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Quantum Chern–Simons vortices on a sphere

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Abstract

The quantisation of the reduced first-order dynamics of the nonrelativistic model for Chern–Simons vortices introduced by Manton is studied on a sphere of given radius. We perform geometric quantisation on the moduli space of static solutions, using a Kähler polarisation, to construct the quantum Hilbert space. Its dimension is related to the volume of the moduli space in the usual classical limit. The angular momenta associated with the rotational $SO(3)$ symmetry of the model are determined for both the classical and the quantum systems. The results obtained are consistent with the interpretation of the solitons in the model as interacting bosonic particles.

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1 Introduction

Over the past decade, much attention has been given to ($1+2$)-dimensional field theories including a Chern–Simons term. The pure Chern–Simons gauge theory, although still interesting both from the mathematical and the physical points of view, has no dynamics by itself. However, many interesting models for field dynamics can be obtained by adding to the Chern–Simons action Maxwell or Yang–Mills terms and/or interactions with other fields [5]. Some of these models have been shown to admit classical solitonic solutions (vortices), at least for critical or "self-dual" values of the parameters in the lagrangian. These objects can be regarded as smeared-out particles which retain a characteristic size and superpose nonlinearly; unlike some types of solitons, they can also be assigned a pointlike core individually. In specific models, vortices often turn out to possess rather exotic properties, which may be relevant in applications. For example, models with abelian vortices have been important in attempts to explain phenomena in condensed matter theory such as superconductivity and the fractional quantum Hall effect.

In [16], Manton constructed a nonrelativistic lagrangian for a $U(1)$ gauge field minimally coupled to a complex scalar on the plane which describes vortex dynamics. The action for the gauge field includes a Chern–Simons term and the purely spatial part of the Maxwell action. The equations of motion for the field theory are first-order in time, and they admit the well-known Bogomol'nyĭ vortices [13] of the Ginzburg–Landau theory as static solutions, for special values of the parameters. Mathematically, Bogomol'nyĭ vortices are rather well understood, even though on general surfaces they cannot be constructed analytically. Their space of gauge equivalence classes splits into disjoint sectors $\mathcal{M}_N$ labelled by an integer vortex number $N \in \mathbb{Z}$, each $\mathcal{M}_N$ (the moduli space of N vortices) being a smooth $2|N|$-dimensional manifold. In his paper, Manton explored the dynamics of time-dependent fields by explicitly reducing the field theory lagrangian to an effective (finite-dimensional) mechanical system on the moduli space $\mathcal{M}_N$. The lagrangian equations of motion for the reduced system are again first-order in time, and so the moduli space is to be regarded as the phase space where a noncanonical hamiltonian dynamics takes place. The symplectic form defining the dynamics, determined by the kinetic term, contains nontrivial information about the system; of course, it may still be written in canonical form locally, but not in a natural way. Time evolution is determined by the potential energy alone, which is supposed to be small so that the field configurations are still approximately Bogomol'nyĭ vortices. In the case of two vortices, the reduced system describes namely a rigid uniform rotation of the two vortex cores about their midpoint, with an angular velocity which is maximal when the distance between the two is roughly a vortex diameter.

If we place the vortices on a compact surface Σ, with a riemannian metric and an orientation, rather than on the plane, the moduli spaces $\mathcal{M}_N$ also become compact. The metric on Σ fixes a complex structure, which in turn induces a complex structure on $\mathcal{M}_N$. This complex structure can be shown to be compatible with the symplectic form relevant for the dynamics, and so each $\mathcal{M}_N$ becomes a Kähler manifold. Compact Kähler phase spaces are optimal stages for geometric quantisation (cf. [20], [24]). The complex geometry supplies a natural (Kähler) polarisation, for which the corresponding quantum Hilbert space turns out to be finite-dimensional. This approach to the quantisation of the vortex system is to be included in a more general framework, pioneered in [7] in the context of BPS monopoles. The idea is to probe the quantum behaviour of solitons through geometric quantisation of the reduced dynamics on the moduli space of static solutions, when such a space is available. In the more familiar situation of the abelian Higgs model [19], where the reduced system is of second order, there is a canonical hamiltonian description of the classical dynamics and the quantisation can be carried out using the vertical polarisation of $T^*\mathcal{M}_N$, which leads to a truncated Schrödinger representation of the quantum system. The accuracy of the approximation involved is very difficult to assess, and the study of an example where the Schrödinger representation is not available, as is the case here, is of considerable interest. From the point of view of geometric quantisation itself, it is fortunate that Manton's system seems to provide us with a nontrivial example where it may be put to work rather directly.

The main aim of the present work is to discuss the geometric quantisation of Manton's reduced system of periodic vortex dynamics, when space is taken to be a sphere of a given radius R. In particular, we shall determine the dimension of the Hilbert space and construct the quantum operators corresponding to the conserved angular momenta determined by the natural action of $SO(3)$.

Let us summarise how this paper is organised. In section 2, we describe the generalisation of Manton's model to the case where the spatial surface is compact. In section 3, we gather some results concerning the moduli space of Bogomol'nyĭ vortices on a sphere and its use in the study of the field theory dynamics. The effective lagrangian on the moduli space is constructed in section 4. In section 5, we obtain conservation laws for both the field theory and the reduced dynamical system, and show that they are consistent. In section 6, the setup for the geometric quantisation of the reduced dynamics on the moduli space of static solutions is presented. The dimension of the Hilbert space of wavefunctions is computed in section 7, and we show how to construct the quantum angular momentum operators in section 8. Finally, we discuss the results obtained and address some outstanding issues.

2 First-order Chern–Simons vortices

We start by discussing the generalisation of the model introduced by Manton in [16] to the situation where space is compact. We shall consider time-periodic boundary conditions in the formulation of the variational principle, and accordingly we fix space-time to be of the form $S^1 \times \Sigma$, where Σ is a compact and oriented 2-dimensional riemannian manifold. In the remaining sections of this paper we will assume that Σ is a sphere, but for now this restriction is unnecessary. Since in two dimensions any metric is conformally flat, we can introduce a complex coordinate z locally on Σ and write

$$ds^2 = dt^2 - \Omega^2(z,\bar{z})dzd\bar{z} \tag{1}$$

where t parametrises time.

Naively, the lagrangian we would like to consider is

$$\begin{aligned}\mathcal{L}[A,\phi] &= \gamma\left(\frac{i}{2}(\bar{\phi}D_t\phi - \phi\overline{D_t\phi}) - A_t\right)\Omega^2 + \mu\left(BA_t + 2i(E_zA_{\bar{z}} - E_{\bar{z}}A_z)\right) \\ &\quad - \left(\frac{1}{2}B^2\Omega^{-2} + \left(|D_z\phi|^2 + |D_{\bar{z}}\phi|^2\right) + \frac{\lambda}{8}\left(1-|\phi|^2\right)^2\Omega^2\right).\end{aligned} \tag{2}$$

This reduces to Manton's lagrangian [16] without transport current if we take Σ to be the plane and set $\Omega^2 = 1$. Here, $A = A_tdt + A_zdz + A_{\bar{z}}d\bar{z}$ is the real-valued $U(1)$ gauge field, with curvature $F_A = dA = (E_z dz + E_{\bar{z}} d\bar{z}) \wedge dt + \frac{i}{2} B dz \wedge d\bar{z}$, and ϕ the Higgs scalar field. The covariant derivatives are $D_\nu \phi = \partial_\nu \phi - i A_\nu \phi$, where $\nu = t$, z or $\bar{z}$. Of course, $\frac{i}{2}\mathcal{L}$ is to be thought of as the coefficient of a 3-form on some open subset of $S^1 \times \Sigma$ where local expressions for the fields can be given, and the factors of Ω introduced in (2) are imposed by the natural interpretation given to the different terms.

To set up the classical field theory, we must give as global data a principal $U(1)$ bundle P over $S^1 \times \Sigma$. With respect to local trivialisations, the gauge field is interpreted as a connection on this bundle, the Higgs field as a section of the complex line bundle associated to P by the defining representation, and $\bar{\phi}$ is a section of the bundle dual to this one. It is natural to restrict to the situation where P is the pull-back to $S^1 \times \Sigma$ of a $U(1)$ bundle on Σ; in particular, the transition functions will be time-independent. Topologically, $U(1)$ bundles on a compact surface are classified by their first Chern class $N \in \mathbb{Z}$, which can be interpreted as the net number of units of quantised magnetic flux through space at any time,

$$\frac{i}{2} \oint_\Sigma B dz \wedge d\bar{z} = 2\pi N. \tag{3}$$

We may assume without loss of generality that the bundle we are considering over Σ can be trivialised on an open disc $U_1 \subset \Sigma$ and on an open neighbourhood U_2 of its complement, with $U_1 \cap U_2$ being a very narrow annulus which for most purposes can be identified with its retraction ∂U_1. More precisely, we may have to consider sub-patches of U_2 to make sense of local data such as the relevant coordinate z, but this will not affect the discussion of the aspects related to the nontriviality of P which will be our main concern, because $P|_{U_2}$ is trivial. Thus we shall consider P to be defined by the homotopy class of a single transition function $f_{12} : \partial U_1 \to U(1)$ whose degree is N, and we refrain from introducing partitions of unity to keep the discussion as simple as possible.

The term with coefficient μ is the Chern–Simons density $\mu A \wedge dA$. On the overlap of the two trivialising patches U_1, U_2,

$$A^{(1)} \wedge dA^{(1)} = A^{(2)} \wedge dA^{(2)} - i f_{12}^{-1} df_{12} \wedge dA^{(2)}$$

where $A^{(j)}$ denotes the connection 1-form on U_j, and so its values on the trivialising patches do not agree on the overlap $U_1 \cap U_2$. So in general we cannot define an action by just using partitions of unity to patch together pieces of the lagrangian given by (2), as we can do for gauge-invariant lagrangians. Notice that the term proportional to A_t, although gauge-dependent, is unambiguously defined globally since we are assuming that the transition functions are time-independent. The most elegant way to define the Chern–Simons action is as the integral of the gauge-invariant second Chern form $\mu dA \wedge dA$, on any 4-dimensional manifold M with boundary $\partial M = S^1 \times \Sigma$. Here, A is a connection on a principal $U(1)$ bundle on M which restricts to our bundle P on $S^1 \times \Sigma$. There is no obstruction to the existence of such an extension of $P \to S^1 \times \Sigma$, since in our case it would lie in the group $H_3(\mathbb{CP}^\infty; \mathbb{Z})$, which is trivial (cf. [4]). The action should be independent (mod 2π) of the choice of the manifold M and the bundle over it, and this imposes the constraint

$$\mu \in \frac{1}{4\pi}\mathbb{Z} \tag{4}$$

on the Chern–Simons coefficient. We shall write $\kappa := 4\pi\mu$.

The group of gauge transformations $\mathcal{G}$ consists of smooth maps from $S^1 \times \Sigma$ to $U(1)$. The connected component of the identity $\mathcal{G}^0$ is the subgroup of maps homotopic to the identity (the small gauge transformations), and the connected components of $\mathcal{G}$ are labelled by 2-homology classes of space-time, dual to the 1-cycles around which the gauge transformations have nontrivial winding:

$$\mathcal{G}/\mathcal{G}^0 \simeq H_2(S^1 \times \Sigma; \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}^{\oplus 2\mathrm{g}} \tag{5}$$

Here, g is the genus of Σ, and we can choose for the generator σ of the first $\mathbb{Z}$ factor the class of a positively-oriented copy of Σ at a particular instant. It can be shown [3] that a gauge transformation in the connected component of $\mathcal{G}$ labelled by a class whose first component in the above decomposition is $k\sigma$ has the effect of adding the term $2\pi\kappa k N$ to the Chern–Simons action, so that this action is gauge-invariant (mod 2π) if and only if the condition (4) holds.

It is possible to express the Chern–Simons action entirely in terms of the 3-dimensional data by treating carefully the boundary terms of the 4-dimensional Chern action introduced above, as shown in [3]. The result is that we should add a correction to the sum of the integrals of the Chern–Simons bulk term appearing in (2) over U_1 and U_2. The correction term is the double integral

$$\mu i \oint_{S^1 \times \partial U_1} f_{12}^{-1} df_{12} \wedge A^{(1)}. \tag{6}$$

We still have to ensure that the term proportional to A_t in the action is gauge-invariant ($\mathrm{mod}\ 2\pi$). Under a gauge transformation g, A_t changes as

$$A_t \mapsto A_t - ig^{-1}\partial_t g.$$

If the class of g in the first factor of $\mathcal{G}/\mathcal{G}^0$ in (5) is $k\sigma$, then everywhere on Σ

$$i \oint_{S^1} g^{-1}\partial_t g dt = k$$

and the change in the action is

$$-\frac{i}{2}\gamma k \oint_{\Sigma} \Omega^2 dz \wedge d\bar{z} = -\gamma k \,\mathrm{Vol}(\Sigma).$$

This will be in $2\pi\mathbb{Z}$ for all $k \in \mathbb{Z}$ if and only if we impose the constraint

$$\gamma \,\mathrm{Vol}(\Sigma) \in \mathbb{Z}. \tag{7}$$

The action for Manton's model on Σ can then be written as

$$S[A,\phi] = \sum_{j=1}^{2} \int_{S^1 \times U_j} \mathcal{L}[A^{(j)}, \phi^{(j)}] d^3x + \mu i \oint_{S^1 \times \partial U_1} f_{12}^{-1} df_{12} \wedge A^{(1)} \tag{8}$$

where we impose the constraints (4) and (7) to the classical parameters to ensure that e^{iS} is well defined and gauge invariant. To implement the variational principle, we consider variations δA, $\delta\phi$ and $\delta\bar{\phi}$ of the fields which are a 1-form and sections of the bundles associated to P by the fundamental representation and its dual, respectively. As usual, the variation of the first (bulk) term in (8) yields after integration by parts

$$\delta S = \sum_{\psi} \sum_{j=1}^{2} \left\{ \int_{S^1 \times U_j} \left[\frac{\delta \mathcal{L}}{\delta \psi} - \partial_\nu \left(\frac{\delta \mathcal{L}}{\delta \partial_\nu \psi} \right) \right] \delta\psi \, d^3x + \int_{S^1 \times U_j} \partial_\nu \left(\frac{\delta \mathcal{L}}{\delta \partial_\nu \psi} \delta\psi \right) d^3x \right\} \tag{9}$$

Here, ψ is any of A_ν, ϕ or $\bar{\phi}$, and the (j) subscripts have been suppressed. If we define on each U_j the 1-form

$$\Psi := \Omega^{-2} \left(\frac{\delta \mathcal{L}}{\delta \partial_t \psi} \right) dt - \left(\frac{\delta \mathcal{L}}{\delta \partial_{\bar{z}} \psi} \right) dz - \left(\frac{\delta \mathcal{L}}{\delta \partial_z \psi} \right) d\bar{z}$$

and denote the Hodge star of (1) by $*$, then the last integral in the expression (9) can be written as

$$\int_{S^1 \times U_j} \partial_\nu \left(\frac{\delta \mathcal{L}}{\delta \partial_\nu \psi} \delta\psi \right) d^3x = \oint_{S^1 \times \partial U_j} *\Psi \, \delta\psi. \tag{10}$$

It is easy to verify by direct computation that, for the terms in $\mathcal{L}$ which are locally gauge-invariant, the contributions to the components of the 2-form $*\Psi\delta\psi$ are just functions on each trivialising patch. Therefore, their corresponding $j = 1, 2$ contributions in (10) cancel as they should, since the boundaries ∂U_1 and ∂U_2 are very close but carry opposite orientations. For the term proportional to A_t, the values of the contributions on the $j = 1, 2$ patches also cancel because of our assumption of the time-independence of f_{12}. However, the Chern–Simons term has a nonvanishing contribution. Indeed,

$$\delta(A \wedge dA) = 2dA \wedge \delta A - d(A \wedge \delta A)$$

and so the sum of the $j = 1, 2$ contributions to (10) is

$$\mu \oint_{S^1 \times \partial U_1} (A^{(1)} \wedge \delta A - A^{(2)} \wedge \delta A) = -\mu i \oint_{S^1 \times \partial U_1} f_{12}^{-1} df_{12} \wedge \delta A,$$

which exactly cancels the variation of the correction (6) to the Chern–Simons bulk term. So we conclude that the stationarity of the action (8) is exactly expressed by the Euler–Lagrange equations for the naive lagrangian (2) in each trivialising coordinate patch. They read

$$\gamma i D_0 \phi = -(D_z D_{\bar{z}} \phi + D_{\bar{z}} D_z \phi)\Omega^{-2} - \tfrac{\lambda}{4} \left(1 - |\phi|^2\right) \phi \tag{11}$$

$$\partial_z (B\Omega^{-2}) = -iJ_z + 2\mu E_z \tag{12}$$

$$2\mu B = \gamma(1 - |\phi|^2)\Omega^2 \tag{13}$$

where the gauge-invariant supercurrent J_z is defined by

$$J_z := -\frac{i}{2} \left(\bar{\phi} D_z \phi - \phi \overline{D_{\bar{z}} \phi} \right). \tag{14}$$

We are making the assumption that these equations admit nonstatic time-periodic solutions which can be patched together on Σ. Notice that no higher than first-order time derivatives of the fields appear in (11)–(13). Equation (11) is a gauge-invariant nonlinear Schrödinger equation for ϕ, while (12) is a version of Ampère's law and (13) can be interpreted as a magnetic Gauss' law.

3 Static vortices on a sphere

Static configurations are time-independent solutions of the field equations of motion with vanishing A_t. For them, our lagrangian reduces to the Ginzburg–Landau energy functional $\mathcal{E}$, given by the last bracket in (2). Notice that this is also the functional relevant to the discussion of static solutions in the abelian Higgs model. If $\lambda = 1$, one can show that all the critical points of $\mathcal{E}$ satisfy the two-dimensional Bogomol'nyĭ equations [1] on Σ. For configurations with $N > 0$, these read

$$D_{\bar{z}}\phi = 0 \tag{15}$$

$$2B = (1 - |\phi|^2)\Omega^2 \tag{16}$$

and their solutions are known as vortices. If $N < 0$, they are called antivortices and satisfy similar equations, with $D_{\bar{z}}$ replaced by D_z in (15), whereas a minus sign is introduced in (16). Solutions (A, ϕ) to the Bogomol'nyĭ equations for which (3) holds have energy $\mathcal{E} = |N|\pi$. Henceforth, we shall be interested in the $N > 0$ case only. Write $\phi = e^{\frac{1}{2}h + i\chi}$; the function h is gauge-invariant while χ (defined only modulo 2π) is not, and both are real. Equation (15) can be used to obtain A_z in terms of h and χ, and substitution in (16) yields

$$4\partial_z\partial_{\bar{z}}h - (e^h - 1)\Omega^2 = 4\pi \sum_{r=1}^{N} \delta^{(2)}(z - z_r). \tag{17}$$

The solution to this equation provides all the information needed to reconstruct the fields on Σ; χ has to give the required winding properties of ϕ in each patch, but the gauge freedom leaves it otherwise undetermined.

The Bogomol'nyĭ equations were first studied on a compact Riemann surface Σ by Bradlow [2] (cf. also [6] and references therein). He showed existence and uniqueness of a solution satisfying (3) with the Higgs field having exactly N zeros (counted with well-defined multiplicities) in any configuration, provided

$$\mathrm{Vol}(\Sigma) > 4\pi N. \tag{18}$$

Thus, given the topological constraint (3), and if the bound (18) is satisfied, the moduli space of solutions to the Bogomol'nyĭ equations can be described as the symmetric product $\mathcal{M}_N = S^N\Sigma = \Sigma^N/\mathfrak{S}_N$, a smooth $2N$-manifold. These solutions can be interpreted as nonlinear superpositions of N indistinguishable vortices located at the zeros of the Higgs field (the vortex cores), which play the rôle of moduli.

For the rest of this paper, we shall restrict to the situation where Σ is a 2-sphere of radius R, which for later convenience we assume to be centred at the origin of $\mathbb{R}^3$. We choose the open subsets U_1 and U_2 introduced in section 2 to be discs around the North and South poles respectively, where stereographic coordinates can be used. In the next section, we will argue that actually we can shrink U_2 to a point, and focus entirely on a U_1 which covers all of Σ except the South pole. So we can parametrise the position of the vortices in the open dense U_1 by a coordinate z with inverse

$$z \mapsto R\left(\frac{z + \bar{z}}{1 + |z|^2}, -i\frac{z - \bar{z}}{1 + |z|^2}, \frac{1 - |z|^2}{1 + |z|^2}\right). \tag{19}$$

As usual, we write $z = \infty$ to refer to the South pole. In terms of this coordinate, the conformal factor in (1) is just

$$\Omega^2(z, \bar{z}) = \frac{4R^2}{(1+|z|^2)^2}. \tag{20}$$

The positions $z_1, \ldots, z_N$ of N vortices define coordinates almost everywhere on the moduli space. They are regular only in the subset of configurations for which all the zeros of ϕ are simple and none of them occur at the South pole. To parametrise the whole subset $V_0 \subset \mathcal{M}_N$ of static configurations with no zeros of the Higgs field at $z = \infty$, we can introduce instead the elementary symmetric polynomials in the N variables z_r:

$$s_k := s_k^{[N]}(z_1, \ldots, z_N) = \sum_{1 \leq j_1 < \cdots < j_k \leq N} z_{j_1} \cdots z_{j_k}, \qquad 1 \leq k \leq N. \tag{21}$$

More generally, let $V_j \subset \mathcal{M}_N$ denote the subset of configurations with exactly j vortices at $z = \infty$; it is parametrised by the symmetric polynomials $s_k^{[N-j]}$ of the coordinates of the $N - j$ remaining vortices. Clearly, $\mathcal{M}_N = \coprod_{j=0}^N V_j$ gives a decomposition of $\mathcal{M}_N$ into $N + 1$ disjoint $2(N-j)$-cells $V_j \cong \mathbb{C}^{N-j}$, and it is easy to verify that they are glued together so as to give $\mathcal{M}_N = \mathbb{CP}^N$.

Later on, it will be useful to consider the standard decomposition of the moduli space as union of affine pieces $\mathcal{M}_N = \cup_{j=0}^N W_j$, where the W_j are defined in terms of homogeneous coordinates as usual,

$$W_j := \{(Y_0 : Y_1 : \cdots : Y_N) \,|\, Y_j \neq 0\} \cong \mathbb{C}^N.$$

We may set $W_0 = V_0$ say, and identify the s_k in (21) with the inhomogeneous coordinates of W_0,

$$s_k = y_k^{(0)} := \frac{Y_k}{Y_0}, \qquad 1 \leq k \leq N.$$

Letting $s_0 := 1$, we can relate the inhomogeneous coordinates in the other W_j to the s_k by

$$y_k^{(j)} := \frac{Y_k}{Y_j} = \frac{s_k}{s_j}, \qquad 0 \leq k \leq N, \; k \neq j.$$

It is not hard to see that V_j is being identified with the $(N-j)$-plane $y_0^{(j)} = \ldots = y_{j-1}^{(j)} = 0$ in W_j.

We still need to introduce a further piece of notation. Near the position $z = z_r$ of an isolated vortex, $h = \log|\phi|^2$ can be expanded as [21], [19]

$$h(z; z_1, \ldots, z_N) = \log|z - z_r|^2 + a_r + \frac{b_r}{2}(z - z_r) + \frac{\bar{b}_r}{2}(\bar{z} - \bar{z}_r) + O(|z - z_r|^2). \tag{22}$$

Here, a_r and b_r are functions of the positions $z_1, \ldots, z_N$ of all the N vortices. If $N = 1$, spherical symmetry can be used to show that [15]

$$b_1 = -\frac{2\bar{z}_1}{1 + |z_1|^2}.$$

This function describes how the level curves of $|\phi|^2$, which consist of circles centred around the core of the vortex on the sphere, are distorted by the stereographic projection onto the z-plane.

An analogous situation arises when we consider more generally the subvariety $\mathcal{M}_N^{\mathrm{co}} \subset \mathcal{M}_N$ of configurations of N coincident vortices. It is parametrised by the position $z = Z$ of the only zero of the Higgs field, whose modulus squared has a logarithm with an expansion identical to (22) around Z, but with a coefficient N before the logarithmic term; the coefficient of $\frac{1}{2}(z - Z)$ is then

$$b = -\frac{2N\bar{Z}}{1 + |Z|^2}. \tag{23}$$

When the positions of the N vortices do not coincide, there is an additional distortion to the $|\phi|^2$ contours caused by the mutual interactions, and this leads to nontrivial b_r coefficients in (22). It turns out that the functions b_r contain all the information about the interactions relevant for the kinetic term of the reduced mechanical system. This was also the case in Samols' analysis of the abelian Higgs model [19].

It will be useful later to relate the functions b_r to their pullbacks

$$T^* b_r(z_1, \ldots, z_N) := b_r(T(z_1), \ldots, T(z_N))$$

under an isometry T of the sphere. This relationship is easy to obtain directly from the expansion (22) by requiring that h be invariant:

$$T^* b_r = \frac{b_r}{T'(z_r)} - \frac{T''(z_r)}{(T'(z_r))^2}. \tag{24}$$

If we interpret T as a (passive) change of coordinates, this equation can be regarded as the formula for the transformation of Christoffel symbols for an affine connection, as pointed out in [17]. In fact, we shall define a natural unitary connection on a line bundle over the moduli space of static vortices. In the next section, we show that this connection is the essential ingredient of the reduced lagrangian for Manton's model. The functions b_r are part of the coefficients of the connection 1-form, and they incorporate the interactions among the vortices at zero potential energy into the reduced dynamics.

4 The effective mechanical system

Ultimately, we are interested in understanding the dynamics of the field configurations, and this is a much harder problem than the study of static solutions. However, it is natural to expect that, for λ close to 1, slow-varying solutions of the field theory should be well approximated by static solutions evolving along the directions defined by the linearised Bogomol'nyĭ equations. This idea was introduced by Manton in the context of Yang–Mills–Higgs monopoles [14] and has proven very fruitful in many situations. For the abelian Higgs model, a careful analysis by Stuart [22] showed that this so-called adiabatic approximation is exponentially accurate in the limit of small velocities, and that it holds even when the coupling differs slightly from the self-dual value $\lambda = 1$. The field dynamics of N vortices is reduced to an effective mechanical system on the moduli space $\mathcal{M}_N$. The relevant lagrangian can be obtained by evaluating the lagrangian for the field theory at static solutions with time-dependent moduli and then integrating out the space dependence. Samols showed in [19] that the reduced dynamics at self-dual coupling corresponds to geodesic motion on the moduli space, with respect to a Kähler metric that encodes information about the local behaviour of the Higgs field at its zeros; we shall explain this more precisely at the end of this section. For general values of $\lambda \simeq 1$, the geodesic motion is distorted by conservative forces, which are absent at self-dual coupling.

In what follows, we shall study Manton's model within the adiabatic approximation, in the regime $\lambda \simeq 1$ and $\gamma = \mu$. The latter assumption enables one to obtain a neat expression for the reduced kinetic energy term, as will be shown below. Notice that, when $\gamma = \mu$, equation (13) reduces to (16), and this raises our hope that the adiabatic picture is a good approximation to the field theory in this model, as Manton pointed out in [16]. Another check is provided by the existence of consistent conservation laws, which we shall explore in section 5.

We shall follow the analysis in [16] to obtain the Lagrangian for the reduced mechanical system on the moduli space. When γ and μ are set to be equal, consistency of the conditions (4) and (7) is expressed by

$$\frac{\kappa}{4\pi}\mathrm{Vol}(\Sigma) = \kappa R^2 \in \mathbb{Z}. \tag{25}$$

The kinetic energy consists of the terms in (8) which contain time derivatives and A_t. After using (16), it can be written in the form

$$T = \frac{i\gamma}{2}\sum_{j=1}^{2}\int_{U_j} \mathrm{Im}\left(4A_{\bar z}\dot A_z - \bar\phi\dot\phi\Omega^2\right) dz\wedge d\bar z,$$

where the overdots denote time derivatives and z is the relevant stereographic coordinate on each of the discs U_j. Notice that the correction to the naive Chern–Simons action has now been cancelled by a boundary term coming from the bulk. In terms of the functions h and χ introduced in section 3, we can write

$$T = \frac{i\gamma}{2}\sum_{j=1}^{2}\int_{U_j} \left(2i(\partial_{\bar z}\zeta_z - \partial_z\zeta_{\bar z}) + \partial_t(\partial_z h\partial_{\bar z}\chi + \partial_{\bar z}h\partial_z\chi) - \dot\chi\Omega^2\right) dz\wedge d\bar z \tag{26}$$

where ζ_z, $\zeta_{\bar z}$ are the components of the 1-form

$$\zeta = \dot\chi(d\chi + \star dh) + \frac{1}{4}\dot h\, dh$$

in each trivialising coordinate patch; here, $\star$ is the Hodge star of the metric on Σ. To evaluate (26), we cut discs of small radius ϵ around each vortex position (where the integrand is singular) and apply Stokes' theorem, letting $\epsilon \to 0$ at the end.

The only contributions to the integrals around ∂U_j come from the first term in (26), yielding

$$-i\gamma\oint_{\partial U_2}\dot\chi f_{12}^{-1}df_{12},$$

where now χ denotes the argument of $\phi^{(2)}$. If we choose U_2 small enough so that it does not overlap with any of the trajectories of the vortices, χ is globally defined on U_2. So this term is a total time derivative and can be discarded from the kinetic energy. No other contribution coming from the nontriviality of the bundle P arises, and hence we may safely shrink U_2 to the South pole, while $U_1 \cong \mathbb{C}$ becomes dense in Σ. Henceforth, z shall always denote the coordinate in (19).

To describe the contributions coming from the neighbourhood of the vortices, we write near vortex r (cf. [16])

$$\chi = \theta_r + \psi_r, \tag{27}$$

where θ_r is the polar angle in the z-plane with respect to z_r, and ψ_r is a function of the position of the vortices only. Of course, equation (27) assumes that the gauge freedom has been reduced in the neighbourhood of the vortices. The analysis in [16] goes through unchanged to conclude that the contributions from the first two terms in (26) add up to

$$\pi\gamma \sum_{r=1}^{N} \left(2\dot{\psi}_r + i b_r \dot{z}_r - i\bar{b}_r \dot{\bar{z}}_r \right). \tag{28}$$

It follows from the expansion (22) that the coefficient b_r has a singularity when vortex r approaches another vortex $s \neq r$,

$$b_r(z_1, \dots, z_N) = 2 \sum_{s \neq r} \frac{1}{z_r - z_s} + \tilde{b}_r(z_1, \dots, z_N), \tag{29}$$

where $\tilde{b}_r$ is a smooth function. However, a gauge can be chosen for which each ψ_r is given by

$$\psi_r(z_1, \dots, z_N) = \arg \prod_{s \neq r}^{N} (z_r - z_s) \qquad (\mathrm{mod}\ 2\pi)$$

and then

$$\sum_{r=1}^{N} \dot{\psi}_r = -i \sum_{r=1}^{N} \sum_{s \neq r} \left(\frac{\dot{z}_r}{z_r - z_s} - \frac{\dot{\bar{z}}_r}{\bar{z}_r - \bar{z}_s} \right)$$

exactly cancels the singularity from the b_r coefficients.

The last term in (26) yields again a contribution from the metric on Σ. Its time integral gives $-2\pi\gamma$ times the signed area enclosed by the trajectories of the vortices $z_r(t)$. In our local coordinate z, the area form can be expressed as

$$\omega_\Sigma = 2iR^2 \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2} = iR^2 d \left(\frac{z d\bar{z} - \bar{z} dz}{1+|z|^2} \right) =: d\vartheta.$$

Therefore, we may write

$$-2\pi\gamma \oint_{S^1} \left(\int_\Sigma \dot{\chi} \omega_\Sigma \right) dt = -2\pi\gamma \oint_{S^1} \sum_{r=1}^{N} z_r^* \vartheta = 2i\pi\gamma R^2 \oint_{S^1} \sum_{r=1}^{N} \frac{\bar{z}_r \dot{z}_r - z_r \dot{\bar{z}}_r}{1+|z_r|^2} dt \tag{30}$$

and interpret the integrand in the last expression above as the relevant contribution to T.

Putting (28) and (30) together, we conclude that the kinetic energy term in the effective lagrangian is given by

$$T^{\mathrm{red}} = \pi i \gamma \sum_{r=1}^{N} \left[\left(2R^2 \frac{\bar{z}_r}{1+|z_r|^2} + \tilde{b}_r \right) \dot{z}_r - \left(2R^2 \frac{z_r}{1+|z_r|^2} + \bar{\tilde{b}}_r \right) \dot{\bar{z}}_r \right]. \tag{31}$$

Unfortunately, the potential energy is harder to deal with. It can be written as

$$\begin{aligned} V^{\mathrm{red}} &= N\pi + \frac{i}{16}(\lambda-1)\int_\Sigma \left(1-e^h\right)^2 \Omega^2 dz\wedge d\bar z \\ &= N\pi\lambda - \frac{\pi}{2}(\lambda-1)R^2 + \frac{i}{16}(\lambda-1)\int_\Sigma e^{2h}\Omega^2 dz\wedge d\bar z. \end{aligned}$$

It does not seem possible to simplify the integral involving e^{2h} to a local expression in the moduli. Stuart has shown in [23] that, for $N=2$, V^{red} can be approximated by a rational function of the distance between the two vortices, in the limit where Bradlow's bound (18) comes close to the equality.

The reduced lagrangian $L^{\mathrm{red}} = T^{\mathrm{red}} - V^{\mathrm{red}}$ is first-order in the time derivatives, just as the field theory lagrangian. So, as mentioned in the Introduction, the equations of motion already arise in hamiltonian form and the moduli space is to be interpreted as a phase space. The potential energy V^{red} is the only contribution to the hamiltonian, while the kinetic term determines a symplectic potential $\mathcal{A}$ in our coordinate patch W_0 for a noncanonical symplectic form on $\mathcal{M}_N$. Indeed, $T^{\mathrm{red}} = \mathcal{A}(\frac{d}{dt})$ with $\frac{d}{dt}$ the vector field determining time evolution, and $\mathcal{A}$ the real 1-form

$$\mathcal{A} = \pi i\gamma \sum_{r=1}^N \left[\left(2R^2\frac{\bar z_r}{1+|z_r|^2} + \tilde b_r\right)dz_r - \left(2R^2\frac{z_r}{1+|z_r|^2} + \bar{\tilde b}_r\right)d\bar z_r\right]. \tag{32}$$

This 1-form is regular throughout W_0. The equations of motion can be cast as

$$\iota_{\frac{d}{dt}}\omega = -dV^{\mathrm{red}},$$

where the (global) symplectic form $\omega = d\mathcal{A}$ is given on W_0 by

$$\begin{aligned} -\frac{4i}{\kappa}\omega &= d\sum_{r=1}^N \left[\left(2R^2\frac{\bar z_r}{1+|z_r|^2} + \tilde b_r\right)dz_r - \left(2R^2\frac{z_r}{1+|z_r|^2} + \bar{\tilde b}_r\right)d\bar z_r\right] \\ &= \sum_{r=1}^N\sum_{s\neq r}\left(\frac{\partial \tilde b_s}{\partial z_r}dz_r\wedge dz_s - \frac{\partial \bar{\tilde b}_s}{\partial \bar z_r}d\bar z_r\wedge d\bar z_s\right) - \\ &\quad - \sum_{r,s=1}^N\left(\frac{\partial b_r}{\partial \bar z_s} + \frac{\partial \bar b_s}{\partial z_r} + \frac{4R^2\delta_{rs}}{(1+|z_r|^2)^2}\right)dz_r\wedge d\bar z_s. \end{aligned}$$

Notice that we are allowed to replace b_r by $\tilde b_r$ in expressions like $\frac{\partial b_r}{\partial \bar z_s}$, since it is know from (29) that $b_r - \tilde b_r$ is holomorphic. Either of the two arguments given in [19] shows that locally

$$\tilde b_r = \frac{\partial \mathcal{B}}{\partial z_r} \tag{33}$$

for a real function $\mathcal{B}$. Hence

$$\omega = -\frac{i\kappa}{2}\sum_{r,s=1}^N \left(\frac{2R^2\delta_{rs}}{(1+|z_r|^2)^2} + \frac{\partial \bar b_s}{\partial z_r}\right)dz_r\wedge d\bar z_s. \tag{34}$$

It is easily verified that equation (33) is equivalent to ω being a Kähler form on W_0 with respect to the complex structure on the moduli space induced by the one on Σ. Moreover, we conclude from (34) that it is proportional to the Kähler form $\omega_{\rm Sam}$ of Samols' metric [19],

$$\omega = -\frac{\kappa}{2}\,\omega_{\rm Sam}. \tag{35}$$

It is a global $(1,1)$-form on $\mathcal{M}_N$ with respect to the complex structure defined by our coordinates. It becomes apparent that $\omega_{\rm Sam}$ (or ω) is a central object in the reduced dynamics of both the aqbelian Higgs model and Manton's model; but we should emphasise that it plays completely different rôles in the two contexts. In the abelian Higgs model, $\omega_{\rm Sam}$ is the $(1,1)$-form corresponding to a Kähler metric on the configuration space $\mathcal{M}_N$. In the hamiltonian picture, the dynamics takes place on the cotangent bundle $T^*\mathcal{M}_N$ with its canonical (tautological) symplectic form, and time evolution is determined by the laplacian of Samols' metric, possibly with an extra potential term if we allow $\lambda \neq 1$. On the other hand, in Manton's system ω is the symplectic form of a hamiltonian system on $\mathcal{M}_N$ itself, which has no time evolution unless the potential $V^{\rm red}$ is switched on. We can also interpret ω as the curvature of the 1-form $\mathcal{A}$ in (32), which represents a unitary connection on a hermitian line bundle over $\mathcal{M}_N$ in a local orthonormal frame. This point of view leads directly to the geometric quantisation of Manton's system using the natural Kähler polarisation, as we shall see in section 6.

5 Symmetries and conserved quantities

In this section, we analyse the symmetries of Manton's model on the sphere, following the similar analysis in [18] for the plane. More precisely, we will study the isometries of the metric (1). We shall make standard use of Noether's theorem to obtain the corresponding conserved quantities in the lagrangian formulations of both the field theory and the reduced mechanical system. The conserved quantities reduced to static solutions are interesting observables of the effective classical system of vortices, and later we will be concerned with their quantisation.

5.1 Symmetries in the field theory

Here, we shall be concerned with the lagrangian (2). When computing the Lie derivatives of the different terms along a vector field ξ, one obtains gauge-dependent quantities in general. However, following [18], we can supplement the field variations under the flow of ξ by a gauge transformation by $e^{-i\alpha A(\xi)}$, where α is the flow parameter, so as to obtain gauge-invariant variations. The whole operation can be interpreted as a covariant Lie derivative, in the spirit of the discussion in the Appendix of reference [23].

The simplest symmetry of the model is time translation, generated by ∂_t. The $O(\alpha)$ variation of the lagrangian is

$$\alpha\delta\mathcal{L} = \alpha\left[\partial_t(\mathcal{L} + \gamma A_t \Omega^2 - \mu B A_t) + 2\mu i\left(\partial_{\bar z}(A_t E_z) - \partial_z(A_t E_{\bar z})\right)\right],$$

where we included a gauge transformation by $e^{-i\alpha A_t}$ in the fields. Noether's theorem then gives the conserved density

$$\begin{aligned} j^t &= \sum_\psi \frac{\delta\mathcal{L}}{\delta\partial_t\psi}\delta\psi - \partial_t(\mathcal{L} + \gamma A_t\Omega^2 - \mu B A_t) \\ &= \frac{1}{2}B^2\Omega^{-2} + (|D_z\phi|^2 + |D_{\bar z}\phi|^2) + \frac{\lambda}{8}(1-|\phi|^2)^2\Omega^2. \end{aligned}$$

This is the density of potential energy, so we learn that V is a constant of motion. This result does not depend on the particular form (20) for the conformal factor of the metric, and so it is also valid for more general Σ.

The $SO(3)$ action on the sphere Σ by rotations about axes through the origin of $\mathbb{R}^3$ provides conservation laws for angular momentum. In our coordinate z, this action is described by elliptic Möbius transformations with antipodal fixed points,

$$z \mapsto \frac{(e^{i\alpha} + |a|^2)z + a(1-e^{i\alpha})}{\bar a(1-e^{i\alpha})z + (1+|a|^2 e^{i\alpha})}, \tag{36}$$

where $a \in \mathbb{C}$ and $\alpha \in \mathbb{R}$. We shall consider the effect of rotations $R_\alpha^{(j)}$ ($j = 1,2,3$) by an angle α about the three cartesian axes, which correspond to taking $a = 1, i, 0$.

The variation of the lagrangian under the rotation $R_\alpha^{(1)}$, to be supplemented by the gauge transformation $e^{-\frac{\alpha}{2}\left((1-z^2)A_z - (1-\bar z^2)A_{\bar z}\right)}$, is given up to $O(\alpha)$ by

$$\begin{aligned} \alpha\delta_{(1)}\mathcal{L} &= \frac{\alpha}{2} i \left[\partial_t\left[(\mu B - \gamma\Omega^2)\left((1-z^2)A_z - (1-\bar z^2)A_{\bar z}\right)\right]\right. \\ &\quad - \partial_z\left[(1-z^2)\mathcal{L} + 2\mu i\left((1-z^2)A_z E_{\bar z} - (1-\bar z^2)A_{\bar z}E_{\bar z}\right)\right] \\ &\quad \left. +\partial_{\bar z}\left[(1-z^2)\mathcal{L} + 2\mu i\left((1-\bar z^2)A_{\bar z}E_z - (1-z^2)A_z E_z\right)\right]\right], \end{aligned}$$

and the variations $\delta_{(2)}\mathcal{L}$, $\delta_{(3)}\mathcal{L}$ are given by similar expressions. The densities of the conserved quantities can then be shown to be

$$\begin{aligned} j^t_{(1)} &= \frac{\gamma i}{2}\left((1-z^2)(J_z + A_z) - (1-\bar z^2)(J_{\bar z} + A_{\bar z})\right)\Omega^2 \\ j^t_{(2)} &= -\frac{\gamma}{2}\left((1+z^2)(J_z + A_z) + (1+\bar z^2)(J_{\bar z} + A_{\bar z})\right)\Omega^2 \\ j^t_{(3)} &= -\gamma i\left(z(J_z + A_z) - \bar z(J_{\bar z} + A_{\bar z})\right)\Omega^2. \end{aligned}$$

where J_z is the supercurrent defined in (14). These densities are still not gauge invariant. As in [18], we can remedy this by adding to the vector field $X_{(k)}$ in $\delta_{(k)}\mathcal{L} =: \partial_\nu X^\nu_{(k)}$ a divergenceless vector field. So we substitute $X^\nu_{(k)}$ by $\tilde X^\nu_{(k)}$ given by

$$\begin{aligned} \tilde X^t_{(k)} &= X^t_{(k)} - \partial_z(\Lambda_{(k)}A_{\bar z}) + \partial_{\bar z}(\Lambda_{(k)}A_z) \\ \tilde X^z_{(k)} &= X^z_{(k)} + \partial_t(\Lambda_{(k)}A_{\bar z}) - \partial_{\bar z}(\Lambda_{(k)}A_t) \\ \tilde X^{\bar z}_{(k)} &= X^{\bar z}_{(k)} - \partial_t(\Lambda_{(k)}A_z) + \partial_z(\Lambda_{(k)}A_t), \end{aligned}$$

using

$$\left(\Lambda_{(1)}, \Lambda_{(2)}, \Lambda_{(3)}\right) = -2i\gamma R^2\left(\frac{z+\bar z}{1+|z|^2}, -i\frac{z-\bar z}{1+|z|^2}, \frac{1-|z|^2}{1+|z|^2}\right).$$

The new densities $\tilde{j}^t_{(k)}$ are now gauge invariant, and their space integrals are the conserved quantities

$$\begin{aligned}
M_1 &= -\frac{\gamma}{4}\int_{\mathbb{C}}\left[\left((1-z^2)J_z-(1-\bar z^2)J_{\bar z}\right)\Omega^2+2iR^2\frac{z+\bar z}{1+|z|^2}B\right]dz\wedge d\bar z\\
M_2 &= \frac{\gamma i}{4}\int_{\mathbb{C}}\left[\left((1+z^2)J_z+(1+\bar z^2)J_{\bar z}\right)\Omega^2-2iR^2\frac{z-\bar z}{1+|z|^2}B\right]dz\wedge d\bar z\\
M_3 &= \frac{\gamma i}{2}\int_{\mathbb{C}}\left[i\left(zJ_z-\bar zJ_{\bar z}\right)\Omega^2+2R^2\frac{1-|z|^2}{1+|z|^2}B\right]dz\wedge d\bar z
\end{aligned}$$

which can be interpreted as angular momenta around the three independent axes. Although Noether's theorem determines the conserved quantities corresponding to a given symmetry generator only up to an additive constant, the requirement that M_k be the components of a hamiltonian moment of $SO(3)$ removes this ambiguity.

Just as in [18], the quantities M_k can be neatly written as moments of the vorticity of the system. This is defined to be the gauge-invariant real quantity

$$\mathcal{V}=2i(\partial_{\bar z}J_z-\partial_z J_{\bar z})+B,$$

which is well-defined and smooth everywhere on Σ. Typically, it approaches zero away from the vortex cores, where both the magnetic field and the supercurrent (and its derivatives) become negligible. We obtain

$$M_1 = \frac{i\gamma}{2}R^2\int_{\mathbb{C}}\frac{z+\bar z}{1+|z|^2}\mathcal{V}\,dz\wedge d\bar z \tag{37}$$

$$M_2 = \frac{i\gamma}{2}R^2\int_{\mathbb{C}}\frac{(-i)(z-\bar z)}{1+|z|^2}\mathcal{V}\,dz\wedge d\bar z \tag{38}$$

$$M_3 = \frac{i\gamma}{2}R^2\int_{\mathbb{C}}\frac{1-|z|^2}{1+|z|^2}\mathcal{V}\,dz\wedge d\bar z \tag{39}$$

The expressions in the integrands should be compared with the cartesian coordinates on the sphere as given by equation (19). We can anticipate that $\mathbf{M}=(M_1,M_2,M_3)$ is a vector in $\mathbb{R}^3$ which contains information about a centre of mass of the vortex configurations. In particular, when N vortices become coincident, we expect $\mathbf{M}$ to point in the direction of the core, given the circular symmetry of the fields.

5.2 Symmetries in the effective mechanical system

There is an action of $SO(3)$ on $\mathbb{CP}^N$ by simultaneous rotation of the vortex positions z_r as in (36). It is generated by the vector fields

$$\xi_{(1)} = -\frac{i}{2}\sum_{r=1}^{N}\left((1-z_r^2)\frac{\partial}{\partial z_r}-(1-\bar z_r^2)\frac{\partial}{\partial \bar z_r}\right), \tag{40}$$

$$\xi_{(2)} = \frac{1}{2}\sum_{r=1}^{N}\left((1+z_r^2)\frac{\partial}{\partial z_r}+(1+\bar z_r^2)\frac{\partial}{\partial \bar z_r}\right), \tag{41}$$

$$\xi_{(3)} = i\sum_{r=1}^{N}\left(z_r\frac{\partial}{\partial z_r}-\bar z_r\frac{\partial}{\partial \bar z_r}\right), \tag{42}$$

which can be seen to extend smoothly to $\mathbb{CP}^N$ after changing coordinates from the z_r to the s_k defined in (21). The rotational symmetry yields three independent relations among the functions b_r, which we shall now derive. Notice that the fluxes of the vector fields $\xi_{(j)}$ are given by acting on each z_r by the rotations $R_\alpha^{(j)}$ introduced in section 5.1. The Lie derivatives of the b_r can be computed by making use of (24),

$$\pounds_{\xi_{(j)}} b_r = \lim_{\alpha\to 0} \frac{R_\alpha^{(j)*} b_r - b_r}{\alpha}.$$

We then find the following relations:

$$-\frac{i}{2}\sum_{s=1}^N \left((1-z_s^2)\frac{\partial b_r}{\partial z_s} - (1-\bar{z}_s^2)\frac{\partial b_r}{\partial \bar{z}_r}\right) = \pounds_{\xi_{(1)}} b_r = -i(1+z_r b_r) \tag{43}$$

$$\frac{1}{2}\sum_{s=1}^N \left((1+z_s^2)\frac{\partial b_r}{\partial z_s} + (1+\bar{z}_s^2)\frac{\partial b_r}{\partial \bar{z}_s}\right) = \pounds_{\xi_{(2)}} b_r = -(1+z_r b_r) \tag{44}$$

$$i\sum_{s=1}^N \left(z_s\frac{\partial b_r}{\partial z_s} - \bar{z}_s\frac{\partial b_r}{\partial \bar{z}_s}\right) = \pounds_{\xi_{(3)}} b_r = -ib_r \tag{45}$$

Using (33), equations (43) and (44) can be written as

$$\sum_{s=1}^N \left((1\mp z_s^2)\frac{\partial b_s}{\partial z_r} \mp (1\mp \bar{z}_s^2)\frac{\partial \bar{b}_s}{\partial z_r}\right) \mp 2(1+z_r b_r) = 0$$

which together imply that the quantity $\sum_{s=1}^N(2z_s + z_s^2 b_s + \bar{b}_s)$ is constant for all vortex configurations. To find what this constant is, we remark that all the singular parts in (29) cancel in pairs in the sum over s, and that in the limit of coincidence of the vortices the functions $\tilde{b}_r$ in (29) tend to b in (23). In particular, when all the vortices are at $Z=0$,

$$\sum_{r=1}^N b_r(0,\ldots,0) = 0. \tag{46}$$

So we conclude that

$$\sum_{s=1}^N (2z_s + z_s^2 b_s + \bar{b}_s) = 0. \tag{47}$$

Similarly, equation (45) and its conjugate imply that the quantity $\sum_{r=1}^N(z_s b_s - \bar{z}_s\bar{b}_s)$ is independent of the vortex positions. Using the explicit formula (23) for N coincident vortices, we then deduce that this constant has to be zero, obtaining

$$\sum_{s=1}^N z_s b_s \in \mathbb{R}. \tag{48}$$

We remark that equations (47) and (48) are analogues of the statement that the sum of the b_r vanishes for any vortex configuration on the plane, as found by Samols [19] as a consequence of translational symmetry. In fact, this statement follows from (47) in the limit where all vortices approach the origin, and (46) may be regarded as a special case of it. Equation (48) is also valid for vortices on the plane, and is a consequence of the $SO(2)$ symmetry, but it has not been noted before in the literature.

The $SO(3)$ action on $\mathbb{CP}^N$ leaves the symplectic form (34) invariant, i.e.

$$\pounds_\xi \omega = d\iota_\xi \omega = 0 \tag{49}$$

for all ξ generating a rotation. To establish this, we make use of the relations (47) and (48). For example, $\xi_{(3)}$ satisfies (49) if and only if

$$\frac{\partial}{\partial \bar{z}_q}\left[\sum_{s=1}^{N}\left(\bar{z}_s\frac{\partial \bar{b}_s}{\partial z_r} - z_s\frac{\partial b_s}{\partial z_r}\right) - b_r\right] = 0$$

for all q and r, and this follows from (48) or the weaker statement (45). Since $H^1(\mathbb{CP}^N;\mathbb{R})$ is trivial, (49) implies that there exist globally defined functions $M_j^{\rm red}$ satisfying

$$\iota_{\xi_{(j)}}\omega = -dM_j^{\rm red}. \tag{50}$$

The functions $M_j^{\rm red}$ are determined from (50) only up to a constant, and the $\xi_{(j)}$ are their corresponding hamiltonian vector fields. We can fix this constant by requiring that the $M_j^{\rm red}$ are the components of a moment map, i.e. that the $SO(3)$ action is hamiltonian (cf. [9]),

$$\{M_i^{\rm red}, M_j^{\rm red}\} := \omega(\xi_{(i)},\xi_{(j)}) = -\sum_{k=1}^{3}\epsilon_{ijk}M_k^{\rm red}. \tag{51}$$

The $M_j^{\rm red}$ turn out to be the conserved quantities corresponding to the rotational symmetry in the reduced mechanical system. Recall that the connection 1-form $\mathcal{A}$ in (32) is a symplectic potential for ω, and thus equation (50) is equivalent to

$$\pounds_{\xi_{(j)}}\mathcal{A} = dW_j \tag{52}$$

with

$$W_j = \mathcal{A}(\xi_{(j)}) - M_j^{\rm red}. \tag{53}$$

Equation (52) is the statement of rotational invariance of the $U(1)$ connection represented by $\mathcal{A}$ (see [12] for a discussion in the more general situation of connections on bundles with nonabelian structure group). The reduced lagrangian $L^{\rm red}$ has a kinetic term (31) of the form

$$T^{\rm red} = \sum_{r=1}^{N}\mathcal{A}_r\dot{z}_r + \text{c.c.}$$

and a rotationally-invariant potential. Using (52), we can establish that

$$\pounds_{\xi_{(j)}}L^{\rm red} = \partial_t W_j \tag{54}$$

and so Noether's theorem implies that M_j^{red} as given by (53) is a conserved quantity. Notice that W_j depends on the choice of symplectic potential for ω, whereas M_j^{red} does not — cf. equations (52) and (50). We can determine M_j^{red} by integrating (50), or alternatively from (54) and (53). We shall follow the latter route, which provides a direct proof of the spherical symmetry of the reduced system. Using equations (43)–(45), we find that

$$\begin{aligned} \pounds_{\xi_{(1)}} L^{\mathrm{red}} &= -\pi\gamma(R^2-N)\partial_t \sum_{r=1}^N (z_r+\bar{z}_r) \\ \pounds_{\xi_{(2)}} L^{\mathrm{red}} &= \pi i\gamma(R^2-N)\partial_t \sum_{r=1}^N (z_r-\bar{z}_r) \\ \pounds_{\xi_{(3)}} L^{\mathrm{red}} &= 0 \;=\; 2\pi\gamma\partial_t\left(N(N-R^2)\right). \end{aligned}$$

The constant term after the time derivative in the last equation was chosen so that the conserved quantities M_j^{red} obey (51). Making use of the relations (47) and (48) for the functions b_r, they can be written as

$$M_1^{\mathrm{red}} = \frac{\kappa}{4}\sum_{r=1}^N \left(2R^2\frac{z_r+\bar{z}_r}{1+|z_r|^2}+b_r+\bar{b}_r\right), \tag{55}$$

$$M_2^{\mathrm{red}} = \frac{i\kappa}{4}\sum_{r=1}^N \left(-2R^2\frac{z_r-\bar{z}_r}{1+|z_r|^2}+b_r-\bar{b}_r\right), \tag{56}$$

$$M_3^{\mathrm{red}} = \frac{\kappa}{2}\sum_{r=1}^N \left(R^2\frac{1-|z_r|^2}{1+|z_r|^2}-(z_rb_r+1)\right). \tag{57}$$

A consistency check of the reduction procedure can be made by comparing the conserved angular momenta in the two pictures. To do this, we shall write the quantities M_k in section 5.1 for static solutions in terms of the moduli. This is most easily done from equations (37)–(39), expressing the fields in terms of the function h and making use of equations (17) and (22) to reduce each M_k to the moduli space, similarly to what we did for the lagrangian in section 4. For example, to obtain the expression for M_1 we start by writing

$$\begin{aligned} \frac{z+\bar{z}}{1+|z|^2}\mathcal{V} &= \frac{2}{R^2}\frac{z+\bar{z}}{1+|z|^2}\partial_z\left(\frac{\partial_{\bar{z}}h\,\partial_z\partial_{\bar{z}}h}{1+|z|^2}\right) \\ &= \frac{4}{R^2}\partial_z\left((z+\bar{z})\partial_z\left(\frac{(\partial_{\bar{z}}h)^2}{1+|z|^2}\right)-\frac{(\partial_{\bar{z}}h)^2}{1+|z|^2}\right)+\text{c.c.} \end{aligned}$$

and use Stokes' theorem to evaluate (37) as a sum of contour integrals along small discs C_r of radius ϵ around the vortex positions:

$$M_1 = \frac{\gamma i}{4}\sum_{r=1}^N \oint_{C_r} \left((z+\bar{z})\left(\bar{z}(\partial_{\bar{z}}h)^2+(1+|z|^2)\partial_{\bar{z}}h\partial_z\partial_{\bar{z}}h\right)-(1+|z|^2)(\partial_{\bar{z}}h)^2\right)d\bar{z}+\text{c.c.}$$

Assuming that the vortices are isolated, we may write on C_r

$$\begin{aligned}
\partial_z \partial_{\bar z} h &= -\frac{R^2}{(1+|z_r|^2)^2} + O(\epsilon^2) \\
\partial_{\bar z} h &= \frac{e^{i\theta_r}}{\epsilon} + \frac{\bar b_r}{2} + O(\epsilon)
\end{aligned}$$

and then obtain in the limit $\epsilon \to 0$

$$\begin{aligned}
M_1 &= \frac{\pi\gamma}{2} \sum_{r=1}^{N} \left(4R^2 \frac{z_r + \bar z_r}{1+|z_r|^2} + \left(1 - z_r^2\right) b_r + \left(1 - \bar z_r^2\right) \bar b_r - 2(z_r + \bar z_r) \right) \\
&= \frac{\kappa}{4} \sum_{r=1}^{N} \left(2R^2 \frac{z_r + \bar z_r}{1+|z_r|^2} + b_r + \bar b_r \right),
\end{aligned}$$

where we made use of (47). Similarly, we find

$$\begin{aligned}
M_2 &= \frac{\pi\gamma i}{2} \sum_{r=1}^{N} \left(-4R^2 \frac{z_r - \bar z_r}{1+|z_r|^2} + \left(1 + z_r^2\right) b_r - \left(1 + \bar z_r^2\right) \bar b_r + 2(z_r - \bar z_r) \right) \\
&= \frac{\kappa i}{4} \sum_{r=1}^{N} \left(-2R^2 \frac{z_r - \bar z_r}{1+|z_r|^2} + b_r - \bar b_r \right)
\end{aligned}$$

and

$$\begin{aligned}
M_3 &= \pi\gamma \sum_{r=1}^{N} \left(2R^2 \frac{1-|z_r|^2}{1+|z_r|^2} - \left(z_r b_r + \bar z_r \bar b_r\right) - 2 \right) \\
&= \frac{\kappa}{2} \sum_{r=1}^{N} \left(R^2 \frac{1-|z_r|^2}{1+|z_r|^2} - (z_r b_r + 1) \right).
\end{aligned}$$

So each M_j agrees with M_j^{red}.

It is instructive to compare the conserved quantities that we have found for the sphere with the ones obtained for the plane in [18]. On the plane, space isometries are described by the euclidean group $E(2)$. Convenient generators are the translations along the x_1 and x_2 axes and the rotation about the origin, and their conserved quantities in the reduced picture were determined to be

$$P_1 = -\pi\gamma i \sum_{r=1}^{N} (Z_r - \bar Z_r), \tag{58}$$

$$P_2 = \pi\gamma \sum_{r=1}^{N} (Z_r + \bar Z_r), \tag{59}$$

$$M = 2\pi\gamma \sum_{r=1}^{N} \left(\frac{1}{2}|Z_r|^2 + B_r Z_r + \bar B_r \bar Z_r + 1 \right), \tag{60}$$

where Z_r denote the positions of the vortex cores, B_r the coefficients in an expansion equivalent to (22), and we removed the 'red' superscripts. In the limit where the radius R is large and the vortices are close together, say in a small neighbourhood of the North pole, one should expect that our M_k should be well approximated by quantities directly related to the ones in (58)–(60). Indeed, identifying $2Rz_r = Z_r$ we obtain from (55)–(57)

$$M_1 = -RP_2 + \pi\gamma R \sum_{r=1}^{N} (B_r + \bar{B}_r) + O(|z_s|) \tag{61}$$

$$M_2 = RP_1 + \pi\gamma i R \sum_{r=1}^{N} (B_r - \bar{B}_r) + O(|z_s|) \tag{62}$$

$$M_3 = 2\pi\gamma N R^2 - M + O(|z_s|). \tag{63}$$

Since it is known from [19] that $\sum_{r=1}^{N} B_r = 0$ (cf. equation (46)), we see that M_1, M_2 and P_1, P_2 are related as expected, but the equation (63) for M_3 is rather surprising. It means that, as $R \to \infty$, M_3 becomes infinite, and we should subtract from it the quantity $2\pi\gamma N R^2$, itself infinite in the limit, to be able to compare it with the angular momentum in the plane, M. Notice that when (61)–(63) are inserted in (51), we obtain

$$\{P_1, P_2\} = 2\pi\gamma N - \frac{1}{R^2} M + O(|z_s|).$$

Thus when $R \to \infty$ we recover the nonvanishing classical Poisson brackets for the linear momenta in the plane as calculated in [11].

For N coincident vortices, we can make use of (23) to obtain the angular momentum vector in closed form:

$$\mathbf{M}^{[N]} = 2\pi\gamma N(R^2 - N) \left(\frac{Z + \bar{Z}}{1 + |Z|^2}, -i\frac{Z - \bar{Z}}{1 + |Z|^2}, \frac{1 - |Z|^2}{1 + |Z|^2} \right). \tag{64}$$

As before, Z denotes the position of the common core. If we take $N = 1$, (64) implies that a single vortex on Σ should be assigned a nonzero $\mathbf{M}$ vector. Its direction gives just the vortex position, and its conservation implies that the single vortex does not move (even if $\lambda \neq 1$), as expected. The length of $\mathbf{M}^{[1]}$ can be interpreted as a nonzero intrinsic angular momentum of the single vortex at rest. This was also a feature of the model in the plane, where a single vortex was found to have an intrinsic momentum $-2\pi\gamma$. This agrees with our result, provided that we subtract the "tail" momentum $2\pi\gamma N R^2$ as discussed above. It should be noted that on the sphere this intrinsic momentum is quantised, from the considerations in section 2 — it is a half integer in units of $\hbar = 1$. On the plane, this feature is not apparent, since γ could take any value. More generally, a configuration of N coincident vortices has angular momentum $-2\pi\gamma N^2$ after subtraction of $2\pi\gamma N R^2$, and this is consistent with the results in [18].

6 Ingredients for geometric quantisation

We would like to investigate the quantum version of the reduced mechanics in the framework of geometric quantisation. We shall follow the conventions in [24] and refer to [8] for background on complex geometry. To construct the quantum system, we need to supplement the classical theory (specified by the phase space $\mathcal{M}_N = \mathbb{CP}^N$, endowed with the symplectic form ω) with a hermitian line bundle L over $\mathcal{M}_N$. The wavefunctions in the quantum Hilbert space are particular sections of L.

To start with, we should verify whether our phase space is quantisable at all. This is equivalent to the integrality of the class represented by the closed form $\frac{1}{2\pi}\omega$ in de Rham cohomology,

$$\frac{1}{2\pi}[\omega] \in H^2(\mathcal{M}_N;\mathbb{Z}) \subset H^2(\mathcal{M}_N;\mathbb{R}). \tag{65}$$

In general, this requirement leads to nontrivial constraints on the parameters of the classical theory — the Weil (pre)quantisation conditions. If they are satisfied, we may regard $\frac{1}{2\pi}[\omega]$ as the first Chern class of a smooth complex line bundle over $\mathcal{M}_N$, which is what we call the prequantum line bundle L.

Recall that $H^2(\mathbb{CP}^N;\mathbb{R})$ is cyclic and we can take as generator the first Chern class $\eta \in H^2(\mathbb{CP}^N;\mathbb{Z})$ of the hyperplane bundle of $\mathbb{CP}^N$. Then $[\omega] = 2\pi\ell\eta$ for suitable $\ell \in \mathbb{R}$. To determine ℓ, we can refer to equation (35) and use the formula for the cohomology class of ω_{Sam} obtained by Manton in [15]. (This formula has been generalised in [17] for Σ of arbitrary genus.) For the benefit of the reader, we reproduce Manton's argument here. Let $\mathcal{M}_N^{\mathrm{co}} \subset \mathcal{M}_N$ be the subvariety of configurations of N coincident vortices. This is a projective line parametrised by the position Z of the zero of the Higgs field. Equation (23) implies that ω restricts to it as

$$\omega|_{\mathcal{M}_N^{\mathrm{co}}} = -i\kappa\frac{N(R^2-N)}{(1+|Z|^2)^2}dZ\wedge d\bar{Z}. \tag{66}$$

It is readily seen that $\mathcal{M}_N^{\mathrm{co}}$ is embedded as a projective curve of degree N in $\mathcal{M}_N$, and is thus homologous to $\pm N[\mathbb{CP}^1]$. Here, we denote by $[\mathbb{CP}^1]$ the homology class of a projective line inside $\mathcal{M}_N$, which is dual to η and a generator of $H_2(\mathcal{M}_N;\mathbb{Z})$. The integral of (66) over $\mathcal{M}_N^{\mathrm{co}}$ is just $-2\pi\kappa N(R^2-N)$, and so we conclude that $\ell = -\kappa(R^2-N)$,

$$\frac{1}{2\pi}[\omega] = \ell\eta = -\kappa(R^2-N)\eta. \tag{67}$$

Equation (65) is equivalent to ℓ being an integer,

$$\kappa(R^2-N)\in\mathbb{Z}$$

and this is weaker than the conditions $\kappa, \kappa R^2 \in \mathbb{Z}$ that we already had to impose in (4) and (25) from considerations of gauge invariance ($\mathrm{mod}\ 2\pi$) of the classical field theory action. We conclude that no further constraints arise from prequantisation.

In geometric quantisation, the prequantum line bundle L is to be equipped with a hermitian metric and a unitary connection. The fact that $\mathbb{CP}^N$ is simply-connected implies that in our case L is uniquely determined as a smooth bundle by the symplectic structure, and so is the hermitian metric and the connection D. The basic idea in the standard construction of L is to interpret (real) symplectic potentials of ω as local expressions for the connection, and then use parallel transport to define local sections and construct the bundle (cf. [24]). A given symplectic potential determines a unique local section σ of L up to a phase of modulus one. The hermitian metric is introduced by requiring that each σ is a local orthonormal frame,

$$\langle\sigma,\sigma\rangle = 1. \tag{68}$$

This is unambiguous since two symplectic potentials must differ by the exterior derivative of a real function u, and then the corresponding local sections are related by the factor e^{-iu}.

The wavefunctions in geometric quantisation are defined as the L^2 polarised sections of L. By L^2 we mean square-integrable with respect to the hermitian product (68) on the fibres and the symplectic measure $\frac{\omega^N}{N!}$ on the base $\mathcal{M}_N$. Roughly speaking, polarised means that they only depend on half of the real coordinates of the phase space, just as the wavefunctions in the Schrödinger representation of quantum mechanics only depend on the position and not on the momentum. More precisely, a polarisation $\mathcal{P}$ is defined as a lagrangian (i.e. maximally isotropic) integrable subbundle of the complexification $T_{\mathbb{C}}\mathcal{M}_N$ of the tangent bundle of the phase space, and the condition

$$\mathcal{D}_{\bar{X}}\psi = 0, \qquad \forall\, X \in \Gamma(\mathcal{M}_N, \mathcal{P}) \tag{69}$$

defines what is meant for a section ψ to be $\mathcal{P}$-polarised. When the classical dynamics takes place in a Kähler phase space, as is our case, there is a natural choice of polarisation $\mathcal{P}$ — namely, the one determined by the i-eigenspaces of the compatible complex structure. It is generated by the holomorphic vector fields in the local complex coordinates. The introduction of a Kähler polarisation can be interpreted naturally in terms of complex geometry as follows. A connection on the prequantum line bundle defines a holomorphic structure for L: By definition, the holomorphic sections are the ones which are annihilated by the part of $\mathcal{D}$ that takes values in $\Omega^{(0,1)}(\mathcal{M}_N, L)$, which is defined from the complex structure in the base. But such sections are precisely the ones satisfying the condition (69) for the Kähler polarisation. Thus, polarised sections of L are nothing but holomorphic sections with respect to the holomorphic structure on L induced by the unitary connection $\mathcal{D}$.

7 The quantum Hilbert space

The Picard variety of $\mathbb{CP}^N$ is trivial, and this implies that L is uniquely determined as a holomorphic line bundle by its first Chern class, which can be read off from (67). A classical result on the sheaf cohomology of $\mathbb{CP}^N$ establishes that L admits nontrivial global holomorphic sections if and only if $\ell > 0$ (i.e. $\kappa < 0$), and then they form the vector space (cf. [10])

$$H^0(\mathbb{CP}^N, \mathcal{O}(L)) \cong \mathbb{C}[Y_0, \ldots, Y_N]_\ell \tag{70}$$

where the right-hand side denotes the homogeneous polynomials of degree ℓ in the $N+1$ variables Y_j. This gives a concrete way to realise L and its sections (up to multiplication by a constant in $\mathbb{C}^\times$). Recall that the local symplectic potential $\mathcal{A}$ in (32) for the connection $\mathcal{D}$ determines a nonvanishing local section $\sigma : W_0 \to L$. It is not holomorphic though, as $\mathcal{A}$ has a nonzero component in $\Omega^{(0,1)}(W_0)$. But we can obtain a holomorphic local section from it by using a nonunitary gauge transformation: Since

$$\mathcal{A} = 2\pi\gamma i \sum_{r=1}^{N} \left(2R^2 \frac{\bar{z}_r}{1+|z_r|^2} + \tilde{b}_r \right) dz_r - 2\pi\gamma i d \left(\frac{1}{2}\mathcal{B} + R^2 \sum_{r=1}^{N} \log\left(1+|z_r|^2\right) \right), \tag{71}$$

where $\mathcal{B}$ is defined up to an additive real constant by (33), we can define on W_0

$$\sigma^{(0)}(z_1 \ldots, z_N) := \sigma(z_1 \ldots, z_N) e^{-2\pi\gamma\left(\frac{1}{2}\mathcal{B} + R^2 \sum_{r=1}^{N} \log\left(1+|z_r|^2\right)\right)}; \tag{72}$$

this is a holomorphic section of L on W_0. It is uniquely determined from σ up to a positive real constant, and thus from A up to a constant in $\mathbb{C}^\times$. It extends to a global section of L, vanishing in the complement of W_0; we identify it with the homogeneous polynomial Y_0^ℓ in (70). From it, we can define the sections

$$\sigma^{(j)} := \left(\frac{Y_j}{Y_0}\right)^\ell \sigma^{(0)} = \left(y_j^{(0)}\right)^\ell \sigma^{(0)}$$

which trivialise L on each W_j, and determine the line bundle through the transition functions

$$\begin{array}{rcl} \varphi_{ij} \;:\; W_i \cap W_j & \longrightarrow & \mathbb{C}^\times \\ (y_1^{(i)}, \ldots, y_N^{(i)}) & \longmapsto & \left(y_j^{(i)}\right)^\ell = \left(\dfrac{s_j}{s_i}\right)^{-\kappa(R^2-N)} . \end{array}$$

On each W_j, global holomorphic sections of L are given by multiplying $\sigma^{(j)}$ by polynomials in the $y_k^{(j)}$ of degree less than or equal to ℓ.

The quantum Hilbert space $\mathcal{H}_\mathcal{P}$ is the space of holomorphic sections of L which are normalisable with respect to the inner product defined by the symplectic measure of $\mathcal{M}_N$ and the product on the fibres given by (68), as we said in section 6. This inner product can be easily written down as an integral over the open dense W_0, where L is trivialised by $\sigma^{(0)}$, by making use of (34), (68) and (72). Since we are dealing with a compact phase space, all the holomorphic sections have finite norm, so the Hilbert space $\mathcal{H}_\mathcal{P}$ is $H^0(\mathbb{CP}^N, \mathcal{O}(L))$ itself, with dimension

$$\dim \mathcal{H}_\mathcal{P} = \begin{pmatrix} N+\ell \\ \ell \end{pmatrix}. \tag{73}$$

All these quantum states belong to a single degenerate energy level when $\lambda = 1$. Recall that in this situation the hamiltonian vanishes and no motion occurs at the classical level.

We may interpret the expression (73) as giving the number of states in a quantum system of N interacting bosons. By interacting, we mean that the area available for the dynamics on the sphere is affected by the space which the vortices themselves occupy. Recall that Bradlow's bound (18) establishes that N vortices can only live on a sphere which has an area exceeding $4\pi N$. Heuristically, we can say that a single vortex occupies 4π units of area. So we can regard (73) as the formula for the number of states for a system of N bosons which can be assigned to any of the $\frac{|\kappa|}{4\pi}(4\pi R^2 - 4\pi N)$ states corresponding to the room available on the sphere, after the total area of the vortices has been discounted. (For $\kappa = -1$, there is a similiar interpretation for (73) as the number of states of a system of N noninteracting fermions, but it breaks down for $\kappa \neq -1$.)

From the formula (67), it is immediate to compute the volume of the moduli space determined by the Kähler form ω:

$$\mathrm{Vol}_\omega(\mathcal{M}_N) = \frac{\left(2\pi|\kappa|(R^2-N)\right)^N}{N!} = \frac{(2\pi\ell)^N}{N!}.$$

It is of course proportional to the volume determined by Samols' metric, as first computed by Manton (cf. [15],[17]). This volume has been used to deduce the thermodynamics of an ideal gas of abelian Higgs vortices at $\lambda = 1$ in the framework of Gibbs' classical statistical mechanics. In

Manton's model at $\lambda = 1$, there is only a ground state as we noted above, and its degeneracy, in Gibbs' approximation, is given by

$$d_{\text{Gibbs}} = \frac{1}{(2\pi\hbar)^N}\text{Vol}(\mathcal{M}_N) = \frac{\ell^N}{N!}. \tag{74}$$

Notice that Planck's constant is $2\pi\hbar = 2\pi$ in our units. Gibbs' partition function is simply $Z_{\text{Gibbs}} = d_{\text{Gibbs}}e^{-\beta N\pi}$. At $\lambda \neq 1$, the degeneracy is lifted but the formula above is still to be interpreted as the total number of states of the system. It is of interest to study the quotient

$$Q := \frac{\dim \mathcal{H}_{\mathcal{P}}}{d_{\text{Gibbs}}}$$

which gives information about how appropriate Gibbs' estimate for the number of states of the quantum system is. From (73) and (74), we find

$$Q = \frac{(N+\ell)!}{\ell^N \ell!}.$$

Using Stirling's formula for the gamma function, we obtain

$$Q = \left(1 + \frac{N+1}{\ell}\right)^N \left(1 + \frac{N}{\ell+1}\right)^{-\frac{1}{2}} \left[\left(1 + \frac{N}{\ell+1}\right)^{\ell+1} e^{-N}\right] e^{J(N+\ell+1)-J(\ell+1)}, \tag{75}$$

where J is the asymptotic series

$$J(z) = \sum_{n=1}^{\infty} \frac{B_n}{(2n-1)2n}\frac{1}{z^{2n-1}}.$$

In the context of Chern–Simons theories, the classical approximation is described as the limit $|\kappa| \to \infty$; this is equivalent to keeping the coupling μ as constant and letting $\hbar \to 0$. So we keep N fixed and let $\ell \to \infty$ in the expression (75), and this gives indeed $Q \to 1$. We might also try to obtain a classical regime in a thermodynamical limit, where both N and the area of the sphere become very large, but keeping a finite (possibly small) density, which we might want to define as

$$\nu := \frac{|\kappa| N}{\ell} = \frac{\frac{N}{R^2}}{(1 - \frac{N}{R^2})}.$$

But it follows from (75) that in this limit Q is infinite, however small ν is taken to be.

8 Quantum angular momenta

From the prequantisation data, it is possible to construct prequantum operators $\mathcal{P}(f)$ for any classical observable $f \in C^\infty(\mathcal{M}_N)$ as

$$\mathcal{P}(f) := -i\mathcal{D}_{\xi_f} + f. \tag{76}$$

Here, ξ_f is the hamiltonian vector field of f with respect to ω, defined by

$$d\iota_{\xi_f}\omega = -df. \tag{77}$$

Equation (50) is of course a special case of (77), with $\xi_{M_j^{\rm red}} = \xi_{(j)}$. In general, the linear operator $\mathcal{P}(f)$ does not map polarised sections of L to polarised sections. It is easy to show that it does if and only if ξ_f preserves the polarisation:

$$[\xi_f, \Gamma(\mathcal{M}_N, \mathcal{P})] \subset \Gamma(\mathcal{M}_N, \mathcal{P}). \tag{78}$$

Then we may interpret $\mathcal{P}(f)$ as the quantum operator corresponding to the observable f. In the Kähler case, (78) can be seen to be equivalent to ξ_f being the real part of a holomorphic vector field. This condition is true for the hamiltonian vector fields (40)–(42) of the angular momenta in (55)–(57).

We can determine explicitly the action of the quantum operators on the wavefunctions Ψ in the quantum Hilbert space $\mathcal{H}_{\mathcal{P}} = H^0(\mathcal{M}_N, \mathcal{O}(L))$. In the holomorphic frame on W_0 provided by $\sigma^{(0)}$, one can write $\Psi = \Psi^{(0)}\sigma^{(0)}$ with

$$\Psi^{(0)}(z_1, \ldots, z_N) = \sum_{j_1+\cdots+j_N=0}^{\ell} \alpha_{j_1\ldots j_N} \prod_{k=1}^{N} s_k^{[N]}(z_1, \ldots, z_N)^{j_k} \tag{79}$$

with $\alpha_{j_1\ldots j_N} \in \mathbb{C}$. The 1-form representing D with respect to this frame can be read off from (71) to be

$$\mathcal{A}^{(0)} = 2\pi\gamma i \sum_{r=1}^{N} \left(2R^2 \frac{\bar{z}_r}{1+|z_r|^2} + \tilde{b}_r \right) dz_r.$$

Substitution in (76) now gives the local representatives of the quantum operators in the local frame $\sigma^{(0)}$. For example, for M_3 we obtain

$$\begin{aligned}
\mathcal{P}(M_3) &= -i\left(\iota_{\xi_{(3)}} d - i\mathcal{A}^{(0)}(\xi_{(3)})\right) + M_3 \\
&= -i\iota_{\xi_{(3)}} d + \frac{\kappa}{2} N(R^2-1) - \kappa \sum_{r=1}^{N} \sum_{s\neq r}^{N} \frac{z_r}{z_r - z_s} \\
&= -i\iota_{\xi_{(3)}} d - \frac{N\ell}{2}.
\end{aligned}$$

Acting on Ψ as in (79), this yields

$$\mathcal{P}(M_3)\Psi^{(0)} = \sum_{j_1+\cdots+j_N=0}^{\ell} \left(j_1 + 2j_2 + \cdots + N j_N - \frac{N\ell}{2} \right) \alpha_{j_1\ldots j_N} \prod_{k=1}^{N} s_k^{[N]}(z_1, \ldots, z_N)^{j_k}. \tag{80}$$

From this expression, it is easy to read off the eigenvalues of $\mathcal{P}(M_3)$ as

$$-\frac{N\ell}{2}, -\frac{N\ell}{2}+1, \ldots, \frac{N\ell}{2},$$

together with their multiplicities. The same spectrum is obtained for $\mathcal{P}(M_1)$ and $\mathcal{P}(M_2)$.

For $N=1$ and a given negative $\kappa \in \mathbb{Z}$, we see that the Hilbert space $\mathcal{H}_{\mathcal{P}}$ yields the irreducible (projective) $(\ell+1)$-dimensional representation of $SO(3)$ through the action of the generators $M_j^{\rm red}$. The situation here is exactly equivalent to the geometric quantisation of the spin degrees of freedom of a particle of spin $\frac{\ell}{2}$, which are described classically by a 2-sphere of half-integer radius $\frac{\ell}{2}$ and the standard Fubini–Study symplectic form. More generally, for any N, it follows from (80) that the representation of $SO(3)$ carried by $\mathcal{H}_{\mathcal{P}}$ is the Nth symmetric power $\mathrm{Sym}^N(\boldsymbol{\ell}+\mathbf{1})$; notice that ℓ itself depends on N. This indicates once again that the vortices in our model can be regarded as interacting bosons, as we have put forward in section 7. It is worthwhile to emphasise how our approach differs from the usual treatment of a system of indistinguishable bosons in quantum mechanics. In the latter context, the N-particle sector of the Fock space is constructed as the Nth symmetric power of the Hilbert space of a single particle. In our situation, the N-particle sector is constructed directly from the quantisation of a classical N-particle phase space.

9 Discussion

In this paper, we have investigated an effective quantisation of Manton's model of first-order Chern–Simons vortices on a sphere Σ of radius R. We have seen that the nontrivial topology of the space manifold leads to the integer constraints (4) and (7) on the parameters γ and μ in the lagrangian. The periodic motion in the classical field theory was reduced to a hamiltonian system on the moduli space of N vortices imposing the condition $\gamma=\mu$. At the self-duality point $\lambda=1$, the effective dynamics is frozen, whereas for $\lambda\simeq 1$ the vortices move slowly, preserving their energy and angular momenta. The energy is purely potential and depends on the relative position of the vortices only.

The angular momenta along the three cartesian axes have been computed in section 5, both for the field theory and the reduced dynamics, and the two results were shown to be consistent. In the latter context, the expressions for the angular momenta can be simplified using the relations (47) and (48) for the functions b_r, which we derived from rotational symmetry. The angular momenta along the three independent directions fit together to form a moment map $\mathbf{M}^{\mathrm{red}}$ which we can regard as taking values in an $\mathfrak{so}(3)^*\cong\mathbb{R}^3$ where the sphere Σ is embedded. The direction of the vector $\mathbf{M}^{\mathrm{red}}$ gives a point on the sphere that can be interpreted as the centre of mass of the configuration of N vortices. Notice that there is no natural notion of centroid for a configuration of N points on a sphere (unless they lie on the same great circle and are not equidistant). We might be tempted to define it for generic configurations as the direction of the sum of the points, regarded as vectors in an $\mathbb{R}^3$ containing the sphere. (This definition is better behaved if we replace the target sphere by the elliptic plane by identifying antipodal points.) For a configuration of vortices, this centroid does not coincide with the direction of the angular momentum, as can be seen from our formula for $\mathbf{M}^{\mathrm{red}}$ in (55)–(57). We believe that, for configurations where the vortices are not symmetrically distributed, the areas of the sphere where vortices are most close together give a contribution to the angular momentum which is smaller than the one corresponding to taking the sum of the vortex positions; this is based on the fact that the total angular momentum of N coincident vortices is proportional to $N(R^2-N)$ rather than to N, as was shown at the end of section 5.2. On the plane [18], the total linear momentum in Manton's model is proportional to the ordinary centroid in $\mathbb{R}^2$ of the positions of the vortices.

A rather unexpected feature of the analysis in section 5 is that the angular momentum of a given number of vortices grows with the square of the radius R of the sphere. In the limit where the vortices are kept close together and $R \to \infty$, the modulus of the angular momentum blows up, and it was found necessary to subtract the constant $2\pi\gamma NR^2$, which becomes infinite in the limit, in order to compare it with the angular momentum of the system of vortices on the plane. This constant was seen to be related to the central charge for the linear momentum Poisson algebra for Manton's model on the plane.

The geometric quantisation of the reduced model is rather straightforward to set up. The prequantum line bundle is uniquely determined by the Kähler structure on the moduli space defined by the kinetic energy term. To construct it, we made use of an argument of Manton [15] to obtain the cohomology class of Samols' Kähler 2-form on the moduli space of Bogomol'nyĭ vortices, which appears in the study of the abelian Higgs model. It is presumed that the quantum system we have obtained approximates a finite truncation of the quantum field theory, in which most of the excitations are kept in the ground state. However, it is not clear how one should assess the validity of this approximation. For $\lambda = 1$, the quantisation of the reduced system yields a single degenerate energy level; this degeneracy is lifted when the potential becomes nontrivial, and in principle its spectrum can be determined using degenerate perturbation theory. In section 7, we have computed the dimension of the quantum Hilbert space and it was shown that it approaches Gibbs' estimate for the number of quantum states, as determined by the volume of the moduli space, in the classical limit of large Chern–Simons coefficient. Another result which comes from the analysis of the quantised effective system is that the solitons in the model should be interpreted as interacting bosons with the characteristic size 4π, as explained in section 7. The bosonic character of the vortices is also apparent from the analysis of the representations of $SO(3)$ arising in the algebra of the quantum angular momentum operators. In section 8, we found that for N vortices the Hilbert space is the Nth symmetric power of an irreducible representation of $SO(3)$. This irreducible representation is the same as the one obtained from quantising a single vortex on a sphere whose area is the one of the original sphere minus the total area occupied by N vortices.

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