

Localization of metric fluctuations on scalar branes

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Abstract

The localization of metric fluctuations on scalar brane configurations breaking spontaneously five-dimensional Poincaré invariance is discussed. Assuming that the four-dimensional Planck mass is finite and that the geometry is regular, it is demonstrated that the vector and scalar fluctuations of the metric are not localized on the brane.

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If internal dimensions are not compact [1, 2, 3, 4] all the fields describing the fundamental forces of our four-dimensional world should be localized on a higher dimensional topological defect [1, 2]. Localized means that the various fields should exhibit normalizable zero modes with respect to the bulk coordinates parameterizing the geometry of the defect in the extra-dimensional space. Among the various interactions an important rôle is played by gravitational forces [5, 6] since the localization of the metric fluctuations can lead to computable and measurable deviations of Newton law at short distances [7]. In [6] it has been shown that the zero mode related to the tensor fluctuations of the geometry is localized, provided the four-dimensional Planck mass is finite. For a recent review see [8].

If the four-dimensional world is Poincaré-invariant, the higher-dimensional geometry will have not only tensor modes but also scalar and vector fluctuations. The localization of the various modes of the geometry will then be the subject of the present analysis. On top of four-dimensional Poincaré invariance, the invariance of the fluctuations under infinitesimal coordinate transformations, i.e. gauge-invariance [9], can be used in order to simplify the problem. Gauge invariance guarantees that the equations for the fluctuations of the geometry do not change when moving from one coordinate system to the other. The tensor modes of the geometry are gauge-invariant but the scalar and vector modes are not. Hence, gauge-invariant fluctuations corresponding to these modes should be constructed and analyzed. The general derivation of these equations allows a model-independent discussion of the normalizability properties of the zero modes related to the various fluctuations of the metric.

The following five-dimensional action ¹.

$$S = \int d^5x \sqrt{|G|} \left[-\frac{R}{2\kappa} + \frac{1}{2} G^{AB} \partial_A \varphi \partial_B \varphi - V(\varphi) \right], \quad (1)$$

can be used in order to describe the breaking of five-dimensional Poincaré symmetry. Consider a potential which is invariant under the $\varphi \rightarrow -\varphi$ symmetry. Then, non-singular domain-wall solutions can be obtained, for various potentials, in a metric

$$ds^2 = \overline{G}_{AB} dx^A dx^B = a^2(w) [dt^2 - d\vec{x}^2 - dw^2]. \quad (2)$$

For instance solutions of the type

$$a(w) = \frac{1}{\sqrt{b^2 w^2 + 1}}, \quad (3)$$

$$\varphi = \varphi(w) = \sqrt{6} \arctan bw, \quad (4)$$

is found for trigonometric potentials [10, 11, 12] like $V(\varphi) \propto (1 - 5 \sin^2 \varphi / \sqrt{6})$. Solutions like Eqs. (3)–(4) represent a smooth version of the Randall-Sundrum scenario [5, 6]. The assumptions of the present analysis will now be listed.

(i) The five-dimensional geometry is regular (in a technical sense) for any value of the bulk coordinate w . This implies that singularities in the curvature invariants are absent.

¹Latin (uppercase) indices run over the five-dimensional space-time whereas Greek indices run over the four-dimensional space-time. Notice that $\kappa = 8\pi G_5 = 8\pi/M_5^3$. Natural gravitational units will be often employed by setting $2\kappa = 1$.

(ii) Five-dimensional Poincaré invariance is broken through a smooth five dimensional domain-wall solution generated by a potential $V(\varphi)$ which is invariant under $\varphi \rightarrow -\varphi$. The warp factor $a(w)$ will then be assumed symmetric for $w \rightarrow -w$.

(iii) Four-dimensional Planck mass is finite because the following integral converges

$$M_P^2 \simeq M^3 \int_{-\infty}^{\infty} dw a^3(w). \quad (5)$$

(iv) Five-dimensional gravity is described according to Eq. (1) and, consequently, the equations of motion for the warped background generated by the smooth wall are, in natural gravitational units,

$$\varphi'^2 = 6(\mathcal{H}^2 - \mathcal{H}'), \quad (6)$$

$$V a^2 = -3(3\mathcal{H}^2 + \mathcal{H}'). \quad (7)$$

where the prime denotes derivation with respect to w and $\mathcal{H} = a'/a$.

Under these assumptions it will be shown that the gauge-invariant fluctuations corresponding to scalar and vector modes of the geometry are not localized on the wall. On the contrary, tensor modes of the geometry will be shown to be localized. This program will be achieved in two steps. In the first step decoupled equations for the gauge-invariant variables describing the fluctuations of the geometry will be obtained for general brane backgrounds without assuming any specific solution. In the second step the normalizability of the zero modes will be addressed using only the assumptions (i)–(iv).

Let us start by perturbing the five-dimensional Einstein equations and of the equation for the scalar field

$$\delta R_{AB} = \frac{1}{2} \partial_A \varphi \partial_B \chi + \frac{1}{2} \partial_A \chi \partial_B \varphi - \frac{1}{3} \frac{\partial V}{\partial \varphi} \chi \bar{G}_{AB} - \frac{V}{3} \delta G_{AB}, \quad (8)$$

$$\delta G^{AB} \left(\partial_A \partial_B \varphi - \bar{\Gamma}_{AB}^C \partial_C \varphi \right) + \bar{G}^{AB} \left(\partial_A \partial_B \chi - \bar{\Gamma}_{AB}^C \partial_C \chi - \delta \Gamma_{AB}^C \partial_C \varphi \right) + \frac{\partial^2 V}{\partial \varphi^2} \chi = 0, \quad (9)$$

where the metric and the scalar field have been separated into their background and perturbation parts:

$$\begin{aligned} G_{AB}(x^\mu, w) &= \bar{G}_{AB}(w) + \delta G_{AB}(x^\mu, w), \\ \varphi(x^\mu, w) &= \varphi(w) + \chi(x^\mu, w). \end{aligned} \quad (10)$$

In Eqs. (8)–(9), $\delta \Gamma_{AB}^C$ and δR_{AB} are, respectively, the fluctuations of the Christoffel connections and of the Ricci tensors, whereas $\bar{\Gamma}_{AB}^C$ are the values of the connections computed using the background metric (2). According to Eq. (8) and (9), the fluctuations of the scalar brane, χ are coupled to the scalar fluctuations of the geometry $\delta G_{AB}(x^\mu, w)$ whose modes can be decomposed using Poincaré invariance in four dimensions as

$$\delta G_{AB} = a^2(w) \begin{pmatrix} 2h_{\mu\nu} + (\partial_\mu f_\nu + \partial_\nu f_\mu) + 2\eta_{\mu\nu} \psi + 2\partial_\mu \partial_\nu E & D_\mu + \partial_\mu C \\ D_\mu + \partial_\mu C & 2\xi \end{pmatrix}. \quad (11)$$

On top of $h_{\mu\nu}$ which is divergence-less and trace-less (i.e. $\partial_\mu h^\mu_\nu = 0$, $h^\mu_\mu = 0$) there are four scalars (i.e. E , ψ , ξ and C) and two divergence-less vectors (D_μ and f_μ). For infinitesimal coordinate transformations $x^A \rightarrow \tilde{x}^A = x^A + \epsilon^A$ the tensors $h_{\mu\nu}$ are invariant whereas the vectors and the scalars transform non-trivially. The four-dimensional part of the infinitesimal shift $\epsilon_A = a^2(w)(\epsilon_\mu, -\epsilon_w)$, can be decomposed as $\epsilon_\mu = \partial_\mu \epsilon + \zeta_\mu$ where ζ_μ is a divergence-less vector and ϵ is a scalar.

It is then clear that the transformations for the scalars involve two gauge functions ϵ and ϵ_w . The transformations for the vectors involve ζ_μ . Two scalars and one divergence-less vector can be gauged-away by fixing the scalar and the vector gauge functions. In a different perspective [9], since there are two scalar gauge functions and four scalar fluctuations of the metric (11), two gauge-invariant (scalar) variables can be defined. In the present case the gauge-invariant scalar variables can be chosen to be:

$$\begin{aligned}\Psi &= \psi - \mathcal{H}(E' - C), \\ \Xi &= \xi - \frac{1}{a}[a(C - E')]'.\end{aligned}\tag{12}$$

By shifting infinitesimally the coordinate system from x^A to \tilde{x}^A the metric fluctuations change as

$$\delta G_{AB}(x^\mu, w) \rightarrow \delta \tilde{G}_{AB} = \delta G_{AB} - \nabla_A \epsilon_B - \nabla_B \epsilon_A,\tag{13}$$

where the covariant derivatives are computed using the background metric of Eq. (2). In spite of this, $\tilde{\Psi} = \Psi$, $\tilde{\Xi} = \Xi$ and $\tilde{h}_{\mu\nu} = h_{\mu\nu}$. The physical interpretation of Ξ and Ψ is clear if a specific gauge choice is selected. In the longitudinal gauge (i.e. $E = 0$, $C = 0$ and $f_\mu = 0$) $\Xi = \xi$ and $\Psi = \psi$. Thus, in the longitudinal gauge, Ψ and Ξ are identical to the metric perturbations in a conformally Newtonian coordinate system. The scalar field fluctuation of Eq. (10) is not gauge-invariant and the gauge-invariant variable associated with it is:

$$X = \chi - \varphi'(E' - C).\tag{14}$$

Since there is one vector gauge function, i.e. ζ_μ , one gauge-invariant variable can be constructed out of D_μ and f_μ :

$$V_\mu = D_\mu - f'_\mu.\tag{15}$$

In terms of the variables defined in Eqs. (12)–(14) and (15) the perturbed system of Eqs. (8)–(9) can be written in a fully gauge-invariant way. The equation for the tensors, as expected, decouples from the very beginning:

$$\mu''_{\mu\nu} - \partial_\alpha \partial^\alpha \mu_{\mu\nu} - \frac{(a^{3/2})''}{a^{3/2}} \mu_{\mu\nu} = 0.\tag{16}$$

where $\mu_{\mu\nu} = a^{3/2} h_{\mu\nu}$ is the canonical normal mode of the of the action (1) perturbed to second order in the amplitude of tensor fluctuations.

The scalar variables (12) and (14) form a closed system consisting of the diagonal components of Eq. (8)

$$\Psi'' + 7\mathcal{H}\Psi' + \mathcal{H}\Xi' + 2(\mathcal{H}' + 3\mathcal{H}^2)\Xi + \frac{1}{3}\frac{\partial V}{\partial \varphi}a^2X - \partial_\alpha \partial^\alpha \Psi = 0,\tag{17}$$

$$-\partial_\alpha \partial^\alpha \Xi - 4[\Psi'' + \mathcal{H}\Psi'] - 4\mathcal{H}\Xi' - \varphi'X' - \frac{1}{3}\frac{\partial V}{\partial \varphi}a^2X + \frac{2}{3}Va^2\Xi = 0, \quad (18)$$

$$(19)$$

supplemented by the gauge-invariant version of the perturbed scalar field equation (9)

$$\partial_\alpha \partial^\alpha X - X'' - 3\mathcal{H}X' + \frac{\partial^2 V}{\partial \varphi^2}a^2X - \varphi'[4\Psi' + \Xi'] - 2\Xi(\varphi'' + 3\mathcal{H}\varphi') = 0, \quad (20)$$

and subjected to the constraints

$$\partial_\mu \partial_\nu [\Xi - 2\Psi] = 0, \quad (21)$$

$$6\mathcal{H}\Xi + 6\Psi' + X\varphi' = 0. \quad (22)$$

coming from the off-diagonal components of Eq. (8). From Eq. (8), the evolution of the gauge-invariant vector variable (15) is

$$\partial_\alpha \partial^\alpha \mathcal{V}_\mu = 0, \quad \mathcal{V}'_\mu + \frac{3}{2}\mathcal{H}\mathcal{V}_\mu = 0, \quad (23)$$

where $\mathcal{V}_\mu = a^{3/2}V_\mu$ is the canonical normal mode of the action (1) perturbed to second order in the amplitude of vector fluctuations of the metric.

Using repeatedly the constraints of Eqs. (21)–(22), together with the background relations (6)–(7), the scalar system can be reduced to the following two equations

$$\Phi'' - \partial_\alpha \partial^\alpha \Phi - z\left(\frac{1}{z}\right)''\Phi = 0, \quad (24)$$

$$\mathcal{G}'' - \partial_\alpha \partial^\alpha \mathcal{G} - \frac{z''}{z}\mathcal{G} = 0, \quad (25)$$

where

$$\Phi = \frac{a^{3/2}}{\varphi'}\Psi, \quad \mathcal{G} = a^{3/2}\Psi - zX. \quad (26)$$

The same equation satisfied by Ψ is also satisfied by Ξ by virtue of the constraint (21). In Eq. (25) and (26) the background dependence appears in terms of the “universal” function $z(w)$

$$z(w) = \frac{a^{3/2}\varphi'}{\mathcal{H}}. \quad (27)$$

It should be appreciated that these equations are completely general and do not depend on the specific background but only upon the general form of the metric (2) and of the action (1). In fact, in order to derive Eqs. (24)–(25) and (26)–(27) no specific background has been assumed, but only Eqs. (6)–(7) which come directly from Eq. (1) and hold for any choice of the potential generating the scalar brane configuration.

The effective “potentials” appearing in the Schrödinger-like equations (24) and (25) are dual with respect to $z \rightarrow 1/z$. It can be shown that the gauge-invariant function \mathcal{G} is the normal

mode of the action perturbed to second order in the amplitude of the scalar fluctuations of the metric analogous to the normal modes one can obtain in the case of compact internal dimensions [13] of Kaluza-Klein type.

Let us now discuss the localization of the zero modes of the various fluctuations and enter the second step of the present discussion. The lowest mass eigenstate of Eq. (16) is $\mu(w) = \mu_0 a^{3/2}(w)$. Hence, the normalization condition of the tensor zero mode implies

$$|\mu_0|^2 \int_{-\infty}^{\infty} a^3 dw = 2|\mu_0|^2 \int_0^{\infty} a^3(w) dw = 1. \quad (28)$$

where the assumed $w \rightarrow -w$ symmetry of the background geometry has been exploited. Using assumptions (i), (ii) and (iii) the tensor zero mode is then normalizable [5, 6].

Let us now move to the analysis of vector fluctuations. Eq. (23) shows that the vector fluctuations are always massless and the corresponding zero mode is $\mathcal{V}(w) \sim \mathcal{V}_0 a^{-3/2}$. Consequently, the normalization condition will be

$$2|\mathcal{V}_0|^2 \int_0^{\infty} \frac{dw}{a^3(w)} = 1, \quad (29)$$

which cannot be satisfied if assumption (i), (ii) and (iii) hold. If $a^3(w)$ converges everywhere, $1/a^3(w)$ will not be convergent. Therefore, if the four-dimensional Planck mass is finite the tensor modes of the geometry are normalizable and the vectors are not.

From Eq. (24) the lowest mass eigenstate of the metric fluctuation Φ corresponds to $\Phi(w) = \Phi_0 z^{-1}(w)$ and the related normalization condition reads

$$2|\Phi_0|^2 \int_0^{\infty} \frac{dw}{z^2(w)} = 1. \quad (30)$$

The integrand appearing in Eq. (30) will now be shown to be non convergent at infinity if the geometry is regular. In fact according to assumption (i)

$$\begin{aligned} R &= \frac{4}{a^2}(2\mathcal{H}' + 3\mathcal{H}^2), & R_{MN}R^{MN} &= \frac{4}{a^4}(4\mathcal{H}^4 + 6\mathcal{H}'\mathcal{H}^2 + 5\mathcal{H}'^2), \\ R_{MNAB}R^{MNAB} &= \frac{8}{a^4}(2\mathcal{H}'^2 - 5\mathcal{H}^4), \end{aligned} \quad (31)$$

should be regular for any w and, in particular, at infinity. The absence of poles in the curvature invariants guarantees the regularity of the five-dimensional geometry. Eq. (31) rules then out, in the coordinate system of (2), warp factors decaying at infinity as e^{-dw} or $e^{-d^2w^2}$: these profiles would lead to divergences in Eqs. (31) at infinity². Since $a(w)$ must converge at infinity, $a(w) \sim w^{-\gamma}$ with $1/3 \leq \gamma \leq 1$. Notice that $\gamma \geq 1/3$ comes from the convergence (at infinity) of the integral of Eq. (5)³ and that $\gamma \leq 1$ is implied by Eqs. (31) since, at infinity,

²In order to avoid confusions it should be stressed that exponential warp factors naturally appear in non-conformal coordinate systems related to the one of Eq. (2) as $a(w)dw = dy$.

³In this sense the power γ measures only the degree of convergence of a given integral.

$R_{MN}R^{MN} \sim R_{MNAB}R^{MNAB} \sim w^{4(\gamma-1)}$ should converge. Using Eq. (27) and Eq. (6) the integrand of Eq. (30) can be written as

$$\frac{1}{z^2} = \frac{\mathcal{H}^2}{a^3 \varphi'^2} = \frac{1}{6a^3} \left(\frac{\mathcal{H}^2}{\mathcal{H}^2 - \mathcal{H}'} \right). \quad (32)$$

The behavior at infinity of Eq. (32) can be now investigated assuming the regularity of Eqs. (31), i.e. $a(w) \sim w^{-\gamma}$ with $0 < \gamma \leq 1$. In this limit

$$\lim_{w \rightarrow \infty} \left(\frac{\mathcal{H}^2}{\mathcal{H}^2 - \mathcal{H}'} \right) \sim \frac{\gamma^2}{\gamma^2 - \gamma}. \quad (33)$$

and $1/z^2$ diverges *at least* as a^{-3} . In fact, if $\gamma = 1$, $1/z^2$ diverges even more as it can be argued from Eq. (33) which has a further pole for $\gamma^2 = \gamma$. The example given in Eqs. (3)–(4) corresponds to a behavior at infinity given by $\gamma = 1$. Direct calculations show that $1/z^2$ diverges, in this case, as w^5 .

Consequently, if the four-dimensional Planck mass is finite and if space-time is regular the gauge-invariant (scalar) zero mode is not normalizable and not localized on the brane. For sake of completeness it should be mentioned that, for the lowest mass eigenvalue, there is a second (linearly independent) solution to Eq. (24) which is given by $z^{-1}(w) \int^w z^2(x) dx$ which has poles at infinity and for $w \rightarrow 0$. The poles appearing for $w \rightarrow 0$ will now be discussed since they are needed in order to prove that the zero modes of Eq. (25) are not localized. As far as the poles at infinity are concerned it is interesting to consider what happens to $z^{-1}(w) \int^w z^2(x) dx$ in the case of the solution (3)–(4). In this case, by direct use of Eqs. (3)–(4) and (27) we have that the second solution diverges, at infinity, as $(1 + b^2 w^2)^{1/4} (1 + 2b^2 w^2)$.

Noticing the duality connecting the effective potentials of Eqs. (24) and (25) it can be verified that the lowest mass eigenstate of Eq. (25) is given by $\mathcal{G}(w) = \mathcal{G}_0 z(w)$. Provided the assumptions (i)–(iv) are satisfied, it will now be demonstrated that the integral

$$2|\mathcal{G}_0|^2 \int_0^\infty z^2 dw, \quad (34)$$

is divergent not because of the behavior at infinity but because of the behavior of the solution close to the core of the wall, i.e. $w \rightarrow 0$. Bearing in mind Eq. (31), assumption (i) and (ii) imply that $a(w)$ and φ should be regular for any w . More specifically close to the core of the wall φ should go to zero and $a(w)$ should go to a constant because of $w \rightarrow -w$ symmetry and the following regular expansions can be written for small w

$$a(w) \simeq a_0 - a_1 w^\beta + \dots, \quad \beta > 0, \quad (35)$$

$$\varphi(w) \simeq \varphi_1 w^\alpha + \dots, \quad \alpha > 0, \quad (36)$$

for $w \rightarrow 0$. Inserting the expansion (35)–(36) into Eq. (6) the relations can be obtained:

$$\beta = 2\alpha, \quad \alpha^2 \varphi_1^2 = 6 \frac{a_1}{a_0} \beta(\beta - 1). \quad (37)$$

Inserting now Eqs. (35)–(36) into Eq. (27) and exploiting the first of Eqs. (37) we have

$$\lim_{w \rightarrow 0} z^2(w) \simeq w^{2(\alpha-\beta)} = w^{-2\alpha}. \quad \alpha > 0 \quad (38)$$

Using Eqs. (35) into Eqs. (31), $R_{AB}R^{AB} \sim R_{MNAB}R^{MNAB} \sim w^{2\beta-4}$, which implies $\beta \geq 2$ in order to have regular invariants for $w \rightarrow 0$. Since, from Eq. (37), $\beta = 2\alpha$, in Eq. (38) it must be $\alpha \geq 1$. As in the case of Eq. (24) also eq. (25) has a second (linearly independent) solution for the lowest mass eigenvalue, namely $z(w) \int^w dx z^{-2}(x)$ which has poles at infinity. In fact, a direct check shows that, at infinity, this quantity goes as $w^{\frac{3}{2}\gamma+1}$ where, as usual, $1/3 \leq \gamma \leq 1$ for the convergence of the Planck mass and of the curvature invariants at infinity.

In conclusion, it has been demonstrated that under assumptions (i)–(iv), the scalar and vector fluctuations of the five-dimensional metric decouple from the wall. Heeding experimental tests [7], the present results suggest that, under the assumptions (i)–(iv), no vector or scalar component of the Newtonian potential at short distances should be expected.

It is a pleasure to thank M. E. Shaposhnikov for important comments.

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