HU-EP-00/22
April 25, 2020

A Small Cosmological Constant and Backreaction of Non-Finetuned Parameters

Axel Krause[1]

Humboldt-Universität, Institut für Physik, D-10115 Berlin, Germany

Abstract

We include the backreaction on the warped geometry induced by non-finetuned parameters for a recently proposed mechanism to obtain an exponentially small cosmological constant Λ_4. It is shown that by separating two domain-walls by a distance $2l$ the cosmological constant appears exponentially suppressed with suppression-length l. Thus no huge hierarchy is required to obtain a realistic Λ_4. Moreover, we find a smooth connection to the limit with finetuned parameters.

PACS: 4.50.+h, 11.10.Kk
Keywords: Cosmological Constant, Warp-Geometry, Extra Dimensions
Los Alamos Database Number: hep-th/0007233

[1]E-mail: axkrause@physik.hu-berlin.de

Recently, a mechanism for obtaining the small observed value of the cosmological constant $\Lambda_4 \simeq 10^{-47}\text{GeV}^4$ has been proposed in [1]. It requires a five-dimensional (or higher) set-up consisting of two four-dimensional positive-tension T domain-walls (there is no need for either the bulk or the walls to be supersymmetric) separated by a distance $2l$ along the fifth noncompact dimension. Together with bulk gravity and a non-positive bulk cosmological constant $\Lambda(x^5) \leq 0$ the set-up is described by the action

$$S = -\int d^5x \left(\sqrt{G}\left[M^3 R(G) + \Lambda(x^5)\right] + \sqrt{g^{(4)}}T\left[\delta(x^5+l) + \delta(x^5-l)\right]\right) . \tag{1}$$

Neither of the walls is conceived as hidden but instead they are together responsible for the remaining three forces beyond gravity. By a string-embedding of the set-up and the realization of the domain-walls as two stacks of D3-branes, one can think of the Standard-Model gauge group SU(3) arising from one stack and the SU(2) × U(1) from the other [1]. For finetuned parameters the set-up leads to a warped geometry containing a flat 4-dimensional spacetime

$$\begin{aligned} ds^2 &= e^{-A(x^5)}\eta_{\mu\nu}dx^\mu dx^\nu + (dx^5)^2 \;\; ; \;\; \mu,\nu = 1,\ldots,4 \\ A(x^5) &= \frac{k}{2}\left(|x^5+l| + |x^5-l|\right) \;\; , \;\; k = \sqrt{-\Lambda_e/3M^3} \end{aligned} \tag{2}$$

with the bulk cosmological constant $\Lambda(x^5)$ and wall-tension T given by

$$\Lambda(x^5) = \begin{cases} \Lambda_e & , \; |x^5| > l \\ \Lambda_e/4 & , \; |x^5| = l \\ 0 & , \; |x^5| < l \end{cases} \quad , \qquad T = \sqrt{-3M^3\Lambda_e} \, . \tag{3}$$

A non-vanishing Λ in the interior region $|x^5| < l$ could be turned on if the tension of both walls would no longer be equal [1]. The exponential warp-factor eventually accounts for an exponential suppression by e^{-kl} of the effective 4-dimensional Λ_4 if the finetuning is suspended to such a degree that the backreaction on the warp-factor is negligible. The more we pull the two walls apart (enlarge the thickness of our world) the more we are able to lower Λ_4. Due to the exponential factor no huge hierarchy is required and a distance $2l \simeq 1/M_{\text{GUT}}$ is already enough to obtain a realistic Λ_4. Note, that unlike other approaches to achieve a zero/small cosmological constant (e.g. [2],[3]), we do not need the addition of a further bulk scalar field.

In this letter, we want to determine the full backreaction of non-finetuned parameters on the warped geometry and demonstrate that the resulting Λ_4 still comes out exponentially suppressed. Therefore, the mechanism to obtain a hierarchically small Λ_4 by separating the two walls sufficiently far from each works without the ususal finetuning between fundamental bulk parameters and wall tensions.

To this aim, we have to determine the resulting 5-dimensional geometry for general non-positive $\Lambda \leq 0$ and positive $T > 0$. Let us start with a D-dimensional warped geometry

$$ds^2 = G_{MN} dx^M dx^N = f(x^D) g_{\mu\nu}(x^\rho) dx^\mu dx^\nu + (dx^D)^2 \ , \tag{4}$$

with $\mu, \nu, \rho = 1, \ldots, D-1$ and the warp-factor $f(x^D)$. The induced metric on a $(D-1)$-dimensional section defined by $x^D = \text{const}$, will be denoted by $g^{(D-1)}_{\mu\nu}(x^\rho, x^D) = f(x^D) g_{\mu\nu}(x^\rho)$. Eventually, we want to solve the Einstein equation to determine the lower-dimensional Λ_4 for the case $D = 5$. Therefore, we decompose the D-dimensional Ricci-tensor R_{MN} into its μ and D components

$$\begin{aligned} R_{\mu\nu}(G) &= R_{\mu\nu}(g) + \frac{1}{4} g_{\mu\nu} \left(2f'' + (D-3) f \left[(\ln f)' \right]^2 \right) \\ R_{\mu D}(G) &= 0 \\ R_{DD}(G) &= \frac{1}{4}(D-1) \left(2\frac{f''}{f} - \left[(\ln f)' \right]^2 \right) \ . \end{aligned} \tag{5}$$

This allows to decompose the D-dimensional Einstein-tensor $E_{MN}(G) = R_{MN} - \frac{1}{2} R(G) G_{MN}$ as

$$\begin{aligned} E_{\mu\nu}(G) &= E_{\mu\nu}(g) + g_{\mu\nu} \frac{(D-2)}{2} \left[\left(1 - \frac{(D-1)}{4} \right) f \left[(\ln f)' \right]^2 - f'' \right] \\ E_{\mu D}(G) &= 0 \\ E_{DD}(G) &= -\frac{1}{2f} R(g) - \frac{(D-1)(D-2)}{8} \left[(\ln f)' \right]^2 \ . \end{aligned} \tag{6}$$

Let us now restrict to $D = 5$, where the expressions simplify to

$$\begin{aligned} E_{\mu\nu}(G) &= E_{\mu\nu}(g) - \frac{3}{2} g_{\mu\nu} f'' \\ E_{\mu 5}(G) &= 0 \\ E_{55}(G) &= -\frac{1}{2f} R(g) - \frac{3}{2} \left[(\ln f)' \right]^2 \ . \end{aligned} \tag{7}$$

For the action (1) specifying the set-up, the gravitational sources consist of a non-positive bulk cosmological constant $\Lambda(x^5) \leq 0$ and walls with tension T placed at $x^5 = l$ and $x^5 = -l$, which amounts to the following energy-momentum tensor

$$T_{MN} = -\Lambda(x^5) G_{MN} - T \left[\delta(x^5 + l) + \delta(x^5 - l) \right] g^{(4)}_{\mu\nu} \delta^\mu_M \delta^\nu_N \ . \tag{8}$$

Decomposing the 5-dimensional Einstein-equation, $E_{MN}(G) = -T_{MN}/(2M^3)$, with the help of (7) into its μ and 5 components, we receive from the $\mu\nu$ part the 4-dimensional Einstein-equation

$$E_{\mu\nu}(g) = \left[\frac{3}{2}f'' + \frac{f}{2M^3}\left[\Lambda(x^5) + T\delta(x^5+l) + T\delta(x^5-l)\right]\right] g_{\mu\nu} \; . \tag{9}$$

From the 55 part follows an expression for the 4-dimensional curvature scalar

$$R(g) = -f\left[3\left[(\ln f)'\right]^2 + \frac{\Lambda(x^5)}{M^3}\right] \; , \tag{10}$$

whereas the $\mu 5$ part is satisfied trivially.

Contraction of $E_{\mu\nu}(g)$ with $g^{\mu\nu}$ gives $E^\mu{}_\mu(g) = \frac{3-D}{2}R(g) \to -R(g)$ and therefore leads to the following consistency equation among (9) and (10)

$$2\frac{f''}{f} - \left[(\ln f)'\right]^2 = -\frac{1}{3M^3}\left[\Lambda(x^5) + 2T\delta(x^5+l) + 2T\delta(x^5-l)\right] \; . \tag{11}$$

It is evident that the right-hand-sides of (9) and (10) must be piecewise constant with respect to x^5, since both left-hand-sides are at least piecewise independent of x^5. This is a consequence of the simple warp-factor Ansatz. It means that the 4-dimensional sections Σ_4, defined by $x^5 = const$, must be spacetimes of constant curvature. For $R(g) < 0$ we have de Sitter (dS_4) and for $R(g) > 0$ Anti-de Sitter (AdS_4) spacetime. Since this already determines the solution to the Einstein equation up to a scalar quantity – the curvature – the equations (9),(10),(11) become linear dependent and it suffices to solve only two of them.

When we foliate the 5-dimensional spacetime into sections Σ_4, we see that the Einstein-equations (9),(10) also follow from the 4-dimensional action on Σ_4

$$S_{D=4}(x^5) = -\int_{\Sigma_4} d^4x\sqrt{g}\left(M_{\text{eff}}^2 R(g) + \Lambda_4(x^5)\right) \tag{12}$$

if we make the following identifications[2]

$$\frac{3}{2}f'' + \frac{f}{2M^3}\left[\Lambda(x^5) + T\delta(x^5+l) + T\delta(x^5-l)\right] = \frac{\Lambda_4(x^5)}{2M_{\text{eff}}^2} \tag{13}$$

$$-f\left[3\left[(\ln f)'\right]^2 + \frac{\Lambda(x^5)}{M^3}\right] = -\frac{2\Lambda_4(x^5)}{M_{\text{eff}}^2} \; . \tag{14}$$

[2]The 4-dimensional sections exhibit

$$E_{\mu\nu}(g) = \frac{\Lambda_4(x^5)}{2M_{\text{eff}}^2}g_{\mu\nu} \; , \qquad R(g) = -2\frac{\Lambda_4(x^5)}{M_{\text{eff}}^2} \; ,$$

with $dS_4 : \; R(g) < 0, \Lambda_4(x^5) > 0$ and $AdS_4 : \; R(g) > 0, \Lambda_4(x^5) < 0$.

Here $M_{\rm eff}$ is the effective Planck-scale, as obtained by integrating the 5-dimensional action (1) over x^5

$$M_{\rm eff}^2 = M^3 \int dx^5 f(x^5) \ . \tag{15}$$

The Einstein equations (9),(10) now become replaced by (13),(14).

To recognize the relation between the cosmological constant $\Lambda_4(x^5)$ on sections Σ_4 and the final effective Λ_4 obtained by integrating out the fifth direction of (1), we note that Λ_4 is given by [1]

$$\Lambda_4 = \int dx^5 f^2 \left(M^3 \big[[(\ln f)']^2 + 4\frac{f''}{f}\big] + \big[\Lambda(x^5) + T\delta(x^5+l) + T\delta(x^5-l)\big] \right) \ . \tag{16}$$

Using (13) for the second term in square brackets, we obtain the simple relationship

$$\Lambda_4 = f'f|_{x_L^5}^{x_R^5} + \langle \Lambda_4(x^5) \rangle \ , \tag{17}$$

where x_R^5, x_L^5 denote the right and left boundary of the x^5 integration region and the mean is weighted with the profile of the warp-factor

$$\langle \Lambda_4(x^5) \rangle \equiv \frac{\int dx^5 f \Lambda_4(x^5)}{\int dx^5 f} \ . \tag{18}$$

Since we will see that the total derivative contribution $f'f|_{x_L^5}^{x_R^5}$ will vanish in our case of interest, we learn that the 4-dimensional effective action $S_{D=4}$ is related to the sectionwise action by taking the mean, $S_{D=4} = \langle S_{D=4}(x^5) \rangle$.

Since only two of the equations (11),(13),(14) are independent, it is most convenient to choose (11) to determine the warp-factor in terms of the fundamental "input" parameters $\Lambda(x^5)$, M and T. In a further step, we will then obtain $\Lambda_4(x^5)$ from (14). Expressing the warp-factor through $f = e^{-A(x^5)}$ and denoting $Y(x^5) = A'(x^5)$, we can write (11) as

$$-2Y' + Y^2 + \frac{\Lambda(x^5)}{3M^3} = -\frac{2T}{3M^3}\left[\delta(x^5+l) + \delta(x^5-l)\right] \ , \tag{19}$$

With the signature-function defined by $\mathrm{sign}(x) = -1$ if $x \le 0$ and $\mathrm{sign}(x) = 1$ if $x > 0$, the solution to this differential equation is given by

$$Y(x^5) = -\frac{k}{2}\big(\mathrm{sign}(x^5+l) + \mathrm{sign}(x^5-l)\big) \coth\left(\frac{k}{4}\left[|x^5+l| + |x^5-l| - 2a\right]\right) \tag{20}$$

together with the following $\Lambda(x^5)$ profile with arbitrary but non-positive $\Lambda_e \le 0$

$$\Lambda(x^5) = \begin{cases} \Lambda_e & , \ |x^5| > l \\ \Lambda_e/4 \le 0 & , \ |x^5| = l \\ 0 & , \ |x^5| < l \end{cases} \tag{21}$$

and the wall-tension

$$\frac{T}{3M^3} = k \coth\left(\frac{k}{2}(a-l)\right) . \tag{22}$$

Here, as in the introduction, $k = \sqrt{-\Lambda_e/3M^3}$ and a is an integration constant. The last relation which determines a through the bulk cosmological constant Λ_e and the wall-tension T has been gained by satisfying the boundary conditions at the wall-locations, which are encoded in the δ-function terms in (19). A matching of the δ-function terms arising from Y' with those proportional to T leads to (22). The symmetry of the set-up – caused by the equality of both wall-tensions – forces the bulk cosmological constant between them to be zero. A non-vanishing value can be obtained, if wished, if we introduce an asymmetry of the set-up through unequal wall-tensions. A further integration of Y yields the warp-function

$$A(x^5) = -2\ln\left|\sinh\left(\frac{k}{4}\left[|x^5+l| + |x^5-l| - 2a\right]\right)\right| + b , \tag{23}$$

where b is a second integration constant. Note, that the above solution is valid for the parameter-range $T \geq 3M^3k$ as can be easily recognized from (22). If $T < 3M^3k$, we have to substitute a "tanh" for the "coth" appearing in (20) and (22), while (21) remains the same. This amounts to a change from "sinh" to "cosh" in (23) Since we assume a positive wall-tension $T > 0$, the integration constant a is constrained through (22) over the whole parameter-region, $T > 0$, $\Lambda_e \leq 0$, by the lower bound $a > l$.

An important observation is that the warp-factor $f = e^{-A(x^5)}$ vanishes at $x^5 = \pm a$. If $Q < 0$ (which later will turn out to be the AdS_4 case, whereas the physically more relevant – since observations point to a positive Λ_4 – dS_4 case is free of singularities) this gives rise to a singular 5-dimensional curvature at these points

$$\lim_{x^5\to\pm a} R(G) \to \frac{24\Theta(-Q)}{(|x^5|-a)^2} , \qquad Q = \frac{T-3M^3k}{T+3M^3k} , \tag{24}$$

where the Heaviside step-function is defined by $\Theta(x) = 0, x < 0$ and $\Theta(x) = 1, x > 0$. Due to the vanishing of the warp-factor at these points we expect a tremendous red-shift in signals originating there. Indeed, let us conceive a wave signal emitted with frequency ν_e at $x^5 = \pm a$. Then that wave will be observed in the interior region $x^5 \in (-a, a)$ with frequency ν_o given by

$$\frac{\nu_o}{\nu_e} = \sqrt{\frac{G_{11}(x^5 = \pm a)}{G_{11}(|x^5| < a)}} = 0 , \tag{25}$$

due to the vanishing of the warp-factor at $x^5 = \pm a$. Hence, an infinite redshift makes it impossible for the region $|x^5| \geq a$ to communicate to our world (at least via electromagnetic radiation). Therefore, we should restrict the x^5 integration region to the causally connected interval $x^5 \in (-a, a)$.

Since recently there has been a discussion in the literature [4],[5],[6] about which singularities are permissible and which have better to be avoided, it is interesting to see the verdict on our singularities in the case of $Q < 0$. In [4] it has been argued that in a gravitational system exhibiting a 4-dimensional flat solution together with bulk scalars, only those singularities are allowed, which leave the scalar potential bounded from above. In our case, where we do not have any scalars, the role of the scalar potential is played by the bulk cosmological constant Λ_e (together with the tension T at the wall-positions), which is clearly bounded from above. If the criterion of [4] generalizes to the case where the 4-dimensional metric deviates slightly (since in the end Λ_4 turns out to be exponentially small) from the flat case, we would conclude that the singularities encountered above for $Q < 0$ are of the permissible type.

Furthermore, in [6] a consistency condition has been derived which should hold for the effective cosmological constant obtained by integration over the causally connected x^5-region. We will now demonstrate that this consistency condition is a simple consequence of (13),(14) and the expression (16), which defines Λ_4. Starting with (16) and employing (13),(14) to eliminate the derivatives $[(\ln f)']^2$ and f'', (16) becomes

$$\Lambda_4 = 2\langle\Lambda_4\rangle - \frac{1}{3}\int_{-a}^{a} dx^5 f^2 \left(2\Lambda(x^5) + T\delta(x^5 + l) + T\delta(x^5 - l)\right) \ . \tag{26}$$

Noticing that $f'f(x^5 = \pm a) = 0$, we use (17) to obtain

$$\begin{aligned}\Lambda_4 &= \frac{1}{3}\int_{-a}^{a} dx^5 f^2 \left(2\Lambda(x^5) + T\delta(x^5 + l) + T\delta(x^5 - l)\right) \\ &= -\frac{1}{3}\int_{-a}^{a} dx^5 f^2 \left(T_1{}^1 + T_5{}^5\right) \ ,\end{aligned} \tag{27}$$

which is nothing but the consistency condition of [6]. Since our solution has been derived from (11),(14) which are equivalent to (13),(14) and we will furthermore only require (16) to obtain Λ_4, we conclude that the consistency condition (27) of [6] should be satisfied for our solution.

After this short intermezzo on singularities, let us proceed by inverting (22), to express a explicitly through the input values T and Λ_e

$$a = -\frac{1}{k}\ln|Q| + l \ , \tag{28}$$

which is valid for both $T \geq 3M^3k$ and $T < 3M^3k$. This shows how the parameters T, M, Λ_e influence the width of the x^5 domain.

In order to determine $\Lambda_4(x^5)$, note that to obey the Einstein equations, we have to fulfill (14). This can be used to derive the following expressions for $\Lambda_4(x^5)$

$$\Lambda_4(x^5) = \pm\frac{3}{2}e^{-b}M_{\text{eff}}^2 \begin{cases} k^2 & , \ |x^5| > l \\ k^2/4 & , \ x^5 = \pm l \\ 0 & , \ |x^5| < l \end{cases} \tag{29}$$

Here, the plus-sign applies to the case $T \geq 3M^3k$, whereas the minus-sign applies to the complementary case in which $T < 3M^3k$. Since we do not want to use $\Lambda_4(x^5)$ as an input to determine b, but rather focus on the opposite, we are looking for an additional constraint, which allows for a determination of the constant b. This extra constraint comes from considering a smooth transition to the flat solution (2) with $\Lambda_4 = 0$. As can be seen from (3), we reach the flat limit by sending $T \to 3M^3k$. Via (28) this limit corresponds to sending the constant $a \to \infty$. Thus we see, that the integration region $x^5 \in (-a, a)$ extends over the whole real line in this limit and the warp-function (23) becomes

$$A(x^5) \to \frac{k}{2}\left(|x^5 + l| + |x^5 - l|\right) + 2\ln 2 - ka + b\ . \tag{30}$$

Thus, to guarantee a smooth transition to the flat solution (2), we have to identify the integration constants a and b as follows

$$b = -2\ln 2 + ka\ . \tag{31}$$

This, together with (28) and (29) yields the following expression for $\Lambda_4(x^5)$ in terms of physical "input" parameters

$$\Lambda_4(x^5) = 6e^{-kl}QM_{\text{eff}}^2 \begin{cases} k^2 & , \ |x^5| > l \\ k^2/4 & , \ x^5 = \pm l \\ 0 & , \ |x^5| < l \end{cases} \tag{32}$$

Notice, that this formula is valid for both parameter-regions $T \geq 3M^3k$ and $T < 3M^3k$.

Finally, to obtain the observable four-dimensional Λ_4, we have to take the mean of $\Lambda_4(x^5)$. Again using that $f'f(x^5 = \pm a) = 0$, we employ (17) and arrive at

$$\Lambda_4 = \frac{\int_{-a}^{a} dx^5 e^{-A(x^5)}\Lambda_4(x^5)}{\int_{-a}^{a} dx^5 e^{-A(x^5)}} = 12e^{-2kl}M^3kQF(|Q|)\ , \tag{33}$$

where we defined $F(|Q|) = 1 - |Q|^2 + 2|Q| \ln |Q|$. In addition we obtain the following effective Planck-scale

$$M_{\rm eff}^2 = M^3 \int_{-a}^{a} dx^5 e^{-A(x^5)} = 2e^{-kl} M^3 \left(l(1-|Q|)^2 + \frac{F(|Q|)}{k} \right) . \tag{34}$$

There is an exponential-factor occuring in Λ_4 which is the square of the one occuring in $M_{\rm eff}^2$. At the classical level (classical in the bulk of the five-dimensional spacetime – the field-theories on the walls are however considered quantum mechanically) an overall constant e^{-kl} multiplying the whole effective 4-dimensional action $S_{D=4} = -\int d^4x \sqrt{g}(M_{\rm eff}^2 R(g) + \Lambda_4) = -e^{-kl} \int d^4x \sqrt{g}(\tilde{M}_{\rm eff}^2 R(g) + \tilde{\Lambda}_4) = -e^{-kl} \tilde{M}_{\rm eff}^2 \int d^4x \sqrt{g}(R(g) + \lambda_4)$ is immaterial – it simply drops out of the field equation. Therefore, we can neglect the overall factor e^{-kl}. The physically observable cosmological constant – invariant under any overall rescaling – is given by $\lambda_4 = \Lambda_4 / M_{\rm eff}^2 = \tilde{\Lambda}_4 / \tilde{M}_{\rm eff}^2$. With (33) and (34) we thus obtain

$$\lambda_4 = e^{-kl} \left(\frac{6k^2 Q F(|Q|)}{kl(1-|Q|)^2 + F(|Q|)} \right) . \tag{35}$$

Let us make some comments about this formula. The physical range of the parameter Q lies between $0 \leq |Q| \leq 1$, where we presuppose a non-negative wall-tension $T > 0$. The lower bound corresponds to the finetuned flat $\Lambda_4 = 0$ limit, while the upper bound is reached for vanishing bulk cosmological constant $\Lambda_e = 0$. Over that domain we have $1 \geq F(|Q| < 1) > 0$, $F(1) = 0$. Hence, we recognize that starting with some fundamental values for $\Lambda_e \leq 0, M, T > 0$ we obtain a positive or negative λ_4 depending on the sign of Q. For $T > \sqrt{-3M^3\Lambda_e}$ the 4-dimensional spacetime will be dS_4, whereas for $T < \sqrt{-3M^3\Lambda_e}$ it will be AdS_4. Furthermore, we see a smooth connection to the case with flat 4-dimensional Minkowski spacetime for finetuned parameters $T = \sqrt{-3M^3\Lambda_e} \Leftrightarrow Q = 0$. Most importantly, there is no need for a finetuning of the fundamental parameters to receive a small λ_4. By adapting the distance $2l$ between both walls, one arrives at a huge enough suppression through the exponential factor such that the observed value could be accounted for. Moreover, thanks to the exponential suppression this does not amount to an extremely large hierarchy between the fundamental scale M and the separation-scale $1/2l$.

Acknowledgements

We are grateful to R. Blumenhagen, G. Curio and particularly A. Karch for discussions.

References

[1] A. Krause, *A Small Cosmological Constant, Grand Unification and Warped Geometry*, hep-th/0006226;

[2] N. Arkani-Hamed, S. Dimopoulos, N. Kaloper and R. Sundrum, *A Small Cosmological Constant from a Large Extra Dimension*, hep-th/0001197;

[3] S. Kachru, M. Schulz and E. Silverstein, *Selftuning Flat Domain Walls in 5-D Gravity and String Theory*, hep-th/0001206;

[4] S.S. Gubser, *Curvature Singularities: The Good, the Bad, and the Naked*, hep-th/0002160;

[5] S. Förste, Z. Lalak, S. Lavignac and H.P. Nilles, *A Comment On Selftuning and Vanishing Cosmological Constant in the Brane World*, *Phys. Lett.* **B 481** (2000) 360, hep-th/0002164;

[6] S. Förste, Z. Lalak, S. Lavignac and H.P. Nilles, *The Cosmological Constant Problem from a Brane-World Perspective*, hep-th/0006139;