On the ground state of a free massless (pseudo)scalar field in two dimensions

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Abstract

We investigate the ground state of a free massless (pseudo)scalar field in 1+1–dimensional space–time. We argue that in the quantum field theory of a free massless (pseudo)scalar field without infrared divergences (Eur. Phys. J. C **24**, 653 (2002)) the ground state can be represented by a tensor product of wave functions of the fiducial vacuum and of the collective zero–mode, describing the motion of the "center of mass" of a free massless (pseudo)scalar field. We show that the bosonized version of the BCS wave function of the ground state of the massless Thirring model obtained in (hep–th/0210104) describes the ground state of the free massless (pseudo)scalar field.

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1 Introduction

The problem which we study in this paper is related to our investigations of the massless Thirring model [1], where we have found a new phase with a wave function of the BCS–type and massive quasiparticles. The (pseudo)scalar collective excitations $\vartheta(x)$ of these massive quasiparticles are bound by a Mexican hat potential and described by the Lagrangian

$$\mathcal{L}(x) = \frac{1}{2}\,\partial_\mu\vartheta(x)\partial^\mu\vartheta(x), \tag{1.1}$$

invariant under field translations

$$\vartheta(x) \to \vartheta\,'(x) = \vartheta(x) + \alpha, \tag{1.2}$$

where α is an arbitrary parameter $\alpha \in \mathbb{R}^1$. The parameter α is related to the chiral phase $\alpha_{\rm A}$ of chiral rotations of the massless Thirring fermion fields $\alpha = -2\alpha_{\rm A}$ [1, 2].

The continuous symmetry (1.2) can be described in terms of the total charge operator $Q(x^0)$ defined by [2–4]

$$Q(x^0) = \int\limits_{-\infty}^{\infty} dx^1\, j_0(x^0, x^1) = \int\limits_{-\infty}^{\infty} dx^1\, \frac{\partial\vartheta(x)}{\partial x^0} = \int\limits_{-\infty}^{\infty} dx^1\, \Pi(x^0, x^1), \tag{1.3}$$

where $j_0(x)$ is the time–component of the conserved current $j_\mu(x) = \partial_\mu\vartheta(x)$ and $\Pi(x) = j_0(x)$ is the conjugate momentum of the ϑ–field obeying the canonical commutation relation

$$[\Pi(x^0, x^1), \vartheta(x^0, y^1)] = -i\delta(x^1 - y^1). \tag{1.4}$$

From this canonical commutation relation follows

$$\vartheta\,'(x) = e^{+i\alpha Q(x^0)}\,\vartheta(x)\,e^{-i\alpha Q(x^0)} = \vartheta(x) + \alpha. \tag{1.5}$$

Acting with the operator $e^{-i\alpha\,Q(0)}$ on the vacuum wave function $|\Psi_0\rangle$ we get the wave function

$$|\alpha\rangle = e^{-i\alpha\,Q(0)}|\Psi_0\rangle. \tag{1.6}$$

This wave function is normalized to unity and possesses all properties of the vacuum state [5].

The average value of the ϑ–field calculated for the wave functions $|\alpha\rangle$ is equal to

$$\langle\alpha|\vartheta(x)|\alpha\rangle = \alpha. \tag{1.7}$$

The same result can be obtained for the vacuum expectation value of the $\vartheta\,'$–field (1.2)

$$\langle\Psi_0|\vartheta\,'(x)|\Psi_0\rangle = \langle\Psi_0|\vartheta(x)|\Psi_0\rangle + \alpha = \alpha. \tag{1.8}$$

This testifies that the parameter α describes the position of the "center of mass". Hence, it is related to the collective zero–mode of the free massless (pseudo)scalar field $\vartheta(x)$ [2–4].

As has been shown in [2–4] the quantum field theory of the free massless (pseudo)scalar field $\vartheta(x)$ with the Lagrangian (1.1) is well–defined if the collective zero–mode, describing the motion of the "center of mass" of the field $\vartheta(x)$, is removed from the states which can be excited by an external source $J(x)$ in the generating functional of Green functions [2–4]

$$\begin{aligned} Z[J] &= \Big\langle \Psi_0 \Big| \mathrm{T}\Big(e^{i\int d^2x\, \vartheta(x) J(x)} \Big) \Big| \Psi_0 \Big\rangle = \\ &= \int \mathcal{D}\vartheta\, e^{i\int d^2x \left[\frac{1}{2}\partial_\mu\vartheta(x)\partial^\mu\vartheta(x) + \vartheta(x)J(x)\right]}, \end{aligned} \tag{1.9}$$

where T is the time–ordering operator. According to the analysis [2–4] the collective zero–mode cannot be excited by any perturbation of the external source $J(x)$ if the external source obeys the constraint [2–4]

$$\int d^2x\, J(x) = \tilde{J}(0) = 0. \tag{1.10}$$

The same constraint one needs for the perturbative renormalization of the sine–Gordon model [6]. It has been shown that the sine–Gordon model is the bosonized version of the massive Thirring model with fermion fields quantized in the chirally broken phase [1].

As has been shown in [2–4] the existence of the chirally broken phase for the massless Thirring model with a non–vanishing fermion condensate [1] and the spontaneously broken field–shift symmetry (1.2) of the free massless (pseudo)scalar field $\vartheta(x)$, characterized by a non–vanishing spontaneous magnetization [2–4], does not contradict both the Mermin–Wagner–Hohenberg theorem [7] and Coleman's theorem [8]. The irrelevance of the Mermin–Wagner–Hohenberg theorem [7] to the problem of the existence of the chirally broken phase in the massless Thirring model is rather straightforward. Indeed, the Mermin–Wagner–Hohenberg theorem [7], proved for non–zero temperature, tells nothing about spontaneous breaking of continuous symmetry in 1+1–dimensional quantum field theories at temperature zero [2–4]. Since the chirally broken phase of the massless Thirring model [1] has been found at temperature zero, the Mermin–Wagner–Hohenberg theorem [7] does not suppress this phase.

The absence of spontaneous breaking of continuous symmetry and Goldstone bosons in 1+1–dimensional quantum field theories at zero–temperature one connects with Coleman's theorem [8]. Following Wightman's axioms [9], demanding the definition of Wightman's observables on test functions from the Schwartz class $\mathcal{S}(\mathbb{R}^2)$, Coleman has argued that there are no Goldstone bosons, massless (pseudo)scalar fields [8]. In turn, the absence of Goldstone bosons [10] can be interpreted as the absence of spontaneous breaking of continuous symmetry [11]. Coleman's assertion is an extension of the well–known statement of Wightman [9] that a non–trivial quantum field theory of a free massless (pseudo)scalar field does not exist in 1+1–dimensional space–time in terms of Wightman's observables defined on the test functions from $\mathcal{S}(\mathbb{R}^2)$.

Such a strict conclusion concerning the non–existence of a 1+1–dimensional quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$ has been drawn from the logarithmic divergences of the two–point Wightman functions [9].

The massless (pseudo)scalar field $\vartheta(x)$ has the following expansion into plane waves [1–4]

$$\vartheta(x) = \int_{-\infty}^{\infty} \frac{dk^1}{2\pi}\, \frac{1}{2k^0} \Big(a(k^1)\, e^{-i\,k\cdot x} + a^\dagger(k^1)\, e^{i\,k\cdot x} \Big), \tag{1.11}$$

where $a(k^1)$ and $a^\dagger(k^1)$ are annihilation and creation operators and obey the standard commutation relation [1–4]

$$[a(k^1), a^\dagger(q^1)] = (2\pi)\, 2k^0\, \delta(k^1 - q^1). \tag{1.12}$$

For the free massless (pseudo)scalar field (1.11) one can define the Wightman function

$$\begin{aligned} D^{(+)}(x;\mu) &= \langle\Psi_0|\vartheta(x)\vartheta(0)|\Psi_0\rangle = \\ &= \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{dk^1}{2k^0}\, e^{-\,i\,k\cdot x} = -\frac{1}{4\pi}\,\ell n[-\mu^2x^2 + i\,0\cdot\varepsilon(x^0)], \end{aligned} \tag{1.13}$$

where $\varepsilon(x^0)$ is the sign function, $x^2 = (x^0)^2 - (x^1)^2$, $k\cdot x = k^0x^0 - k^1x^1$, $k^0 = |k^1|$ is the energy of free massless (pseudo)scalar quantum with a momentum k^1 and μ is the infrared cut–off reflecting the infrared divergences of the Wightman functions (1.13).

According to Wightman's axioms [9] a well–defined quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$ should be formulated in terms of Wightman's observables

$$\vartheta(h) = \int d^2x\, h(x)\vartheta(x) \tag{1.14}$$

determined on the test functions $h(x)$ from the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ [9]. In terms of Wightman's observables (1.14) one can define a quantum state $|h\rangle$ [9]

$$|h\rangle = \vartheta(h)|\Psi_0\rangle = \int d^2x\, h(x)\vartheta(x)|\Psi_0\rangle, \tag{1.15}$$

where $|\Psi_0\rangle$ is the wave function of the ground state. The squared norm of this quantum state is equal to [2]

$$\|h\|^2 = \langle h|h\rangle = \iint d^2xd^2y\, h^*(x)D^{(+)}(x-y;\mu)h(y) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{dk^1}{2k^0}\,|\tilde{h}(k^0,k^1)|^2. \tag{1.16}$$

The momentum integral is logarithmically divergent, since the Fourier transform $\tilde{h}(k^0,k^1)$ of the test function $h(x)$ from the Schwartz class $\mathcal{S}(\mathbb{R}^2)$ has a support at $k^0 = k^1 = 0$, i.e $\tilde{h}(0,0) \neq 0$.

Recently [3, 4] we have analysed the physical meaning of the test functions $h(x)$ in Wightman's observables (1.14). We have shown that the test functions can be interpreted as *apparatus functions* characterizing the device used by the observer for detecting quanta of the free massless (pseudo)scalar field. This interpretation of test functions agrees with our results obtained in Ref. [2], where we have shown that a quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$ can be constructed without infrared divergences if one removes from the ϑ–field the collective zero–mode, describing the motion of the "center of mass". We have shown that the collective zero–mode does not affect the evolution of the other modes of the free massless (pseudo)scalar field $\vartheta(x)$. The removal of the collective zero–mode has been carried out within the path–integral approach in terms of the generating functional of Green functions defined by (1.9). As has been shown in [2] the generating functional of Green functions (1.9) with $\tilde{J}(0) \neq 0$ vanishes identically, $Z[J] = 0$. This agrees well with Wightman's statement [9] about the non–existence of a

quantum field theory of a free massless (pseudo)scalar field defined on test functions $h(x)$ from $\mathcal{S}(\mathbb{R}^2)$ with $\tilde{h}(0,0) \neq 0$.

Hence, the removal of the collective zero–mode of the $\vartheta(x)$-field implies the immeasurability of this state in terms of Wightman's observables. The insensitivity of the detectors to the collective zero–mode can be obtained by the constraint $\tilde{h}(0,0) = 0$ [3,4]. Mathematically this means that the test functions $h(x)$ should belong to the Schwartz class $\mathcal{S}_0(\mathbb{R}^2) = \{h(x) \in \mathcal{S}(\mathbb{R}^2); \tilde{h}(0,0) = 0\}$ [2–4]. As has been shown in [3, 4] the quantum field theory of a free massless (pseudo)scalar field $\vartheta(x)$ defined on the test functions from $\mathcal{S}_0(\mathbb{R}^2)$ is unstable under spontaneous breaking of continuous symmetry (1.2). Quantitatively the symmetry broken phase is characterized by a non–vanishing spontaneous magnetization $\mathcal{M} = 1$ [2–4]. Goldstone bosons are the quanta of a free massless (pseudo)scalar field [2–4]. Coleman's theorem reformulated for the test functions from $\mathcal{S}_0(\mathbb{R}^2) = \{h(x) \in \mathcal{S}(\mathbb{R}^2); \tilde{h}(0,0) = 0\}$ does not refute this statement.

The paper is organized as follows. In Section 2 we describe the collective zero–mode by a rigid rotor. In Section 3 we calculate the generating functional of Green functions and show that the infrared divergences of the free massless (pseudo)scalar field are due to the classical evolution of the collective zero–mode from infinite past to infinite future. In Section 4 we construct the wave function of the ground state of the free massless (pseudo)scalar field for the bosonized version of the massless Thirring model which is of the BCS–type. In the Conclusion we discuss the obtained results.

2 Collective zero–mode

The collective zero–mode of the free massless (pseudo)scalar field $\vartheta(x) = \vartheta(x^0, x^1)$, describing the "center of mass" motion, is a mode orthogonal to all vibrational modes. Due to this we can treat the field $\vartheta(x)$ in the form of the following decomposition

$$\vartheta(x) = \vartheta_0(x^0) + \vartheta_{\rm v}(x), \tag{2.1}$$

where $\vartheta_{\rm v}(x)$ is the field of all vibrational modes. In terms of the plane wave expansion the field $\vartheta_{\rm v}(x)$ can be defined by

$$\vartheta_{\rm v}(x) = {\rm P.v.} \int_{-\infty}^{\infty} \frac{dk^1}{2\pi} \frac{1}{2k^0} \Big(a(k^1)\, e^{-i\,k\cdot x} + a^{\dagger}(k^1)\, e^{i\,k\cdot x}\Big), \tag{2.2}$$

where $a(k^1)$ and $a^{\dagger}(k^1)$ are expansion coefficients and P.v. defines a "principle value", the value of the momentum integral without which deletes formally the contribution of the infinitesimally small region around $k^1 = 0$. The field $\vartheta_{\rm v}(x)$ can be quantized. In this case the coefficients $a^{\dagger}(k^1)$ and $a(k^1)$ are creation and annihilation operators of a quantum with momentum k^1 and obey the commutation relations (1.16). The Lagrangian of the $\vartheta_{\rm v}(x)$–field is given by

$$\mathcal{L}_{\rm v}(x) = \frac{1}{2}\Big(\frac{\partial\vartheta_{\rm v}(x)}{\partial x^0}\Big)^2 - \frac{1}{2}\Big(\frac{\partial\vartheta_{\rm v}(x)}{\partial x^1}\Big)^2. \tag{2.3}$$

Unlike the oscillator modes of a free massless (pseudo)scalar field the collective zero–mode $\vartheta_0(x^0)$ is defined by the Lagrangian

$$\mathcal{L}_0(x^0) = \frac{1}{2}\,\dot{\vartheta}_0^2(x^0), \tag{2.4}$$

which does not have the form of the Lagrangian of a vibrational mode with a pair of squared terms $(\dot{\vartheta}_0^2 - \vartheta_0^{'2})/2$, where $\vartheta_0^{'}$ is a spatial derivative of the ϑ_0–field. The Lagrangian of the "center of mass" motion (2.4) does not contain a "potential energy", i.e. $\vartheta_0^{'2}/2$, responsible for a restoring force as in the vibrational modes. Hence, the collective zero–mode cannot be quantized in terms of annihilation and creation operators $a(0)$ and $a^{\dagger}(0)$.

Our assertion concerning the decomposition of the free massless (pseudo)scalar field $\vartheta(x)$ into a collective zero–mode $\vartheta_0(x^0)$ and vibrational modes $\vartheta_{\rm v}(x)$ can be justified as follows. As has been shown in [12] a free massless (pseudo)scalar field $\vartheta(x)$ described by the Lagrangian (1.1) is equivalent to a one–dimensional linear chain of N oscillators with equal masses, equal equilibrium separations and a potential energy taking into account only nearest neighbors. Their motion can be described in terms of displacements $q_i(x^0)\,(i = 1, \ldots, N)$. The Lagrange function can be written as [12]

$$L(x^0) = \frac{1}{2}\sum_{i=1}^{N} \dot{q}_i^2(x^0) + \frac{1}{2}\sum_{i>j}^{N}(q_i(x^0) - q_j(x^0))^2. \tag{2.5}$$

In normal coordinates $Q_n(x^0)\,(n = 0, 1, \ldots, N-1)$ the Lagrange function (2.5) reads

$$L(x^0) = \frac{1}{2}\,\dot{Q}_0^2(x^0) + \frac{1}{2}\sum_{n=1}^{N-1}(\dot{Q}_n^2(x^0) - \omega_n^2 Q_n^2(x^0)), \tag{2.6}$$

where $Q_0(x^0)$ is the collective zero–mode, describing the motion of the "center of mass" of the system [2], $Q_n(x^0)$ are vibrational normal modes with frequencies ω_n. In the limit $N \to \infty$ the Lagrange function (2.6) reduces to the form [12]

$$\begin{aligned} L(x^0) &= \int_{-L/2}^{L/2} dx^1\,\frac{1}{2}\,\dot{\vartheta}_0^2(x^0) + \frac{1}{2}\int_{-L/2}^{L/2} dx^1\,\partial_\mu\vartheta_{\rm v}(x)\partial^\mu\vartheta_{\rm v}(x) = \\ &= \frac{L}{2}\,\dot{\vartheta}_0^2(x^0) + \frac{1}{2}\int_{-L/2}^{L/2} dx^1\,\partial_\mu\vartheta_{\rm v}(x)\partial^\mu\vartheta_{\rm v}(x). \end{aligned} \tag{2.7}$$

This is exactly the continuum limit of a one–dimensional chain of N oscillators with nearest neighbour coupling [12].

For finite volume L the discretized form of the ϑ–field with the expansion of the $\vartheta_{\rm v}$–field into plane waves reads

$$\vartheta(x) = \vartheta_0(x^0) + \sum_{n\in\mathbb{Z}, n\neq 0}\Big(a_n\,e^{-ik_n^0x^0 + ik_n^1x^1} + a_n^{\dagger}\,e^{+ik_n^0x^0 - ik_n^1x^1}\Big). \tag{2.8}$$

The creation and annihilation operators $a_n^{\dagger}$ and a_n obey the commutation relations

$$\begin{aligned} &[a_n, a_{n'}^{\dagger}] = \delta_{nn'}, \\ &[a_n^{\dagger}, a_{n'}^{\dagger}] = [a_n, a_{n'}] = 0, \end{aligned} \tag{2.9}$$

where $k_n = (k_n^0, k_n^1)$ is the 2–dimensional momentum defined by $k_n = (2\pi|n|/L, 2\pi n/L)$ for $n \in \mathbb{Z}$ and $0 \leq x^1 \leq L$. The annihilation operators act on the vacuum state as $a_n|\Psi_0\rangle = 0$.

The Lagrange function of the collective zero–mode $\vartheta_0(x^0)$ is equal to

$$L_0(\vartheta_0, \dot{\vartheta}_0) = \int_{-L/2}^{L/2} dx^1\, \mathcal{L}_0(x) = \frac{L}{2}\, \dot{\vartheta}_0^2(x^0). \tag{2.10}$$

Such a Lagrange function(2.10) can be used to describe a mechanical system, a rigid rotor, for which $\vartheta_0(x^0)$ is a periodic angle $\vartheta_0(x^0) = \vartheta_0(x^0) + 2\pi \,(\mathrm{mod}\, 2\pi)$, $\dot{\vartheta}_0(x^0)$ is the angular velocity and L can be interpreted as the moment of inertia.

The classical equation of motion $\ddot{\vartheta}_0(x^0) = 0$ has the general solution

$$\vartheta_0(x^0) = \Omega_0 x^0 + \alpha, \tag{2.11}$$

where Ω_0 is the angular velocity or the frequency of the rotation of the rigid rotor. Setting $\Omega_0 = 0$ we get $\vartheta_0(x^0) = \alpha$.

The classical conjugate momentum of the collective zero–mode $\vartheta_0(x^0)$ is equal to

$$\pi_0(x^0) = \frac{\partial L_0(\vartheta_0, \dot{\vartheta}_0)}{\partial \dot{\vartheta}_0} = L\, \dot{\vartheta}_0 \tag{2.12}$$

and the Hamilton function is defined by

$$h_0(\vartheta_0, \pi_0) = \frac{\pi_0^2(x^0)}{2L}. \tag{2.13}$$

Substituting (2.8) in (1.3) one can show that the conjugate momentum $\pi_0(x^0)$ coincides with the total charge operator $Q(x^0)$, i.e. $Q(x^0) = \pi_0(x^0)$. This agrees with the analysis of the ground state of the massive Schwinger model by Kogut and Susskind [13].

For the quantum mechanical description of the rigid rotor we use the ϑ_0–representation. In this case the conjugate momentum is defined by $\hat{\pi}_0 = -id/d\vartheta_0$ and the Hamilton operator reads

$$\hat{h}_0(\vartheta_0, \hat{\pi}_0) = \frac{\hat{\pi}_0^2(x^0)}{2L} = -\frac{1}{2L}\frac{d^2}{d\vartheta_0^2}. \tag{2.14}$$

The wave function $\psi(\vartheta_0)$ of the collective zero–mode in the ϑ_0–representation is the solution of the Schrödinger equation

$$-\frac{1}{2L}\frac{d^2\psi(\vartheta_0)}{d\vartheta_0^2} = E_0\psi(\vartheta_0). \tag{2.15}$$

Imposing periodic boundary conditions $\psi(\vartheta_0) = \psi(\vartheta_0 + 2\pi)$ the normalized solutions of this equation read

$$\psi_m(\vartheta_0) = {}_{\mathrm{o}}\langle \vartheta_0 \mid m\rangle_{\mathrm{o}} = \frac{1}{\sqrt{2\pi}}\, e^{+i\, m\, \vartheta_0}, \quad m = 0, \pm 1, \pm 2, \ldots, \tag{2.16}$$

where m is the "magnetic" quantum number, $m \in \mathbb{Z}$. The wave functions $\psi_m(\vartheta_0)$ are also eigenfunctions of the conjugate momentum $\hat{\pi}_0$ and the total charge operator $\hat{Q} = \hat{\pi}_0 = -id/d\vartheta_0$ with the eigenvalues $m \in \mathbb{Z}$

$$Q\psi_m(\vartheta_0) = m\, \psi_m(\vartheta_0) \Longleftrightarrow \hat{Q}|m\rangle_{\mathrm{o}} = m|m\rangle_{\mathrm{o}}. \tag{2.17}$$

The energy spectrum is defined by

$$E_0^{(m)} = \frac{m^2}{2L}. \tag{2.18}$$

The wave function of the free massless (pseudo)scalar field $\vartheta(x)$ can be represented in the form of direct product of the collective zero–mode and vibrational modes

$$|\Psi\rangle = |m\rangle_0 \otimes |n_1\rangle_1 \otimes |n_2\rangle_2 \otimes \cdots \otimes |n_k\rangle_k \otimes \cdots, \tag{2.19}$$

where $|m\rangle_0$ is the state of the zero–mode and $|n_k\rangle_k$ is the wave function for the k-th vibrational mode with n_k quanta.

The total Hamilton and momentum operators of the free massless (pseudo)scalar field $\vartheta(x)$, defined by (2.8), is equal to

$$\begin{aligned} \hat{h}[\vartheta] &= \frac{\hat{\pi}_0^2}{2L} + \frac{2\pi}{L}\sum_{n\in\mathbb{Z}} |n|\, a_n^\dagger a_n, \\ \hat{\pi}[\vartheta] &= \hat{\pi}_0 + \frac{2\pi}{L}\sum_{n\in\mathbb{Z}} n\, a_n^\dagger a_n. \end{aligned} \tag{2.20}$$

It is well–known that the wave function of the ground state should be eigenfunction of the total Hamilton and momentum operators with eigenvalue zero. For finite L this requirement is fulfilled only for the wave function

$$|\Omega_0\rangle = |0\rangle_0 \otimes |\Psi_0\rangle, \qquad |\Psi_0\rangle = |0\rangle_1 \cdots \otimes |0\rangle_k \otimes \cdots \tag{2.21}$$

where $|0\rangle_0$ is the eigenfunction of the operator (2.14) with eigenvalue $m = 0$. In the ϑ_0–representation the wave function $|0\rangle$ is equal to ${}_0\langle\vartheta_0 \mid 0\rangle_0 = \psi_0(\vartheta_0) = 1/\sqrt{2\pi}$.

Now we can show that the infrared divergences of the free massless (pseudo)scalar field theory are the quantum field theoretical problem and they are not the problem at all. Indeed, in reality these quantum field theoretic divergences are related to the classical motion of the collective zero–mode from the infinite past at $x^0 = -\infty$ to the infinite future at $x^0 = +\infty$.

3 Generating functional of Green functions and the nature of infrared divergences

The generating functional of Green functions for the free massless (pseudo)scalar field is defined by (2.20) and reads

$$\begin{aligned} \mathcal{Z}[J] &= \langle\Omega_0|\mathrm{T}\Big(e^{i\int d^2x\,\vartheta(x)J(x)}\Big)|\Omega_0\rangle = \\ &= \lim_{L,T\to\infty} Z_0[J_0;L,T]\langle\Psi_0|\mathrm{T}\Big(e^{i\int d^2x\,\vartheta_{\rm v}(x)J(x)}\Big)|\Psi_0\rangle = \lim_{L,T\to\infty} Z_0[J_0;L,T]\; Z[J]. \end{aligned} \tag{3.1}$$

The factor $Z[J]$ in this product concerns the vibrational modes and coincides with (1.9). The other factor

$$\begin{aligned} Z_0[J_0;L,T] :=& e^{iW_0[J_0;L,T]} = \\ =& N^{-1}(L,T)\int \mathcal{D}\vartheta_0\, \exp\Big\{i\int_{-T}^{T} dx^0\Big[\frac{L}{2}\,\dot{\vartheta}_0^2(x^0) + \vartheta_0(x^0)J_0(x^0)\Big]\Big\} \end{aligned} \tag{3.2}$$

is the generating functional for the collective zero–mode, which we describe as the motion of a rigid rotor; the external source $J_0(x^0)$ is the integral of the external source $J(x) = J(x^0, x^1)$ over $x^1 \in \mathbb{R}^1$. The normalization factor $N(L,T)$ is the inverse of the path integral without external sources

$$N(L,T) = \int \mathcal{D}\vartheta_0 \, \exp\Big\{ i\frac{L}{2} \int_{-T}^{T} dx^0 \, \dot{\vartheta}_0^2(x^0) \Big\} \tag{3.3}$$

In the limit $T \to \infty$ we perform a Fourier transformation

$$\vartheta_0(x^0) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, \tilde{\vartheta}_0(\omega) \, e^{-i\omega x^0} \tag{3.4}$$

and get

$$Z_0[J_0; L] = N^{-1}(L) \int \mathcal{D}\tilde{\vartheta}_0 \, \exp\Big\{ i\frac{L}{2} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Big[\omega^2 \tilde{\vartheta}_0(\omega)\tilde{\vartheta}_0(-\omega) + \frac{2}{L} \, \tilde{\vartheta}_0(\omega)\tilde{J}_0(-\omega) \Big] \Big\}. \tag{3.5}$$

By quadratic extension $\tilde{\vartheta}_0(\omega) = \tilde{\varphi}(\omega) - \tilde{J}(\omega)/\omega^2 L$ we reduce this to the form

$$Z_0[J_0; L] = \exp\Big\{ -\frac{i}{4\pi L} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2} \mid \tilde{J}_0(\omega) \mid^2 \Big\}. \tag{3.6}$$

For $\tilde{J}_0(\omega = 0) \neq 0$ the functional $W_0[J_0; L]$ has an infrared divergence of first power, which by dimensional reasons is stronger than the logarithmic divergence of the Wightman function (1.1). Due to the infrared divergence $\mathcal{W}[J_0; L]$ becomes infinite and $Z_0[J_0; L]$ vanishes. This agrees with Wightman's assertion concerning the non–existence of a well–defined quantum field theory of a free massless (pseudo)scalar field in 1+1–dimensional space–time which we discussed above. In our treatment of the collective zero–mode of the free massless (pseudo)scalar field the infrared divergence appears at the quantum mechanical level and admits a simple physical interpretation, as we will discuss below. The divergence does not appear due to unbounded quantum fluctuations. It is just the opposite, there is only one contribution to the generating functional $Z_0[J_0; L, T]$, the classical trajectory which is defined by the initial conditions. The correlation functions gets unbounded due to the laws of classical mechanics.

To determine the path–integral (3.2) we decompose $\vartheta_0(x^0)$ into a classical part $\bar{\vartheta}_0(x^0)$ and a fluctuating part $\varphi(x^0)$, $\vartheta_0(x^0) = \bar{\vartheta}_0(x^0) + \varphi(x^0)$. With the standard conditions $\varphi(-T) = \varphi(+T) = 0$ and choosing the classical field $\bar{\vartheta}_0(x^0)$ as a special solution of the "2nd axiom of Newton"

$$\ddot{\bar{\vartheta}}_0(x^0) = \frac{1}{L} \, J_0(x^0) \tag{3.7}$$

we get a decoupling of the quantum fluctuations $\varphi(x^0)$. The rigid rotor behaves completely classical, the generating functional depends only on the classical field $\bar{\vartheta}_0(x^0)$ and the action has the same shape as in (3.2) for the original field ϑ_0

$$Z_0[J_0; L, T] := e^{iW_0[J_0; L, T]} = \exp\Big\{ i \int_{-T}^{T} dx^0 \Big[\frac{L}{2} \, \dot{\bar{\vartheta}}_0^2(x^0) + \bar{\vartheta}_0(x^0) J_0(x^0)) \Big] \Big\}. \tag{3.8}$$

It is important to emphasize that due to this decoupling of quantum and classical degrees of freedom the external source $J_0(x^0)$ can excite the classical degrees of freedom ϑ_0 only.

Integrating by parts in the exponent of (3.8) and using (3.7) we get for the action integral

$$W_0[J_0; L, T] = \frac{1}{2}\int_{-T}^{+T} dx^0\, \bar{\vartheta}_0(x^0) J_0(x^0) + \frac{L}{2}\Big[\bar{\vartheta}_0(x^0)\dot{\bar{\vartheta}}_0(x^0)\Big]_{-T}^{+T}. \tag{3.9}$$

Obviously, this integral and therefore the generating functional (3.2) depends on the initial condition. This is the usual situation for classical systems. Now we insert in Eq. (3.9) a solution of Eq. (3.7). The general solution of (3.7) reads

$$\bar{\vartheta}_0(x^0) = \frac{1}{2L}\int_{-T}^{+T} dy^0 |x^0 - y^0| J_0(y^0) + C_1 x^0 + C_2. \tag{3.10}$$

For the initial conditions $\bar{\vartheta}_0(-T) = \dot{\bar{\vartheta}}_0(-T) = 0$ the integration constants are

$$\begin{aligned} C_1 &= \frac{1}{2L}\int_{-T}^{+T} dy^0 J_0(y^0), \\ C_2 &= -\frac{1}{2L}\int_{-T}^{+T} dy^0 y^0 J_0(y^0), \end{aligned} \tag{3.11}$$

which lead to a special solution of (3.9)

$$\bar{\vartheta}_0(x^0) = \frac{1}{2L}\int_{-T}^{+T} dy^0\, (|x^0 - y^0| + (x^0 - y^0)) J_0(y^0) = \frac{1}{L}\int_{-T}^{x^0} dy^0\, (x^0 - y^0) J_0(y^0). \tag{3.12}$$

Substituting this expression into (3.9) we get

$$\begin{aligned} W_0[J_0; L, T] =& \frac{1}{4L}\int_{-T}^{+T} dx^0 \int_{-T}^{T} dy^0 J_0(x^0)|x^0 - y^0| J_0(y^0) + \\ &+ \frac{1}{2L}\int_{-T}^{+T} dy^0\, (T - y^0) J_0(y^0) \int_{-T}^{+T} dx^0 J_0(x^0). \end{aligned} \tag{3.13}$$

For $\tilde{J}(0) \neq 0$ the functional $W_0[J_0; L, T]$ increases with the time T. This gives a strongly oscillating phase of the generating functional $Z_0[J_0; L, T]$ providing its vanishing in the limit $T \to \infty$. This occurs even if the external source generates an arbitrary small fluctuation of the collective zero–mode.

This can also be seen defining the classical field $\langle \vartheta_0(x^0) \rangle$ in terms of the generating functional $Z_0[J_0; L, T]$. According to the standard definition the classical field $\langle \vartheta_0(x^0) \rangle$ is given by

$$\langle \vartheta_0(x^0) \rangle = \frac{1}{i}\frac{\delta \ell n Z_0[J_0; L, T]}{\delta J_0(x^0)} = \frac{\delta W_0[J_0; L, T]}{\delta J_0(x^0)}. \tag{3.14}$$

This defines the linear response of the system to an external force

$$\langle \vartheta_0(x^0) \rangle = \frac{1}{2L}\int_{-T}^{+T} dy^0 [|x^0 - y^0| + (T - x^0) + (T - y^0)] J_0(y^0), \tag{3.15}$$

but the response is quadratic

$$W_0[J_0; L, T] = \frac{1}{2}\int_{-T}^{+T} dx^0 J_0(x^0)\langle\vartheta_0(x^0)\rangle, \tag{3.16}$$

as it is well-known for free motion.

In the limit $T \to \infty$ the response $\langle\vartheta_0(x^0)\rangle$ does not vanish for infinitesimally small external perturbation. From this behaviour we can conclude that the divergence prohibiting a quantum field theoretic description of a free massless (pseudo)scalar field [9], corresponds to a quite natural motion which is well–known in classical and quantum mechanics of free particles and rigid rotors. An infinitely small kick at $-T = -\infty$ can lead to finite translations or rotational angles at T as shown by (3.14).

The correlation

$$\langle\vartheta_0(x^0)\vartheta_0(y^0)\rangle = \frac{\delta^2 W_0[J_0; L, T]}{\delta J_0(x^0)\delta J_0(y^0)} = \frac{1}{2L}[|x^0 - y^0| + (T - x^0) + (T - y^0)] \tag{3.17}$$

diverges also with T indicating that the system is not stabilized by any potential. Only under the constraint $\tilde{J}(0,0) = 0$, agreeing well with the definition of the quantum field theory of the free massless (pseudo)scalar field on the Schwartz class $\mathcal{S}_0$, the correlation (3.17) remains finite

$$\langle\vartheta_0(x^0)\vartheta_0(y^0)\rangle = \frac{1}{2L}|x^0 - y^0| \qquad \text{for} \qquad \tilde{J}_0(0) = 0. \tag{3.18}$$

Since

$$\langle\vartheta_0(x^0)\rangle = \frac{1}{2L}\int_{-T}^{+T} dy^0 |x^0 - y^0| J_0(y^0) \qquad \text{for} \qquad \tilde{J}_0(0) = 0, \tag{3.19}$$

the response $\langle\vartheta_0(x^0)\rangle$ vanishes for $J_0(y^0) = 0$ and Eq. (3.18) shows that the center of mass motion does not decorrelate for large time differences $|x^0 - y^0|$.

The above discussion demonstrates that the divergence of the correlation (3.17) is not due to large quantum fluctuations, it is due to the the sensitivity of a free system for external perturbations. The center of mass motion does not show any fluctuations, it evolves along the classical trajectory only, see Eq. (3.8). This explains also why the correlation does not vanish for large time intervals.

In the Schwinger formulation of the quantum field theory [14] the generating functional $Z_0[J_0; L, T = \infty]$ defines the amplitude for the transition from the ground state of the center of mass motion at $x^0 = -\infty$ to the ground state at $x^0 = +\infty$ caused by the external force $J_0(x^0)$. For vanishing perturbation this amplitude does not converge to unity and gives the impression that the evolution of the system is ill defined. But this is not the case. Let the state of the center of mass at $x^0 = 0$ be described by $\vartheta_0 = \alpha$ and $\dot{\vartheta}_0 = 0$. Then the classical evolution of the system guarantees that the system will remain in this state for any finite time and the corresponding transition amplitude is unity.

Due to the quadratic extension of the exponent $\vartheta_0(x^0) = \bar{\vartheta}_0(x^0) + \varphi(x^0)$ in the generating functional of Green functions $Z_0[J_0; L, T]$ and the condition (3.7), the quantum degrees of freedom do not couple to the external source $J_0(x^0)$. As a result, the generating

functional of Green functions $Z_0[J_0; L, T]$ is defined by one classical trajectory (3.8) and depends for vanishing source, $J_0 = 0$, on the temporal boundary conditions only

$$W_0[J_0; L, T] = \frac{L}{2}\Big[\bar{\vartheta}_0(x^0)\dot{\bar{\vartheta}}_0(x^0)\Big]_{-T}^{+T}. \tag{3.20}$$

We can take into account 2π-periodicity of ϑ_0 assuming that the paths $\bar{\vartheta}_0(+T)-\bar{\vartheta}_0(-T) = \Delta$ and $\bar{\vartheta}_0(+T) - \bar{\vartheta}_0(-T) = \Delta + 2\pi m$, where $m \in \mathbb{Z}$, are equivalent and indistinguishable. Then, we obtain the generating functional $Z_0[0; L, T]$ in the following form

$$\begin{aligned} Z_0[0; L, T] &= \sqrt{\frac{L}{4\pi i T}} \sum_{m\in\mathbb{Z}} \exp\Big(i\,\frac{L}{4T}\,[\Delta + 2\pi m]^2\Big) = \\ &= \sqrt{\frac{L}{4\pi i T}}\,\exp\Big(i\,\frac{L\Delta^2}{4T}\Big)\,\theta_3\Big(\frac{\pi L\Delta}{2T}, \frac{\pi L}{T}\Big), \end{aligned} \tag{3.21}$$

where $\theta_3(z, t)$ is the Jacobi theta–function defined by [15]

$$\theta_3(z, t) = \sum_{m\in\mathbb{Z}} e^{\,i\pi\, t\, m^2 + i\, 2\, m\, z} \tag{3.22}$$

with $z = \pi L\Delta/2T$ and $t = \pi L/T$. The normalization factor [15] in front of the sum over equivalent paths is chosen in such a way that the path integral for given initial position $\vartheta_0(-T)$ and arbitrary final position $\vartheta_0(+T)$ gives unity. Using the property of the Jacobi theta–function [15]

$$\theta_3(z, t) = \sqrt{\frac{i}{t}}\,\exp\Big(-\frac{iz^2}{\pi t}\Big)\,\theta_3\Big(\frac{z}{t}, -\frac{1}{t}\Big) \tag{3.23}$$

we transcribe the functional $Z_0[0; L, T]$ into the form

$$\begin{aligned} Z_0[0; L, T] &= \frac{1}{2\pi}\,\theta_3\Big(\frac{\Delta}{2}, -\frac{T}{\pi L}\Big) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} \exp\Big(i\, m\,\Delta - i\,\frac{T}{L}\, m^2\Big) = \\ &= \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{\,i\, m\,[\vartheta_0(+T) - \vartheta_0(-T)] - i\, E_m\, 2T}. \end{aligned} \tag{3.24}$$

The r.h.s. contains a sum over the stationary quantum states of the rigid rotor [15]

$$Z_0[0; L, T] = \sum_{m\in\mathbb{Z}} {}_0\langle\vartheta_0(+T)|m\rangle_{00}\langle m|\vartheta_0(-T)\rangle_0 = {}_0\langle\vartheta_0(+T)|\vartheta_0(-T)\rangle_0 \tag{3.25}$$

describing the amplitude for the transition $\vartheta_0(-T) \to \vartheta_0(+T)$. For the reduction of the r.h.s. of (3.25) we have used the notations

$$\begin{aligned} {}_0\langle\vartheta_0(+T)|m\rangle_0 &= \frac{1}{\sqrt{2\pi}}\, e^{+i\, m\,\vartheta_0(+T) - i\, E_m\, T}, \\ {}_0\langle m|\vartheta_0(-T)\rangle_0 &= \frac{1}{\sqrt{2\pi}}\, e^{-i\, m\,\vartheta_0(-T) + i\, E_m\,(-T)} \end{aligned} \tag{3.26}$$

with $|\vartheta_0(x^0)\rangle$ as eigenstate of the Heisenberg operator $\hat{\vartheta}_0(x^0)$ and the completeness condition

$$\sum_{m\in\mathbb{Z}} |m\rangle_{00}\langle m| = 1. \tag{3.27}$$

In the limit $T \to \infty$ the excited stated with $m \neq 0$ in the sum (3.24) are dying out and only the ground state with $m = 0$ survives. This yields $Z_0[0; L, \infty] = 1$.

4 Wave function of the ground state of the free massless (pseudo)scalar field

As has been shown in [16] the wave function of the ground state of the free massless (pseudo)scalar field, describing the bosonized version of the massless Thirring model, quantized in the chirally broken phase with the BCS wave function of the ground state, takes the form

$$|\Omega(0)\rangle_{BCS} = \exp\Big(i\,\frac{\pi}{2}\,\frac{M}{g}\int_{-\infty}^{\infty} dx^1\,\sin(\beta\hat{\vartheta}(0,x^1))\Big)|\Psi_0\rangle. \tag{4.1}$$

where M is the dynamical mass of the massless Thirring fermion field quantized in the chiral broken phase, and g is the Thirring coupling constant [1]. The parameter β in the definition of the wave function (4.1), used in [16], $\sin\beta\vartheta(0,x^1)$, can be removed rescaling the ϑ–field. Indeed, the Lagrangian (2.9) can be transcribed as follows

$$L_0(\vartheta_0,\dot{\vartheta}_0) = \frac{L}{2\beta^2}\,(\beta\dot{\vartheta}_0(x^0))^2. \tag{4.2}$$

The moment of inertia is now defined as L/β^2. The Hamilton operator reads

$$\hat{h}_0(\vartheta_0,\hat{\pi}_0) = -\frac{\beta^2}{2L}\frac{d^2}{d\vartheta_0^2}. \tag{4.3}$$

The energy spectrum $E_0^{(m)}$ is given by

$$E_0^{(m)} = \frac{\beta^2}{2L}\,m^2. \tag{4.4}$$

For the ϑ_0–field with the moment of inertia L/β^2 the total Hamilton and momentum operators are defined by

$$\begin{aligned}\hat{h}[\vartheta] &= \frac{\beta^2}{2L}\,\hat{\pi}_0^2 + \frac{2\pi}{L}\sum_{n\in\mathbb{Z}}|n|\,a_n^\dagger a_n,\\ \hat{\pi}[\vartheta] &= \hat{\pi}_0 + \frac{2\pi}{L}\sum_{n\in\mathbb{Z}} n\,a_n^\dagger a_n.\end{aligned} \tag{4.5}$$

Using the discretized form for the ϑ–field (2.8) the wave function (4.1) can be transcribed into the form

$$|\Omega(0)\rangle_{BCS} = e^{+\,i\,\lambda\,\sin\hat{\vartheta}_0}\,|\Omega_0\rangle = e^{+\,i\,\lambda\,\sin\hat{\vartheta}_0}\,|0\rangle_0\otimes|\Psi_0\rangle. \tag{4.6}$$

where we have denoted $\lambda = \pi ML/2g$. The r.h.s. of (4.6) should be taken in the limit $\lambda \to \infty$ that corresponds to $L \to \infty$. It is obvious that the wave function (4.6) is not invariant under the symmetry transformations (1.2). Hence, it should describe the ground state of the free massless (pseudo)scalar field $\vartheta(x)$ in the symmetry broken phase. The contribution of the $\vartheta_{\rm v}$–field is of order of $O(1/L)$ and smaller compared with the contribution of the zero–mode. It can be dropped in the limit $L \to \infty$. In the ϑ_0–representation the wave function (4.6) reads

$$
{}_0\langle\vartheta_0|\Omega(0)\rangle_{BCS} = \frac{1}{\sqrt{2\pi}}\, e^{+\,i\,\lambda\,\sin\vartheta_0} \otimes |\Psi_0\rangle. \tag{4.7}
$$

The wave function (4.7) can be expanded into the eigenfunctions (2.16) of the Hamilton operator (2.15). The result reads

$$
|\Omega(0)\rangle_{BCS} = \sum\limits_{m=-\infty}^{\infty} J_m(\lambda)\,|m\rangle_0 \otimes |\Psi_0\rangle, \tag{4.8}
$$

where $J_m(\lambda)$ are Bessel functions [17] and the limit $\lambda \to \infty$ is assumed. The normalization of the wave function (4.8) to unity is caused by [18]

$$
\sum\limits_{m=-\infty}^{\infty} J^2_m(\lambda) = 1. \tag{4.9}
$$

Under the field–shift (1.2) the wave function (4.8) transforms into the wave function

$$
|\Omega(\alpha)\rangle_{BCS} = \sum\limits_{m=-\infty}^{\infty} J_m(\lambda)\, e^{\,im\alpha}\,|m\rangle_0 \otimes |\Psi_0\rangle \tag{4.10}
$$

at $\lambda \to \infty$. The orthogonality relation for $\alpha\,' \neq \alpha$ is defined by

$$
\begin{aligned}
{}_{BCS}\langle\Omega(\alpha\,')|\Omega(\alpha)\rangle_{BCS} &= \lim\limits_{\lambda\to\infty} \sum\limits_{m=-\infty}^{\infty} J^2_m(\lambda)\, e^{-im(\alpha\,'\,-\,\alpha)} = \\
&= \lim\limits_{\lambda\to\infty} J_0\Bigl(2\lambda\,\sin\Bigl(\frac{\alpha\,'-\alpha}{2}\Bigr)\Bigr) = \delta_{\alpha\,'\alpha},
\end{aligned} \tag{4.11}
$$

where we used the formula [19]

$$
\sum\limits_{m=-\infty}^{\infty} J^2_m(\lambda)\, e^{-im(\alpha\,'\,-\,\alpha)} = J_0\Bigl(2\lambda\,\sin\Bigl(\frac{\alpha\,'-\alpha}{2}\Bigr)\Bigr). \tag{4.12}
$$

Now let us show that the BCS wave function (4.7) describes a quantum state with energy zero.

For this aim we notice that the wave–function (4.7) is not an eigenfunction of the Hamilton operator (4.3) and the momentum operator $\hat{\pi}_0 = -id/d\vartheta_0$. It is well–known that the wave–function of the ground state should be the eigenfunction of the Hamilton operator of the quantum field under consideration with eigenvalue zero [9]. Let us show that this requirement can be satisfied for the BCS wave function (4.7) by a canonical

transformation [20]. First, we act with the operator $\hat{\pi}_0 = -id/d\vartheta_0$ on the wave function (4.7) and get

$$\Big(- i\frac{d}{d\vartheta_0} - \lambda\, \cos\vartheta_0\Big)_0\langle\vartheta_0|\Omega(0)\rangle_{BCS} = 0. \tag{4.13}$$

The operator in the l.h.s. of (4.13) is the conjugate momentum operator $\hat{\Pi}_0$ in the ϑ_0–representation related to the conjugate momentum $\hat{\pi}_0 = -id/d\vartheta_0$ by the unitary transformation [20]

$$\hat{\Pi}_0 = U\,\hat{\pi}_0\,U^\dagger, \tag{4.14}$$

where U is defined by

$$U = e^{+i\,\lambda\,\sin\hat{\vartheta}_0}. \tag{4.15}$$

The unitary operator U relates the wave functions $|\Omega_0\rangle$ and $|\Omega(0)\rangle_{BCS}$

$$|\Omega(0)\rangle_{BCS} = U\,|\Omega_0\rangle. \tag{4.16}$$

The operator $\hat{\Pi}_0$ is equal to

$$\hat{\Pi}_0 = \hat{\pi}_0 + i\lambda\,[\sin\hat{\vartheta}_0, \hat{\pi}_0] = \hat{\pi}_0 - \lambda\,\cos\hat{\vartheta}_0, \tag{4.17}$$

where we have used the canonical commutation relation $[\hat{\vartheta}_0, \hat{\pi}_0] = i$. The operator $\hat{\Pi}_0$, given by (4.17), coincides with the differential operator in the l.h.s. of (4.13) in the ϑ_0–representation.

The unitary transformation (4.16) is canonical, since it retains the canonical commutation relations

$$\begin{aligned} [\hat{\vartheta}_0, \hat{\Pi}_0] &= \Big[U\,\hat{\vartheta}_0\,U^\dagger, U\,\hat{\pi}_0\,U^\dagger\Big] = [\hat{\vartheta}_0, \hat{\pi}_0] = i, \\ [\hat{\Pi}_0, \hat{\Pi}_0] &= \Big[U\,\hat{\pi}_0\,U^\dagger, U\,\hat{\pi}_0\,U^\dagger\Big] = [\hat{\pi}_0, \hat{\pi}_0] = 0. \end{aligned} \tag{4.18}$$

According to Anderson [20] the transformations (4.15) can be called *similarity (gauge)* transformations.

Due to the canonical transformation (4.14) the field operator $\hat{\vartheta}_0$ does not change but the Hamilton operator transforms as follows

$$\hat{h}_0(\hat{\vartheta}_0, \hat{\pi}_0) \to \hat{H}_0(\hat{\vartheta}_0, \hat{\Pi}_0) = U\,\hat{h}_0(\hat{\vartheta}_0, \hat{\pi}_0)\,U^\dagger = \frac{\beta^2}{2L}\,\hat{\Pi}_0^2. \tag{4.19}$$

Equation (4.13) can be rewritten as

$$\hat{\Pi}_0\,{}_0\langle\vartheta_0|\Omega(0)\rangle_{BCS} = 0. \tag{4.20}$$

This means that the wave function (4.7) is the eigenfunction of the momentum operator $\hat{\Pi}_0$ and the Hamilton operator $\hat{H}_0$ with eigenvalue zero.

The same result can be obtained using

$$\begin{aligned}\hat{h}[\vartheta]|\Omega_0\rangle &= \Big(\frac{\beta^2}{2L}\,\hat{\pi}_0^2 + \frac{2\pi}{L}\sum_{n\in\mathbb{Z}}|n|\,a_n^\dagger a_n\Big)|\Omega_0\rangle = 0,\\ \hat{\pi}[\vartheta]|\Omega_0\rangle &= \Big(\hat{\pi}_0 + \frac{2\pi}{L}\sum_{n\in\mathbb{Z}} n\,a_n^\dagger a_n\Big)|\Omega_0\rangle = 0.\end{aligned} \tag{4.21}$$

By the canonical transformation (4.16) we transcribe (4.21) as follows

$$\begin{aligned}U\,\hat{h}[\vartheta]\,U^\dagger|\Omega(0)\rangle_{BCS} &= \Big(\frac{\beta^2}{2L}\,\hat{\Pi}_0^2 + \frac{2\pi}{L}\sum_{n\in\mathbb{Z}}|n|\,a_n^\dagger a_n\Big)|\Omega(0)\rangle_{BCS} = 0,\\ U\,\hat{\pi}[\vartheta]\,U^\dagger|\Omega(0)\rangle_{BCS} &= \Big(\hat{\Pi}_0 + \frac{2\pi}{L}\sum_{n\in\mathbb{Z}} n\,a_n^\dagger a_n\Big)|\Omega(0)\rangle_{BCS} = 0.\end{aligned} \tag{4.22}$$

This proves that the wave function $|\Omega(0)\rangle_{BCS}$ describes the ground state of the free massless (pseudo)scalar field $\vartheta(x)$ defined by the Lagrangian (1.1). This is the non–perturbative ground state describing the phase of the spontaneously broken continuous symmetry (1.2) related to the chiral symmetry of the massless Thirring model [1,2]. The wave functions (4.10) obey the same equations (4.22)

$$\begin{aligned}&\Big(\frac{\beta^2}{2L}\,\hat{\Pi}_0^2 + \frac{2\pi}{L}\sum_{n\in\mathbb{Z}}|n|\,a_n^\dagger a_n\Big)|\Omega(\alpha)\rangle_{BCS} = 0,\\ &\Big(\hat{\Pi}_0 + \frac{2\pi}{L}\sum_{n\in\mathbb{Z}} n\,a_n^\dagger a_n\Big)|\Omega(\alpha)\rangle_{BCS} = 0.\end{aligned} \tag{4.23}$$

Finally, we would like to show that the "magnetic" quantum number m defines the chirality of the fermionic state. In order to prove this we suggest to use the results obtained by Nambu and Jona–Lasinio [21]. This concerns the analysis of the BCS wave function in terms of the wave functions with a certain chirality X, the eigenvalue of the γ^5 operator, $X = 0, \pm 1, \pm 2, \ldots$. The BCS wave function of the ground state of the massless Thirring model is defined by [1,16]

$$|\Omega(0)\rangle_{BCS} = \prod_{k^1}[u_{k^1} + v_{k^1}\,a^\dagger(k^1)b^\dagger(-k^1)]\,|\Psi_0\rangle, \tag{4.24}$$

where the coefficients u_{k^1} and v_{k^1} have the properties: (i) $u_{k^1}^2 + v_{k^1}^2 = 1$ and (ii) $u_{-k^1} = u_{k^1}$ and $v_{-k^1} = -v_{k^1}$ [1,16], $a^\dagger(k^1)$ and $b^\dagger(k^1)$ are creation operators of fermions and antifermions with momentum k^1. According to Nambu and Jona–Lasinio [21] the wave function (4.24) should be a linear superposition of the eigenfunctions $|\Omega_{2n}\rangle$ with eigenvalues $X_n = 2n\,, n \in \mathbb{Z}$, i.e.

$$|\Omega(0)\rangle_{BCS} = \sum_{n\in\mathbb{Z}} C_{2n}|\Omega_{2n}\rangle. \tag{4.25}$$

For chiral rotations of fermion fields with a chiral phase $\alpha_{\rm A}$ the wave function (4.24) changes as follows [1]

$$|\Omega(\alpha_A)\rangle_{BCS} = \prod_{k^1}[u_{k^1} + v_{k^1}\,e^{-2i\varepsilon(k^1)\alpha_A}\,a^\dagger(k^1)b^\dagger(-k^1)]\,|\Psi_0\rangle, \tag{4.26}$$

where $\varepsilon(k^1)$ is a sign function. In terms of $|\Omega(\alpha_A)\rangle$ the products $C_{2n}|\Omega_{2n}\rangle$ are defined by

$$C_{2n}|\Omega_{2n}\rangle = \int_0^{2\pi} \frac{d\alpha_A}{2\pi}\, e^{+2in\alpha_{\rm A}}\, |\Omega(\alpha_A)\rangle_{BCS}. \tag{4.27}$$

Substituting (4.27) in (4.25) and using the identity [22]

$$\sum_{n\in\mathbb{Z}} e^{\,2in\alpha_{\rm A}} = \pi \sum_{k\in\mathbb{Z}} \delta(\alpha_{\rm A} - 2k\pi) \tag{4.28}$$

one arrives at the BCS wave function (4.24).

The bosonized version of the eigenfunctions $C_{2n}|\Omega_{2n}\rangle$ can be found in analogy with (4.8) and reads [16]

$$\begin{aligned} C_{2n}|\Omega_{2n}\rangle \to \int_0^{2\pi} \frac{d\alpha_A}{2\pi}\, e^{+2in\alpha_{\rm A}}\, e^{\,i\lambda\sin(\hat{\vartheta}_0 - 2\alpha_{\rm A})}|\Omega_0\rangle = \\ = \sum_{m\in\mathbb{Z}} J_m(\lambda)|m\rangle_0 \otimes |\Psi_0\rangle \int_0^{2\pi} \frac{d\alpha_A}{2\pi}\, e^{+2i(n-m)\alpha_{\rm A}} = J_n(\lambda)|n\rangle_0 \otimes |\Psi_0\rangle. \end{aligned} \tag{4.29}$$

This completes the proof. Hence, in the treatment of the collective zero–mode as a rigid rotor the "magnetic" quantum number m defines the chirality $X_m = 2m$ of the fermionic state in the massless Thirring model.

The fact that the BCS wave function is not an eigenstate of chirality testifies that chiral symmetry is spontaneously broken. In order to clarify this assertion we would like to draw a similarity between chirality in the massless Thirring model with triality in QCD. In QCD there exist no triality changing transitions, this means a dynamical change of triality is impossible. It is well–known that the confined phase in QCD is $Z(3)$ symmetric. Triality zero states are screened, and triality non–zero states are confined. This means that states with different triality behave differently. Whereas in the high–temperature phase of QCD all triality states behave in the same way, they get screened. This is guaranteed by the spontaneous breaking of $Z(3)$ symmetry. In our case the situation is similar to the deconfined phase. In the massless Thirring model there are no chirality changing transitions. The ground state is of BCS-type, defining a condensate of fermion-antifermion pairs with different chiralities. In order to get a ground state with properties independent on the exact value of the total chirality of all fermion–antifermion pairs, we need spontaneous breaking of chiral symmetry similar to the the spontaneous breaking of $Z(3)$ symmetry in QCD. Such a spontaneous breaking of chiral symmetry is realized by the BCS wave function.

5 Conclusion

We have shown that the ground state of the free massless (pseudo)scalar field, the bosonized version of the massless Thirring model in the non–trivial phase, can be defined by a direct product of the fiducial vacuum $|\Psi_0\rangle$ and a BCS–type wave function (4.1). We have demonstrated that the BCS wave function is related to the collective zero–mode described by a rigid rotor (4.8). BCS wave functions differing in the values of the field–shifts (1.2) are orthogonal ${}_{BCS}\langle\Omega(\alpha')|\Omega(\alpha)\rangle_{BCS} = \delta_{\alpha\,\alpha}$.

We have analysed the generating functional of Green functions $Z[J]$. We have shown that for $\tilde{J}(0) \neq 0$ the infrared divergences have a simple physical interpretation in terms of the classically moving rigid rotor acquiring an infinite angle for an infinite interim even if its motion has been initiated by an infinitesimal external perturbation. These divergences can be removed by the constraint on the external source $\tilde{J}(0) = 0$. As a result the collective zero–mode cannot be excited and the correlation functions are determined by the contribution of the vibrational modes $\vartheta_{\rm v}(x)$ only. These modes are quantized relative to the fiducial vacuum $|\Psi_0\rangle$. According to [3, 4] this testifies that the quantum field theory of the free massless (pseudo)scalar field $\vartheta_{\rm v}(x)$, in the Wightman sense [9], deals with Wightman's observables defined on the test functions from $\mathcal{S}_0(\mathbb{R}^2)$.

The BCS type wave function (4.8) of the ground state is not invariant under the continuous symmetry (1.2) and behaves as (4.10). Hence, according to the Goldstone theorem [10] the continuous symmetry (1.2) is spontaneously broken. As has been shown in [2–4] the phase of spontaneously broken continuous symmetry (1.2) is characterized quantitatively by the non–vanishing spontaneous magnetization $\mathcal{M} = \langle\Psi_0|\cos\beta\vartheta_{\rm v}(x)|\Psi_0\rangle = 1$. This confirms the non–vanishing value of the fermion condensate in the massless Thirring model with fermion fields quantized in the chirally broken phase [1]. Hence, the massless Thirring model possesses a chirally broken phase as has been pointed out in [1–4, 16, 23].

In the rigid rotor treatment of the collective zero–mode the variation $\delta\vartheta(x)$ of the free massless (pseudo)scalar field $\vartheta(x)$, caused by the field–shift transformation (1.2), is defined by a canonical quantum mechanical commutator

$$\delta\vartheta(x) = \alpha\, i[Q(x^0), \vartheta(x)] = \alpha\, i[\pi_0, \vartheta_0] = \alpha,$$

which can never be equal to zero [8]. This result does not depend on whether the ground state of the free massless (pseudo)scalar field is invariant or non–invariant under symmetry transformations (1.2).

Since the removal of the collective zero–mode from the observable modes of the free massless (pseudo)scalar field $\vartheta(x)$ agrees with the definition of Wightman's observable on the test functions from $\mathcal{S}_0(\mathbb{R}^2)$ the obtained non–vanishing of the variation $\delta\vartheta(x)$ does not contradict Coleman's theorem valid only for Wightman's observables defined on the test functions from $\mathcal{S}(\mathbb{R}^2)$ [3, 4].

References

[1] M. Faber and A. N. Ivanov, Eur. Phys. J. C **20**, 723 (2001), hep-th/0105057.

[2] M. Faber and A. N. Ivanov, Eur. Phys. J. C **24**, 653 (2002), hep–th/0112184.

[3] M. Faber and A. N. Ivanov, *Comments on Coleman's paper "There are no Goldstone Bosons in Two Dimensions*, hep–th/0204237.

[4] M. Faber and A. N. Ivanov, *Quantum field theory of a free massless (pseudo)scalar field in 1+1–dimensional space–time as a test for the massless Thirring model*, hep–th/0206244.

[5] K. Yoshida, Nucl. Phys. B **105**, 272 (1976).

[6] M. Faber and A. N. Ivanov, *On the solution of the massless Thirring model with fermion fields quantized in the chiral symmetric phase*, IK-TUW-Preprint 0111401, hep–th/0112183.

[7] N. D. Mermin and H. Wagner, Phys. Rev. Lett. **17**, 1133 (1966); P. C. Hohenberg, Phys. Rev. **158**, 383 (1967); N. D. Mermin, J. Math. Phys. **8**, 1061 (1967).

[8] S. Coleman, Comm. Math. Phys. **31**, 259 (1973).

[9] A. S. Wightman, *Introduction to Some Aspects of the Relativistic Dynamics of Quantized Fields*, in *HIGH ENERGY ELECTROMAGNETIC INTERACTIONS AND FIELD THEORY*, Cargèse Lectures in Theoretical Physics, edited by M. Levy , 1964, Gordon and Breach, 1967, pp.171–291; R. F. Streater and A. S. Wightman, in *PCT, SPIN AND STATISTICS, AND ALL THAT*, Princeton University Press, Princeton and Oxford, Third Edition, 1980.

[10] J. Goldstone, Nuovo Cimento **19**, 154 (1961); J. Goldstone, A. Salam, and S. Weinberg, Phys. Rev. **127**, 965 (1962).

[11] S. Coleman, Phys. Rev. D **11**, 2088 (1975).

[12] A. L. Fetter and J. D. Walecka, in *THEORETICAL MECHANICS OF PARTICLES AND CONTINUA*, McGraw–Hill Publishing Co., New York, 1980, pp.86–130; K. Huang, in *QUANTUM FIELD THEORY, From Operators to Path Integrals*, John Willey & Sons, Inc., New York, 1998, pp.1–7.

[13] J. Kogut and L. Susskind, Phys. Rev. D **11**, 3594 (1975).

[14] J. Schwinger in *PARTICLES AND SOURCES*, Gordon and Breach, New York 1969 and *PARTICLES, SOURCES AND FIELDS*, Addison–Wesley Publishing Co., Massachusetts 1970

[15] L. S. Schulman,Phys. Rev. **176**, 1558 (1968); Phys. Rev. **188**, 1139 (1969); J. Math. Phys. **12**, 304 (1971); L. S. Schulman, in *TECHNIQUES AND APPLICATIONS OF PATH INTEGRATION*, John & Sons, New York, 1981, pp.190–213.

[16] M. Faber and A. N. Ivanov, *Bosonic vacuum wave functions from the BCS–type wave function of the ground state of the massless Thirring model*, hep–th/0210104.

[17] *HANDBOOK OF MATHEMATICAL FUNCTIONS, with Formulas, Graphs, and Mathematical Tables*, ed. by M. Abramowitz and I. E. Stegun, U.S. Department of Commerce, National Bureau of Standards, Applied Mathematics Series • 55, 1972, p.361 formula (9.1.41).

[18] I. S. Gradshtein and I. M. Ryzhik, in *TABLE OF INTEGRALS, SERIES AND PRODUCTS*, Academic Press, New York and London, 1965, p.980 formula (8.536.3).

[19] (see [16] p.979 formula (8.531.3))

[20] A. Anderson, Ann. of Phys. **232**, 292 (1994)

[21] Y. Nambu and G. Jona–Lasinio, Phys. Rev. **122**, 345 (1960).

[22] I. M. Gel'fand and G. E. Shilov, in *GENERALIZED FUNCTIONS*, Vol.1, Academic Press, New York, 1964, p.331.

[23] M. Faber and A. N. Ivanov, *Is the energy density of the ground state of the sine–Gordon model unbounded from below for* $\beta^2 > 8\pi$ *?*, hep–th/0205249; *Massless Thirring model in the boson field representation.* hep–th/0206034.