Fuzzy-*pp* Waves

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Abstract

We present a noncommutative version of a plane-wave solution to the gravitational field equations. We start with a given classical solution, admittedly rather simple, and construct an algebra and a differential calculus which supports the metric. In the particular solution presented as an example the 1-forms do not anticommute, to a degree which depends on the amplitude of the deviation of the metric from the standard Minkowski metric.

1 Motivation

A noncommutative generalization of a classical solution to Einstein's equation would require first of all a noncommutative version of a smooth manifold. Although noncommutative geometry has its roots in several attempts by physicists to overcome the problems of ultraviolet divergences in field theory it is to A. Connes that is due the first construction [1] of a smooth structure. The version of this which we shall use here [2] is a slight, even 'smoother', modification which permits the use of a metric of Minkowski signature. This metric has also a locality property which is most easily expressible in terms of a 'frame' in the sense of Cartan. A smooth manifold V with a metric can be defined as a subspace of an euclidean space $\mathbb{R}^m$ of sufficiently high dimension m and the embedding can be chosen so that the induced metric coincides with that of the manifold. If the manifold is of dimension n then generically in this case m must satisfy $m \geq n(n-1)/2$. The embedding is equivalent to a projection

$$\mathcal{C}(\mathbb{R}^m) \to \mathcal{C}(V)$$

of the algebra of functions $\mathcal{C}(\mathbb{R}^m)$ onto the algebra $\mathcal{C}(V)$ of smooth functions on V and the latter can be formally defined in terms of generators and relations. This is convenient for the noncommutative generalizations.

Classical metrics have often been defined [3, 4] using embeddings and although it is possible in the present case to avoid this construction [5] it will perhaps in general be convenient. Other metrics with curvature have been proposed [6, 7] in the past but none has been found to be Einstein. The method we use is for the moment limited to what could be considered noncommutative analogues of parallelizable manifolds. Another popular approach does not have this limitation but is restricted to metrics of euclidean signature; we refer to two recent books [8, 9] for an exhaustive description and reference to the original literature.

Within the general framework which we here consider, the principal difference between the commutative and noncommutative cases lies in the spectrum of the operators which we use to generate the noncommutative algebra which replaces the algebra of functions. This in turn depends not only on the structure of this algebra as an abstract algebra but on the representation of it which we choose to consider. The ultimate aim is the construction of a noncommutative generalization of the theory of general relativity which will remain "smooth', but become essentially noncommutative, in regions where the commutative limit would be singular. An explicit example has been constructed [10] recently based on the Kasner metric.

2 The general formalism

For the purpose of the present exposition the expression 'noncommutative space-time' will designate a $*$-algebra $\mathcal{A}$ with trivial center and $n(=4)$ hermitian generators x^μ as well as a differential calculus $\Omega^*(\mathcal{A})$ over $\mathcal{A}$ which has certain 'parallelizability' properties; notably the $\mathcal{A}$-module $\Omega^1(\mathcal{A})$ is free. This will be discussed in more detail below. The x^μ will be referred to as 'position generators'. We shall suppose also that there is a set of $n(=4)$ antihermitian 'momentum generators' λ_α and a 'Fourier transform'

$$F : x^\mu \longrightarrow \lambda_\alpha = F_\alpha(x^\mu)$$

which takes the position generators to the momentum generators. Let ρ be a representation of $\mathcal{A}$ as an algebra of linear operators on some Hilbert space. For every $k_\mu \in \mathbb{R}^4$ one can construct a unitary element $u(k) = e^{ik_\mu x^\mu}$ of $\mathcal{A}$ and one can consider the weakly closed algebra $\mathcal{A}_\rho$ generated by the image of the $u(k)$ under ρ. The momentum operators λ_α are also unbounded but using them one can construct also a set of 'translation' operators $\hat{u}(\xi) = e^{\xi^\alpha \lambda_\alpha}$ whose image under ρ belongs also to $\mathcal{A}_\rho$. In general $\hat{u}u \neq u\hat{u}$; if the metric which we introduce is the flat metric then we shall see that $[\lambda_\alpha, x^\mu] = \delta^\mu_\alpha$ and in this case we can write the commutation relations as $\hat{u}u = qu\hat{u}$ with $q = e^{ik_\mu \xi^\mu}$; the 'Fourier transform' is the simple linear transformation

$$\lambda_\alpha = \frac{1}{i\kbar}\theta^{-1}_{\alpha\mu} x^\mu \tag{2.1}$$

for some symplectic structure $\theta^{\alpha\mu}$. As a measure of noncommmutativity, and to recall the many parallelisms with quantum mechanics, we use the symbol $\kbar$, which will designate the square of a real number whose value could lie somewhere between the Planck length and the proton radius.

Using λ_α we construct the derivation

$$e_\alpha f = [\lambda_\alpha f].$$

That is, the derivations are related to the momenta as usual. In the flat case mentioned above we find that

$$e_\alpha x^\nu = \frac{1}{i\kbar}\theta^{-1}_{\alpha\mu}[x^\mu, x^\nu] = \delta^\nu_\alpha$$

provided we introduce the commutation relations

$$[x^\mu, x^\nu] = i\kbar\theta^{\mu\nu}$$

or equivalently the Poisson structure

$$\{x^\mu, x^\nu\} = \lim_{\kbar\to 0} \frac{1}{i\kbar}[x^\mu, x^\nu] = \theta^{\mu\nu}$$

in the commutative limit.

Since the center is trivial the algebra can also be considered as the 'phase space' and F will be (implicitly) assumed to be an automorphism of $\mathcal{A}$:

$$\widehat{u_1 * u_2} = \hat{u}_1 \mathbin{\hat{*}} \hat{u}_2.$$

We designate here for a moment the product with a star so we can place a hat on it. In the commutative limit F is an extension to phase space of the ordinary Fourier transform. As a simple example, one can consider the two Pauli matrices $(x^\mu) = (\sigma^1, \sigma^2)$ as 'coordinates' for the algebra of 2×2 matrices and the linear combination $(\lambda_\alpha) = (-\sigma_2, \sigma_1) = \epsilon_{\mu\nu}\sigma^\nu$ as 'momenta'. The derivations $e_\alpha = \mathrm{ad}\,\lambda_\alpha$ almost behave as partial derivatives: $e_\alpha x^\mu = \delta^\mu_\alpha \sigma^3$. In this case there is no commutative limit.

In general the representation is not necessarily irreducible but we shall assume that it is a direct sum of an arbitrary number of copies of some representation which is irreducible. The subalgebra $\mathcal{A}'_\rho$ of those elements algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$ which commute with $\mathcal{A}_\rho$ is therefore in general nontrivial. There does not seem to be a direct role to be played by the $\mathcal{A}'_\rho$ except for the constraint that it must grow as the algebra becomes commutative if this limit is not to be singular. Typically then, as the algebra becomes increasingly commutative the representation becomes increasingly reducible.

We shall suppose that $\mathcal{A}$ has a commutative limit which is an algebra $\mathcal{C}(V)$ of smooth functions on a space-time V endowed with a globally defined moving frame θ^α and thus a metric. By parallelizable we mean that the module $\Omega^1(\mathcal{A})$ has a basis θ^α which commutes with the elements of $\mathcal{A}$. For all $f \in \mathcal{A}$

$$f\theta^\alpha = \theta^\alpha f. \tag{2.2}$$

This is the correspondence principle we use. We shall see that it implies that the metric components must be constants, a condition usually imposed on a moving frame. The frame θ^α allows one [2] to construct a representation of the differential algebra from that of $\mathcal{A}$. The frame elements belong in fact to the algebra $\mathcal{A}'_\rho$. Following strictly what one does in ordinary geometry, we shall introduce the set of derivations e_α to be dual to the frame θ^α, that is with

$$\theta^\alpha(e_\beta) = \delta^\alpha_\beta.$$

We define the differential exactly as did E. Cartan in the commutative case. If e_α is a derivation of $\mathcal{A}$ then for every element $f \in \mathcal{A}$ we define df by the constraint $df(e_\alpha) = e_\alpha f$. The construction of the differential calculus from the structure of the resulting module of 1-forms is essentially due to Connes [1]; in the present notation it is to be found elsewhere [2]. It follows from general arguments that the momenta λ_α must satisfy the consistency condition [11, 12]

$$2\lambda_\gamma\lambda_\delta P^{\gamma\delta}{}_{\alpha\beta} - \lambda_\gamma F^\gamma{}_{\alpha\beta} - K_{\alpha\beta} = 0. \tag{2.3}$$

The $P^{\gamma\delta}{}_{\alpha\beta}$ define the product π in the algebra of forms:

$$\theta^\alpha\theta^\beta = P^{\alpha\beta}{}_{\gamma\delta}\theta^\gamma\theta^\delta. \tag{2.4}$$

This product is defined to be the one with the least relations which is consistent with the module structure of the 1-forms. The $F^\gamma{}_{\alpha\beta}$ are related to the 2-form $d\theta^\alpha$ through the structure equations:

$$d\theta^\alpha = -\frac{1}{2}C^\alpha{}_{\beta\gamma}\theta^\beta\theta^\gamma.$$

In the noncommutative case the structure elements are defined as

$$C^\alpha{}_{\beta\gamma} = F^\alpha{}_{\beta\gamma} - 2\lambda_\delta P^{(\alpha\delta)}{}_{\beta\gamma}. \tag{2.5}$$

It follows that

$$e_\alpha C^\alpha{}_{\beta\gamma} = 0. \tag{2.6}$$

This must be imposed then at the commutative level and can be used as a gauge-fixing condition.

Finally, to complete the definition of the coefficients of the consistency condition (2.3) we introduce the special 1-form $\theta = -\lambda_\alpha\theta^\alpha$. In the commutative, flat limit

$$\theta \to i\partial_\alpha dx^\alpha.$$

It is referred to as a 'Dirac operator' [1]. As an (antihermitian) 1-form θ defines a covariant derivative on an associated $\mathcal{A}$-module with local gauge transformations given by the unitary elements of $\mathcal{A}$. The $K_{\alpha\beta}$ are related to the curvature of θ:

$$d\theta + \theta^2 = -\frac{1}{2}K_{\alpha\beta}\theta^\alpha\theta^\beta.$$

All the coefficients lie in the center $\mathcal{Z}(\mathcal{A}_{\hbar})$ of the algebra. With no restriction of generality we can impose the conditions

$$C^{\epsilon}{}_{\gamma\delta} = P^{\alpha\beta}{}_{\gamma\delta} C^{\epsilon}{}_{\alpha\beta}, \qquad K_{\gamma\delta} = P^{\alpha\beta}{}_{\gamma\delta} K_{\alpha\beta}. \tag{2.7}$$

Equation (2.5) is the correspondence principle which associates a differential calculus to a metric. On the left in fact the quantity $C^{\alpha}{}_{\beta\gamma}$ determines a moving frame, which in turn fixes a metric; on the right are the elements of the algebra which fix to a large extent the differential calculus. A 'blurring' of a geometry proceeds via this correspondence. It is evident that in the presence of curvature the 1-forms cease to anticommute. On the other hand it is possible for flat 'space' to be described by 'coordinates' which do not commute.

The correspondence principle between the commutative and noncommutative geometries can be also described as the map

$$\tilde{\theta}^{\alpha} \mapsto \theta^{\alpha} \tag{2.8}$$

with the product satisfying the condition

$$\tilde{\theta}^{\alpha}\tilde{\theta}^{\beta} \mapsto P^{\alpha\beta}{}_{\gamma\delta}\theta^{\gamma}\theta^{\delta} \tag{2.9}$$

The tilde on the left is to indicate that it is the commutative form. The condition can be written also as

$$\tilde{C}^{\alpha}{}_{\beta\gamma} \mapsto C^{\alpha}{}_{\eta\zeta} P^{\eta\zeta}{}_{\beta\gamma}$$

or as

$$\lim_{\hbar\to 0} C^{\alpha}{}_{\beta\gamma} = \tilde{C}^{\alpha}{}_{\beta\gamma}.$$

A solution to these equations would be a solution to the problem we have set. It would be however unsatisfactory in that no smoothness condition has been imposed. This can at best be done using the inner derivations. We shall construct therefore the set of momentum generators. The procedure we shall follow is not always valid; a counter example has been constructed [13] for the flat metric on the torus.

We write $P^{\alpha\beta}{}_{\gamma\delta}$ in the form

$$P^{\alpha\beta}{}_{\gamma\delta} = \frac{1}{2}\delta^{[\alpha}_{\gamma}\delta^{\beta]}_{\delta} + i\hbar Q^{\alpha\beta}{}_{\gamma\delta}\,. \tag{2.10}$$

It will be convenient also to introduce the notation

$$I_{\alpha\beta} = K_{\alpha\beta} + \lambda_{\gamma} I^{\gamma}{}_{\alpha\beta}, \qquad I^{\gamma}{}_{\alpha\beta} = F^{\gamma}{}_{\alpha\beta} - 2i\hbar\lambda_{\delta} Q^{\gamma\delta}{}_{\alpha\beta} \tag{2.11}$$

so we can write (2.3) in the form

$$[\lambda_{\alpha}, \lambda_{\beta}] = I_{\alpha\beta} - i\hbar[\lambda_{\gamma}, \lambda_{\delta}] Q^{\gamma\delta}{}_{\alpha\beta}.$$

If we replace the commutator on the right-hand side by its value we obtain the equation

$$[\lambda_{\alpha}, \lambda_{\beta}] = I_{\alpha\beta} - i\hbar Q^{\gamma\delta}{}_{\alpha\beta}(K_{\gamma\delta} + I_{\gamma\delta}) + (i\hbar)^2 Q^{\eta\zeta}{}_{\alpha\beta} Q^{\gamma\delta}{}_{\eta\zeta}[\lambda_{\gamma}, \lambda_{\delta}].$$

This can be simplified by a redefinition

$$K_{\alpha\beta} - i\hbar Q^{\gamma\delta}{}_{\alpha\beta} K_{\gamma\delta} \mapsto K_{\alpha\beta}, \qquad F^{\epsilon}{}_{\alpha\beta} - i\hbar Q^{\gamma\delta}{}_{\alpha\beta} F^{\epsilon}{}_{\gamma\delta} \mapsto F^{\epsilon}{}_{\alpha\beta}$$

of the coefficients $K_{\alpha\beta}$ and $F^\gamma{}_{\alpha\beta}$. Iterating this proceedure one obtains the consistency condition in the form

$$[\lambda_\alpha, \lambda_\beta] = I_{\alpha\beta} \tag{2.12}$$

as well as the condition

$$Q^{[\gamma\delta]}{}_{\alpha\beta} = 0$$

on the coefficients. The condition that $P^{\gamma\delta}{}_{\alpha\beta}$ be an idempotent now implies a similar condition on $i\bar{k}Q^{(\gamma\delta)}{}_{(\alpha\beta)}$. Unless we are in a non-perturbative regime we can set then the latter equal to zero.

We introduce the derivations $e_{\alpha\beta} = \mathrm{ad}\,(\lambda_\gamma\lambda_\delta)$ so that one can write the identity

$$i\bar{k}e_{\alpha\beta} = i\bar{k}\lambda_{(\alpha}e_{\beta)} - i\bar{k}e_\beta e_\alpha. \tag{2.13}$$

The first term on the right remains in the commutative limit as a vector field; the second term vanishes. The second term resembles a second derivative, but as a derivation. It is easy to see here the relation between an expression from noncommutative geometry and an effective commutative limit with higher-order derivative corrections.

In the commutative case a moving frame is dual to a set of derivations $\tilde{e}_\alpha$ which satisfy the commutation relations

$$[\tilde{e}_\alpha, \tilde{e}_\beta] = \tilde{C}^\gamma{}_{\alpha\beta}\tilde{e}_\gamma \tag{2.14}$$

If we use the correspondence principle and the expression (2.5) we find that this becomes

$$[e_\alpha, e_\beta] = F^\gamma{}_{\alpha\beta}e_\gamma - 2i\bar{k}Q^{\gamma\delta}{}_{\alpha\beta}e_{\gamma\delta} = F^\gamma{}_{\alpha\beta}e_\gamma - 2i\bar{k}Q^{\gamma\delta}{}_{\alpha\beta}(\lambda_{(\gamma}e_{\delta)} - e_{\gamma\delta}) \tag{2.15}$$

The last term on the right-hand side vanishes in the commutative limit.

The Jacobi identity with two momenta and one position generator can be written as

$$e_{[\alpha}e^\mu_{\beta]} = e^\mu_\gamma I^\gamma{}_{\alpha\beta}. \tag{2.16}$$

Because of the symmetries of the indices $I_{\alpha\beta}$ can be considered as a field strength for a gauge potential which takes its values in the formal Lie algebra associated to the group of unitary elements of the associative algebra $\mathcal{A}$. Although the algebra is generally of infinite dimension the relation (2.3) acts as a finiteness condition and one can consider the λ_α as generators of a Lie algebra with commutation relations given [11, 14] by the bracket (2.12). It is convenient to introduce the (right) dual

$$Q^{*\zeta\eta\gamma\delta} = \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}Q^{\zeta\eta}{}_{\alpha\beta}$$

of $Q^{\zeta\eta}{}_{\alpha\beta}$ and to define the Jacobi anomaly for three momenta as

$$A^\delta_J = \frac{1}{4}\epsilon^{\alpha\beta\gamma\delta}[\lambda_\alpha, [\lambda_\beta, \lambda_\gamma]].$$

The Jacobi identities can be written then as

$$A^\alpha_J = 0. \tag{2.17}$$

With the Ansatz we are using we find that

$$A^\alpha_J = -2i\bar{k}K_{\beta\gamma}Q^{*\beta\delta\gamma\alpha}\lambda_\delta + (-2i\bar{k})^2 Q^{(\delta\gamma}{}_{\beta\gamma}Q^{*\beta\eta)\gamma\alpha}\lambda_\delta\lambda_\gamma\lambda_\eta.$$

Equation (2.17) must be verified in each example. Because of the relations amongst the momenta, sufficient but not necessary conditions for the left-hand side to vanish are

$$K_{\beta\gamma} Q^{*\beta\delta\gamma\alpha} = 0, \tag{2.18}$$

$$Q^{(\delta\gamma}{}_{\beta\gamma} Q^{*\beta\eta)\gamma\alpha} = 0. \tag{2.19}$$

In the example we shall find that both these equations place restrictions on the coefficients.

It is necessary [15] to introduce a flip operation

$$\sigma: \ \Omega^1(\mathcal{A}_{\bar{k}}) \otimes \Omega^1(\mathcal{A}_{\bar{k}}) \to \Omega^1(\mathcal{A}_{\bar{k}}) \otimes \Omega^1(\mathcal{A}_{\bar{k}})$$

to define the reality condition and the Leibniz rules. If we write

$$S^{\alpha\beta}{}_{\gamma\delta} = \delta^\beta_\gamma \delta^\alpha_\delta + i\bar{k} T^{\alpha\beta}{}_{\gamma\delta}$$

we find that a choice [11] of connection which is torsion-free, and satisfies all Leibniz rules is given by

$$\omega^\alpha{}_\beta = \frac{1}{2} F^\alpha{}_{\gamma\beta} \theta^\gamma + i\bar{k} \lambda_\gamma T^{\alpha\gamma}{}_{\delta\beta} \theta^\delta. \tag{2.20}$$

We can compare this with the expression (2.5) for the structure elements.

The relation

$$\pi \circ (1 + \sigma) = 0$$

must hold [11, 12] to assure that the torsion be a bilinear map. In terms of the coefficients $P^{\alpha\beta}{}_{\gamma\delta}$ it can be written in the form

$$T^{\alpha\beta}{}_{\eta\zeta} P^{\eta\zeta}{}_{\gamma\delta} + Q^{(\alpha\beta)}{}_{\gamma\delta} = 0.$$

The symmetric part $T^{(\alpha\beta)}{}_{(\gamma\delta)}$, which is here arbitrary, will be fixed by the condition that the connection be metric. It follows that

$$\omega^\alpha{}_{\eta\zeta} P^{\eta\zeta}{}_{\beta\gamma} = \frac{1}{2} C^\alpha{}_{\beta\gamma}. \tag{2.21}$$

This is the usual relation between the Ricci-rotation coefficients and the Levi-Civita connection. Using it one can deduce (2.5) from (2.20). Using the vanishing-torsion condition one can write the expression for the commutator of the derivations as

$$[e_\alpha, e_\beta] - \omega^\gamma{}_{[\alpha\beta]} e_\gamma = -2i\bar{k} Q^{\gamma\delta}{}_{\alpha\beta} (e_\gamma e_\delta - \omega^\epsilon{}_{\gamma\delta} e_\epsilon). \tag{2.22}$$

A second form of the correspondence principle is the map

$$\tilde{\omega}^\alpha{}_\beta \tilde{\theta} \mapsto \omega^\alpha{}_\beta \theta \tag{2.23}$$

between the commutative and noncommutative connection forms.

We shall suppose that $\mathcal{A}_{\bar{k}}$ has a metric, which we define as a bilinear map

$$g: \ \Omega^1(\mathcal{A}_{\bar{k}}) \otimes \Omega^1(\mathcal{A}_{\bar{k}}) \to \mathcal{A}_{\bar{k}}. \tag{2.24}$$

In terms of the frame one can define the metric by the condition that

$$g(\theta^\alpha \otimes \theta^\beta) = g^{\alpha\beta}. \tag{2.25}$$

The $g^{\alpha\beta}$ are taken to be deformed versions $g^{\alpha\beta} \mapsto g^{\alpha\beta} + i\bar{k}h^{\alpha\beta}$ of the components of the standard euclidean or lorentzian metric $g^{\alpha\beta}$ on $\mathbb{R}^n$ which satisfy [16] the symmetry condition

$$P^{\alpha\beta}{}_{\gamma\delta}(g + i\bar{k}h)^{\gamma\delta} = 0. \tag{2.26}$$

Together with the reality condition

$$(g + i\bar{k}h)^{\beta\alpha} + i\bar{k}T^{\alpha\beta}{}_{\gamma\delta}(g + i\bar{k}h)^{\gamma\delta} = ((g + i\bar{k}h)^{\beta\alpha})^*$$

this means that in fact $h^{\alpha\beta} = 0$ and also that the flip satisfies the constraint

$$T^{\alpha\beta}{}_{\gamma\delta}g^{\gamma\delta} = 0.$$

We suppose then that the coefficients of the metric are the usual commutative ones. The various reality conditions [17, 16] imply also that

$$(Q^{\alpha\beta}{}_{\gamma\delta})^* = Q^{\alpha\beta}{}_{\gamma\delta} + o(i\bar{k}), \qquad (T^{\alpha\beta}{}_{\gamma\delta})^* = T^{\alpha\beta}{}_{\gamma\delta} + o(i\bar{k}).$$

The connection is compatible with the metric if

$$T^{\alpha\gamma}{}_{\delta\epsilon}g^{\epsilon\beta} + T^{\beta\gamma}{}_{\delta\epsilon}g^{\alpha\epsilon} + i\bar{k}T^{\beta\gamma}{}_{\epsilon\zeta}g^{\eta\zeta}T^{\alpha\epsilon}{}_{\delta\eta} = 0. \tag{2.27}$$

To first order this simplifies to the usual condition

$$T^{(\alpha\gamma}{}_{\delta}{}^{\beta)} = o(i\bar{k}). \tag{2.28}$$

The index was lifted here with the metric.

If we introduce formally the 'covariant derivative' $D_\alpha X^\beta$ of a 'vector' X^α by the formula

$$D_\alpha X^\beta = \omega^\beta{}_{\alpha\gamma}X^\gamma$$

and the covariant derivative $D_\alpha(X^\beta Y^\gamma)$ of the product of two such 'vectors' as

$$D_\alpha(X^\beta Y^\gamma) = D_\alpha X^\beta Y^\gamma + S^{\beta\delta}{}_{\alpha\epsilon}X^\epsilon D_\delta Y^\gamma,$$

since there is a 'flip' as the index on the derivation crosses the index on the first 'vector', then the condition (2.27) that the connection be metric can be written

$$D_\alpha g^{\beta\gamma} = 0$$

as usual. More satisfactory origins of the condition (2.27) can be given [18].

The 'position' commutation relations are of the form

$$[x^\mu, x^\nu] = i\bar{k}J^{\mu\nu}. \tag{2.29}$$

The right-hand side is not necessarily in the center. The system of operator equations we must solve consists of Equation (2.5), Equation (2.12), the metric compatibility equation (2.28) as well as the constraints on the coefficients given above. Using the same formal manipulations as above we can find the commutation relations for the position generators as solution to a set of differential conditions. From (2.29) we see that

$$i\bar{k}e_\gamma J^{\mu\nu} = [x^{[\mu}, e_\gamma^{\nu]}]. \tag{2.30}$$

We choose the symplectic structure $\theta^{\alpha\beta}$ such that

$$\theta^{-1}_{\alpha\beta} = -i\bar{k}K_{\alpha\beta} + o(\omega^\gamma_{\alpha\beta})$$

and such that

$$J^{\mu\nu} = e^\mu_\alpha e^\nu_\beta \theta^{\alpha\beta}. \tag{2.31}$$

This cannot be generally possible unless $\theta^{(\alpha\beta)} \neq 0$. It will allow us to define (formally) a divergence $D_\mu J^{\mu\nu}$ by setting

$$D_\mu J^{\mu\nu} = e^\nu_\beta D_\alpha \theta^{\alpha\beta} \tag{2.32}$$

and using the connection form (2.20). The condition

$$D_\mu J^{\mu\nu} = 0$$

establishes a link [19] between the symplectic structure and the connection. We cannot make any general statements concerning this condition in the present context.

It is tempting to suppose that to lowest order at least, in a semiclassical approximation, there is an analogue of Darboux's lemma and that it is always possible to choose generators which satisfy commutation relations of the form (2.29) with the right-hand in the center. However the example we shall examine in detail shows that this is not always the case. Having fixed the generators, the manifestations of curvature would be found then in the form of the frame. The two sets of generators x^μ and λ_α satisfy, under the assumptions we make, three sets of equations. The commutation relations (2.29) for the position generators x^μ and the associated Jacobi identities permit one definition of the algebra. The commutation relations for the momentum generators permit a second definition. The conjugacy relations assure that the two descriptions concern the same algebra. We shall analyze these identities later using the example to show that they have interesting non-trivial solutions.

There are two sets of equations to be solved simultaneously in the four variables λ_α. The first set is the set (2.5), a system of $(4-1)\times 6 = 18$ equations which are to be read now from right to left as equations in the λ_α with given 'sources' $C^\alpha{}_{\beta\gamma}$. This gives the Fourier transform as a functional of the gravitational field. The boundary conditions are obscure but could be taken, for lack of something better, as Equations (2.1); this is then to be the form of the map from x^μ to λ_α when there is no source. The second set of equations is the set of consistency conditions (2.12). In the commutative limit the Fourier transformation becomes a set of four functionals, which could be considered as a special coordinate transformation, and in this limit the consistency conditions give rise to differential equations. In general however the transformation will not be a local one, a simple point to point map, but a more elaborate morphism of the entire algebra of functions into itself. It would be analogous to the unitary operator of field theory which takes the values of the fields at minus infinity to their values at some fixed finite time. The former would be the λ_α and the latter the x^μ. Since the algebra is noncommutative one might suppose this morphism to be also inner and write its inverse as

$$x^\mu \to \lambda_\alpha = \frac{1}{i\hbar} \theta^{-1}_{\alpha\mu} U^{-1} x^\mu U$$

for some U. The map defined by Equations (2.1) is not however of this form. We shall return to this briefly in Section 6.

All this can be clarified somewhat by repeating it in the linear approximation, by which we mean a Fourier transformation 'close' to that given by (2.1). Equation (2.5) can be written as

$$C^\alpha{}_{\beta\gamma}(x^\mu) = F^\alpha{}_{\beta\gamma} - 4i\hbar\lambda_\delta Q^{\alpha\delta}{}_{\beta\gamma}.$$

To the lowest order the right-hand side is linear in the metric components and the Fourier map is linear. This is a simple problem in linear algebra and easy to solve, but the solution might be the trivial one. Beyond the linear approximation the complication is due mainly to the fact that, although the right-hand side remains linear in λ_α, the left-hand side becomes very complicated because of the Fourier transformation. The $C^\alpha{}_{\beta\gamma}$ are given functions of x^μ and in general simple; as functions of the λ_α they are quite complicated.

A commutative gauge transformation is an element of the algebra $M_4(\mathcal{C}(\mathbb{R}^4))$. The product in the algebra of forms is not necessarily anti-commutative but the extra terms will not appear in the expression for the differential of a form, because of the relation (2.7) and the form of the Ansatz we are using. We shall adopt the rule that the gauge transformation must be made before 'quantization', not after. This is for many reasons, principally the fact that the 'after-quantization' gauge 'group' [14] is more difficult to define.

We start then with a given frame and associated rotation coefficients and we transform the frame to a new one, to whose rotation coefficients we give the name $C'^\alpha{}_{\beta\gamma}$. As an additional simplification we shall calculate all quantities retaining only terms first order in the gauge elements. The rotation coefficients are then given by

$$C'^\alpha{}_{\beta\gamma} = C^\alpha{}_{\beta\gamma} + D_{[\beta} H^\alpha_{\gamma]}$$

for some first-order gauge transformation whose form must be found as part of the problem. We have here introduced the notation

$$D_\beta H^\alpha_\gamma = e_\beta H^\alpha_\gamma + [C,H]^\alpha{}_{\beta\gamma}, \qquad [C,H]^\alpha{}_{\beta\gamma} = C^\alpha{}_{\beta\delta} H^\delta_\gamma - H^\alpha_\delta C^\delta{}_{\beta\gamma}$$

The use we make of the letter H is motivated in Section 6. We have chosen the original frame more or less arbitrarily. It will determine a symplectic form and a metric and in general it must be gauged so that these two objects are correctly related. The metric is of course Lorentz invariant but the symplectic form not.

From (2.5) one finds that

$$C^\alpha{}_{\beta\gamma} + D_{[\beta} H^\alpha_{\gamma]} = F^\alpha{}_{\beta\gamma} - 4i\bar{k}\lambda_\delta Q^{\alpha\delta}{}_{\beta\gamma}. \tag{2.33}$$

Only the position generators x^μ appear on the left-hand side and only the momentum generators $\lambda_\pm$ appear on the right-hand side. Finally the structure equations can be written as the commutation relations

$$[\lambda_\beta, \lambda_\gamma] = K_{\beta\gamma} + \frac{1}{2}\lambda_\alpha (F^\alpha{}_{\beta\gamma} + C^\alpha{}_{\beta\gamma} + D_{[\beta} H^\alpha_{\gamma]}). \tag{2.34}$$

In the example we give we shall render more explicit this equation. We could have introduced $\lambda_0 = 1$ and written the constant term as $\lambda_0 F^0{}_{\alpha\beta} = K_{\alpha\beta}$. It could be thought of as a spontaneous shift à la Brout-Englert-Higgs-Kibble in the eigenvalues of the operators λ_α, We must also deal with the fact that we have imposed no associativity condition on the product. For example the quantities $P^{\alpha\beta}{}_{\gamma\delta}$ or $S^{\alpha\beta}{}_{\gamma\delta}$ have not been required to satisfy the Yang-Baxter equation. We refer to previous work [16] for a discussion of the problems this might entrain concerning reality conditions within the present formalism. The right-hand side of Equation (2.34) must at least satisfy the Jacobi identities. We shall focus attention on this point in the example.

The problem of gauge invariance and the algebra of observables is a touchy one upon which we shall not dwell. We notice in Section 6 that formally a gauge transformation in noncommutative geometry is the same as the time evolution in quantum field theory. This is somewhat the analog of the result from mechanics to the effect that the hamiltonian generates that particular symplectic transformation which is the time evolution.

It is obvious that not all of the elements of $\mathcal{A}$ are gauge invariant but not that all observables are gauge-invariant. A gauge transformation is a local rotation

$$\theta^\alpha \mapsto \theta'^\alpha = \Lambda^{-1}{}^\alpha_\beta \theta^\beta. \tag{2.35}$$

This formula is slightly misleading since we have written θ'^α as though it belonged to the module of 1-forms of the original calculus whereas in fact it is a frame for a different differential calculus. One should more properly use a direct-sum notation and consider Λ^α_β and its inverse as maps between the two terms; this would complicate considerably the formulae. The structure elements transform classically as

$$\tilde{C}'^\alpha{}_{\beta\gamma} = \tilde{\Lambda}^{-1}{}^\alpha_\delta \tilde{C}^\delta{}_{\epsilon\zeta} \tilde{\Lambda}^\epsilon_\beta \tilde{\Lambda}^\zeta_\gamma + \tilde{\Lambda}^{-1}{}^\alpha_\delta \tilde{e}_\epsilon \tilde{\Lambda}^\delta_{[\gamma} \tilde{\Lambda}^\epsilon_{\beta]}.$$

This is a familiar formula but it cannot be taken over as is to the noncommutative case; under the gauge transformation the calculus transforms also. This is apparent in the conditions (2.7). Within the context of the present calculi the nearest one can come is the transformation

$$C'^\alpha{}_{\beta\gamma} = \Lambda^{-1}{}^\alpha_\delta C^\delta{}_{\epsilon\zeta} \Lambda^\epsilon_\beta \Lambda^\zeta_\gamma + \delta C^\alpha{}_{\beta\gamma} \qquad \delta C^\alpha{}_{\beta\gamma} = -2 e_\delta \Lambda^{-1}{}^\alpha_\epsilon \Lambda^\delta_\zeta \Lambda^\epsilon_\eta P'^{\zeta\eta}{}_{\beta\gamma}.$$

If P and P' are of the form considered here one can write

$$\begin{aligned} C'^\alpha{}_{\beta\gamma} = {} & \Lambda^{-1}{}^\alpha_\delta C^\delta{}_{\epsilon\zeta} \Lambda^\epsilon_\beta \Lambda^\zeta_\gamma - 2 e_\delta \Lambda^{-1}{}^\alpha_\epsilon [\Lambda^\delta_\zeta, \Lambda^\epsilon_\eta] P'^{\zeta\eta}{}_{\beta\gamma} \\ & + \Lambda^{-1}{}^\alpha_\delta e_\epsilon \Lambda^\delta_{[\beta} \Lambda^\epsilon_{\gamma]} + \Lambda^{-1}{}^\alpha_\delta e_\epsilon \Lambda^\delta_\zeta \Lambda^\epsilon_\eta P'^{(\zeta\eta)}{}_{\beta\gamma}. \end{aligned}$$

In the linear approximation, with $\Lambda^\alpha_\beta \simeq \delta^\alpha_\beta + H^\alpha_\beta$, the inhomogeneous term simplifies to the expression

$$-2 e_\delta \Lambda^{-1}{}^\alpha_\epsilon \Lambda^\delta_\zeta \Lambda^\epsilon_\eta P'^{\zeta\eta}{}_{\beta\gamma} = e_{[\beta} H^\alpha_{\gamma]} + e_\delta H^\alpha_\epsilon P^{(\delta\epsilon)}{}_{\beta\gamma}. \tag{2.36}$$

Since

$$\theta'^\alpha(e_\beta) = \Lambda^{-1}{}^\alpha_\beta$$

in a certain formal way one can imagine that

$$e'_\alpha \sim \Lambda^\beta_\alpha e_\beta$$

except that the right-hand-side is not a derivation in general. It makes sense however to introduce $\bar{\lambda}'_\alpha = \Lambda^\beta_\alpha \lambda_\beta$ and to compare it with the elements λ'_β defined to be those which generate the derivations dual to θ'^α. This means that we must consider the identity

$$\Lambda^{-1}{}^\gamma_\beta \theta^\beta_\mu [\lambda'_\alpha, x^\mu] = \theta^\gamma_\mu [\Lambda^{-1}{}^\beta_\alpha \bar{\lambda}'_\beta, x^\mu]$$

as an equation for e'_α in terms of $\bar{e}'_\alpha = \operatorname{ad} \bar{\lambda}'_\alpha$ and solve. There results a rather complicated situation because of the change in Poisson structure which must accompany a gauge transformation in general. We shall attempt to elucidate this with the example.

3 The commutative plane wave

Plane-fronted gravitational waves with parallel rays (*pp*-waves) have a metric [20] which can be defined [5] by the moving frame $\tilde\theta^a = dx^a$, $x^a = (x, y)$ and

$$\tilde\theta^+ = \frac{1}{\sqrt{2}}(\tilde\theta^0 + \tilde\theta^3) = \tilde\Lambda^{-1}(du + \tilde h dv), \qquad \tilde\theta^- = \frac{1}{\sqrt{2}}(\tilde\theta^0 - \tilde\theta^3) = \tilde\Lambda dv. \tag{3.1}$$

The generators of $\mathcal{A}$ will be chosen $x^\mu = (u, v, x, y)$, with u and v null coordinates in the commutative limit. The $\tilde\Lambda$ is for the moment an arbitrary invertible element of $\mathcal{A}$. When $\tilde h = 0$ the space-time is Minkowski and the vector ∂_v is tangent to the bicharacteristics along the characteristic surfaces $u =$ constant. If $\tilde h = \tilde h(v)$ one can introduce an element $I\tilde h(v) \in \mathcal{A}$ such that $\partial_v I\tilde h(v) = \tilde h(v)$ and use it to define $u' = u + I\tilde h(v)$; the space-time is still Minkowski. With $\tilde\Lambda = 1$ then one has $\tilde\theta^+ = du'$ and $\tilde\theta^- = dv$ The map defined by $(uv) \mapsto (u', v)$ does not respect the structure of $\mathcal{A}$ as an algebra but it does leave invariant the commutator: $[u', v] = [u, v]$. Consider the derivations $\tilde e_\alpha$ dual to the moving frame. In the commutative limit we have $\tilde e_1 = \partial_x$, $\tilde e_2 = \partial_y$ and

$$\tilde e_+ = \tilde\Lambda \partial_u, \qquad \tilde e_- = \tilde\Lambda^{-1}(\partial_v - \tilde h \partial_u).$$

The metric-compatible connection form is given by

$$\tilde\omega^\alpha{}_\beta = \begin{pmatrix} \tilde\omega^+{}_+ & 0 & \tilde\omega^+{}_b \\ 0 & \tilde\omega^-{}_- & 0 \\ 0 & \tilde\omega^a{}_- & 0 \end{pmatrix}$$

with

$$\tilde\omega^+{}_+ = \tilde\Lambda^{-2}\tilde e_+ \tilde h \tilde\theta^- + \tilde\Lambda^{-1} d\tilde\Lambda, \qquad \tilde\omega^+{}_a = \tilde\Lambda^{-1}\tilde e_a \tilde h \tilde\theta^-.$$

One of the expressions of the zero-torsion condition is the identity

$$\tilde D_{[\alpha} \tilde D_{\beta]} \tilde x^\mu = 0$$

which can also be written as

$$\tilde e_{[\alpha} \tilde e_{\beta]} \tilde x^\mu = \tilde\omega^\gamma{}_{[\alpha\beta]} \tilde e_\gamma x^\mu. \tag{3.2}$$

This is the same as Equation (2.14.

When $\tilde\Lambda = 1$ the moving frame satisfies the commutation relations

$$\begin{aligned} &[\tilde e_+, \tilde e_-] = -\tilde e_+ \tilde h \tilde e_+, && [\tilde e_-, \tilde e_a] = \tilde e_a \tilde h \tilde e_+, \\ &[\tilde e_+, \tilde e_a] = 0, && [\tilde e_a, \tilde e_b] = 0. \end{aligned}$$

If one includes a possible gauge transformation $\tilde\Lambda$ then one finds that the non-zero structure coefficients are given by

$$\begin{aligned} \tilde C^+{}_{a-} &= -\tilde\Lambda^{-2} \tilde e_a \tilde h, \\ \tilde C^+{}_{+-} &= -\tilde\Lambda^{-2} \tilde e_+ \tilde h - \tilde\Lambda^{-1} \tilde e_- \tilde\Lambda, \\ \tilde C^\pm{}_{\alpha\pm} &= \pm\tilde\Lambda^{-1} \tilde e_\alpha \tilde\Lambda. \end{aligned} \tag{3.3}$$

In writing this we have used the fact that the connection form commutes with the coordinates. This will no longer be true in general and extra terms can appear.

4 *pp*-algebras

The noncommutative generalization will involve essentially finding an equivalent of Equation (2.14). It will be a commutation relation of the form (2.22) and better expressed in terms of the momentum generators. This will fix the algebra as well the calculus. That is, the correspondence principle which we shall actually use is a modified version of the map

$$\tilde{e}_\alpha \mapsto \lambda_\alpha$$

which is, of course, that introduced by Bohr. This representation is valid only in the limit $\hbar \to \infty$. The commutation relations for the $\tilde{e}_\alpha$ must be 'lifted' to commutation relations between the λ_α. For this we notice that the L^2-completion of the algebra of functions on the space-time can be identified with $\mathcal{H} = L^2(\mathbb{R}^4)$. We define then for smooth $\psi \in \mathcal{H}$

$$\lambda_\alpha \psi = \tilde{e}_\alpha \psi. \tag{4.1}$$

Once we have implemented the correspondence principle then this set of momentum generators will satisfy the consistency Equation (2.3), however with $K_{\alpha\beta} = 0$. To obtain a non-zero value for the central extension one must modify the representation. We do this in our analysis of another metric [10]; here we restrict our considerations to formal algebras.

According to the correspondence principle as expressed by (2.8) we introduce a frame θ^α which has the commutative frame as a limit and is as 'near' to it in form as possible for all values of $\hbar$. We shall assume in this example that the noncommutative frame has the same functional form as its limit. When $h = 0$, we have noticed, the space-time is flat; the noncommutative version is also flat. The frame is the noncommutative form of (3.1) which we write in the same way as

$$\theta^+ = \frac{1}{\sqrt{2}}(\theta^0 + \theta^3) = \Lambda^{-1}(du + h dv), \qquad \theta^- = \frac{1}{\sqrt{2}}(\theta^0 - \theta^3) = \Lambda dv, \qquad \theta^a = dx^a. \tag{4.2}$$

This defines also the differential of the generators of the algebra in terms of the basis of the module of 1-forms. It defines therefore the conjugacy relations

$$\begin{aligned} &[\lambda_+, u] = \Lambda, && [\lambda_+, v] = 0, \\ &[\lambda_-, u] = -h\Lambda^{-1}, && [\lambda_-, v] = \Lambda^{-1}, \end{aligned} \tag{4.3}$$

as well as

$$[\lambda_a, x^b] = \delta^b_a, \qquad [\lambda_\pm, x^b] = 0 \qquad [\lambda_a, u] = 0 \qquad [\lambda_a, v] = 0.$$

From (4.3) and the reality conditions it follows that

$$h^*\Lambda = \Lambda^* h, \qquad \Lambda^* = \Lambda.$$

Therefore $h^* = h$. In the linearized approximation $\Lambda^{\pm 1} = 1 \pm H$.

We shall restrict our consideration to the linear approximation, The problem is to find an algebra with a set of 4 generators x^μ satisfying commutation rules as well as a Fourier transformation to a new set λ_α which satisfies Equations (2.34). We use the map (2.8). In the commutative case the geometry is completely determined (to within integration constants) by (2.14). We seek an analogous equation in the noncommutative case. This means that we must rewrite the noncommutative version of Equation (2.14) in terms of the momenta and compare it with the Equation (2.33).

According to the correspondence principle the form of the $C^\alpha{}_{\beta\gamma}$ as functionals of h are determined in the commutative limit. A stronger condition is to assume that they have the same functional form when $\kappa\!\!\!{}^{-} \neq 0$. We satisfy then automatically the 'Bohr-Ehrenfest' condition

$$\lim_{\bar{k}\to 0} h = \tilde{h}. \tag{4.4}$$

The non-vanishing structure elements are given by Equation (2.9) as

$$e_\alpha h = -C^+{}_{\alpha -}, \qquad C^\alpha{}_{[\beta\gamma]} + 2i\bar{k} C^\alpha{}_{(\eta\zeta)} P^{\eta\zeta}{}_{\beta\gamma} = C^\alpha{}_{\beta\gamma}. \tag{4.5}$$

These should be read as equations for h in terms of the commutator $[\lambda_a, \lambda_+]$ if one believes that the noncommutative structure determines the gravitational field. If they are read from right to left then they are equations for the structure of the algebra in terms of the gravitational field. There is considerable ambiguity in the Equations (4.5). They are, strictly speaking, only true in the limit $\bar{k} \to 0$. We suppose however that this ambiguity can be absorbed in a 'gauge' transformation.

We shall look for a solution with only

$$H^+_+ = -H^-_- = H \neq 0.$$

We have then

$$\theta^+ = du + hdv - Hdu, \qquad \theta^- = dv + Hdv$$

as well as

$$C^+{}_{a-} = -e_a h, \qquad C^+{}_{a+} = -C^-{}_{a-} = e_a H.$$

This in turn implies the relations

$$F^+{}_{a+} = -F^-{}_{a-}, \qquad Q^{\pm +}{}_{a+} = -Q^{\pm -}{}_{a-}. \tag{4.6}$$

To be solved are two sets of equations. The first is Equation (2.5); for $\alpha \neq -$

$$F^+{}_{\alpha -} - 4i\bar{k}\lambda_r Q^{+r}{}_{\alpha -} = -e_\alpha h, \qquad r = +, - \tag{4.7}$$

$$F^-{}_{\alpha -} - 4i\bar{k}\lambda_r Q^{-r}{}_{\alpha -} = -e_\alpha H, \tag{4.8}$$

These are the actual equations; they define the algebra in terms of the rotation coefficients. The second set is Equation (2.12)

$$I_{+-} = K_{+-} + \lambda_r F^r{}_{+-} - 2i\bar{k}\lambda_r\lambda_s Q^{rs}{}_{+-}, \tag{4.9}$$

$$I_{a+} = K_{a+} + \lambda_r F^r{}_{a+} - 2i\bar{k}\lambda_r\lambda_s Q^{rs}{}_{a+}, \tag{4.10}$$

$$I_{a-} = K_{a-} + \lambda_r F^r{}_{a-} - 2i\bar{k}\lambda_r\lambda_s Q^{rs}{}_{a-}, \tag{4.11}$$

$$I_{ab} = K_{ab} + \lambda_r F^r{}_{ab}. \tag{4.12}$$

From Equation (4.7) we deduce that

$$e_a e_b h = 4i\bar{k} I_{ar} Q^{+r}{}_{b-}.$$

Since we are working only to first order we should compare the relative magnitudes of the various terms in the equations. One can draw the following table:

$$\begin{array}{rclcrclcl} [K] &=& L^{-2} & | & [2i\bar{k}K] &=& L^0 & \text{real} & \\ [F] &=& L^{-1} & | & [2i\bar{k}\lambda F] &=& L^0 & \text{real} & | \quad F \sim 2i\bar{k}\kappa K \\ [Q] &=& L^{-2} & | & [(2i\bar{k}\lambda)^2 Q] &=& L^0 & \text{real} & | \quad Q \sim 2i\bar{k}\kappa^2 K \end{array}$$

We write then

$$F^r{}_{at} = 2i\hbar\kappa D^{rs}{}_t K_{as}, \qquad Q^{rs}{}_{at} = 2i\hbar\kappa^2 D^{rsu}{}_t K_{au}$$

with the coefficients $D^{rs}{}_t$ and $D^{rst}{}_u$ real and without dimension. We have included the factor $2i\hbar$ to indicate the scaling behaviour and a characteristic gravitational curvature κ to keep the physical dimensions correct and to allow for a variation in the ratio of the two coefficients. The parameters $2i\hbar K_{\alpha\beta}$ are real, without dimension and do not necessarily vanish with the curvature. We have here implicitly supposed they do not. This point has been discussed elsewhere [10]. From the conditions (4.6) we see that the relations

$$D^{+s}{}_+ = -D^{-s}{}_-, \qquad D^{+rs}{}_+ = -D^{-rs}{}_-.$$

We would like the derivative of a physical quantity f to satisfy $e_a f \sim \kappa f$, which implies that

$$(2i\hbar K) \sim \kappa(2i\hbar\lambda).$$

Therefore the order of magnitude of the wave is related to the two length scales by

$$h \sim \kappa(2i\hbar\lambda)(2i\hbar K) \sim (2i\hbar K)^2.$$

If we use the expansion in terms of the coefficients $D^{rs}{}_t$ we find that

$$I_{at} = K_{at} - 2i\hbar\kappa\lambda_r D^{rs}{}_t K_{as} + o(\kappa^2).$$

To simplify the equations, but without other justification, we choose

$$K_{+-} = 0, \qquad K_{ab} = 0, \qquad F^\alpha{}_{ab} = 0, \qquad F^\alpha{}_{rs} = 0, \qquad Q^{\alpha\beta}{}_{rs} = 0.$$

The integrability conditions for the equations for h and H are respectively

$$F^r{}_{ab} e_r h = 4i\hbar I_{[ar} Q^{+r}{}_{b]-},$$
$$F^r{}_{ab} e_r H = 4i\hbar I_{[ar} Q^{-r}{}_{b]-}.$$

These can be written as

$$D^{rst}{}_- K_{[as} K_{b]t} = 0, \qquad D^{ps}{}_u D^{uqt}{}_- K_{[as} K_{b]t} = 0$$

and it implies therefore that

$$D^{r[st]}{}_- = 0, \qquad D^{p[s}{}_q D^{uqt]}{}_- = 0.$$

But $D^{rst}{}_-$ must also be symmetric in the first two indices. We conclude therefore that it is totally symmetric in the first three indices.

The second set of equations gives rise to Jacobi identities:

$$\epsilon^{\alpha\beta\gamma\delta}[\lambda_\alpha, I_{\beta\gamma}] = 0.$$

The Jacobi identities can be written then as

$$I_{[as} Q^{\pm s}{}_{b]-} = 0.$$

These are the same conditions as for the existence of h and H.

We notice also that $h(x^a)$ is harmonic if

$$g^{ab} e_a e_b h = 4i\hbar g^{ab} I_{ar} Q^{+r}{}_{b-} = 0. \tag{4.13}$$

This can be written as

$$g^{ab} D^{+st}{}_{-} K_{as} K_{bt} = 0, \qquad g^{ab} D^{ps}{}_{q} D^{+qt}{}_{-} K_{as} K_{bt} = 0$$

To actually find a (formal) solution is more difficult. One easily sees that

$$\begin{aligned} 4i\hbar\lambda_r\lambda_s Q^{rs}{}_{a+} &= \lambda_+ F^{+}{}_{a+} - \lambda_+ e_a H + 4i\hbar\lambda_-\lambda_r Q^{-r}{}_{a+}, \\ 4i\hbar\lambda_r\lambda_s Q^{rs}{}_{a-} &= \lambda_r F^{r}{}_{a-} + \lambda_+ e_a h + \lambda_- e_a H. \end{aligned}$$

Using the first set, the second set of equations can be replaced by the system

$$I_{a+} = K_{a+} + \frac{1}{2}\lambda_+ e_a H + \frac{1}{2}\lambda_r F^{r}{}_{a+} - \frac{1}{2}\lambda_-(4i\hbar\lambda_r Q^{-r}{}_{a+} - F^{-}{}_{a+}), \tag{4.14}$$

$$I_{a-} = K_{a-} - \frac{1}{2}(\lambda_+ e_a h + \lambda_- e_a H) + \frac{1}{2}\lambda_r F^{r}{}_{a-}. \tag{4.15}$$

This in turn can be written as

$$\begin{aligned} [\lambda_a, \lambda_+(1 - \frac{1}{2}H)] = (1 - \frac{1}{2}H)K_{a+} + i\hbar\kappa\lambda_r D^{rs}{}_{+} K_{as} \\ - i\hbar\kappa\lambda_-(4i\hbar\kappa\lambda_r D^{-rs}{}_{+} - D^{-s}{}_{+})K_{as}, \end{aligned} \tag{4.16}$$

$$\begin{aligned} [\lambda_a, \lambda_-(1 + \frac{1}{2}H) + \lambda_+\frac{1}{2}h] = (1 + \frac{1}{2}H)K_{a-} + i\hbar\kappa\lambda_r D^{rs}{}_{-} K_{as} \\ + \frac{1}{2}hK_{a+}. \end{aligned} \tag{4.17}$$

If we suppose that there is a solution f to the equation

$$e_a f = -F^{-}{}_{a+} + 4i\hbar\lambda_r Q^{-r}{}_{a+}$$

then we can write the second set of equations finally as

$$\begin{aligned} [\lambda_a, \lambda_+(1 - \frac{1}{2}H) + \lambda_-\frac{1}{2}f] = (1 - \frac{1}{2}H)K_{a+} \\ + \frac{1}{2}fK_{a-} + i\hbar\kappa\lambda_r D^{rs}{}_{+} K_{as}, \end{aligned} \tag{4.18}$$

$$\begin{aligned} [\lambda_a, \lambda_-(1 + \frac{1}{2}H) + \lambda_+\frac{1}{2}h] = (1 + \frac{1}{2}H)K_{a-} \\ + \frac{1}{2}hK_{a+} + i\hbar\kappa\lambda_r D^{rs}{}_{-} K_{as}. \end{aligned} \tag{4.19}$$

Here, as in the previous version we have used the linearization assumption to replace, for example, on the right-hand side hI_{a+} by hK_{a+}. By the same assumption we find that

$$h = -2i\hbar\kappa D^{+s}{}_{-}\lambda_s, \qquad H = -2i\hbar\kappa D^{-s}{}_{-}\lambda_s, \qquad f = -2i\hbar\kappa D^{-s}{}_{+}\lambda_s.$$

Since there are only $\lambda_\pm$ on the right-hand side, the extra element f which we have introduced must be a linear combination of h and H. Taking the successive derivatives we find that

$$\begin{aligned} e_a h &= -2i\hbar\kappa D^{+s}{}_{-} I_{as} \simeq -2i\hbar\kappa D^{+s}{}_{-} K_{as} \\ e_a e_b h &= -2i\hbar\kappa D^{+s}{}_{-}[\lambda_a, I_{bs}] \simeq -(2i\hbar\kappa)^2 D^{+s}{}_{-} D^{rt}{}_{s} K_{ar} K_{bt}. \end{aligned}$$

But we have another expression, Equation (4.8), for $e_a h$ and the two must be equal. The leading terms are equal by our choice of normalization. If we compare the next terms, those which dominate the second derivatives, we find that

$$2D^{+rs}{}_{-} = D^{+t}{}_{-} D^{rs}{}_{t}.$$

In Appendix I one reads that the metric component $h(\equiv \tilde{h})$ is a linear function of u, harmonic in the x^a and more or less arbitrary as a function of the v. The term linear in u is in fact a coordinate artifact and can be eliminated by replacing u and v by new coordinates while maintaining the general metric form.

Suppose that h has an expansion around a solution of the form of a vacuum plane wave solution defined by Equation (I.14):

$$h = h_a x^a + \frac{1}{2} h_{ab} x^a x^b + \cdots$$

From (4.7) we find that

$$F^{+}{}_{a-} = -h_a, \qquad h_{ab} x^b = 4i\bar{k}\lambda_r Q^{+r}{}_{a-} = 2(2i\bar{k}\kappa)^2 \lambda_r D^{+rs}{}_{-} K_{as}.$$

The relation between the momenta and coordinates is linear as in the flat-space case. If we take the derivative we find the relations

$$h_{ab} = 4i\bar{k} K_{br} Q^{+r}{}_{a-} = 2(2i\bar{k}\kappa)^2 D^{+rs}{}_{-} K_{ar} K_{bs}.$$

Suppose that the boost H has a similar expansion

$$H = H_a x^a + \frac{1}{2} H_{ab} x^a x^b + \cdots.$$

From (4.8) we find that

$$F^{-}{}_{a-} = -H_a, \qquad H_{ab} x^b = 4i\bar{k}\lambda_r Q^{-r}{}_{a-} = 2(2i\bar{k}\kappa)^2 \lambda_r D^{-rs}{}_{-} K_{as}.$$

If we take the derivative we find the relations

$$H_{ab} = 4i\bar{k} K_{br} Q^{-r}{}_{a-} = 2(2i\bar{k}\kappa)^2 D^{-rs}{}_{-} K_{ar} K_{bs}.$$

The most interesting relations are those which follow from (4.3). In the present context they become

$$\begin{aligned} &[\lambda_+, u] = 1 + \tfrac{1}{2} H_{ab} x^a x^b, &&[\lambda_+, v] = 0,\\ &[\lambda_-, u] = -\tfrac{1}{2} h_{ab} x^a x^b, &&[\lambda_-, v] = 1 - \tfrac{1}{2} H_{ab} x^a x^b. \end{aligned}$$

The fact that the right-hand side is quadratic is related to the approximation we are using and not directly to the quadratic relations satisfied by the momenta.

When $h = 0$, or rather in the limit when $h \to 0$, there is a natural set of conjugacy relations given by $e_\pm x^a = 0$; $e_a x^\pm = 0$, that is, with $K_{\pm a} = 0$. When $h \neq 0$ this can no longer be the case. From the conjugacy relations (4.3) we find that

$$[\lambda_-, x^+] = K_{-a}[x^a, x^+] = -h.$$

From (4.20) below it follows that

$$K_{-a} J^{a+} = -h.$$

Therefore with $h \neq 0$ it is consistent to have $K_{+a} = 0$ as long as $K_{-a} \neq 0$. We shall examine this possibility in some detail.

We recall first that Equation (4.9) – (4.12) should be read from left to right as usual; the h is considered as been given. We start with the most general possible $h = h(x^\alpha)$. Only for certain choices of this function is the metric a *pp*-wave metric. A *pp*-metric arises (by definition) whenever there is a covariantly constant null vector, for example when the metric is Ricci flat. To obtain commutation relations for the position generators one must use the conjugacy relations (4.3). We use Equation (2.30) to obtain differential equations for $J^{\mu\nu}$:

$$\begin{aligned} i\hbar dJ^{+-} &= [v, h]\theta^-, & i\hbar dJ^{+b} &= [x^b, h]\theta^-, \\ i\hbar dJ^{a-} &= 0, & i\hbar dJ^{ab} &= 0. \end{aligned}$$

The Jacobi identities (2.30) reduce then to the equations

$$e_- J^{+-} = [v, h], \qquad e_- J^{+a} = [x^a, h].$$

The $J^{\mu\nu}$ must be of the form

$$J^{+\nu} = J^{+\nu}(u, x^a), \qquad J^{-a} = \theta^{-a}, \qquad J^{ab} = \theta^{ab}$$

and they must satisfy the constraint

$$[v, J^{a+}] = [x^a, J^{-+}]$$

which follows from the Jacobi identities. If Equation (2.31) is to hold we must have

$$J^{+a} = \theta^{+a} - h\theta^{-a}, \qquad h = h(u, v, \lambda_+). \tag{4.20}$$

We have already recalled the fact that in the commutative limit the commutation relations define a symplectic structure. This is one of the foundation facts of quantum mechanics. What is new here is the fact that the extension of these relations to the differential calculus leaves also a gravitation field as 'shadow'. We have constructed the differential calculus (formally) so that it would define the *pp* metric in the limit. The two structures have a common origin and this fact should be apparent. The *pp* metric is of Petrov type N. We have found that the matrix $K_{\alpha\beta}$ - considered as an electromagnetic field or as a B-field - is of type I. Both structures define a common principle null vector, defined by the propagation of the wave. There remains much to be clarified on the relation between the two classes of principle null vectors in more general situations.

We can now consider the algebra and calculus as given and use previously employed methods to find the metric.

5 Curvature

The linear-connection coefficients are defined in terms of the structure elements by the formula (2.21), which can be written in the form

$$\omega^\alpha{}_{[\beta\gamma]} = C^\alpha{}_{\beta\gamma} - i\hbar\omega^\alpha{}_{(\eta\zeta)}Q^{\eta\zeta}{}_{\beta\gamma}.$$

The case which we shall consider is such however that the second term vanishes and so we have the usual formula for the Levi-Civita connection. The associated covariant derivative is defined by the actions $D\theta^\alpha = -\omega^\alpha{}_\beta\theta^\beta$ with the right-hand side defined using (2.20).

The product in the algebra of forms is not necessarily anti-commutative but the extra terms will not appear in the expression for the differential of a form, because of the relation (2.7) and the form of the Ansatz we are using. To calculate the curvature we must know the structure of the algebra of forms, given by Equation (2.4). The relations are the usual ones except for

$$\theta^{(+}\theta^{-)} = 4i\hbar Q^{+-}{}_{ra}\theta^r\theta^a, \qquad (\theta^+)^2 = 2i\hbar Q^{++}{}_{ar}\theta^a\theta^r. \tag{5.1}$$

Because of the symmetries of the rotation coefficients these products do not contribute to the exterior derivative.

Although not a satisfactory object, we define the curvature exactly as in the commutative case, as the left-linear map $\mathrm{Curv} = -D^2$ from $\Omega^1(\mathcal{A})$ into $\Omega^2(\mathcal{A}) \otimes \Omega^1(\mathcal{A})$, with associated Ricci map from $\Omega^1(\mathcal{A})$ into $\Omega^1(\mathcal{A})$ given by

$$\mathrm{Ric}(\theta^+) = -(\Delta h + g^{ab}(e_a h e_b H + e_a H e_b h))\theta^-, \qquad \mathrm{Ric}(\theta^-) = 0, \qquad \mathrm{Ric}(\theta^a) = 0.$$

The only non-vanishing component of the Ricci tensor is therefore

$$R^+{}_- = -\Delta h - g^{ab}(e_a h e_b H + e_a H e_b h).$$

We have not perhaps found the most general *pp*-algebra; in fact we did not even correctly characterize a *pp*-algebra; but we have shown there is at least one algebra with a differential calculus whose compatible metric is a plane wave.

6 Heisenberg picture

One could also consider the problem of finding the metric as an evolution equation in field theory in the sense that one can pass from the Schrödinger picture to the Heisenberg picture with the help of an evolution hamiltonian. In quantum field theory a scalar field evolves in time according to the equation

$$\phi(t) = U(t)^{-1}\phi(0)U(t) = \phi'(0), \qquad U(t) = e^{iHt}.$$

We have chosen a very particular gauge transformation $\tilde\Lambda$ which in the commutative limit is one element of the local Lorentz transformations and we have, in the linear approximation, 'lifted' $\tilde\Lambda$ to a morphism Λ of the differential calculus which is such that the covariant derivative transforms as it should. We have then on the one hand the transformation of the rotation coefficients given by Equations (4.5) and on the other hand what would be the quantum version of a gauge transformation

$$D\phi \mapsto D'\phi' = D'(U^{-1}\phi U) = (U^{-1}DU)(U^{-1}\phi U).$$

There is a strong similarity between the quantum gauge transformation

$$D' = U^{-1}DU$$

and the noncommutative change of frame (4.5), which supports the point of view that gravity is a manifestation of the noncommutative structure of space-time. The evolution of a (free) quantum field in a gravitational background is the same as, or at least similar to, a change of frame in a noncommutative geometry.

7 Embeddings

In this section we would like to consider the *pp*-wave metrics as perturbations of flat metrics and also consider them as noncommutative embeddings [12, 24]. To distinguish them we shall place a prime on perturbed quantities. The commuting generators can be considered as the coordinates of an $\mathbb{R}^4$. If $h = 0$ one can set $\lambda_a = -K_{a\pm}x^{\pm}$ to define an embedding $\mathbb{R}^2 \hookrightarrow \mathbb{R}^4$. Consider now the 8-dimensional phase-space associated to the $\mathbb{R}^4$ obtained by adding 4 extra variables $(x^a, \lambda_\pm)$ and 'quantize' this space by imposing the commutation relations

$$[\lambda_\alpha, x^\mu] = i\bar{k}\delta^\mu_\alpha$$

We write the corresponding relations Equation (4.3) in the form

$$[\lambda'_\alpha, x'^\mu] = i\bar{k}e'^\mu_\alpha$$

and we consider the latter as a perturbation

$$e'^\mu_\alpha = \delta^\mu_\alpha + i\bar{k}f^\mu_\alpha$$

of the former. If we perturb

$$x'^\mu = x^\mu + i\bar{k}\xi^\mu, \qquad \lambda'_\alpha = \lambda_\alpha + i\bar{k}l_\alpha$$

both the position and the momentum generators we see that the relation

$$f^\mu_\alpha = [\lambda_\alpha, \xi^\mu] + [l_\alpha, x^\mu]$$

holds. The induced perturbation of the frame is given by (2.35) with

$$\Lambda^\alpha_\beta \simeq \delta^\alpha_\beta - i\bar{k}f^\alpha_\beta$$

no longer necessarily a gauge transformation.

If λ_α and its perturbed equivalent are to generate differential calculi then they both must satisfy (2.3) with in general different coefficients. We set

$$P'^{\alpha\beta}{}_{\gamma\delta} = P^{\alpha\beta}{}_{\gamma\delta} + i\bar{k}Q^{\alpha\beta}{}_{\gamma\delta}\,, \qquad F'^\alpha{}_{\gamma\delta} = F^\alpha{}_{\gamma\delta} + i\bar{k}\phi^\alpha{}_{\gamma\delta}, \qquad K'_{\gamma\delta} = K_{\gamma\delta} + i\bar{k}\kappa_{\gamma\delta}$$

and we shall restrict our attention to the special case

$$P^{\alpha\beta}{}_{\gamma\delta} = \frac{1}{2}\delta^{[\alpha}_\gamma\delta^{\beta]}_\delta, \qquad F^\alpha{}_{\gamma\delta} = 0$$

which is a perturbation of flat space. Comparing then the two sets of equations we find the relation

$$\begin{aligned} e_{[\alpha}l_{\beta]} = \kappa_{\alpha\beta} &+ \phi^\gamma{}_{\alpha\beta}\lambda_\gamma - i\bar{k}l_{[\alpha}l_{\beta]} - 4i\bar{k}Q^{\gamma\delta}{}_{\alpha\beta}l_\gamma\lambda_\delta \\ &- 2\lambda_\gamma\lambda_\delta Q^{\gamma\delta}{}_{\alpha\beta} - 2(i\bar{k})^2 l_\gamma l_\delta Q^{\gamma\delta}{}_{\alpha\beta} - 2i\bar{k}Q^{\gamma\delta}{}_{\alpha\beta}e_\delta l_\gamma \end{aligned}$$

If the metric is a *pp*-wave then

$$i\bar{k}f^\mu_\alpha = \begin{pmatrix} 0 & -h & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The embedding 'space' described by an algebra $\mathcal{B}$ has a natural differential calculus with a module of 1-forms of rank 8 freely generated by the elements dy^i, with

$$y^i = (\theta^{\mu\alpha}\lambda_\alpha, x^\nu)$$

A basis of the 1-forms of the embedded 'space' $\mathcal{A}$ would have four elements θ^α and would have to be in some sense a restriction of the 1-forms over $\mathcal{B}$. Associated to the inclusion of $\mathcal{A}$ in $\mathcal{B}$ there is an inclusion map

$$\iota^* : \Omega^1(\mathcal{A}) \hookrightarrow \Omega^1(\mathcal{B})$$

between the respective module of 1-forms. If θ^α is a frame and f an element of $\mathcal{A}$ then we must have the relation

$$0 = \iota^*(f\theta^\alpha - \theta^\alpha f) = [\iota^* f, \theta^\alpha_i] dy^i.$$

This implies then the relations

$$[\iota^* f, \theta^\alpha_i] = 0.$$

There are $2n^2$ unknowns and an equal number of relations.

We started with four generators λ_α of an algebra $\mathcal{A}$ and an unknown functional h of a second set of four generators x^μ of the same algebra. Consider the subalgebra $\mathcal{B}$ generated by the x^a. With the choice we have made for the form of the matrix $K_{\alpha\beta}$ we can identify $\mathcal{B}$ as a function algebra and therefore by the commutation relations (4.3) we can conclude that the $\lambda_\pm$ are functions of the x^a. Likewise we identify the λ_a as functions of the $x^\pm$. The four functions constitute an embedding of $\mathcal{A}$ as a subalgebra of a 'flat' algebra $\mathcal{B}$ with twice the generators. Because of the simple structure of the problem, two of these functions are just linear transformations.

8 Discussion

It would seem that it is possible to find a noncommutative geometry, of Minkowski signature in 'dimension' four. We have started with a given commutative metric, admittedly rather degenerate, and constructed an algebra and a differential calculus which supports the metric. The algebra is a 'q-deformed' version of commutative $\mathbb{R}^4$; the differential calculus differs from de Rham's in that the relations (5.1) hold.

Our calculations have been all carried out on the level of formal algebra. The real problem lies in fact in finding a representation of the quadratic algebra (2.22). We have given a certain number of necessary conditions for such a representation to exist; they are not necessarily sufficient. To obtain a concrete solution we must find a representation in terms of operators on a Hilbert space. To each representation corresponds a trace and therefore an action.

The Riemann tensor is of null type. Its relation with $J^{\mu\nu}$ would be expressed in terms of the action of $J^{\mu\nu}$ on the principle null direction. Let k_μ be this vector. We must compare k^μ with $J^{\mu\nu}k_\nu$. In the limit $\hbar \to 0$ we recover the commutative plane-wave solution plus in addition a symplectic structure $J^{\mu\nu}$. The relation of J to the Riemann tensor in general is a subject of interest which we leave also to a future publication.

Appendix I

For completeness we give here a brief review of plane-fronted metrics using the 'principal null vector' approach, a formalism which is a bit more elaborate than is essential but which explicitly identifies, from the beginning, the important vector $k = \tilde{e}_+$, with components k^α. In special cases k can be identified as a principal null vector of the Riemann or Weyl (conformal) curvature tensors. We shall consider n-metrics since to do so is no more complicated than 4-metrics ; when $n = 4$ these metrics are included in the Kerr-Schild class of metrics [20]. We are considering only classical commutative geometry here so we can omit, without ambiguity, the tilde notation used in previous discussions of classical metrics. We recall also that the metric components g_{ab} are real constants.

The line element is given in the frame formalism by

$$ds^2 = \theta^- \otimes \theta^+ + \theta^+ \otimes \theta^- + g_{ab}\theta^a \otimes \theta^b, \tag{I.1}$$

The class of metrics to be considered will be called *plane-fronted*, and can be defined in local coordinates x^α by a frame

$$\theta^\alpha = (\delta^\alpha_\beta + hk^\alpha k_\beta)dx^\beta, \tag{I.2}$$

where the k^α are constants given by

$$k^\alpha = \delta^\alpha_+, \tag{I.3}$$

so that $\partial_\beta(k^\alpha) = 0$. Furthermore $g_{\alpha\beta}k^\alpha k^\beta = 0$, $g_{\alpha\beta}k^\beta = k_\alpha = \delta^-_\alpha$, where, in the natural coordinate basis $(x^+ = u, x^- = v, x^a)$, the flat metric takes the form

$$g_{\alpha\beta}dx^\alpha \otimes dx^\beta = 2dudv + g_{ab}dx^a dx^b,$$

and h is a function. Hence the line element (I.1) of a *plane-fronted metric* g is given in Kerr-Schild form by

$$ds^2 = 2du \otimes dv + g_{ab}dx^a \otimes dx^b + 2h(dv)^2.$$

Note that the vector field $k^\alpha e_\alpha = e_+$ is a null vector with respect to these metrics.

The Levi-Civita connection 1-forms, in components with respect to the above bases $(e_\alpha, \theta^\alpha)$ are given by

$$\omega^\alpha{}_\beta = (k^\alpha \partial_\beta h - \partial^\alpha h k_\beta)k_\gamma dx^\gamma = (k^\alpha \partial_\beta h - \partial^\alpha h k_\beta)\theta^-.$$

The components of the curvature 2-forms, relative to the same bases, are given by

$$R^\alpha{}_{\beta\gamma\delta} = k_\gamma k_\beta \partial_\delta \partial^\alpha h - k_\delta k_\beta \partial_\gamma \partial^\alpha h + k^\alpha k_\delta \partial_\beta \partial_\gamma h - k^\alpha k_\gamma \partial_\beta \partial_\delta h.$$

Consequently

$$R^\alpha{}_{\beta\gamma\delta}k^\delta = k_\gamma k_\beta \partial_\delta \partial^\alpha h k^\delta - k^\alpha k_\gamma \partial_\beta \partial_\delta h k^\delta,$$

and the Ricci tensor has components

$$R_{\beta\delta} = R^\alpha{}_{\beta\alpha\delta} = k_\alpha k_\beta \partial_\delta \partial^\alpha h - k_\delta k_\beta \partial_\alpha \partial^\alpha h + k^\alpha k_\delta \partial_\beta \partial_\alpha h.$$

Hence

$$R_{\beta\delta}k^\delta = 0 \quad \text{iff} \quad \partial_\beta \partial_\delta h k^\delta k^\beta = 0.$$

The results above can be used to prove the following theorems which illustrate some of the properties of the class of metrics being considered here [20, 25, 26, 27, 28].

Theorem I.1 *Let $D_\alpha$denote the Levi-Civita covariant derivative (components with respect to the moving frame). Then*

1. *The following equality holds*

$$D_\alpha k_\beta = k_\alpha k_\beta \partial_\gamma h k^\gamma. \tag{I.4}$$

2. $k^\alpha D_a k_\beta = 0$, *and $k^\alpha e_\alpha$ is tangent to affinely parametrized null geodesics.*

3. $D_{[\alpha} k_{\beta]} = D_\alpha k^\alpha = 0$; $\quad D_\alpha k_\beta = D_{(\alpha} k_{\beta)} \quad$ *and* $\quad D_\alpha k_\beta D^\alpha k^\beta = 0$.

This justifies the teminology 'plane-fronted', since these conditions applied to vacuum Lorentzian 4-metrics of the form considered here, define the so-called *plane-fronted metrics*

Theorem I.2 $D_a k_\beta = 0$ *if and only if*

$$\partial_+ h = \partial_\beta h k^\beta = 0. \tag{I.5}$$

Once again using the previously mentioned Lorentzian 4-metric terminology, when this equation is satisfied, the plane fronted metrics which admit a covariantly constant null vector field are said to be *plane-fronted with parallel rays (pp waves).*

Theorem I.3 *Consider* plane-fronted *metrics. Two useful results are the following.*

1. $R^\alpha{}_{\beta\gamma\delta} k^\delta = 0$ *if and only if*

$$\partial_\alpha \partial_\beta h k^\beta = F k_\alpha, \tag{I.6}$$

for some function F, and then the Ricci tensor must take the form

$$R_{\alpha\beta} = k_\alpha k_\beta (2F - \partial_\alpha \partial^\alpha h). \tag{I.7}$$

2. *Let $Q_{\alpha\beta} = k_\alpha q_\beta - k_\beta q_\alpha$ where $k^\alpha q_\alpha = 0$ and the q_α are also constants. That is, $\partial_\beta q_\alpha = 0$. Then $Q_{\alpha\beta} k^\beta = 0$ and, if f is any function,*

$$D_\alpha (f Q_{\beta\gamma}) = Q_{\beta\gamma} (\partial_\alpha f - h k_\alpha \partial_\beta f k^\beta + f k_\alpha \partial_\beta h k^\beta). \tag{I.8}$$

Hence when $f Q_{\beta\gamma}$ is non-zero $D_\alpha (f Q_{\beta\gamma}) = 0$ if and only if the equation

$$\partial_\alpha \log |f| + k_\alpha \partial_\beta h k^\beta = 0, \tag{I.9}$$

admits a non zero solution f.

By using the results above it is a straight-forward matter to obtain further results about particular classes of solutions of Einstein's equations (see further comment below).

Theorem I.4 1. *In any dimension greater than two, plane-fronted metrics are Ricci flat, $R_{\alpha\beta} = 0$, if and only if the two equations*

$$\partial_\alpha \partial_\beta k^\beta h = F k_\alpha, \qquad \partial_\alpha \partial^\alpha h = 2F$$

for some function F, are satisfied.

Hence it follows from the above results that when the latter two equations are satisfied, the Weyl conformal tensor satisfies the equation $C^\alpha_{\beta\gamma\delta} k^\delta = 0$ and, in the four dimensional Lorentzian case, is said to be of Petrov type N with $k^\alpha e_\alpha$ being a repeated principal null vector.

2. *Using the local coordinate expressions further it follows immediately that plane-fronted metrics are Ricci flat if and only if the function h is given by*

$$h = L(v)u + G(v, x^a), \tag{I.10}$$

where $L(v)$ satisfies $\partial_v L = F$ and the function G satisfies the (Laplace-type) equation

$$g^{ab}\partial_a\partial_b G = 0. \tag{I.11}$$

3. *In the four dimensional Lorentzian case, writing $z = x + iy$, it follows from these equations that plane fronted metrics g is Ricci flat if and only if*

$$G = K(v, z) + c.c., \qquad h = Lu + G(v, x^a). \tag{I.12}$$

for any complex K holomorphic in z.

It is a straightforward matter to show that the term Lu in (8.10) and (8.12) is merely an artifact of the coordinate choice. It can be eliminated, while retaining the general form of the metric, by choosing new u, v coordinates. Hence the Ricci-flat solutions are *pp*-waves. As an immediate corollary follows the

Theorem I.5 *In the four-dimensional Lorentzian case, using the notation of Theorem I.4, it therefore follows that* Ricci flat *pp* waves *are determined by an arbitrary complex function K where*

$$h = G(v, x^a) = K(v, z) + c.c. \tag{I.13}$$

In summary

1. Functions $h = h(v)$ give flat metrics as do functions linear in the coordinates x^α.
2. It is clear from the form of the Ricci tensor given above that is is a straightforward matter to produce plane-fronted (or more particularly *pp*-wave) metrics which are solutions of the Einstein field equations with matter sources such as a massless scalar field ($R_{\alpha\beta} = \kappa e_\alpha(\varphi)e_\beta(\varphi)$, with say $\varphi = \varphi(v)$) and gauge fields (for example the null electromagnetic field with components $F_{\alpha\beta} = k_\alpha p_\beta - k_\beta p_\alpha$, with $k_\alpha p^\alpha = 0$, so that $F_{\alpha\beta}F^{\alpha\beta} = 0$ and the energy momentum tensor is (modulo sign conventions) given by $k_\alpha k_\beta(p_\gamma p^\gamma)$).
3. One final specialization made in the literature which is relevant here is from *pp* waves to *plane waves.*. This corresponds to making particular choices of solutions of the equations above. In the vacuum case , the solutions of Equation (I.11) given by

$$h = \frac{1}{2}h_{ab}(v)x^a x^b, \qquad g^{ab}h_{ab} = 0, \tag{I.14}$$

are *vacuum plane wave solutions.* This terminology can be justified by computing the Weyl tensor and noting the planar symmetry. In addition, the notions of amplitude and polarisation can be introduced.

Appendix II

An alternative form of a plane wave metric, derivable as the Penrose limit of any 4-metric, is currently of some interest in the string-theory community [22]. It can be obtained by the following argument. Consider an arbitrary metric in null coordinates $x^\mu = (u, v, x^i)$. The hypersurfaces $u = u_0$ are assumed to be null and v to be the affine parameter along the corresponding bicharacteristics. The line element can be written in the form

$$ds^2 = 2dudv + g_{++}du^2 + 2g_{+i}dvdx^i + g_{ij}dx^i dx^j.$$

Since $(0, 1, 0, 0)$ is a null vector we have $g_{--} = 0$. We have also used the local coordinate freedom to set

$$g_{+-} = 1, \qquad g_{-i} = 0.$$

The remaining metric components are general functions of all coordinates. We now rescale

$$g_{\mu\nu} \mapsto \omega^{-2} g_{\mu\nu}, \qquad u \mapsto \omega^2 u, \qquad x^i \mapsto \omega x^i.$$

The line element becomes

$$ds^2 = 2dudv + \omega^2 g_{++}du^2 + 2\omega g_{+i}dudx^i + g_{ij}dx^i dx^j$$

with the metric components now functions of the form

$$g_{\mu\nu} = g_{\mu\nu}(\omega^2 u, v, \omega x^i).$$

As a singular limit when $\omega \to 0$ one obtains [21, 22] the plane wave with line element

$$ds^2 = 2dudv + g_{ij}(v)dx^i dx^j.$$

If the initial metric is described by a frame of the form

$$\theta^+ = du, \qquad \theta^- = dv + \theta^-_+ du + \theta^-_i dx^i, \qquad \theta^a = \theta^a_i dx^i$$

then the scaled metric is described by

$$\theta^+ = du, \qquad \theta^- = dv, \qquad \theta^a = \theta^a_i(v)dx^i. \tag{II.1}$$

We shall refer to (II.1) as the Rosen frame.

In the 'quasi-commutative' limit the line element is given by

$$ds^2 = 2d\tilde{u}d\tilde{v} + 2\tilde{h}d\tilde{v}^2 - d\tilde{x}^2 - d\tilde{y}^2.$$

In the Rosen frame it is of the form

$$ds^2 = 2dudv - (\theta^1)^2 - (\theta^2)^2.$$

This form of the metric has a certain similarity to the Kasner metric, which has also [10] a noncommutative extension.

The dual frame is such that the only nontrivial commutation relations are given by

$$[e_-, e_a] = (e^{-1}\dot{e})^b{}_a e_b, \qquad \theta^a_i e^i_b = \delta^a_b. \tag{II.2}$$

The structure functions $C^\alpha{}_{\beta\gamma}$ are seen to be with

$$C^a{}_{-b} = -C^a{}_{b-} = (e^{-1}\dot{e})^b{}_a.$$

the only non-vanishing components.

The transformation from the Rosen frame to the previous frame for the tilded metric is made using the transformation of coordinates

$$u = \tilde{u} + \frac{1}{2}F_{ab}(\tilde{v})\tilde{x}^a\tilde{x}^b, \qquad v = \tilde{v}, \qquad x^i = e^i_a(\tilde{v})\tilde{x}^a.$$

In the old coordinates the Rosen frame is of the form $\theta^\alpha = \Lambda^\alpha{}_\beta\tilde{\theta}^\beta$ with

$$\Lambda^\alpha{}_\beta = \begin{pmatrix} 1 & 0 & -F_{bc}\tilde{x}^c \\ 0 & 1 & 0 \\ 0 & (e^{-1}\dot{e})^a{}_c\tilde{x}^c & 1 \end{pmatrix}$$

The matrix $\Lambda^\alpha{}_\beta$ is a local Lorenz transformation if

$$g^{+a} = \Lambda^+{}_\alpha\Lambda^a{}_\beta g^{\alpha\beta} = \Lambda^a{}_- + \Lambda^+{}_b g^{ab} = (e^{-1}\dot{e} - F)^a{}_b\tilde{x}^b = 0.$$

That is, if

$$F = e^{-1}\dot{e}$$

Here the dot denotes derivative with respect to v. It should be noted that, because of the special form of h one obtains, this applies to the special case of plane waves.

The matrix $F^a{}_b(v)$ is an arbitrary real element of the algebra M_2 of complex 2×2 matrices, which we write

$$F = S + A$$

as the sum of a symmetric part and an antisymmetric part. The latter defines a rotation in the plane transverse to the null coordinates. The non-vanishing components of the connection form are given by

$$\omega^a{}_- = -S^a{}_b\theta^b, \qquad \omega^+{}_a = S_{ab}\theta^b, \qquad \omega^a{}_b = A^a{}_b\theta^-.$$

The curvature 2-form has

$$\Omega^a{}_- = -(\dot{S}^a{}_b + [A,S]^a{}_b - (S^2)^a{}_b)\theta^-\theta^b, \qquad \Omega^a{}_b = 0$$

as non-vanishing components. The non-vanishing components of the Riemann tensor are given by

$$R^a{}_{--b} = -(\dot{S} + [A,S] - S^2)^a{}_b$$

as well as those obtained from symmetries. The only non-vanishing component of the Ricci tensor is

$$R_{--} = -\mathrm{Tr}(\dot{S} + [A,S] - S^2).$$

So the field equations reduce to the condition that the trace of a matrix be constant. The solution has three arbitrary functions; two dynamical ones in S and a gauge degree of freedom in A.

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