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Towards Noncommutative Integrable Systems

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Abstract

We present a strong method to generate various equations which have the Lax representations on noncommutative $(1+1)$ and $(2+1)$-dimensional spaces. The generated equations contain noncommutative integrable equations obtained by using the bicomplex method and by reductions of the noncommutative (anti-)self-dual Yang-Mills equation. This suggests that the noncommutative Lax equations would be integrable and be derived from reductions of the noncommutative (anti-)self-dual Yang-Mills equations, which implies the noncommutative version of Richard Ward conjecture.

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1 Introduction

Non-Commutative (NC) gauge theory has been studied intensively for the last several years and succeeded in revealing various aspects of gauge theories in the presence of background magnetic fields [1]. Especially, NC solitons play crucial roles in the study of D-brane dynamics, such as tachyon condensation [2].

NC spaces are characterized by the noncommutativity of the coordinates:

$$[x^i, x^j] = i\theta^{ij}, \tag{1.1}$$

where θ^{ij} are real constants. This relation looks like the canonical commutation relation in quantum mechanics and leads to "space-space uncertainty relation." Hence the singularity which exists on commutative spaces could resolve on NC spaces. This is one of the distinguished features of NC theories and gives rise to various new physical objects. For example, even when the gauge group is $U(1)$, instanton solutions still exist [3] because of the resolution of the small instanton singularities of the complete instanton moduli space [4].

NC gauge theories are naively realized from ordinary commutative theories just by replacing all products of the fields with star-products. In this context, NC theories are considered to be deformed theories from commutative ones and look very close to the commutative ones.

Under the deformation, the (anti-)self-dual (ASD) Yang-Mills equations could be considered to preserve the integrability in the same sense as in commutative cases [5, 6]. On the other hand, with regard to typical integrable equations such as the Kadomtsev-Petviasfvili (KP) equation, the naive NC extension generally destroys the integrability. There is known to be a method, the *bicomplex method*, to yield NC integrable equations which have many conserved quantities [7, 8].

In this paper, we discuss NC extensions of wider class of integrable equations which are expected to preserve the integrability. First, we present a strong method to give rise to NC Lax pairs and construct various NC Lax equations. Then we discuss the relationship between the generated equations and the NC integrable equations obtained from the bicomplex method and from reductions of the NC ASD Yang-Mills equations. All the results are consistent and we can expect that the NC Lax equations would be integrable. Hence it is natural to propose the following conjecture which contains the NC version of Ward conjecture: *many of NC Lax equations would be integrable and be obtained from reductions of the NC ASD Yang-Mills equations.*

2 Noncommutative Lax Equations

2.1 The Lax-Pair Generating Technique

In commutative cases, Lax representations are common in many of known integrable equations and fit well to the discussion of reductions of the ASD Yang-Mills equations. Here we look for the Lax representations on NC spaces. First we introduce how to find Lax representations on commutative spaces.

An integrable equation which has the Lax representation can be rewritten as the following equation:

$$[L, T + \partial_t] = 0, \tag{2.1}$$

where $\partial_t := \partial/\partial t$. This equation and the pair of operators (L, T) are called the *Lax equation* and the *Lax pair*, respectively.

The NC version of the Lax equation (2.1), the *NC Lax equation*, is easily defined just by replacing the product of L and T with the star product. The star product is defined by

$$f \star g(x) := \exp\left(\frac{i}{2}\theta^{ij}\partial_i^{(x')}\partial_j^{(x'')}\right) f(x')g(x'')\Big|_{x'=x''=x.} \tag{2.2}$$

The star product has associativity: $f \star (g \star h) = (f \star g) \star h$, and reduces to the ordinary product in the limit $\theta^{ij} \rightarrow 0$. The modification of the product makes the ordinary coordinate "noncommutative," which means : $[x^i, x^j]_\star := x^i \star x^j - x^j \star x^i = i\theta^{ij}$.

In this paper, we look for the NC Lax equation whose operator L is a differential operator. In order to make this study systematic, we set up the following problem :

Problem : For a given operator L, find the corresponding operator T which satisfies the Lax equation (2.1).

This is in general very difficult to solve. However if we put an ansatz on the operator T, then we can get the answer for wide class of Lax pairs including NC case. The ansatz for the operator T is of the following type:

Ansatz for the operator T :

$$T = \partial_i^n L + T'. \tag{2.3}$$

Then the problem for T reduces to that for T'. This ansatz is very simple, however, very strong to determine the unknown operator T', that is, the Lax pair (L, T), which is called, in this paper, the *Lax-pair generating technique*.

In order to explain it more concretely, let us consider the Korteweg-de-Vries (KdV) equation on commutative $(1+1)$-dimensional space where the operator L is given by $L_{\text{KdV}} := \partial_x^2 + u(t,x)$.

The ansatz for the operator T is given by

$$T = \partial_x L_{\text{KdV}} + T', \tag{2.4}$$

which corresponds to $n=1$ and $\partial_i = \partial_x$ in the general ansatz (2.3). This factorization was first used to find more wider class of Lax pairs in higher dimensional case [9].

The Lax equation (2.1) leads to the equation for the unknown operator T':

$$[\partial_x^2 + u, T'] = u_x\partial_x^2 + u_t + uu_x, \tag{2.5}$$

where $u_x := \partial u/\partial x$ and so on. Here we want to delete the term $u_x\partial_x^2$ in the RHS of (2.5) so that this equation finally reduces to a differential equation. Therefore the operator T' could be taken as

$$T' = A\partial_x + B, \tag{2.6}$$

where A, B are polynomials of $u, u_x, u_t, u_{xx},$ etc. Then the Lax equation becomes $f\partial_x^2 + g\partial_x + h = 0$. From $f=0, g=0$, we get[3]

$$A = \frac{u}{2}, \quad B = -\frac{1}{4}u_x + \beta, \tag{2.7}$$

that is,

$$T = \partial_x^3 + \frac{3}{4}u_x + \frac{3}{2}u\partial_x. \tag{2.8}$$

Finally $h=0$ yields a Lax equation, the KdV equation:

$$u_t + \frac{3}{2}uu_x + \frac{1}{4}u_{xxx} = 0. \tag{2.9}$$

In this way, we can generate wide class of Lax equations including higher dimensional integrable equations [9]. For example, $L_{\text{mKdV}} := \partial_x^2 + v(t,x)\partial_x$ and $L_{\text{KP}} := \partial_x^2 + u(t,x,y) + \partial_y$ give rise to the modified KdV equation and the KP equation, respectively by the same ansatz (2.4) for T. If we take $L_{\text{BCS}} := \partial_x^2 + u(t,x,y)$ and the modified ansatz $T = \partial_y L_{\text{BCS}} + T'$, then we get the Bogoyavlenskii-Calogero-Schiff (BCS) equation [10].[4]

Good news here is that this technique is also applicable to NC cases.

[3]Exactly speaking, an integral constant should appear in A as $A = u/2 + \alpha$. This constant α is unphysical and can be absorbed by the scale transformation $u \to u + 2\alpha/3$. Hence we can take $\alpha = 0$ without loss of generality.

[4]The multi-soliton solution is found in [11].

2.2 Some Results

We present some results by using the Lax-pair generating technique. First we focus on NC $(2+1)$-dimensional Lax equations. Let us suppose that the noncommutativity is basically introduced in the space directions.

- The NC KP equation [12] :

  The Lax operator is given by

$$L_{\text{KP}} = \partial_x^2 + u(t,x,y) + \partial_y := L'_{\text{KP}} + \partial_y. \tag{2.10}$$

  The ansatz for the operator T is the same as commutative case:

$$T = \partial_x L'_{\text{KP}} + T'. \tag{2.11}$$

  Then we find

$$T' = \frac{1}{2}u\partial_x - \frac{1}{4}u_x - \frac{3}{4}\partial_x^{-1}u_y, \tag{2.12}$$

  and the NC KP equation:

$$u_t + \frac{1}{4}u_{xxx} + \frac{3}{4}(u_x \star u + u \star u_x) + \frac{3}{4}\partial_x^{-1}u_{yy} + \frac{3}{4}[u, \partial_x^{-1}u_y]_\star = 0, \tag{2.13}$$

  where $\partial_x^{-1}f(x) := \int^x dx' f(x'),\ u_{xxx} = \partial^3 u/\partial x^3$ and so on. This coincides with that in [12]. There is seen to be a nontrivial deformed term $[u, \partial_x^{-1}u_y]_\star$ in the equation (2.13) which vanishes in the commutative limit. In [12], the multi-soliton solution is found by the first order to small θ expansion, which suggests that this equation would be considered as an integrable equation.

  The modified version of the NC KP equation is also found in the similar way. This is new.

- The NC modified KP equation :

  For a given operator $L_{\text{mKP}} = \partial_x^2 + v(t,x,y)\partial_x + \partial_y := L'_{\text{mKP}} + \partial_y$, we can take the ansatz $T = \partial_x L'_{\text{mKP}} + T'$, which yields the NC modified KP equation:

$$\begin{aligned} &v_t + \frac{1}{4}v_{xxx} - \frac{3}{8}v \star v_x \star v + \frac{3}{8}([v, v_x]_\star)_x - \frac{3}{4}(\partial_x^{-1}v_y) \star v_x \\ &+ \frac{3}{4}\partial_x^{-1}v_{yy} + \frac{3}{8}[v, v_{xx} + v_y]_\star = 0, \end{aligned} \tag{2.14}$$

  where the operator T' are determined as

$$T' = \frac{1}{2}v\partial_x^2 + \left(-\frac{1}{4}v_x + \frac{3}{8}v \star v - \frac{3}{4}\partial_x^{-1}v_y\right)\partial_x. \tag{2.15}$$

  Nontrivial terms are also seen in the equation (2.14).

- The NC BCS equation :

  This is obtained by the same steps as in commutative case. The new equation is

$$
\begin{aligned}
&u_t + \frac{1}{4}u_{xxy} + \frac{1}{2}(u_y \star u + u \star u_y) + \frac{1}{4}u_x \star (\partial_x^{-1}u_y) \\
&+ \frac{1}{4}(\partial_x^{-1}u_y) \star u_x + \frac{1}{4}[u, \partial_x^{-1}[u, \partial_x^{-1}u_y]_\star]_\star = 0,
\end{aligned} \tag{2.16}
$$

  whose Lax pair and the ansatz are

$$
\begin{aligned}
L_{\mathrm{BCS}} &= \partial_x^2 + u(t,x,y), \\
T &= \partial_y L_{\mathrm{BCS}} + T', \qquad (2.17) \\
T' &= \frac{1}{2}(\partial_x^{-1}u_y)\partial_x - \frac{1}{4}u_y - \frac{1}{4}\partial_x^{-1}[u, \partial_x^{-1}u_y]_\star. \qquad (2.18)
\end{aligned}
$$

  This time, a non-trivial term is found even in the operator T.

We can generate many other NC Lax equations in the same way. Moreover if we introduce the noncommutativity into time coordinate as $[t,x] = i\theta$, we can construct NC $(1+1)$-dimensional integrable equations.

For example, the NC KdV equation is

$$
u_t + \frac{3}{4}(u_x \star u + u \star u_x) + \frac{1}{4}u_{xxx} = 0, \tag{2.19}
$$

which coincides with that derived by using the bicomplex method [13] and by the reduction from NC KP equation (2.13) setting the fields y-independent: $\partial_y u = 0$ and reintroducing the noncommutativity as $[t,x] = i\theta$, this time.[5] (This is true of the NC modified KdV equation.) We also find the NC KdV hierarchy [15].

As one of the new Lax equations, the NC Burgers equation is obtained:

$$
u_t - \alpha u_{xx} + (1 - \alpha - \beta)u \star u_x + (1 + \alpha - \beta)u_x \star u = 0. \tag{2.20}
$$

We succeed in linearizing it by the NC Cole-Hopf transformation [16].

All of the NC integrable equations derived from the bicomplex method are also obtained by our method. The bicomplex method guarantees the existence of the many conserved topological quantities. These results suggest that NC Lax equations would possess the integrability.

[5] We note that this reduction is formal and the noncommutativity here contains subtle points in the derivation from the $(2+2)$-dimensional NC ASD Yang-Mills equation by reduction because the coordinates (t,x,y) originate partially from the parameters in the gauge group of the NC ASD Yang-Mills theory [14]. We are grateful to T. Ivanova for pointing out this point to us.

Here we comment on the multi-soliton solutions. First we note that if the field is holomorphic, that is, $f = f(x - vt) = f(z)$, then the star product reduces to the ordinary product:

$$f(x - vt) \star g(x - vt) = f(x - vt)g(x - vt). \tag{2.21}$$

Hence the commutative multi-soliton solutions where all the solitons move at the same velocity always satisfy the NC version of the equations. Of course, this does not mean that the equations possess the integrability.

The comprehensive list and the more detailed discussion are reported later soon.

3 Comments on the Noncommutative Ward Conjecture

In commutative case, it is well known that many of integrable equations could be derived from symmetry reductions of the four-dimensional ASD Yang-Mills equation [14], which is first conjectured by R.Ward [17].

Even in NC case, the corresponding discussions would be possible and be interesting. The NC ASD Yang-Mills equations also have the Yang's forms [18, 6] and many other similar properties to commutative ones [5]. The simple reduction to three dimension yields the NC Bogomol'nyi equation which has the exact monopole solutions and can be rewritten as the non-Abelian Toda lattice equation [6, 19]. It is interesting that a discrete structure appears. Moreover M.Legaré [20] succeeded in some reductions of the $(2+2)$-dimensional NC ASD Yang-Mills equations which coincide with our results and those by using the bicomplex method, which strongly suggests that the noncommutative deformation would be unique and integrable and the Ward conjecture would still hold on NC spaces.

In four-dimensional Yang-Mills theory, the NC deformation resolves the small instanton singularity of the (complete) instanton moduli space and gives rise to a new physical object, the $U(1)$ instanton. Hence the NC Ward conjecture would imply that the NC deformations of lower-dimensional integrable equations might contain new physical objects because of the deformations of the solution spaces in some case.

4 Conclusion and Discussion

In the present paper, we found a powerful method to find NC Lax equations which is expected to be integrable. The simple, but mysterious, ansatz (2.3) plays an important role and actually gives rise to various new NC Lax equations. Finally we pointed out that some reductions of the NC ASD Yang-Mills equations give rise to NC integrable equations including our results.

Now there would be mainly three methods to yield NC integrable equations:

- Lax-pair generating technique
- Bicomplex method
- Reduction of the ASD Yang-Mills equation

The interesting point is that all the results are consistent at least with the known NC Lax equations, which suggests the existence and the uniqueness of the NC deformations of integrable equations which preserve the integrability.

Though we can get many new NC Lax equations, there need to be more discussions so that such study should be fruitful as integrable systems. First, we have to clarify whether the NC Lax equations are really good equations in the sense of integrability, that is, the existence of many conserved quantities or of multi-soliton solutions, and so on. All of the previous studies including our works strongly suggest that this would be true. Second, we have to reveal the physical meaning of such equations. If such integrable theories can be embedded in string theories, there would be fruitful interactions between the both theories, just as between the (NC) ASD Yang-Mills equation and D0-D4 brane system (in the background of NS-NS B field).

The systematic and co-supplement studies of them would pioneer a new area of integrable systems and perhaps string theories.

Note added

We were informed by O. Lechtenfeld that the $(2+2)$-dimensional NC ASD Yang-Mills equation and some reductions of it can be embedded [21, 22] in $N=2$ string theory [23], which guarantees that such directions would have a physical meaning and might be helpful to understand new aspects of the corresponding string theory.

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