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On the Representation Theory of Negative Spin

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Abstract

We construct a class of negative spin irreducible representations of the $su(2)$ Lie algebra. These representations are infinite-dimensional and have an indefinite inner product. We analyze the decomposition of arbitrary products of positive and negative representations with the help of generalized characters and write down explicit reduction formulae for the products. From the characters, we define effective dimensions for the negative spin representations, find that they are fractional, and point out that the dimensions behave consistently under multiplication and decomposition of representations.

1 Introduction

The representation theory of $su(2)$ is familiar to most physicists and mathematicians [1]. The finite dimensional representations are labelled by their spin, which is non-negative and can take either integer or half-integer values.

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Recently, we found that a certain infinite-dimensional representation of $su(2)$ of negative spin, with value $-\frac{1}{2}$, appears in the description of a ghost particle complementary to a spin $\frac{1}{2}$ degree of freedom [2]. Although this Lie algebra representation did not exponentiate to give a representation of the whole $SU(2)$ group, we found it interesting to investigate in its own right.

We found that the product of the spin $\frac{1}{2}$ and spin $-\frac{1}{2}$ representations was just the spin 0 representation, modulo a well-defined reduction procedure in the product space. We showed that this cancellation of spins could be reflected in character formulae and argued that the spin $-\frac{1}{2}$ representation effectively had a fractional dimension equal to $\frac{1}{2}$.

In this paper, we extend these results to representations of arbitrary negative spin. These representations are all defined on infinite-dimensional indefinite inner product spaces.

We analyze the decomposition of arbitrary products of positive and/or negative representations. In general, we will see that the product representation is well-behaved only after carrying out a well-defined reduction procedure to the cohomology of a nilpotent operator Q constructed in terms of the Casimir $\mathbf{J}^2$.

We then define characters for the negative spin representations and, with the help of these, we write down explicit formulae for the reduction of general product representations.

From the characters, we define effective dimensions for the negative spin representations, find that they are fractional, and point out that the dimensions behave consistently under multiplication and decomposition of representations.

2 The Lie algebra $su(2)$

In this section we summarize some basic formulas regarding $su(2)$ that will be useful in what follows.

The commutation relations of $su(2)$ are [1]

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y.$$

We can define ladder operators

$$\begin{aligned} J_+ &\equiv J_x + iJ_y, \\ J_- &\equiv J_x - iJ_y \end{aligned} \tag{1}$$

satisfying

$$[J_+, J_-] = 2J_z.$$

The Casimir operator

$$\begin{aligned}
\mathbf{J}^2 &= J_x^2 + J_y^2 + J_z^2 \\
&= J_+ J_- + {J_z}^2 - J_z \\
&= J_- J_+ + {J_z}^2 + J_z
\end{aligned} \tag{2}$$

commutes with J_x, J_y and J_z and serves to label irreducible representations by their spin j defined as $\mathbf{J}^2 = j(j+1)$.

The lowest weight state, annihilated by J_-, of a spin j representation has J_z eigenvalue $m = -j$ and we denote it by $\phi^{(j)}_{-j}$. Labelling basis elements by the spin j and J_z eigenvalue m, the full set of basis vectors is generated by applying the ladder operator J_+ repeatedly to the lowest weight state to give the sequence

$$\phi^{(j)}_{-j+k} \equiv (J_+)^k \, \phi^{(j)}_{-j}.$$

3 Negative spin

Let us consider the possibility of constructing hermitian representations of the $su(2)$ Lie algebra that have negative spin j, where j is either half-integral or integral.

As above, we start from a lowest weight state $\phi^{(j)}_{-j}$ annihilated by J_- and consider the ladder of states

$$\phi^{(j)}_{-j} \xrightarrow{J_+} \phi^{(j)}_{-j+1} \xrightarrow{J_+} \phi^{(j)}_{-j+2} \xrightarrow{J_+} \cdots \tag{3}$$

obtained by repeatedly applying J_+.

First, we show that negative spin representations require an indefinite inner product space. In a hermitian representation, the ladder operator J_- is conjugate to J_+, so that we can write the inner product of the state $\phi^{(j)}_{-j+k}$ with itself as follows

$$\left\langle \phi^{(j)}_{-j+k} \,\middle|\, \phi^{(j)}_{-j+k} \right\rangle = \left\langle (J_+)^k \phi^{(j)}_{-j} \,\middle|\, (J_+)^k \phi^{(j)}_{-j} \right\rangle = \left\langle \phi^{(j)}_{-j} \,\middle|\, (J_-)^k (J_+)^k \phi^{(j)}_{-j} \right\rangle .$$

Commuting the J_- operators to the right, this becomes

$$\left\langle \phi^{(j)}_{-j+k} \,\middle|\, \phi^{(j)}_{-j+k} \right\rangle = k! \, (2j) \, (2j-1) \cdots (2j-k+1) \, \left\langle \phi^{(j)}_{-j} \,\middle|\, \phi^{(j)}_{-j} \right\rangle . \tag{4}$$

Since $j < 0$, we see that squared norm of the sequence of states in (3) have alternating signs. In other words, the inner product is indefinite.

We now choose the squared norm of the negative spin j lowest weight state $\phi^{(j)}_{-j}$ to be equal to 1. Defining the normalized state

$$e^{(j)}_{-j+k} \equiv \frac{1}{i^k\sqrt{k!\,|2j|\,|2j-1|\cdots|2j-k+1|}}\,\phi^{(j)}_{-j+k} \tag{5}$$

so that

$$\left\langle e^{(j)}_{-j+k} \,\middle|\, e^{(j)}_{-j+m} \right\rangle = \delta_{km}\,(-)^k, \tag{6}$$

it is straightforward to check, using the commutation relations and taking careful account of signs and factors of i, that

$$\begin{aligned} J_+\,e^{(j)}_m &= i\,\{(j+m+1)\,|j-m|\}^{\frac{1}{2}}\,e^{(j)}_{m+1} \qquad m=-j,-j+1,\ldots \\ &= \{(j+m+1)\,(j-m)\}^{\frac{1}{2}}\,e^{(j)}_{m+1}, \qquad (-1)^{\frac{1}{2}} \equiv +i \end{aligned} \tag{7}$$

and

$$\begin{aligned} J_-\,e^{(j)}_{m+1} &= i\,\{(j+m+1)\,|j-m|\}^{\frac{1}{2}}\,e^{(j)}_m \qquad m=-j,-j+1,\ldots \\ &= \{(j+m+1)\,(j-m)\}^{\frac{1}{2}}\,e^{(j)}_m, \qquad (-1)^{\frac{1}{2}} \equiv +i. \end{aligned} \tag{8}$$

These formulae are the continuation to negative j of the corresponding formulae for positive spin representations.

Notice that when j is negative, the sequence of states $e^{(j)}_{-j+k}$ does not terminate. These representations are therefore infinite dimensional. However, we shall see later that one can assign finite effective dimensions to the negative spin representations in a meaningful way.

A peculiar feature of these negative spin representations is their asymmetry with respect to interchange of J_- and J_+. In particular, there is no highest weight state annihilated by J_+.

With the indefinite inner product (6), the state spaces are examples of what is known in the literature as Kreĭn spaces (see [3], [4], [5] and references therein).

4 The spin $-\frac{1}{2}$ ghost

In [2] we constructed an $su(2)$ representation of spin equal to $-\frac{1}{2}$ on the state space of a bosonic ghost complementary to a single two-state spinor degree of freedom. In this section we briefly review this construction.

The ghost creation and annihilation operators $a^\dagger$ and a satisfy

$$[a, a^\dagger] = -1.$$

The negative sign on the right hand side gives rise to an indefinite inner product on the Fock space generated by applying creation operators to the ground state $\left|\frac{1}{2}\right\rangle_g$ annihilated by a.

A set of hermitian $su(2)$ generators is given by

$$\begin{aligned} J_x &\equiv \frac{i}{2}\left(\sqrt{N+1}\,a - a^\dagger\sqrt{N+1}\right), \\ J_y &\equiv -\frac{1}{2}\left(\sqrt{N+1}\,a + a^\dagger\sqrt{N+1}\right), \\ J_z &\equiv N+\frac{1}{2}, \end{aligned} \tag{9}$$

where $N \equiv -a^\dagger a$. It is easy to check that these generators satisfy the $su(2)$ algebra

$$[J_x, J_y] = iJ_z, \quad [J_y, J_z] = iJ_x, \quad [J_z, J_x] = iJ_y.$$

The ladder operators are

$$\begin{aligned} J_+ &\equiv J_x + iJ_y = -i\,a^\dagger\sqrt{N+1}, \\ J_- &\equiv J_x - iJ_y = \quad i\sqrt{N+1}\,a, \end{aligned}$$

and satisfy $J_\pm^\dagger = J_\mp$ and

$$[J_+, J_-] = 2J_z.$$

It is straightforward to calculate

$$\mathbf{J}^2 = \frac{1}{2}\left(J_+J_- + J_-J_+\right) + {J_z}^2 = -\frac{1}{4}.$$

Since $\mathbf{J}^2 = j(j+1)$, where j denotes the spin, we see that

$$j = -\frac{1}{2}.$$

In other words, the representation that we have constructed has negative spin.

Note that, since the entire state space is generated by applying the raising operator J_+ to the vacuum state, the representation is irreducible. Also, in contrast to the positive spin irreducible representations, it is infinite-dimensional.

The lowest weight state is the ground state $e_{\frac{1}{2}}^{(\frac{1}{2})} = \left|\frac{1}{2}\right\rangle_g$ which has J_z eigenvalue $m = 1/2 = -j$. There is no highest weight state.

Taking the following phase convention for the basis elements

$$e_m^{(-\frac{1}{2})} = |m\rangle_g \equiv (-)^{m-\frac{1}{2}}\frac{1}{\sqrt{(m-\frac{1}{2})!}}\,(a^\dagger)^{m-\frac{1}{2}}\left|\tfrac{1}{2}\right\rangle_g, \qquad m = \tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2}, \ldots \tag{10}$$

it is straightforward to check explicitly that the relations (7) and (8) are satisfied by this representation. Here we have introduced the shorthand $|m\rangle_g \equiv e_m^{(-\frac{1}{2})}$ for the spin $-\frac{1}{2}$ basis vectors.

5 Representation of finite $SU(2)$ elements

Since the negative spin Lie algebra representations have no highest weight state, we do not expect to be able to exponentiate these representations to obtain a representation of the full $SU(2)$ group. In particular, it is easy to see that rotations by π that take the unit vector $\mathbf{z}$ to $-\mathbf{z}$ are not representable on the negative spin state space, since such rotations would normally exchange lowest and highest weight states.

The conclusion is that the Lie algebra representation of the previous sections does not exponentiate to a give a representation of the full group.

However, it is still possible to represent a restricted class of $SU(2)$ group elements. For the spin $-\frac{1}{2}$ case, this was done in reference [2].

6 Multiplication of positive spin representations

Given two positive spin irreducible representations of spins j_1 and j_2, their tensor product is itself a representation which can be written as a direct sum of irreducible representations as follows [1]

$$R^{(j_1)} \otimes R^{(j_2)} = \sum_{J=|j_1-j_2|}^{j_1+j_2} R^{(J)}, \qquad j_1, j_2 \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots\}. \tag{11}$$

The characters of the positive spin representations of $SU(2)$ are given by

$$\chi^{(j)}(\theta) = \mathrm{Tr}\, e^{i\theta J_z} = e^{-ij\theta}(1 + e^{i\theta} + \cdots + e^{2ij\theta}) = \frac{\sin\left(j+\frac{1}{2}\right)\,\theta}{\sin\frac{1}{2}\theta}, \tag{12}$$

and satisfy the orthogonality relation

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta\, \chi^{j_1}(\theta)\, \chi^{j_2}(\theta)\, (1-\cos\theta) = \delta_{j_1 j_2}. \tag{13}$$

We remind the reader that a character is a function on the conjugacy classes of a group. For $SU(2)$, all rotations through the same angle θ are in the same class, irrespective of the direction of their axes. The direct sum decomposition is reflected in the following algebraic relationship satisfied by the corresponding characters

$$\chi^{j_1}\,\chi^{j_2} = \sum_{J=|j_1-j_2|}^{j_1+j_2} \chi^{(J)}, \qquad j_1, j_2 \in \{0, \tfrac{1}{2}, 1, \tfrac{3}{2}, 2, \ldots\}. \tag{14}$$

which, given that the dimensions of the representations are $D^{(j)} = \chi^{(j)}(\mathbf{1})$, is consistent with

$$D^{(j_1)\otimes(j_2)} = D^{(j_1)}\, D^{(j_2)} = \sum_{J=|j_1-j_2|}^{j_1+j_2} D^{(J)}. \tag{15}$$

In the following, we will show that the analysis of the direct sum decomposition in terms of characters can be generalized to include the negative spin representations.

7 Combining spin $\frac{1}{2}$ and spin $-\frac{1}{2}$ representations

As a warmup, we study the product of the spin $\frac{1}{2}$ and spin $-\frac{1}{2}$ representations [2]. We will discuss the sense in which the positive and negative spin representations cancel to give an effectively trivial representation.

First, notice that there is a lowest weight state in the product state space given by

$$|0\rangle \equiv \left|-\tfrac{1}{2}\right\rangle_s \otimes \left|\tfrac{1}{2}\right\rangle_g ,$$

where we have introduced the shorthand $\left|\pm\frac{1}{2}\right\rangle_s \equiv e^{(\frac{1}{2})}_{\pm\frac{1}{2}}$ for the spin $\frac{1}{2}$ basis vectors. By definition, this state is annihilated by $J_- \equiv J_-^{(\frac{1}{2})} + J_-^{(-\frac{1}{2})}$. Using the relation

$$\mathbf{J}^2 \;\; = \;\; J_+J_- + J_z^2 - J_z$$

it trivially follows that

$$\mathbf{J}^2 \; |0\rangle = 0.$$

We are therefore dealing with an $su(2)$ representation of spin 0 in the product space. What is unusual about this representation, however, is that $J_+\,|0\rangle \neq 0$. However, this state has zero norm. In fact, this is true for all higher states in the ladder since

$$J_+^n\,|0\rangle = i^n\,\sqrt{n}\,(n-1)!\;|n\rangle\,,$$

where the states

$$|n\rangle \;\; \equiv \;\; \sqrt{n}\,\left(i\,\left|-\tfrac{1}{2}\right\rangle_s \otimes \left|n+\tfrac{1}{2}\right\rangle_g + \left|\tfrac{1}{2}\right\rangle_s \otimes \left|n-\tfrac{1}{2}\right\rangle_g\right), \tag{16}$$

are null.

In general, we would like to truncate the ladder of states generated by J_+ as soon as we reach a null state. The proper way of doing this is by noticing that the Casimir operator $\mathbf{J}^2$ is nilpotent. We can therefore regard it as a BRST operator (see [6] and [7]) and calculate its cohomology, which defines a natural reduction procedure on the state space. For an overview of the elements of BRST technology that we will use, see appendix A.

To see that $\mathbf{J}^2$ is nilpotent, note that in terms of the basis consisting of $|n\rangle$ and the additional null states

$$|\tilde{n}\rangle \equiv \frac{1}{2\sqrt{n}}\left(-i\left|-\tfrac{1}{2}\right\rangle_s \otimes \left|n+\tfrac{1}{2}\right\rangle_g + \left|\tfrac{1}{2}\right\rangle_s \otimes \left|n-\tfrac{1}{2}\right\rangle_g\right), \tag{17}$$

it is not hard to calculate

$$\mathbf{J}^2\,|\tilde{n}\rangle = |n\rangle\,, \qquad \mathbf{J}^2\,|n\rangle = 0,$$

leading to the matrix representation

$$\mathbf{J}^2 = \begin{pmatrix} 0 & & & \\ & \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & & \\ & & \boxed{\begin{matrix} 0 & 1 \\ 0 & 0 \end{matrix}} & \\ & & & \ddots \end{pmatrix}. \tag{18}$$

In this basis, the inner product is represented by the matrix

$$G \equiv \begin{pmatrix} 1 & & & \\ & \boxed{\begin{matrix} & 1 \\ 1 & \end{matrix}} & & \\ & & \boxed{\begin{matrix} & -1 \\ -1 & \end{matrix}} & \\ & & & \ddots \end{pmatrix}$$

It is easily checked that $(\mathbf{J}^2)^2 = 0$ and that $\mathbf{J}^2$ is hermitian with respect to the inner product G, as follows from

$$\mathbf{J}^2 = (\mathbf{J}^2)^\dagger = G(\mathbf{J}^2)^+G,$$

where the dagger denotes hermitian conjugation with respect to the indefinite inner product $\langle\cdot\,|\,\cdot\rangle$ on the state space and the $+$ sign denotes the usual matrix adjoint.

We now specialize the BRST analysis to the case at hand, taking $Q = \mathbf{J}^2$. Since the generators J_i all commute with Q, they are physical operators in the sense of appendix A.

We can therefore consistently reduce these operators to the cohomology $\ker Q/\operatorname{im} Q$ of Q and use (42) to define the induced operators $[J_+]$, $[J_-]$ and $[J_z]$ on the quotient space by

$$[J_i]\,[|\phi\rangle] = [J_i\,|\phi\rangle], \qquad |\phi\rangle \in \ker Q.$$

The cohomology consists of the single class $[|0\rangle]$, and is a one-dimensional, positive definite Hilbert space. The induced generators $\{[J_+], [J_-], [J_z]\}$ on this space are zero, corresponding to the trivial representation of $su(2)$.

8 Operators on indefinite inner product spaces

In order to address the product decomposition in the general case, we need to understand the behavior of the Casimir operator $\mathbf{J}^2$ a little better. With this in mind, we review a few general facts regarding hermitian operators on indefinite inner product spaces, also known as pseudo-hermitian operators (see [3], [4], [5]).

It is important to be aware that not all results that are valid for positive definite spaces are valid when the inner product is not positive definite. For example, not all pseudo-hermitian operators are diagonalizable. A good counterexample is precisely the operator $Q = \mathbf{J}^2$ above. In addition, not all eigenvalues are necessarily real. In particular, a pseudo-hermitian operator may have complex eigenvalues that come in conjugate pairs.

For our purposes, we will restrict consideration to pseudo-hermitian operators with real eigenvalues, of which $\mathbf{J}^2$ will be the relevant example. The domain of such an operator A can always be decomposed into a direct sum of pairwise orthogonal subspaces, in each of which we can choose a basis such that A has the so-called Jordan normal block form

$$\begin{pmatrix} \lambda & 1 & & \\ & \lambda & 1 & \\ & & \ddots & \ddots \\ & & & \lambda \end{pmatrix} \tag{19}$$

and the inner product has the form

$$\pm \begin{pmatrix} & & & 1 \\ & & 1 & \\ & \dots & & \\ 1 & & & \end{pmatrix}. \tag{20}$$

Notice that only the first vector in the subspace is an eigenvector of A with eigenvalue λ. If the Jordan block has dimension larger than 1, this eigenvector is null. The vectors v in this subspace are called principal vectors belonging to the eigenvalue λ and satisfy

$$(A-\lambda)^m v = 0$$

for some integer $m \geq 1$. The sequence of vectors v_i spanning this subspace satisfying

$$Av_i = \lambda v_i + v_{i-1}$$

is called a Jordan chain.

9 BRST analysis of general product representations

We now generalize the method of section 7 to the analysis of a general product representation.

In appendix B we prove the result, important in what follows, that the Casimir operator $\mathbf{J}^2$ can be decomposed into Jordan blocks of dimension at most two. Since the one and two-dimensional Jordan blocks of $\mathbf{J}^2$ are of the forms (λ_i) and

$$\begin{pmatrix} \lambda_i & 1 \\ 0 & \lambda_i \end{pmatrix}, \tag{21}$$

we can build a BRST operator in the product space of two arbitrary representations in terms of $\mathbf{J}^2$ as the orthogonal direct sum

$$Q = Q_{\lambda_1} \oplus Q_{\lambda_2} \oplus \cdots , \tag{22}$$

where Q_{λ_i} is defined on the principal vector subspace V_{λ_i} belonging to the eigenvalue λ_i of $\mathbf{J}^2$ by

$$Q_{\lambda_i} = \mathbf{J}^2|_{V_{\lambda_i}} - \lambda_i. \tag{23}$$

Since Q_{λ_i} consists of one- and two-dimensional blocks of the forms (0) and

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{24}$$

Q is nilpotent and is a valid BRST operator.

Taking the cohomology of Q discards any Jordan blocks of dimension two in the decomposition of $\mathbf{J}^2$, so that all principal vectors of the reduced

$[\mathbf{J}^2]$ will be eigenvectors. Equivalently, $[\mathbf{J}^2]$ can be decomposed into Jordan blocks of dimension 1.

Let us see how Q affects the analysis of a general product representation. As in the example of section 7, positive spin j representations may be present in the product in the form

$$\phi^{(j)}_{-j} \underset{J_-}{\overset{J_+}{\rightleftarrows}} \phi^{(j)}_{-j+1} \underset{J_-}{\overset{J_+}{\rightleftarrows}} \cdots \underset{J_-}{\overset{J_+}{\rightleftarrows}} \phi^{(j)}_{j} \underset{J_-}{\overset{J_+}{\rightleftarrows}} \boxed{\begin{matrix}\phi^{(j)}_{j+1}\\ \tilde\phi^{(j)}_{j+1}\end{matrix}} \underset{J_-}{\overset{J_+}{\rightleftarrows}} \boxed{\begin{matrix}\phi^{(j)}_{j+2}\\ \tilde\phi^{(j)}_{j+2}\end{matrix}} \underset{J_-}{\overset{J_+}{\rightleftarrows}} \cdots \tag{25}$$

The boxes represent two-dimensional subspaces spanned by null vectors $\phi^{(j)}_{j+n}$ and $\tilde\phi^{(j)}_{j+n}$ on which $\mathbf{J}^2$ has the Jordan normal form (21) and Q has the form (24). We see that just as in the example of section 7, the ladder of states generated from the lowest weight $\phi^{(j)}_{-j}$ consists of null states for $m > j$.

Because of the additional states to the right of $\phi^{(j)}_{-j}$, this representation is not in the original class of irreducible positive or negative spin representations that we started out with. In other words, the the original class of irreducible representations is not closed under multiplication, unless we can get rid of the extra states.

Taking the cohomology with respect to Q discards the boxed subspaces, and on the cohomology classes the representation has the familiar form

$$[\phi^{(j)}_{-j}] \underset{J_-}{\overset{J_+}{\rightleftarrows}} [\phi^{(j)}_{-j+1}] \underset{J_-}{\overset{J_+}{\rightleftarrows}} \cdots \underset{J_-}{\overset{J_+}{\rightleftarrows}} [\phi^{(j)}_{j}].$$

Henceforth, when we analyze product representations, it will always be assumed that we are working in the cohomology with respect to the associated operator Q. This amounts to a redefinition of the product as the tensor product followed by the BRST reduction.

10 Generalized characters

The analysis of the decomposition of product representations may be greatly simplified by reformulating it as an algebraic problem in terms of characters.

We would like the expression for the character of a representation to be invariant under the above BRST reduction to the cohomology of Q. Since the discarded Jordan subspaces have zero metric signature, an expression for the character of a group element U that ignores these blocks is

$$\chi(U) = \sum_n \mathrm{sig}\,(V_{\lambda_n})\lambda_n \tag{26}$$

where V_{λ_n} is the principal vector subspace corresponding to the eigenvalue λ_n of U and $\mathrm{sig}\,(\cdot)$ denotes the signature. Since the signature of a subspace is invariant under unitary transformations V [4], this gives a basis invariant expression invariant under conjugation $U \to VUV^{-1}$.

Using the following properties of the signature

$$\begin{aligned} \mathrm{sig}\,(V \otimes W) &= \mathrm{sig}\,V \cdot \mathrm{sig}\,W, \qquad (27)\\ \mathrm{sig}\,(V \oplus W) &= \mathrm{sig}\,V + \mathrm{sig}\,W, \qquad (28) \end{aligned}$$

it follows that the characters satisfy the following important algebraic properties

$$\begin{aligned} \chi(U_1 \otimes U_2) &= \chi(U_1) \cdot \chi(U_2), \qquad (29)\\ \chi(U_1 \oplus U_2) &= \chi(U_1) + \chi(U_2). \qquad (30) \end{aligned}$$

These are exactly the properties that make the characters useful for analyzing the decomposition of a product representation into a direct sum of irreducible representations. The first property ensures that the character of a product representation is simply the product of the characters of the individual representations. In other words,

$$\chi^{R_1 \otimes R_2} = \chi^{R_1} \chi^{R_2}.$$

The second property then ensures that if $R_1 \otimes R_2 = \sum_i n_i R_i$, then

$$\chi^{R_1 \otimes R_2} = \sum_i n_i \chi^{R_i},$$

where the weight n_i denotes the degeneracy of the representation R_i in the product. In the indefinite metric case, the weight n_i will be negative if the inner product on R_i is of opposite sign to the usual conventions as in (6), or equivalently, if the lowest weight state has negative norm squared. This is due to the inclusion of the signature in the definition (26) above.

Specializing to $U = e^{i\theta J_z}$, we can now compute the characters of the negative dimensional representations. We find, for $j < 0$

$$\begin{aligned} \chi^{(j)}(\theta) &= e^{ij\theta}\,(1 - e^{i\theta} + e^{2i\theta} - \cdots)\\ &= \frac{e^{ij\theta}}{1 + e^{i\theta}}. \qquad (31) \end{aligned}$$

Note that the singularity at $\theta = \pi$ is to be expected. Indeed, as we remarked in section 5, certain group elements in the conjugacy class of rotations by π are not representable on the negative spin state space. The singularity in the character reflects this fact.

11 General product decompositions

Let us first consider the combination of two negative spin representations. For $j_1, j_2 < 0$, we have

$$\begin{aligned}
\chi^{(j_1)}\chi^{(j_2)} &= e^{i(j_1+j_2)\theta}\left(\frac{1}{1+e^{i\theta}}\right)^2 \\
&= e^{i(j_1+j_2)\theta}\,\frac{1}{1+e^{i\theta}}\,(1 - e^{i\theta} + e^{2i\theta} - \cdots) \\
&= \chi^{(j_1+j_2)} - \chi^{(j_1+j_2-1)} + \chi^{(j_1+j_2-2)} - \cdots \,.
\end{aligned} \tag{32}$$

This then implies that, for $j_1, j_2 < 0$, we have

$$R^{(j_1)} \otimes R^{(j_2)} = \sum_{J=-\infty}^{j_1+j_2} (-)^{j_1+j_2-J} R^{(J)}. \tag{33}$$

The analysis of the product of positive and negative representations is most easily done by expanding in powers of $e^{i\theta}$ as in (12) and (31), performing the multiplication and collecting terms. This is straightforward, and we state the results.

When $j_1 > 0$, $j_2 < 0$ and $|j_2| > j_1$, we have

$$\chi^{(j_1)}\chi^{(j_2)} = \chi^{(-j_1+j_2)} + \chi^{(-j_1+j_2+1)} + \cdots + \chi^{(j_1+j_2)}, \tag{34}$$

so that

$$R^{(j_1)} \otimes R^{(j_2)} = \sum_{J=-j_1+j_2}^{j_1+j_2} R^{(J)}. \tag{35}$$

When $j_1 > 0$, $j_2 < 0$ and $|j_2| \leq j_1$, and $j_1 - j_2$ is integral, then

$$\begin{aligned}
\chi^{(j_1)}\chi^{(j_2)} &= \chi^{(j_1+j_2)} - \chi^{(j_1+j_2-1)} + \cdots \pm \chi^{(0)} \\
&\quad + \chi^{-j_1+j_2} + \chi^{-j_1+j_2+1} + \cdots + \chi^{-j_1-j_2-2},
\end{aligned} \tag{36}$$

so that

$$R^{(j_1)} \otimes R^{(j_2)} = \sum_{J=-j_1+j_2}^{-j_1-j_2-2} R^{(J)} + \sum_{J=0}^{j_1+j_2} (-)^{j_1+j_2-J} R^{(J)}. \tag{37}$$

When $j_1 > 0$, $j_2 < 0$ and $|j_2| < j_1$, and $j_1 - j_2$ is half-integral, then

$$\begin{aligned}\chi^{(j_1)}\chi^{(j_2)} &= \chi^{(j_1+j_2)} - \chi^{(j_1+j_2-1)} + \cdots \pm \chi^{(-\frac{1}{2})} \\ &\quad + \chi^{-j_1+j_2} + \chi^{-j_1+j_2+1} + \cdots + \chi^{-j_1-j_2-2},\end{aligned} \tag{38}$$

so that

$$R^{(j_1)} \otimes R^{(j_2)} = \sum_{J=-j_1+j_2}^{-j_1-j_2-2} R^{(J)} + \sum_{J=-\frac{1}{2}}^{j_1+j_2} (-)^{j_1+j_2-J} R^{(J)}. \tag{39}$$

These are the main results of this paper.

12 Dimensions

The state spaces of the negative spin representations are infinite-dimensional vector spaces. However, a more useful concept of dimension may be obtained by defining the effective dimension of a representation R to be the character $\chi^{(R)}(\mathbf{1})$ evaluated on the unit matrix. While this coincides with the usual dimension in the case of positive spin representations, in the case of negative spin, taking $\theta = 0$ in (31) gives a fractional dimension

$$\frac{e^0}{1+e^0} = \frac{1}{2}$$

for all the negative spin representations. Since $\chi^{(R)}(\mathbf{1})$ is, by our definition of the character, just the signature of the representation space, this may also be thought of as

$$1 - 1 + 1 - 1 + \cdots = \frac{1}{2},$$

which coincides with the Abel regularization $\lim_{z\to 1} \sum z^n$ of this alternating series.

From the formulae of the previous section, we see that whenever a representation appears with inner product of sign opposite to the usual convention (6), we should think of its dimension as negative.

Since the dimensions are defined in terms of the characters, the results of the previous sections guarantee that the dimensions behave in the expected way under multiplication and summation of representations. For example, for $j_1, j_2 < 0$, we had

$$R^{(j_1)} \otimes R^{(j_2)} = \sum_{J=-\infty}^{j_1+j_2} (-)^{j_1+j_2-J} R^{(J)}. \tag{40}$$

The dimension of the tensor product on the left hand side is $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$. The sum of the dimensions on the right hand side is

$$\frac{1}{2} - \frac{1}{2} + \frac{1}{2} - \cdots = \frac{1}{2}(1 - 1 + 1 - \cdots) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4},$$

as expected.

As another example, consider

$$R^{(\frac{5}{2})} \otimes R^{(-\frac{3}{2})} = R^{(1)} - R^{(0)} + R^{(-3)} + R^{(-4)}.$$

The dimension of the tensor product on the left hand side is $6 \cdot \frac{1}{2} = 3$. Summing the dimensions of the representations on the right hand side, we obtain

$$3 - 1 + \frac{1}{2} + \frac{1}{2},$$

again as expected.

13 Example: $R^{(1)} \otimes R^{(-\frac{1}{2})}$

We conclude by working out the following example from (39)

$$R^{(1)} \otimes R^{(-\frac{1}{2})} = R^{(\frac{1}{2})} - R^{(-\frac{1}{2})}. \tag{41}$$

Let us introduce the notation $|m\rangle_v \equiv e_m^{(1)}$ for the vector representation. From the above decomposition formula, we expect to find a spin $\frac{1}{2}$ and a spin $-\frac{1}{2}$ lowest weight state in the product representation. The spin $\frac{1}{2}$ lowest weight state is just

$$\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle \equiv |-1\rangle_v \otimes \left|\tfrac{1}{2}\right\rangle_g.$$

This state is trivially annihilated by J_-. Furthermore, using (7) we find

$$J_+ \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle = \sqrt{2}\,|0\rangle_v \otimes \left|\tfrac{1}{2}\right\rangle_g + i\,|-1\rangle_v \otimes \left|\tfrac{3}{2}\right\rangle_g,$$

which is not null. However, applying J_+ twice, we get the state

$$(J_+)^2 \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle = 2\left(|1\rangle_v \otimes \left|\tfrac{1}{2}\right\rangle_g + i\sqrt{2}\,|0\rangle_v \otimes \left|\tfrac{3}{2}\right\rangle_g - |-1\rangle_v \otimes \left|\tfrac{5}{2}\right\rangle_g\right),$$

which is null.

A spin $-\frac{1}{2}$ lowest weight state in the product is given by

$$\left|-\tfrac{1}{2}, \tfrac{1}{2}\right\rangle \equiv i\sqrt{2}\,|-1\rangle_v \otimes \left|\tfrac{3}{2}\right\rangle_g + |0\rangle_v \otimes \left|\tfrac{1}{2}\right\rangle_g.$$

This lowest state has negative norm squared, as expected from the negative sign in (41). In addition, it is orthogonal to the spin $\frac{1}{2}$ state $J_+ \left|\frac{1}{2}, -\frac{1}{2}\right\rangle$ with the same m-value. By repeatedly applying J_+, we get an infinite tower of states filling out the spin $-\frac{1}{2}$ representation.

Each of the subspaces with a specific value of J_z eigenvalue m is three-dimensional. One of these dimensions is taken up by the semi-infinite spin $-\frac{1}{2}$ tower. The remaining two dimensions belong to the spin $\frac{1}{2}$ representation, which has the form (25). Taking the cohomology with respect to Q gets rid of the unwanted higher m states in the spin $\frac{1}{2}$ representation, and we are left with the classes

$$[\left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle], \qquad [J_+ \left|\tfrac{1}{2}, -\tfrac{1}{2}\right\rangle].$$

Now $[J_+]^2 \, [\left|\frac{1}{2}, -\frac{1}{2}\right\rangle] = 0$, and we have an ordinary spin $\frac{1}{2}$ representation on the cohomology. The spin $-\frac{1}{2}$ representation remains unaffected by taking the cohomology.

The dimension of the tensor product is $3 \cdot \frac{1}{2} = \frac{3}{2}$. The sum of the dimensions of the decomposition on the right hand side is $2 - \frac{1}{2} = \frac{3}{2}$, so the two sides match up as expected.

14 Conclusion

We showed that the representation theory of $su(2)$ may be extended to include a class of negative spin representations. We showed that arbitrary positive and negative representations may be multiplied and decomposed into irreducible representations after reducing the product to the cohomology of a certain nilpotent BRST-like operator Q. We defined characters and effective dimensions for the negative spin representations and wrote down explicit formulae for the product decomposition in the general case.

The program initiated here may be extended in various directions. First, it would be interesting to work out the Clebsch-Gordan coefficients, 3-j and 6-j symbols in the general case. Second, although we have argued that the negative spin Lie algebra representations do not exponentiate to representations of the whole group, it is possible to write down explicit forms for a subclass of finite $SU(2)$ transformations (see [2]). It would be interesting to understand somewhat better the status of these transformations.

Finally, it may be possible to apply some of the ideas developed here to a larger class of Lie algebras.

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Appendices

A BRST cohomology

In the BRST formalism (see [6], [7] and references therein), the analysis of physical states and operators is carried out in terms of an operator Q that is hermitian and nilpotent. In other words,

$$Q^\dagger = Q, \qquad Q^2 = 0.$$

States are called physical if they satisfy

$$Q\,|\phi\rangle = 0,$$

and are regarded as equivalent if they differ by a Q-exact state. In other words,

$$|\phi\rangle \sim |\phi\rangle + Q\,|\chi\rangle\,,$$

where $|\chi\rangle$ is an arbitrary state. More formally, physical states are elements of the cohomology of Q, defined as the quotient vector space

$$\ker Q \,/\, \operatorname{im} Q$$

with elements

$$[|\phi\rangle] \equiv |\phi\rangle + \operatorname{im} Q.$$

The inner product on this quotient space may be defined in terms of the original inner product by noting that all elements of $\operatorname{im} Q$ are orthogonal to all elements of $\ker Q$, so that the induced inner product defined on equivalence classes in the cohomology by

$$\langle\, \phi + \operatorname{im} Q \,|\, \phi' + \operatorname{im} Q\rangle \equiv \langle\, \phi \,|\, \phi'\rangle\,, \qquad \phi, \phi' \in \ker Q$$

is well defined.

A hermitian operator A is regarded as physical if $[A, Q] = 0$. This ensures that A leaves $\operatorname{im} Q$ invariant, so that the reduced operator $[A]$ defined on the cohomology classes by

$$[A]\,[|\phi\rangle] = [A\,|\phi\rangle] \tag{42}$$

is well-defined.

B Jordan decomposition of $\mathbf{J}^2$

In this appendix we prove that $\mathbf{J}^2$ can be decomposed into Jordan blocks of dimension at most two.

First, note that, since $\mathbf{J}^2$ commutes with J_z, we can decompose $\mathbf{J}^2$ into Jordan blocks in each eigenspace of J_z. Consider such a Jordan block on a subspace V_m consisting of principal vectors of $\mathbf{J}^2$ belonging to an eigenvalue $j(j+1)$ and with J_z eigenvalue m. Taking any vector v in V_i, by applying J_- to it repeatedly we will eventually obtain zero, since our product representations are bounded below in J_z. This gives a lowest weight state $J_-^k v$, and we can use (2) to obtain $j(j+1)$ in terms of the J_z eigenvalue $m-k$ of this lowest weight state. Since m is integer or half-integer, the possible values of j are also integer or half-integer, either positive or negative. In the following, we shall take the positive solution $j > 0$.

Now consider the sequence of subspaces

$$V_m \xrightarrow{J_-} V_{m-1} \xrightarrow{J_-} V_{m-2} \xrightarrow{J_-} \cdots .$$

Since $\mathbf{J}^2$ commutes with J_-, we see that $\mathbf{J}^2$ takes each of the subspaces V_i to itself. Furthermore, as long as $i \notin \{-j, j+1\}$, J_- cannot change the dimension of these subspaces since that would imply that $\ker J_-$ is not empty, so there would be lowest weight states at values of i inconsistent with j. In other words, the dimensions of the above sequence of spaces V_i can at most jump at $i \in \{-j, j+1\}$. As a corollary, the above sequence terminates at either $i = -j$ or $i = j+1$.

Furthermore, $\mathbf{J}^2$ consists of a single Jordan block on each of the subspaces V_i. By assumption, this is true for the first element V_m of the sequence. In general, assume that $\mathbf{J}^2$ consists of a single Jordan block on V_i and consider V_{i-1}. Since from (2) we have that $J_+V_{i-1} = J_+J_-V_i = (\mathbf{J}^2 - J_z^2 + J_z)V_i$, we see that $J_+V_{i-1} \subseteq V_i$. Now if $\mathbf{J}^2$ were to consist of more than one Jordan block on V_{i-1}, each of these blocks would contain an eigenvector of $\mathbf{J}^2$. Since J_+ commutes with $\mathbf{J}^2$, all these eigenvectors will be taken by J_+ to eigenvectors in V_i, of which there is only one by assumption. Therefore J_+ is not one to one, its kernel on V_{i-1} is nonempty, and there is a highest weight state in V_{i-1}, which is inconsistent with j unless $i \in \{-j, j+1\}$. This proves the assertion when $i \notin \{-j, j+1\}$.

Now consider the case $i = j+1$. The case $i = -j$ is similar. Choose $\tilde{v}_j \in V_j$ such that $J_+\tilde{v}_j = 0$. Since $V_j = J_-V_{j+1}$, there is a $\tilde{v}_{j+1} \in V_{j+1}$ such that $\tilde{v}_j = J_-\tilde{v}_{j+1}$. Then $0 = J_+J_-\tilde{v}_{j+1} = (\mathbf{J}^2 - J_z^2 + J_z)\tilde{v}_{j+1}$, which implies that $\tilde{v}_{j+1}$ is an eigenvector of $\mathbf{J}^2$ and therefore is proportional to the unique eigenvector v_{j+1} in V_{j+1}. Now note that J_+v_j cannot be zero for all eigenvectors v_j in V_j, because if that were the case, then by the above argument there would be more than one linearly independent eigenvector of $\mathbf{J}^2$ in V_j. Therefore we can find a v_j such that $J_+ v_j = v_{j+1}$. Then $J_- J_+ v_j = (\mathbf{J}^2 - J_z^2 - J_z) v_j = [j(j+1) - j(j+1)]\, v_j = 0$, or $J_- v_{j+1} = 0$. But we had $0 \neq \tilde{v}_j = J_- \tilde{v}_{j+1}$, and $\tilde{v}_{j+1} \propto v_{j+1}$, which is a contradiction. This proves the assertion when $i = j + 1$.

We have proved that $\mathbf{J}^2$ consists of a single Jordan block on each V_i. This means that each V_i contains one and only one eigenvector. Consequently, since elements in the kernel of J_- are automatically eigenvectors, the dimension of the V_i can be reduced by at most one at each of the two transition points $V_{j+1} \xrightarrow{J_-} V_j$ and $V_{-j} \xrightarrow{J_-} V_{-j-1}$. Since the sequence terminates at either V_{j+1} or V_{-j}, the initial space V_m can be at most two-dimensional. This completes the proof that the Jordan blocks of $\mathbf{J}^2$ are at most two-dimensional.

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