The "Bootstrap Program" for Integrable Quantum Field Theories in 1+1 Dim *

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Abstract

The purpose of the "bootstrap program" is to construct integrable quantum field theories in 1+1 dimensions in terms of their Wightman functions explicitly. As an input the integrability and general assumptions of local quantum field theories are used. The object is to be achieved in tree steps: 1) The S-matrix is obtained using a qualitative knowledge of the particle spectrum and the *Yang-Baxter equations*. 2) Matrix elements of local operators are calculated by means of the "form factor program" using the S-matrix as an input. 3) The Wightman functions are calculated by taking sums over intermediate states. The first step has been performed for a large number of models and also the second one for several models. The third step is unsolved up to now. Here the program is illustrated in terms of the sine-Gordon model alias the massive Thirring model. Exploiting the "off-shell" Bethe Ansatz we propose general formulae for form factors. For example the n-particle matrix element for all higher currents are given and in particular all eigenvalues of the higher conserved charges are calculated. Furthermore quantum operator equations are obtained in terms of their matrix elements, in particular the quantum sine-Gordon field equation. Exact expressions for the finite wave function and mass renormalization constants are calculated.

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1 Introduction

More than fifty years ago, Heisenberg [1] pointed out the importance of studying analytic continuations of scattering amplitudes into the complex momentum plane. The first concrete investigations in this direction were carried out by Jost [2] and Bargmann [3], initially for non-relativistic scattering processes. The original ideas turned out to be very fruitful and lead to interesting results on-shell, i.e. for the S-matrix [4], as well as off-shell, that is for the two-particle form factors [5]. Once one restricts ones attention to 1+1 dimensional integrable theories, the n-particle scattering matrix factories into two particle S-matrices and the approach, now usually referred to as the 'bootstrap program', reveals its full strength.

The 'bootstrap program [6]. for integrable quantum field theories in 1+1-dimensions' does not start with any classical Lagrangian, but this program classifies integrable quantum field theoretic models and in addition it provides their explicit exact solutions in term of all Wightman functions. We have contact with the classical models only, when at the end we compare our exact results with Feynman graph expansions which are based on the Lagrangians.

One of the authors (M.K.) et al. [7]. formulated the on-shell program i.e. the exact determination of the scattering matrix using the Yang-Baxter equations. Off-shell quantities, namely form factors were first investigated by Vergeles and Gryanik [8] in the sinh-Gordon model and by Weisz [9] in the sine-Gordon model. The concept of generalized form factors was introduced by one of the authors (M.K.) et al. [10]. In this article consistency equations were formulated which are expected to be satisfied by these objects. Thereafter this approach was developed further and studied in the context of several explicit models by Smirnov [11] who proposed the form factor equations $(i)-(v)$ (see below) as extensions of similar formulae in the original article [10]. The formulae were proven by the authors et al. [12]. There is a large number of more recent papers [13]-[49] on form factors. Also there is a nice application [50, 51] of form factors in condensed matter physics. The one dimensional Mott insulators can be described in terms of the quantum sine-Gordon model.

Finally the Wightman functions are obtained by taking sums and integrals over intermediate states. The explicit evaluation[1] of all these integrals and sums remains an open challenge for almost all theories, except the Ising model [52, 53]. A progress towards a solution of this problem has recently been achieved by Korepin et al. [54].

An entirely different method, the Bethe Ansatz [55], was initially formulated in order to solve the eigenvalue problem for certain integrable Hamiltonians. The approach has found applications in the context of numerous models and has led to a detailed study of various mass spectra and S-matrices. The original techniques have been refined into several directions, of which in particular the so-called "off-shell" Bethe Ansatz, which

[1]Of course one may also adopt a very practical point of view and resort to the well-known fact that the first terms of the series expansion of correlation functions in terms of form factors decrease very rapidly. Consequently correlations functions may be approximated very often quite well by simply including the low particle number form factor into the expansion.

was originally formulated by one of the authors (H.B.) [56, 57], will be exploited for our purposes. This version of the Bethe Ansatz paves the way to extend the approach to the off-shell physics and opens up the intriguing possibility to merge the two methods, that is the form factor approach and the Bethe Ansatz. The basis for this opportunity lies in the observation, that the "off-shell" Bethe Ansatz captures the vectorial structure of Watson's equations. These are matrix difference equations giving rise to a matrix Riemann-Hilbert problem which is solved by an "off-shell" Bethe Ansatz.

2 The "bootstrap program"

The 'bootstrap program for integrable quantum field theories in 1+1-dimensions' *classifies* integrable quantum field theoretic models and in addition it provides their explicit exact solutions in term of all Wightman functions. The results are obtained in three steps:

1. The S-matrix is calculated by means of general properties such as unitarity and crossing, the Yang-Baxter equations (which are a consequence of integrability) and the additional assumption of 'maximal analyticity'. This means that the two-particle S-matrix is an analytic function in the physical plane (of the Mandelstam variable $(p_1 + p_2)^2$) and possesses only those poles there which are of physical origin. The only input which depends on the model is the assumption of a particle spectrum. Usually it belongs to representations of a symmetry[2].

2. Generalized form factors which are matrix elements of local operators

$$^{out}\langle\, p'_m, \ldots, p'_1\, |\mathcal{O}(x)|\, p_1, \ldots, p_n\,\rangle^{in}$$

   are calculated by means of the S-matrix. More precisely, the equations $(i) - (v)$ given below are solved. These equations follow from LSZ-assumptions and again the additional assumption of 'maximal analyticity'.

3. The Wightman functions are obtained by inserting a complete set of intermediate states. In particular the two point function for a hermitian operator $\mathcal{O}(x)$ reads

$$\langle\, 0\, |\mathcal{O}(x)\, \mathcal{O}(0)|\, 0\,\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \cdots \int \frac{dp_1 \ldots dp_n}{(2\pi)^n 2\omega_1 \ldots 2\omega_n} \times \left| \langle\, 0\, |\mathcal{O}(0)|\, p_1, \ldots, p_n\,\rangle^{in} \right|^2 e^{-ix\sum p_i}.$$

   Up to now a proof that these sums converge does not exists.

[2] Typically there is a correspondence of fundamental representations with multiplets of particles.

3 Integrability

Integrability in (quantum) field theories means that there exits ∞-many conservation laws

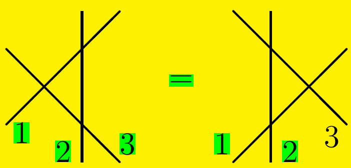
$$\partial_\mu J^\mu_L(t,x) = 0 \quad (L = \pm 1, \pm 3, \dots)\,.$$

A consequence of such conservation laws in 1+1 dimensions is that the n-particle S-matrix is a product of 2-particle S-matrices

$$S^{(n)}(p_1,\dots,p_n) = \prod_{i<j} S_{ij}(p_i,p_j)\,.$$

If there exist backward scattering the 2-particle S-matrices will not commute and one has to specify the order. In particular for the 3-particle S-matrix there are two possibilities

$$\mathbf{S^{(3)} = S_{12}S_{13}S_{23} = S_{23}S_{13}S_{12}}$$



which yield the **"Yang-Baxter Equation".**

Examples of integrable models in 1+1 - dimensions are the **sine-Gordon** model defined by the classical field equation

$$\ddot\varphi(t,x) - \varphi''(t,x) + \frac{\alpha}{\beta}\sin\beta\varphi(t,x) = 0$$

and the massive Thirring model defined by the classical Lagrangian

$$\mathcal{L} = \bar\psi(i\gamma\partial - m)\psi - \tfrac{1}{2}\,g\;\bar\psi\gamma^\mu\psi\,\bar\psi\gamma_\mu\psi\,.$$

Coleman [58] proved that both models are equivalent on the quantum level.

Further integrable models are: Z_N-Ising models, nonlinear σ-models, Gross-Neveu models, Toda models etc. In the following most of the formulae and explicit solutions are given for the sine-Gordon alias massive Thirring model although often corresponding results exist also for other models.

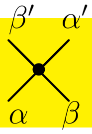
4 The S-matrix

For the Sine-Gordon alias massive Thiring model the particle spectrum consists of: soliton, anti-soliton and breathers (as soliton anti-soliton bound states). Since backward scattering can only appear for particles with the same mass, the two-particle S-matrix is of the form

$$S(\theta) = \begin{pmatrix} u & & & & & & \\ & t & r & & & & \\ & r & t & & & & \\ & & & u & & & \\ & & & & S_{sb} & & \\ & & & & & S_{bb} & \\ & & & & & & \ddots \end{pmatrix}$$

where the rapidity difference $\theta = |\theta_1 - \theta_2|$ is defined by
$p_i = m_i(\cosh\theta_i, \sinh\theta_i)$.

We start with the soliton (anti-soliton) S-matrix:

$$S^{\beta'\alpha'}_{\alpha\,\beta} =$$  $$: \quad S^{ss}_{ss} = u, \quad S^{\bar{s}s}_{s\bar{s}} = t, \quad S^{s\bar{s}}_{s\bar{s}} = r$$

$s =$ soliton, $\bar{s} =$ anti-soliton. As input conditions we have:

1. Unitarity $S(-\theta)S(\theta) = 1$

$$\begin{aligned} u(-\theta)u(\theta) &= 1 \\ t(-\theta)t(\theta) + r(-\theta)r(\theta) &= 1 \\ t(-\theta)r(\theta) + r(-\theta)t(\theta) &= 0 \end{aligned}$$

2. ‘Crossing’

$$u(i\pi - \theta) = t(\theta) \ , \quad r(i\pi - \theta) = r(\theta)$$

3. Yang-Baxter

$$r(\theta_{12})u(\theta_{13})r(\theta_{23}) + t(\theta_{12})r(\theta_{13})t(\theta_{23}) = u(\theta_{12})r(\theta_{13})u(\theta_{23})$$

4. ‘Maximal analyticity’:

   $S(\theta)$ is meromorphic and all poles in the ‘physical strip’ $0 \le \operatorname{Im}\theta \le \pi$ have a physical interpretation; in particular all bound states correspond to simple poles. An S-matrix satisfying this condition is also called ‘minimal’. For couplings $g < 0$ (in the language of the massive Thirring model) there are no soliton anti-soliton bound states. Therefore in this region of the coupling constant $S(\theta)$ is holomorphic in the ‘physical strip’ $0 \le \operatorname{Im}\theta \le \pi$.

The S-matrix bootstrap using the Yang-Baxter relations was proposed by Karowski, Thun, Truong and Weisz [7]. It was shown in this article that the ’minimal’ general solution of these equations is

$$u(\theta, \nu) = \exp \int_0^\infty \frac{dt}{t} \frac{\sinh \frac{t}{2}(1-\nu)}{\sinh \frac{\nu t}{2} \cosh \frac{t}{2}} \sinh t \frac{\theta}{i\pi} \, .$$

This S-matrix was first obtained by Zamolodchikov [59] from the extrapolation of semi-classical expressions. It has been checked in perturbation theory. The free parameter ν is related to the coupling constants by

$$\frac{1}{\nu} = \frac{8\pi}{\beta^2} - 1 = 1 + \frac{2g}{\pi}\,.$$

The second equation is due to Coleman and the first one is obtained by analyzing the pole structure of the amplitude $u(\theta,\nu)$. The assumption of 'maximal analyticity' and comparison with the known semi-classical bound state spectrum implies the identification of the parameter ν (see below).

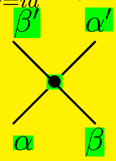
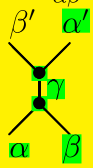
Bound states

Let γ be a bound state of the particles α and β with mass

$$m_\gamma = \sqrt{m_\alpha^2 + m_\beta^2 + 2m_\alpha m_\beta \cos a}\ , \qquad (0 < a < \pi)$$

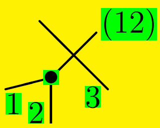
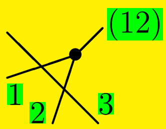
then the 2 particle S-matrix has a pole such that

$$i\operatorname*{Res}_{\theta=ia} \dot S_{\alpha\beta}^{\beta'\alpha'}(\theta) = \Gamma_\gamma^{\beta'\alpha'}\,\Gamma_{\alpha\beta}^\gamma$$

[5mm]i Res  = 

where a is called the fusion angle and $\Gamma_{\alpha\beta}^\gamma$ is the 'bound state intertwiner' [60, 48]. The **bound state S-matrix,** that is the scattering matrix of the bound state (12) with a particle 3, is obtained by the "bootstrap equation" [60]

$$S_{(12)3}(\theta_{(12)3})\,\Gamma_{12}^{(12)} = \Gamma_{12}^{(12)}\,S_{13}(\theta_{13})S_{23}(\theta_{23})$$

[5mm]  = 

where we use the usual short notation of matrices acting in the spaces corresponding to the particles 1, 2, 3 and (12).

As an example we consider the sine-Gordon alias massive Thirring model. For $\nu > 1$ (i.e. $g < 0$) there are no soliton anti-soliton bound states and the S-matrix $S(\theta)$ is analytic in the physical strip $0 \le \operatorname{Im}\theta \le \pi$. For $\nu < 1$ there are poles of the S-matrix element $S_{s\bar s}(\theta)$ at $\theta = i\pi(1-k\nu)$ for $k = 1, \dots < 1/\nu$ which correspond to bound states with masses $m_k = 2M\sin\frac{\pi}{2}\nu k$ where M is the soliton mass. This mass spectrum coincides with the semiclassical [61] breather spectrum if the parameter ν is related to the sine-Gordon coupling constant β as noted above. The pole at $\theta = i\pi(1-\nu)$ correspond in particular to

$$\text{soliton} + \text{anti-soliton} \longrightarrow \text{lowest breather.}$$

The “bootstrap equations” yield the breather-soliton S-matrix [62]

$$S_{bs}(\theta) = t(\theta + \tfrac{1}{2}i\pi(1-\nu))\, u(\theta - \tfrac{1}{2}i\pi(1-\nu)) = \frac{\sinh\theta + i\sin\frac{1}{2}\pi(1+\nu)}{\sinh\theta - i\sin\frac{1}{2}\pi(1+\nu)}$$

and in a second step the 2-breather S-matrix [62]

$$S_{bb}(\theta) = S_{bs}(\theta + \tfrac{1}{2}i\pi(1-\nu))\, S_{bs}(\theta - \tfrac{1}{2}i\pi(1-\nu)) = \frac{\sinh\theta + i\sin\pi\nu}{\sinh\theta - i\sin\pi\nu}$$

5 Form factors

Definition 1 *For a local operator $\mathcal{O}(x)$ the generalized form factors [10] are defined as*

$$\mathcal{O}_{\alpha_1\ldots\alpha_n}(\theta_1,\ldots,\theta_n) = \langle\, 0\,|\,\mathcal{O}(0)\,|\,p_1,\ldots,p_n\,\rangle^{in}_{\alpha_1\ldots\alpha_n}$$

for $\theta_1 > \cdots > \theta_n$ (for other orders of the rapidities they are defined by analytic continuation). The index α_i denotes the type of the particle with momentum p_i. We also use the short notations $\mathcal{O}_{\underline{\alpha}}(\underline{\theta})$ or $\mathcal{O}_{1\ldots n}(\underline{\theta})$.

For the sine Gordon model $\alpha =$ soliton, anti-soliton or breathers. First we restrict ourselves to solitonic states. Similar as for the S-matrix ‘maximal analyticity’ for generalized form factors means again that they are meromorphic and all poles in the ‘physical strips’ $0 \le \operatorname{Im}\theta_i \le \pi$ have a physical interpretation. Together with the usual LSZ-assumptions [63] of local quantum field theory the form factor equations can be derived

(i) Watson’s equations:

$$\mathcal{O}_{\ldots ij\ldots}(\ldots,\theta_i,\theta_j,\ldots) = \mathcal{O}_{\ldots ji\ldots}(\ldots,\theta_j,\theta_i,\ldots)\, S_{ij}(\theta_i-\theta_j)$$

(ii) Crossing relations:

$$\begin{aligned} {}_{\bar\alpha_1}\langle\, p_1\,|\,\mathcal{O}(0)\,|\,p_2,\ldots,p_n\,\rangle^{in,conn.}_{\alpha_2\ldots\alpha_n} &= \mathcal{O}_{\alpha_1\alpha_2\ldots\alpha_n}(\theta_1+i\pi,\theta_2,\ldots,\theta_n) \\ &= \mathcal{O}_{\alpha_2\ldots\alpha_n\alpha_1}(\theta_2,\ldots,\theta_n,\theta_1-i\pi) \end{aligned}$$

(iii) Recursion relations:

$$\operatorname*{Res}_{\theta_{12}=i\pi} \mathcal{O}_{1\ldots n}(\theta_1,\ldots) = 2i\,\mathbf{C}_{12}\,\mathcal{O}_{3\ldots n}(\theta_3,\ldots)\,(\mathbf{1} - S_{2n}\ldots S_{23})$$

where $\mathbf{C}_{12}$ is the charge conjugation matrix

(iv) Bound state form factors equation:

$$\operatorname*{Res}_{\theta_{12}=ia} \mathcal{O}_{123\ldots n}(\underline{\theta}) = \mathcal{O}_{(12)3\ldots n}(\theta_{(12)},\underline{\theta}')\,\sqrt{2}\,\Gamma^{(12)}_{12}$$

where a is the fusion angle.

(v) Lorentz invariance:

$$\mathcal{O}_{1\ldots n}(\theta_1+u,\ldots,\theta_n+u)=e^{su}\,\mathcal{O}_{1\ldots n}(\theta_1,\ldots,\theta_n)$$

where s is the "spin" of $\mathcal{O}$.

These equations have been proposed by Smirnov [11] as generalizations of equations derived in the original articles [10, 53, 6]. They have been proven [12] by means of LSZ-assumptions and 'maximal analyticity'. They hold in this form for bosons; for fermions or more generally for anyons there are some additional phase factors. If we write the form factors as a formal sum of Feynman graphs and use the additional rule that a line changing the 'time' direction also changes a particle to an anti-particle and changes the rapidity $\theta\to\theta\pm i\pi$ we may depicted the equations $(i)-(iv)$ as in figure 1.

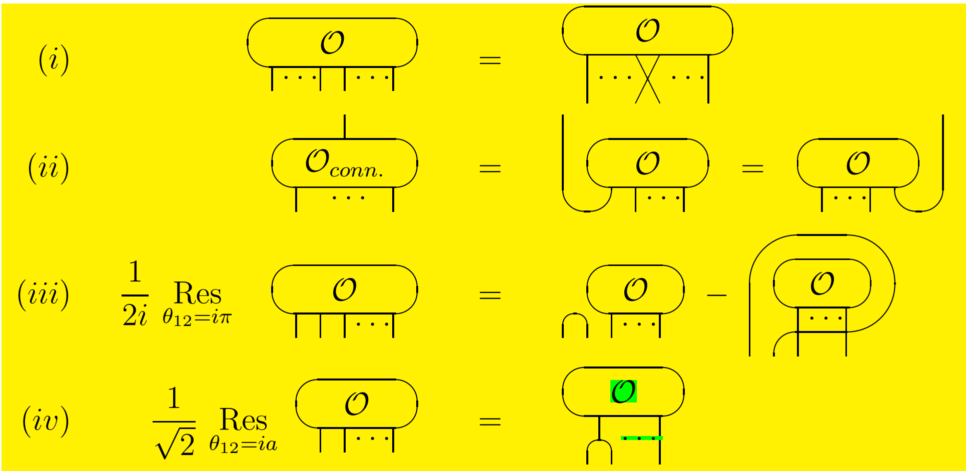


Figure 1: *The form factor equations.*

Locality

It has been proven [65][3]. that the properties $(i)-(iv)$ of the form factors together with a general crossing formula [48] imply locality in the form of

$$^{in}\langle\,\phi\,|\;[\mathcal{O}(x),\mathcal{O}(y)]\;|\,\psi\,\rangle^{in}=0$$

for all matrix elements, if $x-y$ is space like. In the proof one assumes the convergence of the sum over all intermediate states.

[3] For the case of no bound states this was proven before by Smirnov [11] and Lashkevich [64]

Two-particle form factors

For the two-particle form factors the form factor equations are easily understood. The usual assumptions of local quantum field theory yield

$$\langle\, 0\,|\,\mathcal{O}(0)\,|\,p_1,p_2\rangle^{in/out} = F\left((p_1+p_2)^2 \pm i\varepsilon\right) = \; F\left(\pm\theta_{12}\right)$$

where the rapidity difference is defined by $p_1p_2 = m^2\cosh\theta_{12}$. For integrable theories one has particle number conservation which implies (for any eigenstate of the two-particle S-matrix)

$$\langle\, 0\,|\,\mathcal{O}(0)\,|\,p_1,p_2\rangle^{in} = \langle\, 0\,|\,\mathcal{O}(0)\,|\,p_2,p_1\rangle^{out}\, S\left(\theta_{12}\right)\ .$$

Crossing means

$$\langle\, p_1\,|\,\mathcal{O}(0)\,|\,p_2\rangle = F\left(i\pi-\theta_{12}\right)$$

where for one-particle states in- and out-states coincide. Therefore Watson's equations follow

$$\begin{aligned} F\left(\theta\right) &= F\left(-\theta\right) S\left(\theta\right) \\ F\left(i\pi-\theta\right) &= F\left(i\pi+\theta\right)\ . \end{aligned}$$

For general theories Watson's [66] equations only hold below particle production thresholds. However, for integrable theories there is no particle production and therefore they hold for all complex values of θ. It has been shown [10] that these equations together with "maximal analyticity" have a unique solution. As an example we write the sine-Gordon (alias massive Thirring model) two-soliton form factor [10]

$$F(i\pi-\theta) = \cosh\tfrac{1}{2}\theta\,\exp\int_0^\infty\frac{dt}{t}\,\frac{\sinh\frac{t}{2}(1-\nu)}{\sinh\frac{\nu t}{2}\,\cosh\frac{t}{2}\sinh t}\,\sin^2 t\frac{\theta}{2\pi}\,.$$

A formula for generalized form factors

We are looking for solutions of the form factor equations $(i)-(v)$. For generalized form factors for arbitrary numbers of solitons and anti-solitons we make the Ansatz[4]

$$\boxed{\mathcal{O}_{\underline{\alpha}}(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} dz_1\cdots\int_{\mathcal{C}_{\underline{\theta}}} dz_m\, h(\underline{\theta},\underline{z})\, p_n^{\mathcal{O}}(\underline{\theta},\underline{z})\,\Psi_{\underline{\alpha}}(\underline{\theta},\underline{z})} \qquad (1)$$

which transforms the equations $(i)-(v)$ for the co-vector valued function $\mathcal{O}_{\underline{\alpha}}(\underline{\theta})$ into simple equations $(i')-(v')$ for scalar functions $p_n^{\mathcal{O}}(\underline{\theta},\underline{z})$ where n is the number of particles. The later equations are easily solved (see below). To capture the vectorial structure we use the "off-shell" Bethe Ansatz state $\Psi_{\underline{\alpha}}(\underline{\theta},\underline{z})$ (see below). For all integration variables z_j $(j=1,\ldots,m)$ the integration contours $\mathcal{C}_{\underline{\theta}}$ consists of several pieces (see figure 2). The

[4]This formula using the so called 'off-shell Bethe Ansatz' has been introduced by the authors et al. [12]. Similar integral representation have been also introduced by Smirnov [11].

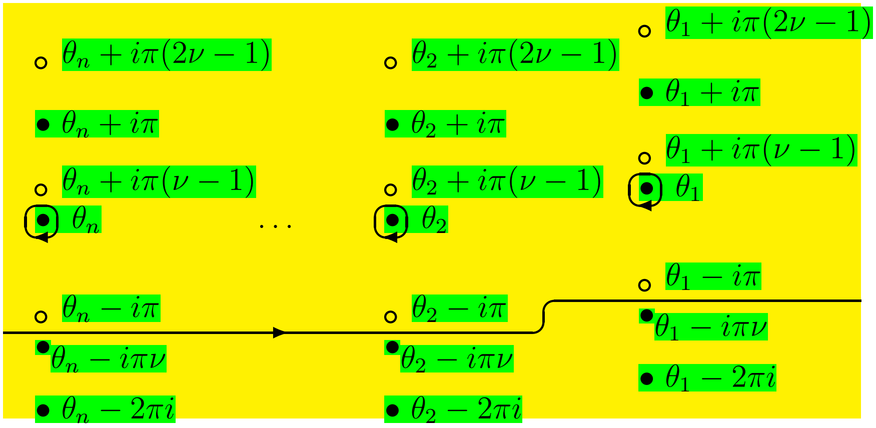


Figure 2: *The integration contour* $\mathcal{C}_{\underline{\theta}}$ *(for the repulsive case* $\nu > 1$*). The bullets belong to poles of the integrand resulting from* $u(\theta_i - u_j)\,\phi(\theta_i - u_j)$ *and the small open circles belong to poles originating from* $t(\theta_i - u_j)$ *and* $r(\theta_i - u_j)$.

number of integrations m depends on the charge of the operator $q = n - 2m$. The scalar function $h(\underline{\theta}, \underline{z})$ is uniquely determined by the S-matrix

$$h(\underline{\theta}, \underline{z}) = \prod_{1 \le i < j \le n} F(\theta_{ij}) \prod_{i=1}^{n} \prod_{j=1}^{m} \phi(\theta_i - z_j) \prod_{1 \le i < j \le m} \tau(z_i - z_j)\,,$$

with

$$\phi(z) = \frac{1}{F(z)\,F(z+i\pi)}\,, \qquad \tau(z) = \frac{1}{\phi(z)\,\phi(-z)} \propto \sinh z \sinh z/\nu$$

where $F(\theta)$ is the soliton-soliton form factor above. The dependence of the operator $\mathcal{O}(x)$ enters only through the scalar p-functions $p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$ (see below). We consider p-functions which satisfy the following conditions:

(i') $p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$ is symmetric with respect to the θ's and the z's.

(ii') $p_n^{\mathcal{O}}(\underline{\theta}, \underline{z}) = p_n^{\mathcal{O}}(\dots, \theta_i - 2\pi i, \dots, \underline{z})$ and it is a polynomial in $e^{\pm z_j}$ $(j = 1, \dots, m)$.

(iii')
$$\begin{cases} p_n^{\mathcal{O}}(\theta_1 = \theta_n + i\pi, \underline{\tilde{\theta}}, \theta_n; \underline{\tilde{z}}, z_m = \theta_n) = \dfrac{\varkappa}{m}\, p_{n-2}^{\mathcal{O}}(\underline{\tilde{\theta}}, \underline{\tilde{z}}) + \tilde{p}^{(1)}(\underline{\theta}) \\ p_n^{\mathcal{O}}(\theta_1 = \theta_n + i\pi, \underline{\tilde{\theta}}, \theta_n; \underline{\tilde{z}}, z_m = \theta_1) = \dfrac{\varkappa}{m}\, p_{n-2}^{\mathcal{O}}(\underline{\tilde{\theta}}, \underline{\tilde{z}}) + \tilde{p}^{(2)}(\underline{\theta}) \end{cases}$$

where $\underline{\tilde{\theta}} = (\theta_2, \dots, \theta_{n-1})$, $\underline{\tilde{z}} = (z_1, \dots z_{m-1})$ and [12] $\varkappa = -\left(F'(0)\right)^2/\pi$. The functions $\tilde{p}^{(1,2)}(\underline{\theta})$ are non-vanishing only for charge less operators and they are independent of the zs.

(iv') The bound state p-functions are investigated below.

(v') $p_n^{\mathcal{O}}(\underline{\theta} + \mu, \underline{z} + \mu) = e^{s\mu} p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$ where s is the 'spin' of the operator $\mathcal{O}(x)$.

Again these equations hold in this form for bosons; for fermions or more generally for anyons there are some additional phase factors.

Theorem 1 *Let generalized form factors be given by the Ansatz (1). They satisfy the form factor equations* $(i)-(v)$ *if the p-functions* $p_n^{\mathcal{O}}(\underline{\theta},\underline{z})$ *satisfies the equations* $(i')-(v')$.

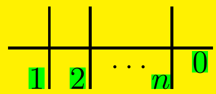
This theorem has bee proven for odd [12] and for even [48] number of solitonic particles.

The "off-shell Bethe Ansatz" state

As usual one defines the 'monodromy matrix'

$$T_{1\dots n,0}(\underline{\theta},\theta_0) \;=\; S_{10}(\theta_1-\theta_0)\,S_{20}(\theta_2-\theta_0)\cdots S_{n0}(\theta_n-\theta_0)$$

$$=$$



acting in the tensor product of the 'quantum space' and the 'auxiliary space' $V^{1\dots n}\otimes V_0$ with $V^{1\dots n}=V_1\otimes\cdots\otimes V_n$ (for the sine-Gordon model which correspond to the quantum group $sl_q(2)$ all $V_i\cong\mathbb{C}^2$). Further one defines as usual the sub-matrices A,B,C,D by

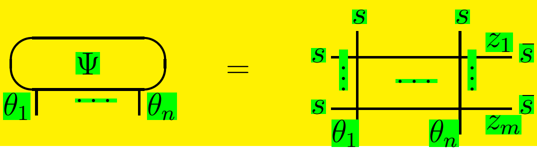
$$T_{1\dots n,0}(\underline{\theta},z)\equiv\begin{pmatrix} A_{1\dots n}(\underline{\theta},z) & B_{1\dots n}(\underline{\theta},z)\\ C_{1\dots n}(\underline{\theta},z) & D_{1\dots n}(\underline{\theta},z)\end{pmatrix}.$$

The "pseudo-vacuum" consists only of solitons

$$\Omega_{1\dots n}=s\otimes\cdots\otimes s\,.$$

A Bethe Ansatz co-vector in $V_{1\dots n}$ is defined by

$$\Psi_{1\dots n}(\underline{\theta},\underline{z}) \;=\; \Omega_{1\dots n}C_{1\dots n}(\underline{\theta},z_1)\cdots C_{1\dots n}(\underline{\theta},z_m)\,.$$



The conventional [55] application of the Bethe Ansatz is to solve an eigenvalue problem of a spin chain Hamiltonian or of a transfer matrix. The eigenstates are then given by a discrete set of Bethe Ansatz vectors where the parameters z_i have to satisfy the Bethe Ansatz equations. In the "off-shell Bethe Ansatz" [56] the parameter z_i are summed or integrated over as above in our Ansatz for the form factors. For other models which correspond to groups or quantum groups of higher rank one has to apply a nested off-shell Bethe Ansatz [57].

Examples of p-functions

Form factors for arbitrary numbers of solitons and anti-solitons are given by the Ansatz above which involves the p-functions. For the sine-Gordon alias massive Thiring-model there have been proposed the p-functions for several local operators.

Examples of operators and their p-functions (up to normalizations) [48]

$$\mathcal{O}(x) \leftrightarrow p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$$

The fundamental fermi field (charge $q = n - 2m = 1$) [12]

$$\psi(x) \leftrightarrow \exp \pm \left(\sum_{j=1}^{m} z_j - \tfrac{1}{2} \sum_{i=1}^{n} \theta_i \right).$$

The fundamental breather field (charge $q = n - 2m = 0$) [48]

$$\varphi(x) \leftrightarrow \frac{1}{\sum e^{\theta} \sum e^{-\theta}} \left(\sum e^{-\theta_i} \sum e^{z_j} + \sum e^{\theta_i} \sum e^{-z_j} \right).$$

The current $j^\mu = \overline{\psi}\gamma^\mu\psi(x)$ [48]

$$\overline{\psi}\gamma^{\pm}\psi(x) \leftrightarrow \frac{\pm 1}{\sum e^{\mp\theta_i}} \left(\sum e^{-\theta_i} \sum e^{z_j} + \sum e^{\theta_i} \sum e^{-z_j} \right).$$

The energy momentum tensor (with $\rho, \sigma = +, -$) [48]

$$T^{\rho\sigma}(x) \leftrightarrow \rho \frac{\sum e^{\rho\theta_i}}{\sum e^{-\sigma\theta_i}} \left(\sum e^{-\theta_i} \sum e^{z_j} - \sum e^{\theta_i} \sum e^{-z_j} \right).$$

The ∞-many conserved currents [48]

$$J_L^{\pm} \leftrightarrow \sum_{i=1}^{n} e^{\pm\theta_i} \sum_{j=1}^{m} e^{L z_j} \quad (L = \pm 1, \pm 3, \dots).$$

It is easy to check that these p-functions satisfy the equations $(i') - (v')$ consistently with the quantum numbers of the operators.

Identification of the operators

Several additional checks [12, 48] have been performed to justify the correspondences of operators and p-functions. The Feynman graph expansion of the matrix element has been compared with the expansion of the exact result given by the integral representation and always agreement has been found. For example the 4-particle form factor of the sine Gordon φ-field can be calculated in lowest order in g by the Feynman graphs of figure 3 using Coleman's [58] formula $\epsilon^{\mu\nu}\partial_\nu\varphi = -\frac{2\pi}{\beta}\overline{\psi}\gamma^\mu\psi$. The result is

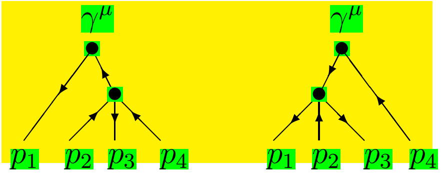


Figure 3: *Feynman graphs*

$$\langle\, 0 \,|\, \varphi\,(0) |\, p_1, \ldots, p_4 \,\rangle^{in}_{\bar{s}s\bar{s}s} = \frac{2\pi i g}{\beta} \frac{\sinh\frac{1}{2}\theta_{13} \sinh\frac{1}{2}\theta_{24} \sinh\frac{1}{2}(\theta_{12}+\theta_{34})}{\prod_{i<j}\cosh\frac{1}{2}\theta_{ij}} + O(g^2)$$

which agrees with the exact result. Another check is to calculate the eigenvalues of charges corresponding to local operators. For example it can be shown [12] that with the p-functions above the higher charges satisfy the eigenvalue equation

$$\left(\int dx J_L^0(x) - \sum_{i=1}^{n} e^{L\theta_i}\right) |\, p_1, \ldots, p_n\rangle^{in} = 0$$

as is expected from the classical higher conservation laws.

Breather form factors

It has been shown it the original article [10] that for particles which do not possess backward scattering the generalized form factors can be written as

$$\mathcal{O}_n(\theta_1, \ldots, \theta_n) = K_n^{\mathcal{O}}(\underline{\theta}) \prod_{1\le i<j\le n} F(\theta_{ij}) \tag{2}$$

where $F(\theta)$ is the two-particle form factor. It satisfies Watson's equations

$$F(\theta) = F(-\theta)S(\theta) = F(2\pi i - \theta)$$

where $S(\theta)$ is the diagonal two-particle S-matrix. For the lowest breathers of the sine-Gordon model the S-matrix is given above and the two-particle form factor is [10]

$$F_{bb}(\theta) = N \exp \int_0^\infty \frac{dt}{t} \frac{\cosh\frac{1}{2}t - \cosh(\frac{1}{2}+\nu)t}{\cosh\frac{1}{2}t \sinh t} \left(1 - \cosh t \left(1 - \frac{\theta}{i\pi}\right)\right)$$

(normalized such that $F_{bb}(\infty) = 1$). The K-function $K_n^{\mathcal{O}}(\underline{\theta})$ depends on the operator and it satisfies Watson's equations for $S = 1$. For simple cases the K-functions have been proposed in the original article [10]. Smirnov [11] used the bound state fusion method to obtain a breather form factor formula. For the sinh-Gordon model in several articles [20, 21, 24, 44] K-functions where proposed for various operators.

Starting with the general formula (1) for soliton form factors and using the bound state fusion method we derive [48] soliton-breather and pure breather form factor formulae. The K-function for the case of the lowest breathers turns out to be of the form

$$K_n^{\mathcal{O}}(\underline{\theta}) = \sum_{l_1=0}^{1} \cdots \sum_{l_n=0}^{1} (-1)^{l_1+\cdots+l_n} \prod_{1\le i<j\le n} \left(1 + (l_i - l_j)\frac{i \sin \pi\nu}{\sinh \theta_{ij}}\right) p_n^{\mathcal{O}}(\underline{\theta}, \underline{l}) \tag{3}$$

where the breather p-function $p_n^{\mathcal{O}}(\underline{\theta}, \underline{l}) = p_{sol,2n}^{\mathcal{O}}(\underline{\tilde{\theta}}, \underline{z})$ is obtained from the solitonic one with $\tilde{\theta}_{2i-1} = \theta_i + \frac{1}{2} i\pi(1-\nu)\,, \tilde{\theta}_{2i} = \theta_i - \frac{1}{2} i\pi(1-\nu)a\,, z_i = \theta_i - \frac{1}{2} i\pi\left(1 - (-1)^{l_1}\nu\right)$. In this way we [49] obtain the breather p-functions for the fundamental breather field $\varphi(x)$, the energy momentum tensor and the ∞-many conserved currents from the solitonic p-functions above.

However, one may also assume a different point of view. Namely, one can consider the form (3) of the K-functions as an Ansatz and look for breather p-functions such that the general form factor equations $(i) - (v)$ are satisfied. Doing this one will obtain a wider class of p-functions corresponding to operators which are local with respect to the breather field, but not necessarily local with respect to solitonic field. For example we propose [47][5] the breather p-function corresponding to the normal ordered exponentials of the field $:e^{i\gamma\varphi}:(x)$ for generic real γ

$$:e^{i\gamma\varphi}: \leftrightarrow p_n^{(q)}(\underline{l}) = N_n^{(q)} \prod_{i=1}^{n} q^{(-1)^{l_i}} \text{ with } q = \exp\left(i\frac{\pi\nu}{\beta}\gamma\right). \tag{4}$$

Here and in the following $:\cdots:$ denotes normal ordering with respect to the physical vacuum which means in particular for the vacuum expectation value $\langle\, 0\,|\!:\!\exp i\gamma\varphi\!:\!(x)|\, 0\,\rangle = 1$. This breather p-function is not related to a solitonic p-function of any local operator (at least for generic γ). One easily calculates the examples for $n = 1, 2$ explicitly

$$\begin{aligned} \left[e^{i\gamma\varphi}\right]_1(\theta) &= N_1^{(q)}\,(q - 1/q) \\ \left[e^{i\gamma\varphi}\right]_1(\theta_1, \theta_2) &= N_2^{(q)}\,(q - 1/q)^2\, F_{bb}(\theta_{12}) \end{aligned}$$

Expanding these relation in powers of γ one obtains the p-functions of all normal ordered powers of the field φ

$$:\varphi^N: \leftrightarrow p_n^{(N)}(\underline{l}) = N_n^{(N)} \left(\sum_{i=1}^{n} (-1)^{l_i}\right)^N \text{ with } N_n^{(N)} = \left(\frac{\pi\nu}{\beta}\right)^N N_n^{(q)} \tag{5}$$

(since $N_n^{(q)}$ is independent on q as we will see below). In particular for $N = 1$ and $n = 1, 2$ one calculates

$$\begin{aligned} \langle\, 0\,|\, \varphi(0)\,|\, p_1\,\rangle &= 2N_1^{(1)} \\ \langle\, 0\,|\, \varphi(0)\,|\, p_1, p_2\,\rangle^{in} &= 0\,. \end{aligned}$$

[5] For the sinh-Gordon model an analogous representation as (2) together with this p-function was obtained previously [44] by different methods.

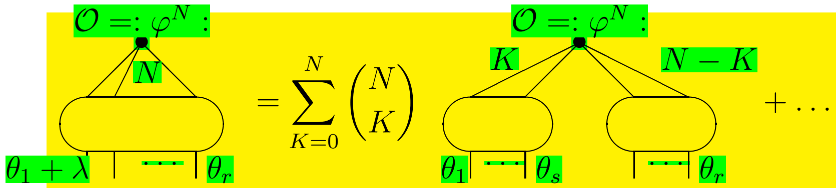
Note that this breather p-function for $\varphi(x)$ is not the same as that obtained from the solitonic one above using bound state fusion. However, it can be proven [49] that the form factors are the same in both cases. To justify the proposal (4) and to calculate the normalization constants we investigate the asymptotic behavior of form factors.

Asymptotic behavior of form factors for $:e^{i\gamma\varphi}:$

Let $\mathcal{O} =: \varphi^N :$ be the normal ordered power of a bosonic field. Set the rapidities as $\underline{\theta} = \lambda\theta'_1, \ldots, \lambda\theta'_m, \theta''_1, \ldots, \theta''_{n-m}$ and let $\lambda \to \infty$ then the asymptotic behavior of the n-boson form factor is

$$\left[\varphi^N\right]_n (\underline{\theta}) = \sum_{K=0}^{N} \binom{N}{K} \left[\varphi^K\right]_m (\underline{\theta}') \left[\varphi^{N-K}\right]_{n-m} (\underline{\theta}'') + O(e^{-\lambda})$$

if the interaction is pure bosonic. This can be proven in any order of perturbation theory as follows. The matrix element on the left hand side may be written in terms of Feynman graphs as



where all other graphs not drawn have lines which connect both parts. Weinbergs power counting theorem for bosonic Feynman graphs implies that these contributions decrease for $\lambda \to \infty$ as $O(\lambda^k e^{-\lambda})$. This behavior is also assumed to hold for the exact form factors (the fact is that the 'logarithmic terms' λ^k do not show up for the exact expressions since the K-functions are meromorphic in the e^{θ_i}). Therefore for the exponentials of the boson field $:e^{i\gamma\varphi}:$ we have the asymptotic behavior

$$\left[e^{i\gamma\varphi}\right]_n (\underline{\theta}) = \left[e^{i\gamma\varphi}\right]_m (\underline{\theta}') \left[e^{i\gamma\varphi}\right]_{n-m} (\underline{\theta}'') + O(e^{-\lambda}).$$

It is easy to see [47, 49] that our proposal (4) together with (2) and (3) satisfies this asymptotic behavior.[6] The asymptotic behavior of other form factors is more complicated [48] in particular if fermions are involved.

Normalization of form factors

The normalization constants are obtained in the various cases by the following observations:

a) The recursion relation (iii) relates N_{n+2} and N_n. For a typical p-function as that for the exponentials of the field this means

$$N_n^{(q)} = N_{n-2}^{(q)} \frac{2}{\sin \pi\nu F_{bb}(i\pi)} \quad (n \geq 3). \tag{6}$$

[6] This type of arguments has been also used before [10, 20, 21, 24].

b) For a field annihilating a one-particle state the normalization is given by the vacuum one-particle matrix element, in particular for the fundamental breather field one has

$$\langle\, 0 \,|\, \varphi(0) \,|\, p \,\rangle = \sqrt{Z^{\varphi}}$$

where Z^{φ} is the finite wave function renormalization constant. For the sine-Gordon field it has been calculated in the original article [10]

$$Z^{\varphi} = (1+\nu)\frac{\frac{\pi}{2}\nu}{\sin\frac{\pi}{2}\nu}\exp\left(-\frac{1}{\pi}\int_0^{\pi\nu}\frac{t}{\sin t}dt\right).$$

c) If a local operator is related to an observable like a charge $Q = \int dx\, \mathcal{O}(x)$ we use the relation

$$\langle\, p' \,|\, Q \,|\, p \,\rangle = q \langle\, p' \,|\, p \,\rangle.$$

This may be applied for example to the higher conserved charges.

d) We use Weinberg's power counting theorem for bosonic Feynman graphs. As discussed above this yields in particular the asymptotic behavior for the exponentials of the boson field $\mathcal{O} = :e^{i\gamma\varphi}:$

$$\mathcal{O}_n(\theta_1,\theta_2,\ldots) = \mathcal{O}_1(\theta_1)\,\mathcal{O}_{n-1}(\theta_2,\ldots) + O(e^{-\operatorname{Re}\theta_1})$$

as $\operatorname{Re}\theta_1 \to \infty$ in any order of perturbation theory. This behavior is also assumed to hold for the exact form factors. Applying this formula iteratively we obtain from (3) relations for the normalization constants.

Examples:

1. For the normalization factors for exponentials of the field given by (4) one uses a) and d) [47, 49]

$$\left.\begin{aligned} N_n^{(q)} &= N_{n-2}^{(q)}\frac{2}{\sin\pi\nu F_{bb}(i\pi)} \\ N_n^{(q)} &= N_1^{(q)} N_{n-1}^{(q)} \end{aligned}\right\} \Rightarrow N_n^{(q)} = \left(\sqrt{Z^{\varphi}}\frac{\beta}{2\pi\nu}\right)^n$$

where the identity $Z^{\varphi} F(i\pi)\beta^2 \sin\pi\nu = 8\,(\pi\nu)^2$ has been used. Note that $N_n^{(q)}$ is independent of q.

2. For the normalization factors for the field given by (5) (for $N = 1$) one uses b) which implies

$$\langle\, 0 \,|\, \varphi(0) \,|\, p \,\rangle = 2N_1^{(1)} = \sqrt{Z^{\varphi}}.$$

This is consistent with our proposals (4) and (5) and justifies the identification $q = \exp\left(i\frac{\pi\nu}{\beta}\gamma\right)$.

6 Some Results

The quantum sine-Gordon equation

We start with the local operator $:\sin\gamma\varphi:(x) = \frac{1}{2i}:(e^{i\gamma\varphi} - e^{-i\gamma\varphi}):(x)$. For the exceptional value $\gamma = \beta$ we find [48, 49] that also $\Box^{-1}:\sin\beta\varphi:(x)$ is local. Moreover the quantum sine-Gordon field equation

$$\Box\varphi(x) + \frac{\alpha}{\beta}:\sin\beta\varphi:(x) = 0 \tag{7}$$

holds for all matrix elements, if the "bare" mass $\sqrt{\alpha}$ is related to the renormalized mass by[7]

$$\alpha = m^2 \frac{\pi\nu}{\sin\pi\nu} \tag{8}$$

where m is the physical mass of the fundamental boson.

This is a sketch of the proof [49] which uses induction and Liouville's theorem. Consider the K-functions of the left hand side of (7)

$$f_n(\underline{\theta}) = -\sum e^{\theta_i}\sum e^{-\theta_i}K_n^{(1)}(\underline{\theta}) + \frac{\pi\nu}{\beta\sin\pi\nu}\frac{1}{2i}\left(K_n^{(q)}(\underline{\theta}) - K_n^{(1/q)}(\underline{\theta})\right).$$

The results of the previous section imply $f_1(\theta) = f_2(\underline{\theta}) = 0$ for $q = e^{i\pi\nu}$. As induction assumption we take $f_{n-2}(\underline{\theta}'') = 0$. The function $f_n(\underline{\theta})$ is meromorphic in terms of the $x_i = e^{\theta_i}$ with at most simple poles at $x_i = \pm x_j$ since $\sinh\theta_{ij} = (x_i + x_j)(x_i - x_j)/(2x_i x_j)$. The residues of the poles at $x_i = x_j$ vanish because of the symmetry under the exchange of $x_i \leftrightarrow x_j$. The residues at $x_i = -x_j$ are proportional to $f_{n-2}(\underline{\theta}'')$ because of the recursion relation (iii). Furthermore it can be shown [49] that $f_n(\underline{\theta}) \to 0$ for $x_i \to \infty$. Therefore $f_n(\underline{\theta})$ vanishes identically by Liouville's theorem.

The factor $\frac{\pi\nu}{\sin\pi\nu}$ in (8) modifies the classical equation and has to be considered as a quantum correction. For the sinh-Gordon model an analogous quantum field equation has been obtained previously [24][8]. Note that in particular at the 'free fermion point' $\nu \to 1$ $(\beta^2 \to 4\pi)$ this factor diverges, a phenomenon which is to be expected by short distance investigations [69]. For fixed bare mass square α and $\nu \to 2, 3, 4, \ldots$ the physical mass goes to zero. These values of the coupling are known to be specific: 1) the Bethe Ansatz vacuum in the language of the massive Thirring model shows phase transitions [70] and 2) the model at these points is related [71, 72, 73] to Baxters RSOS-models which correspond to minimal conformal models with central charge $c = 1 - 6/(\nu(\nu+1))$.

[7]Before such formula was found [67, 68] by different methods.

[8]It should be obtained from (7) by the replacement $\beta \to ig$. However the relation between the bare and the renormalized mass differs from the analytic continuation of (8) by a factor.

The trace of the energy momentum tensor

As a further operator equation we find [47, 49] that the trace of the energy momentum tensor satisfies

$$T^{\mu}_{\ \mu}(x) = -2\frac{\alpha}{\beta^2}\left(1-\frac{\beta^2}{8\pi}\right)(:\cos\beta\varphi:(x)-1)\,. \tag{9}$$

Again this operator equations is to be understood as equations of all its matrix elements. The equation is modified compared to the classical one by a quantum correction $(1-\beta^2/8\pi)$. As a consequence of this fact the model will be conformal invariant in the limit $\beta^2 \to 8\pi$ for fixed bare mass square α. This is related to a Berezinski-Kosterlitz-Thouless [74] phase transition.

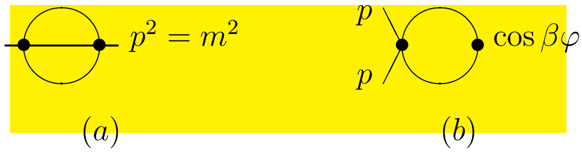


Figure 4: Bosonic Feynman graphs

All the results may be checked in perturbation theory by Feynman graph expansions. In particular in lowest order the relation between the bare and the renormalized mass (8) is given by figure 4 (a). It had already been calculated in the original article [10]. The result is

$$m^2 = \alpha\left(1-\frac{1}{6}\left(\frac{\beta^2}{8}\right)^2 + O(\beta^6)\right)$$

which agrees with the exact formula above. Similarly we check the quantum corrections of the trace of the energy momentum tensor (9) by calculating the Feynman graph of figure 4 (b) with the result [10]

$$\langle\, p \,|\, :\cos\beta\varphi:(0) - 1 \,|\, p\,\rangle = -\beta^2\left(1+\frac{\beta^2}{8\pi}\right) + O(\beta^6).$$

This again agrees with the exact formula above since the usual normalization for the energy momentum given by c) implies $\langle\, p \,|\, T^{\mu}_{\ \mu} \,|\, p\,\rangle = 2m^2$.

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