SUBSYSTEM ENTROPY EXCEEDING BEKENSTEIN'S BOUND *

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Abstract

If Bekenstein's conjectured bound on the microcanonical entropy, $S \leq 2\pi ER$, is applied to a closed subsystem of maximal linear size R and excitation energy up through E, it can be violated by an arbitrarily large factor by a scalar field having a symmetric potential allowing domain walls, and by the electromagnetic field modes in a coaxial cable.

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Motivated by considerations of the generalized second law of thermodynamics for processes in which objects are lowered into black holes, Bekenstein [1] conjectured that the entropy S of a system confined to radius R or less and energy E or less would obey the inequality

$$S \leq 2\pi ER. \tag{1}$$

He and colleagues found many arguments and examples supporting this inequality [1-14], though many counterarguments and counterexamples have also been noted [15-28].

Part of the difficulty of determining whether the bound (1) is correct or not is the ambiguity of what systems are to be considered, and what definitions are to be assigned to S, R, and E. To give a concrete resolution of the first ambiguity, here I shall consider only nongravitational quantum field theories of a single quantum field in Minkowski spacetime, avoiding counterexamples with large numbers of fields [16, 17, 25] (unless it is assumed that the fields have enough positive vacuum energy included in E to save the bound [3, 10, 12]).

For the definitions of S, R, and E, Bekenstein initially [1, 2] took S to be the von Neumann entropy $-Tr(\rho \ln \rho)$ of the system, $R = (A/4\pi)^{1/2}$ with "area A of that (quasi) spherical surface which circumscribes the system," and E to be the mean regularized energy, $Tr(H\rho)$, of the "complete system," which was meant to include any walls needed to keep the system within the (quasi) spherical surface of area $A = 4\pi R^2$. With these definitions, the truth of the conjectured bound (1) for nongravitational systems of a suitably small number of quantum fields in Minkowski spacetime seems to be an open question, since it is difficult to give a detailed description of walls that may be needed to confine the system. If the walls are themselves made up of quantum fields that are not held in place by yet other walls, I would conjecture that no complete stationary system can be totally confined to be within any finite radius R of Minkowski spacetime, in which case the right hand side of (1) is infinite for any system with positive energy, making the bound trivially true for complete systems with a nondegenerate vacuum that has nonnegative energy.

Therefore, to give (1) a nontrivial content, one might prefer to apply it to a subsystem of the universe, say a quantum field inside some boundary circumscribed by a sphere of radius R. Indeed, this is an approach that Schiffer and Bekenstein have advocated as follows [7]:

"But there is no gainsaying the conceptual clarity gained when the bound (1.1) [(1) above] is regarded as applying to the field in the cavity and only the field. This motivates an alternative approach to bound (1.1) which abandons the canonical method in favor of the microcanonical one (entropy is the logarithm of the number of available microstates), interprets E as the available energy above the vacuum state, and ignores the walls of the cavity. ...

"If $\Omega(E)$ denotes the number of quantum states accessible to the field system with energy up to and including E, then the microcanonical definition of entropy is $S(E) = \ln \Omega(E)$."

This approach avoids counterexamples to (1) [15, 16, 20] from negative Casimir energies by simply redefining the ground state energy of the field in the cavity to be zero. By using the microcanonical entropy $\ln \Omega(E)$ instead of the canonical or von Neumann entropy $-Tr(\rho \ln \rho)$, it also avoids counterexamples to (1) [18, 25, 28] from very low temperatures when $E = 0$ for the ground state, as it is defined to be here.

Using these definitions of S, R, and E, I now wish to show two types of counterexamples for the conjectured entropy bound (1) for a single quantum field within a cavity. For each the ratio of the left hand side to the right hand side,

$$B \equiv \frac{S}{2\pi E R}, \tag{2}$$

can be made arbitrarily large, thereby violating the conjectured entropy bound (1) by an arbitrarily large factor.

The first counterexample uses a field with nontrivial self-interaction [28], which old arguments [16] had suggested could be used to violate (1).

In particular, consider a scalar field ϕ whose potential energy density $V(\phi)$ is symmetric in ϕ and has its global minima at $\phi = \pm\phi_m \neq 0$, and impose the condition $\phi = 0$ at the boundary of the region under consideration, say a cavity of radius R. If the region is large enough, the energy of a classical configuration with $\phi = 0$ at the boundary will have a global minimum for a configuration in which ϕ is near ϕ_m over most of the region and then drops smoothly to 0 at the boundary. (The region must be large enough that the reduction in the potential energy density from $V(0)$ to $V(\phi_m)$ integrates to more than the increase in the "kinetic" or gradient energy density from the spatial gradients of ϕ near the boundary. For a spherical region, the potential energy reduction is of the order of $[V(0) - V(\phi_m)]R^3$, whereas the gradient energy increase is of the order of at least $(\phi_m/R)^2 R^3$, so the former definitely dominates if

$$R \gg \phi_m / \sqrt{V(0) - V(\phi_m)}, \tag{3}$$

allowing a nonuniform $\phi(x^i)$ to minimize the energy.)

Classically, there is another global energy minimum of exactly the same energy with $\phi(x^i)$ replaced by $-\phi(x^i)$. But quantum mechanically, there will be some tiny tunneling rate between these two classical configurations, so the quantum ground state, of energy E_0, will be a symmetric superposition of the two classical energy minima (plus quantum fluctuations of all the other modes). However, there will also be an excited state of energy $E_1 > E_0$ which is the antisymmetric superposition of the two classical energy minima (plus other fluctuations). For a large region, the energy excess $E_1 - E_0$ of this excited state will be exponentially tiny. The number of orthogonal quantum states with energy up to and including $E = E_1$ will be $\Omega(E_1) = 2$, so the microcanonical entropy of that energy is $S(E_1) = \ln\Omega(E_1) = \ln 2$.

When the two classical extrema configurations $\phi(x^i)$ and $-\phi(x^i)$ are well separated, we can estimate that the excitation energy $E_1 - E_0$ is given by some energy scale multiplied by e^{-I}, where I is the Euclidean action of an instanton that tunnels between the two classical extrema configurations $-\phi(x^i)$ and $+\phi(x^i)$. This instanton will be a solution of the Euclidean equations of motion of the field ϕ that obeys the boundary condition $\phi = 0$ at spatial radius $r = R$ for all Euclidean times τ, but which for $r < R$ interpolates between $-\phi(x^i)$ and $+\phi(x^i)$ as τ goes from $-\infty$ to $+\infty$. When the strong inequality (3) applies, the static configuration $+\phi(x^i)$ that applies asymptotically for large positive τ is very near ϕ_m over almost all of the spatial volume (except very near $r = R$), and the Euclidean instanton is essentially a domain wall concentrated at some Euclidean time that can be chosen to be $\tau = 0$.

The energy-per-area or action-per-three-volume of the domain wall is

$$\varepsilon = \int_{-\phi_m}^{+\phi_m} \sqrt{2V(\phi)}\, d\phi, \tag{4}$$

and the three-volume of the Euclidean section at $\tau = 0$ across the ball $r \leq R$ is $4\pi R^3/3$, so the Euclidean action of the instanton is

$$I \approx \frac{4\pi}{3} R^3 \varepsilon. \tag{5}$$

A suitable energy scale to multiply e^{-I} is $R^2\varepsilon$. At our level of approximation, it does not help to try to get the numerical coefficient of the energy scale correct, since our estimate (5) of the Euclidean action, though being the dominant piece when it is large in comparison with unity, has smaller corrections that are also large in comparison with unity. Therefore, using $\sim$ to mean that the logarithms of the two sides are approximately equal (up to differences that are small in comparison with the logarithms themselves but which may actually be large in comparison with unity, so that the ratio of the two sides themselves may be much different from unity), we get

$$E_1 - E_0 \sim R^2 \varepsilon e^{-I} \sim R^2 \varepsilon \exp\left(-\frac{4\pi}{3} R^3 \varepsilon\right). \tag{6}$$

When the strong inequality (3) holds, classically the energy E_0 is essentially the energy of a half-thickness of domain wall at the radius R, approximately $2\pi R^2 \varepsilon$. As Bekenstein has noted [14], for large I this classical energy (and also the energy of any wall outside R needed to keep $\phi = 0$ at R) dominates over $E_1 - E_0$ (though, incidentally, it does not go to zero as $R \to \infty$, as he also claimed). However, here I am only considering the field subsystem inside R (and so am not counting the walls), and I am also following the approach of Schiffer and Bekenstein [7] in taking "E as the available energy above the vacuum state" (which avoids counterexamples to the conjectured bound for other closed subsystems with negative Casimir energy).

Therefore, taking $E = E_1 - E_0$ gives

$$B \equiv \frac{S}{2\pi E R} = \frac{\ln 2}{2\pi(E_1 - E_0)R} \sim \frac{e^I}{I} \sim \frac{\exp\left(\frac{4\pi}{3}R^3\varepsilon\right)}{R^3\varepsilon} \sim \exp\left(\frac{4\pi}{3}R^3\varepsilon\right), \tag{7}$$

which can be made arbitrarily large by making R arbitrarily large. Indeed, $B - 1$, the violation of Bekenstein's conjectured bound (1) if it is positive, grows large very rapidly with R large enough to obey the inequality (3).

For instance, take a toy model in which

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - \phi_m^2)^2 = \frac{\lambda}{4}\phi^4 - \frac{1}{4}m^2\phi^2 + \frac{m^4}{16\lambda} \tag{8}$$

where $\lambda \ll 1$ and m is the mass of small field oscillations around the potential minima at $\phi = \pm\phi_m = \pm m/\sqrt{2\lambda}$. (If it is objected that this model is actually a trivial quantum field theory [29, 14], then use instead a well-defined supersymmetric model in which the bosonic sector has a scalar field with this potential.) Then

$$\varepsilon = \frac{2}{3}\sqrt{2\lambda}\,\phi_m^3 = \frac{m^3}{3\lambda}, \tag{9}$$

so for $R \gg \phi_m/\sqrt{V(0) - V(\phi_m)} = 2\sqrt{2}/m$, one gets

$$I \approx \frac{8\sqrt{2}\,\pi}{9}\lambda^{1/2}\phi_m^3 R^3 = \frac{4\pi}{9\lambda}m^3R^3 \gg 1. \tag{10}$$

When this Euclidean tunneling action is inserted into Eq. (7), one gets that the quantity B, which Bekenstein has conjectured to be bounded above by unity, is

$$\begin{aligned} B \;&\sim\; \frac{e^I}{I} \sim e^I \sim \exp\left(\frac{8\sqrt{2}\,\pi}{9}\lambda^{1/2}\phi_m^3 R^3\right) = \exp\left(\frac{4\pi}{9\lambda}m^3R^3\right) \\ &\sim 10^{10^{100}} \exp\left\{\left(10^{100}\ln 10\right)\left[\left(\frac{10^{-12}}{\lambda}\right)\left(\frac{m}{10^{16}\text{ GeV}}\right)^3\left(\frac{R}{0.502269\text{cm}}\right)^3 - 1\right]\right\}. \end{aligned} \tag{11}$$

This is thus larger than a googolplex for

$$R > 0.50227\text{ cm}\left(\frac{10^{16}\text{GeV}}{m}\right)\left(\frac{\lambda}{10^{-12}}\right)^{1/3}. \tag{12}$$

The second counterexample to Bekenstein's conjectured entropy bound (1) in the microcanonical approach uses simply the free electromagnetic field inside the annular region of a long coaxial cable that forms a closed loop of length $L \gg R$ coiled up inside the sphere of radius R.

For simplicity, take the inner and outer coaxial cable cylinders to provide perfectly conducting boundary conditions for the electromagnetic field between them, and let this region between the perfectly conducting cylinders be vacuum (except for the electromagnetic field itself), rather than the dielectric used to keep the inner and outer conducting cylinders apart in realistic coaxial cables. As Schiffer and Bekenstein advocated [7], we shall again take the energy E to be the available electromagnetic field energy over that of the electromagnetic vacuum within the long annular cavity, ignoring the energy of the walls of the cavity itself (the hypothetical perfectly conducting cylinders).

When the coaxial cable lies along the z-axis, and $\rho = \sqrt{x^2 + y^2}$ is the cylindrical radial coordinate, the transverse electromagnetic (TEM) wave mode [30], in the vacuum annular region $\rho_1 < \rho < \rho_2$ between the inner perfectly conducting cylinder at radius ρ_1 and the outer perfectly conducting cylinder at radius ρ_2, has an electromagnetic field two-form

$$\mathbf{F} = \mathbf{d}[f(t-z) + g(t+z)] \wedge \mathbf{d} \ln \rho, \tag{13}$$

a radial electric field and azimuthal magnetic field whose strengths vary as $1/\rho$ in the radial direction and have a wave behavior in the time and z-direction. Here $f(t-z)$ gives a TEM wave moving at the speed of light in the positive z-direction, and $g(t+z)$ gives a TEM wave moving at the speed of light in the negative z-direction.

When the coaxial cable is coiled up so that it lies entirely within the sphere of radius R, the TEM wave forms will be altered by a fractional amount that is small when the radius of bending of the cable is much greater than its cylindrical radius ρ_2, which will be assumed to be the case here. Then TEM waves will still move along the cable with very nearly the speed of light. If the two ends of the cable are connected together to form a closed loop of length L, the angular frequency of TEM modes in the loop will be very nearly

$$\omega_n = 2\pi|n|/L \tag{14}$$

for each nonzero integer n. (Each eigenfrequency has two modes, $n = \pm|n|$, one for each direction the wave can go around the loop.)

Now if we take the energy $E = \omega_1 = 2\pi/L$, then $\Omega(E)$, the number of quantum states accessible to this electromagnetic field system with energy up to and including E, is 3, since there is the electromagnetic vacuum state with energy defined to be zero, the state with one photon of energy E in the mode $n = 1$, and the state with one photon of energy E in the mode $n = -1$ (going in the opposite direction around the loop). Therefore, the microcanonical definition of the entropy for this energy is $S(E) = \ln \Omega(E) = \ln 3$, so

$$B \equiv \frac{S}{2\pi ER} = \frac{(\ln 3)L}{4\pi^2 R}. \tag{15}$$

This exceeds unity if

$$L > \frac{4\pi^2}{\ln 3} R \approx 35.9348\, R. \tag{16}$$

For example, suppose that we coil up the coaxial cable of cylindrical radius $r = \rho_2$ into a close-packed roll, so that in a small cross-section locally perpendicular to the coiled cable, each piece of the cable intersecting this cross-section takes up an hexagonal area $A = \sqrt{12}r^2$. If we fill the sphere of radius R (and hence volume $V = 4\pi R^3/3$) with these coiled up hexagonal cylinders that circumscribe the circular cylinders of the cable itself, so $LA = V$, then $L = 2\pi R^3/(3\sqrt{3}r^2)$, giving

$$B = \frac{\ln 3}{6\sqrt{3}\pi}\left(\frac{R}{r}\right)^2. \tag{17}$$

This obviously can be made arbitrarily large by making the radius R of the sphere enclosing the electromagnetic field system arbitrarily larger than the cylindrical radius r of the outer perfectly conducting cylindrical boundary of the coaxial cable loop containing the electromagnetic field system.

This example appears to be not only a counterexample to Bekenstein's conjectured entropy bound $B \leq 1$ in the microcanonical form for a closed subsystem of maximal linear size R, but also a counterexample to what is claimed to be proved in [7], that the microcanonical bound holds "for a generic system consisting of a non-interacting quantum field in three space dimensions confined to a cavity of arbitrary shape and topology," "any free quantum field which is described by a Hermitian, positive-definite Hamiltonian H. Simple examples are the scalar, electromagnetic, and Dirac fields."

The question now arises as to whether Bekenstein's conjectured entropy bound can be differently interpreted so that it is still viable. As Bekenstein has emphasized [14] in his rebuttal to an earlier paper of mine [28] that gave similar counterexamples to various forms of his conjecture, it may be best to restrict the conjecture to complete systems [3, 10]. If the conjecture is then to be nontrivial, the challenge would be to find a definition of what it means for a complete system to be circumscribed by finite radius R. Work on this is reported elsewhere [31].

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