Relativity of $\hbar$

José M. Isidro
Department of Theoretical Physics,
1 Keble Road, Oxford OX1 3NP, UK
and
Instituto de Física Corpuscular (CSIC-UVEG)
Apartado de Correos 22085, 46071 Valencia, Spain
isidro@thphys.ox.ac.uk

April 25, 2020

Abstract

Looking for a quantum–mechanical implementation of duality, we formulate a relation between coherent states and complex–differentiable structures on classical phase space $\mathcal{C}$. A necessary and sufficient condition for the existence of locally–defined coherent states is the existence of an almost complex structure on $\mathcal{C}$. A necessary and sufficient condition for globally–defined coherent states is a complex structure on $\mathcal{C}$.

The picture of quantum mechanics that emerges is conceptually close to that of a geometric manifold covered by local coordinate charts. Instead of the latter, quantum mechanics has local coherent states. A change of coordinates on $\mathcal{C}$ may or may not be holomorphic. Correspondingly, a transformation between quantum–mechanical states may or may not preserve coherence. Those that do not preserve coherence are duality transformations. A duality appears as the possibility of giving two or more, apparently different, descriptions of the same quantum–mechanical phenomenon. Coherence becomes a local property on classical phase space. Observers on $\mathcal{C}$ not connected by means of a holomorphic change of coordinates need not, and in general will not, agree on what is a semiclassical effect *vs.* what is a strong quantum effect.

Keywords: Coherent states, almost complex manifolds, duality.
2001 Pacs codes: 03.65.Bz, 03.65.Ca, 03.65.-w.

Contents

1 Introduction

1.1 Motivation

Coherent states are quantum–mechanical states that enjoy semiclassical properties [1]. One definition of coherent states uses the Heisenberg inequality $\Delta q\Delta p \geq \hbar/2$. The latter is saturated precisely by coherent states. Planck's constant $\hbar$ can be interpreted as a parameter measuring how far quantum mechanics deviates from classical mechanics.

Along a different line of developments, the notion of duality plays a key role in recent breakthroughs in the quantum theories of fields, strings and branes [2, 3, 4]. Broadly speaking, under duality one understands a transformation of a given theory, in a certain regime of the variables and parameters that define it, into a physically equivalent theory with different variables and parameters. Often, what appears to be a highly nontrivial quantum excitation in a given theory turns out to be a simple perturbative correction from the viewpoint of a theory dual to the original one. This suggests that what constitutes a quantum correction may be a matter of convention: the notion of classical *vs.* quantum is relative to which theory the measurement is made from. In this way we arrive at the conclusion that may $\hbar$ depend on the observer. This, in turn, implies that coherence is also theory–dependent, or observer–depedent. What one observer calls coherent need not be coherent to another observer.

The standard formulation of quantum mechanics [5] does not allow for such a relativity of the concept of a quantum. Somewhat imprecisely, we will call this concept *relativity of* $\hbar$. The limit $\hbar \to 0$ is the *semiclassical regime*, and the limit $\hbar \to \infty$ is the *strong quantum regime*. Under the usual formulation of quantum mechanics, if one observer calls a certain phenomenon *semiclassical*, then so will it be for all other observers. If one observer calls a certain phenomenon *strong quantum*, then so will it be for all other observers.

In view of these developments, a framework for quantum mechanics is required that can accommodate such a relativity of $\hbar$ [4]. Generalising the geometric approach of ref. [6], in ref. [7] we have explicitly developed one such framework. There remains the alternative, though equivalent, possibility of addressing the relativity of $\hbar$ through an analysis of coherent states. Other geometric approaches to quantum mechanics dealing with this and related topics are refs. [8, 9, 10, 11, 12, 13, 14].

1.2 Summary of main results

The purpose of this article is to present a framework for quantum mechanics in which coherent states are defined locally on classical phase space $\mathcal{C}$, but not necessarily globally. This is an explicit implementation of a relativity for $\hbar$. Globally–defined coherent states are the rule in the standard presentations of quantum mechanics. As such they preclude any possible observer–dependence for $\hbar$.

We will analyse the relationship between complex–differentiable structures on $\mathcal{C}$ and coherent states of the corresponding quantum mechanics. We will elaborate on a suggestion presented in ref. [7], according to which coherent states may not be globally defined for all observers on $\mathcal{C}$, and we will relate this observation to the (in)existence of complex structures on $\mathcal{C}$.

To our purposes it is enough to realise that there exist finite–dimensional symplectic manifolds $\mathcal{C}$ admitting no complex structure [15]. Using the symplectic structure, we will construct local coherent states around a certain point on $\mathcal{C}$. Upon a nonholomorphic change of Darboux coordinates to another point on $\mathcal{C}$, those states will cease to be coherent.

Our results may be summarised as follows. Coherent states can always be defined locally, *i.e.*, in the neighbourhood of any point on $\mathcal{C}$. This is merely a restatement, in physical terms, of Darboux's theorem for symplectic manifolds [16]. When there is a complex structure $\mathcal{J}_{\mathcal{C}}$, coherence becomes a global property on $\mathcal{C}$. In the absence of a complex structure, however, the best we can do is to combine Darboux coordinates q, p as $q+{\rm i}p$. Technically this only defines an almost complex structure $J_{\mathcal{C}}$ on $\mathcal{C}$ [17]. Since the combination $q+{\rm i}p$ falls short of defining a complex structure, quantities depending on $q+{\rm i}p$ on a certain coordinate patch will generally also depend on $q-{\rm i}p$ when transformed to another coordinate patch. This proves that coherence remains a local property on classical phase space: observers not connected by means of a holomorphic change of coordinates need not, and in general will not, agree on what is a semiclassical effect *vs.* what is a strong quantum effect.

1.3 Outline

This article is organised as follows. Section 2 sets the scene with a quick review of the geometry of classical and quantum phase spaces, denoted $\mathcal{C}$ and $\mathcal{Q}$ respectively. The construction of coherent states from a complex structure $\mathcal{J}_{\mathcal{C}}$ on $\mathcal{C}$ is recalled in section 3; further issues discussed in this section are the uniqueness of the vacuum and the global character of the coherent states so constructed. In section 4 we relax the complex structure $\mathcal{J}_{\mathcal{C}}$ to an almost complex structure $J_{\mathcal{C}}$. In so doing we observe that the vacuum state is only locally defined on $\mathcal{C}$ and that, under a nonholomorphic change of coordinates on $\mathcal{C}$, global coherence is lost. In section 5 we prove the coherence of the global states constructed in section 3, and the coherence of the local states constructed in section 4.

The conditions under which a complex structure $J_{\mathcal{C}}$ lifts to a complex structure $\mathcal{J}_{\mathcal{C}}$ are known in the mathematical literature; we recast them in physical terms in section 6. In section 7 we introduce the concept of a holomorphic foliation $\mathcal{F}$ of $\mathcal{C}$. The leaves of $\mathcal{F}$ are the maximal holomorphic submanifolds of $\mathcal{C}$ on which coherence remains a globally–defined property. Nonholomorphic coordinate changes between different holomorphic leaves of $\mathcal{F}$ are interpreted as duality transformations of the quantum theory on $\mathcal{Q}$. Some geometric examples serve to illustrate our proposal. Finally, section 8 discusses some physical and mathematical aspects of our construction.

2 Classical and quantum phase spaces

Our starting point is an infinite–dimensional, complex, separable Hilbert space of quantum states, $\mathcal{H}$, that is most conveniently viewed as a real vector space equipped with a complex structure $\mathcal{J}$. Correspondingly, the Hermitian inner product can be decomposed into real and imaginary parts,

$$\langle\phi,\psi\rangle = g(\phi,\psi) + {\rm i}\omega(\phi,\psi), \tag{1}$$

with g a positive–definite, real scalar product and ω a symplectic form. The metric g, the symplectic form ω and the complex structure $\mathcal{J}$ are related as

$$g(\phi,\psi) = \omega(\phi,\mathcal{J}\psi), \tag{2}$$

which means that the triple $(\mathcal{J}, g, \omega)$ endows the Hilbert space $\mathcal{H}$ with the structure of a Kähler space [17].

Let $\mathcal{Q}$ denote the space of unit rays in $\mathcal{H}$, and let $\omega_{\mathcal{Q}}$ be the restriction to $\mathcal{Q}$ of the symplectic form ω on $\mathcal{H}$. The space $\mathcal{Q}$, called *quantum phase space*, is an infinite–dimensional symplectic manifold. Classical phase space $\mathcal{C}$ is a $2n$–dimensional symplectic manifold; let $\omega_{\mathcal{C}}$ denote its symplectic form. If q^l, p_l, $l = 1,\ldots,n$, are local Darboux coordinates on $\mathcal{C}$, we have

$$\omega_{\mathcal{C}} = \sum_{l=1}^{n} {\rm d}p_l \wedge {\rm d}q^l. \tag{3}$$

It should be observed that, while both $\mathcal{C}$ and $\mathcal{Q}$ are symplectic manifolds, the latter is always Kähler, while the former need not be Kähler. However, in the framework of geometric [18] and deformation [19, 20, 21] quantisation it is customary to consider the case when $\mathcal{C}$ is a compact Kähler manifold. In this context one introduces the notion of a quantisable, compact, Kähler phase space $\mathcal{C}$. This means that there exists an associated quantum line bundle (L, g, ∇) on $\mathcal{C}$, where L is a holomorphic line bundle, g a Hermitian metric on L, and ∇ a connection compatible with the complex structure and the Hermitian metric. Furthermore, the curvature F of the connection ∇ and the Kähler form $\omega_{\mathcal{C}}$ are required to satisfy

$$F = -{\rm i}\omega_{\mathcal{C}}. \tag{4}$$

It turns out that quantisable, compact Kähler manifolds are projective algebraic manifolds and viceversa [20]. For the purpose of introducing duality transformations, however, the previous assumptions are too restrictive. At most we will require $\mathcal{C}$ to support a complex structure $J_{\mathcal{C}}$ compatible with the symplectic structure $\omega_{\mathcal{C}}$, as in the next section.

3 Global coherent states from a complex structure

Let us assume that $\mathcal{C}$ admits a complex structure $\mathcal{J}_{\mathcal{C}}$. Furthermore let $\mathcal{J}_{\mathcal{C}}$ be compatible with the symplectic structure $\omega_{\mathcal{C}}$. This means that the real and imaginary parts of the holomorphic coordinates z^l for $\mathcal{J}_{\mathcal{C}}$ are Darboux coordinates for the symplectic form $\omega_{\mathcal{C}}$:

$$z^l = q^l + \mathrm{i}p_l, \qquad l = 1, \ldots, n. \tag{5}$$

The set of all z^l so defined provides a holomorphic atlas for $\mathcal{C}$. Upon quantisation, the Darboux coordinates q^l and p_l become operators Q^l and P_l on $\mathcal{H}$ satisfying the Heisenberg algebra

$$[Q^j, P_k] = \mathrm{i}\delta^j_k. \tag{6}$$

Define the annihilation operators

$$A^l = Q^l + \mathrm{i}P_l, \qquad l = 1, \ldots, n. \tag{7}$$

Quantum excitations are measured with respect to a vacuum state $|0\rangle$. The latter is defined as that state in $\mathcal{H}$ which satisfies

$$A^l|0\rangle = 0, \qquad l = 1, \ldots, n, \tag{8}$$

and coherent states $|z^l\rangle$ are eigenvectors of A^l, with eigenvalues given in equation (5) above:

$$A^l|z^l\rangle = z^l|z^l\rangle, \qquad l = 1, \ldots, n. \tag{9}$$

How do the vacuum state $|0\rangle$ and the coherent states $|z^l\rangle$ transform under a canonical coordinate transformation on $\mathcal{C}$? Call the new coordinates q'^l, p'_l. As they are Darboux, they continue to satisfy equation (3). Upon quantisation the corresponding operators Q'^l, P'_l continue to satisfy the Heisenberg algebra (6). Then the combinations

$$z'^l = q'^l + \mathrm{i}p'_l, \qquad l = 1, \ldots, n \tag{10}$$

continue to provide holomorphic coordinates for $\mathcal{C}$, and the transformation between the z^l and the z'^l is given by an n–variable holomorphic function f,

$$z' = f(z), \qquad \bar\partial f = 0. \tag{11}$$

We can write as above

$$A'^l = Q'^l + \mathrm{i}P'_l, \qquad l = 1, \ldots, n, \tag{12}$$

$$A'^l|0\rangle = 0, \qquad l = 1, \ldots, n, \tag{13}$$

$$A'^l|z'^l\rangle = z'^l|z'^l\rangle, \qquad l = 1, \ldots, n. \tag{14}$$

There is no physical difference between equations (7), (8) and (9), on the one hand, and their holomorphic transforms (12), (13) and (14), on the other. Under the transformation (11), the vacuum state $|0\rangle$ is mapped into itself, and the coherent states $|z^l\rangle$ are mapped into the coherent states $|z'^l\rangle$. Therefore the notion of coherence is global for all observers on $\mathcal{C}$, *i.e.*, any two observers will agree on what is a coherent state *vs.* what is a noncoherent state. A consequence of this fact is the following. Under symplectomorphisms (or, equivalently, holomorphic diffeomorphisms) of $\mathcal{C}$, the semiclassical regime of the quantum theory on $\mathcal{Q}$ is mapped into the semiclassical regime, and the strong quantum regime is mapped into the strong quantum regime.

Conversely, one can reverse the order of arguments in this section. Start from the assumption that one can define global coherent states $|z^l\rangle$ and a global vacuum $|0\rangle$ on the symplectic manifold $\mathcal{C}$. *Globality* here does not mean that one can cover all of $\mathcal{C}$ with just one coordinate chart (which is impossible if $\mathcal{C}$ is compact). Rather it means that, under all symplectomorphisms of $\mathcal{C}$, the vacuum is mapped into itself, and coherent states are always mapped into coherent states. Then the coordinates z^l defined by the eigenvalue equations (9) provide a local chart for $\mathcal{C}$. Collecting together the set of all such possible local charts we obtain an atlas for $\mathcal{C}$. This atlas is holomorphic thanks to the property of globality.

To summarise, the existence of a complex structure $\mathcal{J}_{\mathcal{C}}$ is equivalent to the existence of a globally defined vacuum and globally defined coherent states.

4 Local coherent states from an almost complex structure

We now relax the conditions imposed on $\mathcal{C}$. In this section we will assume that $\mathcal{C}$ carries an almost complex structure $J_{\mathcal{C}}$ compatible with the symplectic structure $\omega_{\mathcal{C}}$. (When $\mathcal{C}$ is compact, the symplectic structure $\omega_{\mathcal{C}}$ automatically leads to an almost complex structure $J_{\mathcal{C}}$, so the assumption of the existence of a $J_{\mathcal{C}}$ compatible with $\omega_{\mathcal{C}}$ is in fact unnecessary; in that case, the symplectic structure $\omega_{\mathcal{C}}$ suffices [15]).

Specificallly, an almost complex structure is defined as a tensor field $J_{\mathcal{C}}$ of type $(1,1)$ such that, at every point of $\mathcal{C}$, $J_{\mathcal{C}}^2 = -\mathbf{1}$ [17]. Using Darboux coordinates q^l, p_l on $\mathcal{C}$ let us form the combinations

$$w^l = q^l + \mathrm{i} p_l, \qquad l = 1, \ldots, n. \tag{15}$$

Compatibility between $\omega_{\mathcal{C}}$ and $J_{\mathcal{C}}$ means that we can take $J_{\mathcal{C}}$ to be

$$J_{\mathcal{C}}\left(\frac{\partial}{\partial w^l}\right) = \mathrm{i}\frac{\partial}{\partial w^l}, \qquad J_{\mathcal{C}}\left(\frac{\partial}{\partial \bar{w}^l}\right) = -\mathrm{i}\frac{\partial}{\partial \bar{w}^l}. \tag{16}$$

Unless $\mathcal{C}$ is a complex manifold to begin with, equations (15) and (16) fall short of defining a complex structure $\mathcal{J}_{\mathcal{C}}$. The set of all such w^l does not provide a holomorphic atlas for $\mathcal{C}$. There exists at least one canonical coordinate transformation between Darboux coordinates, call them (q^l, p_l) and (q'^l, p'_l), such that the passage between $w^l = q^l + \mathrm{i}p_l$ and $w'^l = q'^l + \mathrm{i}p'_l$ is given by a nonholomorphic function g in n variables,

$$w' = g(w, \bar{w}), \qquad \bar{\partial} g \neq 0. \tag{17}$$

Mathematically, nonholomorphicity implies the mixing of w^l and $\bar{w}^l$. Quantum–mechanically, the loss of holomorphicity has deep physical consequences.

One would write, in the initial coordinates w^l, a defining equation for the vacuum state $|0\rangle$

$$a^l|0\rangle = 0, \qquad l = 1, \ldots, n, \tag{18}$$

where $a^l = Q^l + \mathrm{i}P_l$ is the corresponding local annihilation operator. However, one is just as well entitled to use the new coordinates w'^l and write

$$a'^l|0'\rangle = 0, \qquad l = 1, \ldots, n, \tag{19}$$

where we have primed the new vacuum, $|0'\rangle$. Are we allowed to identify the states $|0\rangle$ and $|0'\rangle$? We could identify them if w'^l were a holomorphic function of w^l; such was the case in section 3. However, now we are considering a nonholomorphic transformation, and we cannot remove the prime from the state $|0'\rangle$. This is readily proved. We have

$$a' = G(a, a^\dagger), \tag{20}$$

with G a quantum nonholomorphic function corresponding to the classical nonholomorphic function g of equation (17). As $[a^j, a_k^\dagger] = \delta_k^j$, ordering ambiguities will arise in the construction of G from g, that are usually dealt with by normal ordering. Normal ordering would appear to allow us to identify the states $|0\rangle$ and $|0'\rangle$. However this is not the case, as there are choices of g that are left invariant under normal ordering, such as the sum of a holomorphic function plus an antiholomorphic function, $g(w, \bar{w}) = g_1(w) + g_2(\bar{w})$. Under such a transformation one can see that the state $|0\rangle$ satisfying eqn. (18) will not satisfy eqn. (19). We conclude that, in the absence of a complex structure on classical phase space, the vacuum depends on the observer. The state $|0\rangle$ is only defined locally on $\mathcal{C}$; it cannot be extended globally to all of $\mathcal{C}$.

Similar conclusions may be expected for the coherent states $|w^l\rangle$. The latter are defined only locally, as eigenvectors of the local annihilation operator, with eigenvalues given in equation (15):

$$a^l|w^l\rangle = w^l|w^l\rangle, \qquad l = 1, \ldots, n. \tag{21}$$

Due to $[a^j, a_k^\dagger] = \delta_k^j$, under the nonholomorphic coordinate transformation (17), the local coherent states $|w^l\rangle$ are *not* mapped into the local coherent states satisfying

$$a'^l|w'^l\rangle = w'^l|w'^l\rangle, \qquad l = 1, \ldots, n \tag{22}$$

in the primed coordinates. No such problems arose for the holomorphic operator equation $A' = F(A)$ corresponding to the holomorphic coordinate change $z' = f(z)$ of equation (11), because the commutator $[A^j, A_k^\dagger] = \delta_k^j$ played no role. Thus coherence becomes a local property on classical phase space. In particular, observers not connected by means of a holomorphic change of coordinates need not, and in general will not, agree on what is a semiclassical effect *vs.* what is a strong quantum effect.

As in section 3, one can reverse the order of arguments. Start from the assumption that, around every point on $\mathcal{C}$, one can define local coherent states $|w^l\rangle$ and a local vacuum $|0\rangle$, that however fall short of being global. This means that there exists at least one symplectomorphism of $\mathcal{C}$ that does *not* preserve the globality property. Local coordinates w^l around any point are defined by the eigenvalue equations (21). Collecting together the set of all such possible local charts we obtain an atlas for $\mathcal{C}$. However, unless the local coherent states $|w^l\rangle$ are actually global, this atlas is nonholomorphic. This defines an almost complex structure $J_{\mathcal{C}}$.

To summarise, the existence of an almost complex structure $J_{\mathcal{C}}$ is equivalent to the existence of a locally–defined vacuum and locally–defined coherent states. When the latter are actually global, then $J_{\mathcal{C}}$ lifts to a complex structure $\mathcal{J}_{\mathcal{C}}$, whose associated almost complex structure is $J_{\mathcal{C}}$ itself.

5 Proof of coherence

We have called *coherent* the states constructed in previous sections. However, we have not verified that they actually satisfy the usual requirements imposed on coherent states [1]. What ensures that the states so constructed are actually coherent is the following argument.

We have made no reference to coupling constants or potentials, with the understanding that the Hamilton–Jacobi method has already placed us, by means of suitable coordinate transformations, in a coordinate system on $\mathcal{C}$ where all interactions vanish. At least under the standard notions of classical *vs.* quantum, this is certainly always possible at the classical level. At the quantum level, the approach of ref. [9], which contains the standard quantum mechanics used here as a limiting case, rests precisely on the possibility of transforming between any two quantum–mechanical states by means of diffeomorphisms.

Then any dynamical system with n independent degrees of freedom that can be transformed into the freely–evolving system can be further mapped into the n–dimensional harmonic oscillator. The combined transformation is canonical. Moreover it is locally holomorphic when $\mathcal{C}$ is an almost complex manifold. Thus locally on $\mathcal{C}$, our global states $|z^l\rangle$ of section 3 coincide with the coherent states of the n–dimensional harmonic oscillator. Mathematically this fact reflects the structure of a complex manifold: locally it is always holomorphically diffeomorphic to (an open subset of) $\mathbf{C}^n$. Physically this fact reflects the decomposition into the creation and annihilation modes of perturbative quantum mechanics and field theory. In this way, the mathematical problem of patching together different local coordinate charts (U_α, z^l_α) labelled by an index α may be recast in physical terms. It is the patching together of different local expansions into creators $A^\dagger_\alpha$ and annihilators A_α, for different values of α.

In particular, we can write the resolution of unity on $\mathcal{H}$ associated with a holomorphic atlas on $\mathcal{C}$ consisting of charts (U_α, z^l_α):

$$\sum_\alpha \sum_{l=1}^n \int_{\mathcal{C}} \mathrm{d}\mu_{\mathcal{C}}\, |z^l_\alpha\rangle\langle z^l_\alpha| = \mathbf{1}, \tag{23}$$

where $\mathrm{d}\mu_{\mathcal{C}}$ is an appropriate measure (an (n,n)–differential) on $\mathcal{C}$.

Analogous arguments are also applicable to the local states $|w^l\rangle$ of section 4. In particular, every almost complex manifold is locally a complex manifold [17]. Every holomorphic coordinate chart on $\mathcal{C}$ is diffeormorphic to (an open subset of) $\mathbf{C}^n$, so the $|w^l\rangle$ look locally like the coherent states of the n–dimensional harmonic oscillator. However, the loss of holomorphicity of $\mathcal{C}$ alters equation

(23) in one important way. We may write as above

$$\sum_{\alpha} \sum_{l=1}^{n} \int_{\mathcal{C}} \mathrm{d}\mu_{\mathcal{C}} \, |w_{\alpha}^{l}\rangle\langle w_{\alpha}^{l}|, \tag{24}$$

but we can no longer equate this to the identity on $\mathcal{H}$. The latter is a *complex* vector space, while eqn. (24) allows one at most to expand an arbitrary, real–analytic function on $\mathcal{C}$, since the latter is just a real–analytic manifold. Hence we cannot equate (24) to $\mathbf{1}_{\mathcal{H}}$. We can only equate it to the identity on the *real* Hilbert space of real–analytic functions on $\mathcal{C}$. This situation is not new; coherent states without a resolution of unity have been analysed in ref. [22], where they have been related to the choice of an inadmissible fiducial vector. It is tempting to equate this latter choice with the viewpoint advocated here about the vacuum state.

6 When are local coherent states also global?

This question can be recast mathematically as follows: when does an almost complex structure $J_{\mathcal{C}}$ lift to a complex structure $\mathcal{J}_{\mathcal{C}}$?

6.1 The Newlander–Nirenberg theorem

The almost complex structure $J_{\mathcal{C}}$ is said *integrable* when the Lie bracket $[Z,W]$ of any two holomorphic vector fields Z, W on $\mathcal{C}$ is holomorphic. A necessary and sufficient condition for $J_{\mathcal{C}}$ to be integrable is the following. Define the tensor field N

$$N(Z,W) = [Z,W] - [J_{\mathcal{C}}Z, J_{\mathcal{C}}W] + J_{\mathcal{C}}\,[Z, J_{\mathcal{C}}W] + J_{\mathcal{C}}\,[J_{\mathcal{C}}Z, W]\,. \tag{25}$$

Now the almost complex structure $J_{\mathcal{C}}$ lifts to a complex structure $\mathcal{J}_{\mathcal{C}}$ if and only if the tensor N vanishes identically [23].

6.2 Integrable, almost complex structures

When $J_{\mathcal{C}}$ is integrable, the set of all holomorphic vector fields defines an integrable holomorphic distribution whose integral manifold is $\mathcal{C}$ itself [17]. A knowledge of this integrable distribution of holomorphic vector fields amounts to determining the manifold $\mathcal{C}$. Let us see how this comes about. Assume for simplicity that $\mathcal{C}$ is connected, and let (U_b, ϕ_b) be a holomorphic chart centred around a basepoint $b\in\mathcal{C}$. Such a holomorphic chart always exists locally. The map

$$\phi_b : U_b \to \mathbf{C}^n \tag{26}$$

provides local holomorphic coordinates around b whose real and imaginary parts can be taken to be Darboux coordinates (q^l, p_l), thanks to the assumption of compatibility between $J_{\mathcal{C}}$ and $\omega_{\mathcal{C}}$. Let Z be a holomorphic vector field defined on U_b. We can interpret Z as mapping the chart (U_b, ϕ_b) into another chart $(U_{Z(b)}, \phi_{Z(b)})$ centred around an infinitesimally close basepoint $Z(b)$. Similarly let W be another holomorphic vector field mapping (U_b, ϕ_b) into $(U_{W(b)}, \phi_{W(b)})$. Integrability of $J_{\mathcal{C}}$ means that the Lie bracket $[Z,W]$ maps (U_b, ϕ_b) into another holomorphic chart $(U_{[Z,W](b)}, \phi_{[Z,W](b)})$. Were $J_{\mathcal{C}}$ not integrable, there would exist a holomorphic chart (U_c, ϕ_c) centred around a basepoint $c \in \mathcal{C}$, and pair of holomorphic vector fields Z, W on U_c, such that the chart $(U_{[Z,W](c)}, \phi_{[Z,W](c)})$ would not be holomorphic.

Proceeding as described when $J_{\mathcal{C}}$ is integrable, we succeed in covering $\mathcal{C}$ with a set of holomorphic charts, the transformations between them being holomorphic symplectic diffeomorphisms. Physically, holomorphicity ensures that the passage from one observer to another respects the globality of the notion of coherence and the uniqueness of the vacuum. In section 7 we will relax the complex structure $\mathcal{J}_{\mathcal{C}}$ to an almost complex structure $J_{\mathcal{C}}$. In so doing we will interpret a nonholomorphic mapping such as

$$(U_c, \phi_c) \to (U_{[Z,W](c)}, \phi_{[Z,W](c)}) \tag{27}$$

as a duality transformation of the quantum theory.

We can turn things around and recast the Newlander–Nirenberg theorem in physical terms: when the commutator of any two (infinitesimal) canonical transformations on $\mathcal{C}$ maps coherent states into coherent states, then $\mathcal{C}$ admits a complex structure. The latter is the lift of the almost complex structure $J_{\mathcal{C}}$ defined by $q^l + \mathrm{i}p_l$ in terms of Darboux coordinates q^l, p_l. Conversely, if a canonical transformation on $\mathcal{C}$ maps coherent states into noncoherent, or viceversa, then $J_{\mathcal{C}}$ does not lift to a complex structure.

7 Duality transformations

When $J_{\mathcal{C}}$ is nonintegrable the above construction breaks down. This gives rise to duality transformations of the quantum theory.

7.1 Holomorphic foliations of classical phase space

Let us consider the case when $\mathcal{C}$ admits a certain foliation $\mathcal{F}$ by holomorphic, symplectic submanifolds $\mathcal{L}$ called *leaves* [24]. For simplicity we will make a number of technical assumptions. First, the leaves $\mathcal{L}$ have constant real dimension $2m$, where $0 < 2m < 2n$; m is called the *rank* of the foliation $\mathcal{F}$. We will use the notation $\tilde{\mathcal{L}}$ to denote the $2(n-m)$–dimensional complement of the $\mathcal{L}$ in $\mathcal{C}$. We will assume maximality of the rank m, *i.e.*, no holomorphic leaf exists with dimension greater than $2m$. Second, we suppose that the restrictions $\omega_{\mathcal{L}}$ and $\omega_{\tilde{\mathcal{L}}}$ of the symplectic form $\omega_{\mathcal{C}}$ render the leaves $\mathcal{L}$ and their complements $\tilde{\mathcal{L}}$ symplectic. Third we assume that, on the $\mathcal{L}$, the complex structure is compatible with the symplectic structure as in section 3. Fourth, the complement $\tilde{\mathcal{L}}$ is also assumed to carry an almost complex structure compatible with $\omega_{\tilde{\mathcal{L}}}$ as in section 4.

All these assumptions amount to a decomposition of $\omega_{\mathcal{C}}$ as a sum of two terms,

$$\omega_{\mathcal{C}} = \omega_{\mathcal{L}} + \omega_{\tilde{\mathcal{L}}}, \tag{28}$$

where in local Darboux coordinates around a basepoint $b \in \mathcal{C}$ we have

$$\omega_{\mathcal{L}} = \sum_{k=1}^{m} \mathrm{d}p_k \wedge \mathrm{d}q^k, \qquad \omega_{\tilde{\mathcal{L}}} = \sum_{j=m+1}^{n} \mathrm{d}p_j \wedge \mathrm{d}q^j. \tag{29}$$

Furthermore the combinations $z^k = q^k + \mathrm{i}p_k$, $k = 1, \ldots, m$, are holomorphic coordinates on the $\mathcal{L}$, while the combinations $w^j = q^j + \mathrm{i}p_j$, $j = m+1, \ldots, n$, are coordinates on $\tilde{\mathcal{L}}$. In this way a set of coordinates around b is

$$z^1, \ldots, z^m, w^{m+1}, \ldots, w^n. \tag{30}$$

The holomorphic leaf $\mathcal{L}$ passing through b may be taken to be determined by

$$w^{m+1} = 0, \ldots, w^n = 0, \tag{31}$$

and spanned by the remaining coordinates z^k, $k = 1, \ldots, m$.

The construction of the previous sections can be applied as follows. Coherent states $|z^k; w^j\rangle$ can be defined locally on $\mathcal{C}$. They cannot be extended globally over all of $\mathcal{C}$, as the latter is not a complex manifold. However the foliation $\mathcal{F}$ consists of holomorphic submanifolds $\mathcal{L}$. On each one of them there exist global coherent states specified by equations (30), (31), *i.e.*,

$$|z^k; w^{m+1} = 0, \ldots, w^n = 0\rangle. \tag{32}$$

Physically, this case corresponds to a fixed splitting of the n degrees of freedom in such a way that the first m of them give rise to global coherent states on the holomorphic leaves $\mathcal{L}$. On the latter there is no room for nontrivial dualities. On the contrary, the last $n-m$ degrees of freedom are only locally holomorphic on $\mathcal{C}$. Holomorphicity is lost globally on $\mathcal{C}$, thus allowing for the possibility of nontrivial duality transformations between different holomorphic leaves $\mathcal{L}$.

Let us analyse the resolution of unity in terms of the states $|z^k; w^j\rangle$. With the notations of section 5, the expansion

$$\sum_\alpha \sum_{k=1}^m \sum_{j=m+1}^n \int_{\mathcal{C}} \mathrm{d}\mu_{\mathcal{C}}\, |z^k_\alpha; w^j_\alpha\rangle\langle w^j_\alpha; z^k_\alpha| \tag{33}$$

cannot be equated to the identity, for the same reasons as in section 5. However, integrating over the w^l, the expansion

$$\sum_\alpha \sum_{k=1}^m \int_{\mathcal{L}} \mathrm{d}\mu_{\mathcal{L}}\, |z^k_\alpha\rangle\langle z^k_\alpha| \tag{34}$$

can be equated to the identity. The integral extends over any one leaf $\mathcal{L}$ of the foliation $\mathcal{F}$. On the contrary, integrating over the z^l in (33) would not give a resolution of the identity.

7.2 Examples of duality groups

The previous sections illustrate a possible mechanism to realise quantum–mechanical duality transformations between different vacua and between the coherent states built around them. Holomorphic foliations $\mathcal{F}$ such as those of section 7.1 allow for both continuous and discrete duality transformations. Assume that the w^l span a real, $2(n-m)$–dimensional manifold invariant under a certain group $\mathcal{D}$ of nonholomorphic transformations. Then $\mathcal{D}$ becomes a duality group of the quantum theory on $\mathcal{Q}$. In principle, appropriate choices of the holomorphic foliations $\mathcal{F}$ will allow to obtain any given duality group.

A simple example of a holomorphic foliation (that also happens to be a symplectic foliation of a Poisson manifold) is given by the Kirillov form [25] for the Lie algebra su(2). Using coordinates x, y, z, the latter is spanned by generators T_x, T_y, T_z satisfying the commutation relations

$$[T_i, T_j] = \epsilon_{ijk} T_k. \tag{35}$$

There is the Casimir operator $T_x^2 + T_y^2 + T_z^2$ on the enveloping algebra of su(2). We have a Poisson tensor P

$$P = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}, \tag{36}$$

and the Poisson bracket of any two functions f, g on su(2) reads

$$\{f, g\} = x\left(\partial_y f \partial_z g - \partial_z f \partial_y g\right)$$
$$- y\left(\partial_x f \partial_z g - \partial_z f \partial_x g\right) + z\left(\partial_x f \partial_y g - \partial_y f \partial_x g\right). \tag{37}$$

Now $\det P = 0$ everywhere. Away from the origin $x = y = z = 0$, the Poisson matrix P always contains a 2×2 nonsingular submatrix; this corresponds to the existence of the Casimir function $f(x, y, z) = x^2 + y^2 + z^2$. Hence we have a symplectic foliation of su(2) by symplectic leaves which are concentric spheres $x^2 + y^2 + z^2 = R^2$ of increasing radii $R > 0$. Passing through each point of su(2) there is exactly one such symplectic leaf; only the origin is met by no leaf, as the foliation has zero rank there. In standard spherical coordinates r, θ, φ one finds a Poisson bracket on the leaves

$$\{f, g\} = \frac{1}{r \sin\theta}\left(\partial_\theta f \partial_\varphi g - \partial_\varphi f \partial_\theta g\right) \tag{38}$$

and a Kirillov symplectic form

$$K = r \sin\theta \, \mathrm{d}\varphi \wedge \mathrm{d}\theta. \tag{39}$$

To us, the above example is interesting not because the leaves are symplectic, but because they are holomorphic. They provide a holomorphic foliation of su(2). Of course, the latter cannot be a symplectic manifold, but one can imagine embedding these holomorphic leaves into a nonholomorphic, symplectic manifold.

8 Discussion

8.1 Recapitulation

Most physical systems admit a complex structure $\mathcal{J}_{\mathcal{C}}$ on their classical phase spaces $\mathcal{C}$. Prominent among them is the 1–dimensional harmonic oscillator. Mathematically, the corresponding $\mathcal{C}$ supports the simplest holomorphic structure, that of the complex plane. Physically, canonical quantisation rests on the decomposition of a field into an infinite number of oscillators. The notions that the vacuum state is unique, and that coherence is a universal property independent of the observer, follow naturally. However, as summarised in section 1, recent breakthroughs in quantum field theory and M–theory suggest the need for a framework in which duality transformations can be accommodated at the elementary level of quantum mechanics, before considering field theory or strings. This, in turn, would help to understand better the dualities underlying quantum fields, strings and branes.

The formalism presented here can accommodate duality transformations in a natural way. In the absence of a complex structure $\mathcal{J}_{\mathcal{C}}$, all our statements concerning the vacuum state and the property of coherence are necessarily local in nature, *i.e.*, they do not hold globally on $\mathcal{C}$. A duality transformation of the quantum theory on $\mathcal{Q}$ will thus be specified by a nonholomorphic coordinate transformation on $\mathcal{C}$.

However, the question immediately arises: do we not have an overabundance of vacua? Does every imaginable nonholomorphic transformation induce a *physical* duality? A judicious application of physical symmetries can vastly restrict this apparent overabundance. Usually dualities appear under the form of a group $\mathcal{D}$. Rather than taking every imaginable nonholomorphic transformation to define a physical duality we must assume, as is the case in M–theory, a knowledge of the duality group $\mathcal{D}$, or perhaps even a finite subgroup thereof, and restrict ourselves to those nonholomorphic transformations that actually realise it.

8.2 The notion of duality

Duality is not to be understood as a transformation between different physical phenomena. Rather, it is to be understood as a transformation between different descriptions of the same quantum physics. Similarly, the statement that $\hbar$ depends on the observer is to be understood as meaning that one given quantum phenomenon may be described by different observers on $\mathcal{C}$ as corresponding to different regimes in a series expansion in powers of $\hbar$. Thus the semiclassical regime is given by a truncation of this series to order $\hbar$, while the strong quantum regime requires the whole infinite expansion.

That coherence equals holomorphicity has been known for long [26]. Here we have proved that noncomplex structures (such as almost complex structures) allow to implement duality transformations. The picture that emerges is conceptually close to that of a geometric manifold covered by local coordinate charts. Instead of the latter, quantum mechanics has local coherent states. A change of Darboux coordinates on $\mathcal{C}$ may or may not be holomorphic. Correspondingly, a transformation between quantum–mechanical states may or may not preserve coherence. Those that do not preserve coherence are duality transformations. A duality appears when it is possible to give two or more, apparently different, descriptions of the same quantum–mechanical phenomenon. Coherence thus becomes a local property on classical phase space $\mathcal{C}$. Observers on $\mathcal{C}$ not connected by means of a holomorphic change of coordinates need not, and in general will not, agree on what is a semiclassical effect *vs.* what is a strong quantum effect.

8.3 Holomorphic foliations *vs.* symplectic foliations

Some authors take the view that classical phase space $\mathcal{C}$ requires no more structure than a Poisson bracket as the latter becomes, under quantisation, the commutator of quantum operators [27]. A more geometric viewpoint [16, 18] is to take $\mathcal{C}$ to be not just Poisson, but symplectic. This is also useful in analysing physical issues such as constrained dynamics [28].

In implementing duality transformations we have relaxed the complex structure $\mathcal{J}_{\mathcal{C}}$ to an almost complex structure $J_{\mathcal{C}}$. In foliating $\mathcal{C}$ by holomorphic leaves, our approach is reminiscent of the symplectic foliations of Poisson manifolds [29]. Our conclusion is that, just as symplectic foliations of Poisson manifolds implement constraints, holomorphic foliations of symplectic manifolds implement dualities.

In our analysis, the symplectic structure of classical phase space $\mathcal{C}$ plays a key role. Moreover, we claim that $\omega_{\mathcal{C}}$ also has a quantum–mechanical role to play, too. In ref. [30] we have put forward a starting point for a formulation of quantum mechanics that is compatible with the relativity of $\hbar$. This approach is based on the symplectic structure $\omega_{\mathcal{C}}$. In fact Darboux's theorem falls short (by $\hbar$) of being a quantisation, as symplectic geometry does not know about Planck's constant $\hbar$. Once supplemented with the physical input $\hbar$, Darboux's theorem truly becomes a quantisation.

8.4 Final comments

In using coherent states our approach has been geometric. Indeed coherent states have been applied to geometric quantisation [31]; closely related issues have been studied recently in refs. [32, 33, 34, 35]. However there are a number of alternative viewpoints in order to analyse duality in quantum mechanics. Planck's constant $\hbar$ can be interpreted as the only modulus existing in quantum mechanics. It is precisely this parameter that tells *classical* from *quantum*, so duality in quantum mechanics necessarily refers to $\hbar$. This interpretation is especially natural in deformation quantisation [19, 20, 21]. It has gained renewed interest in the physical community due to its links with noncommutative theories, nontrivial Neveu–Schwarz B–fields and branes. It would be very interesting to analyse duality from this perspective.

Along other lines, interesting points to explore in this context are the higher–order generalisations of Poisson structures [36] and supersymmetric quantum mechanics [37]. Of course, quantum gravity is always an important testing ground for geometric theories [38, 39, 40, 41]. Inasmuch as we are relativising the concept of a quantum we may be said to be quantising gravity too—only in reverse.

Acknowledgements

Support from PPARC (grant PPA/G/O/2000/00469) and DGICYT (grant PB 96-0756) is acknowledged.

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