Geometric Construction of Killing Spinors and Supersymmetry Algebras in Homogeneous Spacetimes

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Abstract

We show how the Killing spinors of some maximally supersymmetric supergravity solutions whose metrics describe symmetric spacetimes (including AdS, $AdS \times S$ and Hpp-waves) can be easily constructed using purely geometrical and group-theoretical methods. The calculation of the supersymmetry algebras is extremely simple in this formalism.

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Introduction

In theories with local supersymmetry (supergravity and superstring theories), the maximally supersymmetric solutions are usually identified as vacua, although vacua with less unbroken supersymmetry can also be interesting. The vacuum supersymmetry algebra, together with Wigner's theorem, determine which fields can be defined on it, their conserved (quantum) numbers, the particle spectrum etc. Thus, the supersymmetry algebra is a very important piece of information.

The calculation of the supersymmetry algebra of a solution (see, e.g. Ref. [1]), involves the calculation of its Killing vectors and Killing spinors, and the computation of bilinears and Lie derivatives of the Killing spinors which can sometimes be difficult or tedious, since their functional form has no manifest geometrical meaning.

However, most known maximally supersymmetric solutions have the spacetime metric of some symmetric space that can be identified with a coset G/H[4]. Our main result is that, quite generally, the Killing spinor equation in maximally supersymmetric solutions can be put in the form

$$(d + u^{-1}du)\kappa = 0\,, \tag{0.1}$$

which, written in the form $u^{-1}d(u\kappa) = 0$ tells us that the Killing spinors are given by

$$\kappa = u^{-1}\kappa_0\,, \tag{0.2}$$

where κ_0 is a constant Killing spinor. u is a coset representative in the spinorial representation. Then, the bilinears $\bar{\kappa}\gamma^\mu\kappa$ can be easily decomposed into Killing vectors and the Lie-Lorentz derivative of the Killing spinors with respect to the Killing vectors are also easily computed. This simplifies dramatically the calculation of the supersymmetry algebras of these maximally supersymmetric solutions.

In Section 1 we give a extremely sketchy review of the theory of symmetric spaces needed to prove the above general result in the examples that will follow. In Section 2 we use the machinery just introduced to give a construction of the metric of several well-known maximally supersymmetric supergravity solutions (all of them corresponding to symmetric, but, in general, not maximally symmetric spacetimes) and to show how the general rule for the construction of the Killing spinors works in practice. We start with the simplest non-trivial example: AdS_4 in $N = 1, d = 4$ (AdS) supergravity (Section 2.1). Then we consider the next non-trivial example: the Robinson-Bertotti solution with geometry $AdS_2 \times S^2$ (Section 2.2) which we then generalize to other known maximally supersymmetric solutions with geometries of the type $AdS \times S$ (Section 2.3). Finally, we consider in Section 2 the last kind of known maximally supersymmetric solutions: the Kowalski-Glikman solutions with Hpp-wave geometries. Section 3 contains our conclusions perspectives for future work.

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[4]The only exception known to us could be the near-horizon metric of the 5-dimensional rotating black hole.

1 Symmetric Spaces

Let us consider[5] the $(p+q)$-dimensional Lie group G, its p-dimensional subgroup H and the q-dimensional space of right cosets $G/H = \{gH\}$. The Lie algebra $\mathfrak{g}$ of G is spanned by the generators T_I $(I = 1, \cdots, p+q)$ with Lie algebra

$$[T_I, T_J] = f_{IJ}{}^K T_K \,. \tag{1.1}$$

The Lie algebra of H is generated by the subalgebra $\mathfrak{h} \subset \mathfrak{g}$ spanned by the generators M_i $(i = 1, \cdots, p)$ with Lie brackets

$$[M_i, M_j] = f_{ij}{}^k M_k \,. \tag{1.2}$$

The vector subspace spanned by the remaining generators, denoted by P_a, $(a, b = 1, \cdots, q)$ is denoted by $\mathfrak{k}$ and, as vector spaces we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{h}$. Exponentiating the generators of $\mathfrak{k}$ we can construct a coset representative $u(x) = u(x^1, \cdots, x^q)$. We will always construct the coset representative as a product of generic elements of the 1-dimensional subgroups generated by the P_as:

$$u(x) = e^{x^1 P_1} \cdots e^{x^q P_q} \,. \tag{1.3}$$

Under a left transformation $g \in G$ u transforms into another element of G which only becomes a coset representative $u(x')$ after a right transformation with an element $h \in H$, which is a function of g and x:

$$gu(x) = u(x')h \,. \tag{1.4}$$

The adjoint representation of $\mathfrak{g}$ has as representation space $\mathfrak{g}$ and can be defined by its action on its generators: for any $T \in \mathfrak{g}$

$$\Gamma_{\mathrm{Adj}}(T)(T_I) \equiv [T, T_I] \,, \quad \Rightarrow \Gamma_{\mathrm{Adj}}(T_I)^K{}_J = f_{IJ}{}^K \,. \tag{1.5}$$

Exponentiating the generators of the Lie algebra $\mathfrak{g}$ in the adjoint representation, we get the adjoint representation of the group G

$$\Gamma_{\mathrm{Adj}}(g(x)) = \exp\left\{x^I \Gamma_{\mathrm{Adj}}(T_I)\right\} \,. \tag{1.6}$$

that acts on the Lie algebra generators

$$T'_J = T_L \Gamma_{\mathrm{Adj}}(g)^L{}_J \,. \tag{1.7}$$

Actually, in any representation r, the adjoint action of G on $\mathfrak{g}$ is given by

$$\Gamma_r(g)\Gamma_r(T_I)\Gamma_r(g^{-1}) = \Gamma_r(T_J)\Gamma_{\mathrm{Adj}}(g)^J{}_I \,. \tag{1.8}$$

The *Killing metric* K_{IJ} is defined by

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[5]Two Physics-oriented general references are [2] and [3].

$$K_{IJ} \equiv \text{Tr}[\Gamma_{\text{Adj}}(T_I)\Gamma_{\text{Adj}}(T_J)], \tag{1.9}$$

and by construction it is invariant under the adjoint action of G, due to the cyclic property of the trace.

The homogeneous space G/H can be used to construct a symmetric space if the pair $(\mathfrak{k}, \mathfrak{h})$ is a *symmetric pair* satisfying

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h},$$

$$[\mathfrak{k}, \mathfrak{h}] \subset \mathfrak{k}, \tag{1.10}$$

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}.$$

The first condition is always satisfied for homogeneous spaces since $\mathfrak{h}$ is a subalgebra. The second condition says that $\mathfrak{k}$ is a representation of H. The two components of a symmetric pair are mutually orthogonal with respect to the Killing metric which is block-diagonal.

The first step is the construction of the left-invariant Lie-algebra valued *Maurer-Cartan 1-form V*

$$V \equiv -u^{-1}du = e^a P_a + \vartheta^i M_i, \tag{1.11}$$

that we have decomposed in *horizontal* e^a and *vertical* components ϑ^i. By construction, V satisfies the *Maurer-Cartan equations*

$$dV - V \wedge V = 0, \quad \Rightarrow \quad \begin{cases} de^a - \vartheta^i \wedge e^b f_{ib}{}^a & = 0, \\[2mm] d\vartheta^i - \frac{1}{2}\vartheta^j \wedge \vartheta^k f_{jk}{}^i - \frac{1}{2}e^a \wedge e^b f_{ab}{}^i & = 0. \end{cases} \tag{1.12}$$

The horizontal components e^a provide us with a co-frame for G/H. Under left multiplication by a constant element $g \in G$ $u(x') = gu(x)h^{-1}$, which implies for the Maurer-Cartan 1-form components

$$\begin{cases} e^a(x') & = (he(x)h^{-1})^a = \Gamma_{\text{Adj}}(h)^a{}_b e^b(x), \\[2mm] \vartheta^i(x') & = (h\vartheta(x)h^{-1})^i + (h^{-1}dh)^i. \end{cases} \tag{1.13}$$

The second step to construct a symmetric space is the construction of the metric. With a symmetric bilinear form B_{ab} in $\mathfrak{k}$ we can construct a Riemannian metric

$$ds^2 \sim B_{ab}e^a \otimes e^b, \tag{1.14}$$

that will be invariant under the left action of G if B is:

$$f_{i(a}{}^c B_{b)c} = 0. \tag{1.15}$$

This is guaranteed if $B_{mn} = K_{mn}$, the projection on $\mathfrak{k}$ of the Killing metric, but sometimes this is singular and another one has to be used . The resulting Riemannian metric contains

G in its isometry group (which could be bigger) and must admit $p + q$ Killing vector fields $k_{(I)}$. The Killing vectors $k_{(I)}$ and the so-called *H-compensator* $W_I{}^i$ are defined through the infinitesimal version of the transformation rule $gu(x) = u(x')h$ with

$$
\begin{aligned}
g &= 1 + \sigma^I T_I\,, \\
h &= 1 - \sigma^I W_I{}^i M_i\,, \\
x^{\mu\prime} &= x^\mu + \sigma^I k_{(I)}{}^\mu\,.
\end{aligned}
\tag{1.16}
$$

Using the above equations into

$$
u(x + \delta x) = u(x) + \sigma^I k_{(I)} u\,,
\tag{1.17}
$$

we get

$$
T_I u = k_I u - u W_I{}^i M_i\,.
\tag{1.18}
$$

Acting with u^{-1} on the left and using the definitions of the adjoint action and the Maurer-Cartan 1-forms , we get

$$
T_J \Gamma_{\mathrm{Adj}}(u^{-1})^J{}_I = -k_{(I)}{}^a P_a - (k_{(I)}{}^\mu \vartheta^i{}_\mu + W_I{}^i) M_i\,,
\tag{1.19}
$$

which, projected on the horizontal and vertical subspaces gives the following expressions for the tangent space components of the Killing vector fields and the H-compensator

$$
k_{(I)}{}^a = -\Gamma_{\mathrm{Adj}}(u^{-1}(x))^a{}_I\,,
\tag{1.20}
$$

$$
W_I{}^i = -k_{(I)}{}^\mu \vartheta^i{}_\mu - \Gamma_{\mathrm{Adj}}(u^{-1}(x))^i{}_I\,.
\tag{1.21}
$$

H-Covariant Derivatives

According to the second of Eqs. (1.13) the vertical components θ^i transform as an $\mathfrak{h}$-valued connection. In fact, comparing the Maurer-Cartan equation for the horizontal components e^a with the Cartan structure equation for the co-frame and (torsionless) spin connection

$$
de^a - \omega^a{}_b \wedge e^b = 0\,,
\tag{1.22}
$$

we find that the spin connection is given by

$$
\omega^a{}_b = \vartheta^i f_{ib}{}^a = \vartheta^i \Gamma_{\mathrm{Adj}}(M_i)^a{}_b\,.
\tag{1.23}
$$

We use these results to define the *H-covariant derivative* that acts on any object that transforms contravariantly $\phi' = \Gamma_r(h)\phi$ or covariantly $\psi' = \psi\Gamma_r(h^{-1})$ (for instance, $u(x)$ itself) in the representation r of H:

$$\mathcal{D}_\mu \phi \equiv \partial_\mu \phi - \vartheta_\mu{}^i \Gamma_r(M_i)\phi\,, \qquad \mathcal{D}_\mu \psi \equiv \partial_\mu \psi + \psi \vartheta_\mu{}^i \Gamma_r(M_i) \tag{1.24}$$

In particular, the Maurer-Cartan equations tell us that

$$\mathcal{D}_{[\mu} e^a{}_{\nu]} = 0\,. \tag{1.25}$$

By definition, the Levi-Cività connection is given by

$$\Gamma_{\mu\nu}{}^a \equiv \mathcal{D}_\mu e^a{}_\nu\,. \tag{1.26}$$

Finally, let us introduce the *H-covariant Lie derivative with respect to the Killing vectors* $k_{(I)}$[6] on objects that transform contravariantly (ϕ) or covariantly y (ψ) in the representation r of H:

$$\mathbb{L}_{k_{(I)}} \phi \equiv \mathcal{L}_{k_{(I)}} \phi + W_I{}^i \Gamma_r(M_i)\phi\,, \qquad \mathbb{L}_{k_{(I)}} \psi \equiv \mathcal{L}_{k_{(I)}} \psi - \psi W_I{}^i \Gamma_r(M_i)\,. \tag{1.27}$$

This Lie derivative has, among other properties

$$[\mathbb{L}_{k_{(I)}}, \mathbb{L}_{k_{(J)}}] = \mathbb{L}_{[k_{(I)}, k_{(J)}]}\,, \tag{1.28}$$

$$\mathbb{L}_{k_{(I)}} e^a = 0 \tag{1.29}$$

$$\mathbb{L}_{k_{(I)}} u = \mathcal{L}_{k_{(I)}} u - u W_I{}^i M_i = T_I u\,, \tag{1.30}$$

where the last property follows from Eqs. (1.21) and (1.18).

2   Killing Spinors in Symmetric Spacetimes

Most maximally supersymmetric solutions of supergravity theories have the metric of some symmetric spacetime. In some cases (Minkowski and AdS) the spacetime is also maximally symmetric but in other cases ($AdS \times S$ and KG spacetimes) it is not, but we can always use the procedure explained in the previous section to construct the metric, spin connection and Killing vectors. We are going to see, example by example, that, when we construct in that way the metric, the Killing spinor equation always takes the form Eq. (0.1). It is, nevertheless, convenient to give a brief overview of how we arrive to the general result.

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[6]*H*-covariant Lie derivatives can be defined with respect to any vector, although the Lie bracket property Eq. (1.28) is only satisfied for Killing vectors. The spinorial Lie derivative [4, 5, 6] or the Lie-Lorentz derivative that naturally appear in calculations of supersymmetry algebras [1, 7] can actually be seen as particular examples of this more general operator (see e.g. Ref. [8]), and, actually, are identical objects when acting on Killing spinors of maximally supersymmetric spacetimes, as we are going to show.

Then, we are going to show how the general result can be exploited to calculate the commutators of the supersymmetry algebra.

In all supergravity theories, the Killing spinor equation is of the the form

$$(\nabla_\mu + \Omega_\mu)\kappa = 0 \,, \tag{2.1}$$

where the form of Ω depends on specific details of the theory. Multiplying by dx^μ, it takes the form

$$\left(d - \tfrac{1}{4}\omega_{ab}\gamma^{ab} + \Omega\right)\kappa = 0 \,. \tag{2.2}$$

If we construct the symmetric space as in the previous section, then the spin connection 1-form ω_{ab} is given by Eq. (1.23) and takes values in the vertical Lie subalgebra $\mathfrak{h}$. Further,

$$\Gamma_s(M_i) \equiv \tfrac{1}{4}f_{ia}{}^b\gamma_b{}^a \,, \tag{2.3}$$

provides a (spinorial) representation of $\mathfrak{h}$ and the Killing spinor equation becomes

$$\left(d - \vartheta^i\Gamma_s(M_i) + \Omega\right)\kappa = 0 \,. \tag{2.4}$$

In all the cases that we are going to examine

$$\Omega = -e^a\Gamma_s(P_a) \,, \tag{2.5}$$

where the matrices $\Gamma_s(P_a)$ are products of a number of Dirac gamma matrices (and, possibly, of other matrices in extended supergravities). Thus, on account of the definition of the Maurer-Cartan 1-forms Eq. (1.11), the Killing spinor equation can be written in the form Eq. (0.1)

$$\left(d - e^a\Gamma_s(P_a) - \vartheta^i\Gamma_s(M_i)\right)\kappa = \left(d + \Gamma_s(u^{-1})d\Gamma_s(u)\right)\kappa = 0 \,, \tag{2.6}$$

with

$$\Gamma_s(u) = e^{x^1\Gamma_s(P_1)}\cdots e^{x^q\Gamma_s(P_q)} \,, \tag{2.7}$$

and the solution can be written in the form

$$\kappa^\alpha = \Gamma_s(u^{-1})^\alpha{}_\beta\kappa_0{}^\beta \,, \tag{2.8}$$

for an arbitrary constant spinor $\kappa_0{}^\beta$ (we have written explicitly the spinor indices here). Since there will be as many independent Killing spinors as components has a real spinor[7], we can use a spinorial index α to label a basis of Killing spinors:

$$\kappa_{(\alpha)}{}^\beta = \Gamma_s(u^{-1})^\beta{}_\alpha \,. \tag{2.9}$$

Killing spinors and Killing vectors are used to find the supersymmetry algebra of supergravity backgrounds (see, e.g. [9, 1, 7]). Killing spinors are related to supercharges and

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[7]We are considering only Majorana spinors.

Killing vectors to bosonic charges. The anticommutator of two supercharges gives bosonic charges and, correspondingly the bilinears $-i\bar{\kappa}\gamma^{\mu}\kappa$ of Killing spinors are Killing vectors. To calculate the anticommutator of any two supercharges $\{Q_{(\alpha)}, Q_{(\beta)}\}$ associated to the Killing spinors $\kappa_{(\alpha)}$ we have to decompose the bilinears into linear combinations of the Killing vectors $k_{(I)}$

$$-i\bar{\kappa}_{(\alpha)}\gamma^{\mu}\kappa_{(\beta)}\partial_{\mu} = c_{\alpha\beta}{}^{I}k_{(I)}\,, \tag{2.10}$$

finding the coefficients $c_{\alpha\beta}{}^{I}$. Now, using the above general form of the Killing spinors, the bilinears take the form

$$-i\bar{\kappa}_{(\alpha)}\gamma^{\mu}\kappa_{(\beta)}\partial_{\mu} = -i\Gamma_{s}(u^{-1})_{\alpha}{}^{\gamma}\mathcal{C}_{\gamma\delta}(\gamma^{a})^{\delta}{}_{\epsilon}\Gamma_{s}(u^{-1})^{\epsilon}{}_{\beta}\,, \tag{2.11}$$

where $\mathcal{C}$ is the charge conjugation matrix $\mathcal{C}^{-1}\gamma^{a\,T}\mathcal{C} = -\gamma^{a}$. Now, in most cases[8],the matrices γ^{a} happen to be proportional to the the dual[9] P^{a} of a Lie algebra generator P_{a} $\Gamma_{s}(P^{a})$

$$\gamma^{a} = \mathcal{S}\Gamma_{s}(P^{a})\,, \tag{2.12}$$

for some matrix $\mathcal{S}$ that depends on the case we are considering. The combination $\tilde{\mathcal{C}} \equiv \mathcal{C}\mathcal{S}$ acts as a charge conjugation matrix in the subspace spanned by the horizontal generators in the spinorial representation[10]

$$\tilde{\mathcal{C}}^{-1}\Gamma_{s}(P^{a})^{T}\tilde{\mathcal{C}} = -\Gamma_{s}(P^{a})\,, \tag{2.13}$$

so

$$\Gamma_{s}(u^{-1})^{T}\mathcal{C}\gamma^{a} = \Gamma_{s}(u^{-1})^{T}\tilde{\mathcal{C}}\Gamma_{s}(P^{a}) = \tilde{\mathcal{C}}\Gamma_{s}(u)\Gamma_{s}(P^{a})\,. \tag{2.14}$$

and, thus,

$$-i\bar{\kappa}_{(\alpha)}\gamma^{\mu}\kappa_{(\beta)}\partial_{\mu} = -i\tilde{\mathcal{C}}_{\alpha\gamma}\Gamma_{s}(u)^{\gamma}{}_{\delta}\Gamma_{s}(P^{a})^{\delta}{}_{\epsilon}\Gamma_{s}(u^{-1})^{\epsilon}{}_{\beta}e_{a}\,. \tag{2.15}$$

In this expression we can recognize $uP^{a}u^{-1}$ in the spinorial representation, which is the coadjoint action of the coset element u on P^{a}

$$-i\bar{\kappa}_{(\alpha)}\gamma^{\mu}\kappa_{(\beta)}\partial_{\mu} = -i\tilde{\mathcal{C}}_{\alpha\gamma}\Gamma_{s}(T^{I})^{\gamma}{}_{\beta}\Gamma_{\text{Adj}}(u^{-1})^{a}{}_{I}e_{a} = -i\tilde{\mathcal{C}}_{\alpha\gamma}\Gamma_{s}(T^{I})^{\gamma}{}_{\beta}k_{(I)}\,, \tag{2.16}$$

where we have used Eq. (1.20). The superalgebra structure constants $c_{\alpha\beta}{}^{I}$ can be readily identified with $-i\tilde{\mathcal{C}}_{\alpha\gamma}\Gamma_{s}(T^{I})^{\gamma}{}_{\beta}$.

To complete all the commutation relations of the supersymmetry algebra, we need the commutators of the bosonic charges and the supercharges, which are determined by the spinorial or Lie-Lorentz derivative of the Killing vectors on the Killing spinors $\mathbb{L}_{k_{(I)}}\kappa_{(\alpha)}$

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[8]The exception seems to be the Kowalski-Glikman Hpp-wave spacetimes.

[9]It is always possible to find the dual of a representation that uses (unitary) gamma matrices.

[10]We thank P. Meessen for pointing this out to us.

[1, 7], since this operation preserves the supercovariant derivative (at least in the ungauged supergravities that we are going to consider) and transforms Killing spinors into Killing spinors

$$\mathbb{L}_{k_{(I)}}\kappa_{(\alpha)} = c_{\alpha I}{}^{\beta}\kappa_{(\beta)}\,, \quad \Rightarrow [Q_{(\alpha)}, P_{(I)}] = c_{\alpha I}{}^{\beta}Q_{(\beta)}\,. \tag{2.17}$$

The Lie-Lorentz derivative acting on a (contravariant) spinor ψ is given by [5, 6]

$$\mathbb{L}_{k_{(I)}}\psi = k_{(I)}{}^{\mu}\nabla_{\mu}\psi + \tfrac{1}{4}\nabla_{a}k_{(I)}^{b}\gamma^{a}{}_{b}\psi\,. \tag{2.18}$$

On a symmetric space G/H,

$$k_{(I)}{}^{\mu}\nabla_{\mu}\psi = k_{(I)}{}^{\mu}\partial_{\mu}\psi - k_{(I)}{}^{\mu}\vartheta^{i}{}_{\mu}\Gamma_{s}(M_{i})\psi\,, \qquad \nabla_{\mu}k_{(I)}^{b} = \partial_{\mu}k_{(I)}^{b} - \vartheta^{i}{}_{\mu}f_{ic}{}^{b}k_{(I)}^{c}\,. \tag{2.19}$$

Furthermore

$$
\begin{aligned}
\partial_{\mu}k_{(I)}{}^{b} &= -\partial_{\mu}\Gamma_{\text{Adj}}(u^{-1})^{b}{}_{I} \\[4pt]
&= \Gamma_{\text{Adj}}(u^{-1})^{b}{}_{J}\partial_{\mu}\Gamma_{\text{Adj}}(u)^{J}{}_{K}\Gamma_{\text{Adj}}(u^{-1})^{K}{}_{I} \\[4pt]
&= -V^{J}{}_{\mu}f_{JK}{}^{b}\Gamma_{\text{Adj}}(u^{-1})^{K}{}_{I} \\[4pt]
&= -e^{a}{}_{\mu}f_{ai}{}^{b}\Gamma_{\text{Adj}}(u^{-1})^{i}{}_{I} - \vartheta^{i}{}_{\mu}f_{ic}{}^{b}\Gamma_{\text{Adj}}(u^{-1})^{c}{}_{I} \\[4pt]
&= e^{a}{}_{\mu}f_{ia}{}^{b}\Gamma_{\text{Adj}}(u^{-1})^{i}{}_{I} + \vartheta^{i}{}_{\mu}f_{ic}{}^{b}k_{(I)}{}^{c}\,,
\end{aligned} \tag{2.20}
$$

so

$$\tfrac{1}{4}\nabla_{a}k_{(I)}^{b}\Gamma^{a}{}_{b} = \tfrac{1}{4}f_{ia}{}^{b}\Gamma_{\text{Adj}}(u^{-1})^{i}{}_{I}\gamma^{a}{}_{b} = -\Gamma_{\text{Adj}}(u^{-1})^{i}{}_{I}\Gamma_{s}(M_{i})\,, \tag{2.21}$$

and

$$
\begin{aligned}
\mathbb{L}_{k_{(I)}}\psi &= k_{(I)}{}^{\mu}\partial_{\mu}\psi - k_{(I)}{}^{\mu}\vartheta^{i}{}_{\mu}\Gamma_{s}(M_{i})\psi - \Gamma_{\text{Adj}}(u^{-1})^{i}{}_{I}\Gamma_{s}(M_{i})\psi \\[4pt]
&= \mathcal{L}_{k_{(I)}}\psi + W^{i}_{I}\Gamma_{s}(M_{i})\psi\,.
\end{aligned} \tag{2.22}
$$

Then, the Lie-lorentz derivative coincides withe the H-covariant Lie derivative. On the inverse coset representative

$$\mathbb{L}_{k_{(I)}}\Gamma_{s}(u^{-1}) = -\Gamma_{s}(u^{-1})[\mathbb{L}_{k_{(I)}}\Gamma_{s}(u)]\Gamma_{s}(u^{-1}) = -\Gamma_{s}(u^{-1})\Gamma_{s}(T_{I})\,, \tag{2.23}$$

on account of Eq. (1.30), which implies the commutators

$$[Q_{(\alpha)}, T_{I}] = -Q_{(\beta)}\Gamma_{s}(T_{I})^{\beta}{}_{\alpha}\,. \tag{2.24}$$

9

2.1 AdS_4 in $N = 1, d = 4$ AdS Supergravity

AdS_4 is the maximally supersymmetric vacuum of $N = 1, d = 4$ AdS supergravity: the integrability conditions of the Killing spinor equations vanish identically, which implies that 4 independent solutions exist. They are not hard to find (see e.g. Ref [10]), but the expressions one gets in most coordinate systems are difficult to make sense of and they are difficult to work with to find supersymmetry algebras.

AdS_4 can be identified with the coset $SO(2,3)/SO(1,3)$. We introduce $SO(2,3)$ indices $\hat{a}, \hat{b}, \cdots = -1, 0, 1, 2, 3$. The metric is $\hat{\eta}^{\hat{a}\hat{b}} = \text{diag}(++---)$ and $\mathfrak{g} = so(2,3)$ the Lie algebra of $SO(2,3)$ can be written in the general form

$$\left[\hat{M}_{\hat{a}\hat{b}}, \hat{M}_{\hat{c}\hat{d}}\right] = -\hat{\eta}_{\hat{a}\hat{c}}\hat{M}_{\hat{b}\hat{d}} - \hat{\eta}_{\hat{b}\hat{d}}\hat{M}_{\hat{a}\hat{c}} + \hat{\eta}_{\hat{a}\hat{d}}\hat{M}_{\hat{b}\hat{c}} + \hat{\eta}_{\hat{b}\hat{c}}\hat{M}_{\hat{a}\hat{d}}. \tag{2.25}$$

We can now perform a $1 + 4$ splitting of the indices $\hat{a} = (-1, a)$, $a = 0, 1, 2, 3$ and define a new basis

$$\hat{M}_{ab} = M_{ab}, \qquad\qquad \hat{M}_{a-1} = -g^{-1}P_a, \tag{2.26}$$

where we have introduced the dimensionful parameter g related to the AdS_4 radius R and to the cosmological constant Λ by

$$R = 1/g = \sqrt{-3/\Lambda}. \tag{2.27}$$

In terms of the new basis, the $so(2,3)$ algebra reads

$$[M_{ab}, M_{cd}] = -\eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc} + \eta_{bc}M_{ad},$$

$$[P_c, M_{ab}] = -2P_{[a}\eta_{b]c}, \qquad [P_a, P_b] = -g^2 M_{ab}. \tag{2.28}$$

The M_{ab}s generate the subalgebra $\mathfrak{h} = so(1,3)$ of the Lorentz subgroup. The complement is $\mathfrak{k} = \{P_a\}$ and the above commutation relations tell us that we have a symmetric pair. Following the general recipe, we construct the coset representative

$$u(x) = e^{x^3 P_3} e^{x^2 P_2} e^{x^1 P_1} e^{x^0 P_0}, \tag{2.29}$$

and the Maurer-Cartan 1-forms e^a, that we are going to use as Vierbeins are[11]

$$e^0 = -dx^0, \quad e^1 = -\cos x^0 dx^1, \quad e^2 = -\cos x^0 \text{ch } x^1 dx^2, \quad e^3 = -\cos x^0 \text{ch } x^1 \text{ ch } x^2 dx^2, \tag{2.32}$$

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[11]In this and similar calculations one has to use the formula

$$e^{xX}Ye^{-xX} = \cos x Y + \sin x Z, \tag{2.30}$$

where

$$[X, Y] = Z, \qquad [Y, Z] = X, \qquad [Z, X] = Y. \tag{2.31}$$

and using the Killing metric $(+ - --)$ we get the AdS_4 metric in somewhat unusual coordinates

$$ds^2 = (dx^0)^2 - \cos^2 x^0 \{(dx^1)^2 + \text{ch}^2 x^1 [(dx^2)^2 + \text{ch}^2 x^1 (dx^3)^2]\}. \tag{2.33}$$

We do not need the explicit form of the vertical 1-forms ϑ^{ab}, but we need to know how they enter the spin connection. According to the general formula Eq. (1.23)

$$\omega^a{}_b = \tfrac{1}{4} e^{cd} f_{cd}{}_{-1b}{}^{-1a} = \tfrac{1}{2} e \vartheta^{ac} \eta_{cb}. \tag{2.34}$$

The Killing spinor equation is

$$(d - \tfrac{1}{4}\omega_{ab}\gamma^{ab} - \tfrac{ig}{2}\gamma_a e^a)\kappa = 0, \tag{2.35}$$

and takes immediately the form of Eq. (0.1) with

$$\Gamma_s(P_a) = \tfrac{ig}{2}\gamma_a, \qquad \Gamma_s(M_{ab}) = \tfrac{1}{2}\gamma_{ab}, \tag{2.36}$$

and the Killing spinors are of the general form $\kappa_{(\alpha)}{}^\beta = (u^{-1})^\beta{}_{(\alpha)}$.

We define the dual generators $\Gamma_s(P^a)$ by

$$\text{Tr}\,[\Gamma_s(P^a)\Gamma_s(P_b)] = \delta^a{}_b, \quad \Rightarrow \quad \Gamma_s(P^a) = \tfrac{-i}{2g}\gamma^a,$$

$$\text{Tr}\,[\Gamma_s(M^{ab})\Gamma_s(P_{cd})] = \delta^{ab}{}_{cd}, \quad \Rightarrow \quad \Gamma_s(M^{ab}) = -\tfrac{1}{2}\gamma^{ab}. \tag{2.37}$$

The bilinears are, then $(\mathcal{S} = 1)$

$$
\begin{aligned}
-i\bar{\kappa}_{(\alpha)}\gamma^a\kappa_{(\beta)}e_a &= 2g\Gamma_s(u^{-1})^T \mathcal{C}\Gamma_s(P^a)\Gamma_s(u^{-1})e_a \\[2mm]
&= g\mathcal{C}\Gamma_s(\hat{M}^{\hat{b}\hat{c}})\Gamma_{\text{Adj}}(u^{-1})^a{}_{\hat{b}\hat{c}}e_a \\[2mm]
&= g\mathcal{C}\Gamma_s(\hat{M}^{\hat{b}\hat{c}})k_{(\hat{b}\hat{c})},
\end{aligned}
\tag{2.38}
$$

and the anticommutator of the supercharges takes the well-known form

$$\{Q_{(\alpha)}, Q_{(\beta)}\} = g[\mathcal{C}\Gamma_s(\hat{M}^{\hat{a}\hat{b}})]_{\alpha\beta}\hat{M}_{\hat{a}\hat{b}} = -i(\mathcal{C}\gamma^a)_{\alpha\beta}P_a - \tfrac{g}{2}(\mathcal{C}\gamma^{ab})_{\alpha\beta}M_{ab}, \tag{2.39}$$

that reduces to the Poincaré supersymmetry algebra in the $g \to 0$ limit.

The commutators $[Q_{(\alpha)}, \hat{M}_{\hat{a}\hat{b}}]$ are given by the general formula (2.24):

$$[Q_{(\alpha)}, \hat{M}_{\hat{a}\hat{b}}] = -Q_{(\beta)}\Gamma_s(\hat{M}_{\hat{a}\hat{b}})^\beta{}_\alpha. \tag{2.40}$$

The generalization to higher dimensions[12] and to spheres, described as cosets $SO(n + 1)/SO(n)$ is evident. As a matter of fact, the coset structure underlies the calculation of Killing spinors in S^n of Ref. [10] but only after this is realized the calculation of bilinears etc. becomes really simple.

---

[12]Maximally supersymmetric AdS vacua arise in gauged supergravities in $d \leq 7$.

2.2 The Robinson-Bertotti Solution in $N = 2, d = 4$ Supergravity

The Robinson-Bertotti solution of $N = 2, d = 4$ supergravity [11, 12] can be obtained as the near-horizon limit of the extreme Reissner-Nordström black hole and is known to be maximally supersymmetric [13, 14], although, to the best of our knowledge, no explicit expression of its 8 real Killing spinors is available in the literature. The metric is that of the direct product of that of AdS_2 with radius R_2 and that of S^2 with radius R_2

$$
\begin{cases}
ds^2 & = & R_2^2 \, d\Pi_{(2)}^2 - R_2^2 \, d\Omega_{(2)}^2 \,, \\
\\
F & = & -\frac{2}{R_2}\omega_{AdS_2} \,,
\end{cases}
\tag{2.41}
$$

where $d\Pi_{(2)}^2$ stands for the metric of the AdS_2 spacetime of unit radius, $d\Omega_{(2)}^2$ for the metric of the unit 2-sphere S^2 and ω_{AdS_2} for the volume 2-form of radius R_2. Both AdS_2 and S^2 are symmetric spacetimes $SO(2,1)/SO(2)$ and $SO(3)/SO(2)$ and we can construct them using the procedure explained in Section 1.

The Lie algebra of $SO(2,1)$ can be written in the form

$$
[T_I, T_J] = -\epsilon_{IJK}\mathsf{Q}^{KL}T_L \,, \quad I, J, \cdots = 1, 2, 3, \,, \qquad \mathsf{Q} = \text{diag}\,(+ + -) \,,
\tag{2.42}
$$

and the Lie algebra of $SO(3)$ can be written in the form

$$
[\tilde{T}_I, \tilde{T}_J] = -\epsilon_{IJK}\tilde{T}_K \,, \quad I, J, \cdots = 1, 2, 3, \,.
\tag{2.43}
$$

We choose the subalgebra $\mathfrak{h}$ to be generated by T_1 and $\tilde{T}_3$ so $\mathfrak{k}$ is generated by T_2, T_3 and $, \tilde{T}_1, \tilde{T}_2$. We perform the following redefinitions

$$
\begin{aligned}
T_1 & = & M_1 \,, & & \tilde{T}_1 & = & R_2 P_3 \,, \\
\\
T_2 & = & R_2 P_1 \,, & & \tilde{T}_2 & = & R_2 P_2 \,, \\
\\
T_3 & = & R_2 P_0 \,, & & \tilde{T}_3 & = & M_2 \,,
\end{aligned}
\tag{2.44}
$$

and the coset representative is the product of two mutually commuting coset representatives $u, \tilde{u}$ with

$$
u = e^{R_2\phi P_0}e^{R_2\chi P_1} \,, \qquad \tilde{u} = e^{R_2\varphi P_3}e^{R_2(\theta - \frac{\pi}{2})P_2} \,.
\tag{2.45}
$$

We get

$$
\begin{aligned}
e^0 & = & -R_2\text{ch}\,\chi\, d\phi \,, & & e^2 & = & -R_2 d\theta \,, \\
\\
e^1 & = & -R_2 d\chi \,, & & e^3 & = & -R_2 \sin\theta d\varphi \,, \\
\\
\vartheta^1 & = & -\text{sh}\,\chi\, d\phi \,, & & \vartheta^2 & = & -\cos\theta d\varphi \,,
\end{aligned}
\tag{2.46}
$$

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that lead, using $B = \text{diag}\,(+---)$ to the above $AdS_2 \times S^2$ metric with

$$
\begin{cases}
d\Pi^2_{(2)} & \equiv \ \text{ch}^2\chi\,d\phi^2 - d\chi^2\,, \\[2mm]
d\Omega^2_{(2)} & \equiv \ d\theta^2 + \sin^2\theta d\varphi^2\,.
\end{cases}
\tag{2.47}
$$

Contracting with dx^μ the $N = 2, d = 4$ Killing spinor equation

$$
(\nabla_\mu + \tfrac{1}{8}\,\slashed{F}\gamma_\mu\sigma^2)\kappa = 0\,,
\tag{2.48}
$$

we immediately see that it takes the form

$$
[d + (u\tilde{u})^{-1}d(u\tilde{u})] = 0\,,
\tag{2.49}
$$

where the Lie algebra generators are represented by

$$
\begin{aligned}
\Gamma_s(P_0) &= \ \tfrac{1}{2R_2}\gamma^1\sigma^2\,, & \Gamma_s(P_2) &= \ \tfrac{1}{2R_2}\gamma^0\gamma^1\gamma^3\sigma^2\,, \\[2mm]
\Gamma_s(P_1) &= \ -\tfrac{1}{2R_2}\gamma^0\sigma^2\,, & \Gamma_s(P_3) &= \ \tfrac{1}{2R_2}\gamma^0\gamma^1\gamma^2\sigma^2\,, \\[2mm]
\Gamma_s(M_1) &= \ \tfrac{1}{2}\gamma^0\gamma^1\,, & \Gamma_s(M_2) &= \ \tfrac{1}{2}\gamma^2\gamma^3\,.
\end{aligned}
\tag{2.50}
$$

The Killing spinors are, then

$$
\kappa = (u\tilde{u})^{-1}\kappa_0 = e^{-\frac{1}{2}\phi\gamma^1\sigma^2}e^{\frac{1}{2}\chi\gamma^0\sigma^2}e^{-\frac{1}{2}\varphi\gamma^0\gamma^1\gamma^2\sigma^2}e^{-\frac{1}{2}(\theta-\frac{\pi}{2})\gamma^0\gamma^1\gamma^3\sigma^2}\kappa_0\,.
\tag{2.51}
$$

Let us now consider the bilinears $-i\bar{\kappa}\gamma^\mu\kappa$ and define the duals $\Gamma_s(P^a)$ by

$$
\text{Tr}[\Gamma_s(P^a)\Gamma_s(P_b)] = \delta^a{}_b\,.
\tag{2.52}
$$

Then

$$
\gamma^a = -\tfrac{4}{R_2}\mathcal{S}\Gamma_s(P^a)\,, \qquad \mathcal{S} = \gamma^0\gamma^1\sigma^2\,,
\tag{2.53}
$$

and we can see that the modified charge conjugation matrix $\tilde{\mathcal{C}} = \mathcal{C}\mathcal{S}$ has the required property

$$
\tilde{\mathcal{C}}^{-1}\Gamma_s(P_a)^T\tilde{\mathcal{C}} = -\Gamma_s(P_a)\,, \ \ \Rightarrow (u\tilde{u})^{-1\,T}\tilde{\mathcal{C}} = \tilde{\mathcal{C}}u\tilde{u}\,,
\tag{2.54}
$$

that allows us to express the bilinears in the form

$$
-i\bar{\kappa}_{(\alpha i)}\gamma^a\kappa_{(\beta j)} = \tfrac{4i}{R_2}\left\{\tilde{\mathcal{C}}[\Gamma_s(T^I)k_{(I)} + \Gamma_s(\tilde{T}^I)\tilde{k}_{(I)}]\right\}_{(\alpha i\,\beta j)}\,,
\tag{2.55}
$$

where the $k_{(I)}$s are the Killing vectors of AdS_2 and the $\tilde{k}_{(I)}$s are those of S^2. This translates into the anticommutator

$$
\{Q_{(\alpha i)}, Q_{(\beta j)}\} = -i\delta_{ij}(\mathcal{C}\gamma^a)_{\alpha\beta}P_a + \tfrac{i}{R_2}\mathcal{C}_{\alpha\beta}\epsilon_{ij}M_1 + \tfrac{1}{R_2}(\mathcal{C}\gamma_5)_{\alpha\beta}\epsilon_{ij}M_2\,.
\tag{2.56}
$$

The commutators of the supercharges and the bosonic generators are given by the general formula (2.24).

2.3 Other $AdS \times S$ Solutions

There are some other maximally supersymmetric vacua of supergravity theories with metrics which are the direct product of AdS_n and S^m spacetimes. They typically arise in the near-horizon limit of p-brane solutions that preserve only a half of the supersymmetries [15] and can be used in Freund-Rubin compactifications [16], with S^m as internal space, to get gauged supergravities in n dimensions with gauge group $SO(n+1)$. The known cases are $AdS_4 \times S^7$ and $AdS_7 \times S^4$ in $N = 1, d = 11$ supergravity, $AdS_5 \times S^5$ in $N = 2B, d = 10$ supergravity, $AdS_3 \times S^3$ in $N = 2, d = 6$ supergravity, $AdS_2 \times S^3$ [17] and $AdS_3 \times S^2$ [18] in $N = 2, d = 5$ supergravity and the Robinson-Bertotti solution $AdS_2 \times S^2$ in $N = 2, d = 4$ that we have just studied and that can be taken as prototype.

The Killing spinors of all these solutions can be obtained in similar forms. The only complications that arise are due to the symplectic-Majorana nature of supergravity spinors in $4 < d < 8$. We are going to see next how the Killing spinors and vectors the supersymmetry algebras of $AdS_4 \times S^7$ and $AdS_7 \times S^4$ in $N = 1, d = 11$ supergravity and $AdS_5 \times S^5$ in $N = 2B, d = 10$ supergravity can be quickly obtained.

2.3.1 $AdS_4 \times S^7$ in $N = 1, d = 11$ Supergravity

This solution is given by

$$\begin{cases} ds^2 &= R_4^2 \, d\Pi_{(4)}^2 - (2R_4)^2 \, d\Omega_{(7)}^2 \, , \\[2mm] G &= \frac{3}{R_4} \omega_{AdS_4} \, , \quad \Rightarrow G_{0123} = \frac{3}{R_4} \, , \end{cases} \tag{2.57}$$

where $d\Pi_{(4)}^2$ stands for the metric of the AdS_4 spacetime of unit radius, $d\Omega_{(7)}^2$ for the metric of the unit 7-sphere S^7 and ω_{AdS_4} for the volume 4-form of radius R_4.

We construct AdS_4 as in Section 2.1 with $g = 1/R_4$ and this gives us the first four Elfbeins e^a associated to the generators $P_a \, a = 0, 1, 2, 3$ and the first 6 1-forms $\vartheta^{ab} = -\vartheta^{ba}$ associated to the first 4 generators of the 11-dimensional Lorentz group $M_{ab} \, a, b = 0, 1, 2, 3$. The detailed expressions of these 1-forms is really not necessary.

To construct the sphere of radius $2R_4$ we split the $SO(8)$ Lie algebra generators

$$\left[\tilde{M}_{\tilde{a}\tilde{b}}, \tilde{M}_{\tilde{c}\tilde{d}} \right] = \delta_{\tilde{a}\tilde{c}} \tilde{M}_{\tilde{b}\tilde{d}} + \delta_{\tilde{b}\tilde{d}} \tilde{M}_{\tilde{a}\tilde{c}} - \delta_{\tilde{a}\tilde{d}} \tilde{M}_{\tilde{b}\tilde{c}} - \delta_{\tilde{b}\tilde{c}} \tilde{M}_{\tilde{a}\tilde{d}} \, . \tag{2.58}$$

into

$$\tilde{M}_{8i} = 2R_4 P_i \, , \qquad \tilde{M}_{ij} = M_{i+3 \, j+3} \, , \quad i, j = 1, \cdots 7 \, , \tag{2.59}$$

and provide the last 7 P_a's and Lorentz generators $M_{ab} \, a, b = 4, \ldots, 8$. The standard procedure also gives us the associated 7 Elfbeins e^a and 1-forms $\vartheta^{ab} \, a, b = 4, \ldots, 8$. Again, the detailed expressions are not necessary. The metric in Eq. (2.57) is obtained using the Killing metric of both factors $(+ - \cdots -)$.

The general arguments given at the beginning of this section ensure that

$$dx^\mu \nabla_\mu = d - \sum_{a<b} \vartheta^{ab} \Gamma_s(M_{ab}), \qquad \Gamma_s(M_{ab}) = \tfrac{1}{2}\Gamma_{ab}, \tag{2.60}$$

and a straightforward calculation gives for the second piece of the Killing spinor equation

$$\tfrac{i}{288}\left(\Gamma^{abcd}{}_f e^f - 8\Gamma^{abc} e^d\right) G_{abcd} = -e^a \Gamma_s(P_a), \tag{2.61}$$

where

$$\Gamma_s(P_a) = \begin{cases} \frac{i}{2R_4}\Gamma^{0123}\Gamma_a, & a \le 3, \\[2mm] -\frac{i}{4R_4}\Gamma^{0123}\Gamma_a, & a > 3. \end{cases} \tag{2.62}$$

The Killing spinor equation takes the general form Eq. (0.1) and is solved as usual. The specific form of the solution depends on the specific choice of coset representative, but it is unimportant in what follows.

Now, let us consider the bilinears $-i\bar{\kappa}_{(\alpha)}\Gamma^a \kappa_{(\beta)}$. Let us define generators $\Gamma_s(P^a)$ dual to the $\Gamma_s(P_a)$ that are exponentiated to construct the coset representative

$$\mathrm{Tr}\left[\Gamma_s(P^a)\Gamma_s(P_b)\right] = \delta^a{}_b. \tag{2.63}$$

They are given by

$$\Gamma_s(P^a) = \begin{cases} -\frac{iR_4}{16}\Gamma^{0123}\Gamma^a, & a \le 3, \\[2mm] -\frac{iR_4}{8}\Gamma^{0123}\Gamma^a, & a > 3. \end{cases} , \tag{2.64}$$

$$\Gamma_s(M^{ab}) = -\tfrac{1}{16}\Gamma^{ab}.$$

The gamma matrices that appear in the bilinears are related to these by

$$\Gamma^a = \tfrac{-16i}{R_4}\mathcal{S}\Gamma_s(P^a), \qquad a \le 3,$$
$$\Gamma^a = \tfrac{-8i}{R_4}\mathcal{S}\Gamma_s(P^a), \qquad a > 3, \qquad \mathcal{S} = \Gamma^{0123}, \tag{2.65}$$

and, since the modified charge conjugation matrix $\tilde{\mathcal{C}} = \mathcal{C}\mathcal{S}$ has the required property

$$\tilde{\mathcal{C}}^{-1}\Gamma_s(P^a)^T \tilde{\mathcal{C}} = -\Gamma_s(P^a), \tag{2.66}$$

the bilinears can be written in the form (suppressing the indices α, β)

$$-i\bar{\kappa}\Gamma^a \kappa = \tfrac{-8}{R_4}\tilde{\mathcal{C}}[\Gamma_s(\hat{M}^{\hat{a}\hat{b}})k_{(\hat{a}\hat{b})} + \tfrac{1}{2}\Gamma_s(\tilde{M}^{\tilde{a}\tilde{b}})k_{(\tilde{a}\tilde{b})}], \tag{2.67}$$

where hatted generators and Killing vectors belong to the AdS_4 factor and the tilded ones to the S^7 factor. The anticommutator of two supercharges can be immediately read in this expression and the commutator of supercharges and bosonic charges is given by the general formula Eq. (2.24).

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2.3.2 $AdS_7 \times S^4$ in $N = 1, d = 11$ Supergravity

This solution is given by

$$
\begin{cases}
ds^2 & = \ R_7^2 \, d\Pi_{(7)}^2 - (R_7/2)^2 \, d\Omega_{(4)}^2 \,, \\[2mm]
G & = \ \frac{6}{R_7} \omega_{S^4} \,, \quad \Rightarrow G_{789\,10} = \frac{6}{R_7} \,,
\end{cases}
\tag{2.68}
$$

where we use the same notation as in the preceding cases and ω_{S^4} stands for the volume of the sphere of radius $R_7/2$. The definitions of the P_a and M_{ab} generators and the construction of the Elfbeins etc. is almost identical to that of the preceding case and we immediately arrive at

$$
dx^\mu \nabla_\mu = d - \sum_{a<b} \vartheta^{ab} \Gamma_s(M_{ab}) \,, \qquad \Gamma_s(M_{ab}) = \tfrac{1}{2}\Gamma_{ab} \,.
\tag{2.69}
$$

The 1-forms ϑ^{ab} have a different form now, but we do not need to know it. The second piece of the Killing spinor equation takes the form

$$
\tfrac{i}{288} \left(\Gamma^{abcd}{}_f e^f - 8\Gamma^{abc} e^d \right) G_{abcd} = -e^a \Gamma_s(P_a) \,,
\tag{2.70}
$$

where now

$$
\Gamma_s(P_a) =
\begin{cases}
\frac{i}{2R_7} \Gamma^{789\,10} \Gamma_a \,, & a \le 6 \,, \\[2mm]
-\frac{i}{R_4} \Gamma^{789\,10} \Gamma_a \,, & a > 6 \,.
\end{cases}
\tag{2.71}
$$

The Elfbeins are also different, but, yet again, we do not need to know their detailed expressions. The dual generators are defined as usual and are given by

$$
\Gamma_s(P^a) =
\begin{cases}
-\frac{iR_7}{16} \Gamma^{789\,10} \Gamma^a \,, & a \le 6 \,, \\[2mm]
-\frac{iR_7}{32} \Gamma^{789\,10} \Gamma^a \,, & a > 6 \,.
\end{cases}
\tag{2.72}
$$

$$
\Gamma_s(M^{ab}) = -\tfrac{1}{16}\Gamma^{ab} \,.
$$

and

$$
\begin{aligned}
\Gamma^a & = \ \frac{16i}{R_7} \mathcal{S} \Gamma_s(P^a) \,, & a \le 6 \,, \\[2mm]
\Gamma^a & = \ \frac{-32i}{R_7} \mathcal{S} \Gamma_s(P^a) \,, & a > 6 \,,
\end{aligned}
\qquad \mathcal{S} = \Gamma^{789\,10} \,.
\tag{2.73}
$$

The modified charge conjugation matrix has the property Eq. (2.66) and we get, suppressing again $\alpha\beta$ indices

$$
-i\bar{\kappa}\Gamma^a\kappa = \frac{-8}{R_7} \tilde{\mathcal{C}}[\Gamma_s(\hat{M}^{\hat{a}\hat{b}})k_{(\hat{a}\hat{b})} - 2\Gamma_s(\tilde{M}^{\tilde{a}\tilde{b}})k_{(\tilde{a}\tilde{b})}] \,,
\tag{2.74}
$$

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where hatted generators and Killing vectors belong to the AdS_7 factor and the tilded ones to the S^4 factor. Again, the anticommutator of two supercharges can be immediately read in this expression and the commutator of supercharges and bosonic charges is given by the general formula Eq. (2.24).

2.3.3 $AdS_5 \times S^5$ in $N = 2B, d = 10$ Supergravity

The solution is given in the string frame by

$$
\begin{cases}
ds^2 & = & R_5^2 d\Pi_{(5)}^2 - R_5^2 d\Omega_{(5)}^2 \,, \\
\\
G^{(5)} & = & \frac{4e^{-\varphi_0}}{R_5}(\omega_{AdS_5} + \omega_{S^5}) \,, \quad \Rightarrow G_{01234}^{(5)} = G_{56789}^{(5)} = \frac{4e^{-\varphi_0}}{R_5} \,, \\
\\
\varphi & = & \varphi_0 \,.
\end{cases}
\tag{2.75}
$$

This case is exactly analogous to the previous ones. The normalization in the splitting of the generators of $SO(2,4)$ and $SO(6)$ is now, respectively:

$$
\begin{aligned}
\hat{M}_{a-1} & = -R_5 P_a \,, \quad (a = 0, ..., 4) \\
\\
\tilde{M}_{6a} & = R_5 P_a \,, \quad (a = 5, ..., 9) \,.
\end{aligned}
\tag{2.76}
$$

Once again we do not need to know the explicit form of the Zehnbeins. From the covariant derivative term in the gravitino supersymmetry transformation (the variation of the dilatino vanishes automatically) we get the generators of $SO(1,4)$ and $SO(5)$ in the spinor representation. From the remaining piece:

$$
-\tfrac{1}{16 \cdot 5!} e^{\varphi_0} G_{bcdef}^{(5)} \Gamma^{bcdef} \Gamma_a i\sigma^2 = -e^a \Gamma_s(P_a) \,,
\tag{2.77}
$$

we read the spinor representation for the generators P_a [13]

$$
\Gamma_s(P_a) =
\begin{cases}
\frac{i}{2R_5} \sigma^2 \Gamma^{01234} \Gamma_a \,, \quad (a = 0, \cdots, 4) \\
\\
-\frac{i}{2R_5} \sigma^2 \Gamma^{01234} \Gamma_a \,. \quad (a = 5, \cdots, 9)
\end{cases}
\tag{2.78}
$$

The dual generators are

$$
\Gamma_s(P^a) = \tfrac{iR_5}{32} \sigma^2 \Gamma^{01234} \Gamma^a \,, \quad \Rightarrow \Gamma^a = \tfrac{32i}{R_5} \mathcal{S} \Gamma_s(P^a) \,, \qquad \mathcal{S} = \sigma^2 \Gamma^{01234} \,,
\tag{2.79}
$$

and the modified charge conjugation matrix has the required property Eq. (2.66) that leads to

$$
-i\bar{\kappa}\Gamma^a \kappa = \tfrac{32}{R_5} \tilde{\mathcal{C}} [\Gamma_s(\hat{M}^{\hat{a}\hat{b}}) k_{(\hat{a}\hat{b})} + \Gamma_s(\tilde{M}^{\tilde{a}\tilde{b}}) k_{(\tilde{a}\tilde{b})}] \,.
\tag{2.80}
$$

---

[13]Here there is another (completely equivalent, since we are dealing with chiral spinors) possibility, consisting in replacing Γ^{01234} by $-\Gamma^{56789}$.

2.4 Hpp-wave Spacetimes and the $KG4, 5, 6, 10, 11$ Solutions

Although maximally supersymmetric pp-wave solutions were discovered long time ago by Kowalski-Glikman in $N = 2, d = 4$ and $N = 1, d = 11$ supergravity [19, 20], only recently they have received wide attention. This renewed interest has been accompanied with the discovery of new maximally supersymmetric solutions of the same kind (henceforth KG solutions) in $N = 2B, d = 10$ supergravity [21] and in $N = 2, d = 5, 6$ supergravities [22], and by the realization that they can be obtained by taking a Penrose limit [23, 24] of the known $AdS \times S$ maximally supersymmetric solutions [25, 26].

The KG solutions are particular examples of homogeneous pp-wave spacetimes (Hpp-waves), symmetric spacetimes to which we can apply our formalism. Let us review briefly the coset construction that leads to them [27, 28].

The generators of $\mathfrak{g}$ in Hpp-wave spacetimes are $\{T_-, T_+, T_i, T_{\star i}\}$ $i = 1, \cdots, d - 2$ and their non-vanishing Lie brackets are

$$[T_-, T_i] = T_{\star i}, \qquad [T_-, T_{\star i}] = A_{ij}T_j, \qquad [T_i, T_{\star j}] = A_{ij}T_+, \qquad A_{ij} = A_{ji}. \qquad (2.81)$$

T_+ is central in this Lie algebra. The subalgebra $\mathfrak{h}$ is generated by the $T_{\star i} \equiv M_i$ and $\mathfrak{k}$ is generated by $T_- \equiv P_-$, $T_+ \equiv P_+$, $T_i \equiv P_i$, and the coset representative is chosen to be

$$u = e^{x^- P_-} e^{x^+ P_+} e^{x^i P_i}, \qquad (2.82)$$

which lead to the Maurer-Cartan 1-form

$$V = u^{-1}du = -dx^- P_- - (dx^+ + \tfrac{1}{2}x^i x^j A_{ij}dx^-)P_+ - dx^i P_i - x^i dx^- M_i. \qquad (2.83)$$

Since $\mathfrak{g}$ is not semisimple, its Killing metric is singular and cannot be used to construct a G-invariant metric. Instead, we choose[14]

$$B_{+-} = 1, \qquad B_{ij} = +\delta_{ij}, \qquad (2.84)$$

and we get the general Hpp-wave metric

$$ds^2 = 2dx^-(dx^+ + \tfrac{1}{2}x^i x^j A_{ij}dx^-) + dx^i dx^i. \qquad (2.85)$$

Different Hpp-wave metrics are characterized by the matrix A_{ij} up to $SO(d - 2)$ rotations. On the other hand (and this is an important difference with the previous cases), the Hpp-wave metric can have more isometries: all possible rotations of the x^i that preserve the matrix A_{ij}. These rotations do not belong to $\mathfrak{g}$ and the corresponding Killing vectors cannot be found by applying Eq. (1.20).

---

[14]The metric $B_{+-} = 1$, $B_{ij} = -\delta_{ij}$ is not invariant under the action of $\mathfrak{h}$ on $\mathfrak{k}$. Thus, we are forced to work with mostly plus signature in this section.

Let us now consider the $KG11$ solution. Its metric is of the above general Hpp-wave form, with A_{ij} and the 4-form field strength given by

$$G_{-123} = \lambda\,, \qquad A_{ij} = \begin{cases} -\frac{1}{9}\lambda^2\delta_{ij} & i,j = 1,2,3\,, \\[2mm] -\frac{1}{36}\lambda^2\delta_{ij} & i,j = 4,\cdots,9\,. \end{cases} \tag{2.86}$$

This solution is additionally invariant under rotations in the subspaces parametrized by x^1, x^2, x^3 and $x^4, \cdots, x^9$. Let us now consider the Killing spinor equation. According to the general construction, we only need to compute the Ω part that involves the 4-form field strength. This can be written in the form $-e^a\Gamma_s(P_a)$ with

$$
\begin{aligned}
\Gamma_s(P_-) &= \tfrac{\lambda}{12}(\Gamma^-\Gamma^+ + 1)\Gamma^{123}\,, \\[2mm]
\Gamma_s(P_+) &= 0\,, \\[2mm]
\Gamma_s(P_i) &= \begin{cases} -\frac{\lambda}{6}\Gamma^{-123}\Gamma_i\,, & i = 1,2,3\,, \\[2mm] -\frac{\lambda}{12}\Gamma^{-123}\Gamma_i\,, & i = 4,\cdots,9\,, \end{cases}
\end{aligned}
\tag{2.87}
$$

and the Killing spinor has the same form as usual, the only difference being that one of the P_a generators (P_+) is represented by zero and does not contribute to the coset representative u. The general formula Eq. (2.24) can be used to calculate the commutators of supercharges and bosonic generators. We see that P_+ is a central charge also in the superalgebra [28]. The calculation of the anticommutators of supercharges is more complicated, though, basically because we can construct duals

$$\Gamma_s(P^a) \sim \Gamma^{+123}\Gamma^a\,, \tag{2.88}$$

but the matrix Γ^{+123} is singular and the relation cannot be inverted. This is related to the existence of the extra rotational Killing vectors $k_{(ij)}$ that do appear in the bilinear $-i\bar{\kappa}_{(\alpha)}\Gamma^a\kappa e_a$ [28]. The above equation can in fact be used to relate the Killing vectors $k_{(I)}$ to some of all the possible bilinears. The additional Killing vectors $k_{(ij)}$ appear in the other bilinears (associated to the anticommutators $\{Q_-, Q_-\}$ in the notation of Ref. [28]).

3   Conclusions

In this paper we have checked in almost all known maximally supersymmetric backgrounds that the Killing spinor equation can be set in the form Eq. (0.1) and we have shown how this can be exploited to calculate their supersymmetry algebras using results from the theory of symmetric spaces.

There are two exceptional cases: the KG spaces, for which it is not easy to compute all the possible anticommutators $\{Q_{(\alpha)}, Q_{(\beta)}\}$ and the maximally supersymmetric solution that can be obtained by taking the near-horizon limit of the 5-dimensional extreme rotating black hole [29, 9, 30] whose description as symmetric space is not known.

The obvious extension of this work is to backgrounds with less supersymmetry, like those that can be obtained by replacing the sphere in $AdS \times S$ solutions by another homogeneous space with the right curvature [31, 32, 33]. Work in this direction is in progress.

Note added in Proof: the authors are indebted to the Referee for pointing out to us the actual group-theoretical reason due to which, in the case of Hpp-wave geometries, not all the Killing vectors can be found by using eq.(1.20). This is because the complete isometry group of the considered Hpp-wave solutions is a semi-direct product of the group defined in eq.(2.81) and the group of additional isometries and, since the coset space only belongs to one of the factors in this semi-direct product, (1.20) does not produce the Killing vectors corresponding to the other factor (the rotational isometries of the wave-front).

Acknowledgments

References

[1] J. M. Figueroa-O'Farrill, Class. Quant. Grav. **16** (1999) 2043 [arXiv:hep-th/9902066].

[2] L. Castellani, R. D'Auria and P. Fré, *Supergravity And Superstrings: A Geometric Perspective. Vol. 1: Mathematical Foundations*, World Scientific, Singapore (1991).

[3] R. Coquereaux and A. Jadczyk, *Riemannian Geometry, Fiber Bundles, Kaluza-Klein Theories And All That*, World Sci. Lect. Notes Phys. **16** (1988) 1.

[4] A. Lichnerowicz, *Spineurs harmoniques C. R. Acad. Sci. Paris* **257** (1963) 7-9.

[5] Y. Kosmann, *Annali di Mat. Pura Appl.* **(IV) 91** (1972) 317-395.

[6] Y. Kosmann, *C. R. Acad. Sci. Paris* Sér. A-B, **262** (1966) A289-92.

[7] T. Ortin, Class. Quant. Grav. **19** (2002) L143 [arXiv:hep-th/0206159].

[8] M. Godina and P. Matteucci [arXiv:math.DG/0201235].

[9] J. P. Gauntlett, R. C. Myers and P. K. Townsend, Class. Quant. Grav. **16** (1999) 1 [arXiv:hep-th/9810204].

[10] H. Lu, C. N. Pope and J. Rahmfeld, J. Math. Phys. **40** (1999) 4518 [arXiv:hep-th/9805151].

[11] I. Robinson, *Bull. Acad. Polon. Sci.* **7**(1959), 351;

[12] B. Bertotti, *Phys. Rev.* **116** (1959) 1331.

[13] G. W. Gibbons, *Three lectures given at GIFT Seminar on Theoretical Physics, San Feliu de Guixols, Spain, Jun 4-11, 1984.*

[14] R. Kallosh, Phys. Lett. B **282** (1992) 80 [arXiv:hep-th/9201029].

[15] G. W. Gibbons and P. K. Townsend, Phys. Rev. Lett. **71** (1993) 3754 [arXiv:hep-th/9307049].

[16] P. G. Freund and M. A. Rubin, Phys. Lett. B **97** (1980) 233.

[17] A. H. Chamseddine, S. Ferrara, G. W. Gibbons and R. Kallosh, Phys. Rev. D **55** (1997) 3647 [arXiv:hep-th/9610155].

[18] G. W. Gibbons, G. T. Horowitz and P. K. Townsend, Class. Quant. Grav. **12** (1995) 297 [arXiv:hep-th/9410073].

[19] J. Kowalski-Glikman, Phys. Lett. B **134** (1984) 194.

[20] J. Kowalski-Glikman, Phys. Lett. B **150** (1985) 125.

[21] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, JHEP **0201** (2002) 047 [arXiv:hep-th/0110242].

[22] P. Meessen, Phys. Rev. D **65** (2002) 087501 [arXiv:hep-th/0111031].

[23] R. Penrose, in *Differential Geometry and Relativity*, Reidel, Dordrecht (1976) 271-275.

[24] R. Güven, Phys. Lett. B **482** (2000) 255 [arXiv:hep-th/0005061].

[25] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, arXiv:hep-th/0201081.

[26] M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos, arXiv:hep-th/0202111.

[27] M. Cahen and N. Wallach, *Bull. Am. Math. Soc.* **76** (1970) 585-591.

[28] J. Figueroa-O'Farrill and G. Papadopoulos, JHEP **0108** (2001) 036 [arXiv:hep-th/0105308].

[29] R. Kallosh, A. Rajaraman and W. K. Wong, Phys. Rev. D **55** (1997) 3246 [arXiv:hep-th/9611094].

[30] M. Cvetič and F. Larsen, Nucl. Phys. B **531** (1998) 239 [arXiv:hep-th/9805097].

[31] L. Castellani, L. J. Romans and N. P. Warner, ' Nucl. Phys. B **241** (1984) 429.

[32] M. J. Duff, H. Lu, C. N. Pope and E. Sezgin, Phys. Lett. B **371** (1996) 206 [arXiv:hep-th/9511162].

[33] L. Castellani, A. Ceresole, R. D'Auria, S. Ferrara, P. Fré and M. Trigiante, Nucl. Phys. B **527** (1998) 142 [arXiv:hep-th/9803039].