BOUND STATES BY A PSEUDOSCALAR COULOMB POTENTIAL IN ONE-PLUS-ONE DIMENSIONS

Antonio S. de Castro

UNESP - Campus de Guaratinguetá
Departamento de Física e Química
Caixa Postal 205
12516-410 Guaratinguetá SP - Brasil

Electronic mail: castro@feg.unesp.br

Abstract

The Dirac equation is solved for a pseudoscalar Coulomb potential in a two-dimensional world. An infinite sequence of bounded solutions are obtained. These results are in sharp contrast with those ones obtained in 3+1 dimensions where no bound-state solutions are found.

The Coulomb potential of a point electric charge in a 1+1 dimension, considered as the time component of a Lorentz vector, is linear and so it provides a constant electric field always pointing to, or from, the point charge. This problem is related to the confinement of fermions in the Schwinger and in the massive Schwinger models [1]-[2] and in the Thirring-Schwinger model [3]. It is frustrating that, due to the tunneling effect (Klein´s paradox), there are no bound states for this kind of potential regardless of the strength of the potential [4]-[5]. The linear potential, considered as a Lorentz scalar, is also related to the quarkonium model in one-plus-one dimensions [6]-[7]. Recently it was incorrectly concluded that even in this case there is solely one bound state [8]. Later, the proper solutions for this last problem were found [9]-[11]. However, it is well known from the quarkonium phenomenology in the real 3+1 dimensional world that the best fit for meson spectroscopy is found for a convenient mixture of vector and scalar potentials put by hand in the equations (see, *e.g.*, [12]). The mixed vector-scalar linear potential in 1+1 dimensions was recently considered [13]. There it was found that there are analytical bound-state solutions on condition that the scalar component of the potential is of sufficient strength compared to the vector component ($|V_s| \geq |V_t|$). As a by-product, that approach also showed that there exist relativistic confining potentials providing no bound-state solutions in the nonrelativistic limit. Although the discussion was confined to the vector-scalar mixing, the inclusion of a pseudoscalar potential could also be allowed.

In a recent paper, McKeon and Van Leeuwen [16] considered a nonconserving-parity pseudoscalar Coulomb (NCPPC) potential ($V = \lambda/r$) in 3+1 dimensions and concluded that there are no bounded solutions for the reason that the different parity eigenstates mix. Furthermore, they asserted that *the absence of bound states in this system confuses the role of the* π*-meson in the binding of nucleons*. Such an intriguing conclusion sets the stage for the analyses of the confinement of fermions by other sorts of pseudoscalar potentials. A natural question to ask is if the absence of bounded solutions by a NCPPC potential is a characteristic feature of the four-dimensional world. With this in mind, we approach in the present paper the less realistic Dirac equation in one-plus-one dimensions with a NCPPC potential ($V = \lambda|x|$). The confinement of fermions by a pure conserving-parity pseudoscalar double-step potential [14] and their scattering by a pure nonconserving-parity pseudoscalar step potential [15] have already been analyzed in the literature providing the opportunity to find some quite interesting results.

The two-dimensional Dirac equation can be obtained from the four-dimensional one with the mixture of spherically symmetric scalar, vector and anomalous magnetic interactions. If we limit the fermion to move in the x-direction ($p_y = p_z = 0$) the four-dimensional Dirac equation decomposes into two equivalent two-dimensional equations with 2-component spinors and 2×2 matrices [17]. Then, there results that the scalar and vector interactions preserve their Lorentz structures whereas the anomalous magnetic interaction turns out to be a pseudoscalar interaction. Furthermore, in the 1+1 world there is no angular momentum so that the spin is absent. Therefore, the 1+1 dimensional Dirac equation allow us to explore the physical consequences of the negative-energy states in a mathematically simpler and more physically transparent way.

Let us begin by presenting the Dirac equation in 1+1 dimensions. In the presence of a time-independent potential the 1+1 dimensional time-independent Dirac equation for a fermion of rest mass m reads

$$\mathcal{H}\Psi = E\Psi \tag{1}$$

$$\mathcal{H} = c\alpha p + \beta mc^2 + \mathcal{V} \tag{2}$$

where E is the energy of the fermion, c is the velocity of light and p is the momentum operator. α and β are Hermitian square matrices satisfying the relations $\alpha^2 = \beta^2 = 1$, $\{\alpha, \beta\} = 0$. From the last two relations it steams that both α and β are traceless and have eigenvalues equal to -1, so that one can conclude that α and β are even-dimensional matrices. One can choose the 2×2 Pauli matrices satisfying the same algebra as α and β, resulting in a 2-component spinor Ψ. The positive definite function $|\Psi|^2 = \Psi^\dagger\Psi$, satisfying a continuity equation, is interpreted as a probability position density and its norm is a constant of motion. This interpretation is completely satisfactory for single-particle states [18]. We use $\alpha = \sigma_1$ and $\beta = \sigma_3$. For the potential matrix we consider

$$\mathcal{V} = 1V_t + \beta V_s + \alpha V_e + \beta\gamma^5 V_p \tag{3}$$

where 1 stands for the 2×2 identity matrix and $\beta\gamma^5 = \sigma_2$. This is the most general combination of Lorentz structures for the potential matrix because there are only four linearly independent 2×2 matrices. The subscripts for the terms of potential denote their properties under a Lorentz transformation: t and e for the time and space components of the 2-vector potential, s and p for the scalar and pseudoscalar terms, respectively. It is worth to note that the Dirac equation is covariant under $x \to -x$ if $V_e(x)$ and $V_p(x)$ change sign whereas $V_t(x)$ and $V_s(x)$ remain the same. This is because the parity operator $P = \exp(i\theta)P_0\sigma_3$, where θ is a constant phase and P_0 changes x into $-x$, changes sign of α and $\beta\gamma^5$ but not of 1 and β.

Defining the spinor ψ as

$$\psi = \exp\left(\frac{i}{\hbar}\Lambda\right)\Psi \tag{4}$$

where

$$\Lambda(x) = \int^x dx' \frac{V_e(x')}{c} \tag{5}$$

the space component of the vector potential is gauged away

$$\left(p + \frac{V_e}{c}\right)\Psi = \exp\left(\frac{i}{\hbar}\Lambda\right)p\psi \tag{6}$$

so that the time-independent Dirac equation can be rewritten as follows:

$$H\psi = E\psi \tag{7}$$

$$H = \sigma_1 cp + \sigma_2 V_p + \sigma_3\left(mc^2 + V_s\right) + 1V_t \tag{8}$$

showing that the space component of a vector potential only contributes to change the spinors by a local phase factor.

Provided that the spinor is written in terms of the upper and the lower components

$$\psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{9}$$

the Dirac equation decomposes into :

$$\begin{aligned}\left(V_t - E + V_s + mc^2\right)\phi(x) &= i\hbar c\chi'(x) + iV_p\chi(x) \\ \left(V_t - E - V_s - mc^2\right)\chi(x) &= i\hbar c\phi'(x) - iV_p\phi(x)\end{aligned} \tag{10}$$

where the prime denotes differentiation with respect to x. In terms of ϕ and χ the spinor is normalized as $\int_{-\infty}^{+\infty} dx\,(|\phi|^2 + |\chi|^2) = 1$, so that ϕ and χ are square integrable functions. It is clear from the pair of coupled first-order differential equations (10) that both $\phi(x)$ and $\chi(x)$ must be discontinuous wherever the potential undergoes an infinite jump and have opposite parities if the Dirac equation is covariant under $x \to -x$. In the nonrelativistic approximation (potential energies small compared to the rest mass) Eq. (10) loses all the matrix structure and becomes

$$\chi = \frac{p}{2mc}\phi \tag{11}$$

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}+V_t+V_s\right)\phi=\left(E-mc^2\right)\phi \tag{12}$$

Eq. (11) shows that χ if of order $v/c<<1$ relative to ϕ and Eq. (12) shows that ϕ obeys the Schrödinger equation without any contribution from the pseudoscalar potential.

Now, let us choose $V_t=V_s=0$ and the intrinsically relativistic NCPPC potential $V_p=\lambda|x|$, with $\lambda>0$. Defining

$$\eta = \sqrt{\frac{2\lambda}{\hbar c}}\,x \tag{13}$$

$$\nu = -1+\frac{E^2-m^2c^4}{2\hbar c\lambda} \tag{14}$$

the Dirac equation (10) turns into the Schrödinger-like differential equations

$$\begin{aligned}
-\frac{d^2\phi}{d\eta^2}+\frac{\eta^2}{4}\phi &= \begin{cases}(\nu+1/2)\,\phi, & x>0\\[2mm] (\nu+3/2)\,\phi, & x<0\end{cases}\\
-\frac{d^2\chi}{d\eta^2}+\frac{\eta^2}{4}\chi &= \begin{cases}(\nu+3/2)\,\chi, & x>0\\[2mm] (\nu+1/2)\,\chi, & x<0\end{cases}
\end{aligned} \tag{15}$$

The second-order differential equations (15) have the form

$$y''(z)-\left(\frac{z^2}{4}+a\right)y(z)=0, \tag{16}$$

whose solution is a parabolic cylinder function [19]. The solutions $D_{-a-1/2}(z)$ and $D_{-a-1/2}(-z)$ are linearly independent unless $n=-a-1/2$ is a nonnegative integer. In that special circumstance $D_n(z)$ has the peculiar property that $D_n(-z)=(-1)^n D_n(z)$, and it is proportional to $\exp\left(-z^2/4\right)H_n(z/\sqrt{2})$, where $H_n(z)$ is a Hermite polynomial. The solutions of (15) do not exhibit this parity property so that we should not expect nonnegative integer values for ν. The physically acceptable solutions for bound states must vanish in the asymptotic region $|\eta|\to\infty$ and are expressed as

$$\phi = \begin{cases} C^{(+)}D_\nu(\eta), & x>0\\[2mm] C^{(-)}D_{\nu+1}(\eta), & x<0\end{cases}$$

$$\eta \quad = \quad \begin{cases} D^{(+)} D_{\nu+1}(\eta), & x > 0 \\ \\ D^{(-)} D_{\nu}(\eta), & x < 0 \end{cases} \tag{17}$$

where $C^{(\pm)}$ and $D^{(\pm)}$ are normalization constants. Substituting the solutions (17) into the Dirac equation (10) and making use of the recurrence formulas

$$\begin{aligned} \frac{d}{dz} D_{\nu}(z) - \frac{z}{2} D_{\nu}(z) + D_{\nu+1}(z) &= 0 \\ \\ \frac{d}{dz} D_{\nu}(z) + \frac{z}{2} D_{\nu}(z) - \nu D_{\nu-1}(z) &= 0 \end{aligned} \tag{18}$$

one has as a result

$$\begin{aligned} \left[\frac{C^{(+)}}{D^{(+)}} \right]^2 &= -\left(\nu+1\right) \frac{E+mc^2}{E-mc^2} \\ \\ \left[\frac{C^{(-)}}{D^{(-)}} \right]^2 &= -\left(\frac{1}{\nu+1} \right) \frac{E+mc^2}{E-mc^2} \end{aligned} \tag{19}$$

The continuity of the wavefunctions (17) at $x = 0$ furnishes

$$\frac{C^{(+)}}{D^{(+)}} = \frac{C^{(-)}}{D^{(-)}} \left[\frac{D_{\nu+1}(0)}{D_{\nu}(0)} \right]^2 \tag{20}$$

Together, (19) and (20) lead to the quantization condition

$$D^2_{\nu+1}(0) = \pm \left(\nu+1\right) D^2_{\nu}(0) \tag{21}$$

This last result combined with (14) shows that the use of the minus sign demands that $-\left(1 + m^2c^4/2\hbar c\lambda\right) < \nu < -1$. If, on the other side, one uses the plus sign then $-1 < \nu < +\infty$. Because the normalization of the spinor is not important for the calculation of the spectrum, one can arbitrarily choose $D_{\nu}(0) = 1$.

The numerical computation of (21) is substantially simpler when $D_{\nu+1}$ is written in terms of the derivative of D_{ν}:

$$\frac{d}{d\eta} D_{\nu}(\eta)\left.\right|_{\eta=0} = \pm\sqrt{\pm\left(\nu+1\right)} D_{\nu}(0) \tag{22}$$

By solving the quantization condition (22) for ν one should impose that the wavefunctions (17) vanish for $|\eta| \to \infty$, as pointed above. By using a fourth-fifth order Runge-Kutta method [20] no solutions are found for $\nu < -1$. On the other side, for $\nu > -1$ an infinite sequence of allowed values of ν are found corresponding to each sign in front of the square root symbol in (22). The ten lowest states are given in Table 1. By inspection of Table 1 one sees that for $\nu \to \infty$ their values come equally spaced ($\Delta\nu = 2$) whatever the sign of ν is.

The energy levels are obtained by inserting those allowed values of ν in (14):

$$E = \pm\sqrt{m^2c^4 + 2\hbar c\lambda\,(\nu + 1)} \tag{23}$$

The spectrum has a dependence on ν in such a way that $\Delta E = 0$ as ν tends to $\pm\infty$. One should realize that the energy levels are symmetrical about $E = 0$. It means that the potential couples to the positive-energy component of the spinor in the same way it couples to the negative-energy component. In other words, this sort of potential couples to the mass of the fermion instead of its charge so that there is no atmosphere for the production of particle-antiparticle pairs. No matter the intensity of the coupling parameter (λ), the positive- and the negative-energy solutions never meet. There is always an energy gap greater or equal to $2mc^2$, thus there is no room for transitions from positive- to negative-energy solutions. This all means that Klein´s paradox does not come to the scenario.

It is worthwhile to note that the Dirac equation with a nonvector potential, or a vector potential contaminated with some scalar or pseudoscalar coupling, is not invariant under $V \to V + const.$, this is so because only the vector potential couples to the positive-energies in the same way it couples to the negative-ones, whereas nonvector contaminants couple to the mass of the fermion. Therefore, if there is any nonvector coupling the absolute values of the energy will have physical significance and the freedom to choose a zero-energy will be lost.

As stated in the third paragraph of this work, the anomalous magnetic interaction in the four-dimensional world turns into a pseudoscalar interaction in the two-dimensional world. The anomalous magnetic interaction has the form $-i\mu\beta\vec{\alpha}.\vec{\nabla}\phi(r)$, where μ is the anomalous magnetic moment in units of the Bohr magneton and ϕ is the electric potential, *i.e.*, the time component of a vector potential [18]. In one-plus-one dimensions the anomalous magnetic interaction turns into $\sigma_2\mu\phi'$, then one might suppose that the pseudoscalar Coulomb potential is due to an electric potential proportional to $sign(x)x^2/2$, where $sign(x)$ stands for the sign function. The oddness of the electric potential (under $x \to -x$) does not present problem to the confinement of a fermion because its effective mass, an x-dependent mass which always increases as the fermion goes away from the origin, depends on $(\phi')^2$, an result independent of the sign of $sign(x)$ [14]-[15].

For short, in addition to their intrinsic importance, the above conclusions renders a contrast to the result found in [16]. There are bound-state solutions for fermions interacting by a pseudoscalar Coulomb potential in 1+1 dimensions notwithstanding the spinor is not an eigenfunction of the parity operator. Therefore, the quadri-dimensional version of this problem requires clarification for the absence of bounded solutions. One might ponder that the underlying reason is the way the spinors are affected by the behavior of the potentials at the origin as well as at infinity because this is the radical difference between the potentials in those two dissimilar worlds.

A word should be said about the role of the π-meson field. The Lagrangian density describing the pion-nucleon interaction $\mathcal{L} = -i\lambda\overline{\psi}\gamma_5\psi\pi$ is parity-invariant because the π-meson is a pseudoscalar field, *i.e.*, $\pi(\vec{r},t) \to -\pi(-\vec{r},t)$ under the parity transformation. Nonetheless, the interaction matrix term present in the Dirac equation as written by McKeon and Van Leeuwen [16], $i\beta\gamma_5 V(r)$, supposed to be due to the π-meson field is not parity-invariant due to the reason that the potential function, $V(r)$, is parity-invariant but the potential matrix is not. Moreover, that matrix potential does not commute with the total angular momentum. These arguments expose the inadequacy of the interaction term in the Dirac equation as proposed in [16]. Therefore, any conundrum about the role of the π-meson should be consistently presented by taking into account a quintessential parity-invariant and spherically symmetric potential. Furthermore, in order to correspond to the physical reality one should be aware that the massive π-meson field gives rise to a Yukawa potential instead of a Coulomb potential.

Acknowledgments

This work was supported in part through funds provided by CNPq and FAPESP.

References

[1] S. Coleman, R. Jackiw and L. Susskind, Ann. Phys. (N.Y.) 93 (1975) 267.

[2] S. Coleman, Ann. Phys. (N.Y.) 101 (1976) 239.

[3] J. Fröhlich and E. Seiler, Helv. Phys. Acta 49 (1976) 889.

Table 1: The lowest solutions of Eq. (22) for $\nu > -1$ (plus sign inside the radical).

ν for negative square root	ν for positive square root
-0.654531	0.548571
1.468582	2.522304
3.482395	4.514353
5.487785	6.510727
7.490650	8.508354
9.492433	10.506935
11.493638	12.505887
13.494521	14.505158
15.495179	16.504544
17.495703	18.504078

[4] A. Z. Capri and R. Ferrari, Can. J. Phys. 63 (1985) 1029.

[5] H. Galić, Am. J. Phys. 56 (1988) 312.

[6] G. ´t Hooft, Nucl. Phys. B 75 (1974) 461.

[7] J. Kogut and L. Susskind, Phys. Rev. D 9 (1974) 3501.

[8] R. S. Bhalerao and B. Ram, Am. J. Phys. 69 (2001) 817.

[9] A. S. de Castro, Am. J. Phys. 70 (2002) 450.

[10] R. M. Cavalcanti, Am. J. Phys. 70 (2002) 451.

[11] J. R. Hiller, Am. J. Phys. 70 (2002) 522.

[12] W. Lucha, F. F. Schöberl and D. Gromes, Phys. Rep. 200 (1991) 127 and references therein.

[13] A.S. de Castro, Phys. Lett. A 305 (2002) 100.

[14] A.S. de Castro and W.G. Pereira, Phys. Lett. A 308 (2003) 131.

[15] A.S. de Castro, Phys. Lett. A, to be published.

[16] D.G.C. McKeon and G. Van Leeuwen, Mod. Phys. Lett. A 17 (2002) 1961.

[17] P. Strange, Relativistic Quantum Mechanics, Cambridge University Press, Cambridge, 1998.

[18] B. Thaller, The Dirac Equation (Springer-Verlag, Berlin, 1992).

[19] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions (Dover, Toronto, 1965).

[20] G. E. Forsythe, M. A. Malcolm and C. B. Moler, Computer Methods for Mathematical Computation (Prentice-Hall, New Jersey, 1977).