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Notes on Fluctuations and Correlation Functions in Holographic Renormalization Group Flows

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Abstract

We study the coupled equations describing fluctuations of scalars and the metric about background solutions of $\mathcal{N} = 8$ gauged supergravity which are dual to boundary field theories with renormalization group flow. For the case of a kink solution with a single varying scalar, we develop a procedure to decouple the equations, and we solve them in particular examples. However, difficulties occur in the calculation of correlation functions from the fluctuations, presumably because the AdS/CFT correspondence has not yet been properly implemented in the coupled scalar-gravity sector. Some new examples of correlators of operators dual to simpler uncoupled bulk scalars are given and are satisfactory. As byproducts of our study we make some observations relevant to the stability of domain walls in the brane-world scenario and to the Hamilton-Jacobi formulation of holographic RG flows.

1 Introduction

In these notes we study correlation functions in certain four-dimensional field theories by examining fluctuations about kink solutions of gauged 5-dimensional $\mathcal{N}=8$ supergravity [1, 2]. In light of the AdS/CFT correspondence [3, 4, 5], these gravitational backgrounds have been interpreted as the duals of $\mathcal{N}=4$ Super Yang-Mills theory either perturbed by various relevant operators [6, 7, 8, 9, 10], or given expectation values for some scalar fields [11]. Two-, three- and four-point correlation functions in pure $\mathcal{N}=4$ SYM are by now well-understood [4, 5, 12, 13], while in kink backgrounds a few two-point functions have been calculated for operators whose dual scalars are inert [11]. Here we focus on the more complex case of field theory operators whose dual supergravity scalars are varying radially in the kink background, although we obtain some new results for inert scalars also.

Implicit in our use of the AdS/CFT correspondence is the assumption that 5D maximally supersymmetric gauged supergravity arises as a consistent truncation of ten-dimensional Type IIB supergravity on $AdS_5 \times S^5$. In the case where the dual field theory is on the Coulomb branch, the "lifts" of certain five-dimensional solutions to ten-dimensional type IIB supergravity are known and correspond to continuous distributions of D3-branes in the compact extra dimensions [11, 14]. Other ten-dimensional configurations of non-coincident D-branes have also been studied [15, 16, 17]. However, we shall largely take a five-dimensional viewpoint.

Five-dimensional $\mathcal{N}=8$ gauged supergravity possesses 42 scalar fields φ^I, and a potential $V(\varphi)$ which depends on 40 of them. The scalars fall into various representations of the $SO(6)$ gauge group: there is a $\mathbf{20'}$ representation dual to the dimension 2 primary operators $\mathrm{Tr}\, X^2$, a $\mathbf{10} \oplus \overline{\mathbf{10}}$ representation corresponding to their dimension 3 descendents, and finally the singlet axion-dilaton, which does not enter into $V(\varphi)$ and is dual to the $\mathcal{N}=4$ Lagrangian $\mathcal{L} = \mathrm{Tr}\, F^2 + \cdots$, and thus corresponds to the exactly marginal coupling $\theta + 4\pi i/g^2$.

The part of the supergravity action essential for our concerns is

$$S = \int d^5x \sqrt{g} \left[-\frac{1}{4} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \varphi^I \partial_\nu \varphi^I - V(\varphi) \right] , \tag{1}$$

where the signature of the bulk metric is $(+ - - - -)$. In these conventions the scalar fields are dimensionless. Tractable kink solutions typically have 1 or 2 "active" scalars $\varphi^I(r)$, which depend on the radial coordinate r, while the remaining "inert" scalars φ^α are constant (and typically vanish). The bulk geometry has an r-dependent scale factor, and the kink background takes the form

$$\begin{aligned} ds^2 &= e^{2A(r)} (\eta_{ij}\, dx^i\, dx^j) - dr^2 , \\ \varphi^I &= \varphi^I(r) , \end{aligned} \tag{2}$$

which is invariant under 4-dimensional Poincaré transformations of the $x^i, i = 0, 1, 2, 3$. The geometries are asymptotically like AdS_5, *i.e.* $A(r) \sim r/L$ as $r \to \infty$. In most cases there is a curvature singularity in the interior, *i.e.* $A(r) \to -\infty$ as $r \to r_s$. The significance of the singularity will be briefly discussed below.

We are primarily interested in fluctuations and correlation functions of SYM operators $\mathcal{O}_I(x)$ dual to the bulk scalars $\varphi^I(x)$. Due to the reduced symmetry in the kink geometries, only two-point correlators are currently tractable to analysis. For inert scalars φ^α, the procedure is straightforward in principle. One must obtain solutions of the equations of motion of the fluctuations $\tilde{\varphi}^\alpha(r,x)$, governed by the action (1) expanded to second order in $\tilde{\varphi}^\alpha$. Notice that the effective mass $U(\varphi^I)$ depends on the active scalars:

$$S_\alpha[\varphi_\alpha] = \frac{1}{2}\int d^5x\sqrt{g}\,\left(g^{\mu\nu}\partial_\mu\tilde{\varphi}^\alpha\partial_\nu\tilde{\varphi}^\alpha - U(\varphi^I(r))\,\tilde{\varphi}^{\alpha 2}\right). \tag{3}$$

The solution $\tilde{\varphi}^\alpha$ must be chosen to vanish at the singularity, $r = r_s$, and obey Dirichlet boundary conditions $\tilde{\varphi}^\alpha(R,x) = \Phi_\alpha(x)$ at some large cutoff radius R. The action (3) is then interpreted as the generating functional for the field theory. Evaluated on the solutions to the equations of motion it reduces to a boundary term, and the correlation function is obtained from the $R \to \infty$ limit,

$$\tilde{S}_\alpha[\varphi_\alpha] = \frac{1}{2}\int d^4x\sqrt{g}\,g^{rr}\,\tilde{\varphi}^\alpha(R,x)\,\partial_r\,\tilde{\varphi}^\alpha(R,x)\,, \tag{4}$$

as prescribed for two-point functions in pure AdS by [4, 12].

Since $U(\varphi^I(r))$ vanishes for the dilaton, the dilaton two-point function is usually the simplest to compute. Analyticity properties (in the momentum p^i conjugate to x^i) reveal the spectrum of the boundary theory in the large N, large $\lambda = g^2N$ limit. For the various Coulomb branch flows of [11, 14], one finds either a mass gap and continuous spectrum or an infinite discrete spectrum of poles. Since the supergravity description formally breaks down at the singularity, it is arguable whether these are actual features of the field theory dynamics or artifacts of a poor approximation. We have little to add to these arguments except to point out that a similar discrete spectrum has also been found for a distribution of D3-branes with non-singular metric [16]. Some new examples of correlators of the dilaton and other inert scalars will be given below.

For active scalars the situation is more complex because the fluctuations $\tilde{\varphi}^I(r,x)$ are coupled by the equations of motion to metric perturbations $h_{ij}(r,x)$, defined by

$$\begin{aligned} ds^2 &= e^{2A(r)}\,(\eta_{ij} + h_{ij}(r,x))\,dx^i dx^j - dr^2\,, \\ \varphi^I &= \bar{\varphi}^I(r) + \tilde{\varphi}^I(r,x)\,. \end{aligned} \tag{5}$$

It is especially interesting that the $\tilde{\varphi}_I$ couple to trace components of h_{ij}, *i.e.*

$$h_{ij} = \frac{1}{4}\,h(r,x)\,\eta_{ij} - \frac{\partial}{\partial x^i}\frac{\partial}{\partial x^j}\,H(r,x)\,. \tag{6}$$

This is expected if the flow $\bar{g}_{\mu\nu}(r), \bar{\varphi}^I(r)$ describes a relevant deformation of $\mathcal{N} = 4$ SYM. In this case the stress tensor satisfies the operator relation

$$T_i^i(x) = \sum_I \beta^I\,\mathcal{O}_I(x)\,, \tag{7}$$

where $\mathcal{O}_I(x)$ is the relevant operator dual to φ^I, and β^I is the beta-function for its coupling. Hence it is natural that the trace of h_{ij}, which is dual to T^i_i, should not be independent of the φ^I. One might hope that the present line of investigation interfaces with more general studies of holographic RG flows [18, 19] and perhaps with ideas concerning the c-theorem, either holographic [6, 8, 23] or field-theoretic [20, 21, 22].

On the other hand for a flow describing the Coulomb branch of $\mathcal{N}=4$ SYM, one expects that $T^i_i(x)=0$, since the trace vanishes as an operator if the Lagrangian contains only marginal couplings. One then hopes that despite the fact that $\tilde{\varphi}^I$ and h^i_i are coupled, h^i_i will not excite T^i_i. Since it is well-known that scalars dual to vevs fall off more rapidly on the boundary than those dual to operator perturbations, it is reasonable to expect that the excitation of h^i_i caused by a fluctuation $\tilde{\varphi}^I$ scales away too quickly to excite an operator in the field theory.

In the next section we derive the coupled linear fluctuation equations of $\tilde{\varphi}^I$, h, and H. Other components of h_{ij} decouple consistently; the transverse traceless modes are known each to obey the same wave equation as the dilaton, and the remaining components can be set to zero by a choice of coordinates. We discuss a pure diffeomorphism solution to the equations, the universal solution. A more detailed analysis of the coupled fluctuation equations is then undertaken. In the case where the background flow involves a single active scalar $\bar{\varphi}(r)$, we derive an uncoupled third order differential equation for its fluctuation $\tilde{\varphi}(r,x)$. The universal solution allows us to apply reduction of order methods to study this equation. It is also instructive to convert the fluctuation equations to equivalent Schrödinger form where positivity properties of supersymmetric quantum mechanics can be applied. Indeed it turns out that the role of SUSY QM is ubiquitous.

In the following two sections we solve the fluctuation equations in detail for two examples of RG flows in which the background is explicitly known. In section 3 we treat the flow obtained in [9], which was interpreted as a relevant perturbation of $\mathcal{N}=4$ SYM leading to pure $\mathcal{N}=1$ SYM theory in the infrared. In section 4, we discuss the Coulomb branch flow of [11] corresponding to a distribution of D3-branes in a 2-dimensional disc. In both cases we discuss the two-point correlators of the dilaton[1] and of a second inert scalar, and obtain a consistent picture of the spectrum of excitations in the boundary field theory. The fluctuation equation for active scalars is reduced to a hypergeometric equation and solved in both cases, and the metric component $h(r,x)$ is also determined.

At this point we attempt to determine the two-point correlation function of the active scalars from (4) by the standard procedure of identifying the most singular term as $R\to\infty$ which is non-analytic in the momentum p_i conjugate to x^i, but encounter difficulties. For the $\mathcal{N}=1$ flow, the standard procedure does not have the expected structure of leading non-analytic term plus more singular polynomials in p^2. This remains true even if an integration constant initially chosen so that the fluctuation vanishes at the curvature singularity is allowed to vary. For the Coulomb branch flow, it appears that if the integration constant is chosen so

[1]The dilaton correlator for the Coulomb branch flow was calculated in [11], while for the N=1 flow it was obtained in the very recent [23] which appeared during the preparation of this manuscript.

the active scalar fluctuation is regular, the metric perturbation h_i^i does not vanish on the boundary as the argument above for vanishing T_i^i suggests. If the integration constant is chosen to make h_i^i behave as expected on the boundary, then we can extract a correlator which is physically reasonable except for $p^2 = 0$ poles. However, regularity of the fluctuations in the interior then fails.

It is conceivable that the interior curvature singularity is the source of the difficulties above, but it is also possible that the standard procedure used to calculate correlators must be modified in the coupled active scalar/graviton sector. We discuss our attempts to do this in Sec 5. This includes an evaluation of the on-shell action (1), which reduces to boundary terms linear and quadratic in fluctuations. Expected supplementary boundary terms in the gravitational action are added. An issue of diffeomorphism invariance is also discussed. None of this leads to a resolution of the problem, which is left for future work, perhaps by people who can approach it with fresh energy and ideas.

It is worthwhile to point out two byproducts of our analysis of the fluctuation equations. First, the boundary values of the metric and active scalar fluctuations are not independent, as shown in equation (29). It is not clear that this constraint has been incorporated in a recent Hamilton-Jacobi analysis of holographic RG flows [19]. Second, the SUSY QM analysis of the final uncoupled active scalar equations appears to be applicable to the stability of domain walls in the brane-world scenario [24, 25, 26, 27].

2 The coupled graviton/scalar system

We first review the general structure of kink solutions of 5D $\mathcal{N} = 8$ gauged supergravity which preserve partial supersymmetry. We then discuss the equations obeyed by linear fluctuations of the kink backgrounds with particular attention to the equations which couple fluctuations $\tilde{\varphi}^I(r, x)$ of the active scalars $\varphi^I(r)$. Fluctuations of inert scalars and transverse traceless metric fluctuations satisfy uncoupled equations of a simpler structure.

2.1 Background Flows

The scalar field manifold of the supergravity theory can be described as the coset $E_{6(6)}/USp(8)$. The potential $V(\varphi)$ is invariant under $SO(6) \times SL(2, R) \subset E_{6(6)}$, where $SO(6) \sim SU(4)$ is the gauge group and the $SL(2, R)$ factor corresponds to the axion-dilaton. The potential is a very complicated function of 40 scalars, and progress has come from restrictions to smaller sets of fields which preserve some subgroup $H \subset SO(6)$. Symmetry properties insure that such restrictions are consistent with the full dynamics [28]. The full theory also has a complicated coset metric $G_{IJ}(\varphi)$ which simplifies to δ_{IJ} on the restricted scalar subspaces. This metric was therefore omitted in (1).

The critical points of $V(\varphi)$ are significant for the problem of RG flows, and those extrema which preserve at least an $SU(2)$ subgroup of $SO(6)$ have been classified [29]. The potential takes negative values at these extrema, so the solutions of the equations include exact anti-de Sitter geometries with scalars fixed at their critical values. In particular there is a critical point with full $SO(6)$ symmetry at the origin of field space, and the corresponding AdS_5 solution with $SO(4,2)$ isometry group is the well known holographic dual of the conformal phase of $\mathcal{N}=4$, d=4 SYM theory. Kink solutions of the form (2) are topologically like AdS_5 but have a smaller isometry group. As r approaches the boundary at $r=\infty$, the kink metric approaches that of the maximally symmetric AdS_5 solution, and scalars vanish at rates determined by the scale dimensions of their operator duals in the SYM theory. As r decreases toward the interior of the geometry certain known kinks [8] flow toward a second critical point of the potential with more negative cosmological constant. The field theory interpretation is that of $\mathcal{N}=4$ SYM perturbed with a relevant operator such that the perturbed theory has a non-trivial IR fixed point. It would be very desirable to study the fluctuations about such flows, but this cannot be done at present because the explicit form of these kinks is not yet known[2]. All explicitly known flows have a curvature singularity at some finite $r=r_s$. This is associated with a breakdown of the supergravity description. The standard prescription is to impose the boundary condition that fluctuations vanish at the singularity and to proceed to extract correlation functions from the on-shell action (4). This procedure leads to problematic results as we will see, but we do not yet have a better alternative to propose.

All explicitly known flow solutions are supersymmetric in the sense that there are associated Killing spinors in the supergravity theory. It is known that the potential $V(\varphi)$ can be derived from a superpotential $W(\varphi)$,

$$V(\varphi)=\frac{g^2}{8}\sum_I\left(\frac{\partial W(\varphi)}{\partial\varphi^I}\frac{\partial W(\varphi)}{\partial\varphi^I}\right)-\frac{g^2}{3}W(\varphi)^2 . \tag{8}$$

on restricted subsets of the field space which include the active scalars $\varphi^I(r)$. Here g is the $SU(4)$ gauge coupling of the 5D supergravity theory. It has dimensions of 1/length, and it is related to the length scale L of the boundary AdS_5 space and the cosmological constant Λ by

$$g=2/L\,,\qquad \Lambda=-12/L^2=4V(\varphi=0). \tag{9}$$

The main simplification of supersymmetric flows is that any solution of the first order Killing spinor conditions [8]

$$\frac{dA(r)}{dr}=-\frac{g}{3}W(\varphi)\,,\qquad \frac{\partial\varphi^I(r)}{\partial r}=\frac{g}{2}\frac{\partial W(\varphi)}{\partial\varphi^I}\,, \tag{10}$$

automatically gives a Poincaré-invariant kink solution of the more complicated second order field equations of the action (1).

[2]Very recently the 10D lift of the fixed point itself has been constructed by [30].

It is worth noting that non-supersymmetric solutions of the form (2) can also be obtained by solving (10), with a choice of $W(\varphi)$ other than the true superpotential of the theory. It was conjectured in [27] that all solutions of the field equations of the form (2) can be obtained this way. Recently this was explained in terms of Hamilton-Jacobi theory [19]. However, our examples will be flows where $W(\varphi)$ is in fact the true superpotential.

2.2 Fluctuation equations

We now consider fluctuations of the scalars and the metric around such a background geometry. We use the freedom to choose coordinates to fix axial gauge,

$$h_{\mu 5} = 0\,, \tag{11}$$

where $\mu = 0, 1, 2, 3, 5$. With this choice the metric, including fluctuations, has the form

$$ds^2 \;=\; e^{2A(r)}\left((\eta_{ij} + h_{ij}(x,r))\, dx^i\, dx^j\right) - dr^2\,. \tag{12}$$

There is still gauge freedom remaining. We find two classes of residual diffeomorphisms $\delta g_{\mu\nu} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$ preserving axial gauge (11). There are four-dimensional diffeomorphisms

$$\begin{aligned} \epsilon_i &= e^{2A(r)}\,\omega_i(x), \qquad \epsilon_5 = 0\,, \\ \delta h_{ij} &= \partial_i\,\omega_j(x) + \partial_j\,\omega_i(x)\,. \end{aligned} \tag{13}$$

Additionally, there is the transformation [31, 32]

$$\begin{aligned} \epsilon_5 &= \epsilon_5(x)\,, \qquad \epsilon_i = -(\partial_i \epsilon_5(x))\left(e^{2A(r)} \int^r dr'\, e^{-2A(r')}\right)\,, \\ \delta h_{ij} &= -2A'(r)\,\eta_{ij}\,\epsilon_5(x) - 2\,(\partial_i\partial_j\epsilon_5(x))\left(\int^r dr'\, e^{-2A(r)}\right)\,. \end{aligned} \tag{14}$$

Here we are viewing the result of an infinitesimal coordinate transformation on the background as a metric fluctuation.

Both active and inert scalars can fluctuate around their backgrounds: $\varphi^I(x,r) = \bar{\varphi}^I(r) + \tilde{\varphi}^I(x,r)$. For ease of notation, we shall in the future drop the bar and use $\varphi(r)$ to denote the classical background. When we discuss inert scalars, whose background values are zero, we will drop the tilde on the fluctuation.

Einstein's equations for linear fluctuations were derived in [27] as equations relating the first order Ricci tensor to its scalar source. These equations are

$$\begin{aligned} \text{“}R^{(1)}_{ij}\text{”} = e^{2A}\left(\frac{1}{2}\partial_r^2 + 2A'\partial_r\right) h_{ij} + \frac{1}{2}\,\eta_{ij}\,e^{2A} A'\,\partial_r(\eta^{kl} h_{kl}) - \frac{1}{2}\,\Box\, h_{ij} - \\ \frac{1}{2}\,\eta^{kl}\,(\partial_i\partial_j h_{kl} - \partial_i\partial_k h_{jl} - \partial_j\partial_k h_{il}) = -\,\frac{4}{3}\,e^{2A}\,\frac{\partial V(\varphi)}{\partial \varphi^I}\,\tilde{\varphi}^I\,\eta_{ij}\,, \end{aligned} \tag{15}$$

$$R^{(1)}_{55} = -\frac{1}{2}(\partial_r^2 + 2A'\partial_r)\,\eta^{kl}h_{kl} = 4\,\varphi^{I\prime}\tilde\varphi^{I\prime} + \frac{4}{3}\frac{\partial V(\varphi)}{\partial\varphi^I}\tilde\varphi^I\,, \tag{16}$$

$$R^{(1)}_{j5} = \frac{1}{2}\eta^{kl}\,\partial_r\,(\partial_k h_{jl} - \partial_j h_{kl}) = 2\,\varphi'^I\,\partial_j\tilde\varphi^I\,. \tag{17}$$

The notation "$R^{(1)}_{ij}$" indicates that a simplification of the actual $R^{(1)}_{ij}$ equation has been made in (15) as in [27]. The scalar fluctuation equation is

$$e^{-2A}\,\Box\,\tilde\varphi^I - \tilde\varphi^{I\prime\prime} - 4A'\tilde\varphi^{I\prime} + \frac{\partial^2 V(\varphi)}{\partial\varphi^I\partial\varphi^J}\,\tilde\varphi^J = \frac{1}{2}\varphi^{I\prime}\eta^{ij}h'_{ij}\,. \tag{18}$$

A prime above denotes ∂_r. In general primes on fields $h_{ij},\tilde\varphi$ and backgrounds A,φ will denote a derivative with respect to the radial coordinate, while primes on $V(\varphi)$ or $W(\varphi)$ refer to φ-derivatives.

For inert scalars $\tilde\varphi^\alpha$, the right-hand side of (18) vanishes, and consequently they do not couple to the graviton. Any coupling to the active scalars due to the potential term must vanish, otherwise the inert scalars could not have been zero in the background:

$$\frac{\partial^2 V(\varphi)}{\partial\varphi^\alpha\partial\varphi^I} = 0\,. \tag{19}$$

Thus they satisfy decoupled second-order equations, several of which we consider explicitly in sections 3 and 4.

It is clear from the equations above that active scalar fluctuations couple to the graviton. To simplify the coupled system we consider an arbitrary graviton fluctuation, decomposed in a complete momentum-space basis. We use the vectors

$$p^i = (p,0,0,0)\,, \qquad \varepsilon^0 = (0,0,0,1)\,, \qquad \varepsilon^\pm = (0,1,\pm i,0)/\sqrt{2}\,, \tag{20}$$

and the decomposition

$$\begin{aligned} h_{ij}(r,p) \;=\;& \varepsilon^+_i\varepsilon^+_j\,h^{++}(r,p) + \varepsilon^-_i\varepsilon^-_j\,h^{--}(r,p) + (\varepsilon^+_i\varepsilon^0_j + \varepsilon^0_i\varepsilon^+_j)\,h^{+0}(r,p) + \\ & (\varepsilon^-_i\varepsilon^0_j + \varepsilon^0_i\varepsilon^-_j)\,h^{-0}(r,p) + (\varepsilon^+_i\varepsilon^-_j + \varepsilon^-_i\varepsilon^+_j - 2\varepsilon^0_i\varepsilon^0_j)\,h^{00}(r,p) + \\ & (p_i\varepsilon^+_j + \varepsilon^+_i p_j)\,a^+(r,p) + (p_i\varepsilon^-_j + \varepsilon^-_i p_j)\,a^-(r,p) + (p_i\varepsilon^0_j + \varepsilon^0_i p_j)\,a^0(r,p) + \\ & \eta_{ij}\,h(r,p)/4 + p_i\,p_j\,H(r,p)\,. \end{aligned} \tag{21}$$

In this decomposition, the five h^{xx} are transverse traceless, the a^x are traceless and longitudinal, and h and H are the trace components.

The h^{xx} contribute only in (15), and it can be shown that each h^{xx} satisfies the same uncoupled equation as a free massless scalar in the background (2), as is well-known [27, 33, 34]. The h^{xx} modes are the expected five physical components of a graviton in five bulk dimensions, and they have the same fluctuation spectrum as the inert dilaton. This can be obtained from an equivalent Schrödinger equation with supersymmetric potential [27]. These modes do not couple to the active scalars.

Further, we can examine equation (17) and see that the a^x must be r-independent. One may then gauge them to zero using the residual coordinate invariance (13). One is still free to use the transformation (14).

We are left with the components h and H, which do couple to active scalar fluctuations. Before we present the equations of this system, consider the action of the remaining gauge freedom (14). A transformation parameterized by $\epsilon_5(x) = e^{ipx}\,E(p)$ modifies the fields by

$$\delta h = -8A'(r)\,E(p)\,, \quad \delta H' = 2\,e^{-2A(r)}\,E(p)\,, \quad \delta\tilde{\varphi} = -\varphi'(r)\,E(p)\,. \tag{22}$$

One may check that (22) solves the equations of motion (15–18). We will refer to this pure gauge solution as the *universal solution*. It will have a vital technical role in our study of the coupled system.

The fields that remain in our reduced system are

$$h_{ij}(r,x) = e^{ipx}\left(\frac{1}{4}\,h(r,p)\,\eta_{ij} + p_i\,p_j H(r,p)\right)\,, \quad \tilde{\varphi}^I(r,x) = e^{ipx}\tilde{\varphi}^I(r,p)\,. \tag{23}$$

and we now begin the process of simplifying the coupled equations which they satisfy.

Substituting the ansatz (23) into the R_{j5} equation (17), H drops out and we are left with

$$h'(r) = -\frac{16}{3}\,\varphi^{I\prime}\,\tilde{\varphi}^I\,, \tag{24}$$

where we have canceled a uniform factor of p_j.

We now consider linear combinations of the equations (15) traced with the tensors η^{ij} and $p^i\,p^j$. It turns out that scalars decouple in the difference of the η^{ij} trace and the $p^i\,p^j$ trace. Dividing by p^2 leaves a simple equation relating H and h:

$$H'' + 4A'H' = -\frac{1}{2}\,e^{-2A}\,h\,. \tag{25}$$

One can also show that an independent linear combination of the two traces is trivial given the R_{j5} equation (24), so the only new relation obtained is (25).

Substituting the ansatz (23) into the R_{55} equation (16), we obtain

$$-\frac{1}{2}(h'' + 2A'h' + p^2H'' + 2p^2A'H') = 4\,\varphi^{I\prime}\,\tilde{\varphi}^{I\prime} + \frac{4}{3}\,\partial_I V(\varphi)\,\tilde{\varphi}^I\,. \tag{26}$$

Combining (25) and (26), we can eliminate H'' and obtain an algebraic relation for H'. Further removing derivatives of h in favor of $\tilde{\varphi}^I$ using (24) and substituting φ-derivatives of W for φ' and $V'(\varphi)$, we obtain:

$$2A'H' = -\frac{1}{2}\,e^{-2A}\,h + \frac{2g\,\partial_I W(\varphi)}{3p^2}\left[2\,\tilde{\varphi}^{I\prime} - g\,\partial_I\partial_J W(\varphi)\,\tilde{\varphi}^J\right]\,. \tag{27}$$

The equations (24), (25), (27), together with the Klein-Gordon equation (18), constitute $n+3$ equations for the $n+2$ fields h, H and $\tilde{\varphi}^I$. However, one can show that the gravitational equations are related by the Bianchi identity: the expected combination of derivatives of (24), (25), (27) vanishes identically if the Klein-Gordon equation is satisfied.

It also turns out that we can construct an algebraic equation for h which will be of later use. The Klein-Gordon equation (18) becomes

$$-\,p^2\,e^{-2A}\,\tilde{\varphi}^I - \tilde{\varphi}^{I\prime\prime} - 4A'\tilde{\varphi}^{I\prime} + \frac{\partial^2 V(\varphi)}{\partial\varphi^I\partial\varphi^J}\,\tilde{\varphi}^J = \frac{1}{2}\varphi^{I\prime}\left(h' + p^2 H'\right). \tag{28}$$

The left-hand side is just derivatives of $\tilde{\varphi}^I$. On the right, h' can be written in terms of $\tilde{\varphi}^I$ using (24), while we can eliminate H' in favor of $\tilde{\varphi}$ and a factor of h using (27). Thus h is determined solely by $\tilde{\varphi}^I$ and derivatives. To avoid tedious notation, we write only the case of a single active scalar, although the general case is straightforward:

$$\begin{aligned} 3p^2 e^{-2A}\,h = \frac{-\,16W}{W'}\tilde{\varphi}'' + g\left(\frac{64W^2}{3W'} + 8W'\right)\tilde{\varphi}' + \\ 4g^2\left(\frac{WW''^2}{W'} + WW'''(\tilde{\varphi}) - \frac{8W^2W''}{3W'} - W'W'' - \frac{4p^2W}{g^2W'}e^{-2A}\right)\tilde{\varphi}\,. \end{aligned} \tag{29}$$

This equation shows that the boundary values of $h(r,p)$ and $\tilde{\varphi}(r,p)$ are not independent, a fact which may have implications for the formulation of Hamilton-Jacobi dynamics proposed in [19].

Unfortunately our equations are still coupled. The next step is to derive a third order equation involving only the $\tilde{\varphi}^I$. In the case of flows with only a single active scalar, this becomes an uncoupled equation which is the key to the present exploration and can be solved for some specific flows.

The Klein-Gordon equation (18) has the conventional form of a scalar field in the curved background, with an additional source term involving the graviton fluctuation. We can express the source in terms of scalars by noting that the R_{55} equation (16) can be written

$$-\,\frac{1}{2}e^{-2A}\partial_r(e^{2A}\eta^{ij}h'_{ij}) = 4\varphi^{I\prime}\tilde{\varphi}^{I\prime} + \frac{4}{3}\frac{\partial V(\varphi)}{\partial\varphi^I}\,\tilde{\varphi}^I\,, \tag{30}$$

which can be integrated to obtain

$$\eta^{ij}h'_{ij} = -8\,e^{-2A}\int^r dr'\,e^{2A(r')}\left(\varphi^{I\prime}\,\tilde{\varphi}^{I\prime} + \frac{1}{3}\,\partial_I V(\varphi)\,\tilde{\varphi}^I\right). \tag{31}$$

Substituting into (18) and taking an r-derivative, we find the third-order equation

$$\begin{aligned} \partial_r\left\{\frac{\varphi^{I\prime}}{|\varphi^{K\prime}|^2}\,e^{2A}\,\left(\left(\partial_r^2 + 4A'\partial_r - e^{-2A}\,\Box\right)\delta_{IJ} - \partial_I\partial_J V(\varphi)\right)\tilde{\varphi}^J\right\} = \\ 4\,e^{2A}\left(\varphi^{J\prime}\tilde{\varphi}^{J\prime} + \frac{1}{3}\partial_J V(\varphi)\,\tilde{\varphi}^J\right). \end{aligned} \tag{32}$$

Although a simplification, (32) still couples all active scalar fluctuations. Since several known flows involve only one active scalar, we specialize to the case of a single active scalar φ. Then (32) reduces to an uncoupled equation,

$$\partial_r \left\{ \frac{1}{\varphi'} e^{2A} \left(\partial_r^2 + 4A'\partial_r - V''(\varphi) - e^{-2A} \Box \right) \tilde{\varphi} \right\} = 4\, e^{2A} \left(\varphi' \tilde{\varphi}' + \frac{1}{3} V'(\varphi)\, \tilde{\varphi} \right) . \tag{33}$$

In the case of a single active scalar, it is also possible to combine (24, 25, 27) to obtain an uncoupled third order equation for h, but we have chosen to work with (33) in applications to specific flows.

One may easily verify that the universal solution, $\tilde{\varphi} \propto W'(\varphi)\, e^{ipx}$, satisfies (33) by using (8) and (10) to relate $V(\varphi)$, $\varphi'(r)$ and $A'(r)$ to the superpotential $W(\varphi)$ and its derivatives. This allows us to use the method of reduction of order to obtain a second-order equation for the remaining solutions. Writing

$$\tilde{\varphi} = W'(\varphi)\, e^{ipx} \int^r dr'\, R(r', p) \,, \tag{34}$$

for some unknown $R(r,p)$, and using the properties of the flow (8) and (10), we find

$$\begin{aligned} R''(r,p) + g\,(W'' - 2W)\, R'(r,p) \quad &+ \\ g^2 \left(-\frac{2}{3} W W'' + \frac{8}{9} W^2 + \frac{1}{2} W' W''' - W'^2 \right) R(r,p) \quad &+ \quad p^2\, e^{-2A}\, R(r,p) = 0\,. \end{aligned} \tag{35}$$

We remind the reader that primes on $R(r)$ are r-derivatives while those on W are with respect to φ. To determine the boundary scaling rates of the three independent solutions of (33), we may approximate $W(\varphi) = -3/2 + \rho\, \varphi^2/2$ where ρ is related to the operator scale dimension Δ by $\rho = \Delta - 4$ for operator deformations and $\rho = -\Delta$ for vacuum expectation values. Standard Frobenius analysis then gives the exponential rates

$$\tilde{\varphi}(r) \sim \exp(\rho r/L), \quad \exp((\rho - 2) r/L), \quad \exp((-\rho - 4) r/L) \,, \tag{36}$$

for the universal solution, and the two independent solutions of (35), respectively.

By solving (35), we can obtain the solution to $\tilde{\varphi}$ from (34). The integration constant is just the freedom to add a multiple of the universal solution to our result for $\tilde{\varphi}$. This will be important later on for assuring regularity of the solutions at the singularity.

2.3 Supersymmetric Quantum Mechanics

We now discuss the transformation of the fluctuation equations into Schrödinger form, which gives considerable intuition [11] into the nature of the fluctuation spectrum. For the dilaton and for the transverse traceless metric fluctuations [27, 35] the Schrödinger potential is that of a supersymmetric quantum mechanics, and this gives a simple proof that the spectrum of normalizable fluctuations contains no tachyons. We will see that SUSY-QM provides a useful framework in which to consider all fluctuations.

The Schrödinger form obtains after a change of radial coordinate r to the horospherical coordinate z, defined by the line element

$$ds^2 = e^{2A(z)} \left[(\eta_{ij} + h_{ij}(x,z))\, dx^i dx^j - dz^2 \right] , \tag{37}$$

and thus the Jacobian must be

$$\frac{dz}{dr} = \pm\, e^{-A} . \tag{38}$$

We choose the $-$ sign for the present application, so that the boundary region $r \to \infty$ is mapped to $z = 0$ and the deep interior to $z \to \infty$.

Consider first the fluctuation equation for an inert scalar $\tilde{\varphi}(r,p)$ obtained from (3),

$$-\,(\Box + U(\varphi))\, \tilde{\varphi}(r,p) = \tilde{\varphi}''(r,p) + 4A'(r)\, \tilde{\varphi}'(r,p) - U(\varphi(r))\, \tilde{\varphi}(r,p) + e^{-2A}\, p^2\, \tilde{\varphi}(r,p) = 0 . \tag{39}$$

Performing the field transformation $\tilde{\varphi}(r,p) = \exp(-3A/2)\, \psi(z,p)$ as well as the change of coordinate $r \to z$, one finds an equation in Schrödinger form

$$-\,\psi''(z,p) + \mathcal{V}(z)\, \psi(z,p) = p^2 \psi(z,p) , \tag{40}$$

with potential

$$\mathcal{V}(z) = \left(\frac{3}{2} A'(z) \right)^2 + \frac{3}{2} A''(z) + e^{2A(z)}\, U(\varphi(z)) , \tag{41}$$

in which $A'(z) = dA/dz = \pm \exp(A)\, A'(r)$. (Note that either sign choice in (38) leads to the same form for (40) and (41).) The first two terms in (41) are in the form of a supersymmetric quantum mechanics potential $\mathcal{V}(z) = \mathcal{U}(z)^2 + \mathcal{U}'(z)$ derived from a prepotential $\mathcal{U}(z) = (3/2)\, A'(z)$, while $U(\varphi(z))$ is determined by the coupling of inert and active scalars in the supergravity potential. If $U(\varphi(z))$ is positive (or absent as it is for the dilaton fluctuation), then one has an immediate argument that normalizable fluctuations occur only for $p^2 \geq 0$. In general, the behavior of the potential (41) near the limits $z \to 0$ and $z \to \infty$ is usually enough information to ascertain whether the fluctuation spectrum is discrete or continuous, with or without gap [11].

In our examples in sections 3 and 4, the two inert fluctuations with nonzero $U(\varphi(z))$ turn out to have $U(\varphi(z)) < 0$, so that positivity of p^2 is not obvious with the potential in the form (41). However in both cases we are able to rewrite the potentials in an exact SUSY QM form with a modified $\mathcal{U}$. The two examples work differently, and we leave the question of the existence of an exact SUSY QM potential for general coupled inert scalars for the future.

The norm is a potentially delicate issue. The Hamiltonian in (40) is self-adjoint with respect to the "Schrödinger norm" $\int dz\, \psi(z)^2$, but not with respect to the transformed covariant norm

$\int dz \, \exp(2A)\, \psi(z)^2$ which is correct in the present setting. (This results because a factor of $\exp(-2A)$ was dropped in passing from (39) to (40).) Because of this delicacy, solutions which have infinite Schrödinger norm but finite covariant norm would not necessarily have $p^2 > 0$.

The treatment of active scalar fluctuations is more complex. The goal is to transform the second order equation (35) to Schrödinger form. The change from r to z produces

$$-R''(z) + ge^A \left(W'' - \frac{5}{3}W\right) R'(z) + g^2 e^{2A} \left(W'^2 + \frac{2}{3}WW'' - \frac{8}{9}W^2 - \frac{1}{2}W'W'''\right) R = p^2 R \,, \tag{42}$$

in which R' and R'' are derivatives with respect to z, but W', W'' etc. indicate φ-derivatives, as has been our practice. The further transformation $R(z) = \exp(S(z))\, \psi(z)$, with

$$\frac{dS}{dz} = \frac{1}{2}\, g\, e^{A(z)} \left(W''(\varphi(z)) - \frac{5}{3}W(\varphi(z))\right) , \tag{43}$$

produces the Schrödinger form (40) with the (ugly) potential

$$\mathcal{V}(z) = \frac{1}{4}\, g^2 e^{2A(z)} \left[\frac{1}{3}W^2 + \frac{7}{3}W'^2 + (W'')^2 - \frac{4}{3}WW'' - W'W'''\right] . \tag{44}$$

Unpromising as it seems, one may try to express (44) in SUSY-QM form perhaps with a simple remainder. To do this we use the ansatz

$$\mathcal{U} = e^A \left(aW + bW' + cW''\right) , \tag{45}$$

where a, b, c are free parameters. We find that an exact SUSY form cannot be achieved, but analogously to (41) one may write

$$\mathcal{V}(z) = \mathcal{U}(z)^2 + \mathcal{U}'(z) + \frac{1}{3}\, g^2 e^{2A(z)} \, (W')^2 \,, \tag{46}$$

with prepotential

$$\mathcal{U}(z) = \frac{1}{2}\, g\, e^{A(z)} \left(W'' - W\right) . \tag{47}$$

The discussion below (41) applies here as well, but here it is manifest that the additional term beyond the SUSY-QM structure $\Delta\mathcal{V} = (1/3)\, e^{2A}\, (W')^2$ is positive-definite. This indicates that there are no tachyons in the normalized fluctuation spectrum of the active scalar. In particular examples one must be careful to use the correct norm which is the transformation of $\int dr \, e^{4A(r)}\, \tilde{\varphi}^2$ with $\tilde{\varphi}(r,p)$ given by (34).

Two further comments can be made. Although in general $\mathcal{V}(z)$ only can be cast in the form (46), in specific examples it may well have an exact SUSY form with a prepotential different from (47). An example of this appears in section 4.3. Additionally, it is also clear that the analysis above has applications to the stability of domain walls in the brane world scenario [24, 25] which we plan to pursue.

3 The $\mathcal{N}=1$ Super Yang-Mills flow

Several kink solutions involving a single real flowing scalar have been considered in the literature. There are only two inequivalent single real scalars transforming in the $\mathbf{10}\oplus\overline{\mathbf{10}}$ of $SU(4)$, as there are two Weyl orbits in the $\mathbf{10}$. Representative scalars in the longer and shorter orbits, respectively, have been called σ and m, and we will use this convention. There are three inequivalent choices in the $\mathbf{20}'$, one of which we will discuss in the next section.

Interesting renormalization group flows have been considered in [9] in which the scalar manifold of bulk $\mathcal{N}=8$ gauged supergravity was restricted to a subspace of two particular inequivalent scalars in the $\mathbf{10}\oplus\overline{\mathbf{10}}$ of $SU(4)$. A kink solution with only $m(r)$ active was obtained and interpreted as the holographic dual of a perturbation of $\mathcal{N}=4$ SYM theory by a dimension 3 operator which leads to a field theoretic RG flow to pure $\mathcal{N}=1$ SYM theory at long distances. The 10-dimensional lift of this solution is not known, and the 5-dimensional geometry has an interior curvature singularity. The significance of this is not clear, but a scenario for its stringy resolution has been outlined [36].

We will use the kink background of [9] as a testing ground for computations of fluctuations and correlation functions as discussed above. For this purpose we review the background solution in the next subsection. We then go on to study fluctuations and correlators of two inert scalars, and finally focus on perturbations $\tilde{m}(r,p)$ of the active scalar.

3.1 The background flow

In the consistent sub-sector of $\mathcal{N}=8$ supergravity considered in [9], the two scalar fields m and σ have canonical kinetic terms and superpotential and potential given by:

$$W(m,\sigma) = -\frac{3}{4}\left[\cosh\left(\frac{2m}{\sqrt{3}}\right)+\cosh(2\sigma)\right], \tag{48}$$

$$V(m,\sigma) = -\frac{3g^2}{8}\left[\frac{1}{4}\cosh^2\left(\frac{2m}{\sqrt{3}}\right)+\cosh\left(\frac{2m}{\sqrt{3}}\right)\cosh(2\sigma)-\frac{1}{4}\cosh^2(2\sigma)+1\right]. \tag{49}$$

Since the σ field appears quadratically in $V(m,\sigma)$ there is a supersymmetric flow in which it vanishes. This flow is a solution of (10) with superpotential

$$W(m) = -\frac{3}{4}\left[1+\cosh\left(\frac{2m}{\sqrt{3}}\right)\right]. \tag{50}$$

In the following, we set $g=2/L$, and W always refers to $W(m)$. The kink solution is

$$m(r) = \frac{\sqrt{3}}{2}\log\frac{1+e^{-r/L}}{1-e^{-r/L}}, \tag{51}$$

$$A(r) = \frac{1}{2}\left(\frac{r}{L}+\log 2\sinh\frac{r}{L}\right). \tag{52}$$

There is a singularity at finite proper distance, which by choice of an additive integration constant we have located at $r_s = 0$.

The horospheric coordinate z is obtained from $dr/dz = -e^A$, which can be solved to give

$$e^{2A(z)} = \cot^2\left(\frac{z}{L}\right) = e^{\frac{2r}{L}} - 1\,. \tag{53}$$

Neither r nor z proves useful for solving the various equations. Instead we are able to make progress by using a variable u in which the boundary is mapped to $u_b = 1$ and the singularity to $u_s = 0$. This is achieved by

$$u \equiv \cos^2\left(\frac{z}{L}\right)\,. \tag{54}$$

The equations for fluctuations become hypergeometric in u as we will see, and the quantities which enter these equations can be expressed as:

$$\begin{aligned}
W(u) &= -\frac{3}{2u}\,, \qquad W'(u) = -\sqrt{3}\,\frac{\sqrt{1-u}}{u}\,, \\
W''(u) &= \frac{u-2}{u}\,, \qquad W'''(u) = -\frac{4}{\sqrt{3}}\,\frac{\sqrt{1-u}}{u}\,, \\
e^{2A(u)} &= \frac{u}{1-u}\,, \qquad \frac{du}{dr} = \frac{2}{L}\,(1-u)\,.
\end{aligned} \tag{55}$$

3.2 Correlators of inert scalars

Let us begin our study of fluctuations in this geometry by calculating the two-point function for the dilaton ϕ. The dilaton couples to $\mathcal{O}_\phi = \mathrm{Tr}\,F^2 + \cdots$ and hence provides information about the glueball spectrum. The Klein-Gordon equation for this field is

$$-\frac{1}{\sqrt{g}}\,\partial_\mu\left(\sqrt{g}\,g^{\mu\nu}\partial_\nu\right)\phi = \left(\frac{\partial}{\partial r} + 4A'(r)\right)\frac{\partial\phi}{\partial r} + p^2 e^{-2A}\phi = 0. \tag{56}$$

In terms of u this becomes

$$\phi''(u) + \frac{1}{1-u}\left(\frac{2}{u} - 1\right)\phi'(u) + \frac{p^2L^2}{4}\,\frac{1}{u(1-u)}\,\phi(u) = 0\,. \tag{57}$$

This equation is hypergeometric; the solution regular in the deep interior ($u = 0$) is

$$\phi(u) = (u-1)^2\,F\left(2 - \frac{pL}{2}, 2 + \frac{pL}{2}; 2; u\right)\,. \tag{58}$$

This is a simpler, but presumably equivalent, form of the solution very recently obtained in [23], where a different independent variable was used.

We can calculate the two-point function in the established way [4, 12]. Specifically, the p-space correlator is obtained by imposing a cutoff at $z = \epsilon$ in the horospheric coordinate. One then finds

$$\langle \mathcal{O}(p)\, \mathcal{O}(-p) \rangle = -\lim_{\epsilon \to 0} \left(\frac{1}{\epsilon^{2(\Delta-4)}} \right) \left[\frac{1}{z^3} \frac{d}{dz} \ln(\phi(z,p)) \right]_{z=\epsilon} , \tag{59}$$

where an energy-momentum conserving δ-function has been dropped, and Δ is the UV dimension of the operator $\mathcal{O}(x)$. The only term which is kept in the limit is the most singular term in ϵ which is non-analytic in p^2. Other more singular terms are multiplied by polynomials in p^2, and are dropped because they correspond to counter terms. This procedure is equivalent to calculating the second variation of the on-shell action (4). Field theory considerations imply that correlators behave as $(-p)^{2\nu} \ln(-p^2)$ for large spacelike momentum, where $\nu = \Delta - 2$. Our calculations lead directly to correlation functions which are normalized so that the leading term has the same coefficient as in the AdS_5 geometry. The analysis of the Appendix of [12], redone for Bessel functions with integer $\nu > 2$, gives

$$\langle \mathcal{O}(p)\, \mathcal{O}(-p) \rangle = -\left[\frac{2\nu}{\Gamma(\nu)\Gamma(\nu+1)4^\nu} \right] (-p)^{2\nu} \ln(-p^2) \tag{60}$$

For the dilaton this procedure gives (with $s = p^2$)

$$\langle \mathcal{O}_\phi(p)\, \mathcal{O}_\phi(-p) \rangle = -\frac{1}{8}\, s \left(s - \frac{4}{L^2} \right) \left[\psi \left(2 + \frac{L\sqrt{s}}{2} \right) + \psi \left(2 - \frac{L\sqrt{s}}{2} \right) \right] , \tag{61}$$

where $\psi(z) \equiv d \ln \Gamma(z)/dz$. This result agrees with [23]. The correlator has a discrete spectrum of poles at $s = 4(n+2)^2/L^2$ as is consistent with the expected glueball spectrum in a confining theory.

We can also consider the two-point function of the inert scalar σ. One must now take into account the coupling $U(m)$ to the active scalar that results from the potential (49). The wave equation is (39) with

$$U(m) \equiv \left. \frac{\partial^2 V(m,\sigma)}{\partial \sigma^2} \right|_{\sigma=0} = -\frac{3g^2}{4} \left[2\cosh\left(\frac{2m}{\sqrt{3}} \right) - 1 \right] , \tag{62}$$

which leads to the hypergeometric wave equation

$$\sigma''(u) + \frac{1}{1-u} \left(\frac{2}{u} - 1 \right) \sigma'(u) + \frac{1}{(1-u)^2} \left(\frac{p^2L^2}{4} \frac{1-u}{u} + \frac{3}{4} \frac{4-3u}{u} \right) \sigma(u) = 0\, . \tag{63}$$

The solution regular at the horizon is

$$\sigma(u) = (1-u)^{3/2} F\left(\frac{3}{2} + \frac{1}{2}\sqrt{9 + L^2p^2}, \frac{3}{2} - \frac{1}{2}\sqrt{9+L^2p^2}; 2; u \right) . \tag{64}$$

The corresponding two-point function is

$$\langle \mathcal{O}_\sigma(p)\, \mathcal{O}_\sigma(-p) \rangle = \frac{s + 8/L^2}{2} \left[\psi \left(\frac{3}{2} + \frac{1}{2}\sqrt{9 + L^2 p^2} \right) + \psi \left(\frac{3}{2} - \frac{1}{2}\sqrt{9 + L^2 p^2} \right) \right] . \tag{65}$$

There is a discrete spectrum of poles at $s = 4n(n+3)/L^2, n = 0, 1, 2...$. We note the presence a massless state, which we shall not attempt to reconcile with the interpretation [9]that the flow describes a confining field theory.

These results may be compared with expectations based on the form of the Schrödinger potentials (41) discussed in section 2. For both fields, the one discovers that $\mathcal{V}(z) \to +\infty$ in both the boundary and deep interior limits, which is indicative of the discrete spectrum found above. In section 2 we showed that the Schrödinger potential of an inert dilaton always possesses a SUSY QM form $\mathcal{V}(z) = \mathcal{U}(x)^2 + \mathcal{U}'(z)$, where

$$\mathcal{U}_\phi(z) = \frac{1}{2} g\, e^A\, W\,, \tag{66}$$

independent of the flow. Thus we can argue that the spectrum is positive, and we are in the situation of broken supersymmetry since the candidate 0-mode (which is just $\phi = const$) is not normalizable. The σ field has an additional term proportional to $U(m(z))$, which turns out to be negative; the m-flow sits atop a ridge in the σ-direction of field space. Thus SUSY quantum mechanics does not manifestly apply with the potential written as in (41). However, one may show that $\mathcal{V}_\sigma(z)$ can be rewritten in an exact SUSY QM form with prepotential

$$\mathcal{U}_\sigma(z) = \frac{1}{2} g\, e^A\, (W + 3)\ , \tag{67}$$

and in this case the zero-mode can be shown to be normalizable, consistent with the spectrum of the correlator (65).

3.3 The graviton/active scalar system

It is straightforward to show that (35) becomes

$$R''(u) + \frac{1}{u(1-u)} R'(u) + \frac{1}{u(1-u)} \left(\frac{2u - 1}{u(1-u)} + p^2 L^2/4 \right) R(u) = 0\,, \tag{68}$$

which has the solution

$$R(u) = u(1-u) F\left(\frac{3}{2} + \frac{1}{2}\zeta, \frac{3}{2} - \frac{1}{2}\zeta; 3; u \right)\ , \tag{69}$$

where $\zeta = \sqrt{1 + p^2 L^2}$. We reconstruct $\tilde{m}$ following (34). Using the relations (55), we integrate to find

$$\tilde{m} = \frac{\sqrt{1-u}}{u} \left[4\, F_1\,(p; u) - f_0 + p^2\, L^2\, u\, F_2\,(p; u) \right]\ , \tag{70}$$

where $F_n(p;u)$ is the hypergeometric function

$$F_n(p;u) \equiv F\left(n - \frac{3}{2} + \frac{1}{2}\zeta, n - \frac{3}{2} - \frac{1}{2}\zeta; n; u\right) . \tag{71}$$

Here f_0 is an integration constant, corresponding to the addition of a multiple of the universal solution. The choice $f_0 = 4$ ensures regularity at the singularity $u = 0$, but we will keep f_0 as a parameter in our further discussion. The second solution to (68) is singular at $u = 0$ and cannot be made regular. Near the boundary $u = 1$, we observe $\tilde{m} \sim \sqrt{1-u} \sim z$ which is the proper scaling for an operator with dimension $\Delta = 3$.

Let us check to see whether the form found for the fluctuation agrees with the intuition based on Schrödinger form (42) of (68), with potential $\mathcal{V}(z)$ given by (44). Near $z = 0$, the leading term of the potential is $\mathcal{V}(z) = -1/4z^2$, which is the limiting strength of an allowed attractive $1/z^2$ potential. Approaching the singularity at $z \to \pi/2$, there is a repulsive $1/(\pi/2 - z)^2$ behavior. Thus one expects a positive discrete spectrum, and this is reflected in the values of ζ which give a terminating hypergeometric series in (69), namely $\zeta = 2n+3$ or $s = p^2 = 4(n+1)(n+2)$ for $n = 0, 1, 2 \ldots$. Examining (70) at these values of p^2, we find the behavior near the boundary

$$\tilde{m}(u,p)|_{\zeta=2n+3} \approx -\sqrt{1-u}\left[f_0 + c_n(1-u) + \mathcal{O}(1-u)^2\right] , \tag{72}$$

where c_n is a constant. These functions are normalizable (in the covariant norm $\int dr\, e^{4A}\,(\tilde{m})^2$) only if $f_0 = 0$.

We now attempt to obtain a correlation function by applying the previously used standard procedure. Hypergeometric analytic continuation formulae [37] give the boundary behavior

$$\begin{aligned} \tilde{m}(u,p) \;&\approx\; \frac{\sqrt{1-u}}{u} \frac{1}{\Gamma\left(\frac{3+\zeta}{2}\right)\Gamma\left(\frac{3-\zeta}{2}\right)} \left[4 + p^2 + \frac{f_0 \pi p^2}{4\cos(\pi\zeta/2)}\right. \\ &\quad + \left.\frac{p^4(1-u)}{4}\left(h_0'' - \ln(1-u)\right) + \mathcal{O}(1-u)^2\right] , \end{aligned} \tag{73}$$

where $h_0'' = \psi(1) + \psi(2) - \psi((3+\zeta)/2) - \psi((3-\zeta)/2)$. Applying (59) we see that the "would-be correlator" should be the most singular term (in $(1-u)$) that is non-analytic in p^2 in

$$\text{“}\langle\mathcal{O}(p)\mathcal{O}(-p)\rangle\text{”} = \frac{s^2}{4s + 16/L^2 + \frac{f_0 \pi s}{\cos(\pi\zeta/2)}}\left[\psi\left(\frac{3}{2} + \frac{1}{2}\zeta\right) + \psi\left(\frac{3}{2} - \frac{1}{2}\zeta\right) - \ln\left((1-u)e^{2\gamma}\right)\right] . \tag{74}$$

At this point we hit the first major barrier of our program; we see no way to extract the leading non-analytic term. For example the coefficient of $\ln(1-u)$ should be a polynomial in s in order to be interpreted as a contact term. However here we have (for general f_0) an entire function of s with essential singularity at infinity, while at $f_0 = 0$ there is a pole at the space-like value $s = -4/L^2$. One might nevertheless try to identify the correlator by simply dropping the log term. However for general f_0 the expected poles of the discrete spectrum, at $\zeta = 2s+3$, cancel between the numerator and denominator, leaving a function with no singularities in the finite s-plane. For $f_0 = 0$ one regains the expected poles, but there are also unphysical tachyon poles.

We are rather sure that our treatment of the fluctuation equations leading to (34), (35) is correct. The solution $\tilde{m}(u,p)$, especially at $f_0 = 4$, has the properties expected in the AdS/CFT correspondence, both at the boundary and the interior singularity. It is reassuring that (35) admits solutions with unphysical behavior in both limits, yet the actual solution $\tilde{m}(r,p)$ behaves correctly at both ends. It thus appears that it is the procedure to obtain the correlation function from the fluctuation which must be modified.

We can additionally calculate the associated graviton fluctuations using equations (27) and (29). We find

$$\begin{aligned} h &= \frac{1}{3\sqrt{3}u}\Big(24f_0 - 96F_1 - 24L^2p^2uF_2 + 24L^2p^2u(1-u)F_3 \\ &+ L^2p^2u^2(1-u)(8-L^2p^2)F_4\Big)\,, \end{aligned} \tag{75}$$

$$\begin{aligned} H' &= -\frac{L(1-u)}{12\sqrt{3}u}\Big(24f_0 - 96F_1 - 24L^2p^2uF_2 + 12L^2p^2u(1-u)F_3 \\ &+ L^2p^2u^2(1-u)(8-L^2p^2)F_4\Big)\,. \end{aligned} \tag{76}$$

Only $H'(r)$ is determined by the equations. Note that if $f_0 = 4$, $\tilde{m}$, h and H' are all regular at the singularity $u = 0$.

In principle, one would like to use (75) and (76) to calculate correlation functions of the trace of the energy-momentum tensor, such as $\langle T^i_i(p)\, T^j_j(-p)\rangle$, which is related to renormalization group flows and proposals for c-theorems [21]. The fact that the solutions for $\tilde{m}$, h and H are all determined by a single set of boundary data is consistent with the field theory reality that the operator T^i_i is not independent of the operator $\mathcal{O}_m$, as in (7). Unfortunately, correlators constructed from the solutions (75), (76) seem to suffer from the same pathologies as (74).

The reader may justifiably complain that two-point functions associated to tensor fluctuations should require a more complex treatment than the inert scalar case, since the actions are different. This is further evidence that the procedure (4) must be generalized in the case of the graviton/active scalar system. In section 5, we evaluate the full gravity + scalar action to second order in fluctuations, in the hope of finding such a generalization.

4 The Coulomb branch

Flows involving a single scalar from the $\mathbf{20'}$ of $SO(6)$, which preserve 16 of the 32 bulk supercharges, were discussed in [11, 14]. Unlike the flow involving m, which is dual to an operator deformation of the Lagrangian of $\mathcal{N}=4$ SYM, these flows change the vacuum of the field theory, moving it out onto the Coulomb branch. The distinction can be perceived by examining the scaling of the scalar profile near the boundary. The $\mathbf{20'}$ scalars are dual to operators with $\Delta = 2$, and the Coulomb branch profiles scale as z^2 rather than the $z^2 \ln z$ associated with an operator flow.

The five flows discussed in [11] all have known lifts to 10D configurations, namely the geometries produced by discs of D3-branes of various dimensionalities. There are three inequivalent single scalars in the $\mathbf{20'}$, and two of these have distinct flows in the positive and negative directions in field space, for five in all. We consider the flow called $n = 2$ in [11], and examine fluctuations of two inert scalars and the graviton/active scalar system.

4.1 The background flow

The scalar field φ involved in the $n = 2$ Coulomb branch flow also played a role in a two-scalar flow to an $\mathcal{N} = 1$ superconformal fixed point [8], where it was denoted φ_3. The other scalar, there called φ_1, is a member of the same $\mathbf{10} \oplus \overline{\mathbf{10}}$ Weyl orbit as σ. The potential and superpotential of this two-scalar subspace is

$$W(\varphi,\sigma) = -\frac{1}{4}\, e^{-2\varphi/\sqrt{6}} \left[e^{\sqrt{6}\varphi}(3 - \cosh(2\sigma)) + 2(\cosh(2\sigma) + 1)\right] , \tag{77}$$

$$V(\varphi,\sigma) = -\frac{g^2}{4}\, e^{2\varphi/\sqrt{6}} \left[e^{-\sqrt{6}\varphi} \left(\frac{3}{4} + \frac{1}{2}\cosh(2\sigma) - \frac{1}{4}\cosh^2(2\sigma)\right) + (1 + \cosh(2\sigma)) + \frac{1}{16} e^{\sqrt{6}\varphi}(1 - \cosh^2(2\sigma))\right] . \tag{78}$$

In the notation of [11], $\varphi = -\mu$ is the active scalar in the $n = 2$ Coulomb branch flow, while σ is inert. The superpotential for φ alone is

$$W(\varphi) = -e^{-2\varphi/\sqrt{6}} - \frac{1}{2}\, e^{4\varphi/\sqrt{6}} . \tag{79}$$

The solution to the equations (10) involving (79) can be obtained, but they are not of direct use since it is more convenient to use a radial coordinate that is a function of the scalar itself:

$$v \equiv e^{\sqrt{6}\,\varphi} . \tag{80}$$

The boundary is at $v = 1$. For this flow $\varphi \to -\infty$, so $v \in [0, 1]$ with a curvature singularity at $v_s = 0$. We have the relations

$$\begin{aligned} W &= -\frac{1}{2}\,\frac{v+2}{v^{1/3}} , & W' &= \frac{2}{\sqrt{6}}\,\frac{1-v}{v^{1/3}} , \\ W'' &= -\frac{2}{3}\,\frac{1+2v}{v^{1/3}} , & W''' &= \frac{4}{3\sqrt{6}}\,\frac{1-4v}{v^{1/3}} , \\ e^{2A} &= \frac{\ell^2}{L^2}\,\frac{v^{2/3}}{1-v} , & \frac{\partial v}{\partial r} &= \frac{2}{L}\, v^{2/3}\,(1-v) . \end{aligned} \tag{81}$$

One can calculate the horospheric variable z in terms of v and the relationship is

$$v = \mathrm{sech}^2\left(\frac{z\,\ell}{L^2}\right). \tag{82}$$

Following [11] we have introduced the length ℓ, the radius of the disc of D3-branes in ten dimensions. Taking $\ell/L \to 0$ with z/L fixed removes the flow and restores pure anti-de Sitter space. This definition is reminiscent of that for the variable u in the $\mathcal{N}=1$ flow (54); one difference is that for (82) the singularity is at $z \to \infty$, and is thus at infinite proper distance.

4.2 Correlators of inert scalars

The two-point function of the dilaton ϕ has been calculated previously in this background [11]. The Klein-Gordon equation becomes

$$\phi''(v) + \frac{2-v}{v(1-v)}\phi'(v) + \frac{p^2L^4}{4\ell^2}\frac{1}{v^2(1-v)}\phi(v) = 0\,, \tag{83}$$

which has the solution

$$\phi(v) = v^a F(a, a; 2+2a; v)\,, \tag{84}$$

where

$$a \equiv -\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{L^4p^2}{\ell^2}}\,. \tag{85}$$

This solution is regular at the singularity for spacelike p^2; the second solution of (83) has a leading v^{-a-1} and is not regular. The result for the 2-point function calculated from (59) is

$$\langle \mathcal{O}_\phi(p)\,\mathcal{O}_\phi(-p)\rangle = -\frac{1}{4}p^4\psi\left(\frac{1}{2} + \frac{1}{2}\sqrt{1-\frac{L^4p^2}{\ell^2}}\right)\,, \tag{86}$$

which as remarked in [11] has a branch cut along the positive real axis, indicating a continuous spectrum with gap or threshold at $m^2_{gap} = \ell^2/L^4$.

For the σ field, there is a $U(\varphi)$ term in the equation of motion (39) owing to the potential (78),

$$U(\varphi) \equiv \left.\frac{\partial^2 V(\varphi,\sigma)}{\partial\sigma^2}\right|_{\sigma=0} = -\frac{g^2}{4}\,e^{2\varphi/\sqrt{6}}\left[4 - e^{\sqrt{6}\varphi}\right]\,, \tag{87}$$

and the wave equation becomes

$$\sigma''(v) + \frac{2-v}{v(1-v)}\sigma'(v) + \frac{1}{v(1-v)}\left(\frac{p^2L^4}{4\ell^2}\frac{1}{v} + \frac{(4-v)}{4(1-v)}\right)\sigma(v) = 0\,, \tag{88}$$

which has the regular solution

$$\sigma(v) = v^a \sqrt{1-v} F(a, a+1; 2+2a; v) \,, \tag{89}$$

with a as (85). The correlation function is

$$\langle \mathcal{O}_\sigma(p)\, \mathcal{O}_\sigma(-p)\rangle = p^2 \psi \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{L^4 p^2}{\ell^2}} \right) \,, \tag{90}$$

This correlator also has a continuous spectrum, with the same gap as (86). The momentum-dependence is appropriate for an operator of $\Delta = 3$.

We may check that this behavior is consistent with the SUSY QM viewpoint. Calculating the Schrödinger potentials $\mathcal{V}_\phi(z)$ and $\mathcal{V}_\sigma(z)$ (41), we find that $\mathcal{V} \to \infty$ at the boundary for both, while they asymptote to ℓ^2/L^4 in the deep interior. This is precisely the expected behavior for a potential with a continuous spectrum and a gap at $m_{gap} = \ell/L^2$. (For the dilaton, this was already noticed in [11].)

As always, the dilaton wave equation can be rewritten in Schrödinger form with an exact SUSY QM potential given by the prepotential (66). Normalizability fails near the boundary for the dilaton zero-mode. As in the $\mathcal{N} = 1$ case, we find that the Coulomb branch flow is along a ridge of the potential (78) in the σ-direction, and thus $U(\varphi)$ contributes negatively to the σ Schrödinger potential (41). However, again we find that the potential can be cast into exact SUSY QM form, this time with the modified prepotential

$$\mathcal{U}_\sigma(z) = -\frac{1}{2} g\, e^A W \,. \tag{91}$$

Unlike the $\mathcal{N} = 1$ case (67), this doesn't have a new term beyond that of (66), but the coefficient is modified instead. In this case normalizability of the zero-mode fails at the singularity.

4.3 The graviton/active scalar system

One can show that (35) becomes in this case

$$R''(v) + \frac{2}{v} R'(v) + \frac{p^2 L^2}{\ell^2} \frac{1}{v^2(1-v)} R(v) = 0 \,. \tag{92}$$

This particularly simple form results because the potential-type term $(2/3)WW'' + (8/9)W^2 + (1/2)W'W''' - W'^2$ evaluates to zero; we have no ready explanation for this cancellation. The relevant solution to (92) is

$$R(v) = v^a \, (1-v) \, F\, (1+a, 2+a; 2+2a; v) \,. \tag{93}$$

The second solution also has v^{-a-1} behavior and cannot be made regular in the interior. One can integrate (93) to obtain the solution for $\tilde{\varphi}$:

$$\tilde{\varphi}(v) = v^a (1-v)\, {}_3F_2\left(1+a, 2+a, \frac{1}{3}+a; 2+2a, \frac{4}{3}+a; v\right) . \tag{94}$$

We could add a multiple of the universal solution to (94), but this would destroy regularity at the interior singularity at $v=0$.

One can see that (92) has a constant zero-mode solution, and that (93) reduces to a constant as $p^2 \to 0$ which is the same as $a \to 0$. Although this constant mode is not normalizable, one might suspect that supersymmetric quantum mechanics is again at work. Indeed one can again look back to (35) in the present case where the potential-type term vanishes. The equivalent Schrödinger equation is then supersymmetric with prepotential $U(z) = g/2\, e^A (W'' - 5W/3)$.

We may use (27),(29) to determine the graviton modes associated with (94):

$$\begin{aligned} h &= -\frac{4\sqrt{2}v^a}{3\sqrt{3}L^4p^2(a+1/3)}\Big[4\ell^2(1+3a)(-2+4v+v^2+a(-2+v+v^2))\times && (95)\\ &\quad F(1+a,2+a;2+2a;v) - 2\ell^2 v(1-v)(2+v)(2+7a+3a^2)F(2+a,3+a,3+2a,v)\\ &- 3\,(2+v)L^4p^2\, {}_3F_2\left(1+a,2+a,\frac{1}{3}+a;2+2a,\frac{4}{3}+a;v\right)\Big] ,\\ H' &= -\frac{\sqrt{2}v^{a-1/3}(1-v)}{3\sqrt{3}L\ell^2p^2(a+1/3)}\Big[4\ell^2(1+3a)(-2+5v+3a(v-1))\times && (96)\\ &\quad F(1+a,2+a;2+2a;v) + 6\ell^2 v(1-v)(2+7a+3a^2)F(2+a,3+a,3+2a,v)\\ &+ 3L^4p^2\, {}_3F_2\left(1+a,2+a,\frac{1}{3}+a;2+2a,\frac{4}{3}+a;v\right)\Big] . \end{aligned}$$

Our analysis of the fluctuations (94), (95), (96), all of which contain the generalized hypergeometric function ${}_3F_2$, is hampered because the relevant analytic continuation formulae are not in the literature. In particular we need the expansion of (94) near the boundary, *i.e.* a series in $(1-v)$. We proceed by expanding (93) in series first, and then integrate to obtain a series for $\tilde{\varphi}$. Doing this we see the leading logarithmic singularity:

$$\tilde{\varphi} = \frac{(a+1/3)\Gamma(2+2a)}{\Gamma(1+a)\Gamma(2+a)}(1-v)\left(-\ln(1-v) + \tilde{\varphi}_0(a) + \mathcal{O}(1-v)\right) , \tag{97}$$

and while all $\mathcal{O}(1-v)$ terms could be calculated from higher order terms in the expansion of (93), the integration constant $\tilde{\varphi}_0$ remains undetermined. In principle it is fixed by our choice not to add a multiple of the universal solution to (94), but it is nontrivial to calculate it.

Let us consider h. We expect a graviton mode not to be singular on the boundary, and indeed $1/(1-v)$ and $\ln(1-v)$ terms cancel between the various terms in (95). The series expansion of $h(v)$ is then

$$h(v) = -\frac{16\sqrt{2}\ell^2}{\sqrt{3}L^4p^2}\frac{\Gamma(2+2a)}{\Gamma(1+a)\Gamma(a)}\left(3\tilde{\varphi}_0(a) - \frac{2}{a(1+a)} + 6\gamma + 6\psi(1+a)\right) + \mathcal{O}(1-v)^2 \,. \tag{98}$$

What form do we expect for $h(v)$? In the field theory, conformal invariance is only spontaneously broken, and consequently $T_i^i = 0$ continues to hold as an operator equation on the Coulomb branch. As a result, one might expect $h(v)$ to fall off more rapidly on the boundary than in the case of an operator deformation, so that it does not excite a dual field theory operator. A hint of this behavior is present in the fact that no $\mathcal{O}(1-v)$ term appears in (98). We are therefore led to postulate that the constant term vanishes as well, and

$$\tilde{\varphi}_0(a) = \frac{2}{3a(1+a)} - 2\gamma - 2\psi(1+a)\,. \tag{99}$$

This is a tempting assumption, since (97) then produces the two-point function[3]

$$\langle \mathcal{O}_\varphi(p)\,\mathcal{O}_\varphi(-p)\rangle = \psi\left(\frac{1}{2} + \frac{1}{2}\sqrt{1 - \frac{L^4p^2}{\ell^2}}\right) + \frac{4L^4}{3p^2\ell^2} \tag{100}$$

In addition to the expected branch point there is a pole at $p^2 = 0$. This is a problem for the formalism since the associated constant zero-mode is non-normalizable, so it should not show up in the correlator. It is also a problem for the field theory interpretation. To see this recall that an extra factor of N^2 must be inserted to agree with the short distance form $\langle \mathcal{O}_\varphi(x)\,\mathcal{O}_\varphi(0)\rangle \sim N^2/x^4$ in field theory. Although the field theory contains massless Goldstone bosons from the breaking of $SO(6)$ flavor symmetry to $SO(4)\times SO(2)$ by the disc of D3-branes, it appears very unlikely that these states should couple to $\mathcal{O}_\varphi = \mathrm{Tr}\,X^2$ with strength N^4.

This motivates us to test the assumption (99) numerically. Specifically we took the defining power series ${}_3F_2 = \sum_n c_n v^n$ about $v = 0$ which is logarithmically divergent at $v = 1$ and subtracted the explicitly summable series of the large n limit of the c_n (obtained using Stirling's formula). The result is a convergent series whose value at $v = 1$ gives the unknown constant $\tilde{\varphi}_0(a)$. The numerical series agrees remarkably well with (98) for parameter values $a \gg 1$, but agreement fails for small a since the poles of the power series coefficients for $a \leq 0$ do not coincide with those of (98). We must therefore conclude that the physically motivated assumption above does not agree with the properties of ${}_3F_2$. It should also be observed that the same correlator (100) can be obtained by subtracting a multiple of the universal solution from $h(v)$ and $\tilde{\varphi}(v)$, so as to impose the condition $h(1) = 0$. This diffeomorphism has exactly the effect of inserting (99) in (97). However the diffeomorphism also makes $h(v)$ and $\tilde{\varphi}(v)$ singular at the origin, and we cannot presently justify it.

The situation may be summarized as follows. We imposed the condition that $h(v)$ vanish on the boundary because it seems to be physically required for Coulomb branch flows. This

[3]Formulas (59), (60) are modified for a dimension $\Delta = 2$ operator. See (25) of [38].

[4]We thank S. Gubser and A. Hanany for discussions of this issue.

leads to a correlation function with apparently unphysical zero-mode poles. Further, the condition $h(1) = 0$ does not seem to be compatible with the properties of ${}_3F_2$ obtained by numerical study. Perhaps the analytic form of the constant $\tilde{\varphi}(a)$ could illuminate the situation.

5 Calculation of correlation functions

Our main purpose here is to discuss some of our attempts to determine the correct prescription for the calculation of active scalar and graviton correlators. First we will outline a calculation of the on-shell supergravity action (1) through second order in the fields $\tilde{\varphi}$, h, and H which are coupled by the fluctuation equations of Sec 2. Because of the coupling, a single set of boundary data determines the boundary behavior of all the fields. Accordingly, factors of h and H in the action will contribute to the correlation functions of the active scalars. Hence to calculate two-point functions we should keep terms quadratic in any of $\tilde{\varphi}$, h and H. One then varies the on-shell action with respect to the boundary data to obtain the correlation function.

The bulk action is given in (1). In addition, it is well-known that this should be supplemented with certain boundary terms [39, 40]. First among these is the Gibbons-Hawking term, which is included to generate a well-posed Hamiltonian formalism [41]:

$$\begin{aligned} S_{GH} &= \frac{1}{2}\int_\partial d^4x\sqrt{g_4}\,\mathcal{K} = \frac{1}{2}\int_\partial d^4x\sqrt{g_4}\,\nabla_\mu n^\mu = \frac{1}{2}\int_\partial \partial_r\sqrt{g_4} \qquad (101) \\ &= \frac{1}{2}\int_\partial e^{4A}\left[\frac{1}{2}h_i^{i\prime} + \frac{1}{4}h_i^i h_j^{j\prime} - \frac{1}{2}h^{ij}h'_{ij} + 4A'\left(1 + \frac{1}{2}h_i^i + \frac{1}{8}(h_i^i)^2 - \frac{1}{4}h^{ij}h_{ij}\right) + \cdots\right], \end{aligned}$$

where $\mathcal{K} = \nabla_\mu n^\mu$ is the trace of the second fundamental form of the boundary, with n^μ a normal to the boundary; in the second line we expand this term to second order in the fluctuations. In addition, a boundary cosmological term has been recommended for canceling the leading volume divergence:

$$S_{vol} = \beta\int_\partial \sqrt{g_4}\,A' \qquad (102)$$

$$= \beta\int_\partial e^{4A}\,A'\left(1 + \frac{1}{2}h_i^i + \frac{1}{8}(h_i^i)^2 - \frac{1}{4}h^{ij}h_{ij} + \cdots\right), \qquad (103)$$

for some β which we will fix momentarily. Note that $A' = 1/L$ is a constant on boundary, which we include explicitly to simplify the formulas that follow.

We proceed to expand the bulk action to second order in fluctuations, making use of the equations of motion as necessary. After a remarkably tedious calculation, we find that as in simpler cases, the bulk action can be reduced entirely to a set of boundary terms:

$$S_{bulk} = S_0 + S_1 + S_2\,,$$

$$
\begin{aligned}
S_0 &= \int d^4x\, e^{4A}\left(-\frac{1}{2}A'\right), \qquad (104)\\
S_1 &= \int d^4x\, e^{4A}\left(-\frac{1}{4}h_i^{i\prime} - \frac{1}{4}A'h_i^i - \varphi'\tilde{\varphi}\right),\\
S_2 &= \int d^4x\, e^{4A}\left(-\frac{1}{2}\tilde{\varphi}\tilde{\varphi}' - \frac{1}{4}\varphi'\tilde{\varphi}h_i^i + \frac{3}{16}h^{ij}h_{ij}' - \frac{1}{16}h_i^i(h_j^j)' + \frac{1}{8}A'h^{ij}h_{ij} - \frac{1}{16}A'h_i^ih_j^j\right),
\end{aligned}
$$

evaluated on the boundary $r = R$. Here we have organized the action by the order of the fluctuations. The zeroth-order term is the volume divergence, which also receives contributions from the boundary terms (101) and (102); it is canceled by the choice $\beta = 1/2$, which in fact removes all terms proportional to A'. The total action then reduces to

$$
S_{tot} = \int d^4x\, e^{4A}\left(-\varphi'\tilde{\varphi} - \frac{1}{2}\tilde{\varphi}\tilde{\varphi}' - \frac{1}{4}\phi'\tilde{\varphi}h_i^i + \frac{1}{16}h_i^ih_j^{j\prime} - \frac{1}{16}h^{ij}h_{ij}'\right). \qquad (105)
$$

We can express this in momentum space in terms of h and H,

$$
\begin{aligned}
S_{tot} &= -\varphi'(R)\,\tilde{\varphi}(R, p=0) + \int d^4p\, e^{4A}\left(-\frac{1}{2}\tilde{\varphi}(R,p)\,\tilde{\varphi}'(R,p)\right. \qquad (106)\\
&+ \left.\frac{3}{32}h(R,p)\,h'(R,p) + \frac{3}{32}p^2H(R,p)\,h'(R,p) + \frac{3}{64}p^2h(R,p)\,H'(R,p)\right).
\end{aligned}
$$

The first term in (106) is linear in the fluctuation $\tilde{\varphi}$, and is thus suggestive of a one-point function. It does not clearly discriminate between the $\mathcal{N}=1$ flow, where no one-point function is expected, and the Coulomb branch flow, where one is. We find this puzzling.

On the other hand, the quadratic terms suggest a modified calculation of the scalar correlator in which h and H are related to the boundary data for $\tilde{\varphi}$. This is straightforward but complicated. We have performed such a calculation for the case of the $\mathcal{N}=1$ active scalar, but ultimately encountered the same difficulties as in section 3. The resolution of the problem presumably involves understanding (106) better, but other pieces of the puzzle may still be missing.

A further uncertainty is the issue of the diffeomorphism invariance. We refer particularly to (14) which has been interpreted as describing [31, 32] bending of the cutoff surface and horizon. It is unsettling that the calculation of the active correlator in the Coulomb branch flow was so markedly changed by such a diffeomorphism. The on-shell action must be diffeomorphism invariant, and it is not clear to us that this is manifest in (106). In particular, a term H with no derivatives appears; this quantity is absent from the equation of motion and its constant term is thus not determined, but has the form of a pure diffeomorphism (13). A better understanding will have to involve coming to terms with these issues.

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