A Brief Summary of the Group-Variation Equations

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Abstract

A brief summary is given of the Group-Variation Equations and the island diagram confinement mechanism, with an explanation of the prediction that the cylinder-topology minimal-area spanning surface term in the correlation function of two Wilson loops at large N_c, when it exists, must have a pre-exponential factor, which for large area A of the minimal-area cylinder-topology spanning surface, decreases with increasing A at least as fast as $1/\ln(\sigma A)$, where σ is the area law parameter. This prediction is expected to be testable in lattice calculations.

1 Description of the Group-Variation Equations

A Group-Variation Equation [1] expresses the derivative with respect to the coupling constant g, of a coefficient in the $1/N$ expansion [2] of a VEV or correlation function of Wilson loops, in terms of a sum of modified Feynman diagrams, related to the ordinary Feynman diagrams contributing to that VEV or correlation function.

The modifications to the Feynman diagrams are:

(1) Each window in an ordinary planar diagram is weighted by the VEV of the Wilson loop that forms its perimeter. The gluon and FP propagators are first resolved into sums over paths, so that the window weights become weights in the sums over paths.

(2) A planar diagram can have "islands", which look like planar vacuum bubbles, drawn within its windows. The non-simply connected window that surrounds one or more islands, is weighted by the correlation function of the two or more Wilson loops that form its perimeter. Again, the gluon and FP propagators are first resolved into sums over paths, so that the window weights become weights in the sums over paths.

(3) Each planar diagram is multiplied by a numerical coefficient, which is the derivative, at $M = 1$, of the "chromatic polynomial", $\mathbf{C}(M)$, of the diagram. The

chromatic polynomial, $\mathbf{C}(M)$, is by definition the number of distinct ways of colouring the windows of the diagram, with M different colours available, subject to the "map-colouring" rule, that no two windows that share a common border, can be coloured the same colour.

For vast classes of diagrams, $\mathbf{C}(M)$ has two or more factors of $(M-1)$, so the numerical coefficient vanishes, and the diagram makes no contribution. In particular, no contributing diagram can have more than one island, and if a diagram has an island, then it can have no propagators that do not form part of that island.

In the RHS of the Group-Variation Equation for the leading term in the $1/N$ expansion of the VEV of a single Wilson loop, only two classes of diagrams survive:

(a) Non-island diagrams. These have no island, and if you rub out the Wilson loop, what remains must have only one connected component.

(b) Island diagrams. These have exactly one island, and no propagators that do not form part of the island. Thus the window weight, for the one non-simply connected window, consists of the correlation function of the LHS Wilson loop, and the Wilson loop that forms the outer boundary of the island, after the gluon and FP propagators in the island have been resolved into sums over paths.

The Group-Variation Equations are closed and complete, in the sense that the ordinary Feynman diagram expansions, for the VEVs and correlation functions, can be recovered by developing their solutions in powers of g, with appropriate "boundary conditions" at $g = 0$, due to the derivative with respect to g in the LHS.

The equations for the coefficients at a common non-vanishing order, in the $1/N$ expansions of the VEVs and correlation functions, close among themselves. In particular, the equations for the leading non-vanishing terms in the $1/N$ expansions for the VEV of one Wilson loop, and the correlation functions of two or more Wilson loops, close among themselves.

By means of the Renormalization Group [3], the equations may be transformed into equations for the derivative of the VEV or correlation functions with respect to $\ln L$, where L is a common scaling parameter, as the sizes and separations, of all the Wilson loops involved in the VEV or correlation function, are uniformly re-scaled to a different size. The equations may then be integrated with respect to L, starting from "boundary conditions" at small L, as given by renormalization-group improved perturbation theory, and continuing to arbitrarily large L, because the correct large-distance behaviour, specifically the Wilson area law [4] for the VEV of one Wilson loop, and massive glueball saturation of the correlation functions, solves the equations self-consistently at large L.

The derivation of the Group-Variation Equations is outlined in Section 8.

2 The Island Diagram Confinement Mechanism

The demonstration that the correct large-L behaviour solves the Group-Variation Equation for the VEV of a single Wilson loop, at large L, consists of the following observations:

(i) the principal effect of the window weights is to generate an effective mass of at least $1.3\sqrt{\sigma}$ for each gluon and FP propagator, where σ is the area-law parameter.

(ii) the effective mass suppresses configurations with long propagators. Thus each non-island diagram gives a contribution, at large L, proportional to the perimeter of the LHS Wilson loop.

(iii) the contribution of each island diagram, at large L, is dominated by the contributions of islands of a fixed size, of order $\frac{1}{\sqrt{\sigma}}$.

(iv) the correlation function of a Wilson loop of the fixed size $\frac{1}{\sqrt{\sigma}}$, and a Wilson loop of much larger size, of order L, is strongly peaked at configurations where the smaller Wilson loop lies close to the minimal-area spanning surface of the larger Wilson loop.

(v) near the configurations mentioned in (iv), the correlation function of the two Wilson loops approximately factorizes into a factor equal to the VEV of the large Wilson loop, and a factor that depends on the shape and orientation of the small loop, and its perpendicular distance from the minimal-area spanning surface of the large loop, but not on the position of the perpendicular projection of the small loop, onto the minimal-area spanning surface of the large loop.

(vi) the factor in (v) that depends on the shape and orientation of the small loop, and the perpendicular distance of the small loop from the minimal-area spanning surface of the large loop, is expected to be approximately independent of the shape and size of the large loop. This is exactly true for the term in the correlation function that has the form of the lightest glueball, propagating by the shortest path, between the separate minimal-area spanning surfaces of the two loops, but is slightly violated by the cylinder-topology minimal-area spanning surface term in the correlation function, when the cylinder-topology term exists. (This is why the cylinder-topology term has to have a pre-exponential factor.)

(vii) thus each island diagram gives a contribution, at large L, approximately equal to a constant, times the area of the minimal-area spanning surface of the left-hand side Wilson loop, times the VEV of the left-hand side Wilson loop.

(viii) in consequence of (ii) and (vii), the large-L behaviour, of the left-hand side Wilson loop, is completely determined by the island diagrams.

(ix) since the derivative, with respect to $\ln L$, of $e^{-\sigma a L^2}$, where a is the area of the minimal-area spanning surface of the left-hand side Wilson loop, when $L = 1$, is $-2\sigma a L^2 e^{-\sigma a L^2}$, the left-hand side and the right-hand side have the same dependence on a and L.

(x) comparing the constant factors in the LHS and the RHS, σ is found to be equal to $\left|\frac{\beta(g)}{g}\right|$, times a power series in g^2, that begins with a term independent of g.

(xi) each coefficient in the power series, is equal to σ, times a numerical coefficient. Thus σ cancels out, and an equation for the critical value of g^2, $g^2(\sqrt{\sigma})$, at which g^2 stops evolving, is obtained. This in turn fixes $\sqrt{\sigma}/\Lambda_s$.

(xii) g^2 stops evolving, at $g^2(\sqrt{\sigma})$, because the large-distance behaviour, of all physical quantities, is completely determined by islands of the fixed size, $\frac{1}{\sqrt{\sigma}}$.

It is not obvious that the sign of the island diagram contributions comes out correct. The net sign, of the contributions of the leading island diagrams, has to be opposite to what would be obtained, if a scalar propagator, rather than gluon and FP propagators, was involved. I have verified that the required sign reversal occurs in the simplest non-trivial calculation, namely the change in the contribution of an island diagram, when the renormalization point is changed. The calculation exactly parallels, step by step, the calculation of the leading β-function coefficient, in a gauge-covariant background field method, so in this instance, the sign change is a direct consequence of the sign of the β-function.

If the sign comes out correct for the net contributions of the island diagrams, at each order in the explicit powers of g^2 in the right-hand sides of the Group-Variation Equations, then the island diagram mechanism works in exactly the same way at each finite order, including at leading order, as it does for the full sum of all the terms in the right-hand sides of the Group-Variation Equations. Evidence that these sums converge is given in Section 6.

3 The Pre-Exponential Factor in the Cylinder-Topology Term

If the correlation function of two Wilson loops contains a term $e^{-\sigma A}$, where A is the area of the cylinder-topology minimal-area spanning surface of the two loops, if it exists, then step (vi) goes slightly wrong. The exponent that determines the rate of decrease of the correlation function, as the small loop moves away from the minimal-area spanning surface of the large loop, has a factor $\ln(\sigma A_L)$ in the denominator, where A_L is the area of the minimal-area spanning surface of the large loop, which results in the contribution of an island diagram getting an extra, unwanted, factor of $\ln(\sigma A_L)$. This extra factor of $\ln(\sigma A_L)$ would result in the VEV of a single Wilson loop decreasing slightly too fast as the area, A_L, of its minimal-area spanning surface increased, and in fact, violating the Seiler bound [5], [6], so it must be cancelled. The simplest solution is to assume that the cylinder-topology term, in the correlation function of two Wilson loops, gets a pre-exponential factor, that decreases at least as fast as $1/\ln(\sigma A)$, as A increases.

The term in the correlation function, that has the form of the lightest glueball, propagating by the shortest possible path, between the separate minimal-area spanning surfaces of the two loops, always gives a contribution of the correct form.

4 The Group-Variation Equations for the Correlation Functions

The demonstration that the correct large-L behaviour solves the Group-Variation Equations for the correlation functions of two or more Wilson loops, at large L, is similar to the case of the VEV of a single Wilson loop, in that the dominant contributions, to the right-hand side, are given by the island diagrams. Instead of minimal-area spanning surfaces, the calculation involves the minimal-length spanning tree of the configuration of well-separated loops. A self-consistency requirement is found, namely that the mass of the lightest glueball must be strictly less than twice the effective mass generated for the gluon lines by the window weights.

If the pre-exponential factor in the cylinder-topology term, in the correlation function of two Wilson loops, decreases strictly faster than $1/\ln(\sigma A)$, then the cylinder-topology term makes no contribution to the leading behaviour, at large L, of the right-hand side of the Group-Variation Equation for the VEV of a single Wilson loop. In this case σ, and the square of the mass $m_{0^{++}}$ of the lightest glueball, are given by series whose terms differ only by simple numerical coefficients, so that if the contributions of all but the leading order island diagrams are neglected, and the ratio is taken, then the zeroth-order estimate $m_{0^{++}}/\sqrt{\sigma} = 2.38$ is obtained, which is about 33% less than the best lattice value of 3.56 [7].

5 Implication of the Lattice Value of $m_{0^{++}}$, for the Critical Value of α_s

Combining the zeroth-order estimate of $m_{0^{++}}/\sqrt{\sigma}$, with the self-consistency constraint, that $m_{0^{++}}$ must be strictly less than twice the effective mass generated for the gluon lines by the window weights, does not provide any new information about the effective mass, which is estimated in reference [1] to be at least $1.3\sqrt{\sigma}$. If we use, instead, the best lattice value of $m_{0^{++}}$, then we find that the effective mass must be strictly greater than $1.78\sqrt{\sigma}$. If we estimate the reciprocal of the typical island size, or in other words, the mass at which g^2 stops evolving, more closely as twice the effective mass, (since stretching an island, in any direction, elongates at least two propagators), then we find that the mass, at which g^2 stops evolving, must be strictly greater than $3.56\sqrt{\sigma}$. Since the experimental value of $\sqrt{\sigma}$ is about 0.44 GeV [8], this means that the mass, at which g^2 stops evolving, must be strictly greater than 1.57 GeV, which is not much smaller than $m_\tau = 1748$ MeV, which is the smallest mass for which α_s is known experimentally [9]. Another way of looking at this, is that the reciprocal of the typical island size, which is the mass at which g^2 stops evolving, must be strictly greater than the mass of the lightest glueball.

6 Convergence of the Sums in the Right-Hand Sides of the Group-Variation Equations

't Hooft has demonstrated, in reference [10], that the sums of the planar Feynman diagrams, in large-N_c QCD, converge geometrically, for a sufficiently small value of g^2, if one throws away all the divergent subdiagrams, and furthermore, in reference [11], that a similar result holds in the presence of the divergent subdiagrams, if one uses a suitably generalized running coupling, and gives the gluons a mass, to cut off the large-distance growth of the running coupling. It is therefore reasonable to suppose that the sums in the right-hand sides of the Group-Variation Equations will converge geometrically, for a sufficiently small value of g^2. It would also seem reasonable to suppose that in a natural renormalization scheme, such as $\overline{MS}$ [12], the convergence behaviour of the large-N_c limit of $\frac{\beta(g)}{g}$, as a power series in g^2, will be neither better, nor worse, than the convergence behaviour of the sums in the right-hand sides of the Group-Variation Equations. The large-N_c limit of $\frac{\beta(g)}{g}$, as a power series in g^2, may therefore also be expected to converge geometrically, for a sufficiently small value of g^2. The known expansion coefficients, in the large-N_c limit of $\frac{\beta(g)}{g}$, all have the same sign, in $\overline{MS}$, and it is reasonable to expect this trend to continue. We may therefore expect that the direction of fastest growth, in the complex g^2 plane, of the large-N_c limit of $\left|\frac{\beta(g)}{g}\right|$, in $\overline{MS}$, will be along the positive real axis. The critical value of g^2 is essentially determined, by (x) and (xi) above, as the point at which $\left|\frac{\beta(g)}{g}\right|$ reaches a critical value. Therefore if these suppositions are correct, the critical value of g^2 is strictly smaller than the radius of convergence of the large-N_c limit of $\frac{\beta(g)}{g}$, in $\overline{MS}$, as a power series in g^2, so the large-N_c limit of $\frac{\beta(g)}{g}$, in $\overline{MS}$, converges geometrically, as a power series in g^2, at the critical value of g^2, and the sums in the right-hand sides of the Group-Variation Equations, in $\overline{MS}$, also converge geometrically, at the critical value of g^2.

7 Implication of the β-Function to Four Loops

Study of the large-N_c limit of the general result for $\frac{\beta(g)}{g}$, in $\overline{MS}$, to four loops, given in reference [13], shows that the ratios of successive pairs of coefficients in the expansion are increasing, but at a decreasing rate, and indicates that the series is likely to diverge for a value of g^2 that corresponds to α_s lying somewhere in the range 0.43 to 0.85, and most likely, near the lower end of this range. Since the critical value of α_s will be strictly less than the value of α_s, that corresponds to the value of g^2 at which the series diverges, and $\alpha_s(m_\tau)$ is equal to 0.35 [9], this is further evidence that the critical value of α_s is not much larger than $\alpha_s(m_\tau)$.

8 Derivation Of The Group-Variation Equations

The Group-Variation Equations are derived by expressing the VEVs and correlation functions of SU(NM) Yang-Mills theory, first in terms of the VEVs and correlation functions of $(\mathrm{SU}(N))^M$ Yang-Mills theory, using general expansions which express the VEVs and correlation functions of a group, in terms of those of a subgroup, then in terms of the VEVs and correlation functions of SU(N) Yang-Mills theory, using the factorization properties of the VEVs and correlation functions of $(\mathrm{SU}(N))^M$ Yang-Mills theory. Substituting in the $1/N$ expansions of the VEVs and correlation functions, and equating coefficients of powers of $1/N$, equations for the coefficients in the $1/N$ expansions of the VEVs and correlation functions are obtained, in which the M-dependence of the left-hand sides is through an overall power of M, and the replacement of the coupling constant g^2 by g^2M, and the M-dependence of the right-hand sides is through the chromatic polynomial factors $\mathbf{C}(M)$ of the diagrams. Taking the derivative with respect to M, at $M = 1$, and in the left-hand sides, expressing the derivative with respect to M, in terms of the derivative with respect to g^2, gives the Group-Variation Equations.

I would like to thank D. Leinweber, J. Negele, and H. Shanahan for emails about the possibility that the pre-exponential factor, in the cylinder-topology term, can be tested in lattice calculations, B. Alles for an email noting the practical difficulties of such a test, and M. Karliner and Z. Sroczynski for suggesting I write a brief summary.

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