Integrability of the C_n and BC_n Ruijsenaars-Schneider models

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Abstract: We study the C_n and BC_n Ruijsenaars-Schneider(RS) models with interaction potential of trigonometric and rational types. The Lax pairs for these models are constructed and the involutive Hamiltonians are also given. Taking nonrelativistic limit, we also obtain the Lax pairs for the corresponding Calogero-Moser systems.

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I Introduction

Ruijsenaars-Schneider(RS) and Calogero-Moser(CM) models as integrable many-body models recently have attracted remarkable attention and have been extensively studied. They describe one-dimensional n-particle system with pairwise interaction. Their importance lies in various fields ranging from lattice models in statistics physics[1, 2], the field theory to gauge theory[3, 4]. e.g. to the Seiberg-Witten theory [5] et al. Recently, the Lax pairs for the elliptic CM models in various root system have been given by Olshanetsky et al[6], Bordner et al[7, 8, 9, 10] and D'Hoker et al[11] respectively, while the commutative operators for RS model based on various type Lie algebra given by Komori [12, 13], Diejen[14, 15] and Hasegawa[1, 16]et al. An interesting result is that in Ref. [17], the authors show that for the sl_2 trigonometric RS and CM models exist the same non-dynamical r -matrix structure compared with the usual dynamical ones. On the other hand, similar to Hasegawa's result that A_{N-1} RS model is related to the Z_n Sklyanin algebra, the integrability of CM model can be depicted by sl_N Gaudin algebra[18].

As for the C_n type RS model, commuting difference operators acting on the space of functions on the C_2 type weight space have been constructed by Hasegawa et al in Ref. [16]. Extending that work, the diagonalization of elliptic difference system of that type has been studied by Kikuchi in Ref. [19]. Despite of the fact that the Lax pairs for CM models have

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been proposed for general Lie algebra even for all of the finite reflection groups[10], however, the Lax integrability of RS model are not clear except only for A_{N-1} -type[20, 2, 21, 22, 23, 24] and for C_2 by the authors by straightforward construction[25], i.e. the general Lax pairs for the RS models other than A_{N-1} -type have not yet been obtained.

Extending the work of Ref. [25], the main purpose of the present paper is to provide the Lax pairs for the C_n and BC_n Ruijsenaars-Schneider(RS) models with the trigonometric and rational interaction potentials. The key technique we used is Dirac's method on the system imposed by some constraints. We shall give the explicit forms of Lax pairs for these systems. It is turned out that the C_n and BC_n RS systems can be obtain by Hamiltonian reduction of A_{2n-1} and A_{2n} ones. The characteristic polynomial of the Lax matrixes leads to a complete set of involutive Hamiltonians associated with the root system of C_n and BC_n. In particular, taking their non-relativistic limit, we shall recover the systems of corresponding CM types.

The paper is organized as follows. The basic materials about A_{N-1} RS model are reviewed in Sec. II. We also give a Lax pair associating with Hamiltonian which has a reflection symmetry with respect to the particles in the origin. The main results are showed in Secs. III and IV. In Sec. III, we present the Lax pairs of C_n and BC_n RS models by reducing from that of A_{N-1} RS model. The explicit forms for the Lax pairs are given in Sec. IV. The characteristic polynomials, which gives the complete sets of involutive constant motions for these systems, will also be given there. Sec. V, is devoted to derive the nonrelativistic limits of these systems which coincide with the forms given in Refs. [6] and [7]. The last section is brief summary and some discussions.

II A_{N-1}-type Ruijsenaars-Schneider model

As a relativistic-invariant generalization of the A_{N-1}-type nonrelativistic Calogero-Moser model, the A_{N-1}-type Ruijsenaars-Schneider systems are completely integrable whose integrability are first showed by Ruijsenaars[20, 26]. The Lax pairs for this model have been constructed in Refs. [20, 2, 21, 22, 23, 24]. Recent progress have showed that the compactification of higher dimension SUSY Yang-Mills theory and Seiberg-Witten theory can be described by this model[5]. Instanton correction of prepotential associated with sl_2 RS system have been calculated in Ref. [27].

II.1 The Lax operator for A_{N-1} RS model

Let us briefly give the basics of this model. In terms of the canonical variables p_i, $x_i (i, j = 1, \ldots, N)$ enjoying in the canonical Poisson bracket

$$\{p_i, p_j\} = \{x_i, x_j\} = 0, \qquad \{x_i, p_j\} = \delta_{ij}, \tag{II.1}$$

we give firstly the Hamiltonian of A_{N-1} RS system

$$H_{A_{N-1}}=\sum_{i=1}^{N}\left(e^{p_i}\prod_{k\neq i}f(x_i-x_k)+e^{-p_i}\prod_{k\neq i}g(x_i-x_k)\right). \tag{II.2}$$

Notice that in Ref. [20] Ruijsenaars used another "gauge" of the momenta such that two are connected by the following canonical transformation:

$$x_i\longrightarrow x_i,\qquad p_i\longrightarrow p_i+\frac{1}{2}ln\prod_{j\neq i}^{N}\frac{f(x_{ij})}{g(x_{ij})}. \tag{II.3}$$

The Lax operator for this model has the form(for the trigonometric case)

$$L_{A_{N-1}}=\sum_{i,j=1}^{N}\frac{\sin\gamma}{\sin(x_i-x_j+\gamma)}\,\exp(p_j)\;b_jE_{ij},$$

and for the rational case

$$L_{A_{N-1}}=\sum_{i,j=1}^{N}\frac{\gamma}{x_i-x_j+\gamma}\exp(p_j)\;b_jE_{ij}. \tag{II.4}$$

where

$$\begin{aligned}
b_j\quad &:\;=\prod_{k\neq j}f(x_j-x_k),\qquad b_j^{'}:=\prod_{k\neq j}g(x_j-x_k),\qquad (E_{ij})_{kl}=\delta_{ik}\delta_{jl},\\
f(x)\quad &:\;=\begin{cases}\frac{\sin(x-\gamma)}{\sin(x)}, & \text{trigonometric case,}\\ \frac{x-\gamma}{x}, & \text{rational case,}\end{cases}\\
g(x)\quad &:\;=f(x)|_{\gamma\rightarrow-\gamma},\qquad x_{ik}:=x_i-x_k,
\end{aligned} \tag{II.5}$$

and γ denotes the coupling constant.

It is shown in Ref. [23] that the Lax operator satisfies the quadratic fundamental Poisson bracket

$$\{L_1,L_2\}=L_1\,L_2\,a_1-a_2\,L_1\,L_2+L_2\,s_1\,L_1-L_1\,s_2\,L_2, \tag{II.6}$$

where $L_1=L_{A_{N-1}}\otimes 1,L_2=1\otimes L_{A_{N-1}}$ and the four matrices read as

$$\begin{aligned}
a_1\;&=\;a+w,\quad s_1=s-w,\\
a_2\;&=\;a+s-s^*-w,\quad s_2=s^*+w.
\end{aligned} \tag{II.7}$$

The forms of a,s,w are

$$\begin{aligned}
a\;&=\;\sum_{k\neq j}\cot(x_k-x_j)E_{jk}\otimes E_{kj},\\
s\;&=\;-\sum_{k\neq j}\frac{1}{\sin(x_k-x_j)}E_{jk}\otimes E_{kk},\\
w\;&=\;\sum_{k\neq j}\cot(x_k-x_j)E_{kk}\otimes E_{jj},
\end{aligned} \tag{II.8}$$

for the trigonometric case and

$$
\begin{aligned}
a &= \sum_{k\neq j} \frac{1}{x_k - x_j} E_{jk} \otimes E_{kj}, \\
s &= -\sum_{k\neq j} \frac{1}{x_k - x_j} E_{jk} \otimes E_{kk}, \\
w &= \sum_{k\neq j} \frac{1}{x_k - x_j} E_{kk} \otimes E_{jj},
\end{aligned}
\tag{II.9}
$$

for the rational case. The $*$ symbol means $r^* = \Pi r \Pi$ with $\Pi = \sum_{k,j=1}^{N} E_{kj} \otimes E_{jk}$.

Noticing that

$$
(L^{-1}_{A_{N-1}})_{ij} = \begin{cases} \sum_{i,j=1}^{N} \frac{-\sin\gamma}{\sin(x_i - x_j - \gamma)} \exp(-p_i)\, b_j' E_{ij}, & \text{for trigonometric case,} \\ \sum_{i,j=1}^{N} \frac{-\gamma}{x_i - x_j - \gamma} \exp(-p_i)\, b_j' E_{ij}, & \text{for rational case,} \end{cases}
\tag{II.10}
$$

one can get the characteristic polynomials of $L_{A_{N-1}}$ and $L^{-1}_{A_{N-1}}$[28]

$$
\det(L_{A_{N-1}} - v \cdot Id) = \sum_{j=0}^{N} (-v)^{n-j} (H_j^+)_{A_{N-1}}, \tag{II.11}
$$

$$
\det(L^{-1}_{A_{N-1}} - v \cdot Id) = \sum_{j=0}^{N} (-v)^{n-j} (H_j^-)_{A_{N-1}}, \tag{II.12}
$$

where $(H_0^{\pm})_{A_{N-1}} = (H_N^{\pm})_{A_{N-1}} = 1$ and

$$
(H_i^+)_{A_{N-1}} = \sum_{\substack{J\subset\{1,\ldots,N\} \\ |J|=i}} \exp\left(\sum_{j\in J} p_j\right) \prod_{\substack{j\in J \\ k\in\{1,\ldots,N\}\setminus J}} f(x_j - x_k), \tag{II.13}
$$

$$
(H_i^-)_{A_{N-1}} = \sum_{\substack{J\subset\{1,\ldots,N\} \\ |J|=i}} \exp\left(\sum_{j\in J} -p_j\right) \prod_{\substack{j\in J \\ k\in\{1,\ldots,N\}\setminus J}} g(x_j - x_k). \tag{II.14}
$$

Define

$$
(H_i)_{A_{N-1}} = (H_i^+)_{A_{N-1}} + (H_i^-)_{A_{N-1}}, \tag{II.15}
$$

from the fundamental Poisson bracket Eq.(II.6), we can verify that

$$
\{(H_i)_{A_{N-1}}, (H_j)_{A_{N-1}}\} = \{(H_i^{\varepsilon})_{A_{N-1}}, (H_j^{\varepsilon'})_{A_{N-1}}\} = 0, \qquad \varepsilon, \varepsilon' = \pm, \qquad i,j = 1,\ldots,N. \tag{II.16}
$$

In particular, the Hamiltonian Eq.(II.2) can be rewritten as

$$H_{A_{N-1}} = (H_1^+)_{A_{N-1}} + (H_1^-)_{A_{N-1}} = \sum_{j=1}^{N}(e^{p_j}b_j + e^{-p_j}b_j') = Tr(L_{A_{N-1}} + L_{A_{N-1}}^{-1}). \qquad \text{(II.17)}$$

It should be remarked the set of integrals of motion Eq.(II.15) have a reflection symmetry which is the key property for the later reduction to C_n and BC_n cases. i.e. if we set

$$p_i \longleftrightarrow -p_i, \qquad x_i \longleftrightarrow -x_i, \qquad \text{(II.18)}$$

then the Hamiltonians flows $(H_i)_{A_{N-1}}$ are invariant with respect to this symmetry.

II.2 The construction of Lax pair for the A_{N-1} RS model

As for the A_{N-1} RS model, a generalized Lax pair has been given in Refs. [20, 2, 21, 22, 23, 24]. But there is a common character that the time-evolution of the Lax matrix $L_{A_{N-1}}$ is associated with the Hamiltonian H_+. We will see in the next section that the Lax pair can't reduce from that kind of forms directly. Instead, we give a new Lax pair which the evolution of $L_{A_{N-1}}$ are associated with the Hamiltonian $H_{A_{N-1}}$

$$\dot{L}_{A_{N-1}} = \{L_{A_{N-1}}, H_{A_{N-1}}\} = [M_{A_{N-1}}, L_{A_{N-1}}], \qquad \text{(II.19)}$$

where $M_{A_{N-1}}$ can be constructed with the help of (r,s) matrices as follows

$$M_{A_{N-1}} = Tr_2((s_1 - a_2)(1 \otimes (L_{A_{N-1}} - L_{A_{N-1}}^{-1}))). \qquad \text{(II.20)}$$

The explicit expression of entries for $M_{A_{N-1}}$ is

$$\begin{aligned}
(M_{A_{N-1}})_{ij} &= \frac{\sin\gamma\cot(x_{ij})}{\sin(x_{ij}+\gamma)}e^{p_j}b_j + \frac{\sin\gamma\cot x_{ij}}{\sin(x_{ij}-\gamma)}e^{-p_i}b_j', \qquad i \neq j,\\
(M_{A_{N-1}})_{ii} &= -\sum_{l\neq i}\left(\frac{\sin\gamma}{\sin(x_{il})\sin(x_{il}+\gamma)}e^{p_l}b_l + \frac{\sin\gamma}{\sin(x_{il})\sin(x_{il}-\gamma)}e^{-p_i}b_l'\right),
\end{aligned} \qquad \text{(II.21)}$$

for trigonometric case and

$$\begin{aligned}
(M_{A_{N-1}})_{ij} &= \frac{\gamma}{x_{ij}(x_{ij}+\gamma)}e^{p_j}b_j + \frac{\gamma}{x_{ij}(x_{ij}-\gamma)}e^{-p_i}b_j', \qquad i \neq j,\\
(M_{A_{N-1}})_{ii} &= -\sum_{l\neq i}\left(\frac{\gamma}{x_{il}(x_{il}+\gamma)}e^{p_l}b_l + \frac{\gamma}{x_{il}(x_{il}-\gamma)}e^{-p_i}b_l'\right),
\end{aligned} \qquad \text{(II.22)}$$

for rational case.

III Hamiltonian reductions of C_n and BC_n RS models from A_{N-1}-type ones

Let us first mention some results about the integrability of Hamiltonian (II.2). In Ref. [26] Ruijsenaars demonstrated that the symplectic structure of C_n and BC_n type RS systems can be proved integrable by embedding their phase space to a submanifold of A_{2n-1} and A_{2n} type RS ones respectively, while in Refs. [14, 15] and [13], Diejen and Komori, respectively, gave a series of commuting difference operators which led to their quantum integrability. However, there are not any results about their Lax representations so far, i.e. the explicit forms of the Lax matrixes L, associated with a M (respectively) which ensure their Lax integrability, haven't been proposed up to now except for the special case of C_2[25]. In this section, we concentrate our treatment to the exhibition of the explicit forms for general C_n and BC_n RS systems. Therefore, some previous results, as well as new results, could now be obtained in a more straightforward manner by using the Lax pairs.

For the convenience of analysis of symmetry, let us first give vector representation of A_{N-1} Lie algebra. Introducing an N dimensional orthonormal basis of $\mathbb{R}^N$

$$e_j \cdot e_k = \delta_{j,k}, \quad j,k = 1,\ldots,N. \tag{III.1}$$

Then the sets of roots and vector weights are:

$$\begin{aligned}\Delta &= \{e_j - e_k : \quad j,k = 1,\ldots,N\}, && \text{(III.2)}\\ \Lambda &= \{e_j : \quad j = 1,\ldots,N\}. && \text{(III.3)}\end{aligned}$$

The dynamical variables are canonical coordinates $\{x_j\}$ and their canonical conjugate momenta $\{p_j\}$ with the Poisson brackets of Eq.(II.1) . In general sense, we denote them by N dimensional vectors x and p,

$$x = (x_1,\ldots,x_N) \in \mathbb{R}^N, \quad p = (p_1,\ldots,p_N) \in \mathbb{R}^N,$$

so that the scalar products of x and p with the roots $\alpha \cdot x$, $p \cdot \beta$, etc. can be defined. The Hamiltonian Eq.(II.2) can be rewritten as

$$H_{A_{N-1}} = \sum_{\mu\in\Lambda}\left(\exp(\mu\cdot p)\prod_{\Delta\ni\beta=\mu-\nu} f(\beta\cdot x) + \exp(-\mu\cdot p)\prod_{\Delta\ni\beta=-\mu+\nu} g(\beta\cdot x)\right), \tag{III.4}$$

in which $f(x)$ and $g(x)$ is given in Eq.(II.5) for various choices of potentials. Here, the condition $\Delta \ni \beta = \mu - \nu$ means that the summation is over roots β such that for $\exists \nu \in \Lambda$

$$\mu - \nu = \beta \in \Delta.$$

So does for $\Delta \ni \beta = -\mu + \nu$.

III.1 C_n model

The set of C_n roots consists of two parts, long roots and short roots:

$$\Delta_{C_n} = \Delta_L \cup \Delta_S, \tag{III.5}$$

in which the roots are conveniently expressed in terms of an orthonormal basis of $\mathbb{R}^n$:

$$\begin{aligned} \Delta_L &= \{\pm 2e_j : \quad j = 1, \dots, n\}, \\ \Delta_S &= \{\pm e_j \pm e_k, : \quad j, k = 1, \dots, n\}. \end{aligned} \tag{III.6}$$

In the vector representation, vector weights Λ are

$$\Lambda_{C_n} = \{e_j, -e_j : \quad j = 1, \dots, n\}. \tag{III.7}$$

The Hamiltonian of C_n model is given by

$$H_{C_n} = \frac{1}{2} \sum_{\mu \in \Lambda_{C_n}} \left(\exp(\mu \cdot p) \prod_{\Delta_{C_n} \ni \beta = \mu - \nu} f(\beta \cdot x) + \exp(-\mu \cdot p) \prod_{\Delta_{C_n} \ni \beta = -\mu + \nu} g(\beta \cdot x) \right). \tag{III.8}$$

From the above data, we notice that either for A_{N-1} or C_n Lie algebra, any root $\alpha \in \Delta$ can be constructed in terms with vector weights as $\alpha = \mu - \nu$ where $\mu, \nu \in \Lambda$. By simple comparison of representation between A_{N-1} or C_n, one can found that if replacing e_{j+n} with $-e_j$ in the vector weights of A_{2n-1} algebra, we can obtain the vector weights of C_n one. Also does for the corresponding roots. This hints us it is possible to get the C_n model by this kind of reduction.

For A_{2n-1} model let us set restrictions on the vector weights with

$$e_{j+n} + e_j = 0, \quad \text{for} \quad j = 1, \dots, n, \tag{III.9}$$

which correspond to the following constraints on the phase space of A_{2n-1}-type RS model with

$$\begin{aligned} G_i &\equiv (e_{i+n} + e_i) \cdot x = x_i + x_{i+n} = 0, \\ G_{i+n} &\equiv (e_{i+n} + e_i) \cdot p = p_i + p_{i+n} = 0, \quad i = 1, \dots, n, \end{aligned} \tag{III.10}$$

Following Dirac's method[29], we can show

$$\{G_i, H_{A_{2n-1}}\} \simeq 0, \quad \text{for} \quad \forall i \in \{1, \dots, 2n\}, \tag{III.11}$$

i.e. $H_{A_{2n-1}}$ is the first class Hamiltonian corresponding to the above constraints Eq. (III.10). Here the symbol $\simeq$ represents that, only after calculating the result of left side of the identity, could we use the conditions of constraints. It should be pointed out that the most necessary condition ensuring the Eq. (III.11) is the symmetry property Eq. (II.18) for the Hamiltonian Eq. (II.2). So that for arbitrary dynamical variable A, we have

$$\begin{aligned} \dot{A} &= \{A, H_{A_{2n-1}}\}_D = \{A, H_{A_{2n-1}}\} - \{A, G_i\}\Delta_{ij}^{-1}\{G_j, H_{A_{2n-1}}\} \\ &\simeq \{A, H_{A_{2n-1}}\}, \qquad\qquad i,j = 1,\ldots,2n, \end{aligned} \tag{III.12}$$

where

$$\Delta_{ij} = \{G_i, G_j\} = 2\begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}, \tag{III.13}$$

and the $\{,\}_D$ denote the Dirac bracket. By straightforward calculation, we have the nonzero Dirac brackets of

$$\begin{aligned} \{x_i, p_j\}_D &= \{x_{i+n}, p_{j+n}\}_D = \frac{1}{2}\delta_{i,j}, \\ \{x_i, p_{j+n}\}_D &= \{x_{i+n}, p_j\}_D = -\frac{1}{2}\delta_{i,j}. \end{aligned} \tag{III.14}$$

Using the above data together with the fact that $H_{A_{N-1}}$ is the first class Hamiltonian (see Eq. (III.11), we can directly obtain Lax representation of C_n RS model by imposing constraints G_k on Eq. (II.19)

$$\begin{aligned} \{L_{A_{2n-1}}, H_{A_{2n-1}}\}_D &= \{L_{A_{2n-1}}, H_{A_{2n-1}}\}|_{G_k, k=1,\ldots,2n}, \\ &= [M_{A_{2n-1}}, L_{A_{2n-1}}]|_{G_k, k=1,\ldots,2n} = [M_{C_n}, L_{C_n}], \end{aligned} \tag{III.15}$$

$$\{L_{A_{2n-1}}, H_{A_{2n-1}}\}_D = \{L_{C_n}, H_{C_n}\}, \tag{III.16}$$

where

$$\begin{aligned} H_{C_n} &= \frac{1}{2}H_{A_{2n-1}}|_{G_k, k=1,\ldots,2n}, \\ L_{C_n} &= L_{A_{2n-1}}|_{G_k, k=1,\ldots,2n}, \\ M_{C_n} &= M_{A_{2n-1}}|_{G_k, k=1,\ldots,2n}, \end{aligned} \tag{III.17}$$

so that

$$\dot{L}_{C_n} = \{L_{C_n}, H_{C_n}\} = [M_{C_n}, L_{C_n}]. \tag{III.18}$$

Nevertheless, the H_+ is not the first class Hamiltonian, so the Lax pair given by many authors previously can't reduce to C_n case directly by this way.

III.2 BC_n model

The BC_n root system consists of three parts, long, middle and short roots:

$$\Delta_{BC_n} = \Delta_L \cup \Delta \cup \Delta_S, \tag{III.19}$$

in which the roots are conveniently expressed in terms of an orthonormal basis of $\mathbb{R}^n$:

$$\begin{array}{rcll} \Delta_L &=& \{\pm 2e_j : & j = 1, \dots, n\}, \\ \Delta &=& \{\pm e_j \pm e_k : & j, k = 1, \dots, n\}, \\ \Delta_S &=& \{\pm e_j : & j = 1, \dots, n\}. \end{array} \tag{III.20}$$

In the vector representation, vector weights Λ can be

$$\Lambda_{BC_n} = \{e_j, -e_j, 0 : \quad j = 1, \dots, n\}. \tag{III.21}$$

The Hamiltonian of BC_n model is given by

$$H_{BC_n} = \frac{1}{2} \sum_{\mu \in \Lambda_{BC_n}} \left(\exp(\mu \cdot p) \prod_{\Delta_{BC_n} \ni \beta = \mu - \nu} f(\beta \cdot x) + \exp(-\mu \cdot p) \prod_{\Delta_{BC_n} \ni \beta = -\mu + \nu} g(\beta \cdot x) \right). \tag{III.22}$$

By similar comparison of representations between A_{N-1} or BC_n, one can found that if replacing e_{j+n} with $-e_j$ and $e_{2n+1}$with 0 in the vector weights of A_{2n} Lie algebra, we can obtain the vector weights of BC_n one. Also does for the corresponding roots. So by the same procedure as C_n model, it is expected to get the Lax representation of BC_n model.

For A_{2n} model, we set restrictions on the vector weights with

$$\begin{array}{rcll} e_{j+n} + e_j &=& 0, \quad \text{for} & j = 1, \dots, n, \\ e_{2n+1} &=& 0, & \end{array} \tag{III.23}$$

which correspond to the following constraints on the phase space of A_{2n}-type RS model with

$$\begin{array}{rcl} G_i' &\equiv& (e_{i+n} + e_i) \cdot x = x_i + x_{i+n} = 0, \\ G_{i+n}' &\equiv& (e_{i+n} + e_i) \cdot p = p_i + p_{i+n} = 0, \quad i = 1, \dots, n, \\ G_{2n+1}' &\equiv& e_{2n+1} \cdot x = x_{2n+1} = 0, \\ G_{2n+2}' &\equiv& e_{2n+1} \cdot p = p_{2n+1} = 0. \end{array} \tag{III.24}$$

Similarly, we can show

$$\{G_i, H_{A_{2n}}\} \simeq 0, \quad \text{for} \quad \forall i \in \{1, \dots, 2n+1, 2n+2\}. \tag{III.25}$$

i.e. $H_{A_{2n}}$ is the first class Hamiltonian corresponding to the above constraints Eq. (III.24). So L_{BC_n} and M_{BC_n} can be constructed as follows

$$\begin{aligned} L_{BC_n} &= L_{A_{2n}}|_{G'_k,k=1,\ldots,2n+2}, \\ M_{BC_n} &= M_{A_{2n}}|_{G'_k,k=1,\ldots,2n+2}, \end{aligned} \tag{III.26}$$

while H_{BC_n} is

$$H_{BC_n} = \frac{1}{2} H_{A_{2n}}|_{G_k,k=1,\ldots,2n+2}, \tag{III.27}$$

due to the similar derivation of Eq.(III.12-III.18).

IV Lax representations of C_n and BC_n RS models

IV.1 C_n model

The Hamiltonian of C_n RS system is Eq.(III.8), so the canonical equations of motion are

$$\dot{x}_i = \{x_i, H\} = e^{p_i} b_i - e^{-p_i} b_i^{'}, \tag{IV.1}$$

$$\begin{aligned} \dot{p}_i &= \{p_i, H\} = \sum_{j\neq i}^{n} \{ e^{p_j} b_j (\frac{f^{'}(x_{ji})}{f(x_{ji})} - \frac{f^{'}(x_j + x_i)}{f(x_j + x_i)}) \\ &+ e^{-p_j} b_j^{'} (\frac{g^{'}(x_{ji})}{g(x_{ji})} - \frac{g^{'}(x_j + x_i)}{g(x_j + x_i)}) \} \\ &- e^{p_i} b_i (2\frac{f^{'}(2x_i)}{f(2x_i)} + \sum_{j\neq i}^{n} (\frac{f^{'}(x_{ij})}{f(x_{ij})} + \frac{f^{'}(x_i + x_j)}{f(x_i + x_j)})) \\ &- e^{-p_i} b_i^{'} (2\frac{g^{'}(2x_i)}{g(2x_i)} + \sum_{j\neq i}^{n} (\frac{g^{'}(x_{ij})}{g(x_{ij})} + \frac{g^{'}(x_i + x_j)}{g(x_i + x_j)})), \end{aligned} \tag{IV.2}$$

where

$$\begin{aligned} f^{'}(x) &:= \frac{df(x)}{dx}, \qquad g^{'}(x) := \frac{dg(x)}{dx}, \\ b_i &= f(2x_i) \prod_{k\neq i}^{n} (f(x_i - x_k) f(x_i + x_k)), \\ b_i^{'} &= g(2x_i) \prod_{k\neq i}^{n} (g(x_i - x_k) g(x_i + x_k)). \end{aligned} \tag{IV.3}$$

The Lax matrix for C_n RS model can be written in the following form for the rational case

$$(L_{C_n})_{\mu\nu} = e^{\nu\cdot p} b_\nu \frac{\gamma}{(\mu-\nu)\cdot x+\gamma}, \tag{IV.4}$$

which is a $2n \times 2n$ matrix whose indices are labelled by the vector weights, denoted by $\mu, \nu \in \Lambda_{C_n}$, M_{C_n} can be written as

$$M_{C_n} = D + Y, \tag{IV.5}$$

where

$$\begin{aligned} Y_{\mu\nu} &= e^{\nu\cdot p} b_\nu \frac{\gamma}{((\mu-\nu)\cdot x)((\mu-\nu)\cdot x+\gamma)} \\ &\quad + e^{-\mu\cdot p} b'_\nu \frac{\gamma}{((\mu-\nu)\cdot x)((\mu-\nu)\cdot x-\gamma)}, \end{aligned} \tag{IV.6}$$

$$\begin{aligned} D_{\mu\mu} &= -\sum_{\nu\neq\mu}(e^{\nu\cdot p} b_\nu \frac{\gamma}{((\mu-\nu)\cdot x)((\mu-\nu)\cdot x+\gamma)} \\ &\quad + e^{-\mu\cdot p} b'_\nu \frac{\gamma}{((\mu-\nu)\cdot x)((\mu-\nu)\cdot x-\gamma)}) \\ &= -\sum_{\nu\neq\mu} Y_{\mu\nu}, \end{aligned} \tag{IV.7}$$

and

$$\begin{aligned} b_\mu &= \prod_{\Delta_{C_n}\ni\beta=\mu-\nu} f(\beta\cdot x), \\ b'_\mu &= \prod_{\Delta_{C_n}\ni\beta=\mu-\nu} g(\beta\cdot x). \end{aligned} \tag{IV.8}$$

For the trigonometric case, we have

$$(L_{C_n})_{\mu\nu} = e^{\nu\cdot p} b_\nu \frac{\sin\gamma}{\sin((\mu-\nu)\cdot x+\gamma)}, \tag{IV.9}$$

and

$$M_{C_n} = D + Y, \tag{IV.10}$$

where

$$\begin{aligned} Y_{\mu\nu} &= e^{\nu\cdot p} b_\nu \frac{\sin\gamma\cot((\mu-\nu)\cdot x)}{\sin((\mu-\nu)\cdot x+\gamma)} + e^{-\mu\cdot p} b'_\nu \frac{\sin\gamma\cot((\mu-\nu)\cdot x)}{\sin((\mu-\nu)\cdot x-\gamma)}, \\ D_{\mu\mu} &= -\sum_{\nu\neq\mu}(e^{\nu\cdot p} b_\nu \frac{\sin\gamma}{\sin((\mu-\nu)\cdot x)\sin((\mu-\nu)\cdot x+\gamma)} \\ &\quad + e^{-\mu\cdot p} b'_\nu \frac{\sin\gamma}{\sin((\mu-\nu)\cdot x)\sin((\mu-\nu)\cdot x-\gamma)}) \end{aligned} \tag{IV.11}$$

$$= \quad -\sum_{\nu\neq\mu} \frac{Y_{\mu\nu}}{\cos((\mu-\nu)\cdot x)}, \tag{IV.12}$$

where b_μ, b'_μ take the value as Eq.(IV.8) with the trigonometric forms of $f(x)$ and $g(x)$.

The L_{C_n}, M_{C_n} satisfies the Lax equation

$$\dot{L}_{C_n} = \{L_{C_n}, H_{C_n}\} = [M_{C_n}, L_{C_n}], \tag{IV.13}$$

which equivalent to the equations of motion Eq.(IV.1) and Eq.(IV.2). The Hamiltonian H_{C_n} can be rewritten as the trace of L_{C_n}

$$H_{C_n} = trL_{C_n} = \frac{1}{2}\sum_{\mu\in\Lambda_{C_n}} (e^{\mu\cdot p}b_\mu + e^{-\mu\cdot p}b'_\mu). \tag{IV.14}$$

The characteristic polynomial of the Lax matrix L_{C_n} generates the involutive Hamiltonians

$$\det(L_{C_n} - v\cdot Id) = \sum_{j=0}^{n-1}(-1)^j(v^j + v^{2n-j})(H_j)_{C_n} + (-v)^n\,(H_n)_{C_n}, \tag{IV.15}$$

where $(H_0)_{C_n} = 1$, and $(H_i)_{C_n}$ Poisson commute

$$\{(H_i)_{C_n}, (H_j)_{C_n}\} = 0, \qquad i,j = 1,...,n. \tag{IV.16}$$

This can be deduced by verbose but straightforward calculation to verify that the $(H_i)_{A_{2n-1}}, i = 1,...2n$ is the first class Hamiltonian with respect to the constraints Eq.(III.10), using Eq.(II.16), (III.12) and the first formula of Eq.(III.17).

The explicit form of $(H_l)_{C_n}$ are

$$(H_l)_{C_n} \quad = \sum_{\substack{J\subset\{1,\ldots,n\},\,|J|\le l \\ \varepsilon_j=\pm1,\,j\in J}} \exp(p_{\varepsilon J})\,F_{\varepsilon J;\,J^c}\,U_{J^c,l-|J|}, \qquad l = 1,\ldots,n, \tag{IV.17}$$

with

$$\begin{aligned}
p_{\varepsilon J} &= \sum_{j\in J} \varepsilon_j\,p_j, \\
F_{\varepsilon J;\,K} &= \prod_{\substack{j,j'\in J\\ j<j'}} f^2(\varepsilon_j x_j + \varepsilon_{j'}x_{j'}) \prod_{\substack{j\in J\\ k\in K}} f(\varepsilon_j x_j + x_k) f(\varepsilon_j x_j - x_k) \prod_{j\in J} f(2\varepsilon_j x_j), \\
U_{I,p} &= \sum_{\substack{I'\subset I\\ |I'|=[p/2]}} \prod_{\substack{j\in I'\\ k\in I\backslash I'}} f(x_{jk}) f(x_j + x_k) g(x_{jk}) g(x_j + x_k) \begin{cases} 0, & (p \text{ odd}), \\ 1, & (p \text{ even}). \end{cases}
\end{aligned} \tag{IV.18}$$

Here, $[p/2]$ denotes the integer part of $p/2$. As an example, for C_2 RS model, the independent Hamiltonian flows $(H_1)_{C_2}$ and $(H_2)_{C_2}$ generated by the Lax matrix L_{C_2} are[25]

$$
\begin{aligned}
(H_1)_{C_2} &= H_{C_2} = e^{p_1} f(2x_1)\, f(x_{12}) f(x_1+x_2) \\
&\quad + e^{-p_1} g(2x_1)\, g(x_{12}) g(x_1+x_2) \\
&\quad + e^{p_2} f(2x_2)\, f(x_{21}) f(x_2+x_1) \\
&\quad + e^{-p_2} g(2x_2)\, g(x_{21}) g(x_2+x_1), \qquad \text{(IV.19)} \\
(H_2)_{C_2} &= e^{p_1+p_2} f(2x_1)\, (f(x_1+x_2))^2 f(2x_2) \\
&\quad + e^{-p_1-p_2} g(2x_1)\, (g(x_1+x_2))^2 g(2x_2) \\
&\quad + e^{p_1-p_2} f(2x_1)\, (f(x_{12}))^2 f(-2x_2) \\
&\quad + e^{p_2-p_1} g(2x_1)\, (g(x_{12}))^2 g(-2x_2) \\
&\quad + 2 f(x_{12})\, g(x_{12})\, f(x_1+x_2) g(x_1+x_2). \qquad \text{(IV.20)}
\end{aligned}
$$

IV.2 BC_n model

The Hamiltonian BC_n model is expressed in Eq.(III.22), so the canonical equations of motion are

$$
\begin{aligned}
\dot{x}_i &= \{x_i, H\} = e^{p_i} b_i - e^{-p_i} b_i^{'}, \qquad \text{(IV.21)} \\
\dot{p}_i &= \{p_i, H\} = \sum_{j\neq i}^{n} \{ e^{p_j} b_j (\frac{f^{'}(x_{ji})}{f(x_{ji})} - \frac{f^{'}(x_j+x_i)}{f(x_j+x_i)}) \\
&\quad + e^{-p_j} b_j^{'} (\frac{g^{'}(x_{ji})}{g(x_{ji})} - \frac{g^{'}(x_j+x_i)}{g(x_j+x_i)}) \} \\
&\quad - e^{p_i} b_i (\frac{f^{'}(x_i)}{f(x_i)} + 2\frac{f^{'}(2x_i)}{f(2x_i)} + \sum_{j\neq i}^{n} (\frac{f^{'}(x_{ij})}{f(x_{ij})} + \frac{f^{'}(x_i+x_j)}{f(x_i+x_j)})) \\
&\quad - e^{-p_i} b_i^{'} (\frac{g^{'}(x_i)}{g(x_i)} + 2\frac{g^{'}(2x_i)}{g(2x_i)} + \sum_{j\neq i}^{n} (\frac{g^{'}(x_{ij})}{g(x_{ij})} + \frac{g^{'}(x_i+x_j)}{g(x_i+x_j)})) \\
&\quad - b_0 (\frac{f^{'}(x_i)}{f(x_i)} + \frac{g^{'}(x_i)}{g(x_i)}), \qquad \text{(IV.22)}
\end{aligned}
$$

where

$$
\begin{aligned}
b_i &= f(x_i) f(2x_i) \prod_{k\neq i}^{n} (f(x_i-x_k) f(x_i+x_k)), \\
b_i^{'} &= g(x_i) g(2x_i) \prod_{k\neq i}^{n} (g(x_i-x_k) g(x_i+x_k)),
\end{aligned}
$$

$$b_0 = \prod_{i=1}^{n}(f(x_i)g(x_i)). \tag{IV.23}$$

The Lax pair for BC_n RS model can be constructed as the form of Eq.(IV.4)-(IV.12) where one should replace the matrices labels with $\mu, \nu \in \Lambda_{BC_n}$, and roots with $\beta \in \Delta_{BC_n}$.

The Hamiltonian H_{BC_n} can be rewritten as the trace of L_{BC_n}

$$H_{BC_n} = trL_{BC_n} = \frac{1}{2}\sum_{\mu\in\Lambda_{BC_n}}(e^{\mu\cdot p}b_\mu + e^{-\mu\cdot p}b'_\mu). \tag{IV.24}$$

The characteristic polynomial of the Lax matrix L generates the involutive Hamiltonians

$$\det(L_{BC_n} - v\cdot Id) = \sum_{j=0}^{n}(-1)^j(v^j - v^{2n+1-j})(H_j)_{BC_n}, \tag{IV.25}$$

where $(H_0)_{BC_n} = 1$ and $(H_i)_{BC_n}$ Poisson commute

$$\{(H_i)_{BC_n}, (H_j)_{BC_n}\} = 0, \qquad i, j = 1, ..., n. \tag{IV.26}$$

This can be deduced similarly to C_n case to verify that the $(H_i)_{A_{2n}}, i = 1, ...2n$ is the first class Hamiltonian with respect to the constraints Eq.(III.24).

The explicit form of $(H_l)_{BC_n}$ are

$$(H_l)_{BC_n} = \sum_{\substack{J\subset\{1,\dots,n\},\,|J|\le l\\ \varepsilon_j=\pm1,\,j\in J}} \exp(p_{\varepsilon J})\, F_{\varepsilon J;\,J^c}\, U_{J^c,\,l-|J|}, \qquad l = 1,\dots,n, \tag{IV.27}$$

with

$$\begin{aligned}
p_{\varepsilon J} &= \sum_{j\in J}\varepsilon_j\, p_j,\\
F_{\varepsilon J;K} &= \prod_{\substack{j,j'\in J\\ j<j'}} f^2(\varepsilon_j x_j + \varepsilon_{j'}x_{j'}) \prod_{\substack{j\in J\\ k\in K}} f(\varepsilon_j x_j + x_k) f(\varepsilon_j x_j - x_k) \prod_{j\in J} f(2\varepsilon_j x_j) \prod_{j\in J} f(\varepsilon_j x_j),\\
U_{I,p} &= \sum_{\substack{I'\subset I\\ |I'|=[p/2]}} \prod_{\substack{j\in I'\\ k\in I\setminus I'}} f(x_{jk}) f(x_j + x_k) g(x_{jk}) g(x_j + x_k) \left\{ \begin{array}{ll} \prod_{i\in I\setminus I'} f(x_i)g(x_i), & (p \text{ odd}),\\ \prod_{i'\in I'} f(x_{i'})g(x_{i'}), & (p \text{ even}). \end{array}\right.
\end{aligned} \tag{IV.28}$$

V Nonrelativistic limit to the Calogero-Moser system

V.1 Limit to C_n CM model

The Nonrelativistic limit can be achieved by rescaling $p_i \longmapsto \beta p_i$, $\gamma \longmapsto \beta\gamma$ while letting $\beta \longmapsto 0$, and making a canonical transformation

$$\left\{ \begin{array}{ll} p_i \longmapsto p_i + \gamma\{\frac{1}{2x_i} + \sum_{k\neq i}^{n}(\frac{1}{x_{ik}} + \frac{1}{x_i+x_k})\}, & \text{rational case,} \\ p_i \longmapsto p_i + \gamma\{\cot(2x_i) + \sum_{k\neq i}^{n}(\cot(x_{ik}) + \cot(x_i+x_k))\}, & \text{trigonometric case,} \end{array} \right. \tag{V.1}$$

such that

$$L \longmapsto Id + \beta L_{CM} + O(\beta^2), \tag{V.2}$$

$$M \longmapsto 2\beta M_{CM} + O(\beta^2), \tag{V.3}$$

and

$$H \longmapsto 2n + 2\beta^2 H_{CM} + O(\beta^2). \tag{V.4}$$

L_{CM} can be expressed as

$$L_{CM} = \begin{pmatrix} A_{CM} & B_{CM} \\ -B_{CM} & -A_{CM} \end{pmatrix}, \tag{V.5}$$

where

$$\begin{array}{rcll} (A_{CM})_{ii} & = & p_i, & (B_{CM})_{ij} = \dfrac{\gamma}{x_i+x_j}, \\ (A_{CM})_{ij} & = & \dfrac{\gamma}{x_{ij}}, & (i\neq j), \end{array} \tag{V.6}$$

for the rational case, and

$$\begin{array}{rcll} (A_{CM})_{ii} & = & p_i, & (B_{CM})_{ij} = \dfrac{\gamma}{\sin(x_i+x_j)}, \\ (A_{CM})_{ij} & = & \dfrac{\gamma}{\sin(x_{ij})}, & (i\neq j), \end{array} \tag{V.7}$$

for the trigonometric case.

M_{CM} is

$$M_{CM} = \begin{pmatrix} \mathcal{A}_{CM} & \mathcal{B}_{CM} \\ \mathcal{B}_{CM} & \mathcal{A}_{CM} \end{pmatrix}, \tag{V.8}$$

as for the rational case

$$
\begin{aligned}
(\mathcal{A}_{CM})_{ii} &= -\sum_{k\neq i}^{n}\Big(\frac{\gamma}{x_{ik}^2}+\frac{\gamma}{(x_i+x_k)^2}\Big)-\frac{\gamma}{(2x_i)^2}, \qquad (\mathcal{B}_{CM})_{ij}=\frac{\gamma}{(x_i+x_j)^2},\\
(\mathcal{A}_{CM})_{ij} &= \frac{\gamma}{x_{ij}^2}, \qquad (\ i\neq j),
\end{aligned}
\tag{V.9}
$$

which identified with the results of Refs. [6] and [7],
and for the trigonometric case

$$
\begin{aligned}
(\mathcal{A}_{CM})_{ii} &= -\sum_{k\neq i}^{n}\Big(\frac{\gamma}{\sin^2 x_{ik}}+\frac{\gamma}{\sin^2(x_i+x_k)}\Big)-\frac{\gamma}{\sin^2(2x_i)}, \qquad (\mathcal{B}_{CM})_{ij}=\frac{\gamma\cos(x_i+x_j)}{\sin^2(x_i+x_j)},\\
(\mathcal{A}_{CM})_{ij} &= \frac{\gamma\cos(x_{ij})}{\sin^2 x_{ij}}, \qquad (\ i\neq j),
\end{aligned}
\tag{V.10}
$$

which coincide with the form given in Ref. [6] up to a diagonalized matrix together with a suitable choose of coupling parameters.

The Hamiltonian of C_n-type CM model can be given by

$$
\begin{aligned}
H_{CM} &= \frac{1}{2}\sum_{k=1}^{n}p_k^2-\gamma^2\sum_{k<i}^{n}\Big(\frac{1}{x_{ik}^2}+\frac{1}{(x_i+x_k)^2}\Big)-\frac{\gamma^2}{2}\sum_{i=1}^{n}\frac{1}{(2x_i)^2}\\
&= \frac{1}{4}trL^2, \qquad \text{for the rational case,}
\end{aligned}
\tag{V.11}
$$

$$
\begin{aligned}
H_{CM} &= \frac{1}{2}\sum_{k=1}^{n}p_k^2-\gamma^2\sum_{k\neq i}^{n}\Big(\frac{1}{\sin^2 x_{ik}}+\frac{1}{\sin^2(x_i+x_k)}\Big)-\frac{\gamma^2}{2}\sum_{i=1}^{n}\frac{1}{\sin^2(2x_i)}\\
&= \frac{1}{4}trL^2, \qquad \text{for the trigonometric case.}
\end{aligned}
\tag{V.12}
$$

The L_{CM}, M_{CM} satisfies the Lax equation

$$
\dot{L}_{CM}=\{L_{CM},H_{CM}\}=[M_{CM},L_{CM}]. \tag{V.13}
$$

V.2 Limit to BC_n CM model

The Nonrelativistic limit of BC_n model can also be achieved by rescaling $p_i\longmapsto\beta p_i$, $\gamma\longmapsto\beta\gamma$ while letting $\beta\longmapsto 0$, and making the following canonical transformation

$$
\begin{cases}
p_i\longmapsto p_i+\gamma\{\frac{1}{x_i}+\frac{1}{2x_i}+\sum_{k\neq i}^{n}(\frac{1}{x_{ik}}+\frac{1}{x_i+x_k})\}, & \text{rational case,}\\
p_i\longmapsto p_i+\gamma\{\cot(x_i)+\cot(2x_i)+\sum_{k\neq i}^{n}(\cot(x_{ik})+\cot(x_i+x_k))\}, & \text{trigonometric case,}
\end{cases}
\tag{V.14}
$$

such that

$$\begin{aligned} L &\longmapsto Id + \beta L_{CM} + O(\beta^2), \quad &\text{(V.15)} \\ M &\longmapsto 2\beta M_{CM} + O(\beta^2), \quad &\text{(V.16)} \end{aligned}$$

and

$$H \longmapsto (2n+1) + 2\beta^2 H_{CM} + O(\beta^2). \tag{V.17}$$

L_{CM} can be expressed as

$$L_{CM} = \begin{pmatrix} A_{CM} & B_{CM} & E_{CM} \\ -B_{CM} & -A_{CM} & -E_{CM} \\ -(E_{CM})^t & (E_{CM})^t & G_{CM} \end{pmatrix}, \tag{V.18}$$

where

$$\begin{aligned} (A_{CM})_{ii} &= p_i, \qquad (B_{CM})_{ij} = \frac{\gamma}{x_i + x_j}, \qquad (E_{CM})_{i1} = \frac{1}{x_i}, \qquad G_{CM} = 0, \\ (A_{CM})_{ij} &= \frac{\gamma}{x_{ij}}, \qquad (i \neq j), \end{aligned} \tag{V.19}$$

for the rational case, and

$$\begin{aligned} (A_{CM})_{ii} &= p_i, \qquad (B_{CM})_{ij} = \frac{\gamma}{\sin(x_i + x_j)}, \qquad (E_{CM})_{i1} = \frac{1}{\sin x_i}, \qquad G_{CM} = 0, \\ (A_{CM})_{ij} &= \frac{\gamma}{\sin(x_{ij})}, \qquad (i \neq j), \end{aligned} \tag{V.20}$$

for the trigonometric case.

M_{CM} is

$$M_{CM} = \begin{pmatrix} \mathcal{A}_{CM} & \mathcal{B}_{CM} & \mathcal{E}_{CM} \\ \mathcal{B}_{CM} & \mathcal{A}_{CM} & \mathcal{E}_{CM} \\ (\mathcal{E}_{CM})^t & (\mathcal{E}_{CM})^t & \mathcal{G}_{CM} \end{pmatrix}, \tag{V.21}$$

where t denotes the transposition. As for the rational case, the forms of $\mathcal{A}, \mathcal{B}, \mathcal{E}, \mathcal{G}$ are

$$\begin{aligned} (\mathcal{A}_{CM})_{ii} &= -\sum_{k \neq i}^{n} \left(\frac{\gamma}{x_{ik}^2} + \frac{\gamma}{(x_i + x_k)^2}\right) - \frac{\gamma}{(2x_i)^2} - \frac{\gamma}{(x_i)^2}, \\ (\mathcal{B}_{CM})_{ij} &= \frac{\gamma}{(x_i + x_j)^2}, \qquad (\mathcal{E}_{CM})_{i1} = \frac{\gamma}{(x_i)^2}, \\ (\mathcal{A}_{CM})_{ij} &= \frac{\gamma}{x_{ij}^2}, \qquad (i \neq j), \qquad \mathcal{G}_{CM} = -\sum_{k=1}^{n} \frac{2\gamma}{x_k^2}, \end{aligned} \tag{V.22}$$

which identified with the results of Refs. [6] and [7],
and for the trigonometric case

$$\begin{aligned}
(\mathcal{A}_{CM})_{ii} &= -\sum_{k\neq i}^{n}(\frac{\gamma}{\sin^2 x_{ik}}+\frac{\gamma}{\sin^2(x_i+x_k)})-\frac{\gamma}{\sin^2(2x_i)}-\frac{\gamma}{\sin^2(x_i)},\\
(\mathcal{B}_{CM})_{ij} &= \frac{\gamma\cos(x_i+x_j)}{\sin^2(x_i+x_j)}, \qquad (\mathcal{E}_{CM})_{i1}=\frac{\gamma\cos x_i}{\sin^2 x_i},\\
(\mathcal{A}_{CM})_{ij} &= \frac{\gamma\cos(x_{ij})}{\sin^2 x_{ij}}, \quad (\, i\neq j), \quad \mathcal{G}_{CM}=-\sum_{k=1}^{n}\frac{2\gamma}{\sin^2 x_k},
\end{aligned} \tag{V.23}$$

which coincide with the form given in Ref. [6] up to a diagonalized matrix together with a suitable choose of coupling parameters.

The Hamiltonian of BC_n-type CM model can be given by

$$\begin{aligned}
H_{CM} &= \frac{1}{2}\sum_{k=1}^{n}p_k^2-\gamma^2\sum_{k<i}^{n}(\frac{1}{x_{ik}^2}+\frac{1}{(x_i+x_k)^2})-\frac{\gamma^2}{2}\sum_{i=1}^{n}(\frac{1}{(2x_i)^2}+\frac{2}{(x_i)^2})\\
&= \frac{1}{4}trL^2, \qquad \text{for the rational case,}
\end{aligned} \tag{V.24}$$

$$\begin{aligned}
H_{CM} &= \frac{1}{2}\sum_{k=1}^{n}p_k^2-\gamma^2\sum_{k<i}^{n}(\frac{1}{\sin^2 x_{ik}}+\frac{1}{\sin^2(x_i+x_k)})-\frac{\gamma^2}{2}\sum_{i=1}^{n}(\frac{1}{\sin^2(2x_i)}+\frac{2}{\sin^2(x_i)})\\
&= \frac{1}{4}trL^2, \qquad \text{for the trigonometric case.}
\end{aligned} \tag{V.25}$$

The L_{CM}, M_{CM} satisfies the Lax equation

$$\dot{L}_{CM}=\{L_{CM},H_{CM}\}=[M_{CM},L_{CM}]. \tag{V.26}$$

VI Summary and discussions

In this paper, we have proposed the Lax pairs for rational, trigonometric C_n and BC_n RS models. Involutive Hamiltonians are showed to be generated by the characteristic polynomial of the corresponding Lax matrix. In the nonrelativistic limit, the system leads to CM systems associated with the root systems of C_n and BC_n which are known previously. There are still many open problems, for example, it seems to be a challenging subject to carry out the Lax pairs with as many independent coupling constants as independent Weyl orbits in the set of roots, as done for the Calogero-Moser systems[6, 7, 8, 9, 10, 11]. What is also interesting may generalize the results obtained in this paper to the systems associated with all of other Lie Algebras even to those associated with all the finite reflection groups[10] which including models based on the non-crystallographic root systems and those based on crystallographic root systems.

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