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THE CHALLENGE OF LIGHT-FRONT QUANTISATION: RECENT RESULTS *

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Abstract

We explain what is the challenge of light-front quantisation, and how we can now answer it because of recent progress in solving the problem of zero modes in the case of non-Abelian gauge theories. We also give a description of the light-front Hamiltonian for $SU(2)$ finite volume gluodynamics resulting from this recent solution to the problem of light-front zero modes.

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1. Introduction: What is the challenge of light-front quantisation?

The idea of light-front quantisation (or *light-cone* quantisation) of relativistic dynamical systems is more than fifty years old. It was introduced by Dirac [1]. In a nutshell, it suggests taking a point of view of a massless observer flying with the speed of light. The picture of the relativistic dynamics such an observer would have is much different from the conventional one. In the most general terms, the challenge of light-front quantisation is to exploit the possible advantages of this light-like observer's point of view.

Light-front quantisation has been extensively studied and used for fifty years. For a recent review, see Ref. [2]. Despite those massive efforts, the challenge is still not answered properly. To explain why, we first explain what is potentially the most prominent advantage of the light-front description.

The major complication of a complete (nonperturbative) treatment in the conventional equal-time quantisation of relativistic dynamical systems (e.g., Lorentz invariant field theories) is that it is not possible to single out a finite "set of characters" taking part in the events. If we single out some set of particles, new particles are created through interactions, and the number of particles is always potentially infinite. At the moment, the only practical way out of that intractable situation is the lattice regularisation. Under that regularisation, the situation is put under control by keeping the number of degrees of freedom in the description proportional to the number of the vertices of the lattice that replaces the continuum space. The biggest promise of the light-front quantisation is that it may give an alternative way to overcome the problem of an infinite number of interacting degrees of freedom. Therefore, the challenge of light-front quantisation is to develop it to a stage where it would be a serious rival to the lattice field theory. Evidently, it is nowhere close to that stage of development.

The way the light-front keeps the infinite number of interacting degrees of freedom under control is very different from lattice regularisation. It is less severe: the initial theory is distorted not at small and large distances, as is the case with the lattice regularisation, but only at large distances. Because of that, the total number of degrees of freedom the light-front formulation deals with is infinite, in sharp contrast to the lattice regularisation. Instead of cutting the total number of degrees of freedom, light-front quantisation gives a possibility to break the total set of degrees of freedom into subsets, each of the subsets finite, in such a way that there would be no interaction between the degrees of freedom from different subsets. One may say that the light-front promises to slice the complete theory into an infinite set of independent subtheories in such a way that each subtheory constitutes a quantum mechanics with a finite number of degrees of freedom.

Each subtheory is singled out by a value of an additive semi-positive conserving quantum number. The availability of such a quantum number is the major characteristic feature of the light-front quantisation. Let us explain what this quantum number is.

Let the light-like observer be flying along the third axis in the positive direction. The natural "time" (i.e., the coordinate parametrising the world-line of the observer) is then $x^+ \equiv (x^0 + x^3)/\sqrt{2}$ (the normalising square root above is a matter of later convenience). Therefore, the hyperplane in the space-time where the light-like observer is setting the initial conditions for all the dynamics is singled out by the condition $x^+ = const$. In analogy with the equal-time formulation, the components of the momentum generating the shifts of the system at fixed x^+ are kinematical, i.e., even in the presence of an interaction between the parts of the system, the total momentum is a sum of the momenta of the subsystems. In other words, the operators corresponding to the kinematical variables are quadratic in the creation–annihilation operators and easy to diagonalise even in the presence of an interaction. To reiterate, the components of the momentum $P_- = (P_0 - P_3)/\sqrt{2}$, $P_\perp$ are kinematical in the light-front quantisation ($P_\perp$ denotes the set of the space-like momentum components perpendicular to the third direction). This is quite similar to the kinematical character of P_3 and $P_\perp$ in the equal-time quantisation. What is not similar is the fact that P_- is *non-negative* ($P_- = (P_\perp^2 + M^2)/(2P_+)$, where M is the mass of the state).

This qualitative difference between the equal-time and light-front quantisations enables the following consideration: Suppose we make an infrared regularisation of the system that discards all excitations whose P_- is smaller than some regularisation parameter, and let P_- be conserving for the regularised system as it is for the original system. Then the consideration of the dynamics can be restricted to the sectors of fixed P_-, and every such sector cannot accommodate more excitations than its P_- divided by the minimal P_- allowed. That is the case because adding one more excitation would increase P_- at least by the minimal value allowed for P_-. Notice that it is not the case for the equal-time quantisation: because the negative values of momentum are allowed in that case, it is always possible to add more excitations to any state without changing its momentum.

We summarise that the promise of the light-front quantisation is high: it promises to give a non-perturbative definition of quantum field theories that may compete and complement the lattice formulation.

In what follows, we will see how this promise fails in general, and how the non-Abelian gauge theory escapes this failure because of recent findings. The rest of the paper is organised as follows. Section 2 discusses the problem of zero modes, which is the biggest threat to the light-front division into finite sectors; Section 3 specifies

the discussion to non-Abelian gauge theories; Section 4 describes the recent findings restoring the hope for realising of the light-front separation into sectors; Section 5 contains a description of the Hamiltonian of $SU(2)$ gluodynamics reduced to the smallest non-trivial sector of fixed P_-; Section 5 contains the conclusion and outlook.

2. The Zero Modes

It is time to specify the infrared regularisation needed to cut P_- from below. It should respect the symmetries of the theory, in particular, the gauge symmetry (as for the lattice formulation, the major application for the light-front quantisation should be to the non-Abelian gauge theories). The only evident way to achieve this is to compactify the x^- direction. Therefore, in what follows we consider all the fields to be periodic in the x^- direction: $A(x^- = L/2) = A(x^- = -L/2)$. In that case, the spectrum of P_- is discrete, and the smallest possible non-zero value of P_- is $2\pi/L$. This approach, with finite x^- span and discrete P_- is known as *discretised light-cone quantisation* (DLCQ) [3, 4].

What about the degrees of freedom that have zero P_-? Potentially they endanger our program, because it may be possible to add any number of such excitations to any state without changing its P_-. Such degrees of freedom correspond to the field configurations independent of x^-, $\partial_- A = 0$. This is the infamous problem of light-front zero modes. Because of it, the promise of light-front quantisation has yet to be realised.

The problem of the light-front zero modes was first analysed in Ref. [4]. It was pointed out that, by Lorentz invariance, the time derivative of a field A may enter the action in the combination $\partial_+ A \partial_- A$. Therefore, in the cases where the time derivative enters in this combination alone, no time derivative of the zero modes enter the action, because their ∂_- is zero. Thus, varying the action over the zero modes gives an equation without time derivatives. In other words, the zero modes are not among the dynamical degrees of freedom. The classical equations of motion allow one to express them in terms of the real dynamical degrees of freedom at any moment of time, without referring to dynamical evolution. After that, substituting them back into the action yields a theory without zero modes, and the promise of the light-front quantisation survives. That is, it survives in principle. But practically, the equation for the zero modes is a nonlinear equation in partial derivatives. We do not know how to solve it. It is instructive to analyse an example of the ϕ^4 theory to see what is the characteristic appearance of the zero modes equation (to obtain it, integrate the equation for the field over x^-).

We conclude that in general the light-front quantisation is stuck at the problem of zero modes. It remains to consider particular cases: what if in a case of interest

the zero mode problem can be solved?

Our message is that it is indeed the case: we are lucky. The most interesting case of non-Abelian gauge theories has a zero mode equation which can be explicitly solved, and the Hamiltonians can be explicitly obtained for particular sectors of fixed value of P_-.

In what follows, we consider the simplest relevant case of $SU(2)$ gluodynamics.

3. Light-Front Gluodynamics

The first complication with gluodynamics is that in general there are time derivatives of the field configurations independent of x^- in the action. That is the case because there is a four-vector apart from ∂_μ at our disposal (it is the gauge field A_μ). Evidently, combinations like $(\partial_+\phi)A_-\phi$, where ϕ is a matter field, or transverse components of the gauge field, are present in the action. Their presence complicates the zero mode issues as well as the Hamiltonian treatment, because for the Hamiltonian treatment we want the time derivatives to enter the action in the combinations $p_i\dot{q}_i$, where p are the momenta, and q are their conjugate coordinates.

Because of that, probably starting from Ref. [5], it is customary to consider the light-front formulation in the light-cone gauge, $A_- = 0$. It is important that in this gauge the Euler-Lagrange equation that follows from variation of the action of gluodynamics with respect to A_- is implied by the rest of the equations. Thus, it is consistent to set $A_- = 0$ right in the action. Variation over A_+ gives an equation for A_+ without time derivatives. The action at $A_- = 0$, and at A_+ excluded by the equations of motion is amenable to the Hamiltonian treatment.

It was noticed in Ref. [6] that at finite span of x^- the light-cone gauge is inaccessible. This is the case because at finite volume there is a gauge invariant quantity depending only on A_-. It is the trace of the large Wilson loop embracing the hole span of the x^- direction: $W = \mathrm{Tr}P\exp\left(ig\int_{-L/2}^{L/2} dx^- A_-\right)/2$. If W deviates from unity, no gauge transformation can change this fact. The closest one can get to the light-cone gauge at finite volume is to keep A_- diagonal and independent of x^-. If we take this gauge (we will call it light-cone gauge even if its A_- is non-zero), the problem of the Hamiltonian treatment is non-trivial. In particular, it was studied in Ref. [6], and in a number of subsequent papers. The results of those studies were summarised in Ref. [7]. We extract from Ref. [7] the conclusion relevant for our consideration: the problem of zero modes is very complicated in the case under consideration. Other studies [8] seem to agree with that conclusion.

Our recent work [9] sets the whole picture in a new perspective. The major conclusion is opposite: the zero mode equation is *linear* with respect to the zero modes, it is quite possible to solve it, and to write down explicit expressions for the

Hamiltonians at fixed values of P_-.

The reason for this qualitative difference in conclusions lies in the choice of variables: in Ref. [7], the formulation is made in the traditional light-cone gauge, while in Ref. [9], there is no particular gauge choice, and the determination of the canonical variables, zero modes, Gauss law, etc. is made prior to any gauge choice. This enables a direct approach to solving the problem with fully retained zero modes.

In the next Section, we sketch the results of Ref. [9], and then, in Section 5, give a description of the Hamiltonian in the sector of $P_- = 2\pi/L$.

4. Canonical Variables and Zero Modes of $SU(2)$ Gluodynamics

In Ref. [9], the problem of Hamiltonian treatment of the light-front gluodynamics has been reconsidered using the Faddeev-Jackiw approach [10] to constrained systems. In this Section, we give a simplified version of the treatment of Ref. [9].

There are two key steps to determine the canonical structure of the light- front gluodynamics [9]. The first step is made in analogy with the equal-time treatment. It is related with the determination of the variable canonically conjugate to A_-. The analogy hinges on the fact that the field equations for A_- have second time derivatives, which are related to the terms of the action whose structure is $\partial_+ A_- \partial_+ A_-$. Notice that in this respect A_- is similar to all the space components of the gauge field if the dynamics of the latter is considered in the equal-time approach. Therefore, by this analogy, one of the canonical variables is A_-, and its canonical conjugate is $E = F_{+-} + \ldots$. The dots denote the terms whose appearance is due to the non-trivial dependence of the rest of the canonical variables on A_- (see below). Because of that dependence, time derivatives of $A_\perp$, when expressed in terms of the canonical variables, contain terms proportional to the time derivatives of A_-, thus extra contributions to E appear.

The second step is related to the treatment of the terms of the action containing time derivatives of $A_\perp$. They are

$$\partial_+ A_\perp D_- A_\perp. \tag{1}$$

Because of the form of the term (1), it is natural to expand the field $A_\perp$ over the eigenfunctions of D_-:

$$A_\perp = B_\perp \chi_0 + \sum_{p>0} \left[\frac{\chi_p}{\sqrt{2p}} (a_p^\dagger)_\perp + \frac{\chi_p^\dagger}{\sqrt{2p}} (a_p)_\perp \right], \tag{2}$$

where $D_- \chi_p = ip\chi_p$, $\chi_p^\dagger = \chi_{-p}$, and p is the real eigenvalue of D_-/i. To avoid misunderstanding, χ_p on the right-hand side of Eq. (2) are matrices like $A_\perp$ on the

left-hand side; $B_\perp$ is a real field independent of x^-, and $(a_p^\dagger)_\perp$, $(a_p)_\perp$ are complex, conjugated to one another, independent of x^-. There is a scalar product with respect to which D_-/i is a Hermitian operator, and χ_p are its eigenfunctions (see Refs. [9, 11]).

If this expansion of $A_\perp$ is substituted in the action, the obtained form of the action shows that $a_\perp^\dagger$, $a_\perp$ are the creation-annihilation operators, while $B_\perp$ is the zero mode (there are no time derivatives of $B_\perp$ in the action). Simultaneously, extra terms in E are generated replacing the dots above (see Ref. [9] for details).

The good news is that the action is quadratic in $B_\perp$, and the equation for the zero modes is a *non-singular linear* equation. Yet there is one more component of A_μ which we have not mentioned: the A_+ component. There are no time derivatives of that component in the action, and the action is linear in it. This is quite similar to the way the A_0 component enters the action in the case of the equal-time treatment. Also similar is its role: variation over it yields the Gauss law. It is crucial that the Gauss law does not contain $B_\perp$, and the equation for $B_\perp$ does not contain A_+: they do not meet in the action.

So, the general structure of the light-front theory is quite similar to the general structure of the equal-time theory: there are canonical variables determined regardless of the gauge (like the canonical pair $\mathbf{E}$, $\mathbf{A}$ in the equal-time theory), and there is a Gauss law. The only complication is that we need to solve for $B_\perp$.

Now turn back to our program: Does the promise of the light-front formulation persist? Can we slice the theory into independent quantum mechanical sectors, each with finite number of the degrees of freedom? The answer is in the affirmative, but there are subtleties. To see them, we need to give a description of the excitations we have in the formulation, in particular, to trace how the total P_- is built up from contributions of the separate excitations.

First of all, using the Gauss law, we can demonstrate that only a diagonal part of A_-, independent of the location on the transverse plane and of x^-, is a true dynamical quantity. This is so because the non-diagonal, or x^--dependent part of A_- can be removed by a gauge transformation, and a diagonal part dependent on the location on the transverse plane can be expressed in terms of the rest of the dynamical variables through the Gauss law. The global contribution to A_- does not contribute to P_-. Therefore, restricting P_- does not restrict the dynamical component of A_-. Thus, the wave functions of the sectors with fixed P_- will depend on this variable. It is important that this is not a field but a number variable, i.e., there is only one degree of freedom related to this variable. It is also important that the wave functions should be periodic with respect to this variable because of the presence of "large" gauge transformations (see Refs. [9, 12] for details). Thus, A_- does not threaten our program of slicing the theory into quantum mechanics.

Next are the "transverse gluons" excited by $(a_p^\dagger)_\perp$. First of all, to really count them, we need to describe the spectrum p of D_-. Generally, each p is a function of location on the transverse plane, and a functional of A_-. It is not very convenient to count degrees of freedom by a function (not to say about a functional), but we really do not need to do this because we can use the natural ordering of p: for $SU(2)$, we can describe the positive part of the spectrum by two parameters; one is a non-negative integer n, and another is a variable $\sigma = -1, 0, +1$. Then n will determine $2\pi n/L$ minimising the difference $|p - 2\pi n/L|$, and σ will determine the sign of the deviation of p off $2\pi n/L$. With such a definition, we gain the possibility to number the modes by n and σ. Note that when $n = 0$ only $\sigma = +1$ is available, because $(a_{0,-})^\dagger = a_{0,+}$, and the term of Eq. (2) corresponding to $n = 0$, $\sigma = 0$ is the zero mode $B_\perp$. It is important to note that $p_{n,\sigma}$ is a non-smooth functional of A_-:

$$p_{n,\sigma} = \frac{2\pi n}{L} + \sigma \mathrm{Dev}(gA_-), \tag{3}$$

where $\mathrm{Dev}(*)$ is a function whose value is equal to the smallest absolute deviation of its argument off an integer multiple of $2\pi/L$. Therefore, in the Hamiltonian, we expect a non-smooth dependence on A_- (see below).

After we know the numbering of the excitations, it is time to ask what is the contribution of the excitations to the total P_- of the system. Using explicit expressions for P_- [9] it is easy to verify that the contribution of the (n, σ) excitation to P_- is $2\pi n/L$. Therefore, there is an excitation whose contribution to P_- vanishes: it is the $(0, +)$ excitation.

That endangers our program: there is a zero-momentum physical degree of freedom. But the general idea of the light-front split survives. This is the case because σ coincides with an Abelian charge of the excitations, and there is a component of the Gauss law requesting the total Abelian charge on the transverse plane to vanish all the time. So, if we add to an admissible state a single excitation $(0, +)$, we have to add one more excitation whose $\sigma = -1$, to keep the total charge zero. And the smallest P_- of such an excitation is $2\pi/L$ (this is the momentum of the $(1, -)$ excitation).

We conclude that the combination of the Gauss law and zero mode analysis keeps the light-front split alive. In the next Section, we look at a simplest non-trivial quantum mechanics related to light-front $SU(2)$ gluodynamics.

5. The $P_- = 2\pi/L$ Sector

All this is very nice, but does it really work? What are the quantum mechanics appearing in the sectors of fixed P_-? The real interesting physics is in the sectors of

large $n \sim L$. But the sectors of low n should also contain valuable information. For example, they should contain ultraviolet divergences from which we can deduce the non-perturbative running coupling. This idea that the two-dimensional quantum mechanics contains dimensional transmutation was introduced in Ref. [13].

So, the first test of the approach of Ref. [9] is to try it on the sectors of small P_-. The sector of $P_- = 0$ was considered in Ref. [9]. It is trivial in the sense that there are no ultraviolet divergences in that sector. However, it is quite interesting in its own right. The objects one finds in this sector are hybrids of "fluxes" known from finite volume equal-time formulation [14], and non-Abelian plane waves of Coleman [15]. Considering this sector allows the conclusion that qualitative features of the infinite volume gluodynamics (like the presence of the mass gap) depend on the way the infinite volume limit is taken.

The next in complexity is the sector of $P_- = 2\pi/L$. In what follows, we describe the Hamiltonian reduced to this sector.

5.0. The Structuring

The description we give is structured: there are three levels. The first level involves the synthetic quantities constructed from the fields, like components of the field strength, F_{kl}. The second level resolves the first expressing the Hamiltonian in terms of the creation–annihilation operators. The last, third, level specifies the description to a sector of a fixed number of quanta of the longitudinal momentum P_-. The notations of Ref. [9] are used in the description.

5.1. The Synthetic Level

The Hamiltonian consists of the following terms:

$$H = K + D + F + B. \tag{4}$$

K is the kinetic energy of the global angle variable q, $0 \le q \le 1$ ($K = -g^2 L/(2V(2\pi)^2)(\partial/\partial q)^2$, V is the volume of the transverse space). D, F, and B are functionally dependent on the transverse components of the gluon field, A_k, and on q. D is the characteristic light-front term containing the inverse of D_-, F is the most conventional term—the square of the transverse components of the field strength, and B comes from the zero modes.

Here are the explicit expressions for the terms:

$$D = \frac{1}{2}\sum_{p\neq 0}\left|\left(\frac{1}{D_-}D_k F_{-k}\right)^p\right|^2; \tag{5}$$

$$F = \frac{1}{4} < F_{lm}|F_{lm} >; \tag{6}$$

$$B = -\frac{1}{2}J_k M^{-1} J_k. \tag{7}$$

Here M is an operator acting in the space of the zero modes, i.e., functions depending only on the transverse coordinates and x^+:

$$M =< \chi_0| - D_k^2\chi_0 >, \tag{8}$$

which is a sum of two terms; first of them is just the transverse Laplacian, and the second is quadratic in the transverse components A_k. Also, throughout this section, $A_\perp$ is given by Eq. (2) with $B_\perp$ set to zero. The current J_k in Eq. (7) is as follows:

$$J_k =< \chi_0|ig\left[A_k, \frac{1}{D_-}D_l F_{-l}\right] + D_l F_{lk} > . \tag{9}$$

On the right-hand sides of Eqs. (5) - (7), there is an integration over the transverse coordinates, $\int \prod_k dx^k$, which is not explicitly shown. J_k is a function of the location in the transverse space.

The inverse of M in Eq. (7) is understood in the sense of expansion in powers of the fields. It involves the inversion of the transverse Laplacian. Here and in what follows, this inversion is understood as an operator which annihilates the zero mode of the function it acts on.

5.2. The Creation–Annihilation Operators Level

Now we need to resolve the synthetic quantities of the previous section in terms of the creation–annihilation operators. First we list the relations we will need:

$$(a_k^p)^\dagger = \sqrt{2p}A_k^p\,, p > 0; \tag{10}$$

$$A_k^{-p} = (A_k^p)^\dagger; \tag{11}$$

$$[\chi_{p_1}, \chi_{p_2}] = \epsilon(\sigma_1, \sigma_2)\frac{\chi_{p_1+p_2}}{\sqrt{L}}; \tag{12}$$

$$A_- = \frac{1}{\Delta_\perp}\frac{g}{L}\sum_{p>0}\sigma_p (a_k^p)^\dagger a_k^p; \tag{13}$$

$$p = \frac{2\pi n}{L} + \sigma \mathrm{Dev}\Big[\frac{2\pi q}{L} + gA_-\Big]. \tag{14}$$

In the above relations, the sign ϵ depending on the two signs of deviations of the p-eigenvalue off an integer multiple of $2\pi/L$ is (i) antisymmetric in its arguments, and (ii) its non-trivial values are as follows: $\epsilon(+,-) = -$, $\epsilon(0,+) = -$, and $\epsilon(0,-) = +$ (0 in the arguments of ϵ appears when the corresponding eigenvalue is an integer multiple of $2\pi/L$); $\Delta_\perp$ is the transverse Laplacian, whose inversion is understood as above; the function Dev[$*$] in Eq. (14) is by definition as follows:

$$\mathrm{Dev}[x] = \min_n \Big|x - \frac{2\pi n}{L}\Big|, \tag{15}$$

i.e., it is the magnitude of the deviation of x from its nearest integer multiple of $2\pi/L$ (see also Eq. (3)). To avoid misunderstanding, A_- above is not a matrix as before. It is a number field, and the right-hand side can be treated as its definition; it is related to A_- as it was before: using the notations of Ref. [9], now $A_- = (\tilde{A}^0_- - \int dx^\perp \tilde{A}^0_-/V)/\sqrt{L}$.

With the equipment of the above relations, we resolve the "square root" of the D-term as follows:

$$\begin{aligned}\Big(\frac{1}{D_-}D_k F_{-k}\Big)^p &= \frac{1}{ip}\Big(\partial_k(ipA^p_k) - \frac{ig}{\sqrt{L}}\sum_{p'\neq 0} ip'\epsilon(\sigma_{p-p'},\sigma_{p'})A^{p-p'}_k A^{p'}_k \\ &+ ig\epsilon(\sigma_p,0)A^p_k\partial_k A_-\Big).\end{aligned} \tag{16}$$

The rest of the terms in the Hamiltonian are resolved in the same way.

5.3. Specification for a Fixed Value of P_-

Specification for a particular sector of fixed longitudinal momentum is the most involved part of the description of the Hamiltonian. We are not ready to give it for the general case of arbitrary fixed P_-. We consider the simplest non-trivial case.

5.3.1. Specification for $P_- = 2\pi/L$

First of all, three kinds of excitations are involved:

(i) the excitation with the lowest possible eigenvalue of D_-/i; we will call it a-excitation, its creation–annihilation operators will be denoted by $a^\dagger_k$, a_k;

(ii) the excitation with the next-to-lowest possible eigenvalue of D_-/i, the b-excitation ($b_k^\dagger$, b_k); and

(iii) the excitation whose D_-/i coincides with the value of $P_- = 2\pi/L$, the c-excitation ($c_k^\dagger$, c_k).

So, for example, a c-excitation can decay into a pair of a and b without breaking the conservation of the longitudinal momentum (a carries no longitudinal momentum, and b carries the same quantum of the longitudinal momentum $2\pi/L$ as c does).

The reduction of the above Hamiltonian to the sector under consideration can be performed by retaining only terms up to fourth order in the creation–annihilation operators, and containing only the $a-$, $b-$ and $c-$ operators.

The B-term above is the most amenable with respect to this reduction. So we start the reduction with

5.3.2. B-Term Reduction

The B-term is a term of the kind $J_k M^{-1} J_k$, and both J_k and M^{-1} are complicated (non-polynomial) functionals of the creation–annihilation operators. The first step in reduction is to notice that the expansion of J_k in the creation–annihilation operators starts from the quadratic term. Therefore, the leading term in the expansion of the B-term in powers of the creation–annihilation operators is the only one we need for our reduction, and we can replace in the leading term M^{-1} by $1/(-\Delta_\perp)$ (we recall that there is no ambiguity in the action of this operator on a constant function).

The next step is to reveal the leading quadratic contribution to the current J_k (see Eq. (9)). At the moment, J_k is expressed in terms of A_k. A_k are expandable in powers of the creation–annihilation operators, and the expansion starts from the linear term; J_k in turn is expandable in powers of A_k, and the leading term is quadratic. So, we start from retaining the leading term of the expansion of J_k in powers of A_k:

$$J_k = ig < \chi_0 | 2\partial_l [A_k, A_l] + [A_l, \partial_k A_l] > . \qquad (17)$$

All we need to obtain the desired reduction of the B-term now is to substitute in Eq. (17) the expansion $A_k = a_k^\dagger \chi_{(0,+)}/\sqrt{2p_a} + b_k^\dagger \chi_{(1,-)}/\sqrt{2p_b} + c_k^\dagger \chi_{(1,0)}/\sqrt{2p_c} + h.c.$, calculate the commutator involved in Eq. (17) using Eq. (12), and compute the scalar product of the commutator with χ_0 keeping in mind that the eigenvectors χ

form the orthonormal set. Here is the outcome of these manipulations:

$$\begin{aligned} J_k &= \frac{2ig}{\sqrt{L}}\partial_l\Big(\frac{b_k^\dagger b_l - b_l^\dagger b_k}{p_b} - \frac{a_k^\dagger a_l - a_l^\dagger a_k}{p_a}\Big) \\ &+ \frac{ig}{\sqrt{L}}\Big(\frac{b_l^\dagger \partial_k b_l - (\partial_k b_l^\dagger) b_l}{p_b} - \frac{a_l^\dagger \partial_k a_l - (\partial_k a_l^\dagger) a_l}{p_a}\Big). \end{aligned} \tag{18}$$

Here $p_a = \mathrm{Dev}[2\pi q/L]$, $p_b = 2\pi/L - \mathrm{Dev}[2\pi q/L]$, and $p_c = p_a + p_b$.

If we substitute the above expression of the J_k in the B-term, it will generate a number of contributions which can be classified as b-b interaction, a-a interaction, and a-b interaction. In our sector, we need only the a-b contribution (because we have only one a-excitation, and only one b-excitation). Retaining the a-b contribution we obtain

$$B = \frac{g^2}{L}\frac{1}{p_a p_b} j_k(b)\frac{1}{\Delta_\perp} j_k(a), \tag{19}$$

where

$$j_k(a) = 2\partial_l(a_k^\dagger a_l - a_l^\dagger a_k) + (a_l^\dagger \partial_k a_l - (\partial_k a_l^\dagger) a_l), \tag{20}$$

$j_k(b)$ is obtained from $j_k(a)$ by the substitution $a \to b$. The last thing to notice is that the ugly non-smooth Dev-function featuring the expressions for p_a, p_b can be dropped out of Eq. (19). To see how it works, consider a small value of q and observe that here $p_a p_b = (2\pi/L)^2 q(1-q)$ which is symmetric with respect to the reflection $q \to 1-q$. Therefore, the exact expression with Dev, which works also for the values of q exceeding $1/2$, is identical in this case with the above naive representation without Dev, which in general holds only at $q < 1/2$.

Therefore, our final expression for the reduced B-term is as follows:

$$B = \frac{g^2}{(2\pi)^2}\frac{L}{q(1-q)} j_k(b)\frac{1}{\Delta_\perp} j_k(a). \tag{21}$$

5.3.3. Reduction of the D- and F-terms

To reduce the D- and F- terms we notice that they both are infinite sums over the spectrum of D_-/i. Only finite number of terms of these sums contribute when the action of the Hamiltonian is reduced to the sector $P_- = 2\pi/L$. In fact, an inspection reveals that only the terms whose eigenvalues p satisfy $p \le P_-$ contribute. Therefore, the D-term contribution is in fact $|D^a|^2 + |D^b|^2 + |D^c|^2$, where, for example, $D^a = ((1/D_-)D_k F_{-k})^a$; and the F-term contribution is $(|F^0_{lm}|^2 + |F^a_{lm}|^2 + |F^b_{lm}|^2 + |F^c_{lm}|^2)/2$, where, for example, $F^0_{lm} = -ig([A_l, A_m])^0$.

When the transverse components A_k are replaced in those expressions by the sum of the creation–annihilation operators divided by the square roots of the corresponding eigenvalues, non-polynomial functions of the creation–annihilation operators appear, because the square roots of the eigenvalues downstairs contain A_-, which is quadratic in the creation–annihilation operators (see Eq. (13)). For our reduction, we need to expand these functions in A_- and to retain only the linear terms in A_-. A Characteristic example is as follows:

$$\frac{1}{\sqrt{2p}} \simeq \frac{1}{\sqrt{2\bar{p}}}\Big(1 - \frac{g\sigma\epsilon(q)}{2\bar{p}}A_-\Big), \tag{22}$$

where $\bar{p}$ is p at zero A_-, σ is the sign of the deviation of p off the multiple integer of $2\pi/L$, and $\epsilon(q) = +1$ when $q < 1/2$ and -1 otherwise.

When Eq. (13) is used to express A_- in terms of the creation–annihilation operators, a characteristic "potential" generated by the a- or b- charges enters the formulas. We introduce a dedicated notation for it:

$$V_a = \frac{1}{-\Delta_\perp}a_k^\dagger a_k,\; V_b = \frac{1}{-\Delta_\perp}b_k^\dagger b_k. \tag{23}$$

Also, we will use another useful notation:

$$\mathcal{A}_k = \frac{a_k}{\sqrt{2p_a}}, \tag{24}$$

and similarly for other creation–annihilation operators (for example, $\mathcal{C}_k^\dagger = c_k^\dagger/\sqrt{2p_c}$).

With these notations, the reduced expressions for D^a, D^b, ..., F^c are as follows (by reduced we mean that some terms have been omitted because they are vanishing in the sector under consideration):

$$D^a = \Big(1 - \frac{g^2\epsilon(q)}{2Lp_a}V_b\Big)\partial_k\mathcal{A}_k^\dagger + \Big(1 + \frac{\epsilon(q)}{2}\Big)\frac{g^2}{Lp_a}(\partial_k V_b)\mathcal{A}_k^\dagger + \frac{-ig}{\sqrt{L}}\frac{p_c + p_b}{p_a}\mathcal{C}_k^\dagger\mathcal{B}_k; \tag{25}$$

$$D^c = \partial_k\mathcal{C}_k^\dagger + \frac{ig}{\sqrt{L}}\frac{p_b - p_a}{p_c}\mathcal{B}_k^\dagger\mathcal{A}_k^\dagger; \tag{26}$$

$$F_{lm}^0 = \frac{-ig}{\sqrt{L}}(\mathcal{A}_l\mathcal{A}_m^\dagger - \mathcal{A}_m\mathcal{A}_l^\dagger) + \frac{ig}{\sqrt{L}}(\mathcal{B}_l\mathcal{B}_m^\dagger - \mathcal{B}_m\mathcal{B}_l^\dagger); \tag{27}$$

$$\begin{aligned} F_{lm}^a &= \partial_l\Big(\mathcal{A}_m^\dagger\Big(1 - \frac{g^2V_b\epsilon(q)}{2Lp_a}\Big)\Big) - \partial_m\Big(\mathcal{A}_l^\dagger\Big(1 - \frac{g^2V_b\epsilon(q)}{2Lp_a}\Big)\Big) \\ &+ \frac{-ig}{\sqrt{L}}(\mathcal{B}_l\mathcal{C}_m^\dagger - \mathcal{B}_m\mathcal{C}_l^\dagger); \end{aligned} \tag{28}$$

$$F^c_{lm} = \partial_l \mathcal{C}^\dagger_m - \partial_m \mathcal{C}^\dagger_l - \frac{ig}{\sqrt{L}}(\mathcal{B}^\dagger_l \mathcal{A}^\dagger_m - \mathcal{B}^\dagger_m \mathcal{A}^\dagger_l). \tag{29}$$

The two missing expressions for D^b and F^b_{lm} are obtained from the expressions for D^a and F^a_{lm} by the substitutions $\mathcal{A} \to \mathcal{B}$, $a \to b$, and $-i \to i$ for the imaginary unit.

With these expressions at our disposal, the reduced Hamiltonian is derived by taking the sum of their magnitudes squared, and by omitting the excessive terms.

5.3.4. The three-gluon vertex

The terms bilinear in the creation-annihilation operators are easily obtainable; they are as they should be, $a^\dagger_k(-\Delta_\perp)/(2p_a)a_k + b^\dagger_k(-\Delta_\perp)/(2p_b)b_k + c^\dagger_k(-\Delta_\perp)/(2p_c)c_k$.

The next in complexity is the three-gluon vertex describing the decay of a c-excitation into a pair of a- and b- excitations. We will express it in terms of the Fourier modes of the above operators, $\mathcal{A}^\dagger_k(x) \equiv \sum_{k^a}(\tilde{A}^\dagger_k(k^a)\exp ik^a x)/\sqrt{V}$, etc., where x is the location in the transverse space, k^a is the transverse momentum of the a-excitation, and the V is the volume of the transverse space. The term of the Hamiltonian we are looking for is representable as follows:

$$G_3 = \sum_{k^a,k^b,k^c} \delta(k^c - k^a - k^b)\tilde{A}^\dagger_a(k^a)\tilde{B}^\dagger_b(k^b)\tilde{C}_c(k^c)V_{abc}(k^a,k^b,k^c) + h.c. \tag{30}$$

Now our task is to get an expression for V_{abc} (summation over the transverse vectorial indices abc is implied above).

A calculation along the above lines gives

$$V_{abc}(k^a,k^b,k^c) = \frac{2g}{\sqrt{LV}}\left[\frac{\delta_{bc}}{p_a}(k^b_a p_a - k^a_a p_b) + \frac{\delta_{ac}}{p_b}(k^b_b p_a - k^a_b p_b) - \frac{\delta_{ab}}{p_c}(k^b_c p_a - k^a_c p_b)\right]. \tag{31}$$

We have a loose notation in the above formula: e.g., the subscript a denotes both vectorial index of the a-excitation, and the label on the D_-/i-eigenvalue. To avoid misunderstanding, it is a vectorial index when it hangs on the transverse momenta, or on a Kronecker's delta-symbol.

The only term we are to determine in the reduced Hamiltonian $H_{red} = K + G_2 + G_3 + G_4$ is the G_4-term. Partly we know it, because we specified above reduction of the B-term, which gives a contribution to G_4.There are quite a number of terms in G_4, and we will not write them down explicitly. We hope that the general pattern is clear, and the reader can recover the rest of the terms in G_4.

Comparison with Ref. [13] shows that the dimensional transmutation is implied by this Hamiltonian. The crucial check of the whole construction can be given by a calculation of the numerical coefficient by the leading inverse logarithm of the ultra-violet cut-off in the running coupling, because its value is known from conventional

perturbation theory. This calculation is in progress.

6. Conclusion and Outlook

We described a promising approach to a non-perturbative description of non-Abelian gauge theories. It results from previous efforts to answer the challenge of the light-front quantisation of the gauge theories (see Ref. [2]), and from recent analysis of the light-front formulation [9]. The first result we expect from this approach is an alternative non-perturbative definition of the running coupling that can be obtained from the quantum mechanics in the sector $P_- = 2\pi/L$.

In more general terms, we are at the very beginning of a long road: we need to generalise to $SU(N)$, to include fermions, and, the most interesting, to go to the sectors of large P_-.

We note also that the finite volume light-front formulation may play an important role for string theory, where one has to quantise a compactified theory. Since we have obtained a light-front formulation without a gauge fixing and in finite volume our results can stimulate a deeper understanding of a relation with novel M-theory developments [16].

It is too early to make a conclusion about the approach we presented, but we believe in its promising future.

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