A Geometric Algorithm to construct new solitons in the $O(3)$ Nonlinear Sigma Model

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Abstract

The $O(3)$ nonlinear sigma model with boundary, in dimension two, is considered. An algorithm to determine all its soliton solutions that preserve a rotational symmetry in the boundary is exhibited. This nonlinear problem is reduced to that of clamped elastica in a hyperbolic plane. These solutions carry topological charges that can be holographically determined from the boundary conditions. As a limiting case, we give a wide family of new soliton solutions in the free $O(3)$ nonlinear sigma model.

MSC: 53C40; 53C50
PACS: 11.10.Lm; 11.10.Ef; 11.15.-q; 11.30.-j; 02.30.-f; 02.40.-k
Keywords: $O(3)$ nonlinear sigma model; Elastica; Boundary value problem; Soliton.

1 Introduction

It is well known the existence of a close link between the differential geometry of surfaces in Euclidean three-space and a wide variety of non-linear phenomena in physics and mathematics. This setting including problems in geometric analysis, continuum mechanics, sigma models, string theories, theory of biological membranes etc. (see as an example [6, 9, 11] and references therein). Many of those problems are related with functionals of the type

$$\mathcal{B}(M) = \int_M F(dN)dA,$$

where M is a surface in $\mathbb{R}^3$ with Gauss map N, F is a certain function and dA is the element of area, on M, of the induced metric from the Euclidean one.

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Now, people are interested in variational problems related with these functionals. In particular, in minimizing them in a given class, $\mathcal{M}$, of surfaces satisfying certain constraints either topological (free Willmore, free $O(3)$ nonlinear sigma model, genus one biological membranes, constant mean curvature surfaces etc.) or/and boundary conditions (Plateau, Willmore with boundary, $O(3)$ nonlinear sigma model with boundary etc.)

On the other hand, the $O(3)$ nonlinear sigma model (NSM) is ubiquitous in physics (see for example [10, 21] and references therein). It is used in a wide ranging of fields from condensed matter physics, (see for example [8, 16]), to high energy physics (see for instance [1, 2]). The NSM, in dimension two, plays an important role in string theories where the model description is applicable. However, it has its own interest in connection with differential geometry, for example, it naturally leads to the appearance of partially integrable almost product structures, [1, 2].

In this note we deal with the two-dimensional NSM, in general with non trivial boundary values. We obtain all the solutions which preserve a considered rotational symmetry in the boundary conditions. To do it, we exhibit an algorithm with two main phases. In the first, that could be called the reduction phase and culminates in the theorem 2, we phase out the search of symmetric solitons in the NSM with rotational symmetric boundary to that of clamped elasticae in a hyperbolic plane. In a second phase, we give the description of the whole space of clamped elasticae in the hyperbolic plane, [13]. The topological charges, that these solitons carry, can be holographically determined from the boundary conditions (see the corollary).

As a limiting case, we discover an ample family of new soliton solutions in the free NSM. In particular, the qualitative behaviour of those solitons obtained by rotation of wavelike elastica, can be explained in the following points:

1. The solutions are topologically twice punctured spheres obtained as surfaces of revolution whose profile curves are wavelike elasticae in the hyperbolic plane.
2. They oscillate along meridians of twice punctured round spheres.
3. The topological charge of these solitons vanishes identically, even though they can be completed to obtain Zoll's surfaces that present two singularities.

2 The boundary free NSM

The elementary fields in the two dimension, boundary free, NSM are $\mathbb{R}^3$-valued unit vector fields on surfaces without boundary. An interesting approach to study this model, in connection with the differential geometry of surfaces in three dimensional Euclidean space, was considered in [17]. In this context, one identifies the unit normal vector field, or, more correctly, the Gauss map of a surface in $\mathbb{R}^3$ with the dynamical variable of the NSM. To be precise, let M be a surface and denote by $I(M, \mathbb{R}^3)$ the space of immersions of M in the Euclidean space, $(\mathbb{R}^3, g =<,>)$. For any $\phi \in I(M, \mathbb{R}^3)$, we have its Gauss map, $N_\phi : M \to \mathbb{S}^2$. Therefore, dN_ϕ denotes the shape operator of ϕ. Now, the field configuration of the NSM can be identified with $I(M, \mathbb{R}^3)$ and the Lagrangian that governs the dynamics of the model,

$\mathcal{D} : I(M, \mathbb{R}^3) \to \mathbb{R}$, measures the total energy of the Gauss mappings, that is,

$$\mathcal{D}(\phi) = \int_M | \, dN_\phi \, |^2 \, dA_\phi,$$

where dA_ϕ denotes the element of area of $(M, \phi^* g)$. The known solutions to the boundary free NSM can be obtained from the theory of surfaces with constant mean curvature. For example, the solitons discovered by A.A.Belavin and A.M.Polyakov, [8], correspond with those surfaces whose Gauss maps are conformal (round spheres and minimal surfaces). Also, the solutions given by S.Purkait and D.Ray, [20], are induced by the family of constant mean curvature helicoids studied by M.P.Do Carmo and M.Dajzer, [12].

We denote by H_ϕ the mean curvature function of $\phi \in I(M, \mathbb{R}^3)$ and put $G_\phi = det(dN_\phi)$ to name the Gaussian curvature of $(M, \phi^*(g))$. The following relationship is classical

$$| \, dN_\phi \, |^2 = 4H_\phi^2 - 2G_\phi.$$

When M is assumed to be compact then, one can use the Gauss-Bonnet theorem to obtain

Theorem 1 *Let M be a compact surface then, $\phi \in I(M, \mathbb{R}^3)$ is a soliton of the NSM if and only if (M, ϕ) is a Willmore surface, that is ϕ a critical point of the action $\mathcal{W} : I(M, \mathbb{R}^3) \to \mathbb{R}$, given by*

$$\mathcal{W}(\phi) = \int_M H_\phi^2 dA_\phi.$$

This result can be used to generate wide families of compact soliton solutions of the free NSM (combine, for example with the classes of Willmore surfaces obtained in [5, 7, 14, 15, 19]).

3 The NSM with boundary

The Willmore functional and the NSM action, both in the free boundary approach, are particular cases of functional $\mathcal{B}(\phi) = \int_M F(dN_\phi) dA_\phi$. From now on, we will deal with the following problem, which can be considered as the NSM with boundary conditions. It has been widely considered along the literature (see for example [3, 4] and references therein).

THE FIRST ORDER BOUNDARY CONDITIONS. We consider the boundary conditions (Γ, N_o), where $\Gamma = \{\gamma_1, \gamma_2, \cdots, \gamma_n\}$ is a finite set of regular closed curves in $\mathbb{R}^3$ with $\gamma_i \cap \gamma_j = \emptyset$ if $i \neq j$, given $x \in \Gamma$, we put $\Gamma'(x) = \gamma_j'(x)$ if $x \in \gamma_j$ to denote the tangent vector field of Γ. Now, N_o is a unit normal vector field along Γ and such that $< N_o(x), \Gamma'(x) >= 0$, $\forall x \in \Gamma$. In this setting, we have a vector field, ν along Γ determined by $\Gamma'(x) \wedge \nu(x) = N_o(x)$, $\forall x \in \Gamma$.

THE BOUNDARY VALUE PROBLEM. Let M be a differentiable surface with boundary $\partial M = c_1 \cup c_2 \cdots \cup c_n$. We denote by $I_\Gamma^D(M, \mathbb{R}^3)$ the space of immersions, $\phi : M \to \mathbb{R}^3$, that satisfy the following boundary conditions

1. *$\phi(\partial M) = \Gamma$, or $\phi(c_j) = \gamma_j$, $1 \leq j \leq n$, and*

2. *$d\phi_p(T_pM)$ is orthogonal to $N_o(\phi(p))$, $\forall p \in \partial M$.*

Now the problem is to study the dynamics of the model $\mathcal{D} : I_\Gamma^D(M, \mathbb{R}^3) \to \mathbb{R}$ given by

$$\mathcal{D}(\phi) = \int_M \mid dN_\phi \mid^2 dA_\phi.$$

Roughly speaking, if we identify each immersion $\phi \in I_\Gamma^D(M, \mathbb{R}^3)$ with its graph $\phi(M)$, viewed as a surface with boundary in $\mathbb{R}^3$, then we propose the study of the Lagrangian $\mathcal{D}$ in the class of surfaces with the same boundary and with the same Gauss map along the common boundary.

The amazing fact is that the NSM with boundary and the Willmore problem with boundary, [22], are equivalent. A result similar to that obtained in the theorem 1 for closed (compact and boundary free) case. To prove this claim we observe that, via the Gauss-Bonnet formula, the NSM action can be written as

$$\mathcal{D}(\phi) = \int_M \mid dN_\phi \mid^2 dA_\phi = 4 \int_M H_\phi^2 dA_\phi + 2 \sum_{i=1}^n \int_{\gamma_i} \kappa_i^\phi ds,$$

where κ_i^ϕ stands for the curvature function of γ_i in $\phi(M)$ endowed with the g-induced metric. However, the functional $\mathcal{L} : I_\Gamma^D(M, \mathbb{R}^3) \to \mathbb{R}$ given by

$$\mathcal{L}(\phi) = \int_{\phi(\partial M)} \kappa^\phi ds = \sum_{i=1}^n \int_{\gamma_i} \kappa_i^\phi ds,$$

is constant on the whole $I_\Gamma^D(M, \mathbb{R}^3)$. In fact, this follows from the stated boundary conditions which imply that the curvature, κ^ϕ, of $\phi(\partial M) = \Gamma$ in $\phi(M)$, endowed with the g-induced metric, actually does not depend on $\phi \in I_\Gamma^D(M, \mathbb{R}^3)$ because κ^ϕ comes from the projection of the boundary acceleration, Γ'', on $d\phi_p(T_pM)$, which is the tangent plane of each $\phi(M)$ because all the immersions have the same Gauss map along the common boundary Γ.
Consequently, the NSM with boundary is equivalent to the following Willmore problem with boundary, [22]

$$\mathcal{W}(\phi) = \int_M H_\phi^2 dA_\phi + \int_{\phi(\partial M)} \kappa^\phi ds.$$

In particular, the class of soliton solutions in the NSM with boundary coincides with the class of Willmore soliton surfaces with boundary.

4 The NSM with rotational symmetry in the boundary

The above boundary conditions are invariant under a rotational group of symmetries, $G = SO(2)$, with axis L if and only if: (1) The boundary, Γ is made up of two circles, $\{\gamma_1, \gamma_2\}$, in parallel planes orthogonal to L. (2) Moreover, the angle that ν (equivalently N_o) makes with the axis, L, is constant along each component of Γ. We denote by $\theta_i \in [0, \pi]$, the angle that ν makes with L along γ_i, $1 \leq i \leq 2$. These are nothing but the angles that the tangent plane, $d\phi_p(T_pM)$, along each admissible immersion makes with the axis. Therefore, the admissible immersions, $\phi \in I^D_\Gamma(M, \mathbb{R}^3)$, that have the same boundary and the same Gauss map along the common boundary, now also satisfy that $N_\phi(\Gamma)$ lie in two circles of $\mathbb{S}^2$.

Reduction of symmetry. We consider the cylinder $M = \mathbb{S}^1 \times [a_1, a_2]$, then $\partial M = c_1 \cup c_2$, where $c_i = \mathbb{S}^1 \times \{a_i\}$, $1 \leq i \leq 2$. In this case, G acts naturally on $I^D_\Gamma(M, \mathbb{R}^3)$ by $(f, \phi) \mapsto f \circ \phi$, $\forall f \in G$. Furthermore, the NSM Lagrangian $\mathcal{D} : I^D_\Gamma(M, \mathbb{R}^3) \to \mathbb{R}$ is obviously G-invariant, i.e., $\mathcal{D}(f \circ \phi) = \mathcal{D}(\phi)$, $\forall f \in G$ and $\forall \phi \in I^D_\Gamma(M, \mathbb{R}^3)$.

The principle of symmetric criticality, [18], can be applied in this setting. It works in the following sense. Let Σ_G be the space of immersions, $\phi \in I^D_\Gamma(M, \mathbb{R}^3)$, which are G-invariant, that is $f \circ \phi = \phi$, $\forall f \in G$, we will refer these immersions as symmetric points. Then, a symmetric point, $\phi \in \Sigma_G$, is a solution in the NSM with rotational symmetric boundary if and only if it is a critical point of $\mathcal{D} : \Sigma_G \to \mathbb{R}$. In other words, the G-invariant solutions of the field equations coincide with the solutions of the G-reduced field equations.

To compute this restriction, first, we need to identify the space Σ_G. For better understanding, we consider the following framework. Let P be a half-plane in $\mathbb{R}^3$ with boundary the straight line L. Put $m_i = \gamma_i \cap P$, $1 \leq i \leq 2$. Now, choose any regular curve, $\alpha : [t_1, t_2] \to P$, with $\alpha(t_i) = m_i$ and $\alpha'(t_i) = \nu(m_i)$, $1 \leq i \leq 2$, and denote by M_α the surface of revolution, in $\mathbb{R}^3$, obtained when rotate α around L. The immersion $\phi \in I^D_\Gamma(M, \mathbb{R}^3)$ such that $\phi(M) = M_\alpha$ obviously lies in Σ_G. Conversely, if $\phi \in \Sigma_G$, then we can regard its image, $\phi(M)$, as a surface of revolution in $\mathbb{R}^3$ obtained when rotate a certain curve, with the obvious first order boundary data, in P around the axis L. Hence, the space Σ_G can be identified with the following class of revolution surfaces

$$\Sigma_G \equiv \{M_\alpha \,/\, \alpha : [t_1, t_2] \to P, \quad \alpha(t_i) = m_i,\, \alpha'(t_i) = \nu(m_i),\, 1 \leq i \leq 2\}.$$

We have proved that the NSM is equivalent to the Willmore one, under any boundary conditions. This, in particular holds for boundary conditions with rotational symmetry. On the other hand, the Willmore action is obviously G-invariant. Hence, both problems are also equivalent when reduced, via G, to the space of symmetric immersions, Σ_G. Then, we need to characterize the critical points of $\mathcal{W} : \Sigma_G \to \mathbb{R}$

Using conformal invariance to reduce variables. The next idea is to exploit the extrinsic conformal invariance of the Willmore model with boundary. We take L to be the z-axis and choose P as the right-half-plane in the $\{x, z\}$-plane. The Euclidean space $(\mathbb{R}^3 - L, g)$ can be viewed as the warped product, $P \times_h \mathbb{S}^1$, here P is endowed with its Euclidean metric, g_o and the warping function $h : P \to \mathbb{R}$ measures the distance to the axis, L. To characterize the critical points in the NSM with rotational symmetry in the boundary, we use the conformal invariance in the Willmore variational one. Therefore, in $(\mathbb{R}^3 - L, g)$, we make the conformal change with conformal factor $\frac{1}{h^2}$, that is, $\bar{g} = \frac{1}{h^2} g = \frac{1}{h^2} g_o + dt^2$. Then, $(\mathbb{R}^3 - L, \bar{g})$ is the Riemannian product of the hyperbolic plane, $(P, \frac{1}{h^2} g_o)$ and the unit circle, $\mathbb{S}^1$. We denote with overbars the corresponding objects in the new metric, for example M_α is a surface of revolution with boundary $(\mathbb{R}^3 - L, g)$ while $\bar{M}_\alpha$ is a tube (a Riemannian product) with boundary in $(\mathbb{R}^3 - L, \bar{g})$. Then

$$\mathcal{W}(M_\alpha) = \int_{M_\alpha} H_\alpha^2 dA_\alpha + \int_{\partial M_\alpha} \kappa^\alpha ds = \bar{\mathcal{W}}(\bar{M}_\alpha) = \int_{\bar{M}_\alpha} \left(\bar{H}_\alpha^2 + \bar{R}_\alpha\right) d\bar{A}_\alpha + \int_{\partial \bar{M}_\alpha} \bar{\kappa}^\alpha d\bar{s},$$

where $\bar{R}_\alpha$ denotes the sectional curvature of $\bar{g}$ along the surface $\bar{M}_\alpha$ (notice that the corresponding term in the original metric vanishes identically because g is Euclidean and so flat) and $\bar{\kappa}^\alpha$ is the curvature of $\partial \bar{M}_\alpha$ in $\bar{M}_\alpha$.

Now, $\bar{R}_\alpha = 0$ because it is a part of the mixed sectional curvature in a Riemannian product and this vanishes identically. On the other hand, the parallels of tubes are geodesics and so $\bar{\kappa}^\alpha = 0$. Therefore, we have

$$\mathcal{W}(M_\alpha) = \int_{M_\alpha} H_\alpha^2 dA_\alpha + \int_{\partial M_\alpha} \kappa^\alpha ds = \bar{\mathcal{W}}(\bar{M}_\alpha) = \int_{\bar{M}_\alpha} \bar{H}_\alpha^2 d\bar{A}_\alpha = \frac{\pi}{2} \int_\alpha \bar{\kappa}^2 d\bar{s},$$

where $\bar{\kappa}$ is nothing but the curvature function of α in the hyperbolic plane P. Notice that to obtain the last equality we have used that parallels are also geodesics in $(\mathbb{R}^3 - L, \bar{g})$.

Hence, the NSM with rotational symmetry in the boundary is reduced to that for clamped elastica in the hyperbolic plane. To be precise, in the hyperbolic plane, $(P, \frac{1}{h^2} g_o)$ we choose two points, m_1 and m_2, unit vectors $\nu_i(m_i) \in T_{m_i} P$ and the space of clamped curves, $\Lambda = \{\alpha : [t_1, t_2] \to P, \alpha(t_i) = m_i, \alpha'(t_i) = \nu_i(m_i), 1 \le i \le 2\}$, and then the variational problem associated with the total elastic energy, $\mathcal{E} : \Lambda \to \mathbb{R}$, given by

$$\mathcal{E}(\alpha) = \int_\alpha \bar{\kappa}^2 ds.$$

Theorem 2. *The solitons, in the NSM with rotational symmetry in the boundary, that preserve this symmetry, correspond with the surfaces of revolution obtained by rotation of clamped elastic curves in the hyperbolic plane, P. That is curves that are solutions of the field equations associated with the boundary valued problem $\mathcal{E} : \Lambda \to \mathbb{R}$.*

Corollary. *The solitons in the NSM with rotational symmetry in the boundary carry topological charges that can be determined, holographically from the boundary conditions, to be $2\pi(\sin\theta_1 + \sin\theta_2)$.*

Proof. With the choice $\xi = (0, 0, 1)$ and P as the right-half-plane in the $\{x, z\}$-plane, we have

$$\gamma_i(s) - p_i = (r_i \cos \frac{s}{r_i}, r_i \sin \frac{s}{r_i}, 0).$$

Now, from $\cos\theta_i =< \nu_i(s), \xi >$, we obtain

$$\nu_i(s) = (\sin\theta_i \cos \frac{s}{r_i}, \sin\theta_i \sin \frac{s}{r_i}, \cos\theta_i).$$

In particular, we have $m_i = (r_i, 0, 0)$, $\nu_i(m_i) = (\sin\theta_i, 0, \cos\theta_i)$ and so the curvature of the parallel γ_i in M_α is $\kappa_i^\alpha = -\frac{\sin\theta_i}{r_i}$. Consequently the topological charge carried by M_α is

$$Q(M_\alpha) = \int_{M_\alpha} G_\alpha dA_\alpha = -\sum_{i=1}^{2} \int_{\gamma_i} \kappa_i^\alpha(s) ds = 2\pi(\sin\theta_1 + \sin\theta_2).$$

5 Clamped elasticae in Hyperbolic plane

In this section we conclude the algorithm by describing the moduli space of clamped elastic curves in the hyperbolic plane.

In $(P, \frac{1}{h^2} g_o)$, assumed to have Gaussian curvature -1, we consider the space of clamped curves, $\Lambda = \{\alpha : [t_1, t_2] \to P,\, \alpha(t_i) = m_i,\, \alpha'(t_i) = \nu(m_i),\, 1 \le i \le 2\}$, and the action $\mathcal{E} : \Lambda \to \mathbb{R}$, given by

$$\mathcal{E}(\alpha) = \int_\alpha \kappa^2 ds.$$

We use the standard terminology (see [13] for details) to obtain the following first variation formula for $\mathcal{E}$

$$\delta\mathcal{E}(\alpha)[W] = \int_\alpha < \Omega(\alpha), W > ds + [\mathcal{R}(\alpha, W)]_{t_1}^{t_2},$$

where $\Omega(\alpha)$ and $\mathcal{R}(\alpha, W)$ stand for the Euler-Lagrange and Boundary operators, respectively, and they are given by

$$\Omega(\alpha) = 2\nabla_T^3 T + 3\nabla_T \kappa^2 T - 2\nabla_T T,$$

$$\mathcal{R}(\alpha, W) = 2 < \nabla_T W, \nabla_T T > - < W, \nabla_T^2 T + 3\kappa^2 T >,$$

where ∇ denotes the Levi-Civita connection of the metric $<,>= \frac{1}{h^2} g_o$ in P, T is unit tangent vector field of α and $W \in T_\alpha\Lambda$.

On the other hand, we can make the following computations along a curve, $\bar{\alpha}$, in Λ with first order data (α, W)

$$W = d\bar{\alpha}(\partial_r), \qquad \nabla_T W = fT + d\bar{\alpha}(\partial_r T),$$

where $f = \partial_r(\log | V |)$. Then, we evaluate these formulas along the curve α by making $r = 0$ and use the first order boundary data to obtains the following values at the endpoints

$$W(t_i) = 0, \qquad \nabla_T W(t_i = f(t_i)\nu(m_i), \qquad 1 \le i \le 2.$$

As a consequence, the boundary operator drops out, $[\mathcal{R}(\alpha, W)]_{t_1}^{t_2} = 0$.

Then, $\alpha \in \Lambda$ is a critical point of the variational problem $\mathcal{E} : \Lambda \to \mathbb{R}$ if and only if $\Omega(\alpha) = 0$ and it happens if and only if the curvature function of α is a solution of the following second order differential equation

$$2\frac{d^2\kappa}{ds^2} + \kappa(\kappa^2 - 2).$$

These curves will be called clamped elasticae in the hyperbolic plane and we will briefly describe them using the free boundary case which was given in [13]. First, notice that this equation admit a couple of constant solutions, geodesics ($\kappa = 0$) and geodesic circles with $\kappa = 2$. The former, when rotate the limiting case, provides a twice punctured round sphere which can be completed to obtain the sphere as solution to the $O(3)$-model, while the later gives an anchor-ring with ratio $\sqrt{2}$.

When searching for non constant solutions, observe that the equation admits a first integral, which after the change of variable $u = \kappa^2$ can be written as follows

$$(u')^2 = P(u), \qquad P(u) = -u(u^2 - 4u - 4A),$$

where $u' = \frac{du}{ds}$. Moreover, a non constant solution, $u = \kappa^2$, must take on values at which $P(u) > 0$. Thus, $P(u)$ has three real roots, say $-a_1, a_2, a_3$, which satisfy $-a_1 \leq a_2 \leq a_3$. Now, the general solution of the equation is expressed in terms of elliptic functions

$$u = u(s) = a_3(1 - q^2 \mathrm{sn}^2(rs, p)),$$

where

$$p^2 = \frac{a_3 - a_2}{a_3 + a_1}, \qquad q^2 = \frac{a_3 - a_2}{a_3}, \qquad r = \frac{\sqrt{a_3 + a_1}}{2}.$$

We distinguish the following possibilities:

(A) If $-a_1 = 0 < a_2 < a_3$, then $0 < p = q = \frac{\sqrt{a_3 - a_2}}{\sqrt{a_3}} < 1$ and $r = \frac{\sqrt{a_3}}{2}$ and then $u(s) = \kappa^2(s) = a_3 \mathrm{dn}^2(rs, p)$. An elastica with this curvature function is said to be orbitlike. The qualitative behaviour of these elasticae was obtained in [13]. They oscillate between two concentric geodesic circles. Every piece of an orbitlike elastica gives a clamped elastica which, when rotate, provides a solution to the NSM. In particular, the class of orbitlike elasticae admits a rational one-parameter subclass of closed elasticae. They provide, by rotation, genus one compact solutions of the free NSM, [14].

(B) If $-a_1 = a_2 = 0 < a_3$, then $p = q = 1$ and $r = 1$ because the Gaussian curvature is -1. Therefore, the elliptic function providing the curvature becomes into a hyperbolic one, i.e. $\kappa(s) = 2\mathrm{sech}(s)$. This elastica is called asymptotically geodesic, it never closes and it has an integral horocycle. Each piece of an asymptotically geodesic elastica gives a clamped elastica which provides a surface of revolution being a solution to the NSM.

(C) If $-a_1 < a_2 = 0 < a_3$, then $\frac{\sqrt{2}}{2} < p < 1$, $q^2 = 1$ and $r = \frac{1}{2}\sqrt{a_3 + a_1}$. In this case the curvature is $\kappa = \sqrt{a_3} \mathrm{cn}(rs, p)$ and the elastica is said to be wavelike. A wavelike elastica can never be closed, however, it oscillates along an axial geodesic. Pieces of wavelike elasticae give clamped elasticae which through rotation provide solutions to the NSM. In particular, in the limit, they give twice punctured genus zero surfaces of revolution which are new solutions to the free NSM. They can be completed to get Zoll surfaces with a couple of antipodal singularities.

6 Conclusions

In this note, we have developed a geometric algorithm to obtain the whole moduli space of rotational solitons in the NSM with rotational symmetry in the boundary. This criterion reduces the search of those solitons to that of clamped elastic curves in a hyperbolic plane.

The main ideas in this method involve the principle of symmetric criticality, the Gauss-Bonnet formula, the extrinsic conformal invariance of the NSM with boundary and the theory of elasticae in a hyperbolic plane.

The algorithm works as follows: We choose the boundary axis of symmetry, L, as the boundary of the hyperbolic plane, P, regarded as the Poincare half plane model with Gaussian curvature -1. Let $\omega : I \subset \mathbb{R} \to \mathbb{R}$, defined to be one of the following functions

$$\begin{aligned}
\omega(s) &= 2a\,\mathrm{dn}(as,p), \quad a \in \mathbb{R}, \quad \text{and} \quad 0 < p < 1, \quad \text{or} \\
\omega(s) &= 2\,\mathrm{sech}(s), \quad \text{or} \\
\omega(s) &= a\,\mathrm{cn}(rs,p), \quad a, r \in \mathbb{R}, \quad \text{and} \quad \sqrt{2}/2 < p < 1.
\end{aligned}$$

We choose $\alpha : I \subset \mathbb{R} \to P$ to be a curve with curvature function ω. For any $[s_1, s_2] \subset I$, we put $\alpha(s_i) = m_i$, $\alpha'(s_i) = v_i$ and θ_i to denote the angle that v_i makes with the axis L, $1 \leq i \leq 2$. In this context, let $M_{s_1}^{s_2}(\alpha)$ be the surface of revolution obtained when rotate $\alpha([s_1, s_2])$ around L. It is obvious that its boundary is $\partial M_{s_1}^{s_2}(\alpha) = \Gamma = \{\gamma_1, \gamma_2\}$, γ_i being the parallel pictured by m_i. It is evident too that the tangent plane $T_p(M_{s_1}^{s_2}(\alpha))$, $p \in \gamma_i$, makes an angle θ_i with L, $1 \leq i \leq 2$. Then, **the Gauss map of $M_{s_1}^{s_2}(\alpha)$ is a soliton of the NSM with boundary $(\Gamma, \theta_1, \theta_2)$. Moreover all the solitons preserving the boundary rotational symmetry are obtained in this way. The charges of these solitons are computed to be $2\pi(\sin\theta_1 + \sin\theta_2)$**.

As limiting cases we also obtain solitons of the free NSM when rotate the whole elastica defined in $\mathbb{R}$. Even if the elastica closes, we get compact solitons as in [14].

Finally, it should be noticed that the NSM with boundary is invariant under conformal changes of the surrounding gravitational field because it is equivalent to the Willmore problem with boundary. This fact can be used to construct other solitons with different symmetry. For example one can obtain Hopf tubes with boundary in the three sphere by lifting clamped elasticae in the two sphere and then to project them, via a suitable stereographic map, to obtain solitons in the NSM with boundary. The details of this construction so as other related ideas will be developed in a forthcoming paper.

Acknowledgements
This work was partially supported by MCYT Grant BFM2001-2871 with FEDER funds.

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