}$$

Using quaternion properties $(ab)_S = (ba)_S$, $(\overline{a})_S = (a)_S$ and $(\overline{a})_V = -(a)_V$ gives

$$(\nabla\{U\}\mathbf{n}_3\overline{U})_S = (U\mathbf{n}_3\{\overline{U}\}\nabla)_S = (U\nabla\{\mathbf{n}_3\overline{U}\})_S = 1/2(\nabla\{U\mathbf{n}_3\overline{U}\})_S = 0.$$

So, we have a single constraint on quaternion U, namely, vector-quaternion $\mathbf{f} = U\mathbf{n}_3\overline{U}$ is a solenoidal one: $(\nabla\mathbf{f})_S = -(\nabla\mathbf{f}) = 0$, $\mathbf{f} = [\nabla\mathbf{F}]$.

The expressions (5) are the quaternion form of those obtained in [8] (see also [3], proposition 1). There are many ways of dealing with these equations. For instance, if $(\nabla\{U\}\mathbf{n}_3\overline{U})_S = (\nabla\{U\}\overline{U}\mathbf{n}_3)_S = 0$ then from the second equation of (5) we get

$$\nabla\{U\}\overline{U} = \omega_1\mathbf{n}_1 + \omega_2\mathbf{n}_2 + \omega_3 e_0, \tag{6}$$

where ω_1, ω_2 and ω_3 are arbitrary functions. The simplest case corresponding to the choice $\omega_1 = \omega_2 = 0$ was already considered in [8]. So,

$$\nabla\{U\} = \omega U, \tag{7}$$

$(\omega = \omega_3/U\overline{U})$ and the vector potential $\mathbf{A}$ takes form

$$\mathbf{A} = \omega\frac{U\mathbf{n}_3\overline{U}}{U\overline{U}}.$$

3. However, our main goal is to build a general zero modes Ansatz without solving equations (6) or (7). Indeed, it would be quite enough to find $U(\mathbf{f})$ from the algebraic equation

$$\mathbf{f} = U\mathbf{n}_3\overline{U}. \tag{8}$$

It can be done in at least two rather similar ways.

The first one starts from the vector-quaternion expression for planar $SO(3)$ rotation of $\mathbf{n}_3$ to $\mathbf{f}$ up to $SO(2)$ isotropy subgroup of $\mathbf{n}_3$ [4]. The second way is based on the similarity of equation (8) and Kustaanheimo-Stiefel bundle transformation $\dot{R}^4 \to \dot{R}^3$ (the latter being isomorphic to Hopf one $S^3 \to S^2$). The quaternion bundle sections were constructed up to $U(1)$ fiber transformations in [5]. In both the cases quaternion U can be written as:

$$U = \sqrt{f}\frac{1+\mathbf{c}}{\sqrt{1+\mathbf{c}^2}},$$

where vector-quaternion $\mathbf{c}$ is

$$\mathbf{c} = \frac{[\mathbf{n}_3\,\mathbf{f}]}{f + (\mathbf{n}_3\mathbf{f})},$$

or

$$U = \frac{\sqrt{f + (\mathbf{n}_3\mathbf{f})}}{\sqrt{2}}\Big(1 + \frac{[\mathbf{n}_3\,\mathbf{f}]}{f + (\mathbf{n}_3\mathbf{f})}\Big). \tag{9}$$

Substituting (9) into (5) we get vector-potential

$$\mathbf{A} = \frac{[\nabla\mathbf{f}]}{2f} + \frac{f - (\mathbf{n}_3\mathbf{f})}{2f}\nabla\{\hat{\mathbf{c}}\}\hat{\mathbf{c}}\mathbf{n}_3. \tag{10}$$

Here

$$f = \sqrt{\mathbf{f}^2}, \quad \hat{\mathbf{c}} = \frac{[\mathbf{n}_3\mathbf{f}]}{\sqrt{f^2 - (\mathbf{n}_3\mathbf{f})^2}}, \quad \mathbf{f} = [\nabla\mathbf{F}].$$

We note that transformations of isotropy subgroup of vector $\mathbf{n}_3$ or fiber in the Kustaanheimo-Stiefel bundle describe the gauge freedom of quaternions $2U\Pi_1$ ($\Psi\Pi_1$) and $\mathbf{A}$ satisfying the first biquaternion static Pauli equation of (2):

$$\Psi\Pi_1 \to \exp(i\alpha)\Psi\Pi_1 = 2U\exp(-\alpha\mathbf{n}_3)\Pi_1 = 2U\frac{1-\beta\mathbf{n}_3}{\sqrt{1+\beta^2}}\Pi_1\,, \quad \alpha = \arctan\beta\,,$$

$$\mathbf{A} \to \mathbf{A} + \nabla\{\arctan\beta\}.$$

The general solutions (9) and (10) of static Pauli equation

$$\sigma_l(\partial_l - iA_l)\psi = 0$$

can be easily written in spinor and vector notations:

$$\begin{aligned} \psi &= \frac{1}{\sqrt{2(f + (\mathbf{f}\mathbf{n}_3))}}\begin{pmatrix} f + (\mathbf{f}\mathbf{n}_3) \\ (\mathbf{f}\mathbf{n}_1) + i(\mathbf{f}\mathbf{n}_2)\end{pmatrix}, && (11)\\ \mathbf{A} &= \frac{\mathrm{curl}\,\mathbf{f}}{2f} + \frac{f - (\mathbf{f}\mathbf{n}_3)}{2f}\mathrm{grad}\Big(\arctan\frac{(\mathbf{f}\mathbf{n}_2)}{(\mathbf{f}\mathbf{n}_1)}\Big), && (12)\\ \mathbf{f} &= \mathrm{curl}\mathbf{F}, \end{aligned}$$

where $\mathbf{F}$ is an arbitrary vector-function.

The solutions (11) and (12) reproduce well-known special cases. For instance, let

$$\mathbf{F} = \frac{1}{4(1+r^2)^2}(2[\mathbf{n}_3\mathbf{r}] + 2(\mathbf{n}_3\mathbf{r})\mathbf{r} + (1-r^2)\mathbf{n}_3).$$

Then the spinor is

$$\psi = \frac{1 - i(\mathbf{n}_3\mathbf{r})}{\sqrt{1 + (\mathbf{n}_3\mathbf{r})^2}}\frac{1}{(1+r^2)^{3/2}}\begin{pmatrix} 1 + i(\mathbf{n}_3\mathbf{r}) \\ -(\mathbf{n}_2\mathbf{r}) + i(\mathbf{n}_1\mathbf{r})\end{pmatrix}$$

and vector-potential is

$$\mathbf{A} = 12\mathbf{F} - \nabla\{\arctan(\mathbf{n}_3\mathbf{r})\}.$$

By choosing $\mathbf{n}_3$ along the z axis, i.e., $\mathbf{n}_3 = (0, 0, 1)$ we get the solution [2] up to the gauge transformation mentioned above,

$$\psi = \frac{1}{(1+r^2)^{3/2}} \begin{pmatrix} 1+iz \\ -y+ix \end{pmatrix},$$

$$\mathbf{A} = \frac{3}{(1+r^2)^2}(2xz - 2y,\, 2yz + 2x,\, z^2 + 1 - x^2 - y^2).$$

The next example having the monopole-like structure is

$$\mathbf{F} = \frac{1}{r}\frac{[\mathbf{n}_3\mathbf{r}]}{r + (\mathbf{n}_3\ \mathbf{r})}\ .$$

Substituting this equation into (9) and (10) we obtain quaternion function

$$U = \frac{\sqrt{r + (\mathbf{n}_3\mathbf{r})}}{r\sqrt{2r}}\left(1 + \frac{[\mathbf{n}_3\mathbf{r}]}{r + (\mathbf{n}_3\mathbf{r})}\right),$$

spinor

$$\psi = \frac{1}{r\sqrt{2r(r + (\mathbf{n}_3\mathbf{r}))}}\begin{pmatrix} r + (\mathbf{n}_3\mathbf{r}) \\ (\mathbf{n}_1\mathbf{r}) + i(\mathbf{n}_2\mathbf{r}) \end{pmatrix}$$

and vector-potential

$$\mathbf{A} = \frac{1}{2r}\frac{[\mathbf{n}_3\mathbf{r}]}{r + (\mathbf{n}_3\ \mathbf{r})}\ .$$

The choice of $\mathbf{n}_3$ along the z axis leads to the solution obtained in [9].

It should be emphasized that monopole potential is a connection on the Kustaanheimo-Stiefel bundle. Moreover, quaternion U includes a monopole-like term. So, this potential appears both as a connection and a section of bundle. This fact was investigated in more detail in [5].

In addition, we note that if $\sqrt{|\mathrm{curl}\mathbf{F}|}$ is a square-integrable function, i.e., $\sqrt{|\mathrm{curl}\mathbf{F}|} \in L^2$ then so is the spinor ψ. The latter fact follows from equation $|\psi|^2 = |\mathrm{curl}\mathbf{F}|$.

We note in conclusion the examples of zero modes proposed in [3] can be also represented in form (11) and (12). The corresponding vector-function $\mathbf{F}$ depends on two scalar function connected by subsidiary conditions.

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