Fermion propagator for QED_3 in the IR domain

C. D. Fosco and A. López [*]

Department of Physics, Theoretical Physics
University of Oxford
Oxford OX1 3NP, United kingdom.

April 25, 2020

Abstract

We evaluate the fermion propagator in parity-conserving QED_3 with N flavours, in the context of an IR domain approximation. This provides results which are non-perturbative in the loopwise expansion sense. We include fermion-loop effects, and show that they are relevant to the chiral symmetry breaking phenomenon, that can be understood in this context.

Parity-conserving QED in $2+1$ dimensions, in the context of the large-N approximation, is an effective theory description for many interesting models of condensed matter theory [1]. Moreover, it provides a simple yet non-trivial model where to study chiral symmetry breaking. Unlike in $1+1$ dimensions, there is a truly dynamical gauge field; besides, the model shares some important properties with its $3+1$ dimensional counterpart. For example, the dimensionful coupling constant for the $2+1$ dimensional model plays an analogous role to the renormalization scale of the $3+1$ dimensional theory. It is, however, a much simpler framework where to study chiral symmetry breaking, and it allows for the use of controlled approximations. Many detailed studies have been carried out using the truncated Schwinger-Dyson equations for the gap function Σ, both at zero [2]-[4] and finite [5]-[10] temperature. Results (which are consistent with the continuum calculations) have also been obtained by using numerical lattice calculations [11]-[15].

---

[*]On leave from Centro Atómico Bariloche, 8400 S. C. de Bariloche, Argentina.

It is well known that QED_3 has a non trivial IR structure, where the *a priori* severe IR divergences can be consistently treated thanks to the model superrenormalizability [16]. Indeed, the leading and first sub-leading logarithms have been summed to all orders for the fermion self-energy [17].

The relevant degrees of freedom driving the chiral phase transition are the low-momentum ones, with an effective dynamical cutoff set up by the (square of the) coupling constant. It then makes sense to attempt approximations based on that property. In this note, we calculate the full fermion propagator in parity-conserving large-N QED_3, in the context of the IR domain technique presented in [18], and comment on the relevance of this result for the chiral phase transition.

The model may be defined in terms of an Euclidean generating functional $\mathcal{Z}(j, \bar{\eta}, \eta)$, which in the path-integral representation is given by

$$\mathcal{Z}(j, \bar{\eta}, \eta) = \int [\mathcal{D}A]\mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\left[-S(\bar{\psi}, \psi, A) + \int d^3x (j_\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta)\right] , \tag{1}$$

where the Euclidean action S is,

$$S(\bar{\psi}, \psi, A) = S_f(\bar{\psi}, \psi, A) + S_g(A) \tag{2}$$

with S_f and S_g denoting the fermion and gauge field actions, respectively. For S_f, we assume a Dirac-like form:

$$S_f(\bar{\psi}, \psi, A) = \int d^3x \sum_{a=1}^{N} \bar{\psi}_a(\slashed{\partial} + ie_N \slashed{A})\psi_a \tag{3}$$

while the gauge field has a standard Maxwell action:

$$S_g(A) = \int d^3x \frac{1}{4} F_{\mu\nu} F_{\mu\nu} , \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{4}$$

Here $e_N = eN^{-1/2}$, where e is a coupling constant (with the dimensions of the square root of a mass). It is the only dimensionful parameter in the model. The $N^{-1/2}$ factor is introduced, as usual in the large-N expansion, in order to have a finite limit for the leading term in the expansion, when $N \to \infty$.

The Dirac γ-matrices are in a (reducible) 4×4 representation of the Dirac algebra:

$$\gamma_0 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} , \quad \gamma_1 = \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix} , \quad \gamma_2 = \begin{pmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{pmatrix} \tag{5}$$

2

where σ_j, $j = 1, 2, 3$ are the standard Pauli matrices. In this representation, the model has an extra 'chiral' symmetry, generated by a 'γ_5' matrix

$$\gamma_5 \;=\; \left(\begin{array}{cc} 0 & \mathbf{1}_{2\times 2} \\ \mathbf{1}_{2\times 2} & 0 \end{array} \right) , \tag{6}$$

which together with other generators complete the Lie algebra of an internal $U(2N)$ symmetry group. This symmetry is known to be dynamically broken[2], and a non-vanishing mass for the fermions is generated as a consequence. The $U(2N)$ symmetry breaks down to $U(N) \times U(N)$.

Besides, a fermion mass *does not* break the parity symmetry of the massless model. This means that parity symmetry is not anomalous, since there are regularizations that indeed preserve that symmetry, like the Pauli-Villars method, for example. This guarantees that there is no radiatively induced Chern-Simons term, and that it is perfectly consistent to use a parity-conserving action, since parity is not broken by renormalization effects.

The symbol $[\mathcal{D}A_\mu]$ in (1) denotes the gauge field functional integration measure, plus any gauge-fixing artifacts. An effect of the breaking of chiral symmetry is given by the generation of a non-vanishing mass gap Σ for the fermionic field. This may in turn be observed by evaluating the full fermion propagator G in the theory defined by (1)

$$G(x - y) \;=\; \langle \psi(x)\bar{\psi}(y) \rangle , \tag{7}$$

which, (by integrating out the fermion fields) may be written as

$$G(x - y) \;=\; \frac{1}{\mathcal{N}} \int [\mathcal{D}A] \, \langle x | \not{D}^{-1} | y \rangle \, (\det \not{D})^N \, \exp[-S_g(A)] , \tag{8}$$

where

$$\mathcal{N} \;=\; \int [\mathcal{D}A] \, (\det \not{D})^N \, \exp[-S_g(A)] , \tag{9}$$

and we have adopted Schwinger's notation $\langle x | K | y \rangle$ for the coordinate space matrix elements (or kernel) of an operator K; in the case above the fermion propagator in an external gauge field.

In the large-N limit, the leading contribution from the fermionic determinant comes from its quadratic part, namely,

$$(\det \not{D})^N \;=\; \exp(N \mathrm{Tr} \ln \not{D}) \;=\; \exp[-W^{(2)}(A) + \mathcal{O}(N^{-1})] \tag{10}$$

where

$$W^{(2)}(A) \;=\; -\frac{e^2}{2} \, \mathrm{Tr} \left[\not{\partial}^{-1} \not{A} \not{\partial}^{-1} \not{A} \right] . \tag{11}$$

3

The exact form of the quadratic functional $W^{(2)}(A)$ is well known; it may be written as follows:

$$W^{(2)}(A) \; = \; \frac{e^2}{32} \int d^3x \, F_{\mu\nu} \frac{1}{\sqrt{-\partial^2}} F_{\mu\nu} \tag{12}$$

The operator $\frac{1}{\sqrt{-\partial^2}}$, which has a nonlocal kernel in coordinate space, may be thought of as being defined by its Fourier representation:

$$\langle x| \frac{1}{\sqrt{-\partial^2}} |y\rangle \; = \; \int \frac{d^3k}{(2\pi)^3} e^{ik\cdot(x-y)} \frac{1}{|k|} \, , \tag{13}$$

where it is obviously local.

Thus, in the large-N limit, the gauge field average of (8) has a purely Gaussian weight. The evaluation of the gauge field average may be further simplified by using a convenient gauge-fixing term:

$$[\mathcal{D}A] \; = \; \mathcal{D}A \, \exp[-\frac{1}{2\xi} \int d^3x \, \partial \cdot A \, (1 + \frac{e^2}{8\sqrt{-\partial^2}}) \, \partial \cdot A] \, , \tag{14}$$

where ξ is a real constant. Then

$$G(x-y) \; = \; \frac{1}{\mathcal{N}} \int \mathcal{D}A \, \langle x|\not{D}^{-1}|y\rangle \, \exp[-S_G(A)] \, , \tag{15}$$

where the quadratic functional $S_G(A)$ may be written explicitly as

$$S_G(A) \; = \; \frac{1}{2} \int d^3x \, A_\mu M_{\mu\nu} A_\nu \tag{16}$$

with

$$M_{\mu\nu} \; = \; M(\sqrt{-\partial^2}) \, \left[\delta_{\mu\nu} + (\xi^{-1} - 1)\frac{\partial_\mu \partial_\nu}{\partial^2} \right] \tag{17}$$

and

$$M(\sqrt{-\partial^2}) \; = \; \sqrt{-\partial^2} \, (\sqrt{-\partial^2} + \frac{e^2}{8}) \, . \tag{18}$$

In order to evaluate G, in particular its self-energy part, we follow the approach of [18], although with a difference that turns out to be crucial in our case, namely, keeping the fermion determinant contribution to the effective gauge field action. We remark that it is totally justified to ignore the determinant in [18], since the fermions are assumed to have a non-vanishing renormalized mass, and that mass sets up a scale where to make the relevant IR approximations. In our case, however, the only scale appears in the one-loop contribution. Moreover, there will also be a non-vanishing fermion mass,

but here it will appear in a self-consistent way. This self-consistency depends upon the inclusion of the full one-loop gauge field propagator. Indeed, this is known to be important in any study of chiral symmetry breaking in QED_3 [2], and this calculation is no exception, as it will become clear in what follows.

Following [18], we represent the fermionic propagator in an external field in terms of a path integral over trajectories, as follows

$$\langle x|\not{D}^{-1}|y\rangle = \int_{0^+}^{\infty} dT \int \mathcal{D}x\mathcal{D}p \, \exp[i\int_0^T d\tau p \cdot \dot{x}] \, \mathcal{P}\exp[-i\int_0^T d\tau \not{p}(\tau)]$$

$$\times \exp[-ie_N \int_0^T d\tau \dot{x} \cdot A(x(\tau))] , \qquad (19)$$

where $\mathcal{P}$ denotes path ordering, and the functional integral over x_μ is taken over paths with the boundary condition $x_\mu(0) = y_\mu$ and $x_\mu(T) = y_\mu$. There is no boundary condition for the values of $p_\mu(\tau)$ at 0 and T. It is clear that the functional average over A will only affect the last factor in (19). Indeed,

$$G(x - y) = \int_{0^+}^{\infty} dT \int \mathcal{D}x\mathcal{D}p \, \exp[i\int_0^T d\tau \, p(\tau) \cdot x(\tau)]$$

$$\times \mathcal{P}\exp[-i\int_0^T d\tau \, \not{p}(\tau)] \, \exp(-\mathcal{F}) , \qquad (20)$$

where $\mathcal{F}$ is a functional of the trajectories $x(\tau)$ and a function of T, and is defined by the average:

$$\exp\{-\mathcal{F}\} = \left\langle \exp[-ie_N \int_0^T d\tau \dot{x} \cdot A(x(\tau))]\right\rangle , \qquad (21)$$

with

$$\langle \ldots \rangle = \frac{\int \mathcal{D}A \ldots \exp[-S_G(A)]}{\int \mathcal{D}A \exp[-S_G(A)]} . \qquad (22)$$

Being S_G quadratic, the integral over A can of course be performed exactly. The result for $\mathcal{F}$ may then be written as follows

$$\mathcal{F} = \frac{e_N^2}{2} \int d^3u \int d^3v \, j_\mu(u)\langle u|M_{\mu\nu}^{-1}|v\rangle j_\nu(v) \qquad (23)$$

where we have introduced the current j_μ associated to the particle's worldline:

$$j_\mu(u) = \int_0^T d\tau \, \dot{x}_\mu(\tau) \, \delta[u - x(\tau)] , \qquad (24)$$

5

with $\dot{x}_\mu \equiv \frac{dx}{d\tau}$. Taking into account the fact that the worldline begins at y and ends at x, we see that

$$\partial_\mu j_\mu(u) = \delta(u - y) - \delta(u - x) , \tag{25}$$

and because of this, we may evaluate the contribution of $\mathcal{F}$ due to the longitudinal part of $(M^{-1})_{\mu\nu}$ exactly. Namely, the longitudinal part of the propagator, which is where all the gauge-dependence is contained, will contribute to a term, which we denote $\mathcal{F}_l$, in the following expression for $\mathcal{F}$:

$$\mathcal{F} = \mathcal{F}_t + \mathcal{F}_l \tag{26}$$

where

$$\mathcal{F}_t = e_N^2 \int_0^T d\tau \int_\tau^T d\tau' \, \dot{x}(\tau) \cdot \dot{x}(\tau') \, \langle x(\tau)|M^{-1}|x(\tau')\rangle \tag{27}$$

and

$$\mathcal{F}_l = e_N^2 (\xi - 1) \left[\langle x| \frac{M^{-1}(\sqrt{-\partial^2})}{-\partial^2} |x\rangle - \langle x| \frac{M^{-1}(\sqrt{-\partial^2})}{-\partial^2} |y\rangle \right] . \tag{28}$$

Each separate term in (28), when written in momentum space, is divergent. However, their sum may be evaluated by combining the two integrands under a common momentum integral, what yields a convergent result. The reason is, of course, that the first term in (28) exactly kills the logarithmic divergent of the second.

The momentum integration can then be performed, obtaining:

$$\mathcal{F}_l[r] = \frac{4}{\pi^2 N} (\xi - 1) f_l(r) , \tag{29}$$

where

$$f_l(r) = \left\{ \gamma - 1 + \ln(\frac{8r}{e^2}) + \frac{\pi}{2}[1 - \cos(\frac{8r}{e^2})] \right.$$
$$\left. + \frac{8}{e^2 r}[\cos(\frac{e^2 r}{8})\mathrm{si}(\frac{e^2 r}{8}) - \sin(\frac{e^2 r}{8})\mathrm{ci}(\frac{e^2 r}{8})] \right\} , \tag{30}$$

where γ is the Euler's constant, $r \equiv |x - y|$, and si (ci) denotes the integral sine (cosine) function.

Since $\mathcal{F}_l$ does not depend on the trajectories $x(\tau)$, it is clear that the only contribution relevant to the fermion mass will come from $\mathcal{F}_t$ in (27). To calculate $\mathcal{F}_t$, we follow the procedure described in [18]. After the approximations are implemented, and a 'ribbon' regularization with a small-distance cutoff b is introduced [18], we end up with

$$\mathcal{F}_t = \frac{e^2}{N} \int_0^T d\tau \int_\tau^T d\tau' \, M^{-1}(\sqrt{|\tau - \tau'|^2 + b^2}) \tag{31}$$

6

where $M^{-1}(x) \equiv \langle x|M^{-1}|0\rangle$. The kernel $M(\sqrt{-\partial^2})$ in coordinate space can be computed, yielding

$$M^{-1}(x) = \frac{1}{2\pi^2 r}\left[\frac{\pi}{2}\cos(\frac{e^2 r}{8}) + \mathrm{ci}(\frac{e^2 r}{8})\sin(\frac{e^2 r}{8}) - \mathrm{si}(\frac{e^2 r}{8})\cos(\frac{e^2 r}{8})\right] \quad (32)$$

where $r \equiv |x|$. It is evident that, if we use the exact expression for $M^{-1}(x)$ in (31), we cannot evaluate the integral over τ and τ' analytically, something that we do need in order to obtain the fermion self-energy. One can easily check that the small-r behaviour of (32) corresponds to the case of a pure Maxwell gauge field action, namely,

$$M^{-1}(x) \sim \frac{1}{4\pi r} , \quad r \sim 0 , \quad (33)$$

while it yields the pure one-loop action dependence for large-r:

$$M^{-1}(x) \sim \frac{4}{\pi^2 e^2 r^2} , \quad r >> e^{-2} . \quad (34)$$

It is possible to use a simple analytic approximation to (32), which however captures both (33) and (34) exactly and does not differ significantly from the exact function in between. We use an approximant to $M^{-1}(x)$, which we denote by $\mathcal{M}^{-1}(x)$, and is given by the expression:

$$\mathcal{M}^{-1}(x) = \frac{1}{2\pi^2 r(\frac{e^2}{8}r + \frac{2}{\pi})} . \quad (35)$$

Inserting this expression in place of the exact one, M^{-1} in (31), one can evaluate exactly the integral over τ and τ'. Of course, not all the dependence in b is interesting, since, in the end, we will renormalize and take the limit $b \to 0$. Thus we may just collect the leading terms when $T >> b$. Then

$$\mathcal{F}_t[x] \sim \frac{e^2}{4\pi N}\left[T\ln(\frac{32}{\pi}\frac{1}{e^2 b}) - \frac{8}{e^2}\ln(\frac{\pi e^2 T}{16})\right] \quad (36)$$

where terms that tend to zero when $b/T \to 0$ have been neglected. It is then possible to insert (36) and (28) into (20), and integrate over T, to obtain:

$$G(x-y) = \kappa\,(\frac{e^2}{8})^{\frac{2}{\pi N}}\,\exp\left\{-\frac{4}{\pi^2 N}(\xi-1)f_l(r)\right\}$$

$$\times \int \frac{d^3 p}{(2\pi)^3}\,e^{ip\cdot(x-y)}\,(i\,\not{p}+\Sigma)^{-(1+\frac{2}{\pi N})} , \quad (37)$$

7

where

$$\Sigma = \frac{e^2}{4\pi N} \ln\left(\frac{32}{\pi}\frac{1}{e^2 b}\right) . \tag{38}$$

and κ is a numerical constant.

It should be evident from the above that Σ, modulo a renormalization that we shall explain in what follows, signals the existence of a mass for the fermionic field. It may be argued that a fermion mass appears just because the ribbon regularization breaks chiral symmetry. However, this only explains the existence of a perturbative mass, and in fact can (and will) be taken into account by a careful choice of the finite counterterms. The situation is analogous to the one in QED_4 with a non gauge-invariant regulator: in general, a mass counterterm is usually required in order to have a gauge invariant renormalized theory.

On the other hand, a non-zero (renormalized) mass which is non-perturbative in the coupling constant shall be an *a posteriori* justification for the use of this IR approach.

Σ determines the position of the singularity in momentum space, and it is evident from (38) that it needs a renormalization, since it diverges when $b \to 0$. The renormalized mass, which we shall denote by m, will differ from Σ by a subtraction corresponding to a mass counterterm:

$$m = \Sigma - \Sigma_0 \tag{39}$$

where Σ_0 is the counterterm. Now such a divergence in this superrenormalizable theory does only occur for the self-energy diagram which is of order e^2/N. Noting on the other hand that Σ may be rewritten as

$$\Sigma = \frac{e^2}{4\pi N} \ln(\frac{1}{b\mu}) + \frac{e^2}{4\pi N} \ln(\frac{32}{\pi}\frac{\mu}{e^2}) \tag{40}$$

where μ is a mass scale, we see that the divergent part $\frac{e^2}{4\pi N}\ln(\frac{1}{b\mu})$ is indeed of order $\frac{e^2}{N}$, as expected in this superrenormalizable theory. Then we fix the divergent part of the mass counterterm to be such that

$$\Sigma_0^{div} = \frac{e^2}{4\pi N} \ln(\frac{1}{b\mu}) \tag{41}$$

so that

$$m = m_0 + \frac{e^2}{4\pi N} \ln(\frac{32}{\pi}\frac{\mu}{e^2}) . \tag{42}$$

where m_0 is a *finite* counterterm, which has to be fixed by a normalization condition. It should of course be of the form:

$$m_0 = \frac{e^2}{4\pi N} C , \tag{43}$$

8

where C is a number of order 1. We do not want to introduce an extra mass, scale into the theory, so we have to use $\mu = m + \delta m$ as the renormalization scale:

$$m \; = \; m_0 \; + \; \frac{e^2}{4\pi N} \; \ln(\frac{32}{\pi}\frac{m+\delta m}{e^2}) \; . \tag{44}$$

The reason why μ cannot be just equal to m is the following: $m = 0$ has to be one of the possible solutions to equation (44), so that chiral symmetry is not explicitly broken. But setting $m = 0$ in (44) certainly requires a non vanishing δm:

$$\delta m \; = \; \frac{\pi e^2}{32} \; \exp(-4\pi N\frac{m_0}{e^2}) \; , \tag{45}$$

what determines the relation between δm and m_0. This is in fact the renormalization condition that guarantees that the chiral symmetry is not explicitly broken by the renormalization procedure. Using relation (45), we may finally write the expression for m as a transcendental equation:

$$m \; = \; \frac{\pi e^2}{32} \; e^{-C} \; \left[\exp(4\pi N\frac{m}{e^2}) - 1\right] \; . \tag{46}$$

where C cannot be determined within the approximation made in obtaining (31). Indeed, in this approximation the induced mass is a constant rather than a momentum dependent function. Therefore it impossible to decide which is the value of the constant C, related to the dynamical cutoff ($\sim \frac{e^2}{8N}$), which is observed in the Schwinger-Dyson equations approach [3].

However, from a simple analysis of (46), we see that besides the trivial solution $m = 0$, there will be a non-vanishing mass as long as N remains smaller than a critical value N_c, given by:

$$N_c \; = \; \frac{8}{\pi^2} \; e^C \; . \tag{47}$$

This result is consistent with the ones obtained in more standard settings; for example, the value $N_c = \frac{32}{\pi^2}$ of [3], corresponds to $C = \ln 4 \sim 1.39$. It is interesting to compare this result with the one one would have obtained by ignoring the one-loop correction to the photon propagator. As already mentioned above, there is no induced mass term in such an approximation, since one finds no term linear in T in (36).

We conclude by stressing that this IR approximation scheme looks promising as a tool to study the chiral phase transition in QED_3 like models, in a gauge invariant setting. However, in order to obtain the full momentum dependence of the fermion mass, a higher order approximation than the one used here may be required. Indeed, it is plausible that fermion recoil effects are necessary to sense that momentum dependence. This effect can not be

appreciated in the approximation used here since the motion of the fermions is not perturbed by their interaction with photons [18].

Acknowledgements

References

[1] W. Rantner, and W. G. Wen, arXiv: cond-mat/0105540; W. Rantner, and W. g. Wen, Phys. Rev. Lett. **86**, 3871 (2001); J. Ye, Phys. Rev. Lett. **87**, 227003 (2001); M. Franz, and Z. Tesanovik, Phys. Rev. Lett. **87**, 257003 (2001); I. F. Herbut, Phys. Rev. Lett. **88**, 047006 (2002); D. V. Khveshchenko, Phys. Rev. B **65**, 235111 (2002); V. P. Gusynin, D. V. Khveshchenko, M. Reenders, Phys. Rev. B **67**, 052502 (2003).

[2] R. D. Pisarski, Phys. Rev. D **29**, 2423 (1984).

[3] T. Appelquist, D. Nash and L. C. Wijewardhana, Phys. Rev. Lett. **60**, 2575 (1988).

[4] D. Nash, Phys. Rev. Lett. **62**, 3024 (1989).

[5] N. Dorey and N. E. Mavromatos, Nucl. Phys. B **386**, 614 (1992).

[6] N. Dorey and N. E. Mavromatos, Phys. Lett. B **266**, 163 (1991).

[7] I. J. Aitchison, N. Dorey, M. Klein-Kreisler and N. E. Mavromatos, Phys. Lett. B **294**, 91 (1992) [arXiv:hep-ph/9207246].

[8] I. J. Aitchison and M. Klein-Kreisler, Phys. Rev. D **50**, 1068 (1994) [arXiv:hep-ph/9402213].

[9] I. J. Aitchison, Z. Phys. C **67**, 303 (1995).

[10] D. J. Lee, Phys. Rev. D **58**, 105012 (1998) [arXiv:hep-th/9803115].

[11] E. Dagotto, J. B. Kogut and A. Kocic, Phys. Rev. Lett. **62**, 1083 (1989).

[12] E. Dagotto, A. Kocic and J. B. Kogut, Nucl. Phys. B **334**, 279 (1990).

[13] S. Hands and J. B. Kogut, Nucl. Phys. B **335**, 455 (1990).

[14] V. Azcoiti and X. Q. Luo, Mod. Phys. Lett. A **8**, 3635 (1993) [arXiv:hep-lat/9212011].

[15] S. J. Hands, J. B. Kogut, L. Scorzato and C. G. Strouthos, arXiv:hep-lat/0209133.

[16] R. Jackiw and S. Templeton, Phys. Rev. D **23**, 10, 2291 (1981).

[17] S. Templeton, Phys. Rev. D **24**, 12, 3134 (1981).

[18] A. I. Karanikas, C. N. Ktorides and N. G. Stefanis, Phys. Rev. D **52**, 5898 (1995).