Selected topics in integrable models

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Abstract

In these talks, I discuss a few selected topics in integrable models that are of interest from various points of view. Some open questions are also described.

1 Introduction:

The subject of integrable models now encompasses a very large area of research involving many seemingly different topics. It is not at all possible to give a detailed exposition of the subject in just a few lectures. Therefore, when the organizers of the André Swieca summer school asked me to choose a few topics on which to speak at the school, I agreed with a lot of trepidation. In order to make the lectures self-complete, I, of course, had to start with some basics. The subsequent topics that I talked about, naturally, represent a personal choice according to my interests. There are many other interesting areas that are being pursued by many active groups, but I simply could not have done justice to all, in the limited time available. Similarly, the literature on the subject is vast and it would have been totally impossible for me to even pretend to have a complete list of references. Consequently, I have only chosen a handful of references, for my talks, that I have absolutely used in the preparation of my lectures. I apologize to all whose works I have not been able to mention in my talks or references that I have not been able to list.

2 Historical development:

Let me begin with some introduction to the historical development of the subject. Let us begin with the simple, working definition of an integrable model as a physical system that is described by nonlinear partial differential equations, which can be exactly solved. We will make the definition more precise as we go along. There are quite a few systems of equations of this kind that arise in various physical theories. For example, in $0+1$ dimensions, the Toda lattice is described by the set of equations

$$\begin{aligned}
\dot{Q}_i &= P_i, \qquad i = 1,2,\cdots,N\\
\dot{P}_1 &= -e^{-(Q_2-Q_1)}\\
\dot{P}_N &= e^{-(Q_N-Q_{N-1})}\\
\dot{P}_\alpha &= e^{-(Q_\alpha-Q_{\alpha-1})} - e^{-(Q_{\alpha+1}-Q_\alpha)}, \qquad \alpha = 2,3,\cdots,N-1
\end{aligned} \tag{1}$$

and consists of a chain of N particles on a one dimensional lattice at the coordinates Q_i, with P_i representing the conjugate momenta.

In $1+1$ dimensions, similarly, there is the celebrated KdV equation described by

$$\frac{\partial u(x,t)}{\partial t} = u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \tag{2}$$

where u, for example, may describe the height of a water wave from the normal surface. The variables of the KdV equation can be scaled to have arbitrary coefficients in front of all the terms. As a consequence, another form in which the KdV equation is also known corresponds to

$$\frac{\partial u}{\partial t} = 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \tag{3}$$

In $1+1$ dimensions, there is also the non-linear Schrödinger equation described by

$$\begin{aligned}\frac{\partial\psi(x,t)}{\partial t} &= -\frac{\partial^2\psi}{\partial x^2} - 2\kappa|\psi|^2\psi \\ \frac{\partial\overline{\psi}(x,t)}{\partial t} &= \frac{\partial^2\overline{\psi}}{\partial x^2} + 2\kappa|\psi|^2\overline{\psi}\end{aligned} \tag{4}$$

where κ is a constant measuring the strength of the nonlinear interaction and can be both positive or negative corresponding to an attractive or repulsive interaction. There are several other integrable systems in $1+1$ dimensions, but these two are the ones that have been widely studied. There are fewer integrable systems in $2+1$ dimensions, which include the Kadomtsev-Petviashvelli (KP) equations and the Davey-Stewartson (DS) equations.

One of the most important features that all integrable models have is that they possess soliton solutions to the equations of motion. Solitons are defined as localized, non-dispersive solutions that maintain their shape even after being scattered. Historically, of course, research in this area grew out of J. Scott Russel's observation, in 1834, of a solitary wave travelling for miles maintaining its shape. It was only in 1895 that Korteweg and de Vries gave a mathematical description of such shallow water waves, which is known as the KdV equation. Being nonlinear and difficult to solve, these equations, however, did not generate a lot of interest. In 1965, Kruskal and Zabusky undertook a "computer" experiment, namely, they wanted to numerically study the evolution of the solutions of the KdV equation. What they found was impressive, namely, when certain solutions of the KdV equation were scattered off each other, they maintained their shape even after going through the scattering region. Kruskal coined the term "solitons" for such solutions in 1969 and the presence of such solutions generated an enormous interest in such systems from then on.

Non-dispersive solutions:

Most physical linear equations have dispersive solutions and the presence of non-dispersive solutions in such systems is quite interesting. To appreciate the origin of such solutions and to see their relation to the nonlinear interactions of the theory, let us analyze the KdV equation,

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \tag{5}$$

As we have noted, such a system describes shallow water waves, where we can think of $u(x,t)$ as representing the height of the water wave from the normal surface of water. The first term on the right hand side represents the nonlinear term. Let us, for a moment, look at the KdV equation without the nonlinear term, namely,

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} \tag{6}$$

It is easy to write down the dispersion relation following from this equation,

$$E(k) = k^3 \tag{7}$$

This immediately tells us that the phase and the group velocities, associated with a wave packet, in this case, are different, namely,

$$v_{\text{phase}} = \frac{E(k)}{k} = k^2, \qquad v_{\text{group}} = \frac{dE(k)}{dk} = 3k^2 \tag{8}$$

Thus, we see that if the KdV equation contained only the linear term on the right hand side, solutions will disperse.

On the other hand, let us next assume that the KdV equation does not contain the linear term on the right hand side, namely,

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} \tag{9}$$

This is also known as the Riemann equation. This can be solved by the method of characteristics and the solution has the general form

$$u(x,t) = f(x + ut) \tag{10}$$

which is quite interesting, for it says that the velocity of propagation is directly proportional to the height of the wave. Namely, the higher points of the wave will travel faster than those at a lower height. This is what leads to the breaking of waves etc. However, from our point of view, we see that this has a localizing effect, opposite of what the linear term leads to. The linear and the nonlinear terms, on the right hand side of the KdV equation, therefore, have opposing behavior and if they can balance each other exactly, then, we can have solutions that will travel without any dispersion. The presence of nonlinear interactions, therefore, is quite crucial to the existence of non-dispersive solutions.

In the KdV equation, this indeed happens and we have non-dispersive solutions. For example, let us consider

$$u(x,t) = 3v\text{sech}^2\frac{\sqrt{v}}{2}(x + vt) \tag{11}$$

This gives

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -3v^{\frac{5}{2}}\text{sech}^2\frac{\sqrt{v}}{2}(x+vt)\tanh\frac{\sqrt{v}}{2}(x+vt) \\
u\frac{\partial u}{\partial x} &= -9v^{\frac{5}{2}}\text{sech}^4\frac{\sqrt{v}}{2}(x+vt)\tanh\frac{\sqrt{v}}{2}(x+vt) \\
\frac{\partial^3 u}{\partial x^3} &= -3v^{\frac{5}{2}}\text{sech}^2\frac{\sqrt{v}}{2}(x+vt)\tanh^3\frac{\sqrt{v}}{2}(x+vt) + 6v^{\frac{5}{2}}\text{sech}^4\frac{\sqrt{v}}{2}(x+vt)\tanh\frac{\sqrt{v}}{2}(x+vt)
\end{aligned}$$

With trigonometric identities, it is easy to check now that

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$$

holds so that this represents a solution of the KdV equation. From the form of the solution, it is clear that it maintains its shape as it travels (non-dispersive) and such a solution is known as a one soliton solution. One can also construct multi-soliton solutions through what is known as Backlünd transformations, which I will not go into.

Conserved charges:

Given the KdV equation, one can immediately construct three conserved charges, namely, it is easy to check that

$$\begin{aligned} H_1 &= \int dx\, u \\ H_2 &= \frac{1}{2}\int dx\, u^2 \\ H_3 &= \int dx\, \left(\frac{1}{3!}u^3 - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2\right) \end{aligned} \tag{12}$$

are conserved under the evolution of the KdV equation. Several people had also constructed up to 13 conserved charges for the system when Kruskal conjectured that the KdV system has an infinite number of functionally independent conserved charges. This was subsequently proved and the conserved charges constructed through the Miura transformation (as well as through the method of inverse scattering). However, in retrospect, the presence of an infinite number of conserved charges associated with a system possessing soliton solutions is intuitively quite clear. As we have noted, solitons scatter through each other maintaining their shape. This implies that there must be conservation laws which prevents the solution from deformations. Since the soliton is an extended solution, there must, therefore, be an infinity of such conservation laws for the solution to maintain its shape through collisions. The important thing is that such a system has an infinite number of conserved charges, which also means that the system is integrable.

Bi-Hamiltonian structure:

Integrable systems are Hamiltonian systems. For example, in the case of the KdV equation, we note that if we define

$$\{u(x), u(y)\}_1 = \partial\delta(x-y) = \frac{\partial}{\partial x}\delta(x-y) \tag{13}$$

then, the KdV equation can be written in the Hamiltonian form as

$$\frac{\partial u}{\partial t} = \{u(x), H_3\}_1 = \frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + \frac{\partial^2 u}{\partial x^2}\right) = u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \tag{14}$$

However, what is even more interesting is the fact that the same set of equations can also be written in the Hamiltonian form if we define

$$\{u(x), u(y)\}_2 = \left(\frac{\partial^3}{\partial x^3} + \frac{1}{3}\left(\frac{\partial}{\partial x}u + u\frac{\partial}{\partial x}\right)\right)\delta(x-y) \tag{15}$$

so that

$$\frac{\partial u}{\partial t} = \{u(x), H_2\}_2 = \left(\frac{\partial^3}{\partial x^3} + \frac{1}{3}\left(\frac{\partial}{\partial x}u + u\frac{\partial}{\partial x}\right)\right)u = u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \tag{16}$$

Namely, the KdV equation is Hamiltonian with respect to at least two distinct Hamiltonian structures.

Representing the two Hamiltonian structures as (operators acting on delta function)

$$\begin{aligned} \mathcal{D}_1 &= \partial \\ \mathcal{D}_2 &= \partial^3 + \frac{1}{3}(\partial u + u\partial) \end{aligned} \tag{17}$$

we note that the first structure is what is normally called the Abelian current algebra, while the second structure is known as the Virasoro algebra. As a result, we do not have to worry about these structures satisfying the Jacobi identity, which they do. This is another general feature of integrable models, namely, the Hamiltonian structures of integrable models are generally associated with symmetry algebras. In fact, some of the nonlinear algebras, such as the W algebras, were studied from the point of view of integrable models.

Looking at the structure of the two Hamiltonian structures of the KdV equation, it is clear that not only does this system have two distinct Hamiltonian structures, but that that

$$\mathcal{D} = \mathcal{D}_2 + \alpha\mathcal{D}_1 \tag{18}$$

also represents a genuine Hamiltonian structure (not necessarily of the system). This is a nontrivial statement, considering that a structure must satisfy the Jacobi identity - a nontrivial relation - in order to qualify as a Hamiltonian structure. In this case, it follows from the fact that $\mathcal{D}_2$ is a genuine Hamiltonian structure for any variable u and that

$$\mathcal{D}_2(u + \frac{3}{2}\alpha) = \mathcal{D}_2(u) + \alpha\mathcal{D}_1 \tag{19}$$

When two Hamiltonian structures have such a relation (namely, if two structures are Hamiltonian, then, a linear combination of the two is also), they are said to be compatible. When a Hamiltonian system can be described by two distinct Hamiltonian structures that are compatible, the system is said to be a bi-Hamiltonian system.

The existence of two Hamiltonian descriptions for the same equation, of course, implies that

$$\frac{\partial u}{\partial t} = \mathcal{D}_2\frac{\delta H_2}{\delta u} = \mathcal{D}_1\frac{\delta H_3}{\delta u} \tag{20}$$

This is a prototype of the recursion relation that exists between conserved charges in such systems. One can define a recursion operator

$$\mathcal{R} = \mathcal{D}_1^{-1}\mathcal{D}_2 \tag{21}$$

which will relate the successive conserved charges as

$$\frac{\delta H_{n+1}}{\delta u} = \mathcal{R}\frac{\delta H_n}{\delta u} \tag{22}$$

Furthermore, if the two Hamiltonian structures are compatible, one can further show that these conserved charges are also in involution with respect to either of the Hamiltonian structures, thereby proving that the system is integrable.

The phase space geometry of integrable systems is quite interesting. These are, of course, symplectic manifolds, but because there are at least two distinct Hamiltonian structures (symplectic structures), these are very special symplectic manifolds. Let us call the two symplectic structures as ω_1 and ω_2. Then, on this manifold, one can naturally define a nontrivial $(1,1)$ tensor as

$$S = \omega_1^{-1}\omega_2 \tag{23}$$

The evolution of this equation can be thought of as the Lax equation and, therefore, this gives a geometrical meaning to the Lax equation. Furthermore, one can show that if the Nijenhuis torsion tensor, associated with this $(1,1)$ tensor, vanishes, then, the conserved charges will be in involution. Consequently, the vanishing of the Nijenhuis torsion tensor can be thought of as a sufficient condition for integrability in this geometrical description.

Let me also note here that since the KdV system has an infinite number of conserved quantities $H_n, n = 1, 2, \cdots$, each of this can be thought of as a Hamiltonian and will lead to a flow as

$$\frac{\partial u}{\partial t_k} = \mathcal{D}_1 \frac{\delta H_{k+1}}{\delta u} = \mathcal{D}_2 \frac{\delta H_k}{\delta u} \tag{24}$$

Thus, with every integrable system is a hierarchy of flows and these represent the higher order flows of the system. The entire hierarchy of flows shares the same infinite set of conserved quantities and are integrable.

Initial value problem:

An interesting question, in connection with these nonlinear integrable systems, is how can one solve the initial value problem. Namely, given the initial values of the dynamical variables, in such systems, how does one determine their values at any later time. In linear systems, we are familiar with techniques such as the Fourier transformation or the Laplace transformation, which help by transforming differential equations into algebraic ones. However, these methods are not very useful in dealing with nonlinear equations. The method that is useful (and, therefore, which can be thought of as the analog of the Fourier transformation in the case of nonlinear equations) is the method of inverse scattering. Let me explain this in some detail.

Let us consider the linear Schrödinger equation

$$\left(\partial^2 + \frac{1}{6}u(x,t)\right)\psi = \lambda\psi \tag{25}$$

where ∂ stands for $\frac{\partial}{\partial x}$ and $u(x,t)$ is the dynamical variable of the KdV equation. Here t is just a parameter that the potential in the Schrödinger equation depends on and not the evolution parameter of the Schrödinger equation. Since the potential $u(x,t)$ depends on the parameter t, it follows that both ψ and λ will depend on t as well. However, what Gardner, Greene, Kruskal and Miura observed was that, if $u(x,t)$ satisfied the KdV equation, then, the eigenvalues, λ, were independent of t, namely, the evolution is isospectral in such a case, or,

$$\lambda_t = 0 \tag{26}$$

Furthermore, in such a case, the dependence of ψ on t is very simple, namely,

$$\psi_t = -\left(\frac{1}{6}u_x + \alpha\right)\psi + \left(4\lambda + \frac{1}{3}u\right)\psi_x \tag{27}$$

In such case, the evolution of the scattering data, such as the reflection coefficient, the transmission coefficient etc, with t is easy to determine and, in fact, take a simple form. Thus, the strategy for solving the initial value problem can be taken as follows. Let us choose the linear Schrödinger equation with the potential $u(x,0)$ and determine the scattering data. Next determine the scattering data at an arbitrary value of t from the simple evolution of the scattering data. Once we have the scattering data for an arbitrary t, we can ask what is the potential, $u(x,t)$, which would give rise to those scattering data. This is essentially the method of inverse scattering. The reconstruction of the potential from the scattering data is done through the Gel'fand-Levitan-Marchenko equation,

$$K(x,y) + B(x,y) + \int_x^\infty dz\, K(x,z)B(y+z) = 0, \qquad y \geq x \tag{28}$$

where

$$B(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, R(k)e^{ikx} + \sum_{n=1}^{N} c_n e^{-\kappa_n x} \tag{29}$$

Here, R is the coefficient of reflection, κ_n's represent the eigenvalues for the bound states and c_n's correspond to the normalization constants for the bound state wave functions. Once the solution of the Gel'fand-Levitan-Marchenko equation is known, the potential is determined from

$$\frac{1}{6}u(x) = 2\frac{\partial K(x,x)}{\partial x} \tag{30}$$

Let us see explicitly how the method works, in an example. However, let me also note that, while one can, in principle, find a solution to the GLM equation, in practice it may be difficult unless the starting potential were very special. One such class of potentials are solitonic potentials which are known to be reflectionless. In such a case,

$$R = 0 \tag{31}$$

and this makes calculations much simpler.

Let us, therefore, consider

$$u(x,0) = 12\text{sech}^2 x \tag{32}$$

We recognize, from our earlier discussion, that this is the one soliton solution of the KdV equation. Such a potential leads to no reflection. It supports only one bound state, for which

$$\kappa = 1, \qquad \psi(x,0) = \frac{1}{2}\mathrm{sech}x \tag{33}$$

Therefore, we obtain

$$c(0) = \left(\int_{-\infty}^{\infty} dx\, \psi^2(x,0)\right)^{-1} = 2 \tag{34}$$

From the equation for the "time" evolution of the wave function, it is easy to determine that

$$c(t) = c(0)e^{-8t} = 2e^{-8t} \tag{35}$$

so that we have

$$B(x,t) = c(t)e^{-x} = 2e^{-8t-x} \tag{36}$$

In this case, the GLM equation becomes,

$$K(x,y,t) + 2e^{-8t-x-y} + 2\int_x^{\infty} dz\, K(x,z,t)e^{-8t-y-z} = 0 \tag{37}$$

This determines

$$\begin{aligned} K(x,y,t) &= -\frac{2e^{-8t-x-y}}{1+e^{-8t-2x}} \\ \text{or,}\quad u(x,t) &= 12\frac{\partial K(x,x)}{\partial x} = 12\mathrm{sech}^2(x+4t) \end{aligned} \tag{38}$$

This is, of course, the solution that we had determined earlier (corresponding to a specific choice of v) and this explains how the method of inverse scattering works. (It is worth noting here that the inverse scattering method was also independently used by Faddeev and Zakharov to solve the KdV equation.)

The Lax equation:

In some sense, the Lax equation is a formal generalization of the ideas of Gardener, Greene, Kruskal and Miura. Let us consider a linear operator, $L(t)$ that depends on a parameter t through the potential. Let us assume the eigenvalue equation

$$L(t)\psi = \lambda\psi \tag{39}$$

along with the evolution of the wave function

$$\frac{\partial \psi}{\partial t} = B\psi \tag{40}$$

where B represents an anti-symmetric operator. It follows now that

$$\begin{aligned} \frac{\partial L(t)}{\partial t}\psi + L(t)\frac{\partial \psi}{\partial t} &= \lambda_t\psi + \lambda\frac{\partial \psi}{\partial t} \\ \text{or,}\quad \frac{\partial L}{\partial t}\psi + LB\psi &= \lambda_t\psi + \lambda B\psi = \lambda_t\psi + BL\psi \\ \text{or,}\quad \frac{\partial L}{\partial t}\psi &= [B,L]\,\psi + \lambda_t\psi \end{aligned} \tag{41}$$

It follows, therefore, that

$$\lambda_t = 0 \tag{42}$$

provided

$$\frac{\partial L}{\partial t} = [B, L] \tag{43}$$

This is known as the Lax equation and L, B are called the Lax pair. What this equation says is that the evolution of the linear equation with respect to the parameter t will be isospectral, provided the Lax equation is satisfied (λ is commonly referred to as the spectral parameter.). Furthermore, if a Lax pair is found such that the Lax equation yields a given nonlinear equation, then, this says that one can associate a linear Schrödinger equation with it and the method of inverse scattering can be carried out for this system leading to the integrability of the nonlinear system. This is the power of the Lax equation and, as we have mentioned earlier, the geometrical meaning of the Lax equation is that it represents the evolution of a special $(1,1)$ tensor in the phase space (symplectic manifold) of the system.

As an example, let us analyze the KdV equation in some detail. Let us note that if we choose,

$$\begin{aligned} L(t) &= \partial^2 + \frac{1}{6}u(x,t) \\ B(t) &= 4\partial^3 + \frac{1}{2}(\partial u + u\partial) \end{aligned} \tag{44}$$

then, with some straight forward computation, we can determine that

$$[B, L] = \frac{1}{6}\left(u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) \tag{45}$$

so that the Lax equation

$$\frac{\partial L}{\partial t} = [B, L]$$

leads to

$$\begin{aligned} \frac{1}{6}\frac{\partial u}{\partial t} &= \frac{1}{6}\left(u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) \\ \text{or,}\quad \frac{\partial u}{\partial t} &= u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} \end{aligned} \tag{46}$$

We recognize this to be the KdV equation and, having a Lax representation for the equation, then, immediately determines the linear Schrödinger equation associated with it, which we have described earlier, in connection with the method of inverse scattering. This is, however, a general procedure that applies to any integrable model and that is why the Lax equation plays an important role in the study of integrable systems.

Zero curvature formalism:

There is an alternate method of representing integrable systems, which brings out some other properties associated with the system quite nicely. Let us continue with the example of the KdV equation and consider the following vector potentials.

$$\begin{aligned} \bar{A}_0 &= \begin{pmatrix} \frac{1}{6}C_x + \frac{\sqrt{\lambda}}{3}C & -\frac{1}{6}C_{xx} - \frac{1}{18}uC - \frac{\sqrt{\lambda}}{3}C_x \\ \frac{1}{3}C & -\frac{1}{6}C_x - \frac{\sqrt{\lambda}}{3}C \end{pmatrix} \\ \bar{A}_1 &= \begin{pmatrix} \sqrt{\lambda} & -\frac{1}{6}u \\ 1 & -\sqrt{\lambda} \end{pmatrix} \end{aligned} \tag{47}$$

There are several things to note here. First, $C = C[u, \lambda]$ and that the vector potentials belong to the Lie algebra of $SL(2, \mathbf{R})$, namely, $\bar{A}_0, \bar{A}_1 \in SL(2, \mathbf{R})$.

The curvature (field strength) associated with these potentials can be easily calculated and one recognizes that the vanishing of the curvature yields the equations associated with the KdV hierarchy. This is seen as follows.

$$F_{01} = \partial_0 \bar{A}_1 - \partial \bar{A}_0 - [\bar{A}_0, \bar{A}_1] = 0 \tag{48}$$

gives

$$u_t = C_{xxx} + \frac{1}{3}\left(\partial u + u\partial\right) C - 4\lambda C_x \tag{49}$$

For $\lambda = 0$ and $C = u$, this coincides with the KdV equation. In general, we can expand in a power series of the form

$$C = \sum_{n=0}^{N} (4\lambda)^{N-n} C_n[u] \tag{50}$$

Substituting this into the equation and matching the corresponding powers of (4λ), we obtain

$$\begin{aligned} C_0 &= 1 \\ \left(\partial^3 + \frac{1}{3}\left(\partial u + u\partial\right)\right) C_n &= \partial C_{n+1}, \qquad n = 0, 1, 2, \cdots, N-1 \\ u_t &= \left(\partial^3 + \frac{1}{3}\left(\partial u + u\partial\right)\right) C_N \end{aligned} \tag{51}$$

We recognize these as giving the recursion relation between the conserved charges (which we have discussed earlier) as well as the N th equation of the hierarchy. This is known as the zero curvature representation of the integrable system and brings out the recursion relation between the conserved charges, the current algebra etc quite nicely.

Drinfeld-Sokolov formalism:

Thus, we see that an integrable model can be represented as a scalar Lax equation as well as a matrix zero curvature condition. The natural question that arises is whether there is any connection between the two.

To analyze this question, let us note that the scalar Lax equation for KdV is described by the Lax pair which leads to the linear equations

$$\begin{aligned} L\psi &= \left(\partial^2 + \frac{1}{6}u\right)\psi = \lambda\psi \\ \frac{\partial\psi}{\partial t} &= B\psi = \left(4\partial^3 + \frac{1}{2}\left(\partial u + u\partial\right)\right)\psi \end{aligned} \tag{52}$$

The scalar Lax equation can be thought of as the compatibility condition for these two equations when the spectral parameter is independent of t. We note that, while the second equation may appear to be third order in the derivatives, with the use of the Schrödinger equation, the higher order derivatives can, in fact, be reduced. As a result, this pair of equations appears to be at the most quadratic in the derivatives.

We know that a second order equation can be written in terms of two first order equations. Keeping this in mind, let us define

$$\psi_1 = (\partial\psi) \tag{53}$$

as well as a two component column matrix wavefunction

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi \end{pmatrix} \tag{54}$$

It is clear now that the linear Schrödinger equation

$$L\psi = \left(\partial^2 + \frac{1}{6}u\right)\psi = \lambda\psi$$

can be written in the matrix form as

$$\partial_x\Psi = A_1\Psi = \begin{pmatrix} 0 & \lambda - \frac{1}{6}u \\ 1 & 0 \end{pmatrix}\Psi \tag{55}$$

Similarly, the time evolution equation (depending on B) can also be written as a matrix equation of the form

$$\partial_t\Psi = A_0\Psi \tag{56}$$

The compatibility of these two matrix linear equations leads to the zero curvature condition

$$\partial_t A_1 - \partial_x A_0 - [A_0, A_1] = 0 \tag{57}$$

These potentials, however, do not resemble the potentials that we studied earlier. However, it is easy to check that the two sets of potentials are related by a global (λ is a constant independent of t) similarity transformation. For example, note that

$$\bar{A}_1 = S^{-1}A_1 S, \qquad S = \begin{pmatrix} 1 & -\sqrt{\lambda} \\ 0 & 1 \end{pmatrix} \tag{58}$$

This, therefore, establishes the connection between the two formalisms and tells us how to go from one to the other or *vice versa*.

3 Pseudo-differential operators:

With the basics of the previous section, we are now ready to discuss some of the integrable models in some detail. The first thing that we note is that the Lax formalism and the Lax pair is quite crucial in the study of integrable models. However, finding a Lax pair, for a given integrable model, seems like a formidable task. This is where the Gel'fand-Dikii formalism comes to rescue.

Let us consider a general operator of the form

$$P = \sum_i a_i \partial^i \tag{59}$$

where

$$\partial = \frac{\partial}{\partial x}, \qquad a_i = a_i(x) \tag{60}$$

If $i \geq 0$, namely, if the operator P only contains non-negative powers of ∂, then, it is a differential operator ($i = 0$ term is a multiplicative operator) On the other hand, if i takes also negative values, then, the operator P is known as a pseudo-differential operator (Formally, ∂^{-1} is defined from $\partial\partial^{-1} = 1 = \partial^{-1}\partial$). There are some standard nomenclature in using pseudo-differential operators. Thus, for example,

$$P_+ = \left(\sum_i a_i \partial^i\right)_{i\geq 0} = \cdots + a_1\partial + a_0 \tag{61}$$

and, correspondingly,

$$P_- = \left(\sum_i a_i \partial^i\right)_{i<0} = a_1\partial^{-1} + a_2\partial^{-2} + \cdots \tag{62}$$

By construction, therefore, we have

$$P = P_+ + P_- \tag{63}$$

We can also define, in a corresponding manner,

$$(P)_{\geq k} = \left(\sum_i a_i \partial^i\right)_{i\geq k} \tag{64}$$

Let us now note the standard properties of the derivative operator, namely,

$$\begin{aligned} \partial^i \partial^j &= \partial^{i+j} \\ \partial^i f &= \sum_{k=0}^{\infty} \begin{pmatrix} i \\ k \end{pmatrix} f^{(k)} \partial^{i-k} \end{aligned} \tag{65}$$

which holds true for any i, j, positive or negative, where

$$\begin{pmatrix} i \\ k \end{pmatrix} = \frac{i(i-1)(i-2)\cdots(i-k+1)}{k!}, \qquad \begin{pmatrix} i \\ 0 \end{pmatrix} = 1 \tag{66}$$

and $f^{(k)}$ denotes the k th derivative of the function f. It is worth noting here from the above formulae that positive powers of the derivative operator acting to the right cannot give rise to negative powers of the derivative and *vice versa*.

Using these properties of the derivative operators, we note that we can define a multiplication of pseudo-differential operators. The product of two pseudo-differential operators defines a pseudo-differential operator and they define an algebra. We can define a residue of a pseudo-differential operator to be the coefficient of ∂^{-1}, in analogy with the standard residue, namely,

$$\text{Residue} \quad P = \text{Res} \quad P = a_{-1}(x) \tag{67}$$

This allows us to define a concept called the trace of a pseudo-differential operator as

$$\text{Trace} \quad P = \text{Tr} \quad P = \int dx \, \text{Res} \quad P = \int dx \, a_{-1}(x) \tag{68}$$

Let us note that, given two pseudo-differential operators, P, P',

$$\text{Res} \quad [P, P'] = (\partial f(x)) \tag{69}$$

In other words, the residue of the commutator of any two arbitrary pseudo-differential operators is a total derivative. This, therefore, immediately leads to the fact that

$$\text{Tr} \quad PP' = \text{Tr} \quad P'P \tag{70}$$

since the "trace" of the commutator would vanish with the usual assumptions on asymptotic fall off of variables. This shows that the "Trace" defined earlier satisfies the usual cyclicity properties and justifies the name.

Let us also note that, given a pseudo-differential operator

$$P = \sum_i a_i \partial^i \tag{71}$$

we can define a dual operator as

$$Q = \sum_i \partial^{-i} q_{-i} \tag{72}$$

where, the q_i's are independent of the a_i's. This allows us to define a linear functional of the form

$$F_Q(P) = \text{Tr} \quad PQ = \int dx \sum_i a_i q_{i-1} \tag{73}$$

The Lax operators, as we have seen earlier in the case of the KdV equation, have the form of differential operators. However, in general, they can be pseudo-differential operators. Thus, there exist two classes of Lax operators. Operators of the form

$$L_n = \partial^n + u_1 \partial^{n-1} + u_2 \partial^{n-2} + \cdots + u_n \tag{74}$$

are differential operators and lead to a description of integrable models called the generalized KdV hierarchy. On the other hand, Lax operators of the form

$$\Lambda_n = \partial^n + u_1\partial^{n-1} + \cdots + u_n + u_{n+1}\partial^{-1} + \cdots \tag{75}$$

correspond to pseudo-differential operators and lead to a description of integrable models, commonly called the generalized KP hierarchy.

Let us consider the Lax operator for the generalized KdV hierarchy, for the moment. Thus,

$$L_n = \partial^n + u_1\partial^{n-1} + \cdots + u_n \tag{76}$$

We can now formally define the n th root of this operator as a general pseudo-differential operator of the form

$$(L_n)^{\frac{1}{n}} = \partial + \sum_{i=0}^{\infty} \alpha_i(x)\partial^{-i} \tag{77}$$

such that

$$\left(L_n^{\frac{1}{n}}\right)^n = L_n \tag{78}$$

This allows us to determine all the coefficient functions, $\alpha_i(x)$, iteratively and, therefore, the n th root of the Lax operator.

Let us next note that, since

$$\left[L_n^{\frac{k}{n}}, L_n\right] = 0 \tag{79}$$

for any k, it follows that

$$\frac{\partial L_n}{\partial t_k} = \left[\left(L_n^{\frac{k}{n}}\right)_+, L_n\right] = -\left[\left(L_n^{\frac{k}{n}}\right)_-, L_n\right], \qquad k \neq mn \tag{80}$$

defines a consistent Lax equation. This can be seen as follows. First, if $k = mn$, then,

$$\left(L_n^{\frac{k}{n}}\right)_+ = (L_n^m)_+ = L_n^m$$

and, therefore, the commutator will vanish and we will not have a meaningful dynamical equation. For $k \neq mn$, we note, from the structure of the first commutator, that it will, in general, involve powers of the derivative of the forms

$$\partial^{n+k-1}, \partial^{n+k-2}, \cdots, \partial^0$$

On the other hand, the terms in the second commutator will, in general, have powers of the derivative of the forms

$$\partial^{n-2}, \partial^{n-3}, \partial^{n-4}, \cdots$$

However, if the two expressions have to be equal, then, they can only have nontrivial powers of the derivative of the forms

$$\partial^{n-2}, \partial^{n-3}, \cdots, \partial^0$$

This is precisely the structure of the Lax operator (except for the term with ∂^{n-1}), which says that the above equation represents a consistent Lax equation. This equation also will imply that

$$\frac{\partial u_1}{\partial t} = 0$$

which is why, often, this constant is set to zero (as is the case in, say, the KdV equation). This result is very interesting, for once we have a Lax operator, the other member of the pair can now be identified with

$$B_k = \left(L^{\frac{k}{n}}_n\right)_+ \tag{81}$$

up to a multiplicative constant. Such a Lax representation of a dynamical system is known as the standard representation.

Furthermore, we note that the Lax equation also implies that

$$\frac{\partial L^{\frac{1}{n}}_n}{\partial t_k} = \left[\left(L^{\frac{k}{n}}_n\right)_+, L^{\frac{1}{n}}_n\right] \tag{82}$$

It is straight forward to show, using this, that

$$\partial_{t_k}\partial_{t_m} L_n = \partial_{t_m}\partial_{t_k} L_n \tag{83}$$

Namely, different flows commute. This is equivalent to saying that the different Hamiltonians corresponding to the different flows are in involution. Therefore, if we have the right number of conserved charges, the system is integrable.

The construction of the conserved charges, therefore, is crucial in this approach. However, we note from

$$\frac{\partial L^{\frac{1}{n}}_n}{\partial t_k} = \left[\left(L^{\frac{k}{n}}_n\right)_+, L^{\frac{1}{n}}_n\right] \tag{84}$$

that

$$\frac{\partial}{\partial t_k}\mathrm{Tr}\quad L^{\frac{1}{n}}_n = \mathrm{Tr}\quad \left[\left(L^{\frac{k}{n}}_n\right)_+, L^{\frac{1}{n}}_n\right] = 0 \tag{85}$$

which follows from the cyclicity of the "trace". Therefore, we can identify the conserved quantities of the system (up to multiplicative factors) with

$$H_m = \frac{n}{m}\mathrm{Tr}\quad \left(L^{\frac{m}{n}}_n\right), \qquad m \neq ln \tag{86}$$

This naturally gives the infinite number of conserved charges of the system, which, as we have shown before, are in involution. Therefore, in this description, integrability is more or less automatic.

Hamiltonian structures:

In the Lax formalism, we can also determine the Hamiltonian structures of the system in a natural manner. These are known in the subject as the Gel'fand-Dikii brackets and they are determined from the observation that the Lax equation looks very much like Hamilton's equation, with $\left(L_n^{\frac{k}{n}}\right)_+$ playing the role of the Hamiltonian and the commutator substituting for the Hamiltonian structure. Analyzing this further, one ends up with two definitions of Gel'fand-Dikii brackets, which give rise to the two Hamiltonian structures of the system. With the notation of the linear functional defined earlier, they can be written as

$$\begin{aligned}
\{F_Q(L_n), F_V(L_n)\}_1 &= \mathrm{Tr} \quad (L_n [V, Q]) \\
\{F_Q(L_n), F_V(L_n)\}_2 &= \mathrm{Tr} \quad \Big(L_n Q (L_n V)_+ - Q L_n (V L_n)_+\Big)
\end{aligned} \tag{87}$$

The second bracket is particularly tricky if the Lax operator has a constrained structure and the modifications, in such a case, are well known and I will not get into that.

It is worth noting that these brackets are, by definition, anti-symmetric as a Hamiltonian structure should be. While the first bracket is manifestly anti-symmetric, the second is not. However, it is easy to see that the second is also anti-symmetric in the following way.

$$\begin{aligned}
\{F_Q(L_n), F_V(L_n)\} &= \mathrm{Tr} \quad \Big(L_n Q (L_n V)_+ - Q L_n (V L_n)_+\Big) \\
&= \mathrm{Tr} \quad \Big(L_n Q L_n V - L_n Q (L_n V)_- - Q L_n V L_n + Q L_n (V L_n)_-\Big) \\
&= \mathrm{Tr} \quad \Big(-(L_n Q)_+ (-L_n V)_- + (Q L_n)_+ (V L_n)_-\Big) \\
&= \mathrm{Tr} \quad \Big(-(L_n Q)_+ L_n V + (Q L_n)_+ V L_n\Big) \\
&= -\mathrm{Tr} \quad \Big(L_n V (L_n Q)_+ - V L_n (Q L_n)_+\Big) \\
&= -\{F_V(L_n), F_Q(L_n)\}
\end{aligned} \tag{88}$$

Thus, the two brackets indeed satisfy the necessary ant-symmetry property of Hamiltonian structures. Furthermore, it can also be shown (I will not go into the details) that these brackets satisfy Jacobi identity as well and, therefore, constitute two Hamiltonian structures of the system.

Without going into details, I would like to make some general remarks about the Lax operators of the KP type. Let us consider a Lax operator of the type

$$\Lambda_n = \partial^n + u_1 \partial^{n-1} + \cdots + u_n + u_{n+1}\partial^{-1} + \cdots \tag{89}$$

This is a pseudo-differential operator, unlike the earlier case, and, as we have already remarked, such Lax operators describe generalized KP hierarchies. In this case, it can be shown that a Lax equation of the form

$$\frac{\partial \Lambda_n}{\partial t_k} = \left[\left(\Lambda_n^{\frac{k}{n}}\right)_{\geq m}, \Lambda_n\right] \tag{90}$$

is consistent, only for $m=0,1,2$. This is, therefore, different from the generalized KdV hierarchy that we have already studied. For $m=0$, the Lax equation is called, as before, a standard representation, while for $m=1,2$, it is known as a non-standard representation. All the ideas that we had developed for the standard representation go through for the non-standard representation as well and we will return to such an example later.

Example:

As an application of these ideas, let us analyze some of the integrable models from this point of view. First, let us consider the KdV hierarchy. In this case, we have already seen that

$$L = L_2 = \partial^2 + \frac{1}{6}u \tag{91}$$

As we had noted earlier, we note that the coefficient of the linear power of ∂ has been set to zero (which is consistent with the Lax equation). In this case, we can determine the square root of the Lax operator, following the method described earlier, and it has the form,

$$L^{\frac{1}{2}} = \partial + \frac{1}{12}u\partial^{-1} - \frac{1}{24}u_x\partial^{-2} + \frac{1}{48}\left(u_{xx} - \frac{1}{6}u^2\right)\partial^{-3} + \cdots \tag{92}$$

It is easy to check that the square of this operator leads to L up to the particular order of terms.

In this case, we have

$$\left(L^{\frac{1}{2}}\right)_+ = \partial \tag{93}$$

which gives

$$\begin{aligned} \frac{\partial L}{\partial t_1} &= \left[\left(L^{\frac{1}{2}}\right)_+, L\right] = [\partial, L] \\ \text{or,} \quad \frac{\partial u}{\partial t_1} &= \frac{\partial u}{\partial x} \end{aligned} \tag{94}$$

This is the chiral boson equation and is known to be the lowest order equation of the KdV hierarchy. Let us also note that

$$\left(L^{\frac{3}{2}}\right)_+ = \left(LL^{\frac{1}{2}}\right)_+ = \partial^3 + \frac{1}{4}u\partial + \frac{1}{8}u_x = \partial^3 + \frac{1}{8}(\partial u + u\partial) = \frac{1}{4}B \tag{95}$$

where B is the second member of the Lax pair for the KdV equation that we had talked about earlier. It is clear, therefore, that

$$\frac{\partial L}{\partial t} = 4\left[\left(L^{\frac{3}{2}}\right)_+, L\right] \tag{96}$$

will lead to the KdV equation. Similarly, one can derive the higher order equations of the KdV hierarchy from the higher fractional powers of the Lax operator.

We note from the structure of the square root of L that

$$\text{Tr} \quad L^{\frac{1}{2}} = \frac{1}{12}\int dx\, u(x) \tag{97}$$

Similarly,

$$\begin{aligned}\text{Tr}\quad L^{\frac{3}{2}} &= \int dx\,\left[\frac{1}{48}\left(u_{xx} - \frac{1}{6}u^2\right) + \frac{1}{72}u^2\right]\\ &= \frac{1}{96}\int dx\, u^2(x)\end{aligned} \tag{98}$$

Up to multiplicative constants, these are the first two conserved quantities of the KdV hierarchy and the higher order ones can be obtained similarly from the "trace" of higher fractional powers of the Lax operator.

From the form of the Lax operator, in this case,

$$L = \partial^2 + \frac{1}{6}u$$

we note that we can define the dual operators

$$Q = \partial^{-2}q_2 + \partial^{-1}q_1, \qquad V = \partial^{-2}v_2 + \partial^{-1}v_1 \tag{99}$$

Here q_i and v_i are supposed to be independent of the dynamical variable u, so that the linear functionals take the forms

$$F_Q(L) = \text{Tr}\quad LQ = \frac{1}{6}\int dx\, uq_1, \qquad F_V(L) = \frac{1}{6}\int dx\, uv_1 \tag{100}$$

In this case, we can work out

$$\{F_Q(L), F_V(L)\}_1 = \frac{1}{36}\int dx\; dy\, q_1(x)v_1(y)\{u(x), u(y)\}_1 \tag{101}$$

On the other hand,

$$\text{Tr}\quad L\,[V, Q] = \int dx\;(q_1 v_{1,x} - q_{1,x}v_1) = -2\int dx\, q_{1,x}v_1 \tag{102}$$

Thus, comparing the two expressions, we obtain

$$\{u(x), u(y)\}_1 = 72\frac{\partial}{\partial x}\delta(x-y) \tag{103}$$

We recognize this to be the correct first Hamiltonian structure for the KdV equation (except for a multiplicative factor). The derivation of the second Hamiltonian structure is slightly more involved since the structure of the KdV Lax operator has a constrained structure (the linear power of ∂ is missing). However, the construction through the Gel'fand-Dikii brackets, keeping this in mind, can be carried through and gives the correct second Hamiltonian structure for the theory.

Let me note in closing this section that the generalization of the method of inverse scattering as well as the generalization of the Lax formalism (or the Gel'fand-Dikii formalism) to higher dimensions is not as well understood and remain open questions.

4 Two boson hierarchy:

In this section, I will describe another integrable system in 1+1 dimensions, which is very interesting. The study of this system is of fundamental importance, since this system can reduce to many others under appropriate limit/reduction. It is described in terms of two dynamical variables and has the form

$$\begin{aligned} \frac{\partial u}{\partial t} &= \left(2h + u^2 - \alpha u_x\right)_x \\ \frac{\partial h}{\partial t} &= (2uh + \alpha h_x)_x \end{aligned} \tag{104}$$

Here α is an arbitrary constant parameter and we can think of h as describing the height of a water wave from the surface, while u describes the horizontal velocity of the wave. This equation, therefore, describes general shallow water waves.

This system of equations is integrable and, as we have mentioned, reduces to many other integrable systems under appropriate limit/reduction. To name a few, let us note that, when the parameter $\alpha = 0$, this system reduces to Benney's equations, which also represents the standard, dispersionless long water wave equation. For $\alpha = -1$ and $h = 0$, this gives us the Burger's equation. When $\alpha = 1$ and we identify

$$u = -\frac{q_x}{q}, \qquad h = \bar{q}q$$

we obtain the non-linear Schrödinger equation from the two boson equations. Similarly, both the KdV and the mKdV equations are contained in this system as higher order flows. Thus, the study of this system is interesting because once we understand this system,,properties of all these systems are also known.

Conventionally, the two boson equation is represented with the identifications

$$\alpha = 1, \qquad u = J_0, \qquad h = J_1 \tag{105}$$

which is the notation we will follow in the subsequent discussions. In these notations, therefore, the two boson equations take the form

$$\begin{aligned} \frac{\partial J_0}{\partial t} &= \left(2J_1 + J_0^2 - J_{0,x}\right)_x \\ \frac{\partial J_1}{\partial t} &= (2J_0 J_1 + J_{1,x})_x \end{aligned} \tag{106}$$

Being integrable, this system of equations can be described by a Lax equation. Let us consider the Lax operator

$$L = \partial - J_0 + \partial^{-1} J_1 \tag{107}$$

so that it is a pseudo-differential operator. Let us note that, for this operator,

$$\begin{aligned}\left(L^2\right)_{\geq 1} &= \partial^2 - 2J_0\partial \\ \left(L^3\right)_{\geq 1} &= \partial^3 - 3J_0\partial^2 + 3(J_1 + J_0^2 - J_{0,x})\partial\end{aligned} \tag{108}$$

and so on. It is now straight forward to compute and show that

$$\left[L, \left(L^2\right)_{\geq 1}\right] = -\left(2J_1 + J_0^2 - J_{0,x}\right)_x + \partial^{-1}\left(2J_0J_1 + J_{1,x}\right) \tag{109}$$

It is, therefore, clear that the non-standard Lax equation

$$\frac{\partial L}{\partial t} = \left[L, \left(L^2\right)_{\geq 1}\right] \tag{110}$$

gives as consistent equations the two boson equations.

The higher order flows of the two boson (TB) hierarchy can be obtained from

$$\frac{\partial L}{\partial t_k} = \left[L, \left(L^k\right)_{\geq 1}\right] \tag{111}$$

Of particular interest to us is the next higher order equation coming from

$$\frac{\partial L}{\partial t_3} = \left[L, \left(L^3\right)_{\geq 1}\right] \tag{112}$$

These have the forms

$$\begin{aligned}\frac{\partial J_0}{\partial t} &= -J_{0,xxx} - 3\left(J_0J_{0,x}\right)_x - 6\left(J_0J_1\right)_x - \left(J_0^3\right)_x \\ \frac{\partial J_1}{\partial t} &= -\left(J_{1,xx} + 3\left(J_0J_1\right)_x + 3\left(J_1(J_1 + J_0^2 - J_{0,x})\right)\right)_x\end{aligned} \tag{113}$$

This equation is interesting, for we note that if we set $J_0 = 0$ and identify $J_1 = \frac{1}{6}u$, the equations reduce to the KdV equation. That is, as we had mentioned earlier, the KdV equation is contained in the higher flows of the TB hierarchy. Furthermore, it is also interesting to note that this provides a nonstandard representation of the KdV equation, unlike the earlier example where it was described by a standard Lax equation.

Given the Lax representation, the conserved charges are easily constructed by the standard procedure from

$$H_n = \mathrm{Tr}\quad L^n \tag{114}$$

so that we have

$$\begin{aligned}H_1 &= \mathrm{Tr}\quad L = \int dx\, J_1 \\ H_2 &= \mathrm{Tr}\quad L^2 = \int dx\, J_0J_1 \\ H_3 &= \mathrm{Tr}\quad L^3 = \int dx\, \left(J_1^2 - J_{0,x}J_1 + J_1J_0^2\right)\end{aligned} \tag{115}$$

and so on. It is clear that if we set $J_0 = 0$ and identify $J_1 = \frac{1}{6}u$ all the even conserved charges vanish and the odd ones coincide with the conserved charges of the KdV equation.

Hamiltonian structures:

Let us denote the generic Hamiltonian structure associated with this system as

$$\begin{pmatrix} \{J_0, J_0\} & \{J_0, J_1\} \\ \{J_1, J_0\} & \{J_1, J_1\} \end{pmatrix} = \mathcal{D}\delta(x-y) \tag{116}$$

Then, it can be shown that the TB equation has three Hamiltonian structures. But, let me only point out the first two here.

$$\begin{aligned} \mathcal{D}_1 &= \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \\ \mathcal{D}_2 &= \begin{pmatrix} 2\partial & \partial J_0 - \partial^2 \\ J_0\partial + \partial^2 & \partial J_1 + J_1\partial \end{pmatrix} \end{aligned} \tag{117}$$

so that we can write the two boson equations as

$$\partial_t \begin{pmatrix} J_0 \\ J_1 \end{pmatrix} = \mathcal{D}_1 \begin{pmatrix} \frac{\delta H_3}{\delta J_0} \\ \frac{\delta H_3}{\delta J_1} \end{pmatrix} = \mathcal{D}_2 \begin{pmatrix} \frac{\delta H_2}{\delta J_0} \\ \frac{\delta H_2}{\delta J_1} \end{pmatrix} \tag{118}$$

It is now easily checked that, under $J_0 \to J_0 + \alpha$, where α is an arbitrary constant,

$$\mathcal{D}_2 \to \mathcal{D}_2 + \alpha\mathcal{D}_1 \tag{119}$$

which proves that these two Hamiltonian structures are compatible and that the system is integrable (which we already know from the Lax description of the system).

Normally, the second Hamiltonian structure of an integrable system is related to some symmetry algebra. To see this connection, in this case, let us redefine (this is also known as changing the basis)

$$J(x) = J_0(x), \qquad T(x) = J_1 - \frac{1}{2}J_{0,x}(x) \tag{120}$$

In terms of these new variables, the second Hamiltonian structure takes the form

$$\begin{aligned} \{J(x), J(y)\}_2 &= 2\partial_x\delta(x-y) \\ \{T(x), J(y)\}_2 &= J(x)\partial_x\delta(x-y) \\ \{T(x), T(y)\}_2 &= (T(x) + T(y))\,\partial_x\delta(x-y) + \frac{1}{2}\partial_x^3\delta(x-y) \end{aligned} \tag{121}$$

We recognize this as the Virasoro-Kac-Moody algebra, which is the bosonic limit of the twisted $N = 2$ superconformal algebra.

Non-linear Schrödinger equation:

Since the TB system reduces to the non-linear Schrödinger equation, we can also find a Lax description for that system from the present one. We note that with the identification

$$J_0 - q_x|overq, \qquad J_1 = \bar{q}q \tag{122}$$

the Lax operator for the TB system becomes,

$$\begin{aligned} L &= \partial - J_0 + \partial^{-1} J_1 \\ &= \partial + \frac{q_x}{q} + \partial^{-1}\bar{q}q \\ &= q^{-1}\left(\partial + q\partial^{-1}\bar{q}\right) q \\ &= G\tilde{L}G^{-1} \end{aligned} \tag{123}$$

where we have defined

$$G = q^{-1}, \qquad \tilde{L} = \partial + q\partial^{-1}\bar{q} \tag{124}$$

Namely, the two Lax operators L and $\tilde{L}$ are related by a gauge transformation. The adjoint of this transformed Lax operator is determined to be

$$\mathcal{L} = \tilde{L}^\dagger = -\left(\partial + \bar{q}\partial^{-1}q\right) \tag{125}$$

It is straightforward to check that both the standard Lax equations

$$\frac{\partial \tilde{L}}{\partial t} = \left[\tilde{L}, \left(\tilde{L}^2\right)_+\right] \tag{126}$$

and

$$\frac{\partial \mathcal{L}}{\partial t} = \left[\left(\mathcal{L}^2\right)_+, \mathcal{L}\right] \tag{127}$$

give rise to the non-linear Schrödinger equation. (However, supersymmetry seems to prefer the second representation.)

Furthermore, if we identify

$$\bar{q} = q = u \tag{128}$$

then, the standard Lax equations

$$\frac{\partial \tilde{L}}{\partial t} = \left[\tilde{L}, \left(\tilde{L}^3\right)_+\right] \tag{129}$$

and

$$\frac{\partial \mathcal{L}}{\partial t} = \left[\left(\mathcal{L}^3\right)_+, \mathcal{L}\right] \tag{130}$$

give the mKdV equation.

Thus, we see that the TB system is indeed a rich theory to study.

5 Supersymmetric equations:

Given a supersymmetric integrable system, we can ask if there also exist supersymmetric integrable systems corresponding to it. It turns out to be a difficult question in the sense that the supersymmetrization turns out not to be unique and we do not yet fully understand how to classify all possible supersymmetrizations of such systems. Let me explain this with an example.

Let us consider the KdV equation

$$\frac{\partial u}{\partial t} = 6uu_x + u_{xxx}$$

Then, a supersymmetric generalization of this system that is also integrable is given by

$$\begin{aligned} \frac{\partial u}{\partial t} &= 6uu_x + u_{xxx} - 3\psi\psi_{xx} \\ \frac{\partial \psi}{\partial t} &= 3\,(u\psi)_x + \psi_{xxx} \end{aligned} \tag{131}$$

Here ψ represents the fermionic superpartner of the bosonic dynamical variable u of the KdV equation. It is easy to check that these equations remain invariant under the supersymmetry transformations,

$$\delta\psi = \epsilon u, \qquad \delta u = \epsilon\psi_x \tag{132}$$

where ϵ is a constant Grassmann parameter (fermionic parameter) of the transformation, satisfying $\epsilon^2 = 0$.

We can, of course, determine the supersymmetry charge (generator of supersymmetry) associated with this system and it can be shown that the supersymmetry algebra satisfied by this charge is

$$[Q, Q]_+ = 2P \tag{133}$$

This can also be checked by taking two successive supersymmetry transformations in opposite order and adding them. There are several things to note from this system. First, the second Hamiltonian structure associated with this system, $\mathcal{D}_2$, is the superconformal algebra, which is the supersymmetrization of the Virasoro algebra. Second, just as this represents a $N = 1$ supersymmetric extension of the KdV equation, there also exists a second $N = 1$ supersymmetric extension,

$$\begin{aligned} \frac{\partial u}{\partial t} &= 6uu_x + u_{xxx} \\ \frac{\partial \psi}{\partial t} &= 6u\psi_x + \psi_{xxx} \end{aligned} \tag{134}$$

which is integrable. This second supersymmetric extension was originally discarded as being a “trivial” supersymmetrization, since the bosonic equations do not change in the presence of fermions. However, it generated a lot of interest after it was realized that it is this equation that arises in a study of superstring theory from the point of view of matrix models. Such a supersymmetric extension now has the name -B supersymmetrization. Thus, we see that even at the level of $N = 1$ supersymmetrization, there is no unique extension of the integrable model. The problem becomes more and more severe as we go to higher supersymmetrizations. For $N = 2$, it is known that there are at least four distinct "nontrivial" supersymmetrizations of the KdV equation that are integrable. Understanding how many distinct supersymmetrizations are possible for a given integrable equation, therefore, remains an open question. In addition, there are also fermionic extensions of a given integrable model (not necessarily supersymmetric) that are also integrable and there does not exist any unified description of them yet.

Lax description:

Since the supersymmetric KdV equation (super KdV) is an integrable system, let us determine a Lax description for it. The simplest way to look for a Lax description is to work on a superspace, which is the natural manifold to study supersymmetric systems.

Let us consider a simple superspace parameterized by (x, θ), a single bosonic coordinate x and a single real fermionic coordinate θ, satisfying $\theta^2 = 0$. Supersymmetry, then, can be shown to correspond to a translation, in this space, of the fermionic coordinate θ. On this space, one can define a covariant derivative

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x} \tag{135}$$

which transforms covariantly under a supersymmetry transformation and can be seen to satisfy

$$D^2 = \partial \tag{136}$$

On the superspace, a function is called a superfield and, since the fermionic coordinate is nilpotent, has a simple representation of the form (in the present case)

$$\Phi(x, \theta) = \psi(x) + \theta u(x) \tag{137}$$

The Grassmann parity of the components is completely determined by the parity of the superfield. For our discussion of the super KdV system, let us choose the superfield Φ to be fermionic so that we can think of u as the bosonic dynamical variable of the KdV equation and ψ as its fermionic superpartner.

In terms of this superfield, the super KdV equations can be combined into one single equation of the form

$$\frac{\partial\Phi}{\partial t} = \left(D^6\Phi\right) + 3\left(D^2\left(\Phi(D\Phi)\right)\right) \tag{138}$$

It is now easy to check that if we choose, as Lax operator on this superspace,

$$L = D^4 + D\Phi \tag{139}$$

then, the Lax equation

$$\frac{\partial L}{\partial t} = \left[\left(L^{\frac{3}{2}}\right)_+, L\right] \tag{140}$$

gives the super KdV equation. The structure of this Lax operator and the Lax equation is, of course, such that they reduce to the KdV equation in the bosonic limit, which is nice.

It is worth making a few remarks about operators on the superspace. First, a pseudo-differential operator on this space is defined with powers of D. Correspondingly, various decompositions are done with respect to powers of D. Thus, we define

$$\begin{aligned} \text{super Residue} \quad P &= \text{sRes} \quad P = \text{coefficient of } D^{-1} \\ \text{super Trace} \quad P &= \text{sTr} \quad P = \int dx\, d\theta\, \text{sRes} \quad P \end{aligned} \tag{141}$$

The conserved quantities, for the super KdV system, for example, are obtained as

$$H_n = \text{sTr} \quad L^{\frac{2n+1}{2}} \tag{142}$$

These are all bosonic conserved quantities and there is an infinite number of them. They all reduce to the infinite set of conserved charges of the bosonic integrable model in the bosonic limit. The Gel'fand-Dikii brackets can be generalized to this space as well and lead to the correct Hamiltonian structures of the theory. An interesting feature of the Lax description on a superspace is that the same integrable system can be described in terms of a Lax operator that is either bosonic or fermionic. Furthermore, in the supersymmetric models we can define non-local conserved charges from the Lax operator by taking, say, for example in the super KdV case, powers of quartic roots of the Lax operator. Finally, let me also point out that a supersymmetric -B system has fermionic Hamiltonians (conserved charges) with corresponding odd (fermionic) Hamiltonian structures.

Super TB hierarchy:

As we have seen, the TB hierarchy consists of two dynamical variables, J_0, J_1. Therefore, the supersymmetrization of this system will involve two fermionic partners, say ψ_0, ψ_1. From our experience with the super KdV system, let us combine them into two fermionic superfields of the forms

$$\Phi_0 = \psi_0 + \theta J_0, \qquad \Phi_1 = \psi_1 + \theta J_1 \tag{143}$$

With a little bit of algebra, we can check that if we choose the Lax operator

$$L = D^2 - (D\Phi_0) + D^{-1}\Phi_1 \tag{144}$$

then, we obtain

$$\left(L^2\right)_{\geq 1} = D^4 - 2(D\Phi_0)D^2 - 2\Phi_1 D \tag{145}$$

which gives

$$\left[L, \left(L^2\right)_{\geq 1}\right] = \left(D\left((D^4\Phi_0) - D(D\Phi_0)^2 - 2(D^2\Phi_1)\right)\right) + D^{-1}\left((D^4\Phi_1) + 2D^2(\Phi_1(D\Phi_0))\right) \tag{146}$$

It is clear, therefore, that the Lax equation

$$\frac{\partial L}{\partial t} = \left[L, \left(L^2\right)_{\geq 1}\right] \tag{147}$$

leads to consistent equations and gives

$$\begin{aligned}\frac{\partial\Phi_0}{\partial t} &= -(D^4\Phi_0)+2(D\Phi_0)(D^2\Phi_0)+2(D^2\Phi_1)\\ \frac{\partial\Phi_1}{\partial t} &= (D^4\Phi_1)+2\left(D^2(\Phi_1(D\Phi_0))\right)\end{aligned} \tag{148}$$

The Lax operator as well as the Lax equation (and the equations following from it are easily seen to reduce to the TB equations in the bosonic limit. These, therefore, represent a supersymmetric extension of the TB hierarchy that is integrable. The higher order flows of the hierarchy are obtained from

$$\frac{\partial L}{\partial t_k} = \left[L, \left(L^k\right)_{\geq 1}\right] \tag{149}$$

Once we have the Lax description of the system, we can immediately construct the conserved charges from

$$H_n = \mathrm{sTr}\quad L^n = \int dx\, d\theta\, \mathrm{sRes}\quad L^n, \qquad n = 1, 2, 3, \cdots \tag{150}$$

Explicitly, we can construct the lower order conserved charges as

$$\begin{aligned}H_1 &= -\int dx\, d\theta\, \Phi_1\\ H_2 &= 2\int dx\, d\theta\, (D\Phi_0)\Phi_1\\ H_3 &= 3\int dx\, d\theta\, \left((D^3\Phi_0)-(D\Phi_1)-(D\Phi_0)^2\right)\Phi_1\end{aligned} \tag{151}$$

and so on. Thus, we see that the system has an infinite number conserved charges, which are in involution (follows from the Lax description) and, therefore, is integrable.

Hamiltonian structures:

Just like the bosonic system, the super TB equations also possess three Hamiltonian structures, although they are not necessarily local unlike the structures in the bosonic case. Let me only describe the first two here.

Defining, as in the bosonic case, a generic Hamiltonian structure as,

$$\begin{pmatrix} \{\Phi_0,\Phi_0\} & \{\Phi_0,\Phi_1\}\\ \{\Phi_1,\Phi_0\} & \{\Phi_1,\Phi_1\}\end{pmatrix} = \mathcal{D}\delta(z-z') = \mathcal{D}\delta(x-x')\delta(\theta-\theta') \tag{152}$$

we note that

$$\mathcal{D}_1 = \begin{pmatrix} 0 & -D\\ -D & 0\end{pmatrix} \tag{153}$$

as well as

$$\mathcal{D}_2 = \begin{pmatrix} -2D-2D^{-1}\Phi_1 D^{-1}+D^{-1}(D^2\Phi_0)D^{-1} & D^3-D(D\Phi_0)+D^{-1}\Phi_1 D\\ -D^3-(D\Phi_0)D-D\Phi_1 D^{-1} & -D^2\Phi_1-\Phi_1 D^2\end{pmatrix} \tag{154}$$

define the first two Hamiltonian structures of the super TB hierarchy. These structures have the necessary symmetry property and it can be checked using the method of prolongation that these structures satisfy the Jacobi identity. They give rise to super TB equation, say for example, through

$$\begin{pmatrix} \frac{\partial\Phi_0}{\partial t} \\ \frac{\partial\Phi_1}{\partial t} \end{pmatrix} = \mathcal{D}_1 \begin{pmatrix} \frac{\delta H_3}{\delta\Phi_0} \\ \frac{\delta H_3}{\delta\Phi_1} \end{pmatrix} = \mathcal{D}_2 \begin{pmatrix} \frac{\delta H_2}{\delta\Phi_0} \\ \frac{\delta H_2}{\delta\Phi_1} \end{pmatrix} \tag{155}$$

Although it is not obvious, the second Hamiltonian structure corresponds to the twisted $N=2$ superconformal algebra, which can be seen as follows. Let us look at the second Hamiltonian structure in the component variables. Let us define

$$\xi = \frac{1}{2}\psi_1, \qquad \bar{\xi} = -\frac{1}{2}\left(\psi_{0,x} - \psi_1\right) \tag{156}$$

In terms of these variables, the nontrivial elements of the second Hamiltonian structure take the local form

$$\begin{aligned}
\{J_0(x), J_0(y)\}_2 &= 2\partial_x\delta(x-y) \\
\{J_0(x), J_1(y)\}_2 &= \partial_x\left(J_0\delta(x-y)\right) - \partial_x^2\delta(x-y) \\
\{J_1(x), J_1(y)\}_2 &= (J_1(x)+J_1(y))\partial_x\delta(x-y) \\
\{J_0(x), \xi(y)\}_2 &= \xi\delta(x-y) \\
\{J_0(x), \bar{\xi}(y)\}_2 &= -\bar{\xi}\delta(x-y) \\
\{J_1(x), \xi(y)\}_2 &= (\xi(x)+\xi(y))\partial_x\delta(x-y) \\
\{J_1(x), \bar{\xi}(y)\}_2 &= \bar{\xi}(x)\partial_x\delta(x-y) \\
\{\bar{\xi}(x), \xi(y)\}_2 &= -\frac{1}{4}J_1\delta(x-y) + \frac{1}{4}\partial_x\left(J_0\delta(x-y)\right) - \frac{1}{4}\partial_x^2\delta(x-y)
\end{aligned} \tag{157}$$

We recognize this to be the $N=2$ superconformal algebra.

$\mathbf{N=2}$ supersymmetry:

Although the super TB system that we have constructed, naively appears to have $N=1$ supersymmetry, in fact, it does possess a $N=2$ supersymmetry. This is already suggested by the fact that the second Hamiltonian structure of this system corresponds to the $N=2$ superconformal algebra. Explicitly, this can be checked as follows.

Let us note that the super TB equations, in terms of the redefined components, take the forms

$$\begin{aligned}
\frac{\partial J_0}{\partial t} &= -J_{0,xx} + 2J_0J_{0,x} + 2J_{1,x} \\
\frac{\partial J_1}{\partial t} &= J_{1,xx} + 2\left(J_0J_1\right)_x + 8\left(\xi\bar{\xi}\right)_x \\
\frac{\partial \xi}{\partial t} &= \xi_{xx} + 2\left(\xi J_0\right)_x \\
\frac{\partial \bar{\xi}}{\partial t} &= \bar{\xi}_{xx} + 2\left(\bar{\xi} J_0\right)_x
\end{aligned} \tag{158}$$

It is now straight forward to check that this system of equations is invariant under the following two sets of supersymmetric transformations,

$$\begin{aligned}
\delta J_0 &= 2\epsilon\xi \\
\delta J_1 &= 2\epsilon\xi_x \\
\delta\xi &= 0 \\
\delta\bar{\xi} &= -\frac{1}{2}\epsilon\left(J_{0,x} - J_1\right)
\end{aligned} \tag{159}$$

and

$$\begin{aligned}
\bar{\delta} J_0 &= 2\bar{\epsilon}\bar{\xi} \\
\bar{\delta} J_1 &= 0 \\
\bar{\delta}\xi &= -\frac{1}{2}\bar{\epsilon} J_1 \\
\bar{\delta}\bar{\xi} &= 0
\end{aligned} \tag{160}$$

Nonlocal charges:

As we have already noted, in supersymmetric integrable systems, in addition to the local bosonic conserved charges, we also have nonlocal conserved charges. Let us study this a little within the context of the super TB hierarchy.

Let us recall that the Lax operator for the system is given by

$$L = D^2 - (D\Phi_0) + D^{-1}\Phi_1$$

and the infinite set of local, bosonic conserved charges are obtained as

$$H_n = \mathrm{sTr}\quad L^n, \qquad n = 1, 2, 3, \cdots \tag{161}$$

On the other hand, let us also note that in this system, we can also define a second infinite set of conserved charges as

$$Q_{\frac{2n-1}{2}} = \mathrm{sTr}\quad L^{\frac{2n-1}{2}}, \qquad n = 1, 2, 3, \cdots \tag{162}$$

There are several things to note from this definition. First of all, unlike the earlier set, these conserved charges are fermionic in nature. Second, they are nonlocal. Let me write down a few lower order charges of this set.

$$\begin{aligned}
Q_{\frac{1}{2}} &= -\int dx\, d\theta\, (D^{-1}\Phi_1) = -\int dz\, (D^{-1}\Phi_1) \\
Q_{\frac{3}{2}} &= -\int dz\, \left[\frac{3}{2}(D^{-1}\Phi_1)^2 - \Phi_0\Phi_1 - \left(D^{-1}((D\Phi_0)\Phi_1)\right)\right] \\
Q_{\frac{5}{2}} &= -\int dz\, \left[\frac{1}{6}(D^{-1}\Phi_1)^3 - \left(5(D^{-2}\Phi_1)\Phi_1 - 2\Phi_0\Phi_1 - 3(D\Phi_1) - (D^{-1}\Phi_1)^2\right)(D\Phi_0)\right. \\
&\qquad \left. + \left(D^{-1}\left((D\Phi_1)\Phi_1 + \Phi_1(D\Phi_0)^2 - (D\Phi_1)(D^2\Phi_0)\right)\right)\right]
\end{aligned} \tag{163}$$

and so on.

There are several things to be noted about these charges. First, as we have already mentioned and as can be explicitly seen, these charges are fermionic in nature and are conserved. Second, even though they are defined on the superspace, they are not invariant under supersymmetry (it is the nonlocality that is responsible for this problem). This infinite set of charges satisfies an algebra which appears to have the structure of a Yangian algebra, which arises in the study of supersymmetric nonlinear sigma models, namely,

$$\begin{aligned}
\{Q_{\frac{1}{2}}, Q_{\frac{1}{2}}\}_1 &= 0 \\
\{Q_{\frac{1}{2}}, Q_{\frac{3}{2}}\}_1 &= H_1 \\
\{Q_{\frac{1}{2}}, Q_{\frac{5}{2}}\}_1 &= H_2 \\
\{Q_{\frac{3}{2}}, Q_{\frac{3}{2}}\}_1 &= 2H_2 \\
\{Q_{\frac{3}{2}}, Q_{\frac{5}{2}}\}_1 &= \frac{7}{3}H_3 + \frac{7}{24}H_1^3 \\
\{Q_{\frac{5}{2}}, Q_{\frac{5}{2}}\}_1 &= 3H_4 - \frac{5}{8}H_2H_1^2
\end{aligned} \tag{164}$$

and son on. The role of the fermionic nonlocal charges as well as the meaning of the Yangian algebra, however, are not fully understood.

6 Dispersionless integrable systems:

Let us consider again the KdV equation as an example.

$$\frac{\partial u}{\partial t} = 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}$$

As we have noted earlier, it is the linear term on the right hand side that is the source of dispersion for the solutions. Let us, therefore, get rid of the dispersive term, in which case, the equation becomes

$$\frac{\partial u}{\partial t} = 6u\frac{\partial u}{\partial x} \tag{165}$$

This is known as the Riemann equation and we see that it corresponds to the dispersionless limit of the KdV equation. Given a nonlinear, bosonic equation, the systematic way in which the dispersionless limit is obtained is by scaling

$$\frac{\partial}{\partial t} \rightarrow \epsilon\frac{\partial}{\partial t}, \qquad \frac{\partial}{\partial x} \rightarrow \epsilon\frac{\partial}{\partial x} \tag{166}$$

in the equation and then taking the limit $\epsilon \rightarrow 0$. This leads to the dispersionless limit of the original system of equations. It is important to note here that the dynamical variable is not scaled (although in supersymmetric systems, as we will see, it is necessary to scale the fermionic variables to maintain supersymmetry).

Let us recall that the Lax description for the KdV equation is obtained from the Lax operator

$$L = \partial^2 + u$$

through the Lax equation

$$\frac{\partial L}{\partial t} = 4\left[\left(L^{\frac{3}{2}}\right)_+, L\right]$$

As we will now see, the Lax description for the dispersionless model is obtained in a much simpler fashion. Let us consider a Lax function, on the classical phase space, of the form

$$L(p) = p^2 + u \tag{167}$$

Here p represents the classical momentum variable on the phase space and, therefore, this Lax function only consists of commuting quantities. However, we can think of this as consisting of a power series in p and formally calculate

$$\left(L^{\frac{3}{2}}\right)_+ = p^3 + \frac{3}{2}up \tag{168}$$

where the projections are defined with respect to the powers of p. It is now simple to check, with the standard canonical Poisson bracket relations,

$$\{x, p\} = 1, \qquad \{x, x\} = 0\{p, p\}$$

that the Lax equation

$$\frac{\partial L}{\partial t} = 4\{L, \left(L^{\frac{3}{2}}\right)_+\} \tag{169}$$

leads to the Riemann equation, which is the dispersionless limit of the KdV equation.

Thus, we see that given a Lax description of an integrable model in terms of Lax operators, the dispersionless limit is obtained from a simpler Lax description on the classical phase space through classical Poisson brackets. Let us also note that the conserved quantities of the dispersionless model can be obtained from this Lax function as (in this model)

$$H_n = \text{Tr} \quad L^{\frac{2n+1}{2}} = \int dx, \text{Res} \quad L^{\frac{2n+1}{2}}, \qquad n = 0, 1, 2, \cdots \tag{170}$$

where "Res" is defined as the coefficient of the p^{-1} term. All of our discussions in connection with pseudo-differential operators carries through to this case where the Lax function has a polynomial structure in the momentum variables. (When I gave these talks, I had mentioned that the construction of the Hamiltonian structure from the Gel'fand-Dikii was an open question. Since then, this problem has been solved and we know now that these can be constructed rather easily from a Moyal-Lax representation of integrable models.)

Let me say here that dispersionless models encompasses a wide class of systems such as hydrodynamic equations, polytropic gas dynamics, Chaplygin gas, Born-Infeld equation, Monge-Ampère equation, elastic medium equation etc, some of which show up in the study of string theory, membrane theory as well as in topological field theories.

Let us study an example of such systems in some detail, namely, the polytropic gas dynamics. These are described by a set of two equations

$$\begin{aligned} u_t + uu_x + v^{\gamma-2}v_x &= 0, \qquad \gamma \neq 0,1 \\ v_t + (uv)_x &= 0 \end{aligned} \tag{171}$$

These equations are known to be Hamiltonian with

$$H = \int dx \left(-\frac{1}{2}u^2 v - \frac{v^{\gamma-1}}{\gamma(\gamma-1)}\right) \tag{172}$$

and

$$\mathcal{D} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix} \tag{173}$$

so that we can write the polytropic gas equations as

$$\begin{pmatrix} u_t \\ v_t \end{pmatrix} = \mathcal{D} \begin{pmatrix} \frac{\delta H}{\delta u} \\ \frac{\delta H}{\delta v} \end{pmatrix} \tag{174}$$

In fact, this system has three distinct Hamiltonian structures, but I will not get into the details of this.

Let us next consider a Lax function, on the classical phase space, of the form

$$L = p^{\gamma-1} + u + \frac{v^{\gamma-1}}{(\gamma-1)^2} p^{-(\gamma-1)} \tag{175}$$

Then, it is straight forward to check that the classical Lax equation

$$\frac{\partial L}{\partial t} = \frac{\gamma-1}{\gamma}\{\left(L^{\frac{\gamma}{\gamma-1}}\right)_{\geq 1}, L\} \tag{176}$$

gives rise to the equations for the polytropic gas. The higher order equations of the hierarchy are similarly obtained from

$$\frac{\partial L}{\partial t_n} = c_n\{\left(L^{n+\frac{1}{\gamma-1}}\right)_{\geq 1}, L\} \tag{177}$$

Thus, we see that the polytropic gas dynamics is obtained as a nonstandard Lax description on the classical phase space. (At the time of the school, this was the only Lax description for the system that was known. Subsequently, a standard Lax description has been obtained, which brings out some interesting connection between this system and the Lucas polynomials.)

Once we have the Lax description, we can, of course, obtain the conserved quantities from the "Trace". However, in this case, unlike in the case of Lax operators, we observe an interesting feature, namely, the residue can be obtained from expanding around $p = 0$ or around $p = \infty$. Thus, there are two series of conserved charges that we can construct for this system. Expanding around $p = \infty$, we obtain

$$H_{n+1} = C_{n+1}\mathrm{Tr}\quad L^{n+1-\frac{1}{\gamma-1}}, \qquad n = 0, 1, 2, \cdots \tag{178}$$

where C_{n+1}'s are normalization constants. Explicitly, the first few of the charges have the forms

$$\begin{aligned} H_1 &= \int dx\, u \\ H_2 &= \int dx \left(\frac{1}{2!}u^2 + \frac{v^{\gamma-1}}{(\gamma-1)(\gamma-2)}\right) \\ H_3 &= \int dx \left(\frac{1}{3!}u^3 + \frac{uv^{\gamma-1}}{(\gamma-1)(\gamma-2)}\right) \end{aligned} \tag{179}$$

and so on. On the other hand, an expansion around $p = 0$ leads to

$$\tilde{H}_n = \tilde{C}_n \mathrm{Tr}\quad L^{n+\frac{1}{\gamma-1}}, \qquad n = 0, 1, 2, \cdots \tag{180}$$

The first few charges of this set have the explicit forms,

$$\begin{aligned} \tilde{H}_0 &= \int dx\, v \\ \tilde{H}_1 &= \int dx\, uv \\ \tilde{H}_2 &= \int dx \left(\frac{1}{2!}u^2 v + \frac{v^{\gamma}}{\gamma(\gamma-1)}\right) \end{aligned} \tag{181}$$

and so on. The two sets of conserved quantities can, in fact, be expressed in closed forms. Let us also note that if we define two functions as

$$\begin{aligned} \chi &= \lambda^{-\frac{1}{\gamma-1}}\left\{\left[\left(\frac{u+\lambda}{2}\right)^2 - \frac{v^{\gamma-1}}{(\gamma-1)^2}\right]^{\frac{1}{2}} + \frac{u+\lambda}{2}\right\}^{\frac{1}{\gamma-1}} \\ \tilde{\chi} &= \lambda^{-\frac{1}{\gamma-1}}\left\{\left[\left(\frac{u+\lambda}{2}\right)^2 - \frac{v^{\gamma-1}}{(\gamma-1)^2}\right]^{\frac{1}{2}} - \frac{u+\lambda}{2}\right\}^{\frac{1}{\gamma-1}} \end{aligned} \tag{182}$$

where λ is an arbitrary constant parameter, it can be shown that these two functions generate the two sets of conserved quantities as the coefficients of distinct powers of λ. Let me also note that the second Hamiltonian structure for the polytropic gas has the form

$$\mathcal{D}_2 = \begin{pmatrix} \partial v^{\gamma-2} + v^{\gamma-2}\partial & \partial u + (\gamma-2)u\partial \\ (\gamma-2)\partial u + u\partial & \partial v + v\partial \end{pmatrix} \tag{183}$$

Dispersionless supersymmetric KdV:

Let us recall that the super KdV equation can be described by the Lax operator

$$L = D^4 + D\Phi$$

and the Lax equation

$$\frac{\partial L}{\partial t} = 4\left[\left(L^{\frac{3}{2}}\right)_+, L\right]$$

Here,

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x}$$

represents the covariant derivatiove on the superspace. In trying to obtain the dispersionless limit of this supersymmetric system, let us recall what we have learnt from the dispersionless limit of a bosonic model. We noted that the Lax operator goes over to the Lax function with $\partial \to p$. However, our Lax operator, in the supersymmetric case, is described in terms of super covariant derivative D. Therefore, the natural question is what this object goes over to in the dispersionless limit.

Let us note that the classical super phase space is parameterized by (x, θ, p, p_θ) . From these we can define a variable

$$\Pi = -(p_\theta + \theta p) \tag{184}$$

whose action on any phase space variable, through the Poisson brackets is

$$\{\Pi, A\} = (DA), \qquad \{\Pi, \Pi\} = -2p \tag{185}$$

Therefore, it would seem natural to let $D \to \Pi$ in the dispaersionless limit. However, this leads to a serious problem. For example, we know that $D^2 = \partial \to p$ whereas $\Pi^2 = 0$ since it is a classical fermionic variable.

To analyze this problem a little more, let us recall that the dispersionless limit is obtained by scaling

$$\partial_t \to \epsilon\partial_t, \qquad \partial \to \epsilon\partial$$

Therefore, since $D^2 = \partial$, consistency would require that we scale

$$D = \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial x} \to \epsilon^{\frac{1}{2}} D \tag{186}$$

This implies that the fermionic coordinate needs to be scaled as

$$\theta \to \epsilon^{-\frac{1}{2}}\theta \tag{187}$$

On the other hand, let us recall that the basic fermionic superfield of the theory is given as

$$\Phi(x, \theta) = \psi + \theta u \tag{188}$$

Since the dynamical variable u does not scale and θ scales, it follows that supersymmetry can be maintained under such a scaling only if

$$\psi \to \epsilon^{-\frac{1}{2}}\psi, \qquad \Phi \to \epsilon^{-\frac{1}{2}}\Phi \tag{189}$$

This shows that, unlike in the bsoonic theory, in a supersymmetric theory, fermions scale in going to the dispersionless limit. With this scaling, we can go to the super KdV equation and note that, in the dispersionless limit, the equation becomes

$$\Phi_t = 3D^2\left(\Phi(D\Phi)\right) \tag{190}$$

which can be thought of as the super Riemann equation.

Obtaining a Lax function and, therefore, a Lax description of a supersymmetric theory reamins an open question. However, through brute force construction it is known that the Lax function

$$L = p^2 + \frac{1}{2}(D\Phi) + \frac{p^{-2}}{16}\left((D\Phi)^2 - 2\Phi\Phi_x\right) - \frac{p^{-4}}{32}\Phi(D\Phi)\Phi_x \tag{191}$$

leads through the classical Lax equation (the projection is with respect to powers of p)

$$\frac{\partial L}{\partial t} = 4\{L, \left(L^{\frac{3}{2}}\right)_+\} \tag{192}$$

gives the dispersionless limit of the super KdV equation (super Riemann equation). It is worth emphasizing here that, although the Lax description for afew supersymmetric dispersionless models have been constructed through brute force, a systematic understanding of them is still lacking.

The conserved charges can be obtained from this Lax description immediately. Thus,

$$H_n = C_n \mathrm{Tr} \quad L^{\frac{2n+1}{2}} = \int dz\, \left(\Phi(D\Phi)^n - n\Phi(D\Phi)^{n-1}\Phi_x\right) \tag{193}$$

where $n = 0, 1, 2, \cdots$ and it is clear that these are bosonic conserved charges. The supersymmetric dispersionless model has the Hamiltonian structure

$$\mathcal{D} = -\frac{1}{2}\left(3\Phi D^2 + (D\Phi)D + 2(D^2\Phi)\right) \tag{194}$$

which can be recognized as the centerless superconformal algebra. With this Hamiltonian structure, it is straight forward to check that the conserved charges are in involution,

$$\{H_n, H_m\} = \int dz\, \frac{\delta H_n}{\delta\Phi}\mathcal{D}\frac{\delta H_m}{\delta\Phi} = 0 \tag{195}$$

which also follows from the Lax description of the system.

Let me note in closing that this supersymmetric model has two infinite sets of nonlocal charges of the forms

$$\begin{aligned} F_n &= \int dz\, (D^{-1}\Phi)^n \\ G_n &= \int dz\, \Phi(D^{-1}\Phi)^n \end{aligned} \tag{196}$$

where $n = 1, 2, \cdots$. (Note that $G_0 = H_0$.) These charges have been constructed by brute force, since it is not clear how to obtain nonlocal quantities from a classical Lax function. This remains an open question. Second, of the two sets of nonlocal charges, we see that F_n is fermionic while G_n is bosonic. Furthermore, all these conserved charges satisfy a very simple algebra,

$$\begin{aligned} \{H_n, H_m\} &= 0 = \{F_n, H_m\} = \{G_n, H_m\} \\ \{F_n, G_m\} &= 0 = \{G_n, G_m\} \\ \{F_n, F_m\} &= nmG_{n+m-2} \end{aligned} \tag{197}$$

In connection with dispersionless supersymmetric integrable models, several questions remain. For example, it is not clear how to systematically construct the Lax description for them. It is not at all clear how nonlocal charges can be obtained from a classical Lax function. Neither is it clear what is the role played by these charges within the context of integrability.

It is a pleasure to thank the organizers of the Swieca school for the warm hospitality. This work was supoported in part by US DOE Grant no. DE-FG-02-91ER40685.

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