Velocity of particles in Doubly Special Relativity

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Abstract

Doubly Special Relativity (DSR) is a class of theories of relativistic motion with two observer-independent scales. We investigate the velocity of particles in DSR, defining velocity as the Poisson bracket of position with the appropriate hamiltonian, taking care of the nontrivial structure of the DSR phase space. We find the general expression for four-velocity, and we show further that the three-velocity of massless particles equals 1 for all DSR theories. The relation between the boost parameter and velocity is also clarified.

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1 Introduction

There is a rapidly growing interest in a research programme, which might be called "Quantum Special Relativity", i.e. a class of theories of kinematics that differ in their predictions from that of Special Relativity in the regime of ultra-high energies. These possible differences might be understood as traces of still unknown Quantum Gravity theory present even in the regime of negligible gravitational field. It is feasible that predictions of such generalizations of Special Relativity, like dependence of the speed of massless particles on momentum they carry, might be experimentally tested in the near future experiments.

Two alternative classes of "Quantum Special Relativity" have been recently attracting attention. In one, one argues that Lorentz invariance is broken at high energies due to string [1], loop-quantum gravity [2] effects, or due to existence of the cosmological preferred frame [3]. In the second, proposed by Amelino-Camelia [4], [5], called Doubly Special Relativity (DSR), one assumes that the ten dimensional algebra of physical symmetries (rotations, boosts, and translations) is still present, but is deformed in a way so as to possess two observer-independent scales. In this paper we will concentrate on this second possibility only.

The construction of the first specific model of DSR, called DSR1, presented in [6] and [7] borrowed a lot from the earlier investigations in quantum Hopf deformations of Poincaré algebra, the so-called κ-Poincaré algebra (see e.g., [8], [9]). It turned out however that there exist another DSR models: the first one, called nowadays DSR2 was formulated by Magueijo and Smolin [10], and it was soon realized that there exist a whole class of DSR theories [11], [12].

In the majority of the (especially early) literature devoted to DSR, Doubly Special Relativity is defined only as a theory based on energy-momentum sector. This is *not* what we understand by DSR in this paper. In our view, a DSR theory is defined by a set of commutators describing the whole of the phase space of the system. There are two systematic and equivalent ways of deriving such a set in any particular model[1]. The first is based on the Hopf algebra structure that can be built on the energy–momentum algebra, and the use of the so called Heisenberg double construction [13], [14], [12]. Equiv-

[1]Some other methods of deriving consistent phase space may exist, of course. We are not aware however of any nontrivial alternative to the procedure we make use of.

alently, one can get the same phase space by making use of the geometric, de Sitter picture of DSR, [15], [16].

Doubly Special Relativity is a theory of particle kinematics, and the proper understanding of the concept of velocity in such a theory is of course an important step towards full understanding of it. The first attempt to analyze the notion of velocity has been made already in the early days of the κ-Poincaré theory in [17], from the DSR perspective this problem has been investigated, among others, in [18], [19], [20], [21], [22].

The starting point of our investigations reported here consists of two major assumptions: that velocity is defined as the Poisson bracket of position with deformed relativistic hamiltonian (see also [17]), and that to compute this bracket one must take into account the nontrivial phase space structure of DSR theories. We show that in all DSR theories four velocities transform as standard Lorentz vectors, and that the three velocities of massless particles equal one. This general statement will be justified in Section 3; before turning to that, we present our method on the specific example of DSR1.

2 Particle velocity in DSR1

Before starting our investigations let us state clearly what our assumptions are. We define four velocity as the Poisson bracket, based on a DSR phase space structure, of positions with an appropriate hamiltonian. Let us explain why we decided to use Poisson brackets instead of commutators, usually employed in the DSR literature. The reason is quite simple: the use of commutators implies that the relevant objects are operators acting on some Hilbert space. Since no investigations of such operators' properties (functional analysis) has been performed so far, working with the commutator algebra is equivalent to phrasing results in terms of Poisson brackets. Since we will make use of the expression for three-velocity as a function of four velocities $v^i = \dot{x}^i/\dot{x}^0$ whose meaning for x^0 being an operator is not clear, it is just safer to work with Poisson brackets. For this reason we will confirm our discussion to classical particles and not quantum matter waves. In our investigations in this section we will also postulate a particular form of relativistic hamiltonian. We will argue that such form is natural in the next section devoted to general properties shared by all DSR theories.

In all the DSR theories (contrary to the statements that can be sometimes found in the literature), the Lorentz algebra of rotations M_i and boosts N_i is exactly the same as in Special Relativity[2]

$$[M_i, M_j] = \epsilon_{ijk} M_k, \quad [M_i, N_j] = \epsilon_{ijk} N_k,$$

$$[N_i, N_j] = -\epsilon_{ijk} M_k, \tag{1}$$

In the DSR1 theory the momenta transform under action of boosts as follows

$$[N_i, p_j] = \delta_{ij} \left(\frac{\kappa}{2} \left(1 - e^{-2p_0/\kappa} \right) + \frac{1}{2\kappa} \vec{p}^{\,2} \right) - \frac{1}{\kappa} p_i p_j, \tag{2}$$

and

$$[N_i, p_0] = p_i. \tag{3}$$

The first Casimir of this theory equals

$$\mathcal{C} = \left(2\kappa \sinh \left(\frac{p_0}{2\kappa} \right) \right)^2 - \vec{p}^{\,2}\, e^{p_0/\kappa} = m^2. \tag{4}$$

Let us now turn to description of the phase space of DSR1. As for all DSR theories we have

$$[x_0, x_i] = -\frac{1}{\kappa}\, x_i, \quad [x_i, x_j] = 0. \tag{5}$$

As shown in [16] there are infinitely many phase spaces compatible with the DSR1 boost transformations (3), (4) and the brackets (5). Here we will describe only the two most simple ones for whose the cross brackets take the following form

$$[p_0, x_0] = -1, \quad [p_i, x_0] = \frac{1}{\kappa}\, p_i,$$

$$[p_i, x_j] = \delta_{ij}\, e^{-2p_0/\kappa} - \frac{1}{\kappa^2} \left(\vec{p}^{\,2}\, \delta_{ij} - 2 p_i p_j \right), \quad [p_0, x_i] = -\frac{2}{\kappa}\, p_i \tag{6}$$

or, following [9], [17]

$$[p_0, x_0] = 1, \quad [p_i, x_0] = -\frac{1}{\kappa}\, p_i, \quad [p_i, x_j] = -\delta_{ij}, \quad [p_0, x_i] = 0. \tag{7}$$

We choose the hamiltonian to be

$$\mathcal{H} = \kappa^2\, \cosh \frac{p_0}{\kappa} - \frac{\vec{p}^{\,2}}{2}\, e^{\frac{p_0}{\kappa}}. \tag{8}$$

[2]Let us stress again that $[*, *]$ denotes the Poisson bracket and *not* the commutator.

This hamiltonian has the large κ limit

$$\mathcal{H} \sim \kappa^2 + \frac{1}{2}\left(p_0^2 - \vec{p}^{\,2}\right) + \ldots$$

i.e., it reduces in this limit to the standard hamiltonian of relativistic particle (up to the irrelevant constant shift). Let us now define the four velocities in the standard way as the bracket

$$u_0 \equiv \dot{x}_0 = [x_0, \mathcal{H}], \quad u_i \equiv \dot{x}_i = [x_i, \mathcal{H}] \tag{9}$$

In calculating the brackets in (9) one should carefully take care of the non-trivial phase space structure of DSR1 (6) or (7) (in accordance with the scheme presented in [17])

$$[x_0, \mathcal{H}] \equiv \frac{\partial \mathcal{H}}{\partial p_0}\,[x_0, p_0] + \frac{\partial \mathcal{H}}{\partial p_i}\,[x_0, p_i],$$

$$[x_k, \mathcal{H}] \equiv \frac{\partial \mathcal{H}}{\partial p_0}\,[x_k, p_0] + \frac{\partial \mathcal{H}}{\partial p_i}\,[x_k, p_i]. \tag{10}$$

The second set of Hamilton equations is quite simple

$$\dot{p}_\mu = 0 \tag{11}$$

because in all DSR theories the hamiltonian depends on momenta only, and momenta have vanishing bracket among themselves. Notice that this property guarantees that in DSR free motion is uniform. One should note at this point that for this reason it seems that the so called "twisted phase spaces", investigated recently in [23], which would lead to non-uniform motion of free particles, are likely not to be physical.

It is easy to derive the expression for four velocity, following our general prescription (10). We find

$$\begin{aligned} u_0 &= \kappa \sinh \frac{p_0}{\kappa} + \frac{\vec{p}^{\,2}}{2\kappa}\, e^{\frac{p_0}{\kappa}} \\ u_i &= p_i\, e^{\frac{p_0}{\kappa}} \end{aligned} \tag{12}$$

in the case of phase space (6) and

$$\begin{aligned} u_0 &= -\kappa \sinh \frac{p_0}{\kappa} - \frac{\vec{p}^{\,2}}{2\kappa}\, e^{\frac{p_0}{\kappa}} \\ u_i &= -p_i\, e^{\frac{p_0}{\kappa}} \end{aligned} \tag{13}$$

for the phase space (7)[3]. We see that the four velocities differ by the overall sign, but that in both cases the expression for three-velocity is exactly the same and reads

$$v_i = \frac{u_i}{u_0} = p_i \left(\frac{\kappa}{2} \left(1 - e^{-2p_0/\kappa} \right) + \frac{\vec{p}^{\,2}}{2\kappa} \right)^{-1} \tag{14}$$

Using the expansion of the mass-shell condition for massless particles (cf. eq. (4))

$$0 = \kappa^2 \left(1 - e^{-p_0/\kappa} \right)^2 - \vec{p}^{\,2} \tag{15}$$

we find

$$v_i^{(m=0)} = \frac{p_i}{\kappa} \left(1 - e^{-p_0/\kappa} \right)^{-1} . \tag{16}$$

Using (15) again we find that the speed of massless particles

$$v^{(m=0)} = \left| v_i^{(m=0)} \right| = 1 \tag{17}$$

as in Special Relativity. Of course, there are deviations from Special Relativistic results in the case of massive particles. Indeed, in the massive case we have with the help of (4)

$$v_i^{(m)} = p_i \left[\kappa \left(1 - e^{-p_0/\kappa} \right) - \frac{m^2 e^{-p_0/\kappa}}{2\kappa} \right]^{-1} \tag{18}$$

Note that the speed-of-light of a massive particle, i.e., the maximal speed of massive particle carrying infinite energy equals again

$$v^{(\infty)} = 1 \tag{19}$$

because $p_0 = \infty$ corresponds to $|\vec{p}| = \kappa$, as it follows simply from (4).

It is worth noticing that if one derives the transformation rules for four velocities (12), (13) under action of boosts, using expressions (2), (3), one finds that they transform as standard Lorentz four vectors of Special Relativity. This is not an accidental property of DSR1, in fact it holds for all DSR theories. Let us turn therefore to general formulation and properties of such theories.

[3] In this particular case our result agrees with the one reported in [19], [21].

3 Velocity in DSR theories – generalities

The results derived in the previous section turn out to be valid not only for DSR1, but in fact for all DSR theories. In order to see that let us recall the geometric, de Sitter space formulation of the DSR theories presented in [15]. The starting point here is a five dimensional manifold of Minkowski signature

$$ds^2 = g^{AB} d\eta_A d\eta_B = -d\eta_0^2 + d\eta_i^2 + d\eta_4^2. \tag{20}$$

In this space the four dimensional de Sitter space is imbedded by

$$-\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2 + \eta_4^2 = \kappa^2, \tag{21}$$

We split the ten dimensional algebra of isometries of de Sitter space (21) into the six dimensional Lorentz algebra (1) and the remainder, which we identify with positions x_μ satisfying (5). The remaining brackets are

$$[M_i, \eta_j] = \epsilon_{ijk}\, \eta_k, \quad [M_i, \eta_0] = 0, \quad [M_i, \eta_4] = 0 \tag{22}$$

$$[N_i, \eta_j] = \delta_{ij}\, \eta_0, \quad [N_i, \eta_0] = \eta_i, \quad [N_i, \eta_4] = 0 \tag{23}$$

$$[x_0, \eta_4] = \frac{1}{\kappa}\, \eta_0, \quad [x_0, \eta_0] = \frac{1}{\kappa}\, \eta_4, \quad [x_0, \eta_i] = 0, \tag{24}$$

$$[x_i, \eta_4] = \frac{1}{\kappa}\, \eta_i, \quad [x_i, \eta_0] = \frac{1}{\kappa}\, \eta_i, \quad [x_i, \eta_j] = \frac{1}{\kappa}\, \delta_{ij}(\eta_0 - \eta_4), \tag{25}$$

The main result of [15], based on the previous investigations reported in [11] and [12] is that any DSR theory can be represented as a particular coordinate system on the de Sitter space (21), i.e., the mapping from the space of physical momenta p_μ to η_A satisfying (21). For example, in the case of the DSR1 theories with phase spaces (6), (7) this mapping takes the form

$$\begin{aligned}
\eta_0 &= \pm\left(\kappa \sinh\frac{p_0}{\kappa} + \frac{\vec{p}^{\,2}}{2\kappa}\, e^{\frac{p_0}{\kappa}}\right) \\
\eta_i &= \pm\left(p_i\, e^{\frac{p_0}{\kappa}}\right) \\
\eta_4 &= \kappa \cosh\frac{p_0}{\kappa} - \frac{\vec{p}^{\,2}}{2\kappa}\, e^{\frac{p_0}{\kappa}}.
\end{aligned} \tag{26}$$

Let us observe now that in view of eqs. (21) and (23) $\kappa\, \eta_4$ is the most natural candidate for relativistic hamiltonian. Indeed it is by construction Lorentz-invariant, and reduces to the standard relativistic particle hamiltonian in the large κ limit. Indeed, using the fact that for p_μ small compared to κ, in any DSR theory $\eta_\mu \sim p_\mu + O(1/\kappa)$ we have

$$\kappa\eta_4 = \kappa^2\sqrt{1+\frac{p_0^2-\vec{p}^{\,2}}{\kappa^2}} \sim \kappa^2 + \frac{1}{2}\left(p_0^2-\vec{p}^{\,2}\right) + O\left(\frac{1}{\kappa^2}\right) \tag{27}$$

Then it follows from eqs. (24), (25) that $\eta_\mu = [x_\mu, \kappa\eta_4]$ can be identified with four velocities u_μ. The Lorentz transformations of four velocities are then given by eq. (23) and are with those of Special Relativity. Moreover, since

$$u_0^2 - \vec{u}^{\,2} \equiv \mathcal{C} = m^2 \tag{28}$$

by the standard argument the three velocity equals $v_i = u_i/u_0$ and the speed of massless particle equals 1. Let us stress here once again that this result is DSR model independent, though, of course, the relation between three velocity of massive particles and energy they carry depends on a particular DSR model one uses.

Thus if in the time-of-flight experiment (see e.g., [24] for recent review), will measure time gap between arrivals of photons emitted from a distant source, and carrying different energies, this would falsify the construction of the DSR models presented above.

Note finally that as the result of eq. (28) and the definition of three velocity as $v_i = u_i/u_0$, the rule of adding the latter in any DSR theory is identical with the standard rule of Special Relativity. It is however an open question if this can be consistently extended to the rule of addition of momenta, following the considerations of Lukierski and Nowicki [25] and Judes and Visser [26].

4 Comments and concluding remarks

The main result of our investigations reported here is that if the phase space of DSR theories is constructed in the way suggested by geometric picture of de Sitter space, the speed of massless particle equals 1. There is a number of simple observations one can make. First, in the scheme adopted here, for any DSR theory the velocity is just

$$v_i = \frac{u_i}{u_0} = \frac{\eta_i(p)}{\eta_0(p)} \tag{29}$$

which provides the velocity–momentum relation for an arbitrary DSR theory, since in any DSR theory the variables η_μ are functions of momenta, given by its definition. It follows also that the boost parameter ξ is related to velocity in exactly the same way as in the Special Relativity

$$\tanh \xi = v \tag{30}$$

This can be easily seen by realizing that the brackets (23) are equivalent to the equations (for boost acting in 3rd direction, say)

$$\frac{d\eta_3}{d\xi} = \eta_0, \quad \frac{d\eta_0}{d\xi} = \eta_3, \tag{31}$$

from which (30) immediately follows. To obtain the corresponding equation for dependence of momenta p_μ on rapidity in a particular DSR theory, defined by a particular functions $\eta_A(p_\mu)$, one uses eq. (31) and the Leibnitz rule. But this does not change the relation (30) where the right hand side can be taken a given function of momenta.

It is interesting to compare our result with the calculation of the group velocity of wave packets presented in [20]. In this paper the authors perform their computations using essentially only the noncommutative structure of κ-Minkowski space-time (5), the particular form of non-commutative differential calculus, and a natural ordering of plane waves. Their result, that the group velocity $v^{(g)} = \partial p_0/\partial p$ (where the derivative is taken on the mass-shell, i.e. assuming that eq. (4) holds,) should be therefore valid for all DSR theories. The clear disagreement of their result with the one presented in this paper deserves further studies, since it seems to indicates that in the framework of DSR the behavior of matter waves may differ from that of particles. Specifically, this discrepancy means that either naively constructed wave packets could not represent point particles, so that the latter are represented by a non-linear combination of plane waves, or that one of these two notions (linear wave packet and/or point particles) just does not make physically sense in the DSR theories.

To conclude let us stress that the final judgement on possible dependence of velocity on energy will be made by near future experiments. We would like to note however that if it turns out that our result is correct, i.e., if in the DSR theories we still have to do (as in Special Relativity) with universal speed of physical signals carried by massless particles, this will constitute an important information, making it easier to formulate Doubly Special Relativity operationally.

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