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Star Products from Open Strings in Curved Backgrounds

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Abstract

We define a non-commutative product for arbitrary gauge and B-field backgrounds in terms of correlation functions of open strings. While off-shell correlations are, of course, not conformally invariant, it turns out that, at least to first derivative order, our product has the trace property and is associative up to surface terms if the background fields are put on-shell. No on-shell conditions for the inserted functions are needed, but it is essential to include the full contribution of the Born-Infeld measure. We work with a derivative expansion and avoid any topological limit, which would effectively constrain H.

Keywords: Non-commutative geometry, D-branes, open strings

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1 Introduction and summary

Noncommutative geometry as it turned out to arise from open strings in a background B-field [1–8] has attracted much attention. Most of this and subsequent work was done in the context of a constant background. In terms of D-brane physics this corresponds to an embedding of a flat brane into a flat background. It is well known that in this situation the implications of the B field background can effectively be described by replacing the ordinary (commutative) product of functions on the world volume of the D-brane by a noncommutative product, the Moyal product.

There have been several attempts to generalize this picture to the situation of a non-constant background. This lead to a physical interpretation of Kontsevich's formula [9], originally derived in the context of deformation quantization of the algebra of functions on Poisson manifolds, in terms of the perturbative expansion of the path integral of a topological model of bosonic open strings [10]. A typical example of a Poisson manifold is provided by a symplectic manifold, i.e., a differentiable manifold naturally equipped with a closed non degenerate two form, $d\omega = 0$. In terms of string theory this is related to a background B-field with vanishing field strength, $H = dB = 0$.

In this paper we address the problem of open strings in general backgrounds, in particular B field backgrounds with nonvanishing field strength [11–13]. In the terminology of D-branes this is related to the embedding of curved branes into curved backgrounds. It has been argued that in this case the algebra of functions on the D-brane world volume becomes non-associative, with the non-associativity controlled by the field strength H [12]. Nevertheless, it turned out that the product is still described in terms of Kontsevich's formula.

Following a similar strategy as the authors of ref. [12] we will expand about background fields to extract the star product from correlation functions computed on the disk. We will work with a derivative expansion and will nowhere use a zero slope limit in our arguments. Furthermore, we do not choose any gauge conditions for the background gauge fields. Here our setting deviates vitally from the one used in [12], where radial gauge was imposed on the two form gauge potential B. This choice of gauge allows them to extract the part of the nonassociative star product deformation due to H, while the part due to $F = dA$ is given by Kontsevich's formula. Instead we prefer to work in a manifestly gauge invariant way. Furthermore, we only perform a perturbation expansion around the constant zero modes, and not an additional expansion around constant backgrounds, as done in [12]. This keeps the full zero mode dependence of the background fields and simplifies the calculations.

Our main concern in this paper will be to discuss the properties of the product obtained by the procedure described above. Although this product is noncommutative and even nonassociative, we will show that associativity of the product of three functions and the trace property for the integrated product of an arbitrary number of functions hold, at least to first order in the derivative expansion, up to surface terms. This is achieved by including the full Born-Infeld measure and the equations of motion of the spacetime background fields. However, no on-shell condition is needed for the functions inserted in the product! We find this result remarkable in view of a possible application of our product for the description of correlation functions. Some work in this direction is in progress.

Finally, we comment on the relation to the recent work of Cornalba and Schiappa [12]. Using the topological limit $g_{\mu\nu} \to 0$ they found that with the choice of radial gauge it is possible to adjust the integration measure in such a way that the integral still acts as a trace. We rather use the measure that arises from the string theory correlations. In this approach it turns out that the trace property of the integral is maintained when the background fields are put on-shell. This holds independent of the gauge and even away from the topological limit.

The paper is organized as follows. In section 2 we introduce the setup for the models under consideration. We give the derivative expansions of the background fields in terms of Riemannian normal coordinates and introduce the additional interaction vertices. In section 3 we review the calculations of [5] for the free field theory defined by the constant parts of the background fields and identify the effective open string parameters G and Θ. The vacuum amplitude of the free theory on the disk is computed in section 4. It contributes the "Born-Infeld" measure to the integration over the zero modes in the path integral. The relevant disk correlators are then presented in section 5, with some technical details given in the appendix. In section 6 we extract a noncommutative and nonassociative Kontsevich-type product from these correlators and discuss its properties. In particular we show that the trace property of the two-point function holds due to the equations of motion of the background fields. The "Born-Infeld" measure exactly cancels the additional contributions arising from partial integration. By the same mechanism the product of three functions does not depend on the way one introduces brackets, i.e. the nonassociativity is a surface term. This, in turn, implies the trace property for an arbitrary number of functions. We finish this section with some comments on the relations of our approach to the recent work of Cornalba and Schiappa. In particular we examine the implications of the radial gauge and the consistency of the topological limit used in [12]. In the last section we conclude with comments on some open questions and an outlook on further work.

2 The open string sigma model

The starting point of our considerations is the nonlinear sigma model of the bosonic open string [15–17]

$$\begin{aligned} S &= \frac{1}{4\pi\alpha'}\int_\Sigma \mathrm{d}^2\sigma\sqrt{h}\Big(h^{ab}\partial_a X^\mu \partial_b X^\nu g_{\mu\nu}(X) + i\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X)\Big) \\ &+ i\int_{\partial\Sigma}\mathrm{d}s\Big(\partial_s X^\mu A_\mu(X)\Big), \end{aligned} \tag{1}$$

which includes the spacetime metric $g_{\mu\nu}(X)$, the 2-form gauge potential $B_{\mu\nu}(X)$ and the 1-form gauge potential $A_\mu(X)$. h_{ab} denotes the Euclidean metric on the world sheet Σ and $\mathrm{d}s$ is the induced line element on the boundary.

In (1) the boundary term with the 1-form gauge potential A can be rewritten as a bulk term

$$\int_{\Sigma} \mathrm{d}^2\sigma \sqrt{h}\, i\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu F_{\mu\nu}(X), \tag{2}$$

where $F = \mathrm{d}A$ is the corresponding 2-form field strength.

Both, the 1-form potential A and the 2-form potential B, are associated with spacetime gauge invariances. For the former the gauge transformation

$$\delta A = \mathrm{d}\lambda \tag{3}$$

leaves the action (1) invariant. In open string theory there does not exist a gauge transformation for the 2-form potential B alone, because surface terms require a combined transformation

$$\begin{aligned} \delta B &= \mathrm{d}\Lambda, \\ \delta A &= -\frac{\Lambda}{2\pi\alpha'} \end{aligned} \tag{4}$$

that does not change the action (1). From (3) and (4) one can see that the combination $\mathcal{F} = B + 2\pi\alpha' F = B + 2\pi\alpha'\mathrm{d}A$ is invariant under both gauge symmetries. Therefore, gauge invariant expressions contain the 2-form $\mathcal{F}$ and the 3-form field strength $H = \mathrm{d}\mathcal{F} = \mathrm{d}B$.

If one considers a brane that is not spacetime filling, the gauge field A and hence $\mathcal{F}$, as well as the resulting noncommutative geometry, are only defined on that brane. Furthermore, in topologically nontrivial backgrounds, the gauge potentials A and B may not be globally defined. These issues are, however, irrelevant in the present context.

In the classical approximation of open string theory the world sheet Σ is a disk. Taking advantage of the conformal invariance of the theory, we map the disk to the upper half plane $\mathbb{H}$ and perform our calculations there. Furthermore, we choose the conformal gauge and use complex coordinates $z = \sigma^1 + i\sigma^2$. Thus the world sheet metric becomes $h_{z\bar{z}} = e^{2\omega(z,\bar{z})}\delta_{z\bar{z}}$ and the invariant line element at the boundary is $\mathrm{d}s = e^{\omega}\mathrm{d}\tau$. The derivatives tangential and normal to the boundary are $\partial_\tau = (\partial + \bar{\partial})$ and $\partial_n = i(\bar{\partial} - \partial)$, respectively. In this parametrization the action (1) is given by

$$S = \frac{1}{2\pi\alpha'} \int_{\mathbb{H}} \mathrm{d}^2 z\, \partial X^\mu \bar{\partial} X^\nu \Big(g_{\mu\nu}(X) + \mathcal{F}_{\mu\nu}(X) \Big), \tag{5}$$

and the corresponding mixed boundary condition along the brane is

$$g_{\mu\nu}(X)(\partial - \bar{\partial})X^\nu - \mathcal{F}_{\mu\nu}(X)(\partial + \bar{\partial})X^\nu \Big|_{\bar{z}=z} = 0. \tag{6}$$

Following the standard procedure we expand the field $X^\mu(z,\bar{z})$ around the constant zero mode contribution x [15],

$$X^\mu(z,\bar{z}) = x^\mu + \zeta^\mu(z,\bar{z}), \tag{7}$$

so that the path integral over the field $X^\mu(z,\bar z)$ splits into an ordinary integral over the constant zero modes x^μ and a path integral over the quantum fluctuations $\zeta^\mu(z,\bar z)$

$$\begin{aligned}\langle :f_1[X(z_1)]: \;\ldots\; :f_N[X(z_N)]:\rangle \quad &= \\ &= \int [\mathrm{d}X]\, e^{-S[X]} f_1[X_1]\ldots f_N[X_N] \quad = \\ &= \int \mathrm{d}^D x \int [\mathrm{d}\zeta]\, e^{-S[x+\zeta]} f_1[x+\zeta_1]\ldots f_N[x+\zeta_N]\,, \end{aligned} \tag{8}$$

where the functions $f_i[X(z_i)]$ denote arbitrary insertions into the path integral. Before we expand the action $S[X]=S[x+\zeta]$ around the zero modes we simplify our computation by choosing Riemannian normal coordinates [18, 19],

$$g_{\mu\nu}(x+\zeta) \;=\; \eta_{\mu\nu} - \frac{1}{3}R_{\mu\rho\nu\sigma}(x)\zeta^\rho\zeta^\sigma + \mathcal{O}(\zeta^3), \tag{9}$$

$$\mathcal{F}_{\mu\nu}(x+\zeta) \;=\; \mathcal{F}_{\mu\nu}(x) + \partial_\rho\mathcal{F}_{\mu\nu}(x)\zeta^\rho + \frac{1}{2}\partial_\rho\partial_\sigma\mathcal{F}_{\mu\nu}(x)\zeta^\rho\zeta^\sigma + \mathcal{O}(\zeta^3). \tag{10}$$

In contrast to [12] we do not choose radial gauge for $\mathcal{F}_{\mu\nu}(X)$. In that case (10) would split into two separate expansions for B and F, where the non-constant part of the B expansion contains only the field strength H. With (9) and (10) we are able to write the action (5) as

$$\begin{aligned} S = \frac{1}{2\pi\alpha'}\int_{\mathbb{H}} \mathrm{d}^2 z \quad \Big\{ &\partial\zeta^\mu\bar\partial\zeta^\nu(\eta_{\mu\nu}+\mathcal{F}_{\mu\nu}) + \partial\zeta^\mu\bar\partial\zeta^\nu\zeta^\rho\,\partial_\rho\mathcal{F}_{\mu\nu} + \\ &+ \quad \partial\zeta^\mu\bar\partial\zeta^\nu\zeta^\rho\zeta^\sigma\,(\frac{1}{2}\partial_\rho\partial_\sigma\mathcal{F}_{\mu\nu} - \frac{1}{3}R_{\mu\rho\nu\sigma}) + \mathcal{O}(\partial_\rho^3)\Big\}. \end{aligned} \tag{11}$$

In the following we will restrict our considerations to terms of at most first order in derivatives of the spacetime background fields.

3 The free propagator

As a warm up for later calculations and to set up the relevant techniques of our approach let us first calculate the propagator for the free field theory defined by the Gaussian part of (11) in the path integral,

$$S_{\text{free}} = \frac{1}{2\pi\alpha'}\int_{\mathbb{H}} \mathrm{d}^2 z\,\partial\zeta^\mu\bar\partial\zeta^\nu\eta_{\mu\nu} + \frac{i}{4\pi\alpha'}\oint_{\partial\mathbb{H}} \mathrm{d}\tau\,\zeta^\mu\partial_\tau\zeta^\nu\mathcal{F}_{\mu\nu}. \tag{12}$$

Here $\partial\mathbb{H}$ denotes the boundary of the upper half plane, i.e., the real line.[1] The second term contributes to the boundary condition which takes the same form as (6) with $\eta_{\mu\nu}$ and $\mathcal{F}_{\mu\nu}(x)$ replacing the full metric $g_{\mu\nu}(X)$ and $\mathcal{F}_{\mu\nu}(X)$, respectively. The boundary term can be regarded as a perturbative correction [5] to the free propagator

$$\langle\zeta^\mu(u,\bar u)\,\zeta^\nu(w,\bar w)\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|u-w|^2 - \frac{\alpha'}{2}\eta^{\mu\nu}\ln|u-\bar w|^2. \tag{13}$$

[1] We have used the divergence theorem for complex coordinates, which reads $\int_\Sigma \mathrm{d}^2 z\,(\partial_z v^z \pm \partial_{\bar z} v^{\bar z}) = i\oint_{\partial\Sigma}(\mathrm{d}\bar z v^z \mp \mathrm{d}z v^{\bar z})$.

The homogeneous (image charge) part accounts for the Neumann boundary condition $\partial_n \zeta^\mu|_{\partial\mathbb{H}} = 0$ of the theory without perturbation. The propagator of the perturbed theory is then given by the (connected) 2-point correlation

$$\langle \zeta^\mu(u,\bar u)\,\zeta^\nu(w,\bar w)\rangle_{\mathcal{F}} = \langle \zeta^\mu(u,\bar u)\,\zeta^\nu(w,\bar w) \mathrm{e}^{-\frac{i}{4\pi\alpha'}\oint_{\partial\mathbb{H}} \mathrm{d}\tau\, \zeta^\rho \partial_\tau \zeta^\sigma \mathcal{F}_{\rho\sigma}} \rangle. \tag{14}$$

Disconnected loop contributions will only contribute to the measure (see below). Expanding in a perturbation series the term of order n

$$\langle \zeta^\mu(u,\bar u)\zeta^\nu(w,\bar w)\frac{1}{n!}\Big\{\frac{i}{4\pi\alpha'}\Big[\oint_{\partial\mathbb{H}} \mathrm{d}z\,\partial\zeta^\rho\zeta^\sigma\mathcal{F}_{\rho\sigma} + \oint_{\partial\mathbb{H}} \mathrm{d}\bar z\,\bar\partial\zeta^\rho\zeta^\sigma\mathcal{F}_{\rho\sigma}\Big]\Big\}^n\rangle \tag{15}$$

gives two slightly different contributions, depending on whether n is even or odd. By using the derivative of the propagator (13) it is straightforward to obtain the result[2]

$$\begin{aligned}\frac{i}{2\pi}(\mathcal{F}^n)_{\lambda\kappa}\Big\{(-1)^{n-1}\oint_{\partial\mathbb{H}} \mathrm{d}z\,\eta^{\mu\lambda}\frac{1}{\bar u - z}\langle\zeta^\kappa\zeta^\nu(w,\bar w)\rangle \\ + \oint_{\partial\mathbb{H}} \mathrm{d}\bar z\,\eta^{\mu\lambda}\frac{1}{u-\bar z}\langle\zeta^\kappa\zeta^\nu(w,\bar w)\rangle\Big\}\quad .\end{aligned} \tag{16}$$

The remaining divergent integrals are regularized by differentiating with respect to w and $\bar w$, respectively. This yields a finite result plus an infinite additive constant $C^{\mu\nu}_{(\infty)}$,

$$\alpha'(\mathcal{F}^n)^{\mu\nu}\Big\{(-1)^{n-1}\ln(\bar u - w) - \ln(u-\bar w)\Big\} + C^{\mu\nu}_{(\infty)}. \tag{17}$$

Now it is possible to sum up all orders in a geometrical series, which finally gives the desired propagator [16, 17]

$$\begin{aligned}\langle \zeta^\mu(u,\bar u)\,\zeta^\nu(w,\bar w)\rangle_{\mathcal{F}} = -\alpha'\Big\{&\eta^{\mu\nu}(\ln|u-w| - \ln|u-\bar w|) \\ &+ G^{\mu\nu}\ln|u-\bar w|^2 - \Theta^{\mu\nu}\ln\Big(\frac{\bar w - u}{\bar u - w}\Big)\Big\} + C^{\mu\nu}_{(\infty)},\end{aligned} \tag{18}$$

where we have introduced the quantities[3]

$$G^{\mu\nu} := \Big(\frac{1}{g-\mathcal{F}}\; g\; \frac{1}{g+\mathcal{F}}\Big)^{\mu\nu} \quad \text{and} \quad \Theta^{\mu\nu} := -\Big(\frac{1}{g-\mathcal{F}}\;\mathcal{F}\;\frac{1}{g+\mathcal{F}}\Big)^{\mu\nu}. \tag{19}$$

The integration constant $C^{\mu\nu}_{(\infty)}$ plays no essential role and can be set to a convenient value, e.g. $C^{\mu\nu}_{(\infty)} = 0$ [8]. When restricted to the boundary ($u = \bar u = \tau$ and $w = \bar w = \tau'$) the propagator reduces to the simple form

$$\begin{aligned}\alpha' i\pi\Delta^{\mu\nu}(\tau,\tau') &:= \langle\zeta^\mu(\tau)\,\zeta^\nu(\tau')\rangle_{\mathcal{F}} \\ &= -\alpha' G^{\mu\nu}\ln(\tau-\tau')^2 - \alpha' i\pi\Theta^{\mu\nu}\epsilon(\tau-\tau').\end{aligned} \tag{20}$$

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[2] In this calculation there appear integrals of the form $\oint_{\partial\mathbb{H}} \mathrm{d}z\frac{1}{\bar u - z}\frac{1}{\bar z - w}$. The part along the real axis $\mathbb{R}$ is $\int_{\mathbb{R}} \mathrm{d}r\frac{1}{\bar u - r}\frac{1}{r-w}$, whereas the integral along the semicircle in the upper half plane with infinite radius is zero. Therefore, the original integral can be written as

$$\oint_{\partial\mathbb{H}} \mathrm{d}z\frac{1}{\bar u - z}\frac{1}{\bar z - w} = \oint_{\partial\mathbb{H}} \mathrm{d}z\frac{1}{\bar u - z}\frac{1}{z-w},$$

which can be evaluated using the residue theorem.

[3] For later reference we have expressed $G^{\mu\nu}$ and $\Theta^{\mu\nu}$ using the full bulk metric $g_{\mu\nu}$, whereas the terms in (18) contain, of course, the Minkowski metric $\eta_{\mu\nu}$ because of the Riemannian normal coordinates.

As discussed in [8] the boundary propagator (20) suggests to interpret $G_{\mu\nu}$ as an effective metric seen by the open strings, in contrast to $g_{\mu\nu}$, which is to be viewed as the closed string metric in the bulk.

For later purposes we elaborate on the distinction between the open string quantities $G^{\mu\nu}$ and $\Theta^{\mu\nu}$ and the closed string quantities $g_{\mu\nu}$ and $B_{\mu\nu}$. In order to make a clear distinction between the bulk and the boundary quantities, we mark all expressions that refer to boundary quantities with bars. To this end we define

$$\bar{G}_{\mu\nu} := (g - \mathcal{F}^2)_{\mu\nu} \quad \text{and} \quad \bar{\Theta}^{\mu}{}_{\nu} := -\mathcal{F}^{\mu}{}_{\nu}. \tag{21}$$

The first of the above definitions is equivalent to setting $\bar{G}^{\mu\nu} = G^{\mu\nu}$ and requiring $\bar{G}_{\mu\nu}$ to be its inverse. The second definition follows from setting $\bar{\Theta}^{\mu\nu} = \Theta^{\mu\nu}$ and pulling indices with $\bar{G}_{\mu\nu}$. In an analogous way we label all expressions that are built out of these quantities with bars, e.g. the Christoffel symbol $\bar{\Gamma}_{\mu\nu}{}^{\rho}$ and the covariant derivative $\bar{D}_\mu$ compatible with the open string metric $\bar{G}_{\mu\nu}$.

4 Vacuum amplitude and integration measure

Let us now consider loop contributions arising from an even number of insertions of the boundary perturbation of (12)[4]. In this calculation there appear divergences when the insertion points approach the boundary. We regularize these terms by keeping a fixed distance d_0 with respect to the metric in conformal gauge to the boundary $\partial\mathbb{H}$, i.e., we impose $|z-\bar{z}| \geq 2\mathrm{Im}(z) \geq e^{-\omega} d_0$. To make this more explicit let us consider the one loop contribution of the $\mathcal{F}^2$ term,

$$\frac{1}{2}\langle\Big(\frac{-i}{4\pi\alpha'}\Big)^2 \oint_{\partial\mathbb{H}} \mathrm{d}\tau\, \zeta^\mu \partial_\tau \zeta^\nu \mathcal{F}_{\mu\nu} \times \oint_{\partial\mathbb{H}} \mathrm{d}\tau'\, \zeta^\rho \partial'_\tau \zeta^\sigma \mathcal{F}_{\rho\sigma}\rangle_{1-\mathrm{loop}}. \tag{22}$$

Using the same techniques as for the chains (15) gives the divergent contribution

$$\Big(\frac{1}{4\pi d_0}\int \mathrm{d}s\Big)\frac{1}{2}\mathcal{F}_\mu{}^\nu \mathcal{F}_\nu{}^\mu, \tag{23}$$

where $\mathrm{d}s = \mathrm{d}\tau e^{\omega}$ is the invariant line element in conformal gauge. Summing up all powers of $\mathcal{F}$ in the 1-loop contribution yields

$$\Big(\frac{1}{4\pi d_0}\int \mathrm{d}s\Big)\sum_{n=1}^{\infty}\frac{1}{2n}\mathcal{F}^{2n} = -\Big(\frac{1}{4\pi d_0}\int \mathrm{d}s\Big)\frac{1}{2}\ln(\det(\delta - \mathcal{F}^2)_\mu{}^\nu). \tag{24}$$

As observed in [20,21] this linear divergence is in fact regularization scheme dependent and can be absorbed into the tachyon by a field redefinition. But a finite constant part

$$b_0 \ln(\det(\delta - \mathcal{F}^2)_\mu{}^\nu) \tag{25}$$

[4] Odd powers vanish because of the antisymmetry of $\mathcal{F}_{\mu\nu}$

may remain after subtraction of appropriate counter terms. The analysis given in [15, 21] determined the constant b_0 to be $\frac{1}{4}$ in order to yield the Born-Infeld action for a vanishing tachyon field.

In (25) we have added up all powers of $\mathcal{F}$ contributing to the connected vacuum graphs. Taking into account all disconnected one loop graphs to all orders of the interaction leads to the Born-Infeld Lagrangian

$$\sum_{n=0}^{\infty} \frac{1}{n!} \big(\ln(\det(\delta - \mathcal{F}^2)_\mu{}^\nu)^{\frac{1}{4}} \big)^n = \sqrt[4]{\det(\delta - \mathcal{F}^2)_\mu{}^\nu} = \sqrt{\det(\delta - \mathcal{F})_\mu{}^\nu} \ . \tag{26}$$

Here we used the antisymmetry of $\mathcal{F}_{\mu\nu}$ to change the sign in the determinant. Expression (26) can also be interpreted as a contribution to the measure of the integration over the zero modes in the path integral. Although we make use of Riemannian normal coordinates for the perturbation expansion, we can write the measure in a covariant way by including the term $\sqrt{\det g_{\mu\nu}}$. Therefore, if there are no operator insertions in the path integral (8), we obtain the Born-Infeld action

$$\begin{aligned} \int \mathrm{d}^D x \sqrt{\det g_{\mu\nu}} \sqrt[4]{\det(\delta - \mathcal{F}^2)_\mu{}^\nu} &= \int \mathrm{d}^D x \sqrt[4]{\det g_{\mu\nu}} \sqrt[4]{\det \bar{G}_{\mu\nu}} = \\ &= \int \mathrm{d}^D x \sqrt{\det(g - \mathcal{F})_{\mu\nu}}, \end{aligned} \tag{27}$$

where $\bar{G}_{\mu\nu}$ is the boundary metric as defined in (21).

So far we have taken into account all possible diagrams of the boundary insertion of (12). Therefore, we can now work with the full propagator (18) for all higher order interaction terms. For the remainder of the paper we make use of the abbreviations $g = \det g_{\mu\nu}$ and $\int_x = \int \mathrm{d}^D x \sqrt{g - \mathcal{F}} = \int \mathrm{d}^D x \sqrt[4]{g} \sqrt[4]{G}$. Furthermore, we set $2\pi\alpha' = 1$.

5 Correlation functions

In string theory interactions of different particles of the string spectrum are calculated by inserting the corresponding vertex operators into the path integral. Our goal is to extract a noncommutative product of functions from the open string theory correlation functions [12]. To this end we do not restrict ourselves to on-shell vertex operators, but investigate the correlator of two general functions $f[X(\tau)]$ and $g[X(\tau')]$ allowed to be off-shell. To simplify the calculations we take the order of insertions to be $\tau < \tau'$. Since the functions are composed operators, one has to introduce an appropriate normal ordering. As shown in appendix A.4 it is given by

$$\begin{aligned} :\zeta^\mu(\tau)\, \zeta^\nu(\tau'): \ &= \ \zeta^\mu(\tau)\, \zeta^\nu(\tau') \\ &+ \frac{1}{2\pi} G^{\mu\nu} \ln(\tau - \tau')^2 + \frac{1}{2\pi} \partial_\rho G^{\mu\nu} \ln(\tau - \tau')^2 \, \zeta^\rho(\frac{\tau + \tau'}{2}). \end{aligned} \tag{28}$$

Taking into account the subtractions (28) the free propagator (20) yields [5]

$$\begin{aligned}
&\langle :f[X(\tau)]::g[X(\tau')]:\rangle_{\mathcal{F}} = \\
&= \int_x \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{i}{2}\Big)^n \Delta^{\mu_1\nu_1}\ldots\Delta^{\mu_n\nu_n}\,\partial_{\mu_1}\ldots\partial_{\mu_n} f(x)\,\partial_{\nu_1}\ldots\partial_{\nu_n} g(x) \\
&= \int_x \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{-1}{2\pi}\Big)^n G^{\mu_1\nu_1}..\,G^{\mu_n\nu_n}\ln^n(\tau-\tau')^2\,\partial_{\mu_1}..\,\partial_{\mu_n} f(x)\,*\,\partial_{\nu_1}..\,\partial_{\nu_n} g(x).
\end{aligned} \tag{29}$$

In the last line we have summarized all $\Theta^{\mu\nu}$-dependent contributions in the product

$$\begin{aligned}
f*g &= \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{i}{2}\Big)^n \Theta^{\mu_1\nu_1}(x)\ldots\Theta^{\mu_n\nu_n}(x)\,\partial_{\mu_1}\ldots\partial_{\mu_n} f(x)\,\partial_{\nu_1}\ldots\partial_{\nu_n} g(x) \\
&= e^{\frac{i}{2}\Theta^{\mu\nu}(z)\partial_{x^\mu}\partial_{y^\nu}} f(x)g(y)\Big|_{x=y=z},
\end{aligned} \tag{30}$$

which we will refer to as "Moyal like" part of the final non-commutative product. It has the well known structure of the Moyal product and reduces to it if $\Theta^{\mu\nu}$ is constant. In this case (30) is clearly associative and satisfies the trace property. This is, however, no longer true, if $\Theta^{\mu\nu}$ is a generic field.

Going one step further in the derivative expansion we have to take into account the contribution to the non-commutative product arising from the interaction term

$$\frac{1}{2\pi\alpha'}\int_{\mathbb{H}} \mathrm{d}^2 z\,\partial\zeta^\mu\bar{\partial}\zeta^\nu\zeta^\rho\,\partial_\rho\mathcal{F}_{\mu\nu}. \tag{31}$$

The rather cumbersome calculations are explained in the appendix. Using (66) and (67) we obtain

$$\begin{aligned}
\langle :f[X(\tau)]::g[X(\tau')]:\rangle_{\partial\mathcal{F}} &= \\
&-\frac{1}{12}\int_x \Theta^{\mu\rho}\partial_\rho\Theta^{\nu\sigma}\big(\partial_\mu\partial_\nu f*\partial_\sigma g+\partial_\sigma f*\partial_\mu\partial_\nu g\big) \\
&-\frac{i}{8\pi}\int_x \Theta^{\mu\rho}\partial_\rho G^{\nu\sigma}\big(\partial_\nu\partial_\sigma f*\partial_\mu g-\partial_\mu f*\partial_\nu\partial_\sigma g\big)\ln(\tau-\tau')^2 \\
&-\frac{i}{4\pi}\int_x G^{\nu\sigma}\partial_\sigma\Theta^{\rho\mu}\big(\partial_\nu\partial_\rho f*\partial_\mu g-\partial_\mu f*\partial_\nu\partial_\rho g\big)\ln(\tau-\tau')^2 \\
-\frac{1}{16\pi^2}\int_x\big(G^{\mu\rho}\partial_\rho G^{\nu\sigma}&-2G^{\nu\rho}\partial_\rho G^{\sigma\mu}\big)\big(\partial_\nu\partial_\sigma f*\partial_\mu g+\partial_\mu f*\partial_\nu\partial_\sigma g\big)\ln^2(\tau-\tau')^2+\ldots,
\end{aligned} \tag{32}$$

where we only kept the $\Theta^{\mu\nu}$ terms from the contributions of the free propagator (18), since the $G^{\mu\nu}$ parts are irrelevant for our further discussion, as we shall see shortly. Only the first line of (32) will contribute to our non-commutative product. The partial derivatives of the fields imply that the whole expression (32) vanishes for constant background fields.

We define now the non-commutative product as

$$\sqrt{g-\mathcal{F}}\,f(x)\,\circ\,g(x) \;:=\; \int[\mathrm{d}\zeta]\,e^{-S[x+\zeta]}f[X(0)]\,g[X(1)]\,. \tag{33}$$

The choice of the distance $\tau' - \tau = 1$ is such that the scale dependent contributions of (29) and (32) are removed.[5] The resulting non-commutative product is the scale and translation invariant part of the 2-point correlation. This product is independent of $G^{\mu\nu}$, and we will see that only this part of the correlation has appropriate off-shell properties (as long as the background fields are on-shell). The full off-shell correlations will, of course, also have $G^{\mu\nu}$-dependent contributions.

From (29) and (32) we see that, up to first order in derivatives of $\Theta^{\mu\nu}$, the product reads[6]

$$\begin{aligned} f(x) \circ g(x) &= f * g - \frac{1}{12}\Theta^{\mu\rho} D_\rho \Theta^{\nu\sigma} \Big(D_\mu D_\nu f \; * D_\sigma g + D_\sigma f \; * D_\mu D_\nu g \Big) + \\ &\quad + \; \mathcal{O}\big((D\Theta)^2, D^2\Theta\big). \end{aligned} \tag{34}$$

Here we have reintroduced the covariant notation. This is justified because in Riemannian normal coordinates the Christoffel symbol vanishes and (34) contains no derivatives thereof. The same is true for the $G^{\mu\nu}$-dependent parts.

A comparison of (34) with the formula given in [9] shows that, apart from the covariant derivatives, the non-commutative product (33) coincides with the Kontsevich formula. We do not require, however, that the field $\Theta^{\mu\nu}$ defines a Poisson structure.

6 Properties of the non-commutative product

In the limit $\alpha' \to 0$ the correlator of an arbitrary number of functions in the presence of a closed B-field background can be evaluated by integration of the non-commutative product of these functions. On the disk the $SL(2,\mathbb{R})$ invariance of the correlators requires the product to satisfy the trace property.

The non-commutative product (33) defined without the use of this limit, however, does not describe the full correlation functions, because the $G^{\mu\nu}$-dependent contractions give additional contributions. Even so, we will show in this section that the trace property can be maintained for the product (34) if one imposes the equations of motion for the background fields, while the inserted functions are allowed to stay completely generic.

In string theory the background field equations of motion are related to the renormalization group β functions which probe the breaking of Weyl invariance (and hence the conformal invariance) of the theory. Since we perform our calculations up to first order in derivatives of the background fields, we expect that we only have to account for the generalization of the Maxwell equation [17, 22],

$$G^{\rho\sigma} D_\rho \mathcal{F}_{\sigma\mu} - \frac{1}{2}\Theta^{\rho\sigma} H_{\rho\sigma\lambda} \mathcal{F}^{\lambda}{}_{\mu} = 0. \tag{35}$$

[5]The value 1 is due to our choice of the infrared cut-off, i.e., the constant $C^{\mu\nu}_{(\infty)}$ in (18).

[6]Subsequently we abbreviate $\mathcal{O}\big((D\Theta)^2, D^2\Theta\big)$ by $\mathcal{O}(D^2)$.

To show our proposition we rewrite (35) in a more appropriate way,

$$\begin{aligned}
\partial_\mu\Big(\sqrt{g-\mathcal{F}}\;\Theta^{\mu\nu}\Big) &= \sqrt{g}D_\mu\Big(\frac{\sqrt[4]{\bar{G}}}{\sqrt[4]{g}}\;\Theta^{\mu\nu}\Big) = \\
&= -\sqrt{g-\mathcal{F}}\Big(G^{\rho\sigma}D_\rho\mathcal{F}_{\sigma\mu}-\frac{1}{2}\Theta^{\rho\sigma}H_{\rho\sigma\lambda}\mathcal{F}^\lambda{}_\mu\Big)G^{\mu\nu}=0\;,
\end{aligned} \tag{36}$$

where we have used the relation

$$\Theta^{\rho\sigma}D_\mu\mathcal{F}_{\rho\sigma}=-\frac{1}{2}\bar{G}^{\rho\sigma}D_\mu\bar{G}_{\rho\sigma}=\Gamma_{\mu\lambda}{}^\lambda-\bar{\Gamma}_{\mu\lambda}{}^\lambda, \tag{37}$$

and the fact that the quotient $\frac{\sqrt[4]{\bar{G}}}{\sqrt[4]{g}}$ is a scalar. We introduce the usual notation $\approx$ for equivalence up to equations of motion. Note, furthermore, that in the following all relations are valid only to first order in derivatives of $\Theta^{\mu\nu}$.

We start with the product of two functions and show that (34) is symmetric under the integral

$$\int \mathrm{d}^Dx\sqrt{g-\mathcal{F}}\;f\;\circ\;g\;\;\approx\;\;\int \mathrm{d}^Dx\sqrt{g-\mathcal{F}}\;g\;\circ\;f. \tag{38}$$

This relation holds due to (35) and (36), because then the first order term in $\Theta^{\mu\nu}$ of (34) becomes a total divergence,

$$\int \mathrm{d}^Dx\sqrt{g-\mathcal{F}}\;\Theta^{\mu\nu}\;D_\mu f\;D_\nu g\approx\int \mathrm{d}^Dx\;\partial_\mu\Big(\sqrt{g-\mathcal{F}}\;\Theta^{\mu\nu}\;f\;D_\nu g\Big)=0, \tag{39}$$

and the remaining antisymmetric parts can be written as contributions of second order in derivatives. Notice that here and in the subsequent relations it is essential that the constant b_0 in the integration measure takes the value $\frac{1}{4}$ in order to produce the total divergence.

For a general field $\Theta^{\mu\nu}$ the product (34) is not associative. But, applying (35,36) again, associativity up to surface terms is ensured for the product of three functions. To see this we calculate $(f\circ g)\circ h-f\circ(g\circ h)$. Using the formula

$$\begin{aligned}
\partial_\rho\big(f*g\big)(x) &= (\partial_{x^\rho}+\partial_{y^\rho}+\partial_{z^\rho})e^{\frac{i}{2}\Theta^{\mu\nu}(z)\partial_{x^\mu}\partial_{y^\nu}}f(x)g(y)\Big|_{x=y=z} \\
&= \partial_\rho f*g+f*\partial_\rho g+\frac{i}{2}\partial_\rho\Theta^{\mu\nu}\partial_\mu f*\partial_\nu g.
\end{aligned} \tag{40}$$

for the product (30) we obtain

$$\begin{aligned}
(f*g)*h &= [f*g*h]+\frac{1}{4}\Theta^{\mu\sigma}D_\sigma\Theta^{\nu\rho}D_\nu f*D_\rho g*D_\mu h+\mathcal{O}(D^2), \\
f*(g*h) &= [f*g*h]-\frac{1}{4}\Theta^{\mu\sigma}D_\sigma\Theta^{\nu\rho}D_\mu f*D_\nu g*D_\rho h+\mathcal{O}(D^2),
\end{aligned} \tag{41}$$

where $[f*g*h]$ denotes the part with no derivatives acting on $\Theta^{\mu\nu}$. So the non-associativity reads

$$\begin{aligned}
(f\circ g)\circ h-f\circ(g\circ h) &= \\
&= \frac{1}{6}\big(\Theta^{\mu\sigma}D_\sigma\Theta^{\nu\rho}+(\mathrm{cycl.}^{\mu\nu\rho})\big)\,D_\mu f*D_\nu g*D_\rho h+\mathcal{O}(D^2) \\
&= \frac{1}{6}\Theta^{\mu\sigma}\Theta^{\nu\lambda}\Theta^{\rho\kappa}\bar{H}_{\sigma\lambda\kappa}\,D_\mu f*D_\nu g*D_\rho h+\mathcal{O}(D^2).
\end{aligned} \tag{42}$$

In the last line we have introduced the 3-form field $\bar{H} = \mathrm{d}(\Theta^{-1})$, that is associated with the inverse of $\Theta^{\mu\nu}$,

$$(\Theta^{-1})_{\mu\nu} = -(g - \mathcal{F})_{\mu\rho}(\mathcal{F}^{-1})^{\rho\sigma}(g + \mathcal{F})_{\sigma\nu} = (\mathcal{F} - g\mathcal{F}^{-1}g)_{\mu\nu}. \tag{43}$$

Therefore, associativity is obtained (even off-shell) if

$$\bar{H}_{\mu\nu\rho} = 0. \tag{44}$$

At this point we want to stress that we nowhere have employed the limit $\alpha' \to 0$ in our considerations, so that the "full" $\Theta^{\mu\nu}$ occurs in all the relations. This means that (44) is a generalization of the well known property that in the limit $\alpha' \to 0$ the product becomes associative if $H = 0$.

However, open string theory does not require such a restriction and we investigate again the effects of the equation of motion (35). From (42) we obtain immediately that

$$\begin{aligned} &\int \mathrm{d}^D x \sqrt{g - \mathcal{F}} \, \big((f \circ g) \circ h - f \circ (g \circ h)\big) = \qquad (45) \\ = \;& \frac{1}{6} \int \mathrm{d}^D x \sqrt{g - \mathcal{F}} \, \big(\Theta^{\mu\sigma}\Theta^{\nu\lambda}\Theta^{\rho\kappa}\bar{H}_{\sigma\lambda\kappa} \, D_\mu f * D_\nu g * D_\rho h\big) + \mathcal{O}(D^2) \approx \\ \approx \;& \frac{1}{6} \int \mathrm{d}^D x \partial_\mu \big(\sqrt{g - \mathcal{F}} \ldots\big)^\mu + \mathcal{O}(D^2) \; = \; 0, \end{aligned}$$

so that we are allowed to omit the brackets (note that $\bar{H}$ is already of order $\mathcal{O}(D\Theta)$).

For more than three functions we are allowed to omit the outermost bracket. In the case of four functions we obtain, for instance, the relation

$$\int \mathrm{d}^D x \sqrt{g - \mathcal{F}} \, (f \circ g) \circ h \circ l = \int \mathrm{d}^D x \sqrt{g - \mathcal{F}} \; f \circ g \circ (h \circ l). \tag{46}$$

Finally, taking into account (38) and (45), we immediately see that the trace property holds for an arbitrary number of functions,

$$\begin{aligned} &\int \mathrm{d}^D x \sqrt{g - \mathcal{F}} \, \Big((\ldots(f_1 \circ \ldots)) \circ f_{N-1}\Big) \circ f_N \approx \\ \approx \;& \int \mathrm{d}^D x \sqrt{g - \mathcal{F}} \; f_N \circ \Big((\ldots(f_1 \circ \ldots)) \circ f_{N-1}\Big) \approx \\ \approx \;& \int \mathrm{d}^D x \sqrt{g - \mathcal{F}} \, \Big(f_N \circ (\ldots(f_1 \circ \ldots))\Big) \circ f_{N-1} \approx \; \ldots \quad . \end{aligned} \tag{47}$$

We close this section with a remark on the relation to the recent work of Cornalba and Schiappa [12]. They considered the special case of a slowly varying background field B in radial gauge, i.e., $B_{\mu\nu}(x) = B_{\mu\nu} + \frac{1}{3}H_{\mu\nu\rho}x^\rho + \mathcal{O}(x^2)$, and a vanishing field strength F for their path integral analysis. Taking the topological limit, $g_{\mu\nu} \sim \epsilon \to 0$,[7] the above properties of the product were achieved by adjusting a constant $\mathcal{N}$ in the integration measure $\sqrt{B}\,(1+\mathcal{N}(B^{-1})^{\mu\nu}H_{\mu\nu\rho}x^\rho)$.

[7] Note that this limit is similar to the limit $\alpha' \to 0$ of Seiberg and Witten [8].

Using consistency arguments they determined the appropriate value of the constant to be $\mathcal{N} = \frac{1}{3}$.

However, dropping the radial gauge and repeating the calculations[8] of [12] for the trace property one obtains

$$-i \int \sqrt{B}[\mathcal{N}(B^{-1})^{\rho\sigma}(B^{-1})^{\mu\nu} - (B^{-1})^{\mu\sigma}(B^{-1})^{\rho\nu}]\partial_\mu B_{\sigma\rho}\, f \partial_\nu g. \tag{48}$$

This expression in general does not vanish for any $\mathcal{N}$. Thus the trace property cannot be restored by an appropriate choice for the constant $\mathcal{N}$ as it is possible for radial gauge, where $\partial_\mu B_{\sigma\rho}$ is replaced with $H_{\mu\sigma\rho}$.[9]

On the other hand, expanding $B_{\mu\nu}(x)$ around its constant value and taking the topological limit in our setting, the Born-Infeld measure reduces to $\sqrt{B(x)} = \sqrt{B}\ (1 + \frac{1}{2}(B^{-1})^{\mu\nu}\partial_\rho B_{\nu\mu}x^\rho)$. Then (48) can be recast into

$$\begin{aligned} &-i \textstyle\int \sqrt{B}[\frac{1}{2}(B^{-1})^{\rho\sigma}(B^{-1})^{\mu\nu}\partial_\mu B_{\sigma\rho} - \frac{1}{2}(B^{-1})^{\mu\sigma}(B^{-1})^{\nu\rho}\partial_\rho B_{\mu\sigma} \\ &+\tfrac{1}{2}(B^{-1})^{\mu\sigma}(B^{-1})^{\nu\rho}H_{\mu\sigma\rho}]f\partial_\nu g = \tfrac{i}{2}\textstyle\int \sqrt{B}[(B^{-1})^{\mu\sigma}H_{\mu\sigma\rho}](B^{-1})^{\rho\nu} f \partial_\nu g. \end{aligned} \tag{49}$$

The last expression in square brackets in the second line is exactly what remains from the generalized Maxwell equation (35) in the topological limit, namely the constraint $(B^{-1})^{\rho\sigma}H_{\rho\sigma\lambda} = 0$. So again, the trace property holds when the background fields are on-shell! Nevertheless, taking the topological limit mutilates the on-shell conditions in the sense that no dynamics is left and only a highly restrictive nonlinear constraint remains. In dimensions up to four this constraint already implies the vanishing of the field strength H. Moreover, in the next order one has to take into account the beta function for the background metric, namely the Einstein equation, which imposes the even stronger restriction

$$R_{\mu\nu} - \frac{1}{4}H_{\mu\rho\sigma}H^{\rho\sigma}{}_\nu \sim -\frac{1}{4}H_{\mu\rho\sigma}H^{\rho\sigma}{}_\nu + \mathcal{O}(\epsilon^0) = 0, \tag{50}$$

which enforces $H_{\mu\rho\sigma} = 0$ for any dimension (cf. [14]).[10] Hence the topological limit only seems to make sense in the symplectic case.

7 Conclusion

On the world volume of a D-brane the product of functions (34) represents a nonassociative deformation of a star product.[11] Nevertheless, it enjoys the properties that the integral acts

[8]Note that in this paragraph $B_{\sigma\rho}$ denotes the constant part of $B_{\sigma\rho}(x)$ and all dependencies on the zero modes are explicitly written.

[9]It can be restored by including a term proportional to H into the measure, but such a term cannot arise from a string vacuum amplitude.

[10]This can be seen by first setting $\mu = \nu = 0$ so that $H^2_{00} = 0$ and using the antisymmetry of H. This yields $H_{0ij} = 0$ for $i, j \neq 0$. The condition for purely spatial components follows immediately.

[11]Note that in the limit of vanishing gauge fields, $\mathcal{F}_{\mu\nu} \sim \epsilon \to 0$, the product reduces to the "ordinary" product of functions and the measure reduces to $\sqrt{g}$.

as a trace and the product of three functions is associative up to total derivatives. This is accomplished by the equations of motion of the background fields (36) and the Born-Infeld measure. No on-shell conditions have to be imposed on the inserted functions! Note, however, that the product of four or more functions inserted into an integral is ambiguous if the brackets are omitted. This is due to the fact that associativity for three functions is valid only up to total derivatives. Only the outermost lying bracket may be omitted, but this suffices to ensure the trace property for an arbitrary number of functions.

Our results are complete to first order in the derivative expansion of the background fields. In this approximation the influence of gravity amounts to the use of covariant derivatives in the generalized product (34), but the structure is still that of the formula given by Kontsevich. It would be interesting to investigate whether gravity induces a deviation from this structure at higher orders of the derivative expansion. One should expect that higher order terms of the generalized Maxwell equation have to be used, and also the equations of motion of other background fields may be required to maintain the properties of the product. However, a first attempt in this direction showed that these calculations lead to generalized polylogarithms and therefore turn out to be very cumbersome.

In the near future we plan to address the question of how to use the open string noncommutative product and a perturbative operator product expansion in order to calculate correlation functions in general backgrounds. The property that the product of four or more functions is not unique without brackets seems related to the fact that these products are not independent of the moduli of the insertion points. For instance, in the case of four functions there are two distinct possibilities where to put the brackets, which coincides with the number of independent crossratios. This suggests that for higher n-point correlation functions one has to use linear combinations of the various orderings of the brackets weighted with coefficients depending on the moduli [13]. Also the relation of these correlators to A_∞ algebras [23], the fundamental structure underlying open-closed string field theory [24], needs further clarification.

Furthermore, it would be interesting to investigate how the noncommutative differential calculus is generalized in the case of a nonassociative algebra [25].

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Appendix: 3-point correlations and coincidence limits

In the following we give an explicit calculation of the tree level contribution of the interaction term (31), i.e., the 3-point Greens function $\langle\!\langle\zeta^\mu(\tau_i)\zeta^\nu(\tau_j)\zeta^\rho(\tau_k)\rangle\!\rangle$, which is needed in section 5. There we derive the correlator of two functions f and g. These functions contain an arbitrary power of quantum fluctuations ζ^μ. Therefore, the correlator has also contributions from 3-point Greens functions with two coinciding quantum fields ζ^μ, i.e., $\lim_{\tau_j\to\tau_i}\langle\zeta^\mu(\tau_i)\zeta^\nu(\tau_j)\zeta^\rho(\tau_k)\rangle$ or $\lim_{\tau_j\to\tau_k}\langle\zeta^\mu(\tau_i)\zeta^\nu(\tau_j)\zeta^\rho(\tau_k)\rangle$. The coincidence limits consist of both finite and divergent terms, which need different treatments. The divergent ones must be compensated by appropriate subtractions, which are accounted for in the normal ordering of the inserted functions. The finite terms contribute explicitly to the correlator.

We will start the appendix with the introduction of convenient notations following [12]. Thereafter, we derive the Greens function $\langle\zeta^\mu(\tau_i)\zeta^\nu(\tau_j)\zeta^\rho(\tau_k)\rangle$, which needs a regularization similar to the propagator (14). We will see that the result is a generalization of the one in [12], because we do not use the limit $\alpha'\to 0$ and the radial gauge. Finally, we perform the coincidence limits to obtain the correct normal ordering and the finite contributions to the correlator.

A.1 Convenient notations and useful relations

The free propagator (18) with one side connected to the boundary is

$$\langle\zeta^\mu(\tau_i)\,\zeta^\nu(z,\bar z)\rangle = -\frac{1}{2\pi}\big(G^{\mu\nu}\mathcal{S}(\tau_i,z) - \Theta^{\mu\nu}\mathcal{A}(\tau_i,z)\big), \tag{51}$$

where $\mathcal{A}_i$ and $\mathcal{S}_i$ are defined as

$$\mathcal{A}_i = \mathcal{A}(\tau_i,z) = \ln\Big(\frac{\bar z-\tau_i}{\bar\tau_i - z}\Big) \quad\text{and}\quad \mathcal{S}_i = \mathcal{S}(\tau_i,z) = \ln|\tau_i-\bar z|^2. \tag{52}$$

Note that $\mathcal{A}_i$ is an antisymmetric function in τ_i and z, whereas $\mathcal{S}_i$ is symmetric, i.e., $\mathcal{A}(\tau_i,z) = -\mathcal{A}(z,\tau_i)$ and $\mathcal{S}(\tau_i,z)=\mathcal{S}(z,\tau_i)$. From (52) we see that $\mathcal{A}_i$ and $\mathcal{S}_i$ satisfy the relations $\partial\mathcal{S}_i = -\partial\mathcal{A}_i$ and $\bar\partial\mathcal{S}_i = \bar\partial\mathcal{A}_i$. Therefore we get

$$\begin{aligned}
\langle\zeta^\mu(\tau_i)\,\partial\zeta^\nu(z,\bar z)\rangle &= \frac{1}{2\pi}\big(\Theta^{\mu\nu}+G^{\mu\nu}\big)\,\partial\mathcal{A}_i\\
\langle\zeta^\mu(\tau_i)\,\bar\partial\zeta^\nu(z,\bar z)\rangle &= \frac{1}{2\pi}\big(\Theta^{\mu\nu}-G^{\mu\nu}\big)\,\bar\partial\mathcal{A}_i.
\end{aligned} \tag{53}$$

Furthermore we introduce the functions

$$f_A(\tau_a,\tau_b,\tau_c) = \int_{\mathbb{H}} \mathrm{d}^2z\;\partial\mathcal{A}_a\bar\partial\mathcal{A}_b\mathcal{A}_c \tag{54}$$

$$f_S(\tau_a,\tau_b,\tau_c) = \int_{\mathbb{H}} \mathrm{d}^2z\;\partial\mathcal{S}_a\bar\partial\mathcal{S}_b\mathcal{S}_c = -\int \mathrm{d}^2z\;\partial\mathcal{A}_a\bar\partial\mathcal{A}_b\mathcal{S}_c, \tag{55}$$

which are finite except for an infinite constant. So the computation of (54) and (55) will need a regularization. With the above abbreviations and the relations

$$\begin{aligned}
D_\rho G^{\mu\nu} &= -G^{\mu\lambda}D_\rho\mathcal{F}_{\lambda\sigma}\Theta^{\sigma\nu} - \Theta^{\mu\lambda}D_\rho\mathcal{F}_{\lambda\sigma}G^{\sigma\nu}\\
D_\rho\Theta^{\mu\nu} &= -\Theta^{\mu\lambda}D_\rho\mathcal{F}_{\lambda\sigma}\Theta^{\sigma\nu} - G^{\mu\lambda}D_\rho\mathcal{F}_{\lambda\sigma}G^{\sigma\nu}\quad,
\end{aligned} \tag{56}$$

the tree level amplitude of (31) reads

$$\begin{aligned}
&\langle \zeta^{\kappa_i}(\tau_i)\,\zeta^{\kappa_j}(\tau_j)\,\zeta^{\kappa_k}(\tau_k)\,\Big\{-\int_{\mathbb{H}} \mathrm{d}^2 z\,\partial\zeta^\mu\bar\partial\zeta^\nu\zeta^\rho\,\partial_\rho\mathcal{F}_{\mu\nu}\Big\}\rangle_{\rm tree} \;= \\
&= \;-\frac{1}{(2\pi)^3}\,\Big\{+\Theta^{\kappa_k\rho}\partial_\rho\Theta^{\kappa_i\kappa_j}\,\big(f_A(\tau_i,\tau_j,\tau_k)-f_A(\tau_j,\tau_i,\tau_k)\big)\\
&\qquad\qquad +\Theta^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j}\,\big(f_A(\tau_i,\tau_j,\tau_k)+f_A(\tau_j,\tau_i,\tau_k)\big)\\
&\qquad\qquad +G^{\kappa_k\rho}\partial_\rho\Theta^{\kappa_i\kappa_j}\,\big(f_S(\tau_i,\tau_j,\tau_k)-f_S(\tau_j,\tau_i,\tau_k)\big)\\
&\qquad\qquad +G^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j}\,\big(f_S(\tau_i,\tau_j,\tau_k)+f_S(\tau_j,\tau_i,\tau_k)\big)\\
&\qquad\qquad +\big(\text{cycl. perm. (ijk)}\,\big)\Big\}.
\end{aligned} \tag{57}$$

For the subsequent computation of (57) we take the order $\tau_i < \tau_j < \tau_k$ on the real axis.

A.2 Regularization of $f_A(\tau_a,\tau_b,\tau_c)$ and $f_S(\tau_a,\tau_b,\tau_c)$

To regularize f_A and f_S we differentiate the integral representations (54) and (55) with respect to τ_a, τ_b and τ_c, respectively. Then we can perform the integration over the upper half plane $\mathbb{H}$. This can be done by the well known method of a transformation into a contour integral and using the residue theorem. The pole prescriptions on the real axis are obtained by slightly shifting the insertion points τ_i into the upper half plane, so that

$$\mathcal{A}_i = \ln\Big(\frac{\bar z-\tau_i-i\epsilon}{\tau_i-i\epsilon-z}\Big) \quad\text{and}\quad \mathcal{S}_i = \ln\big((\tau_i-i\epsilon-z)(\tau_i+i\epsilon-\bar z)\big). \tag{58}$$

The appearance of the logarithm needs a selection of a cut and it turns out that the negative real axis is a convenient choice. Finally, we determine the integrals with respect to τ_a, τ_b and τ_c. Then the infinity is contained in the integration constant.

Thus we get

$$\begin{aligned}
f_A(\tau_a,\tau_b,\tau_c) &= 2\pi\int_0^t \mathrm{d}x\Big(\frac{\ln(x\pm i\epsilon')}{1-x}+\frac{\ln(1-x\pm i\epsilon')}{x}\Big)+C^A_{(\infty)},\\
f_S(\tau_a,\tau_b,\tau_c) &= 2\pi\int_0^t \mathrm{d}x\Big(-\frac{\ln(x\pm i\epsilon')}{1-x}+\frac{\ln(1-x\pm i\epsilon')}{x}\Big)\\
&\quad -\frac{\pi}{2}\ln^2(\tau_b-\tau_a)^2+i\pi^2\epsilon(\tau_b-\tau_a)\ln(\tau_b-\tau_a)^2+C^S_{(\infty)},
\end{aligned} \tag{59}$$

where the $\pm$ in the logarithm abbreviate the sign function $+\epsilon(\tau_b-\tau_a)$. In (59) we have introduced the parameter t which is defined as the combination $t=\frac{\tau_c-\tau_a}{\tau_b-\tau_a}$. The shift ϵ' is needed to integrate along the correct side of the cut for negative arguments of the logarithm. This selection is determined by the pole prescription explained above.

The integrals in (59) lead to expressions containing the dilogarithm which is defined as

$$\mathrm{Li}_2(m) = -\int_0^m \mathrm{d}x\frac{\ln(1-x)}{x} \qquad \text{for}\quad 0<m<1. \tag{60}$$

with the modulus

$$m = \frac{\tau_j - \tau_i}{\tau_k - \tau_i}, \tag{61}$$

which is restricted to $0 < m < 1$ because of our order $\tau_i < \tau_j < \tau_k$.

A.3 The tree level amplitude

What is left is to use (59) to bring together all combinations of the functions f_A and f_S in (57). This leads to the rather lengthy result

$$\begin{aligned}
&-2\pi^2 \langle \zeta^{\kappa_i}(\tau_i)\, \zeta^{\kappa_j}(\tau_j)\, \zeta^{\kappa_k}(\tau_k) \int_{\mathbb{H}} \mathrm{d}^2 z\, \partial\zeta^\mu \bar{\partial}\zeta^\nu \zeta^\rho\, \partial_\rho \mathcal{F}_{\mu\nu} \rangle_{\text{tree}} \;= \\
&\Big\{ + \Theta^{\kappa_k\rho}\partial_\rho\Theta^{\kappa_i\kappa_j}\big(\mathrm{Li}_2(1-m) - \mathrm{Li}_2(m) + \frac{\pi^2}{3}\big) \\
&\quad + \Theta^{\kappa_i\rho}\partial_\rho\Theta^{\kappa_j\kappa_k}\big(\mathrm{Li}_2(1-m) - \mathrm{Li}_2(m) - \frac{\pi^2}{3}\big) \\
&\quad + \Theta^{\kappa_j\rho}\partial_\rho\Theta^{\kappa_k\kappa_i}\big(\mathrm{Li}_2(1-m) - \mathrm{Li}_2(m) \qquad\big) \\
&+ i\pi\Theta^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j}\big(\ln(m)\big) \\
&- i\pi\Theta^{\kappa_i\rho}\partial_\rho G^{\kappa_j\kappa_k}\big(\ln(1-m)\big) \\
&- i\pi G^{\kappa_k\rho}\partial_\rho\Theta^{\kappa_i\kappa_j}\big(\ln(\tau_k - \tau_i)\big) \\
&- i\pi G^{\kappa_i\rho}\partial_\rho\Theta^{\kappa_j\kappa_k}\big(\ln(\tau_k - \tau_i)\big) \\
&+ i\pi G^{\kappa_j\rho}\partial_\rho\Theta^{\kappa_k\kappa_i}\big(\ln(\tau_k - \tau_i)\big) \\
&\quad + G^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j}\big(\ln(m)\ln(1-m) - \ln^2(\tau_k - \tau_i) + 2\ln(\tau_j - \tau_i)\ln(\tau_k - \tau_i)\big) \\
&\quad + G^{\kappa_i\rho}\partial_\rho G^{\kappa_j\kappa_k}\big(\ln(m)\ln(1-m) - \ln^2(\tau_k - \tau_i) + 2\ln(\tau_k - \tau_j)\ln(\tau_k - \tau_i)\big) \\
&\quad - G^{\kappa_j\rho}\partial_\rho G^{\kappa_k\kappa_i}\big(\ln(m)\ln(1-m) - \ln^2(\tau_k - \tau_i)\big) \quad \Big\},
\end{aligned} \tag{62}$$

where we have set the integration constants of (59) to a convenient value, which can be done since they play no essential role (cf. equation (18)).

All terms containing the boundary metric $G^{\mu\nu}$ vanish in the limit $\alpha' \to 0$. If we use, furthermore, the radial gauge and a vanishing gauge field A, the terms $\pm\frac{\pi^2}{3}$ in the first two lines disappear. Due to relation $\mathrm{Li}_2(1-m) - \mathrm{Li}_2(m) = \frac{\pi^2}{6}(1 - 2L(m))$, where $L(m)$ is the normalized Rogers dilogarithm, we thus recover the result of [12].

A.4 The coincidence limits

In section 5 we calculate the correlator of two functions. For that purpose we have to consider the coincidence limits $\tau_j \to \tau_i$ and $\tau_j \to \tau_k$ of (62).

Singular Terms In these limits there appear logarithmic singularities which can be regularized by a cut-off parameter Λ, i.e., $\lim_{\tau_j\to\tau_i}\ln(\tau_j-\tau_i)\to\ln\Lambda$. In terms of Λ we get

$$\begin{aligned}
-\langle\zeta^{\kappa_i}(\tau_i)\,\zeta^{\kappa_j}(\tau_i)\quad\zeta^{\kappa_k}(\tau_k)\quad\int_{\mathbb{H}}\mathrm{d}^2z\,\partial\zeta^\mu\bar\partial\zeta^\nu\zeta^\rho\,\partial_\rho\mathcal{F}_{\mu\nu}\rangle_{\text{tree,sing}} &= \\
&= +\frac{i}{2\pi}\Theta^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j}\ln\Lambda+\frac{1}{\pi^2}G^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j}\ln\Lambda\ln(\tau_k-\tau_i)\\
&= -\frac{1}{\pi}\partial_\rho G^{\kappa_i\kappa_j}\ln\Lambda\langle\zeta^\rho(\tau_i)\,\zeta^{\kappa_k}(\tau_k)\rangle
\end{aligned}\tag{63}$$

for $\tau_j\to\tau_i$ and

$$\begin{aligned}
-\langle\zeta^{\kappa_i}(\tau_i)\quad\zeta^{\kappa_j}(\tau_k)\,\zeta^{\kappa_k}(\tau_k)\int_{\mathbb{H}}\mathrm{d}^2z\,\partial\zeta^\mu\bar\partial\zeta^\nu\zeta^\rho\,\partial_\rho\mathcal{F}_{\mu\nu}\rangle_{\text{tree,sing}} &= \\
&= -\frac{i}{2\pi}\Theta^{\kappa_i\rho}\partial_\rho G^{\kappa_j\kappa_k}\ln\Lambda+\frac{1}{\pi^2}G^{\kappa_i\rho}\partial_\rho G^{\kappa_j\kappa_k}\ln\Lambda\ln(\tau_k-\tau_i)\\
&= -\frac{1}{\pi}\partial_\rho G^{\kappa_j\kappa_k}\ln\Lambda\langle\zeta^{\kappa_i}(\tau_i)\,\zeta^\rho(\tau_k)\rangle
\end{aligned}\tag{64}$$

for $\tau_j\to\tau_k$. The singularities (63) and (64) must be compensated by appropriate counter terms. The correct subtractions can easily be read off from (63,64). Together with the singular part of the propagator (20) we get

$$\begin{aligned}
\zeta^\mu(\tau)\,\zeta^\nu(\tau') &= -\frac{1}{2\pi}G^{\mu\nu}\ln(\tau-\tau')^2-\frac{1}{2\pi}\partial_\rho G^{\mu\nu}\ln(\tau-\tau')^2\,\zeta^\rho(\frac{\tau+\tau'}{2})\\
&+\quad(\text{ regular terms }).
\end{aligned}\tag{65}$$

Finite Terms In our limits equation (62) contains also finite parts which read

$$\begin{aligned}
&-\langle\,\zeta^{(\kappa_i}(\tau_i)\,\zeta^{\kappa_j)}(\tau_i)\quad\zeta^{\kappa_k}(\tau_k)\quad\int_{\mathbb{H}}\mathrm{d}^2z\,\partial\zeta^\mu\bar\partial\zeta^\nu\zeta^\rho\,\partial_\rho\mathcal{F}_{\mu\nu}\rangle_{\text{tree,fin}} =\\
&= -\frac{1}{12}\big(\Theta^{\kappa_i\rho}\partial_\rho\Theta^{\kappa_j\kappa_k}-\Theta^{\kappa_j\rho}\partial_\rho\Theta^{\kappa_k\kappa_i}\big)\\
&\quad-\frac{i}{2\pi}\big(\Theta^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j}+G^{\kappa_j\rho}\partial_\rho\Theta^{\kappa_i\kappa_k}+G^{\kappa_i\rho}\partial_\rho\Theta^{\kappa_j\kappa_k}\big)\ln(\tau_k-\tau_i)\\
&\quad-\frac{1}{2\pi^2}\big(G^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j}-G^{\kappa_i\rho}\partial_\rho G^{\kappa_j\kappa_k}-G^{\kappa_j\rho}\partial_\rho G^{\kappa_k\kappa_i}\big)\ln^2(\tau_k-\tau_i)
\end{aligned}\tag{66}$$

for $\tau_j\to\tau_i$ and

$$\begin{aligned}
&-\langle\,\zeta^{\kappa_i}(\tau_i)\quad\zeta^{(\kappa_j}(\tau_k)\,\zeta^{\kappa_k)}(\tau_k)\quad\int_{\mathbb{H}}\mathrm{d}^2z\,\partial\zeta^\mu\bar\partial\zeta^\nu\zeta^\rho\,\partial_\rho\mathcal{F}_{\mu\nu}\rangle_{\text{tree,fin}} =\\
&= +\frac{1}{12}\big(\Theta^{\kappa_k\rho}\partial_\rho\Theta^{\kappa_i\kappa_j}-\Theta^{\kappa_j\rho}\partial_\rho\Theta^{\kappa_k\kappa_i}\big)\\
&\quad+\frac{i}{2\pi}\big(\Theta^{\kappa_i\rho}\partial_\rho G^{\kappa_j\kappa_k}+G^{\kappa_j\rho}\partial_\rho\Theta^{\kappa_k\kappa_i}+G^{\kappa_k\rho}\partial_\rho\Theta^{\kappa_j\kappa_i}\big)\ln(\tau_k-\tau_i)\\
&\quad-\frac{1}{2\pi^2}\big(G^{\kappa_i\rho}\partial_\rho G^{\kappa_j\kappa_k}-G^{\kappa_j\rho}\partial_\rho G^{\kappa_k\kappa_i}-G^{\kappa_k\rho}\partial_\rho G^{\kappa_i\kappa_j}\big)\ln^2(\tau_k-\tau_i)
\end{aligned}\tag{67}$$

for $\tau_j\to\tau_k$. We have taken into account only the symmetric part of the limit, since the antisymmetric one does not contribute in section 5.

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