M-Theory on a Supermanifold

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Abstract

A conjectured finite M-theory based on eleven-dimensional supergravity formulated in a superspace with a non-anticommutative $\Diamond$-product of field operators is proposed. Supermembranes are incorporated in the superspace $\Diamond$-product formalism. When the deformed supersymmetry invariant action of eleven-dimensional supergravity theory is expanded about the standard supersymmetry invariant action, the spontaneously compactified M-theory can yield a four-dimensional de Sitter spacetime inflationary solution with dark energy described by the four-form F-fields. A fit to the present cosmological data for an accelerating universe is obtained from matter fields describing the dominant dark matter and the four-form F-field dark energy. Chiral fermions are obtained from the M-theory by allowing singularities in the compact internal seven-dimensional space. The possibility of obtaining a realistic M-theory containing the standard model is discussed.

The great tragedy of Science – the slaying of a beautiful hypothesis by an ugly fact.

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1 Introduction

For almost thirty years the idea that there exists a supersymmetry between fermion and boson particles has been one of the basic motivations for constructing a unified theory of gravitation and particle interactions. This culminated in the formulation of string theories and their dual relationships and M-theory, from which the five basic string theories emanate by compactification [1, 2].

The description of the fermions of the standard model with the group structure $SU(3) \times SU(2) \times U(1)$ in Kaluza-Klein and supergravity theories became problematic due to Witten's application of the Atiyah-Hirzbruch chiral index theorem [3, 4, 5, 6, 7]. The possibility of using a smooth internal space with continuous isometries was prevented by a no-go theorem, and spelt the end of interest in a conventional interpretation of Kaluza-Klein theories and eleven-dimensional supergravity theory. Superstring theory in ten dimensions now came to the forefront and began to dominate the particle physics and unification scene, since the compactifications of string theory did not suffer the fate of the chiral index no-go theorem, and promised to resolve the problem of the consistency of quantum gravity by producing finite calculations of graviton scattering amplitudes.

The advent of duality theorems and the discovery that an eleven-dimensional M-theory with N=1 supersymmetry should constitute the fundamental unified theory with string theory evolving from a compactification of the eleventh dimension, raised again the question as to why nature would stop at ten dimensions in its use of supersymmetry. The maximum supersymmetry was obtained in eleven dimensions – not ten – as in string theory. Beyond eleven dimensions, there ceased to be a supersymmetry and if one entertained two time dimensions severe causality problems would ensue. M-theory has eleven-dimensional supergravity theory as a low energy limit with its associated type IIA string theory and mass spectrum, so eleven-dimensional supergravity rose like the 'phoenix from the ashes' and became again a subject of intensive research.

Now the questions arise: What constitutes an M-theory? How many degrees of freedom does M-theory possess? One important fact must be considered, namely, that *there are no strings* in eleven-dimensional M-theory, because the only boson components of the action are the three-form potential A_{MNP} where $M, N, P = 0, 1...10$ and the four-form F_{MNPQ} obtained from the curl of A_{MNP}. So we cannot rely on the finiteness of string theory to guarantee a finite eleven-dimensional M-theory. However, supermembranes can couple to A_{MNP} and F_{MNPQ}, so these higher-dimensional objects can play a fundamental role in eleven dimensions. But such branes cannot be quantized in perturbation theory, because the standard method of string perturbation theory quantization does not work. Each term in string perturbation theory corresponds to a two-dimensional worldsheet with an increasing number of holes. A sum over all topologies of the world sheet is performed. But for surfaces with more than two dimensions– as encountered with branes– we do not know how to do this. Powerful theorems in mathematics preclude the possibility of doing this, since we are unable to classify surfaces with $p > 3$. Attempts have been made to quantize the supermembrane in a non-covariant light front gauge [8]. The results indicate that the supermembrane has a continuous spectrum leading to instability.

Treating branes as fundamental objects raises the unpleasant question of how to perform non-perturbative quantization. So the issue of how to implement *finiteness* of calculations of amplitudes in M-theory and quantum gravity has returned in a new guise. The question arises whether the resolution of the finiteness problem in eleven dimensions would lead to new physics beyond string theory on compactification of the correct M-theory.

Hořava and Witten [9] performed an orbifold compactification of M-theory along a finite length of the eleventh dimension and obtained a ten-dimensional bulk with two (9+1) branes, each with an $E8$ heterotic string gauge structure. It was shown that this picture could be compacitfied to a five-dimensional bulk-brane scheme and this led to the phenomenological Randall-Sundrum scenario [10]. Due to the existence of boundaries in the bulk world and associated discontinuities at the junction of bulk-brane, the chiral no-go theorem was avoided. However, we are still left with the unanswered question of the meaning of the original eleven-dimensional M-theory.

One suggestion for an M-theory was to introduce noncommuting eleven-dimensional coordinates in the form of $N \times N$ matrices and construct a Lagrangian with noncommuting coordinates in a light-front gauge and then take the $N \to \infty$ limit to obtain a super Yang-Mills gauge structure [11]. However, this theory is tied to the specific light-front gauge and it is not clear how it produces a finite eleven-dimensional theory. A rigorous proof that it produces a gauge-free eleven-dimensional supergravity theory at low energies is lacking.

We shall propose that M-theory is structured on an eleven-dimensional supermanifold of superspace coordinates with both commuting 'boson' and anticommuting 'fermion' Grassmann coordinates [12, 13, 14, 15]. An associated operator Hilbert space has noncommuting and non-anticommuting coordinates and we map this operator space to the space of ordinary superspace coordinates and fields by a $\circ$-product of field operators or a $\diamondsuit$-product of field operators. The eleven-dimensional supergravity action is now invariant under a generalized 'deformed' group of supersymmetry transformations. This formalism leads to a finite and unitary perturbation theory in eleven dimensions. The noncommutative quantum field theory, based on the familiar 'bosonic' commuting coordinates x^M, cannot by itself lead to a renormalizable or finite M-theory.

Recent observations of supernovae and the cosmic microwave background have led to the mounting evidence that the universe is presently undergoing an accelerating expansion [18, 19]. This means that either there is a positive, small cosmological constant or a slowly varying quintessence field energy (dark energy) [20] that dominates the present universe. This has led to a crisis in string theory and M-theory, since it is difficult to invoke a positive cosmological constant and the associated de Sitter (dS) spacetime in supersymmetric theories that produce stable and consistent solutions. This is particularly true for eleven-dimensional M-theory and supergravity and ten-dimensional string theory. Moreover, eternal quintessence acceleration and dS spacetimes lead to future cosmological event horizons which prevent the construction of an S-matrix theory – a serious problem for string theory. In early constructions of eleven-dimensional supergravity, it was realized that supersymmetric vacua preferred a spontaneous compactification resulting in a four-dimensional anti-de Sitter (AdS) spacetime and a compact internal seven-dimensional compact space [21, 22, 23, 40]. This compacitification naturally selected the dimensions of spacetime to be four.

A spontaneous compactification of our M-theory to a product space $M_4 \times M_7$, where M_4 is a maximally symmetric Friedmann-Robertson-Walker spacetime and M_7 is a compact Einstein space, can produce solutions of the cosmological field equations which possess AdS and dS solutions in four dimensions as well as a solution corresponding to a zero four-dimensional and seven-dimensional cosmological constant. We postulate that the four-form field strengths of supergravity describe the 'dark energy' associated with quintessence and the vacuum energy.

When we take the limit of standard eleven-dimensional supergravity and its associated supersymmetry transformations, we obtain on compactification the familiar result of the product of an AdS spacetime and a positively curved seven-dimensional Einstein space. By including matter fields obtained from a compactification of our M-theory to four dimensions and the vacuum dark energy solution with a small positive cosmological constant λ_4, we can obtain a fit to the present cosmological data describing an accelerating universe.

An old problem in eleven-dimensional supergravity is obtaining a satisfactory isometry group on compactification that can include the standard model $SU(3) \times SU(2) \times U(1)$. In the next section, we shall review the problem of obtaining chiral fermion representations and non-Abelian gauge fields from compactifications of eleven-dimensional M-theory.

2 The Chiral Fermion Problem

Let us review the origin of the chiral fermion no-go theorem. Compact internal spaces are characterized by an index [3] which determines the number of chiral fermions. The nature and application of this index in the context of dimensional reduction has been discussed by Witten and Wetterich [4, 5] and in the context of non-Riemannian geometry in ref. [7]. Consider an operator D acting on spinors containing first derivatives of the spinor and non-derivative terms and that this operator commutes with existing gauge transformations and anticommutes with Γ_d, the d-dimensional internal space gamma matrix. The chiral index is defined by

$$N_C(D) = n_C^+ - n_C^- - n_{\bar{C}}^+ + n_{\bar{C}}^-, \tag{1}$$

where n_C^+ denotes the number of zero modes in the d-dimensional internal space of the Weyl spinor ψ^+ belonging to the complex representation C and with the same kind of meaning for $n_C^-, n_{\bar{C}}^+$ etc. We suppose that we perform a dimensional reduction on the $(d+4)$ space to a Dirac operator $\mathcal{D}$, so that D is the mass operator and N_C counts the number of chiral four-dimensional left-handed fermion generations in the C representation.

We denote by e_M^A the D-dimensional vielbein where M, N denote the world space coordinates and A, B the tangent space coordinates. If the viebein has an inverse

$$e_A^M e_M^B = \delta_A^B, \quad e_M^A e_A^N = \delta_M^N, \tag{2}$$

then we have

$$\Gamma^M D_M \psi = g^{MN} e_{NA} \Gamma^A D_M \psi, \tag{3}$$

where the metric g_{MN} may also have an inverse

$$g_{MN} g^{NP} = \delta_M^P. \tag{4}$$

The internal d-dimensional Dirac operator is

$$\Gamma^m D_m \psi = g^{\Sigma\Delta} e_{\Delta m} \Gamma^m D_\Sigma \psi. \tag{5}$$

If the vielbein e^m_Δ, where m, n and Δ, Σ denote the internal tangent space and space indices, respectively, is invertible everywhere, then the operator $\Gamma^\Delta D_\Delta$ is an elliptic operator. The index N_C remains invariant under continuous changes of the metric, the vielbein and the connections for elliptic operators that preserve the isometries of the space. On compact spaces we have [4]

$$N_C(D) = N_C(D + aO). \tag{6}$$

Here, a is arbitrary and O is an operator satisfying

$$[\mathcal{G}, O] = 0, \quad \{\Gamma_d, O\} = 0, \tag{7}$$

where $\mathcal{G}$ denotes the generators of the isometry group. For everywhere invertible vielbeins the chiral index $N_C(D) = 0$ and spinors have a vector-like behaviour in the compact internal space.

We must allow for singularities in our internal compact space in order to obtain chiral fermions on dimensional reduction. This has led recently to a study of G_2 holonomy spaces which contain conical singularites [24, 25, 26].

There are no complex spinor representations in an odd-dimensional space, and in particular in the eleven-dimensional space of M-theory. The chiral fermions must arise from a compactification of the eleven-dimensional M-theory space $M_{11} = M_4 \times M_7$. For the choice $M_7 = S^7$, we cannot permit a singularity within the smooth seven sphere S^7.

If we gauge the supergravity internal space by allowing for non-compact groups such as $SO(p, r)$, then we find that there can exist singular points in the group space manifold and the vielbeins are not invertible at these singular points, leading to a non-vanishing chiral index and chiral fermions. These possibilities were recognized soon after the discovery of the no-go theorems [5, 6, 7] but they were not pursued in earnest, because superstrings again became the subject of dominant interest. The non-compact versions of internal spaces suffer from instabilities and the lack of a physical mass spectrum.

Superstrings have extra gauge fields within their higher-dimensional spaces and these automatically guarantee a non-vanishing chiral index N_C. Indeed, conventional Kaluza-Klein theories can be extended to include extra gauge fields [27] but such models were considered unattractive, because they ruined the pure higher-dimensional gravity approach of Kaluza-Klein theories and supergravity theories.

The proposed solution of the chiral fermion problem in M-theory by Acharya and Witten [24] compactifies M-theory on a seven manifold X of G_2 holonomy that preserves the supersymmetric ground state. It leads naturally to a four-dimensional theory with $N = 1$ supersymmetry. When the manifold X is smooth, then one obtains Abelian gauge groups only without chiral fermions. However, the singular manifolds of G_2 holonomy offer the possibility of generating non-Abelian gauge groups and chiral fermions. In codimension six, M-theory singularities can occur in Calabi-Yau threefolds, which can be embedded in a manifold of G_2 holonomy. There are no known complete classifications of G_2 holonomy spaces. Nevertheless, the known isolated singularities of G_2-manifolds are conical singularities, and some of them generate chiral fermions and certain special models of this kind lead to anomaly cancellation. Acharya and Witten [24] have investigated examples constructed by taking the quotient of a conical hyper-Kahler eight-manifold by a $U(1)$ symmetry group. The compactification to four-dimensional spacetime leads to a flat Minkowski spacetime with vanishing cosmological constant. At present, it is not known how to compactify a seven manifold to a four-dimensional de Sitter spacetime with chiral fermions and non-Abelian gauge fields.

3 The Accelerating Universe and de Sitter Spaces

Much of the research in particle physics over the past thirty years has focussed on fundamental interactions and their unification in relation to supersymmetry. This is certainly true for string theory and M-theory. We know that supersymmetry must be badly broken for energies less than about $1-10$ TeV, so we are already confronted with the need to break supersymmetry at low energies. There is no well-defined theory for doing this and this has been a thorn in the side of string theory research for a long time. The issue of supersymmetry breaking has been exacerbated recently with the cosmological observations based on supernovae and of the anisotropy of the cosmic microwave background, which suggest that the universe entered a stage of accelerated expansion a few billion years after the big bang [18, 19]. Standard inflationary models are based on the idea that the energy-momentum tensor is dominated by the potential energy of a scalar field, $V(\phi)$, with $V > 0$ [28]. The scalar field is assumed to be slowly rolling with a flat potential with $\dot{\phi}^2 \ll V(\phi)$ and the limiting case $\dot{\phi} = 0$ corresponds to a cosmological constant with $V > 0$. When M/string theory is compactified on an internal compact space, then a supersymmetric vacuum will produce a stable AdS (3+1) space (with negative cosmological constant) and an Einstein seven-dimensional space with positive curvature. When we attempt to extend these solutions to (3+1) dS spacetimes and Einstein spaces with negative curvature, then the solutions are unstable for supersymmetric vacua and even for broken supersymmetric vacua [29, 30].

Recently, investigations have centered on obtaining stable dS spacetimes in extended four-dimensional supergravity theories. Whereas such solutions may be obtainable, they do not help us in making progress in string theory or higher-dimensional unification theories such as M-theory. This has led to a renewed interest in studying gauged higher-dimensional supergravity theories with non-compact internal spaces and finite volume [31]. However, such theories immediately run into serious difficulties. To be physically realistic and correspond to a stable universe, they must possess a discrete mass spectrum with a mass gap. Such internal non-compact spaces have been studied by Nicolai and Wetterich [32], who found that it is possible to obtain normalizable wave functions associated with a discrete spectrum when appropriate boundary conditions are imposed. However, a more serious problem arises due to the kinetic energies of scalar and antisymmetric gauge fields possessing the *wrong sign*, as is to be expected for non-compact spaces. These anomalous kinetic terms violate unitarity and cause the theories to be unstable and unphysical. One might argue that at the scale of the Hubble distance H_0^{-1}, such violations of unitarity can be ignored but this is courting trouble! What is inherently unphysical will stay to haunt the innocent researcher and invariably lead to unpleasant unphysical behaviour.

4 Complementary Chiral Symmetry and de Sitter Space Problems

In our search for a consistent unified description of fundamental forces including gravity based on an M-theory in eleven dimensions, we are currently caught in a dilemma. To guarantee a physical chiral fermion mass spectrum, we would prefer to compactify the M-theory spontaneously within a compact space with orbifold singularities or isolated singularities such as conical singularities. However, when we attempt to obtain stable dS solutions within a supersymmetric scheme or within a broken supersymmetric framework (dS solutions in M-theory would break most or all of the supersymmetry), then we fail to do so within a conventional compact internal space compactification. We could avoid the chiral index no-go theorem by attempting gauged supergravity models with non-compact internal group structures, thereby, finding dS and inflationary solutions but these may still not be stable for broken supersymmetry ground states and they suffer from serious unitarity violations with their related instabilities.

There is still the third problem that if we succeed in deriving a stable dS solution in M-theory with chiral fermions associated with the compactification, then we are faced with potential generic future cosmological horizons which make any attempt to formulate a physical S-matrix doomed to failure [33, 34]. Since string theory is a first quantized S-matrix theory and any conceivable M-theory formulated within scenarios based on modern physics is only sensibly formulated within standard S-matrix theory, then we are confronted with an uncontrollable foundation for the theory of everything!

The prospect of having to formulate a theory of quantum gravity in de Sitter space is unpleasant to say the least. It is hard to define meaningful physical operators in de Sitter space, there is no unique vacuum in de Sitter space and there is the problem of constructing a field theory with only a finite number of degrees of freedom. A glance at a possible quantum gravity theory based on perturbation theory, shows that spin 2 propagators are unwieldy in x-space and it is hard to make sense of Green's functions in momentum space, for Fourier transforms have to be performed on operator spaces that do not possess a time-like or null-like infinity and the operators do not form a complete set of observables in the usual sense. Much of the progress in physics since Newton, has been made using some form of perturbation theory. Little or no progress has been made in studies of non-perturbative quantum field theory. However, it might seem that ultimately a non-perturbative route towards a workable M-theory is needed.

5 Supermanifolds

In general, we can parameterize the superspace coordinates as $\rho^{\bar{M}} = (x^y, \beta^\alpha)$ where the x^y are the commuting spacetime coordinates that belong to a smooth topological manifold and the β^α is an anticommuting spinor field, $\{\beta^\alpha, \beta^\sigma\} = 0$.

Associated with our supervector space $\mathcal{S}$ is the $\mathcal{S}$-parity mapping $\mathcal{P} : \mathcal{S} \to \mathcal{S}$ acting on an arbitrary supervector $\vec{X} = \vec{X}_c + \vec{X}_a$ by the rule [35, 36]:

$$\mathcal{P}(\vec{X}) = \vec{X}_c - \vec{X}_a. \tag{8}$$

For a pure supervector $\vec{X}$, we have

$$\mathcal{P}(\vec{X}) = (-1)^{\epsilon(\vec{X})}\vec{X}, \tag{9}$$

where $\epsilon(\vec{X}) = 0$ for a c-number commuting supervector and $\epsilon(\vec{X}) = 1$ for an anticommuting supervector.

For real superspaces $\Re^{p,q}$ where p and q denote the dimensions of the x and β coordinates, we can introduce general coordinate transformations. Consider the one-to-one mapping of $\Re^{p,q}$ on itself. This is

$$\rho^M \to \rho'^M = f^M(\rho). \tag{10}$$

We shall restrict ourselves to supersmooth transformations and the condition for an invertible transformation is that the supermatrix

$$H^N_M = \partial_M f^N(\rho) \tag{11}$$

is non-singular at each point $\rho^M \in \Re^{p,q}$. The set of all invertible supersmooth transformations (11) forms a group called the supergroup of diffeomorphism transformations. We can also have superalgebras $\mathcal{A}$ which are Z_2-graded linear spaces $\mathcal{S} = \mathcal{S}_c \oplus \mathcal{S}_a$ with the multiplication law

$$[A, B\} = A \cdot B - (-1)^{\epsilon(A)\epsilon(B)} B \cdot A, \tag{12}$$

where A and B are arbitrary pure elements and $[..., ...\}$ is the graded Lie superbracket. The super-Jacobi identities are

$$(-1)^{\epsilon(A)\epsilon(C)}[A, [B, C\}\} + (-1)^{\epsilon(B)\epsilon(A)}[B, [C, A\}\} + (-1)^{\epsilon(C)\epsilon(B)}[C, [A, B\}\} = 0. \tag{13}$$

For arbitrary elements A, B and C we have

$$[A, B\} = [A_c, B_c] + [A_c, B_a] + [A_a, B_c] + \{A_a, B_a\}. \tag{14}$$

For super Lie algebras $\mathcal{A}$ one can connect a super Lie group $\mathcal{G}$ by the exponential rule: $\mathbf{g}(\xi^i) = \exp(\mathbf{a})$ where $\mathbf{a} = \xi^i e_i$, e_i is the basis for $\mathcal{G}$, $\mathbf{a}$ are the super Lie algebra elements and the ξ^i $(i = 1, 2, ..., \dim \mathcal{G})$ are the components of the Lie algebra elements which play the role of local coordinates in the neighborhood of the identity of the group manifold [36]. We have a generalized Poincaré supergroup and superalgebra which are extensions of the corresponding standard Poincaré group and algebra.

6 M-Theory Based on Noncommutative and Non-Anticommutative Geometry

If we believe that there exists a physical, unified description of fundamental interactions described by an eleven-dimensional M-theory, then we must seek an eleven-dimensional structure that produces a finite M-theory, compactifies to a (3+1) spacetime and a seven-dimensional space with chiral fermions and contains solutions to the field equations which describe a stable de Sitter space. Such a scenario should lead to a finite quantum gravity in four dimensions and give a realistic description of cosmology and contain the standard model of quarks and leptons. The superspace contemplated can allow a generalization of the standard supersymmetry transformations associated with eleven-dimensional supergravity and a generalized supersymmetric ground state. These new supersymmetry transformation laws can be considered as deformations of the standard classical supersymmetry transformations. We shall show that such a generalization of the supersymmetric invariance group of supergravity can allow four-dimensional de Sitter solutions.

Instead of pursuing a non-perturbative quantization of supermembranes in eleven dimensions, we shall introduce a superspace with supercoordinates [12, 13, 14, 15]

$$\rho^M = x^M + \beta^M, \tag{15}$$

where the x^M are eleven-dimensional commuting coordinates and β^M are Grassmann anticommuting coordinates, so they are even and odd elements of a Grassmann algebra, respectively.

Both noncommutative and non-anticommutative geometries can be unified within the superspace formalism using the $\circ$-product of two operators $\hat{f}$ and $\hat{g}$ [12, 13, 14, 15]:

$$\begin{aligned}(\hat{f} \circ \hat{g})(\rho) &= \Big[\exp\Big(\frac{1}{2}\omega^{MN}\frac{\partial}{\partial\rho^M}\frac{\partial}{\partial\eta^N}\Big)f(\rho)g(\eta)\Big]_{\rho=\eta} \\ &= f(\rho)g(\rho) + \frac{1}{2}\omega^{MN}\partial_M f(\rho)\partial_N g(\rho) + O(|\omega|^2),\end{aligned} \tag{16}$$

where $\partial_M = \partial/\partial\rho^M$ and ω^{MN} is a nonsymmetric tensor

$$\omega^{MN} = \tau^{MN} + i\theta^{MN}, \tag{17}$$

with $\tau^{MN} = \tau^{NM}$ and $\theta^{MN} = -\theta^{NM}$. Moreover, ω^{MN} is Hermitian symmetric $\omega^{MN} = \omega^{\dagger MN}$, where $\dagger$ denotes Hermitian conjugation. The familiar commutative coordinates of spacetime are replaced by the superspace operator relations

$$[\hat{\rho}^M, \hat{\rho}^N] = 2\beta^M\beta^N + i\theta^{MN}, \tag{18}$$

$$\{\hat{\rho}^M, \hat{\rho}^N\} = 2x^M x^N + 2(x^M\beta^N + x^N\beta^M) + \tau^{MN}. \tag{19}$$

In the limits that $\beta^M \to 0$ and $|\tau^{MN}| \to 0$, we get the familiar expression for noncommutative coordinate operators

$$[\hat{x}^M, \hat{x}^N] = i\theta^{MN}. \tag{20}$$

In the limits $x^M \to 0$ and $|\theta^{MN}| \to 0$, we obtain the Clifford algebra anticommutation relation

$$\{\hat{\beta}^M, \hat{\beta}^N\} = \tau^{MN}. \tag{21}$$

We shall use the simpler non-anticommutative geometry obtained when $\theta^{MN} = 0$ to construct the M-theory, because it alone can lead to a finite and unitary quantum field theory and quantum gravity theory [12, 13, 14, 15]. In the non-anticommutative field theory formalism, the product of two operators $\hat{f}$ and $\hat{g}$ has a corresponding $\diamondsuit$-product

$$(\hat{f}\diamondsuit\hat{g})(\rho) = \biggl[\exp\biggl(\frac{1}{2}\tau^{MN}\frac{\partial}{\partial\rho^M}\frac{\partial}{\partial\eta^N}\biggr)f(\rho)g(\eta)\biggr]_{\rho=\eta}$$

$$= f(\rho)g(\rho) + \frac{1}{2}\tau^{MN}\partial_M f(\rho)\partial_N g(\rho) + O(\tau^2). \tag{22}$$

We choose as the basic action of the M-theory, the CJS [38] action for eleven-dimensional supergravity, replacing all products of field operators with the $\diamondsuit$-product. The action is invariant under the generalized $\diamondsuit$-product supersymmetry gauge transformations of the vielbein e^A_M, the spin 3/2 field ψ_M and the three-form field A_{MNQ}. Our M-theory has as a low energy limit the CJS supergravity theory when $|\tau^{MN}| \to 0$ and $\beta^M \to 0$.

Although we derive many of our basic results for our field theory formalism in the superspace configuration space, in practice all calculations of amplitudes using our generalized Feynman rules will be performed in momentum space. To this end, we must make a simplifying ansatz that defines the Fourier transform of an operator $\hat{f}(\rho)$:

$$\hat{f}(\rho) = \frac{1}{(2\pi)^D}\int d^D p \exp(ip\rho)\tilde{f}(p). \tag{23}$$

When this Fourier transform of the operator $\hat{f}(\rho)$ is invertible, then we can calculate matrix elements in momentum space without having to concern ourselves with additional volume factors that can occur due to integrations over the Grassmann coordinates β. These additional volume factors occur in standard superspace integrations [35].

We have for the $\diamondsuit$-product the exponential function rule

$$\exp(ik\rho)\diamondsuit\exp(iq\rho) = \exp[i(k+q)]\exp\Big[\frac{1}{2}(k\tau q)\Big], \tag{24}$$

where $k\tau q = k_M\tau^{MN}q_N$. Then for the $\diamondsuit$-product of two operators $\hat{f}$ and $\hat{g}$ we get

$$(\hat{f}\diamondsuit\hat{g})(\rho) = \frac{1}{(2\pi)^{2D}}\int d^Dkd^Dq\tilde{f}(k)\tilde{g}(q)\exp\Big[\frac{1}{2}(k\tau q)\Big]\exp[i(k+q)\rho], \tag{25}$$

where

$$\tilde{f}(k) = \frac{1}{(2\pi)^D}\int d^D\rho\exp(-ik\rho)\hat{f}(\rho). \tag{26}$$

It was demonstrated in previous work [13, 14, 15], that scalar quantum field theory and weak field quantum gravity are finite to all orders of perturbation theory, and the S-matrix for these theories is expected to obey unitarity. The regularization of the ultraviolet divergence of the standard local point-like quantum field theory is caused by an exponential damping of the Feynman propagator. The modified Feynman propagator $\bar{\Delta}_F$ is defined by the vacuum expectation value of the time-ordered $\diamondsuit$-product

$$\begin{aligned} i\bar{\Delta}_F(\rho-\eta) &\equiv \langle 0|T(\phi(\rho)\diamondsuit(\eta))|0\rangle \\ &= \frac{i}{(2\pi)^D}\int\frac{d^Dk\exp[-ik(\rho-\eta)]\exp[-\frac{1}{2}(k\tau k)]}{k^2+m^2-i\epsilon}. \end{aligned} \tag{27}$$

Since we shall be performing all the calculations of Feynman graphs in momentum space, the limit $\rho\to\eta$ in (27) is not rigorously applied.

In momentum space we choose the canonical value for the symmetric tensor $\tau^{\mu\nu} = \delta^{\mu\nu}/\Lambda^2$ where Λ denotes an energy scale. We shall set Λ equal to the eleven-dimensional Planck mass $M_{\rm PL}^{(11)}$ and since the eleven-dimensional gravitational constant, $G^{(11)}$, is proportional to Newton's constant i.e. the inverse of the Planck mass squared, then our M-theory is free of arbitrary parameters. We obtain in momentum space

$$i\bar{\Delta}_F(p) = \frac{i\exp\Big(-\frac{p^2}{2\Lambda^2}\Big)}{p^2+m^2-i\epsilon}, \tag{28}$$

where p denotes the Euclidean D-momentum vector. This reduces to the standard commutative field theory form for the Feynman propagator

$$i\Delta_F(p) = \frac{i}{p^2+m^2-i\epsilon} \tag{29}$$

in the limit $|\tau^{\mu\nu}| \to 0$ and $\Lambda \to \infty$.

An analysis of scattering amplitudes shows that due to the numerator of the modified Feynman propagator being an entire function of p^2, there are no unphysical singularities in the finite complex p^2 plane and the Cutkosky rules for the absorptive part of the scattering amplitude are satisfied, leading to unitarity of the S-matrix [14].

An analysis of the higher-dimensional field theories, including supersymmetric gauge theories generalized to the non-anticommutative formalism, shows that they will also be finite to all orders of perturbation theory. This result is mainly due to the fundamental length scale ℓ in the theory that owes its existence to a quantization of spacetime. When $\ell \to 0$ ($\Lambda \to \infty$) the non-anticommutative field theories reduce to the standard local point particle field theories which suffer the usual difficulties of infinities and non-renormalizable quantum gravity.

Our finite field theory formalism is based on a nonlocal field theory, due to the infinite number of derivatives in the exponential function that occurs in our $\circ$-product or $\diamondsuit$-product of field operators. The nonlocal modification of standard local point field theory occurs at short distances or high energies where accelerator physics has not been probed.

It should be emphasized that the promise of significant progress in noncommutative quantum field theory has not been fulfilled [37, 16, 17]. The planar Feynman graphs of perturbative noncommutative scalar and Yang-Mills field theories, in which the actions are described by Groenwald-Moyal $\star$-products of field operators, are as divergent as in local point field theory, so that the ultraviolet behaviour of scattering amplitudes is no better than in the standard renormalizable theories, albeit that the non-planar graphs do exhibit an oscillatory damping of ultraviolet divergences. This is also true of noncommutative weak field quantum gravity [16, 17] which continues to be nonrenormalizable. On the other hand, the non-anticommutative quantum field theories in superspace are perturbatively finite and unitary. However, all these quantum field theories are nonlocal at short distances and further study of such nonlocal field theories is required.

We can now contemplate an M-theory which is finite to all orders of perturbation theory and contains an invariance of the eleven-dimensional supergravity under a *generalized supersymmetry transformation* with a ground state that reduces to the standard supersymmetric ground state of CJS in the limit that the energy scale $\Lambda \to \infty$.

7 Superspace M-Theory Action

The field content consists of the vielbein e_M^A, a Majorana spin $\frac{3}{2}$ ψ_M, and of a completely antisymmetric gauge tensor field A_{MNP}. The metric is $(- + + + ...+)$ and the eleven-dimensional Dirac matrices satisfy

$$\{\Gamma_A, \Gamma_B\} = -2\eta_{AB}, \tag{30}$$

where η_{AB} denotes the flat Minkowski tangent space metric. Moreover, $\Gamma^{A_1 \dots A_N}$ denotes the product of $N \Gamma$ matrices completely antisymmetrized.

Our superspace M-theory Lagrangian, using the $\diamondsuit$-product has the form [38, 39, 40]:

$$\mathcal{L}_{\rm SG} = -\frac{1}{4\kappa^2} e\diamondsuit R(\omega)_\diamondsuit - \frac{i}{2} e\diamondsuit \bar{\psi}_M \diamondsuit \Gamma^{MNP} D_N\Big(\frac{\omega+\hat{\omega}}{2}\Big)\diamondsuit\psi_P - \frac{1}{48} e\diamondsuit F_{MNPQ}\diamondsuit F^{MNPQ}$$
$$+\frac{\kappa}{192} e\diamondsuit(\bar{\psi}_M\diamondsuit\Gamma^{MNOPQR}\psi_N + 12\bar{\psi}^P\diamondsuit\Gamma^{OR}\psi^Q)\diamondsuit(F_{PQOR}+\hat{F}_{PQOR})$$
$$+\frac{2\kappa}{(144)^2}\epsilon^{O_1O_2O_3O_4P_1P_2P_3P_4MNR}F_{O_1O_2O_3O_4}\diamondsuit F_{P_1P_2P_3P_4}\diamondsuit A_{MNR}, \tag{31}$$

where $R(\omega)_\diamondsuit$ is the scalar contraction of the curvature tensor

$$R_{MNAB} = \partial_M\omega_{NAB} - \partial_N\omega_{MAB} + \omega_{MA}{}^C\diamondsuit\omega_{NCB} - \omega_{NA}{}^C\diamondsuit\omega_{MCB}, \tag{32}$$

and F_{MNOP} is the field strength defined by

$$F_{MNOP} = 4\partial_{[M}A_{NOP]}, \tag{33}$$

with [...] denoting the antisymmetrized sum over all permutations, divided by their number.

The covariant derivative is

$$D_N(\omega)\psi_M = \partial_N\psi_M + \frac{1}{4}\omega_{NAB}\diamondsuit\Gamma^{AB}\psi_M. \tag{34}$$

The spin connection ω_{MAB} is defined by

$$\omega_{MAB} = \omega^0_{MAB}(e) + T_{MAB}, \tag{35}$$

where T_{MAB} is the spin torsion tensor.

The transformation laws are

$$\delta e^A_M = -i\kappa\bar{\epsilon}\diamondsuit\Gamma^A\psi_M, \tag{36}$$

$$\delta\psi_M = \frac{1}{\kappa}D_M(\hat{\omega})\diamondsuit\epsilon + \frac{i}{144}(\Gamma^{OPQR}{}_M - 8\Gamma^{PQR}\delta^O_M)\epsilon\diamondsuit\hat{F}_{OPQR} \equiv \frac{1}{\kappa}\hat{D}_M\epsilon, \tag{37}$$

$$\delta A_{MNP} = \frac{3}{2}\bar{\epsilon}\diamondsuit\Gamma_{[MN}\psi_{P]}, \tag{38}$$

where

$$\hat{F}_{MNPQ} = F_{MNPQ} - 3\kappa\bar{\psi}_{[M}\diamondsuit\Gamma_{NP}\psi_{Q]}. \tag{39}$$

We also have

$$\hat{\omega}_{MAB} = \omega_{MAB} + \frac{i\kappa^2}{4}\bar{\psi}_O\diamondsuit\Gamma_{MAB}{}^{OP}\psi_P. \tag{40}$$

In the limit $|\tau^{MN}| \to 0$ and $\beta^M \to 0$, (31) reduces to the CJS eleven-dimensional supergravity Lagrangian [38], which should be the correct low energy limit of an M-theory. The finiteness and gauge invariance of the M-theory is guaranteed by the non-anticommutative field theory [13, 14, 15]. The symmetric tensor τ^{MN} can be written as

$$\tau^{MN} = \ell^2 s^{MN} = \frac{1}{\Lambda^2} s^{MN}, \tag{41}$$

where Λ is a fundamental energy scale chosen to be $\Lambda = M_{PL}^{(11)}$. Thus, there are no free parameters in our M-theory.

8 Supermembranes

The coordinates of a curved superfield superspace are given by

$$Z^{\bar{M}} = (\rho^M, \theta^\alpha) \tag{42}$$

where θ^α denote superfield superspace spinors. We also introduce the superelfbeins $E_{\bar{M}}^{\bar{A}}(Z)$ where $\bar{A} = (A, \alpha)$ are tangent space indices and the pull-back on the world volume coordinates $\xi^i = (\tau, \sigma_1, \sigma_2)$ (i=0,1,2) is

$$E_i^{\bar{A}} = \partial_i Z^{\bar{M}} E_{\bar{M}}^{\bar{A}}. \tag{43}$$

The supermembrane action using the $\diamondsuit$-product rule is given by [1, 8]:

$$S_{\rm SM} = \int d^3\xi E \diamondsuit \Big[-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij} E_i^{\bar{M}} \diamondsuit E_j^{\bar{N}} \diamondsuit g_{\bar{M}\bar{N}}$$
$$+\frac{1}{2}\sqrt{-\gamma} + \frac{1}{3!}\epsilon^{ijk} E_i^{\bar{M}} \diamondsuit E_j^{\bar{N}} \diamondsuit E_k^{\bar{P}} \diamondsuit A_{\bar{M}\bar{N}\bar{P}} \Big]. \tag{44}$$

The transformation rules are

$$\delta Z^{\bar{M}} \diamondsuit E_{\bar{M}}^{A} = 0, \quad \delta Z^{\bar{M}} E_{\bar{M}}^{\alpha} = \kappa^\beta (1+\Gamma)^\alpha{}_\beta, \tag{45}$$

where $\kappa^\beta(\xi)$ is an anticommuting spacetime spinor and

$$\Gamma^\alpha{}_\beta = \frac{1}{3!}[\sqrt{-\gamma}\epsilon^{ijk} E_i^{\bar{A}} \diamondsuit E_j^{\bar{B}} \diamondsuit E_k^{\bar{C}} \Gamma_{\bar{A}\bar{B}\bar{C}}]^\alpha{}_\beta. \tag{46}$$

The Γ_a are Dirac matrices and $\Gamma_{abc} = \Gamma_{[abc]}$.

The required κ symmetry [1, 8] is achieved by certain constraints satisfied by the four-form field strength $F_{\bar{M}\bar{N}\bar{P}\bar{Q}}$ and the supertorsion. These constraints amount to saying that the field variables in the CJS action satisfy the eleven-dimensional supergravity equations of motion.

9 M-theory Bosonic Field Equations

The bosonic action of the M-theory takes the form

$$S_{\mathrm{SGB}} = \int d^{(11)}\rho\sqrt{g^{(11)}}\diamondsuit\Big[-\frac{1}{2}R_{\diamondsuit} - \frac{1}{48}F_{MNPQ}\diamondsuit F^{MNPQ} + \Big[\frac{\sqrt{2}}{6\cdot(4!)^2}\Big]\Big(\frac{1}{\sqrt{g^{(11)}}}\Big)$$

$$\times\diamondsuit\epsilon^{M_1M_2\ldots M_{11}}F_{M_1M_2M_3M_4}\diamondsuit F_{M_5M_6M_7M_8}\diamondsuit A_{M_9M_{10}M_{11}}\Big]. \tag{47}$$

The metric is $(-+++...+)$, $\epsilon^{0123\ldots}=+1$ and $F_{MNPQ}=4!\partial_{[M}A_{NPQ]}$ and we have set $8\pi G^{(11)}=c=1$, where $G^{(11)}$ is the eleven-dimensional gravitational coupling constant.

The field equations are

$$R_{MN} - \frac{1}{2}g_{MN}\diamondsuit R_{\diamondsuit} = -T_{F\,MN}, \tag{48}$$

$$\frac{1}{\sqrt{g^{(11)}}}\diamondsuit\partial_M(\sqrt{g^{(11)}}\diamondsuit F^{MNPQ}) = -\Big[\frac{\sqrt{2}}{2\cdot(4!)^2}\Big]\Big(\frac{1}{\sqrt{g^{(11)}}}\Big)\diamondsuit\epsilon^{M_1\ldots M_8NPQ}$$

$$\times F_{M_1M_2M_3M_4}\diamondsuit F_{M_5M_6M_7M_8}, \tag{49}$$

where

$$T_{F\,MN} = \frac{1}{48}(8F_{MPQR}\diamondsuit F_N{}^{PQR} - g_{MN}\diamondsuit F_{SPQR}\diamondsuit F^{SPQR}). \tag{50}$$

The bosonic sector of the supermembrane action has the form

$$S_{\mathrm{SMB}} = \int d^3\xi\Big[-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i\rho^M\partial_j\rho^N g_{MN}(\rho)$$

$$+\frac{1}{2}\sqrt{-\gamma} + \frac{1}{3!}\epsilon^{ijk}\partial_i\rho^M\partial_j\rho^N\partial_k\rho^P A_{MNP}(\rho)\Big]. \tag{51}$$

10 Cosmological Solutions

For cosmological purposes, we shall restrict our attention to an eleven-dimensional metric of the form

$$g_{MN} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & a_4^2(t)\tilde{g}_{rs} & 0 \\ 0 & 0 & a_7^2(t)\tilde{g}_{\Delta\Sigma} \end{pmatrix}. \tag{52}$$

Here, $\tilde{g}_{rs}$ (r,s=1,2,3) and $\tilde{g}_{\Delta\Sigma}$ $(\Delta,\Sigma=5,...,11)$ are the maximally symmetric three and seven spacelike spaces, respectively, and a_4 and a_7 are the corresponding time dependent cosmological scale factors. We have assumed for simplicity that the seven extra spacelike dimensions form a maximally symmetric space, although there is no a priori reason that this be the case. We shall assume that we are working at energy scales well below the eleven-dimensional Planck mass, $M_{\rm PL}^{(11)}$, so that the Grassmann coordinates β^M are small compared to x^M, $\rho^M \approx x^M$.

The non-vanishing components of the Christoffel symbols are

$$\Gamma^0_{rs} = \frac{\dot{a}_4}{a_4} g_{rs}, \quad \Gamma^0_{\Delta\Sigma} = \frac{\dot{a}_7}{a_7} g_{\Delta\Sigma}, \quad \Gamma^r_{s0} = \frac{\dot{a}_4}{a_4}\delta^r{}_s,$$

$$\Gamma^\Gamma_{\Delta 0} = \frac{\dot{a}_7}{a_7}\delta^\Gamma{}_\Delta, \quad \Gamma^r_{st} = \tilde{\Gamma}^r_{st}, \quad \Gamma^\Gamma_{\Delta\Sigma} = \tilde{\Gamma}^\Gamma_{\Delta\Sigma}, \tag{53}$$

where $g_{rs} = a_4^2 \tilde{g}_{rs}$, $g_{\Delta\Sigma} = a_7^2 \tilde{g}_{\Delta\Sigma}$ and $\tilde{\Gamma}^r_{st}$ and $\tilde{\Gamma}^\Gamma_{\Delta\Sigma}$ are the Christoffel symbols formed from the $\tilde{g}_{rs}$, $\tilde{g}_{\Delta\Sigma}$ and their derivatives.

The non-vanishing components of the Ricci tensor are

$$R_{00} = 3\frac{\ddot{a}_4}{a_4} + 7\frac{\ddot{a}_7}{a_7},$$

$$R_{rs} = -\Big[\frac{2k_4}{a_4^2} + \frac{d}{dt}\Big(\frac{\dot{a}_4}{a_4}\Big) + \Big(3\frac{\dot{a}_4}{a_4} + 7\frac{\dot{a}_7}{a_7}\Big)\frac{\dot{a}_4}{a_4}\Big] g_{rs},$$

$$R_{\Delta\Sigma} = -\Big[\frac{2k_7}{a_7^2} + \frac{d}{dt}\Big(\frac{\dot{a}_7}{a_7}\Big) + \Big(3\frac{\dot{a}_4}{a_4} + 7\frac{\dot{a}_7}{a_7}\Big)\frac{\dot{a}_7}{a_7}\Big] g_{\Delta\Sigma}, \tag{54}$$

where k_4 and k_7 are the curvature constants of four-dimensional and seven-dimensional space. Positive and negative values of k_4 and k_7 correspond to the sphere and the pseudosphere, respectively, while vanishing values of k_4 and k_7 correspond to flat spaces. The energy-momentum tensor can be expressed in a perfect fluid form

$$T_{MN} = {\rm diag}(-\rho, p g_{rs}, p' g_{\Delta\Sigma}), \tag{55}$$

where $\rho = \rho_m + \rho_F, p = p_m + p_F$ and $p' = p'_m + p'_F$.

We now adopt the Freund-Rubin ansatz for which all components of the four-form field F_{MNPQ} vanish except [21, 22]:

$$F_{\mu\nu\rho\sigma} = m f(t) \frac{1}{\sqrt{-g^{(4)}}} \epsilon_{\mu\nu\rho\sigma}, \tag{56}$$

where $\mu, \nu = 0, 1, 2, 3$ and m is a constant. With this ansatz, the trilinear contributions in A_{MNP} and its derivatives in the action vanish. The three form potential field A_{MNP} is required to live on a three manifold M_3 (i.e. to be a maximally form-invariant tensor on M_3) :

$$A_{MNP} \equiv A_{rst} = m A(t) \sqrt{g^{(3)}} \epsilon_{rst}, \tag{57}$$

where $A(t)$ is a function of time and $g^{(3)}$ and ϵ_{rst} are the metric determinant and the Levi-Civita symbol on M_3, respectively, and we have $f(t) = -\dot{A}(t)$.

Let us assume that the four-form tensor F_{MNPQ} dominates, so that we neglect the effects of the matter fields with density ρ_m. We shall use (22) to expand the products of the F-tensors in small values of $|\tau^{MN}|$. We neglect the contributions from the $\diamondsuit$-products of the metric g_{MN} and its derivatives compared to the $\diamondsuit$-products of the F-tensors. Using the results that $\epsilon_{\mu\alpha\rho\sigma}\epsilon_{\nu}{}^{\alpha\rho\sigma} = 6m^2 f^2(t) g_{\mu\nu}(-g^{(4)})$ and $\epsilon_{\mu\nu\rho\sigma}\epsilon^{\mu\nu\rho\sigma} = 24m^2 f^2(t)(-g^{(4)})$, we find to first order in $|\tau|$:

$$T_{F\,MN} = \frac{1}{2}\epsilon m^2\Big(f^2 - \frac{\dot{f}^2}{2\Lambda^2}\Big)g_{MN}, \tag{58}$$

where we have chosen

$$\tau^{00} = -\frac{1}{\Lambda^2}. \tag{59}$$

Moreover, $\epsilon = +1$ for $M, N = \mu, \nu$ and -1 for $M, N = \Delta, \Sigma$. Eq.(49) becomes

$$\frac{d}{dt}\Big[(a_7(t))^7 f(t)\Big] = 0. \tag{60}$$

Our spontaneous compactification leads to

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \lambda_4 g_{\mu\nu}, \tag{61}$$

and

$$R_{\Delta\Sigma} - \frac{1}{2}g_{\Delta\Sigma}R = \lambda_7 g_{\Delta\Sigma}, \tag{62}$$

where

$$\lambda_4 = -\frac{1}{2}m^2\Big(f^2 - \frac{\dot{f}^2}{2\Lambda^2}\Big), \quad \lambda_7 = \frac{1}{2}m^2\Big(f^2 - \frac{\dot{f}^2}{2\Lambda^2}\Big). \tag{63}$$

Let us assume that

$$C \approx f^2 - \frac{\dot{f}^2}{2\Lambda^2}, \tag{64}$$

where C is a constant. In the limit, $\Lambda \to \infty$, we obtain the standard supersymmetric vacuum result

$$\lambda_4 = -\frac{1}{2}m^2 f^2, \quad \lambda_7 = \frac{1}{2}m^2 f^2. \tag{65}$$

The eleven-dimensional space becomes a product of a four-dimensional Einstein AdS space with negative cosmological constant and a seven-dimensional Einstein space with positive cosmological constant [40]. If we require the vacuum to be supersymmetric by demanding covariantly constant spinors θ for which

$$\delta\psi = \bar{D}_M\theta = 0, \tag{66}$$

where

$$\bar{D}_M = D_M + \frac{i\sqrt{2}}{288}(\Gamma_M^{NPQR} - 8\Gamma^{PQR}\delta_M{}^N)F_{NPQR}, \tag{67}$$

then for $m \neq 0$, $N = 8$ supersymmetry uniquely chooses $AdS \times S^7$ with an $SO(8)$-invariant metric on S^7 [23, 40].

If we choose $f^2 = \dot{f}^2/2\Lambda^2$, we get $\lambda_4 = \lambda_7 = 0$ and flat $(3+1)$ and seven-dimensional spaces. On the other hand, if we choose $f^2 < \dot{f}^2/2\Lambda^2$, then $\lambda_4 > 0$ and $\lambda_7 < 0$ and we obtain a positive cosmological constant in $(3+1)$ spacetime, corresponding to a dS universe, and a seven-dimensional space with negative curvature.

From (61) and (62), we obtain

$$3\ddot{a}_4 + 7\frac{\ddot{a}_7 a_4^2}{a_7} = \lambda_4 a_4^2, \tag{68}$$

$$2k_4 + \ddot{a}_4 a_4 + 2\dot{a}_4^2 + 7\frac{\dot{a}_7\dot{a}_4 a_4}{a_7} = \lambda_4 a_4^2, \tag{69}$$

$$2k_7 + \ddot{a}_7 a_7 + 6\dot{a}_7^2 + 3\frac{\dot{a}_4\dot{a}_7 a_7}{a_4} = \lambda_7 a_7^2. \tag{70}$$

Combining (68) and (69) gives

$$\left(\frac{\dot{a}_4}{a_4}\right)^2 + \frac{k_4}{a_4^2} - \frac{7}{6}\frac{\ddot{a}_7}{a_7} + \frac{7}{2}\frac{\dot{a}_7\dot{a}_4}{a_4 a_7} = \frac{1}{3}\lambda_4. \tag{71}$$

For solutions in which the scale factor a_4 expands faster than a_7, we get the standard four-dimensional Friedmann equation:

$$\left(\frac{\dot{a}_4}{a_4}\right)^2 + \frac{k_4}{a_4^2} = \frac{1}{3}\lambda_4. \tag{72}$$

This has the dS inflationary solution for $k_4 = 0$ and $\lambda_4 > 0$:

$$a_4 = B\exp\left(\sqrt{\frac{\lambda_4}{3}}t\right), \tag{73}$$

where B is a constant.

Let us now consider a time-dependent solution including matter density ρ_m, obtained from additional matter fields generated by our compactification to a four-dimensional spacetime. We write

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = -T_{\mu\nu}, \tag{74}$$

where

$$T_{\mu\nu} = T_{F\mu\nu} + T_{M\mu\nu}, \tag{75}$$

and

$$T_{F\mu\nu} = \frac{1}{2}m^2 S(t) g_{\mu\nu}. \tag{76}$$

$T_{M\mu\nu}$ denotes the energy-momentum contribution in four dimensions from the matter fields. We also have

$$S = f^2 - \frac{\dot{f}^2}{2\Lambda^2}. \tag{77}$$

The $T_{M\mu\nu}$ can be written for a comoving cosmological fluid in the form

$$T_{M\mu\nu} = \mathrm{diag}(-\rho_m, p_m, p_m, p_m), \tag{78}$$

where the ρ_m and p_m denote the density and pressure of matter fields, respectively. We rewrite (74) as

$$R_{\mu\nu} = -\tilde{T}_{\mu\nu}, \tag{79}$$

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T_\lambda{}^\lambda. \tag{80}$$

We express the energy-momentum tensor $\tilde{T}_{\mu\nu}$ as an effective perfect fluid

$$\tilde{T}_{\mu\nu} = (\rho + p)u_\mu u_\nu + \frac{1}{2}(\rho - p)g_{\mu\nu}, \tag{81}$$

where the four-velocity has the components $u^0 = 1$, $u^m = 0$ and $\rho = \rho_m + \rho_F$. We have

$$\rho_F = -\frac{3}{2}m^2 S, \quad p_F = \frac{3}{2}m^2 S, \tag{82}$$

so that $\rho_F = -p_F$ which is the equation of state for the vacuum density ρ_{vac}.

From (54) we obtain

$$\left(\frac{\dot{a}_4}{a_4}\right)^2 + \frac{k_4}{a_4^2} = \frac{1}{3}\rho + \frac{1}{3}\lambda_4 - \frac{7}{2}\frac{\dot{a}_7}{a_7}\frac{\dot{a}_4}{a_4} + \frac{7}{6}\frac{\ddot{a}_7}{a_7}, \tag{83}$$

$$\frac{\ddot{a}_4}{a_4} = -\frac{1}{6}(\rho + 3p) + \frac{1}{3}\lambda_4 - \frac{7}{3}\frac{\ddot{a}_7}{a_7}, \tag{84}$$

$$\frac{\ddot{a}_7}{a_7} + \frac{2k_7}{a_7^2} = -\left[6\left(\frac{\dot{a}_7^2}{a_7^2}\right) + 3\frac{\dot{a}_4\dot{a}_7}{a_4 a_7}\right]. \tag{85}$$

Let us again assume that f and $\dot{f}$ are slowly varying and choose the solution $\lambda_4 > 0$, $f^2 < \dot{f}^2$ and $\rho_F = -p_F$. We also assume the equation of state $p_m = w\rho_m$ and that for the present universe $w \approx 0$. Then, we have for a four-dimensional universe expanding faster than the seven-dimensional compact space:

$$\left(\frac{\dot{a}_4}{a_4}\right)^2 + \frac{k_4}{a_4^2} = \frac{1}{3}\rho_m + \frac{1}{3}\lambda_4 \tag{86}$$

$$\frac{\ddot{a}_4}{a_4} = -\frac{1}{6}(\rho_m + 3p_m) + \frac{1}{3}\lambda_4. \tag{87}$$

We can write (86) in the form

$$\Omega_M + \Omega_{k_4} + \Omega_{\lambda_4} = 1, \tag{88}$$

where

$$\Omega_M = \frac{\rho_m}{3H^2}, \quad \Omega_{k_4} = -\frac{k_4}{a_4^2 H^2}, \quad \Omega_{\lambda_4} = \frac{\lambda_4}{3H^2}. \tag{89}$$

By choosing

$$\Omega_M = 0.32, \quad \Omega_{k_4} = 0, \quad \Omega_{\lambda_4} = 0.68, \tag{90}$$

we can obtain a fit to all the current observational data [18, 19].

Generally, phenomenological matter contributions are not permitted in the eleven-dimensional M-theory supergravity, because of the restrictions incurred by the generalized supersymmetric invariance of the action. However, when we perform our spontaneous compactification the fermionic gravitino contributions and the elvbeins will generate matter fields in the $M_4 \times M_7$ product space, which can describe the '*dark matter*' and the baryons and neutrinos. The F-field contributions, on the other hand, describe the '*dark energy*' associated with the vacuum and the cosmological constant.

11 A Realistic M-Theory of Particles?

We now turn to the important question as to whether we can derive a realistic description of particle unification in our M-theory. Early attempts to obtain an isometry group from the M^{pqr} spaces introduced by Witten [41] failed to give a realistic descriptions of quarks and leptons [40]. Witten starts from the eleven-dimensional model and assumes that the seven-dimensional internal space is a coset space G/H. Choosing the minimal H with $G = SU(3) \times SU(2) \times U(1)$ allows a nontrivial action of each of the factors $SU(3), SU(2)$ and $U(1)$ of G on G/H and this space is exactly seven-dimensional. Decomposing g_{MN} into the product space $M_4 \times M_7$, one finds that the action can gauge $SU(3) \times SU(2) \times U(1)$ in $D = 4$. However, the lepton-quark part of the fermion decomposition does not fit into the required $SU(3) \times SU(2) \times U(1)$ group representations.

Another attempt to obtain a description of particles consists of a spontaneous compactification of the type using the Freund-Rubin ansatz for the four-form field F_{MNPQ} and a round seven sphere S^7 or squashed seven sphere [23]. Such a compactification does not permit singularities in the smooth internal space S^7 and therefore cannot yield a chiral fermion spectrum and non-Abelian gauge fields within $SO(8)$, which does not contain the standard model group $SU(3) \times SU(2) \times U(1)$.

Some time ago it was speculated that some of the gauge bosons and fermions are composite as in the model of Ellis, et al. [42]. The expectation that there are two kinds of gauge bosons, the "elementary" and the "composite" is a consequence of Kaluza-Klein theories and not due to an ad hoc assumption. Kaluza-Klein theories unify gravity and Yang-Mills theories in two ways. First D-dimensional general covariance of our superspace $\rho^M \to \rho'^M(\rho)$ and then secondly D-dimensional local super Lorentz invariance $\delta e^A_M(\rho) = L^A_B(\rho) e^B_M(\rho), \delta\omega^{AB}_M = -D_M L^{AB}(\rho)$ give in eleven dimensions a superspace orthogonal group. Thus, the elfbeins e^A_M are the gauge bosons of superspace general covariance and the superspace spin connections are the gauge bosons of super Lorentz invariance. Whereas the e^A_M are the "elementary" fields, the ω^{AB}_M are the "composite" fields. Since the ω^{AB}_M have no kinetic terms of their own, we have to express them in terms of the elfbein and its derivatives. The Kaluza-Klein compactification gives rise to both elementary and composite gauge bosons in $D = 4$.

At energies well below the Planck mass, our superspace can be approximated by the ordinary four-dimensional spacetime coordinates $\rho^\mu \sim x^\mu$, and the elementary gauge bosons $A_\mu(x)$ come from the the $e^a_\mu(x, y)$ and correspond to a gauge group given by the isometry group of the extra dimensions, $SO(8)$ for the round S^7, while the composite gauge fields $C_\mu(x)$ come from the $\omega^{ab}_\mu(x, y)$ and correspond to the tangent space group of the extra internal dimensions, $SO(7)$, for the case of $d = 7$. Cremmer and Julia [43] showed that when the three form field A_{MNP} is accounted for, the hidden symmetry could be larger than $SO(7)$ and may be as large as $SO(8)$ or $SU(8)$. Now there are many ways to embed $SU(3)\times SU(2)\times U(1)$ in 'elementary' $SO(8)\times$ 'composite' $SU(8)$. However, there still remain unresolved questions about the physical properties of these bound state representations of gauge bosons and fermions, and whether they permit chiral representations in four dimensions.

Perhaps the most promising way to obtain a realistic description of particles in four dimensions from a compactification of M-theory, with complex chiral fermion representations and non-abelian gauge fields, is to seek a compactification on a seven manifold that contains singularities. As shown by Acharya and Witten [24], we should seek a hyper-Kahler manifold compactification with conical singularities that generate the desired chiral fermions and non-Abelian gauge fields within a four-dimensional supersymmetric unified model, which contains the standard $SU(3) \times SU(2) \times U(1)$ group. Such a compactification scheme could be generalized to our supermanifold noncommutative and non-anticommutative geometry.

12 Conclusions

Our M-theory describes a finite quantum field theory in an eleven-dimensional super manifold in which the action is constructed from a $\diamondsuit$-product of field operators based on eleven-dimensional CJS supergravity theory. The quantum gravity and quantum gauge field parts of the action will be finite for $\Lambda < \infty$ due to the finiteness of the non-anticommutative quantum field theory. In the limits $\Lambda \to \infty$, $\beta^M \to 0$ and $|\tau^{MN}| \to 0$, we obtain the low energy limit of eleven-dimensional CJS supergravity. The compactified version of this theory has the same massless ten-dimensional particle spectrum as type-IIA superstring theory and is connected to the latter theory by a duality transformation.

The M-theory eleven-dimensional field equations are invariant under generalized $\diamondsuit$-product supersymmetric gauge transformations, which can be thought of as classical deformations of the standard supersymmetric gauge transformations of CJS supergravity. The generalized $\diamondsuit$-product supersymmetric vacuum leads to de Sitter space solutions and thus provides an empirical basis for a realistic cosmology. The no-go theorems in [29, 30] do not apply in this case, because they are derived from standard supersymmetric theories. Our M-theory with generalized supersymmetric equations predicts an inflationary period in the early universe. As the universe expands, f^2 can tend towards $\dot{f}^2/2\Lambda^2$ and produce a small four-dimensional cosmological constant with $\lambda_4 > 0$ and an accelerating universe at present. We have described the dominant dark matter and dark energy by the energy-momentum matter field tensor and the four-form F-field energy-momentum tensor, respectively.

The problem of chiral fermions can be resolved by permitting isolated singularities in our seven-dimensional compact space M_7. To obtain a realistic mass spectrum, which contains the standard model $SU(3) \times SU(2) \times U(1)$, we should seek a compactification on a seven-dimensional hyper-Kahler type manifold and generate four-dimensional dS solutions through our generalized supersymmetric cosmological solutions.

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