Non local parity transformations and anomalies

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Abstract

We present an alternative derivation of the parity anomaly for a massless Dirac field in $2+1$ dimensions coupled to a gauge field. The anomaly functional, a Chern-Simons action for the gauge field, is obtained from the non-trivial Jacobian corresponding to a non local symmetry of the Pauli-Villars regularized action. That Jacobian is well-defined, finite, and yields the standard Chern-Simons term when the cutoff tends to infinity.

1 Introduction

Because of the presence of ultraviolet divergences, most quantum field theory models require the use of a regularization scheme at some intermediate step in the renormalization program. The regularization procedure, to be effective,

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will modify the large momentum behaviour of the theory. In some cases, there are symmetries which are particularly sensitive to that modification, leading to the existence of anomalies, namely, symmetries which remain broken even after the regularization is removed. In other words, performing a symmetry transformation and renormalizing are non-commuting operations.

In the context of the functional integral quantization, anomalies are traditionally attributed to the non invariance of the integration measure under a classical symmetry transformation. The non-invariance of the integration measure shows up in that the Jacobian corresponding to the classical symmetry transformation is non trivial and, in general,ill-defined. When a proper regularization is introduced, this Jacobian gives rise to an anomalous term in the effective action [1].

In reference [2], an alternative procedure to study anomalies within the functional integral approach was presented. It amounts to considering the functional integral for a *regularized* action, and to restrict the study to symmetry transformations that leave that action invariant. For the cases of the conformal and chiral anomalies, we have shown that those symmetry transformations do exist, and that they tend to the usual ones when the cutoff is removed. The transformations are necessarily non local, but on a length scale of the order of the inverse of the cutoff. The outcome of performing those symmetry transformations is that, while the regularized classical action is indeed invariant (even for a finite cutoff), the integration measure is not. Moreover, this non invariance is explicit, since the corresponding Jacobian determinant is *well defined and different from* 1, for any value of the cutoff. Contact with the usual anomalies is made by taking the infinite cutoff limit, what reproduces the known results.

In this letter, we will point out that this procedure can also be applied to the case of the parity anomaly in 2+1 dimensions, since the regularized action *does* have a non local symmetry, which coincides with the classical parity symmetry in the infinite cutoff limit. Moreover, the associated Jacobian for this discrete transformation is *finite*, and reproduces the parity anomaly, a Chern-Simons term in the effective action [3]-[4]. This shows that the procedure we have developed in ref. [2] can also be applied to the case of a *discrete* symmetry.

The organization of this letter is as follows: In section 2, we discuss the classical parity transformations and its generalization to the (quantum) regularized case. The generalized transformations are then applied to the calculation of the parity anomaly in section 3. Section 4 contains our conclusions.

2 Generalized parity transformations

Let us consider massive Dirac fermions coupled to an external Abelian gauge field A_μ, in $2+1$ dimensions. The (Minkowski spacetime) action for the system is

$$S_f[A] \;=\; \int d^3x\,\mathcal{L}_f \tag{1}$$

where the Lagrangian density $\mathcal{L}_f$ is given by

$$\mathcal{L}_f \;=\; \bar{\psi}(x)(i\not{\partial} - e\not{A}(x) - m)\psi(x)\;. \tag{2}$$

In our conventions, $g_{\mu\nu} = \mathrm{diag}(1,-1,-1)$, and the Dirac matrices verify $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, and $\gamma_\mu^\dagger = \gamma^\mu$.

We recall that, in $2+1$ dimensions, parity transformations are, in fact, tantamount to spatial reflections:

$$x = (x^0, x^1, x^2) \xrightarrow{\mathcal{P}} x^P = (x^0, -x^1, x^2) \tag{3}$$

since a spatial inversion is a rotation in π when the number of space coordinates is even.

The gauge and spinor fields transform according to the rules:

$$\begin{aligned} A(x) = (A^0(x), A^1(x), A^2(x)) \;\to\; A^P(x^P) = (A^0(x), -A^1(x), A^2(x)) \\ \psi(x) \;\to\; \psi^P(x^P) = \gamma_1\psi(x) \quad \bar{\psi}(x) \;\to\; \bar{\psi}^P(x^P) = \bar{\psi}(x)\gamma_1\;. \end{aligned} \tag{4}$$

Hence, under $\mathcal{P}$, the classical action transforms as follows:

$$\begin{aligned} S[A] \to S_f^P[A] \;&=\; \int d^3x^P\, \bar{\psi}_P(x_P)(i\not{\partial}_P - e\not{A}_P(x_P) - m)\psi_P(x_P) \\ &=\; \int d^3x\, \bar{\psi}(x)(i\not{\partial} - e\not{A}(x) + m)\psi(x)\;. \end{aligned} \tag{5}$$

Writing the mass dependence of the action explicitly, we have the relation:

$$S_f^P(m) = S_f(-m)\;. \tag{6}$$

This indicates, of course, that the massless classical theory is parity invariant.

Regarding the regularized action, although not every regularization method can be implemented in terms of an action, there are many important cases where this can be done. Examples of those are the Euclidean cutoff, lattice, and Pauli-Villars (PV) regularization methods. We shall use the PV method, since it maintains most of the symmetries, except parity (see [5] for a thorough discussion in odd-dimensional spaces). This greatly simplifies the discussion. In our $2+1$ dimensional example, the functional integral is rendered convergent by the addition of just one bosonic regulator spinor field ϕ, whose mass Λ plays the role of a cutoff. The regularized action S_f^r is:

$$S_f^r = i \int d^3x \left[\bar{\psi}(x)(\not{\partial} + ie\not{A})\psi(x) + \overline{\phi}(x)(\not{\partial} + ie\not{A} + i\Lambda)\phi(x) \right] . \tag{7}$$

It is possible to write the regularized action (7) in terms of just one fermionic field Ψ, although at the expense of equipping that field with a non local action. To that end, we consider the regularized functional integral $\mathcal{Z}^r[A]$ corresponding to (7):

$$\mathcal{Z}^r[A] \;=\; \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}\phi \mathcal{D}\bar{\phi} \, \exp\left\{ - \int d^3x [\bar{\psi}(x) \not{D} \psi(x) + \bar{\phi}(x)(\not{D} + i\Lambda)\phi(x)] \right\} \tag{8}$$

where $\not{D} \equiv (\not{\partial} + ie\not{A})$.

We first integrate out the regulator in (8):

$$\mathcal{Z}^r[A] \;=\; \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \, \overset{-1}{\det}[1 - \frac{i\not{D}}{\Lambda}] \, \exp\left(- \int d^3x \bar{\psi} \not{D} \psi \right) \tag{9}$$

(we have neglected an irrelevant constant factor). We then make the change of variables:

$$\psi(x) \;=\; \left[1 - \frac{i\not{D}}{\Lambda} \right]^{-\frac{1}{2}} \Psi(x) \tag{10}$$

$$\bar{\psi}(x) \;=\; \bar{\Psi}(x) \left[1 - \frac{i\not{D}}{\Lambda} \right]^{-\frac{1}{2}} \tag{11}$$

under which the measure transforms according to:

$$\mathcal{D}\psi \mathcal{D}\bar{\psi} = \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \det[1 - \frac{i\not{D}}{\Lambda}] \, . \tag{12}$$

The result of the two previous steps is that we may rewrite (9) in the equivalent form:

$$\mathcal{Z}^r[A] = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \exp(-S_f^{nl}) \tag{13}$$

where S_f^{nl} denotes a non-local form of the regularized action,

$$S_f^{nl} \;=\; \int d^3x \, d^3y \, \bar{\Psi}(x) \mathcal{D}(x,y) \Psi(y) \, . \tag{14}$$

with

$$\mathcal{D}(x,y) = \left[\frac{\not{D}}{1 - \frac{i\not{D}}{\Lambda}} \right] (x,y) \, . \tag{15}$$

We have found it convenient to adopt a 'bracket' like notation to write the action (14),

$$S_f^{nl} \;=\; \langle\Psi|\mathbf{D}|\Psi\rangle \;\;,\;\; \mathbf{D} \;=\; \frac{\not{D}}{1-\frac{i\not{D}}{\Lambda}}\;, \tag{16}$$

since it avoids writing operators kernels and integrations explicitly. Note that the 'bra' includes the γ_0 factor for the adjoint field.

To understand the behaviour of (14) under parity transformations, we previously need to know the $\mathbf{D}$ operator transformation properties. Denoting by $\not{D}(x,y)$ the kernel of $\not{D}$, we see that

$$\gamma_1 \not{D}^P(x^P,y^P)\,\gamma_1 \;=\; \not{D}(x,y)\;. \tag{17}$$

We can also express this result as:

$$\gamma_1 \not{D}^P\,\gamma_1 \;=\; \not{D}\;, \tag{18}$$

what in turn implies for $\mathbf{D}$:

$$\gamma_1 \mathbf{D}^P(\Lambda)\gamma_1 \;=\; \mathbf{D}(-\Lambda)\,. \tag{19}$$

The parity transformed non local regularized action becomes

$$S_f^{nlP} = \langle\Psi^P|\mathbf{D}^P(\Lambda)|\Psi^P\rangle = \langle\Psi^P|\gamma_1\mathbf{D}(-\Lambda)\gamma_1|\Psi^P\rangle\;, \tag{20}$$

where we need to plug in the parity transformed of Ψ and $\bar{\Psi}$. The transformation rules for the new fields, can be obtained as follows: from the relation between Ψ and ψ, we learn that

$$\psi^P(x^P) \;=\; \left[1-\frac{i\not{D}^P}{\Lambda}\right]^{-\frac{1}{2}}\;\Psi^P(x^P) \tag{21}$$

and

$$\psi^P(x^P) = -\gamma_1\left[1+\frac{i\not{D}}{\Lambda}\right]^{-\frac{1}{2}}\;\gamma_1\Psi^P(x^P)\;. \tag{22}$$

On the other hand, we have the relation

$$\psi^P(x^P) = \gamma_1\psi(x) \;=\; \gamma_1\left(1-\frac{i\not{D}}{\Lambda}\right)^{-\frac{1}{2}}\;\Psi(x)\;. \tag{23}$$

Then,

$$\Psi^P(x^P) = \gamma_1\frac{\sqrt{1+\frac{i\not{D}}{\Lambda}}}{\sqrt{1-\frac{i\not{D}}{\Lambda}}}\Psi(x) = \gamma_1\langle x|\frac{\mathbf{D}(\Lambda)}{\sqrt{\mathbf{D}(\Lambda)\mathbf{D}(-\Lambda)}}|\Psi\rangle\;. \tag{24}$$

We obtain an analogous relation for $\bar{\Psi}(x)$. Using the compact notation, we see that

$$|\Psi^P\rangle = \gamma_1 \frac{\mathbf{D}(\Lambda)}{\sqrt{\mathbf{D}(\Lambda)\mathbf{D}(-\Lambda)}} |\Psi\rangle \tag{25}$$

$$\langle\Psi^P| = \langle\Psi| \frac{\mathbf{D}(\Lambda)}{\sqrt{\mathbf{D}(\Lambda)\mathbf{D}(-\Lambda)}} \gamma_1 \;, \tag{26}$$

and these are the generalized parity transformations we were looking for. We note that they tend to the standard parity transformations when the cutoff is removed ($\Lambda \to \infty$).

It is immediate to verify that, with these transformation rules, the non-local form of the regularized action remains invariant:

$$S_f^{nlP} = S_f^{nl} \;. \tag{27}$$

Being the Pauli-Villars fields integrated out exactly, the effective action (14) incorporates the complete effect of the regularization, in the sense that the contribution to S_f^{nl} of the determinant associated with the Pauli-Villars field is totally included. For example, were one to use this effective action to compute the vacuum polarization, the corresponding diagrams will automatically be convergent to all orders.

3 Parity anomaly

Although the regularized action is invariant under the generalized parity transformations, the fermionic measure acquires a non trivial Jacobian:

$$\mathcal{D}\Psi_P \mathcal{D}\bar{\Psi}_P = \mathcal{D}\Psi\mathcal{D}\bar{\Psi}\;\mathcal{J} \tag{28}$$

where

$$\mathcal{J} \;=\; \det\left[-\left(\frac{\sqrt{D(\Lambda)D(-\Lambda)}}{D(\Lambda)}\right)^2\right] = \det\left(\frac{i\not{D} - \Lambda}{i\not{D} + \Lambda}\right) \;. \tag{29}$$

The parity anomaly is obtained from the infinite-Λ limit of this Jacobian. This result is known, and coincides of course with (twice) the leading term in a derivative expansion of the effective action $I_f^{\rm eff}[A, \Lambda]$: We see that:

$$\exp(iI_{\rm eff}[A,\Lambda] \;-\; iI_{\rm eff}[A,-\Lambda]) \;=\; \det\left(\frac{\not{D} + i\Lambda}{\not{D} - i\Lambda}\right) \;=\; \mathcal{J} \;. \tag{30}$$

$$I_{\rm eff}[A,\Lambda] = -i\ln\det(\not{D} + i\Lambda). \tag{31}$$

Hence the Jacobian is expressed as a function af the effective action. In the $\Lambda \to \infty$ one obtains [6]-[7]:

$$I_{\rm eff}[A,\Lambda] \ \to \ \frac{\Lambda}{|\Lambda|} S_{C-S} \ , \tag{32}$$

where the $C-S$ action S_{C-S} is defined by:

$$S_{C-S} \ = \ \frac{e^2}{4\pi}\int d^3x\, \epsilon^{\mu\nu\rho} A_\mu(x)\partial_\nu A_\rho(x) \ . \tag{33}$$

Thus we can write for the Jacobian:

$$\mathcal{J} \ = \ \exp\left\{\pm 2iS_{C-S}[A]\right\} \ , \tag{34}$$

which is the parity anomaly result. It is not renormalized by higher order correction, as a consequence of the Coleman-Hill theorem [8].

For the case of a continuous symmetry transformation, the Fujikawa Jacobian resulting from the associated infinitesimal change of variables can be used to calculate the potentially anomalous divergence of the corresponding Noether current. This procedure does not, of course, have a direct parallel in the present, discrete symmetry case. Nevertheless, from the knowledge of the Jacobian (34), one can make explicit the anomalous behavior of the fermion current under a parity transformation. Indeed, from eq.(13), we can write

$$\mathcal{Z}^r[A^P] \equiv \exp\left(-I^r_{eff}[A^P]\right) = \int \mathcal{D}\Psi\mathcal{D}\bar{\Psi}\exp(-S^{nl}_f[A^P,\bar{\Psi},\Psi]) \ . \tag{35}$$

Now, changing the fermion variables as in (25)-(26), and using the invariance (27) of the non local action, we see that

$$\mathcal{Z}^r[A^P] = \mathcal{J}[A^P]\int \mathcal{D}\Psi^P\mathcal{D}\bar{\Psi}^P\exp(-S^{nl}_f[A^P,\bar{\Psi}^P,\Psi^P]) = \mathcal{J}[A^P]\mathcal{Z}^r[A] \tag{36}$$

or

$$\exp\left(-I^r_{eff}[A^P] + I^r_{eff}[A]\right)) = \mathcal{J}[A^P] \ . \tag{37}$$

Differentiation of the regularized effective action with respect to the gauge field A yields the ground state current $\langle j_\mu[A]\rangle$ so that one has

$$-e\langle j_\mu[A^P]\rangle + e\langle j_\mu[A]\rangle = \frac{\delta \log \mathcal{J}[A^P]}{\delta A^\mu} \ . \tag{38}$$

Using relations (4) for the l.h.s. and (34) for the r.h.s., we can then write the anomalous, parity-odd vacuum current:

$$\langle j_\mu[A]\rangle^{odd} \;=\; \frac{1}{2e}\frac{\delta \log \mathcal{J}[A^P]}{\delta A^\mu} = \pm i\frac{e}{4\pi}\int d^3x\,\epsilon^{\mu\nu\rho}\partial_\nu A_\rho(x)\;. \qquad (39)$$

which is the well-known result first obtained in [4].

It is immediate to verify that all the steps leading the parity anomaly for the Abelian case can be generalized to the non-Abelian case as well, with the only modification that the Jacobian is now related to the non-Abelian massive determinant. The result is of course that this Jacobian is the exponential of ($2i$ times) the non-Abelian Chern-Simons action.

4 Conclusions

We have shown that the parity anomaly in 2+1 dimensions in a theory of massless fermions coupled to an external gauge field can be obtained from the Jacobian for a generalized symmetry transformation of the regularized theory. This procedure has the virtue of disentangling the symmetries and infinities, by looking for transformations which are symmetries of the *regularized* action. This avoids involving modes with an arbitrarily high momentum into the symmetry transformation, and the bonus is that the resulting Jacobian automatically takes care of the anomaly. It is worth noting that it is precisely the fact that the regularization breaks the symmetry what renders the symmetry transformations of the regularized action non-local. Non-anomalous local symmetries have, in this setting, a distinctive property: they are always local, even when acting on the regularized action.

Our calculations required the use of a non-local form of the regularized action and we used 'regularized' fermion fields Ψ. The non-locality of the action, however, is hidden in the non-physical region of momenta above the cutoff. For example, when $A = 0$, the $\mathbf{D}$ operator has a kernel:

$$\not{D}(x,y) = \not\partial\left(1 + \frac{i\not\partial}{\Lambda}\right)\int \frac{d^3k}{(2\pi)^3}\,\frac{e^{-ik(x-y)}}{1-(k/\Lambda)^2} \qquad (40)$$

which is non-local on a scale $1/\Lambda$.

Acknowledgements

C. D. F. would like to acknwledge Prof. F. D. Mazzitelli, for useful conversations and correspondence.

M. L. C. is supported by a CNEA fellowship. C. D. F. is a member of CONICET, and is partially supported by ANPCyT (PICT 97/1040) and Fundación Antorchas grants. F. A. S is partially supported by CICBA as Investigador, and through grants CONICET (PIP 4330/96), and ANPCyT (PICT 97/2285).

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