

Aspects of noncommutative descriptions of planar systems in high magnetic fields.

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April 25, 2020

Abstract

We study some aspects of recent proposals to use the noncommutative Chern-Simons theory as an effective description of some planar condensed matter models in strong magnetic fields, such as the Quantum Hall Effect. We present an alternative justification for such a description, which may be extended to other planar systems where a uniform magnetic field is present.

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1 Introduction

Noncommutative field theories have recently attracted renewed attention, mostly because of their relevance for the understanding of some phenomena in the context of string theory, like the low energy limit of open strings in the presence of some special background field configurations [1, 2].

In the condensed matter physics context, non-commutative Chern-Simons (NCCS) theories have recently been proposed as effective descriptions of the Laughlin states in the Quantum Hall Effect [3, 4, 5]. Noncommutative field theories have also been used to describe the skyrmionic excitations of the Quantum Hall ferromagnet at $\nu = 1$ [6, 7].

The physics of a bidimensional system of particles in the presence of an external magnetic field has a very rich structure, a phenomenon which is partly due to the particularities of the Landau level spectrum for a particle in an external field. In particular, it is a well known fact that when the system is restricted to the lowest Landau level (LLL), area preserving diffeomorphisms become a symmetry of the system [8, 9, 10]. The restriction to the LLL is usually invoked as a consequence of the existence of a large gap between the lowest and higher Landau Levels [11]. However, this restriction cannot be defined as a smooth limit of the full (all level) system, since there is a change in the number of physical degrees of freedom, an effect that has been known since the early studies on Chern-Simons quantum mechanics [12], and entirely analogous to a similar reduction from the Maxwell-Chern-Simons theory into the pure Chern-Simons one [8, 12]. The change in the number of degrees of freedom means that one of the physical variables (a coordinate, for the particle) is transformed into the canonical momenta of the remaining variable. Thus the usual invariance under canonical transformations acquires a much greater relevance, since it becomes a spacetime symmetry. As canonical transformations preserve the phase space volume, the symmetry of the reduced system can be analogously thought of as invariance under area preserving diffeomorphisms.

The quantum version of these symmetry transformations necessarily has to cope with operator ordering problems, since they involve canonical conjugate variables that do not commute in the quantum theory. In the operatorial (canonical) quantization method the use of the Weyl quantization prescription, is the natural way to introduce the Moyal product for phase space functions [1]. Of course, the same phenomenon can be studied in the path integral framework, for example by means of the mid-point prescription [13] to define the matrix elements of Weyl-ordered products, when they appear inside the path integral.

In this work we consider systems described by an action with the general

structure:

$$S = S_m + S_g + S_{int} , \quad (1)$$

where S_m denotes the free action for a system of particles (either in its first or second quantized representations), S_g is the action for a vector (gauge) field A_μ , and S_{int} corresponds to the coupling between the particles and the gauge field. A distinctive feature of the systems we will analyze is that the free action S_m will be negligible for the dynamics, due to the presence of a strong external magnetic field (defined as part of S_{int}). This is usually stated as the ‘freezing’ of the kinetic energy, and it is a fundamental requisite for the emergence of a noncommutative description. We shall argue that the noncommutativity is, for the kind of systems we are considering, a property of the *description* used rather than a fundamental symmetry. For the noncommutative theory corresponding to a system in the presence of an external magnetic field there is, as we shall see, also a freedom in the choice of the deformation parameter. A variation in this parameter may be compensated by the introduction of a constant noncommutative magnetic field. The usefulness of the noncommutative description will be that it might simplify the treatment of problems that are difficult to deal with in the usual commutative setting.

The organization of this paper is as follows: In section 2 we review some properties of a planar system of particles coupled to a strong magnetic field. In particular, we discuss the emergence of area preserving diffeomorphisms as symmetry transformations, and the necessity of the introduction of a noncommutative geometry if a consistent representation of the algebra of classical symmetries is required. In particular, we discuss how the quantum version of those symmetries imply the noncommutativity of the gauge transformations.

In section 3, the noncommutative description is introduced as a tool to change the part of the gauge field dynamics compatible with those symmetries. Finally, it is argued that a noncommutative theory may be used as an effective description of the Quantum Hall Effect, alternative to the usual CS commutative approach.

Some technical aspects of the path integral version of the Moyal product, which are recalled in the main part of the article are presented in Appendix A. Also, the apparently different way to introduce the NCCS theory, based in the incompressible fluid picture is discussed in Appendix B.

2 Matter current coupled to an external field

In order to study the symmetries of the full system, as defined by S in equation (1), it is useful to begin with the simpler case of a conserved matter

current coupled only to an *external* gauge field. The latter is assumed to correspond to a strong uniform magnetic field \mathbf{B} , whose strength is supposed to be large when compared with the interactions, in such a way that the dynamics can be safely restricted to the lowest Landau level. This assumption will be crucial in all our subsequent developments.

2.1 Symmetries in the Lowest Landau level

In a first quantized description, the form of the interaction term in the action, S_{int} , is supposed to be of the minimal type:

$$S_{int} = \int d^3x A_\mu(x) j^\mu(x), \quad (2)$$

where A_μ denotes the gauge field corresponding to the purely external (i.e., non dynamical) magnetic field. Later on we shall also include a fluctuating part a_μ , so that in (2) we will make the replacement: $A_\mu \rightarrow A_\mu + a_\mu$. The most important part of the gauge field, determining the spectrum of the theory will be assumed to be A_μ , while a_μ is, from the time being, assumed to be perturbative in character.

To fix the ambiguity in the gauge field configuration corresponding to the constant magnetic field \mathbf{B} , we will adopt the Weyl gauge ($A_0 = 0$), and a symmetric gauge choice for A_j :

$$A_j(\vec{x}) = -\frac{1}{2} B \epsilon_{jk} x^k. \quad (3)$$

On the other hand, (2) also involves the spatial part of the matter field current, which for a system of N particles may be written as:

$$j^k(x^0, \vec{x}) = e \int dt \sum_{a=1}^N \frac{dx_a^k(t)}{dt} \delta(x^0 - t) \delta^{(2)}(\vec{x} - \vec{x}_a(t)) \quad (4)$$

where $t \rightarrow \vec{x}_a(t), a = 1, \dots, N$ defines the particles' trajectories. Then,

$$S_{int} = e \sum_{a=1}^N \int dt A_k(\vec{x}_a(t)) \frac{dx_a^k(t)}{dt} = \frac{b}{2} \int dt \sum_{a=1}^N \dot{x}_a^j(t) \epsilon_{jk} x_a^k(t) \quad (5)$$

where $b \equiv eB$, and $k, j = 1, 2$. Using the first-order action to define the canonical momenta (or taking into account the second-class constraints that follow from this first-order action) one sees that the Poisson (Dirac) brackets are:

$$\{x_a^j, x_b^k\} = \theta \delta_{ab} \epsilon^{jk}, \quad \theta = -b^{-1}, \quad (6)$$

while for arbitrary functions f, g of the coordinates one has

$$\{f, g\} = \theta \sum_{a=1}^N \frac{\partial f}{\partial x_a^j} \epsilon^{jk} \frac{\partial g}{\partial x_a^k}. \quad (7)$$

Since the theory is invariant under reparametrizations, the canonical Hamiltonian vanishes, and the theory is then also invariant under the full group of ‘canonical’ transformations, namely, transformations that leave the bracket (6) invariant. The infinitesimal version of these transformations may be written as:

$$\delta_\Lambda x^i = \eta \{x^i, \Lambda(x)\} \quad (8)$$

where η is an infinitesimal constant, and $\Lambda(x)$ is an arbitrary function of the particles’ coordinates \vec{x}_a . These transformations, and their finite counterparts, are symmetries of the classical action¹. Alternatively, they may be interpreted as time independent gauge transformations of the gauge field (the remaining gauge freedom in the $A_0 = 0$ gauge). Indeed, under a standard (time independent) gauge transformation, the Lagrangian changes in a total time derivative:

$$\delta_\Lambda A_j(x) = \partial_j \Lambda(x) \Rightarrow \delta_\Lambda S_{int} = e \int dt \frac{d}{dt} \sum_{a=1}^N \Lambda(\vec{x}_a(t)), \quad (9)$$

implying that Λ is the infinitesimal generator of the canonical transformations (8) of the coordinates.

To understand the quantum realization of these symmetries, in the canonical quantization approach, one imposes the fundamental commutator

$$[\hat{x}_a^j, \hat{x}_b^k] = i \hbar \theta \delta_{ab} \epsilon^{jk}, \quad (10)$$

where we have written \hbar explicitly, in order to trace the quantum effects.

To the classical canonical transformations there correspond the quantum counterparts:

$$\hat{x}^i \rightarrow \hat{x}_U^i = \hat{U}^\dagger \hat{x}_i \hat{U}, \quad (11)$$

where \hat{U} is an arbitrary unitary operator. When the transformation defined by \hat{U} is connected to the identity, the infinitesimal version of (11) is, of course,

$$\hat{x}^i \rightarrow \hat{x}^i + \delta_\Lambda \hat{x}^i, \quad \delta_\Lambda \hat{x}^i = \eta [\hat{x}^i, \hat{\Lambda}]. \quad (12)$$

To represent the classical symmetry generator by a quantum operator, one must adopt an ordering prescription. The product of operators thus ordered

¹This is of course valid only if we ignore S_m .

is not, however, compatible with the standard classical composition rule for the product of functions. This is a well known fact, usually presented in the context of the Weyl ordering prescription [14]. Therefore, the noncommutativity in the *classical* theory arises as a consequence of the usual operator ordering problems of quantum mechanics, if one wants the mapping between classical and quantum transformations to be consistently defined.

One way to see this, is to write the Weyl-ordered operator $\hat{\mathcal{O}}(f)$ associated to a classical function f of the coordinates in a ‘Fourier’ representation:

$$\hat{\mathcal{O}}(f) = \int \left[\prod_{a=1}^N \frac{dp_1^a dp_2^a}{2\pi\hbar} \right] \tilde{f}(p) \exp\left[\frac{i}{\hbar} \sum_{a=1}^N (p_1^a \hat{x}_a^1 + p_2^a \hat{x}_a^2)\right] \quad (13)$$

where

$$\tilde{f}(p) = \int \left[\prod_{a=1}^N \frac{dx_a^1 dx_a^2}{2\pi\hbar} \right] f(x) \exp\left[-\frac{i}{\hbar} \sum_{a=1}^N (p_1^a x_a^1 + p_2^a x_a^2)\right] \quad (14)$$

is the Fourier transform of $f(x)$. Recalling that the Weyl-order of a product of operators is defined as the sum over all permutations of the operators, it follows that (13) is Weyl-ordered. Notice that the exponentials are Weyl-ordered (as can be checked by using their series expansions), and that a linear combination of Weyl-ordered operators is also Weyl-ordered. Thus, expression (13) may be thought of as a convenient way to unambiguously assign an operator $\hat{\mathcal{O}}(f)$ to a given classical function of the coordinates, f . As already advanced, by bringing the product of two Weyl-ordered operators back to Weyl-order, one sees that the resulting operator is not the one corresponding to the usual, commutative, product of the two classical functions, but rather to:

$$\hat{\mathcal{O}}(f) \hat{\mathcal{O}}(g) = \hat{\mathcal{O}}(f \star g) \quad (15)$$

where \star denotes the Moyal product:

$$f(x) \star g(x) = \exp\left(\frac{i}{2}\hbar \theta \epsilon^{jk} \frac{\partial}{\partial \eta_j} \frac{\partial}{\partial \xi_k}\right) f(x + \eta) g(x + \xi) \Big|_{\eta \rightarrow 0, \xi \rightarrow 0}. \quad (16)$$

For infinitesimal transformations, we may use the expansion: $\tilde{f}(p) = 1 + \eta \tilde{\Lambda}(p) + \mathcal{O}(\eta^2)$, so that unitary operators may also be expanded as

$$\hat{U}(f) = \hat{I} + \eta \hat{T}_\Lambda + \mathcal{O}(\eta^2) \quad (17)$$

where

$$\hat{T}_\Lambda = \int \left[\prod_{a=1}^N \frac{dp_1^a dp_2^a}{2\pi\hbar} \right] \tilde{\Lambda}(p) \exp\left[\frac{i}{\hbar} \sum_{a=1}^N (p_1^a \hat{x}_a^1 + p_2^a \hat{x}_a^2)\right]. \quad (18)$$

Thus, for the composition of two infinitesimal transformations, the change in \hat{x}^j to the first order in each of the respective infinitesimal parameters $\eta_{\Lambda,2}$ shall be given by

$$\delta \hat{x}^j = \eta_1 \eta_2 [\hat{x}^j, [\hat{T}_{\Lambda_1}, \hat{T}_{\Lambda_2}]] . \quad (19)$$

By (15), we see that

$$[\hat{T}_{\Lambda_1}, \hat{T}_{\Lambda_2}] = \hat{T}_{\Lambda_1 \star \Lambda_2 - \Lambda_2 \star \Lambda_1} . \quad (20)$$

This shows that, in order to have a consistent unitary representation of the symmetry transformations, the Moyal commutator should replace the Poisson bracket in the classical theory. In particular, the infinitesimal transformation of the coordinates is now

$$\delta_{\Lambda} x_a^i = \frac{1}{i\hbar} (x_a^i \star \Lambda - \Lambda \star x_a^i) \quad (21)$$

which reduces to the Poisson bracket only in the $\hbar\theta \rightarrow 0$ limit. The combination $\hbar\theta = \frac{\hbar}{eB}$ can be interpreted as the area per particle that results from dividing the total area of the system by the degeneracy of the Landau levels. The dimensionless combination which may be used to give a meaning to the $\hbar\theta \rightarrow 0$ limit is the ratio $\hbar\theta/L^2$, where L is the typical scale of variation of the functions that appear in the Moyal bracket. Therefore, if the functions are smooth on the scale of $\hbar\theta$, the Moyal product is approximately the regular one.

We see then that the usual commutative product between functions of the coordinates is replaced by the Moyal product. The latter appears naturally in the Weyl quantization prescription, and is a simple reflection of the non-commutativity of the spatial coordinates. The existence of a minimal volume in the plane is, in this case, due to the non-vanishing commutation relation between the coordinates, which play the role of conjugate variables (in the canonical sense). The cyclotron length sets the scale of the minimal area for this problem.

Knowing what the transformation rules for the classical x_i functions should be, it is clear that they do not correspond to a standard gauge transformation of the gauge field. As we have already mentioned, a standard (time independent) gauge transformation changes the Lagrangian by a total time derivative, and therefore it is equivalent to a canonical transformation of the coordinates (8). On the other hand, the gauge field variation corresponding to the transformations (21) is:

$$\delta_{\Lambda} A_j(x) = \partial_j \Lambda(x) + \frac{1}{i\hbar} (A_j(x) \star \Lambda(x) - \Lambda(x) \star A_j(x)) , \quad (22)$$

namely, they are $U(1)$ non-commutative gauge transformations. These are the gauge transformations we were looking for, and the gauge field action must, therefore, be constructed using this symmetry as a criterion. It is worth remarking that this results agrees with the somewhat different (but obviously related) approach of [3], if the full noncommutative version of the latter is used.

It is important to realize that the previous discussion on the noncommutativity of the coordinates, and hence the ‘deformation’ of the ordinary product of classical functions into the Moyal product is independent of the gauge choice adopted for A_j . Indeed, had we used a gauge field in a general gauge (subject only to the condition $\partial_1 A_2 - \partial_2 A_1 = B$) in the action S_{int} :

$$S_{int} = e \sum_{a=1}^N \int dt A_k(\vec{x}_a(t)) \frac{dx_a^k(t)}{dt} \quad (23)$$

the canonical Poisson brackets would have been:

$$\{x_a^k, eA_k(x_b)\} = \delta_{ab} \quad (24)$$

(no sum over k). Then, the use of the standard properties of the Poisson bracket:

$$\begin{aligned} \{x_a^k, eA_k(x_b)\} &= e\{x_a^k, x_b^j\} \partial_j A_k(x_b) \\ &= \frac{e}{2} \{x_a^k, x_b^j\} (\partial_j A_k(x_b) - \partial_k A_j(x_b)) = -\frac{1}{2\theta} \epsilon_{jk} \{x_a^k, x_b^j\} \end{aligned} \quad (25)$$

allows us to derive the same bracket as for the symmetric gauge choice, namely,

$$\{x_a^j, x_b^k\} = \theta \delta_{ab} \epsilon^{jk} . \quad (26)$$

2.2 LLL projection and non-commutative description

Let us now turn to the construction of the Hilbert space for the one-particle first quantized system. This step is required to implement the second quantization, since the one-particle states are indeed the building blocks of the Fock space. In canonical quantization, one sees that the theory has, in Dirac’s terminology, two primary second-class constraints χ_1, χ_2 :

$$\chi_1 = \pi_1 - eA_1(x) \approx 0 \quad , \quad \chi_2 = \pi_2 - eA_2(x) \approx 0 . \quad (27)$$

where $\pi_j = -i\partial_j$. Of course, there are many different ways to construct the quantum theory for this system, depending on the way to implement these constraints. We have found it convenient to use an approach which follows

closely the physical situation corresponding to a non-relativistic particle of mass m in the presence of an external magnetic field, when that magnetic field becomes very large. One begins from the observation that \hat{H} , the Hamiltonian for a single particle of mass m in a constant magnetic field \mathbf{B} may be written as

$$\hat{H} = \frac{1}{2m}[(\pi_1 - eA_1)^2 + (\pi_2 - eA_2)^2] = \frac{1}{2m}(\chi_1^2 + \chi_2^2). \quad (28)$$

The constraints χ_1 and χ_2 are equivalent to the two complex combinations: $\chi = (\chi_1 - i\chi_2)/\sqrt{2}$, $\chi^* = (\chi_1 + i\chi_2)/\sqrt{2}$, which in the quantum theory become a pair of mutually adjoint operators:

$$\hat{\chi} = \frac{(\hat{\chi}_1 - i\hat{\chi}_2)}{\sqrt{2}}, \quad \hat{\chi}^\dagger = \frac{(\hat{\chi}_1 + i\hat{\chi}_2)}{\sqrt{2}} \quad (29)$$

verifying the commutation relation:

$$[\hat{\chi}, \hat{\chi}^\dagger] = \frac{\hbar}{\theta}, \quad (30)$$

which is independent of the gauge choice adopted for A_j . These two second class constraints may also be thought of as a pair composed by a first-class constraint (χ , say) plus its gauge fixing ($\hat{\chi}^\dagger$). This allows us to treat the constraints differently, by using an alternative interpretation. For example, one may just use Dirac's method for first class constraints, and demand the physical subspace \mathcal{H}_{phys} of the full Hilbert space \mathcal{H} (i.e., the one constructed out of the unconstrained system) to be annihilated by the first class constraint

$$\hat{\chi} |\psi\rangle = 0 \quad \forall |\psi\rangle \in \mathcal{H}_{phys}. \quad (31)$$

Thus, the definition of the physical Hilbert space can be conveniently defined as a 'reduction' from the one corresponding to the usual Hamiltonian for a particle in an external magnetic field. This treatment of the constraints is of course the most convenient when one is indeed considering a physical situation described by the Hamiltonian \hat{H} , since not only it describes the physical Hilbert space (as a 'vacuum'), but also it allows for the consideration of the possible corrections due to the fact that the reduction is a simplification of the real physical situation. Indeed, while the constrained manifold is defined by (31), corrections due to the kinetic term will be contained in higher states, built upon the 'vacuum' \mathcal{H}_{phys} .

To make this more explicit, one may introduce the operators:

$$\hat{a} = \sqrt{\frac{\theta}{\hbar}} \hat{\chi} \quad \hat{a}^\dagger = \sqrt{\frac{\theta}{\hbar}} \hat{\chi}^\dagger, \quad (32)$$

which verify the standard creation and annihilation algebra,

$$[\hat{a}, \hat{a}^\dagger] = 1, \quad (33)$$

while the Hamiltonian \hat{h} becomes:

$$\hat{h} = \hbar\omega_c \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right), \quad (34)$$

where $\omega_c = -\frac{eB}{m} = \frac{1}{m\theta}$ is the cyclotron frequency. The lowest Landau level of the Hamiltonian \hat{h} is of course annihilated by \hat{a} , and the higher states may be generated by repeated application of \hat{a}^\dagger : $|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle$. All these states are, however, degenerated. To treat this degeneracy one introduces the operators \hat{x}_0^1, \hat{x}_0^2 , which classically correspond to the motion of the center of the trajectory and are usually called guiding center coordinates. They are defined by:

$$\begin{aligned} \hat{x}_0^1 &= \hat{x}^1 - \theta \hat{\chi}^2 \\ \hat{x}_0^2 &= \hat{x}^2 + \theta \hat{\chi}^1, \end{aligned} \quad (35)$$

and verify the commutation relations:

$$[\hat{x}_0^1, \hat{x}_0^2] = i\hbar\theta, \quad (36)$$

rather than the usual commutativity, which holds between \hat{x}^1 and \hat{x}^2 :

$$[\hat{x}^1, \hat{x}^2] = 0. \quad (37)$$

Besides, both \hat{x}_0^1 and \hat{x}_0^2 commute with \hat{a} and \hat{a}^\dagger .

It is clear that the physical Hilbert space is the lowest Landau level of the Hamiltonian \hat{h} . Let us now consider how to define physical operators, also in the Dirac approach. Being this a first-class system, physical operators are to be defined as those that commute with the first class constraints, i.e., they are gauge invariant. Thus what we need now is a procedure to assign a gauge invariant operator to a given classical function of the coordinates. This ‘reduction’ mechanism, and its relation to the Moyal product is now conveniently studied in terms of an arbitrary classical function $f(x)$ of the coordinates, and its corresponding operator $\hat{\mathcal{O}}(f)$. We begin by introducing a correspondence between functions and operators which is valid *before* reducing to the physical subspace, and then make the necessary changes. If $f(x)$ is represented in terms of its Fourier transform in momentum space, $\tilde{f}(p)$:

$$\hat{\mathcal{O}}(f) = \int \frac{dp_1 dp_2}{2\pi\hbar} \tilde{f}(p) \exp\left[\frac{i}{\hbar}(p_1 \hat{x}^1 + p_2 \hat{x}^2)\right] \quad (38)$$

then the product between classical functions is commutative, since (37) implies that there are no ordering problems in the definition of $\hat{\mathcal{O}}(f)$. The noncommutativity arises when writing \hat{x}^i in terms of \hat{x}_0^i , so that (38) becomes:

$$\hat{\mathcal{O}}(f) = \int \frac{dp_1 dp_2}{2\pi\hbar} \tilde{f}(p) \exp\left[\frac{i}{\hbar}(p_1 \hat{x}_0^1 + p_2 \hat{x}_0^2)\right] \exp[\alpha \hat{a}^\dagger - \alpha^* \hat{a}] \quad (39)$$

where \hat{a} and \hat{a}^\dagger are the operators defined in (32), and

$$\alpha(p) = \frac{p_1 - ip_2}{\sqrt{2m\hbar\omega_c}} \quad \alpha^*(p) = \frac{p_1 + ip_2}{\sqrt{2m\hbar\omega_c}}. \quad (40)$$

It is obvious that, in general, a function so defined will not be gauge invariant, since there are operators that do not commute with the constraint (which is proportional to \hat{a}). Indeed, the gauge non-invariance of $\hat{\mathcal{O}}(f)$ is due to the presence of the unitary operator $\hat{D}(\alpha, \alpha^*)$, defined by

$$\hat{D}(\alpha, \alpha^*) = e^{\alpha \hat{a}^\dagger - \alpha^* \hat{a}}, \quad (41)$$

which produces shifts in \hat{a} when acting on a coherent state characterized by a complex number λ :

$$\hat{D}(\alpha, \alpha^*)|\lambda\rangle = |\lambda + \alpha\rangle, \quad \hat{a}|\lambda\rangle = \lambda|\lambda\rangle. \quad (42)$$

On the other hand, $\hat{D}(\beta, \beta^*)$ is, indeed, the unitary operator that realizes the gauge transformations generated by the first class constraint, so that we may project $\hat{D}(\alpha, \alpha^*)$ into its gauge invariant part by taking the average with respect to the gauge group:

$$\hat{D}_0(\alpha, \alpha^*) = \frac{1}{\pi^2} \int d\beta d\beta^* \hat{D}^\dagger(\beta, \beta^*) \hat{D}(\alpha, \alpha^*) \hat{D}(\beta, \beta^*). \quad (43)$$

It is simple to check that:

$$\hat{D}^\dagger(\beta, \beta^*) \hat{D}(\alpha, \alpha^*) \hat{D}(\beta, \beta^*) = \hat{D}(\alpha, \alpha^*) \exp[2i\text{Im}(\alpha\beta^*)] \quad (44)$$

and this implies, after integrating over β and β^* , that:

$$\hat{D}_0 = 1 \quad (45)$$

where 1 denotes the identity operator. Thus, we see that the physical operator corresponding to $\hat{\mathcal{O}}(f)$ is

$$\hat{\mathcal{O}}_0(f) = \int \frac{dp_1 dp_2}{2\pi\hbar} \tilde{f}(p) \exp\left[\frac{i}{\hbar}(p_1 \hat{x}_0^1 + p_2 \hat{x}_0^2)\right], \quad (46)$$

which, in view of the noncommutativity between the x_0 coordinates, will imply the Moyal product for the classical functions. It is important to realize that this reduction has been presented here entirely in terms of the constrained system, and not in the context of an approximation to the real situation where there are more levels than just the vacuum. Had we wanted to keep the full Hilbert space, then the projection would have to be understood as an operation that *changes the number of physical degrees of freedom*. Still, the reduced operator could now be defined by taking the vacuum expectation value of (39) on the lowest Landau level. This is a partial average, affecting only the annihilation and creation operators that go from one Landau level to the next one, leaving a dependence on the operators that take care of the degeneracy. Under this reduction in the number of degrees of freedom, the operator $\hat{O}(f)$ becomes $\hat{O}_r(f)$, defined by

$$\hat{O}_r(f) = \mathcal{N} \int \frac{dp_1 dp_2}{2\pi\hbar} \tilde{f}(p) \exp\left[\frac{i}{\hbar}(p_1 \hat{x}_0^1 + p_2 \hat{x}_0^2)\right] \exp\left(-\frac{1}{2}|\alpha(p)|^2\right), \quad (47)$$

where \mathcal{N} denotes a normalization constant, defined as

$$\mathcal{N}^{-1} = \int \frac{dp_1 dp_2}{2\pi\hbar} \exp\left(-\frac{1}{2}|\alpha|^2\right), \quad (48)$$

and introduced by reasons that will become clear later on. Then the correspondence between functions and operators should be defined by

$$f(x) \rightarrow \hat{O}_r(f) = \int \frac{dp_1 dp_2}{2\pi\hbar} \tilde{f}_r(p) \exp\left[\frac{i}{\hbar}(p_1 \hat{x}_0^1 + p_2 \hat{x}_0^2)\right] \quad (49)$$

where

$$\tilde{f}_r(p) = \mathcal{N} \tilde{f}(p) \exp\left(-\frac{1}{2}|\alpha(p)|^2\right) \quad (50)$$

is a ‘smoothed’ version of \tilde{f} . Indeed, in coordinate space, \tilde{f}_r corresponds to \tilde{f} convoluted with a Gaussian window of size equal to the cyclotron length for each coordinate. Of course, the Moyal product will now appear for the functions \tilde{f}_r , and not for the original ones, \tilde{f} . This is to be expected, since the model with all the Landau levels as physical states is commutative, and some modifications are to be expected when comparing with the purely non-commutative model. The normalization \mathcal{N} is included in order to preserve the probability, when the reduction in the number of degrees of freedom is implemented.

Summarizing, we have shown that the proper treatment of the constrained system naturally leads to the consideration, at the classical level, of a non-commutative theory. It should be noted that the original, commuting coordinates are mapped into the guiding center coordinates.

It is worth mentioning that everything we discussed here has its analog formulation in the path integral quantization scheme, if the proper translations are used. In particular, the Weyl ordering may be implemented by using the ‘mid-point prescription’. The emergence of a noncommutative theory may also be shown to happen in the path-integral setting, as shown for point-splitting regularization in string theory [15]. This holds true also for the general case of quantization deformation of a Poisson structure [16]. We apply this to the case at hand in Appendix A, using the ‘magnetic’ language, and particularizing to the system of interest.

3 Effective description for large magnetic fields

As a description of a system with a large but finite magnetic field, a noncommutative formulation should, by the previous reasoning, be a good approximation. However, it is unpleasant to realise that, in fact, as soon as we assume that the gap between the lowest Landau level and the upper ones is finite, the coordinates commute. This discontinuous behaviour would seem to forbid any attempt to use the noncommutative approach as a good starting point to deal with the case of a finite gap. The main reason for this discontinuous behaviour is of course that the number of physical degrees of freedom is different for the finite and infinite gap cases. In this sense the phenomenon is analogous to the CS quantum mechanics model of [12]. We could attempt, however, an intermediate approach: the noncommutative theory could be introduced with a θ parameter corresponding to a strong magnetic field (not necessarily equal to the real external one), but with the non-commutative theory still containing a (noncommutative) external magnetic field. Indeed, for the single particle action in an external field,

$$S_{int} = e \int dt A_k(\vec{x}(t)) \frac{dx^k(t)}{dt} \quad (51)$$

we may now assume that A_k corresponds to a magnetic field \mathbf{B} , which can always be represented as

$$\mathbf{B} = \mathbf{B}^\theta + \mathbf{B} \quad (52)$$

where for some reason that depends on the physical problem one is dealing with, \mathbf{B}^θ is such that it results convenient to use the noncommutative description, and $\mathbf{B} = \mathbf{B} - \mathbf{B}^\theta$. Thus the idea is to go from the commutative description, where there is a constant magnetic field \mathbf{B} , to a noncommutative one with a noncommutative parameter

$$\theta = -\frac{1}{eB^\theta} \quad (53)$$

and with a constant noncommutative magnetic field $\vec{\mathcal{B}}$, related to \vec{B} , as we shall see. Indeed, splitting also the gauge field \vec{A} (which verifies $\vec{\nabla} \times \vec{A} = \vec{B}$), in two parts: \vec{A}^θ and $\vec{\mathcal{A}}$, such that $\vec{\nabla} \times \vec{A}^\theta = \vec{B}^\theta$ and $\vec{\nabla} \times \vec{\mathcal{A}} = \vec{\mathcal{B}}$, we have the action describing the interaction:

$$S_{int} = e \int dt A_k(\vec{x}(t)) \frac{dx^k(t)}{dt} = S^\theta + \mathcal{S} \quad (54)$$

where

$$S^\theta = e \int dt A_k^\theta(\vec{x}(t)) \frac{dx^k(t)}{dt} \quad (55)$$

and

$$\mathcal{S} = e \int dt \mathcal{A}_k(\vec{x}(t)) \frac{dx^k(t)}{dt}. \quad (56)$$

Then, the part of the action corresponding to \vec{A}^θ is used to introduce the noncommutativity, while the part proportional to $\vec{\mathcal{A}}$, is treated as an external field for the remaining theory. However, in the noncommutative theory, this remaining field is not precisely equal to $\vec{\mathcal{A}}$: When we consider gauge transformations for $\vec{\mathcal{A}}$, with \vec{A}^θ fixed, the classical action is invariant, since these transformations change the Lagrangian by a total derivative. However, the quantum theory will not have this symmetry, by the same reason that made the action (5) invariant under (22) rather than under the usual commutative Abelian gauge transformations. Being a noncommutative gauge field, we should write $\hat{\vec{\mathcal{A}}}$ rather than $\vec{\mathcal{A}}$ for the remaining gauge field in the noncommutative theory, so that the action (54) taking into account quantum effects is now written as:

$$\hat{S}_{int} = e \int dt \hat{A}_k(\vec{x}(t)) \frac{dx^k(t)}{dt}. \quad (57)$$

These quantum effects may be introduced by the device of using $S^\theta = e \int dt A_k^\theta \frac{dx^k}{dt}$ as the ‘free’ action, which then defines the canonical structure and its associated Weyl ordering. A possible way to accomplish this can be to use the path integral framework to derive the action \hat{S}_{int} as the ‘effective’ action that results from a (partial) integration of the degrees of freedom, namely,

$$e^{\frac{i}{\hbar} \hat{S}_{int}[\hat{A}]} = \langle e^{\frac{i}{\hbar} S} \rangle^\theta \quad (58)$$

where

$$\langle \dots \rangle^\theta = \int \mathcal{D}x \dots e^{\frac{i}{\hbar} S^\theta} \quad (59)$$

with the path integral evaluated in a semiclassical expansion, defined in the same way as in Appendix A. The resulting $\widehat{\mathcal{S}}$ action is of course noncommutative, since when expanding \mathcal{S} in (58), each product is replaced by its Moyal analog. We note that also a perturbative field \mathbf{a} (not necessarily corresponding to a magnetic field but for instance to an external probe) will be transformed into a noncommutative one by this device.

Of course, this procedure is not exact, since, had we used the full gauge field as the free action, the canonical theory would have been different. Besides, there would be no remnant field for this different noncommutative theory, since in this case, we would have traded all the magnetic field \mathbf{B} by \mathbf{B}_θ . There is then an interplay between the θ parameter and the strength of the remaining noncommutative field, which of course corresponds to a constant noncommutative field strength F_{ij} .

To see this, we realize that to the usual $U(1)$ gauge orbits of the classical theory there will correspond gauge orbits of the noncommutative $U(1)$ theory, so that the relation

$$\widehat{\delta_\lambda \mathcal{A}} = \widehat{\delta_\lambda \mathcal{A}} \quad (60)$$

which is the expression that leads to the Seiberg-Witten mapping between commutative and noncommutative theories [15]. Thus, if not all the uniform magnetic field \mathbf{B} is traded by \mathbf{B}_θ in the noncommutative description there is an extra constant noncommutative magnetic field. To find a quantitative expression of this interplay, we may recall that a constant commutative magnetic field is mapped, via the Seiberg-Witten equations, to a noncommutative constant field $\widehat{\mathcal{B}}$, with the relation:

$$\frac{1}{e\mathcal{B}} = \frac{1}{e\widehat{\mathcal{B}}} - \theta \quad (61)$$

which is an exact solution of the SW relations, valid for the case of a constant magnetic field [15].

Equation (61) shows that if θ vanishes, the noncommutative description reduces to the usual commutative theory in a continuous way. On the other hand, if all the magnetic field \mathbf{B} is traded by \mathbf{B}_θ , there is no remaining magnetic field in the noncommutative theory. In this case, the limit of vanishing θ does not reduce to the original commutative theory anymore. This is of course consistent with the fact that for the Landau problem, the projection onto the LLL is not a continuous process since it implies a change in the Hilbert space of the system.

Based in the previously derived relations between the strong magnetic field Hamiltonian and a noncommutative theory, it should be noted that the classical action (to be used in second quantization) should contain the Moyal

product whenever products of functions of the spatial coordinates appear. This is of course valid also for every other term in the action, including a pair interaction term. In particular, it can be shown that an ultra-local pair interaction term in the noncommutative theory, can be mapped into the Hamiltonian for a free particle in a uniform magnetic field determined by θ , with an effective mass proportional to the strength of the pair potential. This term will play the role of an effective kinetic term for the *projected* theory.

The standard second quantization action for a bidimensional system of non interacting particles in the presence of an external magnetic field (before reducing to the lowest Landau level) would be:

$$S_s = \int dt dx^1 dx^2 \psi^\dagger(t, x) [i\hbar \partial_t - e a_0 + \mu - \frac{1}{2m}(-i\hbar \vec{\nabla} - e \vec{A}_\theta - e \vec{\mathcal{A}} - e \vec{a})^2] \psi(t, x) \quad (62)$$

where \vec{A}_θ and $\vec{\mathcal{A}}$ where defined above, and a_μ corresponds to an external probe.

Then the non-commutative description *with B_θ determining the noncommutativity* is introduced, as a reduction to the first Landau level for B_θ , passing from the action (62) to the noncommutative one

$$S_{nc} = \int dt dx^1 dx^2 [\psi^\dagger(t, x) \star (i\hbar \partial_t + \mu) \psi(t, x) - e \psi^\dagger(t, x) \star a_0(t, x) \star \psi(t, x) - \frac{1}{2m} \psi^\dagger(t, x) \star (-i\hbar \vec{\nabla} - e(\vec{\mathcal{A}} + \vec{a})) \star (-i\hbar \vec{\nabla} - e(\vec{\mathcal{A}} + \vec{a})) \star \psi(t, x)] \quad (63)$$

where the B_θ field part has disappeared from the action (i.e., it is in \blacksquare), since it has been traded for the noncommutativity of the coordinates:

$$\theta = -\frac{1}{e B_\theta} . \quad (64)$$

An alternative way of justifying the introduction of the noncommutative description in this context is as follows. We can try to decouple the matter fields from the uniform magnetic field by performing a singular gauge transformation. In principle we can write

$$\begin{aligned} \psi(t, x) &= G_c(x) \psi_c(t, x) \\ \psi^\dagger(t, x) &= \psi_c^\dagger(t, x) G_c^\dagger(x) \end{aligned} \quad (65)$$

where

$$G_c(x) = \exp \frac{ie}{\hbar} \int_{\mathcal{C}(x)} d\vec{y} \cdot \vec{A}_\theta(y) \quad (66)$$

with $\mathcal{C}(x)$ denoting a curve that starts at spacial infinity and ends at the point x . In this way the new fields are free, but at the cost of being dependent on the curve \mathcal{C} . However, this dependence on the curve could be get rid off if, for any path Γ , the condition

$$\frac{e}{\hbar} \int_{\Gamma} d\vec{y} \cdot \vec{A}_{\theta}(y) = \frac{eB_{\theta}}{\hbar} S(\Gamma) = 2\pi n \quad (67)$$

with $n \in \mathbb{Z}$ were satisfied. In this expression, $S(\Gamma)$ denotes the area enclosed by the curve Γ . Thus, the gauge transformation that eliminates the external magnetic field would be independent of the path only if the area enclosed by an arbitrary path Γ were quantized, i.e., if $S(\Gamma) = \hbar\theta 2\pi n$. However, the quantization of the area is difficult to justify, unless we work in the context of non commutative geometry, were there is an uncertainty relation for the two spatial coordinates. Notice that in the Landau problem the natural scale for the ‘quantum’ of area is set by the cyclotron length $l_0 = \sqrt{\hbar\theta}$.

For a system of non-relativistic fermions in the presence of a commutative gauge field with a part that corresponds to a uniform magnetic field B and a fluctuation a_{μ} , the fermionic determinant can be calculated [17] when the ratio between the average density and the magnetic field is such that there is an integer number of Landau levels filled. In this case, the leading order term of the effective action for a_{μ} has the Chern-Simons form, and its coupling constant is proportional to the ratio between the magnetic field and the average density (or the inverse of the filling fraction).

According to our previous discussion, we can apply the Seiberg-Witten transformation to this gauge field, with a θ parameter defined by B_{θ} , so that the constant magnetic field B is transformed into B_{θ} , through the relation (61). Then the commutative CS action is transformed into the noncommutative one [18] for the field \hat{a}_{μ} which is related to a_{μ} through the Seiberg-Witten relation as well. We know that for vanishing θ the CS action becomes the commutative one with a coupling constant proportional to the inverse of the filling fraction. On the other hand, if all the uniform magnetic field is traded by B_{θ} , there is no induced Chern-Simons action. We will return to the problem of the coupling constant for an arbitrary θ elsewhere [28].

To finish this section we discuss briefly a possible realization of this approach in the context of the QHE problem. It is well known that in the presence of a strong perpendicular magnetic field, a system of oppositely charged particles (such as a neutral dipole) moves in a straight line perpendicular to the vector connecting them, even though its size grows with its momentum [19]. Such dipoles are the objects described by non-commutative field theories. In particular, it has been shown that a set of local gauge invariant operators in noncommutative gauge theories can be constructed by using

straight Wilson lines with momentum p_μ such that the distance between the end points of the line is $l^\nu = p_\mu \theta^{\mu\nu}$ [20, 21]. Given a local operator $\mathcal{O}(x)$ in an ordinary gauge theory (in the adjoint representation) its noncommutative generalization is [20]

$$\tilde{\mathcal{O}}(k) = Tr \int d^3x \mathcal{O}(x) * P_* \exp(iq \int_C d\lambda^\mu A_\mu(x + \lambda)) * e^{ikx} \quad (68)$$

where C is a straight path $\lambda^\mu(\sigma) = k_\mu \theta^{\mu\nu} \sigma$, $0 \leq \sigma < 1$, and P_* denotes path ordering with respect to the star product. The tilde is used as a reminder that there is a Wilson line attached to the operator. The Wilson line is extended in the direction perpendicular to the momentum. For small k or l the length of the Wilson line goes to zero and $\tilde{\mathcal{O}}$ reduces to the corresponding operator in the commutative field theory.

In the context of the FQHE, it was argued in reference [22] that for the half filled state, the true low-energy quasiparticles in the fermion Chern-Simons theory obtained upon screening of the magnetoplasmon mode, are electrically neutral (see also [22, 23, 24, 25, 26]). Based on trial wave functions in the LLL, Read noticed that the electron and the correlation hole are separated from one another by a distance proportional and perpendicular to the canonical momentum \vec{k} of these low energy quasiparticles. Therefore, these neutral quasiparticles carry an electric dipole moment $e l^2 \hat{z} \times \vec{k}$ with l the magnetic length. Thus, if we choose the deformation parameter θ such that all the external magnetic field is traded by B_θ (i.e. $B_\theta = B$), the effective theory (63) (including a pair potential term, not written explicitly in that expression) will be an appropriate description for this problem, since it naturally describes the elementary quasiparticles of the half-filled state. There are a couple of results that support our proposal. In a similar model studied in reference [27] the authors show that the corresponding ground state wave function has the shifting between the particle and the correlation hole discussed by Read [22]. We also know [28] that this model breaks parity without the presence of an explicit Chern-Simons term, in coincidence with the description of reference [22].

4 Conclusions

In this work we have studied different aspects of the description of two dimensional systems in high magnetic fields using noncommutative theories. We began by reviewing the problem of a particle coupled to a magnetic field whose magnitude is large enough to neglect the kinetic energy. In this case, the spatial coordinates are canonical conjugate to each other, and the system

is invariant under area preserving diffeomorphisms of the plane. Thus, at the quantum level the Moyal bracket should replace the Poisson bracket for infinitesimal coordinate transformations. Alternatively, one may think in terms of gauge transformations for the gauge field coupled to the particle. In this case, the usual gauge transformations are replaced by their noncommutative version. Therefore, the gauge field action must be constructed being invariant under this noncommutative gauge symmetries. In references [29, 30] it was argued using general hydrodynamical arguments that the effective action for an incompressible state of a system of charged particles in two dimensions in the presence of a strong magnetic field must be a Chern-Simons action. Analogously, and using the fact that the *correct* symmetry for the gauge fields in the LLL is the noncommutative gauge symmetry, the *natural* effective description should be given by the noncommutative Chern-Simons action.

As we have already mentioned, the Hilbert space is not the same in the case that all the Landau levels are taken into account, than if only the LLL can be occupied. In particular, the space coordinates commute in the first case, and they do not in the latter, and the number of degrees of freedom is different. In this sense the *projection* onto the LLL can not be made in a continuous way. We have argued that if the only allowed state is the LLL, the correct description is a noncommutative free theory. Therefore the obvious question is how to make compatible the noncommutative description with a problem in which some Landau level mixing is present. We argued that in this case it should be used a somehow intermediate approach. The noncommutative theory could be introduced with a θ parameter corresponding to a magnetic field B_θ (not necessarily equal to the external one), but with the non-commutative theory still containing a (noncommutative) uniform magnetic field \mathcal{B} , in such a way that \mathcal{B} is related to B through the Seiberg-Witten relation, and the external uniform magnetic field is $B = B_\theta + \mathcal{B}$. Then we argued that once the fermionic determinant is calculated for this theory, the leading order term in a derivative expansion will be given by a NCCS action for the external probe whose coupling constant will be a function of θ and \mathcal{B} .

To conclude, we mention that noncommutative field theories have an unusual perturbative behaviour. This is due to the fact that the Moyal product generates phases appearing in the perturbative structure that induce an interplay between the infrared and the ultraviolet regimes. It can be argued that since spacial non-commutativity is a short distance property, it would be surprising that some effect related to it could show up in the low energy effective theory. However, it was shown that for some non-commutative field theories [31, 32, 33] the noncommutativity of the coordinates modifies the critical behaviour of the theory, since the long distance behavior is entangled

to the short distance one due to the presence of the Moyal phases. This interplay between short and long distance behaviour therefore changes the critical properties of the noncommutative theories compared to their commutative counterparts. In this context, we believe it might prove useful to explore the alternative *noncommutative descriptions* of bidimensional systems in high magnetic fields described in this work, to approach problems where their commutative counterparts fail.

Acknowledgments

We thank Prof. M. Aurelio for enlightening discussions. This work is supported by CONICET (Argentina), by ANPCyT through grant No. **03-03924** (AL), and by Fundación Antorchas (Argentina).

Appendix A: Path integral representation of the Moyal product

The Moyal product of two functions of the coordinates f, g may be represented, following [16], in terms of a quantum mechanical path integral with a topological action. This action becomes particularly simple when one considers the deformation quantization of a Poisson structure defined by a symplectic form, and this is, indeed, the case at hand.

For this simple case, the expression for the Moyal product may be written as

$$(f \star g)(x) = \int_{\gamma(\pm\infty)=x} \mathcal{D}\gamma f(\gamma(1))g(\gamma(0)) e^{\frac{i}{\hbar}S[\gamma]} \quad (69)$$

where $\gamma : \mathbb{R} \rightarrow \mathbb{R}^2$ denotes a plane curve, and the action $S[\gamma]$ is defined by

$$S[\gamma] = \frac{b}{2} \int_{-\infty}^{+\infty} dt \dot{\gamma}^j(t) \epsilon_{jk} \gamma^k(t) , \quad (70)$$

with $b = -\theta^{-1}$. It is also adopted as a prescription that the functional integral should be evaluated semiclassically, around the ‘classical’ configuration $\dot{\gamma}^j(t) = x^j = \text{constant}$. This path integral formula may also be thought of as a concrete realization of Kontsevich’s result on the expression of the star product in a Feynman-like perturbation expansion [34].

In the case at hand, the above definition may be applied to two functions f and g more directly if they are written in terms of their Fourier transforms:

$$f(x) = \int \frac{d^2k}{(2\pi)^2} \tilde{f}(k) e^{i\vec{k} \cdot \vec{x}}$$

$$g(x) = \int \frac{d^2k}{(2\pi)^2} \tilde{g}(k) e^{i\vec{k}\cdot\vec{x}} \quad (71)$$

so that (69) becomes

$$\begin{aligned} & \int_{\gamma(\pm\infty)=x} \mathcal{D}\gamma f(\gamma(1))g(\gamma(0)) e^{\frac{i}{\hbar}S[\gamma]} = \int \frac{d^2k}{(2\pi)^2} \frac{d^2l}{(2\pi)^2} \\ & \times \tilde{f}(k)\tilde{g}(l) \int_{\gamma(\pm\infty)=x} \mathcal{D}\gamma \exp\left\{\frac{i}{\hbar}S[\gamma] + i \int_{-\infty}^{+\infty} dt \gamma^j(t) [k^j\delta(t-1) + l^j\delta(t)]\right\}, \end{aligned} \quad (72)$$

where the plane wave parts of the Fourier transforms have been included in the source term of $\gamma^j(t)$. We then make a shift in the integration variables: $\gamma^j(t) \rightarrow x^j + \xi^j(t)$, so that the measure is now $\mathcal{D}\xi$, and ξ vanishes at $\pm\infty$:

$$\begin{aligned} & \int_{\gamma(\pm\infty)=x} \mathcal{D}\gamma f(\gamma(1))g(\gamma(0)) e^{\frac{i}{\hbar}S[\gamma]} = \int \frac{d^2k}{(2\pi)^2} \frac{d^2l}{(2\pi)^2} \tilde{f}(k)\tilde{g}(l) e^{i(\vec{k}\cdot\vec{x} + \vec{l}\cdot\vec{x})} \\ & \times \int_{\xi(\pm\infty)=0} \mathcal{D}\xi \exp\left\{\frac{i}{\hbar}S[\xi] + i \int_{-\infty}^{+\infty} dt \xi^j(t) [k^j\delta(t-1) + l^j\delta(t)]\right\}. \end{aligned} \quad (73)$$

Thus the integral over ξ is a Gaussian and we may write its result explicitly:

$$\begin{aligned} & \int_{\xi(\pm\infty)=0} \mathcal{D}\xi \exp\left\{\frac{i}{\hbar}S[\xi] + i \int_{-\infty}^{+\infty} dt \xi^j(t) [k^j\delta(t-1) + l^j\delta(t)]\right\} \\ & = \exp\left\{-\frac{i\hbar}{2b} \int dt_1 \int dt_2 [k^i\delta(t_1-1) + l^i\delta(t_1)] K_{ij}(t_1-t_2) [k^j\delta(t_2-1) + l^j\delta(t_2)]\right\} \end{aligned} \quad (74)$$

where $K_{ij}(t)$ is the inverse of the operator defining the quadratic form in the action, namely,

$$-\epsilon^{ij} \frac{d}{dt} K_{jk}(t) = \delta(t) \delta_j^i \quad (75)$$

which has the solution

$$K_{ij}(t) = \frac{1}{2} \epsilon_{ij} \text{sign}(t). \quad (76)$$

This propagator is uniquely defined, since it has to be Bose symmetric: $K_{ij}(t) = K_{ji}(-t)$, and moreover it is also consistent with the canonical commutator of Equation (10). The last condition may be verified by a direct application of the BJL limit [35] to derive the equal time commutator between γ^j and γ^k :

$$[\gamma^j(t), \gamma^k(t)] = \left(\lim_{\epsilon \rightarrow 0^+} - \lim_{\epsilon \rightarrow 0^-} \right) \langle \gamma^j(t+\epsilon) \gamma^k(t) \rangle \quad (77)$$

where $\langle \gamma^j(t_1) \gamma^k(t_2) \rangle$ is the propagator derived from (69). This is of course proportional to the inverse of K_{ij} :

$$\langle \gamma^j(t_1) \gamma^k(t_2) \rangle = i \frac{\hbar}{2b} \epsilon^{jk} \text{sign}(t_1 - t_2) , \quad (78)$$

and when inserted in (77) reproduces the commutator we had obtained by canonical means in (10) for the coordinates of the particles in an external field.

Using now the explicit form for K in (74), we find

$$\int_{\xi(\pm\infty)=0} \mathcal{D}\xi \exp\left\{ \frac{i}{\hbar} S[\xi] + i \int_{-\infty}^{+\infty} dt \xi^j(t) [k^j \delta(t-1) + l^j \delta(t)] \right\} = e^{-\frac{i\hbar}{2b} \epsilon_{ij} k^i l^j} . \quad (79)$$

which inserted in (73) yields

$$\int_{\gamma(\pm\infty)=x} \mathcal{D}\gamma f(\gamma(1)) g(\gamma(0)) e^{\frac{i}{\hbar} S[\gamma]} = \left[\exp\left(\frac{i\hbar\theta}{2} \frac{\partial}{\partial x^j} \frac{\partial}{\partial y^k} \right) f(x) g(y) \right]_{y \rightarrow x} \quad (80)$$

where the last expression is of course one of the possible ways to define the Moyal product $(f \star g)(x)$, with a parameter $\theta = -b^{-1}$.

Appendix B: Fluid representation

We discuss here some aspects of a somewhat different approach to the introduction of a noncommutative CS theory, this time in terms of a fluid representation. Although this is not the path to the NCCS theory that we have followed in the main part of this article, we have nevertheless included it here, for the sake of completeness. Besides, we consider here a different version of the approach developed in ref.[3], which is applicable to the more general case of bosonized theories in $2+1$ dimensions, supplemented by an incompressibility constraint. Our starting point is the expression for the bosonized action $S_B[A]$, which in the leading approximation in a derivative expansion is given by

$$S_B[A] = S_{CS}[A] \quad (81)$$

where S_{CS} denotes the CS action:

$$S_{CS}[A] = \frac{\kappa}{2} \int d^3x \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda . \quad (82)$$

This gauge field is related to the vacuum expectation value (VEV) of the bosonized matter current by

$$\langle J^\mu(x) \rangle = j^\mu(x) = \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda(x) . \quad (83)$$

In the $A_0 = 0$ gauge, the spatial components of the current are

$$j^k(t, \vec{x}) = -\epsilon^{kl} \frac{\partial}{\partial t} A_l(t, \vec{x}) . \quad (84)$$

To go to the fluid interpretation, one regards the spatial current as a density ρ times the fluid's velocity \vec{v} . From this expression, we may formally write the equation that determines the fluid flux lines:

$$\frac{\partial x^k}{\partial t} = -\frac{1}{\rho} \epsilon^{kl} \frac{\partial A_l}{\partial t} \quad (85)$$

and its solution shall be of the form:

$$x^k = x^k(t, \vec{y}) \quad (86)$$

where \vec{y} denotes the initial conditions for a given line. Namely, a given value of \vec{y} determines one line from its initial point at $t = t_0$. Note that, in principle, both ρ and A may be functions of the space and time coordinates. In order to proceed from equation (85), we need to make further use of the continuity equation, and introduce the area preserving diffeomorphisms symmetry assumption. A convenient way to do this is by defining a 2-form Ω by

$$\Omega = \rho(dx^1 - v^1 dt) \wedge (dx^2 - v^2 dt) \quad (87)$$

which, by some elementary algebraic steps can be shown to verify:

$$d\Omega = \left[\frac{\partial}{\partial t} \rho + \vec{\nabla} \cdot \vec{j} \right] dt \wedge dx^1 \wedge dx^2 = 0 , \quad (88)$$

as a consequence of the continuity equation (an assumption of the bosonization approach). When writing Ω in terms of the formal solutions (86), one makes use of $dx^i = \frac{\partial x^i}{\partial t} dt + \frac{\partial x^i}{\partial y^j} dy^j$ and $v^i = \frac{\partial x^i}{\partial t}$ to obtain:

$$\Omega = \rho \frac{\partial(x^1, x^2)}{\partial(y^1, y^2)} dy^1 \wedge dy^2 . \quad (89)$$

Equation (88) holds for any choice of coordinates, and in this set implies:

$$\frac{\partial}{\partial t} \left[\rho \frac{\partial(x^1, x^2)}{\partial(y^1, y^2)} \right] = 0 . \quad (90)$$

We now impose the area preservation requirement to the system, namely,

$$\frac{\partial(x^1, x^2)}{\partial(y^1, y^2)} = 1 \quad (91)$$

so that $\rho = \rho_0$. Choosing the \vec{y} coordinates in order to have a constant ρ_0 , we have a uniform and constant density. With this in mind, (85) can be integrated, yielding

$$x^k(t, \vec{y}) = y^k - \frac{1}{\rho_0} \epsilon^{kl} A_l(t, \vec{y}) , \quad (92)$$

which is, indeed, the relation introduced in ref(). Up to now, we have used just the bosonization rule that yields the VEV of the current in terms of the curl of the gauge field A_μ , without actually using the explicit form of the bosonized action. It turns out that the invariance under area preserving diffeomorphisms is not compatible with the standard Chern-Simons action. This may be seen from the relation (92), which, when applied to the area element, yields

$$dx^1 \wedge dx^2 = [1 - \frac{1}{\rho_0} \mathcal{B}] dy^1 \wedge dy^2 = dy^1 \wedge dy^2 \quad (93)$$

where

$$\mathcal{B} = \partial_1 A_2 - \partial_2 A_1 - \frac{1}{\rho_0} \{A_1, A_2\} \quad (94)$$

with $\{A, B\} = \epsilon^{jk} \partial_j A \partial_k B$. Thus, one must impose the constraint $\mathcal{B} = 0$. As discussed in ref[3], this constraint, together with the ‘kinetic’ term for the fluid may be written in a way which is tantamount to the first non-trivial approximation to the noncommutative Chern-Simons action. Thus we may certainly conclude that the bosonization mapping between the current and the gauge field leads naturally to a fluid interpretation, and that this fluid may be described by a non linear Chern-Simons like action which is an approximation to the full noncommutative theory. However, it is easy to see that, even in the context of this approximate bosonization, the noncommutativity is bound to arise when including quantum effects. Indeed, one way to see this is from the fact that the fluid coordinates x^i will be correlated by quantum (loop) effects. Since the coordinates are proportional to the components of A , the existence of a nontrivial correlation between the two different components of A ’s in the quantum theory will be translated into a non trivial correlation for the corresponding coordinates. By the BJL limit, this correlation implies the noncommutativity of the coordinates in the quantum version of the theory, and hence the noncommutativity of the CS action. The correlation of the A ’s, on the other hand, is due to the fermion loop, and in this approximation is given by the (commutative) Chern-Simons action.

References

- [1] L. Castellani, Class. Quant. Grav. 17 (2000) 3377 [hep-th/0005210], and references therein.
- [2] M. Douglas and N. Nekrasov, hep-th/0106048, and references therein.
- [3] L. Susskind, hep-th/0101029.
- [4] A. P. Polychronakos, JHEP **0104** 011 (2001) [hep-th/0103013].
- [5] D. Karabali and B. Sakita, hep-th/0106016
- [6] V. Pasquier, Phys. Lett. **B 490** (2000) 258; V. Pasquier, cond-mat/0012207.
- [7] B. H. Lee, K. Moon, and C. Rim, hep-th/0105127. skyrmions.
- [8] I. I. Kogan, Int. J. Mod. Phys. A9 (1994) 3887.
- [9] B. Sakita, Phys. Lett. B315 (1993) 124; Interactions Rashmi Ray and B. Sakita, Ann. Phys. 230 (1994) 131.
- [10] A. Capelli, C. Trugenberger and G. Zemba, Nucl. Phys **B396** (1993) 465.
- [11] S. M. Girvin, and T. Jach, Phys. Rev. **B 29** (1984) 5617.
- [12] G. V. Dunne, R. Jackiw, and C. A. Trugenberger. Phys. Rev **D41** (1990) 661.
- [13] W. Greiner, and J Reinhardt, “Field Quantization”, Springer-Verlag (1996).
- [14] V. Chari and A. Presley, “A guide to Quantum Groups”, Cambridge University Press (1994).
- [15] N. Seiberg, and E. Witten, JHEP 09 (1999) 032. [hep-th/9908142]
- [16] A. S. Cattaneo and G. Felder, formula,” Commun. Math. Phys. **212** (2000) 591 [math.qa/9902090].
- [17] A. Lopez and E. Fradkin, Phys. Rev. **B44**, 5246 (1991).
- [18] N. Grandi and G. A. Silva, hep-th/0010113.
- [19] D. Bigatti and L. Susskind, Phys. Rev. **D62** 066004 (2000).

- [20] D. J. Gross, A. Hashimoto and N. Itzhaki, hep-th/0008075.
- [21] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, Nucl. Phys. **B573**, 573 (2000).
- [22] N. Read, Semi. Cond. Sci. Tech. **9**, 1859 (1994); Surf. Sci. **361/362** 7 (1996), Phys. Rev. **B 58** (1998) 16262.
- [23] G. Murthy and R. Shankar in “Composite Fermions”, ed. by O. Heinonen, (North Holland, 1998).
- [24] V. Pasquier, and F. D. M. Haldane, Nucl. Phys. **B516** 719 (1998).
- [25] D. H. Lee, Phys. Rev. Lett **80** 4547 (1998).
- [26] A. Stern, B. I. Halperin, F. von Oppen, and S. Simon, Phys. Rev. **B 59** (1999) 12547.
- [27] D. Bak, S. K. Kim, K. S. Soh, and J. H. Yee, Phys. Rev. Lett. **85**, 3087, (2000).
- [28] C. D. Fosco, and A. Lopez, unpublished.
- [29] J. Frolich, and A. Zee, Nucl. Phys. **B364** 517 (1991).
- [30] J. Frolich, and T. Kerler, Nucl. Phys. **B354** 369 (1991).
- [31] A. Matusis, L. Susskind, N. Toumbas, JHEP 0012 (2000) 002
- [32] J. Gomis, K. Landsteiner, and E. Lopez, hep-th/0004115.
- [33] G. H. Chen, and Y. S Wu, hep-th/0103020.
- [34] M. Kontsevich, “Deformation quantization of Poisson manifolds, I,” q-alg/9709040.
- [35] J. D. Bjorken, Phys. Rev. **148**, 1467 (1966); K. Johnson and F. E. Low, Prog. Theoret. Phys.(Kyoto), Suppl.**37-38**, 74 (1966).