Low-Energy Dynamics of Noncommutative CP^1 Solitons in 2+1 Dimensions

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Abstract

We investigate the low-energy dynamics of the BPS solitons of the noncommutative CP^1 model in 2+1 dimensions using the moduli space metric of the BPS solitons. We show that the dynamics of a single soliton coincides with that in the commutative model. We find that the singularity in the two-soliton moduli space, which exists in the commutative CP^1 model, disappears in the noncommutative model. We also show that the two-soliton metric has the smooth commutative limit.

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1 Introduction

Noncommutative geometry appears in M-theory, string theory and condensed matter physics [1]. Noncommutative field theories are known to describe the low-energy effective theory of D-branes in a background B-field [2, 3, 4]. (2+1)-dimensional noncommutative theories have applications to the quantum Hall effect [5]. In spite of the nonlocality and entangled UV/IR mixing in perturbation theory, the appearance of these theories in string theory suggests that some class of noncommutative field theories is a sensible deformation of ordinary field theories.

In view of understanding nonperturbative effects in noncommutative field theories, noncommutative solitons and instantons have been investigated. In four-dimensional noncommutative Yang-Mills theory, there exists a U(1) instanton which is nonsingular due to the position-space uncertainty [6]. The moduli space of instantons are also smooth [7, 8]. In noncommutative scalar field theories, there exist solitons (GMS solitons) [9] at the limit of large noncommutativity parameter θ; they cannot exist in the commutative counterpart.

Low-energy dynamics of solitons can be approximated by geodesic motion on the moduli space of static solutions [10]. In commutative theories, scattering properties of solitons in gauge theories and nonlinear sigma models were studied extensively using this approximation [11, 12, 13]. A common feature is that the scattering of two solitons occurs at right angle for the head-on collision. In noncommutative theories, the scatterings of solitons in the GMS model, Yang-Mills theories and integrable models were investigated [14, 15, 16].

In this letter we investigate the low-energy dynamics of BPS solitons of the noncommutative CP^1 model in 2+1 dimensions. It was shown that the noncommutative CP^N model has the BPS solutions as the commutative model does [17]. In the commutative CP^N model, the moduli space of solitons is known to be a Kähler manifold [18]. We show that the moduli space of the noncommutative CP^N model is a Kähler manifold too. We calculate the θ-dependence of the moduli space metric in the CP^1 case.

This letter is organized as follows. In section 2, we review the BPS solutions of the noncommutative CP^N model. In section 3, we give the compact form of the Kähler potential of the moduli space of these solutions. In section 4, we investigate the one-soliton metric of the noncommutative CP^1 model and show that the motion of a single soliton is the same as that of the commutative model. In section 5, we study the two-soliton metric of the model and find that the singularity which exists in the commutative model disappears in the noncommutative model. Furthermore, we show that

the two-soliton metric has the smooth commutative limit.

2 BPS solution of the noncommutative CP^N model

We recapitulate the (2+1)-dimensional noncommutative CP^N model, closely following Lee, Lee and Yang [17].

We consider the (2+1)-dimensional field theory on the noncommutative space. The noncommutativity is introduced by

$$[x, y] = i\theta, \quad \theta > 0. \tag{1}$$

We set

$$z = \frac{1}{\sqrt{2}}(x + iy), \quad \bar{z} = \frac{1}{\sqrt{2}}(x - iy). \tag{2}$$

Then the equation (1) becomes

$$[z, \bar{z}] = \theta, \tag{3}$$

or

$$[a, a^\dagger] = 1, \quad a = \frac{z}{\sqrt{\theta}}, \quad a^\dagger = \frac{\bar{z}}{\sqrt{\theta}}. \tag{4}$$

This is the algebra of the creation and annihilation operators. We use the Fock space of the quantum harmonic oscillator for the representation of the algebra (1).

The derivative and the integral on the noncommutative space are given by

$$\partial_x \Phi = i\theta^{-1}[y, \Phi], \quad \partial_y \Phi = -i\theta^{-1}[x, \Phi], \tag{5}$$

$$\int d^2 x \mathcal{O} \to \mathrm{Tr}\mathcal{O} = 2\pi\theta \sum_{n \geq 0} \langle n|\mathcal{O}|n\rangle, \tag{6}$$

where $|n\rangle \ (n = 0, 1, 2, \cdots)$ are the basis of the Fock space.

The Lagrangian of the noncommutative CP^N model [17] is defined by

$$L = \mathrm{Tr}[D_\mu \Phi^\dagger D^\mu \Phi + \lambda(\Phi^\dagger \Phi - 1)], \tag{7}$$

$$D_\mu = \partial_\mu \Phi - i\Phi A_\mu, \quad A_\mu = -i\Phi^\dagger \partial_\mu \Phi, \tag{8}$$

where the field $\Phi = {}^t(\phi_1, \phi_2, \ldots, \phi_{N+1})$ is the complex $(N+1)$-component vector, and λ is the Lagrange multiplier field which gives the constraint

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$\Phi^\dagger \Phi = 1$. This theory has the global $SU(N+1)$ symmetry and $U(1)$ gauge symmetry $\Phi(x) \to \Phi(x)g(x)$, where $g(x) \in U(1)$.

The energy functional is

$$E = \mathrm{Tr}(|D_0\Phi|^2 + |D_z\Phi|^2 + |D_{\bar{z}}\Phi|^2). \tag{9}$$

We have the Bogomolnyi bound

$$E \geq \mathrm{Tr}(|D_0\Phi|^2) + 2\pi|Q|, \tag{10}$$

where Q is the topological charge,

$$Q = \frac{1}{2\pi}\mathrm{Tr}(|D_z\Phi|^2 - |D_{\bar{z}}\Phi|^2). \tag{11}$$

The BPS equation of the static solution is

$$D_{\bar{z}}\Phi = 0 \quad \text{for self-dual solution}, \tag{12}$$
$$D_z\Phi = 0 \quad \text{for anti-self-dual solution}. \tag{13}$$

We parametrize Φ as

$$\Phi = W(W^\dagger W)^{-1/2}, \tag{14}$$

where W is the complex $(N+1)$-component vector. We define the projection operator P by

$$P = 1 - W(W^\dagger W)^{-1}W^\dagger. \tag{15}$$

Then (7) and (11) are

$$L = \mathrm{Tr}\left[\frac{1}{\sqrt{W^\dagger W}}\partial_\mu W^\dagger P \partial^\mu W \frac{1}{\sqrt{W^\dagger W}}\right], \tag{16}$$

$$Q = \frac{1}{2\pi}\mathrm{Tr}\left[\frac{1}{\sqrt{W^\dagger W}}(\partial_{\bar{z}}W^\dagger P \partial_z W - \partial_z W^\dagger P \partial_{\bar{z}}W)\frac{1}{\sqrt{W^\dagger W}}\right]. \tag{17}$$

The BPS equation (12) becomes

$$D_{\bar{z}}\Phi = P(\partial_{\bar{z}}W)(W^\dagger W)^{-1/2} = 0. \tag{18}$$

This equation is equivalent to $\partial_{\bar{z}}W = WV$, where V is an arbitrary scalar. In the commutative case, using the N-component vector w, we set $W = {}^t(w,1)$ by the gauge transformation

$$W \to W' = W\Delta(z,\bar{z}), \tag{19}$$

where $\Delta(z, \bar{z})$ is an arbitrary scalar. Then (18) becomes $\partial_{\bar{z}} w = 0$, namely, w is holomorphic. In the noncommutative case, The Lagrangian (16) is invariant under the transformation (19) when Δ is invertible.

We will be mainly concerned with the one- and two-soliton solutions of the noncommutative CP^1 model. The one- and two-soliton solutions of the commutative CP^1 model are respectively given by [12, 13]

$$w = \lambda + \frac{\mu}{z - \nu}, \tag{20}$$

$$w = \alpha + \frac{2\beta z + \gamma}{z^2 + \delta z + \epsilon}, \tag{21}$$

where $\alpha, \beta, \cdots \in C$ are the moduli parameters. We may set the moduli parameters λ and α to zero, using the global $SU(2)$ symmetry.

In the noncommutative case, $W = (z, 1)$ and $W' = W z^{-1} = (1, z^{-1})$ are gauge inequivalent, where z^{-1} is defined to be $z^{-1} \equiv (\bar{z}z)^{-1}\bar{z} \equiv \bar{z}(\bar{z}z + \theta)^{-1}$. z^{-1} satisfies $zz^{-1} = 1$ and $z^{-1}z = 1 - |0\rangle\langle 0|$. W satisfies the BPS equation (18) but W' does not. In general, W satisfies the BPS equation (18) if the components of W are polynomials of z. The BPS one- and two-soliton solutions in the noncommutative CP^1 model corresponding to (20) and (21) are respectively given by

$$W = \begin{pmatrix} z - \nu \\ \mu \end{pmatrix}, \tag{22}$$

$$W = \begin{pmatrix} z^2 + \delta z + \epsilon \\ 2\beta z + \gamma \end{pmatrix}. \tag{23}$$

3   Moduli space metric

In the following section, we consider the scattering of BPS solutions of the noncommutative CP^N model. In the low-energy limit (near the Bogomolnyi bound), it is a good approximation that only the moduli parameters depend on the time [10]. Their time evolution is determined by minimizing the action $S = \int dt \, L$. It amounts to dealing with the kinetic energy T (the term in the Lagrangian including the time derivative only), since the rest of the Lagrangian gives the topological charge. The kinetic energy is given by

$$\begin{aligned} T &= \text{Tr} \left(\frac{1}{\sqrt{W^\dagger W}} \partial_t W^\dagger P \partial_t W \frac{1}{\sqrt{W^\dagger W}} \right) \\ &= \frac{1}{2} \text{Tr}(\partial_t P')^2, \end{aligned} \tag{24}$$

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where P' is defined by

$$P' = 1 - P = W(W^\dagger W)^{-1} W^\dagger. \tag{25}$$

T is written as $T = \frac{1}{2}(ds/dt)^2$, where ds^2 is the line element of the moduli space. The dynamics of solitons is given by the geodesic line in the moduli space. We denote generically the moduli parameters by ζ^a. We then have

$$T = \frac{1}{2} g_{ab} \frac{d\zeta^a}{dt} \frac{d\zeta^b}{dt}, \tag{26}$$

where g_{ab} is the moduli space metric.

It is convenient to express P' in the Fock space language,

$$P' = \sum_{n,m} |\psi_n\rangle h^{nm} \langle \psi_m|, \quad |\psi_n\rangle = W|n\rangle, \tag{27}$$

$$h_{nm} = \langle \psi_n | \psi_m \rangle, \quad h^{nm} = (h_{nm})^{-1}. \tag{28}$$

The BPS solution W (or $|\psi_n\rangle$) is a holomorphic function of the moduli parameters. It was shown that the moduli space in the case where $|\psi_n\rangle$ is holomorphic is a Kähler manifold [9]. Hence, we write

$$T = \frac{1}{2} g_{\bar{a}b} \frac{d\zeta^{\bar{a}}}{dt} \frac{d\zeta^b}{dt}, \quad g_{\bar{a}b} = \frac{\partial}{\partial \zeta^{\bar{a}}} \frac{\partial}{\partial \zeta^b} K, \tag{29}$$

where the Kähler potential K is given by

$$K = \operatorname{Tr} \ln(h_{nm}) = \operatorname{Tr} \ln(W^\dagger W). \tag{30}$$

4 One-soliton metric

The BPS one-soliton solution of the noncommutative CP^1 model is given by (22). We substitute (22) into (24), then we have

$$
\begin{aligned}
T = \operatorname{Tr} \Bigg[& \frac{1}{\sqrt{(\bar{z}-\bar{\nu})(z-\nu)+|\mu|^2}} \partial_t \left(\bar{z}-\bar{\nu} \quad \bar{\mu} \right) \\
& \times \left\{ 1 - \binom{z-\nu}{\mu} \frac{1}{(\bar{z}-\bar{\nu})(z-\nu)+|\mu|^2} \left(\bar{z}-\bar{\nu} \quad \bar{\mu} \right) \right\} \\
& \times \partial_t \binom{z-\nu}{\mu} \frac{1}{\sqrt{(\bar{z}-\bar{\nu})(z-\nu)+|\mu|^2}} \Bigg].
\end{aligned}
\tag{31}
$$

The $\dot{\bar{\mu}}\dot{\mu}$ term in T is

$$2\pi\theta\dot{\bar{\mu}}\dot{\mu}\sum_{n\geq 0}\frac{1}{\theta n+|\mu|^2}\left[\frac{\theta n}{\theta n+|\mu|^2}+\frac{|\nu|^2}{\theta(n+1)+|\mu|^2}\right]. \tag{32}$$

The first term in (32) diverges. We set μ to a constant in the low-energy approximation. Calculating the trace in (31) with $\dot{\mu}=0$, we obtain

$$T=2\pi\frac{d\bar{\nu}}{dt}\frac{d\nu}{dt}, \quad \text{or} \quad ds^2=4\pi d\bar{\nu}d\nu. \tag{33}$$

This is θ-independent and coincides with the commutative case. A single soliton moves straight without changing the size.

The result (33) can be explained in a more general framework. In the low-energy limit, the action $S=\int dt\, L$ is invariant under the Galilean transformation

$$\begin{aligned} t &\to t \\ x &\to x+v_x t \\ y &\to y+v_y t \;, \end{aligned} \tag{34}$$

since this transformation does not change the commutation relation (1). Under the Galilean transformation, the kinetic energy in the center-of-mass frame T_{cm} is transformed as

$$T_{\text{cm}}\to\frac{1}{2}Mv^2+T_{\text{cm}}\;, \tag{35}$$

where M is the total mass of the solitons; $M=2\pi|Q|$. From (35), it follows that the contribution of the center-of-mass coordinates to the kinetic energy is the same as in the commutative case. From now on, we restrict the moduli parameters to the center-of-mass frame.

5   Two-soliton metric

The BPS two-soliton solution of the noncommutative CP^1 model in the center-of-mass frame (*i.e.* $\delta=0$.) is

$$W=\begin{pmatrix} z^2+\epsilon \\ 2\beta z+\gamma \end{pmatrix}. \tag{36}$$

Computing the kinetic energy in the low-energy limit, we can see that the contribution of the $\dot{\bar{\beta}}\dot{\beta}$ term to the kinetic energy diverges. In the low-energy approximation, we set β to a constant. We consider the case of

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$\beta = 0$ for simplicity. In the commutative model, the moduli space metric is calculated by Ward [12]. There exists a singularity at $(\epsilon, \gamma) = (0, 0)$. In the next subsection, we will see the disappearance of this singularity in the noncommutative model.

The Kähler potential corresponding to (36) is

$$K = \text{Tr} \ln \left[(\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon) + \bar{\gamma}\gamma \right] \, , \tag{37}$$

This is a formal expression since the trace in (37) diverges. A finite Kähler potential is obtained by subtracting the divergent terms using the Kähler transformation

$$K(\gamma, \epsilon; \bar{\gamma}, \bar{\epsilon}) \rightarrow K(\gamma, \epsilon; \bar{\gamma}, \bar{\epsilon}) - f(\gamma, \epsilon) - \bar{f}(\bar{\gamma}, \bar{\epsilon}) \tag{38}$$

In (37), only the terms with the same number of z^2 and $\bar{z}^2$ contribute to the trace. Hence, K is the function of $\bar{\epsilon}\epsilon$ and $\bar{\gamma}\gamma$ only; $K = K(\bar{\epsilon}\epsilon, \bar{\gamma}\gamma)$. It then follows that the moduli space is manifestly invariant under the following three kinds of transformations: i) $(\epsilon, \gamma) \leftrightarrow (\bar{\epsilon}, \bar{\gamma})$, ii) $\epsilon \rightarrow e^{i\phi}\epsilon$, and iii) $\gamma \rightarrow e^{i\chi}\gamma$.

The moduli space metric is

$$g_{\bar{\gamma}\gamma} = \text{Tr} \left[\frac{1}{\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)} \left(1 - \frac{\bar{\gamma}\gamma}{\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)} \right) \right], \tag{39a}$$

$$g_{\bar{\epsilon}\gamma} = -\text{Tr} \left[\bar{\gamma}(z^2 + \epsilon) \frac{1}{[\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)]^2} \right], \tag{39b}$$

$$g_{\bar{\gamma}\epsilon} = -\text{Tr} \left[\gamma \frac{1}{[\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)]^2} (\bar{z}^2 + \bar{\epsilon}) \right], \tag{39c}$$

$$g_{\bar{\epsilon}\epsilon} = \text{Tr} \left[\frac{1}{\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)} \frac{\bar{\gamma}\gamma}{\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon) + 4\theta\bar{z}z + 2\theta^2} \right]. \tag{39d}$$

It is difficult to compute the trace of (39) exactly, but we can investigate the moduli space metric of the two solitons in the case of $|\gamma|, |\epsilon| \ll \theta$ or $|\gamma|, |\epsilon| \gg \theta$.

5.1   The case of $|\gamma|, |\epsilon| \ll \theta$

In the case of $|\gamma|, |\epsilon| \ll \theta$, we write (39a) as

$$g_{\bar{\gamma}\gamma} = \frac{1}{\theta^2} \text{Tr} \left[\frac{1}{\bar{\gamma}\gamma/\theta^2 + (a^{\dagger 2} + \bar{\epsilon}/\theta)(a^2 + \epsilon/\theta)} \right.$$
$$\left. \times \left(1 - \frac{\bar{\gamma}\gamma/\theta^2}{\bar{\gamma}\gamma/\theta^2 + (a^{\dagger 2} + \bar{\epsilon}/\theta)(a^2 + \epsilon/\theta)} \right) \right], \tag{40}$$

The operator $a^2 + \epsilon/\theta$ has two zero eigenstates $e^{\pm i\sqrt{\frac{\epsilon}{\theta}}a^\dagger}|0\rangle$. These states do not contribute to the trace in (40). Then, we can easily compute the trace in the lowest order of θ^{-1}. For (39b), (39c) and (39d), we can calculate the trace similarly. Then we obtain

$$ds^2 = \frac{2\pi}{\theta}\left(d\bar{\gamma}d\gamma + \frac{2}{3}d\bar{\epsilon}d\epsilon\right) + O(\theta^{-2}). \tag{41}$$

Therefore, the metric is flat. The singularity which exists in the commutative model at $(\epsilon, \gamma) = (0, 0)$ disappears in the noncommutative model. The same phenomena are known in noncommutative Yang-Mills theories [7, 8]. Since the relative coordinate of the solitons is $i\epsilon^{1/2}$, it seems that the geodesic which connects $\epsilon = \epsilon_0$ and $\epsilon = -\epsilon_0$ represents the right angle scattering. However, (41) is valid only in the region $|\epsilon| \ll \theta$. To investigate the scattering, a further study of the moduli space is needed.

5.2　The case of $|\gamma|, |\epsilon| \gg \theta$

In the case of $|\gamma|, |\epsilon| \gg \theta$, it is convenient to use the $\star$-product formalism to compute (39) rather than the operator formalism. The $\star$-product is defined by

$$f(z, \bar{z}) \star g(z, \bar{z}) = f(z, \bar{z}) \exp\left[\frac{\theta}{2}(\overleftarrow{\partial_z}\overrightarrow{\partial_{\bar{z}}} - \overleftarrow{\partial_{\bar{z}}}\overrightarrow{\partial_z})\right] g(z, \bar{z}). \tag{42}$$

We use the formulae

$$\frac{1}{\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)} = \int_0^\infty du \exp\left[-u\{\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)\}\right]$$

$$\rightarrow \int_0^\infty du \exp_\star\left[-u\{\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon}) \star (z^2 + \epsilon)\}\right], \tag{43}$$

$$\frac{1}{[\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)]^2} = \int_0^\infty duu \exp\left[-u\{\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)\}\right]$$

$$\rightarrow \int_0^\infty duu \exp_\star\left[-u\{\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon}) \star (z^2 + \epsilon)\}\right], \tag{44}$$

where $\exp_\star$ is defined by

$$\exp_\star(A) = 1 + A + \frac{1}{2!}A \star A + \frac{1}{3!}A \star A \star A + \cdots. \tag{45}$$

For small θ, we have the following relation

$$\exp_\star(A) = \exp(A) + O(\theta^2). \tag{46}$$

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Using the formulae (43)-(46), the moduli space metric up to the order θ is

$$
g_{\bar{\gamma}\gamma} = \int d^{2x}\left[\frac{1}{\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)}\left(1 - \frac{\bar{\gamma}\gamma}{\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)}\right)\right.
$$
$$
\left. + \theta\frac{2\bar{z}z}{[\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)]^2}\left(1 - \frac{2\bar{\gamma}\gamma}{\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)}\right)\right] + O(\theta^2), \quad (47\text{a})
$$

$$
g_{\bar{\epsilon}\gamma} = -\int d^{2x}\left[\frac{\bar{\gamma}(z^2 + \epsilon)}{[\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)]^2}\right.
$$
$$
\left. + \theta\frac{4\bar{\gamma}\bar{z}z(z^2 + \epsilon)}{[\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)]^3}\right] + O(\theta^2), \quad (47\text{b})
$$

$$
g_{\bar{\gamma}\epsilon} = -\int d^{2x}\left[\frac{\gamma(\bar{z}^2 + \bar{\epsilon})}{[\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)]^2}\right.
$$
$$
\left. + \theta\frac{4\gamma\bar{z}z(\bar{z}^2 + \bar{\epsilon})}{[\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)]^3}\right] + O(\theta^2), \quad (47\text{c})
$$

$$
g_{\bar{\epsilon}\epsilon} = \int d^{2x}\frac{\bar{\gamma}\gamma}{[\bar{\gamma}\gamma + (\bar{z}^2 + \bar{\epsilon})(z^2 + \epsilon)]^2} + O(\theta^2). \quad (47\text{d})
$$

The integrals appearing in the coefficients of θ in (47) converge. Hence, the moduli space metric has the smooth commutative limit $\theta \to 0$ with γ and ϵ fixed.

Acknowledgements

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