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Graviphoton and graviscalars delocalization in brane world scenarios

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Abstract

A manifestly gauge-invariant theory of gravitational fluctuations of brane-world scenarios is discussed. Without resorting to any specific gauge choice, a general method is presented in order to disentangle the fluctuations of the brane energy-momentum from the fluctuations of the metric. As an application of the formalism, the localization of metric fluctuations on scalar branes breaking spontaneously five-dimensional Poincaré invariance is addressed. Only assuming that the four-dimensional Planck mass is finite and that the geometry is regular, it is demonstrated that the vector and scalar fluctuations of the metric are not localized on the brane.

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1 Introduction

Fields of various spin can be localized [1] on higher dimensional topological defects [2, 3, 4]. A field is localized if it exhibits a normalizable zero mode with respect to the bulk coordinates parameterizing the geometry of the defect in the extra-dimensional space. Moreover, if the four-dimensional Planck mass is finite, the *tensor* modes of the geometry itself can be localized on a higher dimensional topological defect living, respectively, in six, seven or eight dimensions [5]. An example of this phenomenon is provided by five-dimensional AdS space with a brane source [6, 7].

The question which will be addressed in this paper is the fate of the other modes of the geometry itself, namely the scalar and vector fluctuations. In order to make the discussion more concrete let us concentrate on the case of a $D = d + 2$-dimensional geometry characterized only by one bulk coordinate w and whose line element can be written as

$$ds^2 = \overline{G}_{AB} dx^A dx^B = a^2(w)[dt^2 - dx_1^2 - ... - dx_d^2 - dw^2], \tag{1.1}$$

where $a(w)$ is the warp factor and $\overline{G}_{AB}$ denotes the background metric. For a review on the various implications of warped geometries see also [8].

The analysis of gravitational fluctuations in brane-world models may be rather complicated. Given a physical brane configuration characterized by a thickness and a profile, the fluctuations of the geometry will be necessarily entangled with the fluctuations of the energy-momentum tensor of the brane. Furthermore, the fluctuations of the metric (and of the brane energy-momentum tensor) depend upon the coordinate system. Both problems may a have a crucial impact on the analysis of the fluctuations. If the normal modes of the system are not properly analyzed, their localization cannot be discussed.

Since the meaning of the fluctuations may change in different coordinate systems, spurious gauge modes could make the whole analysis unreliable. Similar caveats should be borne in mind in the analysis of metric fluctuations on a given cosmological background in four-dimensions. Here the situation is similar but also crucially different: because of the five-dimensional nature of the geometry, the metric perturbations have to be classified according to the symmetry of the problem. If five-dimensional Poincaré invariance is broken (either spontaneously or explicitly), the various modes of the geometry should be classified according to four-dimensional Poincaré transformations. In cosmology fluctuations are classified according to the group of rotations in three dimensions.

In order to discuss the metric fluctuations of scalar-tensor actions, the *Bardeen formalism* [9] has been very useful in four-dimensional backgrounds. The main idea is to parameterize the metric fluctuations by defining a suitable set of gauge-invariant variables which do not change for infinitesimal coordinate transformations. The Bardeen formalism is also rather effective in order to identify the coordinate systems where the gauge-invariant variables take a simple form. In the following it will be shown that a non-trivial generalization of the formalism is allowed by requiring that the geometry has more than four dimensions and by classifying the fluctuations with respect to four-dimensional

Poincaré transformations [10].

Having generalized the Bardeen formalism to the case of five-dimensional warped geometries, the localization properties of metric fluctuations can be investigated in a fully gauge-invariant manner. If the zero mode of a given fluctuation is not normalizable, then it will not be localized on the brane and it will not affect the four-dimensional physics. Since all the equations for the fluctuations will be reduced to second order (partial) dfferential equations, the tower of mass eigenstates can be analyzed using known methods borrowed from supersymmetric quantum mechanics [12].

Explicit physical (thick) brane solutions have been derived in different numbers of transverse dimensions. In five dimensions physical branes can be obtained in terms of a scalar domain-wall whose scalar field depends upon the bulk radius [13, 14, 15, 16]. In [17, 18] the stability of scalar domain walls (inspired by gauged supergravity theories) has been analyzed. In more than five dimensions, physical brane solutions including also background gauge fields have been recently discussed [19, 20]. The formalism presented in this paper can be generalized to the case where the transverse space contains more than one bulk coordinate. This formalism has been also applied in the discussion of the radion wave-function in connection with the self-tuning problem [21] and it has been recently generalized to the case where the Einstein-Hilbert action is supplemented by quadratic curvature corrections parametrized in the Euler-Gauss-Bonnet (EGB) form [22]. Here a self-contained derivation, heavily based on gauge-invariance, will be presented.

The plan of this paper is the following. In Section II gauge-invariance will be exploited in order to obtain a set of evolution equations which are independent on the specific coordinate system. In Section III the general formalism will be applied to the case of thick brane configurations. Section IV contains our concluding remarks.

2 Exploiting gauge-invariance

The equations describing the brane configuration can be written, in general terms, as

$$R_{A}^{B} = 8\pi G_{D}\ \tau_{A}^{B}, \tag{2.1}$$

where R_A^B is the Ricci tensor and

$$\tau_{A}^{B} = 8\pi G_{D}\ (T_{A}^{B} - \frac{T}{d}\delta_{A}^{B}),\quad T = T_{A}^{A}, \tag{2.2}$$

is constructed from the energy-momentum tensor of the brane T_A^B.

For the background metric of Eq. (1.1) describing a warped space-time the Ricci tensors are *

$$\overline{R}_{\mu}^{\nu} = \frac{1}{a^{2}}(\mathcal{H}' + d\mathcal{H}^{2})\delta_{\mu}^{\nu},$$

*The Greek indices run over the $d+1$-dimensional sub-space whereas the Latin indices run over the full D-dimensional space-time.

$$\overline{R}_w^w = \frac{(d+1)}{a^2}\mathcal{H}', \tag{2.3}$$

where $\mathcal{H} = a'/a$ and the prime denotes derivation with respect to w. Then the background equations can be written as

$$\overline{R}_A^B = 8\pi G_D \overline{\tau}_A^B, \tag{2.4}$$

where $\overline{\tau}_A^B$ is the background energy-momentum tensor. In spite of the specific form of the action describing the domain-wall solution, its energy energy-momentum tensor should satisfy the following symmetry properties

$$\begin{aligned}
&\overline{\tau}_\mu^w = \overline{\tau}_w^\mu = 0,\\
&\overline{\tau}_\mu^\nu \propto \delta_\mu^\nu,\\
&\overline{\tau}_w^w \propto \delta_w^w.
\end{aligned} \tag{2.5}$$

which are obtained from Eqs. (2.4) by requiring that the form of τ_A^B is compatible with the specific form of the Ricci tensor given in Eq. (2.3).

To first order in the fluctuations of the metric

$$G_A^B(x^\mu, w) = \overline{G}_A^B(w) + \delta G_A^B(x^\mu, w), \tag{2.6}$$

the fluctuations of the Ricci tensors and of the τ_A^B can be written as

$$\begin{aligned}
&R_A^B(x^\mu, w) = \overline{R}_A^B(w) + \delta R_A^B(x^\mu, w),\\
&\tau_A^B(x^\mu, w) = \overline{\tau}_A^B(w) + \delta\tau_A^B(x^\mu, w),
\end{aligned} \tag{2.7}$$

where δ denotes all the terms linear in the metric and matter fluctuations. The equations of motion for small metric fluctuations linearized around the background are

$$\delta R_A^B = 8\pi G_D \delta\tau_A^B. \tag{2.8}$$

In general the fluctuations of the Ricci tensor and of the energy-momentum tensor will have scalar, vector and tensor modes which can be classified according to the $(d+1)$-dimensional Poincaré invariance of the metric (1.1). Without assuming any specific gauge the fluctuations of the Ricci tensor can be obtained, after a tedious calculation involving the repeated use of Palatini identities (see the Appendix). In particular it can be obtained that

$$\begin{aligned}
\delta R_w^w &= \frac{(d+1)}{a^2}\Big\{\psi'' + \mathcal{H}(\xi' + \psi') + 2\mathcal{H}'\xi \\
&- \partial_\alpha\partial^\alpha[(C - E')' + \mathcal{H}(C - E') - \xi]\Big\},
\end{aligned} \tag{2.9}$$

$$\delta R_\mu^w = \frac{1}{a^2}\Big\{d\partial_\mu(\mathcal{H}\xi + \psi') + \frac{1}{2}\partial_\alpha\partial^\alpha(D_\mu - f_\mu')\Big\}, \tag{2.10}$$

$$\begin{aligned}
\delta R_{\mu}^{\nu} &= \frac{1}{a^2}\Big\{ h_{\mu}^{\nu\prime\prime} + d\mathcal{H} h_{\mu}^{\nu\prime} - \partial_{\alpha}\partial^{\alpha} h_{\mu}^{\nu} \\
&+ \delta_{\mu}^{\nu}[\psi'' + (2d+1)\mathcal{H}\psi' - \partial_{\alpha}\partial^{\alpha}\psi \\
&+ \mathcal{H}\xi' + 2(\mathcal{H}' + d\mathcal{H}^2)\xi - \mathcal{H}\partial_{\alpha}\partial^{\alpha}(C - E')] \\
&+ \partial_{\mu}\partial^{\nu}[(E' - C)' + d\mathcal{H}(E' - C) + \xi - (d-1)\psi] \\
&+ [(\partial_{(\mu} f^{\nu)})'' + d\mathcal{H}(\partial_{(\mu} f^{\nu)})'] - [(\partial_{(\mu} D^{\nu)})' + d\mathcal{H}(\partial_{(\mu} D^{\nu)})]\Big\}.
\end{aligned} \tag{2.11}$$

The various functions appearing in Eqs. (2.9)–(2.10) come from the perturbed form of the metric

$$\delta G_{AB} = a^2(w) \begin{pmatrix} 2h_{\mu\nu} + (\partial_{\mu} f_{\nu} + \partial_{\nu} f_{\mu}) + 2\eta_{\mu\nu}\psi + 2\partial_{\mu}\partial_{\nu}E & D_{\mu} + \partial_{\mu}C \\ D_{\mu} + \partial_{\mu}C & 2\xi \end{pmatrix}. \tag{2.12}$$

In Eq. (2.12) $h_{\mu\nu}$ is a divergence-less and trace-less [i.e. $\partial_{\mu} h_{\nu}^{\mu} = 0$, $h_{\mu}^{\mu} = 0$] rank-two tensor in the $(d+1)$-dimensional Poincaré invariant space-time. The vectors f_{μ} and D_{μ} are both divergence-less [i.e. $\partial_{\mu} D^{\mu} = \partial_{\nu} f^{\nu} = 0$]. The other functions are scalars under Poincaré transformations. The number of independent functions parameterizing the fluctuations of the metric is then $(d+2)(d+3)/2$.

The scalar, vector and tensor fluctuations of the geometry defined in Eq. (2.12) are not invariant under infinitesimal coordinate transformations of the type

$$x^{A} \to \tilde{x}^{A} = x^{A} + \epsilon^{A}, \tag{2.13}$$

where ϵ^{A} stands for the $(d+2)$ functions parameterizing the gauge transformation. Similarly to what happens for the metric fluctuations, the perturbations of the Ricci tensor change for infinitesimal coordinate transformations. Hence, the logic will now be the following. From the transformation properties of the metric fluctuations under the gauge shift of Eq. (2.13) gauge-invariant functions will be defined. This definition should be simple (i.e. linear in the fluctuations) and moreover it should allow to write the fluctuations of the Ricci tensor in terms of a gauge-invariant part supplemented by a gauge-dependent part. It will be shown, in general, that the gauge-dependent part of the Ricci fluctuation is identically compensated by the gauge-dependent part of the fluctuation of the brane energy-momentum tensor. This observation allows to write, in general terms, a gauge-invariant form of the evolution equations of the fluctuations. The obtained system, will be, by construction, independent on the specific coordinate system.

Under the transformation of Eq. (2.13) the fluctuations of the metric transform according to the usual expression involving the Lie derivative in the direction of the vector ϵ^{A}

$$\delta\tilde{G}_{AB} = \delta G_{AB} - \nabla_{A}\epsilon_{B} - \nabla_{B}\epsilon_{A}, \tag{2.14}$$

where $\epsilon_{A} = a^2(w)(\epsilon_{\mu}, -\epsilon_{w})$. We can always use $(d+1)$-dimensional Poincaré invariance in order to classify the transformation properties of the gauge functions. In pratical terms,

$$\epsilon_{\mu} = \partial_{\mu}\epsilon + \zeta_{\mu}, \tag{2.15}$$

with $\partial_\mu \zeta^\mu = 0$. Hence, while ζ_μ transforms as a pure (i.e. divergence-less vector), ϵ and ϵ_w are scalars.

The tensor mode of the metric, i.e. $h_{\mu\nu}$ are automatically invariant under the transformation (2.13). The vectors and the scalars are not gauge-invariant and this is the source of the lack of gauge-invariance of Eqs. (2.9)–(2.11). Since there are two scalar gauge functions [i.e. ϵ and ϵ_w], and one vector gauge function [i.e. ζ_μ], two gauge-invariant scalar functions and one gauge-invariant vector function can be defined. The gauge-invariant scalars are

$$\Psi = \psi - \mathcal{H}(E' - C), \tag{2.16}$$

$$\Xi = \xi - \frac{1}{a}[a(C - E')]'. \tag{2.17}$$

The gauge-invariant vector is

$$V_\mu = D_\mu - f'_\mu. \tag{2.18}$$

By noticing that, under (2.13) the scalar fluctuations of (2.12) transform as

$$E \to \tilde{E} = E - \epsilon, \tag{2.19}$$

$$\psi \to \tilde{\psi} = \psi - \mathcal{H}\epsilon_w, \tag{2.20}$$

$$C \to \tilde{C} = C - \epsilon' + \epsilon_w, \tag{2.21}$$

$$\xi \to \tilde{\xi} = \xi + \mathcal{H}\epsilon_w + \epsilon'_w, \tag{2.22}$$

and the vector fluctuations transform as

$$f_\mu \to \tilde{f}_\mu = f_\mu - \zeta_\mu, \tag{2.23}$$

$$D_\mu \to \tilde{D}_\mu = D_\mu - \zeta'_\mu, \tag{2.24}$$

the gauge-invariance of Eqs. (2.16)–(2.17) and (2.18) can be explicitly shown.

Using Eqs. (2.16)–(2.18) into Eqs. (2.9)–(2.11) the fluctuations of the Ricci tensor can be written in a fully gauge-invariant manner

$$\delta R_w^w = \delta^{(\mathrm{gi})} R_w^w - [\overline{R}_w^w]'(C - E'), \tag{2.25}$$

$$\delta R_\mu^w = \delta^{(\mathrm{gi})} R_\mu^w - \overline{R}_w^w \partial_\mu (C - E') + \overline{R}_\mu^\nu \partial_\nu (C - E'), \tag{2.26}$$

$$\delta R_\mu^\nu = \delta^{(\mathrm{gi})} R_\mu^\nu - [\overline{R}_\mu^\nu]'(C - E'), \tag{2.27}$$

where $\delta^{(\mathrm{gi})}$ denotes a variation which preserves gauge-invariance and where

$$\delta^{(\mathrm{gi})} R_w^w = \frac{1}{a^2}\Big\{(d+1)[\Psi'' + \mathcal{H}(\Psi' + \Xi') + 2\mathcal{H}'\Xi] + \partial_\alpha \partial^\alpha \Xi\Big\} \tag{2.28}$$

$$\delta^{(\mathrm{gi})} R_\mu^w = \frac{1}{a^2}\Big\{d\partial_\mu[\mathcal{H}\Xi + \Psi'] + \frac{1}{2}\partial_\alpha\partial^\alpha V_\mu\Big\}, \tag{2.29}$$

$$\begin{aligned}\delta^{(\mathrm{gi})} R_\mu^\nu &= \frac{1}{a^2}\Big\{\partial_\mu \partial^\nu [\Xi - (d-1)\Psi] \\ &+ \delta_\mu^\nu [\Psi'' + (2d+1)\mathcal{H}\Psi' - \partial_\alpha\partial^\alpha \Psi + \mathcal{H}\Xi' + 2(\mathcal{H}' + d\mathcal{H}^2)\Xi] \\ &- [(\partial_{(\mu} V^{\nu)})' + d\mathcal{H}(\partial_{(\mu} V^{\nu)})] + h_\mu^{\nu\,\prime\prime} + d\mathcal{H} h_\mu^{\nu\,\prime} - \partial_\alpha \partial^\alpha h_\mu^\nu\Big\}.\end{aligned} \tag{2.30}$$

Bearing in mind the symmetry properties of the background energy-momentum tensor we can also write the fluctuations of τ_A^B in gauge-invariant terms:

$$\delta\tau_w^w = \delta^{(\mathrm{gi})}\tau_w^w - [\overline{\tau}_w^w]'(C - E'), \tag{2.31}$$
$$\delta\tau_\mu^w = \delta^{(\mathrm{gi})}\tau_\mu^w - \overline{\tau}_w^w\partial_\mu(C - E') + \overline{\tau}_\mu^\nu\partial_\nu(C - E'), \tag{2.32}$$
$$\delta\tau_\mu^\nu = \delta^{(\mathrm{gi})}R_\mu^\nu - [\overline{\tau}_\mu^\nu]'(C - E'). \tag{2.33}$$

Inserting now Eqs. (2.25)–(2.27) and Eqs. (2.31)–(2.33) into Eqs. (2.8) we obtain, after the use of Eqs. (2.4), the gauge-invariant form of the perturbed equations, namely

$$(d+1)[\Psi'' + \mathcal{H}(\Psi' + \Xi') + 2\mathcal{H}'\Xi] + \partial_\alpha\partial^\alpha\Xi = 8\pi G_D a^2\delta^{(\mathrm{gi})}\tau_w^w \tag{2.34}$$
$$d\partial_\mu[\mathcal{H}\Xi + \Psi'] + \frac{1}{2}\partial_\alpha\partial^\alpha V_\mu = 8\pi G_D a^2\delta^{(\mathrm{gi})}\tau_\mu^w \tag{2.35}$$
$$\begin{aligned} & h_\mu^{\nu\,\prime\prime} + d\mathcal{H}h_\mu^{\nu\,\prime} - \partial_\alpha\partial^\alpha h_\mu^\nu \\ + \ & \delta_\mu^\nu[\Psi'' + (2d+1)\mathcal{H}\Psi' - \partial_\alpha\partial^\alpha\Psi + \mathcal{H}\Xi' + 2(\mathcal{H}' + d\mathcal{H}^2)\Xi] \\ + \ & \partial_\mu\partial^\nu[\Xi - (d-1)\Psi] - [(\partial_{(\mu}V^{\nu)})' + d\mathcal{H}(\partial_{(\mu}V^{\nu)})] = 8\pi G_D a^2\delta^{(\mathrm{gi})}\tau_\mu^\nu. \end{aligned} \tag{2.36}$$

The gauge invariant variation of the brane energy-momentum tensor can be obtained once the brane action is specified and specific examples will be provided in the following Section. In spite of this it should be noticed that the derivation presented in this paper is completely independent on the specific form of the brane energy-momentum tensor. It is only required that the generic brane configuration is a solution of the five-dimensional extension of the Einstein-Hilbert theory.

3 Localization of the modes of the geometry

The formalism discussed in the previous section is based on the construction of a gauge-invariant fluctuation of the evolution equations describing the brane configuration. The following five-dimensional action †.

$$S = \int d^5x\sqrt{|G|}\Big[-\frac{R}{2\kappa} + \frac{1}{2}G^{AB}\partial_A\varphi\partial_B\varphi - V(\varphi)\Big], \tag{3.1}$$

can be used in order to describe the breaking of five-dimensional Poincaré symmetry. Consider a potential which is invariant under the $\varphi \to -\varphi$ symmetry. Then, non-singular domain-wall solutions can be obtained, for various potentials. For instance solutions of the type

$$a(w) = \frac{1}{\sqrt{b^2w^2 + 1}}, \tag{3.2}$$
$$\varphi = \varphi(w) = \sqrt{2d}\,\arctan bw, \tag{3.3}$$

†Notice that $\kappa = 8\pi G_5 = 8\pi/M_5^3$. Natural gravitational units will be often employed by setting $2\kappa = 1$.

can be found for different classes of symmetry-breaking potentials [15, 13, 17]. Solutions like Eqs. (3.2)–(3.3) represent a smooth version of the Randall-Sundrum scenario [6, 7]. The assumptions of the present analysis will now be listed.

(i) The five-dimensional geometry is regular (in a technical sense) for any value of the bulk coordinate w. This implies that singularities in the curvature invariants are absent.

(ii) D-dimensional Poincaré invariance is broken through a smooth five dimensional domain-wall solution generated by a potential $V(\varphi)$ which is invariant under $\varphi \to -\varphi$. The warp factor $a(w)$ will then be assumed symmetric for $w \to -w$.

(iii) Four-dimensional Planck mass is finite because the following integral converges

$$M_P^2 \simeq M^d \int_{-\infty}^{\infty} dw a^d(w). \tag{3.4}$$

(iv) Five-dimensional gravity is described according to Eq. (3.1) and, consequently, the equations of motion for the warped background generated by the smooth wall are, in natural gravitational units,

$$\varphi'^2 = 2d(\mathcal{H}^2 - \mathcal{H}'), \tag{3.5}$$

$$V a^2 = -d(d\mathcal{H}^2 + \mathcal{H}'). \tag{3.6}$$

where the prime denotes derivation with respect to w and $\mathcal{H} = a'/a$.

Under these assumptions it will be shown that the gauge-invariant fluctuations corresponding to scalar and vector modes of the geometry are not localized on the wall. On the contrary, tensor modes of the geometry will be shown to be localized. This program will be achieved in two steps. In the first step decoupled equations for the gauge-invariant variables describing the fluctuations of the geometry will be obtained for general brane backgrounds without assuming any specific solution. In the second step the normalizability of the zero modes will be addressed using only the assumptions (i)–(iv).

Recalling that, in the case of the action (3.1)

$$\tau_A^B = \partial_A\varphi\partial^B\varphi - 2\frac{V}{d}\delta_A^B \tag{3.7}$$

the formalism discussed in the previous Section can be applied. Consider now, for sake of simplicity, the case $d = 3$ (the same can be however discussed for a generic d). In this case, following the notations of the previous Section we get :

$$\begin{aligned} \overline{\tau}_\mu^\nu &= -\frac{2}{3} V \delta_\mu^\nu, \\ \overline{\tau}_w^w &= -\frac{\varphi'^2}{a^2} - \frac{2}{3} V, \\ \overline{\tau}_\mu^w &= 0. \end{aligned} \tag{3.8}$$

The gauge-invariant fluctuation of τ_A^B will instead be:

$$\begin{aligned}
\delta^{(gi)}\tau_\mu^\nu &= -\frac{2}{3}\frac{\partial V}{\partial\varphi}X\delta_\mu^\nu,\\
\delta^{(gi)}\tau_w^w &= -\frac{2}{a^2}\Xi - \frac{2}{a^2}\varphi' X' - \frac{2}{3}\frac{\partial V}{\partial\varphi}X,\\
\delta^{(gi)}\tau_\mu^w &= -\frac{\varphi'}{a^2}\partial_\mu X,
\end{aligned} \tag{3.9}$$

where

$$X = \chi - \varphi'(E' - C). \tag{3.10}$$

is the gauge-invariant fluctuation of the scalar field φ. Hence, using Eqs. (2.31) and (2.32) into Eqs. (2.34)–(2.36) we get

$$\Psi'' + 7\mathcal{H}\,\Psi' + \mathcal{H}\,\Xi' + 2(\mathcal{H}' + 3\mathcal{H}^2)\,\Xi + \frac{1}{3}\frac{\partial V}{\partial\varphi}a^2 X - \partial_\alpha\partial^\alpha\Psi = 0, \tag{3.11}$$

$$-\,\partial_\alpha\partial^\alpha\,\Xi - 4\,[\Psi'' + \mathcal{H}\,\Psi'] - 4\mathcal{H}\,\Xi' - \varphi' X' - \frac{1}{3}\frac{\partial V}{\partial\varphi}a^2 X + \frac{2}{3}V a^2\Xi = 0. \tag{3.12}$$

Eqs. (3.11) and (3.12) are subjected to the constraints

$$\partial_\mu\partial_\nu[\Xi - 2\Psi] = 0, \tag{3.13}$$

$$6\mathcal{H}\,\Xi + 6\Psi' + X\,\varphi' = 0. \tag{3.14}$$

These equations should be supplemented by the gauge-invariant version of the perturbed scalar field equation

$$\delta G^{AB}\left(\partial_A\partial_B\varphi - \overline{\Gamma}_{AB}^C\,\partial_C\varphi\right) + \overline{G}^{AB}\left(\partial_A\partial_B\chi - \overline{\Gamma}_{AB}^C\partial_C\chi - \delta\Gamma_{AB}^C\partial\varphi\right) + \frac{\partial^2 V}{\partial\varphi^2}\chi = 0, \tag{3.15}$$

where $\delta\Gamma_{AB}^C$ are the fluctuations of the connections. The explicit (gauge-invariant) version of Eq. (3.15) is

$$\partial_\alpha\partial^\alpha X - X'' - 3\mathcal{H}X' + \frac{\partial^2 V}{\partial\varphi^2}a^2 X - \varphi'[4\Psi' + \Xi'] - 2\Xi\,(\varphi'' + 3\mathcal{H}\varphi') = 0. \tag{3.16}$$

The evolution of the gauge-invariant vector variable (2.18) is

$$\partial_\alpha\partial^\alpha\mathcal{V}_\mu = 0, \quad \mathcal{V}_\mu' + \frac{3}{2}\mathcal{H}\mathcal{V}_\mu = 0, \tag{3.17}$$

where $\mathcal{V}_\mu = a^{3/2}V_\mu$ is the canonical normal mode of the action (3.1) perturbed to second order in the amplitude of vector fluctuations of the metric.

The equation for the tensors, as expected, decouples from the very beginning:

$$\mu_{\mu\nu}'' - \partial_\alpha \partial^\alpha \mu_{\mu\nu} - \frac{(a^{3/2})''}{a^{3/2}} \mu_{\mu\nu} = 0. \tag{3.18}$$

where $\mu_{\mu\nu} = a^{3/2} h_{\mu\nu}$ is the canonical normal mode of the of the action (3.1) perturbed to second order in the amplitude of tensor fluctuations.

Using repeatedly the constraints of Eqs. (3.13)–(3.14), together with the background relations (3.5)–(3.6), the scalar system can be reduced to the following two equations [10, 11]

$$\Phi'' - \partial_\alpha \partial^\alpha \Phi - z \left(\frac{1}{z}\right)'' \Phi = 0, \tag{3.19}$$

$$\mathcal{G}'' - \partial_\alpha \partial^\alpha \mathcal{G} - \frac{z''}{z} \mathcal{G} = 0, \tag{3.20}$$

where

$$\Phi = \frac{a^{3/2}}{\varphi'} \Psi, \qquad \mathcal{G} = a^{3/2} X - z\Psi. \tag{3.21}$$

The same equation satisfied by Ψ is also satisfied by Ξ by virtue of the constraint (3.13). In Eq. (3.20) and (3.21) the background dependence appears in terms of the "universal" function $z(w)$

$$z(w) = \frac{a^{3/2} \varphi'}{\mathcal{H}}. \tag{3.22}$$

Notice, incidentally, that $\mathcal{G}$ represent the canonical normal modes of the action. If the action (3.1) is perturbed to second order in the amplitude of the scalar fluctuations of the system, then its form, up to total derivatives, is

$$\delta^{(2)} S_S = \int d^4x dw \frac{1}{2} \left[\eta^{\alpha\beta} \partial_\alpha \mathcal{G} \partial_\beta \mathcal{G} - \mathcal{G}'^2 - \frac{z''}{z} \mathcal{G}^2 \right]. \tag{3.23}$$

The results [10] is non trivial [10]. These scalar normal modes are analogous to the ones we would obtain in the case of compact extra-dimensions [23].

It should be appreciated that these equations are completely general and do not depend on the specific background but only upon the general form of the metric and of the action (3.1). In fact, in order to derive Eqs. (3.19)–(3.20) and (3.21)–(3.22) no specific background has been assumed, but only Eqs. (3.5)–(3.6) which come directly from Eq. (3.1) and hold for any choice of the potential generating the scalar brane configuration.

The effective "potentials" appearing in the Schrödinger-like equations (3.19) and (3.20) are dual with respect to $z \to 1/z$. This property can be used in order to discuss the properties of the massive spectrum [10]. Here it is sufficient to notice that the effective potentials can be related to their supersymmetric partner potentials [10] usually defined in the context of supersymmetric quantum mechanics [12]. Under the duality transformation

$z \to 1/z$ the superpotentials related to the equation for Φ and $\mathcal{G}$ go one into the other [10].

Let us now discuss the localization of the zero modes of the various fluctuations and enter the second step of the present discussion. The lowest mass eigenstate of Eq. (3.18) is $\mu(w) = \mu_0 a^{3/2}(w)$. Hence, the normalization condition of the tensor zero mode implies

$$|\mu_0|^2 \int_{-\infty}^{\infty} a^3 \, dw = 2|\mu_0|^2 \int_0^{\infty} a^3(w)\, dw = 1. \tag{3.24}$$

where the assumed $w \to -w$ symmetry of the background geometry has been exploited. Using assumptions (i), (ii) and (iii) the tensor zero mode is then normalizable [6, 7].

Let us now move to the analysis of vector fluctuations. Eq. (3.17) shows that the vector fluctuations are always massless and the corresponding zero mode is $\mathcal{V}(w) \sim \mathcal{V}_0 a^{-3/2}$. Consequently, the normalization condition will be

$$2|\mathcal{V}_0|^2 \int_0^{\infty} \frac{dw}{a^3(w)} = 1, \tag{3.25}$$

which cannot be satisfied if assumption (i), (ii) and (iii) hold. If $a^3(w)$ converges everywhere, $1/a^3(w)$ will not be convergent. Therefore, if the four-dimensional Planck mass is finite the tensor modes of the geometry are normalizable and the vectors are not.

From Eq. (3.19) the lowest mass eigenstate of the metric fluctuation Φ corresponds to $\Phi(w) = \Phi_0 z^{-1}(w)$ and the related normalization condition reads

$$2|\Phi_0|^2 \int_0^{\infty} \frac{dw}{z^2(w)} = 1. \tag{3.26}$$

The integrand appearing in Eq. (3.26) will now be shown to be non convergent at infinity if the geometry is regular. In fact according to assumption (i)

$$\begin{aligned} &R = \frac{4}{a^2}(2\mathcal{H}' + 3\mathcal{H}^2), \\ &R_{MN}R^{MN} = \frac{4}{a^4}(4\mathcal{H}^4 + 6\mathcal{H}'\mathcal{H}^2 + 5\mathcal{H}'^2), \\ &R_{MNAB}R^{MNAB} = \frac{8}{a^4}(2\mathcal{H}'^2 - 5\mathcal{H}^4), \end{aligned} \tag{3.27}$$

should be regular for any w and, in particular, at infinity. The absence of poles in the curvature invariants guarantees the regularity of the five-dimensional geometry. Eq. (3.27) rules then out, warp factors decaying at infinity as e^{-dw} or $e^{-d^2w^2}$: these profiles would lead to divergences in Eqs. (3.27) at infinity [‡]. Since $a(w)$ must converge at infinity, $a(w) \sim w^{-\gamma}$ with $1/3 \leq \gamma \leq 1$. Notice that $\gamma \geq 1/3$ comes from the convergence (at

[‡]In order to avoid confusions it should be stressed that exponential warp factors naturally appear in non-conformal coordinate systems related to the one of Eq. (1.1) as $a(w)dw = dy$.

infinity) of the integral of Eq. (3.4) [§] and that $\gamma \leq 1$ is implied by Eqs. (3.27) since, at infinity, $R_{MN}R^{MN} \sim R_{MNAB}R^{MNAB} \sim w^{4(\gamma-1)}$ should converge. Using Eq. (3.22) and Eq. (3.5) the integrand of Eq. (3.26) can be written as

$$\frac{1}{z^2} = \frac{\mathcal{H}^2}{a^3\varphi'^2} = \frac{1}{6a^3}\left(\frac{\mathcal{H}^2}{\mathcal{H}^2 - \mathcal{H}'}\right). \tag{3.28}$$

The behavior at infinity of Eq. (3.28) can be now investigated assuming the regularity of Eqs. (3.27), i.e. $a(w) \sim w^{-\gamma}$ with $0 < \gamma \leq 1$. In this limit

$$\lim_{w\to\infty}\left(\frac{\mathcal{H}^2}{\mathcal{H}^2 - \mathcal{H}'}\right) \sim \frac{\gamma^2}{\gamma^2 - \gamma}. \tag{3.29}$$

and $1/z^2$ diverges *at least* as a^{-3}. In fact, if $\gamma = 1$, $1/z^2$ diverges even more as it can be argued from Eq. (3.29) which has a further pole for $\gamma^2 = \gamma$. The example given in Eqs. (3.2)–(3.3) corresponds to a behavior at infinity given by $\gamma = 1$. Direct calculations show that $1/z^2$ diverges, in this case, as w^5.

Consequently, if the four-dimensional Planck mass is finite and if space-time is regular the gauge-invariant (scalar) zero mode is not normalizable and not localized on the brane. For sake of completeness it should be mentioned that, for the lowest mass eigenvalue, there is a second (linearly independent) solution to Eq. (3.19) which is given by $z^{-1}(w)\int^w z^2(x)\,dx$ which has poles at infinity and for $w \to 0$. The poles appearing for $w \to 0$ will now be discussed since they are needed in order to prove that the zero modes of Eq. (3.20) are not localized. As far as the poles at infinity are concerned it is interesting to consider what happens to $z^{-1}(w)\int^w z^2(x)\,dx$ in the case of the solution (3.2)–(3.3). In this case, by direct use of Eqs. (3.2)–(3.3) and (3.22) we have that the second solution diverges, at infinity, as $(1+b^2w^2)^{1/4}(1+2b^2w^2)$.

Noticing the duality connecting the effective potentials of Eqs. (3.19) and (3.20) it can be verified that the lowest mass eigenstate of Eq. (3.20) is given by $\mathcal{G}(w) = \mathcal{G}_0 z(w)$. Provided the assumptions (i)–(iv) are satisfied, it will now be demonstrated that the integral

$$2|\mathcal{G}_0|^2\int_0^\infty z^2\,dw, \tag{3.30}$$

is divergent not because of the behavior at infinity but because of the behavior of the solution close to the core of the wall, i.e. $w \to 0$. Bearing in mind Eq. (3.27), assumption (i) and (ii) imply that $a(w)$ and φ should be regular for any w. More specifically close to the core of the wall φ should go to zero and $a(w)$ should go to a constant because of $w \to -w$ symmetry and the following regular expansions can be written for small w

$$a(w) \simeq a_0 - a_1\,w^\beta + ..., \qquad \beta > 0, \tag{3.31}$$
$$\varphi(w) \simeq \varphi_1\,w^\alpha + ..., \qquad \alpha > 0, \tag{3.32}$$

[§]In this sense the power γ measures only the degree of convergence of a given integral.

for $w \to 0$. Inserting the expansion (3.31)–(3.32) into Eq. (3.5) the relations can be obtained:

$$\beta = 2\alpha, \quad \alpha^2\varphi_1^2 = 6\frac{a_1}{a_0}\,\beta(\beta-1). \tag{3.33}$$

Inserting now Eqs. (3.31)–(3.32) into Eq. (3.22) and exploiting the first of Eqs. (3.33) we have

$$\lim_{w\to 0} z^2(w) \simeq w^{2(\alpha-\beta)} = w^{-2\alpha}. \;\; \alpha > 0 \tag{3.34}$$

Using Eqs. (3.31) into Eqs. (3.27), $R_{AB}R^{AB} \sim R_{MNAB}R^{MNAB} \sim w^{2\beta-4}$, which implies $\beta \geq 2$ in order to have regular invariants for $w \to 0$. Since, from Eq. (3.33), $\beta = 2\alpha$, in Eq. (3.34) it must be $\alpha \geq 1$. As in the case of Eq. (3.19) also eq. (3.20) has a second (linearly independent) solution for the lowest mass eigenvalue, namely $z(w)\int^w dx\, z^{-2}(x)$ which has poles at infinity. In fact, a direct check shows that, at infinity, this quantity goes ar $w^{\frac{3}{2}\gamma+1}$ where, as usual, $1/3 \leq \gamma \leq 1$ for the convergence of the Planck mass and of the curvature invariants at infinity.

We showed that the graviphoton and the graviscalars are delocalized under the assumptions (i)–(iv). The delocalization of a given zero mode may also be interpreted as a break-down of perturbation theory since the lowest mass eigenstate is divergent for such a mode [¶]. The interpretation of this phenomenon is that zero-modes whose wave functions are singular decouple from the four-dimensional effective theory. One can wonder if these divergences could be regularized in a gauge-invariant way. A standard way of implementing this regularization procedure is to include quadratic corrections to the Einstein-Hilbert term hoping that the divergences of the zero modes will disappear because of the different equations of the fluctutaions. The same analysis performed here can be extended to the context of theories with higher derivatives [22]. In these theories, brane solutions can be indeed found both in five [22, 25] and in six dimensions[20, 26]. The conclusions obtained are similar to the ones reported in the present analysis.

4 Localization of the modes of the geometry

As far as the localization properties of the various modes of the geometry are concerned, the convergence of the following integrals should be checked:

$$I_{\rm tens} = \int_0^{+\infty} a^d(w)dw, \tag{4.1}$$

$$I_{\rm vec} = \int_0^{\infty} \frac{dw}{a^d(w)}, \tag{4.2}$$

$$I_{\Phi} = \int_0^{+\infty} \frac{dw}{z^2(w)}, \tag{4.3}$$

$$I_{\mathcal{G}} = \int_0^{+\infty} z^2(w)dw, \tag{4.4}$$

[¶]I thank G. Veneziano for stressing this point [24].

where

$$z(w) = \frac{a^{d/2}\varphi'}{\mathcal{H}}, \tag{4.5}$$

and with $d = 3$ in the case of a four-dimensional Poincaré invariant world. It has been demonstrated that under assumptions (i)–(iv), the scalar and vector fluctuations of the five-dimensional metric decouple from the wall. Eqs. (4.1)–(4.4) do not assume any *specific* background solution but only the form (1.1) of the metric together with the background equations. Similarly the obtained results have been discussed and obtained in *general terms*. General means that in order to assess the conclusions presented here no particular solution has been used. The analysis of the fluctuations is independent on the specific coordinate system. The gauge-invariant method proposed in this context can certainly be applied to other (related) contexts.

Heeding experimental tests [27], the present results suggest that, under the assumptions (i)–(iv), no vector or scalar component of the Newtonian potential at short distances should be expected.

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