Anomalies and Star Products

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Abstract

A formulation of anomalies in terms of star products is suggested which promises insight from an alternative and unifying point of view.

1   Overview

Anomalies correspond to the fact that sometimes symmetries of a classical theory cannot be implemented in the corresponding quantum theory. The study of field theoretical anomalies has a long history. They were originally observed as unexpected terms in the commutators of quantum currents among each others (Schwinger terms) or with products of field operators (anomalous Ward identities). Much insight was subsequently obtained in the nature of anomalies by topological and cohomological methods.

One line of development lead to a description of gauge anomalies as local cocycles on the gauge group expressed in terms of field operators. Relating anomalies to the lack of invariance of the determinant of operators of Dirac type yielded an understanding in terms of characteristic classes of determinant bundles over the space of connections.

These characteristic classes are all obtainable from first quantization of a Dirac type particle in an external classical gauge field, either in a Lagrangian or in a Hamiltonian formulation. Sometimes it is also appropriate to consider bundles of

Fock spaces over the space of connections ([7, 11]). In any case, the information on anomalies is related to the bundles of null spaces of families of Dirac operators. The anomalies are traces of products of inverse powers of the Dirac operator D and certain vertex operators, just those expressions also obtainable from one-loop Feynman integrals, where D^{-1} corresponds to the propagator of the Dirac field.

In yet another language, the anomalies are obtained from the symbol calculus of differential operators on the Dirac bundle. Further insight is gained from non-commutative geometry. Starting from the algebra of differential operators on the Dirac bundle, non commutative characteristic classes can be found, which are just the non commutative counterpart of the anomalies and which, evaluated on suitable non-commutative cycles, just give the characteristic numbers.

These results can also be looked at from the point of view of star products or deformation quantization ([3, 4, 5]). The algebra $\mathfrak{A}$ of symbols of differential operators on the Dirac bundle can be considered as the algebra of observables of a system consisting of a Dirac particle in an external gauge field. The symbol calculus in this algebra is thus a special example of a star product on $\mathfrak{A}$. More precisely, one has to go over to the algebra $\mathfrak{A}[[\lambda]]$ of formal power series in an indeterminate λ (eventually to be substituted by $\hbar$) with coefficients in $\mathfrak{A}$.

This restriction to formal power series is by no means only a shortcoming, but it also has definite advantages.

It opens the way to a concentration on a better understanding of algebraic aspects of the quantization procedure. In particular, there is the powerful notion of equivalence of star products that are related, for example, by changing ordering prescriptions in the quantization procedure. Anomalies, anyhow, are always of low order in $\hbar$ and should be treatable without any loss of information in terms of formal power series in λ. In the language of star product quantization, anomalies are objects in the non-commutative geometry of the algebra $\mathfrak{A}[[\lambda]]$ endowed with the star product. The trace functional on $\mathfrak{A}[[\lambda]]$, applied to non-commutative characteristic classes on $\mathfrak{A}[[\lambda]]$ give characteristic numbers. Uniqueness theorems on the trace functional and non commutative cohomologies can successfully be applied. The theory of star products on finite dimensional phase spaces like the phase space of the Dirac system is well defined. Anomalies are expected also to arise as imperfections in invariance properties of the star product.

2   A mechanical example

Let $\mathcal{M}$ be a classical phase space of finite dimension. Let a group $\mathcal{G}$ with Lie algebra $\mathfrak{g}$ act on $\mathcal{M}$ in a Hamiltonian way with equivariant momentum map ([1])

$$\mathcal{J} : \mathcal{M} \to \mathfrak{g}^*,$$

such that for every $\xi \in \mathfrak{g}$, $f \in C^\infty(\mathcal{M})$

$$L_\xi f = -\{f, \mathcal{J}_\xi\} \quad \text{and} \quad \{\mathcal{J}_\xi, \mathcal{J}_\eta\} = \mathcal{J}_{[\xi,\eta]},$$

where $\mathcal{J}_\xi(p) = \langle \mathcal{J}(p), \xi \rangle$.

A star product $\star$ on $C^\infty(\mathcal{M})[[\lambda]]$ is called (see, e.g., [2])

1. covariant, if $[\mathcal{J}_\xi, \mathcal{J}_\eta]_\star = \mathcal{J}_\xi \star \mathcal{J}_\eta - \mathcal{J}_\eta \star \mathcal{J}_\xi = i\lambda\mathcal{J}_{[\xi,\eta]}$,

2. invariant, if $\{\mathcal{J}_\xi, f \star g\} = \{\mathcal{J}_\xi, f\} \star g + f \star \{\mathcal{J}_\xi, g\}$,

3. strongly invariant, if $\mathcal{J}_\xi \star f - f \star \mathcal{J}_\xi = i\lambda\{\mathcal{J}_\xi, f\}$.

Clearly, the last property implies the two others. The second equation is the infinitesimal version of the requirement of invariance under canonical transformations Φ:

$$\Phi^*(f \star g) = \Phi^* f \star \Phi^* g.$$

Schwinger terms should correspond to a violation of the covariance condition (1.), whereas anomalous terms in Ward identities are expected to reflect themselves in a lack of strong invariance (3.).

The group of all classical canonical transformations, for example, cannot be implemented in quantum theory without anomalies. Indeed, for this large group, all observables are momentum maps, conditions (1.) and (2.) become identical and the impossibility of a star product to transform Poisson brackets to star commutators directly follows from the theorem of Groenewald and van Hove. These anomalies are the reason for the notorious incompabilities of quantization procedures and canonical transformations. If conditions (1.), (2.) or (3.) are not fulfilled, one may try to save the situation by adding non leading terms to $\mathcal{J}_\xi$,

$$\mathbb{J}_\xi = \mathcal{J}_\xi + \lambda \cdot (\ldots),$$

such that $\mathbb{J}_\xi$, the so-called quantum momentum map ([12]), shows the above properties.

For example, quantum covariance means

$$\mathbb{J}_\xi \star \mathbb{J}_\eta - \mathbb{J}_\eta \star \mathbb{J}_\xi = i\lambda\mathbb{J}_{[\xi,\eta]}.$$

One easily convinces oneself, that covariance holds for a star product $\star$ if and only if it holds for all equivalent star products. Hence, the transformation

$$T : \mathcal{J}_\xi \to \mathbb{J}_\xi$$

rendering a non-covariant $\mathcal{J}_\xi$ covariant cannot be an equivalence transformation but must depend on ξ. The redefinition of $\mathcal{J}_\xi$ is ad hoc rather than global.

As an illustration we can consider a mechanical system in one space dimension. Galilei invariant systems are not appropriate, because there is no equivariant momentum map. The generators of translations and boosts p and q commute as Lie algebra elements, but as generators of canonical transformations we have $\{p, q\} = 1$. The Galilei group can be realized by canonical transformations only after central extension.

The Poincaré group does not have this defect. The Ruijsenaars-Schneider model ([9]) is a mechanical system in one space dimension with Poincaré invariance and non trivial interaction.

Let us denote the generators of time and space translations by $\mathfrak{h}$ and $\mathfrak{p}$ and the boost generator by $\mathfrak{b}$. In the Lie algebra of the Poincaré group we have

$$[\mathfrak{h}, \mathfrak{p}] = 0, \quad [\mathfrak{b}, \mathfrak{h}] = \mathfrak{p}, \quad [\mathfrak{b}, \mathfrak{p}] = \frac{1}{c^2}\mathfrak{h}.$$

In the Ruijsenaars-Schneider model, the same relations hold,

$$\{\mathfrak{h}, \mathfrak{p}\} = 0, \quad \{\mathfrak{b}, \mathfrak{h}\} = \mathfrak{p}, \quad \{\mathfrak{b}, \mathfrak{p}\} = \frac{1}{c^2}\mathfrak{h},$$

where $\mathfrak{h}$, $\mathfrak{p}$, and $\mathfrak{b}$ now are certain functions over phase space. This is not in contradiction with the no-interaction theorem, because the translations are realised in the conventional way only to order $\frac{1}{c^2}$. Moreover, in a canonical quantization one has

$$[\mathbb{H}, \mathbb{P}] = 0, \quad [\mathbb{B}, \mathbb{H}] = -\frac{i}{\hbar}\mathbb{P}, \quad [\mathbb{B}, \mathbb{P}] = -\frac{i}{\hbar c^2}\mathbb{H},$$

or (the factor $-i$ can be absorbed in the formal parameter λ)

$$[\hat{\mathfrak{h}}, \hat{\mathfrak{p}}]_\star = 0, \quad [\hat{\mathfrak{b}}, \hat{\mathfrak{h}}]_\star = \hat{\mathfrak{p}}, \quad [\hat{\mathfrak{b}}, \hat{\mathfrak{p}}]_\star = \frac{1}{c^2}\hat{\mathfrak{h}}$$

for the standard ordered star product $\star$. Here, $\hat{\mathfrak{h}}$, $\hat{\mathfrak{p}}$, and $\hat{\mathfrak{b}}$ differ from the corresponding classical functions only in higher orders of λ. In addition the classical Ruijsenaars-Schneider system is completely integrable. It has a series of conserved quantities $\mathscr{J}_k$ in involution,

$$\{\mathscr{J}_k, \mathscr{J}_l\} = 0.$$

In the quantized theory one has, as shown by Ruijsenaars ([10]), that a modification $\mathscr{J}_k \to \mathbb{J}_k$ is possible, such that quantum integrability

$$[\mathbb{J}_k, \mathbb{J}_l]_\star = 0$$

holds, where $\star$ again is the standard ordered star product.

3   Field theory

One would, of course, like to apply the star product formalism directly to field theory, not only to first quantization. Unfortunately, for this case, the theory of star products is not yet fully developed.

One reason lies in the fact that even for the classical algebra of fields products like $\varphi_1(x_1)\varphi_2(x_2)$ are in general undefined for $x_1 = x_2$, this deficiency beeing even more serious for quantum fields. The normal way out is to consider (quantum) fields as (operator valued) distributions. For objects

$$\varphi_f = \int d^n x_1 \ldots d^n x_r \, \varphi(x_1) \cdots \varphi(x_r) \, f(x_1, \ldots, x_r)$$

with suitable test functions f, products are now defined ([8] and references therein). For the same reason, one expects star products $\varphi_f \star \varphi_g$ to be well defined.

In quantum field theory, one normally concentrates on expectation values like

$$w(x_1, \ldots, x_n) = \langle 0|\varphi(x_1) \ldots \varphi(x_n)|0\rangle.$$

This should also be a promising strategy for field theory in star product formulation. One can interprete the distribution

$$w(x_1, \ldots, x_n) \quad = \langle 0|\varphi(x_1) \ldots \varphi(x_n)|0\rangle$$
$$= \mu\left(\varphi(x_1) \star \ldots \star \varphi(x_n)\right)$$

as the value of a state on the algebra $\mathcal{A}[[\lambda]]$ of classical fields endowed with a star product. In fact, w even resembles a trace functional on $\mathcal{A}[[\lambda]]$. w has to be interpreted as a formal series in λ, whose coefficients are just the quasi-classical expansion of the quantum correlation functions. This will result in considerable technical simplifications as compared to quantum field theory. The star product algebra can now be reconstructed by a generalized GNS construction in the sense of the method introduced by M. Bordemann and S. Waldmann ([6]). Recently, part of this program has been implemented by K. Fredenhagen and M. Dütsch in [8].

For the discussion of anomalies, the momentum map $\mathcal{J}_\xi$ has to be replaced by the conserved current $\mathcal{J}_\xi^\mu$. Adding higher terms to find a quantum momentum map corresponds to the need of regularization. Indeed, by using the canonical (anti)commutation relations the classical expression for normal ordered currents is found formally only to order $\hbar$.

Conditions similar to (1.), (2.), and (3.) can be formulated, and their violation will be largely analogous to anomalous Ward identities in quantum field theory but in the sense of formal power series. Violations of the above conditions should be characterized cohomologically.

There should be a formulation of anomalies in terms of the non-commutative geometry in the field algebra $\mathcal{A}[[\lambda]]$ and not just in the algebra $\mathcal{A}_D[[\lambda]]$ of a Dirac particle.

The Dirac Operator plays a fundamental rôle in the non-commutative geometry of $\mathcal{A}_D[[\lambda]]$. D can also be interpreted as a derivation in $\mathcal{A}$, which can also be written in the form

$$\mathcal{A}[[\lambda]] \ni a \mapsto [Q, a]_\star \in \mathcal{A}[[\lambda]],$$

with

$$Q = \int d^n x\, \bar{\varphi} D \varphi.$$

A more ambitious task would be to formulate anomalies as algebraic properties on the algebra $\mathcal{B}[[\lambda]]$ of observables of a field theory rather than on the larger field algebra $\mathcal{A}[[\lambda]]$. The Dirac operator will still be a derivation in $\mathcal{B}[[\lambda]]$. Gauge anomalies primarily concern non observable quantities, their effect should be hidden more deeply in $\mathcal{B}[[\lambda]]$.

Acknowledgements

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