Consistent noncommutative quantum gauge theories?

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Abstract

A new noncommutative model invariant with respect to $U(1)$ gauge group is proposed. The model is free of nonintegrable infrared singularities. Its commutative classical limit describes a free scalar field. Generalization to $U(N)$ models is also considered.

1 Introduction

Perturbative aspects of noncommutative quantum theories were recently a subject of numerous investigations. These studies revealed some peculiar feathures of noncommutative models. Noncommutativity introduces naturally nonlocality of interaction, which serves as an ultraviolet regulator. However the regularization is not complete and ultraviolet divergencies do not disappear completely. Planar diagrams of a noncommutative theory require renormalization similar to the procedure used in commutative models. Nonplanar diagrams become ultraviolet convergent due to the presence of phase factors, however the corresponding integrals have pole singularities in external momenta leading to infrared divergency of higher order diagrams ([1], [2], [3], [4], [5]). In particular these infrared singularities are present in noncommutative $U(1)$ gauge theory, and although planar diagrams may be renormalized in a gauge invariant way [6], the model is inconsistent ([7], [8], [9]).

Another unusual property of noncommutative gauge theories is related to the fact that noncommutative $SU(N)$ algebra is not closed and one is forced to consider $U(N)$ models which include a $U(1)$ sector and hence are also inconsistent ([10], [11]).

One could try to cure this desease by introducing nonlocal counterterms which cancel the infrared singularities. However it would lead to a drastic modification of the original action and nobody was able to prove that such a procedure may be carried out in a consistent and gauge invariant way.

Experience obtained in commutative theories suggests that appearance of divergencies, which cannot be removed by renormalizing charges, masses and wave functions is a signal that the underlying classical theory is not complete and should be modified in such a way that possible divergent structures have the same form as the terms present in the classical action.

Motivated by this observation we propose a modified noncommutative $U(1)$ invariant action which does not lead to infrared divergencies. All divergencies are ultraviolet and may be removed by a standard renormalization procedure. The classical theory in the limit when the noncommutativity parameter (ξ) tends to zero reduces, contrary to naive expectations, not to free electrodynamics, but to a free scalar field theory. A similar procedure may be applied to $U(N)$ noncommutative gauge models, where the commutative classical limit describes $SU(N)$ vector bosons and $U(1)$ scalar particle.

2 Noncommutative gauge invariant models.

We start by reminding the basic facts about the conventional noncommutative $U(1)$ theory.

The model is described by the action

$$S = \int d^4x \{-\frac{1}{4} F_{\mu\nu} F_{\mu\nu}\} \tag{1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu * A_\nu] \tag{2}$$

The star product is defined as follows

$$f(x) * g(x) = \exp\{i\xi\theta_{\mu\nu}\partial^x_\mu \partial^y_\nu\} f(x) g(y)_{y=x} \tag{3}$$

where $\theta_{\mu\nu}$ is a real antisymmetric matrix and ξ is a noncommutativity parameter. In the limit $\xi \rightarrow 0$ the action (1) obviously reduces to the free electromagnetic action.

The gauge transformations look similar to nonabelian gauge transformations

$$\delta A_\mu = \partial_\mu \epsilon - ig(A_\mu * \epsilon - \epsilon * A_\mu) \tag{4}$$

Note that although for a general skewsymmetric matrix $\theta_{\mu\nu}$ the interaction (1) is nonlocal, models with $\theta_{i0} = 0$ introduce only spatial nonlocality and the standard Hamiltonian formalism may be applied. In what folows we assume that $\theta_{i0} = 0$ and Hamiltonian formalism may be used. Without loss of generality one may take $\theta_{12} = -\theta_{21} = 1; \theta_{13} = \theta_{23} = 0$.

One sees that the $U(1)$ noncommutative theory is nonabelian, and the Feynman rules look similar to the usual Yang-Mills theory. In particular Faddeev-Popov ghosts, parametrizing $\det(\partial_\mu D_\mu)$, where D_μ is the covariant derivative, are present. The free propagators coincide with the propagators of Yang-Mills theory, and the vertex functions are obtained substituting Lie algebra structure constants by the phase factors. For example the three point gauge vertex with momenta p, q, k and indices μ, ν, ρ looks as follows

$$2ig \sin(\xi p\tilde{q})[(p-q)_\rho \delta_{\mu\nu} + (q-k)_\mu \delta_{\nu\rho} + (k-p)_\nu \delta_{\mu\rho}] \tag{5}$$

Here we use the notation $\tilde{p}_\mu = \theta_{\mu\nu} p_\nu$.

The gauge field polarization operator has ultraviolet divergent part corresponding to the planar diagrams, and the convergent nonplanar part, which contains the term singular at $p = 0$. Explicit calculation gives

$$\Pi_{\mu\nu}(p) = \frac{g^2}{2\pi^2} \frac{\tilde{p}_\mu \tilde{p}_\nu}{\xi^2 (\tilde{p}^2)^2} + \dots \tag{6}$$

where $\dots$ denotes less singular terms. One sees that $\Pi_{\mu\nu}$ has a pole singularity at $p = 0$ and the limit $\xi \to 0$ does not exist. The diagrams which have several insertions of $\Pi_{\mu\nu}$ into gauge field lines are infrared divergent.

Similar singularities appear in the three point function, which looks as follows

$$\Gamma_{\mu\nu\rho}(p, q) \sim \cos(\xi p \tilde{q}) \{ \frac{\tilde{p}_\mu \tilde{p}_\nu \tilde{p}_\rho}{\xi (\tilde{p}^2)^2} + sym \} + \dots \tag{7}$$

where $\dots$ again stands for less singular terms and sym means symmetrization

$$p \to q, \mu \to \nu; \quad p \to -(p+q), \mu \to \rho; \quad q \to -(p+q), \nu \to \rho. \tag{8}$$

In general infrared pole singularities arise in the diagrams which in the absence of phase factors would be quadratically or linearly ultraviolet divergent. Logarihmically divergent diagrams produce only logarithmic infrared singularities which do not spoil integrability. In the commutative case gauge invariance prevents the appearance of linear and quadratic divergencies, but in the noncommutative theory they do appear. The only possible exception known so far is presented by supersymmetric gauge theories ([12], [13], [14]).

To avoid infrared divergencies one may try to subtract nonlocal counterterms (6, 7). However the meaning of such subtraction is not clear, as it does not correspond to renormalization of any parameter present in the original Lagrangian and the subtraction procedure is ambigous. Moreover, as was mentioned above, nobody proved that such subtraction can be done in a consistent and gauge invariant way.

We consider the appearance of singular terms proportional to $\tilde{p}_\mu A_\mu$ as a signal that original action must be modified to include the terms of this type.

The action (1) is not the only gauge invariant expression one can write in the noncommutative $U(1)$ theory. The most general gauge invariant action, which corresponds to a power counting renormalizable theory, posesses passive Lorentz invariance (i.e. is invariant if the tensor $\theta_{\mu\nu}$ also undergoes Lorentz transformations), and introduces nonlocality only via star product has a form

$$S = \int d^4x \{ -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} + \beta \lambda(x) \theta_{\mu\nu} F_{\mu\nu}(x) + \gamma (\theta_{\mu\nu} F_{\mu\nu}(x))^2 \} \tag{9}$$

Here β and γ are arbitrary parameters and the Lagrange multiplier $\lambda(x)$ transforms according to adjoint representation of the gauge group.

We choose $\beta = 1$. Then obviously the last term is irrelevant and one may put $\gamma = 0$. The action (9) describes a constrained system and to study its physical content one has to formulate the Hamiltonian dynamics. A natural requirement for a noncommutative theory is the condition that in the limit $\xi \to 0$ the Lorentz invariance is restored. The commutative limit of the action (9) is

$$S_0 = \int d^4x\{-\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \lambda(x)\tilde{\partial}_i A_i(x) \tag{10}$$

We remind that we consider the case when the only nonzero elements of the matrix $\theta_{\mu\nu}$ are $\theta_{12} = -\theta_{21} = 1$. In this case $\sum_i \tilde{\partial}_i^2 = \sum_{i=1,2} \partial_i^2 = \tilde{\partial}^2$.

At first sight the Lorentz invariance is broken even in the commutative limit. The proper Hamiltonian analysis shows however that the action(10) describes a usual free scalar field.

Let us rewrite the equation (10) as the action of a generalized Hamiltonian system:

$$S_0 = \int d^4x\{p_i\dot{A}_i - \frac{p_i^2}{2} - \frac{1}{4}(\partial_i A_j - \partial_j A_i)^2 + A_0\partial_i p_i + \lambda\tilde{\partial}_i A_i\} \tag{11}$$

To fix the gauge we choose the Coulomb condition $\partial_i A_i = 0$. Apart from the first class constraint $\partial_i p_i = 0$ the action (10) includes the additional constraint $\tilde{\partial}_i A_i = 0$. The commutator of this constraint with the Hamiltonian is different from zero:

$$[\int dx\frac{p_i^2}{2}, \theta_{ij}\partial_i A_j] = \tilde{\partial}_i p_i \tag{12}$$

That means the secondary constraint must be included and the complete action looks as follows

$$\begin{aligned} S_0 = \int d^4x\{p_i\dot{A}_i - \frac{p_i^2}{2} - \frac{1}{4}(\partial_i A_j - \partial_j A_i)^2 + \\ A_0\partial_i p_i + \lambda(x)\tilde{\partial}_i A_i + \mu(x)\tilde{\partial}_i p_i \end{aligned} \tag{13}$$

Let us parametrize the field A_i and momentum p_i as follows

$$\begin{aligned} A_i &= \partial_i\chi + \tilde{\partial}_i\psi + \epsilon^{ijk}\tilde{\partial}^{-2}\partial_j\tilde{\partial}_k\phi \\ p_i &= \partial_i p_\chi + \tilde{\partial}_i p_\psi + \epsilon^{ijk}\tilde{\partial}^{-2}\partial_j\tilde{\partial}_k p_\phi \end{aligned} \tag{14}$$

The Coulomb gauge condition and the constraint $\tilde{\partial}_i A_i = 0$ insure that $\chi = \psi = 0$. The remaining constraints nullify the corresponding momenta.

After solution of the constraints and gauge condition the eq.(13) may be written in the following form

$$S_0 = \int d^4x\{p_\phi\dot{\phi} - \frac{p_\phi^2}{2} - \frac{\partial_i\phi\partial_i\phi}{2}\} \tag{15}$$

which is the Lorentz invariant action for the scalar field. Contrary to naive expectations the limit $\xi \to 0$ of the modified electromagnetic action describes a scalar field.

Now we turn to the analysis of infrared singularities. We ignore possible local determinants which appear due to the second class constraints, assuming that some gauge invariant regularization (e.g. dimensional) is used, in which these factors are absent. The Feynman rules differ from the usual noncommutative $U(1)$ theory by the presence of propagators of λ-fields, mixed propagators λA_μ and the new vertex $g\lambda[A_\mu * A_\nu]\theta_{\mu\nu}$. The corresponding elements of diagram technique are:

The propagator λ, λ

$$k^2\tilde{k}^{-2} \tag{16}$$

The propagator λ, A_μ

$$\tilde{k}_\mu(\tilde{k})^{-2} \tag{17}$$

The vertex λ, A^2

$$ig\sin(\xi p\tilde{q})\theta_{\mu\nu} \tag{18}$$

The propagator of the Yang-Mills field is also modified. In the diagonal Feynman gauge it is

$$\frac{1}{k^2}(g^{\mu\nu} - \frac{\tilde{k}_\mu\tilde{k}_\nu}{\tilde{k}^2}) \tag{19}$$

We start with the one loop polarization operator. To study the leading singularities we may put the external momentum equal to zero everywhere except for the phase factors. Thus we have

$$\Pi^{sing}_{\mu\nu}(p) = \int d^4k\sin^2(\xi p\tilde{k})P_{\mu\nu}(k) \tag{20}$$

where $P_{\mu\nu}$ is a rational function with the dimention k^{-2}. To separate the infrared singular contribution one presents the phase factor as

$$\sin^2(\xi p\tilde{k}) = \frac{1}{2}(1 - \cos(2\xi p\tilde{k})) \tag{21}$$

The constant term corresponds to the planar contribution and produces ultraviolet divergency which is removed by the usual wave function renormalization, whereas the term proportional to $\cos(2\xi p\tilde{k})$ gives the infrared singular contribution

$$\Pi^{sing}_{\mu\nu}(p) \sim \frac{Ag^{\mu\nu}\tilde{p}^2 + B\tilde{p}^\mu\tilde{p}^\nu}{\xi^2(\tilde{p}^2)^2} \tag{22}$$

The polarization operator must satisfy ST-identities, which in this case reduce to transversality condition

$$p_\mu p_\nu\Pi^{sing}_{\mu\nu} = 0 \rightarrow A = 0 \tag{23}$$

Hence the pole singularity is proportional to $\tilde{p}_\mu\tilde{p}_\nu$. Recalling that the free propagator (19) of the Yang-Mills field A_μ is transversal with respect to $\tilde{p}_\mu$, we conclude that the infrared singularity is irrelevant if the polarization operator is connected to other part of a diagram by the gauge field propagator. The next possible singular term is $\sim \ln(p^2)$ and does not lead to nonintegrable singularity.

In our model there is also a mixed propagator $A_\mu\lambda$. Due to this mixing the singular part of $\Pi_{\mu\nu}$ may contribute to the amplitude with four external gauge field lines, obtained by connecting the polarization operator $\Pi_{\mu\nu}$ with the vertices (18) by the mixed propagators. The corresponding contribution is proportional to

$$\frac{\sin^2(\xi p\tilde{q})}{(\tilde{p}^2)^2\xi^2} \tag{24}$$

and does not lead to infrared divergencies at small p. Note that the limit $\xi \to 0$ is also nonsingular.

There are also one-loop polarization operators $\Pi_\mu(p)$, corresponding to diagrams with one external A_μ-line and one λ-line, and $\Pi(p)$, corresponding to diagrams with two λ-lines. All these diagrams in the absence of phase factors diverge at most logarithmically and therefore are infrared safe.

However if the the polarization operator of λ-field, $\Pi(p)$ required renormalization, that would mean that the action (9) was not complete and new counterterms $\sim \lambda^2$ have to be introduced.

Moreover the complete λ-field propagator may include subsequent insertions of several polarization operators $\Pi_{\mu\nu}, \Pi_\mu, \Pi$ connected by the mixed propagator $A_\mu \lambda$. For example

$$D_{\lambda\lambda}(p) = D_{\lambda A_\mu}(p)\Pi_{\mu\nu}(p)D_{A_\nu \lambda}(p)\Pi(p)D_{\lambda A_\rho}(p)\Pi_{\rho\sigma}(p)D_{A_\sigma \lambda}(p) \tag{25}$$

Obviously a diagram containing such propagator may produce infrared singularities due to accumulation of the $\Pi_{\mu\nu}$ poles.

Let us study the infrared behaviour of $\Pi(p)$ more closely. In the lowest order the polarization operator $\Pi(p)$ is equal to

$$\Pi(p) = \int d^4k[\sin(p\tilde{k}\xi)\theta_{\mu\nu}(g_{\mu\alpha} - \frac{\tilde{k}_\mu \tilde{k}_\alpha}{\tilde{k}^2})k^{-2} \times$$
$$(g_{\nu\beta} - \frac{(\tilde{p}+\tilde{k})_\nu(\tilde{p}+\tilde{k})_\beta}{(\tilde{p}+\tilde{k})^2})(p+k)^{-2}\sin(p\tilde{k}\xi)\theta_{\alpha\beta}] \tag{26}$$

Performing the multiplication explicitely we can rewrite this expression in the form

$$\Pi(p) = \int d^4k \sin^2(p\tilde{k}\xi)[\theta_{\alpha\beta} - \frac{k_\beta \tilde{k}_\alpha}{\tilde{k}^2} + \frac{(p+k)_\alpha(\tilde{k}+\tilde{p})_\beta}{(\tilde{k}+\tilde{p})^2} +$$
$$\frac{(\tilde{p}k)\tilde{k}_\alpha(\tilde{k}+\tilde{p})_\beta}{\tilde{k}^2(\tilde{p}+\tilde{k})^2}]k^{-2}(p+k)^{-2}\theta_{\alpha\beta} \tag{27}$$

According to our choice of $\theta_{\mu\nu}$ in this equation $\alpha, \beta = 1, 2$. The most singular terms vanish after summation over α, β, providing absolute convergence of the integral (27).

$$\Pi(p) = \int d^4k \sin^2(p\tilde{k}\xi)\frac{(\tilde{p}k)^2}{\tilde{k}^2(\tilde{p}+\tilde{k})^2}k^{-2}(p+k)^{-2} \tag{28}$$

To estimate the infrared behaviour of $\Pi(p)$ let us take $|p_1| = |p_2| = p$. Rescaling the integration variables $p\xi k \to x$ we get

$$\Pi(p) = p^4\xi^2 f(p^2\xi) \tag{29}$$

where the function $f(p^2\xi)$ has a logarithmic singularity at the origin $f(p^2\xi)_{p\sim 0} \sim \ln(p^2\xi)$. Therefore up to logarithmic corrections $\Pi(p)$ vanishes at $p = 0$ as p^4. It compensates the infrared singularity of the operator $\Pi_{\mu\nu}$ and guarantees the infrared convergence.

A general polarization operator $\Pi(p)$ may be analyzed in a similar way. It may be presented in the following form

$$\Pi(p) = \int d^4k \ldots d^4s\{\sin(p\tilde{k}\xi)\theta_{\alpha\beta}(g_{\alpha\mu} - \frac{\tilde{k}_\mu\tilde{k}_\alpha}{\tilde{k}^2})(g_{\nu\beta} - \frac{(\tilde{p}+\tilde{k})_\nu(\tilde{p}+\tilde{k})_\beta}{(\tilde{p}+\tilde{k})^2}) \times$$
$$\Pi_{\mu\nu\rho\sigma}(p,k,s)(g_{\rho\lambda} - \frac{\tilde{s}_\rho\tilde{s}_\lambda}{\tilde{s}^2})(g_{\sigma\kappa} - \frac{(\tilde{s}+\tilde{p})_\sigma(\tilde{s}+\tilde{p})_\kappa}{(\tilde{s}+\tilde{p})^2})\theta_{\lambda\kappa}\sin(p\tilde{s}\xi)\} \quad (30)$$

We consider the case when the outer vertices are connected with the internal part of the diagram by the gauge field propagators. There are also the diagrams where some of these propagators are replaced by the mixed propagators $A_\mu\lambda$. They are considered in a similar way and we shall not present the analysis here. The function $\Pi_{\mu\nu\rho\sigma}$ in the eq.(30) may be separated into parts symmetric and antisymmetric with respect to $\mu\nu$ and $\rho\sigma$. The antisymmetric parts are proportional to $\theta_{\mu\nu}$ and $\theta_{\rho\sigma}$ respectively. Performing the summation over all indices one sees that the first non-vanishing term is proportional to p^2. Therefore the integral is absolutely convergent and to study its behaviour at $\xi \sim 0$ we may put $\xi \sim 0$ in the integrand. In this way one sees that the $\Pi(p,\xi)$ and its first derivative over ξ vanish at $\xi = 0$. Possible asymptotics of $\Pi(p,\xi)$ at $\xi \sim 0$ have the form

$$\Pi(p,\xi)_{\xi\sim 0} = \xi^n \ln^m(\xi) \quad (31)$$

Therefore $\Pi(p,\xi)$ for small ξ is proportional to $\xi^2\ln^m(\xi)$. By dimensinal reasons at $p \to 0$, $\Pi(p) \sim p^4\xi^2\ln^m(p^2\xi)$, in accordance with the lowest order result.

Insertion of the mixed polarization operator $\Pi_\mu(p)$ does not change our analysis. This operator vanishes at $p = 0$ as $|p|$. At the same time the mixed propagator $A_\mu\lambda$ has a singularity $\sim |p|^{-1}$. So the product is not singular. It is important to note that to have two insertions of $\Pi_{\mu\nu}(p)$ which might produce nonintegrable infrared singularity one needs at least one insertion of the operator $\Pi(p)$, which cancels the singularity.

Now we turn to the study of three point gauge field vertex. It satisfies the ST-identity, which we take in the original form ([15]):

$$< A_\mu(x)A_\nu(y)\partial_\rho A_\rho(z) > = < \partial_\mu M^{-1}_{xz}A^b_\nu(y) > +$$
$$g < A_\mu(x)M^{-1}_{xz}A_\nu(y) > + (\mu \to \nu, x \to y). \quad (32)$$

Here M^{-1}_{xy} is the Green function of ghost field in the external gauge field.

We start again with the one-loop diagrams. To pass to the proper vertex function we must amputate external propagators, which include both $A_\mu A_\nu$-propagators and mixed propagators $A_\mu\lambda$. However mixed propagators and mixed proper vertex functions do not have pole singularities and being interested in the leading singular terms, we may drop them. The gauge-ghost vertices and ghost propagators which enter the r.h.s. of eq.(32) also have no pole singularities as in the absence of phase factors the corresponding integrals diverge logarithmically. Keeping only the terms which may have pole singularities we rewrite the eq(32) in the form

$$\frac{(p+q)_\rho}{(p+q)^2}(g_{\mu\alpha} - \frac{\tilde{p}_\mu\tilde{p}_\alpha}{\tilde{p}^2})(g_{\nu\beta} - \frac{\tilde{q}_\nu\tilde{q}_\beta}{\tilde{q}^2})\Gamma^{1,sing}_{\alpha\beta\rho}(p,q) = 0 \quad (33)$$

where $\Gamma^{1,sing}_{\mu\nu\rho}$ is the pole singular part of the one-loop proper vertex function. In deriving this equation we used the transversality of the free gauge field propagators with respect to $\tilde{p}_\mu$ and established earlier fact that the singular part of $\Pi_{\mu\nu}(p)$ is proportional to $\tilde{p}_\mu \tilde{p}_\nu$. By the same reasonings as above the singular part of $\Gamma^1_{\mu\nu\rho}(p,q)$ depends only on $\tilde{p}, \tilde{q}$. The only possible structure which has a proper symmetry and dimension, and satisfies the identity (33) is

$$\Gamma^{1,sing}_{\mu\nu\rho}(p,q) \sim \{\frac{\tilde{p}_\mu \tilde{p}_\nu \tilde{p}_\rho}{\xi^2 |\tilde{p}|^4} + (p \to q) + (p \to -(p+q))\} \tag{34}$$

Due to transversality of the free gauge field propagator $\Gamma^{1,sing}_{\mu\nu\rho}$ does not contribute to the vertex function with three external gauge lines. It might give a nonzero contribution to the diagram with four external gauge lines obtained by connecting $\Gamma^{1,sing}_{\mu\nu\rho}$ with the vertex (18) by the mixed propagators $A_\mu \lambda$. However as in the case of the two point polarization operator this diagram does not produce infrared singularities. Proper vertex functions with at least one external λ-line in the absence of the phase factors diverge logarithmically and do not produce infrared divergencies.

To analyze higher loop diagrams one should perform carefully the renormalization and check if our estimates remain valid. It is not done in the present paper. Assuming that renormalization does not introduce new problems we may basically repeat our arguments for arbitrary multiloop diagrams.

The singular part of the polarization operator again may be calculated by taking external momentum equal to zero everywhere except for the phase factors. (Of course we assume that necessary ultraviolet subtractions of divergent subgraphs are done in accordance with R-operation). Therefore the singular part of polarization operator at arbitrary order depends only on $\tilde{p}$. Gauge invariance and dimensional reasons fix the form of the singular part to $\tilde{p}_\mu \tilde{p}_\nu |\tilde{p}|^{-4}$. Hence the arguments given above to prove the absence of infrared singularities may be applied directly.

The singular part of the three point function depends only on $\tilde{p}, \tilde{q}$ and by gauge invariance must satisfy the identity (32). An analogue of the eq.(33) for a singular part of $\Gamma^n_{\mu\nu\rho}$ will now include the terms

$$\frac{(p+q)_\rho}{(p+q)^2} \sum_{m,l=1}^{n-1} D^m_{\mu\alpha}(p) D^l_{\nu\beta}(q) \Gamma^{n-m-l,sing}_{\mu\nu\rho}(p,q) \tag{35}$$

We proved that the two point Green functions and proper one-loop three-point Green functions have no pole singularities at zero momenta. Assuming that it is true for all $m < n$, we see that these terms do not produce singular contributions to $\Gamma^n_{\mu\nu\rho}(p,q)$ and its structure is also given by the eq.(34). It completes the induction.

A similar modification allows to formulate a consistent noncommutative $U(N)$ gauge theory. The standard noncommutative $U(N)$ Yang-Mills action is

$$S = \int d^4x \, \mathrm{tr} \, [-\frac{1}{8} F_{\mu\nu} * F_{\mu\nu}] \tag{36}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + g[A_\mu * A_\nu] \tag{37}$$

and A_μ belongs to $U(N)$ Lie algebra. Due to noncommutativity of the star product this action mixes the $U(1)$ and the $SU(N)$ gauge bosons, leading to UV/IR mixing analogous to the pure $U(1)$ case. The one loop diagrams in this theory were analyzed in ([10]). It appears that the planar diagrams in this theory may be renormalized in a gauge invariant way. Nonplanar diagrams with the $U(1)$ boson external lines are infrared singular, whereas the nonplanar diagrams with only $SU(N)$ boson external lines do not exibit infrared singularities.

The infrared pole singularities may be eliminated in analogy with the $U(1)$ case. Let us consider the modified $U(N)$ action:

$$S = \int d^4x \, \mathrm{tr}\,[-\frac{1}{8}F_{\mu\nu} * F_{\mu\nu}] + \lambda(x)\, \mathrm{tr}\,[\theta_{\mu\nu}F_{\mu\nu}] \tag{38}$$

where the Lagrange multiplier $\lambda(x)$ belongs to the adjoint representation of the $U(1)$ group, and $F_{\mu\nu}$ is the $U(N)$ curvature tensor.

The free action consists of the usual $SU(N)$ part and modified $U(1)$ action considered above. So the spectrum includes vector $SU(N)$ bosons and the scalar particle assosiated with the $U(1)$ group. The analysis of infrared singularities given above was based on the gauge invariance, power counting and the explicit form of $U(1)$ propagators. Therefore it may be applied directly to the diagrams with at least one external $U(1)$ line and leads to the same conclusion. These diagrams are free of infrared pole singularities. The one loop diagrams with only $SU(N)$ external lines were shown to be infrared safe. If this property holds at higher loops, the noncommutative $U(N)$ theory has the same infrared properties as the commutative one and is renormalizable.

3 Discussion

In this paper I wanted to show that consistent noncommutative quantum gauge theories free of infrared singularities may exist even in the absense of supersymmetry. The crucial observation which allows to construct such models is a possibility to describe by noncommutative gauge models not only vector but also scalar fields.

Several questions may be raised in this connection.

We did not consider carefully the ultraviolet renormalization of the theory. It seems very plausible that the ultraviolet renormalization preserves the invariance of the model and does not change our estimates of asymptotics, but it would be good to demonstrate it explicitely.

We concentrated in this paper on pole singularities, which lead to infrared divergency. However the logarithmic singularities, which do not cause infrared problems, may be present. Commutative limit of quantum theory deserves further investigation.

Our proof of existence of the noncommutative quantum $U(N)$ model assumed the absence of infrared singularities in the diagrams with pure $SU(N)$ external lines. To my knowleadge explicit proof of this fact has been given for one-loop diagrams ([10]). Although it is likely to be true for a general diagram, a careful study would be useful.

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