A Cohomological Interpretation of the Migdal-Makeenko Equations

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Abstract

The equations of motion of quantum Yang - Mills theory (in the planar 'large N' limit), when formulated in Loop-space are shown to have an anomalous term, which makes them analogous to the equations of motion of WZW models. The anomaly is the Jacobian of the change of variables from the usual ones i.e. the connection one form A, to the holonomy U. An infinite dimensional Lie algebra related to this change of variables (the Lie algebra of loop substitutions) is developed, and the anomaly is interpreted as an element of the first cohomology of this Lie algebra. The Migdal-Makeenko equations are shown to be the condition for the invariance of the Yang-Mills generating functional Z under the action of the generators of this Lie algebra. Connections of this formalism to the collective field approach of Jevicki and Sakita are also discussed.

1 Introduction

The purpose of the present paper is to investigate certain geometric and algebraic properties of the Migdal-Makeenko equations. What is of special interest is an anomaly contained in these equations, for which we provide a natural geometric interpretation. To elaborate more on our motivation, we briefly review the loop-space formulation of QCD in the rest of this section.

The natural gauge invariant variable for describing Yang-Mills theory, the Wilson loop ($W(\gamma)$), is defined as

$$W(\gamma) = \langle trU(\gamma)\rangle = \left\langle trPe^{\int_\gamma A}\right\rangle; \tag{1}$$

where, γ is a loop in the base manifold of the principal bundle, and $U(\gamma)$ is the corresponding parallel transport operator. In geometric terms, changing variables from the connection one form A to the parallel transport operator corresponding to loops $U(\gamma)$, amounts to talking in terms of differential

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forms on the loop space of a manifold instead of the connections on the manifold. Considerable literature has been devoted in physics[?, ?, ?, ?, ?, ?, ?, ?] towards understanding Yang-Mills using the language of loop spaces, and in mathematics[?, ?, ?] towards constructing the theory of differential forms on loops spaces. The analogue of $A_\mu(x)$ in loop space is $\mathcal{F}\mu(\gamma(s))$, defined as;

$$\mathcal{F}_\mu(\gamma(s)) = \frac{\delta}{\delta\sigma_{\mu\nu}(\gamma(s))} U(\gamma)\dot{\gamma}_\nu(s) = U(\gamma_{0s}) F_{\mu\nu}\dot{\gamma}_\nu(s) U(\gamma_{s0}). \tag{2}$$

In the above equation $\frac{\delta}{\delta\sigma_{\mu\nu}(\gamma(s))}$ is the 'area derivative' (in the sense of Migdal[?, ?, ?]), γ is a loop beginning and ending at $\gamma(0)$, and $U(\gamma_{0s})(U(\gamma_{s0}))$ is the parallel transport operator associated with the path starting at $\gamma(0)(\gamma(s))$ and ending at $\gamma(s)(\gamma(0))$.
Classical Yang-Mills theory can now be recast in the language of $\mathcal{F}_\mu$, which is to be thought of as a one form in loop space. The classical equations of motion for Yang-Mills theory in this language are,

$$\frac{\delta}{\delta\gamma_\mu(s)} \mathcal{F}_\mu(\gamma(s)) = 0, \tag{3}$$

along with the identity,

$$\frac{\delta}{\delta\gamma_\nu(t)} \mathcal{F}_\mu(\gamma(s)) - \frac{\delta}{\delta\gamma_\mu(t)} \mathcal{F}_\nu(\gamma(s)) + [\mathcal{F}_\mu(\gamma(s)), \mathcal{F}_\nu(\gamma(t))] = 0. \tag{4}$$

In the above equations, $\frac{\delta}{\delta\gamma_\mu(s)}$ is the usual variational derivative at $\gamma(s)$.
It was pointed out by Polyakov [?], that these equations are very reminiscent of the classical equations of motion for chiral fields.

Interesting phenomena happen at the quantum level, where equation (3) no longer continues to be true as an equation for expectation values. The quantum analog of (3) (in the planar large N limit) is the Migdal-Makeenko equation (MME), which is,

$$\left\langle tr \frac{\delta}{\delta\gamma_\mu(s)} \mathcal{F}_\mu(\gamma(0)) \right\rangle = \frac{\partial^2 W(\gamma)}{\partial\gamma_\mu^2(0)} = -e^2 \int \delta(\gamma(s) - \gamma(0)) W(\gamma_{0s}) W(\gamma_{s0}) \dot{\gamma_\mu}(0)\dot{\gamma_\mu}(s) ds \tag{5}$$

In the equation above, the left hand side denotes the action of the loop-space Laplacian on the Wilson loop. The Loop-space Laplacian [?, ?] is a second order differential operator satisfying the Leibnitz rule, and is defined as follows.

$$\frac{\partial^2}{\partial\gamma_\mu^2(s)} = \lim_{\epsilon\to 0} \int_{-\epsilon}^{\epsilon} dt \frac{\delta^2}{\delta\gamma_\mu(s+t/2)\delta\gamma_\mu(s-t/2)} \tag{6}$$

The delta function on the right hand side is a delta function on the underlying manifold, and it implies that the right hand side of the equation is non-zero if the loop has a self intersection at $\gamma(s) = \gamma(0)$.

Note: Since the Wilson loops themselves do not have any marked points, we can always identify the point at which the loop-Laplacian acts as the initial point of the loop. This convention leads to some notational simplifications, and hence we shall adhere to it in the rest of the paper.

This equation is usually derived by requiring the invariance of $W(\gamma)$ under a linear change of variables. In other words one considers the equation defining the Wilson loop, which is,

$$W(\gamma) = \int D[A_\mu] e^{\frac{1}{2e^2}\int trF^2 dx} P\left(tre^{\int_\gamma A}\right), \tag{7}$$

and requires the invariance of the right hand side under $A_\mu \to A_\mu + \delta A_\mu$. This change induces two effects; a change in the action (proportional to $\nabla_\mu F_{\mu\nu}$), and a change in the holonomy associated with the loop γ, which involves the product of the holonomies associated with the closed loops γ splits into if it happens to have a self intersection. The cancellation of these two changes generates the Migdal-Makeenko equation.

In the MME, the information about the Gauge theory is encoded in the Loop-space Laplacian appearing in (4) because this operator represents the change in the action under an infinitesimal variation of the gauge field. The term that appears on the right hand side of the MME is a sort of universal piece that is independent of the gauge theory that one might be studying. This universal term is a measure of the violation of the chiral equations of motion at the quantum level, its absence would lead us back to classical Yang-Mills theory. Hence it is necessary to understand the nature of this universal term, without which we can have no hope of understanding phenomena like confinement in quantum Yang-Mills theory.

In the present paper, we interpret this universal term as an anomaly [**?**] arising form the change in the Yang-Mills measure under a transformation that enables us to recast the theory directly in the language of objects defined on the loop space (e.g. $U(\gamma)$). Hence the structure of Yang-Mills theories in loop spaces is reminiscent of WZW models, with the right hand side of (5) representing the WZ term.

We accomplish the change of variables $A \to U(\gamma)$ through the action of some abstract Lie algebra generators $L(\gamma)$ on the connection one form; and we show that the MME is the condition for the invariance of the Yang-Mills generating functional $Z(=\int D[A\mu] e^{\frac{1}{2e^2}\int trF^2})$ under the action of these generators, i.e. $L(\gamma)Z = 0 \Rightarrow$ MME. We show further, that the expectation value of $L(\gamma)$ on the Yang-Mills measure produces the universal term present in the MME and we also observe that this anomalous term has a geometric meaning as an element of the first cohomology of this Lie algebra (the algebra of loop substitutions). We also prove that the effect of these $L(\gamma)$s on the gauge fixing and ghost part of the action is equivalent to a total BRS variation, which is an explanation for why no explicit gauge fixing is required in the MME.

2 The Change of Variables, and the Lie Algebra of Loop Substitutions

The change in the Gauge field that we talked about in the preceding section, can be thought of as the action of an abstract operator $L(\gamma)$(associated with the loop γ) on A_μ, defined as,

$$L(\gamma)A_\mu(x) = -\epsilon\delta(x - \gamma(0))\dot{\gamma}_\mu(0)U(\gamma) \tag{8}$$

The composition of two such generators is also defined if the two loops have a space-time point in common:

$$L(\gamma_2)L(\gamma_1) = -L(\gamma_{2,0t} \circ \gamma_1 \circ \gamma_{2,t0})\dot{\gamma_2}(t)\dot{\gamma_1}(0)\delta(\gamma_2(t) - \gamma_1(0))dt \tag{9}$$

The composition above is well defined if $\gamma_2(t) = \gamma_1(0)$ for some value of t. When the condition is satisfied then the product of the generators corresponds the generator associated with the composed curve starting at $\gamma_2(0)$, going up to the point $\gamma_1(0)$ along γ_2, then coming back to $\gamma_1(0)$ by going around γ_1, and then back to $\gamma_2(0)$ along γ_2.

So these abstract generators form an algebra. In-fact, they form a Lie Algebra (a generalization of the free Lie algebra [?]) with the following lie bracket:

$$\begin{aligned}[L(\gamma_2), L(\gamma_1)] \quad = \quad & \int_0^1 dt[L(\gamma_{1,0t} \circ \gamma_2 \circ \gamma_{1,t0})\dot{\gamma_1}(t)\dot{\gamma_2}(0)\delta(\gamma_1(t) - \gamma_2(0)) - \\ & L(\gamma_{2,0t} \circ \gamma_1 \circ \gamma_{2,t0})\dot{\gamma_2}(t)\dot{\gamma_1}(0)\delta(\gamma_2(t) - \gamma_1(0))]\end{aligned} \tag{10}$$

It is easy to check that given the above Lie bracket,the Jacobi identity is also satisfied by the generators.

The generators have a natural action on the Wilson Loops, which is given below.

$$L(\gamma_1)W(\gamma) = \int_0^1 dtW(\gamma_{0t} \circ \gamma_1 \circ \gamma_{t0})\delta(\gamma(t) - \gamma_1(0))\dot{\gamma}_\mu(t)\dot{\gamma}_{1\mu}(0)dt \tag{11}$$

3 Migdal-Makeenko Equation Revisited

Given the Yang-Mills generating functional, we now show that, the Migdal-Makeenko equation is equivalent to the condition $L(\gamma)Z = 0$, and the universal term in the loop-space equations is the expectation value of the change in the measure of integration in Z under the action of $L(\gamma)$.

The change of variables we are concerned with here is,

$$A^a_{\mu b}(\gamma(0)) \to A'^a_{\mu b}(\gamma(0)) = A^a_{\mu b}(\gamma(0) - \epsilon\left(L(\gamma)A^a_{\mu b}(\gamma(0))\right) = A^a_{\mu b}(\gamma(0)) + \epsilon\dot{\gamma}_\mu(0)U(\gamma)^a_b \tag{12}$$

Or to express the same thing in words, only the gauge field at the space-time point corresponding to the initial point of the loop γ undergoes a change proportional to the parallel transport operator associated with the loop γ. The gauge fields at all the other space-time points are left unchanged. Now;

$$L(\gamma)Z \quad = \int (L(\gamma)D[A\mu])\, e^{\left(\frac{1}{2e^2}\int trF^2dx\right)} + \left\langle L(\gamma)\frac{1}{2e^2}\int trF^2dx\right\rangle$$

$$= \langle detJ - 1\rangle + \left\langle L(\gamma)\frac{1}{2e^2}\int trF^2 dx\right\rangle, \quad (13)$$

Where J is the Jacobian corresponding to the change of variables (11). The Jacobian can be written explicitly as follows,

$$\begin{aligned} J^{ac\gamma(0)\nu}_{bdx\mu} &= \frac{\delta A'^a_{\mu b}(\gamma(0))}{\delta A^d_{\nu c}(x)} \\ &= \delta(\gamma(0)-x)\delta^\nu_\mu\delta^a_d\delta^c_b + \epsilon\dot\gamma_\mu(0)\int{}^1_0 dt U(\gamma_{0t})^a_d\delta(\gamma(t)-x)U(\gamma_{t0})^c_b\dot\gamma_\nu(t)dt \end{aligned} \quad (14)$$

In deriving (14), we have used the relation,

$$\frac{\delta A^a_{\mu,b}(x)}{\delta A^d_{\nu,c}(y)} = \delta(x-y)\delta^\nu_\mu\left(\delta^a_d\delta^c_b + \frac{1}{N}\delta^a_b\delta^d_c\right), \quad (15)$$

and have dropped the $O(1/N)$ terms.

Suppressing the color, Lorentz and space-time indices, it is possible to think of J as a tensor, i.e. $J = 1+\epsilon R$, where R corresponds to the tensor in (13) that involves the product of the two parallel transport operators.
Hence,

$$\begin{aligned} \langle detJ - 1\rangle &= \left\langle e^{trlogJ} - 1\right\rangle = -\epsilon\langle trR\rangle = -\epsilon\left\langle R^{ab\mu\gamma(0)}_{ba\mu\gamma(0)}\right\rangle \\ &= -\epsilon\int_0^1 W(\gamma_{0t})W(\gamma_{t0})\delta(\gamma(t)-\gamma(0))\dot\gamma_\mu(t)\dot\gamma_\mu(0)dt, \end{aligned} \quad (16)$$

which is nothing but the universal term in the MME. This calculation shows that it is indeed an anomaly (in the sense of Fujikawa [?]) because it arises from the change in the measure of integration under a non-linear change of variables.

The second term in (13) can be evaluated using the action of $L(\gamma)$ on the gauge field, as given in (12). It follows easily that,

$$\left\langle L(\gamma)\frac{1}{2e^2}\int trF^2dx\right\rangle = -\frac{\epsilon}{e^2}\langle tr(\nabla_\mu F_{\mu\nu}(\gamma(0))\dot\gamma_\nu(0)U(\gamma))\rangle = -\frac{\epsilon}{e^2}\frac{\partial^2}{\partial\gamma^2_\mu(0)}W(\gamma) \quad (17)$$

So substituting the results of equations (16) and (17) back in (13), we see that the Migdal Makeenko equation is the condition for the invariance of the Yang-Mills generating functional under the action of the generators of loop substitutions.

4 Cohomology

In this section we shall try to extract the geometric meaning of the anomaly in the Loop-space equations. The Migdal-Makeenko equations represent the invariance of the Yang-Mills generating functional under the action of the Lie algebra generators $L(\gamma)$. One may regard these generators as vector fields, and think of the anomalous term as resulting from the action of an abstract one form ω (the anomaly one form) on these generators. Or more explicitly,

$$\omega(L(\gamma)) = \int_0^1 dt W(\gamma_{0t}) W(\gamma_{t0}) \delta(\gamma(0) - \gamma(t)) \dot{\gamma}_\mu(t) \dot{\gamma}_\mu(0). \tag{18}$$

We claim that γ is an element of the first cohomology of the lie algebra described above. We first note that ω is closed, i.e. $d\omega = 0$. Or in other words;

$$L(\gamma_2)\omega(L(\gamma_1)) - L(\gamma_1)\omega(L(\gamma_2)) = \omega\left([L(\gamma_2), L(\gamma_1)]\right) \tag{19}$$

Proof:
We start with the right hand side of (19).

$$\begin{aligned}&\omega\left([L(\gamma_2), L(\gamma_1)]\right) = \\ &\omega\left(\int_0^1 ds L(\gamma_{1,0s} \circ \gamma_2 \circ \gamma_{1,s0}) \delta(\gamma_2(0) - \gamma_1(s)) \dot{\gamma}_{1\mu}(s) \dot{\gamma}_{2\mu}(0) ds\right) - (\gamma_1 \to \gamma_2)\end{aligned} \tag{20}$$

Now looking at the first term on the right hand side of the equation above we notice that ω is being evaluated on the generator corresponding to $\gamma_{1,0s} \circ \gamma_2 \circ \gamma_{1,0s}$; assuming that $\gamma_1(s) = \gamma_2(0)$, for some s, (this has to be true for the equation to be non-trivial). So by the definition of ω, this will be non-zero if the composed curve self intersects itself at $\gamma_1(0)$. Hence we require that either $\gamma_1(t) = \gamma_1(0)$, or that $\gamma_2(t) = \gamma_1(0)$ for some value of the parameter t. It is easy to see that the product of Wilson loops obtained when the second condition holds good is symmetric under $\gamma_1 \to \gamma_2$and hence does not contribute to the commutator. Hence we shall focus only on the first case, i.e. $\gamma_1(t) = \gamma_1(0)$. Now depending on whether $t < s$ or $t > s$, we have;

$$\begin{aligned}&\omega\left([L(\gamma_2), L(\gamma_1)]\right) = \\ &\int_0^1 dt \int_0^1 ds \left(\delta(\gamma_(t) - \gamma_1(0)) \delta(\gamma_1(s) - \gamma_2(0))\right) \dot{\gamma}_{1\mu}(t) \dot{\gamma}_{1\mu}(0) \dot{\gamma}_{1\nu}(s) \dot{\gamma}_{2\nu}(0)\right) \\ &(W(\gamma_{1,0t}) W(\gamma_{1,ts} \circ \gamma_2 \circ \gamma_{1,st}) \theta(t < s) + \\ &W(\gamma_{1,0s} \circ \gamma_2 \circ \gamma_{1,s0}) W(\gamma_{1,0t}) \theta(t > s)) - \\ &(\gamma_1 \to \gamma_2).\end{aligned} \tag{21}$$

Now we see that the left hand side of (19) is

$$\begin{aligned}&L(\gamma_2)\omega(L(\gamma_1)) - (\gamma_1 \to \gamma_2) = \\ &\int_0^1 dt (\delta(\gamma_1(t) - \gamma_1(0)) \dot{\gamma}_{1\nu}(t) \dot{\gamma}_{1\nu}(0) \\ &\left((L(\gamma_2) W(\gamma_{1,0t})\right) W(\gamma_{1,t0}) + W(\gamma_{1,0t}) \left(L(\gamma_2) W(\gamma_{1,t0})\right)) - \\ &(\gamma_1 \to \gamma_2)\end{aligned} \tag{22}$$

Now using the natural action of the generators on the Wilson loops (as given in (10)), we obtain,

$$L(\gamma_2)\omega(L(\gamma_1)) - (\gamma_1 \to \gamma_2) =$$

$$\int_0^1 dt \int_0^1 ds \left(\delta(\gamma_(t) - \gamma_1(0))\delta(\gamma_1(s) - \gamma_2(0))\right)\dot{\gamma}_{1\mu}(t)\dot{\gamma}_{1\mu}(0)\dot{\gamma}_{1\nu}(s)\dot{\gamma}_{2\nu}(0))$$
$$(W(\gamma_{1,0t})W(\gamma_{1,ts} \circ \gamma_2 \circ \gamma_{1,st})\theta(t < s) +$$
$$W(\gamma_{1,0s} \circ \gamma_2 \circ \gamma_{1,s0})W(\gamma_{1,0t})\theta(t > s)) -$$
$$(\gamma_1 \to \gamma_2). \tag{23}$$

Noting the equality of the right hand sides of (21) and (22), we conclude that $d\omega = 0$.

Even without constructing an explicit proof, it is easy to convince oneself that ω cannot be exact because of the presence of the logarithm in its definition; $\omega =< \exp(tr \log J) - 1 >$.

Hence the one form ω related to the anomalous term in the Migdal-Makeenko equation is an element of the first cohomology of the Lie algebra of Loop substitutions.

Comments on the Connection to the collective field formalism: This approach towards understanding the loop equations is closely tied to previous advances made by Jevicki and Sakita [**?**, **?**, **?**, **?**] towards formulating Yang-Mills theory on loop spaces. The loop space theory in their approach was defined by the generating functional (Z_{JS}) defined by[**?**];

$$Z_{JS} = \int [DW\gamma] e^{-S + \log J_{JS}}, \tag{24}$$

Where J_{JS} is the Jacobian arising from transforming to loop space variables. This Jacobian is different from the ones considered in the present paper because the changes of variables considered in the collective field approach are not necessarily infinitesimal. This Jacobian was formally shown to satisfy the following equation.

$$\frac{\delta \log J_{JS}}{\delta W(\gamma)} = -\Sigma_{\gamma'} \Omega^{-1}(\gamma, \gamma') w(\gamma'), \tag{25}$$

where,

$$\Omega(\gamma, \gamma') = \int d^4x \frac{\delta W(\gamma)}{\delta A_\mu^a(x)} \frac{W(\gamma')}{\delta A_\mu^a(x)}, \tag{26}$$

and,

$$w(\gamma) = -\int d^4x \frac{\delta^2 W(\gamma)}{\delta A_\mu(x)^2}. \tag{27}$$

The Migdal - Makeenko equations can now be thought of as the condition for the invariance of Z_{JS} under the action of $L(\gamma)$. Or in more precise terms, the relation between the Jacobian appearing in the collective field formalism and the anomaly one form may be expressed formally as the following equation,

$$\omega(L(\gamma)) = - < L(\gamma) \log J_{JS} > . \tag{28}$$

Although the precise definition of this Jacobian in the continuum remains a reasonably open question, (though it has been understood in simpler lattice like theories [**?**, **?**, **?**]), its deformation under the action of the algebra introduced in this paper is a much better behaved quantity (namely the anomaly one form), for which we now have a cohomological interpretation.

5 Loop substitution generator, and BRS invariance

It is well known that the loop space equations do not require explicit gauge fixing. In this final section we realize this very desirable feature of the loop equations in the formalism set up in this paper by showing that the effect of the generator of loop-substitution on the Gauge-fixing and Ghost part of the action is a total BRS variation.

The gauge fixed form of the Yang-Mills action is,

$$S = \int \left[-\frac{1}{4} trF^2 - \frac{1}{2\xi} tr(\partial^\mu A_\mu)^2 - itr(\partial^\mu \bar{C})(\nabla_\mu C) \right]. \tag{29}$$

The action is invariant under the BRS transformations,

$$\delta A = \lambda \nabla_\mu C, \tag{30}$$

$$\delta C = \frac{i\lambda}{2} g[\bar{C}, C]_+, \delta \bar{C} = -i\lambda \frac{1}{\xi} \partial^\mu A_\mu, \tag{31}$$

where, λ is the BRS parameter.

It is easy to see that,

$$\delta U(\gamma) = ig\lambda [C(\gamma(0)), U(\gamma)]. \tag{32}$$

Now using the definition of the action of $L(\gamma)$, we see that,

$$\begin{aligned}
&-i\lambda L(\gamma)(S_{Gauge-Fixing} + S_{Ghost}) = \\
&i\lambda L(\gamma) \int \left[\frac{1}{2\xi} tr(\partial^\mu A_\mu)^2 + itr(\partial^\mu \bar{C})(\nabla_\mu C) \right] = \\
&-i\lambda tr \left[\frac{1}{\xi} \partial^\nu [\partial^\mu A_\mu(\gamma(0))] U(\gamma) + g[\partial_\nu \bar{C}(\gamma(0))][C(\gamma(0)), U(\gamma)] \right] \dot{\gamma}_\nu(0) = \\
&\delta tr \left(\partial^\nu \bar{C}(\gamma(0)) U(\gamma) \right) \dot{\gamma}_\nu(0),
\end{aligned} \tag{33}$$

which is a total BRS variation.

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