Symmetries of fluid dynamics with polytropic exponent*

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Abstract

The symmetries of the general Euler equations of fluid dynamics with polytropic exponent are determined using the Kaluza-Klein type framework of Duval et *al*. In the standard polytropic case the recent results of O'Raifeartaigh and Sreedhar are confirmed. Similar results are proved for polytropic exponent $\gamma = -1$, which corresponds to the dimensional reduction of d-branes. The relation between the duality transformation used in describing supernova explosion and Cosmology is explained.

1 Introduction

The amazing similarity of supernova explosion and plasma implosion has been explained not less amazingly by Drury and Mendonça [1], who pointed out that the two situations can be related by the "duality" transformation $\Sigma\ : t \to -1/t,\ \mathbf{x} \to \mathbf{x}/t$. This strange-looking transformation belongs to the $\mathrm{SL}(2, \mathbf{R})$ group generated by the dilatations, $\mathcal{D} : t \to \delta^2 t, \mathbf{x} \to \delta\mathbf{x}$, expansions, $\mathcal{K}\ : t \to t/1 + \kappa t,\ \mathbf{x} \to \mathbf{x}(1 + \kappa t)^{-1}$, and time-translation, $\mathcal{H} : t \to t + \epsilon$, which are indeed symmetries of a free non-relativistic particle [2, 3]. In fact, $\Sigma = \mathcal{H}_{-1} \circ \mathcal{K}_1 \circ H_{-1}$.

Motivated by the results of Drury and Mendonça, O'Raifeartaigh and Sreedhar [4] performed a systematic study of the symmetries of the Euler

*Dedicated to the memory of Lochlainn O'Raifeartaigh, our late friend and teacher.

equations of fluid dynamics,

$$
\begin{aligned}
D\rho &= -\rho\vec{\nabla}\cdot\mathbf{u}, &(1.1)\\
\rho\, D\mathbf{u} &= -\Lambda(\gamma-1)\vec{\nabla}(\chi\rho^{\gamma})+\mathbf{V}, &(1.2)\\
D\chi &= 0, &(1.3)
\end{aligned}
$$

where D is the convective derivative, $D=\partial_t+\mathbf{u}\cdot\vec{\nabla}$, and the fields ρ and $\mathbf{u}$ are the density and the velocity. $\mathbf{V}$ is the viscosity term, with components

$$
V_i=\partial_j\left(\eta\left(\partial_j u_i+\partial_i u_j-\frac{2}{d}\delta_{ij}\partial_k u_k\right)\right)+\partial_i\left(\xi\partial_k u_k\right), \tag{1.4}
$$

where d is the spatial dimension, ξ and η represent the bulk and shear viscosity fields, respectively. γ is the polytropic exponent and Λ is the coupling constant of a potential, $U(\rho)=\Lambda\rho^{\gamma}$. The field χ is related to the energy density ϵ by $\epsilon=\chi\rho^{\gamma}$.

O'Raifeartaigh and Sreedhar consider first the sub-class of (1.1)-(1.3) characterised by *(i)* the absence of viscosity terms, $\mathbf{V}=0$; *(ii)* the dynamical field χ is choosen to be $\chi=1$; *(iii)* the motion is assumed irrotational, $\mathrm{rot}\,\mathbf{u}=0$. Then they show that when the polytropic exponent takes the standard value $\gamma=1+2/d$, the equations (1.1)-(1.3) are invariant w.r.t. Schrödinger transformations, composed of Galilei transformations, augmented by dilatations and expansions [2]. When the conditions *(i)*, *(ii)* and *(iii)* are relaxed, the expansions are generally broken by the viscosity term; dilatations remain, however, symmetries [4].

Similar questions were investigated by Bordemann and Hoppe, and Jevicki [5], and by Jackiw, Polychronakos, and Bazeia [6, 7], who found that the dimensional reduction of d-brane theory yields a viscosity–free, isentropic and irrotational hydrodynamical model called the Chaplygin gas, eqns. (1.1)-(1.3) with $\mathbf{V}=0$ and $\chi=1$ and with effective potential $U\propto 1/\rho$. Remarkably, their system admits a hidden Poincaré symmetry [5, 6, 7], composed of the Galilei transformations, augmented by $(d+1)$ generators we called time-dilatation and antiboost [8].

In this Letter, we combine and generalize these results in a unified framework. First, we confirm the results of O'Raifeartaigh et *al.* by dropping condition *(iii)* right on from the beginning. Then we extend the d-brane results in [5, 6, 7] showing that, for $U\propto 1/\rho$, the symmetries of the general equations (1.1)-(1.3) with conditions *(i)* and *(ii)* alone still admit a Poincaré symmetry. Viscosity breaks part of this large symmetry. There remains, however, time-dilatation, $\Delta\ : t\to e^{\alpha}t,\ \mathbf{x}\to\mathbf{x}$, analogous to dilatations, $\mathcal{D}$, in the standard case.

The relation of the duality transformation Σ and newtonian cosmology is also explained. Although our results could also be obtained in a classical approach [2, 4, 7], we found it more convenient to use Duval's Kaluza Klein–type framework [3], which sheds a new light on the arisal of these symmetries.

2 Symmetries of the Euler equations

The simplest way to confirm the result of O'Raifeartaigh and Sreedhar [4], is to consider [8], Sect. 2, p. 224 (see also [9]), the stress–energy tensor $T^{\alpha\beta}$. In the absence of viscosity, $\mathbf{V}=0$ and for $\chi=1$, they are given, e. g., in Eq. (2.2) in the first reference of [6], as

$$T^{00}=\rho\frac{\mathbf{u}^2}{2}+U(\rho),\qquad T^{ij}=\rho u^i u^j-\delta^{ij}(U-\rho\partial_\rho U),\tag{2.1}$$

where $\partial_\rho U$ is the enthalpy[1]. Next recall (e. g. [10], Eq. (2.261)) the criterion of Schrödinger symmetry:

$$2T^{00}=\sum_i T^{ii},\tag{2.2}$$

which replaces, in the non-relativistic context, the familiar condition for relativistic conformal invariance, *viz.* $T^\mu_{\ \mu}=0$. With the above expression for T^{00} and T^{ij}, we get a differential equation for U, namely $\rho\partial_\rho U=(2/d+1)U$ or $U=\Lambda\rho^{1+2/d}$, which is the result in [4].

More generally, let us first consider the sub-class of (1.1)-(1.3) with conditions (i) and (ii) alone. Using the Clebsch parametrization [11], $\mathbf{u}=\vec\nabla\phi-\nu\vec\nabla\theta$, provides us with a local lagrangian theory [4]. Then eliminating the Lagrange multiplier ν yields the equations of motion

$$(E_\gamma)\quad\left\{\begin{array}{l}
\partial_t\rho+\partial_k\left(\rho\partial_k\phi+\frac{\rho}{\ }|\vec\nabla\theta|^2\partial_k\theta\left(\partial_t\theta+\vec\nabla\theta\cdot\vec\nabla\phi\right)\right)=0,\\
[3mm]\partial_t\phi+\frac{1}{2}|\vec\nabla\phi|^2+\frac{1}{\ }2|\vec\nabla\theta|^2\left(\partial_t\theta+\vec\nabla\theta\cdot\vec\nabla\phi\right)^2-\gamma\Lambda\rho^{\gamma-1}=0,\\
[3mm]\partial_t\left(\frac{\rho}{|\vec\nabla\theta|^2}\left(\partial_t\theta+\vec\nabla\theta\cdot\vec\nabla\phi\right)\right)+\\
[2mm]\qquad\partial_k\left(\frac{\rho\,\partial_k\phi}{\ }|\vec\nabla\theta|^2\left(\partial_t\theta+\vec\nabla\theta\cdot\vec\nabla\phi\right)-\frac{\rho\,\partial_k\theta}{\ }|\vec\nabla\theta|^4\left(\partial_t\theta+\vec\nabla\theta\cdot\vec\nabla\phi\right)^2\right)=0.
\end{array}\right.\tag{2.3}$$

The velocity field $\mathbf{u}$ here is expressed in terms of θ and ϕ by $\mathbf{u}=\vec\nabla\phi-(\vec\nabla\theta/|\vec\nabla\theta|^2)\left(\partial_t\theta+\vec\nabla\theta\cdot\vec\nabla\phi\right)$.

Below we analyse the symmetries of (2.3) in the Kaluza-Klein type framework of [3]. Non-relativistic space-time, Q, has coordinates $(\mathbf{x},t)$, and can also be obtained from one higher dimensional manifold M with coordinates $(\mathbf{x},t,s)$, when the coordinate s is factored out. M is endowed with the flat Lorentz metric $d\mathbf{x}^2+2dtds$; $\Xi=\partial_s$ light-like vector field. M is a relativistic spacetime, upon which we consider the real fields R, Θ and Φ. Inspired by

[1] It is worth noting that, although it has been derived assuming irrotationality, (2.1) actually provides us with a conserved energy-momentum tensor in the general case, as it can be verified by a directly, using the Euler equations.

(2.3), we postulate

$$
(\mathcal{E}_\gamma) \quad \left\{ \begin{array}{l} \partial_\mu \left(\frac{R}{2}\,\partial^\mu \Phi + \frac{R\,\partial^\mu \Theta}{}(\partial_\sigma \Theta)(\partial^\sigma \Theta)\partial_\nu \Theta\,\partial^\nu \Phi \right) = 0 \\ [3,5mm]\partial_\mu \Phi\,\partial^\mu \Phi + \frac{1}{(\partial_\mu \Theta)(\partial^\mu \Theta)}\,(\partial_\nu \Phi\,\partial^\nu \Theta)^2 - \gamma \Lambda R^{\gamma-1} = 0, \\ [3,5mm]\partial_\mu \left(R\,\partial^\mu \Phi\,\frac{\partial_\sigma \Phi\,\partial^\sigma \Theta}{}(\partial_\nu \Theta\,\partial^\nu \Theta) - \frac{R}{2}\,\partial^\mu \Theta\,\frac{(\partial_\sigma \Phi\,\partial^\sigma \Theta)^2}{}(\partial_\nu \Theta\,\partial^\nu \Theta)^2 \right) = 0. \end{array} \right. \tag{2.4}
$$

To complete our Kaluza-Klein framework, we need to establish a correspondance between the systems (2.3) and (2.4). Below we define, for both critical values of γ, a judicious (and different) relation between the fields on M and those on Q, such that the relativistic system $(\mathcal{E}_\gamma)$ projects to the non-relativistic one (E_γ). Then the symmetries of the latter arise by projection.

• Let us first consider the standard case, $\gamma = 1 + 2/d$. If the fields R, Θ and Φ are of the particular form

$$
R(\mathbf{x},t,s) = \rho(\mathbf{x},t), \quad \Theta(\mathbf{x},t,s) = \theta(\mathbf{x},t) \quad \Phi(\mathbf{x},t,s) = \phi(\mathbf{x},t) + s \tag{2.5}
$$

(which is in fact the usual equivariance condition [3]), then the equations $(\mathcal{E}_{1+2/d})$ project to $(E_{1+2/d})$.

Now we determine the symmetries. One shows readily that if the fields R, Φ and Θ are solutions of equations $(\mathcal{E}_{1+2/d})$, then their images under a conformal transformation of M, $\varphi^\star g = \Omega^2 g$, implemented as $\tilde{R} = \Omega^d\,\varphi^\star R$, $\tilde{\Phi} = \varphi^\star \Phi$ and $\tilde{\Theta} = \varphi^\star \Theta$, also satisfy the same equations. They are hence symmetries for (2.4). To make the transformed fields equivariant in the sense (2.5), however, we must restrict ourselves to transformations which preserve the "vertical" vector field Ξ. Their action on M,

$$
\left\{ \begin{array}{l} \tilde{\mathbf{x}} = \vec{\gamma} - \vec{\beta} t + \frac{\delta\,\mathcal{R}\mathbf{x}}{(1+\kappa t)}, \\ [2mm]\tilde{t} = \frac{\epsilon + \delta^2\,t}{(1+\kappa t)}, \\ [2mm]\tilde{s} = s + \lambda(t,\mathbf{x}), \qquad \lambda(t,\mathbf{x}) \equiv \vec{\beta}\cdot\mathbf{x} - \frac{1}{2}|\vec{\beta}|^2\,t + \frac{\kappa}{2}\frac{|\mathbf{x}|^2}{(1+\kappa t)}, \end{array} \right. \tag{2.6}
$$

(where $\mathcal{R} \in so(2)$, $\vec{\beta}, \vec{\gamma}, \epsilon, \kappa$ and δ are interpreted as rotation, boost, space translation, time translation, expansion and dilatation) projects into non-relativistic space-time, Q, according to the classical Schrödinger transformations [2, 3]. The action on fields are obtained by using the previous relations. Setting $M = (\partial \tilde{x}_i / \partial x_j)$, we get

$$
\left\{ \begin{array}{l} \tilde{\rho}(t,\mathbf{x}) = \frac{\delta^d}{(1+\kappa t)^d}\,\rho(\tilde{t},\tilde{\mathbf{x}}) = \det(M)\,\rho(\tilde{t},\tilde{\mathbf{x}}), \\ [2.8mm]\tilde{\phi}(t,\mathbf{x}) = \phi(\tilde{t},\tilde{\mathbf{x}}) + \lambda(t,\mathbf{x}), \\ [2mm]\tilde{\theta}(t,\mathbf{x}) = \theta(\tilde{t},\tilde{\mathbf{x}}). \end{array} \right. \tag{2.7}
$$

Since the Ξ-preserving symmetries of (2.4) project to symmetries, we conclude that, in the viscosity–free case $\xi = \eta = 0$, the (not necessarily irrotational) system has a full Schrödinger symmetry, as stated above.

Another way of reaching this result, closer in spirit to our first proof, is to observe that Eqns. (2.4) derive from the relativistic Action

$$S = \int \Big(R\partial_\mu \Phi \, \partial^\mu \Phi + \frac{R}{\partial_\mu \Theta \, \partial^\mu \Theta} \left(\partial_\sigma \Phi \, \partial^\sigma \Theta\right)^2 - 2\Lambda R^\gamma \Big) \sqrt{g} d^{d+2} x, \tag{2.8}$$

where, for convenience, we moved to a general Lorentz metric $g_{\mu\nu}$ on M. The associated energy-momentum tensor $\mathcal{T}_{\mu\nu} = 2\delta S/\delta g^{\mu\nu}$, i. e.,

$$\begin{aligned} \mathcal{T}_{\mu\nu} &= R\,\partial_\mu \Phi \,\partial_\nu \Phi - \frac{R}{2}(\partial_\sigma \Phi \,\partial^\sigma \Phi)\, g_{\mu\nu} + \Lambda\, R^\gamma g_{\mu\nu} \\ &+ \frac{R}{\partial_\sigma \Theta \,\partial^\sigma \Theta} \left(\partial_\mu \Phi \,\partial_\nu \Theta + \partial_\mu \Theta \,\partial_\nu \Phi\right) \left(\partial_\sigma \Theta \,\partial^\sigma \Phi\right) \\ &- R\,\partial_\mu \Theta \,\partial_\nu \Theta \frac{(\partial_\sigma \Theta \,\partial^\sigma \Phi)^2}{(\partial_\sigma \Theta \,\partial^\sigma \Theta)^2} - \frac{R}{2} g_{\mu\nu} \frac{(\partial_\sigma \Theta \,\partial^\sigma \Phi)^2}{} \left(\partial_\sigma \Theta \,\partial^\sigma \Theta\right), \end{aligned} \tag{2.9}$$

(which generalizes the expression given in [8]) is seen to be symmetric and conserved. Relativistic conformal invariance requires the vanishing of its trace,

$$\sum_\mu \mathcal{T}^\mu_\mu = \Lambda d\, R^\gamma \left(\gamma - [1 + \frac{2}{d}]\right) = 0, \tag{2.10}$$

which yields the correct polytropic exponent $\gamma = 1 + 2/d$ once again. To conclude, the Schrödinger group is the Ξ-preserving part of the (relativistic) conformal group. It is worth mentionning that the ti, it and ij components of the relativistic $\mathcal{T}^{\mu\nu}$ are related to the non-relativistic $T^{\alpha\beta}$ by surface terms, and that the non-relativistic trace condition (2.2) follows from $-T^{00} = \mathcal{T}^s_s = \mathcal{T}^t_t$.

Let us now return to the general equations (1.1)-(1.3) including viscosity. We first determine how $\mathbf{u}$ transforms. Let us define on M an $\mathbf{s}$-independent vector $(k_\nu) \equiv (k_t, \mathbf{u}, k_s)$,

$$k_\nu = \partial_\nu \Phi - \frac{\partial_\nu \Theta}{(\partial_\sigma \Theta \,\partial^\sigma \Theta)} \left(\partial_\mu \Theta \,\partial^\mu \Phi\right). \tag{2.11}$$

Using the transformation rule on M of this vector, $\tilde{k}_\mu = (\partial \tilde{x}^\nu / \partial x^\mu)\, k_\nu$, the action on $\mathbf{u}$, the space component of k_ν, is obtained, namely

$$\tilde{\mathbf{u}}\,(t, \mathbf{x}) = [\mathcal{R}\,(\det M)^{1/d}]\, \mathbf{u}\,(\tilde{t}, \tilde{\mathbf{x}}) + \vec{\nabla} \lambda. \tag{2.12}$$

It is interesting to observe that the restriction (ii), *viz.* $\chi = 1$, can actually be relaxed: the viscosity–free Euler equations are invariant w.r.t. transformations (2.6) and (2.7), whenever $\tilde{\chi} = \chi$. The first term in (1.2),

$\rho D\mathbf{u}$, transforms in fact into $(\det M)^{1+3/d}\rho D\mathbf{u}$, and if $\tilde{\chi}=\chi$, then the term $\vec{\nabla}(\chi\rho^{1+2/d})$ becomes $(\det M)^{1+3/d}\vec{\nabla}(\chi\rho^{1+2/d})$ so that eqn. (1.2) merely gets multiplied by an overall factor. The other equations are plainly invariant.

Now, if $\tilde{\eta}=(\det M)\,\eta$ and $\tilde{\xi}=(\det M)\,\xi$, the viscosity term transforms as

$$\begin{aligned}&V_i\to\tilde{V}_i=(\det M)^{1+\frac{3}{d}}\,V_i+\\&(\det M)^{1+\frac{1}{d}}\left[\tilde{\partial}_i(\xi\Delta\lambda)+\tilde{\partial}_j\left(\eta[2\partial_i\partial_j\lambda-\tfrac{2}{d}\delta_{ij}\Delta\lambda]\right)\right].\end{aligned}\tag{2.13}$$

Invariance of Eqn. (1.2) requires the second term here to vanish. For λ in (2.6), this is automatical for the shear viscosity field η. The bulk viscosity field, ξ, however, breaks the expansions, leaving us with dilatational symmetry only. For time-independent fields one also have time-translations. (This is consistent, owing to $\{\mathcal{H},\mathcal{D}\}=\mathcal{H}$). When the viscosity fields only depend on time, though, the residual symmetry includes the expansions but break the time-translational invariance. These results confirm the conclusion of [4] obtained in a rather different way.

• Next, we consider the d-brane potential, $\gamma=-1$. The "non-relativistic conformal symmetries" (i. e. dilatations and expansions) are plainly broken. However, when the motion is irrotational and viscosity–free, this $(d+1)$ dimensional non-relativistic model admits the $(d+1,1)$-dimensional Poincaré group as symmetry [5, 6, 7]. Generalising the results and the procedure presented in [8], now we show that the not necessarily irrotational but still viscosity–free system (E_{-1}) is Poincaré symmetric. Our previous equivariance condition (2.5) is seen to be be too restrictive and we propose to relate instead the fields defined on M and Q according to

$$\left\{\begin{array}{l}\rho(\mathbf{x},t)=R\left(\mathbf{x},t,-\phi(\mathbf{x},t)\right)\,\partial_s\Phi\left(\mathbf{x},t,-\phi(\mathbf{x},t)\right),\\ {[1,2mm]}\Phi\left(\mathbf{x},t,-\phi(\mathbf{x},t)\right)=0,\\ {[1.2mm]}\Theta(\mathbf{x},t,s)=\theta(\mathbf{x},t).\end{array}\right.\tag{2.14}$$

Here the point $(t,\mathbf{x},-\phi(t,\mathbf{x}))$ in M is defined as a zero of the field $\Phi=0$. Note that R can depend on the s variable; however, ρ is already defined Q. It is easy to see that this condition is more general than classical equivariance (2.5). As previously, $(\mathcal{E}_{-1})$ with the constraint (2.14), project into Q as (E_{-1}). Let us insist that this projection is only possible for the d-brane potential [8]. The advantage of the general equivariance is that, now, we can consider transformations which do not necessarily preserve Ξ. But the particular form of our potential restricts ourselves to consider only isometric transformations. These latter are symmetries of equations $(\mathcal{E}_{-1})$ coupled to the constraint (2.14). The action of the Ξ-preserving isometries lead to the extended Galilei transformations. The non-preserving part is composed by

$(d+1)$ generators whose action on M is given by [8] :

$$\left\{ \begin{array}{l} \tilde{\mathbf{x}} = \mathbf{x} - \vec{\omega} s \\ \tilde{t} = e^{\alpha}\left(t + \vec{\omega}\cdot\mathbf{x} - \frac{1}{2}|\vec{\omega}|^2 s\right), \\ \tilde{s} = e^{-\alpha} s, \end{array} \right. \tag{2.15}$$

where α and $\vec{\omega}$ are the parameters associated with time dilatation and antiboost, respectively. Our transformations act on fields naturally, as $\tilde{R}(x,t,s) = R(\tilde{\mathbf{x}},\tilde{t},\tilde{s})$, etc. The projection into Q yields [6, 7]

$$\left\{ \begin{array}{l} \tilde{\mathbf{x}} = \mathbf{x} + \vec{\omega}\,\phi(\tilde{\mathbf{x}},\tilde{t}), \\ {[2mm]}\tilde{t} = e^{\alpha}\left(t + \frac{1}{2}\vec{\omega}\cdot(\mathbf{x}+\tilde{\mathbf{x}})\right) \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} \tilde{\rho}(\mathbf{x},t) = \rho(\tilde{\mathbf{x}},\tilde{t})\, J^{-1} \\ {[2mm]}\tilde{\phi}(\mathbf{x},t) = e^{\alpha}\phi(\tilde{\mathbf{x}},\tilde{t}) \\ {[2mm]}\tilde{\theta}(\mathbf{x},t) = \theta(\tilde{\mathbf{x}},\tilde{t}) \end{array} \right. \tag{2.16}$$

where J is the Jacobian of the transformation given by

$$J = e^{\alpha}\left[1 - \sum_k \omega_k\, \tilde{\partial}_k\phi(\tilde{\mathbf{x}},\tilde{t}) - \frac{1}{2}|\vec{\omega}|^2\, \partial_{\tilde{t}}\phi(\tilde{\mathbf{x}},\tilde{t})\right]^{-1}. \tag{2.17}$$

As in the standard case, the vector k_μ (2.11) can be used to determine the transformation on the velocity. But now because of this particular equivariance, the velocity is equal to $\mathbf{u} = (\mathbf{k}/\partial_s\Phi)(t,\mathbf{x},-\phi(t,\mathbf{x}))$ and a similar calculation yields instead

$$\tilde{\mathbf{u}}(t,\mathbf{x}) = J\left[\mathbf{u}(\tilde{t},\tilde{\mathbf{x}}) + \vec{\omega}\left(\tilde{\partial}_t\phi - \frac{\partial_{\tilde{t}}\theta}{\sum(\tilde{\partial}_k\theta)^2}\left(\partial_{\tilde{t}}\theta + \tilde{\partial}_m\theta\,\tilde{\partial}_m\phi\right)\right)\right]. \tag{2.18}$$

As in the standard case, the viscosity term breaks most of the symmetry. A rather tedious calculation shows in fact that, under a Poincaré transformation, the viscosity term (1.4) transforms as

$$\tilde{V}_i = e^{\alpha}\, V_i + F(\vec{\omega},\xi,\eta), \tag{2.19}$$

where $F(\vec{\omega},\xi,\eta)$ is a complicated expression which vanishes for $\vec{\omega},\ \xi,$ or η equal zero. For non-trivial viscosity, this means that the antiboosts are broken. Eq. (1.2) is, however, merely multiplied by e^{α} under $\Delta\ : t \to e^{\alpha} t$: time (rather then non-relativistic) dilatation, Δ, is a residual symmetry.

3 Explosion/implosion duality and cosmology

The clue of Drury and Mendonça [1] is to map, using the "duality transformation" $\Sigma : \tilde{t} = -1/t,\ \tilde{\mathbf{x}} = \mathbf{x}/t$, supernova *explosion* at time $t = 0$ into an *implosion* starting at $\tilde{t} = -\infty$ and evolving to $\tilde{t} = 0$. Then they find that, implementing Σ on the fields as $\tilde{\rho} = a^3\rho$ and $\tilde{\mathbf{u}} = a\,\mathbf{u} - \dot{a}\,\mathbf{x}$, the equations of viscosity–free polytropic hydrodynamical system with $\chi = 1$ are invariant when $a(t) \propto t$ and $\gamma = 5/3$.

Curiously, their Σ appeared before in cosmology. The relation is explained as follows. In the uniformly expanding newtonian cosmological model [12], the gravitational acceleration has the form $\mathbf{g} = -(B/a^3)\mathbf{x}$, where B is a constant related to the scale factor $a(t)$ as $B = -a^2\ddot{a}$. The Hubble constant is $H = \dot{a}/a$, and $\mathbf{g}$ satisfies $\vec{\nabla}\cdot\mathbf{g} = -4\pi G\rho$ (rather than the Einstein equations, as in relativity). Combining this with $\dot{\mathbf{x}} = H\mathbf{x}$ and $\ddot{\mathbf{x}} = \mathbf{g}$ yields

$$(\dot{a})^2 = \frac{2B}{a} - K \qquad \text{and} \qquad \rho = \frac{3B}{4\pi G a^3}, \tag{3.1}$$

where K is another constant now unrelated to space curvature. This non-relativistic model is, however, equivalent to the relativistic Friedmann universe with constant curvature K [13]. The model is also conveniently described [3] by the ("Kaluza–Klein") 5-metric

$$d\mathbf{x}^2 + 2dtds - \frac{B\,\mathbf{x}^2}{a^3}dt^2, \tag{3.2}$$

whose gravitational field equation requires indeed $\triangle(B\mathbf{x}^2 a(t)^{-3}) = 8\pi G\rho$ as above. Now this metric can be conformally mapped to flat space with metric $d\tilde{\mathbf{x}}^2 + 2d\tilde{t}d\tilde{s}$, using

$$\tilde{\mathbf{x}} = \frac{\mathbf{x}}{a}, \qquad \tilde{t} = \int \frac{dt}{a^2}, \qquad \tilde{s} = s + \tfrac{1}{2}H\mathbf{x}^2. \tag{3.3}$$

The (inverse of) (3.3) carries the flat-space hydrodynamical equations into those valid in the expanding universe.

For the choice of Drury and Mendonça $B = 0$, so that their expanding metric (3.2) is flat and has therefore little cosmological interest since then also $\rho = 0$. Ignoring this aspect, we note that the transformation (3.3), which becomes now precisely Σ completed with $s \to s + \mathbf{x}^2/2t$, is a conformal transformation of flat space into itself. The invariance of the Euler equations under Σ follows. This is of course consistent with Σ belonging to the $\mathrm{SL}(2, \mathbf{R})$ invariance group of the free system discussed above. Unfortunately, this symmetry is broken by the viscosity.

Interestingly, the map Σ has also been used to solve planetary motion when the gravitational constant changes inversely with time [14, 3]. It is worth mentionning also that a Friedmann metric containing a perfect fluid with equation of state $p = (\gamma - 1)\rho$ has also been studied [15].

4 Schrödinger fields and the Madelung fluid

Let us conclude with a remark on the well-known Schrödinger invariance of the non-linear Schrödinger equation $i\partial_t\psi = -\triangle\psi/2 + \lambda\,|\psi|^{4/d+1}\psi$. Decomposing the Schrödinger field into module and phase, $\psi = \sqrt{\rho}\,e^{i\phi}$, yields in fact the hydrodynamical system referred to as the Madelung fluid [16],

$$\partial_t\rho \quad + \quad \vec{\nabla}\cdot(\rho\vec{\nabla}\phi) = 0, \tag{4.1}$$

$$\partial_t\phi \quad + \quad \frac{1}{2}|\vec{\nabla}\phi|^2 = -\frac{1}{4\rho}\left[\frac{1}{2}\frac{|\vec{\nabla}\rho|^2}{\rho} - \Delta\rho\right] + \partial_\rho U, \tag{4.2}$$

where $U = \lambda\rho^{(2/d+1)}$. Eqns (3.1) and (3.2) can be obtained from the irrotational and viscosity–free Euler equations choosing the field χ non-trivially,

$$\chi = \frac{d}{8\Lambda\rho^{2/d+1}}\left[\frac{|\vec{\nabla}\rho|^2}{2\rho} - \triangle\rho\right]. \tag{4.3}$$

Now, as seen above, the general Euler equations with the standard polytropic exponent $\gamma = 1 + 2/d$, are Schrödinger invariant whenever $\tilde{\chi} = \chi$. Using (2.7), we can show that our χ transforms precisely in this way. Therefore, the Madelung equations are Schrödinger invariant.

In is worth noting that for the membrane potential $\gamma = -1$ one can still choose such a χ. However, owing to the bracketed term, $\tilde{\chi} \neq \chi$, so that the Poincaré symmetry is broken. The non-relativistic conformal symmetries are also broken, and we are left with a mere Galilei symmetry.

Note added. After this paper has been accepted, we became aware of a paper by Bordemann and Hoppe [17], which offers yet another way to derive the Schrödinger invariance. For simplicity, we only spell this out in the irrotational case $\theta = 0$. Expressing ρ from the second equation in (2.3) and inserting into the first one yields the so-called "Steichen equation", which in fact derives from the Lagrangian

$$\mathcal{L} = \left[\partial_t\phi + \frac{1}{2}(\vec{\nabla}\phi)^2\right]^{\frac{\gamma}{\gamma-1}}. \tag{4.4}$$

Under a non-relativistic dilatation $\mathcal{L}$ scales as $\mathcal{L} \to \delta^{2\gamma/1-\gamma}\,\mathcal{L}$; taking into account the scaling of the volume element, invariance is obtained precisely when $\gamma = 1+2/d$. Note that for the Chaplygin value $\gamma = -1$, (4.4) becomes the Lagrangian used by Jackiw and Polychronakos [6].

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