hep-th/0301085

BRST-ANTI-BRST SYMMETRIC CONVERSION OF SECOND-CLASS CONSTRAINTS

I.A. BATALIN AND M.A. GRIGORIEV

Abstract. A general method of the BRST–anti-BRST symmetric conversion of second-class constraints is presented. It yields a pair of commuting and nilpotent BRST-type charges that can be naturally regarded as BRST and anti-BRST ones. Interchanging the BRST and anti-BRST generators corresponds to a symmetry between the original second-class constraints and the conversion variables, which enter the formalism on equal footing.

1. Introduction

For first-class constraints, the well-known BFV–BRST quantization method provides an adequate description of general constrained systems at the classical and quantum levels [1]. But for second-class systems, a proper counterpart of the BFV–BRST method is not yet known. Most of the well-established approaches to second-class systems are based on the construction of an effective first-class system that is equivalent to the original second-class system [2, 3, 4, 5, 6]. Among these methods, the most developed one is the so-called *conversion method* [2, 3, 4, 7, 9]. In this approach, one introduces additional variables ϕ, called *conversion variables*, and extends second-class constraints by ϕ-dependent terms such that the extended constraints become first-class.

The conversion method has proved a powerful tool for description and quantization of second-class systems. But the general conversion is too ambiguous, and it seems natural to reduce this ambiguity by imposing additional conditions on the conversion procedure. A well-known restriction consists in requiring the effective gauge algebra to be of a fixed rank; a common choice is to take a 0-rank (Abelian) effective gauge algebra. This *Abelian* conversion is well studied; in particular, the existence theorem and a constructive procedure are known for the effective constraints [7].

In this paper, we take another route and reduce the ambiguity by requiring the conversion procedure to respect extra symmetries. Namely, we treat the original second-class constraints and conversion variables on equal footing. Technically, this implies that the conversion variables appear as constraints $\bar\theta^\alpha$ dual to the original second-class constraints θ_α. We then proceed with the conversion keeping the symmetry between θ_α and $\bar\theta^\alpha$ explicit.

The conversion is carried out in the framework of the BFV–BRST approach. At the level of the effective gauge system (converted system), the symmetry between θ^α and $\bar\theta^\alpha$ results in the BRST–anti-BRST invariance. The BRST and anti-BRST charges are associated to the appropriately extended original constraints θ_α and the extended dual constraints $\bar\theta^\alpha$ respectively.

The paper is organized as follows: in Sec. **2**, we develop basics of our approach to conversion and show that the resulting gauge system is naturally described by a pair of nilpotent and commuting generators identified as the BRST and anti-BRST generators. In Sec. **3**, generating equations of the BRST–anti-BRST symmetric conversion are reformulated as a certain type of master equations with respect to the *quantum antibracket* [11]. A more restricted version of the formalism is considered in Sec. **4**. In this more special approach, generating operators satisfy stronger conditions. These conditions are also shown to allow the existence of a conversion procedure for arbitrary original constraints θ_α. In Sec. **5**, we consider construction of a unitarizing Hamiltonian in the BRST–anti-BRST symmetric formulation.

2. Basics

Let a second-class constrained system be determined by constraints $\theta^\alpha, \alpha = 1, \dots, 2N$. Their Dirac matrix is given by

$$i\hbar\,\Delta^0_{\alpha\beta} = [\theta_\alpha, \theta_\beta]\,, \tag{2.1}$$

and is assumed to be invertible. To each constraint θ^α we associate its dual $\bar\theta^\alpha$. We choose $\varepsilon(\bar\theta^\alpha) = \varepsilon(\theta_\alpha)$ where $\varepsilon(f)$ denotes Grassmann parity of f. Dual constraints plays a role of extra degrees of freedom and therefore are defined on the appropriate extension of the original phase space. A basic example is provided by taking $\bar\theta^\alpha = e^\alpha_\beta\phi^\beta$ where ϕ^α are conversion variables and e^α_β depend on phase space variables only.

In what follows we prefer to work in terms of generating functions (operators) and not in terms of constraints. To this end let us introduce ghost variables $\mathcal{C}^\alpha$ and $\bar{\mathcal{P}}_\alpha$ associated either to constraints θ_α or $\bar\theta^\alpha$. One assignees the following Grassmann parities to the ghost variables:

$$\varepsilon(\mathcal{C}^\alpha) = \varepsilon(\bar{\mathcal{P}}_\alpha) = \varepsilon(\theta_\alpha) + 1 = \varepsilon(\bar\theta^\alpha) + 1\,, \qquad \alpha = 1, \dots, 2N\,.$$

We also choose standard commutation relations and ghost number gradings for variables $\mathcal{C}^\alpha$ and $\bar{\mathcal{P}}_\alpha$

$$[\mathcal{C}^\alpha, \bar{\mathcal{P}}_\beta] = i\hbar\,\delta^\alpha_\beta\,, \qquad \mathrm{gh}(\mathcal{C}^\alpha) = 1\,, \quad \mathrm{gh}(\bar{\mathcal{P}}_\alpha) = -1\,.$$

The ghost number operator is then assumed to have the following form:

$$G = \frac{1}{2}(\mathcal{C}^\alpha\bar{\mathcal{P}}_\alpha(-1)^{\varepsilon_\alpha} - \bar{\mathcal{P}}_\alpha\mathcal{C}^\alpha)\,.$$

Generating operators encoding constraints θ_α and $\bar\theta^\alpha$ are denoted by $\mathbf{\Theta}$ and $\bar{\mathbf{\Theta}}$ respectively; their expansions with respect to ghost variables start as

$$\mathbf{\Theta} = \mathcal{C}^\alpha\theta_\alpha + \dots\,, \qquad \bar{\mathbf{\Theta}} = \bar\theta^\alpha\bar{\mathcal{P}}_\alpha(-1)^{\varepsilon_\alpha} + \dots\,,$$

and are subjected to the following ghost number prescriptions:

$$[G, \mathbf{\Theta}] = i\hbar\,\mathbf{\Theta}\,, \qquad [G, \bar{\mathbf{\Theta}}] = -i\hbar\,\bar{\mathbf{\Theta}}\,. \tag{2.2}$$

We are interested in constructing first-class constrained (gauge) system equivalent to the original second-class one and entering constraints θ and $\bar{\theta}$ in a symmetric way. To this end we are looking for a pair of generators Ω and $\bar{\Omega}$ satisfying

$$
[\Omega, \Omega] = 0\,, \qquad [\bar{\Omega}, \bar{\Omega}] = 0\,, \qquad [\Omega, \bar{\Omega}] = 0\,. \tag{2.3}
$$

Ω and $\bar{\Omega}$ are to be understood as appropriate extensions of the original generating operators $\mathbf{\Theta}$ and $\bar{\mathbf{\Theta}}$ respectively.

It is useful to introduce the following condensed notations:

$$
\begin{aligned}
&\mathbf{\Theta}^a\,, \quad a = 1,2 \qquad \mathbf{\Theta}^1 = \mathbf{\Theta}\,, \quad \mathbf{\Theta}^2 = \bar{\mathbf{\Theta}}\,,\\
&\Omega^a\,, \quad a = 1,2 \qquad \Omega^1 = \Omega\,, \quad \Omega^2 = \bar{\Omega}\,.
\end{aligned} \tag{2.4}
$$

The ghost number assignments then take the form

$$
[G, \mathbf{\Theta}^a] = i\hbar\, g^a_b \mathbf{\Theta}^b\,, \qquad [G, \Omega^a] = i\hbar\, g^a_b \Omega^b\,, \tag{2.5}
$$

where the only nonvanishing components of g^a_b are $g^1_1 = 1$ and $g^2_2 = -1$. We also introduce operator Δ^{ab} defined by

$$
2i\hbar\, \Delta^{ab} = [\mathbf{\Theta}^a, \mathbf{\Theta}^b]\,.
$$

In terms of new notations conditions (2.3) read as

$$
[\Omega^a, \Omega^b] = 0\,. \tag{2.6}
$$

In order to find Ω^a satisfying (2.6) consider the following anzatz for Ω^a

$$
\Omega^a = \mathbf{\Theta}^a + (i\hbar)^{-1}[Z, \mathbf{\Theta}^a]\,,
$$

where an operator Z has been introduced. Equations (2.6) take then the form:

$$
\begin{aligned}
2[\Omega^a, \Omega^b] = [\Omega^{\{a}, \Omega^{b\}}] &= \\
= [\mathbf{\Theta}^{\{a}, \mathbf{\Theta}^{b\}} &- (i\hbar)^{-2}[Z,[Z,\mathbf{\Theta}^{b\}}]]] + \\
&+ 4(i\hbar)^{-1}[Z,\Delta^{ab}] + 2(i\hbar)^{-2}[Z,[Z,\Delta^{ab}]] = 0\,.
\end{aligned} \tag{2.7}
$$

It is natural to make this conditions satisfied by imposing the following stronger conditions

$$
\begin{aligned}
&[Z,[Z,\mathbf{\Theta}^a]] = (i\hbar)^2 \mathbf{\Theta}^a\,,\\
&\frac{1}{2}[Z,[Z,\Delta^{ab}]] = -i\hbar\,[Z,\Delta^{ab}]\,,
\end{aligned} \tag{2.8}
$$

so that second and third lines in (2.7) vanish separately. It follows from (2.8) that Z acts on $\Omega^a = \mathbf{\Theta}^a + (i\hbar)^{-1}[Z, \mathbf{\Theta}^a]$ as projector:

$$
[Z, \Omega^c] = i\hbar\Omega^c\,. \tag{2.9}
$$

Given $\mathbf{\Theta}^a$ and Z such that (2.8) holds one arrives at a pair of commuting and nilpotent BRST-like charges Ω^a. They can be understood as BRST and anti-BRST generators. To explain this in more details let us consider explicitly lowest order equations coming from (2.3). Introducing the lowest order structure functions according to

$$\begin{aligned} \Omega = \Omega^1 &= \mathcal{C}^\alpha T_\alpha + \frac{1}{2}\mathcal{C}^\beta\mathcal{C}^\alpha U^\gamma_{\alpha\beta}\bar{\mathcal{P}}_\gamma(-1)^{\varepsilon_\beta+\varepsilon_\gamma} + \dots\,, \\ \bar{\Omega} = \Omega^2 &= \bar{T}^\alpha\bar{\mathcal{P}}_\alpha(-1)^{\varepsilon_\alpha} + \frac{1}{2}\mathcal{C}^\gamma\bar{U}^{\alpha\beta}_\gamma\bar{\mathcal{P}}_\beta\bar{\mathcal{P}}_\alpha(-1)^{\varepsilon_\beta} + \dots\,, \end{aligned} \tag{2.10}$$

one arrives at the following explicit form of the lowest order equations:

$$\bar{T}^\alpha T_\alpha = 0\,, \tag{2.11}$$

$$[T_\alpha, T_\beta] = i\hbar\, U^\gamma_{\alpha\beta}T_\gamma\,, \qquad [\bar{T}^\alpha, \bar{T}^\beta] = i\hbar\,\bar{T}^\gamma\bar{U}^{\alpha\beta}_\gamma\,, \tag{2.12}$$

$$[T_\alpha, \bar{T}^\beta] = i\hbar\,(\bar{U}^{\beta\gamma}_\alpha T_\gamma + \bar{T}^\gamma U^\beta_{\gamma\alpha}) + O(\hbar^2)\,. \tag{2.13}$$

These relations coincide with those familiar from the standard BRST-anti-BRST symmetric formulation.

Let us also note that the algebra formed by Ω^a and Z coincide with that in the $sp(2)$ symmetric formalism with Z being a counterpart of the "new ghost number" operator [14]

3. Formulation in terms of the quantum antibracket

Generating equations (2.8) can be interpreted as a type of master equations formulated in terms of the so-called *quantum antibracket*. To show that consider the following bracket operation:

$$(f,\, g)_Q = \frac{1}{2}\left([f, [Q, g]] + (-1)^{(\varepsilon(f)+\varepsilon(Q))(\varepsilon(g)+\varepsilon(Q))+\varepsilon(Q)}[g, [Q, f]]\right)\,. \tag{3.1}$$

In the case where $\varepsilon(Q) = 1$ this structure is know as the quantum antibracket [11]. In terms of this structure equations (2.8) take the following form:

$$\begin{aligned} (Z,\, Z)_{\mathbf{\Theta}^a} &= -(i\hbar)^2\mathbf{\Theta}^a\,, \\ \frac{1}{2}(Z,\, Z)_{\Delta^{ab}} &= -i\hbar\,[\Delta^{ab}, Z]\,, \end{aligned} \tag{3.2}$$

One can also reformulate these equations in terms of a projection operator

$$S^a_b = g^a_b G + \delta^a_b Z$$

so that one can express Ω^a as follows

$$\Omega^a = [S^a_b, \mathbf{\Theta}^b]\,.$$

Indeed, it follows from (3.2) that S^a_b satisfies

$$(S^{\{a}_{\{c},\, S^{b\}}_{d\}})_{\mathbf{\Theta}^d} = i\hbar\left(g^{\{a}_{\{c}g^{b\}}_{d\}} + \delta^{\{a}_{\{c}\delta^{b\}}_{d\}}\right)[\mathbf{\Theta}^e, S^d_e] \tag{3.3}$$

$$\frac{1}{2}(S^{\{a}_{\{c},\, S^{b\}}_{d\}})_{\Delta^{cd}} = i\hbar\, g^{\{a}_{\{c}g^{b\}}_{d\}}[\Delta^{ed}, S^c_e]\,. \tag{3.4}$$

4. More special formulation

As we have seen conditions (2.8) ensure nilpotency and mutual commutativity of $\boldsymbol{\Omega}^a$. One can either consider $\boldsymbol{\Theta}^a$ and $\boldsymbol{Z}$ as independent quantities to be defined by (2.8) or try to reduce the number of unknowns. The later option leads a priory to a more restricted class of solutions to (2.8). We choose this option and express $\boldsymbol{Z}$ through $\boldsymbol{\Theta}^a$ by the following anzatz

$$i\hbar\, Z = [\boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}] = g_{ab}\Delta^{ab}\,. \tag{4.1}$$

where $g_{ab} = \epsilon_{ac} g^c_b$ with ϵ_{ab} being antisymmetric and $\epsilon_{12} = -1$. As a consequence operators $\boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}$,$\Delta^{ab}$ and $\boldsymbol{Z}$ satisfy the following relations:

$$[\Delta, \bar{\boldsymbol{\Theta}}] = [\boldsymbol{\Theta}, Z]\,, \qquad [\bar{\Delta}, \boldsymbol{\Theta}] = [\bar{\boldsymbol{\Theta}}, Z]\,, \tag{4.2}$$

where notations

$$2i\hbar\,\Delta = 2i\hbar\,\Delta^{11} = [\boldsymbol{\Theta}, \boldsymbol{\Theta}]\,, \qquad 2i\hbar\,\bar{\Delta} = 2i\hbar\,\Delta^{22} = [\bar{\boldsymbol{\Theta}}, \bar{\boldsymbol{\Theta}}]\,,$$

are introduced.

In the case where $\boldsymbol{Z}$ is given by (4.1) conditions (2.8) can be satisfied by imposing the following stronger ones:

$$\begin{gathered} [Z, \Delta] = 0\,, \qquad [Z, \bar{\Delta}] = 0\,, \\ [\Delta, \bar{\Delta}] = i\hbar\, G\,. \end{gathered} \tag{4.3}$$

These are to be understood as equations on $\boldsymbol{\Theta}^a$ ensuring nilpotency and mutual commutativity of $\Omega^a = \boldsymbol{\Theta}^a + (i\hbar)^{-1}[Z, \boldsymbol{\Theta}^a]$. It also follows from Eq. (4.3) that

$$[\boldsymbol{\Theta}^{\{a}, [\boldsymbol{\Theta}^{b\}}, Z]] = 0\,.$$

This in turn implies that one can consider the following a little bit more general expression for $\boldsymbol{\Omega}^a$:

$$\Omega = \boldsymbol{\Theta} + \alpha(i\hbar)^{-1}[Z, \boldsymbol{\Theta}]\,, \qquad \bar{\Omega} = \bar{\boldsymbol{\Theta}} + \bar{\alpha}(i\hbar)^{-1}[Z, \bar{\boldsymbol{\Theta}}]\,,$$

which are nilpotent provided $\alpha = \bar{\alpha} = 1$ or $\alpha = \bar{\alpha} = -1$.

It is instructive to rewrite equations (4.3) in a condensed form directly in terms of Δ^{ab}:

$$4[\Delta^{ab}, \Delta^{cd}] = i\hbar g^{\{a\{c}\epsilon^{b\}d\}}G\,, \tag{4.4}$$

where

$$g^{\{a\{c}\epsilon^{b\}d\}} = g^{ac}\epsilon^{bd} + g^{bc}\epsilon^{ad} + g^{ad}\epsilon^{bc} + g^{bd}\epsilon^{ac}\,,$$

and $g^{ab} = g^a_c \epsilon^{cb}$ with $\epsilon^{12} = 1$. One can then check that this equations does contain all relations (4.3).

Contracting equation (4.4) with $g_{ab} = \epsilon_{ac} g^c_b$ and ϵ_{ab} one arrives at

$$\epsilon_{bc}[\Delta^{ab}, \Delta^{cd}] = -i\hbar g^{ad} G \tag{4.5}$$

and

$$g_{bc}[\Delta^{ab}, \Delta^{cd}] = i\hbar \epsilon^{ad} G \tag{4.6}$$

respectively. The later equation is not equivalent to (4.4) while the former is in fact equivalent. It can also be rewritten in terms of $\Delta_a^b = \epsilon_{ac}\Delta^{cb}$ as

$$[\Delta_a^b, \Delta_b^d] = i\hbar g_a^d G\,. \tag{4.7}$$

It follows from nilpotency (2.6) and ghost number grading (2.5) of Ω that it can be considered as a BRST charge of a gauge system equivalent to the original second class one. At the same time $\bar{\Omega}$ can be understood as respective anti-BRST charge. To illustrate the idea consider the simplest case where original second-class constraints are “Abelian” i.e. their commutators form a constant operator-valued matrix $[\theta_\alpha, \theta_\beta] = \omega_{\alpha\beta} = \mathrm{const}_{\alpha\beta}$. In this case one can identify constraints $\bar{\theta}^\alpha$ with conversion variables ϕ^α. The simplest choice is to require the following commutation relations of variables ϕ:

$$[\phi^\alpha, \phi^\beta] = \omega^{\alpha\beta}(-1)^{\varepsilon_\beta}\,, \qquad \omega_{\alpha\gamma}\omega^{\gamma\beta} = \delta_\alpha^\beta\,.$$

One can then check that $\boldsymbol{\Theta}$ and $\bar{\boldsymbol{\Theta}}$ given by

$$\boldsymbol{\Theta} = \mathcal{C}^\alpha\theta_\alpha\,, \qquad \bar{\boldsymbol{\Theta}} = \phi^\alpha\bar{\mathcal{P}}_\alpha \tag{4.8}$$

are such that equations (4.3) hold. Indeed, this fact can be easily seen from the explicit expressions of $\Delta, \bar{\Delta}$ and Z

$$\Delta = \frac{1}{2}\mathcal{C}^\alpha\omega_{\alpha\beta}\mathcal{C}^\beta\,, \qquad \bar{\Delta} = \frac{1}{2}\bar{\mathcal{P}}_\alpha\omega^{\alpha\beta}\bar{\mathcal{P}}_\beta(-)^{\varepsilon_\beta}\,, \\ Z = \theta_\alpha\phi^\alpha\,. \tag{4.9}$$

The respective BRST and anti-BRST charges Ω and $\bar{\Omega}$ have the form

$$\Omega = \boldsymbol{\Theta} + (i\hbar)^{-1}[Z, \boldsymbol{\Theta}] = \mathcal{C}^\alpha(\theta_\alpha - \omega_{\alpha\beta}\phi^\beta)\,, \\ \bar{\Omega} = \bar{\boldsymbol{\Theta}} + (i\hbar)^{-1}[Z, \bar{\boldsymbol{\Theta}}] = (\phi^\alpha + \theta_\beta\omega^{\beta\alpha}(-1)^{\varepsilon_\alpha})\bar{\mathcal{P}}_\alpha\,. \tag{4.10}$$

Let us also consider explicitly the lowest order equation on the constraints and structure functions encoded in Eqs. (4.3). For simplicity we consider the Poisson bracket version (classical limit) of equations (4.3). We also take all constraints θ_α and $\bar{\theta}_\alpha$ to be bosonic. The most general form of $\boldsymbol{\Theta}^a$ allowed by the ghost number and Grassmann parity prescriptions reads as:

$$\boldsymbol{\Theta}^1 = \mathcal{C}^\alpha\theta_\alpha + \frac{1}{2}\mathcal{C}^\beta\mathcal{C}^\alpha\theta_{\alpha\beta}^\gamma\bar{\mathcal{P}}_\gamma + \ldots\,, \tag{4.11}$$

$$\boldsymbol{\Theta}^2 = \bar{\theta}^\alpha\bar{\mathcal{P}}_\alpha + \frac{1}{2}\mathcal{C}^\gamma\bar{\theta}_\gamma^{\alpha\beta}\bar{\mathcal{P}}_\beta\bar{\mathcal{P}}_\alpha + \ldots\,. \tag{4.12}$$

The expansion for Δ^{ab} then has the form

$$\Delta = -\frac{1}{2}\mathcal{C}^\beta\mathcal{C}^\alpha\Delta_{\alpha\beta} + \ldots\,, \qquad \Delta_{\alpha\beta} = \{\theta_\alpha, \theta_\beta\} - \theta_{\alpha\beta}^\gamma\theta_\gamma \tag{4.13}$$

$$\bar{\Delta} = -\frac{1}{2}\bar{\Delta}^{\alpha\beta}\bar{\mathcal{P}}_\beta\bar{\mathcal{P}}_\alpha + \ldots\,, \qquad \bar{\Delta}^{\alpha\beta} = \{\bar{\theta}^\alpha, \bar{\theta}^\beta\} - \bar{\theta}^\gamma\bar{\theta}_\gamma^{\alpha\beta}\,, \tag{4.14}$$

and

$$
\begin{aligned}
&Z = Z_0 + \mathcal{C}^\alpha Z^\beta_\alpha \bar{\mathcal{P}}_\beta + \dots\,, \\
&Z_0 = \bar{\theta}^\alpha \theta_\alpha\,, \quad Z^\beta_\alpha = \{\theta_\alpha, \bar{\theta}^\beta\} - \bar{\theta}^{\beta\gamma}_\alpha \theta_\gamma - \bar{\theta}^\gamma \theta^\beta_{\gamma\alpha}\,.
\end{aligned} \tag{4.15}
$$

In the lowest orders in ghost variables equations (4.3) then imply

$$
\Delta_{\alpha\beta}\bar{\Delta}^{\beta\gamma} = \delta^\alpha_\gamma\,, \tag{4.16}
$$

$$
\{Z_0, \Delta_{\alpha\beta}\} + Z^\gamma_{[\alpha}\Delta_{\gamma\beta]} = 0\,, \quad \{\bar{\Delta}^{\alpha\beta}, Z_0\} + \bar{\Delta}^{[\alpha\gamma}Z^{\beta]}_\gamma = 0\,. \tag{4.17}
$$

To complete the section let we also present relations between $\Omega^a = \boldsymbol{\Theta}^a + (i\hbar)^{-1}[Z, \boldsymbol{\Theta}^a]$, $\boldsymbol{\Theta}^a$, and Δ^{ab} which are simple consequences of the basic equations (4.3). First, it is easy to see that

$$
[\Omega, \bar{\boldsymbol{\Theta}}] = i\hbar(Z - G)\,, \qquad [\bar{\Omega}, \boldsymbol{\Theta}] = i\hbar(Z + G)\,, \tag{4.18}
$$

$$
[\Omega, \boldsymbol{\Theta}] = 2\,i\hbar\Delta\,, \qquad [\bar{\Omega}, \bar{\boldsymbol{\Theta}}] = 2\,i\hbar\bar{\Delta}\,, \tag{4.19}
$$

or in the short-hand notations

$$
[\Omega^a, \boldsymbol{\Theta}^b] = i\hbar(2\Delta^{ab} - \epsilon^{ab}G)\,.
$$

One further arrives at

$$
[Z, \Omega^a] = i\hbar\Omega^a \tag{4.20}
$$

$$
[\Delta, \Omega] = 0\,, \qquad [\bar{\Delta}, \bar{\Omega}] = 0\,. \tag{4.21}
$$

and

$$
[\Omega, \bar{\Delta}] = i\hbar\bar{\Omega}\,, \qquad [\bar{\Omega}, \Delta] = i\hbar\Omega\,. \tag{4.22}
$$

An important consequence of the above relations is the fact that operator algebra generated by $\boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}$ is a finite-dimensional Lie superalgebra that contains all the generating operators. As a basis in the algebra one can take the following operators:

$$
\boldsymbol{\Theta}, \bar{\boldsymbol{\Theta}}, G, \Delta, \bar{\Delta}, Z, \Omega, \bar{\Omega}\,.
$$

5. Hamiltonian

Besides BRST generators Ω^a formulation of the quantum theory requires specification of a unitarizing Hamiltonian. The first step is to introduce pre-Hamiltonian $\mathcal{H}$ that satisfies

$$
[\Omega^a, \mathcal{H}] = 0\,, \qquad \varepsilon(H) = 0\,, \quad \mathrm{gh}(\mathcal{H}) = 0 \tag{5.1}
$$

and whose expansion with respect to ghost variables starts as

$$
\mathcal{H} = H + \mathcal{C}^\alpha V^\beta_\alpha \bar{\mathcal{P}}_\beta (-1)^{\varepsilon_\beta} + \dots\,.
$$

Here, the ghost-independent term H is an "original Hamiltonian". In the lowest orders in ghost variables classical limit of Eq. (5.1) implies

$$
\{T_\alpha, H\} = V^\beta_\alpha T_\beta\,, \qquad \{\bar{T}^\alpha, H\} = -\bar{T}^\beta V^\alpha_\beta\,.
$$

Here, we made use explicit expansions (2.10) of Ω^1 and Ω^2 with respect to the ghost variables.

A next step in gauge fixing is to introduce nonminimal variables $\bar{\mathcal{C}}_\alpha, \mathcal{P}^\alpha$ and $\lambda^\alpha, \pi_\alpha$. They are assumed to have the standard Grassmann parities and ghost numbers

$$\begin{gathered} \varepsilon(\mathcal{P}^\alpha) = \varepsilon(\bar{\mathcal{C}}_\alpha) = \varepsilon_\alpha + 1\,, \qquad \varepsilon(\lambda^\alpha) = \varepsilon(\pi_\alpha) = \varepsilon_\alpha \\ \mathrm{gh}(\mathcal{P}^\alpha) = 1\,, \quad \mathrm{gh}(\bar{\mathcal{C}}_\alpha) = -1\,, \qquad \mathrm{gh}(\lambda^\alpha) = 0\,, \quad \mathrm{gh}(\pi_\alpha) = 0\,, \end{gathered} \tag{5.2}$$

and are also subjected to the following commutation relations

$$[\mathcal{P}^\alpha, \bar{\mathcal{C}}_\beta] = i\hbar\, \delta^\alpha_\beta\,, \qquad [\lambda^\alpha, \pi_\beta] = i\hbar\, \delta^\alpha_\beta\,. \tag{5.3}$$

Nonminimal BRST charges are introduced according to

$$Q = \Omega + \pi_\alpha \mathcal{P}^\alpha\,, \qquad \bar{Q} = \bar{\Omega} + \bar{\mathcal{C}}_\alpha \lambda^\alpha\,. \tag{5.4}$$

We use unified notation Q^a with $Q^1 = Q$ and $Q^2 = \bar{Q}$. Unlike the $sp(2)$ symmetric formalism [14] these nilpotent charges do not commute:

$$[Q, \bar{Q}] = \frac{1}{2} g_{ab} [Q^a, Q^b] = i\hbar\,(\pi_\alpha \lambda^\alpha + \bar{\mathcal{C}}_\alpha \mathcal{P}^\alpha)\,. \tag{5.5}$$

The constrained system under consideration can be described just by nilpotent BRST charge Ω and BRST-anti-BRST-invariant Hamiltonian $\mathcal{H}$. (it can be also equivalently described by $\bar{\Omega}$ and $\mathcal{H}$). Then there exists two natural proposals for unitarizing Hamiltonian

$$\mathcal{H}_{complete} = \mathcal{H} + (i\hbar)^{-1}[Q, \bar{\Psi}] \qquad \text{and} \qquad \bar{\mathcal{H}}_{complete} = \mathcal{H} + (i\hbar)^{-1}[\bar{Q}, \Psi] \qquad ,$$

where Ψ and $\bar{\Psi}$ are respective gauge fixing fermions. In order to remove gauge degeneracy it is sufficient to take Ψ and $\bar{\Psi}$ in the "minimal" form:

$$\bar{\Psi} = \bar{\mathcal{C}}_\alpha \chi^\alpha + \bar{\mathcal{P}}_\alpha \lambda^\alpha\,, \qquad \Psi = \bar{\chi}_\alpha \bar{\mathcal{P}}^\alpha + \pi_\alpha \mathcal{C}^\alpha\,, \tag{5.6}$$

where χ and $\bar{\chi}$ are gauge conditions which are supposed to be nondegenerate in the following sense: matrices

$$\{\chi^\alpha, T_\beta\}\,, \qquad \{\bar{\chi}_\alpha, \bar{T}^\beta\}$$

are nondegenerate. One can see that χ and $\bar{\chi}$ are to be understood as gauge fixing conditions associated to constraints T and $\bar{T}$ respectively.

In spite of the fact that each pair $Q, \bar{\Psi}$ or $\bar{Q}, \Psi$ can be used to describe correct physics we are interested in the class of unitarizing Hamiltonians (and gauge fixing conditions) respecting the symmetry between Q and $\bar{Q}$. To this end let us consider a class of $\mathcal{H}_{complete}$ corresponding to $\bar{\Psi}$ of the form $\bar{\Psi} = (i\hbar)^{-1}[\bar{Q}, F]$ for some even operator F of ghost number -2. The expression for $\mathcal{H}_{complete}$ then takes the form

$$\mathcal{H}_{complete} = \mathcal{H} + (i\hbar)^{-2}[Q, [\bar{Q}, F]]\,. \tag{5.7}$$

It can be represented as

$$\begin{aligned} \mathcal{H}_{complete} &= \mathcal{H} - \frac{1}{2}(i\hbar)^{-2}\epsilon_{ab}[Q^a, [Q^b, F]] + \frac{1}{4}(i\hbar)^{-2} g_{ab}[[Q^a, Q^b], F] = \\ &= \mathcal{H} - \frac{1}{2}(i\hbar)^{-2}\epsilon_{ab}[Q^a, [\bar{Q}^b, F]] + \frac{1}{2}(i\hbar)^{-2} g_{ab}[\pi_\alpha \lambda^\alpha + \bar{\mathcal{C}}_\alpha \mathcal{P}^\alpha, F]\,. \end{aligned} \tag{5.8}$$

We note that besides the third term the expression for $\mathcal{H}_{complete}$ coincides with that in the $sp(2)$-symmetric formalism. However, presence of this (actually unavoidable) term reflects breaking down of the $sp(2)$ symmetry. The residual symmetry is that preserving both ϵ_{ab} and g_{ab} and its algebra can be shown to be isomorphic to $u(1)$.

We also note that discussion of the unitarizing Hamiltonian also applies to any commuting and nilpotent BRST charges Ω^a with appropriate ghost numbers and boundary conditions. In particular, the same expression works for a wider class of BRST charges considered in Sec. **2**.

6. Conclusion

We have constructed a conversion scheme that results in the BRST–anti-BRST symmetric formulation of the effective first-class constrained (gauge) system. This is achieved by requiring an explicit symmetry between the original second-class constraints θ_α and the extra degrees of freedom (conversion variables). These degrees of freedom enter the formalism as new constraints $\bar{\theta}^\alpha$ dual to the original constraints θ_α. In the BFV–BRST framework, the two sets of constraints are encoded in a pair of generators $\boldsymbol{\Theta}$ and $\bar{\boldsymbol{\Theta}}$. At the level of the converted system, they are promoted to nilpotent and commuting generators $\boldsymbol{\Omega}$ and $\bar{\boldsymbol{\Omega}}$ that are identified as BRST and anti-BRST charges. The symmetry between the constraints θ and $\bar{\theta}$ then appears as a BRST-anti-BRST invariance of the effective system.

It seems natural that generating equations of the BRST-anti-BRST symmetric conversion can be recast to a master equation form if one makes use of the quantum antibracket structure [11] (see also [15]). Indeed, a quantum antibracket master equation encodes projector properties of the "master action" (either $\boldsymbol{Z}$ or S^a_b in the paper), which allow constructing nilpotent generators $\boldsymbol{\Omega}$ and $\bar{\boldsymbol{\Omega}}$ from the original generators $\boldsymbol{\Theta}$ and $\bar{\boldsymbol{\Theta}}$. From this general standpoint, the conversion can be understood as a certain projection procedure.

The existence of the BRST–anti-BRST symmetric conversion follows from the explicit solution (4.10) given in Sec. **4** for the "Abelian" constraints (in the general situation, one can always bring constraints to an "Abelian" basis and then apply the inverse transformation to the generating operators. Therefore, there always exists a solution to the generating equations that correctly describes the physical sector). But there remains an open problem to describe the structure of a general solution to the generating equations. The difficulties are here related to a rather nonstandard structure of the equations.

As a final remark, it is worth mentioning another point of view on the method developed in the paper. Instead of treating the dual constraints $\bar{\theta}_\alpha$ as extra variables, we can assume that the second-class system at hand is determined by a set of second-class constraints that splits into two halves θ_α and $\bar{\theta}_\alpha$ such that each half is itself a set of second-class constraints. In this case, our approach results in a BRST–anti-BRST symmetric effective gauge system that involves the constraints θ and $\bar{\theta}$ in a symmetric way. In this approach, however, the constraints θ and $\bar{\theta}$ are required to satisfy some additional relations coming from generating equations.

Acknowledgments. We are grateful to A.M. Semikhatov and I.V. Tyutin for illuminating discussions. The work of I.A. Batalin was supported by the President grant 00-15-96566, RFBR grant 02-02-16944, and INTAS grant 00-00262. The work of M.A. Grigoriev was supported by the RFBR grant 02-01-00930, INTAS grant 00-00262, and RFBR grant 02-01-06096 for support of young scientists.

References

[1] Batalin I.A. and Vilkovisky G.A., *Relativistic S-matrix of dynamical systems with boson and fermion constraints,* Phys. Lett. **B 69** (1977) 309-312.
Fradkin E.S. and Vilkovisky G.A. *Quantization of relativistic systems with constraints,* Phys. Lett. **B 55** (1975) 224-226.
Batalin I.A. and Fradkin E.S., *A generalized canonical formalism and quantization of reducible gauge theories,* Phys.Lett. **B 122** (1983) 157-164.
Batalin I.A. and Fradkin E.S. *Operatorial quantization of relativistic dynamical systems subject to first class constraints,* Phys.Lett. **B 128** 303-308 (1983).
Batalin I.A., Fradkin E.S. *Operatorial quantization of dynamical systems subject to constraints. A further study of the construction,* Ann Inst. Henri Poincaré **49** (1988) 145-214.

[2] L.D. Faddeev, S.L. Shatashvili, *Realization of the Schwinger term in the Gauss law and the possibility of correct quantization of a theory with anomalies*, Phys. Lett. **B 167**, 225 (1986)

[3] I.A. Batalin and E.S. Fradkin, *Operator Quantization Of Dynamical Systems With Irreducible First And Second Class Constraints,* Phys. Lett. **B180**, 157 (1986).

[4] I.A. Batalin and E.S. Fradkin, *Operatorial Quantization Of Dynamical Systems Subject To Second Class Constraints,* Nucl. Phys. **B279**, 514 (1987).

[5] I. A. Batalin, S. L. Lyakhovich and I. V. Tyutin, *"Split Involution And Second Class Constraints,"* Mod. Phys. Lett. A **7**, 1931 (1992).

[6] I. Batalin and R. Marnelius, *"Gauge theory of second class constraints without extra variables,"* Mod. Phys. Lett. A **16**, 1505 (2001) [arXiv:hep-th/0106087].

[7] I.A. Batalin and I.V. Tyutin, *Existence theorem for the effective gauge algebra in the generalized canonical formalism with Abelian conversion of second class constraints,* Int. J. Mod. Phys. **A6**, 3255 (1991).

[8] I.A. Batalin, E.S. Fradkin and T.E. Fradkina, *Generalized Canonical Quantization Of Dynamical Systems With Constraints And Curved Phase Space,* Nucl. Phys. **B332**, 723 (1990).

[9] I. A. Batalin, M. A. Grigoriev and S. L. Lyakhovich, *"Star product for second class constraint systems from a BRST theory,"* Theor. Math. Phys. **128**, 1109 (2001) [Teor. Mat. Fiz. **128**, 324 (2001)] [arXiv:hep-th/0101089].

[10] I. Batalin, S. Lyakhovich and R. Marnelius, Phys. Lett. B **534**, 201 (2002) [arXiv:hep-th/0112175].

[11] I. Batalin and R. Marnelius, *"General quantum antibrackets,"* Theor. Math. Phys. **120**, 1115 (1999) [Teor. Mat. Fiz. **120**, 358 (1999)] [arXiv:hep-th/9905083]. *"Quantum antibrackets,"* Phys. Lett. B **434**, 312 (1998) [arXiv:hep-th/9805084].

[12] I. A. Batalin and I. V. Tyutin, Phys. Lett. B **302**, 47 (1993).

[13] I. A. Batalin and I. V. Tyutin, Phys. Lett. B **317**, 354 (1993) [arXiv:hep-th/9305167].
Canonical Formalism With Abelian Conversion Of Second Class

[14] I. A. Batalin, P. M. Lavrov and I. V. Tyutin, *"Extended Brst Quantization Of Gauge Theories In The Generalized Canonical Formalism,"* J. Math. Phys. **31**, 6 (1990).

[15] Igor Batalin and Igor Tyutin, *"BRST-Invariant Constraint Algebra in Terms of Commutators and Quantum Antibrackets"*, [arXiv:hep-th/0301043].

Lebedev Physics Institute, RAS, Leninskiy 53, Moscow 117924, Russia