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Nonpertubative Effects of Extreme Localization in Noncommutative Geometry

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Abstract

"Extremely" localized wavefunctions in noncommutative geometry have disturbances that are localized to distances smaller than $\sqrt{\theta}$, where θ is the "area" parameter that measures noncommutativity. In particular, distributions such as the sign function or the Dirac delta function are limiting cases of extremely localized wavefunctions. It is shown that Moyal star products of extremely localized wavefunctions cannot be correctly computed perturbatively in powers of θ. Nonperturbative effects as a function of θ are explicitly displayed through exact computations in several examples. In particular, for distributions, star products end up being functions of θ^{-1} and have no expansion in positive powers of θ. This result provides a warning for computations in noncommutative space that often are performed with perturbative methods. Furthermore, the result may have interesting applications that could help elucidate the role of noncommutative geometry in several areas of physics.

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1 Star-commutators with distributions

For simplicity we will limit our discussion in this note to a two dimensional noncommutative plane (generalizations are immediate). The two noncommutative coordinates are denoted as $x_1 = x$ and $x_2 = p$ as a reminder of the close relation between noncommutative geometry and quantum mechanics, but we have in mind various applications of noncommutative geometry in physics, including the quantum Hall effect, strings in large background fields, and string field theory. The noncommutativity parameter θ has dimensions of "area", i.e. units of x times units of p, and its meaning depends on the specific physical application. As we will make explicit, localization to distances shorter than $\sqrt{\theta}$ produce nonperturbative effects as a function of the parameter θ in computations involving the noncommutative geometry.

Consider functions in the noncommutative plane $\Lambda(p, x)$. In particular, consider the sign step-function, $\varepsilon(p) = \frac{p}{|p|}$, which takes the values ± 1 for $p \gtrless 0$ respectively. Its derivative is the Dirac delta function

$$\frac{\partial}{\partial p}\varepsilon(p) = 2\delta(p). \tag{1}$$

As is well known, it can be represented as an integral

$$\varepsilon(p) = \int_{-\infty}^{\infty} \frac{dq}{\pi i} e^{iqp} \left(P\frac{1}{q}\right) \tag{2}$$

where $\left(P\frac{1}{q}\right)$ is the principal value.

Now consider the Moyal star-commutator of $\varepsilon(p)$ with any function $\Lambda(p, x)$. It is given by

$$\begin{aligned}
& \varepsilon(p) \star \Lambda(p, x) - \Lambda(p, x) \star \varepsilon(p) && (3)\\
= & \left\{ e^{\frac{i\theta}{2}\left(\partial_x \partial'_p - \partial_p \partial'_x\right)} \left[\varepsilon(p)\Lambda(p', x') - \Lambda(p, x)\varepsilon(p')\right]\right\}_{p=p';x=x'} && (4)\\
= & \ \varepsilon\left(p - \frac{i\theta\partial}{2\partial x}\right)\Lambda(p, x) - \varepsilon\left(p + \frac{i\theta\partial}{2\partial x}\right)\Lambda(p, x), && (5)\\
= & \ \Lambda\left(p', x - \frac{i\theta\partial}{2\partial p}\right)\varepsilon(p) - \Lambda\left(p', x + \frac{i\theta\partial}{2\partial p}\right)\varepsilon(p), && (6)
\end{aligned}$$

where in the last line one sets $p' = p$ after the derivatives are performed.

If one expands any of the expression in Eqs.(4-6) in a power series in θ one finds that the result is zero if $p \neq 0$. This is intuitively understandable, since for $p \neq 0$, one is trying to commute $+1$ or -1 with some function, and therefore zero appears as a reasonable result. However, quantum mechanics (or equivalently, noncommutative geometry) can be tricky because there is a probability distribution for the values ± 1. More precisely, every term in the power series expansion of (4-6) is proportional to the delta function $\delta(p)$ or its derivatives (odd number of derivatives of $\varepsilon(p)$); therefore, it seems that, if there is any support for a non-zero result, it is only at $p=0$. Away from $p=0$ the result of the power expansion is apparently zero.

This result correctly applies when $\Lambda(p,x)$ involves simple powers of x. Indeed, it is straightforward to use the form of Eq.(6) to evaluate the commutator when $\Lambda = x, x^2, x^3,$ etc. In such cases the dependence on θ is necessarily of the perturbative form. However, it is shown in this note that for more general functions $\Lambda(p,x)$ the perturbative computation described in the previous paragraph surprisingly misses nonperturbative effects in θ which are not zero even when $p \neq 0$. The result of the commutator turns out to be a smooth function of (p,x,θ) that involves only the inverse powers of θ.

By using the integral representation, the expression in Eq.(5) is evaluated as follows

$$
\begin{aligned}
& [\varepsilon(p), \Lambda(x,p)]_\star && (7)\\
= & \int_{-\infty}^{\infty} \frac{dq}{\pi i}\left(P\frac{1}{q}\right)\left(e^{iq\left(p-\frac{i\theta\partial}{2\partial x}\right)} - e^{iq\left(p+\frac{i\theta\partial}{2\partial x}\right)}\right)\Lambda(x,p), && (8)\\
= & \int_{-\infty}^{\infty} \frac{dq}{\pi i}\left(P\frac{1}{q}\right) e^{iqp}\left(\Lambda\left(x+\frac{\theta q}{2},p\right) - \Lambda\left(x-\frac{\theta q}{2},p\right)\right). && (9)
\end{aligned}
$$

The integral is well defined if $\Lambda(x,p)$ goes to zero (or even to a constant) at $x \to \infty$.

Consider the example $\Lambda(x,p) = f(p)\left(1+x^2\right)^{-1}$, with any function $f(p)$. Then, according to (5)

$$
\begin{aligned}
& \left[\varepsilon(p), \frac{f(p)}{1+x^2}\right]_\star && (10)\\
= & \int_{-\infty}^{\infty} \frac{dq}{\pi i} e^{iqp}\left(\frac{f(p)\left(P\frac{1}{q}\right)}{1+\left(x+\frac{\theta q}{2}\right)^2} - \frac{f(p)\left(P\frac{1}{q}\right)}{1+\left(x-\frac{\theta q}{2}\right)^2}\right) && (11)\\
= & \int_{-\infty}^{\infty} \frac{dq}{\pi i} e^{iqp} \frac{-2\theta x q\left(P\frac{1}{q}\right) f(p)}{\left(1+\left(x+\frac{\theta q}{2}\right)^2\right)\left(1+\left(x-\frac{\theta q}{2}\right)^2\right)} && (12)\\
= & \frac{2i\theta x f(p)}{\pi} \int_{-\infty}^{\infty} \frac{dq\, e^{iqp}}{\left(1+\left(x+\frac{\theta q}{2}\right)^2\right)\left(1+\left(x-\frac{\theta q}{2}\right)^2\right)}. && (13)
\end{aligned}
$$

The integral is evaluated by using complex integration, noting that there are poles in the complex q plane at $\frac{2}{\theta}(x \pm i), \frac{2}{\theta}(-x \pm i)$. Closing the contour in the upper half plane (for p positive or zero), or in the lower half plane (for p negative or zero), and evaluating the residues of the poles enclosed in either contour, gives the following result

$$\left[\varepsilon\left(p\right),\frac{f\left(p\right)}{1+x^{2}}\right]_{\star} = if\left(p\right)\left[\frac{e^{\frac{2}{\theta}i(x+i)|p|}}{(x+i)}+\frac{e^{-\frac{2}{\theta}i(x-i)|p|}}{(x-i)}\right] \tag{14}$$

$$= \frac{2if\left(p\right)e^{-\frac{2|p|}{\theta}}}{1+x^{2}}\left[x\cos\left(\frac{2px}{\theta}\right)+\sin\left(\frac{2\left|p\right|x}{\theta}\right)\right]. \tag{15}$$

The significance of this simple exercise is that the result is nonperturbative in θ. First of all, it is not zero even when $p\neq 0$. Second, it is not a power series with positive powers of θ. It is still true that as $\theta\rightarrow 0$ the commutator vanishes, but this happens exponentially, not linearly. A perturbative computation, in which the Moyal star product in Eq.(4) is evaluated through a series expansion in θ, misses this result completely as shown above.

Using the Weyl-Moyal correspondence, one may consider the operator image of $\varepsilon\left(p\right)$. Then the computation presented above corresponds to computing the commutators of this operator with other operators that act in a quantum mechanical space. Since θ can be regarded as $\hbar$ in quantum mechanics, what we have obtained is a non-perturbative quantum mechanical effect as a function of $\hbar$, which approaches a classical limit as $\hbar\rightarrow 0$, not at the usual linear rate, but at an exponential rate. There is no usual semi-classical limit. Certainly this is surprising since a commutator usually (but of course, not necessarily as seen here) is a power series in $\hbar$ which starts with the first power.

It should be emphasized that $\varepsilon\left(p\right)$ does not decay at infinity in the noncommutative plane and it is not an infinitely differentiable smooth function that belongs to C^{∞}. Rather, $\varepsilon\left(p\right)$ is a distribution. So, it does not correspond to a bounded or to a compact quantum operator. The Dirac delta function $\delta\left(p\right)$, again a distribution, does vanish at infinity. Its star commutator can be obtained by differentiating the star commutators of $\varepsilon\left(p\right)$ before one sets $p'=p$. By differentiating the results for the example above one obtains

$$\left[\delta\left(p\right),\frac{f\left(p\right)}{1+x^{2}}\right]_{\star}=-\frac{2i}{\theta}f\left(p\right)e^{-\frac{2|p|}{\theta}}\sin\left(\frac{2}{\theta}x\left|p\right|\right).$$

We see that the star commutators of $\delta\left(p\right)$ are also non-perturbative in θ, and non-vanishing even when $p\neq 0$. By contrast, the perturbative expansion would have produced a vanishing result when $p\neq 0$.

2 Extremely localized wavefunctions

To understand better how the nonperturbative effects arise, it is instructive to analyze smeared distributions. Thus, let us consider the following C^∞ functions

$$\delta_{\varepsilon_1}(p) = \frac{e^{-p^2/\varepsilon_1}}{\sqrt{\pi\varepsilon_1}}, \quad \delta_{\varepsilon_2}(x) = \frac{e^{-x^2/\varepsilon_2}}{\sqrt{\pi\varepsilon_2}}. \tag{16}$$

As long as $\varepsilon_1, \varepsilon_2$ are positive and finite, these are well behaved, infinitely differentiable, and are representatives of bounded operators in a quantum Hilbert space according to the Weyl correspondence. Their star product can be computed by using an integral representation of the star product [1][2], and the result is a special case of star products of multidimensional gaussians with matrix insertions and shifts given in [3],

$$\delta_{\varepsilon_1}(p) \star \delta_{\varepsilon_2}(x) = \frac{1}{\pi\sqrt{\theta^2+\varepsilon_2\varepsilon_1}} \exp\left(-\left[\frac{x^2\varepsilon_1 + 2ixp\theta + p^2\varepsilon_2}{\theta^2+\varepsilon_2\varepsilon_1}\right]\right), \tag{17}$$

or

$$\delta_{\varepsilon_2}(x) \star \delta_{\varepsilon_1}(p) = \frac{1}{\pi\sqrt{\theta^2+\varepsilon_2\varepsilon_1}} \exp\left(-\left[\frac{x^2\varepsilon_1 - 2ixp\theta + p^2\varepsilon_2}{\theta^2+\varepsilon_2\varepsilon_1}\right]\right), \tag{18}$$

and their commutator is

$$\left[\delta_{\varepsilon_1}(p), \delta_{\varepsilon_2}(x)\right]_\star = \frac{-2i\sin\left(\frac{2xp\theta}{\theta^2+\varepsilon_2\varepsilon_1}\right)}{\pi\sqrt{\theta^2+\varepsilon_2\varepsilon_1}} \exp\left(-\left[\frac{x^2\varepsilon_1 + p^2\varepsilon_2}{\theta^2+\varepsilon_2\varepsilon_1}\right]\right). \tag{19}$$

$\delta_{\varepsilon_1}(p)$ becomes the Dirac delta function $\delta(p)$ when ε_1 approaches zero. Similarly one may consider independently an ε_2 limit to reach the Dirac delta function $\delta(x)$.

A perturbative expansion of the results above around $\theta = 0$ are possible. However, these expressions become invalid (not convergent) as soon as the product $\varepsilon_1\varepsilon_2$ is smaller than θ^2. Indeed, when $\varepsilon_1 = 0$, for any finite ε_2, the expressions above contain only θ^{-1} and cannot have a perturbative expansion with positive powers of θ. This shows that nonperturbative star products are unavoidable for wavefunctions localized to noncommutative space regions of distances smaller than $\sqrt{\theta}$. In particular, distributions such as $\varepsilon(p)$, $\delta(p)$, etc., necessarily have nonperturbative star products as given in examples above.

As an aside, note that from the expressions above we learn how to star-multiply Dirac delta functions for mutually noncommuting variables (when both $\varepsilon_1 = \varepsilon_2 = 0$)

$$\delta(p) \star \delta(x) = \frac{1}{\pi|\theta|} \exp\left(-\frac{2ixp}{\theta}\right), \quad \delta(x) \star \delta(p) = \frac{1}{\pi|\theta|} \exp\left(\frac{2ixp}{\theta}\right). \tag{20}$$

Similarly, the two dimensional Dirac delta function in noncommutative space can be obtained from the smeared distribution

$$\delta_{\varepsilon}^{(2)}(x,p)=\delta_{\varepsilon}(x)\,\delta_{\varepsilon}(p)=\frac{1}{\pi\varepsilon}\exp\left(-\frac{x^2+p^2}{\varepsilon}\right). \tag{21}$$

The star product of two such gaussians with different $\varepsilon_1,\varepsilon_2$ was given in [4][1] and again is a special case of the results given in [3]

$$\delta_{\varepsilon_1}^{(2)}(x,p)\star\delta_{\varepsilon_2}^{(2)}(x,p)=\frac{1}{\pi^2\left(\theta^2+\varepsilon_2\varepsilon_1\right)}\exp\left(-\frac{\varepsilon_2+\varepsilon_1}{\theta^2+\varepsilon_2\varepsilon_1}\left(x^2+p^2\right)\right). \tag{22}$$

As before, if $\varepsilon_1\varepsilon_2$ is small compared to θ^2, nonperturbative effects take over. In particular, for $\varepsilon_1=0$, at any ε_2, the result is purely a function of θ^{-1}. Their commutator $\left[\delta_{\varepsilon_1}(x,p),\delta_{\varepsilon_2}(x,p)\right]_\star=0$ is evidently zero for all $\varepsilon_1,\varepsilon_2$, θ. A byproduct of this exercise is the following formula for the star product of two 2-dimensional delta functions in noncommutative space (for $\varepsilon_1=\varepsilon_2=0$)

$$\delta^{(2)}(x,p)\star\delta^{(2)}(x,p)=\frac{1}{\pi^2\theta^2} \tag{23}$$

Multi-dimensional generalizations, and more complicated examples can be easily computed by using the general star product formulas for generating functions given in [3].

3 Comments

Various distributions may well play a role in a physical setting that involves noncommutative geometry, just as they do in commutative geometry. We have learned in this note that one should expect nonperturbative behavior in the star products of the distributions $\varepsilon(p)$, $\delta(p)$, $\delta(x)$, etc., and of course, this would extend to their derivatives. The star algebra of distributions with functions and with other distributions can be computed by using similar methods, and we expect to find nonperturbative behavior in general in such star products.

In the noncommutative geometry that arises in string theory [5], θ is proportional to the inverse of a large background antisymmetric field. In the quantum Hall effect [6], θ is proportional to the inverse of the background magnetic field. In string field theory, having realized that the string star product is basically the Moyal star product [3], we see that θ is determined by the fundamental string length (since $\Delta p\sim\Delta x/\alpha'$ in string theory). Also, as already mentioned, θ is $\hbar$ in quantum mechanics. Nonperturbative behavior in such parameters would be of great interest, and we expect it to be relevant when wavefunctions probe distances shorter than $\sqrt{\theta}$.

The non-perturbative effect discussed here is intriguing, and one wonders if it has interesting applications in various areas of physics? If so, it could help elucidate the content and role of noncommutative geometry in physics.

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