

# $n=3$ Differential calculus and gauge theory on a reduced quantum plane.

M. EL BAZ<sup>1</sup>, A. EL HASSOUNI<sup>2</sup>, Y. HASSOUNI<sup>3</sup>  
and E.H. ZAKKARI<sup>4</sup>.

Laboratory of Theoretical Physics  
PO BOX 1014, University Mohammed V  
Rabat, Morocco.

**Abstract:** We discuss the algebra of  $N \times N$  matrices as a reduced quantum plane. A 3-nilpotent deformed differential calculus involving a complex parameter  $q$  is constructed. The two cases,  $q$  3<sup>rd</sup> and  $N^{th}$  root of unity are completely treated. As application, a gauge field theory for the particular cases  $n=2$  and  $n=3$  is established.

**Keywords:** reduced quantum plane, non-commutative differential calculus  $n=3$ , gauge theory.

---

<sup>1</sup>moreagl@yahoo.co.uk

<sup>2</sup>lhassoun@fsr.ac.ma

<sup>3</sup>y-hassou@fsr.ac.ma

<sup>4</sup>hzakkari@hotmail.com

# 1 Introduction:

An adequate way leading to generalizations of the ordinary exterior differential calculus arises from the graded differential algebra [1 – 3]. These generalizations are not universal as far as we know, and many technics have been used to introduce differential calculus that corresponds to non commutative calculi. The latters involve a complex parameter that satisfies some conditions allowing the obtention of a consistent generalized differential calculus. It is usually called  $q$ -differential calculus. In ref [1 – 3], it is seen as a graded  $q$ -differential algebra which is the sum of  $k$ -graded subspaces, where  $k \in \{0, 1, 2 \dots m-1\}$ . The relevant differential operator is an endomorphism  $d$  satisfying  $d^m = 0$  and the  $q$ -Leibniz rule:

$$d(AB) = (dA)B + qAd(B).$$

The most important property of this calculus is that it contains not only first differentials  $dx^i, i = 1 \dots n$ , but also it involves the higher-order differentials  $d^j x^i, j = 1 \dots m-1$ .

On the other hand, the differential calculi ( $d^2 = 0$ ) on noncommutative spaces was also studied by different authors, see for exemple [4 – 9]. The common property of these calculi is the covariance of these latters under some symmetry quantum group.

In this paper, we construct a covariant differential calculus  $d^3 = 0$  on the algebra  $M$  of  $3 \times 3$  matrices considered as a quantum plane. We will show that our differential calculus is covariant under the algebra of transformations with a quantum group structure. The complex deformation parameter  $q$  ( $3^{rd}$ -root of unity) will play an important role in constructing the differential calculus that we introduce. As it is done in the literature of deformed differential calculus [4, 5], this case implies a non trivial study. As application, we treat the gauge field theory.

The paper is organized as follows:

We start in section 2 by defining the algebra of  $N \times N$  matrix as a reduced quantum plane, where the deformation parameter  $q$  is  $N$ -th root of unity. We also give a matrix realization in the case  $N = 3$ . In section 3 we construct the covariant differential calculus  $d^3 = 0$ , on two dimensional reduced quantum plane as in ref [1 – 3]. The new objects,  $d^2 x$  and  $d^2 y$ , appearing in this construction are seen as the analogous of the differential elements  $dx$  and  $dy$  in the ordinary differential calculus. In section 4, we generalize this result by considering a complex deformation parameter  $q$   $N^{th}$  root of unity.

In section 5, we study the application of this new differential calculus ( $N = 3$ ) to the gauge field theory on  $M_3(C)$ . We recall in section 6 the differential calculus  $d^2 = 0$  [6–9], and we apply it to the gauge theory on  $M_3(C)$ .

## 2 Preliminaries about the algebra $M_3(C)$ of $N \times N$ matrices as a reduced quantum plane.

The associative algebra of  $N \times N$  matrices is generated by two elements  $x$  and  $y$  [10] satisfying the relations:

$$xy = qyx \quad (1)$$

and

$$x^N = y^N = 1, \quad (2)$$

where  $1$  is the unit matrix and  $q$  ( $q \neq 1$ ) is a complex parameter  $N^{th}$  root of unity.

In the case  $N = 3$ , an explicit matrix realization of generators  $x$  and  $y$  [6, 11] is given by:

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & q^{-1} & 0 \\ 0 & 0 & q^{-2} \end{pmatrix} \quad (3)$$

$$y = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad (4)$$

and  $q$  satisfies the relation:

$$1 + q + q^2 = 0. \quad (5)$$

The associative algebra, noted by  $C_q[x, y] := C_q$ , of formal power series defined over the two dimensional quantum plane is generated by  $x$  and  $y$  with a single quadratic relation  $xy = qyx$ . It is clear that  $C_1[x, y]$  coincides with the algebra of polynomials over commuting variables  $x, y$ .

Note that if the generators  $x, y$  do not satisfy any additional relations, then  $C_q$  is infinite dimensional. In the case of the algebra  $M_3(C)$  of  $3 \times 3$  matrices over complex numbers, the generators  $x, y$  satisfy the above quadratic relation and the cubic ones  $x^3 = y^3 = 1$ . Thus it is generated by the following set:  $\{1, x, y, x^2, y^2, xy, x^2y, xy^2, x^2y^2\}$ . In this case, the algebra

$M_3(\mathbb{C})$  appears as the associative quotient algebra  $C_q^0$  by the bilateral ideal generated by  $x^3 - 1 = 0$  and  $y^3 - 1 = 0$ . Here  $C_q^0$  is the unital extension of  $C_q$ . That is, in the sense of [6, 11], the  $3 \times 3$  matrices over  $\mathbb{C}$  are seen as a reduced quantum plane.

We note that the functions of  $x$  and  $y$  are seen as formal power series with a maximum degree 3; this property will be extremely useful in what follows. In fact, the set of those functions is an associative algebra that is used to introduce a gauge field theory on the reduced quantum plane. This idea will be developed in sections 5 and 7.

### 3 Differential calculus with nilpotency $n = 3$ on reduced quantum plane, case $q^3 = 1$

The aim of this section is to construct a covariant  $n = 3$  nilpotent differential calculus by mixing two approaches; namely we adapt to the reduced quantum plane an idea originally proposed by Kerner [1–3], and we use Couquereaux's technics [6] to ensure covariance. We denote by  $\Omega$  the differential algebra generated by  $x, y, dx, dy, d^2x$  and  $d^2y$ , where the "2- forms"  $d^2x$  and  $d^2y$  are the second differentials of the basic variables  $x$  and  $y$ .

Let us introduce the differential operator  $d$  that satisfies the following conditions:

Nilpotency,

$$d^3 = 0. \quad (6)$$

Leibniz rule,

$$d(uv) = d(u)v + q^n u d(v), \quad (7)$$

where  $u$  is a form of degree  $n$  and  $q$  is  $3^{rd}$  root of unity.

By applying the Leibniz rule on the 1-form we obtain:

$$d(f(x) dx) = (df(x)) dx + f(x) d^2x, \quad (8)$$

$f(x)$  are the 0-form in the algebra  $\Omega$ . The notion of covariance is necessary for the consistency of every differential calculus. The set of transformations leaving covariant our differential calculus is  $F \subset Fun(SL_q(2, \mathbb{C}))$  and the covariance is described by the left coaction. We start by explaining this coaction [12].

The left coaction of the group  $H$  on the reduced quantum plane is the linear transformation of coordinates given by:

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} x \\ y \end{pmatrix}.$$

We introduce also the line vectors with coordinate functions:

$$(x^1, y^1) = (x, y) \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where the matrix elements  $a, b, c$  and  $d$  do not commute with each others. We require that the quantities  $x_1, y_1, x^1, y^1$  obtained in the above relations satisfy the same relations as  $x$  and  $y$ . The two constraints  $x_1 y_1 = q y_1 x_1$  and  $x^1 y^1 = q y^1 x^1$  lead to the relations:

---


$$ac = qca \quad bd = qdb$$


---

$$ab = qba \quad cd = qdc$$


---

$$bc = cb \quad ad - da = (q - q^{-1})bc.$$

The algebra generated by  $a, b, c$  and  $d$  is usually denoted  $\text{Fun}(GL_q(2, C))$ . The  $q$ -determinant  $D = da - q^{-1}bc$  is in the center of  $\text{Fun}(GL_q(2, C))$ . If we set it to be equal to 1, we define the algebra  $\text{Fun}(SL_q(2, C))$ . Assuming that the supplementary conditions  $(x)^3 = 1$  and  $(y)^3 = 1$  are also verified by the coordinates  $x_1, y_1$  (and  $x^1, y^1$ ),  $((x_1)^3 = (y_1)^3 = 1, ((x^1)^3 = (y^1)^3 = 1))$ , implies  $a^3 = 1, b^3 = 0, c^3 = 0, d^3 = 1$ . These new cubic relations on  $\text{Fun}(SL_q(2, C))$ , yields a new algebra that we denote  $F$ . It is also a Hopf algebra. Indeed, it has a coalgebra structure (coproduct) which is compatible with the algebra one (product), this defines a bialgebra structure. An antipode and a co-unit are also defined. For further details on such structures on  $F$ , see, for example [6].

The mixture of Kerner's idea and Coquereaux's technics allows us to construct the left covariant differential algebra  $\Omega = \{x, y, dx, dy, d^2x, d^2y\}$ , see appendix. The commutation relations between the generators of  $\Omega$  are as follows:

$$x dx = q^2 dx x \tag{9}$$

$$x dy = q dy x + (q^2 - 1) dx y \tag{10}$$

$$y dx = q dx y \tag{11}$$

$$y \, dy = q^2 dy \, y \quad (12)$$

$$dy \, dx = q^2 dx \, dy \quad (13)$$

$$x \, d^2 x = q^2 d^2 x \, x \quad (14)$$

$$y \, d^2 x = q d^2 x \, y \quad (15)$$

$$y \, d^2 y = q^2 d^2 y \, y \quad (16)$$

$$x \, d^2 y = q d^2 y \, x + (q^2 - 1) d^2 x \, y \quad (17)$$

$$dx \, d^2 y = d^2 y \, dx + q(1 - q) d^2 x \, dy \quad (18)$$

$$dy \, d^2 x = d^2 x \, dy \quad (19)$$

$$dx \, d^2 x = q d^2 x \, dx \quad (20)$$

$$dy \, d^2 y = q d^2 y \, dy \quad (21)$$

$$d^2 y \, d^2 x = q^2 d^2 x \, d^2 y. \quad (22)$$

As in the standard way, we define the partial derivatives in directions  $\mathbf{x}$  and  $\mathbf{y}$  through:

$$d = \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy = \partial_x dx + \partial_y dy. \quad (23)$$

Consistency conditions as in [9] yield:

$$\partial_x \partial_y = q \partial_y \partial_x \quad (24)$$

$$\partial_x x = 1 + q^2 x \partial_x + (q^2 - 1) y \partial_y \quad (25)$$

$$\partial_x y = q y \partial_x \quad (26)$$

$$\partial_y y = 1 + q^2 y \partial_y \quad (27)$$

$$\partial_y y = 1 + q^2 y \partial_y \quad (28)$$

$$(dx)^3 = (dy)^3 = 0. \quad (29)$$

The last equality [eq\(29\)](#) can be related to the nilpotency relation encountered in the description of the fractional statistics. More precisely, we recover the description of physical systems that generalize fermions. In a forthcoming paper [\[13\]](#), we reintroduce these systems using this new differential calculus by establishing an adequate correspondence between our differential calculus and some deformed Heisenberg algebras, as it is done in [\[14\]](#) for the particular case  $(d^2 = 0)$ .

Now, we generalize our differential calculus by considering the case  $q^N = 1$ .

## 4 Differential calculus on a reduced quantum plane, case $q^N = 1$ .

A two dimensional reduced quantum plane is an associative algebra generated by  $x$  and  $y$  with relations [\(1\)](#) and [\(2\)](#). One can always define the differential operator " $d$ " satisfying  $d^3 = 0$ ,  $(d^2 \neq 0)$ , and the Leibniz rule:

$$d(uv) = d(u)v + (j)^n u d(v), \quad (30)$$

$u \in \Omega^n$  and  $v \in \Omega^m$ , where  $\Omega^n$  and  $\Omega^m$  are the spaces of  $n$  and  $m$  forms on reduced quantum plane respectively.

Note that in contrast to [eq\(7\)](#), one have to distinguish between the deformation parameter  $q$  and the  $j$  parameter,  $j^3 = 1$ ,  $(j \neq 1)$ , in [eq\(30\)](#).

Following the same method of section [3](#), we get the covariant differential calculus:

$$x dx = j^2 dx x \quad (31)$$

$$x dy = -\frac{jq}{1+q^2} dy x + \frac{j^2 q^2 - 1}{1+q^2} dx y \quad (32)$$

$$y dx = \frac{j^2 - q^2}{1+q^2} dy x - \frac{jq}{1+q^2} dx y \quad (33)$$

$$y \, dy = j^2 dy \, y \quad (34)$$

$$dx \, dy = q dy \, dx \quad (35)$$

$$x \, d^2 x = j^2 d^2 x \, x \quad (36)$$

$$x \, d^2 y = -\frac{jq}{1+q^2} d^2 y \, x + \frac{j^2 q^2 - 1}{1+q^2} d^2 x \, y \quad (37)$$

$$y \, d^2 x = \frac{j^2 - q^2}{1+q^2} d^2 y \, x - \frac{jq}{1+q^2} d^2 x \, y \quad (38)$$

$$y \, d^2 y = j^2 d^2 y \, y \quad (39)$$

$$dx \, d^2 x = j d^2 x \, dx \quad (40)$$

$$dx \, d^2 y = -\frac{q}{1+q^2} d^2 y \, dx + \frac{jq^2 - j^2}{1+q^2} d^2 x \, dy \quad (41)$$

$$dy \, d^2 x = \frac{j - j^2 q^2}{1+q^2} d^2 y \, dx - \frac{q}{1+q^2} d^2 x \, dy \quad (42)$$

$$dy \, d^2 y = j d^2 y \, dy \quad (43)$$

$$d^2 x \, d^2 y = q d^2 y \, d^2 x. \quad (44)$$

We recover the differential calculus obtained in section 3, if  $q = j$ . As an application of this new differential calculus  $d^3 = 0$  on the reduced quantum plane, we construct in the section below a gauge field theory on  $M_3(C)$ .

## 5 Gauge theory on $M_3(C)$ as a reduced quantum plane with $d^3 = 0$

In this section, we use the  $n = 3$  differential calculus constructed in section 3 to establish a gauge theory on the reduced quantum plane.

As in the ordinary case, the covariant differential is defined by:

$$D\Phi(x, y) = d\Phi(x, y) + A(x, y)\Phi(x, y), \quad (45)$$



where the field  $\Phi(x, y)$  is a function on  $M_3(C)$  and the gauge field  $A(x, y)$  is a 1-form valued in the associative algebra of functions on the reduced quantum plane  $M_3(C)$ .

We have assumed that the algebra of functions on  $M_3(C)$  is a bimodule over the differential algebra  $\Omega$ .

As usual, the covariant differential  $D$  must satisfy:

$$DU^{-1}\Phi(x, y) = U^{-1}D\Phi(x, y), \quad (46)$$

where  $U$  is an endomorphism defined on  $Fun[M_3(C)]$ .

This leads to the following gauge field transformation:

$$A(x, y) \rightarrow U^{-1}A(x, y)U + U^{-1}dU. \quad (47)$$

In general, the 1-form gauge field  $A(x, y)$  can be written as:

$$A(x, y) = A_x(x, y)dx + A_y(x, y)dy. \quad (48)$$

The differential calculus  $n=3$  allows to define the curvature  $R$  as follows [2, 15]:

$$D^3\Phi(x, y) = R\Phi(x, y). \quad (49)$$

Direct computations show that  $R$  is a "three-form" given by:

$$R = d^2A(x, y) + dA^2(x, y) + A(x, y)dA(x, y) + A^3(x, y) \quad (50)$$

$$= d^2A(x, y) + (dA(x, y))A(x, y) + (1+q)A(x, y)dA(x, y) + A^3(x, y) \quad (51)$$

$$= d^2A(x, y) + (dA(x, y))A(x, y) - q^2A(x, y)dA(x, y) + A^3(x, y). \quad (52)$$

One has to express the curvature written above in terms of 3-forms constructed from basic generators  $dx, dy, d^2x$  and  $d^2y$  of the differential algebra  $\Omega$ . Since we are dealing with a non-commutative space (reduced quantum plane), this task is not straightforward. In fact, the non-commutativity prevents us from rearranging the different terms in eq(52) adequately. To overcome this technical difficulty we require that the components of the gauge field  $A_x(x, y)$  and  $A_y(x, y)$  are expressed as formal power series of the space coordinates  $x$  and  $y$  [16–19]. The condition eq(2) in section 2 ( $N=3$ ) is extremely useful, in the sense that it limits the power series to finite ones rather than infinite:

$$A_x(x, y) = a_{mn}x^m y^n; \quad m, n = 0, 1, 2 \quad (53)$$

$$A_y(x, y) = b_{kl}x^k y^l; \quad k, l = 0, 1, 2. \quad (54)$$

Using the formulae (1, 31–44, 52–54), and after technical computations, the desired expression of the curvature arises as:

$$\begin{aligned} R = & \left[ R_{xxy} + qR_{yxx} + q^2R_{xyx} + \right. \\ & (1-q)\left\{ \partial_y A_x(x, y) + q\partial_x A_y(x, y) + \partial_y A_Y(x, y)((1-q)f_2(y) - f_1(x, y)) \right. \\ & + qf_4(x, y)f_0(x, y) - q^2f_6(x, y) + A_y(x, y)(f_5(x, y) + \\ & A_y(x, y)A_y(q^2x, y)((1-q)f_2(y) - f_1(x, y))A_y(x, y)A_x(q^2x, y)f_4(x, y) + \\ & qA_x(x, y)A_y(qx, q^2y)f_0(x, y) + q^2A_y(x, y)f_4(x, y)A_y(x, y) + \\ & \left. \left. A_y(x, y)f_3(x, y)A_y(q^2x, qy) \right\} \right] dx dx dy \\ & + \left[ R_{yyx} + qR_{yxy} + q^2R_{xyy} + (1-q)\left\{ -q^2\partial_y A_y(x, y)f_0(x, y) - q^2A_y(x, y)f_7(x, y) - \right. \right. \\ & \left. \left. A_y(x, y)A_y(q^2x, qy)f_8(x, y) \right\} \right] dy dy dx + qF_{xy}^q d^2x dy + F_{yx}^{q^2} d^2y dx, \end{aligned}$$

where:

$$\begin{aligned} R_{xxy} = & \partial_x \partial_x A_y(x, y) + \partial_x A_x(x, y)A_y(q^2x, qy) - q^2A_x(x, y)\partial_x A_y(qx, q^2y) + \\ & A_x(x, y)A_x(qx, q^2y)A_y(q^2x, qy) \\ R_{yxx} = & \partial_y \partial_x A_x(x, y) + \partial_y A_x(x, y)A_x(x, y) - q^2A_y(x, y)\partial_x A_x(qx, q^2y) + \\ & A_y(x, y)A_x(q^2x, qy)A_x(x, y) \\ R_{xyx} = & \partial_x \partial_y A_x(x, y) + \partial_x A_y(x, y)A_x(x, y) - q^2A_x(x, y)\partial_y A_x(qx, q^2y) + \\ & A_x(x, y)A_y(qx, q^2y)A_x(x, y) \\ R_{yyx} = & \partial_y \partial_y A_x(x, y) + \partial_x A_y(x, y)A(x, y) - q^2A_x(x, y)\partial_y A_x(qx, q^2y) + \\ & A_y(x, y)A_y(q^2x, qy)A_x(qx, q^2y) \\ R_{yxy} = & \partial_y \partial_x A_y(x, y) + \partial_y A_x(x, y)A_x(x, y) - q^2A_y(x, y)\partial_x A_y(qx, q^2y) + \\ & A_y(x, y)A_x(q^2x, qy)A_y(x, y) \\ R_{xyy} = & \partial_x \partial_y A_y(x, y) + \partial_x A_y(x, y)A_y(x, y) - q^2A_x(x, y)\partial_y A_y(qx, q^2y) + \\ & A_x(x, y)A_y(qx, q^2y)A_y(x, y) \end{aligned} \quad (55)$$

$$\begin{aligned}
F_{xy}^q &= \partial_x A_y(x, y) - q \partial_y A_x(x, y) + A_x(x, y) A_y(qx, q^2 y) - q A_y(x, y) A_x(q^2 x, qy) \\
F_{yx}^{q^2} &= \partial_y A_x(x, y) - q^2 \partial_x A_y(x, y) + A_y(x, y) A_x(q^2 x, qy) - q^2 A(x, y) A(qx, q^2 y)
\end{aligned} \tag{56}$$

$$\begin{aligned}
f_0(x, y) &= -b_{11}y^2 - qb_{10}y + q^2b_{22}x + b_{20}xy + qb_{21}xy^2 - b_{21} \\
f_1(x, y) &= -a_{11}y^2 - a_{10}y + a_{22}x + a_{20}xy + a_{21}xy^2 - a_{12} \\
f_2(x, y) &= -b_{20}y^2 - q^2b_{21}y - q^2b_{22}y \\
f_3(x, y) &= -q^2a_{11}y^2 - a_{10}y + q^2a_{22}x + qa_{20}xy + a_{21}xy^2 - qa_{12} \\
f_4(x, y) &= -q^2b_{11}y^2 - b_{10}y + q^2b_{22}x + qb_{20}xy + b_{21}xy^2 - qb_{12} \\
f_5(x, y) &= -qb_{21}y^2 - b_{20}y - qb_{22} \\
f_6(x, y) &= +qa_{12}y^2 + -a_{11}y + qa_{21}xy - qa_{22}xy^2 \\
f_7(x, y) &= +qb_{21}y^2 - b_{11}y + qb_{21}xy^2 - qb_{22}xy^2 \\
f_8(x, y) &= -b_{11}y^2 - b_{10}y + b_{22}x - b_{20}xy + b_{21}xy^2 - b_{12}.
\end{aligned} \tag{57}$$

The expression of the curvature components [eq\(55\)](#) and the deformed field strength [eqs\(56\)](#) are formally the same as those obtained by Kerner [\[2, 15\]](#). The functions [\[f<sub>i</sub>\(x, y\)\]](#) [i = 0, ..., 8](#) can be interpreted as a direct consequence of the non-commutativity property of the space.

The covariant [n = 3](#) differential calculus constructed in section [3](#) and [4](#) respectively for [q](#) [3<sup>rd</sup>](#) and [N<sup>th</sup>](#)-root of unity can be seen as a generalization of the case [n = 2](#). However, one cannot see [d<sup>2</sup> = 0](#) as a certain limit of [d<sup>3</sup> = 0](#) case. In the next section, we remind the differential calculus [d<sup>2</sup> = 0](#).

## 6 Differential calculus with nilpotency [n = 2](#) on a reduced quantum plane.

We recall that the exterior differential ["d"](#) on the reduced quantum plane satisfies usual properties [\[6 – 9\]](#), namely

- [i/](#) linearity,
- [i i /](#) Nilpotency,

$$d^2 = 0. \tag{58}$$

- [i i i /](#) Leibniz rule,

$$d(uv) = d(u)v + (-1)^n u d(v), \tag{59}$$

where

$$u \in \Omega^n, v \in \Omega^m \text{ and, } d(x) = dx, d(y) = dy, d1 = 0. \quad (60)$$

The deformed differential calculus satisfies:

$$x dx = q^2 dx x \quad (61)$$

$$x dy = q dy x + (q^2 - 1) dx y \quad (62)$$

$$y dx = q dx y \quad (63)$$

$$y dy = q^2 dy y \quad (64)$$

$$dy dx = -q^2 dx dy \quad (65)$$

$$(dx)^2 = (dy)^2 = 0. \quad (66)$$

So, the differential algebra  $\Omega$  is generated by  $x, y, dx$  and  $dy$ ,  $\Omega = \{x, y, dx, dy\}$ .

Using the standard realization of the differential  $d$ :

$$d = \frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy = \partial_x dx + \partial_y dy, \quad (67)$$

one can prove that:

$$\partial_x x = 1 + q^2 x \partial_x + (q^2 - 1) y \partial_y \quad (68)$$

$$\partial_y x = q x \partial_x \quad (69)$$

$$\partial_x y = q y \partial_x \quad (70)$$

$$\partial_y y = 1 + q^2 y \partial_y. \quad (71)$$

We apply this covariant differential calculus to study the related gauge field theory on  $M_3(C)$ .

## 7 Gauge field theory on $M_3(\mathbb{C})$ as reduced quantum plane with $d^2=0$

Similarly, the covariant differential is defined as in section 6:

$$D\Phi(x, y) = d\Phi(x, y) + A(x, y)\Phi(x, y). \quad (72)$$

The expression of the curvature is:

$$D^2\Phi(x, y) = (dA(x, y) + A(x, y)A(x, y))\Phi(x, y) = R\Phi(x, y). \quad (73)$$

The differential realization of "d" eqs(67–71) allows to rewrite the expression of the curvature R:

$$R = (\partial_x A_y(x, y) - q\partial_y A_x(x, y))dx dy + A_x(x, y)dx A_y(x, y)dy. \quad (74)$$

Using the differential calculus eqs(61–71) on the reduced quantum plane and the expressions of  $A_x(x, y)$ ,  $A_y(x, y)$  eqs(53, 54) as formal power series, it is easy to establish:

$$R = [\partial_x A_y(x, y) - q\partial_y A_x(x, y) + A_x(x, y)A_y(qx, q^2y) - qA_y(x, y)A_x(q^2x, qy) +$$

$$(1-q)A_y(x, y)\{-qb_{12} - b_{10}y + q^2b_{22}x - q^2b_{11}y^2 + qb_{20}xy + b_{21}xy^2\}]dx dy, \quad (75)$$

this permit us to define

$$\begin{aligned} F_{xy}^q &= \partial_x A_y(x, y) - q\partial_y A_x(x, y) + A_x(x, y)A_y(qx, q^2y) - qA_y(x, y)A_x(q^2x, qy) \\ &= -q\{\partial_y A_x(x, y) - q^2\partial_x A_y(x, y) + A_x(q^2x, qy)A_y(x, y) - q^2A_x(x, y)A_y(q^2x, qy)\} \\ &= -qF_{yx}^{q^2}, \end{aligned} \quad (76)$$

which is the  $q$ -deformed antisymmetric field strength.

The comparison of the two expressions of curvature ( $d^3=0$  section 5 and  $d^2=0$ ) will be given in the following section.

## 8 Discussions and concluding remarks

In this paper, we have constructed a differential calculus  $n=3$  nilpotent on the reduced quantum plane by mixing Kerner's idea and Coquereaux's technics. The notion of covariance for this differential calculus is also given

and we show that there is a quantum group structure behind this covariance. As an application, we have constructed a gauge theory based on this calculus.

In the case  $n = 3$ , the expressions of curvature contain additionnal terms [eqs\(55, 57\)](#) compared with [eq\(75\)](#). These terms can be interpreted as a generic consequence of the extension of the differential calculus  $d^2 = 0$  to the higher order  $d^3 = 0$ .

We can also compare our results with those of [Kerner & al \[2, 15\]](#). In fact, [eqs\(55, 56\)](#) are formally the same as in [\[2, 15\]](#), they differ only by the appearance of the deformation parameter  $q$ . However, there is no analogous of [eq\(57\)](#) in [\[2, 15\]](#). It is a direct consequence of the noncommutativity of the space considered here.

In a forthcoming paper, we shall treat in a mathematical way the correspondance between this calculus and the Heisenberg algebra. This correspondance is based on the bargman Fock representation and will give a new oscillator algebra. To study the minimization of incertitude principal in this case, we will try to find the eigenvectors of the annihilation operator in the way to construct the corresponding Klauder s coherent states [\[13\]](#).

## Acknowledgments

The authors Y. Hassouni and E. H. Zakkari wish to thank the Abdus Salam International Centre for Theoretical Physics for financial and scientific support, where a great part of this work has been done. This work was done within the framework of Associateship and Federation Arrangement Schemes of the Abdus Salam ICTP.

## Appendix:

We start by writting a priori  $xdx$ ,  $xdy$ ,  $ydx$  and  $ydy$  in terms of  $dxx$ ,  $dyx$ ,  $dxy$  and  $dyy$  ie:

$$xdx = a_1dxx + b_1dyx + c_1dxy + d_1dyd \quad (77)$$

$$xdy = a_2dxx + b_2dyx + c_2dxy + d_2dyd \quad (78)$$

$$ydx = a_3dxx + b_3dyx + c_3dxy + d_3dyd \quad (79)$$

$$ydy = a_4dxx + b_4dyx + c_4dxy + d_4dyd. \quad (80)$$

Differentiating the commutation relation  $xy = qyx$  and replacing  $xdx$  and  $xdy$  by their expressions in the formulae above, permit us to fix three unknown coefficients. Actually, we have *nine* independant parameters.

The left coaction of  $F$  on a quantum plane is defined by:

$$x_1 = a \otimes x + b \otimes y$$

$$y_1 = c \otimes x + d \otimes y.$$

Hence

$$dx_1 = a \otimes dx + b \otimes dy$$

$$dy_1 = c \otimes dx + d \otimes dy.$$

We impose that the relations between  $x_1$ ,  $y_1$  and  $dx_1$ ,  $dy_1$  be the same as the relations between  $x$ ,  $y$  and  $dx$ ,  $dy$ ; these conditions yields to:

$$a_2 = a_3 = a_4 = b_1 = b_4 = c_1 = c_4 = d_1 = d_2 = d_3 = 0 \text{ and } d_1 = a_4,$$

So, the unknown coefficients  $b_2$ ,  $b_3$ ,  $c_2$ , and  $c_3$  can be expressed in the terms of one unknown coefficient  $a_1$ . Indeed:

$$b_2 = \frac{q(1 + a_1)}{1 + q^2} \quad c_2 = \frac{a_1q^2 - 1}{1 + q^2}$$

$$b_3 = \frac{a_1 - q^2}{1 + q^2} \quad c_3 = \frac{q(1 + a_1)}{1 + q^2}.$$

Differentiating the relations (74–77) and noticing that  $dxdx$ ,  $d^2x$ ,  $dydy$  and  $d^2y$  are independant, we find  $a_1 = q^2$ . The left covariant differential calculus on a reduced quantum plane is hence constructed *eqs(9–22)*.

### References:

- [1]- V. Abramov and R. Kerner, J. Math. Phys, **41**, (2000) 5598.
- [2]- V. Abramov and R. Kerner, hep-th/9607143 (1996).
- [3]- R. Kerner, Math-ph/0011023 (2000).
- [4]- M. Daoud and Y. Hassouni, Int. Jour. Theor. Phys. **36**, (1996) 37.
- [5]- A. Elhassouni, Y. Hassouni and E. H. Tahri, Int. Jour. Theor. Phys. **35** (1996) 2517.
- [6]- R. Coquereaux, A. O. Garcia and R. Trincherro, Math-ph/9807012 (1998).
- [7]- R. Coquereaux, A. O. Garcia and R. Trincherro, Phys. Lett. B. **443** (1998) 221.
- [8]- T. Brezinski and J. Rembielinski, J. Phys. **A25** (1992) 1945.
- [9]- J. Wess and B. Zumino, Nucl. Phys. B (Proc Suppl) **18** B (1990) 302.
- [10]- H. Weyl, *The thory of groups and quantum mechanics* (Dover Publications, 1931).
- [11]- R. Coquereaux, A. O. Garcia and R. Trincherro, math-hep/9811017 (1998).
- [12]- R. Coquereaux and G. E. Schieber, *Action of finite quantum group on the algebra of complex  $N \times N$  matrices*. A.I.P Conference Proceedings. **45**.
- [13]- Y. Hassouni and E. H . Zakkari, in preparation.
- [14]- A. K. Mishra and G. Rajasekaran, J. Math. Phys. **38** (1997) 1.
- [15]- R. Kerner, math-ph/0004031 (2000).
- [16]- J. Madore, S. Schraml and J. Wess, Eur. phys. J . C **16** (2000) 161.
- [17]- B. Jurcco, P. Schupp and J. Wess, Nucl. phys. B. **584**. (2000). 784.
- [18]- B. Jurcco, S. Schraml, P. Schupp and J. Wess, Eur. phys. J C **17** (2000) 521.
- [19]- B. Jurcco, P. Schupp and J. Wess, Mod. Phys. Lett. A. **16**. (2001) 343.