A GENERALIZATION OF WITTEN'S CONJECTURE RELATING DONALDSON AND SEIBERG–WITTEN INVARIANTS

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Abstract. We generalize Witten's conjectured formula relating Donaldson and Seiberg–Witten invariants to manifolds of non-simple type, via equivariant localization techniques. This approach does not use the theory of non-abelian monopoles, but works directly on the Donaldson–Witten and Seiberg–Witten moduli spaces. We give a formal derivation of Witten's conjecture and its generalization, making use of an infinite dimensional version of the abelian localization theorem.

1. Introduction

Topological quantum field theories (TQFT) emerged in the late 1980s as part of the renewed relationship between differential geometry/topology and physics. In the 1990s, developments in TQFT gave unexpected results in differential topology and symplectic and algebraic geometry.

One striking feature of physicists' approach to TQFT is the use of mathematically non-rigorous Feynman path integrals to produce new topological invariants of manifolds, which appear as the physical observables of the TQFT. For example, the first TQFT, formulated by Witten [37] in 1988 (Donaldson–Witten theory), gives a quantum field theory representation of Donaldson invariants. Witten formulated two additional TQFT: 2D topological sigma models [40], and 3D Chern-Simons gauge theory [37]. These theories are related, respectively, to Gromov–Witten invariants, and to knot and link invariants (Jones polynomials and generalizations). Seiberg and Witten [32] extended these ideas to introduce the Seiberg–Witten invariants of 4–manifolds. These invariants have revealed many deep properties of 3– and 4–manifolds and of symplectic manifolds. Witten's papers have stimulated a tremendous amount of mathematical research to make his work mathematically rigorous (e.g. Axelrod and Singer [4], Kronheimer and Mrowka [18], Ruan and Tian [30], Taubes [35]).

This paper makes use of the Mathai–Quillen formalism [26] to study properties of Donaldson–Witten and Seiberg–Witten theories, and relations between them. Basically, the Mathai–Quillen formalism can be applied to any TQFT, with topological invariants defined by functional integrals of a certain Euler class (or wedge products of the Euler class with other forms). This approach was first applied by Atiyah and Jeffrey [2] to Donaldson theory and the Casson invariant, and was further developed in e.g. [7],[41]. This approach is only formal at present, because ordinary finite dimensional geometric constructions are applied to infinite dimensional path integrals. While a mathematically rigorous understanding of path integrals is still unknown, their predictive power is well known in physics, and their predictions in mathematics are equally striking.

In this paper, the Mathai–Quillen formalism for Donaldson–Witten and Seiberg–Witten theories is stated in an equivariant setup, following the work of Constantinescu [7], and the relevant invariants are obtained from a functional integral point of view. Our main result is a generalization of Witten's formula relating Donaldson and Seiberg–Witten invariants. We state the main result as a conjecture, since the argument makes use of an infinite dimensional version of the abelian localization theorem.

In more detail, Witten [39] conjectured on physical grounds that Donaldson–Witten invariants can be written in terms of Seiberg–Witten invariants. This conjecture states the following. Consider a compact, connected, simply-connected, smooth 4–manifold X with $b_2^+ > 1$. Let $\mathcal{A}_E^*$ be the space of irreducible connections on an $SU(2)$–bundle E over X, with second Chern class $c_2(E) = k$, and let $\mathcal{G}_E$ be the group of unitary gauge transformations of E. Then, if $\mu(\Sigma) \in H^2(\mathcal{A}_E^*/\mathcal{G}_E)$ and $\mu(\nu) \in H^4(\mathcal{A}_E^*/\mathcal{G}_E)$ are the so-called μ–classes, the total Donaldson polynomial is given by:

$$\mathbb{D}_X(\Sigma_1^{\alpha_1}, \ldots, \Sigma_{b_2}^{\alpha_{b_2}}, \nu^\beta) = \sum_{k\in\mathbb{Z}} \int_{\mathcal{M}_k} \mu(\Sigma_1)^{\alpha_1} \ldots \mu(\Sigma_{b_2})^{\alpha_{b_2}} \mu(\nu)^\beta,$$

where $\mathcal{M}_k$ is the (Uhlenbeck compactification of the) moduli space of Donaldson theory, consisting basically of anti-self-dual connections modulo the gauge group. Following Witten's notation, let us introduce formal variables $q_1, \ldots, q_{b_2}, \lambda$, and write the generating function for the total Donaldson polynomial as

$$\mathbb{D}_X\left(e^{\sum q_a\Sigma_a + \lambda\nu}\right) = \sum \frac{\mathbb{D}_X((q_1\Sigma_1)^{\alpha_1}, \ldots, (q_{b_2}\Sigma_{b_2})^{\alpha_{b_2}}, (\lambda\nu)^\beta)}{\alpha_1! \cdots \alpha_{b_2}! \cdot \beta!}.$$

Let $SW(x)$ denote the Seiberg–Witten invariant for each isomorphism class x of spin^c–structures on X, which counts the number of points in the zero dimensional SW moduli space. For $v = \sum_a q_a\Sigma_a$, the conjectured equality for manifolds of simple type is [39]

$$\begin{aligned}\mathbb{D}_X(e^{\sum_a q_a\Sigma_a + \lambda\nu}) &= \\ 2^{1+\frac{1}{4}(7\chi+11\sigma)} &\cdot \left[e^{\left(\frac{v^2}{2}+2\lambda\right)} \sum_x SW(x)\cdot e^{v\cdot x} + i^{\frac{\chi+\sigma}{4}} e^{\left(-\frac{v^2}{2}-2\lambda\right)} \sum_x SW(x)\cdot e^{-iv\cdot x} \right],\end{aligned} \tag{1.1}$$

where χ, σ denote the Euler characteristic and signature of X, respectively, and the sums run over all basic classes x.

While this conjecture checks in all known examples, Witten's derivation [39] of the formula uses physical arguments whose mathematical content is unclear at present (see [7], [16] for other physical derivations). In a series of papers [10], Feehan and Leness propose a rigorous derivation of this conjecture, following the work of [29]. These authors have made excellent progress towards the conjecture, although the proofs are quite different from Witten's original intuition. In particular, the authors work on the moduli space of non-abelian monopoles.

This paper takes a different approach towards Witten's conjecture. Some steps of this argument are still formal at present. In particular, following [7], equivariant localization techniques are used to reduce Witten's conjecture to computations on the fixed point set of a group action. This is closer in spirit to recent proofs [14], [24] of the mirror symmetry conjecture. In general, it seems that equivariant cohomology is a more natural domain for TQFT observables than the usual cohomology groups, since the configuration spaces of TQFT come with group actions.

In contrast of working on the moduli spaces of non-abelian monopoles, we introduce group actions on the moduli spaces of Seiberg-Witten solutions (abelian monopoles) and anti-self-dual connections. The relevant equivariant path integrals can be compared by localization techniques, since the fixed point sets of the group actions coincide, even though the total moduli spaces differ.

The main result of this paper is a derivation of a generalization of Witten's formula to equivariant invariants and manifolds of non-simple type, via equivariant localization techniques. Let $\mathbb{D}_X\left(e^{\Sigma+\lambda\nu}\right)(m)$ be the equivariant Donaldson generating series, and $SW(c,m)$ be the equivariant SW invariant for a spin^c-structure c, as defined in §3.1, and §3.2, respectively. Let $2s(c)$ be the dimension of the SW moduli space for the corresponding spin^c-structure c. We obtain the following conjectured formula relating equivariant Donaldson and Seiberg–Witten invariants, equality which holds in $\mathbb{C}[m]$, the S^1-equivariant cohomology ring of a point:

$$\begin{aligned}\mathbb{D}_X\left(e^{\Sigma+\lambda\nu}\right)(m) &= 2^{1+\frac{7\chi+11\sigma}{4}}\sum_c 2^{8s(c)}\left(\frac{2\pi}{m}\right)^{2s(c)} SW(c,m)\cdot e^{\left(\frac{m}{2\pi}\right)(c\cdot\Sigma+\frac{1}{2}\Sigma^2)}\cdot e^{\left(\frac{m}{2\pi}\right)^2(2\lambda)} \\ &+2^{1+\frac{7\chi+11\sigma}{4}} i^{\frac{\chi+\sigma}{4}}\sum_c 2^{8s(c)}\left(\frac{2\pi}{m}\right)^{2s(c)} SW(c,m)\cdot e^{\left(\frac{m}{2\pi}\right)(-ic\cdot\Sigma-\frac{1}{2}\Sigma^2)}\cdot e^{\left(\frac{m}{2\pi}\right)^2(-2\lambda)}.\end{aligned} \tag{1.2}$$

(1.2) reduces to Witten's formula (1.1), identifying the degree zero components (the expression being regular at $m=0$), and assuming that X has simple type. This derivation assumes that finite dimensional equivariant localization formulas extend to infinite dimensions, so the argument has a formal part. However, it is possible that the equivariant path integrals used can be made rigorous, which would provide an alternative to the unsuccessful search for good measures on infinite dimensional spaces of connections.

Since this paper covers topics in both mathematics and physics, we have assembled material from both fields (some well-known and some quite specialized) in the initial sections. Section 2 covers standard mathematical material on Mathai–Quillen formalism, specialized results on extensions to the equivariant setting (following the work of [7]), and a discussion of the physicists' version of Mathai–Quillen formalism in infinite dimensions. Section 3 discusses the application of the Mathai–Quillen construction to the infinite dimensional settings for Donaldson–Witten and Seiberg–Witten theories, in the spirit of [7]. Section 4 is the main part of this paper, where abelian localization techniques are applied to the setups of section 3, and new descriptions of Donaldson polynomial and Seiberg–Witten invariants are obtained. Section 5 is devoted to a complete derivation our generalization of Witten's conjecture, and a derivation of Witten's formula (1.1) from (1.2).

2. The Mathai-Quillen Formalism

The Mathai–Quillen (MQ) formalism [26] can be applied to any TQFT. This approach was first implemented by Atiyah and Jeffrey [2], and later further developed in [6], [7], [8]. The basic idea behind this formalism is the extension to the infinite-dimensional case of ordinary finite-dimensional geometric constructions. MQ formalism gives a unified description of many Cohomological Field Theories (CFT) and Supersymmetric Quantum Mechanics (SQM), and it also provides some insight into the mechanism of the localization of path integrals in SQM and TQFT.

2.1. **The Mathai–Quillen Construction.** Let us start with a summary of definitions and properties of equivariant cohomology, which will be used later. For details one should see [1], [5], [7], [8].

Let M be a manifold, and G a connected compact Lie group with Lie algebra $\mathfrak{g}$, such that G acts on M. The equivariant cohomology $H^*_G(M)$ is defined to be the ordinary cohomology $H^*(M^G)$ of the space $M^G = EG \times_G M$, where EG is the universal bundle over BG, the classifying space of G. Equivariant cohomology is a contravariant functor from G-spaces to modules over the base ring $H^*_G(\mathrm{pt}) = H^*(BG)$.

If $H \subset G$ is a closed subgroup, then $H^*_G(G/H) = H^*_H(\mathrm{pt})$. Consider the inclusion $i : M \hookrightarrow M^G$, via the fiber over a base point of BG, which induces a natural homomorphism from equivariant cohomology to ordinary cohomology

$$i^* : H^*_G(M) \longrightarrow H^*(M).$$

For T a maximal torus in G, we have

$$H^*_G(M) \stackrel{\simeq}{\longrightarrow} H^*_T(M)^W, \tag{2.1}$$

where the right hand side consists of elements of $H^*_T(M)$ which are invariant under the Weyl group W. (2.1) reflects the fact that the study of compact Lie groups, which reduces to the behavior of the Weyl group acting on a maximal torus of G, also applies to the equivariant theory [1, p. 4].

There are also algebraic models for defining the equivariant cohomology. The most common are the Weil model and the Cartan model. Although they are equivalent (see e.g. [5], [8]), the Cartan model is more natural in TQFT. We summarize it below. Consider the graded algebra of equivariant differential forms

$$\Omega_G(M) = [S(\mathfrak{g}^*) \otimes \Omega(M)]^G \otimes \mathbb{C},$$

where the $\mathbb{Z}$-grading is the usual on $\Omega(M)$, and twice the grading on $S(\mathfrak{g}^*)$, $[\,\cdot\,]^G$ is the subalgebra of G-invariant elements, and the action of G on $\Omega(M)$ is obtained from the action of G as diffeomorphisms on M, and on $S(\mathfrak{g}^*)$ is obtained from symmetric powers of the coadjoint action. The equivariant deRham graded differential of degree $+1$ on $\Omega_G(M)$ is defined by

$$d_G\alpha(X) = d[\alpha(X)] - \iota_{\rho(X)}[\alpha(X)],$$

where we extend α to be in $S^i(\mathfrak{g}^*) \otimes \Omega^j(M)$, so $d\alpha \in S^i(\mathfrak{g}^*) \otimes \Omega^{j+1}(M)$. Moreover, $\iota_{\rho(\cdot)} \in S^{i+1}(\mathfrak{g}^*) \otimes \Omega^{j-1}(M)$. Here d denotes the usual deRham differential, ι is the interior product, and $\rho : \mathfrak{g} \longrightarrow \mathrm{Vect}(TM)$ is the infinitesimal action of $\mathfrak{g}$. One can show that

$d_G^2 = 0$. Then the G-equivariant cohomology of M with complex coefficients is defined as the cohomology of the complex above, and this cohomology equals $H_G^*(M;\mathbb{C})$.

If α is equivariantly closed for all $X \in \mathfrak{g}$, i.e. $d_G\alpha = 0$, and if we decompose α into its homogenous components, then

$$\iota_{\rho(X)}\alpha(X)_{[i]} = d\alpha(X)_{[i-2]}.$$

Moreover, one can show that $H_G^*(\mathrm{pt}) = S(\mathfrak{g})^G \otimes \mathbb{C}$. In particular, for $G = S^1$, $H_{S^1}^*(\mathrm{pt}) = \mathbb{C}[m]$. Then integration of G-equivariant forms, $\displaystyle\int : \Omega_G^*(M) \longrightarrow S(\mathfrak{g})^G$, is given by the Berezin integral, as defined in Definition 2.

For $G = S^1$, the coadjoint action is trivial, so

$$\Omega_{S^1}(M) = \mathbb{C}[m] \otimes \Omega(M)^{S^1},$$

where m is a generator of $\mathfrak{u}(1)^*$, the dual of the Lie algebra of S^1, and

$$\Omega(M)^{S^1} = \{\omega \in \Omega(M) | \mathcal{L}_X\omega = 0\},$$

where X is the vector field corresponding to the generator of the Lie algebra of S^1, dual to m. Then if $\omega \in \Omega^{S^1}(M)$ and $k \in \mathbb{Z}$,

$$d_{S^1}(m^k\omega) = m^k d\omega - m^{k+1}\iota_X\omega.$$

Remark 1. The BRST operator of a TQFT, as defined in the physics literature, can be interpreted as the differential d_G for a G-equivariant cohomology of a certain space of fields. Note that in some models G might be trivial.

The Mathai–Quillen (MQ) formalism was first defined in [26] and then refined in a series of later papers and books (e.g. [2]). The setup for the MQ construction is the following (the organization of the material and the notation below is based on [7]). Let M be a compact oriented manifold, G a compact Lie group, P a principal G-bundle over M, V an oriented inner product space of $\dim V = 2m$, and $\rho_V : G \longrightarrow SO(V)$ a representation of G. Let $E = P \times_{\rho_V} V$ be the associated vector bundle. One can identify P with the bundle of all orthonormal oriented frames on E. The MQ construction produces *universal Thom forms* $U_t \in \Omega_G(V)$ for $t \in \mathbb{R}_+$.

Definition 2. Consider A a commutative superalgebra, and $A\otimes\Lambda(V)$ the graded tensor product. Choose a (non-zero) volume element $\mathrm{vol} \in \Lambda^{\mathrm{top}}(V)$. Then the *Berezin integral* is the map

$$\mathcal{B}_\chi : A \otimes \Lambda(V) \longrightarrow A$$

given by $\mathcal{B}_\chi(a) =$ the coefficient of a^{top}, where $a^{\mathrm{top}} \in A \otimes \Lambda^{\mathrm{top}}(V) \simeq A \otimes \mathbb{R}\cdot \mathrm{vol}$.

In what will follow we will consider the case $A = S(\mathfrak{g}^*) \otimes \Lambda(V)$.

Definition 3. For all $t \in \mathbb{R}_+$, the *universal Thom form* $U_t \in S(\mathfrak{g}^*)\otimes\Lambda(V)$ is defined by

$$U_t = (2\pi t)^{-m}\mathcal{B}_\chi\left(e^{\frac{t}{2}(-\|x\|^2 - 2i\sum_i dx_i\cdot\chi_i + \sum_{i,j}\chi_i\cdot\phi_V\chi_j)}\right)$$

where the notation is explained as follows:

a) x_i are coordinates on V dual to χ_i.

b) $\phi_V \in S^1(\mathfrak{g}^*) \otimes so(V)$ is obtained as follows. Let
$$\phi \in S^1(\mathfrak{g}^*) \otimes \mathfrak{g} \simeq \mathfrak{g}^* \otimes \mathfrak{g} \simeq \mathrm{End}(\mathfrak{g}),$$
be the universal Weil element corresponding to the identity. Apply $d\rho : \mathfrak{g} \longrightarrow \mathrm{End}(V)$ to the $\mathfrak{g}$ part of ϕ, and call the result ϕ_V. We may regard ϕ_V as the G-universal curvature matrix with entries $(\phi_V)_{jk}$. Here ϕ_V acts on the fibers of E.

c) $\chi_i \cdot \phi_V \otimes \chi_j \in S^1(\mathfrak{g}^*) \otimes \Lambda(V) \otimes \Lambda(V)$, where $\cdot$ denotes the exterior product in $\Lambda(V)$.

d) $\sum_i dx_i \cdot \chi_i = \sum_i dx_i \otimes \mathbf{1} \otimes \chi_i \in S(\mathfrak{g}^*) \otimes \Lambda(V) \otimes \Lambda(V)$.

e) The exponential factor in the definition above lies in $S(\mathfrak{g}^*) \otimes \Lambda(V) \otimes \Lambda(V)$, so its Berezin integral is in $S(\mathfrak{g}^*) \otimes \Omega(V)$.

We can use the shorthand notation
$$U_t = (2\pi t)^{-m} e^{-\frac{tx^2}{2}} \mathcal{B}_\chi \left(e^{\frac{t}{2}(-2idx\chi + \chi\phi\chi)} \right),$$
where we regard x, dx, χ as vectors and ϕ as a matrix. The factor $e^{-\frac{tx^2}{2}}$ guarantees the rapid decay of U_t as differential forms on V.

The Thom forms U_t are G-invariant (i.e. $U_t \in \Omega_G(V)$, $d_G U_t = 0$), and $\int_V U_t = 1$ (see [7, p. 26-28]). Moreover, for all $t \in \mathbb{R}_+$ and sections s, $U_t(s) = s^* U_t$ represents the cohomology class in $H^{2m}_G(V)$ corresponding to $\mathbf{1} \in H^0_G(\mathrm{pt})$. Here we view V as a G-equivariant bundle over a point.

Via the universal Thom forms U_t, the Thom forms on E are constructed as follows. The horizontal projection composed with the Chern-Weil map [7, p. 15]
$$\Omega_G(V) \xrightarrow{CW} \Omega(P \times V)^G \xrightarrow{\mathrm{Hor}} \Omega(E)$$
is a chain map with respect to d_G on $\Omega_G(V)$ and d on $\Omega(E)$. Thus we define the Thom forms on E by
$$Th_t(E) = \mathrm{Hor} \circ CW(U_t),$$
One can prove that $Th_t(E) \in \Omega^{2m}(E)$, $dTh_t(E) = 0$, and $\int_{E_x} Th_t(E) = 1$.

The Euler classes $e_t(E, s)$ are obtained form a section $\hat{s} : M \longrightarrow E$ (or equivalently, from a G-equivariant map $s : P \longrightarrow V$), using the following commutative diagram:
$$\begin{tikzcd}
U_t \in \Omega_G(V) \arrow[r, "\mathrm{Hor} \circ CW"] \arrow[d, "s^*"'] & \Omega(E) \ni Th_t(E) \arrow[d, "\hat{s}^*"] \\
U_t(s) \in \Omega_G(P) \arrow[r, "\mathrm{Hor} \circ CW"] & \Omega(M) \ni e_t(E, s)
\end{tikzcd}$$
In the diagram above
$$U_t(s) = (2\pi t)^{-m} \mathcal{B}_\chi \left(e^{\frac{t}{2}(-\|s\|^2 - 2ids\chi + \chi\phi\chi)} \right).$$

If $\dim V = \dim M$, then the Euler number is given by

$$\chi(E) = \int_M e_t(E,s),$$

for all t and s. Moreover, one can show that

$$\chi(E) = (2\pi t)^{-m} \int_M \mathrm{Hor} \circ CW(\alpha),$$

where $\alpha = (2\pi t)^m U_t(s)$ is an equivariantly closed form in $\Omega_G(P)$, and therefore defines a class in $H^*_G(P)$.

Since the action of G on P is free, $\mathrm{Hor} \circ CW_\theta = (\pi^*)^{-1}$, where $P \xrightarrow{\pi} P/G = M$. Then

$$\int_M \mathrm{Hor} \circ CW_\theta(s^*(U_t)) = \int_M (\pi^*)^{-1}(s^*(U_t)) = \int_M \int_{TM} \mathcal{D}\eta \, (\pi^*)^{-1}(s^*(U_t)),$$

where η is a fermionic variable on TM which integrates to one. Let us write $\Omega = d\theta + \frac{1}{2}[\theta,\theta]$. The horizontal part is

$$\Omega_{\mathrm{hor}} = \phi = \mathrm{Hor}\left(d\theta + \frac{1}{2}[\theta,\theta]\right),$$

where ϕ is defined as above. Moreover, one can show that $\chi\phi\chi = \dfrac{1}{2} R^{ij}_{kl}\eta^k\eta^l\chi_i\chi_j$, so one obtains the following expression for $\chi(E)$, appearing in physics literature (see e.g. [20]):

$$\chi(E) = (2\pi t)^{-m} \int_M dx \int_{TM} \mathcal{D}\eta \, \mathcal{B}_\chi \left(e^{-\frac{t}{2}\|s(x)\|^2 + \frac{t}{2}(-2i\nabla_i s^j(x)\eta^i\chi_j + \frac{1}{2}R^{ij}_{kl}\eta^k\eta^l\chi_i\chi_j)} \right).$$

In the definition of $\chi(E)$, there is no dependence on t or on the section s. Then for $t \longrightarrow \infty$ we obtain

$$\chi(E) = \sum_{\text{zeros of } s} \pm 1,$$

and for $t \longrightarrow 0$, we recover the definition of the Euler number:

$$\chi(E) = (2\pi)^{-m} \int \mathrm{Pf}(\Omega),$$

For details, one should see e.g. [8, p. 111].

2.2. **The Mathai-Quillen Construction for Equivariant Vector Bundles.** In the framework of the MQ formalism, an equivariant extension of the Thom form with respect to a vector field action, has been constructed by Labastida and Mariño [23]. In this paper, the authors analyze in detail the cases of topological sigma models and non-abelian monopoles on four-dimensional manifolds. In a similar context, we summarize below an extension of the MQ construction to equivariant vector bundles, following [7, Secs. 2.3, 2.6].

As before, let M be a compact oriented manifold. Let H be a subgroup of G, and P be a H-equivariant G-bundle over M, i.e. H acts on P and M, and the action of H on P commutes with the action of G on P. Here G and H are compact Lie groups, with

H connected (such that the action on E below preserves the orientation). Let V be a $2m$-dimensional oriented vector space with inner product and representations:

$$\rho : G \longrightarrow SO(V), \qquad \lambda : H \longrightarrow SO(V).$$

Set $E = P \times_G V$, where the action of H on E is induced by the action of G, so E is naturally an H-equivariant bundle over M.

One needs to replace the Chern-Weil map CW by an equivariant version. If we take an H-invariant connection on P, and consider its H-equivariant 1-form $\theta \in \Omega^1(P) \otimes \mathfrak{g}$, i.e. $\mathcal{L}_Y \theta = 0$, for all $Y \in \mathfrak{h} = \mathrm{Lie}(H)$, then its curvature $\Omega \in \Omega^2(P) \otimes \mathfrak{g}$. Then one can define the equivariant moment $\mathcal{I}$ of the connection θ (analogous to the moment map in symplectic geometry), given by

$$\mathcal{I}(Y) = \iota_Y \theta \in C^\infty(P) \otimes \mathfrak{g},$$

for all $Y \in \mathfrak{h}$, where ι_Y is the vector field on P determined by $Y \in \mathfrak{h}$ through the infinitesimal action of $\mathfrak{h}$ on P. Then the H-equivariant curvature is given by

$$\Omega_H = \Omega - \mathcal{I} \in S(\mathfrak{g}^*) \otimes \Omega(P) \otimes \mathfrak{g}.$$

One can show the following:

Proposition 4. *For all $X \in \mathfrak{g}$, we have*

$$\mathcal{L}_X(\Omega_H) = -ad_X \circ \Omega_H.$$

Since $\mathcal{L}_Y \theta = 0$, for all $Y \in \mathfrak{h}$, $\theta \in \Omega^1_H(P) \otimes \mathfrak{g}$, and

$$\Omega_H = d_H \theta + \frac{1}{2}[\theta, \theta]$$

(the equivariant Maurer-Cartan equations). For D the covariant derivative of θ, we define D_H on $\Omega_H(P)$ by:

$$D_H \omega(Y) = D\omega(Y) - \iota_Y[\omega(Y)]$$

for $Y \in \mathfrak{h}$. Then $D_H \Omega_H = 0$ (the equivariant Bianchi identity).

The H-equivariant CW map is the algebra homomorphism $CW_{H,\theta} : S(\mathfrak{g}^*) \longrightarrow \Omega_H(P)$ defined by:

$$CW_{H,\theta}(X^*) = X^*(\Omega_H) \in \Omega_H(P),$$

for all $X^* \in \mathfrak{g}^*$. Note that $CW_{H,\theta}$ extends to a map $CW_{H,\theta} : \Omega_{G\times H}(P) \longrightarrow \Omega_H(P)^G$, inducing the H-equivariant CW homomorphism:

$$\mathrm{Hor} \circ CW_{H,\theta} : \Omega_{G\times H}(P) \longrightarrow \Omega_H(M).$$

Proposition 5. [7, p. 31] *The following hold:*

1. *$Hor \circ CW_{H,\theta}$ is a chain map with respect to $d_{G\times H}$ on $\Omega_{G\times H}(P)$, and d_H on $\Omega_H(M)$, so $Hor \circ CW_{H,\theta}$ descends to a map*
$$H^*_{G\times H}(P) \longrightarrow H^*_H(M).$$
2. *$Hor \circ CW_{H,\theta} : H^*_{G\times H}(P) \longrightarrow H^*_H(M)$ is independent of θ.*
3. *$Hor \circ CW_{H,\theta} : H^*_{G\times H}(P) \xrightarrow{\simeq} H^*_H(M)$ is an isomorphism.*

The MQ construction can now be extended to equivariant vector bundles. We define as before a form $U_t \in \Omega_{G\times H}(V)$, and let $Th_{H,t}(E) = \mathrm{Hor} \circ CW_H(U_t) \in \Omega_H(E)$, where

$$\Omega_{G\times H}(V) \xrightarrow{CW_H} \Omega_H(P\times V)^G \xrightarrow{\mathrm{Hor}} \Omega_H(E).$$

Then $\deg(Th_{H,t}) = 2m$, $d_H Th_{H,t} = 0$, and $\int_{E_x} Th_{H,t} = 1$.

We have the following commutative diagram:

$$\begin{tikzcd} U_t \in \Omega_{G\times H}(V) \arrow[r, "\mathrm{Hor}\circ CW_H"] \arrow[d, "s^*"'] & \Omega_H(E) \ni Th_{H,t}(E) \arrow[d, "\hat{s}^*"] \\ U_{H,t}(s) \in \Omega_{G\times H}(P) \arrow[r, "\mathrm{Hor}\circ CW_H"] & \Omega_H(M) \ni e_{H,t}(E,s) \end{tikzcd}$$

where $s : M \longrightarrow E$ is an H-equivariant section. More explicitly,

$$U_{H,t}(s) = \mathcal{B}_\chi\left((2\pi t)^{-m} e^{\frac{t}{2}(-\|s\|^2 - 2ids\chi + \chi(\phi_G+\phi_H)\chi)}\right).$$

2.3. **Refinements.** In this section we give an alternative version of the MQ integral, which expresses the Euler number as an integration over P instead of over M. This section is based on [7, Sec. 2.4, 2.6].

Choose a vertical volume element v (volume element along the fibers of $\pi : P \longrightarrow M$) normalized such that its integral over the fibers is 1. Then for all $\beta \in \Omega^{\mathrm{top}}$ we have

$$\int_P \pi^*\beta \wedge v = \int_M \beta.$$

Definition 6. The *equivariant vertical volume element* $\gamma_G \in \Omega(P)\otimes S(\mathfrak{g})$ is constructed as follows. Let $\lambda_1, \ldots, \lambda_{\dim \mathfrak{g}}$ be an orthonormal basis of $\mathfrak{g}$ with respect to an inner product (e.g. the Killing form for a semisimple Lie group), normalized such that $\mathrm{vol}(G) = 1$. Choose a G-invariant connection θ on P with curvature Ω. If $\eta = (\eta_1, \ldots, \eta_{\dim \mathfrak{g}})$ is a second basis of $\mathfrak{g}$, we set

$$\gamma_G = e^{\sum_\alpha \Omega_\alpha \otimes \lambda_\alpha} \mathcal{B}_\eta\left(e^{\sum_\alpha \theta_\alpha \otimes \eta_\alpha}\right).$$

If we write $\theta = \theta_\alpha \otimes \lambda_\alpha$, and $\Omega = \Omega_\alpha \otimes \lambda_\alpha$, then

$$e^{\sum_\alpha \Omega_\alpha \otimes \lambda_\alpha} \in \Omega(P)\otimes S(\mathfrak{g}), \qquad \mathcal{B}_\eta\left(e^{\sum_\alpha \theta_\alpha\otimes\eta_\alpha}\right) \in \Omega(P)\otimes \Lambda(\mathfrak{g}),$$

and we can write

$$\gamma_G = e^{\Omega\otimes\lambda}\mathcal{B}_\eta\left(e^{\theta\otimes\eta}\right).$$

Proposition 7. *The following hold:*

(1) *The linear functional on $\Omega_G(P)$*

$$\alpha \mapsto \int_P \langle \alpha \wedge \gamma_G\rangle$$

(contraction of the polynomial parts of α and γ_G) vanishes on d_G-exact forms, and hence descends to a map $H^*_G(P) \longrightarrow H^*_G(pt)$.

(2) *Let* $\alpha^P \in H^*_G(P)$ *and* $\alpha^M = Hor \circ CW(\alpha^P) \in H^*(M)$. *Then*

$$\int_P \langle \alpha^P \wedge \gamma_G \rangle = \int_M \alpha^M.$$

For a proof of this proposition, see [3]. Note that $\alpha^P \in \Omega(P) \otimes S(\mathfrak{g}^*)$, and $\gamma_G \in \Omega(P) \otimes S(\mathfrak{g})$, so the pairing $\langle \, , \, \rangle$ is the pairing between $S(\mathfrak{g}^*)$ and $S(\mathfrak{g})$. Moreover, if $\alpha^P = \pi^* \beta$, with $\beta \in \Omega^{\mathrm{top}}(M)$, then by a degree count, the only part of γ_G involved is the usual vertical volume element. The pairing between α and γ_G substitutes the curvature in the polynomial part of α. The result is then just multiplied by the vertical volume element.

Applying Proposition 7 to the MQ construction, one obtains:

$$\begin{aligned} \chi(E) &= (2\pi t)^{-m} (2\pi)^{-\dim \mathfrak{g}} \int_P e^{-\frac{t}{2}\|s\|^2} \\ &\quad \cdot \int_{\lambda \in \mathfrak{g}} \int_{\phi \in \mathfrak{g}} e^{i\langle \lambda, \phi \rangle} e^{-i\Omega \otimes \lambda} \mathcal{B}_\chi \left(e^{\frac{t}{2}(-2ids\chi + \chi\phi\chi)} \right). \end{aligned} \tag{2.2}$$

In order to be rigorous, one should introduce a convergence factor $e^{-\epsilon\langle \phi, \phi \rangle}$, and take the limit when $\epsilon \longrightarrow 0$.

More explicitly, let us choose a G-invariant metric on P, which induces a connection θ with curvature Ω, whose horizontal distribution is given by the orthogonal complements to the tangent space to the G-orbits. Now let $C : \mathfrak{g} \longrightarrow TP$ be the infinitesimal action of $\mathfrak{g}$, and $C^* \in \Omega^1(P) \otimes \mathfrak{g}$ its adjoint with respect to the inner products on $\mathfrak{g}$ and TP. Then $\theta = (C^*C)^{-1}C^*$, and

$$\Omega_{\mathrm{hor}} = \mathrm{Hor}\left(d\theta + \frac{1}{2}[\theta, \theta] \right) = (C^*C)^{-1} dC^*.$$

In the formula (2.2), Ω_{hor} can be used instead of Ω, since γ_G is a top degree vertical form. Finally one obtains:

$$\begin{aligned} \chi(E) &= (2\pi t)^{-m} (2\pi)^{-\dim \mathfrak{g}} \int_P e^{-\frac{t}{2}\|s\|^2} \mathcal{B}_\eta \left(e^{(C^*, \eta)} \right) \cdot \\ &\quad \cdot \int_{\lambda, \phi \in \mathfrak{g}} e^{i\langle \phi, C^*C\lambda \rangle} e^{-idC^* \otimes \lambda} \mathcal{B}_\chi \left(e^{-itds\chi + \frac{t}{2}\chi\phi\chi} \right) \end{aligned} \tag{2.3}$$

We set:

$$\Gamma_G = \mathcal{B}_\eta \left(e^{(C^*, \eta)} \right) e^{i\langle \phi, C^*C\lambda \rangle - idC^* \otimes \lambda}.$$

Then (2.3) can be written as follows:

$$\chi(E) = (2\pi t)^{-m} (2\pi)^{-\dim \mathfrak{g}} \int_P e^{-\frac{t}{2}\|s\|^2} \int_{\lambda, \phi \in \mathfrak{g}} \Gamma_G \wedge \mathcal{B}_\chi \left(e^{-itds\chi + \frac{t}{2}\chi\phi\chi} \right) \tag{2.4}$$

Integration over λ yields a delta function in ϕ, centered at $(C^*C)^{-1}dC^* = \Omega_{\mathrm{hor}}$. This means that ϕ can be replaced by Ω_{hor}.

As computed by Constantinescu [7, Sec. 2.4, 2.6], the formula (2.4) can be refined by replacing Γ_G with

$$\Gamma_G(t) = \mathcal{B}_\eta \left(e^{t(C^*, \eta)} \right) e^{it\langle \phi, C^*C\lambda \rangle - itdC^* \otimes \lambda}$$

which introduces the "coupling constant" t. Changing the variables: $\lambda \mapsto t\frac{\lambda}{2}$, $\eta \mapsto t\frac{\eta}{2}$, one obtains:

$$\begin{aligned}\chi(E) &= (2\pi)^{-\dim G}(2\pi t)^{-m}\int_P\int_{\lambda,\phi\in\mathfrak{g}}\mathcal{B}_\eta\mathcal{B}_\chi\left(e^{\frac{t}{2}(-\|s\|^2+(C^*,\eta)+i\langle\phi,C^*C\lambda\rangle-dC^*\otimes\lambda-2ids\chi+\chi\phi\chi)}\right)\\ &= (2\pi)^{-\dim G}\int_{P,\mathfrak{g}}\Gamma_G(t)\wedge U_t(s).\end{aligned}$$

which agrees with the results of [2].

An important aspect, which will be crucial in §4, is that one can write the same refinements for groups G which split as $G_0\times S^1$, with the assumption that G_0 acts freely on P (see [7, Sec 2.6]). Then $H^*_{G_0\times S^1}(P)\simeq H^*_{S^1}(M)$.

Definition 8. The (G_0,S^1)–equivariant vertical volume element

$$\gamma_{G_0,S^1}\in\Omega(P)\otimes S(\mathfrak{g}_0)\otimes\mathbb{C}[[m]],$$

is defined by

$$\gamma_{G_0,S^1}=e^{(\Omega-\mathcal{I})\otimes\lambda}\mathcal{B}_\eta\left(e^{\theta\otimes\eta}\right),$$

where θ is a S^1-equivariant G-connection on P, $\Omega_{S^1}=\Omega-\mathcal{I}$ is the S^1-equivariant curvature, and $\mathcal{I}=m\theta(X)$, where X is the vector field corresponding to a generator m of the Lie algebra $\mathfrak{u}(1)$ of S^1.

Proposition 9. *The following hold:*

(1) *The functional* $\Omega_{G_0,S^1}(P)\longrightarrow\mathbb{C}[[m]]$ *defined by*

$$\alpha\mapsto\int_P\langle\alpha\wedge\gamma_{G_0,S^1}\rangle,$$

descends to $H^*_{G_0,S^1}(P)$.

(2) *If* $\alpha^P\in H^*_{G_0,S^1}(P)$ *and* $\alpha^M\in H^*_{S^1}(M)$ *satisfy* $\alpha^M = Hor\circ CW_H(\alpha^P)$, *then*

$$\int_P\langle\alpha^P\wedge\gamma_{G_0,S^1}\rangle=\int_M\alpha^M.$$

A detailed proof of the proposition above is given in [7, p. 38-39].

Remark 10. We can regard $m\in\mathfrak{u}(1)^*$ as a real number, so we can define $d_{G_0,m}$ on $\Omega_{G_0}(P)^{S^1}$ by:

$$d_{G_0,m}(\omega)=d\omega-m\iota_X\omega.$$

Then $d^2_{G_0,m}=0$, and we denote the corresponding cohomology group by $H^*_{G_0,m}(P)$. In this context, we define the corresponding vertical volume element by:

$$\gamma_{G_0,m}\in\Omega(P)\otimes S(\mathfrak{g}_0).$$

Proposition 11. [7, p. 39] *The functional* $\Omega_{G_0\times S^1}(P)\longrightarrow\mathbb{C}[m]$, *defined by*

$$\alpha\mapsto\int_P\langle\alpha\wedge\gamma_{G_0,m}\rangle,$$

induces a functional integral $\int : H^*_{G_0,m}(P)\longrightarrow\mathbb{C}[m]$.

2.4. **Intersection numbers and localization on the moduli space.** In this section we describe the fundamental formulas which express the relationship between the MQ formalism applied in a TQFT setup and topological invariants of manifolds. We note here that the TQFT frameworks make use of infinite dimensional configuration spaces, therefore one has to formally extend the MQ construction to infinite dimensions.

Let $N \stackrel{i}{\hookrightarrow} M$ be a submanifold of a given finite-dimensional manifold M, and $\alpha \in H^*(M)$. Then

$$\int_N i^*\alpha = \int_M \alpha \wedge \eta,$$

where η is the Poincaré dual of the inclusion of N in M. If $N_1, \ldots, N_m$ are m submanifolds of M which intersect transversally, and the sum of the codimensions of N_i in M equals the dimension of M, then:

$$\int_M \eta_1 \wedge \cdots \wedge \eta_m = \#(N_1 \cap \cdots \cap N_m).$$

For a generic section s, the *localization principle* on the moduli space $\mathcal{Z}(s)$ states the following [8]:

$$\int_M s^*(Th_t(E)) \wedge \mathcal{O} = \int_{\mathcal{Z}(s)} i^*\mathcal{O}, \tag{2.5}$$

where $\mathcal{O}$ is a product of observables (differential forms) of a given TQFT. This follows from a Poincaré duality argument.

Proposition 12 ([8])**.** *The following facts hold:*

(1) $Th_t(E)$ *does not depend on* t.
(2) *If* $t \longrightarrow 0$ *the support of* s^*Th_t *becomes concentrated on* $\mathcal{Z}(s)$.
(3) *The stationary phase approximation of the integral*

$$\int_M s^*(Th_t(E)) \wedge \mathcal{O}$$

is given by an integral over $\mathcal{Z}(s)$ *and a Gaussian integral in the normal directions. By* 1.*, the Gaussian approximation is exact.*

From the equivariant setup of the MQ formalism, the correlation functions of the theory, for observables $\mathcal{O}_i$, $1 \leq i \leq k$, are described by:

$$\begin{aligned} \langle \mathcal{O}_1 \ldots \mathcal{O}_k \rangle &= (2\pi)^{-\dim \mathfrak{g}} (2\pi t)^{-m} \int_P \hat{\mathcal{O}}_1 \wedge \cdots \wedge \hat{\mathcal{O}}_k \wedge e^{-\frac{t}{2}\|s\|^2} \mathcal{B}_\eta \left(e^{(C^*,\eta)} \right) \cdot \\ &\quad \cdot \int_{\lambda,\phi \in \mathfrak{g}} e^{i\langle \phi, C^*C\lambda \rangle - idC^* \otimes \lambda} \mathcal{B}_\chi \left(e^{-itds\chi + \frac{t}{2}\chi\phi\chi)} \right) \\ &= (2\pi)^{-\dim \mathfrak{g}} \int_{P,\mathfrak{g}} \hat{\mathcal{O}}_1 \wedge \cdots \wedge \hat{\mathcal{O}}_k \wedge \Gamma_{G,t} \wedge U_t(s), \end{aligned}$$

where $\hat{\mathcal{O}}_i$ are the images of the regular cohomology classes $\mathcal{O}_i$ on $M = P/G$ in the equivariant cohomology of P, assuming that G acts freely.

The main obstruction to the MQ formalism in infinite dimensions is that $e(E)$ is not well defined. Using the Mathai-Quillen formalism, one defines an analogous Euler class for E. The outcome of the construction is called a *regularized Euler number* for the bundle E [2]. Unfortunately, it depends explicitly on the section chosen for the construction, so it is important to make a good choice. In this context, topological invariants are invariants of numbers like $\chi_s(E)$ under deformations of certain data entering into their calculation. It is in this sense that the Donaldson invariants of four-manifolds, which arise as correlation functions of the TQFT considered in [37], are topological, as they are independent of the metric which enters into the definition of the instanton moduli space. As is well known, they have the remarkable property of distinguishing differentiable structures on four-manifolds.

3. The Mathai–Quillen Formalism in TQFT

The Mathai–Quillen (MQ) formalism can be used to give formal geometric interpretations of the partition functions of certain TQFT (see e.g. [7], [21], [22], [25], [41]). The general idea is that the partition function of TQFT is a path-integral representation of the equivariant Euler number of a certain infinite-dimensional bundle, and the path integrals describing the correlation functions can be understood as representing intersection numbers of equivariant cohomology classes. Below we summarize the MQ formalism for Donaldson–Witten (DW) and Seiberg–Witten (SW) TQFT.

3.1. **Donaldson–Witten Theory via MQ.** The framework for Donaldson-Witten theory is the following. Let X^4 be a simply-connected oriented 4-manifold with a Riemannian metric g. Let E be a rank 2 hermitian vector bundle over X with structure group $SU(2)$ and second Chern class $c_2(E) = k$. We denote by $\mathcal{A}_E$ the affine space of unitary connections on E, and by $\mathcal{G}_E$ the group of unitary gauge transformations of E with Lie algebra $\mathfrak{g}_E = su(E)$, the bundle of traceless skew-Hermitian endomorphisms of E.

The setup for Mathai–Quillen formalism applied to Donaldson–Witten theory is the following. The notation below was explained in §2.

a) $\mathcal{P} = \mathcal{A}_E$.
b) $\mathcal{G} = \mathcal{G}_E$.
c) $\mathcal{V} = \Omega^2_+(\mathfrak{g}_E)$.
d) $\mathcal{E} = \mathcal{A}_E \times_{\mathcal{G}_E} \Omega^2_+(\mathfrak{g}_E)$, an infinite dimensional bundle over $\mathcal{P}/\mathcal{G}$.
e) $s : \mathcal{P} \longrightarrow \mathcal{V}, \quad s(A) = F^+_A$, which descends to a section $s : \mathcal{P}/\mathcal{G} \longrightarrow \mathcal{E}$.

The Mathai-Quillen formalism expresses the $\mathcal{G}$-equivariant Euler number $\chi_{\mathcal{G}}(\mathcal{E}, s)$ in terms of the section s. The formal expression for the equivariant Euler number associated to DW theory is given by:

$$
\begin{aligned}
\chi_{\mathcal{G}_E}(\mathcal{E}, s) &= \text{const} \cdot \int_{\mathcal{A}_E} e^{-\frac{t}{2}\|F^+_A\|^2} \mathcal{B}_\eta \left(e^{-\langle D^*_S\psi, \eta\rangle}\right) \cdot \\
&\quad \cdot \int\int_{\lambda,\phi\in\mathfrak{g}} e^{i\langle\phi, D^*_A D_A\lambda\rangle} \cdot e^{-i[\psi,\psi]_0\lambda} \cdot \mathcal{B}_\chi \left(e^{-it\langle(D_A\psi)^+,\chi\rangle + \frac{t}{2}\chi[\phi,\chi]}\right) \\
&= \text{const} \cdot \int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\eta\mathcal{D}\lambda\mathcal{D}\phi\mathcal{D}\chi \cdot e^{-\int_X \operatorname{Tr}\mathcal{L}_{DW}}, \qquad (3.1)
\end{aligned}
$$

where

$$
\mathcal{L}_{DW} = \frac{t}{2}(F^+_A)^2 + \psi \wedge *D_A\eta - i * \psi D^*_A D_A\lambda + i * [\psi, \psi]_0\lambda + itD_A\psi \wedge \chi - \frac{t}{2}\chi \wedge [\phi, \chi]
$$

is the Lagrangian of the theory. For details see [7, Sec 3.1]. The equivariant Euler number (3.1) is the equivariant partition function $Z_k(m)$ of the theory. One can show that the constant in the above formula is

$$\text{const } = (2\pi)^{-\dim \Omega^0(su(E))/2} \cdot (2\pi)^{-\dim \Omega^2_+(su(E))/2}.$$

It also follows from the MQ formalism that (3.1) can be written as

$$Z_k(m) = \int_{\mathcal{A}_E/\mathcal{G}_E} e_{\mathcal{G}_E}(\mathcal{E}, s) \wedge \Gamma_{\mathcal{G}_E}, \tag{3.2}$$

where $\Gamma_{\mathcal{G}_E}$ is a vertical volume element along the fibers, and $e_{\mathcal{G}_E}(\mathcal{E}, s)$ is the equivariant Euler class.

Remark 13. The MQ formalism for the DW theory is constructed for $b_2^+(X) > 1$. Such a condition on X assures that the localization on the moduli space formula (see §2.4) holds. For $b_2^+(X) = 1$ there are choices involved in the construction of the DW moduli spaces, which are reflected in the path integrals over the configuration spaces $\mathcal{A}_E/\mathcal{G}_E$. The MQ formalism will depend on these choices (e.g. metric chambers) and correction terms need to be added, but one still hopes to obtain similar formulas for the DW partition function. We have not yet worked out the details for this case.

We enumerate below some important facts from Donaldson theory.

Proposition 14. *If the Betti numbers satisfy $b_2 > 0$ and $b_1 = 0$, then for a generic metric on X, there are no reducible anti-self-dual (ASD) connections in $\mathcal{M}_k$, the moduli space of instantons on E, which is a finite dimensional manifold of dimension*

$$\dim \mathcal{M}_k = 8k - 3(1 + b_2^+) = 8k - \frac{3}{2}(\chi + \sigma),$$

where χ and σ are the Euler characteristic and the signature of X, respectively.

Remark 15. The reducible connections satisfy the following:

(1) $A \in \mathcal{A}_E$ is reducible if and only if preserves a splitting

$$E = \lambda_1 \oplus \lambda_2.$$

This is equivalent to either $\mathrm{Stab}(A)/\mathbb{Z}_2 \simeq U(1) \simeq S^1$, where $\mathrm{Stab}(A)$ is the stabilizer of A and $\mathbb{Z}_2$ is the center of $\mathcal{G}_E$, or $\ker d_A$ is 1-dimensional. We denote by $\mathcal{A}_E^*$ the space of irreducible connections. Then $\mathcal{G}_E/\mathbb{Z}_2$ acts freely on $\mathcal{A}_E^*$.

(2) For E an $SU(2)$-bundle, A is reducible if and only if A preserves a splitting $E = l \oplus l^{-1}$, so $\mathfrak{g}_E = \mathbb{R} \oplus l^2$. Moreover we get an action of $\mathrm{Stab}(A)$ on $\mathcal{A}_E$ by the square of the standard action. Note that

$$E = l \oplus l^{-1} \Rightarrow c_2(E) = -c_1(l)^2.$$

(3) There is an elliptic complex

$$0 \longrightarrow \Omega^0(\mathrm{ad}\,\mathcal{E}) \xrightarrow{d_A} \Omega^1(\mathrm{ad}\,\mathcal{E}) \xrightarrow{P_\pm d_A} \Omega^2_\pm(\mathrm{ad}\,\mathcal{E}) \longrightarrow 0,$$

so $\ker d_A = H^0_A$ of the above complex. The elliptic complex is $\mathrm{Stab}(A)$-equivariant, so $\mathrm{Stab}(A)$ acts on H^*_A, where H^*_A is the cohomology of this complex. Moreover, $\mathrm{Stab}(A)$ acts trivially on H^0_A.

(4) The set of irreducible ASD connections modulo the gauge group $\mathcal{G}_E$ is in 1-1 correspondence with the set

$$\{(\pm c, \pm c) \mid 0 \neq c \in H^2(X;\mathbb{Z}), c^2 = -c_2(E)\},$$

with holonomy group S^1 (see [9, Prop 4.2.15]).

The DW polynomial invariants are defined as follows. Let $\Sigma \in H_2(X,\mathbb{Z})$, and define

$$\hat{\mu} = \hat{\mu}^{2,0}(\Sigma) + \hat{\mu}^{0,1}(\Sigma) \in \Omega^2(\mathcal{A}_E) \oplus \left(\Omega^0(\mathcal{A}_E) \otimes S^1(\mathfrak{g}_E^*)\right),$$

where

$$\hat{\mu}^{2,0}(\Sigma)(\psi_1,\psi_2) = \frac{1}{8\pi^2}\int_\Sigma \mathrm{Tr}(\psi_1 \wedge \psi_2), \quad \text{for } \psi_1,\psi_2 \in T_A\mathcal{A}_E,$$

$$\hat{\mu}^{0,1}(\Sigma)(A,\phi) = \frac{1}{4\pi^2}\int_\Sigma \mathrm{Tr}(\phi F_A), \quad \text{for } \phi \in \mathfrak{g}_E, A \in \mathcal{A}_E.$$

For $\nu \in H_4(X,\mathbb{Z})$ define:

$$\hat{\mu}(\nu) \in S^2\left(\mathfrak{g}_E^*\right), \quad \hat{\mu}(\nu) = \frac{1}{8\pi^2}\int_\nu \mathrm{Tr}(\phi^2), \quad \text{for } \phi \in \mathfrak{g}_E.$$

One can show that $\hat{\mu}(\Sigma), \hat{\mu}(\nu)$ are $\mathcal{G}_E$-invariant, and hence define equivariant differential forms in $\Omega^2_{\mathcal{G}}(\mathcal{A}_E)$, and $\Omega^4_{\mathcal{G}}(\mathcal{A}_E)$, respectively. Moreover, $\hat{\mu}(\Sigma), \hat{\mu}(\nu)$ are equivariantly closed, and thus define classes in $H^*_{\mathcal{G}}(\mathcal{A})$.

Remark 16. When restricted to $\mathcal{A}_E^*$, the space of irreducible connections, $\hat{\mu}(\Sigma)$ and $\hat{\mu}(\nu)$ determine classes in $H_{\mathcal{G}}(\mathcal{A}_E^*)$. The action of $\mathcal{G}_E$ on $\mathcal{A}_E^*$ is free, so the Chern–Weil map

$$CW : H_{\mathcal{G}}(\mathcal{A}_E^*) \longrightarrow H(\mathcal{A}_E^*/\mathcal{G}_E)$$

applied to $\hat{\mu}(\Sigma)$ and $\hat{\mu}(\nu)$ yields

$$\mu(\Sigma) \in H^2(\mathcal{A}_E^*/\mathcal{G}_E), \qquad \mu(\nu) \in H^4(\mathcal{A}_E^*/\mathcal{G}_E).$$

Note that $\mu(\nu) = c_2(\mathcal{U})\,|_{\mathcal{A}_E/\mathcal{G}_E}$, and $\mu(\Sigma)$ is the slant product of $[\Sigma]$ and $c_2(\mathcal{U})$, where $\mathcal{U}$ is the universal bundle over $X \times \mathcal{A}_E/\mathcal{G}_E$.

The k-th Donaldson polynominal, $\mathbb{D}_k : \mathbb{R}[\Sigma_1,\ldots,\Sigma_{b_2},\nu]_{(k)} \longrightarrow \mathbb{R}$, where Σ_i have degree 2, ν has degree 4, and (k) denotes polynomials of total degree

$$d(k) = 8k - 3(1 + b_2^+),$$

is defined by:

$$\mathbb{D}_k(\Sigma_1^{\alpha_1},\ldots,\Sigma_{b_2}^{\alpha_{b_2}},\nu^\beta) = \int_{\mathcal{M}_k} \mu(\Sigma_1)^{\alpha_1}\ldots\mu(\Sigma_{b_2})^{\alpha_{b_2}}\mu(\nu)^\beta. \tag{3.3}$$

$\mathbb{D}_k$ is a polynomial of degree $d(k)$ on $H^2 \oplus H^0$. To be rigorous, one has to consider the Uhlenbeck compactification of the moduli spaces $\mathcal{M}_k$, choose suitable representatives for μ's which extend to the compactification, prove the independence of the choices, etc.

The total Donaldson polynomial is defined by

$$\mathbb{D}_X(\Sigma_1^{\alpha_1},\ldots,\Sigma_{b_2}^{\alpha_{b_2}},\nu^\beta) = \sum_{k\in\mathbb{Z}}\int_{\mathcal{M}_k} \mu(\Sigma_1)^{\alpha_1}\ldots\mu(\Sigma_{b_2})^{\alpha_{b_2}}\mu(\nu)^\beta.$$

Remark 17. Strictly speaking, the instanton number k must be positive. Formally, allowing k to have also negative values, comes down to a factor of 2 in the definition of the Donaldson invariant. The difference comes from a normalization argument (see e.g. [38, (2.18)]). From this point on, we omit the factor of 2, and we consider $k \in \mathbb{Z}$. In Section 5 we will add the factor of 2, in order to agree to the usual topological conventions, and we will refer to this remark.

If we introduce formal variables $q_1, \ldots, q_{b_2}, \lambda$, we can write

$$\mathbb{D}_X((q_1\Sigma_1)^{\alpha_1}, \ldots, (q_{b_2}\Sigma_{b_2})^{\alpha_{b_2}}, (\lambda\nu)^{\beta}) = \sum_{k\in\mathbb{Z}} q_1^{\alpha_1} \ldots q_{b_2}^{\alpha_{b_2}} \lambda^{\beta} \cdot \int_{\mathcal{M}_k} \mu(\Sigma_1)^{\alpha_1} \ldots \mu(\Sigma_{b_2})^{\alpha_{b_2}} \mu(\nu)^{\beta}.$$

Using generating function notation [37], we set

$$\mathbb{D}_X\left(e^{\sum q_a\Sigma_a + \lambda\nu}\right) = \sum \frac{\mathbb{D}_X((q_1\Sigma_1)^{\alpha_1}, \ldots, (q_{b_2}\Sigma_{b_2})^{\alpha_{b_2}}, (\lambda\nu)^{\beta})}{\alpha_1! \cdots \alpha_{b_2}! \cdot \beta!}.$$

For simplicity, we will work from now on with observables $\mathcal{O}$ containing only one Σ. Then the generating function can be written in terms of a sum over the dimension $d = d(k)$ of the moduli spaces $\mathcal{M}_k$. By (3.3), we obtain:

$$\mathbb{D}_X\left(e^{\Sigma + \lambda\nu}\right) = \sum_d \sum_{a+2b=2d} \frac{(2\lambda)^b}{a!b!} \int_{\mathcal{M}_k} \mu(\Sigma)^a \mu(\nu)^b. \tag{3.4}$$

If $\mathcal{O}$ is product of μ classes, the MQ formalism provides the following expression for the total equivariant Donaldson invariant:

$$\mathbb{D}_X(\mathcal{O})(m) = \text{ const } \cdot \sum_{k\in\mathbb{Z}} \int_{\mathcal{A}_k} \int_{\mathfrak{g}_k} \Gamma_{\mathcal{G}_k} \wedge U_{\mathcal{G}_k,t}(s) \wedge \mathcal{O},$$

where for simplicity of notation, we denote $\mathcal{A}_{E_k}$ by $\mathcal{A}_k$, $\mathcal{G}_{E_k}$ by $\mathcal{G}_k$, etc. More concretely:

$$\mathbb{D}_X(\Sigma^{\alpha}, \nu^{\beta})(m) = \text{ const } \cdot \sum_{k\in\mathbb{Z}} \int_{\mathcal{A}_k, \mathfrak{g}_k} \mathcal{D}(\text{fields}) \wedge \hat{\mu}(\Sigma)^{\alpha} \wedge \hat{\mu}(\nu)^{\beta} \cdot e^{-\int_X \operatorname{Tr} \mathcal{L}_{DW}}.$$

Summarizing, one obtains the formal equality of generating series:

$$\begin{aligned}\mathbb{D}_X(e^{q\Sigma + p\nu})(m) \quad &= \quad \text{const } \cdot \sum_{k\in\mathbb{Z}} \int_{\mathcal{A}_k, \mathfrak{g}_k} \mathcal{D}(\text{fields}) \cdot \\ &\cdot e^{-\int_X \operatorname{Tr} \mathcal{L}_D W + \frac{p}{8\pi^2}\int_X \operatorname{Tr}\phi^2 + \frac{q}{8\pi^2}\int_\Sigma \operatorname{Tr}(\psi\wedge\psi + 2\phi F_A)}.\end{aligned}$$

The Donaldson simple type condition is defined as follows:

Definition 18. For $b_2^+ \geq 2$ and odd, X is said to be of *D-simple type* if for every $z \in A(X) = S^*(H_0(X) \oplus H_2(X))$,

$$\mathbb{D}_X(u^2 z) = 4\mathbb{D}_X(z).$$

Remark 19. The condition of D-simple type above can be reformulated in the form that $\mathbb{D}_X$ annihilates the ideal in $A(X)$ generated by $u^2 - 4$. Equivalently,

$$\mathbb{D}_X(e^{\lambda u} z) = e^{2\lambda}\mathbb{D}_X\left(\left(1 + \frac{u}{2}\right) z\right) + e^{-2\lambda}\mathbb{D}_X\left(\left(1 - \frac{u}{2}\right) z\right),$$

for $z \in A(X)$ and $\lambda \in \mathbb{Z}$.

Assuming the D-simple type condition, the generating series can defined as in [17]:

$$\mathbb{D}_X(\exp(\mathcal{O})) = \sum_{d'} \frac{q_{d'}}{(d')!},$$

where the Donaldson polynomials of degree $d' = d'(k)$ are defined by

$$q_{d'-2} = \langle \mu(\Sigma)^{d'-2}\nu, [\mathcal{M}_{d'}]\rangle.$$

Here d' is considered as

$$d' \equiv \frac{1}{4}(\chi + \sigma) \mod 2,$$

rather than mod 4. Equivalently, one can consider $k \in \frac{1}{2}\mathbb{Z}$.

3.2. **Seiberg-Witten Theory via MQ.** The framework is the following. Let X^4 be a closed oriented manifold with Riemannian metric g, equipped with a spin^c-structure c. Let $W^{\pm}$ be the corresponding rank 2 hermitian vector bundles (the spin bundles), and L the hermitian line bundle over X with $c_1(L) = c$, and $\det(W^+) = L$. Let $\rho : \Lambda^2_+(X) \longrightarrow su(W^+)$ be a bundle isomorphism, and $\eta \in \Omega^2_+(X)$ a perturbation. We denote by $\mathcal{A}_L$ the space of unitary connections on L.

The Seiberg-Witten (SW) equations for a connection $A \in \mathcal{A}_L$ and a spinor $\psi \in \Gamma(W^+)$ are:

$$\begin{aligned} \not{D}_A\psi &= 0 \\ F_A^+ + i(\psi \otimes \bar{\psi})_{oo} &= 0, \end{aligned}$$

where $\not{D}_A$ is the Dirac operator associated to the spin^c-structure and the connection A, $i(\psi \otimes \bar{\psi})_{oo} \in \Gamma(su(W^+))$ is the traceless part of $\psi \otimes \bar{\psi}$, and $su(W^+)$ and $\Lambda^2_+(X)$ are isomorphic as Clifford algebras, via ρ. One can consider a perturbation of the second equation with the form η.

The MQ formalism for the SW theory has the following setup:

a) $\mathcal{P} = \mathcal{A}_L \times \Gamma(W^+)$.
b) $\mathcal{V} = \Omega^2_+ \oplus \Gamma(W^-) =: \mathcal{V}_1 \oplus \mathcal{V}_2$.
c) $\mathcal{G}_L = \mathrm{Aut}(L) = \mathrm{Map}(X, U(1))$.
d) $\mathcal{E} = \mathcal{P} \times_{\mathcal{G}} \mathcal{V}$, and $\mathcal{E}_1 = \mathcal{P} \times_{\mathcal{G}} \mathcal{V}_1$, $\mathcal{E}_2 = \mathcal{P} \times_{\mathcal{G}} \mathcal{V}_2$.
e) $s : \mathcal{P} \longrightarrow \mathcal{V}$, $\quad s(A,\psi) = (F_A^+ + i(\psi \otimes \bar{\psi}), \not{D}_A\psi) =: (s_1, s_2)$.

Here $\mathcal{G}_L$ acts by the usual multiplication by $e^{i\theta}$ in the fibers of W^+, and $\mathcal{G}_L$ acts on $\mathcal{A}_L$ through the gauge transformation $e^{2i\theta}$, i.e.

$$g \cdot (A, \psi) = (A - 2g^{-1}dg, g\psi).$$

We enumerate below some important aspects of SW theory. The moduli space $\mathcal{M}_c$ is the space of solutions of the SW equations modulo the gauge group $\mathcal{G}_L$, and it is proven to be a finite dimensional, compact, oriented, smooth manifold for a generic metric and regular perturbation, of dimension

$$\dim \mathcal{M}_c = \frac{c^2 - 2\chi - 3\sigma}{4} \tag{3.5}$$

The rolled-up SW elliptic complex is

$$0 \longrightarrow \Lambda^1 \oplus \Gamma(X, W^+ \otimes L) \overset{\not{D}_A + d^+ + d^*}{\longrightarrow} \Lambda^0 \oplus \Lambda^{2+} \oplus \Gamma(X, W^- \otimes L) \longrightarrow 0.$$

The SW equations are gauge invariant, and so s is $\mathcal{G}_L$-equivariant. The action of $\mathcal{G}_L$ is not free. If $\psi \neq 0$ the stabilizer of (A, ψ) is trivial, and if $\psi = 0$ the stabilizer of $(A, 0)$ is S^1. For $x_0 \in X$ we set

$$\mathcal{G}_L^0 = \{g \in \mathcal{G}_L \mid g \cdot x_0 = 1\}.$$

Then $\mathcal{G}_L = \mathcal{G}_L^0 \times S^1$ and $\mathcal{G}_L^0$ acts freely on $\mathcal{A}_L \times \Gamma(W^+)$.

The SW invariant $SW(c)$ for $\dim \mathcal{M}_c = 0$ is defined to be the sum of ± 1 over the points in the moduli space, according to their orientation. For $\dim \mathcal{M}_c > 0$, the SW invariant is defined as follows. Consider the principal S^1 bundle $\mathcal{M}_{\mathcal{G}_L^0} \longrightarrow \mathcal{M}_c$ given by

$$\mathcal{M}_{\mathcal{G}_L^0} = s^{-1}(0)/\mathcal{G}_L^0 \subset \mathcal{P}/\mathcal{G}_L^0,$$

for a generic metric, and perturbation. Equivalently, we can consider the associated universal line bundle

$$\mathcal{L} \longrightarrow X \times P$$

restricted to $x_0 \times P$. Then [31]

$$e(\mathcal{M}_{\mathcal{G}_L^0} \longrightarrow \mathcal{M}_c) = c_1(\mathcal{L}\big|_{x_0}).$$

If $\dim \mathcal{M}_c = 2s(c)$, the SW invariant is defined to be

$$SW(c) = \int_{\mathcal{M}_c} c_1(\mathcal{L}\big|_{x_0})^s$$

The computation of the quantities in the MQ construction for the unperturbed equations can found in [7, Sec. 3.2]. One can write:

$$\begin{aligned} U_t(s) \quad = \quad \text{const} \cdot \int \mathcal{D}\chi \mathcal{D}T e^{t\left(\int_X -\frac{1}{2}(F_A^+)^2 - F_A^+ \wedge i(\psi\otimes\bar{\psi})_0 - \frac{1}{2}\|i(\psi\otimes\bar{\psi})_0\|^2 - \frac{1}{2}*\|\not{D}_A\psi\|^2\right)} \cdot \\ e^{t\left(\int_X -i\left(d^+\alpha + i(\psi\otimes\bar{\sigma} + \sigma\otimes\bar{\psi})_0\right)\chi - i\langle \not{D}_A\sigma + \frac{1}{2}\mathrm{cl}(\alpha)\psi, T\rangle + \frac{1}{2}\langle T, \phi T\rangle\right)} \end{aligned}$$

for all $t > 0$. The constant is given by

$$\text{const} \ = (2\pi)^{-\dim \Omega^0/2}(2\pi t)^{-\dim(\Omega_+^2\otimes\Gamma(W^-))/2}.$$

Finally, the equivariant Euler number (the SW partition function) of the theory is given by the following formula:

$$Z_c(m) = \ \text{const} \ \cdot \int_{\mathcal{A}_L\times\Gamma(W^+)\times \mathrm{Map}(X,u(1))} \mathcal{D}A\mathcal{D}\psi\mathcal{D}\phi \ \Gamma_{\mathcal{G}_L} \wedge U_t(s),$$

where $\Gamma_{\mathcal{G}_L}$ is a vertical volume element along the fibers. By integrating over the fiber, one gets the equivalent expression

$$SW(c, m) = \int_{\mathcal{P}/\mathcal{G}} e_{\mathcal{G}_L}(\mathcal{E}, s) \wedge \Gamma_{\mathcal{G}_L}.$$

Remark 20. For $c^2 = 2\chi + 3\sigma$, the partition function gives a formal expression for the equivariant SW invariant $SW(c, m)$. For $\dim \mathcal{M}_c = 2s(c) > 0$, we can define the equivariant SW invariants in terms of the equivariant first Chern class of the universal bundle $\mathcal{L}\big|_{x_0}$:

$$SW(c, m) = \int_{\mathcal{P}/\mathcal{G}} e_{\mathcal{G}}(\mathcal{E}, s) \wedge \Gamma_{\mathcal{G}_L} \wedge c_{1,\mathcal{G}_L}(\mathcal{L}\big|_{x_0})^s.$$

4. Equivariant Localization

We recall below the abelian localization theorem, in the case of an S^1 action on a given manifold, and then we formally apply an infinite version of it to the DW and SW theories. We note here that there are more general versions of localization formulas for non-abelian group actions (see e.g. [5],[15]).

The setup is the following. Let M be a compact oriented manifold equipped with an S^1 action and a Riemannian metric g invariant under the action. Let $\alpha \in \Omega_{S^1}(M)$ be an equivariantly closed form, and m a generator of $\mathfrak{u}(1)^*$ which induces the vector field X on M. Denote by M_0 the fixed point set of the S^1 action (the zero set of X). The abelian localization theorem [5, Theorem 7.13] asserts:

Theorem 21. *M_0 is a submanifold of M whose normal bundle ν_{M_0} is orientable and even dimensional. Moreover,*

$$\int_M \alpha = \int_{M_0} \frac{\alpha\big|_{M_0}}{e_{S^1}(\nu_{M_0})}. \tag{4.1}$$

where $\alpha\big|_{M_0} = i^*\alpha$ for the inclusion map $i : M_0 \hookrightarrow M$.

On the left hand side of (4.1), $\alpha \in \mathbb{C}[m] \otimes \Omega(M)^{S^1}$, so

$$\int_M \alpha \in \mathbb{C}[m].$$

On the right hand side, we have:

$$e_{S^1}(\nu_{M_0}) = m^{\frac{k}{2}} e^0 + m^{\frac{k}{2}-1} e^2 + \cdots + e^k \in H^{k/2}_{S^1}(M_0),$$

where $k = \mathrm{rk}(\nu_{M_0})$ and e^i are elements of $H^i(M_0)$. Then

$$e_{S^1}(\nu_{M_0})^{-1} = \frac{1}{m^{\frac{k}{2}} e^0} \left(1 + \sum_{k \geq 1} \left(\frac{1}{e^0} (m^{-1} e^2 + \cdots + m^{-\frac{k}{2}} e^k) \right)^k \right)$$

is a well defined homogenous element of degree $-k/2$ in

$$\Omega_{S^1}(\nu_{M_0})_m = \mathbb{C}[m, m^{-1}] \otimes \Omega(M_0)^{S^1}.$$

Here $\deg m = 2$ and $\alpha\ |_{M_0} \cdot e_{S^1}(\nu_{M_0})^{-1}$ is a homogenous element of degree $(\deg \alpha - k)/2$. Note that $\int_{M_0}$ picks up those terms whose *usual* degree as a form is $\dim M_0$. The result is a multiple of $m^{(\deg \alpha - k)/2}$.

It is important to remark here that we can regard m as a real number (see Remark 10). In order to keep track of the degree 2 of m, we must think of m as actually being a square of a real number.

4.1. **Abelian Localization of Donaldson–Witten theory.**

The equivariant DW partition function $Z_k(m)$ for $c_2(E) = k$ can be written as an integral over the configuration space:

$$Z_k(m) = \int_{\mathcal{A}_E/\mathcal{G}_E} e_{\mathcal{G}_E}(\mathcal{E}, s) \wedge \Gamma_{\mathcal{G}_E} \in \mathbb{C}[m]. \tag{4.2}$$

Consider the based gauge group $\mathcal{G}_0 = \{g \in \mathcal{G}_E | gx = \mathrm{id}\}$, for fixed $x \in X$. We have the following exact sequence:

$$1 \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G}_E \longrightarrow SU(2) \longrightarrow 1.$$

Using the action of a maximal torus S^1 of $SU(2)$ we can rewrite (4.2) in a form more suitable for abelian localization. At this moment we assume that (4.1) holds in infinite dimensions, as well. We get:

$$Z_k(m) = \int_{\mathcal{A}_E/\mathcal{G}_E} e_{S^1}(\mathcal{E}, s) \wedge \Gamma_{S^1} = \int_{M_0/\mathcal{G}_E} e_{S^1}(\mathcal{E}, s)\big|_{M_0} \wedge \Gamma_{S^1}\big|_{M_0} \wedge e_{S^1}(\nu_{M_0})^{-1}, \quad (4.3)$$

where we first apply (4.1) and then mod out by $\mathcal{G}_E$. Note that S^1 acts on $\Omega^1(\mathfrak{g}_E)$ by the square of the standard action, and $\mathcal{G}_E/\mathbb{Z}_2$ acts freely on $\mathcal{A}^*_E$. The fixed point set for the S^1 action is given by the *reducible* connections. For $E = l \oplus l^{-1}$ we have

$$\mathfrak{g}_E = \mathbb{R} \oplus l^2 \quad \text{and} \quad d_{A_E} = d \oplus d_{A_{l^2}}. \quad (4.4)$$

The original DW elliptic complex is

$$0 \longrightarrow \Omega^0(\mathfrak{g}_E) \xrightarrow{d_{A_E}} \Omega^1(\mathfrak{g}_E) \xrightarrow{d^+_{A_E}} \Omega^{2+}(\mathfrak{g}_E) \longrightarrow 0,$$

and by (4.4) the elliptic complex splits in two elliptic complexes (see [13, p. 70]):

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^0(l^2) & \xrightarrow{d_A} & \Omega^1(l^2) & \xrightarrow{d^+_A} & \Omega^{2+}(l^2) & \longrightarrow & 0 \\
 & & \oplus\big\downarrow & & \oplus\big\downarrow & & \oplus\big\downarrow & & \\
0 & \longrightarrow & \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d^+} & \Omega^{2+} & \longrightarrow & 0
\end{array}$$

Let $\mathcal{A}_l$ be the space of connections on l. Then the fixed point set of the S^1-action, is

$$M_0 = \bigcup_{E=l\oplus l^{-1}} \mathcal{A}_{l^2} = \bigcup_{\substack{y\in H^2(X;\mathbb{Z})\\ y^2=-k}} \left(\bigcup_{c_1(l)=2y} \mathcal{A}_l \right),$$

where we use (4.4) to identify a reducible connection on $\mathfrak{g}_E$ with a connection on l^2, and hence on l. The last equality uses the fact that X is simply connected, and hence $H^2(X, \mathbb{Z})$ has no 2-torsion. Now we define

$$\mathcal{R}_x = \bigcup_{c_1(l)=x} \mathcal{A}_l.$$

The spaces $\mathcal{R}_x$ are $\mathcal{G}_E$-invariant. Moreover $\Gamma_{S^1}\big|_{M_0} = 1$ [7, p. 67]. Then (4.3) becomes

$$\begin{aligned}
Z_k(m) &= \int_{M_0/\mathcal{G}_E} e_{S^1}(\mathcal{E}, s) \wedge e_{S^1}(\nu_{M_0 \subset \mathcal{A}_E/\mathcal{G}_E})^{-1} \\
&= \sum_{\substack{x=2y\\ y^2=-k}} \int_{\mathcal{R}_x/\mathcal{G}_E} e_{S^1}(\mathcal{E}, s) \wedge e_{S^1}(\nu_{M_0 \subset \mathcal{A}_E/\mathcal{G}_E})^{-1}.
\end{aligned}$$

From this point on we will write the sum in the formula above as a sum over x, with $x^2 = -4k$. The diffeomorphism

$$i : \mathcal{A}_{l_x}/\mathcal{G}_{l_x} \approx \mathcal{R}_x/\mathcal{G}_E$$

yields the following expression for the partition function:

$$Z_k(m) = \sum_{x^2=-4k} \int_{\mathcal{A}_{l_x}/\mathcal{G}_{l_x}} i^* \left(e_{S^1}(\mathcal{E}, s) \wedge e_{S^1}(\nu_{M_0})^{-1} \right). \tag{4.5}$$

Let $\nu_{\mathcal{R}_x}$ be the normal bundle of $\nu_{\mathcal{R}_x}$ in $\mathcal{A}_E$. At a point $A \in \mathcal{A}_E$, the fiber of $\nu_{\mathcal{R}_x}$ is isomorphic to

$$\{a \in \Omega^1(l^2) | d_A^* a = 0\}.$$

Here $\nu_{\mathcal{A}_l} \subset \mathcal{A}_E$, and because $\mathfrak{g}_E \simeq \mathbb{R} \oplus l^2$, we have $\nu_{\mathcal{A}_l} \simeq \Omega^1(l^2)$. The structure group of E and ν_{M_0} is $\mathcal{G}_E$. When pulled back by i^* to $\mathcal{A}_{l_x}/\mathcal{G}_{l_x}$, their structure group reduces to $\mathcal{G}_{l_x}$. Then, for every $l = l_x$, we get:

$$\begin{aligned} i^*\mathcal{E} &= \mathcal{A}_l \times_{\mathcal{G}_l} \Omega^2_+(\mathfrak{g}_E) = \mathcal{A}_l \times_{\mathcal{G}_l} \left(\Omega^2_+ \oplus (\Omega^2_+ \otimes l^2) \right) \\ i^*(\nu_{M_0}) &= \mathcal{A}_l \times_{\mathcal{G}_l} \nu_{\mathcal{A}_{l_x}} = \mathcal{A}_l \times_{\mathcal{G}_l} \{a \in \Omega^1(l^2) | d_A^* a = 0\} = \mathcal{A}_l \times_{\mathcal{G}_l} \Omega^1(l^2)/d_A\Omega^0(l^2). \end{aligned}$$

We decompose $i^*\mathcal{E}$ as follows:

$$i^*\mathcal{E} = \left(\mathcal{A}_l \times_{\mathcal{G}_l} \Omega^2_+ \right) \oplus \left(\mathcal{A}_l \times_{\mathcal{G}_l} (\Omega^2_+ \otimes l^2) \right) =: \mathcal{E}_1 \oplus \mathcal{E}_2$$

For $\mathcal{G}_l^0 \subset \mathcal{G}_l$ the based group, we have

$$1 \longrightarrow \mathcal{G}_l^0 \longrightarrow \mathcal{G}_l \longrightarrow S^1 \longrightarrow 1,$$

and because S^1 acts trivially on $\mathcal{A}_l/\mathcal{G}_l$, we get the isomorphism

$$\mathcal{A}_l/\mathcal{G}_l \simeq \mathcal{A}_l/\mathcal{G}_l^0.$$

Moreover, S^1 acts on $\mathcal{E}_1$ trivially, and S^1 acts on $i^*\nu$ with weight 2. Then

$$e_{S^1}(\mathcal{E}_1, s) = e(\mathcal{E}_1, s),$$

so (4.5) becomes

$$Z_k(m) = \sum_{x^2=-4k} \int_{\mathcal{A}_{l_x}/\mathcal{G}_{l_x}} e(\mathcal{E}_1, s) \wedge e_{S^1}(\mathcal{E}_2, s) \wedge e_{S^1}(i^*\nu_{M_0})^{-1}.$$

More precisely,

$$\begin{aligned} Z_k(m) = \sum_{x^2=-4k} \int_{\mathcal{A}_{l_x}/\mathcal{G}_{l_x}} & e(\mathcal{A}_l \times_{\mathcal{G}_l} \Omega^2_+, s) \wedge e_{S^1}(\mathcal{A}_l \times_{\mathcal{G}_l} (\Omega^2_+ \otimes l^2), s) \\ & \wedge e_{S^1}(\mathcal{A}_l \times_{\mathcal{G}_l} \Omega^1(l^2)/d_A\Omega^0(l^2))^{-1}. \end{aligned} \tag{4.6}$$

Since the kernel and cokernel of the operator

$$d_A^* + d_A^+ : \ker d_A^* \longrightarrow \Omega^{2+}(l^2)$$

are isomorphic to the kernel and cokernel of the elliptic operator $d_A^* + d_A^+$ defined on $\Omega^1(l^2)$, we can compute the quotient of Euler classes in (4.6) in terms of the total Chern class of (minus) the index bundle of $d_A^* + d_A^+$. For a similar argument see [7, Prop 4.4]. Assuming that this finite dimensional formula holds in infinite dimensions, we obtain:

$$\frac{e_{S^1}(\mathcal{A}_l \times_{\mathcal{G}_l} (\Omega^2_+ \otimes l^2), s)}{e_{S^1}(\mathcal{A}_l \times_{\mathcal{G}_l} \Omega^1(l^2)/d_A\Omega^0(l^2))} = \left(-\frac{4m}{2\pi}\right)^{-\mathrm{Ind}(d_A^+ + d_A^*)} c_{\mathrm{tot}}\left(-\mathrm{ind}(d_A^+ + d_A^*)\right)\left(-\frac{2\pi}{4m}\right),$$

where m is the generator of $\mathfrak{u}(1)^*$, Ind denotes the integer index, and the factor of 4 comes from the S^1 action of L^2. The splitting of the elliptic complex into two elliptic complexes shows that

$$\mathrm{Ind}(d_A^+ + d_A^*) = -\mathrm{Ind}(d^+ + d^*) + 2d(k),$$

where $2d(k) = \dim \mathcal{M}_k$. When $2d(k) = 0$, we can check that the (real) indexes agree:

$$\mathrm{Ind}(d_A^+ + d_A^*) = 8k - (\chi + \sigma) = \frac{1}{2}(\chi + \sigma) = -\mathrm{Ind}(d^+ + d^*),$$

since $0 = 2d(k) = 8k - \frac{3}{2}(\chi + \sigma)$.

Remark 22. It important to notice here the following formal aspect. The degrees of the forms inside the integrals over the infinite dimensional spaces $\mathcal{A}_{l_x}/\mathcal{G}_{l_x}$ are considered to sum to the (finite) dimension of the respective Donaldson moduli spaces. Therefore, one can consider the *formal dimension* of the integrals in question to equal the dimension of the corresponding moduli spaces. For this reason we only pick the degree zero component of the total Chern class above, denoted by $\mathbf{1}$.

Using the remark above, we obtain:

$$Z_k(m) = 2^{-(\chi+\sigma)} \sum_{x^2=-4k} \int_{\mathcal{A}_{l_x}/\mathcal{G}_{l_x}} e(\mathcal{A}_l \times_{\mathcal{G}_l} \Omega^2_+, s) \wedge \left(\frac{m}{2\pi}\right)^{-\frac{\chi+\sigma}{2}} \cdot \mathbf{1} \tag{4.7}$$

where we notice that -1 is raised to an even power, so it equals 1.

The localization of (the equivariant version of) the observables of the theory is given by:

$$\mu(\Sigma)\big|_{M_0} = -\langle c_1(l), \Sigma\rangle \cdot h,$$

where h is the two-dimensional generator of the integer cohomology of BS^1 (see [9, p. 187]). But we have seen that $H^*(BS^1) \cong H^*_{S^1}(pt) = \mathbb{C}[m]$. Therefore, h can be identified with $\left(\frac{m}{2\pi}\right)^2 \cdot \mathbf{1}$. This gives

$$\mu(\Sigma)\big|_{M_0} = -\langle c_1(l), \Sigma\rangle \cdot \left(\frac{m}{2\pi}\right)^2 \cdot \mathbf{1}. \tag{4.8}$$

Moreover, it is not hard to show that the restriction of $\mu(\nu)$ to the fixed point set is given by (minus) the generator of the four-cohomology of BS^1, thus

$$\mu(\nu)\big|_{M_0} = -\left(\frac{m}{2\pi}\right)^4 \cdot \mathbf{1}, \tag{4.9}$$

Recall that the equivariant k-th Donaldson polynomial for $\mathcal{O} = \Sigma^a \wedge \nu^b$ is defined as follows:

$$\mathbb{D}_k(\mathcal{O})(m) = \int_{\mathcal{A}_E/\mathcal{G}_E} e_{\mathcal{G}_E}(\mathcal{E}, s) \wedge \Gamma_{\mathcal{G}_E} \wedge \mu(\Sigma)^a \wedge \nu^b, \tag{4.10}$$

where $2d = 2d(k) = a + 2b$. Now we compute the localization formula for the corellation functions of DW theory, applying (4.8) and (4.9), obtaining a formula for the equivariant Donaldson polynomials in $\mathbb{C}[m]$. The dependence of m will denoted in parentheses, as in the partition function case.

$$\begin{aligned}\mathbb{D}_k(\mathcal{O})(m) = \sum_{x^2=-4k} \int_{\mathcal{A}_{l_x}/\mathcal{G}_{l_x}} e(\mathcal{A}_{l_x} \times_{\mathcal{G}_{l_x}} \Omega^2_+, F^+_A) \wedge \left(\frac{4m}{2\pi}\right)^{-\frac{\chi+\sigma}{2}-2d} \cdot \mathbf{1} \\ \wedge(-1)^a \langle x, \Sigma\rangle^a \cdot \left(\frac{m}{2\pi}\right)^{2a} \mathbf{1} \wedge (-1)^{2b} \left(\frac{m}{2\pi}\right)^{4b} \cdot \mathbf{1} \\ = \sum_{x^2=-4k} 2^{-(\chi+\sigma)-4d} \langle x, \Sigma\rangle^a \int_{\mathcal{A}_{l_x}/\mathcal{G}_{l_x}} e(\mathcal{A}_{l_x} \times_{\mathcal{G}_{l_x}} \Omega^2_+, F^+_A) \wedge \left(\frac{m}{2\pi}\right)^{-\frac{\chi+\sigma}{2}+2d} \cdot \mathbf{1}\end{aligned} \tag{4.11}$$

Remark 23. If we heuristically compare (4.11) with (4.7), we can formally write the k-th (equivariant) Donaldson polynomial $\mathbb{D}_k(\mathcal{O})(m)$ in terms of the partition function $Z_k(m)$, which represents the zero dimensional Donaldson invariant q_0. If we formally replace (modulo 2π) m by $m^{2d(k)}$, then we can write the equivariant Donaldson–Witten generating function (3.4) as follows:

$$\mathbb{D}_X(e^{\Sigma+\lambda\nu})(m^{2d(k)}) = \sum_k e^{\langle x,\Sigma\rangle+2\lambda} Z_k(m). \tag{4.12}$$

This formula is similar to the one obtained by Witten in [38, (2.36)]. A more detailed version of (4.12) is the following:

$$\mathbb{D}_X(e^{\Sigma+\lambda\nu})(m) = \sum_k 2^{-4d(k)} e^{\left(\frac{m}{2\pi}\right)\langle x,\Sigma\rangle} e^{\left(\frac{m}{2\pi}\right)^2 \cdot 2\lambda} Z_k(m).$$

4.2. **Abelian Localization of Seiberg–Witten theory.** Recall that the equivariant SW partition function is given by

$$Z_c(m) = \int_{\mathcal{A}_L \times_{\mathcal{G}_L} \Gamma(W^+)} e_{\mathcal{G}}(\mathcal{E}_1, s_1) \wedge e_{\mathcal{G}}(\mathcal{E}_2, s_2) \wedge \Gamma_{\mathcal{G}},$$

where

$$\begin{aligned}\mathcal{E}_1 &= \big(\mathcal{A}_L \times \Gamma(W^+)\big) \times_{\mathcal{G}_L} \Omega^2_+(X), \\ \mathcal{E}_2 &= \big(\mathcal{A}_L \times \Gamma(W^+)\big) \times_{\mathcal{G}_L} \Gamma(W^-).\end{aligned}$$

S^1 acts on $\mathcal{A}_L \times \Gamma(W^+)$ by scalar multiplication on the spinors $\Gamma(W^+)$, and we can consider the based gauge group $\mathcal{G}^0_L$ in the split exact sequence

$$1 \longrightarrow \mathcal{G}^0_L \longrightarrow \mathcal{G}_L \longrightarrow S^1 \longrightarrow 1,$$

with $\mathcal{G}^0_L$ acting freely on $\mathcal{A}_L \times \Gamma(W^+)$, and S^1 acting on $\mathcal{A}_L \times \Gamma(W^+)/\mathcal{G}^0_L$. The fixed point set of the S^1 action is:

$$M_0 = \mathcal{A}_L/\mathcal{G}^0_L.$$

As in §4.1, we write the partition function in terms of S^1-equivariant Euler classes and then formally apply the abelian localization theorem. Note that $\Gamma_{S^1}\big|_{M_0} = 1$ as before. We get

$$SW(c,m) = \int_{\mathcal{A}_L/\mathcal{G}_L^0\times 0} e(\mathcal{E}_1, s_1) \wedge \frac{e_{S^1}(\mathcal{E}_2, s_2)}{e_{S^1}(\nu_{\mathcal{A}_L/\mathcal{G}_L^0\subset \mathcal{A}_L\times_{\mathcal{G}_L^0}\Gamma(W^+)}, 0)},$$

where the Euler classes are restricted to $\mathcal{A}_L/\mathcal{G}_L^0$. Moreover notice that

$$\begin{aligned} e_{S^1}(\mathcal{E}_1, s_1)\big|_{\mathcal{A}_L/\mathcal{G}_L^0\times 0} &= e(\mathcal{E}_1, F_A^+)\big|_{\mathcal{A}_L/\mathcal{G}_L^0}, \\ e_{S^1}(\mathcal{E}_2, s_2)\big|_{\mathcal{A}_L/\mathcal{G}_L^0\times 0} &= e_{S^1}(\mathcal{E}_2, 0)\big|_{\mathcal{A}_L/\mathcal{G}_L^0}. \end{aligned}$$

We compute the following quotient of Euler classes as in the previous section (see also [7, Prop. 69]). We have:

$$\frac{e_{S^1}(\mathcal{E}_2, 0)}{e_{S^1}(\nu, 0)} = \left(\frac{m}{2\pi}\right)^{-\mathrm{Ind}(\not{D}_A)} c_{\mathrm{tot}}(-\mathrm{ind}(\not{D}_A))\left(\frac{2\pi}{m}\right).$$

Here

$$\mathrm{Ind}(\not{D}_A) = -\mathrm{Ind}(d^+ + d^*) + 2s(c), \tag{4.13}$$

where $2s(c) = \dim \mathcal{M}_c$. In the partition function case we have $2s(c) = 0$, thus,

$$\mathrm{Ind}(\not{D}_A) = -\mathrm{Ind}(d^+ + d^*) = \frac{\chi+\sigma}{2}. \tag{4.14}$$

Then the expression for the equivariant SW partition function becomes

$$Z_c(m) = \int_{\mathcal{A}_L/\mathcal{G}_L} e(\mathcal{A}_L \times_{\mathcal{G}_L} \Omega^2_+, F_A^+) \wedge \left(\frac{m}{2\pi}\right)^{-\frac{\chi+\sigma}{2}} \cdot \mathbf{1}, \tag{4.15}$$

from the same reasons of dimensionality as in the Remark 22.

Remark 24. Note that, using physics arguments, one has to introduce a massive term in the MQ expression of the SW integral, which corresponds to twisting the $N = 2$ SUSY Yang-Mills theory coupled with a massive hypermultiplet. For details, one can see e.g. [16], [39]. This comes down to a perturbation of the SW equations, equivalently one can consider spin^c–structures $\tilde{c} = c + 2l$, where $l^2 < 0$ and

$$c \cdot l = -\mathrm{Ind}\, \not{D}_A + \frac{\chi+\sigma}{2} = -2s(c).$$

Furthermore, if one chooses a basis $\Sigma_1, \ldots, \Sigma_{b_2}$ of $H_2(X)$, and then write l in this basis, then there is a constant α such that

$$(c + 2l)\cdot \Sigma_i = c\cdot \Sigma_i + 2\alpha\Sigma_i^2.$$

In the case when X has simple type, the SW invariant for the perturbed spin^c–structure will equal the unperturbed invariant, or it will be zero. For details, one can see [33]. This remark will be very useful in §5.

Recall that the observables of SW theory are obtained by wedging with the first Chern class of the line bundle $\mathcal{L}\big|_{pt}$. Using the same arguments for the localization of $\mu(\nu)$ in the Donaldson theory, we get:

$$c_{1,\mathcal{G}_L}(\mathcal{L}\big|_{pt})\big|_{M_0} = -\left(\frac{m}{2\pi}\right)^4 \cdot \mathbf{1} \tag{4.16}$$

Then the formula for the equivariant SW invariant is:

$$
\begin{aligned}
SW(c,m) &= \int_{\mathcal{A}_L/\mathcal{G}_L} e(\mathcal{A}_L \times_{\mathcal{G}_L} \Omega^2_+, F_A^+) \wedge \left(\frac{m}{2\pi}\right)^{-\frac{\chi+\sigma}{2}-2s} \cdot \mathbf{1} \wedge (-1)^{2s} \left(\frac{m}{2\pi}\right)^{4s} \cdot \mathbf{1} \\
&= \int_{\mathcal{A}_L/\mathcal{G}_L} e(\mathcal{A}_L \times_{\mathcal{G}_L} \Omega^2_+, F_A^+) \wedge \left(\frac{m}{2\pi}\right)^{-\frac{\chi+\sigma}{2}+2s} \cdot \mathbf{1}
\end{aligned}
\tag{4.17}
$$

Remark 25. A heuristic comparison of (4.17) with the formula for the SW partition function (4.15) suggests one can, in some cases, perturb the spin^c–structure c, for which $2s(c) > 0$, with a line bundle l, such that $(c+2l)^2-(2\chi+3\sigma) = 8s(c+2l) = 0$, in order to obtain a zero dimensional moduli space. In this case, if we formally replace (modulo 2π) m with $m^{2s(c)}$, we can conjecture the following formula which relates the SW invariants for positive dimensional moduli spaces with SW invariants for zero dimensional moduli spaces:

$$SW(c, m^{2s}) = SW(c+2l, m)$$

The above formula is an analog of the formula obtained in [33, Theorem 1.3], for $l = \Sigma$ of negative square, together with certain conditions on Σ. Note that in general there are obstructions for finding l, and this perturbation cannot be done in complete generality. Without assuming that the moduli space for the spin^c–structure $c+2l$ has dimension zero, we can still write:

$$SW(c,m) = \left(\frac{m}{2\pi}\right)^{2s(c)-2s(c+2l)} SW(c+2l, m) \tag{4.18}$$

5. Witten's Conjecture

In this section we compare the formulas previously obtained by abelian localization of DW and SW theories, and give a formal derivation for Witten's formula. Witten's conjecture states the following:

Conjecture 26. *Let X be a compact simply connected smooth 4-manifold with b_2^+ odd and greater than or equal to 3. Then X has Donaldson simple type if and only if it has SW simple type. Moreover, if X has simple type then the Kronheimer and Mrowka basic classes* [17] *agree with the SW basic classes and*

$$
\begin{aligned}
\mathbb{D}_X\left(\left(1+\frac{u}{2}\right)e^{\Sigma}\right) &= 2^{2+\frac{7\chi+11\sigma}{4}} e^{\Sigma\cdot\Sigma/2} \sum_{\substack{c\\ \text{basic classes}}} SW(c)\cdot e^{c\cdot\Sigma} \\
\mathbb{D}_X\left(\left(1-\frac{u}{2}\right)e^{\Sigma}\right) &= 2^{2+\frac{7\chi+11\sigma}{4}} i^{\frac{\chi+\sigma}{4}} e^{-\Sigma\cdot\Sigma/2} \sum_{\substack{c\\ \text{basic classes}}} SW(c)\cdot e^{-ic\cdot\Sigma}
\end{aligned}
\tag{5.1}
$$

for $\Sigma \in H_2(X)$ and u the generator of $H_0(X)$. Note that (5.1) *is equivalent to* (1.1) *(see e.g.* [17]*).*

We do not prove that D-simple type is equivalent to SW-simple type. We only give a derivation of a similar formula relating DW and SW invariants without imposing the simple type conditions. If we assume both simple type conditions, we obtain exactly Witten's formula (1.1). The partition functions case was studied in [36].

5.1. **A direct derivation.** The derivation of the general conjecture relating the Donaldson generating series with the SW invariants is as follows. Recall that the k-th equivariant Donaldson polynomial for observables $\mathcal{O} = \Sigma^a \wedge \nu^b$ is given by (4.11), and the SW invariant for a spin^c–structure c is given by (4.17).

In order to identify the integrals inside both formulas (4.11) and (4.17), we must match up the formal dimension of the spaces $\mathcal{A}_l/\mathcal{G}_l$, as defined in Remark 22. Using Remark 23, we write the Donaldson generating series as a formal sum of the partition functions $Z_k(m)$. The integral obtained will have formal dimension zero, with respect to the Donaldson moduli space.

However, $x^2 = l_x^2$ does not equal $2\chi + 3\sigma$, so we cannot identify x's with spin^c–structures for the SW partition function. Thus we first identify the DW generating series with SW invariants for positive dimension moduli spaces, and we obtain a general formula relating the invariants, without assuming any simple type conditions. Then, assuming both simple types for our manifold X, the general formula specializes to Witten's conjectured formula (1.1).

Using (4.11), we write the Donaldson generating series (3.4) as follows:

$$\begin{aligned}\mathbb{D}_X(e^{\Sigma+\lambda\nu})(m) &= \sum_d \sum_{a+2b=2d} \frac{(2\lambda)^b}{a!b!} \sum_{x^2=-4k} 2^{-(\chi+\sigma)-4d} \langle x, \Sigma\rangle^a \left(\frac{m}{2\pi}\right)^{2d-2s} \\ &\quad \cdot \int_{\mathcal{A}_{l_x}/\mathcal{G}_{l_x}} e(\mathcal{A}_{l_x} \times_{\mathcal{G}_{l_x}} \Omega^2_+, F_A^+) \wedge \left(\frac{m}{2\pi}\right)^{-\frac{\chi+\sigma}{2}+2s} \cdot \mathbf{1}\end{aligned} \tag{5.2}$$

Here $2d = 2d(k) = a + 2b$ is the dimension of the Donaldson moduli space $\mathcal{M}_k$, and $2s = 2s(c)$ is the dimension of the Seiberg–Witten moduli space $\mathcal{M}_c$.

A useful approach is to consider perturbations of the line bundles l_x for the integral in (5.2), as this integral is similar to the integral in (4.17). This leads us to replace the cohomology classes $x = 2y$ above, which have square $x^2 = -4k$, with classes of the form $l + 2y$, for a fixed spin^c–structure l, as mentioned in Remark 24. Here $l^2 \equiv 0$ mod 2. Using a blowup/blowdown argument (see e.g. [11], [12]), we claim that we actually can take $l^2 = 0$. Therefore, by Remark 24, the integral in (5.2) can be identified with $SW(y, m)$ in (4.17). Under the change of variables $x = l + 2y$, the sum $\sum_{x^2=-4k}$ in (5.2) becomes $\sum_{y^2=-k}$. Moreover, the factor $\langle x, \Sigma\rangle$ becomes $2 \cdot \left(\langle y, \Sigma\rangle + \frac{1}{2}\Sigma^2\right)$ (see again Remark 24). Thus (5.2) becomes

$$\begin{aligned}\mathbb{D}_X(e^{\Sigma+\lambda\nu})(m) &= \sum_d \sum_{a+2b=2d} \frac{(2\lambda)^b}{a!b!} \sum_{y^2=-k} 2^{-(\chi+\sigma)-4d+a} \left\langle y, \Sigma + \frac{1}{2}\Sigma^2 \right\rangle^a \cdot \\ &\quad \cdot \left(\frac{m}{2\pi}\right)^{2d-2s} SW(y, m).\end{aligned} \tag{5.3}$$

As in Remark 17, we now adopt the topological convention that k is positive, which multiplies the right hand side of (5.3) by two. Replacing y by $-iy$, and m by $-m$, we can include also spin^c–structures y of positive square, so we can rewrite (5.3) as a sum over all spin^c–structures, i.e. over all classes $c \in H^2(X)$. Specifically, we obtain

$$\begin{aligned}\mathbb{D}_X\left(e^{\Sigma+\lambda\nu}\right)(m) &= \sum_{c\in H^2(X)} 2^{1-(\chi+\sigma)-4d+a}\left(\frac{2\pi}{m}\right)^{2s(c)} SW(c,m)\cdot e^{\left(\frac{m}{2\pi}\right)(c\cdot\Sigma+\frac{1}{2}\Sigma^2)}\cdot e^{\left(\frac{m}{2\pi}\right)^2(2\lambda)}\\ &+i^{\frac{\chi+\sigma}{4}}\sum_{c\in H^2(X)} 2^{1-(\chi+\sigma)-4d+a}\left(\frac{2\pi}{m}\right)^{2s(c)} SW(c,m)\cdot e^{\left(\frac{m}{2\pi}\right)(-ic\cdot\Sigma-\frac{1}{2}\Sigma^2)}\cdot e^{\left(\frac{m}{2\pi}\right)^2(-2\lambda)}\end{aligned} \tag{5.4}$$

In (5.4) we have used

$$SW(-ic,-m) = i^{\frac{\chi+\sigma}{4}}\cdot SW(c,m).$$

The degree $2b$ of the observable $\mu(\nu)$ from Donaldson theory can be identified with the dimension $2s$ of the Seiberg–Witten moduli space, so $a+2s=2d$. Thus

$$2^{1-(\chi+\sigma)-4d(k)+a} = 2^{1-(\chi+\sigma)-2d(k)-2s(l+2y)}. \tag{5.5}$$

Furthermore, by Proposition 14 and (3.5), one easily checks that

$$2d(k) = -2s(l+2y) - 8s(y) - (\chi+\sigma) - \frac{7\chi+11\sigma}{4}.$$

Therefore (5.5) equals

$$2^{8s(y)+1+\frac{7\chi+11\sigma}{4}}. \tag{5.6}$$

Combining (5.4), (5.5), and (5.6) lead us to the following conjectured formula, which expresses the Donaldson generating series in terms of Seiberg–Witten invariants. Note that the formula below does not assume any of the simple type conditions.

Conjecture 27. *For X a compact simply connected smooth 4-manifold with b_2^+ odd, and $b_2^+\geq 3$, the equivariant DW generating series can be expressed via equivariant SW invariants as follows:*

$$\begin{aligned}\mathbb{D}_X\left(e^{\Sigma+\lambda\nu}\right)(m) &= 2^{1+\frac{7\chi+11\sigma}{4}}\sum_{c} 2^{8s(c)}\left(\frac{2\pi}{m}\right)^{2s(c)} SW(c,m)\cdot e^{\left(\frac{m}{2\pi}\right)(c\cdot\Sigma+\frac{1}{2}\Sigma^2)}\cdot e^{\left(\frac{m}{2\pi}\right)^2(2\lambda)}\\ &+2^{1+\frac{7\chi+11\sigma}{4}}i^{\frac{\chi+\sigma}{4}}\sum_{c} 2^{8s(c)}\left(\frac{2\pi}{m}\right)^{2s(c)} SW(c,m)\cdot e^{\left(\frac{m}{2\pi}\right)(-ic\cdot\Sigma-\frac{1}{2}\Sigma^2)}\cdot e^{\left(\frac{m}{2\pi}\right)^2(-2\lambda)}\end{aligned} \tag{5.7}$$

The sums in (5.7) run over all $c\in H^2(X)$. For these sums to be finite, we must assume e.g. that X has finite type in the sense of [19] or [28]. In this case, identifying the degree zero components in (5.7) (the expression being regular at $m=0$), one obtains a formula for the Donaldson generating function for manifolds of finite type, formula that looks similar with the ones proved in [28], or described in [19].

Remark 28. It is important to mention here also that, because of lack of examples, (5.7) cannot be checked for accuracy.

If X has D-simple type, we replace d by d', where

$$d' \equiv \frac{1}{4}(\chi+\sigma) \mod 2.$$

This allows us to consider $k \in \frac{1}{2}\mathbb{Z}$. A direct computation, starting from (5.2), shows that the D-simple type condition reduces $\sum\limits_{a+2b=2d}$ in (5.2) to the cases $b=0$ or $b=1$; the $b=1$ case can be reduced further to $s(c)=0$, by dimensional arguments and relations between Donaldson invariants. This simplifies the computation above, in the sense that the integrals inside the DW generating function (5.2) can be directly identified with SW partition functions (4.17). Moreover, assuming also that X has SW simple type, the terms $\sum\limits_{c}$ in (5.7) are finite sums over all SW basic classes. In this case, we obtain:

$$\begin{aligned}\mathbb{D}_X\left(e^{\Sigma+\lambda\nu}\right)(m) &= 2^{1+\frac{7\chi+11\sigma}{4}} \sum_{c \text{ basic}} SW(c,m)\cdot e^{\left(\frac{m}{2\pi}\right)(c\cdot\Sigma+\frac{1}{2}\Sigma^2)}\cdot e^{\left(\frac{m}{2\pi}\right)^2(2\lambda)} \\ &+ 2^{1+\frac{7\chi+11\sigma}{4}} i^{\frac{\chi+\sigma}{4}} \sum_{c \text{ basic}} SW(c,m)\cdot e^{\left(\frac{m}{2\pi}\right)(-ic\cdot\Sigma-\frac{1}{2}\Sigma^2)}\cdot e^{\left(\frac{m}{2\pi}\right)^2(-2\lambda)}\end{aligned} \tag{5.8}$$

Identifying the degree zero components in (5.8), we derive Witten's conjectured formula:

$$\begin{aligned}\mathbb{D}_X\left(e^{\Sigma+\lambda\nu}\right) &= \\ 2^{1+\frac{7\chi+11\sigma}{4}} &\cdot \left[e^{\left(\frac{1}{2}\Sigma^2+2\lambda\right)}\sum_c SW(c)\cdot e^{c\cdot\Sigma} + i^{\frac{\chi+\sigma}{4}} e^{\left(-\frac{1}{2}\Sigma^2-2\lambda\right)}\sum_c SW(c)\cdot e^{-ic\cdot\Sigma}\right].\end{aligned}$$

Note that the factor $e^{\Sigma^2/2}$ in Witten's formula usually appears via the polarization identity for K3 surfaces (see [17, p. 690]), or by performing Gaussian integrals on the functional integral [16],[38],[39]. In our computation it comes from perturbations of spin^c–structures.

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