An O(N) symmetric extension of the Sine-Gordon Equation

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Abstract

We discuss an $O(N)$ exension of the Sine-Gordon (S-G)equation described by the Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu \vec{\phi})^2$ + N $\frac{\alpha_0}{\beta^2} \cos \beta \sqrt{\rho}$, where $\rho = \frac{\vec{\phi} \cdot \vec{\phi}}{N}$ which allows us to perform an expansion around the leading order in large-N result using Path-Integral methods. In leading order we show our methods agree with the results of a variational calculation at large-N. We discuss the striking differences for a non-polynomial interaction between the form for $\langle V[\phi] \rangle$ in the Gaussian approximation that one obtains at large-N when compared to the $N = 1$ case. This is in contrast to the case when $V[\phi]$ is a polynomial and no such drastic differences occur. We find for our large-N extension of the Sine-Gordon model that the unbroken ground state is unstable as one increases the coupling constant (as it is for the original S-G equation) and we find in leading order that the unbroken symmetry vacuum is stable as long as $\beta^2 \leq 24\pi$.

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I. INTRODUCTION

Although the Sine-Gordon equation has previously been studied using variational methods, [1] [2] a systematic expansion around the mean field theory result has been lacking. Here we use the method of auxiliary fields to study going beyond mean field theory using a systematic expansion based on the the parameter N obtained when we consider a particular O(N) extension of the orginal S-G model. Using the methodology of [3] [4] we first extend the problem to having O(N) symmetry and then introduce an auxiliary field $\rho = \frac{\phi_i \phi_i}{N}$ by the insertion of a functional delta function into the path integral. The large-N expansion is then obtained by integrating out exactly the ϕ field and then performing a steepest descent evaluation of the remaining path integral. Unlike the case of polynomial interactions, at leading order in the Gaussian approximation there is a distinct difference in the equations of motion when $N = 1$ and N large. We display this by comparing the ϕ^6 results with those of the S-G equation. As in the original S-G equation, our $O(N)$ extended model has an unstable vacuum as a function of the coupling constant. We find that in leading order in large-N that the vacuum is unstable for $\beta^2 \geq 24\pi$.

II. TIME DEPENDENT VARIATIONAL APPROACH

One method for obtaining a time dependent variational approximation to a quantum field theory is to start with Dirac's Variational principle [5] [6]for obtaining the functional Schrodinger Equation. By using wave functionals which become "exact" in the large-N limit– namely Gaussians we will obtain the time-dependent Hartree approximation to the exact field equations.

Dirac's variational principle

$$\Gamma = \int dt < \Psi | i \frac{\partial}{\partial t} - H | \Psi > \tag{1}$$

leads directly to the Schrodinger equation:

$$\delta\Gamma = 0 \to \{ i \frac{\partial}{\partial t} - H \} | \Psi >= 0 \tag{2}$$

In the φ representation

$$\Psi[\varphi, t] =< \varphi | \Psi >$$

and

$$H = \int d^n x[-\frac{1}{2}\delta^2/[\delta\varphi(x)]^2 + \frac{1}{2}\nabla\varphi(x)\nabla\varphi(x) + V[\varphi]] \tag{3}$$

In this representation one can choose a Gaussian trial wave functional:

$$\begin{aligned} &< \varphi|\Psi_v >= \psi_v[\varphi, t] = A\exp\{-\int_{x,y}\{[\varphi(x) - \phi_c(x,t)] \\ &[G^{-1}(x,y,t)/4 - i\Sigma(x,y,t)][\varphi(y) - \phi_c(y,t)]\} + i\int_x \pi_c(x,t)[\varphi(x) - \phi_c(x,t)]\}, \end{aligned} \tag{4}$$

where $\phi_c(x,t) = \langle\Psi_v|\varphi(x)|\Psi_v\rangle$; $\pi_c(x,t) = \langle\Psi_v| - i\delta/\delta\varphi(x)|\Psi_v\rangle$ and

$$G(x,y,t) = \langle\Psi_v|\varphi(x)\varphi(y)|\Psi_v\rangle - \phi_c(x,t)\phi_c(y,t). \tag{5}$$

Then the effective action for the trial wave functional is

$$\begin{aligned} \Gamma(\phi_c, \pi_c, G, \Sigma) &= \int dt < \Psi_v|i\partial/\partial t - H|\Psi_v > \\ &= \int dtdx[\pi_c(x,t)\partial\phi_c(x,t)/\partial t + \int dtdxdy\Sigma(x,y;t)\partial G(x,y;t)/\partial t] \\ &- \int dt < H > \end{aligned} \tag{6}$$

where

$$\begin{aligned} < H > &= \int dx\{\frac{1}{2}\pi^2 + 2[\Sigma G\Sigma](x,x) + \frac{1}{8}G^{-1}(x,x) + \frac{1}{2}(\nabla\varphi)^2 \\ &+ \frac{1}{2}\lim_{x\to y}\nabla_x\nabla_y G(x,y) + < V >\}. \end{aligned} \tag{7}$$

$< H >$ is a constant of the motion and is a first integral of the motion. The equations one gets by varying the effective action with respect to the variational parameters are:

$$\begin{aligned} \dot{\pi}_c(x,t) &= \nabla^2\phi_c(x,t) - \frac{\partial < V >}{\partial\phi_c}; \\ \dot{\phi}_c(x,t) &= \pi_c(x,t) \\ \dot{G}(x,y;t) &= 2\int dz[\Sigma(x,z)G(z,y) + G(x,z)\Sigma(z,y)] \\ \dot{\Sigma}(x,y;t) &= \int dz[-2\Sigma(x,z)\Sigma(z,y) + \frac{1}{8}G^{-1}(x,z)G^{-1}(z,y)] \\ &+ [\frac{1}{2}\nabla_x^2 - \frac{\partial < V >}{\partial G}]\delta(x-y) \end{aligned}$$

For calculating expectation values,

$$P[\phi] = A^2\exp\{-\int_{x,y}[\varphi(x) - \phi_c(x,t)][\frac{G^{-1}(x,y;t)}{2}[\varphi(y) - \phi_c(y,t)]] \tag{8}$$

with the normalization A determined from the conservation of probability:

$$\int \mathcal{D}\phi \quad P[\phi] = 1 \tag{9}$$

To get a general expression for $\langle V \rangle$ one can use the method of the generating functional:

$$Z[j] = \langle \exp[\int j(x,t)\{\phi(x,t) - \phi_c(x,t)\}] \rangle = \exp[\frac{1}{2}\int j(x,t)G(xy,t)j(y,t)]. \tag{10}$$

In the single field case (N=1) for example one has

$$\langle (\phi(x,t) - \phi_c(x,t))^{2n} \rangle = \frac{\delta^{2n} Z[j]}{\delta j^{2n}(x)}|_{j=0}. \tag{11}$$

and we obtain the simple result

$$\langle (\phi(x,t) - \phi_c(x,t))^{2n} \rangle = (2n-1)!!G(x,x;t)^n \tag{12}$$

To obtain the expectation value of an arbitrary potential we assume that the potential has a Taylor series expansion about the classical field $\phi_c = \langle \phi \rangle$ and the expectation value is taken with respect to the Gaussian trial wave function. At N=1 we obtain the result:

$$\langle V[\phi] \rangle = \sum_n \frac{1}{(2n)!} \frac{d^{2n} V[\phi_c]}{d\phi_c^{2n}} \langle (\phi - \phi_c)^{2n} \rangle = \sum_n \frac{1}{2^n n!} G^n \frac{d^{2n} V[\phi_c]}{d\phi_c^{2n}} \tag{13}$$

For the O(N) extension of the potential, $\phi \to \phi_i; \quad i = 1 \ldots N$ and the Green's function G is now an $N \times N$ matrix. One then finds making the Gaussian ansatz and ignoring non-leading terms in $1/N$ that (see [7])

$$\langle (\frac{\vec{\phi} \cdot \vec{\phi}}{N})^n \rangle \to \langle \rho \rangle^n \tag{14}$$

where

$$\langle \rho \rangle = \frac{\langle \vec{\phi} \cdot \vec{\phi} \rangle}{N} = G + \frac{\vec{\phi}_c \cdot \vec{\phi}_c}{N} \equiv G + \phi_c^2. \tag{15}$$

Thus at leading order in large-N the expectation value of the potential becomes

$$\langle V[\frac{\vec{\phi} \cdot \vec{\phi}}{N}] \rangle = V_{cl}[\phi_c^2 + G] \tag{16}$$

A. ϕ^6 Model

First let us look at a typical polynomial model such as the ϕ^6 model described by the classical potential

$$V_{cl}[\phi] = \frac{\mu^2}{2} \vec{\phi} \cdot \vec{\phi} + \frac{\lambda_0}{4N} (\vec{\phi} \cdot \vec{\phi})^2 + \frac{\eta}{6N^2} (\vec{\phi} \cdot \vec{\phi})^3 \tag{17}$$

For $N=1$ one gets the Hartree approximation result

$$\begin{aligned}\langle V\rangle &= \frac{\mu^2}{2}[\phi^2+G]+\frac{\lambda_0}{4}[\phi^4+6\phi^2G+3G^2] \\ &+\frac{\eta}{6}[\phi^6+15\phi^4G+45\phi^2G^2+15G^2]\end{aligned} \tag{18}$$

where in the above $G=G(x,x)$ and $G(x,y;t)$ is the variational function.

To obtain the phase structure one calculates the effective potential which is the expectation value of the Hamiltonian in the Gaussian trial state for spatially homoogenous time independent fields:

$$V_{eff}=\frac{G^{-1}(x,x)}{8}+\frac{1}{2}lim_{x\to y}\nabla_x\nabla_y G(x,y)+\langle V\rangle. \tag{19}$$

We can change variables from $G(x,y)$ to the effective mass χ for constant (homogeneous) external sources and in that case parametrize $G(x,y)$ via

$$G(x,y)=\int\frac{dk}{2\pi}\quad e^{ik(x-y)}\quad\frac{1}{2\sqrt{k^2+\chi}}, \tag{20}$$

So that if we introduce a cutoff Λ

$$G(x,x)=\frac{1}{4\pi}\ln\frac{\Lambda^2}{\chi} \tag{21}$$

The first two terms in the effective potential coming from the kinetic energy can now be written as

$$\langle KE\rangle=\int\frac{dk}{2\pi}[\frac{\sqrt{k^2+\chi}}{4}+\frac{k^2}{4\sqrt{k^2+\chi}}] \tag{22}$$

This has an infinite χ independent term which is the cosmological constant and needs to be subtracted by hand.

In general V_{eff} is a functional of ϕ, G or ϕ,χ. Using the chain rule we have

$$\frac{\partial V}{\partial\chi}=\frac{\partial V}{\partial G}\frac{\partial G}{\partial\chi} \tag{23}$$

with

$$\frac{\partial G}{\partial\chi}=-\frac{1}{4\pi\chi} \tag{24}$$

Here

$$\begin{aligned}\frac{\partial V}{\partial G} &= \frac{1}{2}[-\chi+\mu^2+3\lambda\{\phi^2+G[\chi]\} \\ &+5\eta\{\phi^4+6\phi^2G[\chi]+3G^2[\chi]\}]\end{aligned} \tag{25}$$

Another way of writing the Kinetic energy which removes the cosmological constant term is used in [7] and leads to a result independent of the cutoff. There one lets $\chi \to m^2$ in the definition of G and sets

$$K[\chi] = -\frac{1}{2}\int_0^\chi m^2 \frac{\partial G[m^2]}{\partial m^2} = \frac{\chi}{8\pi}. \tag{26}$$

The potential energy also has divergences which are related to mass and coupling constant renormalization if one does not do any normal ordering. Here we will first use conventional renormalization and then show it is equivalent (in 1+1 dimensions) to normal ordering the unrenormalized result with respect to the renormalized mass. First let us look at the case $N = 1$. We define the renormalized mass as the value of χ determined from the gap equation obtained from setting $\frac{\partial V}{\partial G}|_{\phi=0} = 0$ i.e.

$$m^2 = \mu^2 + 3\lambda_0 G[m^2] + 15\eta G^2[m^2] \tag{27}$$

The gap equation for χ is fully renormalized if we solve the above for μ in terms of m and also perform the coupling constant renormalization

$$\lambda_r = \lambda_0 + 10\eta G[m^2] \tag{28}$$

Defining

$$G_r[\chi] = G[\chi] - G[m^2] = \frac{1}{4\pi}\ln\frac{m^2}{\chi} \tag{29}$$

one then obtains the renormalized equation

$$\begin{aligned}\frac{\partial V}{\partial G} &= \frac{1}{2}[-\chi + m^2 + 3\lambda_r\{\phi^2 + G_r[\chi]\} \\ &\quad + 5\eta\{\phi^4 + 6\phi^2 G_r[\chi] + 3G_r^2[\chi]\}]\end{aligned} \tag{30}$$

We notice here by comparing Eq. 25 and Eq. 30 that the *only* difference between the unrenormalized and renormalized gap equations are the replacement of:

$$\mu \to m; \quad \lambda \to \lambda_r; G \to G_r. \tag{31}$$

This is an example of the result in two space-time dimensions (see also [8]) that if one did *no* mass and coupling constant renormalization, but normal ordered the two point function with respect to the renormalized mass parameter m the theory would be rendered finite.

To obtain the renormalized effective potential one needs to do an integration of the renormalized equaition for $\frac{\partial V}{\partial \chi}$ and add the pure ϕ dependent terms.

We have

$$\begin{aligned}\frac{\partial V}{\partial \chi} &= -\frac{1}{8\pi\chi}[-\chi + m^2 + 3\lambda_r\{\phi^2 + G_r[\chi]\} \\ &+ 5\eta\{\phi^4 + 6\phi^2 G_r[\chi] + 3G_r^2[\chi]\}]\end{aligned} \tag{32}$$

Integrating and adding the pure ϕ dependent terms we obtain:

$$\begin{aligned}V[\phi,\chi] &= \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda_r\phi^4 + \frac{1}{6}\eta\phi^6 \\ &\frac{1}{8\pi}[\chi - m^2 + (m^2 + 3\lambda_r\phi^2 + 5\eta\phi^4)\ln\frac{m^2}{\chi}] \\ &\frac{3}{64\pi^2}(\lambda_r + 10\eta\phi^2)\ln^2\frac{m^2}{\chi} + \frac{5}{128\pi^3}\eta\ln^3\frac{m^2}{\chi}.\end{aligned} \tag{33}$$

B. ϕ^6at large N

At leading order in large N one has as discussed before that

$$\langle V\rangle/N = \frac{1}{2}\mu^2[\phi_c^2 + G(x,x)] + \frac{\lambda_0}{4}[\phi_c^2 + G(x,x)]^2 + \frac{\eta}{6}[\phi_c^2 + G(x,x)]^3, \tag{34}$$

where here we use the shorthand $\phi_c^2 \equiv \frac{\vec{\phi}\cdot\vec{\phi}}{N}$. Performing the mass renormalization we find now:

$$m^2 = \mu^2 + \lambda_0 G(x,x;m^2) + \eta_0 G^2(x,x;m^2). \tag{35}$$

The equation for the coupling constant renormalization is also slightly changed and reads:

$$\lambda_r = \lambda_0 + 2\eta G(x,x;m^2) \tag{36}$$

then the renormalized gap equation is

$$\frac{\partial V}{\partial G} = \frac{1}{2}[-\chi + m^2 + \lambda_r\{\phi^2 + G_r(x,x;\chi)\} + \eta\{\phi^2 + G_r(x,x;\chi)\}^2]. \tag{37}$$

Thus apart from the factors in front of λ and η reflecting $3 = 1+2/N$ etc, the renormalization at large N is quite similar to what happens in the Hartree approximation. We also find

$$\frac{\partial H}{\partial \chi} = \frac{\partial H}{\partial G}\frac{\partial G}{\partial \chi} \tag{38}$$

which after renormalization can be written (using V_{eff} for H at constant fields)

$$\frac{\partial V_{eff}}{\partial \chi} = \frac{1}{8\pi\chi}\left[\chi - m^2 - \lambda_r\left(\phi^2 + \frac{1}{4\pi}\ln[\frac{m^2}{\chi}\right) - \eta\left(\phi^2 + \frac{1}{4\pi}\ln[\frac{m^2}{\chi}]\right)^2\right] \tag{39}$$

This result again could have been obtained if we did *NOT* renormalize the coupling constant or the mass and instead used m^2 for the mass parameter and everywhere made the subtraction:

$$G(xx,\chi) \to G(xx,\chi) - G(xx,m^2) \equiv G_r(x,x;\chi). \tag{40}$$

Integrating as before we find the renormalized effective potential can be written in the form

$$\begin{aligned} V[\phi,\chi]/N = \quad & \frac{1}{8\pi}[\chi - m^2] + \frac{1}{2}m^2[\phi^2 + G_r[\chi]] \\ & + \frac{1}{4}\lambda_r[\phi^2 + G_r]^2 + \frac{1}{6}\eta[\phi^2 + G_r]^3 \end{aligned} \tag{41}$$

which has the advertised form.

C. sine-Gordon equation

The sine-Gordon equation is quite interesting since the classical equation has a kink solution which remains after quantization and there is also a phase transition as we increase the effective coupling constant as discussed by Coleman [1]. The usual sine-Gordon equation is described by the Hamiltonian:

$$H = \int dx[\frac{1}{2}\pi(x,t)^2 + \frac{1}{2}(\nabla\varphi(x,t))^2 - \frac{\alpha_0}{\beta^2}\cos\beta\varphi + \gamma] \tag{42}$$

where in the "φ" representation

$$\pi(x) = -i\frac{\delta}{\delta\varphi}$$

Taking the expectation value of this Hamiltonian in the trial Gaussian wave functional state one now has

$$< V > = < -\frac{\alpha_0}{\beta^2}\cos\beta\varphi > = -\frac{\alpha_0}{\beta^2}\cos\beta\phi \quad e^{-\frac{\beta^2}{2}G(x,x)} \tag{43}$$

Varying the action we obtain the following field equations for the variational functions:

$$\ddot{\phi} - \nabla^2\phi + \frac{\alpha_0}{\beta}\sin\beta\phi \quad e^{-\frac{\beta^2}{2}G(x,x)} = 0, \tag{44}$$

$$\dot{G}(x,y) = 2\int dz\{G(x,z)\Sigma(z,y) + \Sigma(x,z)G(z,y)\} \tag{45}$$

$$\begin{aligned} \dot{\Sigma}(x,y;t) \quad &= \int dz[-2\Sigma(x,z)\Sigma(z,y) + \frac{1}{8}G^{-1}(x,z)G^{-1}(z,y)] \\ & + [\frac{1}{2}\nabla_x^2 - \frac{1}{2}\alpha_0\cos\beta\phi(x,t) \quad e^{-\frac{\beta^2}{2}G(x,x)}]\delta(x-y) \end{aligned} \tag{46}$$

In the Heisenberg picture, making the same the Hartree approximation gives the following covariant equation for the nonequaltime correlation function:

$$[\Box + \langle\frac{\partial^2 V}{\partial\phi^2}\rangle]\mathcal{G}(x,x') = \delta^2(x-x') \tag{47}$$

where

$$\langle\frac{\partial^2 V}{\partial\phi^2}\rangle = \alpha_0 \cos\beta\phi(x,t) \quad e^{-\frac{\beta^2}{2}G(x,x)} \equiv \chi(x). \tag{48}$$

Again the connection between the covariant and noncovariant Green's functions at equal spatio-temporal points is: $\frac{\mathcal{G}(x,x;t,t)}{i} = G(x,x;t)$.

When χ is independent of x one can again introduce the Fourier Transform:

$$G(x,y) = \frac{1}{2\pi}\int dk\tilde{G}(k)e^{ik(x-y)} \tag{49}$$

where now

$$\tilde{G}(k) = \frac{1}{2(k^2+\chi)^{1/2}}$$

In particular, when $\phi = 0$ one has the gap equation:

$$\chi = \alpha_0 \quad e^{-\frac{\beta^2}{2}G(x,x;\chi)} = \alpha_0(\frac{\Lambda^2}{\chi})^{-\frac{\beta^2}{8\pi}} \tag{50}$$

This equation tells one how to choose α_0 as a function of the cutoff Λ to insure that the physical mass χ is finite.

To study the phase transition in this theory in this approximation we follow [1] [2]. We determine for what values of the coupling constant one can have $\phi = 0$ be a minimum of the energy.

In the vacuum sector we have for the Hamiltonian Density:

$$< \mathcal{H} >= \frac{1}{8}G^{-1}(xx;\chi) + \frac{1}{2}\lim_{x\to y}\nabla_x\nabla_y G(x,y;\chi) - \frac{\alpha_0}{\beta^2} \quad e^{-\frac{\beta^2}{2}G(x,x;\chi)} \tag{51}$$

Inserting a momentum space cutoff Λ

$$\mathcal{H}(\mu^2) = \frac{1}{8\pi}\int_{-\Lambda}^{\Lambda} dk\frac{2k^2+\mu^2}{(k^2+\mu^2)^{1/2}} - \frac{\alpha_0}{\beta^2}(\frac{\Lambda^2}{\mu^2})^{-\beta^2/8\pi} \tag{52}$$

Let us define an arbitrary renormalized mass squared parameter α_r by

$$\alpha_r(m^2) = \alpha_0(\frac{m^2}{\Lambda^2})^{\beta^2/8\pi} \tag{53}$$

which has the property that

$$\alpha_r(\chi) = \chi \tag{54}$$

we find that the once subtracted (at m^2) energy density is given by

$$\mathcal{H}(\chi) - \mathcal{H}(m^2) = \frac{1}{8\pi}(\chi - m^2) - \frac{\alpha_r}{\beta^2}(\frac{\chi}{m^2})^{\beta^2/8\pi} \tag{55}$$

The first derivative is given by:

$$\frac{\partial H}{\partial \chi} = \frac{1}{8\pi}[1 - \frac{\alpha_r(m^2)}{m^2}(\frac{\chi}{m^2})^{\beta^2/8\pi - 1}] \tag{56}$$

The minimum for $\phi = 0$ is at

$$(\frac{\chi}{m^2})^{\beta^2/8\pi - 1} = \frac{m^2}{\alpha_r(m^2)} \tag{57}$$

We notice that if we choose

$$m^2 = \alpha_r(m^2)$$

then the extrema of the energy is at

$$\chi = m^2.$$

Since the second derivative of the energy is given by:

$$\frac{\partial^2 \mathcal{H}}{\partial^2 \chi} = -\frac{1}{8\pi m^2}(\frac{\beta^2}{8\pi} - 1)(\frac{\chi}{m^2})^{(\beta^2/8\pi) - 2} \tag{58}$$

we find that a stable ground state with unbroken symmetry ($\phi = 0$) exists only for

$$\frac{\beta^2}{8\pi} < 1 \tag{59}$$

which is in accord with the result of Coleman [1].

The resulting renormalized effective potential is

$$V[\phi, \chi] = \frac{1}{8\pi}[\chi - m^2] - \frac{m^2}{\beta^2}(\frac{\chi}{m^2})^{(\beta^2/8\pi)} \cos\beta\phi \tag{60}$$

For the Schrodinger picture update equations we can defined a finite spatially and temporally varying renormalized mass $M^2(x,t)$ via

$$M^2(x,t) \equiv m^2 \exp[-\frac{\beta^2}{2}\{G(xx;\chi) - G(xx;m^2)\}] = m^2 \exp[-\frac{\beta^2}{2} G_r(xx;\chi)] \tag{61}$$

This mass renormalization will render the resulting update equations for the time evolution problem finite. In terms of the renormalized mass $M^2(x,t)$ the renormalized TDHF equations are :

$$\ddot{\phi} - \nabla^2\phi + \frac{M^2(x,t)}{\beta}\sin\beta\phi = 0, \tag{62}$$

$$\dot{G}(x,y) = 2\int dz\{G(x,z)\Sigma(z,y) + \Sigma(x,z)G(z,y)\} \tag{63}$$

$$\begin{aligned}\dot{\Sigma}(x,y;t) \quad &= \int dz[-2\Sigma(x,z)\Sigma(z,y) + \frac{1}{8}G^{-1}(x,z)G^{-1}(z,y)] \\ &+ [\frac{1}{2}\nabla_x^2 - \frac{1}{2}M^2(x,t)\cos\beta\phi(x,t)]\delta^d(x-y)\end{aligned} \tag{64}$$

In the Heisenberg picture, the covariant time dependent equations in terms of $M^2(x,t)$ are

$$[\Box + M^2(x,t)\cos\beta\phi(x)]\mathcal{G}(x,x') = \delta^2(x-x') \tag{65}$$

thus the space and time dependent effective mass is

$$\chi(x) = m^2\cos\beta\phi(x)e^{-\frac{\beta^2\mathcal{G}_r(x,x;\chi)}{2i}} = M^2(x,t)\cos\beta\phi(x) \tag{66}$$

The expectation value of the field obeys the equation

$$\Box\phi(x) + M^2(x,t)\sin\beta\phi(x) = 0. \tag{67}$$

The subscript r in Eq. 66 means that one uses the once subtracted covariant Green's function. Here we see that the main effect of the Hartree approximation on the kink evolution equation is to replace m^2 by a self consistently determined $M^2(x,t)$. We notice that the "classical" field evolves quite differently than the propogator. As we shall see next, making the Gaussian approximation but keeping only leading order terms at large-N dramatically alters the evolution equation in a manner that can be interpreted as a change in the behavior of the periodicity itself, but with the evolution of ϕ and $\mathcal{G}$ being similar. For the N-component field we will consider instead the classical potential:

$$V_{cl} = -N\frac{\alpha_0}{\beta^2}\cos\beta\sqrt{\rho}; \qquad \rho \equiv \frac{\vec{\phi}\cdot\vec{\phi}}{N}. \tag{68}$$

Using the previous result at large N that $\langle H\rangle = K[\chi] + V_{cl}\,[\phi_c^2 + G[\chi]]$, we have that the effective potential is now:

$$V_{eff}/N = \frac{\chi}{8\pi} - \frac{\alpha_0}{\beta^2}\cos\beta\sqrt{\phi_c^2 + G[\chi]}. \tag{69}$$

The renormalized effective potental is thus given by

$$V_{eff}/N = \frac{\chi}{8\pi} - \frac{\alpha}{\beta^2}\cos\beta\sqrt{\phi_c^2 + G_r[\chi; m^2]}. \tag{70}$$

with m^2 an arbitrary subtraction point. If we choose the extremum of energy when $\phi = 0$ to be at $\chi = m^2$ than the renormalized effective potential can be written as:

$$V[\phi,\chi] = \frac{1}{8\pi}[\chi - m^2] - \frac{m^2}{\beta^2}\cos\left(\beta\sqrt{\phi^2 + \frac{1}{4\pi}\ln[\frac{m^2}{\chi}]}\right) \tag{71}$$

We notice now that the effective mass (pole in the propagator) is at

$$\chi = m^2 \frac{\sin\left(\beta\sqrt{\phi^2 + \frac{1}{4\pi}\ln[\frac{m^2}{\chi}]}\right)}{\beta\sqrt{\phi^2 + \frac{1}{4\pi}\ln[\frac{m^2}{\chi}]}} \tag{72}$$

which is functionally totally different than the $N = 1$ Hartree result that

$$\chi(x) = m^2 \cos\beta\phi(x) e^{-\frac{\beta^2 \mathcal{G}_r(x,x;\chi)}{2i}} = M^2(x,t)\cos\beta\phi(x) \tag{73}$$

III. PATH INTEGRAL APPROACH

To obtain the large-N expansion for an arbitrary even polynomial interaction we follow the work of Eyal et. al. [4]. Starting from the usual path integral for the generating functional

$$Z[j] = \int D\phi \;\; \mathrm{Exp}\left[i\{\frac{1}{2}[\partial_\mu \phi_i]^2 - NV[\frac{\vec{\phi}\cdot\vec{\phi}}{N}] + \vec{j}\cdot\vec{\phi}\}\right] \tag{74}$$

one introduces the composite field ρ using :

$$1 = \int d\chi \delta(\rho - \frac{\vec{\phi}\cdot\vec{\phi}}{N}) = \int d\rho d\chi \;\; \exp^{\left[iN\frac{\chi}{2}(\rho - \frac{\vec{\phi}\cdot\vec{\phi}}{N})\right]} \tag{75}$$

This allows us to rewrite the generating functional (also adding sources for χ and ρ) as

$$Z = \int D\phi D\chi D\rho \mathrm{Exp}\left[i\{\frac{1}{2}[\partial_\mu \phi_i]^2 - \frac{1}{2}\chi\vec{\phi}\cdot\vec{\phi} - NV[\rho] + \frac{N}{2}\rho\chi + \vec{j}\cdot\vec{\phi} + NS\chi + NJ\rho\}\right] \tag{76}$$

The equations resulting from the second form of the action are

$$[\Box + \chi]\phi_i = j_i \;; \quad \chi = -2J - 2V'[\rho] \;; \quad \rho = \vec{\phi}\cdot\vec{\phi}/N - 2S \tag{77}$$

The large-N expansion is obtained by integrating out the ϕ fields and, recognizing that the resulting action is proportional to N performing the remaining integral by Stationary Phase. The first integration yields

$$Z[j,S,J] = \int D\chi D\rho e^{iNS_{eff}[\rho,\chi,j,S,J]} \tag{78}$$

where

$$S_{eff} = i\{[\int \frac{\vec{j}(x)\cdot\vec{j}(y)}{2N}\mathcal{G}[x,y;\chi] - V[\rho] + \frac{1}{2}\rho\chi + S\chi + J\rho + \frac{i}{2}Tr\mathrm{Ln}\mathcal{G}^{-1}[x,y,\chi]\}\Big] \tag{79}$$

and here

$$\mathcal{G}^{-1}[x,y,\chi] = [\Box + \chi]\delta^2(x-y) \tag{80}$$

is now the covariant two time Green function. The stationary phase conditions are:

$$\begin{aligned} \frac{1}{i}\frac{\delta S}{\delta\chi} &= S + \frac{\rho}{2} + \frac{i}{2}\mathrm{Tr}\mathcal{G} - \frac{1}{2}\phi_c^2 = 0, \\ \frac{1}{i}\frac{\delta S}{\delta\rho} &= \frac{\chi}{2} - V'[\rho] + J = 0 \end{aligned} \tag{81}$$

where $\vec{\phi}_c = \int \mathcal{G}\vec{j}$ and $\phi_c^2 = \vec{\phi}\cdot\vec{\phi}/N$. In the absence of sources, this leads to the constraint equation for ρ

$$\rho = \phi_c^2 + \frac{1}{i}\mathrm{Tr}\mathcal{G} \tag{82}$$

and the equation for the effective mass (gap equation) in leading order:

$$\chi = -2V'[\rho]. \tag{83}$$

At leading order we also have from the definition of $\phi_c = \frac{1}{1}\frac{\delta lnZ}{\delta j}$ that

$$[\Box + \chi]\vec{\phi}_c = \vec{j} \tag{84}$$

so that unlike the Hartree case the mass for the classical field ϕ is the same as the mass entering into the propagator. The Gaussian fluctuations about this mean field (Stationary point of the effective action) is obtained from the second derivative of the effective action at the stationary point. This matrix is actually the inverse propagator of the two component field made up of χ and ρ. The stability of the large-N approximation is related to the eigenvalues of this matrix being positive. Critical behavior is related to the deteminant of the matrix being zero. The second derivatives are given by

$$\frac{1}{i}\frac{\delta^2 S}{\delta\chi(x)\delta\chi(y)} = -\frac{i}{2}\mathcal{G}(x,y)\mathcal{G}(y,x) + \phi_c(x)\mathcal{G}(x,y)\phi_c(y) \equiv \frac{\Sigma(x,y)}{2} \tag{85}$$

$$\frac{1}{i}\frac{\delta^2 S}{\delta\chi(x)\delta\rho(y)} = \frac{1}{i}\frac{\delta^2 S}{\delta\chi(x)\delta\rho(y)} = \frac{1}{2}\delta(x-y) \tag{86}$$

$$\frac{1}{i}\frac{\delta^2 S}{\delta\rho(x)\delta\rho(y)} = -V"[\rho]\delta(x-y). \tag{87}$$

For constant fields, the Fourier transform of the inverse propagator is given by

$$D^{-1}[p;\chi,\rho,\phi] = \begin{pmatrix} \frac{\Sigma(p)}{2} & \frac{1}{2} \\ \frac{1}{2} & -V"[\rho] \end{pmatrix} \tag{88}$$

Expanding S about the stationary phase point, keeping up to Gaussian fluctuations and Legendre transforming one has the effective action to $1/N$ is

$$\frac{\Gamma}{N} = \int dx \mathcal{L}_{cl}[\phi,\rho,\chi] + \frac{i}{2} Tr \ln \mathcal{G}^{-1}[x,y,\chi]\} + \frac{i}{2N} Tr \ln D^{-1}[x,y,\chi,\rho,\phi]\} \tag{89}$$

and the inverse matrix propagator $D^{-1}$is defined in terms of the second derivatives of the effective action at the stationary phase point with respect to the fields ρ and χ.

In 1+1 dimensions, $\Sigma(q)$ is finite and (for the case of zero classical field) given by the integral

$$\Sigma(q;\chi) = \frac{1}{i}\int \frac{d^2p}{(2\pi)^2} \frac{1}{(-p^2+\chi)(-(p-q)^2+\chi)} = \int \frac{d^2p_E}{(2\pi)^2} \frac{1}{(p_E^2+\chi)(p_E-q_E)^2+\chi)} \tag{90}$$

where in the last equality we have made the Wick rotation $p_0 \to ip_{0E}$. The value of $\Sigma(0)$ is important in studying the critical points of the theory where the determinant of the quadratic fluctuations around the mean field becomes zero and one has a massless excitation. At that point the large-N expansion breaks down and a resummation technique is needed as discussed in [4]. Explicitly

$$\Sigma(q=0,\chi) = \frac{1}{4\pi\chi}. \tag{91}$$

and the condition for the determinant to vanish becomes

$$\Sigma(0)V''[\rho] + \frac{1}{2} = 0. \tag{92}$$

A. ϕ^6 model at large-N

We start with the usual Lagrangian for the ϕ^6 model is (see [7])

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 - \frac{1}{2}\mu^2 \vec{\phi}\cdot\vec{\phi} - \frac{\lambda_0}{4N}(\vec{\phi}\cdot\vec{\phi})^2 - \frac{\eta}{6N^2}(\vec{\phi}\cdot\vec{\phi})^3 \tag{93}$$

and $i = 1 \ldots N$.

Introducing the functional δ function that $\rho = \frac{\vec{\phi}\cdot\vec{\phi}}{N}$ we get the second form of the Lagrangian

$$\frac{\mathcal{L}}{N} = \frac{1}{2N}(\partial_\mu \phi_i)^2 - \frac{1}{2}\mu^2\rho - \frac{\lambda_0}{4}\rho^2 - \frac{\eta}{6}\rho^3 + \frac{\chi}{2}(\rho - \frac{\vec{\phi}\cdot\vec{\phi}}{N}). \tag{94}$$

The effective potential obtained at leading order in large N for ϕ^6 theory written in terms of the auxiliary fields χ, ρ is simply (scaling out the N)

$$V_{eff} = \frac{1}{2}\chi(\phi^2 - \rho) + \frac{1}{2}\mu^2\rho + \frac{\lambda_0}{4}\rho^2 + \frac{\eta}{6}\rho^3 + \frac{1}{2iN}\mathrm{trln}\mathcal{G}^{-1}[\chi] \tag{95}$$

where here $\mathcal{G}$ is the covariant two time Green function

$$\mathcal{G}^{-1}(x, y; \chi) = [\Box + \chi(x)]\delta^2(x - y). \tag{96}$$

The gap equation is obtained from

$$\frac{\partial V_{eff}}{\partial \rho} = \frac{1}{2}[\mu^2 - \chi + \lambda_0\rho + \eta\rho^2] = 0 \tag{97}$$

and ρ can be eliminated in favor of χ using

$$\frac{\partial V_{eff}}{\partial \chi} = 0 \rightarrow \rho = \phi^2 + \frac{1}{i}\mathcal{G}(x, x; \chi) \tag{98}$$

However, $\frac{1}{i}\mathcal{G}(x, x; \chi) = G(x, x; \chi) = \langle\{\phi(x) - \langle\phi(x)\rangle\}^2\rangle$ where $G(x, y)$ was defined earlier in the variational approach. This is seen by considering

$$\frac{1}{i}\mathcal{G}(x, x; \chi) = \frac{1}{(2\pi)^2}\int\frac{d^2k_E}{(k_E^2 + \chi)} \equiv \frac{1}{(2\pi)}\int\frac{dk}{2\sqrt{k^2 + \chi}} \tag{99}$$

Once we rewrite V_{eff} solely in terms of χ as $V_{eff}[\phi^2, \chi, \rho[\chi]]$, then we obtain that

$$\frac{\partial V_{eff}}{\partial \chi}|_\phi = \frac{\partial V}{\partial \rho}\frac{\partial \rho}{\partial \chi} \tag{100}$$

This yields the unrenormalized equation:

$$\frac{\partial V_{eff}}{\partial \chi}|_\phi = \frac{1}{2}[\mu^2 - \chi + \lambda_0\rho + \eta\rho^2][-\frac{1}{4\pi\chi}] \tag{101}$$

using the same mass and coupling constant renormalization as in the variational aproach we get exactly the same result:

$$\frac{\partial V_{eff}}{\partial \chi} = \frac{1}{8\pi\chi}\left[\chi - m^2 - \lambda_r\left(\phi^2 + \frac{1}{4\pi}\ln[\frac{m^2}{\chi}]\right) - \eta\left(\phi^2 + \frac{1}{4\pi}\ln[\frac{m^2}{\chi}]\right)^2\right] \tag{102}$$

Integrating this with respect to χ we obtain the renormalized effective potential:

$$\begin{aligned} V_{eff}[\phi, \chi] \quad &= \frac{1}{2}m^2\phi^2 + \frac{1}{4}\lambda_r\phi^4 + \frac{1}{6}\eta\phi^6 \\ &\frac{1}{8\pi}[\chi - m^2 + (m^2 + \lambda_r\phi^2 + \eta\phi^4)\ln\frac{m^2}{\chi}] \\ &\frac{1}{64\pi^2}(\lambda_r + 2\eta\phi^2)\ln^2\frac{m^2}{\chi} + \frac{1}{384\pi^3}\eta\ln^3\frac{m^2}{\chi}. \end{aligned} \tag{103}$$

We can rewrite this as:

$$V_{eff}[\phi,\chi] = \frac{1}{8\pi}[\chi - m^2] + \frac{1}{2}m^2[\phi^2 + G_r[\chi]] + \frac{1}{4}\lambda_r[\phi^2 + G_r]^2 + \frac{1}{6}\eta[\phi^2 + G_r]^3 \tag{104}$$

This is exactly what we got from the variational method. However now we can systematically improve on the variational result in terms of a series in $1/N$.

To study the critical behavior of ϕ^6 and the stability of the large-N expansion one needs to study the condition for the determinant of the Gaussian fluctuations to vanish, namely:

$$\Sigma(0)V''[\rho] + \frac{1}{2} = 0. \tag{105}$$

where

$$\Sigma(q=0,\chi) = \frac{1}{4\pi\chi}. \tag{106}$$

Here the potential in terms of ρ is given by

$$V[\rho] = \frac{\mu^2}{2}\rho + \frac{\lambda_0}{4}\rho^2 + \frac{\eta}{6}\rho^3 \tag{107}$$

The second derivative is to be evaluated at the stationary point where $\chi = m^2[\phi]$, $\rho = \phi^2 + G[\chi,\phi]$

We can rewrite everything in terms of renormalized parameters by having $\mu \to m$; $\lambda_0 \to \lambda_r$ and $\rho -> \phi^2 + G_r$ where everything is normal ordered with respect to the renormalized mass of the ϕ field m. Taking the second derivative at the stationary phase point we obtain when $\phi \to 0$ and $\chi \to m^2$ that (since $G_r[\chi = m] = 0$)

$$V''[\rho] = \lambda_r + \eta\rho \to \lambda_r \tag{108}$$

. Thus the critical condition for the existence of a zero in the inverse propagator is

$$\frac{\lambda_r}{4\pi m^2} + \frac{1}{2} = 0. \tag{109}$$

In ϕ^6 we can have the renormalized λ_r negative and still have a positive definite theory as long as $\eta > 0$.

B. O(N) Sine-Gordon

Here the Lagrangian divided by N written in terms of the auxiliary fields χ, ρ is given by

$$\frac{1}{2N}[\partial_\mu \phi_i]^2 - \frac{1}{2N}\chi\vec{\phi}\cdot\vec{\phi} - V[\rho] \tag{110}$$

where

$$V[\rho] = -\frac{\alpha_0}{\beta^2}\cos\beta\sqrt{\rho} \tag{111}$$

Following what we did for ϕ^6 field theory we have that the effective potential is now:

$$V_{eff} = \frac{1}{2}\chi(\phi^2 - \rho) - \frac{\alpha_0}{\beta^2}\cos\beta\sqrt{\rho} + \frac{1}{2i}Tr\mathrm{Ln}\mathcal{G}^{-1}[\chi] \tag{112}$$

where again $\mathcal{G}$ is the covariant Green's function with

$$\mathcal{G}^{-1} = [\Box + \chi]\delta^2(x-y). \tag{113}$$

The gap equation is obtained from

$$\frac{\partial V_{eff}}{\partial \rho} = \frac{1}{2}[\alpha_0 \frac{\sin\beta\sqrt{\rho}}{\beta\sqrt{\rho}} - \chi] = 0 \tag{114}$$

and ρ again can be eliminated in favor of χ using

$$\frac{\partial V_{eff}}{\partial \chi} = 0 \rightarrow \rho = \phi^2 + \frac{1}{i}\mathcal{G}(x,x;\chi) \tag{115}$$

As in the polynomial potential cases, we can render the theory finite by just regulating ρ with respect to an arbitrary mass parameter m_1

$$\rho_R = \phi^2 + \frac{1}{i}[\mathcal{G}(x,x;\chi) - \mathcal{G}(x,x;m_1^2)] \tag{116}$$

so that

$$\rho_R = \phi^2 + \frac{1}{4\pi}\ln[\frac{m_1^2}{\chi}] \tag{117}$$

The gap equation for χ at $\phi = 0$ gives the renormalized mass m^2:

$$m^2 = \alpha_0 \frac{\sin\left(\beta\sqrt{\frac{1}{4\pi}\ln[\frac{m_1^2}{m^2}]}\right)}{\beta\sqrt{\frac{1}{4\pi}\ln[\frac{m_1^2}{m^2}]}} \tag{118}$$

so that choosing our subtraction point to be $m_1 = m$ we obtain that

$$\alpha_0 = m^2 \tag{119}$$

Next we have again

$$\frac{\partial V}{\partial \chi}|_\phi = \frac{\partial V}{\partial \rho}\frac{\partial \rho}{\partial \chi} \tag{120}$$

Using our renormalization scheme $\rho \to \rho_R$, we obtain

$$\frac{\partial V_{eff}}{\partial \chi} = \frac{1}{8\pi\chi}\left[\chi - m^2 \frac{\sin\left(\beta\sqrt{\phi^2 + \frac{1}{4\pi}\ln[\frac{m^2}{\chi}]}\right)}{\beta\sqrt{\phi^2 + \frac{1}{4\pi}\ln[\frac{m^2}{\chi}]}}\right] \tag{121}$$

Integrating this with respect to χ gives the renormalized effective potential

$$V[\phi,\chi] = \frac{1}{8\pi}[\chi - m^2] - \frac{m^2}{\beta^2}\cos\left(\beta\sqrt{\phi^2 + \frac{1}{4\pi}\ln[\frac{m^2}{\chi}]}\right) \tag{122}$$

which again displays the usual result at large-N that the renormalized expectation of the Classical potential has the property:

$$\langle V[\phi^2]\rangle \to V[\phi^2 + G_r[\chi]] \tag{123}$$

The second derivative of V_{eff} with respect to χ at the minimum $\chi = m^2$ (for the unbroken vacuum $\phi = 0$) is

$$-\frac{\beta^2 - 24\pi}{192\pi^2\chi}. \tag{124}$$

Thus the vacuum is stable as long as

$$\beta^2 \leq 24\pi. \tag{125}$$

Let us now show that this result is the same as demanding that the large-N expansion be an expansion about a minimum by considering the determinant of the fluctuations around the minimum. Critical behavior is given by the vanishing of the determinant (Eq. 92). We have, using our renormalization scheme,

$$V[\rho] = -\frac{\alpha_0}{\beta^2}\cos\beta\sqrt{\rho} \to V_r[\rho_r] \tag{126}$$

where now

$$V_r[\rho_r] = -\frac{m^2}{\beta^2}\cos\beta\sqrt{\rho_r} \tag{127}$$

and $\rho_r = \phi^2 + G_r$ with the subtraction being made at the physical mass m. Taking two derivatives we find that

$$V''[\rho] = m^2\frac{\beta^2}{4x^2}\left[\cos x - \frac{\sin x}{x}\right] \tag{128}$$

where $x = \beta\sqrt{\rho_r}$, $\rho_r = \frac{1}{4\pi}\ln\frac{m^2}{\chi}$ At the stationary phase pont we need the limit of V'' as $x \to 0$ This is

$$V''[\rho] \to -m^2\frac{\beta^2}{12} \tag{129}$$

The condition for the determinant of the fluctuations to be positive is

$$\Sigma(0)V''[\rho] + \frac{1}{2} > 0, \tag{130}$$

where

$$\Sigma(q = 0, \chi) = \frac{1}{4\pi\chi}. \tag{131}$$

Eq. 130 leads to the condition (when $\chi \to m^2$)

$$\beta^2 \leq 24\pi. \tag{132}$$

This is the same result as that we found for the stability of the vacuum

To go beyond the lowest order approximation one first includes the Gaussian fluctuations and considers ρ and χ as components of a new two component field with matrix inverse propagator D^{-1}. The one particle irreducible generating functional now gets a contribution

$$\frac{i}{2N}TrLn[D^{-1}] \tag{133}$$

However this naive $1/N$ expansion is secular for time evolution problems as shown in [9]. A better approximation which avoids secularity is a resummed next to leading order in 1/N expansion obtained from the two-particle irreducible formalism [10] written in terms of the fields ϕ, χ, ρ and propagators $\mathcal{G}$ and D. The self-consistent $1/N$ correction is found from the Graph in $\Gamma_2[\Phi, \mathcal{G}]$

$$\Gamma_2 = \frac{i}{4}Tr\int dxdy\mathcal{G}(x,y)D(x,y)\mathcal{G}(y,x) \tag{134}$$

which will give the same Schwinger Dyson equations as the standard 1-PI 1/N expansion except with FULL propagators and not leading order ones in the self energy graphs. This is discussed in detail in [11] [12].

IV. CONCLUSIONS

In this paper we have shown how to generalize the Sine-Gordon equation to the case of O(N) symmetry. This then allows us to go beyond the Gaussian approximation (at large-N) in a systematic, controlable way. We found that at leading order in large N, the pole in the propagator looks functionally different from that found in the Hartree approximation for $N = 1$. Thus the combinatorics of large-N for non-polynomial potentials leads to expressions for the evolution of the one and two point function which look qualitatively different from what is found for polynomial potentials. We also found that the stability of the mean-field vacuum found using a Hamiltonian approach leads to the same condition as the stability of the large-N approximation found by studying Gaussian fluctuations about the leading order. We briefly discussed the fact that the naive $1/N$ expansion is secular beyond leading order and has to be replaced by a resummed $1/N$ expansion obtainable from a two particle irreducibility approach.

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