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Boundary Fermions, Coherent Sheaves and D-branes on Calabi-Yau manifolds

Suresh Govindarajan
Department of Physics, Indian Institute of Technology, Madras, Chennai 600 036, India
Email: suresh@chaos.iitm.ernet.in

and

T. Jayaraman
The Institute of Mathematical Sciences, Chennai 600 113, India
Email: jayaram@imsc.ernet.in

Abstract

We construct boundary conditions in the gauged linear sigma model for B-type D-branes on Calabi-Yau manifolds that correspond to coherent sheaves given by the cohomology of a monad. This necessarily involves the introduction of boundary fields, and in particular, boundary fermions. The large-volume monodromy for these D-brane configurations is implemented by the introduction of boundary contact terms. We also discuss the construction of D-branes associated to coherent sheaves that are the cohomology of complexes of arbitrary length. We illustrate the construction using examples, specifically those associated with the large-volume analogues of the Recknagel-Schomerus states with no moduli. Using some of these examples we also construct D-brane states that arise as bound states of the above rigid configurations and show how moduli can be counted in these cases.

1 Introduction

As in the case of closed string compactifications on Calabi-Yau manifolds, the gauged linear sigma model description appears to be the natural starting point in the context of D-branes wrapped on supersymmetric cycles in Calabi-Yau manifolds, particularly for the study of the dependence of D-brane physics on the Kähler moduli.(For a list of references on D-branes on CY manifolds see [1–4].) In an earlier paper [5], we had taken the first steps towards such a description studying in particular the case of the six-brane wrapped on the full Calabi-Yau manifold. Subsequently, following on the work of [6] and [7] we had demonstrated a systematic procedure using the techniques of helices and mutations whereby we could construct the large volume analogues of all the $\sum_a l_a = 0$ Recknagel-Schomerus states as restriction of exceptional sheaves in the ambient variety to the Calabi-Yau hypersurface [8]. While the procedure used the intuition of the GLSM we had not provided an explicit field-theoretic method of construction of these D-brane configurations in the GLSM. This paper is devoted to such explicit constructions.

The paper in essence extends the techniques developed in the context of heterotic strings to describe vector bundles on Calabi-Yau manifolds to the case of world-sheets with boundary. In the heterotic case these essentially involved using the monad constructions of vector bundles in the case where all the elements in the complex involved only line bundles. In the first instance in this paper we extend this technique to the case of worldsheets with boundary.

But since this provides a extremely limited class of configurations (and not even the full set of Recknagel-Schomerus states) we are led naturally to constructions of D-brane configurations that are described by complexes of length greater than two. Such constructions are also important in the light of the role that the derived category of coherent sheaves[1] is expected to play in understanding issues related to B-type D-branes especially in the stringy regime as argued by Douglas in [10, 11]. The derived category of coherent sheaves is precisely the description of all sheaves on the CY manifold by complexes that are generically longer than monads. It would be of some importance to extend the construction involving complexes of length greater than two to the heterotic string.

The description of the formation of bound states of D-brane configurations in terms of sequences of coherent sheaves, as first explained by Harvey and Moore [12], is easily implemented in the GLSM construction. We provide examples of such description of bound states in the context of the D-brane configurations that we consider as examples in this paper.

The organisation of the paper is as follows: In sec. 2 we remark on the use of boundary fermions in relation to vector bundles. In sec. 3 we set up our notation

[1]For some earlier work where the derived category of coherent sheaves appeared in the context of string theory, see [9].

and discuss the decomposition of bulk multiplets under B-type supersymmetry. Section 4 describes the mathematical construction of bundles using complexes with an emphasis on the monad construction. Section 5 gives an explicit realisation, in the GLSM with boundary, of bundles that are described by monads in sec. 4. In sec. 6, we discuss how the large volume monodromy action on the vector bundles is realised in the GLSM. In sec. 7, we discuss the GLSM construction for arbitrary complexes. In sec. 8, we discuss how D-branes corresponding to bound states may be realised by our methods using specific examples. We conclude in sec. 9 with a brief discussion.

The possibility of using boundary fermions in the GLSM to describe D-brane configurations has been discussed in a talk by S. Kachru [13]. In the course of our work a paper appeared [14] that uses boundary fermions in relation to B-type D-branes but in a very limited context. Lastly, as this manuscript was being completed for publication another paper [15] appeared, that has some overlap with the work of sec. 7 of our paper.

2 Some remarks on boundary fermions

In an earlier paper [5] where we discussed the GLSM with boundary, we did not introduce boundary fermions but nevertheless obtained some consistent boundary conditions. By consistent boundary conditions, we mean boundary conditions in the GLSM which have a proper NLSM limit. As was shown in that context, this imposes a rather stringent condition on the possible boundary conditions. In that construction two problems arose in the context of fairly simple examples:

(a) One has to find appropriate boundary conditions for the p-field which is set to zero in the NLSM limit. The boundary condition proposed in [5] was to impose $p = 0$. This is acceptable in the CY-phase but clearly is in conflict with the bulk ground state condition in the LG-phase.

(b) For the case of "mixed"-boundary conditions, i.e., for branes such as the $D2-$ and $D4-$ brane, the boundary conditions on the fields in the vector multiplet were somewhat contrived and unnatural unlike the case of the D6-brane, i.e., the brane wrapping the full CY manifold.

In this paper we obtain a simple resolution to these two problems by introducing boundary fermi supermultiplets. The boundary fermions impose the condition $P = 0$ for case (a) and $f(\phi) = 0$ for case (b) as a ground state condition. In the process one finds that one can obtain natural boundary conditions on the fields in the vector multiplet from the NLSM limit.

For the case of coincident D-branes, the boundary fermions can be interpreted as objects carrying the Chan-Paton index. More precisely, they are objects carrying indices associated with the vector bundle of the corresponding D-brane. In the GLSM, these have non-trivial interactions with the bulk multiplets. We note that boundary fermions were introduced early on as carriers of the Chan-Paton index by Marcus and Sagnotti [16](see also [17]). More recently, they have been introduced in the study of tachyon condensation and non-BPS states beginning with the work of [18, 19].

With the introduction of boundary fermions, the worldsheet Witten index computation becomes completely analogous to the original calculations of the index theorem using supersymmetric quantum mechanics. Consider the case, when the worldsheet is topologically a cylinder. We will be interested in the case, when the two boundaries end on D-branes corresponding to non-trivial vector bundles. The D-brane configurations are assumed to preserve B-type supersymmetry. The Witten index associated with the BRST charge $Q \equiv \overline{Q}_+ + \eta\overline{Q}_-$ can be seen to be equal to

$$\chi(E_1, E_2) \equiv \int_M \mathrm{ch}(E_1)\mathrm{ch}(E_2^*)\mathrm{Td}(M) \quad ,$$

(see [20]) where M is the target manifold and E_1 and E_2 are the vector bundles on the two boundaries.

It is known that the contributions $\mathrm{ch}(E_1)$ and $\mathrm{ch}(E_2^*)$ from the two boundaries can be realised by introducing fermions living on the boundaries. More precisely, the path-ordered integral

$$P\left(\exp\left[\int dx^0 \partial_0\phi^\mu A^r_\mu(\phi)T^r\right]\right)$$

(where T^r are in the fundamental representation) is equivalent to the path-integral of anti-periodic (complex) fermions in the fundamental representation of E provided we restrict to *one-particle states* [21]. The action is given by

$$S_{b.f.} = \int dx^0[\overline{\pi}_a D_0 \pi_a] \quad ,$$

where $D_0\pi = (\partial_0 + \partial_0\phi^\mu A^r_\mu(\phi)T^r)\pi$. If one however restricts to n-particle states, then the path integral leads to gauge fields on the n-th anti-symmetric power of E. On the other hand, if one allows all states, then one obtains gauge fields on the *spinor bundle* over E[2].

In what follows in the rest of the paper, we will assume that such a restriction to one-particle states is in operation always (this can be trivially done by using a Lagrange multiplier [22])except where we explicitly consider situations to the contrary.

2.1 Boundary fermions and the GLSM

For supersymmetric D-brane configurations, the boundary preserves a linear combination of the bulk $(2, 2)$ supersymmetry. Thus, any boundary multiplet will nec-

[2]We thank K. Hori for bringing this issue to our attention.

essarily be a supermultiplet of the unbroken linear combination. We will loosely refer to these multiplets as $(0,2)$ multiplets. While this construction closely parallels $(0,2)$ constructions for vector bundles in the GLSM without boundary [23], there are also important differences. For instance, some conditions like the D-term constraint appear in the low energy analysis by continuity from the bulk. Another important difference is that one need not impose the vanishing of the first Chern class of the vector bundle. This is related to the fact that the fields associated with the vector bundle live on the boundary of the worldsheet and do not play a role in the bulk R-anomaly cancellation. We will comment on other differences later on in the text, where appropriate.

The strategy in this paper is to construct boundary actions involving $(0,2)$ multiplets such that together with the bulk action, the boundary conditions on the bulk fields preserve the unbroken worldsheet B-type supersymmetry. This results in the coupling of the boundary multiplets to the boundary values of the bulk fields. The boundary actions, together with appropriate bulk-boundary couplings, that we construct lead in the low-energy limit, in a natural fashion, to the monad constructions used for vector bundles. As in [5], we will require that our boundary conditions satisfy the following constraints:

(i) cancellation of ordinary and supersymmetric variations modulo equations of motion;

(ii) the set of boundary conditions are closed under the action of the unbroken supersymmetry;

(iii) the boundary conditions (especially on the fields in vector multiplets) have a consistent NLSM limit. The theta term in the bulk requires the addition of a *contact term* which correctly implements the large volume monodromy action on the vector bundle. This generalises the contact terms which appeared earlier [5, 20].

Along the way, we carefully work out the decomposition of bulk multiplets in terms of boundary $(0,2)$ multiplets. This reorganises the bulk fields in suitable fashion and motivates the bulk-boundary couplings.

A large class of D-brane configurations can be constructed using the techniques of this paper. We illustrate this for the large volume analogues of the $\sum_a l_a = 0$ Recknagel-Schomerus (RS) states [1] at the Gepner point. The monads associated with these states are precisely those which appear as mutations of helices associated with the six-brane. Unlike in most heterotic constructions, the constituents of the monads are vector bundles in general. Thus, one can in fact construct vector bundles which are the cohomology of complexes of line bundles with length greater than two, in the GLSM.

Our construction can be applied to the cases of D-branes wrapping lower dimensional cycles. Some of these branes can be obtained from the transverse intersection of hypersurfaces with the Calabi-Yau manifold. These fit naturally into the general framework. This is related to the fact that the cohomology of complexes closely related to the monad can give rise to sheaves. In such situations, one can also reinterpret the result as a bound state of branes and anti-branes [12]. We also consider the bound state of two $\sum_a l_a = 0$ RS states to obtain a $\sum_a l_a = 1$ RS state.

3 Background and Notation

3.1 The GLSM

In this section, we will consider the GLSM with $(2,2)$ supersymmetry on a worldsheet with a boundary which preserves half of the bulk supersymmetry. We will mostly consider the case of B-type boundary conditions where the unbroken supersymmetry is given by

$$\epsilon_- = \eta\epsilon_+ \equiv \frac{\epsilon}{\sqrt{2}} \quad ,$$

where $\eta = \pm 1$.

In order to fix the notations and conventions used in this paper, we review the Lagrangian and supersymmetries of the GLSM following [24]. We work in Minkowski space with the metric $(-,+)$. We are interested in describing compactifications of string theory with eight supercharges; the worldsheet conformal field theory must then have $\mathcal{N} = (2,2)$ superconformal symmetry. We expect that a nonconformal theory with such an infra-red fixed point should have $\mathcal{N} = 2$ supersymmetry as well.

The theory can be obtained by dimensional reduction from $d = 4, \mathcal{N} = 1$ abelian gauge theory with chiral multiplets. It contains s $U(1)$ vector multiplets, described by the vector superfields $V^a(a = 1, \cdots, s)$ and k chiral multiplets described by the chiral superfields $\Phi_i(i = 1, \cdots, k)$. Written in components, the vector multiplet consists of the vector fields $v^a_\alpha(\alpha = 0, 1)$, the complex scalar field σ^a, complex chiral fermions $\lambda^a_\pm$, and the real auxiliary field D^a. The chiral multiplet consists of a complex scalar ϕ_i, complex chiral fermions $\psi_{\pm i}$, and a complex auxiliary scalar field F_i. They are charged under the $U(1)$s with charge Q^a_i. In component notation, the supersymmetry transformations of the vector multiplet are:

$$\begin{aligned}
\delta v^a_0 &= i\left(\bar\epsilon_+\lambda^a_+ + \bar\epsilon_-\lambda^a_- + \epsilon_+\bar\lambda^a_+ + \epsilon_-\bar\lambda^a_-\right), \\
\delta v^a_1 &= i\left(\bar\epsilon_+\lambda^a_+ - \bar\epsilon_-\lambda^a_- + \epsilon_+\bar\lambda^a_+ - \epsilon_-\bar\lambda^a_-\right), \\
\delta\sigma^a &= -i\sqrt{2}\bar\epsilon_+\lambda^a_- - i\sqrt{2}\epsilon_-\bar\lambda^a_+, \\
\delta\bar\sigma^a &= -i\sqrt{2}\epsilon_+\bar\lambda^a_- - i\sqrt{2}\bar\epsilon_-\lambda^a_+, \\
\delta D^a &= -\bar\epsilon_+(\partial_0 - \partial_1)\lambda^a_+ - \bar\epsilon_-(\partial_0 + \partial_1)\lambda^a_-
\end{aligned} \tag{3.1}$$

$$
\begin{aligned}
& + \quad \epsilon_+(\partial_0 - \partial_1)\overline{\lambda}^a_+ + \epsilon_-(\partial_0 + \partial_1)\overline{\lambda}^a_-, \\
\delta\lambda^a_+ \quad &= \quad i\epsilon_+ D^a + \sqrt{2}(\partial_0 + \partial_1)\overline{\sigma}^a\epsilon_- - v^a_{01}\epsilon_+, \\
\delta\lambda^a_- \quad &= \quad i\epsilon_- D^a + \sqrt{2}(\partial_0 - \partial_1)\sigma^a\epsilon_+ + v^a_{01}\epsilon_-, \\
\delta\overline{\lambda}^a_+ \quad &= \quad -i\overline{\epsilon}_+ D^a + \sqrt{2}(\partial_0 + \partial_1)\sigma^a\overline{\epsilon}_- - v^a_{01}\overline{\epsilon}_+, \\
\delta\overline{\lambda}^a_- \quad &= \quad -i\overline{\epsilon}_- D^a + \sqrt{2}(\partial_0 - \partial_1)\overline{\sigma}^a\overline{\epsilon}_+ + v^a_{01}\overline{\epsilon}_- \ ,
\end{aligned}
$$

where $\epsilon_\pm$ and $\overline{\epsilon}_\pm$ are the Grassman parameters for SUSY transformations. The transformation rules for the chiral multiplet are:

$$
\begin{aligned}
\delta\phi_i \quad &= \quad \sqrt{2}(\epsilon_+\psi_{-i} - \epsilon_-\psi_{+i}), \\
\delta\psi_{+i} \quad &= \quad i\sqrt{2}(D_0 + D_1)\phi_i\overline{\epsilon}_- + \sqrt{2}\epsilon_+ F_i - 2Q^a_i\phi_i\overline{\sigma}^a\overline{\epsilon}_+, \\
\delta\psi_{-i} \quad &= \quad -i\sqrt{2}(D_0 - D_1)\phi_i\overline{\epsilon}_+ + \sqrt{2}\epsilon_- F_i + 2Q^a_i\phi_i\sigma^a\overline{\epsilon}_-, \qquad (3.2) \\
\delta F_i \quad &= \quad -i\sqrt{2}\overline{\epsilon}_+(D_0 - D_1)\psi_{+i} - i\sqrt{2}\overline{\epsilon}_-(D_0 + D_1)\psi_{-i} \\
& \quad + 2Q^a_i(\overline{\epsilon}_+\overline{\sigma}^a\psi_{-i} + \overline{\epsilon}_-\sigma^a\psi_{+i}) + 2iQ^a_i\phi_i(\overline{\epsilon}_-\overline{\lambda}^a_+ - \overline{\epsilon}_+\overline{\lambda}^a_-) \qquad (3.3)
\end{aligned}
$$

The supersymmetric bulk action can be written as a sum of four terms,

$$
S = S_{ch} + S_{gauge} + S_W + S_{r,\theta} \qquad (3.4)
$$

The terms on the right hand side are, respectively: the kinetic term for the chiral superfields; the kinetic terms for the vector superfields; the superpotential interaction; and the Fayet-Iliopoulos and theta terms. S_{ch} is:

$$
\begin{aligned}
S_{ch} \quad = \quad & \sum_i \int d^2x \Big\{ - D_\alpha\overline{\phi}_i D^\alpha\phi_i + i\overline{\psi}_{-i}(\overleftrightarrow{D}_0 + \overleftrightarrow{D}_1)\psi_{-i} \ + i\overline{\psi}_{+i}(\overleftrightarrow{D}_0 - \overleftrightarrow{D}_1)\psi_{+i} \\
& + \ |F_i|^2 \ - \ 2\sum_a \overline{\sigma}^a\sigma^a(Q^a_i)^2\overline{\phi}_i\phi_i \ - \ \sqrt{2}\sum_a Q^a_i(\overline{\sigma}^a\overline{\psi}_{+i}\psi_{-i} \ + \ \sigma^a\overline{\psi}_{-i}\psi_{+i}) \\
& + \ D^a Q^a_i\overline{\phi}_i\phi_i - i\sqrt{2}\sum_{aQ^a_i}\overline{\phi}_i(\psi_{-i}\lambda^a_+ \ - \ \psi_{+i}\lambda^a_-) \\
& - \ i\sqrt{2}Q^a_i\phi_i(\overline{\lambda}^a_-\overline{\psi}_{+i} \ - \ \overline{\lambda}^a_+\overline{\psi}_{-i})\Big\} \qquad (3.5)
\end{aligned}
$$

where

$$
A \overleftrightarrow{D_i} B \equiv \frac{1}{2}(AD_iB - (D_iA)B) \ . \qquad (3.6)
$$

This symmetrized form of the fermion kinetic term is Hermitian in the presence of a boundary. Meanwhile, S_{gauge} is:

$$
\begin{aligned}
S_{gauge} \quad = \quad & \sum_a \frac{1}{e^2_a}\int d^2x \left\{ \frac{1}{2}(v^a_{01})^2 + \frac{1}{2}(D^a)^2 - \partial_\alpha\sigma^a\partial^\alpha\overline{\sigma}^a \right. \\
& \left. + \ i\overline{\lambda}^a_+(\overleftrightarrow{\partial}_0 - \overleftrightarrow{\partial}_1)\lambda^a_+ + i\overline{\lambda}^a_-(\overleftrightarrow{\partial}_0 + \overleftrightarrow{\partial}_1)\lambda^a_- \right\} \qquad (3.7)
\end{aligned}
$$

The superpotential term is:

$$S_W = -\int d^2x \left(F_i \frac{\partial W}{\partial \phi_i} \;+\; \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \psi_{-i}\psi_{+j} \;+\; \overline{F}_i \frac{\partial \overline{W}}{\partial \overline{\phi}_i} \;-\; \frac{\partial^2 \overline{W}}{\partial \overline{\phi}_i \partial \overline{\phi}_j} \overline{\psi}_{-i}\overline{\psi}_{+j} \right) \; . \tag{3.8}$$

Finally, the Fayet-Iliopoulos D-term and theta term are:

$$S_{r,\theta} = -r_a \int d^2x D^a + \frac{\theta_a}{2\pi} \int d^2x v_{01}^a \; . \tag{3.9}$$

The bosonic potential energy is given by

$$U = \sum_i |F_i|^2 + \sum_a \left(\frac{D^a}{2e^2} + 2|\sigma^a|^2 \sum_i Q_i^{a2} |\phi_i|^2 \right) \quad . \tag{3.10}$$

The auxiliary fields D and F_i can be eliminated by their equations of motion:

$$\begin{aligned} D^a &= -e^2 \left(\sum_{iQ_i^a|} \phi_i|^2 - r^a \right) \\ F_i^* &= \frac{\partial W}{\partial \phi_i} \; , \end{aligned} \tag{3.11}$$

In this paper, we will be mostly considering the case of a single $U(1)$ gauge field though the generalisation to many $U(1)$'s is obvious. Consider $(n+1)$ chiral superfields Φ_i of positive charge Q_i ($i = 1, \ldots, n+1$) and one superfield $\Phi_0 \equiv P$ of charge $Q_0 = Q_p = -\sum_{i \neq 0} Q_i$. In the absence of a superpotential, in the NLSM limit, the target space is a non-compact Calabi-Yau manifold given by the total space of a line bundle $\mathcal{O}(Q_p)$ over the weighted projective space $\mathbb{P}^{Q_1,\ldots,Q_{n+1}}$. A quasi-homogeneous superpotential of $W = PG(\Phi)$, where G is a degree $|Q_p|$ polynomial (satisfying certain transversality conditions [24]) involving chiral fields other than P gives rise to a compact Calabi-Yau manifold which is the hypersurface $G = 0$ in the weighted projective space.

3.2 Bulk and Boundary Supermultiplets

When A- or B-type boundary conditions are imposed, one half of the bulk $(2,2)$ supersymmetry is broken. Let $Q_\pm$ and $\overline{Q}_\pm$ be the generators of the $(2,2)$ supersymmetry algebra. The unbroken generators are given by the linear combinations

$$Q \equiv \frac{1}{\sqrt{2}}(Q_- - \eta \overline{Q}_+)$$

for A-type boundary conditions and by

$$Q \equiv \frac{1}{\sqrt{2}}(Q_- + \eta Q_+)$$

for B-type boundary conditions. They satisfy the $0+1$ dimensional supersymmetry algebra

$$\{Q,\overline{Q}\} = 2P_0 \quad ,$$

where P_0 is the generator of translations along x^0 and the other anticommutators are vanishing.

3.2.1 Boundary Superspace description

In superspace with coordinates $(x^0,\theta,\overline{\theta})$, the supersymmetry generators have the following representation[3]:

$$Q = \frac{\partial}{\partial\theta} + i\bar{\theta}\partial_0 \quad , \quad \overline{Q} = -\frac{\partial}{\partial\bar{\theta}} - i\theta\partial_0 \quad , \tag{3.12}$$

where $\partial_0 \equiv \partial/\partial x^0$. The superderivatives which commute with the supersymmetry generators are

$$D = \frac{\partial}{\partial\theta} - i\bar{\theta}\partial_0 \quad , \quad \overline{D} = -\frac{\partial}{\partial\bar{\theta}} + i\theta\partial_0 \quad . \tag{3.13}$$

<u>The Gauge Multiplet</u>

Gauge fields are introduced in superspace by means of gauge covariant derivatives $\mathcal{D}$, $\overline{\mathcal{D}}$ and $\mathcal{D}_0$ satisfying the constraints

$$\begin{aligned} \mathcal{D}^2 &= \overline{\mathcal{D}}^2 = 0 \\ \{\mathcal{D},\overline{\mathcal{D}}\} &= 2i\mathcal{D}_0 \quad . \end{aligned} \tag{3.14}$$

One can solve for the above constraints by introducing a Lie algebra valued real superfield V such that

$$\begin{aligned} \mathcal{D} &= e^{-V}\, D\, e^{V} \\ \overline{\mathcal{D}} &= e^{V}\, \overline{D}\, e^{-V} \quad . \end{aligned} \tag{3.15}$$

In the analogue of the Wess-Zumino gauge, $V = \theta\overline{\theta}v_0$ with

$$\begin{aligned} \mathcal{D}_0 &= \partial_0 + iv_0 \\ \mathcal{D} &= \frac{\partial}{\partial\theta} - i\bar{\theta}\mathcal{D}_0 \\ \overline{\mathcal{D}} &= -\frac{\partial}{\partial\bar{\theta}} + i\theta\mathcal{D}_0 \end{aligned} \tag{3.16}$$

[3] The Grassmann parameters θ and $\bar{\theta}$ are related to the bulk Grassmann parameters. For example, $\theta^- = \eta\theta^+ = \sqrt{2}\theta$ on the boundary for B-type boundary conditions. This can also be viewed as the definition of the boundary in superspace.

In the Wess-Zumino gauge, $\delta v_0 = 0$ i.e., the gauge field is invariant under supersymmetry transformations (such that the Wess-Zumino gauge is preserved). Further, there is no kinetic energy term for the gauge field since we are in $0+1$ dimensions.

Chiral Multiplets

Chiral multiplets (with $U(1)$ charge Q) satisfy

$$\overline{\mathcal{D}}\Phi = 0$$

and have the following component expansion

$$\Phi = \phi + \sqrt{2}\theta\tau - i\theta\overline{\theta}D_0\phi \quad , \tag{3.17}$$

where $D_0\phi = (\partial_0 + iQv_0)\phi$ is the usual covariant derivative. The supersymmetry transformation of the component fields are

$$\begin{aligned} \delta\phi &= \sqrt{2}\epsilon\tau & (3.18) \\ \delta\tau &= -\sqrt{2}i\overline{\epsilon}D_0\phi & (3.19) \end{aligned}$$

The kinetic energy term for chiral superfields is given by

$$\begin{aligned} S &= -\frac{i}{2}\int dx^0 d^2\theta \left(\overline{\Phi}D_0\Phi\right) & (3.20) \\ &= \int dx^0 \left(|D_0\phi|^2 + i\overline{\tau}D_0\tau\right) & (3.21) \end{aligned}$$

We will also need to consider fermi multiplets Π with components (π, l) satisfying the constraint

$$\overline{\mathcal{D}}\Pi = \sqrt{2}E(\Phi) \quad , \tag{3.22}$$

where $E(\phi)$ is a function of chiral superfields Φ_i. The component expansion of the superfield Π is

$$\Pi = \pi - \sqrt{2}\theta l - \overline{\theta}\sqrt{2}E(\phi) + \theta\overline{\theta}\left(-iD_0\pi + 2\tau_i\frac{\partial E}{\partial\phi_i}\right) \quad . \tag{3.23}$$

When $E = 0$, this reduces to the expansion of a chiral superfield. The supersymmetry transformation of the fields in the Fermi multiplet are

$$\begin{aligned} \delta\pi &= -\sqrt{2}\epsilon l - \sqrt{2}\overline{\epsilon}E(\phi) & (3.24) \\ \delta l &= i\sqrt{2}\overline{\epsilon}D_0\pi - \sqrt{2}\overline{\epsilon}\tau_i\frac{\partial E}{\partial\phi_i} & (3.25) \end{aligned}$$

Consider Fermi superfields satisfying

$$\overline{\mathcal{D}}\Pi_a = E_a(\Phi) \tag{3.26}$$

where E_a are chiral superfields. The supersymmetric action for Fermi multiplets is

$$S_F = -\frac{1}{2}\int dx^0 d^2\theta\ \overline{\Pi}_a \Pi_a \tag{3.27}$$

The component expansion of the above action is

$$S_F = \int dx^0 \left(i\overline{\pi}_a D_0 \pi_a + |l_a|^2 - |E_a(\phi)|^2 - \overline{\pi}_a \frac{\partial E_a}{\partial \phi_i}\tau_i - \frac{\partial \overline{E}_a}{\partial \overline{\phi}_i}\overline{\tau}_i \pi_a \right) \tag{3.28}$$

Further, let $J^a(\Phi)$ be chiral superfields satisfying the condition

$$E_a J^a = 0 \quad . \tag{3.29}$$

One can then introduce a boundary superpotential of the form

$$S_J = -\frac{1}{\sqrt{2}}\int dx^0 d\theta (\Pi_a J^a)|_{\overline{\theta}=0} - \text{h.c.} \tag{3.30}$$

The component expansion is

$$S_J = -\int dx^0 \left(l_a J^a(\phi) + \pi_a \frac{\partial J^a}{\partial \phi_i}\tau_i + \overline{l}_a \overline{J}^a(\overline{\phi}) + \frac{\partial \overline{J}^a}{\partial \overline{\phi}_i}\overline{\tau}_i \overline{\pi}_a \right) \tag{3.31}$$

On eliminating the auxiliary fields l_a using their equations of motion, we obtain the boundary Lagrangian

$$\begin{aligned} S_F + S_J = \int dx^0 \Big(& i\overline{\pi}_a \widetilde{D}_0 \pi_a - |J^a(\phi)|^2 - |E_a(\phi)|^2 \\ & - \overline{\pi}_a \frac{\partial E_a}{\partial \phi_i}\tau_i - \frac{\partial \overline{E}_a}{\partial \overline{\phi}_i}\overline{\tau}_i \pi_a - \pi_a \frac{\partial J^a}{\partial \phi_i}\tau_i - \frac{\partial \overline{J}^a}{\partial \overline{\phi}_i}\overline{\tau}_i \overline{\pi}_a \Big) \end{aligned} \tag{3.32}$$

3.2.2 Decomposition of bulk multiplets: B-type

We now study how $(2,2)$ multiplets in the bulk decompose as boundary multiplets. We will consider the case of B-type boundary conditions. The boundary supersymmetry parameter ϵ is related to the bulk parameters by

$$\epsilon = \sqrt{2}\epsilon_- = \sqrt{2}\eta\epsilon_+ \quad .$$

In addition, the boundary supercoordinates are related to the bulk ones by

$$\sqrt{2}\theta = \theta^- = \eta\theta^+$$

<u>$(2,2)$ **Vector Multiplets**</u>

The vector multiplet decomposes into the following combinations:

1. $\widetilde{v}_0 = v_0 + \eta\frac{(\sigma+\overline{\sigma})}{\sqrt{2}}$ transforms as a singlet under the unbroken supersymmetry and is the $(0,2)$ vector multiplet in the Wess-Zumino gauge.

2. The bulk twisted chiral superfield $\boldsymbol{\Sigma}$ becomes an unconstrained complex $(0,2)$ superfield with the following expansion

$$\Sigma = \sigma - 2i\eta\theta\overline{\lambda}_+ - 2i\overline{\theta}\lambda_- + 2\sqrt{2}\theta\overline{\theta}\eta(\widetilde{D} - i\widetilde{v}_{01}) \tag{3.33}$$

   where $\widetilde{D} \equiv D + i\eta\partial_1\frac{\sigma-\overline{\sigma}}{\sqrt{2}}$ and $\widetilde{v}_{01} = \partial_0 v_1 - \partial_1\widetilde{v}_0$.

3. It is also useful to note that the combination $\widetilde{v}_1 \equiv v_1 - \eta\frac{(\sigma-\overline{\sigma})}{\sqrt{2}}$, $(\lambda_+ - \eta\lambda_-)$ form a chiral superfield (we will call it V_1) if we choose $\partial_1\widetilde{v}_0 = 0$ on the boundary.

$(2,2)$ Chiral Multiplets

A $(2,2)$ chiral superfield $\boldsymbol{\Phi}$ decomposes into two $(0,2)$ multiplets:

1. A chiral multiplet $\boldsymbol{\Phi'}$ with components (ϕ, τ) with $\tau \equiv (\psi_- - \psi_+)/\sqrt{2}$. The boundary Lagrangian for chiral multiplets Φ'_i given by

$$S_{\rm pert} = \frac{1}{2}\int dx^0 d^2\theta F^{i\overline{j}}\Phi'_i\overline{\Phi}'_j \tag{3.34}$$

   corresponds to turning on a constant field strength $F^{i\overline{j}}$ in the worldvolume of the brane. It has the following component expansion:

$$S_{\rm pert} = \int dx^0 F^{i\overline{j}}\left[\frac{i}{2}(\phi_i\widetilde{D}_0\overline{\phi}_j - \overline{\phi}_j\widetilde{D}_0\phi_i) + \tau_i\overline{\tau}_j\right] \tag{3.35}$$

2. A fermi multiplet $\boldsymbol{\Xi}$ with components $(\xi \equiv \frac{\psi_- + \psi_+}{\sqrt{2}}, -F)$ satisfying

$$\overline{\mathcal{D}}\Xi = -i\sqrt{2}\widetilde{D}_1\Phi' \quad .$$

   where $\widetilde{D}_1\Phi' \equiv (\partial_1 + iQV_1)\Phi'$. On the boundary, we treat $\partial_1\Phi'$ as an independent $(0,2)$ chiral superfield with components $(\partial_1\phi, \partial_1\tau)$.

4 Holomorphic vector bundles from complexes

In anticipation of the fact that in the low-energy limit, we expect our construction to reduce to the construction of vector bundles as the cohomology of complexes such as monads(which are complexes of length two), we now discuss in this section some relevant aspects of the monad construction followed by the a discussion of the cases where the complexes have length greater than two. We also discus in some detail the monads associated with the $\sum_a l_a = 0$ RS states for the case of Calabi-Yau manifolds given by hypersurfaces in weighted projective space.

Monads are a construction originally due to Horrocks and used extensively by Beilinson for constructing holomorphic vector bundles on $\mathbb{P}^n$. (For a readable introduction to this subject, see [25].) The basic idea is to consider the following complex (monad) of holomorphic vector bundles A, B and C

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0 \quad , \tag{4.1}$$

which is exact at A (equivalently, the map a is injective) and C (equivalently, the map b is surjective) such that Im$(a) \subset B$ and $b \cdot a = 0$. The holomorphic vector bundle

$$E = \ker b / \mathrm{Im}\ a$$

is the cohomology of the monad. Some topological properties of E are

$$\mathrm{rk}\ E = \mathrm{rk}\ B - \mathrm{rk}\ A - \mathrm{rk}\ C \tag{4.2}$$
$$\mathrm{ch}(E) = \mathrm{ch}(B) - \mathrm{ch}(A) - \mathrm{ch}(C) \tag{4.3}$$

We will also consider the exact sequences

$$0 \to A \xrightarrow{a} B \to E \to 0 \quad ,$$

(with $\mathrm{rk}\ E = \mathrm{rk}\ B - \mathrm{rk}\ A$ and $\mathrm{ch}(E) = \mathrm{ch}(B) - \mathrm{ch}(A)$) and

$$0 \to E \to B \xrightarrow{b} C \to 0 \quad ,$$

(with $\mathrm{rk}\ E = \mathrm{rk}\ B - \mathrm{rk}\ C$ and $\mathrm{ch}(E) = \mathrm{ch}(B) - \mathrm{ch}(C)$) in order to construct holomorphic vector bundles.

The field theoretic construction of these vector bundles is as follows [24]. Consider fermions π_a ($a = 1, \ldots, \mathrm{rk}\ B$) and κ_i ($i = 1, \ldots, \mathrm{rk}\ A$) which are sections of B and A respectively. The map a (represented by $E^i_a(\phi)$) is realised as the gauge invariance

$$\pi_a \sim \pi_a + E^i_a(\phi)\kappa_i \quad .$$

This gauge invariance is fixed by the gauge choice

$$\overline{E}^i_a(\phi)\pi_a = 0 \tag{4.4}$$

The map b (represented by $J^a_m(\phi)$) imposes the holomorphic constraints

$$J^a_m(\phi)\pi_a = 0 \quad m = 1, \ldots, \mathrm{rk}\ C \tag{4.5}$$

The remaining fermions, i.e., those not set to zero by (4.4) and (4.5), are sections of the holomorphic vector bundle E given by the monad construction.

Given a vector bundle E as the cohomology of a monad with constituents A, B, C as above, one can verify that the vector bundle $E(n) = E \otimes \mathcal{O}(n)$ is given by the cohomology of the following monad

$$0 \to A(n) \xrightarrow{a} B(n) \xrightarrow{b} C(n) \to 0 \quad , \tag{4.6}$$

In the field theoretic construction that we will pursue, the operation of tensoring with $\mathcal{O}(n)$ corresponds to shifting the charges of all the fermions by n units.

One may encounter situations where the map a is not injective on some sub-manifold Σ. Then, one obtains a sheaf rather than a vector bundle whose singularity set is Σ. A simple example which illustrates this is to consider a single fermion which is a section of $\mathcal{O}$. We choose $E = \phi_1$. Then, the condition

$$\overline{\phi}_1 \pi = 0$$

sets $\pi = 0$ on all points except the hyperplane $\phi_1 = 0$. Thus, the fermion π is non-vanishing on the hyperplane and is a section of the sheaf of functions with support on the hyperplane. We will see that this is useful in constructing lower dimensional branes i.e., D-branes wrapping some holomorphic sub-cycle of the Calabi-Yau manifold rather than the whole Calabi-Yau manifold.

4.1 Vector bundles for $\mathbb{P}^n$

For the case of $\mathbb{P}^n$, we would like to construct the bundles corresponding to the $l = 0$ orbit. The homogeneous coordinates on $\mathbb{P}^n$ are given by $(\phi_1, \ldots, \phi_{n+1})$. Introduce $\binom{n+1}{m+1}$ fermions $\pi_{[i_1,\cdots,i_{m+1}]}$ $(i_1, \cdots, i_{m+1} = 1, \ldots, n+1)$ subject to the conditions

$$\overline{E}^{[j_1,\cdots,j_m]}_{[i_1,\cdots,i_{m+1}]} \pi_{[i_1,\cdots,i_{m+1}]} = 0 \quad , \tag{4.7}$$

where

$$E^{[j_1,\cdots,j_m]}_{[i_1,\cdots,i_{m+1}]} = \frac{1}{(m+1)!} \sum_{\text{all perms}} (-)^p \phi_{i_{p(1)}} \delta^{j_1}_{i_{p(2)}} \cdots \delta^{j_m}_{i_{p(m+1)}} \quad .$$

For example, when $m = 1$, $E^k_{ij} = (\phi_i \delta^k_j - \phi_j \delta^k_i)/2$. One can verify that the following is true

$$\phi_{j_1} E^{[j_1,\cdots,j_m]}_{[i_1,\cdots,i_{m+1}]} = 0$$

Thus, the number of independent E's are $\binom{n}{m}$ and hence the number of fermions remaining after we impose the conditions (4.7) is equal to $\binom{n}{m+1}$. The remaining fermions transform as a section of $\mathbb{T}^{m+1}(-m-1)$ – the $(m+1)$-th exterior power of the tangent bundle tensored by $\mathcal{O}(-m-1)$. When $m = 0$, this is seen by considering the Euler sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{O}^{\oplus(n+1)} \to \mathbb{T}_{\mathbb{P}^n}(-1) \to 0 \tag{4.8}$$

The general case is given by the exact sequence (derivable from the Euler sequence)

$$0 \to \mathbb{T}^{p-1}_{\mathbb{P}^n}(-p) \xrightarrow{E_i} \mathcal{O}^{\oplus\binom{n+1}{p}} \to \mathbb{T}^{p}_{\mathbb{P}^n}(-p) \to 0 \qquad (4.9)$$

where $\mathbb{T}^p_{\mathbb{P}^n} \equiv \wedge^p \mathbb{T}\mathbb{P}^n$ is the p-th exterior power of the tangent bundle to $\mathbb{P}^n$. Now, consider the exact sequence which is dual to the above one

$$0 \to \Omega^p_{\mathbb{P}^n}(p) \to \mathcal{O}^{\oplus\binom{n+1}{p}} \to \Omega^{p-1}_{\mathbb{P}^n}(p) \to 0 \qquad (4.10)$$

where $\Omega_{\mathbb{P}^n}$ is the cotangent bundle to $\mathbb{P}^n$. This bundle is constructed by choosing conditions imposed by J's instead of the E's. For example, to obtain the cotangent bundle, we consider $(n+1)$ fermions π^i subject to the (holomorphic) constraint

$$\phi_i \pi^i = 0 \quad .$$

Thus, given a holomorphic vector bundle E as the cohomology of a monad, its dual E^* is given by a monad where the gauge conditions are exchanged with the constraints.

Vector bundle E	$\leftrightarrow$	Its dual E^*
Gauge conditions	$\leftrightarrow$	Holomorphic constraints
Holomorphic constraints	$\leftrightarrow$	Gauge conditions

As we will see, in the field theoretic realisation, this has a very simple relation.

4.2 Long sequences and $\Omega^2(2)$

As we have seen in the previous subsection, the monad construction seems to lead us to sequences of length two (i.e., those involving three bundles). Further, monads involving vector bundles may be represented by longer sequences involving only line bundles. This is best illustrated by considering the case of $\Omega^2(2)$. By combining the monad used for $\Omega^1(1)$, one can obtain the following sequence of length three which leads to $\Omega^2(2)$.

$$0 \to \Omega^2(2) \to \mathcal{O}^{\oplus 10} \to \mathcal{O}^{\oplus 5}(1) \to \mathcal{O}(2) \to 0 \qquad (4.11)$$

As we will see, in the field theoretic construction, it is more natural for the fermions to be sections of direct sums of line bundles and thus, we will need to implement a sequence of length three in order to obtain $\Omega^2(2)$. We will postpone the precise details to the next section.

More generally, from the work of Beilinson [26], all coherent sheaves on $\mathbb{P}^n$ arise from sequences of length less than or equal to n. Thus, it is of interest to be able to deal with sequences which are of length greater than two.

4.3 Vector bundles for weighted projective spaces

Weighted projective spaces are typically singular. We will be interested in situations where the Calabi-Yau is embedded as an hypersurface in some weighted projective space. There are two possible scenarios in this context: (i) the hypersurface does not inherit any of the singularities of the ambient weighted projective space; (ii) the weighted projective space does inherit some of the singularities of the ambient projective space. For case (ii) we assume that one can make the Calabi-Yau hypersurface non-singular by blowing up the singularities. This process converts case (ii) to that of case (i).

In a recent paper [8], we have constructed rigid sheaves on weighted projective spaces by suitably generalising the mutations of helices on these spaces (see also [27]). This construction has the advantage of being adaptable to the GLSM construction of vector bundles pursued in this paper. We will illustrate this for the case of of a degree six hypersurface in $\mathbb{P}^{1,1,1,1,2}$. Let ϕ_i for $i = 1, \ldots, 5$ be the quasi-homogeneous coordinates of the weighted projective space with ϕ_5 having weight two. The $\sum_a l_a = 0$ bundles S_i, $i = 1, \ldots, 6$ are related to each other under the quantum $\mathbb{Z}_6$ symmetry. $S_1 = \mathcal{O}$.

The bundle $-S_2$ (the minus sign reflects the K-theory class) is defined by the left mutation

$$0 \to (-S_2) \to \mathcal{O}^{\oplus 4} \xrightarrow{J} \mathcal{O}(1) \to 0 \tag{4.12}$$

where $J^i = \phi_i$ for $i = 1, 2, 3, 4$. This sequence is similar to the Euler sequence associated with $\mathbb{P}^3$ (with homogeneous coordinates $\phi_1, \ldots, \phi_4$) and is a bundle of rank three. Thus $(-S_2)$ is closely related to the cotangent bundle of the $\mathbb{P}^3$ (i.e., $\Omega(1)$) in the chart $\phi_5 = 0$. The next bundle is given by the exact sequence

$$0 \to S_3 \to \mathcal{O}^{\oplus 5} \to (-S_2) \otimes \mathcal{O}(1) \to 0 \tag{4.13}$$

We will instead consider the sequence (similar to the one associated with $\Omega^2(2)$ for $\mathbb{P}^3$)

$$0 \to S_3' \to \mathcal{O}^{\oplus 6} \to (-S_2) \otimes \mathcal{O}(1) \to 0 \tag{4.14}$$

We consider six fermions $\pi^{[ij]}$ ($i, j = 1, 2, 3, 4$) subject to the holomorphic constraints $J^k_{ij}\pi^{[ij]} = 0$, where $J^k_{ij} = \frac{1}{2}(\phi_i\delta^k_j - \phi_j\delta^k_i)$. S_3' is thus a rank three vector bundle. The bundle of interest S_3 is in the K-theory class $(S_3' - \mathcal{O})$ as can be verified by comparing the Chern classes [8]. The other three bundles are given by Serre duality: $S_i \simeq S^*_{7-i} \otimes \mathcal{O}(-1)$. The restriction of these bundles to the degree six hypersurface gives rise to bundles V_i which can be shown to reproduce the charges associated with the $\sum_a l_a = 0$ D-branes [8].

4.4 Vector bundles for hypersurfaces in $\mathbb{P}^n$

We will be interested in vector bundles for Calabi-Yau manifolds given by hypersurfaces or transverse intersection of hypersurfaces in $\mathbb{P}^n$. For simplicity, we will only consider the case of a hypersurface M given by a homogeneous polynomial $G(\phi)$ of degree Q_p. One simple class of vector bundles are given by the restriction $E|_M$ of vector bundles on $\mathbb{P}^n$ to the hypersurface X. Thus, the $L=0$ bundles are restrictions to bundles on M.

The tangent bundle $\mathbb{T}_M$ is obtained by considering $(n+1)$ fermions subject to the conditions given by $E_i=\phi_i$ and $J^i=(\partial G/\partial\phi_i)$. The tangent bundle is thus given by the monad

$$0\to\mathcal{O}\xrightarrow{\otimes E_i}\mathcal{O}(1)^{\oplus n+1}\xrightarrow{\otimes J^i}\mathcal{O}(Q_p)\to 0$$

5 The monad construction in the GLSM with boundary

In addition to the standard Lagrangian for the bulk GLSM, on the boundary, we introduce fermi multiplets Π_a satisfying[4]

$$\overline{\mathcal{D}}\Pi_a=\sqrt{2}\Sigma' E_a(\Phi') \quad , \tag{5.1}$$

where Φ' is the bulk $(0,2)$ chiral multiplet and Σ' is a boundary chiral multiplet with components (ς,β). We will also introduce a superpotential coupling $P'J^a(\Phi')$, where P' is a boundary chiral multiplet with components (p',γ). The boundary Lagrangian is

$$\begin{aligned} S^1_{\text{bdry}} \;=\; & \int dx^0\,\Big(i\overline{\pi}_a\widetilde{D}_0\pi_a-|p'|^2|J^a(\phi)|^2-|\varsigma|^2|E_a(\phi)|^2 \\ & -E_a\overline{\pi}_a\beta-\overline{E}_a\overline{\beta}\pi_a-a_1\overline{\pi}_a\varsigma\frac{\partial E_a}{\partial\phi_i}\tau_i-a_1\overline{\varsigma}\frac{\partial\overline{E}_a}{\partial\overline{\phi}_i}\overline{\tau}_i\pi_a \\ & -\overline{J}^a\overline{\gamma}\overline{\pi}_a-J^a\pi_a\gamma\;-a_2\pi_a p'\frac{\partial J^a}{\partial\phi_i}\tau_i-a_2\overline{p}'\frac{\partial\overline{J}^a}{\partial\overline{\phi}_i}\overline{\tau}_i\overline{\pi}_a\Big) \end{aligned} \tag{5.2}$$

where a_1 and a_2 are two real constants which are determined by the condition that the boundary terms in the ordinary and supersymmetric variation of the Lagrangian $S_{\text{bulk}}+S_{\text{bdry}}$ vanish. The choice $a_1=a_2=1$ gives the standard $(0,2)$ Lagrangian as discussed in sec. 3.2. Further, the covariant derivative given by $\widetilde{D}_0=\partial_0+iQ\widetilde{v}_0$ involves the combination $\widetilde{v}_0=v_0+\eta\frac{\sigma+\overline{\sigma}}{\sqrt{2}}$ as is appropriate from the boundary decomposition of the bulk vector multiplet as discussed in sec 3.2. In the heterotic $(0,2)$ constructions, the the fermions π_a are chosen to have charge to equal the degree of E_a. We will initially choose the fermions to be charge neutral and as a consequence, the Σ' field has charge $-Q_a$ (where Q_a is the degree of the E_a). However, in what follows, we will always use the covariant derivatives on the boundary fermions since we will eventually discuss the cases when they are charged.

[4] This constraint is interpreted as the gauge fixing of a fermionic gauge invariance. In this paper, we always work in this gauge-fixed formulation. For details, see [23, 24].

We shall choose a first-order kinetic term for the Σ' and P' superfields associated with the gauge-invariance and holomorphic constraints. This is a departure from the standard $(0,2)$ construction in the context of the heterotic string. Our motivation is two-fold:

(i) A first-order action is (almost) unavoidable for the case of complexes of length greater than two. (see sec. 7)

(ii) The large-volume monodromy associated with the bundles is obtained in a simple manner. (see sec. 6)

The action that we choose is

$$S^2_{\rm bdry} = \int dx^0 \left(\frac{i}{2}(\varsigma \widetilde{D}_0\overline{\varsigma} - \overline{\varsigma}\widetilde{D}_0\varsigma) + \beta\overline{\beta} + \frac{i}{2}(p'\widetilde{D}_0\overline{p}' - \overline{p}'\widetilde{D}_0 p') + \gamma\overline{\gamma}\right) \tag{5.3}$$

5.1 $\theta = 0$ and the low-energy limit

We will now consider the low-energy limit of this field theory when the bulk is its geometric phase (which can be obtained by the usual $e^2 r \to \infty$ limit). The coupling $\overline{E}_a\overline{\beta}\pi_a$ is a mass-term for the fermionic combination $\overline{E}_a\pi_a$ and hence one obtains $\overline{E}_a\pi_a = 0$ at energy scales much smaller than this mass scale. Similarly, one obtains the holomorphic constraint $J^a\pi_a = 0$ at low-energies from the mass-term $J^a\pi_a\gamma$. Further, when $\sum_a |E_a|^2$ is non-vanishing (which we assume for now), the ground state condition requires $\varsigma = 0$. Similarly, $p' = 0$ at low energies. These arguments parallel those for the $(0,2)$ constructions for the heterotic string [23].

The corresponding analysis for the bulk fields is standard and we will not repeat them here [24]. We recall that fields in the vector multiplet become Lagrange multipliers enforcing constraints (These constraints have been explicitly presented in ref. [5]). We shall quote some of the relevant ones

$$\frac{\sigma - \overline{\sigma}}{\sqrt{2}} = \frac{1}{2K[\phi]}\sum_i Q_i\left[\overline{\tau}_i\xi_i - \overline{\xi}_i\tau_i\right] \tag{5.4}$$

$$\frac{\sigma + \overline{\sigma}}{\sqrt{2}} = -\frac{1}{2K[\phi]}\sum_i Q_i\left[\overline{\xi}_i\xi_i - \overline{\tau}_i\tau_i\right] \tag{5.5}$$

$$\widetilde{v}_0 = \frac{1}{K[\phi]}\sum_i Q_i\left[\frac{i}{2}(\overline{\phi}_i\partial_0\phi_i - \phi_i\partial_0\overline{\phi}_i) + \overline{\tau}_i\tau_i\right] \tag{5.6}$$

where $K[\phi] = \sum_i Q_i^2|\phi_i|^2$.

We will follow the strategy pursued in [5] in obtaining boundary conditions. We first obtain boundary conditions for the fields in the matter multiplets in the NLSM limit as well as equations of motion for the boundary fermi multiplets such that all boundary terms which appear on the ordinary and supersymmetric variations (involving these fields) of the action vanish. Next, one obtains conditions on the fields in the vector multiplet and the equations of motion for the Σ' and P' fields. Finally, these conditions are lifted to the GLSM by requiring that the boundary terms of order $1/e^2$ also vanish.

The ordinary variation of the action gives rise to the following boundary terms (where we have for the moment ignored the terms arising from the ordinary variations of the fields in the vector multiplet as well as the fields in the Σ' and P' multiplets)

$$
\begin{aligned}
\delta_{\rm ord}(S_{\rm bulk}+S_{\rm bdry}) = \int dx^0 \Bigg[& \frac{i}{2}\left(\tau_i\delta\overline{\xi}_i+\overline{\tau}_i\delta\xi_i\right) \\
-\Bigg(& D_1\overline{\phi}_i+|\varsigma|^2\frac{\partial E_a}{\partial\phi_i}\overline{E}_a - a_1\varsigma\frac{\partial^2 E_a}{\partial\phi_i\partial\phi_j}\tau_j\overline{\pi}_a+\frac{\partial E_a}{\partial\phi_i}\overline{\pi}_a\beta \\
& +|p'|^2\frac{\partial J^a}{\partial\phi_i}\overline{J}^a - a_2 p'\frac{\partial^2 J^a}{\partial\phi_i\partial\phi_j}\tau_j\pi_a+\frac{\partial J^a}{\partial\phi_i}\pi_a\gamma\Bigg)\delta\phi_i \\
-\Bigg(& D_1\phi_i+|\varsigma|^2\frac{\partial\overline{E}_a}{\partial\overline{\phi}_i}E_a - a_1\overline{\varsigma}\frac{\partial^2\overline{E}_a}{\partial\overline{\phi}_i\partial\overline{\phi}_j}\pi_a\overline{\tau}_j+\frac{\partial\overline{E}_a}{\partial\overline{\phi}_i}\overline{\beta}\pi_a \\
& +|p'|^2\frac{\partial\overline{J}^a}{\partial\overline{\phi}_i}J^a - a_2\overline{p}'\frac{\partial^2\overline{J}^a}{\partial\overline{\phi}_i\partial\overline{\phi}_j}\overline{\pi}_a\overline{\tau}_j+\frac{\partial\overline{J}^a}{\partial\overline{\phi}_i}\overline{\gamma}\overline{\pi}_a\Bigg)\delta\overline{\phi}_i \\
& +\left(\frac{i}{2}\overline{\xi}_i - a_1\overline{\pi}_a\varsigma\frac{\partial E_a}{\partial\phi_i} - a_2\pi_a p'\frac{\partial J^a}{\partial\phi_i}\right)\delta\tau_i \\
& +\left(\frac{i}{2}\xi_i + a_1\pi_a\overline{\varsigma}\frac{\partial\overline{E}_a}{\partial\overline{\phi}_i} + a_2\overline{\pi}_a\overline{p}'\frac{\partial\overline{J}^a}{\partial\overline{\phi}_i}\right)\delta\overline{\tau}_i \\
& +\delta\overline{\pi}_a\left(i\widetilde{D}_0\pi_a - E_a\beta+\overline{J}^a\overline{\gamma} - a_1\varsigma\frac{\partial E_a}{\partial\phi_i}\tau_i + a_2\overline{p}'\frac{\partial\overline{J}^a}{\partial\overline{\phi}_i}\overline{\tau}_i\right) \\
& -\delta\pi_a\left(-i\widetilde{D}_0\overline{\pi}_a - \overline{E}_a\overline{\beta}+J^a\gamma + a_1\overline{\varsigma}\frac{\partial\overline{E}_a}{\partial\overline{\phi}_i}\overline{\tau}_i - a_2 p'\frac{\partial J^a}{\partial\phi_i}\tau_i\right)\Bigg]
\end{aligned}
\tag{5.7}
$$

Boundary conditions consistent with supersymmetry and the ordinary variation of the total action are obtained when $a_1 = a_2 = \frac{1}{2}$. We also add an additional boundary contact term

$$
S_1^c = \int dx^0 i\left(\frac{\sigma-\overline{\sigma}}{\sqrt{2}}\right)\left(\sum_i Q_i|\phi_i|^2 - r\right) \tag{5.8}
$$

The coefficient of r is added for the cancellation of the boundary terms in the vector multiplet sector which we consider later. In the matter sector, the boundary conditions that we obtain are

$$
\xi_i - i\overline{\varsigma}\frac{\partial\overline{E}_a}{\partial\overline{\phi}_i}\pi_a - i\overline{p}'\frac{\partial\overline{J}^a}{\partial\overline{\phi}_i}\overline{\pi}_a = 0 \tag{5.9}
$$

$$
\begin{aligned}
&\widetilde{D}_1\phi_i + \frac{\partial}{\partial\overline{\phi}_i}\left(|\varsigma|^2|E_a|^2 + |p'|^2|J^a|^2\right) \\
&\qquad - \overline{\varsigma}\frac{\partial^2\overline{E}_a}{\partial\overline{\phi}_i\partial\overline{\phi}_j}\pi_a\overline{\tau}_j - \overline{p}'\frac{\partial^2\overline{J}^a}{\partial\overline{\phi}_i\partial\overline{\phi}_j}\overline{\pi}_a\overline{\tau}_j + \frac{\partial\overline{E}_a}{\partial\overline{\phi}_i}\overline{\beta}\pi_a + \frac{\partial\overline{J}^a}{\partial\overline{\phi}_i}\overline{\gamma}\overline{\pi}_a = 0 \qquad (5.10)\\
&\qquad \overline{F}_i - i\varsigma p'\frac{\partial}{\partial\phi_i}\left(E_aJ^a\right) = 0 \quad , \qquad (5.11)
\end{aligned}
$$

where $\widetilde{D}_1\phi_i \equiv [\partial_1 + iQ_iv_1 - iQ_i\frac{\sigma-\overline{\sigma}}{\sqrt{2}}]\phi_i$. The last equation can be integrated to obtain the condition

$$
i\varsigma p' E_aJ^a = W \qquad (5.12)
$$

when one has a superpotential. This is different from the condition $E_aJ^a = 0$ seen in the corresponding $(0,2)$ construction of vector bundles for the heterotic string. We deal with this condition by introducing a chargeless spectator boundary fermi multiplet $\widehat{\Pi}$ satisfying

$$
\overline{\mathcal{D}}\widehat{\Pi} = 1
$$

with superpotential given by $\widehat{J} = W$. Note that we *do not* introduce any ς or p' fields with this boundary fermion. This is a solution proposed by Warner in the LG context. Note that the 1 above is actually a dimensionful scale μ which we have set to one. Thus the rest of the boundary fermions will satisfy $E\cdot J = 0$ as in the $(0,2)$ vector bundle constructions for the heterotic string. Since this spectator multiplet occurs for all examples, we will assume that this has been introduced in all examples that we consider in this paper.

The equation of motion for the boundary fermions are

$$
\widetilde{D}_0\pi_a + i\varsigma\frac{\partial E_a}{\partial\phi_i}\tau_i - i\overline{p}'\frac{\partial\overline{J}^a}{\partial\overline{\phi}_i}\overline{\tau}_i + iE_a\beta - i\overline{J}^a\overline{\gamma} = 0 \qquad (5.13)
$$

One can now verify that all the boundary terms vanish in the low-energy limit provided the boundary fermions are uncharged as we assumed earlier.

We now consider the fields in the vector multiplet. On substituting for the ξ_i using the boundary condition as derived earlier, we obtain

$$
\frac{\sigma-\overline{\sigma}}{\sqrt{2}i} = \frac{1}{2K[\phi]}\sum_i Q_i\left[\overline{\varsigma}\frac{\partial\overline{E}_a}{\partial\overline{\phi}_i}\overline{\tau}_i\pi_a - \varsigma\frac{\partial E_a}{\partial\phi_i}\tau_i\overline{\pi}_a + \overline{p}'\frac{\partial\overline{J}^a}{\partial\overline{\phi}_i}\overline{\tau}_i\overline{\pi}_a - p'\frac{\partial J^a}{\partial\phi_i}\tau_i\pi_a\right] \qquad (5.14)
$$

It is easy to see that this leads to the boundary condition

$$
\sigma - \overline{\sigma} = 0 \qquad (5.15)
$$

on using the low-energy condition $\varsigma = p' = 0$. The above equation and its supersymmetric partners will be the boundary conditions on the fields in the vector multiplet in the GLSM.

The equation of motion for the fields ς and β are

$$\beta + \overline{E}_a \pi_a = 0 \tag{5.16}$$

$$i\tilde{D}_0 \varsigma + \varsigma |E_a|^2 + \frac{\partial \overline{E}_a}{\partial \overline{\phi}_i} \overline{\tau}_i \pi_a = 0 \tag{5.17}$$

Similarly, the equations of motion for p' and γ are

$$i\tilde{D}_0 p' + p' |J^a|^2 + \frac{\partial \overline{J}^a}{\partial \overline{\phi}_i} \overline{\tau}_i \overline{\pi}_a = 0 \tag{5.18}$$

$$\gamma + \overline{J}^a \overline{\pi}_a = 0 \tag{5.19}$$

5.1.1 Examples

1. A six-brane wrapping a Calabi-Yau threefold given by a hypersurface $G = 0$ in weighted projective space. This is given by a single boundary fermi multiplet with $J = P'P$. This fermion has support on the space $p = 0$. The restriction to the hypersurface (in the NLSM) is implemented by continuity from the bulk.

2. A four-brane given by a holomorphic equation $f(\phi) = 0$ is described by an additional holomorphic constraint by $J_1 = f(\Phi)$ over and above the six-brane condition discussed above. This J_1 comes with a P_1' superfield. Other lower dimensional branes can be obtained by the transverse intersection of holomorphic conditions. This will introduce as many P'-fields as there are conditions. The condition in the vector multiplet remains unchanged from that of the six-brane[5]. This follows from the fact that π vanishes when $f \neq 0$ and as a consequence, p' vanishes when $f \neq 0$. It follows that the combination $p'f$ vanishes in the NLSM limit.

3. The restriction of $\Omega^1(1)$ to the quintic hypersurface. where we have chosen the J's as in section 4.1.

4. Consider the case of a vector bundle $(-V_2)$ (in then notation of sec. 4.3) on a degree six hypersurface in $\mathbb{P}^{1,1,1,1,2}$. In this case $J_i = \phi_i$ (in the notation of section 4.3).

5.2 $\theta = 0$ in the GLSM

In order to lift the boundary conditions of the previous subsection to the GLSM, we have to deal with the boundary terms given by the ordinary variations of

[5]This is a more satisfactory description of lower-dimensional branes than the one chosen in [5] where one chooses $f(\phi) = 0$ as the boundary condition. In this construction, this occurs as a low-energy condition.

fields in the bulk vector multiplet. We also have to deal with the contributions to the boundary action of $\mathcal{O}(\frac{1}{e^2})$. The boundary terms involving fields in the vector multiplet that arise from the ordinary variations of the bulk as well as S_1^c are

$$\begin{aligned}
\delta_{\text{ord}}(S_{\text{bulk}} + S_1^c) \;=\; & \frac{1}{e^2}\int dx^0 \Bigg[-(v_{01} + \frac{\theta e^2}{2\pi})\,\delta v_0 \\
& - \partial_1\left(\frac{\sigma+\overline{\sigma}}{\sqrt{2}}\right)\delta\left(\frac{\sigma+\overline{\sigma}}{\sqrt{2}}\right) - i\widetilde{D}\delta\left(\frac{\sigma-\overline{\sigma}}{\sqrt{2}}\right) \\
& + \frac{i}{2}\Bigg(\left(\frac{\overline{\lambda}_- - \overline{\lambda}_+}{\sqrt{2}}\right)\delta\left(\frac{\lambda_- + \lambda_+}{\sqrt{2}}\right) + \left(\frac{\overline{\lambda}_+ + \overline{\lambda}_-}{\sqrt{2}}\right)\delta\left(\frac{\lambda_- - \lambda_+}{\sqrt{2}}\right) \\
& + \left(\frac{\lambda_- - \lambda_+}{\sqrt{2}}\right)\delta\left(\frac{\overline{\lambda}_+ + \overline{\lambda}_-}{\sqrt{2}}\right) + \left(\frac{\lambda_+ + \lambda_-}{\sqrt{2}}\right)\delta\left(\frac{\overline{\lambda}_- - \overline{\lambda}_+}{\sqrt{2}}\right)\Bigg)\Bigg]
\end{aligned}$$

There is no contribution to the above from S^1_{bdry} since we have chosen π_a to be chargeless. However, there is a contribution to $\delta\tilde{v}_0$ coming from S^2_{bdry} i.e., the kinetic part of the boundary action involving the Σ' and P' superfields.

First, we choose

$$\sigma - \overline{\sigma} = 0 \tag{5.20}$$

$$\lambda_+ - \eta\overline{\lambda}_- = 0 \tag{5.21}$$

$$v_{01} - \eta\partial_1\left(\frac{\sigma+\overline{\sigma}}{\sqrt{2}}\right) = 0 \tag{5.22}$$

With the above choice of boundary conditions, most of boundary terms in the ordinary variations of $S_{\text{bulk}} + S_1^c$ vanish except for the terms involving $\delta\tilde{v}_0$. The coefficient of this term is

$$-\frac{v_{01}}{e^2} + Q_\varsigma|\varsigma|^2 + Q_{p'}|p'|^2 \quad .$$

Thus, the final boundary condition is

$$\frac{v_{01}}{e^2} = Q_\varsigma|\varsigma|^2 + Q_{p'}|p'|^2 \quad . \tag{5.23}$$

These boundary conditions when combined with the boundary conditions on the matter fields and equations of motion for the boundary fields (as obtained earlier in the low-energy limit) lead to a complete cancellation of all boundary terms which occur in both the ordinary as well as supersymmetric variation of the action. The case of multiple P' fields easily generalises and we shall not discuss them any more.

It is useful to comment here that the equations of motion for all the boundary fields (π_a, p' and σ) can all be obtained from superfield actions. This reflects the fact that all boundary terms which appear in the supersymmetric variation of the complete bulk and boundary actions vanish. For instance, for the π field we had chosen $a_1 = a_2 = 1/2$ when the superfield expansion gives $a_1 = a_2 = 1$ in the action. The equation of motion for π however is as if we had chosen the latter.

5.3 An intriguing observation

So far, we have implicitly restricted our attention to the one-particle sector of the boundary fermion Fock space as we indicated in the introduction. This restriction was essential to obtain the vector bundle of interest rather than all its antisymmetric powers. One may wonder whether sectors of other fermion number have any meaning. In this regard, consider the case of the cotangent bundle to $\mathbb{P}^4$ that we discussed earlier. The boundary state associated with it can be obtained as

$$|B\rangle \sim \int [D\pi]\ e^{iS_{\text{bdry}}}\ \mathcal{P}_1\ |B\rangle_0 \otimes |0\rangle_\pi \tag{5.24}$$

where we have schematically indicated the realisation of the boundary state associated with the cotangent bundle as the path-integral over all the boundary fields (symbolically indicated by $[D\pi]$) subject to the restriction to one-particle states (shown by the projector $\mathcal{P}_1$). By $|B\rangle_0$, we mean the state associated with Neumann boundary conditions on all matter fields.

If instead, in the boundary state, we change the restriction to fermion number p, the index computation suggests that on obtains the boundary state associated to the vector bundle $\Omega^p(p)$. These states form the large-volume analogue of the Recknagel-Schomerus states in the $\sum_a l_a = 0$. Pushing this further, to more complicated examples involving weighted projective spaces such as $\mathbb{P}^{1,1,1,1,2}$, we expect to recover the $\sum_a l_a = 0$ orbit in such cases provided one suitably modifies the projection on states with definite particle number.

However, under large-volume monodromy as implemented in this paper, it is not clear that all $\Omega^p(p)$ will behave in an appropriate fashion. Hence, we prefer to realise $\Omega^p(p)$ only via the one-particle projection involving a GLSM realisation of complexes of suitable length.

6 Charged fermions, $\theta \neq 0$ and large volume monodromy

As has been shown in a simple example in [5, 20], the inclusion of the θ term requires the addition of contact terms (derivable in the NLSM). This modified the boundary conditions in a manner which was compatible with supersymmetry. For the case of vector bundles on Calabi-Yau manifolds, we will need an an additional condition: Under $\theta \to \theta + 2\pi$, the monodromy of the B-branes around the singularity at large volume should be implemented correctly. It turns out that the monodromy corresponds to tensoring the bundle with a line bundle ($\mathcal{O}(-1)$ for the quintic). This process does not affect the stability and the moduli space of the bundle at large volume. As we shall see, this has a simple and elegant realisation in the NLSM limit.

In the monad construction, the large-volume monodromy action corresponds to shifting the charges of all boundary fermions by one unit. Thus, we can anticipate that dealing with the case of charged boundary fermions should be closely related to turning on the theta term. When the fermions are charged, the the boundary terms in the ordinary variation do not vanish in the NLSM limit. This is due to the term of the form $Q_\pi \overline{\pi}\pi\delta\widetilde{v}_0$. This is very similar to the boundary term which appears when one turns on the theta term i.e., $\theta\delta v_0/2\pi$. Hence, we expect a contact term involving bilinears of the boundary fermions playing the role of θ. Such contact terms, among other things, modify the equations of motion of the boundary fermions.

6.1 $\theta \neq 0$ and the low-energy limit

From large-volume monodromy considerations, when $\theta = -2\pi n$, we expect the fermion to have charge $Q_\pi = n$ and the vector bundle E becomes $E(n) = E \otimes \mathcal{O}(n)$. This corresponds to turning on a $U(1)$ gauge field on the worldvolume of the brane. The coupling of the boundary fermions to the gauge field takes the form (similar terms appear in the NLSM considerations in the context of the heterotic string [28])

$$S_{\text{gauge field}} = -\int dx^0 \left\{ \frac{in}{2r}\overline{\pi}_a\pi_a \sum_i (\phi_i\partial_0\overline{\phi}_i - \overline{\phi}_i\partial_0\phi_i) \right\} \tag{6.1}$$

One can, in fact, verify that this gauge field shifts the first Chern class of the bundle in an appropriate fashion (see [20] for the case when E is a line bundle). The boundary fermions also couple to the bulk fermions τ_i and $\overline{\tau}_i$.

Let us add the above term to the boundary action. The modification to boundary condition on the bulk scalar fields is clear from above. One obtains the following boundary condition from the cancellation of boundary terms proportional to $\delta\overline{\phi}$ in the ordinary variation

$$\widetilde{D}_1\phi_i + \frac{in}{r}\overline{\pi}_a\pi_a\partial_0\phi_i + \frac{in}{2r}\partial_0(\overline{\pi}_a\pi_a)\phi_i + \cdots = 0 \tag{6.2}$$

where the ellipsis denotes terms which are theta independent. Such a term can arise from the supersymmetric variation of the following boundary condition on the matter fermions of the form

$$\xi_i - \frac{in}{2r}\overline{\pi}_a\pi_a\tau_i + \cdots = 0 \tag{6.3}$$

whose supersymmetric variation leads to

$$\widetilde{D}_1\phi_i + \frac{in}{r}\overline{\pi}_a\pi_a\widetilde{D}_0\phi_i - \frac{in}{2r}\delta_{susy}(\overline{\pi}_a\pi_a)\tau_i + \cdots = 0 \tag{6.4}$$

where by $\delta_{susy}(\overline{\pi}_a\pi_a)$, we mean the term proportional to $\overline{\epsilon}$ in the supersymmetric variation of $\overline{\pi}_a\pi_a$. In order for the two equations (6.2) and (6.4) to match[6], we need

$$\delta_{susy}(\overline{\pi}_a\pi_a) = 0 \quad \text{and} \quad \partial_0(\overline{\pi}_a\pi_a) = 0$$

It turns out that both expectations are true in the low-energy/NLSM limit on using the equations of motion of π_a which turn out to be of the form

$$i\partial_0\pi_a - \frac{in}{r}\sum_i(\phi_i\partial_0\overline{\phi}_i - \overline{\phi}_i\partial_0\phi_i) + \frac{n}{r}\sum_i\overline{\tau}_i\tau_i + \cdots = 0$$

The above equation clearly reflects the fact that we have indeed turned on a gauge field on the worldvolume of the brane. In the NLSM limit, this is in fact equivalent to the following equation

$$i\widetilde{D}_0\pi_a + \cdots = 0$$

where we have introduced a covariant derivative corresponding to a fermion of charge n. This shows that *gauge invariance* as well as *supersymmetric invariance* necessarily forces the change of boundary fermion charge and matches the monad construction for the vector bundle $E(n)$.

Keeping in mind that we are interested in the GLSM, we rewrite the boundary term corresponding to turning on a gauge field in the NLSM as

$$S_{\text{gauge field}} = \int dx^0 \left\{ -\frac{in}{2r}\overline{\pi}_a\pi_a \sum_i(\phi_i\widetilde{D}_0\overline{\phi}_i - \overline{\phi}_i\widetilde{D}_0\phi_i)\right\} \tag{6.5}$$

with an additional condition that the charge of the boundary fermion is shifted by n by appropriate covariant derivatives in the boundary action.

The boundary term which we add in the NLSM for non-zero theta takes the form

$$S^{NLSM}_{\text{boundary}} = \int dx^0 \left\{ i\frac{\Theta}{2\pi r}\sum_i(\phi_i\widetilde{D}_0\overline{\phi}_i - \overline{\phi}_i\widetilde{D}_0\phi_i)\right\} \tag{6.6}$$

where

$$\frac{\Theta}{2\pi r} \equiv \left[\frac{\theta_f}{2\pi r} + \frac{[\theta/2\pi]}{2r}\overline{\pi}_a\pi_a\right] \quad .$$

Here $[\theta/2\pi]$ is the integer part of $\theta/2\pi$ and $\theta_f/2\pi$ is the fractional part of $\theta/2\pi$.

The boundary conditions (for non-zero θ) is modified to

$$\xi_i - i\frac{\Theta}{2\pi r}\tau_i - i\overline{\varsigma}\frac{\partial\overline{E}_a}{\partial\overline{\phi}_i}\pi_a - i\overline{p}'\frac{\partial\overline{J}^a}{\partial\overline{\phi}_i}\overline{\pi}_a = 0 \tag{6.7}$$

[6] In the NLSM limit, terms which lead to the $\widetilde{v}_0$ term in (6.4) vanish in the ordinary variations on using $\delta(\sum_i\overline{\phi}_i\phi_i = 0)$.

$$\widetilde{D}_1\phi_i + i\frac{\Theta}{2\pi r}\widetilde{D}_0\phi_i + \frac{\partial}{\partial\overline{\phi}_i}\left(|\varsigma|^2|E_a|^2 + |p'|^2|J^a|^2\right)$$
$$- \overline{\varsigma}\frac{\partial^2\overline{E}_a}{\partial\overline{\phi}_i\partial\overline{\phi}_j}\pi_a\overline{\tau}_j - \overline{p}'\frac{\partial^2\overline{J}^a}{\partial\overline{\phi}_i\partial\overline{\phi}_j}\overline{\pi}_a\overline{\tau}_j + \frac{\partial\overline{E}_a}{\partial\overline{\phi}_i}\overline{\beta}\pi_a + \frac{\partial\overline{J}^a}{\partial\overline{\phi}_i}\overline{\gamma}\overline{\pi}_a = 0 \quad (6.8)$$
$$\overline{F}_i - i\varsigma p'\frac{\partial}{\partial\phi_i}\left(E_a J^a\right) = 0 \quad , \quad (6.9)$$

The equation of motion for the boundary fermions get modified to

$$\widehat{D}_0\pi_a + i\varsigma\frac{\partial E_a}{\partial\phi_i}\tau_i - i\overline{p}'\frac{\partial\overline{J}^a}{\partial\overline{\phi}_i}\overline{\tau}_i + iE_a\beta - i\overline{J}^a\overline{\gamma} = 0 \quad (6.10)$$

where $\widehat{D}_0\pi_a = (\partial_0 + i[\frac{\theta}{2\pi}]\widetilde{v}_0)\pi_a$ reflects the shift in the charge of π_a from 0 to $[\frac{\theta}{2\pi}]$. Further, from eqn. (5.6) we can see that $\widetilde{v}_0$ is the pullback of a spacetime $U(1)$ gauge field of constant field strength to the worldvolume of the D-brane.

One can now verify, that for the above choice of boundary conditions and eqn. of motion for the boundary fermion, all boundary terms in the ordinary as well as supersymmetric variation vanish (in the NLSM limit). We will now obtain boundary conditions on the fields in the bulk vector multiplet. The boundary conditions on the the fields σ and $\overline{\sigma}$ are

$$\frac{\sigma - e^{-2i\gamma}\overline{\sigma}}{\sqrt{2}i} = 0 \quad (6.11)$$

where $\tan\gamma = -\frac{\Theta}{2\pi r}$.

6.2 $\theta \neq 0$ in the GLSM

In the NLSM limit, the consistency of the boundary conditions with both the ordinary and supersymmetric variation required $\delta_{susy}(\overline{\pi}_a\pi_a) = 0$ and $\partial_0(\overline{\pi}_a\pi_a) = 0$ which was true in the NLSM on using the equations of motion for π. We will show that something similar occurs in the GLSM. We will require a $\mathcal{J}$ which has similar properties i.e.,

$$\delta_{susy}(\mathcal{J}) = 0 \quad ; \quad \partial_0(\mathcal{J}) = 0 \quad .$$

It is easy to see using the equations of motion (of the boundary fields) that $\mathcal{J}$ is given by[7]

$$\mathcal{J} \equiv \left(\overline{\pi}_a\pi_a - |\varsigma|^2 + |p'|^2\right) \quad (6.12)$$

This reflects the fact that we expect that the only change in the boundary equations of motion is through the change in the charge of the fields. Such changes obviously do not effect $\partial_0\mathcal{J} = 0$ since it has zero charge.

[7]We note here that our choice of a first-order action for ς and p' fields leads to a linear coupling to $\widetilde{v}_0$ for all boundary fields. This plays an important role in ensuring a simple and closed form of $\mathcal{J}$ with the required properties.

We define

$$\frac{\Theta}{2\pi r} \equiv \left[\frac{\theta_f}{2\pi r} + \frac{[\theta/2\pi]}{2r}\mathcal{J}\right] \quad .$$

and choose the theta dependent boundary (contact) term to be

$$S^{GLSM}_{\text{boundary}} = \int dx^0 \left\{ i\frac{\Theta}{2\pi r}\sum_i (\phi_i \widetilde{D}_0\overline{\phi}_i - \overline{\phi}_i\widetilde{D}_0\phi_i)\right\} \tag{6.13}$$

in addition to the *appropriate shift in charge* for the boundary fields π, p' and ς.

It is straightforward to obtain the equations of motion for the boundary fields. For example, the equation of motion for π_a is

$$\widehat{D}_0\pi_a + i\varsigma\frac{\partial E_a}{\partial\phi_i}\tau_i - i\overline{p}'\frac{\partial\overline{J}^a}{\partial\overline{\phi}_i}\overline{\tau}_i + iE_a\beta - i\overline{J}^a\overline{\gamma} + \mathcal{O}(1/e^2) = 0 \tag{6.14}$$

where

$$\widehat{D}_0\pi_a = \partial_0\pi_a + i[\theta/2\pi]\left(\widetilde{v}_0 + \frac{i}{2}\sum_i Q_i(\phi_i\widetilde{D}_0\overline{\phi}_i - \overline{\phi}_i\widetilde{D}_0\phi_i) - \overline{\tau}_i\tau_i\right)\pi_a \quad .$$

There are three new contributions which arise in comparison with the $\theta = 0$ case. The $\widetilde{v}_0$ is the piece arising from the change in charge, the $\overline{\tau}_i\tau_i$ piece comes from from the ordinary variations of $\delta\xi$ and $\delta\overline{\xi}$. The third piece comes from the boundary term that we add. It is easy to see that in the NLSM limit, on using the D-term constraint, that the three pieces collapse precisely to the pull-back of a constant gauge field as in the line-bundle case [20]. The $\mathcal{O}(1/e^2)$ indicates potential contributions from the vector multiplet sector which we have not included.

The boundary conditions on the bulk fields in the vector multiplet now follow straightforwardly from the calculations of [5], with the sole modification that the θ of the earlier paper is replaced appropriately by the Θ that we have defined above. We write down the relevant equations below:

$$(\bar{\sigma} - e^{2i\gamma}\sigma)|_{x^1=0} = 0 \tag{6.15}$$

$$(\lambda_+ - \eta e^{2i\gamma}\lambda_-)|_{x^1=0} = 0 \tag{6.16}$$

$$\left(v_{01} + \frac{\Theta}{2\pi r}D\right)\Big|_{x^1=0} = Q_\varsigma|\varsigma|^2 + Q_{p'}|p'|^2 + Q_\pi\overline{\pi}_a\pi_a \tag{6.17}$$

$$\left(\partial_1\frac{\overline{\sigma} + e^{2i\gamma}\sigma}{\sqrt{2}} - \eta\frac{\Theta}{2\pi r}e^{i\gamma}D\right)\Big|_{x^1=0} = Q_\varsigma|\varsigma|^2 + Q_{p'}|p'| + Q_\pi\overline{\pi}_a\pi_a \tag{6.18}$$

where $Q_\pi = [\theta/2\pi]$, $Q_\varsigma = Q_\pi - \deg(E)$ and $Q_{p'} = -Q_\pi - \deg(J)$.

7 Vector bundles from complexes of length > 2

In the field theoretic construction, it seems most natural to consider complexes whose elements are direct sums of line bundles rather than vector bundles. This is however not a very restrictive condition. For example, all coherent sheaves on $\mathbb{P}^n$ (and those on weighted projective spaces) can be obtained as the cohomology of such complexes. This is a result due to Beilinson.

Before getting to the most general situation, it is useful to study an example which naturally illustrates how one deals with longer complexes. The simplest one is that of $\Omega^2(2)$ which we shall discuss now.

7.1 $\Omega^2(2)$ and $\mathbb{T}^2(-2)$

As was discussed earlier, the following exact sequence gives rise to $\Omega^2(2)$.

$$0 \to \Omega^2(2) \to \mathcal{O}^{\oplus 10} \xrightarrow{J^k_{[ij]}} \Omega^1(2) \to 0 \tag{7.1}$$

Following the discussion in the earliers sections, we consider ten fermi multiplets $\Pi^{[ij]}$ with the superpotential

$$S_J = -\frac{1}{\sqrt{2}} \int dx^0 d\theta \left(\Pi^{[ij]} J^k_{[ij]}(\Phi') P'_k\right)|_{\overline{\theta}=0} - \text{h.c.} \tag{7.2}$$

where $J^k_{[ij]}(\phi) = (\phi_i \delta^k_j - \phi_j \delta^k_i)$ and P'_k are five boundary chiral multiplets.

It is easy to see that the above superpotential admits the gauge invariance (with bosonic gauge parameter b)

$$p'_k \sim p'_k + b\phi_k \tag{7.3}$$

which is implied by the identity: $\phi_k J^k_{[ij]}(\phi) = 0$. Thus, even though the superpotential gives mass to four fermions in the fermi multiplet, the gauge invariance indicates that one linear combination of the P'_k remain massless. One has to fix this gauge invariance which can be done by imposing the following constraint of the superfield P'_k

$$\overline{\mathcal{D}} P'_k = \sqrt{2} N \Phi'_k \tag{7.4}$$

where N is a chiral fermi superfield(with lowest component n). It is easy to see that P'_k is now a section of $\mathbb{T}^1(-2) = (\Omega^1(2))^*$ given by tensoring the Euler sequence with $\mathcal{O}(-1)$:

$$0 \to \mathcal{O}(-2) \to \mathcal{O}^{\oplus 5}(-1) \to \mathbb{T}^1(-2) \to 0 \tag{7.5}$$

Thus, in the GLSM construction, one is implementing the following exact sequence

$$0 \to \Omega^2(2) \to \mathcal{O}^{\oplus 10} \to \mathcal{O}^{\oplus 5}(1) \to \mathcal{O}(2) \to 0 \tag{7.6}$$

One can now verify that P'_k being a section of $\mathbb{T}^1(-2)$ is consistent with the superpotential S_J being a scalar. This also explains how a holomorphic constraint (in the sequence given above) appears as a gauge invariance in the GLSM construction.

An important point to note that we need to choose a *first order action* for the bosonic field p'_k. This is essential for obtaining the right number of massless fermions in the NLSM limit. Thus, the kinetic energy that we choose for the superfield P'_k is

$$S_{P'_k} = \int dx^0 d^2\theta \overline{P}'_k P'_k \tag{7.7}$$

which leads to the mass term $\gamma_k \overline{n} \overline{\phi}_k$ for the remaining massless fermion in P'_k. If we had chosen a standard second order action for the bosons, one can verify that the expected mass term does not appear. Thus, one is forced to choose the first order action. We also include a standard action for the N superfield.

The p'_k fields thus behave like "ghost fields" that appear whenever one has nested gauge-invariances. Their action is like that of a fermion but their statistics are bosonic. The determinant associated with the partition function of one such bosonic ghost is cancelled by that a fermion. It is useful to "count" the boundary chiral superfields that we have introduced: there are eleven fermi superfields and five bosonic chiral superfields which leads effectively to six (massless) fermi superfields (this is equal to the dimension of $\Omega^2(2)$).

A related example is the construction for $\mathbb{T}^2(-2)$. This is obtained by considering ten fermi superfields $\Pi_{[ij]}$ subject to the constraint

$$\overline{\mathcal{D}}\Pi_{[ij]} = \sqrt{2}\Sigma_k E^k_{[ij]}(\Phi') \quad .$$

where Σ_k are five superfields subject to the constraint

$$\overline{\mathcal{D}}\Sigma_k = \sqrt{2} N E_k(\Phi') \quad .$$

Here we have introduced a chiral fermi superfield N. The consistency condition between the two constraints is

$$\sum_k E^k_{[ij]}(\Phi')\ E_k(\Phi') = 0$$

which is statement that the composition of two consecutive maps in the following complex vanish.

$$0 \to \mathcal{O}(-2) \xrightarrow{E_k} \mathcal{O}(-1)^{\oplus 5} \xrightarrow{E^k_{[ij]}} \mathcal{O}^{\oplus 10} \to \mathbb{T}^2(-2) \to 0 \tag{7.8}$$

7.2 The general case

In the general situation, the complex may have longer length and also the cohomology may occur in more than one place. First, the generalisation to complexes of arbitrary length is fairly straightforward. One begins by introducing fermi multiplets at the point where the cohomology occurs. At other points one introduces bose or fermi multiplets depending on the position in the complex. The charges of the fields are fixed by the line bundles which occur at the each point. Finally, one has to fix whether a given map is implemented through a superpotential (holomorphic constraint) or through the gauge fixing of a gauge invariance.

The situation, where the cohomology occurs at two places in the complex (separated by even number of terms) also goes through in a similar fashion. The massless fermions that appear will arise from fields introduced at these two places instead of one as in the examples that we considered.

It clearly of interest to extend this construction to the case of vector bundles in the heterotic string. The major difference is the change in dimension – here we were dealing with a quantum mechanical situation while in the heterotic string we have a $1+1$ dimensional case. Thus, naively our first order actions do not work in this case. But naive arguments do not constitute a no-go theorem and the issue remains open.

8 Bound states

8.1 D4-brane

Let us recall the the construction for a (complex) codimension one brane i.e., the D4-brane for the quintic. Let us consider the case when the D4-brane is given by $\phi_1 = 0$. In the construction of this paper, one considers the following sequence

$$0 \to \mathcal{O} \xrightarrow{\phi_1} \mathcal{O}(1) \to \mathcal{O}_H \to 0 \tag{8.1}$$

where $\mathcal{O}_H$ is the sheaf with support on the hyperplane $\phi_1 = 0$. Since $\mathrm{ch}(\mathcal{O}_H) = \mathrm{ch}\mathcal{O}(1) - \mathrm{ch}\mathcal{O}$, using the fact that Chern character preserves K-theory classes, one can reinterpret this construction to be the *bound state* of the D-brane corresponding to $\mathcal{O}(1)$ and the anti-brane corresponding to $\mathcal{O}$. This observation (to our knowledge) first appeared in [12]. For more recent work, see [14, 19, 29] and references therein In our construction, the boundary Lagrangian for this is given by (excluding kinetic energy pieces for ς and β)

$$S_{\mathrm{bdry}} = \int dx^0 \left(i\overline{\pi}\widetilde{D}_0\pi - |\varsigma|^2|\phi_1|^2 - \phi_1\overline{\pi}_a\beta - \overline{\phi}_1\overline{\beta}\pi - a_1\overline{\pi}\varsigma\tau_1 - a_1\overline{\varsigma}\overline{\tau}_1\pi \right) \tag{8.2}$$

The terms that appear in this Lagrangian are reminiscent of the Atiyah-Bott-Shapiro construction as discussed in [14, 19, 29]. Following the discussion in [2], we can see that a general hyperplane condition of the form $\sum_i a_i\phi_i = 0$ has four parameters. These can be considered to be four possible deformations of the $\phi_1 = 0$ condition for the D4-brane.

More generally, suppose a brane associated with a vector bundle E_3 is expected to arise as the bound state of a brane (associated with vector bundle E_1) and an anti-brane (of the brane associated with bundle E_2). (We assume that all three vector bundles have rank greater than zero.) First, conservation of RR charge implies that $\mathrm{ch}E_3 = \mathrm{ch}E_1 - \mathrm{ch}E_2$. This can occur in one of the following two sequences

$$\begin{array}{c} 0 \to E_3 \to E_1 \to E_2 \to 0 \\ 0 \to E_2 \to E_1 \to E_3 \to 0 \end{array}$$

This ambiguity is resolved by using the condition that the bundle E_1 is semi-stable [12].

8.2 $\sum_a l_a = 1$ as a bound state

We have so far considered the large-volume analogues of $\sum_a l_a = 0$ Recknagel-Schomerus states [2]. We will now consider the $\sum_a l_a = 1$ states. At the Gepner point, these states are bound states of two $\sum_a l_a = 0$ RS states. Let us consider the vector bundle $\mathcal{B}$ given by the complex

$$0 \to \mathcal{B} \to \Omega^1(1) \to \mathcal{O} \to 0 \tag{8.3}$$

The D-brane corresponding to $\mathcal{B}$ has four moduli. Our strategy will be to first construct the bound state in $\mathbb{P}^4$ and then restrict to the Calabi-Yau manifold. We further assume that no new moduli appear on restriction (this can be actually proven).

In the GLSM, we constructed $\Omega^1(1)$ by considering five fermi multiplets Π^i subject to the degree one holomorphic constraint $\phi_i \pi^i = 0$. In order to obtain the bound state $\mathcal{B}$, we must impose an additional degree zero constraint:

$$a_i \pi^i = 0 \quad , \tag{8.4}$$

where a_i are five constants. Given the overall scaling of the above relation this leads to four independent parameters which we identify with the moduli. We will show that $\mathcal{B}$ is a sheaf on $\mathbb{P}^4$. Consider the case when $a_1 \neq 0$. We fix the scaling by setting $a_1 = 1$. In the chart $\phi_1 = 1$, it is easy to see that at the point $(1, a_2, a_3, a_4, a_5)$, the two conditions collapse to one. However, this point does not lie on the (generic) quintic hypersurface and hence on restriction to the quintic hypersurface, we obtain a vector bundle[8]. The extension of this to other cases involves longer sequences and can be worked out using the methods discussed in this paper.

[8]We thank M. Douglas for a useful discussion on this point

9 Summary

We briefly recapitulate here the main results of the paper. First we have given an explicit GLSM description of B-type D-brane configurations that mathematically correspond to the monad construction of vecctor bundles. Secondly, we have shown how the correct large-volume monodromy action on these configurations requires the modification of the contact term at the boundary. Third, we extend the techniques of (0,2) type constructions in the GLSM to the case where we have complexes of length greater than two. Thus we have a complete description of all D-brane configurations that are the large-volume analogues of the $\sum_a l_a = 0$ Recknagel-Schomerus states. Finally we show how D-branes configurations that are bound states can be naturally described in the GLSM and we show how the appropriate moduli can be described.

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