External Fields as Intrinsic Geometry

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Abstract

There is an interesting dichotomy between a space-time metric considered as external field in a flat background and the same considered as an intrinsic part of the geometry of space-time. We shall describe and compare two other external fields which can be absorbed into an appropriate redefinition of the geometry, this time a noncommutative one. We shall also recall some previous incidences of the same phenomena involving bosonic field theories. It is known that some such theories on the commutative geometry of space-time can be re-expressed as abelian-gauge theory in an appropriate noncommutative geometry. The noncommutative structure can be considered as containing extra modes all of whose dynamics are given by the one abelian action.

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1 Introduction and motivation

It is known that some bosonic field theories on the commutative geometry of space-time can be re-expressed as abelian-gauge theory in an appropriate noncommutative geometry. This fact is quite the analogue of the dichotomy in general relativity between the components of a metric considered as external fields in a flat background and the same components considered as defining the metric and therefore a non-flat geometry. In the next section we mention very briefly a certain number of examples which have been considered in the past and which exhibit the property of an external field which can be incorporated into a redefinition of the basic geometry. The noncommutative structure can be considered as containing extra modes all of whose dynamics are given by the one abelian action. An example is afforded by the Yang-Mills-Higgs-Kibble action of the standard model [?, ?]. Somewhat analogous results are also known, for example, for non-relativistic hamiltonians and classical spin. Some of the most illuminating examples are taken from the field of simple hamiltonian mechanics. Complicated non-local non-polynomial hamiltonians can be considered [?, ?] as the free-particle hamiltonian in appropriately chosen geometries. An important dynamical variable which can also be considered as part of the space-time geometry is classical spin; a relativistic spinning particle can be described [?] as an ordinary particle in a noncommutative geometry.

We shall be mainly concerned with a further example of this sort, involving an external field B which can be absorbed into an appropriate redefinition of the commutation relations of a noncommutative geometry [?]. When considered as part of the geometry the field B changes the structure of the gauge group, indirectly because of the way the commutation relations of the algebra depend on it. A Yang-Mills potential A has one gauge group in the presence of a B field considered as external field and its noncommutative counterpart $\hat{A}$ has another. Since the physics cannot depend on the interpretation of the field there must be a well-defined map $\hat{A} = \hat{A}(A, B)$ which reduces to the identity when $B = 0$. In the third section we shall interpret this map as a map between covariant derivatives. We also mention the Kaluza-Klein interpretation. The set of noncommutative structures over space-time is in many aspects similar to a Kaluza-Klein extension. This is particularly clear when the noncommutativity is due to a matrix algebra [?]. The B field acts then as a set of extra coordinates which parametrize the extra dimensions. This is implicit in earlier work [?, ?] where the role of the B field is played by the spin. In fact by simply counting indices one can conclude that extra variables are necessary. If an algebra has 4 generators then the set of all commutators has 6 elements. The smallest algebra one can consider is the associative algebra of dimension $10 = 4 + 6$ which is a representation of the Lie algebra of the de Sitter group. In the last section we present a finite model which illuminates some of the aspects of the map. In the Appendix we recall some basic facts about the particular version of noncommutative geometry which will be used. We shall set a tilde on a quantity when it is necessary to distinguish the commutative limit. Words in quotes are ill-defined.

2 Paleoparadigmata

A free particle in motion in a curved space-time can be considered as a particle in a flat space-time moving under the influence of an external field. There is an analogous example in noncommutative geometry. Consider an interaction hamiltonian $H = H_0 + V$ on the real line $\mathbb{R}$ with time added or not. Then for appropriate V these hamiltonians are equivalent [?, ?]

to free hamiltonians acting on often exotic noncommutative structures. Such phenomena exist also in field theory. There have been in the recent literature several models which can be either considered as unified field theories on flat space-time or as abelian gauge theory on an appropriate noncommutative geometry. We mention these models first as examples of the phenomenon which we wish to investigate here because they can also be interpreted from another closely related point of view, that of dimensionally reduced Kaluza-Klein theories. There is a version of this theory which involves a matrix geometry in the hidden dimensions and so an abelian-gauge theory in the noncommutative geometry appears as an U_n gauge theory including the associated Higgs-Kibble scalars, when regarded traditionally as an external field problem in a plain, flat geometry. Two simple examples can be given to illustrate how the abelian-gauge action over a noncommutative geometry contains supplementary fields when reinterpreted in terms of ordinary geometry. These examples involve noncommutative extensions of the algebra of functions on space-time. The extra modes are hidden in the extra structure. For simplicity of presentation we shall replace space-time by a point and consider only the extra noncommutative geometry.

As a first example [?] write $\mathbb{C}^2 = \mathbb{C}^1 \oplus \mathbb{C}^1$ and decompose accordingly the algebra of 2×2 matrices $M_2 = M_2^+ \oplus M_2^-$ into diagonal and off-diagonal parts. The commutative algebra M_2^+ is the algebra of functions on 2 points. Introduce a graded derivation $d\alpha$ of $\alpha \in M_2$ by

$$d\alpha = -[\eta, \alpha], \qquad \eta \in M_2^-.$$

The bracket is graded and η is anti-hermitian. We find that $d\eta = -2\eta^2$ and that $d^2\alpha = [\eta^2, \alpha]$. If we choose η such that $\eta^2 = -1$ then $d^2 = 0$. Then $\Omega_\eta^* = M_2$ is a differential calculus over M_2^+. Notice that

$$d\eta + \eta^2 = 1. \tag{2.1}$$

Choose $\psi \in M_2^+$. A covariant derivative is given by

$$D_{(0)}\psi = -\eta\psi. \tag{2.2}$$

We recall that a covariant derivative must satisfy a left-Leibniz rule. Because of the definition of d one sees that this is indeed the case:

$$D_{(0)}(f\psi) = -\eta f\psi = df\psi - f\eta\psi.$$

The most general D is necessarily of the form

$$D\psi = -\eta\psi - \psi\phi$$

where ϕ defines a left-module morphism of M_2^+. If one introduce the map

$$d\psi = -[\eta, \psi]$$

one can write $D\psi = d\psi - \psi\omega$ in terms of a 'connection form' $\omega = \eta + \phi$ which transforms as

$$\omega' = g^{-1}\omega g + g^{-1}dg, \quad g \in U_1 \times U_1.$$

Since in particular $\eta' = \eta$ one finds that

$$\phi' = g^{-1}\phi g. \tag{2.3}$$

The curvature is

$$\Omega = d\omega + \omega^2 = 1 + \phi^2 = 1 - |\phi|^2$$

and the analogue of the abelian-gauge action is given by

$$V(\phi) = \frac{1}{4}\mathrm{Tr}(1 - |\phi|^2)^2.$$

We emphasize the fact that it is abelian-gauge theory; the geometry has changed not the theory being studied. Because of the exotic geometry however the result looks more like abelian Higgs theory.

As another example [?, ?] we consider the algebra M_n of $n \times n$ complex matrices with an anti-hermitian basis λ_a of SU_n and define the frame

$$\theta^a = \lambda_b \lambda^a d\lambda^b.$$

The structure of the algebra $\Omega^*(M_n)$ is given by the relations $\theta^a\theta^b = -\theta^b\theta^a$. These relations can be rewritten in the form (5.4) in the special case (5.8). It is easily seen that

$$d\theta^a = -\frac{1}{2}C^a{}_{bc}\theta^b\theta^c$$

from which follows that

$$d\theta + \theta^2 = 0, \qquad \theta = -\theta^a\lambda_a. \tag{2.4}$$

A special covariant derivative is given by

$$D_{(0)}\psi = -\theta\psi$$

and the most general one is of the form

$$D\psi = -\theta\psi - \psi\phi.$$

If one introduce the map

$$d\psi = -[\theta\, , \psi] \tag{2.5}$$

one can write again $D\psi = d\psi - \psi\omega$ in terms of a 'connection form' $\omega = \theta + \phi$ which transforms as

$$\omega' = g^{-1}\omega g + g^{-1}dg, \quad g \in U_n.$$

Since in particular $\theta' = \theta$ one finds again (2.3). The curvature is

$$\Omega = d\omega + \omega^2 = \frac{1}{2}\Omega_{ab}\theta^a\theta^b$$

where

$$\Omega_{ab} = [\phi_a\, , \phi_b] - C^c{}_{ab}\, \phi_c.$$

The $C^c{}_{ab}$ is a sort of 'Christoffel symbol'; the algebra M_n with the present differential calculus is 'curved' as a geometry. The analogue of the electromagnetic action is

$$V(\phi) = \frac{1}{4}\mathrm{Tr}(\Omega_{ab}\Omega^{ab}).$$

Again, as above, this action describes 'abelian-gauge' theory on a noncommutative 'space'. By radically changing the 'space' we have radically changed the aspect of a well-known theory.

We have presented these two examples in some detail since they illustrate well the definition of a covariant derivative. In both cases the module is a bimodule over the algebra. The covariant derivative however uses only the right-module structure and satisfies a right-Leibniz rule. The left-module structure is reserved for the action of the gauge group which we identify as a subset of the algebra. We shall encounter similar calculations in the next section.

As examples of noncommutative extensions of space-time we shall choose algebras which are deformations of the algebra of smooth functions on Minkowski space. Let $\tilde{x}^\mu$ be cartesian coordinates. As has been done previously [?, ?, ?] we replace $\tilde{x}^\mu$ by four hermitian generators x^μ, elements of an abstract $*$-algebra $\mathcal{A}$ which do not commute:

$$[x^\mu, x^\nu] = i\kappa\!\!\!^{-} J^{\mu\nu}, \qquad x^{\mu *} = x^\mu. \tag{2.6}$$

The parameter $\bar{k}$ is so chosen so that $J^{\mu\nu}$ has no dimensions. We shall set $\bar{k} = 1$ by a choice of units. A natural Ansatz which respects all reflection symmetries would be

$$x^\mu = \tilde{x}^\mu + \kappa J^\mu, \qquad J^\mu = \bar{z}\gamma^\mu z. \tag{2.7}$$

We shall impose on z the following commutation relations:

$$[z, z] = 0, \quad [z, \bar{z}] = 1, \quad [\bar{z}, \bar{z}] = 0. \tag{2.8}$$

The unit on the right-hand side of these equations is the tensor product of the unit in the Clifford algebra and the unit in the operator algebra. Written out in terms of components of the Dirac spinors Equations (2.8) become

$$[z^\alpha, z^\beta] = 0, \quad [z^\alpha, \bar{z}_\beta] = \delta^\alpha_\beta, \quad [\bar{z}_\alpha, \bar{z}_\beta] = 0.$$

If we introduce

$$S^{\mu\nu} = \bar{z}\sigma^{\mu\nu} z, \qquad \sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

then from the commutation relations (2.8) follow the commutation relations (2.6) for the generators with

$$S^{\mu\nu} = -J^{\mu\nu}, \qquad \bar{k} = 2\kappa^2.$$

We can consider the Dirac spinor as an element of the quantized version of an algebra of functions over the classical phase space $(z, \bar{z})$ with Poisson bracket $\{z, \bar{z}\} = i$. There are therefore two distinct quantization procedures, the ordinary one involving $\hbar$ and this new one. As a mathematical simplification we shall 'dequantize' z and consider the classical phase space $(z, \bar{z})$. Introduce C^λ by

$$C^\lambda = \frac{i}{2}((\bar{z}\gamma^\lambda)_\alpha \bar{\partial}^\alpha - (\gamma^\lambda z)^\alpha \partial_\alpha), \quad \partial_\alpha = \partial/\partial z^\alpha$$

and consider the condition

$$\partial_\lambda C^\lambda f = 0. \tag{2.9}$$

This is of second order in all the derivatives but of first order in ∂_λ. So it resembles a constraint. If f depends only on the quantity x^λ defined in (2.7) then (2.9) is identically satisfied. However, the converse is not true. To the $(\tilde{x}^\mu, z, \bar{z})$ we add p_λ to form a phase space. We extend the bracket by requiring that $(p_\lambda, \tilde{x}^\mu)$ Poisson-bracket-commute with $(z, \bar{z})$. It is not this full phase space which interests us but rather the reduced phase space given by the $(p_\lambda, x^\mu, z, \bar{z})$ which satisfy the constraints (2.9). This reduced phase space describes the motion of a spinning particle. Define S^λ by

$$S^\lambda = \bar{z}\gamma^\lambda\gamma^5 z.$$

Then the constraints (2.9) are equivalent to the conditions

$$\begin{aligned}
&p^2 - \mu^2 = 0, \quad p_\mu S^\mu = 0, \qquad\qquad \bar{z}\gamma^5 z = 0,\\
&\mu J^\lambda = \bar{z}zp^\lambda, \quad \mu S^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} p_\rho S_\sigma.
\end{aligned}$$

The parameter μ is a mass parameter.

Models can be constructed using the tensor product, for example using the algebras introduced in (3.4). We shall need to slightly change our notation since the situation we consider here is very similar to the situation of the next section in which $\mathcal{A}$ and $\hat{\mathcal{A}}$ describe noncommutative versions of flat space-time or of a brane and the matrix factor is a modified Kaluza-Klein extension [?]. Let $\mathcal{A}$ and $\mathcal{B}$ be two algebras with differential calculi $\Omega^*(\mathcal{A})$ and $\Omega^*(\mathcal{B})$. Then there is a natural differential calculus over the $\mathcal{A}\otimes\mathcal{B}$ given by

$$\Omega^*(\mathcal{A}\otimes\mathcal{B}) = \Omega^*(\mathcal{A})\otimes\Omega^*(\mathcal{B}). \tag{2.10}$$

If $\alpha\in\Omega^*(\mathcal{A})$, $\beta\in\Omega^p(\mathcal{B})$, $\gamma\in\Omega^q(\mathcal{A})$ and $\delta\in\Omega^*(\mathcal{B})$ then the product in $\Omega^*(\mathcal{A})\otimes\Omega^*(\mathcal{B})$ is given by

$$(\alpha\otimes\beta)(\gamma\otimes\delta) = (-1)^{pq}\alpha\gamma\otimes\beta\delta. \tag{2.11}$$

Equation (2.10) does not define the only choice of differential calculus over the product algebra. Consider the module of 1-forms

$$\Omega^1(\mathcal{A}\otimes\mathcal{B}) = \mathcal{A}\otimes\Omega^1(\mathcal{B})\oplus\Omega^1(\mathcal{A})\otimes\mathcal{B}.$$

It can be used to construct another differential calculus $\Omega^*(\mathcal{A}\otimes\mathcal{B})$ over the tensor product of the two algebras which is in a sense the largest which is consistent with the module structure. This extension is in general larger than the tensor product. If θ^α is a frame for $\Omega^1(\mathcal{A})$ and θ^a is a frame for $\Omega^1(\mathcal{B})$ then

$$(\theta^\alpha, \theta^a) = (\theta^\alpha\otimes 1, 1\otimes\theta^a)$$

is a frame for $\Omega^1(\mathcal{A}\otimes\mathcal{B})$. The commutation relations for each factor can be extended to the entire frame by the rule (2.11). In this case both constructions yield the same algebra of forms. We are interested in the case with $\mathcal{B} = M_n$. Then if we define

$$\Omega^1_h = \Omega^1(\mathcal{A})\otimes M_n, \qquad \Omega^1_v = \mathcal{A}\otimes\Omega^1(M_n),$$

we can write $\Omega^1(\mathcal{A}\otimes M_n)$ as a direct sum:

$$\Omega^1(\mathcal{A}) = \Omega^1_h\oplus\Omega^1_v.$$

The differential df of an element f of $\mathcal{A}$ is given by

$$df = d_h f + d_v f.$$

We have written it as the sum of two terms, the horizontal and vertical parts, using notation from Kaluza-Klein theory. The algebra $\Omega^*(\mathcal{A})$ of differential forms is given in terms of the differential forms of each factor by the formula:

$$\Omega^p(\mathcal{A} \otimes M_n) = \bigoplus_{i+j=p} \Omega^i(\mathcal{A}) \otimes \Omega^j(M_n). \tag{2.12}$$

Consider two elements $f,\, g \in \mathcal{A} \otimes M_n$. Let x^μ be the generators of $\mathcal{A}$ and use the Gell-Mann matrices λ_a as a basis of M_n, as described in the Appendix. If we expand $f = f^0 + f^a \lambda_a$ and $g = g^0 + g^a \lambda_a$ then we find that the commutator is given by

$$[f\,, g] = \frac{1}{2}[f^a\,, g^b] F^c{}_{ab} \lambda_c + \frac{1}{2}[f^a\,, g^b] D^c{}_{ab} \lambda_c + \frac{1}{n}[f^a\,, g^b] g_{ab} + ([f^0\,, g^a] - [g^0\,, f^a])\lambda_a.$$

As a set of generators for the algebra we can choose the tensor products $x^\mu \otimes 1$ and $1 \otimes \lambda^a$. These would correspond respectively in Kaluza-Klein theory to the space-time coordinates and the internal coordinates. The commutation relations for the two sets follow immediately from (3.10), with an appropriate change of notation.

An interesting example can be found [?] using group manifolds. A group manifold M_G can be embedded as a submanifold of its Lie algebra considered as an euclidean space. Let x^i be the coordinates of this space and consider the Poisson bracket defined by the Lie bracket. The procedure of star quantization will yield once again the Lie bracket. If the group is compact all irreducible representations will be of finite dimension; there are an infinite number indexed by Casimir operators c_i, each with a well-defined dimension d_i. If we set $\mathcal{A} = \mathcal{C}(M_G)$ then we can write

$$\hat{\mathcal{A}} = \bigoplus_{c_i} M_{d_i}.$$

This situation generalizes to arbitrary Kähler manifolds [?]. We are especially interested in situations which at least in some formal sense we can identify

$$\underline{\hat{g}} \sim \bigoplus_{i} \underline{su}_{n_i}. \tag{2.13}$$

If the algebra $\hat{\mathcal{A}}$ contains a matrix algebra M_n then one can consider $\underline{su}_n$ as a subalgebra of $\underline{\hat{g}}$.

3 Noncommutativity versus Field Theory

Consider again the formal algebra $\mathcal{A}$ of the previous section defined less precisely in terms of commutation relations of the form (2.6) but with the right-hand side a non-specified set of elements of the algebra. Consider also a second algebra $\hat{\mathcal{A}}$ which has the same number of generators $\hat{x}^i$ but in general a different set of elements $\hat{J}^{ij}$ on the right-hand side of the commutation relations. We shall suppose that both of these algebras can be represented as subalgebras of the algebra of differential operators $\tilde{\mathcal{A}}$ on some space of smooth functions. In the Appendix such a representation is given explicitly in a special but important case. We designate the product in $\mathcal{A}$ by $*$ and in $\hat{\mathcal{A}}$ by $\hat{*}$.

We assume that there is an algebra homomorphism

$$\hat{\mathcal{A}} \xrightarrow{\rho} \mathcal{A} \tag{3.1}$$

of $\hat{\mathcal{A}}$ onto $\mathcal{A}$ which can be formally defined by the action

$$x^i = \rho(\hat{x}^i) = \Lambda^i(\hat{x}^j)$$

on the generators. By assumption then

$$\rho(\hat{x}^i \hat{*} \hat{x}^j) = x^i * x^j = \rho(\hat{x}^i) * \rho(\hat{x}^j). \tag{3.2}$$

The kernel of ρ is a 2-sided ideal so $\hat{\mathcal{A}}$ cannot in any sense of the word be 'simple'. If $\hat{\mathcal{A}}$ is commutative then so obviously is $\mathcal{A}$; if on the other hand $\mathcal{A}$ is commutative then the kernel of ρ contains necessarily the ideal generated by the commutators. If $\hat{J}$ is non-degenerate then this can again by identified with $\mathcal{A}$ and so $\rho = 0$. In the special case with J and $\hat{J}$ constant non-degenerate matrices we can choose $F^i(\hat{x}^j) = F^i_j \hat{x}^j$ a linear transformation. We have then

$$x^i * x^i = F^i_k F^j_l \hat{x}^k \hat{*} \hat{x}^l, \qquad J^{ij} = F^i_k F^j_l \hat{J}^{kl}.$$

In general the relation between the generators is much more complicated. If we can write for example $F^i(\hat{x}^j) = \hat{x}^i - \xi^i(\hat{x}^j)$ as a linear perturbation then

$$i\hbar J^{ij} = [x^i, x^j] = [\rho(\hat{x}^j), \rho(\hat{x}^j)] = i\hbar \hat{J}^{ij} - \hbar[x^{[i}, \xi^{j]}].$$

which we write in the form

$$\hat{J}^{ij} = J^{ij} + \theta^{ij}, \qquad i\hbar\theta^{ij} = -i[x^{[i}, \xi^{j]}]. \tag{3.3}$$

If we suppose that J^{ij} is constant then using the $\tilde{\lambda}_a$ of the Appendix and writing ξ^i as $\xi^i = i\hbar J^{ia} a_a$ we find that

$$\theta^{ij} = \hbar J^{ia} J^{jb} e_{[a} a_{b]}.$$

In this case the perturbation of the commutation relations is related to the exact form

$$f = da, \qquad a = a_b \theta^b, \qquad f = \frac{1}{2} e_{[a} a_{b]} \theta^a \theta^b.$$

We show in the Appendix that $d\theta^a = 0$. We refer to the literature [?] for a description of the relation between θ^{ij} and the B field.

One might be tempted to consider the $F^i(\hat{x}^j)$ as a 'change of coordinates'. But the change is in the 'phase space' of which $\hat{\mathcal{A}}$ is the structure algebra and so when one looks for a similar transformation in ordinary geometry one must imagine not only a change of coordinates but also a shift in the position because of the term in the definition of the generators which depends on the momentum. What can more properly be considered as a change of coordinates is an automorphism of the algebra, for example the inner automorphism

$$\hat{x}^i = \Lambda^{-1} x^i \Lambda.$$

In this case the product is conserved.

It is perhaps preferable to consider ρ as a change of product on one fixed vector space. We drop then the hat on the generators and distinguish the two products by putting a hat on one of them. In the case of a linear perturbation Equation (3.2) becomes

$$\rho(x^i \hat{*} x^j) = x^i * x^j + x^i * \xi^j + \xi^i * x^j$$

The requirement that the new product be associative places restrictions [?] on the ξ^i.

In general one can consider the set S_0 of all products on the vector space $\mathcal{A}$. There is a subset $S_1 \subset S_0$ in which the product is associative; this is the set which interests us here. Let π be a given product and consider the orbit $S_2 \subset S_1$ of π under the group of all possible maps ρ. This group has a subgroup of automorphisms of $\mathcal{A}$, which leave the product invariant. In a formal sense S_2 can be identified with the quotient of the two groups. In general S_1 will be a union of orbits of different products of non-isomorphic algebras. If we assume that there are no relations other than the commutation relations (2.6) then the set S_2 will be parameterized by the J^{ij}. To pass from stratum of S_1 to another would require a singular variation in J. A familiar example from the theory of Lie algebras is furnished by the embedding $SU_n \hookrightarrow SO_{n^2-1}$. If $\{\lambda_i\}$ is a set of generators of the Lie algebra of SU_n then so is the set $\{\hat{\lambda}_i\}$ with $\hat{\lambda}_i = g^{-1}\lambda_i g$ for $g \in SU_n$. One can write then $\{\hat{\lambda}_i\} = \Lambda_i^j \lambda_j$ where the transformation coefficients are complex numbers. It is the analog of those transformations of SO_n which do not respect the Lie algebra structure which interests us here.

As a limiting case with singular ρ one consider an algebra $\hat{\mathcal{A}}$ with a non-degenerate $\hat{J}$ and an algebra $\mathcal{A}$ with $J = 0$. In the latter case we can identify x^i with $\tilde{x}^i$, the 'space' coordinates of $\tilde{\mathcal{A}}$. The 'lift' by the inverse of ρ is a quantization procedure, a way of associating an operator to a function. One such method is the Weyl-Moyal quantization procedure [?, ?] which furnishes a 'natural' right inverse for ρ which lifts an element $f \in \mathcal{A}$ to an element $\hat{f} \in \hat{\mathcal{A}}$. This is a map between two different strata of S_1.

Let $\mathcal{H}$ be a right $\mathcal{A}$-module and $\hat{\mathcal{H}}$ be a right $\hat{\mathcal{A}}$-module. We shall place a hat on an element of $\mathcal{H}$ whenever it is necessary to distinguish the $\hat{\mathcal{A}}$-module structure. For simplicity we shall suppose that both modules are free over their respective algebras and so the map ρ can be extended to a map

$$\hat{\mathcal{H}} \xrightarrow{\rho} \mathcal{H}$$

between the two of them. We shall simplify even further and suppose that the module is of rank one. It can be identified therefore with the respective algebra and each identification is equivalent to a choice of gauge. We choose $\psi_0 \in \mathcal{H}$ as basis of $\mathcal{H}$ as both $\mathcal{A}$-module and $\hat{\mathcal{A}}$-module and we write $\psi = \psi_0 * f$ and $\hat{\psi} = \psi_0 \hat{*} f$. This defines the map ρ in terms of the products. We shall suppose that the potential A lies in the Lie algebra $\underline{g}$ of a Lie (pseudo)group $\mathcal{G}$ which we shall take to be a subgroup of the unitary elements of $\mathcal{A}$ and likewise that $\hat{A}$ lies in the Lie algebra $\hat{\underline{g}}$ of a Lie (pseudo)group $\hat{\mathcal{G}}$. We shall suppose that the gauge group acts on the left. The left action of $\mathcal{G}$ on $\mathcal{H}$ is compatible with the algebra action from the right. This condition is automatic in normal Yang-Mills theory where the two actions always commute. Since the derivative is covariant from the left one has also

$$D(g^{-1}\psi) = g^{-1}D\psi, \qquad g \in \mathcal{G}.$$

If $g \simeq 1 + h$ then one can write this in the form of a left Leibniz rule for h.

In ordinary geometry the case we are considering would be called an abelian gauge theory. This is in fact more general since gauge theory with unitary groups can be incorporated simply by the replacements

$$M_n \otimes \mathcal{A} \mapsto \mathcal{A}, \qquad M_n \otimes \hat{\mathcal{A}} \mapsto \hat{\mathcal{A}}. \tag{3.4}$$

It is only important that the matrix factor be the same for both algebras since otherwise the map ρ in general would not be interesting. If we choose the differential calculus given by (2.12) and make the replacement (3.4) then we can consider Equation (3.11) below to be valid also in the product case. The bracket must be chosen to be that of the product algebra.

We suppose finally that there is a differential calculus $\Omega^*(\mathcal{A})$ over $\mathcal{A}$ and a differential calculus $\hat{\Omega}^*(\hat{\mathcal{A}})$ over $\hat{\mathcal{A}}$ and that the map ρ can be extended to an algebra morphism

$$\hat{\Omega}^*(\hat{\mathcal{A}}) \xrightarrow{\rho} \Omega^*(\mathcal{A})$$

of the latter onto the former. As important special cases we mention the calculi whose modules of 1-forms are free with a special basis (frame) θ^a and $\hat{\theta}^a$ as given in the Appendix. We have then the identifications

$$\Omega^1(\mathcal{A}) = \bigoplus_1^d \mathcal{A}, \qquad \hat{\Omega}^1(\hat{\mathcal{A}}) = \bigoplus_1^d \hat{\mathcal{A}}.$$

The integer d here is the 'dimension' and must be the same in both cases. The extension of ρ can be defined by setting

$$\rho(d\hat{f}) = d\rho(\hat{f}). \tag{3.5}$$

This is a natural extension but it is not necessarily compatible with the identification of a form with its components. The image of a free module is not necessarily free.

Let D and $\hat{D}$ be covariant derivatives defined on respectively $\mathcal{H}$ and $\hat{\mathcal{H}}$. We introduce the gauge potentials as usual by the conditions

$$D\psi_0 = \psi_0 * A, \qquad \hat{D}\psi_0 = \psi_0 \,\hat{*}\, \hat{A}.$$

These define D and $\hat{D}$ on all of $\mathcal{H}$ either by the Leibniz rule or by the gauge covariance. If $f \simeq 1 + h$ then to first order in h we can write

$$D\psi = \psi * (A + Dh), \qquad \hat{D}\psi = \psi \,\hat{*}\, (\hat{A} + \hat{D}h).$$

We have here introduced the covariant derivatives

$$Dh = dh + [A\,, h], \qquad \hat{D}h = \hat{d}h + [\hat{A}\,\hat{,}\, h]$$

of an element $h \in \mathcal{A}$ ($\hat{\mathcal{A}}$), with

$$[A\,, h] = A * h - h * A, \qquad [\hat{A}\,\hat{,}\, h] = \hat{A} \,\hat{*}\, h - h \,\hat{*}\, \hat{A}.$$

Conversely, given A and $\hat{A}$ one can construct a map [?]

$$\mathrm{SW}: \; D \longrightarrow \hat{D}$$

between the two derivatives by assuring that the two Leibniz rules are satisfied. The map SW becomes then an equation because of integrability conditions; it must be well-defined on all of $\mathcal{H}$.

If ρ is an automorphism then $\hat{D} - D$ is a (right) module morphism. One can neglect the distinction between the two products and write

$$\hat{D}h = Dh + [\Gamma, h] \tag{3.6}$$

with $\Gamma = \hat{A} - A$. If we define the variation

$$\delta_h \Gamma = \hat{D}h - Dh \tag{3.7}$$

of Γ under multiplication by $f \simeq 1 + h$, we see that it is given by

$$\delta_h \Gamma = [\Gamma, h]. \tag{3.8}$$

This is the well-known formula which expresses the gauge covariance of the difference between two connections. The map SW is a generalization of this formula to situations where the two connections in question are with respect to two different gauge groups.

In general, if ρ is not an automorphism, then Equation (3.8) will have no solution and we cannot define Γ as we have done. Since ρ is surjective we can introduce a function $\gamma(h)$ with values in $\mathcal{A}$ such that

$$\psi_0 \hat{*} (1 + h) = \psi_0 * (1 + h)(1 + \gamma).$$

This implies that $\psi_0 \hat{*} dh = \psi_0 * d(h + \gamma)$ and therefore that

$$\hat{D}\psi = \psi * \hat{D}(h + \gamma[h]).$$

Using the definition of $\delta_h \Gamma$ given above this can be written as

$$\delta_h \Gamma = D\gamma + \hat{D}h - Dh = D\gamma + [\Gamma, h] + [\hat{A}, h] - [\hat{A}, h]. \tag{3.9}$$

If ρ is not an automorphism then to compensate for the difference between ρ and an automorphism we have introduced an element $\gamma \in \underline{g}$. This is equivalent to an interpretation of the modification of the product by a change of gauge. We have in fact identified the gauge group as the unitary elements of the algebra. When we change the structure of the algebra this entails necessarily a change in the structure of the gauge group and hence of the Lie algebra. In certain cases the change involves a finite number of parameters in the commutation relations. As an example of this one can consider (3.3) with the θ^{ij} real numbers. A gauge transformation which depends on these extra parameters is equivalent to a local gauge transformation in a Kaluza-Klein extension of the theory with the θ^{ij} as the local coordinates of the extra dimensions. The variation described in Equation (3.9) is however for fixed 'Kaluza-Klein' parameters and gives only the variation of Γ under change of gauge. Having found the solution explicitly in terms of the extra parameters one could calculate also their variation.

Both D and $\hat{D}$ can be extended to the entire differential calculus; in general however there is no extension of SW. In the special cases we are considering here both of the differential calculi can be written in the form

$$\Omega^*(\mathcal{A}) = \mathcal{A} \otimes \bigwedge^*$$

where the second factor is the deformed exterior algebra over the vector space spanned by the frame. If

$$\bigwedge^* = \hat{\bigwedge}^*$$

then both ρ and SW can be extended to the exterior algebra. We can write

$$D\psi = \theta^a D_a \psi, \qquad \hat{D}\psi = \hat{\theta}^a \hat{D}_a \psi.$$

We shall restrict our attention here to the important special case with the projector $P^{ab}{}_{cd}$, defined in the Appendix, given by the expression (5.8). We have then

$$[\lambda_a \, , \lambda_b] = \lambda_c F^c{}_{ab} + K_{ab}, \qquad [\lambda_a \stackrel{*}{,} \lambda_b] = \lambda_c \hat{F}^c{}_{ab} + \hat{K}_{ab}. \tag{3.10}$$

It follows from (5.4) that the product structure of the frame is the same with or without hat. One finds from (5.16), to lowest order, the expression

$$\hat{e}_a f = e_a f + i\bar{k}\theta^{bc}[\lambda_b \stackrel{*}{,} \lambda_a] \hat{*} \hat{e}_c f$$

for the 'partial derivatives'. As seen by comparing (3.10) with (5.19), this is an identity. The frame is gauge invariant: $\delta_h \theta^a = 0$. Because of the special properties of the frame Equation (3.9) can be written using components as

$$\delta_h \Gamma_a = D_a \gamma + [\Gamma_a \, , h] + \frac{1}{2}\theta^{bc}[e_b A_a \, , e_c h] + +o(\bar{k}^2). \tag{3.11}$$

The solution is difficult to find in general but if the deformation parameter $\bar{k}$ which defines the algebra $\hat{\mathcal{A}}$ in terms of $\mathcal{A}$ is small a formal Taylor-series expansion can be given [?]. In the limit then $J^{ab} \to 0$ Equation (3.11) can be written using only ordinary derivatives as

$$\delta_h \Gamma^a = \theta^{aj} D_j \gamma + [\Gamma^a \, , h] - \frac{1}{2}\theta^{kl}[\partial_k h \, , \partial_l(\theta^{aj} a_j)], \qquad \Gamma^a = \theta^{ab}\Gamma_b. \tag{3.12}$$

To emphasize the special status of this case we have written the potential using a lower-case letter: $A_i \mapsto a_i$.

In principle the preceding must be generalized to the case where the covariant derivative includes a gravitational contribution. We have changed the structure of the algebra without changing that of the differential calculus and this is not always possible. With the formalism we have used, based on the existence of a frame we have essentially assumed that the differential calculus is not gauge dependent. In general this will not be true since the gauge group depends on the structure of the algebra and the differential calculus depends on the latter. The pair (γ, Γ) of external fields depends through Equation (3.9) on the Poisson structure θ which in turn can be identified with the B field. One can say then that the map SW is another example of the equivalence between the point of view which considers geometry as an essential given aspect of space-time and the point of view which considers geometry as a convenient description of an external field on a conventional space-time. In other words we are lead to interpret SW as a correspondence between on the one hand some physical situation with external fields and on the other the same physics but with the extra variables considered as an intrinsic part of a noncommutative geometry.

4 Neoparadigma

In this section we shall consider an example of the map SW constructed using the first two examples of Section 2. This will consist in a contraction of the second model onto the first [?]. The algebras are respectively

$$\hat{\mathcal{A}} = M_2, \qquad \mathcal{A} = M_2^+.$$

One can think of the limit as the classical limit of a quantum spin or as a contraction of a gauge group. The 'local' gauge group of the algebra M_2 is the group U_2 and that of $M_1 \times M_1$ is $U_1 \times U_1$. Associated to the latter are two gauge potentials, the photon γ and a massive neutral vector boson Z_0; the former has also a massive charged W. The contraction can be implemented by letting the W mass tend to infinity. The role of the B-field is played by the charged W-boson. In this example there is no obvious interpretation of the commutation relations of $\hat{\mathcal{A}}$ in terms of a B-field, unless it be the fact that the W-boson takes its values in the complement of $U_1 \times U_1$ in U_2. The passage from $\mathcal{A}$ to $\hat{\mathcal{A}}$ is here an example of a map between algebras which is not a deformation quantization.

We introduce ρ_ϵ by the action

$$\rho_\epsilon(\hat{\lambda}^1) = \epsilon\lambda^1, \qquad \rho_\epsilon(\hat{\lambda}^2) = \epsilon\lambda^2, \qquad \rho_\epsilon(\hat{\lambda}^3) = \lambda^3$$

on the Pauli matrices. Therefore the structure constants rescale as

$$C^1{}_{23} = \hat{C}^1{}_{23}, \qquad C^2{}_{31} = \hat{C}^2{}_{31}, \qquad C^3{}_{12} = \epsilon^{-2}\hat{C}^3{}_{12}$$

and the metric as $g^{ab} = \mathrm{diag}(\epsilon^2, \epsilon^2, 1)$. For all $\epsilon > 0$ this is a redefinition of the product of M_2 such that ρ_ϵ is an isomorphism and for $\epsilon = 0$ it is a singular contraction. We define ρ_0 to be the singular limit as $\epsilon \to 0$. If we decompose $\hat{f} = \hat{f}^+ + \hat{f}^-$ then we have $\rho_\epsilon(\hat{f}) = f^+ + \epsilon f^-$ and

$$\rho_\epsilon(\hat{f} \,\hat{*}\, \hat{g}) = f^+ * g^+ + o(\epsilon).$$

It follows that the image of ρ_0 contains nilpotent elements. This accounts for the difference in the dimensions of $\hat{\mathcal{A}}$ and $\mathcal{A}$. Except for a rescaling the frame remains invariant under the contraction and the extension (3.5) is given simply by

$$\theta^1 = \epsilon\hat{\theta}^1, \qquad \theta^2 = \epsilon\hat{\theta}^2, \qquad \theta^3 = \hat{\theta}^3.$$

The differential remains invariant:

$$\rho_\epsilon(\hat{d}\hat{f}) = d\rho_\epsilon(\hat{f}).$$

We choose $\psi_0 = 1$, the unit matrix of M_2 and we set $\hat{D} \cdot 1 = \hat{A} = \hat{A}_a\hat{\theta}^a$. The image $\hat{A}$ under ρ_ϵ must be of the form $\rho_\epsilon(\hat{A}) = A_3(\lambda^3)\theta^3 + o(\epsilon)$. The remaining two modes become infinitely heavy in the limit and decouple. With the identifications it follows that near the identity matrix we can write $\hat{h} = h + \gamma$. We can therefore write

$$\hat{D}\hat{h} = d(h + \gamma) + [\hat{A}, \gamma] + [\hat{A}, h], \qquad Dh = dh$$

and (3.9) becomes the equation

$$\delta_h\Gamma = d\gamma + [\hat{A}, \gamma] + [\hat{A}, h].$$

Since h defines a gauge transformation of $\mathcal{A}$ it must be of the form $h = h_3\lambda^3$. If therefore $\hat{A} = \hat{A}(\hat{\lambda}^3)$ then a solution is given by $\gamma = 0$, $\Gamma = 0$. One can consistently choose $\hat{A} = A$. If on the other hand

$$\hat{A} = \hat{A}_3(\hat{\lambda}^1, \hat{\lambda}^2)\hat{\theta}^3,$$

for example, then the equation becomes the equation

$$\delta_h \Gamma_3 = e_3\gamma + [A_3, \gamma] + [\hat{A}_3 \,\hat{,}\, h] \tag{4.1}$$

for the third component. The source term $[\hat{A}_3 \,\hat{,}\, h]$ now is not equal to zero and the external fields, the difference between the potentials Γ_3 as well as the 'scalar' γ, cannot vanish. We are free to interpret them as components in a noncommutative geometry or as external fields in a commutative (albeit discrete) one.

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5 Appendix

Let $\mathcal{A}$ be a noncommutative algebra with a differential calculus $\Omega^*(\mathcal{A})$. A large class of differential calculi, but not all, are such that the module $\Omega^1(\mathcal{A})$ is free as a left or right $\mathcal{A}$-module and has a special frame θ^a with

$$[f\,, \theta^a] = 0, \qquad 1 \leq a \leq n \tag{5.1}$$

which is dual to a set of derivations $e_a = \mathrm{ad}\,\lambda_a$:

$$df = e_a f\theta^a = [\lambda_a\,, f]\theta^a = -[\theta\,, f], \qquad \theta = -\lambda_a\theta^a. \tag{5.2}$$

The set of θ^a is the noncommutative equivalent of a Cartan moving frame and in ordinary geometry the derivations e_a would be called Pfaffian derivatives. The 'Dirac operator' θ generates $\Omega^1(\mathcal{A})$ as a bimodule; it is not a free bimodule. The λ_a must satisfy the consistency condition [?]

$$2\lambda_c\lambda_d P^{cd}{}_{ab} - \lambda_c F^c{}_{ab} - K_{ab} = 0. \tag{5.3}$$

It has been shown recently [?] that this can be interpreted as a vanishing-curvature condition.

The $P^{cd}{}_{ab}$ define the product in the algebra of forms:

$$\theta^a\theta^b = P^{ab}{}_{cd}\theta^c\theta^d. \tag{5.4}$$

The $F^c{}_{ab}$ are related to the 2-form $d\theta^a$ through the structure equations:

$$d\theta^a = -\frac{1}{2}C^a{}_{bc}\theta^b\theta^c, \qquad C^a{}_{bc} = F^a{}_{bc} - 2\lambda_e P^{(ae)}{}_{bc}. \tag{5.5}$$

The K_{ab} are related to the curvature of θ:

$$d\theta + \theta^2 = \frac{1}{2}K_{ab}\theta^a\theta^b.$$

All the coefficients lie in the center $\mathcal{Z}(\mathcal{A})$ of the algebra. With no restriction of generality we can impose the conditions

$$F^e{}_{cd} = P^{ab}{}_{cd}F^e{}_{ab}, \qquad K_{cd} = P^{ab}{}_{cd}K_{ab}. \tag{5.6}$$

Define

$$C^{ab}{}_{cd} = \delta^a_c\delta^b_d - 2P^{ab}{}_{cd}.$$

Then from the fact that $P^{cd}{}_{ab}$ is a projector we find that $C^{ab}{}_{cd}C^{cd}{}_{ef} = \delta^a_e\delta^b_f$. We can write then the first term of Equation (5.3),

$$2\lambda_d\lambda_e P^{de}{}_{bc} = \lambda_b\lambda_c - \lambda_d\lambda_e C^{de}{}_{bc} \equiv [\lambda_b\,,\lambda_c]_C,$$

as a sort of deformed bracket and Equation (5.3) can be rewritten in the form

$$[\lambda_b\,,\lambda_c]_C = \lambda_a F^a{}_{bc} + K_{bc}. \tag{5.7}$$

If $P^{ab}{}_{cd}$ is given by

$$P^{ab}{}_{cd} = \frac{1}{2}(\delta^a_c\delta^b_d - \delta^a_d\delta^b_c) \tag{5.8}$$

then we have

$$C^{ab}{}_{cd} = \delta^b_c\delta^a_d.$$

Equation (5.7) defines a ‘twisted’ Lie algebra with a central extension and the $F^a{}_{bc}$ must satisfy a set of modified Jacobi identities. From (5.7) one derives immediately the relations

$$[e_a\,,e_b]_C = C^c{}_{ab}e_c. \tag{5.9}$$

between the first and second derivatives. When $P^{ab}{}_{cd}$ is of the form (5.8) the derivations form a Lie algebra.

As an example we recall the case of the matrix algebra M_n. Let λ_a, for $1 \leq a \leq n^2-1$ be an anti-hermitian frame of the Lie algebra of the special unitary group SU_n. The product $\lambda_a\lambda_b$ can be written in the form

$$\lambda_a\lambda_b = \frac{1}{2}F^c{}_{ab}\lambda_c + \frac{1}{2}D^c{}_{ab}\lambda_c - \frac{1}{n}g_{ab}. \tag{5.10}$$

The components g_{ab} of the Killing metric can be defined in terms of the structure constants by the equation

$$g_{ab} = -\frac{1}{2n}F^c{}_{ad}F^d{}_{bc}.$$

One lowers and raises indices with g_{ab} and its inverse g^{ab}.

We suppose that $\mathcal{A}$ is a formal algebra with n generators x^i which satisfy commutation relations of the form

$$[x^j, x^k] = i\hbar J^{jk}, \qquad J^{jk} \in \mathcal{A}, \qquad (J^{jk})^* = J^{jk}. \tag{5.11}$$

If the right-hand is considered as given then it must satisfy the constraints

$$[x^i, J^{jk}] + [x^j, J^{ki}] + [x^k, J^{ji}] = 0$$

which follow from the Jacobi identities. If J^{ij} is non-degenerate then the center of $\mathcal{A}$ is trivial. The inverse J^{-1}_{ij} exists in the sense that

$$J^{-1}_{ij} J^{jk} = \delta^k_i, \qquad J^{-1}_{ij} \in \mathcal{A}.$$

The algebra has as well n generators λ_a which satisfy the quadratic relations (5.7). The commutation relations between the two sets determines the differential calculus through the relations (5.1). Consider first the case with J^{ij} central elements of the algebra and with λ_a defined by (5.18). This means that $P^{ab}{}_{cd}$ is given by (5.8) and that $F^a{}_{bc} = 0$. The associated geometry is flat. Consider also the smooth manifold $V = \mathbb{R}^n$ and the algebra $\tilde{\mathcal{A}}$ generated by the coordinates $\tilde{x}^i$ and the conjugate momenta p_j. We shall use the convention of distinguishing between the operator p_j and the result $i\tilde{\partial}_j f$ of the action of p_j on f. There is a simple representation of $\mathcal{A}$ as a subalgebra of the algebra of (pseudo-)differential operators $\tilde{\mathcal{A}}$, given by the identification

$$x^i = \tilde{x}^i + \frac{1}{2} \kbar J^{ij} p_j. \tag{5.12}$$

From this it follows immediately that

$$f(x^i) = f(\tilde{x}^i) + \frac{1}{2} \kbar J^{jk} p_k \partial_j f + o(\kbar^2) = f(\tilde{x}^i) + \frac{1}{2} \kbar J^{jk} \partial_j f p_k + o(\kbar^2)$$

and from this 'Taylor' expansion in phase space we can deduce the commutation relations

$$[f, x^j] = i\kbar J^{ij} \partial_i f + o(\kbar^2)$$

and hence

$$[f, g] = i\kbar J^{ij} \partial_i f \partial_j g + o(\kbar^2).$$

This can be considered as part of an expression which defines a noncommutative '$\ast$-product' on an algebra of functions [?, ?] using a formal expression which is an exponential in the partial derivatives. If the J^{ij} are not central then by introducing the vector fields $J^i = J^{ij}\partial_j$ we can write the commutation relations as

$$[x^i, x^j] = \frac{1}{2} i\kbar J^{[ij]} + \frac{1}{4} \kbar^2 [J^i, J^j]. \tag{5.13}$$

In this case it is convenient to write (5.12) differently. We introduce n vector fields p_a on $\tilde{\mathcal{A}}$ such that p_a is the operator which yields $p_a\tilde{f} = ie_a\tilde{f}$ when acting on $\tilde{f}$ and the $e_a\tilde{f} = e^i_a(\tilde{x}^k)\tilde{\partial}_i\tilde{f}$ are the commutative limits of the elements $e_a f \in \mathcal{A}$. We define also

$$J^{ij} = J^{ib} e_b x^j = J^{ab} e_a x^i e_b x^j, \qquad \hat{J}^{ij} = \hat{J}^{ib} \hat{e}_b x^j = \hat{J}^{ab} \hat{e}_a x^i \hat{e}_b x^j$$

and we suppose that J^{ab} is an hermitian central matrix which satisfies (5.19). Since

$$e_a J^{ia} = -e_a e_b x^i J^{ab} = -J^{ab} F^c{}_{ab} e_c x^i = 0$$

the operators x^i are hermitian provided $F^c{}_{ab} = 0$. This result relies on the particular form of the product we have chosen within the algebra of forms.

If we have two $*$-products as in Section 3 and derivations e_a and $\hat{e}_a$ then we can write equivalently Equation (5.12) in the form

$$x^i = \tilde{x}^i + \frac{1}{2}\hbar J^{ia} p_a, \qquad \hat{x}^i = \tilde{x}^i + \frac{1}{2}\hbar \hat{J}^{ia} \hat{p}_a. \tag{5.14}$$

To lowest order this and the perturbed equivalent simplify to respectively

$$[x^i, x^j] = \frac{1}{2} i\hbar J^{[ij]}(x^i), \qquad [x^i \,\hat{,}\, x^j] = \frac{1}{2} i\hbar \hat{J}^{[ij]}(x^i). \tag{5.15}$$

If we define θ^{ab} by the identities

$$\hat{J}^{ij} = J^{ij} + \theta^{ij}, \qquad \theta^{ij} = e_a x^i e_b x^j \theta^{ab}$$

then we can write the difference between the commutators as

$$[f \,\hat{,}\, g] = [f, g] + i\hbar \theta^{ab} e_a f * e_b g + o(\hbar^2). \tag{5.16}$$

In general one would expect that the λ_a generate also the algebra and that each x^i can be expressed as a formal power series in the λ_a. The algebra depends then on the coefficients in the Equation (5.7) for λ_a. In fact the whole differential calculus depends on these coefficients:

$$\mathcal{A} = \mathcal{A}(P, F, K), \qquad \Omega^*(\mathcal{A}) = \Omega^*(\mathcal{A})(P, F, K). \tag{5.17}$$

We do not imply here that $(P^{cd}{}_{ab}, F^c{}_{ab}, K_{ab})$ are the only parameters. An explicit representation would introduce more. In the simplest case with J^{ij} a central non-degenerate matrix we can choose $P^{ab}{}_{cd}$ of the form (5.8) and set $F^c{}_{ab} = 0$. We find that x^i is linear in λ_a and the relation can be inverted:

$$\lambda_a = \frac{1}{i\hbar} J^{-1}_{ai} x^i, \qquad \hat{\lambda}_a = \frac{1}{i\hbar} \hat{J}^{-1}_{ai} x^i. \tag{5.18}$$

We find that K_{ab} is given by the expression

$$K_{ab} = -\frac{1}{i\hbar} J^{-1}_{ab}, \qquad i\hbar K_{ac} J^{cb} = -\delta^b_a. \tag{5.19}$$

In this case we can write also

$$\mathcal{A} = \mathcal{A}(K).$$

The λ_a are represented by

$$\lambda_a = \frac{1}{2i} p_a - K_{aj} \tilde{x}^j.$$

To a certain extent in this case one might expect that formally at least the algebra depends only on K_{ab}. It is equivalent to a quantized phase space. In general we suppose that the commutator is defined in terms of the C-commutator defined above. That is we write

$$[x^i, x^j] = [x^i(\lambda_a), x^j(\lambda_a)]$$

and use (5.7) to calculate J^{ij} in terms of $(P^{cd}{}_{ab}, F^c{}_{ab}, K_{ab})$. In certain cases it might be more convenient to use a representation of the λ_a and from them construct a representation of the x^i considered as a secondary set of generators. For example if we set

$$x^i = i\hbar J^{ia}_0 \lambda_a, \qquad J^{ib}_0 = \delta^i_a J^{ab}, \quad K_{0,ib} = \delta^a_i K_{ab}$$

then we find that

$$[x^i, x^j] = i\bar{k}(J_0^{ij} + F_0^{ij}{}_k x^k), \qquad F_0^{ij}{}_k = F^c{}_{ab} J_0^{ia} J_0^{jb} K_{0,kc}.$$

We have here constructed a nonconstant $J^{ij} = J_0^{ij} + F_0^{ij}{}_k x^k$ directly from the λ_a, which can be considered as comprising the first two terms an an infinite multipole expansion. More eleborate forms can be obtained by chossing

$$e_a \tilde{x}^i = \delta_a^i + \Lambda_a^i(\tilde{x}^k).$$

One obtains then

$$x^i = \tilde{x}^i + \frac{1}{2}\bar{k}(J_0^{ib} + \Lambda_a^i(\tilde{x}^k) J^{ab}) p_b. \tag{5.20}$$

We can choose x^i to be the operator obtained by setting $\Lambda_a^i(\tilde{x}^k) = 0$ and denote $\hat{x}^i$ the operator with generic $|\Lambda_a^i(\tilde{x}^k)| \ll \delta_a^i$. Equation (5.20) can be written as (3.3) if we write $\Lambda_a^i = \delta_a^j \Lambda_j^i$ and set

$$\xi^i(x^k) = \Lambda_j^i(x^k)(x^j - \tilde{x}^j).$$

Here the variables $\tilde{x}^a$ are to be considered as parameters. We deduce, to lowest order, the 'Taylor' expansion

$$f(\hat{x}^i) = f(x^i) + \frac{1}{2}\bar{k}(\hat{J}^{ab} e_a f \hat{p}_b - J^{ab} e_a f p_b).$$

If as in Section 3 we write $\hat{\lambda}_b = \lambda_b + a_b$ then from (3.10) we find that a_b must satisfy the equation

$$e_{[a} a_{b]} = \hat{K}_{ab} + \lambda_c \hat{F}^c{}_{ab} - K_{ab}.$$

This can also be written as an equality of 2-forms: $da = d\theta + \theta^2 - \hat{\theta}^2$.

The forms K_{ab} and $\hat{K}_{ab}$ obviously break Lorentz invariance, as do the vectors $F_a = \epsilon_{abcd} F^{bcd}$ and $\hat{F}_a = \epsilon_{abcd} \hat{F}^{bcd}$. We shall consider these effects to be of the same order of magnitude as the gravitational effects. In particular, from this point of view Minkowski space-time is a degenerate limit. We would prefer to identify the absence of gravitational field as the commutative limit but it is more convenient to consider this state as a 'regular' cellular structure. The price to be paid for this assumption is a ground state which is not Lorenz invariant. This is unfortunate since Lorenz invariance was the original motivation of noncommutative structure [?].