

# $\kappa$ -deformed Wigner construction of relativistic wave functions and free fields on $\kappa$ -Minkowski space\*

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## Abstract

We describe the extension of the Wigner's infinite-dimensional unitary representations of Poincaré group to the case of  $\kappa$ -deformed Poincaré group. We show that the corresponding coordinate wave functions on noncommutative space-time are described by free field equations on  $\kappa$ -deformed Minkowski space. The cases of Klein–Gordon, Dirac, Proca and Maxwell fields are considered. Finally some aspects of second quantization are also discussed.

## 1 Introduction

Recently, there has been much interest in field theory on noncommutative space-time see e.g. [1-3]. Most authors deal with the simplest case when coordinates commute to a c-number; this implies that classical Poincaré symmetries are not modified, Lorentz symmetry however is explicitly broken. A slightly more complicated structure is obtained if one assumes that the noncommutative space-time coordinates form a Lie algebra, with “quantum” time and commutative space coordinates [4]. It appears then that, although more complicated, such quantum Minkowski space admits 10-parameter symmetry with noncommuting quantum group parameters. Such a symmetry is given by particular deformation of Poincaré group called  $\kappa$ -Poincaré quantum group  $P_\kappa$  with classical subgroup of nonrelativistic  $O(3)$  rotations [5]. Due to the fact that in such a case the deformation parameter is dimensionful and can be related to Planck length

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( $\ell_p \simeq 10^{-33}$  cm) one can speculate that  $\kappa$ -Poincaré symmetry could describe the space-time structure at Planck scale (see e.g. [6]). We suppose, therefore, that it is interesting to elaborate in more detail the consequences of the assumption that  $P_\kappa$  is a generalization of standard space-time symmetry at very high energies. First steps towards the construction of quantum field theory based on  $P_\kappa$  were made recently in [7]. This paper has as its aim the group-theoretic derivation of  $\kappa$ -deformed free field theories.

We would like to discuss here a particular aspect of  $P_\kappa$  symmetry. In standard field theory one starts with free fields which provide a building block for the construction of Lagrangeans, defining propagators etc.; they are basic elements of the theory on perturbative level [5]. In order to define the free fields one has to solve the basically group-theoretical problem: to construct the intertwinners between the representations of Poincaré group acting on fields and states. These intertwinners are simply the wave functions of particles of definite mass and spin and are related with Wigner theory of unitary representations of Poincaré group [8]. It is customary to define them through the appropriate wave equations describing free fields.

Below we solve the same problem in the  $\kappa$ -deformed case. The solution appears to be quite simple and, basically, reduces to the one for the nondeformed ( $\kappa = \infty$ ) case. It is important to stress that our problem is of group-theoretical origin. In such a case we speak about coordinate wave functions rather than quantum fields. Here we shall consider  $\kappa$ -deformed classical fields, with deformation originating from  $\kappa$ -deformation of Minkowski space. To obtain quantum  $\kappa$ -deformed fields one has only to replace their Fourier transforms describing momentum amplitudes by relevant creation/annihilation operators. We would like to point out that solving the problems of quantum (second-quantized)  $\kappa$ -deformed free fields is outside of the scope of this talk.

To conclude this section let us define our basic notions: the quantum Minkowski space  $M_\kappa$  and  $\kappa$ -Poincaré group  $P_\kappa$ .

$M_\kappa$  is a  $\kappa$ -algebra generated by four Hermitian elements  $x^\mu$  obeying

$$[x^\mu, x^\nu] = \frac{i}{h}(\delta_0^\mu x^\nu - \delta_0^\nu x^\mu).$$

We shall use in the sequel the following "normal" ordering which gives the correspondence between  $M_\kappa$  and standard Minkowski space. The product of  $x^\mu$ 's is called normally ordered if all  $x^0$  factors stand leftmost; we denote normal product by : :. To any analytic function  $f(x)$  we can ascribe a (formal) element  $\hat{f}$  of  $M_\kappa$  by

$$\hat{f} =: f(x) :$$

The quantum  $\kappa$ -Poincaré group  $P_\kappa$  [2] is a  $\kappa$ -Hopf algebra generated by hermitean elements  $\Lambda^\mu_\nu$  and  $a^\mu$  obeying

$$\begin{aligned} g_{\mu\nu} \Lambda^\mu_\alpha \Lambda^\nu_\beta &= g_{\alpha\beta} I \\ [a^\mu, a^\nu] &= \frac{i}{\kappa}(\delta_0^\mu a^\nu - \delta_0^\nu a^\mu) \\ [\Lambda^\mu_\nu, a^\rho] &= -\frac{i}{\kappa}((\Lambda^\mu_0 - \delta_0^\mu) \Lambda^\rho_\nu + (\Lambda^0_\nu - \delta^0_\nu) g^{\mu\rho}) \\ [\Lambda^\mu_\nu, \Lambda^\alpha_\beta] &= 0 \end{aligned}$$

$$\begin{aligned}
\Delta(\Lambda^\mu{}_\nu) &= \Lambda^\mu{}_\alpha \otimes \Lambda^\alpha{}_\nu \\
\Delta(a^\mu) &= \Lambda^\mu{}_\nu \otimes a^\nu + a^\mu \otimes I \\
S(\Lambda^\mu{}_\nu) &= \Lambda^\mu{}_\nu \\
S(a^\mu) &= -\Lambda^\mu{}_\nu a^\nu \\
\varepsilon(\Lambda^\mu{}_\nu) &= \delta^\mu{}_\nu I \\
\varepsilon(a^\mu) &= 0 \quad ;
\end{aligned}$$

here  $g_{\mu\nu} = \text{diag}(+ - - -)$  is numerical metric tensor.

## 2 Unitary representations of $\kappa$ -Poincaré group

Let us recall the construction of induced representations [9]. One starts with some subgroup  $H \subset G$  of the group under consideration. The relevant Hilbert space is the space of square integrable functions on  $G$  taking their values in the vector space carrying some representation of  $H$ . The group action is defined to be, say, right action:  $g_0 : f(g) \rightarrow f(gg_0)$ . The essence of the method is the selection of invariant subspace by imposing coequivariance condition; in many cases the invariant subspace obtained in this way carries an irreducible representation of  $G$ . The more explicit characterization of the representation is achieved by solving explicitly the coequivariance condition which gives rise to the description of the representation in terms of Hilbert space of functions defined on the right coset space  $H \backslash G$ . This is especially effective for the case of semidirect products in which one factor is abelian. In particular, in the case of Poincaré group we obtain Wigner's construction.

The whole construction can be generalized in a rather straightforward way to the  $\kappa$ -deformed case [10]. The actual construction for the case of  $\kappa$ -Poincaré group has been given in Ref. [11]. The results read:

(a) the massive case

The carrier space is the space of square-integrable (with respect to the invariant measure  $d\bar{q}/q^0$ ) functions on hyperboloid  $(q^0)^2 - \bar{q}^2 = m^2$ ,  $q^0 > 0$ , taking their values in the representation space of spin  $\mathbf{s}$  representation of rotation group ( $\mathbf{s}$  should be integer as we are dealing with standard “vectorial” Poincaré group). The right coaction of  $P_\kappa$  is then given by

$$\begin{aligned}
f(q, \sigma) &\rightarrow \sum_{\sigma'} D_{\sigma\sigma'}^{(\mathbf{s})}(R(q \otimes I, I \otimes \Lambda)) \\
&\quad e^{-iP_0(q) \otimes a^0} e^{-iP_k(q) \otimes a^k} f(q_\nu \otimes \Lambda^\nu{}_\mu, \sigma');
\end{aligned} \tag{1}$$

where

$$P_0(q) = \kappa \ln(\text{ch}(\frac{m}{\kappa}) + \frac{q_0}{m} \text{sh}(\frac{m}{\kappa})) \tag{2}$$

$$P_k(q) = \frac{\kappa \text{sh}(\frac{m}{\kappa}) q_k}{m \text{ch}(\frac{m}{\kappa}) + q_0 \text{sh}(\frac{m}{\kappa})} \tag{3}$$

and  $D^{(\mathbf{s})}(R(q \otimes I, I \otimes \Lambda))$  is the spin  $\mathbf{s}$  representation of classical Wigner rotation written in appropriate tensor product form.

(b) the massless case

The carrier space is now the space of  $\mathbb{C}$ -valued functions on upper light cone,  $(q^0)^2 - \vec{q}^2 = 0$ ,  $q^0 > 0$ , square-integrable with respect to the same measure  $d^3\vec{q}/q^0$ . The right coaction of  $P_\kappa$  is now given by

$$f(q) \rightarrow e^{i\lambda\Theta(q\otimes I, I\otimes\Lambda)} e^{-iP_0(q)\otimes a^0} e^{-iP_k(q)\otimes a^k} f(q_\nu \otimes \Lambda^\nu_\mu); \quad (4)$$

$\lambda$  is the (integer) helicity,  $\Theta(q\otimes I, I\otimes\Lambda)$  the classical rotation angle of the little group  $E(2)$  while

$$P_0(q) = \kappa \ln\left(\frac{q_0}{k}(e^{\frac{k}{\kappa}} - 1) + 1\right) \quad (5)$$

$$P_k(q) = \frac{\kappa(e^{\frac{k}{\kappa}} - 1)q_k}{q_0(e^{\frac{k}{\kappa}} - 1) + k} \quad (6)$$

and  $k$  parametrises the standard fourvector  $(k, 0, 0, k)$ . It is easy to check [12] that the infinitesimal form of the above representation is the representation of the  $\mathfrak{m}$ -Poincaré algebra in bicrossproduct basis with classical Lorentz algebra sector [13]. It has been shown in [11] that the  $\mathfrak{m}$ -Poincaré group and  $\mathfrak{m}$ -Poincaré algebra are related by Hopf algebra duality; the explicit form of duality relations in M-R basis is given by

$$\begin{aligned} \langle P_\mu, : f(a) : \rangle &= i \frac{\partial f}{\partial a^\mu} \Big|_{a=0} \\ \langle M_{\mu\nu}, g(\Lambda) \rangle &= i \left( \frac{\partial g}{\partial \Lambda^{\mu\nu}} - \frac{\partial g}{\partial \Lambda^{\nu\mu}} \right) \Big|_{\Lambda=I} \end{aligned} \quad (7)$$

for translations and Lorentz transformations, respectively, extended suitably with the help of bicrossproduct structure [14]. The infinitesimal generators are then defined as follows: denote by  $\varrho_R$  the right coaction (1); for any element  $X$  of  $\mathfrak{m}$ -Poincaré algebra we define

$$\hat{X}f \equiv \sum_\alpha \langle X, \varphi_\alpha \rangle f_\alpha \quad (8)$$

where  $\varrho_R(f) = \sum_\alpha f_\alpha \otimes \varphi_\alpha$ . Simple calculation gives

$$\begin{aligned} \hat{P}_0 &= \kappa \ln\left(\text{ch}\left(\frac{m}{\kappa}\right) + \frac{q_0}{m} \text{sh}\left(\frac{m}{\kappa}\right)\right) \\ \hat{P}_k &= \frac{\kappa \text{sh}\left(\frac{m}{\kappa}\right) q_k}{m \text{ch}\left(\frac{m}{\kappa}\right) + q_0 \text{sh}\left(\frac{m}{\kappa}\right)} \\ \hat{M}_{ij} &= i \left( q_i \frac{\partial}{\partial q^j} - q_j \frac{\partial}{\partial q^i} \right) + \varepsilon_{ijk} S_k \\ \hat{M}_{i0} &= -i q_0 \frac{\partial}{\partial q^i} + \varepsilon_{ijk} \frac{q_j S_k}{q_0 + m}; \end{aligned} \quad (9)$$

here  $\{S_k\}$  (spin matrices) span the representation of the little group. Similar results are obtained in the case of massless representation, Eq. (4).

In the bicrossproduct basis the Lorentz algebra acts nonlinearly in momentum space. This action can be linearized (providing thereby the equivalence of

the algebraic sector with that of the classical Poincaré algebra) by the following deformation map [15,16]

$$P_0(q) = \kappa \ln\left(\frac{q_0 + C}{C - A}\right) \quad (10)$$

$$P_k(q) = \frac{\kappa q_k}{q_0 + C} \quad (11)$$

where  $A$  and  $C$  are functions of  $m^2 = q^2 = q_0^2 - q_k^2$  obeying  $A^2 - 2AC + m^2 = 0$ . The expressions appearing on the right hand side of Eqs. (2), (3) provide a special case of deformation map with  $A = m(cth(\frac{m}{\kappa}) - \frac{1}{sh(\frac{m}{\kappa})})$ ,  $C = mcth(\frac{m}{\kappa})$  (those given by Eqs. (5), (6) correspond to  $A = 0$ ,  $C = \frac{k}{e^{\frac{k}{\kappa}} - 1}$ ). Using the properties of deformation map (10) and (11) one can easily show that by replacing  $P_\mu(q)$  given by Eqs. (2),(3),(5) and (6) by more general ones (10), (11) one obtains equivalent representations. More precisely, for two representations, given by Eqs. (1)(or(4)) and (10), (11), determined by the functions  $A$  and  $C$ , which depends on the mass  $m^2 = q^2$  and deformation parameter  $\kappa$ , this equivalence is given by

$$f'(q, \sigma) = \left(\frac{C'}{C}\right) f\left(\frac{C}{C'} q, \sigma\right) \quad (12)$$

Because

$$P_\mu\left(\frac{C}{C'} q_\mu, A', C'\right) = P_\mu(q_\mu, A, C) \quad (13)$$

the inverse formula to (10–11) satisfies the relation

$$q'_\mu = q_\mu(P, A', C') = \frac{C'}{C} q_\mu(P, A, C) \quad (14)$$

It follows from (14) that replacement  $C \rightarrow C'$  implies the change of the mass parameter  $q_\mu^2 = m^2 \rightarrow q'^2_\mu = \frac{C'^2}{C^2} m^2 = m'^2$ . Therefore, by a particular choice of the function  $C$  one can relate the deformed fourmomenta described by deformed mass shell condition (parametrized by  $\kappa$ -Poincaré-invariant mass  $M$ )

$$\left(2\kappa \sinh \frac{P_0}{2\kappa}\right)^2 - e^{\frac{P_0}{\kappa}} p^2 = M^2 \quad (15)$$

to the fourmomenta  $q_\mu$ , obeying on classical relativistic mass shell condition  $q_0^2 - \vec{q}^2 = m^2$  with arbitrary choice of positive mass parameter  $m$ .

To deal with halfinteger spin case one has only to replace  $\kappa$ -Poincaré group by  $\kappa$ -deformed counterpart of  $ISL(2, C)$  group described in [17]. The whole construction can be repeated almost without modifications; one has only to replace Wigner rotation by its classical  $SL(2, C)$  element so that  $D^{(\kappa)}$  becomes a function of  $q_\mu \otimes I$  and  $I \otimes A, A \in SL(2, C)$ .

### 3 Covariant wave functions

In the standard case (i.e. with nondeformed symmetry) the building blocks for the construction of local interactions are the covariant fields linear in creation/annihilation operators [8]. They are obtained by solving the following

problem: given particles of a definite mass and spin we know the transformation properties of the relevant momentum amplitudes, i.e. the wave functions in momentum space or, after second quantization, momentum space creation/annihilation operators. In the following we shall construct the appropriate coordinate space wave functions carrying the same amplitudes and transforming according to a given representation of Lorentz group

$$\Phi_l(x) \rightarrow D_{ll'}(\Lambda) \Phi_{l'}(\Lambda^{-1}(x - a)) \quad (16)$$

Quantum fields are then obtained by replacing momentum space amplitudes by creation or annihilation operators.

We attempt here to solve the same problem in  $\kappa$ -deformed case. Due to the rather mild character of  $\kappa$ -deformation as seen in the form of unitary representations given in previous section, it is not surprising that this problem can be solved in a quite straightforward way. Let us recall that  $P_\kappa$  acts covariantly on  $M_\kappa$  from the left as given by:

$$x^\mu \rightarrow \Lambda^\mu{}_\nu \otimes x^\nu + a^\mu \otimes I \quad (17a)$$

However, we prefer to consider the right coaction of  $\kappa$ -Poincaré which corresponds to the classical representations rather than antirepresentations. It appears that, in order to get a covariant action on  $M_\kappa$ , one has to consider the group  $P_{-\kappa}$ , the  $\kappa$ -Poincaré group with  $\kappa$  replaced by  $-\kappa$ . Therefore, we define the action of  $P_\kappa$  on  $M_\kappa$  as follows

$$x^\mu \rightarrow x^\nu \otimes \Lambda_\nu{}^\mu - I \otimes a^\nu \Lambda_\nu{}^\mu \quad (17b)$$

As a next step we define the coordinate wave functions. Let us select some representation  $D^{(A,B)}$  of classical Lorentz group and let  $\mathbf{C}^n$ ,  $n = (2A+1)(2B+1)$  be the carrier space of this representation. The coordinate wave function  $\Phi_l(x)$  is an element of  $M_\kappa \otimes \mathbf{C}^n$ . It is then easy to check that the following action of  $P_{-\kappa}$  is well defined

$$\begin{aligned} \Phi_l(x) &\rightarrow \Phi_{l'}(x^\nu \otimes \Lambda_\nu{}^\mu - I \otimes a^\nu \Lambda_\nu{}^\mu) \\ &\quad (I \otimes D_{ll'}^{(AB)}(\Lambda)) \end{aligned} \quad (18)$$

The following result forms the basis for the whole subsequent discussion. Let  $\varphi(x) \in M_\kappa$  be an element given by

$$\varphi(x) = \int \frac{d^3 \vec{q}}{q_0} a(q) : e^{-iP_\mu(q)x^\mu} : \quad (19)$$

where  $a(q)$  is a  $\mathbf{C}$ -valued function on the positive hyperboloid  $(q^0)^2 - \vec{q}^2 = m^2$  while  $P_\mu(q)$  is given by (2,3).

Then the following basic identity holds

$$\begin{aligned} &\int \frac{d^3 \vec{q}}{q_0} a(q) e^{-iP_0(q)(x^\nu \otimes \Lambda_\nu{}^0 - I \otimes a^\nu \Lambda_\nu{}^0)} \\ &\quad e^{-iP_k(q)(x^\nu \otimes \Lambda_\nu{}^k - I \otimes a^\nu \Lambda_\nu{}^k)} = \\ &= \int \frac{d^3 \vec{q}}{q_0} e^{-iP_0(q)(I \otimes a^0)} e^{-iP_k(q)(I \otimes a^k)} \\ &\quad a(q_\nu(I \otimes \Lambda^\nu{}_\mu)) (: e^{-iP_\mu(q)x^\mu} : \otimes I) \end{aligned} \quad (20)$$

The proof of the above identity is straightforward but tedious. It can be also extended to massless case.

The basic identity (20) allows us to formulate the following general result. Let us put

$$\Phi_l(x) \equiv \sum_{\sigma} \int \frac{d^3 \vec{q}}{q_0} U_l(q, \sigma) a(q, \sigma) : e^{-iP_{\mu}(q)x^{\mu}} : \quad (21)$$

where the wave functions  $U_l(q, \sigma)$  obey the intertwining conditions

$$D_{ll'}^{(AB)}(\Lambda) U_{l'}(q, \sigma) = D_{\sigma'\sigma}^{(S)}(R(q, \Lambda)) U_l(\Lambda q, \sigma'). \quad (22)$$

Then  $\Phi_l(x)$  transforms according to the formula (18). The massless counterpart reads:

$$\Phi_l(x) = \int \frac{d^3 \vec{q}}{q_0} U_l(q, \lambda) a(q) : e^{-iP_{\mu}(q)x^{\mu}} : \quad (23)$$

with  $P_{\mu}(q)$  given by Eqs. (5, 6) transforms covariantly if

$$D_{ll'}^{(AB)}(\Lambda) U_{l'}(q, \lambda) = e^{i\lambda\Theta(q, \Lambda)} U_l(\Lambda q, \lambda) \quad (24)$$

It is crucial that the conditions (22) and (24) are purely classical. This allows to write out immediately the deformed counterpart of free field theory once the classical problem is solved. Finally, let us remark that it is not difficult to show that above reasoning is valid also if one uses a general deformation map (10, 11).

## 4 Free field equations

In general there is some redundancy in the description in terms of coordinate space wave functions. In fact, the representation  $D^{(AB)}$  can, in the massive case, describe all particles of spin  $s$  obeying  $|A - B| \leq s \leq A + B$ . Therefore, apart from the Klein-Gordon (KG) condition,  $(q^2 - m^2)U_l(q, \sigma) = 0$  there are other conditions of the form

$$\Pi_{ll'}(q) U_{l'}(q, \sigma) = 0 \quad (25)$$

singling out a definite spin. These conditions can be converted into constant coefficients matrix differential equations in coordinate space. The basic problem of the theory of wave equations in undeformed as well as  $\kappa$ -deformed case is to find a single equation from which K-G equation as well as additional conditions (25) follow and to derive it from some Lagrangian. This problem has particular solutions depending on  $\kappa$  and  $(A, B)$ . It is considered as a preliminary step towards interacting field theory. In fact, such an equation defines propagators which are necessary to formulate the perturbative version of the theory.

The result of Sec. 3 imply the following procedure. Let us invert the relations (10, 11) which yield

$$\begin{aligned} q_0(P) &= (C - A)e^{P_0/\kappa} - C \\ q_k(P) &= (C - A)e^{P_0/\kappa} \frac{P_k}{\kappa} \end{aligned} \quad (26)$$

For any  $f \in M_\kappa$  given by

$$f = : f(x) : \quad (27)$$

we define the derivative operators as follows

$$i\hat{\partial}_\mu f = : q_\mu(i\frac{\partial}{\partial x^\mu})f(x) : \quad (28)$$

Now, let

$$F_{AB}(\partial)\Phi_B = 0 \quad (29)$$

be the relevant classical wave equation on commutative Minkowski space. Its  $\kappa$ -deformed counterpart has the same form, only the standard space-time derivatives  $\partial_\mu$  are replaced by the vector fields  $\hat{\partial}_\mu$  on  $\kappa$ -deformed Minkowski space (see (23)).

$$F_{AB}(\hat{\partial})\Phi_B = 0 \quad (30)$$

This is the general prescription for constructing wave equations on  $\kappa$ -Minkowski space.

Let us write the Fourier transform for  $\Phi_A \in M_\kappa$  as follows (compare with (17))

$$\Phi_A(\hat{x}) = \int \frac{d^3 p}{\omega_\kappa(p)} \tilde{a}_A(p) : e^{-ip_\kappa \hat{x}^\kappa} : \quad (31)$$

where

$$\begin{aligned} \tilde{a}_A(p) &= a_A(q(p)) \\ \omega_\kappa(p) &= q_0(p) \cdot \det \left( \frac{\partial q_i}{\partial p_j} \right) \end{aligned} \quad (32)$$

one can replace the algebra of fields (29) on noncommutative Minkowski space by homomorphic algebra of fields on commutative Minkowski space with the multiplication described by the  $\kappa$ -deformed star-product (see [7,18])

$$\begin{aligned} \Phi_A(\hat{x}) \cdot \Phi_B(\hat{x}) &\longleftrightarrow \varphi_A(y) \star \varphi_B(y) = \\ &= \varphi_A(y + \xi_1) \exp i y_\mu \\ &\quad \Gamma^\mu \left( \frac{\overleftarrow{\partial}}{\partial \xi_1}, \frac{\overrightarrow{\partial}}{\partial \xi_2} \right) \varphi_B(y + \xi_2) \Big|_{\xi_1 = \xi_2 = 0} \end{aligned} \quad (33)$$

where

$$\begin{aligned} \Gamma_o(\eta, \xi) &= \eta^0 + \xi^0 \\ \Gamma^i(\eta, \xi) &= \frac{f_\kappa(\eta^0) e^{\frac{\xi_0}{\kappa}} \eta^i + f_\kappa(\xi^0) e^{\frac{\eta_0}{\kappa}} \xi_i}{f_\kappa(\eta^0 + \xi^0)} \end{aligned} \quad (34)$$

and  $f_\kappa(\alpha) = \frac{\kappa}{\alpha} (e^{\frac{\alpha}{\kappa}} - 1)$ .

The field equation (30) on noncommutative space is translated into the  $\kappa$ -deformed field equation on standard Minkowski space as follows

$$F_{AB} \left( q_\mu \left( i \frac{\partial}{\partial y^\mu} \right) \right) \phi_B(y) = 0 \quad (35)$$



Further we shall consider the following simple choice of the formulae (26), corresponding to  $C = (m^2 + \kappa^2)^{\frac{1}{2}}$ , leading to (see e.g. [15])

$$\begin{aligned} q_0(P_\mu) &= \kappa \sinh \frac{P_0}{\kappa} + \frac{1}{2\kappa} e^{\frac{P_0}{\kappa}} \vec{p}^2 \\ q_i(P_\mu) &= e^{\frac{P_0}{\kappa}} P_i \end{aligned} \quad (36)$$

Below we shall consider the  $\kappa$ -deformed counterpart of the Klein–Gordon, Dirac, Proca and Maxwell equations.

(a) Klein–Gordon equation.

On  $\kappa$ -deformed Minkowski space the KG field equation looks as follows:

$$\left( \hat{\partial}_\mu \hat{\partial}^\mu - m_0^2 \right) \Phi(\hat{x}) = 0 \quad (37)$$

If we observe that

$$q_\mu q^\mu = \mathcal{M}^2(p) \left( 1 + \frac{\mathcal{M}^2}{4\kappa^2} \right) \quad (38)$$

where

$$\mathcal{M}^2(p) = \left( 2\kappa \sinh \frac{p_0}{2\kappa} \right)^2 - e^{\frac{p_0}{\kappa}} \vec{p}^2 \quad (39)$$

describes the  $\kappa$ -deformed mass Casimir in bicrossproduct basis, we obtain the following counterpart of the  $\kappa$ -deformed KG equation on standard Minkowski space

$$\mathcal{M}^2 \left( \frac{1}{i} \partial_\mu \right) \left( 1 + \frac{\mathcal{M}^2(\frac{1}{i} \partial_\mu)}{4\kappa^2} \right) \varphi(y) = m_0^2 \varphi(y) \quad (40)$$

We see that the  $\kappa$ -deformed KG operator contains additional tachyon of the mass  $-2\kappa^2(1 + (1 + m^2/\kappa^2)^{\frac{1}{2}})$ .

(b) Dirac equation.

On  $\kappa$ -deformed Minkowski space, due to lack of deformation in Lorentz sector the algebra of Dirac matrices is not deformed, i.e. we get

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \quad (41)$$

The Dirac equation on  $\kappa$ -deformed Minkowski space

$$\left( \gamma^\mu \hat{\partial}_\mu - m_0 \right)_{AB} \Psi_B(\hat{x}) = 0 \quad (42)$$

has the following counterpart on standard Minkowski space (see also (36))

$$\begin{aligned} & \left[ \gamma^0 \left( \kappa \sinh \frac{\partial_0}{\kappa} - \frac{1}{2\kappa} e^{i\frac{\partial_0}{\kappa}} \Delta \right) \right. \\ & \left. - e^{i\frac{\partial_0}{\kappa}} \gamma^i \partial_i + m_0 \right]_{AB} \Psi_B = 0 \end{aligned} \quad (43)$$

The square of the  $\kappa$ -deformed Dirac operator provides the  $\kappa$ -deformed K.G. operator (39).

The equation (43) has been firstly given in [19,20]

(c) Proca and Maxwell equation.

The Proca equation on  $\kappa$ -deformed Minkowski space takes the form:

$$\left( \hat{\partial}_\mu \hat{\partial}^\mu \delta_\nu^\rho - \hat{\partial}_\nu \hat{\partial}^\rho \right) U_\rho(\hat{x}) = m_0^2 U_\nu(\hat{x}) \quad (44)$$

or equivalently (we observe that  $[\hat{\partial}_\mu, \hat{\partial}_\nu] = 0$ )

$$\hat{\partial}^\mu F_{\mu\nu}(\hat{x}) = m_0^2 U_\nu(\hat{x}) \quad (45)$$

where

$$F_{\mu\nu}(\hat{x}) = \hat{\partial}_\mu U_\nu(x) - \hat{\partial}_\nu U_\mu(x) \quad (46)$$

If  $m = 0$  we get

$$\hat{\partial}^\mu F_{\mu\nu}(\hat{x}) = 0 \quad (47)$$

and the theory is invariant under the following local  $U(1)$  gauge transformations (we denote  $U_\mu|_{m_0=0} = A_\mu$ ):

$$A'_\mu(\hat{x}) = A_\mu(\hat{x}) - \hat{\partial}_\mu \alpha(\hat{x}) \quad (48)$$

The formula (48) has the following counterpart on commutative Minkowski space:

$$\begin{aligned} A'_i(\vec{y}, y_0) &= A_i(\vec{y}, y_0) - \partial_i \alpha\left(\vec{y}, y_0 - \frac{i}{\kappa}\right) \\ A'_0(\vec{y}, y_0) &= A_0(\vec{y}, y_0) - \kappa \left( \alpha\left(\vec{y}, y_0 + \frac{i}{\kappa}\right) \right. \\ &\quad \left. - \alpha\left(\vec{y}, y_0 - \frac{i}{\kappa}\right) + \frac{1}{2\kappa} \Delta \alpha\left(\vec{y}, y_0 + \frac{i}{\kappa}\right) \right) \end{aligned} \quad (49)$$

One can calculate that the product of two  $\kappa$ -deformed  $U(1)$  gauge transformations is given by the formula:

$$: e^{i\alpha(\hat{x})} : : e^{i\alpha'(\hat{x})} = e^{i\alpha''(\hat{x})} : \quad (50)$$

where

$$\begin{aligned} \hat{\alpha}''_\kappa(\vec{x}, \hat{x}_0) &= \alpha'(\vec{x}, \hat{x}_0) \\ &\quad + \alpha \left( \exp i \left[ \alpha' \left( \vec{x}, \hat{x}_0 - \frac{i}{\kappa} \right) \right. \right. \\ &\quad \left. \left. - \alpha' \left( \vec{x}, \hat{x}_0 \right) \right] \vec{x}, \hat{x}_0 \right) \end{aligned} \quad (51)$$

## 5 On the second quantization of $\kappa$ -deformed free fields

Let us consider as an example the second quantization of  $\kappa$ -deformed KG field (see (40)) using the commutative space-time framework. The relativistic wave equation (40) can be written as the following product of standard and tachyonic deformed KG factors:

$$\begin{aligned} &\left( \mathcal{M}^2 \left( \frac{1}{i} \partial_\mu \right) - m_+^2 \right) \\ &\cdot \left( \mathcal{M}^2 \left( \frac{1}{i} \partial_\mu \right) - m_-^2 \right) \varphi(x) = 0 \end{aligned} \quad (52)$$

where

$$\begin{aligned} m_+^2 &= m_0^2 - \frac{m_0^4}{4\kappa^2} + \mathcal{O}\left(\frac{1}{\kappa^4}\right) \\ m_-^2 &= -4\kappa^2 - m_0^2 + \frac{m_0^4}{4\kappa^2} + \mathcal{O}\left(\frac{1}{\kappa^4}\right) \end{aligned} \quad (53)$$

We see that  $\varphi(x)$  is the linear combination of two solutions  $\varphi_+, \varphi_-$  satisfying the equation

$$\left(\mathcal{M}^2\left(\frac{1}{i}\partial_\mu\right) - m_\pm^2\right)\varphi_\pm(x) = 0 \quad (54)$$

The physical solution of (54) with  $m_+^2 > 0$  can be quantized. Let us write

$$\begin{aligned} \hat{\varphi}_\pm(x) &= \frac{1}{(2\pi)^2} \int d^4 p \delta(\mathcal{M}^2(p) - m_+^2) \\ &\quad \cdot \varphi_+(p) e^{ipx} \\ &= \frac{1}{(2\pi)^2} \int \frac{d^3 \vec{p}}{\omega_\kappa(\vec{p})} \\ &\quad \cdot \left(a(\vec{p}) e^{i\vec{p}\vec{x} - \omega_\kappa(\vec{p})t} + H.C.\right) \end{aligned} \quad (55)$$

where  $p_0 = \omega(\vec{p})$  describes the dispersion relation for  $\kappa$ -deformed relativistic particle (see (39)). As the first attempt let us introduce the following  $\kappa$ -deformed creation and annihilation relations

$$[a^+(\vec{p}), a(\vec{p}')] = 2\omega_\kappa(p) \delta^3(\vec{p} - \vec{p}'). \quad (56)$$

If we define the Fock vacuum as follows

$$a^+(p)|0\rangle = 0 \quad (57)$$

one gets

$$\begin{aligned} \langle 0 | \hat{\varphi}_\pm(x), \hat{\varphi}_\pm(x') | 0 \rangle &= \\ &= \frac{1}{(2\pi)^4} \int \frac{d^3 \vec{p}}{2\omega_\kappa(\vec{p})} e^{i\omega_\kappa(\vec{p})(t-t')} \\ &= \frac{1}{(2\pi)^4} \int d^4 p \theta(p_0) \delta(\mathcal{M}^2(p) - m_+^2) e^{ip(x-x')} \end{aligned} \quad (58)$$

We see that the formula (58) describes the  $\kappa$ -deformation of free scalar Wightmann function. It should be mentioned here that the  $\kappa$ -deformed Green functions in standard basis, obtained by the redefinition  $p_i \rightarrow e^{-\frac{p_0}{\kappa}} p_i$ , were discussed in [21].

It should be stressed, however, that the relation (56) is not consistent with the  $\kappa$ -covariant definition of two-particle state  $|p_1; p_2\rangle$  introduced as  $|p_1; p_2\rangle = a(p_1)a(p_2)|0\rangle$ . It is easy to see that the relation  $|p_1; p_2\rangle = |p_2, p_1\rangle$  is not consistent with the nonsymmetric coproduct  $\Delta^{(2)}(\vec{p})$  describing the threemomenta of two-particle states. We get

$$\begin{aligned} \vec{P}|\vec{p}_1, E_1; \vec{p}_2, E_2\rangle &= \left(\vec{p}_1 e^{\frac{E_2}{\kappa c}} + \vec{p}_2\right) \\ &\quad |\vec{p}_1, E_1; \vec{p}_2, E_2\rangle \end{aligned} \quad (59a)$$

$$\vec{P}|\vec{p}_2, E_2; \vec{p}_1, E_1\rangle = \begin{pmatrix} \vec{p}_2 e^{\frac{E_1}{\kappa c}} + \vec{p}_1 \\ |\vec{p}_1, E_1; \vec{p}_2 \vec{E}_2\rangle \end{pmatrix}. \quad (59b)$$

It is easy to see that both formulae (59a-b) give the same value of total two-particle threemomentum if we assume

$$a(\vec{p}_1, E_1) a(\vec{p}_2, E_2) = \begin{pmatrix} a\left(\vec{p}_2 e^{-\frac{E_1}{\kappa c}}, E_2\right) \\ a\left(\vec{p}_1 e^{-\frac{E_2}{\kappa c}}, E_1\right) \end{pmatrix}. \quad (60)$$

The flip operation (60) can be represented by the action of quantum  $\mathbb{R}$ -matrix, which for the quantum  $\mathfrak{m}$ -Poincaré algebra is not known.

Further considerations leading to the  $\mathfrak{m}$ -Poincaré covariant Fock space and corresponding formulation of second-quantized fields on noncommutative Minkowski space is under investigation.

## 6 Conclusions

Let us summarize our results. Firstly we have defined the covariant wave functions on  $\mathfrak{m}$ -deformed Minkowski space  $M_{\mathfrak{m}}$ . Their transformation properties are defined by the right coacting of  $P_{-\mathfrak{m}}$  -  $\mathfrak{m}$ -Poincaré group with the parameter  $\mathfrak{m}$  replaced by  $-\mathfrak{m}$ . We solved the problem of finding the intertwiners between this representation acting on coordinate wave functions and the unitary representation of  $P_{\mathfrak{m}}$  acting in momentum space. These intertwiners are actually the wave functions of particles of definite mass and spin. The main result here is that once this problem is solved in the nondeformed case it can be similarly solved also for nonvanishing deformation parameter.

In second part of this report we define the basic deformed free fields on noncommutative Minkowski as well as on standard Minkowski space. Due to star product technique one can represent the effect of noncommutativity of  $\mathfrak{m}$ -Minkowski space by nonlocal deformation of the derivatives defining free field equations (see (36)). It should be stressed that  $\mathfrak{m}$ -deformation implies the deformation of free equations and the nonlocality in time.

In order to define the  $\mathfrak{m}$ -deformed second-quantized field theory one should quantize the Fourier transforms  $\tilde{a}(q)$  (see (19)) or  $\tilde{a}_A(p)$  (see (31)). This problem was briefly discussed in Sect. 5 and requires for its further understanding the  $\mathfrak{m}$ -covariant structure of multiparticle states and consistency with nonAbelian addition law for  $\mathfrak{m}$ -deformed threemomenta.

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