Liénard-Wiechert Potentials of a Non-Abelian Yang Mills Charge

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Given the path of a point particle, one can relate its acceleration and, in general, its kinematics to the curvature scalars of its trajectory. Using this, a general Ansatz is made for the Yang Mills connection corresponding to a non-Abelian point source. The Yang Mills field equations are then solved outside the position of the point source under physically reasonable constraints such as finite total energy flux and finite total color charge. The solutions contain the Trautman solution; moreover two of them are exact whereas one of them is found using a series expansion in $1/R$, where R is the retarded distance. These solutions are new and, in their most general form, are not gauge equivalent to the original Trautman solution.

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I. INTRODUCTION

In classical Maxwell electrodynamics, it is well known that accelerated charges emit electromagnetic radiation. Using Liénard-Wiechert (LW) potentials, one is able to calculate the field strength for a point charge (or a system of point charges) and relate the radiation from the accelerated charge to its motion and the geometry of its trajectory [1], [2]. One may, of course, ask the same question in the case of non-Abelian Yang Mills (YM) theory, i.e. the existence of a LW potential such that the emitted radiation and the trajectory of the charge are interrelated. An analysis of this problem was done long ago [3] and it was found that the color of a single point particle remains constant although there is a transfer of energy, provided that the gauge group is compact and semi-simple. Later it was also shown that the total gauge-invariant color of an external line source in YM theory could change as a result of color radiation, depending on the YM waves considered [4].

Here we want to examine whether one can generalize the LW potential of [3] in the light of the recent results we found relating the generalized acceleration scalars (a, a_k) of a point particle to the curvature scalars of its trajectory [5]. To this effect, we make a new Ansatz for the YM vector potential and find new solutions to the source-free YM field equations. Two of the solutions we present are exact solutions whereas one of them is found using a series expansion in $1/R$, the reciprocal of the retarded distance R. [redacted]ned by using physically acceptable constraints such as demanding that the total energy flux N_{YM} and the total color charge Q are finite at large R values. We also examine whether the solutions presented are gauge equivalent to the Trautman solution and show that they are not in their full generality.

In Section II, we give a brief review of some basic elements regarding the geometry of a trajectory curve. A new Ansatz for the YM connection is presented in Section III. The total energy flux of the YM field is defined and its behavior at large distances is examined. Demanding the flux to be finite further constrains the form of the Ansatz studied. In Section IV, we give the general form of the source free YM field equations. We also examine in detail the existence of special trajectories for which YM field equations are satisfied. We then give the series expansion of these equations in powers of $1/R$. In Section V, the total color charge is defined and its form is derived for the $1/R$ expansion of the YM connection Ansatz under study. In Section VI, we present two new exact solutions as well as one approximate solution that is obtained by solving the $1/R$ expanded field equations under the finite total color charge constraint. In Section VII, we examine whether the solutions presented are gauge equivalent to the Trautman solution [3]. In two Appendices, we give the explicit form of some complicated equations that are needed in the text.

II. THE GEOMETRY OF A CURVE

Let $z^\mu(\tau)$ define a smooth curve C in a flat spacetime with Minkowski metric $\eta_{\mu\nu}$[11]. One can, in general, define two times using an arbitrary point x^μ outside the curve. Now let τ denote *the retarded time* in the usual sense – that one obtains by looking at the roots of $(x^\mu - z^\mu(\tau))\,(x_\mu - z_\mu(\tau)) = 0$ – and define the null vector

$$\lambda_\mu \equiv \frac{\partial\,\tau}{\partial\,\mu} = \frac{x_\mu - z_\mu(\tau)}{R}, \tag{1}$$

where $R \equiv \dot{z}^\mu(\tau)\,(x_\mu - z_\mu(\tau))$ gives *the retarded distance*. Here and from now on a dot over a letter denotes differentiation with respect to the retarded time τ. Differentiating λ_μ and R, one finds that

$$\lambda_{\mu,\nu} = \frac{1}{R}\left[\eta_{\mu\nu} - \dot{z}_\mu\,\lambda_\nu - \dot{z}_\nu\,\lambda_\mu - (R\,a - \epsilon)\,\lambda_\mu\lambda_\nu\right], \tag{2}$$

$$R_{,\mu} = (R\,a - \epsilon)\,\lambda_\mu + \dot{z}_\mu \tag{3}$$

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where *the acceleration* of the particle is $a \equiv \frac{1}{R}\,\ddot{z}^{\mu}\,(x_{\mu}-z_{\mu}(\tau))$ and $\dot{z}^{\mu}\,\dot{z}_{\mu}=\epsilon\equiv 0,\pm 1$. [We choose $\epsilon=-1$ for time-like curves and $\epsilon=0$ for null curves.] Moreover, one also finds that $\lambda^{\mu}\,\dot{z}_{\mu}=1\,,\lambda^{\mu}\,R_{,\mu}=1$ and $\lambda^{\mu}\,a_{,\mu}=0$.

In fact, one notices that one can define[12]

$$a_k \equiv \lambda_{\mu}\,\frac{d^k\,\ddot{z}^{\mu}}{d\tau^k}, \quad k=1,2,\cdots,D-1$$

which satisfy

$$\lambda^{\mu}\,a_{k,\mu}=0\,, \tag{4}$$

for all k. Here $a_0\equiv a$ and D denotes the dimension of the spacetime. [From now on we will take $D=4$.] The scalars a_k, which are generalizations of the acceleration a of the point particle, are related to the curvature scalars of the smooth curve C [6], [5].

By using the curve C and its kinematics, one can construct more general solutions to the classical source free YM field equations than the one found by Trautman [3].

III. THE TOTAL ENERGY FLUX

Making use of the curve kinematics that has been briefly discussed in Section II, we now make the following Ansatz for the LW potential

$$A_{\mu}=H\,\dot{z}_{\mu}+G\,\lambda_{\mu}\,, \tag{5}$$

where H and G are differentiable functions of R and some R-independent functions $c_i\ (i=1,2,\cdots)$ such that $\lambda^{\mu}\,c_{i,\mu}=0$ for all i[13]. It is clear that due to the property (4) of a_k, all of these functions (c_i) are functions of the scalars a and $a_k\ (k=1,2,3)$ and the retarded time τ. For the original Trautman solution, $G=0$ and $H=q/R$ where $q=q(\tau)$ only [3]. We are motivated to choose the more general Ansatz above (5) by our recent results concerning the D-dimensional Einstein Maxwell theory with a null perfect fluid in the Kerr-Schild geometry [5].

In close analogy to how it is defined in the Maxwell case [1], *the total energy flux* of the YM field is given by[14]

$$N_{YM}=-\int_S \dot{z}_{\mu}\,T^{\mu\nu}\,n_{\nu}\,R\,d\Omega\,. \tag{6}$$

Here $T_{\mu\nu}=F_{\mu\alpha}\,F_{\nu}{}^{\alpha}-\frac{1}{4}\,\eta_{\mu\nu}\,F_{\alpha\beta}\,F^{\alpha\beta}$ is the YM energy momentum tensor (with again the trace over the internal group indices suppressed on the YM field strength F.) The vector n_{ν} is orthogonal to the velocity vector field $\dot{z}_{\mu}$ and is defined through

$$\lambda_{\mu}=\epsilon\dot{z}_{\mu}+\epsilon_1\,\frac{1}{R}\,n_{\mu} \quad ;\ n^{\mu}\,n_{\mu}=-\epsilon R^2\,. \tag{7}$$

(Here $\epsilon_1=\pm 1$.) One can consider S in the rest frame of the point particle as a sphere S^2 of very large radius R. $d\Omega$, of course, denotes the solid angle.

Substituting for A_{μ} (5) in the YM field strength[15]

$$F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}+[A_{\mu},A_{\nu}]\,, \tag{8}$$

one finds that

$$F_{\mu\nu}=\lambda_{\mu}\chi_{\nu}-\lambda_{\nu}\chi_{\mu}+H_{,\mu}\dot{z}_{\nu}-H_{,\nu}\dot{z}_{\mu}$$

where for convenience we have defined $\chi_{\nu}\equiv H\ddot{z}_{\nu}-G_{,\nu}-[H,G]\dot{z}_{\nu}$.

Using the orthogonality of the velocity vector field $\dot{z}_{\mu}$ and the vector n_{μ}, one finds that

$$\dot{z}_{\mu}\,T^{\mu\nu}\,n_{\nu}=(F_{\mu\alpha}\,\dot{z}^{\mu})\,(F_{\nu}{}^{\alpha}\,n^{\nu})\,,$$

or substituting for $F_{\mu\nu}$ in this expression, that

$$R\,\dot{z}_{\mu}\,T^{\mu\nu}\,n_{\nu}=-\epsilon\,\epsilon_1\,R^2\,\mathrm{Tr}\,(Y^{\alpha}\,J_{\alpha})$$

where

$$\begin{aligned} Y_{\alpha} &\equiv H\ddot{z}_{\alpha}+\dot{z}_{\alpha}\,(H_{,\mu}\dot{z}^{\mu}-[H,G]) \\ &\quad +\lambda_{\alpha}\,(G_{,\mu}\dot{z}^{\mu}+\epsilon\,[H,G])-G_{,\alpha}-\epsilon\,H_{,\alpha} \qquad \text{and} \\ J_{\alpha} &\equiv H\ddot{z}_{\alpha}+\dot{z}_{\alpha}\,(H_{,\mu}\dot{z}^{\mu}-[H,G]-\epsilon\,H') \\ &\quad +\lambda_{\alpha}\,(G_{,\mu}\dot{z}^{\mu}+\epsilon\,(a\,H-G'))-G_{,\alpha}\,. \end{aligned}$$

Here and from now on a prime over a letter denotes partial differentiation with respect to R.

Notice that when $G=G(R,c_i(\tau,a,a_k))\quad (k=1,2,3)$ in its full generality,

$$\begin{aligned} G_{,\alpha} &= G'R_{,\alpha}+G_{,c_i}c_{i,\alpha} \\ &= G'R_{,\alpha}+G_{,c_i}(\dot{c}_i\lambda_{,\alpha}+c_{i,a}a_{,\alpha}+c_{i,a_k}a_{k,\alpha}) \end{aligned}$$

and similarly for $H=H(R,c_i(\tau,a,a_k))\quad (k=1,2,3)$. The derivative of the acceleration a is

$$a_{,\alpha}=\frac{1}{R}\ddot{z}_{\alpha}-\frac{a}{R}\dot{z}_{\alpha}+(a_1-a^2+\epsilon\,\frac{a}{R})\lambda_{\alpha} \tag{9}$$

and similarly one can also calculate the derivatives of a_k for $k=1,2,3$. As an example, the derivative of a_1 is given in the Appendix A, (A1). It is easy to see that putting the derivatives of a_k, one ends up with higher order τ derivatives of $\dot{z}_{\mu}$ and more complicated expressions for $G_{,\alpha}$ (and likewise for $H_{,\alpha}$) and hence N_{YM}. So to keep the angular integrations that appear in the expression of N_{YM} and also the calculations that one encounters in the solution of the YM field equations simple, from now on we assume that the functions c_i, whose properties were described above, are just functions of τ and a. Hence one has $G=G(R,c_i(\tau,a))$ and similarly $H=H(R,c_i(\tau,a))$ from now on. With these assumptions, substituting the derivative of a and the derivative of the retarded distance R (3) in the relevant expressions above, one finds that Y_{α} and J_{α} can, respectively, be put in the form

$$\begin{aligned} Y_{\alpha} &= \ddot{z}_{\alpha}Y_2+\dot{z}_{\alpha}Y_1+\lambda_{\alpha}Y_0 \\ J_{\alpha} &= \ddot{z}_{\alpha}J_2+\dot{z}_{\alpha}J_1+\lambda_{\alpha}J_0\,, \end{aligned}$$

with six coefficients $Y_2, J_2, Y_1, J_1, Y_0, J_0$ to be determined. Notice that this yields

$$
\begin{aligned}
\mathrm{Tr}\,(Y^\alpha J_\alpha) &= \ddot{z}^\alpha \ddot{z}_\alpha\, \mathrm{Tr}\,(Y_2 J_2) + a\, \mathrm{Tr}\,(Y_2 J_0 + Y_0 J_2) \\
&\quad + \epsilon\, \mathrm{Tr}\,(Y_1 J_1) + \mathrm{Tr}\,(Y_1 J_0 + Y_0 J_1)
\end{aligned}
$$

using the properties of the velocity vector field $\dot{z}_\mu$ and λ_μ.

Notice at this point that it is still very difficult to work out the general form of the energy flux formula N_{YM} let alone to find solutions of the YM field equations in this general setting. However demanding that N_{YM} be finite at very large values of R, one can in general assume that a series expansion of H and G can be made in powers of $1/R$ (with $R \neq 0$, of course) as

$$
\begin{aligned}
H &\equiv \alpha + \frac{\beta}{R} + \frac{\gamma}{R^2} + O(R^{-3}) \quad \text{and} \\
G &\equiv \sigma + \frac{\omega}{R} + \frac{\delta}{R^2} + O(R^{-3})\,.
\end{aligned}
$$

Here α, β, γ and σ, ω, δ are the functions c_i that we have described above and are just functions of the retarded time τ and the acceleration a.

With these assumptions then, one finds that

$$
\begin{aligned}
J_2 &\equiv H - \frac{1}{R} G_{,c_i}\, c_{i,a} \\
&= \alpha + \frac{1}{R}(\beta - \sigma_{,a}) + \frac{1}{R^2}(\gamma - \omega_{,a}) + O(R^{-3}) \quad \text{and} \\
Y_2 &\equiv H - \frac{1}{R}(G_{,c_i} + \epsilon H_{,c_i})\, c_{i,a} \\
&= \alpha + \frac{1}{R}(\beta - \sigma_{,a} - \epsilon\, \alpha_{,a}) + \frac{1}{R^2}(\gamma - \omega_{,a} - \epsilon \beta_{,a}) \\
&\quad + O(R^{-3})\,,
\end{aligned}
$$

which in turn implies that $\mathrm{Tr}\,(Y_2 J_2) = \mathrm{Tr}\,(\alpha^2) + O(1/R)$ and one has to set $\alpha = 0$ to get a finite energy flux as one takes the limit $R \to \infty$ in the expression (6) for N_{YM}. So one is now left with

$$
H \equiv \frac{\beta}{R} + \frac{\gamma}{R^2} + O(R^{-3}) \quad \text{and} \tag{10}
$$

$$
G \equiv \sigma + \frac{\omega}{R} + \frac{\delta}{R^2} + O(R^{-3})\,, \tag{11}
$$

where the five remaining coefficients $\beta, \gamma, \sigma, \omega$ and δ (functions of τ and a) are to be determined by the YM field equations.

Carrying out the calculations of the remaining coefficients Y_1, J_1, Y_0 and J_0 in the same manner to $O(R^{-3})$ and using these, one finds that

$$
\begin{aligned}
\mathrm{Tr}\,(Y_2 J_2) &= \frac{1}{R^2} \mathrm{Tr}\,(\beta - \sigma_{,a})^2 + O(R^{-3})\,, \\
a\, \mathrm{Tr}\,(Y_2 J_0) &= \epsilon \frac{a^2}{R^2} \mathrm{Tr}\,(\beta - \sigma_{,a})^2 + O(R^{-3})\,, \\
a\, \mathrm{Tr}\,(Y_0 J_2) &= -\mathrm{Tr}\,(Y_1 J_0) \\
&= \epsilon \frac{a}{R^2}\, \mathrm{Tr}\,\{([\beta, \sigma] + a(\beta - \sigma_{,a}) - \dot{\beta} \\
&\quad - (a_1 - a^2)\beta_{,a}) \times (\beta - \sigma_{,a})\} + O(R^{-3})\,, \\
\epsilon\, \mathrm{Tr}\,(Y_1 J_1) &= -\mathrm{Tr}\,(Y_0 J_1) \\
&= \frac{\epsilon}{R^2} \mathrm{Tr}\,\{[\beta, \sigma] + a(\beta - \sigma_{,a}) - \dot{\beta} \\
&\quad - (a_1 - a^2)\beta_{,a}\}^2 + O(R^{-3})\,,
\end{aligned}
$$

and hence

$$
R\, \dot{z}_\mu\, T^{\mu\nu}\, n_\nu = -\epsilon\, \epsilon_1\, (\ddot{z}^\alpha \ddot{z}_\alpha + \epsilon a^2)\, \mathrm{Tr}\,(\beta - \sigma_{,a})^2 + O(1/R)\,.
$$

So taking the $R \to \infty$ limit, one finds

$$
N_{YM} = \int_S d\Omega\, \epsilon\, \epsilon_1\, (\ddot{z}^\alpha \ddot{z}_\alpha + \epsilon a^2)\, \mathrm{Tr}\,(\beta - \sigma_{,a})^2 \tag{12}
$$

for the energy flux.

Notice that the YM field equations have not been solved yet and at this stage N_{YM} contains only the first terms that appear in the series expansion of H and G in powers of $1/R$ (10), (11). So to observe any outgoing radiation at large distances, one has to keep either β, the coefficient of the $1/R$ term in H, and/or σ, the constant term in G of the YM connection (5).

We will come back to the discussion of the energy flux N_{YM} after we find solutions of the YM field equations using the YM connection (5) with H and G given by (10) and (11).

IV. THE SOURCE FREE YM EQUATIONS

The source free YM field equations simply read

$$
D^\mu\, F_{\mu\nu} = \partial^\mu\, F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = 0\,. \tag{13}
$$

The field strength is given by (8), of course. Taking A_μ as in (5) with $H = H(R, c_i(\tau, a))$ and $G = G(R, c_i(\tau, a))$, calculating $F_{\mu\nu}$ and using these in (13), one finds that in the general case $D^\mu F_{\mu\nu}$ can be put in the form

$$
D^\mu\, F_{\mu\nu} = X\, z_\nu^{(3)} + Y\, \ddot{z}_\nu + K\, \dot{z}_\nu + L\, \lambda_\nu = 0 \tag{14}
$$

where $z_\nu^{(3)}$ denotes $d^3 z_\nu / d\tau^3$. The explicit form of $D^\mu F_{\mu\nu}$ is given in the Appendix A. One immediately recognizes that it is extremely difficult to find exact solutions to $D^\mu F_{\mu\nu} = 0$ (13) in this general setting. Even though this is the case, in what follows we are going to try to consider every possible case in detail, do what can be said and done and leave out only the most complicated equations which we found too hard to solve.

A. A Closer Look at the Trajectories

Notice at this point that one can ask whether $D^\mu F_{\mu\nu} = 0$ is satisfied identically without imposing any conditions on H and/or G in A_μ; i.e. whether there are any special trajectories that a point particle can follow so that the YM equations identically hold outside its position.

Contracting $D^\mu F_{\mu\nu}$ with $\lambda^\nu, \dot{z}^\nu, \ddot{z}^\nu$, etc. one finds that $D^\mu F_{\mu\nu}$ can in general be written as

$$D^\mu F_{\mu\nu} = X(z^{(3)}_\nu + p\ddot{z}_\nu + q\dot{z}_\nu + r\lambda_\nu)$$

where

$$\begin{aligned} p &\equiv \frac{1}{\Delta}(\ddot{z}^\mu\, z^{(3)}_\mu + a(\epsilon\, a_1 + \ddot{z}^\mu\, \ddot{z}_\mu))\ , \\ q &\equiv -\frac{1}{\Delta}(a\, \ddot{z}^\mu\, z^{(3)}_\mu + \ddot{z}^\mu\, \ddot{z}_\mu\, (a^2 - a_1))\ , \\ r &\equiv \frac{1}{\Delta}(\epsilon\, a\, \ddot{z}^\mu\, z^{(3)}_\mu - (\epsilon\, a_1 + \ddot{z}^\mu\, \ddot{z}_\mu)\, \ddot{z}^\alpha\, \ddot{z}_\alpha)\ , \\ \Delta &\equiv -(\epsilon\, a^2 + \ddot{z}^\mu\, \ddot{z}_\mu) \neq 0\ . \end{aligned}$$

Now setting $X = 0$ would imply that $H_{,c_i} c_{i,a} = 0$. Since $H_{,c_i} = 0$ necessarily excludes the curve kinematics that we want to introduce into the picture, this leaves one with $c_{i,a} = 0$ or $c_i = c_i(\tau)$. [Notice that G still may depend on c_i with $c_i = c_i(\tau, a)$.] However in that case $D^\mu F_{\mu\nu}$ now takes the form

$$D^\mu F_{\mu\nu} = M_2 \ddot{z}_\nu + M_1 \dot{z}_\nu + M_0 \lambda_\nu$$

where, for example,

$$M_2 = H' + \frac{H}{R} - \frac{1}{R}(G'_{,c_i} + [H, G_{,c_i}])c_{i,a}\ .$$

One is then either forced to take the trajectory curve C as a straight line or set $M_2 = M_1 = M_0 = 0$ for a nontrivial curve C. However these three equations $M_2 = M_1 = M_0 = 0$ are again highly nonlinear and very complicated to work with.

For $X \neq 0$, one has to put

$$z^{(3)}_\nu + p\ddot{z}_\nu + q\dot{z}_\nu + r\lambda_\nu = 0\ . \tag{15}$$

However making use of the Serret-Frenet frame in four dimensions [5], one finds that

$$p = -\frac{\dot{\kappa}_1}{\kappa_1}\ \ ,\ \ q = -\kappa_1^2 \ \text{ and }\ r = 0$$

and that (15) is satisfied identically when $i)\, \kappa_1 = 0$, i.e. $a = 0$ which implies that the trajectory curve C is a straight line or $ii)$ $\kappa_2 = 0$ but $\kappa_1 \neq 0$. The first case again gives a "trivial" solution, whereas the second case implies that one has to both constrain the trajectory curve C and find the corresponding H and G which satisfy

$$Y = -\frac{\dot{\kappa}_1}{\kappa_1}X\ \ ,\ \ K = -\kappa_1^2 X \ \text{ and }\ L = 0\ .$$

These equations are again very difficult to solve. So instead of working out these complicated conditions on the trajectory curve C, we now concentrate on what one can do with the YM equations themselves.

B. $1/R$ Expansion of Source Free YM Equations

Since it is very hard to find exact solutions to $D^\mu F_{\mu\nu} = 0$ (14) in its full generality, we look for approximate solutions by using the series expansion of H and G in powers of $1/R$ (10), (11). Substituting these in the expressions for X, Y, K and L (see equations (A2), (A3), (A4) and (A5)), one finds to order R^{-3} that

$$\begin{aligned} D^\mu F_{\mu\nu} \;=\; & \frac{1}{R}\lambda_\nu(L_0) \\ & + \frac{1}{R^2}(X_1\, z^{(3)}_\nu + Y_1\, \ddot{z}_\nu + K_1\, \dot{z}_\nu + L_1\, \lambda_\nu) \\ & + O(R^{-3})\ , \end{aligned}$$

where

$$
\begin{aligned}
L_0 \;&\equiv\; a\{\dot\beta + (a_1 - a^2)\beta_{,a} - [\beta,\sigma]\} + 2[\dot\beta + (a_1-a^2)\beta_{,a},\sigma] + [\beta,\dot\sigma + (a_1-a^2)\sigma_{,a}] + [\sigma,[\beta,\sigma]] \\
&\quad - \beta_{,a}(a_2 - 3a_1 a + 2a^3) - 2\,(a_1-a^2)\,\dot\beta_{,a} - (a_1-a^2)^2\,\beta_{,aa} - \ddot\beta\;, \qquad (16)\\
L_1 \;&\equiv\; 2a\omega - \dot\omega - (a_1-a^2)\,\omega_{,a} + 2\epsilon\, a\sigma_{,a} + (\epsilon\, a^2 + \ddot z^\alpha\,\ddot z_\alpha)\,\sigma_{,aa} + (a_1 - 3a^2)\gamma - \epsilon\,\dot\beta - \epsilon\, a\dot\beta_{,a} \\
&\quad - \epsilon(2a_1 - 3a^2 + \epsilon\,\ddot z^\alpha\,\ddot z_\alpha)\beta_{,a} - \epsilon\, a(a_1-a^2)\beta_{,aa} - (a_1-a^2)^2\,\gamma_{,aa} - 2(a_1-a^2)\dot\gamma_{,a} - \ddot\gamma + 3a\dot\gamma \\
&\quad - (a_2 - 6a_1 a + 5a^3)\,\gamma_{,a} + \epsilon\,[\beta,\sigma] - 3a[\gamma,\sigma] - 2a[\beta,\omega] + 2[\dot\beta + (a_1-a^2)\,\beta_{,a},\omega] \\
&\quad + 2[\dot\gamma + (a_1-a^2)\,\gamma_{,a},\sigma] + [\gamma,\dot\sigma + (a_1-a^2)\,\sigma_{,a}] + [\beta,\dot\omega + (a_1-a^2)\,\omega_{,a}] + \epsilon\,[\beta,[\beta,\sigma]] \\
&\quad - [\sigma,\omega] + [\sigma,[\beta,\omega]] + [\omega,[\beta,\sigma]] + [\sigma,[\gamma,\sigma]] - \epsilon\,[\beta,\dot\beta + (a_1-a^2)\,\beta_{,a}] - \epsilon\, a[\sigma,\beta_{,a}] - \epsilon\, a[\beta,\sigma_{,a}]\;, \qquad (17)\\
K_1 \;&\equiv\; \dot\beta + (2a_1 - 3a^2)\beta_{,a} + a(\dot\beta_{,a} + (a_1-a^2)\,\beta_{,aa}) + [\sigma,\beta] + [\beta,\dot\beta] + (a_1-a^2)\,[\beta,\beta_{,a}] - [\beta,[\beta,\sigma]] \\
&\quad + a([\sigma,\beta_{,a}] + [\beta,\sigma_{,a}])\;, \qquad (18)\\
Y_1 \;&\equiv\; -[\sigma,\beta_{,a}] - [\beta,\sigma_{,a}] - \dot\beta_{,a} - (a_1-a^2)\,\beta_{,aa} + 2\,a\beta_{,a}\;, \qquad (19)\\
X_1 \;&\equiv\; -\beta_{,a}\;. \qquad (20)
\end{aligned}
$$

Before dwelling on the solutions of the equations above, we now make a digression and briefly review the definition of a gauge-invariant total color charge. Demanding the total color charge to be finite at large R values constrains the system of differential equations to be solved considerably and one is, at least, able to talk about the behavior of solutions.

V. TOTAL COLOR CHARGE Q

In the presence of sources, the YM field equations are

$$D^\mu F_{\mu\nu} \;=\; \partial^\mu F_{\mu\nu} \;+\; [A^\mu, F_{\mu\nu}] \;=\; J_\nu\;. \qquad (21)$$

Even though $D^\mu J_\mu = 0$, using this, one can not define a conserved or a gauge-invariant charge in the usual sense. However, one can make use of the fact that

$$\partial^\nu\,\partial^\mu\,F_{\mu\nu} = \partial^\nu\,(J_\nu - [A^\mu, F_{\mu\nu}]) \;=0$$

and define the total color as

$$I = \int d^3x\,(J_0 - [A^i, F_{i0}])\,. \qquad (22)$$

According to [4], considering gauge transformations that are independent of space-time coordinates at large distances; i.e. taking into account YM connections A_μ that go to zero faster than $1/R$ at large R, I turns out to be gauge covariant and a gauge invariant *total color charge* can be defined as[16] (see the discussions in [7] and [4] as well)

$$Q \;=\; \sqrt{\mathrm{Tr}\, I^2}\;. \qquad (23)$$

However it is also claimed in [8], [4] that the total color (22) is not well suited for determining the color exchange between an external source and the YM waves. The reason they give is that neither (22) nor (23) can be written in a gauge independent manner as a sum of the gauge invariant color of the external source and that of the YM field.

Here we are not going to take part in this discussion. Whether it can be written in a gauge independent manner or not, for (23) to make sense as the definition of some charge, (22) must, first of all, be finite when the integration is carried on the full space. This is especially essential for us in this work, since the only source here is a point source moving along a trajectory. However some attention is also needed here: One has to do the integration on such a surface that it respects the motion of the point source.

Notice that similar considerations led [1] to the expression for the total energy flux N_{YM} in (6). So in complete analogy, one can define the total color as

$$I = \int_S d\Omega\, R^2\,(J_\nu - [A^\mu, F_{\mu\nu}])\,\dot z^\nu\;, \qquad (24)$$

or simply as

$$I = \int_S d\Omega\, R^2\,(\partial^\mu\, F_{\mu\nu})\,\dot z^\nu\;, \qquad (25)$$

so that the motion of the point source is taken into account. The surface S can again be thought of as a sphere S^2 of very large radius R in the rest frame of the point source. One can again use (23) as the definition of the total color charge then.

So we need to examine $\partial^\mu F_{\mu\nu}$ for the A_μ Ansatz (5) that we have. Similar to what we did for $D^\mu F_{\mu\nu}$, substituting H and G in powers of $1/R$ (10), (11) in the expression for $\partial^\mu F_{\mu\nu}$ (see Appendix B), one finds to order R^{-3} that

$$
\begin{aligned}
\partial^\mu F_{\mu\nu} \;&=\; \frac{1}{R}\lambda_\nu(S_0) \\
&\quad + \frac{1}{R^2}(B_1\, z^{(3)}_\nu + C_1\,\ddot z_\nu + P_1\,\dot z_\nu + S_1\,\lambda_\nu) \\
&\quad + O(R^{-3})\,,
\end{aligned}
$$

where

$$
\begin{aligned}
S_0 \;\equiv\;& a\{\dot\beta + (a_1 - a^2)\beta_{,a} - [\beta,\sigma]\} + [\dot\beta + (a_1 - a^2)\beta_{,a},\sigma] + [\beta,\dot\sigma + (a_1 - a^2)\sigma_{,a}] - \beta_{,a}(a_2 - 3a_1 a + 2a^3) \\
& - 2(a_1 - a^2)\dot\beta_{,a} - (a_1 - a^2)^2\,\beta_{,aa} - \ddot\beta\,, \qquad (26)\\
S_1 \;\equiv\;& 2a\omega - \dot\omega - (a_1 - a^2)\,\omega_{,a} + 2\epsilon\, a\sigma_{,a} + (\epsilon\, a^2 + \ddot z^\alpha\,\ddot z_\alpha)\,\sigma_{,aa} + (a_1 - 3a^2)\gamma - \epsilon\,\dot\beta - \epsilon\, a\dot\beta_{,a} \\
& - \epsilon(2a_1 - 3a^2 + \epsilon\,\ddot z^\alpha\,\ddot z_\alpha)\beta_{,a} - \epsilon\, a(a_1 - a^2)\beta_{,aa} - (a_1 - a^2)^2\,\gamma_{,aa} - 2(a_1 - a^2)\dot\gamma_{,a} - \ddot\gamma + 3a\dot\gamma \\
& - (a_2 - 6a_1 a + 5a^3)\,\gamma_{,a} - 2a[\gamma,\sigma] - 2a[\beta,\omega] + [\dot\beta + (a_1 - a^2)\,\beta_{,a},\omega] + [\dot\gamma + (a_1 - a^2)\,\gamma_{,a},\sigma] \\
& + [\beta,\dot\omega + (a_1 - a^2)\,\omega_{,a}] + [\gamma,\dot\sigma + (a_1 - a^2)\,\sigma_{,a}]\,, \qquad (27)\\
P_1 \;\equiv\;& \dot\beta + (2a_1 - 3a^2)\beta_{,a} + a(\dot\beta_{,a} + (a_1 - a^2)\,\beta_{,aa}) + [\sigma,\beta]\,, \qquad (28)\\
C_1 \;\equiv\;& -\dot\beta_{,a} - (a_1 - a^2)\,\beta_{,aa} + 2\,a\beta_{,a}\,, \qquad (29)\\
B_1 \;\equiv\;& -\beta_{,a}\,. \qquad (30)
\end{aligned}
$$

Since $\lambda^\nu \dot z_\nu = 1$, one is forced to set $S_0 = 0$ so that I in (25) and hence Q become finite for very large R. Hence taking the $R \to \infty$ limit, the total color charge Q is defined through I in (25) which turns out to be

$$I = \int_S d\Omega\,(-B_1\,\ddot z^\alpha\,\ddot z_\alpha + \epsilon\,P_1 + S_1)$$

for the special form of the A_μ Ansatz (5) that we are using.

VI. SOLUTIONS

In this Section we look for solutions to the source free YM equations. We first start with the original Trautman solution to remind the reader about its properties and also to check the calculations that have been done so far. We then give two new exact solutions and finally present the "approximate" one which is obtained by using a series expansion in $1/R$.

A. Trautman Solution

In the case when $G = 0$, i.e. $\sigma = \omega = \delta = \gamma = 0$ and when A_μ is of the form

$$A_\mu \;=\; \frac{\beta}{R}\,\dot z_\mu\,, \qquad (31)$$

one should of course find the original Trautman solution which is the first example of a non-Abelian LW potential. Indeed, one finds in this case that the YM field equations $D^\mu F_{\mu\nu} = 0$ are satisfied *exactly* provided $i)\,\beta = \beta(\tau)$, $ii)\,\ddot\beta - a\,\dot\beta = 0$ and $iii)\,\dot\beta + [\beta,\dot\beta] = 0$, as found by Trautman [3]. In this case since $\sigma = 0$, the total energy flux formula N_{YM} is just a simple generalization of the corresponding expression for the ordinary Abelian Maxwell theory, that is obtained by replacing the square of the electric charge by $\mathrm{Tr}\beta^2$ [1], [3], [5]. Moreover, one also finds that $S_0 = 0, S_1 = -\epsilon\,\dot\beta, P_1 = \dot\beta, B_1 = C_1 = 0$ and hence $(\partial^\mu F_{\mu\nu})\,\dot z^\nu = 0$ identically in this case. So the total color charge $Q = 0$ and automatically conserved as expected from [3].

B. $\beta = \gamma = 0\,,\delta = 0$

For the special choice of $\beta = \gamma = 0$ and $\delta = 0$, i.e. when A_μ is of the form

$$A_\mu \;=\; (\sigma + \frac{\omega}{R})\,\lambda_\mu\,, \qquad (32)$$

one finds that the YM field equations $D^\mu F_{\mu\nu} = 0$ are satisfied *exactly* provided that σ and ω satisfy

$$
\begin{aligned}
\omega \;&=\; \omega(\tau), \qquad (33)\\
\partial_a\,(\sigma_{,a}(\ddot z^\alpha\ddot z_\alpha + \epsilon\,a^2) + a^2\,\omega) - \dot\omega - [\sigma,\omega] \;&=\; 0\,. \qquad (34)
\end{aligned}
$$

Notice that in this case the total energy flux N_{YM} expression (12) has only $\beta - \sigma_{,a} = -\sigma_{,a}$ in it, which is considerably different than the Trautman solution in character. The dependence of σ on the acceleration a of the trajectory turns out to determine the form of N_{YM}. Moreover, in this case one finds that $B_1 = C_1 = P_1 = 0$ as well as $S_0 = 0$, of course, but $S_1 = [\sigma,\omega]$. Hence I now becomes $I = \int_S d\Omega\,[\sigma,\omega]$. $Q = 0$ iff $[\sigma,\omega] = 0$, of course.

A trivial solution to (34) is given by $\sigma_{,aa} = 0\,,\epsilon\,\sigma_{,a} + \omega = 0$ and $\dot\omega + [\sigma,\omega] = 0$. Then $\sigma = ak(\tau) + l(\tau)\,,\omega = -\epsilon k(\tau)$ and the arbitrary functions $k(\tau)$ and $l(\tau)$ satisfy $\dot k + [l,k] = 0$. (If one chooses $k = -q$ and $l = -\dot q$, then $\sigma = -aq(\tau) - \dot q(\tau)$ and $\omega = \epsilon\,q$, and this yields the same condition that one obtains in the case of the Trautman solution. One, of course, expects that this trivial solution is gauge equivalent to Trautman's original solution.) Notice also that then $I = \int_S d\Omega\,(-\epsilon\,[l,k]) = \int_S d\Omega\,\epsilon\,\dot k$ and $\beta - \sigma_{,a} = -\sigma_{,a} = -k(\tau)$ in N_{YM} (12).

C. $\gamma = 0\,,\delta = 0$

For the special choice of $\gamma = 0$ and $\delta = 0$, i.e. when A_μ is of the form

$$A_\mu = \frac{\beta}{R}\,\dot z_\mu + (\sigma + \frac{\omega}{R})\,\lambda_\mu\,, \tag{35}$$

one finds again that the YM field equations $D^\mu F_{\mu\nu} = 0$ are satisfied *exactly* provided that β, σ and ω satisfy

$$\begin{aligned}
\beta &= \beta(\tau), & (36)\\
\dot\beta - [\beta,\sigma] &= 0, & (37)\\
\omega + g(\tau) - [\beta,\omega] &= 0, & (38)\\
[\beta, g] &= 0, & (39)\\
\dot g + \partial_a\,(\sigma_{,a}(\ddot z^\alpha \ddot z_\alpha + \epsilon\, a^2) - a^2 g) + [\sigma, g] &= 0, & (40)\\
\partial_a\,(\omega_{,a}(\ddot z^\alpha \ddot z_\alpha + \epsilon\, a^2)) + \epsilon\,(\omega + g) + [\omega, g] &= 0. & (41)
\end{aligned}$$

Here $g = g(\tau)$ is an arbitrary function of τ that is obtained in one of the integrations in the mid steps of the calculation. For this case one finds that N_{YM} maintains its most general form and that $B_1 = C_1 = P_1 = 0$, as well as $S_0 = 0$, of course, and $S_1 = -\epsilon\,\dot\beta - [\sigma, g]$ which gives $I = \int_S d\Omega\,(-\epsilon\,\dot\beta - [\sigma, g])$. Again $Q \neq 0$ for the general case and can be chosen to depend on τ.

A trivial solution to equations (40) and (41) are given by choosing

$$\sigma_{,aa} = 0\,, \epsilon\,\sigma_{,a} - g = 0\,, \dot g + [\sigma, g] = 0\,, \omega_{,a} = 0\,,$$

$$\epsilon\,(\omega + g) + [\omega, g] = 0\,.$$

All of the conditions (36) to (41) are satisfied identically provided

$$g(\tau) = \epsilon\,\beta\,, \sigma = a\beta + \sigma_0(\tau)\,, \omega = \omega(\tau)\,, \dot\beta + [\sigma_0, \beta] = 0$$

and $[\beta,\omega] = \omega + \epsilon\,\beta$. (Here $\sigma_0(\tau)$ is an arbitrary function and if one chooses $\sigma_0 = -\dot\beta$ and $\beta = q$, this yields the same condition that one finds for q in the Trautman solution.) Notice also that then $\beta - \sigma_{,a} = 0$ in N_{YM} (12) and $I = 0$, and hence $Q = 0$.

D. General Case

In this case we take A_μ to be of the form (5) with H and G given by (10) and (11), respectively. We look for solutions of the YM field equations to order R^{-3} by setting $L_0 = L_1 = K_1 = Y_1 = X_1 = 0$ (16), (17), (18), (19), (20). However due to the discussion at the end of Section V, we also need to set $S_0 = 0$ (26) for a finite total color charge. Hence we now have to solve these six equations simultaneously for the five unknown coefficients $\beta, \gamma, \sigma, \omega$ and δ (which are, remember, only functions of τ and a).

In this case, one finds that $L_0 = L_1 = K_1 = Y_1 = X_1 = 0$ and $S_0 = 0$ are satisfied provided that

$$\begin{aligned}
\beta &= \beta(\tau), & (42)\\
\gamma &= \gamma(\tau), & (43)\\
\dot\beta - [\beta,\sigma] &= 0, & (44)\\
a\gamma - \omega - \tilde g(\tau) + [\beta,\omega] + [\gamma,\sigma] &= 0, & (45)\\
\partial_a\,(\sigma_{,a}(\ddot z^\alpha \ddot z_\alpha + \epsilon\, a^2) - a^2(\tilde g - \dot\gamma)) & &\\
+\,\partial_\tau\,(\tilde g - \dot\gamma) + [\sigma, \tilde g - \dot\gamma] &= 0. & (46)
\end{aligned}$$

Here $\tilde g = \tilde g(\tau)$ is an arbitrary function of τ that appears in one of the integrations in the mid steps of the calculation. In this case one finds that $I = \int_S d\Omega\,(-[\sigma, \tilde g - \dot\gamma] - \epsilon\,\dot\beta)$ and hence for the general case $Q \neq 0$ and, in fact, may be chosen to depend on τ. However notice that when one chooses the integration function $\tilde g(\tau)$ as $\tilde g(\tau) = \epsilon\,\beta + \dot\gamma$, (46) becomes

$$\sigma_{,a}(\ddot z^\alpha \ddot z_\alpha + \epsilon\, a^2) - \epsilon\,\beta\, a^2 = n(\tau)$$

where $n(\tau)$ is a new arbitrary function of τ, and $I = 0$ and hence $Q = 0$. One can further choose $\sigma_{,a} = \beta$, i.e. $\sigma = a\beta(\tau) + \sigma_0(\tau)$, and the arbitrary function $n(\tau)$ suitably such that this is also identically satisfied. So now (44) implies $\dot\beta - [\beta, \sigma_0] = 0$ and a trivial solution to (45) is provided by demanding that $i)\ \gamma + [\gamma, \beta] = 0$ and $ii)\ [\beta,\omega] + [\gamma, \sigma_0] - \omega - \tilde g = 0$.

VII. GAUGE EQUIVALENCE

One natural question to ask at this stage is, of course, whether the solutions that have been found so far are gauge equivalent to the Trautman solution [3], also derived in Subsection VI A. To answer this question, one has to examine whether there exist any gauge potentials Φ (we again suppress internal group indices on Φ) which locally satisfy

$$A_\mu^{Trautman} = A_\mu^{newsol} + \partial_\mu\,\Phi + [A_\mu^{newsol}, \Phi] = \frac{q}{R}\,\dot z_\mu\,. \tag{47}$$

(Here we take the original form of the Trautman solution, i.e. one has q in place of β.)

Notice that substituting the general form of our Ansatz $A_\mu = H\dot z_\mu + G\lambda_\mu$ (5) and solving for $\partial_\mu\Phi$, one finds that in general $\partial_\mu\Phi$ is of the form

$$\partial_\mu\,\Phi = X\,\dot z_\mu + Y\,\lambda_\mu\,. \tag{48}$$

Demanding that Φ has continuous second order derivatives and that $\partial_\mu\partial_\nu\Phi = \partial_\nu\partial_\mu\Phi$, one finds

$$X_{,\nu}\dot z_\mu - X_{,\mu}\dot z_\nu + X(\ddot z_\mu\lambda_\nu - \ddot z_\nu\lambda_\mu) + Y_{,\nu}\lambda_\mu - Y_{,\mu}\lambda_\nu = 0\,. \tag{49}$$

Contracting this with λ^μ and $\dot z^\mu$, one obtains two equations which then can be solved for $X_{,\nu}$ and $Y_{,\nu}$ to yield

$$\begin{aligned}
X_{,\nu} &= \dot z_\nu(\lambda^\mu X_{,\mu}) + \lambda_\nu(\lambda^\mu Y_{,\mu} - a\,X)\,, & (50)\\
Y_{,\nu} &= \ddot z_\nu X + \dot z_\nu(\dot z^\mu X_{,\mu} - \epsilon\,\lambda^\mu X_{,\mu})\\
&\quad + \lambda_\nu(\dot z^\mu Y_{,\mu} - \epsilon\,\lambda^\mu Y_{,\mu} - \epsilon\,a\,X)\,. & (51)
\end{aligned}$$

Substituting these into (49), one finally finds that

$$(\lambda^\mu Y_{,\mu} - a\,X - \dot z^\mu X_{,\mu} + \epsilon\,\lambda^\mu X_{,\mu})(\dot z_\mu \lambda_\nu - \dot z_\nu \lambda_\mu) = 0\,.$$

For a nontrivial gauge potential Φ, one has to demand that

$$\lambda^\mu Y_{,\mu} - a\,X - \dot z^\mu X_{,\mu} + \epsilon\,\lambda^\mu X_{,\mu} = 0 \tag{52}$$

for the coefficients X and Y in (48).

So now we look for the existence of such gauge potentials for each of the solutions presented in Section VI in order of their appearance.

A. $\beta = \gamma = 0\,, \delta = 0$

For this *exact* solution, X and Y in (48) turns out to be

$$X = \frac{q}{R}\;\;,\;\; Y = -(\sigma + \frac{\omega}{R}) - [\sigma + \frac{\omega}{R}, \Phi]$$

and remember that $q = q(\tau), \sigma = \sigma(\tau, a)$ and $\omega = \omega(\tau)$, and these satisfy (34) in this case. So imposing (52), one gets

$$\frac{1}{R}(-\dot q - [\sigma, q]) + \frac{1}{R^2}(\omega - \epsilon\,q + [\omega, \Phi] - [\omega, q]) = 0\,.$$

Since the coefficients q, σ and ω are R independent, one has to set

$$\begin{aligned} \dot q + [\sigma, q] &= 0\,, & (53)\\ \omega - \epsilon\,q + [\omega, \Phi] - [\omega, q] &= 0\,. & (54)\end{aligned}$$

A trivial solution is provided by $\sigma = -aq - \dot q, \omega = \epsilon\,q$ (see the end of Subsection VI B) and $\Phi = kq$, where k is an arbitrary real number. In that case (33) and (34) are also identically satisfied since q, now being part of the Trautman solution, obeys $\dot q + [q, \dot q] = 0$. In this case $A_\mu^{newsol} = (-\dot q - a\,q + \epsilon\,\frac{q}{R})\lambda_\mu$ and this is gauge equivalent to the Trautman solution.

However in the general case when $\sigma = \sigma(\tau, a)$, it is not easy to find a simultaneous solution to (33), (34), (53) and (54), and hence this class of exact solutions is not necessarily gauge equivalent to the Trautman solution.

B. $\gamma = 0\,, \delta = 0$

For this *exact* solution, following similar steps as in Subsection VII A, one finds that

$$X = \frac{q - \beta}{R} - [\frac{\beta}{R}, \Phi]\;\;,\;\; Y = -(\sigma + \frac{\omega}{R}) - [\sigma + \frac{\omega}{R}, \Phi]$$

and for this case remember that $\beta = \beta(\tau), \sigma = \sigma(\tau, a)$ and $\omega = \omega(\tau, a)$, and these satisfy (36), (37), (38), (39), (40) and (41). Imposing (52), one finds that

$$\begin{aligned}&\frac{1}{R}\{\dot\beta - \dot q - [\sigma, q] + [\sigma, [\beta, \Phi]] + [\dot\beta, \Phi] - [\beta, [\sigma, \Phi]]\}\\ &\quad + \frac{1}{R^2}\{\omega - \epsilon\,(q - \beta) - [\omega, q] + [\omega + \epsilon\,\beta, \Phi]\\ &\qquad + [\omega, [\beta, \Phi]] - [\beta, [\omega, \Phi]]\} \;=\; 0\,.\end{aligned}$$

Since the coefficients β, σ, ω and q are R independent, one has to set

$$\begin{aligned} \dot\beta - \dot q - [\sigma, q] + [\sigma, [\beta, \Phi]] + [\dot\beta, \Phi] - [\beta, [\sigma, \Phi]] &= 0\,, & (55)\\ \omega - \epsilon\,(q - \beta) - [\omega, q] + [\omega + \epsilon\,\beta, \Phi] &\\ + [\omega, [\beta, \Phi]] - [\beta, [\omega, \Phi]] &= 0\,. & (56)\end{aligned}$$

If one considers the trivial solution described at the end of Subsection VI C with $q = \beta$, uses $[\beta, \omega] = \omega + \epsilon\,\beta$ in (56) and the Jacobi identity, one gets $\omega - [\omega, \beta] = 0$ which yields $\omega = -\epsilon\,\beta/2$. Then using the Jacobi identity in (55) implies

$$[\beta, \sigma] + [\dot\beta, \Phi] + [[\sigma, \beta], \Phi] = 0$$

and using $\dot\beta + [\sigma_0, \beta] = 0$ gives $[\beta, \sigma] = [\beta, \sigma_0] = 0$ or $\sigma_0 = c\beta(\tau)$ for c an arbitrary real constant, independent of the choice of the gauge potential Φ. However, this in turn implies that $\dot\beta = 0$ or $\beta =$constant, the Trautman solution. Hence, once again this trivial solution is gauge equivalent to the Trautman solution.

Notice that in the general case when $\sigma = \sigma(\tau, a)$ and $\omega = \omega(\tau, a)$, it is not easy to find a simultaneous solution to both the conditions (36), (37), (38), (39), (40), (41), coming from the $D^\mu F_{\mu\nu} = 0$ equations, and the gauge conditions (55), (56) above. Hence this class of exact solutions is not necessarily gauge equivalent to the Trautman solution.

C. General Case

Remember that the solutions in this class were found by solving the $D^\mu F_{\mu\nu} = 0$ YM equations to order R^{-3} and by simultaneously setting the term of order R^{-1} in $\partial^\mu F_{\mu\nu}$, i.e. S_0 (26), to zero. Hence these solutions are *approximate* in character and for that reason we now examine the question of whether these are "approximately" gauge equivalent to the Trautman solution. So we assume that the gauge potential Φ locally has a well defined series expansion in powers of $1/R$ as ($R \neq 0$)

$$\Phi = \psi + \frac{1}{R}\varphi + \frac{1}{R^2}\zeta + O(R^{-3})$$

where the coefficients ψ, φ and ζ are, of course, R independent now and we assume them to be only functions of τ and a. With these in mind, one can write X and Y

in (48) to order R^{-3} as

$$\begin{aligned} X &= \frac{1}{R}(q-\beta-[\beta,\psi]) \\ &\quad + \frac{1}{R^2}(-\gamma-[\beta,\varphi]-[\gamma,\psi]) + O(R^{-3}), \\ Y &= -\sigma-[\sigma,\psi]+\frac{1}{R}(-\omega-[\sigma,\varphi]-[\omega,\psi]) \\ &\quad + \frac{1}{R^2}(-\delta-[\sigma,\zeta]-[\omega,\varphi]-[\delta,\psi]) + O(R^{-3}). \end{aligned}$$

Remember that at this stage all the coefficients above are only functions of τ and a. So imposing (52) and carefully collecting the coefficients of the powers of $1/R$ to order R^{-3}, one gets

$$\begin{aligned} \dot{z}^\mu\partial_\mu(q-\beta-[\beta,\psi]) &= 0, \\ \omega+[\sigma,\varphi]+[\omega,\psi]-a(\gamma+[\beta,\varphi]+[\gamma,\psi]) & \\ +\,\dot{z}^\mu\partial_\mu(\gamma+[\beta,\varphi]+[\gamma,\psi])-\epsilon\,(q-\beta-[\beta,\psi]) &= 0. \end{aligned}$$

One now has to solve these two conditions simultaneously with (42), (43), (44), (45) and (46) for gauge equivalence of this class of solutions to the Trautman solution. Obviously, this is not an easy task if one is to stay in the most general case and we conjecture that the class of "approximate" solutions we found are not "approximately" gauge equivalent to the Trautman solution.

Hence when one considers the solutions presented in Section VI in their full generality, one can assert that they are not gauge equivalent to the Trautman solution.

VIII. CONCLUSIONS

We have found new solutions to the source free YM field equations which generalize the LW potential of Trautman. Two of the solutions are exact whereas one of them is approximate and obtained through a $1/R$ series expansion in the YM field equations. For each solution the total energy flux N_{YM} and the total color charge Q have been constrained to be finite. It has also been shown that the solutions are not gauge equivalent to the Trautman solution in their most general form.

In [9], Trautman's original solution was shown to exist in the setting of Robinson-Trautman metrics in General Relativity. After the seminal work of [10], there has also been an ongoing interest in the particle like solutions of Einstein-YM theory. It would be interesting to study Einstein-YM theory in the Kerr-Schild geometry using the general Ansatz for the YM connection (5) presented here.

Appendix A: The Explicit Form of $D^\mu F_{\mu\nu}$

In this Appendix, we show explicitly how one obtains the YM field equations starting with the general Ansatz for the YM connection A_μ as

$$A_\mu = H(R,c_i(\tau,a))\,\dot{z}_\mu + G(R,c_i(\tau,a))\,\lambda_\mu\,.$$

$R_{,\mu}$ (3), $a_{,\mu}$ (9) and the derivative of a_1

$$a_{1,\mu} = \frac{1}{R}z^{(3)}_\mu - \frac{a_1}{R}\dot{z}_\mu + \{a_2 + \frac{1}{R}(\ddot{z}^\alpha\ddot{z}_\alpha) - a_1 a + \epsilon\,\frac{a_1}{R}\}\lambda_\mu \tag{A1}$$

are expressions that are needed in the calculation of $D^\mu F_{\mu\nu}$.

After lengthy calculations one obtains that

$$D^\mu\,F_{\mu\nu} = X\,z^{(3)}_\nu + Y\,\ddot{z}_\nu + K\,\dot{z}_\nu + L\,\lambda_\nu$$

where

$$X = -\frac{1}{R} H_{,c_i} c_{i,a} \ , \tag{A2}$$

$$Y = H' + \frac{H}{R} - \frac{1}{R}\{RaH'_{,c_i} + H_{,c_i}^{\ ;c_i} c_{i,\mu}\dot{z}^\mu + aH_{,c_i} + \epsilon[H, H_{,c_i}] + [G, H_{,c_i}] + G'_{,c_i} + [H, G_{,c_i}]\}c_{i,a}$$
$$- \frac{1}{R} H_{,c_i}(\dot{c}_{i,a} + c_{i,aa}(a_1 - a^2) - 2ac_{i,a}) \ , \tag{A3}$$

$$K = (Ra - \epsilon)H'' + (3a - 2\frac{\epsilon}{R})H' + \frac{a}{R}H - G'' - \frac{2}{R}G' + H'_{,c_i} c_{i,\mu}\dot{z}^\mu + H_{,c_i}^{\ ;c_i} c_{i,\mu}c_i^{\ ,\mu} + H_{,c_i}c_{i,\mu}^{\ ,\mu}$$
$$+ [G, H'] + 2[G', H] + \frac{2}{R}[G, H] + (Ra - \epsilon)[H, H'] + [H, H_{,c_i} c_{i,\mu}\dot{z}^\mu] - [H, [H, G]]$$
$$+ \frac{1}{R} H_{,c_i}\{a_1 c_{i,a} + a(\dot{c}_{i,a} + c_{i,aa}(a_1 - a^2) - 2ac_{i,a})\}$$
$$+ \frac{a}{R}\{RaH'_{,c_i} + H_{,c_i}^{\ ;c_i} c_{i,\mu}\dot{z}^\mu + aH_{,c_i} + \epsilon[H, H_{,c_i}] + [G, H_{,c_i}] + G'_{,c_i} + [H, G_{,c_i}]\}\, c_{i,a} \ , \tag{A4}$$

$$L = RaG'' + 2G'_{,c_i} c_{i,\mu}\dot{z}^\mu + G_{,c_i}^{\ ;c_i} c_{i,\mu}c_i^{\ ,\mu} + G_{,c_i}c_{i,\mu}^{\ ,\mu} - \frac{2}{R} G_{,c_i} c_{i,\mu}\dot{z}^\mu - a_1(H + RH')$$
$$+ a(Ra - \epsilon)(\frac{H}{R} - RH'' - H') - (Ra - \epsilon)H'_{,c_i} c_{i,\mu}\dot{z}^\mu + (2Ra - \epsilon)[H', G] + [H_{,c_i} c_{i,\mu}\dot{z}^\mu, G]$$
$$+ (Ra + \epsilon)[H, G'] + 2[H, G_{,c_i} c_{i,\mu}\dot{z}^\mu] + a[H, G] + \epsilon\,[H, [H, G]] + [G, G'] + [G, [H, G]] - \epsilon\,(Ra - \epsilon)[H, H']$$
$$- \{RaH'_{,c_i} + H_{,c_i}^{\ ;c_i} c_{i,\mu}\dot{z}^\mu + aH_{,c_i} + \epsilon[H, H_{,c_i}] + [G, H_{,c_i}] + G'_{,c_i} + [H, G_{,c_i}]\}(\dot{c}_i + c_{i,a}(a_1 - a^2 + \epsilon\,\frac{a}{R}))$$
$$- H_{,c_i}\{c_{i,a}(a_2 + \frac{1}{R}(\ddot{z}^\alpha\ddot{z}_\alpha) - a_1 a + \epsilon\,\frac{a_1}{R}) + (a_1 - a^2 + \epsilon\,\frac{a}{R})(\dot{c}_{i,a} + c_{i,aa}(a_1 - a^2) - 2ac_{i,a})$$
$$+ \ddot{c}_i + \dot{c}_{i,a}(a_1 - a^2)\} \ . \tag{A5}$$

Appendix B: The Explicit Form of $\partial^\mu F_{\mu\nu}$

In this Appendix we give the explicit form of $\partial^\mu F_{\mu\nu}$ that is needed in the definition of a total color charge. Following steps similar to those of Appendix A, one finds that

$$\partial^\mu F_{\mu\nu} = B\, z_\nu^{(3)} + C\,\ddot{z}_\nu + P\,\dot{z}_\nu + S\,\lambda_\nu$$

where

$$B = -\frac{1}{R} H_{,c_i} c_{i,a} \ , \tag{B1}$$

$$C = H' + \frac{H}{R} - \frac{1}{R}\{RaH'_{,c_i} + H_{,c_i}^{\ ;c_i} c_{i,\mu}\dot{z}^\mu + aH_{,c_i} + G'_{,c_i}\}c_{i,a} - \frac{1}{R} H_{,c_i}(\dot{c}_{i,a} + c_{i,aa}(a_1 - a^2) - 2ac_{i,a}) \ , \tag{B2}$$

$$P = (Ra - \epsilon)H'' + (3a - 2\frac{\epsilon}{R})H' + \frac{a}{R}H - G'' - \frac{2}{R}G' + H'_{,c_i} c_{i,\mu}\dot{z}^\mu$$
$$+ H_{,c_i}^{\ ;c_i} c_{i,\mu}c_i^{\ ,\mu} + H_{,c_i}c_{i,\mu}^{\ ,\mu} + [G, H'] + [G', H] + \frac{2}{R}[G, H]$$
$$+ \frac{a}{R}\{RaH'_{,c_i} + H_{,c_i}^{\ ;c_i} c_{i,\mu}\dot{z}^\mu + aH_{,c_i} + G'_{,c_i}\}\, c_{i,a} + \frac{1}{R} H_{,c_i}\{a_1 c_{i,a} + a(\dot{c}_{i,a} + c_{i,aa}(a_1 - a^2) - 2ac_{i,a})\} \ , \tag{B3}$$

$$S = RaG'' + 2G'_{,c_i} c_{i,\mu}\dot{z}^\mu + G_{,c_i}^{\ ;c_i} c_{i,\mu}c_i^{\ ,\mu} + G_{,c_i}c_{i,\mu}^{\ ,\mu} - \frac{2}{R} G_{,c_i} c_{i,\mu}\dot{z}^\mu - a_1(H + RH')$$
$$+ a(Ra - \epsilon)(\frac{H}{R} - RH'' - H') - (Ra - \epsilon)H'_{,c_i} c_{i,\mu}\dot{z}^\mu + Ra[H', G] + Ra[H, G'] + [H_{,c_i} c_{i,\mu}\dot{z}^\mu, G]$$
$$+ [H, G_{,c_i} c_{i,\mu}\dot{z}^\mu] - \{RaH'_{,c_i} + H_{,c_i}^{\ ;c_i} c_{i,\mu}\dot{z}^\mu + aH_{,c_i} + G'_{,c_i}\}(\dot{c}_i + c_{i,a}(a_1 - a^2 + \epsilon\,\frac{a}{R}))$$
$$- H_{,c_i}\{c_{i,a}(a_2 + \frac{1}{R}(\ddot{z}^\alpha\ddot{z}_\alpha) - a_1 a + \epsilon\,\frac{a_1}{R}) + (a_1 - a^2 + \epsilon\,\frac{a}{R})(\dot{c}_{i,a} + c_{i,aa}(a_1 - a^2) - 2ac_{i,a})$$
$$+ \ddot{c}_i + \dot{c}_{i,a}(a_1 - a^2)\} \ . \tag{B4}$$

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[11] A detailed and careful analysis of the material presented here can be found in [5]. Here we only give a brief summary of things that will be used and needed in this article.
[12] Please refer to [5] again for a detailed discussion of a_k.
[13] Here we suppress the internal group indices on A_μ.
[14] Please refer to [5] for details.
[15] In this work we set the gauge-field coupling constant equal to one.
[16] The internal group indices on Q and I are suppressed throughout.