Elliptic Families of Solutions of the Kadomtsev – Petviashvili Equation and the Field Elliptic Calogero – Moser System*

A. Akhmetshin[**] I. Krichever[†] Yu. Volvovski[‡]

[**,‡]Department of Mathematics,
Columbia University,
2990 Broadway, Mail Code 4406, New York, NY 10027, USA

[†]Department of Mathematics,
Columbia University,
and
Landau Institute for Theoretical Physics,
Kosygina 2, 117334 Moscow, Russia,
and
Institute for Theoretical and Experimental Physics,
B. Cheremushkinskaja 25, 117259 Moscow, Russia

e-mail:
[**]alakhm@math.columbia.edu
[†]krichev@math.columbia.edu
[‡]yurik@math.columbia.edu

Abstract

We present the Lax pair for the field elliptic Calogero – Moser system and establish a connection between this system and the Kadomtsev – Petviashvili equation. Namely, we consider elliptic families of solutions of the KP equation, such that their poles satisfy a constraint of being *balanced*. We show that the dynamics of these poles is described by a reduction of the field elliptic CM system.

We construct a wide class of solutions to the field elliptic CM system by showing that any N-fold branched cover of an elliptic curve gives rise to an elliptic family of solutions of the KP equation with balanced poles.

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1 Introduction

The main goal of this paper is to establish a connection between the field analog of the elliptic Calogero – Moser system (CM) introduced in [11] and the Kadomtsev – Petviashvili equation (KP). This connection is a next step along the line which goes back to the work [1] where it was found that dynamic of poles of the elliptic (rational or trigonometric) solutions of the Korteweg – de Vries equation (KdV) can be described in terms of commuting flows of the elliptic (rational or trigonometric) CM system.

In the earlier work of one of the authors [8] it was shown that constrained correspondence between a theory of the elliptic CM system and a theory of the elliptic solutions of the KdV equation becomes an isomorphism for the case of the KP equation. It turns out that a function $u(x, y, t)$ which is an elliptic function with respect to the variable x satisfies the KP equation if and only if it has the form

$$u(x, y, t) = -2 \sum_{i=1}^{N} \wp(x - q_i(y, t)) + c, \tag{1.1}$$

and its poles q_i as functions of y satisfy the equations of motion of the elliptic CM system. The latter is a system of N particles on an elliptic curve with pairwise interactions. Its Hamiltonian has the form

$$H_2 = \frac{1}{2} \sum_{i=1}^{N} p_i^2 - 2 \sum_{i \neq j} \wp(q_i - q_j),$$

where $\wp(q)$ is the Weierstrass $\wp$-function. The dynamics of the particles q_i with respect to t coincides with the commuting flow generated by the third Hamiltonian H_3 of the system. Recall, that the elliptic CM system is a completely integrable system. It admits the Lax representation $\dot{L} = [L, M]$, where $L = L(z)$ and $M = M(z)$ are $(N \times N)$ matrices depending on a spectral parameter z [4]. The involutive integrals H_n are defined as $H_n = n^{-1} \operatorname{Tr} L^n$.

An explicit theta-functional formula for algebro-geometric solutions of the KP equation provides an *algebraic* solution of the Cauchy problem for the elliptic CM system [8]. Namely, the positions $q_i(y)$ of the particles at any time y are roots of the equation

$$\theta(\vec{U} q_i + \vec{V} y + \vec{Z} \,|\, B) = 0,$$

where the theta-function $\theta(z \,|\, B)$ is the Riemann theta-function constructed with the help of the matrix of b-periods of the holomorphic differentials on a *time-independent* spectral curve Γ. The spectral curve is given by $R(k, z) = \det(kI - L(z)) = 0$ and the vectors $\vec{U}$, $\vec{V}$, $\vec{Z}$ are defined by the initial data.

The correspondence between finite-dimensional integrable systems and poles systems of various soliton equation has been extensively studied in [2, 9, 10, 12, 13]. A general scheme of constructing such systems using a specific inverse problem for linear equations with elliptic coefficents is presented in [9].

A problem we address in this paper is as follows. The KP equation

$$\frac{3}{4} u_{yy} = \frac{\partial}{\partial x} \left(u_t - \frac{1}{4} u_{xxx} - \frac{3}{2} u u_x \right) \tag{1.2}$$

is the first equation of a hierarchy of commuting flows. A general solution of the whole hierarchy is known to be of the form

$$u(x, y, t, t_4 \ldots) = 2\frac{\partial^2}{\partial x^2} \ln \tau(x, y, t, t_4 \ldots), \qquad x = t_1,\ y = t_2,\ t = t_3,$$

where τ is the so-called KP *tau*-function. We consider solutions u that are elliptic function with respect to some variable t_k or a linear combination of times $\lambda = \sum_k \alpha_k t_k$.

It is instructive to consider first the algebraic-geometrical solutions of the KP equation. According to [7] any smooth algebraic curve Γ with a puncture defines a solution of the KP hierarchy by the formula

$$u = 2\frac{\partial^2}{\partial x^2} \ln \theta \left(\sum\nolimits_k \vec{U}_k t_k + \vec{Z} \,\middle|\, B\right), \qquad x = t_1, \tag{1.3}$$

where as before B is the matrix of b-periods of the normalized holomorphic differentials on Γ and $\vec{Z}$ is the vector of Riemann constants. The vectors $\vec{U}_k$ are the vectors of b-periods of certain meromorphic differentials on Γ. The algebraic-geometrical solution is elliptic with respect to some direction if there is a vector $\vec{\Lambda}$ which spans an elliptic curve $\mathcal{E}$ embedded in the Jacobian $J(\Gamma)$. This is a nontrivial constraint and the space of corresponding algebraic curves has codimension $g-1$ in the moduli space of all the curves. If the vector $\vec{\Lambda}$ does exist then the theta-divisor intersects the shifted elliptic curve $\mathcal{E} + \sum_k \vec{U}_k t_k$ at a finite number of points $\lambda_i(t_1, t_2, \ldots)$.

It can be shown directly that if $u(x, y, t, \lambda)$ is an elliptic family of solutions of the KP equation, then it has the form

$$u = -2\sum_{i=1}^{N} \left[\lambda_{i\,x}^2 \wp(\lambda - \lambda_i) - \lambda_{i\,xx} \zeta(\lambda - \lambda_i)\right] + c(x, y, t), \qquad \lambda_i = \lambda_i(x, y, t). \tag{1.4}$$

The sum of the residues vanishes for an elliptic function u. Therefore, $\sum_i \lambda_{ixx} = 0$. We shall consider only solutions u with the poles λ_i satisfying an additional constraint. Namely, we say that the poles λ_i, $i = 1, \ldots, N$ are *balanced* if they can be presented in the form

$$\lambda_i(x, y, t) = q_i(x, y, t) - hx, \qquad \sum_{i=1}^{N} q_i(x, y, t) = const, \tag{1.5}$$

where h is an arbitrary non-zero constant. We prove that if the poles of u are balanced, then the functions $q_i(x, y)$ satisfy the following equations:

$$\begin{aligned} q_{i\,yy} = &- \left\{\frac{q_{i\,y}^2}{h - q_{i\,x}}\right\}_x + \frac{1}{Nh}(h - q_{i\,x}) \sum_{k=1}^{N} \left\{\frac{q_{k\,y}^2}{h - q_{k\,x}}\right\}_x + \\ &+ 2(h - q_{i\,x})\frac{\delta U(q)}{\delta q_i} - \frac{2}{Nh}(h - q_{i\,x}) \sum_{k=1}^{N} (h - q_{k\,x})\frac{\delta U(q)}{\delta q_k}, \qquad 1 \le i \le N, \end{aligned} \tag{1.6}$$

where

$$\begin{aligned} U(q) = \sum_{i=1}^{N} \frac{q_{i\,xx}^2}{4(h - q_{i\,x})} &- \frac{1}{2}\sum_{j\neq i} \left[(h - q_{j\,x}) q_{i\,xx} - (h - q_{i\,x}) q_{j\,xx}\right] \zeta(q_i - q_j) + \\ &+ \frac{1}{2}\sum_{j\neq i} \left[(h - q_{j\,x})^2 (h - q_{i\,x}) + (h - q_{j\,x})(h - q_{i\,x})^2\right] \wp(q_i - q_j). \end{aligned} \tag{1.7}$$

Here $\delta/\delta q_i$ is the variational derivative. Since $U(q)$ depends only on q_i and its two derivatives, we have

$$\frac{\delta U(q)}{\delta q_i} = \frac{\partial U(q)}{\partial q_i} - \frac{d}{dx}\frac{\partial U(q)}{\partial q_{i\,x}} + \frac{d^2}{dx^2}\frac{\partial U(q)}{\partial q_{i\,xx}}, \qquad 1 \le i \le N,$$

Equations (1.6) can be identified with a reduction of a particular case of the Hamiltonian system introduced in [11]. We call the latter system a field analog of the elliptic Calogero – Moser sytem. The phase space for this system is the space of functions $q_1(x), \dots, q_N(x)$, $p_1(x), \dots, p_N(x)$, the Poisson brackets are given by

$$\{q_i(x), p_j(\tilde{x})\} = \delta_{ij}\delta(x - \tilde{x}).$$

and the Hamiltonian equals

$$\widehat{H} = \int H(x)\, dx, \qquad H = \sum_{i=1}^{N} p_i^2(h - q_{i\,x}) - \frac{1}{Nh}\left(\sum_{i=1}^{N} p_i(h - q_{i\,x})\right)^2 - \widetilde{U}(q), \tag{1.8}$$

where

$$\widetilde{U}(q) = U(q) + \frac{\partial}{\partial x}\left(\frac{h}{2}\sum_{i\neq j}(q_{i\,x} - q_{j\,x})\zeta(q_i - q_j)\right).$$

The corresponding equations of motion are presented in section 3, see (3.1). Note, that if q_i do not depend on x, then (1.8) reduces to the Hamiltonian of the elliptic CM system.

In particular, for $N = 2$ the hamiltonian reduction of this system corresponding to the constraint $\sum_i q_i = 0$ is a hamiltonian system on the space of two functions $q(x)$, $p(x)$, where we set

$$q = q_1 = -q_2, \qquad \frac{1}{h}p(h^2 - q_x^2) = p_1(h - q_x) = -p_2(h - q_x),$$

The Poisson brackets are canonical, i.e. $\{q(x), p(\tilde{x})\} = \delta(x - \tilde{x})$, while the Hamiltonian density H in the coordinates $\{p, q\}$ may be rewritten as

$$H = \frac{2}{h}p^2(h^2 - q_x^2) - h\frac{q_{xx}^2}{2(h^2 - q_x^2)} - 2h(h^2 - 3q_x^2)\wp(2q).$$

It was noticed by A. Shabat that the equations of motion given by this Hamiltonian are equivalent to Landau – Lifshitz equation. This case $N = 2$ was independently studied in [14].

The paper is organized as follows. In sections 2 and 3 we show that the field analog of the elliptic CM system describes a solution of the inverse Picard type problem for the linear equation

$$\left(\frac{\partial}{\partial y} - \mathcal{L}\right)\psi(x, y, \lambda) = 0, \qquad \mathcal{L} = \frac{\partial^2}{\partial x^2} + u(x, y, \lambda), \tag{1.9}$$

which is one of the equations in the auxiliary linear problem for the KP equation. Namely, it turns out that if equation (1.9) with a family of elliptic in λ potentials of the form (1.4) has N linearly independent meromorphic in λ *double-Bloch* solutions, then the variables $q_i = \lambda_i + hx$ satisfy the equations of motion generated by the Hamiltonian (1.8). As in [8] this inverse problem provides the Lax representation for the hamiltonian system (3.1).

In section 4 we show that if $u(x, y, t, \lambda)$ is an elliptic family of solutions of the KP equation with balanced poles, then the corresponding family of operators $\partial/\partial y - \mathcal{L}$ has infinitely many double-Bloch solution. Consequently, the dynamics of $q_i(x,y,t)$ with respect to y coincides with the equations of motion of the field elliptic CM system. We are quite sure that the dynamics of q_i with respect to all the times of the KP hierarchy coincides with the hierarchy of commuting flows for the system (3.1), but up to now this question remains open. We plan to investigate it elsewhere.

In the last section we consider the finite-gap solutions of the KP hierarchy corresponding to an algebraic curve which is an N-fold branched cover of the elliptic curve. We show that they are elliptic with respect to a certain linear combination λ of the times t_k. Moreover, as a function of λ these solutions have precisely N poles. Therefore, they provide a wide class of exact algebraic solutions of the field elliptic CM system.

The definitions and properties of classical elliptic functions and the Riemann θ-function are gathered in the appendix.

2 Generating problem

Let us choose a pair of periods $2\omega_1, 2\omega_2 \in \mathbb{C}$, where $\Im(\omega_2/\omega_1) > 0$. A meromorphic function $f(\lambda)$ is called *double-Bloch* if it satisfies the following monodromy properties:

$$f(\lambda + 2\omega_a) = B_a f(\lambda), \qquad a = 1, 2.$$

The complex constants B_a are called *Bloch multipliers*. Equivalently, $f(\lambda)$ is a section of a linear bundle over the elliptic curve $\mathcal{E} = \mathbb{C}/\mathbb{Z}[2\omega_1, 2\omega_2]$.

We consider the non-stationary Schrödinger operator

$$\partial_y - \mathcal{L} = \partial_y - \partial_{xx}^2 - u(x,y,\lambda), \qquad \partial_x = \partial/\partial x, \quad \partial_y = \partial/\partial y,$$

where the potential $u(x,y,\lambda)$ is a double-periodic function of the variable λ. We do not assume any special dependence with respect to the other variables. Our goal is to find the potentials $u(x,y,\lambda)$ such that the equation

$$(\partial_y - \mathcal{L})\,\psi(x,y,\lambda) = 0 \tag{2.1}$$

has *sufficiently many* double-Bloch solutions. The existence of such solutions turns out to be a very restrictive condition (see the discussion in [9]).

The basis in the space of the double-Bloch functions can be written in terms of the fundamental function $\Phi(\lambda,z)$ defined by the formula

$$\Phi(\lambda,z) = \frac{\sigma(z-\lambda)}{\sigma(z)\sigma(\lambda)} e^{\zeta(z)\lambda}. \tag{2.2}$$

This function is a solution of the Lamè equation

$$\Phi''(\lambda,z) = \Phi(\lambda,z)\big[\wp(z) + 2\wp(\lambda)\big]. \tag{2.3}$$

From the monodromy properties of the Weierstrass functions it follows that $\Phi(\lambda,z)$ is double-periodic as a function of z though it is not elliptic in the classical sense due to the essential singularity at $z = 0$ for $\lambda \neq 0$. It also follows that $\Phi(\lambda, z)$ is double-Bloch as a function of λ, namely

$$\Phi(\lambda + 2\omega_a, z) = T_a(z)\Phi(\lambda, z), \qquad T_a(z) = \exp\left[2\omega_a\zeta(z) - 2\eta_a z\right], \quad a = 1, 2.$$

In the fundamental domain of the lattice defined by the periods $2\omega_1$, $2\omega_2$ the function $\Phi(\lambda, z)$ has a unique pole at the point $\lambda = 0$ with the following expansion in the neighborhood of this point:

$$\Phi(\lambda, z) = \lambda^{-1} + O(\lambda). \tag{2.4}$$

Let $f(\lambda)$ be a double-Bloch function with Bloch multipliers B_a. The gauge transformation

$$f(\lambda) \longmapsto \widetilde{f}(\lambda) = f(\lambda)e^{k\lambda}$$

does not change the poles of f, and produces the double-Bloch function $\widetilde{f}(\lambda)$ with the Bloch multipliers $\widetilde{B_a} = B_a e^{2k\omega_a}$. The two pairs of Bloch multipliers B_a and $\widetilde{B_a}$ connected by such a relation are called equivalent. Note that for all the equivalent pairs of Bloch multipliers the product $B_1^{\omega_2}B_2^{-\omega_1}$ is a constant depending only on the equivalence class. Further note that any pair of Bloch multipliers may be represented in the form

$$B_a = T_a(z)e^{2\omega_a k}, \qquad a = 1, 2,$$

with an appropriate choice of the parameters z and k.

There is no differentiation with respect to the variable λ in the equation (2.1). Thus, it will be sufficient to study the double-Bloch solutions $\psi(x, t, \lambda)$ with Bloch multipliers B_a such that $B_a = T_a(z)$ for some z.

It follows from (2.4) that a double-Bloch function $f(\lambda)$ with simple poles λ_i in the fundamental domain and with Bloch multipliers $B_a = T_a(z)$ can be represented in the form

$$f(\lambda) = \sum_{i=1}^{N} s_i \Phi(\lambda - \lambda_i, z), \tag{2.5}$$

where s_i is the residue of the function $f(\lambda)$ at the pole λ_i. Indeed, the difference of the left and right hand sides in (2.5) is a double-Bloch function with the same Bloch multipliers as $f(\lambda)$. It is also holomorphic in the fundamental domain. Therefore, it equals zero since any non-zero double-Bloch function with at least one of the Bloch multipliers distinct from 1 has at least one pole in the fundamental domain.

Now we are in position to present the generating problem for the equations (1.6).

Theorem 1. *The equation* (2.1) *with the potential given by*

$$u(x, y, \lambda) = -2\sum_{i=1}^{N}\left[(\lambda_{i\,x})^2\wp(\lambda - \lambda_i) + \lambda_{i\,xx}\,\zeta(\lambda - \lambda_i)\right] + c(x, y), \tag{2.6}$$

and the balanced set of poles (1.5), *has* N *linearly independent double-Bloch solutions with Bloch multipliers* $T_a(z)$, *that is, solutions of the form* (2.5), *if and only if*

$$c(x, y) = \frac{2}{Nh}U(q) - \frac{1}{2Nh}\sum_{i=1}^{N}\frac{q_{i\,y}^2}{h - q_{i\,x}}, \tag{2.7}$$

and the functions $q_i(x,y)$ *satisfy* (1.6).

If (2.1) *has* N *linearly independent solutions of the form* (2.5) *for some* z, *then they exist for all values of* z.

Proof. We begin with a remark. In fact, if $u(x,y,\lambda)$ is an elliptic function with a balanced set of poles then it has to be of the form (2.6) provided there exist N linearly independent double-Bloch solutions of (2.1) for all values of the parameter z in a neighborhood of $z=0$.

Indeed, let us substitute (2.5) into (2.1). First of all, we conclude that the potential u has at most double poles at the points λ_i. Thus, the potential is of the form

$$u(\lambda,x,y)=\sum_{i=1}^{N}\big[a_i\wp(\lambda-\lambda_i)+b_i\zeta(\lambda-\lambda_i)\big]+c(x,y)$$

with some unknown coefficients $a_i=a_i(x,y)$ and $b_i=b_i(x,y)$. Now, the coefficients of the singular part of the right hand side in (2.1) must equal zero. The vanishing of the triple poles $(\lambda-\lambda_i)^{-3}$ implies $a_i=-2(\lambda_{i\,x})^2$. The vanishing of the double poles $(\lambda-\lambda_i)^{-2}$ gives the equalities

$$2s_{i\,x}\lambda_{i\,x}=s_i(\lambda_{i\,y}-\lambda_{i\,xx}-b_i)-\sum_{j\neq i}s_ja_i\Phi(\lambda_i-\lambda_j,z). \tag{2.8}$$

Finally, the vanishing of the simple poles $(\lambda-\lambda_i)^{-1}$ leads to the equalities

$$\begin{aligned}s_{i\,y}-s_{i\,xx}=s_i\Big(\lambda_{i\,x}^2\wp(z)+\sum_{j\neq i}\big[a_i\wp(\lambda_i-\lambda_j)+b_j\zeta(\lambda_i-\lambda_j)\big]+c\Big)+\\ +\sum_{j\neq i}s_j\big(a_i\Phi'(\lambda_i-\lambda_j,z)+b_j\Phi(\lambda_i-\lambda_j,z)\big).\end{aligned} \tag{2.9}$$

The equations (2.8) and (2.9) are linear equations for $s_i=s_i(x,y,z)$. If we introduce the vector $\vec{S}=(s_1,\dots,s_N)$ and the matrices $L=(L_{ij})$, $A=(A_{ij})$ with matrix elements

$$L_{ij}=\delta_{ij}\xi_i+(1-\delta_{ij})\lambda_{i\,x}\Phi(\lambda_i-\lambda_j,z),\qquad \text{where}\qquad \xi_i=\frac{\lambda_{i\,y}-\lambda_{i\,xx}-b_i}{2\lambda_{i\,x}}, \tag{2.10}$$

and

$$\begin{aligned}A_{ij}=\delta_{ij}\Big(\lambda_{i\,x}^2\wp(z)+\sum_{j\neq i}\big[-2\lambda_{i\,x}^2\wp(\lambda_i-\lambda_j)+b_j\zeta(\lambda_i-\lambda_j)\big]+c\Big)+\\ +(1-\delta_{ij})\big(-2\lambda_{i\,x}^2\Phi'(\lambda_i-\lambda_j,z)+b_j\Phi(\lambda_i-\lambda_j,z)\big).\end{aligned}$$

then the equations (2.8) and (2.9) can be written in the form

$$\vec{S}_x=L\vec{S},\qquad \vec{S}_y=\vec{S}_{xx}+A\vec{S}=(L^2+L_x+A)\vec{S}. \tag{2.11}$$

Let $M=L^2+L_x+A$, then the compatibility of the equations (2.11) is equivalent to the zero-curvature equation for L and M, i.e.

$$L_y-M_x+[L,M]=0. \tag{2.12}$$

The matrix elements of M can be computed with the help of the identities (A.2):

$$\begin{aligned} M_{ii} &= \lambda_{i\,x}\Big(\sum_{k=1}^{N}\lambda_{k\,x}\Big)\wp(z) + m_i^0, \\ M_{ij} &= -\lambda_{i\,x}\Big(\sum_{k=1}^{N}\lambda_{k\,x}\Big)\Phi'(\lambda_i-\lambda_j,z) + m_{ij}\Phi(\lambda_i-\lambda_j,z), \qquad i\neq j \end{aligned} \tag{2.13}$$

where

$$\begin{aligned} m_i^0 &= \xi_i^2 + \xi_{i\,x} - \sum_{k\neq i}\lambda_{k\,x}(2\lambda_{k\,x}^2+\lambda_{i\,x})\wp(\lambda_i-\lambda_k) + \sum_{k\neq i} b_k\zeta(\lambda_i-\lambda_k) + c, \\ m_{ij} &= \lambda_{i\,x}(\xi_i+\xi_j) + \lambda_{i\,xx} + b_i + \sum_{k\neq i,j}\lambda_{i\,x}\lambda_{k\,x}\,\eta(\lambda_i,\lambda_k,\lambda_j). \end{aligned}$$

The coefficients b_i can be determined from the off-diagonal part of the zero curvature equation. The left-hand side of the equation corresponding to a pair of indexes $i\neq j$ is a double-periodic function of z. It is holomorphic except at $z=0$, where it has the form $O(z^{-3})\exp\big[(\lambda_i-\lambda_j)\zeta(z)\big]$. Such a function equals zero if and only if the corresponding coefficients at z^{-3}, z^{-2} and z^{-1} vanish. A direct computation shows that the coefficient at z^{-3} vanishes identically, while the coefficient at z^{-2} equals

$$\Big(\sum_{k=1}^{N}\lambda_{k\,x}\Big)(b_i + 2\lambda_{i\,xx}).$$

Since our assumption prevents the first factor from vanishing we conclude that $b_i = -2\lambda_{i\,xx}$. Given this, another direct computation shows that the coefficient at z^{-1} also vanishes identically.

The zero-curvature equation (2.12) is not only a necessary but also a sufficient condition for (2.1) to have solutions of the form (2.5). The following lemma now completes the proof of the theorem.

Lemma 1. *Let $L=(L_{ij}(x,y,z))$ and $M=(M_{ij}(x,y,z))$ be defined by the formulas* (2.10) *and* (2.13)*, where $b_i=-2\lambda_{i\,xx}$ and the set of $\lambda_i(x,y)$, $i=1,\ldots,N$ is balanced. Then L and M satisfy the equation* (2.12) *if and only if $c(x,y)$ is given by* (2.7) *and the functions $q_i(x,y)$ solve* (1.6).

Proof. It was mentioned above that all the off-diagonal equations in (2.12) become identities if $b_i=-2\lambda_{i\,xx}$. The diagonal part of the zero-curvature equation (2.12) simplifies with the help of the identities (A.2) and (A.3). Under a change of variables $\lambda_i = q_i - hx$ it takes the form

$$\begin{aligned} q_{i\,yy} &= -2(h-q_{i\,x})c_x + \left\{\frac{q_{i\,xx}^2 - q_{i\,y}^2}{h-q_{i\,x}} + q_{i\,xxx}\right\}_x + \\ &+ 4(h-q_{i\,x})\sum_{j\neq i}\big[(h-q_{j\,x})^3\wp'(q_i-q_j) - 3(h-q_{j\,x})q_{j\,xx}\,\wp(q_i-q_j) + q_{j\,xxx}\,\zeta(q_i-q_j)\big] \end{aligned} \tag{2.14}$$

Now consider the sum of the equations (2.14) for all i from 1 to N. Since the poles are balanced, the left hand side vanishes and the coefficient at c_x becomes $-2Nh$. The other terms in the right hand side can be written as

$$\frac{\partial}{\partial x}\Big(-\sum_{i=1}^{N}\frac{q_{i\,y}^2}{h-q_{i\,x}} + 4U(q)\Big).$$

Therefore, c is given by (2.7) up to an arbitrary function of y, which does not affect the equations (2.14). Finally, substituting (2.7) into (2.14) we arrive at (1.6). $\square$

3 Field analog of the elliptic Calogero – Moser system

In this section we show that equations (1.6) can be obtained as a reduction of the field elliptic CM system.

In [5] the elliptic CM system was identified with a particular case of the Hitchin system on an elliptic curve with a puncture. In [11] a Hamiltonian theory of zero-curvature equations on algebraic curves was developed and identified with infinite-dimentional field analogs of the Hitchin system. In particular, it was shown that the zero-curvature equation on an elliptic curve with a puncture can be seen as a field generalization of the elliptic CM system.

The field elliptic CM system is a Hamiltonian system on the space of functions $\{q_i(x), p_i(x)\}_{i=1}^N$ with the canonical Poisson brackets

$$\{q_i(x), q_j(\tilde{x})\} = \{p_i(x), p_j(\tilde{x})\} = 0, \quad \{q_i(x), p_j(\tilde{x})\} = \delta_{ij}\,\delta(x-\tilde{x}), \qquad 1 \le i,j \le N,$$

Its Hamiltonian is given by (1.8). Note, that $\widetilde{U}(q)$ is elliptic function of each of the variables q_i, $i = 1, \ldots, N$. Substituting the definition of $\widetilde{U}(q)$ into (1.8) we obtain the following expression for the hamiltonian density:

$$\begin{aligned} H = {} & \sum_{i=1}^N p_i^2 (h - q_{i\,x}) - \frac{1}{Nh}\left(\sum_{i=1}^N p_i (h - q_{i\,x})\right)^2 - \\ & - \sum_{i=1}^N \frac{q_{i\,xx}^2}{4(h-q_{i\,x})} - \frac{1}{2}\sum_{i\neq j}\big[q_{i\,x} q_{j\,xx} - q_{j\,x} q_{i\,xx}\big]\zeta(q_i - q_j) + \\ & + \frac{1}{2}\sum_{i\neq j}\big[(h-q_{i\,x})^2(h-q_{j\,x}) + (h-q_{i\,x})(h-q_{j\,x})^2 - h(q_{i\,x}-q_{j\,x})^2\big]\wp(q_i-q_j). \end{aligned}$$

The equations of motion are

$$\begin{aligned} \dot{q}_i = {} & 2p_i(h - q_{i\,x}) - \frac{2}{Nh}\sum_{k=1}^N p_k (h - q_{k\,x})(h - q_{i\,x}), \\ \dot{p}_i = {} & -2p_i p_{i\,x} + \frac{2}{Nh}\left\{\sum_{k=1}^N p_i p_k (h - q_{k\,x})\right\}_x + \left\{\frac{q_{i\,xxx}}{2(h-q_{i\,x})} + \frac{q_{i\,xx}^2}{4(h-q_{i\,x})^2}\right\}_x + \\ & + 2\sum_{j\neq i}\big[q_{j\,xxx}\zeta(q_i-q_j) - 3(h-q_{j\,x})q_{j\,xx}\wp(q_i-q_j) + (h-q_{j\,x})^3\wp'(q_i-q_j)\big] \end{aligned} \tag{3.1}$$

Let us make a remark on the notations. Throughout this section by dots we mean derivatives with respect to the variable y, which we treat as a time variable. In view of the connection with the KP equation this time variable corresponds to the second time of the KP hierarchy, for which y is a standard notation.

It is easy to check that the subspace $\mathcal{N}$ defined by the constraint

$$\sum_{i=1}^{N} q_i(x) = const, \tag{3.2}$$

is invariant for the system (3.1). On that subspace the first two terms of the Hamiltonian density H can be represented in the form

$$H = \frac{1}{2Nh}\left(\sum_{i\neq j}(p_i - p_j)^2(h - q_{i\,x})(h - q_{j\,x})\right) - \widetilde{U}(q). \tag{3.3}$$

Therefore, the Hamiltonian (1.8) restricted to $\mathcal{N}$ is invariant under the transformation

$$p_i(x) \to p_i(x) + f(x), \tag{3.4}$$

where $f(x)$ is an arbitrary function. The constraint (3.2) is the Hamiltonian of that symmetry. The canonical symplectic form is also invariant with respect to (3.2). Therefore, the Hamiltonian system (3.1) restricted to $\mathcal{N}$ can be reduced to a factor space. The reduction can be described as follows.

Let us define the variables $\ell_i = p_i + \kappa$, $i = 1, \ldots, N$, where

$$\kappa = -\frac{1}{Nh}\sum_{k=1}^{N} p_k(h - q_{k\,x}). \tag{3.5}$$

They are invariant with respect to the symmetry (3.4), and satisfy the equation

$$\sum_{k=1}^{N} \ell_k(h - q_{k\,x}) = 0. \tag{3.6}$$

A direct substitution shows that equations (3.1) imply the system of equation

$$\begin{aligned} \dot{q}_i &= 2\ell_i(h - q_{ix}), \\ \dot{\ell}_i &= -2\ell_i\ell_{i\,x} + \frac{2}{Nh}\left\{\sum_{k=1}^{N}\ell_k^2(h - q_{kx}) - U(q)\right\}_x + \left\{\frac{q_{i\,xxx}}{2(h - q_{i\,x})} + \frac{q_{i\,xx}^2}{4(h - q_{i\,x})^2}\right\}_x + \\ &\quad + 2\sum_{j\neq i}\left[(h - q_{j\,x})^3\wp'(q_i - q_j) - 3(h - q_{j\,x})q_{j\,xx}\,\wp(q_i - q_j) + q_{j\,xxx}\,\zeta(q_i - q_j)\right], \end{aligned} \tag{3.7}$$

Theorem 2. *Equations* (1.6) *are equivalent to the restriction of the system* (3.7) *to the subspace* $\mathcal{M}$ *defined by the constraints* (3.2) *and* (3.6).

Proof. Let us show that equations (1.6) imply (3.7). The first equations can be regarded as the definition of ℓ_i, $i = 1, \ldots, N$. Taking their derivative we obtain

$$\ddot{q}_i = 2\dot{\ell}_i(h - q_{i\,x}) - 2\ell_i\big(2\ell_{i\,x}(h - q_{i\,x}) - 2\ell_i q_{i\,xx}\big). \tag{3.8}$$

Therefore,

$$\dot{\ell}_i = 2\ell_i \ell_{i\,x} - 2\ell_i^2 \frac{q_{i\,xx}}{h - q_{i\,x}} + \frac{\ddot{q}_i}{2(h - q_{i\,x})}.$$

To obtain the second equations of (3.7) it is now sufficient to substitute the right hand side of (2.14) for $\ddot{q}_i$ and use formula (2.7).

The equation (3.8) can also be used to derive (1.6) from (3.7) in a straightforward manner. □

Note that a solution of (3.7) restricted to the the subspace $\mathcal{M}$ defines a solution of (3.1) uniquely up to initial data. Namely, it can be checked directly that if $\kappa(x, y)$ as a solution of the equation

$$\dot{\kappa} = \left\{ -\kappa^2 + \frac{2}{Nh} \sum_{k=1}^{N} \ell_i^2 (h - q_{kx}) - \frac{2}{Nh} U(q) \right\}_x , \tag{3.9}$$

and ℓ_i, q_i is a solution of (3.7) on $\mathcal{M}$, then q_i, $p_i = \ell_i - \kappa$ is a solution of (3.1).

Our final goal for this section is to present the Lax pair for the field elliptic CM system.

Theorem 3. *System* (3.1) *admits the zero-curvature representation, i. e. it is equivalent to the matrix equation*

$$\widetilde{L}_y - \widetilde{M}_x + [\widetilde{L}, \widetilde{M}] = 0,$$

with the Lax matrices $\widetilde{L} = (\widetilde{L}_{ij})$ *and* $\widetilde{M} = (\widetilde{M}_{ij})$ *of the form*

$$\begin{aligned} \widetilde{L}_{ij} &= -\delta_{ij} p_i + (1 - \delta_{ij}) \alpha_i \alpha_j \Phi(q_i - q_j, z), \\ \widetilde{M}_{ij} &= \delta_{ij} \big[-Nh\alpha_i^2 \wp(z) + \widetilde{m}_i^0 \big] + (1 - \delta_{ij}) \alpha_i \alpha_j \big[Nh\, \Phi'(q_i - q_j, z) - \widetilde{m}_{ij} \Phi(q_i - q_j, z) \big], \end{aligned} \tag{3.10}$$

where $\alpha_i^2 = q_{i\,x} - h$,

$$\begin{aligned} \widetilde{m}_i^0 &= p_i^2 + \frac{\alpha_{ixx}}{\alpha_i} + 2\kappa p_i - \sum_{j \neq i} \big[\alpha_j^2 (2\alpha_i^4 + \alpha_j^2) \wp(q_i - q_j) + 4\alpha_i \alpha_{i\,x} \zeta(q_i - q_j) \big], \\ \widetilde{m}_{ij} &= p_i + p_j + 2\kappa + \frac{\alpha_{i\,x}}{\alpha_i} - \frac{\alpha_{j\,x}}{\alpha_j} + \sum_{k \neq i,j} \alpha_k^2 \eta(q_i, q_k, q_j), \end{aligned}$$

and κ *is given by* (3.9).

Proof. If we apply to the matrices L and M given by (2.10) and (2.13) a gauge transformation

$$L \longmapsto g_x g^{-1} + gLg^{-1}, \qquad M \longmapsto g_y g^{-1} + gMg^{-1},$$

where g is a diagonal matrix, $g = (g_{ij})$, $g_{ij} = \delta_{ij} (\lambda_{i\,x})^{-1/2}$, and then substitute $\lambda_i = q_i - hx$ and $\lambda_{i\,y}/2\lambda_{i\,x} = \ell_i$, that would give us a Lax pair for system (3.7). To obtain (3.10) we apply another gauge tranformation with $g = e^K I$ and substitute $\ell_i = p_i + \kappa$, $i = 1, \dots, N$. Here $K = K(x, y) = \int^x \kappa(\widetilde{x}, y)\, d\widetilde{x}$. Note that $K_y = -\kappa^2 - c$ due to (3.9) and (2.7). □

4 Elliptic families of solutions of the KP equation

The KP equation (1.2) is equivalent to the commutation condition

$$[\partial_y - \mathcal{L}, \partial_t - \mathcal{A}] = 0, \qquad \partial_y = \partial/\partial y, \quad \partial_t = \partial/\partial t, \tag{4.1}$$

for the auxiliary linear differential operators

$$\mathcal{L} = \partial_{xx}^2 + u(x,y,t), \qquad \mathcal{A} = \partial_{xxx}^3 + \frac{3}{2}u\partial_x + w(x,y,t), \qquad \partial_x = \partial/\partial x.$$

We use this representation in order to derive our main result.

Theorem 4. *Let $u(x,y,t,\lambda)$ be an elliptic family of solution to the KP equation that has a balanced set of poles $\lambda_i(x,y,t) = q_i(x,y,t) - hx$, $i = 1,\dots,N$. Then $u(x,y,t,\lambda)$ has the form* (1.4) *and the dynamics of the functions $q_i(x,y,t)$ with respect to y is described by the system* (1.6).

Proof. Substituting u into (1.2) we immediately conclude that u may have poles in λ of at most second order. Moreover, comparing the coefficients of the expansions of the left and right hand sides in (1.2) near the pole λ_i, we deduce that the principal part of the solution u coincides with the one given by (1.4).

Next step is to show that operator equation (4.1) implies the existence of double-Bloch solutions for the equation $(\partial_y - \mathcal{L})\psi(x,y,t,\lambda) = 0$.

Let us define a matrix $S(x,y,t,z)$ to be a solution of the linear differential equation $\partial_x S = LS$, where $L = (L_{ij})$,

$$L_{ij} = \delta_{ij}\left(\frac{\lambda_{i\,y} + \lambda_{i\,xx}}{2\lambda_{i\,x}}\right) + (1-\delta_{ij})\lambda_{i\,x}\,\Phi(\lambda_i - \lambda_j, z),$$

with the initial conditions $S(0,y,t,z) = S_0(y,t,z)$, a non-singular matrix. By Φ we denote the row-vector $(\Phi(\lambda-\lambda_1, z),\dots,\Phi(\lambda-\lambda_N,z))$. It follows immediately that the vector $(\partial_y - \mathcal{L})\Phi S$ has at most simple poles at λ_i, $i = 1,\dots,N$. Therefore, it is equal to ΦD for some matrix D. The commutation relation (4.1) implies that $D_x = LD$. To show this, consider the vector

$$(\partial_t - \mathcal{A})\Phi D = (\partial_t - \mathcal{A})(\partial_t - \mathcal{L})\Phi S = (\partial_y - \mathcal{L})(\partial_t - \mathcal{A})\Phi S.$$

It has the poles of the at most third order and therefore the vector $(\partial_t - \mathcal{A})\phi S$ has at most simple poles. In this case, however, the vector

$$(\partial_t - \mathcal{A})(\partial_t - \mathcal{L})\Phi S = (\partial_y - \mathcal{L})(\partial_t - \mathcal{A})\Phi S = (\partial_t - \mathcal{A})\Phi D$$

has the poles of the at most second order. Vanishing of the poles of the third order in the expression $(\partial_t - \mathcal{A})\Phi D$ is equivalent to the equation $D_x = LD$.

Since S and D are solutions to the same linear differential equation in x they differ by an x-independent matrix, namely $D(x,y,t,z) = S(x,y,t,z)T(y,t,z)$. Let us define a matrix $F(y,t,z)$ from the equation $\partial_y F + TF = 0$ and the initial condition $F(0,t,z) = I$. Here I is the identity matrix. Let $\widetilde{S} = SF$, then

$$(\partial_y - \mathcal{L})\phi\widetilde{S} = (\partial_y - \mathcal{L})\phi SF = \phi DF + \phi SF_y = \phi S\,(TF + F_y) = 0,$$

and the components of the vector $\phi\widetilde{S}$ are independent double-Bloch solutions to (2.1).

To conclude the proof it now suffices to apply Theorem 1. $\square$

5 The algebraic-geometric solutions

According to [7], a smooth genus g algebraic curve Γ with fixed local coordinate w at a puncture P_0 defines solutions of the entire KP hierarchy by the formula

$$u(t) = 2\frac{\partial^2}{\partial x^2}\ln\theta\left(\sum_k \vec{U}_k t_k + \vec{Z} \mid B\right) + const.$$

Here $B = (B_{jk})$ is a matrix of the b-periods of normalized holomorphic differentials ω_k^h

$$\oint_{a_i} \omega_j^h = \delta_{ij}, \qquad\qquad B_{ij} = \oint_{b_i} \omega_j^h, \tag{5.1}$$

while the vectors $\vec{U}_k = (\vec{U}_k^j)$ are vectors of the b-periods

$$\vec{U}_k^j = \frac{1}{2\pi i}\oint_{b_j} d\Omega_k, \qquad\qquad \oint_{a_j} d\Omega_k = 0,$$

of the normalized meromorphic differentials of the second kind $d\Omega_k$, defined by their expansions

$$d\Omega_k = dw^{-k} + O(1)dw \tag{5.2}$$

in the neighborhood of P_0.

Let Γ be a N-fold branched cover of an elliptic curve $\mathcal{E}$:

$$\rho\colon \Gamma \longrightarrow \mathcal{E}.$$

Then the induced map of the Jacobians defines an embedding of $\mathcal{E}$ into $J(\Gamma)$, i.e. $\rho^*\mathcal{E} \subset J(\Gamma)$. Therefore, any N-fold cover of $\mathcal{E}$ defines an elliptic family of solutions of the KP equation. The following assertion shows that the corresponding solutions have exactly N poles. Moreover, if the local coordinate w at the puncture P_0 is $\rho^*(\lambda)$, then the poles are balanced. Here λ is a flat coordinate on $\mathcal{E}$.

Theorem 5. *Let Γ be a smooth N-fold branched cover of the elliptic curve $\mathcal{E}$, and let $P_0 \in \Gamma$ be a preimage of the point $\lambda = 0$ on $\mathcal{E}$. Let $d\Omega_k$ be a normalized meromorphic differential on Γ with the only pole at P_0 of the form* (5.2)*, where $w = \rho^*(\lambda)$, and let $2\pi i\vec{U}$ and $2\pi i\vec{V}$ be the vectors of b-periods of the differentials $d\Omega_1$ and $d\Omega_2$ respectively. Then the equation*

$$\theta(\vec{\Lambda}\lambda + \vec{U}x + \vec{V}y \mid B) = 0 \tag{5.3}$$

has N balanced roots $\lambda_i(x,y) = q_i(x,y) - x/N$, $\sum_i q_i(x,y) = 0$, and the functions q_i satisfy system (1.6)*.*

Proof. Let $2\omega_1$, $2\omega_2$ be the periods of $\mathcal{E}$, such that $\Im(\tau) = \Im(\omega_2/\omega_1) > 0$. The Jacobian $J(\Gamma)$ is the factor of $\mathbb{C}^g$ over the lattice $\mathcal{B}$, spanned by the basis vectors $\vec{e}_i \in \mathbb{C}^g$, $i = 1, \dots, g$ and the columns $\vec{B}_i = (B_{ij}) \in \mathbb{C}^g$, $i = 1, \dots, g$, of the matrix B. Let $\vec{\Lambda}$ be a vector in $\mathbb{C}^g$ that spans $\rho^*\mathcal{E} \subset J(\Gamma)$. Note that not only $\vec{\Lambda} \in \mathcal{B}$, but also $\tau\vec{\Lambda} \in \mathcal{B}$.

The function $\theta(\sum_k \vec{U}_k t_k + \vec{\Lambda}\lambda + \vec{Z} \mid B)$ as a function of λ has a finite number D of zeros. Its monodromy properties (A.4) imply that it can be written as

$$\theta\left(\sum\nolimits_k \vec{U}_k t_k + \vec{\Lambda}\lambda + \vec{Z} \mid B\right) = f(t)e^{c_1\lambda + c_2\lambda^2} \prod_{i=1}^{D} \sigma(\lambda - \lambda_i(t)),$$

where c_1, c_2 are constants.

Note, that the λ_i's are defined modulo the periods of $\mathcal{E}$. In order to count them we integrate $d \ln \theta$ along the boundary of the fundamental domain of $\rho^*\mathcal{E}$ in $\mathbb{C}^g$.

The embedding of $\mathcal{E}$ in $J(\Gamma)$ is defined by equivalence classes of the divisors $\rho^*(z) - \rho^*(0)$, where $\rho^*(z)$ is the divisor of preimages on Γ of a point $z \in \mathcal{E}$. Preimages on Γ of a and b-cycles of $\mathcal{E}$ are some linear combination of the basis cycles on Γ, i.e.

$$\rho^* a = \sum_{k=1}^{g} n_k a_k + m_k b_k, \qquad \rho^* b = \sum_{k=1}^{g} n'_k a_k + m'_k b_k.$$

Therefore, the vector $\vec{\Lambda}$ equals

$$\vec{\Lambda} = \sum_{k=1}^{g} n_k \vec{e}_k + m_k \vec{B}_k, \qquad \tau\vec{\Lambda} = \sum_{k=1}^{g} n'_k \vec{e}_k + m'_k \vec{B}_k.$$

The usual residue arguments imply

$$2\pi i D = \oint_{\partial(\rho^*\mathcal{E})} d\ln\theta = \int_{\tau\vec{\Lambda}} \left(\int_{\vec{\Lambda}} d\ln\theta\right) - \int_{\vec{\Lambda}} \left(\int_{\tau\vec{\Lambda}} d\ln\theta\right)$$

The monodromy properties of the theta-function imply

$$D = \sum_{k=1}^{g} (n_k m'_k - n'_k m_k).$$

The right hand side in the last formula is the intersection number of the cycles $\rho^* a$ and $\rho^* b$, i. e.

$$D = (\rho^* a) \cap (\rho^* b) = N\,(a \cap b) = N,$$

so the the theta-function has exactly N zeros λ_i, $i = 1, \ldots, N$.

Now let us show that the set of λ_i's is balanced. In a way similar to the residue argument above we find

$$-2\pi i \sum_{j=1}^{N} \frac{\partial\lambda_j}{\partial t_k} = \oint_{\partial(\rho^*\mathcal{E})} (\partial_{t_k} \ln\theta)\, d\lambda = \int_b d\lambda \left(\int_{\rho^* a} d\Omega_k\right) - \int_a d\lambda \left(\int_{\rho^* b} d\Omega_k\right) \tag{5.4}$$

Let $\mathrm{Tr}\, d\Omega = \rho_*(d\Omega_k)$ be the sum of $d\Omega_k$ on all the sheets of Γ over a point $\lambda \in \mathcal{E}$. It is a meromorphic differential on $\mathcal{E}$. Since the local coordinate w near the puncture is defined by the projection ρ we have

$$\mathrm{Tr}\, d\Omega_k = \frac{(-1)^k}{(k-1)!} \wp^{(k-1)}(\lambda)\, d\lambda + r_k d\lambda, \tag{5.5}$$

where r_k is a constant. The right hand side in (5.4) can be written as $2\pi i\, \text{res}_{\lambda=0}(\text{Tr}\,\Omega_k)\, d\lambda$. When $k > 1$ it equals zero, while for $k = 1$ we have

$$\text{res}_{\lambda=0}(\text{Tr}\,\Omega_1)\, d\lambda = \text{res}_{\lambda=0}\, \zeta(\lambda)\, d\lambda = 1.$$

Therefore, we obtain

$$\sum_{i=1}^{N} \frac{\partial \lambda_j}{\partial x} = -1, \qquad \sum_{i=1}^{N} \frac{\partial \lambda_j}{\partial t_k} = 0, \quad k > 1. \tag{5.6}$$

and consequently the set λ_i, $i = 1, \ldots, N$ satisfy (1.5). Note, that our choice of a local coordinate near the puncture corresponds to $h = 1/N$. An arbitrary non-zero value of h may be obtained by setting $w = \rho^*(\lambda/Nh)$. Theorem 5 is proved. □

Remark. If $q_i(x, y)$, $i = 1, \ldots, N$ are periodic functions of x, then the algebraic curve Γ can be identified with the spectral curve for the equation $(\partial_x - L)\vec{S} = 0$ (see [11]).

A Appendix

A.1 Elliptic functions

Here we list the definitions and basic properties of the classical elliptic functions (see [3] for details). Let $2\omega_1, 2\omega_2 \in \mathbb{C}$ be a pair of periods, $\Im(\omega_2/\omega_1) > 0$. The Weierstrass sigma-function is defined by the infinite product

$$\sigma(z) = z \prod\nolimits_{m^2+n^2 \neq 0} \left(1 - \frac{z}{\omega_{mn}}\right) \exp\left\{\frac{z}{\omega_{mn}} + \frac{z^2}{2\omega_{mn}^2}\right\}, \qquad \omega_{mn} = 2m\omega_1 + 2n\omega_2.$$

The product converges for every z to an entire function with simple zeros at the points $z = \omega_{mn}$. The Weierstrass zeta-function and $\wp$-function are then defined by

$$\zeta(z) = \frac{\sigma'(z)}{\sigma(z)}, \qquad \wp(z) = -\zeta'(z).$$

It follows directly from this definition that $\sigma(z)$ and $\zeta(z)$ are odd functions while $\wp(z)$ is an even function. Under shifts of the periods the Weierstrass functions transform as follows:

$$\sigma(z + 2\omega_a) = e^{2\eta_a(z+\omega_a)}\sigma(z), \quad \zeta(z + 2\omega_a) = \zeta(z) + 2\eta_a, \qquad a = 1, 2,$$

where $\eta_a = \zeta(\omega_a)$ and $\eta_1\omega_2 - \eta_2\omega_1 = \pi i/2$. The $\wp$-function is double-periodic

$$\wp(z + 2\omega_1) = \wp(z + 2\omega_2) = \wp(z) = \wp(-z)$$

and can be regarded as a function on the elliptic curve $\Gamma = \mathbb{C}/\mathbb{Z}[2\omega_1, 2\omega_2]$ where it has the only (double) pole at $z = 0$. It is useful to write the Laurent expansions of the Weierstrass functions in the neighborhood of $z = 0$:

$$\sigma(z) = z + O(z^5), \qquad \zeta(z) = \frac{1}{z} + O(z^3), \qquad \wp(z) = \frac{1}{z^2} + O(z^2).$$

A.2 Identities with the function $\Phi(\lambda, z)$

Here we collect some useful identities involving the function $\Phi(\lambda, z)$, which is defined by (2.2).

The derivative of the function $\Phi(\lambda, z)$ with respect to the variable λ equals

$$\Phi'(\lambda, z) = \Phi(\lambda, z)\big[\zeta(z) - \zeta(\lambda) - \zeta(z-\lambda)\big] \tag{A.1}$$

We also have the following product identities:

$$\begin{aligned}
&\Phi(\lambda-\mu, z)\Phi(\mu-\lambda, z) = \wp(z) - \wp(\lambda-\mu),\\
&\Phi(\lambda-\nu, z)\Phi(\nu-\mu, z) = -\Phi'(\lambda-\mu, z) + \Phi(\lambda-\mu, z)\eta(\lambda, \nu, \mu),
\end{aligned} \tag{A.2}$$

where in the second equation we use the notation

$$\eta(\lambda, \nu, \mu) = \zeta(\lambda-\nu) + \zeta(\nu-\mu) - \zeta(\lambda-\mu).$$

Note that η is a completely antisymmetric function of its arguments. To complete the list of the identities required for our computations we differentiate formulas (A.2) to get

$$\begin{aligned}
&\Phi'(\lambda-\mu, z)\Phi(\mu-\lambda, z) - \Phi(\lambda-\mu, z)\Phi'(\mu-\lambda, z) = -\wp'(\lambda-\mu),\\
&\Phi'(\lambda-\nu, z)\Phi(\nu-\mu, z) - \Phi(\lambda-\nu, z)\Phi'(\nu-\mu, z) = -\Phi(\lambda-\mu)\big[\wp(\lambda-\nu) - \wp(\nu-\mu)\big].
\end{aligned} \tag{A.3}$$

A.3 Riemann θ-function

Let Γ be a genus g algebraic curve with fixed basis of cycles a_i, b_i, $i \le 1 \le g$ with intersections $a_i \circ b_j = \delta_{ij}$. Let B be the matrix of normalized holomorphic differentials ω_i^h, see (5.1). Then B is a Riemann matrix, i.e. a symmetric $g \times g$ matrix with positive definite imaginary part $\Im B < 0$.

The Riemann θ-function, associated with the curve Γ is an analitic function of g complex variables $\vec{z} = (z_1, \ldots, z_g)$, defined by its Fourier expansion

$$\theta(\vec{z}\,|\,B) = \sum\nolimits_{\vec{n}\in\mathbb{Z}^g} e^{2\pi i(\vec{m},\vec{z})+\pi i(B\vec{m},\vec{m})}\ .$$

The Riemann θ-function has the following monodromy properties with respect to the lattice $\mathcal{B}$, spanned by the basis vectors Let $\vec{e}_i \in \mathbb{C}^g$, $i = 1, \ldots, g$ and the columns $B_i \in \mathbb{C}^g$ of the matrix B:

$$\begin{aligned}
&\theta(\vec{z} + \vec{n}\,|\,B) = \theta(\vec{z}\,|\,B),\\
&\theta(\vec{z} + B\vec{n}\,|\,B) = \exp\big[-2\pi i(\vec{n},\vec{z}) - \pi i(B\vec{n},\vec{n})\big]\theta(\vec{z}\,|\,B).
\end{aligned} \tag{A.4}$$

Here $\vec{n}$ is a vector with integer components.

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