Discrete symmetries of functional determinants

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Abstract

We study discrete (duality) symmetries of functional determinants. An exact transformation of the effective action under the inversion of background fields $\beta(x) \to \beta^{-1}(x)$ is found. We show that in many cases this inversion does not change functional determinants. Explicitly studied models include a matrix theory in two dimensions, the dilaton-Maxwell theory in four dimensions on manifolds without a boundary, and a two-dimensional dilaton theory on manifolds with boundaries. Our results provide an exact relation between strong and weak coupling regimes with possible applications to string theory, black hole physics and dimensionally reduced models.

1   Introduction

There is an enormous amount of literature devoted to calculations of the one-loop effective action on various backgrounds. It is well known that the divergent part of the effective action is local and can be calculated exactly, at least in principle. The finite part of the effective action is typically non-local and usually cannot be obtained in a closed form. The famous exceptions when the effective action is known exactly are the Polyakov action [1] and the WZNW action [2,3]. There is also an example of a spinorial action [4] for which the finite part of the effective action can be calculated exactly.

In some cases one can obtain powerful relations between functional determinants and spectral functions of relevant operators even though the functional determinants (effective actions) themselves are not known exactly. These cases are governed by the Index Theorem (see [5,6] for a detailed review). The idea behind the Index Theorem is quite simple. Let the two operators

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Δ_1 and Δ_2 be represented in a factorized form: $\Delta_1 = Q_1 Q_2$, $\Delta_2 = Q_2 Q_1$. In such a case, the eigenvalues of Δ_1 and Δ_2 coincide up to zero modes. Consequently, differences of all spectral functions of Δ_1 and Δ_2 (e.g. the zeta function or the heat kernel) come from the zero mode contributions and typically can be expressed in terms of topological invariants.

In this paper we study relations between the functional determinants of

$$\Delta_+ = -(\beta^{-1}\partial_\mu \beta)(\beta \partial_\mu \beta^{-1}) \quad \text{and} \quad \Delta_- = -(\beta \partial_\mu \beta^{-1})(\beta^{-1}\partial_\mu \beta)\,, \tag{1}$$

where $\beta(x)$ is a matrix valued function or a positive scalar field, $\beta = e^{-\phi}$. Below we demonstrate that in certain cases $\log \det \Delta_+ - \log \det \Delta_-$ can be calculated exactly. In most cases this difference is just zero. The two operators are related by the transformation $\beta \to \beta^{-1}$. Hence, the relation between the determinants that has been advertised above is a kind of duality relation between effective actions in strong and weak coupling limits. The key difference between the problem we study here and that of the standard Index Theorem is the summation over the index μ in operators (1). Because of this summation the eigenvalues of the operators Δ_+ and Δ_- generically are different. Nevertheless, we find a relation between the functional determinants of these operators even though this property cannot be extended to other spectral functions (such as higher heat kernel coefficients).

At first sight the operators (1) look a little bit artificial, but in fact they are quite general and appear in a number of physical applications. Consider, e.g., a (spherical) reduction of a field theory from higher dimensions (see [7] for a recent review). Let ψ be a minimal scalar field living on a D-dimensional manifold that has the structure of a semi-direct product:

$$(ds)^2 = g_{\mu\nu}(x)dx^\mu dx^\nu + e^{-\frac{4}{D-2}\phi(x)}(d\Omega)^2\,. \tag{2}$$

Here $g_{\mu\nu}(x)$ is the two-dimensional metric and $(d\Omega)^2$ is the line element on a $(D-2)$-dimensional manifold. Both the dilaton $\phi(x)$ and the metric $g_{\mu\nu}(x)$ are assumed to depend only on the first two coordinates x^μ. Then the action for the scalar field reduces to the 2-dimensional action,

$$\int d^D x \sqrt{g^D} g_{ik}^D (\partial^i \psi)(\partial^k \psi) \; \to \; \int d^2 x \sqrt{g} g^{\mu\nu} e^{-2\phi}(\partial_\mu \psi)(\partial_\nu \psi)\,. \tag{3}$$

To quantize the theory one should also fix an inner product of the quantum fields. The reduced inner product may also depend on the dilaton field:

$$< \psi, \psi' > = \int d^2 x \sqrt{g} e^{-2\varphi(\phi)} \psi(x)\psi'(x)\,. \tag{4}$$

If we start with the natural choice of the measure $\sqrt{g^D}$ for the inner product in D dimensions, then the preferable choice for the scalar function $\varphi(\phi)$ in $2D$ is, of course, $\varphi(\phi) = \phi$. For simplicity we have omitted in Eqs.(3-4) an unimportant constant factor of the volume of the $(D-2)$-dimensional manifold. By performing the functional integration over ψ with the measure defined by (4) we obtain the effective action

$$W[\varphi - \rho, \phi] = \frac{1}{2} \log \det(-e^{\varphi-\rho}\partial_\mu e^{-2\phi}\partial_\mu e^{\varphi-\rho})\,, \tag{5}$$

where the conformal gauge for the two-dimensional metric $g_{\mu\nu} = e^{2\rho}\delta_{\mu\nu}$ has been used. The indices in (5) are contracted with the Kroneker symbol. The dependence of the effective action

(5) on $\varphi - \rho$ is governed by the scale anomaly which can be easily calculated for arbitrary $\varphi - \rho$. The corresponding effective action can be obtained by integration of the anomaly [8, 9]. This means, in particular, that

$$W[\varphi - \rho, \phi] = W[\phi, \phi] + W^{\text{scale}}. \tag{6}$$

where W^{scale} can be easily calculated by the methods of [8, 10–12]. An explicit expression will be given below (see eq. (70)). Thus, the problem of the calculation of the effective action (5) is reduced to the evaluation of the determinant of Δ_+ with $\beta = e^{-\phi}$.

One of the most interesting applications of the reduced theories is the evaluation of the s-wave Hawking radiation [9] (see [7] for more references). Typically, $\phi \to -\infty$ corresponds to the asymptotically flat region and $\phi \to +\infty$ to the region near the black hole singularity. Thus the transformation $\phi \to -\phi$ indeed maps the strong coupling region to the weak coupling one. Of course, reduction and quantization do not commute. The difference is the so called dimensional-reduction anomaly [13–15] (see also [16]). The reduction of other models (as, e.g., that of the Maxwell theory in D-dimensions to the dilaton-Maxwell theory in 4 dimensions) can be considered along the same lines.

In fact, the operator Δ_+ with $\beta = e^{-\phi}$ is a representation of the scalar Laplacian with a potential $V_+ = -(\partial^2 \phi) + (\partial \phi)^2$. Indeed, we have traded one scalar function V for another scalar function ϕ. In a recent paper [17] it has been noted that 3rd order terms in ϕ are surprisingly absent in the expansion of the effective action in powers of ϕ. In the present paper we explain this result and extend it to all odd orders of ϕ.

Another important model where the operators (1) appear is the bosonic string. Consider a model with the action

$$S = \int d^2 x \, G_{ab}(x) \partial_\mu \xi^a \partial^\mu \xi^b. \tag{7}$$

The natural inner product for the field ξ^a reads:

$$< \xi_1, \xi_2 > = \int d^2 x \, G_{ab} \xi_1^a \xi_2^b. \tag{8}$$

Performing the path integral over ξ with the measure defined by (8) we arrive at $\det \Delta_+$ with $(\beta^2)_{ab} = G_{ab}$. Note that if one expands the ordinary bosonic string action in small fluctuations of the string coordinates one more term proportional to the target space Riemann tensor appears in (7). Therefore, (7) describes the one-loop partition function for the bosonic string only in the case of a flat background metric G_{ab} which is perhaps not of particular interest. Our analysis is rather motivated by general ideas of string duality (see [18, 19] for introductory reviews). The target space duality transformation changes the size of the target manifold and is used to relate strong and weak coupling limits (usually with respect to small and large compactification radii). We go further by considering arbitrary matrix transformations with arbitrary dependence on the world-sheet coordinates, and still find a duality symmetry in the quantum determinants! Although we cannot give immediately an exact relation to known formulations of the T-duality, we note that the one-component case is just the dilaton-shift problem in the T-dual models [20]. Also, our transformations interchange Dirichlet and Neumann boundary conditions (see sec. 3), just as it happens in the open string duality [21–26].

Throughout this paper we use zeta function regularization [27, 28]. In the next section we demonstrate that functional determinants for the operators (1) coincide on a two-dimensional

manifold without boundary. Our method is similar to that used in [20] for the one-component case. We also clarify some subtle mathematical points, such as non-standard logarithmic terms in the heat kernel expansion and non-locality of some of the heat kernel asymptotics. In Sec. 3 we extend our formalism to manifolds with boundaries. For simplicity, the one-component case is considered. This time the difference of the effective actions is non-zero. It is given by a surface integral plus a non-local contribution of the zero modes. In Sec. 4 we go to four dimensions and demonstrate that the the effective action in the dilaton-Maxwell theory is independent of the sign in front of the background dilaton field. In that section we extensively use the technique of the multidimensional Darboux transform [29].

2 Determinants of matrix operators in 2D

Consider the first order differential operators

$$
\begin{aligned}
A_\mu &= \partial_\mu + \Phi_\mu = \beta^{-1}\partial_\mu\beta \,, \\
B_\mu &= -A_\mu^\dagger = \partial_\mu - \Phi_\mu^\dagger = \beta\partial_\mu\beta^{-1} \,,
\end{aligned}
\tag{9}
$$

where β is an hermitian matrix-valued field, $\beta^\dagger = \beta$. Φ_μ is expressed in terms of β :

$$
\Phi_\mu = \beta^{-1}(\partial_\mu\beta) \,.
\tag{10}
$$

From the two first-order operators (9) we construct two "Laplacians"

$$
\Delta_+ = -A_\mu B_\mu \,, \qquad \Delta_- = -B_\mu A_\mu \,.
\tag{11}
$$

Note that Δ_+ is mapped to Δ_- under $\beta \to \beta^{-1}$.

Our aim is to compare functional determinants (one-loop effective actions) for the operators Δ_+ and Δ_-. For a general elliptic operator the regularized effective action reads

$$
W^{\mathrm{reg}} = \frac{1}{2}\log\det(D)_{\mathrm{reg}} = -\frac{\mu^{2s}}{2}\int_0^\infty dt\, t^{s-1}\mathrm{Tr}(\exp(-tD)) \,,
\tag{12}
$$

where we have introduced the (zeta-function) regularization parameter s, which should be set to zero after calculations, and a massive parameter μ needed to make the action dimensionless. To proceed further we need a definition of the ζ-function of the operator D:

$$
\zeta^D(s) = \mathrm{Tr}(D^{-s}) = \frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\mathrm{Tr}(\exp(-tD)) \,.
\tag{13}
$$

The equations (12) and (13) are valid for general elliptic (pseudo) differential operators of positive order. From now on we use the fact that D is an operator of Laplace type, i.e. D is an elliptic second-order differential operator with a scalar leading symbol. For such operators the ζ function is regular at $s = 0$ (see e.g. [5]) and, therefore, the regularized effective action is

$$
W^{\mathrm{reg}} = -\frac{\mu^{2s}}{2}\Gamma(s)\zeta^D(s) = -\frac{1}{2s}\zeta^D(0) - \frac{1}{2}\zeta^D(0)' - \log(\mu)\zeta^D(0) + O(s) \,,
\tag{14}
$$

where prime denotes differentiation with respect to s. We have used $\Gamma(s) = 1/s - \gamma_E + \ldots$ and have absorbed the Euler constant γ_E in a redefinition of μ. The pole part of (14) should be canceled by a counter-term. The renormalized effective action reads

$$W^{\mathrm{ren}} = -\frac{1}{2}\zeta^D(0)' - \log(\mu)\zeta^D(0)\,. \tag{15}$$

The $\log(\mu)$ term describes the renormalization ambiguity. By means of the Mellin transformation one can demonstrate that $\zeta^D(0) = a_1(1, D)$ where a_1 is defined through the heat kernel asymptotics

$$K(f, t, D) = \mathrm{Tr}(f\exp(-tD)) \cong \sum_{n=0}^{\infty} t^{n-1}a_1(f, D) \tag{16}$$

as $t \to +0$ valid for arbitrary smooth matrix-valued function f. The coefficient a_1 is locally computable for operators of Laplace type. $\zeta^D(0)'$ is non-local and in general no closed expression for this quantity is available.

Any operator of Laplace type can be represented in the form

$$D = -(\hat{g}^{\mu\nu}\nabla_\mu\nabla_\nu + \hat{E}) \tag{17}$$

with a suitable auxiliary metric $\hat{g}$, covariant derivative $\nabla = \partial + \omega$ and potential $\hat{E}$. For the operator Δ_+ (11) we obtain

$$\hat{g}^{\mu\nu} = \delta^{\mu\nu}, \qquad \omega_\mu = \frac{1}{2}\left(\Phi_\mu - \Phi_\mu^\dagger\right),$$
$$\hat{E} = -\frac{1}{2}\partial_\mu\left(\Phi_\mu + \Phi_\mu^\dagger\right) - \frac{1}{4}\left(\Phi_\mu + \Phi_\mu^\dagger\right)^2 - \frac{1}{2}[\Phi_\mu, \Phi_\mu^\dagger]\,. \tag{18}$$

The coefficient a_1 reads:

$$a_1(f, D) = \frac{1}{24\pi}\mathrm{tr}\int d^2x\sqrt{-\hat{g}}f(\hat{R} + 6\hat{E})\,. \tag{19}$$

$\hat{R}$ is the scalar curvature of $\hat{g}_{\mu\nu}$ and tr denotes ordinary trace over all matrix indices.

Consider the variation of the zeta function ζ^+ of the operator Δ_+:

$$\delta\zeta^+(s) = 2s\mathrm{Tr}\left((\delta\phi)A_\mu B^\mu\Delta_+^{-s-1} - (\delta\phi)B_\mu\Delta_+^{-s-1}A_\mu\right)\,. \tag{20}$$

where we have defined

$$\delta\phi = -\frac{1}{2}((\delta\beta)\beta^{-1} + \beta^{-1}(\delta\beta))\,. \tag{21}$$

In the first term in (20) the operators re-combine in a power of Δ_+, giving the generalized ζ-function:

$$2s\mathrm{Tr}\left((\delta\phi)A_\mu B^\mu\Delta_+^{-s-1}\right) = -2s\zeta^+(\delta\phi, s). \tag{22}$$

The contribution of this term to the variation of the effective action (15) can therefore be treated exactly. Indeed, since $\zeta^+(f, s)$ is regular at $s = 0$, we obtain

$$-\frac{1}{2}\partial_s\left[2s\mathrm{Tr}\left((\delta\phi)A_\mu B^\mu\Delta_+^{-s-1}\right)\right]_{s=0} = \zeta^+(\delta\phi, 0) = a_1(\delta\phi, \Delta_+)\,. \tag{23}$$

This reflects a general property of functional determinants. If the variation of an elliptic operator D has the form $\delta D = (\delta\sigma_1)D + D(\delta\sigma_2)$ for some local operators $\delta\sigma_{1,2}$, then the variation of the zeta function reads: $\delta\zeta^D(s) = -s\mathrm{Tr}((\delta\sigma_1 + \delta\sigma_2)D^{-s})$. Consequently, the variation of ζ' can be expressed through the heat kernel $\delta\zeta^D(0) = -a_1((\delta\sigma_1 + \delta\sigma_2), D)$.

The second term in (20) requires more care because the operators under the trace cannot be recombined in a power of Δ_+:

$$
\begin{aligned}
\mathrm{Tr}\left((\delta\phi)B_\mu\Delta_+^{-s-1}A_\mu\right) &= \frac{1}{\Gamma(s+1)}\int_0^\infty dt\, t^{-s}\mathrm{Tr}\left((\delta\phi)B_\mu\exp(-t\Delta_+)A_\mu\right) \\
&= -\frac{1}{\Gamma(s+1)}\int_0^\infty dt\, t^{-s}\mathrm{Tr}\left((\delta\phi)F\exp(-tF)\right) \\
&= -\mathrm{Tr}\left((\delta\phi)F^{-s}\right)\,.
\end{aligned}
\tag{24}
$$

Here F is a matrix operator having both space-time vector indices and "internal" indices of the matrix field $\beta(x)$:

$$
F_{\mu\nu} = -B_\mu A_\nu\,.
\tag{25}
$$

In eq. (24) the trace is taken over all matrix indices.

It is clear from the construction that the operator F should be understood as acting on the space $\mathcal{V}^L$ of B-longitudinal vector fields which can be represented as $v_\mu = B_\mu\psi$, where ψ is a scalar function. Such fields satisfy the condition

$$
B_\mu\epsilon^{\mu\nu}v_\nu = 0\,.
\tag{26}
$$

The spectrum of the operator F on $\mathcal{V}^L$ can be constructed easily. Let ψ^λ be a normalized eigenfunction of the operator Δ_+ with a non-zero eigenvalue λ. Then the functions $v_\mu^\lambda = B_\mu(\Delta_+)^{-1/2}\psi^\lambda$ are normalized eigenfunctions of the operator F. Accordingly, we write the off-diagonal heat kernel

$$
\begin{aligned}
K(F, x, y; t)_{\mu\nu} &= \sum_\lambda v_\mu^\lambda(x)v_\nu^\lambda(y)e^{-t\lambda} \\
&= \int d^2x'\int d^2y'\, B_\mu^x(\Delta_+)_{xx'}^{-1/2}B_\mu^y(\Delta_+)_{yy'}^{-1/2}K(\Delta_+, x', y'; t)\,.
\end{aligned}
\tag{27}
$$

For the traced heat kernel this relation simplifies to

$$
\mathrm{Tr}(\exp(-tF)) = \mathrm{Tr}(\exp(-t\Delta_+))\,.
\tag{28}
$$

The operator F contains a non-local projector on the space $\mathcal{V}^L$. Therefore, F is a *pseudo* differential operator rather than a differential one. According to the general theory [5], the generalized zeta function for such operator can have a simple pole at $s = 0$:

$$
\zeta^F(f, s) = \frac{1}{s}\zeta_{\mathrm{pole}}^F(f) + \zeta_0^F(f) + O(s)\,.
\tag{29}
$$

By substituting (29) into (20) one obtains

$$
\delta\zeta^+(0)' = -2\zeta^+(\delta\phi, 0) + 2\zeta_0^F(\delta\phi)\,.
\tag{30}
$$

Next we relate the coefficient $\zeta_0^F(f)$ to the heat kernel asymptotics. We invert the obvious relation

$$\zeta^F(f, s) = \text{Tr}(fF^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \text{Tr}(f \exp(-tF)) \tag{31}$$

to show that

$$K(f, t, F) = \text{Tr}(f \exp(-tF)) = \frac{1}{2\pi i} \oint ds \, \Gamma(s) \zeta^F(f, s) t^{-s}, \tag{32}$$

where the integration contour encircles all poles of the integrand. The contribution of the pole at $s = 0$ reads

$$\zeta_0^F(f) - (\gamma_E + \log t) \zeta_{\text{pole}}^F(f). \tag{33}$$

The Euler constant γ_E can be absorbed again in re-definition of the normalization scale μ which is not shown explicitly in (31) and (32). Variation (30) is again defined by the t^0 term in the heat kernel asymptotics.

We see that the heat kernel $K(f, t, F)$ has a quite unusual log-term in the small t asymptotics. Even though this term does not contribute to the variation of the effective action, it is instructive to calculate its value. Translating equation (27) from kernels to the operator notation we obtain:

$$
\begin{aligned}
K(f, t, F) &= \text{Tr}\left[f B_\mu (\Delta_+)^{-1/2} \exp(-t\Delta_+) \left(B_\mu (\Delta_+)^{-1/2} \right)^\dagger \right] \\
&= -\text{Tr}\left[A_\mu f B_\mu \Delta_+^{-1} \exp(-t\Delta_+) \right] \\
&= \text{Tr}\left[f \exp(-t\Delta_+) \right] - \text{Tr}\left[f_{,\mu} B_\mu \Delta_+^{-1} \exp(-t\Delta_+) \right],
\end{aligned} \tag{34}
$$

where

$$f_{,\mu} = \partial_\mu f + [\Phi_\mu, f]. \tag{35}$$

The first term on the last line in (34) obviously does not contain $\log t$ terms. The second term can be represented as an integral over the proper time:

$$-\text{Tr}\left[f_{,\mu} B_\mu \Delta_+^{-1} \exp(-t\Delta_+) \right] = -\int_t^\infty ds \, \text{Tr}\left[f_{,\mu} B_\mu \exp(-s\Delta_+) \right], \tag{36}$$

To evaluate (36) we use the method of [30]. Consider the operator $\Delta_+^\epsilon = \Delta_+ - \epsilon f_{,\mu} B_\mu$. Then

$$\text{Tr}\left[f_{,\mu} B_\mu \exp(-s\Delta_+) \right] = \frac{1}{s} \partial_\epsilon \text{Tr}\left[\exp(-s\Delta_+^\epsilon) \right] \Big|_{\epsilon=0}, \tag{37}$$

The $1/s$ term in the small s asymptotics of (37) is immediately calculated with the help of (19):

$$-\frac{1}{s} \frac{1}{8\pi} \int d^2x \, \text{tr}\left(f_{,\mu} (\Phi_\mu + \Phi_\mu^\dagger) \right), \tag{38}$$

This term generates the $\log t$ term in the small-t asymptotics of (34) and (36). We conclude that

$$\zeta_{\text{pole}}^F(f) = \frac{1}{8\pi} \int d^2x \, \text{tr}\left(f_{,\mu} (\Phi_\mu + \Phi_\mu^\dagger) \right), \tag{39}$$

Note that this term is local. This supports the quite general observation that the leading singularity ($\log t$ in the present case) is usually local , while the subleading terms (t^0) are non-local.

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In order to be able to apply standard expressions for the heat kernel coefficients it is convenient to extend the operator F to the space $\mathcal{V}$ of *all* vector fields (which also carry the internal indices). The space $\mathcal{V}$ splits into a direct sum

$$\mathcal{V} = \mathcal{V}^L \oplus \mathcal{V}^T \,, \tag{40}$$

where the space $\mathcal{V}^T$ consists of the vector fields $u^\mu = \epsilon^{\mu\nu} A_\nu \psi$. It is easy to check that the decomposition (40) is orthogonal, i.e. $\int d^2 x\, u^\mu v_\mu = 0$ for $v \in \mathcal{V}^L$ and $u \in \mathcal{V}^T$. It is clear that F is zero on $\mathcal{V}^T$. Define the operator

$$\tilde{F}_{\mu\nu} = -\epsilon_{\mu\mu'} \epsilon_{\nu\nu'} A^{\mu'} B^{\nu'} \tag{41}$$

that is zero on $\mathcal{V}^L$.

It can be demonstrated that the zero modes of the map $\psi \to v_\mu \in \mathcal{V}^L$ (resp. $\psi \to u_\mu \in \mathcal{V}^T$) coincide with the zero modes of Δ_+ (resp. Δ_-) and that the operators F and $\tilde{F}$ have no more zero modes on $\mathcal{V}^L$ and $\mathcal{V}^T$ respectively. Suppose that the underlying manifold is $\mathbf{R}^2$ and $\beta(x)$ goes to a non-degenerate constant matrix as $|x| \to \infty$. In such a case the zero modes $\psi_\pm^0(x) = \beta(x)^{\pm 1} c$ (c is a constant vector in the "internal" space) of the operators $\Delta_\pm$ are non-normalizable and, therefore, can be neglected. Zero-modes of the decomposition (40) (analogs of harmonic one-forms) are also absent.

The operator $F + \tilde{F}$ is an elliptic operator of Laplace type,

$$\tilde{F} + F = \delta_{\mu\nu} \Delta_+ - \partial_\nu \Phi_\mu^\dagger - \partial_\mu \Phi_\nu - [\Phi_\nu, \Phi_\mu^\dagger] \,. \tag{42}$$

The coefficient a_1 for this operator can be found with the help of (19):

$$a_1(f, \tilde{F} + F) = -\frac{1}{8\pi} \int d^2 x \, \mathrm{tr}(f(\Phi_\mu + \Phi_\mu^\dagger)^2) \,. \tag{43}$$

For the heat kernel of the operator $\tilde{F}$ there is a representation similar to (27),

$$
\begin{aligned}
K(\tilde{F}, x, y; t)_{\mu\nu} &= \int d^2 x' \int d^2 y' \epsilon_{\mu\mu'} A_{\mu'}^x (\Delta_-)_{xx'}^{-1/2} \\
&\quad \times \epsilon_{\nu\nu'} A_{\mu'}^y (\Delta_-)_{yy'}^{-1/2} K(\Delta_-, x', y'; t) \,.
\end{aligned}
\tag{44}
$$

From equation (44) follows an important observation: the trace over vector indices $K(\tilde{F}, x, y; t)_{\mu\mu}$ is obtained from $K(F, x, y; t)_{\mu\mu}$ if we exchange β and β^{-1}. By repeating the calculations above for the operator Δ_- we obtain

$$\delta \zeta^-(0)' = 2\zeta^-(\delta\phi, 0) - 2\zeta_0^{\tilde{F}}(\delta\phi) \,. \tag{45}$$

Now we subtract (45) from (30) to demonstrate that

$$\delta\left(\zeta^+(0)' - \zeta^-(0)'\right) = -2\zeta^+(\delta\phi, 0) - 2\zeta^-(\delta\phi, 0) + 2\zeta^{\tilde{F}+F}(\delta\phi, 0) \,. \tag{46}$$

All zeta functions on the right-hand side of (46) correspond to elliptic operators. We can replace them by the heat kernel coefficients:

$$\delta\left(\zeta^+(0)' - \zeta^-(0)'\right) = 2\left(-a_1(\delta\phi, \Delta_+) - a_1(\delta\phi, \Delta_-) + a_1(\delta\phi, \tilde{F} + F)\right) = 0 \,. \tag{47}$$

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This equation demonstrates that the difference $\zeta^+(0)' - \zeta^-(0)'$ does not depend on β. Since this difference is zero for $\beta = \beta^{-1} = \mathbf{1}$,

$$\zeta^+(0)' = \zeta^-(0)', \tag{48}$$

or

$$W^{\mathrm{ren}}[\beta] = W^{\mathrm{ren}}[\beta^{-1}]. \tag{49}$$

In a particular (one-component) case this relation has been obtained in [20] where the method of [31] was used. The equation (48) can be also formulated as a Generalized Index Theorem:

$$\log \det(-A_\mu B_\mu) = \log \det(-B_\mu A_\mu). \tag{50}$$

Normally, the Index Theorem does not contain summations under the determinant but contains contributions of the zero modes which are absent in the topologically trivial situation considered here.

There is an interesting special case for the one-component matrix $\beta = e^{-\phi}$ where ϕ is an external scalar field (dilaton). Let the dilaton depend on one coordinate only, say x^1. In this case

$$\Delta_\pm = -\partial_2^2 + H_\pm, \qquad H_\pm = Q_\pm Q_\mp, \tag{51}$$

where $Q_\pm = \pm\partial_1 + (\partial_1\phi)$ is (up to a multiplier) the supercharge used in supersymmetric quantum mechanics (for an introduction see [32]). Obviously, the operators $Q_\pm$ are intertwining the Hamiltonians $H_\pm$, leading to pairing of their non-zero eigenvalues. This pairing is the key ingredient of the Witten index construction [33] which governs dynamical supersymmetry breaking. Since in this particular case the operators $H_\pm$ commute with ∂_2^2, the eigenvalues of $\Delta_\pm$ coincide up to zero modes with those of $H_\pm$. Consequently, if we neglect the zero mode contributions, as we have agreed to do in this chapter, all traced spectral functions of Δ_+ and Δ_- should coincide, including $\zeta^\pm(s)$ and $K(1, t, \Delta_\pm)$, for *arbitrary* value of their arguments (s and t respectively). Here comes the main difference from our result. If the dilaton ϕ depends on both coordinates, the operators Δ_- and Δ_+ have, in general, different eigenvalues. This can be confirmed for instance by calculation of the next heat kernel coefficient $a_2(1, \Delta_\pm)$ which differs for Δ_+ and Δ_-.

3 Manifolds with boundaries

Let us extend the results of the previous section to manifolds with boundaries. For simplicity we consider here the one-component case only, $\beta = e^{-\phi}$. Accordingly,

$$\Delta_+ = -A^\mu B_\mu, \qquad A_\mu = D_\mu - \phi_{,\mu} = e^\phi D_\mu e^{-\phi},$$
$$B_\mu = -A_\mu^\dagger = D_\mu + \phi_{,\mu} = e^{-\phi} D_\mu e^\phi. \tag{52}$$

Since we will use a curved coordinate system near the boundary we have restored upper and lower indices and introduced the Riemannian covariant derivative D_μ.

As we will see in the next section, the structure of the elliptic complex plays an important role in our construction. Thus, we could use one of the admissible sets of boundary conditions for that complex [5]. However, we prefer a more direct (though also more lengthy) way to derive

the boundary operators. Let the operator Δ_+ act on the fields satisfying Dirichlet boundary conditions

$$\psi|_{\partial M} = 0. \tag{53}$$

Obviously, $\Delta_+\psi|_{\partial M} = -A^\mu B_\mu \psi|_{\partial M} = 0$. Therefore, the vector $v_\mu = B_\mu \psi$ (see (26)) should satisfy

$$A^\mu v_\mu|_{\partial M} = 0. \tag{54}$$

To proceed further we need some notations. Let N^μ be the inward-pointing unit normal at the boundary. Let τ be a coordinate on the boundary. We can extend this coordinate system to a neighborhood of the boundary keeping orthogonality $(N^\tau = 0)$. The extrinsic curvature k is defined as $k = -N^\mu \partial_\mu g_{\tau\tau} = \Gamma^N_{\tau\tau}$. We can simplify the analysis by choosing a coordinate system in which $g_{\tau\tau} = const.$ along the boundary (this relation cannot be extended inside the manifold).

In the coordinate frame defined above

$$A^\mu v_\mu|_{\partial M} = e^\phi \left[(\partial_N - k)(e^{-\phi}v_N) + g^{\tau\tau}\partial_\tau(e^{-\phi}v_\tau) \right]|_{\partial M}. \tag{55}$$

A local extension of the condition (54) on the space of all vector fields $\mathcal{V}$ (40) can be obtained by requiring that v_N and v_τ satisfy these boundary conditions independently:

$$(\partial_N - k)(e^{-\phi}v_N)|_{\partial M} = 0, \qquad v_\tau|_{\partial M} = 0. \tag{56}$$

In particular, the vectors

$$u_\mu = \sqrt{g}\epsilon_{\mu\nu}A^\nu \chi \in \mathcal{V}^T \tag{57}$$

should satisfy (56). This leads to Neumann boundary conditions for the scalar field χ:

$$\partial_N e^{-\phi} \chi|_{\partial M} = 0. \tag{58}$$

In our procedure the field χ forms a functional space for the operator Δ_-. Therefore, the conditions (58) become the boundary conditions for Δ_-. Since (58) depends on the dilaton, the variation of $\det \Delta_-$ contains two contributions, one coming from variation of the operator, the other from the variation of the functional space on which this operator acts. At this time we do not know how to evaluate the second contribution. To avoid difficulties with the ϕ-dependence of (58) we impose a Neumann boundary condition on the dilaton:

$$\partial_N \phi|_{\partial M} = 0. \tag{59}$$

Due to (59) the conditions (53), (56) and (58) are equivalent to the following ϕ-independent set:

$$\begin{aligned} &\psi|_{\partial M} = 0, \\ &(\nabla_N - k)v_N|_{\partial M} = 0, \qquad v_\tau|_{\partial M} = 0, \\ &\nabla_N \chi|_{\partial M} = 0, \end{aligned} \tag{60}$$

where one can recognize the relative boundary conditions for the de Rham complex [5].

Let us discuss now the zero mode structure of the theory. To be more precise let us restrict ourselves to compact manifolds M with the topology of a two-dimensional disk. The only zero

mode of Δ_+ is eliminated by the Dirichlet boundary condition. There is a zero mode $\psi_-^0 = e^\phi$ of Δ_-. There are no zero modes in the vector sector. This last statement can be derived from [5] or from the simpler analysis of [34].

The boundary conditions (60) are mixed. This means that one of the components of the vector satisfies a Dirichlet and the other one a Neumann boundary condition. The heat kernel expansion for this type of the boundary conditions has been studied in [34–37]. Let us write the boundary conditions for a multi-component field v in the form:

$$(\Pi_D v + (\nabla_N + \mathcal{S})\Pi_N v)\,|_{\partial M} = 0\,, \tag{61}$$

where $\Pi_{D,N}$ are two complementary projectors and $\mathcal{S}$ is some function on the boundary which can be matrix-valued. Clearly, the mixed boundary conditions (61) contain Dirichlet and Neumann conditions as particular cases. The boundary term

$$a_1(f,D) = \frac{1}{24\pi}\mathrm{tr}\int dy\sqrt{h}\,(f(2k + 12\mathcal{S}\Pi_N) + 3f_{,N}(\Pi_N - \Pi_D)) \tag{62}$$

should be added to the volume part (19). Here y is a coordinate on the boundary and h is the determinant of the induced metric.

Obviously, the contribution of the zero mode $\psi_-^0 = e^\phi$ of the operator Δ_- to the heat kernel reads

$$K_0(\Delta_-, x, y; t) = \frac{\psi_-^0(x)\psi_-^0(y)}{\int d^2 z\sqrt{g}(\psi_-^0(z))^2}\,. \tag{63}$$

There is no t-dependence on the right hand side of (63). Therefore we can easily derive the zero mode contribution to $a_1(f, \Delta_-)$:

$$a_1^{(0)}(f, \Delta_-) = \frac{\int d^2 x\sqrt{g}fe^{2\phi}}{\int d^2 z\sqrt{g}e^{2\phi}}\,. \tag{64}$$

By acting exactly as in the previous section we obtain

$$
\begin{aligned}
\frac{1}{2}\left(\delta\zeta^+(0)' - \delta\zeta^-(0)'\right) &= -a_1^+(\delta\phi) - (a_1^-(\delta\phi) - a_1^{(0)}(f, \Delta_-)) + a_1^{F+\tilde{F}}(\delta\phi) \\
&= -\frac{1}{2\pi}\int_{\partial M} dy\sqrt{h}k\delta\phi + \frac{\int d^2 x\sqrt{g}(\delta\phi)e^{2\phi}}{\int d^2 z\sqrt{g}e^{2\phi}}\,.
\end{aligned} \tag{65}
$$

This equation can be integrated to give (with the help of (15))

$$W^{\mathrm{ren}}(\Delta_+) - W^{\mathrm{ren}}(\Delta_-) = \frac{1}{2\pi}\int_{\partial M} dy\sqrt{h}k\phi - \frac{1}{2}\log\left(\int d^2 z\sqrt{g}e^{2\phi}\right) + W^{[0]}\,. \tag{66}$$

Equation (66) represents the main result of this section. Here $W^{[0]}$ describes the difference between the two effective actions for $\phi = const$. Since the boundary conditions for Δ_+ and Δ_- are different there is no reason to believe that this difference is zero. For the standard Euclidean disk $W^{[0]}$ has been calculated in [38, 39]. The non-local term in (66) is typical for the effective action on a compact manifold [40].

As a consistency check let us show that if $\phi = \phi_c = const$, the right hand side of (66) does not depend on ϕ_c. The first term reads $\chi\phi_c$ where χ is the Euler characteristic. The ϕ_c

dependent part of the second term is $-\phi_c$. Since we have assumed that M has the topology of the 2-disk ($\chi = 1$) the two contributions cancel against each other.

To make our discussion self-contained we conclude this section with a calculation of the change in the effective action under a conformal transformation of the external metric. Since any metric in two dimensions is conformally trivial this will provide an extension of our previous results to arbitrary curved manifolds. We use the methods of Refs. [8, 10–12]. Consider the operator $\Delta_+^{[\hat{\rho}]} = e^{-2\hat{\rho}}\Delta_+$, which is unitarily equivalent to the operator (5) $\hat{\rho} = \rho - \varphi + \phi$. It is clear that under variation of $\hat{\rho}$ the effective action changes as

$$\delta W^{\mathrm{ren}}(\Delta_+^{[\hat{\rho}]}) = -a_1(\delta\hat{\rho}, \Delta_+^{[\hat{\rho}]})\,. \tag{67}$$

The coefficient a_1 on the right hand side of (67) is given by equations (19), (62) where all quantities should be calculated with the effective metric $\hat{g}_{\mu\nu} = e^{2\hat{\rho}}g_{\mu\nu}$:

$$\sqrt{\hat{g}}\hat{E} = \sqrt{g}(\phi_{,\mu\mu} - \phi_{,\mu}\phi_{,\mu})\,, \qquad \sqrt{\hat{g}}\hat{R} = -2\sqrt{g}\partial^2\hat{\rho}\,,$$
$$\hat{k}\sqrt{\hat{h}} = \sqrt{h}(k - \partial_N\hat{\rho})\,. \tag{68}$$

All indices in (68) are contracted with $g_{\mu\nu}$. For Neumann boundary conditions we choose $\mathcal{S} = 0$. We obtain

$$
\begin{aligned}
\delta W^{\mathrm{ren}}(\Delta_+^{[\hat{\rho}]}) &= -\frac{1}{24\pi}\left[\int_M d^2x\sqrt{g}\delta\hat{\rho}(-2\partial^2\hat{\rho} + 6(\partial^2\phi) - 6(\partial\phi)^2)\right. \\
&\quad\left. + \int_{\partial M} dy\sqrt{h}(2\delta\hat{\rho}(k - \partial_N\hat{\rho}) \mp 3\partial_N(\delta\hat{\rho}))\right] \\
&\quad + a_1^{(0)}(\delta\hat{\rho}, \Delta_+^{[\hat{\rho}]})\,.
\end{aligned}
\tag{69}
$$

The $\mp$ sign in the surface integral corresponds to Dirichlet ($-$) or Neumann ($+$) boundary conditions. The last term in (69) describes the contribution of the zero modes. It is present for Neumann boundary conditions only and is given by (64) with $\phi \to -\phi$ and $\sqrt{g} \to e^{2\hat{\rho}}\sqrt{g}$. From this equation we easily obtain

$$
\begin{aligned}
W^{\mathrm{ren}}(\Delta_+^{[\hat{\rho}]}) - W^{\mathrm{ren}}(\Delta_+) &= -\frac{1}{24\pi}\left\{\int_M d^2x\sqrt{g}[(\partial\hat{\rho})^2 + 6\hat{\rho}(\partial^2\phi - (\partial\phi)^2)]\right. \\
&\quad\left. + \int_{\partial M} dy\sqrt{h}(2\hat{\rho}\hat{k} \mp 3\partial_N\hat{\rho})\right\} + \frac{1}{4}(1 \mp 1)\log\frac{\int_M d^2x e^{2(\hat{\rho}-\phi)}}{\int_M d^2x e^{-2\phi}}\,.
\end{aligned}
\tag{70}
$$

Here again the upper sign corresponds to Dirichlet and the lower to Neumann boundary conditions. The action W^{scale} in (6) is given by the right hand side of the same equation (70) with $\hat{\rho} = \rho - \varphi + \phi$. Now it is clear which modifications should be made in (66) for arbitrary curved 2D manifolds (for a non-zero conformal factor $\hat{\rho}$):

$$
\begin{aligned}
W^{\mathrm{ren}}(\Delta_+^{[\hat{\rho}]}) - W^{\mathrm{ren}}(\Delta_-^{[\hat{\rho}]}) &= \frac{1}{4\pi}\int_M d^2x\sqrt{\hat{g}}\hat{R}\phi + \frac{1}{2\pi}\int_{\partial M} dy\sqrt{\hat{h}}\hat{k}\phi \\
&\quad - \frac{1}{2}\log\left(\int d^2z\sqrt{\hat{g}}e^{2\phi}\right) + W^{[0]}\,.
\end{aligned}
\tag{71}
$$

It is interesting to note that in the volume and surface integrals the constant mode of the dilaton again couples to the Euler characteristic.

4 Dilaton Maxwell theory in 4D

Consider the dilaton-Maxwell system in flat four dimensional Euclidean space,

$$S = \frac{1}{4} \int d^4x \, e^{-2\phi} \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} , \tag{72}$$

where $\mathcal{F}$ is an abelian field strength, $\mathcal{F}_{\mu\nu} = \partial_\mu a_\nu - \partial_\mu a_\nu$. Such an interaction appears in extended supergravities [41], string theory, and after a reduction from higher dimensions. Mechanisms for the generation of the action (72) and its classical properties were considered e.g. in [42–47] (see also [48–50] for some quantum calculations).

If the action (72) is obtained by a reduction from higher dimensions, the inner products on the spaces of vector and scalar (ghost) fields should also undergo the reduction and should thus contain the dilaton field

$$
\begin{aligned}
< a, a' > &= \int d^4x \, e^{-2\phi} a_\mu a_\mu{}', \\
< \sigma, \sigma' > &= \int d^4x \, e^{-2\phi} \sigma \sigma' .
\end{aligned}
\tag{73}
$$

It is convenient to choose the gauge condition as

$$(\partial_\mu - \phi_{,\mu}) e^{-\phi} a_\mu = 0 . \tag{74}$$

In this gauge the action for a_μ and the Faddeev-Popov ghosts σ is

$$
\begin{aligned}
S = \frac{1}{2} \int d^4x \, \big[a_\nu e^{-\phi} \left[-(\partial_\mu - \phi_{,\mu})(\partial_\mu + \phi_{,\mu})\delta_{\nu\rho} + 2\phi_{,\nu\rho} \right] e^{-\phi} a_\rho \\
- \sigma e^{-\phi}(\partial_\mu - \phi_{,\mu})(\partial_\mu + \phi_{,\mu}) e^{-\phi} \sigma \big] .
\end{aligned}
\tag{75}
$$

Performing the functional integration over a_μ and σ with the measure (73) we arrive at the effective action

$$W = \frac{1}{2} \log \frac{\det(\Delta_+ \delta_{\mu\nu} + 2\phi_{,\mu\nu})}{\det(\Delta_+)} , \tag{76}$$

where $\Delta_+ = -(\partial_\mu - \phi_{,\mu})(\partial_\mu + \phi_{,\mu})$ and the vector determinant in (76) is restricted to the fields $e^{-\phi} a_\mu$ satisfying the gauge condition (74).

The aim of this section is to study transformations of the effective action (76) under change of the sign in front of the dilaton field. We adapt to this problem the technique of the multi-dimensional Darboux transform [29].

Let us introduce the multi-index Kroneker symbol,

$$\delta^{\nu_1...\nu_n}_{\mu_1...\mu_n} = \frac{1}{(4-n)!} \epsilon_{\mu_1...\mu_n \rho_{n+1}...\rho_4} \epsilon^{\nu_1...\nu_n \rho_{n+1}...\rho_4} , \tag{77}$$

and the "supercharges"

$$Q^{(n)-}_{[\mu][\nu]} = \delta^{\nu_1...\nu_{n-1}\rho}_{\mu_1...\mu_n} B_\rho = \left(Q^{(n)+}_{[\nu][\mu]} \right)^\dagger . \tag{78}$$

The operator $Q^{(n)-}$ maps $(n-1)$-forms to n-forms and $Q^{(n)+}$ maps n forms to $(n-1)$-forms. With the help of these operators we define the matrix "Hamiltonians"

$$H^{(1)}_{\mu|\nu} = -B_\mu A_\nu \,, \qquad \tilde{H}^{(1)}_{\mu|\nu} = \frac{1}{2} Q^{(2)+}_{\mu|\rho_1\rho_2} Q^{(2)-}_{\rho_1\rho_2|\nu} \,,$$

$$H^{(2)}_{\mu_1\mu_2|\nu_1\nu_2} = \frac{1}{2} Q^{(2)-}_{\mu_1\mu_2|\rho} Q^{(2)+}_{\rho|\nu_1\nu_2} \,, \qquad \tilde{H}^{(2)}_{\mu_1\mu_2|\nu_1\nu_2} = \frac{1}{12} Q^{(3)+}_{\mu_1\mu_2|\rho_1\rho_2\rho_3} Q^{(3)-}_{\rho_1\rho_2\rho_3|\nu_1\nu_2} \,,$$

$$H^{(3)}_{\mu_1\mu_2\mu_3|\nu_1\nu_2\nu_3} = \frac{1}{12} Q^{(3)-}_{\mu_1\mu_2\mu_3|\rho_1\rho_2} Q^{(3)+}_{\rho_1\rho_2|\nu_1\nu_2\nu_3} \,. \tag{79}$$

These operators replace $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ of Sec. 2. The operator $\tilde{H}^{(3)}$ is constructed in a similar way. $H^{(n)}$ and $\tilde{H}^{(n)}$ act on the space $\mathcal{V}_n$ of the n-forms. We observe that

$$Q^{(n)-} Q^{(n-1)-} = Q^{(n-1)+} Q^{(n)+} = 0 \,. \tag{80}$$

Thus Q^- and Q^+ have the properties of the d and δ operators of an elliptic complex. There is an orthogonal decomposition $\mathcal{V}_n = \mathcal{V}_n^+ \oplus \mathcal{V}_n^-$ such that $\mathcal{V}_n^+ = Q^{(n-1)+}\mathcal{V}_{n+1}$ and $\mathcal{V}_n^- = Q^{(n)-}\mathcal{V}_{n-1}$. The spaces $\mathcal{V}_n^+$ (resp. $\mathcal{V}_n^-$) are spanned by the zero modes of $H^{(n)}$ (resp. $\tilde{H}^{(n)}$). If $\phi \to const.$ as $|x| \to \infty$ there are no more zero modes. In the construction of the spectral zeta functions below we always assume that $H^{(n)}$ and $\tilde{H}^{(n)}$ are restricted to $\mathcal{V}_n^-$ and $\mathcal{V}_n^+$ respectively.

The operators

$$\mathcal{H}^{(n)} = H^{(n)} + \tilde{H}^{(n)} \tag{81}$$

are elliptic on the space of n-forms. The effective action (76) can be represented as

$$W = \frac{1}{2} \log \frac{\det \tilde{H}^{(1)}}{\det \Delta_+} \,. \tag{82}$$

By acting as in Sec.2, we write

$$\frac{1}{2} \delta\zeta^+(0)' = -\zeta_0^+(\delta\phi) + \zeta_0^{H^{(1)}}(\delta\phi) \,. \tag{83}$$

The second term on the right hand side of (83) can be expressed as

$$\zeta_0^{H^{(1)}}(\delta\phi) = \zeta_0^{\mathcal{H}^{(1)}}(\delta\phi) - \zeta_0^{\tilde{H}^{(1)}}(\delta\phi) \,. \tag{84}$$

For the un-smeared ζ-functions we have

$$\zeta^{\tilde{H}^{(1)}}(s) = \zeta^{\mathcal{H}^{(1)}}(s) - \zeta^{H^{(1)}}(s) = \zeta^{\mathcal{H}^{(1)}}(s) - \zeta^+(s) \,. \tag{85}$$

The ζ functions on the right hand side of (85) correspond to elliptic operators of the Laplace type. They are regular at $s = 0$. Therefore, $\zeta^{\tilde{H}^{(1)}}(0)'$ is well defined, and

$$\zeta_0^{\tilde{H}^{(1)}}(\delta\phi) = -\frac{1}{2} \delta\zeta^{\tilde{H}^{(1)}}(0)' + \zeta_0^{H^{(2)}}(\delta\phi) \,. \tag{86}$$

Repeating these steps once again we obtain

$$\zeta_0^{H^{(2)}}(\delta\phi) = \zeta_0^{\mathcal{H}^{(2)}}(\delta\phi) - \zeta_0^{\tilde{H}^{(2)}}(\delta\phi) \,,$$

$$\zeta_0^{\tilde{H}^{(2)}}(\delta\phi) = -\frac{1}{2} \delta\zeta^{\tilde{H}^{(2)}}(0)' + \zeta_0^{H^{(3)}}(\delta\phi) \,. \tag{87}$$

It is convenient to rewrite $H^{(3)}$ in the dual representation:

$$H^{(3)}_{\mu|\nu} = \frac{1}{6}\epsilon^{\mu\mu_1\mu_2\mu_3}\epsilon^{\nu\nu_1\nu_2\nu_3}H^{(3)}_{\mu_1\mu_2\mu_3|\nu_1\nu_2\nu_3} = -\delta_{\mu\nu}B_\rho A_\rho + B_\nu A_\mu \,. \tag{88}$$

We also introduce $\tilde{H}^{(3)}$ and $\mathcal{H}^{(3)}$ in the dual representation:

$$\tilde{H}^{(3)}_{\mu|\nu} = -A_\mu B_\nu \,, \qquad \mathcal{H}^{(3)}_{\mu|\nu} = \delta_{\mu\nu}\Delta_- - 2\phi_{,\mu\nu} \,. \tag{89}$$

Since in the zeta functions in (87) all vector indices are contracted we can evaluate them in the dual representation as well:

$$\zeta_0^{H^{(3)}}(\delta\phi) = \zeta_0^{\mathcal{H}^{(3)}}(\delta\phi) - \zeta_0^{\tilde{H}^{(3)}}(\delta\phi) \,. \tag{90}$$

Next we note that the operators $H^{(3)}$, $\mathcal{H}^{(3)}$ and $\tilde{H}^{(3)}$ can be obtained from $H^{(1)}$, $\mathcal{H}^{(1)}$ and $\tilde{H}^{(1)}$ by changing the sign in front of the dilaton field ϕ. Therefore,

$$\zeta_0^{\tilde{H}^{(3)}}(\delta\phi) = -\frac{1}{2}\delta\zeta^-(0)' + \zeta_0^-(\delta\phi) \,. \tag{91}$$

Combining (83), (84), (85), (86), (87), (90) and (91) we obtain

$$\frac{1}{2}\left(-\delta\zeta^+(0)' + \delta\zeta^{\tilde{H}^{(1)}}(0)' - \delta\zeta^{\tilde{H}^{(2)}}(0)' + \delta\zeta^-(0)'\right)$$
$$= \zeta_0^+(\delta\phi) - \zeta_0^{\mathcal{H}^{(1)}} + \zeta_0^{\mathcal{H}^{(2)}} - \zeta_0^{\mathcal{H}^{(3)}} + \zeta_0^-(\delta\phi) \,. \tag{92}$$

The second line of (92) contains elliptic operators only. For them $\zeta_0^H(\delta\phi) = a_2(\delta\phi, H)$. Consequently the second line of (92) vanishes by the Gauss-Bonnet theorem [5]. One can also check this by direct calculations. Therefore,

$$W^{\mathrm{ren}}[\phi] = W^{\mathrm{ren}}[-\phi] \,. \tag{93}$$

5   Conclusions

In this paper we have found several remarkable duality relations between functional determinants. We have shown how the effective action of quantum fields interacting with the (matrix-valued) scalar background field transforms under inversion of the background field. For flat manifolds without boundaries the effective action is proved to be invariant under this inversion. This property has been explicitly demonstrated for two-dimensional models and for the dilaton-Maxwell theory in four dimensions. The presence of boundaries leads to additional terms that can also be calculated exactly (see Eq.(66)). We have studied in detail the 2D matrix theory, the 4D dilaton-Maxwell system, and the 2D dilaton theory on a manifold with boundary. We have considered the simplest topologies of the field configurations only. Less trivial topologies can be included in our approach by taking into account the zero-mode contributions. The application of our results and methods include duality properties of strings, quantum black holes, and quantum systems obtained after reduction from higher dimensions, to mention a few. We are sure that this list can be considerably extended.

Acknowledgements

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