The Schwinger-DeWitt technique in Gauge-Gravity Theories

April 25, 2020

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Talk presented at DPF (Division of Particle and Fields) 02 conference, May $24-28, 2002$ Williamburg, Virginia[1]

Abstract

We construct the one-loop effective action in Yang-Mills and Pure Quantum Gravity theories with heat kernel(or proper time method), which maintains manifest covariance during and after quantization (gauge and diffeomorphism invariance are always preserved). In this talk, we will basically focus on "What, How, and Why" we prefer heat kernel than the standard Feynman diagram calculation in momentum space at the one loop correction. The beta function of Yang-Mills field in the fixed gravitational background can be more simply obtained. The non-local term which cannot be easily obtained in the expansion method are exactly computed in Yang-Mills in the case of covariantly constant background field. The local term is consistent with asymptotic expansion method or any most standard method. The non-local terms give some physical implication concerning non-perturbative problems such as confinement and instabilities. The modification of this technique to quantum gravity is discussed.

[1]The link of this talk is at www.dpf2002.org/particle-astrophysics.cfm

1 Problems

Progress in gauge-gravity theories depends on the development of covariant formulation of quantum effective action. Background field method is the elegant tool to construct effective action while manifest invariant are always preserved during and after quantization. However, most people follow the procedure on computing Feynman diagrams in momentum space and have to regulate each diagrams separately. This way makes the calculation quite troublesome and even worse if we do not perform expansion over the mean field. In addition, it is uncomfortable to keep track on calculations once the gauge invariant breaks down due to gauge fixing condition or mass cut off. Moreover,the well understood divergent diagrams can only give the local part of effective action. If we want to study the infrared properties of Yang-Mills, we also need the non-local parts to give us some information what and how much we can learn non-perturbative dynamics. Unfortunately, the direct method are too complicated to carry out the summation. For the quantum field in curved space and quantum gravity, the non-local part should contain, for instance, particle creation effects, vacuum polarization, and Hawking radiation in black holes. Using graviton propagator in momentum space brings trouble and calculation can be easily lost.

The aim of this paper is to overcome such problems above (or at least part of it) with an alternative approach called heat kernel or Schwinger-DeWitt technique to construct the invariant formalism of effective action. Heat kernel was first formulated by Schwinger[1] long times ago and then it was later applied in geometrical language by DeWitt[2,3]. Vilkovisky and Barvinsky generalized this technique to nonminimal but causal operators of any order[4]. Recently, Barvinsky and Mukanov[5] derive the non-perturbative and non-local approach to the effective action in space time $n > 2$ by summation of proper time series on the heat kernel and suggest the new method. Our extension is somewhat different. We constructed the exact expression of effective action which the non-local terms are included by modifying the QED Schwinger's results to Yang-Mills in the case of covariantly constant background field. The results give implication about confinement and instabilities. Additionally, the generalization of this technique to quantum gravity is briefly discussed.

2 Yang-Millls in Curved Space

Pure Yang-Mills field in the fixed gravitational background in 4 dimension is considered. The action is

$$S = -\frac{1}{4g_y^2}\int d^4x g^{\frac{1}{2}} F_{a\mu\nu}F^{a\mu\nu}$$

where g_y is the Yang-Mills coupling constant. g is defined as $|detg_{\mu\nu}|$. The indices a,b, and c are raised and lowered by Cartan Killing metric γ_{ab}
It is clear that this action is invariant under infinitesimal gauge transformation and also invariance under diffeomorphism (general coordinate transformation and reparameterization invariance) if the gravitational field is allowed to become dynamics in its own functional.
Since the quadratic term in the mean field expansion is only taken to account at the one-loop level, the operator(called Jacobi Field operator) acting on the fluctuation field can be constructed by simply varying the classical field equation with imposing the differential supplementary condition.

3 Schwinger's representation

The Green's function has the formal representation defined as the inverse of Jocobi field operator. Since the eigenvalues of $\hat{F}$ are real, this can be replaced by a complex Laplace transform or we just simply rotate by 90 degree in the complex s-plane.

$$G = -\frac{1}{\hat{F}+i\epsilon} = i\int_o^\infty e^{i\hat{F}s}ds$$

and similarly for the ghost field operator

$$G_{gh} = -\frac{1}{\hat{F_{gh}}+i\epsilon} = i\int_o^\infty e^{i\hat{F_{gh}}s}ds$$

The good thing is that Schwinger's representation is valid when $\hat{F}$ is the Jacobi field operator of the *nonlinear* bosonic field and also valid when the background is *nonstationary* or in the arbitrary background[1,2].

4 Unique Effective Action

The effective action is formally defined as the Legendre transform of the full connected one particle irreducible diagrams(1PI). In usual formulation of gauge theories and quantum gravity, there are two main problems. First, fixing the gauge of the quantized field automatically fixes the gauge of the mean field and this leads to non-gauge-invariant effective action. However, it is possible to fix the gauge of the quantized field leaving the gauge of the mean field arbitrary hence we obtain the gauge invariant effective action. This is understood in background field gauge. The second problem is that the gauge-invariant effective action depends (parametrically) on the choice of background gauge conditions. The theory is sensible only if the physical quantities do not depend on the gauge choice. This problem was solved by DeWitt and Vilkovisky[2,6]. That is why we call unique effective action which is completely gauge independent[2]. To have the inverse Green's function, the supplementary condition is needed. After varying field equation and imposing supplementary condition, we have the Jacobi field operator and ghost field operator.

$$ {}^{a}_{\mu}\hat{F}^{\nu}_{b} = \delta^{a}_{b}(\delta^{\mu}_{\nu} D_{\sigma} D^{\sigma} - R^{\nu}_{\mu}) + 2 f^{a}_{cb} F^{c\nu}_{\mu} $$

$$ \hat{F^{a}_{ghb}} = \delta^{a}_{b} g^{\frac{1}{2}} D_{\mu} D^{\mu} $$

One might ask why we do choose the supplementary condition in such the way that the $D^{\mu}D^{\nu}$ term is cancelled out from the operator obtained from varying field equation. The reason is that we want to preserve the manifest covariance and still have the unique gauge invariant effective action. If the effective action depends on the particular gauge chosen, it would be hard to get the unambiguous and believable result.
Let us recall the path integral quantization. For the one loop quantum correction, we only need the Gaussian Integral or the quadratic term in the expansion.

$$ W^{(1-loop)} = -ilog(detG)^{\frac{1}{2}} + ilog(detG_{gh}) = -\frac{i}{2} TrlogG + iTrlogG_{gh} $$

To construct effective action in term of heat kernel K, we apply the Schwinger's representation.

$$ W^{(1-loop)} = \frac{1}{2} Tr \int_{0}^{\infty} (e^{i\hat{F}s} - 2e^{i\hat{F_{gh}}s}) \frac{ds}{is} $$

Therefore, the w function is written as[3]

$$ w(x) = \frac{1}{2} tr \int_{0}^{\infty} (K(x,x,s) - 2K_{gh}(x,x,s)) \frac{ds}{is} $$

where

$$ W = \int d^4 x w(x) $$

and

$$ K(x,x',s) = e^{i\hat{F}s} \delta(x,x') $$

One can see that Schwinger's representation also gives the heat equation.

$$ i\frac{\partial K}{\partial s} = -\hat{F} K $$

and similar to the ghost kernel.

[2]The gauge and ghost independent in quantum gravity is discussed in Ref[10]

[3]The symbols Tr stands for the functional trace (space-time is included) whereas tr means the trace over group indices.

5 Asymptotic expansion revisited

The harder thing is to figure out what the kernel K is in which it can satisfy heat equations. If it is for the purpose of the local part of effective action, proposing an ansatz kernel K by doing proper time expansion is good enough to investigate ultraviolet divergence such as beta function and anomalies. It is ,therefore,

$$\begin{aligned} K(x,x',s) &= i(4\pi is)^{-2}D^{\frac{1}{2}}(x,x')e^{\frac{i\sigma}{2s}-im^2s}\Lambda(x,x',s) \\ \Lambda(x,x',s) &= \sum_{k=0}^{\infty} a_k(x,x')(is)^k \end{aligned}$$

(This is also similar for the ghost denoted by $K_{gh}$and a_{kgh}. One can rotate by 90 degree to make sure that a_k and a_{ghk} are real) where

$$\Lambda(x,x,s=0)=1, \ \sigma=\frac{1}{2}(x-x')^2$$

and $D^{\frac{1}{2}}(x,x') \equiv |det - \sigma_{;\mu\nu'}|$ is the Van Vleck-Morette determinant. If one plug the kernel K in W, we see that the integral of effective action diverges at the lower limit. How are we going to fix this? This asymptotic expansion also admit the covariant regularization version. We choose dimensional regularization for simplicity. The kernel K will become $K(x,x',s) = i(4\pi is)^{-\frac{n}{2}}e^{\frac{i\sigma}{2s}-im^2s}\Lambda(x,x',s)$ and the integrand of effective action will diverge at the lower limit for all positive value of n. The way out is to pretend that the dimensionality n of space time is a complex number instead of a positive integer and to define $w(x)$ by analytic continuation to the region of the complex n-plane where integral converges to the vicinity of actual physical dimension.

Since only dimensionless quantities can be analytically continued, one must multiply $w(x)$ by μ^{-n}. Inserting kernel K in $w(x)$ and integrating by part 4 times (for 4 dimensions). There will be coefficients up to a_2 needed to be computed. After taking $m \to 0$, we obtain

$$w = -\frac{\sqrt{g}}{32\pi^2}\left(\frac{1}{n-4}-\frac{3}{4}\right)tr(2a_2-4a_{2gh})+\Gamma_{ren}$$

where Γ_{ren} is the renormalized remainder after the local terms have been subtracted.

5.1 Coefficients and Coincidence limit

As we see above, we need to know what the coefficient $a_2(x, x')$ and $a_{gh2}(x, x')$ precisely and take the limit $x \to x'$ to determine the counter term. By inserting kernel K and K_{gh}in the heat equation, we get the recursion relations.[4]

$$\sigma_;^{\mu} a_{0;\mu} = 0$$

$$\sigma_;^{\mu} a_{k;\mu} + k a_k = \Delta^{-\frac{1}{2}} (\Delta^{\frac{1}{2}} a_{k-1})^{\mu}_{;\mu} - (\delta^a_b R^{\mu}_{\nu} - 2 f^a_b F^{c\nu}_{\mu}) a_{k-1}$$

$$\sigma_;^{\mu} a_{ghk;\mu} + k a_{ghk} = \Delta^{-\frac{1}{2}} (\Delta^{\frac{1}{2}} a_{ghk-1})^{\mu}_{;\mu}$$

where $k = 1, 2, ...$ and $\Delta(x, x') \equiv g^{-\frac{1}{2}} D(x, x') g^{-\frac{1}{2}}$
By using the commutation law of the Lie group, differentiating these recursion relation,and taking the coincidence limit $x \to x'$, we obtain

$$tr[a_{gh2}] = l(\tfrac{1}{30} R^{\mu}_{;\mu} + \tfrac{1}{72} R^2 - \tfrac{1}{180} R_{\mu\nu} R^{\mu\nu} + \tfrac{1}{180} R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau}) - \tfrac{1}{12} F_{a\mu\nu} F^{a\mu\nu}$$

$$tr[a_2] = l(\tfrac{n-5}{30} R^{\mu}_{;\mu} + \tfrac{n-12}{72} R^2 - \tfrac{n-90}{180} R_{\mu\nu} R^{\mu\nu} + \tfrac{n-15}{180} R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau}) + \tfrac{24-n}{12} F_{a\mu\nu} F^{a\mu\nu}$$

where l is the dimension of Yang-Mills Lie group and n is the space-time dimension.
The one-loop counter term is simply obtained by minimal subtraction.

$$\delta(\tfrac{1}{g_y^2}) = -\tfrac{11}{24\pi^2} \tfrac{1}{n-4}$$

Notice that the renormalization group is irrelevant with the curvature terms since only the right coefficients of $F_{a\mu\nu} F^{a\mu\nu}$ term contribute beta function.

6 (Exact) Solution in Yang-Mills

Originally, heat kernel is understood as an expansion in the power series of proper time variables. However, the effective action is generically non-local and the calculation of non-local terms requires the summation of the proper-time series. The direct computation of the coefficients a_k is more difficult at higher order of k . *How are we going to fix this?*
We are more interested in the non-local effective action than the local part (as already existed in most literature) to can interpret the physical consequence. The original work was done in QED by Schwinger[1] and in Yang-Mills and

[4] The condense notation $a_{;\mu}$ means covariant derivative with respect to μ

quantum gravity by DeWitt[2]. The more details are referred to these references. At present, we briefly review what and how to get the results.
In fact, Schwinger have never used any kind of regularization to renormalized QED. He computed the kernel $K(x, x', s)$ when $F_{\mu\nu}$ is constant in Minkowski space time.

$$A_\mu = -\frac{1}{2} F^{\mu\nu} x^\nu$$

What about in Yang-Mills?

It would be less confusing if we choose the covariantly constant Yang-Mills field $F^a_{\mu\nu}$ in a particular direction of the Lie algebra vector space. The simple Lie group in the adjoint representation is assumed.

$$F^a_{\mu\nu} = \delta^a_1 F_{\mu\nu}$$

$$A^a_\mu = -\frac{1}{2} \delta^a_1 F_{\mu\nu} x^\nu$$

where $F_{\mu\nu}$ is now acting as the electro-magnetic like (chromo-electromagnetic field).
Therefore the heat equation becomes

$$(i\frac{\partial}{\partial s} + (\partial_\mu + f_1 A^1_\mu)(\partial^\mu + f_1 A^{1\mu}))K_{gh} = 0$$

$$(i\frac{\partial}{\partial s} + (\partial_\mu + f_1 \otimes 1_4 A^1_\mu)(\partial^\mu + f_1 \otimes 1_4 A^{1\mu}) + 2 f_1 \otimes \eta_M F \eta_M^{-1})K_{gh} = 0$$

where $F \equiv F^\mu_\nu$, $\eta_M \equiv \eta_{\mu\nu}$, and f_1 is the anti-symmetric matrix in Lie Algebra. Recall

$$w(x) = \frac{1}{2} \int_0^\infty tr(K - 2K_{gh}) \frac{ds}{is}$$

By diagolizing the matrices of heat equations, we have the generic expression for the effective action.

$$w(x) = \frac{i}{2} \int_0^\infty (4\pi is)^{-2} [2\rho + det(\frac{Fs}{sinhFs})^{\frac{1}{2}} (\frac{1}{2} tr e^{2iG_1 \otimes \eta_M F \eta_M^{-1} s} - 2)] \frac{ds}{is}$$

where $G_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
After computing the determinant, we obtain

$$w(x) = i \int_0^\infty (4\pi is)^{-2} [\rho - \frac{\chi s^2}{Imcosks} (2|cosks|^2 - 1)] \frac{ds}{is}$$

where $k = \sqrt{2(\alpha + i\chi)}$

$$\alpha = \frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} (\mathbf{E}^2 - \mathbf{H}^2)$$

$$\chi = \frac{1}{4} F^*_{\mu\nu} F^{\mu\nu} = \mathbf{E} \cdot \mathbf{H}$$

and ρ is the rank of Lie algebra. We will forget the dimensional regularization and instead follow the Schwinger's way by subtracting the terms which cause the integral divergence at $s \to 0$. The reason why we can subtract is that, in any renormalizable theories, the coupling constant or wave function renormalization can always be absorbed. This subtraction procedure never make us losing consistency and breaking gauge invariance. The quantities in the square brackets in the integrand of effective action near $s = 0$ has an expansion in power s that begins $(\rho + 1) - \frac{11}{3} \alpha s^2 + O(s^4)$ It is therefore our generic result is

$$w = \frac{1}{16\pi^2} \int_0^\infty [\frac{\chi}{sImcosks}(2|cosks|^2 - 1) + \frac{1}{s^3}] - \underbrace{\frac{11}{48\pi^2}(\frac{1}{4} F_{\mu\nu} F^{\mu\nu}) \int_0^\infty \frac{ds}{s}}$$

* Notice that the last term is local and contribute the logarithmic divergence part of effective action. As well known, its coefficient naturally gives one-loop beta function consistent with what is found in most literature or asymptotic expansion method discussed earlier *

The more interesting thing is how to extract the meaning of the non-local effective action. Indeed, this corresponds to the infinite summation of the proper time series. At present, we restrict to the case of pure chromoelectric and pure chromo-magnetic fields. After some algebra, the results are

QCD results:

$$w^{YM}_{Mag.} = \frac{1}{16\pi^2} \int_0^\infty \frac{1}{s^3} (\frac{Hscos2Hs}{sinHs} + 1 - \frac{11}{6} H^2 s^2) ds$$

$$w^{YM}_{Elec.} = \frac{1}{16\pi^2} \int_0^\infty \frac{1}{s^3} (\frac{Escosh2Es}{sinhEs} + 1 + \frac{11}{6} E^2 s^2) ds$$

Compared with

QED results[1,2]:

$$w^{QED}_{Mag.} = \frac{1}{8\pi^2} \int_0^\infty \frac{1}{s^3} (HscotHs - 1 + \frac{1}{3} H^2 s^2) e^{-im^2 s} ds$$

$$w^{QED}_{Elec.} = \frac{1}{8\pi^2} \int_0^\infty \frac{1}{s^3} (EscothEs - 1 - \frac{1}{3} E^2 s^2) e^{-im^2 s} ds$$

Notice the signs of local terms between QED and QCD reverse due to the asymptotic freedom of the Yang-Mills.

7 Physical Interpretations

7.1 Asymptotic Freedom

It is well known that the counter term obtained in the earlier section implies

$$\frac{1}{g^2_{y_{bare}}} = \mu^{n-4}(\frac{1}{g_y^2} - \frac{11}{24\pi^2}\frac{1}{n-4})$$

Instead of solving renormalization group equation, we can simply differentiate this expression with respect to the energy scale μ. Hence we obtain the β function.

$$\beta = -\frac{11}{48\pi^2} g_y^3$$

where β is defined as $\mu\frac{dg_y}{d\mu}$. The running coupling constant is simply obtained.

$$\frac{1}{g^2(\mu)} = \frac{1}{g_0^2} + \frac{11}{24\pi^2} log\frac{\mu}{\mu_0}$$

where g_0 and μ_0 are integration constants.
As known, this is clearly indicated that the coupling constant gets stronger at the high energy limit and weaker at the low energy limit. In fact, this property is analogous to non-linear sigma model on the Riemanian manifold whose target space is the sphere S^N. Suppose we couple the gluon field with the fermionic fileds, it tends to destroy the effect.

7.2 Confinement of pure electric Yang-Mills type

In the case of covariantly constant chromo-electric field, there is no way we can control the divergence. Note that this is *not* UV divergence as usual. The divergence stems from the $cosh2Es$ term which is non-local. This result leads us to conjecture that the theory is sensible(finite) only if the background field of electric Yang-Mills type component are strong enough to can satisfy the condition

$$|\partial^2 E| \gg E^2$$

Suppose there is the regime in which the Yang-Mills field behave like the Coulomb fields,the field will satisfy

$$E \sim \frac{g_y^2}{r^2}$$

But these two equations will be consistent only if $g_y^2 \ll 1$ and indicate that the field cannot behave like the Coulomb filed in the strong coupling regime. The field must therefore "crinkle". At least, this one-loop calculation gives some implication about confinement.
We later found that the model of confinement in chromo-electric flux tube is studied in Ref[7]. It may be worth to consider Yang-Mills in the strong background field to prove the conjecture and study low energy QCD.

7.3 Instabilities of pure magnetic Yang-Mills type

If we compare the results of w_{Mag}^{YM} and w_{Mag}^{QED}, the integrand of w_{Mag}^{QED} has the poles on the real axis. This can be rotated to the negative imaginary axis without picking up any residues, yielding the real valued QED effective action.
However, this is *not* true in Yang-Mills. There are the infinite number of poles on the real axis. We cannot rotate to the imaginary axis of the s-plane as in QED since we have no exponential factor of mass term attached to it. The contribution from the poles at $s = \frac{n\pi}{H}, n = 1, 2, 3, ...$ can be done by running the integration contour just below the positive real axis. The expression w_{Mag}^{YM} will therefore become a principle-value integral plus a sum of integrals over semicircles around the poles. The residues at the poles are a sum of infinite series. One has the question whether this series converges or not. Fortunately,this is not too hard to can compute. We obtain the finite series in the imaginary part. The result is

$$w_{Mag.YM}^{ren.} = \frac{1}{16\pi^2} P \int_0^\infty \left(\frac{Hcos2Hs}{s^2 sinHs} + \frac{1}{s^3} - \frac{11}{6}\frac{H^2}{s}\right) ds + \frac{\pi i}{16\pi^2} \sum_{n=1}^{\infty} Res[...]$$

where $[...]$ stands for the integrand of the integral.

$$\sum_{n=1,3,...} Res[...] = -\frac{H^2}{\pi^2} \sum_{n=1,3,...} \frac{1}{n^2}$$

$$\sum_{n=2,4,...} Res[...] = \frac{H^2}{\pi^2} \sum_{n=2,4,..} \frac{1}{n^2}$$

Hence

$$Imw = -\frac{1}{192\pi} H^2$$

Notice that Nielsen and Olesen[8] found[5] $ImW = -\frac{V}{8\pi} H^2$
The reason why it contributes negative sign, unlike QED, is due to the asymptotic freedom properties of the Yang-Mills fields. If w -function is allowed

[5]The coefficient factor of imaginary part is different but the real part(the logarithmic divergent term) is consistent. Note that we also have the non-local term contributed

to be imaginary, the vacuum state is unstable and the true ground state is far from trivial. That means the effective potential leads to the presence of negative mode function.

We have studied the question of infrared instability of constant chromo-magnetic background on the series of paper of Ref[8]. There are a lot of arguments.

The interpretations are summarized as following

1.It is implicitly Non-Perturbative problem.

2. There is the whole class of quantum runaway fluctuations, corresponding to mode function with negative frequency at the one-loop correction.

3.The vacuum is unstable. The true ground state cannot be found since the energy density has developed an imaginary part.

4. The unstable mode corresponds to a *tachyon* in 1+1 dimension[8]

5. To determine what mode exactly is stable and unstable,we have to solve the eigenvalues equation and study its spectrum.

It is not yet perfectly clear how one precisely can stabilize this chromo-magnetic type. The different authors give different arguments. Flory suggested that the unstable mode can be completely stabilized when one goes beyond one-loop calculation. That means we need to keep the cubic and quartic terms in the background field expansion. Avaraimi[8] concluded that it is impossible to get a stable vacuum of chromo-magnetic type in space-time of $n < 5$. What is the right way, we cannot tell at this moment.

8 One-loop Quantum Gravity

The more interesting thing is to apply this technique to quantum gravity. Let's start with pure quantum gravity action.

$$S = 2\mu^{n-2} \int \sqrt{g} R d^n x$$

where μ is the Planck mass and R is the curvature scalar.

At present, we briefly review DeWitt's work[2]. The purpose is to show that heat kernel can be applied in quantum gravity and is less troublesome than the standard Feyman diagram calculation in momentum space.

We can apply the method described in the former sections to construct the Jacobi field operators and ghost field operator. After varying field equation and imposing supplementary condition, we obtain

$$ {}_{\mu\nu}F^{\sigma\tau} = \frac{1}{2}(\delta^\sigma_\mu \delta^\tau_\nu + \delta^\tau_\mu \delta^\sigma_\nu) D_\rho D^\rho - \frac{1}{2}(\delta^\sigma_\mu \delta^\tau_\nu + \delta^\tau_\mu \delta^\sigma_\nu - \frac{2}{n-2} g_{\mu\nu} g^{\sigma\tau}) R - \frac{2}{n-2} g_{\mu\nu} R^{\sigma\tau} - g^{\sigma\tau} R_{\mu\nu} + R^{\sigma\tau}_{\mu\nu} + R^{\tau\sigma}_{\mu\nu} + \frac{1}{2}(\delta^\sigma_\mu R^\tau_\nu + \delta^\tau_\mu R^\sigma_\nu + \delta^\sigma_\nu R^\tau_\mu + \delta^\tau_\nu R^\sigma_\mu) $$

and

$$F^{\mu}_{gh\nu} = \sqrt{g}(\delta^{\mu}_{\nu} D_{\sigma} D^{\sigma} + R^{\mu}_{\nu})$$

In 4dimension, we need the a_2 coefficients at the coincidence limit.(If we do in 2 dimension,for example, in non-linear sigma model, we only need the a_1 coefficient) to determine one loop counter term. The results are

$$tr[a_{2gh}] = \frac{n+5}{30} R^{\mu}_{;\mu} + \frac{n+12}{72} R^2 - \frac{n-90}{180} R_{\mu\nu} R^{\mu\nu} + \frac{n-15}{180} R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau}$$

and

$$tr[a_2] = -\frac{n(2n-3)}{30} R^{\mu}_{;\mu} - \frac{23n^3+145n^2-262n-144}{144(n-2)} R^2 - \frac{n^3+n^2+718n-720}{360(n-2)} R_{\mu\nu} R^{\mu\nu} + \frac{n^2-29n+480}{360} R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau}$$

Hence, we obtain one-loop w function in quantum gravity

$$w = -\frac{1}{16\pi^2(n-4)} \sqrt{g}(-\frac{19}{15} R^{\mu}_{;\mu} - \frac{341}{36} R^2 - \frac{193}{90} R_{\mu\nu} R^{\mu\nu} + \frac{19}{18} R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau}) - \frac{1}{16\pi^2} \sqrt{g}(\frac{9}{20} R^{\mu}_{;\mu} + \frac{459}{72} R^2 - \frac{103}{90} R_{\mu\nu} R^{\mu\nu} - \frac{31}{36} R_{\mu\nu\sigma\tau} R^{\mu\nu\sigma\tau}) - \frac{i}{64\pi^2} \sqrt{g} \int_0^{\infty} ln(4i\pi\mu^2)(\frac{\partial}{i\partial s})^3 [tr\Lambda(x,x,s) - 2tr\Lambda_{gh}(x,x,s)] ds$$

Since quantum gravity is not perturbatively renormalizable, it has many difficulties.
$1.$ The effective action depend on the arbitrary constant μ. Each different choices of μ gives the different theories. By setting $\mu \sim m_{Plank}$, it should give the roughly physical interpretation at the Planck scale in order of magnitude 10^{19} GeV.
$2.$ It contains the unknown coefficients of a_k at higher order of k. Up to now, partial solution exists. The coefficients have been computed up to a_4[9]
$3.$ Suppose the renormalization is performed by minimal subtraction then this last two lines should be added to the classical action to get the effective action corrected to one-loop order. As discussed earlier, we have the dependent auxiliary mass problem.
Fortunately, there is the miracle in one-loop pure quantum gravity. Because of the metric independence of the Euler-Poincare characteristic, the counter term can be modified and obtained

$$S_{c.t.} = \frac{1}{16\pi^2(n-4)} \int \sqrt{g}(-\frac{429}{36} R^2 + \frac{187}{90} R_{\mu\nu} R^{\mu\nu})$$

Hence, it is (accidentally) one-loop finite on-shell.

It is interesting to see how one can construct (exact) effective action in quantum gravity without repeating on iteration to determine the coefficient a_k. One might need another technique such as in-in formalism which might be a sensible approach for solving quantum initial values problems. We will leave this for investigation in the near future work.

9 Conclusion and Outlook

One should not jump to conclusions as to what is the true physics at the strong coupling limit since what we have done is only based on the one-loop approximation. Due to the fact that the sum over all loops has physical significance, the physics can only be decided by doing this sum. A straight forward approach certainly leads to complexities beyond human power. However, it is possible to generalize this technique to higher order loop. The calculation is more complicated but not worse than the standard calculation in momentum space. Additionally, other tricks on the correspondence between gauge and gravity theories can be found recently[6]. The problems that appear to be intractable on one side may stand a chance of solution on the other side.

In conclusion, we have developed the technique to construct effective action in gauge and gravity theories. We found the infrared instability in the massless chromomagnetic type which stems from the imaginary part of effective action. The confinement in the case of chromoelectric type is conjectured. The one loop pure quantum gravity is finite. The connection between this technique and closed-time path formalism in quantum gravity is under investigation.

10 Acknowledgement

I thank the organizers at DPF02 conference for financial support and the audiences for their interest and stimulating questions.

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