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Enveloping algebra valued gauge transformations for non-abelian gauge groups on non-commutative spaces

B. Jurčo[1], S. Schraml[2,3], P. Schupp[3], J. Wess[2,3]

[1]Max-Planck-Instiut für Mathematik
Vivatgasse 7, D-53111 Bonn

[2]Max-Planck-Institut für Physik
Föhringer Ring 6, D-80805 München

[3]Sektion Physik, Universität München
Theresienstr. 37, D-80333 München

Abstract

An enveloping algebra valued gauge field is constructed, its components are functions of the Lie algebra valued gauge field and can be constructed with the Seiberg-Witten map. This allows the formulation of a dynamics for a finite number of gauge field components on non-commutative spaces.

1 Introduction

Gauge theories on non-commutative spaces [1, 2, 3] cannot be formulated with Lie algebra valued infinitesimal transformations and consequently not with Lie algebra valued gauge fields. In the composition of the infinitesimal transformations commutators and anticommutators of the generators of the gauge group will appear, eventually generating all the higher powers of the generators. Thus the enveloping algebra of the Lie algebra seems to be a proper setting for such a gauge theory. This, however, is not very attractive because the enveloping algebra is infinite-dimensional, requiring an infinite number of coordinate dependent transformation parameters and an infinite number of gauge fields as a consequence.

In this paper we show that enveloping algebra valued infinitesimal transformations as well as enveloping algebra valued gauge fields can be restricted such that they depend on the Lie algebra valued parameters and the Lie algebra valued gauge fields and their space-time derivatives only. This renders the number of independent parameters and gauge fields to be the same as for the Lie algebra valued gauge theories. The coefficient functions of all the higher powers of the generators of the gauge group are functions of the coefficients of the first power. The construction of the dependent coefficients is based on the Seiberg-Witten map [3]. The existence of this map can be proven in general [4, 5, 6], here we demonstrate this map by explicitely calculating the expansion to first order in a parameter that characterizes the deviation from commuting coordinates.

As a method we use the $*$-product formulation of the algebra [7, 8, 9, 10, 11, 12]. The objects are functions of commuting variables, the algebraic non-commutative properties are encoded in the $*$-product. In the following chapter we introduce this formalism, it can be used for all algebras that have the Poincare-Birkhoff-Witt property. In this paper we restrict the algebra to the algebra of the non-commuting coordinates and the Lie algebra of the gauge group.

For non-commuting spaces the concept of a gauge theory can already be introduced by defining covariant coordinates without speaking about derivatives [1]. In general the algebraic setting of the theory will require an extension of the algebra by derivatives. This formalism has been developed for quantum planes [13].

For the canonical structure of the non-commuting coordinates it is shown in this paper that the derivatives can be obtained from the coordinates. Derivatives do not have to be introduced separately. For several examples of quantum planes this is true as well [14].

For the canonical structure integration can be defined as well. This is shown in chapter 4. This allows us to formulate the dynamics with an action. A gauge invariant action for the gauge field can be constructed from the gauge covariant tensors that agrees with the usual gauge invariant action in the limit of commuting spaces.

The approach generalizes to other non-commutative spaces and it is in particular possible to choose another non-commutative internal space instead of the Lie algebra. For an internal canonical structure this scenario has been studied in [15].

2 Non-commutative Spaces and the $*$-Product Formalism

The coordinates $\hat{z}^i$, $(i = 1, \dots N)$ of a non-commutative space structure are subject to relations. We have in mind the relations for a canonical structure

$$[\hat{z}^i, \hat{z}^j] = i\theta^{ij}, \qquad \theta^{ij} \in \mathbb{C}, \tag{2.1}$$

for a Lie structure

$$[\hat{z}^i, \hat{z}^j] = i f^{ij}{}_k \hat{z}^k, \qquad f^{ij}{}_k \in \mathbb{C}, \tag{2.2}$$

or for a quantum plane structure

$$[\hat{z}^i, \hat{z}^j] = i C^{ij}_{kl} \hat{z}^k \hat{z}^l, \qquad C^{ij}_{kl} \in \mathbb{C}. \tag{2.3}$$

The non-commutative space can be defined as the associative algebra over $\mathbb{C}$, which consists of the algebra freely generated by the coordinates and then divided by the ideal $\mathcal{R}$ generated by the relations:

$$\mathcal{A}_z = \frac{\mathbb{C}[[\hat{z}^1, \dots, \hat{z}^N]]}{\mathcal{R}}. \tag{2.4}$$

Formal power series are accepted. Among these algebras we will restrict our attention to those that have a basis with the Poincare-Birkhoff-Witt property (PBW). This means that when considered as a graded algebra the subspace of polynomials of fixed degree has the same dimension as the corresponding subspace of the polynomials of commuting variables. In this case any element of $\mathcal{A}_z$ is defined by its coefficient function and vice versa.

$$\begin{aligned} \hat{f} &= \sum_{L=0}^{\infty} f_{i_1,\dots,i_L} : \hat{z}^{i_1} \cdots \hat{z}^{i_L} : \\ \hat{f} &\sim \{f_i\}. \end{aligned} \tag{2.5}$$

$: \hat{z}^{i_1} \dots \hat{z}^{i_L} :$ denotes an element of the basis defined by some prescribed ordering, e.g., normal order $i_1 \leq i_2 \ldots \leq i_L$ or, e.g., totaly symmetric. The product of two elements will have its own coefficient function, this defines the diamond product

$$\hat{f}\hat{g} = \hat{h} \quad \sim \quad \{f_i\} \diamond \{g_i\} = \{h_i\}. \tag{2.6}$$

The algebraic properties are now all encoded in the $\diamond$ product.

Next we associate a function f of commuting variables with an element of the algebra, say $\hat{f}$, by substituting the commuting variable $z^1, \dots, z^N$ for the non-commuting variables in (2.5)

$$\hat{f} = \sum f_{i_1 \dots i_L} : \hat{z}^{i_1} \cdots \hat{z}^{i_L} : \quad \sim \quad f(z) = \sum f_{i_1 \dots i_L} z^{i_1} \cdots z^{i_L} \tag{2.7}$$

The diamond product leads to a bilinear $*$-product of functions:

$$\{f_i\} \diamond \{g_i\} = \{h_i\} \quad \sim \quad (f * g)(z) = h(z). \tag{2.8}$$

This star product has been discussed in reference [1].

For the canonical structure it is the Moyal-Weyl product [7, 8, 9]:

$$f * g = e^{\frac{i}{2} \frac{\partial}{\partial z^i} \theta^{ij} \frac{\partial}{\partial z'^j}} f(z) g(z') \Big|_{z' \to z}. \tag{2.9}$$

For the Lie structure we have:

$$f * g = e^{\frac{i}{2}x^l g_l(i\frac{\partial}{\partial z'}, i\frac{\partial}{\partial z''})} f(z')g(z'')\Big|_{\substack{z'\to z \\ z''\to z}}, \tag{2.10}$$

where g_l is defined by group multiplication:

$$e^{ik_l\hat{z}^l} e^{ip_l\hat{z}^l} = e^{i\{k_l+p_l+\frac{1}{2}g_l(k,p)\}\hat{z}^l}. \tag{2.11}$$

The first terms are easily calculated from the Baker-Campbell-Hausdorff formula:

$$\begin{aligned} e^A e^B &= e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}([A,[A,B]]+[B,[B,A]])+\dots} \\ g_l(k,p) &= -k_i p_j f^{ij}{}_l + \frac{1}{6}k_i p_j(p_k - k_k) f^{ij}{}_n f^{nk}{}_l + \dots . \end{aligned} \tag{2.12}$$

For the quantum plane structure we have as an example the $*$-product for the Manin plane:

$$\begin{aligned} xy &= qyx \\ f * g &= q^{\frac{1}{2}(-x'\frac{\partial}{\partial x'}y\frac{\partial}{\partial y}+x\frac{\partial}{\partial x}y'\frac{\partial}{\partial y'})} f(x,y)g(x',y')\Big|_{\substack{x'\to x \\ y'\to y}} \end{aligned} \tag{2.13}$$

3 Enveloping algebra valued connection

A non-abelian gauge theory on a non-commutative space carries two algebraic structures, the algebra $\mathcal{A}_x$ discussed above and the non-abelian Lie algebra $\mathcal{A}_T$ of the gauge group with the generators $T^1, \dots, T^M$ and the relations:

$$[T^a, T^b] = if^{ab}{}_c T^c. \tag{3.1}$$

It is natural to treat both algebras on the same footing and to denote the generating elements of the big algebra by $\hat{z}^i$:

$$\begin{aligned} \hat{z}^i &= \{\hat{x}^1, \dots, \hat{x}^N, T^1, \dots, T^M\} \\ \mathcal{A}_z &= \frac{\mathbb{C}[[\hat{z}^1, \dots, \hat{z}^{N+M}]]}{\mathcal{R}}. \end{aligned} \tag{3.2}$$

The $*$-product formalism as developed in the previous chapter can now be applied to the algebra $\mathcal{A}_z$ as well.

We study functions of the commuting variables x^ν, $(\nu = 1, \dots, N)$ and t^a, $(a = 1, \dots, M)$ and define the star product reflecting the algebraic properties of the algebra $\mathcal{A}_z$.

In the case of a canonical structure for the space variables x^ν we have

$$\begin{aligned} &(F * G)(z) = \\ &\quad e^{\frac{i}{2}(\theta^{\mu\nu}\frac{\partial}{\partial x'^\mu}\frac{\partial}{\partial x''^\nu}+t^a g_a(i\frac{\partial}{\partial t'}, i\frac{\partial}{\partial t''}))} \\ &\quad \times F(x',t')G(x'',t'')\Big|_{\substack{x'\to x, x''\to x \\ t'\to t, t''\to t}}. \end{aligned} \tag{3.3}$$

To exemplify the formalism we shall concentrate on this structure in what follows.

To define gauge theories we first define fields. These are elements of the algebra $\mathcal{A}_x$ that form a representation of the T-algebra. Under a gauge transformation they transform as follows:

$$\delta\hat{\psi} = i\hat{\alpha}\hat{\psi}, \quad \hat{\psi} \in \mathcal{A}_x, \quad \hat{\alpha} \in \mathcal{A}_z. \tag{3.4}$$

The action of the generators T of the Lie algebra on $\hat{\psi}$ is defined as $\hat{\psi}$ is supposed to form a representation of $\mathcal{A}_T$. Thus $\delta\hat{\psi} \in \mathcal{A}_x$ despite $\hat{\alpha} \in \mathcal{A}_z$.

Independent of a representation we have defined $\hat{\alpha}$ as an element of the enveloping algebra of the gauge group and not as Lie algebra-valued, as we would have done it for commuting spaces. We say $\hat{\alpha}$ is enveloping algebra-valued. The same will be true for the connection that we introduce to define covariant coordinates [1]

$$\hat{X}^\nu = \hat{x}^\nu + \hat{A}^\nu, \qquad \hat{A}^\nu \in \mathcal{A}_z. \tag{3.5}$$

We demand that $\hat{X}^\nu\hat{\psi}$ transforms covariantly:

$$\delta\hat{X}^\nu\hat{\psi} = i\hat{\alpha}\hat{X}^\nu\hat{\psi} \tag{3.6}$$

and find that this defines the transformation law of the enveloping algebra-valued connection $\hat{A}^\nu$:

$$\begin{aligned}&\delta\hat{A}^\nu = -i[\hat{x}^\nu, \hat{\alpha}] + i[\hat{\alpha}, \hat{A}^\nu],\\ &\hat{A}^\nu \in \mathcal{A}_z, \quad \hat{\alpha} \in \mathcal{A}_z, \quad \delta\hat{A}^\nu \in \mathcal{A}_z.\end{aligned} \tag{3.7}$$

At first sight it seems that an enveloping algebra-valued connection has infinitely many component fields and thus is not very useful. We shall show, however, that all the component fields can be obtained from a Lie algebra-valued connection by a Seiberg-Witten map [3, 4, 6]. This was also observed in [16], where a result in this direction has been obtained for SO(n) and Sp(n). To show this we cast the algebraic setting into the $*$-product formalism. The transformation of the connection is then

$$\delta A^\nu = -i[x^\nu \stackrel{*}{,} \alpha] + i[\alpha \stackrel{*}{,} A^\nu]. \tag{3.8}$$

We treat the canonical case in more detail, the $*$-product in this case is given in Equation (3.3).

For the first term in the variation of A^ν we obtain

$$-i[x^\nu \stackrel{*}{,} \alpha] = \theta^{\nu\rho}\frac{\partial}{\partial x^\rho}\alpha. \tag{3.9}$$

The variation of A^ν itself starts with a linear term in θ, we therefore assume, as in reference [1], that A^ν starts with a linear term in θ as well:

$$\begin{aligned}A^\nu &= \theta^{\nu\rho}V_\rho\\ \delta V_\rho &= \frac{\partial}{\partial x^\rho}\alpha + i[\alpha \stackrel{*}{,} V_\rho].\end{aligned} \tag{3.10}$$

As in reference [1] we expand in θ, but not in g_a:

$$\begin{aligned}&f * g = \left\{1 + \frac{i}{2}\frac{\partial}{\partial x^\nu}\theta^{\nu\mu}\frac{\partial}{\partial x'^\mu} + \ldots\right\} f(x,t') \circledast g(x',t'')\Big|_{\substack{x'\to x\\ t'\to t, t''\to t}}\\ &f(x,t') \circledast g(x',t'') = e^{\frac{i}{2}t^a g_a(i\frac{\partial}{\partial t'}, i\frac{\partial}{\partial t''})} f(x,t')g(x',t'').\end{aligned} \tag{3.11}$$

We first treat Equation (3.10) to zeroth order in θ and show that it can be solved by assuming V and α to be linear in t. This has to be expected as $\theta = 0$ corresponds to the usual gauge theory on commuting spaces where the infinitesimal transformation and the connection are Lie algebra-valued. To zeroth order:

$$\begin{aligned} \alpha &= \alpha_a^1 t^a, \\ V_\rho &= a_{\rho,a}^1 t^a. \end{aligned} \tag{3.12}$$

From (3.10) we obtain, as expected [6]:

$$\delta a_{\rho,a}^1 = \frac{\partial \alpha_a^1}{\partial x^\rho} - f^{bc}{}_a \alpha_b^1 a_{\rho,c}^1. \tag{3.13}$$

We turn to first order in θ in the variation of V_ρ, Equation (3.10). The contributions that come from the zero order terms of α and V_ρ are at most of second order in t. This is the case because $t^a g_a(i\frac{\partial}{\partial t'}, i\frac{\partial}{\partial t''})$ reduces the power of t by at least one. The terms of order zero in g_a actually contribute exactly in order t^2. Their contribution to δV_ρ is

$$\delta V_\rho = \theta^{\nu\mu} \partial_\nu \alpha_a^1 \partial_\mu a_{\rho,b}^1 t^a t^b + \dots. \tag{3.14}$$

If we now assume that the terms of α and V_ρ linear in θ are all of second order in t we get a consistent set of equations. We define to first order in θ:

$$\begin{aligned} \alpha &= \alpha_a^1 t^a + \alpha_{ab}^2 t^a t^b + \dots \\ V_\rho &= a_{\rho,a}^1 t^a + a_{\rho,ab}^2 t^a t^b + \dots \end{aligned} \tag{3.15}$$

α^1 and a^1 are of order zero in θ and α^2 and a^2 of first order. This expansion in t leads to an expansion in g_a of the $\circledast$-product, because higher order t-derivatives vanish.

In the calculation that now follows we have to use g_a to the order given in (2.12). The term with three derivatives, however, vanishes on commutators of $*$-products because it is symmetric under the exchange of k and p.

The result of the calculation is:

$$\begin{aligned} \delta a_{\rho,ab}^2 t^a t^b &= \partial_\rho \alpha_{ab}^2 t^a t^b \\ &\quad - \theta^{\nu\mu} \partial_\nu \alpha_a^1 \partial_\mu a_{\rho,b}^1 t^a t^b \\ &\quad - 2 f^{bc}{}_a \left\{ \alpha_b^1 a_{\rho,cd}^2 + \alpha_{bd}^2 a_{\rho,c}^1 \right\} t^d t^a. \end{aligned} \tag{3.16}$$

This can be brought closer to the form of reference [1]. We introduce the Lie algebra valued ε and the enveloping algebra-valued G_ρ:

$$\varepsilon = \alpha_b^1 T^b, \quad G_\rho = a_{\rho,cd}^2 T^c T^d \tag{3.17}$$

and compute the commutator

$$i[\varepsilon, G_\rho] = -\alpha_b^1 a_{\rho,cd}^2 f^{bc}{}_l \{T^l T^d + T^d T^l\}. \tag{3.18}$$

This is true because $a_{\rho,cd}^2$ is symmetric in c and d. Now we remember that we have used a star product that corresponds to a completely symmetrical version of the monomials of the bases. Thus we have to replace

$$\frac{1}{2}\{T^l T^d + T^d T^l\} \quad \sim \quad t^l t^d \tag{3.19}$$

and obtain from (3.18) the corresponding term in (3.16). The other term derives from the commutator

$$
\begin{aligned}
& i[\gamma, a_\rho], \quad \text{with} \\
& \gamma = \alpha^2_{bd} T^b T^d, \quad a_\rho = a^1_{\rho,l} T^l.
\end{aligned} \tag{3.20}
$$

This is exactly the structure as in reference [1]:

$$
\begin{aligned}
\delta G_\nu &= \partial_\nu \gamma - \frac{1}{2}\theta^{\kappa\lambda} \{\partial_\kappa \varepsilon, \partial_\lambda a_\nu\} \\
&\quad + i[\varepsilon, G_\nu] + i[\gamma, a_\nu].
\end{aligned} \tag{3.21}
$$

It has the solution already found in [1, 3].

$$
\begin{aligned}
\alpha^2_{ab} t^a t^b &= \frac{1}{2}\theta^{\nu\mu} \partial_\nu \alpha^1_a a^1_{\mu,b} t^a t^b \\
a^2_{\rho,ab} t^a t^b &= -\frac{1}{2}\theta^{\nu\mu} a^1_{\nu,a} (\partial_\mu a^1_{\rho,b} + F^1_{\mu\rho,b}) t^a t^b,
\end{aligned} \tag{3.22}
$$

where $F^1_{\nu\rho,b} = \partial_\nu a^1_{\mu,b} - \partial_\mu a^1_{\nu,b} + f^{cd}{}_b a^1_{\nu,c} a^1_{\mu,d}$.

The procedure can now be generalized to all the higher powers of θ. We always assume that α^n and a^n_ρ are of power θ^{n-1} and polynomials in t of degree n. This amounts to a power series expansion in g_a as well and the existence of the Seiberg-Witten map to all orders follows from the existence of the Seiberg-Witten map in the general setting of the previous chapter. This is discussed in detail in reference [6], where the full non-abelian Seiberg-Witten map is constructed.

It is straightforward to generalize this formalism to a $*$-product, with a coordinate dependent θ underlying the non-commutative, e.g. Lie or quantum plane, structure. We give the result:

$$
\begin{aligned}
\alpha &= \alpha^1_a t^a + \frac{1}{2}\theta^{\nu\mu} \partial_\nu \alpha^1_a a^1_{\mu,b} t^a t^b \\
A^\nu &= \theta^{\nu\mu} a^1_{\mu,a} t^a \\
&\quad - \frac{1}{2}\theta^{\sigma\mu} a^1_{\sigma,a} \left(\partial_\mu(\theta^{\nu\rho} a^1_{\rho,b}) + \theta^{\nu\rho} F^1_{\mu\rho,b}\right) t^a t^b
\end{aligned} \tag{3.23}
$$

For the Lie structure:

$$
\theta^{\mu\nu} = f^{\mu\nu}{}_\kappa x^\kappa. \tag{3.24}
$$

For the quantum plane:

$$
\theta^{\mu\nu} = -ihxy, \qquad h = \ln q. \tag{3.25}
$$

Let us consider once more the gauge transformations (3.4). We see that α is not an arbitrary element of the algebra $\mathcal{A}_z$. There are only M (dimension of the Lie group to be gauged) free parameters $\alpha^1_l(x)$, all the higher-order terms in the enveloping algebra can be expressed in terms of these parameters and the gauge field $a^1_{\rho,l}$ and their derivatives. This can be achieved by the Seiberg-Witten map.

To summarize, the Lie algebra-valued term $\varepsilon = \alpha^1_a(x) T^a$ of α determines all the other terms in the enveloping algebra:

$$
\alpha = \alpha^1_a T^a + \frac{1}{4}\theta^{\nu\mu} \partial_\nu \alpha^1_a a^1_{\mu,b} (T^a T^b + T^b T^a) + \dots \tag{3.26}
$$

Thus a gauge transformation is determined by $\alpha^1(x)$ and $a^1_\rho(x)$, it is defined by:

$$\delta_{\alpha^1}\psi = i\alpha(\alpha^1, a^1_\rho) * \psi. \tag{3.27}$$

The composition of two transformations is defined for arbitrary enveloping algebra-valued transformations as follows:

$$\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = \delta_{i(\beta*\alpha-\alpha*\beta)}. \tag{3.28}$$

We show that this is also true for the restricted transformation defined by α^1:

$$\delta_{\alpha^1}\delta_{\beta^1} - \delta_{\beta^1}\delta_{\alpha^1} = \delta_{i(\beta*\alpha-\alpha*\beta)^1}. \tag{3.29}$$

This reflects the compositon of standard Lie algebra-valued gauge transformations. We show this to first order in θ, using the $*$-formalism.

$$\begin{aligned}
\alpha &= \alpha^1_a t^a + \frac{1}{2}\theta^{\nu\mu}\partial_\nu\alpha^1_a a^1_{\mu,b} t^a t^b \\
\beta &= \beta^1_a t^a + \frac{1}{2}\theta^{\nu\mu}\partial_\nu\beta^1_a a^1_{\mu,b} t^a t^b
\end{aligned} \tag{3.30}$$

We compute $[\alpha \stackrel{*}{,} \beta]$ to first order in θ from Equation (3.11):

$$\begin{aligned}
[\alpha \stackrel{*}{,} \beta] = {} & i\alpha^1_a\beta^1_b f^{ab}{}_c t^c \\
& + \theta^{\nu\mu}\Big\{\frac{i}{2}\partial_\nu(\alpha^1_a\beta^1_b f^{ab}{}_d)a_{\mu,c} \\
& \qquad + \frac{i}{2}(\alpha^1_a\partial_\nu\beta^1_d - \beta^1_a\partial_\nu\alpha^1_d)a^1_{\mu,b}f^{ab}{}_c \\
& \qquad + i\partial_\nu\alpha^1_d\partial_\mu\beta^1_c\Big\}t^d t^c.
\end{aligned} \tag{3.31}$$

Now we compute from (3.4)

$$(\delta_{\beta^1}\delta_{\alpha^1} - \delta_{\alpha^1}\delta_{\beta^1})\hat{\psi} = -(\hat{\alpha}\hat{\beta} - \hat{\beta}\hat{\alpha})\hat{\psi} + i((\delta_{\beta^1}\hat{\alpha}) - (\delta_{\alpha^1}\hat{\beta}))\hat{\psi}. \tag{3.32}$$

The second term arises because α depends on the gauge fields a^1_μ that transforms under gauge transformations

$$\delta_{\beta^1}\alpha = \frac{1}{2}\theta^{\rho\sigma}\partial_\rho\alpha^1_a(\partial_\sigma\beta^1_b - f^{cd}{}_b\beta^1_c a^1_{\sigma,d})t^d t^b. \tag{3.33}$$

We are now ready to compute $\delta_{\beta^1}\delta_{\alpha^1} - \delta_{\alpha^1}\delta_{\beta^1}$ and obtain

$$\begin{aligned}
& \delta_{\beta^1}\delta_{\alpha^1} - \delta_{\alpha^1}\delta_{\beta^1} = i(\delta_{\beta^1}\alpha - \delta_{\alpha^1}\beta) - [\alpha \stackrel{*}{,} \beta] \\
& \quad = -\left(i\alpha^1_a\beta^1_b f^{ab}{}_c t^c + \frac{1}{2}\theta^{\nu\mu}\partial_\nu(i\alpha^1_a\beta^1_b f^{ab}{}_d)a^1_{\mu,c}t^d t^c\right).
\end{aligned} \tag{3.34}$$

This is exactly the formula (3.29) that we obtain if we start from the Lie algebra-valued part of $[\alpha \stackrel{*}{,} \beta]$

$$[\alpha \stackrel{*}{,} \beta] = i\alpha^1_a\beta^1_b f^{ab}{}_c T^c + \dots \tag{3.35}$$

This shows that our restricted enveloping algebra-valued form of the parameters is respected by the commutator of two transformations.

In reference [1] we have introduced tensors

$$\hat{T}^{\mu\nu} = [\hat{X}^\mu, \hat{X}^\nu] - i\hat{\theta}^{\mu\nu}, \tag{3.36}$$

that transform

$$\delta\hat{T}^{\mu\nu} = i[\hat{\alpha}, \hat{T}^{\mu\nu}] \tag{3.37}$$

under the general enveloping algebra valued gauge transformation. $\hat{\theta}^{\mu\nu}$ is the respective term for all the three structures, canonical, Lie and quantum plane. Equation (3.37) will also be true for our restricted gauge transformation (3.27).

For the canonical case to exemplify:

$$\begin{aligned} T^{\mu\nu} &= i\theta^{\mu\kappa}\partial_\kappa A^\nu - i\theta^{\nu\lambda}\partial_\lambda A^\mu + A^\mu * A^\nu - A^\nu * A^\mu \\ &= \theta^{\mu\kappa}\theta^{\nu\lambda}\left\{\partial_\kappa V_\lambda - \partial_\lambda V_\kappa + V_\kappa * V_\lambda - V_\lambda * V_\kappa\right\}. \end{aligned} \tag{3.38}$$

It is natural to introduce the field strength

$$F_{\kappa\lambda} = \partial_\kappa V_\lambda - \partial_\lambda V_\kappa + V_\kappa * V_\lambda - V_\lambda * V_\kappa. \tag{3.39}$$

It is easy to compute the first order correction to the classical field strength F^1 defined after Equation (3.22):

$$\begin{aligned} F_{\kappa\lambda,a}t^a &= F^1_{\kappa\lambda,a}t^a \\ &\quad + \theta^{\mu\nu}\Big\{F^1_{\kappa\mu,a}F^1_{\lambda\nu,b} \\ &\qquad - \frac{1}{2}a^1_{\mu,a}\left((D_\nu F^1_{\kappa\lambda})_b + \partial_\nu F^1_{\kappa\lambda,b}\right)\Big\}t^a t^b. \end{aligned} \tag{3.40}$$

In this formula a covariant derivative of F^1 is used: $(D_\nu F^1_{\kappa\lambda})_b = \partial_\nu F^1_{\kappa\lambda,b} + a_{\nu,c}F^1_{\kappa\lambda,d}f^{cd}{}_b$. This expression can also be obtained from reference [3].

Using the transformation law for α (3.33) and a^1 (3.13) we find as expected

$$\delta_{\alpha^1}F_{\kappa\lambda} = i[\alpha \stackrel{*}{,} F_{\kappa\lambda}], \tag{3.41}$$

with the restricted form of α.

4 Gauge covariant dynamics

The $*$-formalism can be used to formulate a dynamics on non-commutative spaces. The coefficient functions $f(x)$ are the objects for which dynamical laws can be defined. In general we have to enlarge the algebra by derivatives for this purpose. For the quantum plane structure such derivatives have been introduced in [13] in a purely algebraic approach. For the algebra extended by derivatives the formalism developed in the second chapter can be used.

For the canonical structure we can define derivates following the same strategy as for the quantum plane structure. Derivatives have to be defined in such a way, that they do not lead to new relations for the coordinates. Proceeding this way we can define a Leibniz rule:

$$\hat{\partial}_\mu\hat{x}^\nu = \delta^\nu_\mu + d^{\nu\rho}_{\mu\sigma}\hat{x}^\sigma\hat{\partial}_\rho, \tag{4.1}$$

where the coefficients $d^{\nu\rho}_{\mu\sigma} \in \mathbb{C}$ have to be choosen in such a way that

$$\begin{aligned}\hat{\partial}_\rho \{[\hat{x}^\mu, \hat{x}^\nu] - i\theta^{\mu\nu}\} &= \\ = \; & \delta^\mu_\rho \hat{x}^\nu - \delta^\nu_\rho \hat{x}^\mu + d^{\mu\nu}_{\rho\kappa}\hat{x}^\kappa - d^{\nu\mu}_{\rho\kappa}\hat{x}^\kappa \\ & + \hat{x}^\kappa \hat{x}^\beta \left\{ d^{\mu\sigma}_{\rho\kappa} d^{\nu\alpha}_{\sigma\beta} - d^{\nu\sigma}_{\rho\kappa} d^{\mu\alpha}_{\sigma\rho} \right\} \hat{\partial}_\alpha \\ & - i\theta^{\mu\nu}\hat{\partial}_\rho \end{aligned} \tag{4.2}$$

does not lead to new relations when $\hat{\partial}$ is brought to the right hand side. This is the case if we define

$$d^{\nu\rho}_{\mu\sigma} = \delta^\nu_\sigma \delta^\rho_\mu, \tag{4.3}$$

or simply

$$\hat{\partial}_\rho \hat{x}^\mu = \delta^\mu_\rho + \hat{x}^\mu \hat{\partial}_\rho. \tag{4.4}$$

If we now compare this with (2.1) we see that

$$\hat{x}^\alpha - i\theta^{\alpha\rho}\hat{\partial}_\rho \tag{4.5}$$

commutes with all coordinates. This allows us to divide the algebra by the ideal generated by the relation (see also [17, 18])

$$\hat{x}^\alpha - i\theta^{\alpha\rho}\hat{\partial}_\rho = 0. \tag{4.6}$$

The star product, known for the coordinates is now defined for the derivations as well

$$\begin{aligned}\partial_\rho * f &= -i\theta^{-1}_{\rho\sigma} x^\sigma * f \\ &= -i\theta^{-1}_{\rho\sigma}\left([x^\sigma \stackrel{*}{,} f] + f * x^\sigma\right) \\ &= \frac{\partial}{\partial x^\rho} f + f * \partial_\rho. \end{aligned} \tag{4.7}$$

We have used (3.9).

For the canonical structure an integral can be defined:

$$\int \hat{f} = \int d^N x \; f(x^1, \dots x^N). \tag{4.8}$$

For the moment it is simpler to consider just functions of x^i that do not depend on the variables t^a as well. The $*$-product simply is the one of (2.9):

$$\begin{aligned} f * g &= e^{\frac{i}{2}\frac{\partial}{\partial x^i}\theta^{ij}\frac{\partial}{\partial x'^j}} f(x) g(x')\Big|_{x' \to x} \\ &= \int d^N x' \; \delta(\vec{x} - \vec{x}') e^{\frac{i}{2}\frac{\partial}{\partial x^i}\theta^{ij}\frac{\partial}{\partial x'^j}} f(x) g(x') \\ &= \; . \end{aligned} \tag{4.9}$$

$\delta(\vec{x} - \vec{x}')$ is the product of N δ-functions:

$$\delta(\vec{x} - \vec{x}') = \delta(x^1 - x'^1) \cdots \delta(x^N - x'^N). \tag{4.10}$$

We now show that

$$\int \hat{f}\hat{g} = \int \hat{g}\hat{f}. \tag{4.11}$$

We use the $*$-formalism:

$$\int \hat{f}\hat{g} = \int d^N x d^N x' \; \delta(\vec{x} - \vec{x}') e^{\frac{i}{2}\frac{\partial}{\partial x^i}\theta^{ij}\frac{\partial}{\partial x'^j}} f(x) g(x'). \tag{4.12}$$

Partial integration:

$$\int \hat{f}\hat{g} = \int d^N x d^N x' \; f(x) g(x') e^{\frac{i}{2}\frac{\partial}{\partial x^i}\theta^{ij}\frac{\partial}{\partial x'^j}} \delta(\vec{x} - \vec{x}'). \tag{4.13}$$

The δ-function depends on $\vec{x} - \vec{x}'$. Thus

$$\frac{\partial}{\partial x^l}\delta(\vec{x} - \vec{x}') = -\frac{\partial}{\partial x'^l}\delta(\vec{x} - \vec{x}'). \tag{4.14}$$

This leads to

$$\int \hat{f}\hat{g} = \int d^N x d^N x' \; f(x) g(x') e^{\frac{i}{2}\frac{\partial}{\partial x'^i}\theta^{ij}\frac{\partial}{\partial x^j}} \delta(\vec{x} - \vec{x}'). \tag{4.15}$$

Partial integration again:

$$\begin{aligned} \int \hat{f}\hat{g} &= \int d^N x d^N x' \; \delta(\vec{x} - \vec{x}') e^{\frac{i}{2}\frac{\partial}{\partial x'^i}\theta^{ij}\frac{\partial}{\partial x^j}} g(x') f(x) \\ &= \int \hat{g}\hat{f}. \end{aligned} \tag{4.16}$$

We also find Stokes theorem

$$\int [\hat{\partial}_l, \hat{f}] = \int d^N x \; [\partial_l \stackrel{*}{,} f] = \int d^N x \frac{\partial}{\partial x^l} f = 0 \tag{4.17}$$

from (4.7) for functions that vanish at the boundary. Partial integration of the $*$-derivative follows now from (4.17) and the Leibniz rule

$$[\partial_l \stackrel{*}{,} (f * g)] = ([\partial_l \stackrel{*}{,} f]) * g + f * ([\partial_l \stackrel{*}{,} g]), \tag{4.18}$$

which is true, because θ is x independent.

With this integral we can define an action. A tensor that transforms as in (3.37):

$$\delta \hat{L} = i[\hat{\alpha}, \hat{L}] \tag{4.19}$$

will lead to a gauge invariant action:

$$W = \int d^N x \; \mathrm{Tr}\hat{L}, \qquad \delta W = 0. \tag{4.20}$$

The trace has to be taken for the group generators. A proper action would be

$$L = \frac{1}{4} F_{\kappa\lambda} F^{\kappa\lambda}, \tag{4.21}$$

where $F_{\kappa\lambda}$ has been defined in (3.39). The first correction term in θ to the classical field strength has been computed in (3.40).

We have thus formulated dynamics on a non-commutative space entirely whithin the standard framework of quantum field theory. The method can be extended to the treatment of matter fields aswell.

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