

## Comments on Quantum Aspects of Three-Dimensional de Sitter Gravity

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### Abstract

We investigate the quantum aspects of three-dimensional gravity with a positive cosmological constant. The reduced phase space of the three-dimensional de Sitter gravity is obtained as the space which consists of the Kerr-de Sitter space-times and their Virasoro deformations. A quantization of the phase space is carried out by the geometric quantization of the coadjoint orbits of the asymptotic Virasoro symmetries. The Virasoro algebras with real central charges are obtained as the quantum asymptotic symmetries. The states of globally de Sitter and point particle solutions become the primary states of the unitary irreducible representations of the Virasoro algebras. It is shown that those states are perturbatively stable at the quantum level. The Virasoro deformations of these solutions correspond to the excited states in the unitary irreducible representations. In view of the dS/CFT correspondence, we also study the relationship between the Liouville field theory obtained by a reduction of the  $SL(2;\mathbb{C})$  Chern-Simons theory and the three-dimensional gravity both classically and quantum mechanically. In the analyses of the both theories, the Kerr-de Sitter geometries with nonzero angular momenta do not give the unitary representations of the Virasoro algebras.

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# 1 Introduction

Quantum gravity on de Sitter space-time attracts much interest owing to not only conceptual problems [1] (see also [2] and references therein) but also recent astronomical observations [3, 4]. Recently it was proposed that the gravity with a positive cosmological constant has a dual description by the Euclidean conformal field theory on the infinitely past or future boundary [5] (see also [6]). Since de Sitter space-time has no everywhere time-like Killing vector, constructions of quantum field theories on that space could be non-trivial. And it is known that the quantum field theories on de Sitter space has some peculiar features, e.g., the existence of various possible vacua [6, 7]. To deeply understand this correspondence, proper investigations of quantum theories of de Sitter gravity are needed because one has no well-defined stringy example of de Sitter space-time. Since in three dimensions a gauge theoretical description of the gravity is known [8, 9], it seems that we can analyze the gravity at the quantum level<sup>1</sup>.

Our aim in this paper is to construct concretely a quantum theory of the three-dimensional gravity with a positive cosmological constant. For this end, we use the similar procedures to those used in Ref.[18] to investigate a quantum anti-de Sitter gravity. We first construct the phase space of this system by using the  $SL(2;\mathbb{C})$  Chern-Simons description of the three-dimensional gravity [19]. The phase space consists of solutions of the equations of motion under the boundary conditions that the space-time should approach to de Sitter space in the asymptotically past [5]. It includes the Kerr-de Sitter geometries [20, 21] as particular solutions. The Virasoro algebras which are reduced symmetries of the gauge symmetry act on this space by their coadjoint forms and their central charges are given by a well-known value  $c = \frac{3l}{2G}$ . Each geometry corresponding to an element on the phase space can be represented as a non-compact region in  $SL(2;\mathbb{C})/SU(1,1)$  and its metric is induced from the Killing metric of  $SL(2;\mathbb{C})/SU(1,1)$ . The actions of the Virasoro algebras on the phase space can be interpreted as deformations of that region. The family which consists of geometries connected with each other by the Virasoro deformations is identified with a product of the coadjoint orbits of the conformal groups labeled by mass and angular momentum of the Kerr-de Sitter geometry. Since the conformal group is not globally well-defined, our discussion of the coadjoint orbit would be formal. Thus we mainly concentrate on the infinitesimal deformations of the Kerr-de Sitter geometries and neighborhood of the elements corresponding to those Kerr-de Sitter geometries on the orbits.

In Section 3 we consider a quantization of the Virasoro deformations which are degrees of freedom of the gravitational system analyzed here. In Ref.[21] it was shown that the Virasoro

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<sup>1</sup>From the viewpoint of the dS/CFT correspondence, the Chern-Simons gravity on three-dimensional de Sitter space-time is also studied in [10, 11, 12, 13, 14, 15, 16]. In more general settings including the de Sitter gravity case, a correspondence between the three-dimensional Chern-Simons theory with boundary and a conformal field theory was earlier discussed in [17] as a generalization of the AdS/CFT correspondence by using the Dirac's canonical method.

algebras with imaginary central charges were acquired as algebras of the conserved charges by a quantization of the corresponding Chern-Simons theory. In the appendix B, we provide some descriptions which state that the Virasoro algebra with an imaginary central charge are also obtained by a different manner of quantization. We would like to show that the unitary irreducible representations of the Virasoro algebra with a *real* central charge can be obtained by geometric quantization of the coadjoint orbit prescribed by an appropriate symplectic form on the orbit. Here strict analyses will be restricted within the perturbative level under the condition  $c \gg 1$ . We consider the unitary representations of the Virasoro algebra adopting the unitarity conditions  $L_m^\dagger = L_{-m}$  for the Virasoro generators which are usually used in conformal field theories. Then we discuss the perturbative stability of vacua by regarding  $L_0$  as the energy function. The unitarity and the stability of vacua require that the conformal weights of the primary states should be real and positive. It means in the sense of the three-dimensional gravity that only the point particle geometries, i.e., without angular momenta, and their Virasoro deformations can provide the unitary and stable quantum theory. The geometries with excess angles are forbidden.

It is known that the Chern-Simons theory on a manifold with boundaries reduces to the chiral WZNW theories on the boundaries [22]. In the case of the Chern-Simons gravity with (negative) positive cosmological constant, it is shown that the chiral WZNW theories furthermore reduce to the Liouville field theory under the asymptotic (anti-)de Sitter boundary condition [23, 10]. In Section 4 we see the explicit correspondence between three-dimensional geometries and solutions of the Liouville theory. The relationship of infinitesimal Virasoro deformations of the Kerr-de Sitter geometries and those of the Liouville theory are provided. At the quantum level, the direct product of the primary states with appropriate conformal weights in the Liouville theory is identified with the state of the Kerr-de Sitter space-time. The states corresponding to the deformed geometries are identified with the excited states which are obtained by acting the Virasoro generators on the primary state.

Finally we provide some discussions in Section 5.

## 2 Three-Dimensional Geometry with Positive Constant Curvature

Solutions of the Einstein equation with a positive cosmological constant  $1/l^2$  are geometries with a positive constant scalar curvature  $6/l^2$ . All solutions are locally equivalent to each other because the three-dimensional gravity has no local degrees of freedom. In this section, we would like to clarify what characterizes each solution. And we provide a phase space of this system under the boundary conditions being asymptotically de Sitter space-time from the perspective of Chern-Simons gravity.

## 2.1 On the Kerr-de Sitter Solution

The Kerr-de Sitter solution is characterized by its ADM-like mass  $M = -\frac{\mu}{8G}$  and angular momentum  $J$ . In the Schwarzschild coordinates  $(t, r, \phi)$ , its metric  $ds_{(\mu, J)}^2$  is given by

$$ds_{(\mu, J)}^2 = -N^2(dt)^2 + N^{-2}(dr)^2 + r^2 \left( d\phi + N^\phi dt \right)^2, \quad (2.1)$$

where

$$N^2 = \mu - \frac{r^2}{l^2} + \frac{16G^2 J^2}{r^2}, \quad N^\phi = \frac{4GJ}{r^2}. \quad (2.2)$$

The space-time has the cosmological horizon at  $r = r_+$ ,

$$r_+ = \frac{l}{2} \left( \sqrt{\mu + i\frac{8GJ}{l}} + \sqrt{\mu - i\frac{8GJ}{l}} \right), \quad (2.3)$$

if  $\mu \in \mathbb{R}$  for  $J \neq 0$  or  $\mu \in \mathbb{R}_{\geq 0}$  for  $J = 0$ . In the case  $J = 0$ , this metric represents a point particle sitting at  $r = 0$  [20],

$$ds_{(\mu, 0)}^2 = - \left( 1 - \frac{\tilde{r}^2}{l^2} \right) (d\tilde{t})^2 + \frac{(d\tilde{r})^2}{1 - \frac{\tilde{r}^2}{l^2}} + \tilde{r}^2 (d\tilde{\phi})^2, \quad (2.4)$$

where we transformed the coordinates as follows,

$$\tilde{t} = \sqrt{\mu} t, \quad \tilde{r} = \frac{r}{\sqrt{\mu}}, \quad \tilde{\phi} = \sqrt{\mu} \phi. \quad (2.5)$$

We have to make the identification  $\tilde{\phi} \sim \tilde{\phi} + 2\pi\sqrt{\mu}$  because  $\phi \sim \phi + 2\pi$ . Thus this geometry has a conical singularity at  $r = 0$  with a deficit angle  $2\pi(1 - \sqrt{\mu})$  if  $0 < \mu < 1$ . On the other hand, we do not have any physical interpretation for space-time for  $\mu > 1$  which has an excess angle  $2\pi(\sqrt{\mu} - 1)$ . In the case of  $J = 0$  and  $\mu < 0$ ,  $N^2$  becomes negative in the coordinate system used in the eq.(2.1). Thus we will omit this case in the most part in this paper unless explicitly mentioned, although this metric can be rewritten in the form of de Sitter metric locally by an appropriate coordinate transformation.<sup>2</sup>

We will mainly use another coordinate system  $(\tau, z, \bar{z})$  which is defined below. By the coordinate transformations,

$$\begin{aligned} \tau &= \frac{t}{l} + f(r) - \frac{1}{2} \ln \frac{\sqrt{b_0 b_0}}{c/6}, \\ z &= e^{\frac{t}{l} + g(r) + i\phi}, \\ \bar{z} &= e^{\frac{t}{l} + \bar{g}(r) - i\phi}, \end{aligned} \quad (2.6)$$

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<sup>2</sup>In fact, although one can transform it into de Sitter metric like (2.4), time-like coordinate becomes periodic in the resulting coordinate system.

the Kerr-de Sitter metric  $ds_{(\mu,J)}^2$  is written in the following form,

$$ds_{(b_0,\bar{b}_0)}^2 = l^2 \left[ -d\tau^2 + \frac{b_0}{c/6} \frac{dz^2}{z^2} + \frac{\bar{b}_0}{c/6} \frac{d\bar{z}^2}{\bar{z}^2} + \left( e^{-2\tau} + \frac{b_0\bar{b}_0}{(c/6)^2 z^2 \bar{z}^2} e^{2\tau} \right) dz d\bar{z} \right], \quad (2.7)$$

where  $c \equiv \frac{3l}{2G}$ , and  $b_0, \bar{b}_0$  are defined by

$$b_0 \equiv \frac{c}{24}(1-\mu) - \frac{i}{2}J, \quad \bar{b}_0 \equiv \frac{c}{24}(1-\mu) + \frac{i}{2}J. \quad (2.8)$$

And  $f(r), g(r)$  and  $\bar{g}(r)$  are obtained by solving the following first order differential equations,

$$l \frac{d}{dr} f(r) = \frac{\sqrt{1-N^2}}{N^2}, \quad l \frac{d}{dr} g(r) = \frac{\frac{r^2}{l^2} + \frac{b_0 - \bar{b}_0}{c/6}}{\frac{r^2}{l^2} N^2 \sqrt{1-N^2}}, \quad l \frac{d}{dr} \bar{g}(r) = \frac{\frac{r^2}{l^2} - \frac{b_0 - \bar{b}_0}{c/6}}{\frac{r^2}{l^2} N^2 \sqrt{1-N^2}}. \quad (2.9)$$

For these coordinate transformations to be well-defined, the range of  $r$  is restricted in

$$\begin{aligned} 0 < r < l\sqrt{\mu}, & \quad \text{for } 0 \leq \mu \leq 1, J = 0 \\ l\sqrt{\mu-1} < r < l\sqrt{\mu}, & \quad \text{for } 1 < \mu, J = 0 \\ l \left| \sqrt{\frac{b_0}{c/6}} - \sqrt{\frac{\bar{b}_0}{c/6}} \right| < r < r_+, & \quad \text{for } \mu \in \mathbb{R}, J \neq 0 \end{aligned} \quad (2.10)$$

Therefore we will consider the region of the Kerr-de Sitter space-time defined by  $-\infty < t < \infty$ ,  $0 \leq \phi < 2\pi$  and the above range of  $r$ .

These geometries can be represented in the coset space  $\text{SL}(2;\mathbb{C})/\text{SU}(1,1)$ .<sup>3</sup> Let us consider a non-compact region in  $\text{SL}(2;\mathbb{C})/\text{SU}(1,1)$  consisting of the elements

$$\mathcal{G}_0 \equiv \begin{pmatrix} \frac{a+1}{2\sqrt{a}} z^{\frac{a-1}{2}} & -\frac{i}{\sqrt{a}} z^{\frac{a+1}{2}} \\ -\frac{i(a-1)}{2\sqrt{a}} z^{-\frac{a+1}{2}} & \frac{1}{\sqrt{a}} z^{-\frac{a-1}{2}} \end{pmatrix} \begin{pmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{pmatrix} \begin{pmatrix} \frac{\bar{a}+1}{2\sqrt{\bar{a}}} \bar{z}^{\frac{\bar{a}-1}{2}} & -\frac{i(\bar{a}-1)}{2\sqrt{\bar{a}}} \bar{z}^{-\frac{\bar{a}+1}{2}} \\ -\frac{i}{\sqrt{\bar{a}}} \bar{z}^{\frac{\bar{a}+1}{2}} & \frac{1}{\sqrt{\bar{a}}} \bar{z}^{-\frac{\bar{a}-1}{2}} \end{pmatrix}, \quad (2.11)$$

where  $a = \sqrt{1 - \frac{b_0}{c/24}}$  and  $\bar{a} = \sqrt{1 - \frac{\bar{b}_0}{c/24}}$  which coincide with  $\sqrt{\mu}$  for  $J = 0$ .  $\mathcal{G}_0$  is characterized by the relation  $\mathcal{G}_0 = \epsilon \mathcal{G}_0^\dagger \epsilon^{-1}$  where  $\epsilon \equiv 2J^0$ . The metric on this region induced from the Killing metric of  $\text{SL}(2;\mathbb{C})/\text{SU}(1,1)$  coincides with the Kerr-de Sitter metric (2.7),

$$ds_{(b_0,\bar{b}_0)}^2 = -\frac{l^2}{2} \text{Tr} (\mathcal{G}_0^{-1} d\mathcal{G}_0)^2. \quad (2.12)$$

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<sup>3</sup> $\text{SL}(2;\mathbb{C})$  generators we use in this paper are

$$J^0 = \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}, \quad J^1 = \begin{pmatrix} 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & -\frac{i}{2} \\ \frac{i}{2} & 0 \end{pmatrix}.$$

These satisfy  $[J^a, J^b] = \epsilon^{abc} J_c$ , where  $\epsilon^{abc}$  is a totally anti-symmetric tensor  $\epsilon^{012} = +1$  and  $sl(2;\mathbb{C})$  indices are lowered (or raised) by  $\eta_{ab} \equiv \text{diag.}(-1, 1, 1)$  (or  $\eta^{ab}$ , the inverse matrix of  $\eta_{ab}$ ). And  $(J^a)^\dagger = J_a$ .

Therefore we can make the following interpretation:  $\mathcal{G}_0$  gives a mapping from the parameter space  $(\tau, z, \bar{z})$  to  $\text{SL}(2;\mathbb{C})/\text{SU}(1,1)$ . Then the Kerr-de Sitter space-time is realized as a region defined by  $\mathcal{G}_0$  in  $\text{SL}(2;\mathbb{C})/\text{SU}(1,1)$ . Thus each solution is characterized by its region. And its metric is induced from the Killing metric of  $\text{SL}(2;\mathbb{C})/\text{SU}(1,1)$ .

## 2.2 $\text{SL}(2;\mathbb{C})$ Chern-Simons Gauge Theory

It is well-known that the Chern-Simons gauge theory with an appropriate gauge group is on-shell equivalent to the three-dimensional gravity[8, 9].<sup>4</sup> Therefore we can provide a quantum theory of the gravity by quantizing a classical phase space of the Chern-Simons theory. In particular, the gravity with a positive cosmological constant can be described by taking  $\text{SL}(2;\mathbb{C})$  as the gauge group.

The Einstein-Hilbert action is written by using the Chern-Simons action as follows,

$$\begin{aligned} S_{EH} &= \frac{ik}{4\pi} \int_M \text{Tr} \left( AdA + \frac{2}{3} A^3 \right) - \frac{ik}{4\pi} \int_M \text{Tr} \left( \bar{A}d\bar{A} + \frac{2}{3} \bar{A}^3 \right) \\ &= -\frac{k}{\pi l} \int_M \text{Tr} \left[ e (d\omega + \omega^2) - \frac{1}{3l^2} e^3 \right] + \frac{k}{2\pi l} \int_{\partial M} \text{Tr} e\omega \\ &= \frac{1}{16\pi G} \int_M d^3x \sqrt{-g} \left( R - \frac{2}{l^2} \right) + \frac{1}{8\pi G} \int_{\partial M} \text{Tr} e\omega, \end{aligned} \quad (2.13)$$

where  $k = \frac{l}{4G}$  and we decomposed  $\text{SL}(2;\mathbb{C})$  connections  $A, \bar{A}$  to dreibein  $e^a$  and spin connection  $\omega^a$ ,

$$A = \frac{i}{l} e + \omega, \quad \bar{A} = -\frac{i}{l} e + \omega. \quad (2.14)$$

The three-dimensional space  $M$  is parametrized by  $\tau, z$ , and  $\bar{z}$ . In this paper we consider a two-dimensional boundary of  $M$  at  $\tau \rightarrow -\infty$  which is parametrized by  $z$  and  $\bar{z}$ . In the above form (2.13), the gravitational action has a boundary contribution. We note the fact that this boundary term has a well-known form for solutions of the equations of motion,

$$\frac{1}{8\pi G} \int_{\partial M} \text{Tr} e\omega = -\frac{i}{8\pi G} \int_{\partial M} dz \wedge d\bar{z} \frac{\sqrt{h}}{l}, \quad (2.15)$$

where  $h$  is the determinant of an induced metric on  $\partial M$ . The equations of motion of the Chern-Simons theory are the flat connection conditions which are interpreted as the torsion-free condition and the Einstein equation with a positive cosmological constant  $\Lambda = 1/l^2$  in terms of dreibein and spin connection. The flat  $\text{SL}(2;\mathbb{C})$  connections can be solved as

$$A = G^{-1} dG, \quad \bar{A} = \bar{G}^{-1} d\bar{G}. \quad (2.16)$$

For the equations (2.14) to hold,  $G$  and  $\bar{G}$  must satisfy the following relation

$$\bar{G} = \epsilon^{-1} (G^{-1})^\dagger \epsilon, \quad (2.17)$$

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<sup>4</sup>Of course, this equivalence holds only under some assumptions, e.g. the reversibility of the dreibein.

where the relation  $\epsilon^{-1}J^a\epsilon = -J_a = -(J^a)^\dagger$  is useful. That is, only if this condition is satisfied, the  $\text{SL}(2;\mathbb{C})$  Chern-Simons theory can be regarded as a gravity.

The correspondence between  $\text{SL}(2;\mathbb{C})$  flat connections and the Kerr-de Sitter geometries can be made by the following ways: first we define a mapping,  $\text{SL}(2;\mathbb{C}) \times \text{SL}(2;\mathbb{C}) \longrightarrow \text{SL}(2;\mathbb{C})$  by

$$(G, \bar{G}) \longmapsto \mathcal{G} = G\bar{G}^{-1}, \quad (2.18)$$

where it is assumed that  $G$  and  $\bar{G}$  satisfy (2.17). Thus  $\mathcal{G}$  satisfies

$$\bar{\mathcal{G}} = \mathcal{G}^{-1} \quad \text{or equivalently} \quad \mathcal{G} = \epsilon \mathcal{G}^\dagger \epsilon^{-1}. \quad (2.19)$$

$\mathcal{G}$  is invariant under the right diagonal  $\text{SU}(1,1)$  transformation which preserves the relation (2.17),

$$G \longmapsto Gg, \quad \bar{G} \longmapsto \bar{G}g, \quad (2.20)$$

Since  $\mathcal{G}$  satisfies (2.19), we can identify  $\mathcal{G}$  with  $\mathcal{G}_0$  (2.11) corresponding to the Kerr-de Sitter space-time. We consider the following decomposition of  $\mathcal{G}_0$ ,

$$\begin{aligned} \mathcal{G}_0 &= G_0 \bar{G}_0^{-1}, \\ G_0 &= \begin{pmatrix} \frac{a+1}{2\sqrt{a}} z^{\frac{a-1}{2}} & -\frac{i}{\sqrt{a}} z^{\frac{a+1}{2}} \\ -\frac{i(\bar{a}-1)}{2\sqrt{a}} z^{-\frac{a+1}{2}} & \frac{1}{\sqrt{a}} z^{-\frac{a-1}{2}} \end{pmatrix} \begin{pmatrix} e^{\frac{1}{2}\tau} & 0 \\ 0 & e^{-\frac{1}{2}\tau} \end{pmatrix}, \end{aligned} \quad (2.21)$$

$$\bar{G}_0 = \begin{pmatrix} \frac{1}{\sqrt{\bar{a}}} \bar{z}^{-\frac{\bar{a}-1}{2}} & \frac{i(\bar{a}-1)}{2\sqrt{\bar{a}}} \bar{z}^{-\frac{\bar{a}+1}{2}} \\ \frac{i}{\sqrt{\bar{a}}} \bar{z}^{\frac{\bar{a}+1}{2}} & \frac{\bar{a}+1}{2\sqrt{\bar{a}}} \bar{z}^{\frac{\bar{a}-1}{2}} \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{2}\tau} & 0 \\ 0 & e^{\frac{1}{2}\tau} \end{pmatrix} \quad (2.22)$$

Then we construct the flat connections (2.16) from  $G_0$  and  $\bar{G}_0$ ,

$$\begin{aligned} A_0 &= G_0^{-1} dG_0 = \begin{pmatrix} \frac{1}{2}d\tau & -ie^{-\tau}dz \\ -\frac{ib_0}{c/6}e^\tau \frac{dz}{z^2} & -\frac{1}{2}d\tau \end{pmatrix}, \\ \bar{A}_0 &= \bar{G}_0^{-1} d\bar{G}_0 = \begin{pmatrix} -\frac{1}{2}d\tau & \frac{i\bar{b}_0}{c/6}e^\tau \frac{d\bar{z}}{\bar{z}^2} \\ ie^{-\tau}d\bar{z} & \frac{1}{2}d\tau \end{pmatrix}. \end{aligned} \quad (2.23)$$

The dreibein obtained from  $A_0$ ,  $\bar{A}_0$  are written in the form  $e = \frac{l}{2i} (A_0 - \bar{A}_0) = \frac{l}{2i} \bar{G}_0^{-1} (\mathcal{G}_0^{-1} d\mathcal{G}_0) \bar{G}_0$ . Thus these  $\text{SL}(2;\mathbb{C})$  flat connections provide descriptions of the Kerr-de Sitter geometry in the Chern-Simons gravity, that is,  $ds^2 = 2\text{Tre}^2 = ds^2_{(b_0, \bar{b}_0)}$ .

### 2.3 Phase Space of de Sitter Gravity

In studies of the three-dimensional anti-de Sitter gravity, a generalization of anti-de Sitter and the BTZ black hole solutions was considered [23, 24, 25, 18]. Applying their methods to the de Sitter case, we find the following solutions,

$$A_b = \begin{pmatrix} \frac{1}{2}d\tau & -ie^{-\tau}dz \\ -\frac{ib(z)}{c/6}e^\tau dz & -\frac{1}{2}d\tau \end{pmatrix}, \quad \bar{A}_{\bar{b}} = \begin{pmatrix} -\frac{1}{2}d\tau & \frac{i\bar{b}(\bar{z})}{c/6}e^\tau d\bar{z} \\ ie^{-\tau}d\bar{z} & \frac{1}{2}d\tau \end{pmatrix}. \quad (2.24)$$

Here  $b(z)$  and  $\bar{b}(\bar{z})$  have to be invariant under  $z \rightarrow e^{2\pi i}z$  to be consistent with the coordinate transformation (2.6). These  $\text{SL}(2;\mathbb{C})$  flat connections are obtained as solutions of the equations of motion under the gauge fixing condition  $A_\tau = -iJ^0$ ,  $\bar{A}_\tau = iJ^0$  and the following conditions which are consistent with the Strominger's boundary conditions to be asymptotically de Sitter [5],

$$A_{\bar{z}} \rightarrow 0, \quad \bar{A}_z \rightarrow 0, \quad (2.25)$$

$$A_z \rightarrow \begin{pmatrix} 0 & -ie^{-\tau} \\ \mathcal{O}(e^\tau) & 0 \end{pmatrix}, \quad \bar{A}_{\bar{z}} \rightarrow \begin{pmatrix} 0 & \mathcal{O}(e^\tau) \\ ie^{-\tau} & 0 \end{pmatrix}, \quad \text{at } \tau \rightarrow -\infty. \quad (2.26)$$

The degrees of freedom of the classical solutions are two functions  $b(z)$  and  $\bar{b}(\bar{z})$ , and thus the phase space of the three-dimensional de Sitter gravity is composed of these functions. The asymptotic Virasoro algebras which act on this phase space are realized as residual gauge transformation,

$$\eta_f = \begin{pmatrix} \frac{1}{2}f' & -ife^{-\tau} \\ -i\left(\frac{1}{2}f'' + \frac{b}{c/6}f\right)e^\tau & -\frac{1}{2}f \end{pmatrix}, \quad \bar{\eta}_{\bar{f}} = \begin{pmatrix} -\frac{1}{2}\bar{f}' & i\left(\frac{1}{2}\bar{f}'' + \frac{\bar{b}}{c/6}\bar{f}\right)e^\tau \\ i\bar{f}e^{-\tau} & \frac{1}{2}\bar{f} \end{pmatrix}, \quad (2.27)$$

where  $f = f(z)$  and  $\bar{f} = \bar{f}(\bar{z})$  should be invariant under  $z \rightarrow e^{2\pi i}z$ , also. These transformations preserve the forms of the flat  $\text{SL}(2;\mathbb{C})$  connections (2.24) and generate the infinitesimal deformations of  $b(z)$  and  $\bar{b}(\bar{z})$ .

$$\delta_f A = \begin{pmatrix} 0 & 0 \\ -\frac{i\delta_f b}{c/6}e^\tau dz & 0 \end{pmatrix}, \quad \delta_{\bar{f}} \bar{A} = \begin{pmatrix} 0 & \frac{i\delta_{\bar{f}} \bar{b}}{c/6}e^\tau d\bar{z} \\ 0 & 0 \end{pmatrix}. \quad (2.28)$$

The deformations of  $b(z)$  and  $\bar{b}(\bar{z})$  are given by the coadjoint actions of the Virasoro algebras with the central charge  $c$ ,

$$\delta_f b = fb' + 2f'b + \frac{c}{12}f''', \quad \delta_{\bar{f}} \bar{b} = \bar{f}\bar{b}' + 2\bar{f}'\bar{b} + \frac{c}{12}\bar{f}'''. \quad (2.29)$$



The diffeomorphism corresponding to these gauge transformations is generated by the following vector fields,

$$\xi^\mu \partial_\mu = \frac{1}{2}(f' + \bar{f}')\partial_\tau + \left\{ f + \frac{\frac{\bar{b}}{c/6}e^{2\tau}f'' - \bar{f}''}{2\left[\frac{b\bar{b}}{(c/6)^2}e^{2\tau} - e^{-2\tau}\right]} \right\} \partial_z + \left\{ \bar{f} + \frac{\frac{b}{c/6}e^{2\tau}\bar{f}'' - f''}{2\left[\frac{b\bar{b}}{(c/6)^2}e^{2\tau} - e^{-2\tau}\right]} \right\} \partial_{\bar{z}}. \quad (2.30)$$

The flat connections (2.24) describe a deformation of the Kerr-de Sitter space-time. Its metric  $ds_{(b,\bar{b})}^2$  is

$$ds_{(b,\bar{b})}^2 = l^2 \left[ -d\tau^2 + \frac{b(z)}{c/6}dz^2 + \frac{\bar{b}(\bar{z})}{c/6}d\bar{z}^2 + \left( e^{-2\tau} + \frac{b(z)\bar{b}(\bar{z})}{(c/6)^2}e^{2\tau} \right) dzd\bar{z} \right]. \quad (2.31)$$

The deformed flat connections (2.24) can be solved in principle as

$$A_b = G_b^{-1}dG_b, \quad \bar{A}_{\bar{b}} = \bar{G}_{\bar{b}}^{-1}d\bar{G}_{\bar{b}}. \quad (2.32)$$

$G_b$  and  $\bar{G}_{\bar{b}}$  are generally complicated functions of  $b(z)$  and  $\bar{b}(\bar{z})$ , respectively.<sup>5</sup> Then one can provide a region in  $\text{SL}(2;\mathbb{C})/\text{SU}(1,1)$  defined by a mapping  $\mathcal{G}_{(b,\bar{b})}(\tau, z, \bar{z}) = G_b \bar{G}_{\bar{b}}^{-1}$ . We will call this region  $Z_{(b,\bar{b})}$ . The space-time corresponding to the deformed solutions  $A_b$  and  $\bar{A}_{\bar{b}}$  is realized as  $Z_{(b,\bar{b})}$  with the induced metric  $ds_{(b,\bar{b})}^2 = -\frac{l^2}{2}\text{Tr} \left( \mathcal{G}_{(b,\bar{b})}^{-1} d\mathcal{G}_{(b,\bar{b})} \right)^2$ . The Virasoro algebras with the central charge  $c$  act on the phase space consisting of  $b(z)$  and  $\bar{b}(\bar{z})$  by coadjoint actions (2.29). Let us consider the coadjoint orbits of  $b = \frac{b_0}{z^2}$  and  $\bar{b} = \frac{\bar{b}_0}{\bar{z}^2}$ . We denote them as  $W_{b_0}$  and  $\bar{W}_{\bar{b}_0}$  respectively. Then  $W_{b_0} \times \bar{W}_{\bar{b}_0}$  can be identified with the family of  $Z_{(b,\bar{b})}$  in which  $b(z)$  and  $\bar{b}(\bar{z})$  lie on  $W_{b_0}$  and  $\bar{W}_{\bar{b}_0}$  respectively. Namely,  $Z_{(b_0/z^2, \bar{b}_0/\bar{z}^2)}$  is deformed to  $Z_{(b,\bar{b})}$  by coadjoint action of the Virasoro algebras. And the metric  $ds_{(b_0,\bar{b}_0)}^2$  is deformed to  $ds_{(b,\bar{b})}^2$ , (2.31).

We again note that each geometry is characterized by its region in  $\text{SL}(2;\mathbb{C})/\text{SU}(1,1)$ , although all geometries are locally equivalent to  $\text{SL}(2;\mathbb{C})/\text{SU}(1,1)$ . Therefore the degrees of freedom of this system can be regarded as ways of taking a region in  $\text{SL}(2;\mathbb{C})/\text{SU}(1,1)$ , in other words, ways of drawing a two-dimensional surface, which is a boundary of a region, in  $\text{SL}(2;\mathbb{C})/\text{SU}(1,1)$ . Thus this degrees of freedom may be interpreted in terms of a two-dimensional theory.

### 3 Quantization of de Sitter Gravity

In the previous section, we clarified that the phase space of the three-dimensional de Sitter gravity consists of the holomorphic and anti-holomorphic functions,  $b(z)$  and  $\bar{b}(\bar{z})$  respectively, with invariance under  $z \rightarrow e^{2\pi i}z$ . Here we first consider an introduction of a symplectic structure into the phase space. It was shown that the Virasoro algebras with *pure imaginary* central charges are obtained by applying the symplectic reduction method to the  $\text{SL}(2;\mathbb{C})$  Chern-Simons

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<sup>5</sup>The explicit forms of  $G_b$  and  $\bar{G}_{\bar{b}}$  will be given in Section 4.

theory [21].<sup>6</sup> We here show that there exists a symplectic form which leads to the Virasoro algebras with *real* central charges after quantization. The similar methods to those used in Ref.[18] for the asymptotically anti-de Sitter space-time, e.g., geometric quantization method [26, 27, 28, 29], are applied to the case of de Sitter gravity. There is an essential difference between these two cases. While the group which acts on the phase space is  $\widehat{Diff}S^1$  in anti-de Sitter case, we now must deal with the conformal group which is not globally well-defined. Thus we concentrate on perturbative analyses around the de Sitter and Kerr-de Sitter geometries. Then we consider the unitary representations of the Virasoro algebra and discuss the perturbative stability of vacua.

### 3.1 Coadjoint Orbit

A general element of the Virasoro algebra is written as  $f(z)\partial_z + ac$  where  $c$  is a central element and  $a$  is a complex number. We denote this as  $(f, a)$ . The commutation relation is given by

$$[(f, a_1), (g, a_2)] = \left( fg' - f'g, \frac{i}{24} \oint \frac{dz}{2\pi i} (f'''g - fg''') \right). \quad (3.1)$$

Under this commutation relation, adjoint vectors  $l_m = (iz^{m+1}, 0)$  satisfy

$$i[l_m, l_n] = (m-n)l_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \quad (3.2)$$

A general coadjoint vector has the form  $b(z)dz^2 + tc^*$  where  $t$  is a complex number and  $c^*$  a dual element of  $c$ . We abbreviate it as  $(b, t)$ . The dual pairing between coadjoint and adjoint vectors is defined as

$$\langle (b, t), (g, a) \rangle = -i \oint \frac{dz}{2\pi i} b(z)g(z) + ta. \quad (3.3)$$

The coadjoint action of the Virasoro group is given by

$$\delta_f(b, t) = \left( fb' + 2f'b + \frac{t}{12}f''', 0 \right). \quad (3.4)$$

The dual pairing is invariant under the action of the Virasoro algebra,  $\delta_f \langle (b, t), (g, a) \rangle = 0$ . A coadjoint orbit is defined as an orbit of a coadjoint vector by coadjoint action of the conformal group. Let us consider an orbit of  $\left(\frac{b_0}{z^2}, t\right)$ , we denote it as  $W_{b_0}$ , where  $b_0$  is a complex number. From (3.4) for  $b = \frac{b_0}{z^2}$  and  $f = \sum_n f_n z^{n+1}$ , we find

$$\delta_f b = 2 \sum_n n f_n \left[ b_0 + \frac{t}{24}(n^2 - 1) \right] z^{n-2}, \quad (3.5)$$

thus  $b_0$  is invariant under coadjoint actions of the Virasoro algebra. There are two classes of orbits. For general  $b_0$ ,  $\delta_f b$  vanishes for  $f = z$ , thus the stability group of this type of orbits

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<sup>6</sup>In appendix B, we show that the Virasoro algebra with a pure imaginary central charge is obtained by a standard canonical quantization of the  $SL(2;\mathbb{C})$  Chern-Simons theory.

is dilatation. If  $b_0 = -\frac{t}{24}(m^2 - 1)$  where  $m \in Z_{>0}$ , there are three solutions  $f = z, z^{1\pm m}$  of  $\delta_f b = 0$ .

For  $z_s = s(z)$ , the integrated form of (3.4) is given by

$$b^s(z_s) = \left(\frac{dz_s}{dz}\right)^{-2} \left[ \frac{b_0}{z^2} - \frac{t}{12} \{z_s, z\} \right], \quad (3.6)$$

where  $\{z_s, z\}$  is the Schwarzian derivative  $\{z_s, z\} \equiv \frac{d^3 z_s / dz^3}{dz_s / dz} - \frac{3}{2} \left( \frac{d^2 z_s / dz^2}{dz_s / dz} \right)^2$ . Then  $b^s$  is an element of  $W_{b_0}$ . The coadjoint orbit representations of generators of the Virasoro algebra  $l_m = (iz^{m+1}, 0)$  are given by

$$L_m = \langle (b^s, t), (iz^{m+1}, 0) \rangle = \oint \frac{dz_s}{2\pi i} b^s(z_s) z_s^{m+1}. \quad (3.7)$$

Since the central element corresponds to  $(0, 1)$ , it has a constant value  $c = t$  on the orbit. Therefore we denote  $t$  as  $c$ . A canonical symplectic structure is introduced on each coadjoint orbit by the dual pairing [28, 29],

$$\begin{aligned} \Omega|_{b^s}(u, v) &= \langle (b^s, t), [(u, 0), (v, 0)] \rangle \\ &= -i \oint \frac{dz_s}{2\pi i} b^s(z_s) (uv' - u'v)(z_s) - \frac{ic}{24} \oint \frac{dz_s}{2\pi i} (uv''' - u'''v)(z_s), \end{aligned} \quad (3.8)$$

where  $u = u(z_s)$  and  $v = v(z_s)$  are tangent vectors at  $b^s$ . It should be noted that the semi-classical analysis is justified in the region of large  $c$  because the symplectic form  $\Omega$  is large in that region. Under this symplectic structure  $L_m$ 's (3.7) satisfy the classical Virasoro algebra

$$i\{L_m, L_n\} = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \quad (3.9)$$

### 3.2 Quantization of Coadjoint Orbits

We first make a perturbative analysis of the orbit  $W_{b_0}$  at the classical level. We set

$$z_s = s(z) = z + i \sum_{n \neq 0} s_n z^{-n+1}, \quad (3.10)$$

where  $s_n$  is a complex number. Then  $b^s$  is expanded as the following form from the formula (3.6)

$$\begin{aligned} b^s(z_s) &= \frac{b_0}{z^2} + 2i \sum_n (n-1) \left[ b_0 + \frac{c}{24} n(n+1) \right] s_n z^{-n-2} \\ &\quad - 3 \sum_{m,n} (m-1)(n-1) \left[ b_0 + \frac{c}{24} (m^2 + n^2 + mn + m + n) \right] s_m s_n z^{-m-n-2} + \mathcal{O}(s^3). \end{aligned} \quad (3.11)$$

The explicit form of the symplectic form is

$$\Omega(s, s') = -i \sum_m \left[ 2mb_0 + \frac{c}{12}(m^3 - m) \right] s_m s'_{-m} + \mathcal{O}(s^3). \quad (3.12)$$

From the inverse matrix of  $\Omega$ , the following Poisson bracket is obtained

$$\{s_m, s_n\} = -i \left[ 2mb_0 + \frac{c}{12}(m^3 - m) \right]^{-1} \delta_{m+n,0} + \mathcal{O}(s). \quad (3.13)$$

The expressions for the generators of the Virasoro algebra are

$$L_m = 2im \left[ b_0 + \frac{c}{24}(m^2 - 1) \right] s_m + \mathcal{O}(s^2), \quad \text{for } m \neq 0, \quad (3.14)$$

$$L_0 = b_0 + \sum_n n^2 \left[ b_0 + \frac{c}{24}(n^2 - 1) \right] s_{-n}s_n + \mathcal{O}(s^3). \quad (3.15)$$

A discussion of the stability of vacua for “energy function”  $L_0$  in the anti-de Sitter gravity was provided by the similar perturbative method [18]. In that case, the reality condition  $(s_n)^* = s_{-n}$  was imposed because the coordinate corresponding to  $z$  was a real parameter which parametrizes  $S^1$ . Therefore  $s_ns_{-n}$  was positive definite and a condition for vacua to be stable was obtained from the requirement that  $L_0$  should be bounded from below. However  $z$  is now a complex coordinate and thus no restriction is imposed on  $s_n$  at the classical level.

We next consider a semi-classical quantization of the orbits. It is performed by replacing the Poisson bracket (3.13) to  $-i$  times the commutator. Then we find that  $s_n$  are essentially creation and annihilation operators up to their normalizations. We are interested in the unitary representations of the Virasoro algebra in which  $(L_m)^\dagger = L_{-m}$ , in particular  $(L_0)^\dagger = L_0$ , holds. This unitarity condition leads to the conditions  $(s_m)^\dagger = s_{-m}$  and  $b_0 \in \mathbb{R}$  up to this order of perturbations.<sup>7</sup> The stability of the classical vacua means that  $L_0$  should be bounded below at  $b(z) = \frac{b_0}{z^2}$ . It holds true only when

$$b_0 \geq 0. \quad (3.16)$$

Therefore only orbits with  $b_0 \in \mathbb{R}_{\geq 0}$  provide the unitary representations of the Virasoro algebra with the stable vacua.

The quantum Virasoro algebra is given by replacing the Poisson bracket to  $-i$  times the commutator in the classical Virasoro algebra (3.9),

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \quad (3.17)$$

Each unitary representation of the Virasoro algebra is labeled by its highest weight state  $|h\rangle$ . The highest weight state satisfies  $L_0|h\rangle = h|h\rangle$  and  $L_n|h\rangle = 0$  for  $n \geq 1$ . In particular  $|0\rangle$  satisfy  $L_{-1}|0\rangle = 0$  in addition to the above conditions. The unitary irreducible representations  $\mathcal{V}_{h=b_0}$  corresponding to  $W_{b_0}$  ( $b_0 > 0$ ) are constructed by acting the Virasoro generators  $L_{-n}$  ( $n \geq 1$ ) on  $|b_0\rangle$ . The unitary irreducible representation  $\mathcal{V}'_{h=0}$  corresponding to  $W_{b_0=0}$  is given by acting  $L_{-n}$  ( $n \geq 2$ ) on  $|0\rangle$ .

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<sup>7</sup>This type of Hermitian conjugation is also considered in view of the dS/CFT correspondence in Ref.[7].

### 3.3 Quantization of Gravity in $dS_3$

The phase space of the three-dimensional de Sitter gravity is composed of de Sitter, Kerr-de Sitter geometries and their Virasoro deformations. The de Sitter space and its Virasoro deformations are identified with the product of the Virasoro coadjoint orbits  $W_{b_0=0} \times \bar{W}_{\bar{b}_0=0}$ . The quantizations of the orbits lead to the unitary irreducible representations of the Virasoro algebras  $\mathcal{V}'_{h=0} \times \bar{\mathcal{V}}'_{\bar{h}=0}$ . The Kerr-de Sitter geometry with  $b = \frac{b_0}{z^2}$  and  $\bar{b} = \frac{\bar{b}_0}{\bar{z}^2}$  and their deformations  $Z_{(b,\bar{b})}$  are identified with the product of the orbits  $W_{b_0} \times \bar{W}_{\bar{b}_0}$ . If  $b_0$  is real, that is  $b_0 = \bar{b}_0$ , and  $b_0 > 0$ , the quantizations of the orbits lead to the unitary irreducible representations of the Virasoro algebras  $\mathcal{V}_{b_0} \times \bar{\mathcal{V}}_{\bar{b}_0}$ . The unitarity condition that  $b_0$  is real means  $J = 0$ , thus  $b_0 = \frac{c}{24}(1 - \mu)$  from (2.8). In addition to that, the stability condition requires  $b_0 \geq 0$ . Therefore  $\mu$  is restricted in the region  $\mu \leq 1$ . Geometries with excess angles ( $\mu > 1$ ) are forbidden by these conditions. As a result, the quantizations of deformations only of de Sitter and conical geometries without angular momenta provide the unitary theories with stable vacua. On the other hand, quantizations of the Kerr-de Sitter space-times with non-vanishing angular momenta and their deformations do not give unitary theories.

We provide some comments for the geometries with  $J = 0$  and  $\mu < 0$ . As noted in the previous section, these geometries seem to be unphysical in the Schwarzschild coordinates. However, according to the discussion in this section, they are naturally included in the phase space constructed here and are represented by the following region in  $SL(2;\mathbb{C})/SU(1,1)$ ,

$$\begin{aligned} \mathcal{G}_{(\mu < 0)} \equiv & \frac{1}{m} \begin{pmatrix} -\frac{i(1+i\sqrt{m})}{2}z^{-\frac{1-i\sqrt{m}}{2}} & -z^{\frac{1+i\sqrt{m}}{2}} \\ \frac{1-i\sqrt{m}}{2}z^{-\frac{1-i\sqrt{m}}{2}} & -iz^{\frac{1-i\sqrt{m}}{2}} \end{pmatrix} \begin{pmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{pmatrix} \\ & \times \begin{pmatrix} \frac{i(1-i\sqrt{m})}{2}\bar{z}^{-\frac{1+i\sqrt{m}}{2}} & -\frac{1+i\sqrt{m}}{2}\bar{z}^{-\frac{1-i\sqrt{m}}{2}} \\ \bar{z}^{\frac{1-i\sqrt{m}}{2}} & i\bar{z}^{\frac{1+i\sqrt{m}}{2}} \end{pmatrix}, \end{aligned} \quad (3.18)$$

with  $m \equiv -\mu$ . The induced metric on this region is given by

$$ds_{(\mu < 0)}^2 = l^2 \left[ -d\tau^2 + \frac{1+m}{4} \frac{dz^2}{z^2} + \frac{1+m}{4} \frac{d\bar{z}^2}{\bar{z}^2} + \left( e^{-2\tau} + \frac{(1+m)^2}{16} \frac{1}{z^2 \bar{z}^2} e^{2\tau} \right) dz d\bar{z} \right], \quad (3.19)$$

that is  $ds_{(\mu < 0)}^2 = ds_{(b_0, \bar{b}_0)}^2$  with  $b_0 = \bar{b}_0 = \frac{c}{24}(1 + m)$ . The Virasoro deformations of these geometries can be defined by the same way as those explained until here. After quantization, furthermore, the states of these geometries are allowed under our consideration about the unitarity and stability at the perturbative level.

## 4 On Three-dimensional de Sitter Gravity and Liouville Field Theory

It is well-known that the Chern-Simons action on a manifold with boundaries reduces to the chiral WZNW actions on the boundaries [22]. It was shown that the action of the three-dimensional anti-de Sitter gravity which is written by two  $\text{SL}(2;\mathbb{R})$  Chern-Simons actions reduces to the non-chiral WZNW action and furthermore it reduces to the Liouville field theory under the boundary conditions being the asymptotically anti-de Sitter space-time [23]. Recently analogous analyses were done in [10] for the three-dimensional de Sitter gravity with the asymptotically past de Sitter space-time.

### 4.1 Reduction to Liouville Field Theory

We briefly review processes of the reduction to the Liouville field theory [10] in order to fix the notations we use here.

The Einstein action is given by the  $\text{SL}(2;\mathbb{C})$  Chern-Simons action (2.13). The reduction is carried out by two stages. At the first stage, we impose the following reduction conditions,

$$A_{\bar{z}} \longrightarrow 0, \quad \bar{A}_z \longrightarrow 0, \quad (4.1)$$

at the infinitely past  $\tau \longrightarrow -\infty$ . These conditions are same as (2.25). Under these conditions and the gauge fixing  $A_\tau = -iJ^0$ ,  $\bar{A}_\tau = iJ^0$ , the flat connections are written as

$$A = G^{-1}dG, \quad \bar{A} = \bar{G}^{-1}d\bar{G}, \quad (4.2)$$

where  $G$  and  $\bar{G}$  are group elements of  $\text{SL}(2;\mathbb{C})$  and behave at  $\tau \longrightarrow -\infty$  as

$$G \longrightarrow g(z, \bar{z}) \begin{pmatrix} e^{\frac{1}{2}\tau} & 0 \\ 0 & e^{-\frac{1}{2}\tau} \end{pmatrix}, \quad \bar{G} \longrightarrow \bar{g}(z, \bar{z}) \begin{pmatrix} e^{-\frac{1}{2}\tau} & 0 \\ 0 & e^{\frac{1}{2}\tau} \end{pmatrix}. \quad (4.3)$$

Substituting these flat connections into the  $\text{SL}(2;\mathbb{C})$  Chern-Simons action, we obtain two chiral WZNW action in terms of  $g(z, \bar{z})$  and  $\bar{g}(z, \bar{z})$  respectively. Then changing the variables,

$$H = G^{-1}\bar{G}, \quad h = g^{-1}\bar{g}, \quad (4.4)$$

two chiral WZNW actions are combined to the non-chiral WZNW action,

$$S = \frac{ik}{4\pi} \int_{\partial M} dz \wedge d\bar{z} \text{Tr} (h^{-1} \partial_z h h^{-1} \partial_{\bar{z}} h) + \frac{ik}{12\pi} \int_M \text{Tr} (H^{-1} dH)^3. \quad (4.5)$$

Here we provide some comments to this WZNW action.  $H$  and  $h$  must satisfy  $H = \epsilon H^\dagger \epsilon^{-1}$  and  $h = \epsilon h^\dagger \epsilon^{-1}$  for this theory to describe the de Sitter gravity. Thus this is actually  $\text{SL}(2;\mathbb{C})/\text{SU}(1,1)$

WZNW theory. Next the composition (4.4) is different from the pairing (2.20) which we defined in the previous section to clarify the connection between the  $SL(2;\mathbb{C})$  flat connections and the space-time metric.

At the second stage of the reduction, conditions which are on-shell equivalent to the equations (2.26) in the previous section are imposed on the conserved currents of the WZNW theory. We can explicitly describe them by using the Gauss decomposition

$$h = \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} \begin{pmatrix} e^{-\frac{1}{2}\tilde{\Phi}} & 0 \\ 0 & e^{\frac{1}{2}\tilde{\Phi}} \end{pmatrix} \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix}. \quad (4.6)$$

Then the constraints are written by

$$\begin{cases} J_z^0 = 2i \left( -\frac{1}{2}\partial_z \tilde{\Phi} - e^{-\tilde{\Phi}} X \partial_z Y \right) = 0 \\ \frac{1}{2} (J_z^1 + iJ_z^2) = -e^{-\tilde{\Phi}} \partial_z Y = -i, \end{cases} \quad \begin{cases} \bar{J}_{\bar{z}}^0 = 2i \left( -\frac{1}{2}\partial_{\bar{z}} \tilde{\Phi} - e^{-\tilde{\Phi}} Y \partial_{\bar{z}} X \right) = 0 \\ \frac{1}{2} (\bar{J}_{\bar{z}}^1 - i\bar{J}_{\bar{z}}^2) = -e^{-\tilde{\Phi}} \partial_{\bar{z}} X = -i, \end{cases} \quad (4.7)$$

where  $J_z = \partial_z h h^{-1}$  and  $\bar{J}_{\bar{z}} = h^{-1} \partial_{\bar{z}} h$ . After applying the constraints, the WZNW action (4.5) reduces to one of the Liouville field theory,

$$S = \frac{ik}{4\pi} \int_{\partial M} dz \wedge d\bar{z} \left( \frac{1}{2} \partial_z \tilde{\Phi} \partial_{\bar{z}} \tilde{\Phi} + 2e^{\tilde{\Phi}} \right). \quad (4.8)$$

Moreover transforming to the conformal gauge of the metric on  $\partial M$  and redefining  $\tilde{\Phi} = \gamma\Phi$ , the action is written as

$$S = \frac{1}{4\pi} \int_{\partial M} d^2x \left[ \frac{1}{2} h^{ij} \partial_i \Phi \partial_j \Phi + \frac{\lambda}{2\gamma^2} e^{\gamma\Phi} + \frac{Q}{2} R\Phi \right], \quad (4.9)$$

where  $Q = \frac{2}{\gamma} = \sqrt{\frac{l}{2G}}$  and  $\lambda = \frac{16}{l^2}$ .

## 4.2 3d gravity and Liouville Field Theory

General solutions of the equation of motion of the Liouville theory are locally given by the following form [30]

$$e^{\gamma\Phi} = \frac{\partial_z F \partial_{\bar{z}} \bar{F}}{(1 - F\bar{F})^2} \quad (4.10)$$

where  $F$  is a holomorphic function  $F = F(z)$  and  $\bar{F}$  is anti-holomorphic  $\bar{F} = \bar{F}(\bar{z})$ . While  $e^{\gamma\Phi}$  must be invariant under  $z \rightarrow e^{2\pi i} z$ ,  $F$  and  $\bar{F}$  are not required to have this invariance.

We give explicit connections between these classical solutions and three-dimensional geometries. From the constraints (4.7) we find

$$X = \frac{i\gamma}{2} \partial_z \Phi, \quad Y = \frac{i\gamma}{2} \partial_{\bar{z}} \Phi. \quad (4.11)$$

Substituting these relations and the classical solutions (4.10) into the Gauss decomposition of  $h$  (4.6),  $h$  can be written as the decomposed form into holomorphic and anti-holomorphic parts,

$$h \equiv g(z)^{-1} \bar{g}(\bar{z}) = \begin{pmatrix} \frac{1}{\sqrt{\partial_z \bar{F}}} & \frac{iF}{\sqrt{\partial_z \bar{F}}} \\ \frac{i}{2} \frac{\partial_z^2 F}{(\partial_z F)^{3/2}} & \frac{(\partial_z F)^2 - \frac{1}{2} F \partial_z^2 F}{(\partial_z F)^{3/2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{\partial_z \bar{F}}} & \frac{i}{2} \frac{\partial_z^2 \bar{F}}{(\partial_z \bar{F})^{3/2}} \\ \frac{i\bar{F}}{\sqrt{\partial_z \bar{F}}} & \frac{(\partial_z \bar{F})^2 - \frac{1}{2} \bar{F} \partial_z^2 \bar{F}}{(\partial_z \bar{F})^{3/2}} \end{pmatrix}. \quad (4.12)$$

By following the inverse procedures of the reduction from the Chern-Simon theory to the Liouville theory, the  $SL(2; \mathbb{C})$  flat connections corresponding to (4.10) can be obtained as

$$A = \begin{pmatrix} \frac{1}{2} d\tau & -ie^{-\tau} dz \\ -\frac{i}{2} \{F, z\} dz & -\frac{1}{2} d\tau \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} -\frac{1}{2} d\tau & \frac{i}{2} \{\bar{F}, \bar{z}\} d\bar{z} \\ ie^{-\tau} d\bar{z} & \frac{1}{2} d\tau \end{pmatrix}. \quad (4.13)$$

We identify  $A$  and  $\bar{A}$  with the generalized solutions  $A_b$  and  $\bar{A}_{\bar{b}}$  (2.24). Then  $b$  ( $\bar{b}$ ) is written by the Schwarzian derivative of  $F$  ( $\bar{F}$ ),

$$b(z) = \frac{c}{12} \{F(z), z\}, \quad \bar{b}(\bar{z}) = \frac{c}{12} \{\bar{F}(\bar{z}), \bar{z}\}. \quad (4.14)$$

That is,  $b(z)$  ( $\bar{b}(\bar{z})$ ) coincides with the stress tensor of the holomorphic (anti-holomorphic) part of the Liouville theory. By using  $F$  and  $\bar{F}$ ,  $G_b$  and  $\bar{G}_{\bar{b}}$  (2.32) are written as  $G_b = g(z)e^{-i\tau J^0}$  and  $\bar{G}_{\bar{b}} = \bar{g}(\bar{z})e^{i\tau J^0}$ . Thus we can give the explicit form of the region corresponding to the deformed space-time,

$$\begin{aligned} \mathcal{G}_{(b, \bar{b})} &= G_b \bar{G}_{\bar{b}}^{-1} \\ &= \begin{pmatrix} \frac{(\partial_z F)^2 - \frac{1}{2} F \partial_z^2 F}{(\partial_z F)^{3/2}} & -\frac{iF}{\sqrt{\partial_z \bar{F}}} \\ -\frac{i}{2} \frac{\partial_z^2 F}{(\partial_z F)^{3/2}} & \frac{1}{\sqrt{\partial_z \bar{F}}} \end{pmatrix} \begin{pmatrix} e^\tau & 0 \\ 0 & e^{-\tau} \end{pmatrix} \begin{pmatrix} \frac{(\partial_z \bar{F})^2 - \frac{1}{2} \bar{F} \partial_z^2 \bar{F}}{(\partial_z \bar{F})^{3/2}} & -\frac{i}{2} \frac{\partial_z^2 \bar{F}}{(\partial_z \bar{F})^{3/2}} \\ -\frac{i\bar{F}}{\sqrt{\partial_z \bar{F}}} & \frac{1}{\sqrt{\partial_z \bar{F}}} \end{pmatrix}, \end{aligned} \quad (4.15)$$

Setting  $F = F_0 \equiv z^a$  and  $\bar{F} = \bar{F}_0 \equiv \bar{z}^{\bar{a}}$ , the region  $\mathcal{G}_0$  (2.11) corresponding to the Kerr-de Sitter solution is obtained.

We would like clearly to see the relationship between the phase space of the three-dimensional gravity and that of the Liouville theory. Under the conformal transformation  $z \rightarrow z_s = s(z)$ ,  $F(z)$  is transformed as scalar,  $F^s(z_s) = F(z)$ . Then  $b(z)$  (4.14) correctly transforms as the stress tensor,

$$\begin{aligned} b(z) \rightarrow b^s(z_s) &= \frac{c}{12} \{F^s(z_s), z_s\} \\ &= \left( \frac{dz_s}{dz} \right)^{-2} \left[ b(z) - \frac{c}{12} \{z_s, z\} \right]. \end{aligned} \quad (4.16)$$



The infinitesimal form of the transformation is  $\delta_f F = F' f$  and  $\delta_f b = f b' + 2f' b + \frac{c}{12} f'''$ , where we took  $z_s = z - f(z)$  and  $f$  is a single valued function on the complex plane. Setting  $F = F_0$  and  $f = -i \sum_n u_n z^{-n+1}$ , in particular, we consider deformations of the solution corresponding to the Kerr-de Sitter geometry,

$$F_f(z) = F_0 + \delta_f F_0 = z^a \left( 1 - i a \sum_n u_n z^{-n} \right). \quad (4.17)$$

For this  $F_f$ , the stress tensor (4.14) becomes

$$\begin{aligned} b_f(z) = & \frac{1}{z^2} \left\{ \frac{c}{24} (1 - a^2) + \frac{ic}{12} \sum_n n (a^2 - n^2) u_{-n} z^n \right. \\ & \left. - \frac{c}{24} \sum_{m,n} (a+m)(a+n) [(m+n)a - m^2 - n^2 - 3mn] u_{-m} u_{-n} z^{m+n} \right\} + \mathcal{O}(u^3). \end{aligned} \quad (4.18)$$

Then we find

$$\begin{aligned} L_m &= \oint \frac{dz}{2\pi i} b(z) z^{m+1} = 2im \left[ b_0 + \frac{c}{24} (m^2 - 1) \right] u_m + \mathcal{O}(u^2), \quad \text{for } m \neq 0 \\ L_0 &= \oint \frac{dz}{2\pi i} b(z) z = b_0 + \sum_n n^2 \left[ b_0 + \frac{c}{24} (n^2 - 1) \right] u_{-n} u_n + \mathcal{O}(u^3), \end{aligned} \quad (4.19)$$

where  $b_0 = \frac{c}{24} (1 - a^2)$ . These coincide with the perturbative expressions, (3.14) and (3.15), of the Virasoro generators if we identify  $u_m$  with  $s_m$ . Therefore we obtained the correspondences between the infinitesimal Virasoro deformations of the Kerr-de Sitter geometries and those of configurations of the Liouville field. Thus the quantization of this Liouville theory is expected to reproduce the same unitary irreducible representations of the Virasoro algebras as those of the three-dimensional gravity obtained in the previous section.

In the quantum Liouville field theory [30, 31], the primary field is given by  $e^{\alpha\Phi}$  ( $\alpha \in \mathbb{C}$ ) with weight  $-\frac{1}{2}\alpha(\alpha - Q)$ . And the primary state is defined by  $|\alpha\rangle \equiv \lim_{z \rightarrow 0} e^{\alpha\Phi}(z)|0\rangle$  which satisfies

$$L_0|\alpha\rangle = -\frac{1}{2}\alpha(\alpha - Q)|\alpha\rangle, \quad L_n|\alpha\rangle = 0 \quad \text{for } n \geq 1, \quad (4.20)$$

where  $|0\rangle$  is the  $\text{SL}(2;\mathbb{C})$  invariant vacuum,  $L_n|0\rangle = 0$  for  $n \geq -1$ . The representations of the Virasoro algebra are constructed by acting  $L_{-n}$  ( $n \geq 1$ ) on  $|\alpha\rangle$  and  $L_{-n}$  ( $n \geq 2$ ) on  $|0\rangle$ , respectively.

The primary state  $|\alpha\rangle \otimes |\alpha\rangle$  is identified with the state of the point particle without angular momentum in the 3d gravity by matching the conformal weight with  $b_0 = \frac{c}{24}(1 - \mu)$ ,

$$\alpha = \sqrt{\frac{c}{12}} \left[ 1 - \sqrt{1 - \frac{b_0}{c/24}} \right], \quad (4.21)$$

where  $0 \leq b_0 \leq \frac{c}{24}$ , thus  $\alpha$  is real and  $0 \leq \alpha \leq \sqrt{\frac{c}{12}}$ . This upper bound for the value of  $\alpha$  corresponds to the Seiberg bound. The state with  $\alpha$  within this range is non-normalizable. The excitations on the primary states can be identified with the excitations by the deformations of the point particle state, as explained in terms of classical perturbations. In the case of the geometries with  $\mu < 0$  and  $J = 0$ , their counterparts in the Liouville field theory are the primary states  $|\alpha\rangle \otimes |\alpha\rangle$  with  $\alpha = \sqrt{c/12} [1 \pm i\sqrt{m}]$  where  $m = -\mu > 0$ . These states are normalizable because  $Re(\alpha) = \sqrt{c/12}$ . These sorts of states are allowed by our discussion about the unitarity and stability in the gravity side. On the other hand, we have concluded that the states of the Kerr-de Sitter geometries ( $J \neq 0$ ) and their deformations do not provide the unitary theory. Those states would correspond to states in representations constructed on the primary states of the Liouville theory with complex value of  $\alpha$  which is given by eq.(4.21) with  $b_0 = \frac{c}{24}(1-\mu) - \frac{i}{2}J$ . It is known from studies of the Liouville theory that these representations with  $Re(\alpha) \neq \sqrt{c/12}$  are not unitary.

## 5 Discussions

We would like to discuss relations between the holonomies of  $SL(2;\mathbb{C})$  flat connections and the zero modes of the Liouville field. The geometries we have considered here have the closed paths  $z \sim e^{2\pi i} z$  and  $\bar{z} \sim e^{-2\pi i} \bar{z}$ . The holonomies of the  $SL(2;\mathbb{C})$  flat connections (2.23), which correspond to the Kerr-de Sitter metric, along these paths are given by

$$\begin{aligned} \text{Tr } Pe^{\oint A} &= \text{Tr } G_0^{-1}(z, \tau) G_0(e^{2\pi i} z, \tau) = -2 \cos(a\pi), \\ \text{Tr } Pe^{\oint \bar{A}} &= \text{Tr } \bar{G}_0^{-1}(\bar{z}, \tau) \bar{G}_0(e^{-2\pi i} \bar{z}, \tau) = -2 \cos(\bar{a}\pi), \end{aligned} \quad (5.1)$$

where  $a = \sqrt{1 - \frac{b_0}{c/24}}$  and  $\bar{a} = \sqrt{1 - \frac{\bar{b}_0}{c/24}}$  as defined before. Thus the Kerr-de Sitter metric can be characterized by these holonomies as well its mass and angular momentum. Substituting the solution corresponding to this geometry in the Liouville theory into the right hand side in the eq.(4.10), we find

$$\Phi \sim \frac{1}{\gamma} \ln a + \frac{(a-1)}{\gamma} \ln z + \frac{1}{\gamma} \ln \bar{a} + \frac{(\bar{a}-1)}{\gamma} \ln \bar{z}. \quad (5.2)$$

Here we ignore the denominator of the right hand side in the eq.(4.10), that is, the effect of the Liouville potential, thus this equality is not strict. However comparing this equation with the following expansion  $\Phi \sim x - ip \ln z + (\text{anti-holomorphic part})$ , we obtain

$$x = \frac{1}{\gamma} \ln a, \quad p = \frac{i(a-1)}{\gamma}. \quad (5.3)$$

These correspond to the relations which used in the Ref.[18] to investigate a quantization of three-dimensional gravity with a negative cosmological constant in the view of the Liouville

field theory. As pointed out in that paper, we should also consider quantization of these global degrees of freedom, holonomies, within the three-dimensional gravity.

We think that more accurate analyses are needed in order to completely understand the quantum nature of the three-dimensional de Sitter gravity. First we implicitly assumed in this paper that the states obtained by quantization of the phase space of the gravity, e.g., de Sitter or point particle geometries, are normalizable. On the other hand the primary states corresponding to the point particle geometries are non-normalizable in the Liouville field theory. To clarify origin of this discrepancy, it is needed to study the global structure of the phase space and to strictly define the inner product between quantum states in the de Sitter gravity. Next we stated in Section 2 that the flat  $SL(2; \mathbb{C})$  connections  $A$  and  $\bar{A}$  can not be independent mutually so that the dreibein and spin connection are real quantities. This requirement connects the holomorphic part of the degrees of freedom of the theory with the anti-holomorphic part, that is,  $\bar{b}(\bar{z})$  should be a complex conjugate function of  $b(z)$ . On the other hand, we imposed  $L_m^\dagger = L_{-m}$ , which are relations among the modes of the holomorphic part  $b(z)$ , as the unitarity condition. It might be possible to adopt other conditions which relate  $b(z)$  to  $\bar{b}(\bar{z})$  by Hermitian conjugation.<sup>8</sup>

In this paper we used the coordinate system corresponding to the planar coordinate of de Sitter space. In general quantum theory might depend on the choice of the coordinate system, e.g., the vacua corresponding to different coordinates could be connected with each other by the Bogoliubov transformations. Therefore it is meaningful to investigate quantum theory of the de Sitter gravity in other coordinate systems, particularly in the global coordinate.

## Acknowledgment

This work of H. U. was supported in part by JSPS Research Fellowships for Young Scientists.

## A Brown-York Stress Tensor

Here we show that the Brown-York stress tensor [32] for general solutions we used coincides with  $b(z)$  and  $\bar{b}(\bar{z})$ .

Since the equation (2.15) is satisfied for solutions of the equations of motion, we consider the action<sup>9</sup>

$$S = \frac{1}{16\pi G} \int_M d^3x \sqrt{-g} \left( R - \frac{2}{l^2} \right) - \frac{i}{8\pi G} \int_{\partial M} dz \wedge d\bar{z} \frac{\sqrt{h}}{l}. \quad (\text{A.1})$$

For the space-time metric

$$ds^2 = l^2 \left[ -dt^2 + h_{zz} dz^2 + h_{\bar{z}\bar{z}} d\bar{z}^2 + 2h_{z\bar{z}} dz d\bar{z} \right], \quad (\text{A.2})$$

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<sup>8</sup>Recently this type of Hermitian conjugations has been discussed in Ref.[15]

<sup>9</sup>This action originated from the Chern-Simons action is on-shell equivalent to the gravitational action including the boundary term and the counter term considered in Ref.[32, 5, 11].

the Brown-York stress tensors are given by

$$T_{zz} = \frac{2}{\sqrt{h}} \frac{\delta S}{\delta h^{zz}} = \frac{1}{8\pi G} \left[ R_{zz} - \frac{1}{2} h_{zz} \left( R - \frac{2}{l^2} \right) \right] + \frac{1}{8\pi G l} \left[ h_{zz} + \frac{1}{2} \partial_t h_{zz} \right]_{\partial M}, \quad (\text{A.3})$$

$$T_{\bar{z}\bar{z}} = \frac{2}{\sqrt{h}} \frac{\delta S}{\delta h^{\bar{z}\bar{z}}} = \frac{1}{8\pi G} \left[ R_{\bar{z}\bar{z}} - \frac{1}{2} h_{\bar{z}\bar{z}} \left( R - \frac{2}{l^2} \right) \right] + \frac{1}{8\pi G l} \left[ h_{\bar{z}\bar{z}} + \frac{1}{2} \partial_t h_{\bar{z}\bar{z}} \right]_{\partial M}, \quad (\text{A.4})$$

$$T_{z\bar{z}} = \frac{1}{\sqrt{h}} \frac{\delta S}{\delta h^{z\bar{z}}} = \frac{1}{8\pi G} \left[ R_{z\bar{z}} - \frac{1}{2} h_{z\bar{z}} \left( R - \frac{2}{l^2} \right) \right] + \frac{1}{8\pi G l} \left[ h_{z\bar{z}} + \frac{1}{2} \partial_t h_{z\bar{z}} \right]_{\partial M}. \quad (\text{A.5})$$

Substituting the general space-time metric (2.31), the bulk terms of the above stress tensors vanish and only boundary contributions remain,

$$T_{zz} = \frac{1}{2\pi l^2} b(z), \quad T_{\bar{z}\bar{z}} = \frac{1}{2\pi l^2} \bar{b}(\bar{z}), \quad T_{z\bar{z}} = 0. \quad (\text{A.6})$$

## B Canonical Quantization of $SL(2; \mathbb{C})$ Chern-Simons Gravity

In this appendix, as mentioned in Section 3, we make a canonical quantization à la Dirac of three-dimensional de Sitter gravity described by the action (2.13). (The case of three-dimensional anti-de Sitter gravity is discussed in Ref.[33, 34, 35, 17, 25].) Then we find the Virasoro algebra with a *pure imaginary* central charge as the asymptotic symmetry.

$SL(2; \mathbb{C})$  Chern-Simons action is defined from (2.13):<sup>10</sup>

$$S_{EH} = S_{CS}(A) - S_{CS}(\bar{A}),$$

$$S_{CS}(A) = \frac{ik}{4\pi} \int_M \text{Tr} \left( A dA + \frac{2}{3} A^3 \right) \quad (\text{B.1})$$

$$= \frac{ik}{4\pi} \int_M d\tau \wedge d\rho \wedge d\sigma \, \epsilon^{\mu\nu\rho} \left( \eta_{ab} A_\mu^a \partial_\nu A_\rho^b + \epsilon_{abc} A_\mu^a A_\nu^b A_\rho^c \right), \quad (\text{B.2})$$

where  $A$  is the  $sl(2; \mathbb{C})$ -valued 1-form related to dreibein and spin connection by (2.14).  $M$  is a three-dimensional manifold whose topology is  $\mathbf{R} \times \mathbf{R}^2$ .  $M$  is parameterized by  $(\tau, \rho, \sigma)$  whose region is  $(-\infty < \tau < \infty, 0 \leq \rho < \infty, 0 \leq \sigma < 2\pi)$ . We concentrate on  $S_{CS}(A)$  part only in the sequel since the discussion on  $S_{CS}(\bar{A})$  part is parallel.

$M$  is a non-compact manifold with a two-dimensional boundary  $\tau \rightarrow -\infty$  and  $\rho \rightarrow \infty$ . Thus we require the boundary condition for  $A$  so that the action (B.2) is differentiable with respect to  $A_\rho$  and  $A_\sigma$ :

$$A_{\bar{z}} = 0 \quad \text{at} \quad \tau \rightarrow -\infty, \quad (\text{B.3})$$

where  $z = e^{\rho+i\sigma}$  and  $A_z = (A_\rho - iA_\sigma)/2z$ ,  $A_{\bar{z}} = (A_\rho + iA_\sigma)/2\bar{z}$ . This boundary condition is the same as (2.25). As for the boundary  $\rho \rightarrow \infty$  we include the boundary term in the Chern-Simons action so that the action is differentiable with respect to  $A_\tau^a$ .

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<sup>10</sup> $\epsilon^{\tau\rho\sigma} = +1$ .

We make a canonical quantization of the Chern-Simons action  $S_{CS}[A]$ .  $S_{CS}[A]$  is invariant under the gauge transformation  $\delta A_\mu^a = D_\mu \lambda^a$  ( $D_\mu \lambda^a = \partial_\mu \lambda^a + \epsilon^a_{bc} A_\mu^b \lambda^c$ ) and thus describes a constrained system. The standard recipe of Hamiltonian formalism of constrained systems yields the following constraints in addition to  $\pi_a^\tau = 0$  and the Gauss law constraint:<sup>11</sup>

$$\phi_a^i \equiv \pi_a^i + \frac{ik}{8\pi} \epsilon^{ij} \eta_{ab} A_j^b = 0 \quad (i, j = \rho, \sigma \text{ and } \epsilon^{\rho\sigma} = -1). \quad (\text{B.4})$$

We fix the gauge  $A_\tau = -iJ^0$  and will explicitly solve the Gauss law constraint later on. Thus the Dirac bracket of the remaining dynamical variable  $A_i^a$  under the constraint (B.4) becomes

$$\left\{ A_i^a(x), A_j^b(y) \right\}_D = \frac{4\pi}{ik} \epsilon_{ij} \eta^{ab} \delta^2(x-y) \quad (\epsilon_{\rho\sigma} = +1). \quad (\text{B.5})$$

By means of the Dirac bracket (B.5), the Dirac bracket between the functionals  $F[A]$  and  $G[A]$  of the dynamical variables  $A_i^a(x)$  is given by

$$\{F[A], G[A]\}_D = \frac{4\pi}{ik} \epsilon_{ij} \eta^{ab} \int_\Sigma d\rho \wedge d\sigma \frac{\delta F[A]}{\delta A_i^a(x)} \frac{\delta G[A]}{\delta A_j^b(x)}, \quad (\text{B.6})$$

where  $\Sigma$  denotes the space-like surface  $\mathbf{R}^2$  of constant  $\tau$ .

By using the bracket (B.6), the algebra of the generators of the gauge transformation  $\delta A_i^a = D_i \lambda^a$  can be obtained. The generator of the gauge transformation is given by the Gauss law constraint:

$$Q(\lambda) = -\frac{ik}{4\pi} \int_\Sigma d\rho \wedge d\sigma \eta_{ab} \lambda^a F_{\rho\sigma}^b + \frac{ik}{4\pi} \int_{\partial\Sigma} d\sigma \eta_{ab} \lambda^a A_\sigma^b, \quad (\text{B.7})$$

where  $F_{\rho\sigma}^a = \partial_\rho A_\sigma^a - \partial_\sigma A_\rho^a + \epsilon^a_{bc} A_\rho^b A_\sigma^c$ .  $\partial\Sigma$  denotes the boundary of  $\Sigma$  at  $\rho \rightarrow \infty$  and the boundary term is supplemented for  $Q(\lambda)$  to be differentiable with respect to  $A_i^a$ . From (B.6) and (B.7), one can easily find the algebra of the generator  $Q(\lambda)$ :

$$\{Q(\lambda), Q(\eta)\}_D = Q([\lambda, \eta]) - \frac{ik}{4\pi} \int_{\partial\Sigma} d\sigma \eta_{ab} \lambda^a \partial_\sigma \eta^b \quad \left( [\lambda, \eta]^a = \epsilon^a_{bc} \lambda^b \eta^c \right). \quad (\text{B.8})$$

As mentioned above, we solve explicitly the Gauss law constraint and the equations of motion of  $A_i^a$  under the boundary condition (B.3) and the gauge fixing condition for  $A_\tau^a$ . The general solutions are given by

$$A_\tau = -iJ^0 = g^{-1} \partial_\tau g, \quad A_{\bar{z}} = 0, \quad \text{and} \quad A_z = g^{-1} \hat{A}(z) g \quad (g = e^{-i\tau J^0}). \quad (\text{B.9})$$

The space of these general solutions is invariant under the gauge function  $\lambda^a = g^{-1} \hat{\lambda}^a(z) g$ . The generator of this gauge transformation gives the global charge of the Chern-Simons theory on

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<sup>11</sup>We take  $\tau$  as time here. Although, corresponding to the radial quantization of conformal field theory on  $\mathbf{R}^2$ , we can formally take  $\rho$  as “time”, the following discussion and the final result do not essentially change.

a manifold with the boundary [33]. On this classical phase space, the generator of the gauge transformation with this gauge function is reduced to

$$Q(\lambda) = g^{-1} \hat{Q}(\hat{\lambda}) g, \quad \hat{Q}(\hat{\lambda}) = \frac{ik}{4\pi} \oint_{\partial\Sigma} dz \, \eta_{ab} \hat{\lambda}^a(z) \hat{A}^b(z). \quad (\text{B.10})$$

We note that the generator is given by the boundary integral only.  $\hat{Q}(\hat{\lambda})$  satisfies the same algebra as that of  $Q(\lambda)$ :

$$\left\{ \hat{Q}(\hat{\lambda}), \hat{Q}(\hat{\eta}) \right\}_D = \hat{Q}([\hat{\lambda}, \hat{\eta}]) - \frac{ik}{4\pi} \oint_{\partial\Sigma} dz \, \eta_{ab} \hat{\lambda}^a \partial_z \hat{\eta}^b. \quad (\text{B.11})$$

In the similar manner to the case of conformal field theory on  $\mathbf{R}^2$ , we expand  $\hat{A}^a(z)$  and  $\hat{\lambda}^a(z)$  respectively:

$$\hat{A}^a(z) = -\frac{2}{k} T^a(z) = -\frac{2}{k} \sum_{n=-\infty}^{\infty} T_n^a z^{-n-1}, \quad \hat{\lambda}^a(z) = \sum_{n=-\infty}^{\infty} \hat{\lambda}_n^a z^{-n}. \quad (\text{B.12})$$

From (B.11), the algebra of the Laurent coefficients of  $\hat{A}^a(z)$  can be obtained as

$$\left\{ T_m^a, T_n^b \right\}_D = \epsilon^{ab}{}_c T_{m+n}^c - \frac{k}{2} m \, \eta^{ab} \delta_{m+n,0}. \quad (\text{B.13})$$

This is nothing but the holomorphic part of  $\text{SL}(2; \mathbb{C})$  current algebra. Thus the global charge of  $\text{SL}(2; \mathbb{C})$  Chern-Simons theory on a manifold with the boundary satisfies  $\text{SL}(2; \mathbb{C})$  current algebra (B.13).

The solutions (B.9), however, do not satisfy the boundary condition to be asymptotically de Sitter (2.26) in general. Thus we need to require further boundary conditions corresponding to (2.26) for the Laurent coefficients  $T_m^a$  such as<sup>12</sup>

$$T_m^0 = 0, \quad T_m^+ = -ik \delta_{m+1,0}. \quad (\text{B.14})$$

One can treat the above conditions as new constraints on  $T^a(z)$  and obtain the resulting algebra of the remaining variable  $L_{m-1} \equiv -T_m^-$  by means of the Dirac bracket under the new constraints:

$$\{L_m, L_n\}_D = i(m-n)L_{m+n} - \frac{c}{12}(m^3 - m)\delta_{m+n,0}, \quad (\text{B.15})$$

where  $c = 6k = \frac{3l}{2G}$ . This is the Virasoro algebra of the asymptotic symmetry on the asymptotically three-dimensional de Sitter space-time.

However, in order to obtain the corresponding quantum algebra acting on the physical Hilbert space, the replacement of the Dirac bracket with  $-i$  times commutator is required. Thus the quantum algebra becomes

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{ic}{12}(m^3 - m)\delta_{m+n,0}. \quad (\text{B.16})$$

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<sup>12</sup> $T_m^\pm = T_m^1 \pm iT_m^2$ .

As promised, one can find the quantum algebra of the asymptotic symmetry is the Virasoro algebra with a *pure imaginary* central charge  $ic$ .

Finally we comment on the relation between this algebra and the Virasoro algebra discussed in Section 3. From the relation  $T^-(z) = -ib(z)$  (see (2.24)),  $T_m^- = -i\tilde{L}_{m-1}$  leads to the corresponding quantum algebra:

$$[\tilde{L}_m, \tilde{L}_n] = -i(m-n)\tilde{L}_{m+n} - \frac{ic}{12}(m^3 - m)\delta_{m+n,0}. \quad (\text{B.17})$$

The central extension term of the algebra in terms of  $\tilde{L}_m$  still remains *pure imaginary*. Two different algebras, (3.17) and (B.17), are based on the Poisson bracket of the same reduced variables. This discrepancy originates from the different choice of the symplectic structure which defines the Poisson bracket between the reduced variables.

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