Photon fields in a fluctuating spacetime

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Abstract

We present a model of interacting quantum fields, formulated in a non-perturbative manner. One of the fields is treated semi-classically, the other is the photon field. The model has an interpretation of an electromagnetic field in a fluctuating spacetime.

The model is equivalent with the quantization of electromagnetism proposed recently by Czachor. Interesting features are that standard photon theory is recovered as a limiting case, and that localized field operators for the electromagnetic field exist as unbounded operators in Hilbert space.

1 Introduction

1.1 Two views on quantum spacetime

At the heart of quantum mechanics are the dual concepts of states and observables. One way to describe quantum spacetime is by means of a non-abelian algebra of functions of position, or by introduction of non-commuting position-time observables. Proposals for the latter go back to the early days of quantum mechanics [1]. This kind of approach is also the basis for application of non-commutative geometry [2]. The alternative, studied in the present paper, assumes that quantum spacetime has a state which is described by a vector Ω in some Hilbert space $\mathcal{H}$. This assumption implies that empty spacetime has some structure, made up by fluctuations imposed on top of Minkowski space. This structure is often referred to as the spacetime foam. The concept of a fluctuating spacetime goes back to the work of Wheeler [3].

Let us assume that a shift of frame of the observer by a vector q in Minkowski space is implemented by a unitary operator $U(q)$. The shift modifies the state vector Ω into the new vector $U(-q)\Omega$. One immediately expects non-trivial autocorrelations of the form

$$w(q) = \langle\Omega|\,\hat{U}(-q)\Omega\rangle. \tag{1}$$

However, since we cannot measure properties of spacetime without putting particles in it, $w(q)$ is not directly observable. In the standard concept of spacetime the vector Ω equals the vacuum vector Ω_0. The latter is invariant under translations. Hence, in that case $w(q)$ is constant equal to 1. If on the other hand a spacetime structure exists on the scale of Planck's length, or some other relevant length scale l, then $w(q)$ is non-trivial. It equals 1 for $q = 0$ and vanishes for displacements, large compared to l. In Fourier space, these short-distance correlations lead to a cutoff at high momenta.

1.2 Discussion

As is well known, the presence of a high-momentum cutoff implies breaking of Poincaré invariance of the field theory. In the present case any Poincaré transformation Λ maps the state Ω of spacetime onto another state $\Lambda\Omega$. This means that Poincaré covariance is maintained. Poincaré invariance is restored at distances large compared to l by the assumption $w(q) \simeq 0$. Experimental evidence for breaking of Poincaré invariance has been analysed using standard theory [4]. If such a symmetry breaking exists it should be extremely small, because a very small symmetry violation can produce dramatic observable effects.

The presence of a spacetime structure on short length scales has effects similar to those of noncommuting position-time operators. Indeed, a high-momentum cutoff implies difficulties for measurements at short length scales. However, here, as in most quantum field theories, it is not meaningful to define position operators Q_μ. Instead, one can determine field strengths at positions q in Minkowsk space (see below). Quantum expectations of these field strengths are affected by spacetime fluctuations. But it is difficult to conclude from inspection of these measurable quantities that spacetime should be noncommutative.

In early attempts to quantize gravity the metric tensor $g_{\mu\nu}$ is treated as a potential, analoguous to the vector potential of eletromagnetism. The gravitons resulting from second quantization are then treated in the same way as other field particles. In particular, gravitons are expected to interact with all other particles. In the present model the interplay between spacetime fluctuations and field particles is more fundamental and cannot be described in terms of interactions. Field particles like photons or electrons cannot exist without spacetime fluctuations. The structure of the total system is far from the usual tensor product description of independent subsystems, which are brought into interaction via lagrangian or hamiltonian constraints.

Another line of reasoning predicts fluctuations in the speed of light due to quantum uncertainties of matter fields. This has been worked out in detail by Ford et al [5, 6]. Here, speed of light is constant equal to 1. However, we still have to investigate the effect of an inhomogeneous state of spacetime on the effective propagation of electromagnetic radiation.

1.3 Czachor's noncanonical photon theory

As noted by several authors, there is some inconsistency between a description of the electromagnetic field by means of state vectors in a bosonic Fock space, with a trivial vacuum state, and the regression to wavefunctions that are ground states of the harmonic oscillator. Czachor et al [7, 8, 9] have proposed to solve this discord by modifying the link between harmonic oscillators and photons. As a first step, they replace all harmonic oscillators by a single oscillator with quantized frequency, the eigenvalues of which are the photon frequencies. In the next step, several copies of this quantized oscillator are introduced. Photons become collective excitations of a spacetime filled with oscillators.

Features of Czachor's theory are that creation and annihilation operators satisfy generalized commutation relations, and that the vacuum state is not any longer unique. The present paper starts from rather different assumptions to arrive at the same theory, and hence, the same physics. In particular, we know from Czachor's work that the standard theory of quantized free electromagnetic fields is a limiting case of the present theory. Another property of this theory is that field operators $\hat{A}(q)$ for photons or electrons exist as genuine operators in Fock space, and not as operator valued distributions, as is the case in the conventional theory. The latter is important for locality of the electron-photon interaction. In the present formalism, the interaction term

$$\int_{\mathbb{R}^4} \mathrm{d}q\, \hat{J}^\mu(q)\hat{A}_\mu(q), \tag{2}$$

with $J_\mu(q)$ the electron current, exists as an operator in Hilbert space.

1.4 Structure of the paper

In the next section we formulate our model guided by symmetry considerations. In section 3 a state of this model is described by means of its correlation functions. Some of its properties are discussed. In section 4 the link is made with the work of Czachor. The appendices contain a definition of covariance systems and some technical proofs.

2 A model of quantum spacetime

2.1 Symmetry considerations

An important symmetry of quantum fields is additivity of the fields. This symmetry is central to the Weyl approach. Let $H, +$ denote the additive group of single-particle wavefunctions. Then Fock space is a projective representation of this group. Indeed, the Weyl operators $\hat{W}(\psi)$ satisfy the Weyl form of the canonical commutation relations

$$\hat{W}(\phi)\hat{W}(\psi) = \hat{W}(\phi+\psi)\zeta(\phi,\psi) \tag{3}$$

with cocycle ζ given by

$$\zeta(\phi,\psi) = \exp\left(i\,\mathrm{Im}\,\langle\phi|\,\psi\rangle\right), \tag{4}$$

and

$$\langle\phi|\,\psi\rangle = \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\,\frac{1}{2|\mathbf{k}|}\overline{\phi(\mathbf{k})}\psi(\mathbf{k}). \tag{5}$$

The field operator $\hat{A}(\psi)$ is now the generator of $\hat{W}(\psi)$

$$\hat{W}(\psi) = \exp\left(i\hat{A}(\psi)\right). \tag{6}$$

The creation and annihilation operators are expressed in terms of the field operators in the usual way

$$\hat{A}_\pm(\psi) = \frac{1}{2}\hat{A}(\psi) \pm \frac{i}{2}\hat{A}(i\psi). \tag{7}$$

In the theory, presented below, additivity of the field of spacetime fluctuations is a broken symmetry, as a consequence of the presence of additional particle fields. Hence, the Weyl approach is not very well suited to describe fields of spacetime fluctuations. This problem refrains us from formulating a theory system based on the product of two additive groups, the group of fields of spacetime fluctuations, and the group of additional particle fields. Instead, we are forced to describe spacetime fluctuations by means of an algebra of observables $\mathcal{A}$. Its choice will be discussed later on. For the description of the electromagnetic field we can continue to use the Weyl operator approach as discussed above.

We do not include the Poincaré group in the initial formulation of the theory. But this implies that Poincaré covariance has to be checked later on a per state basis.

2.2 Covariance systems

It is obvious to assume that the fluctuations of spacetime have to be second-quantized, and that they form a scalar boson field. Alternatively, a vector or tensor field can be postulated. But this would complicate the theory. For the moment we prefer to keep the model as simple as possible.

We use the correlation function approach of [10]. It starts from a covariance system, also known as C^*-dynamical system. States of this system are determined by correlation functions. By means of a generalized G.N.S.-theorem they determine a Hilbert space representation. See Appendix A. The approach is rather analoguous to that of Wightman functions [11]. It has been tested in a number of cases [12, 13]. In particular, it has been used to give a clean treatment of the theory of free photons [14].

As discussed earlier, the spacetime fluctuations will be described by an algebra $\mathcal{A}$ of observables, the electromagnetic field by an additive group G of classical fields, or equivalently (see [14]), of test functions. This two components can be combined into a covariance system by letting G act on $\mathcal{A}$ in a trivial manner. Hence, the covariance system of choice is of the form $(\mathcal{A}, G, \mathbb{I})$.

An obvious choice of algebra $\mathcal{A}$ would be the algebra of bounded operators in the Fock space of a scalar boson. However, for technical reasons we are forced to take the abelian subalgebra of functions of momenta. A consequence is that in the present model the spacetime fluctuations are still described in a semi-classical way.

More precisely, the C^*-algebra $\mathcal{A}$ consists of functions $f^{(n)}(\mathbf{k}_1, \mathbf{k}_2, \cdots \mathbf{k}_n)$ which depend on the number of particles n, and, for a given n, on wavevectors $\mathbf{k}_m$, $m = 1, 2, \cdots n$ in $\mathbb{R}^3$. Because the particles are indistinguishable, invariance under permutation of the arguments $\mathbf{k}_1, \mathbf{k}_2, \cdots \mathbf{k}_n$ is assumed. Each function f of $\mathcal{A}$ is represented as an operator $\hat{f}$ in the Fock space of the scalar boson given by

$$\hat{f}\chi^{(n)}(\mathbf{k}_1, \mathbf{k}_2, \cdots \mathbf{k}_n) \quad = \quad f^{(n)}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_2, \cdots \mathbf{k}_n)\chi^{(n)}(\mathbf{k}_1, \mathbf{k}_2, \cdots \mathbf{k}_n). \tag{8}$$

The norm $||f||$ of f is then equal to the operator norm $||\hat{f}||$.

The symmetry group G is the additive group of test functions of the electromagnetic field. It consists of complex continuous functions $\phi_\mu(\mathbf{k})$, with compact support, depending on wavevectors in $\mathbb{R}^3$, and labeled with an index running from 0 to 3. They satisfy the gauge condition

$$|\mathbf{k}|\phi_0(\mathbf{k}) = \sum_{\alpha=1}^{3} \mathbf{k}_\alpha \phi_\alpha(\mathbf{k}). \tag{9}$$

This group is locally compact in the discrete topology.

3 States of the model system

When describing interactions between particles it is tradition to start from product states and then, to formulate lagrangian or hamiltonian interactions between the particles. However, for the states studied below the interaction is so strong that it seems unlikely that a hamiltonian or lagrangian description should exist.

3.1 Product states

The vacuum-to-vacuum correlation functions of the free electromagnetic field define a state of the covariance system $(\mathbb{C}, G, \mathbb{I})$ by (see [14])

$$\mathcal{F}(\phi;\psi) \quad = \quad \exp\left(-i\,\mathrm{Im}\,\langle\psi|\phi\rangle\right)\ \exp\left(-\frac{1}{2}\langle\psi-\phi|\psi-\phi\rangle\right) \tag{10}$$

with the (degenerate) scalar product given by

$$\langle\psi|\phi\rangle = -\int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\, \frac{1}{2|\mathbf{k}|}\, \overline{\psi^\mu(\mathbf{k})}\phi_\mu(\mathbf{k}) \geq 0. \tag{11}$$

On the other hand, if χ a state vector in the Fock space of a scalar boson field then a state of the C^*-algebra $\mathcal{A}$ is defined by $\mathcal{F}_\chi(f) = \langle\chi|\,\hat{f}\chi\rangle$. In the special case that χ is of the form

$$\chi = \chi^{(0)} \oplus \chi^{(1)} \oplus \frac{1}{\sqrt{2!}}\chi^{(2)} \otimes \chi^{(2)} \oplus \cdots, \tag{12}$$

then the state on $\mathcal{A}$ is given by

$$\begin{aligned}\mathcal{F}_\chi(f) &= f^{(0)}|\chi^{(0)}|^2\\ &\quad+\sum_{n=1}^\infty\frac{1}{n!}\left[\prod_{j=1}^n\int_{\mathbb{R}^3}\,\mathrm{d}\mathbf{k}_j\frac{1}{2|\mathbf{k}_j|}\,|\chi^{(n)}(\mathbf{k}_j)|^2\right]f^{(n)}(\mathbf{k}_1,\mathbf{k}_2,\cdots,\mathbf{k}_n).\end{aligned}\tag{13}$$

The product of (10) and (13) defines a state of the covariance system $(\mathcal{A},G,\mathbb{I})$. However, we are not interested in this product state. In what follows another way of combining (10) and (13) is investigated.

3.2 Interacting states

An alternative way of combining (10) and (13) is obtained by first bringing the integrations in the exponential of (10) in front of the whole expression

$$\begin{aligned}&\exp\left(-i\,\mathrm{Im}\,\langle\psi|\phi\rangle-\frac{1}{2}\langle\psi-\phi|\psi-\phi\rangle\right)\\ \Rightarrow\ &\int_{\mathbb{R}^3}\,\mathrm{d}\mathbf{k}\exp\left(i\,\mathrm{Im}\,\overline{\psi^\mu(\mathbf{k})}\phi_\mu(\mathbf{k})+\frac{1}{2}\overline{(\psi^\mu(\mathbf{k})-\phi^\mu(\mathbf{k}))}(\psi_\mu(\mathbf{k})-\phi_\mu(\mathbf{k}))\right)\end{aligned}\tag{14}$$

and next modifying all integrations in (13) so as to include the exponential integrand of the above expression. As a final modification we introduce real constants c_n which multiply the test functions ϕ in G, and make their amplitude dependend on the number of spacetime fluctuations. All together, the result is

$$\begin{aligned}&\mathcal{F}_\chi(f;\phi;\psi)=f^{(0)}|\chi^{(0)}|^2\\ &+\sum_{n=1}^\infty\frac{1}{n!}\Bigg[\prod_{j=1}^n\int_{\mathbb{R}^3}\,\mathrm{d}\mathbf{k}_j\frac{1}{2|\mathbf{k}_j|}\,|\chi^{(n)}(\mathbf{k}_j)|^2\exp\left(ic_n^2\,\mathrm{Im}\,\overline{\psi^\mu(\mathbf{k}_j)}\phi_\mu(\mathbf{k}_j)\right)\\ &\times\exp\left(\frac{c_n^2}{2}\overline{(\psi^\mu(\mathbf{k}_j)-\phi^\mu(\mathbf{k}_j))}(\psi_\mu(\mathbf{k}_j)-\phi_\mu(\mathbf{k}_j))\right)\Bigg]\\ &\times f^{(n)}(\mathbf{k}_1,\mathbf{k}_2,\cdots,\mathbf{k}_n).\end{aligned}\tag{15}$$

Covariance of the correlation functions can be checked easily. See Appendix B. The multiplier ζ appearing in the Weyl commutation relations (3) for the electromagnetic field is modified into an operator $\hat{\zeta}$, which represents the function

$$\zeta(\phi;\psi)^{(n)}(\mathbf{k}_1,\cdots,\mathbf{k}_n)=\exp\left(ic_n^2\sum_{j=1}^n\mathrm{Im}\,\overline{\psi^\mu(\mathbf{k}_j)}\phi_\mu(\mathbf{k}_j)\right).\tag{16}$$

3.3 Hilbert space representation

The generalized GNS-theorem of [10] implies the existence of a representation of the algebra $\mathcal{A}$ as bounded operators of a Hilbert space $\mathcal{H}$. The operator corresponding with the function $f^{(n)}(\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_n)$ is denoted $\hat{f}$, as before. Note however that the Hilbert space has changed so that $\hat{f}$ is not any longer defined by (8). There exist also unitary operators $\hat{W}(\phi)$ in $\mathcal{H}$, one for each element of the group G, with the property that

$$\hat{W}(\phi)\hat{W}(\psi) = \hat{W}(\phi + \psi)\hat{\zeta}(\phi; \psi) \tag{17}$$

Finally there exists in $\mathcal{H}$ a vector Ω, such that

$$\mathcal{F}_\chi(f; \phi; \psi) \quad = \quad \langle \Omega | \, \hat{W}(\psi)\hat{f}\hat{W}(\phi)^*\Omega \rangle \tag{18}$$

holds for all choices of the fields ϕ and ψ.

Note that the operators $\hat{f}$ commute with all operators $\hat{W}(\phi)$. This is a consequence of the fact that the action of the group G on the algebra $\mathcal{A}$ is trivial. In other words, the algebra of operators $\hat{f}$ belongs to the center of the representation. The representation is reducible, which can be understood because the present model treats the spacetime fluctuations in a semi-classical manner.

From (17) follows that

$$\hat{W}(\phi)\hat{W}(\psi) = \hat{W}(\psi)\hat{W}(\phi)\hat{\zeta}(\psi, \phi)^*\hat{\zeta}(\phi, \psi). \tag{19}$$

Combining this expression with the definition of the field operators (6) one obtains the commutation relations

$$\left[\hat{A}(\psi), \hat{A}(\phi)\right]_- = i\hat{s}(\psi, \phi), \tag{20}$$

with

$$\hat{\zeta}(\phi, \psi) = \exp((i/2)\hat{s}(\psi, \phi)), \tag{21}$$

where $\hat{s}(\psi, \phi)$ is the operator corresponding with the function

$$s^{(n)}(\psi, \phi)(\mathbf{k}_1, \cdots, \mathbf{k}_n) = 2c_n^2 \sum_{j=1}^{n} \mathrm{Im}\, \overline{\psi^\mu(\mathbf{k}_j)}\phi_\mu(\mathbf{k}_j). \tag{22}$$

The commutation relations (20) differ from the traditional result

$$\left[\hat{A}(\psi), \hat{A}(\phi)\right]_- = -2i\, \mathrm{Im}\, \langle \psi | \, \phi \rangle \tag{23}$$

because the r.h.s. of the commutator is not a multiple of the identity operator, but is an operator in the center of the representation. This kind of commutation relations has been studied before, in the context of generalized free fields — see e.g. section 12.5 of [15].

The operator $\hat{A}(\phi)$ is a real linear function of its argument ϕ — see Appendix C. It is also shown there that

$$\hat{A}(\phi)\Omega = i\hat{A}(i\phi)\Omega \tag{24}$$

holds for all ϕ. An immediate consequence is that the annihilation operator $\hat{A}_-(\phi)$, defined by (7), annihilates the state vector Ω. Hence the latter acts as a vacuum for the photon field. But notice that it is not a vacuum for the field of spacetime fluctuations.

3.4 Poincaré covariance

A shift by a vector q of Minkowski space leaves functions of momentum invariant and maps the state vector Ω into the vector $\hat{u}_{-q}\Omega$, where $\hat{u}_q$ is the representation of the function

$$u_q^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n) = e^{-iq_0 \sum_{j=1}^{n|} \mathbf{k}_j|} e^{i\mathbf{q}\cdot\sum_{j=1}^{n} \mathbf{k}_j}. \tag{25}$$

However, a shift in Minkowski space affects also the electromagnetic field. Test functions ϕ in G transform according to

$$\tau_q\phi_\mu(\mathbf{k}) = \phi_\mu(\mathbf{k}) \exp(iq_0|\mathbf{k}|) \exp(-i\mathbf{q}\cdot\mathbf{k}). \tag{26}$$

Therefore, the unitary operator $\hat{U}(q)$, implementing a shift by q in Minkowski space, is defined by

$$\hat{U}(q)\hat{f}\hat{W}(\phi)\Omega = \hat{f}\hat{u}_q\hat{W}(\tau_q\phi)\Omega. \tag{27}$$

It is straightforward to verify that these operators $\hat{U}(q)$ form a unitary representation of $\mathbb{R}^4, +$.

Calculate now

$$\begin{aligned} w(q) &= \langle\Omega|\,\hat{U}(-q)\Omega\rangle \\ &= \mathcal{F}_\chi(u_{-q}; 0; 0) \\ &= |\chi^{(0)}|^2 \\ &\quad + \sum_{n=1}^{\infty} \frac{1}{n!}\left[\int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\frac{1}{2|\mathbf{k}|}\,|\chi^{(n)}(\mathbf{k})|^2 \exp(iq_0|\mathbf{k}|)\exp(-i\mathbf{q}\cdot\mathbf{k})\right]^n. \end{aligned} \tag{28}$$

From this expression it is clear that for large values of q the correlation function $w(q)$ tends to $|\chi^{(0)}|^2$, while for $q=0$ one has $w(0)=1$.

The generators of the group of unitary shift operators are the momentum operators $\hat{K}_\mu$, defined by

$$\hat{U}(q) = \exp(-iq^\mu \hat{K}_\mu). \tag{29}$$

In particular, $\hat{K}_0$ is the energy operator. A short calculation gives

$$\langle\Omega|\,\hat{W}(\psi)\hat{f}\hat{U}(q)\hat{W}(\phi)^*\Omega\rangle \quad = \quad \mathcal{F}_\chi(fu_q;\tau_q\phi;\psi). \tag{30}$$

Linearization in q_0 gives

$$\langle\Omega|\,\hat{W}(\psi)\hat{f}\hat{K}_0\hat{W}(\phi)^*\Omega\rangle \quad = \quad \sum_{n=1}^{\infty}\left[\left[\sum_{l=1}^{n|}\mathbf{k}_l|\left(1-c_n^2\overline{\phi^\mu(\mathbf{k}_l)}\psi_\mu(\mathbf{k}_l)\right)\right]\right]_n \tag{31}$$

with

$$\begin{aligned}[[X]]_n \quad = \quad & \frac{1}{n!}\left[\prod_{j=1}^{n}\int_{\mathbb{R}^3}\,\mathrm{d}\mathbf{k}_j\frac{1}{2|\mathbf{k}_j|}\,|\chi^{(n)}(\mathbf{k}_j)|^2\right.\\
&\times\exp\left(ic_n^2\,\mathrm{Im}\,\overline{\psi^\mu(\mathbf{k}_j)}\phi_\mu(\mathbf{k}_j)\right)\\
&\left.\times\exp\left(\frac{c_n^2}{2}\overline{(\psi^\mu(\mathbf{k}_j)-\phi^\mu(\mathbf{k}_j))}(\psi_\mu(\mathbf{k}_j)-\phi_\mu(\mathbf{k}_j))\right)\right]\\
&\times f^{(n)}(\mathbf{k}_1,\mathbf{k}_2,\cdots,\mathbf{k}_n)\,X.\end{aligned} \tag{32}$$

This result can be used to show that $\hat{K}_0$ is a positive operator. It has two contributions, the energy of the spacetime fluctuations, and that of the electromagnetic field. A similar calculation shows that also the momenta $\hat{K}_\alpha,\alpha=1,2,3$ have two contributions, one of the spacetime fluctuations, the other of the electromagnetic field. Further calculations show that the squared mass operator satisfies $\hat{K}^\mu\hat{K}_\mu\geq 0$. We omit these calculations here.

Under a Lorentz transformation Λ a 3-dimensional wave vector $\mathbf{k}$ transforms into the vector $\mathbf{k}'$ with components

$$\mathbf{k}'_\alpha = (\Lambda^{-1})^0_\alpha|\mathbf{k}| + \sum_{\beta=1}^{3}(\Lambda^{-1})^\beta_\alpha\mathbf{k}_\beta. \tag{33}$$

The state with state vector χ and correlation functions (15) transforms into a state described by the statevector χ' with components

$$\chi'^{(n)}(\mathbf{k}) = \chi^{(n)}(\mathbf{k}'). \tag{34}$$

The Lorentz transformations are not necessarily unitarily implemented in the present representation.

3.5 Infinite number of spacetime fluctuations

Take now $c_n = 1/\sqrt{n}$ and consider a state vector χ of the form (12), with $\chi^{(n)}(\mathbf{k}) = \delta_{n,N}(N!)^{1/2N} Z(\mathbf{k})^{1/2}$, and with $Z(\mathbf{k})$ a normalized density function

$$\int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\, \frac{1}{2|\mathbf{k}|}\, Z(\mathbf{k}) = 1. \tag{35}$$

Then one calculates

$$\begin{aligned}
& \lim_{N\to\infty} \mathcal{F}_\chi(1;\phi;\psi) \\
= & \lim_{N\to\infty} \left[\int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\, \frac{1}{2|\mathbf{k}|}\, Z(\mathbf{k}) \exp\left((i/N)\, \mathrm{Im}\, \overline{\psi^\mu(\mathbf{k})} \phi_\mu(\mathbf{k}) \right) \right. \\
& \left. \times \exp\left(\frac{1}{2N}\, \overline{(\psi^\mu(\mathbf{k}) - \phi^\mu(\mathbf{k}))} (\psi_\mu(\mathbf{k}) - \phi_\mu(\mathbf{k})) \right) \right]^N \\
= & \exp\left(i\, \mathrm{Im} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\, \frac{1}{2|\mathbf{k}|}\, Z(\mathbf{k}) \overline{\psi^\mu(\mathbf{k})} \phi_\mu(\mathbf{k}) \right) \\
& \times \exp\left(\frac{1}{2} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\, \frac{1}{2|\mathbf{k}|}\, Z(\mathbf{k}) \overline{(\psi^\mu(\mathbf{k}) - \phi^\mu(\mathbf{k}))} (\psi_\mu(\mathbf{k}) - \phi_\mu(\mathbf{k})) \right).
\end{aligned} \tag{36}$$

This expression coincides with result (10) of the standard photon theory, except for the appearance of the function $Z(\mathbf{k})$, which acts as a cutoff for large $\mathbf{k}$-values.

4 Explicit construction à la Czachor

4.1 A single spacetime fluctuation

In the approach of [7, 8, 9] photons in presence of a single spacetime fluctuation are described by a pair of harmonic oscillators, the frequency of which has been quantized and has become an operator. The excitations of the oscillators are the photons. Two oscillators are needed because of the two polarizations of the photon. The frequency of the photon can equal any of the eigenvalues of the frequency operator.

Let $\hat{a}$ and $\hat{a}^\dagger$ be the annihilation and creation operators of the standard harmonic oscillator. They satisfy the commutation relations

$$[\hat{a}, \hat{a}^\dagger] = 1. \tag{37}$$

The Hilbert space of wavefunctions consists of functions of the form $\chi(\mathbf{k}, m, n)$, where $\mathbf{k}$ is a wavevector and m and n are quantum numbers of the two harmonic oscillators. Normalization is such that

$$\sum_{m,n=0}^{+\infty} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k} \, \frac{1}{2|\mathbf{k}|} \, |\chi(\mathbf{k}, m, n)|^2 = 1. \tag{38}$$

Any wavefunction of the form

$$\chi(\mathbf{k}, m, n) = \chi(\mathbf{k}) \delta_{m,0} \delta_{n,0} \tag{39}$$

is a vacuum vector for the photons and will be denoted

$$|\chi\rangle \otimes |0\rangle \otimes |0\rangle. \tag{40}$$

For any bispinor

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}, \tag{41}$$

consisting of two test functions $\phi_1(\mathbf{k})$ and $\phi_2(\mathbf{k})$, one defines an annihilation operator $\hat{A}_-(\phi)$ by

$$\hat{A}_-(\phi) = \hat{\phi}_1 \otimes \hat{a} \otimes \mathbb{I} + \hat{\phi}_2 \otimes \mathbb{I} \otimes \hat{a} \tag{42}$$

(the operators $\hat{\phi}_1, \hat{\phi}_2$ are defined as multiplication operators). The creation operator $\hat{A}_+(\phi)$ is the conjugate of the annihilation operator, as usual. These creation and annihilation operators satisfy the commutation relations

$$\left[\hat{A}_-(\psi), \hat{A}_+(\phi)\right] = \left(\hat{\psi}_1 \hat{\phi}_1^* + \hat{\psi}_2 \hat{\phi}_2^*\right) \otimes \mathbb{I} \otimes \mathbb{I}. \tag{43}$$

The field operators $\hat{A}(\phi) = \hat{A}_-(\phi) + \hat{A}_+(\phi)$ satisfy

$$\left[\hat{A}(\psi), \hat{A}(\phi)\right] \quad = \quad \left(\hat{\psi}_1 \hat{\phi}_1^* + \hat{\psi}_2 \hat{\phi}_2^* - \hat{\phi}_1 \hat{\psi}_1^* - \hat{\phi}_2 \hat{\psi}_2^*\right) \otimes \mathbb{I} \otimes \mathbb{I}. \tag{44}$$

Comparison of (44) with (20) is not immediately possible because here two-component spinors are used instead of four-component test functions.

4.2 Localized field operators

Following [9], momentum operators $\hat{K}_\mu$ are defined by

$$\begin{aligned} \hat{K}_0 \phi(\mathbf{k}, m, n) &= |\mathbf{k}| \left(\frac{1}{2} + m\right) \left(\frac{1}{2} + n\right) \phi(\mathbf{k}, m, n) \\ \hat{K}_\alpha \phi(\mathbf{k}, m, n) &= \mathbf{k}_\alpha \left(\frac{1}{2} + m\right) \left(\frac{1}{2} + n\right) \phi(\mathbf{k}, m, n). \end{aligned} \tag{45}$$

They define unitary shift operators $\hat{U}(q)$ by (29).

It is now immedately clear from (42) that the localized annihilation operators $\hat{A}_-^{(\mathrm{l})}(q)$ and $\hat{A}_-^{(\mathrm{r})}(q)$, one for each of the two polarizations of the photon, are given by

$$\begin{aligned} \hat{A}_-^{(\mathrm{l})}(q) &= e^{-iq^\mu \hat{K}_\mu} \otimes \hat{a} \otimes \mathbb{I}, \\ \hat{A}_-^{(\mathrm{r})}(q) &= e^{-iq^\mu \hat{K}_\mu} \otimes \mathbb{I} \otimes \hat{a}. \end{aligned} \tag{46}$$

4.3 Correlation functions

The field operators $\hat{A}(\phi)$ are used to define unitary operators $\hat{W}(\phi)$ by

$$\hat{W}(\phi) = \exp\left(i\hat{A}(\phi)\right). \tag{47}$$

The Weyl operators $\hat{W}(\phi)$ satisfy the product rule

$$\hat{W}(\phi)\hat{W}(\psi) = \left(e^{(i/2)\hat{s}(\phi,\psi)} \otimes \mathbb{I} \otimes \mathbb{I}\right) \hat{W}(\phi + \psi) \tag{48}$$

with

$$s(\phi,\psi) = -2\,\mathrm{Im}\,\left(\phi_1\overline{\psi_1} + \phi_2\overline{\psi_2}\right). \tag{49}$$

The latter result is obtained by means of the Campbell-Baker-Hausdorff formula

$$\exp\left(i(A+B)\right) = \exp\left(iA\right)\exp\left(iB\right)\exp\left((1/2)[A,B]\right), \tag{50}$$

which is valid when B commutes with $[A,B]$.

The action of $\hat{W}(\phi)$ on a vacuum vector is given by

$$\hat{W}(\phi)|\chi\rangle \otimes |0\rangle \otimes |0\rangle(\mathbf{k}) = \chi(\mathbf{k}) e^{i(\phi_1(\mathbf{k})\hat{a} + \overline{\phi_1(\mathbf{k})}\hat{a}^\dagger)}|0\rangle \otimes e^{i(\phi_2(\mathbf{k})\hat{a} + \overline{\phi_2(\mathbf{k})}\hat{a}^\dagger)}|0\rangle. \tag{51}$$

Using the relation

$$e^{\lambda\hat{a} + \mu\hat{a}^\dagger}|0\rangle = e^{(1/2)\lambda\mu} \sum_n \frac{1}{\sqrt{n!}}\mu^n|n\rangle, \tag{52}$$

the above expression can be rewritten as

$$\begin{aligned} \hat{W}(\phi)|\chi\rangle \otimes |0\rangle \otimes |0\rangle(\mathbf{k}) &= \chi(\mathbf{k}) \exp\left(-\frac{1}{2}|\phi_1(\mathbf{k})|^2 - \frac{1}{2}|\phi_2(\mathbf{k})|^2\right) \\ &\quad \times \sum_{m,n} \frac{i^{m+n}}{\sqrt{m!n!}} \overline{\phi_1(\mathbf{k})^m \phi_2(\mathbf{k})^n} |m\rangle \otimes |n\rangle. \end{aligned} \tag{53}$$

The correlation functions are now calculated as follows.

$$\mathcal{F}_\chi(f;\phi;\psi) = \langle\chi| \otimes \langle 0| \otimes \langle 0|\, \hat{W}(\psi)\hat{f}\hat{W}(\phi)^*\chi\rangle \otimes |0\rangle \otimes |0\rangle$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}\frac{1}{2|\mathbf{k}|} f(\mathbf{k})|\chi(\mathbf{k})|^2 \\
&\times \prod_{s=1,2} \exp(-(1/2)|\phi_s(\mathbf{k})|^2 - (1/2)|\psi_s(\mathbf{k})|^2 - \psi_s(\mathbf{k})\overline{\phi_s(\mathbf{k})}).
\end{aligned}
\tag{54}
$$

Again, this formula does not compare directly with (15) because here two-component spinors are used instead of four-component classical wavefunctions.

4.4 Multiple fluctuations

The transition from the one-oscillator to the multi-oscillator formalism is straightforward. Starting from the Hilbert space $\mathcal{H}$ of wave functions $\chi(\mathbf{k}, m, n)$ a Fock space $\underline{\mathcal{H}}$ is constructed in the usual way. The n-th component of $\underline{\mathcal{H}}$ is the space $\otimes_s^n \mathcal{H}$ of these elements of $\mathcal{H}^{\otimes n}$ which are symmetric under permutations. The field operator $\underline{\hat{A}}(\phi)$, acting on $\underline{\mathcal{H}}$, is defined by

$$
\underline{\hat{A}}(\phi) = 0 \oplus \hat{A}(\phi) \oplus \frac{1}{\sqrt{2}}\left(\hat{A}(\phi)\otimes\mathbb{I} + \mathbb{I}\otimes\hat{A}(\phi)\right) + \cdots . \tag{55}
$$

Consider now a vacuum vector in $\mathcal{H}$ of the form

$$
\Omega = |\chi\rangle \otimes |0\rangle \otimes |0\rangle. \tag{56}
$$

Fix a discrete probability distribution p i.e., positive numbers p_n satisfying $\sum_n p_n = 1$. Then a vacuum vector $\underline{\Omega}$ of $\underline{\mathcal{H}}$ is defined by

$$
\underline{\Omega} \quad = \quad \sqrt{p_0} \oplus \sqrt{p_1}\Omega \oplus \sqrt{p_2}\Omega\otimes\Omega \oplus \cdots . \tag{57}
$$

Correlation functions can now be calculated as follows.

$$
\begin{aligned}
& \mathcal{F}_\chi(f;\phi;\psi) \\
=\ & \langle\underline{\Omega}|\, \underline{\hat{W}}(\psi)\underline{\hat{f}}\underline{\hat{W}}(\phi)^*\underline{\Omega}\rangle \\
=\ & p_0 f^{(0)} + \sum_{n=1}^{\infty} p_n \left[\prod_{j=1}^{n} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}_j \frac{1}{2|\mathbf{k}_j|}|\chi(\mathbf{k}_j)|^2\right. \\
\times\ & \left.\prod_{s=1,2} \exp\left(-\frac{1}{2n}(|\phi_s(\mathbf{k}_j)|^2 + |\psi_s(\mathbf{k}_j)|^2 + 2\psi_s(\mathbf{k}_j)\overline{\phi_s(\mathbf{k}_j)})\right)\right] \\
& \times f^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n).
\end{aligned}
\tag{58}
$$

Comparison with (15) shows that here the constants c_n equal $1/\sqrt{n}$. The state vector χ in the Fock space of the scalar boson is of the special form (12), with $\chi^{(n)}(\mathbf{k}) = n! p_n \chi(\mathbf{k})$. In [9], the special choice

$$
p_n = e^{-\alpha}\frac{\alpha^n}{n!} \tag{59}
$$

is discussed. It is argued that it restores Poisson statistics of the photons in the limit of an infinite number of spacetime fluctuations.

Appendix A

The *covariance system* $(\mathcal{A}, G, \sigma)$ consists of a C^*-algebra $\mathcal{A}$, a locally compact group G, and an action σ of G as automorphisms of $\mathcal{A}$. It is assumed that the map $g \in G \rightarrow \sigma_g f$ is continuous for each f in $\mathcal{A}$. A *state* of the covariance system is determined by correlation functions $\mathcal{F}(f; g; g')$ with f in $\mathcal{A}$ and g, g' in G. They should satisfy the following conditions (for simplicity we assume that $\mathcal{A}$ is a C^*-algebra with unit)

(positivity) For all $n > 0$ and for all possible choices of complex λ_j, $g_j \in G$, and $f_j \in \mathcal{A}$, is

$$\sum_{j,k=1}^{n} \lambda_j \overline{\lambda_k} \mathcal{F}(f_k^* f_j; g_j; g_k) \geq 0; \tag{60}$$

(normalization) $\mathcal{F}(1; e; e) = 1$, where 1 is the unit of $\mathcal{A}$, and e is the neutral element of G;

(covariance) there exists a right multiplier ζ with values in $\mathcal{A}$ such that

$$\mathcal{F}(\sigma_{g''} f; g; g') = \mathcal{F}(\zeta(g', g'') f \zeta(g, g'')^*; gg''; g'g''); \tag{61}$$

(continuity) for each f in $\mathcal{A}$ the map $g, g' \rightarrow \mathcal{F}(f; g; g')$ is continuous in a neighborhood of the neutral element e, e.

The first two conditions imply that the map $f \in \mathcal{A} \rightarrow \mathcal{F}(f; e; e)$ defines a state of the C^*-algebra $\mathcal{A}$.

Appendix B

Here we show that the correlation functions(15) define a state of the covariance system $(\mathcal{A}, G, \mathbb{I})$.

Positivity One calculates

$$\begin{aligned} & \sum_{m,l} \lambda_m \overline{\lambda_l} \mathcal{F}_\chi(f_l^* f_m; \phi_m; \phi_l) \\ = \; & |\sum_m \lambda_m f_m^{(0)}|^2 \, |\omega^{(0)}|^2 \end{aligned}$$

$$
\begin{aligned}
&+ \sum_{n=1}^{\infty} \frac{1}{n!} \left[\prod_{j=1}^{n} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}_j \frac{1}{2|\mathbf{k}_j|} |\chi^{(n)}(\mathbf{k}_j)|^2 \right] \\
&\times \sum_{m,l} \mu_m(\mathbf{k}_1, \cdots \mathbf{k}_n) \overline{\mu_l}(\mathbf{k}_1, \cdots, \mathbf{k}_n) \exp\left(-c_n^2 \sum_{j=1}^{n} \overline{\phi_m^{\mu}(\mathbf{k}_j)} \phi_{l,\mu}(\mathbf{k}_j) \right)
\end{aligned} \tag{62}
$$

with

$$
\mu_m^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n) = \lambda_m f_m^{(n)}(\mathbf{k}_1, \cdots, \mathbf{k}_n) \exp\left(\frac{c_n^2}{2} \sum_{m=1}^{n} \overline{\phi_m^{\mu}(\mathbf{k}_m)} \phi_{m,\mu}(\mathbf{k}_m) \right). \tag{63}
$$

Note that the matrix M with elements

$$
M_{j,l} = -c_n^2 \sum_{j=1}^{n} \overline{\phi_m^{\mu}(\mathbf{k}_j)} \phi_{l,\mu}(\mathbf{k}_j) \tag{64}
$$

is positive definite because the test functions $\phi_m(\mathbf{k})$ satisfy (9). Hence, by Schur's lemma, also the matrix with elements $\exp(M_{j,l})$ is positive One concludes that (62) is positive.

Normalization is trivially satisfied provided that the vector χ in Fock space is properly normalized.

Covariance One calculates

$$
\begin{aligned}
&\mathcal{F}(f; \phi+\xi; \psi+\xi) = f^{(0)} |\chi^{(0)}|^2 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\prod_{j=1}^{n} \int_{\mathbb{R}^3} \mathrm{d}\mathbf{k}_j \frac{1}{2|\mathbf{k}_j|} |\chi^{(n)}(\mathbf{k}_j)|^2 \right. \\
&\quad \times \exp\left(i c_n^2 \,\mathrm{Im}\, \overline{(\psi^{\mu}(\mathbf{k}_j) + \xi^{\mu}(\mathbf{k}_j))} (\phi_{\mu}(\mathbf{k}_j) + \xi_{\mu}(\mathbf{k}_j)) \right) \\
&\quad \left. \times \exp\left(\frac{c_n^2}{2} \overline{(\psi^{\mu}(\mathbf{k}_j) - \phi^{\mu}(\mathbf{k}_j))} (\psi_{\mu}(\mathbf{k}_j) - \phi_{\mu}(\mathbf{k}_j)) \right) \right] \\
&\quad \times f^{(n)}(\mathbf{k}_1, \mathbf{k}_2, \cdots, \mathbf{k}_n) \\
&= \mathcal{F}(\zeta(\psi, \xi) f \zeta(\phi, \xi); \phi; \psi).
\end{aligned} \tag{65}
$$

This proves covariance of the correlation functions.

Continuity The continuity requirement is empty because of the discrete topology of the group G.

One concludes that the correlation functions (15) define a state of the covariance system $(\mathcal{A}, G, \mathbb{I})$.

Appendix C

Here we show that the field operator $\hat{A}(\phi)$ is a real linear function of its argument ϕ. We also show that (24) holds.

One calculates

$$\begin{aligned} & \langle\Omega|\, \hat{W}(\psi)\hat{W}(\xi)\hat{f}\hat{W}(\phi)^*\Omega\rangle \\ = & \langle\Omega|\, \hat{W}(\psi+\xi)\hat{\zeta}(\psi,\xi)\hat{f}\hat{W}(\phi)^*\Omega\rangle \\ = & \mathcal{F}_\chi(\zeta(\psi,\xi)f;\phi;\psi+\xi). \end{aligned} \tag{66}$$

First note that the map

$$\lambda\in\mathbb{R}\to\langle\Omega|\, \hat{W}(\psi)\hat{W}(\lambda\xi)\hat{f}\hat{W}(\phi)^*\Omega\rangle \tag{67}$$

is continuous. This suffices, together with

$$\hat{W}(\lambda\xi)\hat{W}(\mu\xi)=\hat{W}((\lambda+\mu)\xi), \tag{68}$$

to apply Stone's theorem and to conclude the existence of self-adjoint operators $\hat{A}(\xi)$ satisfying $\hat{A}(\lambda\xi)=\lambda\hat{A}(\xi)$ for all real λ.

Next, linearize (66) in ξ, using (15). One obtains

$$\begin{aligned} & i\langle\Omega|\, \hat{W}(\psi)\hat{A}(\xi)\hat{f}\hat{W}(\phi)^*\Omega\rangle = f^{(0)}|\chi^{(0)}|^2 \\ & +\sum_{n=1}^\infty\frac{c_n^2}{n!}\left[\prod_{j=1}^n\int_{\mathbb{R}^3}\mathrm{d}\mathbf{k}_j\frac{1}{2|\mathbf{k}_j|}\,|\chi^{(n)}(\mathbf{k}_j)|^2\exp\left(ic_n^2\,\mathrm{Im}\,\overline{\psi^\mu(\mathbf{k}_j)}\phi_\mu(\mathbf{k}_j)\right)\right. \\ & \left.\times\exp\left(\frac{c_n^2}{2}\overline{(\psi^\mu(\mathbf{k}_j)-\phi^\mu(\mathbf{k}_j))}(\psi_\mu(\mathbf{k}_j)-\phi_\mu(\mathbf{k}_j))\right)\right] \\ & \times\sum_{l=1}^n\left[\overline{\xi^\mu(\mathbf{k}_l)}\psi_\mu(\mathbf{k}_l)-\overline{\phi^\mu(\mathbf{k}_l)}\xi_\mu(\mathbf{k}_l)\right] \\ & \times f^{(n)}(\mathbf{k}_1,\mathbf{k}_2,\cdots,\mathbf{k}_n). \end{aligned} \tag{69}$$

This result shows that $\hat{A}(\xi)$ is a real linear function of ξ.

Finally, let us show (24). Take $\psi=0$ in (69). This gives

$$\begin{aligned} & i\langle\hat{A}(\xi)\Omega|\,\hat{f}\hat{W}(\phi)^*\Omega\rangle = f^{(0)}|\chi^{(0)}|^2 \\ & -\sum_{n=1}^\infty\frac{c_n^2}{n!}\left[\prod_{j=1}^n\int_{\mathbb{R}^3}\mathrm{d}\mathbf{k}_j\frac{1}{2|\mathbf{k}_j|}\,|\chi^{(n)}(\mathbf{k}_j)|^2\exp\left(\frac{c_n^2}{2}\overline{\phi^\mu(\mathbf{k}_j)}\phi_\mu(\mathbf{k}_j)\right)\right] \\ & \times\sum_{l=1}^n\overline{\phi^\mu(\mathbf{k}_l)}\xi_\mu(\mathbf{k}_l)f^{(n)}(\mathbf{k}_1,\mathbf{k}_2,\cdots,\mathbf{k}_n). \end{aligned} \tag{70}$$

This implies (24)

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