# Twistor variational principle for null strings

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I present a twistor action functional for null 2-surfaces (null strings) in 4D Minkowski spacetime. The proposed formulation is reparametrization invariant and free of algebraic and differential constraints. Proposed approach results in derivation of evolution equations for the null strings. It is shown that non-geodesic null strings are contained in the presented formalism. A discussion of the problem of minimality for 2-surfaces with degenerate induced metric is given. I also speculate on the possible description of strings (time-like 2-surfaces) and conventional (space-like) 2-surfaces.

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## I. INTRODUCTION

The notion of null 2-surface was put forward by A. Schild as a tensionless (null) string, i.e. a 2-dimensional ruled by a one-parameter family of light-like geodesics submanifold of 4D Minkowski or curved spacetime. The induced metric on such submanifolds is degenerate:

$$\dot{x}^2 \acute{x}^2 - (\dot{x}\acute{x})^2 = 0. \tag{1}$$

Here $x^a(\tau, \sigma)$ is the world-sheet coordinate, the dots and primes denote differentiation with respect to the parameters $\tau$ and $\sigma$, $\dot{x}\acute{x}$ stands for $\dot{x}^a \acute{x}_a$. It should be emphasized that the null property (1) is manifestly reparametrization invariant while the original Schild's variational principle lacked this feature [1].

A different approach was developed in the articles [2, 3]. In the paper [2] J. Stachel chose a null bivector (2-form obeying algebraic constraints

$$p_{ab}(x)^* p^{ab}(x) = p_{ab}(x) p^{ab}(x) = 0, \tag{2}$$

the star stands for duality operation) as the Lagrangian density and showed that Schild's null strings can be described in this way. He also imposed the integrability condition

$$p_{a[b}(x) \nabla_c p_{de]}(x) = 0 \tag{3}$$

for null bivectors (square brackets denote skew-symmetrization, $\nabla_a$ stands for the covariant derivative). It stems from the requirements of differential geometry and was adopted from the book [4]. Recently, O.E. Gusev and A.A. Zheltukhin [3] have solved the algebraic constraints in the physical dimensions of Minkowski spacetime using a fundamental result of spinor calculus of Cartan-Penrose [5, Vol. 1]. According to this result a null bivector can be represented with the aid of a 2-component spinor and the action principle takes the form

$$S = \int (\bar{\pi}_A \bar{\pi}_B \epsilon_{A'B'} + \pi_{A'} \pi_{B'} \epsilon_{AB}) dx^{AA'} \wedge dx^{BB'}. \tag{4}$$

Using this action, they proved the null property (1) of the resulting 2-surfaces and treated the case of geodesic null strings.

D.V. Volkov and his co-authors were interested in applications of twistors to supersymmetric objects (superbranes) and laid the foundations of the superembedding approach [6–8].

In the present contribution I show that the variational principle (4) admits a natural twistor generalization. Additionally to the solutions already found in [3], the corresponding Euler-Lagrange equations contain as their solutions generic, i.e. not necessarily ruled by null geodesics of the ambient spacetime and referred to here as non-geodesic, null 2-surfaces (null strings). I derive a non-linear evolution equation governing the propagation of a generic non-geodesic null string. A heuristic argument in favour of defining the minimal null 2-surfaces as ruled by null geodesics of the ambient spacetime is presented. I also draw attention of the reader to the possibility and advantages of using the proposed formalism for expressing variational principles for strings (time-like 2-surfaces) and conventional (space-like) 2-surfaces of 4D Minkowski spacetime (cf. [3]).

## II. ACTION FUNCTIONAL

By definition, the null bivector, $p_{ab}$, obeys the pair of above stated algebraic constraints. This fact leads to the existence of a pair of 1-forms with components $u_a(x)$ and $v_b(x)$ such that $p_{ab} = (1/2!)(u_a v_b - u_b v_a) \equiv u_{[a} v_{b]}$. Moreover, it follows that without loss of generality one has $u_a u^a = u_a v^a = 0$, $v_a v^a < 0$, and the particular value of the Lorentz norm of $v_a$ is irrelevant for the variational problem in question [4].

Let the spinor fields $\bar{\pi}_A(x)$ and $\bar{\eta}_A(x)$ to constitute a normalized Newman-Penrose dyad ($\bar{\pi}_A \bar{\eta}^A = 1$) in 4D Minkowski spacetime and additionally assume the

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spinor field $\bar{\pi}_A$ to be chosen in such a way as to represent the coincident principal null directions of the null bivector $p_{ab}$. Then, one can write $u_a = \bar{\pi}_A \pi_{A'}$ and $v_a = \bar{\pi}_A \eta_{A'} + \pi_{A'} \bar{\eta}_A$. This representation is, up to an overall functional multiplier, the general solution of the algebraic constraints for the null bivector $p_{ab}(x)$ (see, for example, [5]). To switch between vector and spinor indices the conventions of [5] are used.

If one assembles a pair of 1-forms $u_a dx^a = \bar{\pi}_A \pi_{A'} dx^{AA'}$ and $v_a dx^a = (\bar{\pi}_A \eta_{A'} + \pi_{A'} \bar{\eta}_A) dx^{AA'}$, introduces the null twistors $Z^\alpha \equiv (\bar{\omega}^A, \pi_{A'})$ and $W^\alpha \equiv (\bar{\xi}^A, \eta_{A'})$, expresses the 1-forms in terms of the null twistors and substitutes the results in the formula for $p_{ab} dx^a \wedge dx^b$ then (s)he obtains the following twistor variational principle for a null string in 4D Minkowski spacetime:

$$S = \int \overline{Z}_\alpha dZ^\alpha \wedge \left( \overline{Z}_\beta dW^\beta + \overline{W}_\beta dZ^\beta \right). \qquad (5)$$

Here the spinor fields $\bar{\omega}^A$ and $\bar{\xi}^A$ are given by the usual definitions: $\bar{\omega}^A = i x^{AA'} \pi_{A'}$ and $\bar{\xi}^A = i x^{AA'} \eta_{A'}$. $\overline{Z}_\alpha$ and $\overline{W}_\alpha$ are the conjugate null twistors. The null property of the twistors $Z^\alpha$ and $W^\alpha$ correspond to the Hermitian property of $x^{AA'}$ and leads to the identities $\overline{Z}_\alpha Z^\alpha = \overline{W}_\alpha W^\alpha = \overline{Z}_\alpha W^\alpha = \overline{W}_\alpha Z^\alpha = 0$. It also reflects the reality conditions imposed on the points of 4D Minkowski spacetime. The 2-form in (5) is understood to be restricted to a 2-dimensional submanifold of 4D Minkowski spacetime parametrized by $\tau$ and $\sigma$.

The Lagrangian density of the twistor action functional (5) is multiplied by the factor $q^2$ under the gauge transformations of the form

$$Z^\alpha \to q Z^\alpha, \quad W^\alpha \to q^{-1} W^\alpha + p Z^\alpha. \qquad (6)$$

Here $q(\tau, \sigma)$ is a nowhere vanishing real-valued function and $p(\tau, \sigma)$ is an arbitrary complex-valued function. This is an admissible freedom for a differential form representing a surface [4]. It gives rise to invariance of the Euler-Lagrange equations under the above mentioned transformations. The invariance corresponds to the possibility of rescaling with real multiples of the extent of the null direction tangent to the null string world-sheet $\bar{\pi}^A \to q \bar{\pi}^A$ and to an addition of real multiples of the null direction to the space-like direction tangent to the world-sheet $\bar{\eta}^A \to q^{-1} \bar{\eta}^A + p \bar{\pi}^A$. Thus the gauge freedom of the null string world-sheet comprises the null- and boost-rotations.

## III. EVOLUTION EQUATIONS

The Euler-Lagrange equations for the variational principle (5) were obtained in the author's doctoral thesis [9]. After some tedious but straightforward algebra they lead to the property (1) (cf. [3]). In addition, one can show that the differential constraint does not introduce new equations to those obtained by variational procedure from the action (5). The latter can be proved by expressing the differential constraint in terms of spinor fields $\bar{\pi}^A$ and $\bar{\eta}^A$ (see [9]).

It follows that the twistor action functional (5) describes the null string as a 2-dimensional submanifold of 4D Minkowski spacetime with the degenerate induced metric. It is also remarkable that this formulation is reparametrization invariant and free of additional algebraic and differential constraints present in the spacetime description [2] and of the auxiliary world-sheet quantities artificially introduced in an earlier formulation by I.A. Bandos and A.A. Zheltukhin [10].

One can choose orthogonal gauge for the world-sheet of a null string so that $\dot{x}^2 = 0$ and (1) implies $\dot{x}\acute{x} = 0$. In spinor terms this means that $\dot{x}^{AA'} = r \bar{\pi}^A \pi^{A'}$ and $\acute{x}^{AA'} = i(\zeta \bar{\eta}^A \pi^{A'} - \bar{\zeta} \bar{\pi}^A \eta^{A'})$, where $r(\tau, \sigma)$ is the real-valued flagpole extent and the function $\zeta(\tau, \sigma)$ is complex valued. The equations of motion for the variational principle (5) also imply that $\dot{\bar{\pi}}^A (\zeta - \bar{\zeta}) = 0$. This means either that $\dot{\bar{\pi}}^A = 0$ or the function $\zeta$ is real-valued.

The first opportunity was spotted by O.E. Gusev and A.A. Zheltukhin in the article [3]. It immediately leads to the equation $\ddot{x}^a \propto \dot{x}^a$, which states that integral curves of the vector field $\dot{x}^a$ are (null) geodesics in the sense of the ambient Minkowski spacetime. If the affine parametrization for the null geodesics is chosen then one can write an evolution equation for *geodesic null strings* in the form:

$$\ddot{x}^a = 0. \qquad (7)$$

The second opportunity allows one to take the so-called natural parametrization, $r^{-1}(\tau, \sigma) = \kappa \equiv \bar{\pi}_A \bar{\pi}^B \pi^{B'} \nabla_{BB'} \bar{\pi}^A$ (for the geodesic case the spin-coefficient $\kappa$ vanishes), and results in the following complete set of equations of motion for non-geodesic null strings:

$$\begin{aligned} \dot{x}^{AA'} &= \kappa^{-1} \bar{\pi}^A \pi^{A'}, \ \acute{x}^{AA'} = i\zeta(\bar{\pi}^A \eta^{A'} - \bar{\eta}^A \pi^{A'}), \\ \kappa^{-1}(\bar{\pi}^A \acute{\bar{\pi}}_A + \pi^{A'} \acute{\pi}_{A'}) &= 2i\zeta(\pi^{A'} \dot{\eta}_{A'} - \bar{\pi}^A \dot{\bar{\eta}}_A), \\ \bar{\pi}^A \dot{\bar{\pi}}_A - \pi^{A'} \dot{\pi}_{A'} &= 0. \end{aligned} \qquad (8)$$

The spin-coefficient $\kappa$ reflects existence of interaction between the null string and external fields. Such interactions preserve the null character of the string world-sheet but violate its geodesic property. For an example see the article [12]. It should be pointed out that the set of equations (8) coincides with the system derived in that paper for the non-geodesic (interacting) null string. This proves an equivalence on the classical level of the both approaches.

In the natural parametrization one finds the next identities $\dot{\bar{\pi}}_A \bar{\pi}^A = 1$, and it follows that $\bar{\eta}^A = -\dot{\bar{\pi}}^A$. These results can be used to show that there exists a non-linear evolution equation for *non-geodesic null strings*

$$\ddot{x}^2 \left[ \acute{x}^2 \ddot{x}^a - (\acute{x}\ddot{x}) \acute{x}^a \right] - \acute{x}^2 (\ddot{x}\dddot{x}) \dot{x}^a = 0. \qquad (9)$$

The non-geodesic null string evolution equation is invariant under a subgroup of diffeomorphisms of the null string world-sheet which preserves the orthogonal gauge.

## IV. MINIMAL NULL 2-SURFACES

The results of Hughston and Shaw [11] on the connection between non-interacting (free) strings and minimal time-like 2-surfaces in 4D Minkowski spacetime provide an impetus for attempts of finding a similar correspondence between geodesic (i.e. non-interacting, cf. [12]) null strings and minimal null 2-surfaces. The task of formulating the conditions of minimality for a null 2-surface in 4D Minkowski spacetime seems to be a rather difficult one. The standard approach of the classical geometry of surfaces in the Riemannian space, which uses a suitable variational principle, fails in this case. The problem lies in the degenerate property of the induced metric (the first fundamental form) of such surfaces and, therefore, one cannot easily construct an analogue of area element like in the case of time-like 2-surfaces.

Nevertheless, it is possible to formulate the minimality conditions for a null two-surface in 4D Minkowski spacetime. In order to find them, a limiting procedure which takes the tangent space to a space-like 2-surface element to that of a null 2-surface element was built in [9].

The conditions of minimality for a space-like 2-surface in 4D Minkowski spacetime can be formulated in the same way as those for a 2-surface in the ordinary Riemannian geometry. This has its origin in the non-degenerate property of the first fundamental form of a space-like 2-surface. In particular, the conditions of minimality can be given by the requirement of vanishing of the relevant mean curvatures. The mean curvatures are calculated by taking a trace of the corresponding second fundamental forms.

Taking the limit of the well-defined minimality conditions for the space-like 2-surfaces with the aid of that procedure, one finds that minimal 2-surfaces admit a one parameter family of null geodesics.

The geometry of the minimal null 2-surfaces depends on whether the corresponding line of striction is a null or space-like curve. In the former case the minimal null 2-surface is (locally) developable and the null geodesics of the congruence are strongly incident; in the latter case the null generators of the 2-surface present an example of weakly incident light rays as has been discussed recently by R. Penrose [13].

These results exhibit unusual features connected to the indefinite nature of the Lorentz norm in 4D Minkowski (and curved) spacetime.

## V. ACTION FOR STRINGS AND SPACE-LIKE 2-SURFACES

The method employed for obtaining the action functional (5) of a null string can be in principle used for designing an action for strings and a variational principle for space-like 2-surfaces. The idea is to take a simple bivector

$$p_{ab}{}^{*}p^{ab} = \quad 0 \tag{10}$$

and impose one of the conditions

$$\begin{aligned} p_{ab}p^{ab} &= 1, \\ p_{ab}p^{ab} &= -1. \end{aligned} \tag{11}$$

The first condition in (11) singles out the string while the second corresponds to a space-like 2-surface. It is easy to see that such a procedure uniquely fixes the symmetric second rank spin-tensor in the standard decomposition of an skew-symmetric 4D Minkowski spacetime tensor

$$p_{ab}(x) = \phi_{AB}(x)\varepsilon_{A'B'} + \bar{\phi}_{A'B'}(x)\varepsilon_{AB}. \tag{12}$$

Then, the variational principle

$$S = \frac{1}{2!}\int p_{ab}(x)dx^a \wedge dx^b \tag{13}$$

defines a 2-surface subject to the differential constraint (3). Now, one hopes that the use of spinor decomposition for $p_{ab}(x)$ consistent with the algebraic constraints (10) and either of (11) would provide equations of motion, which automatically incorporate the differential constraint. This assertion is supported by the success of this procedure for the null bivectors outlined in the current contribution. It may also be possible to derive the analogues of the evolution equation (9) for generic (interacting) strings in 4D Minkowski spacetime and curved spacetimes of general relativity where exist explicit spinor constructions.

In the same way one could build twistor action functionals in the both cases and find corresponding objects on the null twistor space for generic time-like and space-like 2-surfaces of 4D Minkowski spacetime. This would accomplish the task of finding of twistor description for 2-surfaces.

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