DUALITY OF THE DIRICHLET AND NEUMANN PROBLEMS IN BRANEWORLD PHYSICS

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Braneworld effective action for two-brane model is constructed by two different methods based respectively on the Dirichlet and Neumann boundary value problems. The equivalence of these methods is shown due to nontrivial duality relations between special boundary operators of these two problems.

1 Introduction

Calculational methods for braneworld phenomena [1] incorporate together with the well-known old formalisms, like the effective action approach, special new features associated with bulk/brane (boundary) ingredients characteristic of the braneworld scenarios. In what follows we focus our attention at the peculiarities of the Dirichlet and Neumann boundary conditions in braneworld setup, the way they arise in the course of calculating braneworld effective action and, in particular, at a sort of duality relation between these problems. In braneworld context this duality manifests itself in the equality of special boundary operators originating from Dirichlet and Neumann boundary value problems and appearing in the braneworld effective action.

Braneworld effective action implicitly incorporates the dynamics of the fields in the bulk and explicitly features the boundary fields (being the functional of those). Such action, on the one hand, arises as a result of integrating out the bulk fields subject to boundary (brane) fields and, on the other hand, generates effective equations of motion for the latter. This situation obviously suggests two strategies of calculating the braneworld action. One strategy consists in its direct calculation and, in the tree-level approximation, reduces to solving the equations of motion in the bulk subject to given boundary values – brane fields – and substituting the result into the fundamental bulk action. Thus, this strategy involves the Dirichlet problem. Another strategy consists in recovering the braneworld effective action from effective equations of motion for brane fields. Since the latter incorporates well known Israel junction conditions on branes, this method relies on the Neumann boundary value problem. Here we show that they really match in view of a nontrivial relation between nonlocal brane operators arising as restriction of (properly differentiated) kernels of Dirichlet and Neumann Green's functions to the boundaries/branes.

2 Dirichlet and Neumann boundary value problems and braneworld action

In this section we demonstrate the construction of the effective action on the example of the two-brane Randall-Sundrum model [2]. The gravitational part of its action is

$$\mathbf{S}[\,G\,] = \int\limits_{\mathbf{B}\times Z_2} d^{(d+1)}X\sqrt{G}\,\big(R(G) - 2\Lambda\big) - \sum_i \int\limits_{\mathbf{b}_i} d^d x\sqrt{g}\,\big(2[K] + \tau_i\big). \qquad (1)$$

The orbifold symmetry implies that the $(d+1)$-dimensional integration runs over two identical copies, $\mathbf{B} \times Z_2$, of the bulk $\mathbf{B}$ which is bounded by two d-dimensional branes $\mathbf{b}_i$ that can be regarded as the boundaries of $\mathbf{B}$. The discrete index enumerating the branes runs over two values $i = \pm$. Here $G = G_{AB}(X)$ $(A = 0, 1, ..., d)$ is the bulk metric and $g = g^{\pm}_{\alpha\beta}(x)$ $(\alpha = 0, 1, ..., d-1)$ denotes the collection of induced metrics on $i = \pm$ branes, K is the trace of the extrinsic curvature on the brane defined as $K = G^{AB}\nabla_A \mathbf{n}_B$, where ∇_A is a covariant $(d+1)$-dimensional derivative and $\mathbf{n}$ is the inward pointing normal and $[K]$ is the sum of the so-defined extrinsic curvatures on both sides of the brane. τ_n are brane tensions.

Braneworld effective action is formally defined [3, 4] as the result of integrating out the bulk metric subject to given values of induced metrics on branes

$$\exp\Big(iS_{\rm eff}[\,g\,]\Big) = \int DG\,\exp\Big(i\mathbf{S}[\,G\,]\Big)\Big|_{G_{\alpha\beta}(\mathbf{b}_\pm)=g^{\pm}_{\alpha\beta}}. \qquad (2)$$

When this integration is done within $\hbar$-expansion the result reads

$$S_{\rm eff}[\,g\,] = \mathbf{S}\big[\,G[\,g\,]\,\big] + O(\hbar), \qquad (3)$$

where the tree-level part $\mathbf{S}\big[\,G[\,g\,]\,\big]$ is a result of substituting in the classical bulk action the solution $G[g]$ of the following Dirichlet boundary value problem

$$\frac{\delta\,\mathbf{S}[\,G\,]}{\delta G_{AB}(X)} = 0, \quad G_{\alpha\beta}(X)\big|_{\mathbf{b}_\pm} = g^{\pm}_{\alpha\beta}(x). \qquad (4)$$

Expression (2) is formal at least because the theory (1) is gauge invariant and one should factor out the gauge group from the path integral. The presence of gauge invariance also manifests itself in the fact that not all components of $G_{AB}(X)$ should be fixed on boundaries – one fixes only induced metrics on both branes $G_{\alpha\beta}(x)$.

To obtain the effective action exactly for given arbitrary metrics $g^{\pm}_{\alpha\beta}(x)$ is obviously impossible. So we will proceed within perturbation theory. First we choose some background – the solution of Einstein equations of motion $G^0_{AB}(X)$ with some background boundary conditions $(g^{\pm}_{\alpha\beta})^0(x)$ and perturbatively expand the action up to the second order in perturbations $h^{\pm}_{\alpha\beta}(x)$ of $g^{\pm}_{\alpha\beta} = (g^{\pm}_{\alpha\beta})^0(x) + h^{\pm}_{\alpha\beta}(x)$ inducing the perturbations $H_{AB}(X)$ of $G_{AB}(X) = G^0_{AB}(X) + H_{AB}(X)$ in the bulk.

Possible background solutions with nonintersecting branes are the configurations with "parallel" branes generalizing the Randall-Sundrum solution. They have the form

$$G^0_{AB}(X)dX^A dX^B = a^2(y)\, g_{\alpha\beta}(x)dx^\alpha dx^\beta + dy^2 \tag{5}$$

in the coordinates $X^A = (x^\alpha, y)$ in which the branes are hypersurfaces of constant y located at some $y = y_+$ and $y = y_-$ with two conformally equivalent background induced metric $(g^{\pm}_{\alpha\beta})^0 = a^2(y_\pm)\, g_{\alpha\beta}(x)$. For various values of tensions the solutions with d-dimensional dS, flat or AdS branes are possible [4, 5] for the homogeneous metrics $g_{\alpha\beta}(x)$ of positive, zero and negative scalar curvature respectively.

For simplicity we shall consider the sector of *transverse-traceless* perturbations of induced metrics $h^{\pm}_{\alpha\beta}(x) = a^2(y_\pm)\gamma^{\pm}_{\alpha\beta}(x)$, $g^{\alpha\beta}\gamma_{\alpha\beta} = 0\,,\ \bigtriangledown^\alpha\gamma_{\alpha\beta} = 0$, and generated by them metric perturbations in the bulk, $H_{AB}(X)\, dX^A dX^B = a^2(y)\gamma_{\alpha\beta}(x, y)dx^\alpha dx^\beta$. Here the notation $\bigtriangledown_\alpha$ stands for the covariant d-dimensional derivative with with respect to the background metric $g_{\alpha\beta}(x)$. The contribution of the scalar sector of metric perturbations can be found in [4].

The gravitational action (1) expanded to quadratic order in such perturbations reads [4]

$$\mathbf{S}[\gamma^\pm] = \frac{1}{2}\int\limits_{\mathbf{B}} d^d x\, dy\, a^d(y)\sqrt{g(x)}\,\gamma\big(\hat{\mathbf{F}}\gamma\big) + \frac{1}{2}\sum_i \int\limits_{\mathbf{b}_i} d^d x \sqrt{g(x)}\,\gamma\big(\hat{\mathrm{W}}\gamma\big), \tag{6}$$

where

$$\hat{\mathbf{F}} = a^{-d}\partial_y a^d \partial_y + a^{-2}\big(\Box - \frac{2}{d(d-1)}R(g)\big) \tag{7}$$

is the inverse propagator of transverse-traceless modes (gravitons) in the bulk, $\Box = g^{\alpha\beta}(x)\nabla_\alpha\nabla_\beta$ is the covariant d'Alembertian on the brane, $R(g)$ – constant scalar curvature of dS^d, $\mathbb{R}^{d-1,1}$ or AdS^d spaces with the metric $g_{\alpha\beta}(x)$. The first-order differential operator $\hat{\mathrm{W}}$ in the surface term is the Wronskian operator corresponding to (7), $\hat{\mathrm{W}} = a^d \mathbf{n}^A \nabla_A$, which participates in the Wronskian relation for $\hat{\mathbf{F}}$ (see below).

In order to find quadratic approximation for $S_{\rm eff}[g]$ of (3) one should solve the linearized boundary value problem (2) on the background (5) with Dirichlet boundary conditions $\gamma^\pm$ and substitute it in (6). The result reads as [4]

$$S_{\rm eff}[\,\gamma\,] = -\frac{1}{2}\int\limits_{\mathbf{b}} d^{d}x\sqrt{g(x)}\int\limits_{\mathbf{b}} d^{d}x'\sqrt{g(x')}\,\sum_{k,l}\gamma^k(x)[\overrightarrow{\mathrm{W}}\mathbf{G}_D\overleftarrow{\mathrm{W}}]_{kl}(x,x')\gamma^l(x'), \tag{8}$$

where the integral kernel of the quadratic form

$$[\overrightarrow{\mathrm{W}}\mathbf{G}_D\overleftarrow{\mathrm{W}}]_{kl}(x,x') = \overrightarrow{\mathrm{W}}\mathbf{G}_D(X,X')\overleftarrow{\mathrm{W}'}\Big|_{X=X_k(x),\;X'=X_l(x')} \tag{9}$$

($\overleftarrow{\mathrm{W}'}$ obviously acting on the second (primed) argument of $\mathbf{G}_D(X,X')$) expresses in terms of the Dirichlet Green's function of $\hat{\mathbf{F}}$ satisfying

$$\left\{\begin{array}{l} \hat{\mathbf{F}}\,\mathbf{G}_D(X,X') = \delta(X,X')\;, \\ \mathbf{G}_D(X,X')\big|_{X=X_k(x)} = 0, \end{array}\right. \tag{10}$$

and $X = X_k(x)$ is the notation for the embedding functions of the k-th brane into the bulk.

There exists an alternative derivation of the effective action suggested in [3, 7]. One can recover the action from the effective equations for brane d-dimensional field. They actually represent the junction conditions on branes rewritten in terms of the brane metric [3, 4]. For the same two-brane scenario this derivation looks as follows. To begin with, supplement the purely gravitational action (1) by the action of matter fields φ located on branes, $\mathbf{S}[\,G\,] \Rightarrow \mathbf{S}[\,G,\varphi\,] = \mathbf{S}[\,G\,] + S_{\rm mat}[\,g,\varphi\,]$. Under this modification the effective action defined by (2) changes by the same trivial additive law, $S_{\rm eff}[\,g\,] \Rightarrow S_{\rm eff}[\,g,\varphi\,] = S_{\rm eff}[\,g\,] + S_{\rm mat}[\,g,\varphi\,]$, because no integration over the fields (φ, g) is done on branes.

The variation of this action involves the sum of bulk part and the surface term, arising from integration by parts, located on branes. The demand of stationarity of the action then reduces to two equations – the requirement that both bulk and brane parts of this variation vanish –

$$\Big(R^{AB}-\frac{1}{2}\,RG^{AB}\Big)(X)-\Lambda G^{AB}(X)=0, \tag{11}$$

$$\Big(\,\big[K^{\alpha\beta}-Kg^{\alpha\beta}\big]+\frac{1}{2}\,T^{\alpha\beta}-\frac{1}{2}g^{\alpha\beta}\tau_i\Big)\Big|_{\mathbf{b}_i}=0, \tag{12}$$

where $K^{\alpha\beta}$ is the extrinsic curvature tensor and $T^{\alpha\beta}$ is the matter stress tensor on a respective brane. The second equation is nothing but Israel junction condition.

These equations give rise to a nonlinear boundary value problem of *Neumann* type, since the Israel junction conditions contain derivatives normal to branes. Again to handle the problem one has use the perturbation procedure. After linearization of (12) on the background (5) one obtains in the transverse-traceless sector the following boundary value problem for transverse-traceless tensor perturbations

$$\hat{\mathbf{F}}\gamma(x,y)=0\,,\quad a^d\nabla_{\mathbf{n}}\gamma(x,y)\,\Big|_{\mathbf{b}_i}=-\frac{1}{2}\,t_i(x), \tag{13}$$

where t_i is the transverse traceless part of the appropriately rescaled matter stress tensor on the i-th brane (for brevity we omit the tensor indices).

The solution of this problem on *branes*

$$\gamma^l(x)\equiv\gamma(x,y_l)=-\frac{1}{2}\sum_n\int\limits_{\mathbf{b}_n}d^dx'\sqrt{g(x')}\,\mathbf{G}_N^{ln}(x,x')\,t_n(x'), \tag{14}$$

is given in terms of the following restriction to these branes

$$\mathbf{G}_N^{ln}(x,x')=\mathbf{G}_N(X,X')\Big|_{X=X_l(x),\;X'=X_n(x')} \tag{15}$$

of the *Neumann* Green's function of $\hat{\mathbf{F}}$ satisfying

$$\left\{\begin{array}{l}\hat{\mathbf{F}}\,\mathbf{G}_N(X,X')=\delta(X,X')\;,\\ \hat{\mathrm{W}}\,\mathbf{G}_N(X,X')\big|_{X=X_k(x)}=0.\end{array}\right. \tag{16}$$

Note that in contrast to (9) the kernel of the integral operation in (14) is built in terms of the Neumann Green's function rather than the Dirichlet one.

Eq. (14) is the effective equation of motion for brane metrics in the presence of matter sources on branes. It should be derivable by variational procedure from the braneworld effective action we are looking for, $S_{\rm eff}[\,g,\varphi\,] = S_{\rm eff}[\,g\,] + S_{{\rm m}at}[\,g,\varphi\,]$. From this observation it is straightforward to recover $S_{\rm eff}[\,g\,] = S_{\rm eff}[\,\gamma\,] + O(\gamma^3)$ in the quadratic approximation by functionally integrating the equation (14) and taking into account a simple fact that in the variational derivative of $S_{\rm eff}[\,g,\varphi\,]$ the stress tensor enters with a local algebraic coefficient 1/2 (this helps one to find the overall integrating factor). The final result reads [3, 4]

$$S_{\rm eff}[\,\gamma\,] = \frac{1}{2}\int\limits_{\mathbf{b}} d^dx\sqrt{g(x)}\,\int\limits_{\mathbf{b}} d^dx'\sqrt{g(x')}\sum_{k,l=\pm}\gamma^k(x)[\,\mathbf{G}_N^{-1}\,]_{kl}(x,x')\,\gamma^l(x), \quad (17)$$

where $[\mathbf{G}_N^{-1}]_{kl}(x,x')$ is the kernel of the integral operation inverse to that of (14).

3 Duality of boundary value problems

The consistency of two alternative derivations of the above type implies the equivalence of the two algorithms (17) and (8) which implies the relation between the nonlocal kernels defined by Eqs. (9) and (15)

$$[\overrightarrow{\rm W}\mathbf{G}_D\overleftarrow{\rm W}]_{kl}(x,x') = -\,[\mathbf{G}_N^{-1}]_{kl}(x,x'). \quad (18)$$

This statement can be proved [4] irrespective of braneworld theory context, since it is the property of Green's functions of the second-order differential operator of a rather general form, acting in space with boundaries. The proof in a slightly more general setting, when the two branes are replaced by the generic set of boundaries of codimension one, looks as follows.

Consider $\mathcal{M}$ – arbitrary (for simplicity connected) manifold with boundaries $\partial\mathcal{M}_l$: $\partial\mathcal{M} = \bigcup_l \partial\mathcal{M}_l$. Let us introduce Φ – some field (or set of fields) in the bulk, and let their free dynamics of be governed by some nondegenerate second order differential operator $\hat{\mathbf{F}}$. Denote boundary values of the bulk field by ϕ: $\Phi\,|_{\partial\mathcal{M}_l} = \phi^l$. Alternatively, if one introduces some coordinate system X on $\mathcal{M}$ and coordinates x on $\partial\mathcal{M}_l$ and embedding functions $X_l(x)$, then $\Phi(X)\,|_{X=X_l(x)} = \phi^l(x)$. Define also the first order differential (Wronskian) operator $\hat{\rm W}$ associated with $\hat{\mathbf{F}}$ by the following Wronskian relation

$$\int\limits_{\mathcal{M}} dV\,\Phi(\hat{\mathbf{F}}\Psi) + \int\limits_{\partial\mathcal{M}} dS\,\Phi(\hat{\rm W}\Psi) = \int\limits_{\mathcal{M}} dV\,\Phi\overleftrightarrow{\rm F}\Psi \quad (19)$$

valid for arbitrary Φ and Ψ. Here dV is the bulk measure in which $\hat{\mathbf{F}}$ is symmetric, dS is some boundary measure and $\overleftrightarrow{\mathrm{F}}$ is understood (in the sense of integrating by parts) as acting on both right and left fields at most with the first-order derivatives (for $\Psi = \Phi$ the combination $\Phi\overleftrightarrow{\mathrm{F}}\Phi$ implies the Lagrangian of the field Φ quadratic in the first-order derivatives $\partial\Phi$). Finally, define the set of sources j_k, each located on a respective boundary $\partial\mathcal{M}_k$.

To demonstrate the duality relation (18) lets us first explicitly pose Neumann and Dirichlet boundary value problems for Φ. The Neumann problem has the following form involving the boundary sources j_k

$$\begin{cases} \hat{\mathbf{F}}\,\Phi(X) = 0\,, \\ \hat{\mathrm{W}}\,\Phi(X)\big|_{X=X_k(x)} = j_k(x). \end{cases} \tag{20}$$

It has the solution

$$\Phi(X) = \sum_k \int\limits_{\partial\mathcal{M}_k} dS'\, \mathbf{G}_N\big(X, X_k(x')\big) j_k(x'), \tag{21}$$

in terms of the Neumann Green function obeying the boundary value problem (16) (in which the normalization of $\delta(X, X')$ is defined by the relation $\int_{\mathcal{M}} dV\, \Phi(X)\delta(X, X') = \Phi(X')$). The restriction of this solution to boundaries reads

$$\phi^l(x) = \sum_k \int\limits_{\partial\mathcal{M}_k} dS'\, \mathbf{G}_N^{lk}(x, x') j_k(x'), \tag{22}$$

where the kernel $\mathbf{G}_N^{lk}(x, x')$ is defined by (15).

On the other hand, $\Phi(X)$ can be treated as a solution of the Dirichlet boundary value problem

$$\begin{cases} \hat{\mathbf{F}}\,\Phi(X) = 0\,, \\ \Phi(X)\big|_{X=X_k(x)} = \phi^k(x)\,, \end{cases} \tag{23}$$

with boundary conditions given by (22). Its solution in terms of the Dirichlet Green's function (10) reads

$$\Phi(X) = -\sum_l \int\limits_{\partial\mathcal{M}_l} dS'\, \mathbf{G}_D(X, X')\overleftarrow{\mathrm{W}'}\Big|_{X'=X_l(x')} \phi^l(x'). \tag{24}$$

Let us act on this solution by the Wronskian operator, restrict the result to the k-th boundary and take into account that $j_k(x) = \hat{\mathrm{W}}\Phi(X)|_{X=X_k(x)}$. This leads to the equation

$$j_k(x) = -\sum_l \int\limits_{\partial\mathcal{M}_l} dS' \, [\overrightarrow{\mathrm{W}}\mathbf{G}_D\overleftarrow{\mathrm{W}}]_{kl}(x,x')\phi^l(x') \tag{25}$$

with the kernel $[\overrightarrow{\mathrm{W}}\mathbf{G}_D\overleftarrow{\mathrm{W}}]_{kl}(x,x')$ defined by Eq. (9).

Comparing (22) and (25) one finds the needed duality relation (18) between the kernels $\mathbf{G}_N^{lk}(x,x')$ and $-[\overrightarrow{\mathrm{W}}\mathbf{G}_D\overleftarrow{\mathrm{W}}]_{kl}(x,x')$– they form inverse to one another integral operations on the full boundary $\partial\mathcal{M}$ in the boundary measure dS

$$\sum_l \int\limits_{\partial\mathcal{M}_l} dS' \, [\overrightarrow{\mathrm{W}}\mathbf{G}_D\overleftarrow{\mathrm{W}}]_{kl}(x,x') \, \mathbf{G}_N^{li}(x',x'') = -\delta_k^i \, \delta(x,x''). \tag{26}$$

4 Conclusions

We illustrated the calculation of the effective action in braneworld scenarios within two different schemes resorting to Dirichlet and Neumann boundary value problems. Their equivalence is guaranteed by a special duality relation between the boundary operators associated respectively with the Dirichlet and Neumann Green's functions of the theory. Irrespective of the braneworld context this relation holds in a rather general setup. Thus, it seems to have a much wider scope of implications in various models relating volume (bulk) and surface phenomena, like graviton localization [2], AdS/CFT correspondence and holography principle [8]. In particular, this relation explains the dual structure of poles and roots of the nonlocal wave operator of the effective theory underlying the positivity of residues at all the poles of its propagator – the property accounting for normal non-ghost nature of all massive Kaluza-Klein modes in two-brane models [3, 9].

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