

Oscillatory Behaviour in Homogeneous String Cosmology Models

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Abstract

Some spatially homogeneous Bianchi type I cosmological models filled with homogeneous “electric” \mathbf{p} -form fields are shown to mimic the never-ending oscillatory behaviour of generic string cosmologies established recently. The validity of the “Kasner-free-flights plus collisions-on-potential-walls” picture is also illustrated in the case of known, non-chaotic, superstring solutions.

1 Introduction

It has been recently pointed out that the *generic inhomogeneous* solution, near a cosmological singularity, of the low-energy bosonic field equations of all superstring models ($D = 10$, IIA, IIB, I, het_F , het_{SO}), as well as those of M -theory ($D = 11$ supergravity), exhibits a never ending oscillatory behaviour [1], of the Belinskii-Khalatnikov-Lifshitz (BKL) type [2]. As a rule, the existence of such an infinitely continued oscillatory behaviour delicately depends both on the full field content of the considered theory, and on the allowance for generic spatial inhomogeneities. Non generic models, obtained, e.g., by truncating the field content, and/or by imposing some homogeneity ansatz, may turn out to exhibit a finite number of oscillations, ending with a monotonic Kasner-like power-law approach to the cosmological singularity. Such solutions are of measure zero within the full set of solutions of the complete theory and do not contradict the finding of Ref. [1] which concerns generic solutions.

The study of specialized cosmological models, in particular spatially homogeneous models, has played an important role in theoretical cosmology. Of particular interest has been the study of the homogeneous Bianchi type-IX model [3], [4], for Einstein's vacuum equations in $D = 4$. This model captures, by means of a simple set of ordinary differential equations (ODE), the essential features of the generic, inhomogeneous BKL oscillatory behaviour [2]. One may then wonder whether there exist also in the superstring context, simple, homogeneous models which capture by means of ODEs the essential new features of the oscillatory behaviour found in [1]. The purpose of the present paper is to answer this question in the affirmative. The simple homogeneous models in question are just the Bianchi type I models with appropriate p -form sources.

Let us first recall the basic results (and the notation) of [1]. We consider models in spacetime dimension D of the general form

$$S = \int d^D x \sqrt{g} \left[R(g) - \partial_\mu \varphi \partial^\mu \varphi - \sum_p \frac{1}{2} \frac{1}{(p+1)!} e^{\lambda_p \varphi} (d A_p)^2 \right]. \quad (1)$$

We have written (1) in the Einstein conformal frame, and used a specific normalization of the kinetic term of the “dilaton” φ . The integer $p \geq 0$ labels the various p -forms $A_p \equiv A_{\mu_1 \dots \mu_p}$ present in the theory, with field strength $F_{\mu_0 \mu_1 \dots \mu_p} = \partial_{\mu_0} A_{\mu_1 \dots \mu_p} \pm p$ permutations. The real parameter λ_p measures the strength of the coupling of the dilaton to the p -form A_p in the Einstein frame¹. In the case of M -theory, the dilaton is absent, and one must set $\varphi \equiv 0$ in (1).

Following [2], we consider a “generalized Kasner” metric $g_{\mu\nu} dx^\mu dx^\nu = -N^2 (dx^0)^2 + \sum_{i=1}^d a_i^2 (\omega^i)^2$, where $d \equiv D - 1$ denotes the spatial dimension and where $\omega^i(x) = e_j^i(x) dx^j$ is a d -bein, whose time-dependence is neglected compared to that of the local scale factor a_i . When working in the proper-time gauge, $N dx^0 = dt$, the leading Kasner-like approximation to the solution of the field equations for $g_{\mu\nu}$ and φ derived from (1) is, as usual [2] $\ln a_i \simeq p_i(x) \ln t + b_i(x)$, $\varphi \simeq p_\varphi(x) \ln t + \psi(x)$. The

¹Actually, the superstring actions (in $D = 10$) are slightly more complicated than Eq. (1), in that they include additional couplings between the form fields (e.g. Yang-Mills couplings for $p = 1$ multiplets, Chern-Simons terms, $(dC_2 - C_0 dB_2)^2$ -type terms in type IIB). However, these additional terms do not qualitatively modify the generic BKL behaviour discussed in [1]. Similarly, there is a Chern-Simons term in $D = 11$ SUGRA.

spatially dependent Kasner exponents $p_i(x)$, $p_\varphi(x)$ must satisfy the famous Kasner constraints (modified by the presence of the dilaton):

$$p_\varphi^2 + \sum_{i=1}^d p_i^2 - \left(\sum_{i=1}^d p_i \right)^2 = 0, \quad \sum_{i=1}^d p_i = 1. \quad (2)$$

The set of parameters satisfying Eqs. (2) is topologically a $(d-1)$ -dimensional sphere: the “Kasner sphere”. When the dilaton is absent, one must simply set p_φ to zero in (2). In that case the dimension of the Kasner sphere is $d-2 = D-3$.

The approximate generalized Kasner solution is obtained by neglecting in the field equations for $g_{\mu\nu}$ and φ : (i) the effect of the spatial derivatives of $g_{\mu\nu}$ and φ , and (ii) the contributions of the various p -form fields A_p . As in the usual BKL approach [see [5] for a summary of the evidence supporting the BKL picture] the approach of [1, 6] assumes that the Kasner-like solution is approximately valid during a sequence of “free flight” evolutions of $g_{\mu\nu}$ and φ , which are interrupted by “collisions” against some “potential walls” associated to the momentarily growing effect of either (i) the spatial derivatives of $g_{\mu\nu}$, or (ii) the “electric” or “magnetic” contributions of one of the p -form fields A_p .

It was found in [1, 6] that the “potential walls” (conjecturally) responsible for the collisions are, in the gauge $N = \sqrt{g}$, of the form $V(\beta^\mu) \sim \sum C_J \exp(2 w_J(\beta^\mu))$, where $w_J(\beta) = w_{J\mu} \beta^\mu$ are some linear forms in $\beta^i \equiv \ln a_i$, $\beta^0 \equiv \varphi$. Each elementary potential wall $V_J(\beta^\mu) \sim \exp(2 w_J(\beta^\mu))$ in the $(N = \sqrt{g})$ Hamiltonian constraint corresponds to one of the potentially “dangerous terms”, when $t \rightarrow 0$, among the neglected contributions of the types (i) or (ii) mentioned above, i.e. among $t^2 \bar{R}_j$, or among $t^2 T_{(A)0}^0$ and $t^2 T_{(A)j}^i$. Here, \bar{R}_j denotes the d -dimensional Ricci tensor whereas $T_{(A)\nu}^\mu$ denotes the stress-energy tensor of the p -form. More precisely, upon replacing the “ingoing” approximate Kasner behaviour, we have $V_J(\beta^\mu) \sim t^{2w_J(p)}$ where the exponent of t^2 is simply $w_J(p) = w_{J0} p_\varphi + w_{Ji} p_i$. This result exhibits the simple link between the dominant elementary potential walls $V_J(\beta^\mu) \sim \exp(2 w_{J\mu} \beta^\mu)$ responsible for the collisions, and the set of “stability exponents” $w_J(p) = w_{J0} p_\varphi + w_{Ji} p_i$ discussed in [1].

It has been known for a while [2], [7] that spatial inhomogeneities in $g_{\mu\nu}$ give rise to the following set of *gravitational exponents* $g_{ijk}(p) = 2 p_i + \sum_{\ell \neq i,j,k} p_\ell$ ($i \neq j$, $i \neq k$, $j \neq k$). Ref. [1] found that the presence of p -forms gave rise (for each type of p -form) to two additional sets of potential walls and stability exponents: the *electric exponents* $e_{i_1 \dots i_p}^{(p)}(p) = p_{i_1} + p_{i_2} + \dots + p_{i_p} - \frac{1}{2} \lambda_p p_\varphi$ (where all the indices i_n are different) and the *magnetic exponents* $b_{j_1 \dots j_{d-p-1}}^{(p)}(p) = p_{j_1} + p_{j_2} + \dots + p_{j_{d-p-1}} + \frac{1}{2} \lambda_p p_\varphi$ (with all indices j_n being different). In the case of the dilaton-free models, one must set p_φ to zero in these expressions.

The essential new result of [1] was the finding that, in all low-energy superstring models ($D = 10$, IIA, IIB, het_E , het_{SO} ; and $D = 11$ SUGRA) there exists no open region of the Kasner sphere (2) where all the exponents $\{w_J(p)\} = \{g_{ijk}(p), e_{i_1 \dots i_p}^{(p)}(p), b_{j_1 \dots j_{d-p-1}}^{(p)}(p)\}$ can be simultaneously strictly positive. [We shall give below a simple direct proof of this fact for $D = 11$ SUGRA.] In other words, the Kasner-like evolution cannot stay monotonic, but will always be “deflected” by at least one potential wall to a new Kasner regime characterized by new Kasner exponents $p'_i(x)$, $p'_\varphi(x)$ (given by the universal collision law of [1]) as well as, in general,

new “Kasner axes” $e_j^i(x)$. This new Kasner solution will again be deflected because there is always one “dangerous” potential term that takes over, since at least one $w_J(p)$ is negative, etc. This never ending oscillatory behaviour is similar (though more complex because of the greater variety of potential walls) to the BKL oscillations in $D=4$ pure gravity, but contrasts very much with the ultimate monotonic Kasner behaviour holding for pure gravity in $D>11$ [7], and for the Einstein-dilaton system in any D [8]. As pointed out in [1] it is the presence of various form fields (e.g. the three form in $D=11$ SUGRA) which provides the crucial source of generic oscillatory behaviour. This leads us to propose in this paper to consider simple homogeneous models which contain no “gravitational walls” but which contain sufficiently many “form walls” to prevent the possibility of an ultimate monotonic Kasner behaviour. We view these models as analogs of the Bianchi IX model, i.e. as toy ODE models for studying in detail the chaos induced by the form fields.

2 M-theory and electric potential walls

We first consider the case of M-theory. In that instance, the toy ODE models which exhibit the required features are in fact the simplest “Bianchi I” type versions of the action (1). These models contain: (i) no dilaton, (ii) a generic spatially flat D -dimensional metric with closed d -bein forms $d\omega^i = 0$, or equivalently $ds^2 = -N^2(x^0)^2 + g_{ij}(x^0) dx^i dx^j$ ($i, j = 1, \dots, d \equiv D-1$); and (iii) a generic, homogeneous p -form potential $A = (1/p!) A_{i_1 \dots i_p}(x^0) dx^{i_1} \wedge \dots \wedge dx^{i_p}$. [In M-theory, $p=3$, but it is of interest to leave p unspecified at this stage.]

These models have several useful features: (a) because the potential is homogeneous, the curvature $(p+1)$ -form $F = dA = \frac{1}{p!} \partial_0 A_{i_1 \dots i_p} dx^0 \wedge dx^{i_1} \wedge \dots \wedge dx^{i_p}$ is purely electric; there is no magnetic field; (b) the vanishing of the structure constants C_{jk}^i of the homogeneity group allows the purely electric $(p+1)$ -form F to automatically satisfy the Gauss constraint (when $C_{jk}^i \neq 0$, see [9]); and (c) when $D=11$ and $p=3$ this model is equivalent to (Bianchi I) SUGRA because the vanishing of the magnetic part of F means that $F \wedge F$ vanishes, so that the addition of the Chern-Simons term $A \wedge F \wedge F$ to (1) has no effect in the field equations.

After discarding a total derivative and a trivial volume factor, the action (1) (reduced by the ansätze (i),(ii) and (iii)) reads, in Hamiltonian form

$$S_H = \int dx^0 (\pi_g^{ij} \dot{g}_{ij} + \frac{1}{p!} \pi_A^{i_1 \dots i_p} \dot{A}_{i_1 \dots i_p} - \tilde{N} \mathcal{H}), \quad (3)$$

where π_g^{ij} and $\pi_A^{i_1 \dots i_p}$ are respectively the momenta canonically conjugate to g_{ij} and $A_{i_1 \dots i_p}$, and where \tilde{N} is the rescaled lapse of weight minus one related to the standard scalar lapse through $\tilde{N} = N/\sqrt{g}$. In (3), \mathcal{H} is explicitly given by $(\pi_{gj}^i \equiv g_{jk} \pi_g^{ik})$

$$\mathcal{H} = \pi_{gj}^i \pi_{gi}^j - \frac{1}{d-1} (\pi_{gi}^i)^2 + \frac{1}{2} \frac{1}{p!} g_{i_1 j_1} \dots g_{i_p j_p} \pi_A^{i_1 \dots i_p} \pi_A^{j_1 \dots j_p}. \quad (4)$$

The dynamics of Bianchi I models admits many first integrals. Indeed the action S is invariant under the following two sets of rigid symmetries: (i) an arbitrary $GL(d)$ transformation, described by a (non symmetric) $d \times d$ matrix Λ_j^i acting by:

$g'_{ij} = \Lambda_i^a \Lambda_j^b g_{ab}$, $A'_{i_1 \dots i_p} = \Lambda_{i_1}^{j_1} \dots \Lambda_{i_p}^{j_p} A_{j_1 \dots j_p}$, and (ii) an arbitrary shift of the \mathbf{p} -form: $A''_{i_1 \dots i_p} = A_{i_1 \dots i_p} + \alpha_{i_1 \dots i_p}$. The corresponding Noether conserved quantities are

$$\mathcal{P}_j^i = 2 \pi_{gj}^i + \frac{1}{(p-1)!} \pi_A^{is_1 \dots s_{p-1}} A_{js_1 \dots s_{p-1}}, \quad \mathcal{E}^{i_1 \dots i_p} = \pi_A^{i_1 \dots i_p}. \quad (5)$$

[In Eq. (5) and the following, we reserve the notation \mathcal{E}^{\dots} to denote a numerically fixed constant of motion, while π_A denotes a conjugate momentum.] By inserting these integration constants into the definition of the momenta in terms of the first-order time derivatives of g_{ij} and $A_{i_1 \dots i_p}$, one gets a *first-order* system in (g, A) of the symbolic form $\partial_\tau g = P_{1,1}(g, A)$, $\partial_\tau A = P_p(g)$, where $P_{1,1}(x, y)$ is a polynomial of first degree in \mathbf{x} , and first degree in \mathbf{y} , and $P_p(x)$ a polynomial of degree p in \mathbf{x} .

In addition to the conserved quantities (5), the dynamics is subject to the constraint $\mathcal{H} = 0$ which one obtains by varying N in the variational principle. This constraint is preserved in time and defines a further constant of the motion, which is generically independent from the previous ones.

Rather than investigating the polynomial equations of motion obtained by eliminating the momenta, it is more instructive for our purposes to analyse the reduced Hamiltonian for the dynamics of the metric variables, after elimination of the A 's (which are “ignorable” coordinates) and given the constants of motion \mathcal{E} . Actually, it suffices to replace π_A by \mathcal{E}^{\dots} in the Hamiltonian (4).

We work as usual in the \mathbf{n} -gauge, defined by $N = 1$, i.e., $d\tau = dt/\sqrt{g}$. We introduce the positive-definite “potential” for g_{ij} (given the constant “electric field strengths” $\mathcal{E}^{i_1 \dots i_p}$)

$$V_{\mathcal{E}}^{(p)}(g) \equiv \frac{1}{2} \frac{1}{p!} g_{i_1 j_1} \dots g_{i_p j_p} \mathcal{E}^{i_1 \dots i_p} \mathcal{E}^{j_1 \dots j_p}. \quad (6)$$

Then the dynamics for g_{ij} follows from the \mathbf{n} -time Hamiltonian

$$H_\tau^{(\mathcal{E})}(g, \pi_g) = \left(g_{ia} g_{jb} - \frac{1}{d-1} g_{ij} g_{ab} \right) \pi_g^{ij} \pi_g^{ab} + V_{\mathcal{E}}^{(p)}(g_{ij}). \quad (7)$$

The initial conditions must be chosen such that the zero-energy condition $H_\tau^{(\mathcal{E})} = 0$ is satisfied initially (it is then preserved by the evolution). Actually, the reduction leading to (7) is far from optimal because we have not taken into account all the symmetries of the problem, i.e. all the constants of motion, but it is convenient for our present purpose which is to give simple examples of the analysis of [1].

Let us qualitatively discuss, à la BKL [2], or rather (as we work in a Hamiltonian framework) à la Misner [4], the dynamics generated by (7). I.e., let us verify the consistency of a picture in which $g_{ij}(\tau)$ evolves by a succession of “free flights” interrupted by “collisions” on the potential wall $V_{\mathcal{E}}^{(p)}(g)$. The free flights are the periods where $V_{\mathcal{E}}^{(p)}(g)$ is much smaller than the kinetic energy terms $\sim \dot{g}^2 \sim \pi_g^2$ in (7). When this is the case, $g_{ij}(\tau)$ undergoes a simple geodesic flow in the supermetric defined by the Hamiltonian $H_\tau^{(0)}$, i.e. (7) without the potential term. The invariance of $H_\tau^{(0)}$ under $\text{GL}(d)$ gives, as above, the constants of motion $\mathcal{P}_{gj}^i = 2 \pi_{gj}^i$. The free flight evolution of g is therefore given by $\partial_\tau g_{ij} = g_{ia} \mathcal{P}_{gj}^a - (d-1)^{-1} g_{ij} \mathcal{P}_{ga}^a$. This is a linear equation in g_{ij} . It is easily solved by diagonalizing the constant matrix \mathcal{P}_{gj}^i , say $\mathcal{P}_{gj}^i = \mathcal{P}_g^i \delta_j^i$ after

some $\text{GL}(d)$ transformation $\mathcal{P}_g \rightarrow \Lambda^{-1} \mathcal{P}_g \Lambda$. The solution for g_{ij} (in the new linear frame) reads $g_{ij}^{\text{free flight}}(\tau) = \exp(2\beta_{(0)}^i + 2v^i \tau) \delta_{ij}$, with $2v^i \equiv \mathcal{P}_g^i - (d-1)^{-1} \sum_s \mathcal{P}_g^s$.

In terms of the cosmological time $t = \int \sqrt{g} d\tau$, this is a Kasner solution with $a_i \propto t^{p_i}$ and Kasner exponents equal to $p_i = v^i / (\sum_s v^s)$. This (approximate) Kasner solution can continue, uninterrupted, if and only if *all* the terms in the potential $V^{(p)}(g)$ tend to zero as $t \rightarrow 0$. [Note that a typical contribution to the kinetic energy terms $\sim \sum_i (\partial_\tau a_i / a_i)^2$ is of order unity, i.e. independent of t as $t \rightarrow \infty$, i.e. $t \rightarrow 0$.]

In the special diagonal basis where we have written the solution, the electric field $\mathcal{E}^{i_1 \dots i_p}$ will have in generic solutions (i.e. excluding special solutions of zero measure), non zero entries for all its components. Therefore the potential (6) is a sum of positive terms of the form $a_{i_1}^2 \dots a_{i_p}^2 (\mathcal{E}^{i_1 \dots i_p})^2 \propto t^{2(p_{i_1} + \dots + p_{i_p})}$. The Kasner stability condition, i.e. the fact that *all* such terms tend to zero with t , is therefore precisely the condition pointed out in [1] that the electric exponents (with $p_\varphi = 0$ in the present dilaton free model) be strictly positive for *all* possible choices of (different) indices i_1, \dots, i_p . This condition is equivalent to $e_{\min}^{(p)}(p) > 0$ for some p on the Kasner sphere, where $e_{\min}^{(p)}(p) \equiv p_1 + p_2 + \dots + p_p$, with $p_1 \leq p_2 \leq \dots \leq p_d$, denotes the *smallest* electric exponent associated with the presence of a p -form.

3 Electric chaos and spacetime dimension

In the pure electric Bianchi I toy model, we have lost all the other stability conditions linked to spatial inhomogeneities in the gravitational field, or to the generic presence of magnetic-type field strengths. Let us, however, delineate for which values of the spacetime dimension $D = d + 1$, and of the degree p of the form the electric stability conditions (without the dilaton term) are *sufficient*, by themselves, to imply a chaotic behaviour, in the sense of a never ending oscillatory behaviour, as $t \rightarrow 0$. We shall prove the following (we consider only $d \geq 3$, i.e. $D \geq 4$, so that the Kasner sphere exist as a continuous manifold, and forms of degree $p \leq 3$):

Theorem 1. Let $p_1 \leq p_2 \leq \dots \leq p_d$ be ordered Kasner exponents running over the Kasner sphere S^{d-2} ($\sum p_i = \sum p_i^2 = 1$), (with $d \geq 3$), and let $e_{\min}^{(p)}(p)$ denote the smallest electric exponent associated to the presence of a p -form:

- (i) in the case of a one-form ($p = 1$): $e_{\min}^{(1)} \equiv p_1$ can never be > 0 on S^{d-2} (“electric chaos” for any $D \geq 4$),
- (ii) in the case of a two-form ($p = 2$): if $D = 4$, $e_{\min}^{(2)} \equiv p_1 + p_2$ can be > 0 on S^{d-2} but if $D \geq 5$, $e_{\min}^{(2)}$ can never be > 0 (“electric chaos” in $D \geq 5$),
- (iii) in the case of a three-form ($p = 3$): if $D \leq 6$, $e_{\min}^{(3)} \equiv p_1 + p_2 + p_3$ can be > 0 on S^{d-2} , but if $D \geq 7$, $e_{\min}^{(3)}$ can never be > 0 (“electric chaos” in $D \geq 7$).

Note that a consequence of (iii) for $D = 11$ is the result announced in [1], namely the fact that the 3-form of SUGRA creates, by its sole electric effect, chaos in $D = 11$ supergravity. [We sketched in [1] another proof (which was our original proof) of that fact. The proof we give below is much simpler.] Note also that the chaotic nature of a one-form in $D = 4$ (and 5) was clearly recognized in Ref. [10].

We shall only prove the more difficult part of the theorem, namely, the assertion (iii). The other claims are proved similarly [6]. First, we note that: (a) if $d = 3$, $e_{\min}^{(3)} = p_1 + p_2 + p_3 = 1$ is always > 0 on S^1 ; (b) if $d = 4$, $e_{\min}^{(3)} = p_1 + p_2 + p_3 = 1 - p_4$ is > 0 almost everywhere on S^2 ; and (c) if $d = 5$, the particular point $(p_i^{(0)}) = \left(-\frac{3}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}, \frac{2}{5}\right)$ on S^3 satisfies $e_{\min}^{(3)}(p^{(0)}) = \frac{1}{5} > 0$, so that there exists an open neighbourhood of $p^{(0)}$ where $e_{\min}^{(3)}(p) > 0$. Let us now prove that, when $d \geq 6$, the assumption $e_{\min}^{(3)} = p_1 + p_2 + p_3 > 0$ leads to a contradiction. With our convention of ordered p_i 's, this assumption implies $0 < p_3 \leq p_4 \leq \dots \leq p_d$. It follows from the Kasner conditions that the double sum $K(p) \equiv \sum_{1 \leq i < j \leq d} p_i p_j = 0$ vanishes. Let us distinguish in this double sum the indices $i, j = 1$ or 2 from the indices ≥ 3 , which we denote by $\bar{\alpha}, \bar{\beta} = 3, 4, \dots, d$. [The indices $\bar{\alpha}, \bar{\beta}$ take $d-2$ values.] After some simple rearrangements, one easily checks that the following algebraic identity holds:

$$\begin{aligned} K(p) \equiv & (p_3 - p_1)(p_3 - p_2) + (p_1 + p_2 + p_3) \left[p_3 + \sum_{3 \leq \bar{\alpha} \leq d} p_{\bar{\alpha}} \right] \\ & - p_3 \left[p_3 + p_3 + \sum_{3 \leq \bar{\alpha} \leq d} p_{\bar{\alpha}} \right] + \sum_{3 \leq \bar{\alpha} < \bar{\beta} \leq d} p_{\bar{\alpha}} p_{\bar{\beta}}. \end{aligned} \quad (8)$$

The first term on the R.H.S. of (8) is clearly > 0 because of the ordering of the p_i 's. The second term is *strictly* positive if the assumption $e_{\min}^{(3)} > 0$ holds somewhere. Now, if the number of terms $((d-2)(d-3)/2)$ in the last double sum is larger or equal to the number of terms $(2 + d - 2 = d)$ in the (expanded) penultimate simple sum, the inequalities $0 < p_3 \leq p_4 \leq \dots \leq p_d$ show that the difference between the two will be > 0 . This would lead to the contradiction $K(p) > 0$. Therefore $e_{\min}^{(3)} > 0$ is contradictory when $(d-2)(d-3)/2 \geq d$, i.e. (as easily checked) $d \geq 6$. Q.E.D.

Theorem 1 tells us which simple “electric Bianchi I” models incorporate enough potential walls to mimic, within a simplified context, the form-induced chaos that Ref. [1] found to hold in all (inhomogeneous) superstring models. The chaotic homogeneous model relevant to M-theory is the dilaton-free Bianchi I model with a 3-form ($p = 3$), in spacetime dimension $D = d + 1 = 11$. The study of this model, with a generic electric field, might teach us something about the nature of the chaos induced by the 3-form in SUGRA₁₁. We emphasize that it is essential to consider a generic solution to have chaos. There exist particular solutions (of measure zero among all solutions) which exhibit only a finite number of oscillations. For instance, solutions with just one collision are given in [9]. Theorem 1 shows that a purely electric, homogeneous 3-form leads to chaotic oscillations in any space dimension $d \geq 6$. Therefore a study of this simple model in $d = 6$ (which has many less variables than its $d = 10$ analog) might be sufficient to learn about the nature of the chaos induced by the 3-form. Actually, Theorem 1 allows us to consider even simpler models (with less dynamical variables) by lowering both the degree of the form and the dimension. The simplest models that one can consider to study the nature of the chaos induced by a form field (and to compare, and/or contrast it, with the Bianchi IX chaos, which is induced by the gravitational field) is a *vector* field $p = 1$, i.e. a usual, Maxwellian electric field, in a d -dimensional Bianchi I model with $d \geq 3$ [10]. The case $d = 3$ has been studied in [11, 12, 13] with the conclusion that, indeed, a generic homogeneous electromagnetic field induces chaos. Note, however, that in $d = 3$ the

electric and gravitational exponents characterizing the walls are proportional so that they lead to identical collision laws. It would therefore be more interesting (to understand the specificities of the form-induced chaos, as well as the effect of increasing the dimension) to study the case $d > 4$.

4 String theories and chaotic cosmological models

From the above findings for $D=11$ SUGRA, one can easily construct homogeneous cosmological models with an infinite number of Kasner oscillations for type IIA string theory. The above models can, indeed, be interpreted as Bianchi I models in ten dimensions, with a general metric, a Kaluza-Klein vector field, a dilaton, as well as homogeneous 2-form and 3-form fields. Thus, Bianchi I models with homogeneous potentials (i.e., only electric fields) exhibit an infinite number of oscillations in type IIA string theory. By T-duality, the results extend to type IIB, but since some of the magnetic fields now do not vanish, there is no action principle with homogeneous p -form potentials from which the equations of motion derive (some of the potentials must be inhomogeneous).

One can also devise Bianchi type I models which exhibit a never-ending set of oscillations for the ($D=10$) heterotic and type I string theories because the gravitational walls related to spatial inhomogeneities (which are absent in the Bianchi I context) are not necessary to induce the oscillations. As shown in [6], the electric stability conditions associated with the 1 -forms and the magnetic stability conditions associated with the 2 -form cannot be simultaneously fulfilled in $D=10$. So, by considering homogeneous sources of this type, one gets cosmological models that exhibit the required behaviour².

5 A non-chaotic model

The dilaton and the p -forms ($p > 0$) have quite different effects on the oscillatory behaviour of cosmological models. While the dilaton modifies the Kasner exponents in a way that they can all be positive - and so, tends to stabilize the Kasner behaviour - the p -forms ($p > 0$) do not change the Kasner relations but induce collisions - and so, tend to destabilize it. This observation is useful even for solutions with a finite number of oscillations (which form a zero-measure set) as it enables one to understand the dynamics of these models in terms of free-flight motions interrupted by collisions. We illustrate this feature with a discussion, for comparison purposes, of the case of a two-form ($p=2$), in presence of a dilaton (coupled in the bosonic-string or, equivalently, heterotic-string way). This case is interesting because it is known to be exactly integrable [15]. We wish to show, on this example, how its dynamics can be described in terms of BKL-like collisions, and how this collision analysis correctly predicts: (i) the presence of only a finite number of oscillations, and (ii) what is the final “out state” after the oscillations have ceased. The model we consider here is described by the action (1), in any spacetime dimension $D = d + 1$, with $p=2$ and

²Magnetically induced chaos was pointed out in a particular cosmological context in [14]. It was also pointed out there that the dilaton makes this magnetic chaos disappear. However, if one brings in also the electric fields of the 1 -forms, one gets chaos back again.

$\lambda_{(p=2)} = -4/\sqrt{D-2}$. This specific value of the dilaton coupling in the Einstein frame follows from starting from a *string frame* action of the form

$$S = \int d^D x \sqrt{G_D} e^{-\phi} \left[R(G) + (\nabla\phi)^2 - \frac{1}{12} H_{\lambda\mu\nu} H^{\lambda\mu\nu} \right], \quad (9)$$

where $G_{\mu\nu} = \exp(2\phi/(D-2)) g_{\mu\nu}$ is the string frame metric, $\phi = \sqrt{D-2} \varphi$ the dilaton (normalized such that the string coupling squared $g_s^2 = e^\phi$), and where $H_{\lambda\mu\nu} = \partial_\lambda B_{\mu\nu} + \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu}$ is the exterior derivative of the 2-form $B_{\mu\nu}$. The action (9) is (in the corresponding critical dimension) a common subsector (NS-NS) of all string theories. [But it is only for the closed bosonic string ($D_c = 26$) that (9) is the full, tree-level bosonic action.] Anyway, we consider here (9) as a toy model to see how dilaton couplings can modify the stability exponents of the p -forms so much as to quench the form-induced chaos. To study the Kasner stability of the model (9) it is convenient to work with the string-frame Kasner exponents, say α_i , $i = 1, \dots, d$, instead of the Einstein-frame ones p_i . The string-frame exponents are defined by writing the Kasner-solution in terms of the string-frame scale factors \bar{a}_i and the string-frame cosmological time \bar{t} ($G_{\mu\nu} dx^\mu dx^\nu = -d\bar{t}^2 + \sum (\bar{a}_i dx^i)^2$): $\bar{a}_i \propto \bar{t}^{\alpha_i}$. They are linked to the p_i 's by $p_i = ((d-1)\alpha_i - \sigma)/(d-1-\sigma)$, $p_\varphi = (\sqrt{d-1}\sigma)/(d-1-\sigma)$, where $\sigma \equiv (\sum_i \alpha_i) - 1$. In terms of the α 's the Kasner sphere S^{d-1} defined by Eqs. (2) becomes simply $\sum_{i=1}^d \alpha_i^2 = 1$. It is easily found, either by transforming the Einstein-frame exponents, or by a direct analysis in the string frame, that the string-frame exponents $\bar{\alpha}$, defined such that the “dangerous terms”, i.e. the potential walls, grow $\propto \bar{t}^{2\bar{\alpha}}$, ($\bar{\alpha} = (1 - \sigma/(d-1)) \bar{\alpha}$), for each corresponding Einstein-frame exponent, α read $\bar{g}_{ijk} = 1 + \alpha_i - \alpha_j - \alpha_k$, $\bar{e}_{ij}^{(2)} = \alpha_i + \alpha_j$ and $\bar{b}_{\ell_1 \dots \ell_{d-3}}^{(2)} = 1 - \alpha_i - \alpha_j - \alpha_k$. Here, $ijk\ell_1 \dots \ell_{d-3}$ is a permutation of $1 \dots d$. We shall show in [6] that, when considering all these exponents, the Kasner stability conditions ($\bar{g}, \bar{e}^{(2)}, \bar{b}^{(2)}$ all > 0) can be satisfied only when $D \geq 11$. This implies, for instance, that the NS-NS sector of type II or heterotic superstring theories in $D = 10$ are chaotic. However, in the present paper we are interested in considering only the simple *purely electric* Bianchi I models. In these models, there are only “electric walls”, so that the only stability condition to satisfy is $\bar{e}_{ij}^{(2)} = \alpha_i + \alpha_j > 0$. Clearly this is always satisfied in some region of the Kasner sphere $\sum_{i=1}^d \alpha_i^2 = 1$ (as long as $d \geq 2$, which is anyway necessary to consider a B_{ij}). Therefore, we conclude that the electric Bianchi I model (9) (in $d \geq 2$) will not be chaotic, and will contain at most a finite number of oscillations.

We can, in fact, be more precise and determine the maximum number of oscillations by considering the “collision law” induced by an electric wall. This law follows from the general collision law given in [1] (Eq. (9) there). The string-frame Kasner exponents are the “velocities” $\alpha_i = d\bar{\beta}^i/d\bar{\tau}$, where $\bar{\beta}^i = \ln \bar{a}_i$, $d\bar{\tau} = \lambda e^{\bar{\phi}} d\bar{t}$, with a normalization constant λ such that $\alpha_0 = d\bar{\phi}/d\bar{\tau} = -1$. In terms of the incoming (α_i) and outgoing (α'_i) exponents, the collision law corresponding to the wall $\propto e^{2(\bar{\beta}_1 + \bar{\beta}_2)} \propto e^{2(\alpha_1 + \alpha_2)\bar{\tau}}$ (see below) reads $\alpha'_1 = -\alpha_2$, $\alpha'_2 = -\alpha_1$, $\alpha'_a = \alpha_a$ ($a = 3, 4, \dots, d$). If one starts at some initial time $\bar{\tau}_0$ with some initial values of the \bar{a}_i and with some electric components \mathcal{E}^{ij} that are all of the same order of magnitude, one generically expects that the first collision encountered as $\bar{\tau}$ decreases³ will be the

³We follow the evolution *toward* the cosmological singularity at $\bar{\tau} = 0$, say starting from $\bar{\tau} < 0$, with $\bar{\tau} \sim -\bar{\phi} \sim +\ln |\bar{t}|$ going to $-\infty$ at the singularity.

one associated to the fastest growing wall (if there are such walls), i.e. to the most negative $\alpha_i + \alpha_j$, because $\bar{a}_i^2 \bar{a}_j^2 \propto e^{2(\beta_i + \beta_j)} \propto e^{2(\alpha_i + \alpha_j)\bar{\tau}}$. We recover the notion of electric 2-form stability exponent $\bar{e}_{ij}^{(2)} = \alpha_i + \alpha_j$. Let us order the Kasner exponents as $\alpha_1 \leq \alpha_2 \leq \dots \leq \alpha_d$. We expect the following generic qualitative evolution: (i) if the smallest $\bar{e}_{ij}^{(2)}$, i.e. $\alpha_1 + \alpha_2$, is ≤ 0 a collision will occur against the growing wall $\propto \bar{a}_1^2 \bar{a}_2^2$, and the outcome of this collision will be the unordered set α'_i given by the above collision law. Note that $\alpha'_1 + \alpha'_2 = -(\alpha_1 + \alpha_2)$ has become > 0 . However, they may remain some ≤ 0 electric exponents associated to the unchanged $\alpha'_a = \alpha_a$; then (ii) if the next smallest $\bar{e}_{ij}^{(2)}$, namely $\alpha_3 + \alpha_4$ is ≤ 0 , a second collision will occur with the effect of reversing the signs of both α_3 and α_4 . This process will continue until all $\bar{e}_{ij}^{(2)} = \alpha_i + \alpha_j$ have become > 0 , which means that at most α_1^{out} can be ≤ 0 , all the other ones (here $\alpha_1^{\text{out}} \leq \alpha_2^{\text{out}} \leq \dots$) ending up being > 0 . If we do the same reasoning, starting from $\bar{\tau}_0$, but going in the sense of increasing $\bar{\tau}$ (i.e. towards $|t| = \infty$) we conclude (mutatis mutandis) that, near $|t| = \infty$, the “incoming” values α_i^{in} must all be ≤ 0 , except maybe α_d^{in} which might be > 0 . As each collision changes the sign of *two* Kasner exponents, the total number of collisions cannot exceed $[d/2]$, where $[\dots]$ denotes the integer part.

The qualitative picture just explained can be confirmed by an exact analysis. Indeed, it happens that the model at hand is exactly integrable [15]. However, the form of the “general solution” given in Refs. [15] is not optimally useful because it contains a constant $(2d) \times (2d)$ matrix A which must satisfy a whole set of non linear constraints. We found, however, a simpler way of writing the general solution. It is enough to use two facts: (i) a particular, exact solution containing $2d - 1$ arbitrary constants is known, namely the Kasner solution

$$G_{ij}^0(\bar{\tau}) = \exp(2\bar{\beta}_i^{(0)} + 2\alpha_i\bar{\tau}) \delta_{ij}, \quad B_{ij}^0 = 0; \quad (10)$$

and, (ii) there is an $O(d, d)$ symmetry transforming solutions into other solutions. Using a $(d \times d)$ block notation for $(2d) \times (2d)$ matrices the $O(d, d)$ group is realized as $U = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix}$, constrained by $U^T \eta U = \eta$, where $\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. This is equivalent to the constraints $W^T Y + Y^T W = 0$, $X^T Z + Z^T X = 0$ and $W^T Z + Y^T X = \mathbf{1}$. The action of $U \in O(d, d)$ on the matrices $G = (G_{ij})$ and $B = (B_{ij})$ is defined by $M^{\text{new}}(\bar{\tau}) = U M^{\text{old}}(\bar{\tau}) U^T$, where the *symmetric* $(2d) \times (2d)$ matrices M are constructed as

$$M = \begin{pmatrix} G^{-1} & -G^{-1} B \\ B G^{-1} & G - B G^{-1} B \end{pmatrix}. \quad (11)$$

By looking at the action of infinitesimal transformations $U = \mathbf{1} + u$, with $u = \begin{pmatrix} w & x \\ y & -w^T \end{pmatrix}$, with w and y antisymmetric, and x arbitrary, one finds that: w induces a trivial global change of linear frame, y induces a rather trivial shift of B_{ij} by constants, while x generates (when acting on (10)) a time-dependent $\delta B_{ij}(\bar{\tau})$ of enough generality to match any initial data $\delta \dot{B}_{ij}(\bar{\tau}_0)$. Going back to finite transformations U , we conclude that it suffices to act by a triangular $O(d, d)$ matrix $U = \begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix}$, with a generic, antisymmetric matrix X , on the generic time-dependent particular solution (10), to generate a solution of the system with sufficient, physical generality. Thus, we conclude that the general solution can be written in terms of the Kasner

one (10) as

$$G^{-1} = G_0^{-1} - X G_0 X, \quad (12)$$

$$B = -G X G_0 = -G_0 X G = -G_0 X G_0 (1 - X G_0 X G_0)^{-1}. \quad (13)$$

We shall focus on the behaviour of the metric, using the explicit form of (12), i.e. (with $X^{ij} = -X^{ji}$)

$$G^{ij}(\bar{\tau}) = G_0^{ij}(\bar{\tau}) + G_{0ab}(\bar{\tau}) X^{ia} X^{jb}. \quad (14)$$

The exact solution (14) looks so simple that it seems to have nothing to do with the multi-collision picture explained above. However we have verified that (14) is fully compatible with the general collision picture, and does indeed contain up to $[d/2]$ collisions, which can, for a general set of initial conditions, be well separated from each other. First, it is instructive to write down explicitly (14) in the case where the matrix X couples only to a 2×2 subblock: $G_0 = \text{diag}(\bar{a}_0^2, \bar{b}_0^2)$ with $\bar{a}_0(\bar{\tau}) \propto e^{\alpha_1 \bar{\tau}}$, $\bar{b}_0(\bar{\tau}) \propto e^{\alpha_2 \bar{\tau}}$. With $X^{12} = x$ one finds $G = \text{diag}(\bar{a}^2, \bar{b}^2)$ with the new scale factors $\bar{a}^2 = \bar{a}_0^2 / (1 + x^2 \bar{a}_0^2 \bar{b}_0^2)$, $\bar{b}^2 = \bar{b}_0^2 / (1 + x^2 \bar{a}_0^2 \bar{b}_0^2)$. If $\bar{a}_0^2 \bar{b}_0^2$ grows toward the singularity (i.e. if it corresponds to a growing potential wall $V_{\mathcal{E}} \propto (\mathcal{E}^{12})^2 \bar{a}_0^2 \bar{b}_0^2$) we see that these equations describe a collision which turns $(\bar{a}^{\text{in}})^2 \simeq \bar{a}_0^2$, $(\bar{b}^{\text{in}})^2 \simeq \bar{b}_0^2$, into $(\bar{a}^{\text{out}})^2 \simeq x^{-2} \bar{b}_0^{-2}$, $(\bar{b}^{\text{out}})^2 \simeq x^{-2} \bar{a}_0^{-2}$. This corresponds precisely to the collision law given above for the (1, 2) block. One can check that this “elementary” collision will approximately take place each time a 2×2 subblock of G_0 separates itself from the other ones by an especially strong growth of $\bar{a}_0^2 \bar{b}_0^2$. We have also done some numerical experiments (for $d = 4$) with a generic, random antisymmetric matrix X and some initial values of the Kasner exponents α_i ensuring that two of them are especially negative, and the two others less negative, and we have found that, in such a case, one indeed witnesses two successive collisions, each one reversing the sign of a pair of Kasner exponents. [We define some “instantaneous” Kasner exponents as the eigenvalues of the matrix $\bar{K} = \frac{1}{2} G^{-1} \partial_{\bar{\tau}} G$.] One should, however, say that when there happens to be no clear hierarchy between the values of the α_i ’s, i.e. no clearly dominant wall among the total potential $V_{\mathcal{E}}(G)$, the evolution of the eigenvalues of \bar{K} looks rather like a single, complex collision whose outcome is to change the sign of all dangerous α_i ’s at once.

In this respect, we note that it is easy to find analytically, from (14), the net overall (S -matrix) effect of the entire collision process (independently of whether it can be viewed as made of several separate, intermediate collisions). Let us first consider the case where d is even and where one chooses all the α_i ’s in (10) to be negative, i.e. where all the $\bar{a}_i = e^{\beta_i}$ grow toward the singularity. It is easy to see from (14) that the incoming state ($|t| \rightarrow \infty$, $\bar{\tau} \rightarrow +\infty$) is given by $G_{ij}^{\text{in}} \simeq G_{0ij} = (\bar{a}_{0i})^2 \delta_{ij}$, while the outgoing state ($|t| \rightarrow 0$, $\bar{\tau} \rightarrow -\infty$) is generically (using $\det X \neq 0$ generically in even d) equivalent (after rediagonalization) to $G_{ij}^{\text{out}} \simeq (\bar{a}_{0i})^{-2} \delta_{ij}$. This shows that $\alpha_i^{\text{out}} = -\alpha_i^{\text{in}}$, in keeping with the result predicted by the “collision” analysis. The more general case where d might be odd and where the ordered seed α_i ’s, $\alpha_1 < \dots < \alpha_d$, might be of both signs can also be seen to confirm our collision analysis.

6 Conclusions

In this letter, we have shown that some spatially homogeneous Bianchi type I models coupled to appropriate p -form fields are sufficiently complex to mimic the never-ending oscillatory behaviour exhibited by generic string cosmologies [1]. This observation should open the door to numerical investigations of the effect of the collisions induced by the p -forms, which are different from the gravitational ones. We have also shown that the picture of free-flight motions interrupted by collisions is useful even for models with a finite number of oscillations, as are the solutions of [15].

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