On interaction in extended particle model on $(M^4 \times M^4) \otimes Z_4$

A. Smida, M. Hachemane, A.H. Hamici et R. Djelid.
Faculté des Sciences (Physiques),
USTHB, B.P. 32 El-Alia Bab-Ezzouar 16111,
Algiers Algeria.

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Abstract

A $(M^4 \times M^4) \otimes Z_4$ model, describing an extended particle composed of two local modes and represented by a field $\psi\,(x, \xi; z)$, is formulated in its most general form ($(x, \xi; z) \in (M^4 \times M^4) \otimes Z_4$). The z argument specifies whether the particle is observable, unobservable, or partially observable (the latter case appears in two forms). In this four-sheeted structure, each sheet posses its own symmetry localized with respect to both space-times M^4 inducing thereby connections in the continuous directions. Connections in the discrete direction describe transitions between observable, unobservable, and partially observable states. Curvatures and propagators are determined.

1 Introduction

The present work is a reconsideration of a previous one,[1] aiming at a description of conversion of external modes into internal ones, and vice-versa, in a geometro-differential conception of extended particles.

Initially,[2] this conception was based on a Hilbert bundle structure $E^D(M, H^D, U^D(G'))$. The base space M is ordinary space-time which may have a G symmetry or be curved. The typical fiber H^D is a Hilbert space carrying an induced representation[3] $U^D\,(G')$ of the internal symmetry group G'. The particle extension stems from the fact that the latter is no more represented by a point $x \in M$, but by a function $\Psi \in H^D$ which depends on another spatio-temporal variable ξ with a G' symmetry. The function Ψ is not the probabilistic wave function but describes the intrinsic properties of the particle (of which x is a partial representation).[4] The probability amplitude role, played by the quantum mechanics wave function $\psi(x)$, is guaranteed by a functional[4] $X[\Psi, t]$ in this conception. In a realistic model, we assumed that $\Psi_x(\xi)$ describes a quantum mode localized at ξ for a particle localized at x (from a partial standpoint). The particle is composed then of two modes.[2]

The probabilistic functional has been chosen as a bilocal function $X[\Psi](x, \xi) = \psi(x, \xi)$ representing an external quantum mode localized at x and an internal one localized at ξ. This function was quantized by applying the induced representation method to both the external and internal symmetries. When interaction is absent, the induced representation method leads to a propagator which is a product of the propagators of each local mode. The external mode propagation is determined by a transition from external configuration induced representation to the external momentum one and back to the external configuration representation. Internal mode propagation is realized in the internal spaces. If an interaction is represented by a gauge field (a connection in the fiber), the semigroup induced representations lead to a path integral propagator.[5, 6]

The assumption that the external mode may transit via internal momentum space and the generalization of this idea to the possibility of transitions between external and internal representations, called mode conversions, led to new physical interpretations and ideas. However, mathematical expressions were deduced by analogy with induced representation results leading thereby to some inconsistencies.[1]

The purpose of the present work is to overcome the latter inconsistencies by abandoning the fiber bundle structure in favor of an $(M^4 \times M^4) \otimes Z_4$, where M^4 is Minkowski space and Z_4 is a discrete space with four elements. The inducing method is applied then between symmetries of the same type only, and connection in the continuous directions is taken into account. Transitions between symmetries of different (external or internal) types is realized by means of connections in the discrete direction.

The idea of discrete structure is drawn from works of Konisi[7] and Kubo[8] who associated connections in the discrete directions with Higgs fields without recourse to noncommutative geometry (NCG). For the sake of definiteness, we mention that our work is neither to be compared with that of Konisi and Kubo nor with those based on NCG. We have just used the discrete structure to provide our idea of conversion with mathematical consistency.

In Sec. 2, the state spaces structure, the connections, and the physical interpretations are presented. In Sec 3, curvatures are calculated following Kubo's work.[8] In Sec. 4, propagators containing both types of connections are deduced and Sec. 5 is devoted to the conclusion.

2 The structure and connections

To describe an extended particle composed of two local modes let us consider a $(M^4 \times M^4) \otimes Z_4$ structure, where M^4 is Minkowski space-time and Z_4 is the discret space with four elemnts. States Ψ^z of the particle belong to Hilbert spaces H^z and are considered as physical wave functions in the sens of providing all the physical properties of the particle but not probabilties. The latter are provided by functionals $X^z[\Psi^z](x,\xi) = \psi(x,\xi;z)$.[4] The variables x and ξ belong, respectively, to the first and second space-time and the variable z is an element of Z_4 taking values p for pure, c for crossed, e for external, and i for internal. Each case corresponds to a certain type of localizability of the extended particle composed of a first mode localized at x and a second mode localized at ξ. In the pure case, the first mode is localized in the external space-time and the second mode in the internal space-time. The crossed case is the reverse of the former. The external and internal cases mean that both modes are localized in external or internal space-time, respectively. In other words, the localizability type z attributes a fixed physical meaning to each space in the product $(M^4 \times M^4)$ as being external or internal space-time. It endows thereby the extended particle with the property of being completely observable as a bilocal object in external space-time (external case), partially observable as a local object in each space-time (pure and crossed cases), or unobservable (internal case).

In the present work, we associate to each type of localizability z a symmetry group $G(z)$ with elements

$$U = \exp iT(z).\theta(x,\xi;z) \tag{1}$$

We can assume that

$$T(z).\theta(x,\xi;z) = T_a(z)\,\theta^a(x,\xi;z) + T_\alpha(z)\,\theta^\alpha(x,\xi;z) \tag{2}$$

where T_a and θ^a are, respectively, generators and parameters of transformations related with the first space-time. In the same way, T_α and θ^α are generators and parameters of transformations related with the second space-time.

These gauge transformations can be either spatio-temporal or unitary and induce a connection corresponding to parallel transport in the continuous directions, i.e. in one or both space-times. In fact, in the covariant derivative[8]

$$\nabla^z\psi(x,\xi;z) = \psi(x+\delta x,\xi+\delta\xi;z) - \psi_{||}(x+\delta x,\xi+\delta\xi;z) \tag{3}$$

the parallel transported field $\psi_{||}(x+\delta x,\xi+\delta\xi;z)$ from (x,ξ) to $(x+\delta x,\xi+\delta\xi)$ can be written

$$\psi_{||}(x+\delta x,\xi+\delta\xi;z) = H(x+\delta x,\xi+\delta\xi;z)\,\psi(x,\xi;z) \tag{4}$$
$$H(x+\delta x,\xi+\delta\xi;z) = 1 - i\omega_i(x,\xi;z)\,\delta x^i - i\omega_\mu(x,\xi;z)\,\delta\xi^\mu \tag{5}$$

The Lie algebra valued connection one-forms corresponding to gauge transformations in each space-time are written in terms of gauge potentials $A(x,\xi;z)$:

$$\omega_i(x,\xi;z) = T_a(z)\,A_i^a(x,\xi;z) \tag{6}$$
$$\omega_\mu(x,\xi;z) = T_\alpha(z)\,A_\mu^\alpha(x,\xi;z) \tag{7}$$

The covariant derivative takes then the following form

$$\nabla^z = \delta x^i \nabla_i^z + \delta\xi^\mu \nabla_\mu^z \tag{8}$$
$$\nabla_i^z = \partial_i + i\omega_i(x,\xi;z) = \partial_i + iT_a(z)\,A_i^a(x,\xi;z) \tag{9}$$
$$\nabla_\mu^z = \partial_\mu + i\omega_\mu(x,\xi;z) = \partial_\mu + iT_\alpha(z)\,A_\mu^\alpha(x,\xi;z) \tag{10}$$

In the same manner, a covariant difference can be defined in the discrete direction. Parallel transport is a transition from one type of localizability (say z') to another (z)

$$\psi_{||}(x,\xi;z) = H(x,\xi;z,z')\,\psi(x,\xi;z') \tag{11}$$

Covariant difference is then written as follows

$$\begin{aligned}\delta_{z'}\psi(x,\xi;z) &= \psi(x,\xi;z) - \psi_{||}(x,\xi;z)\\ &= \psi(x,\xi;z) - H(x,\xi;z,z')\,\psi(x,\xi;z')\end{aligned}$$

but the $H(x,\xi;z,z')$ field has not a conventional expression in terms of one forms and gauge potentials. It can be interpreted as a transition operator from one type of localizability to another. It corresponds then to a conversion of internal modes to external ones and vice-versa. This conversion can be viewed as a creation of the particle when it passes from an unobservable state to a partially or completely observable one. It is viewed as an annihilation in the inverse transitions. It is then natural to define the conjugate of such a conversion by the following relation

$$H^\dagger(x,\xi;z,z') = H(x,\xi;z',z) \tag{12}$$

3 Curvatures

Now, we define and calculate different types of curvatures stemming from the structure considered in this work.

The first type of curvature corresponds to parallel transport along closed paths in the continuous direction and is given by the well known strength field $F_{AB}\,(x,\xi;z) = -i[\nabla_A^z, \nabla_B^z]$ components where the indices A and B take the values i or μ

$$F_{AB}\,(x,\xi;z) = \partial_{[A}\omega_{B]}\,(x,\xi;z) + i[\omega_A(x,\xi;z), \omega_B(x,\xi;z)] \tag{13}$$

Subscript brackets $[,]$ indicates an antisymmetrization over the indices and ordinary ones are commutator of connection forms. If parallel transport takes place in the first space-time, the curvature takes the following form

$$F_{ij}\,(x,\xi;z) = \partial_{[i}\omega_{j]}\,(x,\xi;z) + i[\omega_i(x,\xi;z), \omega_j(x,\xi;z)] \tag{14}$$

For a path in the second space-time, the curvature is

$$F_{\mu\nu}\,(x,\xi;z) = \partial_{[\mu}\omega_{\nu]}\,(x,\xi;z) + i[\omega_\mu(x,\xi;z), \omega_\nu(x,\xi;z)] \tag{15}$$

and a closed path lying in the two spaces corresponds to the following curvature

$$F_{i,\mu}\,(x,\xi;z) = \partial_{[i}\omega_{\mu]}\,(x,\xi;z) + i[\omega_i(x,\xi;z), \omega_\mu(x,\xi;z)] \tag{16}$$

If the symmetry groups of each space-time commute, the latter curvature becomes

$$F_{i,\mu}\,(x,\xi;z) = \partial_{[i}\omega_{\mu]}\,(x,\xi;z) \tag{17}$$

and, if in addition each connection depends only on its corresponding space-time variable, this curvature vanishes identically.

The second type of curvature is concerned with a combination of a parallel transport in the continuous direction with a parallel transport in the discrete direction, Fig.(1).

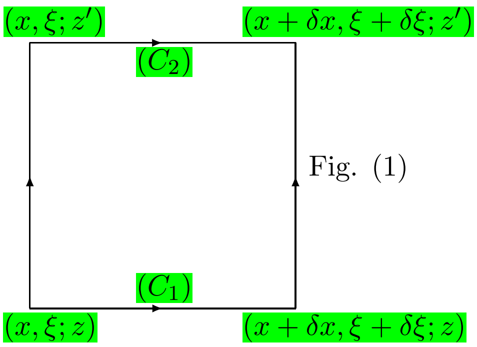

Fig. (1)

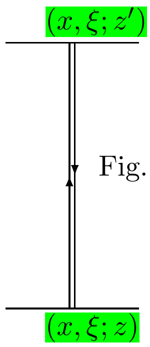

Fig. (2)

The curvature is defined as a difference between two paths C_1 and C_2 where

$$C_1 = H\left(x+\delta x, \xi+\delta\xi; z', z\right) H\left(x+\delta x, \xi+\delta\xi; z\right) \psi\left(x, \xi; z\right) \tag{18}$$
$$C_2 = H\left(x+\delta x, \xi+\delta\xi; z'\right) H\left(x, \xi; z', z\right) \psi\left(x, \xi; z\right) \tag{19}$$

We have

$$C_1 - C_2 = \{\delta x^i F^H_{iz'} + \delta\xi^\mu F^H_{\mu z'}\}\psi\left(x, \xi; z\right) \tag{20}$$

where

$$\begin{aligned} F^H_{iz'}\left(x, \xi; z\right) &= \partial_i H\left(x, \xi; z', z\right) \\ &\quad - iH\left(x, \xi; z', z\right)\omega_i(x, \xi; z) + i\omega_i(x, \xi; z')H\left(x, \xi; z', z\right) \\ F^H_{\mu z'}\left(x, \xi; z\right) &= \partial_\mu H\left(x, \xi; z', z\right) \\ &\quad - iH\left(x, \xi; z', z\right)\omega_\mu(x, \xi; z) + i\omega_\mu(x, \xi; z')H\left(x, \xi; z', z\right) \end{aligned} \tag{21}$$

It is clear that if parallel transport in the continuous direction concerns one space-time, only the corresponding curvature has to be considered.

Parallel transport of the third type curvature links two points in the discrete direction only, Fig. (2). Then

$$F_{z'(z)}\left(x, \xi; z\right) = 1 - H\left(x, \xi; z, z'\right) H\left(x, \xi; z', z\right) \tag{22}$$

For purely discrete curvatures, we adopt the following notation. The initial point z in the diagram is considered as an argument, the intermediate point z' as an index, and the end point z as an index between parenthesis.

There are also parallel transports linking three and four points in the discrete directions depicted in Figs. (3) and (4), respectively.

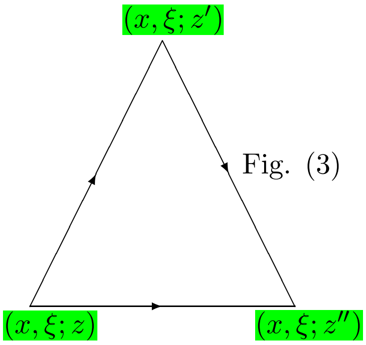

Fig. (3)

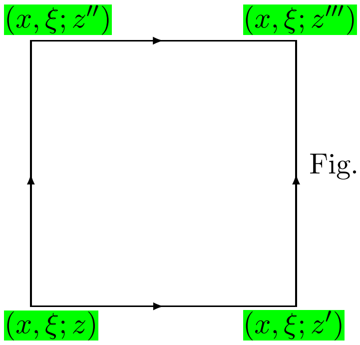

Fig. (4)

They give the curvature of the third type which has the following form

$$F_{z'(z'')}\left(x, \xi; z\right) = H\left(x, \xi; z'', z\right) - H\left(x, \xi; z'', z'\right) H\left(x, \xi; z', z\right) \tag{23}$$

and the curvature of the fourth type which has an analogous expression

$$F_{z'z''(z''')}\,(x,\xi;z) = H\,(x,\xi;z''',z'')\,H\,(x,\xi;z'',z) - H\,(x,\xi;z''',z')\,H\,(x,\xi;z',z) \tag{24}$$

Note that curvature (22) is compatible with (23) since $H\,(x,\xi;z,z) = 1$ and that (24) can be derived from (23)

$$F_{z'z''(z''')}\,(x,\xi;z) = F_{z''(z''')}(z) - F_{z'(z''')}(z) \tag{25}$$

The latter relation shows antisymmetry with respect to z' and z''. Moreover, it is easy to show that

$$F_{\bullet(z')}\,(x,\xi;z) = F^{\dagger}_{\bullet(z)}\,(x,\xi;z') \tag{26}$$

where the dot ($\bullet$) is to be replaced by the adequate arguments of the purely discrete curvatures. Consequently, curvature (22) is hermitic.

All our curvatures are analogous to those of local theories[7, 8] except the continuous curvature which contains extra terms du to extension.

4 Propagators

In order to determine propagators of extended particles described by the $(M^4 \times M^4) \otimes Z_4$ structure, we first consider the case of one sheet with fixed z. The propagator has already been deduced for this case (which is equivalent to $(M^4 \times M^4)$) by means of trajectory semigroups induced representations in each space-time.[6] Use of trajectory semigroups is imposed as far as the continuous connection is to be taken into account in the inducing method of quantization.

A trajectory in M^4 is a class of parallel curves $x(l)$. It is represented by an element

$$[u]_t = \{u(l) \in \mathbf{R}^4 / 0 \le l \le t\}, \qquad t = (s - s'), \qquad u(l) = (\frac{dx}{dl})_{s'+l} \tag{27}$$

where $x' = x(s')$ and $x = x(s)$ are the initial and final points of the curve $x\,(l)$, respectively. Right action of trajectories on space-time points is defined in the following way

$$x[u]_t = x' = x - \int_0^t u(l)\,dl \tag{28}$$

Trajectories $[\gamma]_\tau$ are also defined for curves $\xi\,(\lambda)$ of the second Minkowski space-time in $(M^4 \times M^4) \otimes Z_4$. To each pair of trajectories is associated a translation operator $U([u]_t,[\gamma]_\tau)$

$$[U([u]_t,[\gamma]_\tau)\psi](x,\xi;z) = \psi(x[u]_t,\xi[\gamma]_\tau;z) \tag{29}$$

and a parallel transport operator $H_{([u]_t,[\gamma]_\tau;z)}$, acting jointly with U and taking account of the continuous gauge fields effect

$$[H_{([u]_t,[\gamma]_\tau;z)}U([u]_t,[\gamma]_\tau)\psi](x,\xi;z) = H_{([u]_t,[\gamma]_\tau)}(x,\xi;z)\psi(x[u]_t,\xi[\gamma]_\tau;z) \tag{30}$$

The connection $H_{([u]_t,[\gamma]_\tau)}(x,\xi;z)$ corresponds to a parallel transport from $(x[u]_t,\xi[\gamma]_\tau;z)$ to $(x,\xi;z)$ along two curves belonging to trajectories $[u]_t$ and $[\gamma]_\tau$. Its infinitesimal form is given by relation (5) and its finite form is an ordered path integral over both curves

$$H(x[u]_t,\xi[\gamma]_\tau;z) = P\left[\exp - \int_{(x[u]_t,\xi[\gamma]_\tau;z)}^{(x,\xi;z)} \left(i\omega_i\,(x,\xi;z)\,dx^i + i\omega_\mu\,(x,\xi;z)\,d\xi^\mu\right)\right] \tag{31}$$

In our previous works, $i\omega_i\,(x,\xi;z)\,dx^i$ has been denoted $\Gamma_i\,(x)\,dx^i$ whereas $i\omega_\mu\,(x,\xi;z)\,d\xi^\mu$ was not considered since we used a gauging of the internal symmetry with respect to the external space-time only. The situation is quite different now, we have a symmetry of two space-times gauged with respect to both.

The one sheet propagation operator of functions $\psi\,(x,\xi,z)$ is a path integral expression[6] (θ is the step function)

$$\Pi_z^c = \int dt\,\theta(t)\,\exp(-im_z^2 t)\int d\tau\,\theta(\tau)\,\exp(-im_z'^2\tau)\int d[u]_t\int d[\gamma]_\tau$$
$$\exp(\frac{-i}{4}\int_0^t dl\,u^2(l))\;\exp(\frac{-i}{4}\int_0^\tau d\lambda\,\gamma^2(\lambda))H_{([u]_t,[\gamma]_\tau;z)}\,U([u]_t,[\gamma]_\tau)$$

with measures

$$d[u]_t = \prod_{l=0}^{t} du(l) \qquad\qquad d[\gamma]_\tau = \prod_{\lambda=0}^{\tau} d\gamma(\lambda) \tag{32}$$

Masses (m_z,m_z') correspond to the first and second modes, respectively. The subscript z is a reminder that momentum representations may be different for different sheets. So for masses and spins since in applying the method of induced representations for ordinary symmetry groups, irreducible momentum representations labeled by the mass m and the spin j of each mode must be used. Propagation is determined by a transition from configuration representation to momentum representation (intertwining) and then to configuration representation. The propagator has then to be labeled by the masses (m_z,m_z') and spins (j_z,j_z'), of the first and second mode respectively, corresponding to the sheet where intertwining is realized. In contrast, trajectory (semigroup) momentum representation does not specify masses which are introduced through integration over (s,λ),[6] but they should naturally be labeled by z.

For a pure sheet $z=p$ and $(m_p,m'_p)=(m,\mu)$, the first mode is external with mass m and the second mode is internal with mass μ. For $z=c$ and $(m_c,m'_c)=(\mu,m)$, the first mode is internal while the second is external. For the remaining cases $z=e$ and $z=i$, the respective masses are (m,m) and (μ,μ).

Now, we come to the implementation of our idea of conversion and proceed by comparison with the case where the continuous connection is ignored, that is when groups are used instead of semigroups. A general propagation amounts to a transition from $\psi\left(x',\xi';z'\right)$ to $\psi\left(x,\xi;z\right)$ through momentum representation in a z''-sheet. [?] The function $\psi\left(x',\xi';z'\right)$ is transformed by means of $H\left(x',\xi';z'',z'\right)$ and propagation is realized from $(x',\xi';z'')$ to $(x,\xi;z'')$ in the z''-sheet. Then the result is transformed by $H\left(x,\xi;z,z''\right)$. When the continuous connection is considered (trajectory case), the two H fields representing the discrete connection must be included in the propagator and masses have to be labeled with the intertwining sheet parameter z''.

Hence, the most general expression of the propagator operator is

$$\begin{aligned}\Pi^c_{z''}\left(z,z'\right) = &\int dt\,\theta(t)\,\exp(-im^2_{z''}t)\int d\tau\,\theta(\tau)\,\exp\left(-im'^2_{z''}\tau\right)\\ &\int d[u]_t\int d[\gamma]_\tau\exp(\frac{-\,i}{4}\int_0^t dl\,u^2(l))\;\exp(\frac{-\,i}{4}\int_0^\tau d\lambda\,\gamma^2(\lambda))\\ &H_{([u]_t,[\gamma]_\tau;z'')}\left(z,z'\right)\,U([u]_t,[\gamma]_\tau)\end{aligned}$$

The new operators $H_{([u]_t,[\gamma]_\tau;z'')}\left(z,z'\right)$ defined by

$$\begin{aligned}&\left[H_{([u]_t,[\gamma]_\tau;z'')}\left(z,z'\right)U([u]_t,[\gamma]_\tau\psi\right]\left(x,\xi;z'\right)\\ &=H\left(x,\xi;z,z''\right)H_{([u]_t,[\gamma]_\tau)}(x,\xi;z'')\times\\ &H(x[u]_t,\xi[\gamma]_\tau;z'',z')\psi(x[u]_t,\xi[\gamma]_\tau;z')\end{aligned}\tag{33}$$

contain continuous and discrete connections. We note that the general propagation operator is compatible with the one sheet propagation operator since $H(x,\xi;z,z)=1$. In total, we have sixty-four propagations differing by the values of $(z,z';z'')$. Propagation of fields can be written in the following way

$$\psi(x,\xi;z) = [\Pi^c_{z''}\left(z,z'\right)\psi](x,\xi;z') \tag{34}$$

$$= \int dx'd\xi'\Pi^c_{z''}\left(x,\xi,z;x',\xi',z'\right)\psi(x',\xi';z') \tag{35}$$

The kernel $\Pi^{c}_{z''}(x,\xi,z;x',\xi',z')$ in relation (35) is to be determined after calculation of (34). Such a kernel is interpreted as a spatio-temporal evolution of two modes from (x',ξ') to (x,ξ), which may be accompanied by conversions (if z is different from z' or z'').

5 Conclusion

The $(M^4 \times M^4) \otimes Z_n$ structure seemed interesting in interpreting the geometrical origin of Higgs fields without recourse to noncommutative geometry.[7, 8]

The present work reveals another aspect of this structure, which is not concerned with the Higgs phenomenon. It opens the way to the construction of a theory of extended particles interacting with gauge fields and reaches the determination of a path integral form of the propagators. This work explains in a simple manner the creation of particles by admitting their prior existence in unobservable states and the possibility of their transition to observable ones. The $(M^4 \times M^4) \otimes Z_4$ interest is that it provides the mathematical objects representing these transitions, namely the discrete connections $H(x,\xi;z,z')$. Moreover, the theory allows consideration of gauge fields corresponding to a symmetry localized not only with respect to one space-time (generally, external space-time) but in both space-times.

Symmetries, connections, curvatures, and propagators have been presented in their most general form. Propagators incorporate conversions and effects of gauge fields in the continuous directions. Hence, the particle evolves in space-time and this evolution may be accompanied by its annihilation, its creation, or a transformation of those of its properties which are coupled to the continuous gauge field.

The following step is the adoption of specific physical models and the derivation of equation of motion for $\psi(x,\xi;z)$, $A(x,\xi;z)$, and $H(x,\xi;z,z')$. The study of this question has already been initiated by the determination of curvatures which may lead to a Lagrangian formulation. However, this is to be carefully analyzed since the validity of a bilocal Lagrangian theory is not well established.

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