de Sitter Black Holes with Either of the Two Horizons as a Boundary

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(Dated: February 25, 2003)

The action and the thermodynamics of a rotating black hole in the presence of a positive cosmological constant are analyzed. Since there is no spatial infinity, one must bring in, instead, a platform where the parameters characterizing the thermodynamic ensemble are specified. In the present treatment the platform in question is taken to be one of the two horizons, which is considered as a boundary. If the boundary is taken to be the cosmological horizon one deals with the action and thermodynamics of the black hole horizon. Conversely, if one takes the black hole horizon as the boundary, one deals with the action and thermodynamics of the cosmological horizon. The two systems are different. Their energy and angular momenta are equal in magnitude but have opposite sign. In either case, the energy and the angular momentum are obtained as surface terms on the boundary, according to the standard Hamiltonian procedure. The temperature and the rotational chemical potential are also expressed in terms of magnitudes on the boundary. If, in the resulting expressions, one continues the cosmological constant to negative values, the black hole thermodynamic parameters defined on the cosmological horizon coincide with those calculated at spatial infinity in the asymptotically anti-de Sitter case.

I. INTRODUCTION

The observational evidence for a positive cosmological constant has led to renewed interest in the dynamics of de Sitter space[1]. It is natural in this context to analyze the thermodynamics of de Sitter space in the presence of a black hole. It has been known for a long time that if one uses the Euclidean Schwarzschild–de Sitter solution (or, more generally, the Euclidean Kerr–de Sitter solution) to provide thermodynamical information, one finds that the time periods required to avoid a conical singularity, at both the cosmological and black hole horizons, do not match. This is physically interpreted as indicating that the two horizons are not in thermal equilibrium and that, for example, they both emit Hawking radiation at the corresponding temperatures. An observer somewhere in space would then receive a beam of radiation coming from the black hole and, at the same time, radiation coming from the cosmological horizon would arrive to him from all directions[2].

From the point of view of the action principle, the fact that the Schwarzschild–de Sitter solution has a conical singularity means that the empty space field equations are not satisfied everywhere. If one arranges the period of the time variable so as to have no conical singularity at the cosmological horizon, the field equations will be satisfied there but will not be satisfied at the black hole horizon. Conversely, if the role of the horizons is interchanged, the field equations will not be satisfied at the cosmological horizon.

The main purpose of this article is to point out that this apparent difficulty is rather a virtue, and, at that, one which was to be expected from the point of view of the action principle and thermodynamics. Indeed, if one uses an extremum of the action to evaluate the path integral in the semiclassical approximation, one needs to hold fixed those variables which will become the argument of the partition function once it is evaluated. By the very meaning of "holding fixed", those variables are not varied in the action principle. Thus, for example, for a black hole in asymptotically flat space, one may hold fixed the $1/r$ part of the components of the metric which are determined by the mass. Then, one is dealing with the microcanonical ensemble, where the partition function depends on the total energy. It would be wrong to demand that the partition function thus obtained should have an extremum with respect to variations in the $1/r$ piece of the spatial metric, because then one would obtain a particular value for the mass, i. e., $m = 0$ (and infinite temperature), and thus would not have enough information to develop the thermodynamics of the system.

For the case at hand, there is no notion of spacelike infinity, but the problem itself indicates what to do. One may fix appropriate components of the metric at either the cosmological or black hole horizons. If one chooses the cosmological horizon as the place where the variables are held fixed, there will be no field equations to satisfy there. Then the cosmological horizon will be the analog of spatial infinity in the asymptotically flat case, where the "observer" sits (or, more precisely, the analog of a very large sphere whose radius is eventually allowed to grow without limit). The problem one is solving then will be the thermodynamics of a black hole contained in a space of a given cosmological radius ("box", "boundary"). Conversely, if the variables are fixed at the black hole horizon, it is then that horizon which acts as the boundary. One would then be discussing the thermodynamics of a cosmological horizon with the black hole acting as the boundary. Changes in the black hole variables would then not be subject to dynamics but rather would correspond to changes that the "external observer" decides to make in the environment.

This discussion, which was first given in [3], shows that one should be able to use the Schwarzschild–de Sitter solution (and also, the Kerr–de Sitter solution) as a true extremum of two different action principles which correspond to two different thermodynamical problems. One problem is the thermodynamics of a black hole horizon with a cosmological boundary. The other is the thermodynamics of a cosmological horizon with a black hole boundary. It turns out, as we shall see below, that the physical properties of the two systems have striking differences.

The paper is organized as follows. Section II, which is the bulk of the article, contains the analysis of the Hamiltonian action principle and the identification and evaluation of the energy and the angular momentum, as well as the entropy, as surface terms for a black hole solution in the presence of a positive cosmological constant. The energy U and angular momentum J of Kerr–de Sitter space are evaluated as surface terms at each of the two horizons. The sign of both U and J is reversed if the role of the horizons is interchanged. Interestingly enough, the expressions for the energy U_+ and angular momentum J_+, which are surface terms on the cosmological horizon, turn out to be identical to the surface integrals at spacelike infinity for the Kerr–*anti*–de Sitter solution, if, in those expressions, one replaces the cosmological constant by its negative, or, what is the same, if one continues the anti–de Sitter radius to imaginary values. Next, section III contains a brief discussion of the essential thermodynamical aspects of the system. Finally, the properties of the Kerr–de Sitter geometry that are used in the main text are summarized in Appendix A, while Appendix B contains an alternative treatment of the spherically symmetric case (no rotation).

II. ACTION PRINCIPLE

Since our main interest is gravitational thermodynamics, we shall consider Euclidean line elements, ds^2, and the Euclidean action, I, throughout. It is, however, important to emphasize that the identification of the energy and the angular momentum as surface terms could be achieved in the same way with Lorentzian signature. It is only for the entropy that it is necessary to work with Euclidean signature. Our sign convention for the Euclidean action is that, in the microcanonical ensemble, the semiclassical approximation for the entropy S, coincides with the value of I on–shell. The Euclidean black hole spacetimes have the topology of $S_2 \times S_2$. The first S_2 may be taken to be the ordinary spatial 2–sphere with coordinates θ and ϕ. The second S_2 will be spanned by the coordinates r and t. The radial coordinate r will run between the black hole and cosmological horizons, r_+, r_{++}, which correspond to both geographical poles of the sphere. The coordinate t plays the role of the azimuthal angle. When one horizon is treated as a boundary, that point is removed from the sphere, which becomes a disk. No equations of motion are imposed on the boundary, since the fields are not varied but held fixed. On the other horizon, which is not treated as a boundary, the fields are varied in the action principle and, therefore, the action principle must be such that the Einstein equations hold there.

Boundary terms

We will be specially interested in the surface terms at the horizons which arise in the variation of the action. We will assume that near each of the horizons the metric may be written in the form

$$ds^2 \approx (N^\perp_{,\rho})^2 \rho^2 dt^2 + d\rho^2 + g_{\theta\theta}\, d\theta^2 + g_{\phi\phi}(d\phi + N^\phi dt)^2 \ . \tag{1}$$

In this local coordinate patch, the corresponding horizon is located at $\rho = 0$, where $N^\perp = 0$, and the coefficients $N^\perp_{,\rho}$ and N^ϕ at $\rho = 0$ do not depend on t, θ and ϕ, whereas $g_{\theta\theta}$, and $g_{\phi\phi}$, may depend on θ at $\rho = 0$. As ones moves away from the horizon ρ increases and so does $N^\perp$. Thus $N^\perp_{,\rho}$ at $\rho = 0$, appearing in (1), is positive. As discussed in Appendix A, this form of the line element near the horizon includes the Kerr–de Sitter solution and therefore it suffices for the purpose of this paper. In particular, the independence of the metric components at $\rho = 0$ on t and ϕ is a remnant, imposed as a boundary condition, of the fact that the Kerr solution is stationary and axisymmetric. The off–shell metric away from the horizons is arbitrary.

As shown by Eq. (1) the horizon is at $\rho = 0$, which is the origin of a polar coordinate system in the (ρ, t)–plane. Near $\rho = 0$, the t =constant surface rotates around the horizon during the time interval $t_2 - t_1$ by a "proper angle"

$$\Theta = N^\perp_{,\rho}(t_2 - t_1) \ . \tag{2}$$

Now, consider the Hamiltonian action,

$$I_{\text{HAMILTONIAN}} = \int d^4x \left(\pi^{ij}\dot{g}_{ij} - N^\perp \mathcal{H}_\perp - N^i \mathcal{H}_i\right) \ . \tag{3}$$

If one varies $I_{\text{HAMILTONIAN}}$ one finds both "bulk terms" (integrals over space and time), and surface terms, at each of the horizons. The bulk terms vanish when the Einstein equations hold, and the surface terms take the form[4],

$$-4\pi(t_2 - t_1) \int d\theta \left[N^{\perp}_{,\rho}\delta g^{1/2} \ + N^{\phi}\delta\pi_{\phi}^{\ \rho}\right]\Big|_{\rho=0} \ , \tag{4}$$

where $g = g_{\theta\theta}g_{\phi\phi}$ is the determinant of the angular metric. As we now pass to discuss, the surface term (4) is handled differently depending on whether the horizon at which it is evaluated is treated as dynamical and therefore endowed with thermodynamical properties or as a boundary, where some fields are prescribed.

Horizon dynamical: surface fields not fixed. Entropy.

If the horizon is treated as dynamical, one does not hold fixed anything at $\rho = 0$, and therefore the action should have an extremum with respect to variations of $g^{1/2}$ and π^{ϕ}_{ρ}. The extremization of the hamiltonian action (3) with respect to $g^{1/2}$ would yield the undesirable result $\Theta = 0$ for the opening angle. One wishes instead to have an action principle that gives

$$\Theta = 2\pi \ , \tag{5}$$

which means that there is no conical singularity and therefore, that, on–shell, the manifold is smooth at $\rho = 0$, as it is at any other point. This is accomplished by adding to $I_{\text{HAMILTONIAN}}$ the surface term[4],

$$4\pi A = 8\pi^2 \int d\theta g^{1/2} \ , \tag{6}$$

which is, of course, the horizon entropy, as will be recalled below (we use units in which $16\pi G = 1$, so that the 4π appearing in (6), may also be written as $1/4G$).

It is not necessary to add any additional term to properly account for the extremization with respect to $\pi^{\rho}_{\ \phi}$. Indeed, one obtains from (4)

$$N^{\phi} = 0 \ , \tag{7}$$

which states that the angular coordinate system should be taken as "co–rotating" with the dynamical horizon, as can always be done. Note that the metric given in Appendix A does not obey (7). To that end, one would need to make the change of coordinates,

$$\phi = \phi' - N^{\phi}_{\text{horizon}} t \ . \tag{8}$$

Horizon as boundary: surface fields fixed. Energy and angular momentum

In order to avoid unnecessary cluttering of the equations, we shall take the black hole horizon at r_+ as the dynamical one and the cosmological horizon at r_{++} as the boundary. After the analysis is carried out it will be indicated how the conclusions are changed if the role of the horizons is reversed. Those changes will be simple but significant.

Angular momentum

When the surface fields are fixed, the surface term itself indicates which variables should be held fixed, and, also, gives their physical meaning. The piece of the surface term which contains N^{ϕ} lends itself to immediate treatment. Indeed, that surface term comes from $\mathcal{H}_{\phi}$, which is the generator of purely spatial deformations of the hypersurface t =constant along the direction ϕ. If N^{ϕ} is a constant over the boundary, as it happens in the present case, the deformation at the boundary over the interval $(t_2 - t_1)$ is a spatial rotation by an angle $N^{\phi}(t_2 - t_1)$. The coefficient is, therefore, the variation of the conjugate "global charge", which is the angular momentum J. Thus, we find,

$$J_+ = -\int_{\rho=0} d\theta \pi^{\rho}_{\phi} = \int_{r_{++}} d\theta \pi^{r}_{\phi} \ , \tag{9}$$

for the angular momentum of the horizon at r_+ evaluated as a surface integral at the boundary r_{++}. The change in sign when passing from the coordinate ρ to r arises because π^{i}_{j} is a tensor density and $dr/d\rho$ is negative at r_{++}. When expressed in terms of the Kerr–de Sitter metric parameters m, a (see Appendix A), Eq.(9) gives

$$J_+ = \frac{ma}{Z^2} \ . \tag{10}$$

Energy

Next we turn to the surface term

$$-4\pi(t_2 - t_1)\int d\theta\ N^{\perp}_{,\rho}\delta g^{1/2}\ , \tag{11}$$

which comes from the variation of $\mathcal{H}_\perp$. Normally, *i.e.*, in the asymptotically flat or anti–de Sitter cases, one "pulls the delta through to the left" to achieve the form

$$(t_2 - t_1)\delta U\ . \tag{12}$$

Then U is identified as the total energy because it is conjugate to the time t. [We have used the letter U because it is the customary denomination of the internal energy in thermodynamics. As we will see, U will, in general, not coincide with the mass parameter m of the Kerr–de Sitter, or even Schwarzschild–de Sitter solutions].

In the present case, it will turn out that one can pull the delta through only if one adds to the surface term (11), coming from $\mathcal{H}_\perp$ an additional contribution coming from $\mathcal{H}_\phi$. This means that, on the boundary, what one would like to call a "pure time displacement" is not perpendicular to the t =constant surface, but includes, in addition, a specific spatial rotation. As it will be explained further below, this fact reflects a noteworthy connection between the de Sitter and anti–de Sitter cases.

The analysis proceeds as follows. From the expressions given in Appendix A, one can write the surface term (11) as

$$4\pi(t_2 - t_1)\left\{\frac{H'(r_{++})}{(r_{++}^2 + a^2)}\ \delta\left[\frac{(r_+^2 + a^2)}{Z}\right]\right\}\ . \tag{13}$$

To pull the delta through to the left we observe the identity,

$$4\pi\frac{H'(r_{++})}{(r_{++}^2 + a^2)}\ \delta\left[\frac{(r_{++}^2 + a^2)}{Z}\right] = \delta\left[\frac{m}{Z^2}\right] - \left(\frac{aZ}{r_{++}^2 + a^2} - \frac{a}{l^2}\right)\ \delta\left[\frac{ma}{Z^2}\right]\ , \tag{14}$$

(which also holds if r_{++} is replaced by r_+ throughout) and, hence, we find that the surface term coming from $\mathcal{H}_\perp$ (11) may be written in the form,

$$(t_2 - t_1)\left[\delta U - \left(\frac{aZ}{r_{++}^2 + a^2} - \frac{a}{l^2}\right)\delta J\right]\ , \tag{15}$$

with

$$U = \frac{m}{Z^2}\ . \tag{16}$$

Since in arriving at expressions for the total energy (16) and angular momentum (10) we have only used the form of the Kerr–de Sitter metric near the boundary at r_{++}, these formulas should also hold for any configuration that approaches the Kerr-de Sitter metric at r_{++}. Thus, expressions (16) and (10) have the same content as the standard ADM surface integrals for asymptotically flat space (see, for example [5]) or their generalization to asymptotically anti–de Sitter space[6]. A closer formal analog, but with the same physical content, of the ADM type expressions would be obtained by writing the surface terms for J and M directly in terms of the spatial metric and its conjugate momentum. This has been already achieved for J in expression (9). The corresponding expression for U can be developed easily for the case of spherical symmetry, and it is given in Appendix B. Straightforward efforts to do the same in the case of rotation did not prove successful.

Chemical potential. Relationship with anti–de Sitter space

As mentioned before, the presence of the δJ term in (15), means, geometrically, that the "global time displacement" generated by the total energy U contains a spatial rotation as well as a deformation of the constant time surface normal to itself by an angle

$$(t_2 - t_1)\left(\frac{aZ}{r_{++}^2 + a^2} - \frac{a}{l^2}\right)\ , \tag{17}$$

which may be rewritten in terms of the shift $N^\phi(r_{++})$ of the Kerr–de Sitter metric, given in Appendix A, as

$$-(t_2-t_1)\left(N^\phi(r_{++})+\frac{a}{l^2}\right) \tag{18}$$

One may insert expression (17) in (15) to write the total surface term (4) in the form

$$(t_2-t_1)(\delta U-\Omega\delta J)\ , \tag{19}$$

with,

$$\Omega = N^\phi(r_+)-\frac{a}{l^2}\ . \tag{20}$$

From the point of view of thermodynamics, or, of classical mechanics if one wants, the presence of the extra rotation has the consequence that the conjugate Ω to J is not the relative angular velocity

$$N^\phi(r_+)-N^\phi(r_{++})\ , \tag{21}$$

of the rotating black hole horizon r_+ relative to the cosmological horizon r_{++}, as one might have naively expected. Instead, one may describe the chemical potential Ω given by (20) as the angular velocity of the black hole horizon relative to $r=\infty$, because the term $-a/l^2$ is what one obtains if one sets $r=\infty$ in N^ϕ of the Kerr–de Sitter metric given in (A1). Of course $r=\infty$ is not in the Euclidean section, but one may give meaning to this formal substitution by saying that expression (20) for Ω, and the expressions (16), (10) for U and J are precisely the analytical continuations of the expressions for the anti–de Sitter case obtained in Ref. [6], if one takes the anti–de Sitter radius to be imaginary. It is remarkable that the present treatment, based on identifying the energy and the angular momentum of the black hole horizon as surface terms on the cosmological horizon should give precisely the analytic continuation of the anti–de Sitter expressions with the generators normalized according to the standard $O(4,1)$ Lie algebra structure constants. [Throughout the discussion of the energy, angular momentum and chemical potential, we use the angular coordinate employed in Appendix A, which does not fulfill (7). In this way, the contrast between the actual result (20) and the difference of shifts given by (21), whose value is invariant under the coordinate change (8), is brought out more clearly. In any case, all the surface terms occur at the boundary, which is at r_{++} in this case].

Reversing the roles of the horizons

We now consider the case in which the black hole horizon r_+ is taken as the boundary and the cosmological horizon r_{++} is taken as dynamical. The analysis goes through following the exact same reasoning and steps as before. One finds for the energy U_{++} and angular momentum J_{++}

$$U_{++} = -\frac{m}{Z^2}=-U_+ \tag{22}$$
$$J_{++} = -\frac{ma}{Z^2}=-J_+\ . \tag{23}$$

The sign changes arise ultimately because when the Hamiltonian action is written as an integral over a globally defined radial variable r, the surface term which appears in the variation is a difference of two integrals of the same form, one at r_{++}, the other at r_+. Technically, in the present setting, where for the sake of geometrical clarity we have introduced a local variable ρ, which increases from zero to a positive value as one moves away from each horizon, the changes in sign arise because $dr/d\rho$ is positive at r_+ and it is negative at r_{++}. In the case of the angular momentum this was already mentioned in Eq. (9). For the energy U, the change of sign follows from Eq. (A11) of Appendix A.

The chemical potential (angular velocity) conjugate to J_{++}, Ω_{++}, is now given by

$$\Omega_{++}=N^\phi(r_{++})-\frac{a}{l^2}\ . \tag{24}$$

Again, this Ω is not the angular velocity of the cosmological horizon relative to the black hole horizon. The strict analogy to what was proposed for the black hole horizon would be to say that Ω_{++} is an angular velocity relative to something beyond, *i.e.*, inside the black hole horizon. This, indeed, may be said, because it so happens that $-\frac{a}{l^2}$ also equals N^ϕ at $r=0$. However, in this case, there appears to be no obvious interpretation in terms of an analytic continuation in the de Sitter radius l. Also, the surface $r=0$ is timelike in the Lorentzian section, but, at the moment of this writing, we do not know either, if, and how, the results of this article are related to discussions in terms of asymptotia in time (see, for example, [1] and [7]).

Quite independently of the above issues, which manifest themselves in the presence of rotation, we should emphasize that the consideration of a black hole in de Sitter space leads naturally, as a particular case, to define the energy and angular momentum of de Sitter space by removing the line $r = 0$, corresponding to the worldline of a geodesic observer. Both U and J vanish in this case. One may apply to de Sitter space with the line $r = 0$ removed a general de Sitter transformation. One would obtain then, again, a de Sitter space, but with the line $r = 0$ of another de Sitter coordinate system removed. This would be a different space which would also have $U = J = 0$. In that sense one might say that the "vacuum" is degenerate. Again, it is beyond the scope of this paper to attempt to relate these elementary comments to the ample existing literature on de Sitter vacua[8].

III. THERMODYNAMICS

If one fixes the energy and the angular momentum, one is dealing with the microcanonical ensemble in thermodynamics. In the semiclassical approximation, the value of the action in that ensemble gives the entropy. An important advantage of using the Hamiltonian action, as we have done in this paper, is that the bulk term given by Eq. (3) vanishes on–shell for a stationary solution, such as the Kerr-de Sitter metric. Therefore, one finds from (6) the expected result,

$$S = 4\pi A = \frac{1}{4G} A \ , \tag{25}$$

for the entropy S. One may also use the canonical ensemble, in which case one adds to the action the term $-\beta U$, where the inverse temperature β is now held fixed. If one demands this new action (which is the negative of the Helmholtz free energy) to have an extremum under variations of U, one finds the standard thermodynamical result

$$\beta = \left.\frac{\partial S}{\partial U}\right|_J \ , \tag{26}$$

which combined with Eq. (15) gives

$$\beta = t_2 - t_1 \ , \tag{27}$$

and therefore fixes the time period. It is important to clarify here the geometrical meaning of the time t. Indeed, in asymptotically flat space one is satisfied with the relation (27) between the temperature and the Euclidean time t, because one has assumed that the lapse function $N^{\perp}$ tends to unity at infinity. Here, Eq. (27) holds with the standard form of the Kerr–de Sitter metric, written in Appendix A, for which the rescaled lapse function,

$$N = f^{-1} N^{\perp} \ , \tag{28}$$

introduced in Appendix B, is set equal to unity at the horizon. A similar rescaling of the lapse function comes in anti–de Sitter space. In all three cases, the time t is the "Killing time". In asymptotically flat space t coincides with the proper time displacement but in the de Sitter case it does not coincide with the proper angle Θ introduced in Eq. (2), and, in anti–de Sitter, it does not coincide with the hyperbolic analog of Θ.

Acknowledgments

CECS is a Millennium Science Institute. It is also funded in part by grants from Fundación Andes and the Tinker Foundation. Support from Empresas CMPC is gratefully acknowledged as well. AG gratefully acknowledges support from FONDECYT grant 1010449 and individual support from Fundación Andes. AG and CT acknowledge partial funding under FONDECYT grant 1010446.

Appendix A: The Kerr–de Sitter geometry

The Kerr–de Sitter line element may be written as

$$ds^2 = N^2 dt^2 + \frac{R^2}{H} dr^2 + \frac{R^2}{\Delta} d\theta^2 + g_{\phi\phi}(d\phi + N^{\phi} dt)^2 \ , \tag{A1}$$

with,

$$N^2 = \frac{R^2 H \Delta}{\Delta(r^2+a^2)^2 - H a^2 \sin^2\theta} \ , \tag{A2}$$

$$N^\phi = aZ \frac{H - \Delta(r^2+a^2)}{\Delta(r^2+a^2)^2 - H a^2 \sin^2\theta} \ , \tag{A3}$$

$$g_{\phi\phi} = \frac{\sin^2\theta}{R^2 Z^2}\left[\Delta(r^2+a^2)^2 - H a^2 \sin^2\theta\right] \ , \tag{A4}$$

where

$$H = (r^2+a^2)\left(1 - \frac{r^2}{l^2}\right) - \frac{m}{8\pi} r \ , \tag{A5}$$

$$\Delta = 1 + \frac{a^2}{l^2}\cos^2\theta \ , \tag{A6}$$

$$R^2 = r^2 + a^2 \cos^2\theta \ , \tag{A7}$$

$$Z = 1 + \frac{a^2}{l^2} \ . \tag{A8}$$

The horizon radii r_+, r_{++}, are functions of m and a which solve,

$$H(r) = 0 \ . \tag{A9}$$

Now consider the following change of coordinates,

$$\rho = \pm \frac{R(r_h)}{r_h} \int_{r_h}^{r} \frac{\tilde{r} d\tilde{r}}{\sqrt{H(\tilde{r})}} \ , \tag{A10}$$

where r_h is equal to either r_+ or r_{++}. The sign must be taken to be positive when $r_h = r_+$ and negative when $r_h = r_{++}$. As $\rho \to 0$, to leading order in ρ, the metric takes the form (1), where

$$N^\perp_{,\rho} = \pm \frac{1}{2(r_h^2+a^2)} \left.\frac{dH}{dr}\right|_{r_h} \ . \tag{A11}$$

Here the sign follows the same prescription as above. Expression (A11) was used in the derivation of Eq. (13) in the main text.

Appendix B: Spherical case revisited

We assume that the metric becomes spherically symmetric as it approaches the boundary,

$$ds^2 = N^2(\gamma_{,\rho})^2 dt^2 + d\rho^2 + \gamma^2 d\Omega^2 \quad , \tag{B1}$$

where $d\Omega^2$ is the metric of the 2-sphere. The rescaled lapse N and the coefficient γ are functions of ρ only. The boundary is a horizon, whose radius $\rho = \rho_h$ is defined by,

$$\left.\gamma^2_{,\rho}\right|_{\rho_h} = 0 \ , \tag{B2}$$

which means that at a horizon the area of the 2–sphere is extremal. The lapse function, $N^\perp = N|\gamma_{,\rho}|$ vanishes on the horizon. If we redefine the hamiltonian generator,

$$\tilde{\mathcal{H}}_\perp = \mathcal{H}_\perp |\gamma_{,\rho}| \ , \tag{B3}$$

the associated Lagrange multiplier is N, so that the corresponding term in the hamiltonian is

$$\int d^3x (N^\perp \mathcal{H}_\perp) = \int d^3x (N \tilde{\mathcal{H}}_\perp) \ , \tag{B4}$$

where

$$\tilde{\mathcal{H}}_\perp = 2\sin\theta\, |\gamma_{,\rho}| \left(2\gamma\gamma_{,\rho\rho} - 1 + (\gamma_{,\rho})^2 + \frac{3}{l^2}\gamma^2 \right) \ . \tag{B5}$$

The boundary term in the variation of the Hamiltonian (B4) reads,

$$-2 \int d\phi d\theta N \gamma_{,\rho\rho}\ \delta\gamma^2 \Big|_{r_+}^{r_{++}} \ . \tag{B6}$$

To arrive at (B6) we have taken into account the fact that $\gamma_{,\rho}$ is positive at r_+ and it is negative at r_{++}.
Next, by using the constraint equation,

$$\mathcal{H}_\perp = 0 \ , \tag{B7}$$

we rewrite expression (B6) as,

$$-2 \int d\phi d\theta N \left(1 - \frac{3}{l^2}\gamma^2 - (\gamma_{,\rho})^2 \right) \delta\gamma \Big|_{r_+}^{r_{++}} \ , \tag{B8}$$

and further simplify it, by recalling (B2), to read,

$$-2 \int d\phi d\theta N\ \delta \left(\gamma - \frac{\gamma^3}{l^2} \right) \Big|_{r_+}^{r_{++}} \ . \tag{B9}$$

Finally, if we set $N = 1$ at the boundary, we may write the surface integral at the boundary as

$$-\delta U \ , \tag{B10}$$

where the energy U is given by

$$U_+ = 8\pi \left(\gamma - \frac{\gamma^3}{l^2} \right) \ , \tag{B11}$$

if r_{++} is taken as the boundary, and

$$U_{++} = -8\pi \left(\gamma - \frac{\gamma^3}{l^2} \right) \ , \tag{B12}$$

if the boundary is taken, instead, at r_+.
The energy U is conjugated to the Killing time t. The choice $N = 1$ corresponds to a particular normalization of the Killing vector, and it is analogous to taking $N^\perp = 1$ at spatial infinity in the asymptotically flat case. In this analogy, expressions (B11), (B12) correspond to the ADM surface integral

$$U = \oint (g_{ij,i} - g_{ii,j})\, dS_j \ , \tag{B13}$$

evaluated at spatial infinity in rectangular coordinates.
If one evaluates expressions (B11), (B12) for the Schwarzschild-de Sitter solution one finds

$$U_+ = m \ , \tag{B14}$$

and

$$U_{++} = -m \ , \tag{B15}$$

in agreement with the results stated in the main text.
It would be interesting to develop a discussion along the lines of this appendix in the presence of rotation. Straightforward efforts to do so did not prove successful.

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