

# Twisted partition functions for ADE boundary conformal field theories and Ocneanu algebras of quantum symmetries

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## Abstract

For every ADE Dynkin diagram, we give a realization, in terms of usual fusion algebras (graph algebras), of the algebra of quantum symmetries described by the associated Ocneanu graph. We give explicitly, in each case, the list of the corresponding twisted partition functions.

We dedicate this article to the memory of our friend Prof. Juan A. Mignaco deceased, 6 June 2001.

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# 1 Introduction

For each ADE Dynkin diagram  $G$ , we consider the corresponding Ocneanu graph  $Oc(G)$ , as given by [16], and build explicitly an algebra (the algebra of quantum symmetries of the given Dynkin diagram) whose multiplication table is encoded by this Ocneanu graph. Using this algebra structure, we obtain explicitly – and easily – the expression of all the twisted partition functions that one may associate with the given Dynkin diagram (one for each vertex of its corresponding Ocneanu graph).

Our first purpose is not to deduce the graph  $Oc(G)$  from  $G$ , since that was already done in [16] (actually the details have never been made available in printed form...), but to give a simple presentation of the corresponding algebra of quantum symmetries. In each case, this algebra will be given in terms of a quotient of the tensor square of the graph algebra (fusion algebra) associated with some particular ADE Dynkin diagram. These algebras  $Oc(G)$  are commutative in almost all cases (for  $D_{2n}$  they involve  $2 \times 2$  matrices).

Our other purpose is to use this structure to obtain explicitly the corresponding “toric matrices” (terminology taken from [6]) and the corresponding “twisted partition functions” (terminology taken from [21] and [22]).

The torus structure of all ADE models has been worked out by A. Ocneanu himself several years ago (unpublished). Explicit expressions for the eight toric matrices of dimension  $6 \times 6$  of the  $D_4$  model can be found in [21] (where they are interpreted physically in terms of the 3-state Potts model) and for the twelve toric matrices of dimension  $12 \times 12$  of the  $E_6$  model in [6] (where one can also find a general method of calculation for these quantities). One of our purposes is to give explicit results, in particular for all exceptional cases, by following the method explained in this last reference [6] and summarized in section 2. Starting from Conformal Field Theory (CFT), another general method for obtaining the structure of these twisted partition functions has been described in the subsequent article [22] which contains closed formulae; we do not use this formalism. Actually, the constructions performed in the sequel avoid, deliberately, the use of CFT concepts.

Again, we insist upon the fact that we take for granted the data given by the Ocneanu graphs themselves; otherwise, we should either have to diagonalize the convolution product of the quantum Racah-Wigner bi-algebra associated with the given ADE diagram, or to solve the problem of finding what are the irreducible elements for the set of “connections” that one can define on a pair of graphs (system of generalized Boltzman weights, see also [24]). This was done by Ocneanu himself. The present paper can be read independently of [6] since all the necessary information is gathered in section 2.

As already stressed, our own presentation, which follows [6], uses neither the language nor the techniques of conformal field theory, but the results themselves can be interpreted in terms of CFT. For instance the toric matrices lead to quantities that can be interpreted in terms of partition functions for boundary conformal field theories in presence of defect lines. The reader interested in those CFT aspects should look at the article ([22]) which contains many results of independent interest and is probably the most complete published work on this subject, in relation with conformal field theories.

For every ADE example, the particular toric matrix associated with the “unit vertex” of the corresponding Ocneanu graph is the usual modular invariant for the associated ADE model (in the classification of [4]), i.e., the corresponding sesquilinear form gives the usual modular invariant partition function. The other partition functions (the non trivially “twisted” ones), those associated with the other points of the Ocneanu graph, are not modular invariant.

It is unfortunately almost impossible to provide a unified (or uniform) treatment for

all ADE diagrams; indeed, all of them are “special”, in one way or another. The  $A_n$  are a bit too “simple” (many interesting constructions just coincide in that case), the  $D_{2n}$  are the only ones to give rise to a non abelian algebra of quantum symmetries, the  $E_7$  does not define a positive integral graph algebra and the  $D_{2n+1}$  do not define any integral graph algebra at all; “only”  $E_6$  and  $E_8$  lead, somehow, to a similar treatment.

The structure of the present paper is as follows: after a first section devoted to a general overview of the theory, we examine separately all types of ADE Dynkin diagrams. In each case, i.e., in each section, after having presented the graph algebra associated with the chosen diagram (when it exists), we describe explicitly the structure of an associative algebra that we can associate with its corresponding Ocneanu graph, express it in terms of (usual) graph algebras and deduce, from this algebra structure, the corresponding toric matrices. In order not to clutter the paper with sparse matrices of big size, we list only the sesquilinear forms – i.e., the twisted partition functions – associated with these toric matrices. For pedagogical reasons we prefer to perform this analysis in the following order:  $A_n$ ,  $E_6$ ,  $E_8$ ,  $D_{2n}$ ,  $D_{2n+1}$ ,  $E_7$ .

## 2 Summary of the algebraic constructions

### 2.1 Foreword

To every pair of **ADE** Dynkin diagrams with the same Coxeter number, one may associate (Ocneanu) an algebra of quantum symmetries. Its elements (also called “connections” on the given pair of graphs) can be added and multiplied in a way analogous to what is done for representation of groups; in particular, this algebra has a unit, and one may consider a set of “irreducible” quantum symmetries, that, by definition, build up a basis of linear generators for this algebra (an analogue of the notion of irreducible representations). Using multiplication, we may also single out, in each case, two (algebraic) generators, usually called “chiral left” and “chiral right” generators, playing the role of fundamental representations for groups: all other irreducible elements can be obtained as linear combinations of products of these two generators. The Ocneanu graph precisely encodes this algebraic structure: its number of vertices is equal to the number of irreducible elements and edges encode multiplication by the two generators.

When the two chosen Dynkin diagrams coincide, we can find another interpretation for the Ocneanu graph (and algebra) of quantum symmetries. This is actually the case of interest, for us, in the present paper. Here is a sketch of the theory. One first considers elementary paths (i.e., genuine paths) on the chosen Dynkin diagram **G**; one then build the Hilbert space **Path(G)** of all paths, by taking linear combinations of elementary paths and declaring that elementary paths are orthogonal. This vector space provides a path model for the Jones algebra associated with **G**. The next step is to consider the vector subspace **EssPath(G)** of essential paths: by definition, they span the intersection of kernels of all Jones projectors; in the classical situation, the essential paths starting from the origin would correspond to projectors on the symmetric representations of **SU(2)** (or of finite subgroups of **SU(2)**). We refer to the paper [7] (to be contrasted with [6]) where a geometrical study of the classical binary polyhedral groups [14] (symmetries of Platonic bodies) is performed, using McKay correspondence [15], by studying paths and essential paths on the affine Dynkin diagrams of type  $E_6^{(1)}$ ,  $E_7^{(1)}$  and  $E_8^{(1)}$ . Essential paths start somewhere (**a**), end somewhere (**b**) and have a certain length (**n**). The finite dimensional vector space **EssPath(G)** is therefore graded by the length **n** of the paths:  $\text{EssPath}(G) = \bigoplus_n \text{EssPath}^n(G)$ . Notice that essential paths are usually linear combinations of elementary paths. We may then build the graded algebra  $\mathcal{A} \doteq \bigoplus_n \text{End}(\text{EssPath}^n(G))$  where each summand is the space of endomorphisms of **EssPath<sup>n</sup>(G)**; they can be explicitly written as square matrices. The algebra **A** is not only an algebra (for the obvious composition **■** of endomorphisms) but also a *bi*-algebra: using concatenation of elementary paths together with the existence of a scalar product on **Path(G)**, one can define a convolution product **✱** on **A**. Details concerning this construction, also due to Ocneanu, and about its interpretation in the case of affine **ADE** diagrams (i.e., in the case of **SU(2)** itself and the usual polyhedral groups) will be found in [8]. **A** is therefore a kind of finite dimensional and quantum analogue of the Racah-Wigner bi-algebra. Being semi-simple for both algebra structures **■** and **✱**, we may decompose **A** as a sum of square matrices (blocks) in two different ways. For the first structure (**■**), which is obvious from its very definition, the corresponding projectors are labeled by **n** (the length of essential paths). For the second structure (**✱**), the blocks are labeled by an index, that we shall call **m**; the Ocneanu algebra is then precisely the algebra spanned by those **m**, i.e., by the corresponding projectors: this is an analogue of the table of multiplication of characters (convolution product) for a finite group. The underlying vector

space of  $\mathbf{A}$  possesses two adapted basis, one is expressed in terms of the “double triangles of Ocneanu” (that we prefer to draw as a “fermionic” diffusion graph with a connecting vertical “photon” line labeled by  $\mathbf{a}$ ), the other in terms of diffusion graphs with horizontal “very thick lines” labeled by  $\mathbf{a}$ , the vertices of the Ocneanu graph. The change of basis between the two adapted basis can be thought of as a duality relation; it is a kind of generalized Fourier transform involving quantum Racah symbols at a particular root of unity depending on the chosen Dynkin diagram. It will also be conceptually important to consider the length  $\mathbf{a}$  as labelling a particular vertex of a  $\mathbf{A}_N$  graph (the first vertex to the left being labeled  $\mathbf{0}$ ).

Several constructions used in our paper can certainly be understood in terms of planar algebras [12] (see also [11]), nets of subfactors [2], [3], or in terms of braided categories [13], but we shall not discuss this here. We do not plan, in the present paper, to give any interpretation of these constructions in terms of standard Hopf algebra constructions: this has not been worked out, yet.

## 2.2 Structure of the following sections

### 2.2.1 The diagram (ADE) and its adjacency matrix

We give the diagram  $\mathbf{G}$  itself, choose a particular labelling for vertices and give the adjacency matrix  $\mathbf{G}$  in a specified basis. We consider vertices  $\sigma_n$  of  $\mathbf{G}$  as would-be irreducible representations for a quantum analogue of a group algebra  $\mathcal{H}_{\mathbf{G}}$  that we do not need to define. We also write down the norm  $\beta$  of  $\mathbf{G}$  (the biggest eigenvalue of  $\mathbf{G}$ ) and the Perron-Frobenius eigenvector  $\mathbf{D}$  (i.e., the normalized eigenvector corresponding to  $\beta$ , with its smallest component, associated with the vertex  $\sigma_0$ , normalized to the integer  $\mathbf{1}$ ). The components of  $\mathbf{D}$  give (by definition) the quantum dimensions of the irreducible representations  $\sigma_n$ . In all cases,  $\beta$  is equal to the  $q$ -number<sup>1</sup>  $[2]_q = q + \frac{1}{q} = 2 \cos(\frac{\pi}{n})$ . This value  $\mathbf{n}$  is, by definition, the Coxeter number of the graph. In all cases, the  $q$ -dimension of  $\sigma_0$  (the marked vertex) is  $[1]_q = 1$ . More information can be gathered, for instance, from the book [10].

We should remember the values of Coxeter numbers for the various ADE Dynkin diagrams:

	$A_n$	$D_{n+2}$	$E_6$	$E_7$	$E_8$
Coxeter number	$n + 1$	$2(n + 1)$	12	18	30

### 2.2.2 The graph algebra of the Dynkin diagram and the quantum table of characters

The next step is to associate with the diagram  $\mathbf{G}$ , when possible, a commutative algebra playing the role of an algebra of characters. This algebra is linearly generated, as a vector space, by the vertices  $\sigma_n$  of  $\mathbf{G}$ . As an associative algebra, it admits a unit  $\sigma_0$  and one generator  $\sigma_1$ , with quantum dimension  $[2]_q$ . The relations of this associative algebra are defined by the graph  $\mathbf{G}$  itself, considered as encoding multiplication by the generator  $\sigma_1$ : the irreducible representations appearing in the decomposition of  $\sigma_1 \sigma_n$ , with  $\sigma_n$  a vertex of  $\mathbf{G}$ , are the neighbours of  $\sigma_n$  on the diagram  $\mathbf{G}$ . We impose, furthermore, that the structure constants of this algebra should be positive integers, as it is the case for irreducible representations of groups or, more generally, of Hopf algebras. It is (almost well) known, since [19] that, for ADE diagrams, the solution to the above problem does not

<sup>1</sup>We define  $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$ , where  $q = \exp(\frac{i\pi}{n})$ .

exist for  $E_7$  and  $D_{odd}$ . For all other  $ADE$  diagrams, there exists a unique solution. This algebra is called the the graph algebra associated with  $G$ , or the fusion algebra associated with  $G$  and sometimes ([25]), the dual Pasquier algebra of  $G$ . Such a commutative algebra is also a “positive integral hypergroup”, or simply an hypergroup, when no confusion arises (see [1] and references therein). We shall denote this algebra by the same symbol as the graph itself, and hope that no confusion with the simple Lie group bearing the same name will arise. Practically, we have to build a multiplication table, the first two rows and columns being already known (multiplication by the unit  $\sigma_0$  and by the generator  $\sigma_1$ ). The table is built in a very straightforward way, by imposing associativity. For instance, in the case of the graph  $A_n$ ,  $n > 4$ , let us calculate,

$$\begin{aligned}\sigma_2\sigma_2 &= (\sigma_1\sigma_1 - \sigma_0)\sigma_2 = \sigma_1\sigma_1\sigma_2 - \sigma_0\sigma_2 \\ &= \sigma_1(\sigma_1 + \sigma_3) - \sigma_2 = \sigma_0 + \sigma_2 + \sigma_2 + \sigma_4 - \sigma_2 = \sigma_0 + \sigma_2 + \sigma_4\end{aligned}$$

In every case (except  $E_7$  and  $D_{odd}$ ) we shall give the multiplication table of the graph algebra. When writing down this table, and in order to save space, we shall drop the symbols  $\sigma$  and refer to the different vertices only by their subscript.

The graph matrix algebra of the  $ADE$  diagram  $G$ , with  $n$  vertices, is a matrix algebra linearly generated by  $n$  matrices of size  $n \times n$  providing a faithful realization of the graph algebra spanned by the  $\sigma_a$ 's. Its construction is straightforward: to  $\sigma_0$  one associates the unit matrix (call it  $G_0$ ) and to the generator  $\sigma_1$  we associate a matrix  $G_1$  equal to the adjacency matrix  $G$  of the diagram; to the other vertices  $\sigma_a$ , expressed in terms of  $\sigma_0$  and  $\sigma_1$  we associate the corresponding matrices  $G_a$  given in terms of  $G_0$  and  $G_1$ . Since these last two matrices are already explicitly known, in order to save space, we shall just give the polynomial expressions giving the  $G_j$  in terms of these two.

In the particular case of  $A_N$  graphs, the fusion matrices  $G_i$  will be also called  $N_i$ .

The  $n$  matrices  $G_a$  commute with one another (they are all polynomials in one and the same  $G_1$ ) and can be simultaneously diagonalized thanks to a matrix  $S_G$ . If the  $\sigma_a$ 's were irreducible representations of a finite group, this matrix  $S_G$  would be the table of characters for this finite group, i.e., the result of the pairing between conjugacy classes and irreducible characters (notice that we do not need to build explicitly the conjugacy classes!). In the present situation,  $S_G$  is the quantum analogue of a table of characters. In the case of  $A_N$  graphs, the same matrix, simply denoted by  $S$ , represents one of the generators of the modular group  $SL(2, \mathbb{Z})$  (Verlinde - Hurwitz representation).

### 2.2.3 Essential matrices and paths

The general definition of essential paths on a graph was defined by A. Ocneanu ([16]), but we do not need this precise definition here because we just need to count these particular paths. We shall nevertheless recall this definition in the appendix. It is enough to know that the general notion of essential paths generalizes the notion of symmetric (or  $q$ -symmetric) representations (at least for those paths starting from the origin). Some general comments and particular cases (diagrams  $E_6$  and  $E_6^{(1)}$ ) can be read in [7] and [6]. The number  $E_a[p, b]$  of essential paths of length  $p$  starting at some vertex  $a$  and ending on the vertex  $b$  is given by  $b$ -th component of the row vector  $E_a(p)$  defined as follows:

- $E_a(0)$  is the (line) vector characterizing the chosen initial vertex
- $E_a(1) \doteq E_a(0) \cdot \mathcal{G}$
- $E_a(p) \doteq E_a(p-1) \cdot \mathcal{G} - E_a(p-2)$

The expression of  $E_a(0)$  depends on the chosen ordering of vertices; it is convenient anyway to set  $E_0(0) = (1, 0, 0, \dots)$  for the unit  $\sigma_0$  of  $\mathbf{G}$ , and  $E_1(0) = (0, 1, 0, \dots)$  for the generator  $\sigma_1$ . For a graph with  $\kappa$  vertices, starting from  $E_a(0)$ , we would obtain in this way  $\kappa$  rectangular matrices  $E_a$  with infinitely many rows (labeled by  $p = 0, 1, 2, \dots$ ) and  $\kappa$  columns (labeled by  $b$ ).

The reader can check that, for Dynkin  $ADE$  diagrams, the numbers  $E_a(p)$  are positive integers provided  $0 \leq p \leq \kappa - 2$  ( $\kappa$  being the Coxeter number of the graph), but this ceases to be true as soon as  $p > \kappa - 2$ . For instance, in the case of the  $E_6$  graph, where  $\kappa = 12$ , we get  $E_0(11) = (0, 0, 0, 0, 0, 0)$ ,  $E_0(12) = (0, 0, 0, 0, -1, 0)$ . This reflects the fact (Ocneanu [16]) that essential paths on these graphs, with a length bigger than  $\kappa - 2$ , do not exist. We call “essential matrices” the  $\kappa$  rectangular  $(\kappa - 1) \times \kappa$  matrices obtained by keeping only the first  $\kappa - 1$  rows of the  $E_a(\cdot)$ 's. For every  $ADE$  diagram, these finite dimensional rectangular matrices will still be denoted<sup>2</sup> by  $E_a$ . The components of the rectangular matrix  $E_a$  are denoted by  $E_a[p, b]$ . Matrix elements of these matrices can be displayed as vertices with three edges labeled by  $a, b, p$  (or, dually, as triangles). Warning: the smallest value for  $p$ , the length of essential paths, is  $0$ , and not  $1$ .

In order to save space, we shall not give explicitly all these matrices  $E_a$ , although they are absolutely crucial for obtaining the next results; however their calculation, using the above recurrence formulae, is totally straightforward, once the matrix  $G_1 \equiv \mathcal{G}$  is known. The pattern of non-zero entries of an essential matrix  $E_a$ , associated with some graph  $\mathbf{G}$ , gives a figure expressing “visually” the structure of the space of essential paths starting from  $a$ . These essential matrices were introduced in [6] as a convenient tool, but the geometrical patterns themselves were first obtained by A.Ocneanu (the essential paths starting from all possible vertices are displayed, for all  $ADE$  graphs, in the appendix of [16]). In the coming sections, we shall only display the essential matrix  $E_0$  that encodes essential paths starting from the origin. Let us finally mention that a list of the rectangular matrices  $E_0$ 's (not the  $E_a$ 's), interpreted in the context of RSOS models, can be found in [23].

When the diagram is not an  $ADE$  but an *affine*  $ADE$ , the essential matrices are no longer of finite size: they have infinitely many rows; they can be interpreted in terms of the classical induction/restriction theory for representations of  $SU(2)$  and its finite subgroups (binary polyhedral groups). See the lecture notes [7] for a study of the corresponding classical geometries, along the above lines.

For Dynkin diagrams of type  $A_N$ , we have the relation  $N_i = N_{i-1} \cdot \mathcal{G} - N_{i-2}$ , and from the above definition of essential matrices, we see that there is no difference, in this case, between the fusion graph matrices  $G_i = N_i$  and the essential matrices  $E_i$ .

Before ending this section, we should point out the fact that since rows of the essential matrices associated with a particular Dynkin diagram  $\mathbf{G}$  have labels  $p$  running from  $0$  to  $\kappa - 1$ , they are therefore also indexed by the vertices  $\tau_p$  of the Dynkin diagram  $A_{\kappa-1}$ . In this way, we can interpret these essential matrices as a kind of quantum analogue of the theory of induction/restriction: irreducible representations of  $A_{\kappa-1}$  can be “reduced” to irreducible representations of  $\mathbf{G}$  (essential matrices can be read “horizontally” in this way) and irreducible representations of  $\mathbf{G}$  can “induce” irreducible representations of  $A_{\kappa-1}$  (essential matrices can be read “vertically” in this way). Rather than displaying the essential matrices, or the corresponding spaces of paths, we shall only give, for each vertex of the graph  $\mathbf{G}$ , the list of induced representations of  $A_{\kappa-1}$ . This information can be deduced immediately from the essential matrix  $E_0$ . In other words, we consider, for each

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<sup>2</sup>not to be confused with the symbol used for the exceptional Dynkin diagrams themselves!



vertex  $\sigma_n$  of  $\mathbf{G}$  an associated quantum vector bundle and decompose the space of its sections into irreducible representations of  $A_{k-1}$ .

#### 2.2.4 Dimensions of blocks for the Racah-Wigner-Ocneanu bi-algebras

The Racah-Wigner-Ocneanu bi-algebra  $\mathcal{A}$  is a direct sum of blocks in two different ways (see section 2.1). Its dimension is obtained either by summing the squares  $d_n^2$  where  $d_n$  is the number of essential paths of length  $n$ , or by summing the squares  $d_x^2$ , where the  $d_x$  are the sizes of the Ocneanu blocks. The integers  $d_n$  are obtained by summing all matrix elements of the row  $n+1$  over all essential matrices  $E_n$  (all vertices  $\mathbf{a}$  of a given diagram). This first calculation is relatively easy.

The integers  $d_x$  giving the number of “vertices” labeled by  $(a, b, x)$  can be obtained from the multiplication table of  $\mathcal{H}_{Oc(\mathbf{G})}$ . If the label  $\mathbf{x}$  of an Ocneanu block is of the type  $\mathbf{a} \otimes \mathbf{b}$ , or a linear combination of such elements (the notation  $\otimes$  is introduced later in the text), and when  $\mathcal{H}_{Oc(\mathbf{G})}$  is commutative and contains two (left and right) subalgebras isomorphic with the graph algebra of the Dynkin diagram  $\mathbf{G}$ , the integers  $d_x = d_{\mathbf{a} \otimes \mathbf{b}}$  can be obtained simply by summing all matrix elements  $(\Sigma_x)^c_d$  of the matrices  $\Sigma_x \doteq G_a G_b$ , where  $G_a$  and  $G_b$  are fusion matrices of the Dynkin diagram  $\mathbf{G}$ . This holds in particular for  $A_N$  and for the exceptional cases  $E_6$  and  $E_8$ . The other cases – in particular the case of  $E_7$  – are slightly more involved. We refer to the corresponding sections.

The knowledge of integers  $d_n$ ,  $d_x$  was implicit in the work of Ocneanu, already presented to several audiences years ago (for instance [18]). The values of  $d_n$  and  $d_x$  were first published, for the  $E_6$  case, in [6] (the treatment of the  $E_8$  case being the same). General results, for all cases, were published in [22]. We take advantage of the explicit realization that we find for the bialgebra  $\mathcal{H}_{Oc(\mathbf{G})}$  to recover easily all the results, including the more difficult  $E_7$  case (see the corresponding section).

We give the integers  $d_n$ ,  $d_x$  and the sums  $\sum d_n$ ,  $\sum d_x$  and  $\sum d_n^2 = \sum d_x^2$ . The equality of squares is a direct consequence of the bi-algebra structure. In most cases (not  $D_{even}$ ) one finds also that  $\sum d_n = \sum d_x$ ; this can be understood as coming from a change of basis in the vector space  $EssPath(\mathbf{G})$ . The equality of sums can actually be also achieved for  $D_{even}$  by performing the summation only on particular classes of elements (see the discussion made in [22].)

#### 2.2.5 The Ocneanu graph corresponding to a Dynkin diagram and its algebra

The Ocneanu graph  $Oc(\mathbf{G})$  associated with a Dynkin diagram  $\mathbf{G}$  was already discussed in the present introduction. As already stated, we take it directly from reference [16].

One of our purposes is to give an explicit presentation for the corresponding algebras, that will be called  $\mathcal{H}_{Oc(\mathbf{G})}$ . In most cases it will be obtained from the tensor square of some graph algebra, by taking the tensor product over a particular subalgebra (not over the complex numbers). The multiplication is the natural one, namely:  $(a_1 \otimes b_1) \times (a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ , and we shall identify  $au \otimes b$  and  $a \otimes ub$ , whenever  $u$  belongs to the particular subalgebra over which the tensor product is taken (we use the notation  $\otimes$ ). In other words we take the quotient of the tensor square of the appropriate graph algebra by the two-sided ideal generated by elements  $0 \otimes u - u \otimes 0$ , where  $0$  is the unit of the graph algebra of  $\mathbf{G}$ . In the cases of  $D_{odd}$  and  $E_7$ , the above construction has to be “twisted”: some elements  $au \otimes b$  have to be identified with  $a \otimes \rho(u)b$ , but  $\rho$  is not the identity map.

In most cases, the graph algebra to be used in the above construction is the graph algebra of  $\mathbf{G}$  itself. In the case of the diagram  $E_7$ , however, one has to use the graph



algebra of  $D_{10}$ . For the diagram  $D_{2n+1}$  one has to use the graph algebra of  $A_{4n-1}$ . For the diagram  $D_{2n}$ , elements of  $\mathcal{H}_{Oc(D_{2n})}$  also involve  $2 \times 2$  matrices.

In general, the elements  $u$  that are used to define the appropriate two-sided ideal belong to a subalgebra  $U$  that admits a complementary subspace  $P$  which is invariant by left and right  $U$ -multiplications (a general feature since the algebra  $U$  is semi-simple). This property implies that elements of  $\mathcal{H}_{Oc(G)}$  can be decomposed into linear combinations of only four types of elements belonging to  $0 \otimes U$ ,  $0 \otimes P$ ,  $P \otimes 0$  and  $P \otimes P$ .

Following Ocneanu terminology, we call “chiral left subalgebra” or “chiral right subalgebra” the subalgebras spanned by left or right generators ( $\sigma_1 \otimes \sigma_0$  or  $\sigma_0 \otimes \sigma_1$ ) and “ambichiral” the intersection of the chiral parts. Left and right subalgebra are respectively described on Ocneanu graphs by fat continuous lines, and fat dashed lines. The thin lines (continuous or dashed) represent right or left cosets.

Warning: A given algebra  $\mathcal{H}_{Oc(G)}$  is, in a sense, already defined by its graph  $Oc(G)$  since the later describes multiplication by the two chiral generators. What we do in this paper is to propose, for all Dynkin diagram  $G$ , an explicit realization of these algebras  $\mathcal{H}_{Oc(G)}$ , in terms of usual graph algebras. In turn, this realization allows us in a simple way to determine all the toric matrices associated with a given diagram (see below). We stress the fact that the quantum graphs  $Oc(G)$  are taken from [16], however the proposed realizations for the algebras  $\mathcal{H}_{Oc(G)}$  are ours.

The number of vertices of  $Oc(G)$  depends very much of the choice of  $G$  itself (for instance  $Oc(E_6)$  contains 12 points,  $Oc(E_7)$  contains 17 points,  $Oc(E_8)$  contains 32 points).

## 2.2.6 Modular invariant partition functions and twisted partition functions

To every vertex  $x$  of the Ocneanu graph  $Oc(G)$  of the Dynkin diagram  $G$ , one associates a particular “toric matrix”  $W_x$ . These matrices are related to the study of paths on the Ocneanu graphs: the matrix element  $(W_x)_{i,j}$  of  $W_x$  gives the number of independent paths leaving the vertex  $x$  of  $Oc(G)$  and reaching the origin  $0 \otimes 0$  of  $Oc(G)$  after having performed  $i$  essential steps (resp.  $j$  essential steps) on the left (resp. right) chiral subgraphs. These matrices have other uses and interpretations (in particular in terms of the cell calculus or in terms of the “chiral modular splitting” ([17]) but this will not be discussed here.

As written in the introduction, these toric matrices were defined and obtained by Ocneanu (unpublished but advertised in several conferences since 1995, for instance [18]). The article [22] gives closed formulae for the determination of these objects, in the language of conformal field theory. One of our purposes, in the present paper, is to find them by another method, which consists in a straightforward generalization of the technique introduced in [6]. This method uses explicitly our realization of the algebras  $\mathcal{H}_{Oc(G)}$  in terms of graph algebras.

Our first step is to compute the appropriate essential matrices (those of the graph associated with the graph algebra involved in the previous step); generally, i.e., not for  $E_7$  or  $D_{odd}$ , these are the  $r$  essential matrices of the graph  $G$  itself. As discussed previously, they are rectangular matrices  $E_a$  of size  $(\kappa - 1) \times r$ . We then construct “reduced essential matrices”  $E_a^{red}$  by keeping only those columns associated with the subalgebra over which the tensor product is taken (i.e., we replace the other entries by 0). These are again rectangular matrices of size  $(\kappa - 1) \times r$ .

We then define matrices  $W[a, b]$  associated with elements  $x = a \otimes b$  of the algebra  $\mathcal{H}_{Oc(G)}$  as square matrices of size  $(\kappa - 1) \times (\kappa - 1)$  by setting

$$W[a, b] \doteq E_a \widetilde{E_b}^{red} \equiv (E_a).transpose(E_b^{red}) = (E_a)^{red}.transpose(E_b^{red})$$

The points  $\blacksquare$  of  $Oc(G)$  are, in general, linear combinations of elements of the type  $a \otimes b$ . The toric matrices  $W_x$  associated with points  $x = \sum a \otimes b$  of  $Oc(G)$  are square matrices of size  $(\kappa - 1) \times (\kappa - 1)$ . They are obtained by setting

$$W_x = \sum W[a, b]$$

In the case of  $E_7$  and  $D_{odd}$  the above construction should be slightly twisted (see the relevant sections for details).

There are several ways to display the results: one possibility is to give the collection of all toric matrices  $W_x = W[a, b]$  with matrix elements  $W[a, b](i, j)$ , another one is to fix  $i$  and  $j$  (with  $1 \leq i, j \leq \kappa - 1$ ) and display the Ocneanu graph itself labeled by the entries  $W[a, b](i, j)$ . For physical reasons (at least for traditional reasons) we prefer to display the corresponding (twisted) partition functions: setting  $\chi \doteq \{\chi_0, \chi_1, \chi_2, \dots, \chi_{\kappa-2}\}$ , we associate with  $W_x = W[a, b]$  a partition function:

$$Z_x \equiv Z[a, b] \doteq \overline{\chi} W[a, b] \chi.$$

To ease the reading of the paper we put all these partition functions in tables to be found at the end of the article. The matrix elements of all  $W[a, b]$  are always positive integers. However, in order to display the results in tables, we had sometimes to group together several terms and introduce minus signs that will disappear if the sesquilinear forms are expanded.

These quantities can be interpreted in terms of twisted partition functions for ADE boundary conformal field theories (see<sup>3</sup>[22] and [20]).

The partition function  $Z[0, 0]$  associated with the origin  $\sigma_0 \otimes \sigma_0$  is the usual modular invariant partition function of Itzykson, Capelli, Zuber. The others are not modular invariant. We should remember, at that point, that the representation of the modular group provided by the usual  $S$  and  $T$  matrices, in the representation of Verlinde-Hurwitz, is usually not effective: for instance in the case of  $E_6$ , where  $\kappa = 12$ , on top of relations  $S^4 = (ST)^3 = 1$ , one gets  $T^{4\kappa=48} = 1$  (and  $T^s \neq 1$  for smaller powers of  $T$ ). The representation actually factorizes through a congruence subgroup of  $SL(2, \mathbb{Z})$  and one obtains a representation of  $SL(2, \mathbb{Z}/48\mathbb{Z})$  (one can check that all the defining relations given in [9] are verified).

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<sup>3</sup>This is also discussed in a very recent preprint [5].

### 2.2.7 Summary of notations

- $\mathbf{G}$  is the chosen Dynkin diagram of type  $ADE$ . It has  $\kappa$  vertices. We also call  $\mathbf{G}$  the fusion algebra (graph algebra) of this Dynkin diagram, when it exists.
- $\mathbf{G}$  is the adjacency matrix of  $\mathbf{G}$ .
- $\kappa$  is the Coxeter number of  $\mathbf{G}$ .
- $q$  is a primitive root of unity such that  $q^{2\kappa} = 1$ .
- $A_{\kappa-1}$  is the graph of type  $A$  with same Coxeter number  $\kappa$  as  $\mathbf{G}$ .
- $N_i = (N_i)_b^a$  are the fusion matrices for the graph algebra (fusion algebra) of  $A_{\kappa-1}$ .
- $G_a = (G_a)_c^b$  are the fusion matrices for the graph algebra (fusion algebra) of  $\mathbf{G}$ , when it exists.
- $S_G$  is a  $\kappa \times \kappa$  matrix that (in all cases but  $E_7$  and  $D_{odd}$ ) diagonalizes simultaneously the  $\kappa$  fusion matrices  $G_a$  of the diagram  $\mathbf{G}$ . When the diagram  $\mathbf{G}$  is of type  $A$ , we just call it  $S$ .
- $E_a = (E_a)_b^i$  are the essential matrices for the graph  $\mathbf{G}$ .
- $F_i = (F_i)_b^a \doteq (E_a)_b^i$  provide a representation of the graph algebra of  $A_{\kappa-1}$ . Matrices  $F_i$  (or  $E_a$ ) describe the couplings between vertices  $a, b$  of  $\mathbf{G}$  and the vertex  $i$  of  $A_{\kappa-1}$ .
- $Oc(\mathbf{G})$  is the Ocneanu graph associated with  $\mathbf{G}$ .
- $\Sigma_x = (\Sigma_x)_b^a$  are matrices describing the dual couplings between vertices  $a, b$  of  $\mathbf{G}$  and the vertex  $x$  of the Ocneanu graph  $Oc(\mathbf{G})$ .
- $W_x = (W_x)_i^j$  are the toric matrices (of size  $(\kappa-1) \times (\kappa-1)$ ) associated with the vertices of  $Oc(\mathbf{G})$ .
- $Z_x$  is the twisted partition function associated with  $W_x$ .

### 3 The $A_n$ cases

#### 3.1 $A_4$

The  $A_4$  Dynkin diagram and its adjacency matrix are displayed below, where we use the following order for the basis:  $\{\tau_0, \tau_1, \tau_2, \tau_3\}$ .

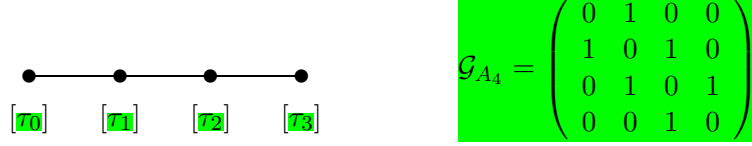


Figure 1: The  $A_4$  Dynkin diagram and its adjacency matrix

Here  $\kappa = 5$  and the norm of the graph is the golden number  $\beta = 2 \cos(\frac{\pi}{5}) = \frac{1+\sqrt{5}}{2}$ , and the normalized Perron-Frobenius vector is  $D = ([1]_q, [2]_q, [2]_q, [1]_q)$ . The  $A_4$  Dynkin diagram determines in a unique way the graph algebra of  $A_4$ , whose multiplication table is displayed below.

	0	1	2	3
0	0	1	2	3
1	1	$0+2$	$1+3$	2
2	2	$1+3$	$0+2$	1
3	3	2	1	0

Table 1: Multiplication table for the  $A_4$  graph algebra

The fusion matrices  $N_i$  are given by the following polynomials:

$$\begin{aligned} N_0 &= Id_4 \text{ (the identity matrix)} & N_2 &= N_1 \cdot N_1 - N_0 \\ N_1 &= \mathcal{G}_{A_4} & N_3 &= N_1 \cdot N_1 \cdot N_1 - 2 \cdot N_1 \end{aligned}$$

They provide a faithful realization of the fusion algebra  $A_4$ . In the chosen basis, they read:

$$N_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad N_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad N_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad N_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

We form the tensor product  $A_4 \otimes A_4$ , whose dimension is 16, but we take it over  $A_4$ . The Ocneanu algebra of  $A_4$  can be realized as the algebra of dimension 4 defined by:

$$\mathcal{H}_{Oc(A_4)} = A_4 \dot{\otimes} A_4 \doteq \frac{A_4 \otimes A_4}{A_4} = A_4 \otimes_{A_4} A_4.$$

It is spanned by a basis with 4 elements:

$$\underline{0} = 0 \dot{\otimes} 0, \quad \underline{1} = 1 \dot{\otimes} 0, \quad \underline{2} = 2 \dot{\otimes} 0, \quad \underline{3} = 3 \dot{\otimes} 0,$$

and is isomorphic to the graph algebra  $A_4$  itself. For this reason, the Ocneanu graph  $Oc(A_4)$  is the same as the Dynkin diagram  $A_4$ . Its elements are of the kind  $m \dot{\otimes} n = 0 \dot{\otimes}$

$mn = mn \otimes 0$ . The dimensions  $d_n$ , with  $n$  in  $(0, 1, 2, 3)$ , for the four blocks of the Racah-Wigner-Ocneanu bi-algebra  $A$  endowed with its first multiplicative law are respectively:  $(4, 6, 6, 4)$ . For its other multiplicative law (convolution), the dimensions  $d_x$  of the four blocks, labeled with  $n$  in the list  $(0 \otimes 0, 1 \otimes 0, 2 \otimes 0, 3 \otimes 0)$  are also respectively:  $(4, 6, 6, 4)$ .

We have of course  $\sum d_n = \sum d_x = 20$  and  $\sum d_n^2 = \sum d_x^2 = 104$  but this observation is trivial in that case.

In the  $A_4$  case (as in all  $A_n$  cases) the essential matrices  $E_i$  happen to be the same as the  $N_i$  matrices. The four toric matrices  $W_{ab}$  of the  $A_4$  model are also equal to the  $N_i$  matrices,  $W_{00} = N_0$  being the modular invariant. We write them as sesquilinear forms (the twisted partition functions given in the appendix).

### 3.2 $A_n$

We display below the Dynkin diagram of  $A_n$ , for  $n > 4$ .

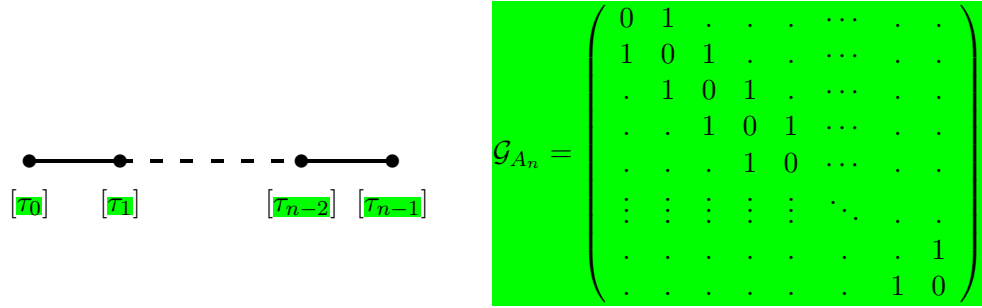


Figure 2: The  $A_n$  Dynkin diagram and its adjacency matrix

In all  $A_n$  cases, the graph algebra is completely determined, in a unique way, by the data of the corresponding Dynkin diagram. The Ocneanu algebra of  $A_n$  can be realized as:

$$\mathcal{H}_{Oc(A_n)} = A_n \otimes A_n \doteq A_n \otimes_{A_n} A_n,$$

and appears to be isomorphic to the graph algebra  $A_n$  itself. Due to this fact, which occurs only in the  $A_n$  cases, the Ocneanu graphs are also equal to the corresponding Dynkin diagrams. The fusion matrices  $N_i$  are given by the following polynomials:

$$\begin{aligned} N_0 &= Id_n \\ N_1 &= \mathcal{G}_{A_n} \\ N_2 &= N_1 \cdot N_1 - N_0 \\ \vdots &= \vdots \\ N_i &= N_{i-1} \cdot N_1 - N_{i-2} \end{aligned}$$

The essential matrices, as well as the  $n$  toric matrices of the  $A_n$  model are equal to these fusion matrices. We just give the modular-invariant in sesquilinear form:

$$A_n : \quad Z_0 = \sum_{i=0}^n |\chi_i|^2 \quad \forall n \geq 3$$

It is easy to see that, for  $A_n$ , the dimensions  $d_p$  of the blocks, for  $p$  from 0 to  $n-1$  are given by  $d_p = (p+1)(n-p)$ .

## 4 The $E_6$ case

The  $E_6$  diagram and its adjacency matrix are displayed below. We use the following order for the vertices:  $\{\sigma_0, \sigma_1, \sigma_2, \sigma_5, \sigma_4, \sigma_3\}$ .

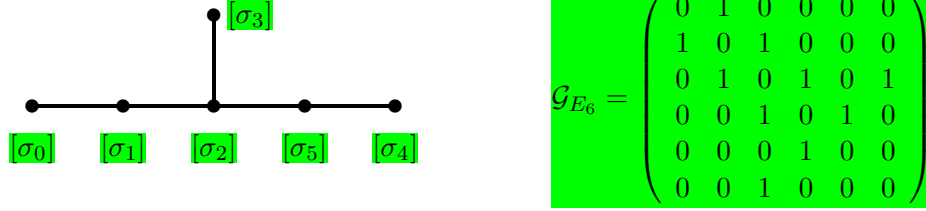


Figure 3: The  $E_6$  Dynkin diagram and its adjacency matrix

Here  $\kappa = 12$ , the norm of the graph is  $\beta = 2 \cos(\frac{\pi}{12}) = \frac{1+\sqrt{3}}{\sqrt{2}}$  and the normalized Perron-Frobenius vector is  $D = ([1]_q, [2]_q, [3]_q, [2]_q, [1]_q, [\frac{3}{2}]_q)$ . The  $E_6$  Dynkin diagram determines in a unique way the multiplication table for the graph algebra of  $E_6$ , displayed below.

	0	1	2	5	4	3
0	0	1	2	5	4	3
1	1	0 + 2	1 + 3 + 5	2 + 4	5	2
2	2	1 + 3 + 5	0 + 2 + 2 + 4	1 + 3 + 5	2	1 + 5
5	5	2 + 4	1 + 3 + 5	0 + 2	1	2
4	4	5	2	1	0	3
3	3	2	1 + 5	2	3	0 + 4

Table 2: Multiplication table for the graph algebra of  $E_6$

The fusion matrices  $G_i$  are given by the following polynomials:

$$\begin{aligned} G_0 &= Id_6 & G_4 &= G_1 \cdot G_1 \cdot G_1 \cdot G_1 - 4G_1 \cdot G_1 + 2G_0 \\ G_1 &= \mathcal{G}_{E_6} & G_5 &= G_1 \cdot G_4 \\ G_2 &= G_1 \cdot G_1 - G_0 & G_3 &= -G_1 \cdot (G_4 - G_1 \cdot G_1 + 2G_0) \end{aligned}$$

Essential matrices have 6 columns and 11 rows. They are labeled by vertices of diagrams  $E_6$  and  $A_{11}$ . They are calculated as explained in section 2.2.3. With the order chosen for vertices (012543) notice that the first row of matrix  $E_5$ , for example, is  $E_5(0) = (000100)$ . The first essential matrix  $E_0$  (essential paths leaving the origin) is given in Fig 4, together with the corresponding induction-restriction graph ( $E_6$  diagram with vertices labeled by  $A_{11}$  vertices).

The subspace  $A_3$  generated by the elements  $\{0, 3, 4\}$  is a subalgebra of the graph algebra of  $E_6$  and leaves invariant (by multiplication) the complementary vector subspace generated by  $\{1, 2, 5\}$ . In other words the subalgebra  $A_3$  of  $E_6$  admits a two-sided  $A_3$ -invariant complement. We form the tensor product  $E_6 \otimes E_6$ , but we take it over the subalgebra  $A_3$  and define the following algebra:

$$\mathcal{H}_{Oc(E_6)} = E_6 \dot{\otimes} E_6 \doteq \frac{E_6 \otimes E_6}{A_3} = E_6 \otimes_{A_3} E_6.$$

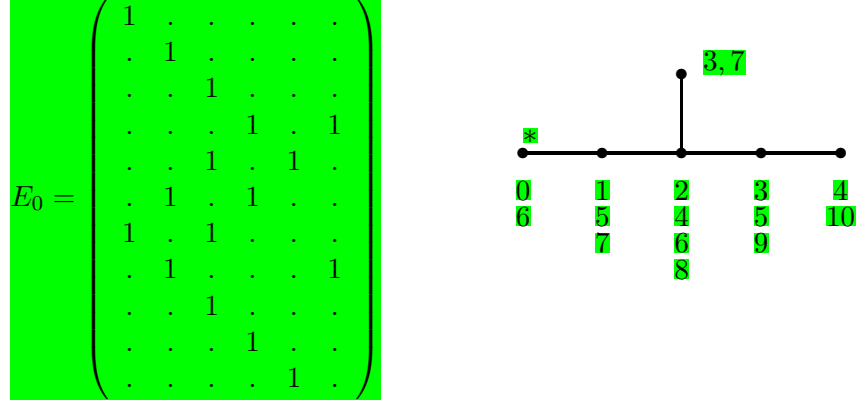


Figure 4: Essential matrix  $E_0$  and Essential Paths from the vertex 0 for the  $E_6$ -model

We have, for example,  $3 \dot{\otimes} 1 = 0 \dot{\otimes} 31 = 0 \dot{\otimes} 2$ , and  $4 \dot{\otimes} 1 = 0 \dot{\otimes} 41 = 0 \dot{\otimes} 5$ .  $\mathcal{H}_{Oc(E_6)}$  is spanned by a basis with 12 elements:

$$\begin{array}{llll} \underline{0} = 0 \dot{\otimes} 0, & \underline{3} = 3 \dot{\otimes} 0, & \underline{1'} = 0 \dot{\otimes} 1, & \underline{31'} = 3 \dot{\otimes} 1, \\ \underline{1} = 1 \dot{\otimes} 0, & \underline{4} = 4 \dot{\otimes} 0, & \underline{11'} = 1 \dot{\otimes} 1, & \underline{41'} = 4 \dot{\otimes} 1, \\ \underline{2} = 2 \dot{\otimes} 0, & \underline{5} = 5 \dot{\otimes} 0, & \underline{21'} = 2 \dot{\otimes} 1, & \underline{51'} = 5 \dot{\otimes} 1. \end{array}$$

The element  $0 \dot{\otimes} 0$  is the identity. The elements  $1 \dot{\otimes} 0$  and  $0 \dot{\otimes} 1$  are respectively the chiral left and right generators; they span separately two subalgebras  $E_6 \otimes 0$  and  $0 \otimes E_6$ , both isomorphic with the graph algebra itself. The ambichiral part is the linear span of  $\{0, 3, 4\}$ . We can easily check that multiplication by generators of  $\mathcal{H}_{Oc(E_6)}$  is indeed encoded by the Ocneanu graph of  $E_6$ , represented in Fig 5. The full lines encode multiplication by the chiral left generator  $\underline{1}$ . For example:  $\underline{1.2} = \underline{1} + \underline{3} + \underline{5}$  and in the  $E_6$  Ocneanu graph the vertices  $\underline{1}, \underline{3}$  and  $\underline{5}$  are joined to the vertex  $\underline{2}$  by a full line. The dashed lines encode multiplication by the chiral right generator  $\underline{1'}$ . For example:  $\underline{1'.4} = \underline{41'}$  and in the  $E_6$  Ocneanu graph the vertices  $\underline{4}$  and  $\underline{41'}$  are joined by a dashed line.

The dimensions  $d_n$ , with  $n$  in  $(0, 1, 2, \dots, 10)$ , for the eleven blocks of the Racah-Wigner-Ocneanu bi-algebra  $\mathcal{A}$  endowed with its first multiplicative law are respectively

$$(6, 10, 14, 18, 20, 20, 20, 18, 14, 10, 6)$$

For its other multiplicative law (convolution), the dimensions  $d_x$  of the twelve blocks, labeled with  $x$  in the list  $(0 \dot{\otimes} 0, 3 \dot{\otimes} 0, 4 \dot{\otimes} 0, 1 \dot{\otimes} 0, 2 \dot{\otimes} 0, 5 \dot{\otimes} 0, 0 \dot{\otimes} 1, 0 \dot{\otimes} 2, 0 \dot{\otimes} 5, 1 \dot{\otimes} 1, 2 \dot{\otimes} 1, 5 \dot{\otimes} 1)$  are respectively

$$(6, 8, 6, 10, 14, 10, 10, 14, 10, 20, 28, 20)$$

Notice that  $\sum d_n = \sum d_x = 156$  and  $\sum d_n^2 = \sum d_x^2 = 2512$ .

The twelve toric matrices  $W_{ab}$  of the  $E_6$  model are obtained as explained in section 2.2.5; for instance  $W_{4 \otimes 1} = E_4 \cdot \widetilde{E_1}^{red}$ . We recall only the matrix expression of  $W_{00}$  (the modular invariant itself). The eleven other matrices<sup>4</sup>, are written as sesquilinear forms in the appendix (they are the twisted partition functions).

<sup>4</sup>They were already given in [6].



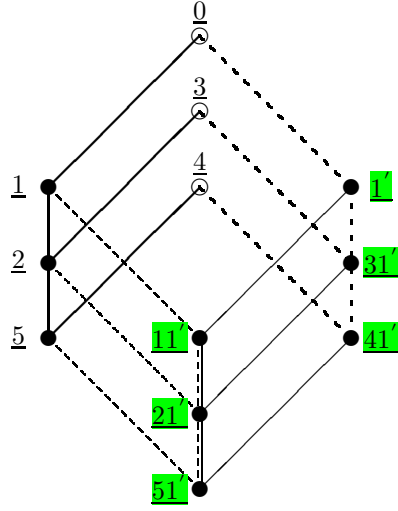


Figure 5: The  $E_6$  Ocneanu graph

$$W_{00} = \begin{pmatrix} 1 & . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & 1 & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . & 1 \\ . & . & . & . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & 1 & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . & 1 \end{pmatrix}$$

## 5 The $E_8$ case

The  $E_8$  Dynkin diagram and its adjacency matrix are displayed below. We use the following order for the vertices:  $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_7, \sigma_6, \sigma_5\}$ .

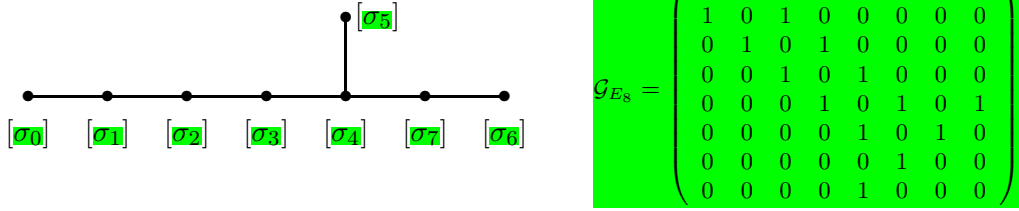


Figure 6: The  $E_8$  Dynkin diagram and its adjacency matrix

Here  $\kappa = 30$ , the norm of the graph is  $\beta = 2 \cos(\frac{\pi}{30})$  and the normalized Perron-Frobenius vector is  $D = ([1]_q, [2]_q, [3]_q, [4]_q, [5]_q, [\frac{7}{2}]_q, [\frac{5}{3}]_q, [\frac{5}{2}]_q)$ . As for the  $E_6$  case, the  $E_8$  Dynkin diagram determines in a unique way the multiplication table for the graph algebra of  $E_8$ , displayed below.

	0	1	2	3	4	7	6	5
0	0	1	2	3	4	7	6	5
1	1	0+2	1+3	2+4	3+5+7	4+6	7	4
2	2	1+3	0+2+4	1+3+5+7	2+4 <sub>2</sub> +6	3+5+7	4	3+7
3	3	2+4	1+3+5+7	0+2+4 <sub>2</sub> +6	1+3 <sub>2</sub> +5+7 <sub>2</sub>	2+4 <sub>2</sub>	3+5	2+4+6
4	4	3+5+7	2+4 <sub>2</sub> +6	1+3 <sub>2</sub> +5+7 <sub>2</sub>	0+2 <sub>2</sub> +4 <sub>3</sub> +6	1+3 <sub>2</sub> +5+7	2+4	1+3+5+7
7	7	4+6	3+5+7	2+4+4	1+3 <sub>2</sub> +5+7	0+2+4+6	1+7	2+4
6	6	7	4	3+5	2+4	1+7	0+6	3
5	5	4	3+7	2+4+6	1+3+5+7	2+4	3	0+4

Table 3: Multiplication table for the graph algebra of  $E_8$

The fusion matrices  $G_i$  are given by the following polynomials:

$$\begin{aligned} G_0 &= Id_8 & G_4 &= G_1.G_1.G_1.G_1 - 3G_1.G_1 + G_0 \\ G_1 &= \mathcal{G}_{E_8} & G_6 &= G_2.G_4 - G_2 - 2G_4 \\ G_2 &= G_1.G_1 - G_0 & G_7 &= G_1.G_6 \\ G_3 &= G_1.G_1.G_1 - 2G_1 & G_5 &= G_2.G_7 - G_3 - G_7 \end{aligned}$$

Essential matrices have 8 columns and 29 rows. They are labeled by vertices of diagrams  $E_8$  and  $A_{29}$ . The first essential matrix  $E_0$  (essential paths leaving the origin) is given in Fig 7, together with the corresponding induction-restriction graph ( $E_8$  diagram with vertices labeled by  $A_{29}$  vertices).

The subspace  $A_2$  generated by the elements  $\{0, 6\}$  is a subalgebra of the graph algebra of  $E_8$  that admits a two-sided  $A_2$ -invariant complement. We form the tensor product  $E_8 \otimes E_8$ , but we take it over the subalgebra  $A_2$ . The Ocneanu algebra of  $E_8$  can be realized as:

$$\mathcal{H}_{Oc(E_8)} = E_8 \dot{\otimes} E_8 \doteq \frac{E_8 \otimes E_8}{A_2} = E_8 \otimes_{A_2} E_8.$$

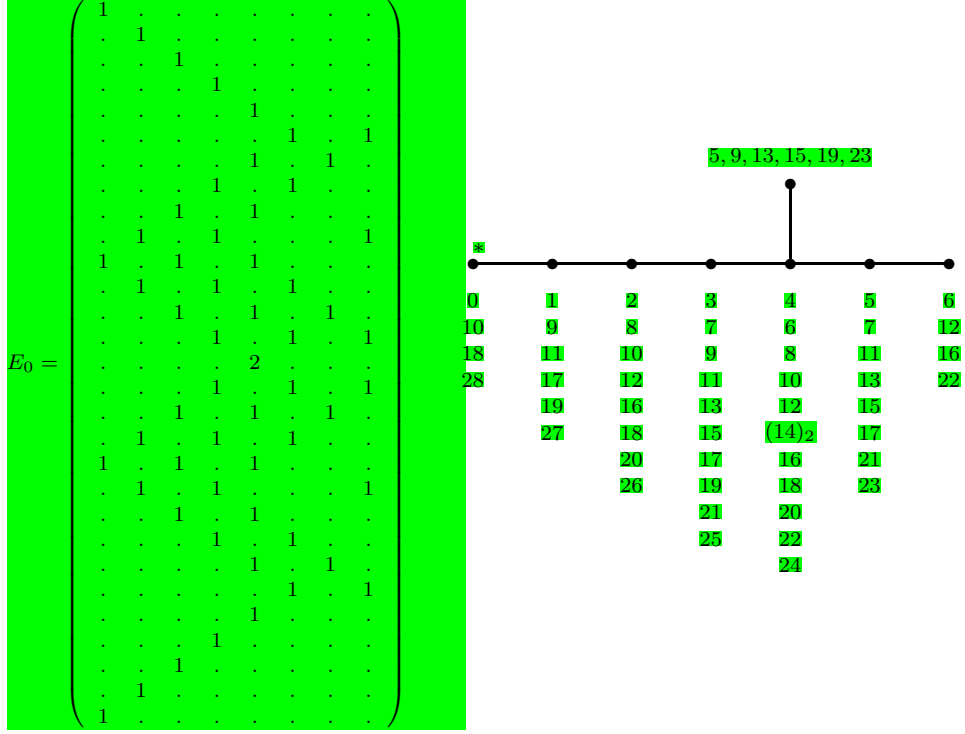


Figure 7: Essential matrix  $E_0$  and Essential Paths from the vertex 0 for the  $E_8$ -model

For instance  $\underline{6} \dot{\otimes} 0 = 0 \dot{\otimes} \underline{6}$ ,  $\underline{6} \dot{\otimes} 1 = 0 \dot{\otimes} \underline{7}$ ,  $\underline{6} \dot{\otimes} 2 = 0 \dot{\otimes} \underline{4}$ ,  $\underline{6} \dot{\otimes} 5 = 0 \dot{\otimes} \underline{3}$ .

$\mathcal{H}_{Oc(E_8)}$  is spanned by a basis with 32 elements:

$\underline{0} = 0 \dot{\otimes} 0,$	$\underline{1}' = 0 \dot{\otimes} 1,$	$\underline{2}' = 0 \dot{\otimes} 2,$	$\underline{5}' = 0 \dot{\otimes} 5,$
$\underline{1} = 1 \dot{\otimes} 0,$	$\underline{11}' = 1 \dot{\otimes} 1,$	$\underline{12}' = 1 \dot{\otimes} 2,$	$\underline{15}' = 1 \dot{\otimes} 5,$
$\underline{2} = 2 \dot{\otimes} 0,$	$\underline{21}' = 2 \dot{\otimes} 1,$	$\underline{22}' = 2 \dot{\otimes} 2,$	$\underline{25}' = 2 \dot{\otimes} 5,$
$\underline{3} = 3 \dot{\otimes} 0,$	$\underline{31}' = 3 \dot{\otimes} 1,$	$\underline{32}' = 3 \dot{\otimes} 2,$	$\underline{35}' = 3 \dot{\otimes} 5,$
$\underline{4} = 4 \dot{\otimes} 0,$	$\underline{41}' = 4 \dot{\otimes} 1,$	$\underline{42}' = 4 \dot{\otimes} 2,$	$\underline{45}' = 4 \dot{\otimes} 5,$
$\underline{5} = 5 \dot{\otimes} 0,$	$\underline{51}' = 5 \dot{\otimes} 1,$	$\underline{52}' = 5 \dot{\otimes} 2,$	$\underline{55}' = 5 \dot{\otimes} 5,$
$\underline{6} = 6 \dot{\otimes} 0,$	$\underline{61}' = 6 \dot{\otimes} 1,$	$\underline{62}' = 6 \dot{\otimes} 2,$	$\underline{65}' = 6 \dot{\otimes} 5,$
$\underline{7} = 7 \dot{\otimes} 0,$	$\underline{71}' = 7 \dot{\otimes} 1,$	$\underline{72}' = 7 \dot{\otimes} 2,$	$\underline{75}' = 7 \dot{\otimes} 5.$

The element  $\underline{0} \dot{\otimes} \underline{0}$  is the identity. The elements  $\underline{1} \dot{\otimes} \underline{0}$  and  $\underline{0} \dot{\otimes} \underline{1}$  are respectively the chiral left and right generators; they span independently the subalgebras  $E_8 \otimes \underline{0}$  and  $\underline{0} \otimes E_8$ . One can easily check that multiplication by these two generators is indeed encoded by the Ocneanu graph of  $E_8$ , represented in Fig 8. Full lines (resp. dashed lines) encode multiplication by the chiral left (resp. chiral right) generator. The ambichiral part is the linear span of  $\{\underline{0}, \underline{6}\}$ .

The dimensions  $d_n$ , with  $n$  in  $(0, 1, 2, \dots, 28)$ , for the twenty nine blocks of the Racah-Wigner-Ocneanu bi-algebra  $\mathcal{A}$  endowed with its first multiplicative law are respectively:

$$(8, 14, 20, 26, 32, 38, 44, 48, 52, 56, 60, 62, 64, 64, 64, 64, 64, 62, 60, 56, 52, 48, 44, 38, 32, 26, 20, 14, 8).$$

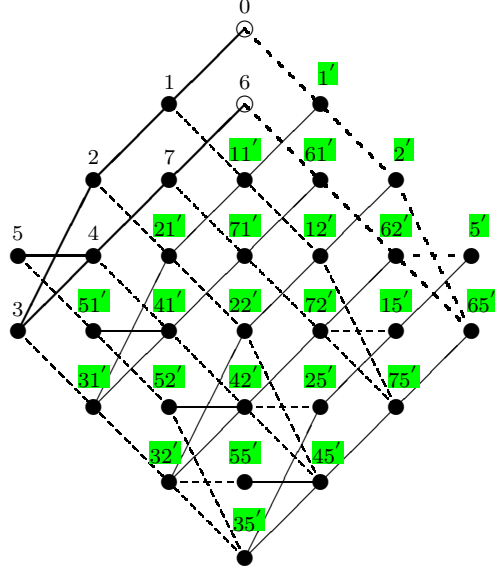


Figure 8: The  $E_8$  Ocneanu graph

For its other multiplicative law (convolution), the dimensions  $d_x$  of the thirty two blocks, labeled with  $x$  in the list  $(0 \otimes 0, 1 \otimes 0, 2 \otimes 0, 3 \otimes 0, 4 \otimes 0, 5 \otimes 0, 6 \otimes 0, 7 \otimes 0, 0 \otimes 1, 1 \otimes 1, 2 \otimes 1, 3 \otimes 1, 4 \otimes 1, 5 \otimes 1, 6 \otimes 1, 7 \otimes 1, 0 \otimes 2, 1 \otimes 2, 2 \otimes 2, 3 \otimes 2, 4 \otimes 2, 5 \otimes 2, 6 \otimes 2, 7 \otimes 2, 0 \otimes 5, 1 \otimes 5, 2 \otimes 5, 3 \otimes 5, 4 \otimes 5, 5 \otimes 5, 6 \otimes 5, 7 \otimes 5)$  are respectively given by:

$(8, 14, 20, 26, 32, 16, 12, 22, 14, 28, 40, 52, 64, 32, 22, 44, 20, 40, 60, 78, 96, 48, 32, 64, 16, 32, 48, 64, 78, 40, 26, 52).$

Notice that  $\sum d_n = \sum d_x = 1240$  and  $\sum d_n^2 = \sum d_x^2 = 63136$ .

The thirty two toric matrices  $W_{ab}$  of the  $E_8$  model are obtained as explained in section 2.2.5 ; for instance  $W_{52'} = W_{5 \otimes 2} = E_5 \cdot \widetilde{E_2}^{red}$ . We recall only the matrix expression of the modular invariant  $W_{00}$ . The partition functions corresponding to all the toric matrices are given as sesquilinear forms in the appendix.

[illegible]

## 6 The $D_{\text{even}}$ case

General formulae valid for all cases of this family are a bit heavy... We therefore only provide a detailed treatment of the cases  $D_4$  and  $D_6$  but generalization is straightforward.

### 6.1 The $D_4$ case

The  $D_4$  diagram and its adjacency matrix are displayed below. We use the following order for the vertices:  $\{\sigma_0, \sigma_1, \sigma_2, \sigma_{2'}\}$ .



Figure 9: The  $D_4$  Dynkin diagram and its adjacency matrix

Here  $\kappa = 6$ , the norm of the graph is  $\beta = 2 \cos(\frac{\pi}{6}) = \sqrt{3}$  and the normalized Perron-Frobenius vector is  $D = \left( [1]_q, [2]_q, \frac{[2]_q}{[2]_q} = 1, \frac{[2]_q}{[2]_q} = 1 \right)$ .

For the  $D_4$  case (as for all the  $D_{2n}$  cases), we have to impose that the structure constants of its graph algebra should be positive integers, in order for the Dynkin diagram to determine in a unique way the multiplication table of the graph algebra, displayed below.

	0	1	2	2'
0	0	1	2	2'
1	1	$0 + 2 + 2'$	1	1
2	2	1	2'	0
2'	2'	1	0	2

Table 4: Multiplication table for the graph algebra of  $D_4$

The fusion matrices  $G_i$  are given by the following polynomials:

$$G_0 = Id_4 \quad G_1 = G_{D_4} \quad G_2 + G_{2'} = G_1 \cdot G_1 - G_0$$

Imposing that entries of  $G_2$  and  $G_{2'}$  should be positive integers leads to a unique solution (up to  $G_2 \leftrightarrow G_{2'}$ ), namely:

$$G_2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad G_{2'} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Essential matrices have 4 columns and 5 rows. They are labeled by vertices of diagrams  $D_4$  and  $A_5$ . The first essential matrix  $E_0$  is given in Fig 10, together with the corresponding induction-restriction graph ( $D_4$  diagram with vertices labeled by  $A_5$  vertices).

The subspace  $J_3$  generated by the elements  $\{0, 2, 2'\}$ , is a subalgebra of the graph algebra of  $D_4$  that admits a two-sided  $J_3$ -invariant complement. We first form the tensor

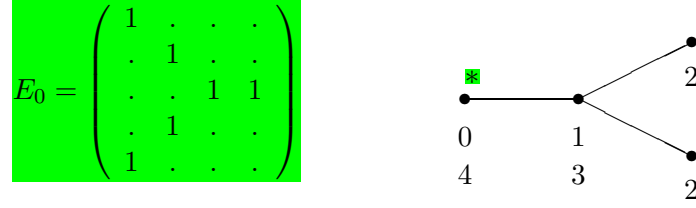


Figure 10: Essential matrix  $E_0$  and Essential Paths from the vertex 0 for the  $D_4$  model

product  $D_4 \otimes D_4$ , but we take it over the subalgebra  $J_3$ . We get the algebra  $D_4^{\dot{\otimes}} = D_4 \otimes D_4 \dot{\otimes} D_4 \otimes_{J_3} D_4$ , spanned by a basis with 6 elements:

$$\underline{0} \dot{\otimes} 0, \quad \underline{1} \dot{\otimes} 0, \quad \underline{2} \dot{\otimes} 0, \quad \underline{2'} \dot{\otimes} 0, \quad \underline{0} \dot{\otimes} 1, \quad \underline{1} \dot{\otimes} 1.$$

The Ocneanu algebra of  $D_4$ ,  $\mathcal{H}_{Oc(D_4)}$ , can be realized as a subalgebra of dimension 8 of the following non-commutative algebra:

$$D_4^{\dot{\otimes}} \oplus M(2, \mathbb{C})$$

The 8 elements of the basis are given by:

$$\begin{aligned} \underline{0} &= 0 \dot{\otimes} 0 & + & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \quad \underline{\epsilon} &= \frac{1}{3}(1 \dot{\otimes} 1) & + & \theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \underline{1} &= 1 \dot{\otimes} 0 & + & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \quad \underline{1\epsilon} &= 0 \dot{\otimes} 1 & + & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \underline{2} &= 2 \dot{\otimes} 0 & + & \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} & \quad \underline{2\epsilon} &= \frac{1}{3}(1 \dot{\otimes} 1) & + & \theta \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \\ \underline{2'} &= 2' \dot{\otimes} 0 & + & \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} & \quad \underline{2'\epsilon} &= \frac{1}{3}(1 \dot{\otimes} 1) & + & \theta \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix} \end{aligned}$$

where:

$$\theta^2 = 0 \quad \text{Grassmann parameter}, \quad \alpha = \frac{-1 + i\sqrt{3}}{2}, \quad \text{and} \quad \beta = \frac{-1 - i\sqrt{3}}{2}.$$

The multiplication in this algebra is defined by:

$$\left( (e1 \dot{\otimes} f1) + A \right) \cdot \left( (e2 \dot{\otimes} f2) + B \right) = (e1.e2) \dot{\otimes} (f1.f2) + A.B,$$

where  $e_1, f_1, e_2, f_2 \in D_4^{\dot{\otimes}}$  and  $A, B \in M(2, \mathbb{C})$ .

The numbers  $\alpha$  and  $\beta$  are determined by the multiplication table of  $\mathcal{H}_{Oc(D_4)}$ . For example, the relations  $\underline{1.1} = \underline{0} + \underline{2} + \underline{2'}$ ,  $\underline{2.2} = \underline{2'}$  and  $\underline{2.2'} = \underline{0}$  lead to the equations:  $\alpha + \beta = -1, \alpha.\beta = 1, \alpha^2 = \beta$  and  $\beta^2 = \alpha$ , that determines uniquely  $\alpha$  and  $\beta$ .

Sketch of our construction: We first define  $D_4^{\dot{\otimes}}$  by quotienting the tensor square of  $D_4$  by the subalgebra  $J_3$  that admits a two-sided  $J_3$ -invariant complement. From the graph  $Oc(D_4)$  taken from [16], we see that  $\underline{1}$  and  $\underline{1\epsilon}$  separately generate the left and right subalgebras isomorphic with the graph algebra of  $D_4$ , therefore we set  $\underline{1} = 1 \dot{\otimes} 0$  and



$\underline{1}\epsilon = 0 \otimes 1$ . We also see that  $\underline{1}\epsilon = \underline{1}\epsilon$ ; this equality implies that the  $\underline{D_4}^{\otimes}$  part of  $\mathfrak{g}$  should be proportional to  $\underline{1} \otimes 1$  since  $(\underline{1} \otimes 0)(\underline{1} \otimes 1) = (0 + 2 + 2') \otimes 1 = 3(0 \otimes 1)$ . The matrix part of  $\mathfrak{g}$  and of the other generators (the coefficients  $\mathfrak{a}$  and  $\mathfrak{b}$ ) can then be determined by imposing that the obtained multiplication table should coincide with the multiplication table constructed from the Ocneanu graph  $\text{Oc}(D_4)$ . Such a construction can be generalized to all  $D_{\text{even}}$  cases.

The element  $\underline{0}$  is the identity. The elements  $\underline{1}$  and  $\underline{1}\epsilon$  are respectively the chiral left and right generators. The multiplication table of this algebra is given in Table 5, and we can check that multiplication by the generators is indeed encoded by the Ocneanu graph of  $D_4$ , represented in Fig 11. Warning: the table is not symmetric (the multiplication is not commutative); for instance  $\underline{2}\epsilon = \underline{2}\epsilon \neq \epsilon\underline{2}$ . The ambichiral part is the linear span of  $\{\underline{0}, \underline{2}, \underline{2}'\}$ .

	$\underline{0}$	$\underline{1}$	$\underline{1}\epsilon$	$\epsilon$	$\underline{2}$	$\underline{2}'$	$\underline{2}\epsilon$	$\underline{2}'\epsilon$
$\underline{0}$	$\underline{0}$	$\underline{1}$	$\underline{1}\epsilon$	$\epsilon$	$\underline{2}$	$\underline{2}'$	$\underline{2}\epsilon$	$\underline{2}'\epsilon$
$\underline{1}$	$\underline{1}$	$\underline{0} + \underline{2} + \underline{2}'$	$\epsilon + \underline{2}\epsilon + \underline{2}'\epsilon$	$\underline{1}\epsilon$	$\underline{1}$	$\underline{1}$	$\underline{1}\epsilon$	$\underline{1}\epsilon$
$\underline{1}\epsilon$	$\underline{1}\epsilon$	$\epsilon + \underline{2}\epsilon + \underline{2}'\epsilon$	$\underline{0} + \underline{2} + \underline{2}'$	$\underline{1}$	$\underline{1}\epsilon$	$\underline{1}\epsilon$	$\underline{1}$	$\underline{1}$
$\epsilon$	$\epsilon$	$\underline{1}\epsilon$	$\underline{1}$	$\underline{\eta}$	$\underline{2}'\epsilon$	$\underline{2}\epsilon$	$\underline{2}'\eta$	$\underline{2}\eta$
$\underline{2}$	$\underline{2}$	$\underline{1}$	$\underline{1}\epsilon$	$\underline{2}\epsilon$	$\underline{2}'$	$\underline{0}$	$\underline{2}'\epsilon$	$\epsilon$
$\underline{2}'$	$\underline{2}'$	$\underline{1}$	$\underline{1}\epsilon$	$\underline{2}'\epsilon$	$\underline{0}$	$\underline{2}$	$\epsilon$	$\underline{2}\epsilon$
$\underline{2}\epsilon$	$\underline{2}\epsilon$	$\underline{1}\epsilon$	$\underline{1}$	$\underline{2}\eta$	$\epsilon$	$\underline{2}'\epsilon$	$\underline{\eta}$	$\underline{2}'\eta$
$\underline{2}'\epsilon$	$\underline{2}'\epsilon$	$\underline{1}\epsilon$	$\underline{1}$	$\underline{2}'\eta$	$\underline{2}\epsilon$	$\epsilon$	$\underline{2}\eta$	$\underline{\eta}$

where  $\underline{\eta} = \frac{1}{3}(\underline{0} + \underline{2} + \underline{2}')$

Table 5: Multiplication table of the Ocneanu algebra of  $D_4$

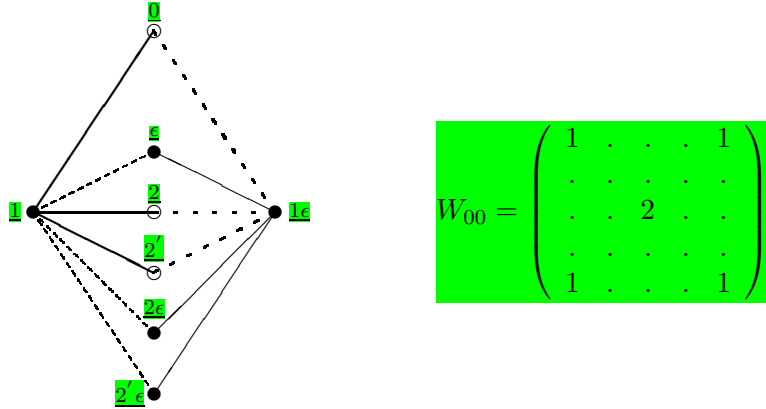


Figure 11: The  $D_4$  Ocneanu graph and the modular invariant matrix

The dimensions  $d_n$ , with  $n = 0, 1, 2, 3, 4$  are respectively  $(4, 6, 8, 6, 4)$ . We find that  $\sum d_n = 28$  and  $\sum d_n^2 = 168$ .

The eight toric matrices  $W_{ab}$  of the  $D_4$  model and the corresponding partition functions are obtained as usual. For instance  $W_\epsilon = W_{\underline{2}\epsilon} = W_{\underline{2}'\epsilon} = \frac{1}{3}E_1\tilde{E}_1^{\text{red}}$ . We recall the matrix expression of the modular invariant  $W_{00}$  and give the others toric matrices as sesquilinear

forms in the appendix<sup>5</sup>.

## 6.2 The $D_6$ case

The  $D_6$  Dynkin diagram and its adjacency matrix are displayed below. We use the following order for the vertices:  $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_{4'}\}$ .

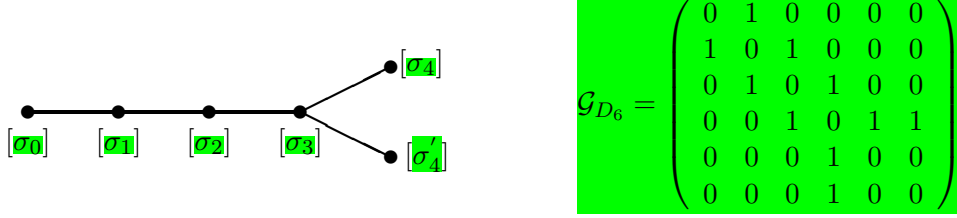


Figure 12: The  $D_6$  Dynkin diagram and its adjacency matrix

Here  $\kappa = 10$ , the norm of the graph is  $\beta = [2]_q = 2 \cos(\frac{\pi}{10}) = \sqrt{\frac{5+\sqrt{5}}{2}}$  and the normalized Perron-Frobenius vector is  $D = ([1]_q, [2]_q, [3]_q, [4]_q, [\frac{4}{2}]_q, [\frac{4}{2}]_q)$ . Imposing positivity, the table of multiplication of the graph algebra of  $D_6$  is completely determined by its Dynkin diagram.

	0	1	2	3	4	4'
0	0	1	2	3	4	4'
1	1	$0+2$	$1+3$	$2+4+4'$	3	3
2	2	$1+3$	$0+2+4+4'$	$1+3+3$	$2+4'$	$2+4$
3	3	$2+4+4'$	$1+3+3$	$0+2+2+4+4'$	$1+3$	$1+3$
4	4	3	$2+4'$	$1+3$	$0+4$	2
4'	4'	3	$2+4$	$1+3$	2	$0+4'$

Table 6: Multiplication table for the graph algebra of  $D_6$

The fusion matrices  $G_i$  are given by the following polynomials:

$$\begin{aligned} G_0 &= Id_6 & G_2 &= G_1 \cdot G_1 - G_0 & G_4 + G_{4'} &= G_1 \cdot G_3 - G_2 \\ G_1 &= \mathcal{G}_{D_6} & G_3 &= G_2 \cdot G_1 - G_1 \end{aligned}$$

Imposing that entries of  $G_4$  and  $G_{4'}$  should be positive integers leads to a unique solution (up to  $G_4 \leftrightarrow G_{4'}$ ), namely:

$$G_4 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad G_{4'} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Essential matrices have 6 columns and 9 rows. They are labeled by vertices of diagrams  $D_6$  and  $A_9$ . The first essential matrix  $E_6$  is given in Fig 13, together with the corresponding induction-restriction graph ( $D_6$  diagram with vertices labeled by  $A_6$  vertices).

<sup>5</sup>The toric matrices of  $D_n$  were already published in [21]

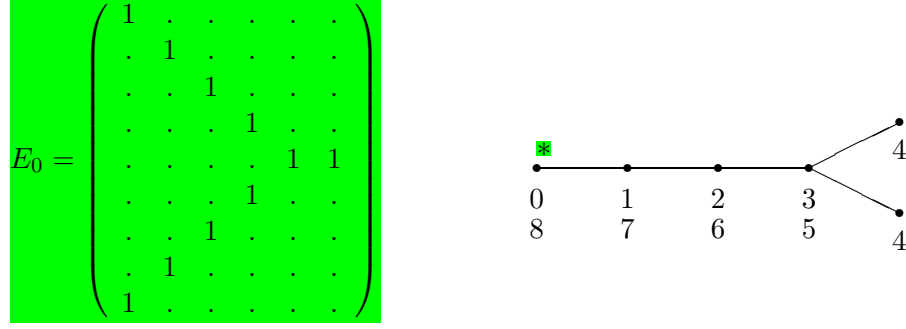


Figure 13: Essential matrix  $E_0$  and Essential Paths from the vertex 0 for the  $D_6$  model

The subspace  $J_4$  generated by the elements  $\{0, 2, 4, 4'\}$  is a subalgebra of the graph algebra of  $D_6$  that admits a two-sided  $J_4$ -invariant complement. We first form the tensor product  $D_6 \otimes D_6$ , but we take it over the subalgebra  $J_4$ . We get the algebra  $D_6^{\otimes} = D_6 \otimes_{J_4} D_6 \doteq D_6 \otimes_{J_4} D_6$ , spanned by a basis with 10 elements:

$$\begin{array}{ccccc} 0 \otimes 0, & 1 \otimes 0, & 2 \otimes 0, & 3 \otimes 0, & 4 \otimes 0, \\ 4' \otimes 0, & 0 \otimes 1, & 0 \otimes 3, & 1 \otimes 1, & 1 \otimes 3. \end{array}$$

The Ocneanu algebra of  $D_6$ ,  $\mathcal{H}_{Oc(D_6)}$ , can be realized as a subalgebra of dimension 12 of the following non-commutative algebra:

$$D_6^{\otimes} \oplus M(2, \mathbb{C})$$

The 12 elements of the basis are given by:

$$\begin{array}{llll} \underline{0} = 0 \otimes 0 & + & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \underline{\epsilon} = \frac{3}{5}(1 \otimes 1) - \frac{1}{5}(3 \otimes 1) & + & \theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \underline{1} = 1 \otimes 0 & + & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \underline{1\epsilon} = 0 \otimes 1 & + & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \underline{2} = 2 \otimes 0 & + & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \underline{2\epsilon} = \frac{2}{5}(1 \otimes 1) + \frac{1}{5}(3 \otimes 1) & + & \theta \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \\ \underline{3} = 3 \otimes 0 & + & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \underline{3\epsilon} = 0 \otimes 3 & + & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \underline{4} = 4 \otimes 0 & + & \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} & \underline{4\epsilon} = -\frac{1}{5}(1 \otimes 1) + \frac{2}{5}(3 \otimes 1) & + & \theta \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \\ \underline{4'} = 4' \otimes 0 & + & \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix} & \underline{4'\epsilon} = -\frac{1}{5}(1 \otimes 1) + \frac{2}{5}(3 \otimes 1) & + & \theta \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix} \end{array}$$

where:

$$\theta^2 = 0, \quad \alpha = \frac{-1 + \sqrt{5}}{2}, \quad \text{and} \quad \beta = \frac{-1 - \sqrt{5}}{2}$$

The element  $\underline{0}$  is the identity. The elements  $\underline{1}$  and  $\underline{1\epsilon}$  are respectively the chiral left and right generators. The multiplication by these generators is encoded by the Ocneanu graph of  $D_6$ , represented in Fig 14. The ambichiral part is the linear span of  $\{0, 2, 4, 4'\}$

The dimensions  $d_n$ , with  $n = 0, 1, 2, \dots, 8$ , are respectively  $(6, 10, 14, 16, 18, 16, 14, 10, 6)$ . Therefore,  $\sum d_n = 110$  and  $\sum d_n^2 = 1500$ .

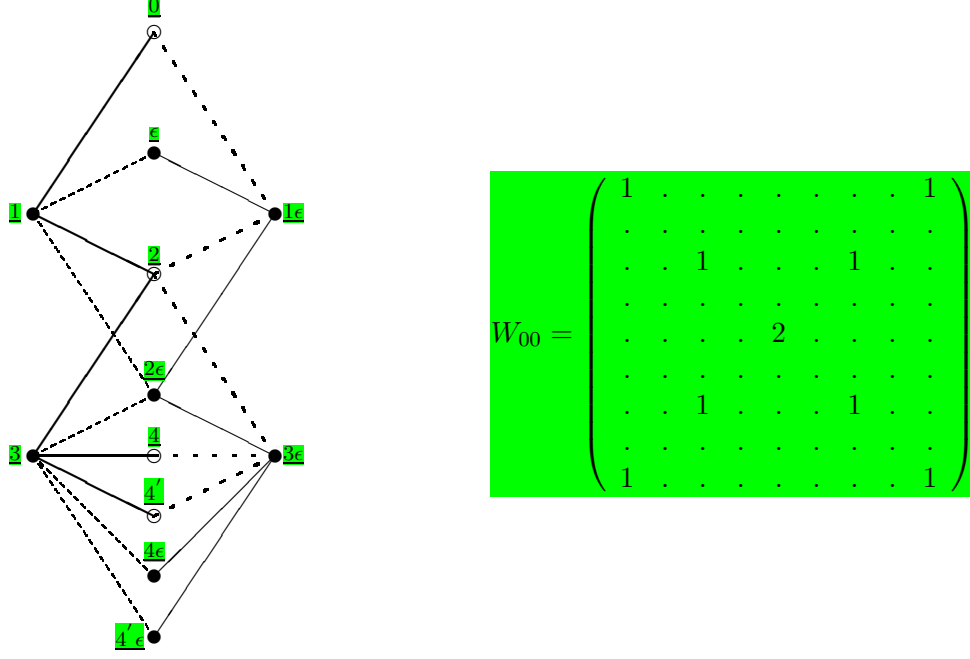


Figure 14: The  $D_6$  Ocneanu graph and the modular invariant matrix

The twelve toric matrices  $W_{ab}$  of the  $D_6$  model and the corresponding partition functions are obtained as usual. For instance  $W_{2\epsilon} = \frac{2}{5}E_1\tilde{E}_1^{red} + \frac{1}{5}E_3\tilde{E}_1^{red}$ . We recall the matrix expression of the modular invariant  $W_{00}$  and give the others as sesquilinears forms in the appendix.

### 6.3 The $D_{even}$ case

In the case of  $D_{2s}$ , we first build  $D_{2s}^\otimes = D_{2s} \otimes D_{2s} / J_{s+1}$ , of dimension  $4s-2$  by dividing the tensor square of  $D_{2s}$  by the two-sided ideal generated by  $u \otimes 0 - 0 \otimes u$ , where  $u$  belongs to the subalgebra  $J_{s+1}$  spanned by  $\{0, 2, 4, 6, \dots, (2s-4), (2s-2), (2s-2)'\}$ . This subalgebra admits a two-sided  $J_{s+1}$ -invariant complement. We then define  $\mathcal{H}_{Oc(D_{2s})}$  as a subalgebra of dimension  $4s$  of  $D_{2s}^\otimes \oplus M(2, \mathbb{C})$ . It is enough to know  $0, 1$  and  $\epsilon$  to build explicitly an algebra  $\mathcal{H}_{Oc(D_{2s})}$  from the graph  $Oc(D_{2s})$ . We fix:

$$0 = 0 \otimes 0 + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad 1 = 1 \otimes 0 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad 1\epsilon = 0 \otimes 1 + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and set:

$$\epsilon = \sum a_\alpha \alpha \otimes 1 + \theta \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where  $\alpha \in \{1, 3, \dots, 2s-3\}$  and where the  $a_\alpha$  are scalars uniquely determined by the (linear) equation  $1\epsilon = 1\epsilon$ . The  $D_{2s}^\otimes$  parts of the other elements are then uniquely fixed.

For the elements  $(2, 3, \dots, (2s-2), (2s-2)')$ , it is:  $(2 \otimes 0, 3 \otimes 0, \dots)$ .

We write the matrix part of  $(2s-2), (2s-2)'$  as:

$$(2s-2) = \dots + \theta \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad (2s-2)' = \dots + \theta \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}$$

By imposing that the obtained multiplication table should coincide with the multiplication table constructed from the Ocneanu graph  $Oc(D_{2s})$ , we determine uniquely the expression of the others elements, and find also the values of  $\theta, \alpha$  and  $\beta$ . In every case  $\theta^2 = 0$ . For  $\alpha$  and  $\beta$  we find:

- s even :  $\alpha$  and  $\beta$  are complexes:

$$\alpha = \frac{-1 + i\sqrt{(2s-1)}}{2} \quad \beta = \frac{-1 - i\sqrt{(2s-1)}}{2} = \bar{\alpha}.$$

- s odd:  $\alpha$  and  $\beta$  are reals:

$$\alpha = \frac{-1 + \sqrt{(2s-1)}}{2} \quad \beta = \frac{-1 - \sqrt{(2s-1)}}{2}.$$

The tables of fusion for the cases s even and s odd have also a different structure, as it is clear from the examples  $D_4$  and  $D_6$  given in the previous sections.

## 7 The $D_{odd}$ case

General formulae valid for all cases of this family are a bit heavy... We therefore only provide a detailed treatment of the cases  $D_5$  and  $D_7$  but generalization is straightforward.

### 7.1 The $D_5$ case

The  $D_5$  Dynkin diagram and its adjacency matrix are displayed below. We use the following order for the vertices:  $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_{3'}\}$ .

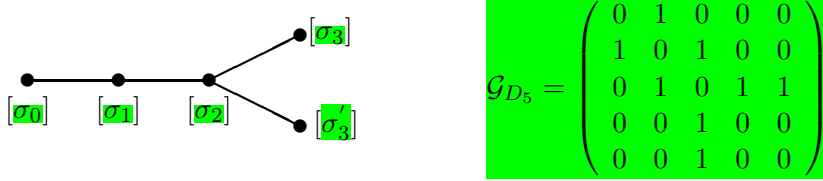


Figure 15: The  $D_5$  Dynkin diagram and its adjacency matrix

Here  $\kappa = 8$ , the norm of the graph is  $\beta = [2]_q = 2 \cos(\frac{\pi}{8}) = \sqrt{2 + \sqrt{2}}$  and the normalized Perron-Frobenius vector is  $D = ([1]_q, [2]_q, [3]_q, [\frac{3}{2}]_q, [\frac{3}{2}]_q)$ .

In the  $D_5$  case, as in all  $D_{odd}$  cases, it is not possible to define a graph algebra at all. Essential matrices of  $D_5$  have 5 columns and 7 rows. They are labeled by vertices of diagrams  $D_5$  and  $A_7$ . The first essential matrix  $E_0$  is given in Fig 16, together with the corresponding induction-restriction graph ( $D_5$  diagram with vertices labeled by  $A_7$  vertices).

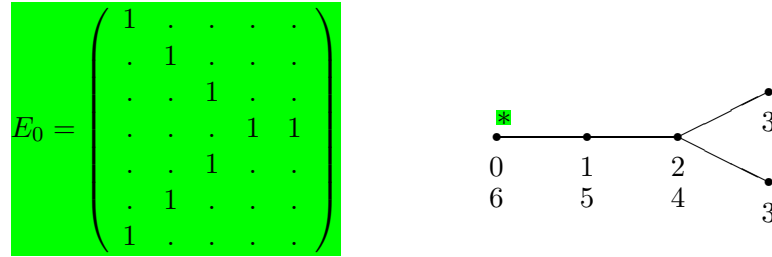


Figure 16: Essential matrix  $E_0$  and Essential Paths from the vertex 0 for the  $D_5$  model

The Ocneanu algebra of  $D_5$  can be realized by using the graph algebra of  $A_7$ . For  $D_{2n+1}$ , we have to use the graph algebra of  $A_{4n-1}$ .

We form the tensor product  $A_7 \otimes A_7$ , and define an application  $\rho : A_7 \rightarrow A_7$  such that:

$$\rho(i) = i \quad \text{for } i \in \{0, 2, 3, 4, 6, 7\} \quad \text{and} \quad \rho(1) = 5, \quad \rho(5) = 1.$$

We take the tensor product over  $\mathbb{Z}$ , and define the Ocneanu algebra of  $D_5$  as:

$$\mathcal{H}_{Oc(D_5)} = A_7 \dot{\otimes} A_7 \doteq \frac{A_7 \otimes A_7}{\rho(A_7)}.$$

For instance  $2 \dot{\otimes} 0 = 0 \dot{\otimes} \rho(2) = 0 \dot{\otimes} 2$ , and  $1 \dot{\otimes} 0 = 0 \dot{\otimes} \rho(1) = 0 \dot{\otimes} 5$ .  $\mathcal{H}_{Oc(D_5)}$  is spanned by a basis with 7 elements:

$$\begin{aligned} \underline{0} &= 0 \dot{\otimes} 0, & \underline{4} &= 4 \dot{\otimes} 0 = 0 \dot{\otimes} 4, \\ \underline{1} &= 1 \dot{\otimes} 0 = 0 \dot{\otimes} 5, & \underline{5} &= 5 \dot{\otimes} 0 = 0 \dot{\otimes} 1, \\ \underline{2} &= 2 \dot{\otimes} 0 = 0 \dot{\otimes} 2, & \underline{6} &= 6 \dot{\otimes} 0 = 0 \dot{\otimes} 6. \\ \underline{3} &= 3 \dot{\otimes} 0 = 0 \dot{\otimes} 3, \end{aligned}$$

$1 \dot{\otimes} 0$  and  $0 \dot{\otimes} 1$  are respectively the chiral left and right generators. The multiplication by these generators is encoded by the Ocneanu graph of  $D_5$ , represented in Fig 17. All the points are ambichiral.

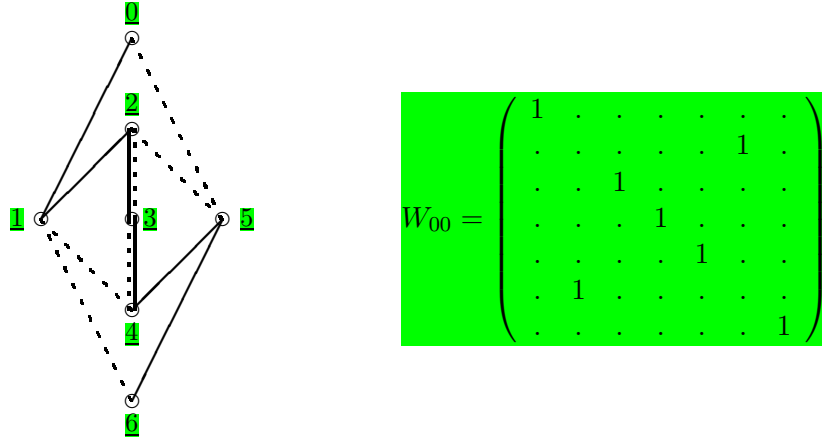


Figure 17: The  $D_5$  Ocneanu graph and the modular invariant matrix

To obtain the toric matrices of the  $D_5$  model, we need the essential matrices  $E_i(A_7)$  of the  $A_7$  case (we recall that in the  $A_n$  cases, the essential matrices are equal to the fusion matrices  $N_i$ ). We define new essential matrices  $E_i^p(A_7)$  by permuting the columns of  $E_i(A_7)$  associated with the vertices 1 and 5. The toric matrices of the  $D_5$  model are then obtained by setting:

$$W[a, b] \doteq E_a(A_7) \cdot (\widetilde{E_b^p(A_7)})$$

We recall the matrix expression of the modular invariant  $W_{00}$  and give the others as sesquilinears forms in the appendix.

## 7.2 The $D_7$ case

The  $D_7$  Dynkin diagram and its adjacency matrix are displayed below. We use the following order for the vertices:  $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_{5'}\}$ .

Here  $\kappa = 12$ , the norm of the graph is  $\beta = [2]_q = 2 \cos(\frac{\pi}{12}) = \frac{1+\sqrt{3}}{\sqrt{2}}$  and the normalized Perron-Frobenius vector is  $D = ([1]_q, [2]_q, [3]_q, [4]_q, [5]_q, \frac{[5]_q}{[2]_q}, \frac{[5]_q}{[2]_q})$ . Essential matrices have 7 columns and 11 rows. They are labeled by vertices of diagrams  $D_7$  and  $A_{11}$ . The first essential matrix  $E_0$  is given in Fig 19, together with the corresponding induction-restriction graph ( $D_7$  diagram with vertices labeled by  $A_{11}$  vertices).



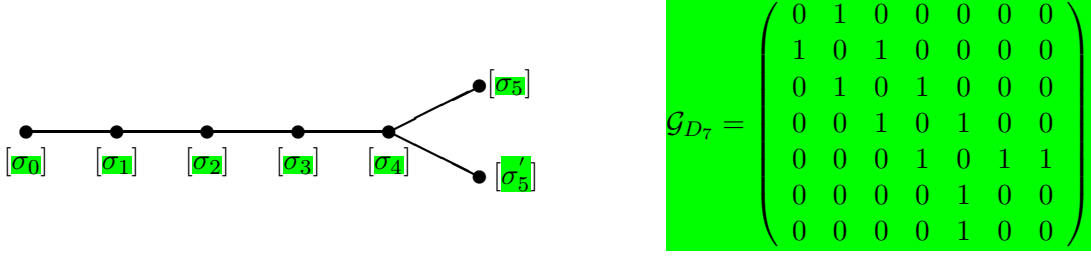


Figure 18: The  $D_7$  Dynkin diagram and its adjacency matrix

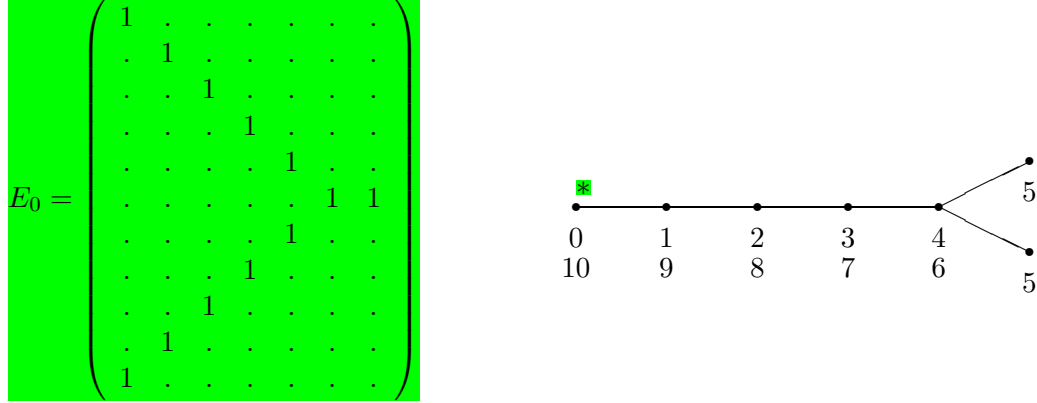


Figure 19: Essential matrix  $E_0$  and Essential Paths from the vertex 0 for the  $D_7$  model

The Ocneanu algebra of  $D_7$  can be realized by using the graph algebra of  $A_{11}$ . We form the tensor product  $A_{11} \otimes A_{11}$ , and define an application  $\rho : A_{11} \rightarrow A_{11}$  such that:

$$\rho(i) = i \quad \text{for} \quad i \in \{0, 2, 4, 5, 6, 8, 10\} \quad \text{and} \quad \rho(1) = 9, \quad \rho(3) = 7, \quad \rho(7) = 3, \quad \rho(9) = 1.$$

We take the tensor product over  $\mathbb{Z}$ , and define the Ocneanu algebra of  $D_7$  as:

$$\mathcal{H}_{Oc(D_7)} = A_{11} \dot{\otimes} A_{11} \doteq \frac{A_{11} \otimes A_{11}}{\rho(A_{11})}.$$

It is spanned by a basis with 11 elements:

$$\begin{array}{lll} \underline{0} = 0 \dot{\otimes} 0 & \underline{4} = 4 \dot{\otimes} 0 = 0 \dot{\otimes} 4 & \underline{8} = 8 \dot{\otimes} 0 = 0 \dot{\otimes} 8 \\ \underline{1} = 0 \dot{\otimes} 0 = 0 \dot{\otimes} 9 & \underline{5} = 5 \dot{\otimes} 0 = 0 \dot{\otimes} 5 & \underline{9} = 9 \dot{\otimes} 0 = 0 \dot{\otimes} 1 \\ \underline{2} = 2 \dot{\otimes} 0 = 0 \dot{\otimes} 2 & \underline{6} = 6 \dot{\otimes} 0 = 0 \dot{\otimes} 6 & \underline{10} = 10 \dot{\otimes} 0 = 0 \dot{\otimes} 10 \\ \underline{3} = 3 \dot{\otimes} 0 = 0 \dot{\otimes} 7 & \underline{7} = 7 \dot{\otimes} 0 = 0 \dot{\otimes} 3 & \end{array}$$

$\underline{1} \dot{\otimes} 0$  and  $0 \dot{\otimes} \underline{1}$  are respectively the chiral left and right generators. The multiplication by these generators is encoded by the Ocneanu graph of  $D_7$ , represented in Fig 20. All the points are ambichiral.

To obtain the toric matrices of the  $D_7$  model, we need the essential matrices  $E_i(A_{11})$  of the  $A_{11}$  case. We define new essential matrices  $E_i^p(A_{11})$  defined by permuting the columns

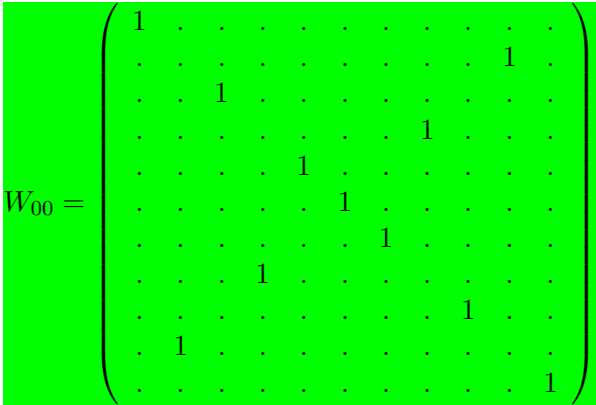


Figure 20: The  $D_7$  Ocneanu graph and the modular invariant matrix

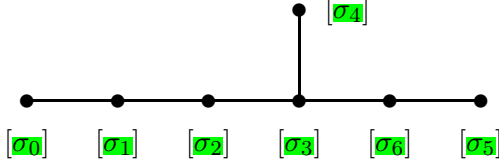
of  $E_i(A_{11})$  associated to the vertices 1 and 9, 3 and 7. The toric matrices of the  $D_7$  model are then obtained by setting:

$$W[a, b] \doteq E_a(A_{11}).(\widetilde{E_b^\rho(A_{11})})$$

We recall the matrix expression of the modular invariant  $W_{00}$  and give the others as sesquilinear forms in the appendix.

## 8 The $E_7$ case

The  $E_7$  Dynkin diagram and its adjacency matrix are displayed below. We use the following order for the vertices:  $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3, \sigma_6, \sigma_5, \sigma_4\}$ .



$$\mathcal{G}_{E_7} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 21: The  $E_7$  Dynkin diagram and its adjacency matrix

Here  $\kappa = 18$ , the norm of the graph is  $\beta = [2]_q = 2 \cos(\frac{\pi}{18})$  and the normalized Perron-Frobenius vector is  $D = ([1]_q, [2]_q, [3]_q, [4]_q, [\frac{6}{2}]_q, [\frac{4}{3}]_q, [\frac{4}{2}]_q)$ . The graph algebra of the Dynkin diagram  $E_7$  is not a positive integral graph algebra. We give it for illustration but it will not be used in the sequel.

$E_7$	0	1	2	3	6	5	4
0	0	1	2	3	6	5	4
1	1	$0 + 2$	$1 + 3$	$2 + 4 + 6$	$3 + 5$	6	3
2	2	$1 + 3$	$0 + 2 + 4 + 6$	$1 + 3_2 + 5$	$2 + 4 + 6$	3	$2 + 6$
3	3	$2 + 4 + 6$	$1 + 3_2 + 5$	$0 + 2_2 + 4 + 6_2$	$1 + 3_2$	$2 + 4$	$1 + 3 + 5$
6	6	$3 + 5$	$2 + 4 + 6$	$1 + 3_2$	$0 + 2 + 6$	$1 + 5$	$2 + 4$
5	5	6	3	$2 + 4$	$1 + 5$	$0 - 4 + 6$	$3 - 5$
4	4	3	$2 + 6$	$1 + 3 + 5$	$2 + 4$	$3 - 5$	$0 + 6$

Table 7: Multiplication table of the graph algebra  $E_7$

The fusion matrices  $G_i$  are given by the following polynomials:

$$\begin{aligned} G_0 &= Id_7 & G_5 &= G_3 \cdot G_2 - G_1 - 2 \cdot G_3 \\ G_1 &= \mathcal{G} & G_4 &= G_5 \cdot G_2 \\ G_2 &= G_1 \cdot G_1 - G_0 & G_6 &= G_5 \cdot G_4 - G_2 \\ G_3 &= G_1 \cdot G_2 - G_1 \end{aligned}$$

Essential matrices of  $E_7$  have 7 columns and 17 rows. They are labeled by vertices of diagrams  $E_7$  and  $A_{17}$ . The first essential matrix  $E_0$  is given below, together with the corresponding induction-restriction graph. To obtain the toric matrices, we also need to know the essential matrices for the  $D_{10}$  case. They are obtained as usual (we also display the essential matrix  $E_6$  of the  $D_{10}$  case below).

We form the tensor product  $D_{10} \otimes D_{10}$ , and identify  $au \otimes b$  with  $a \otimes \rho(u)b$  where

$$\begin{aligned} \rho(0) &= 0, & \rho(4) &= 4, & \rho(8) &= 2, \\ \rho(2) &= 8, & \rho(6) &= 6, & \rho(8') &= 8', \end{aligned}$$

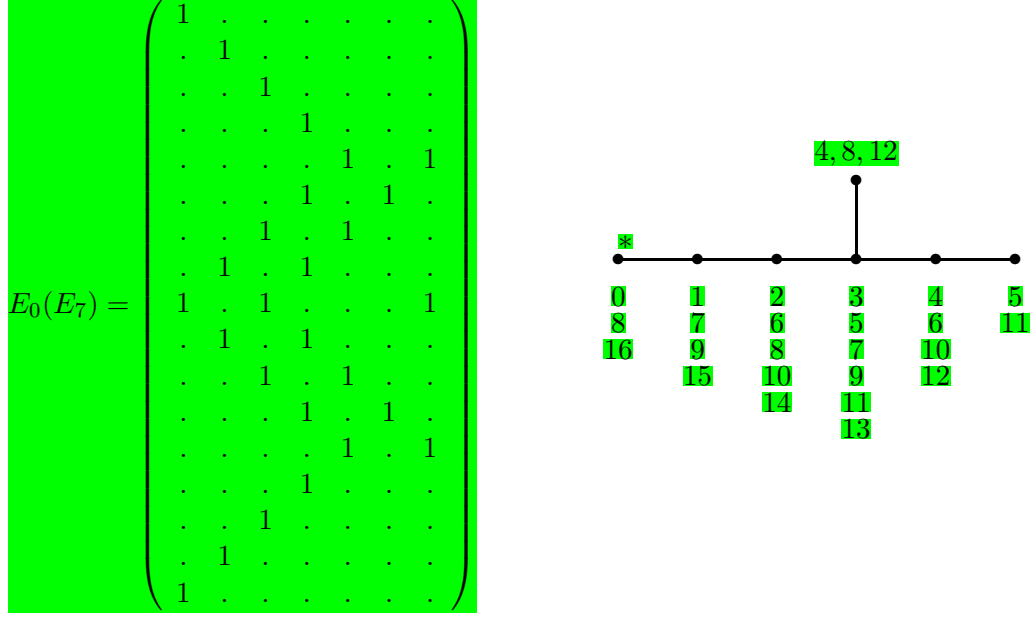


Figure 22: Essential matrix  $E_0$  and Essential Paths from the vertex 0 for the  $E_7$ -model

The Ocneanu algebra of  $E_7$  can be realized as:

$$\mathcal{H}_{Oc(E_7)} = D_{10} \dot{\otimes} D_{10} \doteq \frac{D_{10} \otimes D_{10}}{\rho}.$$

It is spanned by a basis with 17 elements:

$\underline{0} = 0 \dot{\otimes} 0$	$\underline{(0)} = 0 \dot{\otimes} 1$
$\underline{1} = 1 \dot{\otimes} 0$	$\underline{(1)} = 1 \dot{\otimes} 1$
$\underline{2} = 2 \dot{\otimes} 0 = 0 \dot{\otimes} 8$	$\underline{(2)} = 2 \dot{\otimes} 1 = 0 \dot{\otimes} 7$
$\underline{3} = 3 \dot{\otimes} 0$	$\underline{(3)} = 3 \dot{\otimes} 1 = 1 \dot{\otimes} 3$
$\underline{4} = 4 \dot{\otimes} 0 = 0 \dot{\otimes} 4$	$\underline{(4)} = 0 \dot{\otimes} 3$
$\underline{5} = 5 \dot{\otimes} 0$	$\underline{(5)} = 5 \dot{\otimes} 1 - 3 \dot{\otimes} 1$
$\underline{6} = 6 \dot{\otimes} 0 = 0 \dot{\otimes} 6$	$\quad = 1 \dot{\otimes} 5 - 1 \dot{\otimes} 3$
$\underline{7} = 7 \dot{\otimes} 0$	$\underline{(6)} = 0 \dot{\otimes} 5$
$\underline{8} = 8 \dot{\otimes} 0 = 0 \dot{\otimes} 2$	
$\underline{8'} = 8' \dot{\otimes} 0 = 0 \dot{\otimes} 8'$	

$\underline{1}$  and  $\underline{(0)}$  are respectively the left and right generators. The ambichiral part is the linear span of  $\{0, 2, 4, 6, 8, 8'\}$ . The multiplication of the elements of this algebra by the generators is shown in the following table. We can observe on the Ocneanu graph  $Oc(E_7)$  that  $E_7$  does not appear as a subalgebra of  $\mathcal{H}_{Oc(E_7)}$  but as a quotient (there are two such quotients).

The seventeen toric matrices  $W_{ab}$  of the  $E_7$  model are obtained as explained in section 2.2.5, but with a twist: We use the essential matrices  $E_a(D_{10})$ , and replace the matrix

$$E_0(D_{10}) = \begin{pmatrix} 1 & . & . & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & . & 1 & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & . & 1 & 1 & 1 \\ . & . & . & . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & 1 & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . & . \\ . & . & . & 1 & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & . & . & . \\ . & 1 & . & . & . & . & . & . & . & . & . \\ 1 & . & . & . & . & . & . & . & . & . & . \end{pmatrix}$$

Figure 23: Essential matrix  $E_0$  for  $D_{10}$

elements of the columns associated with vertices  $1, 3, 5, 7$  of the graph  $D_{10}$  by  $0$ ; this being done, we permute the columns associated with vertices  $2$  and  $8$  of  $D_{10}$  (with our ordering, these are columns  $3$  and  $9$ ). The reduced and twisted matrix so obtained is called  $E_a^\rho(D_{10})$ . The seventeen toric matrices of the  $E_7$  model are then obtained by setting:

$$W[a, b] \doteq E_a(D_{10}).(\widetilde{E_b^\rho(D_{10})})$$

For instance  $W_{(5)} = W[5, 1] - W[3, 1] = E_5(D_{10}).(\widetilde{E_1^\rho(D_{10})}) - E_3(D_{10}).(\widetilde{E_1^\rho(D_{10})})$ . We recall the matrix expression of the modular invariant  $W_{00}$  and give the others as sesquilinears forms in the appendix.

$$W_{00} = \begin{pmatrix} 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . & . & 1 & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & . & . & . & 1 & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . & . & 1 & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & 1 & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\ 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \end{pmatrix}$$

" $D_{10}$ "	$\underline{1}$	" $D_{10}$ "	$(0)$
$\underline{0}$	$\underline{1}$	$\underline{0}$	$(0)$
$\underline{1}$	$\underline{0} + \underline{2}$	$(0)$	$\underline{0} + \underline{8}$
$\underline{2}$	$\underline{1} + \underline{3}$	$\underline{8}$	$(0) + (4)$
$\underline{3}$	$\underline{2} + \underline{4}$	$(4)$	$\underline{8} + \underline{4}$
$\underline{4}$	$\underline{3} + \underline{5}$	$\underline{4}$	$(4) + (6)$
$\underline{5}$	$\underline{4} + \underline{6}$	$(6)$	$\underline{4} + \underline{6}$
$\underline{6}$	$\underline{5} + \underline{7}$	$\underline{6}$	$(6) + (2)$
$\underline{7}$	$\underline{6} + \underline{8} + \underline{8}'$	$(2)$	$\underline{2} + \underline{6} + \underline{8}'$
$\underline{8}$	$\underline{7}$	$\underline{2}$	$(2)$
$\underline{8}'$	$\underline{7}$	$\underline{8}'$	$(2)$
" $E_7$ "	$\underline{1}$	" $E_7$ "	$(0)$
$(0)$	$(1)$	$\underline{1}$	$(1)$
$(1)$	$(0) + (2)$	$(1)$	$\underline{1} + \underline{7}$
$(2)$	$(1) + (3)$	$\underline{7}$	$(1) + (3)$
$(3)$	$(2) + (4) + (6)$	$(3)$	$\underline{3} + \underline{5} + \underline{7}$
$(4)$	$(3)$	$\underline{3}$	$(3)$
$(6)$	$(3) + (5)$	$\underline{5}$	$(3) + (5)$
$(5)$	$(6)$	$(5)$	$\underline{5}$

Table 8: Multiplication of the elements of the Ocneanu algebra of  $E_7$  by the generators

We conclude this section with the determination of the integers  $d_n$  and  $d_n$ . The integers  $d_n$  associated with endomorphisms of essential paths on the graph  $E_7$  are determined easily by the usual method. For  $n = 1, 2, \dots, 17$  one finds

$$d_n = 7, 12, 17, 22, 27, 30, 33, 34, 35, 34, 33, 30, 27, 22, 17, 12, 7.$$

In order to determine the integers  $d_n$ , we need to know the multiplication table defined by the Ocneanu graph  $Oc(E_7)$ . The full table of  $\mathcal{H}_{Oc(E_7)}$  has the following structure: " $D_{10}$ "  $\times$  " $D_{10}$ "  $\rightarrow$  " $D_{10}$ ", " $D_{10}$ "  $\times$  " $E_7$ "  $\rightarrow$  " $E_7$ ", and " $E_7$ "  $\times$  " $E_7$ "  $\rightarrow$  " $D_{10}$ ", where " $D_{10}$ " is the subalgebra linearly spanned by  $\underline{0}, \underline{1}, \dots, \underline{8}, \underline{8}'$  and " $E_7$ " is the linear subspace linearly spanned by  $(0), (1), \dots, (6)$ . Actually, it is enough, for our purpose, to know the smaller table obtained by restriction to the  $E_7$  quotients, i.e., the " $E_7$ "  $\times$  " $E_7$ "  $\rightarrow$  " $D_{10}$ " table.

We encode this multiplication table by a set of  $10$  matrices  $s_n$  (labeled by  $D_{10}$ , of size  $7 \times 7$  (labeled by  $E_7$ )). The result of a given multiplication, such as  $(4) \times (5) = \underline{3} + \underline{7}$  is indicated by the presence of the integer  $1$  in position  $(4, 5)$  in both matrices  $s_3$  and  $s_7$ .

Since we have an explicit realization of  $\mathcal{H}_{Oc(E_7)}$ , it is not too difficult to find

$$s_0 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad s_1 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$





The sum of matrix elements of the 10 matrices  $s_p$ , for  $p = 0, 1, 2, \dots, 8, 8'$  is 7, 12, 17, 22, 27, 30, 33, 34, 18, 17.

To each linear generator  $x = \sum a \otimes b$  (for instance  $(5) = 5 \otimes 1 - 3 \otimes 1$ ) of the Ocneanu algebra of  $E_7$  (the basis with 17 elements was given previously) we associate a matrix  $\Sigma_x = \sum s_a s_b$  (for instance  $\Sigma_{(5)} = s_5 s_1 - s_3 s_1$ ). The integer  $d_x$  is the sum of matrix elements of the matrix  $\Sigma_x$  (for instance  $d_{(5)} = 16$ ). In particular, the  $d_x$  associated with the " $D_{10}$ " part of the graph are just given by sum of matrix elements of matrices  $s_p$ .

The final list of integers  $d_x$ , associated with blocks  $0, 1, \dots, 8, 8'; (0), (1), \dots, (6)$  is

$$d_x = 7, 12, 17, 22, 27, 30, 33, 34, 18, 17, 12, 24, 34, 44, 22, 16, 30.$$

Note that  $\sum d_n = \sum d_x = 399$  and that  $\sum d_n^2 = \sum d_x^2 = 10905$ .

The above results agree<sup>6</sup> with those obtained by [22].

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<sup>6</sup>The preprint version of [22], available on the web, contains a typing misprint: the last values of  $d_x$  should be read 44, 30, 16, 22 and not 442, 30, 16, 22.

## 9 Acknowledgements

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## A The general notion of essential paths on a graph $\mathbf{G}$

The general definitions given here are adapted from ([16]).

Call  $\beta$  the norm of the graph  $\mathbf{G}$  (the biggest eigenvalue of the adjacency matrix  $\mathbf{G}$ ) and  $\mathbf{D}_i$  the components of the (normalized) Perron-Frobenius eigenvector. Call  $\sigma_i$  the vertices of  $\mathbf{G}$  and, if  $\sigma_j$  is a neighbour of  $\sigma_i$ , call  $\xi_{ij}$  the oriented edge from  $\sigma_i$  to  $\sigma_j$ . If  $\mathbf{G}$  is unoriented (the case for  $\mathbf{ADE}$  and affine  $\mathbf{ADE}$  diagrams), each edge should be considered as carrying both orientations.

An elementary path can be written either as a finite sequence of consecutive (neighbours on the graph) vertices,  $[\sigma_{a_1}\sigma_{a_2}\sigma_{a_3}\dots]$ , or as a sequence  $(\xi(1)\xi(2)\dots)$  of consecutive edges, with  $\xi(1) = \xi_{a_1a_2} = \sigma_{a_1}\sigma_{a_2}$ ,  $\xi(2) = \xi_{a_2a_3} = \sigma_{a_2}\sigma_{a_3}$ , etc. Vertices are considered as paths of length 0.

The length of the (possibly backtracking) path  $(\xi(1)\xi(2)\dots\xi(p))$  is  $p$ . We call  $r(\xi_{ij}) = \sigma_j$ , the range of  $\xi_{ij}$  and  $s(\xi_{ij}) = \sigma_i$ , the source of  $\xi_{ij}$ .

For all edges  $\xi(n+1) = \xi_{ij}$  that appear in an elementary path, we set  $\xi(n+1)^{-1} \doteq \xi_{ji}$ .

For every integer  $n > 0$ , the annihilation operator  $\mathbf{C}_n$ , acting on elementary paths of length  $p$  is defined as follows: if  $p \leq n$ ,  $\mathbf{C}_n$  vanishes and if  $p > n+1$  then

$$\mathbf{C}_n(\xi(1)\xi(2)\dots\xi(n)\xi(n+1)\dots) = \sqrt{\frac{D_{r(\xi(n))}}{D_{s(\xi(n))}}} \delta_{\xi(n),\xi(n+1)^{-1}} (\xi(1)\xi(2)\dots\hat{\xi}(n)\hat{\xi}(n+1)\dots)$$

Here, the symbol “hat” (like in  $\hat{\xi}$ ) denotes omission. The result is therefore either 0 or a linear combination of paths of length  $p-2$ . Intuitively,  $\mathbf{C}_n$  chops the round trip that possibly appears at positions  $n, n+1$ .

A path is called essential if it belongs to the intersection of the kernels of the annihilators  $\mathbf{C}_n$ 's.

For instance, in the case of the diagram  $\mathbf{E}_6$ ,

$$\begin{aligned} C_3(\xi_{01}\xi_{12}\xi_{23}\xi_{32}) &= \sqrt{\frac{1}{[2]}}(\xi_{01}\xi_{12}) \\ C_3(\xi_{01}\xi_{12}\xi_{25}\xi_{52}) &= \sqrt{\frac{[2]}{[3]}}(\xi_{01}\xi_{12}) \end{aligned}$$

The following difference of non essential paths of length 4 starting at  $\sigma_0$  and ending at  $\sigma_2$  is an essential path of length 4 on  $\mathbf{E}_6$ :

$$\sqrt{[2]}(\xi_{01}\xi_{12}\xi_{23}\xi_{32}) - \sqrt{\frac{[3]}{[2]}}(\xi_{01}\xi_{12}\xi_{25}\xi_{52}) = \sqrt{[2]}[0, 1, 2, 3, 2] - \sqrt{\frac{[3]}{[2]}}[0, 1, 2, 5, 2]$$

Here the values of q-numbers are  $[2] = \frac{\sqrt{2}}{\sqrt{3}-1}$  and  $[3] = \frac{2}{\sqrt{3}-1}$ .

Acting on an elementary path of length  $p$ , the creating operators  $\mathbf{C}_n^\dagger$  are defined as follows: if  $n > p+1$ ,  $\mathbf{C}_n^\dagger$  vanishes and, if  $n \leq p+1$  then, setting  $j = r(\xi(n-1))$ ,

$$\mathbf{C}_n^\dagger(\xi(1)\dots\xi(n-1)\dots) = \sum_{d(j,k)=1} \sqrt{\left(\frac{D_k}{D_j}\right)}(\xi(1)\dots\xi(n-1)\xi_{jk}\xi_{kj}\dots)$$

The above sum is taken over the neighbours  $\sigma_k$  of  $\sigma_j$  on the graph. Intuitively, this operator adds one (or several) small round trip(s) at position  $n$ . The result is therefore either 0 or a linear combination of paths of length  $p+2$ .

For instance, on paths of length zero (i.e., vertices),

$$C_1^\dagger(\sigma_j) = \sum_{d(j,k)=1} \sqrt{\frac{D_k}{D_j}} \xi_{jk} \xi_{kj} = \sum_{d(j,k)=1} \sqrt{\frac{D_k}{D_j}} [\sigma_j \sigma_k \sigma_j]$$

Jones' projectors  $e_k$  can be defined (as endomorphisms of  $Path^D$ ) by

$$e_k \doteq \frac{1}{\beta} C_k^\dagger C_k$$

The reader can check that all Jones-Temperley-Lieb relations between the  $e_k$  are satisfied. Essential paths can also be defined as elements of the intersection of the kernels of the Jones projectors  $e_k$ 's.

## B Twisted partition functions for the $ADE$ models

<i>Point</i>	<i>Z</i>
<u>0</u>	$ \chi_0 ^2 +  \chi_1 ^2 +  \chi_2 ^2 +  \chi_3 ^2$
<u>1</u>	$[(\chi_0\overline{\chi_1} + \chi_1\overline{\chi_2} + \chi_2\overline{\chi_3}) + h.c.]$
<u>2</u>	$ \chi_1 ^2 +  \chi_2 ^2 + [(\chi_0\overline{\chi_2} + \chi_1\overline{\chi_3}) + h.c.]$
<u>3</u>	$[(\chi_0\overline{\chi_3} + \chi_1\overline{\chi_2}) + h.c.]$

Table 9: Twisted partition functions for the  $A_4$  model

<i>Point</i>	$\mathcal{Z}$
<u>0</u>	$ \chi_0 + \chi_6 ^2 +  \chi_3 + \chi_7 ^2 +  \chi_4 + \chi_{10} ^2$
<u>3</u>	$(\chi_0 + \chi_4 + \chi_6 + \chi_{10}) \cdot (\overline{\chi_3} + \overline{\chi_7}) + h.c.$
<u>4</u>	$ \chi_3 + \chi_7 ^2 + [(\chi_0 + \chi_6) \cdot (\overline{\chi_4} + \overline{\chi_{10}}) + h.c.]$
<u>11'</u>	$ \chi_1 + \chi_5 + \chi_7 ^2 +  \chi_2 + \chi_4 + \chi_6 + \chi_8 ^2 +  \chi_3 + \chi_5 + \chi_9 ^2$
<u>21'</u>	$(\chi_1 + \chi_3 + 2(\chi_5) + \chi_7 + \chi_9) \cdot (\overline{\chi_2} + \overline{\chi_4} + \overline{\chi_6} + \overline{\chi_8}) + h.c.$
<u>51'</u>	$ \chi_2 + \chi_4 + \chi_6 + \chi_8 ^2 + 2 \chi_5 ^2 + ([(\chi_1 + \chi_7) \cdot (\overline{\chi_3} + \overline{\chi_5} + \overline{\chi_9}) + \chi_3 \overline{\chi_5} + \chi_5 \overline{\chi_9}] + h.c.)$
<u>1</u>	$(\chi_0 + \chi_6) \cdot (\overline{\chi_1} + \overline{\chi_5} + \overline{\chi_7}) + (\chi_3 + \chi_7) \cdot (\overline{\chi_2} + \overline{\chi_4} + \overline{\chi_6} + \overline{\chi_8}) + (\chi_4 + \chi_{10}) \cdot (\overline{\chi_3} + \overline{\chi_5} + \overline{\chi_9})$
<u>1'</u>	$h.c.(Z_1)$
<u>2</u>	$ \chi_3 + \chi_7 ^2 +  \chi_4 + \chi_6 ^2 + (\chi_0 + \chi_{10}) \cdot (\overline{\chi_2} + \overline{\chi_4} + \overline{\chi_6} + \overline{\chi_8}) + (\chi_3 + \chi_7) \cdot (\overline{\chi_1} + 2(\overline{\chi_5}) + \overline{\chi_9}) + (\chi_4 + \chi_6) \cdot (\overline{\chi_2} + \overline{\chi_8})$
<u>31'</u>	$h.c.(Z_2)$
<u>5</u>	$(\chi_0 + \chi_6) \cdot (\overline{\chi_3} + \overline{\chi_5} + \overline{\chi_9}) + (\chi_3 + \chi_7) \cdot (\overline{\chi_2} + \overline{\chi_4} + \overline{\chi_6} + \overline{\chi_8}) + (\chi_4 + \chi_{10}) \cdot (\overline{\chi_1} + \overline{\chi_5} + \overline{\chi_7})$
<u>41'</u>	$h.c.(Z_5)$

Table 10: Twisted partition functions for the  $E_6$  model

Point	$\mathcal{Z}$
<u>0</u>	$ \chi_0 + \chi_{10} + \chi_{18} + \chi_{28} ^2 +  \chi_6 + \chi_{12} + \chi_{16} + \chi_{22} ^2$
<u>6</u>	$ \chi_0 + \chi_6 + \chi_{12} + \chi_{16} + \chi_{22} + \chi_{28} ^2 -  \chi_0 + \chi_{28} ^2 + [(\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}) \cdot (\overline{\chi_{10}} + \overline{\chi_{18}}) + h.c.]$
<u>11'</u>	$ \chi_1 + \chi_9 + \chi_{11} + \chi_{17} + \chi_{19} + \chi_{27} ^2 +  \chi_5 + \chi_7 + \chi_{11} + \chi_{13} + \chi_{15} + \chi_{17} + \chi_{21} + \chi_{23} ^2$
<u>71'</u>	$ \chi_1 + \chi_5 + \chi_7 + \chi_{11} + \chi_{13} + \chi_{15} + \chi_{17} + \chi_{21} + \chi_{23} + \chi_{27} ^2 +  \chi_9 + \chi_{11} + \chi_{13} + \chi_{15} + \chi_{17} + \chi_{19} ^2 -$ $ \chi_1 + \chi_{27} ^2 +  \chi_{11} + \chi_{17} ^2 -  \chi_{13} + \chi_{15} ^2 -  \chi_9 + \chi_{19} ^2 + [(\chi_5 + \chi_7 + \chi_{21} + \chi_{23}) \cdot (\overline{\chi_9} + \overline{\chi_{11}} + \overline{\chi_{17}} + \overline{\chi_{19}}) + h.c.]$
<u>22'</u>	$ \chi_2 + \chi_{26} ^2 +  \sum_{i=2}^{12} (\chi_{2i}) ^2 +  \sum_{i=4}^{10} (\chi_{2i}) ^2 + 2 \chi_{14} ^2 +$ $[(\chi_2 + \chi_{26}) \cdot (\overline{\chi_8} + \overline{\chi_{10}} + \overline{\chi_{12}} + \overline{\chi_{16}} + \overline{\chi_{18}} + \overline{\chi_{20}}) + (\chi_4 + \chi_6 + \chi_{22} + \chi_{24}) \cdot (\overline{\chi_{14}}) + h.c.]$
<u>42'</u>	$ \sum_{i=1}^{13} (\chi_{2i}) ^2 -  \chi_2 + \chi_{24} ^2 +  \sum_{i=2}^{12} (\chi_{2i}) ^2 -  \chi_4 + \chi_6 + \chi_{22} + \chi_{24} ^2 +$ $ \sum_{i=4}^{10} (\chi_{2i}) ^2 + [(\chi_2 + \sum_{i=4}^{10} (\chi_{2i}) + \chi_{26}) \cdot \overline{\chi_{14}} + h.c.]$
<u>55'</u>	$ \chi_5 + \chi_9 + \chi_{13} + \chi_{15} + \chi_{19} + \chi_{23} ^2 +  \chi_3 + \sum_{i=3}^{10} (\chi_{2i+1}) + \chi_{25} ^2$
<u>35'</u>	$ \sum_{i=1}^{12} (\chi_{2i+1}) ^2 +  \sum_{i=3}^{10} (\chi_{2i+1}) ^2 +  \chi_9 + \chi_{13} + \chi_{15} + \chi_{19} ^2 -  \chi_7 + \chi_{11} + \chi_{17} + \chi_{21} ^2 -  \chi_5 + \chi_{23} ^2 +$ $[(\chi_3 + \chi_{25}) \cdot (\overline{\chi_9} + \overline{\chi_{13}} + \overline{\chi_{15}} + \overline{\chi_{19}}) + h.c.]$
<u>1</u>	$(\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}) \cdot (\overline{\chi_1} + \overline{\chi_9} + \overline{\chi_{11}} + \overline{\chi_{17}} + \overline{\chi_{19}} + \overline{\chi_{27}})$ $+ (\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}) \cdot (\overline{\chi_5} + \overline{\chi_7} + \overline{\chi_{11}} + \overline{\chi_{13}} + \overline{\chi_{15}} + \overline{\chi_{17}} + \overline{\chi_{21}} + \overline{\chi_{23}})$
<u>1'</u>	$h.c.(Z_1)$
<u>2</u>	$ \chi_6 + \chi_{12} + \chi_{16} + \chi_{22} ^2 + (\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}) \cdot (\overline{\chi_2} + \overline{\chi_8} + \overline{\chi_{10}} + \overline{\chi_{12}} + \overline{\chi_{16}} + \overline{\chi_{18}} + \overline{\chi_{20}} + \overline{\chi_{26}})$ $+ (\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}) \cdot (\overline{\chi_4} + \overline{\chi_8} + \overline{\chi_{10}} + 2(\overline{\chi_{14}}) + \overline{\chi_{18}} + \overline{\chi_{20}} + \overline{\chi_{24}})$
<u>2'</u>	$h.c.(Z_2)$
<u>3</u>	$(\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}) \cdot \left( \overline{\chi_3} + \sum_{i=3}^{10} (\overline{\chi_{2i+1}}) + \overline{\chi_{25}} \right)$ $+ (\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}) \cdot \left( \sum_{i=1}^{12} (\overline{\chi_{2i+1}}) + \overline{\chi_9} + \overline{\chi_{13}} + \overline{\chi_{15}} + \overline{\chi_{19}} \right)$
<u>65'</u>	$h.c.(Z_3)$
<u>4</u>	$ \chi_6 + \chi_{10} + \chi_{12} + \chi_{16} + \chi_{18} + \chi_{22} ^2 +  \chi_{12} + \chi_{16} ^2 + (\chi_6 + \chi_{22}) \cdot (\overline{\chi_{12}} + \overline{\chi_{16}}) +$ $(\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}) \cdot (\overline{\chi_4} + \overline{\chi_8} + 2(\overline{\chi_{14}}) + \overline{\chi_{20}} + \overline{\chi_{24}}) + (\chi_0 + \chi_{28}) \cdot (\overline{\chi_6} + \overline{\chi_{10}} + \overline{\chi_{12}} + \overline{\chi_{16}} + \overline{\chi_{18}} + \overline{\chi_{22}}) +$ $(\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}) \cdot (\overline{\chi_2} + \overline{\chi_4} + 2(\overline{\chi_8}) + \overline{\chi_{10}} + 2(\overline{\chi_{14}}) + \overline{\chi_{18}} + 2(\overline{\chi_{20}}) + \overline{\chi_{24}} + \overline{\chi_{26}})$
<u>62'</u>	$h.c.(Z_4)$

Table 11: Twisted partition functions for the  $E_8$  model (part 1.)

Point	Z
<u>5</u>	$(\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}).(\overline{\chi_5} + \overline{\chi_9} + \overline{\chi_{13}} + \overline{\chi_{15}} + \overline{\chi_{19}} + \overline{\chi_{23}}) + (\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}).(\overline{\chi_3} + \sum_{i=3}^{10}(\overline{\chi_{2i+1}}) + \overline{\chi_{25}})$
<u>5'</u>	$h.c.(Z_{\underline{5}})$
<u>7</u>	$(\chi_0 + \chi_{10} + \chi_{18} + \chi_{28}).(\overline{\chi_5} + \overline{\chi_7} + \overline{\chi_{11}} + \overline{\chi_{13}} + \overline{\chi_{15}} + \overline{\chi_{17}} + \overline{\chi_{21}} + \overline{\chi_{23}}) +$ $(\chi_6 + \chi_{12} + \chi_{16} + \chi_{22}).(\overline{\chi_1} + \sum_{i=2}^{11}(\overline{\chi_{2i+1}}) + \overline{\chi_{11}} + \overline{\chi_{17}} + \overline{\chi_{27}})$
<u>61'</u>	$h.c.(Z_{\underline{7}})$
<u>21'</u>	$(\chi_1 + \chi_9 + \chi_{11} + \chi_{17} + \chi_{19} + \chi_{27}).(\overline{\chi_2} + \overline{\chi_8} + \overline{\chi_{10}} + \overline{\chi_{12}} + \overline{\chi_{16}} + \overline{\chi_{18}} + \overline{\chi_{20}} + \overline{\chi_{26}})$ $+ (\chi_5 + \chi_7 + \chi_{11} + \chi_{13} + \chi_{15} + \chi_{17} + \chi_{21} + \chi_{23}).(\sum_{i=2}^{12}(\overline{\chi_{2i}}) + \overline{\chi_{14}})$
<u>12'</u>	$h.c.(Z_{\underline{21}'})$
<u>41'</u>	$(\chi_1 + \chi_9 + \chi_{11} + \chi_{17} + \chi_{19} + \chi_{27}).(\sum_{i=2}^{12}(\overline{\chi_{2i}}) + \overline{\chi_{14}}) +$ $(\chi_5 + \chi_7 + \chi_{11} + \chi_{13} + \chi_{15} + \chi_{17} + \chi_{21} + \chi_{23}).(\sum_{i=1}^{13}(\overline{\chi_{2i}}) + \sum_{i=4}^{10}(\overline{\chi_{2i}}))$
<u>72'</u>	$h.c.(Z_{\underline{41}'})$
<u>52'</u>	$(\chi_2 + \chi_8 + \chi_{10} + \chi_{12} + \chi_{16} + \chi_{18} + \chi_{20} + \chi_{26}).(\overline{\chi_5} + \overline{\chi_9} + \overline{\chi_{13}} + \overline{\chi_{15}} + \overline{\chi_{19}} + \overline{\chi_{23}}) +$ $(\sum_{i=2}^{12}(\chi_{2i}) + \chi_{14}).(\overline{\chi_3} + \sum_{i=3}^{10}(\overline{\chi_{2i+1}}) + \overline{\chi_{25}})$
<u>25'</u>	$h.c.(Z_{\underline{52}'})$
<u>32'</u>	$(\chi_2 + \chi_8 + \chi_{10} + \chi_{12} + \chi_{16} + \chi_{18} + \chi_{20} + \chi_{26}).(\overline{\chi_3} + \sum_{i=3}^{10}(\overline{\chi_{2i+1}}) + \overline{\chi_{25}}) +$ $(\sum_{i=2}^{12}(\chi_{2i}) + \chi_{14}).(\sum_{i=1}^{12}(\overline{\chi_{2i+1}}) + \overline{\chi_9} + \overline{\chi_{13}} + \overline{\chi_{15}} + \overline{\chi_{19}})$
<u>45'</u>	$h.c.(Z_{\underline{32}'})$
<u>51'</u>	$(\chi_1 + \chi_9 + \chi_{11} + \chi_{17} + \chi_{19} + \chi_{27}).(\overline{\chi_5} + \overline{\chi_9} + \overline{\chi_{13}} + \overline{\chi_{15}} + \overline{\chi_{19}} + \overline{\chi_{23}}) +$ $(\chi_5 + \chi_7 + \chi_{11} + \chi_{13} + \chi_{15} + \chi_{17} + \chi_{21} + \chi_{23}).(\overline{\chi_3} + \sum_{i=3}^{10}(\overline{\chi_{2i+1}}) + \overline{\chi_{25}})$
<u>15'</u>	$h.c.(Z_{\underline{51}'})$
<u>31'</u>	$ \sum_{i=2}^{11}(\chi_{2i+1}) ^2 + (\chi_1 + \chi_{11} + \chi_{17} + \chi_{27}).(\overline{\chi_3} + \sum_{i=3}^{10}(\overline{\chi_{2i+1}}) + \overline{\chi_{25}}) + (\chi_9 + \chi_{19}).(\overline{\chi_3} - \overline{\chi_5} - \overline{\chi_{23}} + \overline{\chi_{25}})$ $(\chi_5 + \chi_7 + \chi_{11} + \chi_{13} + \chi_{15} + \chi_{17} + \chi_{21} + \chi_{23}).(\overline{\chi_3} + \overline{\chi_9} + \overline{\chi_{13}} + \overline{\chi_{15}} + \overline{\chi_{19}} + \overline{\chi_{25}})$
<u>75'</u>	$h.c.(Z_{\underline{31}'})$

Table 12: Twisted partition functions for the  $E_8$  model (part 2.)



<i>Point</i>	<i>Z</i>
$\underline{0}$	$ \chi_0 + \chi_4 ^2 + 2 \chi_2 ^2$
$\underline{2}, \underline{2}'$	$ \chi_2 ^2 + [(\chi_0 + \chi_4) \cdot \overline{\chi_2} + h.c.]$
$\underline{\epsilon}, \underline{2\epsilon}, \underline{2'\epsilon}$	$ \chi_1 + \chi_3 ^2$
$\underline{1}$	$(\chi_0 + 2(\chi_2) + \chi_4) \cdot (\overline{\chi_1} + \overline{\chi_3})$
$\underline{1\epsilon}$	$h.c.(Z_1)$

Table 13: Twisted partition functions for the  $D_4$  model

<i>Point</i>	<i>Z</i>
$\underline{0}$	$ \chi_0 + \chi_8 ^2 +  \chi_2 + \chi_6 ^2 + 2 \chi_4 ^2$
$\underline{2}$	$ \chi_2 + \chi_6 ^2 + 2 \chi_4 ^2 + [(\chi_0 + 2(\chi_4) + \chi_8) \cdot (\overline{\chi_2} + \overline{\chi_6}) + h.c.]$
$\underline{4}, \underline{4}'$	$ \chi_2 + \chi_4 + \chi_6 ^2 + [(\chi_0 + \chi_8) \cdot \overline{\chi_4} + h.c.]$
$\underline{\epsilon}$	$ \chi_1 + \chi_7 ^2 +  \chi_3 + \chi_5 ^2$
$\underline{2\epsilon}$	$ \chi_1 + \chi_3 + \chi_5 + \chi_7 ^2 +  \chi_3 + \chi_5 ^2$
$\underline{4\epsilon}, \underline{4'\epsilon}$	$ \chi_1 + \chi_3 + \chi_5 + \chi_7 ^2 -  \chi_1 + \chi_7 ^2$
$\underline{1}$	$(\chi_0 + \chi_2 + \chi_6 + \chi_8) \cdot (\overline{\chi_1} + \overline{\chi_7}) + (\chi_2 + 2(\chi_4) + \chi_6) \cdot (\overline{\chi_3} + \overline{\chi_5})$
$\underline{1\epsilon}$	$h.c.(Z_1)$
$\underline{3}$	$(\chi_0 + \chi_2 + \chi_6 + \chi_8) \cdot (\overline{\chi_3} + \overline{\chi_5}) + (\chi_2 + 2(\chi_4) + \chi_6) \cdot (\overline{\chi_1} + \overline{\chi_3} + \overline{\chi_5} + \overline{\chi_7})$
$\underline{3\epsilon}$	$h.c.(Z_3)$

Table 14: Twisted partition functions for the  $D_6$  model

<i>Point</i>	$\mathcal{Z}$
<u>0</u>	$ \chi_0 ^2 +  \chi_2 ^2 +  \chi_3 ^2 +  \chi_4 ^2 +  \chi_6 ^2 + (\chi_1 \overline{\chi_5} + h.c.)$
<u>2</u>	$ \chi_2 + \chi_4 ^2 +  \chi_3 ^2 + [(\chi_0 \overline{\chi_2} + \chi_1 \overline{\chi_3} + \chi_1 \overline{\chi_5} + \chi_3 \overline{\chi_5} + \chi_4 \overline{\chi_6}) + h.c.]$
<u>3</u>	$[(\chi_0 + \chi_2 + \chi_4 + \chi_6) \cdot \overline{\chi_3} + (\chi_1 + \chi_5) \cdot (\overline{\chi_2} + \chi_4) + h.c.]$
<u>4</u>	$ \chi_1 ^2 +  \chi_3 ^2 +  \chi_5 ^2 +  \chi_2 + \chi_4 ^2 + [(\chi_0 \overline{\chi_4} + \chi_1 \overline{\chi_3} + \chi_2 \overline{\chi_6} + \chi_3 \overline{\chi_5}) + h.c.]$
<u>6</u>	$ \chi_1 ^2 +  \chi_3 ^2 +  \chi_5 ^2 + [(\chi_0 \overline{\chi_6} + \chi_2 \overline{\chi_4}) + h.c.]$
<u>1</u>	$(\chi_0 + \chi_2) \cdot \overline{\chi_1} + \chi_5 \cdot (\overline{\chi_0} + \overline{\chi_2}) + \chi_1 \cdot (\overline{\chi_4} + \overline{\chi_6}) + (\chi_4 + \chi_6) \cdot \overline{\chi_5} + [(\chi_2 \overline{\chi_3} + \chi_3 \overline{\chi_4}) + h.c.]$
<u>5</u>	$h.c.(Z_1)$

Table 15: Twisted partition functions for the  $D_5$  model

<i>Point</i>	$\mathcal{Z}$
<u>0</u>	$ \chi_0 ^2 +  \chi_2 ^2 +  \chi_4 ^2 +  \chi_5 ^2 +  \chi_6 ^2 +  \chi_8 ^2 +  \chi_{10} ^2 + [(\chi_1 \overline{\chi_9} + \chi_3 \overline{\chi_7}) + h.c.]$
<u>2</u>	$ \chi_2 + \chi_4 ^2 +  \chi_5 ^2 +  \chi_6 + \chi_8 ^2 + [(\chi_0 \overline{\chi_2} + (\chi_1 + \chi_3) \cdot (\overline{\chi_7} + \overline{\chi_9}) + \chi_3 \overline{\chi_5} + \chi_4 \overline{\chi_6} + \chi_5 \overline{\chi_7} + \chi_8 \overline{\chi_{10}}) + h.c.]$
<u>4</u>	$ \chi_2 + \chi_4 + \chi_6 + \chi_8 ^2 +  \chi_3 + \chi_5 + \chi_7 ^2 + [\chi_0 \overline{\chi_4} + \chi_1 \cdot (\overline{\chi_5} + \overline{\chi_7}) - \chi_2 \overline{\chi_8} + (\chi_3 + \chi_5) \cdot \overline{\chi_9} + \chi_6 \overline{\chi_{10}}) + h.c.]$
<u>5</u>	$[(\chi_0 + \chi_2 + \chi_4 + \chi_6 + \chi_8 + \chi_{10}) \cdot \overline{\chi_5} + (\chi_1 + \chi_3 + \chi_7 + \chi_9) \cdot (\overline{\chi_4} + \overline{\chi_6}) + (\chi_2 + \chi_8) \cdot (\overline{\chi_3} + \overline{\chi_7}) + h.c.]$
<u>6</u>	$ \chi_2 + \chi_4 + \chi_6 + \chi_8 ^2 +  \chi_3 + \chi_5 + \chi_7 ^2 -  \chi_2 ^2 -  \chi_8 ^2 + [(\chi_0 \overline{\chi_6} + \chi_1 \cdot (\overline{\chi_3} + \overline{\chi_5}) + \chi_4 \overline{\chi_{10}} + (\chi_5 + \chi_7) \cdot \overline{\chi_9}) + h.c.]$
<u>8</u>	$ \chi_1 + \chi_3 ^2 +  \chi_4 + \chi_6 ^2 +  \chi_7 + \chi_9 ^2 +  \chi_5 ^2 + [(\chi_0 \overline{\chi_8} + \chi_2 \cdot (\overline{\chi_6} + \overline{\chi_8} + \overline{\chi_{10}}) + \chi_3 \overline{\chi_5} + \chi_4 \overline{\chi_8} + \chi_5 \overline{\chi_7}) + h.c.]$
<u>10</u>	$ \chi_1 ^2 +  \chi_3 ^2 +  \chi_5 ^2 +  \chi_7 ^2 +  \chi_9 ^2 + [(\chi_0 \overline{\chi_{10}} + \chi_2 \overline{\chi_8} + \chi_4 \overline{\chi_6}) + h.c.]$
<u>1</u>	$(\chi_0 + \chi_2) \cdot \overline{\chi_1} + \chi_9 \cdot (\overline{\chi_0} + \overline{\chi_2}) + (\chi_2 + \chi_4) \cdot \overline{\chi_3} + \chi_7 \cdot (\overline{\chi_2} + \overline{\chi_4}) + (\chi_6 + \chi_8) \cdot \overline{\chi_7} + \chi_3 \cdot (\overline{\chi_6} + \overline{\chi_8}) + (\chi_8 + \chi_{10}) \cdot \overline{\chi_9} + \chi_1 \cdot (\overline{\chi_8} + \overline{\chi_{10}}) + [(\chi_4 + \chi_6) \cdot \overline{\chi_5} + h.c.]$
<u>9</u>	$h.c.(Z_1)$
<u>3</u>	$\chi_7 \cdot (\overline{\chi_0} + \overline{\chi_2}) + (\chi_2 + \chi_4) \cdot \overline{\chi_1} + \chi_9 \cdot (\overline{\chi_2} + \overline{\chi_4}) + (\chi_0 + \chi_2) \cdot \overline{\chi_3} + \chi_1 \cdot (\overline{\chi_6} + \overline{\chi_8}) + (\chi_8 + \chi_{10}) \cdot \overline{\chi_7} + \chi_3 \cdot (\overline{\chi_8} + \overline{\chi_{10}}) + (\chi_6 + \chi_8) \cdot \overline{\chi_9} + [(\chi_2 + \chi_4 + \chi_6 + \chi_8) \cdot \overline{\chi_5} + (\chi_3 + \chi_7) \cdot (\chi_4 + \chi_6) + h.c.]$
<u>7</u>	$h.c.(Z_3)$

Table 16: Twisted partition functions for the  $D_7$  model

Point	$\mathcal{Z}$
<u>0</u>	$ \chi_0 + \chi_{16} ^2 +  \chi_4 + \chi_{12} ^2 +  \chi_6 + \chi_{10} ^2 +  \chi_8 ^2 + [(\chi_2 + \chi_{14}) \cdot \overline{\chi_8} + h.c.]$
<u>2</u>	$ \chi_4 + \chi_6 + \chi_8 + \chi_{10} + \chi_{12} ^2 - (\chi_4 + \chi_{12}) \cdot \overline{\chi_8} + \chi_8 \cdot (\overline{\chi_0} + \overline{\chi_{16}}) +$ $(\chi_0 + \chi_4 + \chi_8 + \chi_{12} + \chi_{16}) \cdot (\overline{\chi_2} + \overline{\chi_{14}}) + (\chi_2 + \chi_{14}) \cdot (\overline{\chi_6} + \overline{\chi_8} + \overline{\chi_{10}}) + (\chi_6 + \chi_{10}) \cdot \overline{\chi_8}$
<u>4</u>	$ \chi_2 + \chi_4 + \chi_6 + \chi_8 + \chi_{10} + \chi_{12} + \chi_{14} ^2 -  \chi_2 + \chi_{14} ^2 +  \chi_6 + \chi_8 + \chi_{10} ^2 -  \chi_8 ^2 +$ $[(\chi_0 + \chi_8 + \chi_{16}) \cdot (\overline{\chi_4} + \overline{\chi_{12}}) + h.c.]$
<u>6</u>	$ \chi_2 + \chi_4 + \chi_6 + \chi_8 + \chi_{10} + \chi_{12} + \chi_{14} ^2 +  \chi_4 + \chi_6 + \chi_8 + \chi_{10} + \chi_{12} ^2 -  \chi_4 + \chi_{12} ^2 +  \chi_8 ^2 +$ $[(\chi_0 + \chi_{16}) \cdot (\overline{\chi_6} + \overline{\chi_{12}}) + h.c.]$
<u>8</u>	$ \chi_4 + \chi_6 + \chi_8 + \chi_{10} + \chi_{12} ^2 - \chi_8 \cdot (\overline{\chi_4} + \overline{\chi_{12}}) + (\chi_2 + \chi_{14}) \cdot (\overline{\chi_0} + \overline{\chi_4} + \overline{\chi_8} + \overline{\chi_{12}} + \overline{\chi_{16}}) +$ $(\chi_6 + \chi_8 + \chi_{10}) \cdot (\overline{\chi_2} + \overline{\chi_{14}}) + (\chi_0 + \chi_{16}) \cdot \overline{\chi_8} + \chi_8 \cdot (\overline{\chi_6} + \overline{\chi_{10}})$
<u>8'</u>	$ \chi_2 + \chi_6 + \chi_{10} + \chi_{14} ^2 -  \chi_6 + \chi_{10} ^2 +  \chi_4 + \chi_6 + \chi_8 + \chi_{10} + \chi_{12} ^2 +  \chi_8 ^2 + [(\chi_0 + \chi_{16}) \cdot \overline{\chi_8} + h.c.]$
(1)	$ \chi_1 + \chi_7 + \chi_9 + \chi_{15} ^2 +  \chi_3 + \chi_5 + \chi_7 + \chi_9 + \chi_{11} + \chi_{13} ^2 +  \chi_5 + \chi_{11} ^2$
(3)	$ \sum_{i=0}^7 (\chi_{2i+1}) ^2 +  \sum_{i=1}^6 (\chi_{2i+1}) ^2 +  \sum_{i=2}^5 (\chi_{2i+1}) ^2 -  \chi_1 + \chi_{15} ^2 -  \chi_3 + \chi_{13} ^2 -  \chi_5 + \chi_{11} ^2$
(5)	$ \chi_3 + \chi_7 + \chi_9 + \chi_{13} ^2 +  \chi_5 + \chi_7 + \chi_9 + \chi_{11} ^2 -  \chi_7 + \chi_9 ^2 + [(\chi_1 + \chi_{15}) \cdot (\overline{\chi_5} + \overline{\chi_{11}}) + h.c.]$
<u>1</u>	$(\chi_0 + \chi_8 + \chi_{16}) \cdot (\overline{\chi_1} + \overline{\chi_{15}}) + (\chi_4 + \chi_8 + \chi_{12}) \cdot (\overline{\chi_3} + \overline{\chi_{13}}) + (\chi_4 + \chi_6 + \chi_{10} + \chi_{12}) \cdot (\overline{\chi_5} + \overline{\chi_{11}}) +$ $(\chi_2 + \chi_6 + \chi_8 + \chi_{10} + \chi_{14}) \cdot (\overline{\chi_7} + \overline{\chi_9})$
(0)	$h.c.(Z_{\underline{1}})$
<u>7</u>	$(\chi_2 + \chi_6 + \chi_8 + \chi_{10} + \chi_{14}) \cdot (\sum_{i=0}^7 (\overline{\chi_{2i+1}})) + (\chi_4 + \chi_6 + \chi_{10} + \chi_{12}) \cdot (\sum_{i=1}^6 (\overline{\chi_{2i+1}})) +$ $(\chi_4 + \chi_8 + \chi_{12}) \cdot (\overline{\chi_5} + \overline{\chi_7} + \overline{\chi_9} + \overline{\chi_{11}}) + (\chi_0 + \chi_8 + \chi_{16}) \cdot (\overline{\chi_7} + \overline{\chi_9})$
(2)	$h.c.(Z_{\underline{7}})$
<u>3</u>	$(\chi_4 + \chi_8 + \chi_{12}) \cdot (\sum_{i=0}^7 (\overline{\chi_{2i+1}})) + (\chi_0 + \chi_6 + \chi_{10} + \chi_{16}) \cdot (\overline{\chi_3} + \overline{\chi_{12}}) +$ $(\chi_2 + \chi_6 + \chi_{10} + \chi_{14}) \cdot (\overline{\chi_5} + \overline{\chi_7} + \overline{\chi_9} + \overline{\chi_{11}}) + (\chi_6 + \chi_{10}) \cdot (\overline{\chi_7} + \overline{\chi_9}) + \chi_8 \cdot (\overline{\chi_5} + \overline{\chi_{11}})$
(4)	$h.c.(Z_{\underline{3}})$
<u>5</u>	$(\chi_4 + \chi_6 + \chi_{10} + \chi_{12}) \cdot (\sum_{i=0}^7 (\overline{\chi_{2i+1}})) + (\chi_2 + 2(\chi_8) + \chi_{14}) \cdot (\sum_{i=1}^6 (\overline{\chi_{2i+1}})) +$ $(\chi_0 + \chi_6 + \chi_{10} + \chi_{16}) \cdot (\overline{\chi_5} + \overline{\chi_{11}}) + (\chi_4 + \chi_6 + \chi_{10} + \chi_{12}) \cdot (\overline{\chi_7} + \overline{\chi_9})$
(6)	$h.c.(Z_{\underline{5}})$

Table 17: Twisted partition functions for the  $E_7$  model

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