Cosmological Scaling Solutions and Multiple Exponential Potentials

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Abstract

We present a phase-space analysis of cosmology containing multiple scalar fields with positive and negative exponential potentials. In addition to the well-known assisted inflationary solutions, there exits power-law multi-kinetic-potential scaling solutions for sufficiently flat positive potentials or steep negative potentials, which are the unique late-time attractor whenever they exist. We briefly discuss the physical consequences of these results.

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Scalar field cosmological models are of great importance in modern cosmology. The dark energy is attributed to the dynamics of a scalar field, which convincingly realizes the goal of explaining present-day cosmic acceleration generically using only attractor solutions [1]. A scalar field can drive an accelerated expansion and thus provides possible models for cosmological inflation in the early universe [2]. In particular, there have been a number of studies of spatially homogeneous scalar field cosmological models with an exponential potential. There are already known to have interesting properties; for example, if one has a universe containing a perfect fluid and such a scalar field, then for a wide range of parameters the scalar field mimics the perfect fluid, adopting its equation of state [3]. These scaling solutions are attractors at late times [4]. The inflation [5, 6] and other cosmological effect [7] of multiple scalar fields have also been considered.

The scale-invariant form makes the exponential potential particularly simple to study analytically. There are well-known exact solutions corresponding to power-law solutions for the cosmological scale factor $a \propto t^p$ in a spatially flat Friedmann-Robertson-Walker (FRW) model [8], but more generally the coupled Einstein-Klein-Gordon equations for a single field can be reduced to a one-dimensional system which makes it particularly suited to a qualitative analysis [9, 10]. Recently, adopting a system of dimensionless dynamical variables [11], the cosmological scaling solutions with positive and negative exponentials has been studied [12]. Usually there are many scalar fields with exponential potentials in supergravity, superstring and the generalized Einstein theories, thus multi potentials may be more important. In this paper, we will consider multiple scalar fields with positive and negative exponential potentials. We have assumed that there is no direct coupling between the exponential potentials. The only interaction is gravitational. A phase-space analysis of the spatially flat FRW models shows that there exist cosmological scaling solutions which are the unique late-time attractors, and successful inflationary solutions which are driven by multiple scalar fields with a wide range of each potential slope parameter λ.

We start with more general model with m scalar fields ϕ_i, in which each has an identical-slope potential

$$V_i(\phi_i) = V_{0i} \exp(-\lambda\kappa\phi_i) \tag{1}$$

where $\kappa^2 \equiv 8\pi G_N$ is the gravitational coupling and λ is a dimensionless constant characterising the slope of the potential. Note that there is no direct coupling of the fields, which influence each other only via their effect on the expansion. The evolution equation of each scalar field for a spatially flat FRW model with Hubble parameter H is

$$\ddot{\phi}_i + 3H\dot{\phi}_i^2 + \frac{dV_i(\phi_i)}{d\phi_i} = 0 \tag{2}$$

subject to the Friedmann constraint

$$H^2 = \frac{\kappa^2}{3}\sum_{i=1}^{m}[V_i(\phi_i) + \frac{1}{2}\dot{\phi}_i^2] \tag{3}$$

Defining $2m$ dimensionless variables

$$x_i = \frac{\kappa\dot{\phi}_i}{\sqrt{6}H}, \qquad y_i = \frac{\kappa\sqrt{|V_i|}}{\sqrt{3}H} \tag{4}$$

the evolution equations (2) can be written as an autonomous system:

$$x_i' = -3x_i(1-\sum_{i=1}^{m}x_i^2) \pm \lambda\sqrt{\frac{3}{2}}y_i^2 \tag{5}$$

$$y_i' = y_i(3\sum_{i=1}^{m}x_i^2 - \lambda\sqrt{\frac{3}{2}}x_i) \tag{6}$$

where a prime denotes a deriative with respect to the logarithm of the scalar factor, $N \equiv \ln a$, and the constraint equation (3) becomes

$$\sum_{i=1}^{m}(x_i^2 \pm y_i^2) = 1 \tag{7}$$

where upper/lower signs denote the two distinct cases of $V_i > 0/V_i < 0$. x_i^2 measures the contribution to the expansion due to the field's kinetic energy density, while $\pm y_i^2$ represents the contribution of the potential energy. Critical points correspond to fixed points where $x_i' = 0$, $y_i' = 0$, and there are self-similar solutions with

$$\frac{\dot{H}}{H^2} = -3\sum_{i=1}^{m}x_i^2 \tag{8}$$

This corresponds to a power-law solution for the scalar factor

$$a \propto t^p, \qquad \text{where} \quad p = \frac{1}{3\sum_{i=1}^{m}x_i^2} \tag{9}$$

The system (5) and (6) has at most one m-dimensional sphere embedded in $2m$-dimensional phase-space corresponding to kinetic-dominated solutions, and $(2^m - 1)$ fixed points, one of which is a m-kinetic-potential scaling solution.

In order to analysis the stability of the critical points, we only consider the cosomlogies containing two scalar fields. There are one unit circle and three fixed points as follows:

S: $x_1^2 + x_2^2 = 1$, $y_1 = y_2 = 0$

These kinetic-dominated solutions exist for any form of the potential, which are equivalent to stiff-fluid dominated evolution with $a \propto t^{1/3}$, irrespective of the nature of the potential. In $x_1^2 + x_2^2 = 1$ and $y_1 = y_2 = 0$ sub-space, each fixed point is marginally stable. Linear perturbations $y_1 \to y_1 + \delta y_1$ and $y_2 \to y_2 + \delta y_2$ yield two eigenmodes

$$\delta y_1' = (3 - \lambda\sqrt{\frac{3}{2}}x_1)\delta y_1$$

$$\delta y_2' = (3 - \lambda\sqrt{\frac{3}{2}}x_2)\delta y_2$$

Thus the solutions are stable to potential energy perturbations for $\lambda > \frac{\sqrt{6}}{x_1}$ and $\lambda > \frac{\sqrt{6}}{x_2}$, where $x_1, x_2 > 0$. This result shows that there maybe exit stable points only for sufficiently steep ($\lambda > 2\sqrt{3}$) potential.

A: $x_1 = \frac{\lambda}{\sqrt{6}}, y_1 = \sqrt{\pm(1-\frac{\lambda^2}{6})}, x_2 = y_2 = 0$, or $x_1 = y_1 = 0, x_2 = \frac{\lambda}{\sqrt{6}}, y_2 = \sqrt{\pm(1-\frac{\lambda^2}{6})})$
The two single-potential-kinetic solutions exist for sufficiently flat ($\lambda^2 < 6$) positive potentials or steep ($\lambda^2 > 6$) negative potentials. The power-law exponent, $p = 2/\lambda^2$, depends on the slope of the potential. In order to study the stability of the critical points we perturb lineatly about the critical points and get three eignmodes

$$\begin{aligned} \delta x_1' &= (\lambda^2-6)/2\delta x_1 \\ \delta x_2' &= (\lambda^2-6)/2\delta x_2 \\ \delta y_2' &= \frac{\lambda^2}{2}\delta y_2 \end{aligned}$$

Thus the single-potential-kinetic solutions are unatable for the potive and negative potentials. This indicate that the stability is destroyed by the potential energy perturbations of another scalar field.

B: $x_1 = x_2 = \frac{\lambda}{2\sqrt{6}},\ y_1 = y_2 = \sqrt{\pm(\frac{1}{2}-\frac{\lambda^2}{24})}$
The double-potential-kinetic scaling solution exist for flat ($\lambda^2 < 12$) positive potentials, or steep ($\lambda^2 > 12$) negative potentials. This corresponds to a power-law solution with $a \propto t^{4/\lambda^2}$. Linear perturbations yield three eigenmodes

$$\begin{aligned} \delta x_1' &= -3\delta x_1 \\ \delta x_2' &= (\lambda^2-6)/2\delta x_2 \\ \delta y_2' &= 0 \end{aligned}$$

For positive potentials, the scaling solution is marginally stable for $\lambda^2 < 6$, while it is unstable for $6 < \lambda^2 < 12$. For negative potentials the scaling solution is never stable.

Critical points	Existence	Eigenvalues	Stability
$x_1^2 + x_2^2 = 1,\ y_1 = y_2 = 0$	all λ	$(3-\lambda\sqrt{\frac{3}{2}}x_1)$; $(3-\lambda\sqrt{\frac{3}{2}}x_2)$	stable ($\lambda^2 > 12$) unstable ($\lambda^2 < 12$)
$(\frac{\lambda}{\sqrt{6}}, \sqrt{\pm(1-\frac{\lambda^2}{6})}, 0, 0)$, $(0, 0, \frac{\lambda}{\sqrt{6}}, \sqrt{\pm(1-\frac{\lambda^2}{6})})$	$\lambda^2 < 6(V > 0)$ $\lambda^2 > 6(V < 0)$	$(\lambda^2-6)/2$; $(\lambda^2-6)/2$; $\frac{\lambda^2}{2}$	unstable
$x_1 = x_2 = \frac{\lambda}{2\sqrt{6}},\ y_1 = y_2 = \sqrt{\pm(\frac{1}{2}-\frac{\lambda^2}{24})}$	$\lambda^2 < 12(V > 0)$ $\lambda^2 > 12(V < 0)$	$(\lambda^2-6)/2$; -3; 0	stable ($V > 0, \lambda^2 < 6$) unstable ($V < 0$)

TABLE 2. The properties of the critical points

In summary, we have presented a phase-space analysis of the evolution of a spatially flat FRW unvierse containing two scalar fields with positive and negative exponential potentials. The regions of (λ) parameter space lead to different qualitative evolution.

- For steep positive potentials ($V > 0, \lambda^2 > 12$), only a circle S exists, some kinetic-dominated scaling solutions of which are the late-time attrators.
- For intermediate positive potentials ($V > 0, 6 < \lambda^2 < 12$), a circle S and a fixed point B exist. There exist no stable points.
- For flat positive potentials ($V > 0, \lambda^2 < 6$), all critical points exist. The double-kinetic -potential scaling solution is the unique late-time attractor.
- For steep negative potentials ($V < 0, \lambda^2 > 12$), all critical points exist. some kinetic-dominated scaling solutions of which are the late-time attrators.
- For intermediate negative potentials ($V < 0, 6 < \lambda^2 < 12$), a circle S and two fixed point A exist. There exist no stable points.
- For flat negative potentials ($V < 0, \lambda^2 < 6$), only a circle S exists, which are never stable.

Generalizing above discussion to m scalar fields is straight. The multi-kinetic-potential scaling solution exists for positive potentials ($\lambda^2 < 6m$), or negative potentials ($\lambda^2 < 6m$). As long as each potential satisfies $\lambda^2 < 2m$, this power-law solution is inflationary. For the case $m = 1$, the dimensionless constant λ must be smaller than $\sqrt{2}$ to guarantee power-law inflation [12]. However, presently known theories yield expotential potentials with $\lambda > \sqrt{2}$. In such cases multiple scalar fields may proceed inflation. The reason for this behavior is that while each field experiences the 'downhill' force from its own potential, it feels the friction from all the scalar fields via their contribution to the expansion [5].

We emphasize that we have assumed that there is no direct coupling between these exponential potentials and each scalar field has an identical potential. It is worth studying further the case their potentials have different slopes.

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