

D-BRANES IN GROUP MANIFOLDS AND FLUX STABILIZATION

Christoph Schweigert

LPTHE, Université Paris VI
4 place Jussieu
F – 75 252 Paris Cedex 05

Abstract

We consider D-branes in group manifolds, from the point of view of open strings and using the Born-Infeld action on the brane worldvolume. D-branes correspond to certain integral (twined) conjugacy classes. We explain the integrality condition on the conjugacy classes in both approaches. In the Born-Infeld description, the D-brane worldvolume is stabilized against shrinking by a subtle interplay of quantized $U(1)$ fluxes and the non-triviality of the B-field.

1 D-branes

The role of D-branes in the description of solitonic sectors of string theories is by now well established. Much insight has been gained from the fact that D-branes have two complementary descriptions:

- on the one hand, they correspond to conformally invariant boundary conditions of open strings
- on the other hand, as solitonic objects, they can be described by a worldvolume action of Born-Infeld form.

Here, we are interested in D-branes in backgrounds with non-vanishing values for the metric G and the Kalb-Ramond field B of target space. In particular, we consider a case where B is not even a closed two-form. We will see that the non-triviality of B requires a *quantization* of the possible positions of the D-brane.

Our test case are strings on group manifolds, so-called WZW models; for these backgrounds exact results using two-dimensional conformal field theory are available. Our goal is to see to what extent these results can be derived from *classical* geometry.

For simplicity, we mostly restrict ourselves to the case of $G = SU(2)$. Topologically, $SU(2)$ is a three-sphere S^3 . We write it as a union of two-spheres with coordinates $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ that are parametrized by an angle $0 < \psi < \pi$, with the two elements $+\mathbf{1}$ and $-\mathbf{1}$ of $SU(2)$ added as the “north pole” and the “south pole” of S^3 . In these coordinates, the metric reads

$$ds^2 = k\alpha' \left[d\psi^2 + \sin^2 \psi \left(d\theta^2 + \sin^2 \theta d\phi^2 \right) \right] ,$$

and the Neveu-Schwarz three-form background field is

$$H \equiv dB = 2k\alpha' \sin^2 \psi \sin \theta d\psi d\theta d\phi .$$

The corresponding two-form potential has a Dirac string singularity, which we choose at $\psi = \pi$:

$$B = k\alpha' \left(\psi - \frac{\sin 2\psi}{2} \right) \sin \theta d\theta d\phi . \quad (1)$$

Notice that the Dirac string breaks translation invariance on the group manifold. The two-spheres of “constant latitude” can also be characterized as conjugacy classes: they consist of all elements of $SU(2)$ that are of the form

$$\mathcal{C}(h) = \{ ghg^{-1} \mid g \in SU(2) \}$$

for some fixed diagonal matrix h .

2 Open string analysis

The background we have just described possesses a non-abelian current algebra as its symmetry so that algebraic methods allow to obtain exact results [5]. We wish to describe those conformally invariant boundary conditions for which left moving and right moving currents are connected at the boundary of the world sheet by the action of an automorphism ω of G . This implies that the corresponding boundary state $|\omega, \alpha\rangle$ is constructed from twisted Ishibashi states $|\lambda, \omega\rangle \in \mathcal{H}_\lambda \otimes \mathcal{H}_{\lambda^+}$ that obey

$$[J_n^a \otimes 1 + 1 \otimes \omega(J_{-n}^a)] |\lambda\omega\rangle = 0 .$$

The boundary state has been computed in [3] and reads, in a suitable normalization of the bulk fields:

$$|\omega, \alpha\rangle = \sum \chi_\lambda^\omega(h_\alpha) |\lambda\omega\rangle$$

where χ_λ^ω is the so-called twining character [6] and $h_\alpha = \exp(2\pi i y_\alpha)$. y_α takes its values in a finite set of symmetric weights.

This singles out a finite set of D-branes; their geometry can be directly tested using bulk fields. Consider first the case of flat backgrounds: from the one-point functions on a disc with boundary condition β :

$$G^{ij}(\vec{q}) = \langle \beta | \alpha_{-1}^i \otimes \alpha_{-1}^j | \vec{q} \rangle \quad (2)$$

one obtains [4], after a Fourier transform, the fluctuations of the metric, dilaton and Kalb-Ramond field in the classical D-brane solution.

The symmetries of a free background form an abelian current algebra. The generalization of (2) to the non-abelian case reads

$$G_{\omega,\alpha}^{ab}(v \otimes \tilde{v}) = \langle \omega, \alpha | J_{-1}^a \otimes J_{-1}^b | v \otimes \tilde{v} \rangle$$

where $v \otimes \tilde{v}$ is a vector in $\mathcal{F}_k = \oplus_{\lambda \in P_k} \bar{\mathcal{H}}_\lambda \otimes \bar{\mathcal{H}}_{\lambda^+}$; the sum is over integrable highest weights at level k . We identify \mathcal{F}_k with a subspace of the space $\mathcal{F} = \oplus_\lambda \bar{\mathcal{H}}_\lambda \otimes \bar{\mathcal{H}}_{\lambda^+}$ of functions on G . Then $G_{\omega,\alpha}^{ab}$ can be interpreted [5] as a distribution on G :

$$G_{\omega,\alpha}^{ab}(g) = -\kappa^{ab} \sum_{\lambda \in P_k^\omega} \chi_\lambda^\omega(h_\alpha)^* \chi_\lambda^\omega(g) + \dots$$

where we have dropped antisymmetric terms. Thus the fluctuations of the metric are proportional to the Killing form κ^{ab} and concentrated at the so-called *twined conjugacy class*

$$\mathcal{C}^\omega(h) = \{ gh\omega(g)^{-1} \mid g \in G \}.$$

For a more detailed description of the geometry of these subspaces we refer to [5]; from now on we restrict ourselves to the case of trivial automorphism $\omega = 1$.

3 Space-time analysis

We now consider the (bosonic) Born-Infeld action

$$\int_{\mathcal{B}} d^p \xi \sqrt{\det(G + B + 2\pi\alpha' F)} \quad (3)$$

on the worldvolume \mathcal{B} of the brane. Its form generalizes the action for minimal surfaces: it depends also on the antisymmetric tensor field B . Moreover, one has to choose a connection A with field strength F on \mathcal{B} . We *assume* that classical geometry captures the essential features of the problem; as a consequence the flux $\int_{\mathcal{B}} F$ is quantized.

This point deserves a careful discussion, since one might have tried to impose a quantization of the gauge invariant field strength $\mathcal{F} := B + 2\pi\alpha' F$. A first simple remark is that under the gauge transformations

$$\delta B = 2\pi\alpha' d\Lambda \quad \delta A = -\Lambda$$

integrality of the flux of F is preserved, although its integral value can be changed by large gauge transformations.

One can also see the quantization of F directly from the following worldsheet argument [8]. Apart from the kinetic term, the WZW action on a worldsheet Σ contains a bulk term with the pullback of B and a boundary term with the pull-back of the gauge field A on the brane:

$$\mathcal{L}' = \int_{\Sigma} B - 2\pi\alpha \int_{\partial\Sigma} A. \quad (4)$$

Let us suppose for simplicity that Σ has a single boundary component. We can close Σ by choosing a disc \mathcal{D} in the world volume of the brane. We find

$$\mathcal{L}' = \int_{\Sigma \cup \mathcal{D}} B - \int_{\mathcal{D}} B - 2\pi\alpha' \int_{\mathcal{D}} F = \int_{\mathcal{B}} H - \int_{\mathcal{D}} \mathcal{F},$$

where we have chosen a three-manifold \mathcal{B} with boundary $\Sigma \cup \mathcal{D}$. Our considerations should be independent from the choices of \mathcal{D} and \mathcal{B} . Changing \mathcal{B} , but keeping \mathcal{D} fixed, leads, as usual, to the requirement that the level k should be an integer. On the other hand, changing \mathcal{D} to \mathcal{D}' requires to change \mathcal{B} to $\mathcal{B}' = \mathcal{B} \cup \tilde{\mathcal{B}}$, where $\tilde{\mathcal{B}}$ is a full ball, bounded by $\mathcal{D} \cup (-\mathcal{D}')$. The difference of the action (4) then reads

$$\int_{\tilde{\mathcal{B}}} H - \int_{\partial \tilde{\mathcal{B}}} 2\pi\alpha' F + B = -2\pi\alpha' \int_{\partial \tilde{\mathcal{B}}} F,$$

which, taking into account the correct normalization of the metric, leads to the quantization of F , rather than to a quantization of \mathcal{F} . A similar argument has been given in [1], although in a different order: while we first fix the correct topology and then impose the equations of motion for the Born-Infeld theory, the authors of [1] first imposed (a stronger requirement implying) conformal invariance on the boundary, i.e. the equations of motion of the string-worldsheet theory, to obtain conjugacy classes and then used the topological constraint to find integral conjugacy classes.

To find extrema of the action (3), we use a “mini-superspace” approach: for \mathcal{B} , we consider only submanifolds of the form $\psi = \text{const}$ and we fix the connection to

$$F = dA = -\frac{n}{2} \sin \theta d\theta \wedge \phi,$$

where $n \in \mathbb{Z}$ gives the flux of F . As a function of the single variable ψ , the energy then reads

$$E_n(\psi) = 4\pi k\alpha' (\sin^4 \psi + (\psi - \frac{\sin 2\psi}{2} - \frac{\pi n}{k})^2)^{1/2},$$

which has a minimum for $\psi_n = \pi n/k$. At this minimum, the mass is $M_n = 4\pi k\alpha' T_{(2)} \sin \frac{\pi n}{k}$ which coincides *exactly* with the CFT result. This is truly remarkable, since the theory in question is *not* supersymmetric. We see that the quantization of the possible flux n over S^2 implies a quantization of the position ψ of the D-brane. The only term in E that is not a trigonometric function of ψ comes from the B -field (1); this is the way the non-triviality of B enters in the determination of the position of the brane and conspires with the flux on the D-brane worldvolume to stabilize the D-brane at a finite size.

The charge with respect to the gauge invariant field strength \mathcal{F} turns out to be

$$Q_n = T_{(2)} \int_{S^2} B + 2\pi F = 2\pi k\alpha' T_{(2)} \sin \frac{2\pi n}{k},$$

again in *exact* agreement with the CFT result. Note that these charges are *not* rationally related. Notice, though, that in the limit of large level k (and thus of weak curvature) Q_n approaches the charge of n free D-particles. It is therefore tempting to interpret a D2-brane at $\psi_n = \pi n/k$ as a bound state of n D0-branes. This idea has recently received some attention.

To turn it into a quantitative argument, a careful understanding of the non-abelian Born-Infeld action [7] is needed, though.

One can also compute the spectrum of the quadratic fluctuations. For $j = 0, 1, \dots$, one finds states of

$$m^2 = \frac{j(j+1)}{k\alpha'}$$

in representations of spin $(j-1) \oplus (j+1)$. In particular, one finds a triplet of zero-modes corresponding exactly to the three rotations in $SU(2)$. All other values of m^2 are positive, which confirms the stability of our solution.

A comparison with the CFT results shows that we get the right number of branes at the right locations and with the correct energy. The charge Q_n , moreover, reproduces the correct Ramond-Ramond charge in a supersymmetrized WZW model. There is only a discrepancy for the spectrum of quadratic fluctuations: they should correspond to the open string states which, according to the exact CFT result, come in the affine representation $\oplus_{j=0}^{[k/2]} \mathcal{H}_j$. In both approaches only states with integral spin appear. The Born-Infeld action only sees certain affine descendants. Moreover, the cut-off by the level k is not found in the Born-Infeld approach. One possible explanation is that high spin states, like high momentum states in a free theory, test the ultraviolet structure of space-time. In this space-time analysis, we work with a classical, smooth space-time. It would be, however, interesting to see whether higher order fluctuations could lead to a level-dependent truncation.

4 Conclusions

It is quite remarkable (and still not really understood) why the Born-Infeld action gives essentially the *exact* CFT results, in spite of the fact that the theory is not supersymmetric. In the case of $G = SU(2)$, one might be tempted to find an explanation by embedding the $SU(2)$ -theory into the background describing k NS5-branes.

However, exactness of the results should generalize to arbitrary simple compact Lie groups. A first check is provided by the following result: the number of coordinates needed to fix the position of a brane with $\omega = 1$ has to be equal to the number of independent $U(1)$ -fluxes on the brane worldvolume G/T , where T is a maximal torus of G . This holds indeed true: the fluxes take their values in the lattice $H^2(G/T, \mathbb{Z})$, whose rank equals the rank of G , which is the number of coordinates needed to fix the brane world volume. One can go even further: according to the algebraic theory, possible D-branes correspond to integrable highest weights of G . Indeed, Borel-Bott-Weil theory allows to associate to each integral weight of G a unique line bundle over G/T which should play the role of the gauge bundle on the brane.

Other open questions concern more general conformally invariant boundary conditions. From algebraic investigations, it is known that a WZW theory possesses many more conformally invariant boundary conditions, where, however, left movers and right movers are not any longer connected by an automorphism ω . On the geometric side, one should also consider branes with more general topology, e.g. for $G = SU(2)$ D2-branes of higher genus. Both approaches and their relation remain to be explored.

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