

# An Index Theorem for Domain Walls in Supersymmetric Gauge Theories

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## Abstract

The supersymmetric abelian Higgs model with  $N$  scalar fields admits multiple domain wall solutions. We perform a Callias-type index calculation to determine the number of zero modes of this soliton. We confirm that the most general domain wall has  $2(N - 1)$  zero modes, which can be interpreted as the positions and phases of  $(N - 1)$  constituent domain walls. This implies the existence of moduli for a  $D$ -string interpolating between  $N$   $D5$ -branes in  $IIB$  string theory.

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## I. INTRODUCTION

There exists an intimate connection between supersymmetry and solitons [1]. In many supersymmetric theories, the second-order equations of motion collapse to a first-order Bogomol'nyi equation, with a moduli space of solutions. The physical interpretation of this moduli space is that the competing forces between solitons cancel.

However, in the case of domain walls, supersymmetry does not guarantee the absence of forces [2]. In fact, there is something of a conflict between having domain walls and maximizing supersymmetry. While the former requires isolated vacua, the latter typically implies an extended moduli space of vacua. Recently there has been much interest in models in which the forces do cancel [3]–[7]. In this regard, it is interesting to study the unique class of maximally supersymmetric theories admitting multiple domain wall solutions. This is  $\mathcal{N} = 2$  supersymmetric QED, in which the introduction of a Fayet-Iliopoulos (FI) parameter lifts the Coulomb branch, while non-zero masses lift the Higgs branch.

Domain walls in this theory have been discussed in [8], where it is conjectured that the most general domain wall solution admits  $2(N - 1)$  collective coordinates with the interpretation of the positions and phases of  $(N - 1)$  constituent domain walls. In the strong coupling limit,  $e^2 \rightarrow \infty$ , the model reduces to a massive non-linear sigma model on the Higgs branch of the theory [9], and has a surprisingly rich spectrum of solitons [10]–[16], [8]. In this case, the theory contains only scalar fields, and so one can use Morse theory to determine the number of moduli of the domain walls [17]. This calculation was performed in [15], where it was found that there are indeed no forces and a moduli space of solutions exists.

However, for finite gauge coupling constant, the presence of the gauge field means that Morse theory techniques are not applicable. The purpose of this paper is to examine whether domain wall zero modes remain in the general case. We adapt index theorems of Weinberg (who followed a procedure developed by Callias [18]), and confirm that the system retains all  $2(N - 1)$  zero modes.

This theory finds application in the  $D1$ - $D5$  system of string theory in the presence of a background NS-NS  $B$ -field. The domain wall describes a  $D$ -string interpolating between separated  $D5$ -branes [19, 20]. The index theorem presented here confirms that the  $D$ -string has moduli.

We shall calculate the index by the introduction of a regulator scale,  $M$ . As can be seen from (29), the final answer is dependent on  $M$ . This is in contrast to the index for instantons and vortices, but also occurs for monopoles [21]. In the latter case, the  $M$ -dependence is related to both the quantum mass renormalization of monopoles [22] in four dimensions, and the non-cancellation of determinants in three-dimensional instanton calculations [23]. In the present case, our calculation also contains similar information regarding the mass renormalization of kinks in  $(1+1)$ -dimensional field theories, and the one-loop effects in the background of instantons in supersymmetric quantum mechanics.

The rest of the paper is organized as follows. In Sec. II, we review the gauge theory under consideration and its domain wall solutions, and in Sec. III, we calculate the dimension of the moduli space.

## II. GAUGE THEORY DOMAIN WALLS

The theory under consideration is  $d = 3 + 1$ ,  $\mathcal{N} = 2$  supersymmetric  $U(1)$  gauge theory coupled to  $N$  hypermultiplets. The bosonic part of the Lagrangian is given by:

$$\begin{aligned} \mathcal{L} = & \frac{1}{4e^2}F^2 + \frac{1}{2e^2}|\partial\phi|^2 + \sum_{i=1}^N (|\mathcal{D}q_i|^2 + |\mathcal{D}\tilde{q}_i|^2) - \sum_{i=1}^N |\phi - m_i|^2(|q_i|^2 + |\tilde{q}_i|^2) \\ & - \frac{e^2}{2}(\sum_{i=1}^N |q_i|^2 - |\tilde{q}_i|^2 - \zeta)^2 - \frac{e^2}{2}|\sum_{i=1}^N \tilde{q}_i q_i|^2. \end{aligned} \quad (1)$$

The scalar field  $q_i$  ( $\tilde{q}_i$ ) has charge  $+1$  ( $-1$ ) under the gauge group and complex mass  $m_i$ , whereas  $\phi$  is a neutral complex scalar field. The FI parameter  $\zeta$  must be non-zero to lift the Coulomb branch and can be taken to be positive, without loss of generality.

As in [8], we consider the case of non-zero, distinct masses:  $m_i \neq m_j$  for  $i \neq j$ . We take these to be real and consequently can choose the ordering  $m_{i+1} < m_i$  for all  $i$ . There are then  $N$  isolated vacua, given by:

$$\text{Vacuum } i : \quad \phi = m_i, \quad |q_j|^2 = \zeta\delta_{ij}, \quad |\tilde{q}_j|^2 = 0. \quad (2)$$

Furthermore, certain fields do not appear in the domain wall solutions and so are set to zero now:  $\text{Im}(\phi) = \tilde{q}_i = F = 0$ . Thus, the field  $\phi$  is real.

We choose the domain walls to be oriented in the  $x^2 - x^3$  plane, in which case  $\partial_2 = \partial_3 \equiv 0$ . By completing the square in the Hamiltonian, one finds that the Bogomol'nyi equations that

minimize the potential energy are given by:

$$\partial\phi = e^2\left(\sum_{i=1}^N |q_i|^2 - \zeta\right) \quad (3)$$

$$\mathcal{D}q_i = (\phi - m_i)q_i. \quad (4)$$

Here  $\partial \equiv \partial_1$  and  $\mathcal{D} = \partial - iA$ , where  $A \equiv A_1$  is the gauge potential.

In what follows, we shall concentrate upon domain walls interpolating between the first and  $N$ th vacua. It is conjectured that these kinks will decompose into many kinks, each interpolating between vacuum  $i$  and vacuum  $i + 1$ . We wish to investigate this.

### III. THE DIMENSION OF THE DOMAIN WALL MODULI SPACE

From (3) and (4), the linearized Bogomol'nyi equations are:

$$\partial\dot{\phi} = e^2 \sum_{i=1}^N (\dot{q}_i q_i^\dagger + q_i \dot{q}_i^\dagger) \quad (5)$$

$$\mathcal{D}\dot{q}_i - i\dot{A}q_i = (\phi - m_i)\dot{q}_i + \dot{\phi}q_i, \quad (6)$$

in which dots represent small fluctuations in the fields, i.e.  $\dot{\phi} \equiv \delta\phi$ . These must be supplemented with a gauge-fixing condition. A convenient choice which is compatible with supersymmetry is provided by Gauss' law in  $A_0 = 0$  gauge:

$$\partial\dot{A} = ie^2 \sum_{i=1}^N (q_i \dot{q}_i^\dagger - q_i^\dagger \dot{q}_i). \quad (7)$$

Then (5) and (7) can be combined into:

$$\partial(\dot{\xi}) = 2e^2 \sum_{j=1}^N \dot{q}_j q_j^\dagger, \quad (8)$$

where we have defined  $\xi = \phi + iA$ .

Writing (8) and (6) in matrix form, we have:

$$\mathbb{D}\Psi = 0, \quad (9)$$

where  $\Psi = (\dot{\xi}, \dot{q}_1, \dot{q}_2, \dots, \dot{q}_N)^T$  and

$$\mathbb{D} = \begin{pmatrix} \partial & -2e^2 q_1^\dagger & -2e^2 q_2^\dagger & \cdots & -2e^2 q_N^\dagger \\ -q_1 & \partial - (\xi - m_1) & 0 & \cdots & 0 \\ -q_2 & 0 & \partial - (\xi - m_2) & & \\ \vdots & \vdots & & \ddots & \\ -q_N & 0 & & & \partial - (\xi - m_N) \end{pmatrix}, \quad (10)$$

in which only entries in the first column, first row and main diagonal are non-zero.

Our task, then, is to determine the dimension of the kernel of  $\mathbb{D}$ , which we do by means of an index theorem. Previously, Weinberg has used index theorems to count parameters of multivortex solutions in Ginzburg-Landau theory [24] and multimonopole solutions, first in  $SU(2)$  gauge theory and later generalizing to an arbitrary compact simple gauge group [21, 25]. The index of  $\mathbb{D}$  is given by:

$$\mathcal{I} = \lim_{M^2 \rightarrow 0} \mathcal{I}(M^2), \quad (11)$$

where

$$\mathcal{I}(M^2) = \text{Tr} \left( \frac{M^2}{\mathbb{D}^\dagger \mathbb{D} + M^2} \right) - \text{Tr} \left( \frac{M^2}{\mathbb{D} \mathbb{D}^\dagger + M^2} \right). \quad (12)$$

Since  $\ker(\mathbb{D}) = \ker(\mathbb{D}^\dagger \mathbb{D})$  and  $\ker(\mathbb{D}^\dagger) = \ker(\mathbb{D} \mathbb{D}^\dagger)$ , each zero mode of  $\mathbb{D}$  contributes 1 to the index, while each zero mode of  $\mathbb{D}^\dagger$  contributes  $-1$ . If the continuum parts of the spectra extend to zero, then there is potentially a contribution from this source: this complication does not arise here because the theory has a mass gap. Hence, if there are no zero modes of  $\mathbb{D}^\dagger$ , then the index equals the number of collective coordinates. We now show that this is indeed the case. Consider the kernel of  $\mathbb{D}^\dagger$ . If

$$\mathbb{D}^\dagger \Psi = 0, \quad (13)$$

then (10) implies that the components of  $\Psi$  satisfy:

$$\begin{aligned} \partial \psi_0 + \sum_{i=1}^N q_i^\dagger \psi_i &= 0 \\ 2e^2 q_j \psi_0 + (\partial + (\xi^\dagger - m_j)) \psi_j &= 0, \quad j = 1, 2, \dots, N. \end{aligned} \quad (14)$$

Thus,

$$\begin{aligned}
0 &= \int dx \left( 2e^2 |\partial\psi_0|^2 + \sum_{i=1}^N q_i^\dagger \psi_i|^2 + \sum_{j=1}^N |2e^2 q_j \psi_0 + (\partial + (\xi^\dagger - m_j))\psi_j|^2 \right) \\
&= \int dx \left( 2e^2 |\partial\psi_0|^2 + 2e^2 \sum_{i=1}^N |q_i|^2 |\psi_i|^2 + 4e^4 \sum_{j=1}^N |q_j|^2 |\psi_0|^2 \right. \\
&\quad \left. + \sum_{j=1}^N |(\partial + (\xi^\dagger - m_j))\psi_j|^2 \right), \tag{15}
\end{aligned}$$

where we have used integration by parts and the second Bogomol'nyi equation (4) to show that the cross-terms disappear. Hence,  $\psi_i = 0$ ,  $i = 0, 1, \dots, N$ , and the kernel of  $\mathbb{D}^\dagger$  is trivial.

Note that if  $\mathcal{I}(M^2)$  were independent of  $M$ , then  $M$  could be chosen to be  $M \rightarrow \infty$  to simplify calculations. However, this is the case only if physical fields fall sufficiently rapidly at spatial infinity. Yang-Mills instantons and vortices are in this category, whereas monopoles are not (see [21, Appendix A] for further discussion). Like monopoles, the kink solutions under consideration have an  $M$ -dependent index.

Following the method of [26], let us define:

$$\Theta = \begin{pmatrix} 0 & -\mathbb{D}^\dagger \\ \mathbb{D} & 0 \end{pmatrix}. \tag{16}$$

Also define:

$$\Gamma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \Gamma_5 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \tag{17}$$

Finally, define the matrix  $K(x)$  via:

$$\Theta = \Gamma \partial + K(x). \tag{18}$$

Now,  $\mathcal{I}(M^2)$  can be re-written as:

$$\mathcal{I}(M^2) = \text{Tr} \Gamma_5 \frac{M^2}{-\Theta^2 + M^2} = \int dx \text{tr} \left\langle x \left| \Gamma_5 \frac{M^2}{-\Theta^2 + M^2} \right| x \right\rangle. \tag{19}$$

Let us define the nonlocal current:

$$J(x, y, M) = \text{tr} \left\langle x \left| \Gamma_5 \Gamma \frac{1}{\Theta + M} \right| y \right\rangle. \tag{20}$$

From (18) we have:

$$\begin{aligned}\delta(x-y) &= [\Gamma\partial_x + K(x) + M] \left\langle x \left| \frac{1}{\Theta + M} \right| y \right\rangle \\ &= \left\langle x \left| \frac{1}{\Theta + M} \right| y \right\rangle [-\overset{\leftarrow}{\partial}_y \Gamma + K(y) + M].\end{aligned}\quad (21)$$

Using this and the relations  $\{\Gamma_5, K\} = \{\Gamma_5, \Gamma\} = \{\Gamma_5, \Theta\} = 0$ , we obtain the following:

$$\begin{aligned}(\partial_x + \partial_y)J(x, y, M) &= -2 \operatorname{tr} \left\langle x \left| \Gamma_5 \frac{M}{\Theta + M} \right| y \right\rangle \\ &\quad + \operatorname{tr} \left( [K(x) - K(y)] \Gamma_5 \left\langle x \left| \frac{1}{\Theta + M} \right| y \right\rangle \right).\end{aligned}\quad (22)$$

$$\Rightarrow \partial J(x, x, M) = -2 \operatorname{tr} \left\langle x \left| \Gamma_5 \frac{M^2}{-\Theta^2 + M^2} \right| x \right\rangle. \quad (23)$$

Thus,

$$\mathcal{I}(M^2) = -\frac{1}{2} [J(x, x, M)]_{x=-\infty}^{\infty}. \quad (24)$$

We shall find it more convenient to evaluate  $J(x, x, M)$  from

$$J(x, x, M) = -\operatorname{tr} \left\langle x \left| \Gamma_5 \Gamma \Theta \frac{1}{-\Theta^2 + M^2} \right| x \right\rangle, \quad (25)$$

which follows from the fact that  $\operatorname{tr}(\Gamma_5 \Gamma M / (-\Theta^2 + M^2)) = 0$ .

We now proceed to calculate (24). From (10), lengthy but straightforward calculations yield  $\mathbb{D}^\dagger \mathbb{D}$  and  $\mathbb{D} \mathbb{D}^\dagger$ . These turn out to be somewhat messy matrices. For example, in  $\mathbb{D}^\dagger \mathbb{D}$  the entries  $(i+1, i+1)$ ,  $(1, i+1)$  and  $(i+1, j+1)$ , for  $1 \leq i, j \leq N$ ,  $j \neq i$ , are given by  $(-\partial^2 + 4e^4|q_i|^2 + |\xi - m_i|^2 + \partial\xi - \xi^\dagger\partial)$ ,  $(2e^2\partial q_i^\dagger - q_i^\dagger(\partial - (\xi - m_i)))$  and  $(4e^4q_i q_j^\dagger)$ , respectively.

Since

$$\Gamma_5 \Gamma \Theta \frac{1}{-\Theta^2 + M^2} = \begin{pmatrix} \mathbb{D}(\mathbb{D}^\dagger \mathbb{D} + M^2)^{-1} & 0 \\ 0 & \mathbb{D}^\dagger(\mathbb{D} \mathbb{D}^\dagger + M^2)^{-1} \end{pmatrix}, \quad (26)$$

we wish to be able to invert the matrices  $(\mathbb{D}^\dagger \mathbb{D} + M^2)$  and  $(\mathbb{D} \mathbb{D}^\dagger + M^2)$ . Recall, though, that all that we require are the values of  $J$  at  $\pm\infty$  (corresponding to the  $N$ th and 1st vacua, respectively). In the  $i$ th vacuum, the fields have values given by (2) and

$$\partial q_j = (\xi - m_j)q_j = iAq_i\delta_{ij}, \quad (27)$$

which follows from (4). Applying these boundary conditions greatly simplifies these matrices, each of which then has only two non-zero entries off the main diagonal. Nevertheless, even matrices of this sparse form cannot be inverted if their entries do not commute. In our

case, the problem arises from the presence of the differential operator. However, we can set  $A = 0$ , after which (2) and (27) imply that  $\partial q_j = \partial \xi = 0$  at spatial infinity. Thus, we are able to obtain expressions for  $(\mathbb{D}^\dagger \mathbb{D} + M^2)^{-1}$  and  $(\mathbb{D} \mathbb{D}^\dagger + M^2)^{-1}$  in the first and  $N$ th vacua.

Finally, putting these calculations into (24)–(26), and using

$$\begin{aligned} \left\langle x \left| \frac{m_i - m_j}{-\partial^2 + (m_i - m_j)^2 + M^2} \right| x \right\rangle &= \frac{m_i - m_j}{2\pi} \int \frac{dk}{k^2 + (m_i - m_j)^2 + M^2} \\ &= \frac{1}{2} \frac{m_i - m_j}{\sqrt{(m_i - m_j)^2 + M^2}}, \end{aligned} \quad (28)$$

we obtain:

$$\mathcal{I}(M^2) = \frac{1}{2} \left( \sum_{i=1}^{N-1} \frac{m_i - m_N}{\sqrt{(m_i - m_N)^2 + M^2}} + \sum_{j=2}^N \frac{m_1 - m_j}{\sqrt{(m_1 - m_j)^2 + M^2}} \right). \quad (29)$$

Hence, due to the ordering  $m_{i+1} < m_i$  for all  $i$ ,

$$\mathcal{I} = N - 1. \quad (30)$$

Therefore, we conclude that  $\mathcal{N} = 2$  SQED with  $N$  hypermultiplets admits multi-domain wall solutions having  $N - 1$  complex collective coordinates.

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