Unitary-Matrix Integration on 2D Yang-Mills Action

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Abstract

Using the idea of Itzykson-Zuber integral, unitary-matrix integration of 2D Yang-Mills action is presented. The uniqueness of the solution of heat equation enables us to integrate out the unitary-matrix parts of hermite matrices and to obtain the expression of integration over vectors, also in this case.

Introduction

In 1980, C. Itzykson and J.-B. Zuber presented the method of integration over angular variables on a simplest model coupling two hermite matrices [1]. Here, the angular variable means the unitary matrix which diagonalizes physical quantitiy, i.e. $N{\times}N$ hermite matrix. Their work had a great impact and enabled developments especially on the theories of random matrix and $2D$ gravity ([2] etc.). While, until then, there was Creutz's unitary-matrix integration [3] well known in lattice gauge theory which integrate unitary matrices order by order in high temperature expansion, Itzykson-Zuber integral is epoch-making in the sense that they obtained the expression only using the eigenvalues of hermite matrices,

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after exactly integrating out angular variables, to all orders of exponential. In practice, since it is too complicated to integrate in usual ways, they evaluate the integral with the partial differential equation known as heat equation whose uniqueness of its solution is completely understood. They showed that the expression after integration is equivalent to that before integration using the uniqueness of the solution. After the unitary-matrix integration, the number of variables is reduced to N, i.e. vector, from $N \times N$ matrix, and therefore, the analyses such as $N \to \infty$ planar limit becomes very easy.

In this paper, using their idea, we present a similar expression after integrating out unitary matrices on 2-dimensional Yang-Mills action with intent to prepare to apply this, in future, to 10-dimensional Yang-Mills theory, i.e. IIB Matrix Model [4, 5] which is known to be convergent [6]. On our model, if we introduce an external field as $N \times N$ hermite matrix, the same heat equation as Itzykson-Zuber integral is also found to be satisfied, and therefore, we can evaluate our integral on the basis of the uniqueness of its solution. However, unlike Itzykson-Zuber integral, since a number of coefficients which cannot be determined only from the uniqueness remain, we derive them from the structure of the commutator characteristic of Yang-Mills action. As a result, in case N is even, after integrating out angular variables, we find that the partition function of 2-dimensional Yang-Mills action can be expressed in terms of the remaining eivenvalues.

Unitary-Matrix Integration of 2D Yang-Mills Action

The integral we investigate is the Yang-Mills action in 2 "space-time" dimensions

$$I(C; g) = \int dAdB \, \exp\left\{ g \, tr[A, B]^2 + i \, trC[A, B] \right\}, \tag{1}$$

which exists in the "external field" C. Here, A, B and C are $N \times N$ hermitian matrices, and $iC[A, B]$ is also hermitian. dA (and dB) is the measure

$$dA = \prod_{i=1}^{N} dA_{ii} \prod_{i<j}^{N} d(ReA_{ij})d(ImA_{ij}), \tag{2}$$

which is invariant under the adjoint operation of $U(N)$ matrices

$$d(V^{\dagger}AV) = dA, \qquad V \in U(N).$$

This measure can be written in the diagonalized form as follows:

$$dA = \prod_{i=1}^{N} da_i \, dU \, \Delta^2(\vec{a}), \tag{3}$$

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where A is diagonalized by unitary matrix U such as $A = U^\dagger a U$, $a = diag(a_1, \cdots, a_N)$, and dU is the normalized $U(N)$ Haar measure, $\int dU = 1$. $\Delta(\vec{a})$ is the Vandermonde determinant

$$\Delta(\vec{a}) = (-1)^{\frac{N(N-1)}{2}} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_N \\ a_1^2 & a_2^2 & \cdots & a_N^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{N-1} & a_2^{N-1} & \cdots & a_N^{N-1} \end{vmatrix} = \prod_{i<j}(a_i - a_j). \tag{4}$$

As we can check easily, the Itzykson-Zuber integral [1],

$$\begin{aligned} K(C; g) &= \int dX \exp\left\{ -g\, tr X^2 + tr CX \right\} \\ &= \int d^N \vec{x}\, \frac{\Delta(\vec{x})}{\Delta(\vec{c})} \exp\left\{ -g|\vec{x}|^2 + \vec{c} \cdot \vec{x} \right\}, \end{aligned} \tag{5}$$

obeys the following partial differenial equation,

$$tr\left(\frac{\partial^2}{\partial C^2}\right) K(C; g) = -\frac{\partial}{\partial g} K(C; g), \tag{6}$$

$$tr\left(\frac{\partial^2}{\partial C^2}\right) \equiv \sum_{i,j=1}^{N} \frac{\partial}{\partial C_{ij}} \frac{\partial}{\partial C_{ji}}.$$

Essentially, this equation is the heat equation, and has an unique solution (5) under the boundary condition $K(C; g \to \infty) = 0$.

Similarly, the integral (1) satisfies the same equation

$$tr\left(\frac{\partial^2}{\partial C^2}\right) I(C; g) = -\frac{\partial}{\partial g} I(C; g), \tag{7}$$

and must have an unique solution under the boundary condition $I(C; g \to \infty) = 0$. After integrating over $U(N)$ matrices, we expect that (1) becomes the integral which only contains the integration over two eigenvalues of hermite matrices A and B. Then, we suppose the integral (1) to become like the following form:

$$i(\vec{c}; g) = \int d^N \vec{a} d^N \vec{b} \exp\left\{ -g \sum_{i,j,k,l=1}^{N} R_{ijkl} a_i b_j a_k b_l + \sum_{i,j,k=1}^{N} \Gamma_{ijk} c_i a_j b_k \right\}. \tag{8}$$

Here, the coefficients R_{ijkl} and Γ_{ijk} are independent of $\vec{c} = (c_1, \cdots, c_N)$. Since this integral must obey the partial differential equation

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$$\sum_{j=1}^{N} \frac{\partial^2}{\partial c_j^2} i(\vec{c}; g) = -\frac{\partial}{\partial g} i(\vec{c}; g), \tag{9}$$

R_{ijkl} is determined from Γ_{ijk}:

$$R_{ijkl} = \sum_{m=1}^{N} \Gamma_{mij}\Gamma_{mkl}. \tag{10}$$

Therefore, it is sufficient we obtain Γ_{ijk}, and, from the equation

$$tr\left(\frac{\partial^2}{\partial C^2}\right) = \Delta^{-1}(\vec{c}) \sum_{j=1}^{N} \frac{\partial^2}{\partial c_j^2} \Delta(\vec{c}) + \nabla_{U_C}^2,$$
$$C = U_C^{\dagger} c U_C, \tag{11}$$

where $\nabla_{U_C}^2$ only depends on U_C, the integral (1) becomes

$$I(C; g) = \int d^N\vec{a} d^N\vec{b} \frac{f(\vec{a}, \vec{b})}{\Delta(\vec{c})} \exp\left\{ -g \sum_{i,j,k,l=1}^{N} R_{ijkl} a_i b_j a_k b_l + \sum_{i,j,k=1}^{N} \Gamma_{ijk} c_i a_j b_k \right\}, \tag{12}$$

owing to the uniqueness of the solution of (7). Here, $f(\vec{a}, \vec{b})$ is the undetermined function of $\vec{a}$ and $\vec{b}$. Furthermore, though $\vec{c}$ is exactly the eigenvalue of hermite matrix C, we cannot certain that both $\vec{a}$ and $\vec{b}$ are those of A and B, yet. However, with $g=0$ in (12), the integral

$$\int dU dV \exp\left\{ i\, trC[U^{\dagger}AU, V^{\dagger}BV] \right\}$$
$$= \int dU dV dW \exp\left\{ i\, trW^{\dagger}CW[U^{\dagger}AU, V^{\dagger}BV] \right\}$$
$$= \int dV dW \exp\left\{ i\, trA[V^{\dagger}BV, W^{\dagger}CW] \right\}$$
$$= \int dU dW \exp\left\{ i\, trB[W^{\dagger}CW, U^{\dagger}AU] \right\}$$
$$= \frac{1}{\Delta(\vec{a})\Delta(\vec{b})\Delta(\vec{c})} \exp\left\{ \sum_{i,j,k=1}^{N} \Gamma_{ijk} c_i a_j b_k \right\} \tag{13}$$

enables us to derive $f(\vec{a}, \vec{b}) = \Delta(\vec{a})\Delta(\vec{b})$ and make sure that both $\vec{a}$ and $\vec{b}$ are the eigenvalues of A and B, because all A, B and C stand on the same ground. And, also from the facts

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that $trC[A, B] = trA[B, C] = trB[C, A]$ and $trC[A, B] = -trC[B, A]$, the index of Γ_{ijk} has cyclic symmetry: $\Gamma_{ijk} = \Gamma_{jki} = \Gamma_{kij}$ and anti-symmetry under the interchange of its indices: $\Gamma_{ijk} = -\Gamma_{ikj}$. Therefore, we obtain

$$
\begin{aligned}
I(C; g) &= \int dA dB \, \exp\left\{ g \, tr[A, B]^2 + i \, trC[A, B] \right\} \\
&= \int d^N \vec{a} d^N \vec{b} \, \frac{\Delta(\vec{a})\Delta(\vec{b})}{\Delta(\vec{c})} \, \exp\left\{ -g \sum_{i,j,k,l=1}^{N} R_{ijkl} a_i b_j a_k b_l + \sum_{i,j,k=1}^{N} \Gamma_{ijk} c_i a_j b_k \right\}. \quad (14)
\end{aligned}
$$

The remaining task we must do is to determine the coefficients Γ_{ijk}. Since to have the matrices A and B multiplied by the constants k is equivalent to make the eigenvalues $k\vec{a}$ and $k\vec{b}$ respectively, Γ_{ijk} are also independent of $\vec{a}$ and $\vec{b}$. Then, we can evaluate Γ_{ijk} through the structure peculiar to the commutator $[*, *]$. We can easily check the integral

$$
I(\vec{a}, \vec{b}; g) = \int dU_A dU_B \, \exp\left\{ g \, tr[U_A^\dagger a U_A, U_B^\dagger b U_B]^2 \right\} \quad (15)
$$

$$
= \exp\left\{ -g \sum_{i,j,k,l=1}^{N} R_{ijkl} a_i b_j a_k b_l \right\} \quad (16)
$$

follows from (14), and (15) satisfies the differential equation due to the characteristic structure of the commutator:

$$
\sum_{i=1}^{N} \frac{\partial}{\partial a_i} I(\vec{a}, \vec{b}; g) = 0, \quad (17)
$$

therefore, from (16), we obtain the algebraic equation which R_{ijkl} must obey:

$$
\sum_{i,j,k,l=1}^{N} R_{ijkl} b_j a_k b_l = \sum_{i,j,k,l,m=1}^{N} \Gamma_{mij} \Gamma_{mkl} b_j a_k b_l = 0. \quad (18)
$$

This is the identity about a_i and b_i. By introducing $d_m \equiv \sum_{k,l} \Gamma_{mkl} a_k b_l$, this reduces to the identity

$$
\sum_{i,j,m=1}^{N} \Gamma_{mij} b_j d_m = 0 \quad (19)
$$

about b_j and d_m, and we obtain conditions for Γ_{ijk}:

$$\sum_{i=1}^{N} \Gamma_{ijk} = 0, \qquad \text{for fixed } j, k. \tag{20}$$

When $N = 4$, these equations become

$$\Gamma_{312} + \Gamma_{412} = 0, \quad \Gamma_{123} + \Gamma_{423} = 0,$$
$$\Gamma_{213} + \Gamma_{413} = 0, \quad \Gamma_{124} + \Gamma_{324} = 0,$$
$$\Gamma_{214} + \Gamma_{314} = 0, \quad \Gamma_{134} + \Gamma_{234} = 0,$$

then, except overall factor, all Γ_{ijk}s are determined exactly as

$$\Gamma_{123} = -\Gamma_{124} = \Gamma_{134} = -\Gamma_{234} = 1. \tag{21}$$

When $N = 2$ and $N = 3$, $\Gamma_{ijk} = 0$ for all i, j, k. When $N \geq 4$, from the symmetry of its indices, Γ_{ijk} has $_NC_3$ independent components, but (20) are $_NC_2$ equations at most. So, we cannot determine all Γ_{ijk} from (20) in general. However, as is obvious from (14), all a_i, b_i and c_i are contained in the same weight. That is, except the overall factor which can be absorbed into the integration measure, we can set $|\Gamma_{ijk}| = 1$. Then, for even $N \geq 4$, we can find the solution of (20):

$$\Gamma_{ijk} = (-1)^{i+j+k}, \qquad \text{for} \quad i < j < k. \tag{22}$$

When N is odd, the solution such as $|\Gamma_{ijk}| = 1$ does not exist, because odd numbers of ± 1 cannot add up to zero. In fact, when $N = 5$,

$$\Gamma_{312} + \Gamma_{412} + \Gamma_{512} = 0, \quad \Gamma_{124} + \Gamma_{324} + \Gamma_{524} = 0,$$
$$\Gamma_{213} + \Gamma_{413} + \Gamma_{513} = 0, \quad \Gamma_{125} + \Gamma_{325} + \Gamma_{425} = 0,$$
$$\Gamma_{214} + \Gamma_{314} + \Gamma_{514} = 0, \quad \Gamma_{134} + \Gamma_{234} + \Gamma_{534} = 0,$$
$$\Gamma_{215} + \Gamma_{315} + \Gamma_{415} = 0, \quad \Gamma_{135} + \Gamma_{235} + \Gamma_{435} = 0,$$
$$\Gamma_{123} + \Gamma_{423} + \Gamma_{523} = 0, \quad \Gamma_{145} + \Gamma_{245} + \Gamma_{345} = 0,$$

the solution does not exist. This strange feature that we can use the expression (14) only for even N seems to come from the following characteristic of the commutator $[*, *]$. The algebra of hermite matrices has the subalgebra which consists of real symmetric matrices $A^\dagger = A^T = A$, and, in this subalgebra, the commutator $[A, B]$ always becomes the anti-symmetric matrix $[A, B]^T = -[A, B]$ which has even rank. That is, the commutator $[*, *]$ essentially contains the structure of even rank which is hidden by larger $U(N)$ symmetry before the integration, and then, the integration of $U(N)$ matrices may reveal

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this structure. However, since we need the integral after taking the limit $N \to \infty$, it is sufficient for our purpose to have the result that N is even number.

Lastly, let us line up the result we obtain in this paper.

$$
\begin{aligned}
I(C; g) &= \int dA dB \, \exp\left\{ g \, tr[A, B]^2 + i \, trC[A, B] \right\} \\
&= \int d^N \vec{a} \, d^N \vec{b} \, \frac{\Delta(\vec{a}) \Delta(\vec{b})}{\Delta(\vec{c})} \exp\left\{ -g \sum_{i,j,k,l=1}^{N} R_{ijkl} a_i b_j a_k b_l + \sum_{i,j,k=1}^{N} \Gamma_{ijk} c_i a_j b_k \right\}, \\
R_{ijkl} &= \sum_{m=1}^{N} \Gamma_{mij} \Gamma_{mkl}, \\
\Gamma_{ijk} &= \Gamma_{kij} = -\Gamma_{ikj}, \\
\Gamma_{ijk} &= (-1)^{i+j+k}, \qquad \text{for} \quad i < j < k.
\end{aligned}
$$

Discussions

Using the uniqueness of the solution for heat equation which plays a crucial role on Itzykson-Zuber integral, we derive an expression which contains eigenvalues of hermite matrices only as integration variables after integrationg out angular variables on 2-dimensional Yang-Mills action. Then, characteristic structures of the commutator are important and needed to determine coefficients in the expression. And the expression is possible only for even N, reflecting the structures.

Our expression can be applied to various problems. For example, as the model of Itzykson-Zuber integral which can be solved exactly in $N \to \infty$ planar limit, we can evaluate similarly on 2-dimensional Yang-Mills model. And, as refered in introduction, by extending our result to larger dimensions, for example IIB Matrix Model, we may be able to treat the phenomenon of the phase-transition of space-time, i.e. inflation. However, when we extend to higher dimensions than three, we cannot probably use the uniqueness of the solution of heat equation, so must introduce other methods.

Furthermore, since Vandermonde determinant $\Delta(\vec{x})$, which always emerges in case of integrating over hermite matrices, is the wave function of ground state of Calogero-Sutherland model describing the system of fermions on circle [7], and the orthogonal polynomials containing $\Delta(\vec{x})$ satisfy particular recursion relations [1], we may be able to calculate exactly over remaining eigenvalues.

References

[1] C. Itzykson and J.-B. Zuber, J. Math. Phys. 21(3), March (1980) 411

[2] M. L. Mehta, Boston, Academic Press, c1991

[3] M. Creutz, J. Math. Phys. 19(10), October 1978 2043

[4] N.Ishibashi, H.Kawai, Y.Kitazawa and A.Tsuchiya, hep-th/9612115, Nucl. Phys. B498 (1997) 467

[5] H.Aoki, S.Iso, H.Kawai, Y.Kitazawa, A.Tsuchiya and T.Tada, hep-th/9908038, Prog. Theor. Phys. Suppl. 134 (1999) 47

[6] P.Austing and J.F.Wheater, hep-th/0101071

[7] L. Lapointe and L. Vinet, q-alg/9509003, Commun. Math. Phys. 178 (1996) 425