Radiation reaction for a massless charged particle

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Abstract

We derive effective equations of motion for a massless charged particle coupled to the dynamical electromagnetic field having regard to the radiation back reaction. It is shown that unlike the massive case not all the divergences resulting from the self-action of the particle are Lagrangian, i.e. can be canceled out by adding appropriate counterterms to the original action. Besides, the order of renormalized differential equations governing the effective dynamics turns out to be greater than the order of the corresponding Lorentz-Dirac equation for a massive particle. For the case of homogeneous external field the first radiative correction to the Lorentz equation is explicitly derived via the reduction of order procedure.

1 Introduction

One of the interesting features of the classical electrodynamics is a possibility to account for the self-action of a point charge in a purely local manner by adding certain higher-derivative terms to the usual Lorentz's equations. Physically these terms are responsible for the back reaction of the radiation emitted by an accelerating charge. The first systematic derivation of the effective equations of motion for $d = 4$ relativistic massive particle was given by Dirac [1]. Since the non-relativistic limit of Dirac's equation coincides with the zero-size limit of Lorentz's model of an electron [2] it is often referred to at the Lorentz-Dirac equation. The case of $d = 4$ electrodynamics is not an exceptional one and the analogous effective equations can be obtained for a massive particle in arbitrary dimensions, but the local equations take place only for even d's. The case of $d = 6$ massive particle was first considered in Ref. [3]. The general framework to the Lorentz-Dirac equation in higher dimensions was developed in our recent paper [4].

Although there is a large literature devoted to different aspects of the Lorentz-Dirac equation, its solutions and applications (see [5, 7, 8] for a modern review and further references) the case of a massless particle has somehow escaped consideration. The aim of this paper is to fill this gap in the case of $d = 4$ massless particle. We emphasize that the effective dynamics of massless particle is essentially different from that of the massive one, and it can't be obtained as a massless limit of the Lorentz-Dirac equation.

While massless electrically charged particles (so to speak, massless electrons or charged photons) have not been yet observed experimentally no obvious prohibitions are known against their existence at least from the viewpoint of classical field theory. Moreover, these particles are predicted by supersymmetric gauge theories with an unbroken sector of Abelian gauge symmetry. Perhaps some peculiarities of the effective dynamics, revealed in this paper, do indicate a certain inconsistency of the classical electrodynamics of massless particles with radiation effects included.

This paper may also be viewed as a preparative for study the radiation reaction problem in the case of extended objects like strings and branes universally coupled to the dynamical antisymmetric tensor fields. It is notable that the regularization procedure we use here is closely analogous to the formalism of characteristic (de Rham's) currents intensively exploited for the study of anomalies in 5-branes coupled to 11-dimensional supergravity background [12, 13, 14].

The paper is organized as follows. In Sec. 2 we discuss some peculiarities of the dynamics of a relativistic massless particle coupled to external electromagnetic field and describe a class of isotropic word lines for which the radiation reaction effects can be properly treated. Taking into account the radiation back reaction leads to inevitable infinities resulting from the "pointness" of the particle. The renormalization procedure for these classical infinities is briefly explained in Sec. 3. Here we also present our main result - the effective equations of motion for a massless charged particle, and trace the origin of non-Lagrangian divergences. As for the Lorentz-Dirac equation, our equations involve higher derivatives and thus the reduction-of-order procedure is required to assign them with a proper mechanical interpretation. This is done in Sec. 4. In particular, we explicitly derive the leading correction to the usual Lorentz's equation for the case of homogeneous external fields. In the concluding section we summarize the results. Appendix contains details of the calculations omitted in Sec. 3.

2 Massless particle in the classical electrodynamics

Let $\mathbb{R}^{3,1}$ be four-dimensional Minkowski space with coordinates $\{x^\mu\}$, $\mu = 0, 1, 2, 3$, and the metric $\eta_{\mu\nu} = diag(-1, +1, +1, +1)$. Consider a massless point particle moving in $\mathbb{R}^{3,1}$ and coupled to the electromagnetic field. The dynamics of the whole system (field)+(particle) is governed by the usual action functional[1]

$$S = -\frac{1}{2}\int d\tau e\dot{x}^2 + \int d^4x \left(A_\mu j^{\,\mu} - \frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} \right) . \tag{1}$$

Here $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength tensor of the electromagnetic field, $e \neq 0$ is the Lagrange multiplier and

$$j^\mu(x) = q \int \delta^4(x - x(\tau))\dot{x}^\mu(\tau) d\tau \tag{2}$$

[1] We take unified dimensions for space and time setting $c = 1$.

is the density of the electric current produced by the point charge q moving along a world line $x^\mu(\tau)$ with the four-velocity $\dot{x}^\mu \equiv dx^\mu/d\tau$. The action (1) is invariant under arbitrary world-line reparametrizations $\tau \to \tau'$, provided $e(\tau)$ is transformed as a world-line density (einbein)

$$e(\tau') = \frac{d\tau'}{d\tau} e(\tau)\,. \tag{3}$$

Varying the action (1) with respect to e one gets the standard isotropy condition (Lagrangian constraint) on the four-velocity of the massless particle

$$\dot{x}^2 = 0\,, \tag{4}$$

i.e. a tangent vector to each point of particle's trajectory $x^\mu(\tau)$ lies on the future light cone attached at this point. We will refer to such a trajectory as an isotropic one. The variations of the action with respect to particle's trajectory $x^\mu(\tau)$ and electromagnetic potentials $A_\mu(x)$ lead to the coupled systems of the Lorentz and Maxwell equations

$$e\ddot{x}_\mu + \dot{e}\dot{x}_\mu = qF_{\mu\nu}\dot{x}^\nu, \tag{5}$$

$$\partial^\nu F_{\mu\nu} = 4\pi j_\mu\,. \tag{6}$$

Using the reparametrization invariance of the model and the transformation property (3) one can always bring the einbein e into the form $e = 1$ or, equivalently, one may impose this condition to fix a particular parametrization of the world line. In this gauge the form of the Lorentz equations is quite similar to that for a massive particle,

$$\ddot{x}_\mu = qF_{\mu\nu}\dot{x}^\nu\,, \tag{7}$$

written in a *proper-time parametrization* ($\dot{x}^2 = 1$). The crucial difference, however, is a presence of the additional isotropy condition (4) having no analog in the massive case. In particular this suggests impossibility to introduce the proper-time parametrization for world lines of the massless particle.

Another convenient parametrization to use when working with the isotropic world lines is extracted by the following gauge fixing condition:

$$\ddot{x}^2 = 1\,. \tag{8}$$

One may regard it as a "massless" counterpart of the proper-time parametrization. Contrary to the massive case, however, there are some isotropic trajectories which do not admit such a parametrization. Since for any isotropic curve $\ddot{x}^2 \geq 0$, the trajectories for which $\ddot{x}^2 = 0$ at some τ's may be thought of as *degenerate* ones in the sense that the complementary set of nondegenerate trajectories fills an open and everywhere dense domain in the space of all isotropic curves, so that any degenerate isotropic curve can be made nondegenerate by a small perturbation.

On the other hand, if for some interval $\tau_1 < \tau < \tau_2$ the four-vector of acceleration is know to be isotropic, $\ddot{x}^2 = 0$, then we can immediately conclude that this part of the trajectory is given by a straight segment. Indeed, since the velocity and acceleration of the particle are always orthogonal to each other we get a pair of isotropic and orthogonal vectors $\dot{x}^\mu$ and $\ddot{x}^\mu$, but any such two vectors are known to be proportional to each other in the Minkowski space,

$$\ddot{x}^\mu = \lambda(\tau)\dot{x}^\mu \,. \tag{9}$$

Integrating the last equation we see that the particle does move along a straight line until $\tau_1 < \tau < \tau_2$.

The positive-curvature condition ($\ddot{x}^2 > 0$) for particle's trajectory implies a nonzero value of the external electromagnetic field in the right-hand side of the Lorentz equation (5), otherwise the particle would move along a straight isotropic line for which $\ddot{x}^\mu = 0$. More precisely, squaring both sides of the Lorentz equation and accounting that, by definition, $e \neq 0$ we arrive at the following condition:

$$T_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu > 0 \,, \tag{10}$$

where

$$T_{\mu\nu} = -\frac{1}{4\pi}\left(F_\mu{}^\alpha F_{\alpha\nu} + \frac{1}{4}\eta_{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}\right) \tag{11}$$

is the stress-energy tensor of the electromagnetic field. Passing to an appropriate Lorentz frame it is not hard to see that $T_{\mu\nu}\dot{x}^\mu\dot{x}^\nu \geq 0$ for any field $F_{\mu\nu} = (\mathbf{E}, \mathbf{H})$, and the equality implies either

$$\mathbf{v} \parallel \mathbf{E} \parallel \mathbf{H} \tag{12}$$

or

$$\mathbf{v} \parallel \mathbf{E}\times\mathbf{H} \quad \text{and} \quad \mathbf{E}\cdot\mathbf{H} = 0 \,, \;\; \mathbf{E}^2 = \mathbf{H}^2 \,, \tag{13}$$

$\mathbf{v} = \{\dot{x}^i\}$ being the space part of the four-velocity. The latter situation may occur, for example, for a massless particle moving on a background of free electromagnetic wave.

In this paper we restrict our attention to the self-consistent solutions for the Lorentz-Maxwell equations (5, 6) obeying the gauge fixing condition (8) and the compatibility condition (10). In the next section we will see how these restrictions arise naturally upon calculating the radiation-reaction force.

3 Radiation reaction and renormalization

In this section we follow the approach of work [4] to account for the radiation back reaction to the relativistic motion of a point charge. For this end, we first solve the Maxwell equations (6) expressing the electromagnetic field as a functional of particle's trajectory. The effective equations of motion are then obtained by substituting this solution to the Lorentz equations (5). Inasmuch as the electromagnetic field is singular at the points of particle's trajectory the last step requires a renormalization procedure to remove inevitable infinities. The alternative approach to the problem (also involving some renormalization) is based on the computation of energy radiated by an accelerating charge and identification of this energy with the work of a hypothetical (radiation reaction) force [1, 3, 5, 7].

Let us proceed to the calculations. In the Lorentz gauge $\partial^\mu A_\mu = 0$ the Maxwell equations take the form

$$\Box A_\mu(x) = -4\pi j_\mu(x)\,. \tag{14}$$

Any solution to these equations may be constructed as the sum of their particular solution and a solution to the corresponding homogeneous equations. Although this decomposition is quite ambiguous (from the pure mathematical viewpoint) it has a certain physical meaning if one identifies the former solution with the own field of the point charged particle, treating the latter as an external field describing free electromagnetic waves incident on the particle. The own field of the particle is given by the Liénard-Wiechert potentials

$$A_\mu^{LW}(x) = -4\pi \int G^{ret}(x-z) j_\mu(z) d^4 z\,, \tag{15}$$

where

$$G^{ret}(x) = -\frac{1}{2\pi}\theta(x^0)\delta(x^2) \tag{16}$$

is the retarded Green's function. Taking together Rels. (2), (15), (16) we arrive at the standard expression for the Lorentz force $F_\mu(s) = F_{\mu\nu}(x(s))\dot{x}^\nu(s)$ accounting for the self-action of the particle

$$F_\mu = F_\mu^{ext} + F_\mu^{rr}\,, \qquad F_\mu^{ext}(s) = F_{\mu\nu}^{ext}(x(s))\dot{x}^\nu(s)\,,$$

$$F_\mu^{rr}(s) = F_{\mu\nu}^{LW}(x(s))\dot{x}^\nu(s) = 4q^2 \int\limits_{-\infty}^{s} \delta'(-X^2(s,\tau))\dot{x}_{[\mu}(\tau)X_{\nu]}(s,\tau)\dot{x}^\nu(s)d\tau\,, \tag{17}$$

$$X^\mu(s,\tau) \equiv x^\mu(s) - x^\mu(\tau)\,,$$

where $F_{\mu\nu}^{ext}$ is the strength of the external electromagnetic field and the square brackets stand for the antisymmetrization of indices (without one-half). The abbreviation "rr" labeling the self-action part of the Lorentz force points on the common interpretation of this term as that describing the *radiation reaction*, i.e. the back reaction of the radiation emitted by an accelerating charge.

Note that expression (17) is not very meaningful as it stands since the integral diverges. The divergence arise from the singularity of Green's function (16) at the vertex of the future light cone $x^2 = 0$, $x^0 > 0$. So we need some regularization procedure smoothing the behaviour of Green's function $G(x)$ at $x = 0$. The most simple and efficient way to do this is to replace the δ-function entering the expression for G by the following δ-shaped sequence [4]:

$$\delta(s) = \lim_{a\to+0}\frac{e^{-s/a}}{a}\,, \quad s \geq 0\,. \tag{18}$$

Integrating this expression with a test function on the positive half-line one can obtain the following asymptotic expansion:

$$\delta_a(s) = \frac{e^{-s/a}}{a} = \sum_{n=0}^{\infty}(-a)^n\delta^{(n)}(s)\,. \tag{19}$$

Thus, the regularized expression for the radiation reaction force (17) can be written as

$$F_\mu^{rr}(s,a) = -4q^2\dot{x}^\nu(s)\frac{\partial}{\partial a}\int\limits_0^\infty \frac{d\tau}{a}\, e^{X^2(s,s-\tau)/a}X_{[\nu}(s,s-\tau)\dot{x}_{\mu]}(s-\tau)\,, \tag{20}$$

$$F_\mu^{rr}(s) = \lim_{a\to+0} F_\mu^{rr}(s,a)\,.$$

This integral is localized at the point $\tau=0$ as $a\to+0$, and hence, can be evaluated by the Laplace method [9]. In a small vicinity of the localization point the damping exponential factor behaves as

$$e^{X^2(s,s-\tau)/a} = \exp\left(\dot{x}^2\frac{\tau^2}{a} - \dot{x}\cdot\ddot{x}\frac{\tau^3}{a} + \left[\frac{1}{4}\ddot{x}^2 + \frac{1}{3}\dot{x}\cdot\dddot{x}\right]\frac{\tau^4}{a} + O(\tau^5)\right)\,. \tag{21}$$

Note that for the massive particle $\dot{x}^2<0$ and the asymptotic of the Laplace integral (20) is governed by the leading τ^2-term, while for the massless particle it is determined by τ^4-term. In the latter case

$$\dot{x}^2 = \dot{x}\cdot\ddot{x} = 0\,, \qquad \frac{1}{4}\ddot{x}^2 + \frac{1}{3}\dot{x}\cdot\dddot{x} = -\frac{1}{12}\ddot{x}^2\,, \tag{22}$$

and the coefficient at τ^4 is strongly negative if we demand $\ddot{x}^2=1$. In such a way we recover the positive-curvature condition for the isotropic world lines discussed in the previous section. In the massless case the result of integration should have a form of Laurent's series in $a^{1/4}$ with a finite irregular part. The details of calculations are given in Appendix. The result is

$$\begin{gathered} F_\mu^{rr}(s,a) = -q^2\left[\frac{\alpha_1}{4}\ddot{x}_\mu a^{-3/4} + \frac{\alpha_2}{10}\dddot{x}^2\dot{x}_\mu a^{-1/2} - \frac{\alpha_3}{16}\left(\overset{(4)}{x}_\mu + \frac{11}{10}\dddot{x}^2\ddot{x}_\mu + \frac{11}{5}\dddot{x}\cdot\overset{(4)}{x}\dot{x}_\mu\right)a^{-1/4}\right] - \\ -q^2\frac{2}{5}\left\{\overset{(5)}{x}_\mu + \dddot{x}^2\dddot{x}_\mu + 3\dddot{x}\cdot\overset{(4)}{x}\ddot{x}_\mu + \left(\frac{9}{7}\overset{(4)}{x}{}^2 + \frac{11}{7}\dddot{x}\cdot\overset{(5)}{x} + \frac{2}{5}(\dddot{x}^2)^2\right)\dot{x}_\mu\right\} + \cdots\,, \end{gathered} \tag{23}$$

$$\alpha_n = 12^{n/4}\Gamma\left(1+n/4\right)\,,$$

where dots stand for terms vanishing upon removing regularization. The expression (23) contains three singular terms, two of which are Lagrangian, namely, those at $-3/4$ and $-1/4$ powers of a, while the rest is not. For example, $a^{-3/4}$-term is obtained by varying the following functional:

$$q^2\frac{\alpha_1}{8}a^{-\frac{3}{4}}\int\frac{\dot{x}^2}{\sqrt[4]{\ddot{x}^2}}d\tau\,. \tag{24}$$

As to $a^{-1/4}$-term, the corresponding Lagrangian is given by a rather unwieldy expression. In order to simplify it let us redefine the Lagrangian multiplier e entering the l.h.s. of the Lorentz equation (5) as follows:

$$e \to e - q^2\frac{11}{160}\alpha_3\dddot{x}^2 a^{-1/4}\,. \tag{25}$$

Then the expression for $a^{-1/4}$-term reduces to

$$q^2\frac{\alpha_3}{16}\overset{(4)}{x}_\mu\, a^{-1/4}\,, \tag{26}$$

which can be obtained by varying the following functional:

$$q^2\frac{\alpha_3}{8}a^{-1/4}\int\sqrt[4]{\ddot{x}^2}d\tau \tag{27}$$

A practical recipe to restore both the Lagrangians is as follows. We exclude the electromagnetic field $A_\mu(x)$ from the action functional (1) by the Maxwell equations (6). The result is the Fokker type [10] effective action for the charged massless particle

$$\begin{aligned} S_{eff} &= -\frac{1}{2}\int d\tau e\dot{x}^2 - 2\pi\int d^4x\int d^4y j^\mu(x)G^{ret}(x-y)j_\mu(y) = \\ &= -\frac{1}{2}\int d\tau e\dot{x}^2 + \frac{q^2}{2}\int ds\int d\tau\ \dot{x}^\mu(s)\delta(X^2(s,\tau))\dot{x}_\mu(\tau)\,. \end{aligned} \tag{28}$$

The first term is the usual action of a free massless particle, while the second term describes particle's self-action. Using regularization (18) for the δ-function entering the self-action term we may perform one of two integrations. The calculations are quite similar to those presented in Appendix. The only distinction is that now we cannot use the isotropy condition (4). Instead, we rewrite the exponent (21) in the following way:

$$\exp\left(\left[\frac{1}{4}\ddot{x}^2 + \frac{1}{3}\dot{x}\cdot\dddot{x}\right]\frac{\tau^4}{a}\right)\exp\left(\dot{x}^2\frac{\tau^2}{a} - \dot{x}\cdot\ddot{x}\frac{\tau^3}{a} + O(\tau^5)\right)\,. \tag{29}$$

Then the first multiplier is interpreted as a damping exponential factor of the Laplace integral, while the second is combined with the rest of the integrand and expanded in the Taylor series in τ; in so doing, we may keep only terms which are at most linear in $\dot{x}^2$ and $\dot{x}\cdot\ddot{x}$, since the higher powers will not contribute to the equations of motion $\delta S_{eff}/\delta x = 0$ (more precisely, their contribution will vanish on the shell of the isotropy condition $\dot{x}^2 = \delta S_{eff}/\delta e = 0$). Integrating we get Laurent's series in $a^{1/4}$ without a constant term and with two singular terms giving upon variation two of three foregoing singularities. Clearly, the Lagrangians obtained in such a way are not uniquely determined as one may add to them any local expression proportional to the squares of the Lagrangian constraint $\dot{x}^2$ and its differential consequences.

The reason why performing a self-consistent elimination of the electromagnetic field from the quadratic (in A_μ) action (1) we do not reproduce the whole expression for the radiation reaction force (23) including the finite part and the $a^{-1/2}$-term is as follows: Only symmetric part

$$G^{ret}(x-y) + G^{ret}(y-x) = -\delta((x-y)^2) \tag{30}$$

of the retarded Green's function (16) actually contributes to the effective action (28), while substituting the Liénard-Wiechert potentials to the Lorentz equation (5) we use, in fact, the entire Green's function. The symmetric and antisymmetric parts of the Green function lead to the different groups of terms in (23); the first terms are invariant under reversion of particle's trajectory $\tau \to -\tau$, whereas the second ones acquire the minus sign. Therefore only the invariant terms can arise from the effective action (28).

According to general prescriptions of the renormalization theory the singular coefficients $a^{-3/4}$, $a^{-1/2}$, $a^{-1/4}$ in the regularized expression (23) are replaced by finite constants to be fixed from an experiment. Since there is no evidence for a preferable choice for these constants we may put them to zero to simplify further consideration. Then the radiation reaction force F^{rr}_{μ} is given by the second line in Eq.(23). The non-Lagrangian nature of this force manifests itself in the violation of the time-reversion symmetry, $\tau \to -\tau$, which is consistent with irreversibility of the radiation process.

In conclusion of this section let us note that although the expression (23) for the radiation reaction force was derived in a special gauge, it is not hard to rewrite it in an arbitrary parametrization. For this end one should just treat the overdot derivatives as invariant ones,

$$D = \frac{1}{\sqrt[4]{\ddot{x}^2}}\frac{d}{d\tau}\,, \tag{31}$$

where τ is already an arbitrary evolution parameter. Note that in an arbitrary parametrization the effective equations of motion for a charged massless particle involve a sixth derivative (for comparison, the Lorentz-Dirac equation for d=4 massive particle is of the third order).

4 Reduction of order [2]

Let us write down the effective equation of motion governing the dynamics of a massless charged particle,

$$\begin{aligned} D(\tilde{e}Dx^\mu) = \Omega^\mu_\nu Dx^\nu - \frac{2q}{5}\Big(D^5x^\mu + (D^3x)^2D^3x^\mu + 3D^3x\cdot D^4xD^2x^\mu +\\ + \left[\frac{9}{7}(D^4x)^2 + \frac{11}{7}D^3x\cdot D^5x + \frac{2}{5}((D^3x)^2)^2\right]Dx^\mu\Big)\,, \qquad (Dx)^2 = 0\,, \end{aligned} \tag{32}$$

where $\tilde{e}(\tau) = q^{-1}e(\tau)\sqrt[4]{\ddot{x}^2}$ is a nonzero scalar function and $\Omega^\mu_\nu = \eta^{\mu\lambda}F^{ext}_{\lambda\nu}(x)$.

Since the order of the equation is greater than two, it cannot be assigned with a straightforward mechanical interpretation: In the realm of Newtonian mechanics a state of the particle is unambiguously determined by specifying its position and velocity, that is obviously insufficient to extract a particular solution to our equation. This problem is analogous to that of a mechanical interpretation of the Lorentz-Dirac equation which also has too many solutions, but not all of them are physically meaningful. To extract the subspace of physical solutions we impose an additional selection rule [11]: The physical solutions to the Eq. (32) are only those which have a smooth limit upon switching off the interaction, i.e. when $q \to 0$. In other words, we treat the radiation reaction as a small perturbation deforming the conventional dynamics of the charged particle (i.e. the dynamics without account of the self-action) rather than something introducing extra degrees of freedom to the theory. This treatment is justified by a small parameter q at the higher-derivative terms in the

[2]We are thankful to S.L. Lyakhovich for illuminating discussions on various aspects of the reduction-of-order procedure for higher-derivative Lagrangian systems.

r.h.s. of the Eq.(32). Instead of seeking for solutions which can be smoothly continued to $q = 0$, one may equivalently seek for a second-order differential equation (with a smooth dependence of q) any solution of which would be a solution to the initial equation. The general procedure for obtaining such an equation is known as the *reduction of order*. It consists in successive elimination of the higher derivatives from the equation (32) with the help of all its differential consequences; in so doing, the order of the equation increases at each step of the procedure, but the order of q at higher derivatives increases as well, so that, in the limit, we get a power series in q with coefficients depending on x, Dx and D^2x. The implementation of this procedure to the Lorentz-Dirac equation may be found, for example, in [6, 7, 8]. It should be noted, however, that the resulting equation for the massless particle appears to be unresolved with respect to the highest (second) derivative that poses the questions about existence and uniqueness of its solution given the initial position and velocity of the particle. Without going into detail we just note that for a homogeneous external field ($\Omega = const$) the reduced equation can be perturbatively resolved with respect to D^2x with the right hand side given by a power series in q. Up to the first order in q it reads [3]

$$\ddot{x}^\mu = -\tilde{e}^{-1}(\Omega\dot{x})^\mu +$$
$$+\frac{2q}{5}\left(\frac{(\Omega^4\dot{x})^\mu}{\tilde{e}^5} + \frac{\dot{x}\,\Omega^4\dot{x}}{\tilde{e}^7}(\Omega^2\dot{x})^\mu + \frac{1}{2}\left[\frac{9}{7}\frac{\dot{x}\,\Omega^6\dot{x}}{\tilde{e}^7} + \frac{7}{5}\frac{(\dot{x}\,\Omega^4\dot{x})^2}{\tilde{e}^9}\right]\dot{x}^\mu\right) + O(q^2)\,, \tag{33}$$

where overdot stands for the invariant derivative D and the function $\tilde{e}$, being determined from the identity $\ddot{x}^2 = 1$, is given by

$$\tilde{e}^2 = -\dot{x}\Omega^2\dot{x} + O(q^2)\,. \tag{34}$$

As above we are restricted to initial velocities obeying condition $\tilde{e} \neq 0$ (compare with (10)) which ensures existence of a solution for a sufficiently small time interval. In the gauge $e = 1$ the above equation takes the form

$$\ddot{x}^\mu = q(\Omega\dot{x})^\mu +$$
$$+\frac{2q^4}{5}\left(\frac{(\Omega^4\dot{x})^\mu}{\dot{x}\Omega^2\dot{x}} - \frac{\dot{x}\,\Omega^4\dot{x}}{(\dot{x}\Omega^2\dot{x})^2}(\Omega^2\dot{x})^\mu - \frac{1}{2}\left[\frac{9}{7}\frac{\dot{x}\,\Omega^6\dot{x}}{(\dot{x}\Omega^2\dot{x})^2} - \frac{7}{5}\frac{(\dot{x}\,\Omega^4\dot{x})^2}{(\dot{x}\Omega^2\dot{x})^3}\right]\dot{x}^\mu\right) + o(q^4)\,, \tag{35}$$

where overdot denotes now the ordinary derivative w.r.t. the evolution parameter. Note that although the condition $(\dot{x}\Omega^2\dot{x}) < 0$ has been assumed at each stage of our derivation the resulting equation has the correct *uniform* limit upon switching off the external electromagnetic field (i.e. if we set $\Omega = \varepsilon\Omega'$ and then let $\varepsilon \to 0$). The last statement remains true even with account of higher orders in q, so the limiting equation $\ddot{x}^\mu = 0$ describes the free motion.

Thus, the perturbative treatment of the radiation reaction, being applied to the case at hands, leads to the meaningful second order equation at least for the constant fields.

[3]Here we use matrix notation: $(\Omega\dot{x})^\mu = \Omega^\mu_\nu\dot{x}^\nu$, $\dot{x}\Omega^2\dot{x} = \dot{x}_\nu\Omega^\nu_\sigma\Omega^\sigma_\mu\dot{x}^\mu$, and so on.

5 Conclusion

To summaries, in this paper we have derived the effective equations of motion for a massless charged particle interacting with external electromagnetic field as well as its own one. Although the approach we follow here is quite similar to that used in the case of a massive particle [4] the final results considerably differ from each other. First, not all the divergencies are Lagrangian, i.e. may be canceled out by adding appropriate counterterms to the original action functional; even though such counterterms do exist (for some divergences) they are not uniquely defined. The ambiguity reflects the freedom to add to the Lagrangian any local expression proportional to the square of the isotropy condition and its differential consequences. The non-Lagrangian divergences in turn are recognized as those breaking up the time-reversion symmetry. Second, consistency of the dynamics imposes some restrictions on initial data of the particle and/or the form of external fields. Third, after renormalization we get a fifth order differential equation, instead of third-order equation derived by Dirac for a massive particle.

All these distinctions have a common origin related to the special local structure of the isotropic world lines: The interval between two nearby points $x(\tau)$ and $x(\tau+\delta\tau)$ on an isotropic curve, lying in a general position, is proportional to $\delta\tau^2$, rather than to be linear in $\delta\tau$ as is case for the time-like world lines of massive particles. This leads to the stronger singularities, as compared to the massive case, and as result to a different structure of the radiation reaction force.

Finally, the straightforward application of the reduction-of-order procedure leads to the differential equation in a form unresolved with respect to the second derivative which may lead to some difficulties upon its integration. Besides, it may have no well-defined limit upon switching off an external electromagnetic field as seen from the structure of the first radiative correction (35). The last fact may indicate a certain instability of the classical dynamics and deserve a further study.

It would be interesting to re-derive our equation on the basis of usual energy conservation arguments in order to gain a more physical insight into the problem as well as an independent test for consistency of the formal renormalization procedure we have used.

Appendix

The regularized expression for the self-action force (20) involves the following integral:

$$I_\mu(s,a) = \dot{x}^\nu(s)\frac{\partial}{\partial a}\int\limits_0^\infty \frac{d\tau}{a}\, X_{[\nu}(s,s-\tau)\dot{x}_{\mu]}(s-\tau)e^{X^2(s,s-\tau)/a}\,, \tag{36}$$

Here we apply the Laplace method to obtain the Laurent series for the function I_μ with respect to $a^{1/4}$. More precisely, we are interested only in constant and irregular parts of this series (contributing, respectively, to the finite and the divergent parts the self-action force) since the regular terms vanish upon switching off the regularization.

In order to simplify formulae we use standard notation from the linear algebra for the inner and the exterior (skew-symmetric) products of two vectors: $a\cdot b = a^\mu b_\mu$, $a\wedge b = (a_\mu b_\nu - a_\nu b_\mu)$. Thus, the pre-exponential factor in the integral (36) may be written as

$$\dot{x}(s-\tau)\wedge X(s,s-\tau)=\sum_{m=2}^{\infty}\tau^m b_m(s)\,, \tag{37}$$

where

$$b_m=(-1)^m\sum_{n=1}^{m}\frac{\overset{(n)}{x}\wedge\overset{(m-n+1)}{x}}{n!(m-n)!}\,.$$

In particular,

$$b_2=-\frac{1}{2}\overset{(2)}{x}\wedge\overset{(1)}{x}\,,\qquad b_3=\frac{1}{3}\overset{(3)}{x}\wedge\overset{(1)}{x}\,,$$

$$b_4=-\frac{1}{8}\overset{(4)}{x}\wedge\overset{(1)}{x}-\frac{1}{12}\overset{(3)}{x}\wedge\overset{(2)}{x}\,,\qquad b_5=\frac{1}{30}\overset{(5)}{x}\wedge\overset{(1)}{x}+\frac{1}{24}\overset{(4)}{x}\wedge\overset{(2)}{x}\,,$$

$$b_6=-\frac{1}{144}\overset{(6)}{x}\wedge\overset{(1)}{x}-\frac{1}{80}\overset{(5)}{x}\wedge\overset{(2)}{x}-\frac{1}{144}\overset{(4)}{x}\wedge\overset{(3)}{x}\,,\quad b_7=\frac{1}{840}\overset{(7)}{x}\wedge\overset{(1)}{x}+\frac{1}{360}\overset{(6)}{x}\wedge\overset{(2)}{x}+\frac{1}{360}\overset{(5)}{x}\wedge\overset{(3)}{x}\,.$$

Using the basic identities

$$\dot{x}^2=0\,,\qquad \ddot{x}^2=1$$

as well as their differential consequences

$$\ddot{x}\cdot\dot{x}=0\,,\qquad \dddot{x}\cdot\ddot{x}=0\,,\qquad \dddot{x}\cdot\dot{x}=-1\,,\qquad \overset{(4)}{x}\cdot\dot{x}=0\,,$$

$$\overset{(4)}{x}\cdot\overset{(2)}{x}+\overset{(3)}{x}{}^2=\overset{(5)}{x}\cdot\overset{(1)}{x}-\overset{(3)}{x}{}^2=\overset{(5)}{x}\cdot\overset{(2)}{x}+3\overset{(4)}{x}\cdot\overset{(3)}{x}=\overset{(6)}{x}\cdot\overset{(1)}{x}-5\overset{(4)}{x}\cdot\overset{(3)}{x}=0\,,$$

the argument of the exponent (36) can be written as

$$X^2(s,s-\tau)=(x(s-\tau)-x(s))^2=\sum_{k=4}^{\infty}c_k\tau^k=$$

$$=-\frac{1}{12}\overset{(2)}{x}\cdot\overset{(2)}{x}\,\tau^4+\frac{1}{360}\overset{(3)}{x}\cdot\overset{(3)}{x}\,\tau^6- \tag{38}$$

$$-\frac{1}{360}\overset{(3)}{x}\cdot\overset{(4)}{x}\,\tau^7+\left(\frac{1}{1260}\overset{(3)}{x}\cdot\overset{(5)}{x}+\frac{1}{1344}\overset{(4)}{x}\cdot\overset{(4)}{x}\right)\tau^8+o(\tau^8)\,,$$

where

$$c_k=(-1)^k\sum_{n=1}^{k-1}\frac{\overset{(n)}{x}\cdot\overset{(k-n)}{x}}{n!(k-n)!}\,.$$

Substituting (37) and (38) to the integral (36) and making replacement $t\to a^{1/4}t$ we find that

$$I_\mu(a)=\dot{x}^\nu\frac{\partial}{\partial a}\int_0^\infty d\tau\left(\sum_{m=3}^{7}a^{(m-3)/4}\tau^m(b_m)_{\nu\mu}\right)\exp\left(\sum_{n=4}^{8}a^{(n-4)/4}\tau^n c_n\right)+(\text{regular terms}) \tag{39}$$

The expansion for the exponent reads

$$e^{-\tau^4/12}(1 + a^{1/2}\tau^6 c_6 + a^{3/4}\tau^7 c_7 + a[\tau^8 c_8 + \tau^{12} c_6^2/2]) + o(a)\,. \tag{40}$$

Multiplying this expression on the pre-exponential factor

$$\sum_{m=3}^{7} a^{\frac{m-3}{4}} \tau^m (b_m) = \tau^3 b_3 + a^{1/4}\tau^4 b_4 + a^{1/2}\tau^5 b_5 + a^{3/4}\tau^6 b_6 + a\tau^7 b_7$$

and contracting with $\dot{x}^\nu$ we get

$$\begin{aligned}
&e^{-\tau^4/12}\dot{x}^\nu(\tau^3 b_3 + \\
&+ a^{1/4}\tau^4 b_4 + \\
&+ a^{1/2}[\tau^5 b_5 + b_3 c_6 \tau^9] + \\
&+ a^{3/4}[\tau^6 b_6 + (c_6 b_4 + c_7 b_3)\tau^{10}] + \\
&+ a[\tau^7 b_7 + (c_6 b_5 + c_7 b_4 + c_8 b_3)\tau^{11} + \tfrac{1}{2} c_6^2 b_3 \tau^{15}])_{\nu\mu} + o(a)\,.
\end{aligned}$$

Substituting the last expression into (36) and using the integration formula

$$\int_0^\infty e^{-\tau^4/12}\tau^n d\tau = \frac{12^{\frac{n+1}{4}}}{4}\Gamma\left(\frac{n+1}{4}\right) \tag{41}$$

we finally arrive at (23). It is interesting to note that the resulting expression for $F^{rr}_\mu(s, a)$ turns out to be orthogonal to both $\dot{x}^\mu$ and $\ddot{x}^\mu$.

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References

[1] P.A.M. Dirac, *"Classical Theory of Radiating Electrons"*, Proc. Roy. Soc. London **A167** (1938) 148-169.

[2] H.A. Lorentz, Theory of Electron (B.G. Teubner, Leipzig, 1906; reprinted by Dover Publications, New York, 1952).

[3] B. P. Kosyakov, Teor. Mat. Fiz. V. 119, N. 1 (1999) 119-136 (in russian); Theor. Math. Phys. **199** (1999) 493-505; hep-th/0207217.

[4] P.O. Kazinski, S.L. Lyakhovich and A.A. Sharapov, *Radiation reaction and renormalization in classical electrodynamics of point particle in any dimension*, Phys.Rev., **D66** (2002) 025017.

[5] C. Teitelboim, D. Villarroel and C.C. van Weert, Riv. Nuovo Cimento **3**, (1980) 9.

[6] L.D. Landau and E.M. Lifshitz, The classical theory of fields (Pergammon, Oxford, 1962).

[7] E. Poisson, *An introduction to the Lorentz-Dirac equation*, gr-qc/9912045.

[8] K. T. McDonald, *Limits on the Applicability of Classical Electromagnetic Fields as Inferred from the Radiation Reaction*, physics/0003062.

[9] V. Guillemin and S. Sternberg, Geometric asymptotics, Mathematical surveys, Vol. 14. (Amer. Math. Soc., Providence, Rhode Island, 1977).

[10] A.D. Fokker, Zeits. f. Physik **58** (1929) 386; Physica **9** (1929) 33; Physica **12** (1932) 145.

[11] H.J. Bhabha, Phys.Rev. **70** (1964) 759.

[12] F.R. Harvey and H.B. Lawson Jr., A theory of characteristic currents associated with a singular connections, Asterisque 213 (1993).

[13] K. Lechner and P.A. Marchetti, *"Dirac Branes, Characteristic Currents and Anomaly Cancellations in 5-Branes"*, Nucl.Phys. B (Proc. Suppl.) 102&103 (2001) 94-99.

[14] K. Lechner, P.A. Marchetti and M. Tonin, *"Anomaly free effective action for the elementary M5-brane"*, Phys.Lett. **B524** (2001) 199-207.