Comment on "A new, exact, gauge-invariant RG-flow equation"

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Abstract

We show that the exact RG-flow equation introduced recently in hep-th/ 0207134 can be obtained in the sharp cut-off limit of the well-known ERGE. This can be expected from the fact that in this limit the new scale-dependent effective action coincides with the one which is usually considered.

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The purpose of this short note is to show that the *new* exact RG-flow equation recently proposed in ref.[1], which is obtained by a partial Legendre transform , corresponds to the *sharp* cut-off limit of the *exact* RG equation (ERGE) of ref.[2, 3] [2]. To do this we will start by deriving the ERGE before considering the sharp cut-off limit.

The effective action. The scale-dependent 1PI effective action can be defined in the following way: One introduces the quadratic cut-off functional

$$\mathcal{O}_k[\chi-\varphi] \equiv \exp\left(-\Delta_k S[\chi-\varphi]\right) \equiv \exp\left(-\frac{1}{2}\int(\chi-\varphi)R_k(\chi-\varphi)\right), \tag{1}$$

inside the functional integral which defines the partition function,

$$e^{W[J]} \equiv \int \mathcal{D}\chi\, e^{-S[\chi]+\int J\chi}. \tag{2}$$

We assume that $\lim_{k\to 0} R_k = 0$ and therefore the functional

$$e^{\widehat{W}_k[J,\varphi]} \equiv \int \mathcal{D}\chi\, \mathcal{O}_k[\chi-\varphi]\, e^{-S[\chi]+\int J\chi}, \tag{3}$$

obtained with this procedure converges to $W[J]$ as $k \to 0$. We can now perform a Legendre transformation of W_k with respect to J to obtain

$$\widehat{\Gamma}_k[\phi,\varphi] = -\widehat{W}_k[J,\varphi] + \int J\phi, \quad \phi \equiv \frac{\delta \widehat{W}_k[J,\varphi]}{\delta J}. \tag{4}$$

This means that $\widehat{\Gamma}_k$ is given implicitly by

$$\exp\left(-\widehat{\Gamma}_k[\phi,\varphi]\right) = \int \mathcal{D}\chi\, \mathcal{O}_k[\chi-\varphi+\phi]\, \exp\left(-S[\chi+\phi] + \int \frac{\delta\widehat{\Gamma}_k[\phi,\varphi]}{\delta\phi}\chi\right). \tag{5}$$

Since it is our intention to obtain an effective action which interpolates between the classical action $S[\phi]$ in the UV ($k\to\infty$), and the full effective action $\Gamma[\phi]$ in the IR ($k=0$) we first assign $\mathcal{O}_k[\chi-\varphi]$ the following property

$$\lim_{k\to\infty} \mathcal{O}_k[\chi] \sim \delta[\chi], \tag{6}$$

where the r.h.s. is a δ-functional. This can be obtained if $\lim_{k\to\infty} R_k = \infty$. In this case we have

$$\lim_{k\to\infty} \widehat{\Gamma}_k[\phi,\varphi] = S[\varphi] + \int \frac{\delta\widehat{\Gamma}_k[\phi,\varphi]}{\delta\phi}(\phi-\varphi). \tag{7}$$

[2] For a comprehensive review see [4]

Finally, to obtain the desired property, we set $\varphi = \phi$:

$$\Gamma_k[\phi] \equiv \widehat{\Gamma}_k[\phi, \phi]. \tag{8}$$

One can use now

$$\frac{\delta \widehat{\Gamma}_k[\phi, \varphi]}{\delta \varphi} = -\frac{\delta \widehat{W}_k[J, \varphi]}{\delta \varphi} = R_k(\varphi - \phi), \tag{9}$$

to write $\Gamma_k[\phi]$ as (see eq.(5))

$$\exp\left(-\Gamma_k[\phi]\right) = \int \mathcal{D}\chi\, \mathcal{O}_k[\chi] \exp\left(-S[\chi + \phi] + \int \frac{\delta \Gamma_k[\phi]}{\delta \phi} \chi\right). \tag{10}$$

From this expression it is not dificult to recognize $\Gamma_k[\phi]$ as being given (perturbatively) by the sum of the connected 1PI vacuum graphs with ϕ-dependent vertices and internal lines regulated by the introduced cut-off.

The attentive reader may consider the above formalism reminiscent of the background field method used to ensure the gauge invariance of the 1PI effective action (see [5] and refences therein). And indeed both ideas can be combined to define a scale-dependent, gauge invariant, 1PI effective action [6].

The flow equation. To calculate the exact renormalization group equation (ERGE) for Γ_k let us note that

$$\begin{aligned}\partial_k \Gamma_k[\phi] = \partial_k \widehat{\Gamma}_k[\phi, \phi] = -\partial_k \widehat{W}_k[J, \phi] = \frac{1}{2} \int \partial_k R_k \langle (\chi - \phi)(\chi - \phi) \rangle \\ = \frac{1}{2} \int \partial_k R_k \frac{\delta^2 \widehat{W}_k[J, \phi]}{\delta J \delta J}.\end{aligned} \tag{11}$$

Now, we have

$$\frac{\delta^2 \widehat{W}_k[J, \phi]}{\delta J \delta J} = \left(\frac{\delta^2 \widehat{\Gamma}_k[\phi, \varphi]}{\delta \phi \delta \phi}\Big|_{\varphi = \phi} \right)^{-1}, \tag{12}$$

while

$$\frac{\delta^2 \Gamma_k[\phi]}{\delta \phi \delta \phi} = \frac{\delta^2 \widehat{\Gamma}_k[\phi, \varphi]}{\delta \phi \delta \phi}\Big|_{\varphi = \phi} + 2 \frac{\delta^2 \widehat{\Gamma}_k[\phi, \varphi]}{\delta \phi \delta \varphi}\Big|_{\varphi = \phi} + \frac{\delta^2 \widehat{\Gamma}_k[\phi, \varphi]}{\delta \varphi \delta \varphi}\Big|_{\varphi = \phi}. \tag{13}$$

Using eq.(9) one gets

$$\frac{\delta^2 \widehat{\Gamma}_k[\phi, \varphi]}{\delta \phi \delta \phi}\Big|_{\varphi = \phi} = \frac{\delta^2 \Gamma_k[\phi]}{\delta \phi \delta \phi} + R_k, \tag{14}$$

obtaining in this way the well-known ERGE for Γ_k [2, 3]:

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \int (\partial_k R_k) \frac{1}{\Gamma_k^{(2)}[\phi] + R_k}. \tag{15}$$

Sharp cut-off limit. By sharp cut-off one means a function R_k which diverges for momenta below the scale k and vanishes above this scale. Although one could think that in this limiting case Γ_k is not well defined due to an ill-definition of the Legendre transform of $W_k[J,\varphi]$ we will see that this is not the case.

In this limit $\mathcal{O}_k$ is of the form

$$\mathcal{O}_k[\chi] \sim \prod_{p^2<k^2} \delta(\chi(p)), \tag{16}$$

which means that (using the notation of ref.[1]) we can write

$$\exp W_k[J,\varphi_0^k] = \int \mathcal{D}\chi_k^\Lambda \, \exp\left(-S[\varphi_0^k + \chi_k^\Lambda] + \int J(\varphi_0^k + \chi_k^\Lambda)\right), \tag{17}$$

where φ_0^k contains only Fourier modes with $p < k$ and χ_k^Λ only modes with $p > k$. This differs from the W_k ($\equiv \mathcal{W}_k$) defined in [1] in the following way[3]

$$W_k[J,\varphi_0^k] = \mathcal{W}_k[J_k^\Lambda, \varphi_0^k] + \int J_0^k \varphi_0^k, \tag{18}$$

but, as we will show, the scale dependent effective action Γ_k coincides with the one defined in that work (we will call this one $\mathcal{G}_k$). This follows from the fact that $\Gamma_k[\phi] = \widehat{\Gamma}_k[\phi,\phi]$ in this case is given by

$$\Gamma_k[\phi] = \widehat{\Gamma}_k[\phi, \varphi_0^k], \tag{19}$$

since due to the sharp cut-off we have $\phi = \varphi_0^k$ for $p < k$ and W_k is independent of φ_k^Λ, where $\varphi_k^\Lambda = \varphi$ for $p > k$. Furthermore, we have

$$\Gamma_k[\phi] = -W_k[J,\varphi_0^k] + \int J(\varphi_0^k + \phi_k^\Lambda) = -\mathcal{W}_k[J_k^\Lambda, \varphi_0^k] + \int J_k^\Lambda \phi_k^\Lambda, \tag{20}$$

where

$$\phi_k^\Lambda = \frac{\delta \mathcal{W}_k[J_k^\Lambda, \varphi_0^k]}{\delta J_k^\Lambda}. \tag{21}$$

But this is clearly the definition of $\mathcal{G}_k$, the Legendre transform of $\mathcal{W}_k$ with respect to J_k^Λ, as we intended to show.

The flow equation in the sharp cut-off limit. Let us now see what happens to the flow equation (15) in the sharp cut-off limit. The flow equation can be rewritten as

$$\partial_k \Gamma_k[\phi] = \frac{1}{2} \operatorname{tr} \tilde{\partial}_k \ln(\Gamma_k^{(2)}[\phi] + R_k) = \frac{1}{2} \tilde{\partial}_k \ln\det(\Gamma_k^{(2)}[\phi] + R_k), \tag{22}$$

[3] There is also a irrelevant difference in sign.

where $\tilde{\partial}_k$ only acts upon R_k. Since below k the cut-off fuction R_k diverges we get

$$\partial_k \Gamma_k[\phi] = \lim_{\delta k \to 0} \frac{1}{2\delta k} \left[\ln \det \frac{\delta^2 \Gamma_k}{\delta \phi_k^\Lambda \delta \phi_k^\Lambda} - \ln \det \frac{\delta^2 \Gamma_k}{\delta \phi_{k-\delta k}^\Lambda \delta \phi_{k-\delta k}^\Lambda} \right], \tag{23}$$

where the k in the first term and the $k - \delta k$ in the second one denote the IR cut-offs in the respective determinants and we neglected a ϕ-independent piece. It is not dificult to recognize that this is the flow equation which was obtained by the authors of ref.[1] by other means.

Thus in this paper we showed that the scale-dependent 1PI effective action defined in [1] by a partial Legendre transformation can be considered to be the sharp cut-off limit of the one defined in [2, 3]. As expected, the flow equation of [1] could also be obtained as the same limit of the ERGE of [2, 3].

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