

THE QUBITS OF QUNIVAC

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Abstract

We formulate a theory of quantum processes, extend it to a generic quantum cosmology, formulate a reversible quantum logic for the Quantum Universe As Computer, or Qunivac. Qunivac has an orthogonal group of cosmic dimensionality. It has a Clifford algebra of “cosmonions,” extending the quaternions to a cosmological number of anticommuting units. Its qubits obey Clifford-Wilczek statistics and are associated with unit cosmonions. This makes it relatively easy to program the Dirac equation on Qunivac in a Lorentz-invariant way. Qunivac accommodates a field theory and a gauge theory. Its gauge group is necessarily a quantum group.

1 Overview

Most of us at this symposium suppose that the universe is a computer. If so, then it seems likely that it is a reversible quantum relativistic computer.

To carry quantum theory that far from the spectroscopy laboratory we must package it carefully, especially since there is a virus going around.

Then we must formulate a reversible quantum logic. The familiar quantum logic is based on irreversible filtration operations combined by irreversible lattice operations of \cup and \cap .

Here we express quantum theory in a way suitable for cosmology and provide an antiviral mantra (Part 1), extend this quantum theory from laboratory systems to the cosmos, imitating Laplace (Part 2), and finally specialize to the quantum universe as computer with a useful correspondence principle (Part 3). We formulate a reversible logic and then a reversible quantum logic. It is then straightforward to program the Dirac equation and generic field and gauge theories in a clearly Lorentz-invariant way.

Part I

Quantum systems have no state

An ancient philosophical virus, already described by Francis Bacon, is now at large in the quantum-physics community. Most of us conferees carry it here today, and we unwittingly

infect our students as we were infected by our teachers.

This subtle virus leaves the mathematical part of physics untouched. The calculating engine of the infected theorist runs at full speed and precision. Only the io (input-outtake) function is disturbed, so that the end result is corrupted. Most of those infected by this virus have learned to compensate for it by using one theory and speaking another. At the same time, some physicists express and use quantum theory self-consistently.

As a result, there are today two versions of quantum physics. One is practiced widely and works, but is rarely professed. The other is widely professed and never practiced, not even by its adherents, being inconsistent with actual practice.

The discord appeared in the early days of quantum theory. Shortly after Heisenberg invented matrix mechanics, based on processes or operations, Schrödinger invented a wave mechanics, an ontological theory. This did not work at first, but it could be after-fitted with *ad hoc* rules that made it consistent with experiment and matrix mechanics. The ontological theory propagated more rapidly than the operational one, perhaps because it is visualizable, and now has almost driven the processual one out of the classroom.

But the source of the infection is even older. The virus resides in natural language. Just as natural language conflicts with special relativity in its tense structure, it conflicts with quantum physics in its predication structure. In both cases natural language assumes a non-existent now, an “is” that actually isn’t. Bacon would call this “is” an idol of the tribe [1]. The virus is an ontology; we may as well call it an ontovirus. Mathematical hubris, the belief that mathematical symbols can represent nature faithfully, predisposes one to ontoviral infection.

In this part we juxtapose and align sample formulations of two classes of theories that we call praxic and ontic respectively [10], and which evolved from and generalize matrix mechanics and wave mechanics.

2 Praxic theory

Ideal input and outtake processes represented by vectors $|i\rangle$ and dual vectors $\langle o|$ have the transition probability

$$P = \cos^2 \theta := |\langle o|i\rangle|^2. \quad (1)$$

P is the square of the projection of the intake vector on the outtake vector. Evidently complex phase factors in the two vectors are ignorable.

3 Ontic theory

During an ideal yes-or-no measurement a state represented by a vector $|i\rangle$ changes to a state represented by a vector $|o\rangle$ with probability

$$P = \cos^2 \theta := |\langle o|i\rangle|^2. \quad (2)$$

4 The virus

What a fiendish virus! In their mathematical structures, (1) and (2) are identical. There is no way to tell that one theory is sound, one unsound, from within the mathematical theory. We have to watch them in use to discover this.

In the praxic formulation the vectors represent processes. The ontic theory has mistaken them for states, which then must undergo other processes.

The ontovirus is apparent when [16] speaks of two modes of intervention **1** and **2** into a quantum system, representing measurement and propagation. The indication is that there are

three buildings, not two, at accelerator facilities, housing three modes of intervention, **1, 2, 3**, not two, of inflow, throughflow, and outflow; or beam preparation, target interaction, and counting. They are represented by the three factors in the transition amplitude $\langle o|T|i\rangle$. They exist in the simplest quantum experiment too.

The ontic theory does not count the input process. It miscounts because it mistakes that process for the thing produced and so counts as processes what are actually intervals between processes. A figure-ground reversal has occurred.

Three processes have two intervals; so the ontist imagines a physical process **2** that transforms $|i\rangle$ into $T|i\rangle$, and a process **1** that converts $T|i\rangle$ into $\langle o|$.

No such things occur in the laboratory; $|i\rangle$ does not transform into $T|i\rangle$ which changes into $\langle o|$. $|i\rangle$ is simply followed by T , which is followed by $\langle o|$.

5 Malus

There is no way to tell whether the praxic or ontic theory is right from the mathematical theory. The error is in the semantics. The use of the theory is described incorrectly,

We must watch the working physicist using the theory to describe the use correctly. The earliest quantum experiment suffices. Malus [14] considered a photon that has passed undeflected through one crystal of Iceland spar — that is the input process $|i\rangle$ — and is about to meet another — the outtake process $\langle o|$. (1) is Malus' law for the probability that the photon will again be undeflected.

A photon polarization in flight along an optical bench — say on the z axis — is postulated by the ontic theory to have a state $\psi(z, t)$ at time t , a unit vector of two complex components, with overall phase ignored.

It happens that there already is a familiar physical system that has such a state. In the ontic theory, a photon polarization is merely a particle moving on a sphere with a special first-order dynamical equation; except that unlike such a particle it jumps in a certain probabilistic way when we do what quantum physicists persist in calling a measurement of the polarization along some chosen direction.

This is a misnomer according to the ontic theory, which claims that the process is actually a certain kind of kick of the particle state, not a measurement at all. We never do what the ontist could honestly call observing the particle, which is to measure its state ψ at some time.

5.1 The test

To see what ψ actually is, then, we watch a physicist in action and note what she does; not what she says, of course, since she may carry the ontovirus.

In this *Gedankenexperiment* we lead a trained quantum physicist to the optical bench and ask her to estimate whether a certain photon — say the first after high noon — that has passed undeflected through the first crystal will pass undeflected through the second crystal.

She knows not to make any further measurement on said photon in flight between the polarizers, because that could change it and the outcome. She might measure the angle θ between the input crystal and the outtake crystal, and use Malus' law. She might put a billion other photons through a like process, count the fraction that pass the test, and use that fraction as the probability for the given photon. But in any case her practice is not the astronomer's. He can look at the actual system to tell where it is going and whence it came. She cannot. She looks at the polarizers, not the photon, for the polarizer angle; and nothing she can do to the photon will give her that information.

We should not call that angle the state of the photon because, whether we call it a state or not, it is not of the photon.

Transition probability and $|i\rangle$ and $\langle o|$ are features of the process, not of its product. A ψ does not evolve into a ϕ on the optical bench. We choose both freely when we set up the two crystals. They are not the kind of things that evolve, we just do them or not. Having done

them, we can use Malus' Law to estimate the transition probability, the probability of the **3** going conditional upon the **1** having gone.

It was not a mystery why some took the input process for its product in a first formulation. In classical physics, the input process, the state, and the output process of an allowed transition all determine each other for purposes of prediction and retrodiction. The classical observer could look at any of them to determine the state.

The quantum physicist, however, does not have that choice, but must observe and consider all the processes, which are almost independent of each other in the allowed transitions.

In one breath the ontic theory loads the photon with an infinity of information in its "state," and in the next breath denies that the photon can divulge one bit of that information in a measurement. This is the kind of theory that our fathers warned us against. It feigns a hypothesis.

The mathematical problems of the quantum theory all correspond exactly to problems of the ontic theory, but the ontic theory is wrong for the quantum polarization. The natural-language term "state of the system" has a reserved meaning in physics. We err if we call data gotten by observing the state of Venus "the state of Mars" or "the state of the astronomer." Everywhere else, what we call the state of a system is understood to be something that we can in principle learn from the system itself and use to predict the system's future behavior. Trying to override this fact of natural language has led to serious and costly distortions.

5.2 Quanta have no state

The situation was clearly formulated by Bergmann [4]. The concept of state is inappropriate for quanta.

Quantum theory is a theory of quantum processes. It is no more a theory of a state than special relativity is a theory of a present. This is why Heisenberg called his theory non-objective. and why Blatt and Weisskopf refer to ψ 's as channels, not states [5]. A ψ describes the process, not the product of the process. There is no problem of "collapse" of the state in quantum theory because there is no state to collapse.

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The discord between quantum practice and ontic principle has created unease in the most thoughtful of the affected physicists, and the better the quantum theory works, the greater the unease. The work devoted to resolving this unease has been immense: the quantum potential, the many-world theory, decoherence theory, consistent histories: all manifestations of the ontovirus.

A ψ is neither a quantum nor the state of one. It is a process that can produce one.

We suggest the following mantra for those exposed to the ontovirus:

ψ : process, not object.

Part II

Quantum cosmology

Quantum practitioners have never been troubled by philosophical qualms when they set about making a quantum theory of the universe. Every quantum field theory since Dirac's quantum electrodynamics has been a quantum theory of the universe.

These theories are conspicuously non-operational. No one in the universe can prepare or register it all sharply. Any physical experimenter is made of the very particles and fields of the theory and so is in the field system, not outside it experimenting on it.

At first Bohr objected strenuously to a quantum theory of the universe for this reason, but later he withdrew his objections [6]. It has always been understood, at least tacitly, that to correspond theoretical descriptions to those of a physical experimenter, one simply ignores the variables that the experimenter ignores, including those of the experimenter.

For those who think that a quantum system has a state, however, it is only one step to thinking that the universe has one. This is the ontovirus on a cosmic scale.

The quantum cosmology of field theory and the one we use here corresponds — in the sense of Bohr's correspondence principle — to Laplace's classical cosmology and is just as natural. Both cosmologies metaphorically deify the physicist.

Laplace invented a supreme astronomer who knows the state of the universe. Correspondingly we may imagine a supreme quantum Cosmic Experimenter (CE) who inputs the polarized cosmos with a grand $|I\rangle$ before our experiments and outtakes it with an $\langle O|$ after our experiments. We too need the CE to formulate a cosmology as much as Laplace did, but if we want to do quantum cosmology she has to be a quantum Experimenter.

These cosmologies are not operational. They deal with metaphorical processes carried out by the metaphorical CE. We extract operational predictions from them just as Laplace would.

We describe an actual experimenter as a subsystem of the cosmos with its own algebra, and ignore or average over the degrees of freedom of the cosmos that the actual experimenter ignores, especially those of that very experimenter.

If the CE were to have set us up before the fact to do our little experiments and were to read our notebooks after the fact, then her readings would be consistent with ours [16].

The extramundane CE is our metaphorical way of allowing for all possible experimenter/system interfaces within one embracing theory. The cosmic algebra is not operational but contains operational algebras as quotients. Quantum theory was already the most relativistic theory we have. Quantum cosmology relativizes it further. It relativizes the experimenter.

Those affected by the ontovirus can function just as they do in laboratory applications. Having imagined a cosmic state-vector, they must imagine cosmic observer to collapse it by an observation at the end of time.

Part III

Qunivac

Now we describe the structure of the cosmic computer from this extramundane viewpoint. We begin with the algebra of the universe.

6 Quantization is stabilization

The standard model of the elementary particles has several non-semisimple groups: groups that are reducible but not decomposable. So does Einstein's model of gravity. All such theories are unstable with respect to small variations in their structure tensors [17, 13]. This means that they are singular limits of several deeper, stabler theories that preserve all the basic principles

of quantum theory and relativity, at least asymptotically, and are more unified. One of these stabler theories probably fits experiment better than the present unstable theory [17]. The unstable theory might be right, but this has probability 0.

One of the deeper instabilities — not the deepest — is that of the differential calculus and the space-time continuum [17]. Systematic stabilization therefore quantizes the differential calculus and the space-time continuum. Symmetry and stability arguments like those of [17] led us to postulate an elementary-process hypothesis to replace and unify the continuum hypothesis and the atomic hypothesis:

All physical processes are composed of finitely many finite elementary quantum processes. [8, 18, 10]

We assert this for space-time translations and boosts as well as particle creation and annihilation. We assume that the elementary process — may we call it a pragmon for short? — lasts at least a minimum time $\Delta\tau$, and transfers at most a maximum energy $\Delta\epsilon$.

7 Reversible quantum logic

We assemble qubits into Qunivac by imitating how one assembles bits into a classical computer.

Classical computers are usually defined conceptually using set theory, and set algebra is usually based on operations \cup and \cap without inverse. Since nature is reversible, Qunivac must be reversible in the sense of Bennet and Fredkin. To describe a reversible computation we insist on a reversible set theory and logic, first classical and then quantum.

The only two reversible logical operations on truth values are XOR and its negation NOTXOR. We arbitrarily choose XOR, as the more familiar of the two.

Boole in classical logic and Von Neumann in quantum defined a class by an idempotent process of selection $e^2 = e$. Selection, which is filtration, has no inverse. We define a class by a unipotent process $e^2 = 1$. We may identify this new class-operation with XOR multiplication by the class in the old sense. The empty set is its identity element 1. We still need an element of structure to define the complement, to tell a unit class from its complement, and to define the universal class.

We introduce a logical grade for this purpose. This is just the modulus of the old class logic: 0 for the null set, 1 for unit sets, \therefore .

Our reversible classical logic (or set theory) is thus a graded XOR group. We turn to the quantum logic.

When we go from the classical to the quantum logic we turn on superposition. The reversible quantum class or set algebra is the Clifford algebra generated by the unit classes or sets [9].

Class algebra, reversible or not, is the least interesting part of set theory for computer architects. Since nature has a hierarchic structure, Qunivac must have one. The hierarchy-forming operation of classical set theory is the power set functor $\mathcal{P} : X \mapsto 2^X$. Finite set theory is the theory of the iterated power set functor. We use the power set to organize the computer, to construct its organs.

Our quantum power set functor is the Clifford algebra functor Cliff: $A \mapsto \mathbf{2}^A$ from graded algebras to graded algebras.

[10] still used Grass, not Cliff, to form hierachy, but that theory is unstable and the Clifford logic is its stabilization.

We call a quantum aggregate described by a Clifford algebra over the one-quantum algebra, a *squad*. The qubits of a squad obey a real variant of Wilczek's Clifford statistics [19, 15, 20].

If the qubit of the Qunivac has a (real) algebra A , the algebra of observables of a squad of qubits, say the entire Qunivac, is the Clifford algebra $\mathbf{2}^A$. The grade of this algebra counts elementary computer operations. The elementary operations have grade 1.

The hierarchy of classical sets is built with Peano's unitizer or successor operator $\iota z = \{z\}$, $\iota : X \rightarrow 2^X$. Our quantum $\iota : C \rightarrow \mathbf{2}^C$ is a linear morphism $C \rightarrow \mathbf{2}^C$ that transforms any element $z \in C$ into a first-grade element $\iota z \in C$ and reverses the norm $\|z\| := \Re z^2$:

$$(\iota z)^2 = -\|z\| \cdot \|\iota z\| = -\|z\|. \quad (3)$$

We introduce this sign reversal to generate the indefinite metrics needed for relativity and gauge theory.

The operator ι has no inverse, since many sets are not unit sets, but ι is reversible, in that any unit set has a unique element. Therefore ι has left inverse ι^\perp : $\iota^\perp \iota = \text{Id}$. We fix ι^\perp uniquely by setting it to 0 on all sets of sharp grade other than 1.

Then we may define the reversible quantum logic C as that represented by $\text{Cliff}(\iota)$, the real Clifford algebra generated by ι . Since one calls a Clifford algebra with four units the quaternions, we call $\text{Cliff}(\iota)$ the infinions.

This provides new content to the old surmise [8] that the quantum universe is a quantum computer.

We suppose that in any possible universe a cosmical but finite number N of anticommuting binary variables suffices, generating a finite-dimensional subalgebra of the infinions that can be called the cosmonions.

The Clifford sum provides Qunivac with the famous quantum parallelism that lets it compute so fast.

Iterated Clifford-algebra formation provides a hierarchy-generating, or subprogram-forming, function for Qunivac.

A mode of Qunivac is then represented by a spinor of the cosmonion algebra.

8 Fermions

We have programmed Qunivac for a Dirac particle in a quantum space-time [12]. It respects Lorentz invariance exactly. Its quantification preserves and strengthens the observed spin-statistics correlation, now giving it a purely algebraic origin.

On the other hand Qunivac beats the standard Heisenberg uncertainty relations. Position and momentum are now proportional to angular momentum operators in higher dimensions and so is their commutator η [12]. All three can be exactly 0 at the same time in a singlet channel. We expect that as in quaternion quantum field theory, η contracts to the Higgs field in the limit $\Delta\tau \rightarrow 0, N \rightarrow \infty$ and $i\hbar$ is its effective value in the vacuum. The usual Heisenberg indeterminacy relations appear to be good approximations only for values of η (hopefully, the Higgs field) close to its vacuum value.

At high energy $\sim \Delta\epsilon$, Qunivac also violates the usual continuum-based locality principle. Elementary processes connect events separated not infinitesimally as Einstein postulated but by a time $\Delta\tau$, the chronon. At energies much lower than $\Delta\epsilon = \hbar/\Delta\tau$ this would not show up strongly in the experimental data.

The simplest stabilization of the Dirac equation predicts an upper bound $\Delta\epsilon/c^2$ to the mass of elementary fermions. If we tentatively identify this limit with $M(\text{Top quark})$, we can estimate $\Delta\epsilon$ and $\Delta\tau = \Delta\epsilon/\hbar$. The distance $c\Delta\tau$ is then two or three orders of magnitude smaller than Dehmelt's estimate of the electron size [7].

It is conceivable that both Dehmelt's form factor size and our $\Delta\tau$ are both right. This would, however, imply that the electron is quite composite, as Dehmelt proposes.

The distance $c\Delta\tau$ is many orders of magnitude greater than the Planck length. This discrepancy does not trouble us much. The Planck length comes from a scenario in which it is merely assumed in the absence of evidence to the contrary that nothing limits space-time resolution but black-hole formation. We propose a more serious limit arising from the structure of space-time. In any case, a Clifford algebraic theory like ours makes it more natural to regard gravity as another condensation phenomenon like the Higgs field than a fundamental force.

9 Fields

Field theory begins with a partition of variables into field and space-time. The space-time variables are of the experimenter, the field variables are of the system. The set of fields field is locally an exponential Y^X , where Y is the field fiber and X is the space-time.

To program field theory in Qunivac requires us to define the set exponential Y^X when the field variable space Y and the space-time X are both quantum. We insist on the correspondence principle. Our construction must have a classical limit.

To form a field in Qunivac we must first represent the universe as a squad of events $U = \mathbf{2}^e$. Then the event must reduce to a pair $e = y \otimes X$. This factorization is a condensation or spontaneous symmetry-breaking, reducing the orthogonal group of the event e to a product of two smaller orthogonal groups. The field fiber at each point is a squad of filaments $Y = \mathbf{2}^y$. The field universe is then $U = Y^X := \mathbf{2}^{y \otimes X}$.

This is possible and easy when and apparently only when the field Y at each point is a squad with Clifford algebra $Y = \mathbf{2}^y$. This happens to work for spinor fields.

Since we have formulated this quantum field entirely in Clifford algebra, it is easy to see its classical limit. One simply replaces $\mathbf{2}$ by 2 throughout.

10 Gauge

We turn now to the bosons and the gauge theory of Qunivac, again clinging to the correspondence principle for dear life.

Most of this section represents work done since the Digital Perspectives symposium.

Gauge theory begins like field theory with a division of the variables of the universe into field and space-time variables.

The gauge group of Qunivac must be a quantum group. We reason thus:

Loosely speaking, we recall, a quantum group is a group with quantum parameters. It is defined by an “algebra of observables” with two associative unital products, the usual one for group parameters [10] and an extra one for group elements. If the parameters commute the group is classical. If the elements commute the group is commutative.

The gauge group of Qunivac must be a quantum group because the gauge group element depends on a point of space-time as on a parameter, and any space-time in Qunivac is quantum.

The most recent step towards the gauge theory of Qunivac is to recognize that the quantum gauge group algebra too is a Clifford algebra $G = \mathbf{2}^g \otimes X$.

This follows directly from the fact that a gauge group element is a field of Lie group elements over a quantum space-time.

In an earlier quaternionic theory the varying $i\hbar$ provided the Higgs field and reducing the gauge group. Now the quaternions have spawned a cosmological number of Clifford elements and the question is reopened.

A somewhat fuller exposition is available [3]

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