Deconfinement mechanism in three dimensions for gauge fields coupled to bosonic matter fields with fundamental charge

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We propose a mechanism by which electric charges deconfine in an Abelian Higgs model with matter fields belonging to the fundamental representation of the gauge group. Kosterlitz-Thouless like recursion relations for a scale-dependent stiffness parameter and fugacity are given, showing that for a logarithmic potential between point charges in any dimension, there exists a stable fixed point at zero fugacity, with a dimensionality dependent universal jump in the stiffness parameter at the phase transition.

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It is well known that a pure Maxwell theory with gauge fields arising from a *compact* $U(1)$ gauge group permanently confines electric charges in three dimensions [1]. In such a case, specifically in the absence of matter fields coupled to the gauge fields, the Wilson loop is a good probe for confinement and satisfies the area law, with a linearly confining potential between electric test charges. The periodic character of the gauge fields induces topological defects in the theory. In the case of compact Maxwell theory in three dimensions these defects are magnetic monopoles. This is easily seen as the 2π jumps can be accounted for by writing the field strength as

$$F_{\mu\nu} = f_{\mu\nu} - 2\pi\epsilon_{\mu\nu\lambda}\partial_\lambda \int d^3y D(x-y)n(y), \qquad (1)$$

where $f_{\mu\nu}$ is the non-singular part of the field strength, $n(x) = \sum_i q_i \delta^3(x - x_i)$ is the magnetic charge density and

$$-\partial^2 D(x-y) = \delta^3(x-y). \qquad (2)$$

Thus, the dual field strength $F^*_\mu = \epsilon_{\mu\nu\lambda}F_{\nu\lambda}/2$ satisfies

$$\partial_\mu F^*_\mu(x) = 2\pi n(x). \qquad (3)$$

In the absence of monopoles the potential between electric test charges is given simply by the Coulomb interaction in two space dimensions:

$$V(R) \sim \ln R. \qquad (4)$$

The monopoles of the dual theory, on the other hand, interact through a Coulomb potential in *three dimensions*, i.e., $V_{\rm mon} \sim 1/r$. In order to show that in the compact Maxwell theory electric charges never deconfine, we have to study a classical Coulomb gas of monopoles in three dimensions. Polyakov [1] carried out this study by showing that for low fugacity the Coulomb gas maps onto a sine-Gordon theory, which he solved in a saddle point approximation in the presence of external sources corresponding to the test electric charges. The end result is a potential between electric test charges $V(R) \sim R$ in two space dimensions. Thus, the monopoles act in such a way as to produce an anti-screening effect in the potential between electric charges. The force between the electric charges, which decreases as $1/R$ in two space dimensions for the non-compact Maxwell theory, does not vary with the distance any longer in the case of compact Maxwell theory.

Note that in two space-dimensions, the absence of monopoles in the gauge-field produces a potential between test charges which is given by $V(R) \sim \ln R$. In the presence of fluctuating matter fields in three spacetime dimensions (two space dimensions), this is changed to $V(R) \sim 1/R$, due to the presence of an anomalous scaling dimension of the gauge field. The gauge-field propagator in d space-time dimensions in the presence of critical matter-field fluctuations is given by $D(|x|) \sim 1/|x|^{d-2+\eta_A}$, where η_A is the anomalous scaling dimension of the gauge-field, gauge-invariance dictates that $\eta_A = 4 - d$ [2], and d is the space-time dimensionality. Therefore, we see that $D(|x|) \sim 1/|x|^2$ when $d \in (2, 4]$. This is the same as we would get with or without matter-field fluctuations in four-dimensional space-time. Hence, we see that the role of the anomalous scaling of the gauge field is to essentially produce extra space dimensions. It is as if the test charges were living in three space dimensions, but confined to moving in two.

An alternative derivation of Polyakov's result amounts to showing that a Coulomb gas of monopoles does not undergo any phase transition between a dielectric and metallic phase in three dimensions. The system is always in a "high-temperature" metallic phase. This can be seen by deriving the corresponding recursion relations for a three-dimensional Coulomb gas. The recursion relations for a d-dimensional Coulomb gas were derived by Kosterlitz [3]. The result is

$$\frac{dK^{-1}}{dl} = 4\pi^2 y^2 - (2-d)K^{-1}, \tag{5}$$

$$\frac{dy}{dl} = \left[d - 2\pi^2 f(d)K\right] y, \tag{6}$$

where $f(d) = (d-2)\Gamma[(d-2)/2]/(4\pi)^{d/2}$. Here, $y(l)$ and $K(l)$ are essentially the scale-dependent fugacity and inverse dielectric constant of the d-dimensional Coulomb gas, respectively. These renormalization group equations are therefore basically nothing but self-consistency equations for scale-dependent electrostatics. For $d = 2$, the above equations reduce to the celebrated recursion relations for the Kosterlitz-Thouless (KT) phase transition [4]. In this case a line of fixed points occurs in the flow diagram. However, for $d > 2$ there are no zero-fugacity fixed points to the above recursion relations. For $d > 2$, the second term on the right hand side in Eq. (5) is positive, which means that even at arbitrarily low fugacity, $K^{-1}(l) > 0$ will increase indefinitely as the (logarithmic) length scale l increases. Hence, no matter how large we make the bare value of $K(l)$, it will eventually be reduced enough, by Eq. (5), to make the right hand side of Eq. (6) positive. Hence, it is inevitable that $y(l)$ will eventually start increasing with l, thus destroying the zero-fugacity fixed point well known to exist in the case $d = 2$ [4]. In particular, this result means that no phase transition occurs in $d = 3$ in the ordinary Coulomb gas. Thus, the Coulomb gas of monopoles is always in the plasma phase. This is nothing but a statement which is equivalent to permanent confinement of electric test charges in compact three-dimensional photodynamics [1].

When matter fields are present, the confinement properties of the theory are likely to be changed. There are, however, many subtleties involved when the matter fields are in the so called fundamental representation of compact $U(1)$. To see what are the main points, let us consider the lattice abelian Higgs model, whose action is given by

$$S = -\beta \sum_{x,\mu} \cos[\Delta_\mu \theta(x) - qA_\mu(x)] - \kappa \sum_{x,\mu,\nu} \cos[F_{\mu\nu}(x)], \tag{7}$$

where $q \in \mathbb{N}$ is the charge carried by the Higgs field . The case where the Higgs field carries the fundamental charge ($q = 1$) differs in an essential way from the case $q > 1$. This can be seen by considering the limiting cases $\beta \to \infty$ and $\kappa \to \infty$. Let us consider first the case $q = 1$. First of all, for all values of q the limit $\kappa \to \infty$ corresponds to the $3DXY$ model. This is so because when $\kappa \to \infty$ all gauge-field fluctuations are supressed except those that are indistinguishable from vortex-loop fluctuations in the matter sector. Hence, for all q, the model exhibits a phase transition when $\kappa \to \infty$. The limit $\beta \to \infty$, on the other hand, is trivial when $q = 1$, and there is no phase transition associated with it.

The situation for $q = 2$ is drastically different since in this case the limit $\beta \to \infty$ leads to a Z_2 gauge theory, which exhibits a phase transition in the Ising model universality class when the space-time dimensionality $d = 3$. Thus, when $q = 2$ it is natural to think that there is a critical line in the phase diagram of the $q = 2$ theory that interpolates between the two limiting critical regimes, and this can indeed be demonstrated [5–7]. The case corresponding to the Higgs field with the fundamental charge does not have two asymptotic critical regimes to be interpolated. For this reason, it is generally thought that there is no phase transition in the $q = 1$ three-dimensional lattice Abelian Higgs model and that therefore the theory permanently confines electric test charges [8], as is the case in the pure compact Maxwell theory [1].

Another obstacle against a deconfinement phase transition in the $q = 1$ case comes from Elitzur's theorem [9]. This theorem simply states that averages of non-gauge invariant operators are always zero in the absence of gauge fixing. Only gauge-invariant operators can have a nonzero expectation value. In other words, a local gauge symmetry cannot be spontaneously broken. This is important, since it implies that the Higgs mechanism can only occur upon some gauge fixing. In the lattice action a natural gauge fixing is the unitary gauge parametrization. In the unitary gauge there is a residual *global* gauge symmetry left, which can in turn be broken. In the $q = 2$ case the residual global symmetry corresponds to the Z_2 group. Thus, in this case the Higgs phase can be distinguished from the confinement phase [8]. In the $q = 1$ case, however, the residual symmetry is just the identity group and it is therefore trivial. *The Higgs phase cannot be distinguished from the confinement phase.* Again, there is only one phase and it seems that there is no way out from permanent confinement for fields carrying the fundamental charge.

In the case of continuous global symmetries the Mermin-Wagner theorem [10] forbids spontaneous symmetry breaking in two dimensions. Elitzur's theorem is far more restrictive than Mermin-Wagner's theorem, since it applies to any dimension and to discrete gauge groups. A well known way out from the Mermin-Wagner theorem is the KT transition [4], where a phase transition occurs in the absence of long range order. The KT transition occurs precisely in the case of a *global* $U(1)$ group.

Since the global symmetries do not suffer the severe restriction imposed by Elitzur's theorem, duality transformations where a locally gauge invariant theory is mapped into a globally invariant theory is a powerful tool. We can look for phase transitions there where the symmetry can be spontaneously broken and, in some cases, even to look for phase transitions without spontaneous symmetry breaking, like the KT phase transition at $d = 2$. Unfortunately, it seems to be generally the case that the phase transition in the dual model corresponds to a non-trivial residual symmetry in the original model. Thus, as far as the phase transition is concerned, in those cases it can easily be established in the original model as well. In such cases, the dual model is still a powerful tool to establish the universality class. It would be interesting if a KT-like phase transition could occur in the dual three-dimensional theory, elevating the three-dimensional $q = 1$ Abelian Higgs model to a system sustaining a confinement-deconfinement transition. Recently it was pointed out in Ref. [11] that such a transition indeed occurs in this case. *A deconfinement phase transition is driven by a KT-like phase transition in the monopole plasma.* Note that in general, the topological defects of theory are more complicated objects. The monopoles are generally connected by magnetic vortex lines, and there are also closed vortex loops. However, at the critical point the vortex line looses tension and we have once more an effective description in terms of a gas of magnetic monopoles. The difference to the usual Coulomb gas is that the interaction between the monopoles are no longer $\propto 1/r$ as in the pure compact Maxwell theory. The matter fields induce an anomalous scaling behavior and the monopole-monopole interaction becomes $\propto \ln r$ in three dimensions [11, 12]. This behavior strongly suggests that a KT-like transition may also occur for this *anomalous Coulomb gas* in three dimensions.

For low fugacity the anomalous Coulomb gas can be brought in the form of a sine-Gordon theory with a $1/|p|^3$ free propagator [11, 12]:

$$S = \frac{1}{8\pi^2 K}\int d^3x[\varphi(-\partial^2)^{3/2}\varphi - 2z\cos\varphi], \qquad (8)$$

where $K = 1/g^2$, with g being the gauge coupling. From the above sine-Gordon theory we could in principle raise the following objection to a KT-like behavior at $d = 3$. It could happen that a $\varphi(-\partial^2)\varphi$ term is generated by fluctuation effects. If the theory behaves in such a way, the generated term would clearly dominate at large distances and the $\varphi(-\partial^2)^{3/2}\varphi$ would become irrelevant in the renormalization group (RG) sense. The resulting effective action would just correspond to an ordinary sine-Gordon theory in three dimensions and, therefore, no phase transition occurs in this case. The $\ln r$ interaction is screened into a $1/r$ potential. This argument, if correct, would spoil the deconfinement phase transition for the $q = 1$ case. Let us show that this is not the case by using two different arguments.

The first argument relies on simple power counting. It turns out that $d = 3$ is the lower critical dimension of the problem since the field φ is dimensionless. This means that renormalization proceeds in precisely the same way as in the $d = 2$ well known counterpart of the present problem.

The second argument relies on *exact* scaling and duality properties of the theory. The sine-Gordon action (8) is a result of a duality transformation of the *critical* effective action:

$$S_{\rm eff} \propto \int d^3x F_{\mu\nu}\frac{1}{\sqrt{-\partial^2}}F_{\mu\nu}. \qquad (9)$$

The above corresponds to an *exact* scaling behavior of the gauge field propagator in the three-dimensional Abelian Higgs model [2]. Next, we proceed by *reductio ad absurdum* to prove that the effective sine-Gordon action (8) gives the dominant contribution at large distances. This is seen as follows. The effective action (9) clearly corresponds to the dominant large distance behavior, i.e., the usual Maxwell term $\propto F_{\mu\nu}^2$ is obviously irrelevant in the infrared against the anomalous contribution given in Eq. (9). Here, it is crucial that the sign of coefficient of the anomalous term is positive, to ensure that the usual Maxwell term is indeed irrelevant. A negative sign in front of the anomalous term would make the usual Maxwell term relevant. Next, assume that a $\varphi(-\partial^2)\varphi$ term is generated by fluctuation effects in Eq. (8) and that this term has a positive sign. This clearly implies that the $\varphi(-\partial^2)^{3/2}\varphi$ term becomes irrelevant in the infrared. Therefore, the corresponding effective sine-Gordon action is of the usual type, *thus dualizing back to an ordinary Maxwell theory.* This contradicts the exact result that the dominant effective critical theory is given by Eq. (9).

Next, to further substantiate our scenario, we derive the corresponding recursion relations for a gas of point-charges in three dimensions interacting through a logarithmic pair-potential, i.e., a three-dimensional logarithmic plasma or anomalous Coulomb gas, and for which Eq. (8) is a field-theoretical description [11]. As in the case of Eqs. (5) and (6), we will find it useful to consider the problem in d dimensions. Furthermore, we will consider a more general propagator of the form $1/|p|^\sigma$ for the anomalous sine-Gordon model. Thus, we consider a *bare* potential given by $U_0(r) = -4\pi^2 K V(r)$, where

$$V(r) = \frac{\Gamma\left(\frac{d-\sigma}{2}\right)}{2^\sigma \pi^{d/2}\Gamma(\sigma/2)}[(\Lambda r)^{\sigma-d} - 1], \qquad (10)$$

with Λ being an ultraviolet cutoff. For the particular case $d = \sigma$, corresponding to the lower critical dimension in this generalized problem, we have

$$V(r)|_{d=\sigma} = -\frac{2^{1-\sigma}\pi^{-\sigma/2}}{\Gamma(\sigma/2)} \ln(\Lambda r). \tag{11}$$

The bare electric field is given by $E_0(r) = -4\pi^2 K A(d,\sigma) r^{\sigma-d-1}/r_0^{\sigma-d}$, where $r_0 \equiv 1/\Lambda$ and $A(d,\sigma) = (d-\sigma)\Gamma[(d-\sigma)/2]/[2^\sigma \pi^{d/2}\Gamma(\sigma/2)]$. The *bare* electric field is renormalized by the other dipoles which are treated as a dielectric medium. The *renormalized* electric field is then given by

$$E(r) = -\frac{4\pi^2 K\ A(d,\sigma)\ r^{\sigma-d-1}}{\varepsilon(r)}, \tag{12}$$

where $\varepsilon(r)$ is the scale-dependent dielectric constant of the medium. We can write this in the form $\varepsilon(r) = 1 + S_d\chi(r)$, where $S_d = 2\pi^{d/2}/\Gamma(d/2)$, and the electric susceptibility

$$\chi(r) = N_d \int_{r_0}^{r} n(s,\theta)\alpha(s) s^{d-1} \sin^{d-2}\theta ds d\theta, \tag{13}$$

with $N_d = 2\pi^{(d-1)/2}/\Gamma[(d-1)/2]$. In Eq. (13), $n(r,\theta)$ is the density of pairs and $\alpha(r)$ is the polarizability of a dipole. We have for small separation of the pairs

$$\alpha(r) = \frac{4\pi^2 K r^2}{d} + \mathcal{O}(r^4). \tag{14}$$

For $n(r,\theta)$ we have at lowest order in the bare fugacity y_0

$$n(r) = \frac{y_0^2}{r_0^{2d}} e^{-U(r)}, \tag{15}$$

where $U(r)$ is the effective potential obtained by integrating the renormalized electric field:

$$U(r) = U(r_0) + 4\pi^2 K A(d,\sigma) \int_{r_0}^{r} ds \frac{s^{\sigma-d-1}}{\varepsilon(s)}. \tag{16}$$

The effective stiffness $K_{\rm eff}(l)$ is related to the dielectric constant $\varepsilon(r)$ through

$$\frac{1}{K_{\rm eff}(l)} = \frac{\varepsilon(r_0 \exp l)}{K} e^{-(\sigma-d)l}. \tag{17}$$

Let us define $u(l) = U(r_0 \exp l)$, out of which we obtain

$$u(l) = u(0) + 4\pi^2 A(d,\sigma) \int_0^l dv K_{\rm eff}(v). \tag{18}$$

Thus,

$$\frac{du}{dl} = 4\pi^2 A(d,\sigma) K_{\rm eff}(l). \tag{19}$$

We next define the square of the effective fugacity as follows

$$y^2(l) = \frac{2S_d^2}{d r_0^{\sigma-2}} y_0^2 e^{(2d-\sigma+2)l - u(l)}. \tag{20}$$

From Eqs. (17), (19), and (20) we finally obtain

$$\frac{dK_{\rm eff}^{-1}}{dl} = 4\pi^2 y^2 - (\sigma - d) K_{\rm eff}^{-1}, \tag{21}$$

$$\frac{dy}{dl} = \left[d - \eta_y - 2\pi^2 A(d,\sigma) K_{\rm eff}\right] y, \tag{22}$$

where the *anomalous dimension* of the fugacity is given by $\eta_y = (\sigma-2)/2$. When $\sigma = 2$, we recover the recursion relations (5) and (6), which were originally derived by a completely different method [3]. Such a theory does not exhibit a phase-transition for $d > 2$. The case relevant for the three-dimensional Abelian Higgs model is, on the other hand, $\sigma = d = 3$. In this case Eqs. (21) and (22) are very similar to the usual KT recursion relations, except for the presence of the anomalous scaling dimension of the fugacity, η_y which is nonzero in our case and given by $\eta_y = 1/2$.

By integration of the above recursion relations Eqs. (21), (22), for the case $d = \sigma = 3$, we may compute explicitly the screened effective potential $u(l)$ on the ordered side, the result is [13]

$$u(l) - u(0) = \frac{1}{\omega_-\omega_+}\left[\frac{5}{2}\omega_+ l + \ln\left(\frac{\omega_+ e^{-2\theta} + \omega_-}{\omega_+ e^{-2u} + \omega_-}\right)\right] \tag{23}$$

with $\omega_\pm = 1 \pm \omega$, $u = (5/2)\omega l + \theta$, and ω and θ are integration constants determined from initial conditions on the flow equations. Asymptotically, we have $u(l) \sim l \sim \ln(r/r_0)$, which shows that the effective potential is also logarithmic after screening effects are taken into account. Hence, we conclude that the statement, alluded to above, that the transition is destroyed by screening a bare $\ln(r)$-potential into a $1/r$-potential, is not correct. This can also be seen from a simple Debye-Hückel theory for a $\ln(r)$-potential in d dimensions. Indeed, due to Gauss's theorem in d-dimensions, the field equation in the corresponding Debye-Hückel theory is given by [14] $\nabla \cdot (\mathbf{E} r^{2-d}) = S_d[q\delta^d(\mathbf{r}) + \langle\rho(\mathbf{r})\rangle]$, where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ and $\langle\rho(\mathbf{r})\rangle$ is the variation of the charge density in a plasma with density n_0 [15]. The electric field is $\mathbf{E} = -\nabla U$, where U is the screened effective potential. The high temperature limit corresponds to the Debye-Hückel approximation, where the differential equation for $U(r)$ can be solved exactly [14]. The result is

$$U(r) = \frac{2q}{d} K_0[(r/\lambda_D)^{d/2}] - q \ln \lambda_D, \tag{24}$$

where K_0 is a modified Bessel function of the second kind and the inverse of the Debye screening length λ_D is given by

$$\lambda_D^{-1} = \left(\frac{16 q^2 n_0 \pi^{d/2}}{d^2 \; T\Gamma(d/2)} \right)^{1/d}, \tag{25}$$

where T is the temperature. For $r/\lambda_D \ll 1$, Eq. (24) has the expansion $U(r) = 2q(\ln 2 - \gamma)/d - q \ln r + \mathcal{O}(r^2/\lambda_D^2)$.

The recursion relations Eqs. (21) and (22) thus predict the possibility of a topological phase-transition in the system of point charges interacting with the potential Eq. (10) from a "low-temperature" dielectric phase to a "high-temperature" metallic phase, with a universal jump in the stiffness of the theory given by $K_{\rm eff}^* = 2/5$. This should be contrasted with the universal jump $K_{\rm eff}^* = 2/\pi$ that is found in the two-dimensional case. A specific realization of such a KT-like scenario has recently been suggested in the context of discussing the physics of strongly correlated fermion systems at zero temperature in two spatial dimensions [11].

In summary, we have proposed a mechanism in three space-time dimensions by which a deconfinement transition occurs for $U(1)$ compact gauge fields coupled to bosonic fundamental matter fields. The proposed mechanism relies on a KT-like phase transition in three space-time dimensions. Therefore, no symmetry breaking is involved and Elitzur's theorem is not violated. Instead, a topological phase transition occurs.

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