The Euler-Lagrange Cohomology and General Volume-Preserving Systems

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Abstract

We briefly introduce the conception on Euler-Lagrange cohomology groups on a symplectic manifold $(\mathcal{M}^{2n}, \omega)$ and systematically present the general form of volume-preserving equations on the manifold from the cohomological point of view. It is shown that for every volume-preserving flow generated by these equations there is an important 2-form that plays the analog role with the Hamiltonian in the Hamilton mechanics. In addition, the ordinary canonical equations with Hamiltonian H are included as a special case with the 2-form $\frac{1}{n-1} H\,\omega$. It is studied the other volume preserving systems on $(\mathcal{M}^{2n}, \omega)$. It is also explored the relations between our approach and Feng-Shang's volume-preserving systems as well as the Nambu mechanics.

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1 Introduction

As is well known, in the Hamiltonian mechanics (see, for example, [1], [2]), the canonical equations with Hamiltonian H read

$$\dot{q}^{\,i} = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}. \tag{1}$$

In fact, a Hamiltonian system defined on a symplectic manifold $(\mathcal{M}^{2n}, \omega)$ as its phase space with a symplectic 2-form ω can always be described locally by eqs. (1). The solutions of eqs. (1) are the integral curves of the Hamiltonian vector field

$$\mathbf{X}_H := \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial H}{\partial q^i}\frac{\partial}{\partial p_i} \tag{2}$$

on $\mathcal{M}^{2n}$. It is well known that the above Hamiltonian system preserves both the symplectic form ω and the volume form

$$\tau := \frac{1}{n!}\,\omega^n, \tag{3}$$

where ω^n is the n-fold wedge product of ω.

In the symplectic geometry [7], for a vector field $\mathbf{X}$ on $\mathcal{M}^{2n}$, there is always a 1-form $E_{\mathbf{X}} := -i_{\mathbf{X}}\omega$, where $i_{\mathbf{X}}$ denotes the contraction. The vector is symplectic, if E is closed. On the other hand, for a Hamiltonian vector field $\mathbf{X}_H$ on $\mathcal{M}^{2n}$, the 1-form $E_{\mathbf{X}_H}$ is exact and the ordinary canonical equations (1) can be expressed as

$$-i_{\mathbf{X}_H}\omega = \mathrm{d}H. \tag{4}$$

Thus, a cohomology may be introduced :

$$H_{\mathrm{EL}} := \{E_{\mathbf{X}} | \mathrm{d}E_{\mathbf{X}} = 0\}/\{E_{\mathbf{X}} | E_{\mathbf{X}} = \mathrm{d}\alpha\}. \tag{5}$$

In [3, 4], in order to study its time discrete version in discrete mechanics including the symplectic algorithm, this cohomology has been introduced and is called the Euler-Lagrange cohomology. It is simple but significant to see that by definition the canonical equations (1) are in the image of the cohomology. In addition, it is also straightforward to show that the Euler-Lagrange cohomology is isomorphic to the de Rahm cohomology on $\mathcal{M}^{2n}$ [5, 6] and that both ω and τ in (3) are preserved not only along the phase flow defined by the eqs (1) but also along the symplectic flow. Namely,

$$\mathcal{L}_{\mathbf{X}_H}\omega = 0, \qquad\qquad \mathcal{L}_{\mathbf{X}_H}\tau = 0, \tag{6}$$

$$\mathcal{L}_{\mathbf{X}_S}\omega = 0, \qquad\qquad \mathcal{L}_{\mathbf{X}_S}\tau = 0. \tag{7}$$

However, it should be emphasized that there exist certain important mechanical systems that preserve the volume of the phase space only and cannot be transformed into the ordinary Hamiltonian systems. For example, let us consider the following kind of linear systems

$$\ddot{q}^{\,i} = -k_{ij}\,q^j \tag{8}$$

on $\mathbb{R}^n$ with constant coefficients k_{ij}. Obviously, eqs. (8) can be turned into the form:

$$\dot{q}^{\,i} = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i} - a_{ij}\, q^j \tag{9}$$

with

$$H = \frac{1}{2}\,\delta^{ij}\, p_i p_j + \frac{1}{2}\, s_{ij}\, q^i q^j, \tag{10}$$

$$s_{ij} = \frac{1}{2}\,(k_{ij} + k_{ji}), \qquad a_{ij} = \frac{1}{2}\,(k_{ij} - k_{ji}). \tag{11}$$

When one of a_{ij} is nonzero, the system (9) is not an ordinary Hamiltonian system on $\mathbb{R}^{2n}$. In fact, the corresponding vector field

$$\mathbf{X}_a = \mathbf{X}_H - a_{ij}\, q^j\, \frac{\partial}{\partial p_i}$$

of (9) is not even a symplectic vector field since the Lie derivative

$$\mathcal{L}_{\mathbf{X_a}}\omega = a_{ij}\, \mathrm{d}q^i \wedge \mathrm{d}q^j \neq 0.$$

However, it is easy to verify that the system (9) always preserves the volume form of the phase space, i.e. $\mathcal{L}_{\mathbf{X_a}}(\tau) = 0$.

It is worth while to mention that even some conservative system can be transformed as such kind of non-Hamiltonian linear systems. Consider a system consisting of two 1-dimensional linearly coupled oscillators with different mass:

$$m_1\, \ddot{q}^{\,1} = -k\,(q^2 - q^1), \qquad m_2\, \ddot{q}^{\,2} = -k\,(q^1 - q^2).$$

Obviously, such a system does satisfy Newton's laws. Let $k_{11} = -k/m_1$, $k_{12} = k/m_1$, $k_{21} = k/m_2$ and $k_{22} = -k/m_2$. Then it is a system as described by eqs. (8), with $i, j = 1, 2$. When $m_1 \neq m_2$, we have such a system that $k_{12} \neq k_{21}$.

The volume-preserving systems are very important. This can also be seen as follows. If a system S on $(\mathcal{M}^{2n}, \omega)$ is not volume-preserving, it can be extended as a volume-preserving system S' on $(\mathcal{M}^{2n} \times \mathbb{R}^2, \omega')$ in such a way that the orbits of S are precisely the projection of the orbits of S' onto $\mathcal{M}^{2n}$. As a demonstration, let q^i and p_i $(i = 1, \cdots, n)$ be the Darboux coordinates on $\mathcal{M}^{2n}$ and q^0, p_0 the ones on $\mathbb{R}^2$. Then take $\omega' = \mathrm{d}p_\mu \wedge \mathrm{d}q^\mu = \omega + \mathrm{d}p_0 \wedge \mathrm{d}q^0$, $\mu = 0, 1, \cdots, n$, as the symplectic structure on $\mathcal{M}^{2n} \times \mathbb{R}^2$. Note that ω in the expression should be written as $\pi^*\omega$ in which $\pi : \mathcal{M}^{2n} \times \mathbb{R}^2 \longrightarrow \mathcal{M}^{2n}$ is the projection. But, as a demonstration, we do not need to give a rigorous description, although it can be made.

Suppose that for the system S on $\mathcal{M}^{2n}$, $\mathcal{L}_{\mathbf{X}}(\omega^n) = D\, \omega^n$ where $D = D(q^i, p_i)$ is a function and

$$\mathbf{X} = Q^i(q^1, \ldots, q^n, p_1, \ldots, p_n)\, \frac{\partial}{\partial q^i} + P_i(q^1, \ldots, q^n, p_1, \ldots, p_n)\, \frac{\partial}{\partial p_i}$$

is the vector field of S. Then a system S' on $\mathcal{M} \times \mathbb{R}^2$ corresponding to

$$\mathbf{X}' = Q^i\, \frac{\partial}{\partial q^i} + P_i\, \frac{\partial}{\partial p_i} - D(q^1, \ldots, q^n, p_1, \ldots, p_n) q^0\, \frac{\partial}{\partial q^0}$$

can be constructed. It can be easily checked that S' is a volume-preserving system. And the orbits of S are just the projections of the orbits of S'. If all the properties of system S' are known, so the properties of S.

In this paper, we generalize the Euler-Lagrange cohomology to higher forms on a symplectic manifold $(\mathcal{M}, \omega)$ and present a general form of equations that generate a volume-preserving flow from the cohomological point of view. It is shown that for every volume-preserving flow there is a 2-form that plays an analog role with the Hamiltonian in the Hamilton mechanics. The ordinary Hamiltonian equations with Hamiltonian H are included as a special case with a 2-form $\frac{1}{n-1} H\,\omega$.

The main results on the Euler-Lagrange cohomology will be quoted and outlined without detailed proof. Some of the general content and results can be found in [5] and more details as well as extended topics will be given in the forthcoming paper [6]. In what follows, we first generalize the Euler-Lagrange cohomology to higher order ones in section 2. Then we present the general volume preserving equations from cohomological point of view in section 3. We also show that the canonical equations in the Hamilton mechanics and the kind of linear systems just mentioned are special cases of the general volume preserving equations. In section 4, we compare our general volume-preserving equations with other volume-preserving systems presented before. Especially, we explore the relations with Feng-Shang's volume-preserving systems in their volume-preserving algorithm[9] and with Nambu's mechanics[10]. Finally, we end with some conclusion and remarks.

2 The Euler-Lagrange Cohomology on Symplectic Manifolds

In order to present the general equations for volume-preserving systems from the cohomological point of view, we need first to generalize the Euler-Lagrange cohomology from $H_{\rm EL}$ of 1-forms to higher forms. In this way a sequence of Euler-Lagrange cohomology groups $H_{\rm EL}^{2k-1}(\mathcal{M}, \omega)$ can be introduced. Then we consider the trivial class of the highest one with $k = n$. The general volume-preserving equations will be picked up from it. This is, in fact, a direct generalization for the ordinary Hamilton system, where the canonical equations can be represented by the 1-forms in the trivial class of $H_{\rm EL}$ as defined in eq. (5).

Briefly speaking, a volume-preserving system characterized by a volume-preserving vector field, corresponds to a unique closed $(2n-1)$-form on $\mathcal{M}^{2n}$ and *vise versa*. Such a system that corresponds to an exact $(2n-1)$-form can be determined then by a 2-form up to at least a closed 2-form.

In this section, the concept of Euler-Lagrange cohomology groups $H_{\rm EL}^{2k-1}(\mathcal{M}, \omega)$ for $k = 1, \dots, n$ will be briefly introduced. Some of their important properties will be quoted. But the details on the Euler-Lagrange cohomology groups will be explained in the forthcoming paper [6].

For $k = 1, \ldots, n$, we may define the sets

$$\begin{aligned}
\mathcal{X}_{\mathrm{S}}^{2k-1}(\mathcal{M}^{2n}, \omega) &:= \{ \mathbf{X} \in \mathcal{X}(\mathcal{M}^{2n}) \,|\, \mathcal{L}_{\mathbf{X}}(\omega^k) = 0 \}, \\
\mathcal{X}_{\mathrm{H}}^{2k-1}(\mathcal{M}^{2n}, \omega) &:= \{ \mathbf{X} \in \mathcal{X}(\mathcal{M}^{2n}) \,|\, -i_{\mathbf{X}}(\omega^k) \text{ is exact} \}.
\end{aligned}$$

For simplicity, hereafter $\mathcal{X}(\mathcal{M})$ denotes the space of smooth vector fields on $\mathcal{M}$ of dimension $2n$. The vector fields in $\mathcal{X}_{\mathrm{S}}^1(\mathcal{M}, \omega)$ and $\mathcal{X}_{\mathrm{H}}^1(\mathcal{X}, \omega)$ are known as symplectic vector fields and Hamiltonian vector fields, respectively. Those in $\mathcal{X}_{\mathrm{S}}^{2n-1}(\mathcal{M}, \omega)$ are just volume-preserving vector fields. It is easy to prove that

$$\begin{aligned}
\mathcal{X}_{\mathrm{S}}^1(\mathcal{M}, \omega) \subseteq \ldots \subseteq \mathcal{X}_{\mathrm{S}}^{2k-1}(\mathcal{M}, \omega) \subseteq \mathcal{X}_{\mathrm{S}}^{2k+1}(\mathcal{M}, \omega) \subseteq \ldots \subseteq \mathcal{X}_{\mathrm{S}}^{2n-1}(\mathcal{M}, \omega), \\
\mathcal{X}_{\mathrm{H}}^1(\mathcal{M}, \omega) \subseteq \ldots \subseteq \mathcal{X}_{\mathrm{H}}^{2k-1}(\mathcal{M}, \omega) \subseteq \mathcal{X}_{\mathrm{H}}^{2k+1}(\mathcal{M}, \omega) \subseteq \ldots \subseteq \mathcal{X}_{\mathrm{H}}^{2n-1}(\mathcal{M}, \omega)
\end{aligned}$$

and

$$\mathcal{X}_{\mathrm{H}}^{2k-1}(\mathcal{M}, \omega) \subseteq \mathcal{X}_{\mathrm{S}}^{2k-1}(\mathcal{M}, \omega).$$

In fact, $\mathcal{X}_{\mathrm{S}}^{2k-1}(\mathcal{M}, \omega)$ is a Lie algebra and $\mathcal{X}_{\mathrm{H}}^{2k-1}(\mathcal{M}, \omega)$ is an ideal of $\mathcal{X}_{\mathrm{S}}^{2k-1}(\mathcal{M}, \omega)$ because

$$[\mathcal{X}_{\mathrm{S}}^{2k-1}(\mathcal{M}, \omega), \mathcal{X}_{\mathrm{S}}^{2k-1}(\mathcal{M}, \omega)] \subseteq \mathcal{X}_{\mathrm{H}}^{2k-1}(\mathcal{M}, \omega).$$

The quotient Lie algebra

$$H_{\mathrm{EL}}^{2k-1}(\mathcal{M}, \omega) := \mathcal{X}_{\mathrm{S}}^{2k-1}(\mathcal{M}, \omega) / \mathcal{X}_{\mathrm{H}}^{2k-1}(\mathcal{M}, \omega) \tag{12}$$

is called the $2k-1$-st *Euler-Lagrange cohomology group*[5],[6],[12]. It is called a group because its Lie algebra structure is trivial: For each $k = 1, \ldots, n$, $H_{\mathrm{EL}}^{2k-1}(\mathcal{M}, \omega)$ is an Abelian Lie algebra. For $n = 1$, $H_{\mathrm{EL}}^1(\mathcal{M}, \omega)$ is the Euler-Lagrange cohomology. And it can be proved that $H_{\mathrm{EL}}^1(\mathcal{M}, \omega)$ is linearly isomorphic to the first de Rham cohomology group $H_{\mathrm{dR}}^1(\mathcal{M})$ as was mentioned in the introduction.

For $k = n$, $\mathcal{X}_{\mathrm{S}}^{2n-1}(\mathcal{M}, \omega)$ is the Lie algebra of volume-preserving vector fields, including all the other Lie algebras mentioned above as its Lie subalgebras. Similarly to the case of $k = 1$, the $2n-1$-st Euler-Lagrange cohomology group $H_{\mathrm{EL}}^{2n-1}(\mathcal{M}, \omega)$ is isomorphic to the $(2n-1)$-st de Rham cohomology group $H_{\mathrm{dR}}^{2n-1}(\mathcal{M})$. To prove this, let us first introduce a lemma:

Lemma 1 *For each integer $k = 1, \ldots, n$ $(n \geqslant 1)$ and $x \in \mathcal{M}$, the linear map $\nu_k : T_x\mathcal{M} \longrightarrow \Lambda_{2k-1}(T_x^*\mathcal{M})$ defined by $\nu_k(\mathbf{X}) = -i_{\mathbf{X}}(\omega^k)$ is injective. That is, $i_{\mathbf{X}}(\omega^k) = 0$ iff $\mathbf{X} = 0$.*

The Euler-Lagrange cohomology groups can also be defined as

$$H_{\mathrm{EL}}^{2k-1}(\mathcal{M}, \omega) = \{\theta \in \Omega_{\mathrm{EL}}^{2k-1}(\mathcal{M}, \omega) \,|\, \mathrm{d}\theta = 0\} / \{\theta \in \Omega_{\mathrm{EL}}^{2k-1}(\mathcal{M}, \omega) \,|\, \theta \text{ is exact}\},$$

where $\Omega_{\mathrm{EL}}^{2k-1}(\mathcal{M}, \omega) := \{-i_{\mathbf{X}}(\omega^k) \,|\, \mathbf{X} \in \mathcal{X}(\mathcal{M})\}$. Using the above lemma, it is easy to see that this definition is equivalent to (12).

Since $\dim T_x\mathcal{M} = \dim \Lambda_{2n-1}(T_x^*\mathcal{M})$, lemma 1 implies that $\nu_n : T_x\mathcal{M} \longrightarrow \Lambda_{2n-1}(T_x^*\mathcal{M})$ is a linear isomorphism, which induces a linear isomorphism $\nu_n : \mathcal{X}(\mathcal{M}) \longrightarrow \Omega^{2n-1}(\mathcal{M})$ defined by

$$\nu_n(\mathbf{X}) = -i_{\mathbf{X}}(\omega^n) = -n\,(i_{\mathbf{X}}\omega) \wedge \omega^{n-1}. \tag{13}$$

Hence we obtain

Theorem 1 *The linear map $\nu_n : \mathcal{X}(\mathcal{M}) \longrightarrow \Omega^{2n-1}(\mathcal{M})$ is an isomorphism. Under this isomorphism, $\mathcal{X}_{\mathrm{S}}^{2n-1}(\mathcal{M},\omega)$ and $\mathcal{X}_{\mathrm{H}}^{2n-1}(\mathcal{M},\omega)$ are isomorphic to $Z^{2n-1}(\mathcal{M})$ and $B^{2n-1}(\mathcal{M})$, respectively.*

In the above theorem, $Z^{2n-1}(\mathcal{M})$ is the space of closed $(2n-1)$-forms on $\mathcal{M}$ and $B^{2n-1}(\mathcal{M})$ is the space of exact $(2n-1)$-forms on $\mathcal{M}$. As a consequence, we obtain the following corollary:

Corollary 1.1 *The $2n-1$-st Euler-Lagrange cohomology group $H_{\mathrm{EL}}^{2n-1}(\mathcal{M},\omega)$ is linearly isomorphic to $H_{\mathrm{dR}}^{2n-1}(\mathcal{M})$, the $(2n-1)$-st de Rham cohomology group.*

When $\mathcal{M}$ is closed, $H_{\mathrm{EL}}^{2n-1}(\mathcal{M},\omega)$ is linearly isomorphic to the dual space of $H_{\mathrm{EL}}^{1}(\mathcal{M},\omega)$, because $H_{\mathrm{dR}}^{2k-1}(\mathcal{M}) \cong (H_{\mathrm{dR}}^{2(n-k)+1}(\mathcal{M}))^*$ for such a manifold (see, for example, [8]). If $\mathcal{M}$ is not compact, this relation cannot be assured.

When $n > 2$, there is another lemma:

Lemma 2 *Let $n > 2$. Then, for an arbitrary integer $k = 1, \ldots, n-2$ and a point $x \in \mathcal{M}$, $\alpha \in \Lambda_2(T_x^*\mathcal{M})$ satisfies $\alpha \wedge \omega^k = 0$ iff $\alpha = 0$.*

Again this induces injective linear maps

$$\begin{array}{rcl} \iota_k : \Omega^2(\mathcal{M}) & \longrightarrow & \Omega^{2k+2}(\mathcal{M}) \\ \alpha & \longmapsto & \alpha \wedge \omega^k \end{array}$$

for $k = 1, \ldots, n-2$. Thus we can obtain that

$$\mathcal{X}_{\mathrm{S}}^{1}(\mathcal{M},\omega) = \mathcal{X}_{\mathrm{S}}^{3}(\mathcal{M},\omega) = \ldots = \mathcal{X}_{\mathrm{S}}^{2n-3}(\mathcal{M},\omega) \subseteq \mathcal{X}_{\mathrm{S}}^{2n-1}(\mathcal{M},\omega).$$

When $k = n-2$, it can be easily verified from lemma 2 that

$$\begin{array}{rcl} \iota = \iota_{n-2} : \Omega^2(\mathcal{M}) & \longrightarrow & \Omega^{2n-2}(\mathcal{M}) \\ \alpha & \longmapsto & \alpha \wedge \omega^{n-2} \end{array} \tag{14}$$

is a linear isomorphism. If $n = 2$, we use the convention that $\omega^0 = 1$. Namely, $\iota = \mathrm{id} : \alpha \longmapsto \alpha$. Then we can define a linear map ϕ making the following diagram commutative:

$$\begin{array}{ccc} \Omega^2(\mathcal{M}) & \xrightarrow{\iota} & \Omega^{2n-2}(\mathcal{M}) \\ \phi \big\downarrow & & \big\downarrow \mathrm{d} \\ \mathcal{X}_{\mathrm{H}}^{2n-1}(\mathcal{M},\omega) & \xrightarrow{\nu_n} & B^{2n-1}(\mathcal{M}). \end{array} \tag{15}$$

In general, it can be shown that there exist certain symplectic manifolds on which the $2k-$ 1-st Euler-Lagrange cohomologe group $H_{\rm EL}^{2k-1}(\mathcal{M},\omega),\ k\neq 1,n$, is not isomorphic to either de Rahm cohomology group $H_{\rm dR}^{2k-1}(\mathcal{M})$ or the harmonic cohomology group $H_{\rm har}^{2k-1}(\mathcal{M},\omega)$ on the manifold [6]. Therefore, the Euler-Lagrange cohomology groups for $k\neq 1,n$ are new cohomological property of the symplectic manifold in general.

3 The General Form of Volume Preserving Equations

Now we concentrate on the general form of volume preserving equations from the cohomological point of view. We will first present such kind of equations in the first subsection. Then we illustrate some concrete volume-preserving systems. Especially, we show that the ordinary canonical equations (1) are included as a special case. Finally, we explore some of the properties of important 2-forms for these volume-preserving systems.

3.1 The general volume preserving equations

It is important to see that as shown in the diagram (15), for a given 2-form

$$\alpha=\frac{1}{2}\,Q_{ij}\,\mathrm{d}q^i\wedge\mathrm{d}q^j+A^i_j\,\mathrm{d}p_i\wedge\mathrm{d}q^j+\frac{1}{2}\,P^{ij}\,\mathrm{d}p_i\wedge\mathrm{d}p_j \tag{16}$$

where Q_{ij} and P^{ij} are antisymmetric for each pair of i and j, the vector field

$$\phi(\alpha)=(\nu_n^{-1}\circ\mathrm{d}\circ\iota)(\alpha)=\nu_n^{-1}(\mathrm{d}\alpha\wedge\omega^{n-2})$$

is in $\mathcal{X}_{\rm H}^{2n-1}(\mathcal{M},\omega)$.

For convenience, we set

$$\mathbf{X}=n(n-1)\,\phi(\alpha)=n(n-1)\,\nu_n^{-1}(\mathrm{d}\alpha\wedge\omega^{n-2}),$$

i.e., $i_{\mathbf{X}}(\omega^n)=-n(n-1)\,(\mathrm{d}\alpha)\wedge\omega^{n-2}$. It is easy to obtain that

$$\mathbf{X}=\Big(\frac{\partial P^{ij}}{\partial q^j}+\frac{\partial A^j_j}{\partial p_i}-\frac{\partial A^i_j}{\partial p_j}\Big)\frac{\partial}{\partial q^i}+\Big(\frac{\partial Q_{ij}}{\partial p_j}-\frac{\partial A^j_j}{\partial q^i}+\frac{\partial A^j_i}{\partial q^j}\Big)\frac{\partial}{\partial p_i}. \tag{17}$$

In fact,

$$\begin{aligned}
\mathrm{d}\iota(\alpha)\ &=\ \mathrm{d}\alpha\wedge\omega^{n-2}\\
&=\ \frac{1}{2}\frac{\partial Q_{ij}}{\partial q^k}\,\mathrm{d}q^i\wedge\mathrm{d}q^j\wedge\mathrm{d}q^k\wedge\omega^{n-2}+\frac{1}{2}\frac{P^{ij}}{\partial p_k}\,\mathrm{d}p_i\wedge\mathrm{d}p_j\wedge\mathrm{d}p_k\wedge\omega^{n-2}\\
&\quad+\Big(\frac{1}{2}\frac{\partial Q_{jk}}{\partial p_i}+\frac{\partial A^i_j}{\partial q^k}\Big)\,\mathrm{d}p_i\wedge\mathrm{d}q^j\wedge\mathrm{d}q^k\wedge\omega^{n-2}\\
&\quad+\Big(\frac{1}{2}\frac{\partial P^{ij}}{\partial q^k}-\frac{\partial A^i_k}{\partial p_j}\Big)\,\mathrm{d}p_i\wedge\mathrm{d}p_j\wedge\mathrm{d}q^k\wedge\omega^{n-2}\\
&=\ \Big(\frac{1}{2}\frac{\partial Q_{jk}}{\partial p_i}+\frac{\partial A^i_j}{\partial q^k}\Big)\,\mathrm{d}p_i\wedge\mathrm{d}q^j\wedge\mathrm{d}q^k\wedge\omega^{n-2}\\
&\quad+\Big(\frac{1}{2}\frac{\partial P^{ij}}{\partial q^k}-\frac{\partial A^i_k}{\partial p_j}\Big)\,\mathrm{d}p_i\wedge\mathrm{d}p_j\wedge\mathrm{d}q^k\wedge\omega^{n-2}.
\end{aligned}$$

By virtue of the following two equations

$$\mathrm{d}p_i \wedge \mathrm{d}p_j \wedge \mathrm{d}q^k \wedge \omega^{n-2} = \frac{\delta_j^k}{n-1}\,\mathrm{d}p_i \wedge \omega^{n-1} - \frac{\delta_i^k}{n-1}\,\mathrm{d}p_j \wedge \omega^{n-1}, \tag{18}$$

$$\mathrm{d}p_i \wedge \mathrm{d}q^j \wedge \mathrm{d}q^k \wedge \omega^{n-2} = \frac{\delta_i^j}{n-1}\,\mathrm{d}q^k \wedge \omega^{n-1} - \frac{\delta_i^k}{n-1}\,\mathrm{d}q^j \wedge \omega^{n-1}, \tag{19}$$

we can write $\mathrm{d}\iota(\alpha)$ as

$$\begin{aligned}\mathrm{d}\iota(\alpha) &= \frac{1}{n-1}\Big(\frac{\partial A_j^j}{\partial q^i} - \frac{\partial A_i^j}{\partial q^j} - \frac{\partial Q_{ij}}{\partial p_j}\Big)\,\mathrm{d}q^i \wedge \omega^{n-1} \\ &\quad + \frac{1}{n-1}\Big(\frac{\partial P^{ij}}{\partial q^j} + \frac{\partial A_j^j}{\partial p_i} - \frac{\partial A_j^i}{\partial p_j}\Big)\,\mathrm{d}p_i \wedge \omega^{n-1} \\ &= \frac{1}{n(n-1)}\Big(\frac{\partial A_j^j}{\partial q^i} - \frac{\partial A_i^j}{\partial q^j} - \frac{\partial Q_{ij}}{\partial p_j}\Big)\, i_{\frac{\partial}{\partial p_i}}\omega^n \\ &\quad - \frac{1}{n(n-1)}\Big(\frac{\partial P^{ij}}{\partial q^j} + \frac{\partial A_j^j}{\partial p_i} - \frac{\partial A_j^i}{\partial p_j}\Big)\, i_{\frac{\partial}{\partial q^i}}\omega^n.\end{aligned}$$

Comparing it with

$$\mathrm{d}\iota(\alpha) = \frac{1}{n(n-1)}\,\nu_n(\mathbf{X}) = -\frac{1}{n(n-1)}\, i_{\mathbf{X}}(\omega^n), \tag{20}$$

we can obtain the expression of $\mathbf{X}$, as shown in eq. (17).

Since both ι and ν_n are linear isomorphisms, we can see from the commutative diagram (15) that for each 2-form α on $\mathcal{M}$ as in eq. (16) the vector field $\mathbf{X}$ in eq. (17) belongs to $\mathcal{X}_{\mathrm{H}}^{2n-1}(\mathcal{M},\omega)$ and for each $\mathbf{X} \in \mathcal{X}_{\mathrm{H}}^{2n-1}(\mathcal{M},\omega)$ there exists the 2-form α on $\mathcal{M}$ satisfying eqs. (17). But, there may exist several 2-forms that are mapped to the same vector field $\mathbf{X}$. For example, the vector field $\mathbf{X}$ in (17) is invariant under the transformation

$$\alpha \longmapsto \alpha + \theta \tag{21}$$

where θ is a closed 2-form.

Note that for $\mathbf{X} \in \mathcal{X}_{\mathrm{H}}^{2n-1}(\mathcal{M},\omega)$, the 2-form α is a globally defined differential form on $\mathcal{M}$. If $\mathbf{X} \in \mathcal{X}_{\mathrm{S}}^{2n-1}(\mathcal{M},\omega)$, such a 2-form cannot be found if $H_{\mathrm{dR}}^{2n-1}(\mathcal{M})$ is nontrivial. In this case, α can still be found as a locally defined 2-form, according to the Poincaré lemma. Then the relation between $\mathbf{X}$ and the locally defined 2-form α, eq. (17), is valid only on a certain open subset of $\mathcal{M}$; And on the intersection of two such open subsets, the corresponding 2-forms are not identical. That is, the transformation relation of Q_{ij}, A_j^i as well as P^{ij} is not that of tensors on $\mathcal{M}$. Instead, a transformation like (21) or more complicated should be applied.

No matter whether $\mathbf{X}$ belongs to $\mathcal{X}_{\mathrm{H}}^{2n-1}(\mathcal{M},\omega)$ or $\mathcal{X}_{\mathrm{S}}^{2n-1}(\mathcal{M},\omega)$, i.e., whether the 2-form α is globally or locally defined, the flow of $\mathbf{X}$ can be always obtained provided that the general solution of the following equations can be solved:

$$\dot{q}^i = \frac{\partial P^{ij}}{\partial q^j} + \frac{\partial A_j^j}{\partial p_i} - \frac{\partial A_j^i}{\partial p_j},$$

$$\dot{p}_i \;=\; \frac{\partial Q_{ij}}{\partial p_j} - \frac{\partial A^j_j}{\partial q^i} + \frac{\partial A^j_i}{\partial q^j}. \tag{22}$$

This is just the general form of the equations of a volume-preserving mechanical system on a symplectic manifold $(\mathcal{M}, \omega)$.

3.2 Some concrete volume-preserving systems.

Let us show some concrete volume-preserving systems.

First, since the canonical equations (1) in the Hamiltonian mechanics is volume preserving, it should be a special case of (22). In fact, this is just the case.

For the Hamiltonian H of the canonical system on $\mathcal{M}$, take the 2-form as

$$\alpha = \frac{1}{n-1}\, H\, \omega, \tag{23}$$

then eqs. (22) just turn out to be the canonical equations (1) as it should be.

Secondly, let us recall the kind of linear systems mentioned in §1 on $(\mathbb{R}^{2n}, \omega)$ with $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$.

Taking $Q_{ij} = -a_{ij}\, p_k q^k$, $A^i_j = \frac{1}{n-1}\, H\, \delta^i_j$ and $P^{ij} = 0$ with the constants a_{ij} and the function H as shown in eqs. (10) and (11), namely,

$$\alpha = \frac{1}{n-1}\, H\, \omega - \frac{1}{2}\, p_k q^k\, a_{ij}\, \mathrm{d}q^i \wedge \mathrm{d}q^j, \tag{24}$$

then the eqs (22) turn out to be eqs. (9). Therefore the 2-form α in eq. (24) is one of general forms corresponding to such kind of linear systems. But, it is not the unique.

It is interesting that when $H = 0$ in the above equation, all the coordinates q^i are the first integrals of the linear system. Then all the canonical momenta $p_i = a_{ij}\, q^j t + p_{i0}$ where p_{i0} are constants. This can generalize to an arbitrary symplectic manifold for the equations (22), even though the system is no longer a linear system.

Thirdly, let us now consider some kind of volume preserving systems on a generic symplectic manifold $(\mathcal{M}, \omega)$.

If the 2-form α satisfies on a Darboux coordinate neighborhood $(U; q, p)$ in $\mathcal{M}$

$$i_{\frac{\partial}{\partial p_i}} \alpha = A^i_j\, \mathrm{d}q^j + P^{ij}\, \mathrm{d}p_j = 0 \tag{25}$$

for each $i = 1, \ldots, n$, then $A^i_j = 0$ and $P^{ij} = 0$. Hence, on that coordinate neighborhood U, all the q^i are constant.

3.3 The trace of 2-forms

If we define a function $\mathrm{tr}\,\alpha$ as

$$\alpha \wedge \omega^{n-1} = \frac{\mathrm{tr}\,\alpha}{n}\, \omega^n \tag{26}$$

for each 2-form α, then using the formula

$$\mathrm{d}p_i \wedge \mathrm{d}q^j \wedge \omega^{n-1} = \frac{\delta_i^j}{n}\,\omega^n,$$

we obtain that

$$\operatorname{tr}\alpha = A_i^i. \tag{27}$$

The above expression is obviously independent of the choice of the Darboux coordinates.

Let $\mathbf{X}_{\operatorname{tr}\alpha}$ be the Hamiltonian vector field corresponding to the function $\operatorname{tr}\alpha$. Eq. (17) indicates that

$$\mathbf{X} = \mathbf{X}_{\operatorname{tr}\alpha} + \mathbf{X}' \tag{28}$$

where $\mathbf{X}'$ is the extra part on the right hand side of eq. (17), corresponding to the 2-form

$$\alpha - \frac{\operatorname{tr}\alpha}{n-1}\,\omega. \tag{29}$$

If $f(q,p)$ is a function on $\mathcal{M}$, then the derivative $\dot{f} = \frac{\mathrm{d}}{\mathrm{d}t} f(q(t),p(t))$ along any one of the integral curves of eqs. (22) satisfies the equation

$$\dot{f}\,\omega^n = n(n-1)\,\mathrm{d}\alpha \wedge \mathrm{d}f \wedge \omega^{n-2}. \tag{30}$$

In fact, $\dot{f}(t) = (\mathcal{L}_{\mathbf{X}} f)(q(t),p(t))$. And, since $\mathbf{X}$ is volume-preserving,

$$(\mathcal{L}_{\mathbf{X}} f)\,\omega^n = \mathcal{L}_{\mathbf{X}}(f\,\omega^n) = \mathrm{d}i_{\mathbf{X}}(f\,\omega^n) + i_{\mathbf{X}}\mathrm{d}(f\,\omega^n) = \mathrm{d}(f\,i_{\mathbf{X}}\omega^n).$$

Then according to eq. (20),

$$\begin{aligned}(\mathcal{L}_{\mathbf{X}} f)\,\omega^n &= -n(n-1)\,\mathrm{d}(f\,\mathrm{d}\iota(\alpha)) = -n(n-1)\,\mathrm{d}f \wedge \mathrm{d}(\alpha \wedge \omega^{n-2}) \\ &= -n(n-1)\,\mathrm{d}f \wedge \mathrm{d}\alpha \wedge \omega^{n-2} = n(n-1)\,\mathrm{d}\alpha \wedge \mathrm{d}f \wedge \omega^{n-2}.\end{aligned}$$

Thus eq. (30) has been proved.

Especially, when $\alpha = \frac{H}{n-1}\,\omega$, the system (22) turns out to be the usual Hamiltonian system, as we have mentioned. For such a Hamiltonian system, on the one hand, we can use eq. (30) to obtain

$$\dot{f}\,\omega^n = n\,\mathrm{d}(H\,\omega) \wedge \mathrm{d}f \wedge \omega^{n-2} = n\,\mathrm{d}H \wedge \mathrm{d}f \wedge \omega^{n-1} = \operatorname{tr}(\mathrm{d}H \wedge \mathrm{d}f)\,\omega^n,$$

namely,

$$\dot{f} = \operatorname{tr}(\mathrm{d}H \wedge \mathrm{d}f). \tag{31}$$

On the other hand, $\dot{f}$ can be expressed in terms of the Poisson bracket:

$$\dot{f} = \{f, H\} := \mathbf{X}_H f = \frac{\partial f}{\partial q^i}\frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial H}{\partial q^i}.$$

So we obtain the relation between the Poisson bracket and the trace of 2-forms:

$$\{f, H\} = -\operatorname{tr}(\mathrm{d}f \wedge \mathrm{d}H). \tag{32}$$

4 The Relations with Other Volume Preserving Systems

Let us now consider the relations between our volume-preserving equations and other relevant topics such as Feng-Shang's volume preserving systems and the Nambu mechanics.

4.1 Feng-Shang's volume preserving systems

In order to develop the volume-preserving algorithm, Feng and Shang [9] presented the following lemma.

Theorem 2 (Feng-Shang's lemma) *The volume-preserving vector field $X = (X^1, \dots, X^n)^T$ on $\mathbb{R}^n$ can always be expressed by an antisymmetric tensor a^{ij} on $\mathbb{R}^n$ as*

$$X^i = \frac{\partial a^{ij}}{\partial x^j}, \tag{33}$$

where x^i are the standard coordinates on $\mathbb{R}^n$.

Note that here we have adopted the notations so as to accommodate ours in this paper. In fact, this lemma can be understood as follows: A vector field on $\mathbb{R}^n$, known as a Euclidean space or a Riemannian manifold, is volume-preserving if and only if it is divergence-free. Using the Hodge theory, the divergence of the vector field X reads $\mathrm{div}X = -\delta\tilde{X} = * \, \mathrm{d} * \tilde{X}$, where δ is the codifferential operator and $\tilde{X} = \delta_{ij} \, X^i \, \mathrm{d}x^j$ is the 1-form (covariant vector field) obtained from X by lowering indices with the metric δ_{ij} for $\mathbb{R}^n$. When X is volume-preserving, $\mathrm{div}X = * \, \mathrm{d} * \tilde{X} = 0$. This implies that $*\tilde{X}$ is a closed $(n-1)$-form. Then, according to the Poincaré lemma, there is a 2-form α, say, such that $*\tilde{X} = \mathrm{d} * \alpha$. If α is assumed to be $\frac{1}{2} \, a_{ij} \, \mathrm{d}x^i \wedge \mathrm{d}x^j$ with $a_{ij} = -a_{ji} = \delta_{ik}\delta_{jl} \, a^{kl} = a^{ij}$, we can obtain that $\tilde{X} = (-1)^{n-1} * \mathrm{d} * \alpha = \delta_{ij} \, \frac{\partial a^{jk}}{\partial x^k} \, \mathrm{d}x^i$. Then the vector field X satisfies eq. (33).

According to the precise sense of volume, Feng-Shang's lemma can be grouped into the same class with the approach to the volume-preserving systems proposed by Nambu [10]. This lemma has presented the most general form of volume-preserving systems in this class.

It is worth seeing that the formula (33) is quite similar to eq. (17), only with some slight differences: (1) This formula is a statement on the Euclidean space $\mathbb{R}^n$ with an arbitrary dimension n while eq. (17) is for a symplectic manifold, of $2n$-dimensional. (2) It can be generalized to an n-dimensional Riemannian or pseudo-Riemannian manifold provided that the $(n-1)$-st de Rham cohomology group is trivial, with the ordinary derivatives being replaced by the covariant derivatives. As for eq. (17), when $H^{2n-1}_{\mathrm{dR}}(\mathcal{M}) \neq 0$, not every volume-preserving vector field satisfies it. This is similar to eq. (33). But a covariant derivative is not necessary in eq. (17). (3) When the symplectic manifold $\mathcal{M} = \mathbb{R}^{2n}$, we can set

$$(a_{ij})_{2n\times 2n} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} Q & -A^T \\ A & P \end{pmatrix} \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} = \begin{pmatrix} P & -A \\ A^T & Q \end{pmatrix}$$

where $Q = (Q_{ij})_{n\times n}$, $P = (P^{ij})_{n\times n}$, $A = (A^i_j)_{n\times n}$ and I is the $n\times n$ unit matrix. Accordingly, we set $(x^1, \dots, x^n, x^{n+1}, \dots, x^{2n}) = (q^1, \dots, q^n, p_1, \dots, p_n)$. Then eq. (33) reads

$$X = \Big(\frac{\partial P^{ij}}{\partial q^j} - \frac{\partial A^i_j}{\partial p_j}, \frac{\partial A^j_i}{\partial q^j} + \frac{\partial Q_{ij}}{\partial p_j}\Big)^T, \tag{34}$$

with the index i running form 1 to n. Comparing it with eq. (17), it seems that a condition

$$\operatorname{tr} \alpha = 0 \tag{35}$$

could be imposed on the 2-form α in eq. (17).

4.2 The Nambu mechanics

It should be mentioned that the idea of generalizing the Hamiltonian systems can be traced back to Nambu [10]. In [10], Nambu pointed out that there are various possibilities of generalizing the Hamiltonian systems to the volume-preserving systems. Especially he discussed in detail the volume-preserving system on $\mathbb{R}^{3N}$. His discussion can smoothly generalizes to a $3N$-manifold $M_1 \times M_2 \times \ldots \times M_N$, where M_a for each a is a 3-dimensional Riemannian (or pseudo-Riemannian) manifold. The most exciting is that the quantization can be achieved within such a mechanics. — The ordinary quantization of mechanics, including the geometric quantization [11], is based on the Hamiltonian mechanics.

Our approach presented in this paper, however, differs from Nambu's work in three aspects. First, our interest is the volume-preserving systems on a symplectic manifold. We are not about to generalize the Hamiltonian systems, although certain a generalization has been inevitably resulted in. Secondly, the systems in Nambu's approach preserve a volume induced by the metric on that manifold. In Nambu's paper, in order to perform the vector product, the metric is necessary. The volume preserved in our approach is induced by the symplectic structure. So, such a generalization is different from that of Nambu. Thirdly, as having been discussed, the ordinary Hamiltonian systems have been included as the special cases into our general form of volume-preserving equations. This is not the case in Nambu's approach.

5 Conclusion and Remarks

In this paper, we have briefly introduced the definition and properties of the Euler-Lagrange cohomology groups on a symplectic manifold $(\mathcal{M}, \omega)$ and presented the general form of equations that relate to the image of the highest Euler-Lagrange cohomology group, which generate a volume-preserving flow on the manifold. It has been shown that for every volume-preserving flow there are some 2-forms playing the role similar to the Hamiltonian functions in the Hamilton mechanics and the ordinary canonical equations with Hamiltonian H are included as a special case with the 2-form $\frac{1}{n-1} H\,\omega$. Thus, this is a generalization of the Hamilton mechanics from the cohomological point of view.

Finally, it should be mentioned that the volume-preserving systems play very important role in physics, especially in the both classical mechanics and statistical physics via the famous Liouville theorems in both of them. However, this has not been noticed widely yet. In this paper we have not explained why the volume-preserving systems are so important in statistical physics, but left it as a topic for the forthcoming papers.

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