First Order Formalism for Mixed Symmetry Tensor Fields

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Abstract

In this paper we give explicit first order Lagrangian formulation for mixed symmetry tensor fields $\Phi_{[\mu\nu],\alpha}$, $T_{[\mu\nu\alpha],\beta}$ and $R_{[\mu\nu],[\alpha\beta]}$. We show that such Lagrangians could be written in a very suggestive form similar to the well known tetrad formalism in gravity. Such description could simplify the investigations of possible interactions for these fields. Some examples of interactions are given.

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Introduction

Last times there is a renewed interest in the mixed symmetry high spin tensor fields [1]-[8]. The reason is that such fields naturally appear in a number of physically interesting theories such as superstrings, supergravities and (supersymmetric) high spin theories. One of the technical problems one faces working with such fields is that to get analog of gauge invariant "field strengths" one has to build expressions with more and more derivatives [9], [10]. This problem appears already for the massless spin two field in gravity, but in this case there exist very elegant solution: one goes to the first order formalism and obtains the description in terms of tetrad $e_\mu{}^a$ and Lorentz connection $\omega_\mu{}^{[ab]}$ which play the roles of the gauge fields and "normal" gauge invariant field strengths $T_{[\mu\nu]}{}^a$ and $R_{[\mu\nu]}{}^{[ab]}$ having nice geometrical interpretation as torsion and curvature.

In this paper we follow the same way for the three examples of massless mixed symmetry tensor fields $\Phi_{[\mu\nu],\alpha}$, $T_{[\mu\nu\alpha],\beta}$ and $R_{[\mu\nu],[\alpha\beta]}$. In all three cases we managed to construct first order formalism which is very much similar to the tetrad formalism in gravity and in which the Lagrangians take very simple and suggestive form. We hope that such formalism could simplify the investigation of possible interactions among such fields. Moreover, we give a couple of simple but interesting examples of such interactions.

For completeness, we remind basic properties of the first order formalism in gravity which will be important for us throughout the paper. Usual second order formulation for the massless spin two field uses symmetric second rank tensor field $h_{\{\mu\nu\}}$ with the Lagrangian:

$$\mathcal{L}_0 = \frac{1}{2}\partial_\alpha h^{\mu\nu}\partial_\alpha h_{\mu\nu} - (\partial h)^\mu(\partial h)_\mu + (\partial h)^\mu\partial_\mu h - \frac{1}{2}\partial^\mu h\partial_\mu h \tag{1}$$

where $(\partial h)^\mu = \partial_\nu h^{\nu\mu}$, $h = h_\mu{}^\mu$, which is invariant under the gauge transformations $\delta h_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$. In this, there is no any gauge invariant expression with one derivative analogous to field strength in Yang-Mills theories and the Lagrangian can not be rewritten as expression quadratic in them. As is well known the simplest gauge invariant object is the tensor:

$$R_{[\mu\nu],[\alpha\beta]} = \partial_\mu\partial_\alpha h_{\nu\beta} - \partial_\nu\partial_\alpha h_{\mu\beta} - \partial_\mu\partial_\beta h_{\nu\alpha} + \partial_\nu\partial_\beta h_{\mu\alpha}$$

Now, let us abandon the requirement that the field $h_{\mu\nu}$ is symmetric. Instead, let us consider the most general second order Lagrangian and require that it will be invariant under the gauge transformations $\delta h_{\mu\nu} = \partial_\mu\xi_\nu$. This time, it is a trivial task to construct gauge invariant object which is first order in derivatives: $T_{[\mu\nu],\alpha} = \partial_\mu h_{\nu\alpha} - \partial_\nu h_{\mu\alpha}$ so that the most general gauge invariant Lagrangian could be written as:

$$\mathcal{L} = \frac{1}{8}T^{\mu\nu,\alpha}T_{\mu\nu,\alpha} + \frac{a_1}{2}T^{\mu\nu,\alpha}T_{\mu\alpha,\nu} + \frac{a_2}{2}T^\mu T_\mu \tag{2}$$

where $T_\mu = T_{\mu\nu}{}^\nu$. But in so doing we have introduced additional degrees of freedom, namely antisymmetric part of $h_{\mu\nu}$. Let us decompose our field into symmetric and antisymmetric parts: $h_{\mu\nu} = f_{\{\mu\nu\}} + B_{[\mu\nu]}$. In terms of $f_{\mu\nu}$ and $B_{\mu\nu}$ our Lagrangian takes the form:

$$\begin{aligned} \mathcal{L} \;=\; & \frac{1+2a_1}{4}\partial^\mu h^{\nu\alpha}\partial_\mu h_{\nu\alpha} - \frac{1+2a_1-2a_2}{4}(\partial f)^\mu(\partial f)_\mu - a_2(\partial f)^\mu\partial_\mu f + \frac{a_2}{2}\partial^\mu f\partial_\mu f + \\ & + \frac{1-2a_1}{4}\partial^\mu B^{\nu\alpha}\partial_\mu B_{\nu\alpha} - \frac{1-6a_1-2a_2}{4}(\partial B)^\mu(\partial B)_\mu - \\ & - \frac{1+2a_1+2a_2}{2}(\partial f)^\mu(\partial B)_\mu \end{aligned} \tag{3}$$

We see that in general there exists mixing between $f_{\mu\nu}$ and $B_{\mu\nu}$ components so to understand which physical degrees of freedom such Lagrangian describes one has to go through careful Hamiltonian analysis. But there exists at least one trivial solution [1]: $2a_2 = -(1+2a_1)$ for which mixing term is absent and we get:

$$\begin{aligned} \mathcal{L} &= \frac{1+2a_1}{2}\left[\frac{1}{2}\partial^\mu h^{\nu\alpha}\partial_\mu h_{\nu\alpha} - (\partial f)^\mu(\partial f)_\mu + (\partial f)^\mu\partial_\mu f - \frac{1}{2}\partial^\mu f\partial_\mu f\right] \\ &+ \frac{1-2a_1}{4}\left[\partial^\mu B^{\nu\alpha}\partial_\mu B_{\nu\alpha} - 2(\partial B)^\mu(\partial B)_\mu\right] \end{aligned} \tag{4}$$

One can see that up to normalization we have a sum of two independent Lagrangians: usual gauge invariant Lagrangian for symmetric tensor as well as (also gauge invariant) Lagrangian for antisymmetric field. The well known tetrad formulation of gravity (i.e. massless spin two filed) corresponds to the choice $a_1 = 1/2$. In this, antisymmetric field $B_{\mu\nu}$ completely decouples and does not introduce any additional physical degrees of freedom, while the Lagrangian has the form:

$$\mathcal{L} = \frac{1}{8}T^{\mu\nu,\alpha}T_{\mu\nu,\alpha} + \frac{1}{4}T^{\mu\nu,\alpha}T_{\mu\alpha,\nu} - \frac{1}{2}T^\mu T_\mu \tag{5}$$

It is easy to check that this Lagrangian is invariant not only under the gauge transformations $\delta h_{\mu\nu} = \partial_\mu\xi_\nu$, but under the local shifts $h_{\mu\nu} \to h_{\mu\nu} + \eta_{[\mu\nu]}$ as well[2]. This last invariance suggests the following procedure for transition from second order formalism to the first order one. One introduces auxiliary field $\omega_{\mu,[\alpha\beta]}$ which will play the role of gauge field for the local $\eta_{\mu\nu}$ transformations. Then it is easy to construct the first order Lagrangian

$$\mathcal{L}_I = -\frac{1}{2}\omega^{\mu,\alpha\beta}\omega_{\alpha,\mu\beta} + \frac{1}{2}\omega^\mu\omega_\mu - \frac{1}{2}\omega^{\mu,\alpha\beta}T_{\alpha\beta,\mu} - \omega^\mu T_\mu \tag{6}$$

which is invariant under the gauge transformations $\delta\omega_{\mu,\alpha\beta} = \partial_\mu\eta_{\alpha\beta}$, $\delta h_{\mu\nu} = \eta_{\mu\nu}$ as well as (trivially) under the $\delta h_{\mu\nu} = \partial_\mu\xi_\nu$. The algebraic equations of motion for the ω-field can be solved giving us:

$$\omega_{\mu,\alpha\beta} = \frac{1}{2}[T_{\mu\alpha,\beta} - T_{\mu\beta,\alpha} - T_{\alpha\beta,\mu}] \tag{7}$$

In this, substituting this expression into the first order Lagrangian (6) one obtains exactly the second order Lagrangian (5).

Moreover, if we formally divide all indices into the "local" and "world" ones this Lagrangian could be rewritten in a very simple and suggestive form:

$$\mathcal{L}_I = \frac{1}{2}\left\{ {}^{\mu\nu}_{ab} \right\}\omega_\mu{}^{ac}\omega_\nu{}^{bc} - \frac{1}{2}\left\{ {}^{\mu\nu\alpha}_{abc} \right\}\omega_\mu{}^{ab}\partial_\nu h_\alpha{}^c \tag{8}$$

Here

$$\left\{ {}^{\mu\nu}_{ab} \right\} = \delta_a{}^\mu\delta_b{}^\nu - \delta_a{}^\nu\delta_b{}^\mu$$

and so on. Such form points to the nice geometric interpretation of these fields we have already mentioned and greatly simplify understanding of the possible interactions. Indeed,

[1] Note that in this the Lagrangian written in terms of " torsion" $T_{\mu\nu,\alpha}$ belongs to one parameter family of so called teleparallel Lagrangians.

[2] Clearly, this is a manifestation of $B_{\mu\nu}$ decoupling

if one suppose that interaction Lagrangian has the same general form as the free one, then one easily, by the finite number of iterations can reproduce complete non-linear gravitational interactions.

As it is evident from (8) this construction works for the space-time dimensions $d \geq 3$ only. In the minimal dimension[3] $d = 3$ one can introduce dual variables:

$$f_\mu{}^a = \frac{1}{2}\varepsilon^{abc}\omega_{\mu,bc}, \qquad \eta^a = \frac{1}{2}\varepsilon^{abc}\eta_c \tag{9}$$

and rewrite the Lagrangian in the following simple form:

$$\mathcal{L} = -\frac{1}{2}\left\{ {}^{\mu\nu}_{ab} \right\} f_\mu{}^a f_\nu{}^b + \varepsilon^{\mu\nu\alpha} f_\mu{}^a \partial_\nu h_\alpha{}^a \tag{10}$$

in this, gauge transformations look like:

$$\delta f_\mu{}^a = \partial_\mu \eta^a \qquad \delta h_{\mu a} = -\varepsilon_{\mu ab}\eta^b$$

Then one can introduces rather exotic interaction [11] by adding to the Lagrangian and gauge transformations additional terms:

$$\mathcal{L}_{int} = \frac{\kappa}{6}\varepsilon^{\mu\nu\alpha}\varepsilon_{abc} f_\mu{}^a f_\nu{}^b f_\alpha{}^c \qquad \delta_1 h_{\alpha b} = \kappa \varepsilon_{abc} f_\alpha{}^a \eta^c \tag{11}$$

where κ — arbitrary constant having dimension m^{-2}. Note that in the first order formalism such a model looks very simple, but if one tries to solve algebraic equation of motion for the f-field, one obtains essentially non-linear theory with infinite number of derivatives.

In the following three sections we consider generalization of this approach for massless mixed symmetry tensor fields $\Phi_{[\mu\nu],\alpha}$, $T_{[\mu\nu\alpha],\beta}$ and $R_{[\mu\nu],[\alpha\beta]}$.

1 $\Phi_{[\mu\nu],\alpha}$ tensor

Usual second order formulation for this field uses the tensor $\Phi_{[\mu\nu],\alpha}$ satisfying additional condition $\Phi_{[\mu\nu,\alpha]} = 0$. In this, the free Lagrangian is invariant under two gauge transformations:

$$\delta\Phi_{\mu\nu,\alpha} = \partial_\mu x_{\nu\alpha} - \partial_\nu x_{\mu\alpha} + 2\partial_\alpha y_{\mu\nu} - \partial_\mu y_{\nu\alpha} + \partial_\nu y_{\mu\alpha}$$

where $x_{\{\mu\nu\}}$ is symmetric while $y_{[\mu\nu]}$ antisymmetric. As in the spin two case to construct a gauge invariant object one needs at least two derivatives. Let us abandon additional condition $\Phi_{[\mu\nu,\alpha]} = 0$ and simultaneously join two gauge transformations into: $\delta\Phi_{\mu\nu,\alpha} = \partial_\mu z_{\nu\alpha} - \partial_\nu z_{\mu\alpha}$ where $z_{\mu\nu}$ is arbitrary second rank tensor [12], [13]. Then it is easy to get gauge invariant field strength with one derivative:

$$T_{\mu\nu\alpha,\beta} = \partial_{[\mu}\Phi_{\nu\alpha],\beta}$$

Let us consider the most general second order free Lagrangian invariant under the $z_{\mu\nu}$-transformations:

$$\mathcal{L} = -\frac{1}{6}T^{\mu\nu\alpha,\beta}T_{\mu\nu\alpha,\beta} + \frac{a_1}{4}T^{\mu\nu\alpha,\beta}T_{\mu\nu\beta,\alpha} + \frac{a_2}{4}T^{\mu\nu}T_{\mu\nu} \tag{12}$$

[3] there are no physical degrees of freedom in this case

In discarding additional condition $\Phi_{[\mu\nu,\alpha]} = 0$ we introduce additional degrees of freedom into the theory, so to understand what physical degrees of freedom such a model describes we decompose our field $\Phi_{\mu\nu,\alpha} = \hat{\Phi}_{\mu\nu,\alpha} + C_{\mu\nu\alpha}$, where $\hat{\Phi}_{[\mu\nu,\alpha]} = 0$ and $C_{[\mu\nu\alpha]}$ — completely antisymmetric tensor. In terms of these components the Lagrangian takes the form:

$$
\begin{aligned}
\mathcal{L} \;=\; & -\frac{2-a_1}{4}\partial^\mu\hat{\Phi}^{\alpha\beta,\nu}\partial_\mu\hat{\Phi}_{\alpha\beta,\nu} + \frac{a_2}{4}\partial_\mu\hat{\Phi}^{\alpha\beta,\mu}\partial_\nu\hat{\Phi}_{\alpha\beta,\nu} + \frac{2-a_1}{2}\partial_\mu\hat{\Phi}^{\mu\alpha,\beta}\partial^\nu\hat{\Phi}_{\nu\alpha,\beta} + \\
& + a_2\partial_\mu\hat{\Phi}^{\alpha\beta,\mu}\partial_\alpha\hat{\Phi}_\beta + \frac{a_2}{2}\partial^\alpha\hat{\Phi}^\beta\partial_\alpha\hat{\Phi}_\beta - \frac{a_2}{2}(\partial\hat{\Phi})(\partial\hat{\Phi}) - \\
& - \frac{1+a_1}{2}\partial^\mu C^{\nu\alpha\beta}\partial_\mu C_{\nu\alpha\beta} + \frac{4+7a_1+a_2}{4}(\partial C)_{\alpha\beta}{}^2 + \\
& + \frac{a_1+a_2-2}{2}\partial^\mu\hat{\Phi}^{\alpha\beta,\mu}(\partial C)_{\alpha\beta}
\end{aligned} \tag{13}
$$

Once again we see that there is a mixing of two components so to understand what degrees of freedom such theory describes and how physical they are one needs to go through careful analysis. But there is one evident solution, namely, $a_2 = 2 - a_1$. In this, the Lagrangian becomes a sum of two independent Lagrangians for the fields $\hat{\Phi}_{\mu\nu,\alpha}$ and $C_{\mu\nu\alpha}$. Such a theory could be of some interest, especially in $d = 4$ where $\hat{\Phi}_{\mu\nu,\alpha}$ does not describe any physical degrees of freedom, while $C_{\mu\nu\alpha}$ is equivalent to the vector field. Following the tetrad formulation of gravity in this paper we will adopt the most conservative approach "one field — one object" and we will choose $a_1 = -1$, $a_2 = 3$. Then it is easy to check that the resulting Lagrangian

$$
\mathcal{L} = -\frac{1}{6}T^{\mu\nu\alpha,\beta}T_{\mu\nu\alpha,\beta} - \frac{1}{4}T^{\mu\nu\alpha,\beta}T_{\mu\nu\beta,\alpha} + \frac{3}{4}T^{\mu\nu}T_{\mu\nu} \tag{14}
$$

is invariant not only under the z-transformations, but under the local shifts $\delta\Phi_{\mu\nu,\alpha} = \eta_{[\mu\nu\alpha]}$ as well. This, in turn, suggests the following procedure for transition to first order formalism. One introduces auxiliary field $\Omega_{\mu,[\nu\alpha\beta]}$ antisymmetric in the last three indices which will play a role of gauge field for the η-transformations. Then it is easy to construct a first order Lagrangian

$$
\mathcal{L} = \frac{3}{4}\Omega^{\mu,\nu\alpha\beta}\Omega_{\nu,\mu\alpha\beta} - \frac{3}{4}\Omega^{\alpha\beta}\Omega_{\alpha\beta} - \frac{1}{2}\Omega^{\mu,\nu\alpha\beta}T_{\nu\alpha\beta,\mu} + \frac{3}{2}\Omega^{\alpha\beta}T_{\alpha\beta} \tag{15}
$$

invariant both under the $\delta\Phi_{\mu\nu,\alpha} = \partial_\mu z_{\nu\alpha} - \partial_\nu z_{\mu\alpha}$ and

$$
\delta\Omega_\mu{}^{\nu\alpha\beta} = \partial_\mu\eta^{\nu\alpha\beta} \qquad \delta\Phi_{\mu\nu,\alpha} = \eta_{\mu\nu\alpha} \tag{16}
$$

Now, if we solve the algebraic equation of motion for the Ω field, we get:

$$
\Omega_{\mu,\nu\alpha\beta} = \frac{2}{3}T_{\nu\alpha\beta,\mu} + \frac{1}{3}[T_{\mu\alpha\beta,\nu} + T_{\mu\nu\alpha,\beta} + T_{\mu\beta\nu,\alpha}] \tag{17}
$$

Substituting this expression back into the first order Lagrangian gives us exactly the second order one. Moreover, if we formally divide all indices into the "local" and "world" ones, then this Lagrangian could also be rewritten in a very simple and suggestive form:

$$
\mathcal{L} = -\frac{1}{4}\left\{ {}^{\mu\nu\alpha\beta}_{abcd} \right\}[3\Omega_\mu{}^{aef}\Omega_\nu{}^{bef}\delta_\alpha{}^c\delta_\beta{}^d - \frac{1}{3}\Omega_\mu{}^{abc}T_{\nu\alpha\beta}{}^d] \tag{18}
$$

As we see from the last formula the whole construction works for the space-time dimensions $d \geq 4$ only. In $d = 4$ (where $\Phi_{\mu\nu,\alpha}$ does not describe any physical degrees of freedom) one can introduce dual variables:

$$f_\mu{}^a = \frac{1}{6}\varepsilon^{abcd}\Omega_\mu{}^{bcd}, \qquad \eta^a = \frac{1}{6}\varepsilon^{abcd}\eta_{bcd}$$

Then the first order Lagrangian could be rewritten in the following form:

$$\mathcal{L} = -\frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}[3\varepsilon_{abcd}f_\mu{}^a f_\nu{}^b \delta_\alpha{}^c \delta_\beta{}^d - f_\mu{}^a T_{\nu\alpha\beta}{}^a] \tag{19}$$

in this, the gauge transformations look like:

$$\delta f_\mu{}^a = \partial_\mu \eta^a \qquad \delta\Phi_{\mu\nu,c} = \varepsilon_{\mu\nu cd}\eta^d$$

By analogy with the $d = 3$ case for gravity one can easily construct an example of (rather exotic) interaction by adding to the Lagrangian and to the gauge transformation laws additional terms:

$$\mathcal{L}_1 = \frac{\kappa}{4}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}f_\mu{}^a f_\nu{}^b f_\alpha{}^c \delta_\beta{}^d \qquad \delta_1\Phi_{\mu\nu,c} = \frac{\kappa}{2}f_{[\alpha}{}^a \delta_{\beta]}{}^b \varepsilon_{abcd}\eta^d \tag{20}$$

Computing the commutator of two η-transformations we get:

$$[\delta_1, \delta_2]\Phi_{\alpha\beta,c} = \partial_{[\alpha} z_{\beta]c}, \qquad z_{\beta c} = \frac{\kappa}{2}\varepsilon_{a\beta cd}\eta_1{}^a \eta_2{}^d$$

so that the whole algebra of z and η-transformations closes.

We can proceed and consider next order interaction by introducing quartic terms to the Lagrangian and quadratic ones to the gauge transformation laws:

$$\Delta\mathcal{L} = -\frac{\kappa_2}{8}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{abcd}f_\mu{}^a f_\nu{}^b f_\alpha{}^c f_\beta{}^d \qquad \delta\Phi_{\alpha\beta,c} = \kappa_2 f_\alpha{}^a f_\beta{}^b \varepsilon_{abcd}\eta^d \tag{21}$$

In this, we obtain additional terms for the commutator of two η-transformations:

$$[\delta_1, \delta_2]\Phi_{\alpha\beta,c} = \kappa_2\partial_{[\alpha}\tilde{z}_{\beta]c} - \kappa_2\partial_{[\alpha}f_{\beta]}{}^a \varepsilon_{abcd}\eta_1{}^a \eta_2{}^d \tag{22}$$

where $\tilde{z}_{\beta c} = f_\beta{}^b \varepsilon_{abcd}\eta_1{}^a \eta_2{}^d$.

By adjusting the values of two arbitrary coupling constants κ and κ_2 and adding necessary linear terms to the Lagrangian the whole model could be written in the formally covariant form:

$$\mathcal{L} = \varepsilon^{\mu\nu\alpha\beta}[\frac{1}{32}\varepsilon_{abcd}E_\mu{}^a E_\nu{}^b E_\alpha{}^c E_\beta{}^d - E_\mu{}^a T_{\nu\alpha\beta}{}^a] \tag{23}$$

where $E_\mu{}^a = \delta_\mu{}^a + \kappa f_\mu{}^a$. This Lagrangian is invariant under the following local gauge transformations:

$$\delta E_\mu{}^a = \partial_\mu \eta^a, \qquad \delta\Phi_{\alpha\beta,c} = \partial_{[\alpha} z_{\beta]c} + \frac{1}{2}E_\alpha{}^a E_\beta{}^b \varepsilon_{abcd}\eta^d \tag{24}$$

the commutator of two η-transformations having the form:

$$[\delta_1, \delta_2]\Phi_{\alpha\beta,c} = \frac{1}{2}\partial_{[\alpha}\tilde{z}_{\beta]c} - \frac{1}{2}\partial_{[\alpha}E_{\beta]}{}^b \varepsilon_{abcd}\eta_1{}^a \eta_2{}^d \tag{25}$$

and $\tilde{z}_{\beta c} = E_\beta{}^b \varepsilon_{abcd}\eta_1{}^a \eta_2{}^d$.

2 $T_{[\mu\nu\alpha],\beta}$ tensor

Our next example — mixed tensor $T_{\mu\nu\alpha,\beta}$ antisymmetric on first three indices. This case is very much similar to the previous one so we will be brief. Usual description uses this tensor with additional condition $T_{[\mu\nu\alpha,\beta]} = 0$ and two gauge transformations with parameters $\chi_{[\mu\nu],\alpha}$ such that $\chi_{[\mu\nu,\alpha]} = 0$ and $\eta_{[\mu\nu\alpha]}$. In this, to construct minimal gauge invariant expression one needs at least two derivatives. Let us abandon additional condition $T_{[\mu\nu\alpha,\beta]} = 0$ and simultaneously combine to gauge transformation into unconstrained one:

$$\delta T_{\mu\nu\alpha,\beta} = \partial_{[\mu}\chi_{\nu\alpha],\beta} \tag{26}$$

Then it is trivial task to construct gauge invariant object of the first order:

$$E_{\mu\nu\alpha\beta,\rho} = \partial_{[\mu}T_{\nu\alpha\beta],\rho}$$

Now if we consider the most general (gauge invariant) Lagrangian quadratic in E-tensor and require that this Lagrangian be invariant under the local shifts $\delta T_{\mu\nu\alpha,\beta} = \eta_{\mu\nu\alpha\beta}$ we obtain:

$$\mathcal{L} = \frac{1}{24}E^{\mu\nu\alpha\beta,\rho}E_{\mu\nu\alpha\beta,\rho} + \frac{1}{18}E^{\mu\nu\alpha\beta,\rho}E_{\mu\nu\alpha\rho,\beta} - \frac{2}{9}E^{\mu\nu\alpha}E_{\mu\nu\alpha} \tag{27}$$

Let us make a transition to the first order formalism. First of all we introduce auxiliary field $\Omega_{\mu,[\nu\alpha\beta\rho]}$ antisymmetric on the last four indices which will play the role of gauge field for the η-transformations. Then we construct a first order Lagrangian

$$\mathcal{L}_I = -\frac{2}{9}\Omega^{\rho,\mu\nu\alpha\beta}\Omega_{\mu,\rho\nu\alpha\beta} + \frac{2}{9}\Omega^{\mu\nu\alpha}\Omega_{\mu\nu\alpha} - \frac{1}{9}\Omega^{\rho,\mu\nu\alpha\beta}E_{\mu\nu\alpha\beta,\rho} - \frac{4}{9}\Omega^{\mu\nu\alpha}E_{\mu\nu\alpha} \tag{28}$$

which is invariant not only under the χ-transformations (), but also under

$$\delta\Omega_\rho{}^{\mu\nu\alpha\beta} = \partial_\rho\eta^{\mu\nu\alpha\beta}, \qquad \delta T_{\mu\nu\alpha,\beta} = \eta_{\mu\nu\alpha\beta}$$

If one solves the algebraic equation of motion for the Ω field one get

$$\Omega_{\rho,\mu\nu\alpha\beta} = -\frac{3}{4}E_{\mu\nu\alpha\beta,\rho} - \frac{1}{4}E_{\rho[\nu\alpha\beta,\mu]} \tag{29}$$

In this, substitution of this expression back into the first order Lagrangian () results exactly in the second order Lagrangian (). As in the previous case by dividing all indices into the "local" and "world" ones the first order Lagrangian () could be rewritten in the simple and convenient form:

$$\mathcal{L}_I = \frac{2}{9}\left[\left\{ {}^{\mu\nu}_{ab} \right\}\Omega_\mu{}^{acde}\Omega_\nu{}^{bcde} - \frac{1}{48}\left\{ {}^{\mu\nu\alpha\beta\gamma}_{abcde} \right\}\Omega_\mu{}^{abcd}E_{\nu\alpha\beta\gamma}{}^e\right] \tag{30}$$

This time our construction works for the space-time dimensions $d \geq 5$ only. In the minimal $d = 5$ case one can introduce dual variables:

$$f_\mu{}^a = \frac{1}{24}\varepsilon^{abcde}\Omega_\mu{}^{bcde}, \qquad \eta^a = \frac{1}{24}\varepsilon^{abcde}\eta_{bcde} \tag{31}$$

Then the Lagrangian takes the form:

$$\mathcal{L} = \frac{4}{3} \left\{ {}^{\mu\nu}_{ab} \right\} f_\mu{}^a f_\nu{}^b - \frac{1}{9} \varepsilon^{\mu\nu\alpha\beta\gamma} f_\mu{}^a E_{\nu\alpha\beta\gamma}{}^a \tag{32}$$

while the gauge transformations leaving it invariant look like:

$$\delta f_\mu{}^a = \partial_\mu \eta^a, \qquad \delta T_{\mu\nu\alpha,a} = \varepsilon_{\mu\nu\alpha ab} \eta^b$$

It is not hard to give an example of interaction for this case by adding to the Lagrangian and gauge transformation laws new terms:

$$\mathcal{L}_1 = \kappa \left\{ {}^{\mu\nu\alpha}_{abc} \right\} f_\mu{}^a f_\nu{}^b f_\alpha{}^c, \qquad \delta_1 T_{\alpha\beta\gamma,a} = \kappa \varepsilon_{abc[\alpha\beta} f_{\gamma]}{}^b \eta^c \tag{33}$$

Again this Lagrangian looks very simple in the first order formalism, but it corresponds to highly non-linear second order Lagrangian with non limited number of derivatives.

3 $R_{[\mu\nu],[\alpha\beta]}$ tensor

In this section we consider more interesting and less evident example — tensor $R_{[\mu\nu],[\alpha\beta]}$. Usual description is based on the tensor with constraint $R_{\mu[\nu,\alpha\beta]} = 0$ and gauge transformations with a parameter $\chi_{\mu,[\alpha\beta]}$ also constrained by $\chi_{[\mu,\alpha\beta]} = 0$. We abandon both constraints and consider arbitrary $R_{[\mu\nu],[\alpha\beta]}$ tensor with gauge transformations $\delta R_{\mu\nu,\alpha\beta} = \partial_\mu \chi_{\nu,\alpha\beta} - \partial_\nu \chi_{\mu,\alpha\beta}$, where $\chi_{\mu,\alpha\beta}$ antisymmetric on the last two indices. Then it is trivial to get gauge invariant "field strength":

$$T_{[\mu\nu\alpha],[\beta\gamma]} = \partial_{[\mu} R_{\nu\alpha],\beta\gamma}$$

Now we consider the most general Lagrangian quadratic in $T_{\mu\nu\alpha,\beta\gamma}$ and require that it has to be invariant under the local shifts:

$$R_{\mu\nu,\alpha\beta} \to R_{\mu\nu,\alpha\beta} + \eta_{\mu,\nu\alpha\beta} - \eta_{\nu,\mu\alpha\beta}$$

where $\eta_{\mu,\nu\alpha\beta}$ antisymmetric on the last three indices. We obtain:

$$\begin{aligned} \mathcal{L} &= \frac{1}{6} T^{\mu\nu\alpha,\beta\gamma} T_{\mu\nu\alpha,\beta\gamma} + \frac{1}{2} T^{\mu\nu\alpha,\beta\gamma} T_{\mu\nu\beta,\alpha\gamma} + \frac{1}{2} T^{\mu\nu\alpha,\beta\gamma} T_{\mu\beta\gamma,\nu\alpha} - \\ &\quad - \frac{3}{2} T^{\mu\nu,\alpha} T_{\mu\nu,\alpha} - 3 T^{\mu\nu,\alpha} T_{\mu\alpha,\nu} + \frac{3}{2} T^\mu T_\mu \end{aligned} \tag{34}$$

To make a transition to first order formalism we introduce auxiliary field $\Omega_{[\mu\nu],[\alpha\beta\gamma]}$ antisymmetric on the first two as well as last three indices which will play a role of gauge field for the η-transformations. The following first order Lagrangian:

$$\begin{aligned} \mathcal{L} &= -\frac{3}{2} \Omega^{\mu\nu,\alpha\beta\gamma} \Omega_{\alpha\beta,\mu\nu\gamma} + 6 \Omega^{\mu,\nu\alpha} \Omega_{\nu,\mu\alpha} - \frac{3}{2} \Omega^\mu \Omega_\mu + \\ &\quad + \Omega^{\mu\nu,\alpha\beta\gamma} T_{\alpha\beta\gamma,\mu\nu} + 6 \Omega^{\mu,\nu\alpha} T_{\nu\alpha,\mu} + 3 \Omega^\mu T_\mu \end{aligned} \tag{35}$$

is invariant not only (by construction) under the local χ- transformations, but under the local transformations

$$\delta\Omega_{\mu\nu,\alpha\beta\gamma} = \partial_\mu \eta_{\nu,\alpha\beta\gamma} - \partial_\nu \eta_{\mu,\alpha\beta\gamma} \qquad \delta R_{\mu\nu,\alpha\beta} = \eta_{\mu,\nu\alpha\beta} - \eta_{\nu,\mu\alpha\beta}$$

as well. By solving algebraic equation of motion for the Ω field one obtains:

$$\Omega_{\mu\nu,\alpha\beta\gamma} = \frac{1}{3} T_{\alpha\beta\gamma,\mu\nu} + \frac{1}{6}[T_{\mu\beta\gamma,\alpha\nu} - T_{\nu\beta\gamma,\alpha\mu} + (\alpha,\beta,\gamma)] + \frac{1}{3}[T_{\mu\nu\alpha,\beta\gamma} + (\alpha,\beta,\gamma)] \qquad (36)$$

Substituting this expression back into the first order Lagrangian one reproduces exactly the second order Lagrangian.

As in all previously considered cases if we divide all indices into the "local" and "world" ones it is possible to rewrite our Lagrangian in the following form:

$$\mathcal{L} = -\frac{3}{8} \left\{ {}^{\mu\nu\alpha\beta}_{abcd} \right\} \Omega_{\mu\nu}{}^{abe} \Omega_{\alpha\beta}{}^{cde} + \frac{1}{12} \left\{ {}^{\mu\nu\alpha\beta\gamma}_{abcde} \right\} \Omega_{\mu\nu}{}^{abc} T_{\alpha\beta\gamma}{}^{de} \qquad (37)$$

This time our construction also works for the space-time dimensions $d \geq 5$ only.

Conclusion

Thus we have managed to construct first order formalism for the three examples of mixed symmetry high spin fields. In all three cases the formulation is very much similar to the usual tetrad formalism in gravity so that the Lagrangians have very simple and suggestive form and gauge invariance is almost evident. We hope that such formalism could be useful in investigations of possible interactions for such fields.

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