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Inverse dualisation and non-local dualities between Einstein gravity and supergravities

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Abstract

We investigate non-local dualities between suitably compactified higher-dimensional Einstein gravity and supergravities which can be revealed if one reinterprets the dualised Kaluza-Klein two-forms in $D > 4$ as antisymmetric forms belonging to supergravities. We find several examples of such a correspondence including one between the six-dimensional Einstein gravity and the four-dimensional Einstein-Maxwell-dilaton-axion theory (truncated $N = 4$ supergravity), and others between the compactified eleven and ten-dimensional supergravities and the eight or ten-dimensional pure gravity. The Killing spinor equation of the $D = 11$ supergravity is shown to be equivalent to the geometric Killing spinor equation in the dual gravity. We give several examples of using new dualities for solution generation and demonstrate how p-branes can be interpreted as non-local duals of pure gravity solutions. New supersymmetric solutions are presented including $M2 \subset 5$-brane with two rotation parameters.

1 Introduction

Kaluza-Klein (KK) toroidal reduction is based on a geometric property, originally discovered in the five-dimensional General Relativity, to convert the compactified gravitational degrees of freedom into the scalar-vector theory governed by the dilatonic Maxwell action. In such a way dynamics of the five-dimensional gravity with a non-null Killing symmetry is described by the four-dimensional Einstein-Maxwell-dilaton action with a certain value of the dilaton coupling constant. In the case of the higher-dimensional Einstein gravity with d commuting Killing vectors the number of the KK Maxwell fields in the reduced theory is d, while the dilaton expands into the $d \times d$ matrix of scalar moduli exhibiting the $GL(d, R)$ global symmetry. Similarly, dimensional reduction of multidimensional supergravities gives rise to multiplets of fields transforming under E_n groups [1].

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In revealing hidden symmetries both in Einstein and supergravity theories an important step is the dualisation of antisymmetric form fields. The well-known example is the reduction of Einstein pure gravity to three dimensions. Trading the KK two-forms for scalars via dualisation in three dimensions one discovers the U-duality group $SL(d+1,R)$ (d is the total number of compactified dimensions) instead of the expected T-duality $SL(d,R)$. In four dimensions a $U(1)$ electric-magnetic duality (S-duality) also becomes manifest. U-duality of the three-dimensional theory embeds T and S dualities into a larger group. In supergravities one usually employs dualisation to reduce the rank of antisymmetric forms. We refer the reader to an exhaustive discussion of the role and different ways of implementing the dualisation in dimensionally reduced multidimensional supergravities [2, 3].

Dualisation in the opposite sense (producing the form fields of an increased rank) may also reveal hidden symmetries between different theories. Following conventions of [2, 3], we use in such cases the name "inverse dualisation". Clearly, the lowest dimension where inverse dualisation makes sense is five, where a two-form is dualised to a three form. In $D = 4$ the rank of a two-form is unchanged under dualisation: "neutral dualisation". Dualisation gives rise to several alternative representations of hidden symmetries which maps the space of solutions on itself in a non-trivial way. This can be used for solution generating purposes. For example, dimensional reduction of the $D = 4$ Einstein equations to three dimensions leads to the $SL(2,R)$ Ehlers group [4]. This symmetry is realized on the space of two variables one of which is the component of the four-dimensional metric, and another is the twist potential: a (pseudo)scalar resulting from the dualisation of the KK vector. Going down to two dimensions one finds the infinite-dimensional extension of the Ehlers group to the Geroch group, but here we are interested rather in its finite $SL(2,R)$ subgroup. An alternative realization of the same $SL(2,R)$ symmetry group in two dimensions in known as the Matzner-Misner group, this latter acts directly on two metric functions. A map between these different realizations of the same symmetry (Kramer-Neugebauer map [5]) plays an important role in solution generation techniques of general relativity. Another example, which is closer in spirit to our present considerations, is the Bonnor map [6] between the $SL(2,R)$ symmetry groups of the static Einstein-Maxwell system and the stationary pure gravity (for an extension to dilaton gravity see [7]).

Here we describe new non-trivial realizations of the "Bonnor map" between different gravity and supergravity theories which are related to inverse and neutral dualisation of KK two-forms (some earlier results were presented in [8, 9]). Starting with the compactified pure gravity one can use an inverse dualisation in dimensions $D \geq 5$ to convert the set of the KK two-forms into higher rank forms which are then reinterpreted as belonging to some supergravity theory in another dimension. We are looking for such compactified supergravities which exhibit the same hidden symmetries as a suitable compactified Einstein gravity, so the non-local dualities that we find are essentially of the Bonnor's type. We will show that these dualities relate the p-branes [10] to some pure gravity solutions. To avoid confusion, it is worth noting that our present technique is essentially different from the earlier known method of generating supergravity p-branes via Harrison transformation [11]. This latter is based on U-duality arising in the compactified truncated supergravity action. Here we use a map between both suitably compactified gravity and supergravity (generically in different dimensions) which is a Bonnor-type map between different theories enjoying the same hidden symmetry.

To be explicit, let us consider the D-dimensional action arising from dimensional reduction of the $D+1$ Einstein theory in a space-like dimension

$$S_D = \int d^D x \sqrt{|g|} \left\{ R(g) - \frac{\gamma}{4}(\nabla\psi)^2 - \frac{e^{\gamma\psi}}{4} F^2 \right\}, \tag{1}$$

where

$$\gamma = 1 + \frac{1}{D-2}, \tag{2}$$

$F = dA$, and A, ψ being the KK vector and scalar respectively. Instead of treating the two-form F as a vector field strength, one can dualise it to a $(D-2)$-rank form H:

$$F = e^{-\gamma\psi} \, {}^{*_D} H, \tag{3}$$

where ^{*_D} denotes the D-dimensional Hodge dual. Considering H as an exterior derivative of a $(D-3)$-form, $H = dB$, one can derive the equivalent equations of motion from the action

$$S'_D = \int d^D x \sqrt{|g|} \left\{ R(g) - \frac{\gamma}{4}(\nabla\psi)^2 - \frac{e^{-\gamma\psi}}{2(D-2)!} H^2 \right\}. \tag{4}$$

Continuing the reduction process further one can either generate new antisymmetric forms in lower dimensions via similar dualisations, or treat the appearing new KK fields as vectors, so a variety of alternatives will arise. Note that the reparameterization of antisymmetric forms usually involves the appropriate Chern-Simons terms in lower dimensions.

The action (4) is a typical p-brane producing action with a particular value of the dilaton coupling constant. A crucial point for existence of the Bonnor-type maps we are looking for is the "right" value of the dilaton coupling constant. We will show that this is indeed the case in a number of physically interesting situations. New dualities are essentially non-local, since the relation (3) connecting variables of two theories is a differential equation for the KK potential. Therefore, given a p-brane solution to (4), one has to solve the set of differential equations in order to find the corresponding higher-dimensional metric, and vice versa. The simplest nontrivial example of such a relationship is provided by six-dimensional Einstein theory. After compactification of one dimension and dualisation of the corresponding KK two-form one obtains five-dimensional gravity coupled to a dilaton and a three-form. Therefore, the string solution to the $5D$ gravity with a three-form theory may be seen as a non-local dual to some pure gravity solution in six dimensions [8].

The main purpose of the present paper is to describe non-local dualities involving eleven and ten-diemsnional supergravities. The plan of the paper is as follows. We start with reviewing the standard KK dimensional reduction and discuss particular cases of the resulting four and three-dimensional theories (Sec. 2). Then the dimensional reduction of the six-dimensional relativity to five, four and three dimensions is considered and an alternative matrix representation of the $SL(4,R)$ symmetry is derived (Sec. 3). In Sec. 4 we present the reduction scheme with partial inverse dualisation in arbitrary dimensions. The detailed description of duality between the compactified $D=11$ supergravity and the $D=8$ pure gravity is given in Sec. 5. Then (Sec. 6) several dualities between IIA and IIB supergravities and ten and eight-dimensional Einstein gravities are exhibited. The Sec. 7 is devoted to establishing connection between Killing spinor equation in the compactified $D=11$ supergravity and the geometric Killing spinor equation in eight dimensions. Several examples of using new dualities for solution generation are presented in Sec. 8. We conclude with brief remarks in Sec. 9. Some mathematical details are given in three Appendices.

2 The standard KK reduction

Consider the toroidal compactification of the $D+d$ Einstein gravity to D dimensions starting with the following parameterization of the metric:

$$ds^2_{D+d} = g_{dmn}(d\zeta^m + A^m_M dx^M)(d\zeta^n + A^n_N dx^N) + e^{-\psi/(D-2)}\, ds^2_D, \tag{5}$$

where

$$g_d = ||g_{dmn}||, \qquad e^{\psi} = |\det g_d|. \tag{6}$$

It is assumed that all fields depend only on coordinates in the D-sector. Define a (pseudo)unimodular moduli matrix

$$M_{mn} = g_{dmn} e^{-\psi/d}, \tag{7}$$

so that $\det M = \epsilon$, with $\epsilon = +1$ ($\epsilon = -1$) if the metric g_d has Euclidean (Lorentzian) signature, or equivalently, the metric g_D has Lorentzian (Euclidean) signature.[4] Then one obtains the following action for the reduced theory

$$S_D = \int d^Dx\sqrt{|g|}\Big\{R(g) - \frac{\gamma}{4}(\nabla\psi)^2 - \frac{1}{4}e^{\gamma\psi}F^T M F + \frac{1}{4}g^{MN}\mathrm{Tr}\left(\partial_M M \partial_N M^{-1}\right)\Big\}, \tag{8}$$

where

$$\gamma = \frac{1}{d} + \frac{1}{D-2}. \tag{9}$$

This action is manifestly invariant under T-duality $SL(d,R)\times R$:

$$M \to \Omega^T M \Omega, \quad F \to \Omega^{-1} F, \qquad \Omega \in SL(d,R), \tag{10}$$
$$\psi \to \psi + r, \qquad F \to e^{-\gamma r/2} F, \quad r \in R. \tag{11}$$

The symmetry holds in any dimension $D \geq 2$, and it gets enhanced in $D \leq 4$. Consider a particular case $D=4$. Then $\gamma = (2+d)/2d$ and the equations of motion consist of Einstein equations

$$\begin{aligned} R_{\mu\nu} \;&=\; \frac{2+d}{8d}\nabla_\mu\psi\nabla_\nu\psi - \frac{1}{4}\mathrm{Tr}\left(\nabla_\mu M \nabla_\nu M^{-1}\right) \\ &+\; \frac{1}{2}e^{(d+2)\psi/2d}\left(F^T_{\mu\alpha} M F_\nu{}^\alpha - \frac{1}{4}g_{\mu\nu}F^T M F\right), \end{aligned} \tag{12}$$

[4]We consider here only $\Sigma \times K$ space-times with exactly one 'time' direction.

the dilaton equation

$$\nabla^2\psi = \frac{1}{2}e^{(d+2)\psi/2d}F^T MF, \tag{13}$$

Maxwell equations

$$\nabla_\mu\left(e^{(d+2)\psi/2d}MF^{\mu\nu}\right) = 0, \tag{14}$$

and an equation for the moduli matrix

$$\nabla(M^{-1}\nabla M) = \frac{1}{2}e^{(d+2)\psi/2d}\left(FF^T M - \frac{1}{d}F^T MFI_d\right). \tag{15}$$

One also has the Bianchi identity

$$\nabla_\mu(\tilde{F}^{\mu\nu}) = 0, \tag{16}$$

where $\tilde{F}$ stands for a four-dimensional dual ($\tilde{F} \equiv {}^{*4}F$). In addition to continuous T-duality $SL(d,R)\times R = GL(d,R)$ this system is also symmetric under discrete electric-magnetic duality: the Eqs. (12,13,15) remain invariant, while the Maxwell equation (14) and the Bianchi identity (16) are interchanged under the discrete S-duality transformation

$$\psi \to -\psi, \quad M \to \epsilon M^{-1}, \quad F \to e^{(d+2)\psi/2d}M\tilde{F}. \tag{17}$$

Note that the factor ϵ in the transformation of the moduli matrix is necessary in view of the relation $\tilde{F}_{\mu\nu}\tilde{F}^{\nu\lambda} \equiv -\epsilon F_{\mu\nu}F^{\nu\lambda}$. In the case $\epsilon = 1$ this discrete symmetry is the symmetry of the initial $(D+d)$-dimensional field equations, while for $\epsilon = -1$ it relates theories with the signature $\mathrm{diag}(-,+,...,+;+,...,+)$ (only one time-like direction) and the signature $\mathrm{diag}(+,-,...,-;+,...,+)$ ($d-1$ time-like directions). About dimensional reduction in supergravities with non-standard signatures see [12, 13].

Now let us go down to three dimensions. The resulting action will read

$$S_3 = \int d^3x\sqrt{|h|}\Big\{R_3 - \frac{1}{4}(\nabla\psi)^2 + \frac{1}{4}\mathrm{Tr}(\nabla g_d\nabla g_d^{-1}) - \frac{1}{4}e^{\psi}F^T g_d F\Big\}. \tag{18}$$

It can be transformed into the action for a gravity coupled sigma-model after dualising the Maxwell two-forms as follows

$$e^{\psi}g_d F = {}^{*3}d\Psi, \tag{19}$$

where Ψ is the set of scalars. In terms of new variables the equations of motion may be obtained from the action

$$S_3' = \int d^3x\sqrt{|h|}\Big\{R_3 - \frac{1}{4}(\nabla\psi)^2 + \frac{1}{4}\mathrm{Tr}(\nabla g_d\nabla g_d^{-1}) - \frac{\epsilon}{2}e^{-\psi}\nabla\Psi^T g_d^{-1}\nabla\Psi\Big\}, \tag{20}$$

which simplifies to

$$S_3' = \int d^3x\sqrt{|h|}\left\{R_3 + \frac{1}{4}\mathrm{Tr}(\nabla\mathcal{M}\nabla\mathcal{M}^{-1})\right\}, \tag{21}$$

where the matrix

$$\mathcal{M} = \begin{pmatrix} e^{-\psi} & e^{-\psi}\Psi^T \\ e^{-\psi}\Psi & \epsilon g_d + e^{-\psi}\Psi\Psi^T \end{pmatrix} \tag{22}$$

is a symmetric $(d+1)\times(d+1)$ matrix with $\det\mathcal{M}=\epsilon^{(d+1)}$. It is easy to see that for $\epsilon=1$, $\mathcal{M}\in SL(d+1,R)/SO(d+1)$, while for $\epsilon=-1$, $\mathcal{M}\in SL(d+1,R)/SO(d-1,2)$. Indeed, the matrix $\mathcal{M}$ may be presented in the following vielbien split

$$\mathcal{M}_{MN}=\mathcal{E}^A_M\,\eta_{AB}\,\mathcal{E}^B_N, \tag{23}$$

where

$$\mathcal{E}^A_M=\begin{pmatrix} e^{-\psi/2} & e^{-\psi/2}\Psi_m \\ 0 & \epsilon e^a_m \end{pmatrix}, \qquad \eta_{AB}=\begin{pmatrix} 1 & 0 \\ 0 & \epsilon\eta_{ab} \end{pmatrix}, \tag{24}$$

so that $g_{mn}=e^a_m\eta_{ab}e^b_n$. Therefore for $\epsilon=1$ one has the Euclidean signature metric η_{AB}, while for $\epsilon=-1$ the metric is pseudo-Euclidean. This is the *standard* representation of the three-dimensional reduction of a higher-dimensional pure gravity [14]. Finally, further reduced to $D=2$, the action (21) leads to a completely integrable theory.

3 Alternative reduction of $D=6$ Einstein gravity and EMDA

Our first example of using inverse dualisation to relate pure gravity to (compactified) supergravity starts with six dimensions. A standard dimensional reduction of $D+d=6$ Einstein theory to five dimensions ($d=1$ in notation of the previous section) gives the action (1) with $D=5$. Now, instead of further direct reduction, let us first make an inverse dualisation of KK two-form to get the five-dimensional gravity coupled to a dilaton $\hat{\phi}$ and a three-form field $\hat{H}=d\hat{B}$

$$S_5=\int d^5x\sqrt{-\hat{g}_5}\left\{\hat{R}_5-\frac{1}{2}(\partial\hat{\phi})^2-\frac{e^{-\alpha\hat{\phi}}}{12}\hat{H}^2\right\}, \tag{25}$$

where the dilaton coupling is $\alpha^2=8/3$. This action can be reinterpreted in supergravity terms. To get further insight into the nature of the theories involved we perform reduction to four- and then to the three-dimensional theory. Compactifying first along a space-like dimension

$$d\hat{s}_5^2=e^{-4\varphi}\left(dy+\mathcal{A}_\mu dx^\mu\right)^2+e^{2\varphi}ds_4^2, \tag{26}$$

one obtains for the metric part

$$\sqrt{-\hat{g}_5}\hat{R}_5=\sqrt{-g_4}\Big\{R_4-6g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi-\frac{1}{4}e^{-6\varphi}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}-2\nabla_\mu\left(g^{\mu\nu}\partial_\nu\varphi\right)\Big\}, \tag{27}$$

where $\mathcal{A}=\mathcal{A}_\mu dx^\mu,\ \mathcal{F}=d\mathcal{A}$, while the five-dimensional two-form $\hat{B}$ is decomposed into the four-dimensional two-form B and the one-form A

$$\hat{B}=B-dy\wedge A. \tag{28}$$

Omitting a total divergence we arrive at the following four-dimensional action

$$S_4=\int d^4x\sqrt{-g_4}\Big\{R_4-\frac{1}{2}(\partial\phi)^2-\frac{1}{4}(\partial\psi)^2-\frac{1}{4}e^{\psi-\phi}\mathcal{F}^2-\frac{1}{4}e^{-\psi-\phi}F^2-\frac{e^{-2\phi}}{12}H^2\Big\}, \tag{29}$$

where new scalar fields are introduced

$$\phi = \frac{1}{2}(\alpha\hat{\phi} + 4\varphi), \qquad \psi = \frac{1}{2}(\alpha\hat{\phi} - 8\varphi), \tag{30}$$

and the four-dimensional strength H will include an appropriate Chern-Simons term

$$H = dB - \mathcal{A} \wedge F, \tag{31}$$

where $F = dA$. The value of the dilaton coupling constant corresponding to dimensional reduction of $D = 6$ pure gravity theory $\alpha^2 = 8/3$ coincides with that arising in toroidal of the heterotic string effective action. Assuming this value and dualising the three-form

$$e^{-2\phi} H = -{}^{*_4} d\kappa, \tag{32}$$

one can present the corresponding action as

$$\begin{aligned} S_4' &= \int d^4x\sqrt{-g}\Big\{R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}(\partial\psi)^2 - \frac{1}{2}e^{2\phi}(\partial\kappa)^2 \\ &- \frac{1}{4}e^{\psi-\phi}\mathcal{F}^2 - \frac{1}{4}e^{-\psi-\phi}F^2 - \frac{\kappa}{4}\left(F\tilde{\mathcal{F}} + \mathcal{F}\tilde{F}\right)\Big\}. \end{aligned} \tag{33}$$

Now we reduce the theory to three dimensions using the general partial dualisation procedure described in the Appendix A. With $d_1 = d_2 = 1$ and $\phi = \phi_1 + \phi_2$, our prescription (151)-(156) leads to the following set of three-dimensional fields: two electric v_1, v_2 and two magnetic u_1, u_2 potentials, scale factor f and twist potential χ. Together with already introduced scalars we obtain a three-dimensional σ-model with the nine-dimensional target space $\Phi^A = (f, \chi, \phi, \psi, \kappa, v_1, u_1, v_2, u_2)$ endowed with the following metric

$$\begin{aligned} dl^2 &= \frac{1}{2f^2}\left\{df^2 + \left[d\chi + \frac{1}{2}\left(v_a du_a - u_a dv_a\right)\right]^2\right\} + \frac{1}{2}d\phi^2 + \frac{1}{4}d\psi^2 + \frac{1}{2}e^{2\phi}d\kappa^2 \\ &- \frac{1}{2f}\left[e^{\psi-\phi}dv_1^2 + e^{-\psi+\phi}(du_1 - \kappa dv_2)^2 + e^{-\psi-\phi}(dv_2)^2 + e^{\psi+\phi}(du_2 - \kappa dv_1)^2\right]. \end{aligned} \tag{34}$$

This is the metric of the symmetric space $SL(4,R)/SO(2,2)$ on which the $SL(4,R)$ isometry group acts transitively. As the coset representative one can choose a symmetric $SL(4,R)$ matrix, so that the target space metric will read

$$dl^2 = -\frac{1}{4}\mathrm{Tr}\left(d\mathcal{M}d\mathcal{M}^{-1}\right). \tag{35}$$

The matrix $\mathcal{M}$ is given by the Eq.(163) with the following 2×2 real blocks

$$P_1 = e^{\psi/2}\begin{pmatrix} fe^{-\psi} - (v_1)^2e^{-\phi} & -v_1e^{-\phi} \\ -v_1e^{-\phi} & -e^{-\phi} \end{pmatrix}, \tag{36}$$

$$P_2 = e^{-\psi/2}\begin{pmatrix} fe^{\psi} - (v_2)^2e^{-\phi} & -v_2e^{-\phi} \\ -v_2e^{-\phi} & -e^{-\phi} \end{pmatrix}, \tag{37}$$

$$Q = \begin{pmatrix} \frac{1}{2}\xi - \chi & u_2 - \kappa v_1 \\ u_1 - \kappa v_2 & -\kappa \end{pmatrix}, \tag{38}$$

where

$$\xi = v_1 (u_1 - \kappa v_2) + v_2 (u_2 - \kappa v_1) \, . \tag{39}$$

Matrix $\mathcal{M}$ provides an alternative to (22) representation of the coset $SL(4,R)/SO(2,2)$. For $d=3$ the standard matrix (22) is presented in terms of 3×3, 1×3 blocks and one scalar, whereas (163) is parameterized more symmetrically in terms of 2×2 blocks. New representation provides a direct link to the three-dimensional reduction of the four-dimensional Einstein-Maxwell-Dilaton-Axion (EMDA) theory [15] (truncated $D=4, N=4$ supergravity with only one vector) in which case one has a $Sp(4,R)$ matrix. To obtain the latter from the present model it is enough to identify the electromagnetic potentials $u_1=u_2$, $v_1=v_2$ and set $\psi=0$ (for more details see [16]). Another particular case is the three-dimensional reduction of the EMDA theory with two vector fields in which case the relevant symmetry is $SU(2,2)$ [17] corresponding to the eight-dimensional target space. Note that both $Sp(4,R)$ and $SU(2,2)$ sigma-models are Kähler, while the present theory is not (its target space is odd-dimensional).

The existence of two essentially different parameterizations of the same coset suggests a solution generating technique. Given a solution in terms of one parameterization, one can construct new solution performing the identification of variables in terms of another representation. In the previous paper [8] we obtained a family of five-dimensional string solutions dualising the five-dimensional Kerr metric. Here we give another example taking as a seed the following six-dimensional pure gravity solution [5]:

$$ds_6^2 = F_2^{-1}(dz + U du + a_i dx^i)^2 + F_1 du^2 + 2du(dv - \omega_i dx^i) + F_2 \delta_{ij} dx^i dx^j, \tag{40}$$

with $i,j=1,2,3$, and the harmonic functions

$$\partial_{\vec{x}}^2 F_1 = \partial_{\vec{x}}^2 F_2 = \partial_{\vec{x}}^2 U = 0, \quad \nabla\times\vec{\omega} = \nabla U, \quad \nabla\times\vec{a} = \nabla F_2. \tag{41}$$

This metric is a superposition of two five-dimensional pure gravity solutions: Dobiasch-Maison [18] (whose one-center version corresponds to pp-wave) and Gross-Perry-Sorkin [19, 20] (KK monopole), endowed with NUT. Three harmonic functions F_1, F_2, and U on the three-dimensional transverse space correspond to an electric charge, a momentum and a NUT charge. From here we can derive two different $5D$ string counterparts choosing one or another Killing vector from ∂_z, ∂_u for dimensional reduction from six to five dimensions. Compactifying along ∂_z we obtain an electrically charged NUT-ed string superposed with the Brinkmann wave:

$$\begin{aligned} d\hat{s}_5^2 &= F_1 F_2^{-1/3} du^2 + 2F_2^{-1/3} du(dv - \omega_i dx^i) + F_2^{2/3}\delta_{ij} dx^i dx^j, & (42)\\ \hat{B} &= F_2^{-1} du\wedge(dv - \omega_i dx^i), & (43)\\ e^{\alpha\hat{\phi}} &= F_2^{-4/3}. & (44)\end{aligned}$$

This family of solutions contains an extremal electric string ($U=F_1=1$, $F_2 = 1+2q/r$, $r^2=\delta_{ij}x^i x^j$) and the pure gravity Dobiash-Maison solution (at $U=F_2=1$, $F_1=$

[5] The one-center version of this solution may be considered as plane wave propagating in the Euclidean Taub-NUT background.

$1+2p/r$). Compactifying along ∂_u one obtains the NUT-ed magnetically charged 0-brane superposed with KK monopole along one of the transversal spatial dimensions:

$$\begin{aligned} d\hat{s}_5^2 &= -T^{-2/3}\left\{dv - \omega_i dx^i + UF_2^{-1}(dz + a_j dx^j)\right\}^2 \\ &+ T^{1/3}\left\{F_2^{-1}(dz + a_i dx^i)^2 + F_2\delta_{ij}dx^i dx^j\right\}, \end{aligned} \tag{45}$$

$$e^{\alpha\hat{\phi}} = T^{4/3}, \tag{46}$$

with the three-form

$$\begin{aligned} \hat{H} &= d(\omega_i dx^i)\wedge(dv - \omega_j dx^j) - d(b_i dx^i)\wedge(dz + a_j dx^j) \\ &+ d\left(UF_2^{-1}(dv - \omega_i dx^i)\wedge(dz + a_j dx^j)\right), \end{aligned} \tag{47}$$

where $T = F_1 + U^2F_2^{-1}$ and $\nabla\times\vec{b} = \nabla F_1$. It reduces to the single extremal 0-brane if $U = 0, F_2 = 1, F_1 = 1 + 2p/r$, to the pure pure gravity Gross-Perry-Sorkin monopole for $U = 0, F_1 = 1, F_2 = 1 + 2p/r$ and to NUT solution for $F_1 = F_2 = 1, U \sim n/r$.

Alternatively these solutions can be obtained as *null geodesics* of the target space of the non-linear σ-model (158).[6] To show this it is sufficient to choose as the geodesic generator a degenerate $sl(4,R)$ matrix of the rank one and to construct a solution depending on three harmonic functions. Along these lines one can also construct a four-charge solution. For an asymptotically flat solution one can define the asymptotical charges as

$$\begin{aligned} f &\sim 1 - 2M/r, \quad \phi \sim 2D_\phi/r, \quad \psi \sim 2D_\psi\sqrt{2}/r, \\ v_{1,2} &\sim 2Q_{1,2}/r, \quad u_{1,2} \sim 2P_{1,2}/r, \\ \chi &\sim -2N/r, \quad \kappa \sim 2A/r, \end{aligned} \tag{48}$$

then the null geodesic condition gives the following BPS bound

$$0 = \frac{1}{8}\mathrm{Tr}(\mathcal{B}^2) = M^2 + N^2 + A^2 + D_\phi^2 + D_\psi^2 - Q_1^2 - Q_2^2 - P_1^2 - P_2^2. \tag{49}$$

The signature of the target-space metric (34) tell us that the most general extremal solution which can be generated via harmonic maps should contains four independent harmonic functions, an explicit construction is yet to be done.

4 Higher rank forms from pure gravity

Inverse dualisation of a KK two-form in D dimensions gives an antisymmetric form of the rank $D-2$. With some luck, one can reinterpret this form as a supergravity matter field thus providing a non-local duality between gravity and supergravity. Before passing some realistic possibilities in the context of string/M theory, here we discuss a modified dimensional reduction scheme starting with the standard reduction of $(D+d)$-dimensional

[6] For application of null geodesics method (harmonic maps) in the supergravity context see [11, 21].

pure gravity theory to D dimensions as given in section 2 and then performing dualisation of the part d_1 of all KK two-forms. It is convenient to introduce the following notation:

$$F = dA = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad g_d = \begin{pmatrix} g_1 & g_1 K \\ K^T g_1 & g_2 + K^T g_1 K \end{pmatrix}, \tag{50}$$

with K being an arbitrary $d_1 \times d_2$ real matrix, and g_1, g_2 real symmetric $d_1 \times d_1$ and $d_2 \times d_2$ matrices respectively. Rescaling them as

$$M_a = g_a^{-\varepsilon_a} e^{\varepsilon_a \psi_a / d_a}, \tag{51}$$

where $e^{\psi_a} = \det g_a$, $a = 1, 2$, $\varepsilon_1 = -\varepsilon_2 = 1$, so that

$$\det M_1 = \det M_2 = 1, \tag{52}$$

one obtains

$$M = e^{-\psi/d} \begin{pmatrix} e^{\psi_1/d_1} M_1^{-1} & e^{\psi_1/d_1} M_1^{-1} K \\ e^{\psi_1/d_1} K^T M_1^{-1} & e^{-\psi_2/d_2} M_2 + e^{\psi_1/d_1} K^T M_1^{-1} K \end{pmatrix}, \tag{53}$$

where $d = d_1 + d_2$ and $\psi = \psi_1 + \psi_2$. Such a decomposition of the moduli matrix (53) and the set of gauge fields can be viewed as a two-step standard KK reduction (see Appendix B). In terms of new variables the D-dimensional Lagrangian of the reduced theory (8) can be written as follows

$$\begin{aligned} \mathcal{L}_D &= R_D - \frac{1}{4}\Big\{ d_1(\nabla\phi_1)^2 + d_2(\nabla\phi_2)^2 - \frac{1}{D+d-2}(d_1\nabla\phi_1 - d_2\nabla\phi_2)^2 \Big\} \\ &+ \frac{1}{4}\mathrm{Tr}(\nabla M_1 \nabla M_1^{-1}) + \frac{1}{4}\mathrm{Tr}(\nabla M_2 \nabla M_2^{-1}) - \frac{1}{2} e^{\phi_1+\phi_2}\mathrm{Tr}(\nabla K^T M_1^{-1} \nabla K M_2^{-1}) \\ &- \frac{1}{4}\Big\{ e^{-\phi_2} F_2^T M_2 F_2 + e^{\phi_1}(F_1 + K F_2)^T M_1^{-1}(F_1 + K F_2) \Big\}, \end{aligned} \tag{54}$$

where

$$\phi_a = \varepsilon_a \left(\frac{1}{d_a}\psi_a + \frac{1}{D-2}\psi \right). \tag{55}$$

Now we dualise the part d_1 of the KK two-forms $F_{1[2]}$ to $(D-2)$-forms, $G_{1[D-2]} = dB_{1[D-3]}$:

$$e^{\phi_1} M_1^{-1}(F_1 + K F_2) = {}^{*_D}G_1, \tag{56}$$

then an equivalent D-dimensional action will read

$$\begin{aligned} \mathcal{L}'_D &= R_D - \frac{1}{4}\Big\{ d_1(\nabla\phi_1)^2 + d_2(\nabla\phi_2)^2 - \frac{1}{D+d-2}(d_1\nabla\phi_1 - d_2\nabla\phi_2)^2 \Big\} \\ &+ \frac{1}{4}\mathrm{Tr}(\nabla M_1 \nabla M_1^{-1}) + \frac{1}{4}\mathrm{Tr}(\nabla M_2 \nabla M_2^{-1}) - \frac{1}{2} e^{\phi_1+\phi_2}\mathrm{Tr}(\nabla K^T M_1^{-1} \nabla K M_2^{-1}) \\ &- \frac{1}{4}\Big\{ e^{-\phi_2} F_2^T M_2 F_2 + \frac{2}{(D-2)!} e^{-\phi_1} G_1^T M_1 G_1 + {}^{*_D}G_1^T K F_2 + F_2^T K^T\, {}^{*_D}G_1) \Big\}. \end{aligned} \tag{57}$$

An initial non-dualised action (54) is invariant under global $GL(d,R)$ symmetry (10), (11) which can be decomposed (see Appendix B (174)-(176)) into *central, right* and *left* subgroups. The central transformation $GL(d_1,R)\times GL(d_2,R)$ explicitly reads

$$\begin{aligned} e^{\varepsilon_a\phi_a} &\to e^{\varepsilon_a\phi_a}(\det\mathcal{G}_a)^{-2\varepsilon_a/d_a}, &(58)\\ M_a &\to \mathcal{G}_a^T M_a\mathcal{G}_a(\det\mathcal{G}_a)^{-2/d_a}, &(59)\\ F_a &\to (\mathcal{G}_a)^{\varepsilon_a}F_a, &(60)\\ K &\to \mathcal{G}_1^T K\mathcal{G}_2, &(61)\end{aligned}$$

where $a=1,2$, $\mathcal{G}_a\in GL(d_a,R)$ and $\varepsilon_1=+1$, $\varepsilon_2=-1$. The right transformation is

$$\begin{aligned} K &\to K+\mathcal{R}, &(62)\\ F_1 &\to F_1-\mathcal{R}F_2, &(63)\end{aligned}$$

(other fields inert). The left one in the sector of KK forms reads

$$F_2\to F_2-\mathcal{L}F_1, \tag{64}$$

with F_1 inert, while the scalars undergo rather complicated transformations which are given in the Appendix B. Here $\mathcal{R}$ and $\mathcal{L}$ — are arbitrary real $d_1\times d_2$ and $d_2\times d_1$ parameter matrices respectively. All these are symmetries of the action (54).

The same symmetries apply to the dualised theory (57), with the difference that only the central subgroup remains an off-shell symmetry. Central transformations are the same as in the non-dualised theory with $G_1\to\mathcal{G}_1^{-1}G_1$ replacing the F_1-transformation. Right and left symmetries hold only for the field equations. This asymmetry is the effect of omitting a divergence term when going from (54) to (57) which is not invariant under left and right transformations. The right transformation now is

$$K\to K+\mathcal{R}, \tag{65}$$

with other fields inert, while the left one is

$$\begin{aligned} G_1 &\to G_1+\mathcal{L}^T\Lambda_1, &(66)\\ F_2 &\to F_2+\mathcal{L}\Lambda_2, &(67)\end{aligned}$$

where $\Lambda_1=K^TG_1-e^{-\phi_2}M_2{}^{*_D}F_2$ and $\Lambda_2=KF_2-e^{-\phi_1}M_1{}^{*_D}G_1$ so that $d\Lambda_1=d\Lambda_2=0$ on-shell. Transformations in the scalar sector are the same as in the non-dualised theory, see Appendix B.

Now the question is whether one can reinterpret the dualised action as originating from some supergravity theory with "matter" antisymmetric forms. To get link with M-theory, one has to generate the four-form from the KK two-form. This picks out six dimensions as the dualisation dimension. One has to start from some higher-dimensional pure gravity theory on space-time with Killing symmetries to generate KK two-forms in six dimensions. For IIB theory the seven-dimensional space-time is selected as giving the five-form field strength, the five-diemsnional to get the three-form and so on. One has analyse suitable compactifications of the eleven and ten-dimensional supergravities and check whether the dilaton coupling constants corresponding to reduction to lower dimensions have the same values as given by the Eq.(57).

5 $D = 11$ supergravity/$D = 8$ Einstein gravity

Let us consider a particular case of the dual theory (57) with $d_1 = d_2 = 1$. Then the matrices M_1 and M_2 trivialize, and the matrix K reduces to a (pseudo)scalar axion κ:

$$\begin{aligned}\mathcal{L}_D^{1,1} &= R_D - \frac{e^{2\phi}}{2}(\nabla\kappa)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{D}{8(D-2)}(\nabla\psi)^2 \\ &- \frac{e^{-\phi}}{2}\left\{\frac{e^{\frac{D}{2(D-2)}\psi}}{2!}F_{[2]}^2 + \frac{e^{-\frac{D}{2(D-2)}\psi}}{(D-2)!}G_{[D-2]}^2\right\} \\ &- \frac{\kappa}{2!(D-2)!}E^{\mu_1\mu_2\nu_1\ldots\nu_{D-2}}F_{\mu_1\mu_2}G_{\nu_1\ldots\nu_{D-2}}. \end{aligned} \tag{68}$$

Here suitable linear combinations of the scalar fields are introduced:

$$\phi_1 = \phi + \frac{D}{2(D-2)}\psi, \qquad \phi_2 = \phi - \frac{D}{2(D-2)}\psi. \tag{69}$$

Now we choose $D = 6$, then the field $G_{[D-2]}$ is a four-form. To get a link with eleven-dimensional supergravity we have to consider the dimensional reduction to six dimensions which generates the appropriate moduli fields. This can be done as follows. Starting with the bosonic sector of $D = 11$ supergravity

$$S_{11} = \int d^{11}x\sqrt{-\hat{G}}\left\{\hat{R}_{11} - \frac{1}{2\times 4!}\hat{F}_{[4]}^2\right\} - \frac{1}{6}\int \hat{F}_{[4]}\wedge\hat{F}_{[4]}\wedge\hat{A}_{[3]}, \tag{70}$$

we assume the following $2+3+6$ decomposition of the metric (with time in the six-dimensional sector)

$$d\hat{s}_{11}^2 = e^{\varphi_2}(dy_1^2 + dy_2^2) + e^{\varphi_3}(dy_3^2 + dy_4^2 + dy_5^2) + e^{-(2\varphi_2+3\varphi_3)/4}ds_6^2, \tag{71}$$

and the potential three-form

$$\hat{A}_{\mu_1\mu_2\mu_3} = B_{\mu_1\mu_2\mu_3}(x), \qquad \hat{A}_{\mu y_1 y_2} = A_\mu(x), \qquad \hat{A}_{y_3y_4y_5} = \kappa(x). \tag{72}$$

The fields φ_2, φ_3, $B_{[3]}$, $A_{[1]}$, κ and the six-metric $ds_6^2 = g_{6\mu\nu}dx^\mu dx^\nu$ depend only on coordinates x^μ, parameterizing the six-space. Note that this class of configurations contains $M2$ (delocalised in three directions) and $M5$-branes as well as $M2\subset 5$-brane [22, 23] and some of intersections [24, 25, 26] including their non-static generalizations [9].

The above ansätz leads to the following six-dimensional action:

$$\begin{aligned} S_6 &= \int d^6x\sqrt{|g_6|}\Big\{R_6 - \frac{e^{2\phi}}{2}(\nabla\kappa)^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{3}{16}(\nabla\psi)^2 \\ &- \frac{1}{2}e^{-\phi}\left[\frac{1}{2!}e^{\frac{3}{4}\psi}F_{[2]}^2 + \frac{1}{4!}e^{-\frac{3}{4}\psi}G_{[4]}^2\right]\Big\} - \int \kappa F_{[2]}\wedge G_{[4]} \\ &+ \frac{1}{3}\int d\left[\kappa\left(B_{[3]}\wedge F_{[2]} + G_{[4]}\wedge A_{[1]}\right)\right], \end{aligned} \tag{73}$$

where

$$G_{[4]} = dB_{[3]}, \quad F_{[2]} = dA_{[1]}, \quad \varphi_2 = \frac{1}{3}\phi - \frac{1}{2}\psi, \quad \varphi_3 = -\frac{2}{3}\phi. \tag{74}$$

and the last term can be omitted as a total divergence. The Chern-Simons term is non-trivial in this reduction. Comparing this action with the Eq.(68) for $D=6$ we see that both theories are precisely the same, including the dilaton couplings. Recall that (68) was obtained by dimensional reduction of the eight-dimensional Einstein gravity. Therefore we find a non-local duality between the $2+3+6$ reduction of the $D=11$ supergravity and $D=8$ pure gravity.

Now we can formulate the following generating technique for constructing solutions of the $D=11$ supergravity starting with some solution of the eight-dimensional Einstein equations admitting two commuting Killing vectors. First, one has to choose coordinates adapted to Killing symmetries (generated by $\partial_{\zeta^m},\, m=1,2$):

$$\begin{aligned} ds_8^2 &= g_{mn}\left(d\zeta^m + A^m_\mu dx^\mu\right)\left(d\zeta^n + A^n_\nu dx^\nu\right) + e^{-\psi/4} g_{\mu\nu}dx^\mu dx^\nu, \\ e^\psi &= \det ||g_{mn}||, \end{aligned} \tag{75}$$

and to perform the following identification of the moduli matrix in temrs of the supergravity fields

$$\begin{aligned} g_{mn} &= e^{\psi/2}\begin{pmatrix} e^\phi & \kappa e^\phi \\ \kappa e^\phi & e^{-\phi}+\kappa^2 e^\phi \end{pmatrix}, \\ dA^m_{[1]} &= F^m_{[2]}. \end{aligned} \tag{76}$$

Then the four-form $G_{[4]} = dB_{[3]}$ has to be built via inverse dualisation of the Killing two-forms

$$G_{\alpha_1\ldots\alpha_4} = \frac{1}{2} e^{\phi+3\psi/4} E_{\alpha_1\ldots\alpha_4}{}^{\mu\nu}\left(F^1_{\mu\nu} + \kappa F^2_{\mu\nu}\right). \tag{77}$$

The desired solution of $D=11$ supergravity will read

$$d\hat{s}^2_{11} = g_2^{1/2}\delta_{ab}dz^a dz^b + g_3^{1/3}\delta_{ij}dy^i dy^j + (g_2 g_3)^{-1/4} g_{6\mu\nu}dx^\mu dx^\nu, \tag{78}$$

where

$$\ln g_2 = \frac{2}{3}\phi - \psi, \qquad \ln g_3 = -2\phi, \tag{79}$$

and the $11D$ four-form $\hat{F}_{[4]}$ is given by

$$\hat{F}_{[4]} = G_{[4]} + F^2_{[2]}\wedge\epsilon_{[2]} + d\kappa\wedge\epsilon_{[3]}. \tag{80}$$

Some new supergravity solutions constructing along these lines will be given in the next sections, see also [9, 27, 28].

The $SL(2,R)$ symmetry of the compactified space translate into the following transformations of the field quantities:

$$z \to \frac{az+b}{cz+d}, \qquad z = \kappa + ie^{-\phi}, \quad ad-bc=1, \tag{81}$$

$$\psi \to \psi + \text{const.}, \tag{82}$$

$$\mathcal{F}_{[2]} \to (cz+d)\mathcal{F}_{[2]}, \qquad \mathcal{F}_{[2]} \equiv e^{\psi/2}F_{[2]} + ie^{-\psi/2}\,{}^{*6}G_{[4]}. \tag{83}$$

This symmetry unifies into a single multiplet the $M2$, $M5$- and $M2\subset 5$-branes compatible with the ansätz; all of them have a single pure gravity counterpart: eight-dimensional plane wave solutions delocalized along one axis:

$$ds_8^2 = F(x)d\zeta_1^2 - 2d\zeta_1 dt + d\zeta_2^2 + \delta_{ij}dx^i dx^j, \qquad i,j = 1,...,5, \tag{84}$$

where $\partial_{\vec{x}}^2 F(x) = 0$. More precisely, this solution corresponds to $M5$-brane (or $M2$-brane after $\zeta_{1,2}\to\zeta_{2,1}$) and $M2\subset 5$ solution can be obtained via linear coordinate transformation

$$\zeta_1 \to \zeta_1\cos\xi, \qquad \zeta_2 \to \frac{\zeta_2}{\cos\xi} + \zeta_1\sin\xi. \tag{85}$$

As one straightforward application let us exploit the $SL(2,R)$ symmetry (58-61, 65-67) to generate new non-static supergravity solutions. Starting with the rotating $M5$-brane [29] (entering into the class of the above configurations with $F_{[2]} = \kappa = 0$ and only $G_{[4]} \neq 0$). Than the left transformation can be used to generate $F_{[2]}$- and κ-components of the M-brane (see the Eqs.(177-180) in the Appendix B). As a result we obtain the rotating composite $M2\subset 5$-brane with two rotation parameters

$$\begin{aligned}
d\hat{s}_{11}^2 &= H^{-2/3}H'^{1/3}\Big\{-Hdt^2 + \frac{2m}{r^3}f\left(\cosh\delta dt - l_1\sin^2\theta d\phi_1 - l_2\cos^2\theta\sin^2\psi d\phi_2\right)^2 \\
&+ dy_1^2 + dy_2^2\Big\} + H^{1/3}H'^{-2/3}\left\{dy_3^2 + dy_4^2 + dy_5^2\right\} \\
&+ H^{1/3}H'^{1/3}\left\{f^{-1}\left[\left(1+\frac{l_1^2}{r^2}\right)\left(1+\frac{l_2^2}{r^2}\right) - \frac{2m}{r^3}\right]^{-1}dr^2 + r^2 d\Xi_4^2\right\}.
\end{aligned} \tag{86}$$

The form field reads

$$\hat{F}_{[4]} = \sin\zeta\, dA_{[3]} + \cos\zeta\, {}^{*11}d(A_{[3]}\wedge dy_3\wedge dy_4\wedge dy_5) + \tan\zeta\, d\left(\frac{1-H'}{H'}dy_3\wedge dy_4\wedge dy_5\right). \tag{87}$$

In these expressions

$$H = 1 + \frac{2m\sinh^2\delta}{r^3}f, \qquad H' = 1 + \frac{2m\cos^2\zeta\sinh^2\delta}{r^3}f, \tag{88}$$

$$f^{-1} = \Upsilon\left(1+\frac{l_1^2}{r^2}\right)\left(1+\frac{l_2^2}{r^2}\right), \tag{89}$$

$$A_{[3]} = \frac{1-H^{-1}}{\sinh\delta}(-\cosh\delta dt + l_1\sin^2\theta d\phi_1 + l_2\cos^2\theta\sin^2\psi d\phi_2)\wedge dy_1\wedge dy_2, \tag{90}$$

where

$$\begin{aligned}
\Upsilon &= \cos^2\theta\cos^2\psi + \frac{\sin^2\theta}{1+l_1^2/r^2} + \frac{\cos^2\theta\sin^2\psi}{1+l_2^2/r^2}, \\
d\Xi_4^2 &= \left(1 + \frac{l_1^2\cos^2\theta}{r^2} + \frac{l_2^2\sin^2\theta\sin^2\psi}{r^2}\right)d\theta^2 + \left(1+\frac{l_2^2\cos^2\psi}{r^2}\right)\cos^2\theta\, d\psi^2 \\
&- \frac{2l_2^2\sin\theta\cos\theta\sin\psi\cos\psi}{r^2}d\theta d\psi + \left(1+\frac{l_1^2}{r^2}\right)\sin^2\theta\, d\phi_1^2 + \left(1+\frac{l_2^2}{r^2}\right)\cos^2\theta\sin^2\psi\, d\phi_2^2.
\end{aligned} \tag{91}$$

Electric and magnetic charges for this solution are $Q\propto 2m\sin\zeta\sinh\delta\cosh\delta$, and $P\propto 2m\cos\zeta\sinh\delta\cosh\delta$. The rotating $M2\subset 5$ solution reduces to the rotating $M5$-brane under $\cos\zeta = 1$ and reduces to the two-rotational-parameters $M2$-brane under $\cos\zeta = 0$.

6 Ten-dimensional supergravities

Let us start with the IIB theory. Its bosonic sector contains the NS-NS fields (metric, B-field and dilaton) and the Ramond-Ramond potentials: the scalar and the two-form (combining into $SL(2,R)$ multiplets with the corresponding NS-NS fields) and the four-form with the self-dual five-form field strength. Here there are essentially two possibilities: one is related to seven dimensions as a dualisation platform, another to five dimensions, this latter possibility works for the IIA theory as well. On the other hand, we need the third Killing vector to accommodate for the Ramond-Ramond sector. So the first option is to consider the ten-dimensional Einstein gravity admitting three commuting Killing symmetries

$$\begin{aligned} ds_{10}^2 &= g_{mn}\left(d\zeta^m + A_\mu^m dx^\mu\right)\left(d\zeta^n + A_\nu^n dx^\nu\right) + e^{-\psi/5} g_{7\mu\nu} dx^\mu dx^\nu, \\ e^\psi &= \det ||g_{mn}||, \qquad g_{mn} = g_1^{-1/2} g_2^{-3/4} \begin{pmatrix} 1 & K^T \\ K & g_2 M + K K^T \end{pmatrix}, \end{aligned} \tag{92}$$

where the label m takes values $m = \#, 1, 2$. In the notation of the Appendix B this corresponds to $D = 7, d_1 = 1, d_2 = 2$ with the six-parametric moduli matrix g_{mn} decomposed as $1 + 2$. In the matrix notation

$$\begin{aligned} ||dA_{[1]}^m|| &= ||F_{[2]}^m|| = ||\mathcal{F}_{[2]}, F_{[2]}^a||^T, \quad K = ||K_a||, \\ M &= ||M_{ab}|| = \begin{pmatrix} e^{-\hat\Phi} + \hat\chi^2 e^{\hat\Phi} & e^{\hat\Phi}\hat\chi \\ e^{\hat\Phi}\hat\chi & e^{\hat\Phi} \end{pmatrix}, \end{aligned} \tag{93}$$

so the moduli give two scale factors g_1, g_2, a doublet $K_a, a = 1, 2$ and a unimodular matrix M parameterized by $\hat\Phi, \hat\chi$. The latter are obvious candidates for the dilaton and the R-R scalar of IIB theory, while three one forms $A_{[1]}^\# \equiv \mathcal{A}_{[1]}$ and $A_{[1]}^a, a = 1, 2$ can be used to generate the five-form and the $SL(2,R)$ doublet of three-forms.

This construction is dual to the following dimensional reduction of the IIB theory. The Einstein frame metric splits as $1 + 2 + 7$

$$d\hat s_{10}^2 = g_1 dy^2 + g_2^{1/2} \delta_{ab} dz^a dz^b + (g_1 g_2)^{-1/5} g_{7\mu\nu} dx^\mu dx^\nu, \tag{94}$$

where the scale factors g_1, g_2 and the seven-dimensional metric are the same as in (92). The form fields are constructed from the quantities (93) via

$$\begin{aligned} \hat F_{[3]}^a &= F_{[2]}^a \wedge dy + \epsilon^{ab} dK_b \wedge \epsilon_{[2]}, \\ \hat F_{[5]} &= G_{[5]} + (g_1 g_2)^{4/5} {}^{*_7} G_{[5]} \wedge dy \wedge \epsilon_{[2]}, \\ G_{[5]} &= (g_1 g_2)^{-4/5} {}^{*_7}(\mathcal{F}_{[2]} + K_a F_{[2]}^a). \end{aligned} \tag{95}$$

The ansätz for $\hat F_{[5]}$ is chosen here in a manifestly self-dual form

$$\hat F_{[5]} = (1 + {}^{*_{10}}) G_{[5]}, \qquad G_{[5]} \equiv dB_{[4]}, \tag{96}$$

and one can check that the field equations and the Bianchi identity for the five-form are equivalent to the pure gravity Einstein equations for the metric (92). It is worth noting that in this case both supergravity and dual gravity start from the same dimension ten.

The simplest solutions of the form (94,95) are the $D3$-brane and D-string (the latter being delocalised along $\{z^a\}$). Our class also includes $D1 \subset D3$ solution which gives rise to $D3-$brane in the NS-NS B field, or, in notation of [33], the $NS1 + D1 + D3$ configurations.

The second possibility is inspired by dimensional reduction from $D = 11$ to IIA supergravity. It is relevant for five-brane configurations and works (with some alterations) both for IIA and IIB theories. The Einstein frame supergravity metric is taken as the $2+3+5$ split

$$d\hat{s}^2_{10} = g_2^{1/2}\delta_{ab}dz^a dz^b + g_3^{1/3}\delta_{ij}dy^i dy^j + (g_2 g_3)^{-1/3} g_{5\mu\nu}dx^\mu dx^\nu, \tag{97}$$

where all functions depend on x^μ. This class of solutions can be reconstructed together with the appropriate form fields from the eight-dimensional Ricci-flat metrics admitting three commuting Killing vectors

$$\begin{aligned} ds_8^2 &= g_{mn}\left(d\zeta^m + A^m_\mu dx^\mu\right)\left(d\zeta^n + A^n_\nu dx^\nu\right) + e^{-\psi/3} g_{5\mu\nu}dx^\mu dx^\nu, \\ e^\psi &= \det ||g_{mn}||, \end{aligned} \tag{98}$$

where now the labeling is $m, n = 1, 2, \#$ with $a, b = 1, 2$ corresponding to the first two values the indices m, n. Different identifications of the metric variables as well as different dualisation prescriptions for three Killing two-forms associated with A^m_μ apply to IIA and IIB supergtavities which we list separately (again denoting $A^\#_{[1]} \equiv \mathcal{A}_{[1]}$).

IIA:

$$\begin{aligned} g_{mn} &= e^{-\phi} g_2^{-1} g_3^{-2/3}\begin{pmatrix} g_2^{1/2} g_3 M + K K^T & K \\ K^T & 1 \end{pmatrix}, \\ ||dA^m_{[1]}|| &= ||F^m_{[2]}|| = ||F^a_{[2]}, \mathcal{F}_{[2]}||^T, \quad K = ||K_a||, \\ M &= ||M_{ab}|| = \begin{pmatrix} e^\phi & e^\phi \kappa \\ e^\phi \kappa & e^{-\phi} + e^\phi \kappa^2 \end{pmatrix}. \end{aligned} \tag{99}$$

The dualisation prescription for the four-form $\hat{F}_{[4]} = d\hat{A}_{[3]} - \hat{F}_{[3]} \wedge \hat{A}_{[1]}$ and the three-form $\hat{F}_{[3]} = d\hat{A}_{[2]}$ are as follows (in terms of the first and the third KK two-forms)

$$\begin{aligned} \hat{F}_{[4]} &= -e^\phi g_2^{-1/2} g_3^{-1\, *_5}(dK_2 - \kappa dK_1) + (F^1_{[2]} + \kappa F^2_{[2]}) \wedge \epsilon_{[2]} - dK_1 \wedge \epsilon_{[3]}, \\ \hat{F}_{[3]} &= -e^{-2\phi} g_2^{-5/3} g_3^{-2/3\, *_5}(\mathcal{F}_{[2]} + K_a F^a_{[2]}) + d\kappa \wedge \epsilon_{[2]}, \end{aligned} \tag{100}$$

while the two-form $\hat{F}_{[2]} = d\hat{A}_{[1]}$ is identified with the second KK field

$$\hat{F}_{[2]} = F^2_{[2]}. \tag{101}$$

The $10D$ dilaton is given by

$$e^{\hat{\Phi}} = g_2^{-1} e^{-2\phi}. \tag{102}$$

IIB:

$$\begin{aligned} g_{mn} &= g_2^{-1/2} g_3^{-1/3} \begin{pmatrix} M^{-1} & M^{-1}K \\ M^{-1}K^T & g_2 + K^T M^{-1} K \end{pmatrix}, \\ ||dA^m_{[1]}|| &= ||F^m_{[2]}|| = ||F_{[2]a}, \mathcal{F}_{[2]}||^T, \quad K = ||K_a||, \\ M^{-1} &= ||M^{ab}||. \end{aligned} \tag{103}$$

To generate the $SL(2,R)$ doublet of three-forms $\hat{F}^a_{[3]} = d\hat{A}^a_{[2]}$ (here $a = 1$ and $a = 2$ correspond to NS-NS and R-R fields respectively) and the self-dual five-form

$$\hat{F}_{[5]} = {}^{*10}\hat{F}_{[5]} = d\hat{A}_{[4]} - \frac{1}{2}\epsilon_{ab}\hat{A}^a_{[2]} \wedge \hat{F}^b_{[3]} \tag{104}$$

one has to dualise the KK fields as follows

$$\begin{aligned} \hat{F}^a_{[3]} &= -(g_2 g_3)^{-2/3} M^{ab\,*5}\left(F_{[2]b} + K_b \mathcal{F}_{[2]}\right) + \epsilon^{ab} dK_b \wedge \epsilon_{[2]}, \\ \hat{F}_{[5]} &= \mathcal{F}_{[2]} \wedge \epsilon_{[3]} - g_2 (g_2 g_3)^{-2/3\,*5}\mathcal{F}_{[2]} \wedge \epsilon_{[2]}, \end{aligned} \tag{105}$$

(here $\epsilon_{12} = \epsilon^{12} = +1$). In this case the $10D$ dilaton $\hat{\Phi}$ and the zero-form $\hat{\chi}$ are expressed through the $SL(2,R)/SO(2)$ matrix M_{ab} as

$$||M_{ab}|| = \begin{pmatrix} e^{-\hat{\Phi}} + \hat{\chi}^2 e^{\hat{\Phi}} & \hat{\chi} e^{\hat{\Phi}} \\ \hat{\chi} e^{\hat{\Phi}} & e^{\hat{\Phi}} \end{pmatrix}. \tag{106}$$

A class of $8D$ plane wave solutions delocalised in two directions corresponds to $D0 + D2 + NS5$-brane in IIA theory and $NS1 + D3 + D2$-brane in IIB theory. Hidden supergravity symmetries unifying these non-marginal solutions in the common multiplets with marginal ones are dual to linear coordinate transformations in the compactified subspace ($\zeta^1, \zeta^2, \zeta^\#$) of the corresponding pure gravity theory.

7 Killing spinor equations

In this section we perform dimensional reduction of the $11D$ supergravity Killing spinor equations according to the ansätz (78,80), and try to express the resulting equations in terms of the dual $8D$ pure gravity. We will find that these equations coincide exactly with the geometric Killing spinor equations in eight dimensions, i.e. equations for covariantly constant spinors. Recall that the existence of Killing spinors exhibits unbroken supersymmetry of the bosonic supergravity solutions, so a purely bosonic relationship between $11D$ supergravity and eight-dimensional pure gravity gravity admits a fermionic extension. Let us start with the $11D$ equation for the 32-component Majorana spinors ϵ_{11} expressing the vanishing of the supersymmetry variation of the gravitino:

$$\hat{D}_M \epsilon_{11} + \frac{1}{288}\left(\Gamma_M{}^{NPQR} - 8\delta^N_M \Gamma^{PQR}\right)\hat{F}_{NPQR}\epsilon_{11} = 0. \tag{107}$$

In this section and in the Appendix C we use gamma-matrices with the orthonormal (flat) frame indices as well as the orthonormal components for all other tensor quantities. As usual, multiindex matrices (Γ^{PQR} etc.) are totally antisymmetric products of $11D$ gamma matrices.

Recall that the basic BPS $M2$, $M5$, and $M2 \subset 5$-branes possessing one-half supersymmetry admit Killing spinors with 16 independent components.

For our three block decomposition of the space-time interval it is convenient to introduce the corresponding $2 \times 3 \times 6$ decomposition of the gamma-matrices (more details are given in the Appendix C):

$$\Gamma_a = \gamma_7 \times \rho_a \times \sigma_0, \qquad \Gamma_i = \gamma_7 \times \rho_z \times \sigma_i, \qquad \Gamma_\alpha = \gamma_\alpha \times \rho_0 \times \sigma_0, \tag{108}$$

where ρ_a, σ_i are two sets of the standard 2×2 Pauli matrices, $a = x, y$, $i = x, y, z$, ρ_0 and σ_0 is the unit matrix. Here γ_α are the $6D$ gamma-matrices in the orthonormal frame: $\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta}$, while γ_7 is the chiral operator $\gamma_7 = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_4\gamma_5$ so that $\gamma_7^2 = 1$ and $\{\gamma_7, \gamma_\alpha\} = 0$. Such a representation corresponds to decomposition of the $SO(1,10)$ spinor with respect to the subgroup $SO(1,5) \times SO(2) \times SO(3)$. An arbitrary $11D$ spinor ϵ_{11} splits into the direct product of four $6D$ spinors according to the pattern $\mathbf{32} = (\mathbf{8}, \mathbf{1}^+, \mathbf{2}) + (\mathbf{8}, \mathbf{1}^-, \mathbf{2})$. We write

$$\epsilon_{11} = (\epsilon_1,\ \epsilon_2,\ \epsilon_3,\ \epsilon_4)^T, \tag{109}$$

where $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$ are eight-component six-dimensional spinors. The Majorana condition on ϵ_{11}

$$\epsilon_{11} = \epsilon_{11}^c = C\epsilon_{11}, \tag{110}$$

with the choice of the conjugation matrix C compatible with (108) translates into the following conditions on $D = 6$ spinors (see Appendix C):

$$\epsilon_3 = iP\epsilon_2^*, \qquad \epsilon_4 = iP\epsilon_1^*, \qquad P = (\tau_z \times \tau_x \times \tau_y). \tag{111}$$

Under this decomposition the Eq. (107) splits into the set of three equations, one of which ($M = \alpha$) is the six-dimensional differential spinor equation (D_α is the corresponding spinor covariant derivative):

$$\begin{aligned} &\Big\{12D_\alpha + {\gamma_\alpha}^\beta\partial_\beta(\phi + \psi/2) + e^{3\psi/8+\phi/2}\gamma_7\Big[-2(\mathbf{F}^1_\alpha + \kappa\mathbf{F}^2_\alpha) + (F^1_\alpha + \kappa F^2_\alpha)\Big] \\ &\quad + is_z\Big[e^\phi\gamma_7({\gamma_\alpha}^\beta\partial_\beta\kappa - 2\partial_\alpha\kappa) + e^{3\psi/8-\phi/2}\left(\mathbf{F}^2_\alpha - 2F^2_\alpha\right)\Big]\Big\}\epsilon_i = 0. \end{aligned} \tag{112}$$

This is applicable to to all of four six-dimensional spinors ($i = 1, 2, 3, 4$) with $s_z = +1$ for ϵ_1 and ϵ_3, and $s_z = -1$ for ϵ_2 and ϵ_4 (in what follows we denote these spinors ϵ_6 to distinguish from spinors in other dimensions). Two other are algebraic equations for ϵ_6 (with the same agreement for s_z) which impose consistency conditions on the bosonic background:

$$\Big\{\partial\!\!\!/(\phi - \psi/2) - is_z\left(e^\phi\gamma_7\ \partial\!\!\!/\kappa + e^{3\psi/8-\phi/2}\mathbf{F}^2\right)\Big\}\epsilon_6 = 0, \tag{113}$$

and

$$\Big\{\partial\!\!\!/(\phi + \psi/2) + e^{3\psi/8+\phi/2}\gamma_7\left(\mathbf{F}^1 + \kappa\mathbf{F}^2\right) - is_z e^\phi\gamma_7\ \partial\!\!\!/\kappa\Big\}\epsilon_6 = 0, \tag{114}$$

where

$$\begin{aligned} \partial\!\!\!/ &= \gamma^\alpha\partial_\alpha = \gamma^\alpha e^\mu_\alpha\partial_\mu, \qquad \mathbf{F}^m = \frac{1}{2}F^m_{\alpha\beta}\gamma^{\alpha\beta}, \\ F^m_\alpha &= \frac{1}{2}[\gamma_\alpha, \mathbf{F}^m] = F^m_{\alpha\beta}\gamma^\beta, \qquad \mathbf{F}^m_\alpha = \frac{1}{2}F^m_{\beta\gamma}{\gamma_\alpha}^{\beta\gamma}. \end{aligned} \tag{115}$$

If one multiplies the Eqs. (113) and (114) by γ_α and makes use of the identity $\gamma_\alpha \mathbf{F}^m = \mathbf{F}^m_\alpha + F^m_\alpha$ one can derive the following two relations

$$is_z e^{3\psi/8-\phi/2}\{\mathbf{F}^2_\alpha + F^2_\alpha\}\epsilon_6 = \left\{\partial_\beta(\phi - \psi/2) + is_z e^{\phi}\gamma_7\partial_\beta\kappa\right\}\left(\delta^\beta_\alpha + \gamma_\alpha{}^\beta\right)\epsilon_6, \tag{116}$$

$$\begin{aligned} &e^{3\psi/8+\phi/2}\gamma_7\left\{(\mathbf{F}^1_\alpha + \kappa\mathbf{F}^2_\alpha) + (F^1_\alpha + \kappa F^2_\alpha)\right\}\epsilon_6 \\ &= \left\{\partial_\beta(\phi + \psi/2) + is_z e^{\phi}\gamma_7\partial_\beta\kappa\right\}\left(\delta^\beta_\alpha + \gamma_\alpha{}^\beta\right)\epsilon_6. \end{aligned} \tag{117}$$

Substituting these into the Eqs. (112) we obtain

$$\begin{aligned} &\Big\{4D_\alpha - \partial_\alpha(\phi/3 + \psi/2) - \gamma_\alpha{}^\beta\partial_\beta\psi/4 + e^{3\psi/8+\phi/2}\gamma_7(F^1_\alpha + \kappa F^2_\alpha) \\ &\quad - is_z\left(e^{\phi}\gamma_7\partial_\alpha\kappa + e^{3\psi/8-\phi/2}F^2_\alpha\right)\Big\}\epsilon_6 = 0. \end{aligned} \tag{118}$$

The set of equations (113), (114) and (118) represents the $11D$ Killing spinor equations in terms of the $6D$ quantities. From these, only the last one is a differential equation on the $6D$ spinors, while two others are constraints on the $6D$ background and the chirality of ϵ_6.

Now we wish to demonstrate that these equations reduce to the single equation for covariantly constant spinors in the dual eight-dimensional empty space-time:

$$D_M\epsilon_8 = \partial_M\epsilon_8 + \frac{1}{4}\hat{\Omega}^{AB}{}_M\mathcal{Y}_{AB}\epsilon_8 = 0, \tag{119}$$

in the case of two commuting Killing symmetries (75). Then the $SO(1,7)$ spinor decomposes as 2×6

$$\epsilon_8 = (\epsilon_+, \epsilon_-)^T, \tag{120}$$

where ϵ_+ and ϵ_- are again some $6D$ spinors: they correspond to a decomposition over the subgroup $SO(1,5) \times SO(2) \subset SO(1,7)$, namely $\mathbf{16} = \mathbf{8}^+ + \mathbf{8}^-$. Accordingly, the $8D$ gamma-matrices are presented as

$$\mathcal{Y}_6 = -\rho_x \times I_8, \qquad \mathcal{Y}_7 = -\rho_y \times \gamma_7, \qquad \mathcal{Y}_\alpha = \rho_x \times \gamma_\alpha. \tag{121}$$

It is convenient to introduce the one-form representation for the eight-dimensional metric

$$\begin{aligned} \hat{\theta}^m &= \mathcal{E}^m_n(d\zeta^n + A^n_\nu dx^\nu) = \theta^m, \\ \hat{\theta}^\alpha &= e^{-\psi/8}\theta^\alpha, \qquad \theta^\alpha = e^\alpha_\nu dx^\nu, \end{aligned} \tag{122}$$

where

$$\mathcal{E} = ||\mathcal{E}^m_n|| = e^{\psi/4}\begin{pmatrix} e^{\phi/2} & \kappa e^{\phi/2} \\ 0 & e^{-\phi/2} \end{pmatrix}. \tag{123}$$

Then the spinor connection reads

$$\begin{aligned} \hat{\Omega}^{mn} &= -\partial_\beta\mathcal{E}^m_p g^{pq}\mathcal{E}^n_q\theta^\beta - \frac{1}{2}\mathcal{E}^m_p\mathcal{E}^n_q\partial_\beta g^{pq}\theta^\beta, \\ \hat{\Omega}^{m\alpha} &= -\frac{1}{2}e^{\psi/8}\mathcal{E}^m_p\mathcal{E}^n_q\partial^\alpha g^{pq}\theta_n + \frac{1}{2}e^{\psi/8}F^{m\alpha\beta}\theta_\beta, \\ \hat{\Omega}^{\alpha\beta} &= \Omega^{\alpha\beta} + \frac{1}{8}(\partial^\alpha\psi\theta^\beta - \partial^\beta\psi\theta^\alpha) - \frac{1}{2}e^{\psi/4}F_n{}^{\alpha\beta}\theta^n, \end{aligned} \tag{124}$$

where $F^m = \mathcal{E}^m_n F^n, ||F^n|| = (F^1_{[2]}, F^2_{[2]})^T, ||g_{mn}|| = \mathcal{E}^T\mathcal{E}, ||g^{mn}|| = ||g_{mn}||^{-1}$. Now the $8D$ equation (119) splits into the set of $M = m$ (algebraic) equations, and $M = \mu$ $6D$ spinor equation. The first two coincide with the equations (113) and (114), obtained earlier from the $11D$ theory if we assume $s_z = +1$ for $\epsilon_6 = \epsilon_+$ and $s_z = -1$ for $\epsilon_6 = \epsilon_-$. The six-dimensional $M = \mu$ equation can be also presented in the form which almost coincides with the $11D$ Killing spinor equation (118) except for the second term. But this term can be eliminated via the substitution

$$\epsilon'_6 = e^{-(2\phi+3\psi)/24}\epsilon_6. \tag{125}$$

This means that the bosonic correspondence between three-block truncation of the $11D$ supergravity and eight-dimensional Einstein gravity extends in a natural way to the supersymmetry condition: the $11D$ Killing spinor equation is equivalent to the covariantly constant spinor equation in eight dimensions (119).

Therefore any Ricci-flat $8D$ space-time with two commuting isometries which admits covariantly constant spinors generates some supersymmetric $11D$ supergravity configuration. It is worth noting that space-times admitting covariantly constant spinors are not always Ricci-flat but only Ricci-null (due to integrability condition for the Eq. (119)), see e.g. [34], so one should demand $R_{8MN} = 0$ in addition to the existence of parallel spinors to generate supersymmetric supergravity solutions from eight-dimensional pure gravity ones. For *static* $8D$ space-times this condition is fulfilled automatically. The correspondence between eight and eleven-dimensional Majorana spinors reads explicitly

$$\epsilon_{11} = e^{(2\phi+3\psi)/24}(\epsilon_8, \epsilon'_8)^T. \tag{126}$$

Here ϵ_8 is a covariantly constant $8D$ spinor (which has 16 independent real components as well as the $11D$ Majorana spinor) and ϵ'_8 has to be constructed starting with ϵ_8 through the Majorana conditions (110,111).

This correspondence may be useful also to simplify the check of supersymmetry for $11D$ supergravity solutions. To investigate whether some three-block $11D$ supergravity solution possesses unbroken supersymmetry it is sufficient to check whether the corresponding eight-dimensional pure gravity dual admits covariantly constant spinors. Each (complex) $8D$ covariantly constant spinor correspond exactly to one Majorana Killing spinor in $11D$ theory.

Another possible application is a generating technique for supersymmetric $11D$ backgrounds. Note that all spinor connection forms (124) are invariant under $GL(2, R)$ transformations (10,11) and consequently $6D$ the spinor equations (113),(114),(118) are invariant too. Therefore one can apply the $GL(2, R)$ transformations (81)-(83) to the seed $11D$ supersymmetric solutions of the form (78),(80) in order to obtain new *supersymmetric* ones.

8 Solution generation

Here we present some examples of application of the $11D/8D$ duality for solution generating purposes. Let us start with the five-dimensional generalization of the rotating

Dobiasch-Maison solution [8] endowed with NUT and smeared to eight dimensions as follows

$$
\begin{aligned}
ds_8^2 &= T\left(dy + A_t dt + A_\phi d\phi\right)^2 + dz_1^2 + dz_2^2 + dz_3^2 - \frac{\Delta - a^2\sin^2\theta}{\Sigma T}\left(dt - \cosh\delta\,\omega d\phi\right)^2 \\
&+ \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta\sin^2\theta}{\Delta - a^2\sin^2\theta}d\phi^2\right), \\
A_t &= \frac{\sinh 2\delta\left(mr + an\cos\theta + n^2\right)}{\Sigma T}, \\
A_\phi &= -\frac{2\sinh\delta\left[n\Delta\cos\theta + a\sin^2\theta(mr + n^2)\right]}{\Sigma T},
\end{aligned}
\tag{127}
$$

where

$$\Delta := r^2 - 2mr + a^2 - n^2, \tag{128}$$

$$\Sigma := r^2 + (a\cos\theta + n)^2, \tag{129}$$

$$\omega := \frac{2n\Delta\cos\theta + 2a\sin^2\theta(mr + n^2)}{a^2\sin^2\theta - \Delta}, \tag{130}$$

$$T := 1 + 2\sinh^2\delta(mr + an\cos\theta + n^2)\Sigma^{-1}. \tag{131}$$

First we have to present the seed solution (127) in the desired form (75) and to read off the variables from (76). Here there are two possibilities to identify the variables according to different ordering of two Killing vectors. One possibility is

$$\phi = -\frac{1}{2}\ln T, \quad \psi = \frac{3}{4}\ln T, \quad A^1_{[1]} = A_t dt + A_\phi d\phi. \tag{132}$$

In this case $G_{[4]}$ vanishes and the scale factors g_2 and g_3 are

$$g_2 = T^{-4/3}, \qquad g_3 = T. \tag{133}$$

The resulting $11D$ metric and the four-form field read as follows:

$$
\begin{aligned}
ds_{11}^2 &= T^{-2/3}\left[-\frac{\Delta - a^2\sin^2\theta}{\Sigma}\left(dt - \cosh\delta\,\omega d\phi\right)^2 + dy_1^2 + dy_2^2\right] \\
&+ T^{1/3}\left[\sum_{k=3}^{7} dy_k^2 + \Sigma\left(\frac{dr^2}{\Delta} + d\theta^2 + \frac{\Delta\sin^2\theta}{\Delta - a^2\sin^2\theta}d\phi^2\right)\right], \\
\hat{A}_{t12} &= A_t, \qquad \hat{A}_{\phi 12} = A_\phi.
\end{aligned}
\tag{134}
$$

This is the NUT generalization of the rotating $M2$-brane with one rotation parameter, which is delocalized on five of eight transverse coordinates.

An alternative ordering of Killing vectors gives

$$\phi = \frac{1}{2}\ln T, \quad \psi = \frac{3}{4}\ln T, \quad A^2_{[1]} = A_t dt + A_\phi d\phi. \tag{135}$$

The four-form $G_{[4]}$ is generated via inverse dualisation, the corresponding three-form potential $B_{[3]}$ has the following non-zero components

$$
\begin{aligned}
B_{tz_2z_3} &= 2\sinh\delta\,[m(a\cos\theta+n)-nr]\,\Sigma^{-1}, \\
B_{\phi z_2z_3} &= -\sinh 2\delta\Big\{m\cos\theta+(2n\cos\theta-a\sin^2\theta)\,[nr-m(a\cos\theta+n)]\,\Sigma^{-1}\Big\}.
\end{aligned} \tag{136}
$$

Extracting the scale factors g_2 and g_3

$$
g_2 = T^{-2/3}, \qquad g_3 = T^{-1}, \tag{137}
$$

we finally obtain the following $11D$ supergravity solution:

$$
\begin{aligned}
ds_{11}^2 &= T^{-1/3}\left[-\frac{\Delta-a^2\sin^2\theta}{\Sigma}(dt-\cosh\delta\,\omega d\phi)^2+\sum_{k=1}^{5}dy_k^2\right] \\
&+ T^{2/3}\left[dy_6^2+dy_7^2+\Sigma\left(\frac{dr^2}{\Delta}+d\theta^2+\frac{\Delta\sin^2\theta}{\Delta-a^2\sin^2\theta}d\phi^2\right)\right], \\
\hat{A}_{t67} &= B_{tz_2z_3}, \qquad \hat{A}_{\phi 67} = B_{\phi z_2z_3}.
\end{aligned} \tag{138}
$$

This is the NUT generalization of the rotating $M5$-brane with one rotation parameter, which is delocalized on two of five transverse coordinates.

The same solutions can be obtained applying similar procedure to the five-dimensional rotating NUT-ed Gross-Perry-Sorkin monopole [8] smeared to eight dimensions. Finally, we can apply the *left* transformations (186,187) to the above rotating $M5$-brane with the following parameters: $r=\cot\zeta, l=-\sin\zeta\cos\zeta$ and $s=2\ln(\sin\zeta)$. This leads to NUT-ed rotating composite $M2\subset 5$-brane (dyon):

$$
\begin{aligned}
ds_{11}^2 &= T^{1/3}T'^{1/3}\Big\{T^{-1}\Big[-\frac{\Delta-a^2\sin^2\theta}{\Sigma}(dt-\cosh\delta\,\omega d\phi)^2+dy_1^2+dy_2^2\Big] \\
&+ T'^{-1}\left(dy_3^2+dy_4^2+dy_5^2\right)+dy_6^2+dy_7^2+\Sigma\left(\frac{dr^2}{\Delta}+d\theta^2+\frac{\Delta\sin^2\theta}{\Delta-a^2\sin^2\theta}d\phi^2\right)\Big\}, \\
T' &= 1+2\cos^2\zeta\sinh^2\delta(mr+an\cos\theta+n^2)\Sigma^{-1}, \\
\hat{A}_{t12} &= \sin\zeta\;A_t, \qquad \hat{A}_{\phi 12} = \sin\zeta\;A_\phi, \\
\hat{A}_{345} &= \tan\zeta\left(T'^{-1}-1\right), \\
\hat{A}_{t67} &= \cos\zeta\;B_{tz_2z_3}, \qquad \hat{A}_{\phi 67} = \cos\zeta\;B_{\phi z_2z_3}.
\end{aligned} \tag{139}
$$

This solution reduces to rotating $M5$-brane if $\cos\zeta=1$ and to $M2$-brane when $\cos\zeta=0$.

Now let us explore which of these solutions preserves unbroken supersymmetry. As was shown in the previous section, it is sufficient to check whether the seed $8D$ metric (127) admits covariantly constant spinors. From the Eq. (119) for $M=y$ we obtain in the asymptotic region $r\to\infty$

$$
\sinh\delta\,\{m(\sinh\delta\,\mathcal{Y}_{ry}-\cosh\delta\,\mathcal{Y}_{rt})-n\,\mathcal{Y}_{\theta\phi}\}=0. \tag{140}
$$

This implies the following necessary condition

$$
(m^2+n^2)\sinh^2\delta=0. \tag{141}
$$

So we conclude that $\delta \to \infty$, $me^{2\delta}$, $ne^{2\delta}$ and a are finite and $8D$ covariantly constant spinor takes the form

$$\epsilon_8 = T^{-1/4} e^{-\phi/2} e^{-\varphi \mathcal{Y}_{\theta\phi}/2} \epsilon_0, \qquad \mathcal{Y}_{yt}\, \epsilon_0 = \epsilon_0, \tag{142}$$

where ϵ_0 is constant spinor and $\tan\varphi = r\tan\theta/\sqrt{r^2+a^2}$.

In this case (134) takes the form of extremal $M2$-brane solution with harmonic function $T = 1 + 2q/r, (a = n = 0, m \to 0, m\sinh^2\delta = q)$ presented in polar coordinate system:

$$\begin{aligned} ds^2_{11} &= T^{-2/3}(-dt^2 + dy_1^2 + dy_2^2) + T^{1/3}\left(\sum_{k=3}^{7} dy_k^2 + dr^2 + d\theta^2 + r^2\sin^2\theta d\phi^2\right), \\ \hat{A}_{[3]} &= (1 - T^{-1}) dt \wedge dy_1 \wedge dy_2. \end{aligned} \tag{143}$$

Other two solutions (138,139) represent in this limit extremal $M5$- and $M2 \subset 5$-brane respectively.

Another interesting example is applying the *left* transformation to the following $M5 \perp M5$-brane

$$\begin{aligned} ds^2_{11} &= (H_1H_2)^{2/3}\left\{(H_1H_2)^{-1}\left(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right) + H_1^{-1}\left(dy_1^2 + dy_2^2\right)\right. \\ &+ \left. H_2^{-1}\left(dz_1^2 + dz_2^2\right) + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right\}, \\ \hat{A}_{\phi z_1 z_2} &= -2p_1\cos\theta, \qquad \hat{A}_{\phi y_1 y_2} = -2p_2\cos\theta, \end{aligned} \tag{144}$$

where

$$H_1 = 1 + \frac{2p_1}{r}, \qquad H_2 = 1 + \frac{2p_2}{r}. \tag{145}$$

With the choice of parameters $r = \cot\zeta, l = -\sin\zeta\cos\zeta$ and $s = 2\ln(\sin\zeta)$, one obtains the "dyonic intersecting M–brane"

$$\begin{aligned} ds^2_{11} &= \left(H_1H_2\tilde{H}\right)^{2/3}\left\{-(H_1H_2)^{-1}dt^2 + \tilde{H}^{-1}\left(dx_1^2 + dx_2^2 + dx_3^2\right)\right. \\ &+ \left. H_1^{-1}\left(dy_1^2 + dy_2^2\right) + H_2^{-1}\left(dz_1^2 + dz_2^2\right) + dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right\}, \\ \hat{A}_{ty_1y_2} &= \frac{2p_1\sin\zeta}{r}H_1^{-1}, \qquad \hat{A}_{\phi z_1 z_2} = -2p_1\cos\zeta\cos\theta, \\ \hat{A}_{tz_1z_2} &= \frac{2p_2\sin\zeta}{r}H_2^{-1}, \qquad \hat{A}_{\phi y_1 y_2} = -2p_2\cos\zeta\cos\theta, \\ \hat{A}_{x_1x_2x_3} &= \tan\zeta\left(1 - \tilde{H}^{-1}\right), \end{aligned} \tag{146}$$

where

$$\tilde{H} = \sin^2\zeta + \cos^2\zeta H_1H_2. \tag{147}$$

The same solution can be obtained by applying the same transformation to the $M2\perp M2$-brane. For $\sin\zeta = 0$ the solution reduces to $M2\perp M2$-brane and for $\cos\zeta = 0$ reduces to $M5\perp M5$-brane. This solution was found previously in [26].

9 Discussion

Our aim was to investigate whether one can find new links between gravity and supergravities in higher dimensions using inverse dualisation to generate antisymmetric forms from Kaluza-Klein vectors. Apparently this is an impossible task for the full non-abridged supergravities, but as we have shown, it works for certain consistent truncations. While it is perhaps not surprising that the Maxwell equations and the Bianchi identities for the KK fields translate into similar equations for dual higher rank forms, a non-trivial test is whether the dilatonic exponents in the reduced actions are the same. We have found that it is so in several cases. The most interesting is the correspondence between $2+3+6$ dimensional reduction of the eleven-dimensional supergravity and eight-dimensional Einstein gravity with two commuting Killing vectors. A related duality holds between both (suitably compactified) IIA and IIB ten-dimensional supergravities and eight-dimensional Einstein gravity with three commuting Killing vectors. Another case is the correspondence between the ten-dimensional Einstein gravity and a suitably compactified IIB theory. It is worth noting that all dualities of this sort are non-local in the sense that variables of one theory are related to variables of the dual theory not algebraically, but via solving differential equations.

A remarkable fact is that the $11D$-supergravity/$8D$-gravity duality holds not only in the bosonic sector, but also extends to Killing spinor equations exhibiting unbroken supersymmetries of the $11D$ theory. Namely, the existence of Killing spinors in the supergravity framework is equivalent to the existence of covariantly constant spinors in the dual Einstein gravity. It would be interesting to check whether this correspondence found at the linearized level extends non-linearly, i.e. holds for suitably supersymmetrized $8D$ gravity. A more challenging question is whether classical dualities found here have something to do with quantum theories. Although we were not able to give an answer here, our results concerning the ten-dimensional supergravities look promising in this direction.

As a direct application one can use the above dualities for solution generation purposes similarly to the Bonnor map in general relativity. Fortunately, the truncations considered leave enough space for p-brane solutions including dyonic states, rotating and NUT-ed configurations, as well as various brane intersections. We were able to find hitherto unknown branes (including supersymmetric ones) applying new dualities for generating classical solutions. Recently there was an upsurge of interest to *fluxbranes* or Fp-branes, which are the generalizations of the magnetic fluxtube (Melvin solution). It is worse noting that the first genuine fluxbrane solution with higher rank antisymmetric forms was constructed by a method similar to the present one [11]. The method described here was also used to construct intersecting fluxbranes [27] one of which was later rediscovered by Russo and Tseytlin [50, 51].

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A Alternative reduction to $D=4$ and $D=3$ (partial dualisation)

Here we give details of the alternative reduction scheme based on partial dualisation of KK two-forms to $D=4$ and 3. Let us start with the standard dimensional reduction to four dimensions, using decompositions (50)-(55) with $\gamma=(2+d)/2d$:

$$
\begin{aligned}
\mathcal{L}_4 &= R_4-\frac{1}{4}\Big\{d_1(\nabla\phi_1)^2+d_2(\nabla\phi_2)^2-\frac{1}{d+2}(d_1\nabla\phi_1-d_2\nabla\phi_2)^2\Big\}\\
&+ \frac{1}{4}\mathrm{Tr}(\nabla M_1\nabla M_1^{-1})+\frac{1}{4}\mathrm{Tr}(\nabla M_2\nabla M_2^{-1})-\frac{1}{2}e^{\phi_1+\phi_2}\mathrm{Tr}(\nabla K^TM_1^{-1}\nabla KM_2^{-1})\\
&- \frac{1}{4}\Big\{e^{-\phi_2}F_2^TM_2F_2+e^{\phi_1}(F_1+KF_2)^TM_1^{-1}(F_1+KF_2)\Big\}.
\end{aligned}
\tag{148}
$$

Now we wish to dualise a part d_1 of the full set of d Maxwell two-forms as follows ($\tilde{F}\equiv {}^{*4}F$)

$$
e^{\phi_1}M_1^{-1}(F_1+KF_2)=\tilde{G}_1, \tag{149}
$$

leaving the remaining d_2 forms unchanged. The equivalent (dual) action involves d_2 KK potentials $A^{m_2}_{[1]}$ and d_1 dual potentials $B_{[1]m_1}$, $dB=G_{[2]}$:

$$
\begin{aligned}
\mathcal{L}'_4 &= R_4-\frac{1}{4}\Big\{d_1(\nabla\phi_1)^2+d_2(\nabla\phi_2)^2-\frac{1}{d+2}(d_1\nabla\phi_1-d_2\nabla\phi_2)^2\Big\}\\
&+ \frac{1}{4}\mathrm{Tr}(\nabla M_1\nabla M_1^{-1})+\frac{1}{4}\mathrm{Tr}(\nabla M_2\nabla M_2^{-1})-\frac{1}{2}e^{\phi_1+\phi_2}\mathrm{Tr}(\nabla K^TM_1^{-1}\nabla KM_2^{-1})\\
&- \frac{1}{4}\Big\{e^{-\phi_2}F_2^TM_2F_2+e^{-\phi_1}G_1^TM_1G_1+\tilde{G}_1^TKF_2+F_2^TK^T\tilde{G}_1\Big\}.
\end{aligned}
\tag{150}
$$

Note the presence of Chern-Simons densities in the action. Let us discuss symmetries of the transformed action. In deriving this expression the total divergence was omitted which is not invariant under right and left subgroups of the full $GL(d,R)$ symmetry group. Therefore, only the central subgroup $GL(d_1,R)\times GL(d_2,R)$ will be the symmetry of the action, the rest of the group being the symmetry of the equations of motion.

Now compactify the dualised action (150) further to three dimensions. One can expect to have the σ-model possessing an enhanced symmetry $SL(d+2,R)$ (here $d=d_1+d_2$ refers to four-dimensional theory) as before. Let us write the four-metric as

$$
ds_4^2=g_{\mu\nu}dx^\mu dx^\nu=-f(dt-\varpi_idx^i)^2+f^{-1}h_{ij}dx^idx^j, \tag{151}
$$

and introduce the columns of electric and magnetic potentials

$$
\begin{aligned}
G_{1it} &= \partial_iv_1, \qquad F_{2it}=\partial_iv_2, &(152)\\
M_1G_1^{ij} &= \frac{-fe^{\phi_1}}{\sqrt{h}}\epsilon^{ijk}(\partial_ku_1-K\partial_kv_2), &(153)\\
M_2F_2^{ij} &= \frac{-fe^{\phi_2}}{\sqrt{h}}\epsilon^{ijk}(\partial_ku_2-K^T\partial_kv_1). &(154)
\end{aligned}
$$

Define the twist potential χ via

$$\tau_i = \partial_i \chi + \frac{1}{2} v_a^T \partial_i u_a - \frac{1}{2} u_a^T \partial_i v_a, \quad a = 1, 2, \tag{155}$$

where τ is dual to the $3D$ KK vector ϖ_i

$$\tau^i = \frac{f^2}{\sqrt{h}} \epsilon^{ijk} \partial_j \varpi_k \tag{156}$$

in accordance with the part of Einstein equations

$$R_{4t}{}^i = \frac{f}{2\sqrt{h}} \epsilon^{ijk} \partial_j \tau_k. \tag{157}$$

As a result we arrive at the σ-model

$$S_3 = \int d^3x \sqrt{h} \left\{ R_3 + \frac{1}{4} \mathrm{Tr} \left(\nabla \mathcal{M} \nabla \mathcal{M}^{-1} \right) \right\}, \tag{158}$$

with the following target space metric

$$\begin{aligned} dl_\sigma^2 &= -\frac{1}{4} \left(d\mathcal{M} d\mathcal{M}^{-1} \right) \\ &= \frac{df^2 + d\tilde{\chi}^2}{2f^2} + \frac{1}{4} \Big\{ d_1 d\phi_1^2 + d_2 d\phi_2^2 - \frac{1}{2+d} (d_1 d\phi_1 - d_2 d\phi_2)^2 \Big\} \\ &- \frac{1}{4} \mathrm{Tr}(dM_1 dM_1^{-1}) - \frac{1}{4} \mathrm{Tr}(dM_2 dM_2^{-1}) + \frac{1}{2} e^{\phi_1 + \phi_2} \mathrm{Tr}(dK^T M_1^{-1} dK M_2^{-1}) \\ &- \frac{1}{2f} \Big\{ e^{-\phi_1} dv_1^T M_1 dv_1 + e^{-\phi_2} dv_2^T M_2 dv_2 + e^{\phi_1} dw_1^T M_1^{-1} dw_1 + e^{\phi_2} dw_2^T M_2^{-1} dw_2 \Big\}, \end{aligned} \tag{159}$$

where

$$d\tilde{\chi} = d\chi + \frac{1}{2} v_a^T du_a - \frac{1}{2} u_a^T dv_a, \tag{160}$$

$$dw_1 = du_1 - K dv_2, \tag{161}$$

$$dw_2 = du_2 - K^T dv_1. \tag{162}$$

In deriving this expression we made use of the following $(d_1+1) \times (d_2+1)$ split of the coset matrix

$$\begin{aligned} \mathcal{M} &= \begin{pmatrix} 1 & 0 \\ Q^T & 1 \end{pmatrix} \begin{pmatrix} P_1^{-1} & 0 \\ 0 & P_2 \end{pmatrix} \begin{pmatrix} 1 & Q \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} P_1^{-1} & P_1^{-1} Q \\ Q^T P_1^{-1} & P_2 + Q^T P_1^{-1} Q \end{pmatrix}, \end{aligned} \tag{163}$$

where the P_1, P_2 are the real $d_1 \times d_1$ and $d_2 \times d_2$ symmetric matrices with the same determinant, and Q is the real $d_1 \times d_2$ matrix. The trace decomposes as follows

$$\mathrm{Tr} \left(d\mathcal{M} d\mathcal{M}^{-1} \right) \equiv \mathrm{Tr} \left(dP_1 dP_1^{-1} + dP_2 dP_2^{-1} - 2dQ P_2^{-1} dQ^T P_1^{-1} \right), \tag{164}$$

while the matrices P_1, P_2 and Q in terms of previously introduced variables read

$$
\begin{aligned}
P_1 &= e^{\chi_1}\begin{pmatrix} fe^{\phi_1}-v_1^TM_1v_1 & -v_1^TM_1 \\ -M_1v_1 & -M_1 \end{pmatrix},\\
P_2 &= e^{\chi_2}\begin{pmatrix} fe^{\phi_2}-v_2^TM_2v_2 & -v_2^TM_2 \\ -M_2v_2 & -M_2 \end{pmatrix},\\
Q &= \begin{pmatrix} \frac{1}{2}\xi-\chi & \varphi_2^T \\ \varphi_1 & -K \end{pmatrix},
\end{aligned}
\tag{165}
$$

where

$$\xi = v_a^T\varphi_a, \qquad a=1,2, \tag{166}$$

$$\varphi_1 = u_1-Kv_2, \quad \varphi_2=u_2-K^Tv_1, \tag{167}$$

$$\chi_a = -\phi_a+\frac{1}{d+2}(d_1\phi_1-d_2\phi_2). \tag{168}$$

It is easy to see that the target space (159) is a symmetric space $SL(d+2,R)/SO(d,2)$ as in the standard reduction scheme. But the representation of the coset matrices now is given in terms the blocks $(d_1+1)\times(d_1+1)$, $(d_2+1)\times(d_2+1)$, $(d_1+1)\times(d_2+1)$ and $(d_2+1)\times(d_1+1)$ with arbitrary $0\le d_1,d_2\le d$, instead of the block structure $1\times(d+1)$ in the standard scheme (22).

B Two-step reduction from $D+d_1+d_2$ to D

Our split (50) may be regarded as a two-step dimensional reduction: first from $(D+d_1+d_2)$ dimensions to $(D+d_2)$, second to final D. This corresponds to the following parameterization of the metric

$$
\begin{aligned}
ds^2_{(D+d)} &= g_{m_1n_1}(dz^{m_1}+K^{m_1}_{m_2}dy^{m_2}+\mathcal{A}^{m_1}_\mu dx^\mu)(dz^{n_1}+K^{n_1}_{n_2}dy^{n_2}+\mathcal{A}^{n_1}_\nu dx^\nu)\\
&+ g_{m_2n_2}(dy^{m_2}+\mathcal{B}^{m_2}_\mu dx^\mu)(dy^{n_2}+\mathcal{B}^{n_2}_\nu dx^\nu)+e^{-\psi/(D-2)}g_{D\mu\nu}dx^\mu dx^\nu,
\end{aligned}
\tag{169}
$$

where $m_1,n_1=1,...,d_1$, $m_2,n_2=1,...,d_2$ and $\mu,\nu=1,...,D$. Introducing matrices

$$g_1 = ||g_{m_1n_1}||, \quad g_2=||g_{m_2n_2}||, \quad g_D=||g_{D\mu\nu}||, \tag{170}$$

$$K = ||K^{m_1}_{m_2}||, \quad \mathcal{A}=||\mathcal{A}^{m_1}_\mu||, \quad \mathcal{B}=||\mathcal{B}^{m_2}_\mu||, \tag{171}$$

one can write

$$g_{D+d}=||g_{MN}||=\begin{pmatrix} g_1 & g_1K & g_1\mathcal{A} \\ K^Tg_1 & g_2+K^Tg_1K & g_2\mathcal{B}+K^Tg_1\mathcal{A} \\ \mathcal{A}^Tg_1 & \mathcal{B}^Tg_2+\mathcal{A}^Tg_1K & e^{-\psi/2}g_D+\mathcal{A}^Tg_1\mathcal{A}+\mathcal{B}^Tg_2\mathcal{B} \end{pmatrix}, \tag{172}$$

$$F=dA=d\begin{pmatrix} \mathcal{A}-K\mathcal{B} \\ \mathcal{B} \end{pmatrix}. \tag{173}$$

Consider the action of the $SL(d,R)\times R$ symmetry on this representation. First we combine (10, 11) into the single $GL(d,R)$ transformation

$$\mathcal{M}\to G^T\mathcal{M}G, \qquad F\to G^{-1}F, \tag{174}$$

where

$$\begin{aligned} \mathcal{M} &= e^{\gamma\psi}M\in GL(d,R)/SO(d),\\ G &= e^{\gamma r/2}\Omega\in GL(d,R). \end{aligned} \tag{175}$$

For G it is convenient to use the Gauss decomposition into the product of *left* (triangle), *center* (block-diagonal) and *right* (triangle) matrices

$$G=\begin{pmatrix}1&0\\ \mathcal{L}&1\end{pmatrix}\begin{pmatrix}\mathcal{G}_1^{-1}&0\\ 0&\mathcal{G}_2\end{pmatrix}\begin{pmatrix}1&\mathcal{R}\\ 0&1\end{pmatrix} \tag{176}$$

Then we obtain a natural decomposition of the $GL(d,R)$ symmetry into the following three subgroup:

central: see (58-61), $GL(d_1,R)\times GL(d_2,R)$;

right: $K\to K+\mathcal{R}$, $F_1\to F_1-\mathcal{R}F_2$, other fields inert;

left: $F_2\to F_2-\mathcal{L}F_1$, F_1 being inert and

$$e^{-\phi_2}M_2+e^{\phi_1}K^TM_1^{-1}K \ \text{ invariant}, \tag{177}$$

$$e^{-\phi_1}M_1+e^{\phi_2}KM_2^{-1}K^T \ \text{ invariant}, \tag{178}$$

$$e^{\phi_1}M_1^{-1}\to e^{\phi_1}(1+K\mathcal{L})^TM_1^{-1}(1+K\mathcal{L})+e^{-\phi_2}\mathcal{L}^TM_2\mathcal{L}, \tag{179}$$

$$e^{\phi_1}M_1^{-1}K\to e^{\phi_1}(1+K\mathcal{L})^TM_1^{-1}K+e^{-\phi_2}\mathcal{L}^TM_2. \tag{180}$$

In the dual terms the symmetries are decomposed in a similar way:

central (dual): $G_1\to\mathcal{G}_1^{-1}G_1$, other variables as in (58-61);

right (dual): $K\to K+\mathcal{R}$, other quantities inert;

left (dual): scalar sector transforms as in (177)-(180) and

$$G_1 \ \to\ (1+K\mathcal{L})^TG_1-e^{-\phi_2}\mathcal{L}^TM_2\tilde{F}_2, \tag{181}$$

$$F_2 \ \to\ (1+\mathcal{L}K)F_2-e^{-\phi_1}\mathcal{L}M_1\tilde{G}_1. \tag{182}$$

The Bianchi identities $dG_1=dF_2=0$ hold on shell only. When $d_1=d_2=1$, M_1 and M_2 trivialize and the left (dual) transformations reduce to

$$\kappa \ \to\ \mathcal{D}\left[(1+\kappa l)\kappa+le^{-2\phi}\right], \tag{183}$$

$$e^{-\phi_1} \ \to\ \mathcal{D}e^{-\phi_1}, \tag{184}$$

$$e^{-\phi_2} \ \to\ \mathcal{D}e^{-\phi_2}, \tag{185}$$

where $\mathcal{D}^{-1}=(1+\kappa l)^2+l^2e^{-2\phi}$, $2\phi=\phi_1+\phi_2$. This can be rewritten as

$$\begin{aligned} z^{-1} &\to z^{-1}+l, \qquad z=\kappa+ie^{-\phi},\\ \frac{3}{2}\psi &= \phi_1-\phi_2 \ \text{ invariant}. \end{aligned} \tag{186}$$

Gauge fields transform as follows:

$$\begin{aligned} G_{[4]} &\to (1+\kappa l)G_{[4]}-e^{-\phi_2}l\tilde{F}_{[2]},\\ F_{[2]} &\to (1+\kappa l)F_{[2]}-e^{-\phi_1}l\tilde{G}_{[4]}. \end{aligned} \tag{187}$$

C Majorana spinor conditions

Four six-dimensional spinors which represent a $11D$ spinor are not independent because of the Majorana condition on ϵ_{11} (110). To find the explicit relations we choose the following representation for $6D$ gamma-matrices γ_α in terms of the Pauli matrices (τ_x, τ_y, τ_z):

$$\begin{aligned} \gamma_0 &= i(\tau_x \times \tau_0 \times \tau_0), \qquad \gamma_1 = (\tau_y \times \tau_0 \times \tau_0), \qquad \gamma_2 = (\tau_z \times \tau_x \times \tau_0), \\ \gamma_3 &= (\tau_z \times \tau_y \times \tau_0), \qquad \gamma_4 = (\tau_z \times \tau_z \times \tau_x), \qquad \gamma_5 = (\tau_z \times \tau_z \times \tau_y), \end{aligned} \tag{188}$$

so that $\gamma_7 = (\tau_z \times \tau_z \times \tau_z)$. In this representation the $11D$ gamma matrices read

$$\begin{aligned} \Gamma_\alpha &= \gamma_\alpha \times \rho_0 \times \sigma_0, \qquad \Gamma_6 = \gamma_7 \times \rho_x \times \sigma_0, \qquad \Gamma_7 = \gamma_7 \times \rho_y \times \sigma_0, \\ \Gamma_8 &= \gamma_7 \times \rho_z \times \sigma_x, \qquad \Gamma_9 = \gamma_7 \times \rho_z \times \sigma_y, \qquad \Gamma_{10} = \gamma_7 \times \rho_z \times \sigma_z, \end{aligned} \tag{189}$$

and obey the identities

$$\Gamma_0^+ = -\Gamma_0, \qquad \Gamma_M^+ = \Gamma_M, \qquad \Gamma_M^T = (-1)^{M+1}\Gamma_M. \tag{190}$$

The charge conjugation matrix C satisfying

$$C^{-1}\Gamma_M C = -\Gamma_M^T, \qquad C^T = -C, \qquad C^+C = I, \tag{191}$$

can be chosen as

$$C = -\Gamma_0\Gamma_2\Gamma_4\Gamma_6\Gamma_8\Gamma_{10} = (\tau_y \times \tau_x \times \tau_y) \times \rho_x \times \sigma_y. \tag{192}$$

Then the Majorana condition $\epsilon = \epsilon^c = C\Gamma_0^T\epsilon^j$ translates into the following equation on the $6D$ spinors $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4$:

$$\begin{pmatrix} \epsilon_3 \\ \epsilon_4 \end{pmatrix} = i\rho_x \times P \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix}^* \tag{193}$$

where $P = (\tau_z \times \tau_x \times \tau_y)$. Therefore any $11D$ Majorana Killing spinor ϵ_{11} may be expressed through exactly one $8D$ covariantly constant spinor $\epsilon_8 = (\epsilon_1, \epsilon_2)^T$ via (126).

References

[1] B. Julia, *Group disintegrations*, in: *Superspace and Supergravity*, Eds. S.W. Hawking and M. Rocek (Cambridge Univ. Press, 1981), 331.

[2] E. Cremmer, B. Julia, H. Lü and C.N. Pope, *Dualisation of dualities, I*, *Nucl. Phys.* **B523** (1998) 73-144; hep-th/9710119.

[3] E. Cremmer, B. Julia, H. Lü and C.N. Pope, *Dualisation of dualities, II: Twisted self–duality of doubled fields and superdualities*, *Nucl. Phys.* **B535** (1998) 242-292; hep-th/9806106.

[4] J. Ehlers, in: *Les Theories Relativistes de la Gravitation*, (CNRS, Paris, 1959), 275.

[5] G. Neugebauer and D. Kramer, *Ann. der Physik (Leipzig)* **24** (1969), 62.

[6] W. Bonnor, *Z. Phys.* **161** (1961) 439; *Z. Phys.* **190** (1966) 444.

[7] D. Gal'tsov, A. Garcia and O. Kechkin, *Symmetries of the stationary Einstein-Maxwell-dilaton system Class. Quant. Grav.* **12** (1995) 2887-2903;

[8] C.-M. Chen, D.V. Gal'tsov, K. Maeda and S.A. Sharakin, *$SL(4,R)$ generating symmetry in five–dpmensional gravity coupled to dilaton and three-form*, *Phys. Lett.* **B453** (1999) 7-16; `hep-th/9901130`.

[9] C.-M. Chen, D.V. Gal'tsov and S.A. Sharakin, *Vacuum interpretation for supergravity M-branes*, *Phys. Lett.* **B475** (2000) 269-274; `hep-th/9908133`; *Einstein Gravity — Supergravity Correspondence*, in: *Proceeding of the 4th International Workshop on Gravitation and Astrophysics* (Beijing, China, 10-15 October 1999), eds. L. Liu, J. Luo, X.-Z. Li and J.-P. Hsu, (World Scientific, Singapore, 2000) 244-251; `hep-th/9912127`.

[10] K.S. Stelle, *Lectures on supergravity p–branes*, `hep-th/9710119`.

[11] D.V. Gal'tsov and O.A. Rytchkov, *Generating branes via sigma–models*, *Phys. Rev.* **D58** (1998) 122001; `hep-th/9801160`.

[12] I. Bars and C. Kounnas, *Theories with two times*, *Phys. Lett.* **B402** (1997) 25-32; `hep-th/9703060`

[13] H. Nishino, *Supergravity in $10+2$ dimensions as consistent background for superstring*, *Phys. Lett.* **B428** (1998) 85-94; `hep-th/9703214`

[14] D. Maison, *Ehlers-Harrison type transformations for Jordan's extended theory of gravitation*, *Gen. Rel. Grav.* **10** (1979) 717-723.

[15] D.V. Gal'tsov and O.A. Kechkin, *U-duality and symplectic formulation of dilaton–axion gravity*, *Phys. Rev.* **D54** (1996) 1656-1666; `hep-th/9507005`.

[16] G. Clement and D.V. Gal'tsov, *Bertotti-Robinson type solutions to dilaton-axion gravity*, *Phys. Rev.* **D** (2001) 124011; `hep-th/0102025`.

[17] D.V. Gal'tsov and S.A. Sharakin, *Matrix Ernst potentials for EMDA with multiple vector fields*, *Phys. Lett.* **B399** (1997) 250-257; `hep-th/9702039`.

[18] P. Dobiasch and D. Maison, *Stationary, spherically symmetric solutions of Jordan's unified theory of gravity and electromagnetism*, *Gen. Rel. Grav.* **14** (1982) 231-242.

[19] D.J. Gross and M.J. Perry, *Magnetic monopoles in Kaluza–Klein theories*, *Nucl. Phys.* **B226** (1983) 29-48.

[20] R.D. Sorkin, *Kaluza–Klein monopole*, *Phy. Rev. Lett.* **51** (1983) 87-90.

[21] G. Clement and D.V. Gal'tsov, *Stationary BPS solutions to dilaton-axion gravity*, *Phys. Rev.* **D54** (1996) 6136-6152; `hep-th/9607043`.

[22] M.B. Green, N. Lambert, G. Papadopoulos and P.K. Towsend, *Dyonic p-branes from self-dual $(p+1)$-branes*, *Phys. Lett* **B384** (1996) 86-92; hep-th/9605146.

[23] G. Papadopoulos, *The universality of M-branes*, hep-th/9611029.

[24] M.S. Costa, *Composite M-branes*, *Nucl. Phys.* **B490** (1997) 202-216; hep-th/9609181.

[25] M.S. Costa, *Black composite M-branes*, *Nucl. Phys.* **B495** (1997) 195-205; hep-th/9610138.

[26] M.S. Costa and M. Cvetič, *Non threshold D-brane bound states and black holes with non-zero entropy*, *Phys. Rev.* **D56** (1997) 4834-4843; hep-th/9703204.

[27] C.-M. Chen, D.V. Gal'tsov and S.A. Sharakin, *Intersecting M-fluxbranes*, *Grav. Cosmol.* **5** (1999) 45-48; hep-th/9908132.

[28] C.-M. Chen, *Dyonic wormholes in $5D$ Kaluza-Klein theory*; gr-qc/0009042.

[29] M. Cvetič and D. Youm, *Rotating intersecting M-branes* *Nucl. Phys.* **B499** (1997) 253-282; hep-th/9612229.

[30] N. Itzhaki, J.M. Maldacena, J. Sonnenschein and S. Yankielowicz, *Supergravity and large N limit of theories with sixteen supercharges*, *Phys. Rev.* **D58** (1998) 046004; hep-th/9802042.

[31] A. Hashimoto and N. Itzhaki, *Non-commutative Yang-Mills and the AdS/CFT correspondenece*, *Phys. Lett.* **B465** (1999) 142-147; hep-th/9907166.

[32] J.M. Maldacena and J.G. Russo, *Large N limit of non-commutative gauge theories*, hep-th/9908134.

[33] A.A. Tseytlin, *Composite BPS configurations of p-branes in 10 and 11 dimensions*, *Class. Quant. Grav.* **14** (1996) 2085-2105; hep-th/9702163.

[34] J.M. Figueroa-O'Farrill, *Breaking the M-waves*, *Class. Quant. Grav.* **17** (2000) 2925-2948; hep-th/9904124.

[35] M.J. Duff and K.S. Stelle, *Multimembrane solutions of $D=11$ Supergravity*; *Phys. Lett.* **B253** (1991) 113-118.

[36] R. Güven, *Black p-brane solutions of $D=11$ supergravity theory*; *Phys. Lett.* **B276** (1992) 49-55.

[37] P.K. Townsend, *The eleven-dimensional supermembrane revisited*, *Phys. Lett.* **B350** (1995) 184-187; hep-th/9601068.

[38] D. Rasheed, *The rotating dyonic black holes of Kaluza-Klein theory*, *Nucl. Phys.* **B454** (1995) 379-401; hep-th/9505038.

[39] G. Clément, *Stationary solutions in five-dimensional general relativity*, *Gen. Rel. Grav.* **18** (1986) 137-160.

[40] G. Clément, *Solutions of five-dimensional general relativity without spatial symmetry*, *Gen. Rel. Grav.* **18** (1986) 861-877.

[41] G.W. Gibbons, *Antigravitating black hole solutions with scalar hair in $N = 4$ supergravity*, *Nucl. Phys.* **B207** (1982) 337-349.

[42] G.W. Gibbons and K. Maeda, *Black holes and membranes in higher dimensional theories with dilaton fields*, *Nucl. Phys.* **B298** (1988) 741-775.

[43] P. Breitenlohner and D. Maison, *On nonlinear σ-models arizing in (super-)gravity*, *Commun. Math. Phys.* **209** (2000) 785-810; `gr-qc/9806002`.

[44] P. Breitenlohner, D. Maison and G. Gibbons, *4-Dimensional black holes from Kaluza-Klein theories*, *Commun. Math. Phys.* **120** (1988) 295-333.

[45] H. Lü, C.N. Pope, T.A. Tran and K.-W. Xu, *Classification of p-branes, NUTs, waves and intersections*, *Nucl. Phys.* **B511** (1998) 98-154; `hep-th/9708055`.

[46] G.W. Gibbons and P.K. Townsend, *Vacuum interpolation in supergravity via super p-branes*, *Phys. Rev. Lett.* **71** (1993) 3754-3757; `hep-th/9307049`.

[47] J.H. Horne and G.T. Horowitz, *Rotating dilaton black holes*, *Phys. Rev.* **D46** (1992) 1340-1346; `hep-th/9203083`

[48] V. Frolov, A. Zelnikov and U. Bleyer, *Charged rotating black holes from five-dimensional point of view*, *Ann. der Physik (Leipzig)* **44** (1987) 371-377.

[49] G. Gibbons and D. Wiltshire, *Black holes in Kaluza-Klein theory*, *Ann. Phys.* **167** (1986) 201-223; Erratum: *ibid* **176** (1987) 393.

[50] J.G. Russo and A.A. Tseytlin, *Magnetic backgrounds and tachyonic instabilities in closed superstring theory and M-theory*, `hep-th/0104238`.

[51] A.A. Tseytlin, *Magnetic backgrounds and tachyonic instabilities in closed string theory*, `hep-th/0108140`.