Unconstrained $SU(2)$ Yang-Mills Theory with Topological Term in the Long-Wavelength Approximation

A. M. Khvedelidze [a,b], D. M. Mladenov [c], H.-P. Pavel [c,d], and G. Röpke [d]

[a] *A. Razmadze Mathematical Institute, Tbilisi, 380093, Georgia*

[b] *Laboratory of Information Technologies,*
Joint Institute for Nuclear Research, 141980 Dubna, Russia

[c] *Bogoliubov Laboratory of Theoretical Physics,*
Joint Institute for Nuclear Research, 141980 Dubna, Russia

[d] *Fachbereich Physik der Universität Rostock, D-18051 Rostock, Germany*

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Abstract

The Hamiltonian reduction of $SU(2)$ Yang-Mills theory for arbitrary θ-angle to an unconstrained nonlocal theory of a self-interacting positive definite symmetric 3×3 matrix field $S(x)$ is performed. It is shown that, after exact projection to reduced phase space, the density of the Pontryagin index remains a pure divergence, proving the θ-independence of the obtained unconstrained theory. Expansion of the nonlocal kinetic part of the Hamiltonian in powers of inverse coupling constant and truncation to lowest order, however, leads to violation of the θ-independence of the theory. In order to maintain this property on the level of the local approximate theory, a modified expansion in inverse coupling constant is suggested, which for vanishing θ-angle coincides with the original expansion. The corresponding approximate Lagrangian up to second order in derivatives is derived and the explicit form of the unconstrained analog of the Chern-Simons current linear in derivatives is given. Finally, for the case of degenerate field configurations $S(x)$ with $\mathrm{rank}\|S\| = 1$, a nonlinear σ-type model is obtained, with the Pontryagin topological term reducing to the Hopf invariant of the mapping from the 3-sphere $\mathbb{S}^3$ to the unit 2-sphere $\mathbb{S}^2$ in the Whitehead form.

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1 Introduction

For a complete understanding of the low-energy quantum phenomena of Yang-Mills theory, it is necessary to have a nonperturbative, gauge invariant description of the underlying classical theory including the θ-dependent Pontryagin term [1]-[4]. Several representations of Yang-Mills theory in terms of local gauge invariant fields have been proposed [5]-[24] during the last decades, implementing Gauss law as a generator of small gauge transformations. However, dealing with such local gauge invariant fields special consideration is needed, when the topological term is included, since it is the 4-divergence of a current changing under large gauge transformations. In particular, consistency of constrained and unconstrained formulations of gauge theories with topological term requires to verify that, after projection to the reduced phase space, the classical equations of motion for the unconstrained variables remain θ-independent [1]. Furthermore the question, which trace the large gauge transformations with nontrivial Pontryagin topological index leave on the local gauge invariant fields, has to be addressed.

Having this in mind, we extend in the present paper our approach [22, 27, 28], to construct the unconstrained form of $SU(2)$ Yang-Mills theory, to the case when the topological term is included in the classical action. We generalize the Hamiltonian reduction of classical $SU(2)$ Yang-Mills field theory to arbitrary θ-angle by reformulating the original degenerate Yang-Mills theory as a nonlocal theory of a self-interacting positive definite symmetric 3×3 matrix field. The consistency of the Hamiltonian reduction in the presence of the Pontryagin term is demonstrated by constructing the canonical transformation, well-defined on the reduced phase space, that eliminates the θ-dependence of the classical equations of motion for the unconstrained variables.

With the aim to obtain a practical form of the nonlocal unconstrained Hamiltonian, we perform an expansion in powers of the inverse coupling constant, equivalent to an expansion in the number of spatial derivatives. We find that a straightforward application of the derivative expansion violates the principle of θ-independence of the classical observables. To cure this problem, we propose to exploit the property of chromoelectromagnetic duality of pure Yang-Mills theory, symmetry under exchange of chromoelectric and -magnetic fields. The electric and magnetic fields are subject to dual constraints, the Gauss-law and Bianchi identity, and only when both are fulfilled, the classical equations of motion are θ-independent. Thus any approximation in resolving the Gauss law constraints should be consistent with the Bianchi identity. We show how to use the Bianchi identity to rearrange the derivative expansion in such a way, that the θ-independence is restored to all orders on the classical level.

In order to have a representation of the gauge invariant degrees of freedom suitable for the study of the low energy phase of Yang-Mills theory, we perform a main-axis transformation of the symmetric tensor field and obtain the unconstrained Hamiltonian

[1]The question of consistency of the elimination of redundant variables in theories containing both constraints and pure divergencies, the so-called "divergence problem", has for the first time been analyzed in the context of the canonical reduction of General Relativity by P. Dirac [25] and by R. Arnowitt, S. Deser, C.W. Misner [26].

in terms of the main-axis variables in the lowest order in $1/g$. Carrying out an inverse Legendre transformation to the corresponding unconstrained Lagrangian, we find the explicit form of the unconstrained analog of the Chern-Simons current, linear in the derivatives.

Finally, we consider the case of degenerate symmetric field configurations S with $\mathrm{rank}\|S(x)\| = 1$. We find a non-linear classical theory of a three-dimensional unit-vector $\mathbf{n}$-field interacting with a scalar field. Using typical boundary conditions for the unit-vector field at spatial infinity, the Pontryagin topological charge density reduces to the Abelian Chern-Simons invariant density [4]. We discuss its relation to the Hopf number of the mapping from the 3-sphere $\mathbb{S}^3$ to the unit 2-sphere $\mathbb{S}^2$ in the Whitehead representation[29]. The Abelian Chern-Simons invariant is known from different areas in Physics, in fluid mechanics as “fluid helicity“, in plasma physics and magnetohydrodynamics as “magnetic helicity” [30]-[33]. In the context of 4-dimensional Yang-Mills theory a connection between non-Abelian vacuum configurations and certain Abelian fields with nonvanishing helicity established already in [34, 35].

The paper is organized as follows. In Section II the θ-independence of classical Yang-Mills theory in the framework of the constrained Hamiltonian formulation is revised. Section III is devoted to the derivation of unconstrained $SU(2)$ Yang-Mills theory for arbitrary θ-angle. The consistency of our reduction procedure is demonstrated by explicitly quoting the canonical transformation, which removes the θ-dependence from the unconstrained classical theory. In Section IV the unconstrained Hamiltonian up to order $o(1/g)$ is obtained. Section V presents the long-wavelength classical Hamiltonian in terms of main-axis variables. Performing an inverse Legendre transformation to the corresponding Lagrangian up to second order in derivatives, the unconstrained analog of the Chern-Simons current, linear in the derivatives, is obtained. In Section VI the unconstrained action for degenerate field configurations is considered. Section VII finally gives our conclusions. Several more technical details are presented in the Appendices A, B, and C. Appendix A summarizes our notations and definitions, Appendix B is devoted to the question of the exsistence of the ”symmetric gauge”, and in Appendix C the proof the θ-dependence of the ”naive” $1/g$ approximation is given.

2 Constrained Hamiltonian formulation

Yang-Mills gauge fields are classified topologically by the value of Pontryagin index [2]

$$p_1 = -\frac{1}{8\,\pi^2}\int \mathrm{tr}\, F\wedge F\,. \tag{1}$$

Its density, the so-called topological charge density $Q = -\,(1/8\,\pi^2)\,\mathrm{tr}\, F\wedge F$, being locally exact $Q = d\,C$, can be added to the conventional Yang-Mills Lagrangian with arbitrary

[2] Necessary notations and definitions for $SU(2)$ Yang-Mills theory used in the text have been collected in Appendix A.

parameter θ

$$\mathcal{L} = -\frac{1}{g^2}\,\mathrm{tr}\, F \wedge {}^*F - \frac{\theta}{8\pi^2\, g^2}\,\mathrm{tr}\, F \wedge F\,, \tag{2}$$

without changing the classical equations of motion. In the Hamiltonian formulation, this shifts the canonical momenta, conjugated to the field variables A_{ai},

$$\Pi_{ai} = \frac{\partial \mathcal{L}}{\partial \dot{A}_{ai}} = \dot{A}_{ai} - (D_i(A))_{ac}\, A_{c0} + \frac{\theta}{8\,\pi^2}\, B_{ai}\,, \tag{3}$$

by the magnetic field $(\theta/8\,\pi^2)\, B_{ai}$. As a result, the total Hamiltonian [36, 37] of Yang-Mills theory with θ-angle as a functional of canonical variables (A_{a0}, Π_a) and (A_{ai}, Π_{ai}) obeying the Poisson bracket relations

$$\{A_{ai}(t,\vec{x})\,, \Pi_{bj}(t,\vec{y})\} = \delta_{ab}\,\delta_{ij}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \tag{4}$$

$$\{A_{a0}(t,\vec{x})\,, \Pi_{b}(t,\vec{y})\} = \delta_{ab}\,\delta^{(3)}(\vec{x}-\vec{y})\,, \tag{5}$$

takes the form

$$H_T = \int d^3x \left[\frac{1}{2}\left(\Pi_{ai} - \frac{\theta}{8\pi^2}B_{ai}\right)^2 + \frac{1}{2}\,B^2_{ai} - A_{a0}\,(D_i(A))_{ac}\,\Pi_{ci} + \lambda_a\,\Pi_a\right]\,. \tag{6}$$

Here, the linear combination of three primary constraints

$$\Pi_a(x) = 0\,, \tag{7}$$

with arbitrary functions $\lambda_a(x)$ and the secondary constraints, the non-Abelian Gauss law

$$(D_i(A))_{ac}\,\Pi_{ci} = 0 \tag{8}$$

reflect the gauge invariance of the theory.

Based on the representation (6) for the total Hamiltonian, one can immediately verify that classical theories with different value of the θ-angle are equivalent. Performing the canonical transformation

$$\begin{aligned} A_{ai}(x) \quad &\longmapsto \quad A_{ai}(x)\,, \\ \Pi_{bj}(x) \quad &\longmapsto \quad E_{bj} := \Pi_{bj}(x) - \frac{\theta}{8\pi^2}\,B_{bj}(x)\,, \end{aligned} \tag{9}$$

to the new variables A_{ai} and E_{bj}, and using the Bianchi identity

$$(D_i(A))_{ab}\,B_{bi}(A) = 0\,, \tag{10}$$

one can then see that the θ-dependence completely disappears from the Hamiltonian (6). Note that the canonical transformation (9) can be represented in the form

$$E_{ai} \;=\; \Pi_{ai} - \,\theta\,\frac{\delta}{\delta A_{ai}}\,W[A]\,, \tag{11}$$

where $W[A]$ denotes the winding number functional,

$$W[A] = \int d^3x \, K^0[A] \,, \tag{12}$$

constructed from the zero component of the Chern-Simons current

$$K^\mu[A] = -\frac{1}{16\,\pi^2}\varepsilon^{\mu\alpha\beta\gamma}\,\mathrm{tr}\left(F_{\alpha\beta}\,A_\gamma - \frac{2}{3}\,A_\alpha\,A_\beta\,A_\gamma\right)\,. \tag{13}$$

The question now arises, whether, after reduction of Yang-Mills theory including topological term to the unconstrained system, a transformation analogous to (9) can be found, that correspondingly eliminates any θ-dependence on the reduced level, proving the consistency of the Hamiltonian reduction.

3 Unconstrained Hamiltonian formulation

3.1 Hamiltonian reduction for arbitrary θ-angle

In order to derive the unconstrained form of $SU(2)$ Yang Mills theory with θ-angle we follow the method developed in [22] . We perform the point transformation

$$A_{ai}\,(q,S) = O_{ak}(q)\,S_{ki} + \frac{1}{2g}\,\varepsilon_{abc}\,\left(\partial_i O(q)\,O^T(q)\right)_{bc} \tag{14}$$

from the gauge fields $A_{ai}(x)$ to the new set of three fields $q_j(x),\ j = 1,2,3,$ parameterizing an orthogonal 3×3 matrix $O(q)$ and the six fields $S_{ik}(x) = S_{ki}(x),\ i,k = 1,2,3,$ collected in the positive definite symmetric 3×3 matrix $S(x)$ [3]. Eq. (14) can be seen as a gauge transformation to new field configuration $S(x)$ which satisfy the "symmetric gauge" condition

$$\chi_a(S) := \varepsilon_{abc}\,S_{bc} = 0\,. \tag{15}$$

The complete analysis of the existence and uniqueness of this gauge, i.e. whether any gauge potential A_{ai} can be made symmetric by a unique gauge transformation, is complex mathematical problem. Here we shall consider transformation (14) in a region where the uniqueness and regularity of the change of coordinates can be guaranteed. In Appendix B, we proof the existence and uniqueness of the symmetric gauge for the case of a non-degenerate matrix A using the inverse coupling constant expansion. Furthermore, as an illustration of the obstruction of the uniqueness of the symmetric gauge-fixing (appearance of Gribov copies) for degenerate matrices A, the Wu-Yang monopole configuration, is considered. Although it is antisymmetric in space and color indices, it can be brought into the symmetric form, but there exist two gauge transformations by which this can be

[3]It is necessary to note that a decomposition similar to (14) was used in [11] as generalization of the well-known polar decomposition valid for arbitrary quadratic matrices.

achieved. The case of degenerate matrix field S, $\det \|S\| = 0$, will be discussed for the special situation $\mathrm{rank}\|S\| = 1$ in Section VI.

The transformation (14) induces a point canonical transformation linear in the new momenta $P_{ik}(x)$ and $p_i(x)$, conjugated to $S_{ik}(x)$ and $q_i(x)$, respectively. Their expressions in terms of the old variables $(A_{ai}(x)\,, \Pi_{ai}(x))$ can be obtained from the requirement of the canonical invariance of the symplectic 1-form

$$\sum_{i,a=1}^{3} \Pi_{ai}\, \dot{A}_{ai}\, dt = \sum_{i,j=1}^{3} P_{ij}\, \dot{S}_{ij}\, dt + \sum_{i=1}^{3} p_i\, \dot{q}_i\, dt\,, \tag{16}$$

with the fundamental brackets

$$\{S_{ij}(t,\vec{x})\,, P_{kl}(t,\vec{y})\} = \frac{1}{2}\left(\delta_{ik}\,\delta_{jl} + \delta_{il}\,\delta_{jk}\right)\, \delta^{(3)}(\vec{x}-\vec{y})\,, \tag{17}$$

$$\{q_i(t,\vec{x})\,, p_j(t,\vec{y})\} = \delta_{ij}\, \delta^{(3)}(\vec{x}-\vec{y})\;, \tag{18}$$

for the new canonical pairs $(S_{ij}(x)\,, P_{ij}(x))$ and $(q_i(x)\,, p_i(x))$. The brackets (17) account for the second class symmetry-constraints $S_{ij} = S_{ji}$ and $P_{ij} = P_{ji}$ and therefore are Dirac brackets. As result we obtain the expression

$$\Pi_{ai} = O_{ak}(q)\left[\, P_{\;ki} + g\, \varepsilon_{kin}\, {}^{*}D^{-1}_{nm}(S)\left(\mathcal{S}_m - \Omega^{-1}_{jm} p_j\right)\right]\,, \tag{19}$$

of the old momenta Π_{ai} in terms of the new canonical variables, (for a detailed derivation see [22]). Here ${}^{*}D^{-1}_{mn}(S)$ denotes the inverse of the differential matrix operator [4]

$${}^{*}D_{mn}(S) = \varepsilon_{njc}\left(D_j(S)\right)_{mc}\;, \tag{20}$$

the vector $\mathcal{S}$ is defined as

$$\mathcal{S}_m = \frac{1}{g}\,(D_j(S))_{mn} P_{nj}\;, \tag{21}$$

and the matrix Ω^{-1} the inverse of

$$\Omega_{ni}(q) \;:=\; -\frac{1}{2}\,\varepsilon_{nbc}\left(O^T(q)\,\frac{\partial O(q)}{\partial q_i}\right)_{bc}\,. \tag{22}$$

Here we would like to comment on the geometrical meaning of the above expressions. The vector $\mathcal{S}$ coincides up to divergence with the spin density part of the Noetherian angular momentum after projection to the surface given by the Gauss law constraints. Furthermore, the matrix Ω^{-1} defines the main geometrical structures on $SO(3,R)$ group manifold, namely the three left-invariant Killing vectors fields $\eta_a := \Omega^{-1}_{ja}\frac{\partial}{\partial q_j}$ obeying the $so(3)$ algebra $[\eta_a, \eta_b] = \epsilon_{abc}\eta_c$, and the invariant metric $g := -\mathrm{tr}\left(O^T dO O^T dO\right) = (1/2)\left(\Omega^T\Omega\right)_{ij} dq_i dq_j$

[4]Note that the operator ${}^{*}D_{mn}(S)$ corresponds in the conventional gauge-fixing method to the so-called Faddeev-Popov operator (FP), the matrix of Poisson brackets between the Gauss law constraint (8) and the symmetric gauge (15), $\{(D_i(S))_{mc}\Pi_{ci}(x), \chi_n(y)\} = {}^{*}D_{mn}(S)\delta^3(x-y)$.

as the standard metric on S^3. Since $\det \Omega$ is proportional to the Haar measure on $SO(3,R)$ $\sqrt{\det \|g\|} = |\det \|\Omega(q)\||$, and it is expected to vanish at certain coordinate singularities (see also e.g. discussion in ch. 8 of [38]). In deriving the expression (19) we shall here limit ourselves to the region where the matrix Ω is invertible.

The main advantage of introducing the variables S_{ij} and q_i is, that they Abelianise the non-Abelian Gauss law constraints (8). In terms of the new variables the Gauss's law constraints

$$g\, O_{as}(q)\, \Omega^{-1}_{is}(q)\, p_i = 0\,, \tag{23}$$

depend only on (q_i, p_i), showing that the variables $(S_{ij}\,, P_{ij})$ are gauge-invariant, physical fields. Hence, assuming $\det \Omega(q) \neq 0$ in (19) and (23), the reduced Hamiltonian, defined as the projection of the total Hamiltonian onto the constraint shell, can be obtained from (6) by imposing the equivalent set of Abelian constraints

$$p_i = 0\,. \tag{24}$$

Due to gauge invariance, the reduced Hamiltonian is independent of the coordinates q_i canonically conjugated to p_i and is hence a function of the unconstrained gauge-invariant variables S_{ij} and P_{ij} only

$$H = \int d^3x \left[\frac{1}{2} \left(P_{ai} - \frac{\theta}{8\pi^2}\, B^{(+)}_{ai}(S) \right)^2 + \left(P_a - \frac{\theta}{8\pi^2}\, B^{(-)}_{a}(S) \right)^2 + \frac{1}{2}\, V(S) \right]. \tag{25}$$

Here the P_a denotes the nonlocal functional, according to (19) defined as solution of the system of differential equations

$$^{*}D_{ks}(S) P_s = (D_j(S))_{kn} P_{nj}\,. \tag{26}$$

The nonlocal second term in the Hamiltonian (25) therefore stems from the antisymmetric part of the Π_{ai}, which remains after implementing Gauss's law $p_a = 0$, in terms of the physical P_{ai}. Hence this term contains FP^{-2}, see (26), and is the analogon of the well-known non-local part of Hamiltonian in the Coulomb gauge, see e.g. [9].

Furthermore,

$$B^{(+)}_{ai}(S) := \frac{1}{2}\left(B_{ai}(S) + B_{ia}(S)\right), \qquad B^{(-)}_{a}(S) := \frac{1}{2}\, \varepsilon_{abc}\, B_{bc}(S)\,, \tag{27}$$

denote the symmetric and antisymmetric parts of the reduced chromomagnetic field

$$B_{ai}(S) = \varepsilon_{ijk} \left(\partial_j S_{ak} + \frac{g}{2}\, \varepsilon_{abc}\, S_{bj}\, S_{ck} \right). \tag{28}$$

It is the same functional of the symmetric field S as the original $B_{ai}(A)$, since the chromomagnetic field transforms homogeneously under the change of coordinates (14). Finally the potential $V(S)$ is the square of the reduced magnetic field (28),

$$V(S)\; d^3x = \frac{1}{2}\, \mathrm{tr}\, {}^{*}F^{(3)} \wedge F^{(3)}\,, \tag{29}$$

with the curvature 2-form in 3-dimensional Euclidean space

$$F^{(3)} = dS + S \wedge S\,, \tag{30}$$

in terms of the symmetric 1-form

$$S = g\tau_k\, S_{kl}\, dx_l, \qquad k,l = 1,2,3\,, \tag{31}$$

whose 6 components depend on the time variable as an external parameter. The reduced chromomagnetic field (28) is given in terms of the dual field strength ${}^*F^{(3)}$ as $B_{ai}(S) = \frac{1}{2}\,\varepsilon_{ijk}\,F^{(3)}_{ajk}$.

3.2 Canonical equivalence of unconstrained theories with different θ-angles

For the original degenerate action in terms of the A_μ fields the equivalence of classical theories with arbitrary value of θ-angle has been reviewed in Section 2. Let us now examine the same problem for the derived unconstrained theory considering the analog of the canonical transformation (9) after projection onto the constraint surface,

$$\begin{aligned} S_{ai}(x) &\longmapsto S_{ai}(x)\,, \\ P_{bj}(x) &\longmapsto \mathcal{E}_{bj}(x) := P_{bj}(x) - \frac{\theta}{8\,\pi^2}\, B^{(+)}_{bj}(x)\,. \end{aligned} \tag{32}$$

One can easily check that this transformation to new variables S_{ai} and $\mathcal{E}_{bj}$ is canonical with respect to the Dirac brackets (17). In terms of the new variables S_{ai} and $\mathcal{E}_{bj}$ the Hamiltonian (25) can be written as

$$H = \int d^3x \left[\frac{1}{2}\,\mathcal{E}^2_{ai} + \mathcal{E}^2_a + \frac{1}{2}\,V(S)\right], \tag{33}$$

with $\mathcal{E}_a$ defined as

$$\mathcal{E}_a := P_a - \frac{\theta}{8\,\pi^2}\, B^{(-)}_a\,. \tag{34}$$

Now, if P_a is a solution of equation (26), then $\mathcal{E}_a$ is a solution of the same equation

$${}^*D_{ks}(S)\mathcal{E}_s = (D_j(S))_{kn}\mathcal{E}_{nj}\,, \tag{35}$$

with the replacement $P_{ai} \longmapsto \mathcal{E}_{ai}$, since the reduced field B_{ai} satisfies the Bianchi identity

$$(D_i(S))_{ab}\, B_{bi}(S) = 0\,. \tag{36}$$

Hence we arrive at the same unconstrained Hamiltonian system (33) and (35) with vanishing θ-angle. Note that after the elimination of the three unphysical fields $q_j(x)$ the projected canonical transformation (32) that removes the θ-dependence from the Hamiltonian can be written as

$$\mathcal{E}_{bj}(x) = P_{bj}(x) - \theta \, \frac{\delta}{\delta S_{bj}} \, W[S] \, , \tag{37}$$

which is of the same form as (11) with the nine gauge fields $A_{ik}(x)$ replaced by the six unconstrained fields $S_{ik}(x)$.

In summary, the exact projection to reduced phase space leads to an unconstrained system, whose equations of motion are consistent with the original degenerate theory in the sense that they are θ-independent. Thus if our consideration is restricted only to the classical level of the exact nonlocal unconstrained theory, the generalization to arbitrary θ-angle can be avoided[5]. However, in order to work with such a complicated nonlocal Hamiltonian it is necessary to make approximations, such as for example expansion in the number of spatial derivatives, which we shall carry out in the next section. For these one has to check that this approximation is free of the "divergence problem", that is all terms in the corresponding truncated action containing the θ-angle can be collected into a 4-divergence and all dependence on θ disappears from the classical equations of motion.

4 Expansion of the unconstrained Hamiltonian in $1/g$

Let us now consider the regime when the unconstrained fields are slowly varying in space-time and expand the nonlocal part of the kinetic term in the unconstrained Hamiltonian (25) as a series of terms with increasing powers of inverse coupling constant $1/g$, equivalent to an expansion in the number of spatial derivatives of field and momentum. Our expansion is purely formal and we shall in this work not study the question of its convergence. We shall see, that for nonvanishing θ-angle, a straightforward expansion in $1/g$ leads to the above mentioned "divergence problem", and suggest an improved form of the expansion in $1/g$ of the unconstrained Hamiltonian exploiting the Bianchi identity.

4.1 Divergence problem in lowest-order approximation

According to [22], the nonlocal funtional P_a in the unconstrained Hamiltonian (33), defined as solution of the system of linear differential equations (26), can formally be expanded in powers of $1/g$. The vector P_a is then given as a sum of terms containing an increasing number of spatial derivatives of field and momentum

$$P_s(S,P) = \sum_{n=0}^{\infty} (1/g)^n \, a_s^{(n)}(S,P) \, . \tag{38}$$

The zeroth-order term is

$$a_s^{(0)} = \gamma_{sk}^{-1} \varepsilon_{klm} \, (PS)_{lm} \ , \tag{39}$$

[5]The extension of the proof of θ-independence to quantum theory requires to show the unitarity of the operator corresponding to transformation (32).

with $\gamma_{ik} := S_{ik} - \delta_{ik}\,\mathrm{tr}\,S$, and the first-order term is determined as

$$a_s^{(1)} = -\,\gamma_{sl}^{-1}\,\left[(\mathrm{rot}\,\vec{a}^{(0)})_l + \partial_k P_{kl}\right] \tag{40}$$

from the zeroth-order term. The higher terms are then obtained by the simple recurrence relations

$$a_s^{(n+1)} = -\,\gamma_{sl}^{-1}(\mathrm{rot}\,\vec{a}^{\,(n)})_l\,. \tag{41}$$

Inserting these expressions into (25) we obtain the corresponding expansion of unconstrained Hamiltonian as a series in higher and higher numbers of derivatives.

Let us check, whether the truncation of the expansion (38) to lowest order is consistent with θ-independence, that is, whether all θ-dependent terms can be collected into 4-divergence after Legendre transformation to the corresponding Lagrangian. In $o(1/g)$ approximation (39), the Hamiltonian reads[6]

$$H^{(2)} = \int d^3x \left[\frac{1}{2}\mathrm{tr}\left(P - \frac{\theta}{8\pi^2}\,B^{(+)}\right)^2 + \left(a_s^{(0)}(S,P) - \frac{\theta}{8\,\pi^2}\,B_s^{(-)}\right)^2 + \frac{1}{2}V(S)\right], \tag{42}$$

where $B^{(+)}$ and $B^{(-)}$ denote the symmetric and antisymmetric parts of the chromomagnetic field, defined in (27).

After inverse Legendre transformation of the Hamiltonian (42), the θ-dependent terms in the corresponding Lagrangian cannot be collected to a total 4-divergence, as is shown in Appendix C, and therefore contribute to the unconstrained equations of motion. Hence applying a straightforward derivative expansion to the Yang-Mills theory with topological term after projection to reduced phase space we face the "divergence problem" dicussed above.

4.2 Improved $1/g$ expansion using the Bianchi identity

In order to avoid the "divergence problem" one can proceed as follows. Let us consider additionally to the differential equation (26), which determines the nonlocal term P_a, the Bianchi identity (36) as an equation for determination of the antisymmetric part $B_s^{(-)}$ of the chromomagnetic field

$$^*D_{ks}(S)\,B_s^{(-)} = (D_i(S))_{kl}\,B_{li}^{(+)}\,, \tag{43}$$

in terms of its symmetric part $B_{bc}^{(+)}$. The complete analogy of this equation with (26) expresses the duality of chromoelectric and chromagnetic fields on the unconstrained level. Hence one can write

$$^*D_{ks}(S)\left[P_s - \frac{\theta}{8\pi^2}\,B_s^{(-)}\right] = (D_i(S))_{kl}\left[P_{li} - \frac{\theta}{8\,\pi^2}\,B_{li}^{(+)}\right]. \tag{44}$$

[6]When all spatial derivatives of the fields and momenta are neglected, Yang-Mills theory reduces to the so-called Yang-Mills mechanics and its θ-independence has been shown in [27].

Using the same type of the spatial derivative expansion as before in (39)-(41), we obtain

$$P_s - \frac{\theta}{8\,\pi^2}\, B_s^{(-)} = \sum_{n=0}^{\infty} (1/g)^n\, a_s^{(n)}(S, P - \frac{\theta}{8\pi^2}\, B^{(+)})\,. \tag{45}$$

In this way we achieve a form of the derivative expansion such that the unconstrained Hamiltonian is a functional of field combination $P_{ai} - (\theta/8\,\pi^2)\, B_{ai}^{(+)}$

$$H = \int d^3x \left[\frac{1}{2} \left(P_{ai} - \frac{\theta}{8\pi^2}\, B_{ai}^{(+)} \right)^2 + \left(\sum_{n=0}^{\infty} (1/g)^n\, a_i^{(n)}(S, P - \frac{\theta}{8\pi^2}\, B^{(+)}) \right)^2 + \frac{1}{2}\, V(S) \right], \tag{46}$$

explicitly showing the chromoelectro-magnetic duality on the reduced level and hence free of the "divergence problem". To obtain the unconstrained Hamiltonian up to leading order $o(1/g)$, only the lowest term $a_s^{(0)}(S, P - (\theta/8\,\pi^2)\, B^{(+)})$ in the sum in (46) has to be taken into account, so that

$$H^{(2)} = \frac{1}{2} \int d^3x \left[\operatorname{tr} \left(P - \frac{\theta}{8\pi^2}\, B^{(+)} \right)^2 - \frac{1}{\det^2 \gamma} \operatorname{tr} \left(\gamma\, [S, P - \frac{\theta}{8\,\pi^2}\, B^{(+)}]\, \gamma \right)^2 + V(S) \right]. \tag{47}$$

The advantage of this Hamiltonian compared with (42), derived before, is that the classical equations of motion following from (47) are θ-independent. In order to obtain a transparent form of the corresponding surface term in the unconstrained action, it is useful to perform a main-axis transformation of the symmetric matrix field $S(x)$.

5 Long-wavelength approximation to reduced theory

In this section we shall at first rewrite the unconstrained Hamiltonian (47) in terms of main-axis variables of the symmetric tensor field S_{ij}. The corresponding second-order Lagrangian $L^{(2)}$ is then obtained via Legendre transformation and the form of the corresponding unconstrained total divergence derived in an explicit way.

5.1 Hamiltonian in terms of main-axis variables

In [22] it was shown, that the field $S_{ij}(x)$ transforms as a second-rank tensor under the spatial rotations. This can be used to explicitly separate the rotational degrees of freedom from the scalars in the Hamiltonian (47). Following [22] we introduce the main-axis representation of the symmetric 3×3 matrix field $S(x)$,

$$S(x) = R^T(\chi(x)) \begin{pmatrix} \phi_1(x) & 0 & 0 \\ 0 & \phi_2(x) & 0 \\ 0 & 0 & \phi_3(x) \end{pmatrix} R(\chi(x))\,. \tag{48}$$

The Jacobian of this transformation is

$$J\left(\frac{S_{ij}[\phi, \chi]}{\phi_k, \chi_l} \right) \propto \prod_{i \neq j} \mid \phi_i(x) - \phi_j(x) \mid \;, \tag{49}$$

and thus (48) can be used as definition of the new configuration variables, the three diagonal fields ϕ_1, ϕ_2, ϕ_3 and the three angular fields χ_1, χ_2, χ_3, only if all eigenvalues of the matrix S are different. To have the uniqueness of the inverse transformation we assume here that

$$0 < \phi_1(x) < \phi_2(x) < \phi_3(x) \,. \tag{50}$$

The variables ϕ_i in the main-axis transformation (48) parameterize the orbits of the action of a group element $g \in SO(3,\mathbb{R})$ on symmetric matrices, $S \to S' = g\, S\, g^{-1}$. The configuration (50) belongs to the so-called principle orbit class, whereas all orbits with coinciding eigenvalues of the matrix S are singular orbits [39]. In order to parameterize configurations belonging to a singular stratum one should in principle use a decomposition of the S field different from the above main-axes transformation (48). Alternatively, one can consider the singular orbits as the boundary of the principle orbit-type stratum and study the corresponding dynamics using a certain limiting procedure [7]. In this Section we shall limit ourselves to the consideration of the dynamics on the principle orbits and leave the important case of the singular orbits expected to contain interesting physics for future studies.

The momenta π_i and p_{χ_i}, canonical conjugate to the diagonal elements ϕ_i and χ_i, can be found using the condition of the canonical invariance of the symplectic 1-form

$$\sum_{i,j=1}^{3} P_{ij}\, \dot{S}_{ij}\, dt = \sum_{i=1}^{3} \pi_i\, \dot{\phi}_i dt + \sum_{i=1}^{3} p_{\chi_i}\, \dot{\chi}_i dt \,. \tag{51}$$

The original physical momenta P_{ik}, expressed in terms of the new canonical variables, read

$$P(x) = R^T(x) \sum_{s=1}^{3} \left(\pi_s(x)\, \overline{\alpha}_s + \frac{1}{2} \mathcal{P}_s(x)\, \alpha_s \right) R(x) \,. \tag{52}$$

Here $\overline{\alpha}_i$ and α_i denote the diagonal and off-diagonal basis elements for symmetric matrices with the orthogonality relations $\mathrm{tr}\,(\overline{\alpha}_i\, \overline{\alpha}_j) = \delta_{ij}$, $\mathrm{tr}\,(\alpha_i\, \alpha_j) = 2\, \delta_{ij}$, $\mathrm{tr}\,(\overline{\alpha}_i\, \alpha_j) = 0$, and

$$\mathcal{P}_i(x) = -\frac{\xi_i(x)}{\phi_j(x) - \phi_k(x)} \,, \qquad (\text{cyclic permutations}\, i \neq j \neq k) \,. \tag{53}$$

The ξ_i are the three $SO(3,\mathbb{R})$ right-invariant Killing vector fields given in terms of the angles χ_i and their conjugated momenta p_{χ_i} via

$$\xi_i = M^{-1}_{ji} p_{\chi_j} \,, \tag{54}$$

where the matrix M is

$$M_{ji} := -\frac{1}{2}\, \varepsilon_{jab} \left(\frac{\partial R}{\partial \chi_i}\, R^T \right)_{ab} \,. \tag{55}$$

[7] The relation between an explicit parameterization of the singular strata and their description as a certain limit of the principle orbit stratum has been studied recently in [40] investigating the geodesic motion on the $GL(n,R)$ group manifold.

The physical chromomagnetic field $B(S)$ can be regarded as the components of the curvature 2-form $F^{(3)}$, defined in terms of the symmetric 1-form S in (30). Starting from the coordinate basis expression of S in (31), we observe that the main-axis transformation (48) corresponds to the representation

$$S = \sum_{a=1}^{3} e_a \, \phi_a \, \omega_a \,, \tag{56}$$

with the 1-forms

$$\omega_i := R_{ij} \left(\chi(x)\right) dx_j \,, \qquad i = 1, 2, 3 \tag{57}$$

and the $su(2)$ Lie algebra basis

$$e_a := R_{ab}(\chi(x)) \, \tau_b \,, \qquad a = 1, 2, 3 \,. \tag{58}$$

In this basis the components of the non-Abelian field strength $F^{(3)}$ read

$$F^{(3)}_{aij} = \delta_{aj} \, X_i \, \phi_j - \delta_{ai} \, X_j \, \phi_i + \phi_i \, \Gamma_{aji} - \phi_j \, \Gamma_{aij} + \Gamma_{a[ij]} \, \phi_a + g\varepsilon_{aij} \, \phi_i \, \phi_j \,, \quad \text{(no summation)} \,, \tag{59}$$

with the components of connection 1-form Γ defined as

$$\Gamma_{aib} := \left(X_i R \, R^T\right)_{ab} \,. \tag{60}$$

The vector fields

$$X_i := R_{ij} \, \partial_j \,, \tag{61}$$

are dual to the 1-forms $\omega_j \,, \, \omega_i(X_j) = \delta_{ij}$, and act on the basis elements e_a as

$$X_i \, e_a = -\Gamma_{bia} e_b \,. \tag{62}$$

From the expressions (59) we obtain for the potential (29) (see [22] and Erratum [23]),

$$V(\phi, \chi) = \sum_{i \neq j}^{3} \left(\Gamma_{iij}(\phi_i - \phi_j) - X_j \phi_i\right)^2 + \sum_{cyclic}^{3} \left(\Gamma_{ijk}(\phi_i - \phi_k) - \Gamma_{ikj}(\phi_i - \phi_k) - g\phi_j\phi_k\right)^2 \,. \tag{63}$$

The explicit expressions for the diagonal components β_i and the off-diagonal components b_i of the the symmetric part of the chromomagnetic field

$$B^{(+)} = R^T(\chi) \sum_{i=1}^{3} \left(\beta_i \overline{\alpha}_i + \frac{1}{2} b_i \alpha_i\right) R(\chi) \,, \tag{64}$$

are given in terms of the diagonal fields ϕ_i and the angular fields χ_i in the cyclic form

$$\beta_i = g\phi_j\phi_k - (\phi_i - \phi_j)\Gamma_{ikj} + (\phi_i - \phi_k)\Gamma_{ijk} \,, \tag{65}$$
$$b_i = X_i(\phi_j - \phi_k) - (\phi_i - \phi_j)\Gamma_{ijj} + (\phi_i - \phi_k)\Gamma_{ikk} \,. \tag{66}$$

and the antisymmetric part $B_i^{(-)}$ of the unconstrained magnetic field is

$$B_i^{(-)} = \frac{1}{2}\sum_{cyclic}^{3} R^T_{ia}\left[X_a(\phi_b+\phi_c) + (\phi_b-\phi_a)\Gamma_{abb} + (\phi_c-\phi_a)\Gamma_{acc}\right] . \tag{67}$$

The zeroth-order term of the expansion (45), finally, reads

$$a_i^{(0)} = -\frac{1}{2}\sum_{cyclic}^{3} \frac{R^T_{ia}}{(\phi_b+\phi_c)}\left(\xi_a + \frac{\theta}{8\pi^2}\left(\phi_b-\phi_c\right) b_a\right) \tag{68}$$

Altogether, the $o(1/g)$ Hamiltonian (47), as a functional of main-axis variables, becomes

$$H^{(2)} = \frac{1}{2}\int d^3x\left[\sum_{i=1}^{3}\left(\pi_i - \frac{\theta}{8\pi^2}\beta_i\right)^2 + \sum_{cyclic} k_i\left(\xi_i + \frac{\theta}{8\pi^2}\left(\phi_j-\phi_k\right) b_i\right)^2 + V(\phi,\chi)\right] , \tag{69}$$

with

$$k_i := \frac{\phi_j^2+\phi_k^2}{(\phi_j^2-\phi_k^2)^2} , \quad (\text{cyclic permutations } i\neq j\neq k) . \tag{70}$$

The transformation (32), rewritten in terms of angular and scalar variables,

$$\begin{aligned} &\pi_i \longmapsto \pi_i + \frac{\theta}{8\pi^2}\beta_i , \qquad \phi_i \longmapsto \phi_i , \\ &\xi_i \longmapsto \xi_i - \frac{\theta}{8\pi^2}\left(\phi_j-\phi_k\right) b_i , \end{aligned} \tag{71}$$

excludes the θ-dependence from the Hamiltonian (69) reducing it to the zero θ-angle expression [22]

$$H^{(2)} = \frac{1}{2}\int d^3x\left[\sum_{i=1}^{3}\pi_i^2 + \sum_{cyclic}\xi_i^2\frac{\phi_j^2+\phi_k^2}{(\phi_j^2-\phi_k^2)^2} + V(\phi,\chi)\right] . \tag{72}$$

5.2 Second-order unconstrained Lagrangian

We are now ready to derive the Lagrangian up to second-order in derivatives corresponding to the Hamiltonian (69). Carrying out the inverse Legendre transformation,

$$\dot\phi_i = \pi_i - \frac{\theta}{8\pi^2}\beta_i , \tag{73}$$

$$\dot\chi_a = G_{ab}\left(p_{\chi_b} - \frac{\theta}{8\pi^2}\sum_{cyclic} M^T_{bi}(\phi_j-\phi_k)\, b_i\right) , \tag{74}$$

with the matrix M given in (55), and the 3×3 matrix G

$$G = M^{-1}kM^{-1T} , \tag{75}$$

similar to the diagonal matrix $k = \mathrm{diag}\|k_1, k_2, k_3\|$ with entries k_i of (70), we arrive at the second-order Lagrangian

$$L^{(2)}(\phi,\chi) = \frac{1}{2}\int d^3x \left[\sum_{i=1}^{3} \dot\phi_i^2 + \sum_{i,j=1}^{3} \dot\chi_i G^{-1}_{ij}\dot\chi_j - V(\phi,\chi)\right] - \theta \int d^3x\, Q^{(2)}(\phi,\chi)\,, \tag{76}$$

with all θ-dependence gathered in the reduced topological charge density

$$Q^{(2)} = \frac{1}{8\pi^2}\sum_{a=1}^{3}\left(\dot\phi_a\beta_a + \sum_{cyclic}^{i,j,k} \dot\chi_a M^T_{ai}(\phi_j - \phi_k)\, b_i\right)\,. \tag{77}$$

Using the Maurer-Cartan structure equations for the 1-forms ω_i

$$d\omega_a = \Gamma_{a0c} dt \wedge \omega_c + \Gamma_{abc}\omega_b \wedge \omega_c\ , \tag{78}$$

with the space components of Γ given in (60), and the time components correspondingly defined as

$$\Gamma_{a0b} = \left(\dot R R^T\right)_{ab}\,, \tag{79}$$

Eq. (77) can be rewritten as

$$Q^{(2)} = dC^{(2)}\ , \tag{80}$$

with the 3-form

$$\begin{aligned} C^{(2)} \;&=\; \frac{1}{8\pi^2}\sum_{a<b}^{3}(\phi_a - \phi_b)^2\, \Gamma_{a0b} dt \wedge \omega_a \wedge \omega_b \;- \\ &\quad - \frac{3}{8\pi^2}\sum_{cyclic}^{3}\left[(\phi_a - \phi_b)^2\, \Gamma_{acb} - \frac{2}{3}\,\varepsilon_{abc}\phi_1\phi_2\phi_3\right]\omega_a \wedge \omega_b \wedge \omega_c\,. \end{aligned} \tag{81}$$

This completes our construction of the second-order Lagrangian with all θ-contributions gathered in a total differential (77). The $Q^{(2)}$ in the effective Lagrangian (76) can be represented as the divergence

$$Q^{(2)} = \partial^\mu K^{(2)}_\mu\,, \tag{82}$$

of the 4-vector $K^{(2)}_\mu = (K^{(2)}_0, K^{(2)}_i)$, with the components

$$K^{(2)}_0 = \frac{1}{16\pi^2}\sum_{cyclic}^{3}\left[(\phi_a - \phi_b)^2\Gamma_{acb} - \frac{2}{3}\, g\ \phi_a\phi_b\phi_c\right]\,, \tag{83}$$

$$K^{(2)}_i = \frac{1}{16\pi^2}\sum_{cyclic}^{3} R^T_{ia}(\phi_b - \phi_c)^2\Gamma_{b0c}\,. \tag{84}$$

Thus we have found the unconstrained analog of the Chern-Simons current $K^{(2)}_\mu$, linear in the derivatives. Under the assumption, that the vector part $K^{(2)}_i$ vanishes at spatial infinity, the unconstrained form of the Pontryagin index p_1 can be represented as the difference of the two surface integrals

$$W_{\pm} = \int d^3x \; K_0^{(2)} \left(t \to \pm\infty, \, \vec{x}\right) \,, \tag{85}$$

which are the winding number functional (12) for the physical field $\mathbb{S}$ in terms of main-axis variables (56) at $t \to \pm\infty$ respectively, since $K_0^{(2)}(\phi, \chi)$ of (83) coincides with the full $K_0[S[\phi, \chi]]$ of (13). In the next Section we shall show, how for certain field configurations, it reduces to the Hopf number of the mapping from the $\mathbf{3}$-sphere $\mathbb{S}^3$ to the unit $\mathbf{2}$-sphere $\mathbb{S}^2$.

6 Unconstrained theory for degenerate configurations

The previous study was restricted to the consideration of the domain of configuration space with $\det \|S\| \neq 0$, where the change of variables (14) is well defined. In this Section we would like to discuss the dynamics on the special degenerate stratum (DS) with $\mathrm{rank}\|S\| = 1$, corresponding to the case of two eigenvalues of the matrix $\mathbb{S}$ vanishing. To investigate the dynamics on degenerate orbits it is in principle necessary to use a decomposition of the gauge potential different from our representation (14) and the corresponding subsequent main-axis transformation (48). Instead of this, we shall use here the fact, that the degenerate orbits can be regarded as the boundary of the non-degenerate ones and find the corresponding dynamics taking the corresponding limit from the non-degenerate orbits. Assuming the validity of such an approach we shall analyze the limit when two eigenvalues of the symmetric matrix $\mathbb{S}$ tend to zero [8]. Due to the cyclic symmetry under permutation of diagonal fields it is enough to choose one singular configuration

$$\phi_1(x) = \phi_2(x) = 0 \quad \text{and} \qquad \phi_3(x) \quad \text{arbitrary} \,. \tag{86}$$

Note that for the configuration (86) the homogeneous part of the square of the magnetic field, vanishes and the potential term in the Lagrangian (76) reduces to the expression

$$\begin{aligned} V = \quad & \phi_3^2 \left[(\Gamma_{213})^2 + (\Gamma_{223})^2 + (\Gamma_{233})^2 + (\Gamma_{311})^2 + (\Gamma_{321})^2 + (\Gamma_{331})^2 + (\Gamma_{3[12]})^2\right] \\ & + \left[(X_1\phi_3)^2 + (X_2\phi_3)^2\right] + 2\phi_3 \left[\Gamma_{331} X_1 \phi_3 + \Gamma_{332} X_2 \phi_3\right] \,, \end{aligned} \tag{87}$$

which can be rewritten as [22, 23]

$$V = (\nabla\phi_3)^2 + \phi_3^2 \left[(\partial_i \mathbf{n})^2 + (\mathbf{n} \cdot \mathrm{rot}\; \mathbf{n})^2\right] - (\mathbf{n} \cdot \nabla\phi_3)^2 + \left([\mathbf{n} \times \mathrm{rot}\; \mathbf{n}] \cdot \nabla\phi_3^2\right) \,, \tag{88}$$

introducing the unit vector

$$n_i(x) := R_{3i}(\chi(x)) \,. \tag{89}$$

[8]It can easily be checked that the degenerate stratum with $\mathrm{rank}\|S\| = 1$ is dynamically invariant. Furthermore, it is obvious from the representation (72) of the unconstrained Hamiltonian, that it is necessary to have $\xi_k \to 0$ for some fixed k, in order to obtain a finite contribution of the kinetic term to the Hamiltonian in the limit $\phi_i, \phi_j \to 0$ for $(i, j \neq k)$.

Hence the unconstrained second-order Lagrangian corresponding to the degenerate stratum with $\mathrm{rank}\|S(x)\| = 1$ represents the nonlinear σ-model type Lagrangian

$$\begin{aligned} L_{\mathrm{DS}} &= \frac{1}{2}\int d^3x \Big[(\partial_\mu\phi_3)^2 + \phi_3^2(\partial_\mu \mathbf{n})^2 - \phi_3^2(\mathbf{n}\cdot \mathrm{rot}\ \mathbf{n})^2 + (\mathbf{n}\cdot\nabla\phi_3)^2 \\ &\qquad - ([\mathbf{n}\times \mathrm{rot}\ \mathbf{n}]\cdot\nabla\phi_3^2)\Big] - \theta\int d^3x\ Q_{\mathrm{DS}}\,, \end{aligned} \tag{90}$$

for the unit vector $\mathbf{n}(x)$ - field coupled to the field $\phi_3(x)$. The density of topological term Q_{DS} in the Lagrangian (90) can be represented as the divergence

$$Q_{\mathrm{DS}} = \partial_\mu K^\mu_{\mathrm{DS}} \tag{91}$$

of the 4-vector

$$K^\mu_{\mathrm{DS}} = \frac{1}{16\pi^2}\phi_3^2\left((\mathbf{n}(x)\cdot \mathrm{rot}\ \mathbf{n}(x))\,,\ [\mathbf{n}(x)\times\dot{\mathbf{n}}(x)]\right)\,. \tag{92}$$

If we impose the usual boundary condition that the field $\mathbf{n}$ becomes time-independent at spatial infinity, the contribution from the vector part K^i_{DS} vanishes and the unconstrained form of the Pontryagin topological index p_1 for the degenerate stratum with $\mathrm{rank}\|S\| = 1$ can be represented as the difference

$$p_1 = n_+ - n_- \tag{93}$$

of the surface integrals

$$n_\pm = \frac{1}{16\pi^2}\int d^3x\,(\mathbf{V}_\pm(\vec{x})\cdot \mathrm{rot}\ \mathbf{V}_\pm(\vec{x})) \tag{94}$$

of the fields

$$\mathbf{V}_\pm(\vec{x}) := \lim_{t\to\pm\infty}\ \phi_3(x)\mathbf{n}\,. \tag{95}$$

We shall show now that the surface integrals (94) are Hopf invariants in the representation of Whitehead [29].

Under the Hopf mapping of a 3-sphere to a 2-sphere having unit radius, $N : \mathbb{S}^3 \to \mathbb{S}^2$, the preimage of a point on $\mathbb{S}^2$ is a closed loop. The number Q_H of times, the loops corresponding to two distinct points on $\mathbb{S}^2$ are linked to each other, is the so-called Hopf invariant. According to Whitehead [29], this linking number can be represented by the integral

$$Q_H = \frac{1}{32\pi^2}\int_{S^3} w^1\wedge w^2\,, \tag{96}$$

with the so-called Hopf 2-form curvature $w^2 = H_{ij}dx^i\wedge dx^j$ given in terms of the map N as

$$H_{ij} = \varepsilon_{abc}N_a\,(\partial_i N_b)\,(\partial_j N_c)\ , \tag{97}$$

and the 1-form w^1 related to it via $w^2 = dw^1$. Since the curvature H_{ij} is divergence-free,

$$\varepsilon_{ijk}\partial_i H_{jk} = 0\ , \tag{98}$$

it can be represented as the rotation

$$H_{ij} = \partial_i \mathcal{A}_j - \partial_j \mathcal{A}_i \ , \tag{99}$$

in terms of some vector field $\mathcal{A}_i$ ($i = 1, 2, 3$) defined over the whole of $\mathbb{S}^3$. Thus the Hopf invariant takes the form

$$Q_H = \frac{1}{16\pi^2} \int d^3x \, (\mathcal{A} \cdot \text{rot } \mathcal{A}) \ . \tag{100}$$

Therefore, the surface integrals (94) are just Hopf invariants in the Whitehead representation (100) and the unconstrained form of the topological term $Q^{(2)}$ is an 3-dimensional Abelian Chern-Simons term [4] with "potential" V_i and the corresponding "magnetic field" $\text{rot}\mathbf{V}$. The topological term in the original $SU(2)$ Yang-Mills theory reduces for rank-1 degenerate orbits not to a winding number, but the linking number Q_H of the field lines.

We ask at this place the question whether the obtained unconstrained theory for degenerate field configurations can be treated as the classical counterpart of an effective quantum model relevant to the low energy region of Yang-Mills theory. According to the recent argumentation by Faddeev and Niemi [41] the so-called $O(3)$ Faddeev-Skyrme model can be used as an quantum effective theory for the infrared sector of Yang-Mills theory. The Hopf invariant serves as a topological characteristic of the low energy gluon field configurations [42] [9]. The $O(3)$ Faddeev-Skyrme model represents the theory of a three-dimensional unit vector with an action that includes the standard kinetic part and the so-called Skyrme term, providing the stability of solitonic type solutions (see e.g. discussion in [44]). The degenerate field configurations $\phi_1 = \phi_2 = 0$ with arbitrary ϕ_3 of (86) correspond to the zeros of the homogeneous part of the square of the magnetic field, which is the leading order $g \to \infty$ term in the Hamiltonian, and thus might be interpreted as the classical vacua. As an outlook to a possible quantum description we consider this diagonal field $\phi_3(x)$ as slow varying with a constant non-zero vacuum expectation value

$$\langle \phi_3(x) \rangle = \mu \ . \tag{101}$$

Therefore, neglecting the fluctuations of the field $\phi_3(x)$ around the expectation value (101) in the Lagrangian (90) we arrive at the corresponding effective action

$$S_{eff}(\mathbf{n}) = \frac{\mu^2}{2} \int d^4x \left[(\partial_\mu \mathbf{n})^2 - (\mathbf{n}(x) \cdot \text{rot } \mathbf{n}(x))^2 \right] - \frac{\theta\mu^2}{16\pi^2} \int d^3x \, (\mathbf{n}(x) \cdot \text{rot } \mathbf{n}(x)) \ . \tag{102}$$

The effective action (102) is a non-linear sigma-model-type theory similar to that proposed by Faddeev and Niemi [41]. In difference to the Faddeev-Skyrme model, however, where a standard Skyrme stabilizing term, fourth order in derivatives, is used, in our result (102), the square of the density of the Hopf invariant appears in the Whitehead form (99), linear in derivatives. Furthermore, in the Faddeev-Niemi effective action, the unit-vector is a

[9] See also the discussion in [43], where the representation of gauge fields in terms of the complex two-component $\mathbb{CP}^1$ variables has been exploited.

Lorentz scalar, while the Lorentz transformation properties of the field $\mathbf{n}$ in the action (102) are not standard due to the noncovariance of the symmetric gauge imposed. Their investigation carefully taking into account surface contributions to the unconstrained form of the generators of the Poincaré group is under investigation.

7 Conclusions and remarks

We have generalized the Hamiltonian reduction of $SU(2)$ Yang-Mills gauge theory to the case of nonvanishing θ-angle, and shown that there is agreement between reduced and original constrained equations of motions. We have employed an improved derivative expansion to the non-local kinetic term in the obtained unconstrained Hamiltonian and investigated it in long-wavelength approximation. The corresponding second order Lagrangian has been constructed, with all θ-dependence gathered in a 4-divergence of a current, linear in the derivatives, which is the unconstrained analog of the original Chern-Simons current.

For the degenerate gauge field configurations S with $\mathrm{rank}\|S\| = 1$ we have argued that the obtained long-wavelength Lagrangian reduces to a classical theory with an Abelian Chern-Simons term originating from the Pontryagin topological functional. Therefore the topological characteristic of degenerate configuration is given not by a winding number, but the linking number of the field lines.

Finally let us comment on the Poincaré covariance of our unconstrained version of Yang-Mills theory. It is well-known that the Hamiltonian formulation of degenerate theories reduced with the help of non-covariant gauges destroy the manifest Poincaré invariance. Our "symmetric" gauge condition (15) is not covariant under standard Lorentz transformations. This, however, does not necessarily violate the Poincare invariance of our reduced theory. Such a situation can be found in classical Electrodynamics. After imposing the Coulomb gauge condition the vector potential ceases to be an ordinary Lorentz vector and transforms non-homogeneously under Lorentz transformations. The standard Lorentz boosts are compensated by some additional gauge-type transformation depending on the boost parameters and the gauge potential itself (see e.g. [48, 49, 50]). As for the case of the Coulomb gauge in Electrodynamics, a thorough analysis of the Poincaré group representation for our reduced theory obtained imposing the symmetric gauge condition is required. This problem is technically highly difficult and demands special consideration that is beyond the scope of present article.

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Appendix A: Conventions and notations

In this Appendix, we collect several notations and definitions for $SU(2)$ Yang-Mills theory used in the text following [4].

The classical Yang-Mills action of the $su(2)$-valued connection 1-form A in 4-dimensional Minkowski space-time with a metric $\eta = \mathrm{diag}\|1,-1,-1,-1\|$ reads

$$I = -\frac{1}{g^2}\int \mathrm{tr}\, F \wedge {}^*F - \frac{\theta}{8\pi^2\, g^2}\int \mathrm{tr}\, F \wedge F\,, \tag{103}$$

with the curvature 2-form

$$F = dA + A\,\wedge\,A \tag{104}$$

and its Hodge dual ${}^*F$. The trace in (103) is calculated in the antihermitian $su(2)$ algebra basis $\tau^a = \sigma^a/2\,i$ with Pauli matrices $\sigma^a,\ a = 1,2,3$, satisfying $[\tau_a,\,\tau_b] = \varepsilon_{abc}\,\tau_c$, and $\mathrm{tr}\,(\tau_a\tau_b) = -\frac{1}{2}\,\delta_{ab}$.

In the coordinate basis the components of the connection 1-form A are

$$A = g\,\tau^a\,A^a_\mu\,dx^\mu\,, \tag{105}$$

and the components of the curvature 2-form F are

$$F = \frac{1}{2}\,g\,\tau^a\,F^a_{\mu\nu}\,dx^\mu\,\wedge\,dx^\nu\,, \tag{106}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\,\varepsilon^{abc}A^b_\mu A^c_\nu\,. \tag{107}$$

Its dual ${}^*F$ are given as

$${}^*F = \frac{1}{2}\,g\,\tau^a\,{}^*F^a_{\mu\nu}\,dx^\mu \wedge dx^\nu\,, \tag{108}$$

$${}^*F^a_{\mu\nu} = \frac{1}{2}\,\varepsilon_{\mu\nu\rho\sigma}F^{a\,\rho\sigma}\,, \tag{109}$$

with totally antisymmetric Levi-Civita pseudotensor $\varepsilon_{\mu\nu\rho\sigma}$ using the convention

$$\varepsilon^{0123} = -\,\varepsilon_{0123} = 1. \tag{110}$$

The θ-angle enters the classical action as coefficient in front of the Pontryagin index density

$$Q = -\frac{1}{8\pi^2}\,\mathrm{tr}\,F \wedge F\,. \tag{111}$$

The Pontryagin index density is a closed form $d\,Q = 0$ and thus locally exact

$$Q = d\,C\,, \tag{112}$$

with the Chern 3-form

$$C = -\frac{1}{8\pi^2}\,\mathrm{tr}\left(A\wedge dA + \frac{2}{3}\,A\wedge A\wedge A\right)\,. \tag{113}$$

The corresponding Chern-Simons current K^μ is a dual of the 3-form C,

$$K^\mu = (1/3!)\,\varepsilon^{\mu\nu\rho\sigma}\,C_{\nu\rho\sigma} \;=-\,\frac{1}{16\pi^2}\,\varepsilon^{\mu\alpha\beta\gamma}\mathrm{tr}\left(F_{\alpha\beta}\,A_\gamma - \frac{2}{3}\,A_\alpha A_\beta A_\gamma\right)\,. \tag{114}$$

with the notations $A_\mu := g\,\tau^a\,A^a_\mu$ and $F_{\mu\nu} := g\,\tau^a\,F^a_{\mu\nu}$. The chromomagnetic field is given as

$$B^a_i = \frac{1}{2}\,\varepsilon_{ijk}\,F^a_{jk} \;= \varepsilon_{ijk}\,\left(\partial_j A_{ak} + \frac{g}{2}\,\varepsilon_{abc}\,A_{bj}\,A_{ck}\right)\,, \tag{115}$$

and the covariant derivative in the adjoint representation as

$$\left(D_i(A)\right)_{ac} = \delta_{ac}\partial_i + g\,\varepsilon_{abc}A_{bi}\;. \tag{116}$$

Finally, we frequently use the matrix notations

$$A_{ai} := A^a_i\,, \qquad B_{ai} := B^a_i\,. \tag{117}$$

Appendix B: On the existence of the "symmetric gauge"

In this Appendix we discuss the condition under which the symmetric gauge

$$\chi_a(A) = \varepsilon_{abi}\,A_{bi}(x) = 0\,, \tag{118}$$

exists.

According to the conventional gauge-fixing method (see e.g. [45]), a gauge $\chi_a(A) = 0$ exists, if the corresponding equation

$$\chi_a(A^\omega) = 0 \tag{119}$$

in terms of the gauge transformed potential

$$A^\omega_{ai}\tau_a = U^+(\omega)\left(A_{ai}\tau_a + \frac{1}{g}\frac{\partial}{\partial x_i}\right)U(\omega) \tag{120}$$

has a unique solution for the unknown function $\omega(x)$ [10].

Hence the symmetric gauge (118) exists, if any gauge potential A can be made symmetric by a unique time-independent gauge transformation. The equation that determines

[10]Here we assume that the second gauge condition $A_{a0} = 0$ is fulfilled and the function $\omega(x)$ therefore depends only on the space coordinates.

the gauge transformation $\omega(x)$ which converts an arbitrary gauge potential $A(x)$ into its symmetric counterpart can be written as a matrix equation

$$O^T(\omega)A - A^T O(\omega) = \frac{1}{g}\left(\Sigma(\omega) - \Sigma^T(\omega)\right)\,, \tag{121}$$

with the orthogonal 3×3 matrix related to the $SU(2)$ group element

$$O_{ab}(\omega) = -2\ \mathrm{tr}\left(U^+(\omega)\tau_a U(\omega)\tau_b\right) \tag{122}$$

and the 3×3 matrix Σ

$$\Sigma_{ai}(\omega) := -\frac{1}{4i}\,\varepsilon_{amn}\left(O^T(\omega)\frac{\partial O(\omega)}{\partial x_i}\right)_{mn}\,. \tag{123}$$

We shall now prove the following

Theorem : *For any non-degenerate matrix A equation (121) admits a unique solution in form of a $1/g$ expansion*

$$O(\omega) = O^{(0)}\left[1\ + \sum_{n=1}^{\infty}\left(\frac{1}{g}\right)^n X^{(n)}\right]\,. \tag{124}$$

Proof: In order to prove the statement, we first note that equating coefficients of equal powers in $1/g$ in the orthogonality condition $O^T O = O O^T = I$ of the matrix O, imposes the condition of orthogonality of $O^{(0)}$,

$$O^{(0)T}O^{(0)} = O^{(0)}O^{(0)T} = I\,, \tag{125}$$

as well as the conditions

$$\begin{aligned}
&X^{(1)} + X^{(1)^T} = 0\,,\\
&X^{(2)} + X^{(2)^T} + X^{(1)}X^{(1)^T} = 0\,,\\
&\dots \qquad \dots\,,\\
&X^{(n)} + X^{(n)^T} + \sum_{i+j=n} X^{(i)}X^{(j)^T} = 0\,,\\
&\dots \qquad \dots
\end{aligned} \tag{126}$$

for the unknown functions $X^{(n)}$. Furthermore, plugging expansion (124) into equation (121) and combining the terms of equal powers of $1/g$ we find that the orthogonal matrix $O^{(0)}$ should satisfy equation (121) to leading order in $1/g$

$$O^{(0)^T}A - A^T O^{(0)} = 0\,, \tag{127}$$

and the $X^{(n)}$ should satisfy the infinite set of equations

$$\begin{aligned}
&X^{(1)^T}O^{(0)^T}A - A^T O^{(0)}X^{(1)} = \Sigma^{(0)} - \Sigma^{(0)}\,,\\
&\dots \qquad \dots\,,\\
&X^{(n)^T}O^{(0)^T}A - A^T O^{(0)^T}X^{(n)} = \Sigma^{(n-1)} - \Sigma^{(n-1)^T}\,,\\
&\dots \qquad \dots\,,
\end{aligned} \tag{128}$$

where the corresponding $1/g$ expansion for the matrix $\Sigma(\omega)$

$$\Sigma(\omega) = \sum_{n=0}^{\infty} \left(\frac{1}{g}\right)^n \Sigma^{(n)} \tag{129}$$

has been used. Note that in expansion (129) the n-th order term $\Sigma^{(n)}$ is given in terms of $O^{(0)}$ and $X^{(a)}$ with $a = 1, \cdots, n-1$.

From the structure of equations (125)-(128) one can see that the solution to (121) reduces to an algebraic problem. Indeed, the solution to the first, homogeneous equation (127) is given by the polar decomposition for the arbitrary matrix A,

$$O^{(0)} = AS^{(0)-1}\,, \qquad S^{(0)} = \sqrt{AA^T}\,. \tag{130}$$

This solution is unique only if $\det\|A\| \neq 0$. It follows from the well-known property that the polar decomposition is valid for an arbitrary matrix A, but the orthogonal matrix $O^{(0)}$ is unique only for non-degenerate matrices [47].

To proceed further we use this solution and equations (126) for unknown X to rewrite the remaining equations (128) as

$$\begin{aligned}
&X^{(1)}S^{(0)} + S^{(0)}X^{(1)} = C^{(0)}\,,\\
&\cdots \qquad\quad \cdots\,,\\
&X^{(n)}S^{(0)} + S^{(0)}X^{(n)} = C^{(n-1)}\,,\\
&\cdots \qquad\quad \cdots\,,
\end{aligned} \tag{131}$$

where the n-th order coefficient $C^{(n)}$ is given in terms of $O^{(0)}$ and $X^{(1)}, X^{(2)}, \ldots, X^{(n-1)}$.

Thus starting from the zeroth-order term, the higher-order terms $X^{(n)}$ are given recursively as the solutions of matrix equations of the type $XS^{(0)} + S^{(0)}X = C$ with known symmetric positive definite matrix $S^{(0)} = \sqrt{AA^T}$ and matrix C, expressed in terms of the preceding $X^{(a)}$, $a = 1, \ldots, n-1$. The theory of such algebraic equations is well elaborated, see e.g. [46, 47]. In particular Theorem 8.5.1 in [46] states that for matrix equations for unknown matrix X of the type $XA + BX = C$, there is a unique solution if and only if the matrices A and -B have no common eigenvalues. Based on this theorem one can conclude, that the unique solution to (125)-(128) and hence to our original problem (121) exists always for any non-degenerate matrix A.

It is necessary to emphasize that in order to prove the existence and uniqueness of the representation (14) it should be shown additionally to the above Theorem that the corresponding symmetric matrix field S,

$$S(x) = \sum_{n=0}^{\infty} \left(\frac{1}{g}\right)^n S^{(n)}(x)\,, \tag{132}$$

is sign-definite. Above, the positive-definiteness has been shown only for the zeroth-order term $S^{(0)} = \sqrt{AA^T}$. The study of this problem, as well as an analogous investigation for the degenerate field configurations A with $\det \|A\| = 0$, are beyond the scope of this appendix and will be discussed in detail elsewhere. Here we limit ourselves to the consideration of a specific example, elucidating the generic picture.

In the case that the matrix A is degenerate, we encounter the problem of Gribov's copies. As an illustration of the non-uniqueness of gauge transformation, which turns a given field configuration A into the corresponding symmetric form, we consider the "degenerate" field

$$A_{a0} = 0\,, \qquad A_{ai} = -\frac{1}{gr}\, \varepsilon_{aic}\hat{r}_c\,, \tag{133}$$

known as the non-Abelian Wu-Yang monopole field, with the unit vector $\hat{r}_a = x_a/r$ and $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Performing the gauge transformation

$$S_{ai}\tau_a = U^{+}(\omega)\left(A_{ai}\tau_a + \frac{1}{g}\frac{\partial}{\partial x_i}\right)U(\omega)\,, \tag{134}$$

with $U(\omega) = e^{\omega_a \tau_a}$ parameterized by one time independent spherical symmetric function

$$\omega_a = f(r)\,\hat{r}_a\,, \tag{135}$$

the Wu-Yang monopole configuration (133), antisymmetric in space and color indices, can be brought into the "symmetric form"

$$S^{\pm}_{ai} = \pm\frac{\sqrt{3}}{gr}\left(\delta_{ai} - \hat{r}_a\hat{r}_i\right)\,, \tag{136}$$

if the function $f(r)$ is constant and takes four values

$$f(r) = \begin{cases} \pi/3\,,\ 7\pi/3 & \text{for} \quad (+)\,, \\ 5\pi/3\,,\ 11\pi/3 & \text{for} \quad (-)\,. \end{cases} \tag{137}$$

Here S^+ can be obtained from Wu-Yang monopole configuration (133) applying two different gauge transformation with $f(r) = \pi/3\,, 7\pi/3$

$$U_{1,2} = \pm\left(\frac{\sqrt{3}}{2} - \hat{r}\cdot\tau\right)\,, \tag{138}$$

while the S^- configuration can be reached using $f(r) = 5\pi/3\,, 11\pi/3$

$$U_{3,4} = \mp\left(\frac{\sqrt{3}}{2} + \hat{r}\cdot\tau\right)\,. \tag{139}$$

Here it is in order to make the following comments:

- For the above gauge transformations we have $\lim_{r\to\infty} U \neq \pm I$. Thus they are neither small gauge transformations nor large gauge transformations belonging to any integer n-homotopy class [4].

- The symmetric configurations (136) corresponding to the Wu-Yang monopole lie on the stratum of degenerate symmetric matrices with one eigenvalue vanishing and two eigenvalues equal to each other.
- The symmetric configurations S^+ and S^- in (136) with two-fold Gribov degeneracy are related to each other by parity conjugation.

Appendix C: Proof of θ-dependence of the naive $1/g$ approximation

In this Appendix it is shown that straightforward application of expansion of the nonlocal part P_a of the kinetic term in the unconstrained Hamiltonian to zeroth-order discussed in Section 4.1, leads to the appearance of θ-dependence of the reduced system on the classical level. Expressing the Hamiltonian (42), in terms of the main-axis variables, defined in Section 5, and performing an inverse Legendre transformation, one obtains the Lagrangian density

$$\begin{aligned}\mathcal{L}^{(2)}(\phi,\chi) \;=\; & \frac{1}{2}\left(\sum_{i=1}^{3}\dot{\phi}_i^2+\sum_{i,j=1}^{3}\dot{\chi}_i G^{-1}_{ij}\dot{\chi}_j - V(\phi,\chi)\right)-\frac{1}{2}\left(\frac{\theta}{8\pi^2}\right)^2\sum_{cyclic}\frac{\Delta_i^2}{\phi_j^2+\phi_k^2}\\ & -\frac{\theta}{8\pi^2}\sum_{a=1}^{3}\left(\dot{\phi}_a\beta_a+\sum_{cyclic}\dot{\chi}_a M^T_{ai}(\phi_j-\phi_k)\left(b_i+\frac{(\phi_j-\phi_k)}{\phi_j^2+\phi_k^2}\Delta_i\right)\right)\,, \end{aligned} \tag{140}$$

denoting the difference

$$\Delta_i=\frac{1}{2}(\phi_j-\phi_k)b_i-(\phi_j+\phi_k)\,R_{is}B_s^{(-)}\,, \tag{141}$$

with b_i of (66) and $B_i^{(-)}$ of (67), or explicitly,

$$\Delta_i=-\left[X_i(\phi_j\phi_k)+(\Gamma_{ijj}+\Gamma_{ikk})\phi_j\phi_k-\phi_i(\phi_j\Gamma_{ikk}+\phi_k\Gamma_{ijj})\right]\,. \tag{142}$$

It easy to convince ourselves that the term proportional to θ^2 is not a surface term. Indeed, considering for simplicity configurations of spatially constant angular variables χ_i and $\phi_1=\phi_2=\phi_3=:\phi$,it reduces to

$$-\left(\frac{\theta}{8\pi^2}\right)^2\sum_{i=1}^{3}\partial_i\phi\;\partial_i\phi\,, \tag{143}$$

which is not a 4-divergence. For $\Delta_i=0$ the Lagrangian density (140) reduces to (76), obtained from the improved Hamiltonian (47), free of the divergence problem.

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[23] Erratum to [22]: The last line of (3.42) should be replaced by $\left(\Gamma^1{}_{23}\,\phi_3 + \Gamma^1{}_{32}\,\phi_2 + \Gamma^1{}_{[23]}\,\phi_1 - g\,\phi_2\,\phi_3\right)^2$. Correspondingly, in formula (4.6) the term $\left(\Gamma^3{}_{[12]}\,\phi_3\right)^2$ should be added. Finally, in formulae (4.8), (4.10) and (4.11) the term $(\mathbf{n}\cdot\mathrm{rot}\,\mathbf{n})^2$ should be included. The correct formulae are given in the present work in (63),(87), and (88) correspondingly.

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