First order phase transition and corrections to its parameters in the O(N) - model

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Abstract

The temperature phase transition in the N-component scalar field theory with spontaneous symmetry breaking is investigated using the method combining the second Legendre transform and with the consideration of gap equations in the extrema of the free energy. After resummation of all super daisy graphs an effective expansion parameter, $(1/2N)^{1/3}$, appears near T_c for large N. The perturbation theory in this parameter accounting consistently for the graphs beyond the super daisies is developed. A certain class of such diagrams dominant in $1/N$ is calculated perturbatively. Corrections to the characteristics of the phase transition due to these contributions are obtained and turn out to be next-to-leading order as compared to the values derived on the super daisy level and do not alter the type of the phase transition which is weakly first-order. In the limit N goes to infinity the phase transition becomes second order. A comparison with other approaches is done.

1. Introduction

Investigations of the temperature phase transition in the N-component scalar field theory ($O(N)$-model) have a long history and were carried out by either perturbative or non perturbative methods. This model enters as an important

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part unified field theories and serves to supply masses to fermion and gauge fields via the mechanism of the spontaneous symmetry breaking. A general believe about the type of the symmetry restoration phase transition is that it is of second order for any values of N (see, for instance, the text books [1]-[3]). This conclusion results mainly from non perturbative analytic and numeric methods [4]-[8]. In opposite, a first order phase transition was observed in most perturbative approaches [9]-[14]. An analysis of the sources of this discrepancy was done in different places, in particular, in our previous papers [12]-[14] devoted to the investigation of the phase transition in the $O(N)$-model in perturbation theory (PT). Therein a new method has been developed which combines the second Legendre transform with consideration of the gap equations in the extrema of the free energy functional. This allows for considerable simplification of calculations and for analytic results. The main of them is the discovery in the so-called super daisy approximation (SDA) of an effective expansion parameter $\epsilon = \frac{1}{N^{1/3}}$ near the phase transition temperature T_c. All quantities (the particle masses, the temperatures T_+, T_-) are expandable in this parameter. The phase transition was found to be weakly first order converting into a second order one in the limit $N \to \infty$. The existence of this small parameter improves the status of the resummed perturbative approach making it as reliable as any perturbative calculation in quantum field theory. For comparison we note that formerly there had been two problems with the perturbative calculations near the phase transition. First, unwanted imaginary parts had been observed. Using functional methods we showed in [12, 14] that they disappear after re-summing the super daisy graphs. The second problem was that near T_c the masses become small (although not zero) compensating the smallness of the coupling constant hence making the effective expansion parameter of order one.

In the present paper we construct the PT in the effective expansion parameter $\epsilon = \frac{1}{N^{1/3}}$ near T_c for the $O(N)$-model at large N. It uses as input parameters the values obtained in the SDA. As an application we calculate corrections to the characteristics of the phase transition near T_c which follow from taking into account all BSDA graphs in order $\frac{1}{N}$. Since the masses of particles calculated in the SDA are temperature dependent, we consider in detail the renormalization at finite temperature of the graphs investigated. It will be shown that the counter terms renormalizing these graphs at zero temperature are sufficient to carry out renormalizations at finite temperature.

The paper is organized as follows. In sect.2 we adduce the necessary information on the second Legendre transform and formulate the BSDA PT at temperatures $T \sim T_c$. In the next section we calculate the contribution to the free energy of the graphs called the "bubble chains" having a next-next-to-leading order in $\frac{1}{N}$. In sect.4 the renormalization is discussed. The corrections to the masses and other parameters near T_c are calculated in sect.5. The last section is devoted to discussion.

2. Perturbation theory beyond super daisy approximation

In this section we develop the PT in the effective expansion parameter $\epsilon = \frac{1}{N^{1/3}}$ for the graphs BSDA in the frameworks of the second Legendre transform. Consideration of this problem is quite general and independent of the specific form of the Lagrangian. So, it will be carried out in condensed notations of Refs. [12]-[14].

The second Legendre transform is introduced by representing the connected Green functions in the form

$$W = S[0] + \frac{1}{2}Trlog\beta - \frac{1}{2}\Delta^{-1}\beta + W_2, \tag{1}$$

where $S[0]$ is the tree level action, β is the full propagator of the scalar N-component field, Δ is the free field propagator, W_2 is the contribution of two particle irreducible (2PI) graphs taken out of the connected Green functions and having the functions $\beta(p)$ on the lines. The symbol "Tr" means summation over discrete Matsubara frequencies and integration over a three momentum (see for more details Refs. [12]-[14]).

The propagator is related to the mass operator by the Schwinger-Dyson equation

$$\beta^{-1}(p) = \Delta^{-1}(p) - \Sigma(p), \tag{2}$$

$$\Sigma(p) = 2\frac{\delta W_2}{\delta\beta(p)}. \tag{3}$$

The general expressions (1) and (2) will be the starting point for the construction of the BSDA PT. Calculations in SDA have been carried out already in [13]-[14] and delivered the masses of the fields and the characteristics of the phase transition: T_c - transition temperature, and T_+, T_- - upper and lower spinodal temperatures. These parameters will be used in the new PT as the input parameters (zeroth approximation). Contributions of BSDA diagrams will be calculated perturbatively.

First let us write the propagator in the form

$$\beta(p) = \beta_0(p) + \beta^{'}(p), \tag{4}$$

where β_0 is derived in the SDA and β' is a correction which has to be calculated in the BSDA PT in the small parameter $\epsilon = \frac{1}{N^{1/3}}$ for large N. The 2PI part can be presented as

$$W_2 = W_{SD} + W_2' = W_{SD}[\beta_0 + \beta^{'}] + W_2'[\beta_0 + \beta^{'}] \tag{5}$$

and assuming β' to be small of order ϵ value we write

$$
\begin{align}
W_{SD} &= W_{SD}[\beta_0] + \frac{\delta W_{SD}[\beta_0]}{\delta \beta_0}\beta' + O(\epsilon^2), \tag{6}
\end{align}
$$

$$
\begin{align}
W_2' &= W_2'[\beta_0] + \frac{\delta W_2'[\beta_0]}{\delta \beta_0}\beta' + O(\epsilon^2). \tag{7}
\end{align}
$$

In the above formulas the squared brackets denote a functional dependence on the propagator. The curly brackets as usual mark a parameter dependence. In such a way other functions can be expanded. For β^{-1} we have

$$
\begin{align}
\beta^{-1} &= \beta_0^{-1} - \beta_0^{-1}\frac{\beta'}{\beta_0} \tag{8}
\end{align}
$$

$$
\begin{align}
&= \Delta^{-1} - \Sigma_0[\beta_0] - \Sigma'[\beta_0], \tag{9}
\end{align}
$$

where $\Sigma'[\beta_0] = 2\frac{\delta W_2[\beta_0]}{\delta \beta_0}$ is a correction following from the 2PI graphs, $\Sigma_0[\beta_0](p)$ is the super daisy mass operator. In a high temperature limit within the ansatz adopted in Refs.[12]-[14] ($\beta^{-1} = p^2 + M^2$) it looks as follows

$$
\beta^{-1}(p) = p^2 + M_0^2 - \Sigma'[\beta_0](p), \tag{10}
$$

where M_0^2 is the field mass calculated in the SDA as the solution of the gap equations.

In a similar way, the free energy functional can be presented as

$$
W = W^{(0)} + W' \tag{11}
$$

with

$$
W^{(0)} = S^{(0)} + \frac{1}{2}Trlog\beta_0 - \frac{1}{2}Tr\beta_0\Delta^{-1} + W_{SD}[\beta_0] \tag{12}
$$

and

$$
W' = -\frac{1}{2}Tr\beta'\Delta^{-1} + \frac{\delta W_{SD}[\beta_0]}{\delta \beta_0}\beta' + W_2'[\beta_0] + \frac{1}{2}Tr\beta'\beta_0^{-1} + O(\epsilon^2). \tag{13}
$$

Taking into account that $\beta' = 2\,\beta_0^2\,\frac{\delta W_2'[\beta_0]}{\delta \beta_0}$ one finds

$$
\frac{1}{2}Tr\beta'\beta_0^{-1} = Tr\beta_0\frac{\delta W_2[\beta_0]'}{\delta \beta_0}, \tag{14}
$$

and hence

$$
W' = W_2'[\beta_0]. \tag{15}
$$

Thus, within the representation (4) we obtain for the W functional

$$
W = W_{SD}^{(0)}[\beta_0] + W_2'[\beta_0], \tag{16}
$$

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where $W_{SD}^{(0)}[\beta_0]$ is the expression (1) containing as the $W_2[\beta_0]$ the SDA part only and the particle masses have to be calculated in the SDA. The term $W_2'[\beta_0]$ corresponds to the 2PI graphs taken with the β_0 propagators on lines.

From the above consideration it follows that perturbative calculations in the parameter ϵ derived in the SDA within the second Legendre transform can be implemented in a simple procedure including as the input masses of propagators β the ones obtained in the SDA. Different types of the BSDA diagrams exist. They can be classified as the sets of diagrams having the same orders in $\frac{1}{N}$ from the number of components. So, it is reasonable to account for the contributions of the each class by summing up all diagrams with the corresponding specific power of $\frac{1}{N}$. A particular example of such type calculations will be done below.

3. Expansion near T_c

In this section we shall calculate a first correction in the effective expansion parameter $\epsilon = \frac{1}{N^{1/3}}$ for large N at $T \sim T_c$.

Before to elaborate that, let us take into consideration the main results on the SDA which have to be used as a PT input. In Ref. [14] it was shown that the masses of the Higgs M_η and the Goldstone M_ϕ fields near the phase transition temperature are (for large N)

$$M_\eta^{(0)} = \frac{\lambda T_+}{4\pi}\Big(\frac{1}{(2N)^{1/3}} - \frac{1}{2N} + ...\Big), \qquad (17)$$

$$M_\phi^{(0)} = \frac{\lambda T_+}{2\pi}\Big(\frac{1}{(2N)^{2/3}} - \frac{1}{2N} + ...\Big),$$

where λ is a coupling constant, upper script zero means that in what follows these masses will be chosen as zero approximation, m - initial mass in the Lagrangian. The upper spinodal temperature T_+ is close to the transition temperature $T_c \sim \frac{m}{\sqrt{\lambda}}$. We adduced here the masses for the T_+ case because that delivers simple analytic expressions. Results for other temperatures in between, $T_- \leq T \leq T_+$, are too large to be presented here. Note also that the temperatures T_+, T_- in the SDA are related as (see Refs.[13], [14])

$$\frac{T_+}{T_-} = 1 + \frac{9\lambda}{16\pi^2}\frac{1}{(2N)^{2/3}} + ..., \qquad (18)$$

and $T_- = \sqrt{\frac{12N}{\lambda(N+2)}}m$. Hence it is clear that the transition is of a weakly first-order transforming into a second order one in the limit $N \to \infty$.

With these parameters taken into account an arbitrary graph beyond the SDA having n vertexes can be presented as

$$(\frac{\lambda}{N})^n T^C M^{3C-2L} V_n \sim (\frac{\lambda}{N})^n (1/\sqrt{\lambda})^C (\frac{\lambda}{N^{2/3}})^{3C-2L} V_n = \lambda^1 (\frac{1}{N})^{\frac{1}{3}n+2} V_n. \qquad (19)$$

Here the notation is introduced: C = L - n + 1 - number of loops, L - number of lines. Since we are interested in diagrams with closed loops only, the relation holds: 2n = L. The vertex factor V_n comes from combinatorics. The right-hand-side of Eq.(19) follows when one shifts the three dimensional momentum of each loop, $\vec{p} \rightarrow M\vec{p}$, and substitutes instead of M the mass $M_\phi^{(0)}$ Eq.(17) to have the 'worst case' to consider. In this estimate the static modes ($l = 0$ Matsubara frequency) of propagators were accounted for. One may wonder is it sufficient at T_c? The positive answer immediately follows if one takes into consideration that side by side with the three-momentum rescaling the temperature component of the propagator is also shifted as $T \rightarrow \frac{T}{M} \sim TN^{2/3}$. Hence it is clear that at N goes to infinity a high temperature expansion is applicable and the static mode limit is reasonable.

As it follows from Eq.(19), λ is not a good expansion parameter near T_c whereas $\frac{1}{N^{1/3}}$ is the one because it enters in the power of the number of vertexes of the graph. Of course, we have to consider the graphs beyond the SDA. That is, all diagrams with closed loops of one line (tadpoles) have to be excluded because they were summed up completely already at the SDA level. Note also the important factor $\frac{1}{N^2}$ coming from the rescaling of three-momentum with the mass $M_\phi^{(0)}$.

Before to proceed we note the most important advantages of resummations in the SDA [13],[14]: 1) There is no an imaginary part in the extrema of the free energy. 2) The simple ansatz for the two-particle-irreducible Green function $\beta^{-1} = p^2 + M^2$ is exact in this case. Here, p is a four-momentum, M is a mass parameter which is determined from the solution of gap equations. 3) As it is known for many years, T_c is well determined by this approximation and it is not altered when further resummations are achieved. 4) The existence of the expansion parameter $\epsilon = \frac{1}{N^{1/3}}$ near T_c.

Since the estimate (19) assumes the rescaling of momenta $\vec{p} \rightarrow M\vec{p}$ the same procedure should be fulfilled in perturbation calculations in the parameter ϵ. They are carried out in the following way. First, since λ is not the expansion parameter it can be skipped. Second, only the BSDA diagrams have to be taken into consideration. Moreover, since N is a large number it is convenient to sum up series having different powers in $\frac{1}{N}$. Third, rescaling $\vec{p} \rightarrow M\vec{p}$ can be done before actual calculations. In this case one starts with the expressions like in Eq.(19) (for diagrams of ϕ-sort to consider). When η-particles are included one has to account for the corresponding mass value and rescale the momentum accordingly. Fourth, the temperature is fixed to be T_c. In other words, one has to start with the expressions like in Eq.(19).

To demonstrate the procedure, let us calculate a series of graphs giving leading in $\frac{1}{N}$ contribution F' to the free energy F. Then, the total to be $F = F^{(0)} + F'$, where $F^{(0)}$ is the result of the SDA.

The 'bubble chains' of ϕ-field are the most divergent in N and other diagrams

have at least one power of N less. So, below we discuss the ϕ- bubble chains, only. The contribution of these sequences with the mass $M_\phi^{(0)}$ and various n is given by the series

$$D_\phi = D_\phi^{(2)} + \frac{\lambda^1}{N^2} \sum_{n=3}^{\infty} \frac{1}{n2^n} \frac{V_n}{N^n} Tr_p(6\Sigma_\phi^{(1)}(p)N^{2/3})^n. \tag{20}$$

Here, $D_\phi^{(2)}$ is the contribution of the "basketball" diagram, $\Sigma_\phi^{(1)}(p)$ is the diagram of the type

$$\Sigma_\phi^{(1)}(p) = Tr_k\beta_\phi^{(0)}(k)\beta_\phi^{(0)}(k+p) \tag{21}$$

and the power of the parameter λ is written explicitly.

Now, let us sum up this series for large fixed N. The vertex combinatorial factor for the diagram with n circles is [13]

$$V_n = \frac{N+3}{3}[(\frac{N+3}{3})^{n-1} + (\frac{2}{3})^{n-1}(N-2)]. \tag{22}$$

The leading in N term in the V_n is $\sim (N+1/3)^n$ and we have for the series in Eq.(20)

$$\frac{\lambda^1}{N^2} \sum_{n=3}^{\infty} \frac{1}{n} Tr_p[(-\Sigma_\phi^{(1)}(p)N^{2/3})^n(N+1/N)^n]. \tag{23}$$

To sum up over n we add and subtract two terms: $-\Sigma_\phi^{(1)}(p)N^{2/3}(N+1/N)$, and $\frac{1}{2}[-\Sigma_\phi^{(1)}(p)N^{2/3}(N+1/N)]^2$. We get

$$\begin{aligned} D_\phi &= D_\phi^{(2)} - \frac{\lambda^1}{N^2} Tr_p[log(1 + \Sigma_\phi^{(1)}(p)N^{2/3}(N+1/N)) \\ &- \Sigma_\phi^{(1)}(p)N^{2/3})(N+1/N) + \frac{1}{2}(-\Sigma_\phi^{(1)}(p)N^{2/3}(N+1/N))^2]. \end{aligned} \tag{24}$$

To find all together we insert for the first term

$$D_\phi^{(2)} = \frac{1}{4}\lambda^1 \frac{(N^2-1)}{N^2} Tr_p \frac{(-\Sigma_\phi^{(1)}(p))^2}{N^{2/3}}. \tag{25}$$

Then the limit N goes to infinity has to be calculated. We obtain finally:

$$\begin{aligned} D_\phi &= -\frac{\lambda^1}{N^2} Tr_p log(1 + \Sigma_\phi^{(1)}(p)N^{2/3}) \\ &+ \frac{\lambda^1}{N^{4/3}} Tr_p\Sigma_\phi^{(1)}(p) - \frac{\lambda^1}{4N^{2/3}} Tr_p(\Sigma_\phi^{(1)}(p))^2. \end{aligned} \tag{26}$$

Thus, after summing over n the limit N goes to infinity exists and the series is well convergent. Eq.(26) gives the leading contribution to the free energy calculated in the BSDA in the limit N goes to infinity. This is the final result, if we are interested in the leading in $\frac{1}{N}$ correction.

To obtain the squared mass, M_ϕ^2, one has to sum up the value calculated in the SDA $(M_\phi^{(0)})^2$ and $-\frac{2}{N-1}\delta D_\phi/\delta\beta_\phi(0)$, in accordance with a general expression for the mass [13]. By substituting the propagators one is able to find leading in $\frac{1}{N^{1/3}}$ corrections to free energy and masses at temperatures close to T_c. In this way other quantities can be computed.

The most important observation following from this example is that the limit N goes to infinity does not commute with summing up of infinite series in n at $n \to \infty$. It is seen that the first two terms in the Eq.(26) can be neglected as compared to the last one which is leading in 1/N. As it is occured, this contribution is of order $N^{-5/3}$ that is smaller then the value of $(M_\phi^{(0)})^2 \sim N^{-4/3}$ (17) determined in the SDA. Note that the correction is positive and the complete field mass is slightly increased. We shall calculate the value of the mass in sect.5 simultaneously with other parameters of the phase transition. Contributions of other next-next-to-leading classes of BSDA diagrams can also be obtained perturbatively.

4. Renormalization of vortexes and bubble chains

Calculations carried out in the previous section deal with the unrenormalized functions. But the question may arise: whether the graphs, which are series of the $\Sigma^1(p, M_\phi)$ function with the temperature dependent mass $M_\phi(T)$, are renormalized by the temperature independent counter terms as one expects at finite temperature? We shall consider in detail this question for the non symmetric vertexes $V_{absd}^{(n)}(p)$ and the graphs $S^{(n)}(p)$ of "bubble chain circles " with n $\Sigma^1(p, M_\phi)$ insertions. These are of interest for computations in the previous section. We will show that this is the case for both of these objects. They contain, correspondingly, $n-1$ and n insertions of $\Sigma^{(1)}(p, M_\phi)$, $p = p_a + p_b$ is a momentum incoming in the one-loop vertex $\Sigma_{[\beta_0]}^1(p, M_\phi)$. The functions $S^{(n)}(p)$ are calculated from $V_{absd}^{(n)}(p)$ by means of contracting the indexes and integrating over internal momentum to form one extra $\Sigma^1(p, M_\phi)$ term. Note that the Goldstone field bubble chains give a leading in $\frac{1}{N}$ contribution among the BSDA diagrams.

To carry out actual calculations we adopt the $O(N)$-model with the notations introduced in Refs. [13], [14]. In the restored phase, we have N scalar fields with the same masses $M_r(T)$. In the broken phase, there is one Higgs field η with the mass M_η and $N-1$ Goldstone fields ϕ having the mass M_ϕ (17) derived in the SDA.

Let us first consider the one-loop vertex of the ϕ-field in the s-channel, $s = p^2 = (p_a + p_b)^2$, a, b mark incoming momenta,

$$V_{abc_1d_1}^{(2)} = \frac{\rho^2}{2}\frac{\Sigma^1}{2}(C_1 s_{abc_1d_1} + \frac{2}{3}V_{abc_1d_1}^1), \tag{27}$$

where the notation is introduced: $\rho = \frac{-6\lambda}{N}$ - expansion parameter in the $O(N)$-model, $C_1 = \frac{N+1}{N-1}(\frac{N+1}{3} - \frac{2}{3})$ is a combinatorial factor appearing at the symmetric tensor $s_{abc_1 d_1} = \frac{1}{3}\delta_{ab}\delta_{c_1 d_1}$, $V^1_{abc_1 d_1} = \frac{1}{3}(\delta_{ab}\delta c_1 d_1 + \delta_{ac_1}\delta_{bd_1} + \delta_{ad_1}\delta_{bc_1})$ is the tree vertex in the $O(N)$-model. Subscript 1 in the indexes c_1, d_1 counts the number of loops in the diagram. The diagram with n loops ($\Sigma^1_{[\beta_0]}(p)$ insertions) has the form

$$V^{(n+1)}_{abc_n d_n} = \frac{\rho^{(n+1)}}{2}(\frac{\Sigma^1}{2})^n(C_n s_{abc_n d_n} + (\frac{2}{3})^n V^1_{abc_n d_n}), \tag{28}$$

where now $C_n = \frac{N+1}{N-1}((\frac{N+1}{3})^n - (\frac{2}{3})^n)$ and other notations are obvious.

At zero temperature, the function $\Sigma^1(p)$ has a logarithmic divergence which in a dimensional regularization exhibits itself as a pole $\sim \frac{1}{\epsilon}$, $\epsilon = d - 4$. We denote the divergent part of $\Sigma^1(p)$ as D_1. To eliminate this part we introduce into the Lagrangian the counter term C_2 of order ρ^2:

$$C_2 = -\rho^2 \frac{D_1}{2} v^{(2)}_{abc_1 d_1}, \tag{29}$$

where $v^{(2)}_{abc_1 d_1} = (C_1 s + \frac{2}{3}V^1)_{abc_1 d_1}$. Thus, the renormalized one-loop vertex is

$$V^{(2)}_{abc_1 d_1} = \frac{1}{2}\rho^2 \frac{\Sigma^1_{ren.}}{2} v^{(2)}_{abc_1 d_1} \tag{30}$$

with $\Sigma^1_{ren.} = \Sigma^1 - D_1$.

In order ρ^3 three diagrams contribute. The first comes from the tree vertexes V^1_{abcd} and contains two Σ^1 insertions. Two other graphs are generated by V^1_{abcd} and the counter term vertex C_2. Each of them has one Σ^1 insertion. The sum of these three diagrams is

$$V^{(3)}_{abc_2 d_2} = \frac{1}{2}\rho^3 v^{(3)}_{abc_2 d_2}(\frac{1}{4}((\Sigma^1_{ren.})^2 + 2\Sigma^1_{ren.}D_1 + D_1^2) - \frac{1}{2}D_1\Sigma^1_{ren.} - \frac{1}{2}D_1^2). \tag{31}$$

The terms with the products $\Sigma^1_{ren.}D_1$ cancel in the total. To have a finite expression one has to introduce a new counter term vertex

$$C_3 = \frac{1}{4}\rho^3 D_1^2 v^{(3)}_{abc_2 d_2}. \tag{32}$$

It cancels the last independent of Σ^1 divergence. Thus the renormalized vertex $V^{(3)}$ is given by the first term in the expression (31).

This procedure can be easily continued with the result that in the order $\rho^{(n+1)}$ one has to introduce the counter term of the form

$$C_{n+1} = \rho^{n+1}(\frac{D_1}{2})^n v^{(n+1)}_{abc_n d_n}. \tag{33}$$

The finite vertex calculated with all types of diagrams of the order ρ^{n+1} looks as follows

$$V^{n+1}_{abc_n d_n} = \frac{1}{2}\rho^{n+1}(\frac{\Sigma^1_{ren.}}{2})^n v^{(n+1)}_{abc_n d_n}. \tag{34}$$

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Above we considered the s-channel diagram contributions to the vertex $V^{(n)}$. This is sufficient to study the leading in $\frac{1}{N}$ correction D_ϕ (26) of interest. To have a symmetric renormalized vertex one has to add the contributions of the t- and u-channels and multiply the total by $\frac{1}{3}$.

Now it is easy to show that the counter terms C_n rendering the finiteness of the vertexes at zero temperature are sufficient to renormalize them at finite temperature. Really, $\Sigma^1(p, M(T), T)$ can be divided in two parts, $\Sigma^1(p, M(T), T = 0) = \Sigma^1(p, M(T))$ corresponding to field theory and $\Sigma^1(p, M(T), T)$ - the statistical part. The divergent part D_1 and the counter terms C_n are independent of mass parameters. So, to obtain the renormalized vertex at finite temperature $V^{n+1}_{abc_nd_n}(p, T)$ one has to sum up the same series of the usual and the counter term diagrams as at zero temperature and substitute $\Sigma^1(p, M(T), T) = \Sigma^1(p, M(T)) + \Sigma^1_T(p, M(T), T)$. Again, the terms of the form $[\Sigma^1(p, M(T), T)]^l D_1^{n-l}$ are canceled and we find the finite expression

$$V^{n+1}_{abc_nd_n}(p, T, M(T)) = \frac{1}{2}\rho^{n+1}\Big(\frac{\Sigma^1_{ren.}(p, M(T)) + \Sigma^1_T(p, M(T), T)}{2}\Big)^n v^{(n+1)}_{abc_nd_n}. \quad (35)$$

For this procedure to hold it is important that D_1 is logarithmically divergent and independent of mass. So, the field theoretical part of $V^{n+1}_{abc_nd_n}(p, T, M(T))$ as well as the statistical part is renormalized by the same counter terms as the vertex at zero temperature. In the BSDA PT we have to use the mass $M^{(0)}_\phi(T)$.

Let us turn to the functions $S^{(n)}(p)$. They can be obtained from the vertexes $V^{n+1}_{abc_nd_n}(p, T, M(T))$ in the following way. One has to contract the initial and the final indexes a, b and c_n, d_n and form two propagators β_0 which after integration over the internal line give the term Σ^1. The combinatorial factor of this diagram is $\frac{1}{n+1}$.

We proceed with the function $S^{(3)}$ at zero temperature. In the order ρ^3 two diagrams contribute. One includes three ordinary vertexes V^1_{absd},

$$S^{(3)}_1(p) = 2\frac{1}{3}v^{(3)}_{abab}\Big(\frac{\Sigma^1(p)}{2}\Big)^3, \quad (36)$$

and two other diagrams containing one vertex v^1_{abcd} and the counterterm vertex C_2,

$$2S^{(3)}_2(p) = -\frac{1}{3}\rho^3 v^1_{abcd}v^{(2)}_{cdab}\Big(\frac{\Sigma^1(p)}{2}\Big)^2\frac{D_1}{2}. \quad (37)$$

The contraction of indexes in Eq.(37) gives the factor $v^{(3)} = D^3_s = v^{(3)}_{abab} = \frac{N+1}{3}[(\frac{N+1}{3})^2 + (\frac{2}{3})^2(N-2)]$. Again, substituting $\Sigma^1 = \Sigma^1_{ren.} + D_1$ one can see that the terms with products $(\Sigma^1_{ren.})^l(D_1)^{(3-l)}$ are canceled in the sum of expressions (36) and (37). To obtain a finite $S^{(3)}$ we introduce the counter term of order ρ^3,

$$C^{(3)} = \frac{1}{3}\rho^3\Big(\frac{D_1}{2}\Big)^3 D^3_s. \quad (38)$$

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After that the renormalized circle $S^{(3)}$ is

$$S^{(3)}_{ren.}(p) = \frac{1}{3}\rho^3 \Big(\frac{\Sigma^1_{ren.}(p)}{2}\Big)^3 D_s^3. \tag{39}$$

Note that since $C^{(3)}$ does not depend on any parameter, it can be omitted as well as the divergent term in the expression $S^{(3)}$. In other words, there is no need to introduce new counter terms into the Lagrangian in order to have a finite $S^{(3)}$ and the counter term renormalizing the vertex $V^{(3)}$ are sufficient.

This procedure can be continued step by step for diagrams with any number of Σ^1 insertions. The renormalized graph $S^{(n)}$ looks as follows,

$$S^{(n)}_{ren.}(p) = \frac{1}{n}\rho^n \Big(\frac{\Sigma^1_{ren.}(p)}{2}\Big)^n D_s^n, \tag{40}$$

where the factor coming from the contraction of the $v^{(n)}_{abc_{n-1}d_{n-1}}$ is

$$D_s^n = \frac{N+1}{3}\Big[\Big(\frac{N+1}{3}\Big)^{n-1} + \Big(\frac{2}{3}\Big)^{n-1}(N-2)\Big]. \tag{41}$$

Now, it is a simple task to show that the counter terms renormalizing vertexes $V^{(n)}$ and circles $S^{(n)}$ at zero temperature are sufficient to obtain finite $S^{(n)}(p,T)$ when the temperature is switched on. This is based on the property that the temperature dependent graph $\Sigma^1(p,M,T)$ can be presented as the sum of the zero temperature part and the statistical part independently of the mass term entering. Then it is easy to check that the divergent terms of the form $[\Sigma^1(p,M,T = 0) + \Sigma^1(p,M,T)]^l(D_1)^{n-l}$ are canceled when all the diagrams of order ρ^n forming the circle $S^{(n)}_{ren.}(p,T)$ are summed up.

Thus we have shown, the renormalization counter terms of leading in $1/N$ BSDA graphs calculated at zero temperature being independent of the mass parameter entering Σ^1 renormalize the $S^{(n)}(p,T)$ functions at finite temperature. This gives the possibility to construct a PT based on the solutions of the gap equations. In fact, just the series of Σ^1 functions are of interest at the transition temperature T_c which has to be considered as a fixed given number. Other parameters of the phase transitions can be found by means of some iteration procedure of the investigated already gap equations.

5. Corrections to the parameters of the phase transition

Having obtained the leading correction to the free energy (26) one is able to find perturbatively the characteristics of the phase transition near T_c. We shall do that for the limit $N \to \infty$.

First let us calculate the corrections to the Higgs boson mass, ΔM_η, and the Goldstone boson mass, ΔM_ϕ, due to D_ϕ term (26). The starting point of this calculations is the system of gap equations derived in Ref.[13] (Eqs. (28), (30) and (45)) and Ref. [14] (Eq. (28)). Here we write that in the form when only the term containing D_ϕ is included,

$$
\begin{aligned}
\frac{M_\eta^2}{2} &= m^2 - \frac{3\lambda}{N}\Sigma_\eta^{(0)} - \lambda\frac{N-1}{N}\Sigma_\phi^{(0)}, \\
\frac{M_\phi^2}{2} &= \frac{\lambda}{N}(\Sigma_\phi^{(0)} - \Sigma_\eta^{(0)}) - \frac{1}{N-1}\frac{\delta D_\phi}{\delta\beta_\phi(0)}.
\end{aligned}
\tag{42}
$$

Remind that these equations give the spectrum of mass in the extrema of the free energy in the phase with broken symmetry. The complete system (Eqs.(45) in Ref.[13]) contains other terms which have higher orders in $\frac{1}{N}$ and were omitted. Here m is the mass parameter in the Lagrangian. The functions $\Sigma_\eta^{(0)}, \Sigma_\phi^{(0)}$ are the tadpole graphs with the full propagators β_η, β_ϕ on the lines. For the ansatz adopted in Refs.[13], [14], $\beta_{\eta,\phi}^{-1} = p^2 + M_{\eta,\phi}^2$, they have at high temperature the asymptotic expansion

$$
\Sigma_{\eta,\phi}^{(0)} = Tr\beta_{\eta,\phi} = \frac{T^2}{12} - \frac{M_{\eta,\phi}T}{4\pi} + ...,
\tag{43}
$$

where dots mark next-next-to-leading terms. Within Eqs.(42) - (43) (without the D_ϕ term) the masses (17) have been derived in the limit of large N.

Now we compute the last term in the Eq.(42) for large N. In this case the last term of D_ϕ in the Eq.(26) is dominant and calculating the functional derivative we find

$$
f(M_\phi^{(0)}) = -\frac{2}{N-1}\frac{\delta D_\phi}{\delta\beta_\phi(0)} = \frac{\lambda}{N-1}\frac{1}{N^{2/3}}Tr\beta^3(M_\phi^{(0)}).
\tag{44}
$$

The sunset diagram entering the right-hand-side can be easily computed to give [13]

$$
Tr\beta^3(M_\phi^{(0)}) = \frac{T^2}{32\pi^2}\left(1 - 2\,ln\frac{3M_\phi^{(0)}}{m}\right).
\tag{45}
$$

Thus for $f(M_\phi^{(0)})$ we obtain in the large N limit

$$
f(M_\phi^{(0)}) = \frac{\lambda}{N^{5/3}}\frac{T^2}{32\pi^2}\left(1 - 2\,ln\frac{3\sqrt{3\lambda}}{\pi(2N)^{2/3}}\right),
\tag{46}
$$

where the mass $M_\phi^{(0)}$ (17) was inserted and $T = T_+$ has to be substituted. Here we again turn to the T_+ case to display analytic results.

Since $f(M_\phi^{(0)})$ is small, it can be treated perturbatively when the masses M_η and M_ϕ are calculated. Let as write them as

$$
\begin{aligned}
M_\eta &= M_\eta^{(0)} + x, \\
M_\phi &= M_\phi^{(0)} + y
\end{aligned}
\tag{47}
$$

assuming x, y to be small. Substituting these in the equations (42) - (43) and preserving linear in x, y terms we obtain the system

$$2M_\phi^{(0)} y = \frac{\lambda T}{2\pi N} (x - y) + f(M_\phi^{(0)}), \tag{48}$$

$$2M_\eta^{(0)} x = \frac{3\lambda T}{2\pi N} x + \frac{\lambda(N-1)T}{2\pi N} y$$

of linear equations. Its solutions for large N are

$$x = \frac{1}{3} \frac{T_+}{32\pi} \frac{1}{N^{2/3}} \left(1 - 2 \, ln \frac{3\sqrt{3}\lambda}{\pi(2N)^{2/3}}\right), \tag{49}$$

$$y = \frac{1}{2} \frac{2^{2/3}}{3^{1/2}} \frac{T_+}{32\pi} \frac{1}{N} \left(1 - 2 \, ln \frac{3\sqrt{3}\lambda}{\pi(2N)^{2/3}}\right).$$

As one can see, these corrections are positive numbers smaller than the masses (17) calculated in the SDA, as it should be in a consistent PT. Notice that the correction x is larger as compared to the next to leading term in the SDA $\sim (1/N)$. So, the BSDA graphs deliver the next-to-leading correction. The value y is of the same order as the next-to-leading term in the SDA.

In a similar way the correction to the mass in the restored phase, ΔM_r, can be calculated. In this symmetric case the all components have equal masses which in the SDA within the representation (43) are the solutions of the gap equation (Eq. (36) in Ref.[13] and Eq.(27) in Ref. [14])

$$M_r^2 = -m^2 + \frac{\lambda(N+2)T^2}{12N} - 2M_r \frac{\lambda(N+2)T}{8\pi N}. \tag{50}$$

It has a simple analytic solution

$$M_r^{(0)} = -\frac{\lambda(N+2)T}{8\pi N} + \sqrt{\left(\frac{\lambda(N+2)T}{8\pi N}\right)^2 - m^2 + \frac{\lambda(N+2)T^2}{12N}}, \tag{51}$$

where again the upper script zero means the SDA result. The contribution of the bubble graphs (26) corresponds either to the broken or to the restored phases. The only difference is in the number of contributing field components. In the broken phase this is $N-1$ and in the restored it is N. But it does not matter for large N. So, the function which has to be substituted into Eq. (50) is the one in Eq. (44) with the replacements $N - 1 \to N$ and $M_\phi^{(0)} \to M_r^{(0)}$.

To calculate the mass correction z which is assumed to be small we put $M_r = M_r^{(0)} + z$ into Eq.(50) and find in the limit $N \to \infty$:

$$z = \frac{f(M_r^{(0)})}{2M_r^{(0)} + \frac{\lambda T}{4\pi}}. \tag{52}$$

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We see that the correction has a small positive value. By requiring that M_r to be positive the temperature T_- can be simply calculated. It is clear that $T_- \leq T_-^{(0)}$ and BSDA graphs decrease slightly the lower spinodal temperature.

Now we calculate the correction to T_+. To do that let us turn again to the system (42) with the expressions (43), (46) been substituted. From the second equation of the system we find

$$M_\eta = M_\phi + \frac{2\pi N}{\lambda T}(M_\phi^2 - f(M_\phi^{(0)})) \tag{53}$$

and insert this into the first equation to have

$$\left(\frac{2\pi N}{\lambda T}\right)^2 (M_\phi^4 - 2M_\phi^2 f) + M_\phi^2 + \frac{4\pi N}{\lambda T}(M_\phi^3 - M_\phi f) = \tag{54}$$
$$2m^2 - \frac{\lambda(N+2)T^2}{6N} + \frac{3\lambda T}{2\pi N}\left(M_\phi + \frac{2\pi N}{\lambda T}(M_\phi^2 - f)\right) + \frac{\lambda(N-1)T}{2\pi N}M_\phi,$$

where the linear in f terms are retained. This fourth order algebraic equation can be rewritten in the dimensionless variables $\mu = \frac{2\pi N}{\lambda T}M_\phi$ and $g = (\frac{2\pi N}{\lambda T})^2 f$ as follows:

$$F(\mu) = 0, \tag{55}$$
$$F(\mu) = \mu^4 + 2\mu^3 - 2\mu^2 - h - (N+2)\mu - 2g(\mu^2 + \mu).$$

Here the μ-independent function is

$$h = h^{(0)} - 3g = (2m^2 - \frac{\lambda(N+2)T^2}{6N})(\frac{2\pi N}{\lambda T})^2 - 3g \tag{56}$$

and as before the SDA part is marked by the upper script zero. Remind that this equation determines the Goldstone field masses in the extrema of the free energy. It has two real solutions corresponding to a minimum and a maximum. The equation (55) has roots for any N and T which are too large to be displayed here. To have simple analytic results and investigate the limit of large N we proceed as in Ref.[14] and consider the condition for the upper spinodal temperature. In the case of $T = T_+$ the two solutions merge and we have a second equation,

$$F'(\mu) = \mu^3 + \frac{3}{2}\mu^2 - \frac{1}{2}\mu - (\frac{N}{4} + \frac{1}{2}) - g(\mu + \frac{1}{2}) = 0. \tag{57}$$

It can be easily solved for large N:

$$\mu = \left(\frac{N}{4}\right)^{1/3} - \frac{1}{2} + \dots . \tag{58}$$

Here the first two terms of asymptotic expansion are included that is sufficient for what follows. Notice that since g-dependent terms are of next-next-to-leading

order they do not affect the main contributions. The function h can be calculated from Eq.(55) and its first two terms are

$$h^{(0)} = -3\Big(\frac{N}{4}\Big)^{4/3} - \frac{7}{2}\Big(\frac{N}{4}\Big)^{2/3} + ... \; . \tag{59}$$

The temperature T_+ computed from Eq.(56) can be rewritten in the form

$$T_+ = \sqrt{2}\frac{2\pi N}{\lambda}\Big(\frac{2\pi^2(N+2)N}{3\lambda} + h^{(0)} - 3g\Big)^{-1/2}, \tag{60}$$

where the values of $h^{(0)}$ Eq.(59) and g Eq. (46) have to be substituted. Again, since g is a small next-next-to-leading term the solution can be obtained perturbatively. We find

$$T_+ = T_+^{(0)}\Big(1 + \frac{9\lambda}{32\pi^2}\frac{(1 - 2\, ln\frac{3\sqrt{3\lambda}}{\pi(2N)^{2/3}})}{N^{5/3}}\Big), \tag{61}$$

where the value $T_+^{(0)} = T_-^{(0)}(1 + \frac{9\lambda}{16\pi^2}\frac{1}{(2N)^{2/3}})$ must be inserted. From this result it follows that the upper spinodal temperature is slightly increased due to the BSDA contributions. But this is next-next-to-leading correction to the $T_+^{(0)}$ of the SDA.

Thus, we have calculated the main BSDA corrections to the particle masses and the upper and lower spinodal temperatures in the $(\frac{1}{N})^{1/3}$ PT. We found that T_- is decreased and T_+ is increased as compared to the SDA results due to the leading in $\frac{1}{N}$ BSDA graphs - bubble chains. So, the strength of the first-order phase transition is slightly increased when this contribution is accounted for. However, this is a next-to-leading effect. In such a way we prove the results on the type of the phase transition obtained already in the SDA [13], [14].

6. Discussion

As the carried out calculations showed, the phase transition in the $O(N)$-model at large finite N is weakly of first-order. It becomes a second order one in the limit $N \to \infty$. This conclusion has been obtained in the SDA and was proved in the perturbative calculations in the consistent BSDA PT in the effective expansion parameter $\epsilon = (\frac{1}{N})^{1/3}$. This parameter appears near the phase transition temperature T_c at the SDA level. Let us summarize the main steps of the computation procedure applied and the approximations used to derive that.

In Refs.[12] - [14] as a new method the combination of the second Legendre transform with considering of gap equations in the extrema of a free energy functional was proposed. This has simplified calculations considerably and resulted in transparent formulas for many interesting quantities. Within this approach the phase transitions in the $O(1)$ - and $O(N)$-models were investigated in the super

daisy and beyond approximations. To have analytic expressions a high temperature expansion was systematically applied and the ansatz for the full propagators $\beta^{-1} = p^2 + M^2$ has been used. This ansatz is exact for the SDA which sums up completely the tadpole graphs. Just within these assumptions a first order phase transition was observed and the effective parameter ϵ has been found. In terms of it all interesting characteristics can be expressed in the limit of large N and perturbation theory at $T \sim T_c$ constructed. This solves an old problem on the choice of a zero approximation for perturbative computations near T_c (if such exist). We have shown explicitly here that these are the SDA parameters taken as the input values for BSDA resummations. Clearly, this does not change qualitatively the results obtained in the SDA.

Two important questions should be answered in connection with the results presented. The first is on the relation with other investigations where a second order phase transition was observed (see, for example, the well known literature [1] - [8]), and in fact this is a general opinion. The second is the non zero mass (17) of the Goldstone excitations in the broken phase at finite temperature that was determined in SDA by solving of gap equations [13], [14].

What concerns the first question, we are able to analyze the papers [4], [5] [15], [16],[17], where analytic calculations have been reported. Partially that was done in Refs.[13], [14]. We repeat that here for completeness. First of all we notice that there are no discrepancies for the limit $N \to \infty$ where all the methods determine a second order phase transition. The discrepancy is for large finite N.

In Ref.[15] the renormalization group method at finite temperature has been used and a second order phase transition was observed. Results obtained in this approach are difficult to compare with that found in case of a standard renormalization at zero temperature. This is because a renormalization at finite temperature replaces some resummations of diagrams which remain unspecified basically. In Refs.[16], [17] a non-perturbative method of calculation of the effective potential at finite temperature - an auxiliary field method - has been developed and a second order phase transition was observed in both, the one- and the N-component models. This approach seems to us not self-consistent because it delivers an imaginary part to the effective potential in its minima. This important point is crucial for any calculation procedure as a whole. Really, the minima of an effective potential describe physical vacua of a system. An imaginary part is signaling either the false vacuum or the inconsistency of a calculation procedure used. This is well known beginning from the pioneer work by Dolan and Jackiw [18] who noted the necessity of resummations in order to have a real effective potential. In our method of calculation this requirement is automatically satisfied when the SDA diagrams are resummed in the extrema of free energy. This consistent approximation is widely used and discussed in different aspects in literature nowadays [19]-[21].

Let us discuss in more detail the results of Refs.[4], [5] where an interesting method - the method of average potential - was developed and a second order

phase transition has been observed for any value of N. To be as transparent as possible we consider the equation for the effective potential derived in Ref.[5] (equation (3.13))

$$U^{'}(\rho, T) = \bar{\lambda}\Big(a^2 + \rho - b\sqrt{U^{'}(\rho, T)}\Big), \qquad (62)$$

where the notation is introduced: $U^{'}(\rho, T)$ is the derivative with respect to ρ of $U(\rho, T)$, $\rho = \frac{1}{2}\phi^a\phi^a$ is the condensate value of the scalar field, $\bar{\lambda}$ is the coupling constant. The parameters a^2 and b are:

$$a^2 = (T^2 - T_{cr}^2)\frac{N}{24}, \quad T_{cr}^2 = \frac{24m_R^2}{\lambda_R N}, \qquad (63)$$

$$b = \frac{NT}{8\pi}.$$

Here m_R and λ_R are renormalized values, and $\lambda_R = \bar{\lambda}(1 + \frac{\bar{\lambda}N}{32\pi^2}ln(L^2/M^2))^{-1}$ (see for details the cited paper).

Considering the temperatures $T \sim T_{cr}$ and $\rho << T^2$ the authors have neglected $U^{'}(\rho, T)$ as compared to the $\sqrt{U^{'}(\rho, T)}$ in Eq.(62) and obtained after integration over ρ (formula (3.16) of Ref. [5])

$$U(\rho, T) = \frac{\pi^2}{9}\Big(\frac{T^2 - T_{cr}^2}{T}\Big)^2\rho + \frac{1}{N}\frac{8\pi^2}{3}\frac{T^2 - T_{cr}^2}{T^2}\rho^2 + \frac{1}{N^2}\frac{64\pi^2}{3}\frac{1}{T^2}\rho^3. \qquad (64)$$

Near $T \sim T_{cr}$ this effective potential includes the $\rho^3(\phi^6)$ term and predicts a second order phase transition. That is the main conclusion.

However, this analysis is insufficient to distinguish between a second and a weakly-first-order phase transitions. Actually, the temperature T_{cr} determined from the initial condition $U^{'}(0, T_{cr}) = 0$ in the former case corresponds to the temperature T_- in the latter one. To determine the type of the phase transition one has to verify whether or not a maximum at finite distances from the origin in the ρ-plane exists. To check that the equation (62) has to be integrated exactly, without truncations. This is not a difficult problem if one first solves Eq.(62) with respect to $U^{'}(\rho, T)$,

$$U^{'}(\rho, T) = \bar{\lambda}\Big(a^2 + \frac{\lambda b^2}{2} + \rho - b\sqrt{\bar{\lambda}(a^2 + \rho) + \frac{\bar{\lambda}^2 b^2}{4}}\Big), \qquad (65)$$

and then integrates to obtain

$$U(\rho, T) = \Big(\bar{\lambda}(T^2 - T_{cr}^2)\frac{N}{24} + \frac{\bar{\lambda}^2 T^2 N^2}{128\pi^2}\Big)\rho + \frac{1}{2}\rho^2 \qquad (66)$$

$$- \frac{2}{3}\frac{\bar{\lambda}TN}{8\pi}\Big((T^2 - T_{cr}^2)\frac{N}{24} + \frac{\bar{\lambda}T^2 N^2}{256\pi^2} + \rho\Big)^{3/2},$$

where the values of a and b Eq.(63) are accounted for. We see that in contrast to Eq.(64) this potential includes a cubic term which is responsible for the appearance of a maximum in some temperature interval and a first order phase

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transition, that is quite known. Obviously, expanding the expression (66) at $\rho \to 0$ and retaining three first terms, one reproduces the ρ^3 term of Eq.(64).

This consideration convinces us that there are no discrepancies with the actual results of the average action method. The expression (62) gives the potential (66) predicting a first order phase transition.

Our final remarks are on the Goldstone theorem at finite temperature. In fact, at $T \neq 0$ the Goldstone bosons should not inevitably be massless as it was argued, in particular, in Ref.[22]. Formally, the reason is that at finite temperature Lorentz invariance is broken and therefore the condition $p^2 = 0$ does not mean zero mass of the particle in contrast to as it should be at $T = 0$ (see for details Ref.[22]). We observed in the SDA that this is the case when the first order phase transition happens. In the limit $N \to \infty$ corresponding to a second order phase transition the Goldstone bosons are massless, as it is seen from Eq.(17). The same conclusion follows from the results of Ref.[5] when a second order phase transition is assumed. Probably this problem requires a separate investigation by means of other methods.

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