

Symmetries within domain walls

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The comparison of symmetries in the interior and the exterior of a domain wall is relevant when discussing the correspondence between domain walls and branes, and also when studying the interaction of walls and magnetic monopoles. I discuss the symmetries in the context of an $SU(N) \times Z_2$ model (for odd N) with a single adjoint scalar field. Situations in which the wall interior has less symmetry than the vacuum are easy to construct while the reverse situation requires significant engineering of the scalar potential.

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I. INTRODUCTION

A connection between fundamental branes and field theoretic domain walls was drawn in [1, 2]. Processes occurring in brane scenarios were shown to have direct analogs in topological defect scenarios and vice versa (though often with effects such as confinement taken into account). This connection is noteworthy because defects have been studied for several decades and are relatively well-understood systems. One property that is central to brane scenarios is that matter as we know it is confined to the brane. Only gravitons can propagate in the bulk. The analog of this property in the defect setting is that there are massless excitations on a domain wall and that these excitations cannot escape the wall because they are massive outside. The precise spectrum of massless excitations will depend on the symmetries inside and outside the domain wall.

The spectrum of excitations that live on a topological defect is also very important for determining the interaction of the defect with other defects in the system. As an example, consider the “sweeping scenario” described in [3] where domain walls collide with magnetic monopoles that are produced in the same phase transition. The outcome of the collision depends on whether the magnetic monopole can unwind inside the domain wall [4, 5]. This in turn depends on the symmetry that exists inside the wall and whether it is large enough for the monopole to unwind. If the symmetry is sufficiently large, the walls can sweep away all the monopoles and potentially solve the cosmological magnetic monopole overabundance problem. The interactions of walls and point defects is also relevant in the laboratory where both types of defects are observed.

In this paper, I will investigate the symmetries within domain walls in a model based on a $G_N \equiv SU(N) \times Z_2$ (N =odd) symmetry group with a single adjoint scalar field, Φ . (Symmetries within domain walls in other models have been investigated in [6], while other defects have been considered in [7–9].) The Lagrangian of our model

is:

$$L = \text{Tr}(\partial_\mu \Phi)^2 - V(\Phi) \quad (1)$$

where $V(\Phi)$ is a potential invariant under G_N . The Z_2 factor in G_N takes $\Phi \rightarrow -\Phi$. If the vacuum expectation value (VEV) of Φ is:

$$\Phi_0 = \eta \sqrt{\frac{2}{N(N^2 - 1)}} \begin{pmatrix} n\mathbf{1}_{n+1} & \mathbf{0} \\ \mathbf{0} & -(n+1)\mathbf{1}_n \end{pmatrix}, \quad (2)$$

where $\mathbf{1}_p$ is the $p \times p$ identity matrix and η is an energy scale determined by the minima of the scalar potential $V(\Phi)$, the symmetry breaking pattern is:

$$H_N = [SU(n+1) \times SU(n) \times U(1)]/C. \quad (3)$$

The group C is the center of H and $N \equiv 2n+1$.

The symmetry breaking $G_N \rightarrow H_N$ leads to domain walls since the discrete Z_2 factor in G_N is spontaneously broken. Therefore, if on one side of the wall if $\Phi = \Phi_0$, then on the other side we can have $\Phi = -U^\dagger \Phi_0 U$ for any choice of $U \in SU(N)$. The freedom in choosing U leads to $n+1$ different topological domain wall solutions in the model [10], each of which has different symmetry restoration within. More specifically, let

$$\Phi_- \equiv \Phi(x = -\infty) = \Phi_0 \quad (4)$$

Then, we can choose

$$\Phi_+ \equiv \Phi(x = +\infty) = -\eta \sqrt{\frac{2}{N(N^2 - 1)}} \times \text{diag}(n\mathbf{1}_{n+1-q}, -(n+1)\mathbf{1}_q, n\mathbf{1}_q, -(n+1)\mathbf{1}_{n-q}) \quad (5)$$

where $q \equiv 0, \dots, n$ tells us how many diagonal entries of Φ_- have been permuted in Φ_+ . The case $q \equiv 0$ is when $\Phi_+ = -\Phi_-$ and corresponds to the Z_2 wall in a ϕ^4 model embedded in this more complicated model.

The solution for any q can be written as:

$$\Phi(x) = F_+(x)\mathbf{M}_+ + F_-(x)\mathbf{M}_- + g(x)\mathbf{M}, \quad (6)$$

where

$$\mathbf{M}_+ = \frac{\Phi_+ + \Phi_-}{2}, \quad \mathbf{M}_- = \frac{\Phi_+ - \Phi_-}{2}. \quad (7)$$

The matrix \mathbf{M} is found by requiring that the solution extremize the energy functional. This can be done for an arbitrary polynomial potential $V(\Phi)$ that is invariant under G_N . This gives:

$$\mathbf{M} = \mu \begin{pmatrix} \text{diag}(q(n-q)\mathbf{1}_{n+1-q}, \\ -(n-q)(n+1-q)\mathbf{1}_{2q}, q(n+1-q)\mathbf{1}_{n-q}) \end{pmatrix} \quad (8)$$

with μ being a normalization factor in which we also include the energy scale η for convenience:

$$\mu = \eta[2q(n-q)(n+1-q)\{2n(n+1-q)-q\}]^{-1/2}. \quad (9)$$

The expression for \mathbf{M} does not hold for $q=0$ and $q=n$. In these cases, we have $g(x)=0$ and \mathbf{M} drops out of the solution.

It will be convenient to also explicitly list the values of \mathbf{M}_\pm :

$$\mathbf{M}_+ = \eta \begin{pmatrix} N \sqrt{\frac{1}{2N(N^2-1)}} \\ \text{diag}(0_{n+1-q}, \mathbf{1}_q, -\mathbf{1}_q, 0_{n-q}) \end{pmatrix}, \quad (10)$$

$$\mathbf{M}_- = \eta \begin{pmatrix} \sqrt{\frac{1}{2N(N^2-1)}} \\ \text{diag}(-2n\mathbf{1}_{n+1-q}, \mathbf{1}_{2q}, 2(n+1)\mathbf{1}_{n-q}) \end{pmatrix}. \quad (11)$$

The boundary conditions on the profile functions are: $F_\pm(\pm\infty) = \pm 1$, $F_\pm(\pm\infty) = \pm 1$ and $g(\pm\infty) = 0$. To determine the interior symmetry of the wall, we are interested in finding the values of the profile functions at the center of the wall which we take to be $x=0$.

The antisymmetry of $F_-(x)$, namely $F_-(-x) = -F_+(x)$, follows from the symmetry of the equations of motion (for general potential $V(\Phi)$) and the boundary conditions. This immediately tells us that $F_-(x=0) = 0$. It is also clear that $F_+(x)$ need not vanish on the wall. To determine $g(x=0)$ requires more care since $g(\pm\infty) = 0$. However, from the explicit expressions for the matrices \mathbf{M}_\pm and \mathbf{M} we see that:

$$\text{Tr}(\mathbf{M}_-^{2p+1}\mathbf{M}) \neq 0 \quad (12)$$

for any integer $p \geq 1$. Hence if $V(\Phi)$ contains any term of the form Φ^{2p+2} , then the equation of motion for $g(x)$ will necessarily contain a term of the kind $[F_-(x)]^{2p+1}$. This is odd under $x \rightarrow -x$ and hence $g(x)$ must also satisfy $g(-x) = -g(x)$. This tells us that $g(x=0) = 0$. Putting all this together, we have:

$$\Phi(x=0) = F_+(0)\mathbf{M}_+ \quad (13)$$

It is important to emphasize that this result holds for any choice of polynomial $V(\Phi)$ that gives the desired symmetry breaking.

Once we know the value of Φ inside domain wall, it is straightforward to determine the symmetry since we only need to find those generators of $SU(N)$ that commute with $\Phi(0)$. From the explicit expression for \mathbf{M}_+ , we see that there is a square block of zero entries of size $2n+1-2q = N-2q$, another block proportional to $\mathbf{1}_q$ and another to $-\mathbf{1}_q$. Therefore the symmetry within the wall is:

$$G_{in}^q = \{SU(N-2q) \times [SU(q) \times U(1)]^2\}/C', \quad (14)$$

where $q = 1, \dots, n-1$ and C' denotes the center. The two $U(1)$ factors have been inserted since transformations given by all diagonal generators remain unbroken by $\Phi(x=0)$.

The above result for G_{in}^q holds when $q \neq 0$. This case can be treated individually. If $q=0$, we see that $F_+(x) = 0$, and so we have:

$$G_{in}^0 = SU(N). \quad (15)$$

Next we determine the values of q for which the interior symmetry exceeds the exterior (bulk) symmetry. The number of generators of the exterior symmetry group, z_{out} , are determined from eq. (5):

$$z_{out} = [(n+1)^2 - 1] + [n^2 - 1] + 1 = \frac{1}{2}(N^2 - 1) \quad (16)$$

The number of generators of the interior symmetry group, z_{in} , are found from G_{in}^q :

$$\begin{aligned} z_{in} &= [(N-2q)^2 - 1] + 2[q^2 - 1 + 1] \\ &= 6q^2 - 4qN + (N^2 - 1), \quad q = 0, \dots, n \end{aligned} \quad (17)$$

Therefore the difference in the number of generators of the interior and exterior symmetries are:

$$\Delta z^q \equiv z_{in} - z_{out} = 6q^2 - 4qN + \frac{1}{2}(N^2 - 1), \quad (18)$$

for $q = 0, \dots, (N-1)/2$. The expression for Δz^q is quadratic in q and we can easily find that to get a larger symmetry inside the wall than outside ($\Delta z^q > 0$, $q \leq (N-1)/2$), we need

$$q < \frac{2N - \sqrt{N^2 + 3}}{6} \quad (19)$$

For large values of N , we find $q \lesssim N/6$. As an example, if $N=5$, then only the $q=0$ wall has symmetry restoration in its interior; all the other walls have symmetry reduction in their interior.

We now ask if the $q=0$ wall can be stable for any N for some choice of potential $V(\Phi)$. We can address this question in two ways: first we can check the perturbative stability of the $q=0$ wall, and second, we can compare the mass of the $q=0$ wall to the $q=n$ wall. If the $q=0$ wall is perturbatively stable but more massive than the $q=n$ wall, then we would expect that in a realistic setting, a network of $q=0$ walls will eventually decay

into a network of $q=n$ walls. So both local and global stability are of interest.

If the potential $V(\Phi)$ is quartic, the analysis in [5, 10, 11] shows that the $q=n$ wall is stable while the $q=0$ wall is unstable. Here we would like to consider the more difficult problem of a general potential. We will derive certain conditions that $V(\Phi)$ must satisfy if the $q=0$ wall is to be stable and/or lighter than the $q=n$ wall.

First let us consider perturbative stability of the $q=0$ wall. We perturb the wall solution

$$\Phi(x) = f(x)T_0 + \psi(x)T \quad (20)$$

where $T_0 \equiv \Phi_0/n$, $f(x)$ denotes the wall profile function, $\psi(x)$ is the perturbation profile, and T is a generator of $SU(N)$ that is chosen to be orthogonal to Φ_0 :

$$\text{Tr}(T_0 T) = 0 \quad (21)$$

Then, we find that the perturbation causes a change in the energy

$$\delta E[\psi] = \frac{1}{2} \int_{-\infty}^{+\infty} dx \psi \left[-\frac{d^2}{dx^2} + V_T^{(2)}(f) \right] \psi \quad (22)$$

where $V_T^{(2)}(f)$ denotes the second order variation of $V(\Phi)$ in ψ . In general, we need to determine if the Schrodinger equation

$$\left[-\frac{d^2}{dx^2} + V_T^{(2)}(f) \right] \psi = \epsilon \psi \quad (23)$$

has any negative eigenvalues (bound states). This will depend on the details of the potential V as well as the profile function f .

In the specific case of a quartic potential, the Schrodinger equation has been solved [5, 10] and bound states have been found to be present. The analysis there suggests that it might be possible to derive a simpler criterion for the occurrence of bound states. The idea is that if the perturbation is along the Φ_0 direction, that is, if we choose $T_0 = T$, then the corresponding Schrodinger equation necessarily has a zero mode because we know that translations of the wall do not change the energy. Furthermore the zero mode eigenfunction is proportional to the derivative of $f(x)$. Therefore:

$$\left[-\frac{d^2}{dx^2} + V_0^{(2)}(f) \right] \frac{df}{dx} = 0 \quad (24)$$

where $V_0^{(2)}$ denotes the second order variation of V along the Φ_0 direction. A useful condition for checking for an instability is obtained by evaluating δE in eq. (22) when ψ is the translational zero mode. This leads to:

$$\delta E[f'(x)] = \frac{1}{2} \int_{-\infty}^{+\infty} dx f'(x) [V_T^{(2)}(f) - V_0^{(2)}(f)] f'(x) \quad (25)$$

where a prime denotes derivative with respect to x . The integrand still depends on the profile function and, with another trick, we can eliminate this dependence.

The trick is to write the Bogomolnyi equation for the $q=0$ wall [12]. This is derived by writing the energy functional for the $q=0$ wall in the following way:

$$E[f] = \frac{1}{2} \int dx \left[(f' - \sqrt{2V(f)})^2 + 2\sqrt{2V(f)}f' \right] \quad (26)$$

The last term is a boundary term and its integration yields the energy of the $q=0$ wall:

$$E = \sqrt{2} \int_{-1}^{+1} df \sqrt{V(f)} \quad (27)$$

Minimization of the whole square term in the energy functional yields the Bogomolnyi equation:

$$f'(x) = \sqrt{2V_0(f)} \quad (28)$$

where $V_0(f)$ is the potential evaluated with $\Phi = f(x)T_0$. Note that $f'(x)$ is always positive since we are assuming that the $V(\Phi) = 0$ is a global minimum that occurs only when $\Phi = \Phi_0$ (up to symmetry transformations). Therefore $f(x)$ is monotonic and we can transform the integration variable in eq. (25) to f :

$$\delta E[f'(x)] = \frac{1}{\sqrt{2}} \int_{-\eta}^{+\eta} df [V_T^{(2)}(f) - V_0^{(2)}(f)] \sqrt{V_0(f)} \quad (29)$$

A sufficient condition for the instability of the $q=0$ wall is:

$$\int_{-\eta}^{+\eta} df [V_T^{(2)}(f) - V_0^{(2)}(f)] \sqrt{V_0(f)} < 0 \quad (30)$$

A desirable feature of this instability condition is that it depends only on the potential $V(\Phi)$ and its second order variations. To check for this particular instability, we need not solve any differential equations, but only evaluate the integral. We expect the wall to be most unstable to Goldstone modes as these are massless outside the wall and tachyonic within the wall. Hence we expect that δE will be smallest when $[T, \Phi_0] \neq 0$.

The instability condition in eq. (30) has been written down for a very specific perturbation, namely for $\psi = f'$, while a full blown stability analysis requires that we solve the Schrodinger equation in eq. (23), which also depends on the exact profile function. Hence there is a danger that our instability condition may not be useful and large classes of unstable walls may violate the condition. As we now describe, this fear is not realized, at least in the case of quartic potential but for arbitrary N . So there is hope that the condition will be useful even for more complicated potentials.

The quartic potential for arbitrary N can be written as:

$$V(\Phi) = -m^2 \text{Tr}[\Phi^2] + h(\text{Tr}[\Phi^2])^2 + \lambda \text{Tr}[\Phi^4] - V_0 \quad (31)$$

where, V_0 is a constant chosen so that $V(\Phi) = 0$ at its global minimum. The second order variation, $V_T^{(2)}(f)$, must be of the form:

$$V_T^{(2)}(f) = a_T + b_T f^2 \quad (32)$$

where a_T and b_T may be T -dependent coefficients. Now since the symmetry is completely restored when $f = 0$, $a_T = a$ which is independent of T . (In fact, $a = -m^2$.) Therefore

$$V_T^{(2)}(f) - V_0^{(2)}(f) = (b_T - b_0)f^2 \quad (33)$$

Now if we choose T so that the modes along T are the Goldstone boson modes, then they are massless when $f = \eta$ and $V_T^{(2)}(\eta) = 0$. On the other hand, the modes along T_0 are massive when $f = \eta$ and so $V_0^{(2)}(\eta) > 0$. This shows that we must have $b_T - b_0 < 0$. Hence

$$V_T^{(2)}(f) - V_0^{(2)}(f) < 0, \quad (34)$$

for all values of f . Therefore the integrand in eq. (30) is negative everywhere and the $q = 0$ is perturbatively unstable for quartic $V(\Phi)$ for any value of N .

To understand the instability for quartic potentials, we note that the center of the $q = 0$ wall is at a local maximum of the (multi-dimensional) potential. So the wall is given by a path that goes over the maximum of the potential, and there is at least one direction orthogonal to the path. (It helps to imagine a path over a Mexican hat potential in two dimensions.) So the path can slip from the top of the potential and the energy of the solution can be lowered.

For potentials that are more complicated than quartic, we find:

$$V_T^{(2)}(f) - V_0^{(2)}(f) = f^2 \sum_{n=0}^{\infty} \alpha_n f^{2n} \quad (35)$$

where the α_n are coefficients that depend on T and the choice of potential. As above, for Goldstone modes, by setting $f = \eta$, we find:

$$\sum_{n=0}^{\infty} \alpha_n \eta^{2n} < 0 \quad (36)$$

Other than this condition, the potential needs to have a global minimum with the correct unbroken symmetry. This condition is hard to implement in general. However, it appears that by suitably tuning the parameters in $V(\Phi)$ it should be possible to have a $q = 0$ wall that is perturbatively stable. In other words, if the potential is more complicated than just quartic, the area where the slipping can occur may be very small and gradient energy might be able to prevent the instability.

Even if the $q = 0$ wall is perturbatively stable, it may be more massive than other walls in the model. So we now consider global stability by comparing the energy of the $q = 0$ wall to that of the $q = n$ wall.

First we give a heuristic arguments that indicates that $q = n$ is in many situations likely to be the most stable wall. If we write:

$$\Phi = \text{diag}(f_1, f_2, \dots, f_N) \quad (37)$$

then the boundary conditions for the $q = n$ wall require the least change in the f_i as the wall is crossed. The expressions for Φ_{\pm} (eqs. (4) and (5)) show that only one of the f_i has to change sign and thus pass through zero. The other components merely have to shift a little bit. On the contrary, the boundary conditions for the $q = 0$ wall require that every component f_i change to $-f_i$ and thus pass through zero. In other words, the charge of the wall is given by $Q = \Phi_+ - \Phi_-$ and this has large components for the $q = 0$ wall but small components for the $q = n$ wall. If the energy of the wall is directly related to the charge (for example as $\text{Tr}(Q^2)$) then the $q = n$ wall will have the least energy.

In models motivated by supersymmetry, the energy of the walls can be evaluated explicitly and we can then determine which wall is lightest. In such models, we have:

$$V = \frac{1}{4} \text{Tr} \left| \frac{dW}{d\Phi} \right|^2 \quad (38)$$

where $W(\Phi)$ is the ‘‘superpotential’’. By a slight generalization of the Bogomolnyi derivation described above, we obtain the energy of a q -wall:

$$E_{wall} = \int dx \text{Tr} \left(\frac{d\Phi}{dx} \frac{dW}{d\Phi} \right) = W(\Phi_+^{(q)}) - W(\Phi_0) \quad (39)$$

where we have used eq. (4). Note that the superpotential need not be invariant under $SU(N) \times Z_2$; only the potential $V(\Phi)$, obtained by differentiating W and then squaring, is required to be invariant under the symmetry group. However, if we require that $V(\Phi_0) = 0$, then also $V(\Phi_U) = 0$ where $\Phi_U \equiv U^\dagger \Phi_0 U$, $U \in SU(N)$. Using eq. (38), we get

$$\left. \frac{dW}{d\Phi} \right|_{\Phi_U} = 0 \quad (40)$$

Since U is an element of a continuous group, Φ_0 and Φ_U are continuously related, and, $W(\Phi) = W(\Phi_U)$. Hence $W(\Phi_+^{(q)})$ is independent of q . This implies that the energy of walls in these models does not depend on q .

The stability analysis of $q = 0$ walls and the heuristic argument based on the energy of a wall being proportional to its charge, both suggest that the stable walls in $SU(N) \times Z_2$ models have $q = n$. The arguments using supersymmetry show that there exist potentials for which the $q = 0$ wall may not be unstable to decay into the $q = n$ wall. It is conceivable that this is a limiting case and any departures from supersymmetry might result in the $q = 0$ wall being heavier than the $q = n$ wall.

The $q = n$ walls have an internal symmetry that is smaller than the bulk symmetry. Yet, for the brane analogy and the monopole sweeping scenario, it is desirable

to have large internal symmetry as compared to the bulk symmetry. This can in principle happen if the potential $V(\Phi)$ is sufficiently complicated. The explicit construction of potentials that can lead to stable walls with large internal symmetry, remains an open problem. In Ref. [1, 2] the situation where the exterior non-Abelian symmetries are confining was discussed. In this situation, it is possible that excitations are much more massive in the exterior than in the interior. Another possibility for having enhanced interior symmetry is if the bulk symmetry is broken down, say completely. Then the internal symmetry must necessarily be larger than the bulk symmetry. However it is not clear in this situation if the internal symmetry can include continuous symmetries or whether, as in the case of the \mathbb{Z}_3 kink in the $\lambda\phi^4$ model, only the discrete \mathbb{Z}_3 will be restored within the wall.

The conclusion that the $q = n$ wall is most likely

the stable wall in the model has consequences for the monopole sweeping scenario [5]. Now there are certain monopoles that can unwind within the wall and others that cannot. The ones that cannot unwind, pass through the wall. The monopoles that can unwind will either unwind or get rotated into monopoles that cannot unwind. The interactions of monopoles and walls is an important topic for further investigation.

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- [1] G. Dvali and M. Shifman, Phys. Lett. **B396**, 64 (1997); Erratum, *ibid* **B407**, 452 (1997).
 - [2] G. Dvali and A. Vilenkin, hep-th/0209217 (2002).
 - [3] G. Dvali, H. Liu and T. Vachaspati, Phys. Rev. Lett. **80**, 2281 (1998).
 - [4] L. Pogosian, T. Vachaspati, Phys. Rev. **D62**, 105005 (2000).
 - [5] L. Pogosian and T. Vachaspati, Phys. Rev. **D62**, 123506 (2000).
 - [6] M. Axenides and L. Perivolaropoulos, Phys. Rev. **D56**, 1973 (1997).
 - [7] M. Axenides, L. Perivolaropoulos and T.N. Tomaras, Phys. Rev. **D58**, 103512 (1998).
 - [8] M. Axenides, L. Perivolaropoulos and M. Trodden, Phys. Rev. **D58**, 083505 (1998).
 - [9] B. Carter, R.H. Brandenberger and A.-C. Davis, Phys. Rev. **D65**, 103520 (2002).
 - [10] L. Pogosian and T. Vachaspati, Phys. Rev. **D64**, 105023 (2001).
 - [11] T. Vachaspati, Phys. Rev. **D63**, 105010 (2001).
 - [12] E. B. Bogomolnyi, *Sov. J. Nucl. Phys.* **24** 449 (1976); reprinted in "Solitons and Particles", eds. C. Rebbi and G. Soliani (World Scientific, Singapore, 1984).