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Fusion Rules for $\hat{sl}(2|1;\mathbb{C})$ at Fractional Level $k=-1/2$

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Abstract

We calculate fusion rules for the admissible representations of the superalgebra $\hat{sl}(2|1;\mathbb{C})_k$ at fractional level $k=-1/2$ in the Ramond sector. By representing 3-point correlation functions involving a singular vector as the action of differential operators on the $sl(2|1;\mathbb{C})$ invariant 3-point function, we obtain conditions on permitted quantum numbers involved. We find that in this case the primary fields close under fusion.

1 Introduction

The study of conformal field theory based on affine algebras is one that is well-established. Unitary theories at integer level do not, however, provide the whole story, and it is interesting to also consider theories built on affine algebras at fractional level: these may provide access to many other models via hamiltonian reduction, or indeed (algebraically) define non-unitary models in their own right. In this regard, it is admissible representations which stand out, since their characters form a closed set under modular transformations. The case of $\hat{sl}(2)$ is much studied: in particular, fusion rules for fractional level were investigated in [1, 2, 3, 4], with more abstract analysis carried out in [5] and [6]. This question has also been addressed in the case of $\hat{sl}(3)$ in [7, 8] and in [9], where an overview of the $\hat{sl}(2)$ situation is also given. Conformal blocks for fractional level $\hat{sl}(2)$ have also been studied extensively, for example in [4, 10, 11]. As for superalgebras, the case of $\widehat{osp}(1|2)$ has been considered in [12]. It is clear then that the subject of fractional level is one of continued interest, the present work concerned with extending the techniques of [12] to discuss the superalgebra $\hat{sl}(2|1;\mathbb{C})$. This introduces an additional degree of complexity, arising from the fact that $\hat{sl}(2|1;\mathbb{C})$ is the simplest superalgebra where zero length roots appear, in contrast to $\widehat{osp}(1|2)$. It is worthwhile studying this from an abstract point of view, but also given the intimate link between non-unitary $\hat{sl}(2|1;\mathbb{C})$ and the $N=2$ non-critical superstring [13]. Fractional level $\hat{sl}(2|1;\mathbb{C})$ has also arisen in the study of Gaussian disordered systems [14].

Common to most of the works mentioned above is the understanding that for fractional level representations, one must work with fields which are not only a function of the usual coordinate z, but also of an isotopic coordinate x, representing an internal $sl(2)$ symmetry. This technique was first applied to the unitary $\hat{sl}(2)$ case in [15] and is also developed in [16]. For fractional level, this overcomes the problem of needing to consider general infinite-dimensional representations, which are neither highest nor lowest weight. In [12], the authors extended this approach by including not only the coordinate x but also dependence on a Grassmann coordinate θ to represent the supersymmetry present in $\widehat{osp}(1|2)$: for the case of $\hat{sl}(2|1;\mathbb{C})$ we must additionally augment this by another Grassmann coordinate $\bar{\theta}$, now with two supersymmetries present. Our basic approach will be to determine 3-point correlation functions involving fields with this dependence and then determine this correlator involving a singular vector. We may then rewrite this as an expression involving differential operators acting on the 3-point function, which must equal zero since it involves a singular vector. This will provide relations between the quantum numbers of the three fields present, determining which fusion rules are permitted.

We begin by recalling some essential features of $\hat{sl}(2|1;\mathbb{C})$. We then discuss the relation between highest weight states and primary fields, before going on to determine a realisation of $\hat{sl}(2|1;\mathbb{C})$ in terms of differential operators. This will allow us to determine the $\hat{sl}(2|1;\mathbb{C})$ invariant 3-point function, an issue discussed in the context of $N=2$ superconformal field theory; in examining this issue we will relate the work of [17, 18, 19, 20] on this subject. A brief summary of singular vectors required (as determined in [21]) will then be given, before putting these pieces together in the calculation of fusion rules.

2 Introduction to $\hat{sl}(2|1;\mathbb{C})$

We begin with a brief overview of some of the important features of $\hat{sl}(2|1;\mathbb{C})$. For more details, the references [21], [22] may be consulted.

The affine superalgebra $\hat{sl}(2|1;\mathbb{C})_k$ is made up of the even generators $\{J_n^{\pm}, J_n^3, U_n\}$ and odd generators $\{j_n^{\pm}, j_n^{'\pm}\}$, supplemented by the usual affine generators $\tilde{k}$, the central generator, and d, the derivative operator. This is identified with the generator $-L_0$ of the Virasoro algebra associated to $\hat{sl}(2|1;\mathbb{C})_k$ via the Sugawara construction. The non-zero (anti)commutation relations of $\hat{sl}(2|1;\mathbb{C})_k$ are

$$\begin{aligned}
[J_m^+, J_n^-] &= 2J_{m+n}^3 + 2\tilde{k}m\delta_{m+n,0}, & [U_m, U_n] &= -\tilde{k}m\delta_{m+n,0}, \\
[J_m^3, J_n^{\pm}] &= \pm J_{m+n}^{\pm}, & [J_m^3, J_n^3] &= \tilde{k}m\delta_{m+n,0}, \\
[J_m^{\pm}, j_n^{'\mp}] &= \pm j_{m+n}^{\pm}, & [J_m^{\pm}, j_n^{\mp}] &= \mp j_{m+n}^{'\pm}, \\
[2J_m^3, j_n^{'\pm}] &= \pm j_{m+n}^{'\pm}, & [2J_m^3, j_n^{\pm}] &= \pm j_{m+n}^{\pm}, \\
[2U_m, j_n^{'\pm}] &= \pm j_{m+n}^{'\pm}, & [2U_m, j_n^{\pm}] &= \mp j_{m+n}^{\pm}, \\
\{j_m^{'+}, j_n^{'-}\} &= (U_{m+n} - J_{m+n}^3) - 2\tilde{k}m\delta_{m+n,0}, & & \\
\{j_m^+, j_n^-\} &= (U_{m+n} + J_{m+n}^3) + 2\tilde{k}m\delta_{m+n,0}, & \{j_m^{'\pm}, j_n^{\pm}\} &= J_{m+n}^{\pm}.
\end{aligned} \tag{2.1}$$

In addition, we have that

$$[d, X_n] = nX_n \tag{2.2}$$

and the central generator $\tilde{k}$ commutes with all other generators.

The even generators have mode index $n \in \mathbb{Z}$, whereas the odd generators have $n \in \mathbb{Z}$ in the Ramond sector and $n \in \mathbb{Z} + \frac{1}{2}$ in the Neveu-Schwarz sector. Setting this index to zero recovers the finite $sl(2|1;\mathbb{C})$ algebra. Unless otherwise stated, we work in the Ramond sector.

As usual, we can define a triangular decomposition of the algebra via the generators d and J_0^3, combined as the principal gradation $\tilde{d} = ad(J_0^3) + ad(d)$:

$$\tilde{d}(J_n^{\pm}) = 2n \pm 1, \quad \tilde{d}(J_n^3) = 2n, \quad \tilde{d}(j_n^{\pm}) = 2n \pm \tfrac{1}{2}, \tag{2.3}$$

$$\tilde{d}(j_n^{'\pm}) = 2n \pm \tfrac{1}{2}, \quad \tilde{d}(U_n) = 2n, \quad \tilde{d}(\tilde{k}) = \tilde{d}(d) = 0. \tag{2.4}$$

Denoting the algebra by $\hat{\mathfrak{g}}$, we obtain the decomposition

$$\hat{\mathfrak{g}} = \hat{\mathfrak{g}}_- \oplus \hat{\mathfrak{g}}_0 \oplus \hat{\mathfrak{g}}_+, \tag{2.5}$$

$\hat{\mathfrak{g}}_-$ consisting of those elements having $\tilde{d} < 0$, $\hat{\mathfrak{g}}_0$ of elements with $\tilde{d} = 0$ and $\hat{\mathfrak{g}}_+$ of elements with $\tilde{d} > 0$.

Highest weight states of $\hat{sl}(2|1;\mathbb{C})_k$ are characterised by their conformal weight h, isospin $\frac{1}{2}h_-$ and charge $\frac{1}{2}h_+$:

$$|\Lambda\rangle = |h, \tfrac{1}{2}h_-, \tfrac{1}{2}h_+\rangle \tag{2.6}$$

where

$$L_0|\Lambda\rangle = h|\Lambda\rangle, \quad J_0^3|\Lambda\rangle = \tfrac{1}{2}h_-|\Lambda\rangle, \quad U_0|\Lambda\rangle = \tfrac{1}{2}h_+|\Lambda\rangle. \tag{2.7}$$

$|\Lambda\rangle$ is annihilated by all the raising operators, *i.e.* elements of $\hat{\mathfrak{g}}_+$. This condition is equivalent to the following three:

$$j_0^+|\Lambda\rangle = j_0^{'+}|\Lambda\rangle = J_1^-|\Lambda\rangle = 0 \tag{2.8}$$

and corresponds to a particular choice of simple roots of $sl(2|1;\mathbb{C})$.

The relationship between quantum numbers h, h_- and h_+ is given by

$$h = \frac{1}{4(k+1)}(h_-^2 - h_+^2) \tag{2.9}$$

where k is the level of the representation concerned (taken to be $\frac{1}{2}$ for the purposes of this work). The Kac-Khazdan determinant formula dictates that the Verma module built on a highest weight state with certain specific h_- and h_+ values will contain singular vectors, a full analysis of which was carried out in [21] and [22]. We shall make extensive use of the results of these references for the $k = -\frac{1}{2}$ case.

3 Fields and States

In conformal field theory, the link between highest weight states and primary fields is given by the state-field correspondence, namely

$$\lim_{z,\bar{z}\to 0} \phi(z,\bar{z})\,|0\rangle = |\Lambda\rangle, \tag{3.1}$$

where $|0\rangle$ denotes the vacuum state of the theory. It has been noted by several authors [2, 3, 4, 5, 6, 10, 11] that for the case of admissible $\hat{sl}(2)$ (and $\hat{sl}(3)$ [7, 8, 9]) representations that dependence on a second $sl(2)$ coordinate should be added: this overcomes the difficulty in describing these infinite-dimensional representations, which are in general neither highest nor lowest weight. The case of $\widehat{osp}(1|2)$ was studied in [12] where additionally a Grassmann coordinate θ was introduced to represent the supersymmetry present. For the case of $\hat{sl}(2|1;\mathbb{C})_k$, with two supersymmetries, it is necessary to consider not only the coordinate z, but also an additional $sl(2)$ coordinate x as well as two Grassmann coordinates θ and $\bar{\theta}$. This has been formalised in [23]. Anti-holomorphic counterparts of all these coordinates should also be included, although here we suppress any dependence on these coordinates for clarity. Thus in our case, the state-field correspondence is given by

$$\lim_{z,x,\theta,\bar{\theta}\to 0} \phi_a(z,x,\theta,\bar{\theta})\,|0\rangle = |\Lambda_a\rangle. \tag{3.2}$$

We can represent the action of the $\hat{sl}(2|1;\mathbb{C})$ modes on primary fields via the action of certain differential operators, using the commutation relations

$$\begin{aligned}
[J^{\alpha}_{n},\phi_a(z,x,\theta,\bar{\theta})] &= z^n D^{\alpha}_{a}\phi_a(z,x,\theta,\bar{\theta}),\\
[j^{\alpha}_{n},\phi_a(z,x,\theta,\bar{\theta})] &= z^n d^{\alpha}_{a}\phi_a(z,x,\theta,\bar{\theta}),\\
[j^{'\alpha}_{n},\phi_a(z,x,\theta,\bar{\theta})] &= z^n d^{'\alpha}_{a}\phi_a(z,x,\theta,\bar{\theta}),\\
[U_n,\phi_a(z,x,\theta,\bar{\theta})] &= z^n D^{U}_{a}\phi_a(z,x,\theta,\bar{\theta}).
\end{aligned} \tag{3.3}$$

It should be noted that for the case of the fermionic modes j, $j^{'}$ the commutator should be replaced by an anticommutator as appropriate. In addition, we have the relation with Virasoro modes given by

$$[L_n,\phi_a(z,x,\theta,\bar{\theta})] = \{z^{n+1}\partial_z + (n+1)h^a z^n\}\phi_a(z,x,\theta,\bar{\theta}). \tag{3.4}$$

This leads us to the definition of the action of ϕ_a on the vacuum:

$$\phi_a(z,x,\theta,\bar{\theta})\,|0\rangle = e^{zL_{-1}+xJ^-_0+\theta j^-_0+\bar{\theta}j^{'-}_0}|\Lambda_a\rangle. \tag{3.5}$$

4 Differential Operators for $sl(2|1)$

The equations (3.3) and (3.5) allow the calculation of the explicit forms of the differential operators appearing in (3.3). Considering the $\hat{sl}(2|1;\mathbb{C})$ zero modes, we may write

$$e^{-xJ^-_0-\theta j^-_0-\bar{\theta}j^{'-}_0}X_0 e^{xJ^-_0+\theta j^-_0+\bar{\theta}j^{'-}_0}|\Lambda_a\rangle = e^{-xJ^-_0-\theta j^-_0-\bar{\theta}j^{'-}_0}D^X e^{xJ^-_0+\theta j^-_0+\bar{\theta}j^{'-}_0}|\Lambda_a\rangle, \tag{4.1}$$

where D^X is the differential operator corresponding to the generator X_0. By repeated differentiation of this expression with respect to x and application of the relations (2.1) we obtain the following conjugation formulae:

$$\begin{aligned}
e^{-xJ^-_0-\theta j^-_0-\bar{\theta}j^{'-}_0}J^3_0 e^{xJ^-_0+\theta j^-_0+\bar{\theta}j^{'-}_0}|\Lambda\rangle &= J^3_0 - xJ^-_0 - \tfrac{1}{2}\theta j^-_0 - \tfrac{1}{2}\bar{\theta}j^{'-}_0|\Lambda\rangle\\
e^{-xJ^-_0-\theta j^-_0-\bar{\theta}j^{'-}_0}J^+_0 e^{xJ^-_0+\theta j^-_0+\bar{\theta}j^{'-}_0}|\Lambda\rangle &= 2xJ^3_0 - x^2J^-_0 - x\theta j^-_0 - x\bar{\theta}j^{'-}_0 + \theta\bar{\theta}U_0|\Lambda\rangle\\
e^{-xJ^-_0-\theta j^-_0-\bar{\theta}j^{'-}_0}J^-_0 e^{xJ^-_0+\theta j^-_0+\bar{\theta}j^{'-}_0}|\Lambda\rangle &= J^-_0|\Lambda\rangle\\
e^{-xJ^-_0-\theta j^-_0-\bar{\theta}j^{'-}_0}j^+_0 e^{xJ^-_0+\theta j^-_0+\bar{\theta}j^{'-}_0}|\Lambda\rangle &= -xj^{'-}_0 + x\theta J^-_0 - \theta(J^3_0+U_0) + \tfrac{1}{2}\theta\bar{\theta}j^{'-}_0|\Lambda\rangle\\
e^{-xJ^-_0-\theta j^-_0-\bar{\theta}j^{'-}_0}j^-_0 e^{xJ^-_0+\theta j^-_0+\bar{\theta}j^{'-}_0}|\Lambda\rangle &= j^-_0 - \bar{\theta}J^-_0|\Lambda\rangle\\
e^{-xJ^-_0-\theta j^-_0-\bar{\theta}j^{'-}_0}j^{'+}_0 e^{xJ^-_0+\theta j^-_0+\bar{\theta}j^{'-}_0}|\Lambda\rangle &= xj^-_0 - x\bar{\theta}J^-_0 + \bar{\theta}(J^3_0-U_0) + \tfrac{1}{2}\theta\bar{\theta}j^-_0|\Lambda\rangle\\
e^{-xJ^-_0-\theta j^-_0-\bar{\theta}j^{'-}_0}j^{'-}_0 e^{xJ^-_0+\theta j^-_0+\bar{\theta}j^{'-}_0}|\Lambda\rangle &= j^{'-}_0 - \theta J^-_0|\Lambda\rangle\\
e^{-xJ^-_0-\theta j^-_0-\bar{\theta}j^{'-}_0}U_0 e^{xJ^-_0+\theta j^-_0+\bar{\theta}j^{'-}_0}|\Lambda\rangle &= U_0 + \tfrac{1}{2}\theta j^-_0 - \tfrac{1}{2}\bar{\theta}j^{'-}_0 - \tfrac{1}{2}\theta\bar{\theta}J^-_0|\Lambda\rangle.
\end{aligned} \tag{4.2}$$

Comparing these results with that of calculating

$$e^{-xJ_0^- -\theta j_0^- -\bar{\theta} j_0^{'-}}(a\partial_x + b\theta\partial_x + c\bar{\theta}\partial_x + d\partial_\theta + e\theta\partial_\theta + f\bar{\theta}\partial_\theta + g\partial_{\bar{\theta}} + h\theta\partial_{\bar{\theta}} + l\bar{\theta}\partial_{\bar{\theta}} + m\theta\bar{\theta}\partial_\theta + n\theta\bar{\theta}\partial_{\bar{\theta}})e^{xJ_0^- +\theta j_0^- +\bar{\theta} j_0^{'-}}|\Lambda\rangle = (aJ_0^- + b\theta J_0^- + c\bar{\theta}J_0^- + dj_0^- - \frac{1}{2}d\bar{\theta}J_0^- + e\theta j_0^- - \frac{1}{2}e\theta\bar{\theta}J_0^- + f\bar{\theta}j_0^- + gj_0^{'-} - \frac{1}{2}g\theta J_0^- + h\theta j_0^{'-} + l\bar{\theta}j_0^{'-} + \frac{1}{2}l\theta\bar{\theta}J_0^- + m\theta\bar{\theta}j_0^- + n\theta\bar{\theta}j_0^{'-})|\Lambda\rangle \quad (4.3)$$

we arrive at the following expressions for the differential operator realisation of $sl(2|1;\mathbb{C})$:

$$\begin{aligned}
D^3 &= -x\partial_x - \tfrac{1}{2}\theta\partial_\theta - \tfrac{1}{2}\bar{\theta}\partial_{\bar{\theta}} - \tfrac{1}{2}h_- \\
D^+ &= -x^2\partial_x - x\theta\partial_\theta - x\bar{\theta}\partial_{\bar{\theta}} + xh_- + \tfrac{1}{2}\theta\bar{\theta}h_+ \\
D^- &= \partial_x \\
d^+ &= -x\partial_{\bar{\theta}} + \tfrac{1}{2}x\theta\partial_x - \tfrac{1}{2}\theta(h_- + h_+) + \tfrac{1}{2}\theta\bar{\theta}\partial_{\bar{\theta}} \\
d^- &= \partial_\theta - \tfrac{1}{2}\bar{\theta}\partial_x \\
d^{'+} &= x\partial_\theta - \tfrac{1}{2}x\bar{\theta}\partial_x + \tfrac{1}{2}\bar{\theta}(h_- - h_+) + \tfrac{1}{2}\theta\bar{\theta}\partial_\theta \\
d^{'-} &= \partial_{\bar{\theta}} - \tfrac{1}{2}\theta\partial_x \\
D^U &= \tfrac{1}{2}\theta\partial_\theta - \tfrac{1}{2}\bar{\theta}\partial_{\bar{\theta}} + \tfrac{1}{2}h_+ .
\end{aligned} \quad (4.4)$$

5 $sl(2|1)$ Invariant 3-point Function

In order to discover $\hat{sl}(2|1;\mathbb{C})_k$ fusion rules, we wish to consider quantities such as

$$\langle\Lambda_1^*|\phi_2(z,x,\theta,\bar{\theta})|\omega\Lambda_3\rangle, \quad (5.1)$$

where $\omega\Lambda$ is a singular vector. Then (5.1) will be equal to zero and we may rewrite this expression as

$$(\text{differential operators})\langle\Lambda_1^*|\phi_2(z,x,\theta,\bar{\theta})|\Lambda_3\rangle = 0 \quad (5.2)$$

using the relations (3.3). Evaluating (5.2) we then obtain conditions on possible quantum numbers of ϕ_2 and ϕ_1^* given those of ϕ_3, amounting to a specification of allowed fusings. We see here the appearance of the conjugate field ϕ_1^*, since the fusion rule $\phi_i \times \phi_j = \phi_k$ arises from the non-vanishing of the 3-point function $\langle\phi_k^*\phi_j\phi_i\rangle$. This procedure relies on knowledge of $\hat{sl}(2|1;\mathbb{C})_k$ singular vectors, which we have from [21], and knowledge of the $sl(2|1;\mathbb{C})$ invariant 3-point function, which we now move on to discuss.

The differential operator realisation (4.4) allows us to determine the $sl(2|1;\mathbb{C})$ invariant 3-point function, which is in fact the 3-point function for $N = 2$ superconformal field theory, since $sl(2|1;\mathbb{C})$ is isomorphic to the set of generators of the $N = 2$ super-Möbius group. This has been considered by several authors [17, 18, 19, 20], the interrelation of whose work will be clarified here. We note here that our discussion of the Ramond sector directly parallels the discussion of the Neveu-Schwarz sector in these (and other) works on superconformal field theory: the algebra (2.1) has vanishing central piece for the Ramond sector zero mode subalgebra, whereas the equivalent subalgebra in the usual superconformal discussion is in the Neveu-Schwarz sector. Indeed, there it is Neveu-Schwarz generators which give rise to the super-Möbius group, as discussed in [19]. Although our terminology derives from the fact that we are considering the situation where the fermionic modes have integer labels, when we discuss Ramond fields and states they are more like Neveu-Schwarz rather than Ramond, in the sense that the fields do not introduce branch cuts in the operator product expansion with fermionic currents. This is due to the fact that our fermionic currents are expanded as, for example (see [22]), $J(\mathbf{e}_{\pm\alpha_1})(z) = \sum_n j_n^{'\pm} z^{-n-1}$ whereas a typical fermionic current in superconformal field theory is $G(z) = \sum_n G_n z^{-n-3/2}$. As we will find that the fusion of two Ramond fields gives rise to another Ramond field, this interpretation means that our results are not in conflict with those of (for example) [24] for the $N=1$ superconformal case and [18] for $N=2$, where it was found that the fusion of two Ramond fields produces a Neveu-Schwarz field, whereas the fusion of two Neveu-Schwarz fields gives another Neveu-Schwarz field.

For a 3-point function to be $sl(2|1;\mathbb{C})$ invariant, we require that

$$\langle 0|[X_0, \phi_1]\phi_2\phi_3|0\rangle + \langle 0|\phi_1[X_0, \phi_2]\phi_3|0\rangle + \langle 0|\phi_1\phi_2[X_0, \phi_3]|0\rangle = 0 \tag{5.3}$$

for each of the $sl(2|1;\mathbb{C})$ generators X_0. Using the relations (3.3) yields

$$\sum_{i=1}^{3} D_i^X \langle 0|\phi_1(z_1, x_1, \theta_1, \bar\theta_1)\phi_2(z_2, x_2, \theta_2, \bar\theta_2)\phi_3(z_3, x_3, \theta_3, \bar\theta_3)|0\rangle = 0, \tag{5.4}$$

where D_i^X is the differential operator corresponding to X_0, taking its parameters h_+ and h_- from the primary field ϕ_i. This assumes that the vacuum $|0\rangle$ is annihilated by elements of $sl(2|1;\mathbb{C})$. As is well known, in conformal field theory solving the resulting differential equations determines the 3-point function exactly. In the $N=2$ superconformal case, the 3-point function can depend on the nine variables x_i, θ_i and $\bar\theta_i$. Finding a 3-point function which satisfies the differential equations arising from the generators J_0^+, j_0^- and $j_0^{'-}$ will result in the automatic solution of the remaining equations, from the commutation relations (2.1). With nine parameters but only three independent equations available (although we make use of five equations for simplicity), there will naturally be some ambiguity in the final answer. With this in mind, we proceed with our analysis, closely following the above references and particularly [19].

From the equations for J_0^-, j_0^- and $j_0^{'-}$, it is clear that the 3-point function depends on the following variables:

$$s_{ij} = x_i - x_j - \tfrac{1}{2}\theta_i\bar\theta_j - \tfrac{1}{2}\bar\theta_i\theta_j, \quad \theta_{ij} = \theta_i - \theta_j, \quad \bar\theta_{ij} = \bar\theta_i - \bar\theta_j, \quad i,j = 1,2,3. \tag{5.5}$$

Then from the U_0 equation (suppressing the z dependence for ease of notation)

$$\sum_{i=1}^{3} (\tfrac{1}{2}\theta\partial_\theta - \tfrac{1}{2}\bar\theta\partial_{\bar\theta} + \tfrac{1}{2}h_+)\langle 0|\phi_1(x_1, \theta_1, \bar\theta_1)\phi_2(x_2, \theta_2, \bar\theta_2)\phi_3(x_3, \theta_3, \bar\theta_3)|0\rangle = 0 \tag{5.6}$$

we find that there are several distinct cases to be examined, that is, for which

$$H_+ = \sum_{i=1}^{3} h_{+,i} = 0, \pm 1, \pm 2. \tag{5.7}$$

Consideration of the J_0^+ condition eliminates the cases where $H_+ = \pm 2$, so there are in fact three distinct 3-point functions to be obtained. It remains to solve the J_0^+ equation subject to the conditions $H_+ = 0, \pm 1$. For the case $H_+ = 0$, we find (again suppressing the z dependence)

$$\begin{aligned}\langle 0|\phi_1\phi_2\phi_3|0\rangle = C_{123} s_{12}^{a_3} s_{23}^{a_1} s_{13}^{a_2} \bigg[1 &+ \frac{h_{+,1}\theta_{12}\bar{\theta}_{12}}{2s_{12}} \\ &+ \frac{(h_{+,1}+h_{+,2})\theta_{23}\bar{\theta}_{23}}{2s_{23}} + \frac{h_{+,1}(h_{+,1}+h_{+,2})\theta_{12}\bar{\theta}_{12}\theta_{23}\bar{\theta}_{23}}{4s_{12}s_{23}} \\ &+ \alpha\frac{\theta_{12}\bar{\theta}_{12}s_{23}}{s_{12}s_{13}} - \alpha\frac{(\theta_{12}\bar{\theta}_{23}+\theta_{23}\bar{\theta}_{12})}{s_{13}} + \alpha\frac{\theta_{23}\bar{\theta}_{23}s_{12}}{s_{23}s_{13}} \\ &+ \alpha\frac{h_{+,1}\theta_{12}\bar{\theta}_{12}\theta_{23}\bar{\theta}_{23}}{2s_{23}s_{13}} + \alpha\frac{(h_{+,1}+h_{+,2})\theta_{12}\bar{\theta}_{12}\theta_{23}\bar{\theta}_{23}}{2s_{12}s_{13}} \bigg],\end{aligned} \tag{5.8}$$

where α is an undetermined parameter and $a_1 = \frac{1}{2}(h_{-,2}+h_{-,3}-h_{-,1})$, *etc.* This is essentially the answer of [18] and [20], though now with no restrictions on α. The system of equations given by (5.4) is invariant under permutations of 1,2 and 3: this is not reflected in the solution (5.8) and indeed any permutation of these labels will also give a solution. If we interchange 1 and 3 and add the resulting answer to the one above, we obtain a solution

$$\begin{aligned}\langle 0|\phi_1\phi_2\phi_3|0\rangle = 2C_{123} s_{12}^{a_3} s_{23}^{a_1} s_{13}^{a_2} \bigg[1 &+ \frac{h_{+,1}\theta_{12}\bar{\theta}_{12}}{2s_{12}} + \frac{(h_{+,1}+h_{+,2})\theta_{23}\bar{\theta}_{23}}{2s_{23}} \\ &+ \frac{h_{+,1}(h_{+,1}+h_{+,2})\theta_{12}\bar{\theta}_{12}\theta_{23}\bar{\theta}_{23}}{4s_{12}s_{23}} \bigg],\end{aligned} \tag{5.9}$$

where we have used the fact that $H_+ = 0$ and taken $C_{123} = C_{321}$. Strictly speaking, this is not the exact answer obtained by this procedure, since $s_{ij} = -s_{ji}$ means that the permuted answer differs from (5.8) by a factor $(-1)^{-a_1-a_3-a_2}$. However, when the full dependence on anti-holomorphic variables is included, the overall multiplicative factor in (5.8) is modified to $C_{123}|s_{12}|^{-2a_3}|s_{23}|^{-2a_1}|s_{13}|^{-2a_2}$ (with $h_{-,i} = \bar{h}_{-,i}$) and this discrepancy disappears. The expression (5.9) is in fact a particular case of the solution obtained by Kiritsis [17]. Understanding (5.8) as written for the labelling $\{123\}$ we find that the solution obtained by adding (5.8) written with $\{213\}$ to that with $\{312\}$ is also a particular Kiritsis solution, as is the expression resulting from the addition of $\{132\}$ to $\{231\}$. When all six versions of (5.8) are added together, the solution obtained is precisely that given by Howe and West [19], which is again a specific instance of the solution described by Kiritsis, distinguished by the fact that it is a permutation invariant solution of the equations (5.4).

Before going on to clarify this situation, we consider the other 3-point functions for the cases $H_+ = \pm 1$.

When $H_+ = -1$, we find that

$$\langle 0|\phi_1\phi_2\phi_3|0\rangle = C'_{123} s_{12}^{a_3} s_{23}^{a_1} s_{13}^{a_2+1/2} \left[\frac{\theta_{12}}{s_{12}^{1/2} s_{23}^{-1/2}} - \frac{\theta_{23}}{s_{12}^{-1/2} s_{23}^{1/2}} \right. \\ \left. - \frac{(h_{+,1}\theta_{23}\theta_{12}\bar{\theta}_{12} + (1 - h_{+,1} - h_{+,2})\theta_{12}\theta_{23}\bar{\theta}_{23})}{2 s_{12}^{1/2} s_{23}^{1/2}} \right]; \quad (5.10)$$

when $H_+ = 1$ we have

$$\langle 0|\phi_1\phi_2\phi_3|0\rangle = C''_{123} s_{12}^{a_3} s_{23}^{a_1} s_{13}^{a_2+1/2} \left[\frac{\bar{\theta}_{12}}{s_{12}^{1/2} s_{23}^{-1/2}} - \frac{\bar{\theta}_{23}}{s_{12}^{-1/2} s_{23}^{1/2}} \right. \\ \left. - \frac{(h_{+,1}\bar{\theta}_{23}\theta_{12}\bar{\theta}_{12} + (-1 - h_{+,1} - h_{+,2})\bar{\theta}_{12}\theta_{23}\bar{\theta}_{23})}{2 s_{12}^{1/2} s_{23}^{1/2}} \right]. \quad (5.11)$$

These are identical to the expressions given in [20] and corrected from [18]. Again, they are not invariant under permutations of the field labels. However, we find that (with the proviso discussed above that anti-holomorphic coordinates should be included) the form of these expressions for $\{123\}$ is the same as that for $\{321\}$, *etc.* Indeed, if we sum the resulting three variants in each case, we obtain the solutions found by Howe and West [19].

To proceed with the calculation of fusion rules, we note that the expression (5.2) is in terms of highest weight states rather than fields acting on the vacuum. We have the definition (3.2) to give us the highest weight in state. The global superconformal transformations may be found by exponentiating the generators to be

$$x' = \frac{ax+b}{cx+d} + \frac{e^q\theta((1 - \frac{1}{2}\epsilon_1\bar{\epsilon}_2)\bar{\epsilon}_1 x + (1 + \frac{1}{2}\epsilon_2\bar{\epsilon}_1)\bar{\epsilon}_2)}{(cx+d)^2} + \frac{e^{-q}\bar{\theta}((1 + \frac{1}{2}\epsilon_2\bar{\epsilon}_1)\epsilon_1 x + (1 - \frac{1}{2}\epsilon_1\bar{\epsilon}_2)\epsilon_2)}{(cx+d)^2} \\ + \frac{\theta\bar{\theta}((2d\epsilon_1\bar{\epsilon}_1 - c(\epsilon_1\bar{\epsilon}_2 + \epsilon_2\bar{\epsilon}_1))x + d(\epsilon_1\bar{\epsilon}_2 + \epsilon_2\bar{\epsilon}_1) - 2c\epsilon_2\bar{\epsilon}_2)}{(cx+d)^3},$$

$$\theta' = \frac{\epsilon_1 x + \epsilon_2}{cx+d} + \frac{e^q\theta(1 + \frac{1}{2}(\epsilon_2\bar{\epsilon}_1 - \epsilon_1\bar{\epsilon}_2) - \frac{1}{4}\epsilon_1\bar{\epsilon}_1\epsilon_2\bar{\epsilon}_2)}{cx+d} + \frac{\theta\bar{\theta}(d\epsilon_1 - c\epsilon_2)}{(cx+d)^2},$$

$$\bar{\theta}' = \frac{\bar{\epsilon}_1 x + \bar{\epsilon}_2}{cx+d} + \frac{e^{-q}\bar{\theta}(1 + \frac{1}{2}(\epsilon_2\bar{\epsilon}_1 - \epsilon_1\bar{\epsilon}_2) - \frac{1}{4}\epsilon_1\bar{\epsilon}_1\epsilon_2\bar{\epsilon}_2)}{cx+d} - \frac{\theta\bar{\theta}(d\bar{\epsilon}_1 - c\bar{\epsilon}_2)}{(cx+d)^2}, \quad (5.12)$$

here corrected from [17] (a, b, c and d are the $SL(2)$ parameters $|ad - bc| = 1$, ϵ and $\bar{\epsilon}$ are anticommuting parameters associated with the supersymmetry transformations and q with the transformation arising from U_0).

For a suitable definition of out state $\langle\Lambda|$, we wish to take $x \to \infty$ via the transformation

$$x' = \frac{1}{x}. \quad (5.13)$$

For the global transformation to be of this form, we require that $\epsilon_1 = \epsilon_2 = \bar{\epsilon}_1 = \bar{\epsilon}_2 = 0$. Consequently, the transformations for θ and $\bar{\theta}$ are given by

$$\begin{aligned} \theta' &= \frac{e^q \theta}{x}, \\ \bar{\theta}' &= \frac{e^{-q}\bar{\theta}}{x}. \end{aligned} \tag{5.14}$$

Then the point $(0,0,0)$ is mapped to $(\infty,0,0)$ as its natural inverse, which is then the limit for the out state

$$\langle\Lambda| = \lim_{\substack{x\to 0\\ \theta=\bar{\theta}=0}} x^{h_-}\langle 0|\phi\left(\frac{1}{x},\frac{\theta}{x},\frac{\bar{\theta}}{x}\right). \tag{5.15}$$

The factor x^{h_-} arises from the transformation law for superprimary fields [17]

$$\tilde{\phi}(x,\theta,\bar{\theta}) = \phi(x',\theta',\bar{\theta}')[(\partial_\theta + \tfrac{1}{2}\bar{\theta}\partial_x)\theta']^{(-h_- - h_+)/2}[(\partial_{\bar{\theta}} + \tfrac{1}{2}\theta\partial_x)\bar{\theta}']^{(-h_- + h_+)/2} \tag{5.16}$$

the factor in which reduces to x^{h_-} for the transformation given, evaluated at $\theta = \bar{\theta} = 0$ and with the choice $q = 0$.

Alternatively, we may consider the expansion of a primary field as given by

$$\phi(x,\theta,\bar{\theta}) = \varphi(x) + \theta\psi(x) + \bar{\theta}\bar{\psi}(x) + \theta\bar{\theta}g(x). \tag{5.17}$$

We see that with our previous definition of in state (3.2) we have

$$\varphi(0)|0\rangle = |\Lambda\rangle, \tag{5.18}$$

so then

$$\langle\Lambda| = \lim_{x\to\infty}\langle 0|\varphi(x)x^{-h_-} \tag{5.19}$$

with again $\theta = \bar{\theta} = 0$.

With the definition of out state established, we arrive at the form of 3-point function which we will use for our calculations of fusion rules. For the even case (5.8) for which $h_{+,1} + h_{+,2} + h_{+,3} = 0$, the result is

$$\langle\Lambda_1|\phi(z_2,x_2,\theta_2,\bar{\theta}_2)|\Lambda_3\rangle = C_{123} z_2^{h_1-h_2-h_3} x_2^{(h_{-,2}+h_{-,3}-h_{-,1})/2}\left[1 - \frac{(h_{+,3}-2\alpha)}{2x_2}\theta_2\bar{\theta}_2\right]. \tag{5.20}$$

For the odd case (5.10) for which $h_{+,1} + h_{+,2} + h_{+,3} = -1$ we find

$$\langle\Lambda_1|\phi(z_2,x_2,\theta_2,\bar{\theta}_2)|\Lambda_3\rangle = \tilde{C}'_{123} z_2^{h_1-h_2-h_3} x_2^{(h_{-,2}+h_{-,3}-h_{-,1}-1)/2}\theta_2 \tag{5.21}$$

and the other odd case (5.11), where $h_{+,1} + h_{+,2} + h_{+,3} = 1$, becomes

$$\langle\Lambda_1|\phi(z_2,x_2,\theta_2,\bar{\theta}_2)|\Lambda_3\rangle = \tilde{C}''_{123} z_2^{h_1-h_2-h_3} x_2^{(h_{-,2}+h_{-,3}-h_{-,1}-1)/2}\bar{\theta}_2. \tag{5.22}$$

The z-dependence is determined, as usual, by taking the commutator with L_0: as it will not influence our discussion of fusion rules, we shall generally omit it in what follows.

We should mention at this point that in the limit discussed above, where $x_1 \to \infty$, $x_3 = 0$ and $\theta_1 = \bar{\theta}_1 = \theta_3 = \bar{\theta}_3 = 0$, the odd 3-point functions as given by Howe and West [19] reduce to the expressions (5.21) and (5.22). However, the expression obtained by this procedure from their even 3-point function differs from (5.20). We will show that (5.20) leads to sensible fusion rules, whereas use of the corresponding Howe and West expression only gives these in part. Beyond this, we can give no formal justification of why one might start from a non-permutation invariant expression for the 3-point function (which is thus intrinsically non-local). We might also note that each of the possible ways of writing (5.8) leads to the same expression (5.20), that is

$$\lim_{\substack{x_i\to\infty, x_k=0 \\ \theta_{i,k}=0, \bar{\theta}_{i,k}=0}} x_i^{-h_{-,i}} \langle 0|\phi_i(x_i,\theta_i,\bar{\theta}_i)\phi_j(x_j,\theta_j,\bar{\theta}_j)\phi_k(x_k,\theta_k,\bar{\theta}_k)|0\rangle =$$

$$C_{ijk} x_j^{(h_{-,j}+h_{-,k}-h_{-,i})/2}\left[1 - \frac{(h_{+,k}-2\alpha)}{2x_j}\theta_j\bar{\theta}_j\right], \quad (5.23)$$

where $i, j, k = 1, 2, 3$, $i \neq j \neq k$.

6 Singular Vectors for $k = -1/2$

In the Ramond sector of $\hat{sl}(2|1;\mathbb{C})_k$ at level $k = -1/2$, there are four primary fields $\phi_{m,m'}$, $m, m' = 0, 1$ in the so-called class IV and class V representations, these being the relevant ones since the evidence indicates that their characters form a closed set under modular transformations [25]. The fields $\phi_{0,0}$ and $\phi_{0,1}$ are self-conjugate, while $\phi^*_{1,0} = \phi_{1,1}$. In [21], the authors calculated general expressions for singular vectors using a Malikov-Feigin Fuchs type construction; we will make use of these expressions here. For each of the fields $\phi_{m,m'}$ there are three singular vectors at the first level, from which conditions obtained through the calculation of (5.2) must be simultaneously satisfied. We shall list the appropriate form of singular vectors as given in [21] and go on to make use of them to calculate fusion rules in the next section.

For $|\Lambda_{0,0}\rangle$, with quantum numbers $h_- = h_+ = h = 0$, the three singular vectors are

$$\begin{aligned} &(i) && j_0^-|\Lambda_{0,0}\rangle, \\ &(ii) && j_0^{'-}|\Lambda_{0,0}\rangle, \\ &(iii) && (J_{-1}^+)^{3/2}(j_0^- j_0^{'-} - j_0^{'-} j_0^-)(J_{-1}^+)^{1/2}|\Lambda_{0,0}\rangle. \end{aligned} \quad (6.1)$$

For $|\Lambda_{1,0}\rangle$, $h_- = h_+ = -\frac{1}{2}$, $h = 0$ the singular vectors are

$$\begin{aligned} &(i) && j_0^{'-}|\Lambda_{1,0}\rangle, \\ &(ii) && J_{-1}^+|\Lambda_{1,0}\rangle, \\ &(iii) && (J_0^-)^{1/2}(j_0^- j_0^{'-} - 2j_0^{'-} j_0^-)J_{-1}^+ j_0^- J_{-1}^+ (J_0^-)^{-1/2}(-j_0^{'-} j_0^-)|\Lambda_{1,0}\rangle. \end{aligned} \quad (6.2)$$

The state $|\Lambda_{1,1}\rangle$ has $h_- = -\frac{1}{2}$, $h_+ = \frac{1}{2}$, $h = 0$ and singular vectors

$$\begin{aligned}
&(i) \qquad j_0^-|\Lambda_{1,1}\rangle, \\
&(ii) \qquad J_{-1}^+|\Lambda_{1,1}\rangle, \\
&(iii) \qquad (J_0^-)^{1/2}(3j_0^- j_0^{'-} - J_0^-)J_{-1}^+ j_0^{'-} J_{-1}^+ (J_0^-)^{-3/2}(-j_0^- j_0^{'-} + J_0^-)|\Lambda_{1,1}\rangle
\end{aligned} \tag{6.3}$$

and for $|\Lambda_{0,1}\rangle$ with $h_- = 1$, $h_+ = 0$, $h = \frac{1}{2}$ the singular vectors are

$$\begin{aligned}
&(i) \qquad (\tfrac{3}{2}j_{-1}^+ + j_0^{'-} J_{-1}^+)|\Lambda_{0,1}\rangle, \\
&(ii) \qquad (j_0^- j_0^{'-} - j_0^{'-} j_0^-)|\Lambda_{0,1}\rangle, \\
&(iii) \qquad (-\tfrac{3}{2}j_{-1}^{'+} + j_0^- J_{-1}^+)|\Lambda_{0,1}\rangle.
\end{aligned} \tag{6.4}$$

These expressions for singular vectors may be used in (5.1) to give expressions of the form (5.2), utilising the equations (3.3). The singular vectors generally involve fractional powers of generators, which may be rearranged using

$$AB^a = \sum_{i=0}^{\infty} \binom{a}{i} B^{a-i} [\cdots[[A, \overbrace{B], B], \cdots, B]}^{i} \tag{6.5}$$

to give expressions with integer powers. For the purposes of calculation, it is more convenient to keep the expressions as they stand and modify (3.3) accordingly, using

$$\phi_j(x,\theta,\bar\theta)(X_0)^a = \sum_{i=0}^{\infty} \binom{a}{i} (X_0)^{a-i}(-D_j^X)^i \phi_j(x,\theta,\bar\theta), \tag{6.6}$$

with an overall minus sign as required for the case of fermionic generators and a fermionic field ϕ_j. This results in the differential operators in (5.2) also involving fractional powers, in our case, of the differential operators corresponding to the generators J_0^- and J_0^+. To deal with this situation, we will make use of the following expressions in our calculations:

$$\begin{aligned}
(D^-)^a \theta^\gamma \bar\theta^{\bar\gamma} x^b &= (\partial_x)^a \theta^\gamma \bar\theta^{\bar\gamma} x^b \\
&= \frac{\Gamma(b+1)}{\Gamma(b-a+1)} \theta^\gamma \bar\theta^{\bar\gamma} x^{b-a};
\end{aligned} \tag{6.7}$$

$$\begin{aligned}
(D^+)^a \theta^\gamma \bar\theta^{\bar\gamma} x^b &= (-x^2\partial_x - x\theta\partial_\theta - x\bar\theta\partial_{\bar\theta} + xh_- + \tfrac{1}{2}\theta\bar\theta h_+)^a \theta^\gamma \bar\theta^{\bar\gamma} x^b \\
&= \frac{\Gamma(h_- - b - \gamma - \bar\gamma + 1)}{\Gamma(h_- - b - a - \gamma - \bar\gamma + 1)} \theta^\gamma \bar\theta^{\bar\gamma} x^{b+a} + \frac{a h_+ \Gamma(h_- - b)}{2\Gamma(h_- - b - a + 1)} \theta^{\gamma+1} \bar\theta^{\bar\gamma+1} x^{b+a-1}.
\end{aligned} \tag{6.8}$$

These expressions may be verified as holding for integer values of a, with the validity for fractional values of a following by analytic continuation.

7 Calculation of Fusions

The information presented in the above sections allows us now to calculate fusion rules. As we wish to calculate expressions of the form (5.2), we note that since these are equal to zero, the procedure for deriving (5.2) from (5.1) essentially amounts to replacing the generators by their corresponding differential operators, with any quantum numbers involved in those expressions being the ones associated to the field ϕ_2, through which we are commuting. We may ignore factors of z arising from those generators with mode numbers not equal to zero and the possibility of having to use anticommutators between fields and fermionic generators, since this only gives rise to an overall minus sign. Consider the singular vector (iii) of (6.1). Using (6.6) we have

$$\begin{aligned}\langle\Lambda_1^*|\phi_2(z,x,\theta,\bar{\theta})(J_{-1}^+)^{3/2}(j_0^- j_0^{'-} - j_0^{'-} j_0^-)(J_{-1}^+)^{1/2}|\Lambda_3\rangle = \\ \langle\Lambda_1^*|(J_{-1}^+ - z^{-1}D_2^+)^{3/2}(j_0^- - d_2^-)(j_0^{'-} - d_2^{'-}) - \\ (j_0^{'-} - d_2^{'-})(j_0^- - d_2^-)(J_{-1}^+ - z^{-1}D_2^+)^{1/2}\phi_2(z,x,\theta,\bar{\theta})|\Lambda_3\rangle = 0. \quad (7.1)\end{aligned}$$

This expression may be rearranged using (6.5), with the generators arising from this procedure all such that they annihilate the out state, as can be seen from the mode numbers involved. The only remaining part is the piece made up of the derivative terms, with an overall factor involving powers of (-1) and z, which can be eliminated. The derivative terms may then be "unrearranged" to give the expression with fractional powers and we have

$$\begin{aligned}\langle\Lambda_1^*|\phi_2(x,\theta,\bar{\theta})(J_{-1}^+)^{3/2}(j_0^- j_0^{'-} - j_0^{'-} j_0^-)(J_{-1}^+)^{1/2}|\Lambda_3\rangle \to \\ (D_2^+)^{3/2}(d_2^- d_2^{'-} - d_2^{'-} d_2^-)(D_2^+)^{1/2}\langle\Lambda_1^*|\phi_2(x,\theta,\bar{\theta})|\Lambda_3\rangle = 0. \quad (7.2)\end{aligned}$$

The only instance where some care needs to be taken is for the case of $|\Lambda_{0,1}\rangle$, where there are singular vectors made up of a term involving one generator and a term involving two generators, which will lead to a minus sign difference on commuting with ϕ_2. It remains to apply each of the three singular vectors for each field to the three 3-point functions (5.20), (5.21) and (5.22).

$\underline{\phi_{0,0} : h_- = h_+ = 0}$

We begin by examining $\phi_{0,0}$, the identity field in this context, where we hope the behaviour to be fairly transparent. For the even 3-point function (5.20), where now $H_+ = h_{+,1} + h_{+,2} = 0$ we calculate

$$\begin{aligned}(i) \qquad & d_2^-\langle\Lambda_1^*|\phi_2(x,\theta,\bar{\theta})|\Lambda_{0,0}\rangle = 0 \\ (ii) \qquad & d_2^{'-}\langle\Lambda_1^*|\phi_2(x,\theta,\bar{\theta})|\Lambda_{0,0}\rangle = 0 \\ (iii) \qquad & (D_2^+)^{3/2}(d_2^- d_2^{'-} - d_2^{'-} d_2^-)(D_2^+)^{1/2}\langle\Lambda_1^*|\phi_2(x,\theta,\bar{\theta})|\Lambda_{0,0}\rangle = 0, \quad (7.3)\end{aligned}$$

with

$$\langle\Lambda_1^*|\phi_2(x,\theta,\bar{\theta})|\Lambda_{0,0}\rangle = C[x^{a_1} + \tfrac{1}{2}(h_{+,1} + h_{+,2} + 2\alpha)\theta\bar{\theta}x^{a_1-1}]. \quad (7.4)$$

From the first two equations, we find that $a_1 = \frac{1}{2}(h_{-,2} - h_{-,1}) = 0$ and $\alpha = 0$. Then from the third equation we have two conditions:

$$
\begin{aligned}
&h_{-,2}(h_{-,2} - 1) - 3h_{+,2}^2 = 0, \\
&h_{+,2}(h_{-,2} + \tfrac{1}{2}) = 0.
\end{aligned}
\tag{7.5}
$$

When $h_{+,2} = 0$ we have $h_{-,2} = 0$ or $h_{-,2} = 1$. When $h_{-,2} = -\frac{1}{2}$ we have $h_{+,2} = -\frac{1}{2}$ or $h_{+,2} = \frac{1}{2}$. This unambiguously identifies the following possibilities:

$$
\begin{aligned}
\text{when } \phi_3 = \phi_{0,0}, \text{ then} \quad & \phi_2 = \phi_{0,0} \text{ and } \phi_1^* = \phi_{0,0} \\
\text{or} \quad & \phi_2 = \phi_{1,0} \text{ and } \phi_1^* = \phi_{1,1} \\
\text{or} \quad & \phi_2 = \phi_{1,1} \text{ and } \phi_1^* = \phi_{1,0} \\
\text{or} \quad & \phi_2 = \phi_{0,1} \text{ and } \phi_1^* = \phi_{0,1}.
\end{aligned}
\tag{7.6}
$$

This is the sort of behaviour one would wish for, given that $\phi_{0,0}$ is the identity field. However, when this calculation is performed with the even 3-point function of Howe and West (with limits taken as described), we only find the first and last of these results arising, with no coupling between the identity and $\phi_{1,0}$ or $\phi_{1,1}$.

Repeating the exercise with the odd 3-point functions (5.21) and (5.22) requires that these are identically zero. For example, considering case (i) of (7.3) using (5.21) gives

$$
(\partial_{\theta_2} - \tfrac{1}{2}\bar{\theta}_2\partial_{x_2})\tilde{C}'_{123} x_2^{a_1 - 1/2}\theta_2 = \tilde{C}'_{123}\left(x_2^{a_1-1/2} + \tfrac{1}{2}\left(a_1 - \tfrac{1}{2}\right)\theta\bar{\theta}x_2^{a_1-3/2}\right) = 0 \tag{7.7}
$$

which implies that $\tilde{C}'_{123} = 0$. Hence the even case exhausts all possibilities.

$\underline{\phi_{0,1} : h_- = 1, h_+ = 0}$

The next case we examine is that of $\phi_{0,1}$. The even 3-point function (with $h_{+,1} + h_{+,2} = 0$) yields:

$$
\begin{aligned}
\text{when } \phi_3 = \phi_{0,1}, \quad & \phi_2 = \phi_{0,0} \text{ and } \phi_1^* = \phi_{0,1} \\
\text{or} \quad & \phi_2 = \phi_{0,1} \text{ and } \phi_1^* = \phi_{0,0}.
\end{aligned}
\tag{7.8}
$$

The odd 3-point function (5.21), for which $h_{+,1} + h_{+,2} = -1$ gives:

$$
\text{when } \phi_3 = \phi_{0,1}, \quad \phi_2 = \phi_{1,0} \text{ and } \phi_1^* = \phi_{1,0} \tag{7.9}
$$

while the other odd 3-point function ($h_{+,1} + h_{+,2} = 1$) reveals:

$$
\text{when } \phi_3 = \phi_{0,1}, \quad \phi_2 = \phi_{1,1} \text{ and } \phi_1^* = \phi_{1,1}. \tag{7.10}
$$

$\underline{\phi_{1,0} : h_- = -\frac{1}{2}, h_+ = -\frac{1}{2}}$

Turning now to $\phi_{1,0}$ we notice that for this particular case of quantum numbers, the singular vector in case (iii) of (6.2) gives no additional information over case (i). Once the fact that $j_0^{'-}|\Lambda_{1,0}\rangle = 0$ has been imposed, case (iii) vanishes after the first step and this singular vector need not be considered.

For the even 3-point function, where $h_{+,1} + h_{+,2} - \frac{1}{2} = 0$, we find:

$$h_{-,2} = -h_{+,2} = a_1 = \tfrac{1}{2}\left(h_{-,2} + \left(-\tfrac{1}{2}\right) - h_{-,1}\right). \tag{7.11}$$

While the quantum numbers of ϕ_1^* and ϕ_2 are not given explicitly, we can allow ϕ_2 to take the quantum numbers of all the $\phi_{m,m'}$ in turn and see what results this gives for ϕ_1^*. In fact, since $h_{-,2} = -h_{+,3}$ we are immediately restricted to taking $\phi_2 = \phi_{0,0}$ or $\phi_2 = \phi_{1,1}$. Then

$$\begin{aligned} \text{when } \phi_3 = \phi_{1,0}, \text{ and } \phi_2 = \phi_{0,0} &\text{ then } \phi_1^* = \phi_{1,1} \\ \text{and when } \phi_2 = \phi_{1,1} &\text{ then } \phi_1^* = \phi_{0,0} \end{aligned} \tag{7.12}$$

which is in agreement with results from the $\phi_{0,0}$ calculation (although with different values of the parameter α).

In the case of the odd 3-point function (5.22) we find that this is identically zero. However, the 3-point function (5.21) (for which $h_{+,1} + h_{+,2} - \frac{1}{2} = -1$) gives

$$h_{-,2} = a_1 + \tfrac{1}{2} = \tfrac{1}{2} - h_{-,1}. \tag{7.13}$$

Again, letting ϕ_2 take the quantum numbers of $\phi_{m,m'}$ yields

$$\begin{aligned} \text{when } \phi_3 = \phi_{1,0}, \text{ and } \phi_2 = \phi_{0,0} &\text{ then } h_{-,1}^* = \tfrac{1}{2} \text{ and } h_{+,1}^* = -\tfrac{1}{2} \\ \text{when } \phi_2 = \phi_{1,0} &\text{ then } \phi_1^* = \phi_{0,1} \\ \text{when } \phi_2 = \phi_{1,1} &\text{ then } h_{-,1}^* = 1 \text{ and } h_{+,1}^* = -1 \\ \text{when } \phi_2 = \phi_{0,1} &\text{ then } \phi_1^* = \phi_{1,0}. \end{aligned} \tag{7.14}$$

There are two cases here for which the quantum numbers do not correspond to any of the fields available. However, these are precisely the cases already considered in the even 3-point function and so may be discarded. The last result is as already obtained in the consideration of $\phi_{0,1}$.

$\underline{\phi_{1,1} : h_- = -\frac{1}{2}, h_+ = \frac{1}{2}}$

The situation for $\phi_{1,1}$ is very similar to that for $\phi_{1,0}$. The singular vector (iii) of (6.3) is of the form $(\ldots)(-j_0^- j_0^{'-} + J_0^-)|\Lambda_{1,1}\rangle$ which may be rearranged as $(\ldots)(j_0^{'-} j_0^-)|\Lambda_{1,1}\rangle$. This will again give no additional information over the result of using the singular vector (i) in (6.3), which is $j_0^-|\Lambda_{1,1}\rangle$.

The even 3-point function, with $h_{+,1} + h_{+,2} + \frac{1}{2} = 0$ shows that

$$h_{-,2} = h_{+,2} = a_1 = \tfrac{1}{2}\left(h_{-,2} + \left(-\tfrac{1}{2}\right) - h_{-,1}\right). \tag{7.15}$$

We see that the only options for ϕ_2 are $\phi_{0,0}$ and $\phi_{1,0}$. Hence

$$\begin{aligned} \text{when } \phi_3 = \phi_{1,1}, \text{ and } \phi_2 = \phi_{0,0} &\text{ then } \phi_1^* = \phi_{1,0} \\ \text{and when } \phi_2 = \phi_{1,0} &\text{ then } \phi_1^* = \phi_{0,0}, \end{aligned} \tag{7.16}$$

again with different values of α. As for the odd 3-point functions, it is now (5.21) which is identically zero and (5.22) (where $h_{+,1} + h_{+,2} = \frac{1}{2} = 1$) that gives

$$h_{-,2} = a_1 + \tfrac{1}{2} = \tfrac{1}{2} - h_{-,1}. \tag{7.17}$$

Considering the remaining options for ϕ_2, we find

$$\begin{aligned} \text{when } \phi_3 = \phi_{1,1}, \text{ and } \phi_2 = \phi_{1,1} \text{ then } \phi_1^* = \phi_{0,1} \\ \text{and when } \phi_2 = \phi_{0,1} \text{ then } \phi_1^* = \phi_{1,1}. \end{aligned} \tag{7.18}$$

The first of these results has already been seen in considering $\phi_{1,0}$ while the second echoes the result of the $\phi_{0,1}$ calculation.

To summarise the above results, replacing the fields ϕ_1^* by their relevant conjugates, we have found that the following fusion rules hold for the Ramond fields of $\hat{sl}(2|1;\mathbb{C})_k$ with $k = -\frac{1}{2}$:

$$\begin{aligned} \phi_{0,0} \times \phi_{0,0} &= \phi_{0,0}, & \phi_{1,0} \times \phi_{1,1} &= \phi_{0,0}, \\ \phi_{0,0} \times \phi_{1,0} &= \phi_{1,0}, & \phi_{1,0} \times \phi_{0,1} &= \phi_{1,1}, \\ \phi_{0,0} \times \phi_{1,1} &= \phi_{1,1}, & \phi_{1,1} \times \phi_{1,1} &= \phi_{0,1}, \\ \phi_{0,0} \times \phi_{0,1} &= \phi_{0,1}, & \phi_{1,1} \times \phi_{0,1} &= \phi_{1,0}, \\ \phi_{1,0} \times \phi_{1,0} &= \phi_{0,1}, & \phi_{0,1} \times \phi_{0,1} &= \phi_{0,0}. \end{aligned} \tag{7.19}$$

These fusion rules form an associative algebra, as they should. One immediately obvious statement about these results is that on interchanging $\phi_{1,0}$ and $\phi_{1,1}$ the form of the fusion rules is unchanged. This precisely reflects what was discovered in the investigation of modular invariants in [25]. There we found the permutation invariants

$$Z = \sum \chi_{m,m'} \bar{\chi}_{\Pi(m,m'}, \tag{7.20}$$

involving $\Pi(m,m') = (m, (m-m') \mod u)$. This permutation leaves $\phi_{0,0}$ and $\phi_{0,1}$ unchanged, but interchanges $\phi_{1,0}$ and $\phi_{1,1}$ and so the modular invariant (7.20) would seem to be a consequence of the fusion rule automorphism (though we have not explicitly established the form of fusion rules for the remaining sectors involved in the modular invariant).

As for the Neveu-Schwarz sector, in superconformal field theory it has been found [26] that the fusion of two Ramond fields yields a Neveu-Schwarz field, with the form of fusion rules preserved due to the sector isomorphism. By analogy, we expect to find fusion rules of the form above, but now with $\phi_i^{NS} \times \phi_j^R = \sum N_{ij}{}^k \phi_k^{NS}$: we should replace ϕ_1 and ϕ_3 in (7.19) by their Neveu-Schwarz counterparts. The confirmation of this behaviour remains a task for the future.

8 Conclusion

The study of conformal field theories based on affine algebras at fractional level is one that has been tackled somewhat sporadically over the last decade. As yet, no absolute consensus has been reached even for $\hat{sl}(2)$ as to whether these can actually define *bona fide* conformal field theories in their own right. However, the evidence does seem to suggest that this is possible; in any case, other models may be obtained through hamiltonian reduction or the coset construction. The work of [12] is a first indication that fusion rules are well-defined for fractional level superalgebras, a conclusion which is also borne out by this work. The authors of [12] were able to determine consistent fusion rules for all levels at which admissible representations of $\widehat{osp}(1|2)$ exist. Due to the far more complex nature of singular vectors of $\hat{sl}(2|1;\mathbb{C})_k$, we have thus far only examined a particular case, that of $k = -\frac{1}{2}$, but feel that a complete examination of levels $k = 1/u - 1$, $u \in \mathbb{N} \setminus \{1\}$ may not be out of the question. However, for the situation at hand, we have found that the Ramond fields present do close under fusion. Other methods, particularly that of the Coulomb gas formalism (used in [18] for $N = 2$ superconformal field theory) would seem to be more promising for a more thorough examination of this problem of fusion rules. As yet, the techniques for such a study have not been developed for superalgebras other than $\widehat{osp}(1|2)$ [23]. It will be interesting to deepen the study of results in this area and compare this with the work of [25] on modular transformations of $\hat{sl}(2|1;\mathbb{C})_k$ in the attempt to fully realise a conformal field theory based on fractional level $\hat{sl}(2|1;\mathbb{C})$.

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