Spin zero particle propagator from a random walk in 3-D space

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Abstract

The propagator of a spin zero particle in coordinate space is derived supposing that the particle propagates rectilinearly always at the speed of light and changes its direction in some random points due to a scattering process. The average path between two scatterings is of the order of the Compton length.

1 Introduction

Stochastic problems in physics were extensively studied and this approach has shown its great efficiency since a long time (see, for instance, [1]). The attempts to understand, from stochastical point of view, the quantum behaviour, have a long history (see, e.g., [2, 3]).

The derivation of the electron propagator in a 1-D space from a random walk assumption was carried out in the paper [4], where the motivation of this derivation is explained in detail. It was supposed that electron moves with the speed of light and randomly changes its direction of motion. The average distance between the changes of direction was shown to be exactly equal to the electron Compton length. In a different but, to some extent, analogous approach, the electron propagator was derived in [5].

In the present work we carry out the same calculations in a 3-D space to see what assumptions have to be made to get the correct propagator of a massive zero spin particle.

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2 The propagator derivation

Like in the previous paper [4], we suppose that the particle always moves in straight line at the speed of light. From time to time it encounters on its way something in the space and isotropically scatters with the amplitude f. The process goes on until, at the time t, the particle arrives in the space point $\vec{r}$. Integrating over n scattering points and taking the sum over n, we reproduce the spin zero particle propagator.

We use the term "scattering" for convenience. Indeed, we do not localize the particle between the different scatterings. Therefore one could equally give another interpretation, like in the paper [6], assuming the re-creation of the particle at the random time moments t_i, in the points separated from each other by the distances $r_i = ct_i$. Scatterings on some quantum vacuum fluctuations like in [3] could also be invoked. These issues are beyond the scope of the present paper.

At present, we take our assumptions as postulates. It is instructive to see, how the scattering amplitude looks in order to reproduce the propagator. Namely, we suppose that after i-th scattering the wave function has the form:

$$\psi(r_i) = \frac{f}{4\pi\ell_0^2 r_i}, \tag{1}$$

where r_i is the distance from the i-th scattering center and $\ell_0 = \hbar/mc$ is the Compton length which will be our unit of length throughout.

The factor $1/(4\pi r_i)$ in (1) gives the dependence on the distance from the scattering center. It corresponds to standard decreasing of a spherical wave. The factor $1/\ell_0^2$ is introduced to make the amplitude f dimensionless. After each scattering the amplitude is multiplied by $\psi(r_i)$ and integrated over d^3r_i, so with the factor $1/\ell_0^2$ in (1) the product $\psi(r_i)d^3r_i$ is dimensionless too.

We suppose also that $f = -1$, i.e. that our scattering process mimics the resonance S-wave scattering with $\delta = \pi/2$.

Let us calculate the amplitude I_n over all the possible paths consisting of straight intervals with $n-1$ changes of direction. We denote by $\vec{r}$ the distance between the extremal points in the propagator and t is the necessary time to the particle to pass along that distance.

As mentioned, with our assumptions we have $r_i = ct_i$, the path covered between two changes of direction, t_i the corresponding time interval. We will take $c = 1$ all along and so the Compton length will be $\ell_0 = \hbar/m$ in this system of units.

The amplitude with $n-1$ changes of direction is represented in the form:

$$\begin{aligned} I_n &= (-1)^{n-1}\int \delta^3(\vec{r}_1 + \vec{r}_2 + \cdots + \vec{r}_n - \vec{r}) \\ &\quad \times \delta(r_1 + r_2 + \cdots + r_n - t)\,\frac{1}{4\pi r_1} d^3r_1 \psi(r_2)\, d^3r_2 \psi(r_3)\cdots\psi(r_n) d^3r_n \end{aligned} \tag{2}$$

The function $\delta(r_1 + r_2 + \cdots + r_n - t)$ takes into account the condition that the sum of times $t_i = r_i/c$ gives the total time t. It results from the function $\delta(t_1 + t_2 + \cdots + t_n - t)$ where, for $c = 1$, we can change t_i into r_i.

The first scattering occurs in the point $\vec{r}_1$. The particle arrives in this point without any scattering. It is reflected by the factor $\dfrac{1}{4\pi r_1}$ which does not contain the scattering amplitude f. After the first scattering the particle is described by the wave function $\psi(r_2)$, etc.

The integration in the equation (2) is carried out over the points $\vec{r}_1, \vec{r}_2, \ldots$ and we can say nothing about the particle on the path between these points.

It turns out that we only have to calculate I_2, the amplitude with one change of direction, afterwards all the other amplitudes will be deduced by iteration. We give the details below. First we start with:

$$I_2 = (-1) \int \delta^3(\vec{r}_1 + \vec{r}_2 - \vec{r})\ \delta(r_1 + r_2 - t)\ \frac{d^3 r_1}{4\pi\, r_1}\ \frac{d^3 r_2}{4\pi\, r_2\, {\ell_0}^2} \tag{3}$$

After integration over $d^3 r_1$, we get:

$$I_2 \ = \frac{-1}{(4\pi\, \ell_0)^2} \int \delta(r_1 + r_2 - t)\ \frac{d^3 r_2}{r_1 r_2}$$

with $r_1 = |\vec{r} - \vec{r}_2| = (r^2 + r_2^2 - 2 r r_2 \cos\theta_2)^{1/2}$.

This integral is rewritten as:

$$I_2 = \frac{-1}{8\pi\, {\ell_0}^2} \int \delta\left(r_1 + r_2 - t\right) \frac{r_2\, dr_2 dz_2}{r_1} = \frac{-1}{8\pi\, {\ell_0}^2} \int\limits_0^\infty r_2 dr_2 \int\limits_{-1}^{1} \delta\left(f(z_2)\right) \frac{dz_2}{r_1}$$

where we put $z_2 = \cos\theta_2$ and $f(z_2) = r_2 - t + r_1$; consequently

$$\frac{df(z_2)}{dz_2} \ = -\frac{r\, r_2}{r_1}.$$

The value z_0 for which $f(z_0) = 0$ reads:

$$z_0 \ = \frac{r^2 + 2tr_2 - t^2}{2\, r\, r_2},$$

that gives $r_1 = t - r_2$. The condition $-1 \leq z_0 \leq 1$ restricts the domain of r_2: $t - r < r_2 < t + r$. Integrating over dz_2, we get:

$$I_2 = -\frac{1}{8\pi\, {\ell_0}^2}\, \frac{1}{r} \int\limits_{t-r}^{t+r} dr_2.$$

In this way we obtain the following result:

$$I_2 \ = -\frac{1}{8\pi\, \ell_0^2} \tag{4}$$

We can now look at the I_3 integral. It is given by :

$$I_3 = (-1)^2 \int \delta^3(\vec{r}_1 + \vec{r}_2 - (\vec{r} - \vec{r}_3))\, \delta(r_1 + r_2 - (t - r_3))\, \frac{d^3 r_1}{4\pi\, r_1}\, \frac{d^3 r_2}{4\pi\, r_2\, \ell_0{}^2}\, \frac{d^3 r_3}{4\pi\, r_3\, \ell_0{}^2}$$

So we can write this integral in a simpler form by using the former I_2 result,

$$I_3 = -I_2 \int \frac{d^3 r_3}{4\pi\, r_3\, \ell_0{}^2} = -I_2 \frac{1}{2\,\ell_0^2} \int r_3\, dr_3 dz_3$$

with $z_3 = \cos\theta_3 = \frac{\vec{r}\cdot\vec{r}_3}{r r_3}$. We introduce the variable:

$$(\tau')^2 = (t - r_3)^2 - (\vec{r} - \vec{r}_3)^2 = \tau^2 - 2r_3(t - r z_3)$$

with $\tau^2 = t^2 - r^2$. Then we can replace the variable r_3 by τ'^2 using the formula: $r_3 = \frac{(\tau^2 - \tau'^2)}{2(t - r z_3)}$ which gives $dr_3 = -\frac{d\tau'^2}{2(t - r z_3)}$. The integral I_3 is reduced to:

$$I_3 \;\; = -I_2 \frac{1}{2\ell_0^2} \int_0^{\tau^2} \mathrm{d}\tau'^2 \int_{-1}^{+1} dz_3 \frac{(\tau^2 - \tau'^2)}{4\,(t - r z_3)^2}$$

Taking into account that:

$$\int_{-1}^{+1} \frac{\mathrm{d}z}{(t - r z)^2} = \frac{2}{\tau^2}$$

we represent the I_3 integral in the form:

$$I_3(\tau) = -I_2 \frac{1}{\ell_0^2} \int_0^{\tau^2} \frac{(\tau^2 - \tau'^2)}{4\tau^2} \mathrm{d}\tau'^2$$

Where I_2 has the constant value given in the equation (4), whereas I_3 depends on τ.

The method can be similarly generalized to the integral of order n and gives:

$$I_n(\tau) \;\; = -\frac{1}{\ell_0^2} \int_0^{\tau} I_{n-1}(\tau') \frac{(\tau^2 - \tau'^2)}{2\tau^2} \tau' \mathrm{d}\tau' \tag{5}$$

with as usual $\tau^2 = t^2 - r^2$ and $\tau'^2 = (t - r_n)^2 - (\vec{r} - \vec{r}_n)^2$, without forgetting the first integral:

$$I_1 \;\; = \int \delta^3(\vec{r}_1 - \vec{r})\; \delta\,(r_1 - t)\, \frac{d^3 r_1}{4\pi\, r_1} = \frac{1}{4\pi r} \delta\,(r - t) = \frac{1}{2\pi} \delta\left(\tau^2\right) \tag{6}$$

This integral also contains the factor $1/(4\pi r_1)$ responsible for a decrease of the outgoing wave. But since the first scattering contributes into I_2, the integral I_1 does not contain the scattering amplitude f.

The propagator of the particle will be obtained in summing all the amplitudes corresponding to all the possible numbers of changes of direction, i.e.

$$K(\vec{r},t) = I_1 + I_2 + I_3 + \cdots + I_n + \cdots$$

It is possible to do exactly this summation, the result being:

$$K(\vec{r},t) = \frac{1}{2\pi}\delta\left(\tau^2\right) - \frac{1}{8\pi{\ell_0}^2}\sum_{n=0}^{+\infty}\frac{(-1)^n\left(\tau^2/{\ell_0}^2\right)^n}{(2^n)^2(n+1)(n!)^2}$$

Finally we obtain the propagator in a compact form using J_1 the Bessel function of first kind:

$$K(\vec{r},t) = \frac{1}{2\pi}\delta\left(\tau^2\right) - \frac{1}{4\pi{\ell_0}^2}\frac{\ell_0}{\tau}J_1\left(\frac{\tau}{\ell_0}\right) \tag{7}$$

It coincides with the well-known spin zero particle propagator [7].

One can also obtain this result without explicit calculation of integrals for I_n, but solving the integral equation. Indeed, if we define:

$$S(\tau) = I_2 + I_3(\tau) + I_4(\tau) + \cdots,$$

then using the formula (5), we get:

$$S(\tau) = I_2 - \frac{1}{\ell_0^2}\int_0^\tau \left[I_2 + I_3(\tau') + \cdots\right]\frac{(\tau^2 - \tau'^2)}{2\tau^2}\tau'\mathrm{d}\tau',$$

that is:

$$S(\tau) = -\frac{1}{8\pi\,\ell_0^2} - \frac{1}{\ell_0^2}\int_0^\tau S(\tau')\frac{(\tau^2 - \tau'^2)}{2\tau^2}\tau'\mathrm{d}\tau'. \tag{8}$$

The solution of this equation (which can be transformed into a differential equation) is:

$$S(\tau) = \frac{-1}{4\pi{\ell_0}^2}\frac{\ell_0}{\tau}J_1(\frac{\tau}{\ell_0})$$

which is the same result as in the propagator (7).

3 Mean distance between scatterings

Let us find now the average distance covered by the particle between two scatterings. We repeat for the three dimensions case the same calculation done in [4] for the one-dimensional case. Namely, we will calculate the number of paths $n_k(t)$ with k events during the time t. The total number of paths is:

$$n(t) = \sum_k n_k(t).$$

Then we find the propability to have k events for the time t:

$$w_k(t) = \frac{n_k(t)}{n(t)}.$$

The average number of events for the time t is

$$\bar{k}(t) = \sum_k k w_k(t). \quad (9)$$

The time t divided by the number of events for the time t, that is, the ratio $\bar{t} = t/\bar{k}(t)$ for $t \to \infty$ gives the average time between two events. Since the speed between scatterings is always c, in this way we will find the average path between events $\bar{r} = c\bar{t}$.

At first we find the number of paths $n_k(t)$ with k events for the time t:

$$n_k(t) = \int \delta(t_1 + t_2 + \ldots + t_k + t_{k+1} - t) \frac{d^3 r_1}{\ell_0^3} \ldots \frac{d^3 r_k}{\ell_0^3} \frac{d^3 r_{k+1}}{\ell_0^3} \quad (10)$$

where $t_i = r_i/c$. There are $k+1$ intervals and k scatterings in the points where the intervals are linked with each other. Therefore n_k is determined by $k+1$ integrations. The dimensionless integration measure is $\frac{d^3 r_i}{\ell_0^3}$. Below we put $\ell_0 = 1$ and restore it in the final formulas.

For $k = 0$:

$$n_0(t) = \int \delta(r_1 - t)\, d^3 r_1 = 4\pi \int_0^\infty \delta(r_1 - t) r_1^2 dr_1 = 4\pi t^2 \quad (11)$$

For $k = 1$:

$$\begin{aligned} n_1(t) &= \int \delta(r_1 + r_2 - t)\, d^3 r_1 d^3 r_2 = (4\pi)^2 \int_0^\infty \int_0^\infty \delta(r_1 + r_2 - t) r_1^2 r_2^2 dr_1 dr_2 \\ &= (4\pi)^2 \int_0^t r_1^2 (t - r_1)^2 dr_1 = (4\pi)^2 \frac{t^5}{30} \end{aligned} \quad (12)$$

Note that the function $\delta(r_1+r_2-t)$ does not depend on angles, therefore the angle integrations give simply $(4\pi)^2$.

One can easily find the following recurrence formula:

$$n_k(t) = 4\pi \int_0^t n_{k-1}(t - r_k) r_k^2 dr_k$$

which is represented in the form:

$$n_k(t) = 4\pi \int_0^t n_{k-1}(t')(t - t')^2 dt' \quad (13)$$

From the above equations we can see that $n_k(t)$ has the form:

$$n_k(t) = c_k t^{2+3k} \tag{14}$$

From the recurrence relation (13) one can derive the integral equation for the sum $n(t)$ it gives:

$$\begin{aligned} n(t) &= n_0(t) + n_1(t) + n_2(t) + \ldots \\ &= n_0(t) + 4\pi \int_0^t n_0(t')(t-t')^2 dt' + 4\pi \int_0^t n_1(t')(t-t')^2 dt' + \ldots \\ &= n_0(t) + 4\pi \int_0^t [n_0(t') + n_1(t') + n_2(t') + \ldots](t-t')^2 dt' \end{aligned}$$

Or:

$$n(t) = 4\pi t^2 + 4\pi \int_0^t n(t')(t-t')^2 dt' \tag{15}$$

Calculating the third derivative over t of $n(t)$ defined by eq. (15), we get the equivalent differential equation:

$$n'''(t) = 8\pi n(t) \tag{16}$$

with the initial conditions:

$$n(0) = 0, \quad n'(0) = 0, \quad n''(0) = 8\pi. \tag{17}$$

The solution of eq. (16) satisfying the initial conditions (17) has the form:

$$n(t) = \frac{2\pi^{1/3}}{3} \exp(2\pi^{1/3} t) - \frac{2\pi^{1/3}}{3} \exp(-\pi^{1/3} t) \left[\cos(\sqrt{3}\pi^{1/3} t) + \sqrt{3} \sin(\sqrt{3}\pi^{1/3} t) \right], \tag{18}$$

that can be checked by the direct substitution.

The decomposition of (18) in the Taylor series reads:

$$n(t) = 4\pi t^2 + (4\pi)^2 \frac{t^5}{30} + (4\pi)^3 \frac{t^8}{5040} + \ldots$$

that indeed coincides with the sum $n(t) = n_0(t) + n_1(t) + n_2(t) + \ldots$.

Now we calculate $k(t) = \sum_k k n_k(t)$. Namely, according to eq. (14), $n(t)$ is given by the sum:

$$n(t) = \sum_{k=0}^{\infty} c_k t^{2+3k}$$

and similarly for $k(t)$:

$$k(t) = \sum_{k=0}^{\infty} k n_k(t) = \sum_{k=0}^{\infty} k c_k t^{2+3k}.$$

Calculating the first derivative of $n(t)$ over t, we get:

$$n'(t) = \sum_{k=0}^{\infty}(2+3k)c_k t^{2+3k-1} = \frac{2}{t}\sum_{k=0}^{\infty} c_k t^{2+3k} + \frac{3}{t}\sum_{k=0}^{\infty} kc_k t^{2+3k} = \frac{2}{t}n(t) + \frac{3}{t}k(t).$$

That is:

$$k(t) = \frac{1}{3}\left[tn'(t) - 2n(t)\right] \tag{19}$$

From (9) and (19) we find $\bar{k}(t)$:

$$\bar{k}(t) = \sum_k kw_k(t) = \frac{\sum_k kn_k(t)}{n(t)} = \frac{k(t)}{n(t)} = \frac{1}{3}\left[\frac{tn'(t)}{n(t)} - 2\right]. \tag{20}$$

Substituting here, for $t \to \infty$, the leading term of the expression (18) for $n(t)$:

$$n(t)|_{t\to\infty} \approx \frac{2\pi^{1/3}}{3}\exp(2\pi^{1/3}t)$$

we get:

$$\bar{k}(t)|_{t\to\infty} = \frac{2\pi^{1/3}}{3}t.$$

For large t, the average number of events linearly increases with t, like for the Poisson distribution.

The average time between events is

$$\bar{t} = \frac{t}{\bar{k}(t)} = \frac{3}{2\pi^{1/3}}.$$

From here we obtain the average distance between two events $\bar{r} = c\bar{t}$:

$$\bar{r} = \frac{3}{2\pi^{1/3}}\ell_0. \tag{21}$$

Numerically it gives:

$$\bar{r} = 1.02418\,\ell_0,$$

a value close to the Compton length.

4 Conclusions

In conclusion, the following remarks seem relevant.

It turned out that we had to suppose that the scattering wave v.s. the distance r from the scattering center be decreased as $1/r$, eq. (1). It looks like the diffusion of a spherical wave which seems natural, but, at first glance, it has no any hint to relativity. However, if, instead of $1/r$, we would take another function of r, we would not reproduce the propagator, but also would loose the relativistic invariance. In this case we would obtain an expression depending on t and r taken separately, not in the invariant combination $\tau^2 = t^2 - r^2$. This “no choice” seems unexpected and intriguing.

On the other hand, in the stochastic motion of our particle, at each change of its direction, we supposed that its scattering mimics resonance S-wave scattering with an amplitude $f = -1$ and a $\pi/2$ phase shift. One can say that the "scatterings" which result in the propagator take place at very small distances, so that higher partial waves are suppressed.

Moreover it is interesting to compare the 1-D case with the 3-D one. In the 1-D case the average path between events was found to be exactly the Compton length ℓ_0, whereas in this 3-D study it gets $\frac{3}{2\pi^{1/3}}\ell_0 \approx 1.02418\,\ell_0$. This gives an example, how a fundamental characteristic of elementary particle can be calculated and expressed through simple numbers, like $2, 3, \pi, \ldots$. Other examples can be found in [8].

We have clearly shown that the model is very constrained by our assumptions. We think that obtaining the same propagator for the particle as in standard quantum fields theory is not a pure accident. For sure, one should try to find out physical reasons resulting in the scatterings of the propagating particle on its path. As already said, some hypothesis have been advanced in many papers but it was, at present, beyond the scope of this work. It would be also interesting to see how to introduce the spin property in our 3-D model .

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