Two Families of Geometric Field Theories

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Abstract

We present two families of exterior differential systems (EDS) for causal embeddings of orthonormal frame bundles over Riemannian spaces of dimension $q = 2, 3, 4, 5....$ into orthonormal frame bundles over flat spaces of higher dimension. We calculate Cartan characters showing that these EDS are dynamical field theories. The first family includes a non-isometric embedding EDS for classical Einstein vacuum relativity ($q = 4$). The second, generated only by 2-forms, is a family of classical "stringy" or Kaluza-type ($q = 5$) integrable systems. Cartan forms are found for all these dynamical theories.

1 Introduction

We discuss two families of geometric field theories. The first family is derived by variation of the Einstein-Hilbert action in $q = 3, 4, 5,$ etc. dimensions. The case when $q = 4$ is vacuum general relativity seen as a Ricci-flat orthonormal frame bundle over a 4-space which also carries a second "ghost" metric induced from embedding in flat 10-space. The second family, in dimensions $q = 2, 3, 4, 5,$ etc., has a more elegant string/gauge structure with $q - 1-$dimensional frames that define Riemannian submersions. When $q = 4$ or 5 it offers some alternatives for classical field theory.

By "geometric" we mean that these theories are given as exterior differential systems (EDS) for embedding of q-dimensional submanifolds R^q in flat homogeneous isotropic metric spaces E^N of higher dimension, say N. To formulate these EDS we in fact embed the orthonormal frame bundles over the submanifolds into the orthonormal frame bundles over the flat spaces, that is, into the groups ISO(N), which have dimension $N(N+1)/2$. E. g., the $q = 4$ dimensional EDS are set using the 55 basis 1-forms of ISO(10). The fibers of the embedded bundles are subgroups of the O(N)fibers of ISO(N), thus inducing embedding maps of their q-dimensional bases R^q into the E^N bases of ISO(N).

By "field theories" we mean that each of these various EDS is shown, by an explicit numerical calculation of its associated algebraic structure (or sheaf), to have the property of being "causal". Our technique for analyzing the sheaf of an EDS, using rational arithmetic and special *Mathematica* programs, is explained in Section 2. In Cartan's theory of the sequence of regular integral manifolds (of successively higher dimensions, using the sheaf) of a causal EDS, the final construction of the embedding is determined solely from boundary data, and, at least in the analytic category, Cauchy existence and uniqueness are proved. We believe that, with proper attention to signature, the sets of partial differential equations following from a causal EDS are those of a canonical field theory. Non-trivial embedding EDS that are causal are not common–we have been able to discover just essentially these two families. (There are also simple causal EDS for embedding *flat* geodesic subspaces; these are used in Section 4). A key to their existence may be that for all the EDS we consider here we are also able to find Cartan q-forms from which the EDS may be derived by arbitrary variation.

There is a large literature, beginning with Lepage and Dedecker, on the use of Cartan q-forms and their closure $q+1$-forms ("multisymplectic" forms). These are the multidimensional field-theoretic extensions of classical Hamiltonians and symplectic geometry. A short but essential bibliography can be found in Gotay [1]; cf. also Hermann [2] and Estabrook [3]. The differential geometric setting for that work was for the most part (the structure equations of basis forms on) the first or second jet bundle over a base of q independent variables. Our use instead of basis forms and structure equations from Lie rotation groups, fibers of orthonormal frame bundles over flat metric geometries, appears to be an innovation. It allows application of those techniques of field theory to the movable frames of general relativity, and can lead to interesting extensions.

In both these families of EDS we generalize the method always used in the mathematical literature for isometric embedding, cf. e.g. [4], [5], in that we do *not* require any alignment of the local frames of the solution submanifolds with the orthonormal frames of the embedding space. Perhaps this generalization of the customary isometric embedding can be called "dynamic embedding". The EDS that naturally arise are much more elegant.

The Lie group ISO(N), or one of its signature siblings ISO($N-1,1$) etc., is the isometry group of N-dimensional flat space, E^N. The group space is spanned by $N(N+1)/2$ canonical vector fields, and by a dual basis of left-invariant 1-forms that we denote by $\theta_\mu, \mu = 1...N$, corresponding to translations, and by $\omega_{\mu\nu} = -\omega_{\nu\mu}$, corresponding to rotations. Now the structure equations for general movable frames over an N-dimensional manifold are usually written covariantly (on the second frame bundle) as

$$d\theta^\mu + \omega^\mu_\nu \wedge \theta^\nu = 0 \tag{1}$$

$$d\omega^\mu_\nu + \omega^\mu_\sigma \wedge \omega^\sigma_\nu + R^\mu{}_\nu = 0. \tag{2}$$

These become the Cartan-Maurer equations of ISO(N) or one of its siblings when the curvature 2-forms $R^\mu{}_\nu$ are put equal to zero, and upper indices are systematically lowered using (for signature) a non-singular matrix of constants $\eta_{\mu\nu}$, after which imposing antisymmetry (orthonormality) $\omega_{\mu\nu} = -\omega_{\nu\mu}$. These structure equations then describe $N(N-1)/2$-dimensional rotation groups as fibers over N-dimensional homogeneous spaces E^N.

We will write the two families of EDS using partitions $(n, m), n+m = N$, of the basis forms of ISO(N) into classes labeled respectively by the first n indices i, j, etc. $= 1, 2, ...n$ and the remaining indices A, B, etc. $= n+1, n+2, ...N$. So the basis forms are $\theta_i, \theta_A, \omega_{ij} = -\omega_{ji}, \omega_{AB} = -\omega_{BA}, \omega_{iA} = -\omega_{Ai}$. Summation conventions on repeated indices will be used separately on each partition. The structure equations (1) (2)become (for positive definite signature, $\eta_{\mu\nu} = \delta_{\mu\nu}$)

$$d\theta_i + \omega_{ij} \wedge \theta_j + \omega_{iA} \wedge \theta_A = 0 \tag{3}$$
$$d\theta_A + \omega_{AB} \wedge \theta_B + \omega_{Ai} \wedge \theta_i = 0 \tag{4}$$
$$d\omega_{ij} + \omega_{ik} \wedge \theta_{kj} - \omega_{iA} \wedge \omega_{jA} = 0 \tag{5}$$
$$d\omega_{AB} + \omega_{AC} \wedge \omega_{CB} - \omega_{iA} \wedge \omega_{iB} = 0 \tag{6}$$
$$d\omega_{iA} + \omega_{ij} \wedge \omega_{jA} + \omega_{AB} \wedge \omega_{iB} = 0. \tag{7}$$

It is a classic result [5] that smooth local embedding of Riemannian geometries of dimension $q = 3, 4, 5...$ is always possible into flat spaces of dimension respectively $N = 6, 10, 15...$, which motivates the partitions of our first family, viz. $(n, m) = (3, 3), (4, 6), (5, 10), ...$ The causal EDS we give determine submanifolds of ISO(N) which are themselves $O(n) \otimes O(m)$ bundles fibered over $q = n$-dimensional base spaces, say R^q and induce maps of these into E^N. The n θ_i remain independent when pulled back to a solution bundle, satisfying the structure equations of an orthonormal basis in any cross section, and Equations (5) and (7) express embedding relations that go back to Gauss and Codazzi. The solution bundle metric is the pullback of $\theta_i\theta_i$. We will present in Section 3 the family of Einstein-Hilbert-Cartan forms from which the EDS of this first family are derived by variation. The EDS will require zero torsion for the θ_i but not insist on aligning the solutions with these orthonormal frames (the θ_A are *not* included in the EDS so it is not "isometric", and from the induced map of bundle bases there is also a less interesting "ghost" metric which is the pullback of $\theta_i\theta_i + \theta_A\theta_A$.) The induced curvature 2-forms are required by the EDS to satisfy "horizontality" 3-form conditions and also to have vanishing Ricci $n-1$-forms. We calculate the Cartan characteristic integers describing the sheaves, and showing the EDS to be well set and causal. These "dynamic" embedding maps for low values of N are shown in Figure 1.

The field theories of our second family, of dimension $q = 2, 3, 4, 5...$ also arise from embeddings into flat spaces E^N of dimension $N = 3, 6, 10, 15, ...$ but the EDS use different partitions, viz.., $(n, m) = (1, 2), (2, 4), (3, 7), (4, 11)$, etc. Solutions are ($O(n) \otimes O(m)$ bundles over) geometries of dimension $q = n + 1$ and can be called n-branes. They have rulings that are flat n-spaces. The EDS are generated only by sets of 2-forms (for vanishing torsion of both partitions) and are so-called "integrable systems". Again the embedding is dynamic, the partitioned frames are not required to be aligned with the solution manifolds. In Section 4 we give the EDS and calculate the Cartan characters showing them to be causal. The n θ_i when pulled back into a solution determine a projection and imply a metric $\theta_i\theta_i$ on its quotient space. This is either a theory of relativistic rigidity or of a Kaluza-Klein gravitational field, depending on N and the signature adopted. Cartan forms for those EDS are easily found. The embedding maps into E^N are shown in Figure 2.

As a sole illustration of introduction of coordinates into an EDS, the simplest of these non-isometric geometric field theories, that based on partition $(1, 2)$, is integrated in Section 5. Its solutions turn out to be classically known, in the guise of ruled surfaces in E^3. We have only changed signature to show it as a stringy field causally evolving in time.

2 Sheaf Algebra

We have previously expounded [7] Cartan's construction of the so-called regular integral manifolds of a closed EDS, as a sequence, or "flag", of simple integrations (1) along a line, beginning at a generic point, (2) along a family of lines yielding a surface and containing the first line, (3) along a two-parameter family of lines yielding a 3-space containing the surface, and so on. This construction does not capture all integral manifolds of the EDS, and is in some disrepute mathematically, as it is of little use in proving new existence and uniqueness theorems for p.d.e.'s, being limited to *analytic* solutions for subspaces. Nonetheless Cartan's theory shows the importance of a diagnostically useful algebraic invariant of the EDS, viz. the nested set of homogeneous linear algebras or "sheaf" that arises in the construction. The sheaf elegantly characterizes the analytic solutions of the EDS, and above all can demonstrate whether it is well set as a physical field theory.

At a generic point in a space of dimension, say, M, which includes all independent, dependent and gauge variables, the sheaf of a closed EDS is a nested set of linear homogeneous conditions on the M components of a set of vectors $V_1, V_2, V_3,$ etc. First a vector V_1 is found that annuls all the 1-forms in the EDS. This condition is an underdetermined set of linear equations for the components of V_1 so a number of them, say $M - s_0$, may be chosen arbitrarily. Given that solution, a new vector V_2 (independent of V_1) is required that annuls all the 1-forms and also, together with V_1, annuls all the 2-forms in the EDS. The rank of this second set of linear equations, for the components of V_2, cannot be smaller than that of the first set, so we denote it $s_0 + s_1$, where s_1 is non-negative. $M - s_0 - s_1$ choices of remaining components are made. Given the first two, an independent vector $V_3$is then sought that annuls all 1-forms in the EDS, that with either V_1 or V_2 annuls all 2-forms in the EDS, and that with both now also annuls all 3-forms in the EDS. Again the rank cannot have decreased, so it is denoted $s_0 + s_1 + s_2$. The s's are all non-negative. The equations for new independent solution vectors continue similarly, and the sequence of Cartan characteristic integers s_i calculated. At a generic point, they are a set of numerical concomitants of the EDS. From them the causal and fiber structures of the solutions, integral submanifolds of the EDS, can be inferred.

It is straightforward [7] that the sheaf algebra terminates: there is a maximal generic number of solution vectors, say g. This is Cartan's *genus,* the dimension of the maximal regular solution submanifold. In the following we will write the resulting set of characteristic sheaf integers and the genus in the format $M\{s_0, s_1, s_2, ...s_q, ...s_{g-1}\}g$. The criterion for g is that at the last step $M - \sum_{i=0}^{g-1} s_i \geqslant g$. Some arbitrary choices may remain for $V_g,$ but further construction of independent vectors fails at step $g + 1$. V_g is determined (up to magnitude) if there are no remaining choices for its components and the equality holds; then, at least in principle, the solution of Cartan's construction is uniquely related to suitable conditions set on its boundary. We denote such EDS as being "causal". g will be the number of independent variables, and $M - g$ the number of dependent variables in equivalent sets of partial differential equations.

It often happens that one of the integers (and consequently those following it if there are no generating forms of those higher ranks in the EDS), say s_q, vanishes. Then g becomes the essential number of independent variables in the resulting causal set of partial differential equations, and $g-q$ variables in the solutions are gauge variables.. Geometrically, there are in each solution $g-q-$dimensional fibers. If $g-q$ variables are not explicitly present as basis forms in the EDS, their dual vectors in the M-space are said to be Cartan characteristic vectors [7] of the EDS, and in that event the vanishing of the integer characters $s_q...s_{g-1}$ follows immediately. For the genus g we then write "$q+$ dim. fibers".

To solve for the characteristic sheaf integers s_i explicitly is not however easy, and few sets are to be found in the literature for nontrivial problems. The coefficients in the "linear" equations for new vector components at each step depend on the components of previous vectors, and those components already have been required to jointly annul the EDS at previous steps. Consequently the components being solved for at any step in fact are subject to both linear and non-linear constraint equations, which must be taken account of in determining the rank of the new equations. In general, p-degree forms will lead to pth-order polynomial constraints, so if one attempts to obtain a solution for the s_i algebraically this nested problem is not in fact a linear one.

As an example from what follows, consider an ideal in a space of $M = 55$ dimensions, so each V has 55 components. Calculating the ranks of equations solving an EDS with 2-forms, 3-forms, etc., when we come to consider the equations for, say, V_6, we would have as many as 100 quadratic constraints and 40 or so cubic constraints, involving the 275 components of the previous five vectors $V_1...V_5$! In a general case it is unlikely that even a highly sophisticated algebraic manipulation system would be capable of correctly evaluating the rank of the linear equations for V_6 when subject to such constraints.

This algebraic problem can be finessed by calculating particular solutions with *numerical* components. At each step these vector components are determined to satisfy the appropriate linear equations, and any components left undetermined are assigned *random* numerical values, usually integers. This purely numerical problem leading to a single solution is now linear throughout, and standard techniques to determine the ranks of linear equations can be used, even with a large number of components. The calculations are performed using rational arithmetic; while not really essential, this does avoid the possible problem of having fortuitously small values become zero due to round-off errors.

Clearly, this technique suffers from the difficulty of any Monte-Carlo approach in that it may not give the generic, correct, answer in any single calculation. A particular set of random numerical components may give a lower rank than is true in general. This does in fact happen, however very infrequently, since the large number of random components provides a very large ensemble. Furthermore the calculation only requires from a few seconds to a few minutes, so it can easily be repeated several times to check for accidental degeneracies.

3 Einstein-Hilbert Action

The EDS of our first family arise from Cartan n-forms on ISO(N) expressing the Ricci scalars of $q = n$ dimensional submanifolds of E^N,

$$\Lambda = R_{ij} \wedge \theta_k \wedge ...\theta_p \epsilon_{ijk...p}, \tag{8}$$

where from the Gauss structure equation, Eq. (5), $2R_{ij} = -\omega_{iA} \wedge \omega_{jA}$ is the induced Riemann 2-form. The exterior derivative of the n-form field Λ on ISO(N), using Eq. (3) and (7), is quickly calculated to be the $n+1$-form (closed, cosymplectic)

$$d\Lambda = \theta_A \wedge \omega_{Ai} \wedge R_{jk} \wedge \theta_l \wedge ...\theta_p \epsilon_{ijkl...p}. \tag{9}$$

This $n+1$-form is a sum of products of the m 1-forms θ_A and the m n-forms $\omega_{Ai} \wedge R_{jk} \wedge \theta_l \wedge ...\theta_p \, \epsilon_{ijkl...p}$. A variational isometric embedding EDS is generated by the m θ_A, their exterior derivatives for closure, and the m n-forms, since any vector field that annuls it is sufficient to annul $d\Lambda$, term by term. That is, up to boundary terms, the arbitrary variation of Λ vanishes on solutions. We previously calculated Cartan's characteristic integers for these isometric embedding EDS showing them to be well set and causal [8]. We denoted them as being "constraint free" geometries. Isometric embedding formulations of the Ricci-flat field equations then are obtained by adding in the closed $n-1$-forms for Ricci-flatness as constraints. The augmented EDS are again calculated to be causal. Thus for vacuum general relativity, the partition (4, 6), the constraint-free sheaf algebra was 55 $\{6, 6, 6, 12\}g = 4 + 21$ which, with the augumentation with four 3-forms became 55 $\{6, 6, 10, 8\}g = 4 + 21$. We now see that formulation as nevertheless somewhat unsatisfactory, since the Einstein-Hilbert action appears to have lead to equations which in fact mostly follow from the imposed constraints.

We have however noticed that there is another variational EDS belonging to a different quadratic factoring of the cosymplectic forms $d\Lambda$, Eq. (9). The θ_i will frame a Riemannian metric on an embedded space of dimension n so long as the torsion 2-forms $\omega_{iA} \wedge \theta_A$ vanish, and these factor Eq. (9) term-by-term, as products with the n $n-1$-forms for Ricci-flatness. The exterior derivatives of the torsion terms are 3-forms for symmetries of the Riemann tensor, sometimes called conditions for horizontality. In sum, we have considered the following closed EDS (which now do *not* include the mathematically customary isometric embedding 1-forms θ_A)

$$(\omega_{iA} \wedge \theta_A, R_{ij} \wedge \theta_j, R_{ij} \wedge \theta_k \wedge ...\theta_l \epsilon_{ijk...lp}) \tag{10}$$

The sheaf calculation shows these to be well set and causal systems for embedding of $O(n) \otimes O(m)$ bundles over n space, for the partitions (3, 3), (4, 6), (5, 10) etc. as stated in the introduction. The embedding dimension, the computed Cartan characters, genus and $O(n) \otimes O(m)$ fiber dimension of the solutions for these cases are respectively $21\{0, 6, 3\}g = 3 + 9, 55\{0, 4, 12, 14\}g = 4 + 21, 120\{0, 5, 10, 20, 25\}g = 5 + 55$, etc. These causal dynamic embeddings are shown in Figure 1.

The base spaces of the fibered solution manifolds are spanned by the 1-forms θ_i; evidently a solution is a bundle of orthonormal frames belonging to the Ricci-flat Riemannian connection ω_{ij}, together with a gauge connection ω_{AB}. The metric is $\theta_i\theta_i$. There is also present in the base space R^n another metric pulled back from the induced embedding of it in the base space E^N about which we know little. It is a ghost tensor field, perhaps with only indirect influence. The ideals we are writing are set on ISO(N), and their solutions are frame bundles embedded in ISO(N), and the induced embeddings of the base spaces seem to be of less interest.

The ideal Eq. (10) is contained in the augmented isometric embedding ideal we have previously used, so solutions of the latter will be solutions of the former. This would seem to imply that our new dynamic embedding ideal will have additional solutions; indeed it implies fewer partial differential equations than does the isometric embedding ideal augmented with constraints for Ricci-flat geometry. Perhaps so-called singular solutions of the isometric embedding ideal–solutions which are not regular, that is, obtained by Cartan's sequential integrations–now appear as regular solutions, which could make this new formulation important for local numerical computation from boundaries.

4 Torsion-free n-brane Embedding

We have searched whether the torsion 2-forms induced in *both* the local partitions can together be taken as an EDS: ($\omega_{iA} \wedge \theta_A$, $\omega_{iA} \wedge \theta_i$). It can easily be checked that it is closed, and calculation of the characteristic integers of the sheaf indeed showed that *for just the values of (n, m) of the second family described in the introduction* these EDS are causal, with $q = n+1$ and fibers $O(n) \otimes O(m)$, dim $n(n-1)/2 + m(m-1)/2$. The results for the first five EDS are: $(n, m) = (1, 2)$, $6\{0, 3\}q = 2+1$ dim fiber; $(2, 4)$, $21\{0, 6, 5\}3+7$; $(3, 7)$, $55\{0, 10, 9, 8\}4$ + 24; $(4, 11)$, $120\{0, 15, 14, 13, 12\}5$ + 61; $(5, 16)$, $231\{0, 21, 20, 19, 18, 17]6$ + 130; and the pattern seems evident. The embedding maps are shown in Figure 2.

Well set EDS for geodesic flat dimension n submanifolds of flat N spaces are generated, using the partition (n, m), by the closed ideal of 1-forms (θ_A, ω_{Ai}). For example, if $N = 3$ and $n = 1$ and $m = 2$, geodesic lines in flat 3-space, the Cartan characteristic integers are $6\{4\}1 + 1$. If $N = 4$, for partition $(1, 3)$ we find $10\{6\}1 + 3$ (in all cases ω_{ij} and ω_{AB} give the Cauchy characteristic fibers). Similarly, the EDS for flat 2-dimensional submanifolds of flat N spaces are generated by the 1-forms with partitions $(2, N-2)$. For example if $N = 5$, $(n, m) = (2, 3)$, and the sheaf algebra is $15\{9, 0\}2 + 4$. When $N = 6, (n, m) = (2, 4)$ and $21\{12, 0\}2+7$. These constructions clearly continue. Now the torsion-free EDS ($\omega_{iA} \wedge \theta_A$, $\omega_{Ai} \wedge \theta_i$) is contained in (θ_A, ω_{Ai}), so we see that the q-dimensional solutions of the torsion-free embedding theory must contain flat geodesic fibers of dimension $n = q - 1$. Thus the solutions are *ruled* spaces.

In a solution the θ_i remain independent (are "in involution") but fall short by one of being a complete basis. They define there a vector field, say V, of arbitrary normalization (a congruence), by the relations $V \cdot \theta_i = V \cdot \omega_{ij} = V \cdot \omega_{AB} = 0$. Contracting V on the second torsion 2-form, since the θ_i remain linearly independent, gives also $V \cdot \omega_{iA} = 0$. Thus the Lie derivative with respect to V of all of these (except for θ_A) vanishes. They live in an n dimensional quotient space of the solution, with metric $\theta_i \theta_i$ and Riemann tensor $\omega_{iA} \wedge \omega_{Aj}$. This is in contrast to the rulings, n-dimensional subspaces also carrying the metric $\theta_i \theta_i$ but flat.

In an earlier time we have discussed the problem of defining a rigid body in special and general relativity [6]. The kinematic quotient-space definition of rigidity due originally to Max Born (Riemannian submersion [2]) was shown by Herglotz and Noether to have only three degrees of freedom: the only Born-rigid congruences which were rotating (had vorticity) in Minkowski space were isometries of the space-time without time evolution. We showed this to be the case also for kinematic or "test" rigid bodies moving in vacuum Einstein spaces. It seemed to be impossible then to sensibly discuss the so-called "dynamic" rigid bodies introduced by Pirani, which were to carry their own three dimensional geometry. We are charmed by having now arrived at space-times, using dynamic embedding in the (3, 7) partition, having the greater dynamical freedom allowed by separation of the roles of the induced metrics in the cross sections and quotient space of a solution.

In the (4, 11) partition, the solutions are five dimensional, with a dynamically rigid congruence that projects to a metric 4-space. This is surely a well-posed causal variant of Kaluza-Klein theory, and remains for further investigation.

Closed EDS generated only by "invariant" 2-forms (meaning no explicit functional dependence, as here) have a special structure, inasmuch as they can be equivalent to dual infinite Lie algebras of Kac-Moody type and lead to hierarchies of so-called integrable systems. The prototype of this is the well-known Korteweg-de Vries equation, which both leads to [9], and belongs to the hierarchy of, the infinite Lie algebra ${A_1}^{(1)}$derived from SL(2, R). The Kac-Moody algebras dual to our embedding EDS remain to be worked out.

Although we did not derive these EDS variationally, Cartan forms are easily found. For example, in the (3, 7) theory either the 2-forms $\tau_A = \omega_{Ai} \wedge \theta_i$ or $\sigma_i = \omega_{iA} \wedge \theta_A$ can be used to write a quadratic Cartan form as in some Yang-Mills theories:

$$\Lambda = \tau_A \wedge \tau_A, \text{ so } \; d\Lambda = 2\tau_A \wedge \omega_{Ai} \wedge \sigma_i \tag{11}$$

Every term of $d\Lambda$ contains both a τ_A and a σ_i so arbitrary variation yields the EDS. We also note that $\tau_A \wedge \tau_A + \sigma_i \wedge \sigma_i$ is exact.

5 The Partition (1, 2)

We will set this EDS on the frame bundle ISO(1, 2) over a flat 3-space with signature (-, +, -), so the structure equations of the basis are

$$\begin{aligned}
d\theta_1 + \omega_{12} \wedge \theta_2 + \omega_{31} \wedge \theta_3 &= 0 &(12)\\
d\theta_2 + \omega_{12} \wedge \theta_1 - \omega_{23} \wedge \theta_3 &= 0 &(13)\\
d\theta_3 - \omega_{31} \wedge \theta_1 - \omega_{23} \wedge \theta_2 &= 0 &(14)\\
d\omega_{12} - \omega_{31} \wedge \omega_{23} &= 0 &(15)\\
d\omega_{23} - \omega_{12} \wedge \omega_{31} &= 0 &(16)\\
d\omega_{31} + \omega_{23} \wedge \omega_{12} &= 0, &(17)
\end{aligned}$$

and the EDS to be integrated is generated by the three 2-forms $\omega_{iA} \wedge \theta^A, \omega_{Ai} \wedge \theta^i, i = 1, A = 2, 3$:

$$\begin{aligned}
\omega_{12} \wedge \theta_2 + \omega_{31} \wedge \theta_3 & &(18)\\
\omega_{12} \wedge \theta_1 & &(19)\\
\omega_{31} \wedge \theta_1 . & &(20)
\end{aligned}$$

The characteristic integers of the sheaf are 6{0, 3} g = 2 with O(2) fiber (since ω_{23} is not present). To introduce coordinates - scalar fields - we will successively prolong the EDS with potentials or pseudopotentials, checking at each step that it remains well-set and causal.

First, it is obvious that there is a conservation law, a closed 2-form that is zero mod the EDS, viz. $d\theta_1$. So we adjoin the 1-form

$$\theta_1 + dv, \tag{21}$$

introducing the scalar potential v. The sheaf now is $7\{1,3\}g = 2$ with O(2) fiber. Next we specialize to a particular, convenient, fiber cross-section making a choice of frame or gauge: we introduce two new fields ζ and η while prolonging with three 1-forms taken so that the original 2-forms in the EDS vanish (they have been "factored")

$$\begin{aligned} \omega_{12} - \zeta\theta_1 && (22)\\ \omega_{13} - \eta\theta_1\zeta && (23)\\ \zeta\theta_2 - \eta\theta_3 + (\eta+\zeta)\theta_1. && (24) \end{aligned}$$

To maintain closure, however, three new 2-forms, exterior derivatives of these or algebraically equivalent, must also be adjoined:

$$\begin{aligned} (d\zeta - \eta\omega_{23}) \wedge dv && (25)\\ (d\eta - \zeta\omega_{23}) \wedge dv && (26)\\ (\eta d\zeta - \zeta d\eta) \wedge (\theta_2 + \theta_3) - (\eta+\zeta)\omega_{23} \wedge (\eta\theta_2 - \zeta\theta_3). && (27) \end{aligned}$$

Now we have $9\{4,3\}g = 2$ with no remaining gauge freedom. ω_{23} now appears in the EDS, but is conserved, $d\omega_{23} = 0$ mod EDS. Thus, we can introduce a pseudopotential variable x , and then further find another conserved 1-form and a final pseudopotential u. Which is to say we can adjoin

$$\begin{aligned} \omega_{23} - dx && (28)\\ \theta_2 + \theta_3 - e^x du, && (29) \end{aligned}$$

without adding any 2-forms to the EDS. We have a total of 11 basis 1-forms: six in θ_i, θ_A, ω_{AB}, ω_{iA}, plus $d\zeta, d\eta, dx, du, dv$, and an EDS with $11\{6,3\}g = 2$. The pulled-back original six bases are now all solvable in terms of coordinate fields on the solutions, and can be eliminated: $5\{0,3\}g = 2$. This is equivalent to a set of first order partial differential equations in 3 dependent variables and 2 independent variables.

Taking x and v as independent in the solution, we can solve the first two 2-forms in Eq. (25) and (26) for η and ζ:

$$\begin{aligned} \eta = ae^x + be^{-x} && (30)\\ \zeta = ae^x - be^{-x}, && (31) \end{aligned}$$

where a and b are arbitrary functions of v. The third 2-form then amounts to

$$\begin{aligned} e^x = 1/2(b/a)'\partial_x u, && (32)\\ \text{which integrates to} \quad e^x = 1/2B'(u - A(v)). && (33) \end{aligned}$$

We have put $b/a = B(v)$ and prime is derivation with respect to v.

The two arbitrary functions of v, A and B, give the general solution. On it the pulled-back bases of E^3 (no longer orthonormal or independent)are

$$\theta_1 = -dv \tag{34}$$

$$\theta_2 = dv + \frac{ae^x + be^{-x}}{2a} du \tag{35}$$

$$\theta_3 = -dv + \frac{ae^x - be^{-x}}{2a} du, \tag{36}$$

and the induced 2-metric from E^3 is

$$\begin{aligned} g &= -\theta_1\theta_1 + \theta_2\theta_2 - \theta_3\theta_3 \\ &= Bdu^2 + B'(u-A)dvdu - dv^2. \end{aligned}$$

This is, up to signature, the metric found classically from the construction of ruled surfaces in E^3, cf, e.g., Eisenhardt [10]. The surfaces are intrinsically characterized by a "line of striction", the locus $u - A(v) = 0$, and a "parameter of distribution" $2B/B'$. The geodesic rulings, on which θ_2, θ_3, ω_{12}, and ω_{13} pull back to vanish, are the set of lines $u =$ const. The rigid congruence is the set of lines on which V contracted with θ_1, ω_{12} and ω_{13} vanishes, hence $v =$ const.

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Figure Captions

Figure 1. Einstein-Hilbert field theories for embedding partitions $(n, m), N = n + m$. The causal sheaf algebra is tabulated in each case as dim ISO($\boldsymbol{N}$) $\{\omega_{13}, s_0, ...s_1\}q+$ dim fiber.

Figure 2. Torsion-free field theories for embedding partitions $(n, m), n+m = \boldsymbol{N}$. Sheaf notation as in Fig. 1.