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On D-branes from Gauged Linear Sigma Models

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Abstract

We study both A-type and B-type D-branes in the gauged linear sigma model by considering worldsheets with boundary. The boundary conditions on the matter and vector multiplet fields are first considered in the large-volume phase/non-linear sigma model limit of the corresponding Calabi-Yau manifold, where we find that we need to add a contact term on the boundary. These considerations enable to us to derive the boundary conditions in the full gauged linear sigma model, including the addition of the appropriate boundary contact terms, such that these boundary conditions have the correct non-linear sigma model limit. Most of the analysis is for the case of Calabi-Yau manifolds with one Kähler modulus (including those corresponding to hypersurfaces in weighted projectve space), though we comment on possible generalisations.

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1 Introduction

The improved understanding of non-perturbative aspects of string theory in recent years has shown that all five (perturbative) superstrings appear to be different corners in the moduli space of a single theory [1]. A consequence of this is that while the existence of a perturbative string theory description at these corners singles out strings as fundamental objects, at generic points in this moduli space, there is a certain democracy among all objects, fundamental as well as solitonic, as seen in the perturbative string theory. Thus, the heterotic string appears as a soliton (D1-brane) in the type I theory. A more general analysis indicates that an object which is a soliton at one point in the moduli space can become a fundamental excitation at another point in the moduli space [2].

Another question of interest is the nature of spacetime at short distances. It turns out that the answer is related to the kind of probe which is used. Given the democracy among all objects one can probe spacetime using both fundamental strings as well as solitons such as D-branes. Fundamental strings probe objects which are of the string scale l_s while D-branes in perturbative string theory probe much shorter scales $(g_s)^a l_s$, where g_s is the string coupling constant and a is a positive constant [3]. From earlier studies of closed strings, it is known that strings can propagate in apparently singular spaces such as orbifolds. D-brane probes see these space-time geometries in a manner different from that of closed string theories. For example, it was shown that for D0-brane probes of the orbifold $\mathbb{C}^3/\Gamma$ (where Γ is a discrete subgroup of $SU(3)$), the non-geometric phases seen by the closed string are projected out [4].

While much is known about the nature of fundamental strings probing various space-time geometries, our understanding in the case of D-brane probes is in a much more primitive state, apart from the cases of flat space and toroidal and orbifold backgrounds. In the last couple of years, there has been considerable progress in understanding D-branes in the context of string compactification on Calabi-Yau threefolds [5–18]. Unlike the case of toroidal compactifications, these correspond to fewer unbroken supersymmetries and thus fewer constraints follow. For example, the BPS conditions leave open the possibility of having lines of marginal stability in the moduli space, where a D-brane can decay. A D-brane which fills the non-compact spacetime while also wrapping some cycle of the CY three-fold can possess a non-trivial superpotential in its worldvolume gauge theory. It is of interest to derive this superpotential and its dependence on closed string moduli.

D-branes on CY manifolds fall into two distinct categories: A-type branes are those which wrap special Lagrangian submanifolds while B-type branes wrap holomorphic submanifolds of the CY manifold. In the worldsheet description [5], A-type branes and B-type branes differ in the worldsheet supersymmetry that they preserve. In the open-string channel, A-type branes are compatible with the topological theory obtained with the A-twist and B-type branes are compatible with the B-twist. In the closed-string channel, the roles are reversed due to a change of sign in the boundary conditions on the $U(1)$ currents of the $(2,2)$ worldsheet supersymmetry algebra. In the closed-string case [19], correlation functions of the observables in the topological A-model are independent of complex structure moduli (of the CY) while those in the topological B-model are independent of the Kähler moduli. The *modified geometric hypothesis* proposed in [7, 8] is in a sense the open-string version of this statement. Based on this, one (loosely speaking) expects the lines of marginal stability of A-branes and the superpotential of B-branes to be independent of Kähler moduli and thus calculable in the large volume limit where classical geometry can be applied. (See [8] for a more detailed and careful discussion.)

Tests of the *modified geometric hypothesis* as well as the extended version of mirror symmetry that includes D-branes and their world-volume theories will need a worldsheet description of CY manifolds where both Kähler and complex moduli have simple realisations. The gauged linear sigma model (GLSM) is a suitable worldsheet description in this regard. As shown by Witten, this model has several phases of which the Calabi-Yau phase is one. Thus, the enlarged Kähler cone which is required by mirror symmetry naturally fits into the setup [24]. The price one pays for this choice is that conformal invariance on the worldsheet is obtained only at the infrared fixed point of the GLSM.

One of the advantages of the GSLM in the closed string case is the fact that it unifies the different techniques that are preferred in different regions of Kähler moduli space. In the Calabi-Yau phase or the large volume phase, the GLSM description tends to the non-linear sigma model description of strings moving on CY manifolds. In a Landau-Ginzburg phase, the description would be in terms of N=2 supersymmetric Landau-Ginzburg theory. In particular cases, the description at this point in the moduli space is even more explicit when the LG theory is equivalent to the tensor product of a set of N=2 minimal model conformal field theories. One may hope to see a similar situation in the case of D-branes.

In this paper, as a first step towards the eventual goal described earlier, we study the GLSM with $(2,2)$ supersymmetry on worldsheets with boundary. Due to the presence of a boundary, one has to specify boundary conditions on the various fields in the GLSM such that the appropriate linear combination of supersymmetry is preserved. In order that these boundary conditions correspond to D-branes wrapped around various cycles of the Calabi-Yau manifold, we first construct the boundary conditions in the nonlinear sigma model limit of the GLSM and look for boundary conditions in the GLSM which reduce to sensible ones in the NLSM limit. We also find the need to introduce a boundary (contact) term in order to obtain consistent boundary conditions. This contact term vanishes when the theta-term in the GLSM is turned off and and its presence is justified by considering the NLSM limit.

The organization of the paper is as follows: In section 2 we begin by reviewing the $d = 2$, $N = 2$ supersymmetric GLSM for closed strings following [20]. We then add a boundary and compute the boundary terms generated in computing the equations of motion and the variations of the action under supersymmetry. We also review and extend the work of [9, 10] describing boundary conditions for $d = 2, N = 2$ supersymmetric Landau-Ginzburg (LG) models of conformal field theories: this will be useful in understanding the boundary conditions in LG phases of Calabi-Yau compactifications. We close with a few words about the justification for using the GLSM; in particular we discuss the topological twisting of the GLSM with boundary. In section 3 we construct boundary conditions describing branes on supersymmetric cycles in the $e^2 \to \infty$ of the GLSM, as a guide to understanding the physical meaning of boundary conditions at finite e^2; along the way we will find certain boundary terms that we must add for consistency. In section 4 we finally construct and identify boundary conditions at finite e^2. We also discuss the significance of these boundary conditions. In section 5 we present our conclusions.

Parts of this work have been reported earlier elsewhere [21,22]. While readying this work for publication, other papers [14, 15, 17] have also appeared that, in part, study D-branes in the GLSM approach. While [17] use a combination of the GLSM and world-volume techniques , [15] is closer in spirit to the techniques of this paper and the results of sec. 6 of their paper have some overlap with sec. 4 of this work.

2 The Gauged Linear Sigma Model

In type II string theories compactified on Calabi-Yau threefolds, the moduli space of Kähler classes includes regions where the size of the manifold is of order l_s. Often the CFTs appear non-geometric and are better described via Landau-Ginzburg orbifolds [20, 23]; or they may mediate smooth passage to Calabi-Yau manifolds with different topology [24]. We are particularly interested in studying the physics of D-brane probes as one moves through large distances in the moduli space of such compactifications, across phases or towards singular compactifications. Unfortunately even in the geometric phases of these models the Calabi-Yau metric is not known, and physical objects (such as vertex operator correlation functions) receive corrections from worldsheet instantons. At best there are a few points in the moduli space where the conformal field theory is well understood: in particular at large-radius limits, and exactly solvable Gepner points, the latter being deep in the Landau-Ginzburg region.

The technique introduced in [20] to study motion between these regions was to write a 2d supersymmetric field theory with a known UV Lagrangian whose infrared fixed point is believed to be a Calabi-Yau compactification. (In fact, this technique was an important part of the development of the above picture.) This model is simply a $d = 2$, $N = 2$ supersymmetric gauge theory with some number of vector multiplets and some number of charged chiral multiplets, and is called the gauged linear sigma model (GLSM). This is much in the spirit of using Landau-Ginzburg models as UV Lagrangian descriptions of minimal models: indeed, Landau-Ginzburg orbifolds appear in "non-geometric" phases of the GLSM.

2.1 GLSM for closed strings

For ease of reference, we will review the lagrangian and supersymmetries of the GLSM following [20]. We work in Minkowski space with the metric $(-,+)$. We are interested in describing compactifications of string theory with eight supercharges; the worldsheet conformal field theory must then have $N=(2,2)$ superconformal symmetry. We expect that a nonconformal theory with such an infra-red fixed point should have $N=2$ supersymmetry as well.

Our candidate theory can be obtained by dimensional reduction from $d=4, N=1$ abelian gauge theory with chiral multiplets. It contains s $U(1)$ vector multiplets, described by the vector superfields $V_a(a=1,\cdots,s)$ and k chiral multiplets described by the chiral superfields $\Phi_i(i=1,\cdots,k)$. Written in components, the vector multiplet consists of the vector fields $v^a_\alpha(\alpha=0,1)$, the complex scalar field σ^a, complex chiral fermions $\lambda^a_\pm$, and the real auxiliary field D^a. The chiral multiplet consists of a complex scalar ϕ_i, complex chiral fermions $\psi_{\pm i}$, and a complex auxiliary scalar field F_i. They are charged under the $U(1)$s with charge Q^a_i. In component notation, the supersymmetry transformations of the vector multiplet are:

$$
\begin{aligned}
\delta v^a_0 &= i\left(\bar\epsilon_+\lambda^a_+ + \bar\epsilon_-\lambda^a_- + \epsilon_+\overline{\lambda}^a_+ + \epsilon_-\overline{\lambda}^a_-\right), \\
\delta v^a_1 &= i\left(\bar\epsilon_+\lambda^a_+ - \bar\epsilon_-\lambda^a_- + \epsilon_+\overline{\lambda}^a_+ - \epsilon_-\overline{\lambda}^a_-\right), \\
\delta\sigma^a &= -i\sqrt{2}\bar\epsilon_+\lambda^a_- - i\sqrt{2}\epsilon_-\overline{\lambda}^a_+, \\
\delta\overline{\sigma}^a &= -i\sqrt{2}\epsilon_+\overline{\lambda}^a_- - i\sqrt{2}\bar\epsilon_-\lambda^a_+, \\
\delta D^a &= -\bar\epsilon_+(\partial_0-\partial_1)\lambda^a_+ - \bar\epsilon_-(\partial_0+\partial_1)\lambda^a_- \\
&\quad + \epsilon_+(\partial_0-\partial_1)\overline{\lambda}^a_+ + \epsilon_-(\partial_0+\partial_1)\overline{\lambda}^a_-, \\
\delta\lambda^a_+ &= i\epsilon_+ D^a + \sqrt{2}(\partial_0+\partial_1)\overline{\sigma}^a\epsilon_- - v^a_{01}\epsilon_+, \\
\delta\lambda^a_- &= i\epsilon_- D^a + \sqrt{2}(\partial_0-\partial_1)\sigma^a\epsilon_+ + v^a_{01}\epsilon_-, \\
\delta\overline{\lambda}^a_+ &= -i\bar\epsilon_+ D^a + \sqrt{2}(\partial_0+\partial_1)\sigma^a\bar\epsilon_- - v^a_{01}\bar\epsilon_+, \\
\delta\overline{\lambda}^a_- &= -i\bar\epsilon_- D^a + \sqrt{2}(\partial_0-\partial_1)\overline{\sigma}^a\bar\epsilon_+ + v^a_{01}\bar\epsilon_-\ ,
\end{aligned}
\tag{2.1}
$$

where $\epsilon_\pm$ and $\bar\epsilon_\pm$ are the Grassman parameters for SUSY transformations. The transformation rules for the chiral multiplet are:

$$
\delta\phi_i = \sqrt{2}(\epsilon_+\psi_{-i} - \epsilon_-\psi_{+i}),
$$

$$\begin{aligned}
\delta\psi_{+i} &= i\sqrt{2}(D_0+D_1)\phi_i\bar{\epsilon}_- + \sqrt{2}\epsilon_+ F_i - 2Q_i^a\phi_i\overline{\sigma}^a\bar{\epsilon}_+, \\
\delta\psi_{-i} &= -i\sqrt{2}(D_0-D_1)\phi_i\bar{\epsilon}_+ + \sqrt{2}\epsilon_- F_i + 2Q_i^a\phi_i\sigma^a\bar{\epsilon}_-, \qquad (2.2)\\
\delta F_i &= -i\sqrt{2}\bar{\epsilon}_+(D_0-D_1)\psi_{+i} - i\sqrt{2}\bar{\epsilon}_-(D_0+D_1)\psi_{-i} \\
&\quad + 2Q_i^a(\bar{\epsilon}_+\overline{\sigma}^a\psi_{-i} + \bar{\epsilon}_-\sigma^a\psi_{+i}) + 2iQ_i^a\phi_i(\bar{\epsilon}_-\overline{\lambda}_+^a - \bar{\epsilon}_+\overline{\lambda}_-^a) \qquad (2.3)
\end{aligned}$$

The supersymmetric bulk action can be written as a sum of four terms,

$$S = S_{ch} + S_{gauge} + S_W + S_{r,\theta} \qquad (2.4)$$

The terms on the right hand side are, respectively: the kinetic term for the chiral superfields; the kinetic terms for the vector superfields; the superpotential interaction; and the Fayet-Iliopoulos and theta terms. S_{ch} is:

$$\begin{aligned}
S_{ch} &= \sum_i \int d^2x \Big\{ -D_\alpha\overline{\phi}_i D^\alpha\phi_i + i\overline{\psi}_{-i}(\overleftrightarrow{D}_0 + \overleftrightarrow{D}_1)\psi_{-i} + i\overline{\psi}_{+i}(\overleftrightarrow{D}_0 - \overleftrightarrow{D}_1)\psi_{+i} \\
&\quad + |F_i|^2 - 2\sum_a \overline{\sigma}^a\sigma^a(Q_i^a)^2\overline{\phi}_i\phi_i - \sqrt{2}\sum_a Q_i^a(\overline{\sigma}^a\overline{\psi}_{+i}\psi_{-i} + \sigma^a\overline{\psi}_{-i}\psi_{+i}) \\
&\quad + D^aQ_i^a\overline{\phi}_i\phi_i - i\sqrt{2}\sum_{aQ_i^a}\overline{\phi}_i(\psi_{-i}\lambda_+^a - \psi_{+i}\lambda_-^a) \\
&\quad - i\sqrt{2}Q_i^a\phi_i(\overline{\lambda}_-^a\overline{\psi}_{+i} - \overline{\lambda}_+^a\overline{\psi}_{-i})\Big\} \qquad (2.5)
\end{aligned}$$

where

$$A \overleftrightarrow{D}_i B \equiv \frac{1}{2}(AD_iB - (D_iA)B)\ . \qquad (2.6)$$

This symmetrized form of the fermion kinetic term is Hermitian in the presence of a boundary. Meanwhile, S_{gauge} is:

$$\begin{aligned}
S_{gauge} &= \sum_a \frac{1}{e_a^2}\int d^2x \Big\{\frac{1}{2}(v_{01}^a)^2 + \frac{1}{2}(D^a)^2 - \partial_\alpha\sigma^a\partial^\alpha\overline{\sigma}^a \\
&\quad + i\overline{\lambda}_+^a(\overleftrightarrow{\partial}_0 - \overleftrightarrow{\partial}_1)\lambda_+^a + i\overline{\lambda}_-^a(\overleftrightarrow{\partial}_0 + \overleftrightarrow{\partial}_1)\lambda_-^a\Big\} \qquad (2.7)
\end{aligned}$$

The superpotential term is:

$$S_W = -\int d^2x \left(F_i\frac{\partial W}{\partial\phi_i} + \frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\psi_{-i}\psi_{+j} + \overline{F}_i\frac{\partial\overline{W}}{\partial\overline{\phi}_i} - \frac{\partial^2\overline{W}}{\partial\overline{\phi}_i\partial\overline{\phi}_j}\overline{\psi}_{-i}\overline{\psi}_{+j}\right)\ . \qquad (2.8)$$

Finally, the Fayet-Iliopoulos D-term and theta term are:

$$S_{r,\theta} = -r_a\int d^2y D^a + \frac{\theta_a}{2\pi}\int d^2y v_{01}^a\ . \qquad (2.9)$$

We wish to describe a theory with an $N=(2,2)$ superconformal fixed point. Thus we wish anomaly-free vector and axial $U(1)$ R-symmetries: these may be constructed if $\sum_i Q_i^a = 0$ [20]. These R-symmetries may also be used to topologically twist the theory, as we will discuss below.

Let us review the manifestation of the target space geometry, and of the phase structure of the moduli space of compactifications, in the class of examples which we will use for most of this paper, namely hypersurfaces in weighted projective space with a single Kähler modulus. The spectrum is a single abelian vector multiplet and 5 chiral multiplets. The latter consist of 5 chiral superfields Φ_i with positive charge $Q_{i=1\ldots5}$ and an additional superfield $\Phi_6 = P$ with charge $Q_6 = Q_p = -\sum_{i=1}^5 Q_i$. Furthermore, we choose the superpotential

$$W(\Phi, P) = PG(\Phi) \tag{2.10}$$

where G is a quasi-homogenous transverse polynomial. There are four such examples: a degree five hypersurface in $\mathbb{P}^4_{1,1,1,1,1} = \mathbb{P}^4$ (the quintic); degree six hypersurface in $\mathbb{P}^4_{1,1,1,1,2}$; degree eight hypersurface in $\mathbb{P}^4_{1,1,1,1,4}$ and degree ten hypersurface in $\mathbb{P}^4_{1,1,1,2,5}$.

We wish to find the moduli space of supersymmetric ground states, which should flow to the target space of the infrared CFT. We do so by setting the bosonic potential energy

$$U = \sum_i F_i^2 + \frac{1}{2e^2} D^2 + 2|\sigma|^2 \sum_i Q_i^2 |\phi_i|^2 \tag{2.11}$$

to zero. We substitute the equations of motion for the auxiliary fields D and F_i:

$$\begin{aligned} D &= -e^2 \left(\sum_{iQ_i|} \phi_i|^2 - r \right) \\ F_i^* &= \frac{\partial W}{\partial \phi_i} , \end{aligned} \tag{2.12}$$

to find that:

$$U = |G(\phi_i)|^2 + |p|^2 \sum_i i \left| \frac{\partial G}{\partial \phi_i} \right|^2 + \frac{D}{2e^2} + 2|\sigma|^2 \left(\sum_i Q_i^2 |\phi_i|^2 + Q_p^2 |p|^2 \right) . \tag{2.13}$$

where p and ϕ_i represent the scalar components of P and $\Phi_{i=1\ldots5}$ respectively.

Let us begin with the case $r \gg 0$. The D term requires that the ϕ_i cannot all simultaneously vanish. Thus $\sigma = 0$; transversality of G requires that $p = 0$. One must also set the Fayet-Iliopoulous D-term to zero:

$$\sum_i Q_i |\phi_i|^2 = r . \tag{2.14}$$

This condition together with dividing out the $U(1)$ gauge symmetry means that the ϕ^i describe the weighted projective space $\mathbb{P}^4_{Q_1,\ldots,Q_5}$. Finally, the condition $G = 0$ means that the ϕ live on a degree $|Q_p|$ hypersurface in $\mathbb{P}^4_{Q_1,\ldots,Q_5}$.

For $r \ll 0$, vanishing of the D term requires that $p \neq 0$. Transversality of G then implies that all $\phi_i = 0$. This theory has a unique classical vacuum; the massless excitations are governed by a superpotential with a degenerate critical point, *i.e.* it is a Landau-Ginzburg theory. The residual gauge invariance (for instance, $\mathbb{Z}_5$ in the quintic) in fact implies that it is a Landau-Ginzburg orbifold. At $r \to -\infty$ this is believed to be the exactly solvable Gepner model for the quintic [25] (see [26] for a review and references). In this way the trajectory $r \gg 0 \to r \ll 0$ interpolates between geometric and non-geometric compactifications.

Beginning with the flat metric on $\mathbb{C}^5$ and imposing the gauge invariance and D-term conditions, one can see that r is essentially the size of the ambient projective space. In spacetime this Kähler parameter is complexified by an NS-NS two-form potential; in this model this flows from the theta angle. We will show this explicitly in the next section, but we can note for now that the fact that θ is a periodic variable reflects the periodicity of the 2-form flux. Furthermore one can show that for $\theta \neq 2\pi n$, the GLSM is nonsingular even at $r = 0$ [20].

2.2 GLSM with boundary

Supersymmetric D-branes configurations will preserve four of the eight spacetime supercharges of the compactification. The boundaries of the string worldsheet must therefore preserve half of the $N = (2, 2)$ superconformal symmetry of the closed strings. We take this to mean that boundaries in the corresponding GLSM should also break half of the supersymmetries.

We will work on the half-plane (x_0, x_1) with $x_1 \geq 0$, and impose boundary conditions at $x_1 = 0$. As with the conformal sigma models, the possible boundary conditions fall into two classes [5], "A-type" and "B-type." Roughly these correspond to branes wrapped on special Lagrangian submanifolds and on holomorphic cycles, respectively. In our case, "A-type" boundary conditions correspond to setting $\epsilon_\pm = \eta\bar{\epsilon}_\mp$, where $\eta = \pm 1$; "B-type" conditions correspond to $\epsilon_\pm = \eta\epsilon_\mp$.

The variation of the action in the presence of a boundary will generate boundary terms in addition to the bulk terms proportional to the equations of motion. One chooses boundary conditions on the fields such that these boundary terms vanish. In addition, SUSY variations of the fields will also generate boundary terms; upon choosing the preserved supersymmetries one requires that these boundary terms also vanish. We list these boundary terms here; in the next section we will use them to derive boundary conditions.

1. The boundary terms in the action generated by general variations of the fields are:

$$\delta_{ord} S_{kin} = \int dx^0 \Big\{ -\left[(\partial_1 \phi_i + i Q_i v_1 \phi_i)\delta\bar{\phi}_i + (\partial_1 \bar{\phi}_i - i Q_i v_1 \bar{\phi}_i)\delta\phi_i\right]$$

$$
\begin{aligned}
& \qquad + \frac{i}{2}\left[(\bar\psi_{-i}\delta\psi_{-i} - \psi_{+i}\delta\bar\psi_{+i}) - (\bar\psi_{+i}\delta\psi_{+i} - \psi_{-i}\delta\bar\psi_{-i})\right]\Big\}, \\
\delta_{ord} S_{gauge} &= \frac{1}{e_a^2}\int dx^0 \Big\{ -v_{01}^a \delta v_0^a - [(\partial_1\bar\sigma^a)\delta\sigma^a + (\partial_1\sigma^a)\delta\bar\sigma^a] \\
& \qquad + \frac{i}{2}\left[(\bar\lambda_-^a\delta\lambda_-^a - \lambda_+^a\delta\bar\lambda_+^a) - (\bar\lambda_+^a\delta\lambda_+^a - \lambda_-^a\delta\bar\lambda_-^a)\right]\Big\} \\
\delta_{ord} S_{r,\theta} &= -\frac{\theta_a}{2\pi}\int dx^0 \ \delta v_0^a
\end{aligned} \tag{2.15}
$$

2. The boundary terms generated by the transformation (2.2),(2.3) are:

$$
\begin{aligned}
\delta_{susy} S_{kin} &= \int dx^0 \Big\{ \frac{1}{\sqrt{2}}\left[(D_0\phi_i)(\bar\epsilon_+\bar\psi_{-i} + \bar\epsilon_-\bar\psi_{+i}) - (D_0\bar\phi_i)(\epsilon_+\psi_{-i} + \epsilon_-\psi_{+i})\right] \\
& \qquad + \frac{1}{\sqrt{2}}\left[(D_1\phi_i)(\bar\epsilon_+\bar\psi_{-i} - \bar\epsilon_-\bar\psi_{+i}) - (D_1\bar\phi_i)(\epsilon_+\psi_{-i} - \epsilon_-\psi_{+i})\right] \\
& \qquad + iQ_i^a\left[\bar\phi_i\sigma^a\epsilon_+\psi_{+i} + \phi_i\bar\sigma^a\bar\epsilon_+\bar\psi_{+i} + \bar\phi_i\bar\sigma^a\epsilon_-\psi_{-i} + \phi_i\sigma^a\bar\epsilon_-\bar\psi_{-i}\right] \\
& \qquad + \frac{i}{\sqrt{2}}\left[(\bar\epsilon_+\psi_{+i} - \bar\epsilon_-\psi_{-i})\bar F_i \ + \ (\epsilon_+\bar\psi_{+i} - \epsilon_-\bar\psi_{-i})F_i\right]\Big\} \\
\delta_{susy} S_{gauge} &= \frac{1}{e_a^2}\int dx^0 \Big\{ \frac{i}{\sqrt{2}}\left[(\partial_0\sigma^a)(\epsilon_+\bar\lambda_-^a - \bar\epsilon_-\lambda_+^a) - (\partial_0\bar\sigma^a)(\epsilon_-\bar\lambda_+^a - \bar\epsilon_+\lambda_-^a)\right] \\
& \qquad + \frac{i}{\sqrt{2}}\left[(\partial_1\sigma^a)(\epsilon_+\bar\lambda_-^a + \bar\epsilon_-\lambda_+^a) + (\partial_1\bar\sigma^a)(\epsilon_-\bar\lambda_+^a + \bar\epsilon_+\lambda_-^a)\right] \\
& \qquad - \frac{i}{2}v_{01}^a\left[\epsilon_+\bar\lambda_+^a + \epsilon_-\bar\lambda_-^a + \bar\epsilon_+\lambda_+^a + \bar\epsilon_-\lambda_-^a\right] \\
& \qquad + \frac{D^a}{2}\left[\epsilon_+\bar\lambda_+^a - \epsilon_-\bar\lambda_-^a + \bar\epsilon_-\lambda_-^a - \bar\epsilon_+\lambda_+^a\right]\Big\} \\
\delta_{susy} S_W &= i\sqrt{2}\int dx^0 \left[\frac{\partial W}{\partial\phi_i}(\bar\epsilon_-\psi_{-i} - \bar\epsilon_+\psi_{+i}) + \frac{\partial\bar W}{\partial\bar\phi_i}(\epsilon_-\bar\psi_{-i} - \epsilon_+\bar\psi_{+i})\right] \\
\delta_{susy} S_{r,\theta} &= \frac{-i\theta_a}{2\pi}\int dx^0 \left[\bar\epsilon_+\lambda_+^a + \epsilon_+\bar\lambda_+^a + \bar\epsilon_-\lambda_-^a + \epsilon_-\bar\lambda_-^a\right]
\end{aligned} \tag{2.16}
$$

2.3 Landau-Ginzburg theories with boundary

Finding appropriate boundary conditions is fairly complicated; in addition, we would like to know their physical import. One tool is to try and understand sensible boundary conditions for $r \gg 0$ and $r \ll 0$, where gauge theory/worldsheet instanton corrections are small [20, 26]. The former limit is described by a non-linear sigma model and we will discuss this in the next section. The latter limit is described by a Landau-Ginzburg theory, and we review here supersymmetric boundary conditions for such theories [9, 10, 15].[4]

[4]Previous work on Landau-Ginzburg theories with boundary can be found in [28].

2.3.1 A-type boundary conditions

Consider a Landau-Ginzburg model with n chiral superfields Φ_i and arbitrary superpotential $G(\Phi)$. For A-type boundary conditions, we impose n independent conditions

$$f_a(\phi, \overline{\phi}) = 0 \quad , \tag{2.17}$$

where f_a are real functions. We will use the indices $i, j, \cdots$ to denote the superfields and the indices $a, b, c, \cdots$ to indicate the boundary conditions. Let Σ denote the sub-manifold in $\mathbb{C}^n$ (with complex coordinates ϕ_i and $\overline{\phi}$) obtained by imposing these conditions. We will in addition impose the compatibility condition:

$$\{f_a(\phi, \overline{\phi}), f_b(\phi, \overline{\phi})\}_{PB} = 0 \ , \tag{2.18}$$

where:

$$\{A, B\} = g^{i\bar{j}} \left(\partial_i A \bar{\partial}_{\bar{j}} B - \bar{\partial}_{\bar{j}} A \partial_i B \right) \ . \tag{2.19}$$

We will assume that on Σ, the normals $\vec{n}_a \equiv (\partial_i f_a, \overline{\partial}_{\bar{i}} f_a)$ span the normal bundle $\mathcal{N}\Sigma$. The vanishing of the Poisson bracket can be rewritten as

$$\vec{n}_a \cdot \vec{t}_b = 0 \tag{2.20}$$

where $\vec{t}_b \equiv (\partial_i f_b, -\overline{\partial}_{\bar{i}} f_a)$ are tangent vectors to the curve $f_b = 0$. It follows that they span the tangent bundle $T\Sigma$. Σ is thus a *Lagrangian submanifold* of $\mathbb{C}^n$ by construction [29]. The induced metric (first fundamental form) on Σ is given by

$$h_{ab} = \vec{t}_a \cdot \vec{t}_b = \vec{n}_a \cdot \vec{n}_b \quad . \tag{2.21}$$

The following set of additional boundary conditions on the fields in the LG model are consistent with the vanishing of the boundary terms which occur in the general and supersymmetric variations of the LG Lagrangian. Define $\chi_{\pm a} \equiv \frac{\partial f_a}{\partial \phi_i} \psi_{\pm i}$. Then:

$$\chi_{+a} + \eta \overline{\chi}_{-a} = 0 \quad , \tag{2.22}$$

$$\left(\left[\frac{\partial f_a}{\partial \phi_i} \partial_1 \phi_i - \frac{\partial f_a}{\partial \overline{\phi}_i} \partial_1 \overline{\phi}_i \right] - i K_{abc}\, \chi_-^b \ \overline{\chi}_-^c \right) = 0 \tag{2.23}$$

$$\left\{ f_a(\phi, \overline{\phi}), G(\Phi) - \overline{G}(\overline{\phi}) \right\}_{PB} = 0 \tag{2.24}$$

The complex conjugate conditions are also hold. K_{abc} is the extrinsic curvature tensor (second fundamental form) given by

$$K_{abc} = - \left[\frac{\partial f_c}{\partial \phi_i} \frac{\partial f_b}{\partial \phi_j} \frac{\partial^2 f_a}{\partial \phi_i \partial \phi_j} - \frac{\partial f_c}{\partial \overline{\phi}_i} \frac{\partial f_b}{\partial \phi_j} \frac{\partial^2 f_a}{\partial \phi_i \partial \overline{\phi}_j} - \frac{\partial f_c}{\partial \phi_j} \frac{\partial f_b}{\partial \overline{\phi}_i} \frac{\partial^2 f_a}{\partial \phi_i \partial \overline{\phi}_j} + \frac{\partial f_c}{\partial \overline{\phi}_i} \frac{\partial f_b}{\partial \overline{\phi}_j} \frac{\partial^2 f_a}{\partial \overline{\phi}_i \partial \overline{\phi}_j} \right] \ . \tag{2.25}$$

This full set of boundary conditions is equivalent to requiring that Σ be Lagrangian. Without a superpotential, this corresponds to the microscopic (worldsheet) realisation of situations considered by Harvey and Lawson [29]. In the presence of a superpotential, there is an additional condition that the real conditions F_a have a vanishing Poisson bracket with $(G - \overline{G})$. This suggests that one must choose one of the conditions to be $F = (G - \overline{G}) - ic$ where c is a real constant, as there can only n independent commuting constants of motion in a 2n real-dimensional phase space

2.3.2 B-type boundary conditions

Under B-type boundary conditions, the unbroken $N = 2$ supersymmetry is given by the condition

$$\epsilon_+ = \eta\, \epsilon_- \quad , \tag{2.26}$$

where $\eta = \pm 1$. The following linear boundary conditions were constructed in the LG model [9]

$$\begin{aligned}
(\psi_{+i} + \eta B_i^{\ j} \psi_{-j})|_{x=0} &= 0 \quad , \\
\partial_1(\phi_i + B_i^{\ j}\phi_j)|_{x=0} &= 0 \quad , \\
\partial_0(\phi_i - B_i^{\ j}\phi_j)|_{x=0} &= 0 \quad , \\
\left.\left(\frac{\partial G}{\partial \phi_i} + B_i^{*j}\frac{\partial G}{\partial \phi_j}\right)\right|_{x=0} &= 0 \quad ,
\end{aligned} \tag{2.27}$$

where the boundary condition is specified by a hermitian matrix B which satisfies $B^2 = 1$. Since B squares to one, its eigenvalues are ± 1. An eigenvector of B with eigenvalue of $+1$ corresponds to a Neumann boundary condition and -1 corresponds to a Dirichlet boundary condition. Associated with every eigenvector with eigenvalue $+1$, there is a non-trivial condition involving the superpotential which is given by the last of the above boundary conditions.

More general possibilities are given by boundary conditions corresponding to a holomorphic submanifold $\Sigma \subset \mathbb{C}^n$ defined by the tranverse intersection of the r conditions

$$f_m(\phi) = 0 \quad , \ (m = 1, \ldots, r) \tag{2.28}$$

where f_m are quasi-homogeneous holomorphic functions of the ϕ_i. Under supersymmetric variation with $\epsilon^+ = \eta\epsilon^-$, one obtains the conditions

$$\sum_i n_m^i(\phi)(\psi_{+i} - \eta\psi_{-i}) = 0 \tag{2.29}$$

where $n_m^i \equiv (\partial f_m/\partial\phi_i)$ are the (holomorphic) normals to the surface $f_m = 0$. Let $t_i^a(\phi)$ ($a = 1, \ldots, n - r$) be a basis of tangent vectors to Σ such that in local holomorphic coordinates z_a, $t_i^a = \frac{\partial \phi_i}{\partial z_a}$. One needs to impose further boundary conditions in order to cancel fermionic boundary terms arising in the ordinary variation of the action:

$$\eta^{i\bar{j}} t_i^a(\phi)(\overline{\psi}_{+j} + \eta\overline{\psi}_{-j}) = 0 \tag{2.30}$$

Supersymmetric variation of the above equation leads to the conditions

$$\eta^{i\bar{j}}\left(t_i^a(\phi)\partial_1\overline{\phi}_j + \frac{i}{2}\frac{\partial t_i^a}{\partial\phi_k}(\psi_{+k} - \eta\psi_{-k})(\overline{\psi}_{+j} + \eta\overline{\psi}_{-j})\right) = 0 \tag{2.31}$$

$$t_i^a(\phi)\frac{\partial G}{\partial\phi_i} = 0 \tag{2.32}$$

The first equation can be rewritten in the following form

$$\left(t_i^a(\phi)\eta^{i\bar{j}}\partial_1\overline{\phi}_j\right) - \chi_m^{ab}\ \tau_b\ \overline{\nu}^m = 0 \tag{2.33}$$

where χ_m^{ab} is the extrinsic curvature of the submanifold (second fundamental form) given by (see ref. [30] for a discussion)

$$\chi_m^{ab} \equiv t_i^a t_j^b \frac{\partial^2 f_m}{\partial\phi_i\partial\phi_j}$$

and $h_{a\bar{b}} \equiv \vec{t}_a \cdot \vec{t}_b^*$ is the induced metric (first fundamental form). We have also defined fermionic linear combinations τ_a and ν^m

$$(\psi_{+i} - \eta\psi_{-i}) = t_i^a\tau_a \quad ,$$

$$(\overline{\psi}_{+i} + \eta\overline{\psi}_{-i})\eta^{\bar{i}j} = n_m^j\overline{\nu}^m \quad .$$

These are fermionic combinations which are sections of the tangent bundle and normal bundle respectively. The boundary terms under ordinary variations of the LG action vanish for the above choice of boundary conditions.

The boundary conditions involving the superpotential given by eqn. (2.32) is always satisfied if one chooses one of the boundary conditions to be $G = 0$. For instance, all examples of B-type boundary conditions in LG models considered in [9]) can be seen to imply $G = 0$. This requirement has also been observed independently in [15]. This is the analogue of $(G - \overline{G}) = 0$ condition seen in A-type boundary conditions.

2.4 Topological Aspects of the GLSM

As already mentioned, the GLSM is not conformally invariant and flows to a conformally invariant theory in the infrared (IR) limit [20]. It follows that one must be careful in naively extrapolating results in the GLSM to the conformally invariant fixed point. For example, in the NLSM limit of the GLSM for the quintic, the metric is given by the pullback of the Fubini-Study metric on $\mathbb{P}^4$.

This metric is clearly not the correct one. The fact that both the GLSM and its IR fixed point have worldsheet $(2,2)$ supersymmetry is quite useful in obtaining some control. To be precise, by appropriately twisting the theories, one constructs topological theories whose observables are insensitive to such differences between the two theories and one can make predictions.

There are two possible twists of the (Euclidean) $(2,2)$ model: the A-twist corresponds to the case when the supersymmetry charge: $Q = Q_- + \overline{Q}_+$ (where $Q_\pm = \int G_\pm$ are the supersymmetry charges associated with the supersymmetry generators $G_\pm$.) becomes a scalar and the B-twist is where $Q = \overline{Q}_- + \overline{Q}_+$ is a scalar. Physical states of the topological theory correspond to cohomology classes of Q and the observables are given by correlation functions of vertex operators that are Q-closed. Observables of the topological A-model vary holomorphically in $t = \frac{\theta}{2\pi} + ir$ while those in the B-model are independent of t. Correlation functions in the A-model can receive corrections from gauge theory instantons which do not quite coincide with the worldsheet instantons corrections seen in the conformally invariant NLSM. The difference arises because the instanton moduli space for the GLSM is compact while that of the NLSM is non-compact [20]. However, it has been shown that singularity structure of the moduli space is correctly predicted in the GLSM [20, 27].

For the case of GLSM with boundary, one may hope to apply similar techniques. As has been pointed out earlier [5], A-type boundary conditions are compatible with the A-twist and B-type boundary conditions with the B-twist in the open-string channel. For instance, in the topological A-model, the σ field of the vector multiplet is Q-closed. This can be easily seen by the fact that $\delta\sigma = 0$ under the supersymmetry transformation generated by Q i.e., $\epsilon_+ = \overline{\epsilon}_-$. Further, in the NLSM limit (c.f. sec. 3) we will see that

$$\sigma = -\frac{\sum_i Q_i \overline{\psi}_{+i} \psi_{-i}}{\sqrt{2} K[\phi]}$$

In the topological A-model, $\overline{\psi}_{+i}$ is a $(1,0)$ form (to be precise, a section of $\Phi^*(T^{1,0}(X))$) and ψ_{-i} a $(0,1)$ form and hence σ is proportional to the Kähler form ω on the Calabi-Yau manifold. The proportionality constant $K[\phi]$ can be seen to be non-vanishing everywhere. It is known that A-branes wrap special Lagrangian submanifolds of the Calabi-Yau manifold. Lagrangian submanifolds satisfy the condition that the restriction of the Kähler form ω to the submanifold vanishes. Thus, for A-type boundary conditions, it follows that

$$\sigma|_{x^1=0} = 0 \quad . \tag{2.34}$$

We will see that our analysis in the sequel will be consistent with this condition.

Just as in the closed string case, the interpretation of results in the case of GLSM with boundary should be done with care. For instance, in the case of A-type branes, worldsheet instanton effects lead to a stringy notion of the topology of the cycle which can differ significantly from the topology computed by geometric means [16]. For the case of B-branes, as we move around in the Kähler moduli space, the branes can undergo monodromy transformations; thus, if one writes down boundary conditions far from the large-radius limit and tries to understand it by following that boundary condition out to the large-radius limit, the result can depend on the path one takes. It is also possible that a boundary state at the Gepner point has no stable large-radius analog [8, 12]. Finally, although taking monodromies and lines of marginal stability into account when studying D-branes far from the large radius limit is nontrivial, some progress has been made [7,8,12] It would be interesting to study these effects in the GLSM.

3 The Nonlinear Sigma Model

Our eventual goal is to use the boundary GLSM as a tool for understanding the boundary CFT to which it should flow in the infrared. To begin with, we would like to understand what a given set of boundary conditions for the GLSM might correspond to in the infrared.

As discussed in [20], the theory flows to strong coupling in this limit. It is therefore tempting to simply take the limit $e^2 \to \infty$ and use these results as a physical guide. In particular, in this limit the gauge kinetic terms drop out; upon integrating out the nonpropagating gauge fields one is left with a nonlinear sigma model.

We will now look for consistent boundary conditions for the one-modulus examples in the limit $r \gg 0$. We hope this will provide a simple guide to finding boundary conditions for finite e^2, as well as a crude guide to the infrared physics of the GLSM. We begin by describing the results of integrating out the vector multiplets; along the way we will have to add a contact term to reproduce sensible $N = 2$ NLSM results. Following this we will discuss A-type boundary conditions and Neumann B-type boundary conditions in this limit.

3.1 $e^2 \to \infty$ Limit of the Bulk Linear Sigma Model

In the $e \to \infty$ limit of the GLSM, the kinetic energy terms for the vector multiplet vanish, so that the component fields behave as Lagrange multipliers. This leads to the following constraints for general $U(1)$ charge.

1. The D-term constraint:

$$\sum_i (Q_i|\phi_i|^2 - Q_p|p|^2 - r) = 0 \ , \tag{3.1}$$

When $r \gg 0$, $|p|$ is very massive due to eq. (2.13); we will set $p = 0$ for the remainder of the section.

2. The constraints imposed by integrating out the gauginos are:

$$\sum_i Q_i \overline{\phi}_i \psi_{\pm i} = 0 \tag{3.2}$$

3. The equations of motion for σ and $\overline{\sigma}$:

$$\sigma = -\frac{\sum_i Q_i \overline{\psi}_{+i} \psi_{-i}}{\sqrt{2} K[\phi]} \tag{3.3}$$

$$\overline{\sigma} = -\frac{\sum_i Q_i \overline{\psi}_{-i} \psi_{+i}}{\sqrt{2} K[\phi]} , \tag{3.4}$$

where $K[\phi] \equiv \sum_j Q_j^2 |\phi_j|^2$.

4. The equations of motion for the gauge fields:

$$2K[\phi]\; v_0 = \sum_i Q_i \left[i(\overline{\phi}_i \partial_0 \phi_i - \phi_i \partial_0 \overline{\phi}_i) + \overline{\psi}_{-i} \psi_{-i} + \overline{\psi}_{+i} \psi_{+i} \right] \tag{3.5}$$

$$2K[\phi]\; v_1 = \sum_i Q_i \left[i(\overline{\phi}_i \partial_1 \phi_i - \phi_i \partial_1 \overline{\phi}_i) - \overline{\psi}_{-i} \psi_{-i} + \overline{\psi}_{+i} \psi_{+i} \right] \tag{3.6}$$

The equation for v_0 is simply Gauss' law.

5. Further supersymmetric variation of the above equations leads to

$$\begin{aligned} -\sqrt{2} K[\phi] \lambda_+ &= \sum_i Q_i \left[\overline{\psi}_{-i} (D_0 + D_1) \phi_i + i \overline{F}_i \psi_{+i} \right] + \frac{i\sigma}{K[\phi]} \sum_i Q_i^2 \phi_i \overline{\psi}_{+i} \\ -\sqrt{2} K[\phi] \lambda_- &= \sum_i Q_i \left[-\overline{\psi}_{+i} (D_0 - D_1) \phi_i + i \overline{F}_i \psi_{-i} \right] + \frac{i\sigma}{K[\phi]} \sum_i Q_i^2 \phi_i \overline{\psi}_{-i} \end{aligned} \tag{3.7}$$

Of course we have integrated out λ, but these last equations can be used to take the $e^2 \to \infty$ limit of equations which are functions of λ.

In $N = 2$ supersymmetric type II string compactifications on Calabi-Yau threefolds, the Kähler parameters are complexified by the fluxes of closed NS-NS two-form gauge fields B_{ij}. In the GLSM, this is reflected by the presence of a θ-term for every Fayet-Iliopoulos D-term [20]. This can be seen easily in the $e^2 \to \infty$ limit. Substituting eqs. (3.5),(3.6) into the theta term S_θ of eq. (2.9), we find:

$$\begin{aligned} S_\theta \;&=\; \frac{\theta}{2\pi} \int d^2x \sum_i Q_i \left\{ i \frac{D_1^B \phi_i D_0^B \overline{\phi}_i - D_0^B \phi_i D_1^B \overline{\phi}_i}{K[\phi]} \right. \\ &\left. \quad -(\partial_0 + \partial_1) \left(\frac{\overline{\psi}_{-i} \psi_{-i}}{2K[\phi]} \right) + (\partial_0 - \partial_1) \left(\frac{\overline{\psi}_{+i} \psi_{+i}}{2K[\phi]} \right) \right\} \end{aligned} \tag{3.8}$$

We define v_0^B to the fermion-independent part of v_0 in (3.5); D_0^B is the covariant derivative with v_0 replaced by v_0^B.

In the case of the quintic, $K = r$ and the bosonic part of (3.8) corresponds to:

$$B_{i\bar{j}} = \frac{i\theta}{4\pi r}\eta_{i\bar{j}} \ . \tag{3.9}$$

This is closed and topologically nontrivial in $\mathbb{P}^4$; it should thus correspond to the NS-NS B-field modulus for type II compactification on the quintic. It remains to understand the fermion bilinear term. Clearly it is a boundary term, and it can be discarded in the closed-string case. We claim that in the case at hand it is sensible to subtract this off, by adding an explicit boundary term:

$$S_{\text{boundary}}^{\text{quintic}} = \int dx^0 \frac{\theta}{4\pi r} \sum_i \left(\overline{\psi}_{+i}\psi_{+i} + \overline{\psi}_{-i}\psi_{-i}\right) \tag{3.10}$$

We will now consider the case of more general cases involving weighted projective spaces. In $d = 2, N = 2$ supersymmetric NLSMs, the fermion bilinear terms which scale with B are [31]:

$$\int d^2x \overline{\psi}^{\bar{j}} \psi^i \left(\partial\phi^k H_{\bar{j}ik} + \partial\overline{\phi}^{\bar{k}} H_{\bar{j}i\bar{k}}\right) \ , \tag{3.11}$$

where $H = dB$. In $N = 2$ supersymmetric compactifications of type II string theory on Calabi-Yau threefolds, we must have $H = dB = 0$ and so these bilinear terms must vanish. Clearly $H = 0$ in eq. (3.9). In fact this is true for the B-field arising from more general Q_i. The general expression for the B-field in the NLSM limit is given by

$$\begin{aligned} B_{ij} &= \sum_a \frac{i\theta_a}{2\pi}\left(\frac{[(Q_i^a)^2 Q_j^a - Q_i^a (Q_j^a)^2)]\phi_i\phi_j}{4(K^a[\phi])^2}\right) \ , & (3.12)\\ B_{i\bar{j}} &= \sum_a \frac{i\theta_a}{2\pi}\left(\frac{Q_i^a \eta_{i\bar{j}}}{2K^a[\phi]} - \frac{[(Q_i^a)^2 Q_j^a + Q_i^a (Q_j^a)^2]\phi_i\overline{\phi}_j}{4(K^a[\phi])^2}\right) \ , & (3.13)\end{aligned}$$

where $K^a[\phi] \equiv \sum_i (Q_i^a)^2 |\phi_i|^2$. An explicit calculation of $H_{ij\bar{k}}$ and $H_{i\bar{j}\bar{k}}$ leads to the vanishing of the fermion bilinear in eqn. (3.11) on imposing the NLSM constraints given by (3.1) and (3.2). This implies that H vanishes on the subspace given by the D-term constraint.

For the general case of weighted projective spaces, the B-field corresponds to a non-constant B-field. Further, B_{ij} and $B_{\bar{i}\bar{j}}$ are non-vanishing[5]. Is it possible to find a spacetime gauge tranformation of the form

$$\delta B_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \tag{3.14}$$

[5]One reason to require these to vanish is to observe that for D-branes, gauge invariance dictates that the field strength F and the B-field occur (loosely) in the combination $(F - B)$. For holomorphic connections on B-branes, one has $F_{ij} = F_{\bar{i}\bar{j}} = 0$. Even in the closed string case, it is preferable to work in this gauge since the complex Kähler moduli involve only $B_{i\bar{j}}$.

such that B_{ij} and $B_{\bar{i}\bar{j}}$ vanish? Under such a gauge transformation, the worldsheet Lagrangian transforms as

$$\delta S_{\mathrm{B-field}} = 2\int d^2x \left\{\partial_0(\Lambda_\mu \partial_1 \phi^\mu) - \partial_1(\Lambda_\mu \partial_0 \phi^\mu)\right\} \quad . \tag{3.15}$$

The following choice of gauge transformation does this:

$$\Lambda_i = \sum_a \left(\frac{i\theta_a Q_i^a \phi_i}{8\pi K^a[\phi]} - \frac{i\theta_a \phi_i}{8\pi r_a} \right) \tag{3.16}$$

$$\Lambda_{\bar{i}} = \sum_a \left(-\frac{i\theta_a Q_i^a \overline{\phi}_i}{8\pi K^a[\phi]} + \frac{i\theta_a \overline{\phi}_i}{8\pi r_a} \right) \tag{3.17}$$

Of course, this also modifies $B_{i\bar{j}}$ to the following form

$$B_{i\bar{j}} = \sum_a \left(\frac{i\theta_a}{4\pi r_a} \right) \eta_{i\bar{j}} \ , \tag{3.18}$$

which is a constant B-field as in the quintic! There are total derivative pieces given by the gauge tranformation (see eqn. (3.15). We discard them and this corresponds to a further addition to the existing contact term we obtained by discarding the fermion bilinears. For the one-modulus case, the final contact term takes the following simple form

$$S^{NLSM}_{\text{boundary}} = \frac{i\theta}{4\pi r} \int dx^0 \ \sum_i (\phi_i D_0 \overline{\phi}_i - \overline{\phi}_i D_0 \phi_i) \tag{3.19}$$

where we have used eqn. (3.5) to rewrite the fermion bilinear in terms of the bosonic fields.

The attentive reader may observe that the requirement that B_{ij} and $B_{\bar{i}\bar{j}}$ vanish does not completely fix the required gauge transformation Λ_μ. An additional requirement is that the boundary conditions in the NLSM and the LSM agree. Our choice of gauge transformation given above precisely achieves this. The important point to note is that the choice of boundary conditions in the NLSM is dictated by the form of the B-field, whereas in the LSM it is the contact term (i.e., the terms we discarded in the NLSM limit as total derivatives) which dictates the choice of boundary conditions. This will become clearer in the following sections.

As a final note, such a term arises in an almost identical context in [32]. In that work the issue is the construction of a BRST-invariant vertex operator for the NS-NS B-field on the disc, for D-instantons in flat space. Whether the vertex operator describes a fluctuation δB for which $\delta H = 0$, or one for which δH is nonzero, one must add a "contact term" on the boundary which is a fermion bilinear, in order that the integrated operator is BRST invariant. If one writes this boundary term as a total derivative in the interior of the disc, the total fermion part of the vertex operator has the right symmetry structure in its Lorentz indices to describe a fluctuation of H.

3.2 A-type boundary conditions in the NLSM

In Sec. 2, we considered a LG model with n chiral superfields and arbitrary superpotential. The same techniques give us sensible A-type boundary conditions for the n-chiral superfields in the $e^2 \to \infty$ NLSM. In this limit, the following boundary conditions lead to the vanishing of (2.15) and (2.16), in the limit of infinite gauge coupling:

$$\begin{aligned} F_a(\phi,\overline{\phi}) &= 0 \quad , && (3.20)\\ \chi_{+a} + \eta\overline{\chi}_{-a} &= 0 \quad , && (3.21)\\ \left(\left[\frac{\partial F_a}{\partial \phi_i} D_1\phi_i - \frac{\partial F_a}{\partial \overline{\phi}_i} D_1\overline{\phi}_i \right] - iK_{abc}\, \chi_-^b\, \overline{\chi}_-^c \right) &= 0 && (3.22)\\ \left\{ F_a(\phi,\overline{\phi}), W(\Phi) - \overline{W}(\overline{\phi}) \right\}_{PB} &= 0 && (3.23) \end{aligned}$$

We may use the equations of motion for the vector multiplet to infer sensible boundary condiitions on the component fields, via eqs. (3.3)-(3.7). For σ and λ,

$$\begin{aligned} \sigma|_{x^1=0} = \overline{\sigma}|_{x^1=0} &= 0 && (3.24)\\ (\lambda_+ + \eta\overline{\lambda}_-)|_{x^1=0} &= 0 && (3.25) \end{aligned}$$

This implies the following boundary condition on the twisted chiral superfield (with $\theta^+ = \eta\overline{\theta}^-$)

$$\Sigma|_{x^1=0} = 0 \qquad (3.26)$$

Indeed, it happens that $v_0 = 0$ in the NLSM limit. This is consistent with supersymmetry since $\delta_{susy} v_0 = 0$ using (3.25). Thus, one can choose the gauge condition

$$\delta v_0 = 0 \qquad (3.27)$$

in the LSM.

3.3 B-type boundary conditions in the NLSM

Recall that B-type boundary conditions are defined by requiring that $\epsilon_+ = \eta\epsilon_-$. In this case the θ term complicates the story. We will begin with the case $\theta = 0$.

3.3.1 $\theta = 0$

As we stated above, we will work with fully Neumann boundary conditions in this section. Let us begin with the boundary condition on ψ (justified *a posteriori*):

$$(\psi_+ + \eta\psi_-)|_{x^1=0} = 0\ . \qquad (3.28)$$

The supersymmetric variations of this condition lead to:

$$\left(D_1\phi - i\eta(\frac{\sigma-\overline{\sigma}}{\sqrt{2}})Q\phi \right)\bigg|_{x^1=0} = 0 \qquad (3.29)$$

$$F|_{x^1=0} = \frac{\partial \overline{W}}{\partial \overline{\phi}}\bigg|_{x^1=0} = 0 \ . \qquad (3.30)$$

The vanishing of boundary terms in the ordinary variations of the action requires further that:

$$D_1\phi|_{x^1=0} = 0 \qquad (3.31)$$

$$(\sigma - \overline{\sigma})|_{x^1=0} = 0 \ . \qquad (3.32)$$

It is easy to see that the conditions on σ are consistent with eqs. (3.3),(3.4).

The remaining boundary conditions on the vector multiplet can be derived either from the solutions in eqs. (3.3)-(3.7), or from SUSY variations of (3.32). The results are:

$$(\lambda_+ - \eta\lambda_-)|_{x^1=0} = 0 \qquad (3.33)$$

$$\partial_1(\sigma+\overline{\sigma})|_{x^1=0} = 0 \qquad (3.34)$$

$$v_{01}|_{x^1=0} = 0 \qquad (3.35)$$

Following the notation in [20], these conditions can be written in superfield notation as:

$$(\Sigma - \overline{\Sigma})|_{x^1=0} = 0 \ , \qquad (3.36)$$

where we recall that $\theta^+ = \eta\theta^-$ on the boundary. (θ's are the Grassmann parameters in $d=2$, $N=2$ superspace.) Note that holomorphic boundary conditions on the complex scalars of the chiral superfields leads to reality conditions on the scalars in the twisted chiral multiplet; this might have been anticipated from mirror symmetry, which exchanges A-type and B-type boundary conditions, as well as (roughly) chiral and twisted chiral superfields.

3.3.2 $\theta \neq 0$

In the presence of the theta term, $D_1\phi_i|_{x_1=0} = 0$ is clearly no longer the correct bosonic boundary condition[6]. This is to be expected; even for constant NS-NS B-fluxes through D-branes in flat space, the boundary conditions for the string worldsheet are roughly:

$$\eta_{\mu\nu}\partial_1 X^\nu + B_{\mu\nu}\partial_0 X^\nu = 0 \quad . \qquad (3.37)$$

We find the same effect here in the $e^2 \to \infty$ limit.

[6]Some of the material in this subsection was developed in collaboration with Albion Lawrence.

Ordinary variation of eqn. (3.8) then implies that:

$$\delta S^{ord}_{r,\theta} = -\frac{i\theta}{2\pi r}\int dx^0 \sum_i \left(\partial_0\phi_i\delta\bar{\phi}_i - \partial_0\bar{\phi}_i\delta\phi_i\right) \ , \tag{3.38}$$

where we have ignored the total derivative pieces since they have been cancelled by the addition of the contact term. Adding (3.38) to the boundary term coming from the ordinary variation of the bosonic kinetic energy term in the bulk action, we get the new boundary term,

$$\delta S^{ord}_{kin} = \int dy \sum_i \left[\left(D_1\phi_i + \frac{i\theta}{2\pi r}\partial_0\phi_i\right)\delta\bar{\phi}_i + c.c\right] \tag{3.39}$$

However before we claim that the term in brackets multiplying $\delta\bar{\phi}_i$ provides the boundary conditions for the ϕ's, we note that the boundary conditions may be written in different ways upto the addition of terms that vanish when we use the fact that $\partial_0\left(\sum_i Q_i\bar{\phi}_i\phi_i\right) = \partial_1\left(\sum_i Q_i\bar{\phi}_i\phi_i\right) = 0$, since $\sum_i Q_i\bar{\phi}_i\phi_i = r$ in the bulk. For example, eqn. (3.39) can be rewritten as

$$\delta S^{ord}_{kin} = \int dy \sum_i \left[\left(D_1\phi_i + \frac{i\theta}{2\pi r}D^B_0\phi_i\right)\delta\bar{\phi}_i + c.c\right] \quad , \tag{3.40}$$

where $D^B_0\phi = (\partial_0 + iv^B_0)\phi$. Care should be taken when we write the boundary conditions in the full GLSM where $D \neq 0$. The consistent way to do this is to choose boundary conditions that are closed under supersymmetry. Let us begin with the fermionic boundary conditions that are consistent with with the presence of a constant B-field. In such situations fermions obey the rotated boundary condition

$$(\psi_{+i} + \eta e^{i\gamma}\psi_{-i})|_{x^1=0} = 0 \tag{3.41}$$

where $e^{i\gamma}$ is to be determined later by requiring consistency with the boundary conditions on the bosons. Supersymmetric variation of (3.41) implies

$$D_1\phi_i + \frac{1-e^{i\gamma}}{1+e^{i\gamma}}D_0\phi_i \quad + \quad \eta i\sqrt{2}Q_i\frac{\bar{\sigma}-e^{i\gamma}\sigma}{1+e^{i\gamma}}\phi_i = 0 \tag{3.42}$$

$$F_i \quad = \quad 0 \tag{3.43}$$

where we note that the covariant derivatives involve both the bosonic and fermionic components of v_0 and v_1. The vanishing of the bosonic boundary variation term eqn. (3.39) clearly requires

$$\frac{1-e^{i\gamma}}{1+e^{i\gamma}} = \frac{i\theta}{2\pi r} \tag{3.44}$$

Now by judicious use of the fact that the variation of D is zero and the explicit expressions for the gauge fields and the σ field in the NLSM limit the boundary condition (3.42) indeed reduces to

$$\left(D^B_1\phi_i + \frac{i\theta}{2\pi r}D^B_0\phi_i\right)|_{x^1=0} = 0 \tag{3.45}$$

This boundary condition is indeed the one suggested by the boundary variation terms in eqn (3.40) (with the above fermion boundary conditions $v_1^F = 0$). In the GLSM it is eqn. (3.42) which will be the natural boundary condition on the bosonic boundary fields. Eqn. (3.45) may be obtained by showing that

$$\left[\frac{1-e^{i\gamma}}{1+e^{i\gamma}} i v_0^F + i\sqrt{2}\eta \frac{\bar{\sigma} - e^{i\gamma}\sigma}{1+e^{i\gamma}}\right]|_{x^1=0} = 0 \tag{3.46}$$

where v_0^F is the fermion bilinear part of v_0. With these boundary conditions, in the $e^2 \to \infty$ limit, all of the boundary terms arising in the equations of motion and the supersymmetric variations of the action cancel out.

Finally we wish to use the equations of motion for the vector multiplet to arrive at sensible boundary conditions for the component fields. Eq. (3.3) and (3.4) imply the boundary condition:

$$(\bar{\sigma} - e^{2i\gamma}\sigma)|_{x^1=0} = 0 \tag{3.47}$$

This is consistent with eqn. (3.46) and the boundary conditions for the chiral multiplets. The SUSY variation of (3.47) requires

$$(\lambda_+ - \eta e^{2i\gamma}\lambda_-)|_{x^1=0} = 0 \ , \tag{3.48}$$

which in turn requires

$$\left\{(1-e^{2i\gamma})iD - (1+e^{2i\gamma})v_{01} + \sqrt{2}\eta\partial_1(\bar{\sigma} + e^{2i\gamma}\sigma)\right\}\Big|_{x^1=0} = 0 \quad . \tag{3.49}$$

Our analysis has been for the case of (all Neumann boundary conditions) in one-modulus examples in weighted projective spaces. However, the generalisation to many moduli case is straightforward and we will not enter into the details here. We now turn to the case of "mixed" boundary conditions, i.e., we impose Dirichlet boundary conditions on some of the fields.

3.3.3 "Mixed" boundary conditions

We illustrate the general situation by imposing Dirichlet boundary conditions on one of the fields, say

$$\phi_1 = 0 \quad .$$

Supersymmetric completion requires the condition

$$(\psi_{+1} - \eta\psi_{-1}) = 0 \quad .$$

All other fields have Neumann boundary conditions imposed on them as in sec. 3.3.2. One can check that all boundary terms in the ordinary and supersymmetric variations of the action vanish as they did in the all Neumann case.

What about the boundary conditions implied on the fields in the vector multiplet? By considering the expressions for σ and $\overline{\sigma}$ as given in eqns. (3.3) and (3.4) respectively, it appears that one cannot obtain simple boundary conditions as in (3.47) for the all Neumann case. However, we claim that the problem may be resolved by requiring that on the boundary the bulk expressions for the fields in the vector multiplet have to be *modified*. The modification requires that the fields in the vector multiplet depend only on fields in the chiral multiplet which have Neumann boundary conditions and not on those with Dirichlet boundary conditions(ϕ_1, $\psi_{\pm 1}$ in our case). Note that this is trivially true for the bosonic part and refers only to the fermionic part such as the bilinear expression for σ in the NLSM limit (see eqn. (3.3)). Another way to state this modification is to impose a modified Gauss law constraint on the boundary such that

$$\begin{aligned} J_0 &\equiv \sum_i Q_i \left[i(\overline{\phi}_i D_0 \phi_i - \phi_i D_0 \overline{\phi}_i) + \overline{\psi}_{-i}\psi_{-i} + \overline{\psi}_{+i}\psi_{+i} \right] \\ &= Q_1(\overline{\psi}_{-1}\psi_{-1} + \overline{\psi}_{+1}\psi_{+1}) \end{aligned} \tag{3.50}$$

For the general case, the RHS of the Gauss law constraint is given by a summation over fermions in the Dirichlet directions. Consistency with supersymmetry forces σ as well as $\lambda_{\pm}$ to also depend only on fields with Neumann boundary conditions.

Thus, the boundary conditions on the fields in the vector multiplet are identical to the all Neumann case considered in section 3.3.2 above. This will enable us to now carry over these boundary conditions to the GLSM in a straightforward fashion.

4 Boundary conditions in the GLSM

In this section, we propose to derive supersymmetry preserving boundary conditions for the gauged linear sigma model, that will define appropriate D-branes wrapping supersymmetric cycles of the Calabi-Yau. These boundary conditions that we derive for the GLSM, would have to be consistent with those in the infrared limit, i.e should appropriately reduce to the ones we have described in the last section.

Let us begin by recalling that whereas in the non-linear sigma model limit of the GLSM the fields in the gauge multiplet become very massive and effectively decouple from the theory, this is not the case with the GLSM. As a consequence the analogue of the Dirichlet and Neumann boundary conditions in the GLSM are more general boundary conditions that depend on fields of the gauge multiplet as well. In particular, it is clear that the boundary conditions must be gauge covariant.

4.1 A-type boundary conditions in the GLSM

In order to extend the boundary conditions from the NLSM to the GLSM, we will have to include boundary conditions for the p-field in addition to the ones we obtained for the fields in the vector multiplet. One can check that the following are a consistent set of boundary conditions in the GLSM at finite gauge coupling

$$\begin{aligned} \text{Im } p|_{x^1=0} &= 0 \quad , & (4.1)\\ (\psi_{+p} + \eta\overline{\psi}_{-p})|_{x^1=0} &= 0 \quad , & (4.2)\\ \text{Re } D_1 p|_{x^1=0} &= 0 & (4.3)\end{aligned}$$

All other boundary conditions are as given in the NLSM discussed earlier. It is interesting that the A-type boundary conditions are identical in form in both the GLSM in general (taking into account of course the need to include boundary conditions on the p-field) and in the LG and CY phases. This is however not the case with the B-type as is clear even from the considerations of such boundary conditions in the NLSM limit.

4.2 B-type boundary conditions in the GLSM

4.2.1 $\theta = 0$

In the NLSM limit, we have seen that for both Dirichlet as well as Neumann boundary conditions imposed on the matter fields, the boundary conditions implied on the fields in the vector multiplet are summarised by the simple boundary condition on the twisted chiral superfield

$$(\Sigma - \overline{\Sigma})|_{x^1=0} = 0 \quad , \tag{4.4}$$

where we impose $\theta^+ = \eta\theta^-$ and $\overline{\theta}^+ = \eta\overline{\theta}^-$ as well in the above condition. We will continue to require this set of boundary conditions in the GLSM. However we will also need to impose boundary conditions on the p-field as well as its supersymmetric partners $\psi_{\pm p}$ such that boundary terms in the ordinary as well as supersymmetric variation of the GLSM action vanish. It is useful to observe at this point that once we have fixed the above boundary conditions on the fields in the vector multiplet, the choices of consistent boundary conditions is in fact identical to that of an extended LG model involving the p-field and the fields ϕ_i with superpotential $W = PG(\phi)$ as in section 2 (with the condition that ordinary derivatives in the LG model are replaced by covariant derivatives in the GLSM). This leads to two possible classes of boundary conditions

1. Dirichlet boundary condition on p with $p = 0$. Since the superpotential in the GLSM is given by $W = PG(\phi)$, for this choice, $F_i^* = (\partial W/\partial\phi_i) = p(\partial G/\partial\phi_i) = 0$. Thus, the condition $F_i = 0$ which occurs whenever we impose Neumann boundary conditions on the ϕ_i is trivially satisfied. Thus,

any boundary condition (Neumann or Dirichlet) involving scalar fields other than p goes through subject to the condition that all Dirichlet boundary conditions are specified by homogeneous polynomials. This includes the all Neumann case which did not appear in the LG phase. In the LG phase, (which occurs for large negative r) where p has non-vanishing vev, the boundary condition $p = 0$ cannot be imposed. This also suggests that this boundary condition is acceptable only in the CY phase where $p = 0$ is the ground state condition.

2. Another possibility is that one imposes Neumann boundary condition on the p-field. However, the $F_p = 0$ condition now requires $G = 0$. Further, one is not allowed to choose to impose Neumann boundary conditions on individual fields for the quintic at the Fermat point. A possible consistent choice of boundary conditions at the Fermat point of the quintic is given by $\phi_1 + \phi_2 = 0$ and $\phi_i = 0$ ($i = 3, 4, 5$).

One can verify that for both classes of boundary conditions discussed above, the boundary terms in the ordinary as well as supersymmetric variations vanish.

4.2.2 $\theta \neq 0$

The strategy that we will pursue is to extend the boundary conditions we obtained in the NLSM to that of the GLSM. The ordinary and supersymmetric variations of the kinetic energy terms of the fields in the vector multiplet will now have to be considered. We will first consider the contact term which we *derived* in the NLSM limit as given by eqn. (3.19).

$$S^{NLSM}_{\text{boundary}} = \frac{i\theta}{4\pi r} \int dx^0 \sum_i (\phi_i D_0 \overline{\phi}_i - \overline{\phi}_i D_0 \phi_i) \tag{4.5}$$

However, before we choose this to be the contact term in the GLSM we must remember that there may be a need for other terms which vanish in the NLSM limit. In fact, we do need such a term in order to ensure that there is a smooth NLSM limit to the boundary conditions chosen in the GLSM. The full boundary term that we need in the GLSM turns out be

$$S^{GLSM}_{\text{boundary}} = \int dx^0 \left\{ \frac{i\theta}{4\pi r} \sum_i (\phi_i D_0 \overline{\phi}_i - \overline{\phi}_i D_0 \phi_i) + \eta \frac{\theta}{\sqrt{2\pi r}} \frac{D}{e^2} \sigma e^{i\gamma} \right\}, \tag{4.6}$$

where $\tan(\gamma/2) = -\theta/2\pi r$.

In the presence of this boundary term, we now list the boundary conditions required(using the NLSM as a guide) for cancelling ordinary as well as supersymmetric variations of the action. In the matter sector, these are

$$(\psi_{+i} + \eta e^{i\gamma} \psi_{-i})|_{x^1=0} = 0 \tag{4.7}$$

$$\left\{D_1\phi_i + \frac{1-e^{i\gamma}}{1+e^{i\gamma}}D_0\phi_i + i\sqrt{2}\eta Q_i \frac{\bar{\sigma}-e^{i\gamma}\sigma}{1+e^{i\gamma}}\phi_i\right\}\Big|_{x^1=0} = 0 \tag{4.8}$$

$$p|_{x^1=0} = 0 \tag{4.9}$$

$$(\psi_{+p} - \eta\psi_{-p})|_{x^1=0} = 0 \tag{4.10}$$

where we have included the p-field as well as its fermionic partners $\psi_{\pm p}$.

The boundary conditions on the fields in the vector multiplet in the GLSM are chosen to be

$$(\bar{\sigma} - e^{2i\gamma}\sigma)|_{x^1=0} = 0 \tag{4.11}$$

$$(\bar{\lambda}_+ - \eta e^{2i\gamma}\lambda_-)|_{x^1=0} = 0 \tag{4.12}$$

where, in keeping with the results obtained in the NLSM limit, we have chosen the phase to be twice the phase in the matter boundary conditions. The remaining boundary conditions are

$$\left(\frac{v_{01}}{D} + \frac{\theta}{2\pi r}\right)\Big|_{x^1=0} = 0 \tag{4.13}$$

$$\left(\partial_1(\bar{\sigma} + e^{2i\gamma}\sigma) - \eta\frac{\theta}{\sqrt{2}\pi r}e^{i\gamma}D\right)\Big|_{x^1=0} = 0 \tag{4.14}$$

Notice that the last two boundary conditions are indeed a convenient split of a single equation (given below) arising from the supervariation of eqn. (4.12) in order to make boundary terms in the ordinary variation vanish.

$$\left\{(1-e^{2i\gamma})iD - (1+e^{2i\gamma})v_{01} + \sqrt{2}\eta\partial_1(\bar{\sigma} + e^{2i\gamma}\sigma)\right\}\Big|_{x^1=0} = 0 \tag{4.15}$$

Hence, they are really dictated by our insistence that the rotated boundary conditions on σ and λ have a *consistent* NLSM limit. The full set of boundary conditions on the fields in the vector implies the following boundary condition on the twisted chiral superfield (subject to $\theta^+ = \eta\theta^-$ and $\overline{\theta}^+ = \eta\overline{\theta}^-$)

$$(\Sigma - e^{-2i\gamma}\overline{\Sigma})|_{x^1=0} = 0 \quad , \tag{4.16}$$

There exists another solution to the vanishing of the boundary variation terms that however will however involve rotated boundary conditions on the σ fields that do not agree with the NLSM limit. As always, we begin with the fermions and choose $(\psi_{+i} + \eta e^{i\gamma}\psi_{-i}) = 0$ and then derive other conditions by supersymmetric variation of the condition. The first variation leads to eqn. (4.8) which one can rewrite as two separate conditions

$$\left(D_1\phi_i + \frac{1-e^{i\gamma}}{1+e^{i\gamma}}D_0\phi_i\right)\Big|_{x^1=0} = 0 \tag{4.17}$$

$$\frac{\bar{\sigma}-e^{i\gamma}\sigma}{1+e^{i\gamma}}\Big|_{x^1=0} = 0 \tag{4.18}$$

$$F_i|_{x^1=0} = 0 \tag{4.19}$$

The second equation is clearly in contradiction with the rotations implied on σ in the NLSM. Further supersymmetric variation implies the following boundary condition on the twisted chiral superfield (subject to $\theta^+ = \eta\theta^-$ and $\overline{\theta}^+ = \eta\overline{\theta}^-$)

$$(\Sigma - e^{-i\gamma}\overline{\Sigma})|_{x^1=0} = 0 \quad . \tag{4.20}$$

It is easy to verify that the boundary terms in the ordinary and supersymmetric variations vanish. This solution has also been recently proposed in [15]. As emphasised earlier, this solution does not agree with the boundary conditions on the fields in the vector multiplet as derived in the NLSM limit.

The case of "mixed" boundary conditions (i.e., some fields have Dirichlet boundary conditions imposed on them) also goes through. The boundary contact term is again given by eqn. (4.6). The boundary condition on the fields in the vector multiplet are as in the all Neumann case and one can verify that all boundary terms in the ordinary and supersymmetric variations vanish.

4.3 Discussion

We would now like to discuss some of the implications of the analysis of boundary conditions in the GLSM for the case of the quintic at the Fermat point. In particular we would like to address the question of whether we can identify the branes corresponding to our various choices of boundary conditions.

The choice of all Neumann boundary conditions on the matter fields is clearly suggestive of a D6-brane. However we have to decide what boundary conditions are appropriate for the p field. It is useful to consider for this purpose a slightly different example, without a superpotential, namely that of the $\mathcal{O}(-3)$ line bundle over $\mathbb{P}^2$. In this case, clearly the construction of a D4-brane wrapping the $\mathbb{P}^2$ would require that there be Dirichlet boundary conditions on the charge -3 field. Otherwise we would not obtain a compact D4-brane. A similar situation obtains in the quintic and hence we choose $p = 0$ for the D6-brane in the large volume. The $G = 0$ condition is imposed on the ground state by continuity from the bulk.

Similarly we could identify the cases of the "mixed" Neumann-Dirichlet boundary conditions with lower-dimensional branes while we always maintain Dirichlet boundary conditions on the p field. However the charges of these lower-dimensional branes will not be the minimum charge.

However there could be other situations where we impose Neumann boundary conditions that involve the p field. For instance we could impose conditions of the form $p\phi_1^5 = c$ where c is a complex constant. Associated to this condition would be another Neumann condition involving the p field and the ϕ_1. We can then choose individual boundary conditions on the rest of the fields. The interpretation of these boundary conditions would be different.

Let us now consider the quintic at a point in its Kähler moduli space where it admits a description as a Gepner model. B-type boundary states arising from the Recknagel-Schomerus construction [6] have been discussed in [7]. The important point to note with regard to the Recknagel-Schomerus construction is that the boundary states arise from tensoring boundary states of individual minimal models. Thus, they can only arise in our construction by imposing boundary conditions on individual fields. We will for the moment focus our attention on the boundary states labelled $|00000\rangle_B$ (There are five such states forming a $\mathbb{Z}_5$ orbit.). The analysis in [7] shows that one of these boundary states corresponds to the pure six-brane in the large volume limit.

The Gepner point is in the LG phase (see section 2) where we have seen that the only allowed boundary conditions on individual fields are Dirichlet and hence all RS states (including the one which carries pure six-brane charge) must arise in this class. (See [17] for a related discusson.) Consider now the all Neumann case that we considered in the GLSM with $p = 0$. The boundary condition on the bosonic fields as we have described earlier are (for $i = 1, \ldots, 5$)

$$\left\{ D_1\phi_i + \frac{i\theta}{2\pi r} D_0\phi_i + i\sqrt{2}\eta \frac{\bar{\sigma} - e^{i\gamma}\sigma}{1 + e^{i\gamma}} \phi_i \right\}\Bigg|_{x^1=0} = 0$$

It is of interest to ask what happens to these boundary conditions as $r \to 0$ while keeping θ fixed at some non-zero value. In this limit $e^{i\gamma} \to -1$ and the above boundary conditions tend to a Dirichlet boundary condition.

$$D_0\phi_i + i\eta \frac{(\bar{\sigma} + \sigma)}{\sqrt{2}} \phi_i|_{x^1=0} = 0$$

Interestingly, the σ part in the above equation comes out precisely of the form required because the fermionic boundary condition becomes precisely the Dirichlet combination. However quantum corrections (due to worldsheet instantons) become important for small values of r and further lines of marginal stability may be crossed in taking this limit. Though our classical analysis may thus need to be modified, the above result is suggestively in agreement with the pure six-brane appearing in the LG phase as an all Dirichlet state. The limitations of our argument become clear once we consider the cases of lower branes (such as the pure D4-brane) which do not appear as a RS boundary state at the Gepner point.

We now comment on the extension of the results of this paper to other examples. There are a number of straightforward extensions that require very little beyond the techniques of this paper itself. All examples which are given by complete intersections of hypersurfaces in weighted projective spaces or a tensor product of such spaces can be dealt with. In such cases the analogues of the p-field are such that they can be decoupled in the NLSM limit. Thus the rest of the analysis would proceed much as in the single modulus case.

5 Conclusion

In this paper we have taken the initial steps in what appears to be an useful programme of trying to understand, in a microscopic description, D-branes in large domains of the moduli space of the Calabi-Yau backgrounds in which they live. The explicit nature of the boundary conditions that we impose on the matter fields may be appropriately translated into the more general characterisations of branes in the large-volume phase as zero sections of the corresponding bundles on the CY manifold. Thus contact could be made with other, more geometric techniques for understanding the various properties of these branes in the large volume limit.

The analysis of the GLSM in this paper has been restricted to the open-string channel. More information can be extracted by also investigating the closed-string channel. Some related issues have already been considered in [14, 15].

Another question which needs to be considered is the addition of (marginal) deformations corresponding to gauge fields on the brane as well as its moduli including the introduction of Chan-Paton factors. This will involve describing vector bundles on Calabi-Yau manifolds or their submanifolds. One construction which easily fits the boundary GLSM is the use of monads (as suggested in [7]). It is clear that this will involve techniques which appeared in the context of $(0,2)$ versions of the closed string GLSM [20, 34] given that only half of the $(2,2)$ supersymmetry is preserved on the boundary. For example, one has to introduce boundary fermions which are sections of the appropriate vector bundle [35]. The fermions ν^m which appeared in sec. 2.3.2 are objects of this type (they are sections of the normal bundle).

A related problem is the classification of boundary topological observables in the twisted version of the GLSM. This naturally leads to the next step in this program i.e., the use of the GLSM description of D-branes to determine the superpotential in the worldvolume theory of these branes. We believe that the results of this paper are a useful step in proceeding towards this goal. This (superpotential) can be used as a check of mirror symmetry by computing the same in the mirror manifold. Some aspects of these issues from a slightly different viewpoint have been considered in [13, 16, 33].

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