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On Boundary RG-Flows in Coset Conformal Field Theories

Stefan Fredenhagen* and Volker Schomerus†
Max-Planck-Institute for Gravitational Physics
Am Mühlenberg 1, D-14476 Golm, Germany
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We propose a new rule for boundary renormalization group flows in fixed-point free coset models. Our proposal generalizes the 'absorption of boundary spin'-principle formulated by Affleck and Ludwig to a large class of perturbations in boundary conformal field theories. We illustrate the rule in the case of unitary minimal models.

Renormalization group (RG) flows in models with boundaries or defects are of interest in condensed matter theory, statistical physics, and in string theory where they describe aspects of D-brane dynamics. There exist various tools to investigate flows generated by boundary fields including the Thermodynamic Bethe Ansatz, the Truncated Conformal Space approach and perturbation theory (for an overview see e.g. [1]). These techniques have helped to accumulate a rather extensive knowledge about boundary RG-flows in specific models. On the other hand, with the exception of the 'g-conjecture' [2], we lack model independent principles that could guide us to predict possible boundary flows. Only in the case of WZW models, Affleck and Ludwig found an easy to formulate rule [3]. Here we shall present a generalization of their 'absorption of the boundary spin'-principle which applies to all fixed-point free coset conformal field theories.

Our presentation starts with a precise formulation of the new principle. It is then argued that our proposal is consistent with the perturbative results obtained in [5]. Finally, we apply the proposed rule to coset realizations of unitary minimal models and compare our 'predictions' with known results.

Let G/H be a coset conformal theory with primaries labeled by pairs (l,l') of integrable highest-weight representations l and l' of the affine Lie algebras $\hat{\mathfrak{g}}_k$ and $\hat{\mathfrak{h}}_{k'}$, respectively. Note that in many examples, branching selection rules restrict the admissible pairs and that different pairs may describe the same coset sector. For our purposes, however, there is no need to be more specific about such issues. Boundary conditions (L,L') of so-called *Cardy type* [4] can be labeled by elements from the same set, i.e. by primaries of G/H. It is natural to extend this correspondence between elementary Cardy type boundary conditions and primaries (or their conformal families) such that mixtures (or 'superpositions') of boundary theories are associated with sums of conformal families.

Let us now choose representations σ, L of $\hat{\mathfrak{g}}$ and L' of $\hat{\mathfrak{h}}$. Then our rule predicts the following flow between boundary conditions,

$$(L,\sigma|_{\mathfrak{h}} \hat{\times} L') \longrightarrow (L \hat{\times} \sigma, L') \ . \qquad (1)$$

Here, $\hat{\times}$ denotes the fusion product for representations of the affine Lie algebras $\hat{\mathfrak{g}}$ and $\hat{\mathfrak{h}}$, respectively. The definition of the restriction in $\sigma|_{\mathfrak{h}}$ is not entirely obvious. It is based on restricting the corresponding representation σ_0 of the finite dimensional Lie algebra $\mathfrak{g}$ to its subalgebra $\mathfrak{h}$. The existence of an embedding between the two affine algebras guarantees that all the subrepresentations in $\sigma_0|_{\mathfrak{h}}$ give rise to integrable highest-weight representations of $\hat{\mathfrak{h}}_{k'}$. Their sum is then denoted by $\sigma|_{\mathfrak{h}}$. Note furthermore that in most cases, the boundary labels on both sides of the flow (1) involve reducible representations. To identify the configurations as mixtures of elementary boundary conditions, we have to decompose the representations into irreducibles.

It is certainly of interest to specify which boundary field is responsible for the flow (1). Even though this issue can be analyzed in more detail, we shall content ourselves with some simple statements. They involve the integrable highest-weight representations θ and θ' which are built from the adjoint representations of the Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively. Our main rule asserts that the flows (1) are generated by fields from the coset sectors

$$\mathcal{H}_{(0,l')} \quad , \quad \text{where} \quad l' \subset \theta|_{\mathfrak{h}} \quad . \qquad (2)$$

Moreover, if the initial boundary theory before perturbation contains the sector $\mathcal{H}_{(0,\theta')}$ at most once, then $l' = \theta'$ can be omitted from the list (2).

Let us observe in passing that we recover the principle found by Affleck and Ludwig when we specialize to the example of WZW-models, i.e. to coset models with trivial denominator. Indeed, in this case the flow (1) reduces to

$$\dim(\sigma)\ (L) \ \longrightarrow \ (L \hat{\times} \sigma) \ = \ \bigoplus\nolimits_J N_{\sigma L}{}^J \ (J)$$

where $\dim(\sigma)$ is the dimension of the representation σ_0 of $\mathfrak{g}$ and N denote the fusion rules of the affine Lie algebra $\hat{\mathfrak{g}}$. The perturbing field is given by the product $S^\sigma J(x)$ which describes the coupling of the current $J(x)$ to some boundary spin S^σ. Comparison with [3] shows that we have reproduced the 'absorption of the boundary spin'-principle.

Before we move on to examples and applications of the stated rule, we want to discuss its relation with the perturbative results obtained in [5]. There, we considered

*Electronic address: `stefan@aei.mpg.de`
†Electronic address: `vschomer@aei.mpg.de`

coset models in a limiting regime in which some of the involved levels become large. Using a perturbative approach we identified RG fixed points Q in the vicinity of a chosen boundary condition P. To formulate the main result of [5], let us assign representations P_0 and Q_0 of the finite dimensional Lie algebra $\mathfrak{g} \oplus \mathfrak{h}$ to the boundary conditions P and Q (see above). With this notation, we are able to state that Q appears in the vicinity of P if the associated representations P_0 and Q_0 are equivalent on the diagonally embedded $\mathfrak{h}_{\text{diag}} \subset \mathfrak{g} \oplus \mathfrak{h}$,

$$P_0|_{\mathfrak{h}_{\text{diag}}} \sim Q_0|_{\mathfrak{h}_{\text{diag}}} \quad . \tag{3}$$

Here, we have to assume that P and Q coincide in the directions in which the level is not sent to infinity (see [5] for details). The boundary theories Q satisfying the condition (3) provide candidates for the infrared fixed points of an RG flow which initiates from P.

For comparison, let us now evaluate our rule (1) in the limiting regime, assuming that the representation σ is trivial in the directions belonging to small levels. Under this condition, the fusion products in rel. (1) can be replaced by the usual tensor products of Lie algebra representations. Taking the configurations of both sides of (1) and restricting the associated representations to the diagonally embedded $\mathfrak{h}$, we obtain

$$L|_{\mathfrak{h}} \otimes \sigma|_{\mathfrak{h}} \otimes L' \longrightarrow (L \otimes \sigma)|_{\mathfrak{h}} \otimes L' \ .$$

The two sides are equivalent because the decomposition of representations commutes with taking tensor products.

As an application of our rule, let us consider the unitary minimal models. They can be realized as diagonal coset models of the form $\text{su}(2)_k \oplus \text{su}(2)_1/\text{su}(2)_{k+1}$. Correspondingly, their sectors are labeled by three integers (l, s, l') in the range $l = 0 \dots k$, $s = 0, 1$, $l' = 0 \dots k+1$. Branching selection rules restrict $l+s+l'$ to be even, and there is an identification $(l, s, l') \sim (k-l, 1-s, k+1-l')$ between admissible labels. Our rule (1) predicts flows for a large number of starting configurations. Many of them are superpositions of boundary conditions, but here we will concentrate on perturbations of a single boundary condition (J, S, J'). Let us assume that $1 \leq J' \leq k$. Then we choose the representation σ of the numerator theory as $\sigma = (J', 0)$ and fix L' to be $L' = (0)$. With these choices our rule becomes

$$(J, S, J') \longrightarrow \bigoplus_L N_{J\,J'}{}^L \ (L, S, 0) \tag{4}$$

where N denote the fusion rules of $\text{su}(2)_k$. On the other hand, if we select σ to be $(k+1-J', 0)$ and $L' = (k+1)$, we find

$$(J, S, J') \longrightarrow \bigoplus_L N_{J\,J'-1}{}^L \ (L, 1-S, 0) \quad . \tag{5}$$

The first of these flows can be seen in perturbation theory for large level k [6, 7], whereas the second does not become 'small' in this limit. Nevertheless, both flows are known to exist [8–10]. They are generated by the $(0,0,2)$ field (in standard Kac labels $(1,3)$) and differ by the sign of the perturbation. This is in agreement with our general statements on the boundary fields generating the flow (1).

In the simplest minimal model, the critical Ising model, there are three possible elementary boundary conditions: the free boundary condition $(0,1,1)$, and boundary conditions $(0,0,0), (1,1,0)$ in which the boundary spin is forced to be either up or down. Starting from the free condition, the system can be driven into a theory with fixed spin [11]. These are precisely the two flows (4), (5).

The second model in the unitary minimal series is the tricritical Ising model with central charge $c = 7/10$. Once more, the flows triggered by the ϕ_{13} field [8] are correctly reproduced by (4) and (5). There are, however, more flows known which correspond to a perturbation with other fields [12]. As our rule depends on the specific coset construction, it is possible to find additional flows by choosing different coset realizations of the same theory. For the tricritical Ising model, such an alternative realization does exist. It is given by $(\text{E}_7)_1 \oplus (\text{E}_7)_1/(\text{E}_7)_2$. When we apply our rule to this coset construction, it reproduces the two known flows caused by the ϕ_{33} field. In Kac labels they read

$$(2,2) \longrightarrow (3,1) \ , \quad (2,2) \longrightarrow (1,1) \ .$$

These two flows also appear in higher minimal models [13] where we do not know a coset realization for the ϕ_{33}-perturbations. This may be related to the observation that the tricritical Ising model seems to be the only theory in which the considered perturbations are integrable [13]. Nevertheless, recovering flows from the exceptional E_7 coset construction can be considered as an important check of the conjectured rule.

These examples may help to illustrate the wide applicability of our new rule. It is even possible to generalize the rule in a straightforward way beyond the *Cardy case* when dealing with twisted boundary conditions or with modular invariants not given by charge conjugation. While further checks certainly remain to be done, we hope that our proposal provides an elegant way to summarize results obtained from RG computations and that it will emerge as simple guide to predicting new flows.

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