

Theta functions on Noncommutative T^4

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ABSTRACT

We construct the so-called theta vectors on noncommutative T^4 , which correspond to the theta functions on commutative tori with complex structures. Following the method of Dieng and Schwarz, we first construct holomorphic connections and then find the functions satisfying the holomorphic conditions, the theta vectors. The holomorphic structure in the noncommutative T^4 case is given by a 2×2 complex matrix, and the consistency requires its off-diagonal elements to be the same. We also construct the tensor product of these functions satisfying the consistency requirement.

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I. Introduction

Classical theta functions have played an important role in the string loop calculation [1, 2]. Recently, noncommutative geometry [3] became an important ingredient of string/M theory (for instance, see [4]) starting with the work of [5].

Along this direction, noncommutative torus [6, 7] and its varieties [8, 9, 10], and physics on noncommutative \mathbb{R}^4 [11, 12, 13] have been studied intensively. However, noncommutative tori with complex structures have been rarely studied [14, 15, 16]. Noncommutative geometry with complex structures has been also studied recently with algebraic geometry approach for Calabi-Yau three folds [17, 18, 19], and for K3 surfaces [20].

Classical theta functions can be regarded as state functions over commutative tori with complex structures. Noncommutative generalization of this has been initiated in mathematics in the quantized theta function approach by Manin [14], and with the so-called theta vectors by Schwarz [15]. In the physics literature, this has appeared in the context of noncommutative solitons [21, 22, 23] but mostly in the so-called integral torus case. Recently, Dieng and Schwarz [16] have computed the theta vectors and their tensor products on noncommutative T^2 explicitly without any restriction.

In this paper, we follow the method of Dieng and Schwarz and calculate the theta vectors and their tensor products in the case of noncommutative T^4 . In section II, we construct modules on the noncommutative four torus. In section III, we deal with connections with complex structures. In section IV, we deal with tensor product of these modules. In section V, we conclude with discussion.

II. Modules on noncommutative T^4

In this section, we construct the modules on noncommutative T^4 following the method of Rieffel[7].

Recall that T_θ^d is the deformed algebra of the algebra of smooth functions on the torus T^d with the deformation parameter θ , which is a real $d \times d$ anti-symmetric matrix. This

algebra is generated by operators U_1, \dots, U_d obeying the following relations

$$U_i U_j = e^{2\pi i \theta_{ij}} U_j U_i \quad \text{and} \quad U_i^* U_i = U_i U_i^* = 1, \quad i, j = 1, \dots, d.$$

The above relations define the presentation of the involutive algebra

$$\mathcal{A}_\theta^d = \left\{ \sum a_{i_1 \dots i_d} U_1^{i_1} \cdots U_d^{i_d} \mid a = (a_{i_1 \dots i_d}) \in \mathcal{S}(\mathbb{Z}^d) \right\}$$

where $\mathcal{S}(\mathbb{Z}^d)$ is the Schwartz space of sequences with rapid decay.

Every projective module over a smooth algebra \mathcal{A}_θ^d can be represented by a direct sum of modules of the form $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q \times F)$, the linear space of Schwartz functions on $\mathbb{R}^p \times \mathbb{Z}^q \times F$, where $2p + q = d$ and F is a finite abelian group. The module action is specified by operators on $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q \times F)$ and the commutation relation of these operators should be matched with that of elements in \mathcal{A}_θ^d .

Recall that there is the dual action of the torus group T^d on \mathcal{A}_θ^d which gives a Lie group homomorphism of T^d into the group of automorphisms of \mathcal{A}_θ^d . Its infinitesimal form generates a homomorphism of Lie algebra L of T^d into Lie algebra of derivations of \mathcal{A}_θ^d . Note that the Lie algebra L is abelian and is isomorphic to \mathbb{R}^d . Let $\delta : L \rightarrow \text{Der}(\mathcal{A}_\theta^d)$ be the homomorphism. For each $X \in L$, $\delta(X) := \delta_X$ is a derivation i.e., for $u, v \in \mathcal{A}_\theta^d$,

$$\delta_X(uv) = \delta_X(u)v + u\delta_X(v). \quad (1)$$

Derivations corresponding to the generators $\{e_1, \dots, e_d\}$ of L will be denoted by $\delta_1, \dots, \delta_d$. For the generators U_i 's of T_θ^d , it has the following property

$$\delta_i(U_j) = 2\pi i \delta_{ij} U_j. \quad (2)$$

If E is a projective \mathcal{A}_θ^d -module, a connection ∇ on E is a linear map from E to $E \otimes L^*$ such that for all $X \in L$,

$$\nabla_X(\xi u) = (\nabla_X \xi)u + \xi \delta_X(u), \quad \xi \in E, u \in \mathcal{A}_\theta^d. \quad (3)$$

It is easy to see that

$$[\nabla_i, U_j] = 2\pi i \delta_{ij} U_j. \quad (4)$$

We now consider the endomorphisms algebra of a module over \mathcal{A}_θ^d . Let Λ be a lattice in $G = M \times \widehat{M}$, where $M = \mathbb{R}^p \times \mathbb{Z}^q \times F$ and \widehat{M} is its dual. Let Φ be an embedding map such that Λ is the image of \mathbb{Z}^d under the map Φ . This determines a projective module which will be denoted by E_Λ [7]. The dual lattice of Λ can be defined as

$$\Lambda^\perp := \{(n, \hat{t}) \in M \times \widehat{M} \mid \theta((m, \hat{s}), (n, \hat{t})) = \langle m, \hat{t} \rangle - \langle n, \hat{s} \rangle \in \mathbb{Z}, \text{ for all } (m, \hat{s}) \in \Lambda\}, \quad (5)$$

since in the Heisenberg representation the operators are defined by

$$\mathcal{U}_{(m, \hat{s})}(r) = e^{2\pi i \langle r, \hat{s} \rangle} f(r + m) \quad (6)$$

for $(m, \hat{s}) \in \Lambda$, $r \in M$. Namely, the operators defined in the dual lattice, $\mathcal{U}_{(n, \hat{t})}$ for $(n, \hat{t}) \in \Lambda^\perp$, commute with all the operators defined in the original lattice, $\mathcal{U}_{(m, \hat{s})}$ for $(m, \hat{s}) \in \Lambda$.

It is known that the algebra of endomorphisms on E_Λ , denoted by $\text{End}_{\mathcal{A}_\theta}(E_\Lambda)$, is a C^* -algebra obtained by C^* -completion of the space spanned by operators $\mathcal{U}_{(n, \hat{t})}$, $(n, \hat{t}) \in \Lambda^\perp$. The algebra $\text{End}_{\mathcal{A}_\theta}(E_\Lambda)$ can be identified with a noncommutative torus $\mathcal{A}_{\tilde{\theta}}$, i.e., $\mathcal{A}_{\tilde{\theta}}$ is Morita equivalent to \mathcal{A}_θ [7]. Recall that a C^* -algebra A is said to be Morita equivalent to A' if $A' \cong \text{End}_A(E)$ for some finite projective module E . In general, a noncommutative torus $\mathcal{A}_{\tilde{\theta}}$ is Morita equivalent to \mathcal{A}_θ if θ and $\tilde{\theta}$ are related by $\tilde{\theta} = (A\theta + B)(C\theta + D)^{-1}$, where $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{SO}(d, d|\mathbb{Z})$ [24].

In this paper, we consider a projective module of the form $\mathcal{S}(\mathbb{R}^p \times \mathbb{Z}^q) \otimes \mathcal{S}(F)$ with $p = 2$, $q = 0$.

For the real part, we choose our embedding map as

$$\Phi_{\text{inf}} = \begin{pmatrix} \theta_1 + \frac{n_1}{m_1} & 0 & 0 & 0 \\ 0 & 0 & \theta_2 + \frac{n_2}{m_2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv (x_{ij}), \quad (7)$$

then using the previous expression for the Heisenberg representation with $s_1, s_2 \in \mathbb{R}$

$$(V_i f)(s_1, s_2) = (V_{e_i} f)(s_1, s_2) := \exp(2\pi i(s_1 x_{3i} + s_2 x_{4i})) f(s_1 + x_{1i}, s_2 + x_{2i}),$$

we get

$$\begin{aligned}
(V_1 f)(s_1, s_2) &= f(s_1 + \theta_1 + \frac{n_1}{m_1}, s_2), \\
(V_2 f)(s_1, s_2) &= \exp(2\pi i s_1) f(s_1, s_2), \\
(V_3 f)(s_1, s_2) &= f(s_1, s_2 + \theta_2 + \frac{n_2}{m_2}), \\
(V_4 f)(s_1, s_2) &= \exp(2\pi i s_2) f(s_1, s_2).
\end{aligned}$$

For the finite part, let $F = \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$, where $\mathbb{Z}_{m_i} = \mathbb{Z}/m_i\mathbb{Z}$, ($i = 1, 2$) and consider the space $\mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2}$ as the space of functions on $C(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2})$. For all $m_i \in \mathbb{Z}$ and $n_i \in \mathbb{Z}/m_i\mathbb{Z}$ such that m_i and n_i are relatively prime. We define the operators W_i on $C(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2})$ corresponding to our embedding map

$$\Phi_{\text{fin}} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & \frac{n_1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & \frac{n_2}{m_2} \end{pmatrix} \quad (8)$$

with $k_i \in \mathbb{Z}_{m_i}$ ($i = 1, 2$) as follows

$$\begin{aligned}
(W_1 f)(k_1, k_2) &= f(k_1 - 1, k_2), \\
(W_2 f)(k_1, k_2) &= \exp(2\pi i \frac{n_1 k_1}{m_1}) f(k_1, k_2), \\
(W_3 f)(k_1, k_2) &= f(k_1, k_2 - 1), \\
(W_4 f)(k_1, k_2) &= \exp(2\pi i \frac{n_2 k_2}{m_2}) f(k_1, k_2).
\end{aligned}$$

Thus, we define operators $U_i = V_i \otimes W_i$ acting on the space $E_T := \mathcal{S}(\mathbb{R}^2) \otimes \mathbb{C}^{m_1} \otimes \mathbb{C}^{m_2}$ as

$$\begin{aligned}
(U_1 f)(s_1, s_2, k_1, k_2) &= f(s_1 + \theta_1 + \frac{n_1}{m_1}, s_2, k_1 - 1, k_2), \\
(U_2 f)(s_1, s_2, k_1, k_2) &= e^{2\pi i(s_1 + \frac{n_1 k_1}{m_1})} f(s_1, s_2, k_1, k_2), \\
(U_3 f)(s_1, s_2, k_1, k_2) &= f(s_1, s_2 + \theta_2 + \frac{n_2}{m_2}, k_1, k_2 - 1), \\
(U_4 f)(s_1, s_2, k_1, k_2) &= e^{2\pi i(s_2 + \frac{n_2 k_2}{m_2})} f(s_1, s_2, k_1, k_2).
\end{aligned} \quad (9)$$

One can now see that they satisfy

$$\begin{aligned}
U_2 U_1 &= e^{2\pi i \theta_1} U_1 U_2, \\
U_4 U_3 &= e^{2\pi i \theta_2} U_3 U_4,
\end{aligned} \quad (10)$$

and otherwise $U_i U_j = U_j U_i$.

In order to find operators which commute with the U_i 's, we recall the definition of the dual lattice Λ^\perp :

$$\langle m, \hat{t} \rangle - \langle n, \hat{s} \rangle \in \mathbb{Z}, \quad \text{for all } (m, \hat{s}) \in \Lambda \text{ and } (n, \hat{t}) \in \Lambda^\perp.$$

If we express the embedding map Φ as

$$\Phi = \begin{pmatrix} m & \cdots \\ \hat{s} & \cdots \end{pmatrix}, \quad (11)$$

and the embedding map Ψ for the dual lattice as

$$\Psi = \begin{pmatrix} n & \cdots \\ \hat{t} & \cdots \end{pmatrix}, \quad (12)$$

then the duality condition above can be written as

$$\langle m, \hat{t} \rangle - \langle n, \hat{s} \rangle = \Phi^t J \Psi \in \mathbb{Z}$$

where

$$J = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}. \quad (13)$$

Hence, we obtain the relation between the two embedding maps

$$\Psi = -J\Phi^{-t}\mathbb{Z}. \quad (14)$$

Using the above relation, the dual map for the real part is now given by

$$\Psi_{\text{inf}} = \begin{pmatrix} 0 & \frac{1}{m_1} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{m_2} \\ \frac{1}{m_1\theta_1+n_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m_2\theta_2+n_2} & 0 \end{pmatrix}, \quad (15)$$

and the finite part is given by

$$\Psi_{\text{fin}} = \begin{pmatrix} 0 & -a_1 & 0 & 0 \\ 0 & 0 & 0 & -a_2 \\ \frac{1}{m_1} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{m_2} & 0 \end{pmatrix}. \quad (16)$$

Here, $a_i \in \mathbb{Z}$ and $a_i n_i - b_i m_i = 1$ for some $b_i \in \mathbb{Z}$ ($i = 1, 2$).

The generators of operators corresponding to the embedding map for the dual lattice are thus defined by

$$\begin{aligned}(Z_1 f)(s_1, s_2, k_1, k_2) &= e^{2\pi i(\frac{s_1}{m_1 \theta_1 + n_1} + \frac{k_1}{m_1})} f(s_1, s_2, k_1, k_2), \\(Z_2 f)(s_1, s_2, k_1, k_2) &= f(s_1 + \frac{1}{m_1}, s_2, k_1 - a_1, k_2), \\(Z_3 f)(s_1, s_2, k_1, k_2) &= e^{2\pi i(\frac{s_2}{m_2 \theta_2 + n_2} + \frac{k_2}{m_2})} f(s_1, s_2, k_1, k_2), \\(Z_4 f)(s_1, s_2, k_1, k_2) &= f(s_1, s_2 + \frac{1}{m_2}, k_1, k_2 - a_2).\end{aligned}\tag{17}$$

Here,

$$\begin{aligned}Z_1 Z_2 &= e^{2\pi i \theta'_1} Z_2 Z_1, \\Z_3 Z_4 &= e^{2\pi i \theta'_2} Z_4 Z_3,\end{aligned}\tag{18}$$

where

$$\theta'_i = \frac{a_i \theta_i + b_i}{m_i \theta_i + n_i}, \quad i = 1, 2,\tag{19}$$

and otherwise $Z_i Z_j = Z_j Z_i$. One can check that the Z_i 's commute with the U_i 's, i.e., $U_i Z_j = Z_j U_i$.

III. Connections with complex structures

In the previous section, connections on a projective \mathcal{A}_θ^d -module satisfies the condition (4)

$$[\nabla_i, U_j] = 2\pi i \delta_{ij} U_j.$$

A connection ∇_i is called a constant curvature connection if $[\nabla_i, \nabla_j] = i F_{ij}$ for constants F_{ij} . This condition is satisfied if ∇_i is expressed as $\nabla_i = \partial_i - \frac{i}{2} F_{ij} s_j$ where ∂_i is a derivative with respect to s_i . Note that the condition (4) can be regarded as a compactification condition. This can be seen by considering an operator $\overline{X}_i = -\nabla_i$ with which the condition is expressed as

$$U_j \overline{X}_i U_j^{-1} = \overline{X}_i + 2\pi i \delta_{ij},\tag{20}$$

and this relation is comparable to a compactification with radius R_i , $U_j X_i U_j^{-1} = X_i + 2\pi\delta_{ij}R_i$.

We thus let

$$\begin{aligned} (\bar{X}_i f)(s_1, s_2, k_1, k_2) &= 2\pi i A_{i1} s_1 f(s_1, s_2, k_1, k_2) + 2\pi i A_{i2} s_2 f(s_1, s_2, k_1, k_2) \\ &\quad - A_{i3} \frac{\partial f(s_1, s_2, k_1, k_2)}{\partial s_1} - A_{i4} \frac{\partial f(s_1, s_2, k_1, k_2)}{\partial s_2}, \end{aligned}$$

where $A_{ik} \in \mathbb{R}$ are constants. If we denote the embedding maps as $\Phi_{\text{inf}} \equiv (x_{ij})$ and $\Phi_{\text{fin}} \equiv (y_{ij})$, then U_i action is expressed as

$$(U_i f)(s_1, s_2, k_1, k_2) = e^{2\pi i(s_1 x_{3i} + s_2 x_{4i} + k_1 y_{3i} + k_2 y_{4i})} f(s_1 + x_{1i}, s_2 + x_{2i}, k_1 + y_{1i}, k_2 + y_{2i}).$$

The condition (20) is satisfied if

$$x_{1i}x_{3i} + x_{2i}x_{4i} + y_{1i}y_{3i} + y_{2i}y_{4i} = 0, \quad (21)$$

and

$$A_{ik} = (\Phi_{\text{inf}}^{-1})_{ik}. \quad (22)$$

The embedding maps (7), (8) satisfy the condition (21), and the condition (22) gives

$$(A_{ik}) = \begin{pmatrix} \frac{1}{\theta_1 + \frac{n_1}{m_1}} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \frac{1}{\theta_2 + \frac{n_2}{m_2}} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore the following operators specify a constant curvature connection of right T_θ^4 -module $E_{N,M}$:

$$\begin{aligned} \nabla_1 &= -\frac{2\pi i s_1}{\theta_1 + \frac{n_1}{m_1}}, \\ \nabla_2 &= \frac{\partial}{\partial s_1}, \\ \nabla_3 &= -\frac{2\pi i s_2}{\theta_2 + \frac{n_2}{m_2}}, \\ \nabla_4 &= \frac{\partial}{\partial s_2}. \end{aligned} \quad (23)$$

In general, a constant curvature connection can be obtained by adding some constants: $\nabla_i \rightarrow \nabla_i + d_i$, $i = 1, \dots, 4$, where $d_i \in \mathbb{R}$ are constants.

A complex structure on the module $E_{N,M}$ can be introduced by fixing $\bar{\partial}$ -connection

$$\begin{aligned}\bar{\nabla}_1 &= \lambda_{11} \nabla_1 + \lambda_{12} \nabla_2 + \lambda_{13} \nabla_3 + \lambda_{14} \nabla_4, \\ \bar{\nabla}_2 &= \lambda_{21} \nabla_1 + \lambda_{22} \nabla_2 + \lambda_{23} \nabla_3 + \lambda_{24} \nabla_4,\end{aligned}$$

where $\lambda_{ij} \in \mathbb{C}$. Choosing an appropriate basis such that (λ_{ij}) becomes

$$\begin{pmatrix} \tau_{11} & 1 & \tau_{12} & 0 \\ \tau_{21} & 0 & \tau_{22} & 1 \end{pmatrix} = \begin{pmatrix} \lambda_{12} & \lambda_{14} \\ \lambda_{22} & \lambda_{24} \end{pmatrix}^{-1} \begin{pmatrix} \lambda_{11} & \lambda_{12} & \lambda_{13} & \lambda_{14} \\ \lambda_{21} & \lambda_{22} & \lambda_{23} & \lambda_{24} \end{pmatrix},$$

the (2×2) matrix $\begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{21} & \tau_{22} \end{pmatrix}$, $\tau_{ij} \in \mathbb{C}$ represents the complex structure of T_θ^4 -module and 1-1 corresponds to the complex structure on T_θ^4 via $\bar{\partial}$ -derivative, $\bar{\delta}_i = \sum_j \lambda_{ij} \delta_j$ where δ_j is defined by (2) [15].

Now we consider holomorphic vectors in T_θ^4 -module. A vector $\Theta \in E_{N,M}$ is called holomorphic [15] if it satisfies

$$(\bar{\nabla}_i - c_i)\Theta = 0 \quad \text{for } i = 1, 2, \quad (24)$$

where $c_i \in \mathbb{C}$ are constants. The above holomorphic condition for $f \in E_{N,M}$ now takes the form

$$\begin{aligned}\left(\frac{2\pi i \tau_{11}}{\theta_1 + \frac{n_1}{m_1}} s_1 + \frac{2\pi i \tau_{12}}{\theta_2 + \frac{n_2}{m_2}} s_2 + c_1\right) f &= \frac{\partial f}{\partial s_1}, \\ \left(\frac{2\pi i \tau_{21}}{\theta_1 + \frac{n_1}{m_1}} s_1 + \frac{2\pi i \tau_{22}}{\theta_2 + \frac{n_2}{m_2}} s_2 + c_2\right) f &= \frac{\partial f}{\partial s_2}.\end{aligned} \quad (25)$$

In order for the two equations in (25) to be consistent τ_{ij}, c_i should satisfy

$$\frac{\tau_{12}}{\theta_2 + \frac{n_2}{m_2}} = \frac{\tau_{21}}{\theta_1 + \frac{n_1}{m_1}}. \quad (26)$$

If $\text{Re}\Omega < 0$, Eq.(25) has $m_1 \times m_2$ linearly independent solutions, the so-called theta vectors [15, 16] on noncommutative T^4 ,

$$f_{(\alpha_1, \alpha_2)}(s_1, s_2, k_1, k_2) = \exp\left[\frac{1}{2} S^t \Omega S + C^t S\right] \delta_{\alpha_1}^{k_1} \delta_{\alpha_2}^{k_2} \quad (27)$$

where $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, $S = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$, $c_i \in \mathbb{C}$, $s_i \in \mathbb{R}$, $k_i \in \mathbb{Z}_{m_i}$ ($i = 1, 2$), and $\Omega = \begin{pmatrix} \frac{2\pi i \tau_{11}}{\theta_1 + \frac{n_1}{m_1}} & \frac{2\pi i \tau_{12}}{\theta_2 + \frac{n_2}{m_2}} \\ \frac{2\pi i \tau_{21}}{\theta_1 + \frac{n_1}{m_1}} & \frac{2\pi i \tau_{22}}{\theta_2 + \frac{n_2}{m_2}} \end{pmatrix}$.

IV. Tensor product

In this section we consider a tensor product of two bimodules. Tensor product of a (C, Y) -bimodule E and a (Y, D) -bimodule E' over Y results in a (C, D) -bimodule F for algebras C, Y, D ;

$${}_C E_Y \otimes_Y E'_D = {}_C F_D$$

where the tensor product over Y is obtained by identifying $ey \otimes e' \sim e \otimes ye'$ for $y \in Y$, $e \in E$, $e' \in E'$. Note that in this identification, E behaves as a right Y -module and E' behaves as a left Y -module. Thus, we will denote $E_{N,M}$ as a right T_θ^4 -module and $E'_{K,L}$ as a left $T_{\tilde{\theta}}^4$ -module. Here, we recall that $T_{\tilde{\theta}}^4$ is Morita equivalent to T_θ^4 if θ and $\tilde{\theta}$ are related by $\tilde{\theta} = (A\theta + B)(M\theta + N)^{-1}$ where $\begin{pmatrix} A & B \\ M & N \end{pmatrix} \in \text{SO}(4, 4|\mathbb{Z})$, and $T_\theta^4 \cong \text{End}_{T_\theta^4}(E)$ for some finite projective module E . In this notation, a right module $E_{N,M}$ is identified with a left module $E'_{A,M}$. Let us calculate the tensor product $E_{N,M} \otimes_{T_\theta^4} E'_{K,L}$ which forms a vector space $S(\mathbb{R} \times \mathbb{Z}_{n_1 l_1 + m_1 k_1} \times \mathbb{Z}_{n_2 l_2 + m_2 k_2})$, when each N, M, K, L is reducible into two blocks represented by the values $N \sim (n_1, n_2)$, $M \sim (m_1, m_2)$, $K \sim (k_1, k_2)$, $L \sim (l_1, l_2)$ [25]. For $f(s_1, s_2, \mu_1, \mu_2) \in E_{N,M}$, and $g(t_1, t_2, \nu_1, \nu_2) \in E'_{K,L}$ where $s_i, t_i \in \mathbb{R}$, $\mu_i \in \mathbb{Z}_{m_i}$, $\nu_i \in \mathbb{Z}_{l_i}$ ($i = 1, 2$) the actions of $U_i \in T_\theta^4$ and $Z_i \in T_{\tilde{\theta}}^4$ are given as follows.

The right U_i actions on $E_{N,M}$ are defined as

$$\begin{aligned} (U_1 f)(s_1, s_2, \mu_1, \mu_2) &= f(s_1 + \theta_1 + \frac{n_1}{m_1}, s_2, \mu_1 - 1, \mu_2), \\ (U_2 f)(s_1, s_2, \mu_1, \mu_2) &= e^{2\pi i(s_1 + \frac{n_1 \mu_1}{m_1})} f(s_1, s_2, \mu_1, \mu_2), \\ (U_3 f)(s_1, s_2, \mu_1, \mu_2) &= f(s_1, s_2 + \theta_2 + \frac{n_2}{m_2}, \mu_1, \mu_2 - 1), \\ (U_4 f)(s_1, s_2, \mu_1, \mu_2) &= e^{2\pi i(s_2 + \frac{n_2 \mu_2}{m_2})} f(s_1, s_2, \mu_1, \mu_2). \end{aligned} \quad (28)$$

The left U_i actions on $E'_{K,L}$ are defined as

$$\begin{aligned} (U_1 g)(t_1, t_2, \nu_1, \nu_2) &= g(t_1 - \theta_1 + \frac{k_1}{l_1}, t_2, \nu_1 - 1, \nu_2), \\ (U_2 g)(t_1, t_2, \nu_1, \nu_2) &= e^{2\pi i(t_1 + \frac{k_1 \nu_1}{l_1})} g(t_1, t_2, \nu_1, \nu_2), \\ (U_3 g)(t_1, t_2, \nu_1, \nu_2) &= g(t_1, t_2 - \theta_2 + \frac{k_2}{l_2}, \nu_1, \nu_2 - 1), \\ (U_4 g)(t_1, t_2, \nu_1, \nu_2) &= e^{2\pi i(t_2 + \frac{k_2 \nu_2}{l_2})} g(t_1, t_2, \nu_1, \nu_2). \end{aligned} \quad (29)$$

The left Z_i actions on $E_{N,M}$ are defined as

$$\begin{aligned}
(Z_1 f)(s_1, s_2, \mu_1, \mu_2) &= e^{2\pi i(\frac{s_1}{m_1\theta_1+n_1}+\frac{\mu_1}{m_1})} f(s_1, s_2, \mu_1, \mu_2), \\
(Z_2 f)(s_1, s_2, \mu_1, \mu_2) &= f(s_1 + \frac{1}{m_1}, s_2, \mu_1 - a_1, \mu_2), \\
(Z_3 f)(s_1, s_2, \mu_1, \mu_2) &= e^{2\pi i(\frac{s_2}{m_2\theta_2+n_2}+\frac{\mu_2}{m_2})} f(s_1, s_2, \mu_1, \mu_2), \\
(Z_4 f)(s_1, s_2, \mu_1, \mu_2) &= f(s_1, s_2 + \frac{1}{m_2}, \mu_1, \mu_2 - a_2),
\end{aligned} \tag{30}$$

where $a_i \in \mathbb{Z}$ and $a_i n_i - b_i m_i = 1$ for some $b_i \in \mathbb{Z}$ ($i = 1, 2$).

Following [16], we define the tensor product $f \otimes g \equiv h \in E_{N,M} \otimes_{T_\theta^4} E'_{K,L}$ as

$$\begin{aligned}
h(r_1, r_2, j_1, j_2) &= \sum_{q_1 \in \mathbb{Z}} \sum_{q_2 \in \mathbb{Z}} f((n_1 + m_1\theta_1)r_1 + \frac{n_1 + m_1\theta_1}{m_1}q_1 - \frac{l_1(n_1 + m_1\theta_1)}{m_1(n_1l_1 + m_1k_1)}j_1, \\
&\quad (n_2 + m_2\theta_2)r_2 + \frac{n_2 + m_2\theta_2}{m_2}q_2 - \frac{l_2(n_2 + m_2\theta_2)}{m_2(n_2l_2 + m_2k_2)}j_2, -q_1 + a_1j_1, -q_2 + a_2j_2) \\
&\quad \cdot g((n_1 + m_1\theta_1)r_1 - \frac{k_1 - l_1\theta_1}{l_1}q_1 + \frac{k_1 - l_1\theta_1}{n_1l_1 + m_1k_1}j_1, \\
&\quad (n_2 + m_2\theta_2)r_2 - \frac{k_2 - l_2\theta_2}{l_2}q_2 + \frac{k_2 - l_2\theta_2}{n_2l_2 + m_2k_2}j_2, q_1, q_2)
\end{aligned} \tag{31}$$

for $r_i \in \mathbb{R}$, $j_i \in \mathbb{Z}$ ($i = 1, 2$). Then, one can check that

$$\begin{aligned}
(U_i f) \otimes g &\sim f \otimes (U_i)g, \\
(Z_i h) &\sim (Z_i f) \otimes g, \\
h(r_i, j_i + n_i l_i + m_i k_i) &= h(r_i, j_i)
\end{aligned}$$

for $i = 1, 2$. Notice that in the above calculation Z_i 's act on h as left actions, since h is regarded here as an element of a left module E' . So far, we have only defined the left actions of Z_i on a right module E . Thus, we define left Z_i actions on a left module E' as

$$\begin{aligned}
(Z_1 g) &= (U_2 g), \\
(Z_2 g) &= (U_1 g), \\
(Z_3 g) &= (U_4 g), \\
(Z_4 g) &= (U_3 g),
\end{aligned}$$

where $U_i g$ are defined in (29).

If $E_{N,M}$ is a right module expression of (T_θ^4, T_θ^4) -bimodule and $E'_{K,L}$ is a left module expression of (T_θ^4, T_θ^4) -bimodule, then one can also show that h belongs to $E'_{AK+BL, NL+MK}(\tilde{\theta})$ where $\tilde{\theta} = (A\theta + B)(M\theta + N)^{-1}$ and $\theta = (K\theta' + D)(L\theta' + C)^{-1}$ with $CK - DL \sim 1$, $AN - BM \sim 1$, when each A, B, M, N, K, D, L, C is reducible into 2 blocks in the sense that we described earlier.

Let us consider the tensor product (31) between the two theta vectors, $f \in E_{N,M}$ and $g \in E'_{K,L}$,

$$\begin{aligned} f_{(\alpha_1, \alpha_2)}(s_1, s_2, \mu_1, \mu_2) &= \exp\left[\frac{1}{2}S^t \Omega S + C^t S\right] \delta_{\alpha_1}^{\mu_1} \delta_{\alpha_2}^{\mu_2}, \\ g_{(\beta_1, \beta_2)}(t_1, t_2, \nu_1, \nu_2) &= \exp\left[\frac{1}{2}T^t \Omega' T + C'^t T\right] \delta_{\beta_1}^{\nu_1} \delta_{\beta_2}^{\nu_2}, \end{aligned} \quad (32)$$

where $C = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, $C' = \begin{pmatrix} c'_1 \\ c'_2 \end{pmatrix}$, $S = \begin{pmatrix} s_1 \\ s_2 \end{pmatrix}$, $T = \begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$, $c_i, c'_i \in \mathbb{C}$, $s_i, t_i \in \mathbb{R}$, $\mu_i \in \mathbb{Z}_{m_i}$, $\nu_i \in \mathbb{Z}_{l_i}$ ($i = 1, 2$), $\Omega = \begin{pmatrix} \frac{2\pi i \tau_{11}}{\theta_1 + \frac{n_1}{m_1}} & \frac{2\pi i \tau_{12}}{\theta_2 + \frac{n_2}{m_2}} \\ \frac{2\pi i \tau_{21}}{\theta_1 + \frac{n_1}{m_1}} & \frac{2\pi i \tau_{22}}{\theta_2 + \frac{n_2}{m_2}} \end{pmatrix}$, and $\Omega' = \begin{pmatrix} \frac{2\pi i \tau'_{11}}{-\theta_1 + \frac{k_1}{l_1}} & \frac{2\pi i \tau'_{12}}{-\theta_2 + \frac{k_2}{l_2}} \\ \frac{2\pi i \tau'_{21}}{-\theta_1 + \frac{k_1}{l_1}} & \frac{2\pi i \tau'_{22}}{-\theta_2 + \frac{k_2}{l_2}} \end{pmatrix}$.

The resulting function now takes the form

$$\begin{aligned} h_{\alpha_1, \alpha_2, \beta_1, \beta_2}(r_1, r_2, j_1, j_2) &= \sum_{q_1 \in \mathbb{Z}} \sum_{q_2 \in \mathbb{Z}} \exp\left(\frac{1}{2} \tilde{\mathcal{A}}^t \Omega \tilde{\mathcal{A}} + C^t \tilde{\mathcal{A}}\right) \delta_{\alpha_1, \alpha_2}^{-q_1 + a_1 j_1, -q_2 + a_2 j_2} \\ &\quad \cdot \exp\left(\frac{1}{2} \tilde{\mathcal{A}}'^t \Omega' \tilde{\mathcal{A}}' + C'^t \tilde{\mathcal{A}}'\right) \delta_{\beta_1, \beta_2}^{q_1, q_2} \end{aligned}$$

where $r_i \in \mathbb{R}$, $j_i \in \mathbb{Z}_{n_i l_i + m_i k_i}$ ($i = 1, 2$) and

$$\begin{aligned} \tilde{\mathcal{A}} &= \begin{pmatrix} (n_1 + m_1 \theta_1) r_1 + \frac{n_1 + m_1 \theta_1}{m_1} q_1 - \frac{l_1(n_1 + m_1 \theta_1)}{m_1(n_1 l_1 + m_1 k_1)} j_1 \\ (n_2 + m_2 \theta_2) r_2 + \frac{n_2 + m_2 \theta_2}{m_2} q_2 - \frac{l_2(n_2 + m_2 \theta_2)}{m_2(n_2 l_2 + m_2 k_2)} j_2 \end{pmatrix}, \\ \tilde{\mathcal{A}}' &= \begin{pmatrix} (n_1 + m_1 \theta_1) r_1 - \frac{k_1 - l_1 \theta_1}{l_1} q_1 + \frac{k_1 - l_1 \theta_1}{n_1 l_1 + m_1 k_1} j_1 \\ (n_2 + m_2 \theta_2) r_2 - \frac{k_2 - l_2 \theta_2}{l_2} q_2 + \frac{k_2 - l_2 \theta_2}{n_2 l_2 + m_2 k_2} j_2 \end{pmatrix}. \end{aligned}$$

From the delta function relations, we rewrite q_i as $q_i = p_i + u_i \frac{m_i l_i}{v_i}$, $u_i \in \mathbb{Z}$ ($i = 1, 2$) for some integers p_i where $v_i = \text{g.c.d.}(m_i, l_i)$. Then, h can be written as

$$\begin{aligned} h_{\alpha_1, \alpha_2, \beta_1, \beta_2}(r_1, r_2, j_1, j_2) &= \sum_{u_1 \in \mathbb{Z}} \sum_{u_2 \in \mathbb{Z}} \exp\left(\frac{1}{2}(\mathcal{A} + \mathcal{U})^t \Omega (\mathcal{A} + \mathcal{U}) + C^t \mathcal{U} + C^t \mathcal{A}\right. \\ &\quad \left. + \frac{1}{2}(\mathcal{A}' + \mathcal{U}')^t \Omega' (\mathcal{A}' + \mathcal{U}') + C'^t \mathcal{U}' + C'^t \mathcal{A}'\right), \end{aligned}$$

where

$$\begin{aligned}\mathcal{A} &= \begin{pmatrix} (n_1 + m_1\theta_1)r_1 + \frac{n_1+m_1\theta_1}{m_1}p_1 - \frac{l_1(n_1+m_1\theta_1)}{m_1(n_1l_1+m_1k_1)}j_1 \\ (n_2 + m_2\theta_2)r_2 + \frac{n_2+m_2\theta_2}{m_2}p_2 - \frac{l_2(n_2+m_2\theta_2)}{m_2(n_2l_2+m_2k_2)}j_2 \end{pmatrix}, \\ \mathcal{A}' &= \begin{pmatrix} (n_1 + m_1\theta_1)r_1 - \frac{k_1-l_1\theta_1}{l_1}p_1 + \frac{k_1-l_1\theta_1}{n_1l_1+m_1k_1}j_1 \\ (n_2 + m_2\theta_2)r_2 - \frac{k_2-l_2\theta_2}{l_2}p_2 + \frac{k_2-l_2\theta_2}{n_2l_2+m_2k_2}j_2 \end{pmatrix}, \\ \mathcal{U} &= \begin{pmatrix} \frac{(n_1+m_1\theta_1)l_1}{v_1}u_1 \\ \frac{(n_2+m_2\theta_2)l_2}{v_2}u_2 \end{pmatrix}, \quad \mathcal{U}' = \begin{pmatrix} -\frac{(k_1-l_1\theta_1)m_1}{v_1}u_1 \\ -\frac{(k_2-l_2\theta_2)m_2}{v_2}u_2 \end{pmatrix}.\end{aligned}$$

This can be decompsed into two parts, including the classical theta function,

$$h = \vartheta(\mathcal{T}, \mathcal{Z})\xi(r_1, r_2, j_1, j_2). \quad (33)$$

Here, the classical theta function ϑ is given by

$$\vartheta(\mathcal{T}, \mathcal{Z}) = \sum_{u_1, u_2 \in \mathbb{Z}} \exp(\pi i U^t \mathcal{T} U + 2\pi i \mathcal{Z}^t U),$$

where

$$\begin{aligned}U &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \mathcal{T} = \begin{pmatrix} \mathcal{B}_1 \mathcal{O}_{11} \mathcal{B}_1 + \mathcal{B}'_1 \mathcal{O}'_{11} \mathcal{B}'_1 & \mathcal{B}_1 \mathcal{O}_{12} \mathcal{B}_2 + \mathcal{B}'_1 \mathcal{O}'_{12} \mathcal{B}'_2 \\ \mathcal{B}_2 \mathcal{O}_{21} \mathcal{B}_1 + \mathcal{B}'_2 \mathcal{O}'_{21} \mathcal{B}'_1 & \mathcal{B}_2 \mathcal{O}_{22} \mathcal{B}_2 + \mathcal{B}'_2 \mathcal{O}'_{22} \mathcal{B}'_2 \end{pmatrix}, \\ \mathcal{Z} &= \begin{pmatrix} \mathcal{B}_1 \mathcal{O}_{11} \mathcal{A}_1 + \mathcal{B}_1 \mathcal{O}_{12} \mathcal{A}_2 + \mathcal{B}'_1 \mathcal{O}'_{11} \mathcal{A}'_1 + \mathcal{B}'_1 \mathcal{O}'_{12} \mathcal{A}'_2 \\ \mathcal{B}_2 \mathcal{O}_{21} \mathcal{A}_1 + \mathcal{B}_2 \mathcal{O}_{22} \mathcal{A}_2 + \mathcal{B}'_2 \mathcal{O}'_{21} \mathcal{A}'_1 + \mathcal{B}'_2 \mathcal{O}'_{22} \mathcal{A}'_2 \end{pmatrix} + \frac{1}{2\pi i} \begin{pmatrix} c_1 \mathcal{B}_1 + c'_1 \mathcal{B}'_1 \\ c_2 \mathcal{B}_2 + c'_2 \mathcal{B}'_2 \end{pmatrix}\end{aligned}$$

with $\mathcal{O} = \frac{1}{2\pi i} \Omega$, $\mathcal{O}' = \frac{1}{2\pi i} \Omega'$, $\mathcal{B} = \begin{pmatrix} \mathcal{B}_1 \\ \mathcal{B}_2 \end{pmatrix} = \begin{pmatrix} \frac{(n_1+m_1\theta_1)l_1}{v_1} \\ \frac{(n_2+m_2\theta_2)l_2}{v_2} \end{pmatrix}$, $\mathcal{B}' = \begin{pmatrix} \mathcal{B}'_1 \\ \mathcal{B}'_2 \end{pmatrix} = \begin{pmatrix} -\frac{(k_1-l_1\theta_1)m_1}{v_1} \\ -\frac{(k_2-l_2\theta_2)m_2}{v_2} \end{pmatrix}$, and the function ξ is given by

$$\xi(r_1, r_2, j_1, j_2) = \exp\left(\frac{1}{2} \mathcal{A}^t \Omega \mathcal{A} + \frac{1}{2} \mathcal{A}'^t \Omega' \mathcal{A}' + C^t \mathcal{A} + C'^t \mathcal{A}'\right).$$

Requiring that $E_{N,M}$ and $E'_{K,L}$ have the same complex structure for consistency of the tensor product, i.e., $(\tau_{ij}) = (\tau'_{ij})$, and the consistency condition (26) the resulting function h becomes

$$h_{\alpha_1, \alpha_2, \beta_1, \beta_2}(r_1, r_2, j_1, j_2) = \sum_{\gamma_1, \gamma_2} c_{\alpha_1, \alpha_2, \beta_1, \beta_2}^{\gamma_1, \gamma_2} \varphi_{\gamma_1, \gamma_2}(r_1, r_2, j_1, j_2). \quad (34)$$

Here, the function $\varphi_{\gamma_1, \gamma_2}$ is given by

$$\varphi_{\gamma_1, \gamma_2}(r_1, r_2, j_1, j_2) = \exp(\pi i R^t \tilde{\mathcal{O}} R + (C + C')^t \tilde{R}) \delta_{\gamma_1}^{j_1} \delta_{\gamma_2}^{j_2}$$

with $R = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix}$, $\tilde{\mathcal{O}} = \begin{pmatrix} \frac{\tau_{11}(n_1+m_1\theta_1)(n_1l_1+m_1k_1)}{(k_1-l_1\theta_1)} & \frac{\tau_{12}(n_1+m_1\theta_1)(n_2l_2+m_2k_2)}{(k_2-l_2\theta_2)} \\ \frac{\tau_{21}(n_2+m_2\theta_2)(n_1l_1+m_1k_1)}{(k_1-l_1\theta_1)} & \frac{\tau_{22}(n_2+m_2\theta_2)(n_2l_2+m_2k_2)}{(k_2-l_2\theta_2)} \end{pmatrix}$, $\tilde{R} = \begin{pmatrix} (n_1+m_1\theta_1)r_1 \\ (n_2+m_2\theta_2)r_2 \end{pmatrix}$,
and the constants $c_{\alpha_1, \alpha_2, \beta_1, \beta_2}^{\gamma_1, \gamma_2}$ are given by

$$c_{\alpha_1, \alpha_2, \beta_1, \beta_2}^{\gamma_1, \gamma_2} = \vartheta(\Xi, \Lambda) e^{\mathcal{K}}$$

where

$$\mathcal{K} = \pi i \tilde{Q}^t \tilde{P} + C^t \tilde{M} + C'^t \tilde{L},$$

with

$$\begin{aligned} \tilde{Q} &= \begin{pmatrix} \tau_{11}p_1 + \tau_{12}p_2 - \frac{\tau_{11}l_1j_1}{n_1l_1+m_1k_1} - \frac{\tau_{12}l_2j_2}{n_2l_2+m_2k_2} \\ \tau_{21}p_1 + \tau_{22}p_2 - \frac{\tau_{21}l_1j_1}{n_1l_1+m_1k_1} - \frac{\tau_{22}l_2j_2}{n_2l_2+m_2k_2} \end{pmatrix}, \quad \tilde{P} = \begin{pmatrix} \frac{n_1l_1+m_1k_1}{l_1m_1}p_1 - \frac{j_1}{m_1} \\ \frac{n_2l_2+m_2k_2}{l_2m_2}p_2 - \frac{j_2}{m_2} \end{pmatrix}, \\ \tilde{M} &= \begin{pmatrix} \frac{n_1+m_1\theta_1}{m_1}p_1 - \frac{l_1(n_1+m_1\theta_1)}{m_1(n_1l_1+m_1k_1)}j_1 \\ \frac{n_2+m_2\theta_2}{m_2}p_2 - \frac{l_2(n_2+m_2\theta_2)}{m_2(n_2l_2+m_2k_2)}j_2 \end{pmatrix}, \quad \tilde{L} = \begin{pmatrix} -\frac{k_1-l_1\theta_1}{l_1}q_1 + \frac{k_1-l_1\theta_1}{n_1l_1+m_1k_1}j_1 \\ -\frac{k_2-l_2\theta_2}{l_2}q_2 + \frac{k_2-l_2\theta_2}{n_2l_2+m_2k_2}j_2 \end{pmatrix}, \end{aligned}$$

and

$$\vartheta(\Xi, \Lambda) = \sum_{u_1, u_2 \in \mathbb{Z}} \exp(\pi i U^t \Xi U + 2\pi i \Lambda^t U),$$

with

$$\begin{aligned} U &= \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \Xi = \begin{pmatrix} \frac{\tau_{11}(n_1l_1+m_1k_1)l_1m_1}{v_1^2} & \frac{\tau_{12}(n_1l_1+m_1k_1)l_2m_2}{v_1v_2} \\ \frac{\tau_{21}(n_2l_2+m_2k_2)l_1m_1}{v_1v_2} & \frac{\tau_{22}(n_2l_2+m_2k_2)l_2m_2}{v_2^2} \end{pmatrix}, \\ \Lambda &= \begin{pmatrix} \frac{\tau_{11}}{v_1}((l_1n_1+m_1k_1)p_1 - l_1j_1) + \frac{\tau_{12}}{v_1}(l_1n_1+m_1k_1)(p_2 - \frac{l_2j_2}{l_2n_2+m_2k_2}) \\ \frac{\tau_{21}}{v_2}(l_2n_2+m_2k_2)(p_1 - \frac{l_1j_1}{l_1n_1+m_1k_1}) + \frac{\tau_{22}}{v_2}((l_2n_2+m_2k_2)p_2 - l_2j_2) \end{pmatrix} \\ &\quad + \frac{1}{2\pi i} \begin{pmatrix} c_1\mathcal{B}_1 + c'_1\mathcal{B}'_1 \\ c_2\mathcal{B}_2 + c'_2\mathcal{B}'_2 \end{pmatrix}, \end{aligned}$$

where $c_i, c'_i, \mathcal{B}_i, \mathcal{B}'_i$ are the same as given before.

Notice that the function $\varphi_{\gamma_1, \gamma_2}(r_1, r_2, j_1, j_2)$ is a theta vector (32) belongs to $E'_{AK+BL, NL+MK}(\tilde{\theta})$ with $\tilde{\theta} = \frac{A\theta+B}{M\theta+N}$ as we expected.

V. Discussion

In this paper, we first construct a module on noncommutative T^4 and its dual. Then we define the complex structure on this module and construct the theta vector which is a

solution for a holomorphic connection. We then consider the tensor product of the theta vectors satisfying the consistency requirement.

Here, we want to notice what has not been apparent in the noncommutative T^2 case [16]. When we require the holomorphic condition (24), the symmetry appears not in the complex structure itself but in the Ω -matrix which appears in the theta vector (27), i.e., $\Omega_{12} = \Omega_{21}$ instead of $\tau_{12} = \tau_{21}$ in the commutative 4-torus case. We consider that this difference comes from noncommutativity.

As in the noncommutative T^2 case, the tensor products of modules on the noncommutative 4-torus with complex structures become very restrictive in order to satisfy the consistency requirement. We consider this consistency requirement as another aspect of noncommutativity compared with the commutative case in which there is no restriction.

So far, theta functions on noncommutative tori have not been utilized in the physics literature except for the integral torus case [21, 22, 23]. With the theta functions on noncommutative tori without any restriction, one can hope to explore the physical states on noncommutative tori in more general cases.

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