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BPS Lumps and Their Intersections in $\mathcal{N}=2$ SUSY Nonlinear Sigma Models

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Abstract

BPS lumps in $\mathcal{N}=2$ SUSY nonlinear sigma models on hyper-Kähler manifolds in four dimensions are studied. We present new lump solutions with various kinds of topological charges. New BPS equations and a new BPS bound, expressed by the three complex structures on hyper-Kähler manifolds, are found. We show that any states satisfying these BPS equations preserve 1/8 (1/4) SUSY of $\mathcal{N}=2$ SUSY nonlinear sigma models with (without) a potential term. These BPS states include non-parallel multi-(Q-)lumps.

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1 Introduction

Topological solitons saturating an energy bound, called the BPS bound [1, 2], have played a crucial role in non-perturbative studies of supersymmetric (SUSY) field theories in four dimensions. Domain walls are topological solitons of co-dimension one, which depend on one spatial coordinate and connect two SUSY vacua. Since they preserve half of the original SUSY in $\mathcal{N}=1$ SUSY theories, they are called 1/2 BPS states [3]. Such BPS domain walls were well studied in various models with $\mathcal{N}=1$ SUSY [4, 5]. The intersections or junctions of domain walls preserve 1/4 SUSY in $\mathcal{N}=1$ models [6]–[11]. For the case of $\mathcal{N}=2$ SUSY theories, the nontrivial interactions for hypermultiplets can exist only in nonlinear sigma models. Target spaces of $\mathcal{N}=2$ SUSY nonlinear sigma models must be hyper-Kähler (HK) manifolds [12], and the only possible potential term is given by the square of a tri-holomorphic Killing vector acting on the HK manifold [13, 14]. These models are called "massive HK nonlinear sigma models". Single or parallel domain walls in such models preserve 1/2 SUSY [15, 16], whereas their intersections preserve 1/4 SUSY [17].

On the other hand, there are topological solitons of co-dimension two, called lumps [18]. Lumps can exist in nonlinear sigma models whose target spaces are Kähler manifolds. By the dimensional reductions, along the direction of the lumps, to (2 + 1)- or (2 + 0)-dimensions, the lumps reduce to particle-like solitons [18] or instantons [19], respectively. The degree of the map from two-dimensional space to the Kähler manifold M is expressed by the topological number, taking a value in the second homotopy class, $\pi_2(M)$ [19, 20]. An instanton can exist when $\pi_2(M) \neq 0$, and moreover if $\pi_2(M) = \mathbf{Z}$, multi-instantons with the higher topological number can exist. The configuration of multi-instantons in two dimensions corresponds to the configuration of *parallel* multi-lumps in four dimensions. They preserve half of SUSY in the cases of SUSY nonlinear sigma models. The BPS bound is given by the pull-back of the Kähler form (a linear combination of the triplet of the Kähler forms) in Kähler (HK) manifolds to the real space. In the cases of the massive HK nonlinear sigma models, no static lumps can exist, but stationary lumps, called the "Q-lumps" can exist [21, 22]. They preserve 1/4 SUSY and carry a Noether charge besides the topological charges [23]. Recently, it has also been shown in [23] that lumps can end on a domain wall in massive HK nonlinear sigma models in four dimensions, and this configuration is a 1/4 BPS state. However, no 1/8 BPS state has been found, although HK nonlinear sigma models have eight supercharges. These lumps can be considered a field theory realization of D-strings. Therefore we expect multi-lumps and/or intersections of lumps to exist. In this paper, we concentrate on the (Q-)lumps in (massive) HK nonlinear sigma models in four dimensions. As described above, they are 1/2 BPS states and the BPS bound is given by the topological number represented by the pull-back of a linear combination of the triplet of Kähler forms on the HK manifolds [24]. However, only known solution has been the lump in the Eguchi-Hanson gravitational instanton [25], which wraps around the two-cycle when interpolating two singularities of the manifold.

The purpose of this paper is to work out the BPS equations suitable for the the 1/4 (1/8) BPS states of non-parallel multi-lumps or intersections of lumps in the $\mathcal{N} = 2$ SUSY massless (massive) HK nonlinear sigma models. We find new BPS equations and a new BPS bound given by the sum of three kinds of the pull-backs of the three Kähler forms of the HK manifold to the three *independent* planes in the real space. We also show that any solutions of these BPS equations preserve 1/4 (1/8) SUSY in (massive) HK nonlinear sigma models. This is the first example of a 1/8 BPS state in $\mathcal{N} = 2$ SUSY field theories. Although we do not yet find concrete solutions for such configurations, we expect that these states are realized by the configurations of non-parallel (Q-)lumps or (Q-)lump intersections, because of the following four reasons. First, our new BPS equations admit a parallel configuration of lumps as a particular 1/2 BPS solution. The solutions of our 1/4 (1/8) BPS equations carry three independent topological charges, as required by the non-parallel multi-lump configurations. They preserve the *same* 1/4 (1/8) SUSY as the non-parallel multi-(Q-)lump configurations. We can construct an explicit solution of our 1/4 (1/8) BPS equations as multi-(Q-)lump intersections at least when we consider a direct product of ALE spaces where lumps are not interacting each other. These features suggests that our new BPS equations should contain the non-parallel multi-(Q-)lump configurations as solutions. We find that non-parallel (Q-)lumps requires HK metric with at least three centers and give new lump solutions in multi-center asymptotically locally Euclidean (ALE) space of Gibbons-Hawking [26]. These are not trivial extensions, since lump solutions connecting *different* pairs of centers preserve *different* combinations of the original SUSY. By analyzing the asymptotic behavior of multi-lump configurations, we find that the 1/4 (1/8) SUSY requires that the direction of lumps in base space needs to be correlated with the choice of complex structures.

In sect. 2 we give a brief review of HK nonlinear sigma models and the general properties of the lump in these models. Then we work out the lump solutions in multi-center metric of HK target space, and discuss the relation between the possible configurations of lumps and centers in metric. In sect. 3 we derive the BPS equation and BPS bound for 1/4 (1/8) BPS state in $D = 4$ (massive) HK nonlinear sigma models. Moreover we discuss the SUSY condition for the non-parallel lumps, and we confirm that 1/4 (1/8) SUSY is the same as the new state. Then we discuss that the new states can be expected to be the lump intersections. Summary and discussions are given in sect. 4.

2 1/2 BPS lumps in D=4 hyper-Kähler nonlinear sigma models

2.1 Hyper-Kähler manifold

The target space of the $\mathcal{N}=2$ SUSY nonlinear sigma models in four dimensions must be HK manifolds [12]. Let ϕ^X $(X=1,2,..,4n)$ be the real coordinates of a $4n$-dimensional HK manifold, and $\mathrm{I}^{(s)X}{}_Y$ $(s=1,2,3)$ be the triplet of the complex structures. We use the vielbein formalism to describe a HK manifold. The 4n-beins $f_X^{ia}(\phi)$ have a pair of the tangent indices (ia) and transform as a $(2,2n)$-representation of $Sp_1\times Sp_n$ under the local rotation in the tangent space of the HK manifold. The HK metric can be written in terms of $4n$-beins as

$$g_{XY}(\phi)=f_X^{ia}(\phi)f_Y^{jb}(\phi)\Omega_{ij}\epsilon_{ab}\ , \tag{2.1}$$

where Ω_{ij} and ϵ_{ab} are the real anti-symmetric tensors of Sp_n and Sp_1, respectively. The inverse of the 4n-beins are f_{ia}^X: the equations $f_X^{ia}f_{ia}^Y=\delta_X^Y$ and $f_X^{ia}f_{jb}^X=\delta_j^i\delta_b^a$ hold. Another identity is also valid

$$f_X^{ia}f_{ib}^Y=\frac{1}{2}(\delta_X^Y\delta_b^a-i\vec{\sigma}_b{}^a\cdot\vec{\mathrm{I}}^Y{}_X)\ , \tag{2.2}$$

where $\vec{\sigma}$ is the triplet of Pauli matrices and $\vec{\mathrm{I}}$ is the triplet of complex structures. The triplet of Kähler forms associated with three complex structures, $\Omega_{XY}^{(s)}=g_{XZ}I^{(s)Z}{}_Y$, can be written as

$$\vec{\Omega}=-\frac{i}{2}d\phi^X\wedge d\phi^Y f_X^{ia}f_{Yib}\vec{\sigma}_a{}^b\ . \tag{2.3}$$

2.2 D=4 HK nonlinear sigma models

One can obtain the $D=4$ HK nonlinear sigma model from $D=6$ HK nonlinear sigma model by a dimensional reduction. Let ϕ^X be the scalar fields parameterizing a HK manifold, and χ_α^i be the fermionic partner. Here the index α transforms under $SU(4)^*$. It is known that there cannot exist any potential terms for $D=6$ hypermultiplets [14]. The Lagrangian of the $D=6$ HK nonlinear sigma model is given by [14]

$$\mathcal{L}=-\frac{1}{8}g_{XY}\partial^{\alpha\beta}\phi^X\partial_{\alpha\beta}\phi^Y-i\chi_{\alpha i}\mathcal{D}^{\alpha\beta}\bar{\chi}_\beta^i-\frac{1}{12}R_{ijkl}\chi_\alpha^i\chi_\beta^j\chi_\gamma^k\chi_\delta^l\varepsilon^{\alpha\beta\gamma\delta}\ , \tag{2.4}$$

where $\varepsilon^{\alpha\beta\gamma\delta}$ is the $SU(4)^*$ invariant tensor, and $\mathcal{D}^{\alpha\beta}$ is a covariant derivative. Here, we have taken the Minkowski metric as the 'mostly plus' signature. The SUSY transformation is given by

$$\delta\phi^X = if^X_{ia}\epsilon^{\alpha a}\chi^i_\alpha , \quad (2.5)$$

$$\delta\chi^i_\alpha = f^{ia}_X\partial_{\alpha\beta}\phi^X\epsilon^\beta_a - \delta\phi^X\omega^i_{Xj}\chi^j_\alpha , \quad (2.6)$$

where ϵ^α_a is a constant spinor parameter of $Sp_1 \times SU(4)^*$.

Let us derive the SUSY condition for a bosonic configuration, by requiring the SUSY transformation of the fermion to vanish. Written in terms of the standard Dirac matrices, such a condition becomes

$$\Gamma^m f^{ia}_X \partial_m \phi^X \epsilon_a = 0 , \quad (2.7)$$

where Γ^m $(m = 0, 1, ..., 5)$ are the $D = 6$ Dirac matrices, and ϵ_a is an Sp_1-Majorana and Weyl spinor, satisfying

$$B\epsilon^{*a} = \varepsilon^{ab}\epsilon_b , \quad (2.8)$$

$$\Gamma^{012345}\epsilon = \epsilon . \quad (2.9)$$

Here B and Γ^{012345} are given by $B = -i\rho^3 \otimes \sigma^1 \otimes \tau^2$ and $\Gamma^{012345} = \rho^3 \otimes \sigma^3 \otimes \tau^3$, in the standard representation of the Dirac matrices. Then the spinor parameters ϵ_a of the SUSY transformation are expressed, using the four complex parameters p, q, r, s, by

$$\begin{aligned} \epsilon_1 &= (p, 0, 0, q, 0, r, s, 0) , \\ \epsilon_2 &= (-q^*, 0, 0, p^*, 0, -s^*, r^*, 0) . \end{aligned} \quad (2.10)$$

Multiplying (2.7) by f^Y_{ib} and using the identity (2.2), the SUSY condition is equivalent to [17]

$$\Gamma^m\epsilon\partial_m\phi^Y g_{YX} - i\Gamma^m\vec{\sigma}\epsilon\cdot\vec{\Omega}_{XY}\partial_m\phi^Y = 0 . \quad (2.11)$$

In four dimensions, massless HK nonlinear sigma models can be obtained by the trivial dimensional reduction: we obtain the SUSY condition by simply neglecting the terms labeled by $m = 4, 5$ in Eq. (2.11). On the other hand, massive HK nonlinear sigma models can be obtained by the Scherk-Schwarz reduction [12, 14, 17], given by

$$\partial_4\phi^X = \mu k^X(\phi), \quad \partial_5\phi^X = 0 , \quad (2.12)$$

where $k^X(\phi)$ is a tri-holomorphic Killing vector of the HK manifold, and μ is a mass parameter. We have the scalar potential term

$$V = \frac{\mu^2}{2} g_{XY} k^X(\phi) k^Y(\phi) , \quad (2.13)$$

in addition to the Lagrangian of the massless HK nonlinear sigma model. This type of the potential is the only possibility compatible with $\mathcal{N} = 2$ SUSY [17]. In this case, the SUSY condition can be obtained by substituting (2.12) into the condition (2.11).

2.3 Lumps and BPS condition

The lump is a string-like topological soliton in $D = 4$ HK nonlinear sigma models. First we derive the BPS equation for a static lump in massless HK nonlinear sigma models by minimizing the energy density $\mathcal{E}$. Choosing one complex structure $\mathrm{I}^{(s)}$ and the direction of the lump to be along x^3, we find

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2}g_{XY}(\dot{\phi}^X\dot{\phi}^Y + \partial_1\phi^X\partial_1\phi^Y + \partial_2\phi^X\partial_2\phi^Y + \partial_3\phi^X\partial_3\phi^Y) \\
&= \frac{1}{2}g_{XY}\dot{\phi}^X\dot{\phi}^Y + \frac{1}{2}g_{XY}\partial_3\phi^X\partial_3\phi^Y + \frac{1}{2}g_{XY}(\partial_1\phi^X \mp \mathrm{I}^{(s)X}{}_Z\partial_2\phi^Z)(\partial_1\phi^Y \mp \mathrm{I}^{(s)Y}{}_W\partial_2\phi^W) \\
&\quad \pm\Omega^{(s)}_{XY}\partial_1\phi^X\partial_2\phi^Y \ , \qquad (2.14)
\end{aligned}$$

where dots denote the differentiations with respect to the time coordinate. Therefore the BPS equations are $\partial_1\phi^X = \pm\mathrm{I}^{(s)X}{}_Y\partial_2\phi^Y$, $\partial_3\phi^X = 0$, $\dot{\phi}^X = 0$. If we denote the plane perpendicular to the direction x^k of the lump in base space as (x^i, x^j) and the invariant pseudo-tensor in the plane as ε_{ij}, the BPS equations are generalized to

$$\partial_i\phi^X = \pm\mathrm{I}^{(s)X}{}_Y\partial_j\phi^Y, \quad \partial_k\phi^X = 0, \quad \dot{\phi}^X = 0 \ , \qquad (2.15)$$

and the BPS bound becomes

$$L^{(s)} = \left|\frac{1}{2}\int d^3x\Omega^{(s)}_{XY}\varepsilon_{ij}\partial_i\phi^X\partial_j\phi^Y\right| \ . \qquad (2.16)$$

The integrand in the absolute value in right-hand-side of this equation is the pull-back of the Kähler form, $\Omega^{(s)}_{XY} = g_{XZ}\mathrm{I}^{(s)Z}{}_Y$, to the plane in the real space.

The BPS equations (2.15) satisfy the SUSY condition. We can get these BPS equations by requiring the SUSY conservation (2.11) for the supercharge corresponding to spinor parameter satisfying

$$\Gamma^{ij}\sigma^{(s)}\epsilon = \pm i\epsilon \ , \qquad (2.17)$$

Therefore, a static lump is a 1/2 BPS state. Note that the condition (2.17) for supercharge corresponds to the lump solution which depends on the plane of (x^i, x^j) and whose topological charge is the pull-back of the Kähler form associated with the complex structure $\mathrm{I}^{(s)}$. Consider a lump which is extended along the direction of (θ, ϕ) in the spherical coordinates of the real space, and has the topological charge associated with a linear combination of the three complex structures, $\alpha\mathrm{I}^{(1)} + \beta\mathrm{I}^{(2)} + \gamma\mathrm{I}^{(3)}$. Then, using a three vector $\vec{m} = (\alpha, \beta, \gamma)$, the SUSY condition of this lump is given by

$$\begin{aligned}
\tilde{\Gamma}(\vec{\sigma}\cdot\vec{m})\epsilon &= \tilde{\Gamma}(\alpha\sigma^{(1)} + \beta\sigma^{(2)} + \gamma\sigma^{(3)})\epsilon = \pm i\epsilon \ , \qquad (2.18) \\
\tilde{\Gamma} &\equiv \cos\theta\Gamma^{23} + \sin\theta\cos\phi\Gamma^{31} + \sin\theta\sin\phi\Gamma^{12} \ . \qquad (2.19)
\end{aligned}$$

The topological charge of this lump becomes

$$\alpha L^{(1)} + \beta L^{(2)} + \gamma L^{(3)} \equiv \vec{m} \cdot \vec{L} \ . \tag{2.20}$$

We thus have seen that $\vec{m}$ characterizes the linear combination of the complex structure associated with topological charge of the lump.

Next we discuss the massive HK nonlinear sigma models. In these models, the potential term (2.13) should be added to the energy density (2.14). We show that the BPS equations, the BPS bound and the conserved supercharges also change. The third equation of the BPS equations (2.15) is replaced by [22]

$$\dot{\phi}^X = \pm \mu k^X(\phi) \ , \tag{2.21}$$

using a tri-holomorphic Killing vector of the HK target manifold, $k^X(\phi)$. The BPS bound is replaced to $L^{(s)} + Q$, where Q is a conserved Noether charge, induced by the tri-holomorphic Killing vector:

$$Q = \left| \mu \int d^3x g_{XY} \dot{\phi}^X k^Y(\phi) \right| \ . \tag{2.22}$$

In spite of the time-dependent configuration of this lump, the energy distribution is independent of the time. This situation is called stationary. This lump in massive HK nonlinear sigma models is called a "Q-lump" [22]. Supercharges conserved by a Q-lump is given by Eq. (2.17) and

$$\Gamma^{04}\epsilon = \mp \epsilon \ , \tag{2.23}$$

which corresponds to Eq. (2.21). Therefore we find that a Q-lump is a 1/4 BPS state from Eqs. (2.17) and (2.23).

2.4 Lump solution in multi-center metric

2.4.1 Multi-center models

In this section, we shall consider a special class of SUSY nonlinear sigma models on a four-dimensional ALE space of the multi-center Gibbons-Hawking metric [26]. Let the coordinates be $(\psi, \vec{X})$, with $\vec{X}$ being a three-vector, and $U(X)$ be a harmonic function. Then, the metric is given by

$$ds^2 = U d\vec{X} \cdot d\vec{X} + U^{-1}(\mathcal{D}\psi)^2 \ , \tag{2.24}$$

where $\mathcal{D}_k\psi \equiv \partial_k\psi + \partial_k\vec{X}\cdot\vec{A}$ and $\vec{A}$ is related to U by $\vec{\nabla}\times\vec{A} = \vec{\nabla}U$. The triplet of the Kähler forms (2.3) is expressed by

$$\vec{\Omega} = (d\psi + d\vec{X}\cdot\vec{A})d\vec{X} - \frac{1}{2}Ud\vec{X}\times\vec{X} \ . \tag{2.25}$$

We shall choose the harmonic function as

$$U = \frac{1}{2}\sum_{i=1}^{N}\frac{1}{|\vec{X}-\vec{n}_i|} \ , \tag{2.26}$$

where $\vec{n}_i$ $(i = 1,\cdots N)$ are unit three-vectors. The function U is singular at $\vec{X} = \vec{n}_i$, called the centers, but these are coordinate singularities of the metric if ψ is periodically identified with period 2π. The ALE space can be considered as a S^1-bundle over the three vector space $\vec{X}$, with a fiber being parametrized by ψ. The isometry of $U(1)$ acts on the S^1 coordinate ψ, and its action has fixed points at the centers. Hence, a segment connecting each pair of two centers with ψ parameterize a sphere S^2 as a submanifold. These spheres are called the 2-cycles, and represent a nontrivial element of $\pi_2(M)$. For the function U of Eq. (2.26), we can set without loss of generality (see, e.g., [27])

$$\begin{aligned} A^{(1)} &= \frac{1}{2}\sum_{i=1}^{N}\frac{X^{(2)}-n_i^{(2)}}{|\vec{X}-\vec{n}_i|(X^{(3)}-n_i^{(3)}-|\vec{X}-\vec{n}_i|)} \ , \\ A^{(2)} &= \frac{1}{2}\sum_{i=1}^{N}\frac{-(X^{(1)}-n_i^{(1)})}{|\vec{X}-\vec{n}_i|(X^{(3)}-n_i^{(3)}-|\vec{X}-\vec{n}_i|)} \ , \\ A^{(3)} &= 0 \ , \end{aligned} \tag{2.27}$$

where we have chosen a gauge of $A^{(3)} = 0$. In this multi-center models, SUSY condition of Eq. (2.11) becomes

$$[\Gamma^m\vec{\sigma}\cdot\partial_m\vec{X} + iU^{-1}\Gamma^m\mathcal{D}_m\psi]\epsilon = 0 \ . \tag{2.28}$$

2.4.2 General lump solution as holomorphic map

Let us begin with the lump solution, extended along the x^1-axis (the configuration independent of x^1), and carrying the topological charge associated with complex structure $\mathrm{I}^{(1)}$. This lump conserves 1/2 SUSY for supercharges determined by

$$\Gamma^{23}(\vec{\sigma}\cdot\vec{n})\epsilon = -i\epsilon, \quad \text{for } \vec{n} = (1,0,0) \ . \tag{2.29}$$

From the condition (2.28) for this 1/2 SUSY state, we get the BPS equations for the lump, given by

$$\begin{aligned}
\partial_k X^{(2)} &= \partial_k X^{(3)} = 0 \quad \text{for } k = 1, 2, 3 \;, \\
\partial_1 X^{(1)} &= \mathcal{D}_1 \psi = 0 \;, \\
\mathcal{D}_2 \psi &= U \partial_3 X^{(1)} \;, \\
\mathcal{D}_3 \psi &= -U \partial_2 X^{(1)} \;.
\end{aligned} \tag{2.30}$$

From the first and second of these equations, we find that $X^{(2)} = X^{(3)} =$ constant, and that $X^{(1)}$ and ψ are independent of x^1. Then the third and fourth can be combined together as

$$(\partial_2 + i\partial_3)[u - i(\psi + v)] = 0 \;, \tag{2.31}$$

where we have defined real functions u and v by

$$u(X^{(1)}) \equiv \int dX^{(1)} U(X^{(1)}), \quad v(X^{(1)}) \equiv \int dX^{(1)} A^{(1)}(X^{(1)}) \;. \tag{2.32}$$

Therefore, the BPS equation for a lump can be reinterpreted as a *holomorphic map* from the complex plane $z \equiv x^2 + ix^3$ to the target space. The implicit form of the solution can be written as

$$\exp(u - i(\psi + v)) = Z(z) \;, \tag{2.33}$$

where $Z(z)$ is a holomorphic function. As a class of the solution, we can choose the holomorphic function as $Z = (z - z_0)^{-\alpha}$. Noting the reality of u, v and ψ, we get

$$\begin{aligned}
X^{(1)} &= u^{-1}\left(\frac{\alpha}{2} \log \frac{1}{|z - z_0|^2}\right) \;, \\
\psi &= \alpha \cdot \arg(z - z_0) - v(X^{(1)}) \;.
\end{aligned} \tag{2.34}$$

Since ψ is a periodic variable identification $\psi + 2\pi = \psi$, the parameter α should be an integer, $\alpha \in \mathbf{Z}$. Therefore, we can interpret α as the topological number of the lump, taking values in $\pi_2(M)$. Since $X^{(1)}$ depends only on the absolute value $|z - z_0|$, we can interpret the parameter z_0 as the position of the center of the lump in the z-plane. The BPS bound for this solution can be written in the polar coordinates as

$$E \geq -\pi\alpha \int dx^1 \int dr \partial_r X^{(1)} = -\pi\alpha \int dx^1 [X^{(1)}(r = \infty) - X^{(1)}(r = 0)], \tag{2.35}$$

where $r \equiv \sqrt{(x^2 - x_0^2)^2 + (x^3 - x_0^3)^2}$. We see that the lump has an energy bound proportional to the topological number α.

From Eq. (2.34), we see that $X^{(1)}$ approaches to $u^{-1}(\infty \mathrm{sign}(\alpha))$ in the limit of $r \to 0$, and to $u^{-1}(-\infty \mathrm{sign}(\alpha))$ in the limit of $r \to \infty$. Therefore both of $u^{-1}(\infty)$ and $u^{-1}(-\infty)$ must take finite values for the finiteness of the topological charge of the lump.

Next, we discuss the Q-lump in the massive HK nonlinear sigma models. As a tri-holomorphic Killing vector for the potential (2.13), we can choose the $U(1)$ isometry acting on ψ as a constant shift. Thus additional BPS equation for the Q-lump, Eq. (2.21), becomes

$$\dot{\psi} = \pm \mu. \tag{2.36}$$

Therefore Q-lump solution can be obtained from the static lump-solution of Eq. (2.34), replacing ψ by

$$\psi = \alpha \cdot \arg(z - z_0) - v(X^{(1)}) \pm \mu t \ . \tag{2.37}$$

2.4.3 Lump in 2-center models

Let us consider the simplest case of multi-center models, the 2-center metric. First, we set the two centers $\vec{n}_1 = (1, 0, 0)$ and $\vec{n}_2 = (-1, 0, 0)$. Setting $X^{(2)} = X^{(3)} = \text{constant} = 0$, the harmonic function becomes

$$U = \frac{1}{2}\left(\frac{1}{|X^{(1)} - 1|} + \frac{1}{X^{(1)} + 1}\right) = \frac{1}{1 - X^{(1)2}} \ , \quad A^{(1)} = 0 \ . \tag{2.38}$$

Then the equation $u(X^{(1)}) = \tanh^{-1}(X^{(1)})$ holds, and the solution of the lump can be obtained as [23],

$$\begin{aligned} X^{(1)} &= \tanh\left(\frac{\alpha}{2} \log \frac{1}{|z - z_0|^2}\right) \ , \\ \psi &= \alpha \cdot \arg(z - z_0) \ . \end{aligned} \tag{2.39}$$

Two limits of $X^{(1)}$ correspond to the positions of two centers, and therefore finite: $X^{(1)} = 1$ for $r \to 0$, and $X^{(1)} = -1$ for $r \to \infty$. Hence, the topological charge of the lump is finite. We note that the parameter α determines the size of lump as well as the topological number: we can see from Eq. (2.39) that the size of lump is greater as the topological number gets larger.

It is important to realize that the finiteness of the topological charge imposes a severe constraint on the direction of the lump in base space x^1, x^2, x^3 and the direction of the space-dependent component of the field $\vec{X}$. To illustrate the point, let us consider, for example, a lump solution in the case of two centers placed at $\vec{n}_1 = (0, 1, 0)$ and $\vec{n}_2 = (0, -1, 0)$. The BPS equation (2.30) implies that a lump can be constructed only by choosing $X^{(1)}$ space-dependent. In this case, the space-dependent component of $\vec{X}$ takes the *different* direction in the field space from the vector connecting the two centers in the metric. Setting $X^{(2)} = X^{(3)} = 0$, we obtain

$$U = \log(\sqrt{(X^{(1)})^2 + 1} + X^{(1)}), \quad A^{(1)} = 0 \ . \tag{2.40}$$

The implicit form of the solution of lump can be obtained as

$$\begin{aligned} \frac{\alpha}{2} \log \frac{1}{|z - z_0|^2} &= \log(\sqrt{(X^{(1)})^2 + 1} + X^{(1)}) \ , \\ \psi &= \alpha \cdot \arg(|z - z_0|) \ . \end{aligned} \tag{2.41}$$

However $X^{(1)}$ does not take finite values in the limit of both $r \to 0$ and $r \to \infty$. Therefore the solution gives a divergent topological charge and is unacceptable. To obtain a finite topological charge, it is necessary to align the direction of the lump in the base space with the direction connecting two centers of the metric.

We can understand this situation from the topological point of view. A lump is an axially-symmetric soliton, and the center of the lump ($r = 0$) and the infinity around the lump ($r = \infty$) are mapped to two different points in the target space M, respectively. When these two points coincide with two different centers in the target space respectively as in the case of Eq. (2.39), the map is closed and the lump is wrapped around a 2-cycle. On the other hand, when any one of these two points are not mapped to a center as in the case of Eq. (2.41), the map is not closed. We have shown that if and only if the map is closed, the the lump charge is finite, and takes a value in $\pi_2(M)$. A similar requirement on the finiteness of the topological charge was previously considered before [28, 22].

2.4.4 Lumps in 3-center models

We see that the direction connecting the two centers must be aligned with the direction of the complex structure characterizing the lump. Hence, to study the non-parallel lumps or lump intersections, we need to consider the multi-center model where centers are not aligned. Let us consider 3-center models of the four-dimensional HK metric as the simplest example.

As an example, we can work out an implicit solution of lumps in the case of the metric with three centers st $\vec{n}_1 = (0, 1, 0)$, $\vec{n}_2 = (1, 0, 0)$ and $\vec{n}_3 = (-1, 0, 0)$. We consider two kinds of lumps in this model: The first lump corresponds to the line segment connecting the two centers at $\vec{n}_1$ and $\vec{n}_2$ in the field space $\vec{X}$, and the second lump corresponds to the two centers at $\vec{n}_1$ and $\vec{n}_3$.

Now we find the solution of the first lump. For this purpose, we transform the variable $\vec{X}$ to

$\vec{Y}$ by

$$\begin{pmatrix} Y^{(1)} \\ Y^{(2)} \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} X^{(1)} - X^{(2)} + 1 \\ X^{(1)} + X^{(2)} - 1 \end{pmatrix} . \tag{2.42}$$

In the basis of $\vec{Y}$, lines connecting two centers $\vec{n}_1$ and $\vec{n}_2$ are along the $Y^{(1)}$ axis in the field space. Therefore $Y^{(1)}$ becomes space-dependent for the first lump. Setting $Y^{(2)} = Y^{(3)} = 0$, we have

$$\begin{aligned} U &= \frac{1}{2}\left[\frac{1}{\sqrt{(Y^{(1)})^2}} + \frac{1}{\sqrt{(Y^{(1)} - \sqrt{2})^2}} + \frac{1}{\sqrt{(Y^{(1)})^2 + 2}}\right] \\ &= \frac{1}{2}\left[\frac{2\sqrt{2}}{1 - (\sqrt{2}Y^{(1)} - 1)^2} + \frac{1}{\sqrt{(Y^{(1)})^2 + 2}}\right] , \\ A^{(1)} &= -\frac{1}{2}\frac{\sqrt{2}}{(Y^{(1)})^2 + 2} . \end{aligned} \tag{2.43}$$

Then the functions u and v are given from Eq. (2.32) by

$$u = \tanh^{-1}(\sqrt{2}Y^{(1)} - 1) + \frac{1}{2}\log(Y^{(1)} + \sqrt{(Y^{(1)})^2 + 2}), \quad v = -\frac{1}{2}\tan^{-1}\left(\frac{Y^{(1)}}{\sqrt{2}}\right) . \tag{2.44}$$

We can find the implicit solution of the first lump from Eq. (2.34):

$$\begin{aligned} -\frac{\alpha}{2}\log|z - z_0|^2 &= \tanh^{-1}(X^{(1)} - X^{(2)}) \\ &\quad + \frac{1}{2}\log\left(\frac{1}{\sqrt{2}}(X^{(1)} - X^{(2)} + 1) + \sqrt{\frac{1}{2}(X^{(1)} - X^{(2)} + 1)^2 + 2}\right) , \\ \psi &= \alpha \cdot \arg(z - z_0) + \frac{1}{2}\tan^{-1}\left(\frac{1}{2}(X^{(1)} + X^{(2)} - 1)\right) \end{aligned} \tag{2.45}$$

where $z \equiv x^2 + ix^3$. The second lump is obtained similarly to give

$$\begin{aligned} -\frac{\alpha}{2}\log|z - z_0|^2 &= \tanh^{-1}(X^{(1)} + X^{(2)}) \\ &\quad + \frac{1}{2}\log\left(\frac{1}{\sqrt{2}}(X^{(1)} + X^{(2)} - 1) + \sqrt{\frac{1}{2}(X^{(1)} + X^{(2)} - 1)^2 + 2}\right) , \\ \psi &= \alpha \cdot \arg(z - z_0) + \frac{1}{2}\tan^{-1}\left(\frac{1}{2}(X^{(1)} - X^{(2)} + 1)\right) . \end{aligned} \tag{2.46}$$

We can extend these solutions to lumps oriented to other directions in the multi-center model straightforwardly.

3 BPS equation for coexisting lumps

In the last section, we have found the several kinds of lumps with different topological charges. Each (Q-)lump conserves different 1/2 (1/4) SUSY in massless (massive) HK nonlinear sigma models. In this section, we derive new BPS equations and BPS bound in $D=4$ HK nonlinear sigma models. Any solutions of these BPS equations conserve 1/4 SUSY (or 1/8 SUSY in massive HK nonlinear sigma models). We also derive conditions for coexisting lumps to preserve 1/4 SUSY (or 1/8 in massive theory).

3.1 New BPS bound and BPS equation in HK nonlinear sigma models

The previous 1/2 BPS equation (2.15) used only one complex structure. Since there are three complex structures in the HK manifolds, it should be possible to use all three complex structures to derive a new BPS bound. For that purpose, we consider a configuration in $D=4$ HK nonlinear sigma models which depends on three independent coordinates x^1, x^2, x^3 in base space. The energy density of this solitonic configuration can be rewritten as

$$\begin{aligned}
\mathcal{E} &= \frac{1}{2} g_{XY}(\dot{\phi}^X\dot{\phi}^Y + \partial_1\phi^X\partial_1\phi^Y + \partial_2\phi^X\partial_2\phi^Y + \partial_3\phi^X\partial_3\phi^Y) \\
&= \frac{1}{2} g_{XY}\dot{\phi}^X\dot{\phi}^Y \\
&\quad + \frac{1}{2} g_{XY}(\partial_1\phi^X - \tilde{\mathrm{I}}^{(2)X}{}_Z\partial_3\phi^Z - \tilde{\mathrm{I}}^{(3)X}{}_Z\partial_2\phi^Z)(\partial_1\phi^Y - \tilde{\mathrm{I}}^{(2)Y}{}_W\partial_3\phi^W - \tilde{\mathrm{I}}^{(3)Y}{}_W\partial_2\phi^W) \\
&\quad + \Omega^{(1)}_{XY}\partial_2\phi^Y\partial_3\phi^X + \tilde{\Omega}^{(2)}_{XY}\partial_3\phi^Y\partial_1\phi^X + \tilde{\Omega}^{(3)}_{XY}\partial_2\phi^Y\partial_1\phi^X \\
&\geq \Omega^{(1)}_{XY}\partial_2\phi^Y\partial_3\phi^X + \tilde{\Omega}^{(2)}_{XY}\partial_3\phi^Y\partial_1\phi^X + \tilde{\Omega}^{(3)}_{XY}\partial_2\phi^Y\partial_1\phi^X \ .
\end{aligned} \tag{3.1}$$

where rotated complex structures $\tilde{\mathrm{I}}^{(2)} \equiv \vec{n}_2\cdot\vec{\mathrm{I}}$, $\tilde{\mathrm{I}}^{(3)} \equiv \vec{n}_3\cdot\vec{\mathrm{I}}$, and $\vec{n}_2 \equiv (0,\cos\omega,\sin\omega)$, $\vec{n}_3 \equiv (0,-\sin\omega,\cos\omega)$, are defined with a parameter ω. This BPS inequality is useful for the solitonic configuration which depends on three coordinates. Moreover, this inequality is saturated when the configuration satisfies the following BPS equations

$$\partial_1\phi^X = \tilde{\mathrm{I}}^{(2)X}{}_Y\partial_3\phi^Y + \tilde{\mathrm{I}}^{(3)X}{}_Y\partial_2\phi^Y \ , \quad \dot{\phi}^X = 0 \ . \tag{3.2}$$

In this case, the BPS bound for the soliton in Eq. (3.1) is written as a sum of three kinds of the pull-backs of Kähler forms to three planes in the real space, representing three topological charges.

Next we show that 1/4 SUSY remains unbroken for the solution of Eq. (3.2). By substituting Eq. (3.2) to the SUSY condition (2.11), we obtain

$$
\begin{aligned}
0 \;=\;& \mathrm{I}^{(1)X}{}_Y\partial_2\phi^Y[\Gamma^3 - i\Gamma^2\sigma^{(1)}]\epsilon + \mathrm{I}^{(1)X}{}_Y\partial_3\phi^Y[-\Gamma^2 - i\Gamma^3\sigma^{(1)}]\epsilon \\
&+\tilde{\mathrm{I}}^{(2)X}{}_Y\partial_3\phi^Y[\Gamma^1 - i\Gamma^3(\vec{\sigma}\cdot\vec{n}_2)]\epsilon + \tilde{\mathrm{I}}^{(2)X}{}_Y\partial_1\phi^Y[-\Gamma^3 - i\Gamma^1(\vec{\sigma}\cdot\vec{n}_2)]\epsilon \\
&+\tilde{\mathrm{I}}^{(3)X}{}_Y\partial_2\phi^Y[\Gamma^1 - i\Gamma^2(\vec{\sigma}\cdot\vec{n}_3)]\epsilon + \tilde{\mathrm{I}}^{(3)X}{}_Y\partial_1\phi^Y[-\Gamma^2 - i\Gamma^1(\vec{\sigma}\cdot\vec{n}_3)]\epsilon \\
&-i\mathrm{I}^{(1)X}{}_Y\partial_1\phi^Y[\Gamma^1\sigma^{(1)} + \Gamma^3(\vec{\sigma}\cdot\vec{n}_3)]\epsilon - i\tilde{\mathrm{I}}^{(2)X}{}_Y\partial_2\phi^Y[\Gamma^2(\vec{\sigma}\cdot\vec{n}_2) + \Gamma^3(\vec{\sigma}\cdot\vec{n}_3)]\epsilon\,. \qquad (3.3)
\end{aligned}
$$

The first and the third terms should vanish, giving

$$\Gamma^{23}\sigma^{(1)}\epsilon = -i\epsilon\ , \quad \Gamma^{31}(\vec{\sigma}\cdot\vec{n}_2)\epsilon = -i\epsilon. \qquad (3.4)$$

These two conditions are sufficient to make all the other terms vanish. By calculating Eq. (3.4) for Dirac matrices in the standard representation, the conserved supercharges are given by the following relations for the components of the spinor parameters in Eq. (2.10)

$$q = -p^*,\ r = -s^* \text{ and } p = -e^{-i\omega}s\ . \qquad (3.5)$$

From these relations, we see that 1/4 SUSY remains unbroken.

In the case of massive HK nonlinear sigma models, this soliton can carry the Noether charge (2.22) induced by the isometry of the target space, in addition to the three topological terms of the BPS bound in Eq. (3.1). Correspondingly, the second equation of Eq. (3.2) is replaced by Eq. (2.21), and the condition (2.23) is required as additional constraint for conserved supercharges. Eq. (2.23) is rewritten, in terms of the components of the spinor parameters, as

$$p = \mp i s^*. \qquad (3.6)$$

Eqs. (3.5) and (3.6) together imply that there is only one real parameter for the spinor transformation parameters. Therefore this soliton is 1/8 BPS state and conserves minimal SUSY in $\mathcal{N}=2$ four-dimensional massive HK nonlinear sigma models.

We expect that the solutions of the BPS equation Eq. (3.2) contain the intersecting or coexisting lumps for several reasons. The first reason is that the BPS bound is given by the sum of three kinds of the pull-backs of the Kähler forms to three planes in the real space. Each pull-back corresponds to the topological charge of the lump. The second is that the BPS equation of a single lump can be derived as the special case of Eq. (3.2). If we require the fields to be independent of the x^1, for example, we can recover, from Eq. (3.2), the BPS equation of the lump such as

$$\partial_1\phi^X = 0\ , \quad \partial_2\phi^X = -\mathrm{I}^{(1)X}{}_Y\partial_3\phi^Y. \qquad (3.7)$$

As the third reason, we confirm, in the next section, that the states for coexisting or intersecting lumps can conserve the *same* [1] 1/4 SUSY (1/8 SUSY in massive theory) as the new soliton that we discussed above. The fourth reason is that we can construct an explicit solution of multi-lump intersections as a 1/4 BPS state at least when we consider the direct product of ALE spaces, as discussed below.

3.2 BPS condition for coexistence of lumps

Let us consider generally the BPS condition for coexistence of two lumps here. For simplicity, the first lump is extended along the x^1-direction and is associated with the complex structure $\mathrm{I}^{(1)}$. The SUSY condition for this lump can be written as

$$\Gamma^{23}(\vec{\sigma}\cdot\vec{n})\epsilon = -i\epsilon \quad \text{for} \quad \vec{n} = (1,0,0)\ . \tag{3.8}$$

Next we consider the second lump and look for the condition for the second lump to retain a part of SUSY preserved by the first one. We try to put the second lump in the direction of (θ,ϕ) in the spherical coordinates. Hence, SUSY condition for the second lump is

$$\tilde{\Gamma}(\vec{\sigma}\cdot\vec{m})\epsilon = -i\epsilon \quad \text{for} \quad \vec{m} = (\cos\theta_c, \sin\theta_c\cos\phi_c, \sin\theta_c\sin\phi_c)\ , \tag{3.9}$$

where $\tilde{\Gamma}$ is the linear combination of the Dirac matrices given in Eq. (2.19). Since the coexistence of the two lumps requires the spinor parameter to satisfy the two constraints (3.8) and (3.9). By demanding partial conservation of SUSY, we find that the complex structure has to be correlated with the lump direction: $\theta_c = \theta$, and that the conserved SUSY is given in terms of spinor components in (2.10) as

$$q = -p^*,\ r = -s^* \text{ and } p = -e^{-i(\phi+\phi_c)}s, \tag{3.10}$$

in the standard representation of Dirac matrices. This equation corresponds to Eq.(3.5), by setting $\omega = \phi + \phi_c$. Therefore non-parallel multi-lump configurations with the complex structure satisfying the relation $\theta = \theta_c$ preserves 1/4 SUSY. In other words, a 1/4 SUSY is conserved when several lumps coexist or intersect with arbitrary angles, if the lump extended along the direction of (θ,ϕ) carries the topological charge corresponding to

$$\vec{m} = (\cos\theta, \sin\theta\cos(\omega-\phi), \sin\theta\sin(\omega-\phi)), \tag{3.11}$$

for a fixed parameter ω.

[1] Intersecting domain walls considered in Ref.[17] conserve *different* 1/4 SUSY.

We note that the Eq. (3.2) can allow the many solutions for coexisting lumps which are extended along the arbitrary directions in the real space, if the lumps pick up the particular complex structure as in Eq. (3.11): We can see from Eq. (3.9) that there is a two (real) parameter family (θ, ϕ) of lump configurations conserving the same supercharges as given in Eq. (3.5). In the case of coexisting Q-lumps, 1/8 SUSY corresponding to Eqs.(3.5) and (3.6) is conserved if each of the coexisting (intersecting) Q-lumps picks up the particular complex structure given in Eq. (3.11). Hence the coexisting Q-lumps conserve minimal SUSY in $\mathcal{N} = 2$ four-dimensional massive HK nonlinear sigma models.

3.3 Lump intersections and multi-center metric

We can see from Eq. (3.11) that, if non-parallel lumps coexist, such lumps must have different topological charges. But lumps with various topological charges cannot always exist in general HK nonlinear sigma models. We wish to consider the situation where several lumps with different topological charges can coexist. In the model with multi-center metric, several centers should not be placed on a single line in field space, since several lumps should wrap topologically different 2-cycles of the target space oriented along different directions. Conversely, possible directions of different lumps are determined by the positions of centers for a given model. We think that the 3-center model in previous section is a candidate of the simplest model of four-dimensional HK metric with possibility of two lumps coexisting perpendicularly.

An explicit solution of a 1/4 (1/8) BPS state of massless (massive) nonlinear sigma models can be realized as follows. First, we prepare a $4n$-dimensional target space as a direct product of n ALE spaces. In this case, fields are divided into n sectors, which are not interacting each other. This model admits a solution of n-lump intersection, in which each lump has proper correlation of (3.11). This provides an explicit example of a 1/4 (1/8) BPS state in massless (massive) nonlinear sigma models, although each lump is not interacting with the others. By introducing a constant in the off-diagonal block of the product metric, we may construct a more interesting example of interacting lumps, as was done in a wall intersection [17].

4 Summary and discussions

We have examined several kinds of (Q-)lumps with various topological charges in $D = 4$ hyper-Kähler SUSY nonlinear sigma models. Especially in the model with multi-center metric, we have worked out the implicit solutions for a general lump. In order to obtain a lump with finite energy density, the field $\vec{X}$ has to be aligned along the direction connecting two centers. Otherwise, the configuration does not wrap the 2-cycle connecting the two centers and the topological charge diverges.

Next, we have found the BPS equation and energy bound for a soliton which depends on three spatial coordinates in general HK nonlinear sigma models. The BPS bound is written as a sum of three kinds of the pull-backs of Kähler forms to three planes in the real space. We also show that solutions of the BPS equation corresponds to 1/4 BPS states, and the BPS equation admits a single lump solution preserving 1/2 SUSY as a special case. In massive HK nonlinear sigma models, the soliton conserves 1/8 SUSY.

Moreover, we also have considered the SUSY conditions for general coexisting or intersecting lumps. We found that 1/4 SUSY is conserved by coexisting lumps and 1/8 SUSY is conserved by coexisting Q-lumps, when the directions of lumps correlate properly to the topological charges carried by the lumps. Hence, we expect that new soliton corresponds to the intersecting or coexisting lumps. Partial conservation of SUSY requires that the polar angle θ_c in the space of complex structures has to be the same as the polar angle θ in the real space for the second lump relative to the first. The sum of the azimuthal angle ϕ_c in the space of complex structures and the azimuthal angle ϕ determines the conserved SUSY. Therefore we can consider 1/4 SUSY conservation of arbitrarily many lumps provided the above condition is met with a fixed $\omega = \phi_c + \phi$.

To find an analytic solution of coexisting lumps with different topological charges is a non-trivial problem. When we consider coexisting lumps in multi-center models of four-dimensional HK metric, the directions of lumps should be related to the directions of lines connecting pairs of centers in the multi-center metric. It is very interesting whether the solitons wrapped in different 2-cycles can coexist.

It has been found that lumps can end on the domain wall and that this configuration preserves 1/4 SUSY [23]. In Ref. [23], it has been argued that much of the physics of D-branes can appear as Q-lumps in a purely field theoretical context, in particular a D-string ending on a D-brane. Our results on the non-parallel or intersecting lumps will shed more light on the study of D-branes in a field theoretical context.

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References

[1] E. Bogomol'nyi, *Sov. J. Nucl. Phys.* **B24** (1976) 449; M. K. Prasad and C. H. Sommerfield, *Phys. Rev. Lett.* **35** (1975) 760.

[2] E. Witten and D. Olive, *Phys. Lett.* **B78** (1978) 97.

[3] J. A. de. Azcárraga, J. P. Gauntlett, J. M. Izquierdo and P. K. Townsend, *Phys. Rev. Lett.* **63** (1989) 2443.

[4] M. Cvetic, S. Griffies and S. Rey, *Nucl. Phys.* **B381** (1992) 301 [hep-th/9201007].

[5] G. Dvali and M. Shifman, *Phys. Lett.* **B396** (1997) 64 [hep-th/9612128]; *Nucl. Phys.* **B504** (1997) 127[hep-th/9611213]; A. Kovner, M. Shifman, and A. Smilga, *Phys. Rev.* **D56** (1997) 7978 [hep-th/9706089]; A. Smilga and A. Veselov, *Phys. Rev. Lett.* **79** (1997) 4529 [hep-th/9706217]; B. Chibisov and M. Shifman, *Phys. Rev.* **D56** (1997) 7990; V. Kaplunovsky, J. Sonnenschein, and S. Yankielowicz, *Nucl. Phys.* **B552** (1999) 209 [hep-th/9811195]; B. de Carlos and J. M. Moreno, *Phys. Rev. Lett.* **83** (1999) 2120 [hep-th/9905165]; G. Dvali, G. Gabadadze, and Z. Kakushadze, *Nucl. Phys.* **B493** (1997) 148, [hep-th/9901032]; J. Edelstein, M.L. Trobo, F. Brito and D. Bazeia, *Phys. Rev.* **D57** (1998) 7561 [hep-th/9707016]; M. Naganuma and M. Nitta, *Prog. Theor. Phys.* 105 (2001) 501 [hep-th/].

[6] E. R. C. Abraham and P. K. Townsend, *Nucl. Phys.* **B351** (1991) 313 ; M. Cvetic, F. Quevedo and S. Rey, *Phys. Rev. Lett.* **67** (1991) 1836.

[7] G. Gibbons and P. Townsend, *Phys. Rev. Lett.* **83** (1999) 1727 [hep-th/9905196]; S.M. Carroll, S. Hellerman and M. Trodden, *Phys. Rev.* **D61** (2000) 065001 [hep-th/9905217]; P.M. Saffin, *Phys. Rev. Lett.* **83** (1999) 4249-4252 [hep-th/9907066].

[8] H. Oda, K. Ito, M. Naganuma and N. Sakai, *Phys. Lett.* **B471** (1999) 148 [hep-th/9910095].

[9] K. Ito, M. Naganuma, H. Oda and N. Sakai, *Nucl. Phys.* **B586** (2000) 231 [hep-th/0004188]; *Nucl. Phys. Proc. Suppl.* 101 (2001) 304 [hep-th/0012182].

[10] A. Gorsky and M. Shifman, *Phys. Rev.* **D61** (2000) 085001 [hep-th/9909015]; *Nucl. Phys.* **B586** (2000) 231 [hep-th/0004188]; D. Binosi and T. ter Veldhuis, *Phys. Lett.* **B476** (2000) 124 [hep-th/9912081]; S. Nam and K. Olsen, *JHEP* **0008** (2000) 001 [hep-th/0002176].

[11] M. Naganuma, M. Nitta and N. Sakai, in preparation

[12] L. Alvarez-Gaumé and D. Z. Freedman, *Comm. Math. Phys.* **80** (1981) 443.

[13] L. Alvarez-Gaumé and D. Z. Freedman, *Comm. Math. Phys.* **91** (1983) 87.

[14] G. Sierra and P. K. Townsend, *Nucl. Phys.* B 233 (1984) 289,

[15] E. Abraham and P. K. Townsend, *Phys. Lett.* B 291 (1992) 85.

[16] J. P. Gauntlett, D. Tong and P. K. Townsend, *Phys. Rev.*D 64(2001)025010 [hep-th/0012178]

[17] J. P. Gauntlett, D. Tong and P. K. Townsend, *Phys. Rev.*D 63(2001)085001 [hep-th/0007124],

[18] R. S. Ward, *Phys. Lett.* **158B** (1985) 424.

[19] A. A. Belavin and A. M. Polyakov, *JETP Lett.* **22** (1975) 245.

[20] A. M. Perelomov, *Phys. Rep.*146(1987)135; *Phys. Rep.*174(1989)229,

[21] R. A. Leese, *Nucl. Phys.*B 366(1991)283,

[22] E. Abraham, *Phys. Lett.*B 278(1992)291,

[23] J. P. Gauntlett, R. Portugues, D. Tong and P. K. Townsend, *Phys. Rev.*D 63(2001)085002 [hep-th/0008221],

[24] P. J. Ruback, *Comm. Math. Phys.*116(1988)645,

[25] T. Eguchi and A. J. Hanson, *Phys. Lett.* **74B** (1978) 249.

[26] G. W. Gibbons and S. W. Hawking *Phys. Lett.* **78B** (1978) 430.

[27] T. Nakatsu, N. Ohta and T. Yokono, *Phys. Rev.* **D58** (1998) 26003 [hep-th/9712029].

[28] A. D'adda, M. Lüscher and P. Di Vecchia, Nucl. Phys. **B146** (1978) 63.