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Semiclassical Theory for Two-anyon System

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Abstract

The semiclassical quantization conditions for all partial waves are derived for bound states of two interacting anyons in the presence of a uniform background magnetic field. Singular Aharonov-Bohm-type interactions between the anyons are dealt with by the modified WKB method of Friedrich and Trost. For s-wave bound state problems in which the choice of the boundary condition at short distance gives rise to an additional ambiguity, a suitable generalization of the latter method is required to develop a consistent WKB approach. We here show how the related self-adjoint extension parameter affects the semiclassical quantization condition for energy levels. For some simple cases admitting exact answers, we verify that our semiclassical formulas in fact provide highly accurate results over a broad quantum number range.

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I. INTRODUCTION

In two-dimensional space where the rotation group (SO(2)) is abelian, particles obeying fractional statistics, i.e., anyons, can exist [1–4]. They can be given a concrete mathematical representation in flux-charge composites, since 'charged' bosonic (or fermionic) particles with non-zero magnetic flux attached behave like anyons thanks to the Aharonov-Bohm type interference [5]. Among the physical realizations of anyonic objects, the quasiparticle or quasihole in the Laughlin state of the fractional quantum Hall system corresponds to a particularly interesting case [6]. In this regard, a challenging problem is to settle from the first principles what sort of quantum states would be allowed for anyon matter with some realistic interactions (and possibly a uniform background magnetic field).

Choosing *bosonic* description, quantum dynamics of a system of anyons will be governed by the Hamiltonian of the form [3, 4]

$$H = \sum_n \frac{1}{2m}\left[\vec{p}_n - \alpha\vec{a}_n - e\vec{A}^{\rm ex}(\vec{r}_n)\right]^2 + \frac{1}{2}\sum_{n,m(\neq n)} V(|\vec{r}_n - \vec{r}_m|), \tag{1.1}$$

where $\vec{r}_n = (x_n^1, x_n^2)$, $\vec{A}^{\rm ex}(\vec{r}) = \frac{B}{2}(-x^2, x^1)$ with $B > 0$ is the vector potential appropriate to a uniform background magnetic field, and

$$a_n^i = \epsilon^{ij}\sum_{m(\neq n)} \frac{x_n^j - x_m^j}{|\vec{r}_n - \vec{r}_m|^2}. \tag{1.2}$$

The parameter α characterizes the type of anyons and without loss of generality α may be restricted to the interval (-1, 1]. For $\alpha = 0$ ($\alpha = 1$), we have physical bosons (fermions). This defines a Galilean-invariant system if the background magnetic field, included for generality, is turned off (i.e., for $B = 0$). In the presence of a generic two-body interaction potential $V(|\vec{r}_n - \vec{r}_m|)$ together with the Aharonov-Bohm interactions represented by the point vortex potentials (1.2), the above Hamiltonian leads to a nontrivial energy eigenvalue equation even for the two-body case — the starting point of a systematic n-body study. [See Refs. [3, 4, 7–9] where some related studies are made, usually without a two-body potential term]. For the two-anyon system the time-independent Schrödinger equation, after separating the center-of-mass dynamics (just the problem of a particle with mass $M = 2m$ and charge $2e$ moving in the background magnetic field), reduces to the following equation which has only the relative position $\vec{r} = \vec{r}_2 - \vec{r}_1 \equiv (x, y)$ as independent variables:

$$\left\{-\frac{1}{2\mu}\left[\vec{\nabla}_r - i\alpha\vec{a}(\vec{r}) - \frac{ie}{2}\vec{A}^{\rm ex}(\vec{r})\right]^2 + V(r)\right\}\psi(\vec{r}) = E\psi(\vec{r}), \tag{1.3}$$

where $\mu = \frac{1}{2}m$ and $a^i(\vec{r}) = \epsilon^{ij}\frac{x^j}{r^2}$. [We set $\hbar = 1$ in this paper]. The relative dynamics is effectively that of a single particle moving in the presence of a uniform magnetic field, a point vortex at the origin, and a certain radial potential $V(r)$.

With a nontrivial potential $V(r)$ an exact analysis of (1.3) is usually not possible and hence suitable approximation methods may be sought. In this paper, we shall study quantum bound states of two anyons through the semiclassical or WKB analysis of the radial Schrödinger equations for partial wave amplitudes $\psi_l(r)$ $(l = 0, \pm 2, \pm 4, \cdots)$ which are derived from (1.3). To account for the effects due to the singular point vortex potential (as well as the centrifugal potential term) within the WKB method, the conventional Langer modification [10] of the potential is not adequate; but, the modified method of Friedrich and Trost [11] can be used in an effective way. Based on the latter method, we have obtained the semiclassical quantization condition (containing a nonintegral Maslov index [12])

$$\int_{r_1}^{r_2} dr \sqrt{2\mu(E - V_{\mathit{eff}}(r))} = \left(n + \frac{1}{2} + \frac{1}{2}|l - \alpha| - \frac{1}{2}\sqrt{\eta}\right)\pi, \qquad (n = 0, 1, 2, \cdots) \tag{1.4}$$

where $\eta \equiv (l-\alpha)^2 - \frac{1}{4}$ (assumed to be positive here), $V_{\mathit{eff}}(r)$ is the effective one-dimensional potential

$$V_{\mathit{eff}}(r) = V(r) - \frac{eB}{4\mu}(l - \alpha) + \frac{e^2}{32\mu}B^2 r^2 + \frac{\eta}{2\mu r^2}, \tag{1.5}$$

and r_1 and r_2 refer to the related classical turning points. [The condition (1.4) may be used when the WKB wave function in the region $r > r_2$ contains a decreasing exponential only]. Efficacy of this formula becomes evident once one sees how its predictions compare against the exact values available for some simple cases.

Actually, for the case of the s-wave amplitude $\psi_{l=0}(r)$, there exists an additional complication involving the choice of boundary condition at $r = 0$ (i.e., at the point of two-particle coincidence). We know from the theory of self-adjoint extension that there exist a one-parameter family of acceptable boundary conditions [13], including the so-called hard-core boundary condition [5] as a special case. [Note that there is no *a priori* reason to choose specifically the hard-core boundary condition — a real system under study should determine the relevant boundary condition eventually]. In accordance with this theory, one can represent the s-wave amplitude $\psi_{l=0}(r)$ for small r by the form

$$\psi_{l=0}(r) \propto \left\{ J_{|\alpha|}(kr) + \tan\theta \left(\frac{k}{\rho}\right)^{2|\alpha|} J_{-|\alpha|}(kr) \right\}, \tag{1.6}$$

where $k \equiv \sqrt{2\mu(E - V(0) - \frac{eB}{4\mu}\alpha)}$, θ (the self-adjoint extension parameter) is a real dimensionless number, and ρ a reference scale introduced for convenience. The hard-core boundary condition is associated with a special choice, $\theta = 0$. Needless to say, this boundary-condition-dependent effect should be taken into account in the WKB analysis of the s-wave amplitude. The Langer modification method is simply not applicable here; but, the method of Friedrich and Trost has a natural generalization for this problem. The resulting s-wave semiclassical quantization condition we have obtained for $\frac{1}{2} \leq |\alpha| \leq 1$ has the form

$$\int_{r_1}^{r_2} dr \sqrt{2\mu(E - V_{\mathit{eff}}(r))} = \left(n + \frac{1}{2} + \frac{1}{2}|\alpha| - \frac{1}{2}\sqrt{\alpha^2 - \tfrac{1}{4}}\right)\pi + \Theta(E), \qquad (n = 0, 1, 2, \cdots) \tag{1.7}$$

where $\Theta(E)$, a function of energy E (through $k = \sqrt{2\mu(E - V(0) - \frac{eB}{4\mu}\alpha)}$), is related to the above self-adjoint extension parameter by

$$\tan\Theta = -\frac{\sin\pi|\alpha| \tan\theta\ (k/\rho)^{2|\alpha|}}{1 + \cos\pi|\alpha| \tan\theta\ (k/\rho)^{2|\alpha|}}. \tag{1.8}$$

For $|\alpha| < \frac{1}{2}$, we need some extra consideration and this case is covered by our another formula, given in (3.12).

It might be of some interest to look at our problem also from the viewpoint of field theory. In the field-theoretic description, one can describe (nonrelativistic) anyons by using bosonic Schrödinger fields $\psi(\vec{r}, t),\ \psi^\dagger(\vec{r}, t)$ coupled to an abelian Chern-Simons gauge field $a_\mu(\vec{r}, t)\ (\mu = 0, 1, 2)$ [14]. For the above system specifically, one may consider the Lagrangian density

$$\begin{aligned}\mathcal{L} = &\frac{\kappa}{2}\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \psi^\dagger(iD_t + \frac{1}{2m}\vec{D}^2)\psi - \frac{1}{2}\lambda_B \psi^\dagger\psi^\dagger\psi\psi \\ &- \frac{1}{2}\int d^2\vec{r}\,' \psi^\dagger(\vec{r}, t)\psi^\dagger(\vec{r}\,', t)V(|\vec{r} - \vec{r}\,'|)\psi(\vec{r}\,', t)\psi(\vec{r}, t),\end{aligned} \tag{1.9}$$

where $D_t = \partial_t + iqa_0,\ \vec{D} = \vec{\nabla} - iq\vec{a} - ie\vec{A}^{\mathrm{ex}}$, and the anyon parameter α should be identified with $\frac{q^2}{2\pi\kappa}$. Here the contact interaction term, $-\frac{1}{2}\lambda_B\psi^\dagger\psi^\dagger\psi\psi$, which becomes necessary to ensure the renormalizability of the theory, is responsible for the s-wave boundary condition ambiguity mentioned above [15]; for the precise correspondence between the method of self-adjoint extension in quantum mechanics and the regularization/renormailization procedure in field theory, see Refs. [15–17]. Thus, with a suitable transcription made from the self-adjoint extension parameter to the renormalized contact coupling λ_R (as considered, for instance, in Ref. [17]), our WKB analysis provides appropriate results for the two-particle

bound states of this nonrelativistic field theory system also. But, in this work, such field theoretic language will not be used.

This paper is organized as follows. In Sec.2 we use the modified WKB method to find approximate non-s-wave bound states of the Schrödinger equation (1.3) and especially derive the semiclassical energy level formula (1.4). Then, for some simple cases (e.g., for a circular billiard), we check our WKB-based predictions against the exact results. In Sec.3 the case of s-wave bound states is studied within the framework of the modified WKB method, with special attention given to the dependence on the self-adjoint extension parameter (or contact coupling). Section 4 contains a summary and discussions of our work.

II. NON-S-WAVE SEMICLASSICAL BOUND STATES

The relative dynamics of the two anyon system is governed by the Schrödinger equation (1.3). This equation is analyzed most conveniently in polar coordinates, as the vector potentials in our problem are equal to

$$\vec{a}(\vec{r}) = \frac{1}{r}\hat{\theta} \quad , \quad \vec{A}^{\rm ex}(\vec{r}) = \frac{B}{2} r\hat{\theta}. \tag{2.1}$$

Now, writing $\psi(\vec{r}) = \psi_l(r)e^{il\theta}$ (with l restricted to even integer values in our bosonic description of anyons) in (1.3), one obtains the following radial equation for the partial wave amplitude $\psi_l(r)$:

$$\left\{-\frac{1}{2\mu}\left[\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}(l-\alpha)^2\right] + \tilde{V}(r)\right\}\psi_l(r) = E\psi_l(r), \tag{2.2}$$

where $\tilde{V}(r) \equiv V(r) - \frac{eB}{4\mu}(l-\alpha) + \frac{e^2B^2}{32\mu}r^2$. Introducing the function $R_l(r)$ by

$$\psi_l(r) = \frac{1}{\sqrt{r}} R_l(r), \tag{2.3}$$

(2.2) can be further recast into the form

$$\left\{-\frac{1}{2\mu}\frac{d^2}{dr^2} + V_{\rm eff}(r)\right\} R_l(r) = E R_l(r) \tag{2.4}$$

with the effective one-dimensional potential (given already in (1.5))

$$V_{\rm eff}(r) = \tilde{V}(r) + \frac{\eta}{2\mu r^2}, \qquad (\eta \equiv (l-\alpha)^2 - \frac{1}{4}). \tag{2.5}$$

The eigenvalue problem (2.4) for $l = \pm 2, \pm 4, \cdots$ (and hence $\eta > 0$, assuming $|\alpha| \leq 1$) will be studied by the semiclassical method in this section. The $l = 0$ case is considered separately in the next section.

A naive application of the WKB method with the differential equation (2.4) does not yield very satisfactory results (especially for relatively small l), the reason being that (i) the radial coordinate r runs from 0 to ∞ (instead of $-\infty$ to ∞) and (ii) the potential contains a singular term at the origin, $\dfrac{\eta}{2\mu r^2}$. In fact, the naive WKB wave function does not even have the correct small-r behavior. This defect gets significantly reduced if one introduces, within the usual WKB approach, the so-called Langer modification of the potential [10], effected through replacing the potential (2.5) by

$$V_{\mathit{eff}}^{L}(r) = \tilde{V}(r) + \frac{\eta + \frac{1}{4}}{2\mu r^2}. \tag{2.6}$$

But, to go beyond this simple Langer modification scheme, it becomes necessary to incorporate the *correct phase loss* due to reflection at a classical turning point in the WKB wave function corresponding to the classically allowed region. This leads to the modified WKB method, as described recently by Friedrich and Trost [11]. In this paper, we shall use the latter method for our semiclassical discussion and see how the resulting predictions compare against exact results (and also those obtained with the Langer modification) for some special cases.

We may here suppose that the potential $V_{\mathit{eff}}(r)$, as given by (2.5) (with $\eta > 0$), has a typical shape shown in Fig. 1. Also it may be assumed that the potential $V_{\mathit{eff}}(r)$ near the origin can be approximated by $\tilde{V}(0) + \dfrac{\eta}{2\mu r^2}$, with $\tilde{V}(0) = V(0) - \dfrac{eB}{4\mu}(l - \alpha)$. Then the exact solution of (2.4), which is regular at the origin, should have the small-r behavior as given by

$$R_l^{\mathrm{ex}}(r) \propto \sqrt{kr}\, J_\nu(kr), \tag{2.7}$$

where $k = \sqrt{2\mu(E - \tilde{V}(0))}$, and J_ν is the Bessel function of order $\nu \equiv \sqrt{\eta + \frac{1}{4}} = |l - \alpha|$. We are here interested in finding the WKB bound states in the situation where there are two classical turning points (see Fig. 1), r_1 and r_2. The WKB wave function in the classically allowed region $r_1 < r < r_2$ may be written as

$$R_l^{\mathrm{WKB}}(r) \propto \frac{1}{\sqrt{p(r)}} \cos\left\{ \int_{r_1}^{r} dr' p(r') + \phi \right\}, \tag{2.8}$$

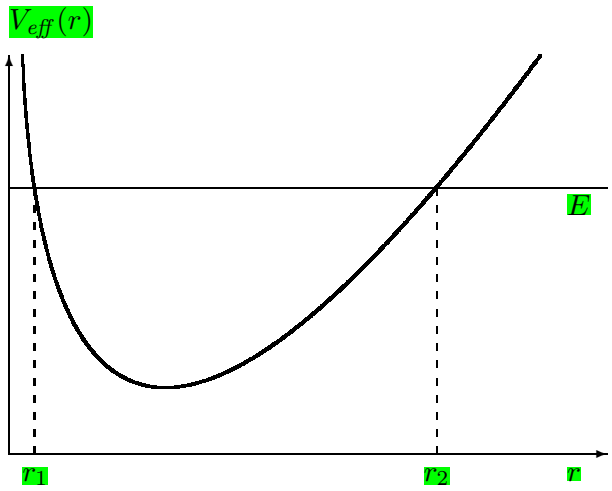

FIG. 1: The shape of our potential $V_{\mathit{eff}}(r)$ with $\eta > 0$.

where $p(r) \equiv \sqrt{2\mu(E - V_{\mathit{eff}}(r))} = \sqrt{k^2 - \frac{\eta}{r^2} - 2\mu(\tilde{V}(r) - \tilde{V}(0))}$, and the phase ϕ is yet undetermined. Note that 2ϕ can be identified as the phase loss of waves due to reflection by the barrier at $r < r_1$. In the conventional WKB method, one then uses the famous connection formula to combine the oscillating wave function in (2.8) with suitable monotonically decreasing wave functions in the classically forbidden regions (i.e., $0 < r < r_1$ and $r > r_2$). But, by the reasons mentioned above, a different strategy must be adopted in our case in dealing with the WKB wave function for small values of r. Here, following Ref. [11], we will simply fix the phase ϕ in (2.8) in such a way that the asymptotic expansion of the expression (2.8) for relatively large kr may match that following from (2.7) (i.e., the correct expression when r is not too large), say, up to terms of order $\frac{1}{(kr)^2}$. On the other hand, in the vicinity of the second turning point $r = r_2$, the normal WKB method with the standard connection formula may well be used.

Using (2.7), the exact solution of the Schrödinger equation near the origin should have the asymptotic expansion

$$R_l^{\mathrm{ex}}(r) \sim \left(1 - \frac{\eta(\eta-2)}{8(kr)^2}\right)\cos\left[kr - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right] - \frac{\eta}{2kr}\sin\left[kr - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2}\right] + O\left(\frac{1}{(kr)^3}\right). \tag{2.9}$$

For the asymptotic expansion of the WKB wave function (2.8), we may use the small-r approximation of $V_{\mathit{eff}}(r)$, i.e., $V_{\mathit{eff}}(r) \simeq \tilde{V}(0) + \frac{\eta}{2\mu r^2}$ in $p(r)$ (and correspondingly the

turning point value $r_1 \simeq \frac{\sqrt{\eta}}{k}$), and then the integral $\int_{r_1}^{r} dr' p(r')$ can be performed explicitly. The result is the following asymptotic behavior:

$$R_l^{\rm WKB}(r) \sim \left(1 - \frac{\eta(\eta-2)}{8(kr)^2}\right) \cos\left[kr - c + \phi\right] - \frac{\eta}{2kr} \sin\left[kr - c + \phi\right] + O\left(\frac{1}{(kr)^3}\right), \tag{2.10}$$

where $c = \sqrt{\eta}\frac{\pi}{2}$. Comparing (2.10) with (2.9), we see that the two expressions agree up to terms of order $\frac{1}{(kr)^2}$ only if we choose the phase ϕ as

$$\phi = c - \left(\nu + \frac{1}{2}\right)\frac{\pi}{2} = \left(\sqrt{\eta} - \sqrt{\eta + \tfrac{1}{4}} - \frac{1}{2}\right)\frac{\pi}{2}. \tag{2.11}$$

[In contrast, we note that the standard WKB method with the Langer modification produces the right argument of sine and cosine in (2.10), but *wrong* coefficients for the terms proportional to $\frac{1}{kr}$ and $\frac{1}{(kr)^2}$ in (2.10)].

The phase ϕ in our WKB wave function (2.8) has been determined now. On the other hand, the WKB wave function obtained by applying the standard connection formula at the second turning point $r = r_2$ has the form

$$R_l^{\rm WKB}(r) \propto \frac{1}{\sqrt{p(r)}} \cos\left(-\int_r^{r_2} dr' p(r') + \frac{\pi}{4}\right). \tag{2.12}$$

The two functions (2.8) and (2.12) must of course be the same. From this follows the quantization condition for energy levels

$$\begin{aligned} \int_{r_1}^{r_2} dr' p(r') &= n\pi - \phi + \frac{\pi}{4} \\ &= \left[n + \frac{1}{2} + \frac{1}{2}\left(\sqrt{\eta + \tfrac{1}{4}} - \sqrt{\eta}\right)\right]\pi. \quad (n = 0, 1, 2, \cdots) \end{aligned} \tag{2.13}$$

This is our formula (1.4). For the sake of comparison, we note that the WKB quantization condition obtained with the help of the Langer modification reads

$$\int_{r_1}^{r_2} dr \sqrt{2\mu\left(E - \tilde{V}(r) - \frac{\eta + \frac{1}{4}}{2\mu r^2}\right)} = \left(n + \frac{1}{2}\right)\pi. \quad (n = 0, 1, 2, \cdots) \tag{2.14}$$

We also remark that, for an effective potential having a qualitatively different form from that shown in Fig. 1, the WKB wave function beyond the point $r = r_2$ might not be given by a decreasing exponential only. In such case the phase $\frac{\pi}{4}$ in the argument of cosine in (2.12)

should be replaced by an appropriate different value, and our formula for the semiclassical energy levels need to be adjusted accordingly.

We will now consider some special cases to see how well the predictions based on our semiclassical formula (2.13) (and (2.14) also for comparison) fare against the exact results. Suppose we have $V(r) = 0$, but $B \neq 0$, i.e., the case of noninteracting anyons in a uniform magnetic field [3, 7]. In this case the solution to the radial equation (2.4), which is regular at the origin, can be expressed in terms of the confluent hypergeometric function [18],

$$R_l(r) = \left(\frac{eB}{4}r^2\right)^{\frac{b}{2}-\frac{1}{4}} e^{-eBr^2/8} F(a, b, \frac{eB}{4}r^2), \tag{2.15}$$

where $a = \frac{1}{2}[1 + |l - \alpha| + (l - \alpha)] - \frac{2\mu E}{eB}$, and $b = 1 + |l - \alpha|$. The large-x asymptotics of the confluent hypergeometric function $F(a, b, x)$ is well-known: $F(a, b, x) \sim \frac{\Gamma(b)}{\Gamma(a)} e^x x^{a-b}$ if $a \neq 0, -1, -2, \cdots$, while $F(a, b, x)$ for $a = 0, -1, -2, \cdots$ reduces to a polynomial (called the associate Laguerre polynomial). Hence, to obtain a normalizable solution from (2.15), the energy E must be restricted to the values

$$E = \left(n + \frac{1}{2} + \frac{1}{2}|l - \alpha| - \frac{1}{2}(l - \alpha)\right)\omega_c, \qquad (n = 0, 1, 2, \cdots) \tag{2.16}$$

where $\omega_c = \frac{eB}{2\mu}$ is the classical cyclotron frequency. These are exact energy levels, exhibiting very different l-dependences for $l > 0$ and for $l < 0$ [3, 7]. On the other hand, within our semiclassical approach, the energy levels follow immediately once we evaluate the integral in the left hand side of (2.13). With $\tilde{V}(r) = -\frac{eB}{4\mu}(l - \alpha) + \frac{e^2B^2}{32\mu}r^2$, the given integral is readily evaluated to yield the condition of the form

$$\begin{aligned} &\frac{2\pi\mu}{eB}\left\{E + \frac{eB}{4\mu}(l - \alpha)\right\} - \sqrt{\eta}\frac{\pi}{2} \\ &\quad = \left[n + \frac{1}{2} + \frac{1}{2}\left(\sqrt{\eta + \tfrac{1}{4}} - \sqrt{\eta}\right)\right]\pi. \qquad (n = 0, 1, 2, \cdots) \end{aligned} \tag{2.17}$$

If we solve this equation for E, the result is precisely (2.16); in this case, our semiclassical formula for energy levels yields the *exact* results. Incidentally, for this special case, we note that the exact results are reproduced even if we use the WKB formula with the Langer modification, i.e., (2.14).

As a particularly simple example with nonzero interaction, let us now consider a circular billiard potential (for general $B > 0$), i.e.,

$$V(r) = \begin{cases} 0 & , \ r < R \\ \infty & , \ r > R. \end{cases} \tag{2.18}$$

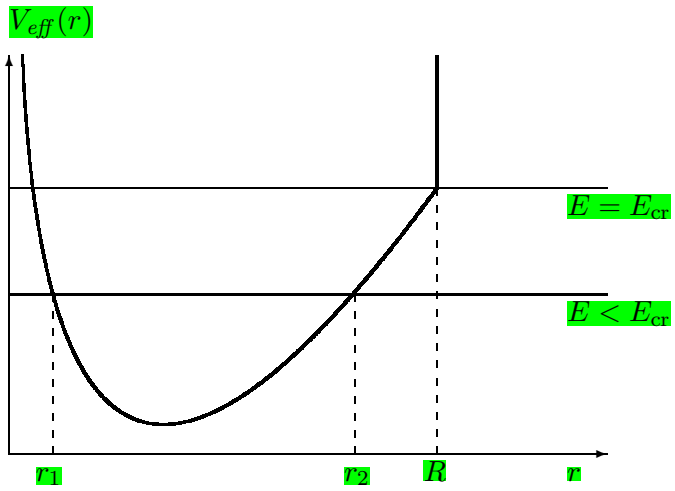

FIG. 2: The shape of $V_{eff}(r)$ for a circular billiard of radius R.

[This type of potential was previously considered in connection with the virial coefficient calculation of the anyon gas [21]]. The corresponding exact solution to (2.4) is then the expression (2.15), subject to the boundary condition $R_l(r = R) = 0$, that is,

$$F\left(a, b, z = \frac{eB}{4}R^2\right) = 0, \tag{2.19}$$

when a and b here represent the same quantities as defined above. Given information on the roots of (2.19), one can determine the exact energy eigenvalues. On the other hand, for the semiclassical energy levels, we cannot use the formula (2.13) blindly — for the effective potential as shown in Fig. 2, the expression (2.12) is not appropriate. That is, instead of (2.12), we must use the form

$$R_l^{\rm WKB}(r) \propto \frac{1}{\sqrt{p(r)}} \cos\left(-\int_r^{r_2} dr' p(r') + \delta\right) \tag{2.20}$$

with the phase $\delta\left(\neq \frac{\pi}{4}\right)$ calculated to suit the present case. This in turn has the consequence that we must add $\left(\delta - \frac{\pi}{4}\right)$ to the right hand side of the quantization condition (2.13) (and of (2.14) also). Here we have two different situations depending on $E \geq E_{\rm cr}$ or $E < E_{\rm cr}$, where $E_{\rm cr} = -\frac{eB}{4\mu}(l - \alpha) + \frac{e^2B^2}{32\mu}R^2 + \frac{\eta}{2\mu R^2}$. For $E \geq E_{\rm cr}$, the second turning point is at $r_2 = R$ (i.e., at the infinite wall) and, obviously, $\delta = \frac{\pi}{2}$ is the correct choice. For $E < E_{\rm cr}$, however, we have $r_2 < R$ (see Fig. 2) and may consider another WKB wave function in the classically forbidden region $r_2 < r < R$,

$$R_l^{\rm WKB}(r) \propto \frac{1}{\sqrt{\kappa(r)}} \left\{ A e^{-\int_{r_2}^{r} dr' \kappa(r')} + B e^{\int_{r_2}^{r} dr' \kappa(r')} \right\}, \quad \kappa(r) \equiv \sqrt{2\mu(V_{eff}(r) - E)} \tag{2.21}$$

TABLE I: Energy levels (E_n) for a circular billiard of radius R with nonzero B. The parameters were chosen as $l = 2$, $\alpha = \frac{1}{2}$, $\mu = \frac{1}{2}$, $R = 1$, $eB = 2$.

n	Langer	Ours	Exact
0	18.4719	18.7389	18.7843
1	57.9961	58.2515	58.2664
2	117.2249	117.4775	117.4850
3	196.1862	196.4378	196.4425
4	294.8844	295.1355	295.1385
5	413.3210	413.5717	413.5745
6	551.4962	551.7468	551.7484
7	709.4104	709.6608	709.6621

with the coefficients A and B chosen such that $R_l^{\rm WKB}(r = R) = 0$ may hold. Then the phase δ in (2.20) should be determined by using the (standard) WKB connection formula with (2.21); this gives the value $\delta = \frac{\pi}{4} + \cot^{-1}\left(2e^{2\int_{r_2}^{R} dr' \kappa(r')}\right)$.

In the presence of the billiard potential (2.18), we have given above all the elements from which the exact and semiclassical energy levels can be found, numerically at least. By studying them, we have observed that the semiclassical predictions, especially the ones based on our method (compared to that based on the Langer modification of the potential), give generically highly accurate results for a broad range of quantum numbers. This should be evident from looking at Table I, where the exact levels and the semiclassical predictions — both ours and that based on the Langer modification — are compared for some representative choice of parameters (for which we have $r_2 = R$ for all $n = 0, 1, 2, \cdots$). [With $R = 5$ (but the same values for the other parameters) we have $r_2 < R$ for small n; still, we have found the excellent overall agreement between the semiclassical and exact energy levels].

III. S-WAVE SEMICLASSICAL BOUND STATES

In this section we specialize to the s-wave bound states of two anyons. The relative wave function then depends on the coordinate r only, i.e., $\psi(\vec{r}) = \frac{1}{\sqrt{r}} R_0(r)$, and from the Schrödinger equation the function $R_0(r)$ should satisfy the differential equation (see (2.4))

$$\left\{ -\frac{1}{2\mu}\frac{d^2}{dr^2} + \frac{\alpha^2 - \frac{1}{4}}{2\mu r^2} + \tilde{V}(r) \right\} R_0(r) = E R_0(r), \tag{3.1}$$

where $\tilde{V}(r) \equiv V(r) + \frac{eB}{4\mu}\alpha + \frac{e^2B^2}{32\mu} r^2$ (the $l = 0$ form of the same function introduced in the previous section). For any potential $\tilde{V}(r)$ which is regular at $r = 0$, one may infer the small-r behavior of the solution $R_0(r)$ by studying the solution of the simpler second-order differential equation, obtained from (3.1) with the replacement of the potential $\tilde{V}(r)$ by the constant $\tilde{V}(0) = V(0) + \frac{eB}{4\mu}\alpha$. In fact, it was an analogous reasoning that picked the form (2.7) as a unique allowed choice for $l \neq 0$. For s-wave states, on the other hand, the form (2.7), i.e., $R_0^{\rm ex}(r) \propto \sqrt{kr} J_{|\alpha|}(kr)$ does not represent the most general small-r behavior allowed for the exact solution of (3.1); here, one finds a *normalizable* wave function even if $R_0(r)$ contains a piece corresponding to another independent solutioin of the just-mentioned second-order differential equation, that is, the form $\sqrt{kr} N_{|\alpha|}(kr)$ ($N_{|\alpha|}$ is the Neumann function). Therefore, for the small-r behavior of the exact solution of (3.1), one may well consider the general form

$$R_0^{\rm ex}(r) \propto \sqrt{kr} \left\{ \cos\Theta \; J_{|\alpha|}(kr) + \sin\Theta \; N_{|\alpha|}(kr) \right\}, \qquad \left(k = \sqrt{2\mu(E - \tilde{V}(0))} \right) \tag{3.2}$$

with an arbitrary r-independent angle Θ.

In quantum mechanics with a singular potential, it is the theory of self-adjoint extension of the related Hamiltonian that determines what boundary conditions may be allowed. [A particularly instructive example here is that of the two-dimensional δ-function potential problem, considered in Ref. [19]]. From this theory follows the one-parameter family of boundary conditions for our s-wave function $\psi_{l=0}(r) = \frac{1}{\sqrt{r}} R_0(r)$ [13],

$$\lim_{r\to 0} \left\{ r^{|\alpha|}\psi_{l=0}(r) - \tan\theta \left(\frac{2}{\rho}\right)^{2|\alpha|} \frac{\Gamma(1+|\alpha|)}{\Gamma(1-|\alpha|)} \frac{d}{d(r^{2|\alpha|})} [r^{|\alpha|}\psi_{l=0}(r)] \right\} = 0, \tag{3.3}$$

where θ, a dimensionless real number, is the self-adjoint extension parameter, and ρ just a (conveniently introduced) reference scale. It is this general boundary condition that leads

to the small-r behavior shown in (1.6). One can now translate the expression (1.6) into the form (3.2) with the help of the identity $J_{-|\alpha|}(kr) = J_{|\alpha|}(kr)\cos\pi|\alpha| - N_{|\alpha|}(kr)\sin\pi|\alpha|$. Between the constant Θ in (3.2) and the self-adjoint extension parameter θ, we then find the connection given in (1.8); this also tells us that Θ depends on k (and hence on energy E).

What would be the correct way of implementing the specific small-r behavior shown in (3.2) with semiclassical bound states? Clearly, applying the standard WKB method with the already-mentioned Langer modification trick would be unjustified here. But we can still make use of the Friedrich-Trost approach — use the WKB wave function in the classically allowed region, i.e.,

$$R_0^{\rm WKB}(r) \propto \frac{1}{\sqrt{p(r)}}\cos\left\{\int_{r_1}^{r} dr' p(r') + \phi\right\}, \qquad \left(p(r) = \sqrt{2\mu(E - V_{\it eff}(r))}\,;\;\; V_{\it eff}(r) = \tilde{V}(r) + \frac{\alpha^2 - \frac{1}{4}}{2\mu r^2}\right) \tag{3.4}$$

with the phase ϕ chosen such that this WKB wave function may lead to an asymptotic form (for relatively large kr) in agreement with that following now from the supposedly correct expression in (3.2). The asymptotic expansion of (3.2) is easily found:

$$R_0^{\rm ex}(r) \;\sim\; \left(1 - \frac{\eta(\eta-2)}{8(kr)^2}\right)\cos\left\{kr - \left(|\alpha| + \frac{1}{2}\right)\frac{\pi}{2} - \Theta\right\} - \frac{\eta}{2kr}\sin\left\{kr - \left(|\alpha| + \frac{1}{2}\right)\frac{\pi}{2} - \Theta\right\} + O\left(\frac{1}{(kr)^3}\right), \tag{3.5}$$

where $\eta \equiv \alpha^2 - \frac{1}{4}$. As regards the WKB wave function (3.4), a new problem arises for $\eta < 0$ (i.e., if $V_{\it eff}(r)$ contains an attractive singular term at the origin) and it thus becomes necessary to discuss the case with $|\alpha| < \frac{1}{2}$ separately from that with $|\alpha| \geq \frac{1}{2}$. See below on this.

If $|\alpha|$ is lager than $\frac{1}{2}$ (i.e., for $\eta > 0$), the present s-wave WKB wave function will have qualitatively the same structure as the non-s-wave WKB function considered in (2.8), and hence its asymptotic behavior by the form (2.10) with the substitution $\eta = \alpha^2 - \frac{1}{4}$ now. Then, as we demand that it match the asymptotic behavior in (3.5), the phase ϕ in the WKB wave function (3.4) will be fixed to have the value

$$\phi = \frac{\pi}{2}\sqrt{\alpha^2 - \frac{1}{4}} - \left(|\alpha| + \frac{1}{2}\right)\frac{\pi}{2} - \Theta, \qquad \left(|\alpha| \geq \frac{1}{2}\right). \tag{3.6}$$

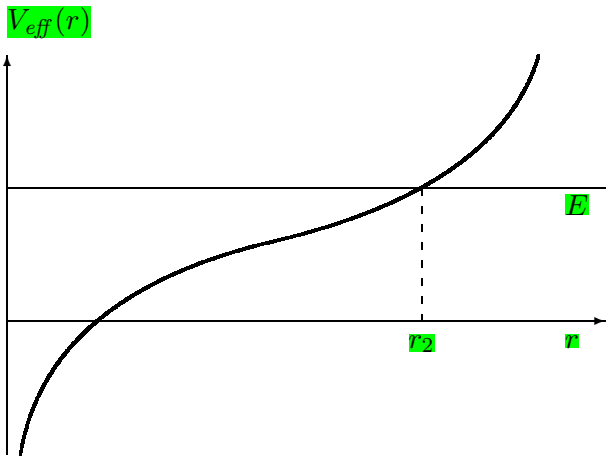

FIG. 3: The shape of $V_{eff}(r)$ with $\eta < 0$.

[Note that, with Θ set to zero, this is nothing but the formula (2.11). Also we have invoked the continuity of our formula to include the case $|\alpha| = \frac{1}{2}$ here]. If $|\alpha| < \frac{1}{2}$ (i.e., for $\eta < 0$), on the other hand, we find the behavior $V_{eff}(r) \to -\infty$ as $r \to 0$ and hence no classical turning point near the origin. (See Fig. 3). Thus, r_1 in the expression (3.4) is ambiguous. For this case, one might be inclined to choose $r_1 = 0$; but, then, the integral $\int_0^r dr' p(r')$ becomes ill-defined. This difficulty is resolved if one chooses the point r_1 to be not the origin but a certain (small) positive number [20]. The asymptotic behavior of the WKB wave function (3.4) can then be described by the form (2.10) again only if the constant c there is now taken to be

$$c = \sqrt{|\eta| + (kr_1)^2} - \sqrt{|\eta|}\ln\left(\sqrt{1 + \frac{|\eta|}{(kr_1)^2}} + \frac{\sqrt{|\eta|}}{kr_1}\right), \qquad \left(\text{for } \eta = \alpha^2 - \frac{1}{4} < 0\right). \tag{3.7}$$

Comparing such asymptotic expressioin with the behavior (3.5), one immediately sees that the phase ϕ in this case should be chosen

$$\phi = c - \left(|\alpha| + \frac{1}{2}\right)\frac{\pi}{2} - \Theta, \qquad \left(|\alpha| < \frac{1}{2}\right). \tag{3.8}$$

The WKB wave function (3.4) for $|\alpha| < \frac{1}{2}$, with this phase choice will have (in the region beyond the point r_1) only a very mild dependence on the cutoff r_1. Actually, we can do away with introducing this ad hoc cutoff by changing our WKB wave function slightly. Let $\bar{V}_{eff}(r)$ denote the small-r approximation of $V_{eff}(r)$, i.e.,

$$\bar{V}_{eff}(r) = \tilde{V}(0) - \frac{|\eta|}{2\mu r^2}. \tag{3.9}$$

Then we would not sacrifice none of the desired properties of the WKB wave function by changing the argument of cosine in (3.4) to the expression

$$\begin{aligned}
&\int_0^r dr' \left\{ \sqrt{2\mu(E - V_{\mathit{eff}}(r'))} - \sqrt{2\mu(E - \bar{V}_{\mathit{eff}}(r'))} \right\} + \int_{r_1}^r dr' \sqrt{2\mu(E - \bar{V}_{\mathit{eff}}(r'))} + \phi \\
&= \int_0^r dr' \left\{ \sqrt{2\mu(E - V_{\mathit{eff}}(r'))} - \sqrt{2\mu(E - \bar{V}_{\mathit{eff}}(r'))} \right\} + \sqrt{|\eta| + (kr)^2} \\
&\quad - \sqrt{|\eta|} \ln \left(\sqrt{1 + \frac{|\eta|}{(kr)^2}} + \frac{\sqrt{|\eta|}}{kr} \right) - \left(|\alpha| + \frac{1}{2} \right) \frac{\pi}{2} - \Theta,
\end{aligned} \tag{3.10}$$

where (3.8) has been used. With $|\alpha| < \frac{1}{2}$, this manifestly cutoff independent WKB wave function shall be used in our discussions below.

In the presence of the second classical turning point $r = r_2$, the WKB wave function found above can be used to obtain the s-wave semiclassical energy levels. This part of argument is entirely parallel to that in the previous section, and hence we may state the result only. Assuming that the potential is smooth near the second turning point and the WKB wave function in the region $r > r_2$ contains a decreasing exponential only, the quantization condition for energy levels reads, for $|\alpha| \geq \frac{1}{2}$ (i.e., $\eta \geq 0$)

$$\int_{r_1}^{r_2} dr \sqrt{2\mu(E - V_{\mathit{eff}}(r))} = \left[n + \frac{1}{2} + \frac{1}{2} \left(|\alpha| - \sqrt{\alpha^2 - \tfrac{1}{4}} \right) \right] \pi + \Theta, \tag{3.11}$$

and, for $|\alpha| < \frac{1}{2}$ (i.e., $-\frac{1}{4} \leq \eta < 0$)

$$\begin{aligned}
&\int_0^{r_2} dr \left\{ \sqrt{2\mu(E - V_{\mathit{eff}}(r))} - \sqrt{2\mu(E - \bar{V}_{\mathit{eff}}(r))} \right\} + \sqrt{|\eta| + (kr_2)^2} \\
&\quad - \sqrt{|\eta|} \ln \left(\sqrt{1 + \frac{|\eta|}{(kr_2)^2}} + \frac{\sqrt{|\eta|}}{kr_2} \right) = \left(n + \frac{1}{2} + \frac{1}{2}|\alpha| \right) \pi + \Theta.
\end{aligned} \tag{3.12}$$

Here it should be remarked that Θ itself is a function of E, being related to the energy variable (and the self-adjoint extension parameter θ) by (1.8), It is through the presence of this function Θ in (3.11) and (3.12) that the θ-dependence enters the semiclassical energy levels.

To check the accuracy of the above s-wave quantization formulas, we may here consider the case $V(r) = 0$, $B \neq 0$ (and with $|\alpha| \geq \frac{1}{2}$ for simplicity). Then the exact solution to (3.1), which satisfies the θ-dependent boundary condition (1.6) at the origin, is

$$\begin{aligned}
R_0(r) &= z^{\frac{b}{2} - \frac{1}{4}} e^{-\frac{z}{2}} F(a, b, z) \\
&\quad + \tan\theta \left(\frac{2}{\rho} \right)^{2|\alpha|} \left(\frac{eB}{4} \right)^{|\alpha|} \frac{\Gamma(1 + |\alpha|)}{\Gamma(1 - |\alpha|)} z^{\frac{3}{4} - \frac{b}{2}} e^{-\frac{z}{2}} F(1 + a - b, 2 - b, z),
\end{aligned} \tag{3.13}$$

TABLE II: Energy levels (E_n) for the s-wave of the eigenstates of $V = 0$, but $B \neq 0$. The parameters were chosen as $\alpha = \frac{2}{3}$, $\mu = \frac{1}{2}$, $eB = 2$, $\theta = \frac{\pi}{4}$.

n	Ours	Exact
-1	1.6667	1.6477
0	3.3468	3.3440
1	5.2337	5.2329
2	7.1796	7.1793
3	9.1480	9.1478
4	11.1270	11.1269
5	13.1120	13.1120
6	15.1007	15.1007

where $z \equiv \frac{eB}{4}r^2$, $a = \frac{1}{2}(1 + |\alpha| + \alpha) - \frac{2\mu E}{eB}$, and $b = 1 + |\alpha|$. For this function to be normalizable, we must further demand from the asymptotic consideration that

$$\frac{\Gamma(b)}{\Gamma(a)} + \tan\theta \left(\frac{2}{\rho}\right)^{2|\alpha|} \left(\frac{eB}{4}\right)^{|\alpha|} \frac{\Gamma(1+|\alpha|)}{\Gamma(1-|\alpha|)} \frac{\Gamma(2-b)}{\Gamma(1+a-b)} = 0. \tag{3.14}$$

Solving this equation for E will give exact energy levels. We may then compare the semiclassical energy levels obtained with the help of (3.11) against these exact values. [With $\theta \neq 0$, the semiclassical formula does not produce exact energy levels]. In Table II, such comparison is made for some representative choice of parameters. We find the high accuracy of our semiclassical predictions, even for small n, very impressive.

IV. SUMMARY AND DISCUSSIONS

In this work we have developed the semiclassical theory of two-anyon bound states, with careful consideration given on the treatment of singular Aharonov-Bohm-type interactions between anyons. The modified WKB method of Friedrich and Trost has proved to be particularly effective for this problem. We have also clarified the role of the self-adjoint extension parameter for s-wave bound states within this semiclassical approach. For some simple cases, we have been able to confirm that our semiclassical formulas provide highly

accurate energy levels over a broad quantum number range. We expect this to be the case with more general classes of interaction potentials. It should also be possible to extend this semiclassical theory to the case of two-anyon scattering states (in the absence of a background magnetic field).

Results of the present semiclassical theory may be applied to study certain features of the anyon gas. In the high-temperature low-density limit, for instance, one usually resorts to the virial expansion to study various thermodynamic properties. In the case of the 'free' anyon gas, the second virial coefficient has been calculated in Refs. [21–23]; especially, in Ref. [23], the effect of the self-adjoint extension parameter on the virial coefficient has been considered also. Based on the semiclassical understanding of two-anyon bound/scattering states, one may extend this discussion to the case of the anyon gas with some nontrivial 2-body potential, and see what new features the presence of such 2-body interaction can give rise to. Some of these issues are under investigation.

Also note that more general kinds of anyons, other than the ones we discussed here, are possible. Anyons obeying so-called matrix (or mutual) statistics [24, 25] may prove to be relevant in the discussion of multi-layered quantum Hall effect, and, theoretically, particles obeying non-abelian statistics [26, 27] can also be contemplated. Bound states of these kind of anyons may be discussed with the help of the semiclassical theory analogous to the one considered in this paper.

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