# Kähler Forms and Cosmological Solutions in Type II Supergravities

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### Abstract

We consider cosmological solutions to type II supergravity theories where the spacetime is split into a FRW universe and a Kähler space, which may be taken to be Calabi-Yau. The various 2-forms present in the theories are taken to be proportional to the Kähler form associated to the Kähler space.

When considering cosmological compactifications in stringy supergravity theories, one usually considers the field strength or the field itself, of some RR- or NS-form to be equivalent to the volume form on some compact manifold [4, 6, 5]. As is well-known, on Kähler manifolds [3] one can define a covariantly constant 2-form, whose components with respect to some coordinate base we denote by $\mathcal{J}_{mn}$, which acts like the root of a volume form defined on the Kähler manifold. Due to this volume-form-like behaviour one can use any 2-form present in a $d$-dimensional theory to trigger the cosmological compactification down to $d - d_k$, where $d_k$ is the, real, dimension of the Kähler manifold (Analogous ideas have been used in spontaneous compactifications, see *e.g.* [8, 9]). Needless to say that when $d_k = 2$ the Kähler form is equivalent to the volume form, so that particular solutions of [6] ought to be recovered.

The idea of using the Kähler form instead of a volume form will be illustrated by discussing some simple cosmological solutions to type II supergravities. A more detailed study will be presented elsewhere. Spacetime will be split into a $d_k$-dimensional Kähler manifold and a $(d - d_k)$-dimensional Friedman-Robertson-Walker manifold. In section (1) we will consider the string common sector, where the Kalb-Ramond field will be taken to be proportional to $\mathcal{J}$, whereas in section (2) we will consider the type IIA RR 2-form field strength to be proportional to $\mathcal{J}$. Note that analogous solutions for the type IIB RR 2-form field can be obtained by applying S-duality to the common sector solution [1].

In the string frame, the equations of motion are, where $H = dB$ ($F_{(2)}$) is the field strength for the Kalb-Ramond (type IIA RR 1-form *resp.*) field,

$$0 \;=\; R_{\mu\nu} - 2\nabla_\mu\nabla_\nu\phi + \tfrac{1}{4}H_\mu{}^{\kappa\rho}H_{\nu\kappa\rho} - \tfrac{1}{2}e^{2\phi}\left[F_{(2)\mu\kappa}F_{(2)\nu}{}^{\kappa} - \tfrac{1}{4}g_{\mu\nu}F_{(2)}^2\right], \tag{1}$$

$$0 \;=\; R + 4\left(\partial\phi\right)^2 - 4\nabla^2\phi + \tfrac{1}{2\cdot 3!}H^2, \tag{2}$$

$$0 \;=\; \nabla_\mu\left(e^{-2\phi}H^{\mu\kappa\rho}\right) \;=\; \nabla_\mu F_{(2)}^{\mu\nu}, \tag{3}$$

and the Bianchi identities are $dH = dF_{(2)} = 0$.

# 1 Kalb-Ramond case

The Ansatz for the metric is taken to be

$$ds^2 = N^2(\sigma)d\sigma^2 - \eta^2(\sigma)\overline{g}_{ij}dx^i dx^j - R^2(\sigma)h_{mn}dy^m dy^n , \qquad (4)$$

where $\overline{g}_{ij}$ is a metric of constant curvature, *i.e.* $R(\overline{g})_{ij} \equiv \lambda\overline{g}_{ij}$, and $h_{mn}$ is a Ricci flat metric[1] on the Kähler manifold. Furthermore, the dimension of the Kähler manifold is $d_k$ and the dimension of the constant curvature part is $D \equiv d - 1 - d_k$, where $d$ is the dimension of spacetime, which will be left arbitrary. Note that it is natural to take the internal space to be compact, which means that it actually is a Calabi-Yau space. For the sake of argument though, we will always refer to the internal space as a Kähler space, since this is the only property that is really needed.

Our Ansatz for the Kalb-Ramond field, in form notation, reads

$$B = f(\sigma)\,\mathcal{J} = \tfrac{1}{2}f(\sigma)\mathcal{J}_{mn}dy^m \wedge dy^n , \qquad (5)$$

where $\mathcal{J}$ is the integrable almost complex structure, the Kähler form, on the Kähler manifold and as such satisfies [3]

$$\mathcal{J}^m{}_p\mathcal{J}^p{}_n = -\delta^m{}_n \quad , \quad \overline{\nabla}_m \mathcal{J}_{np} = 0, \qquad (6)$$

where $\overline{\nabla}$ is the connection on the Kähler manifold. Finally, the dilaton is taken to depend on time, $\sigma$, only.

Calculating the equation of motion for the Kalb-Ramond field, one finds that

$$\dot{f} = \aleph N e^{2\phi}\eta^{-D}R^{4-d_k} , \qquad (7)$$

where $\aleph$ is an arbitrary constant. Defining the field

$$\psi = \phi + \tfrac{1}{2}\log{(N)} - \tfrac{D}{2}\log{(\eta)} - \tfrac{d_k}{2}\log{(R)} , \qquad (8)$$

introducing $M = e^{-2\psi}N$ and changing variables by $e^{2\psi}d\sigma = dt$, one can write the equations of motion as

$$\begin{aligned}
0 &= (\log R)'' + \tfrac{\aleph^2}{2}R^4 , &(9)\\
0 &= (\log\eta)'' - \lambda\eta^{-2}M^2 , &(10)\\
0 &= (\log M)'' - D\lambda\eta^{-2}M^2 , &(11)\\
0 &= \left[(\log M)'\right]^2 - D\left[(\log\eta)'\right]^2 - d_k\left[(\log R)'\right]^2 - \tfrac{d_k\aleph^2}{4}R^4 , &(12)
\end{aligned}$$

where the *prime* indicates derivation with respect to $t$.

The above equations can easily be solved to give $\lambda = 0$, $M = e^{\alpha t}$, $\eta = e^{\beta t}$ and

$$R(t) = R_0 \cosh^{-1/2}\left(\aleph R_0^2 t\right) , \qquad (13)$$

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[1]Most of the equations in what follows can easily be generalized to include a curvature for $h_{mn}$.

which after substitution in Eq. (12) leads to

$$\alpha^2 - D\beta^2 = \frac{d_k \aleph^2 R_0^4}{4} \,.$$

(14)

In these coordinates the dilaton, or equivalently the string coupling squared, is $g_s^2 = e^{2\phi} = M^{-1}\eta^D R^{d_k}$ and the Kalb-Ramond field strength is $H = \aleph R^4 dt \wedge \mathcal{J}$.

Although there is a world of possibilities in the above class of solutions, perhaps the most interesting case is the one where $\beta = 0$, since then the uncompactified part of spacetime is just Minkowski space: Taking $\alpha = 1$ for convenience, one finds that the solution in, string, cosmological time, $\tau$, reads

$$
\begin{aligned}
ds^2 &= d\tau^2 - d\vec{x}_{(D)} - 2R_0^2 B(\tau)^{-1} h_{mn} dy^m dy^n \,, \\
e^{2\phi} &= \left(\sqrt{2}R_0\right)^{d_k} \tau^{-1} B(\tau)^{-d_k/2} \,, \\
H &= 8R_0^2 d_k^{-1/2}\tau^{-1} B(\tau)^{-2}\, d\tau \wedge \mathcal{J} \,, \\
B(\tau) &= \tau^{2/\sqrt{d_k}} + \tau^{-2/\sqrt{d_k}} \,.
\end{aligned}
$$

(15)

(16)

As one can see, this is a completely regular solution, modulo the usual gravitational singularities, which smoothly interpolates between two Kasner-like regions [6]. From the lower dimensional point of view, the Ansatz considered above corresponds to a solution of dilaton-gravity coupled to moduli [10], where the breathing mode, $R$, and $f$ are the scalar fields parameterizing an $SL(2,\mathbb{R})/U(1)$ coset model. When $d_k = 6$, the above solutions can be obtained from the solutions given in [2] by applying an $SL(2,\mathbb{R})$ transformation on the moduli.

## 2 RR case

In much the same way as in the foregoing subsection, we can use the RR two form in type IIA, to trigger compactification. In this case the equations of motion and the Bianchi identity imply that

$$F_{(2)} = \aleph \mathcal{J} = \tfrac{1}{2}\aleph \mathcal{J}_{mn} dy^m \wedge dy^n \,.$$

(17)

Applying the same steps as in the foregoing paragraph, one finds

$$
\begin{aligned}
0 &= (\log R)'' + \tfrac{(d_k - 4)\aleph^2}{8} M\eta^D R^{d_k - 4} \,, \\
0 &= (\log \eta)'' + \tfrac{d_k \aleph^2}{8} M\eta^D R^{d_k - 4} - \lambda\,\eta^{-2} M^2 \,, \\
0 &= (\log M)'' - \tfrac{d_k \aleph^2}{8} M\eta^D R^{d_k - 4} - D\lambda\,\eta^{-2} M^2 \,, \\
0 &= \left[(\log M)'\right]^2 - D\left[(\log \eta)'\right]^2 - d_k \left[(\log R)'\right]^2 - \tfrac{d_k \aleph^2}{4} M\eta^D R^{d_k - 4} - D\lambda M^2 \eta^{-2}
\end{aligned}
$$

(18)

(19)

(20)

(21)

Looking at the above expressions, one sees that they simplify enormously when one considers the case $d_k = 4$: In that case the Kähler breathing mode decouples completely and one has $R = R_0 e^{\alpha t}$. Equating also the powers of $M$ and $\eta$ in the equations, *i.e.* putting $M = \eta^{D+2}$, one necessarily has to impose

$$\lambda = \frac{\aleph^2}{4}(D+3) \,.$$

(22)

The two remaining equations are implied by

$$(\log \eta)' = \pm \left[ \frac{\aleph^2}{4} \eta^{2(D+1)} + \frac{4\alpha^2}{D(D+3)+4} \right]^{1/2} . \tag{23}$$

Now, when $\alpha \neq 0$ the solution to the above equation is complex, but when $\alpha = 0$ one finds that

$$\eta = \left( A \mp \frac{(D+1)\aleph^2}{2} t \right)^{-\frac{1}{D+1}} , \tag{24}$$

where the range of $t$ has to be chosen such that the function is well defined.

For a stringy cosmological observer, *i.e.* introducing a time coordinate $M dt = d\tau$, the above solution is

$$ds^2 = d\tau^2 - \left( \frac{\aleph^2}{2} \tau \right)^2 d\Omega_\lambda^2 - R_0^2 h_{mn} dy^m dy^n , \tag{25}$$

$$e^\phi = 4R_0^2 \aleph^{-2} \tau^{-1} . \tag{26}$$

So, we see that the solution describes a 6-dimensional open FRW with a dilaton such that the string coupling strength goes to zero when $\tau \to \infty$.

In order to find a solution for general $d_k$, we shall follow the same stratagem as above, *i.e.* we put $R = \eta^\alpha$, $M = \eta^\beta$ and will equate the powers on the righthand sides of Eqs. (21). It then follows that $\beta = D + 2 + \alpha(d_k - 4)$, using which one can calculate

$$\lambda = \frac{\aleph^2}{16} \left( d_k(D+3) + (d_k - 4)^2 \right) , \tag{27}$$

$$\alpha = -2 \frac{d_k - 4}{d_k(D+1) + (d_k - 4)^2} , \tag{28}$$

$$\beta = \frac{d_k(D+1)(D+2) + D(d_k - 4)^2}{d_k(D+1) + (d_k - 4)^2} , \tag{29}$$

The dilaton is then fixed to $e^\phi = \eta^{2\alpha - 1}$.

As before, the two remaining equations are implied by

$$(\log \eta)' = \pm B \eta^{\beta - 1} , \tag{30}$$

where

$$B^2 = \frac{\aleph^2}{16} \frac{\left[ d_k(D+1) + (d_k - 4)^2 \right]^2}{d_k(D+1) + (D-1)(d_k - 4)^2} . \tag{31}$$

This then means that

$$\eta = \left[ A \mp (\beta - 1) B t \right]^{\frac{1}{\beta - 1}} , \tag{32}$$

where $A$ is some integration constant and the range of $t$ has to be chosen such that the above function is well-defined.

Choosing the minus-sign in the last equation in order to switch to the cosmological time, $\tau$, one finds that $\eta = B\tau$ and the solution reads

$$ds^2 = d\tau^2 - B^2 \tau^2 d\Omega_\lambda^2 - (B\tau)^{2\alpha} h_{mn} dy^m dy^n , \tag{33}$$

$$e^\phi = (B\tau)^{2\alpha - 1} . \tag{34}$$

4

Seeing Eq. (28), one may wonder whether the breathing mode for the Kähler mode could grow faster than $\pi$, and thus spoil cosmological compactification. This can however happen for $d_k = 2$ only, in which case $\alpha = 2d^{-1}$ which, seeing that we at least must have $d = 3$ in order to apply the Ansatz, is always smaller than 1. A similar analysis for the dilaton shows that it can blow up at large times only when $d_k = 2$, $d = 3$ and is regular in the rest of the cases.

# 3    Conclusions

Although more general solutions are bound to exist, the simple examples discussed in this work show that using a Kähler form instead of a volume form is a viable option when looking for stringy cosmological solutions.

It would be interesting to see whether this kind of compactifications survive when coupling to matter is enabled and whether the dimension of the Kähler manifold will be of any importance. It would also be interesting to generalize the above work to Heterotic strings/M-theory (See *e.g.* [7]) and to investigate cosmological solutions when other type II fields are present. Work in these directions is in progress.

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