

## Coset Construction of Noncompact Spin(7) and $G_2$ CFTs

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### Abstract

We provide a new class of exactly solvable superconformal field theories that corresponds to type II compactification on manifolds with exceptional holonomies. We combine  $\mathcal{N} = 1$  Liouville field and  $\mathcal{N} = 1$  coset models and construct modular invariant partition functions of strings moving on these manifolds. The resulting theories preserve spacetime supersymmetry. Also we explicitly construct chiral currents in these models to realize consistent string theories.

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# 1 Introduction

Recently manifolds with exceptional holonomy are receiving much attention. These are 7-dimensional manifolds with  $G_2$  holonomy and 8-dimensional manifolds with  $\text{Spin}(7)$  holonomy. They provide interesting compactifications in string theories with minimal spacetime supersymmetry and are described by sigma models. To be consistent string vacua, the supersymmetric sigma models on these manifolds must be described by conformal field theories (CFTs). We can make use of worldsheet CFT techniques to describe the dynamics of these models. As is well-appreciated in the context of Calabi-Yau manifolds, conformal field theories and their chiral algebras are powerful tools to explore string theories. Exactly solvable models provide structural information and are certainly important starting points. Worldsheet description of string theories on these manifolds has been discussed and structures of their extended chiral algebras have been clarified.

Starting with the early work of Shatashvili and Vafa [1], there have been several papers about features of the superconformal field theories describing strings moving on manifolds with exceptional holonomies [2–12]. They have derived the extensions of  $\mathcal{N} = 1$  superconformal algebras by the tricritical Ising or Ising algebras from free field representations. After their study of the algebras, it becomes a natural problem to look for explicit realizations. Recent works on the subject include constructions of modular invariant partition functions for strings on (non)compact manifolds with special holonomies. Explicit  $G_2$  manifolds constructed so far are given by certain toroidal orbifolds. Another large class of  $G_2$  manifolds is supposed to result from anti-holomorphic  $\mathbb{Z}_2$  quotients of Calabi-Yau manifolds times a circle. Also  $\text{Spin}(7)$  CFT has been proposed as the  $\mathbb{Z}_2$  involution of Calabi-Yau four-fold. These classes include models constructed by Joyce [13–15].

In this paper we present a new class of exactly solvable superconformal field theories which corresponds to certain point in the moduli space of type II compactification on exceptional holonomy manifolds. We construct modular invariant partition functions of strings compactified on noncompact manifolds with exceptional holonomies. We combine  $\mathcal{N} = 1$  Liouville field and  $\mathcal{N} = 1$  coset models so that resulting theories possess spacetime supersymmetry. Also we explicitly construct sets of extended chiral currents of these models to realize consistent string theories. The purpose of the paper is to propose new explicit examples of rational conformal field theories with exceptional holonomies.

## 2 Modular invariant partition functions

### 2.1 Partition functions of Spin(7) models

In this subsection, we take, as an ansatz of a Spin(7) conformal field theory, a noncompact  $c = 12$  model described by a coset conformal field theory

$$\mathbb{R}_\phi \times \frac{G \times \text{SO}(n)}{H}. \quad (2.1)$$

$\mathbb{R}_\phi$  is the supersymmetric linear dilaton system containing a free boson  $\phi$  and a free fermion  $\psi_\phi$ . This part has  $\mathcal{N} = 1$  superconformal symmetry and associated currents are expressed as the Feigin-Fuchs representation

$$T_L = -\frac{1}{2}(\partial\phi)^2 - \frac{Q}{2}\partial^2\phi - \frac{1}{2}\psi_\phi\partial\psi_\phi, \quad G_L = i\psi_\phi\partial\phi + iQ\partial\psi_\phi, \quad c_L = \frac{3}{2} + 3Q^2. \quad (2.2)$$

$Q$  is the background charge determined by the criticality condition, that is, the total central charge should be 12.

On the other hand, in the coset CFT  $\mathcal{A} := \frac{G \times \text{SO}(n)}{H}$ ,  $G$  is a semi-simple Lie group and  $H$  is a subgroup of  $G$  (for the detail of the coset CFT, see [16] and references there in). When  $n = \dim G - \dim H$  is satisfied, this coset CFT has  $\mathcal{N} = 1$  superconformal symmetry [17]. This condition leads to construct a consistent superstring model. In this subsection we will restrict ourselves to the case  $n = 7$  and study the coset CFT  $\mathcal{A} := \frac{G \times \text{SO}(7)}{H}$ . We shall introduce currents of affine  $G$  as  $J^A(z)$ ,  $A = 1, \dots, \dim G$ , and describe the level 1 affine SO(7) algebra by six free fermions  $\psi^{\bar{a}}(z)$ ,  $\bar{a} = 1, \dots, 7$ . Then currents  $\tilde{J}^a$ ,  $a = 8, \dots, \dim G$  of affine  $H$  are defined as the subalgebra of affine  $G \times \text{SO}(7)$  in the form

$$\tilde{J}^a = J^a + J_f^a, \quad J_f^a := -\frac{i}{2}f_{\bar{a}\bar{b}\bar{c}}\psi^{\bar{b}}\psi^{\bar{c}}. \quad (2.3)$$

$f_{ABC}$ ,  $A, B, C = 1, \dots, \dim G$  is the structure constant of  $G$ . Throughout this subsection, we assume that  $J^a$ 's,  $a = 8, \dots, \dim G$  are affine  $H$  generators. By using these generators of affine algebras, currents of  $\mathcal{N} = 1$  superconformal symmetry are constructed by the standard method

$$\begin{aligned} T_{\mathcal{A}} &= \frac{1}{k} \left[ J^{\bar{a}} J^{\bar{a}} - \frac{\hat{k}}{2} \psi^{\bar{a}} \partial \psi^{\bar{a}} + i f_{\bar{a}\bar{b}\bar{c}} J^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} - \frac{1}{2} f_{\bar{a}\bar{p}\bar{q}} f_{\bar{b}\bar{p}\bar{q}} \psi^{\bar{a}} \partial \psi^{\bar{b}} - \frac{1}{4} f_{\bar{a}\bar{b}\bar{p}} f_{\bar{c}\bar{d}\bar{p}} \psi^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} \psi^{\bar{d}} \right], \\ G_{\mathcal{A}} &= \sqrt{\frac{2}{k}} \left[ \psi^{\bar{a}} J^{\bar{a}} - \frac{i}{6} f_{\bar{a}\bar{b}\bar{c}} \psi^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} \right], \quad c_{\mathcal{A}} = \frac{3\hat{k}}{2k} \dim(G/H) + \frac{1}{2k} f_{\bar{a}\bar{b}\bar{c}} f_{\bar{a}\bar{b}\bar{c}}, \end{aligned} \quad (2.4)$$

where  $\hat{k}$  is the level of affine  $G$ ,  $k$  is supersymmetric level defined by  $k := \hat{k} + h^\vee$ ,  $h^\vee$  is the second Casimir of the adjoint representation of  $G$ , defined by  $f_{ACD} f_{BCD} = h^\vee \delta_{AB}$ .

The character of the coset CFT can be obtained by branching relations. Let  $\Lambda$  be an integrable highest weight of affine  $G$  and  $\chi_\Lambda^G(\tau)$  be the character of the representation with highest weight  $\Lambda$ . We express the representation of  $\text{SO}(7)$  by an index  $s = 0, 1, 2$  which labels basic, spinor and vector representation, respectively. We denote the character of  $\text{SO}(7)$  in the representation  $s$  as  $\chi_s^{\text{SO}(7)}(\tau)$ . As for the affine  $H$ , we write the character of the representation with highest weight  $\lambda$  as  $\chi_\lambda^H(\tau)$ . With these notations, a module of  $\mathcal{A}$  is labelled by three indices  $\Lambda, s, \lambda$  and the character  $\chi_{\Lambda,s,\lambda}^{\mathcal{A}}(\tau)$  of this module is obtained by the branching relation

$$\chi_\Lambda^G(\tau)\chi_s^{\text{SO}(7)}(\tau) = \sum_\lambda \chi_{\Lambda,s,\lambda}^{\mathcal{A}}(\tau)\chi_\lambda^H(\tau).$$

Now, we consider the type II string theory compactified by the CFT in (2.1). Because we are constructing the  $\text{Spin}(7)$  CFT, the resulting theory should be supersymmetric in spacetime, but this claim is generally non-trivial. When does this theory have spacetime supersymmetry? We investigate this problem in the following: Let us denote the affine  $H$  generated by the currents  $J_f^a$  in (2.3) by  $\hat{H}_f$ . For the existence of spacetime supersymmetry the size of this algebra  $\hat{H}_f$  is crucial. Generally,  $\hat{H}_f$  is a subalgebra of  $\text{SO}(7)$ , but it should be smaller to realize supersymmetry in spacetime. In fact we claim a proposition:

If  $\hat{H}_f \subset G_2$ , the theory has spacetime supersymmetry.

We will show this by demonstrating the partition function actually vanishes.

Let us first construct the partition function by the method of [8]. When we take the light-cone gauge, the total theory becomes product of two parts  $\mathbb{R}_\phi \times \mathcal{A}$ . Next we introduce the building block  $F_{\Lambda,\lambda}^{\text{Spin}(7)}(\tau)$  as a combination of characters

$$F_{\Lambda,\lambda}^{\text{Spin}(7)}(\tau) = \chi_{\Lambda,2,\lambda}^{\mathcal{A}}(\tau)\chi_0^{\text{Ising}}(\tau) + \chi_{\Lambda,0,\lambda}^{\mathcal{A}}(\tau)\chi_{1/2}^{\text{Ising}}(\tau) - \chi_{\Lambda,1,\lambda}^{\mathcal{A}}(\tau)\chi_{1/16}^{\text{Ising}}(\tau),$$

where  $\chi_h^{\text{Ising}}(\tau)$ 's with  $h = 0, 1/16, 1/2$  are characters of Ising model and represent contributions of  $\psi_\phi$ . With this block, we obtain the total partition function as

$$Z^{\text{Spin}(7)}(\tau, \bar{\tau}) = \left( \sqrt{\text{Im } \tau} |\eta(\tau)|^2 \right)^{-1} \sum_{\Lambda,\lambda} \left| F_{\Lambda,\lambda}^{\text{Spin}(7)}(\tau) \right|^2,$$

where  $\left( \sqrt{\text{Im } \tau} |\eta(\tau)|^2 \right)^{-1}$  is the contribution of  $\phi$ . Also we can show the partition function  $Z^{\text{Spin}(7)}(\tau, \bar{\tau})$  vanishes. This is equivalent to  $F_{\Lambda,\lambda}^{\text{Spin}(7)}(\tau) = 0$ . Under the assumption  $\hat{H}_f \subset G_2$ ,  $F_{\Lambda,\lambda}^{\text{Spin}(7)}(\tau)$  is rewritten as the form

$$F_{\Lambda,\lambda}^{\text{Spin}(7)} = \sum_{a=0,1} \xi_a^{G_2}(\tau) \chi_{\Lambda,a,\lambda}^{G \times G_2/H}(\tau). \quad (2.5)$$

Here  $\chi_{\Lambda,a,\lambda}^{G \times G_2/H}$ 's are characters of the coset model  $G \times G_2/H$  defined by the branching relation of affine  $G_2$  characters  $\chi_a^{G_2}$ 's

$$\chi_{\Lambda}^G(\tau) \chi_a^{G_2}(\tau) = \sum_{\lambda} \chi_{\Lambda,a,\lambda}^{G \times G_2/H}(\tau) \chi_{\lambda}^H(\tau).$$

On the other hand  $\xi_a^{G_2}$ 's ( $a = 0, 1$ ) are defined by characters  $\chi_h^{\text{Tri}}(\tau)$  of tri-critical Ising model with  $h = 0, 3/2, 7/16, 3/5, 1/10, 3/80$

$$\begin{aligned} \xi_0^{G_2}(\tau) &:= \chi_{1/2}^{\text{Ising}}(\tau) \chi_0^{\text{Tri}}(\tau) + \chi_0^{\text{Ising}}(\tau) \chi_{3/2}^{\text{Tri}}(\tau) - \chi_{1/16}^{\text{Ising}}(\tau) \chi_{7/16}^{\text{Tri}}(\tau) \equiv 0, \\ \xi_1^{G_2}(\tau) &:= \chi_{1/2}^{\text{Ising}}(\tau) \chi_{3/5}^{\text{Tri}}(\tau) + \chi_0^{\text{Ising}}(\tau) \chi_{1/10}^{\text{Tri}}(\tau) - \chi_{1/16}^{\text{Ising}}(\tau) \chi_{3/80}^{\text{Tri}}(\tau) \equiv 0. \end{aligned} \quad (2.6)$$

Actually these vanish identically. It shows that  $F_{\Lambda,\lambda}^{\text{Spin}(7)}(\tau) \equiv 0$  and guarantees the space-time supersymmetry.

We make a remark here: Eqs.(2.6) have typically appeared in the partition functions of  $G_2$  compactifications [4, 5, 18]. Though we consider the  $\text{Spin}(7)$  compactifications, our models contain these factors. This fact is related to singularities of our models and enhanced spacetime superconformal symmetry appears in their dual models as indicated in [18].

A simple example of this type is realized with  $G = \text{SO}(7)$  and  $H = G_2$ . The model proposed in [4] is the special case of this example with restriction  $\hat{k} = 0$ .

## 2.2 Partition functions of $G_2$ models

Let us turn to the  $G_2$  compactifications using linear dilation system and coset CFT. We take the light-cone gauge and consider the transverse theory

$$\mathbb{R} \times \mathbb{R}_{\phi} \times \frac{G \times \text{SO}(6)}{H},$$

with  $\dim G - \dim H = 6$ . The character of  $\mathcal{B} := \frac{G \times \text{SO}(6)}{H}$  is evaluated by the branching relation

$$\chi_{\Lambda}^G(\tau) \chi_s^{\text{SO}(6)}(\tau) = \sum_{\lambda} \chi_{\Lambda,s,\lambda}^{\mathcal{B}}(\tau) \chi_{\lambda}^H(\tau),$$

where the index  $s = 0, 1, 2, 3$  represents the affine  $\text{SO}(6)$  representation and each  $\chi_s^{\text{SO}(6)}$  is the character of affine  $\text{SO}(6)$ . Other notations are the same as the  $\text{Spin}(7)$  case. We introduce a set of building blocks  $F_{\Lambda,\lambda}^{G_2}$  as a combination of characters

$$F_{\Lambda,\lambda}^{G_2}(\tau) = \chi_{\Lambda,2,\lambda}^{\mathcal{B}}(\tau) \chi_0^{\text{SO}(2)}(\tau) + \chi_{\Lambda,0,\lambda}^{\mathcal{B}}(\tau) \chi_2^{\text{SO}(2)}(\tau) - \chi_{\Lambda,1,\lambda}^{\mathcal{B}}(\tau) \chi_1^{\text{SO}(2)}(\tau) - \chi_{\Lambda,3,\lambda}^{\mathcal{B}}(\tau) \chi_3^{\text{SO}(2)}(\tau).$$

$\chi_s^{\text{SO}(2)}$  is the character of affine  $\text{SO}(2)$  and has contributions of  $\psi_\phi$  and the free fermion in the transverse direction in spacetime. By using this building block, we can obtain the partition function as

$$Z^{G_2}(\tau, \bar{\tau}) = (\sqrt{\text{Im } \tau} |\eta(\tau)|^2)^{-2} \sum_{\Lambda, \lambda} |F_{\Lambda, \lambda}^{G_2}(\tau)|^2.$$

The factor  $(\sqrt{\text{Im } \tau} |\eta(\tau)|^2)^{-2}$  represents contributions of  $\phi$  and a spacetime boson in 1 dimensional transverse direction.

As for the spacetime supersymmetry in this compactification, we claim the proposition:

If  $\hat{H}_f \subset \text{SU}(3)$ , this theory has supersymmetry in spacetime.

Actually, when  $\hat{H}_f \subset \text{SU}(3)$  the block  $F_{\Lambda, \lambda}^{G_2}(\tau)$  can be rewritten by using the characters  $\chi_{\Lambda, a, \lambda}^{G \times \text{SU}(3)/H}$ 's of the coset  $G \times \text{SU}(3)/H$ ,

$$F_{\Lambda, \lambda}^{G_2}(\tau) = \sum_{a=-1,0,1} \xi_a^{\text{SU}(3)}(\tau) \chi_{\Lambda, a, \lambda}^{G \times \text{SU}(3)/H}(\tau).$$

$\xi_a^{\text{SU}(3)}$ 's are functions constructed from  $\text{SU}(2)$  classical theta functions  $\Theta_{m,k}$ 's

$$\xi_a^{\text{SU}(3)}(\tau) = \frac{1}{\eta(\tau)^2} \sum_{s \in \mathbb{Z}_4} (-1)^s \Theta_{6+4a-3s,6}(\tau) \Theta_{s,2}(\tau) \equiv 0. \quad (2.7)$$

We can show this set of functions vanishes identically. The most simple example is illustrated with  $G = \text{SU}(2) \times \text{SU}(2) \times \text{SU}(2)$  and  $H = \text{SU}(2)$ . A series of models in [4] is included in this example. Another typical example is realized with  $G = G_2$  and  $H = \text{SU}(3)$ .

## 3 Currents of the extended algebras

### 3.1 Currents of Spin(7) CFT algebra

In this subsection, we will study currents of extended superconformal algebra associated with 8 dimensional Spin(7) manifold. The extended symmetry algebra has been found in paper [1]. In addition to a set of  $\mathcal{N} = 1$  superconformal currents  $(T, G)$ , it contains operators  $(\tilde{X}, \tilde{M})$  with spins  $(2, 3/2)$ .  $\tilde{X}$  is the energy momentum tensor for the  $c = 1/2$  model.

We construct these currents in our coset models discussed in section 2.1. The super-stress tensor  $T$  and  $G$  are given as sums of  $\mathcal{N} = 1$  Liouville parts  $T_L, G_L$  and coset parts  $T_{\mathcal{A}}, G_{\mathcal{A}}$  associated to  $\mathcal{A} = \frac{G \times \text{SO}(7)}{H}$  with  $\dim G - \dim H = 7$

$$T = T_L + T_{\mathcal{A}}, \quad G = G_L + G_{\mathcal{A}}.$$

Their explicit formulae are expressed in Eqs.(2.2)(2.4). Next we propose that the current  $\tilde{X}$  is simply given by combining fermionic fields of Liouville and coset theories

$$\tilde{X} = -\frac{1}{6}\Phi_{\bar{a}\bar{b}\bar{c}}\psi^{\bar{a}}\psi^{\bar{b}}\psi^{\bar{c}}\psi_{\phi} + \frac{1}{24}*\Phi_{\bar{a}\bar{b}\bar{c}\bar{d}}\psi^{\bar{a}}\psi^{\bar{b}}\psi^{\bar{c}}\psi^{\bar{d}} - \frac{1}{2}\psi^{\bar{a}}\partial\psi^{\bar{a}} - \frac{1}{2}\psi_{\phi}\partial\psi_{\phi}, \quad \bar{a}, \bar{b}, \dots = 1, \dots, 7. \quad (3.1)$$

In this equation, we use the structure constants of the octonion  $\Phi_{\bar{a}\bar{b}\bar{c}}$  and its hodge dual  $*\Phi_{\bar{a}\bar{b}\bar{c}\bar{d}} := \frac{1}{6}\epsilon_{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}\bar{f}\bar{g}}\Phi_{\bar{e}\bar{f}\bar{g}}$ . The form of  $\tilde{X}$  in Eq.(3.1) satisfies the following consistency conditions:

- When  $\hat{H}_{\mathfrak{f}} \subset G_2$ ,  $\tilde{X}$  is actually an operator in the coset theory  $\mathcal{A}$ , that is, the condition  $\tilde{J}^a(w)\tilde{X}(z) \sim (\text{regular})$  is satisfied.
- $\tilde{X}$  itself has the OPE of the stress tensor in the critical Ising model. The appearance of this statistical model is essential in the reduction of holonomy from  $SO(8)$  to  $\text{Spin}(7)$  through a relation  $SO(8)/\text{Spin}(7) \cong (\text{Ising model})$ .

The remaining current  $\tilde{M}$  is obtained by the OPE  $G(z)\tilde{X}(w) \sim \frac{G(w)/2}{(z-w)^2} + \frac{\tilde{M}(w)}{z-w}$  as

$$\begin{aligned} \tilde{M} = \frac{1}{2}\sqrt{\frac{2}{k}} \left[ -\Phi_{\bar{a}\bar{b}\bar{c}}J^{\bar{a}}\psi^{\bar{b}}\psi^{\bar{c}}\psi_{\phi} + \frac{1}{3}*\Phi_{\bar{a}\bar{b}\bar{c}\bar{d}}J^{\bar{a}}\psi^{\bar{b}}\psi^{\bar{c}}\psi^{\bar{d}} - J^{\bar{a}}\partial\psi^{\bar{a}} + \partial J^{\bar{a}}\psi^{\bar{a}} \right. \\ \left. + \frac{i}{2}f_{\bar{p}\bar{q}\bar{a}}\Phi_{\bar{p}\bar{c}\bar{d}}\psi^{\bar{a}}\psi^{\bar{b}}\psi^{\bar{c}}\psi^{\bar{d}}\psi_{\phi} - if_{\bar{p}\bar{q}\bar{a}}\Phi_{\bar{p}\bar{q}\bar{b}}\partial\psi^{\bar{a}}\psi^{\bar{b}}\psi_{\phi} - \frac{i}{6}f_{\bar{p}\bar{a}\bar{b}}*\Phi_{\bar{p}\bar{c}\bar{d}\bar{e}}\psi^{\bar{a}}\psi^{\bar{b}}\psi^{\bar{c}}\psi^{\bar{d}}\psi^{\bar{e}} \right. \\ \left. + \frac{i}{2}(f_{\bar{p}\bar{q}\bar{a}}*\Phi_{\bar{p}\bar{q}\bar{b}\bar{c}} - f_{\bar{a}\bar{b}\bar{c}})\partial\psi^{\bar{a}}\psi^{\bar{b}}\psi^{\bar{c}} \right] + \frac{1}{6}\Phi_{\bar{a}\bar{b}\bar{c}}\psi^{\bar{a}}\psi^{\bar{b}}\psi^{\bar{c}}i\partial\phi - \frac{1}{2}\partial\psi_{\phi}i\partial\phi + \frac{1}{2}\psi_{\phi}i\partial^2\phi. \end{aligned} \quad (3.2)$$

We have checked that these currents  $(T, G, \tilde{X}, \tilde{M})$  actually satisfy the  $\text{Spin}(7)$  CFT algebra for the example with  $G = \text{SO}(7)$  and  $H = G_2$ . For general coset cases with  $\hat{H}_{\mathfrak{f}} \subset G_2$ , we propose that the set of these currents also satisfies the  $\text{Spin}(7)$  CFT algebra as a conjecture.

### 3.2 Currents of $G_2$ CFT algebra

We take a seven manifold with a  $G_2$  holonomy. The  $G_2$  structure on this manifold is given by a closed  $G_2$  invariant 3-form  $\Phi$ . By including this operator, an extended algebra has been constructed in the paper [1]. In addition to a set of stress tensor  $T$  and its superpartner  $G$ , the algebra contains sets of chiral currents  $(K, \Phi)$  with spins  $(2, 3/2)$  and  $(X, M)$  with spins  $(2, 5/2)$ . The  $X$  is related with a dual 4-form  $*\Phi$  and  $(X, \Phi)$  is a set of currents of  $\mathcal{N} = 1$  additional superconformal algebra. The extra conformal algebra is isomorphic to the algebra of the tricritical Ising model with central charge  $7/10$  and plays

an essential role in the reduction of holonomy of the manifold from  $SO(7)$  to  $G_2$  through a relation  $SO(7)/G_2 \cong (\text{Tricritical Ising})$ .

In this subsection, we construct these currents in the coset  $G_2$  model discussed in section 2.2. First the set of currents  $(T, G)$  is constructed as combinations of Liouville parts  $T_L, G_L$  and coset parts  $T_{\mathcal{B}}, G_{\mathcal{B}}$  associated with  $\mathcal{B} = \frac{G \times SO(6)}{H}$  ( $\dim G - \dim H = 6$ )

$$T = T_L + T_{\mathcal{B}}, \quad G = G_L + G_{\mathcal{B}}.$$

Next we take the current  $\Phi$  as a combination of a Liouville fermion and fermionic fields of the coset model

$$\Phi = \frac{1}{6} A_{\bar{a}\bar{b}\bar{c}} \psi^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} + \frac{1}{2} H_{\bar{a}\bar{b}} \psi^{\bar{a}} \psi^{\bar{b}} \psi_{\phi}.$$

Other currents  $X, K, M$  are constructed by the OPEs  $\Phi(z)\Phi(w) \sim -\frac{7}{(z-w)^3} + \frac{6}{z-w} X(w)$ ,  $G(z)\Phi(w) \sim \frac{1}{z-w} K(w)$ ,  $G(z)X(w) \sim -\frac{1}{2} \frac{1}{(z-w)^2} G(w) + \frac{1}{z-w} M(w)$  respectively

$$\begin{aligned} X &= -\frac{1}{24} * H_{\bar{a}\bar{b}\bar{c}\bar{d}} \psi^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} \psi^{\bar{d}} + \frac{1}{6} * A_{\bar{a}\bar{b}\bar{c}} \psi^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} \psi_{\phi} + \frac{1}{2} \psi^{\bar{a}} \partial \psi^{\bar{a}} + \frac{1}{2} \psi_{\phi} \partial \psi_{\phi}, \\ K &= \frac{1}{2} \sqrt{\frac{2}{k}} \left[ A_{\bar{a}\bar{b}\bar{c}} J^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} + 2 H_{\bar{a}\bar{b}} J^{\bar{a}} \psi^{\bar{b}} \psi_{\phi} - \frac{i}{2} f_{\bar{p}\bar{a}\bar{b}} A_{\bar{p}\bar{c}\bar{d}} \psi^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} \psi^{\bar{d}} \right. \\ &\quad \left. + i f_{\bar{p}\bar{q}\bar{a}} A_{\bar{p}\bar{q}\bar{b}} \partial \psi^{\bar{a}} \psi^{\bar{b}} - i f_{\bar{p}\bar{a}\bar{b}} H_{\bar{p}\bar{c}} \psi^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} \psi_{\phi} + i f_{\bar{p}\bar{q}\bar{a}} H_{\bar{p}\bar{q}} \partial \psi^{\bar{a}} \psi_{\phi} \right] + \frac{1}{2} H_{\bar{a}\bar{b}} \psi^{\bar{a}} \psi^{\bar{b}} i \partial \phi, \\ M &= \frac{1}{2} \sqrt{\frac{2}{k}} \left[ * A_{\bar{a}\bar{b}\bar{c}} J^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} \psi_{\phi} - \frac{1}{3} * H_{\bar{a}\bar{b}\bar{c}\bar{d}} J^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} \psi^{\bar{d}} + J^{\bar{a}} \partial \psi^{\bar{a}} - \partial J^{\bar{a}} \psi^{\bar{a}} \right. \\ &\quad \left. - \frac{i}{2} f_{\bar{p}\bar{a}\bar{b}} * A_{\bar{p}\bar{c}\bar{d}} \psi^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} \psi^{\bar{d}} \psi_{\phi} + i f_{\bar{p}\bar{q}\bar{a}} * A_{\bar{p}\bar{q}\bar{b}} \partial \psi^{\bar{a}} \psi^{\bar{b}} \psi_{\phi} + \frac{i}{6} f_{\bar{p}\bar{a}\bar{b}} * H_{\bar{p}\bar{c}\bar{d}\bar{e}} \psi^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} \psi^{\bar{d}} \psi^{\bar{e}} \right. \\ &\quad \left. + \frac{i}{2} (-f_{\bar{p}\bar{q}\bar{a}} * H_{\bar{p}\bar{q}\bar{b}\bar{c}} + f_{\bar{a}\bar{b}\bar{c}}) \partial \psi^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} \right] + \frac{1}{2} \partial \psi_{\phi} i \partial \phi - \frac{1}{2} \psi_{\phi} i \partial^2 \phi - \frac{1}{6} * A_{\bar{a}\bar{b}\bar{c}} \psi^{\bar{a}} \psi^{\bar{b}} \psi^{\bar{c}} i \partial \phi, \end{aligned}$$

$$\bar{a}, \bar{b}, \dots = 1, \dots, 6.$$

Here we used notations

$$A_{\bar{a}\bar{b}\bar{c}} := \Phi_{\bar{a}\bar{b}\bar{c}}, \quad H_{\bar{a}\bar{b}} := \Phi_{\bar{a}\bar{b}7}, \quad *A_{\bar{a}\bar{b}\bar{c}} := \frac{1}{6} \epsilon_{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}\bar{f}} A_{\bar{d}\bar{e}\bar{f}}, \quad *H_{\bar{a}\bar{b}\bar{c}\bar{d}} := \frac{1}{2} \epsilon_{\bar{a}\bar{b}\bar{c}\bar{d}\bar{e}\bar{f}} H_{\bar{e}\bar{f}}.$$

For general coset cases, we propose that these currents satisfy the full  $G_2$  CFT algebra as a conjecture when  $\hat{H}_{\text{f}} \subset \text{SU}(3)$  is satisfied.

## 4 Conclusion

In this paper, we construct the Spin(7) and  $G_2$  CFTs combining  $\mathcal{N} = 1$  Liouville and supersymmetric coset models. It provides a new class of exactly solvable superconformal



field theories that corresponds to type II strings compactified on exceptional holonomy manifolds.

We construct modular invariant partition functions. It is shown that they vanish and we make sure that the resulting theories possess spacetime supersymmetry. These noncompact models include factors  $\xi_a^{G_2}, \xi_a^{SU(3)}$  in partition functions as if the models have twice as many supersymmetry as expected. When the target manifolds become singular, their dual field theories become interacting superconformal field theories. It suggests the “holographic dual” theories of these string models are superconformal as indicated in [18] and extra supercharges correspond to superconformal  $S$  generators in the dual theories. It is interesting to apply present approach to investigation of properties in dual field theories.

We also explicitly construct the sets of the Spin(7) and  $G_2$  CFT currents in our models to realize consistent string theories. Among the class of the models considered in subsections 2.1 and 3.1,  $\mathbb{R}_\phi \times (SO(7) \times SO(7))/G_2$  is the most typical example. For  $G_2$  holonomy case,  $\mathbb{R}_\phi \times (G_2 \times SO(6))/SU(3)$  is possible in our coset construction. We will make a remark here: In our paper we mainly discuss exceptional holonomy cases and investigate their coset construction in noncompact models. But our construction is not restricted to these exceptional holonomy manifolds. It is also applicable to other special holonomy cases. For example, we can propose  $\mathbb{R}_\phi \times (SU(3) \times SO(5))/SU(2)$ ,  $\mathbb{R}_\phi \times (SU(2) \times SO(3))$  respectively as  $SU(3)$ ,  $SU(2)$  holonomy models. By considering a string of reduction of holonomies, we can show a cascade of special holonomy groups. Typically it appears in these four cases. Their fermionic parts are described by  $SO$  groups. When the holonomies are reduced, the dimensions of manifolds decrease and the  $SO$  group parts are changed gradually as  $SO(7) \rightarrow SO(6) \rightarrow SO(5) \rightarrow SO(3)$ . On the other hand, bosonic parts of these cosets are transformed as  $SO(7)/G_2 \rightarrow G_2/SU(3) \rightarrow SU(3)/SU(2) \rightarrow SU(2)$ . It represents a string of statistical models indicated in [5],  $SO(7)/G_2 \cong (\text{Tricritical Ising})$ ,  $G_2/SU(3) \cong (\text{3-state Potts})$ ,  $SU(3)/SU(2) \cong U(1)$  and their central charges increase. In these models holonomies are actually reduced as  $\text{Spin}(7) \rightarrow G_2 \rightarrow SU(3) \rightarrow SU(2)$ . It is an interesting problem to study these strings based on sigma models or gauged WZW models that have an explicit picture of target space geometries.

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