

# BOSONIC FIELDS WITH TWO MASS AND SPIN STATES

S.I. Kruglov <sup>1</sup>

*International Educational Centre, 2727 Steeles Ave. W, # 202,  
Toronto, Ontario, Canada M3J 3G9*

## Abstract

We consider the bosonic fields which describe a particle which may exist in states with spins one and zero with different masses. All the linearly independent solutions of the equation for a free particle are obtained in the form of the projection matrix-dyads (density-matrices). The interaction with anomalous magnetic dipole and quadruple electric moments of a particle is studied. The Hamiltonian form of the first order equation has been investigated.

## 1 Introduction

Intermediate vector bosons  $W^\pm$ ,  $Z^0$  and Higgs  $H$  bosons possess spin one and zero, respectively. The standard model (SM), however, does not predict the masses of Higgs scalar particles that is of interest at this time. The quantum field theory allows us to construct the model of a particle which may exist in two spin states (one and zero) with different masses [1]. Possibly such a model can be applied for unified description of  $W^\pm$ ,  $Z^0$  and  $H$  bosons or other vector and scalar particles. Anyway, it is useful to investigate such a model of fields with two spin and mass states for different applications.

The paper is organized as follows. In section 2 we construct the first order (11-dimensional) wave equation for particles with two spin and mass states. Matrix form of equations is considered in Sec. 3. All the linearly independent solutions of the equation for a free particle will be obtained in the form of the projection matrix-dyads in section 4. In Sec. 5 we introduce some terms in a relativistic wave equation which describe specific interactions with external electromagnetic fields such as anomalous magnetic dipole and quadruple electric moments. The Hamiltonian form of an equation is studied in Sec. 6. Section 7 contains the conclusion. In the appendix we write out some matrices entering the equation in the Hamiltonian form.

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<sup>1</sup>E-mail: krouglov@sprint.ca

## 2 Bosonic field equations

A Lagrangian of general form for a charged vector field leading to linear differential equations of not greater than second order has the form (see [1]):

$$\mathcal{L} = \delta_{\mu\nu,\sigma\rho} (\partial_\mu B_\sigma^*) (\partial_\nu B_\rho) + \alpha B_\mu^* B_\mu, \quad (1)$$

where  $\partial_\mu = \partial/\partial x_\mu$ ,  $\delta_{\mu\nu,\sigma\rho} = a\delta_{\mu\nu}\delta_{\sigma\rho} + b\delta_{\mu\sigma}\delta_{\nu\rho} + c\delta_{\mu\rho}\delta_{\nu\sigma}$ , in which  $a$ ,  $b$  and  $c$  are arbitrary constants,  $B_\mu^*$  is the complex conjugated field. From the Lagrangian (1), we obtain field equations of the form

$$a\partial_\mu^2 B_\nu + (b+c)\partial_\nu(\partial_\alpha B_\alpha) - \alpha B_\nu = 0. \quad (2)$$

From the freedom in the choice of the coefficients  $a$ ,  $b$ , and  $c$ , we can consider different cases. The requirement that  $a = -1$  is necessary to have the standard kinetic term in the Lagrangian (1). It can also be derived by the renormalization of the fields  $B_\mu \rightarrow (-a)^{1/2} B_\mu$  (the parameter  $a$  should be negative to have the positive energy of the field  $B_\mu$ ). If  $b+c = 1$  and  $\alpha = -m^2$ , where  $m$  is the rest mass, we arrive at the Proca theory of the vector field [2]. Then it follows from (2) that the Lorentz condition  $\partial_\alpha B_\alpha = 0$  occurs and there is a constraint on the field  $B_\mu$ . The state with spin 0 is excluded and there are only three degrees of freedom which describe the spin projections  $s_z = 0, \pm 1$ .

In the general case, without any constraints, the field  $B_\mu$  realizes the  $(0,0) \oplus (1/2, 1/2)$  representation of the Lorentz group and describes four degrees of freedom which correspond to states with spins  $s = 0$  and  $s = 1$  (with three spin projections  $s_z = 0, \pm 1$ ). To have the Lagrangian formulation of this case when states of the field  $B_\mu$  with spins  $s = 0$  and  $s = 1$  possess the unique rest mass  $m$ , we should impose on the parameters the restrictions  $b+c = 0$ ,  $a = -1$ ,  $\alpha = -m^2$ . Then the field functions  $B_\mu$  will satisfy the Klein-Gordon-Fock equation

$$(\partial_\mu^2 - m^2) B_\nu = 0. \quad (3)$$

With allowance for these conditions, the Lagrangian (1) can be rewritten as follows (within unimportant divergent-type terms):

$$\mathcal{L} = - \left[ (\partial_\mu B_\nu)^* (\partial_\mu B_\nu) + m^2 B_\mu^* B_\mu \right]. \quad (4)$$

The Lagrangian (4) can be connected also with the Stueckelberg formulation of the vector field [3] (see also [4]). A Lagrangian of the form (4) for real fields  $B_\mu$  also was used [5] in a gauge-invariant formulation for a massive neutral vector field.

In the general case of arbitrary parameters  $\alpha$ ,  $\beta$ ,  $\gamma$  and  $\delta$ , we can investigate the spectre of masses of the field  $B_\mu$  with Lagrangian (1) and the equation of motion (2). Without loss of generality, we can also set  $\alpha = -1$ . It is known that a Lagrangian is determined by the accuracy of the divergent terms. It means that if we make the transformation of the Lagrangian  $\mathcal{L} \rightarrow \mathcal{L} + \partial_\mu \Lambda_\mu$ , where  $\Lambda_\mu$  is the arbitrary function, we will get the same equation of motion (2). Therefore only the combination of the parameters  $\beta + \gamma \equiv \beta$  possesses physical meaning. So there are two physical parameters  $\alpha$  and  $\beta$  in the theory. To clear up the connection of these parameters with the masses of the field states, we work in momentum space. Then equation (2) becomes

$$\left[ (p_\lambda^2 - \alpha) \delta_{\mu\nu} - \beta p_\mu p_\nu \right] B_\nu(p) = 0. \quad (5)$$

Introducing the matrix

$$M = (p^2 - \alpha) I_4 - \beta (p.p), \quad (6)$$

where  $p^2 = p_\lambda^2 = \mathbf{p}^2 + p_4^2 = \mathbf{p}^2 - p_0^2$ , the  $I_4$  is the unit  $4 \times 4$ -matrix and  $(p.p)$  is the matrix-dyad with the matrix elements  $(p.p)_{\mu\nu} = p_\mu p_\nu$ , it is possible to find eigenvalues of the matrix  $M$ . The matrix  $M$  obeys the “minimal” equation

$$(M + \alpha - p^2) [M + \alpha - p^2 (1 - \beta)] = 0 \quad (7)$$

with eigenvalues  $\lambda_1 = -\alpha + p^2$ ,  $\lambda_2 = -\alpha + p^2 (1 - \beta)$ . The requirement  $\det M = 0$  defines the masses of the field  $B_\mu$ . This condition is valid if  $\lambda_1 = \lambda_2 = 0$ , which leads to

$$p^2 = \alpha, \quad p^2 = \frac{\alpha}{1 - \beta}. \quad (8)$$

From Eq. (8) we find the squared masses of the field  $B_\mu$  corresponding to the states with spin  $s = 1$  and  $s = 0$ :

$$m^2 = -\alpha, \quad m_0^2 = \frac{\alpha}{\beta - 1}. \quad (9)$$

To have real masses ( $m^2 > 0$ ) we arrive at the constraints:  $\alpha < 0$ ,  $\beta < 1$ . At the particular case of  $\beta = 0$ , there is a degeneracy of mass states and the field  $B_\mu$  possesses the unique squared mass  $m^2 = -\alpha$ . Now it is not difficult to find first order equations which correspond to the second order equation (5) in the coordinate representation, which are given by

$$(\beta - 1) \partial_\mu B_\mu = B,$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu, \quad (10)$$

$$\partial_\nu B_{\mu\nu} - \alpha B_\mu + \partial_\mu B = 0,$$

where  $B$  is a scalar and  $B_{\mu\nu}$  is an antisymmetric tensor of the second order. Inserting the first and second equations into the third of Eq. (10), one gets

$$-\partial_\nu^2 B_\mu + \beta \partial_\mu \partial_\nu B_\nu - \alpha B_\mu = 0. \quad (11)$$

Equation (11) in the momentum representation coincides with (5) and was proposed earlier [6,7]. From (10) we find the second order equation for the scalar field  $B$ :

$$\partial_\nu^2 B - \frac{\alpha}{\beta - 1} B = 0. \quad (12)$$

It follows from (12) that the state of the field with spin  $s = 0$  has the squared mass  $m_0^2 = \alpha/(\beta - 1)$  and therefore the state of the field with spin  $s = 1$  corresponds to the squared mass  $m^2 = -\alpha$ . In this general case when there are no supplementary conditions, the vector field is a mixture of quanta with the mass  $m$  and spin 1 and a quanta with spin 0 and mass  $m_0$ . To have a pure spin 1 one needs to consider the quanta with spin 0 as nonphysical quanta (see [6]). But in general case one can consider a quanta with spin 0 on the same footing as a quanta with spin 1.

### 3 Matrix form of equations

Let us consider the matrix formulation of the first order of the field under consideration which is convenient for constructing the density matrix and for some electrodynamics calculations. We will obtain all the linearly independent solutions of the equation for a free particle in terms of the projection matrix-dyads.

With the use of Eqs. (9) the system of field equations (10) takes the form

$$\partial_\mu B_\mu + \frac{m_0^2}{m^2} B = 0,$$

$$\partial_\nu B_\mu - \partial_\mu B_\nu + B_{\mu\nu} = 0, \quad (13)$$

$$\partial_\nu B_{\mu\nu} + \partial_\mu B + m^2 B_\mu = 0.$$

At  $m_0 = m$  we arrive at the massive Stueckelberg field [3]. It is convenient to introduce the 11-dimensional function

$$\Psi(x) = \{\psi_A(x)\} = \begin{pmatrix} (1/m)B(x) \\ B_\mu(x) \\ (1/m)B_{\mu\nu}(x) \end{pmatrix} \quad (A = 0, \mu, [\mu\nu]), \quad (14)$$

where  $\psi_0(x) = (1/m)B(x)$ ,  $\psi_\mu(x) = B_\mu(x)$ ,  $\psi_{[\mu\nu]}(x) = (1/m)B_{\mu\nu}(x)$ ,  $\mu, \nu = 1, 2, 3, 4$ . Let us consider the elements of the entire matrix algebra

$$\left(\varepsilon^{A,B}\right)_{CD} = \delta_{AC}\delta_{BD}, \quad \varepsilon^{A,B}\varepsilon^{C,D} = \delta_{BC}\varepsilon^{A,D}, \quad (15)$$

where  $A, B, C, D = 0, \mu, [\mu\nu]$ . Thus, the  $\varepsilon^{A,B}$  is the  $11 \times 11$ -matrix with elements which consist of zeroes and only one element is unity where row  $A$  and column  $B$  cross. Using the properties (15) equations (13) can be represented in the form of an equation

$$\begin{aligned} & \partial_\nu \left( \varepsilon^{\mu, [\mu\nu]} + \varepsilon^{[\mu\nu], \mu} + \varepsilon^{\nu, 0} + \varepsilon^{0, \nu} \right)_{AB} \Psi_B(x) + \\ & + \left[ m\varepsilon^{\mu, \mu} + \frac{m_0^2}{m}\varepsilon^{0, 0} + \frac{1}{2}m\varepsilon^{[\mu\nu], [\mu\nu]} \right]_{AB} \Psi_B(x) = 0. \end{aligned} \quad (16)$$

Using 11-dimensional matrices

$$\begin{aligned} \alpha_\nu &= \varepsilon^{\mu, [\mu\nu]} + \varepsilon^{[\mu\nu], \mu} + \varepsilon^{\nu, 0} + \varepsilon^{0, \nu}, \\ P_1 &= \varepsilon^{\mu, \mu} + \frac{1}{2}\varepsilon^{[\mu\nu], [\mu\nu]}, \quad P_0 = \varepsilon^{0, 0} \end{aligned} \quad (17)$$

Eq. (16) transforms into the relativistic wave equation of the first order:

$$\left( \alpha_\mu \partial_\mu + mP_1 + \frac{m_0^2}{m}P_0 \right) \Psi(x) = 0. \quad (18)$$

Matrices  $P_1, P_0$  are the projection matrices (see [8]) which obey the relations:

$$P_1^2 = P_1, \quad P_0^2 = P_0, \quad P_1 P_0 = 0, \quad P_1 + P_0 = I_{11}, \quad (19)$$

where  $I_{11}$  is the unit matrix in  $11$ -dimensional space. Eq. (18) at  $m_1 = m_0$ , after taking into account Eq. (19), becomes (see [4])

$$(\alpha_\mu \partial_\mu + m) \Psi(x) = 0. \quad (20)$$

Eq. (20) represents the Stueckelberg equation for massive fields in the matrix form.

The matrices  $\alpha_\mu$  can be written in the form

$$\begin{aligned} \alpha_\mu &= \beta_\mu^{(1)} + \beta_\mu^{(0)}, \\ \beta_\nu^{(1)} &= \varepsilon^{\mu, [\mu\nu]} + \varepsilon^{[\mu\nu], \mu}, \\ \beta_\nu^{(0)} &= \varepsilon^{\nu, 0} + \varepsilon^{0, \nu}. \end{aligned} \quad (21)$$

The  $10$ - and  $5$ -dimensional matrices  $\beta_\mu^{(1)}$  and  $\beta_\mu^{(0)}$  obey the Petiau-Duffin-Kemmer [9-11] algebra:

$$\beta_\mu \beta_\nu \beta_\alpha + \beta_\alpha \beta_\nu \beta_\mu = \delta_{\mu\nu} \beta_\alpha + \delta_{\alpha\nu} \beta_\mu. \quad (22)$$

The equations for spin- $1$  and spin- $0$  particles are represented as [12]

$$\left( \beta_\mu^{(1)} \partial_\mu + m \right) \Psi^{(1)}(x) = 0, \quad \Psi^{(1)}(x) = \begin{pmatrix} \psi_\mu(x) \\ \psi_{[\mu\nu]}(x) \end{pmatrix}, \quad (23)$$

$$\left( \beta_\mu^{(0)} \partial_\mu + m_0 \right) \Psi^{(0)}(x) = 0, \quad \Psi^{(0)}(x) = \begin{pmatrix} \psi_0(x) \\ \psi_\mu(x) \end{pmatrix}. \quad (24)$$

Eq. (23) is the  $10$ -dimensional Petiau-Duffin-Kemmer equation which is equivalent to the Proca equations [2] for spin- $1$  particles and the  $5$ -dimensional Eq. (24) is equivalent to the Klein-Gordon-Fock equation for scalar particles. The first order  $11$ -dimensional matrix equation (18) describes fields with two spins  $0, 1$  with different masses,  $m_0$  and  $m$ . Using Eqs. (15) one can verify that the  $11$ -dimensional matrices  $\alpha_\mu$  (17) obey the algebra (see [13]):

$$\begin{aligned} \alpha_\mu \alpha_\nu \alpha_\alpha + \alpha_\alpha \alpha_\nu \alpha_\mu + \alpha_\mu \alpha_\alpha \alpha_\nu + \alpha_\nu \alpha_\alpha \alpha_\mu + \alpha_\nu \alpha_\mu \alpha_\alpha + \alpha_\alpha \alpha_\mu \alpha_\nu = \\ = 2 (\delta_{\mu\nu} \alpha_\alpha + \delta_{\alpha\nu} \alpha_\mu + \delta_{\mu\alpha} \alpha_\nu). \end{aligned} \quad (25)$$

Representations of the Petiau-Duffin-Kemmer algebra (22) were investigated in [14-16]. It should be noted that algebra (25) is more complicated than the Petiau-Duffin-Kemmer algebra (22).

## 4 Solutions of first order equations

Let us obtain the solutions to Eq. (18) corresponding to definite values of the energy and momentum of massive particles. Using Fourier transformations, Eq. (18) in the momentum space converts into

$$-i\hat{p}\Psi_p = \varepsilon \left( mP_1 + \frac{m_0^2}{m}P_0 \right) \Psi_p, \quad (26)$$

where  $\hat{p} = \alpha_\mu p_\mu$ ,  $p_\mu = (\mathbf{p}, ip_0)$ . The value of  $\varepsilon = 1$  corresponds to positive energy and  $\varepsilon = -1$  to negative energy and  $\mathbf{p}$  is the momentum of a particle. To extract the states corresponding to pure spin one it is necessary to consider ten-dimensional subspace (23). For this we use the projection operator  $P_1$  (17) with the property (19). Acting on the left and right sides of Eq. (26) by the operator  $P_1$  we get

$$-i\hat{p}^{(1)}\Psi_p^{(1)} = \varepsilon m\Psi_p^{(1)}, \quad (27)$$

where we used the relations

$$P_1 \hat{p} = \hat{p} P_1 = \hat{p}^{(1)}, \quad \hat{p}^{(1)} = p_\mu \beta_\mu^{(1)},$$

$$\Psi_p^{(1)} = P_1 \Psi_p = \begin{pmatrix} 0 \\ \psi_\mu(p) \\ \psi_{[\mu\nu]}(p) \end{pmatrix}. \quad (28)$$

Using Eq. (22) one may verify that the equality

$$\left(\hat{p}^{(1)}\right)^3 = p^2 \hat{p}^{(1)} \quad (29)$$

holds. With the help of the projection operator method [8], we obtain solutions to Eq. (27) in the form

$$M_\varepsilon = \frac{i\hat{p}^{(1)} (i\hat{p}^{(1)} - \varepsilon m)}{2m^2}. \quad (30)$$

The projection matrix  $M_\varepsilon$  obeys the relationship

$$M_\varepsilon^2 = M_\varepsilon. \quad (31)$$

Columns of the matrix  $M_\varepsilon$  are eigenvectors  $\Psi_p$  of Eq. (27) with eigenvalues  $\varepsilon m$ . The matrix  $M_\varepsilon$  may be transformed into the diagonal form with matrix elements containing only ones and zeroes, and the  $M_\varepsilon$  acting on the 11-dimensional vector-column retains components corresponding to the eigenvalue  $\varepsilon m$ .

Now we consider the case of the states with spin-zero. Let us introduce the projection operator extracting the fifth dimensional subspace (24):

$$\bar{P}_0 = \varepsilon^{0,0} + \varepsilon^{\mu,\mu}, \quad (32)$$

obeying the relationships

$$\bar{P}_0^2 = \bar{P}_0, \quad \bar{P}_0 \alpha_\nu = \beta_\nu^{(0)}. \quad (33)$$

After acting on Eq. (26) by the operator  $\bar{P}_0$  one obtains

$$\left(i\varepsilon \hat{p}^{(0)} + m\bar{P} + \frac{m_0^2}{m} P_0\right) \Psi_p^{(0)} = 0, \quad (34)$$

where

$$\bar{P} = \varepsilon^{\mu,\mu}, \quad \Psi_p^{(0)} = \bar{P}_0 \Psi_p = \begin{pmatrix} \psi_0(p) \\ \psi_\mu(p) \\ 0 \end{pmatrix}, \quad (35)$$

where  $\hat{p}^{(0)} = p_\mu \beta_\mu^{(0)}$ . Introducing the the matrix of Eq.(34)

$$K = i\varepsilon \hat{p}^{(0)} + m\bar{P} + \frac{m_0^2}{m}P_0, \quad (36)$$

it is not difficult to verify, using Eq. (15), that the matrix  $K$  obeys the “minimal” equation

$$\left(K^2 - \frac{m^2 + m_0^2}{m}K + m_0^2\right) \left(K^2 - \frac{m^2 + m_0^2}{m}K + m_0^2 + p^2\right) = 0. \quad (37)$$

For the case of the state with spin-zero and mass  $m_0$  of a particle the relation  $p^2 = -m_0^2$  is valid. Then Eq. (37) transforms into

$$K(K - m) \left(K - \frac{m_0^2}{m}\right) \left(K - \frac{m^2 + m_0^2}{m}\right) = 0. \quad (38)$$

With the use of the general technique [8] we find from Eq. (38) the projection operator

$$N_\varepsilon = -\frac{m}{m_0^2(m^2 + m_0^2)}(K - m) \left(K - \frac{m_0^2}{m}\right) \left(K - \frac{m^2 + m_0^2}{m}\right) \quad (39)$$

satisfying the equation  $N_\varepsilon^2 = N_\varepsilon$ . The operator (39) acting on arbitrary non-zero  $11$ -dimensional vector (matrix-column) extracts the solution of Eq. (34) which corresponds to the state with spin-0 and mass  $m_0$ .

To find the states with pure spin and spin projection we have to construct the spin operators. It is not difficult to verify that the generators of the Lorentz group in the  $11$ -dimensional space are given by

$$J_{\mu\nu} = \beta_\mu^{(1)}\beta_\nu^{(1)} - \beta_\nu^{(1)}\beta_\mu^{(1)}. \quad (40)$$

Matrices (40) act in the  $10$ -dimensional subspace  $(\psi_\mu(x), \psi_{[\mu\nu]}(x))$  as the scalar field  $\psi_0(x) = (1/m)B(x)$  is an invariant of the Lorentz transformations. It should be noted that matrices (40) are also generators of the Lorentz group for the case of the Petiau-Duffin-Kemmer equation (23). With the use of the properties (15), it is verified that the commutation relations

$$[J_{\rho\sigma}, J_{\mu\nu}] = \delta_{\sigma\mu}J_{\rho\nu} + \delta_{\rho\nu}J_{\sigma\mu} - \delta_{\rho\mu}J_{\sigma\nu} - \delta_{\sigma\nu}J_{\rho\mu}, \quad (41)$$

$$[\alpha_\lambda, J_{\mu\nu}] = \delta_{\lambda\mu}\alpha_\nu - \delta_{\lambda\nu}\alpha_\mu \quad (42)$$

are valid. Eq. (41) is a commutation relation for generators of the Lorentz group, and relation (42) guarantees that Eq. (18) is form-invariant under



the Lorentz transformations. A Hermitianizing matrix  $\eta$  entering a relativistically invariant bilinear form

$$\bar{\Psi}\Psi = \Psi^\dagger \eta \Psi, \quad (43)$$

possesses the properties [16-18,8]:

$$\eta\alpha_i = -\alpha_i\eta, \quad \eta\alpha_4 = \alpha_4\eta \quad (i = 1, 2, 3). \quad (44)$$

It is verified that the  $\eta$  is given by

$$\eta = -\varepsilon^{0,0} + 2\beta_4^{(1)2} - \varepsilon^{\mu,\mu} - \frac{1}{2}\varepsilon^{[\mu\nu],[\mu\nu]}. \quad (45)$$

The squared Pauli-Lubanski vector (the squared spin operator) reads

$$\sigma^2 = \left( \frac{1}{2m} \varepsilon_{\mu\nu\alpha\beta} p_\nu J_{\alpha\beta} \right)^2 = \frac{1}{m^2} \left( J_{\mu\nu}^2 p^2 - J_{\mu\sigma} J_{\nu\sigma} p_\mu p_\nu \right). \quad (46)$$

The operator  $\sigma^2$  obeys the “minimal” equation

$$\sigma^2 (\sigma^2 - 2) = 0, \quad (47)$$

where the eigenvalues of the squared spin operator  $\sigma^2$  correspond to spin-zero,  $\mathbf{s} = \mathbf{0}$  ( $0 = s(s+1)$ ), and spin-one,  $\mathbf{s} = \mathbf{1}$  ( $2 = s(s+1)$ ). Thus, the fields considered describe fields possessing two spins,  $\mathbf{s} = \mathbf{0}$  and  $\mathbf{s} = \mathbf{1}$ . One may separate these states with the use of the projection operators

$$S_{(0)}^2 = 1 - \frac{\sigma^2}{2}, \quad S_{(1)}^2 = \frac{\sigma^2}{2}. \quad (48)$$

These operators have the properties  $S_{(0)}^2 S_{(1)}^2 = 0$ ,  $(S_{(0)}^2)^2 = S_{(0)}^2$ ,  $(S_{(1)}^2)^2 = S_{(1)}^2$ ,  $S_{(0)}^2 + S_{(1)}^2 = 1$ . According to the general properties of the projection operators, the matrices  $S_{(0)}^2$ ,  $S_{(1)}^2$  acting on the vector-column (the wave function) extract states with pure spin  $\mathbf{0}$  and  $\mathbf{1}$ , respectively. Let us consider the operator of the spin projection on the direction of the momentum  $\mathbf{p}$ :

$$\sigma_p = -\frac{i}{2|\mathbf{p}|} \epsilon_{abc} p_a J_{bc} = -\frac{i}{|\mathbf{p}|} \epsilon_{abc} p_a \beta_b^{(1)} \beta_c^{(1)}, \quad (49)$$

where  $|\mathbf{p}| = \sqrt{\mathbf{p}_1^2 + \mathbf{p}_2^2 + \mathbf{p}_3^2}$ . The spin projection operator obeys the “minimal” matrix equation:

$$\sigma_p (\sigma_p - 1) (\sigma_p + 1) = 0. \quad (50)$$

The projection operators corresponding to spin projection one and zero are

$$\hat{S}_{(\pm 1)} = \frac{1}{2} \sigma_p (\sigma_p \pm 1), \quad \hat{S}_{(0)} = 1 - \sigma_p^2, \quad (51)$$

where operators  $\hat{S}_{(\pm 1)}$ ,  $\hat{S}_{(0)}$  correspond to the spin projections  $s_p = \pm 1$  and  $s_p = 0$ , respectively. One can verify that the commutation relations

$$\begin{aligned} [S_{(0)}^2, \hat{p}] &= [S_{(1)}^2, \hat{p}] = [\hat{S}_{(\pm 1)}, \hat{p}] = [\hat{S}_{(0)}, \hat{p}] = 0, \\ [S_{(0)}^2, \hat{S}_{(\pm 1)}] &= [S_{(1)}^2, \hat{S}_{(\pm 1)}] = [S_{(0)}^2, \hat{S}_{(0)}] = 0 \end{aligned} \quad (52)$$

are valid. Now we may construct the projection matrices extracting states with pure spin, spin projection and energy:

$$\begin{aligned} \Delta_{\epsilon, \pm 1} &= M_\epsilon S_{(1)}^2 \hat{S}_{(\pm 1)} = \frac{i\hat{p}^{(1)} (i\hat{p}^{(1)} - \epsilon m)}{2m^2} \frac{1}{2} \sigma_p (\sigma_p \pm 1), \\ \Delta_\epsilon^{(1)} &= M_\epsilon S_{(1)}^2 \hat{S}_{(0)} = \frac{i\hat{p}^{(1)} (i\hat{p}^{(1)} - \epsilon m)}{2m^2} \frac{\sigma^2}{2} (1 - \sigma_p^2), \\ \Delta_\epsilon^{(0)} &= N_\epsilon S_{(0)}^2 \hat{S}_{(0)} \\ &= -\frac{m(K-m)}{m_0^2(m^2+m_0^2)} \left(K - \frac{m_0^2}{m}\right) \left(K - \frac{m^2+m_0^2}{m}\right) \left(1 - \frac{\sigma^2}{2}\right) (1 - \sigma_p^2). \end{aligned} \quad (53)$$

Here we took into consideration the relationship  $(\sigma^2/2) \sigma_p = \sigma_p$ . The projection operators  $\Delta_{\epsilon, \pm 1}$ ,  $\Delta_\epsilon^{(1)}$  correspond to states with spin  $\mathbf{1}$  and spin projections  $\pm 1$ ,  $0$ , and the  $\Delta_\epsilon^{(0)}$  extracts the spin  $\mathbf{0}$ . The operators  $\Delta_{\epsilon, \pm 1}$ ,  $\Delta_\epsilon^{(1)}$ ,  $\Delta_\epsilon^{(0)}$  are also the density matrices for pure spin states. One can easily construct the impure states by summation of Eqs. (53) over the spin and spin projections. In accordance with the approach in [8], the projection operators for pure states may be represented as matrix-dyads:

$$\begin{aligned} \Delta_{\epsilon, \pm 1} &= \Psi_{\epsilon, \pm 1}(p) \cdot \overline{\Psi}_{\epsilon, \pm 1}(p), \quad \Delta_\epsilon^{(1)} = \Psi_\epsilon(p) \cdot \overline{\Psi}_\epsilon(p), \\ \Delta_\epsilon^{(0)} &= \Psi_\epsilon^{(0)}(x) \cdot \overline{\Psi}_\epsilon^{(0)}(p). \end{aligned} \quad (54)$$

The wave functions  $\Psi_{\epsilon, \pm 1}$ ,  $\Psi_\epsilon$  correspond to spin- $\mathbf{1}$  and spin projections  $\pm 1$  and  $0$ , respectively, and the  $\Psi_\epsilon^{(0)}$  corresponds to the spin- $\mathbf{0}$ . The density matrices (53) and (54) may be used for calculating different electrodynamics processes involving polarized charged particles possessing two spins, one and zero.

## 5 Electromagnetic interactions of bosonic fields

Introducing the interaction with electromagnetic field in the first order equation (18) by the substitution  $\partial_\mu \rightarrow D_\mu = \partial_\mu - ieA_\mu$  ( $A_\mu$  is the four-vector potential of the electromagnetic field), and adding terms in the relativistic manner, which are linear in  $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ , we arrive at the matrix equation

$$\left[ \alpha_\mu D_\mu + \frac{i}{2} (\kappa_0 P_0 + \kappa_1 \bar{P} + \kappa_2 \bar{\bar{P}}) \alpha_{\mu\nu} \mathcal{F}_{\mu\nu} + m P_1 + \frac{m_0^2}{m} P_0 \right] \Psi(x) = 0, \quad (55)$$

where the projection operators  $P_0, \bar{P}, \bar{\bar{P}}$  are given by

$$P_0 = \varepsilon^{0,0}, \quad \bar{P} = \varepsilon^{\mu,\mu}, \quad \bar{\bar{P}} = \frac{1}{2} \varepsilon^{[\mu\nu],[\mu\nu]}, \quad (56)$$

and

$$\alpha_{\mu\nu} = \alpha_\mu \alpha_\nu - \alpha_\nu \alpha_\mu. \quad (57)$$

The projection operators  $P_0, \bar{P}, \bar{\bar{P}}$  extract scalar, vector and tensor parts of the wave function  $\Psi(x)$  respectively.

Using the definition of matrices (15) and Eqs. (55) and (56), we arrive from Eq. (55) at the following system of equations

$$\begin{aligned} D_\mu \psi_\mu + \frac{m_0^2}{m} \psi_0 + i\kappa_0 \mathcal{F}_{\mu\nu} \psi_{\mu\nu} &= 0, \\ m\psi_{\mu\nu} - D_\mu \psi_\nu + D_\nu \psi_\mu + i\kappa_2 (\mathcal{F}_{\nu\rho} \psi_{\mu\rho} - \mathcal{F}_{\mu\rho} \psi_{\nu\rho}) &= 0, \\ D_\nu \psi_{\mu\nu} + D_\mu \psi_0 + m\psi_\mu + 2i\kappa_1 \mathcal{F}_{\mu\nu} \psi_\nu &= 0, \end{aligned} \quad (58)$$

where  $\psi_{\mu\nu} \equiv \psi_{[\mu\nu]}$ . To clear up the physical meaning of constants  $\kappa_0, \kappa_1, \kappa_2$  introduced we consider the second order equation for four-vector  $\psi_\mu$ . For finding such an equation it is necessary to express the antisymmetric tensor  $\psi_{\mu\nu}$  via the four-vector  $\psi_\mu$ . To simplify the problem we imply the smallness of the constant  $\kappa_2$ . With the use of this natural assumption, neglecting  $\kappa_2^2$  and the higher degrees of the constant  $\kappa_2$  we find from the second equation of (58) the approximate expression

$$\begin{aligned} \psi_{\mu\nu} &\approx \frac{1}{m} (D_\mu \psi_\nu - D_\nu \psi_\mu) \\ &+ i \frac{\kappa_2}{m^2} [\mathcal{F}_{\mu\rho} (D_\nu \psi_\rho - D_\rho \psi_\nu) - \mathcal{F}_{\nu\rho} (D_\mu \psi_\rho - D_\rho \psi_\mu)]. \end{aligned} \quad (59)$$

It follows from Eqs. (58) using Eq. (59) the second order equation is

$$\begin{aligned}
& \left( D_\nu^2 - m^2 \right) \psi_\mu + \Delta D_\mu D_\nu \psi_\nu - i(e + 2\kappa_1 m) \mathcal{F}_{\mu\nu} \psi_\nu + i \frac{2\kappa_0 m}{m_0^2} D_\mu \mathcal{F}_{\rho\nu} D_\rho \psi_\nu \\
& - i \frac{\kappa_2}{m} D_\nu [\mathcal{F}_{\mu\rho} (D_\nu \psi_\rho - D_\rho \psi_\nu) - \mathcal{F}_{\nu\rho} (D_\mu \psi_\rho - D_\rho \psi_\mu)] = 0,
\end{aligned} \tag{60}$$

where  $\Delta = (m^2 - m_0^2)/m_0^2$ . Eq. (60) shows that the magnetic moment of a vector state of a particle is  $e/(2m) + \kappa_1$  and gyromagnetic ratio being  $1 + 2m\kappa_1/e$  (see [1]). So,  $\kappa_1$  is anomalous magnetic moment (AMM) of a particle. The constant  $\kappa_2$  is connected with the quadrupole electric moment (KEM) of a field [19]. The constant  $\kappa_0$  also gives the contribution to KEM of a particle but this contribution comes from the scalar state of a field. Indeed, we can consider the pure spin-one field by putting  $m_0 \rightarrow \infty$  in (60), i.e. when the mass of a scalar state is infinity [6]. In this case ( $m_0 \rightarrow \infty$ ) Eq. (60) transforms into

$$\begin{aligned}
& \left( D_\nu^2 - m^2 \right) \psi_\mu - D_\mu D_\nu \psi_\nu - i(e + 2\kappa_1 m) \mathcal{F}_{\mu\nu} \psi_\nu \\
& - i \frac{\kappa_2}{m} D_\nu [\mathcal{F}_{\mu\rho} (D_\nu \psi_\rho - D_\rho \psi_\nu) - \mathcal{F}_{\nu\rho} (D_\mu \psi_\rho - D_\rho \psi_\mu)] = 0.
\end{aligned} \tag{61}$$

Eq. (61) corresponds to the description of a vector particle on the basis of Proca's equation with the additional terms involving AMM and KEM of a particle. As a result Eq. (61) does not contain the constant  $\kappa_0$ . Thus, the consideration of a field with two spins, one and zero, allows us to introduce phenomenologically more constants characterizing the electromagnetic interactions of fields.

## 6 Hamiltonian form of equation

Now we consider the Hamiltonian form of Eq. (55) for particles possessing AMM and KEM in the external electromagnetic field.

To determine the number of dynamical variables of the wave function  $\Psi(x)$ , it is necessary to consider the Hamiltonian form of Eq. (55). For this purpose we write Eq. (55) in the form

$$i\alpha_4 \partial_t \Psi(x) = \left[ \alpha_a D_a + mP_1 + \frac{m_0^2}{m} P_0 + eA_0 \alpha_4 + \right. \tag{62}$$

$$+ \frac{i}{2} \left( \kappa_0 P_0 + \kappa_1 \bar{P} + \kappa_2 \bar{\bar{P}} \right) \alpha_{\mu\nu} \mathcal{F}_{\mu\nu} \Big] \Psi(x).$$

In order to separate the canonical and non-canonical parts of the equation, we introduce the operators:

$$\Lambda = \alpha_4^2, \quad \Pi = 1 - \alpha_4^2, \quad (63)$$

which obey the following relationships

$$\begin{aligned} \Lambda \alpha_4 &= \alpha_4, & \Pi \alpha_4 &= 0, & \Lambda^2 &= \Lambda, \\ \Pi^2 &= \Pi, & \Lambda \Pi &= 0, & \Lambda + \Pi &= 1, \end{aligned} \quad (64)$$

where  $\mathbf{1} \equiv I_{11}$  is a unit  $11 \times 11$ -matrix. Acting on Eq. (62) with the operators  $\alpha_4$  and  $\Pi$ , with the help of the equalities

$$\Lambda \bar{P} = \bar{P}, \quad \Lambda P_0 = P_0, \quad \Pi \alpha_\mu \Pi = \Pi P_0 = \Pi \bar{P} = 0, \quad \Pi P_1 = \Pi \bar{\bar{P}} = \Pi,$$

one obtains

$$\begin{aligned} i\partial_t \varphi(x) &= m \alpha_4 P_1 \Psi(x) + \frac{m_0^2}{m} \alpha_4 P_0 \Psi(x) + e A_0 \varphi(x) + \alpha_4 \left[ \alpha_a D_a + \right. \\ &\quad \left. + \frac{i}{2} \left( \kappa_0 P_0 + \kappa_1 \bar{P} + \kappa_2 \bar{\bar{P}} \right) \alpha_{\mu\nu} \mathcal{F}_{\mu\nu} \right] \Psi(x), \end{aligned} \quad (65)$$

$$m \chi(x) + \Pi \alpha_a D_a \varphi(x) + \frac{i \kappa_2}{2} \Pi \bar{\bar{P}} \alpha_{\mu\nu} \mathcal{F}_{\mu\nu} \Psi(x) = 0, \quad (66)$$

where  $\Lambda \Psi(x) = \varphi(x)$ ,  $\Pi \Psi(x) = \chi(x)$ .

It is seen from Eqs. (65) and (66) that the function  $\varphi(x)$  is the canonical variable, and  $\chi(x)$  is the non-canonical function. To have the Hamiltonian form of Eq. (55) we should exclude the non-canonical function  $\chi(x)$  from Eq. (65). It follows from Eq. (66) that

$$\chi(x) = - (1 + i\gamma)^{-1} \left( \frac{1}{m} \Pi \alpha_a D_a + i\gamma \right) \varphi(x), \quad \gamma = \frac{\kappa_2}{2m} \Pi \bar{\bar{P}} \alpha_{\mu\nu} \mathcal{F}_{\mu\nu}. \quad (67)$$

Neglecting  $\kappa_2^2$  the equality  $\gamma^2 \approx 0$  holds, and as a result

$$(1 + i\gamma)^{-1} \approx (1 - i\gamma). \quad (68)$$

Taking into account Eq. (68), and inserting the expression (67) into Eq. (65), after some transformations (see the appendix), we arrive at the Hamiltonian form of a field equation:

$$i\partial_t \varphi(x) = e A_0 \varphi(x) + \alpha_4 \left\{ \frac{m_0^2 - m^2}{m} P_0 \right.$$

$$\begin{aligned}
& + \left( m + \alpha_a D_a + \frac{i\kappa_0}{2} P_0 \alpha_{\mu\nu} \mathcal{F}_{\mu\nu} \right) (1 - i\gamma) \left( 1 - \frac{1}{m} \Pi \alpha_a D_a \right) \\
& + \frac{i\kappa_1}{2} \overline{P} \alpha_{\mu\nu} \mathcal{F}_{\mu\nu} + \frac{i\kappa_2}{2} \overline{\overline{P}} \alpha_{\mu\nu} \mathcal{F}_{\mu\nu} \left( 1 - \frac{1}{m} \Pi \alpha_a D_a \right) \Big\} \varphi(x).
\end{aligned} \tag{69}$$

We use here the approximate relation  $\kappa_2 \gamma \approx 0$ . It should be noted that we imply only the smallness of the constant  $\kappa_2$ , so  $\kappa_2^2 \approx 0$ , but the constants  $\kappa_0$  and  $\kappa_1$  are arbitrary. The wave function  $\varphi(x)$  has eight non-zero components: six components belong to states with spin  $\frac{1}{2}$  (three spin projections and two values of an energy), and two components belong to states with spin  $0$  (two values of an energy). It should be noted that states with spins  $0$  and  $\frac{1}{2}$  are not separated, i.e., it is impossible on the basis of Eq. (69) to obtain two Hamiltonian equations for functions describing states with spins  $0$  and  $\frac{1}{2}$  separately.

## 7 Conclusion

The theory of fields possessing two spin states, one and zero, AMM and KEM can be considered as effective one. This scheme may be applied for composite systems in particle and nuclear physics (see [20]). It should be noted, however, that the spin-zero state gives the negative contribution to the Hamiltonian of fields under consideration and it is necessary to introduce an indefinite metric to quantize such a field (see [4] where the case  $m = m_0$  was studied). But consideration of fields with two spins is justified for phenomenological applications and possibly for constructing other field schemes.

The density matrices (matrix-dyads) calculated for fields with spins one and zero allow us to make evaluations of different physical quantities in a covariant manner (see [8]).

## APPENDIX

Here we obtain the matrices entering Eq. (69) via elements of entire algebra (15). Using the definitions (15) and with the help of the equality [12]  $\delta_{[\mu\nu][\rho\sigma]} = \delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho}$ , it is easy to find the following expressions

$$\alpha_\mu \alpha_\nu = \varepsilon^{[\alpha\mu],[\alpha\nu]} + \delta_{\mu\nu} \left( \varepsilon^{\alpha,\alpha} + \varepsilon^{0,0} \right) + \varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu} + \varepsilon^{0,[\mu\nu]} + \varepsilon^{[\nu\mu],0},$$

$$\frac{1}{2} P_0 \alpha_{\mu\nu} = \varepsilon^{0,[\mu\nu]},$$

$$\frac{1}{2} \overline{P} \alpha_{\mu\nu} = \varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu},$$

$$\overline{\overline{P}}\alpha_{\mu\nu} = \varepsilon^{[\alpha\mu],[\alpha\nu]} - \varepsilon^{[\alpha\nu],[\alpha\mu]} + \varepsilon^{[\nu\mu],0} - \varepsilon^{[\mu\nu],0},$$

$$\Pi = \frac{1}{2}\varepsilon^{[mn],[mn]} \quad (m, n = 1, 2, 3),$$

$$\gamma = \frac{\kappa_2}{2m} \left( \varepsilon^{[mn],[m\nu]} \mathcal{F}_{n\nu} + \varepsilon^{[nm],0} \mathcal{F}_{mn} \right),$$

$$\overline{P}\alpha_{\mu\nu}\gamma = 0, \quad \Pi\alpha_a = \varepsilon^{[ma],m},$$

$$\gamma\Pi\alpha_a = \frac{\kappa_2}{2m} \left( \varepsilon^{[km],k} \mathcal{F}_{ma} - \varepsilon^{[am],n} \mathcal{F}_{mn} \right).$$

We imply the summation on repeating indexes; Greek and Latin letters take the numbers 1,2,3,4 and 1,2,3, respectively.

## References

- [1] H. C. Corben, J. Schwinger, Phys. Rev. **58**, 953 (1940).
- [2] A. Proca, Compt. Rend. **202**, 1420 (1936).
- [3] E. C. G. Stueckelberg, Helv. Phys. Acta **11**, 299 (1938).
- [4] S. I. Kruglov, Hadronic J. **24**, 167 (2001).
- [5] V. I. Ogievetskii and Polubarinov, Zh. Eksp. Teor. Fiz. **41**, 247 (1961) [Sov. Phys.-JETP **14**, 179 (1962)].
- [6] T. D. Lee, C. N. Yang, Phys. Rev. **128**, 885 (1962).
- [7] V. S. Vanyashin, Zh. Eksp. Teor. Fiz. **43**, 689 (1962) [Sov. Phys.-JETP **16**, 489 (1963)].
- [8] F. I. Fedorov, Zh. Eksp. Teor. Fiz. **35**, 493 (1958) [Sov. Phys. - JETP **35**(8), 339 (1959)]; The Lorentz Group (Nauka, Moscow, 1979) (in Russian).
- [9] E. Petiau, Tesis, Paris, 1936.
- [10] R. J. Duffin, Phys. Rev. **54**, 1114 (1938).
- [11] H. Kemmer, Proc. Roy. Soc. **173**, 91 (1939).
- [12] A. A. Bogush and L. G. Moroz, Introduction to the Theory of Classical Fields (Nauka i Tekhnika, Minsk, 1968) (in Russian).
- [13] K. U. Tzou, J. Phys. et Radium **18**, 619 (1957).

- [14] E. M. Corson, Theory of Tensors, Spinors, and Wave Equations (Benjamin Press., New York, 1953).
- [15] H. Umezawa, Quantum Field Theory (North-Holland, Amsterdam, 1956).
- [16] Harish-Chandra, Proc. Cambr. Phil. Soc. **43**, 414 (1947); Phys. Rev. **71**, 793 (1947).
- [17] I. M. Gel'fand, R. A. Minlos and Z. Ya. Shapiro, Representations of the Rotation and Lorentz Groups and their Applications (Pergamon, New York, 1963).
- [18] M. A. Naimark, Linear Representations of the Lorentz Group (Pergamon, New York, 1964).
- [19] J. A. Joung, S. A. Bludman, Phys. Rev. **131**, 2326 (1963).
- [20] M. Kirchbach, Int. J. Mod. Phys. **A15**, 1435 (2000).