Dynamics of Multiparticle Systems with non – Abelian Symmetry

Ludwik Turko[*1,2] **and Jan Rafelski**[†1]

[1]Department of Physics, University of Arizona
Tucson, AZ 85721, U.S.A.

and

[2]Institute of Theoretical Physics, University of Wroclaw
pl. Maksa Borna 9, 50-204 Wroclaw, Poland

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Abstract

We consider the dynamics governing the evolution of a many body system constrained by an non-abelian local symmetry. We obtain explicit forms of the global macroscopic condition assuring that at the microscopic level the evolution respects the overall symmetry constraint. We demonstrate the constraint mechanisms for the case of $SU(2)$ system comprising particles in fundamental, and adjoint representations ('nucleons' and 'pions').

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1 Introduction and Overview

The consideration of the influence of internal symmetries on the final state of a many body system begun with the pioneering work of Bethe [1]. Much of the subsequent interest in the subject arises from the realization that in the study of hadronic interactions and in particular in studies involving quark confinement, these constraints may be of decisive importance. An important progress in treating equilibrium systems was made employing group projection techniques. This allowed for a consistent treatment of abelian [2] and nonabelian [3, 4, 5, 6, 7] symmetries of compact groups and a consistent formulation of thermodynamics of many particle systems with internal symmetries taken into account [8, 9]. Application of these methods to specific processes demonstrated in which circumstances the presence of symmetry is of physical relevance [10, 11, 12].

However, it is not fully understood how the symmetry-modified properties of the equilibrium system arise from kinetic formulation of the dynamical evolution. When an internal symmetry is not at work, Boltzmann's H theorem in principle assures that the statistical Bose/Fermi/Boltzmann distributions are the asymptotic (equilibrium) distributions, irrespective of the nature of microscopic interaction. However, in presence of exact symmetries the equilibrium distributions are modified, see e.g. [9, 11]. This implies that symmetry constraints introduce effective interactions of potentially far more complex nature than is the usual two body Boltzmann collision term. In fact it can be argued

[*]E-mail: turko@ift.uni.wroc.pl
[†]E-mail: rafelski@physics.arizona.edu

that quantum symmetry constraints are the heart of the nonlocality of quantum physics. However, in the limit of classical Boltzmann equation evolution these are implemented by a strictly local (though non-linear) consideration of Fermi blocking and Bose enhancement in phase space evolution. Our aim in this work is to make a step towards understanding how the microscopic nonabelian symmetry constrains operate within the kinetic master equation description of the time evolution, leading on to the symmetry modified (constrained) macroscopic many particle equilibrium state.

It is first important to convince oneself that an underlying symmetry of microscopic interactions does not lead in general to the desired symmetry properties of a (macroscopic) many body interacting system. To do this, we consider the high energy nuclear (heavy ion) collisions and specifically here two symmetry examples:

a) *SU(2) Isospin symmetry*: The initial state transforms under a given representation of the isospin $SU(2)$ group. All elementary high energy interactions are governed by the strong interaction, which preserves the isotopic symmetry. A final state results as a multiparticle state formed by many individual hadron – hadron collisions. In any of such microprocesses the isospin is conserved. However, proceeding 'as usual' without symmetry constrained treatment of local interactions does not assure that the final multiparticle state (macrostate) transforms under the same representation of the isospin group as the initial state, which is required for symmetry reasons.

b) *SU(3) Colour symmetry*: A similar situation appears in the context of the quark-gluon interactions, especially in case that local deconfinement occurs. The initial state is a colour singlet state, and quark-gluon and gluon-gluon interaction, although invariant under the colour $SU(3)_c$ symmetry group do not assure that during its evolution a (macroscopic) many particle state, once a singlet, always remains a singlet colour state, which, however, it must do because of exact colour symmetry of strong interactions.

As these examples show, in a dynamical (quantum) transport theory description of the approach to equilibrium there must exist a subsidiary condition which should be taken into account by corresponding kinetic equations governing the evolution. This condition is independent from other constraints related to dynamical gauged internal symmetries. For classical fluid dynamics a dynamical evolution equation addressing gauge symmetry has been proposed by Wong [13]. In many current studies of the dynamics of classical non-abelian fields this proposal continues till today to attract considerable interest [14]. However, these are constraints which have no relation to the intrinsic non-locality of the quantum system which we address here.

We first note that in the case of an abelian symmetry there are no additional constraints to consider. Quantum number conservation on a microscopic level is fully equivalent to preservation of all symmetry properties on the macroscopic level. This is easily seen considering the $U(1)$ symmetry related to microscopic particle-antiparticle formation: since the microscopic mechanisms produce equal number of particles and antiparticles (pair production), initial particle-antiparticle number difference is exactly preserved in the macroscopic many body state.

Thus only presence of a nonabelian symmetry poses a true challenge. A suitable mathematical method how to approach this problem is identified considering the previously treated statistical equilibrium case. Here one decomposes a general 'macrostate' consisting of many particles into possible irreducible representations of the symmetry group. Then a projection technique exploiting character function properties of the group is used to constrain the final state. We will here use this approach in order to describe multiparticle evolution applying microscopic kinetic theory scheme. We will show that the behaviour of particle phase - space distribution functions will depend not only on properties of basic interaction, but that it also depends on global properties of the macroscopic system. Those global properties provide subsidiary constraints needed, so that the asymptotic equilibrium state has

properties consistent with the non-abelian constraints.

2 The Projection Method

Let G be a compact internal symmetry group of our system consisting of particles (objects) transforming under irreducible representations of the symmetry group. These representation are denoted as α_i with corresponding dimensions $d(\alpha_i)$. One denotes $f^{(\alpha_i,\nu_i)}_{(\zeta)}(\Gamma,\vec{r},t)$ a distribution function of the particle which belongs to the multiplet α_i of the symmetry group. Members of this multiplet are numbered by indexes ν_i ($\nu_i = 1,\ldots,d(\alpha_i)$) which correspond to given values of charges related to the symmetry group. A subscript ζ denotes other quantum numbers characterizing different multiplets of the same representation α. The variables $(\Gamma,\vec{r})$ denotes a set of the phase - space variables such as $(\vec{p},\vec{r})$ and t is time.

The number of particles of the specie $\{\alpha,\nu_\alpha,\zeta\}$ is:

$$N^{(\alpha)}_{\nu_\alpha;(\zeta)}(t) = \int dV d\Gamma f^{(\alpha,\nu_\alpha)}_{(\zeta)}(\Gamma,\vec{r},t)\,; \tag{1}$$

We consider a system of $\{N^{(\zeta_1)}_{\alpha_1,\nu_{\alpha_1}}(t),\ldots,N^{(\zeta_n)}_{\alpha_n,\nu_{\alpha_n}}(t)\}$ particles at time t. The distribution functions fulfill the generalized Vlasov - Boltzmann kinetic equations, which can be written in the general form:

$$\begin{aligned}\frac{\partial f^{(\alpha_i,\nu_i)}_{(\zeta_i)}(\Gamma_i,\vec{r},t)}{\partial t} &+ \vec{v}\cdot\nabla f^{(\alpha_i,\nu_i)}_{(\zeta_i)}(\Gamma_i,\vec{r},t)\\ &= \sum_{\alpha_j,\alpha_k,\alpha_l}\sum_{\nu_j,\nu_k,\nu_l}\sum_{\zeta_j,\zeta_k,\zeta_l}\int d\Gamma_j d\Gamma_k d\Gamma_l\, \mathcal{W}^{(\zeta_i,\zeta_j;\zeta_k,\zeta_l)}_{\nu_i\nu_j;\nu_k\nu_l}(\Gamma_k,\Gamma_l;\Gamma_j,\Gamma_i)\\ &\Big[\mathcal{F}^{(\alpha_i,\nu_i)}_{(\zeta_i)}(\Gamma_i,\vec{r},t)\mathcal{F}^{(\alpha_j,\nu_j)}_{(\zeta_j)}(\Gamma_j,\vec{r},t)f^{(\alpha_k,\nu_k)}_{(\zeta_k)}(\Gamma_k,\vec{r},t)f^{(\alpha_l,\nu_l)}_{(\zeta_l)}(\Gamma_l,\vec{r},t)\\ &-\mathcal{F}^{(\alpha_k,\nu_k)}_{(\zeta_k)}(\Gamma_k,\vec{r},t)\mathcal{F}^{(\alpha_l,\nu_l)}_{(\zeta_l)}(\Gamma_l,\vec{r},t)f^{(\alpha_i,\nu_i)}_{(\zeta_i)}(\Gamma_i,\vec{r},t)f^{(\alpha_j,\nu_j)}_{(\zeta_j)}(\Gamma_j,\vec{r},t)\Big]\,;\end{aligned} \tag{2}$$

Factors $\mathcal{F}^{(\alpha,\nu)}_{(\zeta)}(\Gamma,\vec{r},t)$ are related to quantum statistics and they are equal to 1 for classical particles, and equal to $[1 \pm f^{(\alpha,\nu)}_{(\zeta)}(\Gamma,\vec{r},t)]$ for bosons/fermions correspondingly. Since by assumption the whole system transforms under given representation Λ of an exact symmetry group, the system under consideration must preserve its transformations properties during its time evolution, provided that it is governed by a symmetry invariant interaction.

We now focus on the case of a quantum system and consider state vectors in particle number representation: $\left|N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right\rangle$. These vectors describe symmetry properties of our systems and all other variables, related to phase-space properties of the system are suppressed here. They transform as a direct product representation of the symmetry group G. This representation is of the form:

$$\alpha_1^{N^{(\alpha_1)}} \otimes \alpha_2^{N^{(\alpha_2)}} \otimes \cdots \otimes \alpha_n^{N^{(\alpha_n)}}\,; \tag{3}$$

A multiplicity $N^{(\alpha_j)}$ of the representation α_j in this product is equal to a number of particles which transform under this representation:

$$N^{(\alpha_j)} = \sum_j \left(\sum_{\zeta_j} N^{(\alpha_j)}_{\nu_{\alpha_j};(\zeta_j)}\right) = \sum_j N^{(\alpha_j)}_{\nu_{\alpha_j}}\,; \tag{4}$$

The representation given by Eq. (3) can be decomposed into direct sum of irreducible representations Λ_k. Corresponding states are denoted as $|\Lambda_k,\lambda_{\Lambda_k};\mathcal{N}\rangle$ where λ_{Λ_k} is an index numbering members of

the representation Λ and $\mathcal{N}$ is a total number of particles

$$\mathcal{N} = \sum_k N^{(\alpha_k)}_{\nu_{\alpha_k}}\,; \tag{5}$$

Each physical state can be decomposed into irreducible representation base states with amplitudes depending on phase space variables Γ:

$$\left|N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}};\Gamma\right\rangle = \sum_k{}^{\oplus}\sum_{\xi_{\Lambda_k}}{}^{\oplus}\,|\Lambda_k,\lambda_{\Lambda_k};\mathcal{N};\xi_{\Lambda_k}\rangle\, a^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}(\xi_{\Lambda_k};\Gamma)\,; \tag{6}$$

Here appear new variables ξ_Λ which are degeneracy parameters required for the full description of a state in the "symmetry space".

Let us define an average weight

$$\overline{P^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}} \;=\; \frac{\sum\limits_{\xi_\Lambda}|a^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}(\xi_\Lambda;\Gamma)|^2}{\sum\limits_{N^{(\alpha_1)}_{\nu_{\alpha_1}}+\cdots+N^{(\alpha_n)}_{\nu_{\alpha_n}}=\mathcal{N}}\;\sum\limits_{\xi_\Lambda}|a^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}(\xi_\Lambda;\Gamma)|^2}\,; \tag{7}$$

This expression gives the probability that $N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}$ particles transforming under the symmetry group representations $\alpha_1,\ldots,\alpha_n$ combine into $\mathcal{N}$ particle state transforming under representation Λ of the symmetry group.

We make the statistical hypothesis that average weights (7) do not depend on phase - space variables and can be calculated alone on basis of symmetry group consideration. This also can be proved under the stronger assumption that in Eq. (6) any state with fixed Λ,λ_Λ has the same weight (see e.g. [3])

Let us consider a projection operator $\mathcal{P}^\Lambda$ on the subspace spanned by all states transforming under representation Λ.

$$\mathcal{P}^\Lambda\left|N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right\rangle = \sum_{\xi_\Lambda}{}^{\oplus}\,|\Lambda,\lambda_\Lambda;\xi_\Lambda\rangle\,\mathcal{C}^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}(\xi_\Lambda)\,; \tag{8}$$

This operator has the generic form (see e.g. [15]):

$$\mathcal{P}^\Lambda = d(\Lambda)\int\limits_G d\mu(g)\bar{\chi}^{(\Lambda)}(g)U(g)\,; \tag{9}$$

Here $\chi^{(\Lambda)}$ is the character of the representation Λ, $d\mu(g)$ is the invariant Haar measure on the group, and $U(g)$ is an operator transforming a state under consideration. We will use the matrix representation:

$$\begin{aligned} & U(g)\left|N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right\rangle \\ = & \sum_{\nu^{(1)}_1,\ldots,\nu^{(N_{\nu_n})}_n} D^{(\alpha_1)}_{\nu^{(1)}_1\nu_1}\cdots D^{(\alpha_1)}_{\nu^{(N_{\nu_1})}_1\nu_1}\cdots D^{(\alpha_n)}_{\nu^{(1)}_n\nu_n}\cdots D^{(\alpha_n)}_{\nu^{(N_{\nu_n})}_n\nu_n}\left|N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right\rangle\,; \end{aligned} \tag{10}$$

$D^{(\alpha_n)}_{\nu,\nu'}$ is a matrix elements of the group element g corresponding to the representation α. Notation convention in Eq. (10) arises since there are $N^{(\alpha_j)}_{\nu_{\alpha_j}}$ states transforming under representation α_j and having quantum numbers of the ν_{α_j}-th member of a given multiplet.

The statistical hypothesis identifies the average weight $\overline{P^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}}$ with a norm of the vector $\mathcal{P}^\Lambda\left|N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right\rangle$. This norm can be written as

$$\left\langle N^{(\alpha_1)}_{\nu_{\alpha_1}},\cdots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right|\mathcal{P}^{\Lambda}\left|N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right\rangle=\sum_{\xi_\Lambda}|\mathcal{C}^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}(\xi_\Lambda)|^2\,; \tag{11}$$

where the relation $(\mathcal{P}^{\Lambda})^2=\mathcal{P}^{\Lambda}$ was used.

Left hand side of this equation can be calculated directly from Eqs.(9) and (10). One gets finally

$$\overline{P^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}}=\mathcal{A}^{\{\mathcal{N}\}}d(\Lambda)\int\limits_G d\mu(g)\bar{\chi}^{(\Lambda)}(g)[D^{(\alpha_1)}_{\nu_1\nu_1}]^{N^{(\alpha_1)}_{\nu_{\alpha_1}}}\cdots[D^{(\alpha_n)}_{\nu_n\nu_n}]^{N^{(\alpha_n)}_{\nu_{\alpha_n}}}\,; \tag{12}$$

where $\mathcal{A}^{\{\mathcal{N}\}}$ is a permutation normalization factor. For particles of the kind $\{\alpha,\zeta\}$ we included in Eq. (12) the permutation factor:

$$\mathcal{A}^{\alpha}_{(\zeta)}=\frac{\mathcal{N}^{(\alpha)}_{(\zeta)}!}{\prod\limits_{\nu_\alpha}\mathcal{N}^{(\alpha)}_{\nu_\alpha;(\zeta)}!}\,; \tag{13}$$

The permutation factor $\mathcal{A}^{\{\mathcal{N}\}}$ is a product of all "partial" factors

$$\mathcal{A}^{\{\mathcal{N}\}}=\prod_j\prod_{\zeta_j}\mathcal{A}^{\alpha_j}_{(\zeta_j)}\,; \tag{14}$$

The permutation factor assures the normalization of state vectors:

$$\left\langle N^{(\alpha_1)}_{\nu_{\alpha_1}},\cdots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right|\left.N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\right\rangle=\mathcal{A}^{\{\mathcal{N}\}}\,; \tag{15}$$

This normalization reflects an invariance of the state vector with respect to permutations which shuffle indistinguishable particles.

3 Incorporation of Symmetry

The expression Eq. (12) is a starting point for further considerations. It provides together with Eq. (1) and Eq. (2) subsidiary constraints on distribution functions $f^{(\alpha_i,\nu_i)}$. These conditions assure that in a dynamical evolution the symmetry of the system is preserved. When symmetry is conserved, then all weights in Eq. (12) are constant in time. In a case of strong interaction and colour symmetry, all weights, except for the weight corresponding to the singlet state, must remain zero.

We now convert the global constraint into a time evolution condition and consider:

$$\frac{d}{dt}\overline{P^{\Lambda,\lambda_\Lambda}_{\{N^{(\alpha_1)}_{\nu_{\alpha_1}},\ldots,N^{(\alpha_n)}_{\nu_{\alpha_n}}\}}}=0\,; \tag{16}$$

Introducing here the result of Eq. (12) one obtains:

$$\begin{aligned}0 &= \frac{d\,\mathcal{A}^{\{\mathcal{N}\}}}{dt}d(\Lambda)\int\limits_G d\mu(g)\bar{\chi}^{(\Lambda)}(g)[D^{(\alpha_1)}_{\nu_1\nu_1}]^{N^{(\alpha_1)}_{\nu_{\alpha_1}}}\cdots[D^{(\alpha_n)}_{\nu_n\nu_n}]^{N^{(\alpha_n)}_{\nu_{\alpha_n}}}\\ &+ \sum_{j=1}^{n}\sum_{\nu_{\alpha_j}}\frac{d\,N^{(\alpha_j)}_{\nu_{\alpha_j}}}{dt}\mathcal{A}^{\{\mathcal{N}\}}d(\Lambda)\int\limits_G d\mu(g)\bar{\chi}^{(\Lambda)}(g)[D^{(\alpha_1)}_{\nu_1\nu_1}]^{N^{(\alpha_1)}_{\nu_{\alpha_1}}}\cdots[D^{(\alpha_n)}_{\nu_n\nu_n}]^{N^{(\alpha_n)}_{\nu_{\alpha_n}}}\log[D^{(\alpha_j)}_{\nu_j\nu_j}]\,;\end{aligned} \tag{17}$$

All integrals which appear in Eq. (12) and Eq. (17) can be expressed explicitly in an analytic form for any compact symmetry group.

To write an expression for the time derivative of the normalization factor $\mathcal{A}^{\{\mathcal{N}\}}$ we perform analytic continuation from integer to continuous values of variables $N^{(\alpha_n)}_{\nu_{\alpha_n}}$. Thus we replace all factorials by the Γ–function of corresponding arguments. We encounter here also the digamma function ψ [16]:

$$\psi(x) = \frac{d\,\log\Gamma(x)}{d\,x}\,; \tag{18}$$

This allows to write for Eq. (17):

$$\frac{d\,\mathcal{A}^{\{\mathcal{N}\}}}{dt} = \mathcal{A}^{\{\mathcal{N}\}}\sum_j\sum_{\zeta_j}\left[\frac{d\mathcal{N}^{(\alpha_j)}_{(\zeta_j)}}{dt}\psi(\mathcal{N}^{(\alpha_j)}_{(\zeta_j)}+1) - \sum_{\nu_{\alpha_j}}\frac{d\mathcal{N}^{(\alpha_j)}_{\nu_{\alpha_j};(\zeta_j)}}{dt}\psi(\mathcal{N}^{(\alpha_j)}_{\nu_\alpha;(\zeta_j)}+1)\right]\,; \tag{19}$$

To get a consistent analytical continuation in the number of particles one should define the time derivatives $d\mathcal{N}^{(\alpha)}_{\nu_\alpha;(\zeta)}/dt$. We define these rates of particle number change from the integrated Boltzmann kinetic equation, Eq. (2), explicitly

$$\begin{aligned}\frac{d\,N^{(\alpha_i)}_{\nu_{\alpha_i}}}{dt} \;=\; & \sum_{\alpha_j,\alpha_k,\alpha_l}\sum_{\nu_j,\nu_k,\nu_l}\sum_{\zeta_j,\zeta_k,\zeta_l}\int dV\,d\Gamma_j d\Gamma_k d\Gamma_l d\Gamma_i\,\mathcal{W}^{(\zeta_i,\zeta_j;\zeta_k,\zeta_l)}_{\nu_i\nu_j;\nu_k\nu_l}(\Gamma_k,\Gamma_l;\Gamma_j,\Gamma_i)\\ & \Big[\mathcal{F}^{(\alpha_i,\nu_i)}_{(\zeta_i)}(\Gamma_i,\vec{r},t)\mathcal{F}^{(\alpha_j,\nu_j)}_{(\zeta_j)}(\Gamma_j,\vec{r},t)f^{(\alpha_k,\nu_k)}_{(\zeta_k)}(\Gamma_k,\vec{r},t)f^{(\alpha_l,\nu_l)}_{(\zeta_l)}(\Gamma_l,\vec{r},t)\\ & -\mathcal{F}^{(\alpha_k,\nu_k)}_{(\zeta_k)}(\Gamma_k,\vec{r},t)\mathcal{F}^{(\alpha_l,\nu_l)}_{(\zeta_l)}(\Gamma_l,\vec{r},t)f^{(\alpha_i,\nu_i)}_{(\zeta_i)}(\Gamma_i,\vec{r},t)f^{(\alpha_j,\nu_j)}_{(\zeta_j)}(\Gamma_j,\vec{r},t)\Big]\,;\end{aligned} \tag{20}$$

Contributions from gradient terms of Eq. (2) vanish due to Gauss law. These terms are transformed in surface integrals and beyond the volume occupied by the system all distribution functions are equal to zero.

Eqs (17,19,20) in fact constitute the global subsidiary condition which should be fulfilled by the microscopic kinetic equations Eq. (2). These are the necessary conditions for preserving the internal symmetry on the macroscopic level. Rates of change $d\mathcal{N}^{(\alpha)}_{\nu_\alpha;(\zeta)}/dt$ are related to "macrocurrents", which are counterparts of "microcurrents" related directly to a symmetry on a microscopic level via the Noether theorem. Eq. (20) can be considered as a set of conditions on macrocurrents to provide consistency with the overall symmetry of the system. Therefore we believe that this equation can also be used as a starting point for multicomponent hydrodynamic equations with internal symmetry properties taken into account. One should notice that this subsidiary condition takes into account also surface effects for the finite volume systems. This is due to the space variables integration which is performed in Eqs (1) and (20).

One easily sees that for the case of abelian symmetry the two constraints Eq. (12) and Eq. (17) do not lead to new results: first we recall that all irreducible representations of abelian group are one-dimensional. Next, let basic particles have "charges" $q_1,\ldots,q_n$, and let the global charge be Q. Then the only consequence of Eq. (12) follows for nonvanishing weight $Q = N_1q_1+\cdots+N_nq_n$, which is a rather obvious result. New results appear only for nonabelian symmetries.

4 Example: Isospin

We now consider as an example the case of the $SU(2)$ symmetry with basic particles transforming under spinor ($\mathbf{\frac{1}{2}}$) (fundamental) and vector ($\mathbf{1}$) (adjoint) representations. This example can be realized by a gas mixture of nucleons and pions. To describe all group elements the three group's parameters α,β,γ are chosen in such a way that diagonal matrix elements have the well known form [15]:
i.) for the fundamental representation ($\mathbf{\frac{1}{2}}$):

$$D^{(1/2)}_{mm}(\alpha,\beta,\gamma) = e^{im(\alpha+\gamma)}\cos\frac{\beta}{2}\,;\qquad m=\pm\frac{1}{2}\,; \tag{21}$$

ii.) and for the adjoint representation (**1**):

$$D^{(1)}_{\pm1,\pm1}(\alpha,\beta,\gamma) = \frac{1}{2}\,e^{\pm im(\alpha+\gamma)}(1+\cos\beta)\,; \tag{22}$$

$$D^{(1)}_{0,0}(\alpha,\beta,\gamma) = \cos\beta\,; \tag{23}$$

The Haar measure for the $SU(2)$ group in this parametrization has the form

$$\int d\mu(g) f[g] = \frac{1}{8\pi^2}\int\limits_0^{2\pi} d\alpha \int\limits_0^{2\pi} d\gamma \int\limits_0^{\pi} d\beta \sin\beta f[g(\alpha,\beta,\gamma)]\,; \tag{24}$$

Any 'macrostate' is made of an arbitrary number: n_n, n_p, n_-, n_0, n_+; where subscripts refer to members of the fundamental representation neutrons, protons; and members of adjoint representations, π^-, π^0, π^+, correspondingly. Let us consider the special case when the macrostate is a $SU(2)$ singlet. The weight of the singlet state is according to Eq. (12):

$$\begin{aligned}&\overline{P^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}} = \\ &\mathcal{A}^{\{\mathcal{N}\}}\frac{1}{8\pi^2}\int\limits_0^{2\pi} d\alpha \int\limits_0^{2\pi} d\gamma \int\limits_0^{\pi} d\beta \sin\beta e^{-\frac{i}{2}(n_n-n_p+2n_--2n_+)(\alpha+\gamma)}\cos^{\mathcal{R}}\frac{\beta}{2}\cos^{n_0}\beta \\ &\qquad\qquad \equiv \mathcal{A}^{\{\mathcal{N}\}}\widetilde{P}^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}\,;\end{aligned} \tag{25}$$

where

$$\mathcal{R} = n_n + n_p + 2n_- + 2n_+\,; \tag{26}$$

The permutation normalization factor is here:

$$\mathcal{A}^{\{\mathcal{N}\}} = \frac{(n_- + n_0 + n_+)!(n_n+n_p)!}{n_-!n_0!n_+!n_n!n_p!}\,; \tag{27}$$

The real nonzero values of the weight is obtained only when the argument of the exponent in Eq.(25) vanishes:

$$n_n - n_p + 2n_- - 2n_+ = 0\,; \tag{28}$$

This is equivalent to the conservation of the third component of the isospin.

Novel behaviour is obtained only when one considers time evolution of the system. Presence of an exact symmetry means that the corresponding weight Eq. (12) is constant, here we consider the expression:

$$\frac{d}{dt}\overline{P^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}} = 0\,; \tag{29}$$

We note that an appropriate analytical continuation should be made. First we evaluate the integral appearing in Eq. (25):

$$\frac{1}{\mathcal{A}^{\{\mathcal{N}\}}}\overline{P^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}} = (-1)^{n_0}\sum_{i=0}^{n_0}(-2)^i\binom{n_0}{i}\frac{1}{\mathcal{R}+1+i}\,; \tag{30}$$

This discrete form is not allowing an analytic continuation which would allow for all necessary differentiations. However, we can write the integral also as the hypergeometric ${}_2F_1$ function [16]:

$$\frac{1}{\mathcal{A}^{\{\mathcal{N}\}}}\overline{P^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}} = (-1)^{n_0}\frac{1}{\mathcal{R}+1}\,{}_2F_1(-n_0,\mathcal{R}+1,\mathcal{R}+2;2)\,; \tag{31}$$

where $\mathcal{R}$ is as defined in Eq. (26). We so obtain:

$$\overline{P^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}} = \frac{\Gamma(n_- + n_0 + n_+ + 1)\Gamma(n_n + n_p + 1)}{\Gamma(n_- + 1)\Gamma(n_0 + 1)\Gamma(n_+ + 1)\Gamma(n_n + 1)\Gamma(n_p + 1)} \frac{\cos n_0\pi}{\Gamma(-n_0)} \sum_{i=0}^{\infty} \frac{\Gamma(-n_0 + i)}{\mathcal{R} + 1 + i} \frac{2^i}{i!}\,; \tag{32}$$

Eqs. (25), (29) and (32), together with the condition (28) result in:

$$\begin{aligned} 0 &= 2\frac{\partial \widetilde{P}^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}}{\partial \mathcal{R}} \left(\frac{dn_n}{dt} + 2\frac{dn_-}{dt}\right) + \frac{\partial \widetilde{P}^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}}{\partial n_0} \frac{dn_0}{dt} \\ &+ \frac{d\,\log \mathcal{A}^{(\mathcal{N})}}{dt} \widetilde{P}^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}\,; \end{aligned} \tag{33}$$

The projection integrals determine the coefficients which are, explicitly:

$$\begin{aligned} \frac{\partial \widetilde{P}^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}}{\partial n_0} &= -\pi \frac{\sin n_0\pi}{\Gamma(-n_0)} \sum_{i=0}^{\infty} \frac{\Gamma(-n_0 + i)}{\mathcal{R} + 1 + i} \frac{2^i}{i!} \\ &\quad + \frac{\cos n_0\pi}{\Gamma(-n_0)} \sum_{i=0}^{\infty} \frac{\Gamma(-n_0 + i)[\psi(-n_0) - \psi(-n_0 + i)]}{\mathcal{R} + 1 + i} \frac{2^i}{i!} \\ &= (-1)^{n_0} \sum_{i=1}^{n_0} \binom{n_0}{i} \frac{[\psi(1 + n_0) - \psi(1 + n_0 - i)]}{\mathcal{R} + 1 + i} (-2)^i\,; \end{aligned} \tag{34}$$

$$\frac{\partial \widetilde{P}^{0,0}_{\{n_n,n_p,n_-,n_0,n_+\}}}{\partial \mathcal{R}} = (-1)^{n_0+1} \sum_{i=0}^{n_0} (-2)^i \binom{n_0}{i} \frac{1}{(\mathcal{R} + 1 + i)^2}\,; \tag{35}$$

and

$$\begin{aligned} \frac{d\,\log \mathcal{A}^{(\mathcal{N})}}{dt} &= \frac{dn_n}{dt}\left[\psi(n_N + 1) - \psi(n_n + 1)\right] + \frac{dn_p}{dt}\left[\psi(n_N + 1) - \psi(n_p + 1)\right] \\ &+ \frac{dn_-}{dt}\left[\psi(n_\pi + 1) - \psi(n_- + 1)\right] + \frac{dn_0}{dt}\left[\psi(n_\pi + 1) - \psi(n_0 + 1)\right] \\ &+ \frac{dn_+}{dt}\left[\psi(n_\pi + 1) - \psi(n_+ + 1)\right]\,; \end{aligned} \tag{36}$$

where $n_N = n_n + n_p$ is the total number of nucleons and $n_\pi = n_- + n_0 + n_+$ is the total number of pions. Eq. (36) can be also written in the form:

$$\begin{aligned} \frac{d\,\log \mathcal{A}^{(\mathcal{N})}}{dt} &= \frac{dn_n}{dt} \sum_{k=1+n_n}^{n_N} \frac{1}{k} + \frac{dn_p}{dt} \sum_{k=1+n_p}^{n_N} \frac{1}{k} \\ &+ \frac{dn_-}{dt} \sum_{k=1+n_-}^{n_\pi} \frac{1}{k} + \frac{dn_0}{dt} \sum_{k=1+n_0}^{n_\pi} \frac{1}{k} + \frac{dn_+}{dt} \sum_{k=1+n_+}^{n_\pi} \frac{1}{k}\,; \end{aligned} \tag{37}$$

Eqs. (33–37) offer the final result for the $SU(2)$ case . Notably, they imply a relation for the number of neutral pions in a system. We thus see, that when the case of the non-abelian symmetry is carefully considered and not ignored, one can get relations determining also the 'neutral' members of multiplets. In the standard approach the multiplicity of neutral particles is obtained by introducing a subsidiary chemical potential which is related to the lack of chemical equilibrium of a system [17] or to the residual interaction with the environment [18].

5 Conclusions and Outlook

We have shown how constraints due to the preservation of the symmetry properties of a multiparticle macroscopic state define the path of evolution of the system. One should notice that results we presented are general and do not depend on the particular choice of the representation of the symmetry group. For different initial symmetry group representations one gets different paths, but they are all of a similar "shape", considering the hypothesis that the global behaviour of a macrosystem (in a sense of statistical physics) should not be altered if a number of particles is changed by a very small ("microscopic") amount.

We have explicitly presented the example how our constraint works in the simplest non - trivial case of $SU(2)$ symmetry group.

Although we have studied and implemented the discreet symmetry using quantum states, whenever we referred here to a dynamical equation we considered the limit of incoherent state evolution described by the Boltzmann equation. The dynamical evolution we consider thus is described in terms of diagonal density matrix. This is the appropriate approach given that our main objective is to arrive at a dynamical derivation of symmetry deformed statistical distribution.

We recall that quantum correlations (without symmetry) alone are responsible for the deformation of the Boltzmann distribution into Bose/Fermi distributions, and that the Boltzmann equation yields this result when we allow for Fermi blocking/Bose enhancement in the collision term. In that line of thought, the next step would be to show that it is possible to obtain now within a dynamical Boltzmann equation calculation the evolution of a many body system into symmetry-deformed statistical equilibrium distribution. We are also exploring the possibility that the methods here presented allow the formulation of a microscopic transport theory which would obey the long range correlations introduced by the macroscopic quantum and symmetry constraints.

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