Around Poincare duality in discrete spaces

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April 25, 2020

Abstract

We walk out the landscape of K-theoretic Poincare Duality for finite algebras, which will pave the way to get continuum Dirac operators from discrete noncommutative manifolds.

1 Generic

This paper is a sequel to [6][1]. There we saw how a family of discrete noncommutative spaces, namely those with intersection matrix

$$q_{ij}^{(n)} = \begin{pmatrix} -1 & 1 & & 0 & 1 \\ 1 & -1 & 1 & & 0 \\ & 1 & \ddots_n & 1 & \\ 0 & & 1 & -1 & 1 \\ 1 & 0 & & 1 & -1 \end{pmatrix}$$

can be arranged to get a one dimensional commutative space, the circle S^1, in the limit $n \to \infty$.

But also we noticed, just solving for the null eigenvectors of q_{ij}, that the intersection matrix is degenerated for size n multiple of six, then putting in question Poincare Duality.

Here we will examine some answers to this small nuisance.

The first one, obviously, is to reject such sizes. We have still subsequences going to infinity with non degenerate q, so we can build the limit without the multiples of 6. This was the approach of the previous paper, but one would like to get a generic procedure, instead of a case-by-case approach.

For the same reason, one worries about the next easier procedure: modify the intersection matrix to get a nondegenerate product while keeping the grading sign. One possibility is to use an simpler spectral triple, at the cost of losing the spatial homogeneity of this one, and even here we need to control possible degenerated forms. It is safer to increase the dimension of the Hilbert spaces in the diagonal, H_{ii}, to 3 or bigger dimension, so the diagonal elements change from -1 to -3, and Poincare duality works. Again, this method is not generic enough, but it is interesting because it forces a increase in the number of particles, just as in happens in Connes-Lot models. On other hand, it sounds strange that even if m_{ij} is diagonal and the algebraic structure does not differ, we have Poincare duality in the later case and degeneracy in the former.

Next step could be fine tuning of Poincare Duality definition. Since the review in [2], Connes has preferred to take as primary definition the existence of an element

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[1]so please print it and staple both in the same folder...

$\beta \in KR_n(A^0 \otimes A)$ such that

$$\beta \otimes_A \mu = 1_{A^0}, \mu \otimes_{A^0} \beta = 1_A$$

(where $\mu \in KR^n(A \otimes A^0)$ is got from our familiar Fredholm module).

Perhaps we can not win enough space with this shift of mind: It seems that in our case this definition implies the same isomorphism that the intersection matrix reflects. I have only seen an sketch of proof in the PhD thesis of H. Emerson. Also, it is remarked by Moscovici that *rational Poincare Duality* based in the intersection form is *weaker* than Poincare duality based in the KK-bifunctor.

2 Specific

To further investigate what is happening inside, lets fix $n = 6$. The degeneracy space is spanned by the vectors $(1, 1, 0, -1 - 1, 0)$ and $(0, 1, 1, 0, -1, -1)$.

We could see it in this way: Suppose an element of $KK(\mathbf{C}, A)$ is given by the projector $(1, 1, 0, 0, 0, 0)$. The product with the intersection matrix drives us to $(0, 0, 1, 0, 0, 1)$. But take now the projector $(0, 0, 0, 1, 1, 0)$: it gives us the same element $(0, 0, 1, 0, 0, 1)$ in $KK(A^0, \mathbf{C})$. So given this element in the K-homology, we can not say which one was the original: Duality is bounded to fail if we only kept the information of the intersection matrix.

Technically, the map between Ramond-Ramond fields and D-Branes... er, between K-theory and K-homology, is given by Kasparov product through the μ element. Regretly, the product is a very complicated operation, described only by a few high level mathematicians. At least, I have been unable to find detailed examples in the (still too modern) bibliography.

Readers of this paper will know that an element of $KK^0(A, B)$ is given by an A-B C^* bimodule, graded, and an bounded operator F. We can get F from an unbounded Dirac operator, D, with F=sign(D). And a third alternative exists, the use of asymptotic morphism. This variety of viewpoints has its origin in the difficulty to calculate the F operator in the product. For a couple of algebras, $a \in KK(A, B)$ and $b \in KK(B, C)$ respectively, Kasparov product $\otimes_B$ is built from the graded tensor product of subjacent Hilbert spaces, with a new operator $F = F_a \oslash F_b$ which I can not explain how to calculate. For the moment, I will try at the level of the graded tensor only. This is no so bad because F is defined up homotopy and modulo compact operators. In fact, the current formulation of classification theorems for finite spectral triples [4, 5] is able to avoid any specific value of F.

Our algebra is $A = \mathbf{C}^6$. Let $a \in KK(\mathbf{C}, A)$ be based in the graded space $H^+ = (p_1 A \oplus p_2 A), H^- = (p_4 A \oplus p_5 A)$. The Dirac element $\mu \in KK(A \otimes A^0, \mathbf{C})$ is based in our old pal[6], the space $\bigoplus E_{ij}$ with grading ΓE_{ij} =sign$(q_{ij})E_{ij}$. And $a \otimes a^0 \in A \otimes A^0$ acts in this space multiplying by $a_i a_j^0$.

With all this data, we only need to promote $KK(\mathbf{C}, A)$ to $KK(A^0, A \otimes A^0)$ with a trivial direct product and than compose the Kasparov groups. The resulting space, in $KK(A^0, \mathbf{C})$ has elements

$$H_{16} \oplus H_{21} \oplus H_{12} \oplus H_{23} \oplus H_{44} \oplus H_{55}$$

in the positive grading, and elements

$$H_{11} \oplus H_{22} \oplus H_{43} \oplus H_{54} \oplus H_{45} \oplus H_{56}$$

in the negative grading.

Under the action of A^0, each part has the same number of vectors in the positive and negative gradings. So the product with the K-theory will give in every case an element homotopic to zero.

In general, we see that a projector p_i of positive grading will give three vectors in the product space: $H_{i,i-1}, H_{i,i+1}$ in the positive side, $H_{i,i}$ in the negative. For the projectors described by $(1,1,0,0,0,0)$ and $(0,0,0,1,1,0)$ we have respectively Hilbert spaces

$$(H_{12} \oplus H_{16} \oplus H_{23} \oplus H_{21}) \oplus (H_{11} \oplus H_{22})$$

and

$$(H_{45} \oplus H_{43} \oplus H_{54} \oplus H_{56}) \oplus (H_{44} \oplus H_{55})$$

If we only look at the action of A^0, we are still in the same situation: both spaces are the same, some information has been lost, and Kasparov product $\beta \otimes_{A^0}$ is unable to bring us back to the original element.

But if we could remember the origin of each vector then we could go back to the right one: the first projector gives the space $H_{23} \oplus H_{16}$ and the second one gives $H_{43} \oplus H_{56}$. But if we want to distinguish between both spaces, the A^0 action is not enough: we need to give again a role to the Fredholm operator F (or to its "unbounded" counterpart, the Dirac operator).

A way to restrict the homotopies of F could be searched in the ambiguity of the Dirac operator. As we told in [6], the limit procedure has an angular freedom, we can define $m_{l-1\,l,ll} = (1/n)\sin\theta$, $m_{ll,l+1\,l} = -(1/n)\cos\theta$ and we still get the same continuous limit. If we study the index of D as θ varies, Index Theorem works and it is conserved. But we can notice that the symmetric derivative, $\theta = \pi/4$, corresponds to a crossing of eigenvalues: just at that point, a pair of eigenvectors of D cross the zero. This could justify us to avoid the symmetric point and to choose a nonsymmetric value for θ. Suppose you choose $\theta < \pi/4$. Then we have a distinction between the Hilbert subspaces $H_{i-1,i}$ and $H_{i+1,i}$. This is more patent if we move to just the backward derivative, $\theta = 0$, where one of the subspaces is directly in the nullspace of D.

Really the extreme cases 0 and $\pi/2$ amount to reduce the spectral triple to a simpler one, so they are no so desiderable. It should be better to kept just a slight asymmetry and to do the Kasparov product keeping this Dirac operator all the way. Thus our conclusion is that more work is needed in Kasparov products to define suitable dualities in finite algebras.

3 Worth mentioning

- When taking the $n \to \infty$ limit, one needs more groups beyond the KK^0. Fortunately they are directly produced by entering Clifford algebras into play, and we get the needed generators in our limit. (for the role of Clifford algebras in the real spectral triples, and all the reduction and unreduction game, see the book [3]).

- It could be interesting to build a relative KK doing contractions of two points to one. It could be interesting also to see all the sequence of discrete spaces as different formulations of discrete derivatives. And if we are really, really interested in this play one should investigate if the Hopf algebra of Connes-Moscovici has a role linking spaces of different size, then aiming to go from a bare series to a renormalized one.

- If you remember the diagonalization procedure of the previous paper, you could note that the vectors $E_{ij} + E_{ji}$ are in the kernel of the Dirac operator, while the vectors $E_{ij} - E_{ji}$ are just the piece one needs to counterweight the

$1/n$ divergence of D, then getting a finite contribution for the one-dimensional derivative. Note also that the E_{ii} vectors, for the same reason, induce a divergent derivative, which is controlled because it goes to an imaginary part: we have "finite $+i$ infinite".

- If the renormalization process forces the survival of some vestige of the approximation procedure, then we will have a spectre of Connes-Lot particles in the continuum. In some sense this is a low-profile approach to the big project of studying quantum groups and general diffeomorfism. Note also that a residual Dirac operator reflects, via Lichnerowitz, a curvature, and then it is a way to proof that the spectrum of particles is bounded away from zero, at least for the particular case coming from discretization and renormalization.
- Of course the physical interest of getting asymmetry into the Dirac operator is because we need diferent masses and mixes in the generations of particles.

References

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(Also included in vol. 2 of the "serie jornades cientifiques" del "Institut d'Estudis Catalans") See also recent reviews in hep-th by this author.

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