

Quadratic Algebra associated with Rational Calogero-Moser Models

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Abstract

Classical Calogero-Moser models with rational potential are known to be superintegrable. That is, on top of the n involutive conserved quantities necessary for the integrability of a system with n degrees of freedom, they possess an additional set of $r-n$ algebraically and functionally independent globally defined conserved quantities. At the quantum level, Kuznetsov uncovered the existence of a quadratic algebra structure as an underlying key for superintegrability for the models based on A type root systems. Here we demonstrate in a universal way the quadratic algebra structure for quantum rational Calogero-Moser models based on any root systems.

I Introduction

Calogero-Moser models [1, 2], [3] with the rational potentials, without the harmonic confining force, have the simplest and best understood dynamical structures among models with the other types of potentials. Their superintegrability, that is the existence of $2r-1$ global, functionally independent conserved quantities (constants of motion) for a system of r degrees of freedom, is one of the most striking features. It was found at the classical level by Wojciechowski [4] and at the quantum level by Kuznetsov [5] and Ujino-Wadati-Hikami [6], for models based on the A type root systems. Kuznetsov [5] uncovered an interesting algebraic structure, the so-called *quadratic algebra* as a hidden symmetry of the superintegrability. Ujino-Wadati-Hikami [6] introduced a similar algebraic structure. The concept of superintegrability is closely related with that of *algebraic linearizability* formulated by Caseiro-Françoise [7] and developed further by Caseiro-Françoise-Sasaki [8] for the models based on any root systems. We follow the notation of our previous paper unless otherwise stated.

In this paper we show, at the quantum level, that the quadratic algebra is ‘universal’, namely, it is enjoyed by all the rational Calogero-Moser models based on *any* root systems including the non-crystallographic ones. The same assertion at the classical level simply follows as the classical limit of replacing the quantum commutator by the Poisson bracket. The generators of the quadratic algebra are the above mentioned conserved quantities of the superintegrable theory. Among them, the involutive subset of r conserved quantities, which characterize the Liouville integrability of the system with r degrees of freedom, constitute the Cartan subalgebra and an ideal among the conserved quantities. Commutators among the additional conserved quantities turn out to be bi-linear (quadratic) combinations of the two types of conserved quantities. This non-linear algebra seems to be closely related to the W -algebras [9], extensions of the Virasoro algebra, or to the algebras related with the R -matrices of integrable theories [10] but the precise relationship remains to be clarified.

Calogero-Moser models for any root systems were formulated by Olshanetsky and Perelomov [11], who provided Lax pairs for the models based on the classical root systems, *i.e.* the A , B , C , D and BC type root systems. A universal *classical* Lax pair applicable to all the Calogero-Moser models based on any root systems including the E_6 and the non-crystallographic root systems was derived by Bordner-Corrigan-Sasaki [12] which unified various types of Lax pairs known at that time [13, 14]. A universal *quantum* Lax pair ap-

plicable to all the Calogero-Moser models based on any root systems and for degenerate potentials was derived by Bordner-Manton-Sasaki [15] which provided the basic tools for the present paper.

The purpose of the present paper is twofold. Firstly, to derive and present the *quadratic algebra* for rational Calogero-Moser models based on any root systems in its fullest universality based on the universal Lax pair [15]. Extracting detailed information from the quadratic algebra to elucidate dynamical properties of each specific system would require formulations suitable for the particular systems. This would not be discussed here. Secondly, we formulate and present the quantum versions of various concepts and formulas related to the algebraic linearizability introduced and developed in [8]. As has been noticed from the earliest days of Calogero-Moser models, the quantum and classical integrability are very closely related. Many formulas related to the algebraic linearizability take the same form at the classical and quantum levels, with some notable exceptions as will be mentioned in the paper.

This paper is organized as follows. In section two we introduce the model and notations with an emphasis on the difference between the quantum and classical versions. The quantum theorem of the algebraic linearizability for the rational model is derived based on the Lax pair formalism. In section three we evaluate fundamental commutation relations which are necessary for the quadratic algebra. This will be carried out with the help of the Dunkl operators, or the so-called \mathcal{D} operators which are equivalent to the quantum \mathcal{D} operator. The quantum theorem of the algebraic linearizability for the higher Hamiltonians of the rational model is derived. In section four the quadratic algebra for rational Calogero-Moser models is derived and presented in its fullest universality. Section five gives the quantum version of the algebraic linearizability of the rational potential model with harmonic confining force. The problem of quantum integrability of rational Calogero-Moser model with quartic interactions is not yet settled. In section six we present a partial result that the quantum equations of motion can be cast into Lax type matrix equations. The existence of quantum conserved quantities, however, does not follow from these matrix equations. In section seven the quantum version of the algebraic linearizability for trigonometric (hyperbolic) Calogero-Sutherland models is given for those models based on root systems which have minimal representations. The final section is for comments on the hermiticity of the algebra generators.

II Quantum Calogero-Moser Models with Rational Potential

Let us start with the Hamiltonian of *quantum* Calogero-Moser model with rational potential based on any root system, which could be any one of the crystallographic root systems, A_r , B_r , C_r , D_r , (BC_r) , E_6 , E_7 , E_8 , F_4 and G_2 or the non-crystallographic H_3 , H_4 and $I_2(m)$, which is the dihedral root system associated with a regular m -gon. The existing works on the quadratic algebras are all for the A_r root system [5, 6, 20]. Let us denote by Δ a root system of rank r . The dynamical variables are the coordinates $q_i, i = 1, \dots, r$ and their canonically conjugate momenta $p_i, i = 1, \dots, r$, with the canonical commutation relations:

$$[q_j, p_k] = i\delta_{jk}, \quad [q_j, q_k] = [p_j, p_k] = 0, \quad j, k = 1, \dots, r. \quad (2.1)$$

As usual the momentum operator p_j acts as a derivative operator on a (wave) function f of q :

$$f \rightarrow p_j f : \quad (p_j f)(q) = -i \frac{\partial f(q)}{\partial q_j}, \quad j = 1, \dots, r.$$

The Hamiltonian for the *quantum* Calogero-Moser model with rational potential is very simple:

$$\mathcal{H} = \frac{1}{2}p^2 + \frac{1}{2} \sum_{\rho \in \Delta_+} \frac{g_{|\rho|}(g_{|\rho|} - 1)|\rho|^2}{(\rho \cdot q)^2}, \quad \Delta_+ : \text{set of positive roots}, \quad (2.2)$$

in which the real coupling constants $g_{|\rho|} > 0$ are defined on orbits of the corresponding finite reflection group, *i.e.* they are identical for roots in the same orbit. The only difference with the classical Hamiltonian is the coupling constant dependence, $g_{|\rho|}(g_{|\rho|} - 1)$ instead of $g_{|\rho|}^2$ in the classical case. The Hamiltonian is invariant under reflections of the phase space variables about a hyperplane perpendicular to any root

$$\mathcal{H}(s_\alpha(p), s_\alpha(q)) = \mathcal{H}(p, q), \quad \forall \alpha \in \Delta, \quad (2.3)$$

with the action of s_α on any vector $\gamma \in \mathbf{R}^r$ defined by

$$s_\alpha(\gamma) = \gamma - (\alpha^\vee \cdot \gamma)\alpha, \quad \alpha^\vee \equiv 2\alpha/|\alpha|^2. \quad (2.4)$$

The integrability is best understood in terms of the quantum Lax pair [15] or the Dunkl operators [16, 17], which are known to be equivalent with the Lax operator [18]. Let us choose a set of \mathbf{R}^r vectors $\mathcal{R} = \{\mu^{(k)} \in \mathbf{R}^r, k = 1, \dots, D\}$, which form a D -dimensional representation of the Coxeter group. That is, they are permuted under the action of the

Coxeter group and they form a single orbit. For example, we can choose the set of vector (minimal) weights for A_r or D_r , or the set of short (long) roots for B_n , C_n or F_4 , G_2 or the set of all roots for E_6 to E_8 . Then the Lax operators are $D \times D$ dimensional matrices

$$\begin{aligned} L(p, q) &= p \cdot \hat{H} + X(q), & X(q) &= i \sum_{\rho \in \Delta_+} g_{|\rho|} \frac{\rho \cdot \hat{H}}{\rho \cdot q} \hat{s}_\rho, \\ M(q) &= -\frac{i}{2} \sum_{\rho \in \Delta_+} g_{|\rho|} \frac{|\rho|^2}{(\rho \cdot q)^2} (\hat{s}_\rho - I), \end{aligned} \quad (2.5)$$

consisting of operators $\{\hat{H}_j\}$, $(j=1, \dots, r)$, $\{\hat{s}_\rho\}$ and the identity operator I . Their matrix elements are defined by:

$$(\hat{H}_j)_{\mu\nu} = \mu_j \delta_{\mu\nu}, \quad (\hat{s}_\rho)_{\mu\nu} = \delta_{\mu, s_\rho(\nu)} = \delta_{\nu, s_\rho(\mu)}, \quad \mu, \nu \in \mathcal{R}. \quad (2.6)$$

The Lax operators are Coxeter covariant:

$$L(s_\alpha(p), s_\alpha(q)) = \hat{s}_\alpha L(p, q) \hat{s}_\alpha, \quad M(s_\alpha(q)) = \hat{s}_\alpha M(q) \hat{s}_\alpha, \quad (2.7)$$

and I is hermitian $I^\dagger = I$ and M is anti-hermitian $M^\dagger = -M$.

We see that the Heisenberg equations of motion are equivalent to a matrix equation [15, 18]

$$\frac{dL}{dt} \equiv \dot{L} = i[\mathcal{H}, L] = [L, M], \quad (2.8)$$

in which the matrix elements are quantum operators. This means that in general the *trace* of the product of two matrix operators $A(p, q)$ and $B(p, q)$ is not commutative, $\text{Tr} AB \neq \text{Tr} BA$, or $\text{Tr}[A, B] \neq 0$, implying that $\text{Tr} L^n$ is not *conserved* in quantum theory. However, thanks to the special property of the above M matrix

$$\sum_{\mu \in \mathcal{R}} M_{\mu\nu} = \sum_{\nu \in \mathcal{R}} M_{\mu\nu} = 0, \quad (2.9)$$

the *total sum* of the powers of the Lax operator L defined by

$$F_j = \text{Ts}(L^j) \equiv \sum_{\nu, \mu \in \mathcal{R}} (L^j)_{\mu\nu}, \quad j = 0, 1, \dots, D-1, \quad (2.10)$$

is conserved:

$$\begin{aligned} \frac{d}{dt} \text{Ts}(L^j) &= \sum_{\mu, \nu \in \mathcal{R}} [(L^j M)_{\mu\nu} - (M L^j)_{\mu\nu}] \\ &= \sum_{\mu, \nu, \kappa \in \mathcal{R}} [L_{\mu\kappa}^j M_{\kappa\nu} - M_{\mu\kappa} L_{\kappa\nu}^j] \\ &= \sum_{\mu, \kappa \in \mathcal{R}} L_{\mu\kappa}^j \left(\sum_{\nu \in \mathcal{R}} M_{\kappa\nu} \right) - \sum_{\nu, \kappa \in \mathcal{R}} \left(\sum_{\mu \in \mathcal{R}} M_{\mu\kappa} \right) L_{\kappa\nu}^j = 0. \end{aligned}$$

It is easy to see from (2.7) that $\{F_j\}$'s are Coxeter invariant. These form the involutive set of conserved quantities of the theory. Not all of them are independent. As is well-known, the independent conserved quantities appear for such j as 1 plus *exponent* of the root system, (see, for example, [18, 19]). For some choice of the set of vectors \mathcal{R} for some root system Δ , F_j can be vanishing for certain j 's. For example, if \mathcal{R} contains a vector μ and $-\mu$ at the same time then $F_{odd} \equiv 0$.

The Hamiltonian (2.2) is proportional to F_2 ,

$$\mathcal{H} = \frac{1}{2C_{\mathcal{R}}} F_2 = \frac{1}{2C_{\mathcal{R}}} \text{Ts}(L^2), \quad (2.11)$$

in which the coefficient $C_{\mathcal{R}}$ is defined by

$$\text{Ts}(\hat{H}_j \hat{H}_k) = \sum_{\mu \in \mathcal{R}} \mu_j \mu_k = \delta_{jk} C_{\mathcal{R}}. \quad (2.12)$$

Following the line of argument of [8] we define

$$Q = q \cdot \hat{H}, \quad G_j = \text{Ts}(QL^j), \quad G_j^{(2)} = \text{Ts}(Q^2 L^j), \quad j = 0, 1, \dots, D-1, \quad (2.13)$$

in which the last quantity $Q^2 L^j$ was introduced by Rañada [21]. Under the reflection, Q transforms in the same way as L and M , (2.7):

$$q \rightarrow s_{\alpha}(q), \quad Q(s_{\alpha}(q)) = \hat{s}_{\alpha} Q(q) \hat{s}_{\alpha}. \quad (2.14)$$

Thus G_j and $G_j^{(2)}$ are Coxeter invariant, too. The time evolution of Q is exactly the same as in the classical case [8]

$$\dot{Q} = [Q, M] + L, \quad (2.15)$$

leading to the corresponding result:

$$\begin{aligned} \dot{G}_j &= \text{Ts}(\dot{Q} L^j + Q \dot{L}^j) \\ &= \text{Ts}(Q M L^j - M Q L^j + L^{j+1} + Q L^j M - Q M L^j) \\ &= \text{Ts}(L^{j+1}) - \sum_{\nu, \kappa \in \mathcal{R}} \left(\sum_{\mu \in \mathcal{R}} M_{\mu\kappa} \right) (Q L^j)_{\kappa\nu} + \sum_{\mu, \kappa \in \mathcal{R}} (Q L^j)_{\mu\kappa} \left(\sum_{\nu \in \mathcal{R}} M_{\kappa\nu} \right) \\ &= \text{Ts}(L^{j+1}) = F_{j+1}. \end{aligned} \quad (2.16)$$

Like $\{F_j\}$'s not all of $\{G_j\}$'s are independent. Independent $\{G_j\}$ appear when $\{j\}$ are the exponents of Δ . This provides the algebraic linearization of the quantum models. Like in the classical theory we have:

Proposition II.1

The quantum Calogero-Moser system (2.2) is superintegrable for any root system.

Proof. On top of the D first integrals F_k which are in involution, we have the $D(D-1)/2$ extra first integrals defined by

$$H_{k,k'} = F_{k+1}G_{k'} - F_{k'+1}G_k, \quad (2.17)$$

$$\dot{H}_{k,k'} = i[\mathcal{H}, H_{k,k'}] = 0. \quad (2.18)$$

Like in our previous paper for the classical systems [8], we do not demonstrate that these $D(D-1)/2$ $\{H_{k,k'}\}$'s contain $r-1$ algebraically independent ones. That would require detailed exhaustive arguments for each root system. We refer to [18] for general arguments of independence of $\{F_i\}$ type conserved quantities.

For the quantum models based on the A type root system, a similar result was derived by Gonera [20] based on a $sl(2, \mathbf{R})$ representation. The time evolution of $G_j^{(2)}$ is slightly complicated:

$$\begin{aligned} \dot{G}_j^{(2)} &= \text{Ts}(\dot{Q}QL^j) + \text{Ts}(Q\dot{Q}L^j) + \text{Ts}(Q^2\dot{L}^j) \\ &= \text{Ts}(LQL^j) + \text{Ts}(QL^{j+1}). \end{aligned} \quad (2.19)$$

Since L and Q do not commute in quantum theory, the classical relation $\dot{G}_j^{(2)} = 2\text{Tr}(QL^{j+1}) = 2G_{j+1}$ does not hold any longer. In quantum theory we have

$$QL - LQ = i\delta_{kl}\hat{H}_k\hat{H}_l + iK, \quad K \equiv \sum_{\rho \in \Delta_+} g_{|\rho|}(\rho \cdot \hat{H})(\rho^\vee \cdot \hat{H})\hat{s}_\rho. \quad (2.20)$$

The right hand side gives the ‘quantum corrections’. Thus we arrive at

$$\dot{G}_j^{(2)} = 2\text{Ts}(QL^{j+1}) - i\delta_{kl}\text{Ts}(\hat{H}_k\hat{H}_lL^j) - i\text{Ts}(KL^j). \quad (2.21)$$

The second term is easy to evaluate, since

$$\delta_{kl}\text{Ts}(\hat{H}_k\hat{H}_lL^j) = \delta_{kl} \sum_{\mu, \nu \in \mathcal{R}} (\mu_k\mu_l(L^j)_{\mu\nu}) = \mu^2\text{Ts}(L^j), \quad (2.22)$$

in which μ^2 is the same for all $\mu \in \mathcal{R}$. The third term reads

$$\text{Ts}(KL^j) = \sum_{\rho \in \Delta_+} \sum_{\mu, \nu \in \mathcal{R}} g_{|\rho|}(\rho \cdot \mu)(\rho^\vee \cdot \mu)(L^j)_{\mu\nu},$$

and for any vector $\mu \in \mathbf{R}^r$ we have

$$\sum_{\rho \in \Delta_+} g_{|\rho|}(\rho \cdot \mu)(\rho^\vee \cdot \mu) = \frac{2}{r}\mu^2 \sum_{\rho \in \Delta_+} g_{|\rho|}, \quad (2.23)$$

in which

$$\frac{2}{r} \sum_{\rho \in \Delta_+} g_{|\rho|} \quad (2.24)$$

can be considered as a *deformed Coxeter number*. For $g_{|\rho|} \equiv 1$ it reduces to the Coxeter number. Thus we arrive at a quantum formula

$$\begin{aligned} \dot{G}_j^{(2)} &= 2G_{j+1} - i\mu^2 \left(1 + \frac{2}{r} \sum_{\rho \in \Delta_+} g_{|\rho|} \right) F_j \\ &= 2G_{j+1} - i\mu^2 \frac{2}{r} \tilde{\mathcal{E}}_0 F_j. \end{aligned} \quad (2.25)$$

Here, the coefficient of the quantum corrections term $\tilde{\mathcal{E}}_0$ is defined by

$$\tilde{\mathcal{E}}_0 = \frac{r}{2} + \sum_{\rho \in \Delta_+} g_{|\rho|}. \quad (2.26)$$

which characterizes the ground state energy of the rational Calogero-Moser model with harmonic confining force, see, for example, (2.21) of [18]. This fact is closely related with the $sl(2, \mathbf{R})$ algebra for rational Calogero-Moser models discussed by many authors, see for example [22]-[24], [17, 20]. We will not discuss $G_j^{(2)}$ any longer in this paper, except for some comments in the final section.

III Basic Commutation Relations

Typical generators of the quadratic algebra are $\{F_j\}$'s (2.10) and $\{H_{k,l}\}$'s (2.17). Namely they are either linear in $\{F_j\}$'s or bi-linear combinations of $\{F_j\}$'s and $\{G_k\}$'s. As will be clear in later discussions, see for example (4.10), the set of $\{F_j\}$'s must be understood in the broadest sense to include the dependent ones. That is, any polynomials in the independent \mathbf{n} involutive conserved quantities are allowed. For example, F_j for $j \neq 1 + \text{exponent}$ or $j > h$ (the Coxeter number) enter into the theory naturally. Likewise, the set of $\{G_j\}$'s include the dependent ones, which are independent ones times any polynomial in $\{F_k\}$'s. In order to explore and present the full content of the quadratic algebra, we need to evaluate the commutators like:

$$[F_j, F_k], \quad [F_j, G_k], \quad [G_j, G_k]. \quad (3.1)$$

For this purpose the Dunkl operators [16] or \mathbf{L} operators which are the *vector* version of the Lax matrix operator \mathbf{L} [18] are useful:

$$\ell_\mu = \ell \cdot \mu = p \cdot \mu + i \sum_{\rho \in \Delta_+} g_{|\rho|} \frac{\rho \cdot \mu}{\rho \cdot q} \check{s}_\rho, \quad \mu \in \mathcal{R}, \quad (3.2)$$

in which another reflection operator \check{s}_ρ acts on a (wave) function f of q as

$$f \rightarrow \check{s}_\rho f : \quad (\check{s}_\rho f)(q) = f(s_\rho(q)). \quad (3.3)$$

The ℓ operator is linear in μ , Coxeter covariant and hermitian:

$$\check{s}_\rho \ell_\mu \check{s}_\rho = \ell_{s_\rho(\mu)}, \quad \ell_\mu = \ell_\mu^\dagger, \quad \forall \rho \in \Delta. \quad (3.4)$$

It is shown [18] that the Hilbert space of any quantum Calogero-Moser system consists of Coxeter invariant wavefunctions. That is, they satisfy

$$\check{s}_\rho \psi = \psi, \quad \forall \rho \in \Delta. \quad (3.5)$$

It is well-known that the ℓ operators for the rational Calogero-Moser models commute:

$$[\ell_\mu, \ell_\nu] = 0, \quad \forall \mu, \nu \in \mathcal{R}. \quad (3.6)$$

The relationship between L and ℓ is simple. For any Coxeter invariant function ψ , $F_k \psi$ and $G_k \psi$ is Coxeter invariant too, and we have [18]:

$$F_k \psi \equiv \text{Ts}(L^k) \psi = \sum_{\mu \in \mathcal{R}} \ell_\mu^k \psi, \quad \forall k \in \mathbf{Z}_+, \quad (3.7)$$

$$G_k \psi \equiv \text{Ts}(QL^k) \psi = \sum_{\mu \in \mathcal{R}} q \cdot \mu \ell_\mu^k \psi, \quad \forall k \in \mathbf{Z}_+. \quad (3.8)$$

The involution of $\{F_j\}$'s is a simple consequence of (3.6) and (3.7):

$$[F_j, F_k] = 0, \quad \forall j, k \in \mathbf{Z}_+, \quad (3.9)$$

which is a well-known result.

For the evaluation of the second and third types of commutators in (3.1) we need to know in general

$$[\ell_\mu^n, q \cdot \nu \ell_\nu^m]. \quad (3.10)$$

It is straightforward to show by induction

$$[\ell_\mu^j, q \cdot \nu] = -i \left[j \mu \cdot \nu \ell_\mu^{j-1} + \sum_{\rho \in \Delta_+} g_{|\rho|}(\rho \cdot \mu)(\rho^\vee \cdot \nu) \frac{\ell_\mu^j - \ell_{s_\rho(\mu)}^j}{\ell_\mu - \ell_{s_\rho(\mu)}} \check{s}_\rho \right], \quad (3.11)$$

starting from

$$[\ell_\mu, q \cdot \nu] = -i \left[\mu \cdot \nu I + \sum_{\rho \in \Delta_+} g_{|\rho|}(\rho \cdot \mu)(\rho^\vee \cdot \nu) \check{s}_\rho \right], \quad (3.12)$$

and

$$[\ell_\mu^2, q \cdot \nu] = -i \left[2(\mu \cdot \nu) \ell_\mu + \sum_{\rho \in \Delta_+} g_{|\rho|}(\rho \cdot \mu)(\rho^\vee \cdot \nu) \frac{\ell_\mu^2 - \ell_{s_\rho(\mu)}^2}{\ell_\mu - \ell_{s_\rho(\mu)}} \check{s}_\rho \right]. \quad (3.13)$$

Here the fraction of operators, $\ell_\mu^j - \ell_{s_\rho(\mu)}^j / \ell_\mu - \ell_{s_\rho(\mu)}$, is well defined since the ℓ operators commute with each other, (3.6). For example, we have $\ell_\mu^2 - \ell_{s_\rho(\mu)}^2 / \ell_\mu - \ell_{s_\rho(\mu)} = \ell_\mu + \ell_{s_\rho(\mu)}$. Thus we arrive at

$$\begin{aligned} [\ell_\mu^j, q \cdot \nu \ell_\nu^k] &= [\ell_\mu^j, q \cdot \nu] \ell_\nu^k \\ &= -i \left[j \mu \cdot \nu \ell_\mu^{j-1} \ell_\nu^k + \sum_{\rho \in \Delta_+} g_{|\rho|}(\rho \cdot \mu)(\rho^\vee \cdot \nu) \frac{\ell_\mu^j - \ell_{s_\rho(\mu)}^j}{\ell_\mu - \ell_{s_\rho(\mu)}} \ell_{s_\rho(\nu)}^k \check{s}_\rho \right]. \end{aligned} \quad (3.14)$$

The second term in the right hand side of (3.14) vanishes when summed over μ :

$$V \equiv \sum_{\mu \in \mathcal{R}} g_{|\rho|}(\rho \cdot \mu)(\rho^\vee \cdot \nu) \frac{\ell_\mu^j - \ell_{s_\rho(\mu)}^j}{\ell_\mu - \ell_{s_\rho(\mu)}} \ell_{s_\rho(\nu)}^k \check{s}_\rho = 0. \quad (3.15)$$

This can be seen as follows. The set \mathcal{R} is Coxeter invariant, *i.e.*, $s_\rho(\mathcal{R}) = \mathcal{R}$. Consider the change of variables $\mu' = s_\rho(\mu)$, then $\mu = s_\rho(\mu')$ and

$$\begin{aligned} V &= \sum_{\mu' \in \mathcal{R}} g_{|\rho|}(\rho \cdot s_\rho(\mu'))(\rho^\vee \cdot \nu) \frac{\ell_{s_\rho(\mu')}^j - \ell_{\mu'}^j}{\ell_{s_\rho(\mu')} - \ell_{\mu'}} \ell_{s_\rho(\nu)}^k \check{s}_\rho \\ &= \sum_{\mu' \in \mathcal{R}} g_{|\rho|}(-\rho \cdot \mu')(\rho^\vee \cdot \nu) \frac{\ell_{\mu'}^j - \ell_{s_\rho(\mu')}^j}{\ell_{\mu'} - \ell_{s_\rho(\mu')}} \ell_{s_\rho(\nu)}^k \check{s}_\rho = -V. \end{aligned}$$

By summing over μ and μ' , we obtain from (3.14)

$$\left[\sum_{\mu \in \mathcal{R}} \ell_\mu^j, \sum_{\nu \in \mathcal{R}} q \cdot \nu \ell_\nu^k \right] = -ij \sum_{\mu, \nu \in \mathcal{R}} (\mu \cdot \nu) \ell_\mu^{j-1} \ell_\nu^k. \quad (3.16)$$

The right hand side is a Coxeter invariant polynomial in ℓ_μ , which corresponds to a polynomial in $\{F_j\}$ to be denoted by $F_{k,j}$:

$$\sum_{\mu, \nu \in \mathcal{R}} (\mu \cdot \nu) \ell_\mu^{j-1} \ell_\nu^k \psi \equiv F_{k,j} \psi, \quad \psi : \text{Coxeter invariant}. \quad (3.17)$$

Thus we arrive at

$$i[F_j, G_k] = jF_{k,j} \quad (3.18)$$

$$[F_n, F_{k,j}] = 0, \quad \forall n \in \mathbf{Z}. \quad (3.19)$$

When the set of vectors \mathcal{R} consists of orthonormal vectors, for example, the vector representation of A_r embedded in an $r+1$ dimensional space, or vector representations of

C_r and D_r , or the set of short roots of B_r , the above $F_{k,j}$ has a simpler expression. In such cases, only $u = \pm \nu$ terms in (3.17) survive and we have

$$\sum_{\mu, \nu \in \mathcal{R}} (\mu \cdot \nu) \ell_\mu^{j-1} \ell_\nu^k = \begin{cases} C_{\mathcal{R}} \sum_{\nu \in \mathcal{R}} \ell_\nu^{j+k-1} \\ 0 \end{cases}, \quad (3.20)$$

in which $C_{\mathcal{R}}$ is defined by (2.12). That is, (3.18) is replaced by a more explicit formula

$$i[F_j, G_k] = j C_{\mathcal{R}} F_{j+k-1}, \quad (3.21)$$

which was reported in Kuznetsov's paper for A_r case [5] ($C_{\mathcal{R}} = 1$). (In the above formula we assume that neither F_j nor G_k vanish.) As for the extra exponent at $r-1$ in D_r theory, the corresponding H and G operators are best expressed by ℓ operators in the orthonormal basis:

$$F_{r'} \leftrightarrow \ell_1 \cdots \ell_r, \quad G_{r'-1} \leftrightarrow \sum_{j=1}^r q_j \ell_1 \cdots \tilde{\ell}_j \cdots \ell_r, \quad (3.22)$$

in which $\tilde{\ell}_j$ means that the factor is missing.

The general commutation relations (3.18), (3.19) provide the algebraic linearization of the Hamiltonian systems generated by the higher conserved quantities $\{F_j\}$.

Proposition III.1

The Hamiltonian system generated by the higher conserved quantity F_j (2.10) of quantum Calogero-Moser system (2.2) is superintegrable for any root system.

Proof. On top of the D first integrals F_k , we have the $D(D-1)/2$ extra first integrals for the Hamiltonian F_j :

$$H_{k,k'}^{(j)} = F_{k,j} G_{k'} - F_{k',j} G_k, \quad (3.23)$$

$$\frac{dH_{k,k'}^{(j)}}{dt_j} = i[F_j, H_{k,k'}^{(j)}] = 0. \quad (3.24)$$

IV Quadratic Algebra

In order to evaluate the commutators among various $\{H_{k,k'}^{(j)}\}$'s we need the knowledge of the third type of commutators in (3.1), that is $[G_j, G_k]$. From (3.14) we have

$$\begin{aligned} & [q \cdot \mu \ell_\mu^j, q \cdot \nu \ell_\nu^k] \\ &= q \cdot \mu [\ell_\mu^j, q \cdot \nu \ell_\nu^k] + [q \cdot \mu, q \cdot \nu \ell_\nu^k] \ell_\mu^j \end{aligned}$$

$$\begin{aligned}
&= -i \left\{ (\mu \cdot \nu) j (q \cdot \mu) \ell_\mu^{j-1} \ell_\nu^k + \sum_{\rho \in \Delta_+} g_{|\rho|} (q \cdot \mu) (\rho \cdot \mu) (\rho^\vee \cdot \nu) \frac{\ell_\mu^j - \ell_{s_\rho(\mu)}^j}{\ell_\mu - \ell_{s_\rho(\mu)}} \ell_{s_\rho(\nu)}^k \check{s}_\rho \right\} \\
&\quad + i \left\{ (\mu \cdot \nu) k (q \cdot \nu) \ell_\nu^{k-1} \ell_\mu^j + \sum_{\rho \in \Delta_+} g_{|\rho|} (q \cdot \nu) (\rho \cdot \nu) (\rho^\vee \cdot \mu) \frac{\ell_\nu^k - \ell_{s_\rho(\nu)}^k}{\ell_\nu - \ell_{s_\rho(\nu)}} \ell_{s_\rho(\mu)}^j \check{s}_\rho \right\}. \quad (4.1)
\end{aligned}$$

As in the previous case (3.15), the coupling constant dependent terms, that is the second and fourth terms in (4.1) cancel with each other when summed over μ and ν :

$$\begin{aligned}
&\left[\sum_{\mu \in \mathcal{R}} (q \cdot \mu) \ell_\mu^j, \sum_{\nu \in \mathcal{R}} (q \cdot \nu) \ell_\nu^k \right] \\
&= -i \sum_{\mu, \nu \in \mathcal{R}} \left\{ j (\nu \cdot \mu) (q \cdot \mu) \ell_\mu^{j-1} \ell_\nu^k - k (\mu \cdot \nu) (q \cdot \nu) \ell_\nu^{k-1} \ell_\mu^j \right\}. \quad (4.2)
\end{aligned}$$

Both terms in the right hand side are Coxeter invariant polynomials in q and ℓ which are linear in q and of degree $j+k-1$ in ℓ . Therefore they are expressible as linear combination of $\{G_l\}$'s or polynomials in $\{F_m\}$'s multiplied on them. This can be checked by direct calculation or by using the Jacobi identity on the left hand side. Thus we express

$$\begin{aligned}
\sum_{\mu, \nu \in \mathcal{R}} (\mu \cdot \nu) (q \cdot \mu) \ell_\mu^{j-1} \ell_\nu^k \psi &\equiv G_{j,k} \psi, \\
\sum_{\mu, \nu \in \mathcal{R}} (\mu \cdot \nu) (q \cdot \nu) \ell_\mu^j \ell_\nu^{k-1} \psi &\equiv G_{k,j} \psi,
\end{aligned} \quad \psi : \text{Coxeter invariant.} \quad (4.3)$$

We arrive at the following general commutation relation

$$i [G_j, G_k] = j G_{j,k} - k G_{k,j}. \quad (4.4)$$

These $\{G_{j,k}\}$'s satisfy the same type of commutation relations as above.

When the set of vectors \mathcal{R} consists of orthonormal vectors, we have

$$\sum_{\mu, \nu \in \mathcal{R}} (\mu \cdot \nu) (q \cdot \mu) \ell_\mu^{j-1} \ell_\nu^k = \sum_{\mu, \nu \in \mathcal{R}} (\mu \cdot \nu) (q \cdot \nu) \ell_\mu^j \ell_\nu^{k-1} = \begin{cases} C_{\mathcal{R}} \sum_{\nu \in \mathcal{R}} (q \cdot \nu) \ell_\nu^{j+k-1} \\ 0 \end{cases}. \quad (4.5)$$

This leads to a simplified commutation relation

$$i [G_j, G_k] = (j - k) C_{\mathcal{R}} G_{j+k-1}, \quad (4.6)$$

which was reported in Kuznetsov's paper for A_r case [5] ($C_{\mathcal{R}} = 1$).

To sum up, we have obtained the following general commutation relations:

$$[F_j, F_k] = 0, \quad (4.7)$$

$$i [F_j, G_k] = j F_{k,j}, \quad (4.8)$$

$$i [G_j, G_k] = j G_{j,k} - k G_{k,j}. \quad (4.9)$$

By using these the operators $\{F_j\}$'s and $\{H_{k,l}^{(m)}\}$'s defined by

$$H_{k,l}^{(m)} = F_{k,m}G_l - F_{l,m}G_k, \quad H_{k,l}^{(m)} = -H_{l,k}^{(m)}, \quad (4.10)$$

generate a quadratic algebra

$$[F_j, F_k] = 0, \quad (4.11)$$

$$i[F_j, H_{k,l}^{(m)}] = j(F_{k,m}F_{l,j} - F_{l,m}F_{k,j}), \quad (4.12)$$

$$i[H_{k,l}^{(m)}, H_{k',l'}^{(m')}] = \text{quadratic in } H_{r,s}^{(n)} \text{ and } F_t. \quad (4.13)$$

This is the quadratic algebra of the quantum rational Calogero-Moser models based on any root systems. For the classical root systems it can be simplified by using the relations (3.21) and (4.6) to the forms given in Kuznetsov's paper [5]. It characterizes the superintegrability structure of quantum models. In applications for specific models, the indices of $\{F\}$'s and $\{G\}$'s and $\{H\}$'s must be chosen properly. This would give more specific forms of the quadratic algebra relations.

V Rational Potential Model with Harmonic Confining Force

The arguments for the algebraic linearization for the *quantum* rational potential model with harmonic confining force go almost parallel with the classical ones. So we present only the key formulas. We have to note the coupling dependence is changed from $g_{|\rho|}^2$ (classical) to $g_{|\rho|}(g_{|\rho|} - 1)$ (quantum) and instead of *trace* (Tr, classical) we need the *total sum* (Ts, quantum). The Hamiltonian is now:

$$\mathcal{H}_\omega = \frac{1}{2}p^2 + \frac{1}{2}\omega^2 q^2 + \frac{1}{2} \sum_{\rho \in \Delta_+} \frac{g_{|\rho|}(g_{|\rho|} - 1)|\rho|^2}{(\rho \cdot q)^2}. \quad (5.1)$$

With the same matrix operators \mathbf{L} , \mathbf{M} and \mathbf{Q} as in the rational case the equations of motion can be expressed in matrix forms:

$$\dot{\mathbf{L}} = [\mathbf{L}, \mathbf{M}] - \omega^2 \mathbf{Q}, \quad \dot{\mathbf{Q}} = [\mathbf{Q}, \mathbf{M}] + \mathbf{L}. \quad (5.2)$$

Introduce the matrices

$$L^\pm = L \pm i\omega Q \quad (5.3)$$

whose time evolution read

$$\dot{L}^{\pm} = [L^{\pm}, M] \pm i\omega L^{\pm}. \quad (5.4)$$

They can be cast into a Lax form for $\mathcal{L} = L^+ L^-$ as

$$\dot{\mathcal{L}} = [\mathcal{L}, M]. \quad (5.5)$$

Consider then the functions:

$$F_k = \text{Ts}(L^+ \mathcal{L}^k), \quad G_k = \text{Ts}(L^- \mathcal{L}^k). \quad (5.6)$$

The time evolution yields

$$\dot{F}_k = i\omega F_k, \quad \dot{G}_k = -i\omega G_k. \quad (5.7)$$

Thus these functions provide the algebraic linearization of the quantum system.

VI Rational Model with a Quartic Potential

As proved by Françoise and Ragnisco [25] for the models based on the A type root systems and by us [8] for the models based on any root systems, the rational Calogero-Moser model can be deformed into an integrable system by adding a quartic potential at the *classical* level. Here we provide a partial result at the quantum level. The equation of motion can be cast into Lax type equations but they fail to produce conserved quantities.

Define again the same matrices L , Q , X and M . Let

$$h(Q) = aQ + bQ^2 \quad (6.1)$$

be a matrix quadratic in Q ; (a, b) are just two new independent parameters. The perturbed Hamiltonian is now:

$$\mathcal{H}_h \propto \text{Ts}(L^2 + h(Q)^2). \quad (6.2)$$

Like in the classical theory, the equations of motion can be cast into matrix forms by defining

$$L^{\pm} = L \pm ih(Q), \quad \mathcal{L}_1 = L^+ L^-, \quad \mathcal{L}_2 = L^- L^+. \quad (6.3)$$

Though care is needed for the quantum non-commutativity, the calculation is essentially the same as in the classical theory and we arrive at the time evolution of L^{\pm} and $\mathcal{L}_1, \mathcal{L}_2$:

$$\dot{L}^{\pm} = [L^{\pm}, M] \pm i\frac{1}{2} \left(h'(Q) L^{\pm} + L^{\pm} h'(Q) \right), \quad (6.4)$$

$$\dot{\mathcal{L}}_1 = [\mathcal{L}_1, M - \frac{i}{2}h'(Q)], \quad \dot{\mathcal{L}}_2 = [\mathcal{L}_2, M + \frac{i}{2}h'(Q)]. \quad (6.5)$$

Because of the added term $\pm \frac{i}{2}h'(Q)$ to the M matrix, it loses the sum up to zero property (2.9) and thus *neither trace nor total sum* of the powers of $\mathcal{L}_{1,2}$ are conserved at the quantum level.

VII Trigonometric Calogero-Sutherland Model

The algebraic linearization of the trigonometric (hyperbolic) *classical* Calogero-Sutherland model was shown in our previous paper [8] for the root systems which have minimal representations, that is A and D series and E_6 and E_7 . These are all simply laced algebras and all the roots have the same length. The quantum Hamiltonian reads

$$\mathcal{H} = \frac{1}{2}p^2 + \frac{1}{2}g(g-1)|\alpha|^2 \sum_{\alpha \in \Delta_+} \frac{1}{\sin^2(\alpha \cdot q)}. \quad (7.1)$$

We consider the Lax matrices:

$$L = p \cdot \hat{H} + X, \quad X = ig \sum_{\rho \in \Delta_+} (\rho \cdot \hat{H}) \frac{1}{\sin(\rho \cdot q)} \hat{s}_\rho, \quad (7.2)$$

$$M = -\frac{ig|\rho|^2}{2} \sum_{\rho \in \Delta_+} \frac{\cos(\rho \cdot q)}{\sin^2(\rho \cdot q)} (\hat{s}_\rho - I), \quad (7.3)$$

and diagonal matrices:

$$R = e^{2iQ}, \quad Q = q \cdot \hat{H}. \quad (7.4)$$

Then, as in the classical case, we obtain

$$\dot{L} = [L, M] \quad (7.5)$$

and

$$\dot{R} = [R, M] + i(RL + LR). \quad (7.6)$$

This is because the main formula of proof in the classical theory, eq(5.8) in [8] is the same in quantum theory. We only have to change the definition of a_k eq(5.4a) and b_k eq(5.4b) in [8] in order to accommodate for the quantum non-commutativity. Define

$$a_j = \text{Ts}(L^j), \quad b_j = \text{Ts} \sum_{k=0}^j \binom{j}{k} L^k R L^{j-k}, \quad (7.7)$$

then we obtain

$$\dot{a}_j = 0, \quad \dot{b}_j = ib_{j+1}. \quad (7.8)$$

This provides the algebraic linearization of quantum trigonometric (hyperbolic) Calogero-Sutherland for the A and D series and E_6 and E_7 root systems. The models with hyperbolic potential can be discussed in a similar way. See also [26, 27] in this connection.

VIII Comments on the Hermiticity of Algebra Generators

In quantum mechanics physical quantities or the observables are described by hermitian operators in Hilbert space [20]. The hermiticity of F_j , (2.10) is obvious from that of L . The original definition of G_j , (2.13) is not hermitian. With the following redefinition of hermitian G_j ,

$$G_j = \text{Ts} \sum_{k=0}^j (L^k Q L^{j-k}) / (j+1), \quad (8.1)$$

it satisfies the same formula (2.16). Whereas the definition of $F_{k,j}$ (3.17) remains the same, that of $H_{k,k'}^{(j)}$ (3.23) should be changed to a hermitian form

$$2H_{k,k'}^{(j)} = F_{k,j}G_{k'} + G_{k'}F_{k,j} - F_{k',j}G_k - G_kF_{k',j}. \quad (8.2)$$

A formulation with explicitly hermitian G_j could have been achieved by

$$G_{j-1} \propto i[q^2, F_j], \quad (8.3)$$

which is closely related with the extension of the $sl(2, \mathbf{R})$ algebra [22]-[24],[6, 17, 20]. This also explains the assertion that independent $\{G_j\}$'s appear at $j = \text{exponent}$.

We chose the current presentation in order to avoid excessively complicated looking formulas and to allow an easy comparison with the original work [5] on the quadratic algebra.

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