

Noncommutative effective theory of vortices in a complex scalar field

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Abstract

We derive a noncommutative theory description for vortex configurations in a complex field in $2+1$ dimensions. We interpret the Magnus force in terms of the noncommutativity, and obtain some results for the quantum dynamics of the system of vortices in that context.

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Noncommutative field theories have recently been the subject of intense research, mostly because of their relevance to the study of some situations arising in string theory, where an antisymmetric tensor field coupled to the world sheet develops a non-zero vacuum expectation value [1, 2]. On the other hand, it has been realized that the noncommutative geometry setting may also be useful in the description of the quantum Hall effect [3, 4, 5, 6], where the verification of noncommutative geometry effects should be more accessible from the experimental point of view than in the previous example.

The emergence of noncommutativity is usually understood as a consequence of the presence of a strong magnetic field, a role which in the string theory context is played by the antisymmetric tensor field, and by the real magnetic field in the quantum Hall effect. It is our purpose in this letter to emphasize that a noncommutative theory may also be a good approximation to the description of vortices in a planar (i.e., two spatial dimensions) complex field. Here, a dual description shall be introduced, such that the ‘magnetic field’ will actually be due to the non-zero vacuum density of the field.

Our starting point is the action for a self-interacting nonrelativistic complex field ϕ :

$$S(\phi^*, \phi) = \int dt d^2x \left[\frac{i\hbar}{2} (\phi^* \partial_0 \phi - \partial_0 \phi^* \phi) - \frac{\hbar^2}{2m} \partial_j \phi^* \partial_j \phi - V(\phi^* \phi) \right] \quad (1)$$

where $V(\phi^* \phi) = \frac{\lambda}{2} (\phi^* \phi - \mu)^2$, with $\mu > 0$, is a potential which favors the emergence of a non zero expectation value for the charge density of the field: $\rho = \phi^* \phi$. To consider vortex configurations, we use the technique and notation that have been used for a similar system in [7, 8]. What follows is an adapted version of that technique to the case at hand.

We thus introduce a more convenient parametrization of the scalar field:

$$\phi(x) = \sqrt{\rho(x)} e^{i\theta(x)} \tilde{\phi}(x) \quad (2)$$

in terms of the density; a field θ which is the regular part of the phase of ϕ ; and $\tilde{\phi}$, which accounts for the singular part of the configuration. The singular part is, of course, constrained to have modulus equal to one.

Introducing the parametrization (2) into (1), we find an equivalent expression for the action, namely:

$$S = \int dt d^2x \left[-\hbar \rho \partial_0 \theta + i\hbar \rho \tilde{\phi}^* \partial_0 \tilde{\phi} - \frac{\hbar^2 \rho}{2m} (\partial_j \theta - i \tilde{\phi}^* \partial_j \tilde{\phi})^2 - \frac{\hbar^2}{8m\rho} \partial_j \rho \partial_j \rho - V(\rho) \right]. \quad (3)$$

To linearize the third term in (3), we introduce an auxiliary vector field \vec{J} , so that

$$S = \int dt d^2x \left[-\hbar \rho \partial_0 \theta + i \hbar \rho \tilde{\phi}^* \partial_0 \tilde{\phi} - \hbar J_k (\partial_k \theta - i \tilde{\phi}^* \partial_k \tilde{\phi}) \right. \\ \left. + \frac{m}{2\rho} J_k J_k - \frac{\hbar^2}{8m\rho} \partial_j \rho \partial_j \rho - V(\rho) \right]. \quad (4)$$

The regular phase θ becomes now a Lagrange multiplier field, which imposes the linear constraint $\partial_0 \rho + \partial_k J_k = 0$, i.e., a continuity equation mixing the density associated to the field ϕ with a ‘current’ defined by the auxiliary field \vec{J} . This constraint may be solved by introducing a field b_μ , such that $\rho = \epsilon_{jk} \partial_j b_k$ and $J_k = \epsilon_{kl} (\partial_l b_0 - \partial_0 b_l)$. This will not, of course, determine b_μ completely, since the equations that give ρ and J_k are invariant under the ‘gauge transformations’: $b_\mu \rightarrow b_\mu + \partial_\mu \alpha$. This freedom should, and indeed will, be taken into account for the derivation of the effective theory, by adopting a gauge fixing condition whenever required (i.e., when inverting operators depending on the quadratic form for b_μ).

After some elementary algebra, the action S may be written in the form:

$$S = \int dt d^2x \left[-\hbar b_0 \tilde{J}_0 - \hbar b_k \tilde{J}_k + \frac{m}{2\rho_b} \tilde{J}_b^2 - \frac{\hbar^2}{8m\rho_b} \partial_j \rho_b \partial_j \rho_b - V(\rho_b) \right], \quad (5)$$

where \tilde{J}_μ is a ‘topological’ vortex current, with components defined by

$$\tilde{J}_0 = \frac{1}{2\pi i} \epsilon_{kl} \partial_k (\tilde{\phi}^* \partial_l \tilde{\phi}) \quad \tilde{J}_k = \frac{1}{2\pi i} \epsilon_{kl} [\partial_l (\tilde{\phi}^* \partial_0 \tilde{\phi}) - \partial_0 (\tilde{\phi}^* \partial_l \tilde{\phi})] \quad (6)$$

and the notation ρ_b and \vec{J}_b has been used to emphasize the fact that those fields are determined by b_μ , since θ has been integrated out, and the corresponding constraint completely solved.

Of course, the topological current can only be non-vanishing when the ϕ -field configuration has singularities. In two spatial dimensions, they can only correspond to isolated points $\vec{x}^{(\alpha)}$, around which the phase winds up an integer number of times, q_α ¹. Thus, an N vortex configuration carries a vortex density and current which may be written as follows:

$$\tilde{J}_0(t, x) = \sum_{\alpha=1}^N q_\alpha \delta[\vec{x} - \vec{x}^{(\alpha)}(t)], \quad \tilde{J}_k(t, x) = \sum_{\alpha=1}^N q_\alpha \dot{x}_k^{(\alpha)} \delta[\vec{x} - \vec{x}^{(\alpha)}(t)], \quad (7)$$

where $\vec{x}^{(\alpha)}(t)$ denotes the actual trajectory of the vortex labeled by the index α . The quantum dynamics of the vortices corresponding to this current shall

¹The sign of q_α is defined as positive when the phase winding is counterclockwise.

be derived by considering the functional integral representation of the vacuum transition amplitude, and integrating out the auxiliary field b_μ . Namely, the action describing the effective dynamics of the vortices, S_{eff} , will be expressed as:

$$\exp\left\{\frac{i}{\hbar}S_{eff}[\tilde{J}_\mu]\right\} = \int [\mathcal{D}b_\mu] \exp\left\{\frac{i}{\hbar}S[b_\mu; \tilde{J}_\mu]\right\}, \quad (8)$$

where the symbol $[\mathcal{D}b_\mu]$ has been used to denote the b_μ field integration measure, including gauge fixing artifacts (i.e., Faddeev-Popov factors). They will be neglected in the following discussion, since the theory is Abelian and hence they factor out for the calculations we are interested in. S_{eff} cannot be evaluated exactly, because of the presence of non-quadratic terms in the action. We may, however, obtain an approximation to S_{eff} by integrating out the quadratic fluctuations around an homogeneous vacuum configuration for b_μ . Taking into account the special form we have assumed for the potential V , we see that the homogeneous vacuum corresponds to a constant density $\rho_b \equiv \rho_0 = \mu$ which may be solved for b_j , by adopting the symmetric gauge: $b_j = -\frac{\rho_0}{2}\epsilon_{jk}x_k$.

Assuming first the vacuum to be isotropic, the current \vec{J}_b must of course vanish. Moreover, as \vec{b} is time-independent, the spatial components of the current are: $J_k = \epsilon_{kl}\partial_k b_0$, and any constant b_0 value becomes compatible with this vacuum configuration. This constant will then have to be integrated out. Denoting by $S_{eff}^{(0)}$ the result of evaluating the action on this vacuum configuration, we see that

$$S_{eff}^{(0)} = \int dt \frac{h\rho_0}{2}\epsilon_{kl}x_l\tilde{J}_k, \quad (9)$$

plus the constraint:

$$Q \equiv \int d^2x J_0 = 0 \quad (10)$$

which follows from the integration of the constant value b_0 ².

We shall now include fluctuations on top of this configuration. To that end, we consider small variations $\delta\rho$ and $\delta\vec{J}$, which may be written in terms of δb_μ by adopting a gauge. We then expand the action S up to the quadratic order in δb_μ , and integrate out δb_μ . In this Gaussian approximation, and ignoring \tilde{J} -independent terms, we see that the effect of these fluctuations

²We are assuming the gauge invariance to hold even at spatial infinity, i.e., the gauge group is not reduced to the identity at infinity. This is the reason why the net charge has to be zero. This, on the other hand, guarantees the finiteness of the static energy.

amounts to adding to $S_{eff}^{(0)}$ a contribution $S_{eff}^{(1)}$, given explicitly by

$$S_{eff}^{(1)} = \frac{h^2 \rho_0}{2m} \int dt d^2 x d^2 x' \tilde{J}_0(t, x) \Delta^{-1}(x, x') \tilde{J}_0(t, x') \\ + \frac{h^2}{2} \int dt dt' d^2 x d^2 x' \tilde{J}_k(t, x) K_{kl}(t - t', x - x') \tilde{J}_l(t', x'), \quad (11)$$

where

$$K_{kl}(t, x) = \delta_{kl} \int \frac{d\omega d^2 p}{(2\pi)^3} \frac{e^{-i\omega t + i\vec{p} \cdot \vec{x}}}{(\frac{h^2 p^2}{4m\rho_0} + \lambda)p^2 - \frac{m}{\rho_0}\omega^2} \quad (12)$$

is an operator containing nonlocalities in time³. To obtain a local theory, we assume that λ is sufficiently large, so that only the first term in (11) (which is λ -independent) remains. This yields a (time local) expression for $S_{eff}^{(1)}$:

$$S_{eff}^{(1)} = \frac{h^2 \rho_0}{2m} \int dt d^2 x d^2 x' \tilde{J}_0(t, x) \Delta^{-1}(x, x') \tilde{J}_0(t, x') \quad (13)$$

where the first correction to this expression is of order $\sim \lambda^{-1}$

$$\mathcal{O}(\lambda^{-1}) = -\frac{h^2}{2\lambda} \int dt d^2 x d^2 x' \tilde{J}_k(t, x) \Delta^{-1}(x - x') \tilde{J}_k(t, x'). \quad (14)$$

This large- λ approximation may be justified if the condition $\frac{mv^2}{\rho_0 \lambda} \ll 1$ (where v denotes the typical velocity of the vortices) holds.

Putting together the first two leading terms contributing to the effective action, we get a tractable approximation to the effective action:

$$S_{eff} = \int dt \left[\frac{h\rho_0}{2} \epsilon_{kl} x_l \tilde{J}_k + \frac{h^2 \rho_0}{2m} \int d^2 x d^2 x' \tilde{J}_0(t, x) \Delta^{-1}(x, x') \tilde{J}_0(t, x') \right], \quad (15)$$

which should be supplemented by the constraint that imposes the vanishing of the total vortex charge. Notice that even for finite λ the second term in (11) will not modify the phase space structure of the theory described by (15). Indeed, its expansion is non-local in time, and there are no quadratic term in the time derivatives.

We will use this expression as a starting point for the noncommutative description of the vortex dynamics in this model. Let us first consider a ‘first quantized’ language: introducing in (15) expression (7) for the vortex

³We are omitting the (implicit) $i\epsilon$ to avoid the pole.

current, we see that the effective action for N vortices becomes

$$S_{eff} = \int dt \left[\frac{h\rho_0}{2} \sum_{\alpha=1}^N q_\alpha \epsilon_{jk} \dot{x}_j^{(\alpha)} x_k^{(\alpha)} + \frac{h^2 \rho_0}{4\pi m} \sum_{\alpha, \beta=1}^N q_\alpha q_\beta \ln \left| \frac{x^{(\alpha)} - x^{(\beta)}}{\xi} \right| \right] \quad (16)$$

where we have written explicitly the inverse of Δ in the coordinate representation, and the global neutrality condition is taken into account by assuming that $\sum_{\alpha=1}^N q_\alpha = 0$. The parameter ξ can be regarded as a minimum length, related to the size of the vortex core, as in reference [11]. This action has a very particular structure since the kinetic term is linear in time derivatives. This means that the piece without time derivatives is (minus) the Hamiltonian; namely, the effective *first order* Lagrangian is:

$$\mathcal{L}_{eff} = \frac{h\rho_0}{2} \sum_{\alpha=1}^N q_\alpha \epsilon_{kl} \dot{x}_k^{(\alpha)} x_l^{(\alpha)} - \mathcal{H} \quad (17)$$

with

$$\mathcal{H} = -\frac{h^2 \rho_0}{4\pi m} \sum_{\alpha, \beta=1}^N q_\alpha q_\beta \ln \left| \frac{x^{(\alpha)} - x^{(\beta)}}{\xi} \right|, \quad (18)$$

which coincides with the one of ref [9, 11], where it was obtained using a hydrodynamic approach for a system of slightly deformed rectilinear vortices in an incompressible fluid.

The special form of the term containing the time derivatives is responsible for the noncommutativity, since the canonical commutators following from (17) are:

$$[x_k^{(\alpha)}, x_l^{(\beta)}] = \frac{i}{2\pi\rho_0 q^{(\alpha)}} \delta_{\alpha\beta} \epsilon_{kl} \equiv \frac{i}{q^\alpha} \theta \delta_{\alpha\beta} \epsilon_{kl} \quad (19)$$

(no sum over α). Contrary to what happens when it is due to a real magnetic field, here the noncommutativity parameter $\theta = (2\pi\rho_0)^{-1}$ is purely classical, i.e., there is no \hbar factor⁴. This is also consistent with dimensional analysis, since q is dimensionless and ρ_0^{-1} has the dimensions of an area. By contrast, in the magnetic field induced noncommutativity, the analogous relation yields a θ proportional to l^2 , where l denotes the cyclotron length, which is proportional to $\hbar^{\frac{1}{2}}$.

The Hamiltonian \mathcal{H} is the one of a two-dimensional neutral Coulomb gas, with the neutrality condition guaranteeing the finiteness of the energy.

⁴Of course, the commutator between x_k and its canonical conjugate is proportional to \hbar . The presence of an \hbar factor in the Lagrangian, however, cancels out the \hbar factor in the commutator between coordinates.

It should be noted that we are here considering a sector corresponding to a (fixed) number (N) of vortices. Had we wanted to compare sectors with different numbers of singularities, the constant terms we have neglected would have been relevant (i.e., to determine the chemical potential of the system). The partition function for the classical system of vortices at finite temperature is then proportional to the one of a (globally neutral) two dimensional Coulomb gas. It is noteworthy that this partition function is not exactly identical to the one of a standard, classical $2 + 1$ dimensional Coulomb gas composed of dynamical charges, since that theory would have a phase space twice as large, because the canonical momenta would be independent from the coordinates, at least assuming normal quadratic kinetic energy terms. The Gaussian integral over the canonical momenta would then, as usual, yield a decoupled factor which modifies the total entropy, as well as the zero of the free energy.

It is, at this level, already possible to make contact with some noncommutative geometry results. In particular, the interpretation of noncommutative field theory as describing elementary dipoles [10] is found by looking at the simplest case: $N = 2$, i.e., a vortex-antivortex pair

$$\mathcal{L}_{eff} = \frac{\hbar \rho_0 q}{2} \epsilon_{kl} [\dot{x}_k^{(1)} x_l^{(1)} - \dot{x}_k^{(2)} x_l^{(2)}] - \frac{\hbar^2 \rho_0 q^2}{2\pi m} \ln \left| \frac{x^{(1)} - x^{(2)}}{\xi} \right|, \quad (20)$$

where $x^{(1)}$ and $x^{(2)}$ denote the vortex ($q_1 = q$) and antivortex ($q_2 = -q$) coordinates, respectively. Introducing the relative (r) and center of mass (R) coordinates, we see that

$$\mathcal{L}_{eff} = \hbar \rho_0 q \epsilon_{kl} \dot{r}_k R_l - \frac{\hbar^2 \rho_0 q^2}{2\pi m} \ln \left| \frac{r}{\xi} \right|, \quad (21)$$

where $r_k = x_k^{(1)} - x_k^{(2)}$ and $R_k = \frac{x_k^{(1)} + x_k^{(2)}}{2}$. This form of the Lagrangian makes it explicit the fact that the center of mass and relative coordinates of the ‘dipole’ built from the vortex-antivortex pair become conjugate canonical variables. We also note that the interaction potential between vortex and antivortex is confining, so it makes sense to attempt a description in terms of the dipole as a single entity, with the noncommutativity taking into account part of the vortex ‘internal structure’.

For more tractable forms of the interaction potential, like a quadratic one, the system is equivalent to a single particle with the coordinates of the center of mass R_k and a mass determined by the parameter of the harmonic interaction potential [10]. In this interpretation, the uncertainty relations

$$\Delta r_1 \Delta R_2 \geq \theta \quad \Delta r_2 \Delta R_1 \geq \theta, \quad (22)$$

relate the ‘size’ of the dipoles to the uncertainty of the center of mass coordinates. The noncommutative description naturally arises when considering the quantum version of the theory defined by the effective action (15).

When a (globally neutral) configuration of the system is such that it can be approximately described as a set of condensed vortex-antivortex pairs, the dipole interpretation may also be introduced. Let us assume that N is even, $N = 2M$, with M vortices of charge $q = +1$ and M with charge $q = -1$. Introducing coordinates $x^{(\alpha)}$ for the vortices, and $y^{(\alpha)}$ for the antivortices, we first arrange the different terms in the Lagrangian in a convenient way:

$$\begin{aligned} \mathcal{L}_{eff} = & \frac{h\rho_0 q}{2} \sum_{\alpha=1}^M \epsilon_{kl} [\dot{x}_k^{(\alpha)} x_l^{(\alpha)} - \dot{y}_k^{(\alpha)} y_l^{(\alpha)}] + \frac{h^2 \rho_0 q^2}{4\pi m} \sum_{\alpha, \beta=1}^M \ln \left| \frac{x^{(\alpha)} - x^{(\beta)}}{\xi} \right| \\ & + \frac{h^2 \rho_0 q^2}{4\pi m} \sum_{\alpha, \beta=1}^M \ln \left| \frac{y^{(\alpha)} - y^{(\beta)}}{\xi} \right| \\ & - \frac{h^2 \rho_0 q^2}{2\pi m} \sum_{\alpha, \beta=1}^M \ln \left| \frac{x^{(\alpha)} - y^{(\beta)}}{\xi} \right|. \end{aligned} \quad (23)$$

Assuming that the situation is such that the system has condensed into vortex-antivortex pairs with $x^{(\alpha)}$ paired to $y^{(\alpha)}$, $\forall \alpha$, it is convenient to introduce the new set of coordinates: $\vec{r}^{(\alpha)} = \vec{x}^{(\alpha)} - \vec{y}^{(\alpha)}$ and $\vec{R}^{(\alpha)} = \frac{\vec{x}^{(\alpha)} + \vec{y}^{(\alpha)}}{2}$. If the dipoles have condensed, the relative distance $|r^{(\alpha)}|$ should be negligible in comparison with the distance between the center of masses of the dipoles. Using a two-dimensional multipole expansion for the interaction potentials, we have of course

$$\mathcal{L}_{eff} \sim h\rho_0 q \sum_{\alpha=1}^M \epsilon_{kl} \dot{r}_k^{(\alpha)} R_l^{(\alpha)} - \frac{h^2 \rho_0 q^2}{2\pi m} \sum_{\alpha=1}^M \ln \left| \frac{r^{(\alpha)}}{\xi} \right|, \quad (24)$$

as the leading contribution. The next to leading term introduces a dipole-dipole interaction potential $V_{\alpha, \beta}$, such that for each pair α, β of dipoles it is given by:

$$V_{\alpha\beta} = \frac{h^2 \rho_0 q^2}{2\pi m} r_j^{(\alpha)} \Omega_{jk}^{(\alpha\beta)} r_k^{(\beta)} \quad (25)$$

where

$$\Omega_{jk}^{(\alpha\beta)} = \frac{2R_j^{(\alpha\beta)} R_k^{(\alpha\beta)} - (R^{(\alpha\beta)})^2 \delta_{jk}}{(R^{(\alpha\beta)})^4}, \quad R^{(\alpha\beta)} \equiv R^{(\alpha)} - R^{(\beta)}. \quad (26)$$

We are interested in the quantum theory corresponding to this system. The convenience of having a ‘field’ description for the many particle case,

should be evident. Indeed, it would be desirable to be able to define the quantum dynamics in terms of a field corresponding to the density J_0 , a function of the coordinates. However, since the coordinates x_1 and x_2 of a vortex are conjugate variables, we see that the density has to be understood as a field defined on a noncommutative space. The situation is entirely analogous to the deformation quantization procedure [12] where one describes the dynamics in terms of the Wigner function, which defines a density on phase space. In the case at hand, the density will also be a function of the phase space coordinates x_1 and x_2 , and the Hamiltonian H_{eff} may be expressed in terms of the Moyal product \star of the corresponding Weyl symbol:

$$A(x_1, x_2) \star B(x_1, x_2) = A(x_1, x_2) e^{i\frac{\theta}{2}(\overleftarrow{\partial}_{x_1} \overrightarrow{\partial}_{x_2} - \overleftarrow{\partial}_{x_2} \overrightarrow{\partial}_{x_1})} B(x_1, x_2) . \quad (27)$$

Using the notation ρ for the Weyl symbol of the density, we see that the Hamiltonian H , as obtained from (15), should be

$$H_{eff} = \frac{h^2 \rho_0}{2m} \int d^2x d^2x' \rho(t, x) \star \Delta^{-1}(x, x') \star \rho(t, x') , \quad (28)$$

where, of course, one of the stars may be deleted. Equation (28) may be understood as the mean energy of a configuration defined by the density ρ ; static solutions that are extrema of (28) should be possible collective states of the system. The equation for those extrema is of course an eigensystem:

$$\frac{1}{4\pi} \int d^2y \ln\left(\frac{|x-y|}{\xi}\right) \star \rho(y) = E \rho(x) \quad (29)$$

or, replacing the \star by the equivalent shift in the corresponding coordinate:

$$\frac{1}{4\pi} \int d^2y \ln(\xi^{-1} \sqrt{(x_1 - i\frac{\theta}{2}D_{y_2})^2 + (x_2 + i\frac{\theta}{2}D_{y_1})^2}) \rho(y) = E \rho(x) \quad (30)$$

where $D_{y_1} = \partial_{y_1} + \frac{2i}{\theta}y_2$ and $D_{y_2} = \partial_{y_2} - \frac{2i}{\theta}y_1$ have the form of covariant derivatives for a constant uniform ‘magnetic field’ $B = \frac{4}{\theta}$. The presence of this covariant derivative is indeed a manifestation of the Magnus force at the purely quantum level, since when considering simplified versions of the interaction potential (like quadratic ones), it clearly induces a dynamical behavior similar to the one of particles in a magnetic field. Of course, the Magnus force is evident at the semiclassical level just from the particular form of the kinetic term in (15) [8].

Let us, for the sake of completeness, also present the result for the case of the non-isotropic vacuum with a non-vanishing constant value J_k^0 for the

auxiliary field J_k . Keeping all the terms which survive for this configuration, the quadratic approximation yields now a correction $S_{eff}^{(1)}$, which is explicitly given by

$$S_{eff}^{(1)} = S_{eff}^{(1)}|_{J=0} + \frac{h^2}{2} \int dt d^2x dt' d^2x' (\tilde{J}_k(t, x) - \frac{J_k^0}{\rho_0} \tilde{J}_0(t, x))$$

$$M_{kl}^{-1}(t, x; t', x') (\tilde{J}_l(t, x) - \frac{J_l^0}{\rho_0} \tilde{J}_0(t, x)) \quad (31)$$

where M_{kl} is non-local in time and depends on the (constant) value of J_0 . In momentum space, \tilde{M}^{-1} , the Fourier transform of M^{-1} , may be written as

$$(\tilde{M}^{-1})_{kl}(p) = - \left\{ \left[\frac{m}{\rho_0} \omega^2 - \left(\frac{\hbar^2 p^2}{4m\rho_0} + \lambda \right) p^2 + \frac{2m\omega}{\rho_0^2} \vec{J}^0 \cdot \vec{p} \right]^{-1} e_{kl} \right.$$

$$\left. + \left[\frac{m}{\rho_0} \omega^2 + \left(\frac{m}{\rho_0^3} (J^0)^2 - \frac{\hbar^2 p^2}{4m\rho_0} - \lambda \right) p^2 + \frac{2m\omega}{\rho_0^2} \vec{J}^0 \cdot \vec{p} \right]^{-1} f_{kl} \right\} \quad (32)$$

where $e_{kl} \equiv e_k e_l$, e_k : unit vector in the direction of \vec{J}^0 , and $f_{kl} \equiv \delta_{kl} - e_{kl}$.

In the same limit were we found an effective theory local in time (15) for the $\vec{J}^0 = 0$ case, we find here for the correction S_1 :

$$S_{eff}^{(1)} \sim \frac{h^2 \rho_0}{2m} \int dt d^2x d^2x' \tilde{J}_0(t, x) \Delta^{-1}(x, x') \tilde{J}_0(t, x')$$

$$- \frac{h^2}{2\lambda} \int dt d^2x d^2x' (\tilde{J}_k - \frac{J_k^0}{\rho_0} \tilde{J}_0)(t, x) \Delta^{-1}(x - x') (\tilde{J}_k - \frac{J_k^0}{\rho_0} \tilde{J}_0)(t, x') . \quad (33)$$

This means that the correction to the previous case is still of order λ^{-1} , and amount to modifying the vortex current by the subtraction of a constant current $\frac{J_k^0}{\rho_0} \tilde{J}_0$.

It is interesting to compare the previous results with the ones that follow by starting from a relativistic model, since there are similarities but also very important differences between the two cases. To that end, we consider now the derivation of the effective theory for vortices in the relativistic scalar field case. The action is now assumed to be

$$S = \int d^3x [\partial_\mu \phi^* \partial_\mu \phi - V(\phi^* \phi)] \quad (34)$$

where V is assumed to have the same structure as in the non-relativistic case, equation (1). Of course, now the parameters in that potential have a different physical meaning, but we keep the same notation for the sake of simplicity.

Introducing in (34) the same decomposition we have used in the non-relativistic case, we see that the action S becomes:

$$S = \int d^3x \left[\rho(\partial\theta + i\tilde{\phi}^*\partial\tilde{\phi})^2 + \frac{1}{4\rho}(\partial\rho)^2 - V(\rho) \right] \quad (35)$$

Again, an auxiliary field is employed to linearize the action. In this case, we need a vector field ξ so that:

$$S = \int d^3x \left[\xi_\mu(\partial^\mu\theta + i\tilde{\phi}^*\partial^\mu\tilde{\phi}) - \frac{1}{4\rho}\xi_\mu\xi^\mu + \frac{1}{4\rho}(\partial\rho)^2 - V(\rho) \right]. \quad (36)$$

Next, the pure gradient $\partial_\mu\theta$ can be replaced by a vector field $\theta_\mu \equiv \partial_\mu\theta$ which is, of course, constrained to verify the zero curl constraint: $\epsilon^{\mu\nu\lambda}\partial_\nu\theta_\lambda = 0$. This constraint may be enforced by adding to the action a new term S_δ :

$$S_\delta = \int d^3x A_\mu \epsilon^{\mu\nu\lambda} \partial_\nu \theta_\lambda, \quad (37)$$

where A_μ is a Lagrange multiplier field.

This action is assumed to be used for the functional quantization of the system, so we may, as usual, perform field redefinitions since they amount to changes in the (functional) integration variables. Using in particular the shift

$$\theta_\mu \rightarrow \theta_\mu - i\tilde{\phi}^*\partial_\mu\tilde{\phi} \quad (38)$$

we find that

$$S = \int d^3x \left[\xi_\mu\theta^\mu - \frac{1}{4\rho}\xi^2 + \frac{1}{4\rho}(\partial\rho)^2 - V(\rho) + A_\mu \epsilon^{\mu\nu\lambda} \partial_\nu \theta_\lambda + 2\pi A^\mu \tilde{J}_\mu \right] \quad (39)$$

where \tilde{J}_μ denotes the topological current

$$\tilde{J}_\mu = \frac{1}{2\pi i} \epsilon_{\mu\nu\lambda} \partial^\nu (\tilde{\phi}^* \partial^\lambda \tilde{\phi}). \quad (40)$$

Finally, we integrate out the vector field θ_μ , obtaining,

$$S = \int d^3x \left[-\frac{1}{4}\left(\frac{1}{2\rho}\right) F_{\mu\nu}(A) F^{\mu\nu}(A) + 2\pi A^\mu \tilde{J}_\mu + \frac{1}{4\rho}(\partial\rho)^2 - V(\rho) \right] \quad (41)$$

where $F_{\mu\nu}(A) = \partial_\mu A_\nu - \partial_\nu A_\mu$. This expression is useful in order to make a comparison with the already considered non-relativistic case. First we note that, depending on the form chosen for the potential, one still has the

possibility of having a vacuum where ρ (now a relativistic scalar) has a non-zero uniform value. This, however, does not necessarily mean that there will be a corresponding uniform vacuum magnetic field that guarantees the desired form for the kinetic term of the action, as in the non-relativistic case. In other words, to get a non-zero vacuum magnetic field for A_μ , one should assume that there is spontaneous breaking of Poincare invariance. If that is the case, then one can assume a constant $B = \epsilon_{jk}\partial_j A_k$, a constant $\rho = \rho_0$, and look for the minima of a ‘potential’ U , resulting from adding to $V(\rho_0)$ the contribution coming from the F^2 term:

$$U(B, \rho_0) = V(\rho_0) - \frac{1}{4\rho_0} B^2. \quad (42)$$

The solution for the minima of this potential will provide for an expression for the uniform magnetic field B_0 in terms of the background density ρ_0 . Thus, the action evaluated in this configuration will look exactly like (9), but with $h\rho_0$ replaced by $2\pi B_0$.

We conclude that a noncommutative geometry description may very well be appropriate to the description of vortices in a planar system defined in terms of a complex scalar field. The dual description allows one to precisely identify all the parameters of the noncommutative theory, and moreover the dipole interpretation for the natural excitations in a noncommutative theory finds here a natural and concrete realization. Also, the introduction of corrections to the effective description is under control, and even the existence of external currents may be taken into account with minor changes in the model.

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