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Zhu's Theorem and an algebraic characterization of chiral blocks

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Abstract

Working in the axiomatic framework recently proposed by Gaberdiel and Goddard, we prove a generalized version of Zhu's Theorem; for any chiral bosonic conformal field theory on the sphere, our result characterizes the chiral blocks in terms of a certain quotient of the Fock space. We also establish, under a finiteness hypothesis closely related to rationality of the theory, that the relevant Knizhnik-Zamolodchikov-type equation admits solutions.

1 Introduction

The problem of determining the chiral blocks in a given conformal field theory is *a priori* a difficult one. In certain specific cases this problem has been completely solved – e.g., for a broad class of well behaved theories, the chiral blocks are understood to arise from "Feynman diagrams" with only 3-valent vertices, with the interaction vertices completely determined by the fusion rules [3]. Furthermore the spaces of chiral blocks have been computed explicitly in certain cases, e.g. the WZW models, for which they have a straightforward algebraic description as spaces of coinvariants [7], [21].

For more general theories (even rational ones), somewhat less is known. The most significant progress was made by Zhu, who in [28] introduced a completely algebraic technique for determining the highest weight representations of a vertex operator algebra; namely, he constructed a functor which associates to a vertex operator algebra V (with conformal weights in $\mathbb{N}$) an associative algebra $A(V)$, such that the irreducible representations of $A(V)$ are in 1-1 correspondence with the irreducible highest weight representations of V. In fact, given a representation of $A(V)$, Zhu constructed the corresponding representation of V by first defining correlation functions on the sphere and then factorizing to obtain the states; so when V is the algebra of fields in the vacuum sector of a chiral conformal field theory, we can interpret Zhu's construction as giving all

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the 2-point chiral blocks. Roughly speaking, $A(V)$ is the algebra of zero modes of V (see [2] for a discussion of this point, [5], [22] for other facets of $A(V)$, and [9], [23], [4] for some explicit calculations). A modification of Zhu's construction allows one to compute, in a similar algebraic fashion, the fusion rules of the theory [9].

This paper is mainly concerned with a generalization of Zhu's construction to the case of k-point conformal blocks. We now turn to a description of its contents.

In Section 2 we fix notation and briefly review the formalism introduced by Gaberdiel and Goddard in [16], which is a convenient framework in which to state the theorems of this paper.

In Section 3 we discuss Zhu's construction and a natural generalization, first mentioned in [16], which associates to V and any $\mathbf{u} = (u_1, \ldots, u_k) \in \mathbb{P}^k$ a vector space $A_{\mathbf{u}}$. This $A_{\mathbf{u}}$ will be obtained as a quotient of the Fock space at $0 \in \mathbb{P}$.

In Section 4 we prove that, when the u_i are distinct, any linear functional $\eta \in (A_{\mathbf{u}})^*$ corresponds to the value of a chiral block at $\mathbf{u}$, in the sense that η induces a consistent prescription for correlation functions $\langle \prod_{i=1}^k \phi_i(u_i) \cdots \rangle$ where the dots indicate arbitrary insertions of the vertex operators in V. So η corresponds to a particular way of coupling some set of V-primary fields ϕ_i placed at the points u_i.

In Section 5 we introduce a certain finiteness condition on V which generalizes Zhu's "condition C." Under this condition, which appears to be closely related to rationality of V, we show that the Knizhnik-Zamolodchikov-type equation governing the $\mathbf{u}$-dependence of the chiral block (obtained essentially by making the substitution $L_{-1} \to \partial$) admits a solution.

Finally, in Section 6 we discuss remaining open questions, and a possible relation between the present work and the Friedan-Shenker vector bundle formalism [10].

2 Hypotheses and notation

We assume the reader is familiar with basic notions of conformal field theory, as found for instance in [3], [13]. Some acquaintance with the language of vertex operator algebras [8], [20], [1] is also helpful.

At all times in this paper we are considering a fixed chiral bosonic conformal field theory on the sphere $\mathbb{P}$. To be completely rigorous, by a "chiral bosonic conformal field theory" we mean an object of the type discussed in [16]. The details of the chosen formalism are generally not essential to following the ideas of this paper, however; what is essential is just that a chiral conformal field theory on $\mathbb{P}$ is regarded as defined by its amplitudes. We write these amplitudes $\langle \prod_{a=1}^k V(\psi_a, z_a) \rangle$, with the vertex operator corresponding to ψ written $V(\psi, z)$ and its modes written

$$V(\psi, z) = \sum_{n \in \mathbb{Z}} V_n(\psi) z^{-n-h_\psi} \tag{1}$$

(so the grading is by conformal weight, which we always represent by the letter h). The only exception to this rule is the Virasoro field which we write $L(z)$, with modes L_n. The space of Virasoro quasiprimary states is denoted by V; it has a grading by conformal weight, which we assume can be taken of the form $V = \oplus_{h=0}^{\infty} V^h$, with each V^h finite-dimensional.

In [16] the role of "space of states" is played by a collection of topological vector spaces denoted $\mathcal{V}^{\mathcal{O}}$, for $\mathcal{O}$ an open set in $\mathbb{P}$ (usually with $\mathbb{P} - \mathcal{O}$ simply connected.) These vector spaces are obtained by factorization from the amplitudes, with two states regarded as equal just if they agree in correlation functions with vertex operators inside $\mathcal{O}$. Roughly speaking, an element $\chi \in \mathcal{V}^{\mathcal{O}}$ is a coherent state (or limit of coherent states) constructed out of fields away from $\mathcal{O}$. In this paper we make explicit reference to $\mathcal{V}^{\mathcal{O}}$ only occasionally; when we do, $\mathcal{O}$ will always have simply connected complement. We will make frequent use of the Fock space $\mathcal{H}$ at 0, which is defined similarly by factorization.

We will often consider meromorphic functions and differentials defined on the Riemann sphere $\mathbb{P}$. It is convenient to use the language of "divisors" (see [17]) to classify the zeros and poles of these functions. So let a divisor on $\mathbb{P}$ be any formal sum of the form

$$D = \sum_{P \in \mathbb{P}} c_P [P], \qquad c_P \in \mathbb{Z}, \qquad \text{finitely many } c_P \neq 0. \tag{2}$$

Divisors can be added in the obvious way. We say $D \geq 0$ if all $c_P \geq 0$. Now let $\nu_P(f)$ denote the order of vanishing of f at P, and define the divisor of f to be

$$\text{div } f = \sum_{P \in \mathbb{P}} \nu_P(f)[P], \tag{3}$$

so that div $f \geq -D$ if the poles of f are "at worst given by D." Clearly div $fg =$ div $f +$ div g. We can similarly define div ω where ω is a meromorphic k-differential on $\mathbb{P}$; explicitly, such an ω can always be written as $\omega = f dz^{\otimes k}$ for some f, and then we have

$$\text{div } \omega = \text{div } f + k \text{div } dz = \text{div } f - 2k[\infty]. \tag{4}$$

The crucial analytic property which the amplitudes of the theory must possess is that for $\langle V(\psi, z) \cdots \rangle dz^{\otimes h_\psi}$ is meromorphic on $\mathbb{P}$ for any $\psi \in V$, and has poles only when z meets the coordinates of the insertions $\cdots$.

3 Zhu's subspace

Zhu in [28] introduced a purely algebraic mechanism for determining the highest weight representations of a chiral theory. This construction, when generalized to k-point functions, amounts to the following: Fix $\mathbf{u} \in (\mathbb{P} - \{0\})^k$ and consider a subspace $O_{\mathbf{u}} \subset \mathcal{H}$ of the Fock space, defined by

$$O_{\mathbf{u}} = \text{Span} \left\{ \oint_0 dz g(z) V(\psi, z) \chi \;\middle|\; \chi \in \mathcal{H}, \psi \in V, \right. \\ \left. \text{div } g dz^{\otimes -h_\psi + 1} \geq -N[0] + \sum_{i=1}^{k} h_\psi [u_i] \text{ for some } N \right\}. \tag{5}$$

This definition can be motivated in the following way: as described in [16], define a "highest weight state at $\mathbf{u}$" to be any state Σ such that

$$\text{div } \left(\langle \Sigma V(\psi, z) \rangle dz^{\otimes h_\psi} \right) \geq \sum_{i=1}^{k} -h_\psi [u_i], \qquad \forall \psi \in \mathcal{H}. \tag{6}$$

(Informally, the idea is that Σ should stand for insertions of primary fields $\phi_1(u_1)\cdots\phi_k(u_k)$; then the requirement (6) just says that each ϕ_i is annihilated by positive modes of $V(\psi, z)$.) If we fix such a Σ and consider $\omega \in O_{\mathbf{u}}$, then $\langle\Sigma\omega\rangle$ must vanish, because substituting the integral appearing in (5) for ω we find that the resulting integrand has been engineered to have no poles on $\mathbb{P}-\{0\}$. $O_{\mathbf{u}}$ therefore represents a space of "states in the Fock space at 0 which are orthogonal to primary fields placed at the points $u_1, \ldots, u_k$."

For most of this paper we consider only the case where all u_i are distinct (though the behavior of $O_{\mathbf{u}}$ when some of the u_i come together is important — in particular, it motivates the finiteness condition we impose in Section 5, and also see Section 6.)

Now suppose given a particular Σ which is a highest weight state at $\mathbf{u}$. Then Σ induces a linear functional $\eta : \mathcal{H} \to \mathbb{C}$ by the rule $\eta(\chi) = \langle\Sigma\chi\rangle$, and we have just seen that this η must vanish on $O_{\mathbf{u}}$. Conversely, in the special case $k = 2$, Zhu essentially showed that any $\eta : \mathcal{H} \to \mathbb{C}$ vanishing on $O_{\mathbf{u}}$ in fact comes from a $\Sigma \in \mathcal{V}^{\mathcal{O}}$ satisfying (6) — in other words, every such η comes from a highest weight representation. So we have a correspondence between linear functionals on $\mathcal{H}/O_{\mathbf{u}}$ and representations of our theory. Actually, different linear functionals can give rise to equivalent representations; the precise formulation in [28] defines an algebra structure on the quotient $\mathcal{H}/O_{\mathbf{u}}$ and shows that the irreducible representations of this algebra are exactly the highest weight representations of the theory. This is a remarkable result as it provides a completely systematic way of constructing the representations, which are *a priori* rather complicated objects from an algebraic standpoint since one needs to specify the action of every field in the theory.

In calculations it will be useful to know that the function g in (5) can be chosen to have an ancillary property, namely, we can choose it to be holomorphic in $\mathbf{u}$: e.g. it is easily checked that

$$g_N(z) = z^{-(N+1+(k-2)h)} \prod_{i=1}^{k} (z - u_i)^h \tag{7}$$

satisfies the condition in (5) for all $N \geq 1$. Furthermore, a straightforward induction shows that it is actually sufficient to use only the g_N in the definition of $O_{\mathbf{u}}$ (essentially because a function satisfying (5) is determined by its singular part at 0.)

4 A generalization of Zhu's Theorem

We will now give a generalization of Zhu's result mentioned above — which can be viewed as a construction of correlation functions corresponding to insertions of representations of $A(V)$ at 2 points — to general k-point functions.

To prove our generalized version of Zhu's Theorem we need to construct correlation functions which induce a given linear $\eta : \mathcal{H} \to \mathbb{C}$ (vanishing on $O_{\mathbf{u}}$). We represent these (putative) correlation functions by the notation

$$\langle\prod_{a=1}^{l} V(\psi_a, z_a)\rangle_\eta, \tag{8}$$

where the subscript η reminds us that these are not the vacuum correlation functions. In a sense we have no choice in defining the functions (8), because to say they are induced from η is exactly to say that η already defines for us their Laurent series; the real question is whether these series converge. However, working with these Laurent series will be somewhat difficult, because the correlation functions are expected to have poles whenever two of the z_i coincide, and the series in question are expanded about the point $\mathbf{z} = (0, \ldots, 0)$. We must therefore be careful about the domain in which we are working. We will use the letter R to denote one of the $l!$ possible permutations of the coordinates $z_1, \ldots, z_l$; by abuse of notation, R is the region $\{\mathbf{z} : |z_{R(1)}| > |z_{R(2)}| > \cdots > |z_{R(l)}|\}$.

So given η, some $(\psi_1, \ldots, \psi_l) \in V^{\otimes l}$, and a region R, we define a formal power series by

$$\langle \prod_{a=1}^{l} V(\psi_a, z_a) \rangle_{\eta,R} = \sum_{\mathbf{j} \in \mathbb{Z}^l} \eta \left(\prod_{i=1}^{l} V_{j_{R(i)}}(\psi_{R(i)}) \right) \mathbf{z}^{-\mathbf{j}-\mathbf{h}} \tag{9}$$

where $\mathbf{j}$ is a multi-index, so e.g. by $\mathbf{z}^{-\mathbf{j}-\mathbf{h}}$ we mean $\prod_{i=1}^{l} z_i^{-j_i - h_i}$. For η which could be induced from primary fields inserted at $\mathbf{u}$ — in other words, η vanishing on the subspace $O_{\mathbf{u}}$ introduced in Section 3 — we will show that the power series (9) are the Laurent expansions of a single function $\langle \prod_{a=1}^{l} V(\psi_a, z_a) \rangle_{\eta}$ in the different regions R. This is the content of Theorem 4, toward which we are working (the next three lemmas are somewhat technical, so the reader may want to flip to the theorem first.)

In order to prove Theorem 4 we first establish that the series (9) obey a formal version of the operator product expansion. To formulate this statement precisely we need one more bit of notation: for a meromorphic function $f(\mathbf{z})$ with poles only at $z_i = z_j$, let $\iota_{i,j} f$ mean "the Laurent series for f around $(0, \ldots, 0)$, expanded in the region where $|z_i| > |z_j|$" (this notation is commonly employed in the study of vertex operator algebras, see e.g. [20], [8]). Then we can state

Lemma 1. *Let $R = \{|z_1| > \cdots > |z_l|\}$ (for simplicity). Then for any $\eta : \mathcal{H} \to \mathbb{C}$ the power series defined by η obey the following "operator product expansion" identities:*

- *For all $m \in [1, l)$, we have the OPE as $z_m \to z_l$:*

$$\begin{aligned} &\langle \prod_{a=1}^{l} V(\psi_a, z_a) \rangle_{\eta,R} = \\ &\quad \sum_{i=m+1}^{l-1} \sum_{n=-h_1-h_i}^{-1} (\iota_{m,i} - \iota_{i,m})(z_m - z_i)^n \langle V(V_{-n-h_m}(\psi_m)\psi_i, z_i) \prod_{a \neq m,i} V(\psi_a, z_a) \rangle_{\eta,R} \\ &\quad + \sum_{n=-h_m-h_l}^{\infty} \iota_{m,l}(z_m - z_l)^n \langle V(V_{-n-h_m}(\psi_m)\psi_l, z_l) \prod_{a \neq m,l} V(\psi_a, z_a) \rangle_{\eta,R}. \end{aligned} \tag{10}$$

- *We also have an OPE as $z_l \to z_{l-1}$:*

$$\begin{aligned} &\langle \prod_{a=1}^{l} V(\psi_a, z_a) \rangle_{\eta,R} = \\ &\quad \sum_{n=-h_{l-1}-h_l}^{\infty} \iota_{l,l-1}(z_l - z_{l-1})^n \langle V(V_{-n-h_l}(\psi_l)\psi_{l-1}, z_{l-1}) \prod_{a \neq l-1,l} V(\psi_a, z_a) \rangle_{\eta,R}. \end{aligned} \tag{11}$$

Proof. The essence of the proof is the observation that checking any coefficient in the above identities involves only a finite computation; this computation amounts to verifying that a certain state $\chi \in \mathcal{H}$ is annihilated by η. But the OPE of the conformal field theory then shows that this same χ is annihilated by the linear functionals induced by correlations with vertex operators inserted away from 0. Then by the factorization property $\chi = 0$, so naturally $\eta(\chi) = 0$. In other words: the operator product expansion is already encoded in the definition of $\mathcal{H}$, so naturally every linear functional on $\mathcal{H}$ must obey it.

More explicitly: first we prove (10). For notational simplicity we consider only the case $m = 1$. Fix a multi-index $\mathbf{j}$ and consider the coefficient of $\mathbf{z}^{\mathbf{j}}$ in (10); we have to show this coefficient receives only finitely many contributions on each side. The left side manifestly has only a single term involving $\mathbf{z}^{\mathbf{j}}$. On the other hand, each term in the sums on the right side can contain at most one contribution to the coefficient of $\mathbf{z}^{\mathbf{j}}$. The double sum contains only finitely many terms, so manifestly makes only a finite contribution. The single sum contains infinitely many terms; indeed, replacing $(z_1 - z_l)^n$ and $\langle V(V_{-n-h_1}(\psi_1)\psi_l, z_l) \prod_{a\neq 1,l} V(\psi_a, z_a)\rangle_{\eta,R}$ by their power series expansions, we find that for each $k \geq 0$ the coefficient of $\mathbf{z}^{\mathbf{j}}$ can receive a contribution proportional to

$$\eta\left(\left(\prod_{i=2}^{l-1} V_{-j_i-h_i}(\psi_i)\right) V_{-j_1-j_l-h_1-h_l}(V_{j_1-h_1-k}(\psi_1)\psi_l)\Omega\right). \tag{12}$$

So we need to show that only finitely many of these terms are nonzero. When $\mathbf{j}$ is fixed, then setting $\alpha = j_1 + j_l + h_1 + h_l, \beta = h_1 + h_l - j_1$, we find that (12) depends linearly on a state of the form $V_{-\alpha}(\chi_k)\Omega$ where χ_k has weight $\beta + k$. But such an expression always vanishes when $k > \alpha - \beta$. (Note that we are taking advantage of a special property of the vacuum, and so this only works because z_l is the coordinate closest to the origin — this is the reason why we restricted ourselves to that special case.) Hence the coefficient of $\mathbf{z}^{\mathbf{j}}$ receives only finitely many contributions on each side of (10), and we can rewrite (10) in the form

$$\eta(\chi_{\mathbf{j}}) = 0 \tag{13}$$

for some $\chi_{\mathbf{j}} \in \mathcal{H}$.

Now choose some $\mathcal{O} \subset \mathbb{P}$ containing 0. Then the space $\mathcal{V}^{\mathcal{O}}$ of limits of coherent states is embedded in $\mathcal{H}^*$ as a dense subspace (in the weak-$*$ topology induced from $\mathcal{H}$ — see [16]). But for η induced from $\Sigma \in \mathcal{V}^{\mathcal{O}}$, the power series we are considering actually do converge — $\langle \prod_{a=1}^{l} V(\psi_a, z_a)\rangle_{\eta,R}$ is exactly the power series expansion of $f(\mathbf{z}) = \langle \Sigma \prod_{a=1}^{l} V(\psi_a, z_a)\rangle$ for $\mathbf{z} \in R$. To determine this expansion we make use of a construction from [12], as follows. Fix $z_2, \ldots, z_l \in \mathcal{O}$ with $|z_2| > \cdots > |z_l|$ and note that for z_1 close enough to z_l we have [3], [13]

$$f(\mathbf{z}) = \sum_{n=-h_1-h_l}^{\infty} (z_1 - z_l)^n \langle \Sigma V(V_{-n-h_1}(\psi_1)\psi_l, z_l) \prod_{a\neq 1,l} V(\psi_a, z_a)\rangle. \tag{14}$$

From the operator product expansion we also know the pole structure of the correlation function, so that if we put

$$g(\mathbf{z}) = \sum_{i=2}^{l-1} \sum_{n=-h_1-h_i}^{-1} (z_1 - z_i)^n \langle \Sigma V(V_{-n-h_1}(\psi_1)\psi_i, z_i) \prod_{a\neq 1,i} V(\psi_a, z_a)\rangle \tag{15}$$

then $f(\mathbf{z})-g(\mathbf{z})$ has no poles as a function of $z_1 \in \mathcal{O}$. Its power series expansion about $z_1 = z_l$ therefore converges on any disc contained in $\mathcal{O}$. So we can write $f(\mathbf{z})$ when $|z_1| > |z_2|$ as (the expansion of $f(\mathbf{z})-g(\mathbf{z})$ for z_1 near z_l) plus (the expansion of $g(\mathbf{z})$ for $|z_1| > |z_2|$). This gives exactly (10) except that we have substituted fixed values for $z_2, \ldots, z_l$; but since those values were arbitrary we get (10) as an identity of functions, which implies the desired identity of power series. So (10), and hence (13), hold for all $\eta \in \mathcal{V}^{\mathcal{O}}$. Then (13) and (10) must hold for all $\eta \in \mathcal{H}^*$, completing the proof of (10) (in the case $m = 1$, but the other m are proven in an exactly analogous way.)

The proof of (11) is similar to the proof of (10) in case $m = l-1$ (which is actually somewhat easier than the general case because there are no poles to be subtracted.) The point is that the necessary finiteness condition will hold on the right side of (11), because after we decompose the field at z_l into modes acting at z_{l-1}, the field at z_{l-1} will be the one closest to the origin; so we can argue as above. ■

Fix k and fix some $\mathbf{u} = (u_1, \ldots, u_k) \in (\mathbb{C} - \{0\})^k$, with all u_i distinct. Our strategy in proving Theorem 4 will be to first establish convergence of modified power series in which we have shifted the poles from $z_i = u_j$ to $z_i = \infty$; once this is established the rest is easy. The next two lemmas concern these modified power series.

Lemma 2. *Fix $l \geq 0$ and let $R = \{|z_1| > \cdots > |z_l|\}$. Suppose given $\eta : \mathcal{H} \to \mathbb{C}$ such that $O_{\mathbf{u}} \subset \ker \eta$, and $(\psi_1, \ldots, \psi_l) \in V^{\otimes l}$. Then in the power series*

$$\left(\prod_{m=1}^{k} (z_1 - u_m)^{h_1}\right) \langle \prod_{a=1}^{l} V(\psi_a, z_a) \rangle_{\eta,R} \tag{16}$$

the coefficient of $\mathbf{z}^{\mathbf{j}}$ vanishes whenever $j_1 > (k-2)h_1$.

Proof. The motivating idea is that as a function of z_1 in the region R, $\langle \prod_{a=1}^{l} V(\psi_a, z_a) \rangle_\eta$ should only have poles at $z_1 = u_i$, of order at most h_1. By multiplying by $\prod_{m=1}^{k}(z_1 - u_m)^{h_1}$ we convert all these poles to a single pole at ∞, still of bounded order; and since all other poles have been removed, the power series expansion in R of the resulting function can be expected to converge up to $z_1 = \infty$. The bounded order of the pole at ∞ will therefore be manifested as a cutoff in the power series.

Explicitly, the proof consists in noting that the coefficient of $\mathbf{z}^{\mathbf{j}}$ in (16) is given by

$$\eta\left(\oint_0 \frac{dz_1}{z_1^{j_1+1}} \prod_{m=1}^{k} (z_1 - u_m)^{h_1} V(\psi_1, z_1) \prod_{a=2}^{l} V_{-j_a - h_a}(\psi_a)\Omega\right) \tag{17}$$

which vanishes by hypothesis for $j_1 > (k-2)h_1$. ■

Now we are in a position to show that our modified power series actually converge.

Lemma 3. *Suppose given $\eta : \mathcal{H} \to \mathbb{C}$ such that $O_{\mathbf{u}} \subset \ker \eta$. Fix $l \geq 0$, and for any $(c_1, \ldots, c_l) \in (\mathbb{Z}^+)^l$, define*

$$\Pi = \prod_{j=1}^{l} \prod_{m=1}^{k} (z_j - u_m)^{c_j}. \tag{18}$$

Then for any $(\psi_1, \ldots, \psi_l) \in V^{\otimes l}$, the c_j can be chosen sufficiently large that the power series

$$\Pi \cdot \langle \prod_{a=1}^{l} V(\psi_a, z_a) \rangle_{\eta,R} \tag{19}$$

is convergent in R. Furthermore, this power series can be continued to a meromorphic function defined on $\mathbb{C}^l$, with poles only at $z_i = z_j$, independent of R. For $l > 1$ this function is given recursively by the formula

$$\begin{aligned}\Pi \cdot \langle \prod_{a=1}^{l} V(\psi_a, z_a) \rangle_{\eta} =&\Pi \cdot \sum_{i=2}^{l} \sum_{n=-h_1-h_i}^{-1} (z_1 - z_i)^n \langle V(V_{-n-h_1}(\psi_1)\psi_i, z_i) \prod_{a\neq 1,i} V(\psi_a, z_a) \rangle_{\eta} \\ &-\Pi \cdot \sum_{i=2}^{l-1} \sum_{n=-h_1-h_i}^{-1} \iota_{i,1}(z_1 - z_i)^n \langle V(V_{-n-h_1}(\psi_1)\psi_i, z_i) \prod_{a\neq 1,i} V(\psi_a, z_a) \rangle_{\eta} \\ &+\Pi \cdot \sum_{n=0}^{\infty} (z_1 - z_l)^n \langle V(V_{-n-h_1}(\psi_1)\psi_l, z_l) \prod_{a\neq 1,l} V(\psi_a, z_a) \rangle_{\eta},\end{aligned} \tag{20}$$

which must be interpreted as follows: the first term is a meromorphic function in z_1, and the second two together define a convergent power series in z_1 (in fact a polynomial.)

Proof. By induction on l. For $l = 1$, choose $c_1 = h_1$; then we are just considering

$$\prod_{i=1}^{k} (z - u_i)^h \langle V(\psi, z) \rangle_{\eta,R} \tag{21}$$

and Lemma 2 says this series can have no power of z exceeding $z^{(k-2)h}$. On the other hand, from the definition we see immediately that it can have no pole at $z = 0$; so in this case the series is just a polynomial.

So take $l \geq 2$ and assume the lemma true for $l - 1$. We use the fact that the power series satisfy a formal OPE as $z_1 \to z_l$ (Lemma 1) to reduce a correlator with l vertex operators to a sum of correlators with $l - 1$ vertex operators. From the case $m = 1$ of (10), we have the expansion

$$\begin{aligned}\langle \prod_{a=1}^{l} V(\psi_a, z_a) \rangle_{\eta,R} =& \sum_{i=2}^{l-1} \sum_{n=-h_1-h_i}^{-1} (\iota_{1,i} - \iota_{i,1})(z_1 - z_i)^n \langle V(V_{-n-h_1}(\psi_1)\psi_i, z_i) \prod_{a\neq 1,i} V(\psi_a, z_a) \rangle_{\eta,R} \\ &+ \sum_{n=-h_1-h_l}^{\infty} \iota_{1,l}(z_1 - z_l)^n \langle V(V_{-n-h_1}(\psi_1)\psi_l, z_l) \prod_{a\neq 1,l} V(\psi_a, z_a) \rangle_{\eta,R},\end{aligned} \tag{22}$$

where $R = \{|z_1| > \cdots > |z_l|\}$. To make the notation more palatable we now define, for $i \in (1, l]$ and $n \in \mathbb{Z}$,

$$g_{i,n}(z_2, \ldots, z_l) = \left(\prod_{j=2}^{l} \prod_{m=1}^{k} (z_j - u_m)^{c_j} \right) \langle V(V_{-n-h_1}(\psi_1)\psi_i, z_i) \prod_{a\neq 1,i} V(\psi_a, z_a) \rangle_{\eta,R}. \tag{23}$$

In addition, we fix the c_j ($j \in (1,l]$) sufficiently large that $g_{i,n}$ is convergent in R for ($i \in (1,l)$, $n \in [-h_1-h_i,-1]$) and for ($i=l, n \in [-h_1-h_i, kh_1-1]$). The inductive hypothesis guarantees that such a choice of the c_j is possible, since we are only requiring convergence of finitely many functions.

From Lemma 2 we know that the left side of $\Pi \cdot (22)$ contains powers of z_1 only up to $z_1^{(k-2)h_1}$. Now we want to isolate all negative powers of z_1 on the right side without disturbing this condition. We therefore rewrite (22) in the following way:

$$\begin{aligned}\langle\prod_{a=1}^{l} V(\psi_a,z_a)\rangle_{\eta,R} =& \sum_{i=2}^{l}\sum_{n=-h_1-h_i}^{-1} \iota_{1,i}(z_1-z_i)^n \langle V(V_{-n-h_1}(\psi_1)\psi_i,z_i)\prod_{a\neq 1,i} V(\psi_a,z_a)\rangle_{\eta,R} \\ &-\sum_{i=2}^{l-1}\sum_{n=-h_1-h_i}^{-1} \iota_{i,1}(z_1-z_i)^n \langle V(V_{-n-h_1}(\psi_1)\psi_i,z_i)\prod_{a\neq 1,i} V(\psi_a,z_a)\rangle_{\eta,R} \\ &+\sum_{n=0}^{\infty}(z_1-z_l)^n\langle V(V_{-n-h_1}(\psi_1)\psi_l,z_l)\prod_{a\neq 1,l} V(\psi_a,z_a)\rangle_{\eta,R},\end{aligned} \tag{24}$$

which on choosing $c_1 = h_1$ and multiplying by Π becomes

$$\begin{aligned}\Pi\cdot\langle\prod_{a=1}^{l} V(\psi_a,z_a)\rangle_{\eta,R} =& \prod_{m=1}^{k}(z_1-u_m)^{h_1}\sum_{i=2}^{l}\sum_{n=-h_1-h_i}^{-1}\iota_{1,i}(z_1-z_i)^n g_{i,n}(z_2,\ldots,z_l) \\ &-\prod_{m=1}^{k}(z_1-u_m)^{h_1}\sum_{i=2}^{l-1}\sum_{n=-h_1-h_i}^{-1}\iota_{i,1}(z_1-z_i)^n g_{i,n}(z_2,\ldots,z_l) \\ &+\prod_{m=1}^{k}(z_1-u_m)^{h_1}\sum_{n=0}^{\infty}(z_1-z_l)^n g_{l,n}(z_2,\ldots,z_l).\end{aligned} \tag{25}$$

On the right side of (25) all of the negative powers of z_1 have now been collected into the first term. This term is convergent in R by our inductive hypothesis on the $g_{i,n}$. So restrict attention to the other two terms; write their sum $f(\mathbf{z})$,

$$\begin{aligned}f(\mathbf{z}) =& -\prod_{m=1}^{k}(z_1-u_m)^{h_1}\sum_{i=2}^{l-1}\sum_{n=-h_1-h_i}^{-1}\iota_{i,1}(z_1-z_i)^n g_{i,n}(z_2,\ldots,z_l) \\ &+\prod_{m=1}^{k}(z_1-u_m)^{h_1}\sum_{n=0}^{\infty}(z_1-z_l)^n g_{l,n}(z_2,\ldots,z_l),\end{aligned} \tag{26}$$

which is still a formal Laurent series.

As remarked earlier, the left side of (25) only contains powers of z_1 up to $z_1^{(k-2)h_1}$; and it is clear that the first term on the right contains powers of z_1 only up to $z_1^{kh_1-1}$; so $f(\mathbf{z})$ is actually a polynomial in z_1, of degree at most kh_1-1. We can therefore expand $f(\mathbf{z})$ as

$$f(\mathbf{z}) = \sum_{s=0}^{kh_1-1}\frac{(z_1-z_l)^s}{s!}\left(\frac{\partial}{\partial z_1}\right)^s\Big|_{z_1=z_l} f(\mathbf{z}). \tag{27}$$

To exploit (27) we must make the formal substitution $z_1 = z_l$ in each term of (26). To verify that this is well defined we need to check that, for fixed $j_2, \ldots, j_{l-1}$ and fixed $\alpha+\beta$, there are only finitely many terms $z_1^\alpha z_2^{j_2} \cdots z_{l-1}^{j_{l-1}} z_l^\beta$ appearing in each term of (26). This in turn amounts to checking that the Laurent series $g_{i,n}$ has only a finite singularity at $z_l = 0$, with order bounded uniformly in n; this is automatic for $i \neq l$, and for $i = l$ it is guaranteed by the the fact that z_l is the coordinate closest to the origin, by an argument similar to that in the proof of Lemma 1. So we can substitute (26) into (27), obtaining finally a polynomial in z_1 whose coefficients are convergent power series in R (by our inductive hypothesis on the relevant $g_{i,n}$.) So $f(\mathbf{z})$ is convergent in R. Since we already dealt with the first term in (25), this proves that $\Pi \cdot \langle \prod_{a=1}^l V(\psi_a, z_a) \rangle_{\eta,R}$ converges to a holomorphic function in R. Call this function $\Pi \cdot \langle \prod_{a=1}^l V(\psi_a, z_a) \rangle_\eta$.

By (24) it is clear that the recursive formula (20) is satisfied. On the other hand, (20) defines a meromorphic function on all of $\mathbb{C}^l$, so we get the required analytic continuation of the power series $\Pi \cdot \langle \prod_{a=1}^l V(\psi_a, z_a) \rangle_{\eta,R}$ to all of $\mathbb{C}^l$. It only remains to check that the resulting function is actually independent of which region R we started with.

First suppose R' is obtained from R by swapping z_i with z_j, for some $i, j \neq l$. Choose any $m \neq l$. Using $\Pi \cdot$ (10) and bringing z_m close to z_l we get the "simple OPE" for the functions analytically continued from R:

$$\Pi \cdot \langle \prod_{a=1}^l V(\psi_a, z_a) \rangle_{\eta,R} = \Pi \cdot \sum_{n=-h_m-h_l}^{\infty} (z_m - z_l)^n \langle V(V_{-n-h_m}(\psi_m)\psi_l, z_l) \prod_{a \neq m,l} V(\psi_a, z_a) \rangle_\eta \quad (28)$$

But note that we would get the same thing on the right side had we started with R' instead of R. The analytic continuations from R and R' therefore agree when z_m is close to z_l, hence everywhere.

On the other hand, suppose R' is obtained from R by swapping z_l with z_{l-1}. In this case we need to use (11) and the case $m = l-1$ of (10); multiplying both equations by Π we get two expressions for $\Pi \cdot \langle \prod_{a=1}^l V(\psi_a, z_a) \rangle_{\eta,R}$ when z_l is near z_{l-1}, namely

$$\Pi \cdot \langle \prod_{a=1}^l V(\psi_a, z_a) \rangle_{\eta,R} = \Pi \cdot \sum_{n=-h_{l-1}-h_l}^{\infty} (z_{l-1} - z_l)^n \langle V(V_{-n-h_{l-1}}(\psi_{l-1})\psi_l, z_l) \prod_{a \neq l-1,l} V(\psi_a, z_a) \rangle_\eta, \quad (29)$$

$$\Pi \cdot \langle \prod_{a=1}^l V(\psi_a, z_a) \rangle_{\eta,R} = \Pi \cdot \sum_{n=-h_{l-1}-h_l}^{\infty} (z_l - z_{l-1})^n \langle V(V_{-n-h_l}(\psi_l)\psi_{l-1}, z_{l-1}) \prod_{a \neq l-1,l} V(\psi_a, z_a) \rangle_\eta. \quad (30)$$

Exchanging the label z_{l-1} for z_l in one of the two equations makes manifest that the functions continued from R and R' agree when z_l is near z_{l-1}, hence everywhere. This completes the proof since we can transform any R to any R' by successive swaps of the types we have considered. ■

Conbining the last two lemmas, now we can finally prove that the original power series are well behaved, thus establishing the existence of the correlation functions:

Theorem 4. *Suppose given $\eta : \mathcal{H} \to \mathbb{C}$ such that $O_{\mathbf{u}} \subset \ker \eta$. Then the power series $\langle \prod_{a=1}^l V(\psi_a, z_a) \rangle_{\eta,R}$ defined by (9) each converge on some domain and can be analytically* *continued to a single function* $\langle\prod_{a=1}^l V(\psi_a, z_a)\rangle_\eta$ *which is meromorphic on* $\mathbb{C}^l$*. This function has poles only at* $z_i = z_j$ *or* $z_i = u_j$*, and* $\langle\prod_{a=1}^l V(\psi_a, z_a)\rangle_\eta dz_i^{\otimes h_i}$ *is nonsingular at* $z_i = \infty$*. Furthermore, for all* $i, j \in [1, l]$ *we have the operator product expansion*

$$\langle\prod_{a=1}^{l} V(\psi_a, z_a)\rangle_\eta = \sum_{n=-h_i-h_j}^{\infty} (z_i - z_j)^n \langle V(V_{-n-h_i}(\psi_i)\psi_j, z_j) \prod_{a\neq i,j} V(\psi_a, z_a)\rangle_\eta. \tag{31}$$

for z_i *sufficiently close to* z_j*.*

Proof. This all follows directly from Lemma 3 except for the behavior at ∞, which is a consequence of Lemma 2. ■

Finally we can establish a limited form of the "representation property" in the sense of [16] (see also [25]):

Theorem 5. *Suppose given* $\eta : \mathcal{H} \to \mathbb{C}$ *such that* $O_{\mathbf{u}} \subset \ker\eta$*. Let* $\mathcal{O}$ *be an open disc* $\{|z| < R\} \subset \mathbb{C}$*, with all* $u_i \notin \mathcal{O}$*. Then there exists a state* $\Sigma \in \mathcal{V}^{\mathcal{O}}$ *which induces* η *in the sense that, for* $\chi \in \mathcal{H}$*,*

$$\eta(\chi) = \langle\Sigma\chi\rangle. \tag{32}$$

Proof. The topological vector space $\mathcal{V}^{\mathcal{O}}$ contains the Fock space at ∞ which we denote $\mathcal{H}_\infty$ (to distinguish it from $\mathcal{H}$ which is the Fock space at 0); the idea of the proof is to build up Σ as a limit of states in $\mathcal{H}_\infty$, using the existence of correlation functions to establish convergence.

We have $\mathcal{H}_\infty \subset \mathcal{H}^*$ via the rule [13]

$$\psi(\chi) = \lim_{z\to 0}\langle V((-z)^{-2L_0} e^{-z^{-1}L_1}\psi, z^{-1}) V(\chi, z)\rangle \qquad \forall\ \psi \in \mathcal{H}_\infty, \chi \in \mathcal{H} \tag{33}$$

(note that this indeed defines an injection — the definition of $\mathcal{H}_\infty$ by factorization guarantees that if any $\psi \in \mathcal{H}_\infty$ annihilates every $\chi \in \mathcal{H}$ then $\psi = 0$.) Both $\mathcal{H}$ and $\mathcal{H}_\infty$ are graded by conformal weight. Writing $\mathcal{H}^{(N)}$ for the space of states with weight $\leq N$, and likewise $\mathcal{H}_\infty^{(N)}$, we have $\dim \mathcal{H}^{(N)} = \dim \mathcal{H}_\infty^{(N)}$ (by assumption as described in Section 2, both dimensions are finite.) Let $P_N : \mathcal{H} \to \mathcal{H}^{(N)}$ denote the projection. Then we claim that its adjoint $P_N^* : \mathcal{H}^* \to \mathcal{H}^*$ actually maps $\mathcal{H}^* \to \mathcal{H}_\infty^{(N)}$. To prove this claim, note that $\mathcal{H}_\infty^{(N)}$ is contained in $P_N^*(\mathcal{H}^*)$, since for any $\psi \in \mathcal{H}_\infty^{(N)}$, we have $\psi(\chi) = 0$ when $h_\chi > N$. On the other hand the two spaces have equal dimension, which proves the claim.

Now write $\Sigma_N = P_N^*\eta$. By the above, the Σ_N are actually elements of $\mathcal{H}_\infty$. We claim that they converge as $N \to \infty$ to some $\Sigma \in \mathcal{V}^{\mathcal{O}}$. To check this we need only verify that, for any $(\psi_1, \ldots, \psi_l) \in V^{\otimes l}$ and $\mathbf{z}$ in some compact $K \subset \mathcal{O}^k$ bounded away from the diagonals,

$$f_N(\mathbf{z}) = \langle\Sigma_N \prod_{a=1}^{l} V(\psi_a, z_a)\rangle \to f(\mathbf{z}) = \langle\prod_{a=1}^{l} V(\psi_a, z_a)\rangle_\eta \tag{34}$$

uniformly on K. So for fixed $\mathbf{z}$, consider the function $\lambda \mapsto f(\lambda\mathbf{z})$ where λ ranges over $\mathbb{C}^\times$. This function is holomorphic on some disc containing $0 < |\lambda| \leq 1$; and now we claim that the $f_N(\mathbf{z})$ are nothing but the partial sums in its Laurent expansion, evaluated at $\lambda = 1$. To prove this claim we note that $f(\mathbf{z})$ and $f_N(\mathbf{z})$ satisfy the same operator product expansion identities; using these identities we can reduce to the case $l = 1$, in which case the claim is straightforward. Hence the $f_N(\mathbf{z})$ converge to $f(\mathbf{z})$, and since the $\mathbf{z}$-dependence in $f(\mathbf{z})$ is of a particularly simple sort (f is a rational function of $\mathbf{z}$, with poles only on the diagonals or at the u_i) it is clear that the convergence of this Laurent expansion is uniform in $\mathbf{z}$ in the required sense. ∎

5 Functional dependence

In the last section we checked that a linear functional $\eta : \mathcal{H} \to \mathbb{C}$ vanishing on $O_{\mathbf{u}}$ is sufficient to determine a set of correlation functions involving k highest weight states fixed at $\mathbf{u}$. Next we want to show that these correlation functions can in fact be extended to general $\mathbf{u}$, in a manner consistent with the "Knizhnik-Zamolodchikov" differential equations imposed by the rule $L_{-1} \mapsto \partial$ [24]. We will find that this can indeed be done, provided that we impose a finiteness condition which in some sense expresses the existence of a null-vector.

By Theorem 4, we know that to determine correlation functions at each point, it is sufficient to give linear functionals $\eta(\mathbf{u}) : \mathcal{H} \to \mathbb{C}$ such that each $\eta(\mathbf{u})$ annihilates $O_{\mathbf{u}}$. We want to arrange that the correlation functions $\langle\rangle_{\eta(\mathbf{u})}$ corresponding to $\eta(\mathbf{u})$ satisfy the appropriate KZ-type equations: explicitly, what we require is that

$$\oint_{u_i} \langle L(z)\chi\rangle_{\eta(\mathbf{u})} dz = \frac{\partial}{\partial u_i} \langle \chi \rangle_{\eta(\mathbf{u})}. \tag{35}$$

This differential equation implies a differential equation for the functionals $\eta(\mathbf{u})$, which we will construct below; the remainder of this section is essentially devoted to checking that this equation (which we view as a kind of "parallel transport" problem for the $\eta(\mathbf{u})$) admits solutions.

First we introduce a bit of notation. Let X denote the open set $\{\mathbf{u} = (u_1, \ldots, u_k) \in \mathbb{P}^k : u_i \neq u_j \ \forall i \neq j, u_i \neq 0 \ \forall i\}$. Let B denote the trivial vector bundle $X \times \mathcal{H}$ over X, and let $\Gamma(U, B)$ denote the space of holomorphic sections of B over any $U \subset X$. (Since B is an infinite-dimensional vector bundle we should say what we mean by a "holomorphic section:" to be exact, we mean a holomorphic section of some finite-dimensional subbundle.) Then for each $\mathbf{u} \in X$, let $O_{\mathbf{u}} \subset B_{\mathbf{u}} \simeq \mathcal{H}$ be the subspace we defined in (5). Then let O denote the sheaf of holomorphic sections of the collection of spaces $O_{\mathbf{u}}$ — in other words, we define

$$\Gamma(U, O) = \left\{ s \in \Gamma(U, B) \ \middle| \ s(\mathbf{u}) \in O_{\mathbf{u}} \ \ \forall \mathbf{u} \in U \right\}. \tag{36}$$

It is not clear *a priori* that O is a vector bundle (for example, different $O_{\mathbf{u}}$ could have different codimensions in $\mathcal{H}$).

What is the relation between the different spaces $O_{\mathbf{u}}$? Let us work informally for a moment to see what we should expect. Suppose we consider some $\chi \in \Gamma(U, B)$, and introduce the notation $W(\phi, u)$ for an insertion of a primary field ϕ (corresponding to some highest weight representation of the theory) at the point $u \in \mathbb{P}$. Then, *once we* *have defined the correlation functions for general* $\mathbf{u} \in X$, we would expect to have

$$\begin{aligned} \frac{\partial}{\partial u_i} \langle \prod_{a=1}^{k} W(\phi_a, u_a) \chi(\mathbf{u}) \rangle = \langle W(L_{-1}\phi_i, u_i) \prod_{a \neq i} W(\phi_a, u_a) \chi(\mathbf{u}) \rangle \\ + \langle \prod_{a=1}^{l} W(\phi_a, u_a) \frac{\partial}{\partial u_i} \chi(\mathbf{u}) \rangle. \end{aligned} \tag{37}$$

(There is a potential notational confusion here: we emphasize that $\chi(\mathbf{u})$ refers to an element of $B_{\mathbf{u}} \simeq \mathcal{H}$, which lives at $0 \in \mathbb{P}$, and not some kind of "field at $\mathbf{u}$.") In particular, suppose in fact that $\chi \in \Gamma(U, O)$. Then by the definition of O the left side of (37) should vanish identically. On the right side, by the usual trick of reversing the contour, we could think of L_{-1} as acting on $\chi(\mathbf{u})$ instead of on ϕ_i. To do this we need to get rid of the contributions from the poles in $(z - u_a)$ for $a \neq i$. We can do this using the highest weight condition, which guarantees that these poles have order at most 2.

So, for $i \in [1, l]$ and $\mathbf{u} \in X$, define an operator $L^i(\mathbf{u}) : \mathcal{H} \to \mathcal{H}$ as follows:

$$L^i(\mathbf{u}) = -\oint_0 dz L(z) f^i_{\mathbf{u}}(z) \tag{38}$$

where $f^i_{\mathbf{u}}(z)dz$ is holomorphic on $\mathbb{P} - \{0\}$, with

$$\nu_{u_j}((f^i_{\mathbf{u}} - \delta^{ij})dz) \geq 2 \tag{39}$$

(so $f^i_{\mathbf{u}} - 1$ has a zero of order 2 at u_i, and $f^i_{\mathbf{u}}$ has a zero of order 2 at all u_j with $j \neq i$.) The definition (38) of $L^i(\mathbf{u})$ depends on which function $f^i_{\mathbf{u}}$ we pick, but from the definition (5) of $O_{\mathbf{u}}$ we see at once that different choices only differ by maps $\mathcal{H} \to O_{\mathbf{u}}$. Now (37) says that

$$0 = \langle \prod_{a=1}^{k} W(\phi_a, u_a) \left(L^i(\mathbf{u}) + \frac{\partial}{\partial u_i} \right) \chi(\mathbf{u}) \rangle, \tag{40}$$

in other words, $\left(L^i(\mathbf{u}) + \frac{\partial}{\partial u_i} \right) \chi(\mathbf{u})$ is orthogonal to all highest weight states. Writing

$$D^i(\mathbf{u}) = L^i(\mathbf{u}) + \frac{\partial}{\partial u_i}, \tag{41}$$

the above considerations lead us to expect:

Lemma 6. *For open sets* $U \subset X$*,*

1. *The operator* $D^i : \Gamma(U, B) \to \Gamma(U, B)$ *maps* $\Gamma(U, O) \to \Gamma(U, O)$.
2. *The operator* $[D^i, D^j]$ *maps* $\Gamma(U, B) \to \Gamma(U, O)$.

Proof. First we fix a particular choice of $f^i_{\mathbf{u}}(z)$ which will make the calculations easier. Namely, we let $f^i_{\mathbf{u}}(z)$ be of the form

$$f^i_{\mathbf{u}}(z) = \frac{1}{z^{3k+1}} \prod_{j \neq i} (z - u_j)^3 (Az^2 + Bz + C) \tag{42}$$

where A, B, C are fixed by requiring that $f^i_{\mathbf{u}}(z)-1$ have a zero of order 3 at $z=u_i$. The point of this choice is that it makes $f^i_{\mathbf{u}}$ holomorphic in $\mathbf{u}$ so long as $\mathbf{u}$ stays in X, and satisfies

$$\nu_{u_j}\left((f^i_{\mathbf{u}}-\delta^{ij})dz\right)\geq 3. \tag{43}$$

Proof of 1. First note we are indeed free to choose $f^i_{\mathbf{u}}(z)$ as above, since different choices of $f^i_{\mathbf{u}}(z)$ satisfying (39) change D^i only by maps into $O_{\mathbf{u}}$. Now $\Gamma(U,O)$ is spanned (over the holomorphic functions on U) by sections of the form

$$s(\mathbf{u})=-\oint dwg(\mathbf{u},w)V(\psi,w)\chi, \tag{44}$$

where $\chi\in\mathcal{H}$, $\psi\in V$ and $g(\mathbf{u},w)$ is holomorphic in $\mathbf{u}$ (as given e.g. by (7)). (Strictly speaking, it is not completely obvious that sections of this form are enough — pathologies are excluded by choosing a maximal set of sections (44) which are linearly independent at a single point $\mathbf{u}$, then noting that the subset of U on which they become degenerate is nowhere dense.) So it is sufficient to check that D^i maps the section (44) into a section of O. We therefore compute

$$\begin{aligned}D^is(\mathbf{u})&=-\left(\frac{\partial}{\partial u_i}-\oint_{|z|>|w|}dzL(z)f^i(\mathbf{u},\mathbf{z})\right)\oint dwg(\mathbf{u},w)V(\psi,w)\chi\\&=-\oint dw\left(\frac{\partial}{\partial u_i}g(\mathbf{u},w)\right)V(\psi,w)\chi+\oint\oint_{|z|>|w|}dwdzf^i(\mathbf{u},z)g(\mathbf{u},w)L(z)V(\psi,w)\chi\end{aligned} \tag{45}$$

By the usual contour manipulation argument, the last term in (45) can be rewritten as

$$\begin{aligned}&\oint_{|w|>|z|}dwg(\mathbf{u},w)V(\psi,w)\left(\oint dzf^i(\mathbf{u},z)L(z)\chi\right)\\&+\oint_0 dwg(\mathbf{u},w)\oint_w dzf^i(\mathbf{u},z)\left(\frac{V(L_0\psi,w)}{(z-w)^2}+\frac{V(L_{-1}\psi,w)}{z-w}+O((z-w)^0)\right)\chi,\end{aligned} \tag{46}$$

where in the second term we have used the OPE between $L(z)$ and $V(\psi,w)$. Now the first term in (46) is manifestly in $O_{\mathbf{u}}(\mathcal{H})$. Evaluating the integral of z around w in the second term we obtain

$$\oint_0 dwg(\mathbf{u},w)\left(f^i(\mathbf{u},w)V(L_{-1}\psi,w)+\frac{\partial}{\partial w}f^i(\mathbf{u},w)V(L_0\psi,w)\right)\chi. \tag{47}$$

Now the second term of (47) is in $O_{\mathbf{u}}(\mathcal{H})$, but the first is not, because $L_{-1}\psi$ has weight $h_\psi+1$ and $gf^i(\mathbf{u},w)$ has a zero of order only h_ψ at $w=u_i$. Combining it with the first term in (45), we see that what remains to be checked is that

$$\oint_0 dwg(\mathbf{u},w)f^i(\mathbf{u},w)V(L_{-1}\psi,w)\chi-\left(\frac{\partial}{\partial u_i}g(\mathbf{u},w)\right)V(\psi,w)\chi \tag{48}$$

belongs to $O_{\mathbf{u}}(\mathcal{H})$. Using the fact that $V(L_{-1}\psi,w)=\frac{\partial}{\partial w}V(\psi,w)$ and integrating by parts, this boils down to the assertion that

$$\frac{\partial}{\partial u_i}g(\mathbf{u},w)+\frac{\partial}{\partial w}\left(g(\mathbf{u},w)f^i(\mathbf{u},w)\right) \tag{49}$$

has a zero of order at least h_ψ at each $w = u_j$. For $j \neq i$ this is clear since each term separately has such a zero. At $w = u_i$ we use the fact that $f^i(\mathbf{u}, w) = 1$ to second order in w. ■

Proof of 2. Using the result of part 1, we see that we are again free to use our convenient choice of $f^i_{\mathbf{u}}(w)$. With this choice we will show that the operators $[L^j, \frac{\partial}{\partial u_i}]$ and $[L^i, L^j]$ *separately* map $\Gamma(B, U) \to \Gamma(O, U)$ (for other choices this would not be the case.)

So take any $\chi(\mathbf{u}) \in \Gamma(B, U)$. We have

$$\begin{aligned}\left[L^j, \frac{\partial}{\partial u_i}\right] \chi(\mathbf{u}) &= \frac{\partial}{\partial u_i} \oint dz L(z) f^i(\mathbf{u}, z)\chi(\mathbf{u}) - \oint dz L(z) f^i(\mathbf{u}, z)\frac{\partial}{\partial u_i}\chi(\mathbf{u}) \\ &= \oint dz L(z) \left(\frac{\partial}{\partial u_i} f^i(\mathbf{u}, z)\right) \chi(\mathbf{u})\end{aligned} \tag{50}$$

which belongs to $O_{\mathbf{u}}$ by our hypothesis (43) on f^i (this is where we are using the fact that the number 3 appears there, instead of the 2 in (39).) Next, for any $\chi \in \mathcal{H}$, we have the purely algebraic fact

$$\begin{aligned}[L^i, L^j]\chi &= \left(\oint_{|z|>|w|} - \oint_{|w|>|z|}\right) dz dw f^j(\mathbf{u}, z) f^i(\mathbf{u}, w) L(z) L(w)\chi \\ &= \oint_0 dw f^i(\mathbf{u}, w) \oint_w dz f^j(\mathbf{u}, z) \left(\frac{c/2}{(z-w)^4} + \frac{2L(w)}{(z-w)^2} + \frac{\partial_w L(w)}{z-w} + O((z-w)^0)\right)\chi \\ &= \oint_0 dw \left(\frac{c}{2} f^i(\mathbf{u}, w)\partial_w^3 f^j(\mathbf{u}, w) + f^i(\mathbf{u}, w)\partial_w f^j(\mathbf{u}, w) L(w) + (i \leftrightarrow j)\right)\chi\end{aligned} \tag{51}$$

and the first term vanishes since f^i, f^j are holomorphic on $\mathbb{P} - \{0\}$, while the last two terms belong to $O_{\mathbf{u}}$. This completes the proof. ■

The D^i as defined by (41) are the components of a connection (covariant derivative) in B, which according to Lemma (6) is well defined and flat modulo sections of O. So define the quotient sheaf $A = B/O$ by $\Gamma(U, A) = \Gamma(U, B)/\Gamma(U, O)$. The D^i then induce a flat connection in A in the obvious way; we will use the letter D for this connection as well.

To exploit the existence of this connection we will need to be able to solve differential equations in the spaces of interest; to guarantee this can always be done, we now impose a strong finiteness condition on the conformal field theory. Namely consider the space $O_{\mathbf{u}}$ at the point $\mathbf{u} = (\infty, \ldots, \infty)$. If we call this space C_k, then (5) becomes

$$C_k = \mathrm{Span}\left\{V_{-N-(k-1)h}(\psi)\chi \;\middle|\; \chi \in \mathcal{H}, \psi \in V^h, N \geq 1\right\}. \tag{52}$$

Note that unlike the generic spaces $O_{\mathbf{u}}$, C_k inherits the grading from $\mathcal{H}$, so we can write $C_k = \oplus_{h \geq 0} C_k^h$.

We digress briefly to discuss the space C_k. In the case $k = 2$ it was originally introduced by Zhu in [28], who proved that the characters of the chiral theory close under modular transformations, under the hypothesis that $\mathcal{H}/C_2$ is finite-dimensional. Zhu conjectured that this hypothesis is equivalent to rationality of the theory. As far as the author is aware, this conjecture is still unproven.

For our purposes the important point is that C_k gives a kind of uniform control over the fibres of O, as we see from the following (essentially contained already in [28] for $k = 2$):

Lemma 7. *Let S_k be a graded subspace of $\mathcal{H}$ with $S_k + C_k = \mathcal{H}$. Then $S_k + O_{\mathbf{u}} = \mathcal{H}$ for any $\mathbf{u} \in X$.*

Proof. First note that the term $V_{-N-(k-1)h}(\psi)\chi$ appearing in the definition (52) is precisely the term of highest conformal weight in the element of $O_{\mathbf{u}}$ obtained by substituting N, ψ, χ in (7), so that any element of C_k equals an element of $O_{\mathbf{u}}$ plus "lower-order corrections." Explicitly, for any M, $C_k^M \subset O_{\mathbf{u}} + \mathcal{H}^{(M-1)}$ (where by $\mathcal{H}^{(M-1)}$ we mean $\oplus_{h=0}^{M-1}\mathcal{H}^M$.)

Now we prove by induction that $\mathcal{H}^{(M)} \subset S_k^{(M)} + O_{\mathbf{u}}$. For $M = -1$ there is nothing to prove. So assume $\mathcal{H}^{(M-1)} \subset S_k^{(M-1)} + O_{\mathbf{u}}$. By assumption we have $\mathcal{H}^M = S_k^M + C_k^M$, so $\mathcal{H}^M \subset S_k^M + O_{\mathbf{u}} + \mathcal{H}^{(M-1)} \subset S_k^M + S_k^{(M-1)} + O_{\mathbf{u}} = S_k^{(M)} + O_{\mathbf{u}}$ as desired. ■

Now we can formulate our key finiteness hypothesis (which is a kind of higher-dimensional analogue of Zhu's "condition C," to which it reduces in case $k = 2$) and our main lemma:

Lemma 8. *Suppose $\mathcal{H}/C_k$ is finite-dimensional. Fix $\mathbf{v} \in X$ and a simply connected neighborhood U of $\mathbf{v}$. Then any $\chi \in A_{\mathbf{v}}$ may be extended to $\tilde{\chi} \in \Gamma(U, A)$ such that $D^i\tilde{\chi} = 0$ for all $i \in [1, k]$.*

Proof. By assumption, we can find a finite-dimensional graded S with $S + C_k = \mathcal{H}$. Let $\{s_1, \ldots, s_d\}$ be a basis for S. From Lemma 7 we know that $S + O_{\mathbf{u}} = \mathcal{H}$ for all $\mathbf{u} \in X$. Let U be any simply connected neighborhood of $\mathbf{v}$ in X. We write

$$\tilde{\chi}(\mathbf{u}) = \sum_{l=1}^{d} f_l(\mathbf{u}) s_l \pmod{O_{\mathbf{u}}} \tag{53}$$

where the f_l are complex-valued functions on U, yet to be determined. Then

$$D^i\tilde{\chi}(\mathbf{u}) = \sum_{l=1}^{d} \frac{\partial f_l}{\partial u_i}(\mathbf{u}) s_l + f_l(\mathbf{u}) L^i(\mathbf{u}) s_l \pmod{O_{\mathbf{u}}}. \tag{54}$$

Writing $L^i(\mathbf{u})s_l = \sum_{m=1}^{d} C_l^{im}(\mathbf{u}) s_m \pmod{O_{\mathbf{u}}}$, to get $D^i\tilde{\chi}(\mathbf{u}) = 0$ it is therefore sufficient to demand that

$$0 = \frac{\partial f_m}{\partial u_i} + \sum_{l=1}^{d} f_l C_l^{im}(\mathbf{u}) \tag{55}$$

for each m. For each i, this is a regular matrix differential equation for the functions f_l; furthermore, the fact that $[D^i, D^j] = 0 \pmod{O_{\mathbf{u}}}$ is exactly the integrability condition for this system of differential equations, so Frobenius's theorem [27] implies they have a common solution with the specified initial condition. ■

Now all the work has been done and we can prove our main theorem, which is essentially a translation of the last result to the dual sheaf A^*.

Theorem 9. *Suppose $\mathcal{H}/C_k$ is finite-dimensional. Fix $\mathbf{v} \in X$ and a simply connected neighborhood U of $\mathbf{v}$. Then any $\eta \in A^*_{\mathbf{v}}$ may be uniquely extended to $\tilde{\eta} \in \Gamma(A^*, U)$ such that for any $\chi \in \mathcal{H}$,*

$$\tilde{\eta}(\mathbf{u})(L^i(\mathbf{u})\chi) = \frac{\partial}{\partial u_i}\tilde{\eta}(\mathbf{u})(\chi). \tag{56}$$

Proof. We define $\tilde{\eta}(\mathbf{u})$ by the following rule: given any $\chi \in A_{\mathbf{u}}$, use Lemma 8 to extend χ to a section $\tilde{\chi}$ of A over U and in particular over $\mathbf{v}$. Then set $\tilde{\eta}(\mathbf{u})(\chi) = \eta(\tilde{\chi}(\mathbf{v}))$. The point of this definition is that it makes $\tilde{\eta}$ "constant on horizontal sections:" for any covariantly constant section $\tilde{\chi}$ of A, we have

$$\tilde{\eta}(\mathbf{u})(\tilde{\chi}(\mathbf{u})) = const. \tag{57}$$

Differentiating (57) we obtain

$$\left(\frac{\partial}{\partial u_i}\tilde{\eta}(\mathbf{u})\right)(\tilde{\chi}(\mathbf{u})) + \tilde{\eta}(\mathbf{u})\left(\frac{\partial}{\partial u_i}\tilde{\chi}(\mathbf{u})\right) = 0, \tag{58}$$

and since $D^i\tilde{\chi}(\mathbf{u}) = 0$, using the definition (41) of D in (58) we obtain (56). ■

In terms of the correlation functions induced from $\tilde{\eta}(\mathbf{u})$ by Theorem 4, the result (56) can be reexpressed as

$$\oint_{u_i} \langle L(z)\chi\rangle_{\tilde{\eta}(\mathbf{u})} dz = \frac{\partial}{\partial u_i}\langle\chi\rangle_{\tilde{\eta}(\mathbf{u})}, \tag{59}$$

using the definition (38) of L^i and the fact that the correlation functions satisfy the highest weight condition. This is the desired functional dependence of the correlation functions on $\mathbf{u}$. So when $\mathcal{H}/C_k$ is finite-dimensional, Theorem 9 (together with Theorem 4) gives a construction of a complete family of k-point functions on simply connected neighborhoods in X, starting from a single linear functional on one space $A_{\mathbf{v}}$.

We remark that there is another approach to this system of differential equations, which gives a slightly different result. Namely, one can use the fact that all correlation functions $\langle\prod_{a=1}^{k} W(\phi_a, u_a)\rangle$ can be computed in terms of correlations between fields ϕ_a belonging to the "special subspaces" [26] of the k relevant representations. When at least $k-3$ of the representations are quasirational, one then finds that the relevant differential equations close on a finite-dimensional space. By a calculation similar to that done in the proof of Lemma 6 one can then show that the integrability conditions are always satisfied, so that the differential equations admit a solution. It would be interesting to find a more explicit connection between this approach and that presented above.

6 Discussion

In this paper we have presented a construction of chiral correlation functions on the sphere. This construction guarantees that the correlation functions are locally single-valued, but tells us nothing about the monodromy when two fields are transported around one another. In generic situations one expects that the correlation function will change at most by a phase under this transformation, but there are examples known in which this is not the case, such as the *logarithmic* conformal field theories [18]. It would be interesting to find a natural condition which implies that logarithms do not occur. In the case of 2-point functions with logarithms one sees at once that the problem is failure of L_0 to act semisimply; more generally it has been suggested [14] that semisimplicity of Zhu's algebra might be sufficient to exclude logarithmic behavior for all correlation functions (finite-dimensionality of $\mathcal{H}/C_2$ is not sufficient, as one sees [14] from the example of [15].)

After this work was completed the author became aware that vector bundles of conformal blocks with connection determined by the stress-energy tensor, similar to the sheaf A^* appearing in Section 5, were introduced by Friedan and Shenker and have been considered previously in e.g. [10], [6], [11]. These constructions are formulated on moduli spaces which are compactified by including configurations in which the marked points come together; the vector bundle of conformal blocks becomes nontrivial, and the connection only projectively flat, when these extra configurations are included. It would be interesting to understand more precisely the relation between the Friedan-Shenker vector bundles and those introduced in this paper; in particular, if there were a canonical way to extend the vector bundle $\mathcal{H}/O_{\mathbf{u}}$ to points in moduli space where the u_i coincide, it might shed light on the conjecture of Zhu mentioned in Section 5, as well as the question of the monodromy of the correlation functions.

It would also be interesting to understand in more detail the relation between the present work and the tensor product theory of [19].

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