

NOTE ON MIRROR SYMMETRY AND COISOTROPIC D-BRANES ON TORI

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ABSTRACT. We describe mirror symmetry on higher dimensional tori, paying special attention to the behaviour of D-branes under mirror symmetry. To find the mirror D-branes the description of mirror symmetry on D-branes due to Ooguri, Oz and Yin is used. This method allows us to deal with the coisotropic D-branes recently introduced by Kapustin and Orlov. We compare this to the description of mirror symmetry on D-branes using the Fourier-Mukai transform of charges.

1. INTRODUCTION

Since its first breakthrough in 1991 (see [1]) mirror symmetry has been developed in many directions. This topic has stimulated a lively exchange of ideas between mathematics and physics. Since the formulation of the homological mirror symmetry conjecture by Kontsevich in [6] and the introduction of D-branes by Polchinsky (see [9, 8]), D-branes have started to play an increasingly important role. We will restrict our attention to BPS-branes on Calabi-Yau spaces. These come in two classes the (special) Lagrangian D-branes and the holomorphic D-branes. These are also often called the A-branes and B-branes, because the Lagrangian branes can also be discussed in the topological A-model, whereas the B-branes can be included into the topological B-model.

Kontsevich proposes to combine each of these classes of D-branes into a category. The holomorphic D-branes should lead to the category which is known among mathematicians as the derived category of coherent sheaves. The Lagrangian D-branes should give rise to the so-called Fukaya category (or better a derived version thereof). The homological mirror symmetry conjecture then says that for a mirror pair of Calabi-Yau manifolds X and Y the derived Fukaya category of X should be equivalent to the derived category of coherent sheaves of Y .

Recently these D-brane categories have been investigated more closely by physicists (see e.g. [2] and references therein). These investigations led to interesting conjectures and results about stability, the relation to Witten's open string field theory etc. Another interesting line of research was initiated by Kapustin and Orlov in [5], where it was shown that the description of A-branes as Lagrangian branes is in general incomplete and extra objects corresponding to so-called coisotropic manifolds have to be introduced. To find these objects Kapustin and Orlov used the description of mirror symmetry introduced in [7].

In this article I want to give a more concrete description of these extra objects. In [5] they were studied from a general perspective. Here we will restrict ourselves to even-dimensional tori as a special example of Calabi-Yau manifolds. This restriction allows us to give a much more concrete description of these still rather mysterious objects. We also investigate some general properties of these objects and check that they match the expectations from mirror symmetry.

2. MIRROR SYMMETRY ON TORI

2.1. Notation. Let us first briefly discuss the notation that we will use throughout this text. We write $X = T^n \times T^n$ for a $2n$ -dimensional torus and $Y = T^n \times \check{T}^n$ for its mirror. We will use coordinates $(x, y) \in \mathbb{R}^n/\mathbb{Z}^n \times \mathbb{R}^n/\mathbb{Z}^n$ on X and $(x, \check{y}) \in \mathbb{R}^n/\mathbb{Z}^n \times \mathbb{R}^n/\mathbb{Z}^n$ on Y . The data describing objects on X will have a tilde, to distinguish them from the data describing objects on Y . The SYZ-fibration is given by projection to the x -coordinates. We will discuss the A-model on X and the B-model on Y .

The tangent bundle is trivial and can be written as $TX = X \times \mathbb{R}^n \times \mathbb{R}^n$. A Ricci flat metric can then be identified with a positive definite $2n \times 2n$ -matrix g , the complex structure with a $2n \times 2n$ -matrix squaring to $-I_{2n}$. The compatibility of the complex structure and the metric can be written as $J^t g J = g$. It follows that $\omega = gJ$ is an antisymmetric matrix defining the Kähler form on X .

The complex structure can also be defined by requiring $z := \tau x + y$ to define complex coordinates on X . Here $\tau = \tau_1 + i\tau_2$ is a complex $n \times n$ -matrix. For simplicity we will assume that $\tau_1 = 0$ and that τ_2 be symmetric. This mirror symmetric to requiring the B-field to vanish. The relation between these two ways of define the complex structure is that in term of τ the matrix J can be written as

$$(2.1) \quad J = \begin{pmatrix} 0 & \tau_2^{-1} \\ -\tau_2 & 0 \end{pmatrix}.$$

On a Calabi-Yau manifold we also have a holomorphic $(n, 0)$ -form Ω . On a torus is can be written as $\Omega = d^n z$.

2.2. R -matrices. We will use the description of D-branes in Calabi-Yau spaces that was given in [7]. This description was also used by Kapustin and Orlov to give a general description of the non-Lagrangian A-branes. In [7] everything is discussed in terms of local coordinates and it is not completely clear how to formulate all of it in a coordinate independent way, especially because in general the mirror manifold and the original manifold are really different manifolds. Fortunately the mirror manifold of a torus is a torus again and in addition we can use global flat coordinates on the tori (as discussed above). So we give a description taylored for tori and ignore subtleties that do not matter for tori.

Let X be a complex torus as discussed above. The worldsheet Σ is a two-dimensional surface with metric h . The fields¹ of the sigma model are the bosonic field x which is a map $\Sigma \rightarrow X$ and the fermionic fields ψ_+ and ψ_- , which are sections of $x^*(TX) \otimes K^{1/2}$ and $x^*(TX) \otimes \bar{K}^{1/2}$ respectively. Here K is the bundle of $(1, 0)$ -forms on the surface Σ .

Let S be a submanifold of X . We will assume that S corresponds to an affine subspace of $\mathbb{R}^n \times \mathbb{R}^n$. This assumption will allow us to describe the D-brane in terms of a matrix. We want to discuss boundary conditions for open strings ending on S . For string theory on a Calabi-Yau manifolds there are two special classes of boundary conditions that preserve half of the supersymmetry, which are called A- and B-type boundary conditions. For D-branes on a torus they can be described using an $2n \times 2n$ -matrix R . On the boundary of Σ the following boundary conditions should hold

$$\partial x^\mu = R^\mu{}_\nu \bar{\partial} x^\nu, \quad \psi_+^\mu = \pm R^\mu{}_\nu \psi_-^\nu.$$

The matrix R should be orthogonal with respect to the metric g on the Calabi-Yau manifold X . As explained in [7] the codimension of the D-brane with R -matrix R is given by the dimension of the eigenspace E_{-1} for eigenvalue -1 of R . The

¹I use the notation from [3]. The fields x , ψ_+ and ψ_- correspond to the fields X , ψ_L and ψ_R in [7].

orthogonal complement E_{-1}^\perp of that eigenspace is the tangent space to the D-brane. The restriction of R to that eigenspace can be written as

$$R|_{E_{-1}^\perp} = (I + F)^{-1}(I - F).$$

Here F is an antisymmetric matrix defining the curvature of the connection on the D-brane.

Whether these boundary conditions are of A- or B-type depends on the additional conditions that we impose on R . The conditions for A-type boundary conditions can be formulated as

$$\begin{aligned} R^t \omega R &= -\omega, \\ R^* \Omega &= \bar{\Omega}. \end{aligned}$$

For B-type boundary conditions R should satisfy

$$\begin{aligned} R^t \omega R &= \omega, \\ R^* \Omega &= e^{i\theta} \Omega, \end{aligned}$$

for some $\theta \in \mathbb{R}$.

2.3. T-duality and mirror symmetry. When the target manifold X has a fibration with a torus as factor, one can apply T-duality and replace the torus by its dual. Because T-duality affects the boundary conditions, we should describe what T-duality does with the R -matrices discussed above. Following [7] we will describe T-duality using a matrix T . This matrix should square to I_{2n} , be symmetric (or equivalently orthogonal) with respect to the metric g and satisfy $TJT = -J$. In terms of this matrix mirror symmetry on D-branes should be described as follows. If we start with a D-brane with R -matrix R , then mirror D-brane should have R -matrix $R' = RT$. However, for this description to be valid there must be a relation between the choice of coordinates on X and on Y (because R, T depend on the coordinates on X and R' on the coordinates on Y). We will suppose this description is valid for coordinates on X such that $g = I_{2n}$ and $J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. We will use appropriate coordinate transformations to transform to this situation. To see how that works, we must have a closer look at the background fields.

In the coordinates that we are using here $T = \text{diag}(I_n, -I_n)$. For this matrix to be symmetric with respect to the metric g , we have to require that g has the following block diagonal form $g = \text{diag}(g_{xx}, g_{yy})$, where g_{xx} and g_{yy} a positive definite symmetric $n \times n$ -matrices. Recall that we consider a complex structure of the block form (2.1). Note that J automatically satisfies $TJT = -J$. The condition that J be orthogonal with respect to g yields $g_{xx} = \tau_2^t g_{yy} \tau_2$. Let S_{xx} be an $n \times n$ -matrix such $g_{xx} = S_{xx}^t S_{xx}$. Then we can introduce new coordinates (x', y') defined by the following equation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = S \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} S_{xx} & 0 \\ 0 & S_{xx} \tau_2^{-1} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

With respect to these new coordinates the metric and the complex structure have the standard form discussed above. In the new coordinates the D-brane is described by the R -matrix $\tilde{R} := SRS^{-1}$. Note that T is invariant under such coordinate transformations. So the mirror D-brane is described by $\tilde{R}' = \tilde{R}T$. However, this description is in terms of certain unknown coordinates (x'', \tilde{y}') on Y . Recalling that \tilde{y} is dual to y , a reasonable guess for the definition of these new coordinates is

$$\begin{pmatrix} x'' \\ \tilde{y}' \end{pmatrix} = \check{S} \begin{pmatrix} x \\ \tilde{y} \end{pmatrix} = \begin{pmatrix} S_{xx} & 0 \\ 0 & S_{xx}^{-t} \tau_2^t \end{pmatrix} \begin{pmatrix} x \\ \tilde{y} \end{pmatrix}.$$

In the coordinates $(x'', \tilde{y}') = (x', \tilde{y}')$, the metric on Y is the standard metric, so in the coordinates (x, \tilde{y}) the metric is

$$\check{g} = \check{S}^t \check{S} = \begin{pmatrix} g_{xx} & 0 \\ 0 & g_{yy}^{-1} \end{pmatrix}.$$

Similarly we find the complex structure

$$\check{J} = \check{S}^{-1} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \check{S} = \begin{pmatrix} 0 & g_{xx}^{-1} \tau_2^t \\ -\tau_2^{-t} g_{xx} & 0 \end{pmatrix}.$$

Comparing to (2.1), we see $\check{\tau}_2 = \tau_2^{-t} g_{xx}$. Together with $\check{g}_{xx} = g_{xx}$ and $\check{g}_{\tilde{y}\tilde{y}} = g_{yy}^{-1}$, this defines the mirror map on the background fields. To compare this with the mirror map in e.g. [3], let us define $k = g_{yy} \tau_2 = \tau_2^{-t} g_{xx}$. One can easily check that the Kähler form can be written as an antisymmetric block matrix

$$\begin{pmatrix} 0 & k^t \\ -k & 0 \end{pmatrix}.$$

So k parametrises the Kähler structure. In this notation the mirror map on the background fields is given by $\check{k} = \tau_2$ and $\check{\tau}_2 = k$, which matches the result in [3].

2.4. D-brane charges. To D-branes one can associate so-called charges. These are easiest to define for B-branes. For a vector bundle E on a Calabi-Yau manifold Y the charge vector is given by the Chern class $\text{ch}(E)$. Instead also the so-called Mukai vector $v(E) = \text{ch}(E) \sqrt{\text{td}(Y)}$ is used quite frequently. Of course for a torus $\text{td}(Y) = 1$, so $v(E) = \text{ch}(E)$.

$$\chi(E, F) = \int_Y \text{ch}(\check{E}) \cup \text{ch}(F) \cup \text{td}(Y) = \int_Y v(\check{E}) \cup v(F).$$

According to the Riemann-Roch theorem we have

$$(2.2) \quad \chi(E, F) = \sum_i (-1)^i h^i(E, F),$$

where $h^i(E, F) := \dim \text{Hom}^i(E, F)$. For the purposes of this article these definitions suffice. However, it has been suggested in the literature that one should really use K-theory instead of cohomology classes.

For A-branes the definition is quite simple in the case of Lagrangian branes, where one can simply use the cohomology class of the Poincaré dual of the Lagrangian submanifold. For more general coisotropic submanifolds $Z \subset X$ with a bundle F defined on Z , we can define the charge as follows

$$c(Z, F) := i_*(\text{ch}(F)),$$

where $i : Z \rightarrow X$ is the embedding. For Lagrangian submanifolds this reproduces the older definition. In the simplest case of two Lagrangian submanifolds L and M , the Euler characteristic can be defined as

$$\chi(L, M) := [L] \cdot [M].$$

The general definition of the Euler characteristic for objects of the Fukaya category is not yet clear, but it should be such that an analogon of (2.2) holds in the Fukaya category.

The mirror map on charge vectors is expected to be given by fibrewise Fourier-Mukai FM : $A_Y^* \rightarrow A_X^*$ transform of differential forms, where A_X^* denotes the space of differential forms on X and similarly for A_Y^* . This map is defined as follows

$$\text{FM}(\alpha) = \int_{T_{\tilde{y}}^n} \alpha e^{-\langle d\tilde{y}, dy \rangle}.$$

Let μ be the mirror map of D-branes, mapping holomorphic D-branes on Y to objects of the Fukaya category on X . Then one expects

$$\text{FM}(\text{ch}(E)) = c(\mu(E)).$$

The charges depend on the support of the D-brane and the curvature of the connection. This is also the information contained in the R -matrix. In both cases we have a description of how mirror symmetry should act. In the sequel we will analyse mirror symmetry using both descriptions and compare the results.

3. MIRROR SYMMETRY FOR D-BRANES

3.1. Line bundles. Let us first consider a line bundle L on Y and its mirror object on X . The first Chern class of this line bundle can be written as

$$c_1(L) = \frac{1}{2}\langle dx, Adx \rangle + \frac{1}{2}\langle d\tilde{y}, Bd\tilde{y} \rangle + \langle Cdx, d\tilde{y} \rangle.$$

Because $c_1(L)$ should be integral, we see that the entries of the $n \times n$ matrices A , B and C have to be integers. Note that the matrices A and B can be chosen to be antisymmetric. To make contact with the usual notation in physics, the curvature matrix F of the line bundle L can, in terms of the matrices A , B and C , be written as $F = \begin{pmatrix} A & C^t \\ -C & B \end{pmatrix}$.

Another requirement is that $c_1(L)$ should be a $(1, 1)$ -form. If $z = \tau x + y$ defines complex coordinates on Y , then $y = (z + \bar{z})/2$ and $x = (2i\tau_2)^{-1}(z - \bar{z})$. So the $(0, 2)$ -part of $c_1(L)$ is given by

$$c_1(L)^{(0,2)} = -\frac{1}{8}\langle d\bar{z}, \tau_2^{-t} A \tau_2^{-1} d\bar{z} \rangle + \frac{1}{8}\langle d\bar{z}, B d\bar{z} \rangle - \frac{i}{4}\langle d\bar{z}, \tau_2^{-t} C d\bar{z} \rangle.$$

So for $c_1(L)$ to be a $(1, 1)$ -form $\tau_2^{-t} C$ should be symmetric and the antisymmetric matrices $\tau_2^{-t} A \tau_2^{-1}$ and B should be equal.

For the SYZ-fibration that we fixed above, mirror symmetry is given by the T-duality matrix $T = \begin{pmatrix} I_n & 0 \\ 0 & -I_n \end{pmatrix}$. As discussed above a very general description of T-duality can be given in terms of the R -matrices. For simplicity we will first assume that the metric is the standard metric given by the matrix $g = I_{2n}$. In that case the R -matrix for the line bundle L is given by $R = (I_{2n} + F)^{-1}(I_{2n} - F)$. The R -matrix of the dual A-brane is then given by $\tilde{R} = RT$. For a geometric interpretation of the B-brane in terms of a Lagrangian submanifold we need that \tilde{R} is a symmetric matrix. Using the antisymmetry of F and the symmetry of T this condition can be written as $(I_{2n} - F)T(I_{2n} - F) = (I_{2n} + F)T(I_{2n} + F)$. A small calculation with block matrices shows that condition is met when A and B vanish. So we find that the geometric case is exactly the case investigated in [3].

In this case \tilde{R} can easily be calculated. We can write $\tilde{R} = \begin{pmatrix} \alpha & \gamma \\ \gamma^t & \beta \end{pmatrix}$. If we write the defining equation for \tilde{R} as $(I_{2n} + F)\tilde{R} = (I_{2n} - F)T$, is a matter of some careful computations with block matrices to find

$$\tilde{R} = \begin{pmatrix} I_n - 2C^t(I_n + CC^t)^{-1}C & 2C^t(I_n + CC^t)^{-1} \\ 2(I_n + CC^t)^{-1}C & -I_n + 2CC^t(I_n + CC^t)^{-1} \end{pmatrix}$$

One can verify that the eigenspace with eigenvalue 1 is $\{ \begin{pmatrix} x \\ Cx \end{pmatrix} \}$ and that its orthogonal complement $\{ \begin{pmatrix} -C^t y \\ y \end{pmatrix} \}$ is the eigenspace with eigenvalue -1 . This is in complete agreement with [3], where the mirror brane was found to be given by the equation $y = Cx + \alpha$ for some shift vector $\alpha \in \mathbb{R}^n$, which we cannot find in this way.

It is interesting to check what this looks like when the metric is not the standard metric. As we did above we will use a coordinate transform given by a block diagonal matrix $S = \text{diag}(S_{xx}, S_{xx}\tau_2^{-1})$ to transform to the standard metric and the standard complex structure (note however, that we are starting on Y and

transforming to X instead of the other way around). We will denote the matrices on the B-side with respect to the new orthogonal coordinates with primes. We find $R' = (I_{2n} + F')^{-1}(I_{2n} - F')$. Again R' will only be symmetric when $F' = S^{-t}FS^{-1}$ has the form $F' = \begin{pmatrix} 0 & C'^t \\ -C' & 0 \end{pmatrix}$, where $C' = S_{xx}^{-t}\tau_2^tCS_{xx}^{-1}$. Because $T' = T$, we can use the result above to find \tilde{R}' . However, this result is still with respect to transformed coordinates. The transformation to the original coordinates on X is given by $\tilde{S} = \text{diag}(S_{xx}, S_{xx}^{-t}\tau_2^t)$. So the eigenspace of \tilde{R} for the eigenvalue 1 can be written as

$$E_1^{\tilde{R}} = \{\tilde{S}^{-1} \begin{pmatrix} x' \\ C'x' \end{pmatrix} \mid x' \in \mathbb{R}^n\} = \left\{ \begin{pmatrix} S_{xx}^{-1}x' \\ CS_{xx}^{-1}x' \end{pmatrix} \mid x' \in \mathbb{R}^n \right\} = \left\{ \begin{pmatrix} x \\ Cx \end{pmatrix} \mid x \in \mathbb{R}^n \right\}.$$

So we see that the mirror brane does not depend on the metric.

The more complicated case is when \tilde{R} is not symmetric. We will first describe this case in terms of charges. Using an orthogonal basis transformation we can write obtain a basis compatible with the decomposition $\mathbb{R}^{2n} = \ker A \oplus \ker B \oplus (\ker A)^\perp \oplus (\ker B)^\perp$. Note that orthogonal and perpendicular is with respect to the standard metric on \mathbb{R}^{2n} (and not the one inducing the metric on the torus). The corresponding coordinates will be denoted with $(x_0, \check{y}_0, x_1, \check{y}_1)$.

Using these coordinates we can write $A = \begin{pmatrix} 0 & 0 \\ 0 & A' \end{pmatrix}$ (w.r.t. x_0, x_1), $B = \begin{pmatrix} 0 & 0 \\ 0 & B' \end{pmatrix}$ (w.r.t. \check{y}_0, \check{y}_1) and $C = \begin{pmatrix} C_{00} & C_{01} \\ C_{10} & C_{11} \end{pmatrix}$ (w.r.t. \check{y}_0, \check{y}_1 and x_0, x_1). In this notation the matrix F can be written as

$$F = \begin{pmatrix} 0 & C_{00}^t & 0 & C_{01}^t \\ -C_{00} & 0 & -C_{10} & 0 \\ 0 & C_{10}^t & A' & C_{11}^t \\ -C_{01} & 0 & -C_{11} & B' \end{pmatrix}$$

The charge on the B-side is given by the Chern character

$$\text{ch}(L) = e^{\frac{1}{2}\langle dv, Fv \rangle},$$

where $v = (x_0, \check{y}_0, x_1, \check{y}_1)$. On the A-side the D-brane is given by a coisotropic submanifold $Z \subset X$. On the submanifold Z a C^∞ vector bundle E with connection ∇ is defined. The charge of such a D-brane should be defined as $i_* \text{ch}(E)$, where $i : Z \rightarrow X$ is the inclusion. A fibrewise Fourier-Mukai transform on differential forms should map a representative of the charge cohomology class on the B-side to one representing the charge on the A-side, i.e.

$$\begin{aligned} \text{FM}(\text{ch}(L)) &= \int_{T_{\check{y}}^n} e^{\frac{1}{2}\langle dv, Fdv \rangle} e^{-\langle d\check{y}_0, dy_0 \rangle - \langle d\check{y}_1, dy_1 \rangle} \\ &= \int_{T_{\check{y}}^n} e^{\frac{1}{2}\langle dx_1, A'dx_1 \rangle + \frac{1}{2}\langle dy_1, B'dy_1 \rangle + \langle C_{00}dx_0 + C_{10}dx_1 - y_0, d\check{y}_0 \rangle} \\ &\quad \times e^{\langle C_{01}dx_0 + C_{11}dx_1 - dy_1, d\check{y}_1 \rangle}. \end{aligned}$$

Writing $T_{\check{y}}^n$ as $T_{\check{y}_0}^{n-p} \times T_{\check{y}_1}^p$ and integrating over $T_{\check{y}_0}^p$, we obtain using (A.2)

$$\begin{aligned} \text{FM}(\text{ch}(L)) &= e^{\frac{1}{2}\langle dx_1, A'dx_1 \rangle} d^p(C_{00}dx_0 + C_{10}dx_1 - y_0) \\ &\quad \times \int_{T_{\check{y}_1}^{n-p}} e^{\frac{1}{2}\langle dy_1, B'dy_1 \rangle + \langle C_{01}dx_0 + C_{11}dx_1 - dy_1, d\check{y}_1 \rangle}. \end{aligned}$$

Using (A.1) we find

$$\begin{aligned} (3.1) \quad \text{FM}(\text{ch}(L)) &= d^p(C_{00}x_0 + C_{10}x_1 - y_0) \\ &\quad \times e^{\frac{1}{2}\langle dx_1, A'dx_1 \rangle} e^{\frac{1}{2}\langle C_{01}dx_0 + C_{11}dx_1 - dy_1, (B')^{-1}(C_{01}dx_0 + C_{11}dx_1 - dy_1) \rangle} \sqrt{\det(B')}. \end{aligned}$$

This is a very nice formula, but the interpretation is a bit complicated. The easy case is when $A = B = 0$, which we already discussed in terms of R -matrices above. In that case we only keep x_0, y_0 and C_{00} . Dropping the indices 0, we obtain the following formula

$$\text{FM}(\text{ch}(L)) = d^n(Cx - y),$$

which is the Poincaré dual of the Lagrangian submanifold given by the equation $y = Cx$. So this is in full agreement with previous results.

In the general case expect this to be of the form $\text{PD}(Z) \text{ch}(F)$, where $Z \subset X$ is a coisotropic submanifold of X and F is a vector bundle defined on Z . If we define

$$Z := \{(x_0, C_{00}x_0 + C_{10}x_1, x_1, y_1) \mid x_0 \in \mathbb{R}^p, x_1, y_1 \in \mathbb{R}^{n-p}\},$$

then the first factor in (3.1) can be identified with the Poincaré dual $\text{PD}(Z)$ of Z . The second line of (3.1) should therefore be interpreted as $\text{ch}(F)$. The degree 0 part of the Chern character is the rank of the vector bundle. So we see that F has rank $\sqrt{B'}$. The 2-form

$$\begin{aligned} & \frac{1}{2} \langle dx_0, C_{01}^t (B')^{-1} C_{01} dx_0 \rangle + \frac{1}{2} \langle dx_1, (A' + C_{11}^t (B')^{-1} C_{11}) dx_1 \rangle \\ & + \frac{1}{2} \langle dy_1, (B')^{-1} dy_1 \rangle - \langle dx_0, C_{01}^t dy_1 \rangle - \langle dx_1, C_{11}^t (B')^{-1} dy_1 \rangle + \langle dx_0, C_{01}^t C_{11} dx_1 \rangle \end{aligned}$$

is not integral on $\mathbb{Z}^p \times \mathbb{Z}^p \times \mathbb{Z}^{n-p} \times \mathbb{Z}^{n-p}$, but it is integral on $\Lambda_{B'} := \mathbb{Z}^p \times \mathbb{Z}^p \times \mathbb{Z}^{n-p} \times B' \mathbb{Z}^{n-p}$. So we can find a line bundle L on the torus $\tilde{Y} = \mathbb{R}^{2n} / \Lambda_{B'}$ with first Chern class given by the above 2-form. Using the isogeny $i : \tilde{Y} \rightarrow Y$, we can define $\tilde{F} := i_* L$. The restriction of \tilde{F} to Z is then the vector bundle on Z that we are looking for.

Checking if this matches the description in terms of R -matrices is rather complicated. In the coordinates discussed above mirror symmetry is given by the matrix $T = \text{diag}(I_p, -I_p, I_{n-p}, -I_{n-p})$. The R -matrix \tilde{R} of the mirror D-brane is given by the equation $(I_{2n} + F)\tilde{R} = (I_{2n} - F)T$. Recall that the R -matrices are orthogonal matrices, so the eigenspaces for different eigenvalues are orthogonal. It is again easiest to use a suitable coordinate transform so that we only have to deal with the standard metric and the standard complex structure. Then one can check that the orthogonal complement of the eigenspace for eigenvalue -1 of \tilde{R} is indeed given by Z . Further checks get increasingly complicated.

3.2. Higher rank bundles. To describe higher rank bundles on the B-side and their mirrors on the A-side, we use the classification of semi-homogeneous vector bundles on tori (see Chapter 4 in [3]). Such vector bundles can be constructed out of line bundles on tori using isogenies and tensor product with flat line bundles. We will discuss these two possibilities in turn.

3.2.1. Tensor product with flat vector bundle. The simplest possibility is tensor product with a flat vector bundle. Flat vector bundles correspond to representations of the fundamental group. For the torus $T^n \times \tilde{T}^n$ on the B-side, we have $\pi_1(T^n \times \tilde{T}^n) = \mathbb{Z}^n \times \mathbb{Z}^n$. We can always use a holomorphic bundle isomorphism to find an equivalent bundle F such that the representation is trivial on the second \mathbb{Z}^n -factor. Such a representation also induces a vector bundle on the base torus T^{2n} and using pull back with the projection on the mirror a vector bundle \tilde{F} on the mirror torus $T^n \times T^n$ on the A-side. One has

$$\mu(E \otimes F) = \mu(E) \otimes F.$$

3.2.2. Isogenies. Another way to construct vector bundles of rank greater than one, is to use isogenies. For a general isogeny it is not clear what the effect on the mirror will be, but there are two special cases which can be described. These two cases are when the isogeny is compatible with the SYZ-fibration.

The easiest of these two special cases is an isogeny $i : \tilde{T}^n \rightarrow T^n$ on the base torus alone. This simply commutes with the mirror map, so we have

$$\mu(i_*(L)) = i_*(\mu(L)).$$

Here we slightly abuse notation by using i for both isogenies $\tilde{T}^n \times T^n \rightarrow T^n \times T^n$ and $\tilde{T}^n \times \tilde{T}^n \rightarrow T^n \times \tilde{T}^n$.

The second case is an isogeny on the fibres alone. Let $i : \tilde{T}^n \rightarrow T^n$ be an isogeny. This induces an isogeny $\check{i} : \check{T}^n \rightarrow \check{T}^n$. The mirror map intertwines \check{i}_* and i^* as follows

$$\mu(\check{i}_*(L)) = i^*(\mu(L)).$$

The isogeny that we used above, can presumably be interpreted in this way.

4. CONCLUSIONS AND OUTLOOK

We have given a preliminary description of the mirror objects of semi-homogeneous vector bundles. These include coisotropic, but not Lagrangian D-branes. We have also investigated some of their properties and the results fit in nicely with the expectations from physics in particular from mirror symmetry. Semi-homogeneous vector bundles are conjectured to generate in a certain sense the full derived category. So we may hope that this provides us with a fairly complete understanding of the mirror map on the level of objects.

To do these calculations we restricted the class of background fields that we considered. In [4] we will discuss an extension to more general background fields. This should also clarify the role of the B-field. In that article we will also give some more details on things that were left rather sketchy in this article.

Another restriction is that we described the mirror map only on objects. However, it should be a functor of D-brane categories, so it should also define a map on morphisms. Here we have made less progress, because it remains difficult to define morphisms in the Fukaya category, especially when non Lagrangian objects are involved. One approach to make progress would be to give a general definition of an Euler characteristic using the charges that we defined. This would at least give some information about the dimensions of the graded pieces of the space of morphisms.

APPENDIX A. GAUSSIAN INTEGRALS WITH DIFFERENTIAL FORMS ON A TORUS

Let T^{2m} be an even dimensional torus and let A be a nondegenerate anti-symmetric $2m \times 2m$ -matrix. We can choose coordinates y_1, \dots, y_{2m} , such that $A = \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix}$. This allows us to write the Gaussian integral, that we want to calculate, as follows

$$\int_{T^{2m}} e^{\frac{1}{2}\langle dy, A dy \rangle} = \int_{T^{2m}} e^{\sum_{i,j} dy_i \wedge dy_j B_{i,j-m}} = \int_{T^{2m}} \frac{1}{m!} \left(\sum_{i,j} dy_i \wedge dy_j B_{i,j-m} \right)^m.$$

Here the sum is over $i = 1, \dots, m$ and $j = m + 1, \dots, 2m$. Because the integral is only nonvanishing when all dy_i and dy_j occur exactly once, we find

$$\begin{aligned} \int_{T^{2m}} e^{\frac{1}{2}\langle dy, Ady \rangle} &= \int_{T^{2m}} \sum_{j_1, \dots, j_m} dy_1 \wedge dy_{j_1} \wedge \dots \wedge dy_m \wedge dy_{j_m} B_{1, j_1-m} \dots B_{m, j_m-m} \\ &= \int_{T^{2m}} \sum_{\sigma \in S^m} \text{sgn}(\sigma) dy_1 \wedge dy_{m+1} \wedge \dots \wedge dy_m \wedge dy_{2m} B_{1, \sigma(1)} \dots B_{m, \sigma(m)} \\ &= \det(B) \text{vol}(T^{2m}) = \sqrt{\det(A)} \text{vol}(T^{2m}). \end{aligned}$$

More general integrals can be computed by ‘completing the square’.

$$\begin{aligned} \int_{T^{2m}} e^{\frac{1}{2}\langle dy, Ady \rangle + \langle a, dy \rangle} &= \int_{T^{2m}} e^{\frac{1}{2}\langle dy - A^{-1}a, A(dy - A^{-1}a) \rangle - \frac{1}{2}\langle A^{-1}a, a \rangle} \\ (A.1) \qquad &= e^{\frac{1}{2}\langle a, A^{-1}a \rangle} \sqrt{\det(A)}. \end{aligned}$$

Here a is a vector of 1-forms vanishing on T^{2m} , so $\langle dy, a \rangle = -\langle a, dy \rangle$. We also use the antisymmetry of A to rewrite $-\langle A^{-1}a, Ady \rangle = \langle a, dy \rangle$. The shift over a does not affect the Gaussian integral, because we have $2m$ different 1-forms $dy_i + a_i$. So wedge products of more than $2m$ factors vanish automatically. The integral picks out the term with only dy_i , because we are in dimension $2m$.

Another simple, but useful formula for integrals of differential forms on a torus is

$$(A.2) \qquad \int_{T^m} e^{\langle a, dy \rangle} = da_1 \wedge \dots \wedge da_m =: d^m a.$$

Here a is a vector of 1-forms vanishing on T^m .

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