# Irregular Hamiltonian Systems

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**Abstract**

Hamiltonian systems with linearly dependent constraints (irregular systems), are classified according to their behavior in the vicinity of the constraint surface. For these systems, the standard Dirac procedure is not directly applicable. However, Dirac's treatment can be slightly modified to obtain, in some cases, a Hamiltonian description completely equivalent to the Lagrangian one. A recipe to deal with the different cases is provided, along with a few pedagogical examples.

## 1 Introduction

Dirac's Hamiltonian analysis provides a systematic method for finding the gauge symmetries present in a theory. The analysis identifies and classifies the constraints, which are local functions of the phase space coordinates. Consistency requires that the constraints be preserved in during the evolution (for the review of the Hamiltonian analysis see Ref.[1]–[4]). However, if the constraints are not functionally independent, then Dirac's procedure is not applicable. The test of functional independence are the so-called regularity conditions, and those systems which fail the test are said to be *irregular*.

Irregular systems are not necessarily intractable or exotic. A simple example is a relativistic massless particle ($p^\mu p_\mu = 0$), which is irregular at the origin of momentum space ($p^\mu = 0$). This point in phase space is exceptional, as it is unclear whether this would be an observable state for a photon, say. On the other hand, we know the configuration $p^\mu = 0$ to be a very important one: the ground state. There are other physical circumstances in which regularity is violated, and not only for isolated states but on large portions of the region in phase space where the system evolves. Chern-Simons theories in $2n + 1$ spacetime dimensions are examples where, for some initial configurations, regularity can fail at all times and one is forced to live with this problem.

Here we discuss the possible ways in which the constraints can fail the test of functional independence, and how the Hamiltonian treatment of Dirac must be modified in each case.

## 2 Regularity Conditions

If we call $z^i \equiv (q,p)$ $(i = 1,\ldots,2n)$ the coordinates in the phase space $\Gamma$, the constraints $\phi^r(z) \approx 0$ $(r = 1,...R)$ define the *constraint surface* $\Sigma$ given by

$$\Sigma = \{\bar{z} \in \Gamma \mid \phi^r(\bar{z}) = 0 \;\; (r = 1,\ldots,R)\,(R \leq 2n)\}\,. \tag{1}$$

**Regularity Conditions** (RCs)*: The constraints $\phi^r \approx 0$ are regular if and only if their small variations $\delta\phi^r$, evaluated on $\Sigma$, are $R$ linearly independent functions of $\delta z^i$.*

To first order in $\delta z^i$, the variation of the constraints are $\delta\phi^r = J_i^r \delta z^i$, where $J_i^r \equiv \left.\frac{\partial \phi^r}{\partial z^i}\right|_\Sigma$. Consequently, the RCs can also be defined as [2]:

*The set of constraints $\phi^r \approx 0$ is regular if and only if the Jacobian $J_i^r \equiv \left.\frac{\partial \phi^r}{\partial z^i}\right|_\Sigma$ has maximal rank: $\Re(J) = R$.*

A simple classical mechanical example of functionally *dependent* constraints occurs in a 2-dimensional phase space with coordinates $(q,p)$ and constraints $\phi^1 \equiv q \approx 0$ and $\phi^2 \equiv pq \approx 0$. In this case, $J = \begin{bmatrix} 1 & p \\ 0 & q \end{bmatrix}_\Sigma$ and $\Re\left[\frac{\partial(\phi^1,\phi^2)}{\partial(q,p)}\right]_{q=0} = 1$. A system of just one constraint can also fail the test of regularity. Consider the constraint $\phi = q^2 \approx 0$ in a 2-dimensional phase space. In this case, $J = \begin{bmatrix} 2q \\ 0 \end{bmatrix}_{q^2=0} = 0$ and $\Re\,(J) = 0$. The same problem occurs with the constraint $q^7 \approx 0$, which has a zero of seventh order at the constraint surface, or with any other constraint which is not linear in $z^i - \bar{z}^i$.

## 3 Classification of Irregular Constraints

Irregular constraints can be classified according to their approximate behavior near the surface $\Sigma$.

**A. Linear constraints**. The Jacobian has constant, non-maximal rank throughout $\Sigma$, and

$$\phi^r \equiv J_i^r \left(z^i - \bar{z}^i\right) \approx 0, \qquad \Re(J) = R' < R. \tag{2}$$

These are regular systems in disguise. Regularity fails simply because $R - R'$ constraints are redundant and should be discarded. The regular system gives the correct description.

**B. Multilinear constraints**. In the vicinity of $\Sigma$, the constraints are of the form

$$\phi \equiv \frac{1}{m!} S_{i_1 \ldots i_m} \left(z^{i_1} - \bar{z}^{i_1}\right) \cdots \left(z^{i_m} - \bar{z}^{i_m}\right) \approx 0 \qquad (m \leq 2n)\,, \tag{3}$$

where the coefficients $S_{i_1 \ldots i_m}$ vanish if any two indices are equal. Thus, $\phi$ has *simple zeros* on surfaces of dimension $2n-1$, and zeros of higher order occur at the intersections of these surfaces.

The RCs fail at the points of intersection, where $\phi$ has multiple zeros. At those intersections, $\phi$ can be replaced by the equivalent[1] set of constraints,

$$\varphi^1 \equiv z^{i_1} - \bar{z}^{i_1} \approx 0, \qquad \cdots \qquad ,\varphi^m \equiv z^{i_m} - \bar{z}^{i_m} \approx 0. \tag{4}$$

At the intersections, the set $\{\varphi^1, ..., \varphi^m\}$ is regular and therefore this provides a recipe for substituting the irregular multilinear constraint $\phi$ by a regular set of linear constraints. For example, the constraint $\phi \equiv qp \approx 0$ is irregular at $(0,0)$ because it admits a linear approximation everywhere except at this point. Replacement $\phi$ by the linear constraints $\{q \approx 0, p \approx 0\}$ at $(0,0)$ regularizes the system.

**C. Higher order constraints.** In the vicinity of $\Sigma$, the constraints do not possess a linear approximation:

$$\phi \equiv C\left(z^s - \bar{z}^s\right)^k \approx 0 \qquad (k > 1)\,. \tag{5}$$

The Jacobian vanishes everywhere on the constraint surface. Naively, it would seem possible to choose $z^s - \bar{z}^s \approx 0$ as an equivalent regular constraint, but it turns out that this could change the dynamics of original theory, as we show below.

These three classes are generic and, in general, there can be combinations of these three types occurring simultaneously near a constraint surface. By selecting a sufficiently small neighborhood of a point on $\Sigma$, one can always expect to be in one and only one of the three situations just described.

In correspondence with the three generic cases in which regularity can fail, the nature of the constraint surface $\Sigma$ falls into one of three categories:

**A.** *The RCs are satisfied on the whole constraint surface.* These are regular systems, either desguised or not.

**B.** *The RCs fail on a submanifold of* $\Sigma$: $\Re\left(J\right) = R$ except on a submanifold $\Sigma_0 \subset \Sigma$ where $\Re\left(J\right) = R' < R$ .

**C.** *The RCs fail everywhere on* $\Sigma$*:* $J$ has constant lower than $R$ rank on $\Sigma$.

The first case is treated in the standard texts and will not be further discussed here.

In the second case, the constraint surface can be decomposed into two non-empty submanifolds, $\Sigma_0$ and $\Sigma_R$ such that $\Sigma = \Sigma_0 \cup \Sigma_R$ and $\Sigma_0 \cap \Sigma_R$ is empty. Then, the rank of the Jacobian jumps from $\Re\left(J\right) = R$ on $\Sigma_R$, to $\Re\left(J\right) = R'$ on $\Sigma_0$. As mentioned above, in this case it is possible to replace $\phi$ at $\Sigma_0$ by a set of regular constraints $\left\{\varphi^1 \approx 0, \cdots, \varphi^m \approx 0\right\}$ which regularize the system at $\Sigma_0$.

The important question is how to proceed in the third case. If the RCs fail everywhere on $\Sigma$, the previous approach is not applicable, because there is no guarantee that the resulting Hamiltonian dynamics will be equivalent to that of the original Lagrangian system.

This can be seen in the example of Lagrangian in a three-dimensional configuration space,

$$L\left(x, y, z\right) = \dot{x}\dot{z} + yz^2. \tag{6}$$

[1] Two sets of constraints are said to be *equivalent* if they define the same constraint surface $\Sigma$.

The general solution of Euler-Lagrange equations describes a system with *one* degree of freedom – a free particle, whose time evolution $\bar{x} = p_0\, t + x_0$ is determined by two initial conditions $p_0$ and $x_0$. The remaining fields are $\bar{y}(t)$, a Lagrange multiplier with indeterminate evolution, and $\bar{z}(t)$, with trivial evolution, $\bar{z}(t) = 0$.

The Hamiltonian approach gives just two (first class) constraints, $p_y \approx 0$, $z^2 \approx 0$ $(R = 2)$, and *one* degree of freedom, as expected from the Lagrangian approach. However, this is only superficially correct, because the Jacobian has rank 1, and not 2. This is because the constraint function $\phi = z^2$ has no linear approximation at $z = 0$.

On the other hand, if we naively take $z \approx 0$, which is regular and equivalent to $z^2 \approx 0$, then the Hamiltonian analysis generates *three* first class constraints $p_y \approx 0$, $z \approx 0$, $p_x \approx 0$, which leave *zero* physical degrees of freedom. This result is not consistent with the Lagrangian description where there is one degree of freedom.

As seen in the previous example, even if two constraints are equivalent in defining the same constraint surface, they may yield different dynamics and should be treated more carefully.

Suppose $\phi \approx 0$ is a constraint of the form (5), which is equivalent to the *regular* constraint

$$\chi \equiv \phi^{1/k} \approx 0. \tag{7}$$

The question is whether $\chi \approx 0$ gives also the correct dynamics. The answer to this question depends on whether $\chi$ is a first or second class constraint. Namely, it makes a difference whether the linear constraint $\chi$ can generate a transformation in phase space that leaves the Hamiltonian action unchanged or not . As shown in [5], if the linearized constraint $\chi$ is second class, then it is not only geometrically equivalent to $\phi$ in the sense that it defines the same $\Sigma$, but the substitution also yields the same dynamical description as the Lagrangian approach. On the other hand, if $\chi$ is first class, then the subtitution generates a system whose dynamics is different from the one obtained from the Euler-Lagrange equations.

# 4 Conclusions

The recipe for treating the non-regular constraints is:

- Every linear or multi-linear set of constraints can be exchanged by an equivalent regular set. It allows to carry out Dirac's procedure in the standard way.

- A higher order constraint $\phi = C\,(z^s - \bar{z}^s)^k \approx 0$, can be exchanged by the equivalent linear constraint $\chi = \phi^{1/k} \approx 0$. If $\chi$ is a second class constraint, the dynamics of the new system is equivalent to the Lagrangian one. If $\chi$ is first class, the substitution yields a system which is not dynamically equivalent to the Lagrangian one. In this latter case, one should view the original Lagrangian as an incomplete, if not a totally inconsistent description for a dynamical system.

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