PERMUTATION ORBIFOLDS AND THEIR APPLICATIONS

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Abstract. The theory of permutation orbifolds is reviewed and applied to the study of symmetric product orbifolds and the congruence subgroup problem. The issue of discrete torsion, the combinatorics of symmetric products, the Galois action and questions related to the classification of RCFTs are also discussed.

1. Introduction

The notion of permutation orbifold had been introduced in [1], followed by a couple of papers investigating their properties, mostly through considerations related to modular invariance [2]. Much later, essentially permutation orbifold techniques have been applied in [3] in the study of second quantized strings. To our knowledge the first systematic study of permutation orbifolds appeared in [4], where the primary field content, the genus one characters, the modular representation and the fusion rules have been worked out for permutation orbifolds with cyclic twist groups. These results have been generalized to arbitrary permutation groups in [5][6], which also gave the geometric interpretation of the results through the theory of covering surfaces. The mathematics of the construction have been clarified in [7] in the framework of Vertex Operator Algebras. Permutation orbifolds have been also investigated in relation to the Orbifold Virasoro Master Equation [8].

The aim of these notes is to give a sketchy overview of the basics of permutation orbifolds, together with applications to symmetric product orbifolds and the proof of the congruence subgroup property of RCFTs. They are not meant to be self contained, the relevant details may be found in the cited papers. Besides presenting standard results like the partition function and the modular representation of permutation orbifolds, we'll touch upon such topics as discrete torsion, the combinatorics of symmetric products, the Galois action and the classification of RCFTs.

2. Permutation orbifolds

Let's begin by recalling the most important results of [5][6]. We consider a permutation group Ω of degree n, and a rational CFT $\mathcal{C}$. The n-fold tensor power of $\mathcal{C}$, i.e. the tensor product of n identical copies, admits the permutations in Ω as symmetries, consequently one may orbifoldize this tensor power by the twist group Ω. We call the resulting theory the Ω permutation orbifold of $\mathcal{C}$, and denote it by $\mathcal{C}\wr\Omega$. The central charge of $\mathcal{C}\wr\Omega$ is just n times the central charge of $\mathcal{C}$. The wreath product notation for permutation orbifolds reflects the following basic fact: if Ω_1 and Ω_2 are two permutation groups, then the Ω_2 permutation orbifold of $\mathcal{C}\wr\Omega_1$ is the same as the $\Omega_1\wr\Omega_2$ orbifold of $\mathcal{C}$, i.e.

$$(\mathcal{C}\wr\Omega_1)\wr\Omega_2=\mathcal{C}\wr(\Omega_1\wr\Omega_2) \tag{1}$$

where $\Omega_1\wr\Omega_2$ denotes the wreath product of the permutation groups Ω_1 and Ω_2 with the standard (imprimitive) action [9]. This fundamental result lies at the heart of most of what follows, e.g. it already enables one to enumerate the primary fields of the permutation orbifold $\mathcal{C}\wr\Omega$: these are in one-to-one correspondence with the orbits of Ω on the set of pairs $\langle p,\phi\rangle$, where p is an n-tuple of primaries of $\mathcal{C}$, i.e. a map $p:\{1,\ldots,n\}\to\mathcal{I}$ (we denote by $\mathcal{I}$ the set of primaries of $\mathcal{C}$), while ϕ is an irreducible character of the double $\mathcal{D}(\Omega_p)$ (cf. [10],[11]) of the stabilizer $\Omega_p=\{x\in\Omega\,|\,xp=p\}$, the action of Ω on the map p being the obvious one, and the action of $x\in\Omega$ on the pair $\langle p,\phi\rangle$ given simply by

$$x\,\langle p,\phi\rangle=\langle xp,\phi^x\rangle \tag{2}$$

where $\phi^x(y,z)=\phi(y^x,z^x)$. A simple counting argument gives the number of primaries of $\mathcal{C}\wr\Omega$:

$$\left|\mathcal{I}^{\Omega}\right|=\frac{1}{|\Omega|}\sum_{(x,y,z)\in\Omega^{\{3\}}}s^{|\mathcal{O}(x,y,z)|} \tag{3}$$

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where $s=|\mathcal{I}|$ denotes the number of primaries of $\mathcal{C}$, $\Omega^{\{k\}}$ is the set of commuting k-tuples from Ω, and $\mathcal{O}(x,y,z)$ denotes the set of orbits on $\{1,\ldots,n\}$ of the group generated by the triple (x,y,z). For example, if $\Omega=S_3$ is the symmetric group on three letters (with its natural permutation action), then

$$\left|\mathcal{I}^{\Omega}\right|=\frac{s^3+21s^2+26s}{6}$$

Once the primaries have been classified, the next task is to determine their genus one characters. In order to do this, the key ingredient is to understand the geometric aspect of orbifoldization, namely its relation to the theory of covering surfaces. Roughly speaking, the value in the orbifold theory of a quantity associated to a given surface is obtained as a (weighted) sum over all n-sheeted coverings of the surface whose monodromy belongs to Ω, the summands corresponding to the contributions of the twisted sectors. To illustrate this principle, let's consider the genus one partition function $Z(\tau)$. In this case the surface under consideration is a torus, and the allowed coverings are characterized by homomorphisms from the fundamental group $\mathbb{Z}\oplus\mathbb{Z}$ of the torus into Ω, i.e. by commuting pairs $(x,y)\in\Omega^{\{2\}}$ corresponding to the images of the generators of the fundamental group. The resulting covering surface is in general not connected, its connected components - which are all tori by the Riemann-Hurwitz formula - being in one-to-one correspondence with the orbits of the image of the homomorphism, i.e. the subgroup generated by x and y. Each $\xi\in\mathcal{O}(x,y)$ may be characterized by a triple of integers $(\lambda_\xi,\mu_\xi,\kappa_\xi)$, where λ_ξ is the common length of all x orbits contained in ξ, μ_ξ is the number of these x orbits (so that $|\xi|=\lambda_\xi\mu_\xi$ gives the length of the orbit ξ), while κ_ξ is the unique non-negative integer less than λ_ξ such that $y^{\mu_\xi}x^{-\kappa_\xi}$ belongs to the stabilizer of ξ. If the modular parameter of the torus under consideration is τ, then the modular parameter τ_ξ of the covering torus corresponding to the orbit ξ may be expressed as

$$\tau_\xi=\frac{\mu_\xi\tau+\kappa_\xi}{\lambda_\xi} \tag{4}$$

According to the general recipe, the genus one partition function Z^{Ω} of the orbifold evaluated at τ equals

$$Z^{\Omega}\left(\tau\right)=\frac{1}{|\Omega|}\sum_{(x,y)\in\Omega^{\{2\}}}\prod_{\xi\in\mathcal{O}(x,y)}Z\left(\tau_\xi\right) \tag{5}$$

In other words, the partition function of the orbifold is a sum over all coverings, and the contribution of each covering equals the product of the partition functions of its connected components. By similar considerations, the genus one character of the primary $\langle p,\phi\rangle$ of $\mathcal{C}\wr\Omega$ reads

$$\chi_{\langle p,\phi\rangle}(\tau)=\frac{1}{|\Omega_p|}\sum_{(x,y)\in\Omega_p^{\{2\}}}\overline{\phi}(x,y)\prod_{\xi\in\mathcal{O}(x,y)}\omega_{p(\xi)}^{-\kappa_\xi/\lambda_\xi}\chi_{p(\xi)}(\tau_\xi) \tag{6}$$

In this last formula, $p(\xi)$ denotes the value of the map p on the orbit ξ, while for a primary $q\in\mathcal{I}$ of $\mathcal{C}$

$$\omega_q=\exp\left(2\pi i(\Delta_q-\frac{c}{24})\right) \tag{7}$$

denotes its exponentiated conformal weight, i.e. the corresponding eigenvalue T_{qq} of the Dehn-twist $T:\tau\mapsto\tau+1$.

One may also express the higher genus partition functions of $\mathcal{C}\wr\Omega$ in terms of those of $\mathcal{C}$ [12]. The result is that the partition function Z^{Ω} of the orbifold evaluated at the surface $\mathbb{H}/G$ reads

$$Z^{\Omega}\left(\mathbb{H}/G\right)=\frac{1}{|\Omega|}\sum_{\varphi:G\to\Omega}\varepsilon(\varphi)\prod_{\xi\in\mathcal{O}(\varphi)}Z\left(\mathbb{H}/G_\xi\right) \tag{8}$$

where $\mathbb{H}$ denotes the upper half-plane, G is the Fuchsian group uniformizing the surface, the sum runs over all homomorphisms $\varphi:G\to\Omega$, and we denote by $\mathcal{O}(\varphi)$ the set of orbits of $\varphi(G)$ on $\{1,\ldots,n\}$, while G_ξ is the inverse image under φ of the stabilizer of the orbit ξ. We note that Eq.(8) is valid more generally, with $\mathbb{H}$ denoting an arbitrary surface and G a discrete subgroup of $Aut\,(\mathbb{H})$, so that in particular it covers the genus one case too (cf. Eq.(5)).

We have also included in Eq.(8) complex phases $\varepsilon(\varphi)$, which run under the name of discrete torsion (cf. [13]) in the orbifold literature. These are determined by a 2-cocycle $\vartheta\in Z^2(\Omega)$ via the following recipe [14]: one may use the homomorphism φ to pull back the 2-cocycle ϑ to a closed 2-form of $\mathbb{H}/G$, and the integral of this 2-form over the surface determines the value of $\varepsilon(\varphi)$. There exists explicit expressions for $\varepsilon(\varphi)$ in terms of ϑ [15][16], e.g. in the genus one case (when the homomorphism is determined by a couple $(x,y)\in\Omega^{\{2\}}$) one has

$$\varepsilon_T\left(x,y\right)=\frac{\vartheta\left(x,y\right)}{\vartheta\left(y,x\right)} \tag{9}$$

The explicit knowledge of the genus one characters allows us to determine the matrix elements of the modular transformations. The exponentiated conformal weights of the primaries of $\mathcal{C} \wr \Omega$ read

$$\omega_{\langle p,\phi\rangle} = \frac{1}{d_\phi} \sum_{x\in\Omega_p} \phi(x,x) \prod_{\xi\in\mathcal{O}(x)} \omega_{p(\xi)}^{\frac{1}{|\xi|}} \tag{10}$$

with $d_\phi = \sum_x \phi(x,1)$, while for the transformation $S : \tau \mapsto \frac{-1}{\tau}$ we have

$$S_{\langle p,\phi\rangle}^{\langle q,\psi\rangle} = \frac{1}{|\Omega_p||\Omega_q|} \sum_{z\in\Omega} \sum_{x,y\in\Omega_p\cap\Omega_{zq}} \overline{\phi}(x,y)\overline{\psi}^z(y,x) \prod_{\xi\in\mathcal{O}(x,y)} \Lambda_{p(\xi)}^{zq(\xi)} \left(\frac{\kappa_\xi}{\lambda_\xi}\right) \tag{11}$$

The matrices $\Lambda(r)$ which appear in this last formula play a fundamental role in the theory. They are defined as follows: if $r = \frac{k}{n}$ is a rational number in reduced form, i.e. with k and n coprime and $n > 0$, then there exists integers x and y such that $kx - ny = 1$, i.e. such that the matrix $m = \begin{pmatrix} k & y \\ n & x \end{pmatrix}$ belongs to $SL(2,\mathbb{Z})$. If we denote by M_{pq} the matrix element of the modular transformation $\tau \mapsto \frac{k\tau+y}{n\tau+x}$ between the primaries p and q, then we define

$$\Lambda_{pq}(r) = \exp\left(2\pi i r\left(\Delta_p - \frac{c}{24}\right)\right) M_{pq} \exp\left(2\pi i r^*\left(\Delta_q - \frac{c}{24}\right)\right) \tag{12}$$

or symbolically $\Lambda(r) = T^r M T^{r^*}$, where $r^* = \frac{x}{n}$. While these matrices are well defined, i.e. they do not depend on the actual choice of the integers x and y, their value depends on the choice of a branch of the complex logarithm, which is involved in the computation of rational powers of T, and which should be kept fixed once for all. One may show that the actual choice of this branch is irrelevant, a different choice would simply amount to a relabeling of the primaries of the orbifold.

The matrices $\Lambda(r)$ enjoy interesting properties, for example $\Lambda(r+1) = \Lambda(r)$, $\Lambda(0) = S$ and $\Lambda_{pq}(r^*) = \Lambda_{qp}(r)$, the last equality generalizing the symmetry of the matrix S. For an integer $n > 0$ we have the explicit expression

$$\Lambda\left(\frac{1}{n}\right) = T^{-\frac{1}{n}} S^{-1} T^{-n} S T^{-\frac{1}{n}}$$

Once we know the matrix elements of the transformation S, we can insert them into Verlinde's formula to compute the fusion rules of the theory. The resulting cumbersome expressions may be found in [6], we just note the apparition of quantities called twisted dimensions, which are defined as

$$\mathcal{D}_g \begin{pmatrix} p_1 & \dots & p_N \\ r_1 & \dots & r_N \end{pmatrix} = \sum_{q\in\mathcal{I}} S_{0q}^{2-2g} \prod_{i=1}^N \frac{\Lambda_{qp_i}(r_i)}{S_{0q}} \tag{13}$$

where $p_1, \dots, p_N \in \mathcal{I}$ are primaries of $\mathcal{C}$, and $r_1, \dots, r_N \in \mathbb{Q}$ are rational numbers. Besides being the basic building blocks of the fusion rules of permutation orbifolds, twisted dimensions also appear as the partition functions of Seifert-manifolds in the 3D topological field theory associated to $\mathcal{C}$, leading to Verlinde-like formulae for the traces of mapping classes of finite order, as explained in [17].

3. Symmetric product orbifolds

The importance of symmetric product orbifolds had been recognized long ago, e.g. for second quantized strings [3],[18] and matrix string theory [19]. From a geometric point of view, they amount to passing from a sigma model describing string propagation on some target manifold X to the sigma model for the Hilbert-scheme of X. In other words, if the propagation of a single string on X is described by the CFT C, the propagation of n identical strings should be described by the permutation orbifold $\mathcal{C} \wr S_n$, where S_n denotes the symmetric group on n letters, i.e. we have to gauge the permutation symmetries. As we don't want to fix the number of identical strings, what we are really interested in is not the value A_n of some quantity A in the permutation orbifold $\mathcal{C} \wr S_n$ for a given n, but rather the generating function

$$\sum_{n=0}^{\infty} A_n p^n$$

where p is a formal variable. For example, a beautiful result of [3] states that

$$\sum_n p^n Z_n(\tau) = \exp\left(\sum_{n=1}^{\infty} p^n T_n Z(\tau)\right) \tag{14}$$

where $Z(\tau)$ is the genus one partition function of $\mathcal{C}$, and the T_n-s are the Hecke-operators (cf. [20]) which act on $Z(\tau)$ via

$$T_n Z(\tau) = \frac{1}{n}\sum_{d|n}\sum_{0\leq k<d} Z\left(\frac{n\tau}{d^2}+\frac{k}{d}\right) \tag{15}$$

There is a general combinatorial identity that lies at the heart of all related computations. It reads [15]

$$\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\phi:G\rightarrow S_n}\prod_{\xi\in\mathcal{O}(\phi)}\mathcal{Z}(G_\xi) = \exp\left(\sum_{H<G}\frac{\mathcal{Z}(H)}{[G:H]}\right) \tag{16}$$

where G is any finitely generated group, while $\mathcal{Z}$ is a function on the set of finite index subgroups of G that takes its values in a commutative ring and is constant on conjugacy classes of subgroups. The second summation on the lhs. runs over the homomorphisms $\phi : G \rightarrow S_n$ from G into the symmetric group S_n. For a given ϕ, we denote by $\mathcal{O}(\phi)$ the orbits of the image $\phi(G)$ on the set $\{1,\ldots,n\}$, and $G_\xi = \{x \in G \,|\, \phi(x)\xi^* = \xi^*\}$ is the stabilizer subgroup of any point $\xi^* \in \xi$ of the orbit ξ - note that the lhs. of Eq.(16) is well defined, since the stabilizers of points on the same orbit are conjugate subgroups. Finally, $[G : H]$ denotes the index of the subgroup $H < G$, and the $n = 0$ term on the lhs. of Eq.(16) equals 1 by convention.As for the proof of Eq.(16), one may first reduce it to the case when G is free by using the one-to-one correspondence between subgroups of a homomorphic image and subgroups that contain the kernel of the homomorphism. Then one may proceed by induction on the rank of G, noticing that in the case when G has rank one, i.e. $G = \mathbb{Z}$, Eq.(16) reduces to a well known formula for the generating function of the cycle indices of symmetric groups.

To understand the relevance of Eq.(16) to our problem, recall from the previous section the expression Eq.(8) for the partition function of the orbifold. As subgroups of finite index of a Fuchsian group are themselves Fuchsian, it follows that for a given Fuchsian group G the quantity

$$\mathcal{Z}(H) = p^{[G:H]} Z(\mathbb{H}/H)$$

makes sense, and has all the desired properties for Eq.(16) to hold (p is a formal variable and Z is the partition function of $\mathcal{C}$). Inserting the above expression into Eq.(16) and using Eq.(8), we get

$$\sum_n p^n Z_n(\mathbb{H}/G) = \exp\left(\sum_{H<G}\frac{p^{[G:H]}}{[G:H]}Z(\mathbb{H}/H)\right)$$

where on the lhs. we recognize the generating function we were looking for. The above argument may be extended to the genus one case as well, recovering the result Eq.(14) after evaluation of the right hand side.

This is not the end of the story, for it was pointed out by Dijkgraaf in [18] that, because $H^2(S_n) = \mathbb{Z}_2$ for $n \geq 4$, it is possible to introduce non-trivial discrete torsion [13] in the above models. Recall from the previous section that the introduction of discrete torsion amounts to modifying Eq.(8) by suitable phases

$$Z_n^{\varepsilon}(\mathbb{H}/G) = \frac{1}{n!}\sum_{\phi:G\rightarrow S_n}\varepsilon(\phi)\prod_{\xi\in\mathcal{O}(\phi)}Z(\mathbb{H}/G_\xi) \tag{17}$$

In the case at hand, i.e. discrete torsion corresponding to the non-trivial cocycle $\vartheta \in H^2(S_n)$ for $n > 3$, there is a simple closed formula for the discrete torsion coefficients on the torus, i.e. the genus one case Eq.(9). To write it down let's introduce the quantities

$$|x| = \sum_{\xi\in\mathcal{O}(x)}(|\xi|-1) \tag{18}$$

and

$$|x,y| = \sum_{\xi\in\mathcal{O}(x,y)}(|\xi|-1) \tag{19}$$

for arbitrary permutations x and y, where as usual, we denote by $\mathcal{O}(x,y)$ the set of orbits of the group generated by x and y, and $|\xi|$ denotes the length of the orbit ξ. Note that $|x|$ determines the parity of the permutation x, i.e. x is even or odd according to whether $|x|$ is even or odd. The discrete torsion coefficients for the torus read [15]

$$\varepsilon_T(x,y) = (-1)^{(|x|-1)(|y|-1)+|x,y|-1} \tag{20}$$

for a pair of commuting permutations $xy = yx$. There is an alternate form of the discrete torsion coefficients that is more suitable for computations, namely

$$\varepsilon_T(x,y) = \frac{(-1)^{|x,y|}}{4} \sum_{\alpha,\beta\in\{\pm1\}} (1-\alpha-\beta-\alpha\beta)\alpha^{|x|}\beta^{|y|} \tag{21}$$

for $xy = yx$.

Armed with the above, we can now compute the generating functions in the presence of discrete torsion. In contrast to the case without discrete torsion, we no longer get an exponential, but rather a combination of four exponential expressions. The final result reads

$$\sum_{n=0}^{\infty} p^n Z_n^{\varepsilon}(\tau) = \frac{1}{4} \sum_{\alpha,\beta\in\{\pm1\}} (1-\alpha-\beta-\alpha\beta) \exp\left\{\sum_{n=1} p^n T_n^{\alpha\beta} Z(\tau)\right\} \tag{22}$$

where for $\alpha,\beta\in\{\pm1\}$ the operators $T_n^{\alpha\beta}$ acting on the partition function $Z(\tau)$ are defined as

$$T_n^{\alpha\beta} Z(\tau) = \frac{-(-\alpha\beta)^n}{n} \sum_{d|n} \sum_{0\le k<d} \alpha^{\frac{n}{d}} \beta^{dk+d+k} Z\left(\frac{n\tau+kd}{d^2}\right) \tag{23}$$

The close analogy with Eq.(14) makes it tempting to interpret the operators $T_n^{\alpha\beta}$ as some kind of generalizations of the usual Hecke-operators. Note that for n odd $T_n^{\alpha\beta}$ equals the usual Hecke-operator T_n, independently of the values of α and β.

4. The congruence subgroup property

A well known property of Rational Conformal Field Theories is that their genus one characters χ_p span a finite dimensional unitary representation of the modular group $\Gamma(1) \cong SL(2,\mathbb{Z})$. In other words, to any matrix $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ there corresponds a unitary representation matrix M such that

$$\chi_p\left(\frac{a\tau+b}{c\tau+d}\right) = \sum_q M_p^q \chi_q(\tau) \tag{24}$$

Of special interest are the matrices T and S representing $t = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $s = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. As s and t generate the modular group $\Gamma(1)$, any representation matrix M may be written in terms of S and T. It follows from the defining relations of $\Gamma(1)$ that $STS = T^{-1}ST^{-1}$ and $S^4 = 1$. Moreover, it is known that S^2 equals the charge conjugation operator, i.e.

$$\left(S^2\right)_p^q = \delta_{p,\overline{q}} \tag{25}$$

where $\overline{q}$ denotes the charge conjugate of the primary q.

The modular representation enjoys some remarkable properties, the most important ones being summarized in the celebrated theorem of Verlinde [22]:

(1) *T is diagonal of finite order.*
(2) *S is symmetric.*
(3) *The quantities*

$$N_{pqr} = \sum_{s\in\mathcal{I}} \frac{S_{ps}S_{qs}S_{rs}}{S_{0s}}$$

are non-negative integers, being the dimension of suitable spaces of holomorphic blocks. Here and in the sequel, the label 0 refers to the vacuum of the theory.

Further properties of the modular representation have been conjectured over the years, e.g. that its kernel

$$\mathcal{K} = \left\{ m \in \Gamma(1) \,|\, M_p^q = \delta_{p,q} \right\}$$

is of finite index in $\Gamma(1)$, culminating in the following conjecture [23][26][27].

Congruence subgroup property*: The kernel $\mathcal{K}$ is a congruence subgroup, i.e. it contains the principal congruence subgroup*

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \,|\, a,d \equiv 1 \pmod N,\ b,c \equiv 0 \pmod N \right\}$$

for some N. Moreover, the matrix entries M_p^q of modular transformation matrices belong to the cyclotomic field $\mathbb{Q}[\zeta_N]$, and N equals the order of the Dehn-twist T.

The truth of this conjecture would have important implications in the study of RCFTs, e.g. it would make available the apparatus of the theory of modular functions. The first result about the conjecture was obtained in [23], where it was shown that the conjecture holds if the order of the Dehn-twist is odd. Unfortunately, this is rather atypical, e.g. for most of the Virasoro minimal models the order of T is even. A proof valid for arbitrary RCFTs had been presented in [28], which relies on the theory of the Galois action and the Orbifold Covariance Principle. Let's sketch these two main ingredients of the proof.

The basic idea in the theory of the Galois action [24][25] is to look at the field F obtained by adjoining to the rationals $\mathbb{Q}$ the matrix elements of all modular transformations. One may show that, as a consequence of Verlinde's theorem, F is a finite Abelian extension of $\mathbb{Q}$. By the celebrated theorem of Kronecker and Weber this means that F is a subfield of a cyclotomic field $\mathbb{Q}[\zeta_n]$ for some integer n, where $\zeta_n = \exp\left(\frac{2\pi i}{n}\right)$ is a primitive n-th root of unity. We'll call the conductor of $\mathcal{C}$ the smallest n which is divisible by the order of the Dehn-twist and for which $F \subseteq \mathbb{Q}[\zeta_n]$. It follows from the above that the elements of the Galois group $\mathrm{Gal}(F/\mathbb{Q})$ are (the restriction to F of) the Frobenius maps $\sigma_l : \mathbb{Q}[\zeta_n] \to \mathbb{Q}[\zeta_n]$ that leave $\mathbb{Q}$ fixed, and send ζ_n to ζ_n^l for l coprime to n.

According to [25], we have

$$\sigma_l\left(S_p^q\right) = \varepsilon_l(q) S_p^{\pi_l q} \tag{26}$$

for some permutation π_l of the primaries and some signs $\varepsilon_l(p)$. In other words, upon introducing the orthogonal monomial matrices

$$\left(G_l\right)_p^q = \varepsilon_l(q)\delta_{p,\pi_l q} \tag{27}$$

and denoting by $\sigma_l(M)$ the matrix that one obtains by applying σ_l to M element-wise, we have

$$\sigma_l(S) = SG_l = G_l^{-1}S \tag{28}$$

Note that for l and m both coprime to the conductor

$$\begin{aligned} \pi_{lm} &= \pi_l \pi_m \\ G_{lm} &= G_l G_m \end{aligned}$$

The Galois action on T is even simpler, for T is diagonal, and its eigenvalues are roots of unity, consequently

$$\sigma_l(T) = T^l \tag{29}$$

We note that the above results follow directly from Verlinde's theorem by simple number theoretic arguments, and they do not require the full strength of Verlinde, for they are true even if we just require that the numbers N_{pqr} belong to $\mathbb{Q}$, there is no need for their integrality nor positivity.

The second ingredient of the proof, the Orbifold Covariance Principle states that, because a permutation orbifold of an RCFT is itself an RCFT, any property shared by all RCFTs should hold in all permutation orbifolds as well [29]. While this might seem tautological, it does lead to interesting results. For example, we have seen that we may express the fusion rules of a permutation orbifold in terms of quantities of the original theory. According to Verlinde's theorem the fusion rule coefficients should be non-negative integers, but the truth of this statement in the orbifold does not follow automatically from the construction, rather it gives interesting arithmetic restrictions on the modular representation.

In the case at hand, the Orbifold Covariance Principle states that the Galois action on the S-matrix elements of the orbifold may be described via suitable permutations π_l of the primaries and signs ε_l. As we can express the S-matrix elements in terms of modular matrices of the original theory, this way we get information about the Galois action on arbitrary modular matrices. A careful study leads to the following results [28]:

(1) For all l coprime to the conductor

$$G_l^{-1} T G_l = T^{l^2} \tag{30}$$

and more generally

$$G_l^{-1} M G_l = \sigma_l^2(M) \tag{31}$$

for any modular matrix M.

(2) If $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ with d coprime to the conductor, then

$$\sigma_d(M) = T^b S^{-1} T^{-c} \sigma_d(S) \tag{32}$$

Eq.(30) had been conjectured in [23], and was used to derive the following results, which in turn were previously conjectured in [26]:

- For l coprime to the conductor,

$$\sigma_l(S) = T^l S T^{\hat{l}} S T^l \tag{33}$$

  where $\hat{l}$ denotes the inverse of l modulo the conductor.
- The conductor equals the order N of T, and $F = \mathbb{Q}[\zeta_N]$.

Eq.(30) has a further implication, which is important for the classification of allowed modular representations: there exists a function $N(r)$ such that the conductor N of an RCFT with r primary fields divides $N(r)$. According to this result, for a given number of primary fields one has only a finite number of consistent choices for the conductor N and the matrix T. Values of the upper bound $N(r)$ for small values of r are given in the following table:

r	$N(r)$
2	240
3	5040
4	10080
5	1441440

As to the second result, Eq.(32) implies that

$$\mathcal{K} \cap \Gamma_1(N) = \Gamma(N)$$

where

$$\Gamma_1(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid a, d \equiv 1 \pmod{N},\ c \equiv 0 \pmod{N} \right\}$$

in particular $\mathcal{K}$ is a congruence subgroup of level N. Moreover, Eq.(32) leads to a simple arithmetic characterization of the kernel which is suitable for explicit computations.

Let's summarize what we have found: the congruence subgroup property holds in any Rational Conformal Field Theory, the kernel of the modular representation may be described by arithmetic conditions, and the conductor is bounded by a function of the number of primary fields. All these results follow ultimately from Verlinde's theorem and the Orbifold Covariance Principle.

Of course, a host of questions remains open. Just to cite a few, it would be interesting to understand the relation between the algebraic geometry of the modular curve $X(\mathcal{K})$ associated to the kernel and standard properties of the RCFT. Another interesting point would be to clarify the implications of the above results on the structure of the associated 3D Topological Field Theory, especially in connection with the trace identities of [17]. Finally, these results pave the way for a systematic enumeration of RCFTs, or at least of the allowed modular representations.

5. Summary

As we have seen, besides being interesting for their own sake, permutation orbifolds have important application both in Conformal Field Theory and String Theory. They provide a consistent framework for constructing new CFTs from old ones, in which any interesting quantity may be expressed in terms of the corresponding quantities of the original theory, thanks to the underlying geometric picture. This fact supplemented by the Orbifold Covariance Principle leads to a powerful tool for investigating deeper properties of CFTs.

Of course, there are many open questions related to permutation orbifolds. For example, one may ask for an algorithm which would recognize whether an RCFT is a permutation orbifold, and output the twist group and the original theory. This could have important computational applications. One may also speculate about the possibility of twisting the construction by a 3-cocycle, in analogy with the case of holomorphic orbifolds [30][14]. The classification of the conformally invariant boundary conditions (cf. [31],[32]) of these models would be a rewarding task too.

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