Vilkovisky-DeWitt Effective Action for Einstein Gravity on Kaluza-Klein Spacetimes $M^4 \times S^N$

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Abstract

We evaluate the divergent part of the Vilkovisky-DeWitt effective action for Einstein gravity on even-dimensional Kaluza-Klein spacetimes of the form $M^4 \times S^N$. Explicit results are given for N=2, 4, and 6. Trace anomalies for gravitons are also given for these cases. Stable Kaluza-Klein configurations are sought, unsuccessfully, assuming the divergent part of the effective action dominates the dynamics.

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I. INTRODUCTION

Appelquist and Chodos [1] were the first to use the effective potential formalism to study the problem of spontaneous compactification in Kaluza-Klein theories. The hope was that quantum effects could explain the smallness of the extra dimensions. It was soon realized that results obtained using the standard effective action theory were dependent on which quantum gauge fixing condition was used [2, 3]. This non-uniqueness was overcome [4, 5] by the introduction of a new effective action formulated by Vilkovisky [6] and modified by DeWitt [7]. It is now known as the Vilkovisky-DeWitt (VD) effective action and has the merit of being gauge choice independent. Progress in compactification, however, immediately slowed to a snail's pace [8–15], because the VD effective action for gravity involves evaluating determinants of complicated non-local operators (even at one-loop). In [16], we were able to make progress with the six-dimensional case of a general background spacetime. We evaluated the divergent part of the VD effective action by extending the four-dimensional methods of Barvinsky and Vilkovisky [17]. Due to the complexity of this calculation, we concluded that it is next to impossible to push this method to higher dimensional general spacetimes.

In this paper we therefore restrict ourselves to specific even-dimensional Kaluza-Klein backgrounds of the form $M^4 \times S^N$. In the next section we briefly review the VD effective action formalism. We then extend the method of Barvinsky and Vilkovisky to these higher dimensional cases and expand the effective action in terms of functional traces of various operators. In this way we can extract the divergent part of the effective action by only considering a finite number of terms.

In Section III, the formalism is applied to Einstein gravity. The eigenvalues of the operators [18, 19] mentioned above are evaluated for gravity fields on $M^4 \times S^N$ backgrounds. Using these eigenvalues, we then extract the divergent parts of the VD effective action. Because the internal geometry is assumed static the effective action only gives the effective potential. In Section IV two applications are made of these results. First, gauge-independent trace anomalies for gravitons are explicitly given for N=2, 4, and 6. Second, if the divergent part of the effective potential dominates the dynamics of the internal spheres, self-consistent stable configurations are shown not to exist. Conclusions are given in Section V, and formulae for the divergent parts of functional traces, relevant to these calculations, are listed in the

Appendix.

II. VILKOVISKY-DEWITT EFFECTIVE ACTION

In this section we briefly review the formalism of the Vilkovisky-DeWitt effective action. We follow closely the method of Barvinsky and Vilkovisky [17], as well as their notation.

Consider first a general gauge theory with the action $S^G[\Phi]$, where Φ^i is the set of fields with i in the condensed notation representing both the spacetime and gauge indices. Let Q^i_α be the generators of gauge transformations,

$$\delta\Phi^i = Q^i_\alpha \epsilon^\alpha, \tag{1}$$

where ϵ^α is the gauge parameter. Since the action S^G is gauge invariant,

$$Q^i_\alpha \frac{\delta S^G}{\delta\Phi^i} = 0. \tag{2}$$

Up to one-loop, the gauge fixing action in the background field gauge is given by

$$S^{GF} = -\frac{1}{2}\chi^\alpha c_{\alpha\beta}\chi^\beta, \tag{3}$$

where χ^α is a linear gauge condition, and $c^{\alpha\beta}$ is a local, invertible matrix. Both χ^α and $c_{\alpha\beta}$ may depend on the background field. The corresponding ghost operator is $Q^i_\alpha(\delta\chi^\beta/\delta\Phi^i)$, and the one-loop effective action can be written as

$$iW = -\frac{1}{2}\text{Trln } F_{ij} + \text{Trln}\left(Q^i_\alpha \frac{\delta\chi^\beta}{\delta\Phi^i}\right), \tag{4}$$

where

$$F_{ij} = \frac{\delta^2 S^G}{\delta\Phi^i\delta\Phi^j} - \frac{\delta\chi^\alpha}{\delta\Phi^i}c_{\alpha\beta}\frac{\delta\chi^\beta}{\delta\Phi^j}. \tag{5}$$

However, the one-loop effective action W is gauge dependent in general, that is, it depends on the choice of the gauge fixing action S^{GF} when the background field is not a solution of the classical equation of motion,

$$\mathcal{E}_i \equiv \frac{\delta S^G}{\delta\Phi^i} = 0. \tag{6}$$

This poses a problem using the effective action formalism in off-shell calculations, for example, in calculating the trace anomalies for gravitons [20], and in studying the spontaneous compactification of Kaluza-Klein spaces [5]. This gauge fixing problem can be resolved by using the VD effective action, since this effective action is manifestly independent to the choice of gauge conditions.

At the one-loop level, the VD effective can be obtained simply by replacing the functional derivative in the ordinary effective action by a covariant functional derivative

$$\begin{aligned} \frac{\delta^2 S^G}{\delta\Phi^i \delta\Phi^j} &\to \frac{D}{\delta\Phi^i}\left(\frac{\delta S^G}{\delta\Phi^j}\right) \\ &= \frac{\delta^2 S^G}{\delta\Phi^i \delta\Phi^j} - \Gamma^k_{ij}\frac{\delta S^G}{\delta\Phi^k}, \end{aligned} \tag{7}$$

where the connection consists of two parts,

$$\Gamma^k_{ij} = \left\{ \begin{array}{c} k \\ i\ j \end{array} \right\} + T^k_{ij}. \tag{8}$$

$\left\{ \begin{array}{c} k \\ i\ j \end{array} \right\}$ is the local Christoffel symbol constructed in the usual manner from a configuration space metric γ_{ij},

$$\left\{ \begin{array}{c} k \\ i\ j \end{array} \right\} = \frac{1}{2}\gamma^{kl}(\gamma_{li,j} + \gamma_{lj,i} - \gamma_{ij,l}), \tag{9}$$

where the derivative in $\gamma_{li,j} = \delta\gamma_{li}/\delta\Phi^j$ represents the ordinary functional derivative. The configuration space metric is the new ingredient in the VD theory. A prescription for defining it has been given by Vilkovisky [6]. The non-local part T^k_{ij} of the connection comes from the gauge constraints,

$$T^k_{ij} = -2Q^k_{\alpha;(i}\gamma_{j)l}N^{\alpha\beta}Q^l_\beta + \gamma_{(il}N^{\alpha\mu}Q^l_\mu Q^m_\alpha Q^k_{\beta;m}\gamma_{j)n}N^{\beta\nu}Q^n_\nu, \tag{10}$$

where the derivative in $Q^k_{\beta;m}$ is the covariant functional derivative defined with the Christoffel symbol $\left\{ \begin{array}{c} k \\ i\ j \end{array} \right\}$, and $N^{\alpha\beta}$ is the inverse of $N^{-1}_{\alpha\beta}$,

$$N^{-1}_{\alpha\beta}N^{\beta\gamma} = \delta^\gamma_\alpha, \tag{11}$$

with

$$N^{-1}_{\alpha\beta} = \gamma_{ij}Q^i_\alpha Q^j_\beta. \tag{12}$$

Here we have used the convention of symmetrization such that $A_{(i}B_{j)} = \frac{1}{2}(A_iB_j + A_jB_i)$. A detailed derivation of T^k_{ij} can be found, for example, in [21]. Therefore, the one-loop VD effective action can be written as

$$iW_{unique} = -\frac{1}{2}\mathrm{Trln}\mathcal{F}_{ij} + \mathrm{Trln}\left(Q^i_\alpha \frac{\delta\chi^\beta}{\delta\Phi^i}\right), \tag{13}$$

where

$$\begin{aligned}\mathcal{F}_{ij} &= \frac{D}{\delta\Phi^i}\left(\frac{\delta S^G}{\delta\Phi^j}\right) - \frac{\delta\chi^\alpha}{\delta\Phi^i}c_{\alpha\beta}\frac{\delta\chi^\beta}{\delta\Phi^j} \\ &= S^G_{,ij} - \frac{\delta\chi^\alpha}{\delta\Phi^i}c_{\alpha\beta}\frac{\delta\chi^\beta}{\delta\Phi^j} - \Gamma^k_{ij}\mathcal{E}_k. \end{aligned} \tag{14}$$

To calculate the divergent part of iW_{unique}, we separate the local and the non-local parts of $\mathcal{F}$. First we define the Green's function $\mathcal{G}$ such that

$$\left\{S^G_{,ij} - \left\{\begin{matrix} k \\ i\ j \end{matrix}\right\}\mathcal{E}_k - \frac{\delta\chi^\alpha}{\delta\Phi^i}c_{\alpha\beta}\frac{\delta\chi^\beta}{\delta\Phi^j}\right\}\mathcal{G}^{jl} = -\delta^l_i. \tag{15}$$

Therefore, using $-\mathcal{G}^{-1}$ to represent the above operator in the parenthesis,

$$\begin{aligned}\mathcal{F}_{ij} &= -\mathcal{G}^{-1}_{ij} - T^k_{ij}\mathcal{E}_k \\ &= -\mathcal{G}^{-1}_{il}(\delta^l_j + \mathcal{G}^{lm}T^k_{mj}\mathcal{E}_k). \end{aligned} \tag{16}$$

The contribution of $\mathcal{F}$ to the VD effective action can then be written as

$$\begin{aligned}-\frac{1}{2}\mathrm{Trln}\mathcal{F} &= -\frac{1}{2}\mathrm{Trln}\mathcal{G}^{-1} - \frac{1}{2}\mathrm{Trln}(1+\mathcal{G}T\mathcal{E}) \\ &= -\frac{1}{2}\mathrm{Trln}\mathcal{G}^{-1} - \frac{1}{2}\mathrm{Trln}M + \frac{1}{4}\mathrm{Trln}M^2 + \cdots, \end{aligned} \tag{17}$$

where $M^l_j = \mathcal{G}^{lm}T^k_{mj}\mathcal{E}_k$. The various traces can be evaluated with the help of the following identities. From Eq. (2),

$$\begin{aligned} &\frac{\delta}{\delta\Phi^j}\left(Q^i_\alpha\frac{\delta S^G}{\delta\Phi^i}\right) = 0 \\ \Rightarrow &\left(\frac{\delta Q^i_\alpha}{\delta\Phi^j}\right)\mathcal{E}_i + Q^i_\alpha S^G_{,ij} = 0. \end{aligned} \tag{18}$$

From the definition of the operator $\mathcal{G}^{-1}$ in Eq. (15),

$$S^G_{,ij} = -\mathcal{G}^{-1}_{ij} + \left\{\begin{matrix} k \\ i\ j \end{matrix}\right\}\mathcal{E}_k + \frac{\delta\chi^\alpha}{\delta\Phi^i}c_{\alpha\beta}\frac{\delta\chi^\beta}{\delta\Phi^j}. \tag{19}$$

If we choose the DeWitt background gauge [22],

$$\frac{\delta\chi^\alpha}{\delta\Phi^i} = -c^{-1\alpha\beta}Q^j_\beta\gamma_{ji}, \tag{20}$$

we can obtain minimal operators [19] with the appropriate choice of $c_{\alpha\beta}$. In the DeWitt gauge,

$$\frac{\delta\chi^\alpha}{\delta\Phi^i} c_{\alpha\beta} \frac{\delta\chi^\beta}{\delta\Phi^j} = \gamma_{ik} Q^k_\alpha c^{-1\alpha\beta} Q^l_\beta \gamma_{lj}, \tag{21}$$

and the ghost operator

$$Q^i_\alpha \frac{\delta\chi^\beta}{\delta\Phi^i} = -N^{-1}_{\alpha\mu} c^{-1\mu\beta}. \tag{22}$$

Therefore, Eq. (18) becomes

$$\begin{aligned} &\left(\frac{\delta Q^i_\alpha}{\delta\Phi^j}\right)\mathcal{E}_i + Q^i_\alpha \left(-\mathcal{G}^{-1}_{ij} + \left\{ \begin{array}{c} k \\ i\ j \end{array} \right\} \mathcal{E}_k + \gamma_{ik} Q^k_\mu c^{-1\mu\nu} Q^l_\nu \gamma_{lj}\right) = 0 \\ \Rightarrow\ & N_{\alpha\beta} c^{-1\beta\mu} Q^i_\mu \gamma_{ij} = Q^i_\alpha \mathcal{G}^{-1}_{ij} - Q^i_{\alpha;j}\mathcal{E}_i. \end{aligned} \tag{23}$$

Multipling both sides by $\mathcal{G}^{jk} N^{\alpha\nu}$, we have

$$Q^i_\nu \gamma_{ij} \mathcal{G}^{jk} = Q^k_\alpha N^{\alpha\beta} c_{\beta\nu} - Q^i_{\alpha;j} \mathcal{E}_i \mathcal{G}^{jk} N^{\alpha\beta} c_{\beta\nu}. \tag{24}$$

This is the needed basic identity. If we multiply both sides by $N^{\mu\gamma} Q^l_{\gamma;k} \mathcal{E}_l$,

$$\begin{aligned} Q^i_\nu \gamma_{ij} \mathcal{G}^{jk} Q^l_{\gamma;k} \mathcal{E}_l N^{\mu\gamma} &= N^{\mu\gamma} Q^l_{\gamma;k} \mathcal{E}_l Q^k_\alpha N^{\alpha\beta} c_{\beta\nu} - N^{\mu\gamma} Q^i_{\gamma;k} \mathcal{E}_i \mathcal{G}^{kl} Q^j_{\alpha;l} \mathcal{E}_j N^{\alpha\beta} c_{\beta\nu} \\ &= {U_1}^\mu_\nu - {U_2}^\mu_\nu, \end{aligned} \tag{25}$$

where

$$ {U_1}^\mu_\nu \equiv N^{\mu\gamma} Q^k_\gamma Q^l_{\alpha;k} \mathcal{E}_l N^{\alpha\beta} c_{\beta\nu}, \tag{26}$$

$$ {U_2}^\mu_\nu \equiv N^{\mu\gamma} Q^i_{\gamma;k} \mathcal{E}_i \mathcal{G}^{kl} Q^j_{\alpha;l} \mathcal{E}_j N^{\alpha\beta} c_{\beta\nu}. \tag{27}$$

We have also used the fact that

$$\begin{aligned} Q^i_\alpha Q^j_{\gamma;i} \mathcal{E}_j &= Q^i_\alpha [(Q^j_\gamma \mathcal{E}_j)_{;i} - Q^j_\gamma \mathcal{E}_{j;i}] \\ &= -Q^i_\alpha Q^j_\gamma \left[S^G_{,ij} - \left\{ \begin{array}{c} k \\ i\ j \end{array} \right\} \mathcal{E}_k \right] \\ &= Q^i_\gamma Q^j_{\alpha,i} \mathcal{E}_j, \end{aligned} \tag{28}$$

is symmetric with respect to the indices α and γ. We can obtain yet another identity by multipling the basic identity, Eq. (24), with $\gamma_{kl} Q^l_\gamma$,

$$Q^i_\nu \gamma_{ij} \mathcal{G}^{jk} \gamma_{kl} Q^l_\gamma = c_{\nu\gamma} - Q^i_{\alpha;j} \mathcal{E}_i \mathcal{G}^{jk} N^{\alpha\beta} c_{\beta\nu} \gamma_{kl} Q^l_\gamma. \tag{29}$$

Applying again the basic identity, Eq. (29) becomes

$$
\begin{aligned}
Q^i_\nu \gamma_{ik} \mathcal{G}^{kl} \gamma_{lj} Q^j_\gamma &= c_{\nu\gamma} - Q^i_{\alpha;j} \mathcal{E}_i N^{\alpha\beta} c_{\beta\nu} (Q^j_\mu N^{\mu\rho} c_{\rho\gamma} - Q^l_{\mu;k} \mathcal{E}_l \mathcal{G}^{jk} N^{\mu\rho} c_{\rho\gamma}) \\
&= c_{\nu\beta} (\delta^\beta_\gamma - U_1{}^\beta_\gamma + U_2{}^\beta_\gamma). \qquad (30)
\end{aligned}
$$

With these identities we are in a position to evaluate the various traces in Eq. (17). First,

$$
\begin{aligned}
\mathrm{Tr} M &= \mathcal{G}^{ij} T^k_{ij} \mathcal{E}_k \\
&= -2(U_1{}^\alpha_\alpha - U_2{}^\alpha_\alpha) + U_1{}^\alpha_\beta (\delta^\beta_\alpha - U_1{}^\beta_\alpha + U_2{}^\beta_\alpha) \\
&= -\mathrm{Tr} U_1 + 2\mathrm{Tr} U_2 - \mathrm{Tr} U_1^2 + \mathrm{Tr} U_1 U_2, \qquad (31)
\end{aligned}
$$

where we have made use of both the identities in Eqs. (25) and (30). Similarly, we can evaluate the traces of higher powers of M.

$$
\begin{aligned}
\mathrm{Tr} M^2 &= -\mathrm{Tr} U_1^2 + 2\mathrm{Tr} U_2 + 2\mathrm{Tr} U_1^3 - 2\mathrm{Tr} U_1 U_2 + \mathrm{Tr} U_1^4 - 6\mathrm{Tr} U_1^2 U_2 + 4\mathrm{Tr} U_2^2 \\
&\quad -2\mathrm{Tr} U_1^3 U_2 + 4\mathrm{Tr} U_1 U_2^2 + O(\mathcal{E}^6), \qquad (32) \\
\mathrm{Tr} M^3 &= 2\mathrm{Tr} U_1^3 - 3\mathrm{Tr} U_1 U_2 - 6\mathrm{Tr} U_1^2 U_2 + 6\mathrm{Tr} U_2^2 \\
&\quad -3\mathrm{Tr} U_1^5 + 9\mathrm{Tr} U_1^3 U_2 - 3\mathrm{Tr} U_1 U_2^2 + O(\mathcal{E}^6), \qquad (33) \\
\mathrm{Tr} M^4 &= -\mathrm{Tr} U_1^4 + 2\mathrm{Tr} U_2^2 - 4\mathrm{Tr} U_1^5 + 16\mathrm{Tr} U_1^3 U_2 - 12\mathrm{Tr} U_1 U_2^2 + O(\mathcal{E}^6), \qquad (34) \\
\mathrm{Tr} M^5 &= -\mathrm{Tr} U_1^5 + 5\mathrm{Tr} U_1^3 U_2 - 5\mathrm{Tr} U_1 U_2^2 + O(\mathcal{E}^6). \qquad (35)
\end{aligned}
$$

Note that we have expanded the various expressions only up to the fifth power of the first functional derivative $\mathcal{E}_i$ of the action. To calculate the divergent part of the VD effective action, one needs only terms up to some power of $\mathcal{E}_i$, depending on the dimensionality of the spacetime that one is considering.

Finally, the VD effective action in Eq. (13) can be expanded in terms of the various traces of the operators U_1 and U_2,

$$
\begin{aligned}
iW_{unique} &= -\frac{1}{2}\mathrm{Trln}\mathcal{F} + \mathrm{Trln} N^{-1} \\
&= -\frac{1}{2}\mathrm{Trln}\mathcal{G}^{-1} + \mathrm{Trln} N^{-1} \\
&\quad +\frac{1}{2}\mathrm{Tr} U_1 + \frac{1}{4}\mathrm{Tr} U_1^2 - \frac{1}{2}\mathrm{Tr} U_2 + \frac{1}{6}\mathrm{Tr} U_1^3 - \frac{1}{2}\mathrm{Tr} U_1 U_2 + \frac{1}{8}\mathrm{Tr} U_1^4 \\
&\quad -\frac{1}{2}\mathrm{Tr} U_1^2 U_2 + \frac{1}{4}\mathrm{Tr} U_2^2 + \frac{1}{10}\mathrm{Tr} U_1^5 - \frac{1}{2}\mathrm{Tr} U_1^3 U_2 + \frac{1}{2}\mathrm{Tr} U_1 U_2^2 + O(\mathcal{E}^6). \qquad (36)
\end{aligned}
$$

Using this expression we evaluate the divergent part of the VD effective action for Einstein gravity on Kaluza-Klein spaces $M^4 \times S^N$ in the following section.

III. EINSTEIN GRAVITY ON KALUZA-KLEIN SPACES $M^4 \times S^N$

Here we shall apply the formalism set up in the last section to the case of Einstein gravity. Because such calculations are very tedious to carry out for general spacetimes (see [16]), we restrict ourselves to Kaluza-Klein backgrounds of the form $M^4 \times S^N$.

A. Einstein gravity

We start with the n-dimensional Einstein-Hilbert action,

$$S^G = \int d^n x \sqrt{-g}(R - 2\Lambda), \tag{37}$$

where Λ is the cosmological constant. The first functional derivative

$$\mathcal{E}^{\mu\nu x} = \frac{\delta S^G}{\delta g_{\mu\nu}(x)} = -(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu}(R - 2\Lambda)). \tag{38}$$

The gauge symmetry is the general coordinate invariance,

$$\delta g_{\mu\nu}(x) = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = Q_{\mu\nu x,\alpha y} \epsilon^\alpha(y), \tag{39}$$

where $\epsilon_\mu(x)$ is the gauge transformation parameter, ∇_μ is the ordinary covariant derivative, and

$$Q_{\mu\nu x,\alpha y} = -(g_{\mu\alpha} \nabla_\nu + g_{\nu\alpha} \nabla_\mu) \delta^n(x - y). \tag{40}$$

Because of this gauge symmetry, the second functional derivative of the action, $S^G_{,ij}$, is a singular operator.

$$\begin{aligned} &\frac{\delta S^G}{\delta g_{\mu\nu}(x) \delta g_{\alpha\beta}(y)} \\ &= C^{\mu\nu,\rho\sigma} \Big[\delta^{\alpha\beta}_{\rho\sigma} \Box - \delta^{(\alpha}_{(\rho} \nabla_{\sigma)} \nabla^{\beta)} - \delta^{(\alpha}_{(\rho} \nabla^{\beta)} \nabla_{\sigma)} + g^{\alpha\beta} \nabla_{(\rho} \nabla_{\sigma)} + 2 R_\rho{}^{(\alpha}{}_\sigma{}^{\beta)} + 2 \delta^{(\alpha}_{(\rho} R^{\beta)}_{\sigma)} \\ &\quad - g^{\alpha\beta} R_{\rho\sigma} - \frac{2}{n-2} g_{\rho\sigma} R^{\alpha\beta} + \frac{1}{n-2} g_{\rho\sigma} g^{\alpha\beta} R - (R - 2\Lambda) \delta^{\alpha\beta}_{\rho\sigma} \Big] \delta^n(x - y)/\sqrt{-g}, \end{aligned} \tag{41}$$

where

$$\delta^{\alpha\beta}_{\rho\sigma} = \delta^\alpha_{(\rho} \delta^\beta_{\sigma)}, \tag{42}$$

$$C^{\mu\nu,\alpha\beta} = \frac{1}{4}(g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} - g^{\mu\nu} g^{\alpha\beta}). \tag{43}$$

To invert (41) we need to pick a gauge, and again as in the last section, we choose the DeWitt gauge. The DeWitt gauge requires a metric for the configuration space of $g_{\mu\nu}(x)$. The one given by Vilkovisky is [6]

$$\gamma^{\mu\nu x,\alpha\beta y} = C^{\mu\nu,\alpha\beta}\sqrt{-g}\ \delta^n(x-y), \tag{44}$$

in which $C^{\mu\nu,\alpha\beta}$ is exactly the same as the factor in front of the operator $S^G_{,ij}$. With this metric γ_{ij} and the choice of $c_{\mu\nu} = \delta_{\mu\nu}$, the DeWitt gauge becomes the harmonic gauge and the corresponding graviton operator in Eq. (4) is minimal [19],

$$\begin{aligned} F^{\alpha\beta}_{\mu\nu} &= \delta^{\alpha\beta}_{\mu\nu}\Box + 2R_{\mu}{}^{(\alpha}{}_{\nu}{}^{\beta)} + 2\delta^{(\alpha}_{(\mu}R^{\beta)}_{\nu)} - g^{\alpha\beta}R_{\mu\nu} \\ &\quad -\frac{2}{n-2}g_{\mu\nu}R^{\alpha\beta} + \frac{1}{n-2}g_{\mu\nu}g^{\alpha\beta}R - (R-2\Lambda)\delta^{\alpha\beta}_{\mu\nu}. \end{aligned} \tag{45}$$

Note that for simplicity we have left out the spacetime coordinate indices, and we shall do so in the subsequent discussion. The ghost operator in Eq. (22) is now simply,

$$N_{\mu\nu} = -(g_{\mu\nu}\Box + R_{\mu\nu}). \tag{46}$$

To find the expression for the local operator $\mathcal{G}^{-1}$ in the VD effective action, we need the Christoffel symbol $\left\{ \begin{matrix} k \\ i\ j \end{matrix} \right\}$. From the metric $\gamma^{\mu\nu,\alpha\beta}$ above,

$$\left\{ \begin{matrix} g_{\rho\sigma} \\ g_{\mu\nu}\ g_{\alpha\beta} \end{matrix} \right\} = \frac{1}{4}g^{\mu\nu}\delta^{\alpha\beta}_{\rho\sigma} + \frac{1}{4}g^{\alpha\beta}\delta^{\mu\nu}_{\rho\sigma} - \frac{1}{2}\delta^{\mu(\alpha}_{\rho\sigma}g^{\beta)\nu} - \frac{1}{2}\delta^{\nu(\alpha}_{\rho\sigma}g^{\beta)\mu} + \frac{1}{n-2}g_{\rho\sigma}C^{\mu\nu,\alpha\beta}. \tag{47}$$

The operator $\mathcal{G}^{-1}$ has the form

$$\mathcal{G}^{-1}{}^{\alpha\beta}_{\mu\nu} = -(\delta^{\alpha\beta}_{\mu\nu}\Box + P^{\alpha\beta}_{\mu\nu}), \tag{48}$$

where

$$\begin{aligned} P^{\alpha\beta}_{\mu\nu} &= 2R_{\mu}{}^{(\alpha}{}_{\nu}{}^{\beta)} - \frac{1}{2}g^{\alpha\beta}R_{\mu\nu} - \frac{1}{n-2}g_{\mu\nu}R^{\alpha\beta} \\ &\quad +\frac{1}{n-2}(\delta^{\alpha\beta}_{\mu\nu} + \frac{1}{2}g_{\mu\nu}g^{\alpha\beta})R - \frac{n}{2(n-2)}(R-2\Lambda)\delta^{\alpha\beta}_{\mu\nu}. \end{aligned} \tag{49}$$

Note that we have left out an overall factor $C^{\mu\nu,\alpha\beta}$ which is irrelevant in this particular calculation.

Next we would like to find the expressions for the operators ${U_1}^\mu_\nu$ and ${U_2}^\mu_\nu$. To do so we need to use the following equations.

$$
\begin{aligned}
& (Q_{\alpha\beta,\mu})^{;\rho\sigma}\mathcal{E}^{\alpha\beta} \\
= & \left[-R^{\nu(\rho}\delta^{\sigma)}_\mu\nabla_\nu + R^{(\rho}_\mu\nabla^{\sigma)} - \frac{1}{2}g^{\rho\sigma}R^\nu_\mu\nabla_\nu + \frac{1}{2}\left(R - \frac{2n}{n-2}\Lambda\right)\delta^{(\rho}_\mu\nabla^{\sigma)}\right. \\
& \left. +\frac{1}{2}(R^{\rho\sigma} - \frac{1}{2}g^{\rho\sigma}R)\nabla_\mu + \frac{n}{2(n-2)}\Lambda g^{\rho\sigma}\nabla_\mu + (\nabla_\mu R^{\rho\sigma}) - \frac{1}{2}g^{\rho\sigma}(\nabla_\mu R)\right], \qquad (50)
\end{aligned}
$$

and

$$
\begin{aligned}
& Q_{\rho\sigma,\mu}(Q_{\alpha\beta,\nu})^{;\rho\sigma}\mathcal{E}^{\alpha\beta} \\
= \; & g_{\mu\nu}R^{\alpha\beta}\nabla_\alpha\nabla_\beta - R_{\mu\nu}\Box - \frac{1}{2}\left(R - \frac{2n}{n-2}\Lambda\right)(g_{\mu\nu}\Box + R_{\mu\nu}) \\
& -R^\alpha_\mu R_{\alpha\nu} + R^{\alpha\beta}R_{\mu\alpha\nu\beta} - (\nabla^\alpha R_{\mu\nu})\nabla_\alpha + (\nabla_\mu R^\alpha_\nu)\nabla_\alpha - (\nabla_\nu R^\alpha_\mu)\nabla_\alpha. \qquad (51)
\end{aligned}
$$

Now, the operator ${U_1}^\mu_\nu$ and ${U_2}^\mu_\nu$ are given by

$$
\begin{aligned}
{U_1}^\mu_\nu \; = \; & N^\mu_\alpha\left[\delta^\alpha_\beta R^{\sigma\lambda}\nabla_\sigma\nabla_\lambda - R^\alpha_\beta\Box - \frac{1}{2}\left(R - \frac{2n}{n-2}\Lambda\right)(\delta^\alpha_\beta\Box + R^\alpha_\beta)\right. \\
& \left. -R^\alpha_\lambda R^\lambda_\beta + R^{\sigma\lambda}R^\alpha{}_{\sigma\beta\lambda} - (\nabla^\lambda R^\alpha_\beta)\nabla_\lambda + (\nabla^\alpha R^\lambda_\beta)\nabla_\lambda - (\nabla_\beta R^{\alpha\lambda})\nabla_\lambda\right] N^\beta_\nu \qquad (52) \\
{U_2}^\mu_\nu \; = \; & N^\nu_\alpha\left\{\frac{1}{2}\left[(\nabla^\alpha R^{\sigma\lambda}) - (\nabla^\sigma R^{\lambda\alpha}) - (\nabla^\lambda R^{\sigma\alpha})\right] + \left[D^{\sigma\lambda,\alpha\mu} - \left(R - \frac{2n}{n-2}\Lambda\right)C^{\sigma\lambda,\alpha\mu}\right]\nabla_\mu\right\} \\
& \mathcal{G}_{\sigma\lambda,\omega\epsilon}\left\{(\nabla_\beta R^{\epsilon\omega}) - \frac{1}{2}g^{\epsilon\omega}(\nabla_\beta R) - \left[D^{\omega\epsilon,}{}_\beta{}^\gamma - \left(R - \frac{2n}{n-2}\Lambda\right)C^{\omega\epsilon,}{}_\beta{}^\gamma\right]\nabla_\gamma\right\}N^\beta_\mu, \qquad (53)
\end{aligned}
$$

where N^μ_α is the ghost Green's function

$$
(\delta^\mu_\alpha\Box + R^\mu_\alpha)N^\alpha_\nu = \delta^\mu_\nu, \qquad (54)
$$

and

$$
D^{\sigma\lambda,\alpha\mu} \equiv \frac{1}{2}(g^{\sigma\lambda}R^{\alpha\mu} - g^{\alpha\mu}R^{\sigma\lambda} + g^{\alpha\sigma}R^{\lambda\mu} - g^{\lambda\mu}R^{\alpha\sigma} + g^{\alpha\lambda}R^{\sigma\mu} - g^{\sigma\mu}R^{\alpha\lambda}. \qquad (55)
$$

We have written down the expressions for the operators $\mathcal{G}^{-1}$, N^{-1}, U_1, and U_2, which are present in the VD effective action. In principle, the divergent part of the VD effective action can now be evaluated by working out the traces of different combinations of these operators. However, the algebra gets exceedingly tedious as one goes to higher dimensions. In [16], we have considered the case of $n = 6$, but going to larger values of n seems to be impossible. In the following analysis, we therefore restrict ourselves to the specific cases of Kaluza-Klein spaces of the form $M^4 \times S^N$.

B. Eigenvalues of various operators on $M^4 \times S^N$

Here we work out the eigenvalues of the operators N^{-1}, $\mathcal{G}^{-1}$, U_1, and U_2. To distinguish between external spacetime and internal space indices, we use Greek letters for external spacetime M^4 and Latin letters for internal space S^N. Since the traces of these operators are divergent in general, a regularization scheme is needed. We adopt the method of dimensional regularization by taking the external spacetime dimension to be a general d and we take $d \to 4-\epsilon$ at the end to extract the divergent parts. Hence in this subsection we first assume the spacetime dimension to be $M^d \times S^N$.

First we consider the ghost operator N^{-1}:

$$\begin{aligned} N^{-1\mu}{}_{\nu} &= -\delta^{\mu}_{\nu}(\Box_d + \Box_N), \quad &(56)\\ N^{-1a}{}_{b} &= -\delta^{a}_{b}[(\Box_d + \Box_N) + R^{a}_{b}] \\ &= -\delta^{a}_{b}\left[(\Box_d + \Box_N) + \frac{1}{r^2}(N-1)\right], \quad &(57) \end{aligned}$$

where $\Box_d$ is the d'Alembertian on M^d, while $\Box_N$, r, and R^a_b are the Laplacian, the radius, and the Ricci tensor on S^N, respectively. The off-diagonal terms $N^{-1\mu}{}_{a}$ and $N^{-1a}{}_{\mu}$ vanish.

The eigenvector η^{ν} of $N^{-1\mu}{}_{\nu}$ is a vector with d components on M^d and a scalar on S^N. Consequently, [18],

$$N^{-1\mu}{}_{\nu}\eta^{\nu} = (k^2 - \Lambda_l)\eta^{\mu}, \tag{58}$$

where k^{μ} is the momentum and

$$\Lambda_l = -\frac{l(l+N-1)}{r^2}, \tag{59}$$

for $l = 0, 1, 2, \cdots$ is the scalar Laplacian eigenvalue on S^N with degeneracy

$$D^{(s)}_l(N) = \frac{(2l+N-1)(l+N-2)!}{l!(N-1)!}. \tag{60}$$

The eigenvectors of $N^{-1a}{}_{b}$ are $\eta^{b\mathrm{T}}$ and $\eta^{b\mathrm{L}}$. $\eta^{b\mathrm{T}}$ is the transverse part of η^b, which is a scalar on M^d and a vector on S^N, with

$$(\nabla_N)_b\eta^{b\mathrm{T}} = 0. \tag{61}$$

The eigenvalue of $\eta^{b\mathrm{T}}$ is

$$N^{-1a}{}_{b}\eta^{b\mathrm{T}} = \left[k^2 - \Lambda_l - \frac{N}{r^2}\right]\eta^{a\mathrm{T}}, \tag{62}$$

for $l = 1, 2, \cdots$ with degeneracy

$$D_l^{(v)}(N) = \frac{l(l+N-1)(2l+N-1)(l+N-3)!}{(N-2)!(l+1)!}. \tag{63}$$

And the eigenvalue of $\eta^{b\mathrm{L}}$, the longitudinal part of η^b, is

$$N^{-1}{}^a_b \eta^{b\mathrm{L}} = \left(k^2 - \Lambda_l - \frac{2}{r^2}(N-1)\right)\eta^{a\mathrm{L}}, \tag{64}$$

for $l = 1, 2, \cdots$, with degeneracy $D_l^{(s)}(N)$.

For the operator $\mathcal{G}^{-1}$, the eigenfunctions are symmetric tensors. First, $\eta_{\alpha\beta}$, which is a symmetric tensor on M^d and a scalar on S^N, can be decomposed [18] into the transverse-traceless (TT) part $\eta^{\mathrm{TT}}_{\alpha\beta}$, the longitudinal-transverse-traceless (LTT) part $\eta^{\mathrm{LTT}}_{\alpha\beta}$, the longitudinal-longitudinal-traceless (LLT) part $\eta^{\mathrm{LLT}}_{\alpha\beta}$, and the trace (Tr) part $\eta^{\mathrm{Tr}}_{\alpha\beta}$. $\eta^{\mathrm{TT}}_{\alpha\beta}$ has $(d-2)(d+1)/2$ components on M^d,

$$\mathcal{G}^{-1}{}^{\alpha\beta}_{\mu\nu}\eta^{\mathrm{TT}}_{\alpha\beta} = \left(k^2 - \Lambda_l + \frac{1}{2r^2}N(N-1) - \frac{N+d}{N+d-2}\Lambda\right)\eta^{\mathrm{TT}}_{\mu\nu}. \tag{65}$$

$\eta^{\mathrm{LTT}}_{\alpha\beta}$ has $(d-1)$ components on M^d,

$$\mathcal{G}^{-1}{}^{\alpha\beta}_{\mu\nu}\eta^{\mathrm{LTT}}_{\alpha\beta} = \left(k^2 - \Lambda_l + \frac{1}{2r^2}(N-1) - \frac{N+d}{N+d-2}\Lambda\right)\eta^{\mathrm{LTT}}_{\mu\nu}. \tag{66}$$

$\eta^{\mathrm{LLT}}_{\alpha\beta}$ has 1 component on M^d,

$$\mathcal{G}^{-1}{}^{\alpha\beta}_{\mu\nu}\eta^{\mathrm{LLT}}_{\alpha\beta} = \left(k^2 - \Lambda_l + \frac{1}{2r^2}(N-1) - \frac{N+d}{N+d-2}\Lambda\right)\eta^{\mathrm{LLT}}_{\mu\nu}. \tag{67}$$

$\eta^{\mathrm{Tr}}_{\alpha\beta}$ also has only 1 component on M^d,

$$\mathcal{G}^{-1}{}^{\alpha\beta}_{\mu\nu}\eta^{\mathrm{Tr}}_{\alpha\beta} = \left(k^2 - \Lambda_l + \frac{1}{2r^2}\frac{N(N-1)(N-2)}{N+d-2} - \frac{N+d}{N+d-2}\Lambda\right)\eta^{\mathrm{Tr}}_{\mu\nu}. \tag{68}$$

They all have degeneracy $D_l^{(s)}(N)$ with $l = 0, 1, 2, \cdots$.

Similarly, η_{ab} is a scalar on M^d and a symmetric tensor on S^N.

$$\mathcal{G}^{-1}{}^{cd}_{ab}\eta^{\mathrm{TT}}_{cd} = \left(k^2 - \Lambda_l + \frac{1}{2r^2}N(N-1) - \frac{N+d}{N+d-2}\Lambda\right)\eta^{\mathrm{TT}}_{ab}, \tag{69}$$

for $l = 2, 3, \cdots$, with degeneracy

$$D_l^{(t)}(N) = \frac{(N+1)(N-2)(l+N)(l-1)(2l+N-1)(l+N-3)!}{2(N-1)!(l+1)!}. \tag{70}$$

$$\mathcal{G}^{-1}{}^{cd}_{ab}\eta^{\rm LTT}_{cd} = \left(k^2 - \Lambda_l + \frac{1}{2r^2}N(N-3) - \frac{N+d}{N+d-2}\Lambda\right)\eta^{\rm LTT}_{ab}, \tag{71}$$

for $l = 2, 3, \cdots$, with degeneracy $D^{(v)}_l(N)$.

$$\mathcal{G}^{-1}{}^{cd}_{ab}\eta^{\rm LLT}_{cd} = \left(k^2 - \Lambda_l + \frac{1}{2r^2}(N-1)(N-4) - \frac{N+d}{N+d-2}\Lambda\right)\eta^{\rm LLT}_{ab}, \tag{72}$$

for $l = 2, 3, \cdots$, with degeneracy $D^{(s)}_l(N)$.

$$\begin{aligned}\mathcal{G}^{-1}{}^{cd}_{ab}\eta^{\rm Tr}_{cd} \;=\; & \left(k^2 - \Lambda_l + \frac{1}{2r^2}\frac{N^3 + N^2(2d-7) - 2N(3d-7) + 4(d-2)}{N+d-2}\right. \\ & \left. - \frac{N+d}{N+d-2}\Lambda\right)\eta^{\rm Tr}_{ab},\end{aligned} \tag{73}$$

for $l = 0, 1, 2, \cdots$, with degeneracy $D^{(s)}_l(N)$. We also have the off-diagonal terms,

$$\mathcal{G}^{-1}{}^{ab}_{\mu\nu}\eta^{\rm Tr}_{ab} \;=\; -\frac{(N-1)(N-2)d}{2r^2(N+d-2)}\eta^{\rm Tr}_{\mu\nu}, \tag{74}$$

$$\mathcal{G}^{-1}{}^{\mu\nu}_{ab}\eta^{\rm Tr}_{\mu\nu} \;=\; \frac{N(N-1)(d-2)}{2r^2(N+d-2)}\eta^{\rm Tr}_{ab}. \tag{75}$$

The eigenfunctions $\eta^{\rm Tr}_{\mu\nu}$ and $\eta^{\rm Tr}_{ab}$ are seen to be coupled together, and $\mathcal{G}^{-1}$ forms a 2×2 matrix $\mathcal{G}^{-1}_{\rm Tr}$ in the subspace of these eigenfunctions. The determinant of this matrix is

$$\begin{aligned}\det\mathcal{G}^{-1}_{\rm Tr} \;=\; & \left[k^2 - \Lambda_l + \frac{1}{2r^2}\frac{N(N-1)(N-2)}{N+d-2} - \frac{N+d}{N+d-2}\Lambda\right] \\ & \left[k^2 - \Lambda_l + \frac{1}{2r^2}\frac{N^3 + N^2(2d-7) - 2N(3d-7) + 4(d-2)}{N+d-2} - \frac{N+d}{N+d-2}\Lambda\right] \\ & + \frac{N(N-1)^2(N-2)d(d-2)}{4r^4(N+d-2)^2}.\end{aligned} \tag{76}$$

There are also eigenfunctions $\eta^{\rm T,T}_{\nu b}$, $\eta^{\rm T,L}_{\nu b}$, $\eta^{\rm L,T}_{\nu b}$, and $\eta^{\rm L,L}_{\nu b}$. $\eta^{\rm T,T}_{\nu b}$ is a transverse vector on M^d as well as on S^N, and so on. $\eta^{\rm T,T}_{\nu b}$ has $(d-1)$ components on M^d,

$$\mathcal{G}^{-1}{}^{\nu b}_{\mu a}\eta^{\rm T,T}_{\nu b} = \frac{1}{2}\left(k^2 - \Lambda_l + \frac{1}{2r^2}(N+1)(N-2) - \frac{N+d}{N+d-2}\Lambda\right)\eta^{\rm T,T}_{\mu a}, \tag{77}$$

for $l = 1, 2, \cdots$, with $D^{(v)}_l(N)$. $\eta^{\rm T,L}_{\nu b}$ has $(d-1)$ components on M^d,

$$\mathcal{G}^{-1}{}^{\nu b}_{\mu a}\eta^{\rm T,L}_{\nu b} = \frac{1}{2}\left(k^2 - \Lambda_l + \frac{1}{2r^2}(N-1)(N-2) - \frac{N+d}{N+d-2}\Lambda\right)\eta^{\rm T,L}_{\mu a}, \tag{78}$$

for $l = 1, 2, \cdots$, with $D^{(s)}_l(N)$. $\eta^{\rm L,T}_{\nu b}$ has 1 components on M^d,

$$\mathcal{G}^{-1}{}^{\nu b}_{\mu a}\eta^{\rm L,T}_{\nu b} = \frac{1}{2}\left(k^2 - \Lambda_l + \frac{1}{2r^2}(N+1)(N-2) - \frac{N+d}{N+d-2}\Lambda\right)\eta^{\rm L,T}_{\mu a}, \tag{79}$$

for $l = 1, 2, \cdots$, with $D_l^{(v)}(N)$. $\eta_{\nu b}^{\mathrm{L,L}}$ has 1 components on M^d,

$$\mathcal{G}^{-1}{}^{\nu b}_{\mu a}\eta_{\nu b}^{\mathrm{L,L}} = \frac{1}{2}\left(k^2 - \Lambda_l + \frac{1}{2r^2}(N-1)(N-2) - \frac{N+d}{N+d-2}\Lambda\right)\eta_{\mu a}^{\mathrm{L,L}}, \tag{80}$$

for $l = 1, 2, \cdots$, with $D_l^{(s)}(N)$.

For $U_1{}^{\mu}_{\nu}$, the eigenfunction η^{ν} is a vector on M^d and a scalar on S^N. η^{ν} has d components on M^d,

$$\begin{aligned} U_1{}^{\mu}_{\nu}\eta^{\nu} \;=\; & \left[\left(\frac{1}{2r^2}N(N-1) - \frac{N+d}{N+d-2}\Lambda\right)(k^2)\right. \\ & \left. - \left(\frac{1}{2r^2}(N-1)(N-2) - \frac{N+d}{N+d-2}\Lambda\right)(\Lambda_l)\right]\left(\frac{1}{k^2-\Lambda_l}\right)^2\eta^{\mu}, \end{aligned} \tag{81}$$

for $l = 0, 1, 2, \cdots$, with degeneracy $D_l^{(s)}(N)$.

For $U_1{}^{a}_{b}$, the eigenfunctions are $\eta^{b\mathrm{T}}$ and $\eta^{b\mathrm{L}}$.

$$\begin{aligned} U_1{}^{a}_{b}\eta^{b\mathrm{T}} \;=\; & \left[\left(\frac{1}{2r^2}(N+2)(N-1) - \frac{N+d}{N+d-2}\Lambda\right)(k^2)\right. \\ & \left. - \left(\frac{1}{2r^2}N(N-1) - \frac{N+d}{N+d-2}\Lambda\right)\left(\Lambda_l + \frac{N}{r^2}\right)\right]\left(\frac{1}{k^2-\Lambda_l-\frac{N}{r^2}}\right)^2\eta^{a\mathrm{T}}, \end{aligned} \tag{82}$$

for $l = 1, 2, \cdots$, with degeneracy $D_l^{(v)}(N)$. And

$$\begin{aligned} U_1{}^{a}_{b}\eta^{b\mathrm{L}} \;=\; & \left[\left(\frac{1}{2r^2}(N+2)(N-1) - \frac{N+d}{N+d-2}\Lambda\right)(k^2)\right. \\ & \left. - \left(\frac{1}{2r^2}N(N-1) - \frac{N+d}{N+d-2}\Lambda\right)\left(\Lambda_l + \frac{2}{r^2}(N-1)\right)\right] \\ & \left(\frac{1}{k^2-\Lambda_l-\frac{2}{r^2}(N-1)}\right)^2\eta^{a\mathrm{L}}, \end{aligned} \tag{83}$$

for $l = 1, 2, \cdots$, with degeneracy $D_l^{(s)}(N)$.

The eigenvalues of U_2 are more complicated. After lengthy calculations, we obtained the following results. For the eigenfunction $\eta^{\nu\mathrm{T}}$, which has $(d-1)$ components on M^d,

$$\begin{aligned} & U_2{}^{\mu}_{\nu}\eta^{\nu\mathrm{T}} \\ = & \left[\left(\frac{1}{2r^2}N(N-1) - \frac{N+d}{N+d-2}\Lambda\right)^2(k^2)\left(\frac{1}{k^2-\Lambda_l+\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\Lambda}\right)\right. \\ & - \left(\frac{1}{2r^2}(N-1)(N-2) - \frac{N+d}{N+d-2}\Lambda\right)^2(\Lambda_l) \\ & \left.\left(\frac{1}{k^2-\Lambda_l+\frac{1}{2r^2}(N-1)(N-2)-\frac{N+d}{N+d-2}\Lambda}\right)\right]\left(\frac{1}{k^2-\Lambda_l}\right)^2\eta^{\mu\mathrm{T}}, \end{aligned} \tag{84}$$

for $l=0,1,2,\cdots$, with degeneracy $D_l^{(s)}(N)$. For eigenfunction $\eta^{b^{\mathrm{T}}}$, which is a scalar on M^d,

$$
\begin{aligned}
& U_{2\,b}^{\ a}\eta^{b^{\mathrm{T}}} \\
= & \left[\left(\frac{1}{2r^2}(N+2)(N-1)-\frac{N+d}{N+d-2}\Lambda\right)^2 (k^2)\left(\frac{1}{k^2-\Lambda_l+\frac{1}{2r^2}(N+1)(N-2)-\frac{N+d}{N+d-2}\Lambda}\right)\right. \\
& -\left(\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\Lambda\right)^2\left(\Lambda_l+\frac{N}{r^2}\right) \\
& \left.\left(\frac{1}{k^2-\Lambda_l+\frac{1}{2r^2}N(N-3)-\frac{N+d}{N+d-2}\Lambda}\right)\right]\left(\frac{1}{k^2-\Lambda_l-\frac{N}{r^2}}\right)^2\eta^{a^{\mathrm{T}}},
\end{aligned} \tag{85}
$$

for $l=1,2,\cdots$, with degeneracy $D_l^{(v)}(N)$.

For the eigenfunctions $\eta^{\nu^{\mathrm{L}}}$ and $\eta^{b^{\mathrm{L}}}$, they are coupled together. For $l=0$,

$$
\begin{aligned}
& U_{2\,\nu}^{\ \mu}\eta^{\nu^{\mathrm{L}}} \\
= & \left[2\left(\frac{d-1}{d}\right)\left(\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\right)^2\left(\frac{1}{k^2}\right)\left(\frac{1}{k^2+\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\Lambda}\right)\right. \\
& -\left(\frac{d-2}{d}\right)\left(\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\Lambda\right)\left(\frac{1}{2r^2}\frac{N(N-1)(N-2)}{N+d-2}-\frac{N+d}{N+d-2}\Lambda\right) \\
& \left(k^2+\frac{1}{2r^2}\frac{N^3+N^2(2d-7)-2N(3d-7)+4(d-2)}{N+d-2}-\frac{N+d}{N+d-2}\Lambda\right)\left(\frac{1}{k^2}\right)\frac{1}{\det\mathcal{G}_{l=0}^{-1}} \\
& -\frac{N(N-1)^2(N-2)(d-2)^2}{4r^4(N+d-2)^2}\left(\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\Lambda\right)\left(\frac{1}{k^2}\right)\frac{1}{(\det\mathcal{G}_{\mathrm{Tr}}^{-1})_{l=0}} \\
& \left.-\frac{N(N-1)(d-2)}{2r^2(N+d-2)}\left(\frac{1}{2r^2}(N-1)(N-2)-\frac{N+d}{N+d-2}\Lambda\right)\frac{1}{(\det\mathcal{G}_{\mathrm{Tr}}^{-1})_{l=0}}\right]\eta^{\mu^{\mathrm{L}}},
\end{aligned} \tag{86}
$$

where $(\det\mathcal{G}_{\mathrm{Tr}}^{-1})_{l=0}$ is the expression $\det\mathcal{G}_{\mathrm{Tr}}^{-1}$ in Eq. (76) evaluated at $l=0$. The other eigenfunctions do not contribute in this case. For $l=1,2,\cdots$,

$$
\begin{aligned}
& U_{2\,\nu}^{\ \mu}\eta^{\nu^{\mathrm{L}}} \\
= & \left[2\left(\frac{d-1}{d}\right)\left(\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\right)^2 (k^2)\right. \\
& \left(\frac{1}{k^2-\Lambda_l}\right)^2\left(\frac{1}{k^2-\Lambda_l+\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\Lambda}\right) \\
& -\left(\frac{1}{2r^2}(N-1)(N-2)-\frac{N+d}{N+d-2}\Lambda\right)^2(\Lambda_l) \\
& \left(\frac{1}{k^2-\Lambda_l}\right)^2\left(\frac{1}{k^2-\Lambda_l+\frac{1}{2r^2}(N-1)(N-2)-\frac{N+d}{N+d-2}\Lambda}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{d-2}{d}\right)\left(\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\Lambda\right)\left(\frac{1}{2r^2}\frac{N(N-1)(N-2)}{N+d-2}-\frac{N+d}{N+d-2}\Lambda\right)(k^2)\\
&\left(k^2-\Lambda_l+\frac{1}{2r^2}\frac{N^3+N^2(2d-7)-2N(3d-7)+4(d-2)}{N+d-2}-\frac{N+d}{N+d-2}\Lambda\right)\\
&\left(\frac{1}{k^2-\Lambda_l}\right)^2\frac{1}{\det\mathcal{G}_{\mathrm{Tr}}^{-1}}\\
&-\frac{N(N-1)^2(N-2)(d-2)^2}{4r^4(N+d-2)^2}\left(\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\Lambda\right)(k^2)\left(\frac{1}{k^2-\Lambda_l}\right)^2\frac{1}{\det\mathcal{G}_{\mathrm{Tr}}^{-1}}\\
&-\frac{N(N-1)(d-2)}{2r^2(N+d-2)}\left(\frac{1}{2r^2}(N-1)(N-2)-\frac{N+d}{N+d-2}\Lambda\right)(k^2)\left(\frac{1}{k^2-\Lambda_l}\right)\frac{1}{\det\mathcal{G}_{\mathrm{Tr}}^{-1}}\Bigg]\eta^{\mu\mathrm{L}},
\end{aligned}
\tag{87}
$$

$$
\begin{aligned}
&U_{2}{}_{\nu}^{a}\eta^{\nu\mathrm{L}}\\
=&\Bigg[\left(\frac{1}{2r^2}(N+2)(N-1)-\frac{N+d}{N+d-2}\Lambda\right)\left(\frac{1}{2r^2}(N-1)(N-2)-\frac{N+d}{N+d-2}\Lambda\right)(k^2)\\
&\left(\frac{1}{k^2-\Lambda_l}\right)\left(\frac{1}{k^2-\Lambda_l-\frac{2}{r^2}(N-1)}\right)\left(\frac{1}{k^2-\Lambda_l+\frac{1}{2r^2}(N-1)(N-2)-\frac{N+d}{N+d-2}\Lambda}\right)\\
&-\left(\frac{1}{2r^2}(N+2)(N-1)-\frac{N+d}{N+d-2}\Lambda\right)\left(\frac{1}{2r^2}\frac{N(N-1)(N-2)}{N+d-2}-\frac{N+d}{N+d-2}\Lambda\right)(k^2)\\
&\left(k^2-\Lambda_l+\frac{1}{2r^2}\frac{N^3+N^2(2d-7)-2N(3d-7)+4(d-2)}{N+d-2}-\frac{N+d}{N+d-2}\Lambda\right)\\
&\left(\frac{1}{k^2-\Lambda_l}\right)\left(\frac{1}{k^2-\Lambda_l-\frac{2}{r^2}(N-1)}\right)\frac{1}{\det\mathcal{G}_{\mathrm{Tr}}^{-1}}\\
&-\frac{N(N-1)^2(N-2)d(d-2)}{4r^4(N+d-2)^2}\left(\frac{1}{2r^2}(N+2)(N-1)-\frac{N+d}{N+d-2}\Lambda\right)(k^2)\\
&\left(\frac{1}{k^2-\Lambda_l}\right)\left(\frac{1}{k^2-\Lambda_l-\frac{2}{r^2}(N-1)}\right)\frac{1}{\det\mathcal{G}_{\mathrm{Tr}}^{-1}}\\
&-\frac{(N-1)(N-2)(d-2)}{2r^2(N+d-2)}\left(\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\Lambda\right)(k^2)\\
&\left(\frac{1}{k^2-\Lambda_l-\frac{2}{r^2}(N-1)}\right)\frac{1}{\det\mathcal{G}_{\mathrm{Tr}}^{-1}}\Bigg]\eta^{a\mathrm{L}},
\end{aligned}
\tag{88}
$$

$$
\begin{aligned}
&U_{2}{}_{b}^{\mu}\eta^{b\mathrm{L}}\\
=&\Bigg[-\left(\frac{1}{2r^2}(N-1)(N-2)-\frac{N+d}{N+d-2}\Lambda\right)\left(\frac{1}{2r^2}(N+2)(N-1)-\frac{N+d}{N+d-2}\Lambda\right)(\Lambda_l)\\
&\left(\frac{1}{k^2-\Lambda_l}\right)\left(\frac{1}{k^2-\Lambda_l-\frac{2}{r^2}(N-1)}\right)\left(\frac{1}{k^2-\Lambda_l+\frac{1}{2r^2}(N-1)(N-2)-\frac{N+d}{N+d-2}\Lambda}\right)\\
&+\left(\frac{1}{2r^2}(N-1)(N-2)-\frac{N+d}{N+d-2}\Lambda\right)
\end{aligned}
$$

$$
\begin{aligned}
&\left(\frac{1}{2r^2}\frac{N(N-1)(N+2d-2)}{N+d-2}-\frac{N+d}{N+d-2}\Lambda\right)(\Lambda_l)\\
&\left(k^2-\Lambda_l+\frac{1}{2r^2}\frac{N(N-1)(N-2)}{N+d-2}-\frac{N+d}{N+d-2}\Lambda\right)\left(\frac{1}{k^2-\Lambda_l}\right)\\
&\left(\frac{1}{k^2-\Lambda_l-\frac{2}{r^2}(N-1)}\right)\frac{1}{\det\mathcal{G}_{\mathrm{Tr}}^{-1}}\\
&+\frac{N(N-1)^2(N-2)d(d-2)}{4r^4(N+d-2)^2}\left(\frac{1}{2r^2}(N-1)(N-2)-\frac{N+d}{N+d-2}\Lambda\right)(\Lambda_l)\\
&\left(\frac{1}{k^2-\Lambda_l}\right)\left(\frac{1}{k^2-\Lambda_l-\frac{2}{r^2}(N-1)}\right)\frac{1}{\det\mathcal{G}_{\mathrm{Tr}}^{-1}}\\
&-\frac{(N-1)(N-2)(d-2)}{2r^2(N+d-2)}\left(\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\Lambda\right)(\Lambda_l)\\
&\left(k^2-\Lambda_l-\frac{2}{r^2}\frac{N^2+N(d-3)-(d-2)}{N+d-2}\right)\left(\frac{1}{k^2-\Lambda_l}\right)\\
&\left.\left(\frac{1}{k^2-\Lambda_l-\frac{2}{r^2}(N-1)}\right)\frac{1}{\det\mathcal{G}_{\mathrm{Tr}}^{-1}}\right]\eta^{\mu\mathrm{L}},
\end{aligned}
\tag{89}
$$

$$
\begin{aligned}
&U_{2\,b}^{\;a}\eta^{b\mathrm{L}}\\
=&\left[-2\left(\frac{N-1}{N}\right)\left(\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\right)^2\left(\Lambda_l+\frac{N}{r^2}\right)\right.\\
&\left(\frac{1}{k^2-\Lambda_l-\frac{2}{r^2}(N-1)}\right)^2\left(\frac{1}{k^2-\Lambda_l+\frac{1}{2r^2}(N-1)(N-4)-\frac{N+d}{N+d-2}\Lambda}\right)\\
&+(\frac{1}{2r^2}(N+2)(N-1)-\frac{N+d}{N+d-2}\Lambda\Big)^2(k^2)\\
&\left(\frac{1}{k^2-\Lambda_l-\frac{2}{r^2}(N-1)}\right)^2\left(\frac{1}{k^2-\Lambda_l+\frac{1}{2r^2}(N-1)(N-2)-\frac{N+d}{N+d-2}\Lambda}\right)\\
&+\left(\frac{N-2}{N}\right)\left(\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\Lambda\right)\\
&\left(\frac{1}{2r^2}\frac{N(N-1)(N+2d-2)}{N+d-2}-\frac{N+d}{N+d-2}\Lambda\right)(\Lambda_l)\\
&\left(k^2-\Lambda_l+\frac{1}{2r^2}\frac{N(N-1)(N-2)}{N+d-2}-\frac{N+d}{N+d-2}\Lambda\right)\left(\frac{1}{k^2-\Lambda_l-\frac{2}{r^2}(N-1)}\right)^2\frac{1}{\det\mathcal{G}_{\mathrm{Tr}}^{-1}}\\
&+\frac{(N-1)^2(N-2)^2d(d-2)}{4r^4(N+d-2)^2}\left(\frac{1}{2r^2}N(N-1)-\frac{N+d}{N+d-2}\Lambda\right)(\Lambda_l)\\
&\left(\frac{1}{k^2-\Lambda_l-\frac{2}{r^2}(N-1)}\right)^2\frac{1}{\det\mathcal{G}_{\mathrm{Tr}}^{-1}}\\
&-\frac{(N-1)(N-2)d}{2r^2(N+d-2)}\left(\frac{1}{2r^2}(N+2)(N-1)-\frac{N+d}{N+d-2}\Lambda\right)(\Lambda_l)
\end{aligned}
$$

$$\left(k^2-\Lambda_l-\frac{2}{r^2}\frac{N^2+N(d-3)-(d-2)}{N+d-2}\right)\left(\frac{1}{k^2-\Lambda_l-\frac{2}{r^2}(N-1)}\right)^2\frac{1}{\det\mathcal{G}_{\mathrm{Tr}}^{-1}}\Bigg]\eta^{b\mathrm{L}}. \tag{90}$$

Both $\eta^{\nu\mathrm{L}}$ and $\eta^{b\mathrm{L}}$ have degeneracy $D_l^{(s)}(N)$.

C. VD effective actions for gravitons on $M^4\times S^N$

With the eigenvalues of the various operators in the last subsection and the divergent parts of their traces listed in the Appendix, we can evaluate the divergent parts of the VD effective action on $M^4\times S^N$.

Let us start with $\mathrm{Trln}N^{-1}$, the ghost contribution to the VD effection action. From Eqs. (58), (62), and (64), we obtain on $M^4\times S^N$,

$$\begin{aligned}\mathrm{Trln}N^{-1}\Big|^{div} &= d\sum_k\sum_{l=0}^{\infty}D_l^{(s)}(N)\ln(k^2-\Lambda_l)+\sum_k\sum_{l=1}^{\infty}D_l^{(v)}(N)\ln\left(k^2-\Lambda_l-\frac{N}{r^2}\right)\\&\quad+\sum_k\sum_{l=1}^{\infty}D_l^{(s)}(N)\ln\left(k^2-\Lambda_l-\frac{2}{r^2}(N-1)\right)\Bigg|^{div}.\end{aligned} \tag{91}$$

If we start the summations over l from $l=0$,

$$\begin{aligned}\mathrm{Trln}N^{-1}\Big|^{div} &= d\sum_k\sum_{l=0}^{\infty}D_l^{(s)}(N)\ln(k^2-\Lambda_l)+\sum_k\sum_{l=0}^{\infty}D_l^{(v)}(N)\ln\left(k^2-\Lambda_l-\frac{N}{r^2}\right)\\&\quad+\sum_k\sum_{l=0}^{\infty}D_l^{(s)}(N)\ln\left(k^2-\Lambda_l-\frac{2}{r^2}(N-1)\right)\\&\quad-\delta_{N2}\sum_k\ln\left(k^2-\frac{N}{r^2}\right)-\sum_k\ln\left(k^2-\frac{2}{r^2}(N-1)\right)\Bigg|^{div}\\&= d\sum_k\sum_{l=0}^{\infty}D_l^{(s)}(N)\ln(k^2-\Lambda_l)\\&\quad+\sum_k\sum_{l=0}^{\infty}D_l^{(v)}(N)\ln(k^2-\Lambda_l)-\sum_k\sum_{l=0}^{\infty}D_l^{(v)}(N)\sum_{q=1}^{\infty}\frac{1}{q}\left(\frac{N}{r^2}\right)^q\left(\frac{1}{k^2-\Lambda_l}\right)^q\\&\quad+\sum_k\sum_{l=0}^{\infty}D_l^{(s)}(N)\ln(k^2-\Lambda_l)-\sum_k\sum_{l=0}^{\infty}D_l^{(s)}(N)\sum_{q=1}^{\infty}\frac{1}{q}\left(\frac{2(N-1)}{r^2}\right)^q\left(\frac{1}{k^2-\Lambda_l}\right)^q\\&\quad-\delta_{N2}\sum_k\ln k^2+\delta_{N2}\sum_k\sum_{q=1}^{\infty}\frac{1}{q}\left(\frac{N}{r^2}\right)^q\left(\frac{1}{k^2}\right)^q\\&\quad-\sum_k\ln k^2+\sum_k\sum_{q=1}^{\infty}\frac{1}{q}\left(\frac{2(N-1)}{r^2}\right)^q\left(\frac{1}{k^2}\right)^q\Bigg|^{div},\end{aligned} \tag{92}$$

where we have Taylor-expanded the logarithmic function. Using the results in the Appendix,

$$\begin{aligned}
&\mathrm{Trln}N^{-1}\Big|^{div} \\
&= \frac{iV_4}{(4\pi r^2)^2\epsilon}\Bigg[F^{(v)}(N)+5F^{(s)}(N)-\sum_{q=1}^{\infty}\frac{1}{q}\left(\frac{N}{r^2}\right)^q G^{(v)}(0,q,N) \\
&\quad -\sum_{q=1}^{\infty}\frac{1}{q}\left(\frac{2(N-1)}{r^2}\right)^q G^{(s)}(0,q,N)+\delta_{N2}\left(\frac{1}{2}\right)\left(\frac{N}{r^2}\right)^2(2r^4)+\left(\frac{1}{2}\right)\left(\frac{2(N-1)}{r^2}\right)^2(2r^4)\Bigg].
\end{aligned} \tag{93}$$

Therefore, for $N=2$,

$$\mathrm{Trln}N^{-1}\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[-\frac{24}{35}\right]. \tag{94}$$

For $N=4$,

$$\mathrm{Trln}N^{-1}\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[-\frac{4640}{189}\right]. \tag{95}$$

For $N=6$,

$$\mathrm{Trln}N^{-1}\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[-\frac{1232816}{10395}\right]. \tag{96}$$

Similarly for $\mathrm{Trln}\mathcal{G}^{-1}$, we have for $N=2$,

$$\mathrm{Trln}\mathcal{G}^{-1}\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[\frac{106}{15}-\frac{257}{10}(\Lambda r^2)+36(\Lambda r^2)^2-\frac{189}{8}(\Lambda r^2)^3\right]. \tag{97}$$

For $N=4$,

$$\mathrm{Trln}\mathcal{G}^{-1}\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[-\frac{7574}{45}+\frac{67552}{315}(\Lambda r^2)-\frac{13504}{135}(\Lambda r^2)^2+\frac{1664}{81}(\Lambda r^2)^3-\frac{128}{81}(\Lambda r^2)^4\right]. \tag{98}$$

For $N=6$,

$$\begin{aligned}
\mathrm{Trln}\mathcal{G}^{-1}\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\quad&\left[\frac{5833069}{3360}-\frac{26312047}{24192}(\Lambda r^2)+\frac{6498535}{24192}(\Lambda r^2)^2-\frac{8375}{256}(\Lambda r^2)^3\right. \\
&\left.+\frac{18125}{9216}(\Lambda r^2)^4-\frac{6875}{147456}(\Lambda r^2)^5\right].
\end{aligned} \tag{99}$$

Next, for the operator U_1, one can evaluate the trace of some general power of the operator. The results are the following. For $N=2$,

$$\mathrm{Tr}U_1|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[\frac{148}{15}-\frac{26}{5}(\Lambda r^2)\right], \tag{100}$$

$$\mathrm{Tr}U_1^2\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[24-40(\Lambda r^2)+18(\Lambda r^2)^2\right], \tag{101}$$

$$\mathrm{Tr}U_1^3\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[\frac{57}{5}-\frac{69}{2}(\Lambda r^2)+\frac{81}{2}(\Lambda r^2)^2-\frac{81}{4}(\Lambda r^2)^3\right]. \tag{102}$$

For $N = 4$,

$$\mathrm{Tr}U_1\big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[\frac{13432}{45} - \frac{95504}{2835}(\Lambda r^2)\right], \tag{103}$$

$$\mathrm{Tr}U_1^2\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[\frac{6908}{5} - \frac{3808}{9}(\Lambda r^2) + \frac{13312}{405}(\Lambda r^2)^2\right], \tag{104}$$

$$\mathrm{Tr}U_1^3\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[\frac{9612}{5} - 996(\Lambda r^2) + \frac{1600}{9}(\Lambda r^2)^2 - \frac{896}{81}(\Lambda r^2)^3\right], \tag{105}$$

$$\begin{aligned}\mathrm{Tr}U_1^4\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}&\left[\frac{29412}{35} - \frac{3136}{5}(\Lambda r^2) + \frac{2752}{15}(\Lambda r^2)^2\right.\\ &\left.-\frac{2048}{81}(\Lambda r^2)^3 + \frac{1024}{729}(\Lambda r^2)^4\right].\end{aligned} \tag{106}$$

For $N = 6$,

$$\mathrm{Tr}U_1\big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[\frac{1670863}{945} - \frac{67301}{756}(\Lambda r^2)\right], \tag{107}$$

$$\mathrm{Tr}U_1^2\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[\frac{803450}{63} - \frac{101855}{63}(\Lambda r^2) + \frac{77815}{1512}(\Lambda r^2)^2\right], \tag{108}$$

$$\mathrm{Tr}U_1^3\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[\frac{193775}{6} - \frac{53575}{8}(\Lambda r^2) + \frac{22475}{48}(\Lambda r^2)^2 - \frac{2125}{192}(\Lambda r^2)^3\right], \tag{109}$$

$$\begin{aligned}\mathrm{Tr}U_1^4\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}&\left[\frac{2197000}{63} - \frac{91375}{9}(\Lambda r^2) + 1125(\Lambda r^2)^2\right.\\ &\left.-\frac{8125}{144}(\Lambda r^2)^3 + \frac{625}{576}(\Lambda r^2)^4\right],\end{aligned} \tag{110}$$

$$\begin{aligned}\mathrm{Tr}U_1^5\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}&\left[\frac{41524375}{3024} - \frac{996875}{192}(\Lambda r^2) + \frac{1611875}{2016}(\Lambda r^2)^2 - \frac{18125}{288}(\Lambda r^2)^3\right.\\ &\left.+\frac{15625}{6144}(\Lambda r^2)^4 - \frac{3125}{73728}(\Lambda r^2)^5\right].\end{aligned} \tag{111}$$

For traces involving the operator U_2, the calculation is more complicated. Following the same procedure as above, we obtain, for $N = 2$,

$$\mathrm{Tr}U_2\big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[10 + \frac{1}{2}(\Lambda r^2) - \frac{45}{2}(\Lambda r^2)^2 + \frac{81}{4}(\Lambda r^2)^3\right], \tag{112}$$

$$\mathrm{Tr}U_1U_2\big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[\frac{151}{12} - \frac{73}{2}(\Lambda r^2) + \frac{81}{2}(\Lambda r^2)^2 - \frac{81}{4}(\Lambda r^2)^3\right]. \tag{113}$$

For $N = 4$,

$$\begin{aligned}\mathrm{Tr}U_2\big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}&\left[\frac{1724}{5} - \frac{328}{3}(\Lambda r^2) + \frac{20432}{405}(\Lambda r^2)^2\right.\\ &\left.-\frac{128}{9}(\Lambda r^2)^3 + \frac{1024}{729}(\Lambda r^2)^4\right],\end{aligned} \tag{114}$$

$$\mathrm{Tr}U_1U_2\big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\left[\frac{5211}{5} - \frac{4976}{15}(\Lambda r^2) - \frac{176}{15}(\Lambda r^2)^2\right.$$

$$+\frac{128}{9}(\Lambda r^2)^3-\frac{1024}{729}(\Lambda r^2)^4\Big], \qquad (115)$$

$$\mathrm{Tr}U_1^2U_2\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\Big[\frac{4419}{5}-\frac{1936}{3}(\Lambda r^2)+\frac{5008}{27}(\Lambda r^2)^2 -\frac{2048}{81}(\Lambda r^2)^3+\frac{1024}{729}(\Lambda r^2)^4\Big], \qquad (116)$$

$$\mathrm{Tr}U_2^2\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\Big[\frac{4657}{5}-\frac{9952}{15}(\Lambda r^2)+\frac{25312}{135}(\Lambda r^2)^2 -\frac{2048}{81}(\Lambda r^2)^3+\frac{1024}{729}(\Lambda r^2)^4\Big]. \qquad (117)$$

For $N=6$,

$$\mathrm{Tr}U_2\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\Big[\frac{2356505}{2016}+\frac{10560265}{32256}(\Lambda r^2)-\frac{12249815}{96768}(\Lambda r^2)^2 +\frac{717125}{36864}(\Lambda r^2)^3-\frac{26875}{18432}(\Lambda r^2)^4+\frac{3125}{73728}(\Lambda r^2)^5\Big], \qquad (118)$$

$$\mathrm{Tr}U_1U_2\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\Big[\frac{26262425}{2304}-\frac{2887775}{1536}(\Lambda r^2)+\frac{3084025}{18432}(\Lambda r^2)^2 -\frac{230125}{12288}(\Lambda r^2)^3+\frac{26875}{18432}(\Lambda r^2)^4-\frac{3125}{73728}(\Lambda r^2)^5\Big], \qquad (119)$$

$$\mathrm{Tr}U_1^2U_2\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\Big[\frac{3026375}{144}-\frac{39073375}{8064}(\Lambda r^2)+\frac{9993125}{32256}(\Lambda r^2)^2 +\frac{43625}{6144}(\Lambda r^2)^3-\frac{26875}{18432}(\Lambda r^2)^4+\frac{3125}{73728}(\Lambda r^2)^5\Big], \qquad (120)$$

$$\mathrm{Tr}U_1^3U_2\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\Big[\frac{5418875}{384}-\frac{3549125}{672}(\Lambda r^2)+\frac{7434625}{9216}(\Lambda r^2)^2 -\frac{387875}{6144}(\Lambda r^2)^3+\frac{15625}{6144}(\Lambda r^2)^4-\frac{3125}{73728}(\Lambda r^2)^5\Big], \qquad (121)$$

$$\mathrm{Tr}U_2^2\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\Big[\frac{609125}{96}+\frac{998125}{1536}(\Lambda r^2)-\frac{2394875}{4608}(\Lambda r^2)^2 +\frac{163625}{2304}(\Lambda r^2)^3-\frac{36875}{9216}(\Lambda r^2)^4+\frac{3125}{36864}(\Lambda r^2)^5\Big], \qquad (122)$$

$$\mathrm{Tr}U_1U_2^2\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\Big[\frac{117011375}{8064}-\frac{38508875}{7168}(\Lambda r^2)+\frac{26252375}{32256}(\Lambda r^2)^2 -\frac{583625}{9216}(\Lambda r^2)^3+\frac{15625}{6144}(\Lambda r^2)^4-\frac{3125}{73728}(\Lambda r^2)^5\Big]. \qquad (123)$$

Putting all these together, we can obtain the divergent parts of the VD effective actions for various $M^4\times S^N$ Kaluza-Klein spaces. For $M^4\times S^2$, we have

$$iW_{unique}^{div} = \mathrm{Tr}\ln N^{-1}-\frac{1}{2}\mathrm{Tr}\ln\mathcal{G}^{-1}+\frac{1}{2}\mathrm{Tr}U_1+\frac{1}{4}\mathrm{Tr}U_1^2-\frac{1}{2}\mathrm{Tr}U_2+\frac{1}{6}\mathrm{Tr}U_1^3-\frac{1}{2}\mathrm{Tr}U_1U_2\Big|^{div} = \frac{iV_4}{(4\pi r^2)^2\epsilon}\Big[-\frac{2249}{840}+\frac{25}{2}(\Lambda r^2)-\frac{63}{4}(\Lambda r^2)^2+\frac{135}{16}(\Lambda r^2)^3\Big]. \qquad (124)$$

This result is consistent with that in ref.[16] where the divergent part of the VD effective action was calculated in a general six-dimensional background spacetime. For $M^4 \times S^4$,

$$
\begin{aligned}
iW^{div}_{unique} &= \mathrm{Trln}N^{-1} - \frac{1}{2}\mathrm{Trln}\mathcal{G}^{-1} + \frac{1}{2}\mathrm{Tr}U_1 + \frac{1}{4}\mathrm{Tr}U_1^2 - \frac{1}{2}\mathrm{Tr}U_2 + \frac{1}{6}\mathrm{Tr}U_1^3 - \frac{1}{2}\mathrm{Tr}U_1U_2 \\
&\quad + \frac{1}{8}\mathrm{Tr}U_1^4 - \frac{1}{2}\mathrm{Tr}U_1^2U_2 + \frac{1}{4}\mathrm{Tr}U_2^2\Big|^{div} \\
&= \frac{iV_4}{(4\pi r^2)^2\epsilon}\Big[\frac{41657}{540} - \frac{7850}{81}(\Lambda r^2) + \frac{6152}{135}(\Lambda r^2)^2 \\
&\quad - \frac{2176}{243}(\Lambda r^2)^3 + \frac{448}{729}(\Lambda r^2)^4\Big]. \qquad (125)
\end{aligned}
$$

Finally for $M^4 \times S^6$ we obtain,

$$
\begin{aligned}
iW^{div}_{unique} &= \mathrm{Trln}N^{-1} - \frac{1}{2}\mathrm{Trln}\mathcal{G}^{-1} + \frac{1}{2}\mathrm{Tr}U_1 + \frac{1}{4}\mathrm{Tr}U_1^2 - \frac{1}{2}\mathrm{Tr}U_2 + \frac{1}{6}\mathrm{Tr}U_1^3 - \frac{1}{2}\mathrm{Tr}U_1U_2 \\
&\quad + \frac{1}{8}\mathrm{Tr}U_1^4 - \frac{1}{2}\mathrm{Tr}U_1^2U_2 + \frac{1}{4}\mathrm{Tr}U_2^2 + \frac{1}{10}\mathrm{Tr}U_1^5 - \frac{1}{2}\mathrm{Tr}U_1^3U_2 + \frac{1}{2}\mathrm{Tr}U_1U_2^2\Big|^{div} \\
&= \frac{iV_4}{(4\pi r^2)^2\epsilon}\Big[-\frac{476483023}{591360} + \frac{10896475}{21504}(\Lambda r^2) - \frac{32112235}{258048}(\Lambda r^2)^2 \\
&\quad + \frac{549625}{36864}(\Lambda r^2)^3 - \frac{10625}{12288}(\Lambda r^2)^4 + \frac{625}{32768}(\Lambda r^2)^5\Big]. \qquad (126)
\end{aligned}
$$

IV. APPLICATIONS

In this section we make two applications of the above effective actions. We give trace anomalies and we search for self-consistent configurations of the internal Kaluza-Klein spheres.

Some explanation of a trace anomaly is in order for gravity. Einstein gravity is not Weyl invariant and the trace part we calculate is NOT a combination of the anomalous and the "normal" contributions. What we calculate below is the "anomalous" part of the trace, as discussed in previous works. In "Trace anomaly for gravitons" [27] Critchley argues that the "anomalous" part of the trace is given by a_2, while the first terms of his (4) gives the "normal" contribution, irrespective to the value of ζ. For Einstein gravity, this is equivalent to his (48), which is the same as our calculation. It is in this sense that we calculate the "anomalous" part of the trace anomaly. This is not the same as the usual trace anomaly in which the original classical action is conformal, while the quantum action is not. For nonconformal theories, Duff [28] in his (29) gave a definition for the anomaly, which we suspect is the same as what we have calculated.

This problem is also discussed in "Quantum fields in curved space" [29]. On page 179, the authors mention that for fields which are not conformally invariant, there will be extra non-anomalous (normal) terms, which are really the same as the first terms of (4) in Critchley's paper.

The paper "Non-conformal renormalized stress tensors in Robertson-Walker space-times", [30] by Bunch and Davies, gave a rather detailed calculation for the stress tensor for a massless, but minimally coupled scalar (therefore a non-conformally invariant field) in a conformal Robertson-Walker spacetime. Their (3.16) gives the total trace of the stress tensor of the quantum theory and (3.18) gives the "normal" part of the trace. Subtracting (3.18) from (3.16) (using Duff's definition) gives (3.19), the "anomalous" trace part which should be equivalent to what we have calculated because (3.19) is proportional to a_2.

In the spirit of the above we can easily determine the gauge-independent VD trace anomaly for gravitons from the divergent part of the effective action of the corresponding Kaluza-Klein space. The appropriate expression is:

$$\begin{aligned} \langle T^{\mu}_{\mu} \rangle_{ren} &= \frac{\epsilon W^{div}_{unique}}{V_4 V_{S^N}}, \\ \text{where} \qquad V_{S^N} &= \frac{2\pi^{(N+1)/2} r^N}{\Gamma\left(\frac{N+1}{2}\right)}, \end{aligned} \tag{127}$$

is the volume of the sphere S^N. Thus, for $M^4 \times S^2$,

$$\langle T^{\mu}_{\mu} \rangle_{ren} = \frac{1}{(4\pi)^3} \left[\frac{135}{16} \Lambda^3 - \frac{63}{4} \left(\frac{\Lambda^2}{r^2} \right) + \frac{25}{2} \left(\frac{\Lambda}{r^4} \right) - \frac{2249}{840} \left(\frac{1}{r^6} \right) \right], \tag{128}$$

for $M^4 \times S^4$,

$$\langle T^{\mu}_{\mu} \rangle_{ren} = \frac{1}{(4\pi)^4} \left[\frac{896}{243} \Lambda^4 - \frac{4352}{81} \left(\frac{\Lambda^3}{r^2} \right) + \frac{12304}{45} \left(\frac{\Lambda^2}{r^4} \right) - \frac{15700}{27} \left(\frac{\Lambda}{r^6} \right) + \frac{41657}{90} \left(\frac{1}{r^8} \right) \right], \tag{129}$$

and for $M^4 \times S^6$,

$$\begin{aligned} \langle T^{\mu}_{\mu} \rangle_{ren} = \frac{1}{(4\pi)^5} \Bigg[& \frac{9375}{8192} \Lambda^5 - \frac{53125}{1024} \left(\frac{\Lambda^4}{r^2} \right) + \frac{336875}{384} \left(\frac{\Lambda^3}{r^4} \right) - \frac{160561175}{21504} \left(\frac{\Lambda^2}{r^6} \right) \\ & + \frac{54482375}{1792} \left(\frac{\Lambda}{r^8} \right) - \frac{476483023}{9856} \left(\frac{1}{r^{10}} \right) \Bigg]. \end{aligned} \tag{130}$$

The pure Λ terms in all three cases can be compared with [20] and the N = 2 case agrees with [16].

Our second use of the VD effective actions computed above is to produce stable configurations of the internal spheres. Because quantum fluctuations of the gravity field itself (about the Kaluza-Klein background) are generating the corrections to the effective potential, any stable configuration should be called a self-consistent dimensionally reduced configuration. For a stable configuration to be of interest it should have a positive renormalized Newton's constant G_0. Most efforts to find such configurations have used the naive effective action (see [2, 25, 26]) and have failed. Stable configurations were found but only with negative gravity constants. Those efforts that attempted to correctly use the VD effective potential have not gotten past the simplest cases: $M^4 \times S^1$ or $M^4 \times S^2$, and $M^4 \times S^1 \times S^1$. For these cases no acceptable configurations were found either [4, 5, 8–11, 15] .

Because of our lack of knowledge of the finite part of the effective potential we will assume that the divergent part ($\propto 1/\epsilon$) dominates at one-loop. We can then easily seek stable configurations for the internal geometry, i.e., configurations where the classical gravity forces balance the quantum gravity (Casimir) pressures. We follow [24] and seek configurations satisfying

$$V(r) = \frac{\partial V}{\partial r} = 0, \quad \frac{\partial^2 V}{\partial r^2} > 0,$$

where the potential $V(r)$ is the negative of the 1-loop corrected effective action:

$$-V(r) \equiv \frac{1}{16\pi G}\left(\frac{N(N-1)}{r^2} - 2\Lambda\right) + \frac{W^{div}_{unique}}{V_4}. \tag{131}$$

The bare value of Newton's 4-dimensional gravity parameter G depends inversely on the sphere's volume V_{S^N} [see (127)], i.e., $G \times V_{S^N}$ is the initial gravity constant in 4+N dimensions. Values for W^{div}_{unique} can be found in (124), (125), and (126).

There are only two static configurations for even dimensions ≤ 6 (see columns 2 and 3 of Table I), one stable and one not.

Columns 4 and 5 relate the bare parameters to their renormalized values. Because Einstein gravity is not renormalizable we are rather unconstrained in our use of the effective actions (potentials). We have done the 1-loop renormalization by rewriting (131) as

$$\begin{aligned} -V(r) &= \frac{1}{16\pi G}\left(\frac{N(N-1)}{r^2} - 2\Lambda\right) \\ &+ \frac{1}{(4\pi)^2\epsilon r^4}\left\{c_0 + c_1(\Lambda r^2) + c_2(\Lambda r^2)^2 + \cdots + c_{N/2+2}(\Lambda r^2)^{N/2+2}\right\}, \end{aligned} \tag{132}$$

TABLE I: The only 2 static configurations for $M^4 \times S^N$ (N = 2, 4, or 6) at 1-loop. Columns 2 and 3 are required by stability ($\partial^2 V/\partial r^2 > 0$), columns 4 and 5 relate the bare parameters G and Λ to the renormalized parameters G_0 and Λ_0, and columns 6-8 are resulting parameter values computed using columns 2-5.

N	Λr^2	$G/\epsilon r^2$	G/G_0	Λ/Λ_0	$r^2\Lambda_0$	$G_0\Lambda_0/\epsilon$	$g^2/G_0\Lambda_0$
2	2.3609	0.17076	-1.3857	4.9834	0.47375	-0.058377	159.15
6	5.6876	-22.931	221.15	2.9948	1.8991	-0.19692	92.636

and collecting the r^N and r^{N-2} terms to get two equations:

$$\frac{1}{16\pi G}(-2\Lambda) + \frac{1}{(4\pi)^2\epsilon}\left\{c_{N/2+2}\Lambda^{N/2+2}r^N\right\} = \frac{1}{16\pi G_0}(-2\Lambda_0)\,, \tag{133}$$

$$\frac{1}{16\pi G}\left(\frac{N(N-1)}{r^2}\right) + \frac{1}{(4\pi)^2\epsilon}\left\{c_{N/2+1}\Lambda^{N/2+1}r^{N-2}\right\} = \frac{1}{16\pi G_0}\left(\frac{N(N-1)}{r^2}\right). \tag{134}$$

The constants $c_{N/2+1}$ and $c_{N/2+2}$ can be read from (124),(125), and (126). Solving these two equations gives:

$$\frac{G}{G_0} = 1 + c_{N/2+1}\left(\Lambda r^2\right)^{N/2+1}(G/\pi\epsilon r^2)/N(N-1), \tag{135}$$

$$\frac{\Lambda}{\Lambda_0} = \frac{1 + c_{N/2+1}\left(\Lambda r^2\right)^{N/2+1}(G/\pi\epsilon r^2)/N(N-1)}{1 - c_{N/2+2}\left(\Lambda r^2\right)^{N/2+1}(G/\pi\epsilon r^2)/2.} \tag{136}$$

Both stationary configurations in the Table have negative values for the external dimensional regularization parameter $\epsilon = 4 - d$ and positive values for the renormalized Newton's and cosmological constants G_0 and Λ_0. The $N = 2$ configuration is stable but requires a negative bare gravity constant G. The $N = 6$ configuration has a positive unrenormalized gravity constant G, but is unstable.

Two input parameters, e.g., G_0 and Λ_0, are required to evaluate all other parameters (see columns 6-8) including the renormalized $O(N+1)$ coupling constant $g^2 = (N+1)8\pi G_0/r^2$, see [24]. As expected, this theory cannot apply to the current phase of the universe where $\Lambda_0 \leq 10^{-52}m^{-2}$ and $G_0 = 2.6 \times 10^{-70}m^2$, but perhaps to one of the pre-inflation phases where $G_0\Lambda_0 \sim 10^{-2}$.

V. CONCLUSIONS

We have demonstrated how to evaluate the divergent part of the VD effective action for Einstein gravity on even-dimensional Kaluza-Klein spacetimes of the form $M^4 \times S^N$. First the effective action is expanded as a series of functional traces of various operators including N^{-1}, $\mathcal{G}^{-1}$, U_1, and U_2 and some of their products. Then the eigenvalues of these operators are obtained by decomposing the corresponding eigenfunctions, vectors or symmetric tensors, into their irreducible parts. Using these eigenvalues, the divergent parts of the traces can be obtained, thus giving the divergent parts of the VD effective action. The formulae used to extract these divergent parts are tabulated in the Appendix for N=2, 4, and 6.

Although the above procedure becomes more and more tedious as one goes to higher dimensions, there is no conceptual difficulty in doing so. One can therefore extend this method to even dimensions with $N \geq 8$, as well as to other more general coset spaces (provided eigenvalues for the corresponding Laplacians are known).

From the divergent parts of the VD effective action, we have obtained the gauge-independent trace anomaly for gravitons on $M^4 \times S^N$. The trace anomaly for gravitons derived from the usual effective action depends on the choice of gauge condition in the off-shell case; however, the VD formalism provides an alternative definition of a unique trace anomaly, even when off-shell [20]. Our final application was an attempt to find self-consistent dimensionally reduced Kaluza-Klein spaces. Only one was found (N=2) and it required that we start with a negative bare Newton's constant. It also possessed much too small of a gauge coupling constant to represent the current stage of the universe.

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In this appendix we evaluate the divergent part of the traces of various operators. We use the dimensional regularization scheme in which the dimension of the external spacetime is taken to be

$$d \to 4 - \epsilon. \tag{137}$$

Then we extract the part which is proportional to $1/\epsilon$ as $\epsilon \to 0$.

The traces that we shall consider are

$$\begin{aligned}
&\sum_k \ln k^2, \\
&\sum_k \left(\frac{1}{k^2}\right)^p, \\
&\sum_k \sum_{l=0}^{\infty} D_l^{(s,v,t)}(N)\ln(k^2 - \Lambda_l(N)) \equiv \frac{iV_4}{(4\pi r^2)^2\epsilon} F^{(s,v,t)}(N), \\
&\sum_k \sum_{l=0}^{\infty} D^{(s,v,t)}(N)(k^2)^p \left(\frac{1}{k^2 - \Lambda_l(N)}\right)^q \equiv \frac{iV_4}{(4\pi r^2)^2\epsilon} G^{(s,v,t)}(p,q,N),
\end{aligned} \tag{138}$$

for $p \geq 0$, $q \geq 1$.

First, using the proper time method, the divergent part of $\sum_k \ln k^2$ can be written as

$$\begin{aligned}
\sum_k \ln k^2 \Big|^{div} &= -\int_0^\infty \frac{d\tau}{\tau} \sum_k e^{-\tau k^2} e^{-\tau m^2} \Big|^{div}_{m\to 0} \\
&= -\frac{iV_d}{(4\pi)^{d/2}} \int_0^\infty d\tau\ \tau^{-d/2-1} e^{-\tau m^2} \Big|^{div}_{m\to 0} \\
&= -\frac{iV_d}{(4\pi)^2} \frac{1}{\epsilon}(m^4) \Big|_{m\to 0} \\
&= 0.
\end{aligned} \tag{139}$$

Similarly,

$$\sum_k \left(\frac{1}{k^2}\right)^p \Big|^{div} = \frac{iV_4}{(4\pi)^2} \frac{1}{\epsilon}(2)\delta_{p2}, \tag{140}$$

that is, the trace has a divergent part only when $p = 2$.

Next, we consider $F^{(s,v,t)}(N)$. For $N = 2$, $D_l^{(t)} = 0$ for $l \geq 1$ and $D_{l=0}^{(t)} = -3$. While

$$D_l^{(v)} = D_l^{(s)} = 2l + 1, \tag{141}$$

and

$$\Lambda_l(2) = -\frac{l(l+1)}{r^2}, \tag{142}$$

for $l \geq 0$. Therefore,

$$\begin{aligned}
\sum_k \sum_{l=0}^{\infty} D_l^{(t)}(2)\ln(k^2 - \Lambda_l(2)) \Big|^{div} &= \sum_k (-3)\ln k^2 \Big|^{div} \\
&= 0,
\end{aligned} \tag{143}$$

and

$$
\begin{aligned}
& \sum_k \sum_{l=0}^{\infty} D_l^{(v)}(2) \ln(k^2 - \Lambda_l(2)) \Big|^{div} \\
= \; & \sum_k \sum_{l=0}^{\infty} (2l+1) \ln\left(k^2 + \frac{l(l+1)}{r^2}\right) \Big|^{div} \\
= \; & -\int_0^{\infty} \frac{d\tau}{\tau} \left(\sum_k e^{-\tau k^2}\right) \left(\sum_{l=0}^{\infty} (2l+1) e^{-\tau l(l+1)/r^2}\right) e^{-\tau m^2} \Big|_{m\to 0}^{div} \\
= \; & -\int_0^{\infty} \frac{d\tau}{\tau} \frac{iV_d}{(4\pi)^{d/2}} \frac{e^{-\tau m^2}}{\tau^{d/2}} \left(\frac{r^2}{\tau} + \frac{1}{3} + \frac{\tau}{15r^2} + \frac{4\tau^2}{315r^4} + \cdots\right) \Big|_{m\to 0}^{div} \\
= \; & \frac{iV_4}{(4\pi r^2)^2} \frac{1}{\epsilon} \left(-\frac{8}{315}\right). \qquad (144)
\end{aligned}
$$

Note that we have used an asymptotic expansion for the summation over l above for small values of τ [23]. Now we have

$$
\begin{aligned}
F^{(t)}(2) \; &= \; 0, & (145) \\
F^{(v)}(2) \; &= \; F^{(s)}(2) = -\frac{8}{315}. & (146)
\end{aligned}
$$

For $N = 4$,

$$
\begin{aligned}
D_l^{(t)} \; &= \; \frac{5}{6}(2l+3)(l+4)(l-1), & (147) \\
D_l^{(v)} \; &= \; \frac{1}{2}(2l+3)(l+3)l, & (148) \\
D_l^{(s)} \; &= \; \frac{1}{6}(2l+3)(l+2)(l+1), & (149) \\
\Lambda_l \; &= \; -\frac{l(l+3)}{r^2}, & (150)
\end{aligned}
$$

for $l \geq 0$, and the asymptotic expansions,

$$
\begin{aligned}
& \sum_{l=0}^{\infty} \frac{5}{6}(2l+3)(l+4)(l-1) e^{-\tau l(l+3)/r^2} \\
= \; & \frac{5r^4}{6\tau^2} - \frac{10r^2}{3\tau} - \frac{91}{18} - \frac{508\tau}{189r^2} - \frac{127\tau^2}{378r^4} + \frac{806\tau^3}{2079r^6} + \frac{21311\tau^4}{81081r^8} + \frac{3416\tau^5}{57915r^{10}} + \cdots, \qquad (151) \\
& \sum_{l=0}^{\infty} \frac{1}{2}(2l+3)(l+3)l e^{-\tau l(l+3)/r^2} \\
= \; & \frac{r^4}{2\tau^2} - \frac{11}{30} - \frac{46\tau}{315r^2} + \frac{19\tau^2}{210r^4} + \frac{388\tau^3}{3465r^6} + \frac{6179\tau^4}{135135r^8} - \frac{268\tau^5}{225225r^{10}} + \cdots, \qquad (152) \\
& \sum_{l=0}^{\infty} \frac{1}{6}(2l+3)(l+2)(l+1) e^{-\tau l(l+3)/r^2} \\
= \; & \frac{r^4}{6\tau^2} + \frac{r^2}{3\tau} + \frac{29}{90} + \frac{37\tau}{189r^2} + \frac{149\tau^2}{1890r^4} + \frac{179\tau^3}{10395r^6} - \frac{1387\tau^4}{405405r^8} - \frac{13162\tau^5}{2027025r^{10}} + \cdots. \qquad (153)
\end{aligned}
$$

We obtain

$$F^{(t)}(4) = \frac{127}{189}, \tag{154}$$

$$F^{(v)}(4) = -\frac{19}{105}, \tag{155}$$

$$F^{(s)}(4) = -\frac{149}{945}. \tag{156}$$

For $N = 6$,

$$D_l^{(t)} = \frac{7}{60}(2l+5)(l+6)(l+3)(l+2)(l-1), \tag{157}$$

$$D_l^{(v)} = \frac{1}{24}(2l+5)(l+5)(l+3)(l+2)l, \tag{158}$$

$$D_l^{(s)} = \frac{1}{120}(2l+5)(l+4)(l+3)(l+2)(l+1), \tag{159}$$

$$\Lambda_l = -\frac{l(l+5)}{r^2}, \tag{160}$$

for $l \geq 0$, and the asymptotic expansions,

$$\begin{aligned} &\sum_{l=0}^{\infty} \frac{7}{60}(2l+5)(l+6)(l+3)(l+2)(l-1)e^{-\tau l(l+5)/r^2} \\ &= \frac{7r^6}{30\tau^3} - \frac{21r^2}{5\tau} - \frac{262}{27} - \frac{4133\tau}{450r^2} - \frac{248\tau^2}{99r^4} + \frac{958729\tau^3}{289575r^6} + \frac{14624\tau^4}{3861r^8} \\ &\quad + \frac{11267779\tau^5}{16409250r^{10}} - \frac{357108736\tau^6}{168358905r^{12}} + \cdots, \end{aligned} \tag{161}$$

$$\begin{aligned} &\sum_{l=0}^{\infty} \frac{1}{24}(2l+5)(l+5)(l+3)(l+2)le^{-\tau l(l+5)/r^2} \\ &= \frac{r^6}{12\tau^3} + \frac{r^4}{4\tau^2} - \frac{1823}{3780} - \frac{487\tau}{1260r^2} + \frac{3671\tau^2}{13860r^4} + \frac{264611\tau^3}{405405r^6} + \frac{1603\tau^4}{4290r^8} \\ &\quad - \frac{9333977\tau^5}{45945900r^{10}} - \frac{13067106599\tau^6}{23570246700r^{12}} + \cdots, \end{aligned} \tag{162}$$

$$\begin{aligned} &\sum_{l=0}^{\infty} \frac{1}{120}(2l+5)(l+4)(l+3)(l+2)(l+1)e^{-\tau l(l+5)/r^2} \\ &= \frac{r^6}{60\tau^3} + \frac{r^4}{12\tau^2} + \frac{r^2}{5\tau} + \frac{1139}{3780} + \frac{833\tau}{2700r^2} + \frac{137\tau^2}{660r^4} + \frac{121442\tau^3}{2027025r^6} - \frac{45251\tau^4}{810810r^8} \\ &\quad - \frac{23068481\tau^5}{229729500r^{10}} - \frac{1974977293\tau^6}{23570246700r^{12}} + \cdots. \end{aligned} \tag{163}$$

We obtain

$$F^{(t)}(6) = \frac{496}{99}, \tag{164}$$

$$F^{(v)}(6) = -\frac{3671}{6930}, \tag{165}$$

$$F^{(s)}(6) = -\frac{137}{330}. \tag{166}$$

For $G^{(s,v,t)}(p,q,N)$, we again use the asymptotic expansions for various dimensions and the same procedure to extract the divergent parts. The results are listed out in the following tables.

$G^{(t)}(p,q,2)$	$q=1$	2	3	4	5	6
$p=0$	0	$-6r^4$	0	0	0	0
1	0	0	$-6r^4$	0	0	0
2	0	0	0	$-6r^4$	0	0
3	0	0	0	0	$-6r^4$	0

$G^{(v)}(p,q,2)$	$q=1$	2	3	4	5	6
$p=0$	$\frac{2r^2}{15}$	$\frac{2r^4}{3}$	r^6	0	0	0
1	$\frac{16}{315}$	$\frac{4r^2}{15}$	$\frac{2r^4}{3}$	$\frac{2r^6}{3}$	0	0
2	$\frac{4}{105r^2}$	$\frac{16}{105}$	$\frac{2r^2}{5}$	$\frac{2r^4}{3}$	$\frac{r^6}{2}$	0
3	$\frac{64}{1155r^4}$	$\frac{16}{105r^2}$	$\frac{32}{105}$	$\frac{8r^2}{15}$	$\frac{2r^4}{3}$	$\frac{2r^6}{5}$

$G^{(s)}(p,q,2)$	$q=1$	2	3	4	5	6
$p=0$	$\frac{2r^2}{15}$	$\frac{2r^4}{3}$	r^6	0	0	0
1	$\frac{16}{315}$	$\frac{4r^2}{15}$	$\frac{2r^4}{3}$	$\frac{2r^6}{3}$	0	0
2	$\frac{4}{105r^2}$	$\frac{16}{105}$	$\frac{2r^2}{5}$	$\frac{2r^4}{3}$	$\frac{r^6}{2}$	0
3	$\frac{64}{1155r^4}$	$\frac{16}{105r^2}$	$\frac{32}{105}$	$\frac{8r^2}{15}$	$\frac{2r^4}{3}$	$\frac{2r^6}{5}$

$G^{(t)}(p,q,4)$	$q=1$	2	3	4	5	6	7	8
$p=0$	$-\frac{1016r^2}{189}$	$-\frac{91r^4}{9}$	$-\frac{10r^6}{3}$	$\frac{5r^8}{18}$	0	0	0	0
1	$-\frac{254}{189}$	$-\frac{2032r^2}{189}$	$-\frac{91r^4}{9}$	$-\frac{20r^6}{9}$	$\frac{5r^8}{36}$	0	0	0
2	$\frac{3224}{693r^2}$	$-\frac{254}{63}$	$-\frac{1016r^2}{63}$	$-\frac{91r^4}{9}$	$-\frac{5r^6}{3}$	$\frac{r^8}{12}$	0	0
3	$\frac{340976}{27027r^4}$	$\frac{12896}{693r^2}$	$-\frac{508}{63}$	$-\frac{4064r^2}{189}$	$-\frac{91r^4}{9}$	$-\frac{4r^6}{3}$	$\frac{r^8}{18}$	0
4	$\frac{54656}{3861r^6}$	$\frac{1704880}{27027r^4}$	$\frac{32240}{693r^2}$	$-\frac{2540}{189}$	$-\frac{5080r^2}{189}$	$-\frac{91r^4}{9}$	$-\frac{10r^6}{9}$	$\frac{5r^8}{126}$

$G^{(v)}(p,q,4)$	$q=1$	2	3	4	5	6	7	8
$p=0$	$-\frac{92r^2}{315}$	$-\frac{11r^4}{15}$	0	$\frac{r^8}{6}$	0	0	0	0
1	$\frac{38}{105}$	$-\frac{184r^2}{315}$	$-\frac{11r^4}{15}$	0	$\frac{r^8}{12}$	0	0	0
2	$\frac{1552}{1155r^2}$	$\frac{38}{35}$	$-\frac{92r^2}{105}$	$-\frac{11r^4}{15}$	0	$\frac{r^8}{20}$	0	0
3	$\frac{98864}{45045r^4}$	$\frac{6208}{1155r^2}$	$-\frac{76}{35}$	$-\frac{368r^2}{315}$	$-\frac{11r^4}{15}$	0	$\frac{r^8}{30}$	0
4	$-\frac{4288}{15015r^6}$	$\frac{98864}{9009r^4}$	$\frac{3104}{231r^2}$	$\frac{76}{21}$	$-\frac{92r^2}{63}$	$-\frac{11r^4}{15}$	0	$\frac{r^8}{42}$

$G^{(s)}(p,q,4)$	$q=1$	2	3	4	5	6	7	8
$p=0$	$\frac{74r^2}{189}$	$\frac{29r^4}{45}$	$\frac{r^6}{3}$	$\frac{r^8}{18}$	0	0	0	0
1	$\frac{298}{945}$	$\frac{148r^2}{189}$	$\frac{29r^4}{45}$	$\frac{2r^6}{9}$	$\frac{r^8}{36}$	0	0	0
2	$\frac{716}{3465r^2}$	$\frac{298}{315}$	$\frac{74r^2}{63}$	$\frac{29r^4}{45}$	$\frac{r^6}{6}$	$\frac{r^8}{60}$	0	0
3	$-\frac{22192}{135135r^4}$	$\frac{2864}{3465r^2}$	$\frac{596}{315}$	$\frac{296r^2}{189}$	$\frac{29r^4}{45}$	$\frac{2r^6}{15}$	$\frac{r^8}{90}$	0
4	$-\frac{210592}{135135r^6}$	$-\frac{22192}{27027r^4}$	$\frac{1432}{693r^2}$	$\frac{596}{189}$	$\frac{370r^2}{189}$	$\frac{29r^4}{45}$	$\frac{r^6}{9}$	$\frac{r^8}{126}$

$G^{(t)}(p,q,6)$	p=1	2	3	4	5	6	7	8	9	10
$q=0$	$-\frac{4133r^2}{225}$	$-\frac{524r^4}{27}$	$-\frac{21r^6}{5}$	0	$\frac{7r^{10}}{360}$	0	0	0	0	0
1	$-\frac{992}{99}$	$-\frac{8266r^2}{225}$	$-\frac{524r^4}{27}$	$-\frac{14r^6}{5}$	0	$\frac{7r^{10}}{900}$	0	0	0	0
2	$\frac{3834916}{96525r^2}$	$-\frac{992}{33}$	$-\frac{4133r^2}{75}$	$-\frac{524r^4}{27}$	$-\frac{21r^6}{10}$	0	$\frac{7r^{10}}{1800}$	0	0	0
3	$\frac{233984}{1287r^4}$	$\frac{15339664}{96525r^2}$	$-\frac{1984}{33}$	$-\frac{16532r^2}{225}$	$-\frac{524r^4}{27}$	$-\frac{42r^6}{25}$	0	$\frac{r^{10}}{450}$	0	0
4	$\frac{90142232}{546975r^6}$	$\frac{1169920}{1287r^4}$	$\frac{7669832}{19305r^2}$	$-\frac{9920}{99}$	$-\frac{4133r^2}{45}$	$-\frac{524r^4}{27}$	$-\frac{7r^6}{5}$	0	$\frac{r^{10}}{720}$	0
5	$-\frac{11427479552}{3741309r^8}$	$\frac{180284464}{182325r^6}$	$\frac{1169920}{429r^4}$	$\frac{15339664}{19305r^2}$	$-\frac{4960}{33}$	$-\frac{8266r^2}{75}$	$-\frac{524r^4}{27}$	$-\frac{6r^6}{5}$	0	$\frac{r^{10}}{1080}$

$G^{(v)}(p,q,6)$	p=1	2	3	4	5	6	7	8	9	10
$q=0$	$-\frac{487r^2}{630}$	$-\frac{1823r^4}{1890}$	0	$\frac{r^8}{12}$	$\frac{r^{10}}{144}$	0	0	0	0	0
1	$\frac{3671}{3465}$	$-\frac{487r^2}{315}$	$-\frac{1823r^4}{1890}$	0	$\frac{r^8}{24}$	$\frac{r^{10}}{360}$	0	0	0	0
2	$\frac{1058444}{135135r^2}$	$\frac{3671}{1155}$	$-\frac{487r^2}{210}$	$-\frac{1823r^4}{1890}$	0	$\frac{r^8}{40}$	$\frac{r^{10}}{720}$	0	0	0
3	$\frac{12824}{715r^4}$	$\frac{4233776}{135135r^2}$	$\frac{7342}{1155}$	$-\frac{974r^2}{315}$	$-\frac{1823r^4}{1890}$	0	$\frac{r^8}{60}$	$\frac{r^{10}}{1260}$	0	0
4	$-\frac{37335908}{765765r^6}$	$\frac{12824}{143r^4}$	$\frac{2116888}{27027r^2}$	$\frac{7342}{693}$	$-\frac{487r^2}{126}$	$-\frac{1823r^4}{1890}$	0	$\frac{r^8}{84}$	$\frac{r^{10}}{2016}$	0
5	$-\frac{104536852792}{130945815r^8}$	$-\frac{74671816}{255255r^6}$	$\frac{38472}{143r^4}$	$\frac{4233776}{27027r^2}$	$\frac{3671}{231}$	$-\frac{487r^2}{105}$	$-\frac{1823r^4}{1890}$	0	$\frac{r^8}{112}$	$\frac{r^{10}}{3024}$

$G^{(s)}(p,q,6)$	p=1	2	3	4	5	6	7	8	9	10
$q=0$	$\frac{833r^2}{1350}$	$\frac{1139r^4}{1890}$	$\frac{r^6}{5}$	$\frac{r^8}{36}$	$\frac{r^{10}}{720}$	0	0	0	0	0
1	$\frac{137}{165}$	$\frac{833r^2}{675}$	$\frac{1139r^4}{1890}$	$\frac{2r^6}{15}$	$\frac{r^8}{72}$	$\frac{r^{10}}{1800}$	0	0	0	0
2	$\frac{485768}{675675r^2}$	$\frac{137}{55}$	$\frac{833r^2}{450}$	$\frac{1139r^4}{1890}$	$\frac{r^6}{10}$	$\frac{r^8}{120}$	$\frac{r^{10}}{3600}$	0	0	0
3	$-\frac{362008}{135135r^4}$	$\frac{1943072}{675675r^2}$	$\frac{274}{55}$	$\frac{1666r^2}{675}$	$\frac{1139r^4}{1890}$	$\frac{2r^6}{25}$	$\frac{r^8}{180}$	$\frac{r^{10}}{6300}$	0	0
4	$-\frac{92273924}{3828825r^6}$	$-\frac{362008}{27027r^4}$	$\frac{971536}{135135r^2}$	$\frac{274}{33}$	$\frac{833r^2}{270}$	$\frac{1139r^4}{1890}$	$\frac{r^6}{15}$	$\frac{r^8}{252}$	$\frac{r^{10}}{10080}$	0
5	$-\frac{15799818344}{130945815r^8}$	$-\frac{184547848}{1276275r^6}$	$-\frac{362008}{9009r^4}$	$\frac{1943072}{135135r^2}$	$\frac{137}{11}$	$\frac{833r^2}{225}$	$\frac{1139r^4}{1890}$	$\frac{2r^6}{35}$	$\frac{r^8}{336}$	$\frac{r^{10}}{15120}$

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