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Nonspherical Giant Gravitons and Matrix Theory

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Abstract

We consider the plane wave limit of the nonspherical giant gravitons. We compute the Poisson brackets of the coordinate functions and find a nonlinear algebra. We show that this algebra solves the supersymmetry conditions of the matrix model. This is the generalization of the algebraic realization of the spherical membrane as the "fuzzy sphere". We describe finite dimensional representations of the algebra corresponding to the fuzzy torus.

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1 Introduction.

Brane polarization [1] provides a remarkable link between algebra and geometry. The simplest example of this phenomenon is the realization of the spherical D2 brane stabilized by the Ramond-Ramond flux as the static configuration of N D0 branes. The worldvolume theory of the N D0 branes is the theory of nine $N \times N$ matrices $X_i(t)$ describing the trajectories of the zero-branes[2]. For the branes moving in flat empty space the equations of motion imply for the static solutions that the matrices $X_i(t)$ for $i = 1, \ldots, 9$ commute with each other. Their common eigenvalues $x_{i,k}(t)$, $k = 1, \ldots, N$ describe the motion of N $D0$ branes. But in the curved space with the Ramond-Ramond fluxes there are static solutions with $[X_i, X_j] \neq 0$. For example, for the space-time with the constant Ramond-Ramond field strength there are static configurations with X_1, X_2, X_3 generating the N-dimensional representation of the algebra $su(2)$:

$$[X_i, X_j] = rX_k \tag{1}$$

These configurations are known as "fuzzy spheres". In the limit $N \to \infty$ with the fixed Nr such solutions are interpreted as spherical D2 branes. Therefore D2 branes are solutions of the worldvolume theory of the D0 branes. Of course, D0 branes can also be represented as the solutions of the worldvolume theory of the D2 branes. We can think of them as spherical D2 branes carrying the topologically nontrivial $U(1)$ bundle.

It turns out that in curved spaces like AdS massless particles can polarize into massive branes [3]. Polarized massless particles are called "giant gravitons". These giant gravitons were originally described as continuous branes of the spherical shape. Recently the authors of [4] found the matrix description of the plane wave geometries which are certain limits of $AdS_7 \times S^4$. It was shown in [4] that the giant gravitons represented by the M2 branes polarized in the direction of S^4 survive in the plane wave limit and are indeed described in the corresponding matrix model as fuzzy spheres (1).

One of the interesting properties of the giant gravitons is that unlike the polarized D0 branes they are deformable. We have found in [5] that spherical giant gravitons are representatives of the families of non-spherical solutions parametrized by the holomorphic functions. In this paper we will argue that the non-spherical giant gravitons correspond in the matrix model to finite-dimensional representations of the nonlinear algebra:

$$\begin{aligned}
&[X, \bar{X}] = 2X_3 \\
&[X_3, X] = X - \bar{\phi}(\bar{X}) \\
&[X_3, \bar{X}] = -\bar{X} + \phi(X)
\end{aligned} \tag{2}$$

where $\phi(X)$ is an arbitrary function. We will start with considering the nonspherical M2 brane preserving $\frac{1}{4}$ of the supersymmetries. We will describe the plane wave limit of this configuration and compute the Poisson brackets of the worldsheet coordinates. We will find that (2) is the algebra of the coordinate functions on the spacial slice of the

M2 brane. We then consider the supersymmetry conditions on the $\frac{1}{4}$ BPS configurations in the matrix theory which were formulated by Dongsu Bak in [6] (see also [7, 8]). We show that the algebra (2) solves these conditions. As an example we consider the matrix realization of a toroidal giant graviton.

2 Giant gravitons in pp wave background.

2.1 Giant gravitons in the Penrose limit.

To study the non-spherical giant gravitons in $AdS_7\times S^4$ it is convenient to embed $AdS_7\times S^4$ in the flat space with the signature $(2,11)$. Consider the flat space $\mathbf{R}^{2+11}$ with the metric

$$ds^2 = \sum_{A=1}^{6} dX_A^2 - dX_7^2 - dX_8^2 + \sum_{I=1}^{5} dY_I^2 \tag{3}$$

We embed $AdS_7 \times S^4$ as the intersection of two quadrics:

$$\begin{array}{r}X_7^2 + X_8^2 - \sum\limits_{i=1}^{6} X_i^2 = 4\\ \sum\limits_{i=1}^{5} Y_i^2 = 1\end{array} \tag{4}$$

We will be interested in the Penrose limit of the non-spherical giant gravitons. We will start with considering the Penrose limit of the embedding (4). It corresponds to the following approximate solution:

$$X_7 + iX_8 = 2\exp\left[i\left(x_+ + \frac{\epsilon^2}{4}x_-\right) + \frac{1}{8}\epsilon^2\sum_{a=1}^{6} x_i^2\right] \tag{5}$$

$$Y_4 + iY_5 = \exp\left[2i\left(x_+ - \frac{\epsilon^2}{4}x_-\right) - \frac{1}{2}\epsilon^2\sum_{a=1}^{3} y_a^2\right] \tag{6}$$

$$X_a = \epsilon x_a, \quad a = 1,\ldots,6, \quad Y_a = \epsilon y_a, \quad a = 1,2,3 \tag{7}$$

The metric is:

$$ds^2 = -4dx_+dx_- - 4(dx_+)^2\left(\sum y_a^2 + \frac{1}{4}\sum x_a^2\right) + \sum dx_a^2 + \sum dy_a^2 \tag{8}$$

We will consider the giant gravitons which expand inside S^4. According to the Section 5 of [5], the $\frac{1}{4}$-BPS giant gravitons in $AdS_7 \times S^4$ correspond to the holomorphic curves. Given the equation of the curve $F(z_1, z_2) = 0$ the equation of the $M2$ brane worldvolume is given by

$$F\left(\frac{Y_4 + iY_5}{(X_7 + iX_8)^2}, \frac{Y_1 + iY_2}{(X_7 + iX_8)^2}\right) = 0 \tag{9}$$

In the Penrose limit this reduces to:

$$\frac{|y|^2 + y_3^2}{2} + ix_- = W(e^{-2ix_+}y) \tag{10}$$

where W is a holomorphic function. It is convenient to consider separately the real and the imaginary parts of this equation:

$$|y|^2 + y_3^2 = W(e^{-2ix_+}y) + \overline{W}(e^{2ix_+}\bar{y}) \tag{11}$$

$$x_- = \frac{1}{2i}(W(e^{-2ix_+}y) - \overline{W}(e^{2ix_+}\bar{y})) \tag{12}$$

This determines the shape of the membrane worldvolume in terms of a single holomorphic function W.

2.2 Poisson brackets

The "spacial slices" $x_+ =$ const of the membrane worldvolume have a naturally defined symplectic structure[9, 10]. This allows to formulate the theory of membranes as the dynamics on the space of functions X_i with the potential expressed in terms of the Poisson brackets. Let us briefly review the construction of the symplectic form. The surfaces $x_+ =$ const of the eleven-dimensional space have null directions, which are generated by the lightlike Killing vector field $\frac{\partial}{\partial x_-}$. The momentum corresponding to this Killing vector field can be computed as the integral

$$P_+ = \int_{M2,\ x_+=const} \omega \tag{13}$$

This defines the two-form ω which should be taken as the symplectic form. We can describe it more explicitly. Suppose that we want to compute the integral of ω over the small element of the membrane area at $x_+ = 0$. We should then take the area of this element and multiply by the x_+-component of its velocity which should be orthogonal to the surface of the membrane. Suppose that $v_\perp$ denotes the vector on the membrane worldvolume which is orthogonal to the slice $x_+ = const$. Then,

$$\omega = \frac{dx_+(v_\perp)}{|v_\perp|}\,\text{Area} \tag{14}$$

where Area is the two-form which measures the area of the membrane.

Let us compute the symplectic structure for the membrane described by the Eq. (10). We will parametrize the worldvolume of the membrane by $(x_+, y, \bar{y})$. The metric on the worldvolume is

$$ds^2|_{M2} = -4(|y|^2 + y_3^2)(dx_+)^2 - \frac{2}{i}(W' d(e^{-2ix_+}y) - \overline{W'} d(e^{2ix_+}\overline{y}))dx_+ + dy d\overline{y} + (dy_3)^2 \tag{15}$$

where $y_3 = \pm\sqrt{W + \overline{W} - |y|^2}$. One can take

$$v_\perp = \frac{\partial}{\partial x_+} - 2ie^{-2ix_+} W' \frac{\partial}{\partial \bar{y}} + 2ie^{2ix_+} \overline{W'} \frac{\partial}{\partial y} \tag{16}$$

with

$$dx_+(v_\perp) = 1, \quad ||v_\perp||^2 = -4(W + \overline{W} + |W'|^2 - yW' - \bar{y}\overline{W'}) \tag{17}$$

The area element of the slice $x_+ = const$ is

$$\text{Area} = \sqrt{1 + 4|\partial y_3/\partial y|^2} dy \wedge d\bar{y} = \frac{\sqrt{W + \overline{W} + |W'|^2 - yW' - \bar{y}\overline{W'}}}{y_3} dy \wedge d\bar{y} \tag{18}$$

This leads to the simple expression for the symplectic form:

$$\omega = \frac{dy \wedge d\bar{y}}{y_3} \tag{19}$$

The Poisson brackets computed with this symplectic form are:

$$\begin{aligned} \{y, \bar{y}\} &= 2iy_3 \\ \{y_3, y\} &= i(y - \overline{W'}(\bar{y})) \\ \{y_3, \bar{y}\} &= -i(\bar{y} - W'(y)) \end{aligned} \tag{20}$$

In the next section we will verify that this algebra satisfies the supersymmetry condition of the Matrix theory.

The time dependence of the worldsheet coordinates can be read from (16):

$$\frac{dy}{dt} = ie^{it}\overline{W'}(e^{it}\bar{y}) \tag{21}$$

where we have denoted $t = 2x_+$. This corresponds to the supersymmetric trajectory of the superparticle in the constant magnetic field. Indeed, let us introduce $z = e^{-it}y$. We can interpret (21) as coming from the Lagrangian:

$$\begin{aligned} S = \left|\frac{dz}{dt}\right|^2 + i\left(z\frac{d}{dt}\bar{z} - \bar{z}\frac{d}{dt}z\right) + |z|^2 + |W'(z)|^2 - zW'(z) - \bar{z}\overline{W'}(\bar{z}) - \\ -iC_{ij}\bar{\xi}^i\frac{d}{dt}\xi^j + C_{ij}\bar{\xi}^i\xi^j - \frac{1}{2}W''(z)\varepsilon_{ij}\xi^i\xi^j + \frac{1}{2}\overline{W''}(\bar{z})\varepsilon_{ij}\bar{\xi}^i\bar{\xi}^j \end{aligned} \tag{22}$$

with the supersymmetry transformations:

$$\begin{aligned} \delta\xi^i &= \left(\tfrac{dz}{dt} + iz\right) C^{ij}\bar{\epsilon}_j + i\overline{W'}(\bar{z})\varepsilon^{ij}\epsilon_j \\ \delta z &= i\epsilon_i\xi^i \end{aligned} \tag{23}$$

where the supersymmetry parameter ϵ_i is subject to the reality condition

$$C^{ij}\bar{\epsilon}_j = -\varepsilon^{ij}\epsilon_j, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{24}$$

It would be interesting to learn if the Lagrangian (22) has any applications to the matrix description of the giant graviton besides giving the supersymmetric trajectories.

3 Matrix Theory.

3.1 Supersymmetry of the massive matrix model.

The matrix model for the M-theory plane waves was introduced in [4]. In this section we will be interested in the solutions of this matrix model with $x_1 = \ldots = x_6 = 0$. The bosonic sector of the model is:

$$\begin{aligned} S = \int\ dt\ \mathrm{Tr}\ & \left[\sum_{j=1}^{3} \tfrac{1}{2}(D_0 y^j)^2 + \tfrac{1}{4}\sum_{j,k=1}^{3}[y^j, y^k]^2 - \right. \\ & \left. -\tfrac{1}{2}\left(\tfrac{\mu}{3}\right)^2 \sum_{j=1}^{3}(y^j)^2 - i\tfrac{\mu}{3}\sum_{j,k,l=1}^{3}\epsilon_{jkl}\mathrm{Tr}\ y^j y^k y^l\right] \end{aligned} \tag{25}$$

The generators of the supersymmetry transformation are the nine-dimensional Majorana spinors ϵ explicitly depending on time:

$$\epsilon(t) = e^{-\frac{\mu}{12}\gamma_{123}t}\epsilon_0 \tag{26}$$

The condition for ϵ to annihilate the solution is:

$$\left(D_0 y^i \gamma^i + \frac{\mu}{6}\sum_{i=1}^{3} y^i\gamma^i\gamma_{123} + \frac{i}{2}[y^i, y^j]\gamma_{ij}\right)\epsilon(t) = 0 \tag{27}$$

The spherical giant graviton preserves all the supersymmetry of the matrix model. It corresponds to constant y^i generating a representation of $su(2)$:

$$[y^i, y^j] = \frac{i\mu}{6}\epsilon^{ijk}y^k \tag{28}$$

We are interested in the solutions which respect only half of the supersymmetry generators. The preserved supersymmetries are generated by the parameters ϵ satisfying the condition

$$\gamma^3\epsilon = \epsilon \tag{29}$$

For such ϵ the condition (27) is equivalent to:

$$D_0 y + i\frac{\mu}{6}y - i[y^3, y] = 0 \tag{30}$$

$$\frac{\mu}{6}y^3 - \frac{1}{2}[y, \bar{y}] = 0 \tag{31}$$

$$D_0 y^3 = 0 \tag{32}$$

We will choose the gauge $A = 0$, $D_0 = \frac{\partial}{\partial t}$. The solution to (30) is

$$y^3(t) = y^3(0) = \frac{3}{\mu}[y(0), \overline{y(0)}] \tag{33}$$

$$y(t) = e^{-it\frac{\mu}{6}} e^{ity^3} y(0) e^{-ity^3} \tag{34}$$

Supersymmetric solutions to the matrix model equations of motion are subject to the constraint

$$\frac{1}{2}([\partial_t y, \bar{y}] + [\partial_t \bar{y}, y]) + [\partial_t y^3, y^3] = 0 \tag{35}$$

The solution (33) satisfies this constraint if and only if

$$[[y^3, y], \bar{y}] = \frac{\mu}{6}[y, \bar{y}] \tag{36}$$

Equations (33) and (36) were derived by Dongsu Bak in [6]. They determine supersymmetric solutions of the matrix model which preserve one half of the supersymmetry. The Poisson brackets (20) correspond to the following commutators[2]:

$$\begin{aligned} &[y, \bar{y}] = \tfrac{\mu}{3} y^3 \\ &[y^3, y] = \tfrac{\mu}{6}(y - \overline{W'}(\bar{y})) \\ &[y^3, \bar{y}] = -\tfrac{\mu}{6}(\bar{y} - W'(y)) \end{aligned} \tag{37}$$

This algebra indeed satisfies (33) and (36) and therefore represents the supersymmetric solution of the matrix model.

3.2 Representations for the toroidal giant graviton.

It would be interesting to study the representations of the algebra (37). Our giant gravitons are compact manifolds, which suggests that one should look for the finite-dimensional representations. For the giant gravitons with the shape close to spherical, it should be possible to realize the algebra (37) on the space of a finite dimensional representation of $su(2)$. Indeed, one can always find three functions S_1, S_2, S_3 with the Poisson brackets $\{S_i, S_j\} = \epsilon_{ijk} S_k$ and the constraint $\sum S_i^2 = R^2$ where R is related to the area of the sphere. (There is no difference between the round sphere and the deformed sphere, from the point of view of the symplectic geometry.) The coordinates y_1, y_2, y_3 can be expressed in terms of S_1, S_2, S_3, which means that they act on the representation space of $su(2)$.

Here we want to consider the example of the toroidal membrane corresponding to $W(y) = A \log y$:

$$\begin{aligned} &[y, \bar{y}] = \tfrac{\mu}{3} y_3 \\ &[y_3, y] = \tfrac{\mu}{6}(y - A\bar{y}^{-1}) \\ &[y_3, \bar{y}] = -\tfrac{\mu}{6}(\bar{y} - A y^{-1}) \end{aligned} \tag{38}$$

[2]We have determined the normalization constant $\frac{\mu}{6}$ in the relation between the Poisson brackets and the commutators so that the spherical giant graviton (28) satisfies the supersymmetry equations.

We will try the following ansatz:

$$y = \sum_{j=1}^{N} q_j E_{j+1,j} \tag{39}$$

where $E_{i,j}$ is the matrix $||E_{i,j}||_{mn} = \delta_{im}\delta_{jn}$ and we put $E_{N+1,N} = E_{1,N}$. For example for $N = 4$:

$$y = \begin{pmatrix} 0 & q_1 & 0 & 0 \\ 0 & 0 & q_2 & 0 \\ 0 & 0 & 0 & q_3 \\ q_4 & 0 & 0 & 0 \end{pmatrix} \tag{40}$$

The operators $y, \bar{y}$ and $y_3 = \frac{3}{\mu}[y, \bar{y}]$ satisfy (38) if

$$2|q_j|^2 - |q_{j-1}|^2 - |q_{j+1}|^2 = \frac{36}{\mu^2}\left(1 - \frac{A}{|q_j|^2}\right) \tag{41}$$

We will consider this equation in the continuous approximation which is possible when $A >> 1$. We denote $\tau_j = \frac{j}{\sqrt{A}}$ and substitute $|q_j|^2 = A(1 + f(\tau_j))$. We will get:

$$f''(\tau) = -\frac{f}{1+f} \tag{42}$$

This has a solution

$$\tau = \pm \int \frac{df}{\sqrt{2(E - f + \log(1+f))}} \tag{43}$$

The "potential energy" $U(f) = f - \log(1+f)$ is a convex function of f with the minimum $U(0) = 0$. This means that for any $E > 0$ $|q_j|^2$ as a function of j will be oscillating around A. For finite A, the correct boundary conditions at the turning points will restrict E to belong to a discreet set. This solution describes the toroidal giant graviton.

It would be interesting to describe finite dimensional representations for other W. A similar problem was solved in [11] for holomorphic membranes in flat space resulting in infinite-dimensional representations.

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