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Hyperkähler Metrics from Periodic Monopoles

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Abstract

Relative moduli spaces of periodic monopoles provide novel examples of Asymptotically Locally Flat hyperkähler manifolds. By considering the interactions between well-separated periodic monopoles, we infer the asymptotic behavior of their metrics. When the monopole moduli space is four-dimensional, this construction yields interesting examples of metrics with self-dual curvature (gravitational instantons). We discuss their topology and complex geometry. An alternative construction of these gravitational instantons using moduli spaces of Hitchin equations is also described.

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1 Introduction

One of the most powerful methods for obtaining hyperkähler manifolds is the Hyperkähler Quotient Construction [1]. Most known hyperkähler manifolds are hyperkähler quotients of affine hyperkähler spaces by a suitable subgroup of tri-holomorphic isometries. For example, all Asymptotically Locally Euclidean four-dimensional hyperkähler manifolds (in other words, ALE gravitational instantons) have been constructed in this way [2]. The affine hyperkähler space is finite-dimensional in this case.

More general hyperkähler manifolds are obtained if one starts with an infinite-dimensional affine hyperkähler space and quotients by an infinite-dimensional subgroup of isometries. Well-known examples of this sort are moduli spaces of instantons on $\mathbb{R}^4$ and moduli spaces of monopoles on $\mathbb{R}^3$. The affine space is the space of connections on a vector bundle on $\mathbb{R}^4$ in the first instance, and the space of pairs $(connection, Higgs\ field)$ in the second instance. The quotienting group is the group of gauge transformations in both instances.

The monopole example is particularly nice, as one can determine the asymptotic behavior of the metric from simple physical considerations [3, 4]. In the asymptotic region the monopoles are well separated, and can be regarded as point particles interacting via long-range scalar and electromagnetic fields. Each particle has an internal degree of freedom living on a circle, which when excited gives the monopole an electric charge (i.e. makes it into a dyon). In the asymptotic region the radius of this circle is a fixed number determined by the vacuum expectation value of the Higgs field at infinity. It follows that asymptotically the moduli space of k $SU(2)$ monopoles looks like a T^k fibration over $(\mathbb{R}^3)^k/S_k$, where we divided by the symmetric group S_k to take into account the indistinguishability of monopoles. Since the electric charges are conserved, the fiberwise action of T^k must be an isometry (in fact, a tri-holomorphic isometry). A more detailed analysis of the long-range interactions of moving monopoles yields the precise metric in the asymptotic regime [3, 4], which turns out to be Asymptotically Locally Flat.

It is customary to quotient the moduli space by the translations of $\mathbb{R}^3$ and the diagonal of T^k, or equivalently to fix the center-of-mass coordinates of the monopoles and the sum of their internal degrees of freedom (phases). The resulting $4(k-1)$-dimensional manifold is again hyperkähler and is called the relative (or centered) moduli space. The relative moduli space of two monopoles is known as the Atiyah-Hitchin manifold [5]. At infinity it looks like a circle of fixed radius fibered over $\mathbb{R}^3/\mathbb{Z}_2$, and the asymptotic metric has the Taub-NUT form.

One can generalize this example somewhat and consider $SU(2)$ monopoles moving in a background of n point-like Dirac monopoles sitting at fixed locations [6]. If the number of $SU(2)$ monopoles is one, then the (uncentered) moduli space is the multi-Taub-NUT space [7, 6]. It is a four-dimensional ALF manifold with a tri-holomorphic $U(1)$ isometry isomorphic as a complex variety to a blow-up of $\mathbb{C}^2/\mathbb{Z}_n$. At infinity it looks like a circle of fixed radius fibered over $\mathbb{R}^3$, and the $U(1)$ action is fiberwise.[1] If the number of $SU(2)$ monopoles is two, then the *relative* moduli space is four-dimensional and ALF, but does not have a tri-holomorphic $U(1)$ isometry. As a complex variety the moduli space is isomorphic to a blow-up of $\mathbb{C}^2/\Gamma$, where Γ is a binary dihedral group [6]. The asymptotic metric has the Taub-NUT form and looks at infinity like a circle of fixed radius fibered over $\mathbb{R}^3/\mathbb{Z}_2$. In particular the asymptotic metric has a tri-holomorphic $U(1)$ isometry which acts fiberwise.

ALE gravitational instantons also have an asymptotic tri-holomorphic $U(1)$ isometry, but the circumference of the orbits grows linearly as a function of the "radius." Finite-dimensional HKQ construction suffices to construct all such manifolds. In the cases when the circumference of the orbits stays fixed at infinity, one needs to resort to the infinite-dimensional HKQ construction, in general.

An obvious generalization is to consider ALF gravitational instantons which asymptotically have a tri-holomorphic T^2 action. We will call such gravitational instantons *ALG manifolds*. Such manifolds previously arose in the physics literature as quantum moduli spaces of $d = 4$ $N = 2$ gauge theories compactified on a circle (see below). No "classical" construction of such manifolds has been known previously. In this paper we will produce examples of ALG manifolds using an infinite-dimensional HKQ construction. More generally, we will show how to construct ALF hyperkähler manifolds of dimension $4(k-1)$ which asymptotically have a tri-holomorphic $T^{2(k-1)}$ isometry. To this end we will consider k $SU(2)$ monopoles on $\mathbb{R}^2 \times \mathbb{S}^1$ with a flat metric. Such "periodic" monopoles have been studied in Refs. [9, 10]. It was shown there that although each periodic monopole has a logarithmically divergent mass, the relative moduli space has a well-defined hyperkähler metric. We expect that this metric is smooth and geodesically complete. The asymptotic behavior of this metric will be determined along the lines of Refs. [3, 4]. We will also consider a more general problem of periodic $SU(2)$ monopoles moving in a background of point-like Dirac monopoles.

In the case $k = 2$ the moduli space is four-dimensional, and we will describe its geometry in some detail using the results of Refs. [9, 10]. In fact,

[1]The multi-Taub-NUT metric can also be obtained by a finite-dimensional HKQ construction [1], or by using the Gibbons-Hawking ansatz [8].

since the number of Dirac singularities n can vary from 0 to $4,$ in this way we obtain five topologically distinct four-dimensional hyperkähler manifolds. We show that they are ALG manifolds. Moreover, we will see that the moduli spaces have a distinguished complex structure in which they look like elliptic fibrations over $\mathbb{C}$. The volume of the elliptic fiber is constant in the asymptotic region of the moduli space. The asymptotic T^2 isometry acts on the fibers in a natural manner. The number and type of singular fibers depends on the parameters of the metric. We will discuss which kinds of singular fibers occur, compute the Betti numbers of the moduli spaces, and in some cases the intersection pairing on the second homology. We will see that the most general ALG gravitational instanton one can get in this way has an intersection form which is the affine Cartan matrix of type D_4. All other gravitational instantons we construct can be regarded as its degenerations.

Finally, we explain an alternative construction of our ALG manifolds using moduli spaces of Hitchin equations [11] on a cylinder. The two constructions are related by a version of Nahm transform [9, 10].

As discussed in Refs. [9, 10], moduli spaces of periodic monopoles are closely related to certain $N = 2,$ $d = 4$ quantum gauge theories. For example, the moduli space of k $SU(2)$ monopoles moving in a background of n Dirac monopoles is isomorphic to the quantum Coulomb branch of $SU(k)$ gauge theory with n fundamental hypermultiplets compactified on a circle. The D_4 gravitational instanton mentioned above corresponds to the $SU(2)$ gauge theory with four hypermultiplets, while its degenerations correspond to the $SU(2)$ gauge theory with three or fewer hypermultiplets. The quantum Coulomb branch of these theories on $\mathbb{R}^4$ has been determined in two celebrated papers by Seiberg and Witten [12, 13]. Our results provide information about the same theories on $\mathbb{R}^3 \times \mathbb{S}^1$. The asymptotic form of the metric on the Coulomb branch has been computed in [14, 15, 16]. The result agrees with the asymptotics of the metric on the moduli space of periodic monopoles computed below. However, if one want the complete metric, the gauge theory realization is not very useful, since the metric is corrected by gauge theory instantons. Such non-perturbative effects lead to exponentially small corrections to the metric which are quite hard to compute. On the other hand, we realized the same manifolds as classical objects, namely as moduli spaces of Bogomolny or Hitchin equations. We hope that the corresponding hyperkähler metrics can be computed in a closed form using twistor methods.

2 Asymptotic Metric on the Monopole Moduli Spaces

2.1 Generalities

We will use the conventions of Refs. [9, 10]. We identify $\mathbb{R}^2 \times \mathbb{S}^1$ with $\mathbb{C} \times \mathbb{S}^1$ and use a complex affine coordinate z on $\mathbb{C}$ and a real coordinate χ on $\mathbb{S}^1$ with an identification $\chi \sim \chi + 2\pi$. For monopoles located at points $a_j = (z_j, \chi_j),\ j = 1, \dots, k,$ the field configuration at a distant point $x = (z, \chi)$ is given in a suitable gauge by

$$\phi(x) = v + \sum_{j=1}^{k} \phi^j(z - z_j), \tag{1}$$

$$A_z = 0, \quad A_\chi = b + \sum_{j=1}^{k} A^j_\chi(z - z_j). \tag{2}$$

When all the distances $|z_i - z_j|$ are large, we interpret these fields as a superposition of the background fields, given by constants b and v, and individual fields of the monopoles ϕ^j and A^j.

When all monopoles are well separated, it is natural to think of their dynamics in terms of motion and interaction of particles on $\mathbb{R}^2 \times \mathbb{S}^1$. The moduli space coordinates are understood as parameterizing the positions of these k particles as well as their internal degrees of freedom valued in $\mathbb{S}^1$ (phases). A particle whose phase is changing with time aquires an electric charge proportional to the rate of the phase change [3]. This is consistent with charge conservation because the rate of phase change is an integral of motion. Motion on the moduli space is thus interpreted as motion of k dyons on $\mathbb{R}^2 \times \mathbb{S}^1$.

So far the discussion parallels that for monopoles on $\mathbb{R}^3$ [3]. But unlike for monopoles on $\mathbb{R}^3$, there is a subtlety here related to the fact that a single periodic monopole has infinite mass, because the integral of the energy density logarithmically diverges at long distances [9, 10]. One might conclude that the kinetic energy associated with the motion on the moduli space is infinite as well. If this were the case, the metric on the moduli space would be ill-defined (divergent), and the positions of the particles would be parameters rather than moduli. In fact, only the coordinates of the center of mass and the total phase are parameters. The kinetic energy of the relative motion is finite, and therefore there is a finite metric on the relative moduli space [10]. To deal with this subtlety, we use the following procedure. In terms of the universal covering space of $\mathbb{R}^2 \times \mathbb{S}^1$, each periodic monopole is an array of infinitely many 't Hooft-Polyakov monopoles. Such an array has an infinite mass per unit length because of the divergence mentioned above. We regularize the problem by replacing each infinite array by a finite array of $2N+1$ monopoles. This way all the masses and fields are finite. At the end of the computation we will send N to infinity. As a result, we indeed recover a finite metric on the relative moduli space and verify that the center of mass and total phase of the configuration are parameters (the kinetic energy associated with them diverges logarithmically as $N \to \infty$).

With this remark in mind, the Higgs field produced by one periodic monopole of charge g located at $z = 0$ at distances large compared to the size of the monopole is

$$\phi^j(x) = \sum_{l=-N}^{N} \frac{-g}{\sqrt{|z|^2 + (\chi - 2\pi l)^2}}. \tag{3}$$

We note for future use that for an 't Hooft-Polyakov monopole $g = 1$, and for a singular Dirac monopole $g = -1/2$ [10]. Since we are going to send N to infinity, we may assume that $|z| \ll N$. In this region the expression for ϕ^j simplifies:

$$\phi^j(x) = \frac{g}{\pi} \log|z| - gC_N + O\left(\frac{1}{|z|}\right). \tag{4}$$

Here C_N a positive constant diverging logarithmicaly with N; it will eventually be absorbed into the constant background v. From now on we shall omit terms decaying as $1/|z|$ or faster when writing the monopole fields.

The connection A^j corresponding to $\phi^j(x)$ is given by

$$A^j_\chi = \frac{g}{\pi} \arg z, \quad A^j_z(x) = 0, \tag{5}$$

in a suitable gauge. To be precise, we should have added an N-dependent constant to A^j_χ, but since it can be absorbed into the constant background b, we did not write it explicitly.

For convenience we define two auxiliary functions:

$$u(z) = \frac{1}{\pi} \log|z| - C_N, \quad w(z) = \frac{1}{\pi} \arg z.$$

Note that the total field $\phi(x)$ of Eq.(1) is given for large z by

$$\phi(x) = v - kgC_N + \frac{kg}{\pi} \log|z|. \tag{6}$$

Since we are interested in field configurations with fixed asymptotics, it is natural to introduce

$$v_{\rm ren} = v - kgC_N$$

and adjust v so that v_{ren} remains fixed when N is sent to infinity. Thus in the limit $N \to +\infty$ we have $v \to +\infty$.

Next we introduce an electric charge q_j for each monopole. The resulting dyons acquire new interactions. The Lagrangian of the k-th dyon is

$$L_k = -4\pi\phi\sqrt{g^2+q_k^2}\sqrt{1-\vec{V}_k^2} + 4\pi q_k \vec{V}_k \cdot \vec{A} - 4\pi q_k A_0 + 4\pi g \vec{V}_k \cdot \vec{\tilde{A}} - 4\pi g \tilde{A}_0.$$

Here $\vec{V}$ is the velocity 3-vector of the k-th dyon with components $(\mathrm{Re}\,\dot{z}, \mathrm{Im}\,\dot{z}, \dot{\chi})$ (dot denotes time derivative). The fields $\phi, A,$ and $\tilde{A}$ are superpositions of the fields produced by other dyons evaluated at the location of the k-th dyon and the constant background. The magnetic charge g couples to the "magnetic" potentials $\vec{\tilde{A}}, \tilde{A}_0$ which are dual to the "electric" potentials $\vec{A}, A_0$ and are defined by

$$\begin{array}{rcrcrcl} \nabla \times \vec{\tilde{A}} & \equiv & \vec{\tilde{B}} & = & -\vec{E} & \equiv & \nabla A_0 + \frac{\partial}{\partial t}\vec{A}, \\ -\nabla \tilde{A}_0 - \frac{\partial}{\partial t}\vec{\tilde{A}} & \equiv & \vec{\tilde{E}} & = & \vec{B} & \equiv & \nabla \times \vec{A}. \end{array} \tag{7}$$

The fields produced by a dyon at rest located at $z = 0$ are

$$\phi^j(x) = \sqrt{g^2+q_j^2}\; u(z), \tag{8}$$

$$\begin{array}{lll} A^j_\chi(x) = gw(z), & A^j_0(x) = -q_j u(z), & A^j_z(x) = 0, \\ \tilde{A}^j_\chi(x) = -q_j w(z), & \tilde{A}^j_0(x) = -gu(z), & \tilde{A}^j_z(x) = 0. \end{array} \tag{9}$$

The fields of a moving dyon are obtained by a Lorentz boost. Keeping terms up to second order in velocities in $\phi, A_0, \tilde{A}_0$ and up to first order in $\vec{A}, \vec{\tilde{A}},$ we get:

$$\begin{array}{rcl} \phi^j(x) & = & \sqrt{g^2+q_j^2 u(z)}\sqrt{1-\vec{V}_j^2}, \\ A^j_\chi(x) & = & -q_j u(z) V_{j\chi} + gw(z), \\ A^j_z(x) & = & -q_j u(z) V_{jz} \\ A^j_0(x) & = & -q_j u(z) + gw(z) V_{j\chi}, \\ \tilde{A}^j_\chi(x) & = & -gu(z)V_\chi - q_j w(z), \\ \tilde{A}^j_z(x) & = & -gu(z) V_{jz}, \\ \tilde{A}^j_0(x) & = & -gu(z) - q_j w(z) V_{j\chi}. \end{array} \tag{10}$$

Following Ref. [4], we omitted certain terms of second order in velocities in $\phi, \tilde{A}_0$ by replacing $1/\sqrt{r^2 - (\mathbf{r}\times\mathbf{V})^2}$ with $1/r$ in the Liénard-Wiechert potentials. This is allowed because such second-order terms enter the kinetic energy with the same coefficients as $1/r$ terms enter the static energy. Since the static interactions cancel, so do the second-order terms of this type.

2.2 Two-Monopole Interactions

Consider the Lagrangian of the k-th dyon in the presence of a dyon with $j=1$. Keeping terms up to second order in electric charges and velocities, we get

$$\begin{aligned} L_k &= -m_k + \frac{1}{2} m_k \vec{V}_k^2 + 2\pi g^2 u(z_k - z_1) \left(\vec{V}_k - \vec{V}_1\right)^2 + \\ & + 4\pi g w(z_k - z_1)\,(q_k - q_1)\,(V_{k\chi} - V_{1\chi}) - \\ & - 2\pi u(z_k - z_1)\,(q_k - q_1)^2 + 4\pi b q_k V_{k\chi}. \end{aligned} \tag{11}$$

Here the dyon's rest mass m_k is given by $4\pi v\sqrt{g^2 + q_k^2}$. Expanding m_k to second order in q_k, omitting a constant term $-4\pi g v$, and symmetrizing with respect to the two particles, we obtain the total Lagrangian for the dyons with $j=1$ and $j=k$:

$$\begin{aligned} \frac{1}{4\pi} L_{1k} &= -\frac{v}{2g} q_1^2 - \frac{v}{2g} q_k^2 + \frac{gv}{2} \vec{V}_1^2 + \frac{gv}{2} \vec{V}_k^2 + \frac{g^2}{2} u(z_k - z_1)(\vec{V}_k - \vec{V}_1)^2 \\ & + \left(\frac{b}{2} + g w(z_k - z_1)\right)(q_k - q_1)(V_{k\chi} - V_{1\chi}) - \\ & - \frac{1}{2} u(z_k - z_1)(q_k - q_1)^2 + \frac{b}{2}(q_1 + q_k)(V_{1\chi} + V_{k\chi}). \end{aligned} \tag{12}$$

Hence the Lagrangian describing the relative motion of the two dyons is

$$\begin{aligned} \frac{1}{4\pi} L_{\mathrm{rel}} &= g^2\left(\frac{v_{\mathrm{ren}}}{4g} + \frac{1}{2\pi} \log|z_k - z_1|\right)(\vec{V}_k - \vec{V}_1)^2 + \\ & + \left(\frac{b}{2} + \frac{g}{\pi} \arg(z_k - z_1)\right)(q_k - q_1)(V_{k\chi} - V_{1\chi}) - \\ & - \left(\frac{v_{\mathrm{ren}}}{4g} + \frac{1}{2\pi} \log|z_k - z_1|\right)(q_k - q_1)^2, \end{aligned} \tag{13}$$

while the Lagrangian for the motion of the center of mass is

$$\frac{1}{4\pi} L_{CM} = \frac{vg}{4}\left(\vec{V}_1 + \vec{V}_k\right)^2 - \frac{v}{4g}(q_1 + q_k)^2 + \frac{b}{2}(q_1 + q_k)(V_{1\chi} + V_{k\chi}).$$

In the limit $N \to \infty,\ v \to \infty$ with v_{ren} and b fixed, the relative Lagrangian stays finite, while the center-of-mass Lagrangian diverges, as expected.

Now we have to extract from L_{rel} the effective metric on the relative moduli space. As explained above, the electric charges q_j are conserved momenta conjugate to phase degrees of freedom t_j associated to monopoles. Since the monopoles are indistinguishable, we may assume that t_j are periodic variables with the same period. The coordinates on the relative moduli space of two monopoles are $z = z_k - z_1$, $\chi = \chi_k - \chi_1$, and $t = t_k - t_1$. To read off the metric on the moduli space, we need to reintroduce the dependence on $\dot{t}_j$ into the Lagrangian. This is achieved by the Legendre transform with respect to $q = q_k - q_1$. We let

$$L'_{\rm rel} = L_{\rm rel} + 4\pi g\, \dot{t} q,$$

solve the algebraic equation of motion for q, and substitute back into $L'_{\rm rel}$. The factor $4\pi g$ in front of $\dot{t}$ is introduced for convenience. The result is

$$\frac{1}{4\pi g^2} L'_{\rm rel} = \frac{\tau_2(z)}{2} \left(|\dot{z}|^2 + \dot{\chi}^2\right) + \frac{1}{2\tau_2(z)} \left(\dot{t} + \tau_1(z)\dot{\chi}\right)^2, \tag{14}$$

where

$$\tau_1(z) = \frac{b}{2g} + \frac{1}{\pi} \arg(z), \tag{15}$$

$$\tau_2(z) = \frac{v_{\rm ren}}{2g} + \frac{1}{\pi} \log|z|. \tag{16}$$

From now on and to the end of this subsection we set $g = 1$, as appropriate for non-abelian monopoles.

From Eq. (14) we read off the asymptotic metric on the moduli space. Setting

$$\tau(z) = \tau_1(z) + i\tau_2(z) = \frac{i}{2}(v_{\rm ren} - ib) + \frac{i}{\pi} \log \bar{z}, \tag{17}$$

we can write the metric as follows:

$$\frac{1}{4\pi} ds^2 = \tau_2(z)|dz|^2 + \frac{1}{\tau_2(z)} \left|dt + \tau(z) d\chi\right|^2. \tag{18}$$

Eq. (18) is a special form of the Gibbons-Hawking ansatz [8] which depends on a harmonic function on $\mathbb{R}^3$. In our case the harmonic function is $\tau_2(z)$. It is well-known that such a metric is hyperkähler and has a tri-holomorphic $U(1)$ isometry generated by $\frac{\partial}{\partial t}$. Since the harmonic function τ_2 does not depend on χ, there is an additional $U(1)$ isometry generated by the vector field $\frac{\partial}{\partial \chi}$. It is easy to check that it is also tri-holomorphic. Thus the asymptotic metric on the moduli space has a tri-holomorphic T^2 isometry, as promised. Moreover, it looks like a T^2-fibration over the z-plane, and the T^2 action is fiberwise. Moreover, there is a distinguished complex structure on this T^2 fibration, defined up to a sign, with respect to which the projection map is holomorphic. This is the complex structure

$$\begin{aligned}
\frac{\partial}{\partial z} &\mapsto -i\frac{\partial}{\partial z}, \\
\frac{\partial}{\partial \bar{z}} &\mapsto i\frac{\partial}{\partial \bar{z}}, \\
\frac{\partial}{\partial t} &\mapsto \frac{1}{\tau_2}\left(\frac{\partial}{\partial \chi} - \tau_1(z)\frac{\partial}{\partial t}\right), \\
\frac{1}{\tau_2}\left(\frac{\partial}{\partial \chi} - \tau_1(z)\frac{\partial}{\partial t}\right) &\mapsto -\frac{\partial}{\partial t}.
\end{aligned} \tag{19}$$

The nice thing about the distinguished complex structure is that it can be computed not only for well-separated monopoles, but everywhere on the moduli space [9]. The geometry of the resulting elliptic fibration is discussed in detail in the next section.

The expressions Eqs. (18,17) do not completely specify the asymptotic metric on the moduli space because we have not fixed the period of t. Let the period be $2\pi/p$, where $p \in (0, +\infty)$. To determine p, we note that $\tau(z)$ is a multi-valued function of z. This is not so surprising, if we realize that, in the distinguished complex structure, $p\tau(z)$ is the Teichmüller parameter of the T^2 fiber at point z, which is only defined up to a $PSL_2(\mathbb{Z})$ transformation. It is not hard to verify that the metric is well-defined if and only if the monodromy of $p\tau(z)$ belongs to $PSL_2(\mathbb{Z})$. Here it is important to remember that z, χ, t are the relative coordinate of two monopoles, and thus the points (z, χ, t) and $(-z, -\chi, -t)$ must be identified. Therefore $\tau(z)$ and $\tau(-z)$ must be related by a $PSL_2(\mathbb{Z})$ transformation. From Eq. (17) it follows that under $z \to -z$ the monodromy is $\tau \to \tau + 1$. This implies that $p \in \mathbb{N}$.

The precise value for p depends on the choice of the topology of the gauge group. One can equally well work with an $SU(2)$ or an $SO(3) = SU(2)/\mathbb{Z}_2$ gauge group. This ambiguity has the following consequence. The coordinate t on the moduli space parametrizes "large" gauge transformations which leave invariant the Higgs field at infinity. Such transformations form a $U(1)$ subgroup of the gauge group. When one passes from an $SU(2)$ gauge group to its $\mathbb{Z}_2$ quotient, the period of t reduces by a factor 2, and therefore the value of p increases by a factor 2. We will see in the next section that when the gauge group is taken to be $SO(3)$, one has $p = 4$. Therefore if the gauge group is taken to be $SU(2)$ (the more standard choice for non-abelian monopoles), we have $p = 2$, and t has period π.

The metric (18,17) is applicable for large $|z|$. If we try to continue it formally to small $|z|$, we encounter a singularity at the hypersurface $\tau_2(z) = 0$. This singularity is at a finite distance, so the metric (18,17) is geodesically incomplete. This is completely analogous to the case of ordinary monopoles on $\mathbb{R}^3$: the exact metric on the relative moduli space of two monopoles (the Atiyah-Hitchin metric) asymptotically looks like a Taub-NUT metric with a “wrong” sign of the Taub-NUT parameter, so that the naive continuation of the asymptotic metric is geodesically incomplete. We expect that the exact metric on the relative moduli space of two periodic monopoles is smooth and complete, just like the Atiyah-Hitchin metric. However, the exact metric cannot have a tri-holomorphic $U(1)$ isometry. This can be seen, for example, from the fact that in the limit $v_{\rm ren} \to \infty,$ when periodic monopoles reduce to ordinary monopoles [10], the exact metric must reduce to the Atiyah-Hitchin metric, which does not have continuous tri-holomorphic isometries.

2.3 Multi-Monopole Interactions

It is obvious how to extend this procedure to interactions of several dyons. The Lagrangian turns out to be

$$\begin{aligned}\frac{1}{4\pi}L \;&=\; \sum_{j=1}^{k}\left(-\frac{v}{2g}q_j^2+\frac{gv}{2}\vec{V}_j^2+bq_jV_{j\chi}\right)+\sum_{1\le i<j\le k}\left(\frac{g^2}{2}u(z_i-z_j)(\vec{V}_i-\vec{V}_j)^2+\right.\\ &\left.+\,gw(z_i-z_j)(q_i-q_j)(V_{i\chi}-V_{j\chi})-\frac{1}{2}u(z_i-z_j)(q_i-q_j)^2\right).\end{aligned}$$

Using an identity

$$k\sum_{j=1}^{k}a_jb_j=\left(\sum_{j=1}^{k}a_j\right)\left(\sum_{j=1}^{k}b_j\right)+\sum_{1\le i<j\le k}(a_i-a_j)(b_i-b_j), \tag{20}$$

we can rewrite L as a sum of the center-of-mass Lagrangian

$$\frac{1}{4\pi}L_{CM}=\frac{vg}{2k}\left(\sum_{j=1}^{k}\vec{V}_j\right)^2-\frac{v}{2gk}\left(\sum_{j=1}^{k}q_j\right)^2+\frac{b}{k}\left(\sum_{j=1}^{k}q_j\right)\left(\sum_{l=1}^{k}V_{l\chi}\right), \tag{21}$$

and the Lagrangian describing the relative motion

$$\begin{aligned}\frac{1}{4\pi}L_{\rm rel} \;&=\; \sum_{1\le i<j\le k}\left\{\left(\frac{gv_{\rm ren}}{2k}+\frac{g^2}{2\pi}\log|z_i-z_j|\right)(\vec{V}_i-\vec{V}_j)^2+\right.\\ &+\left(\frac{b}{k}+\frac{g}{\pi}\arg(z_i-z_j)\right)(q_i-q_j)(V_{i\chi}-V_{j\chi})-\\ &\left.-\left(\frac{v_{\rm ren}}{2gk}+\frac{1}{2\pi}\log|z_i-z_j|\right)(q_i-q_j)^2\right\}.\end{aligned} \tag{22}$$

In the limit $N \to \infty, v \to \infty$ with v_{ren} and b fixed, the relative Lagrangian stays finite, while the center-of-mass Lagrangian diverges.

The relative moduli space of k monopoles on $\mathbb{R}^2 \times \mathbb{S}^1$ has the geometry of a $2(k-1)$-dimensional torus fibered over $\mathbb{R}^{2k-2}$. The torus is parameterized by monopoles' relative positions along $\mathbb{S}^1$ and their relative phases, while the coordinates on the base are given by the monopoles' relative positions on $\mathbb{R}^2$. The general form of the metric is given by the expression

$$\begin{aligned} ds^2 = {} & \frac{1}{2} g_{ij}(dz_i d\bar{z}_j + d\bar{z}_i dz_j) + \tilde{g}_{ij} d\chi_i d\chi_j + \\ & h_{ij}\left[dt_i + W_{ik} d\chi_k + \mathrm{Re}\,(Z_{ik} d\bar{z}_k)\right]\left[dt_j + W_{jl} d\chi_l + \mathrm{Re}\,(Z_{jl} d\bar{z}_l)\right], \end{aligned} \tag{23}$$

restricted to the submanifold $\sum_j z_j = \mu$, $\sum_j \chi_j = \alpha$ and $\sum_j t_j = \beta$ for some constants μ, α, and β. These restrictions are imposed to fix the position of the center of mass and the total phase. The variables t_j, χ_j are periodic with j-independent periods.

To determine the metric coefficients, we note that the asymptotic metric must have $U(1)^{k-1}$ isometry acting fiberwise. Without loss of generality, we may assume that the corresponding Killing vector fields are given by

$$\frac{\partial}{\partial t_{j+1}} - \frac{\partial}{\partial t_j}, \quad j = 1, \dots, k-1.$$

Then all metric coefficients must be independent of t_j. The corresponding integrals of motion must be identified with q_j. Thus we may compute the reduced Lagrangian which is independent of $\dot{t}_j$ by performing the Legendre transform on $\dot{t}_j$. We then compare with Eq. (23) and obtain the following answer:

$$\begin{aligned}
\frac{1}{4\pi}g_{ii} &= \frac{1}{4\pi}\tilde{g}_{ii} = gv_{\rm ren}\frac{k-1}{k} + \frac{g^2}{\pi}\sum_{j\neq i}\log|z_i - z_j|, \\
\frac{1}{4\pi}g_{ij} &= \frac{1}{4\pi}\tilde{g}_{ij} = -\frac{gv_{\rm ren}}{k} - \frac{g^2}{\pi}\log|z_i - z_j|, \ (i \neq j) \\
W_{ii} &= b\frac{k-1}{gk} + \frac{1}{\pi}\sum_{j=i+1}^{k}\arg(z_i - z_j) + \frac{1}{\pi}\sum_{j=1}^{i-1}\arg(z_j - z_i), \\
W_{ij} &= -\frac{b}{gk} - \frac{1}{\pi}\arg(z_i - z_j), \ (i < j) \\
W_{ij} &= -\frac{b}{gk} - \frac{1}{\pi}\arg(z_j - z_i), \ (i > j) \\
Z_{ij} &= 0, \\
4\pi g^2\left(h^{-1}\right)_{ii} &= \frac{v_{\rm ren}}{g}\frac{k-1}{k} + \frac{1}{\pi}\sum_{j\neq i}\log|z_i - z_j|, \\
4\pi g^2\left(h^{-1}\right)_{ij} &= -\frac{v_{\rm ren}}{gk} - \frac{1}{\pi}\log|z_i - z_j|, \ (i \neq j).
\end{aligned}$$

Note that the matrix h^{-1} is not invertible. However, it is invertible on the subspace defined by $\sum_j dt_j = \sum_j d\chi_j = 0,$ and that is all we need. Similarly, the matrix g_{ij} has a one-dimensional kernel, but on the submanifold of interest it is positive-definite if all $|z_i - z_j|$ are large.

This metric is very similar to the one found by Gibbons and Manton for monopoles on $\mathbb{R}^3$. They are both special cases of a common ansatz (Eqs. (23),(28),(29) of Ref. [4]) which is the most general $4(k-1)$-dimensional hyperkähler metric with a tri-holomorphic $U(1)^{k-1}$ isometry [1, 17]. In our case all the metric coefficients are independent of $t_j, \chi_j,$ and therefore we have $2(k-1)$ commuting Killing vector fields

$$\frac{\partial}{\partial t_{j+1}} - \frac{\partial}{\partial t_j}, \quad \frac{\partial}{\partial \chi_{j+1}} - \frac{\partial}{\partial \chi_j}, \quad j = 1, \ldots, k-1. \tag{24}$$

It is easy to check that they are tri-holomorphic. Thus the asymptotic metric on the relative moduli space admits a tri-holomorphic $T^{2(k-1)}$ isometry.

It remains to fix the periodicity of the variables $t_{j+1} - t_j.$ For the metric given by Eq. (23) to be well-defined, the period must be $2\pi/p,$ with $p \in \mathbb{N}.$ When two of the monopoles are far from the rest, the metric must agree with that found in the previous subsection. This implies that p is equal to 2 or 4 depending on whether the gauge group is $SU(2)$ or $SO(3).$

The multi-monopole metric is valid when the separations $|z_j - z_j|$ between all the monopoles are large. If we try to continue the metric to small separations, g_{ij} ceases to be invertible on the submanifold of interest. The resulting singularities indicate that the asymptotic metric is not geodesically complete. The exact metric is expected to be smooth and complete.

2.4 Two Monopoles with Singularities

It is straightforward to derive the moduli space metric in the presence of Dirac-type singularities on $\mathbb{R}^2 \times \mathbb{S}^1$. We will write it down only for the case $k = 2$ (two periodic monopoles). This case is of particular interest because the relative moduli space is four-dimensional, and therefore one obtains new examples of gravitational instantons. As explained in Ref. [10], the number of Dirac singularities n cannot exceed $2k = 4,$ therefore we obtain five gravitational instantons corresponding to $n = 0, 1, \ldots, 4.$ We denote them D_n, $n = 0, \ldots, 4.$ The reason for this nomenclature is the following. Gravitational instanton of type D_n is isomorphic to the Coulomb branch of $N = 2$ supersymmetric $SU(2)$ gauge theory with n hypermultiplets on $\mathbb{R}^3 \times \mathbb{S}^1$ [10]. The latter theory has $SO(2n)$ global flavor symmetry in the ultraviolet.

At long distances the fields created by n singularities and the two non-abelian monopoles are in a $u(1)$ Cartan subalgebra of the $su(2)$ gauge algebra. (This $u(1)$ subalgebra is defined locally by the condition that it leaves invariant the Higgs field.) Each of the singularities has magnetic charge $g_j = -1/2$ [10], while each 't Hooft-Polyakov monopole has magnetic charge 1 [10]. Since the singularities are stationary and have no electric charge, their only effect is to replace the constant background fields v_{ren} and b with $v_{\text{ren}} + \sum_{j=1}^{n} g_j u(z_{1,2} - m_j)$ and $b + \sum_{j=1}^{n} g_j w(z_{1,2} - m_j),$ respectively. Here $m_j,$ $j = 1, \ldots, n,$ are the z-coordinates of the singularities and $z_{1,2}$ are respective positions of the 't Hooft-Polyakov monopoles.

The Lagrangian is given by

$$\begin{aligned}
\frac{1}{4\pi}L &= \left(\frac{v}{2}-\frac{1}{4}\sum_{j=1}^{n}u(z_1-m_j)\right)\vec{V}_1^2+\left(\frac{v}{2}-\frac{1}{4}\sum_{j=1}^{n}u(z_2-m_j)\right)\vec{V}_2^2\\
&+ \frac{1}{2}u(z_1-z_2)\left(\vec{V}_1-\vec{V}_2\right)^2\\
&- \left(\frac{v}{2}-\frac{1}{4}\sum_{j=1}^{n}u(z_1-m_j)\right)q_1^2-\left(\frac{v}{2}-\frac{1}{4}\sum_{j=1}^{n}u(z_2-m_j)\right)q_2^2\\
&- \frac{1}{2}u(z_1-z_2)\left(q_1-q_2\right)^2\\
&+ \left(b-\frac{1}{2}\sum_{j=1}^{n}w(z_1-m_j)\right)q_1V_{1\chi}+\left(b-\frac{1}{2}\sum_{j=1}^{n}w(z_2-m_j)\right)q_2V_{2\chi}\\
&+ w(z_1-z_2)(q_1-q_2)\left(V_{1\chi}-V_{2\chi}\right).
\end{aligned}$$

To get the Lagrangian describing the relative motion, we set $\vec{V}_1+\vec{V}_2=0$ and $q_1+q_2=0$, and obtain:

$$\begin{aligned}
\frac{1}{4\pi}L_{rel} = &\left(\frac{v}{4}+\frac{1}{2}u(z_1-z_2)-\frac{1}{16}\sum_{j=1}^{n(u(z_1-m_j)+u(z_2-m_j))}\right)\left(\vec{V}_1-\vec{V}_2\right)^2\\
&+\left(\frac{b}{2}+w(z_1-z_2)-\frac{1}{8}\sum_{j=1}^{n(u(z_1-m_j)+u(z_2-m_j))}\right)\\
&\times\left(q_1-q_2\right)\left(V_{1\chi}-V_{2\chi}\right)\\
&-\left(\frac{v}{4}+\frac{1}{2}u(z_1-z_2)-\frac{1}{16}\sum_{j=1}^{n(u(z_1-m_j)+u(z_2-m_j))}\right)\left(q_1-q_2\right)^2.
\end{aligned}$$

¿From this expression we immediately see that the divergent constant C_N in the function $u(z)$ can be absorbed into a renormalization of v:

$$v_{ren}=v-\frac{4-n}{2}C_N.$$

This is precisely the same renormalization which makes the Higgs field ϕ finite in the limit $N\to\infty$ with fixed v_{ren}.

We can also read off the metric on the relative moduli space. As in subsection(2.2), we introduce the relative coordinates $z=z_1-z_2, \chi=\chi_1-\chi_2, t=t_1-t_2,$ and set $z_1+z_2=0.$ (More generally, we could set $z_1+z_2=c$ for some $c\in\mathbb{C},$ but the constant c can always be absorbed into a shift of

m_i, so one does not gain anything by considering non-zero c.) The resulting asymptotic metric has the form Eq. (18) with the function $\tau(z)$ given by

$$\tau(z) = i\left(\frac{v_{\rm ren} - ib}{2} + \frac{1}{\pi}\log(\bar{z}) - \frac{1}{8\pi}\sum_{j=1}^{n}\log(\bar{m}_j^2 - \frac{1}{4}\bar{z}^2)\right). \tag{25}$$

In particular, for $n = 4$ $\tau(z)$ has a trivial monodromy around $z = \infty$.

This metric is valid when non-abelian monopoles are far from each other and the Dirac monopoles, i.e. when $|z|$ and $|z \pm 2m_i|$, $i = 1, \ldots, n$, are all large.

Unlike in the previous cases, there is no ambiguity in the choice of gauge group here. Recall that the magnetic charge of the Dirac singularity is $-1/2$. The geometric meaning of this non-integral magnetic charge is that the monopole bundle on $\mathbb{R}^2 \times \mathbb{S}^1$ is an $SO(3)$ bundle which cannot be lifted to an $SU(2)$ bundle [10]. The obstruction is the second Stiefel-Whitney class evaluated on a sphere centered at the Dirac singularity. Thus only $SO(3)$ is a consistent choice of the gauge group.

The monodromy of τ around the points $z = \pm 2m_i$ is given by $\tau \to \tau + 1/4$. On the other hand, if the period of t is $2\pi/p$, then the monodromy of $p\tau$ must be in $PSL_2(\mathbb{Z})$. This implies that $p/4 \in \mathbb{N}$. The minimal value for p is 4, in which case t has period $\pi/2$. In the next section we will show that in the presence of Dirac singularities the minimal choice $p = 4$ is the right one. This is also the right value for p in the absence of singularities, provided that the gauge group is taken to be $SO(3)$.

3 Geometry of New Gravitational Instantons

In the previous section we have constructed five gravitational instantons (D_n, $n = 0, \ldots, 4$) and showed that they are ALG manifolds. In this section we discuss their topology and geometry.

The basic observation is that the distinguished complex structure on the moduli spaces of periodic monopoles is easy to compute using the monopole spectral curve defined in Refs. [9, 10]. Let us specialize the results of Refs. [9, 10] to the present case. To each solution of the $U(2)$ Bogomolny equations (possibly with singularities) one can associate an algebraic curve in $\mathbb{C} \times \mathbb{C}^*$. If the number of non-abelian monopoles is 2, and the z-coordinates of the singularities are given by $m_i, i = 1, \ldots, 4$, then the curve has the form

$$(y - m_1)(y - m_2)w^2 + a(y^2 - u)w + b(y - m_3)(y - m_4) = 0.$$

Here $y \in \mathbb{C}$, $w \in \mathbb{C}^*$ are the coordinates in $\mathbb{C} \times \mathbb{C}^*$, and the parameters $a, b \in \mathbb{C}^*$ can be expressed in terms of the asymptotic behavior of the monopole fields, see Ref. [10]. The complex parameter u is the modulus of the curve (i.e. it is not fixed by the boundary conditions on the monopole fields). Thus there is a map from the monopole moduli space X to the complex u-plane. As explained in the above-cited papers, this map is holomorphic (in the distinguished complex structure), and its fiber is the Jacobian of the curve. Since the curve is elliptic in our case, the fiber coincides with the curve itself. It follows that X is an elliptic fibration over $\mathbb{C}$. The asymptotic coordinate z of the previous section should be identified with $\sqrt{u}$ times a constant factor. We will see below that with our normalizations this constant factor is unity.

It is helpful to note that this elliptic fibration is precisely the Seiberg-Witten fibration for the $N = 2$, $d = 4$ gauge theory with gauge group $SU(2)$ and four fundamental hypermultiplets with masses $m_i, i = 1, \ldots, 4.$[2] This is trivial to see if we use the form of the Seiberg-Witten fibration found in Ref. [18]. Thus we can borrow the results in the physics literature [12, 13, 19] on the geometry of this fibration.

For generic m_i there are six singular fibers each of which is a rational curve with a node (i.e. a singular curve $y^2 = x^2$). In Kodaira's classification of singularities of elliptic fibrations [20], these are type-I_1 singular fibers. The Euler characteristic of an I_1 fiber is 1, so the Euler characteristic of X is 6. It is easy to see that $b_1(X) = b_3(X) = b_4(X) = 0,$ hence $b_2(X) = 5$.

When all m_i are large, four out of six singularities occur near $u = m_i^2,$ i.e. far out in the moduli space. In this region of the moduli space the asymptotic formula (19) for the distnguished complex structure is valid and should agree with the results obtained from the spectral curve approach. Indeed, we see from Eq. (25) that $\tau(z)$ has four singularities at $z = m_i.$ Thus in the asymptotic region $u \simeq z^2,$ as claimed. Moreover, this comparison allows us to infer the precise periodicity of the coordinate t left undetermined by the analysis of the previous section. There, we saw that if the period of t is $2\pi/p,$ then for large $|z|$ the Teichmüller parameter of the elliptic fiber at point z is $p\tau(z),$ where $\tau(z)$ is given by Eq. (25). The asymptotic metric is well-defined if $p/4 \in \mathbb{N}.$ From Eq. (25) we see that the monodromy of $p\tau(z)$ near $z = m_i$ is such that the singularity is of type $I_{p/4}.$ Therefore agreement with the spectral curve approach requires $p = 4.$

If one sets all m_i to zero, then all six singular fibers coalesce into a single singular fiber at $u = 0$, and the j-invariant of the curve becomes u-independent. The singularity at $u = 0$ is of type I_0^* in Kodaira's classification. This means that the singular fiber is a union of five rational curves whose

[2] As explained in Refs. [9, 10], this coincidence follows from very general string-theoretic considerations. In fact Seiberg-Witten solutions for many gauge theories can be derived by considering periodic monopoles for various gauge groups.

intersection matrix is the affine Cartan matrix of type D_4. Since $b_2(X) = 5$, these rational curves span $H_2(X)$, and therefore the intersection form on $H_2(X)$ is the affine D_4 Cartan matrix.[3] From the viewpoint of the quantum $SU(2)$ gauge theory, the I_0^* singularity corresponds to a non-trivial CFT with global $SO(8)$ symmetry [13].

In general, the elliptic fibration corresponding to the D_4 ALG manifold can have 1,3,4,5, or 6 singular fibers [13, 19]. The types of singular fibers that can occur are given by the following list:

$$I_0^*, I_1,\ I_2,\ I_3,\ I_4,\ II,\ III,\ IV.$$

We have already discussed the physical meaning of I_0^* singularity. I_n singularity corresponds to the infrared behavior of $N = 2$ $U(1)$ gauge theory with n massless charge-1 hypermultiplets [13]. Singular fibers of type II, III, and IV correspond to non-trivial CFTs (so-called Argyres-Douglas points) [19]. It is more convenient to use the notation H_1, H_2, and H_3, respectively for these singularities.

Now let us discuss the geometry of the remaining ALG manifolds (D_n with $0 \leq n \leq 3$). It is easy to see what happens when we decrease the number of Dirac monopoles n by taking one or more m_i to infinity. X is still elliptically fibered over $\mathbb{C}$, but the number of singular fibers is now given by $n+2$ for generic m_i. Each of the singular fibers is of type I_1. It follows that the Euler characteristic is $n+2$, and the second Betti number is $n+1$. By Zariski's lemma, the self-intersection number of each singular fiber vanishes, and therefore the rank of the intersection form is at most n. In particular, for $n = 0$ the second homology is one-dimensional and the intersection form vanishes altogether.

For $n < 4$ it is impossible to tune the remaining m_i to bring all $n + 2$ singular fibers together [19]. At most one can bring $n+1$ of them together, so that the elliptic fibration has two singular fibers. It has been shown in Ref. [19] that one of them is an I_1 singularity, while the other one is of type I_1, $H_1(II)$, $H_2(III)$, or $H_3(IV)$, depending on whether $n = 0, 1, 2$, or 3. More generally, an elliptic fibration corresponding to the ALG manifold of type D_n may have from 2 to $n+2$ singular fibers. The types of singular fibers that occur are I_ℓ and H_ℓ, $1 \leq \ell \leq n$.

Note that the intersection form of a singular fiber of type I_n has rank $n-1$ (it is the affine A_{n-1} Cartan matrix). Hence we may conclude that the rank of the intersection form on $H_2(X)$ is either $n-1$ or n. We saw above that for $n = 0$ the intersection form vanishes identically, while for $n = 4$ it

[3] Since X is non-compact, the intersection form need not be non-degenerate. In the present case, the kernel of the intersection form is one-dimensional.

coincides with the affine Cartan matrix of type D_4. It would be interesting to compute the intersection form for the remaining cases ($n = 1, 2, 3$).

4 Realization via Hitchin equations

In Refs. [9, 10] we showed that Nahm transform establishes a one-to-one correspondence between periodic monopoles with Dirac singularities and solutions of Hitchin equations on a cylinder with particular boundary conditions. Furthermore, we showed that the map between the corresponding moduli spaces is bi-holomorphic if one uses the natural complex structures. Both moduli spaces are hyperkähler manifolds, and analogy with the case of monopoles on $\mathbb{R}^3$ suggests that Nahm transform induces an isometry between them. If this is true, then we have an alternative construction of ALG gravitational instantons using the moduli space of Hitchin equations.

Let us focus on the case of ALG manifold of type D_4, when the boundary conditions for Hitchin equations are especially simple. According to Ref. [10], if the number of Dirac singularities is four and the number of non-abelian monopoles is two, then the Hitchin data consist of a $U(2)$ connection $\hat{A}$ and a Higgs field $\hat{\phi}$ on a cylinder $\mathbb{R} \times \mathbb{S}^1$ with two points removed. We will identify $\mathbb{R} \times \mathbb{S}^1$ with a strip $0 \leq \mathrm{Im} s < 1$ in a complex s-plane (this requires picking an orientation on $\mathbb{R} \times \mathbb{S}^1$). The two punctures are located at $s = s_1$ and $s = s_2$. Away from the punctures $\hat{A}$ and $\hat{\phi}$ satisfy the Hitchin equations [11]

$$\bar{\partial}_{\hat{A}} \hat{\phi} = 0, \quad \hat{F}_{s\bar{s}} + \frac{i}{4}\left[\hat{\phi}, \hat{\phi}^\dagger\right] = 0,$$

while near the punctures they have the following behavior:

$$\hat{\phi}(s) \sim \frac{R_i}{s - s_i}, \quad \hat{A}_s \sim \frac{Q_i}{s - s_i}, \quad i = 1, 2.$$

Here R_i and Q_i are rank-one matrices which can be simultaneously diagonalized by a gauge transformation. Their eigenvalues depend on the behavior of the monopole fields near $z = \infty$, see Ref. [10] for details. We should also specify the behavior of $\hat{\phi}$ and $\hat{A}$ at infinity. Let $r = \mathrm{Re}\, s$. Let $m_i \in \mathbb{C}, i = 1, \ldots, 4$, be the z-coordinates of the Dirac singularities, and $\chi_i \in \mathbb{R}/(2\pi\mathbb{Z})$ be their coordinates on $\mathbb{S}^1$. For $|r| \to \infty$ the connection A becomes flat; the asymptotic holonomy is given by

$$\mathrm{diag}\left(e^{i\chi_1}, e^{i\chi_2}\right)$$

for $r \to -\infty$ and

$$\mathrm{diag}\left(e^{i\chi_3}, e^{i\chi_4}\right)$$

for $r \to +\infty$. The eigenvalues of the Higgs field tend to m_1, m_2 for $r \to -\infty$ and to m_3, m_4 for $r \to +\infty$.

It is rather obvious that the moduli space of Hitchin equations is a hyperkähler manifold. Indeed, Hitchin equations on a cylinder can be regarded as moment map equations for an infinite-dimensional HKQ. The quotienting group is the group of gauge transformations, and it acts on the cotangent bundle of an infinite-dimensional affine hyperkähler space, the space of $U(2)$ connections on a cylinder. The residues of the Higgs field and the connection at s_1, s_2 can be regarded as the level of the moment map.

Solutions of Hitchin equations of this kind have been extensively studied by C. Simpson [21] and others. To make explicit the connection with Simpson's work, we make a conformal transformation $w = e^{2\pi s}$ which maps the cylinder with two punctures into a $\mathbb{P}^1$ with four punctures, with w being the coordinate on the North patch of the $\mathbb{P}^1$. Hitchin equations are conformally invariant if we agree that $\hat{\phi}$ transforms as a 1-form, i.e. $\hat{\phi}_s ds = \hat{\phi}_w dw$. Then $\hat{\phi}_w$ has simple poles at all four punctures, with residues which can be simultaneously diagonalized by a gauge transformation. Furthermore, $\hat{A}_w$ also has simple poles, and the residues can be diagonalized simultaneously with the residues of $\hat{\phi}_w$.

One can simplify this further by noting that the trace and traceless parts of $\hat{A}_w, \hat{\phi}_w$ separately satisfy Hitchin equations. Hitchin equations for the trace part simply say that $\mathrm{Tr}\ \hat{\phi}_w$ and $\mathrm{Tr}\ \hat{A}_w$ are holomorphic 1-form and flat connection on a punctured $\mathbb{P}^1$, and therefore are completely determined by their residues. Thus the moduli space of $U(2)$ Hitchin equations can be replaced by the moduli space of $SU(2)$ Hitchin equations. Using the results of Ref. [10], one can easily compute the eigenvalues of the residues of the $SU(2)$ Higgs field and connection in terms of locations of the Dirac singularities and the asymptotic behavior of the monopole fields. In the notation of Ref. [10], the eigenvalues of the residues of the Higgs field at the four punctures are given by

$$\pm\frac{1}{2}(m_1 - m_2),\ \pm\frac{1}{2}(m_3 - m_4),\ \pm\frac{\mu_1}{2},\ \pm\frac{\mu_2}{2},$$

and the eigenvalues of the residues of the connection are given by

$$\pm\frac{i}{2}(\chi_1 - \chi_2),\ \pm\frac{i}{2}(\chi_3 - \chi_4),\ \pm\frac{i\alpha_1}{8\pi},\ \pm\frac{i\alpha_2}{8\pi}.$$

The parameters μ_1, μ_2 and α_1, α_2 can be expressed in terms of the asymptotic behavior of the monopole fields [10].

When the number of Dirac singularities is less than four, the Nahm transform is again given in terms of solutions of Hitchin equations on $\mathbb{P}^1$. But the singularities of $\hat{\phi}$ and $\hat{A}$ are more complex in this case (they are not "tame," in the terminology of Ref. [21]). For more details, see Ref. [10].

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