Spin(7)-manifolds and symmetric Yang–Mills instantons

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Abstract

In this Letter we establish a relationship between symmetric $SU(2)$ Yang–Mills instantons and metrics with Spin(7) holonomy. Our method is based on a slight extension of that of Bryant and Salamon developed to construct explicit manifolds with special holonomies in 1989.

More precisely, we prove that making use of symmetric $SU(2)$ Yang–Mills instantons on Riemannian spin-manifolds, we can construct metrics on the chiral spinor bundle whose holonomy is within Spin(7). Moreover if the resulting space is connected, simply connected and complete, the holonomy coincides with Spin(7).

The basic explicit example is the metric constructed on the chiral spinor bundle of the round four-sphere by using a generic $SU(2)$-instanton of unit action; hence it is a five-parameter deformation of the Bryant–Salamon example, also found by Gibbons, Page and Pope.

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1 Introduction

The classification of holonomy groups of non-symmetric Riemannian manifolds by M. Berger in 1955 [7], is as fundamental and relevant in both physics and mathematics as the classification of simple Lie algebras by E. Cartan.

From the mathematical point of view, Berger's list provides a powerful and effective way to distinguish the main branches of Riemannian geometries. It is certainly not an exaggeration that the main driving force of the latest decades in Riemannian geometry is a trial for construction and understanding the special holonomy manifolds occurring in Berger's list. The classical example is the solution of the Calabi conjecture by Yau, which is nothing but the proof of existence of compact Riemannian manifolds with $SU(n)$ holonomy. After solving the Calabi conjecture, the only cases had remained in doubt were the two exceptional ones: metrics with G_2-holonomy in seven dimensions and those of Spin(7)-holonomy in eight dimensions. Very roughly, the construction of these spaces took three major steps: first Bryant proved the local existence of such metrics on open balls in $\mathbb{R}^7$ and $\mathbb{R}^8$, respectively and also gave explicit examples in 1987 [9]. Secondly non-compact, complete examples were found by Bryant and Salamon in 1989 [10]. These spaces were re-discovered also by Gibbons, Page and Pope in 1990 [13]. The next breakthrough was done by Joyce in 1994 who constructed implicitly such metrics on plenty of compact manifolds and studied the moduli of theses metrics as well (for a general and excellent introduction and outline of the topic see [15]).

From the physical point of view, the understanding of special holonomy manifolds is also important. By the well-known correspondence, existence of special-holonomy metrics on a given manifold provides us various covariantly constant tensor fields on it, which can be interpreted as solutions to field equations of appropriate physical theories defined over the manifold. In simple terms, the large the symmetry of the physical theory is, the small is the holonomy group of the underlying manifold. Therefore, parallel to the constructions of manifolds with more and more special holonomy by mathematicians, physicists are also searching for such spaces for theories with larger and larger symmetries. For example, compact Calabi–Yau spaces are important in describing the supersymmetric ground states of supersymmetric ten dimensional string theories; while recently it turned out that non-compact G_2-spaces are relevant in the understanding of the unbroken $N=1$ low-energy regime of eleven dimensional M-theory (very far from being complete cf. e.g. [1], [2], [3], [4], [5], [17]) while the less studied non-compact Spin(7)-manifolds are useful tools for example in three-dimensional $N=1$ supersymmetric Yang–Mills theory related with M-theory [14] or in brane-theory [11]. Motivated by this, there have been lot of efforts to construct such spaces explicitly. Again without completeness we could mention Gibbons, Page, Pope [13] and more recently a sequence of papers by Cvetič et al. (as a typical example see [11] and the references therein) or [14]. These methods mainly are based on various coset constructions and focus on solving the Ricci-flatness condition. However studying other technics, e.g. based on the fundamental work [10] or more recently on [6] for instance, there are indications that $SU(2)$-instantons may have an intimate relationship with special holonomy manifolds hence would be good to find a natural correspondence between them.

Our paper, which is supposed to be a small step towards this direction, is organized as follows. In Section 1 we present a slight extension of the method of Bryant and Salamon developed in 1989 [10] which allows us to construct local models for metrics whose holonomy is within Spin(7) by using "round" $SU(2)$ Yang–Mills instantons on chiral spinor bundles of suitable four dimensional Riemannian spin manifolds. Here "round" means that the curvatures of these instantons are characterized by only *one* (i.e., not three, as in general) smooth function. The basic example for such instantons is the well-known five-parameter family of unit action over the round four-sphere, hence the name.

In Section 2 we prove via representation theory that if the resulting space is connected, simply connected and complete, then the holonomy group actually coincides with Spin(7).

In Section 3 we turn our attention to the existence of explicit examples. We prove that in the case of the round four-sphere the resulting complete examples are just deformations of the Bryant–Salamon space [10][13] with moduli the open five-ball which is the moduli space of 1-instantons.

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2 Local construction of Spin(7)-metrics

Let us denote by $\mathbb{H}$ the field of quaternions. In order to make our calculations as simple as possible, we will be using quaternionic notation: η, ξ etc. will denote $\mathbb{H}$-valued 1-forms while $\overline{\eta}, \overline{\xi}$ etc. their quaternionic conjugates. Moreover we take the basic identification $\mathfrak{su}(2) \cong \mathrm{Im}\mathbb{H}$.

Let (M, g) be a four dimensional Riemannian spin-manifold. Consider a local chart $U \subset M$ and introduce the quaternion-valued 1-form

$$\xi := \xi^0 \pm \xi^1\mathbf{i} \pm \xi^2\mathbf{j} \pm \xi^3\mathbf{k}$$

on it (the signs are chosen independently), where ξ^i form a local orthonormal frame on U with respect to the metric g. With this forms we can construct various bases for $\mathrm{Im}\mathbb{H}$-valued self-dual 2-forms over (M, g). For example, the standard choice $\xi := \xi^0 + \xi^1\mathbf{i} + \xi^2\mathbf{j} + \xi^3\mathbf{k}$ gives rise to the basis

$$\frac{1}{2}\xi \wedge \overline{\xi} = -\left(\xi^0 \wedge \xi^1 + \xi^2 \wedge \xi^3\right)\mathbf{i} - \left(\xi^0 \wedge \xi^2 - \xi^1 \wedge \xi^3\right)\mathbf{j} - \left(\xi^0 \wedge \xi^3 + \xi^1 \wedge \xi^2\right)\mathbf{k}. \tag{1}$$

Taking into account the splitting $\mathrm{Spin}(4) \cong SU(2) \times SU(2)$, the (complex) chiral spinor bundles $S^\pm M$ may be regarded as $SU(2)$-bundles over M. Assume there is a smooth self-dual $SU(2)$-connection i.e., an $SU(2)$-instanton $\nabla^\pm$ on $S^\pm M$. Then $\nabla^\pm|_U$ can be represented locally by $\mathrm{Im}\mathbb{H}$-valued 1-forms $A^\pm$. Consider the curvature $F^\pm$ of this connection, locally given by $F^\pm = \mathrm{d}A^\pm + \frac{1}{2}[A^\pm, A^\pm] = \mathrm{d}A^\pm + A^\pm \wedge A^\pm$. We make the following restriction.

Definition 2.1 *Let (M, g) be a four dimensional Riemannian spin-manifold. We call an $SU(2)$-instanton $\nabla^\pm$ on the chiral spinor bundle $S^\pm M$* round, *if there is a smooth function $f^\pm : M \to \mathbb{R}$ and a suitable $\mathbb{H}$-valued 1-form ξ, constructed above, such that its curvature can be written over all local charts as*

$$F^\pm = \frac{f^\pm}{4}\xi \wedge \overline{\xi}. \tag{2}$$

The energy-density of an round instanton on U has the shape $|F^\pm|^2_g = 3(f^\pm)^2\xi^0 \wedge \xi^1 \wedge \xi^2 \wedge \xi^3$ consequently self-duality guarantees that if $\nabla^\pm$ is not flat then $f^\pm$ nowhere vanishes. This shows that $f^\pm$ is strictly positive or negative.

Bianchi identity implies that the derivative of the round $F^\pm$ has the shape

$$\mathrm{d}\left(\frac{f^\pm}{4}\xi \wedge \overline{\xi}\right) = -A^\pm \wedge \left(\frac{f^\pm}{4}\xi \wedge \overline{\xi}\right) + \frac{f^\pm}{4}\xi \wedge \overline{\xi} \wedge A^\pm. \tag{3}$$

From now on we will follow the work of Bryant and Salamon (cf. pp. 846-847 of [10]) although we remark that our notations and conventions will differ significantly from theirs.

Consider the chiral spinor bundle $S^\pm M$. This is a non-compact, eight dimensional real manifold possessing the structure of a two-rank complex vector bundle over M. We regard the fibers, all isomorphic to $\mathbb{C}^2$, as copies of $\mathbb{H}$. By introducing the linear coordinate system (y^0, y^1, y^2, y^3) along each fibers, the above identification allows us to introduce the quaternion $\mathbf{q} := y^0 + y^1\mathbf{i} + y^2\mathbf{j} + y^3\mathbf{k}$ and consider the following $\mathbb{H}$-valued object on $S^\pm U \cong U \times \mathbb{H}$:

$$\eta := \mathrm{d}\mathbf{q} + A^\pm\mathbf{q}, \quad \overline{\eta} = \mathrm{d}\overline{\mathbf{q}} - \overline{\mathbf{q}}A^\pm.$$

In coordinates $\eta=\eta^0+\eta^1\mathbf{i}+\eta^2\mathbf{j}+\eta^3\mathbf{k}$, adopted to (1). We can see that under a gauge (i.e., a coordinate) transformation $g: M\rightarrow SU(2)\cong S^3\subset\mathbb{H}$ given by $\mathbf{q}\mapsto g\mathbf{q}$, η behaves as

$$\mathrm{d}\mathbf{q}+A^\pm\mathbf{q}\longmapsto \mathrm{d}(g\mathbf{q})+(gA^\pm g^{-1}+g\mathrm{d}g^{-1})g\mathbf{q}=g\mathrm{d}\mathbf{q}+(\mathrm{d}g+gA^\pm-\mathrm{d}g)\mathbf{q}=g(\mathrm{d}\mathbf{q}+A^\pm\mathbf{q})$$

that is, it transforms as a 1-form. Therefore it is a well-defined $\mathbb{H}$-valued 1-form over the whole chiral spinor bundle. Its derivative is easily calculated:

$$\mathrm{d}\eta=-A^\pm\wedge\eta+\frac{f^\pm}{4}\xi\wedge\overline{\xi}\mathbf{q},\quad \mathrm{d}\overline{\eta}=-\overline{\eta}\wedge A^\pm-\frac{f^\pm}{4}\overline{\mathbf{q}}\xi\wedge\overline{\xi}.$$

In this way the derivative of the other self-dual basis $\frac{1}{2}\eta\wedge\overline{\eta}$ looks like

$$\mathrm{d}\left(\frac{1}{2}\eta\wedge\overline{\eta}\right)=-A^\pm\wedge\left(\frac{1}{2}\eta\wedge\overline{\eta}\right)+\frac{1}{2}\eta\wedge\overline{\eta}\wedge A^\pm+\frac{f^\pm}{8}\left(\xi\wedge\overline{\xi}\wedge\mathbf{q}\overline{\eta}+\eta\overline{\mathbf{q}}\wedge\xi\wedge\overline{\xi}\right).$$

Let us denote by $r^2:=|\mathbf{q}|^2=\mathbf{q}\overline{\mathbf{q}}$ the radial coordinate on the fibers; with this notation we can write $2r\mathrm{d}r=\mathrm{d}\mathbf{q}\,\overline{\mathbf{q}}+\mathbf{q}\,\mathrm{d}\overline{\mathbf{q}}$, implying the following identities:

$$\mathbf{q}\overline{\eta}=r\mathrm{d}r-r^2A^\pm,\quad \eta\overline{\mathbf{q}}=r\mathrm{d}r+r^2A^\pm.$$

These calculations eventually yield

$$\mathrm{d}\left(\frac{1}{2}\eta\wedge\overline{\eta}\right)=-A^\pm\wedge\left(\frac{1}{2}\eta\wedge\overline{\eta}\right)+\frac{1}{2}\eta\wedge\overline{\eta}\wedge A^\pm+\frac{f^\pm}{4}r\xi\wedge\overline{\xi}\wedge\mathrm{d}r-\frac{f^\pm}{8}r^2\left(-A^\pm\wedge\xi\wedge\overline{\xi}+\xi\wedge\overline{\xi}\wedge A^\pm\right).\tag{4}$$

Via the last but one equation we also prove the equality

$$\frac{r^2}{2}A^\pm\wedge\eta\wedge\overline{\eta}=r\mathrm{d}r\wedge\eta\wedge\overline{\eta}.\tag{5}$$

As a next step, we introduce the following real valued 4-forms on $S^\pm M$, which play a crucial role in the determination of the Spin(7)-structure:

$$\Omega_1:=\frac{1}{24}\mathrm{Re}\left(\xi\wedge\overline{\xi}\wedge\overline{\xi\wedge\overline{\xi}}\right)=\xi^0\wedge\xi^1\wedge\xi^2\wedge\xi^3,$$

$$\Omega_2:=\frac{1}{4}\mathrm{Re}\left(\xi\wedge\overline{\xi}\wedge\overline{\eta\wedge\overline{\eta}}\right)=$$

$$\xi^0\wedge\xi^1\wedge\eta^0\wedge\eta^1+\xi^0\wedge\xi^1\wedge\eta^2\wedge\eta^3+\xi^2\wedge\xi^3\wedge\eta^0\wedge\eta^1+\xi^2\wedge\xi^3\wedge\eta^2\wedge\eta^3$$

$$+\xi^0\wedge\xi^2\wedge\eta^0\wedge\eta^2-\xi^0\wedge\xi^2\wedge\eta^1\wedge\eta^3-\xi^1\wedge\xi^3\wedge\eta^0\wedge\eta^2+\xi^1\wedge\xi^3\wedge\eta^1\wedge\eta^3$$

$$+\xi^0\wedge\xi^3\wedge\eta^0\wedge\eta^3+\xi^0\wedge\xi^3\wedge\eta^1\wedge\eta^2+\xi^1\wedge\xi^2\wedge\eta^0\wedge\eta^3+\xi^1\wedge\xi^2\wedge\eta^1\wedge\eta^2,$$

and

$$\Omega_3:=\frac{1}{24}\mathrm{Re}\left(\eta\wedge\overline{\eta}\wedge\overline{\eta\wedge\overline{\eta}}\right)=\eta^0\wedge\eta^1\wedge\eta^2\wedge\eta^3.$$

By the straightforward invariance of the definition, these forms are well-defined on $S^\pm M$ (they are defined by the Killing-form $-\mathrm{tr}(AB)=2\mathrm{Re}(x\overline{y})$ on the Lie algebra $\mathfrak{su}(2)\cong\mathrm{Im}\mathbb{H}$). One has two other expressions for the 4-form $-\Omega_1+\Omega_2-\Omega_3$ (cf. p. 834 of [10]). First we can write $-\Omega_1+\Omega_2-\Omega_3=\xi^0\wedge\zeta+*_\zeta\zeta$ where

$$\zeta=\xi^1\wedge(\xi^3\wedge\xi^2+\eta^0\wedge\eta^1-\eta^3\wedge\eta^2)+\mathrm{Re}\left((\xi^3+\xi^2\mathbf{i})\wedge(\eta^0+\eta^1\mathbf{i})\wedge(\eta^3-\eta^2\mathbf{i})\right).$$

This decomposition enables us to conclude that the 4-form $-\Omega_1+\Omega_2-\Omega_3$ is kept fixed at one hand by the group $\{1\}\times G_2\subset GL_+(8,\mathbb{R})$ where the subspace spanned by ξ^0 is acted on trivially while the form ζ is fixed by the natural action of G_2. On the other hand we observe $-\Omega_1+\Omega_2-\Omega_3=-\frac{1}{2}\alpha\wedge\alpha+\mathrm{Re}\beta$ with

$$\alpha:=\xi^0\wedge\xi^1-\xi^3\wedge\xi^2-\eta^0\wedge\eta^1+\eta^3\wedge\eta^2,$$

$$\beta:=(\xi^0+\xi^1\mathbf{i})\wedge(\xi^3-\xi^2\mathbf{i})\wedge(\eta^0-\eta^1\mathbf{i})\wedge(\eta^3+\eta^2\mathbf{i}). \tag{6}$$

By this representation it is also possible to see that $-\Omega_1+\Omega_2-\Omega_3$ remains invariant under the group $SU(4)\subset GL_+(8,\mathbb{R})$ where the complex structure on the tangent spaces is induced by the complex 4-form β. These observations yield that the full stabilizer of $-\Omega_1+\Omega_2-\Omega_3$ is the group $\mathrm{Spin}(7)\subset GL_+(8,\mathbb{R})$, as it is proved in [10] or in a more detailed way in [9].

First note that taking into account (3), (4) and (5) we have (cf. p. 847 of [10])

$$\mathrm{d}\Omega_1=0,\quad \mathrm{d}\Omega_3=\frac{r}{2}f^\pm\Omega_2\wedge\mathrm{d}r. \tag{7}$$

Moreover by writing $\Omega_2=(f^\pm)^{-1}\mathrm{Re}\left(F^\pm\wedge\overline{\eta\wedge\overline{\eta}}\right)$, one obtains

$$\mathrm{d}\Omega_2=-\frac{\mathrm{d}f^\pm}{(f^\pm)^2}\wedge\mathrm{Re}\left(F^\pm\wedge\overline{\eta\wedge\overline{\eta}}\right)+\frac{1}{f^\pm}\mathrm{dRe}\left(F^\pm\wedge\overline{\eta\wedge\overline{\eta}}\right).$$

But in light of (3), (4) we can write $\mathrm{dRe}\left(F^\pm\wedge\overline{\eta\wedge\overline{\eta}}\right)=3r(f^\pm)^2\Omega_1\wedge\mathrm{d}r$ leading to

$$\mathrm{d}\Omega_2=-\frac{\mathrm{d}f^\pm}{f^\pm}\wedge\Omega_2+3rf^\pm\Omega_1\wedge\mathrm{d}r. \tag{8}$$

(cf. p. 847 of [10]). Moreover we have the two straightforward equalities

$$\Omega_1\wedge\mathrm{d}f^\pm=0,\quad \Omega_3\wedge\mathrm{d}r=0. \tag{9}$$

Remark. We can always assume that $f^\pm$ is positive in (7) and (8) because the transformations $\xi\wedge\overline{\xi}\mapsto-\xi\wedge\overline{\xi}$ and $\eta\wedge\overline{\eta}\mapsto-\eta\wedge\overline{\eta}$ leave Ω_i invariant while one has $f^\pm\mapsto-f^\pm$.

Now consider two functions $\varphi,\psi:S^\pm M\to\mathbb{R}^+$ and assume that they depend on the fiber coordinates y^i only through the radial coordinate r. Take the 4-form

$$\Omega:=-\varphi^2\Omega_1+\varphi\psi\Omega_2-\psi^2\Omega_3$$

and the associated metric g_Ω, locally given by

$$\mathrm{d}s^2:=\varphi\left(\xi_0^2+\xi_1^2+\xi_2^2+\xi_3^2\right)+\psi\left(\eta_0^2+\eta_1^2+\eta_2^2+\eta_3^2\right).$$

Ω is self-dual with respect to the associated metric, moreover at each tangent spaces we can find an isomorphism sending Ω into $-\Omega_1+\Omega_2-\Omega_3$. As it is proved for example in [9],[10] or [15], if $\nabla\Omega=0$ with respect to g_Ω (a non-linear problem!) then this metric has a holonomy group, whose identity component $\mathrm{Hol}^0(g_\Omega)$ is contained within Spin(7). Now we prove that by a suitable choice of the functions φ and ψ we can achieve this.

Proposition 2.2 *Let (M, g) be a Riemannian spin four-manifold and $\nabla^{\pm}$ an round $SU(2)$-instanton on the spinor bundle $S^{\pm}M$. Then there is a metric g_{Ω} on the non-compact eight-manifold $S^{\pm}M$, locally given by*

$$\mathrm{d}s^2 = (1+r^2)^{3/5} f^{\pm} \left(\xi_0^2 + \xi_1^2 + \xi_2^2 + \xi_3^2\right) + \frac{4}{5}(1+r^2)^{-2/5} \left(\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2\right), \tag{10}$$

where $f^{\pm}$ comes from (2), satisfying $\mathrm{Hol}^0(g_{\Omega}) \subseteq \mathrm{Spin}(7)$. The space $(S^{\pm}M, g_{\Omega})$ is complete if (M, g) is compact. If (M, g) is non-compact but complete and

$$\int\limits_0^{\infty} \sqrt{f^{\pm}(\gamma(t))}\, \mathrm{d}t = \infty \tag{11}$$

for each curve $\gamma : \mathbb{R}^+ \to M$ (not contained in any compact set of M) then $(S^{\pm}M, g_{\Omega})$ is also complete.

Proof. By virtue of Theorem 2.3 of [10], it is enough to prove that with a suitable choice of the functions φ, ψ, the 4-form Ω is closed i.e. $\mathrm{d}\Omega = 0$.

Let us denote by (x^0, x^1, x^2, x^3) a local coordinate system on $U \subset M$. Calculating the exterior derivative we get

$$\mathrm{d}\Omega = -2\varphi \left(\frac{\partial \varphi}{\partial r}\mathrm{d}r + \frac{\partial \varphi}{\partial x^i}\mathrm{d}x^i\right) \wedge \Omega_1 - \varphi^2 \mathrm{d}\Omega_1$$

$$+ \left(\varphi\frac{\partial \psi}{\partial r}\mathrm{d}r + \varphi\frac{\partial \psi}{\partial x^i}\mathrm{d}x^i + \psi\frac{\partial \varphi}{\partial r}\mathrm{d}r + \psi\frac{\partial \varphi}{\partial x^i}\mathrm{d}x^i\right) \wedge \Omega_2 + \varphi\psi\, \mathrm{d}\Omega_2$$

$$- 2\psi \left(\frac{\partial \psi}{\partial r}\mathrm{d}r + \frac{\partial \psi}{\partial x^i}\mathrm{d}x^i\right) \wedge \Omega_3 - \psi^2 \mathrm{d}\Omega_3.$$

By using identities (7), (8) and (9) this reduces to

$$\mathrm{d}\Omega = -\left(\frac{\partial \varphi^2}{\partial r} - 3rf^{\pm}\varphi\psi\right)\mathrm{d}r \wedge \Omega_1 + \left(\frac{\partial(\varphi\psi)}{\partial r} - \frac{r}{2}f^{\pm}\psi^2\right)\mathrm{d}r \wedge \Omega_2$$

$$-\frac{\partial \varphi^2}{\partial x^i}\mathrm{d}x^i \wedge \Omega_1 + \frac{\partial(\varphi\psi)}{\partial x^i}\mathrm{d}x^i \wedge \Omega_2 - \varphi\psi\frac{\mathrm{d}f^{\pm}}{f^{\pm}} \wedge \Omega_2 - \frac{\partial \psi^2}{\partial x^i}\mathrm{d}x^i \wedge \Omega_3.$$

The terms of the first row are eliminated by solving the system of ordinary differential equations just appeared in the coefficients. As we have seen we can assume $f^{\pm} > 0$ hence the general solution is (cf. p. 847 of [10])

$$\varphi(r, x^i) = \frac{1}{h_1(x^i)}(h_1(x^i)r^2 + h_2(x^i))^{3/5} f^{\pm}(x^i), \quad \psi(r, x^i) = \frac{4}{5}(h_1(x^i)r^2 + h_2(x^i))^{-2/5}$$

where h_1, h_2 are arbitrary functions of x^i's only.

Now focus on the second row of the above expression. Take simply $h_1 = h_2 = 1$. In this case ψ is independent of x^i consequently the last term of the second row vanishes. Moreover by noticing that with the above choice of h_i we have

$$\frac{\partial \varphi^2}{\partial x^i}\mathrm{d}x^i \wedge \Omega_1 = 2(1+r^2)^{6/5} f^{\pm}\mathrm{d}f^{\pm} \wedge \Omega_1,$$

we can see that by the first equation of (9) the first term vanishes, too. Henceforth the calculation amounts to an expression for the middle terms (after substituting φ, ψ):

$$\mathrm{d}\Omega = \frac{4}{5}(1+r^2)^{1/5}\mathrm{d}f^{\pm}\wedge\Omega_2 - \frac{4}{5}(1+r^2)^{1/5}\mathrm{d}f^{\pm}\wedge\Omega_2 = 0$$

showing $\mathrm{d}\Omega = 0$ with the above choice of the functions φ and ψ. This implies that the associated metric satisfies $\mathrm{Hol}^0(g_\Omega)\subseteq \mathrm{Spin}(7)$.

Now we turn our attention to the geodesic completeness of the resulting metric (10). We can see that this metric is geodesically complete along each fibers because $\eta_0^2+\eta_1^2+\eta_2^2+\eta_3^2$ is complete and

$$\int\limits_0^\infty \frac{\mathrm{d}r}{(1+r^2)^{1/5}} = \infty.$$

Consequently (10) is complete if $f^{\pm}(\xi_0^2+\xi_1^2+\xi_2^2+\xi_3^2)$ is complete; this is valid if M is compact. However it might fail this property if M is not compact and $f^{\pm}$ decays too fast. Suppose M is non-compact and $\xi_0^2+\xi_1^2+\xi_2^2+\xi_3^2$ is complete on it; then the re-scaled metric is complete if and only if (11) is valid. Consequently $(S^{\pm}M, g_\Omega)$ might be incomplete. $\square$

In summary we have found a local form (10) of Riemannian metrics g_Ω with the property $\mathrm{Hol}^0(g_\Omega)\subseteq\mathrm{Spin}(7)$.

3 Proof of Spin(7) holonomy

We have to still find a condition for the holonomy groups actually coincide with $\mathrm{Spin}(7)$. By using a result of Bryant and Salamon, we can prove this.

Proposition 3.1 *Assume that M is connected and simply connected and the associated space $(S^{\pm}M, g_\Omega)$ of the previous proposition is complete. Then $\mathrm{Hol}(g_\Omega)\cong\mathrm{Spin}(7)$ is valid.*

Proof. If M is connected and simply connected then the same is true for $S^{\pm}M$. Therefore, by the aid of Theorem 2.4 of Bryant and Salamon [10] we have to show that there are no non-trivial parallel 1-forms and 2-forms on $(S^{\pm}M, g_\Omega)$ because this implies that the holonomy $\mathrm{Hol}^0(g_\Omega)=\mathrm{Hol}(g_\Omega)$ cannot be smaller than Spin(7).

To achieve this, we list all the possible holonomy groups which are subgroups of Spin(7) (cf. Theorem 10.5.7 in [15]):

(i) Reducible actions:

$\{1\}$	acting on $\mathbb{R}^8$ trivially,
$\{1\}\times SU(2)\cong\{1\}\times\mathrm{Spin}(3)$	acting on $\mathbb{R}^8\cong\mathbb{R}^4\oplus\mathbb{C}^2$ trivially on $\mathbb{R}^4$, as usual on $\mathbb{C}^2$,
$SU(2)\times SU(2)\cong\mathrm{Spin}(4)$	acting on $\mathbb{R}^8\cong\mathbb{C}^2\oplus\mathbb{C}^2$ on each $\mathbb{C}^2$ as usual,
$\{1\}\times SU(3)$	acting on $\mathbb{R}^8\cong\mathbb{R}^2\oplus\mathbb{C}^3$ trivially on $\mathbb{R}^2$, as usual on $\mathbb{C}^3$,
$\{1\}\times G_2$	acting on $\mathbb{R}^8\cong\mathbb{R}\oplus\mathbb{R}^7$ trivially on $\mathbb{R}$, as usual on $\mathbb{R}^7$.

(ii) Irreducible actions:

$Sp(2)\cong\mathrm{Spin}(5)$	acting on $\mathbb{R}^8\cong\mathbb{H}^2$ as usual,
$SU(4)\cong\mathrm{Spin}(6)$	acting on $\mathbb{R}^8\cong\mathbb{C}^4$ as usual,
$\mathrm{Spin}(7)$	acting on $\mathbb{R}^8$ as usual.

Assume now $(S^{\pm}M, g_\Omega)$ is moreover complete. Then (10) is a complete metric on a simply connected manifold and does not split, as one can check. Taking into account the de Rham theorem [16], its holonomy group cannot act reducibly on the tangent spaces. Consequently the groups listed in (i) cannot occur.

Concerning part (ii) of the list, we can proceed as follows: assume the holonomy group is Spin(6) only. The group Spin$(6) \cong SU(4)$ acts irreducibly on $\mathbb{R}^8 \cong \mathbb{C}^4$ i.e. there are no non-trivial parallel 1-forms on the space $(S^{\pm}M, g_\Omega)$ (cf. e.g. Theorem 2.5.2 of [15]). Moreover we have an induced action on $\Lambda^2\mathbb{C}^4$ as well, which gives rise to an action on $\Lambda^2\mathbb{R}^8$. Since this action is nothing but one of the fundamental representations of $SU(4)$, it is also irreducible. Consequently, there are no non-trivial parallel 2-forms, too. But this implies that the holonomy group must be Spin(7), a contradiction.

Now assume that the holonomy group is Spin(5). The action of Spin$(5) \cong Sp(2)$ is also irreducible on $\mathbb{R}^8 \cong \mathbb{H}^2$ i.e. again there are no parallel 1-forms. The induced action on $\Lambda^2\mathbb{H}^2$ is *not* irreducible, however. To see this, we will follow [8], pp. 269-272. Consider the identification $\mathbb{H}^2 \cong \mathbb{C}^4$ with a basis $(e_{\pm 1}, e_{\pm 2})$, and regard the action of $Sp(2)$ as a subgroup of $SU(4)$, leaving the well-known symplectic form invariant. With this notation, the induced reducible representation of $Sp(2)$ on $\Lambda^2\mathbb{C}^4$ splits as $\Lambda^2\mathbb{C}^4 \cong V_1 \oplus V_5$ where the action is trivial on the first summand $V_1 \cong \mathbb{C}$, spanned by the 2-form

$$e_1^* \wedge e_{-1}^* + e_2^* \wedge e_{-2}^*$$

($e_{\pm i}^*$ are the dual basis elements to $e_{\pm i}$), while the five dimensional orthogonal complement V_5 (with respect to the standard Hermitian inner product on $\mathbb{C}^4$) is acted on non-trivially. This induces a splitting of $\Lambda^2\mathbb{R}^8$, too. Therefore we can see that a non-trivial parallel 2-form on $(S^{\pm}M, g_\Omega)$ must be either the real or imaginary part of the 2-form

$$f((\xi^0 + \xi^1\mathbf{i}) \wedge (\xi^3 - \xi^2\mathbf{i}) + (\eta^0 - \eta^1\mathbf{i}) \wedge (\eta^3 + \eta^2\mathbf{i}))$$

where the identification $\mathbb{R}^8 \cong \mathbb{C}^4$ on the tangent spaces is induced by (6); f is some complex valued function on $S^{\pm}M$. But we can check that the only 2-form of the above shape which is parallel with respect to (10) is the zero 2-form. Indeed, since $\nabla(\xi^i \wedge \xi^j) \neq 0$ and depends only on x^i furthermore $\nabla(\eta^i \wedge \eta^j) \neq 0$ and depends on both x^i and $\mathbf{q}$, this implies that f must be zero.

Hence again we have not been able to find non-trivial parallel 1- and 2-forms consequently the holonomy must be Spin(7). $\square$

4 A global example: the round four-sphere

In this section we construct new explicit examples whose holonomy groups are Spin(7).

The most straightforward example is the round four-sphere (S^4, g) with isometry group $SO(5)$ ([12], pp. 99-105). Because of conformal invariance, we may consider the flat $\mathbb{R}^4 \cong \mathbb{H}$ as well. Let $\mathbf{x}, \mathbf{b} \in \mathbb{H}$ and $\lambda > 0$ real. Then the basic instanton together with its curvature looks like

$$A = \mathrm{Im}\frac{\mathbf{x}\mathrm{d}\overline{\mathbf{x}}}{1 + |\mathbf{x}|^2}, \qquad F = \frac{\mathrm{d}\mathbf{x} \wedge \mathrm{d}\overline{\mathbf{x}}}{(1 + |\mathbf{x}|^2)^2}.$$

If we apply the homothety $T_{\lambda,\mathbf{b}} : \mathbf{x} \mapsto \lambda(\mathbf{x} - \mathbf{b})$ then we get the five-parameter family of instantons,

$$T^*_{\lambda,\mathbf{b}}A := A_{\lambda,\mathbf{b}} = \mathrm{Im}\frac{(\mathbf{x} - \mathbf{b})\mathrm{d}\overline{\mathbf{x}}}{\lambda^2 + |\mathbf{x} - \mathbf{b}|^2}, \qquad T^*_{\lambda,\mathbf{b}}F := F_{\lambda,\mathbf{b}} = \frac{\lambda^2\mathrm{d}\mathbf{x} \wedge \mathrm{d}\overline{\mathbf{x}}}{(\lambda^2 + |\mathbf{x} - \mathbf{b}|^2)^2}.$$

Therefore these instantons are round with respect to the self-dual basis (1). Now putting $A_{\lambda,\mathbf{b}}$ into (10) we can produce a five-parameter family of complete metrics g_Ω over $S^\pm S^4$ with holonomy within Spin(7):

$$\mathrm{d}s^2 = \frac{\lambda^2(1+r^2)^{3/5}}{(\lambda^2+|\mathbf{x}-\mathbf{b}|^2)^2}(\mathrm{d}x_0^2+\mathrm{d}x_1^2+\mathrm{d}x_2^2+\mathrm{d}x_3^2)+\frac{4}{5}(1+r^2)^{-2/5}\left(\eta_0^2+\eta_1^2+\eta_2^2+\eta_3^2\right)=$$

$$\frac{\lambda^2(1+r^2)^{3/5}(1+|\mathbf{x}|^2)^2}{(\lambda^2+|\mathbf{x}-\mathbf{b}|^2)^2}\mathrm{d}\Omega_{S^4}^2+\frac{4}{5}(1+r^2)^{-2/5}\left(\eta_0^2+\eta_1^2+\eta_2^2+\eta_3^2\right)$$

where we have used the conformal re-scaling $1/(1+|\mathbf{x}|^2)^2(\mathrm{d}x_0^2+\mathrm{d}x_1^2+\mathrm{d}x_2^2+\mathrm{d}x_3^2)=\mathrm{d}\Omega_{S^4}^2$. Taking the inverse of the homothety $T_{\lambda,\mathbf{b}}$ which sends $A_{\lambda,\mathbf{b}}$ back to A, we just recover the Bryant–Salamon metric on the chiral spinor bundle of the four-sphere [10], also found by Gibbons, Page and Pope [13]:

$$\mathrm{d}s^2=(1+r^2)^{3/5}\mathrm{d}\Omega_{S^4}^2+\frac{4}{5}(1+r^2)^{-2/5}\left(\eta_0^2+\eta_1^2+\eta_2^2+\eta_3^2\right).$$

This procedure intuitively corresponds to the limit $\lambda\to\infty$ i.e., the basic instanton A is "centerless". Consequently these new spaces are deformations of the Bryant–Salamon space with moduli space the five-ball B^5, nothing but the moduli space of $SU(2)$-instantons of unit charge on S^4. In this picture the Bryant–Salamon space corresponds to the centerless instanton represented by the center of B^5. By the previous proposition, these spaces have holonomy Spin(7).

5 Concluding remarks

A very natural question arises whether it is possible to remove the very restrictive "roundness" assumption for the Yang–Mills instantons in this extended Bryant–Salamon construction. If yes, we could establish a correspondence between $SU(2)$ Yang–Mills instantons over compact spin manifolds and Spin(7)-metrics on the chiral spinor bundle. If the underlying spin manifold is non-compact then the geodesic completeness of the associated space would depend on the fall-off properties of the field strength of the instanton.

Of course it would be also interesting to know if the above method can be repeated for the G_2-case. The main difference between the two cases is that while for Spin(7) we have only one non-linear partial differential equation for the existence, in the G_2 case we have two; consequently it is typically more difficult to obey the conditions for the G_2-case.

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