Propagator for a spin 1/2 particle in terms of the unitary representations of the Lorentz group

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Abstract

In this paper we extend our previous result on the description of the partcle motion in a generalized Heisenberg picture to a relativistic fermion. The operators of the Lorentz algebra in this picture may be regarded as field operators. In this approach the transition amplitudes for the particle are constructed in terms of two-component functions in the unitary representations of the Lorentz group.

In [1] it was found that the propagation of a massive relativistic particle may be defined as space-time transition between states with equal eigenvalues of the first and second Casimir operators C_1 and C_2 of the Lorentz algebra in the unitary representations . In addition a vector on the light-cone n $(n_0^2 - \mathbf{n}^2 = 0)$ was used. An integral representation for the transition amplitude for a massive particle with spin 0 has been obtained. This method for describing the particle motion is based on a generalized Heisenberg/Schrödinger picture in which either the analogue of Heisenberg states or the analogue of Schrödinger operators are independent of both time and space coordinates t, $\mathbf{x}$ [2]. There is no $\mathbf{x}$ representation.

In the present work we apply this approach for such a practically important example as spin 1/2 particle. The unitary representations of the Lorentz group correspond to the eigenvalues $1 + \alpha^2 - \lambda^2$ of the operator C_1 and the eigenvalues $\alpha\lambda$ of the operator C_2 $(0 \leq \alpha < \infty, \quad \lambda = -s, ..., s; s = spin)$

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[3]. We find for $\lambda = -1/2, 1/2$ the eigenfunctions of both operators C_1 and C_2 which contain the vector on the light-cone n and use these functions to construct the transition amplitudes for a spin 1/2 particle.

At first for the subsequent presentation we will give a short review for the description of the particle motion in the generalized Heisenberg picture. In this picture the coordinates t, $\mathbf{x}$ occur equal in the description and the operators of the Lorentz algebra ($\mathbf{N}$, $\mathbf{J}$ - are space-time independent operators)

$$\mathbf{N}(x) = S^{-1}(x)\mathbf{N}S(x) = \mathbf{N} + t\mathbf{P} - \mathbf{x}H, \tag{1}$$

$$\mathbf{J}(x) = S^{-1}(x)\mathbf{J}S(x) = \mathbf{J} - \mathbf{x} \times \mathbf{P}, \tag{2}$$

may be considered as field operators. In (1), (2) H and $\mathbf{P}$ are the Hamilton and momentum operators of the particle in the generalized Schrödinger picture and $S(x) = \exp[-i(tH - \mathbf{x} \cdot \mathbf{P})]$. The equations for this field may be written in the form in which the operators H, $\mathbf{P}$ play the source role

$$\nabla_{\mathbf{x}} \times \mathbf{J}(x) = \frac{\partial \mathbf{N}(x)}{\partial t} + \mathbf{P}, \quad \nabla_{\mathbf{x}} \cdot \mathbf{J}(x) = 0, \tag{3}$$

$$\nabla_{\mathbf{x}} \cdot \mathbf{N}(x) = -3H, \quad \nabla_{\mathbf{x}} \times \mathbf{N}(x) = 0. \tag{4}$$

In the classical version of this field along the trajectory of the particle ($\mathbf{x}_t = \mathbf{x}_0 + (t - t_0)\mathbf{P}/H$) for two points one can find ($C_1(x) = \mathbf{N}^2(x) - \mathbf{J}^2(x)$, $C_2(x) = \mathbf{N}(x) \cdot \mathbf{J}(x)$)

$$C_1(x_1) = C_1(x_2), \quad C_2(x_1) = C_2(x_2). \tag{5}$$

The classical analogue of the operators $\mathbf{N}$, $\mathbf{J}$ must be separated from the integrals of motion of the particle. In this case the conversion from the relativistic mechanics to the quantum version takes place. This leads to the expression for the transition amplitude of the particle in terms of the eigenstates $|n, \lambda, \alpha>$ of the operators C_1, C_2 and the vector n

$$K(x_2; x_1, \alpha, \lambda, n) =< \alpha, \lambda, n|S(x_2 - x_1)|n^{'}, \lambda^{'}, \alpha^{'} >_{\alpha,\lambda,n=\alpha',\lambda',n'}. \tag{6}$$

We use the momentum representation (m = mass, $p_0 = \sqrt{m^2 + \mathbf{p}^2}$, $\vec{\sigma}$ - are the Pauli matrices)

$$\mathbf{N} = ip_0\nabla_{\mathbf{p}} - \frac{\vec{\sigma} \times \mathbf{p}}{2(p_0 + m)}, \quad \mathbf{J} = -i\mathbf{p} \times \nabla_{\mathbf{p}} + \frac{\vec{\sigma}}{2}. \tag{7}$$

We look for the solutions of the equations

$$C_1\, \zeta_\lambda(\mathbf{p}; \alpha, \mathbf{n}) = (1 + \alpha^2 - 1/4)\, \zeta_\lambda(\mathbf{p}; \alpha, \mathbf{n}), \tag{8}$$

$$C_2\,\zeta_\lambda(\mathbf{p};\alpha,\mathbf{n}) = \alpha\lambda\,\zeta_\lambda(\mathbf{p};\alpha,\mathbf{n}), \tag{9}$$

in the form

$$\zeta_\lambda(\mathbf{p};\alpha,\mathbf{n}) = A_\lambda(\mathbf{p};\mathbf{n})\,\xi^{(0)}(\mathbf{p};\alpha,\mathbf{n}), \tag{10}$$

where ($\mathbf{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$

$$\xi^{(0)}(\mathbf{p},\alpha,\mathbf{n}) = \frac{1}{(2\pi)^{3/2}}[(pn)/m]^{-1+i\alpha}, \tag{11}$$

are the eigenfunctions of the operator C_1 for the particle with spin zero [4, 5, 6]).

Substituting (10) into equations (8) and (9), we obtain for $A_\lambda(\mathbf{p};\mathbf{n})$

$$C_1\,A_\lambda(\mathbf{p};\mathbf{n}) = -\frac{1}{4}A_\lambda(\mathbf{p};\mathbf{n}), \quad C_2\,A_\lambda(\mathbf{p};\mathbf{n}) = -\imath\lambda A_\lambda(\mathbf{p};\mathbf{n}), \tag{12}$$

$$[m^2\mathbf{n}\nabla_{\mathbf{p}} - (pn)\mathbf{p}\nabla_{\mathbf{p}} - \imath\frac{m\vec{\sigma}\cdot(\mathbf{p}\times\mathbf{n})}{2(p_0+m)}]A_\lambda(\mathbf{p};\mathbf{n}) = 0, \tag{13}$$

$$[m\vec{\sigma}\cdot\mathbf{n} - \frac{pn+m}{p_0+m}\vec{\sigma}\cdot\mathbf{p}]A_\lambda(\mathbf{p};\mathbf{n}) = 2\lambda(pn)A_\lambda(\mathbf{p};\mathbf{n}). \tag{14}$$

The calculations give ($p_- = p_1 - \imath p_2$)

$$A_{1/2}(\mathbf{p};\mathbf{n}) = B\left(\begin{array}{c} (p_0+m-p_3)e^{-i\varphi/2}\cos(\theta/2) - p_-e^{i\varphi/2}\sin(\theta/2) \\ (p_0+m+p_3)e^{i\varphi/2}\sin(\theta/2) - p_+e^{-i\varphi/2}\cos(\theta/2) \end{array}\right), \tag{15}$$

$$A_{-1/2}(\mathbf{p};\mathbf{n}) = B\left(\begin{array}{c} p_-e^{i\varphi/2}\cos(\theta/2) - (p_0+m+p_3)e^{-i\varphi/2}\sin(\theta/2) \\ (p_0+m-p_3)e^{i\varphi/2}\cos(\theta/2) - p_+e^{-i\varphi/2}\sin(\theta/2) \end{array}\right), \tag{16}$$

where

$$B(\mathbf{p};\mathbf{n}) = \frac{1}{\sqrt{2(p_0+m)(pn)}}. \tag{17}$$

As a result we have

$$\zeta_{1/2}(\mathbf{p};\alpha,\mathbf{n}) = A(\mathbf{p};\mathbf{n})\,\xi^{(0)}(\mathbf{p},\alpha,\mathbf{n})\chi_{1/2}, \quad \chi_{1/2} = \left(\begin{array}{c} 1 \\ 0 \end{array}\right), \tag{18}$$

$$\zeta_{-1/2}(\mathbf{p};\alpha,\mathbf{n}) = A(\mathbf{p};\mathbf{n})\,\xi^{(0)}(\mathbf{p},\alpha,\mathbf{n})\chi_{-1/2}, \quad \chi_{-1/2} = \left(\begin{array}{c} 0 \\ 1 \end{array}\right), \tag{19}$$

here $A(\mathbf{p};\mathbf{n})$ is the matrix whose columns are $A_{1/2}(\mathbf{p};\mathbf{n})$ and $A_{-1/2}(\mathbf{p};\mathbf{n})$

$$A^\dagger(\mathbf{p},\mathbf{n})\,A(\mathbf{p},\mathbf{n}) = 1. \tag{20}$$

The completeness and orthogonality relations for the functions $\zeta_\lambda(\mathbf{p};\alpha,\mathbf{n})$ and $\zeta^*_\lambda(\mathbf{p};\alpha,\mathbf{n})$ have the form

$$\int(\alpha^2+1/4)d\alpha\, d\omega_{\mathbf{n}}\,\zeta^*_{\lambda_1}(\mathbf{p};\alpha,\mathbf{n})\,\zeta_{\lambda_2}(\mathbf{p}^{'};\alpha,\mathbf{n})=\delta_{\lambda_1\lambda_2}p_0\delta(\mathbf{p}-\mathbf{p}^{'}), \qquad (21)$$

$$\sum_{\lambda=-1/2}^{1/2}\int\frac{d\mathbf{p}}{p_0}\,\zeta^*_\lambda(\mathbf{p};\alpha,\mathbf{n})\,\zeta_\lambda(\mathbf{p};\alpha^{'},\mathbf{n}^{'})=\frac{\delta(\mathbf{n}-\mathbf{n}^{'})\,\delta(\alpha-\alpha^{'})}{\alpha^2+1/4}. \qquad (22)$$

Using in (6) the functions $\zeta_{1/2}(\mathbf{p};\alpha,\mathbf{n})$ and $\zeta_{-1/2}(\mathbf{p};\alpha,\mathbf{n})$ we obtain for the transition amplitudes with $\lambda=\lambda^{'}=1/2$ and $\lambda=\lambda^{'}=-1/2$ equal integral representation

$$\begin{aligned} K(x_2;x_1,1/2,n) &= \int\frac{d\mathbf{p}}{p_0}\,\zeta^*_{1/2}(\mathbf{p},\alpha,\mathbf{n})\,S(x_2-x_1)\zeta_{1/2}(\mathbf{p},\alpha,\mathbf{n}) \\ &= \frac{1}{(2\pi)^3}\int\frac{d\mathbf{p}}{p_0}\frac{\exp -i[(x_2-x_1)p]}{[(pn)/m]^2}, \end{aligned} \qquad (23)$$

$$\begin{aligned} K(x_2;x_1,-1/2,n) &= \int\frac{d\mathbf{p}}{p_0}\,\zeta^*_{-1/2}(\mathbf{p},\alpha,\mathbf{n})\,S(x_2-x_1)\zeta_{-1/2}(\mathbf{p},\alpha,\mathbf{n}) \\ &= \frac{1}{(2\pi)^3}\int\frac{d\mathbf{p}}{p_0}\frac{\exp -i[(x_2-x_1)p]}{[(pn)/m]^2}. \end{aligned} \qquad (24)$$

These transition amplitudes contain the vector of the light-cone n and have the same form as the transition amplitude for the particle with spin zero.

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