Exploring the second class constraint quantization approach proposed by Batalin and Marnelius

Michael Chesterman*

The Physics Department,

Queen Mary,

University of London,

Mile End Road, E1 4NS, UK

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Abstract

I extend upon the paper by Batalin and Marnelius, in which they show how to construct and quantize a gauge theory from a Hamiltonian system with second class constraints. Among the avenues explored, their technique is analyzed in relation to other well-known methods of quantization and a bracket is defined, such that the operator formalism can be fully developed. I also extend to systems with mixed class constraints and look at some simple examples.

1 Introduction

The subject of this work concerns a paper [?] by Batalin and Marnelius, in which they propose a novel way of constructing a gauge theory from a Hamiltonian system with second class constraints. Their work extends on ideas first developed in [?].

The covariant quantization of general Hamiltonian systems, with second class constraints, is no easy task. Two of the best methods so far are 'constraint conversion' [?, ?, ?] and 'gauge unfixing' [?, ?]. The first introduces additional variables in such a way that the second class constraints are converted into first class ones. The second on the other hand, finds a subset of half of the constraints in involution with each other, and then discards the remaining constraints as 'gauge fixing' conditions, thus leaving a first class constraint surface. The resultant gauge theories can be quantized using the full power of Hamiltonian BRST (BFV) quantization. However, even the above two methods are limited. For example, 'gauge unfixing' is only possible locally in general.

Although the authors of the subject paper [?] constructed the correct path integral following the superfield version of antifield (BV) quantization, there were some

*M.J.Chesterman@qmul.ac.uk

stones left unturned. In particular, an operator formalism was not developed and the question of the position of the technique, in relation both to standard Dirac bracket quantization and the above two approaches, was not fully explored. In the work which follows, I explore these points and others in some detail and also extend to systems with mixed class constraints.

I do not intend to review the subject paper in any depth, as this would be superfluous. Rather, this work is designed to be read in conjunction with the subject paper. However, all equations and identities from the subject paper that I use are also shown here, either in section 2, or as needed in the text. To avoid confusion, I also use the same notation.

2 Some Preliminaries

2.1 The Setting

We start, as in [?], with symplectic supermanifold $\mathbb{M}$ which has co-ordinates x^i, $i = 1, \ldots, 2N$ with $\varepsilon^i = \varepsilon(x^i)$. The non-degenerate symplectic two-form $\omega_{ij}(x)$ is closed

$$d\omega = 0 \Leftrightarrow \partial_i \omega_{jk}(x)(-1)^{(\varepsilon^i+1)\varepsilon^k} + cycle(i,j,k) = 0 \tag{2.1}$$

and has properties

$$\omega_{ij}(x) = \omega_{ji}(x)(-1)^{(\varepsilon^i+1)(\varepsilon^j+1)} \qquad \varepsilon(\omega_{ij}) = \varepsilon^i + \varepsilon^j. \tag{2.2}$$

Since ω_{ij} is non-degenerate, there exists an inverse $\omega^{ij}(x)$, in terms of which the Poisson bracket is defined by

$$\{A(x), B(x)\} = A(x)\overleftarrow{\partial}_i \omega^{ij}(x) \overrightarrow{\partial}_j B(x). \tag{2.3}$$

w^{ij} obeys

$$\omega^{ij}(x) = -\omega^{ij}(x)(-1)^{\varepsilon^i \varepsilon^j} \qquad \varepsilon(\omega^{ij}) = \varepsilon^i + \varepsilon^j. \tag{2.4}$$

The Poisson bracket (2.3) satisfies the Jacobi identities since (2.1) implies

$$\omega^{ij}\overleftarrow{\partial}_l \omega^{lk}(-1)^{\varepsilon^i \varepsilon^k} + cycle(i,j,k) = 0. \tag{2.5}$$

We consider a Hamiltonian $H(x)$, with $2M < 2N$ irreducible second class constraints $\theta^\alpha(x)$, which satisfy regularity condition

$$\text{Rank } \theta^\alpha(x) \frac{\overleftarrow{\partial}}{\partial x^i}|_{\theta=0} = 2M \tag{2.6}$$

and class condition

$$\text{Rank } \{\theta^\alpha(x), \theta^\beta(x)\}|_{\theta=0} = 2M. \tag{2.7}$$

Constraints $\theta^\alpha = 0$ determine a constraint surface Γ, which is itself a symplectic supermanifold of dimension $(2N - 2M)$.

The equations of motion can be calculated from the extended Hamiltonian form of the action

$$S_E[x] = \int dt\, x^i \bar{\omega}_{ij}(x)\dot{x}^j - H(x) - \lambda_\alpha \theta^\alpha(x), \tag{2.8}$$

where [?]

$$\bar{\omega}_{ij}(x) = \int_0^1 \omega_{ij}(\alpha x)\alpha\, d\alpha \Rightarrow \omega_{ij}(x) = (x^k \overrightarrow{\partial}_k + 2)\bar{\omega}_{ij}(x). \tag{2.9}$$

Equations (2.1) and (2.9) yield the required property that

$$\delta \int dt\, x^i \bar{\omega}_{ij}(x)\dot{x}^j = \int dt\, \delta x^i \omega_{ij}(x)\dot{x}^j. \tag{2.10}$$

In general, x^i will be canonical co-ordinates with constant ω^{ij} and $\bar{\omega}^{ij} = \frac{1}{2}\omega^{ij}$.

In the subject paper, the authors parameterize the constraint surface Γ with the original phase space co-ordinates x^i, constructing functions $\bar{x}^i(x)$ satisfying

$$\theta^\alpha(\bar{x}^i(x)) = 0 \tag{2.11}$$

$$\bar{x}^i(\tilde{x}) = \tilde{x}^i \qquad \text{for } \tilde{x}^i \in \Gamma. \tag{2.12}$$

As there are $2M$ more co-ordinates than the dimension of Γ, there are $2M$ gauge symmetries denoted by (2.22). The BM gauge invariant action is

$$S[\bar{x}(x)] = \int dt\, \bar{x}^i \bar{\omega}_{ij}(\bar{x})\dot{\bar{x}}^j - H(\bar{x}). \tag{2.13}$$

2.2 Equations and Identities from the BM paper

$$\theta^\alpha(x)\overleftarrow{\partial}_i Z^i_\beta(x) = \delta^\alpha_\beta \tag{2.14}$$

$$P^i_j(x) = \delta^i_j - Z^i_\alpha(x)(\theta^\alpha(x)\overleftarrow{\partial}_j) \tag{2.15}$$

$$P^i_j(x)P^j_k(x) = P^i_k(x) \tag{2.16}$$

$$P^i_j(x)Z^j_\alpha(x) = 0 \tag{2.17}$$

$$\frac{d\tilde{x}^i}{d\lambda} = Z^i_\alpha(\tilde{x})\theta^\alpha(\tilde{x}) \tag{2.18}$$

$$\bar{x}^i(x)\overleftarrow{\partial}_k = P^i_m(\bar{x})\sigma^m_k(x) \tag{2.19}$$

$$\sigma^m_k(\bar{x}) = \delta^i_k \qquad \det \sigma^i_k(x) \neq 0 \tag{2.20}$$

$$\sigma^m_k(x) \longrightarrow \sigma^m_k(x) + Z^m_\alpha(\bar{x})M^\alpha_k(x) \tag{2.21}$$

$$\bar{x}^i(x)\overleftarrow{\partial}_k G^k_\alpha(x) = 0 \qquad G^k_\alpha(x) = (\sigma^{-1})^k_m Z^m_\alpha(\bar{x}(x)) \tag{2.22}$$

3 Conditions for existence of continuous function $\bar{x}(x)$

In sections 3 and 4 of the subject paper[?], the authors argue that given second class constraint functions $\theta^\alpha(x)$ obeying regularity conditions (2.6) and (2.7), the continuous function $\bar{x}^i(x)$ always exists. In fact, a small modification of their argument is necessary. The mathematics required to cast light on this point is well known [?]. Consider a generic constraint surface Γ, a submanifold of $\mathcal{M}$. If a continuous function $\bar{x} : \mathcal{M} \to \Gamma$ exists, obeying (2.11) and (2.12), then $\bar{x}$ is said to be a retraction and Γ a retract of $\mathcal{M}$. Furthermore, if there exists a continuous map $H : \mathcal{M} \times I \to \mathcal{M}$, where I is the interval $[0, 1]$, such that

$$H(x, 0) = x \quad H(x, 1) \in \Gamma \quad \text{for any } x \in \mathcal{M} \tag{3.1}$$
$$H(x, s) = x \quad \text{for any } x \in \Gamma \quad \text{and any } s \in I \tag{3.2}$$

then Γ is said to be a deformation retract of $\mathcal{M}$. Not all retracts are deformation retracts, however, it's reasonable here to just check for a deformation retract with $H(x, 1) = \bar{x}(x)$. The properties (3.1) and (3.2) of $H(x, s)$ imply that the identity function on $\mathcal{M}$ is homotopic to the function $\bar{x}$. Thus $\mathcal{M}$ and Γ have the same homotopy type. In other words, $\mathcal{M}$ and Γ must have the same fundamental group:

$$\pi_1(\mathcal{M}) = \pi_1(\Gamma). \tag{3.3}$$

This topological condition can be translated into a restriction on the constraint functions. Batalin and Marnelius explicitly construct $\bar{x}^i(x)$ as a deformation retract, via the differential equation (2.18) for continuous function $\tilde{x}^i(\lambda, x)$ in section 4 of the subject paper. Given initial conditions $\tilde{x}^i(0, x) = x^i$, the authors define

$$\bar{x}^i(x) = \lim_{\lambda \to -\infty} \tilde{x}^i(\lambda, x). \tag{3.4}$$

By construction, this definition obeys both (2.11) and (2.12). The differential equation (2.18) uniquely dictates the path in $\mathcal{M}$ that the point x^i takes to $\bar{x}^i(x)$ which is traced by $\tilde{x}^i(\lambda, x)$ over the interval $\lambda = [0, -\infty]$. We can see that $\tilde{x}(\lambda, x)$ is a homotopy as defined in (3.1) and (3.2) if we substitute $s = 1 - e^\lambda$, since

$$\tilde{x}(s = 0, x) = x \qquad \tilde{x}(s = 1, x) = \bar{x}(x). \tag{3.5}$$

The only problem with the Batalin Marnelius definition of $\bar{x}(x)$ is that $Z^i_\alpha(x)$ can be singular off the constraint surface in the general case. The requirement (2.6) on the matrix $\theta^\alpha(x)\overleftarrow{\partial}_i$ only applies on the constraint surface. Away from the surface, $\text{Rank } (\theta^\alpha(x)\overleftarrow{\partial}_i) \leq 2M$, but $Z^i_\alpha(x)$ obeying (2.14) can only be defined when

$$\text{Rank } \theta^\alpha(x)\overleftarrow{\partial}_i = 2M \quad \text{for any } x \in \mathcal{M}. \tag{3.6}$$

The above equation is actually equivalent to (3.3). So long as $Z^i_\alpha(x)$ can be smoothly defined, the differential equation for $\tilde{x}(\lambda, x)$ leads to a unique and smooth function $\bar{x}(x)$ in terms of $Z^i_\alpha(x)$ and $\theta^\alpha(x)$. The points $x \in \mathcal{M}$ where (3.6) doesn't hold correspond to singularities in $Z^i_\alpha(x)$ and discontinuities or singularities in $\bar{x}(x)$. In order to illustrate the connection between (3.3) and (3.6), consider a simple example. Suppose $\mathcal{M} = \mathbb{R}^2 \times \mathbb{R}^2$, with canonical co-ordinates (q^i, p_i), $i = 1, 2$. Also, suppose there's one first class constraint $\theta = (q^1)^2 + (q^2)^2 - R^2 = 0$ where R is a constant. The class of θ isn't important. The constraint surface $\Gamma = S^1 \times \mathbb{R}^2$. If we forget the co-ordinates (p_1, p_2) for now, clearly $\pi_1(S^1) \neq \pi_1(\mathbb{R}^2)$ and one can expect a singular point in $\bar{x}(x)$ anywhere inside the circle, depending on the definition of $\bar{x}(x)$. In fact, the singular point is determined by θ. $\text{Rank } \theta(x)\overleftarrow{\partial}_i = 1$ everywhere, except at singular point $q^1 = q^2 = 0$. The solution is to exclude $q^1 = q^2 = 0$ from $\mathcal{M}$, then $\pi_1(\mathbb{R}^2 - \{0\}) = \pi_1(S^1)$.

One should ask how severe the restriction (3.6) is. In order to use the Dirac bracket for second class constraints $\theta^\alpha(x)$, the matrix $\{\theta^\alpha, \theta^\beta\}$ must be invertible everywhere in $\mathcal{M}$, i.e.

$$\text{Rank } \{\theta^\alpha(x), \theta^\beta(x)\} = 2M \quad \text{for any } x \in \mathcal{M}. \tag{3.7}$$

Now since

$$\{\theta^\alpha, \theta^\beta\} = \theta^\alpha \overleftarrow{\partial}_i \omega^{ij} \overrightarrow{\partial}_j \theta^\beta, \tag{3.8}$$

we can write an inequality for the rank of the matrix $\theta^\alpha \overleftarrow{\partial}_i$ using the following result. For $N \times N$ matrices A_i $i = 1, \ldots, n$ and B where

$$A_1 A_2 \ldots A_n = B, \tag{3.9}$$

it follows that

$$\max(\text{Ker } A_1, \ldots, \text{Ker } A_n) \leq \text{Ker } B \leq \sum_{i=1}^{n} \text{Ker } A_i, \tag{3.10}$$

where $\text{Ker } A_i = N - \text{Rank } A_i$. All matrices in (3.8) can be made square $2N \times 2N$ by filling with zeros as necessary. The resultant inequality is

$$\text{Rank } \theta^\alpha \overleftarrow{\partial}_i \geq \text{Rank } \{\theta^\alpha, \theta^\beta\} \geq \text{Rank } \theta^\alpha \overleftarrow{\partial}_i - \text{Ker } \theta^\alpha \overleftarrow{\partial}_i, \tag{3.11}$$

thus (3.7) implies (3.6). The BM technique can be used for all second class systems where a Dirac bracket can be smoothly defined. Usually $\mathcal{M} \equiv R^n \times R^n$ which has trivial fundamental group $\pi_1(\mathcal{M})$. This might seem to severely limit the possible constraint surfaces that can be dealt with. However, exclusion of strategic point(s), or in general, hyperplanes from $\mathcal{M}$, may sometimes be possible, with drastic effects on $\pi_1(\mathcal{M})$. These points are ones where (3.6) doesn't hold. This is dynamically allowed if the excluded points are fixed points.

4 A suitable bracket on $\mathcal{M}$

In a previous paper [?], a third condition was placed on $\bar{x}^i(x)$

$$\{\bar{x}^i(x), \bar{x}^j(x)\} = \{x^i, x^j\}_D|_{x \to \bar{x}(x)}, \tag{4.1}$$

where $\{,\}$ and $\{,\}_D$ are the Poisson and Dirac bracket respectively on $\mathbb{M}$. In the subject paper [?], condition (4.1) was considered to restrict the choice of gauge theory and was removed. In this spirit, one can instead search for a new bracket $\{,\}_M$ on $\mathbb{M}$ with the property

$$\{A(\bar{x}(x)), B(\bar{x}(x))\}_M = \{A(x), B(x)\}_D|_{x\to\bar{x}(x)}, \tag{4.2}$$

where $A(\bar{x}(x))$ and $B(\bar{x}(x))$ are arbitrary, gauge invariant observables. So

$$\{\bar{x}^i(x), \bar{x}^j(x)\}_M = \{x^i, x^j\}_D|_{x\to\bar{x}(x)}. \tag{4.3}$$

Utilizing (2.19), equation (4.3) may be written as

$$P^i_m(\bar{x})\sigma^m_k(x)\omega^{kl}_M(x)\sigma^n_l(x)P^j_n(\bar{x}) = \omega^{ij}_D(\bar{x}), \tag{4.4}$$

where we define

$$\{a(x), b(x)\}_M = a(x)\overleftarrow{\partial}_i\omega^{ij}_M(x)\overrightarrow{\partial}_j b(x) \tag{4.5}$$

and similarly for the Dirac bracket which is characterized by

$$\omega^{ij}_D(x) = \omega^{ij}(x) - \{x^i, \theta^\alpha\}\{\theta^\alpha, \theta^\beta\}^{-1}\{\theta^\beta, x^j\}. \tag{4.6}$$

Note that $\text{Rank } P^i_m(\bar{x}) = 2N - 2M$, so $\omega^{ij}_M(x)$ isn't uniquely defined and furthermore this is also reflected by the ambiguity in choosing $\sigma^m_k(x)$ shown in (2.21). Firstly, a particular solution for $\omega^{ij}_M(x)$ must be found. Since $\{\theta^\alpha, x^j\}_D = 0$ and making use of the explicit expression for $P^i_j(x)$ in (2.15),

$$P^i_k(x)\omega^{kl}_D(x)P^j_l(x) = \omega^{ij}_D(x). \tag{4.7}$$

It soon becomes clear that a particular solution is

$$\omega^{ij}_M(x) = (\sigma^{-1})^i_k(x)\omega^{kl}_D(\bar{x})(\sigma^{-1})^j_l(x). \tag{4.8}$$

In order to obtain the general solution for ω^{ij}_M we add a term Δ^{ij} such that

$$P^i_k(\bar{x})\Delta^{kl}_\sigma(x)P^j_l(\bar{x}) = 0, \tag{4.9}$$

where $\Delta^{kl}_\sigma = (\sigma\Delta\sigma)^{kl}$ and where we also require that $\Delta^{ij} = (-)(-)^{\varepsilon^i\varepsilon^j}\Delta^{ji}$. The $2M$ vectors $Z^i_\alpha(x)$ form a basis for the kernel of the matrix $P^i_k(x)$ as can be seen from (2.17). The most general such expression is

$$\Delta^{kl}_\sigma(x) = Z^k_\alpha(\bar{x})J^{\alpha l}(x) - (-1)^{\varepsilon^{\alpha l}(1+\varepsilon^k+\varepsilon^l)}J^{\alpha k}(x)Z^l_\alpha(\bar{x}), \tag{4.10}$$

where $J^{\alpha k}(x)$ is some arbitrary expression such that $\varepsilon(J^{\alpha k}) = \varepsilon^k - \varepsilon^\alpha$. Note that freedom in the choice of $\Delta^{ij}(x)$ means freedom in the choice of $\text{Rank } \omega^{ij}_M$. One can have $\text{Rank } \omega^{ij}_M = 2N - 2M$ as in (4.8), or choose $\Delta^{ij}(x)$ such that $\text{Rank } \omega^{ij}_M = 2N$.

The bracket $\{,\}_M$ should also satisfy the Jacobi identity. If only gauge invariant observables of the form $A(\bar{x}(x))$ are ever used inside the bracket we simply require that

$$\begin{aligned}&\{\{A(\bar{x}), B(\bar{x})\}_M, C(\bar{x})\}_M\\ &+ (-)^{\varepsilon^A(\varepsilon^B+\varepsilon^C)}\{\{B(\bar{x}), C(\bar{x})\}_M, A(\bar{x})\}_M\\ &+ (-)^{\varepsilon^C(\varepsilon^A+\varepsilon^B)}\{\{C(\bar{x}), A(\bar{x})\}_M, B(\bar{x})\}_M = 0.\end{aligned} \tag{4.11}$$

Since $P^i_k(\bar{x})\omega^{kj}_D(\bar{x}) = \omega^{ij}_D(\bar{x})$, one can carefully show that

$$\{\{A(\bar{x}), B(\bar{x})\}_M, C(\bar{x})\}_M = \{\{A(x), B(x)\}_D, C(x)\}_D|_{x\to\bar{x}}. \tag{4.12}$$

As the Dirac bracket obeys the Jacobi identity, so (4.11) is true trivially. However, if one wants to allow general functions inside the bracket too, the Jacobi condition is considerably harder to solve and a full solution won't be necessary here.

One notable property of $\{,\}_M$ is

$$\{A(\bar{x}), B(\bar{x})\}_M \overleftarrow{\partial}_i G^i_\alpha(x) = 0, \tag{4.13}$$

since $\{A(\bar{x}), B(\bar{x})\}_M$ is a function of $\bar{x}$ as can be seen in (4.2).

The bracket $\{,\}_M$ actually originates from the uniquely defined, degenerate, symplectic two-form

$$\omega_{M\,ij}(x) = (\overrightarrow{\partial}_i \bar{x}^k)\,\omega_{kl}(\bar{x}(x))\,(\bar{x}^l \overleftarrow{\partial}_j). \tag{4.14}$$

One can see for example that (2.13) defines equations of motion

$$\omega_{M\,ij}(x)\dot{x}^j = \overrightarrow{\partial}_i H(\bar{x}(x)) \tag{4.15}$$

The fact that $\omega_{M\,ij}$ is degenerate means that $\dot{x}^j$ isn't unique and we can check, using the identities in section 2.2 and the explicit expression for the Dirac bracket, that

$$\dot{x}^j = \{x^j, H(\bar{x}(x))\}_M \tag{4.16}$$

defines the general solution.

5 A review of the difficulties with the Dirac bracket

It will be useful to review the explicit reasons why, in general, the Dirac bracket quantization method fails [?]. There are various ways to attempt operator quantization of the system described in section 2.1. One way is to find a quantum representation of the Dirac bracket, with second class constraints imposed as operator equations. Another, possibly simpler, approach is to get rid of the constraint conditions in the classical system before quantization, by parameterizing the constraint surface Γ with $2N - 2M$ independent co-ordinates:

$$y^\mu \qquad \varepsilon(y^\mu) = \varepsilon^\mu, \tag{5.1}$$

where there exists a one-to-one continuous mapping $\bar{x}^i(y)$, obeying

$$\theta^\alpha(\bar{x}(y)) = 0. \tag{5.2}$$

The non-degenerate, induced, symplectic two-form is

$$\omega_{\mu\nu}(y) = (\overrightarrow{\partial}_\mu \bar{x}^i)\,\omega_{ij}(\bar{x}(y))\,(\bar{x}^j \overleftarrow{\partial}_\nu). \tag{5.3}$$

All observables $f(y)$ can be written in the form $F(\bar{x}(y))$. We now have a system with no constraints, action $S_E[\bar{x}(y)]$ and in general, non-trivial Poisson bracket constructed with $\omega^{\mu\nu}(y)$, the inverse of $\omega_{\mu\nu}(y)$. It's well known [?] that

$$\{f(y), g(y)\}_* = \{F(x), G(x)\}_D|_{x\to\bar{x}(y)} \tag{5.4}$$

for

$$f(y) = F(\bar{x}(y)) \qquad g(y) = G(\bar{x}(y)) \tag{5.5}$$

$$\{f(y), g(y)\}_* = f(y)\overleftarrow{\partial}_\mu \omega^{\mu\nu}(y)\overrightarrow{\partial}_\nu g(y). \tag{5.6}$$

We now look for a consistent representation of the bracket (5.7) below, for independent operators $\hat{y}^\mu$:

$$[\hat{y}^\mu, \hat{y}^\nu] = i\hbar\omega^{\mu\nu}(\hat{y}) \tag{5.7}$$

where

$$[\hat{y}^\mu, \hat{y}^\nu] = \hat{y}^\mu\hat{y}^\nu - (-)^{\varepsilon^\mu\varepsilon^\nu}\hat{y}^\nu\hat{y}^\mu. \tag{5.8}$$

The problem is that no such consistent representation exists in general, except in particular cases, e.g. when $\omega^{\mu\nu}$ is constant or where $\hat{y}^\mu$ form an algebra in the bracket $[,]$.

The situation is no easier from the path integral approach. One can write the path integral in terms of y^μ

$$\int [Dy^\mu]\,\mathrm{sdet}(\omega_{\mu\nu})^{1/2} \exp\frac{i}{\hbar}\int dt\, S_E[\bar{x}(y)], \tag{5.9}$$

or in terms of the original co-ordinates x^i,

$$\int [Dx^i][D\lambda^\alpha]\,\mathrm{sdet}(\omega_{ij})^{-1/2}\mathrm{sdet}(\{\theta^\alpha,\theta^\beta\})^{1/2} \exp\frac{i}{\hbar}\int dt\, S_E[x]. \tag{5.10}$$

Both expressions may look deceptively simple. However, in the co-ordinate representation for example, the problem is in the boundary conditions. In the co-ordinate representation, both (5.9) and (5.10) are quantum amplitudes between states determined by the initial and final configuration of fields in the path integral. We need to find a complete set of $N - M$ independent, commuting operators $\hat{Q}^\gamma$, in order to label states.

The problem of finding global, commuting co-ordinates $\hat{Q}^\gamma$ is generally unsolvable and seems to be related to the difficulty in the operator formalism of finding a consistent representation of the bracket in (5.7). This explains why the methods mentioned in the introduction to this paper are more powerful, as they work with the original Poisson bracket ω^{ij}, which furnishes a representation when, as is most often the case, x^i are canonical co-ordinates.

6 Exploring the Batalin Marnelius gauge theory

It's true that two of the most effective approaches, [?, ?, ?] and [?, ?] described in the introduction, to the covariant quantization of Hamiltonian systems with irreducible second class constraints are by converting to an equivalent gauge theory. However, its not easy to covariantly quantize all gauge theories. We should therefore be specific about which gauge theories are desirable. To this purpose, I define two terms:

I define the 'extended Hamiltonian form of the action' to be

$$S_E[x] = \int dt\, x^i\bar{\omega}_{ij}(x)\dot{x}^j - H(x) - \lambda_\alpha\theta^\alpha(x) - \lambda_a g^a(x), \tag{6.1}$$

where $\bar{\omega}_{ij}(x)$ is defined in (2.9) and $\theta^\alpha(x)$ and $g^a(x)$ are the irreducible second and first class constraints respectively.

I define the action to be 'desirable' when it can be written in the form of $S_E[x]$ such that there are no $\theta^\alpha(x)$ terms and where ω_{ij} is constant and thus equal to $2\bar{\omega}_{ij}$. Usually, constant ω_{ij} means that $S_E[x]$ is written in terms of canonical co-ordinates (q^i, p_i).

Note that we can get rid of the $\theta^\alpha(x)$ terms in $S_E[x]$ by parameterizing the second class constraint surface with non-degenerate co-ordinates y^μ, as in section 5. The cost is that $S_E[y]$ will then have in general a symplectic two-form $\omega_{\mu\nu}(y)$, as defined in (5.3), which depends on y^μ. The two previously mentioned methods convert the original action $S_E[x]$, assumed to be in canonical co-ordinates, to a 'desirable' one. 'Desirable' actions can be elegantly covariantly quantized using one of the equivalent antifield (BV) or Hamiltonian BRST (BFV) approaches [?]. As mentioned in section 5, the system can be quantized for some non-constant $\omega_{ij}(x)$'s, but usually, when converting to a gauge theory, we really mean a 'desirable' gauge theory.

The point is that the Batalin Marnelius action, $S[\bar{x}(x)]$, isn't in 'extended Hamiltonian form' because $\omega_M(x)_{km}$, defined in (4.14), is degenerate. It's thus not immediately clear whether it's a 'desirable' gauge theory or not.

One can convert $S[\bar{x}(x)]$ to standard S_E form, by treating x^i as ordinary Lagrangian co-ordinates and then following the standard procedure for formation of the Hamiltonian etc. Assume for simplicity that the system is bosonic and ω_{ij} is constant, as is most often the case, and introduce canonical momenta and Poisson bracket $\{,\}_P$ as follows:

$$p_{x^i} = \frac{\partial}{\partial \dot{x}^i} L(x, \dot{x}) \tag{6.2}$$

$$\{x^i, x^j\}_P = \{p_{x^i}, p_{x^j}\}_P = 0 \qquad \{x^i, p_{x^j}\}_P = \delta^i_j. \tag{6.3}$$

The Hamiltonian is

$$H(x, p_x) = H(\bar{x}(x)) \tag{6.4}$$

and there are $2N$ irreducible, primary constraints:

$$\phi_k = p_{x^k} - \frac{1}{2}\bar{x}^i(x)\omega_{ij}\partial_k \bar{x}^j \qquad \{\phi_k, \phi_m\}_P = \omega_M(x)_{km}, \tag{6.5}$$

with no further constraints obtained from consistency conditions

$$\dot{\phi}_k \approx \{\phi_k, H\}_P + v^m\{\phi_k, \phi_m\}_P \approx 0, \tag{6.6}$$

where $A \approx B$ means $A = B$ on the constraint surface $\phi_k = 0$. Now, $\text{Rank}\ \{\phi_k, \phi_m\}_P = 2N - 2M$ and 2M first class constraints can be projected out

$$\phi_\alpha = \phi_k G^k_\alpha(x) \qquad \{\phi_k, \phi_\alpha\}_P \approx 0. \tag{6.7}$$

As expected, ϕ_α generate the gauge transformations

$$\{x^i, \phi_\alpha\}_P = G^i_\alpha(x) \qquad \{p_{x^i}, \phi_\alpha\}_P = -p_{x^k}\partial_i G^k_\alpha(x) \tag{6.8}$$

and the action is

$$S_E[x, p_x] = \int dt\, p_{x^i}\dot{x}^i - H(\bar{x}(x)) - \lambda^k \phi_k(x, p_x). \tag{6.9}$$

We now parameterize the second class constraint surface using co-ordinates (p_α, x^i),

$$\begin{aligned} x^k(p_\alpha, x^i) &= x^k && (6.10)\\ p_{x^k}(p_\alpha, x^i) &= \frac{1}{2}\bar{x}^i\omega_{ij}\partial_k\bar{x}^j + p_\alpha(\partial_m\theta^\alpha)(\bar{x})\sigma_k^m(x). && (6.11)\end{aligned}$$

So,

$$\begin{aligned} \phi_k(p_\alpha, x^i) &= p_\alpha(\partial_m\theta^\alpha)(\bar{x})\sigma_k^m(x) && (6.12)\\ \phi_\alpha(p_\alpha, x^i) &= p_\alpha, && (6.13)\end{aligned}$$

using some of the identities in section 2.2. Note that on the first class constraints surface, $\phi_k(p_\alpha, x^i) = 0$ as required. We now calculate the corresponding symplectic two-form in these co-ordinates,

$$\omega_{(p_\alpha,x^i),(p_\beta,x^j)} = \frac{\partial X^k}{\partial(p_\alpha, x^i)}\Omega_{km}\frac{\partial X^m}{\partial(p_\beta, x^j)}, \tag{6.14}$$

where $X^k = (x^i, p_{x^i})$ and Ω_{km} is the symplectic two-form corresponding to the bracket $\{,\}_P$.

$$\omega_{(p_\alpha,x^i),(p_\beta,x^j)} = \begin{pmatrix} 0 & \vdots & (\partial_m\theta^\alpha)(\bar{x})\sigma_j^m(x) \\ \cdots & \cdots & \cdots \\ -\sigma_i^m(x)(\partial_m\theta^\beta)(\bar{x}) & \vdots & \omega_{M\,ij}(x) + p_\alpha A_{ij}^\alpha(x) \end{pmatrix}_{(p_\alpha,x^i),(p_\beta,x^j)} \tag{6.15}$$

where

$$A_{ij}^\alpha(x) = \frac{1}{2}\partial_{[i}((\partial_m\theta^\alpha)(\bar{x})\sigma_{j]}^m(x)). \tag{6.16}$$

Special co-ordinates $x^{i\prime} = (\theta^\alpha, y^\mu)$ can always be found such that $\bar{x}^{i\prime}(x') = (0, y^\mu)$. Note that the modified regularity condition (3.6) implies θ^α are adequate co-ordinates locally about any point $x \in \mathcal{M}$. Then,

$$\omega_{(p_\alpha,\theta^\gamma,y^\mu),(p_\beta,\theta^\delta,y^\nu)} = \begin{pmatrix} 0 & \vdots & 1 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -1 & \vdots & 0 & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \vdots & 0 & \vdots & \omega_{\mu\nu}(y) \end{pmatrix}_{(p_\alpha,\theta^\gamma,y^\mu),(p_\beta,\theta^\delta,y^\nu)}. \tag{6.17}$$

We can now see that if no non-degenerate co-ordinates y^μ on the constraint surface Γ can be found such that $\omega_{\mu\nu}(y)$ is constant, then $\omega_{(p_\alpha,x^i),(p_\beta,x^j)}$ isn't constant in any co-ordinate system. Thus, $S[\bar{x}(x)]$ is only 'desirable' when canonical co-ordinates y^μ exist on the constraint surface, this being one of the few examples where Dirac bracket quantization is possible. In other words, we can quantize no more systems using the BM form of the action $S[\bar{x}(x)]$, than by the standard Dirac bracket discussed in section 5.

The repercussions of the above analysis appear on applying the superfield version of the antifield (BV) formalism directly to the BM action $S[\bar{x}(x)]$, as performed in the subject paper [?]. One can directly write down the BRST cohomology and after choosing a suitable gauge fixing fermion, jump straight to the Fadeev-Popov style path integral. This Langranian form of BRST quantization is exactly equivalent to following through the above construction of $S_E[x,p_x]$ and the Hamiltonian BRST (BFV) path integral and then integrating out the momenta, leaving just x^i, Lagrange multipliers and ghosts.

First follows a short recap of this work:

The superfield version of $S[\bar{x}(x)]$ is

$$S'[\bar{x}(x(t,\tau))] = \int dt\, d\tau\, \bar{x}^i \bar{\omega}_{ij}(\bar{x}) D\bar{x}^j - \tau H(\bar{x}), \tag{6.18}$$

where $x^i(t,\tau) = x^i_0(t) + \tau x^i_1(t)$, τ is an odd Grassmann parameter, $\varepsilon(x^i_0) = \varepsilon^i$, $\varepsilon(x^i_1) = \varepsilon^i + 1$ and

$$D = \frac{d}{d\tau} + \tau \frac{d}{dt} \implies D^2 = \frac{d}{dt}. \tag{6.19}$$

The superfield master action is

$$\begin{aligned} S = S'[\bar{x}(x(t,\tau))] + \int \tilde{x}_i\, d\tau\, dt\, G^i_\alpha(x) C^\alpha \\ + \frac{1}{2}\int \tilde{C}_\gamma\, d\tau\, dt\, U^\gamma_{\alpha\beta}(x) C^\beta C^\alpha (-1)^{\varepsilon^\alpha} - \int \tilde{\bar{C}}^\alpha\, d\tau\, dt\, \lambda_\alpha (-1)^{\varepsilon^\alpha} \end{aligned} \tag{6.20}$$

where

$$(S,S) = 0 \qquad \Delta S = 0, \tag{6.21}$$

for superfields $\{x^i(t,\tau), C^\alpha(t,\tau), \bar{C}_\alpha(t,\tau), \lambda_\alpha(t,\tau)\}$ and their super antifields. The superfield measure has the remarkable property [?], of behaving as a scalar under general, superfield co-ordinate transformations, thus providing the natural measure. Equation (6.21) implies that no quantum corrections to the measure are necessary.

The authors choose a satisfactory gauge fixing fermion which determines the super antifields, leaving the gauge fixed action

$$\begin{aligned} S_\psi = S'[\bar{x}(x(t,\tau))] + \int d\tau\, dt\, \bar{C}_\alpha \theta^\alpha(x) \overleftarrow{\partial}_i G^i_\beta(x) C^\beta \\ - \int d\tau\, dt\, \lambda_\alpha \theta^\alpha(x). \end{aligned} \tag{6.22}$$

There are various points to be made at this stage:

Elements of the zero ghost number cohomology $f(x)$ satisfy

$$(f(x), S) = 0 \implies f(x)\overleftarrow{\partial}_i G^i_\alpha(x) = 0, \tag{6.23}$$

where the ghosts play no role in closing the cohomological algebra.

Although the gauge fixed action (6.22) is in standard Fadeev-Popov form, with BRST symmetry

$$\delta x^i = \eta G^i_\alpha(x) C^\alpha \qquad \delta\lambda_\alpha = 0 \tag{6.24}$$

$$\delta C^\alpha = -\frac{1}{2}\eta U^\alpha_{\beta\gamma} C^\beta C^\gamma \qquad \delta\bar{C}_\alpha = \eta\lambda_\alpha, \tag{6.25}$$

for Grassmann parameter η, there's no associated conserved Noether BRST charge. This follows from the fact that the gauge symmetries of $S[\bar{x}(x)]$ in the original phase space $\mathbb{M}$ have no associated Noether charges. If we vary x^i as $\delta x^i = G^i_\alpha(x)\eta^\alpha(t)$, where η^α is a small parameter with Grassmann parity $\varepsilon(\eta^\alpha) = \varepsilon^\alpha$, then the Noether identity

$$\delta S[\bar{x}(x)] = \int dt\, S[\bar{x}] \frac{\overleftarrow{\delta}}{\delta \bar{x}^k(t)} (\bar{x}^k \overleftarrow{\partial}_i) G^i_\alpha(x) \eta^\alpha(t) = 0, \tag{6.26}$$

disappears trivially. The equivalent gauge symmetries in $S_E[x,p]$ yield the first class constraints ϕ_α as Noether charges. These disappear when p_{x^i} are integrated out of the path integral.

The ghost part of the path integral disappears on integrating out C^α and $\bar{C}_\alpha$, which seems to be associated with the non-existence of a BRST operator. The identity $\theta^\alpha(x)\overleftarrow{\partial}_i G^i_\beta(x)|_{\theta=0} = \delta^\alpha_\beta$ used in the subject paper is not needed to show this.

The end result is the superfield version [?], of the standard, problematic 2nd class constraint path integral (5.10). So, in another way, we have come up against the result that $S[\bar{x}(x)]$ isn't a 'desirable' action. This is because, in parameterizing the 2nd class constraint surface $\theta(x) = 0$ degenerately, we haven't escaped from the fact that we are still parameterizing the constraint surface with the same problems as in non-degenerate parameterization, discussed in section 5.

7 Operator Quantization

As already alluded to in section 6, the gauge transformations (2.22) have no corresponding Noether charges within the phase space $\mathbb{M}$ and hence no generators. Specifically, the Batalin Marnelius action $S[\bar{x}(x)]$ isn't in extended Hamiltonian form, as would be required for gauge transformations to be generated by first class constraints. However, to use a parameterized $S_E[x, p_x(x, p_\alpha)]$ with first class constraint functions $\phi_\alpha = p_\alpha$ instead would be no more advantageous, in that no more systems could be quantized than using the approach below. Furthermore, $S_E[x, p_x(x, p_\alpha)]$ is more cumbersome and, with 2M extra phase space co-ordinates p_α, not in the spirit of the subject paper [?].

As we shall see, the following methods, are directly related to the Dirac bracket approach in section 5. Classically, all observables $F(x)$ satisfy

$$F(x)\overleftarrow{\partial}_k G^k_\alpha(x) = 0 \qquad \Rightarrow F(x) = F(\bar{x}(x)) \tag{7.1}$$

$$\dot{F}(\bar{x}) = \{F(\bar{x}), H(\bar{x}(x))\}_M, \tag{7.2}$$

where $\{,\}_M$ is the bracket defined in section 4.

There is a correspondence between the bracket $\{,\}_M$ on degenerate co-ordinates x^i and the bracket $\{,\}_*$ on non-degenerate co-ordinates y^μ. They have the similar properties (4.2) and (5.4), thus we can write

$$\{F(\bar{x}(x)), G(\bar{x}(x))\}_M = \{F(\bar{x}(y)), G(\bar{x}(y))\}_* \tag{7.3}$$

for $\bar{x}^i(x) = \bar{x}^i(y)$. The only difference is that we only allow gauge invariant functions, i.e. functions of the form $F(\bar{x}(x))$, inside the bracket $\{,\}_M$. All functions $f(y)$ can be written in the form $F(\bar{x}(y))$ however.

To emphasize this point, there is *no* equivalent, consistent, quantum gauge invariance condition on observables of the form

$$[\hat{F}(x), \hat{G}_\alpha(x)] = V_\alpha^\beta \hat{G}_\beta, \tag{7.4}$$

as there are no first class constraints within $\mathbb{M}$ associated with $\overleftarrow{\partial}_k G_\alpha^k(x)$. Thus, there is also *no* gauge invariance condition on the states $|s>$ in the Hilbert space of the form

$$\hat{G}_\alpha |s>= 0. \tag{7.5}$$

One quantization procedure is as follows:

Firstly, we choose a basis for the classical observables

$$Y^\mu(\bar{x}(x)) \qquad \mu = 1, \ldots 2N - 2M, \tag{7.6}$$

where $Y^\mu(\bar{x}(x)) = (y^\mu)^{-1}(\bar{x}(x))$ and y^μ are degenerate co-ordinates on the constraint surface $\mathbb{\Gamma}$. So,

$$Y^\mu(\bar{x}(y)) = y^\mu. \tag{7.7}$$

and

$$\begin{aligned} \{Y^\mu(\bar{x}(x)), Y^\nu(\bar{x}(x))\}_M &= \{Y^\mu(\bar{x}(y), Y^\nu(\bar{x}(y))\}_* \\ &= \omega^{\mu\nu}(Y(\bar{x}(x))) \end{aligned} \tag{7.8}$$

All observables $F(\bar{x}(x))$ are written $f(Y(\bar{x}(x)))$. We can remove all mention of $\bar{x}(x)$ and the problem reduces to finding a representation of

$$[\hat{Y}^\mu, \hat{Y}^\nu] = i\hbar\omega^{\mu\nu}(\hat{Y}), \tag{7.9}$$

just as in section 5.

A second procedure is as follows:

Sometimes, we can choose $\omega_M^{ij}(x)$ such that a representation of

$$[\hat{x}^i, \hat{x}^j] = i\hbar\omega_M^{ij}(\hat{x}) \tag{7.10}$$

can be found. In particular, if $\omega_M^{ij} = \omega^{ij}$ and x^i are canonical co-ordinates (q,p), then we can construct the standard co-ordinate representation. The states $|q>$, such that $\hat{q}^i|q>= q^i|q>$ form a basis for the Hilbert space.

It's always possible to make an appropriate choice of $\bar{x}^i(x)$ and $\omega_M^{ij} = \omega^{ij}$ when there exist non-degenerate, canonical co-ordinates y^μ on the constraint surface.

Clearly, unphysical operators and states are present in this representation, and there's no way of projecting them out as in the BRST formalism. By its very nature, all unphysical operators must be removed at the classical stage.

So, we define some consistent normal ordering scheme, which preserves the reality properties of observables $:\hat{F}(\bar{x}(\hat{x})):$. Now, for a co-ordinate representation, we find $N - M$ commuting observables $:\hat{Q}^\gamma(\bar{x}(\hat{x})):$

$$[:\hat{Q}^\gamma(\bar{x}(\hat{x})):, :\hat{Q}^\delta(\bar{x}(\hat{x})):] = 0, \tag{7.11}$$

then states $\{|Q>\}$, such that $\hat{Q}^\gamma|Q>= Q^\gamma|Q>$, form a basis for all physical states.

Essentially, this is the formalism advocated in [?], where the condition (4.1) was actually imposed.

It's not always necessary to write down a basis for all physical operators. We can consider a sub-algebra of operators, so long as we can construct $:\hat{Q}^\gamma:$ from them in order to have a basis for all states. Example 9.2 illustrates this point, a basis for the sub-algebra being $\{\hat{L}_i\}$ and the maximal set of commuting operators $\{\hat{\mathbf{L}}^2, \hat{L}_3\}$.

8 Mixed class constraints

There are various ways that one can envisage introducing first class constraints into this formalism. Supposing we start with the usual irreducible second class constraints $\theta^\alpha(x)$ and some irreducible first class constraints $g^a(x)$,

1. We could introduce good canonical gauge fixing constraints [?] $c^a(x)$, such that $\mathrm{sdet}\ \{g^a(x), c^b(x)\} \neq 0$, then all the constraints are second class and we can apply the BM formalism to this new system.

2. We could calculate $\bar{x}^i(x)$ which parameterizes the second class surface as before, however $g^a(\bar{x}) \neq 0$ now. We then quantize the action $S[\bar{x}(x)]$. This was alluded to in [?].

Method 1 is unlikely to be desirable since first class constraints are far easier to deal with than a BM processing of second class ones, but it is method 2 which is most useful and which I shall further explore below.

So after constructing $\bar{x}^i(x)$ with the usual properties (2.11) and (2.12), one can write down the action. The original extended action is

$$S_E[x] = \int dt\, (x^i\bar{\omega}_{ij}(x)\dot{x}^j - H(x) - \lambda_\alpha\theta^\alpha(x) - \lambda_a g^a(x)). \tag{8.1}$$

The BM gauge invariant action is

$$S[\bar{x}(x)] = \int dt\, (\bar{x}^i\bar{\omega}_{ij}(\bar{x})\dot{\bar{x}}^j - H(\bar{x}) - \lambda_a g^a(\bar{x})). \tag{8.2}$$

As well as the BM gauge symmetries from the degenerate parameterization of $\bar{x}(x)$ as denoted by

$$S[\bar{x}]\frac{\overleftarrow{\delta}}{\delta x^i}G^i_\alpha(x) = 0, \tag{8.3}$$

there are gauge symmetries arising from $g^a(\bar{x})$. An infinitesimal gauge transformation of a BM observable $F(\bar{x})$ is

$$\delta_\eta F(\bar{x}) = \{F(\bar{x}), \eta_a g^a(\bar{x})\}_M \tag{8.4}$$

where η_a are infinitesimal parameters with Grassmann parity $\varepsilon(\eta_a) = \varepsilon(g^a(x))$. The relevant $\{,\}_M$ bracket algebra is

$$\{g_a(\bar{x}), g_b(\bar{x})\}_M = U^c_{ab}(\bar{x})g_c(\bar{x}) \tag{8.5}$$

$$\{H(\bar{x}), g_a(\bar{x})\}_M = V^b_a(\bar{x})g_b(\bar{x}) \tag{8.6}$$

and the gauge symmetry of the action $S[\bar{x}]$ in (8.2) arising from g_a is

$$\delta_\eta g_a(\bar{x}) = \{g_a(\bar{x}), \eta^b g_b(\bar{x})\}_M \tag{8.7}$$
$$\delta_\eta H(\bar{x}) = \{H(\bar{x}), \eta^a g_a(\bar{x})\}_M \tag{8.8}$$
$$\delta_\eta \lambda^a = \dot{\eta}^a + \lambda^c \eta^b U^a_{bc}(\bar{x}) - \eta^b V^a_b(\bar{x}) \tag{8.9}$$

An observable $f(x)$ must be gauge invariant in both senses:

$$f(x)\overleftarrow{\partial}_i G^i_\alpha(x) = 0 \qquad \{f(x), g_a(\bar{x})\}_M = W^b_a(\bar{x}) g_a(\bar{x}) \tag{8.10}$$

The above formalism is similar to that which we would obtain on parameterizing the second class constraint surface with non-degenerate co-ordinates y^μ, as in section 5. The approach is nearly identical to that in [?], with the bracket $\{,\}_*$ replaced by $\{,\}_M$. The relation between the two brackets is evident in equation (7.3).

One can then either pursue the operator formalism as laid out in section 7, or follow the superfield anti-field approach, remembering the extra gauge symmetries from $g_a(\bar{x})$.

9 Examples

Example 9.1 (The simplest example)**.** The simplest example that one can conceive is where there are bosonic, canonical co-ordinates $\{x^i\} = \{q^1, q^2, p_1, p_2\}$ and second class constraints

$$\theta^1 = q^1 \qquad \theta^2 = p_1. \tag{9.1}$$

For further simplicity, we don't consider the Hamiltonian, we just generate the correct quantum representation of the system.

The simplest degenerate parameterization is

$$\bar{q}^1 = 0 \qquad \bar{p}_1 = 0 \tag{9.2}$$
$$\bar{q}^2 = q^2 \qquad \bar{p}_2 = p_2, \tag{9.3}$$

where

$$Z^i_1 \partial_i = G^i_1 \partial_i = \frac{\partial}{\partial q^1} \qquad Z^i_2 \partial_i = G^i_2 \partial_i = \frac{\partial}{\partial p_1}. \tag{9.4}$$

Now, $\bar{q}^2$ and $\bar{p}_2$ form a basis for the observables and in this simple case,

$$\{\bar{x}^i, \bar{x}^j\}_M = \{\bar{x}^i, \bar{x}^j\}_D|_{x \to \bar{x}(x)} \tag{9.5}$$

is solved by taking $\{,\}_M \equiv \{,\}$, where $\{,\}$ is the standard Poisson bracket. So a basis of states is $\{|q^2>\}$ and yields a representation of

$$[\hat{q}^2, \hat{p}_2] = i\hbar \qquad [\hat{q}^2, \hat{q}^2] = [\hat{p}_2, \hat{p}_2] = 0. \tag{9.6}$$

Example 9.2 (A non-relativistic particle on a sphere)**.** This example was both considered briefly in [?], and in detail in [?] in the context of 'gauge-unfixing'.

The action is

$$S = \int dt \, (\frac{m}{2}\dot{\mathbf{q}}^2 + \lambda(\mathbf{q}^2 - R^2)), \tag{9.7}$$

where $\mathbf{q} = (q^1, q^2, q^3)$, R is the radius of the sphere and λ is a Lagrange multiplier. The Hamiltonian is

$$H = \frac{1}{2m}\mathbf{p}^2 - \lambda(\mathbf{q}^2 - R^2), \tag{9.8}$$

where $\mathbf{p} = (p_1, p_2, p_3)$ and p_λ are the canonical momenta conjugate to $\mathbf{q}$ and λ respectively. The constraints are

$$\theta^1 = p_\lambda \qquad \theta^2 = \mathbf{q}^2 - R^2 \tag{9.9}$$

$$\theta^3 = \mathbf{q}.\mathbf{p} \qquad \theta^4 = \lambda - \frac{\mathbf{p}^2}{2mR^2}. \tag{9.10}$$

where $\theta^2, \theta^3, \theta^4$ come from requiring the consistency conditions $\dot{\theta^\alpha} \approx 0$. We can calculate $\det \{\theta^\alpha, \theta^\beta\} \neq 0$ so long as $\mathbf{q}^2 \neq 0$, thus the constraints are second class.

I now compare various different methods.

The 'gauge unfixing' procedure, discussed in [?], was to discard the constraints θ^3, θ^4, leaving first class constraints θ^1 and θ^2.

$$\tilde{H} = \frac{\mathbf{L}^2}{2mR^2} \tag{9.11}$$

is then a suitable first class Hamiltonian i.e. $\{\tilde{H}, \theta^\alpha\}|_{\theta^1=\theta^2=0} = 0$ for $\alpha = 1, 2$, with $L_i = \varepsilon_{ijk} q^j p_k$ and

$$\{L_i, L_j\} = \varepsilon_{ijk} L_k \tag{9.12}$$

as the standard angular momenta algebra.

The total Hilbert space is spanned by $|\lambda; r; \theta, \phi >$, where $r^2 = \mathbf{q}^2$ and θ, ϕ are spherical polar co-ordinates. A basis for physical states is $|p_\lambda = 0; r = R; l, m >$, from which a consistent representation of sub-algebra

$$[\hat{L}_i, \hat{L}_j] = i\hbar \hat{L}_k \tag{9.13}$$

is formed. The states $|l, m >$ are labelled, by employing commuting operators $\hat{\mathbf{L}}^2$ and $\hat{L}_3$, as follows:

$$\hat{\mathbf{L}}^2 |l, m >= \hbar^2 l(l+1) \qquad \hat{L}_3 |l, m >= \hbar m |l, m > . \tag{9.14}$$

The physical states are energy eigenstates, with eigenvalues

$$E_l = \frac{\hbar^2 l(l+1)}{2mR^2}. \tag{9.15}$$

The point here, is that we use the standard Poisson bracket from which a quantum representation is always possible.

A second method is the non-degenerate parameterization technique , discussed in section 5. We can take spherical polar co-ordinates such that

$$\bar{\mathbf{q}} = R\mathbf{e_r} \qquad \bar{\mathbf{p}} = \frac{p_\theta}{R}\mathbf{e}_\theta + \frac{p_\phi}{R\sin\theta}\mathbf{e}_\phi \tag{9.16}$$

where

$$\mathbf{e_r} = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \qquad \mathbf{e}_\theta = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix} \tag{9.17}$$

$$\mathbf{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}. \tag{9.18}$$

Constraints θ^1 and θ^4 are simply parameterized away using their defining equations. Applying equation (5.3), one can show that $\{\theta, \phi, p_\theta, p_\phi\}$ are canonical co-ordinates on the constraint surface, thus a quantum representation can be found. Writing $\mathbf{L} = \bar{\mathbf{q}} \times \bar{\mathbf{p}}$, we find that the Hamiltonian is

$$H = \frac{\mathbf{L}^2(\bar{\mathbf{q}}, \bar{\mathbf{p}})}{2mR^2} \tag{9.19}$$

and using $\{\theta, p_\theta\}_* = 1$ etc. ,

$$\{L_i, L_j\}_* = \varepsilon_{ijk}L_k. \tag{9.20}$$

A quantum representation of (9.13) is constructed. The full Hilbert space of states is spanned by $\{|l, m>\}$, where $\hat{\mathbf{L}}^2$ and $\hat{L}_3$ form the maximal set of commuting operators.

A third method is the second operator formalism of section 7, which in this case, is the same as that in [**?**].

A degenerate parameterization is

$$\bar{\mathbf{q}} = \frac{R}{\sqrt{\mathbf{q}^2}}\mathbf{q} \qquad \bar{\mathbf{p}} = \frac{\sqrt{\mathbf{q}^2}}{R}(\mathbf{p} - \frac{\mathbf{p}.\mathbf{q}}{\mathbf{q}^2}\mathbf{q}), \tag{9.21}$$

where we can see that conditions (2.11) and (2.12) are satisfied. The Hamiltonian is

$$H = \frac{\bar{\mathbf{p}}^2}{2m} = \frac{\mathbf{L}^2(\mathbf{q}, \mathbf{p})}{2mR^2}. \tag{9.22}$$

We can check that $\{,\}_M \equiv \{,\}$ in this case, and also that $\mathbf{L}(\bar{\mathbf{q}}, \bar{\mathbf{p}}) = \mathbf{L}(\mathbf{q}, \mathbf{p})$. Thus the functions L_i are observables and form the closed algebra as in (9.12). We can form a quantum representation of the sub-algebra (9.13), and again label states $|l, m>$ with the maximal commuting sub-algebra of observables, $\hat{\mathbf{L}}^2$ and $\hat{L}_3$.

The point was made in [**?**] of the similarity between the above degenerate parameterization quantization approach, and the 'gauge-unfixing' procedure in the first part of this example. As I have argued in sections 6 and 7 of this paper, the similarities between the degenerate and non-degenerate parameterization are more fundamental, being linked by equation (7.3).

10 Concluding remarks

In the preceding work, it was shown that the regularity conditions on the constraints needed to be modified, and also, the bracket $\{,\}_M$ was introduced from which was developed the operator formalism. We have seen that quantization of the BM theory amounts to Dirac bracket quantization, and lacks the power of other established approaches [?, ?, ?] and [?, ?] mentioned in the introduction.

However, where Dirac bracket quantization is possible, the second of the operator formalisms in section 7 is an attractive alternative to the methods used in section 5. It combines the advantage, as with non-degenerate parameterization, of imposing the constraints at the classical stage whilst avoiding the, often inconvenient, co-ordinates y^μ. Example 9.2, 'a non-relativistic particle on a sphere', illustrates this point.

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