Comments on Two-Loop Four-Particle Amplitude in Superstring Theory

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Abstract

It is shown that the four-particle amplitude of superstring theory at two loops obtained in [1, 2] is equivalent to the previously obtained results in [3, 4, 5]. Here the $\mathbf{Z}_2$ symmetry in hyperelliptic Riemann surface plays an important role in the proof.

In some previous papers [1, 2, 6] we have computed explicitly the two loop n-particle amplitude in superstring theory for all $n \leq 4$ by using the newly obtained measure of D'Hoker and Phong [7, 8, 9, 10] (for a recent review see [11, 12]). The new measure of D'Hoker and Phong is unambiguous and slice-independent. For all $n \leq 3$ we proved that the n-particle amplitudes are identically zero. A simple expression was also obtained for the 4-particle amplitude which is independent of the insertion points of the supermoduli. Our explicit results beautifully verifies the result of D'Hoker and Phong, especially for the non-zero four-particle amplitude.

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A natural question is the relation with the previously obtained result in [3, 4, 5] which had been proved to satisfy all the conditions for a qualified 4-particle amplitude. In particular, the explicit result has also been used by Iengo [13] to prove the vanishing of the 2-loop correction to the R^4 term [14], in agreement with the indirect argument of Green and Gutperle [15], Green, Gutperle and Vanhove [16], and Green and Sethi [17] that the R^4 term does not receive perturbative contributions beyond one loop. Recently, Stieberger and Taylor [18] also used the result of [3, 4, 5] to prove the vanishing of the heterotic two-loop F^4 term. For some closely related works we refer the reader to the reviews [19, 20].

In this paper we will show that the explicit results obtained by using different chiral measure are actually equivalent. This should be the case because they are all derived from first principle. In the proof we used the $\mathbf{Z}_2$ symmetry of the conformal field theory on hyperelliptic Riemann surface.

Our starting point is the chiral integrand obtained in [2]:

$$\mathcal{A} = \langle : \partial X(q_1) \cdot (\partial X(q_1) + \partial X(q_2)) : \prod_{i=1}^{4} \mathrm{e}^{ik_i \cdot X(z_i, \bar{z}_i)} \rangle \prod_{i=1}^{4} \frac{q - z_i}{y(z_i)}. \tag{1}$$

As we said in [2], we need to symmetrize in $q_{1,2}$ to obtain an explicitly q independent result. This symmetrization is justified as follows. First we have

$$\begin{aligned} : \partial X(q_1) \cdot (\partial X(q_1) + \partial X(q_2)) : &= \frac{1}{2} : ((\partial X(q_1))^2 - (\partial X(q_2))^2) : \\ &+ \frac{1}{2} : (\partial X(q_1) + \partial X(q_2)) \cdot (\partial X(q_1) + \partial X(q_2)) : \\ &= -(T(q_1) - T(q_2)) \\ &+ \frac{1}{2} : (\partial X(q_1) + \partial X(q_2)) \cdot (\partial X(q_1) + \partial X(q_2)) : , \end{aligned} \tag{2}$$

where $T(q_i) = -\frac{1}{2} : (\partial X(q_i))^2 :$ is the stress energy tensor. In the above, the last term gives an q independent result (see [2]) and the first term will give an vanishing contribution to the amplitude after integration over z_i as we can prove as follows.

By the $\mathbf{Z}_2$ symmetry of the conformal field theory on hyperelliptic Riemann surface[1] we have

$$\langle T(q_1) \prod_{i=1}^{4} \mathrm{e}^{ik_i \cdot X(z_i, \bar{z}_i)} \rangle = \langle T(q_2) \prod_{i=1}^{4} \mathrm{e}^{ik_i \cdot X(\tilde{z}_i, \bar{\tilde{z}}_i)} \rangle, \tag{3}$$

[1]See [21, 22, 23, 24] for works on conformal field theory on $\mathbf{Z}_n$ Riemann surface.

where $\tilde{z}_i$ is the Z_2 transformed point of z_i, i.e., $\tilde{z}_i$ and z_i are the two identical points on the two different sheets of the hyperelliptic Riemann surface. By using this result, we have

$$
\begin{aligned}
&\int \prod_{i=1}^{4} \mathrm{d}^2 z_i \prod_{i=1}^{4} \frac{q-z_i}{y(z_i)} \langle T(q_1) \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(z_i,\bar{z}_i)}\rangle \times (\text{Right Part}) \\
&\quad = \int \prod_{i=1}^{4} \mathrm{d}^2 z_i \prod_{i=1}^{4} \frac{q-z_i}{y(z_i)} \langle T(q_2) \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(\tilde{z}_i,\bar{\tilde{z}}_i)}\rangle \times (\text{Right Part}) \\
&\quad = \int \prod_{i=1}^{4} \mathrm{d}^2 \tilde{z}_i \prod_{i=1}^{4} \frac{q-\tilde{z}_i}{y(\tilde{z}_i)} \langle T(q_2) \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(\tilde{z}_i,\bar{\tilde{z}}_i)}\rangle \times (\text{Right Part}) \\
&\quad = \int \prod_{i=1}^{4} \mathrm{d}^2 z_i \prod_{i=1}^{4} \frac{q-z_i}{y(z_i)} \langle T(q_2) \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(z_i,\bar{z}_i)}\rangle \times (\text{Right Part}). \qquad (4)
\end{aligned}
$$

In the above we have used the $\mathbf{Z}_2$ invariance of the integration measure $\mathrm{d}^2 z_i$ and in the last step we have changed the dummy integration variables $\tilde{z}_i$'s back into z_i's. We note that in the above proof we have symbolically denoted the contribution of the right part as (Right Part). For specific amplitudes the above reasoning can be justified completely. So effectively we can set $T(q_1) = T(q_2)$ in the chiral integrand, as we have used in [2]. The chiral integrand is then given as follows:

$$
\begin{aligned}
\mathcal{A} &= \frac{1}{2}\langle : (\partial X(q_1) + \partial X(q_2))^2 : \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(z_i,\bar{z}_i)}\rangle \prod_{i=1}^{4} \frac{q-z_i}{y(z_i)} \\
&= -\frac{1}{2}\langle (\partial X^\mu(q_1) + \partial X^\mu(q_2)) k_i \cdot X(z_i)\rangle \\
&\quad \times \langle (\partial X_\mu(q_1) + \partial X_\mu(q_2)) k_j \cdot X(z_j)\rangle \langle \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(z_i,\bar{z}_i)}\rangle \prod_{i=1}^{4} \frac{q-z_i}{y(z_i)} \\
&= -\frac{1}{2}\sum_{i,j=1}^{4} k_i \cdot k_j \frac{1}{q-z_i}\frac{1}{q-z_j} \langle \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(z_i,\bar{z}_i)}\rangle \prod_{i=1}^{4} \frac{q-z_i}{y(z_i)} \\
&= \frac{s(z_1z_2+z_3z_4)+t(z_1z_4+z_2z_3)+u(z_1z_3+z_2z_4)}{2\prod_{i=1}^{4} y(z_i)} \langle \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(z_i,\bar{z}_i)}\rangle . \qquad (5)
\end{aligned}
$$

Now we recall the result obtained in [3, 4, 5] for the 4-particle chiral amplitude. We have

$$
\begin{aligned}
\tilde{\mathcal{A}} &= \Big\{ \langle \partial X(q_1) \cdot \partial X(q_2) \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(z_i,\bar{z}_i)}\rangle \\
&\quad - 4 I_{\mathrm{gh}}(q) \langle \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(z_i,\bar{z}_i)}\rangle \Big\} \prod_{i=1}^{4} \frac{q-z_i}{y(z_i)}, \qquad (6)
\end{aligned}
$$

where

$$
\begin{aligned}
I_{\rm gh}(q) &= -\frac{1}{8}\left(\sum_{i=1}^{6}\frac{1}{q-a_i}-2\sum_{i=1}^{3}\frac{1}{q-b_i}\right)\left(\sum_{i=1}^{6}\frac{1}{q-a_i}-\sum_{k=1}^{4}\frac{1}{q-z_k}\right)\\
&\quad -\frac{1}{32}\left(\sum_{i=1}^{6}\frac{1}{(q-a_i)^2}-2\sum_{i<j}^{6}\frac{1}{q-a_i}\,\frac{1}{q-a_j}+8\sum_{i<j}^{3}\frac{1}{q-b_i}\,\frac{1}{q-b_j}\right), \qquad (7)
\end{aligned}
$$

is the contribution from the ghost supercurrent. In the above expression the b_i's are the three ghost insertion points and should be set to a_i if we choose $a_{i=1,2,3}$ as the three moduli to be integrated.

By using the result of [2], we found that the contribution to the chiral integrand from $\mathcal{X}_1+\mathcal{X}_6$ is given as follows:

$$
\begin{aligned}
\mathcal{A}_1+\mathcal{A}_6 &= \prod_{i=1}^{4}\frac{q-z_i}{y(z_i)}\Big\{\langle\partial X(q_1)\cdot\partial X(q_2)\prod_{i=1}^{4}\mathrm{e}^{ik_i\cdot X(z_i,\bar z_i)}\rangle\\
&\quad -\Big[\partial_{q_1}G_2(q_1,q_2)+\partial_{q_2}G_2(q_2,q_1)\\
&\quad +(G_2(q_1,q_2)+G_2(q_2,q_1))\sum_{k=1}^{4}\frac{1}{q-z_k}\Big]\langle\prod_{i=1}^{4}\mathrm{e}^{ik_i\cdot X(z_i,\bar z_i)}\rangle.\Big\} \qquad (8)
\end{aligned}
$$

The expressions that we will need are given as follows:

$$
\begin{aligned}
G_2(q_1,q_2)+G_2(q_2,q_1) &= -\frac{1}{2}\,\Delta_1(q)+\left[\frac{1}{q-p_1}\,\frac{(q-p_2)(q-p_3)}{(p_1-p_2)(p_1-p_3)}+...\right]\\
&= -\frac{1}{2}\,\Delta_1(q)+\sum_{a=1}^{3}\frac{1}{q-p_a}, \qquad (9)\\
\partial_{q_1}G_2(q_1,q_2)+\partial_{q_2}G_2(q_2,q_1) &= \frac{3}{8}\,\Delta_1^2(q)+\frac{1}{4}\,\Delta_2(q)\\
&\quad +\left[\frac{1}{q-p_1}\left(\frac{1}{q-p_2}+\frac{1}{q-p_3}-\Delta_1(q)\right)\frac{(q-p_2)(q-p_3)}{(p_1-p_2)(p_1-p_3)}+...\right]\\
&= \frac{3}{8}\,\Delta_1^2(q)+\frac{1}{4}\,\Delta_2(q)-\Delta_1(q)\sum_{a=1}^{3}\frac{1}{q-p_a}+\sum_{a<b}^{3}\frac{1}{(q-p_a)(q-p_b)}. \qquad (10)
\end{aligned}
$$

(See [1, 2] for details and notations.) By direct comparison we found the following equality:

$$
\begin{aligned}
\mathcal{A}_1+\mathcal{A}_6 &= \Big\{\langle\partial X(q_1)\cdot\partial X(q_2)\prod_{i=1}^{4}\mathrm{e}^{ik_i\cdot X(z_i,\bar z_i)}\rangle\\
&\quad +4I_{\rm gh}(q)\langle\prod_{i=1}^{4}\mathrm{e}^{ik_i\cdot X(z_i,\bar z_i)}\rangle\Big\}\prod_{i=1}^{4}\frac{q-z_i}{y(z_i)}, \qquad (11)
\end{aligned}
$$

if we identify $p_i = b_i$, the three ghost insertion points. We note the important "+" sign in front of $4I_{\rm gh}$. This shows clearly that $\mathcal{A}_1+\mathcal{A}_6$ is not the complete expression for the chiral integrand.

To prove that $\mathcal{A}$ and $\tilde{\mathcal{A}}$ give identical scattering amplitude, we first note the following:

$$
\begin{aligned}
&\langle \partial X(q_1)\cdot\partial X(q_2)\prod_{i=1}^{4}{\rm e}^{ik_i\cdot X(z_i,\bar z_i)}\rangle \\
&= \langle \partial X(q_1)\cdot\partial X(q_2)\rangle\langle\prod_{i=1}^{4}{\rm e}^{ik_i\cdot X(z_i,\bar z_i)}\rangle \\
&-\sum_{i,j=1}^{4}\langle\partial X^\mu(q_1)k_i\cdot X(z_i)\rangle\langle\partial X_\mu(q_2)k_j\cdot X(z_i)\rangle\langle\prod_{i=1}^{4}{\rm e}^{ik_i\cdot X(z_i,\bar z_i)}\rangle \\
&= \Bigg\{\langle \partial X(q_1)\cdot\partial X(q_2)\rangle - \frac{1}{4}\sum_{i,j=1}^{4}k_i\cdot k_j\frac{1}{q-z_i}\frac{1}{q-z_j} \\
&\qquad +\frac{1}{4}\sum_{i,j=1}^{4}k_i\cdot k_j f(q_1,z_i)f(q_1,z_j)\Bigg\}\langle\prod_{i=1}^{4}{\rm e}^{ik_i\cdot X(z_i,\bar z_i)}\rangle\,,
\end{aligned}
\tag{12}
$$

by using the following explicit expression for the $\langle\partial X(q_{1,2})X(z_i,\bar z_i)\rangle$ correlator:

$$
\langle\partial X_\mu(q_a)X_\nu(z_i,\bar z_i)\rangle = -\frac{\eta_{\mu\nu}}{2}\left(\frac{1}{q_a-z_i}+f(q_a,z_i)\right)\,, \tag{13}
$$

where

$$
f(q_a,z_i)=\frac{1}{T}\int\frac{y(z_i)}{y(q_a)}\frac{1}{q_a-z_i}\frac{(q_a-u)(q_a-v)}{(z_i-u)(z_i-v)}\left|\frac{u-v}{y(u)y(v)}\right|^2{\rm d}^2u{\rm d}^2v. \tag{14}
$$

From eq. (12), we see that the second term will give half of the result of $\mathcal{A}$ when we substitute it into $\tilde{\mathcal{A}}$. It remains to compute the rest terms in $\tilde{\mathcal{A}}$. We denote it as $\mathcal{A}_D$ and it is given as follows:

$$
\begin{aligned}
\mathcal{A}_D \;=\; & \Big\{\langle \partial X(q_1)\cdot\partial X(q_2)\rangle + \frac{1}{4}\sum_{i,j=1}^{4}k_i\cdot k_j f(q_1,z_i)f(q_1,z_j) \\
&\qquad -4I_{\rm gh}(q)\Big\}\langle\prod_{i=1}^{4}{\rm e}^{ik_i\cdot X(z_i,\bar z_i)}\rangle\prod_{i=1}^{4}\frac{q-z_i}{y(z_i)}\,.
\end{aligned}
\tag{15}
$$

As we proved in [3, 4, 5], the above expression is independent of q after integration over all moduli. To simplify the computation we can make a convenient choice for the insertion points $q_{1,2}$ or q. We will take the limit $q_1 \to z_1$. In this limit we have

$$\mathcal{A}_D = \frac{z_{12}z_{13}z_{14}}{\prod_{i=1}^4 y(z_i)} \left\{ \frac{1}{2} \sum_{j=2}^4 k_1 \cdot k_j f(z_1, z_j) \right.$$
$$\left. + \frac{1}{2} \left(\Delta_1(z_1) - 2 \sum_{i=1}^3 \frac{1}{z_1 - b_i} \right) \right\} \langle \prod_{i=1}^4 \mathrm{e}^{ik_i \cdot X(z_i, \bar{z}_i)} \rangle. \quad (16)$$

We will prove that the second term vanishes and the first term gives the other half of $\mathcal{A}$. We note that this is the case only after we make this special choice for q.

First we start with the second term. By using the result of [4], we can change the integration over the moduli $a_{1,2,3}$ and $z_{1,\cdots,4}$ into $a_{1,\cdots,6}$ and z_1 by fixing $z_{2,3,4}$. The chiral integrand doesn't change and it is still given by the following expression:

$$\mathcal{A}_{D2} = \frac{z_{12}z_{13}z_{14}}{2\prod_{i=1}^4 y(z_i)} \left(\sum_{i=1}^6 \frac{1}{z_1 - a_i} - 2 \sum_{i=1}^3 \frac{1}{z_1 - b_i} \right) \langle \prod_{i=1}^4 \mathrm{e}^{ik_i \cdot X(z_i, \bar{z}_i)} \rangle. \quad (17)$$

The important point is the following: the rest part of the integrand is invariant under modular transformation, i.e., symmetric under $a_i \leftrightarrow a_j$. (We note that $b_i = a_i$.) On the other hand, $\mathcal{A}_{D2}$ is antisymmetric under the following modular transformation:

$$a_1 \to a_4, \qquad a_2 \to a_5, \qquad a_3 \to a_6. \quad (18)$$

So after integration over all a_i's, $\mathcal{A}_{D2}$ gives a vanishing result. This result can also be understood from another point of view. Because of modular invariance, each term in the summation in $\mathcal{A}_{D2}$ gives the same result after integration over all a_i's and all of them add to zero.

The other term in (16) is:

$$\mathcal{A}_{D1} = \frac{z_{12}z_{13}z_{14}}{2\prod_{i=1}^4 y(z_i)} \sum_{j=2}^4 k_1 \cdot k_j f(z_1, z_j) \langle \prod_{i=1}^4 \mathrm{e}^{ik_i \cdot X(z_i, \bar{z}_i)} \rangle. \quad (19)$$

In order to compute it explicitly, we first note the result of $T(q_1) = T(q_2)$ proved before. Explicitly we have[2]:

$$0 = \langle (T(q_1) - T(q_2)) \prod_{i=1}^4 \mathrm{e}^{ik_i \cdot X(z_i, \bar{z}_i)} \rangle$$

[2]All equality, such as $T(q_1) = T(q_2)$, should be understood as expressions inserted in the complete amplitude and integration over all moduli is implicit.

$$
\begin{aligned}
&= \frac{1}{4}\sum_{i,j=1}^{4} k_i\cdot k_j \left\{ \frac{1}{q-z_i} f(q_1,z_j) \right.\\
&\qquad \left. + \frac{1}{q-z_j} f(q_1,z_i) \right\} \langle \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(z_i,\bar{z}_i)} \rangle \,, \qquad (20)
\end{aligned}
$$

by using the explicit result for the $\langle \partial X(q_a) X(z_i) \rangle$ correlator given in (13). If we set $q_1 \to z_1$, we have

$$
\begin{aligned}
&\langle (T(q_1) - T(q_2)) \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(z_i,\bar{z}_i)} \rangle \\
&\to \frac{1}{2} \left\{ \frac{1}{q-z_1} \sum_{j=2}^{4} k_1\cdot k_j\, f(z_1,z_j) \right.\\
&\qquad \left. + \frac{1}{q-z_1} \sum_{j=2}^{4} \frac{k_1\cdot k_j}{z_1-z_j} \right\} \langle \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(z_i,\bar{z}_i)} \rangle = 0\,. \qquad (21)
\end{aligned}
$$

Substituting this result into eq. (19), we have

$$
\begin{aligned}
\mathcal{A}_{D1} &= -\frac{z_{12}z_{13}z_{14}}{2\prod_{i=1}^{4} y(z_i)} \sum_{j=2}^{4} \frac{k_1\cdot k_j}{z_1-z_j} \langle \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(z_i,\bar{z}_i)} \rangle \\
&= \frac{s(z_1z_2+z_3z_4) + t(z_1z_4+z_2z_3) + u(z_1z_3+z_2z_4)}{4\prod_{i=1}^{4} y(z_i)} \\
&\quad \times \langle \prod_{i=1}^{4} \mathrm{e}^{ik_i\cdot X(z_i,\bar{z}_i)} \rangle = \frac{\mathcal{A}}{2}. \qquad (22)
\end{aligned}
$$

This proves

$$
\mathcal{A} = \tilde{\mathcal{A}}. \qquad (23)
$$

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