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PERTURBATIVE DYNAMICS ON THE FUZZY S^2 AND $\mathbb{R}\mathrm{P}^2$

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Abstract

By considering scalar theories on the fuzzy sphere as matrix models, we construct a renormalization scheme and calculate the one-loop effective action. Because of UV-IR mixing, the two- and the four-point correlators at low energy are not slowly varying functions of external momenta. Interestingly, we also find that field theories on fuzzy $\mathbb{R}\mathrm{P}^2$ avoid UV-IR mixing and hence are much more like conventional field theories. We calculate the one-loop β-function for the $O(N)$ theory on fuzzy $\mathbb{R}\mathrm{P}^2$ at large N and show that in addition to the trivial one, it has a nontrivial fixed point that is accessible in perturbation theory.

1 Introduction

Noncommutative theories have generated considerable interest since the discovery that they arise in a certain limit of string theory [?]. They occur in the low energy descriptions of a class of brane configurations that have a non-zero B_{NS}-field turned on. Noncommutative manifolds also make their appearance in discussions of brane configurations in external Ramond-Ramond (RR) fields [?].

Theories on noncommutative manifolds also offer a novel method of discretization that is quite different from lattice discretization. In this approach the manifold $\mathcal{M}$ is treated as a phase space and is "quantized". If the manifold is compact, the total number of states is finite and we end up with a matrix model. The continuum physics corresponds to the limit in which the coordinates commute.

Quantum properties of noncommutative theories are often very different from the corresponding conventional ones, however. A striking demonstration of this fact is the peculiar mixing between ultraviolet (UV) and infrared (IR) degrees of freedom that occurs in quantum noncommutative theories [?]. More concretely, the correlation functions show extreme sensitivity to low momenta because of this mixing. In fact, it was shown by [?] that the two-point correlation function develops new poles and branch cuts at zero momentum. This makes the low energy analysis considerably more subtle.

In this article, we will explore the quantum version of a class of noncommutative theories in a controlled setting: scalar theories with quartic interaction on a fuzzy S^2 and $\mathbb{R}\mathrm{P}^2$. Because these manifolds are compact, modes of the scalar field are quantized and labelled by an

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angular momentum label l. Working on a compact manifold also provides a natural infrared cutoff, which gives us a better understanding about the nature of the IR singularities of the correlation functions. Also, since theories on the fuzzy S^2 are simply matrix models, we will be able to develop a scheme for doing renormalization by integrating out "shells" of high momenta. Roughly speaking, integrating out a single high (angular) momentum shell corresponds to writing the theory in terms of a matrix of smaller dimension. Rescaling corresponds to embedding this smaller matrix into a matrix of original size that we started with. In the case of ordinary theories, this procedure of integrating out a high momentum shell allows us to write down the renormalization group (RG) equations for the various couplings. We will show that as a consequence of UV-IR mixing, the low-momentum correlation functions will behave very differently for even and odd (angular) momenta. In other words, there is no smooth way to approach zero momentum.

We also examine field theories on fuzzy $\mathbb{RP}^2$, i.e. theories constructed from fields that take the same value at antipodes of the two-sphere. Interestingly, we find that these theories show no UV-IR mixing and behave much more like conventional field theories. In particular, we will demonstrate that they have a low energy effective Wilsonian description, which will allow us to calculate the 1-loop β-function. For a single component scalar theory, we will find that the non-trivial fixed point is very far away from the zero mass Gaussian fixed point, and hence its existence cannot be trusted. For $O(N)$ theories however, there is a nontrivial fixed point at small coupling if N is sufficiently large, and the perturbative calculation is trustworthy.

The earliest investigations of the fuzzy sphere S_F^2 were by done by [?], followed by works of [?, ?]. Solitons and monopoles in non-linear σ-models were studied by [?] (see also [?]). Topological issues such as instantons, θ-term and derivation of the chiral anomaly on fuzzy S^2 were discussed in [?]. (For an alternate derivation of the chiral anomaly, see [?].) The continuum limit of the fuzzy non-linear σ-model has been discussed in [?]. Interest in S_F^2 has also increased since Myers showed that $D0$-branes in a constant RR field arrange themselves in the form of a fuzzy sphere [?]. There have also been investigations by [?] regarding open string versions of WZW models which naturally lead to S_F^2. Gauge theories on S_F^2 have also been studied by [?, ?]. Continuum limits of gauge theories on fuzzy sphere have also been discussed by [?].

The organization of the paper is as follows: in section 2, we describe scalar field theories on fuzzy S^2 and $\mathbb{RP}^2$, and make notational explanations. The details of the renormalization group procedure, namely decimation (i.e. integrating out a high energy shell) and rescaling, are explained in section 3. In section 4, we apply these techniques to quartic theories on fuzzy S^2 and demonstrate UV-IR mixing. We also look at theories on $\mathbb{RP}^2$ and derive the RG equations in section 5. The generalization to $O(N)$ theories is easy, and we show that there is a nontrivial fixed point at large N. Our conclusions are in section 6.

2 Scalar theory on the fuzzy sphere

Scalar theories on the fuzzy sphere were first discussed by [**?**, **?**]. We will review their work in a notation convenient for our purposes.

The fuzzy sphere S_F^2 is described by three operators X_a subject to the relations

$$\sum_{a=1}^{3} X_a X_a = R^2 \mathbf{1}, \text{and} \quad [X_a, X_b] = \frac{Ri\epsilon_{abc}}{\sqrt{j(j+1)}} X_c. \tag{2.1}$$

The limit $j \to \infty$ reproduces the ordinary sphere $\sum X_a X_a = R^2$. We will henceforth work with $R = 1$.

Functions on an ordinary sphere may be written as

$$f(x_a) = \sum (f_0 + f_a x_a + f_{ab} x_a x_b + \cdots) \equiv \sum_{l,m} f_{lm} Y_{lm}(\theta, \phi). \tag{2.2}$$

where $Y_{lm}(\theta, \phi)$ are the spherical harmonics. Under the replacement $X_a \to (1/\sqrt{j(j+1)}) J_a$, f is not a function anymore, but a $(2j+1) \times (2j+1)$ matrix. The set of all such matrices forms an algebra $\mathcal{A}_j$, which is the analog of the algebra of functions on an ordinary sphere. In particular, hermitian matrices are analogous to real functions. The scalar product on $\mathcal{A}_j$ is defined by

$$(f, g)_j = \frac{1}{2j+1} \mathrm{Tr}(f^\dagger g), \quad f, g \in \mathcal{A}_j. \tag{2.3}$$

An arbitrary "function" (i.e. a matrix) on the fuzzy sphere can be expanded in terms a special basis of matrices, namely the polarization operators $T_{lm}^{(j)}$. These are constructed out of the angular momentum operators J_3, $J_\pm = J_1 \pm iJ_2$ and various powers thereof (see for example [**?**]). For example, $T_{00}^{(j)} = \frac{1}{2j+1}\mathbf{1}, T_{1m}^{(j)} = -\sqrt{\frac{3}{2j(j+1)(2j+1)}} J_\pm$, etc. They are rank l irreducible tensors

$$[J_\pm, T_{lm}^{(j)}] = \sqrt{l(l+1) - m(m \pm 1)}\, T_{l,m\pm1}^{(j)}, \quad [J_3, T_{lm}^{(j)}] = m T_{lm}^{(j)}, \tag{2.4}$$

and satisfy the relations

$$(T_{lm}^{(j)})^\dagger = (-1)^m T_{l,-m}^{(j)}, \quad \mathrm{Tr}[T_{lm}^{(j)} T_{l'm'}^{(j)}] = (-1)^m \delta_{ll'} \delta_{m+m',0}. \tag{2.5}$$

These allow us to expand any matrix Φ as

$$\Phi = \sum_{l=0}^{2j} \sum_{m=-l}^{l} \phi_{lm} [(2j+1)^{1/2} T_{lm}^{(j)}] \equiv \sum \phi_{lm} \Psi_{lm}^{(j)}. \tag{2.6}$$

The $\Psi_{lm}^{(j)}$'s form an orthonormal set with respect to the product (2.3). The inner product of two arbitrary "functions" has the correct $j \to \infty$ limit.

If Φ is hermitian, then $\phi_{l,-m} = (-1)^m \bar{\phi}_{lm}$. Thus for a given l, the total number of independent real parameters is $2l + 1$.

We will study actions for for Φ that are of the form

$$\begin{aligned} S &= \frac{1}{2j+1}\mathrm{Tr}\left[-[J_i,\Phi][J_i,\Phi]+\mu_j^2\Phi^2+\sum_n \frac{\lambda_j^{(n)}}{n!}\Phi^n\right], \\ &= \frac{1}{2j+1}\mathrm{Tr}\left[\Phi[J_i,[J_i,\Phi]]+\mu_j^2\Phi^2+\sum_n \frac{\lambda_j^{(n)}}{n!}\Phi^n\right], \\ &\equiv S_0+S_{int}. \end{aligned} \tag{2.7}$$

As $j\to\infty$, this action goes over to

$$S=\frac{1}{4\pi}\int d\Omega\left[-\mathcal{L}_i\Phi\mathcal{L}_i\Phi+\mu^2\Phi^2+\sum_n\frac{\lambda^{(n)}}{n!}\Phi^n\right], \quad \text{where} \quad \mathcal{L}_i=-i\epsilon_{ijk}x_j\partial_k. \tag{2.8}$$

This is the same as the action for a scalar field on the the sphere.

Using (2.6), we write the free action as

$$S_0=\left[\sum_{l=0}^{2j}\sum_{m=-l}^{l}|\phi_{lm}|^2[l(l+1)+\mu_j^2]\right], \tag{2.9}$$

and the quartic interaction term Φ^4 as

$$\begin{aligned} S_{int} &= \frac{\lambda_j}{4!}\frac{1}{2j+1}\mathrm{Tr}\Phi^4 \\ &= \frac{\lambda_j}{4!}(2j+1)\phi_{l_1m_1}\phi_{l_2m_2}\phi_{l_3m_3}\phi_{l_4m_4}\mathrm{Tr}[\Psi^{(j)}_{l_1m_1}\Psi^{(j)}_{l_2m_2}\Psi^{(j)}_{l_3m_3}\Psi^{(j)}_{l_4m_4}] \\ &\equiv \phi_{l_1m_1}\phi_{l_2m_2}\phi_{l_3m_3}\phi_{l_4m_4}V(l_1,m_1;l_2,m_2;l_3,m_3;l_4,m_4;j). \end{aligned} \tag{2.10}$$

We will henceforth use the shorthand $V(1234;j)$ for the function $V(l_1,m_1;l_2,m_2;l_3,m_3;l_4,m_4;j)$ from now on. It can be conveniently written as

$$\begin{aligned} V(1234;j) &= \frac{\lambda_j}{4!}(2j+1)\prod_{i=1}^{4}(2l_i+1)^{1/2}\times \\ &\sum_{l,m}^{l=2j}\left\{\begin{array}{ccc} l_1 & l_2 & l \\ j & j & j\end{array}\right\}\left\{\begin{array}{ccc} l_3 & l_4 & l \\ j & j & j\end{array}\right\}(-1)^m C^{l_1\ l_2\ l}_{m_1m_2m}C^{l_3\ l_4\ l}_{m_3m_4-m}. \end{aligned} \tag{2.11}$$

The $C^{l_1\ l_2\ l}_{m_1m_2m}$ are the Clebsch-Gordan coefficients and the objects with 6 entries within brace brackets are the $6j$ symbols.

The partition function for the theory is

$$\mathcal{Z}_j = \int\mathcal{D}[\Phi]e^{-(S_0+S_{int})} \quad \text{where} \tag{2.12}$$

$$\mathcal{D}[\Phi] = \prod_{lm}^{l=2j}\frac{d\bar{\phi}_{lm}d\phi_{lm}}{2\pi i}. \tag{2.13}$$

Correlation functions

$$\langle \phi_{l_1 m_1} \cdots \phi_{l_k m_k} \rangle = \int \mathcal{D}[\Phi] \phi_{l_1 m_1} \cdots \phi_{l_k m_k} e^{-(S_0+S_{int})} \tag{2.14}$$

can be calculated using the standard procedure, by adding a source term $\frac{1}{2j+1}\mathrm{Tr}(J\Phi) = \sum_{lm}(\bar{J}_{lm}\phi_{lm} + J_{lm}\bar{\phi}_{lm})$ to the action and then taking derivatives with respect to J_{lm}. In particular, the two-point correlation function or the propagator for the free action is

$$\langle \phi_{l_1 m_1} \phi_{l_2 m_2} \rangle = \frac{\delta_{l_1 l_2} \delta_{m_1+m_2,0} (-1)^{m_2}}{l(l+1) + \mu_j^2}. \tag{2.15}$$

2.1 Scalar theories on fuzzy $\mathbb{R}\mathrm{P}^2$

The manifold $\mathbb{R}\mathrm{P}^2$ is obtained from S^2 by identifying diametrically opposite points $\hat{r}$ and $-\hat{r}$. Functions on $\mathbb{R}\mathrm{P}^2$ are simply a subset of the functions on S^2 that are invariant under this identification. This means that only even l values are allowed when the function is expanded in terms of spherical harmonics, because $Y_{lm}(-\hat{r}) = (-1)^l Y_{lm}(\hat{r})$.

In the noncommutative case, we will be interested in fields that are invariant under $\vec{J} \to -\vec{J}$. Again, since $\Psi^{(j)}_{lm}(-\vec{J}) = (-1)^l \Psi^{(j)}_{lm}(\vec{J})$, we have to restrict l to even values. We stress that it is only in this sense that we use the phrase 'fuzzy $\mathbb{R}\mathrm{P}^2$'.

We can now write the scalar action on the fuzzy $\mathbb{R}\mathrm{P}^2$: it is of the same form as (2.7), with the additional restriction that all the l's be even, and hence j an integer.

2.2 Multicomponent Scalar field theories

It is easy to generalize to $O(N)$ scalar field theories. There are N hermitian scalar fields $\Phi^\alpha, \alpha = 1 \cdots N$. The "linear" σ-model on S_F^2 has the action

$$S = \frac{1}{2j+1} \mathrm{Tr}\left(-[J_i, \Phi^\alpha][J_i, \Phi^\alpha] + \mu_j^2 \Phi\cdot\Phi + V(\Phi\cdot\Phi)\right) \tag{2.16}$$

where $\Phi \cdot \Phi = \sum_{\alpha=1}^N \Phi^\alpha \Phi^\alpha$. We expand the fields Φ^α as

$$\Phi^\alpha = \sum_{l,m}^{l=2j} \phi^\alpha_{lm} \Psi^{(j)}_{lm}. \tag{2.17}$$

The free action can be evaluated to be

$$S_{free} = \sum_\alpha \sum_{l,m} (l(l+1) + \mu_j^2) |\phi^\alpha_{lm}|^2 \tag{2.18}$$

and the propagator is

$$\langle \phi^\alpha_{l_1 m_1} \phi^\beta_{l_2 m_2} \rangle = \frac{\delta^{\alpha\beta} \delta_{l_1 l_2} \delta_{m_1+m_2,0} (-1)^{m_2}}{l(l+1) + \mu_j^2}. \tag{2.19}$$

If $V(\Phi \cdot \Phi) = \frac{\lambda_j}{4!}(\Phi \cdot \Phi)^2$, then we write this quartic interaction as

$$S_{int} = \sum_{\alpha,\beta} \sum_{l_i,m_i} \phi^{\alpha}_{l_1 m_1} \phi^{\alpha}_{l_2 m_2} \phi^{\beta}_{l_3 m_3} \phi^{\beta}_{l_4 m_4} V(1234; j), \tag{2.20}$$

where $V(1234; j)$ is the same as in (2.11).

For fuzzy $\mathbb{RP}^2$ the formulae are the same as above, but with the restriction that all the l's are even and j is an integer.

3 Renormalization Procedure

From our expansion for the matrix Φ, we notice that the "energy" of the mode ϕ_{lm} increases as $l(l+1)$. In other words, modes with large l correspond to high energy fluctuations. Our strategy from renormalization will be very simple and very much in the spirit described by Wilson [?] (there are many excellent reviews as well, see for e.g. [?, ?]). We start with a theory with some large value of the cutoff j. The field Φ is described in terms of $(2j+1) \times (2j+1)$ matrices, and the theory has modes going all the way till $\phi_{2j,m}$. We separate the modes $\phi_{2j,m}$ as the fast degrees of freedom and explicitly integrate out these modes in the partition function, to get an effective action that depends only on the modes $\phi_{00}, \cdots, \phi_{2j-1,-m}, \cdots, \phi_{2j-1,m}$. For large j, the effective action should be describable in terms of the field $\Phi_{slow} = \sum_{lm}^{l=2j-1} \phi_{lm} \Psi^{(j-\frac{1}{2})}_{lm}$, i.e. in terms of $2j \times 2j$ matrices. We now "rescale" the field Φ_{slow} by embedding the $2j \times 2j$ matrix in a $(2j+1) \times (2j+1)$ matrix. This rescaling redefines

$$\phi_{lm} \to \phi'_{l'm'} \quad l' = 0 \cdots 2j, \tag{3.1}$$

$$\mu_j^2 \to \mu'^2_j, \tag{3.2}$$

$$\lambda_j \to \lambda'_j, \tag{3.3}$$

and so on. Comparing the rescaled action to the original one gives us the renormalization group equations for μ_j^2 and λ_j. Since rescaling corresponds to $(2j-1) \to 2j$, the rate of change of the various couplings is given by the equations

$$\frac{d\mu_j^2}{dt} \equiv 2j(\mu'^2_j - \mu_j^2) = M(\mu_j^2, \lambda_j), \tag{3.4}$$

$$\frac{d\lambda_j}{dt} \equiv 2j(\lambda'_j - \lambda_j) = G(\mu_j^2, \lambda_j) \tag{3.5}$$

where dt is given by $(2j-1)/2j = e^{dt}$. In particular the zeroes of $G(\mu_j^2, \lambda_j)$ tells us the fixed points of the β-function.

Let us first understand the scaling of the field Φ. Recall that the free massless action is

$$S = \frac{1}{2j+1} \mathrm{Tr}(\Phi[J_i, [J_i, \Phi]]) = \sum_{l=0}^{2j} \sum_{m=-j}^{j} |\phi_{lm}|^2 [l(l+1)]. \tag{3.6}$$

The action for the slow variables is simply

$$S_{slow} = \sum_{l=0}^{2j-1} \sum_{m=-j+1/2}^{j-1/2} |\phi_{lm}|^2 [l(l+1)] \tag{3.7}$$

which we rewrite as

$$S_{slow} = \frac{1}{2j} \mathrm{Tr}(\Phi_{slow}[J_i, [J_i, \Phi_{slow}]]), \tag{3.8}$$

where

$$\Phi_{slow} = \sum_{l=0}^{2j-1} \phi_{lm} \Psi_{lm}^{(j-\frac{1}{2})}, \tag{3.9}$$

the $\Psi_{lm}^{(j-\frac{1}{2})}$ being $2j \times 2j$ matrices. These $\Psi_{lm}^{(j-\frac{1}{2})}$ are simply the orthonormal basis in terms of which we can expand any $2j \times 2j$ matrix. We embed each $\Psi_{lm}^{(j-\frac{1}{2})}$ in a $(2j+1) \times (2j+1)$ matrix as

$$\begin{bmatrix} \Psi_{lm}^{(j-\frac{1}{2})} & 0 \\ 0 & 0 \end{bmatrix} \equiv \Psi'^{(j)}_{lm} \tag{3.10}$$

allowing us to write

$$\Phi' = \sum_{l=0}^{2j-1} \phi_{lm} \Psi'^{(j)}_{lm}. \tag{3.11}$$

This Φ' is a $(2j+1) \times (2j+1)$ matrix, and hence can be re-expressed using the $\Psi_{lm}^{(j)}$'s:

$$\Phi'^{(j)} = \sum_{l=0}^{2j} \phi'_{lm} \Psi_{lm}^{(j)}. \tag{3.12}$$

This is the rescaling operation in our RG procedure. It gives us the relation between the ϕ_{lm}'s and the ϕ'_{lm}'s:

$$\phi'_{l'm'} = \sum_{l,m}^{l=2j-1} \left[\phi_{lm} \sqrt{\frac{(2l+1)(2l'+1)}{2j(2j+1)}} \left(\sum_{\nu,\nu'} C^{j-\frac{1}{2}}_{\nu}{}^{l}_{m}{}^{j-\frac{1}{2}}_{\nu'} C^{\;j}_{\nu+\frac{1}{2}}{}^{l'}_{m'}{}^{j}_{\nu'+\frac{1}{2}} \right) \right]. \tag{3.13}$$

Thus

$$S_{slow} = \frac{1}{2j} \mathrm{Tr}(\Phi'[J_i, [J_i, \Phi']]) = \frac{2j+1}{2j} \sum_{l=0}^{2j} \sum_{m} |\phi'_{lm}|^2 [l(l+1)] \tag{3.14}$$

The actions S and S_{slow} can now be identified. In order that the kinetic energy term look the same in the original and the rescaled action, the ϕ's must be scaled as

$$\phi_{lm} = \left(\frac{2j+1}{2j} \right)^{1/2} \phi'_{l'm'} \delta_{ll'} \delta_{mm'} = \left(\frac{2j+1}{2j} \right)^{1/2} \phi'_{lm} \tag{3.15}$$

for $l << 2j$. Similar considerations tell us that the mass μ_j^2 scales as:

$$\mu'^2_j = \frac{2j+1}{2j} \mu_j^2. \tag{3.16}$$

Now that we know how to do the rescaling, it is straightforward to write down the effective action. We separate S into S_{slow} and S_{fast}:

$$\mathcal{Z} = \int \mathcal{D}[\Phi] e^{-(S_0+S_{int})} = \int \mathcal{D}[\Phi_{slow}] e^{-(S_0(slow)+S_{int}(slow))} \int \mathcal{D}[\Phi_{fast}] e^{-(S_0(fast)+\delta S[\Phi_{slow},\Phi_{fast}])}. \tag{3.17}$$

We explicitly do the integral over the fast variables $\Phi_{fast} = \sum \phi_{2j,m} \Psi^{(j)}_{2j,m}$:

$$\int \mathcal{D}[\phi_{fast}] e^{-(S_0(fast)+\delta S[\Phi_{slow},\Phi_{fast}])} \equiv \langle e^{-\delta S[\Phi_{slow},\Phi_{fast}]} \rangle_f = e^{-S'_{slow}} \tag{3.18}$$

where $\langle \mathcal{O} \rangle_f$ stands for the expectation value of $\mathcal{O}$ over the fast degrees of freedom.

Finally, we rescale the variables so that the two actions look similar:

$$\int \mathcal{D}[\Phi_{slow}] e^{-(S_0(slow)+S_{int}(slow)+S'_{slow})} = \int \mathcal{D}[\Phi'] e^{-(S_0[\Phi']+S_{int}[\Phi'])}. \tag{3.19}$$

A comparison of the coupling constants of the original theory and the rescaled theory gives us the RG equations.

For fields on fuzzy $\mathbb{RP}^2$, the analysis is almost identical. Since j takes only integer values, integrating out the high energy modes $\phi_{2j,m}$ leaves us with the modes $\phi_{00}, \cdots, \phi_{2j-2,m}$, which are then reassembled into a $(2j-2) \times (2j-2)$ matrix. Rescaling corresponds to $(2j-2) \to 2j$. For $l << 2j$ we find that ϕ_{lm} and μ_j^2 must scale as

$$\phi_{lm} = \left(\frac{2j+2}{2j}\right)^{1/2} \phi'_{lm}, \tag{3.20}$$

$$\mu'^2_j = \frac{2j+2}{2j} \mu_j^2. \tag{3.21}$$

4 Quartic Theories

We are now ready to apply the RG technique described above to quartic interactions. The action cannot be evaluated exactly, and we will resort to perturbation theory.

We separate the interaction piece S_{int} of the full action into slow and fast parts:

$$\begin{aligned} S_{int} &= \phi_1\phi_2\phi_3\phi_4 V(1234; j) \\ &= S^{slow}_{int} + 4\phi_1\phi_2\phi_3\phi_{f_4} V(123f_4) + 4\phi_1\phi_2\phi_{f_3}\phi_{f_4} V(12f_3f_4) \\ &+ 2\phi_1\phi_{f_2}\phi_3\phi_{f_4} V(1f_23f_4) + 4\phi_1\phi_{f_2}\phi_{f_3}\phi_{f_4} V(1f_2f_3f_4) + S^{fast}_{int} \\ &\equiv S^{slow}_{int} + \delta S_1 + \delta S_2 + \delta S_3 + S^{fast}_{int} = S^{slow}_{int} + \delta S. \end{aligned} \tag{4.1}$$

We have used the shorthand

$$\begin{aligned} \phi_1 &= \phi_{l_1m_1}, \quad \phi_{f_1} = \phi_{2j,m_1}, \quad \text{etc} \\ V(123f_4) &= V(l_1, m_1; l_2, m_2; l_3, m_3; 2j, m_4; j) \quad \text{etc.} \end{aligned}$$

Having separated the action into slow and fast parts, we proceed to evaluate (3.18) using perturbation theory. In order to do so, we make use of the cumulant expansion:

$$\left\langle e^{-\delta S}\right\rangle_f = \exp\left(-\langle\delta S\rangle_f + \frac{\langle\delta S^2\rangle_f - \langle\delta S\rangle_f^2}{2} + \cdots\right). \tag{4.2}$$

The first term of the exponent is

$$\begin{aligned}\langle\delta S\rangle_f &= \langle S_{int}^{fast}\rangle_f + 4\phi_1\phi_2\phi_3 V(123f_4)\langle\phi_{f_4}\rangle_f + 4\phi_1 V(1f_2f_3f_4)\langle\phi_{f_2}\phi_{f_3}\phi_{f_4}\rangle_f \\ &+ 4\phi_1\phi_2 V(12f_3f_4)\langle\phi_{f_3}\phi_{f_4}\rangle_f + 2\phi_1\phi_3 V(1f_23f_4)\langle\phi_{f_2}\phi_{f_4}\rangle_f.\end{aligned} \tag{4.3}$$

The second and the third terms are zero, while the first term is an irrelevant constant. The fourth term involves summing over m_3 and m_4, each sum going over the range $-2j$ to $2j$. Similarly the last term involves summing over m_2 and m_4. To calculate these sums, we will make extensive use of identities for summing $3j$- and $6j$ symbols found in [**?**]. Using (2.15), we can write this term as

$$\langle\delta S\rangle_f = |\phi_{lm}|^2\frac{\lambda_j}{4!}\left[\frac{4(4j+1)}{2j(2j+1)+\mu_j^2} + (-1)^l\frac{2(2j+1)(4j+1)}{2j(2j+1)+\mu_j^2}\left\{\begin{array}{ccc} j & j & l \\ j & j & 2j\end{array}\right\}\right]. \tag{4.4}$$

The scaling relation for ϕ's (3.15) gives us the renormalized two-point vertex:

$$G^{-1}(\phi\phi') = \langle\phi_{lm}\phi_{l'm'}\rangle^{-1} = \delta_{ll'}\delta_{m+m',0}(-1)^m[l(l+1) + \mu_j'^2] \tag{4.5}$$

where

$$\mu'^2 = \left(\frac{2j+1}{2j}\right)\left[\mu_j^2 + \frac{2\lambda_j}{4!}A(2j)(2+(-1)^l B(l,2j))\right], \tag{4.6}$$

and

$$A(2j) = \frac{2(2j)+1}{2j(2j+1)+\mu_j^2}, \quad B(l,2j) = \frac{(2j+1)!(2j)!}{(2j-l)!(2j+l+1)!}. \tag{4.7}$$

The term in (4.6) with $(-1)^l$ is due to noncommutative effects, and is not present in ordinary theories.

To calculate the one-loop contribution to the four-point function, we need the next term of the cumulant expansion (4.2). This automatically counts only the connected diagrams. The contribution to the four-point function comes only from δS_2. We find that

$$\begin{aligned}\langle\delta S_2^2\rangle &= 16\phi_1\phi_2\phi_{1'}\phi_{2'}V(12f_3f_4)V(1'2'f_{3'}f_{4'})\langle\phi_{f_3}\phi_{f_4}\phi_{f_{3'}}\phi_{f_{4'}}\rangle_f \\ &+ 16\phi_1\phi_2\phi_{1'}\phi_{3'}V(12f_3f_4)V(1'f_{2'}3'f_{4'})\langle\phi_{f_3}\phi_{f_4}\phi_{f_{2'}}\phi_{f_{4'}}\rangle_f \\ &+ 4\phi_1\phi_3\phi_{1'}\phi_{3'}V(1f_23f_4)V(1'f_{2'}3'f_{4'})\langle\phi_{f_2}\phi_{f_4}\phi_{f_{2'}}\phi_{f_{4'}}.\end{aligned} \tag{4.8}$$

The three terms can be evaluated using various Clebsch-Gordan identities [**?**] to give the following:

$$\frac{1}{2}\left(\langle\delta S^2\rangle-\langle\delta S\rangle^2\right)=\frac{\lambda_j^2}{(4!)^2}\sum_{l_i,m_i}\phi_1\phi_2\phi_3\phi_4A^2(2j)(2j+1)^2\prod_{i=1}^{4}(2l_i+1)^{1/2}\times$$

$$\left[\sum_{l,m}F(l_i,m_i;l)(\Gamma_1(l_i;l,2j)+\Gamma_2(l_i;l,2j)+\Gamma_3(l_i;l,2j)\right], \tag{4.9}$$

$$\equiv\sum_{l_i,m_i}\phi_1\phi_2\phi_3\phi_4\Gamma_4^{(1-loop)}(l_i,m_i;j) \tag{4.10}$$

where $A(2j)$ is from (4.7) and

$$F(l_i,m_i;l) = (-1)^m C^{l_1\ l_2\ l}_{m_1m_2m}C^{l_3\ l_4\ l}_{m_3m_4-m}, \tag{4.11}$$

$$\Gamma_1(l_i;l,2j) = 8(1+(-1)^l)\times \times\left\{\begin{array}{ccc} l_1 & l_2 & l\\ j & j & j\end{array}\right\}\left\{\begin{array}{ccc} l_3 & l_4 & l\\ j & j & j\end{array}\right\}\left\{\begin{array}{ccc} 2j & 2j & l\\ j & j & j\end{array}\right\}^2, \tag{4.12}$$

$$\Gamma_2(l_i;l,2j) = 8(-1)^{2j+l_4}(1+(-1)^l)\times \times\left\{\begin{array}{ccc} l_1 & l_2 & l\\ j & j & j\end{array}\right\}\left\{\begin{array}{ccc} 2j & 2j & l\\ j & j & j\end{array}\right\}\left\{\begin{array}{ccc} l_3 & l_4 & l\\ j & j & 2j\\ j & j & 2j\end{array}\right\}, \tag{4.13}$$

$$\Gamma_3(l_i;l,2j) = 2[(-1)^{l_2+l_4}+(-1)^{l_3+l_4}](-1)^l\times \times\left\{\begin{array}{ccc} l_1 & l_2 & l\\ j & j & 2j\\ j & j & 2j\end{array}\right\}\left\{\begin{array}{ccc} l_3 & l_4 & l\\ j & j & 2j\\ j & j & 2j\end{array}\right\}, \tag{4.14}$$

where the objects with 9 entries within brace brackets are the $9j$-symbols. This tells us the renormalized four-point function and hence the RG equation for the quartic coupling:

$$V'(1234;j)=\left(\frac{2j+1}{2j}\right)^2\left[V(1234;j)-\Gamma_4^{(1-loop)}(l_i,m_i;j)\right] \tag{4.15}$$

4.1 UV-IR mixing

Let us recall how the β-function is calculated for ordinary scalar theories. Integrating out the fast modes gives equations analogous to (4.6,4.15). These renormalized n-point correlation functions are slowly varying functions of external momenta l_i, at least for small momenta: for example at large j, the 2-point function with $l_i=0$ differs only slightly from that at $l_i=1$. If the momenta take continuous values, this translates to saying that the correlation functions are analytic functions of external momenta in the neighborhood of zero external momentum. This allows us to derive a difference (or differential) equation for the mass and the coupling constant.

The situation in our case is clearly different. From either (4.6) or (4.15), it is obvious that the correlation functions are not slowly varying functions of l_i for small values of the external momenta l_i: the relative factors of $(-1)^{l_i}$ make the values of the correlation functions change abruptly as one moves from l_i to l_i+1. This is a clear signature of UV-IR mixing: integrating out a high energy mode has a violent effect on the low energy properties of the correlation functions, and traditional Wilsonian RG cannot be implemented.

5 RG equations on fuzzy $\mathbb{RP}^2$

As explained before, scalar fields on $\mathbb{RP}^2$ come only with even values of l. There are no factors of $(-1)^{l_i}$ in the two- and four-point correlation functions, and so they are indeed slowly varying functions of external momenta. They can thus be thought of as coming from a low-energy Wilsonian action, and can be used to write the RG equations for the mass and the coupling constant.

If we define the square of the mass and the coupling constant as the two- and four-point vertices respectively at zero external momentum, we get the following RG equations:

$$
\begin{aligned}
j(\mu'^2_j - \mu^2_j) &= \mu^2 + \frac{6\lambda_j}{4!}[jA(2j)] + O(1/j), &(5.1)\\
\beta(\lambda_j) \equiv j(\lambda'_j - \lambda_j) &= 2\lambda_j + \frac{\lambda_j}{j} - \frac{4\lambda_j^2}{4!}\frac{18j(4j+1)}{[2j(2j+1)+\mu_j^2]^2}. &(5.2)
\end{aligned}
$$

Let $\epsilon = 1/j$. For ϵ small, these equations can be written as

$$
\begin{aligned}
\frac{\mu'^2_j - \mu^2_j}{\epsilon} &= \mu^2_j + \frac{\lambda_j}{4}, &(5.3)\\
\beta(\lambda_j) &= (2+\epsilon)\lambda_j - \frac{3\lambda_j^2\epsilon^2}{4} + O(\epsilon^3). &(5.4)
\end{aligned}
$$

The interpretation of the β-function equation is standard: the first term tells us that the coupling increases until nonlinear effects (described by the second term) kick in. There exists a critical value λ_c at which the increase in the coupling due to rescaling is compensated by the decrease due to the nonlinear effect. However this non-trivial fixed point occurs at $\lambda_c = 4(2+\epsilon)/3\epsilon^2$ i.e. at large λ, since $\epsilon = 1/j$ is small. This fixed point was deduced using perturbation theory, and so its existence cannot be trusted. In order to know if there is really a non-trivial fixed point, one would need to know the β-function to all orders in λ. We do not know how to do this, but can estimate the two-loop contribution. At large j,

$$
\beta(\lambda_j) = (2+\epsilon)\lambda_j - \frac{3\lambda_j^2\epsilon^2}{4} + C\lambda_j^3\epsilon^5, \qquad (5.5)
$$

where C is a constant of order 1. Although its numerical value changes, the nontrivial fixed point continues to exist. However, as mentioned before, in order to really trust the new fixed point, we will need $\beta(\lambda_j)$ to all orders in λ_j.

5.1 $O(N)$ theories on $\mathbb{RP}^2$

The large N limit of $O(N)$ theories solves the problem we faced regarding the fixed point of the β-function. The RG equations for the mass and coupling constant are

$$\mu'^2_j = \frac{2j+2}{2j}\left[\mu_j^2 + \frac{(2N+2)\lambda_j}{4!}\frac{4j+1}{2j(2j+1)+\mu_j^2}\right], \tag{5.6}$$

$$\lambda'_j = \left(\frac{2j+2}{2j}\right)^2\left[\lambda_j - \frac{(8N+64)\lambda^2}{4!}\frac{4j+1}{[2j(2j+1)+\mu_j^2]^2}\right]. \tag{5.7}$$

From (5.7, 5.6), it is easy to see that the non-trivial fixed point is at

$$\mu_c^2 = -\left(\frac{\frac{1}{2}\frac{N+1}{N+8}\frac{j(2j+1)^2}{j+1}}{1+\frac{1}{4}\frac{N+1}{N+8}\frac{2j+1}{j+1}}\right), \tag{5.8}$$

$$\lambda_c = \frac{3}{N+8}\left(\frac{2j+1}{4j+1}\right)\left(\frac{2j(2j+1)+\mu_c^2}{j+1}\right)^2, \tag{5.9}$$

$$= \frac{3}{N+8}\left(\frac{4j^2(2j+1)^3}{(j+1)^2(4j+1)}\right)\left(\frac{1}{1+\frac{1}{4}\frac{N+1}{N+8}\frac{2j+1}{j+1}}\right)^2. \tag{5.10}$$

Thus if $N+8 >> 32j^2/3$, the new fixed point occurs at small value of λ_j, and its deduction is consistent with perturbation theory.

6 Conclusion and Outlook

We have done a careful perturbative analysis of quartic theories on noncommutative S^2 and $\mathbb{RP}^2$. Although the classical theory on the noncommutative S^2 is the same as the corresponding theory on the ordinary S^2 in the limit of large j, the quantum theories are very different. Even in the limit of large j, the low energy behavior of correlation functions of the quantum theory on noncommutative S^2 shows a mixing between UV and IR that is characteristic of noncommutative theories.

Surprisingly, we also find that if we restrict ourselves field theories on noncommutative $\mathbb{RP}^2$, we can avoid the effects of UV-IR mixing. In the large j-limit, the quartic theory on $\mathbb{RP}^2$ flows away from the zero mass Gaussian fixed point in the infrared. For a single component scalar theory, it is difficult to find the new fixed point in perturbation theory. The situation for $O(N)$ theories is much better. For N sufficiently large, these theories have a fixed point that is "close" to $\lambda = 0$, and hence can be trusted in perturbation theory.

There are several questions that may be studied in light of our results. One can try to look for situations in supergravity/string theory that correspond to branes distributed on a fuzzy $\mathbb{RP}^2$ instead of a fuzzy S^2. If such configurations exist, our results could have significant implications for their low energy dynamics.

Compactification scenarios that use fuzzy torus as extra dimensions have been suggested recently [?]. It may illuminating to understand the implications of compactifying on the fuzzy sphere in light of the non-trivial UV-IR mixing.

Our approach to quantum theories on fuzzy manifolds in this article relies on such theories being expressible as models of finite-dimensional matrices. In particular, the realization of the sphere S^2 as a coadjoint orbit $SU(2)/U(1)$ allows us to make extensive use of $SU(2)$ representation theory. Using the coadjoint orbit method, one can also construct fuzzy versions of $\mathbb{C}\mathrm{P}^2$ and $\frac{SU(3)}{U(1)\times U(1)}$, which in the continuum limit correspond to four- and six-dimensional manifolds respectively [?]. It would be interesting to use techniques analogous to the ones in this article to study explicitly the nature of UV-IR mixing on these fuzzy manifolds, and whether UV-IR mixing can be avoided by imposing restrictions on the modes as we found for the case of fuzzy $\mathbb{R}\mathrm{P}^2$.

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