Maxwell F^N Characteristic Equation Algorithm Applied to Abelian Born-Infeld Action in Dp-branes

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Abstract

An algorithm is devised to generate characteristic identities between Maxwell fieldstrength invariants (traced over Lorentz indices and disregarding ordering) that suffer linear dependence in certain dimensionalities as they have been originally obtained using a Maple routine. These relations between invariants are then applied to simplify the Abelian Born-Infeld (ABI) effective action in arbitrary degree of fieldstrength invariants. I have explicitly displayed the simplified ABI action in 4, 6, 8, 10, and 12 space-time dimensions relevant in Dp-branes.

1 Introduction

Abelian Born-Infeld (ABI) action [1] is the non-linear generalization of the Maxwell action in quantum electrodynamics. It appears as a low energy effective action of open strings and in world-volume effective action in D-branes [3].

In particular, a flat D-brane in type II string theory the bosonic massless degrees of freedom of an open string ending on the D-brane are a U(1) gauge field. Apart from the 9-p neutral scalar fields that describe the transversal excitations of the D-branes which I choose to disregard, this U(1) gauge field in $p+1$ dimensions describes the open string longitudinal fluctuations of the brane.[5] When n D-branes coincide, the massless degrees of freedom of open strings beginning and ending on them are a U(n) gauge field. [9] (A number of the disregarded scalar fields are in the adjoint of the gauge group.)

The present work explores the structure of the ABI action [7]

$$\Gamma_{\mathrm{BI}} = -T_p \int d^{p+1}\sigma \; \sqrt{\det(\delta_{\mu\nu} - 2\pi\alpha' F_{\mu\nu})} \qquad (1)$$

up to all orders in the string length $\sqrt{\alpha'}$ in the limit of slowly varying fieldstrengths, in the assumption of zero derivatives of the fieldstrength. Furthermore, the static gauge is choosen and transverse scalars are disregarded throughout the paper. Only dynamics contributed by the longitudinal excitations are considered as they are described by the U(n) gauge of n several coinciding D-branes. Here in (1), T_p is the Dp-brane tension, $\alpha\epsilon\{0, 1, \ldots, p\}$.

Now in the ABI (1) action, the term can be expanded in powers of F [8]

$$\sqrt{\det(\delta_{\mu\nu} + F_{\mu\nu})} = \sum_{k=0}^{\infty} C^{\mu_1\nu_1\ldots\mu_k\nu_k} F_{\mu_1\nu_1} \cdots F_{\mu_{2k}\nu_{2k}} \qquad (2)$$

where the Lorentz tensors $C^{\mu_1\nu_1\cdots\mu_k\nu_k}$ are defined as coefficients in the $2\pi\alpha' = 1$ setting. Here, such coefficients are set to 1 directing then our attention to the fieldstrength tensor in powers of F, F^{2k}, and investigate its structure.

These powers of F in (2) has been given attention in the work of Delbourgo [2] where characteristic equations were expressed in terms of polynomials over traces of matrix powers. In this work, he developed a Maple routine that computes a set of invariants involving the electromagnetic Maxwell field tensor in arbitrary space-time dimensions specifically for describing multiphoton processes [6].

Here, I devised an algorithm that generates the same set of invariants without using any symbolic software. But I use the results obtained from the latter and constructed a prescription based on the principle of induction. The devised algorithm will be enumerated in Section 2. A test run of the algorithm is performed in Section 3 at the same time compact notations are introduced. After having explicitly displayed the characteristic equations, the same set of equations is used to simplify the powers of F in the ABI action as it has been expanded in Eqn (2). I displayed explicitly in Section 4 the simplified version of the ABI action relevant to Dp-branes in 4, 6, 8, 10, and 12 space-time dimensions.

2 The Algorithm

The following algorithm is devised to construct characteristic equations relating fieldstrength invariants

$$\sum_{e=0}^{d=\max(2e+f)} T_{(2e)}F^f = 0 \tag{3}$$

($T_{(0)} = 1$) that suffer linear dependence in certain dimensionalities. That is,

$$F^d = T_{(2)}F^{d-2} + T_{(4)}F^{d-4} + \ldots + T_{(d)} \tag{4}$$

$$F^d = T_{(2)}F^{d-2} + T_{(4)}F^{d-4} + \ldots + T_{(d-1)}F \tag{5}$$

when d is even (4) and odd (5), respectively. These are subject to the condition that the degree d of F be $d = \max(2e + f)$. Furthermore, $T_{(2e)}$ are the terms the devised algorithm generate.

(a) All of the terms within a characteristic equation are all of degree d. For example:

$$F^{12},\ T_2T_4F^6,\ T_4F^8,\ T_2T_4T_6,\ \text{etc.} \tag{6}$$

are all of degree 12.

(b) Each characteristic equation has one fieldstrength tensor of degree D, say for $D = d$, the relevant tensor is F^d. See example in (a).

(c) With $T_j = \mathrm{Tr}(F^j)$, the rest of the terms in each characteristic equation are products of the form $\prod_k T_{j_k}$ where $j = j_k$ is even only and that $\sum_k j_k = d$ is also even. The subscript here initially is set to $j = d$. See example in (a).

(d) These are only formed when the degree is even also. It is not constructed when the degree is odd. The following are all of degree 13,

$$F^{13},\ T_2T_4F^7,\ T_4F^9,\ T_2T_4T_6F,\ \text{etc.} \tag{7}$$

which is Example in (a) times F.

(e) Each term formed in steps (c)-(d) has a prefactor $(-1)^t\frac{1}{1!j^1}$ with j and d defined in (c). This is applied only when j_k in (c) is unique. Here t denotes the degree of T.

(f) If in $\prod_k T_{j_k}$, j_k is not unique (i.e. j_k appeared as a subscript p times) the term formed in step (e) will have a prefactor $(-1)^t\frac{1}{p!j^p}$. It can be a product $(-1)^d\prod_q \frac{1}{(p_q)!j^{p_q}}$ for every qth T_j term that appeared p_q times. It should be noted that $T_2T_2T_2$ will have a different prefactor with T_2^3. And that the latter expression should be used.

(g) Characteristic equations whose common degree is odd is simply the even degree characteristic equation (obtained from steps (a)-(c)) with all terms unit F multiplied.

(h) All terms in a characteristic equation whose degree $d + 1$ is odd are all carried to $d + 2$ characteristic equation except that F is multiplied to each term. The additional degree is contributed by this unit F multiplication. Since $d+2$ is again even, steps (c)-(f) is repeated.

(i) All terms formed in the above steps are added after following strictly the constraints provided in each step as in (3) or equivalently (4) or (5). These terms comprise the characteristic equation of F^n true only when F is abelian.

It should be noted that this algorithm is constructed in the assumption of zero derivatives of the fieldstrength tensor and no ordering in indices is set. No other constraints about the indices are imposed.

3 Test Run of Algorithm

Applying the algorithm in the Section 2 (particularly steps (c) to (f)), unique traced invariants $T_{(d)}$ for even d (where $d = 2, 4, 6, 8, 10$) are displayed as follows.

$$\begin{aligned}
T_{(2)} &= +\frac{1}{2}T_2,\ \ T_{(4)} = -\frac{1}{8}T_2^2 + \frac{1}{4}T_4\\
T_{(6)} &= +\frac{1}{48}T_2^3 - \frac{1}{8}T_2T_4 + \frac{1}{6}T_6\\
T_{(8)} &= -\frac{1}{384}T_2^4 + \frac{1}{32}T_2^2T_4 - \frac{1}{12}T_2T_6 - \frac{1}{32}T_4^2 + \frac{1}{8}T_8\\
T_{(10)} &= +\frac{1}{3840}T_2^5 - \frac{1}{192}T_2^3T_4 + \frac{1}{48}T_2^2T_6 - \frac{1}{16}T_2T_8\\
&\qquad -\frac{1}{24}T_4T_6 + \frac{1}{10}T_{10}
\end{aligned} \tag{8}$$

where $\sum_k a_k \cdot b_k = d$ for even d in $T_{b_k}^{a_k}$. While $d = 2, 4, 6$ is a reproduction of the results obtained via a Maple routine [2], the same characteristic invariants (3) were obtained following the algorithm devised in Section 2. Here, $T_{(2e)}$ are given in Eqn (8). Explicitly (3) (or equivalently (4) or (5)), these are ($d = 2$ up to $d = 10$)

$$\begin{aligned}
&F^2 - T_{(2)} = 0,\\
&F^3 - T_{(2)}F = 0,\\
&F^4 - T_{(2)}F^2 - T_{(4)} = 0\\
&F^5 - T_{(2)}F^3 - T_{(4)}F = 0,\\
&F^6 - T_{(2)}F^4 - T_{(4)}F^2 - T_{(6)} = 0\\
&F^7 - T_{(2)}F^5 - T_{(4)}F^3 - T_{(6)}F = 0
\end{aligned} \tag{9}$$

$$F^8-T_{(2)}F^6-T_{(4)}F^4-T_{(6)}F^2-T_{(8)}=0$$
$$F^9-T_{(2)}F^7-T_{(4)}F^5-T_{(6)}F^3-T_{(8)}F=0$$
$$F^{10}-T_{(2)}F^8-T_{(4)}F^6-T_{(6)}F^4-T_{(8)}F^2-T_{(10)}=0$$

The case when $d = 8, 9, 10$ is shown explicitly for illustration purposes. In Eqn (9), F^n in odd degrees are now displayed illustrating the application of steps (d), (g) and (i). These correspond to $D = 2, 4, 6, 8, 10, 12, 14, 16, 18, 20$ space-time dimensions, respectively. For my purposes, I consider $D = 2$ up to $D = 12$ space-time dimensions.

4 Application to Abelian Born-Infeld Action

Now that I have the algorithm in place, let me consider the following notations, conventions, and definitions as well as assumptions which will be used throughout the present work.

From now on, I set $2\pi\alpha' = 1$, I ignored the overall factor T_p and an additive constant. The metric is Euclidean and is "mostly plus". That is, $g_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$ and $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$ Indices denoted by $\mu, \nu, \ldots$ run from 1 to $2p$, $i, j, \ldots$ run from 1 to $2p$, $\alpha, \beta, \ldots$ run from 1 to p. There is no distinction between upper and lower indices. All are lowered. Repeated indices are summed over. Ordering of indices are also disregarded. Anti-hermitian matrices for U(n) generators (some subcartan algebra) are chosen, etc. The fieldstrength is given by

$$F_{\alpha\bar{\beta}} = \partial_\alpha A_{\bar{\beta}} - \partial_{\bar{\beta}} A_\alpha \tag{10}$$

Instead of using real spatial coordinates x^μ, complex coordinates z^α. I denote F as $F = F_{\alpha\bar{\beta}}$ unless otherwise indicated. Also, $F_{\alpha\bar{\alpha}} = 0$ considering that I am working on a flat space. Furthermore, $F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0$.

Using (2), the ABI action (1) becomes

$$\Gamma_{\rm BI} = -\int \sum_{d=2}^{6} d^{p+1}\sigma \; C^d F^d \tag{11}$$

and using (3) or (9) with traced invariants given by (8), the simplified ABI action relevant to $D = 4, 6, 8, 10, 12$ space-time dimensions in Dp-branes is given by

$$\Gamma_{\rm BI} = -\int d^{p+1}\sigma \; \left\{ \frac{1}{2}T_2 + \frac{1}{2}T_2 F + \frac{1}{4}\left(\frac{1}{2}T_2^2 + T_4\right) + \frac{1}{4}\left(\frac{T_2^2}{2} + T_4\right)F + \frac{1}{2}\left(\frac{T_2^3}{24} + \frac{T_{24}^{11}}{4} + \frac{T_6}{3}\right)\right\} \tag{12}$$

after setting $C^k = 1$. Hence, this is the most simplified ABI action I can have with the assumption of no ordering, zero derivatives in fieldstrength tensor and using the characteristic equations I generated with the algorithm this work presented.

Deforming $F_{\alpha\bar{\alpha}} = 0$ to

$$\left(1 + \frac{1}{2}T_2 + \frac{1}{4}\left(\frac{T_2^2}{2} + T_4\right)\right)F = 0, \tag{13}$$

the Donaldson-Uhlenbeck-Yau condition acquires an order F^5 correction. After some invariance restoration, the equations of motion integrate to the ABI action (12) with

$$\frac{1}{2}T_2 + \frac{1}{4}\left(\frac{1}{2}T_2^2 + T_4\right) + \frac{1}{2}\left(\frac{T_2^3}{24} + \frac{T_{24}^{11}}{4} + \frac{T_6}{3}\right). \tag{14}$$

Modulo an undetermined overall multiplicative constant, this exhibits the Born-Infeld action through order F^6, leading to a uniquely fixed equation of motion. These results raise the suspicion that the ABI action is the only deformation which allows solutions on Dp-branes at angles of the form $F_{\alpha\bar{\alpha}} = F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0$. [4] This suspicion provides an important tool to probe the structure of the effective action that captures correctly the D-brane dynamics.

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