John Bell's Observations on the Chiral Anomaly and Some Properties of Its Descendants

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Abstract

John Bell's emphasis of the essential ambiguities in anomaly calculations is recalled. Some descendants of the anomaly are reviewed.

1   John Bell and the Chiral Anomaly

I expect that most everyone in this audience knows that John Bell codiscovered the mechanism of anomalous symmetry breaking in quantum field theory. Indeed, our paper on this subject [1] is his (and my) most-cited work. The symmetry breaking in question is a quantum phenomenon that violates the correspondence principle; it arises from the necessary infinities of quantum field theory. Over the years it has become evident that theoretical/mathematical physicists are not the only ones to acknowledge this effect. Nature makes fundamental use of the anomaly in at least two ways: the neutral pion's decay into two photons is controlled by the anomaly [1, 2] and elementary fermions (quarks and leptons) arrange themselves in patterns such that the anomaly cancels in those channels to which gauge bosons – photon, W, Z – couple [3]. (There are also phenomenological applications of the anomaly to collective, as opposed to fundamental, physics – for example, to edge states in the quantum Hall effect.) Beyond physics, in mathematics one finds closely related structures, such as Atiyah-Singer index theory, zero modes of Dirac operators, Chern-Pontryagin characteristics of gauge fields, Chern-Simons secondary characteristics. The mathematical ideas were developed at nearly the same time as the physical ones, and this unexpected conjunction between physics and mathematics seeded joint activity that flourishes to this day.

Once it was appreciated that the anomaly is not merely an obscure pathology of quantum field theory, but reflects an as-yet-to-be-understood wrinkle in the field-theoretic description of Nature, many people wrote many papers providing various and alternative derivations of the result. But not John Bell. It seems that all he had to say on the subject was contained in our first joint paper. He did follow the subsequent developments and elaborations, and he commented on them to me whenever we met in Geneva or Cambridge.

In his characteristically diffident manner, he did not always support the various fancy elaborations, and he remained skeptical about their value. On the contrary, he insisted on one statement, which already appeared in our original paper, but which perhaps had not been forcefully enunciated, so I want to call attention to it here.

Our original analysis concerns the correlation function for three currents – one axial vector and two vector currents – which is given in lowest order by the fermionic triangle graph. With massless fermions, this correlation function should be transverse in all three channels, the axial vector and both vectors, as a consequence of various symmetries in the theory. The anomaly manifests itself in that any evaluation of the relevant diagram is ambiguous up to a local term, owing to the underlying infinities of the quantum field theory. Moreover, no matter how one fixes the ambiguity, the transversality conditions fail – the calculated amplitude is not transverse in all three channels. Thus the full extent of symmetry, which would ensure full transversality, is broken. While specific choices for resolving the ambiguity allow transversality in some (but not all three) channels, John Bell always insisted that there is no intrinsic way to select a "correct" result. And this opinion informed his criticism of alternative derivations of the anomaly, which usually preserve vector transversality at the expense of axial vector transversality. Whenever we discussed yet another new approach to the anomaly, he always wanted to verify that there remained the possibility of obtaining a variety of results. If this freedom were not present, he would dismiss the rederivation as too restrictive.

One alternative viewpoint did appeal to John Bell. The lack of transversality in a particular channel can be ascribed to an anomalous nonconservation of the appropriate "symmetry" current, with the nonvanishing anomalous current divergence determined by the gauge fields with which the fermions interact. But from another perspective, one could state that transversality fails because the commutators between the relevant current operators differ from the corresponding Poisson brackets of their classical antecedents – the commutators contain additional, quantal (anomalous) contributions. (Indeed, this is the point of view one must adopt in the interaction picture, where operator equations do not see the interaction.) The form for the anomalous commutators is uniquely determined by the triangle graph, yet the transversality conditions retain their ambiguity, because they continue to reflect the arbitrariness in the local contribution to the graph. This gives an appealing algebraic characterization of the anomaly [4]. Either point of view signals absence of the expected symmetries and exposes a breakdown of the correspondence principle.

John Bell's insistence that the triangle graph supports a variety of anomalies is not merely a pedantic nicety. In fact it has a physical realization in Gerard 't Hooft's calculation of fermion-

number nonconservation in the standard model [5], where the vectorial fermion number current in the triangle graph carries the anomaly, and the chiral current is anomaly free [6].

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2 Descendants of the Anomaly

The axial anomaly, that is, the departure of transversality of the 3-fermion current correlation function, involves $^*\!FF$, an expression constructed from the gauge fields to which the fermions couple. Specifically, in the Abelian case one encounters

$$^*\!F^{\mu\nu} F_{\mu\nu} = \tfrac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta} = -4 \boldsymbol{E} \cdot \boldsymbol{B} \tag{1}$$

where $F_{\mu\nu}$ is the covariant electromagnetic tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\mu A_\nu \tag{2a}$$

while $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields

$$E^i = F^{io} , \quad B^i = -\tfrac{1}{2} \varepsilon^{ijk} F_{jk} . \tag{2b}$$

The non-Abelian generalization reads

$$^*F^{\mu\nu a}F^a_{\mu\nu} = \tfrac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F^a_{\mu\nu}F^a_{\alpha\beta} \tag{3}$$

where $F^a_{\mu\nu}$ is Yang-Mills gauge field strength

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu \tag{4}$$

and a labels the components of the gauge group, whose structure constants are f^{abc}.

The quantity $^*FF$ is topologically interesting. Its integral over 4-space is quantized, and measures the topological class (labeled by integers) to which the vector potential A belongs. Consequently, the 4-volume integral of $^*FF$ is a topological invariant and we expect that, as befits a topological invariant, it should be possible to present $^*FF$ as a total derivative, so that its 4-volume integral becomes converted by Gauss' law into a surface integral, sensitive only to long distance, global properties of the gauge fields. That the total derivative form for $^*FF$ holds is seen when $F_{\mu\nu}$ is expressed in terms of potentials. In the Abelian case, we use (2a) and find immediately

$$\tfrac{1}{2}{}^*F^{\mu\nu}F_{\mu\nu} = \partial_\mu\left(\varepsilon^{\mu\alpha\beta\gamma}A_\alpha\partial_\beta A_\gamma\right). \tag{5}$$

For the non-Abelian fields, (4) establishes the desired result:

$$\tfrac{1}{2}{}^*F^{\mu\nu a}F^a_{\mu\nu} = \partial_\mu\varepsilon^{\mu\alpha\beta\gamma}\left(A^a_\alpha\partial_\beta A^a_\gamma + \tfrac{1}{3}f^{abc}A^a_\alpha A^b_\beta A^c_\gamma\right). \tag{6}$$

The quantities whose divergence gives $^*FF$ are called Chern-Simons terms. By suppressing one dimension they become naturally defined on a 3-dimensional manifold (they are 3-forms), and we are thus led to consider the Chern-Simons terms in their own right [1]:

$$\mathrm{CS}(A) = \varepsilon^{ijk}A_i\partial_j A_k \qquad\qquad \text{(Abelian)} \tag{7}$$

$$\mathrm{CS}(A) = \varepsilon^{ijk}\left(A^a_i\partial_j A^a_k + \tfrac{1}{3}f^{abc}A^a_i A^b_j A^c_k\right) \qquad \text{(non-Abelian).} \tag{8}$$

The 3-dimensional integral of these quantities is again topologically interesting. When the non-Abelian Chern-Simons term is evaluated on a pure gauge, non-Abelian vector potential

$$A_i = g^{-1}\partial_i g \tag{9}$$

the 3-dimensional volume integral of $\mathrm{CS}(g^{-1}\partial g)$ measures the topological class (labeled by integers) to which g belongs. The integral in the Abelian case – the case of electrodynamics – is called the magnetic helicity $\int \mathrm{d}^3 r\, \boldsymbol{A}\cdot\boldsymbol{B}$, $\boldsymbol{B} = \boldsymbol{\nabla}\times\boldsymbol{A}$, and measures the linkage of magnetic flux lines. An analogous quantity arises in fluid mechanics, with the local fluid velocity $\boldsymbol{v}$ replacing $\boldsymbol{A}$, and the vorticity $\boldsymbol{\omega} = \boldsymbol{\nabla}\times\boldsymbol{v}$ replacing $\boldsymbol{B}$. Then the integral $\int \mathrm{d}^3 r\, \boldsymbol{v}\cdot\boldsymbol{\omega}$ is called kinetic helicity [2].

I shall not review here the many uses to which the Chern-Simons terms, Abelian and non-Abelian, introduced in [1], have been put. The applications range from the mathematical characterization of knots to the physical description of electrons in the quantum Hall effect [3],

vivid evidence for the deep significance of the Chern-Simons structure and of its antecedent, the chiral anomaly.

Instead I pose the following question: Can one write the Chern-Simons term as a total derivative, so that (as befits a topological quantity) the spatial volume integral becomes a surface integral. An argument that this should be possible is the following: The Chern-Simons term is a 3-form on 3-space, hence it is maximal and its exterior derivative vanishes because there are no 4-forms on 3-space. This establishes that on 3-space the Chern-Simons term is closed, so one can expect that it is also exact, at least locally, that is, it can be written as a total derivative. Of course, such a representation for the Chern-Simons term requires expressing the potentials in terms of "pre-potentials", since the formulas (7), (8) show no evidence of derivative structure. [Recall that the total derivative formulas (5), (6) for the axial anomaly also require using potentials to express F.]

There is a physical, practical reason for wanting the Abelian Chern-Simons term to be a total derivative. It is known in fluid mechanics that there exists an obstruction to constructing a Lagrangian for Euler's fluid equations, and this obstruction is just the kinetic helicity $\int \mathrm{d}^3 r\, \boldsymbol{v} \cdot \boldsymbol{\omega}$, that is, the volume integral of the Abelian Chern-Simons term, constructed from the velocity 3-vector $\boldsymbol{v}$. This obstruction is removed when the integrand is a total derivative, because then the kinetic helicity volume integral is converted to a surface integral by Gauss' theorem. When the integral obtains contributions only from a surface, the obstruction disappears from the 3-volume, where the fluid equation acts [4].

It is easy to show that the Abelian Chern-Simons term can be presented as a total derivative. We use the Clebsch parameterization for a 3-vector [5]

$$\boldsymbol{A} = \boldsymbol{\nabla}\theta + \alpha\boldsymbol{\nabla}\beta \,. \tag{10}$$

This nineteenth-century parameterization of the a 3-vector $\boldsymbol{A}$ in terms of the prepotentials $(\boldsymbol{\theta}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an alternative to the usual transverse/longitudinal parameterization. In modern language it is a statement of Darboux's theorem that the 1-form $A_i\,\mathrm{d}r^i$ can be written as $\mathrm{d}\theta + \alpha\,\mathrm{d}\beta$ [6]. With this parameterization for $\boldsymbol{A}$, one sees that the Abelian Chern-Simons term is indeed a total derivative:

$$
\begin{aligned}
\mathrm{CS}(A) &= \varepsilon^{ijk} A_i \partial_j A_k \\
&= \varepsilon^{ijk} \partial_i \theta \partial_j \alpha \partial_k \beta \\
&= \partial_i \big(\varepsilon^{ijk} \theta \partial_j \alpha \partial_k \beta \big) \,.
\end{aligned}
$$

When the Clebsch parameterization is employed for $\boldsymbol{v}$ in the fluid dynamical context, the situation is analogous to the force law in electrodynamics. While the Lorentz equation is written in terms of field strengths, a Lagrangian formulation needs potentials from which the field strengths are reconstructed. Similarly, Euler's equation involves the velocity vector $\boldsymbol{v}$, but in a Lagrangian for this equation the velocity must be parameterized in terms of the prepotentials $\boldsymbol{\theta}$, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$.

In a natural generalization of the above, one asks whether a non-Abelian vector potential can also be parameterized in such a way that the non-Abelian Chern-Simons term (8) becomes a total derivative. We have answered this question affirmatively and we have found appropriate prepotentials that do the job [4, 7, 8], but the details of the construction are too technical to be presented here. We hope that our non-Abelian generalization of the Clebsch parameterization will be as interesting and useful as the Abelian one.

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