OU-HET 370
hep-th/0011158
November 2000

Nonlinear Realization of Lorentz Symmetry

Naoto Yokoi[1]

Department of Physics,
Graduate School of Science, Osaka University,
Toyonaka, Osaka 560-0043, JAPAN

Abstract

We explore a nonlinear realization of the (2+1)-dimensional Lorentz symmetry with a constant vacuum expectation value of the second rank anti-symmetric tensor field. By means of the nonlinear realization, we obtain the low-energy effective action of the Nambu-Goldstone bosons for the spontaneous Lorentz symmetry breaking.

[1]yokoi@het.phys.sci.osaka-u.ac.jp

1 Introduction

Field theories on the space-time with non-commutative coordinates (non-commutative field theories) have been extensively studied for a few years [1, 2, 3]. A construction of the non-commutative field theories has been developed in [3]: the world volume theory of Dp-brane with a constant background NS-NS B-field is equivalent to a (p+1)-dimensional non-commutative field theory whose non-commutative (constant) parameter θ^{ij} is given by the background B-field B_{ij}.

It is a well-known fact that the theory with a constant second-rank anti-symmetric tensor cannot have explicit Lorentz invariance in (p+1)-dimensions for p ≥ 2. In string theory, NS-NS B-field is a dynamical field and its constant background field can be regarded as the vacuum expectation value of the B-field. In this view point, Lorentz symmetry is spontaneously broken by the vacuum expectation value of the second rank anti-symmetric tensor field.

One can ask naturally how Lorentz symmetry is realized in the broken theory and what is the Nambu-Goldstone boson for the spontaneous Lorentz symmetry breaking. In this paper, we study the first problem. We also discuss the low-energy effective action of the corresponding Nambu-Goldstone bosons, which is obtained from only the symmetry argument. The second problem, which is model-dependent, will be studied in the forthcoming paper [4].

This paper is organized as follows. In section 2, we summarize the nonlinear realization of general space-time symmetry. In section 3, we study the nonlinear realization of the Lorentz symmetry and construct the effective action of Nambu-Goldstone bosons and other fields invariant under the nonlinear transformation of the Lorentz symmetry. In section 4, we discuss the related topics.

2 Nonlinear realization of space-time symmetry

It is well-known that if a symmetry is broken spontaneously, the broken symmetry is realized nonlinearly in the effective theory. In this section we summarize the general theory of the nonlinear realization of space-time symmetry, such as conformal symmetry, supersymmetry and so on, following [5] (references therein). We will apply this formalism to the Lorentz symmetry in the next section.

Let G be a group of a space-time symmetry and H be its stability subgroup, i.e., unbroken subgroup[2]. Then we assume that the generators satisfy the following commutation

[2]We denote the corresponding Lie algebra as $\mathcal{G}$ and $\mathcal{H}$ respectively.

relations:

$$
\begin{aligned}
[V_i, V_j] &= f_{ijk} V_k \qquad (i = 1, \cdots, \mathrm{dim} H), && (2.1)\\
[V_i, Z_a] &= f_{iab} Z_b \qquad (a = 1, \cdots, \mathrm{dim}(G/H)), && (2.2)\\
[Z_a, Z_b] &= f_{abc} Z_c + f_{abi} V_i, && (2.3)
\end{aligned}
$$

where $V_i \in \mathcal{H},\ Z_a \in \mathcal{G} - \mathcal{H}$ and f_{ABC} is the structure constant of G. If $f_{abc} = 0$, the coset G/H is called symmetric space.

From the above algebra we can construct a nonlinear realization of G following the standard procedure [6]. There is one peculiarity, however, caused by the fact that G is a space-time symmetry. Since G is a space-time symmetry, G necessarily includes translation generators. According to the formalism in [5], in the case of nonlinear realization of space-time symmetry, one should take the translation generators as the "broken" generators $Z_a \in \mathcal{G} - \mathcal{H}$, even when these symmetries are not broken. One reason is that the generators of G must satisfy reductivity condition Eq. (2.2) and the other is that the coordinate dependence of the Nambu-Goldstone field (NG-field) need to be properly transformed under G.

Considering this fact, we define the representative of the right coset G/H:

$$
L(x, \xi_a(x)) = e^{ix_\mu P^\mu} e^{i\xi_a(x) Z^a}, \tag{2.4}
$$

where P^μ ($\mu = 0, 1, \cdots, D$–1) are the unbroken translation generators and $\xi_a(x)$ is the NG-field of the broken generator Z^a. It is worth noting that the unbroken translation generators P^μ occupies a special place in $L(x, \xi_a(x))$. Then we define the action of G on the NG-fields as the left action:

$$
L(x^{'}, \xi^{'}_a(x^{'})) = gL(x, \xi_a(x)) = e^{ix^{'}_\mu P^\mu} e^{i\xi^{'}_a(x^{'}) Z^a} e^{ih_i(\xi_a(x), g) V^i}, \tag{2.5}
$$

where $g \in G$.[3] One can see easily this action reproduces the transformation of $\xi_a(x)$ under the translation, if we choose $g = e^{ia_\mu P^\mu}$:

$$
x^{'}_\mu = x_\mu + a_\mu, \quad \text{and} \quad \xi^{'}_a(x^{'}) = \xi_a(x). \tag{2.6}
$$

In general, under this G-action the NG-field $\xi_a(x)$ is transformed nonlinearly, $\xi_a(x) \rightarrow \xi^{'}_a(x^{'})$. One can define the G-action on other fields $\psi(x)$ which belong to a linear representation of H as follows:

$$
\psi(x) \rightarrow \psi^{'}(x^{'}) = D\left(e^{ih_i(\xi_a(x), g) V^i}\right) \psi(x), \tag{2.7}
$$

[3]We will discuss only the global symmetry where the transformation parameter is constant.

where $g \in G$ and D means the matrix representation of H which $\psi(x)$ belongs to. This is a nonlinear realization of the space-time symmetry of G.

To construct the effective action invariant under the G-action, we must obtain the quantities that transform covariantly using the NG-fields $\xi_a(x)$. Following the standard recipe [6, 5], we define the Maurer-Cartan 1-form of G:[4]

$$L^{-1}dL(x,\xi_a(x)) = i\mathcal{D}x_\mu P^\mu + i\mathcal{D}\xi_a(x)Z^a + i\mathcal{D}h_i(x)V^i, \tag{2.8}$$

where

$$\begin{aligned} \mathcal{D}x^\mu &= W(\xi_a(x))^\mu{}_\nu dx^\nu, & (2.9)\\ \mathcal{D}\xi_a(x) &= \mathcal{D}_\mu \xi_a(x) dx^\mu, \qquad \mathcal{D}h_i(x) = \mathcal{D}_\mu h_i(x) dx^\mu. & (2.10) \end{aligned}$$

Here, since the commutators between P^μ and Z^a are nonzero in general, nontrivial "vielbein" depending on the NG-fields $\xi_a(x)$ appears in (2.9). Each term of the right hand side of Eq. (2.8) transforms under the G-action as follows:

$$\begin{aligned} \mathcal{D}x'_\mu P^\mu &= e^{ih_i(\xi_a(x),g)V^i}\left(\mathcal{D}x_\mu P^\mu\right)e^{-ih_i(\xi_a(x),g)V^i} & (2.11)\\ \mathcal{D}\xi'_a(x')Z^a &= e^{ih_i(\xi_a(x),g)V^i}\left(\mathcal{D}\xi_a(x)Z^a\right)e^{-ih_i(\xi_a(x),g)V^i} & (2.12)\\ \mathcal{D}h'_i(x')V^i &= e^{ih_i(\xi_a(x),g)V^i}\left(\mathcal{D}h_i(x)V^i\right)e^{-ih_i(\xi_a(x),g)V^i} & (2.13)\\ &\quad -ie^{ih_i(\xi_a(x),g)V^i}de^{-ih_i(\xi_a(x),g)V^i} & (2.14) \end{aligned}$$

One can define the "covariant derivative" of the NG-fields using these quantities:

$$\nabla_\mu \xi_a(x) = \frac{\mathcal{D}\xi_a(x)}{\mathcal{D}x^\mu} = \left(W^{-1}(\xi_a(X))\right)^\nu{}_\mu \mathcal{D}_\nu \xi_a(x). \tag{2.15}$$

One can also define the "covariant derivative" of other fields similarly:

$$\nabla_\mu \psi(x) = \frac{\mathcal{D}\psi(x)}{\mathcal{D}x^\mu} = \left(W^{-1}(\xi_a(x)\right)^\nu{}_\mu \left(\partial_\nu \psi(x) + i\mathcal{D}_\nu h_i(x) D\left(V^i\right)\psi(x)\right), \tag{2.16}$$

where $D\left(V^i\right)$ is a matrix representation of H which $\psi(x)$ belongs to.

Finally we construct the G-invariant effective action by means of these covariant quantities. In order to construct the G-inavriant effective action, one needs the G-invariant integration measure. If H contains the D-dimensional homogeneous Lorentz group, one simply obtains the G-invariant measure by the following replacement:

$$\begin{aligned} dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{D-1} \longrightarrow \quad & \mathcal{D}x^0 \wedge \mathcal{D}x^1 \wedge \cdots \wedge \mathcal{D}x^{D-1} & (2.17)\\ & = (\det W)\, dx^0 \wedge dx^1 \wedge \cdots \wedge dx^{D-1}. & (2.18) \end{aligned}$$

[4] $d = dx^\mu \partial/\partial x^\mu$ acts also on $\xi_a(x)$ implicitly.

Then the resulting G-invariant effective action is given by

$$S_{\text{eff}}(\xi_a(x),\ \psi(x)) = \int d^Dx\, (\det W)\, f(\nabla_\mu \xi_a(x),\ \nabla_\mu\psi(x),\ \psi(x)), \tag{2.19}$$

where f is a H-invariant function.

3 Nonlinear realization of the Lorentz symmetry

In this section, as a simplest example, we consider the nonlinear realization of the Lorentz symmetry in (2+1)-dimensions. In (2+1)-dimensions, the Poincaré algebra ($\sim iso(2,1)$) is given by the Lorentz generators $M_{\mu\nu}$ and the translation generators P_μ:[5]

$$\begin{aligned} [P^\mu, P^\nu] &= 0, & [J^\mu, P^\nu] = -i\epsilon^{\mu\nu\rho} P_\rho, \qquad\qquad (3.1)\\ [J^\mu, J^\nu] &= -i\epsilon^{\mu\nu\rho} J_\rho, & \text{where}\ \ J^\mu \equiv \frac{1}{2}\epsilon^{\mu\nu\rho} M_{\nu\rho}. \qquad (\mu = 0,1,2) \qquad (3.2) \end{aligned}$$

The homogeneous Lorentz subgroup generated by J^μ forms $SO(2,1) \sim SL(2,\mathbf{R})$ and we take a basis as

$$J^0 = \frac{1}{2}\sigma_3, \quad J^1 = \frac{i}{2}\sigma_1, \quad \text{and} \quad J^2 = \frac{i}{2}\sigma_2, \tag{3.3}$$

where σ_i's are Pauli matrices.

We consider the situation that a second rank anti-symmetric tensor field $B_{\mu\nu}$ has a nonzero vacuum expectation value (vev). We assume only the component B_{12} has a constant nonzero vev[6], i.e.,

$$\langle B_{12} \rangle = B \neq 0 \ \ (= \text{const.}), \quad \text{and} \quad \langle B_{0i} \rangle = 0. \tag{3.4}$$

One can easily find the boost generators M_{01} and M_{02} or J^1 and J^2 are broken by the vev of $B_{\mu\nu}$, (3.4). However, since the vev is a constant, the translation generators P^μ are not broken. Thus, in this case, G is the Poincaré group ($ISO(2,1)$) and H is the subgroup generated by the translations and the spatial rotation. Therefore one may expect naively the nonlinear realization of the Lorentz symmetry can be simply constructed from the coset $G/H \sim SO(2,1)/SO(2) \sim SL(2,\mathbf{R})/U(1)$.

[5] $\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag.}(+1,-1,-1)$ and $\epsilon^{012} = +1$.

[6] If the vev of $\tilde{B}^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho}B_{\nu\rho}$ is a time-like constant vector, one can transform it to this form by the Lorentz transformation.

However, according to the discussion in the previous section, we should decompose "broken" and "unbroken" generators in a different manner:

$$\begin{aligned} &G\ :\ ISO(2,1), \qquad H\ :\ U(1) \\ \text{"Unbroken generator"}\ &:\ J^0 \qquad\qquad\qquad \in \mathcal{H}, \\ \text{"Broken generator"}\ &:\ J^1,\ J^2,\ P^\mu \qquad\qquad \in \mathcal{G}-\mathcal{H}. \end{aligned} \tag{3.5}$$

Using this decomposition, the representative of G/H is defined as follows:

$$L(x,\phi_a(x)) \;=\; e^{ix_\mu P^\mu} e^{i\phi_a(x)J^a} = e^{ix^0 P_0 + i\bar{z}P_+ + izP_-} e^{i\bar{\phi}(x)J^+ + i\phi(x)J^-}, \tag{3.6}$$

$$\text{where} \qquad z = x^1 + ix^2, \quad P_\pm = \frac{1}{2}\left(P_1 \pm iP_2\right), \tag{3.7}$$

$$\phi(x) = \phi_1(x) + i\phi_2(x), \quad J^\pm = \frac{1}{2}\left(J^1 \pm iJ^2\right). \tag{3.8}$$

$\phi_1(x)$ and $\phi_2(x)$ are the NG-bosons for the broken generators J^1 and J^2 respectively. The candidates of $\phi_a(x)$ in specific models will be discussed in the next section. The G-action is defined as the left-action:

$$L^{'}(x^{'},\phi_a^{'}(x^{'})) = gL(x,\phi_a(x)). \qquad (g \in G) \tag{3.9}$$

In particular, one can find that this G-action properly reproduces the linear transformation under the unbroken symmetries. If $g = e^{ia_\mu P^\mu}$, the transformation becomes

$$x_\mu^{'} = x_\mu + a_\mu, \quad \phi^{'}(x^{'}) = \phi(x), \quad \bar{\phi}^{'}(x^{'}) = \bar{\phi}(x). \tag{3.10}$$

And under the unbroken spatial rotation, i.e., $g = e^{ibJ^0}$ $(= e^{ibM_{12}})$, the transformation becomes

$$x^{0'} = x^0, \quad z^{'} = e^{-ib}z, \quad \phi^{'}(x^{'}) = e^{-ib}\phi(x), \quad \bar{\phi}^{'}(x^{'}) = e^{ib}\bar{\phi(x)}. \tag{3.11}$$

We can obtain the Maurer-Cartan 1-form explicitly using the representative (3.6).

$$\begin{aligned} L^{-1}dL(x,\phi_a(x)) \;&=\; e^{-i\bar{\phi}(x)J^+ - i\phi(x)J^-}\left(iP_0 dx^0 + iP_+ d\bar{z} + iP_- dz\right) e^{i\bar{\phi}(x)J^+ + i\phi(x)J^-} \\ &\quad + e^{-i\bar{\phi}(x)J^+ - i\phi(x)J^-} d\left(e^{i\bar{\phi}(x)J^+ + i\phi(x)J^-}\right) \qquad (3.12) \\ &\equiv\; i\mathcal{D}x^\alpha P_\alpha + i\mathcal{D}\bar{\phi}(x)J^+ + i\mathcal{D}\phi(x)J^- + i\mathcal{D}h(x)J^0, \qquad (3.13) \end{aligned}$$

where $x^\alpha = (x^0, z, \bar{z})$. The explicit form is given by

$$\mathcal{D}x^\alpha = W^\alpha{}_\beta(\phi,\bar{\phi})dx^\beta, \quad \mathcal{D}\phi(x) = \mathcal{D}_\alpha\phi(x)dx^\alpha, \quad \mathcal{D}h(x) = \mathcal{D}_\alpha h(x)dx^\alpha, \tag{3.14}$$

where

$$W^{\alpha}{}_{\beta}(\phi,\bar\phi)=\begin{pmatrix}\cosh(|\phi|) & -\frac{1}{2}i\bar\phi\frac{\sinh(|\phi|)}{|\phi|} & \frac{1}{2}i\phi\frac{\sinh(|\phi|)}{|\phi|}\\ i\phi\frac{\sinh(|\phi|)}{|\phi|} & \frac{1+\cosh(|\phi|)}{2} & \frac{1}{2}\phi^2\left(\frac{1-\cosh(|\phi|)}{|\phi|^2}\right)\\ -i\bar\phi\frac{\sinh(|\phi|)}{|\phi|} & \frac{1}{2}\bar\phi^2\left(\frac{1-\cosh(|\phi|)}{|\phi|^2}\right) & \frac{1+\cosh(|\phi|)}{2}\end{pmatrix} \tag{3.15}$$

and

$$\mathcal{D}_\alpha\phi(x) = \frac{1}{2}\left(\frac{\phi\partial_\alpha\bar\phi+\bar\phi\partial_\alpha\phi}{\bar\phi}-\frac{(\phi\partial_\alpha\bar\phi-\bar\phi\partial_\alpha\phi)\sinh(|\phi|)}{\bar\phi|\phi|}\right) \tag{3.16}$$

$$\mathcal{D}_\alpha h(x) = \frac{\sinh^2\left(\frac{|\phi|}{2}\right)}{i\;|\phi|^2}\left(\phi\partial_\alpha\bar\phi-\bar\phi\partial_\alpha\phi\right), \tag{3.17}$$

and $\mathcal{D}_\alpha\bar\phi(x)=\overline{\mathcal{D}_\alpha\phi(x)}$, denoting the complex conjugation of (3.16).

Following the recipe explained in the previous section, one can obtain the covariant derivative of the NG-fields $\phi(x)$ and $\bar\phi(x)$.

$$\nabla_\alpha\phi(x)=\left(W^{-1}\right)^{\beta}{}_{\alpha}\mathcal{D}_\beta\phi(x),\quad \nabla_\alpha\bar\phi(x)=\left(W^{-1}\right)^{\beta}{}_{\alpha}\mathcal{D}_\beta\bar\phi(x), \tag{3.18}$$

where

$$\nabla_0\phi = e^{i\theta}\cosh(\rho)(\partial_0\rho+i\sinh(\rho)\partial_0\theta)-ie^{2i\theta}\sinh(\rho)(\partial_z\rho+i\sinh(\rho)\partial_z\theta) +i\sinh(\rho)(\partial_{\bar z}\rho+i\sinh(\rho)\partial_{\bar z}\theta) \tag{3.19}$$

$$\nabla_z\phi = \frac{i}{2}\sinh(\rho)(\partial_0\rho+i\sinh(\rho)\partial_0\theta)+e^{i\theta}\cosh^2(\frac{\rho}{2})(\partial_z\rho+i\sinh(\rho)\partial_z\theta) -e^{-i\theta}\sinh^2(\frac{\rho}{2})(\partial_{\bar z}\rho+i\sinh(\rho)\partial_{\bar z}\theta) \tag{3.20}$$

$$\nabla_{\bar z}\phi = -\frac{i}{2}e^{2i\theta}\sinh(\rho)(\partial_0\rho+i\sinh(\rho)\partial_0\theta)-e^{3i\theta}\sinh^2(\frac{\rho}{2})(\partial_z\rho+i\sinh(\rho)\partial_z\theta) +e^{i\theta}\cosh^2(\frac{\rho}{2})(\partial_{\bar z}\rho+i\sinh(\rho)\partial_{\bar z}\theta), \tag{3.21}$$

and $\nabla_0\bar\phi(x)=\overline{\nabla_0\phi(x)}$, $\nabla_z\bar\phi(x)=\overline{\nabla_{\bar z}\phi(x)}$, $\nabla_{\bar z}\bar\phi(x)=\overline{\nabla_z\phi(x)}$. Here we have introduced $\rho(x)$ and $\theta(x)$ by $\phi(x)\equiv\rho(x)e^{i\theta(x)}$ for simplicity. In order to construct the covariant derivative of other fields $\psi(x)$ from Eq. (2.16), we need the explicit form of

$$\nabla_\alpha h(x)=\left(W^{-1}\right)^{\beta}{}_{\alpha}\mathcal{D}_\beta h(x), \tag{3.22}$$

namely

$$\nabla_0 h(x) = -2\sinh^2(\frac{\rho}{2})\left(\cosh(\rho)\partial_0\theta-ie^{i\theta}\sinh(\rho)\partial_z\theta+ie^{-i\theta}\sinh(\rho)\partial_{\bar z}\theta\right) \tag{3.23}$$

$$\nabla_z h(x) = -2\sinh^2(\frac{\rho}{2})\left(\frac{i}{2}e^{-i\theta}\sinh(\rho)\partial_0\theta+\cosh^2(\frac{\rho}{2})\partial_z\theta -e^{-2i\theta}\sinh^2(\frac{\rho}{2})\partial_{\bar z}\theta\right) \tag{3.24}$$

and $\nabla_{\bar{z}} h(x) = \overline{\nabla_z h(x)}$. For example, if $\psi(x)$ is a real scalar field, the covariant derivative is given by (2.16) with $D(V^i) = D(J^0) = 0$, i.e.,

$$\nabla_\alpha \psi(x) = \left(W^{-1}\right)^\beta{}_\alpha \partial_\beta \psi(x), \tag{3.25}$$

whose explicit forms are

$$\begin{aligned} \nabla_0 \psi(x) &= \left(\cosh(\rho)\partial_0 - ie^{i\theta}\sinh(\rho)\partial_z + ie^{-i\theta}\sinh(\rho)\partial_{\bar{z}}\right)\psi(x), &(3.26)\\ \nabla_z \psi(x) &= \left(\frac{i}{2}e^{-i\theta}\sinh(\rho)\partial_0 + \cosh^2(\frac{\rho}{2})\partial_z - e^{-2i\theta}\sinh^2(\frac{\rho}{2})\partial_{\bar{z}}\right)\psi(x), &(3.27)\end{aligned}$$

and $\nabla_{\bar{z}}\psi(x) = \overline{\nabla_z \psi(x)}$. Similarly, for a spinor field, the covariant derivative is given by (2.16) with $D(J^0) = \sigma_3/2$:

$$\nabla_\alpha \psi_a(x) = \left(W^{-1}\right)^\beta{}_\alpha \partial_\beta \psi_a(x) + i\nabla_\alpha h(x)\left(\frac{\sigma_3}{2}\right)_a{}^b \psi_b(x). \tag{3.28}$$

As discussed in the previous section, the G-invariant integration measure is also needed. In our case, H does not contain the (2+1)-dimensional homogeneous Lorentz group. Hence, we have three G-invariant integration measures, which are 1-dimensional measure $\mathcal{D}x^0$, 2-dimensional measure $\mathcal{D}z \wedge \mathcal{D}\bar{z}$, and (2+1)-dimensional measure $\mathcal{D}x^0 \wedge \mathcal{D}z \wedge \mathcal{D}\bar{z}$. Because we want to obtain the (2+1)-dimensional G-invariant effective action, we take the G-invariant measure $\mathcal{D}x^0 \wedge \mathcal{D}z \wedge \mathcal{D}\bar{z} = (\det W)dx^0 \wedge dz \wedge d\bar{z}$.

Finally, the G-invariant effective action of NG-fields $\phi(x)$ and $\bar{\phi}(x)$ is given by

$$S_{\text{eff}}(\phi(x), \bar{\phi}(x)) = \int d^3x (\det W)\ f\left(\nabla_\alpha \phi(x), \nabla_\alpha \bar{\phi}(x)\right), \tag{3.29}$$

where f is a H-invariant function. Thus we can write down explicitly the effective action up to two derivatives of NG-fields: [7]

$$\begin{aligned} S^{(2)}_{\text{eff}}(\phi(x), \bar{\phi}(x)) &= \int d^3x \left\{ a\nabla_0\phi\nabla_0\bar{\phi} + b\nabla_z\bar{\phi}\nabla_{\bar{z}}\phi + c\nabla_z\phi\nabla_{\bar{z}}\bar{\phi} \right.\\ &\qquad + d\left((\nabla_z\phi)^2 + (\nabla_{\bar{z}}\bar{\phi})^2\right) + ie\left((\nabla_z\phi)^2 - (\nabla_{\bar{z}}\bar{\phi})^2\right)\\ &\qquad \left. + k(\nabla_z\phi + \nabla_{\bar{z}}\bar{\phi}) + il(\nabla_z\phi - \nabla_{\bar{z}}\bar{\phi}) \right\}. &(3.30)\end{aligned}$$

Here we make a comment: the Jacobian factor $\det W = 1$ identically for this Lorentz symmetry, although this factor depends nontrivially on the NG-fields and contributes to the effective action for the conformal symmetry [7], supersymmetry [8] and superconformal

[7] Although one may wonder if the second and third term in (3.30) are independent of each other, the difference of these terms is a topological term, i.e., a nontrivial surface term and cannot be ignored in general.

symmetry [9]. This is because the simple integration measure d^3x is already Lorentz invariant. This G-invariant effective action of the NG-fields up to two derivative terms has not only two derivative terms but also one derivative terms and seven undetermined parameters. This is quite different from the counterpart of internal symmetry, which includes only two derivative terms and one parameter [6].

In the same way, one can construct the G-invariant effective action of other field $\psi(x)$:

$$S_{\text{eff}}(\psi(x)) = \int d^3x \; p\left(\psi(x), \nabla_\alpha \psi(x)\right), \tag{3.31}$$

where $\nabla_\alpha \psi(x)$ is given by (2.16) and (3.22) and p is a H-invariant function. It is worth commenting that this action, even up to two derivatives of fields, necessarily includes derivative couplings between the NG-fields $\phi(x)$ and *any* other fields $\psi(x)$.

Thus we have obtained the effective action invariant under the nonlinear transformation (3.9) of the Lorentz symmetry. This gives the low-energy effective action of the NG-bosons for the spontaneous Lorentz symmetry breaking in (2+1)-dimensions.

4 Discussions

We have discussed the nonlinear realization of the Lorentz symmetry in (2+1)-dimensions and obtained the low-energy effective action invariant under the nonlinear transformation of the Lorentz symmetry.

We comment shortly on the several examples that realize the spontaneous Lorentz symmetry breaking. The first example is gauge invariant field theory of the second rank anti-symmetric tensor field $B_{\mu\nu}$ [10, 11]. In the free field theory, the vev of $B_{\mu\nu}$ can be a non-zero constant, thus the vev (3.4) is realized. In the gauge fixed quantum theory[8], the NG-bosons for this spontaneous Lorentz symmetry breaking are B_{01} and B_{02}. ϕ_1 and ϕ_2 in the previous section correspond to B_{01} and B_{02} respectively. In perturbation theory, the low-energy effective action of B_{01} and B_{02} is given by (3.30) obtained in the previous section. Unfortunately, in (2+1)-dimensions, $B_{\mu\nu}$ does not have the physical degree of freedom.

The second example is the quantum electrodynamics with the Chern-Simons coupling in (2+1)-dimensions. In this theory, spontaneous magnetization occurs and thus spontaneous Lorentz symmetry breaking is realized [12, 13]. In this case, the field strength of

[8]In order to discuss the spontaneous Lorentz symmetry breaking, the gauge fixing condition should be manifestly Lorentz invariant. Thus we take a covariant gauge, say, the Feynman gauge.

photon condenses and the vev's of $F_{\mu\nu}$ become

$$\langle F_{12} \rangle = B \neq 0 \ (= \text{const.}), \quad \text{and} \quad \langle F_{0i} \rangle = 0. \tag{4.1}$$

Then the NG-bosons are F_{01} and F_{02}[9], which correspond to ϕ_1 and ϕ_2 respectively. Thus the low-energy effective action of F_{01} and F_{02} will be given by (3.30) and the coupling of fermions and F_{01} and F_{02} is given by (3.31).

The third example is the field theory of a second rank anti-symmetric tensor field coupled with an abelian gauge field. This example will be extensively studied in the forthcoming paper [4].

The generalization to higher dimensions is straightforward. The construction discussed in this paper can be applied to the spontaneous Lorentz symmetry breaking in higher dimensions. For more realistic example, the nonlinear realization of the Lorentz symmetry in (3+1)-dimensions on the vacuum

$$\langle B_{12} \rangle = B \neq 0 \ (= \text{const.}), \quad \text{and} \quad \text{others} \ = 0 \tag{4.2}$$

can be similarly constructed and the corresponding low-energy effective action can also be constructed.

The generalization to supersymmetric models is also interesting.

Acknowledgements
The author is very grateful to K. Higashijima for the useful suggestions and discussions and careful reading of this manuscript. The author also thanks Y. Hosotani for the useful discussions.

References

[1] A. Connes, M. R. Douglas and A. Schwarz, "Noncommutative Geometry and Matrix Theory: Compactification On Tori", JHEP **9802:003** (1998).

[2] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, "Noncommutative Yang-Mills in IIB Matrix Model", Nucl. Phys. **B565** (2000) 176.

[3] N. Seiberg and E. Witten, "String Theory and Noncommutative Geometry", JHEP **9909:032** (1999).

[4] K. Higashijima and N. Yokoi, in preparation.

[9] In the physical Hilbert space, these two NG-bosons are not independent each other [13].

[5] V. I. Ogievetsky, "Nonlinear Realization of Internal and Space-time Symmetries", in Proc. of 10th Winter School of Theoretical Physics in Karpacz, vol. 1., Wroclaw (1974) 117.

[6] For example, S. Weinberg, "THE QUANTUM THEORY OF FIELDS", section 19 in Vol. II (Cambridge University Press, 1996).

[7] A. Salam and J. Strathdee, "Nonlinear Realizations. II. Conformal Symmetry", Phys. Rev. **184** (1969) 1760.

[8] D. V. Volkov and V. P. Akulov, "Is the Neutrino a Goldstone Particle ?", Phys. Lett. **46B** (1973) 109.

[9] K. Kobayashi and T. Uematsu, "Non-linear Realization of Superconformal Symmetry", Nucl. Phys. **B263** (1986) 309.

[10] P. K. Townsend, "Covariant Quantization of Antisymmetric Tensor Gauge Fields" Phys. Lett. **88B** (1979) 97.

[11] T. Kimura, "Antisymmetric Tensor Gauge Field in General Covariant Gauges", Prog. Theor. Phys. **64** (1980) 357.

[12] Y. Hosotani, "Spontaneously Broken Lorentz Invariance in Three-Dimensional Gauge Theories", Phys. Lett. **319B** (1993) 332.

[13] Y. Hosotani, "Spontaneous Breakdown of Lorentz Invariance", Phys. Rev. **D51** (1995) 2022.