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On the Moduli Space of Noncommutative Multi-solitons at Finite θ

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Abstract

We study the finite θ correction to the metric of the moduli space of noncommutative multi-solitons in scalar field theory in (2+1) dimensions. By solving the equation of motion up to order $O(\theta^{-2})$ explicitly, we show that the multi-soliton solution must have the same center for a generic potential term. We examine the condition that the multi-centered configurations are allowed. Under this condition, we calculate the finite θ correction to the metric of the moduli space of multi-solitons and argue the possibility of the non right-angle scattering of two solitons. We also obtain the potential between two solitons.

Solitons in field theories on noncommutative spacetime are useful for studying non-perturbative effects. These play also an important role in string theory and condensed matter physics(see [1] for reviews). In particular, the noncommutative soliton in $2+1$ dimensional scalar field theory [2] provides an interesting and nontrivial example since it does not exist in scalar field theory on commutative spacetime.

Various aspects of noncommutative scalar solitons have been studied [3, 4, 5, 6, 7, 8, 9]. In ref. [3], the multi-soliton solutions and their moduli space are studied in the limit of large noncommutative parameter θ. The geodesic in the moduli space of two solitons describes the scattering of the soliton in the adiabatic approximation[10]. In refs. [4, 5, 6], it is shown that the right-angle scattering of solitons occurs at the head-on collision. The scattering is also studied for various noncommutative solitons [11].

It is an interesting problem whether noncommutative solitons in scalar field theory exist at finite θ. In this case, we need to consider the attractive or repulsive force between solitons. Recently, Durhuus and Jonsson pointed out that there are no multi-soliton solutions which interpolate smoothly between n overlapping solitons and n solitons with an infinite separation at the lowest order perturbation in θ^{-1}[7]. Therefore multi-solitons at finite θ are in general unstable and decay into infinitely separate or overlapping solitons. This heavily depends on the shape of potential term. In fact, for the potential $V(\phi)$ with $1/V''(0)+1/V''(\lambda)=0$, where $\phi=0$ and λ are two critical points of $V(\phi)$, energies for the above two configurations agree with each other. Hence the moduli space approximation looks still good for such a potential.

In this paper, we study the θ correction to the noncommutative scalar soliton. We solve the static equation of motion explicitly in scalar field theory up to the order $O(\theta^{-2})$. We examine some consistency conditions which appears in the process of solving the equation of motion. From these conditions, we show that noncommutative multi-solitons must have the same center for the potential with $1/V''(0)+1/V''(\lambda)\neq 0$. On the other hand, multi-centered configurations are allowed for the potential with $1/V''(0)+1/V''(\lambda)=0$. Using the perturbed solutions, we may calculate the finite θ correction to the metric of the moduli space for the multi-solitons. We argue the possibility of the non right-angle scattering for scattering of two solitons for finite θ. This would provide interesting physics in contrast with the right-angle scattering in the large θ limit. We also study the force between solitons by calculating the potential and discuss existence of a static multi-soliton solution at finite θ.

We consider two-dimensional noncommutative space coordinates $(\hat{x}, \hat{y})$ satisfying $[\hat{x}, \hat{y}] = i\theta$. We rescale the coordinates by the factor $\sqrt{\theta}$ such that $[\hat{x}, \hat{y}] = i$. Introducing the harmonic oscillators $a = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{y})$ and $a^\dagger = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{y})$, the action of the real scalar field theory is

$$S = \int dt 2\pi \mathrm{Tr} \left\{ \theta(\partial_t \phi)^2 + [a, \phi][\phi, a^\dagger] + \theta V(\phi) \right\}, \tag{1}$$

where the trace is taken over the Fock space $\mathcal{H}$ of the harmonic oscillator. Here we assume that the potential term $V(\phi)$ has critical points at $\phi = 0$ and λ. In the large θ limit, the second term in the action drops out. The static equation of motion becomes $V'(\phi) = 0$. Then it admits the soliton solution of the form[2]

$$\phi_0 = \lambda P, \tag{2}$$

where P is a projection operator $P^2 = P$. The multi-soliton solution [3] is constructed by using the coherent state $|z\rangle \equiv e^{a^\dagger z}|0\rangle$. The n level one solitons centered at points $z_1, \cdots, z_n$ are given by

$$P = |z_i\rangle h^{ij} \langle z_j|, \quad h_{ij} = \langle z_i | z_j \rangle, \tag{3}$$

where h^{ij} is inverse matrix of h_{ij}. One may also construct the level k soliton centered at z in terms of basis $|\psi_i\rangle = (a^\dagger)^i |z\rangle$ $(i = 0, \cdots, k-1)$:

$$P = |\psi_i\rangle H^{ij} \langle \psi_j|, \quad H_{ij} = \langle \psi_i | \psi_j \rangle. \tag{4}$$

Here H^{ij} satisfies $H^{ij} H_{jk} = \delta^i_k$. For large but finite θ, we must consider both the second and the third terms in the action. The energy functional

$$E = 2\pi \mathrm{Tr} \left(\theta V(\phi) + [a, \phi][\phi, a^\dagger] \right) \tag{5}$$

leads to the equation of motion:

$$2[a^\dagger, [a, \phi]] + \theta V'(\phi) = 0. \tag{6}$$

We want to obtain the solution of (6) of the form

$$\phi = \phi_0 + \frac{1}{\theta}\phi_1 + \frac{1}{\theta^2}\phi_2 + \cdots. \tag{7}$$

In ref. [5], ϕ_1 has been constructed. We will calculate the ϕ_2-term.

Substituting (7) into (6), we obtain

$$2\sum_{r=0}^{\infty}\theta^{-r}[a^\dagger,[a,\phi_r]]+\sum_{r_0\geq r_1\geq\geq 0}\theta^{1-\sum_i r_i}\left(V^{(r_0+1)}(\phi_0)\frac{\phi_1}{(r_0-r_1)!}\frac{\phi_2}{(r_1-r_2)!}\cdots\right)_S=0, \quad (8)$$

where $(\mathcal{O}_1\cdots\mathcal{O}_n)_S=\frac{1}{n!}\sum_{\sigma\in S_n}\mathcal{O}_{\sigma(1)}\cdots\mathcal{O}_{\sigma(n)}$ denotes the symmetrized sum of the product $\mathcal{O}_1\cdots\mathcal{O}_n$. S_n is the permutation group of order n. Taking the coefficient of θ^{-r} of the equation of motion, we have a series of equations for ϕ_r's. The first three equations become

$$\begin{aligned}
&V^{(1)}(\phi_0)=0,\\
&2[a^\dagger,[a,\phi_0]]+(V^{(2)}(\phi_0)\phi_1)_S=0,\\
&2[a^\dagger,[a,\phi_1]]+\frac{1}{2!}(V^{(3)}(\phi_0)\phi_1^2)_S+(V^{(2)}(\phi_0)\phi_2)_S=0.
\end{aligned} \quad (9)$$

The first equation in (9) has a solution $\phi_0=\lambda P$. In order to solve the second and the third equations, it is convenient to use the formulas for an operator A:

$$\begin{aligned}
(V^{(n)}(\phi_0)A)_S &= V^{(n)}(\lambda)PAP+V^{(n)}(0)QAQ\\
&\quad+\frac{1}{\lambda}\left(V^{(n-1)}(\lambda)-V^{(n-1)}(0)\right)(PAQ+QAP), \quad (10)\\
\frac{1}{2!}(V^{(n)}(\phi_0)A^2)_S &= \frac{1}{2!}V^{(n)}(\lambda)PAPAP+\frac{1}{2!}V^{(n)}(0)QAQAQ\\
&\quad-\frac{1}{\lambda^2}\left\{V^{(n-2)}(\lambda)-V^{(n-2)}(0)-\lambda V^{(n-1)}(\lambda)\right\}(PAPAQ+PAQAP+QAPAP)\\
&\quad+\frac{1}{\lambda^2}\left\{V^{(n-2)}(\lambda)-V^{(n-2)}(0)-\lambda V^{(n-1)}(0)\right\}(PAQAQ+QAPAQ+QAQAP)
\end{aligned} \quad (11)$$

where $Q=1-P$. Since $V^{(1)}(0)=V^{(1)}(\lambda)=0$, we have

$$(V^{(2)}(\phi_0)A)_S=V^{(2)}(\lambda)PAP+V^{(2)}(0)QAQ. \quad (12)$$

Since the r.h.s. of this equation does not include the off-diagonal parts PAQ and QAP, each equation in (9) gives further constraints on the structure of the solution. For example, from the off-diagonal parts $P(\ldots)Q$ and $Q(\ldots)P$ of the second equation in (9), we can see that ϕ_0 must satisfy

$$P[a^\dagger,[a,\phi_0]]Q=0,\quad Q[a^\dagger,[a,\phi_0]]P=0. \quad (13)$$

These equations are satisfied if the projection operator P obeys the equation $(1-P)aP = 0$, which means $aP\mathcal{H} \subset P\mathcal{H}$. When P is constructed from the coherent states of the form (3), this condition is satisfied. In this case, ϕ_1 is regarded as the leading correction to ϕ_0 and given by

$$\phi_1 = -\frac{2\lambda}{V^{(2)}(\lambda)}P[a^\dagger,[a,P]]P - \frac{2\lambda}{V^{(2)}(0)}Q[a^\dagger,[a,P]]Q + PX_1Q + QX_1^\dagger P, \tag{14}$$

where X_1 is an arbitrary operator. Since P is the projection operator of the form (3), we can put $QaP = 0$ and (14) may be simplified to

$$\phi_1 = -\frac{2\lambda}{V^{(2)}(\lambda)}PaQa^\dagger P + \frac{2\lambda}{V^{(2)}(0)}Qa^\dagger PaQ + PX_1Q + QX_1^\dagger P. \tag{15}$$

The second order correction ϕ_2 can be solved in a similar way. The consistency condition from the third equation in (9) yields

$$2P[a^\dagger,[a,\phi_1]]Q + \frac{1}{\lambda}(V^{(2)}(\lambda)P\phi_1P\phi_1Q - V^{(2)}(0)P\phi_1Q\phi_1Q) = 0 \tag{16}$$

where we have used the second equation in (11). Using $QaP = 0$ and (15), (16) turns out to be

$$\begin{aligned} F_2 \equiv\ & 4\lambda\left(\frac{1}{V^{(2)}(\lambda)} + \frac{1}{V^{(2)}(0)}\right)(Pa^\dagger PaQa^\dagger PaQ - PaQa^\dagger Pa^\dagger PaQ) \\ & +2\left(PaPa^\dagger PX_1Q - PaPX_1Qa^\dagger Q - Pa^\dagger PX_1QaQ + PX_1Qa^\dagger QaQ\right) = 0. \end{aligned} \tag{17}$$

One may ask whether this consistency condition provides further constraints on P and PX_1Q.

When $1/V^{(2)}(\lambda) + 1/V^{(2)}(0) \neq 0$ the first term in (17) is not zero for a multi-soliton projection operator P. In fact, we may evaluate the matrix elements of the operator F_2 on the basis $\{|z_1\rangle,\cdots,|z_n\rangle, Q|z\rangle\ |z\in\mathbf{C}, z\neq z_i\}$ which spans $\mathcal{H}$. Non-trivial elements are given by

$$\langle z_i|F_2Q|z\rangle = A\langle z_i|aQa^\dagger|z_p\rangle h^{pq}(\bar{z}_i - \bar{z}_q)\langle z_q|aQ|z\rangle - 2N_i^q\langle z_q|[PX_1Q,a^\dagger]Q|z\rangle \tag{18}$$

where $A = 4\lambda\left(\frac{1}{V^{(2)}(\lambda)} + \frac{1}{V^{(2)}(0)}\right)$ and $N_i^q = h_{ip}(z - z_p)h^{pq}$. From this formula, we can see that the first term is zero for the level k soliton projection operator (4) but non-zero for the multi-soliton case (3). In the latter case, we are not able to find PX_1Q so that the non-zero first term is canceled by the second one. In the former case, on the other hand, one may choose $PX_1Q = 0$ in order to make F_2 zero. These mean the following: if we take into account the $O(\theta^{-2})$-correction, the multi-soliton configuration cannot be a solution of the equation of motion for generic potential with $1/V^{(2)}(\lambda) + 1/V^{(2)}(0) \neq 0$, but the single level k soliton configuration can. This seems to be consistent with the argument based on the evaluation of the energy functional in [3, 5, 7].

In the case of $1/V^{(2)}(\lambda)+1/V^{(2)}(0) = 0$, however, the consistency condition is satisfied for a multi-soliton projection operator P and X_1 with $PX_1Q = 0$. Hence we assume $PX_1Q = 0$ from now on. We find that ϕ_2 is given by

$$\begin{aligned}
\phi_2 \quad = \quad & -\frac{1}{V^{(2)}(\lambda)}\left(2P[a^\dagger,[a,\phi_1]]P + \frac{1}{2!}V^{(3)}(\lambda)P\phi_1P\phi_1P + \frac{1}{\lambda}V^{(2)}(\lambda)P\phi_1Q\phi_1P\right) \\
& -\frac{1}{V^{(2)}(0)}\left(2Q[a^\dagger,[a,\phi_1]]Q + \frac{1}{2!}V^{(3)}(0)Q\phi_1Q\phi_1Q - \frac{1}{\lambda}V^{(2)}(0)Q\phi_1P\phi_1Q\right) \\
& +PX_2Q + QX_2^\dagger P
\end{aligned} \tag{19}$$

where X_2 is an arbitrary operator. We will assume $X_2 = 0$ for simplicity. Substituting (15) into (19) and using $QaP = 0$, one may obtain an explicit formula for ϕ_2:

$$\begin{aligned}
\phi_2 \quad = \quad & \lambda\Big\{v^2(PaPa^\dagger PaQa^\dagger P - PaPaQa^\dagger Pa^\dagger P - Pa^\dagger PaQa^\dagger PaP + PaQa^\dagger PaPa^\dagger P \\
& \quad -PaQa^\dagger P) + \left(2v^2 - vw + \frac{1}{2}C_1\right) PaQa^\dagger PaQa^\dagger P\Big\} \\
& - \lambda\Big\{w^2(Qa^\dagger PaQ + Qa^\dagger QaQa^\dagger PaQ - QaQa^\dagger PaQa^\dagger Q - Qa^\dagger Qa^\dagger PaQaQ \\
& \quad +Qa^\dagger PaQa^\dagger QaQ) + \left(2w^2 - vw + \frac{1}{2}C_2\right) Qa^\dagger PaQa^\dagger PaQ\Big\},
\end{aligned} \tag{20}$$

where

$$v = -2/V^{(2)}(\lambda), \quad w = 2/V^{(2)}(0), \tag{21}$$

$$C_1 = \frac{\lambda}{2!}v^3V^{(3)}(\lambda), \quad C_2 = \frac{\lambda}{2!}w^3V^{(3)}(0). \tag{22}$$

We have written the equation (20) for generic v and w. But the condition $1/V^{(2)}(\lambda) + 1/V^{(2)}(0) = 0$ implies $v = w$, so we should impose this condition on (20).

Obtaining the solution of the equation of motion, we may calculate the metric of the moduli space and its θ^{-1}-correction. In the case of $\theta = \infty$, the moduli space of n level one soliton solution are parameterized by the coordinates $z_i, \bar{z}_i$ $(i = 1, \cdots, n)$ of the level one solitons. Using the adiabatic approximation [10], the metric of the moduli space is determined by the action of the particles:

$$S=\int dt\left(g_{i\bar{j}}\frac{dz_i}{dt}\frac{d\bar{z}_j}{dt}+g_{ij}\frac{dz_i}{dt}\frac{dz_j}{dt}+g_{\bar{i}\bar{j}}\frac{d\bar{z}_i}{dt}\frac{d\bar{z}_j}{dt}\right). \tag{23}$$

Here the metric is given by the formula

$$\begin{aligned} g_{i\bar{j}} &= \frac{1}{\lambda^2}\mathrm{Tr}\partial_i\phi\bar{\partial}_j\phi, \\ g_{ij} &= \frac{1}{\lambda^2}\mathrm{Tr}\partial_i\phi\partial_j\phi, \\ g_{\bar{i}\bar{j}} &= \frac{1}{\lambda^2}\mathrm{Tr}\bar{\partial}_i\phi\bar{\partial}_j\phi \end{aligned} \tag{24}$$

where $\partial_i=\frac{\partial}{\partial z_i}$ and $\bar{\partial}_i=\frac{\partial}{\partial \bar{z}_i}$. From (7), the metric can be expanded as

$$\begin{aligned} g_{i\bar{j}} &= g^{(0)}_{i\bar{j}}+\frac{1}{\theta}g^{(1)}_{i\bar{j}}+\frac{1}{\theta^2}g^{(2)}_{i\bar{j}}+\cdots, \\ g_{ij} &= g^{(0)}_{ij}+\frac{1}{\theta}g^{(1)}_{ij}+\frac{1}{\theta^2}g^{(2)}_{ij}+\cdots, \\ g_{\bar{i}\bar{j}} &= g^{(0)}_{\bar{i}\bar{j}}+\frac{1}{\theta}g^{(1)}_{\bar{i}\bar{j}}+\frac{1}{\theta^2}g^{(2)}_{\bar{i}\bar{j}}+\cdots. \end{aligned} \tag{25}$$

In the large θ limit, the metric is shown to satisfy the Kähler condition[3]:

$$\partial_k g^{(0)}_{i\bar{j}}=\partial_i g^{(0)}_{k\bar{j}}, \quad g^{(0)}_{\bar{i}\bar{j}}=g^{(0)}_{ij}=0. \tag{26}$$

The Kähler structure comes from the Grassmannian structure of the moduli space [8] which can be regarded as the space of symmetric products of $\mathbf{R}^2$ [3].

We begin with the first order correction to the metric for the multi-solitons. The first order correction to the metric for two solitons has been calculated in [5]. In order to simplify the formula, we use the following differential operators which take values in the space of operators on the Hilbert space $\mathcal{H}$:

$$\partial P=\partial_i P dz_i, \quad \bar{\partial}P=\bar{\partial}_i P d\bar{z}_i. \tag{27}$$

Using $P=|z_i\rangle h^{ij}\langle z_j|$, we have

$$\partial_i P=Qa^\dagger|z_i\rangle h^{il}\langle z_l|, \quad \bar{\partial}_i P=|z_l\rangle h^{li}\langle z_i|aQ. \tag{28}$$

Then it is shown that the Kähler form K takes the form [3]

$$K = \frac{i}{2}\mathrm{Tr}\partial P \wedge \bar{\partial} P = \frac{i}{2} g^{(0)}_{i\bar{j}} dz_i \wedge d\bar{z}_j \tag{29}$$

with

$$g^{(0)}_{i\bar{j}} = h^{ij} M_{ji} \tag{30}$$

and

$$M_{ji} \equiv \langle z_j | a Q a^\dagger | z_i \rangle = \bar{z}_j h_{ji} z_i + h_{ji} - h_{jl} z_l h^{lm} \bar{z}_m h_{mi}. \tag{31}$$

The first order correction to the metric $g^{(1)}$ is

$$g^{(1)}_{i\bar{j}} = \frac{1}{\lambda^2}\mathrm{Tr}[\partial_i \phi_0 \partial_{\bar{j}} \phi_1 + \partial_i \phi_1 \partial_{\bar{j}} \phi_0] \tag{32}$$

and similar expressions for $g^{(1)}_{ij}$ and $g^{(1)}_{\bar{i}\bar{j}}$. Using (15), we have

$$\begin{aligned} g^{(1)}_{i\bar{j}} &= 2v h^{il} M_{lm} h^{mj} M_{ji} - 2w h^{ij} M_{jm} h^{ml} M_{li}, \\ g^{(1)}_{ij} &= g^{(1)}_{\bar{i}\bar{j}} = 0, \end{aligned} \tag{33}$$

where v and w have been defined in (21).

Let us check whether the metric including $O(\theta^{-1})$-correction satisfies the Kähler condition. After some computations, we find

$$\partial_k g^{(1)}_{i\bar{j}} - \partial_i g^{(1)}_{k\bar{j}} = (v-w)\left(h^{il}(\bar{z}_l - \bar{z}_j) M_{lk} h^{kj} M_{ji} - (i \leftrightarrow k)\right). \tag{34}$$

These quantities vanish if $v = w$ or the second factor becomes zero. In the case of $v = w$ i.e. $1/V^{(2)}(\lambda) + 1/V^{(2)}(0) = 0$, the metric $g^{(0)} + \frac{1}{\theta} g^{(1)}$ is Kähler for any multi-soliton configuration. In the case of $v \neq w$ i.e. $1/V^{(2)}(\lambda) + 1/V^{(2)}(0) \neq 0$, on the other hand, the second factor does not vanish for the multi-soliton configuration, thus the metric $g^{(0)} + \frac{1}{\theta} g^{(1)}$ is not Kähler. But it is still a complex manifold because of the property $g_{ij} = g_{\bar{i}\bar{j}} = 0$. For the single level k soliton configuration and the two soliton configuration, the Kähler structure is trivial, even if $v \neq w$.

We next consider the second order correction to the metric. As we have discussed, we need to impose the condition $v = w$ in order that a multi-soliton configuration is allowed as the solution of the equation of motion up to $O(\theta^{-2})$. The metric $g^{(2)}$ is obtained from

$$g^{(2)}_{i\bar{j}} = \frac{1}{\lambda^2}\mathrm{Tr}[\partial_i \phi_0 \bar{\partial}_j \phi_2 + \partial_i \phi_1 \bar{\partial}_j \phi_1 + \partial_i \phi_2 \bar{\partial}_j \phi_0] \tag{35}$$

etc. Using (15) and (20), we get

$$
\begin{aligned}
g^{(2)}_{i\bar{j}} &= \left(5v^2 - 2vw + C_1\right) h^{il} M_{lm} h^{mn} M_{np} h^{pj} M_{ji} \\
&\quad + \left(5w^2 - 2vw + C_2\right) h^{ij} M_{jm} h^{mn} M_{np} h^{pl} M_{li} \\
&\quad +(v^2 - 2w^2)\left\{(\bar{z}_n - \bar{z}_j)(z_m - z_j) - 1\right\} h^{ij} M_{jm} h^{mn} M_{ni} \\
&\quad +(w^2 - 2v^2)\left\{(\bar{z}_l - \bar{z}_j)(z_m - z_j) + 1\right\} h^{il} M_{lm} h^{mj} M_{ji} \\
&\quad +(v - w)^2 h^{il} M_{lm} h^{mj} M_{jn} h^{np} M_{pi} \qquad (36)
\end{aligned}
$$

and

$$
g^{(2)}_{ij} = -(v^2 + w^2)(\bar{z}_l - \bar{z}_m)^2 h^{jl} M_{li} h^{im} M_{mj}. \qquad (37)
$$

In (36) and (37), we have written the formulas for generic v and w. But as we have noted, we must put $v = w$. The condition $v = w$ implies that the Kähler condition holds at the first order, as mentioned before. However, taking into account the second order correction, the metric is neither Kähler nor complex.

We give an example of the metric for two level one solitons ((1, 1)-system). The metric of two-soliton solution and its behavior around the origin are particularly interesting because the geodesic under the metric describes the scattering of two noncommutative solitons. The effect of the first order corrections has been analyzed in ref. [5]. The first order correction does not change the structure of the right-angle scattering property. Let us examine the effect of the second order perturbation. In the center of mass system, i.e. $z_1 = -z_2 = y/2$, we get

$$
\begin{aligned}
g^{(0)}_{y\bar{y}} &= \coth t - \frac{t}{\sinh^2 t}, \qquad (38) \\
g^{(1)}_{y\bar{y}} &= 2(v - w)\left(\coth t - \frac{t}{\sinh^2 t}\right) + 2(v + w)\frac{t}{\sinh^2 t}(1 - t\coth t) \qquad (39)
\end{aligned}
$$

and

$$
\begin{aligned}
g^{(2)}_{y\bar{y}} &= \frac{1}{\sinh^4 t}\Big[v^2\{-t^3(2\cosh^2 t + 3) - 3t^2 \sinh t \cosh t + 5t \sinh^2 t + 3\sinh^3 t \cosh t\} \\
&\quad +vw\{2t^3 - 2t^2 \sinh t \cosh t + 6t \sinh^2 t - 6\sinh^3 t \cosh t\} \\
&\quad +w^2\{-t^3(2\cosh^2 t + 3) + 17t^2 \sinh t \cosh t - 21t \sinh^2 t + 9\sinh^3 t \cosh t\} \\
&\quad +C_1\{-t^3 - t^2 \sinh t \cosh t + t \sinh^2 t + \sinh^3 t \cosh t\}
\end{aligned}
$$

$$+C_2\{-t^3+3t^2\sinh t\cosh t-3t\sinh^2 t+\sinh^3 t\cosh t\}\Big], \tag{40}$$

$$g^{(2)}_{yy} = \frac{(v^2+w^2)\bar{y}^2}{2\sinh^2 t}(1-t\cosh t)^2. \tag{41}$$

Here $t=|y|^2/2$ and $v=w$.

A particularly interesting fact from the above results is that each $g^{(i)}$ have the same conical singularity at the origin $t=0$. This can be explicitly seen when we transform the metric into the Kähler form. We change the coordinates from $(y,\bar{y})$ to $(y',\bar{y}')$ by

$$y'=y+\frac{1}{\theta^2}\delta y(y,\bar{y}), \quad \bar{y}'=\bar{y}+\frac{1}{\theta^2}\delta\bar{y}(y,\bar{y}), \tag{42}$$

where δy satisfies

$$\frac{\partial\delta y}{\partial y}=\frac{1}{2}\frac{g^{(2)}_{yy}}{g^{(0)}_{y\bar{y}}}. \tag{43}$$

The solution of the above equation takes the form

$$\delta y=\frac{1}{2}H(|y|^2)y+\text{holomorphic function of } y \tag{44}$$

where $H(|y|^2)$ satisfies

$$\frac{dH(|y|^2)}{d|y|^2}=\frac{g^{(2)}_{yy}}{g^{(0)}_{y\bar{y}}|y|^2}. \tag{45}$$

After change of the coordinates, the metric for finite θ becomes $ds^2=g_{y'\bar{y}'}dy'd\bar{y}'$ where

$$g_{y'\bar{y}'}=g^{(0)}_{y'\bar{y}'}+\frac{1}{\theta}g^{(1)}_{y'\bar{y}'}+\frac{1}{\theta^2}g'^{(2)}_{y'\bar{y}'}+\cdots \tag{46}$$

and

$$g'^{(2)}_{y'\bar{y}'}=g^{(2)}_{y'\bar{y}'}-g^{(2)}_{y'y'}\frac{|y'|^2}{\bar{y}'^2}-g^{(0)}_{y'\bar{y}'}H(|y'|^2). \tag{47}$$

Near $|y'|^2=0$ we have

$$\begin{aligned} g^{(0)}_{y'\bar{y}'} &= \frac{2}{3}\frac{|y'|^2}{2}+O(|y'|^6), \\ g^{(1)}_{y'\bar{y}'} &= \Big(\frac{2}{3}v-2w\Big)\frac{|y'|^2}{2}+O(|y'|^6), \\ g'^{(2)}_{y'\bar{y}'} &= \Big(\frac{2}{3}v^2-\frac{16}{3}vw+\frac{34}{3}w^2+\frac{2}{3}C_1+2C_2\Big)\frac{|y'|^2}{2}+O(|y'|^6). \end{aligned} \tag{48}$$

From the above results one might expect the metric for $(1,1)$-system for finite θ takes the form

$$ds^2=f(r,\theta)(dr^2+r^2d\varphi^2) \tag{49}$$

in the polar coordinate (r,φ). The conformal factor $f(r,\theta)$ has an expansion

$$f(r,\theta) = f_1(\theta)r^2 + f_3(\theta)r^6 + \cdots \tag{50}$$

If θ satisfies $f_1(\theta)=0$, the right-angle scattering does not occur at the head-on collision. In this case, the leading term in (50) is r^6 term and the $\pi/6$ scattering occurs instead. Indeed, due to symmetry, it is very hard to think about non-right angle scattering for the same type of solitons. In order to see whether this phenomena really happen, we need to obtain the exact solution of the equation of motion for finite θ.

For the study of dynamical aspects of the solitons at finite θ, we also consider the potential $U(|y|)$ for two solitons, which is obtained by substituting the solution (7) into the energy functional (5) [3, 5]. The energy functional E can be expanded as

$$\begin{aligned} E &= 2\pi\theta \mathrm{Tr}\left[\sum_{r=0}^{\infty}\theta^{-r-1}\sum_{m=0}^{r}[a,\phi_{r-m}][\phi_m,a^\dagger]\right. \\ &\quad \left. + \sum_{r_0\geq r_1\geq\cdots\geq 0}^{\infty}\theta^{-\sum_{i=0}^{\infty}r_i}\left(V^{(r_0)}(\phi_0)\frac{\phi_1^{r_0-r_1}}{(r_0-r_1)!}\frac{\phi_2^{r_1-r_2}}{(r_1-r_2)!}\cdots\right)_S\right] \\ &= 2\pi\theta\left(E_0+\frac{1}{\theta}E_1+\frac{1}{\theta^2}E_2+\frac{1}{\theta^3}E_3+\ldots\right), \end{aligned} \tag{51}$$

where

$$\begin{aligned} &E_0[\phi_0] = \mathrm{Tr}V(\phi_0), \\ &E_1[\phi_0] = \mathrm{Tr}[a^\dagger,[a,\phi_0]]\phi_0, \\ &E_2[\phi_0,\phi_1] = \mathrm{Tr}\left[2[a^\dagger,[a,\phi_0]]\phi_1 + (V^{(2)}(\phi_0)\frac{\phi_1^2}{2!})_S\right], \\ &E_3[\phi_0,\phi_1] \\ &= \mathrm{Tr}\left[\left\{2[a^\dagger,[a,\phi_0]] + (V^{(2)}(\phi_0)\phi_1)_S\right\}\phi_2 + [a^\dagger,[a,\phi_1]]\phi_1 + (V^{(3)}(\phi_0)\frac{\phi_1^3}{3!})_S\right] \end{aligned} \tag{52}$$

and so on. Here we have already used $V^{(1)}(\lambda)=V^{(1)}(0)=0$. For a rank k projection operator P, $E_0=kV(\lambda)$. Further assuming $QaP=0$, we have $E_1=\lambda^2 k$. These are constants and are not included in the potential U. For $(1,1)$-system, U takes the form

$$U = \frac{1}{\theta}U^{(1)} + \frac{1}{\theta^2}U^{(2)} + \cdots \tag{53}$$

where

$$\begin{aligned} U^{(1)} &= 2(v-w)\frac{t^2}{\sinh^2 t} \\ U^{(2)} &= -8(v^2-w^2)\frac{t^2(1-t\coth t)}{\sinh^2 t} - 4(13v^2-7w^2-\frac{1}{2}C_1-\frac{1}{2}C_2)\frac{t^2}{\sinh^2 t}. \end{aligned} \tag{54}$$

$U^{(1)}$ is obtained in [5], which comes from

$$E_2[\phi_0,\phi_1] = -\frac{(2\lambda)^2}{2!}\left(\frac{1}{V^{(2)}(\lambda)}+\frac{1}{V^{(2)}(0)}\right)\mathrm{Tr}(PaQa^\dagger P)^2. \tag{55}$$

$U^{(2)}$ comes from

$$\begin{aligned} E_3[\phi_0,\phi_1] &= \lambda^2\mathrm{Tr}\left[\left\{\frac{1}{3}C_1+\frac{1}{3}C_2+2v^2-2vw+2w^2\right\}(PaQa^\dagger P)^3\right. \\ &\quad + 2(v^2-w^2)QaQa^\dagger P(Pa^\dagger PaQa^\dagger PaQ - PaQa^\dagger Pa^\dagger PaQ) \\ &\quad \left. -(v^2-w^2)(PaQa^\dagger P)^2\right]. \end{aligned} \tag{56}$$

Here we have dropped the constant terms coming from E_2 and E_3. The formula (56) is valid for generic v and w, as long as ϕ_0 and ϕ_1 are the consistent solution of the equation of motion up to $O(\theta^{-1})$. Of course we should take $v=w$ for $(1,1)$-system. In this case, $U^{(1)}$ and the first term in $U^{(2)}$ become zero. Assuming that the coefficient of the second term in $U^{(2)}$ does not vanish, the potential U becomes to have the same functional form as $U^{(1)}$ at the leading order. So the force between two solitons will be attractive or repulsive, according to the sign of the coefficient. Thus, even if $1/V^{(2)}(\lambda)+1/V^{(2)}(0)=0$, unless the further condition

$$V^{(3)}(\lambda)+V^{(3)}(0) = -\frac{6\cdot 2!}{\lambda}V^{(2)}(0) \tag{57}$$

is satisfied in order that $U^{(2)}$ vanishes, any static multi-soliton solution does not exist at finite θ. For instance, in a case of ϕ^3-theory with a stable vacuum, we have $V(\phi) = c(\phi^3-\frac{3}{2}\lambda\phi^2)$ $(c<0)$. This satisfies $v=w$ but (57). Then

$$U^{(2)} = -4\cdot\frac{32}{9c^2\lambda^2}\frac{t^2}{\sinh^2 t} \tag{58}$$

and it gives attractive force between two solitons. On the other hand, when the condition (57) is satisfied, it may be determined by examining the E_4 term in (51) whether there is a force between solitons.

In this paper, we have studied the noncommutative scalar soliton at finite θ, starting from the infinite θ limit and taking into account the effect of finite θ in power series of $1/\theta$. First we have explored the consistent solution of the equation of motion up to the second order in $1/\theta$. The allowed configuration depends on the potential term $V(\phi)$ of the theory. Then we have discussed the correction to the moduli space of multi solitons up to that order. The effect destroys the Kähler structure of the metric, which comes from the Grassmannian nature of the moduli space. It would be interesting to study the geometrical meaning of this finite θ correction. Finally we have calculated the potential U in the effective dynamics of two solitons and discussed the force between solitons at finite θ. In most cases, any static multi-soliton solution does not exist, but there seems to be a possibility that static one is allowed when we choose a particular potential $V(\phi)$.

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