# Runge-Lenz operator for Dirac field in Taub-NUT background

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**Abstract**

Fermions in $D = 4$ self-dual Euclidean Taub-NUT space are investigated. Dirac-type operators involving Killing-Yano tensors of the Taub-NUT geometry are explicitly given showing that they anticommute with the standard Dirac operator and commute with the Hamiltonian as it is expected. They are connected with the hidden symmetries of the space allowing the construction of a conserved vector operator analogous to the Runge-Lenz vector of the Kepler problem. This operator is written down pointing out its algebraic properties.

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## 1 Introduction

It is highly non-trivial problem to study the Dirac equation in a curved background. One of the most interesting geometries is that of the Euclidean Taub-NUT space since it has not only usual isometries but also it admits a hidden symmetry of the Kepler type [1, 2]. Moreover in the Taub-NUT

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geometry there are four Killing-Yano tensors, connected with three Stäckel-Killing tensors which represent the components of an analogue to the Runge-Lenz vector of the Kepler problem [2, 3, 4, 5].

In other respects, the Taub-NUT metric is involved in many modern studies in physics [1, 2, 6]. The Dirac equation in the Kaluza-Klein monopole field was studied in the mid eighties [7] but, to our knowledge, the conserved observables of the Dirac theory corresponding to the hidden symmetries of the Taub-NUT geometry have not been well discussed in literature. An attempt to take into account the Runge-Lenz vector of this geometry was done in [8], but their approach is incomplete.

For this reason the present paper is devoted to the construction of the Runge-Lenz vector-operator for the Dirac field in the Taub-NUT background. The method we use is to calculate the Dirac-type operators defined with the help of the Killing-Yano tensors and then to write down the Runge-Lenz operator.

It is known that space-time isometries give rise to constants of motion along geodesics. However not all conserved quantities along geodesics arise from isometries. There are prime integrals of motion related to *hidden* symmetries of the manifold, a manifestation of the existence of the Stäckel-Killing tensors [9, 10]. A Stäckel-Killing tensor of valence $r$ is a tensor $k_{\mu_1\ldots\mu_r}$ which is completely symmetric and which satisfies a generalized Killing equation $k_{(\mu_1\ldots\mu_r;\lambda)} = 0$.

There are geometries where the Stäckel-Killing tensor can have a certain root represented by Killing-Yano tensors. We recall that a tensor $f_{\mu_1\ldots\mu_r}$ is a Killing-Yano tensor of valence $r$ if it is totally antisymmetric and satisfies the equation $f_{\mu_1\ldots(\mu_r;\lambda)} = 0$ [11]. The role of the Killing-Yano tensors becomes important in theories with spin as in the cases of the pseudo-classical spinning manifolds [12, 13] or in quantum theory of fermions [14].

On the other hand, Carter and McLenaghan [15] showed that, in the theory of Dirac fermions, for any isometry with Killing vector $k^\mu$ there is an appropriate operator

$$X_k = -i(k^\mu \hat{\nabla}_\mu - \frac{1}{4}\gamma^\mu\gamma^\nu k_{\mu;\nu}) \tag{1}$$

which *commutes* with the *standard* Dirac operator defined by

$$D_s = \gamma^\rho \hat{\nabla}_\rho \tag{2}$$

where $\hat{\nabla}_\rho$ are the spin covariant derivatives including the spin-connection while $\gamma^\mu$ are the point-dependent Dirac matrices carrying natural indices. The usual Dirac matrices with local (hated) indices will be denoted by $\hat{\gamma}^{\hat{\mu}}$.

We specify that $X_k$ is just the generator of the operator-valued representation according which the Dirac field must transform under the isometry transformations corresponding to $k^\mu$ [16]. Moreover, as stated in [15], each Killing-Yano tensor $f_{\mu\nu}$ of valence 2 produces the non-standard Dirac operators of the form

$$D_f = -i\gamma^\mu (f_\mu{}^\nu \hat{\nabla}_\nu - \frac{1}{6}\gamma^\nu \gamma^\rho f_{\mu\nu;\rho}) \tag{3}$$

which *anticommutes* with the standard Dirac operator $D_s$. We observe that the operators (1) and (3) have specific spin parts.

## 2 The Dirac field in Taub-NUT space

Let us consider the Taub-NUT space and the chart with Cartesian coordinates $x^\mu$ ($\mu, \nu, ... = 1, 2, 3, 4$) having the line element

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \frac{1}{V} dl^2 + V(dx^4 + A_i dx^i)^2 \tag{4}$$

where $dl^2 = (d\vec{x})^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2$ is the Euclidean three-dimensional line element and $\vec{A}$ is the gauge field of a monopole. Another chart suitable for applications is that of spherical coordinates, $x' = (r, \theta, \phi, \chi)$, among them the first three are the spherical coordinates commonly associated with the Cartesian space ones, $x^i$ ($i, j, ... = 1, 2, 3$), while $\chi + \phi = -\mu x^4$. The real number $\mu$ is the parameter of the theory which enters in the form of the function $1/V(r) = 1 + \mu/r$. The unique non-vanishing component of the vector potential in spherical coordinates is $A_\phi = \mu(1 - \cos\theta)$. In Cartesian coordinates we get $\mathrm{div}\vec{A} = 0$ and $\vec{B} = \mathrm{rot}\,\vec{A} = \mu\frac{\vec{x}}{r^3}$.

When one uses Cartesian charts in the Taub-NUT geometry, it is useful to consider the local frames given by tetrad fields, $e(x)$ and $\hat{e}(x)$, as defined in [17]. Their components, have the usual orthonormalization properties and give the components of the metric tensor, $g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} \hat{e}^{\hat{\alpha}}_\mu \hat{e}^{\hat{\beta}}_\nu$ and $g^{\mu\nu} = \eta^{\hat{\alpha}\hat{\beta}} e^\mu_{\hat{\alpha}} e^\nu_{\hat{\beta}}$ where $\eta_{\hat{\alpha}\hat{\beta}} = \delta_{\hat{\alpha}\hat{\beta}}$ is the Euclidean metric tensor. Its gauge group, $G(\eta) = SO(4)$, has the universal covering group $\tilde{G}(\eta) \sim SU(2) \otimes SU(2)$. The four Dirac matrices matrices $\hat{\gamma}^{\hat{\alpha}}$, that satisfy $\{\hat{\gamma}^{\hat{\alpha}}, \hat{\gamma}^{\hat{\beta}}\} = 2\eta^{\hat{\alpha}\hat{\beta}}$, can be taken as

$$\hat{\gamma}^i = -i \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \hat{\gamma}^4 = \begin{pmatrix} 0 & \mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}. \tag{5}$$

In addition we consider the matrix

$$\hat{\gamma}^5 = \hat{\gamma}^1 \hat{\gamma}^2 \hat{\gamma}^3 \hat{\gamma}^4 = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix} \tag{6}$$

which is denoted by $\gamma^0$ in Kaluza-Klein theory explicitly involving the time [14]. These matrices are self-adjoint and give the covariant basis generators of the group $\tilde{G}(\eta)$, denoted by $S^{\hat\alpha\hat\beta} = -i[\hat\gamma^{\hat\alpha}, \hat\gamma^{\hat\beta}]/4$.

The standard Dirac operator is defined as [14]

$$D_s = \hat\gamma^{\hat\alpha}\hat\nabla_{\hat\alpha} = i\sqrt{V}\vec{\hat\gamma}\cdot\vec{P} + \frac{i}{\sqrt{V}}\hat\gamma^4 P_4 + \frac{i}{2}V\sqrt{V}\hat\gamma^4\vec\Sigma^*\cdot\vec{B} \tag{7}$$

where $\hat\nabla_{\hat\alpha}$ are the components of the spin covariant derivatives with local indices

$$\hat\nabla_i = i\sqrt{V}P_i + \frac{i}{2}V\sqrt{V}\varepsilon_{ijk}\Sigma_j^* B_k\,, \quad \hat\nabla_4 = \frac{i}{\sqrt{V}}P_4 - \frac{i}{2}V\sqrt{V}\vec\Sigma^*\cdot\vec{B}\,. \tag{8}$$

These depend on the momentum operators $P_i = -i(\partial_i - A_i\partial_4)$ and $P_4 = -i\partial_4$, which obey the commutation rules $[P_i, P_j] = i\varepsilon_{ijk}B_kP_4$ and $[P_i, P_4] = 0$. The spin matrices giving the spin connection are

$$\Sigma_i^* = S_i + \frac{i}{2}\hat\gamma^4\hat\gamma^i\,, \quad S_i = \frac{1}{2}\varepsilon_{ijk}S^{jk}\,. \tag{9}$$

In our representation of the Dirac matrices, the Hamiltonian operator of the *massless* Dirac field reads [14]

$$H = \hat\gamma^5 D_s = \left(\begin{array}{cc} 0 & V\pi^*\frac{1}{\sqrt{V}} \\ \sqrt{V}\pi & 0 \end{array}\right)\,. \tag{10}$$

This is expressed in terms of the operators $\pi = \sigma_P - iV^{-1}P_4$ and $\pi^* = \sigma_P + iV^{-1}P_4$ where $\sigma_P = \vec\sigma\cdot\vec{P}$ involves the Pauli matrices $\sigma_i$. Denoting by $\nabla_\mu$ the usual covariant derivatives we find that the Klein-Gordon operator has the equivalent forms

$$\Delta = -\nabla_\mu g^{\mu\nu}\nabla_\nu = V\,\pi^*\pi = V\vec{P}^{\,2} + \frac{1}{V}{P_4}^2\,. \tag{11}$$

# 3 Conserved observables

We say that an operator represents a *conserved* observable if this *commutes* with $H$. The specific off-diagonal form of $H$ suggested us to introduce the $\mathcal{Q}$-operators defined as [14]

$$\mathcal{Q}(X) = \left\{H\,, \left(\begin{array}{cc} X & 0 \\ 0 & 0\end{array}\right)\right\} = i\left[Q_0\,, \left(\begin{array}{cc} X & 0 \\ 0 & 0\end{array}\right)\right] \tag{12}$$

where $Q_0 = iD_s = i\hat{\gamma}^5 H$. These operators allow us to associate Dirac operators to the Pauli operators, $\pi, \pi^*$, $\sigma_P$, $\sigma_L = \vec{\sigma}\cdot\vec{L}$ and $\sigma_r = \vec{\sigma}\cdot\vec{x}/r$ [18]. The remarkable property of the $\mathcal{Q}$-operators is that if $[X, \Delta] = 0$ then $\mathcal{Q}(X)$ commutes with $H$ [14]. Other conserved observables can be of diagonal form $T = \mathrm{diag}\,(T^{(1)}, T^{(2)})$ when the condition $[T, H] = 0$ implies

$$T^{(2)}\sqrt{V}\pi = \sqrt{V}\pi T^{(1)}\,, \quad V\pi^*\frac{1}{\sqrt{V}}T^{(2)} = T^{(1)}V\pi^*\frac{1}{\sqrt{V}} \tag{13}$$

and, therefore, $[T^{(1)}, \Delta] = 0$.

The conserved observables may be found among the operators which commute or anticommute simultaneously with $D_s$ and $\hat{\gamma}^5$. An important type of such operators are the generators of the operator-valued representations of the isometry group carried by the spaces of physical fields [15, 16]. The space defined by (4) has the isometry group $G_s = SO(3)\otimes U_4(1)$, of rotations of the coordinates $x^i$ and translations of $x^4$, with the Killing vectors $k_i$ ($i = 1, 2, 3$) and $k_4$ respectively. According to Eq.(1) the $U_4(1)$ generator is $P_4$ while other three Killing vectors give the $SO(3)$ generators which are the components of the whole angular momentum $\vec{J} = \vec{L} + \vec{S}$ as in the flat space-times. The difference is that here the orbital angular momentum is

$$\vec{L} = \vec{x}\times\vec{P} - \mu\frac{\vec{x}}{r}P_4\,. \tag{14}$$

The components of $\vec{J}$ commute with $D_s$ and $\hat{\gamma}^5$ and, therefore, they are conserved, commuting with $H$. Moreover, they satisfy the canonical commutation rules among themselves and with the components of all the other vector operators (e.g. coordinates, momenta, spin etc.).

The first three Killing-Yano tensors of the Taub-NUT space [2]

$$f^i = f^i_{\hat{\alpha}\hat{\beta}}\hat{e}^{\hat{\alpha}}\wedge\hat{e}^{\hat{\beta}} = 2\hat{e}^5\wedge\hat{e}^i + \varepsilon_{ijk}\hat{e}^j\wedge\hat{e}^k \tag{15}$$

are rather special since they are covariantly constant. According to Eq.(3), after some algebra, we obtain the Dirac-type operators

$$Q_i = -if^i_{\hat{\alpha}\hat{\beta}}\hat{\gamma}^{\hat{\alpha}}\hat{\nabla}^{\hat{\beta}} = \mathcal{Q}(\sigma_i) \tag{16}$$

which anticommute with $Q_0$ and $\hat{\gamma}^5$, commute with $H$ and obey the $N = 4$ superalgebra

$$\{Q_A, Q_B\} = 2\delta_{AB}H^2\,, \quad A, B, ... = 0, 1, 2, 3 \tag{17}$$

linked to the hyper-Kähler geometry of the Taub-NUT space.

The fourth Killing-Yano tensor of the Taub-NUT space

$$f^Y = -\frac{x^i}{r} f^i + \frac{2x^i}{\mu V} \varepsilon_{ijk} \hat{e}^j \wedge \hat{e}^k \tag{18}$$

is not covariantly constant having the following non-vanishing field strength components

$$f^Y_{r\theta;\phi} = \frac{2r^2}{\mu V} \sin\theta . \tag{19}$$

Taking into account these covariant derivatives we can calculate its corresponding Dirac-type operator, $Q^Y$, according to the general rule (3), as indicated in Appendix. The result is

$$Q^Y = \frac{r}{\mu} \left\{ H \, , \begin{pmatrix} \sigma_r & 0 \\ 0 & -\sigma_r V^{-1} \end{pmatrix} \right\} = i \frac{r}{\mu} \left[ Q_0 \, , \begin{pmatrix} \sigma_r & 0 \\ 0 & \sigma_r V^{-1} \end{pmatrix} \right] . \tag{20}$$

One can verify that this operator commutes with $H$ and anticommutes with $Q_0$ and $\hat{\gamma}^5$.

# 4 The Runge-Lenz operator

The hidden symmetries of the Taub-NUT geometry are encapsulated in the non-trivial Stäckel-Killing tensors $k_{i\mu\nu}, (i = 1, 2, 3)$. They can be expressed as symmetrized products of Killing-Yano tensors (15) and (18) [4]:

$$k_{i\mu\nu} = -\frac{\mu}{4} ( f^Y_{\ \mu\lambda} f^{i\lambda}_{\ \ \nu} + f^Y_{\ \nu\lambda} f^{i\lambda}_{\ \ \mu} ) + \frac{1}{2\mu} ( k_{4\mu} k_{i\nu} + k_{4\nu} k_{i\mu} ). \tag{21}$$

In fact only the product of Killing-Yano tensors $f^i$ and $f^Y$ leads to non-trivial Stäckel-Killing tensors, the last term in the r.h.s. of (21) being a simple product of Killing vectors. This term is usually added to write the Runge-Lenz vector of the scalar (Schrödinger or Klein-Gordon) theory [2, 19]

$$\vec{K} = -\frac{1}{2} \nabla_\mu \vec{k}^{\mu\nu} \nabla_\nu = \frac{1}{2} ( \vec{P} \times \vec{L} - \vec{L} \times \vec{P} ) - \frac{\mu}{2} \frac{\vec{x}}{r} \Delta + \mu \frac{\vec{x}}{r} {P_4}^2 \tag{22}$$

whose components commute with $\Delta$ and obey

$$[L_i, \, K_j] = i \varepsilon_{ijk} K_k \quad , \quad [K_i, \, K_j] = i ( {P_4}^2 - \Delta ) \varepsilon_{ijk} L_k \, . \tag{23}$$

For the Dirac theory the construction of the Runge-Lenz operators can be done using products among the operators $Q^Y$ and $Q_i$. This procedure represents the only possibility to generate non-trivial conserved operators which should be not related to the Hamiltonian, like in Eq.(17), or the Casimir operators of the symmetry group of the manifold.

Using an analogy with the relation (21) from the Taub-NUT geometry, let us consider the vector operator $\vec{N}$ with the components

$$N_i = \frac{\mu}{4}\left\{Q^Y,\, Q_i\right\} - J_i P_4 \tag{24}$$

which, after simple but tedious calculations based on Eqs. (A.8) and (A.9) from Appendix, can be put in the form

$$\begin{aligned}\vec{N} \;&=\; \frac{1}{2}W\left(\vec{P}\times\vec{J}-\vec{J}\times\vec{P}-i\hat{\gamma}^4\vec{\hat{\gamma}}\times\vec{P}\right)W^{-1}-\frac{\mu}{2}\mathcal{Q}(\vec{x}/r)H\\ &\quad+\left[\vec{S}+\vec{x}\times(\vec{\Sigma}^*\times\vec{B})\right]P_4+\mu\frac{\vec{x}}{r}{P_4}^2\end{aligned} \tag{25}$$

where $W=\mathrm{diag}\,(1,\sqrt{V})$ and $\vec{\Sigma}^*$ is given by (9).

Since $\vec{N}$ commutes with $H$, its diagonal blocks, $\vec{N}^{(1)}$ and $\vec{N}^{(2)}$, satisfy the relations (13). It is straightforward to find that the first block which commutes with $\Delta$,

$$\vec{N}^{(1)} = \vec{K} + \frac{\vec{\sigma}}{2}P_4\,, \tag{26}$$

contains not only the orbital Runge-Lenz operator (22) but a spin term too. Moreover the components of the operator $\vec{N}$ satisfy the following commutation relations:

$$[N_i,\, P_4] = 0\,, \qquad [N_i,\, J_j] = i\varepsilon_{ijk}N_k\,, \tag{27}$$

$$[N_i,\, Q_0] = 0\,, \qquad [N_i,\, Q_j] = i\varepsilon_{ijk}Q_kP_4 \tag{28}$$

and

$$[N_i,\, N_j] = i\varepsilon_{ijk}J_kF^2 + \frac{i}{2}\varepsilon_{ijk}Q_iH\,, \quad F^2 = {P_4}^2 - H^2\,. \tag{29}$$

In order to put the last commutator in a form close to (23) we can redefine the components of the Runge-Lenz operator, $\vec{\mathcal{K}}$, as follows

$$\mathcal{K}_i = N_i + \frac{1}{2}H^{-1}(F-P_4)Q_i \tag{30}$$

These operators satisfy the desired commutation rules

$$[\mathcal{K}_i,\, H] = 0\,, \qquad [\mathcal{K}_i,\, J_j] = i\varepsilon_{ijk}\mathcal{K}_k\,, \tag{31}$$

$$[\mathcal{K}_i,\, P_4] = 0\,, \qquad [\mathcal{K}_i,\, Q_j] = i\varepsilon_{ijk}Q_kF \tag{32}$$

and

$$[\mathcal{K}_i,\, \mathcal{K}_j] = i\varepsilon_{ijk}J_kF^2\,. \tag{33}$$

The explicit form of them makes obvious the spin contribution to the Runge-Lenz vector, re-confirming the result from pseudo-classical approach [4, 20]. We specify that there are no zero modes [14] and, therefore, the operator $\boldsymbol{H}$ is invertible such that our definition (30) of the Runge-Lenz operator is correct. Moreover, as in the scalar case [6, 2], this can be re-scaled in order to recover the familiar dynamical algebras $o(4)$, $o(3,1)$ or $e(3)$, corresponding to different spectral domains of the Kepler-type problems.

Finally, it is worthy to note that commutation relation (28) of the Runge-Lenz operator with the standard Dirac operator remains valid even if a mass term is included in Eq.(2).

## Appendix: The operator $Q^Y$

To evaluate the operator $Q^Y$ we start with the first term of (3)

$$-if^Y_{\hat\alpha\hat\beta}\hat\gamma^{\hat\alpha}\hat\nabla^{\hat\beta}=-\frac{x^i}{r}Q_i+\frac{2i}{\mu\sqrt{V}}\left(\begin{array}{cc}0&\lambda-V\\1-\lambda&0\end{array}\right). \tag{A.1}$$

where $\lambda=\vec\sigma\cdot(\vec x\times\vec P)+1=\sigma_L+1+\mu\sigma_r P_4$ is the operator introduced in [21]. Since the Dirac matrices with spherical indices, $\gamma^\mu(x')=e'^{\mu}_{\hat\alpha}(x')\hat\gamma^{\hat\alpha}$, corresponding to the orthogonal coordinates $r$, $\theta$ and $\phi$ are

$$\begin{aligned}\gamma^r(x')&=\sqrt{V}(\hat\gamma^1\sin\theta\cos\phi+\hat\gamma^2\sin\theta\sin\phi+\hat\gamma^3\cos\theta)\,, &\text{(A.2)}\\ \gamma^\theta(x')&=r^{-1}\sqrt{V}(\hat\gamma^1\cos\theta\cos\phi+\hat\gamma^2\cos\theta\sin\phi-\hat\gamma^3\sin\theta)\,, &\text{(A.3)}\\ \gamma^\phi(x')&=(r\sin\theta)^{-1}\sqrt{V}(-\hat\gamma^1\sin\phi+\hat\gamma^2\cos\phi) &\text{(A.4)}\end{aligned}$$

we find that

$$\gamma^r\gamma^\theta\gamma^\phi=-\gamma^r\gamma^\phi\gamma^\theta=\cdots=r^{-2}(\sin\theta)^{-1}V\sqrt{V}\,\hat\gamma^5\hat\gamma^4 \tag{A.5}$$

is completely antisymmetric in $r$, $\theta$, $\phi$. Then, we calculate the second term of (3), according to (19) and (A.5), and embedding the result with (A.1) we get

$$Q^Y=-\mathcal{Q}(\sigma_r)+\frac{2i}{\mu\sqrt{V}}\left(\begin{array}{cc}0&\lambda\\-\lambda&0\end{array}\right). \tag{A.6}$$

The form (20) and other interesting formulas can be derived using the identities $[\sigma_r,\sigma_P]=2ir^{-1}\lambda$ and

$$\sigma_P\lambda=-\lambda\sigma_P=\frac{i}{2}\vec\sigma\cdot(\vec P\times\vec L-\vec L\times\vec P)-\frac{i\mu}{r}\lambda P_4\,. \tag{A.7}$$

With their help one can demonstrate that the relations

$$Q^Y \mathcal{Q}(X) = -\mathcal{Q}(\sigma_r X)H + \frac{2i}{\mu}\begin{pmatrix} \lambda\pi X & 0 \\ 0 & -\sqrt{V}\lambda X \pi^* \frac{1}{\sqrt{V}} \end{pmatrix}, \tag{A.8}$$

$$\mathcal{Q}(X) Q^Y = -\mathcal{Q}(X\sigma_r)H + \frac{2i}{\mu}\begin{pmatrix} -X\pi^*\lambda & 0 \\ 0 & \sqrt{V}\pi X \lambda \frac{1}{\sqrt{V}} \end{pmatrix} \tag{A.9}$$

hold for any $2\times 2$ matrix operator $X$ which commutes with $\Delta$ and $V$. In particular for $X = 1$ it results $[Q^Y, H] = 0$.

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