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Chiral Condensates in the Light-Cone Vacuum

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Abstract

In light-cone quantization, the standard procedure to characterize the phases of a system by appropriate ground state expectation values fails. The light-cone vacuum is determined kinematically. We show that meaningful quantities which can serve as order parameters are obtained as expectation values of Heisenberg operators in the equal (light-cone) time limit. These quantities differ from the purely kinematical expectation values of the corresponding Schrödinger operators. For the Nambu–Jona-Lasinio and the Gross-Neveu model, we describe the spontaneous breakdown of chiral symmetry; we derive within light-cone quantization the corresponding gap equations and the values of the chiral condensate.

Inherent to the light-cone description of quantum field theories is the triviality of the vacuum. Most of the simplifying features of light-cone quantization as well as foundation and phenomenological success of the quark-parton model are, to a large extent, related to the simplicity of the structure of the vacuum (cf. the reviews on light-cone quantization [1, 2]). The simplicity of the vacuum is independent of dynamics, it is of kinematical origin. In the light-cone formulation, Minkowski space-time is described by the metric

$$g_{\mu\nu} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

and parametrized by the coordinates

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^3) \ , \qquad x^{\perp} = (x^1, x^2) \ .$$

With the form (1) of the metric, the dispersion relation $p^2 = m^2$ leads to the following relation between the *light-cone energy* p_+ and momentum components $p_-, p_\perp$

$$p_+ = \frac{p_\perp^2 + m^2}{2p_-} \ . \tag{2}$$

In contradistinction to the standard parametrization of space-time, the light-cone energy p_+ assigned to a single particle state of a given momentum is unique. The sign of the energy is determined by the sign of the momentum component p_-. Thus in the absence of interactions, the fermionic vacuum consists of occupied states with negative p_- and of empty states with positive p_-. This vacuum structure does not change when turning on interactions between the fermions. No other states with equal momentum are available which could be reached by collisions among the fermions. Thus the structure of the vacuum is independent of interactions.

This triviality of the vacuum poses conceptual problems when applying light-cone quantization to systems which are known to possess a non-trivial vacuum structure induced, for instance, by spontaneous symmetry breakdown, Higgs mechanism or topological properties. While the equivalence of light-cone quantization with more standard quantization has been established perturbatively (cf. [2]) the triviality problem points to a lack of understanding of this quantization scheme in the non-perturbative regime. It remains to be understood how, in light-cone quantization, different phases of a system can be built on a vacuum which is determined kinematically. In particular, vacuum expectation values (VEV) such as the chiral condensate $\langle 0|\bar{\psi}\psi|0\rangle$ are trivial in light-cone quantization and thus cannot serve as order parameters characterizing the realization of symmetries. On the other hand, it is known from the study of low dimensional systems such as the

't Hooft model [3] that light-cone quantization can reproduce correctly spectra which contain Goldstone bosons; furthermore, by using properties of the spectrum, the correct value of the quark condensate could be determined [4] although explicit calculation yields a vanishing VEV.
To clarify the physical relevance of the light-cone vacuum we consider model theories in which spontaneous symmetry breakdown of a continuous symmetry occurs with the ensuing emergence of Goldstone particles and formation of condensates. In the Nambu–Jona-Lasinio model (NJL) [5] and its two dimensional version, the (chiral) Gross-Neveu model (GN) [6], the breakdown of the chiral symmetry is induced by mass generation of the fermions. The Lagrangian of these models has the following structure

$$\mathcal{L} = \bar{\psi}(\mathrm{i}\partial_\mu\gamma^\mu - m)\psi + \mathcal{L}_{\mathrm{int}}(\psi, \bar{\psi}) \; .$$

$\mathcal{L}_{\mathrm{int}}$ is a 4-fermion self interaction. This expression contains implicitly a sum over fermion species ("color") while flavor dependences important in phenomenological applications are of no importance for our discussion. In the following we shall display the formalism for the 3+1 dimensional NJL model and we shall discuss later the necessary modifications for the lower-dimensional GN model. We use a representation of the γ matrices in which γ_5 and the projection operators $\Lambda^\pm$ are given by

$$\gamma_5 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} \; , \quad \Lambda^\pm = \frac{1}{2}(1 \pm \gamma^0\gamma^3) \; , \quad \gamma^0\gamma^3 = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} \; . \tag{3}$$

The projection operators $\Lambda^\pm$ decompose the 4-spinor into 2-spinors

$$\psi = 2^{-1/4} \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$$

and the Lagrangian becomes

$$\mathcal{L} = \mathrm{i}\varphi^\dagger\partial_+\varphi + \mathrm{i}\chi^\dagger\partial_-\chi + \frac{\mathrm{i}}{\sqrt{2}}\left(\varphi^\dagger\tilde{\partial}_m\chi + \chi^\dagger\partial_m\varphi\right) + \mathcal{L}_{\mathrm{int}}(\varphi, \chi) \tag{4}$$

with

$$\mathrm{i}\partial_m = \mathrm{i}\sigma_3\partial_1 - \partial_2 + \sigma_1 m \; , \quad \mathrm{i}\tilde{\partial}_m = \mathrm{i}\sigma_3\partial_1 + \partial_2 + \sigma_1 m \; .$$

Only the spinor φ is dynamical, no time derivative of χ is present. In canonical quantization, χ is treated as a constrained field. This reduction in the number of dynamical degrees of freedom makes the single particle states with given momentum unique and thereby the light-cone vacuum trivial. In the representation (3), chiral rotations are defined by

$$\varphi(x) \to \mathrm{e}^{\mathrm{i}\alpha\sigma_3}\varphi(x) \; , \quad \chi(x) \to \mathrm{e}^{\mathrm{i}\alpha\sigma_3}\chi(x) \; . \tag{5}$$

With the following choice of the 4-fermion interaction,

$$\mathcal{L}_{\mathrm{int}} = \frac{g^2}{2}\left((\bar{\psi}\psi)^2 + (\mathrm{i}\bar{\psi}\gamma_5\psi)^2\right) = \frac{g^2}{4}\left(\left(\varphi^\dagger\sigma_1\chi + \chi^\dagger\sigma_1\varphi\right)^2 + \left(\varphi^\dagger\sigma_2\chi + \chi^\dagger\sigma_2\varphi\right)^2\right) \; , \tag{6}$$

the NJL-Lagrangian is invariant under chiral rotations provided the (bare) mass m vanishes. At this point we do not follow the standard path in employing the canonical formalism; the description in terms of light-cone Schrödinger operators will turn out to be too restrictive. We rather study this model by using functional techniques based on the generating functional

$$Z[\eta,\gamma] = \int D[\varphi,\chi] \mathrm{e}^{\mathrm{i}\int \mathrm{d}^4x\,(\mathcal{L}+\varphi^\dagger\eta+\eta^\dagger\varphi+\chi^\dagger\gamma+\gamma^\dagger\chi)} \; . \tag{7}$$

Since fermionic mass generation is the mechanism which drives the system into the spontaneously broken phase the correlation function related to the chiral condensate for the case of noninteracting ($g = 0$) massive fermions reveals the difficulties in describing non-trivial vacua. We consider

$$\begin{aligned} C(x) &= \langle 0|T(\varphi^\dagger(x)\sigma_1\chi(0))|0\rangle = \mathrm{i}m\, 2^{3/2}\int \frac{\mathrm{d}^4p}{(2\pi)^4}\frac{\mathrm{e}^{\mathrm{i}px}}{p^2-m^2+\mathrm{i}\epsilon} & (8)\\ &= m\sqrt{2}\left(\frac{1}{2\pi}\right)^3\int \mathrm{d}^2p_\perp \int_0^\infty \frac{\mathrm{d}p_-}{p_-}\mathrm{e}^{-\mathrm{i}\frac{p_\perp^2+m^2-\mathrm{i}\epsilon}{2p_-}|x^+|+\mathrm{i}p_\perp x^\perp-\mathrm{i}p_-x^-\epsilon(x^+)} & (9)\\ &= \frac{1}{\sqrt{2}\pi^2}\,\frac{m^2}{\sqrt{-x^2}}\,K_1(m\sqrt{-x^2}) \; . & (10)\end{aligned}$$

As has been noted quite some time ago [7] in a discussion of bosonic theories, values of such correlation functions are actually not well defined. In particular evaluating $C(x)$ for $x^+ = 0$, using Eq. (9) yields

$$C_{\mathrm{S}}(x^-,x^\perp) = m\sqrt{2}\left(\frac{1}{2\pi}\right)^3\int \mathrm{d}^2p_\perp \int_0^\infty \frac{\mathrm{d}p_-}{p_-}\mathrm{e}^{\mathrm{i}p_\perp x^\perp-\mathrm{i}p_-x^-} \tag{11}$$

while using Eq. (10)

$$C_{\mathrm{H}}(x^-,x^\perp) = \frac{1}{\sqrt{2}\pi^2}\,\frac{m^2}{\sqrt{x_\perp^2}}\,K_1(m\sqrt{x_\perp^2}) \; . \tag{12}$$

Expression (11) agrees with the result of the canonical formalism in which Schrödinger operators are used. This expression has only a trivial dependence on m, reflecting the triviality of the vacuum. It is divergent even off the light-cone. On the other hand, the expression (12) is regular for space- or timelike separations and depends non-trivially on the fermion mass. Furthermore it is invariant under Lorentz transformations. The origin of this different behavior is a direct consequence of the light-cone dispersion relation. However small x^+ is chosen, there are always states with sufficiently small p_- available which give rise to oscillations in the integrand in (9) and thereby regularize the $1/p_-$ singularity. In standard coordinates such an effect does not exist, $x^0 = 0$ can be chosen at every level of the calculation and the result agrees with Eq. (12). From these observations we conclude: Expectation values of Schrödinger operators in the light-cone vacuum do not agree with the limit of expectation values of Heisenberg operators

$$\lim_{x^+\to 0} \langle 0|T(\varphi^\dagger(x)\sigma_1\chi(0))|0\rangle \neq \langle 0|\varphi^\dagger(x^+=0^+,x^-,x^\perp)\sigma_1\chi(0)|0\rangle \ . \tag{13}$$

Although we have computed these expectation values for non-interacting fermions, it is easy to see that these arguments are essentially not changed when interactions are present. The triviality of the vacuum implies that VEV's of Schrödinger operators do not change when including interactions; on the other hand the absence of singularities in $C(x)$ for arbitrary small but non-vanishing x^+ and $x^2 \neq 0$ is easily demonstrated by inserting a complete set of states (subtleties may only occur in 1+1 dimensional systems, if massless particles are present.) Furthermore, covariance dictates that in the absence of singularities, vacuum expectation values of Heisenberg operators at given spacelike x^2 are the same for $x^+ \to 0$ and $x^0 = 0$

$$\lim_{x^+\to 0} \langle 0|T(\varphi^\dagger(x)\sigma_1\chi(0))|0\rangle\Big|_{x^2} = \langle 0|\varphi^\dagger(x^0=0^+,\mathbf{x})\sigma_1\chi(0)|0\rangle\Big|_{\mathbf{x}^2=-x^2}$$

and coincide with the VEV of the $x^0=0$ Schrödinger operators. Thus, on the light-cone, VEV's of Heisenberg operators in the equal light-cone limit and not VEV's of Schrödinger operators are physically meaningful quantities; in particular they can serve in the limit $x^2 \to 0$ as order parameters to characterize the phases of a system and properly define for finite x^2 "observable" correlation functions.

We now demonstrate in a schematic light-cone calculation for the NJL model the procedure for computing condensate values. In the first step, the spectrum of the light-cone Hamiltonian has to be determined. In the above model this step is done easily for large N. Replacing in this limit the bilinear $(\chi^\dagger\sigma_1\varphi)$ by a c-number

$$g^2\sum_{i=1}^{N}\chi_i^\dagger(x)\sigma_1\varphi_i(x) = g^2\sum_{i=1}^{N}\varphi_i^\dagger(x)\sigma_1\chi_i(x) \approx \frac{\hat{m}}{\sqrt{2}} \tag{14}$$

yields for $m=0$, to leading order, the NJL-Lagrangian in which only quadratic fluctuations are kept

$$\mathcal{L} = \mathrm{i}\varphi^\dagger\partial_+\varphi + \mathrm{i}\chi^\dagger\partial_-\chi + \frac{\mathrm{i}}{\sqrt{2}}\left(\varphi^\dagger\tilde{\partial}_{\hat{m}}\chi + \chi^\dagger\partial_{\hat{m}}\varphi\right) \ .$$

Integrating out the constrained field χ, the Hamiltonian of a system of non-interacting massive fermions

$$H = \frac{\mathrm{i}}{2}\int \mathrm{d}^3x\, \varphi^\dagger\tilde{\partial}_{\hat{m}}\frac{1}{\partial_-}\partial_{\hat{m}}\varphi \tag{15}$$

is obtained. To determine the unknown mass parameter $\hat{m}$, we require the sum in Eq. (14) to be given by the limit of the vacuum expectation value of the sum over the corresponding Heisenberg operators. In the large N limit, determination of the spectrum and computation of vacuum expectation values of Heisenberg operators is simple. We obviously can use our above results with $m \to \hat{m}$ and obtain, using Eq. (12) and the asymptotics of the Bessel functions in the limit of small spacelike x^2 the well known gap equation of the NJL model

$$\hat{m}\left[\frac{g^2 N}{\pi^2}\left(\Lambda^2 + \frac{\hat{m}^2}{4}\ln\frac{\hat{m}^2}{\Lambda^2}\right) - 1\right] = 0 \tag{16}$$

with the cutoff Λ defined by the point splitting procedure

$$\Lambda^2 = \frac{1}{-x^2}\,.$$

This consistency condition is always solved trivially by $\hat{m} = 0$. Beyond a critical coupling (for fixed cutoff), Eq. (16) has a solution with $\hat{m} \neq 0$ describing the phase with spontaneously broken chiral symmetry. In ordinary coordinates, the solution with the lower energy describes the stable phase. In light-cone quantization with its kinematically determined vacuum, the vacuum energy cannot be determined variationally; stability can be checked either by evaluation of the fluctuations (the NJL meson spectrum [8]) or by calculation of the associated values of the effective potential (cf. [9]). Since the effective potential is a Lorentz scalar, the values obtained in ordinary coordinates are trivially reproduced for the solutions of the gap equation (16).

Identification of the chiral condensate with the limiting VEV of light-cone Heisenberg operators is crucial. Use of VEV's of Schrödinger operators (Eq. (11)) yields

$$\frac{2N\hat{m}}{(2\pi)^3}\int \mathrm{d}^2 p_\perp \int_0^\infty \frac{\mathrm{d}p_-}{p_-} = \frac{\hat{m}}{g^2} \tag{17}$$

which admits only the solution $\hat{m} = 0$.

This procedure also works in the 1+1 dimensional (chiral) GN model with its even more severe infrared problems. Since in two dimensions $x^+ = 0$ denotes points on the light-"cone", VEV's of products of Schrödinger operators are necessarily singular; again they are regularized by point-splitting. The following substitution in Eq. (4)

$$\mathrm{i}\partial_m \to -m\,, \quad x^\perp = 0\,, \quad \mathcal{L}_{\mathrm{int}} = g^2(\varphi^\dagger\chi)(\chi^\dagger\varphi)$$

defines the Gross-Neveu model in terms of the (one component) fields φ, χ. The relevant two-point function for non-interacting massive fermions is

$$\langle 0|T(\varphi^\dagger(x)\chi(0))|0\rangle = \frac{\mathrm{i}m}{\sqrt{2}}\int \frac{\mathrm{d}^2 p}{(2\pi)^2 p_-}\frac{\mathrm{e}^{\mathrm{i}px}}{p_+ - \frac{m^2 - \mathrm{i}\epsilon}{2p_-}} = \frac{m}{\pi\sqrt{2}}K_0(m\sqrt{-x^2})\,.$$

The basic large N limit now reads

$$g^2\sum_{i=1}^N \chi_i^\dagger(x)\varphi_i(x) = g^2\sum_{i=1}^N \varphi_i^\dagger(x)\chi_i(x)) \approx -\frac{\hat{m}}{\sqrt{2}}$$

which yields, following the above arguments, the self-consistency equation

$$\hat{m}\left(1+\frac{Ng^2}{2\pi}\ln\frac{\hat{m}^2}{\Lambda^2}\right)=0 \tag{18}$$

with

$$\Lambda^2=\frac{4\,\mathrm{e}^{-2C}}{-x^2}\ .$$

Eq. (18) again admits apart from $\hat{m}=0$ a non-trivial solution. This solution defines the running of the coupling constant in terms of the physical mass $\hat{m}$; it breaks the 1+1 dimensional chiral symmetry

$$\varphi(x)\to\mathrm{e}^{\mathrm{i}\alpha}\varphi(x)\ ,\quad \chi(x)\to\mathrm{e}^{-\mathrm{i}\alpha}\chi(x)\ . \tag{19}$$

Once more, the solution is selected according to stability. In two dimensions the energy density is a Lorentz scalar which, if regularized as $-x^2\to 0$ limit of Heisenberg operators

$$\begin{aligned}\epsilon(\hat{m}) &= \langle 0|\left[-\mathrm{i}\chi^\dagger(x)\partial_-\chi(0)-g^2(\varphi^\dagger(x)\chi(0))(\chi^\dagger(0)\varphi(x))\right]|0\rangle\\ &= -\frac{\hat{m}^2}{4\pi}\ln\frac{\hat{m}^2}{\Lambda^2}\left(1+\frac{Ng^2}{2\pi}\ln\frac{\hat{m}^2}{}\Lambda^2\right)\ ,\end{aligned} \tag{20}$$

agrees with the values of the effective potential at the stationary points, i.e., when the gap equation is satisfied. In particular one obtains

$$\epsilon(\hat{m})-\epsilon(0)=-\frac{\hat{m}^2}{4\pi}\,.$$

Thus for both the GN and the NJL model, light-cone quantization reproduces the well known results of ordinary quantization. Within these models, the simplicity of the light-cone description is not spoiled by a dynamical symmetry breakdown.

Our resolution of the triviality problem of the light-cone vacuum differs from the outset from previous attempts which have focused on the VEV's of Schrödinger operators. Regularization of VEV's leading to expressions like in Eq. (17) offers the possibility for introducing dynamical dependences into these purely kinematical objects. In the context of the NJL model, rules for regularization have been proposed by which the value of the chiral condensate obtained in ordinary quantization could be reproduced [10, 11, 12]. However it is difficult to see how, by such rules, the difference in the dynamics of broken and unbroken phase could be accounted for or how covariance in the evaluation of the corresponding correlation functions for non-vanishing spacelike separations could be respected (cf. [13]). In the approach we have described, non trivial vacuum properties are associated with products of Heisenberg operators in the equal light-cone time limit. Unlike in standard quantization schemes, VEV's determined in such a limiting procedure do not agree with VEV's of products of the corresponding Schrödinger operators and it is only the latter ones whose VEV's are trivial. It is by this subtle distinction between Schrödinger operators and the equal time limit of Heisenberg operators that condensates serving as order parameters for spontaneously broken symmetries can be defined despite the triviality of the ground state. From this point of view, the successful evaluation of the chiral condensate of the 't Hooft model in [4] becomes plausible; it avoids completely light-cone Schrödinger operators and uses general properties of VEV's which on the light-cone can be attributed only to expectation values of limits of Heisenberg operators. In a similar vein one can understand why the condensate issue could be bypassed in a light-cone calculation of fermion-antifermion scattering and bound states in the GN model [14]. Finally, in the correct determination of the chiral condensate of the Schwinger model in [15], the use of Heisenberg operators and point splitting was an essential element.

With the identification of VEV's of appropriate limits of Heisenberg operators as the relevant quantities for definition of order parameters, the standard tools of analyzing the effects of broken symmetries become available to light-cone quantization [8]. Ward identities can be derived and their consequences such as the Gell-Mann, Oakes, Renner relation [16] can be studied within light-cone quantization; perturbative treatments of explicit symmetry violations become amenable to the light-cone approach. Although our analysis has focused on fermionic theories, the extension to bosons is straightforward unless the VEV to be considered is linear in the field operator. In this particular case, as has been advocated in various studies (cf. [17, 18, 19, 20, 21]) the dynamics of a single (zero) mode may require a special treatment.

For light-cone studies of QCD the distinction between VEV's of Schrödinger operators and of limits of Heisenberg operators will be significant not only for the description of the quark condensate but also for the gluon condensate which is quadratic and of higher order in the gauge fields. Extension to gauge theories introduces a novel dynamical element into the discussion. Definition of non-trivial vacuum expectation values in light-cone quantization requires splitting in light-cone time; in turn, gauge invariance requires, in light-cone gauge $A_- = 0$, associated gauge strings to be introduced whose effects are expected to be enhanced by the infrared ($p_- = 0$) singularity characteristic for light-cone quantization.

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