

Supersymmetry in Singular Quantum Mechanics

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Abstract

The breaking of supersymmetry due to singular potentials in supersymmetric quantum mechanics is critically analyzed. It is shown that, when properly regularized, these potentials respect supersymmetry, even when the regularization parameter is removed.

1 Introduction

Supersymmetry is a beautiful and, simultaneously, a tantalizing symmetry [1-7]. On the one hand, supersymmetry leads to field theories and string theories with exceptional properties [8-9]. On the other hand, supersymmetry also predicts degenerate superpartner states which are not observed experimentally and, consequently, one expects that supersymmetry must be spontaneously (dynamically) broken. However, unlike ordinary symmetries, spontaneous breaking of supersymmetry has so far proved extremely difficult in the conventional framework. Consequently, in the context of supersymmetry, one constantly looks for alternate, unconventional methods of breaking of this symmetry [6-7]. There is, of course, the breaking of supersymmetry due to instanton effects which is well understood. However, several authors, in recent years have suggested that supersymmetry may be broken in the presence of singular potentials or boundaries in a nonstandard manner [10-12]. The examples, where such a breaking has been discussed, are simple quantum mechanical models which nonetheless arise from the non-relativistic limit of some field theories. It is for this reason that, in an earlier paper, we had examined [13] a candidate relativistic $2 + 1$ dimensional field theory to see if the manifestation of such a mechanism was possible in a field theory. However, a careful examination of the theory revealed that supersymmetry prevails at the end although it might appear naively, in the beginning, that supersymmetry would be broken in the nonstandard manner. This prompted us to re-analyze the quantum mechanical models, where this mechanism was demonstrated, more carefully and a systematic and critical examination, once again, reveals

[14] that supersymmetry is manifest even in such singular quantum mechanical models which is the main result of this talk.

Since our discussion would be entirely within the context of one dimensional supersymmetric quantum mechanics, let us establish the essential notations here. Given a superpotential, $W(x)$, we can define a pair of supersymmetric potentials as

$$V_+ = \frac{1}{2} \left(W^2(x) + W'(x) \right), \quad V_- = \frac{1}{2} \left(W^2(x) - W'(x) \right) \quad (1)$$

where “prime” denotes differentiation with respect to x . With $\hbar = 1$ and $m = 1$, we can, then, define a pair of Hamiltonians which describe a supersymmetric system as

$$H_+ = -\frac{1}{2} \frac{d^2}{dx^2} + V_+, \quad H_- = -\frac{1}{2} \frac{d^2}{dx^2} + V_- \quad (2)$$

In fact, defining the supercharges as

$$Q = \frac{1}{\sqrt{2}} \left(-\frac{d}{dx} + W(x) \right), \quad Q^\dagger = \frac{1}{\sqrt{2}} \left(\frac{d}{dx} + W(x) \right) \quad (3)$$

we recognize that we can write the pair of Hamiltonians in eq. (2) also as

$$H_+ = Q^\dagger Q, \quad H_- = Q Q^\dagger \quad (4)$$

All the eigenstates of the two Hamiltonians H_+ and H_- would be degenerate except for the ground state with vanishing energy which would correspond to the state satisfying

$$Q|\psi_+\rangle = 0, \quad \text{or,} \quad Q^\dagger|\psi_-\rangle = 0 \quad (5)$$

For a given superpotential, at most one of the two conditions in eq. (5) can be satisfied (that is, at most, only one of the two conditions in (5) would give a normalizable state). Namely, the ground state with vanishing energy is unpaired and can belong to the spectrum of either H_+ or H_- depending on which of the conditions leads to a normalizable state. This corresponds to the case of unbroken supersymmetry. If, on the other hand, the superpotential is such that neither of the states in eq. (5) is normalizable, then, supersymmetry is known to be broken by instanton effects [6]. In this talk, we carefully analyze the models [10-12] where supersymmetry is thought to be broken because of singular nature of the potentials and show that when carefully analyzed, the systems with singular potentials have manifest supersymmetry.

2 Super “Half” Oscillator

Let us consider a particle moving in the harmonic oscillator potential on the “half” line

$$V(x) = \begin{cases} \frac{1}{2}(\omega^2 x^2 - \omega) & \text{for } x > 0 \\ \infty & \text{for } x < 0 \end{cases} \quad (6)$$

The spectrum of this potential is quite clear intuitively. Namely, because of the infinite barrier in the negative axis, we expect the wave function to vanish at the origin leading to the conclusion that, of all the solutions of the oscillator on the full line, only the odd solutions (of course, on the “half” line there is no notion of even and odd) would survive in this case. While this is quite obvious, let us analyze the problem systematically for later purpose.

First, let us note that singular potentials are best studied in a regularized manner because this is the only way that appropriate boundary conditions can be determined correctly. Therefore, let us consider the particle moving in the regularized potential

$$V(x) = \begin{cases} \frac{1}{2}(\omega^2 x^2 - \omega) & \text{for } x > 0 \\ \frac{c^2}{2} & \text{for } x < 0 \end{cases} \quad (7)$$

with the understanding that the limit $|c| \rightarrow \infty$ is to be taken at the end. The Schrödinger equation can now be solved in the two regions. Since $|c| \rightarrow \infty$ at the end, for any finite energy solution, we have the asymptotically damped solution, for $x < 0$,

$$\psi^{(II)}(x) = A e^{(c^2 - 2\epsilon)^{\frac{1}{2}} x} \quad (8)$$

Since the system no longer has reflection symmetry, the solutions, in the region $x > 0$, cannot be classified into even and odd solutions. Rather, the normalizable (physical) solution would correspond to one which vanishes asymptotically. The solutions of the Schrödinger equation, in the region $x > 0$, are known as the parabolic cylinder functions [15] and the asymptotically damped physical solution is given by

$$\psi^{(I)}(x) = B U(-(\frac{\epsilon}{\omega} + \frac{1}{2}), \sqrt{2\omega} x) \quad (9)$$

It is now straightforward to match the solutions in eqs. (8, 9) and their first derivatives across the boundary at $x = 0$ and their ratio gives

$$\frac{1}{\sqrt{c^2 - 2\epsilon}} = -\frac{1}{2\sqrt{\omega}} \frac{\Gamma(-\frac{\epsilon}{2\omega})}{\Gamma(-\frac{\epsilon}{2\omega} + \frac{1}{2})} \quad (10)$$

It is clear, then, that as $|c| \rightarrow \infty$, this can be satisfied only if

$$-\frac{\epsilon}{2\omega} + \frac{1}{2} \xrightarrow{|c| \rightarrow \infty} -n, \quad n = 0, 1, 2, \dots \quad (11)$$

In other words, when the regularization is removed, the energy levels that survive are the odd ones, namely, (remember that the zero point energy is already subtracted out in (6) or (7)) $\epsilon_n = \omega(2n+1)$. The corresponding physical wave functions are nontrivial only on the half line $x > 0$ and have the form

$$\psi_n(x) = B_n U(-(2n + \frac{3}{2}), \sqrt{2\omega} x) = \tilde{B}_n e^{-\frac{1}{2}\omega x^2} H_{2n+1}(\sqrt{\omega} x) \quad (12)$$

Namely, only the odd Hermite polynomials survive leading to the fact that the wave function vanishes at $x = 0$. Thus, we see that the correct boundary condition naturally arises from regularizing the singular potential and studying the problem systematically.

We now turn to the analysis of the supersymmetric oscillator on the half line. One can define a superpotential [10]

$$W(x) = \begin{cases} -\omega x & \text{for } x > 0 \\ \infty & \text{for } x < 0 \end{cases} \quad (13)$$

which would, naively, lead to the pair of potentials

$$V_{\pm}(x) = \begin{cases} \frac{1}{2}(\omega^2 x^2 \mp \omega) & \text{for } x > 0 \\ \infty & \text{for } x < 0 \end{cases} \quad (14)$$

Since, this involves singular potentials, we can study it, as before, by regularizing the singular potentials as

$$\begin{aligned} V_+(x) &= \begin{cases} \frac{1}{2}(\omega^2 x^2 - \omega) & \text{for } x > 0 \\ \frac{c_+^2}{2} & \text{for } x < 0 \end{cases} \\ V_-(x) &= \begin{cases} \frac{1}{2}(\omega^2 x^2 + \omega) & \text{for } x > 0 \\ \frac{c_-^2}{2} & \text{for } x < 0 \end{cases} \end{aligned} \quad (15)$$

with the understanding that $|c_{\pm}| \rightarrow \infty$ at the end.

The earlier analysis can now be repeated for the pair of potentials in eq. (15). It is straightforward and without going into details, let us simply note the results, namely, that, in this case, we obtain

$$\begin{aligned} \epsilon_{+,n} &= \omega(2n+1) & \psi_{+,n}(x) &= B_{+,n} e^{-\frac{1}{2}\omega x^2} H_{2n+1}(\sqrt{\omega} x) \\ \epsilon_{-,n} &= 2\omega(n+1) & \psi_{-,n}(x) &= B_{-,n} e^{-\frac{1}{2}\omega x^2} H_{2n+1}(\sqrt{\omega} x) \end{aligned} \quad (16)$$

Here $n = 0, 1, 2, \dots$. There are several things to note from this analysis. First, only the odd Hermite polynomials survive as physical solutions since the wave function has to vanish at the origin. This boundary condition arises from a systematic study involving a regularized potential. Second, the energy levels for the supersymmetric pair of Hamiltonians are no longer degenerate. Furthermore, the state with $\epsilon = 0$ no longer belongs to the Hilbert space (since it corresponds to an even Hermite polynomial solution). This leads to the conventional conclusion that supersymmetry is broken in such a case and let us note, in particular, that in such a case, it would appear that the superpartner states do not belong to the physical Hilbert space (Namely, in this case, the supercharge is an odd operator and hence connects even and odd Hermite polynomials. However, the boundary condition selects out only odd Hermite polynomials as belonging to the physical Hilbert space.).

There is absolutely no doubt that supersymmetry is broken in this case. The question that needs to be addressed is whether it is a dynamical property of the system or an artifact of the

regularization (and, hence the boundary condition) used. The answer is quite obvious, namely, that supersymmetry is broken mainly because the regularization (and, therefore, the boundary condition) breaks supersymmetry. In other words, for any value of the regularizing parameters, c_{\pm} (even if $|c_+| = |c_-|$), the pair of potentials in eq. (15) do not define a supersymmetric system and hence the regularization itself breaks supersymmetry. Consequently, the breaking of supersymmetry that results when the regularization is removed cannot be trusted as a dynamical effect.

Regularized Superpotential

Another way to understand this is to note that for a supersymmetric system, it is not the potential that is fundamental. Rather, it is the superpotential which gives the pair of supersymmetric potentials through Riccati type relations. It is natural, therefore, to regularize the superpotential which would automatically lead to a pair of regularized potentials which would be supersymmetric for any value of the regularization parameter. Namely, such a regularization will respect supersymmetry and, with such a regularization, it is, then, meaningful to ask if supersymmetry is broken when the regularization parameter is removed at the end. With this in mind, let us look at the regularized superpotential

$$W(x) = -\omega x \theta(x) + c \theta(-x) \quad (17)$$

Here c is the regularization parameter and we are supposed to take $|c| \rightarrow \infty$ at the end. Note that the existence of a normalizable ground state, namely, the form of the superpotential in eq. (13) selects out $c > 0$ (otherwise, the regularization would have broken supersymmetry through instanton effects as we have mentioned earlier).

The regularized superpotential now leads to the pair of regularized supersymmetric potentials

$$\begin{aligned} V_+(x) &= \frac{1}{2} \left[(\omega^2 x^2 - \omega) \theta(x) + c^2 \theta(-x) - c \delta(x) \right] \\ V_-(x) &= \frac{1}{2} \left[(\omega^2 x^2 + \omega) \theta(x) + c^2 \theta(-x) + c \delta(x) \right] \end{aligned} \quad (18)$$

which are supersymmetric for any $c > 0$. Let us note that the difference here from the earlier case where the potentials were directly regularized (see eq. (15)) lies only in the presence of the $\delta(x)$ terms in the potentials. Consequently, the earlier solutions in the regions $x > 0$ and $x < 0$ continue to hold. However, the matching conditions are now different because of the delta function terms. Carefully matching the wave function and the discontinuity of the first derivative across $x = 0$ for each of the wavefunctions and taking their ratio, we obtain the two conditions

$$\frac{1}{(c^2 - 2\epsilon_+)^{1/2} - c} = -\frac{1}{2\sqrt{\omega}} \frac{\Gamma(-\frac{\epsilon_+}{2\omega})}{\Gamma(-\frac{\epsilon_+}{2\omega} + \frac{1}{2})} \quad (19)$$

$$\frac{1}{(c^2 - 2\epsilon_-)^{1/2} + c} = -\frac{1}{2\sqrt{\omega}} \frac{\Gamma(-\frac{\epsilon_-}{2\omega} + \frac{1}{2})}{\Gamma(-\frac{\epsilon_-}{2\omega} + 1)} \quad (20)$$

It is now clear that, as $c \rightarrow \infty$, (19) and (20) give respectively, $\epsilon_{+,n} = 2\omega n$ and $\epsilon_{-,n} = 2\omega(n+1)$ with $n = 0, 1, 2, \dots$. The corresponding wave functions, in this case, have the forms

$$\begin{aligned}\psi_{+,n}(x) &= B_{+,n} e^{-\frac{1}{2}\omega x^2} H_{2n}(\sqrt{\omega} x) \\ \psi_{-,n}(x) &= B_{-,n} e^{-\frac{1}{2}\omega x^2} H_{2n+1}(\sqrt{\omega} x)\end{aligned}\tag{21}$$

This is indeed quite interesting for it shows that the spectrum of H_+ contains the ground state with vanishing energy. Furthermore, all the other states of H_+ and H_- are degenerate in energy corresponding to even and odd Hermite polynomials as one would expect from superpartner states. Consequently, it is quite clear that if the supersymmetric “half” oscillator is defined carefully by regularizing the superpotential, then, supersymmetry is manifest in the limit of removing the regularization. This should be contrasted with the general belief that supersymmetry is broken in this system (which is a consequence of using boundary conditions or, equivalently, of regularizing the potentials in a manner which violates supersymmetry). Of course, we should worry at this point as to how regularization independent our conclusion really is. Namely, our results appear to follow from the matching conditions in the presence of singular delta potential terms and, consequently, it is worth investigating whether our conclusions would continue to hold with an alternate regularization of the superpotential which would not introduce such singular terms to the potentials. We have done this [14] which shows that our result is regularization independent.

3 Oscillator with $\frac{1}{x^2}$ Potential

In the last section, we showed that, in the presence of one kind of singularity, namely, a boundary, supersymmetry is unbroken. In what follows, we will study another class of supersymmetric models, namely, the supersymmetric oscillator with a $\frac{1}{x^2}$ potential, where there is a genuine singularity in the potential not necessarily arising from a boundary. A naive analysis of this model [11] also shows that supersymmetry is broken by such a singular potential (for certain parameter ranges). However, this conclusion can be understood, again, as a consequence of regularizing the potential which, as we have seen before, does not respect supersymmetry. In stead, we will show through a careful analysis that, when the superpotential is regularized, supersymmetry is manifest in this model as well (with a lot of interesting features). In this section, however, we will systematically analyze only the quantum mechanical system corresponding to an oscillator in the presence of a $\frac{1}{x^2}$ potential (postponing the discussion of the supersymmetric case to the next section). This system has been analyzed by several people [16-18] and the most complete analysis appears to be in ref. [18]. However, we feel that, while the energy levels derived in [18] are correct, the wave functions are not (namely, the extensions of the solutions from the positive to the negative axis are incomplete and the wave functions, of course, become quite crucial when one wants to extend the analysis to a supersymmetric system) and, consequently, we present a careful analysis of this system regularizing the singular potential in a systematic manner. With the supersymmetric system in mind (to follow in the next section), we write the potential for the system as (with $\hbar = m = \omega = 1$)

$$V(x) = \frac{1}{2} \left[\frac{g(g+1)}{x^2} + x^2 - 2g + 1 \right]\tag{22}$$

The singular potential is repulsive for $g > 0$ or $g < -1$ while it is attractive for $-1 < g < 0$. It is also worth noting here that the Schrödinger equation, in this case, is invariant under

$$\begin{aligned} g &\leftrightarrow -(g+1) \\ \epsilon &\leftrightarrow \epsilon + 2g + 1 \end{aligned} \quad (23)$$

This symmetry, of course, would also be reflected in the solutions. Furthermore, the fixed point of this symmetry, namely, $g = -\frac{1}{2}$ separates the two branches (namely, for every value of λ there exist two distinct values of g corresponding to two distinct branches separated at the branch point) in the parameter space.

Regularized Potential

The Schrödinger equation can be solved quite easily for $x > 0$ as was also done in [18]. However, to determine correctly how this wavefunction should be extended to the negative axis, it is more suitable to regularize the potential near the origin and study the problem carefully. Let us consider a potential of the form

$$V(x) = \begin{cases} \frac{1}{2} \left[\frac{g(g+1)}{x^2} + x^2 - 2g + 1 \right] & \text{for } |x| > R \\ \frac{1}{2} \left[\frac{g(g+1)}{R^2} + R^2 - 2g + 1 \right] & \text{for } |x| < R \end{cases} \quad (24)$$

Namely, we have regularized the potential in a continuous manner preserving the symmetry in eq. (23) with the understanding that the regularization parameter $R \rightarrow 0$ at the end. With this regularization, the Schrödinger equation has to be analyzed in three distinct regions. However, since the potential has reflection symmetry, we need to analyze the solutions only in the regions $-R < x < R$ and $x > R$.

The potential is a constant in the region $-R < x < R$ and hence the Schrödinger equation is quite simple here. The solutions can be classified into even and odd ones and take the forms

$$\psi^{(II)even}(x) = A(R) \cosh \kappa x, \quad \psi^{(II)odd}(x) = B(R) \sinh \kappa x \quad (25)$$

where we have defined

$$\kappa = \sqrt{\frac{g(g+1)}{R^2} + R^2 - (2\epsilon + 2g - 1)} \approx \frac{\sqrt{g(g+1)}}{R} \quad (26)$$

Since R is small (and we are to take the vanishing limit at the end), the last equality holds only if $g \neq 0$ or -1 which we will assume. The special values of g corresponding to the absence of a singular potential have to be treated separately and we will come back to this at the end of this section. We note here that the normalization constants, A and B , can, in principle depend on the regularization parameter which we have allowed for in writing down the form of the solutions in eq. (25).

The potential is much more complicated in the region $x > R$. However, the physical solution can be obtained in terms of confluent hypergeometric functions [15] in the form, (for $x > 0$)

$$\begin{aligned} \psi^{(I)}(x) = & C(R) e^{-\frac{1}{2}x^2} \left[\frac{\Gamma(-g - \frac{1}{2})}{\Gamma(\frac{1}{2} - g - \frac{\epsilon}{2})} x^{g+1} M(1 - \frac{\epsilon}{2}, g + \frac{3}{2}, x^2) \right. \\ & \left. + \frac{\Gamma(g + \frac{1}{2})}{\Gamma(1 - \frac{\epsilon}{2})} x^{-g} M(\frac{1}{2} - g - \frac{\epsilon}{2}, -g + \frac{1}{2}, x^2) \right] \end{aligned} \quad (27)$$

Once again, we have allowed for a dependence of the normalization constant, C , on the regularization parameter, R . However, for a nontrivial solution to exist, we require that

$$C(R) \xrightarrow{R \rightarrow 0} C \neq 0$$

So far, we have the general solutions, in the two regions, where energy is not quantized and which should arise from the matching conditions. Furthermore, we have not bothered to evaluate the solution in the region $x < -R$ which clearly would be the same as in the region $x > R$. However, the matching conditions would determine how we should extend the solutions in the region $x > R$ to the region $x < -R$. Therefore, let us now examine the matching conditions systematically since there are two possible cases.

(i) **Even Solution**

We can match the even solution of the region $-R < x < R$ and its derivative with those of the region $x > R$ at $x = R$. Taking the ratio and remembering that R is small (which is to be taken to zero at the end), we obtain to the leading order in R

$$\sqrt{g(g+1)} \tanh \sqrt{g(g+1)} = \frac{(g+1) \frac{\Gamma(-g-\frac{1}{2})}{\Gamma(\frac{1}{2}-g-\frac{\epsilon}{2})} R^{g+1} - g \frac{\Gamma(g+\frac{1}{2})}{\Gamma(1-\frac{\epsilon}{2})} R^{-g}}{\frac{\Gamma(-g-\frac{1}{2})}{\Gamma(\frac{1}{2}-g-\frac{\epsilon}{2})} R^{g+1} + \frac{\Gamma(g+\frac{1}{2})}{\Gamma(1-\frac{\epsilon}{2})} R^{-g}} \quad (28)$$

Since the left hand side is independent of R , for consistency, the right hand side must also be and this can happen in two different ways.

First, for $g > -\frac{1}{2}$, it is clear that relation (28) can be satisfied if (we assume from now on that $n = 0, 1, 2, \dots$)

$$\epsilon_n = 2(n+1) - 2f_1(g)R^{2g+1} \quad (29)$$

with a suitable choice of $f_1(g)$.

On the other hand, for $g < -\frac{1}{2}$, if

$$\epsilon_n = (2n - 2g + 1) - 2f_2(g)R^{-2g-1} \quad (30)$$

relation (28) can be satisfied with a suitable choice of $f_2(g)$. It is clear that the two possible branches of the solution simply reflect the symmetry in eq. (23).

This analysis shows that when the regularization is removed (namely, $R \rightarrow 0$), we have an even extension of the solution of the forms $g > -\frac{1}{2}$, $\epsilon_n = 2(n+1)$ with

$$\psi_n(x) = C_n \frac{\Gamma(-g - \frac{1}{2})}{\Gamma(-g - \frac{1}{2} - n)} e^{-\frac{1}{2}x^2} M(-n, g + \frac{3}{2}, x^2) \begin{cases} x^{g+1} & \text{for } x > 0 \\ |x|^{g+1} & \text{for } x < 0 \end{cases} \quad (31)$$

and $g < -\frac{1}{2}$, $\epsilon_n = 2n - 2g + 1$ with

$$\psi_n(x) = C_n \frac{\Gamma(g + \frac{1}{2})}{\Gamma(g + \frac{1}{2} - n)} e^{-\frac{1}{2}x^2} M(-n, -g + \frac{1}{2}, x^2) \begin{cases} x^{-g} & \text{for } x > 0 \\ |x|^{-g} & \text{for } x < 0 \end{cases} \quad (32)$$

(ii) **Odd Solution** In a similar manner, we can determine the odd solutions which have the forms $g > -\frac{1}{2}$, $\epsilon_n = 2(n+1)$ with

$$\psi_n(x) = C_n \frac{\Gamma(-g - \frac{1}{2})}{\Gamma(-g - \frac{1}{2} - n)} e^{-\frac{1}{2}x^2} M(-n, g + \frac{3}{2}, x^2) \begin{cases} x^{g+1} & \text{for } x > 0 \\ -|x|^{g+1} & \text{for } x < 0 \end{cases} \quad (33)$$

and $g < -\frac{1}{2}$, $\epsilon_n = 2n - 2g + 1$ with

$$\psi_n(x) = C_n \frac{\Gamma(g + \frac{1}{2})}{\Gamma(g + \frac{1}{2} - n)} e^{-\frac{1}{2}x^2} M(-n, -g + \frac{1}{2}, x^2) \begin{cases} x^{-g} & \text{for } x > 0 \\ -|x|^{-g} & \text{for } x < 0 \end{cases} \quad (34)$$

Understanding of the Result

The conclusion following from this analysis, therefore, is that every energy level of this system is doubly degenerate. Both even and odd extensions of the solution are possible for every value of the energy level. The energy levels which we have obtained are, of course, identical to those obtained in [18]. The crucial difference is in the structure of the wave functions, namely, that both even and odd extensions of the solution are possible for every value of the energy (Incidentally, the solutions we have obtained in terms of confluent hypergeometric functions also coincide with generalized Laguerre polynomials as was obtained in ref. [18]). It is crucial, therefore, to ask if such a conclusion is physically plausible. To understand this question, let us recapitulate the results from a simple quantum mechanical model which is well studied. Namely, let us look at a particle moving in a potential of the form

$$V(x) = \begin{cases} \gamma \delta(x) & \text{for } |x| < a \\ \infty & \text{for } |x| > a \end{cases}$$

It is well known that the solutions of this system can be classified into even and odd ones with energy levels ($\hbar = m = 1$)

$$E_n^{even} = \frac{n^2\pi^2}{2(a + \frac{1}{\gamma})^2}, \quad E_n^{odd} = \frac{n^2\pi^2}{2a^2}$$

The even and the odd solutions, of course, have distinct energy values for any finite strength of the delta potential. However, when $\gamma \rightarrow \infty$, both the even and the odd solutions become degenerate in energy. Namely, a delta potential with an infinite strength leads to a double degeneracy of every energy level corresponding to both even and odd solutions. The connection of this example with the problem we are studying is intuitively clear. Namely, we can think of

$$\frac{g(g+1)}{x^2} = \lim_{\eta \rightarrow 0} \frac{g(g+1)}{x^2 + \eta^2} = \lim_{\eta \rightarrow 0} \left(\frac{\pi g(g+1)}{\eta} \right) \left(\frac{1}{\pi} \frac{\eta}{x^2 + \eta^2} \right)$$

It is clear that for $g \neq 0$ or -1 , the singular $\frac{1}{x^2}$ potential behaves like a delta potential with an infinite strength and it is quite natural, therefore, that this system has both even and odd solutions degenerate in energy.

It is also clear from this analysis that it is meaningless to take the $g = 0$ or -1 limit from the results obtained so far simply because the characters of the two problems are quite different. As we have argued, for any finite value of g not coinciding with those special values, the potential behaves, at the origin, like a delta potential of infinite strength while for the special values, there is no such potential. The two cases are related in a drastically discontinuous manner. As a result, one cannot treat the $\frac{g(g+1)}{x^2}$ as a perturbation and obtain the full, correct solution simply because there is nothing perturbative (small) about this potential for any “nontrivial” value of g . Another way of saying this is to re-emphasize what we have already observed following eq. (26), namely, the character of κ and, therefore, the matching conditions change depending on whether or not g differs from the special values $0, -1$.

4 Supersymmetric Oscillator with $\frac{1}{x^2}$ Potential:

The supersymmetric version of the case studied is obtained from a superpotential of the form [11]

$$W(x) = \frac{g}{x} - x \tag{35}$$

In this case, it is easily seen that a normalizable ground state wavefunction exists only for $g > -\frac{1}{2}$.

The superpotential and, therefore, the potential is singular at the origin. Thus, once again, the proper way to study the spectrum of such a system is by regularizing the superpotential. We introduce the regularized superpotential

$$W(x) = \theta(x - |R|) \left(\frac{g}{x} - x \right) + \theta(|R| - x) \left(\frac{g}{R} - R \right) \frac{x}{R} \tag{36}$$

Here, R denotes the regularization parameter which is to be taken to zero at the end. The regularized superpotential is continuous and the resulting pair of potentials take the forms

$$\begin{aligned} V_+ &= \frac{1}{2} \left[\theta(x - |R|) \left(\frac{g(g-1)}{x^2} + x^2 - 2g - 1 \right) + \theta(|R| - x) \left(\left(\frac{g}{R^2} - 1 \right)^2 x^2 + \left(\frac{g}{R^2} - 1 \right) \right) \right] \\ V_- &= \frac{1}{2} \left[\theta(x - |R|) \left(\frac{g(g+1)}{x^2} + x^2 - 2g + 1 \right) + \theta(|R| - x) \left(\left(\frac{g}{R^2} - 1 \right)^2 x^2 - \left(\frac{g}{R^2} - 1 \right) \right) \right] \end{aligned}$$

The solutions of the pair of Hamiltonians can now be studied. Without going into details [14], let us note that the solutions, in this case, again turn out to be confluent hypergeometric functions. There are several interesting features that arise in this case. For example, it turns out that, in the limit $R \rightarrow 0$, H_+ has three sets of normalizable solutions – one even and two odd. The three sets of normalizable solutions of H_- also correspond to one even and two sets of odd solutions. While one of the three sets of solutions correspond to a supersymmetric system, there are additional solutions which apparently have no relation to one another.

The proper understanding of the solutions comes really from recognizing that, given a bosonic system, there is an arbitrariness in supersymmetrizing the system. It is much like the arbitrariness of whether a spin $\frac{1}{2}$ particle belongs to a supersymmetric multiplet $(0, \frac{1}{2})$ or $(\frac{1}{2}, 1)$. The different solutions really correspond to different possible supersymmetrizations and matching has to be done carefully. When analyzed carefully, it turns out that supersymmetry is manifest in the system. In addition, in this case, the problem can be solved algebraically because of a special symmetry in the problem known as shape invariance. The algebraic solution also coincides with the explicit solutions obtained.

5 Conclusion

In this talk, we have discussed systematically two classes of supersymmetric quantum mechanical models - one consisting of a singular boundary and the other with a singular potential. We have shown that, contrary to the conventional understanding [10-12], supersymmetry is manifest in these systems. In particular, for a system with a singular potential such as $\frac{1}{x^2}$, the solution of the Schrödinger equation leads to several distinct solutions corresponding to distinct supersymmetrizations of the system. Consequently, it becomes quite important to identify the appropriate wavefunctions when supersymmetric properties are being investigated. Finally, we would like to conclude by noting that supersymmetry is known to be robust at short distances (high energies). The singularities discussed in the quantum mechanical models occur at short distances and, therefore, it is intuitively quite clear that they are unlikely to break supersymmetry. Our detailed, systematic analysis only reinforces this.

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