Ambiguities of the Seiberg-Witten map in the presence of matter field

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Abstract

The ambiguities of the Seiberg-Witten map for gauge field coupled with fermionic matter are discussed. We find that only part of the ambiguities can be absorbed by gauge transformation and/or field redefinition and thus are negligible. The existence of matter field makes some other part of the ambiguities difficult to be absorbed by gauge transformation or field redefinition.

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1 Introduction

Recently, quantum field theories on noncommutative spaces have received considerable interests.Though historically the concept of space(time) noncommutativity was proposed over 50 years ago[1], most serious study and progress in field theories defined on such spaces was made within the last few years[2, 3, 4].The reason that there has been a renewed interests in noncommutative spaces is due to the studies in string theory[5, 6]. It was found that in the presence of a NS-NS background B field, the quantization of open string would result in spacial noncommutativity at the boundary, i.e. on the D-branes. Among all the important achievements in the study of noncommutative field theories, e.g. existence and construction of nonperturbative solutions (solitons, instantons, monopoles etc.), relation to D-brane and tachyon condensation and so on, the so-called Seiberg-Witten map[6] is a particularly useful idea for exploring various properties of noncommutative field theories and thus an important subject of intensive study.

In their original paper[6], based on the observation that different regularization schemes (point splitting vs. Pauli-Villars) in the field theory limit of string theory leads to either a commutative or a noncommutative field theory, Seiberg and Witten proposed a map which establish an equivalent relationship between an ordinary gauge theory and a noncommutative one. They then argued that the same kind of equivalent relationship should also hold between noncommutative gauge theories with a small deviation in the noncommutative parameter θ. Many authors subsequently studied various properties of Seiberg-Witten map[7, 8, 9, 10], including applying Seiberg-Witten map to different gauge theories, trying to find the accumulated effect of successive Seiberg-Witten maps (i.e. solving the so-called Seiberg-Witten equation)[11, 12], and so on. Some paper revealed that fact that, in general, Seiberg-Witten maps between noncommutative gauge theories of different noncommutative parameters possess ambiguities. Ref.[13], argued that part of the ambiguities of Seiberg-Witten map between noncommutative gauge theories can be absorbed by a gauge transformation, while the rests of the ambiguities can be removed by a field redefinition. In this paper,

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we would like to re-exam the problem of ambiguities of Seiberg-Witten map. However, in contrast to ref.[13], the field theories we consider are noncommutative gauge theories coupled to some noncommutative matter fields (represented by some spinor field in noncommutative space). As we shall see, Seiberg-Witten maps between noncommutative gauge theories coupled to matter fields would also yield some ambiguities, and, although some part of the ambiguities could be absorbed by a gauge transformation, there are some other parts which can neither be absorbed by gauge transformations nor by field redefinitions. In other words, the presence of matter fields spoils the consistency of the kind of argument made in [13], (though it seems that without matter field the argument of ref. [13] makes perfect sense), and therefore the problem of cancellation of ambiguities in Seiberg-Witten map should be considered as still an open question.

2 Seiberg-Witten map for pure gauge theories: a brief review

Before going into the details of Seiberg-Witten map for gauge theories coupled with matter, we would like to first make a brief review on the works on Seiberg-Witten maps for a purely gauge theory. This would not only provide the theoretical setup for our subsequent work, but also fix the notations.

By definition, a noncommutative gauge theory is gauge theory with gauge potential depending on noncommutative space coordinates (spacetime noncommutativity is not considered in most cases). In practice, one often uses the so-called Moyal product algebra of ordinary functions on commutative spaces to represent the algebra of functions on noncommutative spaces. In this formalism, the gauge field strength for a noncommutative gauge theory can be written as

$$\hat{F}_{ij} = \partial_i \hat{A}_j - \partial_i \hat{A}_i - i[\hat{A}_i, \hat{A}_j]_*,$$

where throughout this paper we use hatted notations to represent noncommutative field objects. In the above equation, the $*$-commutator is defined as

$$[\hat{A}, \hat{B}]_* = \hat{A} * \hat{B} - \hat{B} * \hat{A},$$

where for any pair of ordinary functions $f(x)$ and $g(x)$, the Moyal product $*$ is given as[14]

$$f * g(x) = e^{\frac{i}{2}\theta^{ij}\frac{\partial}{\partial\xi^i}\frac{\partial}{\partial\varsigma^j}} f(x+\xi)g(x+\varsigma)|_{\xi=\varsigma=0} = fg + \frac{i}{2}\theta^{ij}\partial_i f \partial_j g + O(\theta^2).$$

The parameter θ is hence forth referred to as noncommutativity parameter. The noncommutative gauge transformations take the same form as ordinary gauge transformations, the only difference is that one has to replace ordinary products between functions by Moyal products:

$$\begin{aligned} \delta_{\hat{\lambda}} \hat{A}_i &= \partial_i \hat{\lambda} + i[\hat{\lambda}, \hat{A}_i], \\ \delta_{\hat{\lambda}} \hat{F}_{ij} &= i[\hat{\lambda}, \hat{F}_{ij}], \end{aligned} \tag{1}$$

where $\hat{\lambda}$ is an infinitesimal noncommutative gauge parameter.

The ingenious observation made by Seiberg and Witten on noncommutative gauge theories can be stated in the following simple sentence: *there exists an equivalent relation between noncommutative gauge theories of noncommutativity parameters θ and $\theta + \delta\theta$, where the equivalence is established by identifying the gauge equivalent classes of both theories.* In terms of mathematical formalism, the last statement can be written as

$$\tilde{A}_i(\hat{A}) + \delta_{\tilde{\lambda}} \tilde{A}_i(\hat{A}) = \tilde{A}_i(\hat{A} + \delta_{\hat{\lambda}} \hat{A}), \tag{2}$$

or more compactly,

$$\delta_{\tilde{\lambda}} \delta \hat{A} = \delta \delta_{\hat{\lambda}} \hat{A}, \tag{3}$$

where $\tilde{A}, \tilde{\lambda}$ are respectively the gauge potential and gauge parameter in the noncommutative gauge theory with noncommutativity parameter $\theta + \delta\theta$, and the operator δ in (3) represents the variation with respect to θ. One thing one has to take notice is that the variation δ not only acts on functions but also on Moyal products:

$$\delta(f * g) = \delta f * g + f * \delta g + \frac{i}{2}\delta\theta^{ij}\partial_i f * \partial_j g. \tag{4}$$

In order to solve (2) or (3), we write $\tilde{A} = \hat{A} + \delta\hat{A}(\hat{A}, \hat{\lambda}, \delta\theta) + O(\delta\theta^2)$, $\tilde{\lambda} = \hat{\lambda} + \delta\hat{\lambda}(\hat{A}, \hat{\lambda}, \delta\theta) + O(\delta\theta^2)$ and expand (3) to first order in $\delta\theta$. It then follows from (4) that

$$\delta_{\hat{\lambda}}\delta\hat{A}_i - \partial_i\delta\hat{\lambda} - i[\delta\hat{\lambda}, \hat{A}_i]_* - i[\delta\hat{A}_i, \hat{\lambda}]_* = -\frac{1}{2}\delta\theta^{kl}\{\partial_k\hat{A}_i, \partial_l\hat{\lambda}\}_*. \tag{5}$$

Following the discussion of [6], one gets a particular solution

$$\begin{aligned} \delta\hat{A}_i &= -\frac{1}{4}\delta\theta^{kl}\{\hat{A}_k, \partial_l\hat{A}_i + \hat{F}_{li}\}_*, \\ \delta\hat{\lambda} &= \frac{1}{4}\delta\theta^{kl}\{\partial_k\hat{\lambda}, \hat{A}_l\}_*. \end{aligned} \tag{6}$$

Notice, however, that in the above process one gets two unknowns from a single equation (5), so naturally there might be other solutions which solve (5) just as well. In ref.[13], such redundant solutions were indeed found (and referred to as ambiguities of the Seiberg-Witten map), which takes the form

$$\begin{aligned} \delta\hat{A}_i &= -\frac{1}{4}\delta\theta^{kl}\{\hat{A}_k, \partial_l\hat{A}_i + \hat{F}_{li}\}_* + \alpha\delta\theta^{kl}\hat{D}_i\hat{F}_{kl} + \beta\delta\theta^{kl}\hat{D}_i[\hat{A}_k, \hat{A}_l]_*, \\ \delta\hat{\lambda} &= \frac{1}{4}\delta\theta^{kl}\{\partial_k\hat{\lambda}, \hat{A}_l\}_* + 2\beta\delta\theta^{kl}[\partial_k\hat{\lambda}, \hat{A}_l]_*, \end{aligned} \tag{7}$$

where α, β are some arbitrary commuting constant parameters. The α, β dependent terms in $\delta\hat{A}_i$ can be regarded as a field dependent gauge transformation, and hence can be neglected because Seiberg-Witten map is an identification between gauge equivalent classes, rather than identification between field configurations.

However, it was argued also in [13] that there are further ambiguities arising from successive applications of Seiberg-Witten maps, for which part can be absorbed by a field-dependent gauge transformation, and the rests could be removed by a field redefinition. We shall make much detailed discussion on this point for the case of gauge theory coupled with matter and find some different conclusion.

3 Ambiguities of Seiberg-Witten map in the presence of matter field

After making a brief review and setting up our notations, now we come to our central subject – Seiberg-Witten map in the presence of fermionic matter field. Assume now that, in addition to the purely gauge field configuration described in (1), our theory contains also some fermionic matter field ψ which transforms under the fundamental representation of the gauge group,

$$\delta_{\hat{\lambda}}\psi = i\hat{\lambda} * \psi. \tag{8}$$

Assume also that the basic concept of Seiberg-Witten map still holds in the presence of matter field, which means that the gauge equivalent classes for two such theories with small deviation

$\delta\theta$ in the noncommutativity parameter can still be identified. Then, in addition to the standard Seiberg-Witten map (2,3) for gauge fields, we have also the following identities for the matter field,

$$\tilde{\psi}(\hat{\psi}) + \delta_{\tilde{\lambda}}\tilde{\psi}(\hat{\psi}, \hat{A}) = \tilde{\psi}(\hat{\psi} + \delta_{\hat{\lambda}}\hat{\psi}, \hat{A} + \delta_{\hat{\lambda}}\hat{A}), \tag{9}$$

$$\delta_{\tilde{\lambda}}\delta\hat{\psi} = \delta\delta_{\hat{\lambda}}\hat{\psi}. \tag{10}$$

In order to solve the above equations, we write, just as in the case of purely gauge theory, the following expansion for the matter field,

$$\tilde{\psi} = \hat{\psi} + \delta\hat{\psi}(\hat{\psi}, \hat{A}, \hat{\lambda}, \delta\theta) + O(\delta\theta^2). \tag{11}$$

Substituting (11) into (10), we get

$$\delta_{\hat{\lambda}}\delta\hat{\psi} - i\hat{\lambda} * \delta\hat{\psi} - i\delta\hat{\lambda} * \hat{\psi} = -\frac{1}{2}\delta\theta^{kl}\partial_k\hat{\lambda} * \partial_l\hat{\psi}. \tag{12}$$

It can be easily checked that(See Refs.[3])

$$\delta\hat{\psi} = -\frac{1}{2}\delta\theta^{kl}\hat{A}_k * \partial_l\hat{\psi} + \frac{i}{4}\delta\theta^{kl}\hat{A}_k * \hat{A}_l * \hat{\psi} \tag{13}$$

is a particular solution to (12).

Now let us consider the ambiguities arisen in the presence of matter fields. According to equation (12), suppose there exits some functions $\hat{\psi}', \hat{\lambda}$ such that

$$\delta_{\hat{\lambda}}\hat{\psi}' - i\hat{\lambda} * \hat{\psi}' - i\hat{\lambda}' * \hat{\psi} = 0, \tag{14}$$

then $\delta\hat{\psi} + \hat{\psi}', \delta\hat{\lambda} + \hat{\lambda}'$ also solve (12). Therefore, any solution $\hat{\psi}', \hat{\lambda}'$ of (14) would give rise to ambiguities for the Seiberg-Witten map (2,9). After some algebra, we find that the solution to (14) is

$$\begin{aligned} \hat{\lambda}' &= 2\beta\delta\theta^{kl}[\partial_k\hat{\lambda}, \hat{A}_l]_*, \\ \hat{\psi}' &= i\alpha\delta\theta^{kl}\hat{F}_{kl} * \hat{\psi} + i\beta\delta\theta^{kl}[\hat{A}_k, \hat{A}_l]_* * \hat{\psi}. \end{aligned}$$

Combining with the corresponding result for the case of gauge fields (see eq. (7) of the last section), we see that the α, β ambiguities in $\delta\hat{A}$ and $\delta\hat{\psi}$ are both of the form of a gauge transformation with gauge parameter $\alpha\delta\theta^{kl}\hat{F}_{kl} + \beta\delta\theta^{kl}[\hat{A}_k, \hat{A}_l]_*$ and therefore can be neglected just as in the case of a pure gauge theory.

Next we come to the case when more ambiguities may arise from successive applications of Seiberg-Witten maps. Consider for instance that we wish to map a theory with noncommutativity parameter θ to a theory with noncommutativity parameter $\theta + \delta\theta_1 + \delta\theta_2$. Of cause we can realize the map by a two-step process: we may either map from θ to $\theta + \delta\theta_1$ and then to $\theta + \delta\theta_1 + \delta\theta_2$ or from θ to $\theta + \delta\theta_2$ and then to $\theta + \delta\theta_1 + \delta\theta_2$. Naively both paths should work the same way. However, a detailed check reveals that the two paths are actually non-equivalent.

Let us now study the details of the new ambiguities arisen from the non-equivalence of the two paths just stated. Along the two paths the fields are mapped as

$$\begin{aligned} &\theta \to \theta + \delta\theta_1 \to \theta + \delta\theta_1 + \delta\theta_2 \\ &\hat{\psi} \to \tilde{\psi} \to \tilde{\tilde{\psi}}, \\ &\hat{A} \to \tilde{A} \to \tilde{\tilde{A}}; \end{aligned}$$

$$\begin{aligned} &\theta \to \theta + \delta\theta_2 \to \theta + \delta\theta_1 + \delta\theta_2 \\ &\hat{\psi} \to \bar{\psi} \to \tilde{\bar{\psi}}, \\ &\hat{A} \to \bar{A} \to \tilde{\bar{A}}. \end{aligned}$$

For the first path, equation (13) yields

$$\tilde{\tilde{\psi}} = \tilde{\psi} - \frac{1}{2}\delta\theta_2^{kl}\tilde{A}_k *' \partial_l\tilde{\psi} + \frac{i}{4}\delta\theta_2^{kl}\tilde{A}_k *' \tilde{A}_l *' \tilde{\psi},$$

where $*'$ is the Moyal product defined with the noncommutativity parameter $\theta + \delta\theta_1$. Applying (13) again, we get

$$\begin{aligned}
\tilde{\tilde{\psi}} = \tilde{\psi} &- \frac{1}{2}(\delta\theta_1 + \delta\theta_2)^{kl}\tilde{A}_k * \partial_l\tilde{\psi} + \frac{i}{4}(\delta\theta_1 + \delta\theta_2)^{kl}\tilde{A}_k * \tilde{A}_l * \tilde{\psi} \\
&+ \frac{1}{4}\delta\theta_2^{kl}\delta\theta_1^{pq}\Big[\hat{A}_k * \partial_l(\hat{A}_p * \partial_q\hat{\psi}) - \frac{i}{2}\hat{A}_k * \partial_l(\hat{A}_p * \hat{A}_q * \hat{\psi}) \\
&+ \frac{1}{2}\{\hat{A}_p, \partial_q\hat{A}_k + \hat{F}_{qk}\}_* * \partial_l\hat{\psi} - i\partial_p\hat{A}_k * \partial_q\partial_l\hat{\psi} \\
&- \frac{i}{4}\hat{A}_k * \{\hat{A}_p, \partial_q\hat{A}_l + \hat{F}_{ql}\}_* * \hat{\psi} - \frac{i}{4}\{\hat{A}_p, \partial_q\hat{A}_k + \hat{F}_{qk}\}_* * \hat{A}_l * \hat{\psi} \\
&- \frac{i}{2}\hat{A}_k * \hat{A}_l * \hat{A}_p * \partial_q\hat{\psi} - \frac{1}{4}\hat{A}_k * \hat{A}_l * \hat{A}_p * \hat{A}_q * \hat{\psi} \\
&- \frac{1}{2}\partial_p\hat{A}_k * \partial_q\hat{A}_l * \hat{\psi} - \frac{1}{2}\partial_p\hat{A}_k * \hat{A}_l * \partial_q\hat{\psi} \\
&- \frac{1}{2}\hat{A}_k * \partial_p\hat{A}_l * \partial_q\hat{\psi}\Big]. \qquad (15)
\end{aligned}$$

The same procedure for the second path would give rise corresponding result for $\tilde{\tilde{\psi}}$, which is basically of the same form as $\tilde{\tilde{\psi}}$, but with $\delta\theta_1$ and $\delta\theta_2$ exchanged:

$$\tilde{\tilde{\psi}} = \tilde{\tilde{\psi}}(\text{with} \quad \delta\theta_1 \leftrightarrow \delta\theta_2).$$

Therefore, to first order in $\delta\theta$, $\tilde{\tilde{\psi}}$ and $\tilde{\tilde{\psi}}$ are equal, while to the order of $\delta\theta_1\delta\theta_2$, we get

$$\begin{aligned}
[\delta_1, \delta_2]\hat{\psi} \equiv\ & \tilde{\tilde{\psi}} - \tilde{\tilde{\psi}} \\
=\ & -\frac{i}{4}\delta\theta_2^{kl}\delta\theta_1^{pq}\hat{F}_{pk} * \hat{D}_l\hat{D}_q\hat{\psi} \\
& - \frac{1}{8}\delta\theta_2^{kl}\delta\theta_1^{pq}\Big[-2\hat{F}_{pk} * (\partial_l\hat{A}_q + i\hat{A}_l * \hat{A}_q) \\
& + i\hat{A}_k * \partial_l(\hat{A}_p * \hat{A}_q) - i\hat{A}_p * \partial_q(\hat{A}_k * \hat{A}_l) \\
& + \partial_p\hat{A}_k * \partial_q\hat{A}_l - \partial_k\hat{A}_p * \partial_l\hat{A}_q \\
& + \frac{i}{2}[\hat{A}_k, \{\hat{A}_p, \partial_q\hat{A}_l + \hat{F}_{ql}\}_*]_* \\
& - \frac{i}{2}[\hat{A}_p, \{\hat{A}_k, \partial_l\hat{A}_q + \hat{F}_{lq}\}_*]_* \\
& + \frac{1}{2}[\hat{A}_k * \hat{A}_l, \hat{A}_p * \hat{A}_q]_*\Big] * \hat{\psi}. \qquad (16)
\end{aligned}$$

In the last calculations, we did not consider the α, β dependent ambiguities which were thought to be trivial. In fact, a careful check on the α, β dependent terms would show that their contribution to the expression $[\delta_1, \delta_2]\hat{\psi}$ is still of the form (α, β dependent terms)$*\hat{\psi}$. So the α, β dependent ambiguities are indeed only a matter of gauge choice and hence can be neglected. The rest parts of $[\delta_1, \delta_2]\hat{\psi}$ are grouped into two terms – the first term is a gauge covariant "local" differential polynomial in the fundamental fields $\hat{A}$ and $\hat{\psi}$, the second term again takes the form of a field-dependent gauge transformation over $\hat{\psi}$.

At first sight, one may tends to make the same conclusion as in [13] that one part of the last ambiguities may be absorbed by a gauge transformation and the other part by a local field redefinition. But actually this is not the case. Let us recall that the action of $[\delta_1, \delta_2]$ on $\hat{A}$ was already given in ref. [13],

$$\begin{aligned}[\delta_1, \delta_2]\hat{A}_i = &\frac{i}{8}\delta\theta_2^{kl}\delta\theta_1^{pq}[\hat{F}_{kp}, \hat{D}_l\hat{F}_{qi} + \hat{D}_q\hat{F}_{li}]_* \\ &+ \frac{1}{16}\delta\theta_2^{kl}\delta\theta_1^{pq}\hat{D}_i\left[\frac{1}{2}\{\hat{A}_k, \{\hat{A}_p, \hat{F}_{lq}\}_*\}_* + \frac{1}{2}\{\hat{A}_p, \{\hat{A}_k, \hat{F}_{lq}\}_*\}_*\right. \\ &\left.+ \frac{1}{2}[[\hat{A}_k, \hat{A}_p]_*, \partial_l\hat{A}_q + \partial_q\hat{A}_l]_* - i[\partial_p\hat{A}_k, \partial_l\hat{A}_q]_* + i[\partial_k\hat{A}_p, \partial_q\hat{A}_l]_*\right] \\ &+ \hat{D}_i(\alpha, \beta \quad \text{dependent} \quad \text{terms}). \end{aligned} \tag{17}$$

Apart from the terms which are "local", gauge covariant differential polynomials of the fundamental fields, the terms which take the form of field dependent gauge transformations in eqs. (16) and (17) have different gauge parameters. Consequently, although one might be able to cancel all the terms on the right hand side of (17) by simultaneous use of gauge transform and field redefinition, one cannot simultaneously remove all the terms appeared in (16) by the same gauge transform. In other words, if initially $\hat{A}_i$ and $\hat{\psi}$ are in the same gauge slice of the theory with noncommutativity parameter θ, the successive application of two Seiberg-Witten maps along two different paths in "θ-space" would not lead to field configurations which live in the same gauge slice in the resulting theory. This path dependence in the "θ-space" might imply that there exist some nontrivial topological obstacles in the space of noncommutativity parameter which prevents one from deforming one noncommutative gauge theory coupled with matter into another in a smooth way. Perhaps a relatively safe manner in studying noncommutative field theories is to keep the noncommutativity parameter fixed, before we have a complete understanding of the mysterious ambiguities described above.

References

[1] H.S. Snyder, Qauntized Space-Time, Phys. Rev. 71 (1947) 38

[2] J. Madore,S. Schraml,P. Schupp and J.Wess, Gauge theory on noncommutative spaces, Eur. Phys,J.C 16(2000) 161, arXiv:hep-th/0001203

[3] B. jurco,L. Moller,S. Schraml,P. Schupp,J. Wess, Construction of non-Abelian gauge theories on noncommutative spaces, arXiv:hep-th/0104153

[4] F. Lizzi, R.J. Szabo, A. Zampini,Geometry of the Gauge Algebra in Noncommutative Yang-Mills Theory, JHEP 0108:032, 2001, arXiv:hep-th/0107115

[5] Alain Connes, Michael R. Douglas, Albert Schwarz,Noncommutative Geometry and Matrix Theory: Compactification on Tori, JHEP 9802 (1998) 003, arXiv:hep-th/9711162

[6] N. Seiberg, E. Witten,String theory and noncommutative geometry, JHEP 9909:032(1999), arXiv:hep-th/9908142

[7] A.A. Bichl, J.M. Grimstrup, H. Grosse, E. Kraus, L. Popp, M. Schweda, R. Wulkenhaar, Noncommutative Lorentz Symmetry and the Origin of the Seiberg-Witten Map, arXiv:hep-th/0108045

[8] A.A. Bichl, J. M. Grimstrup, L. Popp, M. Schweda, R. Wulkenhaar, Perturbative Analysis of the Seiberg-Witten Map, arXiv:hep-th/0102044

[9] Andrei G. Bytsko,Singularities of the Seiberg-Witten map, JHEP 0101 (2001) 020, arXiv:hep-th/0012018

[10] Aristophanes Dimakis, Folkert Muller-Hoissen, Lett. Math. Phys. 54 (2000) 123-135, Moyal Deformation, Seiberg-Witten-Map, and Integrable Models

[11] Thomas Mehen, Mark B. Wise, Generalized *-Products, Wilson Lines and the Solution of the Seiberg-Witten Equations, JHEP 0012 (2000) 008, arXiv:hep-th/0010204

[12] Yuji Okawa, Hirosi Ooguri, An Exact Solution to Seiberg-Witten Equation of Noncommutative Gauge Theory, Phys. Rev. D64 (2001) 046009, arXiv:hep-th/0104036

[13] T. Asakawa,I. Kishimoto, Comments on Gauge Equivalence in Noncommutative Geometry, JHEP 9911:024 (1999), arXiv:hep-th/9909139

[14] A. Connes, Noncommutative Geometry, Academic Press(1994)