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A Mysterious Zero in $AdS_5 \times S^5$ Supergravity

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Abstract

It is shown that all the states in $AdS_5 \times S^5$ supergravity have zero eigenvalue for all Casimirs of its symmetry group $SU(2,2|4)$. To compute this zero in supergravity we refine the oscillator methods for studying the lowest weight unitary representations of $SU(N,M|R,S)$. We solve the reduction problem when one multiplies an arbitrary number of super doubletons. This enters in the computation of the Casimir eigenvalues of the lowest weight representations. We apply the results to $SU(2,2|4)$ that classifies the Kaluza-Klein towers of ten dimensional type IIB supergravity compactified on $AdS_5 \times S^5$. We show that the vanishing of the $SU(2,2|4)$ Casimir eigenvalues for all the states is indeed a group theoretical fact in $AdS_5 \times S^5$ supergravity. By the AdS-CFT correspondence, it is also a fact for gauge invariant states of super Yang-Mills theory with four supersymmetries in four dimensions. This non-trivial and mysterious zero is very interesting because it is predicted as a straightforward consequence of the fundamental local Sp(2) symmetry in 2T-physics. Via the 2T-physics explanation of this zero we find a global indication that these special supergravity and super Yang-Mills theories hide a twelve dimensional structure with (10,2) signature.

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I. INTRODUCTION

In recent sudies it has been shown that the non-linearly realized hidden superconformal symmetries OSp$(N|4)$, SU$(2,2|N)$, F$_4$, OSp$(8^*|N)$ of ordinary superparticle actions in $d = 3,4,5,6$ respectively, become linearly realized and evident symmetries in the 2T-physics formulation of superparticles [1][2] . The 2T-physics method has been extended to 12-dimensional super phase space (X^M, P^M, Θ) with (10+2 signature, and 32 dimensional Weyl spinor Θ) to describe the collection of all the Kaluza-Klein towers in AdS$_5\times$S^5 supergravity, as the quantum states of a single SU$(2,2|4)$ super particle on AdS$_5\times$S^5 [2][3]. This has predicted an interesting property of type IIB supergravity that had not been noticed before. Namely, the Casimir eigenvalues vanish for all the SU$(2,2|4)$ Casimir operators for all the states of supergravity. The mechanism for this outcome in [3] is the fundamental Sp(2) gauge symmetry and kappa supersymmetry in 2T-physics. Namely, the gauge singlet condition (vanishing of 12-dimensional constraints) on physical states predicts the vanishing of the SU$(2,2|4)$ Casimir eigenvalues. From the point of view of supergravity this is rather mysterious and requires verification with other methods. This motivated the present study of representation theory of the supergroup SU$(N,M|R,S)$ to compute the corresponding Casimir eigenvalues with "oscillator methods" that were previously applied in the studies of compactification of type IIB supergravity in 10 dimensions. In this paper all the Casimir eigenvalues are computed and verified to vanish.

The following method for determining lowest weight unitary representations of noncompact supergroups was developed in [4]. Here we refine the method to get a clearer description of the lowest states in unitary representations of SU$(N,M|R,S)$, and to compute the Casimir eigenvalues for such representations, which were not properly computed in previous literature. The correct computation of the Casimir eigenvalues is essential in our application, and is needed to prove that all the AdS$_5\times$S^5 supergravity states have zero SU$(2,2|4)$ Casimir eigenvalues.

Our aim is to study the case of SU$(2,2|4)$ which is the relevant supergroup. In the case of SU$(2,2|4)$ the SU$(2,2)$ subgroup is interpreted as the conformal group SO$(4,2)$ which is the symmetry of AdS$_5$ space. Likewise the SU$(4)=$SO(6) subgroup is interpreted as the symmetry of S^5 space. The SU$(2)\times$SU$(2)\times$SU(4) quantum numbers of the lowest states identify the fields that enter in a field theory. We keep this physical application in mind as we discuss first the more general setting of SU$(N, M|R, S)$, and later specialize to SU$(2, 2|4)$.

Using the "color" terminology of [4], the number of "colors" in this setup is interpreted as a device for solving the reduction problem when one takes direct products of the fundamental unitary representation, which is called super-doubleton for SU$(N, M|R, S)$ for one color. Thus the reduction of the direct product of two doubletons is achieved by taking two colors. For the direct product of three doubletons we take three colors, and so on. It is natural to define a color group SU(C) that acts on C copies of the super-doubleton. This trick of "color" was first used for compact supergroups to derive the results in [5] and later applied to noncompact groups [6] and supergroups [4].

We show that the "color" group of the oscillators cleanly classifies all the lowest states of SU$(N, M|R, S)$ and determines the Casimir eigenvalues. The refinement in the methods of [4] comes from improved use of the properties of the "color" group, which in turn permit the new computation of the Casimir operator for these representations.

After classifying all the SU$(N, M|R, S)$ lowest states that appear in the reduction according to the color group SU(C), we focus on the smaller set of color singlet representations that are of interest in our physical application. We present a new and concise characterization of such color singlet super multiplets in terms of a single lowest state which identifies the supermultiplet of fields in field theory. In the case of SU$(2, 2|4)$ we identify a unique color singlet multiplet for each value of the color C. This unique lowest state identifies all the fields in a Kaluza-Klein tower that occurs in the compactification of 10D type IIB supergravity on AdS$_5\times$S^5. Taking all the values of color $C = 2, 3, \cdots, \infty$ we find the supergravity set of Kaluza-Klein towers which coincide with the quantum states of our superparticle in 12D in 2T-physics [3]. In the 2T-physics formulation of the problem the vanishing of the SU$(2, 2|4)$ Casimir eigenvalues was a direct consequence of Sp(2) and kappa gauge symmetries. In this paper we verify this prediction through representation theory of SU$(2, 2|4)$ in the context of supergravity.

II. OSCILLATOR CONSTRUCTION AND THE CASIMIR

Consider the annihilation operators ($a_n^\alpha, b_{\alpha m}$ = bosons, $\psi_r^\alpha, \chi_{\alpha s}$ = fermions), and the corresponding creation operators (denoted by placing a bar on the symbol). The lower/upper SU(C) labels $\alpha = 1, \cdots, C$ denote copies (named "color" in [4]) and the other indices belong to the fundamental representation of SU$(N)\times$SU$(M)\times$SU$(R)\times$SU(S) as follows

$$n, n' = 1, 2, \dots, N; \quad m, m' = 1, 2, \dots, M; \tag{1}$$

$$r, r' = 1, 2, \dots, R; \quad s, s' = 1, 2, \dots, S. \tag{2}$$

Lower indices are unprimed and upper indices are primed. They indicate the fundamental representation (lower) and the complex conjugate of the fundamental representation (upper) respectively. The generators of the supergroups are constructed from the harmonic oscillators as follows (the color index α is summed over, so the SU$(N, M|R, S)$ generators are color singlets)

$$\begin{pmatrix} a_n \\ \bar{b}^{m'} \\ \psi_r \\ \bar{\chi}^{s'} \end{pmatrix}^{\alpha} \begin{pmatrix} \bar{a}^{n'} & -b_m & \bar{\psi}^{r'} & \chi_s \end{pmatrix}_{\alpha} \tag{3}$$

$$= \begin{pmatrix} a_n \cdot \bar{a}^{n'} & -a_n \cdot b_m & a_n \cdot \bar{\psi}^{r'} & a_n \cdot \chi_s \\ \bar{b}^{m'} \cdot \bar{a}^{n'} & -\bar{b}^{m'} \cdot b_m & \bar{b}^{m'} \cdot \bar{\psi}^{r'} & \bar{b}^{m'} \cdot \chi_s \\ \psi_r \cdot \bar{a}^{n'} & -\psi_r \cdot b_m & \psi_r \cdot \bar{\psi}^{r'} & \psi_r \cdot \chi_s \\ \bar{\chi}^{s'} \cdot \bar{a}^{n'} & -\bar{\chi}^{s'} \cdot b_m & \bar{\chi}^{s'} \cdot \bar{\psi}^{r'} & \bar{\chi}^{s'} \cdot \chi_s \end{pmatrix} \tag{4}$$

$$= J + \mathbf{1}\,\frac{(\Delta + (N - R)\,C)}{N + M - R - S} \tag{5}$$

The column is the fundamental of the supergroup, and the row is the hermitian conjugate of the fundamental times the invariant metric, and the matrix represents the generators in the adjoint. The dot represents summation over the "color" index α. We are distinguishing between upper and lower indices. On the diagonal blocks we separate the traceless and trace parts, and get the SU(N), SU(M), SU(R), SU(S) subgroups and four U(1) subgroups generated by the number operators $N_a = tr\,(\bar{a} \cdot a)$, $N_b = tr\,(\bar{b} \cdot b)$, $N_\psi = tr\,(\bar{\psi} \cdot \psi)$, $N_\chi = tr\,(\bar{\chi} \cdot \chi)$. The supertraceless part of the matrix denoted as J gives the the SU$(N, M|R, S)$ generators. The coefficient of the matrix $\mathbf{1}$ is the supertrace of the matrix, while Δ is given in terms of the number operators by

$$\Delta = N_a - N_b + N_\psi - N_\chi. \tag{6}$$

A SU$(C)\times$U(1) "color" group commutes with all the generators J. The U(1) is generated by Δ and the color SU(C) is generated by the traceless matrix $G_{\alpha}^{\ \beta}$

$$G_{\alpha}^{\ \beta}=\bar{a}_{\alpha}^{n}a_{n}^{\beta}-b_{\alpha m}\bar{b}^{m\beta}+\bar{\psi}_{\alpha}^{r}\psi_{r}^{\ \beta}+\chi_{\alpha s}\bar{\chi}^{s\beta}-\delta_{\alpha}^{\ \beta}\frac{1}{C}\left(\Delta-CM+CS\right). \tag{7}$$

The traceless $G_{\alpha}^{\ \beta}$ has the commutation rule $\left[G_{\alpha}^{\ \beta},\Xi^{\gamma}\right]=-\delta_{\alpha}^{\gamma}\Xi^{\alpha}$ (fundamental of "color") for every component Ξ^{α} in the column matrix given above.

We can compute the quadratic Casimir operator of SU$(N,M|R,S)$, which is given by the supertrace of the square of J, as

$$C_{2}^{(N,M|R,S)}=\frac{1}{2}Str\left(JJ\right). \tag{8}$$

Higher Casimir operators are simply the supertrace of higher powers of J. After some algebra one can verify our result that the the quadratic Casimir operator is a function of only the Casimir operators of SU$(C)\times$U(1)

$$C_{2}^{(N,M|R,S)}=C_{2}^{SU(C)}+\frac{(N+M-R-S)-C}{2\left(N+M-R-S\right)C}\left(\Delta+\left(N-R\right)C\right)\left(\Delta-\left(M-S\right)C\right) \tag{9}$$

where $C_{2}^{SU(C)}$ is the quadratic Casimir for SU(C) given by

$$C_{2}^{SU(C)}=\frac{1}{2}G_{\alpha}^{\ \beta}G_{\beta}^{\ \beta}. \tag{10}$$

The above expression for $C_{2}^{(N,M|R,S)}$ is incorrect when $N+M=R+S$ because in that case the expression in Eq.(5) is not valid due to the fact that one must remove from J one generator proportional to the identity matrix which is supertraceless. The relevant generator is identified as the U(1) in the decomposition SU$(N,M|R,S)\rightarrow$SU$(N|R)\times$SU$(M|S)\times$U(1), which is proportional to $\delta=C+(N_{a}+N_{\psi})/(N-R)-(N_{b}+N_{\chi})/(M-S)$. When $N+M=R+S$ it becomes $C+\Delta/(N-R)$ which commutes with all other generators. Its contribution to the Str$(J^{2})/2$ must be subtracted to obtain the correct Casimir when $N+M=R+S$. This amounts to subtracting the quantity $(\delta)^{2}(N-R)(M-S)/2(N+M-R-S)$ from $C_{2}^{(N,M|R,S)}$ and then taking the limit $N+M=R+S$. When this is taken into account in the computation of the quadratic Casimir, the result is

$$\left.C_{2}^{(N,M|R,S)}\right|_{N+M=R+S}=C_{2}^{SU(C)}+\frac{1}{2C}\left(\Delta+\left(N-R\right)C\right)^{2}+\frac{C}{2}\left(\Delta+\left(N-R\right)C\right). \tag{11}$$

Note that the pole at $N+M=R+S$ has cancelled in this expression. Both expressions in Eqs.(9,11) for the Casimir operators in the oscillator approach for supergroups are new. We will use them in our discussion.

In particular, for SU$(2,2|4)$ color singlet states, which are important in our setup below, Eq.(11) reduces to ($N=M=2,$ $R=4,$ $S=0,$ $C_2^{SU(C)}=0$)

$$C_2^{(2,2|4)}=\frac{1}{2C}\left(\Delta-2C+C^2\right)(\Delta-2C)\,. \tag{12}$$

In the rest of the paper we will show that for all the states in the Kaluza-Klein towers of supergravity on AdS$_5\times$S^5 we must have $\Delta=2C,$ for $C=2,3,\cdots\infty,$ and therefore $C_2^{(2,2|4)}=0$ in AdS$_5\times$S^5 supergravity.

One may check our general expression for the Casimir operator in various limits of the numbers M,N,R,S for which one can find formulas in the literature. For example for SU(N) with one color $C=1$ we know we will obtain only the one row Young tableaux because of the symmetry imposed by the products of bosonic creation operators $\bar{a}^{n_1'}\cdots\bar{a}^{n_k'}|0>$. Our formula in Eq.(9) reduces then to the Casimir for SU(N) when we take the limit $M=R=S=0$ and use $C_2^{SU(1)}=0$

$$C_2^{SU(N)}\Big|_{one-row}=\frac{N-1}{2N}\left(\Delta+N\right)\Delta \tag{13}$$

where $\Delta=N_a$ is the number of boxes. Indeed this is the correct Casimir eigenvalue for the one row Young tableaux representations. In particular for SU(2) we have $N=2,$ and defining $j=\Delta/2$ gives $C_2^{SU(2)}=j\left(j+1\right)$ which is the well known formula.

Another simple example is the case $N=M=S=0$ which leaves SU(R) constructed with fermionic oscillators ψ_r. In the case of one color $C=1$ we can obtain only the one column antisymmetric representations of SU(R) because of the antisymmetry imposed by fermionic creation operators $\bar{\psi}^{r_1'}\cdots\bar{\psi}^{r_k'}|0>$. In this case our formula reduces to (the minus sign is because of the supertrace)

$$-\,C_2^{SU(R)}\Big|_{one\text{-}column}=\frac{R+1}{2R}\left(\Delta-R\right)\Delta \tag{14}$$

where $\Delta=N_\psi$ is the number of boxes in the column with $N_\psi\leq R.$ Indeed this is the correct expression. Such tests show the utility and power of our general formula.

It takes more effort to compute the general formulas for the higher Casimir operators in the oscillator approach. However, as we will see later we only need to compute them for a more restricted subset of representations within the oscillator approach, namely those relevant to AdS$_5\times$S^5 supergravity. Once we define the algebraic restriction that was first discovered in the 2T-physics approach, we will easily determine all Casimir eigenvalues in the last section of this paper, and thus verify the universal zero.

III. LOWEST WEIGHT UNITARY REPRESENTATIONS

The "ground states" that label the lowest weight representations are singular states which are annihilated by the bosonic $a \cdot b$, $\psi \cdot \chi$ and fermionic $a \cdot \chi$, $\psi \cdot b$ generators. A few examples of such ground states are

$$|0>, \quad \bar{a}_\alpha^{n'}|0>, \quad \bar{b}^{m'\beta}|0>, \quad \bar{\psi}_\alpha^{r'}|0>, \quad \bar{\chi}^{s'\alpha}|0>, \tag{15}$$

$$\left(\bar{a}_\alpha^{n'}\bar{b}^{m'\beta} - \frac{1}{C}\delta_\alpha^\beta \bar{a}^{n'} \cdot \bar{b}^{m'}\right)|0>, \quad \bar{a}_{\alpha_1}^{n'_1} \cdots \bar{a}_{\alpha_k}^{n'_k}|0>, \quad etc. \tag{16}$$

In general, the Fock space states that have only upper color indices, or only lower color indices, are always ground states. When there are both upper and lower color indices, if they form a traceless tensor in color space, then the state is also a ground state, provided there are no overall factors of trace multiplying the expression (i.e. in the examples above $\bar{a}^{n'} \cdot \bar{b}^{m'}, \bar{a}^{n'} \cdot \bar{\chi}^{s'}$ etc. should not appear as factors applied on these states). Such ground states are classified as irreducible representations of SU$(C) \times$U(1) and they correspond to Young tableaux for the color group. The traceless color Young tableau combines a Young tableau Y_a or Y_ψ for the color indices on the oscillators a or ψ, with a Young tableau Y_b^* or Y_χ^* for the color indices on the oscillators b or χ, into an irreducible traceless color tableau.

The resulting color-traceless tableau determines an eigenvalue for the quadratic Casimir operator $C_2\left(SU\left(C\right)\right)$ and this number enters in our formulas in Eqs.(9,11). The other number Δ that enters in our formula is determined directly from the total numbers of oscillators N_a, N_b, N_ψ, N_χ in the "ground state".

In addition, the ground states are classified by the Young tableaux of the subgroup S(U$(N) \times$ U$(M) \times$ U$(R) \times$ U(S)), but these are not unrelated to the color-traceless SU(C) tableaux. Namely, for the bosons a, b when the color indices are associated with a given tableau, the SU(N) or SU(M) indices must have the same shape tableau, up to a complex conjugation. Likewise, for the fermions ψ, χ when the color indices are associated with a given tableau, the SU(R) or SU(S) indices must correspond to the reflection of the tableau from its diagonals (interchange rows and columns), again up to a complex conjugation. In addition, the U(1) quantum numbers simply correspond to the numbers of oscillators N_a, N_b, N_ψ, N_χ applied on the vacuum. We see that the color tableaux are directly related to the S(U$(N) \times$U$(M) \times$U$(R) \times$U(S)) tableaux, and this explains why the quadratic Casimir operator of SU$(N, M|R, S)$ is determined by the color Casimir and Δ.

Applying arbitrary polynomials of all the remaining generators on a given SU$(N,M|R,S)$ ground state produces an infinite tower of states that have the same Casimir eigenvalues as the ground state (since the SU$(N,M|R,S)$ generators are SU$(C)\times$U(1) singlets). Therefore these towers form infinite dimensional irreducible representations of SU$(N,M|R,S)$. All states have positive norm since they are Fock states created by positive norm oscillators. Hence these irreducible representations are unitary.

A particular set of ground states that are relevant for classifying the Kaluza-Klein towers of interest in this paper are those that are "color" singlet as we will see below. In this case all the states in the representation are also color singlets since the SU$(N,M|R,S)$ generators that produce the infinite tower are themselves color singlets. In field theoretic applications it is of interest to organize the towers by listing the minimal set of SU$(2,2)$ ground states (both bosonic and fermionic) because their quantum numbers correspond to the labels on fields which form a supermultiplet in AdS$_5$ space in a field theoretic setting. Therefore more generally we will discuss the SU(N,M) ground states that belong to the same supermultiplet of SU$(N,M|R,S)$. The SU(N,M) ground states are those that are annihilated by $a_n\cdot b_m$; this subset of states provide the quantum numbers of the fields that form supermultiplets in field theory.

IV. GROUP THEORY FOR ADS$_5\times$S^5 SUPERGRAVITY

We use the oscillator approach for SU$(2,2|4)$ with generators constructed from oscillators. Thus, we specialize to $R=4$ and $S=0.$ The U(1) in SU$(2,2)$ is interpreted as the AdS "energy" operator

$$E=N_a+N_b+2C. \tag{17}$$

It is obtained in the closure of $\left[a\cdot b,\bar{b}\cdot\bar{a}\right]$, and is determined by the total number of bosons (a's or b's), and the number of "colors" $C.$

The SU$(2,2)$ ground states are constructed by applying any number of creation operators $\bar{a}$ and $\bar{b}$ on the vacuum or other state made up of only the fermions, provided they are in SU(C) color-traceless combinations

$$\left[\left((\bar{a})^{n_a}\left(\bar{b}\right)^{n_b}\right)_{color-traceless}\right]\left(\bar{\psi}\right)^{n_\psi}|0>. \tag{18}$$

The numbers of oscillators n_a,n_b,n_ψ are any integers. The SU$(2)\times$SU(2) indices m,n on the oscillators (which are not shown) are free to be anything; the color indices α can also be anything as long as they are in traceless combinations when contracted with δ_α^β (respecting upper/lower indices). When the ladder down operators $a_m \cdot b_n$ are applied on these states one obtains zero either because there are not enough creation operators of either kind, or because one is forced to contract the color indices but the color trace vanishes.

The overall color of the state can be classified by SU(C) Young tableaux which have definite eigenvalues with respect to the SU(C) Casimir operators. The important factor is to include only those tableau that come from the traceless condition specified withing the a, b oscillators. The total SU(C) Young tableaux thus obtained (including the $\bar\psi' s$) determine the SU(C) quadratic casimir eigenvalues, and through Eq.(11)), also the SU$(2,2|4)$ quadratic casimir eigenvalues.

There are an infinite number of such color-traceless SU$(2,2)$ lowest states. Some examples are

$$\left(\bar a_{n_1}^{\alpha_1} \cdots \bar a_{n_k}^{\alpha_k}\right) \left(\bar\psi^{r_1\beta_1} \cdots \bar\psi^{r_l\beta_l}\right) |0>, \tag{19}$$

$$\left((\bar a)_n^{\alpha_1} \left(\bar b\right)_{m\alpha_2} - \frac{1}{C}\delta_{\alpha_2}^{\alpha_1} \bar a_n \cdot \bar b_m \right) \left(\bar\psi^{r_1\beta_1} \cdots \bar\psi^{r_l\beta_l}\right) |0>, \; etc. \tag{20}$$

The AdS-energy E together with the SU$(2)\times$SU$(2)\times$SU(4) quantum numbers of such lowest states identify the quantum numbers of the corresponding field in supergravity. These quantum numbers are easy to figure out by inspecting the state: first the color indices on the states need to be reduced to irreducible SU(C) Young tableau. This forces the SU$(2)\times$SU$(2)\times$SU(4) quantum numbers to appear in definite Young tableaux for each one of these groups. Then, the AdS-energy is given by the numbers $N_a + N_b$ of $a's$ and $b's$ as in Eq.(17). The SU$(2)\times$SU(2) quantum numbers are also determined as (j_1, j_2) by counting the number of unpaired boxes in each of the SU$(2)\times$SU(2) tableau and dividing by 2. Finally the SU(4) quantum numbers are read off from the corresponding Young tableau.

We want to identify the SU$(2,2|4)$ lowest states, which when decomposed into SU$(2,2)$ lowest states, contain (j_1, j_2) which do not fall outside of the following list

$$\begin{array}{l} (j_1, j_2):\; (0,0)\,,\;\; (1/2,0)\,,\;\; (0,1/2)\,,\;\; (1/2,1/2)\,, \\ \qquad\qquad (1,0)\,,\;\; (0,1)\,,\;\; (1/2,1)\,,\;\; (1,1/2)\,,\;\; (1,1)\,. \end{array}$$

This because the total spin of the field is given by $j_1 + j_2$, and this should not exceed 2. The representation $(j_1, j_2) = (1,1)$ of SU(2)$\times$SU(2) $\subset$ SU$(2,2)$ represents the spin 2 graviton with $(2j_1+1)\times(2j_2+1)=3\times 3=9$ independent components, corresponding to the traceless symmetric field $g_{\mu\nu}$ ($\frac{1}{2}4\times 5-1=9$).

The condition of maximum spin 2 comes from the fact that in the compactification of supergravity higher spinning fields cannot occur. Therefore we need to identify the SU$(2,2|4)$ representations that have the property of Eq.(??). This means that the subgroup SU$(2,2)$ *lowest state* can have at the most $(\bar{a})^2(\bar{b})^2$ in *color-traceless* combination applied on the vacuum (with any number of $\bar{\psi}$). Therefore the lowest state of the full SU$(2,2|4)$ should have the property that it should vanish if more than two powers of the supergenerators $\bar{\psi}\cdot\bar{a}$ or $\bar{b}\cdot\psi$ are applied, *provided the color-traceless condition is imposed* on the a,b color indices. We will identify below a unique color singlet SU$(2,2|4)$ ground state that satisfies these requirements.

To describe it we examine the lowest SU$(2,2|4)$ color singlet constructed by using the color Levi-Civita symbol. We denote this state symbolically by $\bar{\psi}^{2C}|0>$

$$\bar{\psi}^{2C}|0>\sim\left(\varepsilon_{\alpha_1\cdots\alpha_C}\bar{\psi}^{a_1\alpha_1}\cdots\bar{\psi}^{a_C\alpha_C}\right)\left(\varepsilon_{\beta_1\cdots\beta_C}\bar{\psi}^{b_1\beta_1}\cdots\bar{\psi}^{b_C\beta_C}\right)|0>. \tag{21}$$

After taking into account the fact that the $\bar{\psi}^{a\alpha}$ are fermions, one deduces that the SU(4) indices on the $\bar{\psi}'s$ must have the permutation properties of the SU(4) Young tableau with two rows, each having C boxes. The SU(4) Young tableaux labels are therefore $(C,C,0,0)$. Since this is a Lorentz singlet, the tower based on this SU$(2,2)$ lowest state represents the AdS$_5\times$S^5 scalar field with appropriately symmetrized/antisymmetrized indices according to the $(C,C,0,0)$ SU(4) Young tableau

$$\bar{\psi}^{2C}|0>\rightarrow\phi^{(a_1\cdots a_C),(b_1\cdots b_C)}\left(x^\mu,u\right)\sim(C,C,0,0). \tag{22}$$

The maximum number of supergenerators that can be applied on this state before annihilating it are $2C$ for type $\bar{b}\cdot\psi$ supergenerator, and $2C$ for type $\bar{\psi}\cdot\bar{a}$ supergenerator (since products of more than $4C$ fermions $\bar{\psi}$ vanish). However when we apply powers of the supergenerators $(\psi\cdot\bar{a})^k\left(\bar{b}\cdot\bar{\psi}\right)^l$ on the state in Eq.(21) we create SU$(2,2)$ states that may have color trace (i.e. contain the color dot product $\bar{a}\cdot\bar{b}$). These should not be counted among the SU$(2,2)$ lowest states that identify fields since they are descendants of the form $\left(\bar{a}\cdot\bar{b}\right)^n|lowest\ SU\,(2,2)>$. Descendants correspond to derivatives applied on the fields. To identify the fields themselves uniquely one must project to the color traceless states only.

Thus, the following states obtained by applying supergenerators which form color traceless combinations

$$\left((\psi \cdot \bar{a})^k \left(\bar{b} \cdot \bar{\psi} \right)^l \right)_{color\, traceless} \bar{\psi}^{2C} \left|0\right> \tag{23}$$

are also annihilated by $a \cdot b$ thanks to the color-traceless condition. With this condition we find that the limits of Eq.(**??**) for the spins of the fields are indeed satisfied because then the powers k or l are limited to a range $k \leq k_{\max}$, $l \leq l_{\max}$ and $k_{\max} + l_{\max} \leq 4$. This collection of SU$(2,2)$ lowest states - which result from a single SU$(2,2|4)$ lowest state - form supermultiplets that correspond to the fields on the AdS space (x^μ, u).

Including all derivatives of the fields corresponds to including all the states, without the color traceless condition, obtained by applying all the generators with arbitrary powers. In our construction applying all powers of generators without restrictions corresponds to a unitary representation based on the lowest SU$(2,2|4)$ state. By construction, all states are automatically color singlets for any number of colors C, since in Eq.(23) $\bar{\psi}^{2C}$ as well as all SU$(2,2|4)$ generators are color singlets. The towers based on the lowest supermultiplets in Eq.(23) correspond to the fields that form supermultiplets without derivatives on the fields. The quantum numbers on the fields are the SU$(2) \times$SU$(2) \times$SU(4) Young tableau labels of the lowest states. By construction, these fields form irreducible supermultiplets of SU$(2,2|4)$. There is such a supermultiplet for every value of the color $C = 1, 2, 3, \cdots \infty$ with its lowest state being $\bar{\psi}^{2C} \left|0\right>$.

One important fact that should be noted is that the color Casimir eigenvalue $C_2^{SU(C)}$ of these color singlet states is automatically zero!

A. Yang-Mills supermultiplet

We now discuss SU$(2,2|4)$ with only one color $C = 1$. This case is simple because there are no color traceless combinations to discuss. Consider the lowest state $\bar{\psi}^2 \left|0\right>$. More than two powers of the supergenerators $b\bar{\psi}$ annihilate this state since there are only four different $\bar{\psi}^a$. Also more than two powers of $\psi\bar{a}$ annihilate this state since no more than two annihilation operators ψ_a can survive on this state. All the distinct SU$(2,2)$ lowest states are obtained by applying all powers of the supergenerators

$$\bar{\psi}^2|0>, \quad \phi^{[a_1a_2]}, \quad (scalar, 6) \tag{24}$$

$$(\psi\bar{a})\,\bar{\psi}^2|0>, \quad \varepsilon^{a_1a_2a_3a_4}\,\chi_{a_4}, \quad (left\,spinor, 4) \tag{25}$$

$$(\psi\bar{a})^2\,\bar{\psi}^2|0>, \quad F_{(\mu\nu)_+}\,\varepsilon^{a_1a_2a_3a_4}, \quad (self\,dual\,tensor, 1) \tag{26}$$

$$(\bar{b}\bar{\psi})\,\bar{\psi}^2|0>, \quad \xi^a, \quad (right\,spinor, 4^*) \tag{27}$$

$$(\bar{b}\bar{\psi})^2\,\bar{\psi}^2|0>, \quad F_{(\mu\nu)_-}, \quad (anti\;self\,dual\,tensor, 1) \tag{28}$$

These are all annihilated by (ab) since they are of the type (19). The irreducible SU$(2,2)$ modules (infinite towers of descendants) are obtained by applying all powers of $(\bar{b}\bar{a})$ on each of these lowest states. There are no other SU$(2,2)$ lowest states in the Hilbert space based on $\bar{\psi}^2|0>$ because states obtained by applying higher powers of supergenerators, such as $(\psi\bar{a})\,(\bar{b}\bar{\psi})\,\bar{\psi}^2|0>$ can be brought to the form $(\bar{b}\bar{a})\,\bar{\psi}^2|0>$ and therefore already are accounted in the towers including descendants based on $\bar{\psi}^2|0>$.

Thus, the lowest state $\bar{\psi}^2|0>$ fully characterizes the entire SU$(2,2|4)$ supermultiplet. In more detail, this state has the form $\bar{\psi}^{a_1}\bar{\psi}^{a_2}|0>$ and therefore is the 6 dimensional antisymmetric tensor of SU(4), and is a scalar under SU$(2,2)$, as indicated in Eq.(24). Therefore the SU$(2,2)$ tower based on this state represents the scalar fields $\phi^{[a_1a_2]}\,(x^\mu, u)$ on AdS space. The tower corresponds to the derivatives of the field that form a complete basis. Similarly, in terms of the Lorentz group embedded in SU$(2,2)$ the other lowest states give spin 1/2 and spin 1 fields as indicated in the equations above. This collection of states, which make an irreducible multiplet of SU$(2,2|4)$ may be viewed as the scalar supermultiplet of $N=4$ supersymmetry. This supermultiplet does not contain a spin 2 field since the maximum spin among the lowest states is 1. This set of fields correspond to the super Yang-Mills theory.

V. SU$(2,2|4)$ KK TOWERS IN SUPERGRAVITY

For two colors, $C=2$, the Lorentz and SU(4) content of the fields are as follows. The Lorentz part is decribed by $[j_1, j_2]$ that label the SU$(2)\times$SU$(2)\subset$SU$(2,2)$. By construction of the state, $[j_1, j_2]$ are obtained by using angular momentum addition rules that combine the 1/2 spins carried by the labels a,b. The SU(4) content is given by U(4) Young tableaux with 4 rows with (n_1, n_2, n_3, n_4) boxes respectively. The n_i are obtained by combining the Young tableaux of the initial state $\bar{\psi}^{2C}|0>$, which is $(C, C, 0, 0)$ with those of the supergenerators after the color traceless condition on the a, b part. From the Young tableaux one may extract SU(4) Dynkin labels consisting of three integers $\{\nu_1, \nu_2, \nu_3\}$ given by $\nu_1 = n_1 - n_2$, $\nu_2 = n_2 - n_3$ and $\nu_3 = n_3 - n_4$. In what follows we provide the $[j_1, j_2]_E$, and SU(4) Young and Dynkin labels, where AdS energy $E = (k + l) + 2C$ appears as a subscript. We will also give a field theory notation with upper and lower U(4) indices that correspond to traceless tensors, and Lorentz indices including vector and dotted and undotted spinor indices[1].

The $C = 2$ field content obtained with this approach coincides with the graviton supermultiplet in AdS$_5\times$S^5 supergravity written in terms of $d = 4$ Lorentz and spinor labels and with $SU(4)$ representations. The SU$(2, 2)$ lowest states obtained by applying the supergenerators on the SU$(2, 2|4)$ lowest state $\bar{\psi}^{2C}|0>$ as in Eq.(23), satisfy $k + l \leq 4$ and the

[1] The migration from the (j_1, j_2) labels for compact SU$(2)\times$SU(2) $\subset$SU$(2, 2)$ to the Lorentz labels for noncompact SO$(3, 1)$ $\subset$SU$(2, 2)$ corresponds to a change of basis within the same group SU$(2, 2)$ to describe the irreducible representation (the entire tower). In terms of oscillators this is obtained by a non-unitary Bogoliubov transformation [7].

conditions on spin in Eq.(**??**), for $C = 2$, as follows

$$k=0,\quad l=0:\quad [0,0]_4\,(2,2,0,0)\,\{0,2,0\}\,,\quad \phi^{[cd]}_{[ab]} \tag{29}$$

$$k=1,\quad l=0:\quad \left[\frac{1}{2},0\right]_5\,(2,2,1,0)\;\{0,1,1\}\,,\quad (\zeta_\alpha)^c_{[ab]} \tag{30}$$

$$k=0,\quad l=1:\quad \left[0,\frac{1}{2}\right]_5\,(2,1,0,0)\;\{1,1,0\}\,,\quad (\bar{\zeta}_{\dot{\alpha}})^{[cd]}_a \tag{31}$$

$$k=2,\quad l=0:\left\{\begin{array}{ll}[1,0]_6\,(2,2,1,1)\,\{0,1,0\}\,, & B^{[ab]}_{(\mu\nu)_+}\\ [0,0]_6\,(2,2,2,0)\,\{0,0,2\}\,, & \varphi_{(ab)}\end{array}\right. \tag{32}$$

$$k=0,\quad l=2:\left\{\begin{array}{ll}[0,1]_6\,(1,1,0,0)\,\{0,1,0\}\,, & B^{[ab]}_{(\mu\nu)_-}\\ [0,0]_6\,(2,0,0,0)\,\{2,0,0\}\,, & \bar{\varphi}^{(ab)}\end{array}\right. \tag{33}$$

$$k=1,\quad l=1:\quad \left[\frac{1}{2},\frac{1}{2}\right]_6\,(2,1,1,0)\,\{1,0,1\}\,,\quad (A_\mu)^a_b \tag{34}$$

$$k=3,\quad l=0:\quad \left[\frac{1}{2},0\right]_7\,(2,2,2,1)\;\{0,0,1\}\,,\quad (\xi_\alpha)_a \tag{35}$$

$$k=0,\quad l=3:\quad \left[0,\frac{1}{2}\right]_7\,(1,0,0,0)\;\{1,0,0\}\,,\quad (\bar{\xi}_{\dot{\alpha}})^a \tag{36}$$

$$k=1,\quad l=2:\quad \left[\frac{1}{2},1\right]_7\,(1,1,1,0)\;\{0,0,1\}\,,\quad (\Psi_{\mu\dot{\alpha}})_a \tag{37}$$

$$k=2,\quad l=1:\quad \left[1,\frac{1}{2}\right]_7\,(2,1,1,1)\;\{1,0,0\}\,,\quad (\bar{\Psi}_{\mu\alpha})^a \tag{38}$$

$$k=4,\quad l=0:\quad [0,0]_8\,(2,2,2,2)\,\{0,0,0\}\,,\quad \Sigma \tag{39}$$

$$k=0,\quad l=4:\quad [0,0]_8\,(0,0,0,0)\,\{0,0,0\}\,,\quad \bar{\Sigma} \tag{40}$$

$$k=2,\quad l=2:\quad [1,1]_8\,(1,1,1,1)\,\{0,0,0\}\,,\quad g_{\mu\nu} \tag{41}$$

$$k=3,\quad l=1:\quad \textit{excluded by color traceless rule} \tag{42}$$

$$k=1,\quad l=3:\quad \textit{excluded by color traceless rule} \tag{43}$$

Of course, this list of supergravity states is in agreement with previous results of Günaydin and Marcus in [4] that used oscillator methods with a different SU(4) classification of oscillators than ours. Although equivalent, in our approach the entire multiplet is more clearly identified by simply specifying only the $k = l = 0$ state $\bar{\psi}^4|0>$.

In the same manner the supermultiplets in higher Kaluza-Klein towers for all values of color $C = 2, 3, 4, \cdots$ are identified with only the lowest state $\bar{\psi}^{2C}|0>$, and the resulting supermultiplets that follow from them are in agreement with those found by Günaydin and Marcus, and those of Kim, Romans and Van Nieuwenhuizen [8] who used a completely different technique.

The significant point in our analysis is that all the states have been identified as color singlets. This fixes $C_2^{SU(C)} = 0$, while the value of Δ is read off immediately from the lowest state $\bar{\psi}^{2C}|0>$ as being $\Delta = N_\psi = 2C$. The value of Δ is fixed for the entire tower since the SU$(2,2|4)$ generators commute with Δ. This allows us to compute the SU$(2,2|4)$ Casimir eigenvalue for all the states, which is the new point in our investigation.

A. AdS-CFT

It must be noted that one can apply these ideas in the context of the AdS-CFT correspondence. The same set of supermultiplets discussed above can be built by starting from the scalar fields $\phi^{[ab]}$ in N=4 super Yang-Mills theory in four dimensions, as follows.

The scalars $\phi^{[ab]}$ are in the 2-index antisymmetric representation of SU(4) which forms the vector of SO(6), while being in the adjoint of Yang-Mills gauged SU(N). The spacetime and SU(4) quantum numbers of this CFT state should be compared to the $C = 1$ lowest state $\bar{\psi}^2|0>$ of the Yang-Mills supermultiplet given in Eq.(24).

Using the AdS-CFT correspondence, the AdS$_5\times$S^5 supermultiplets that we discussed above would emerge by constructing the lowest states in the super Yang-Mills context. These are obtained by taking the gauge invariant SU(N) trace over C such scalars Tr$\left(\phi^{[a_1b_1]}\dots\phi^{[a_Cb_C]}\right)$, and further demanding that their SU(4) quantum numbers be projected to the Young tableau $(C, C, 0, 0)$. A priori there are gauge singlet combinations of the scalars that do not correspond to the SU(4) tableau $(C, C, 0, 0)$. Our AdS-CFT correspondence identifies only this tableau. Then supersymmetry would correctly generate the remainder of the Kaluza-Klein tower as we discussed above in our oscillator formalism. In identifying the fields in a supermultiplet by applying super conformal symmetry, higher derivatives that result from applying the conformal supergenerators must be dropped (this is the analog of the color traceless condition in Eq.(23).

Note that the first power, $C = 1$, is not included among the AdS$_5\times$S^5 fields since the gauge invariant SU(N) trace is zero for $C = 1$, i.e. Tr$\left(\phi^{[a_1b_1]}\right) = 0$.

VI. THE ZERO

The quadratic Casimir eigenvalues for any color singlet states of SU$(2,2|4)$ are given in Eq.(12). Note that generally these are not zero. However, in our case, since the ground state satisfies $\Delta = N_\psi = 2C,$ we see that the quadratic Casimir vanishes for the ground state $\bar{\psi}^{2C}|0>$ as well as for the entire Kaluza-Klein tower for each $C.$ Therefore all the states of AdS$_5\times$S^5 supergravity, or those obtained through the AdS-CFT correspondence with the prescription given in the previous paragraph, have zero SU$(2,2|4)$ quadratic Casimir eigenvalues.

There is no explanation of this fact within supergravity or AdS-CFT. However this mysterious zero is easily explained in [3] as a consequence of the symmetry structures revealed in 2T-physics. It follows from a relationship between the fundamental 2T-physics gauge symmetry Sp(2) and local kappa symmetry, and the corresponding gauge invariance condition for physical states. The Sp(2) gauge symmetry is the universal mechanism for the projection of $d+2$ dimensional 2T-physics theory to a d dimensional 1T-physics theory. The zero SU$(2,2|4)$ Casimir follows directly from the gauge invariance of physical states under this Sp(2) symmetry, that is

$$X^2 = P^2 = X \cdot P = 0 \tag{44}$$

where X^M, P^M describe 12-dimensional phase space (see [2]), and $X^2, P^2, X \cdot P$ are the Sp(2) generators.

In fact, the vanishing of the quadratic Casimir is a reflection of a more general universal zero that applies to all the Casimir eigenvalues for all the Kakuza-Klein towers in AdS$_5\times$S^5 supergravity. As shown in [3], the generators of SU$(2,2|4)$ that describe these states can be written in the form of an 8×8 supertraceless matrix J constructed from the 12-dimensional super phase space (X^M, P^M, Θ) in a 2T formulation of a superparticle in 12-dimensions, with 32 fermions Θ's. This provides a dynamical particle representation of the J in Eq.(5) after modifying it for $M+N=R+S$ (this means supertraceless J for oscillators is defined by imposing Str$J = \Delta + (N-R)C = 0$). The particle representation of the supermatrix has the algebraic property [3]

$$(J)^2 = \frac{1}{4}\hbar^2 l\,(l+4)\,\mathbf{1} - 2\hbar\,(J)\,, \tag{45}$$

where the first term is proportional to the identity supermatrix $\mathbf{1},$ with $l = 0,1,2,\cdots$, where l labels the harmonics on S^5. The representation of SU$(2,2|4)$ changes as l changes, corresponding to the supergravity Kaluza-Klein towers. We have verified that the oscillator representations described in this paper, in the color singlet sector, satisfy precisely these constraint equations predicted by the 2T-physics 12-dimensional structure. The details will be further elaborated in a future publication.

Thus Eq.(45) is the fundamental algebraic structure that defines the supergravity Kaluza-Klein fields. It selects the correct subset of physical states in the vast Fock space of oscillators. We conjecture that the SU$(2,2|4)$ representations that satisfy these conditions for the generators J are unique in any formalism. These algebraic equations are understood naturally as the physical state constraint equations that follow from local supersymmetries for the 12D superparticle described in [3].

Higher powers of the supermatrix $(J)^n$ can now be computed by repeated applications of this formula. In particular the quadratic and all higher Casimir operators of SU$(2,2|4)$ must vanish in these realizations since the supertrace of $\mathbf{1}$ and the supertrace of J are zero

$$C_n\,(2,2|4) = Str\,(J^n) = 0. \tag{46}$$

These SU$(2,2|4)$ properties arose through constraints (associated with gauge symmetries) on a 12-dimensional super phase space, and hence the properties of these representations reflect the underlying 12-dimensional structure.

In this paper we have seen that the universal zero Casimirs prediction implied by 2T-physics [3] is a group theoretical fact in supergravity or the AdS-CFT correspondence, by using traditional group theoretical methods involving the oscillator representations.

It must be mentioned that the 2T-physics superparticle approach does not predict a similar universal zero for standard AdS$_4\times$S^7 or AdS$_7\times$S^4 supergravities, therefore this should not be expected. However, it does make certain predictions [3] for certain structures related to such spaces which again can in principle be verified.

The 2T-physics approach provides a much simpler view of the SU$(2,2|4)$ and Sp(2) covariant dynamics, and their inter-relationship. The properties of SU$(2,2|4)$ representations discussed in this paper followed from the quantum states of a superparticle, therefore they apply to supergravity, but not to higher superstring states. The extension of the 2T-physics superparticle formulation to superstrings would be very interesting, and would be expected to reveal profound structures.

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[1] I. Bars, C. Deliduman and D. Minic, "Supersymmetric Two-Time Physics", Phys. Rev. **D59** (1999) 125004, hep-th/9812161; "Lifting M-theory to Two-Time Physics", Phys. Lett. **B457** (1999) 275, hep-th/9904063;

I. Bars, "2T formulation of superconformal dynamics relating to twistors and supertwistors", Phys. Lett. **B483** (2000) 248, hep-th/0004090; "A Toy M-model", in preparation (partial results in hep-th/9904063 and hep-th/0008164).

[2] I. Bars, reviews in conference proceedings: "Two-Time Physics", hep-th/9809034; "Survey of Two-Time Physics ", hep-th/0008164; "2T-Physics 2001", hep-th/0106021.

[3] I. Bars, "Hidden 12-dimensional Structures in $AdS_5\times S^5$ and $M^4\times R^6$ Supergravities", hep-th/0208012.

[4] I. Bars and M. Günaydin, Comm. Math. Phys. **91** (1983) 31. M. Günaydin and N. Marcus, Class. and Quant. Grav. **2** (1985) L11; M. Günaydin, D. Minic and M. Zagermann, Nucl. Phys. **B534** (1998) 96; Nucl. Phys. **B544** (1999) 737. M. Günaydin and D. Minic, Nucl. Phys. **B523** (1998) 145.

[5] I. Bars and A.B. Balantekin, J. Math. Phys. **22** (1981) 1149; J. Math. Phys. **22** (1981)1810.

[6] M. Günaydın and C. Saçlıoğlu, Comm. Math. Phys. **87** (1982) 159.

[7] M. Günaydın, "AdS/CFT dualities and ... Wigner versus Dirac", hep-th/0005168.

[8] H.J. Kim, L.J. Romans and P. van Nieuwenhuizen, Nucl. Phys. **B242** (1984) 377.