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Superpotentials for Vector Bundle Moduli

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Abstract

We present a method for explicitly computing the non-perturbative superpotentials associated with the vector bundle moduli in heterotic superstrings and M-theory. This method is applicable to any stable, holomorphic vector bundle over an elliptically fibered Calabi-Yau threefold. For specificity, the vector bundle moduli superpotential, for a vector bundle with structure group $G = SU(3)$, generated by a heterotic superstring wrapped once over an isolated curve in a Calabi-Yau threefold with base $B = \mathbb{F}_1$, is explicitly calculated. Its locus of critical points is discussed. Superpotentials of vector bundle moduli potentially have important implications for small instanton phase transitions and the vacuum stability and cosmology of superstrings and M-theory.

1 Introduction:

The calculation of non-perturbative superpotentials for the moduli of superstrings and M-theory has a considerable literature. The first computations were carried out from the point of view of string worldsheet conformal field theory [1, 2]. Subsequently, a second approach appeared, pioneered in [3, 4], in which the associated worldsheet instantons are viewed as genus-zero holomorphic curves C in the compactification space, and one integrates over their physical oscillations. This latter technique has been used to compute non-perturbative superpotentials in F-theory [5], weakly coupled heterotic string theory on Calabi-Yau manifolds [6], M-theory compactified on seven-manifolds of G_2 holonomy [7], and heterotic M-theory on Calabi-Yau threefolds [8, 9]. The results for both the weakly and strongly coupled heterotic string theories are proportional to a factor involving the Wess-Zumino term, which couples the superstring to the background $SO(32)$ or $E_8 \times E_8$ gauge bundle V [6, 8, 9]. This term can be expressed as the Pfaffian of a Dirac operator twisted by the gauge bundle restricted to the associated holomorphic curve C. It was pointed out in [6] that this Pfaffian, and, hence, the superpotential, will vanish if and only if the restriction of the gauge bundle, $V|_C$, is non-trivial. Furthermore, it is clear that the Pfaffian must be a holomorphic function of the gauge bundle moduli associated with $V|_C$. Although related work has appeared in other contexts [5], neither the vanishing structure of the Pfaffian in heterotic string theories, nor its functional dependence on the vector bundle moduli, has yet appeared in the literature. It is the purpose of this paper to provide explicit solutions to these two problems, within the framework of both weakly and strongly coupled heterotic $E_8 \times E_8$ superstring theories compactified on elliptically fibered Calabi-Yau threefolds.

Our approach to determining the zeros of the Pfaffian is the following. First, we note that the Pfaffian will vanish if and only if the chiral Dirac operator on the holomorphic curve C, in the background of the restricted gauge bundle $V|_C$, has at least one zero mode. Thus, the problem becomes one of determining whether or not the dimension of the kernel of the Dirac operator is non-vanishing. We then show that this kernel naturally lies in a specific exact sequence of cohomologies and will be non-vanishing if and only if the determinant of one of the maps in this sequence vanishes. For a wide range of holomorphic vector bundles on elliptically fibered Calabi-Yau threefolds, we can explicitly compute this determinant as a holomorphic, homogeneous polynomial of the vector bundle moduli associated with $V|_C$. These moduli parameterize a quotient manifold which is the projective space of "transition" moduli introduced and described in [10]. It is then straightforward to determine its zeros and, hence, the zeros of the Pfaffian.

It follows from this that the vanishing structure of the Pfaffian is determined by a holomorphic polynomial function on the space of vector bundle moduli. Note, however, that the Pfaffian must itself be a holomorphic function of the same moduli, and that this function must vanish at exactly the same locus as does the polynomial. Since the moduli space is compact, one can conclude that the Pfaffian is given precisely by the holomorphic polynomial function, perhaps to some positive power, multiplied by an over-all constant. Using the results of [11, 12, 13], one can show that this power must be unity. Therefore, solving the first problem, that is, the zeros of the Pfaffian, automatically solves the second problem, namely, explicitly determining the Pfaffian, and, hence, the superpotential, as a function of the vector bundle moduli.

Specifically, in this paper we do the following. In Section 2 we discuss the definition of the Pfaffian, Pfaff$(\mathcal{D}_-)$, and its relation to the non-perturbative superpotential W. We also show that W is both gauge and local Lorentz anomaly free. We begin our discussion of the structure of the zeros of Pfaff$(\mathcal{D}_-)$ in Section 3. This is accomplished in several steps. First, we show that the Pfaffian will vanish if and only if the dimension of the space of sections of a specific bundle on C, $h^0(C, V|_C \otimes \mathcal{O}_C(-1))$, is non-vanishing. This condition is shown to be identical to the non-triviality of $V|_C$ discussed in [6]. As a second step, we briefly review salient properties of elliptically fibered Calabi-Yau threefolds and stable, holomorphic vector bundles over such spaces. In this paper, for specificity, we will restrict our discussion to a single, non-trivial example, with the base of the Calabi-Yau threefold chosen to be a Hirzebruch surface $\mathbb{F}_1$ and the vector bundle V taken to arise from an irreducible, positive spectral cover with structure group $G = SU(3)$. Thirdly, we choose an isolated curve of genus zero, $\mathcal{S} = \mathbb{P}^1$, in the Calabi-Yau threefold and wrap an $E_8 \times E_8$ heterotic superstring once around it. We explicitly categorize $V|_{\mathcal{S}}$ and its associated moduli in terms of the images on $\mathcal{S}$ of the spectral data. It is shown, using results from [10], that in our example these moduli parameterize a projective space of dimension twelve, $\mathbb{P}^{12}$, which is the projectivization of the thirteen-dimensional linear space of transition moduli. As the fourth and most important step, we derive a simple exact cohomology sequence in which the linear space of zero modes of the Dirac operator on $\mathcal{S}$, $H^0(\mathcal{S}, V|_{\mathcal{S}} \otimes \mathcal{O}_{\mathcal{S}}(-1))$, appears. From this, we establish that its dimension, $h^0(\mathcal{S}, V|_{\mathcal{S}} \otimes \mathcal{O}_{\mathcal{S}}(-1))$, will be non-vanishing if and only if the determinant of a certain linear map, f_D, in the exact sequence vanishes. As a last step, we explicitly compute $det f_D$, finding it to be a polynomial of degree twenty which exactly factors into the fourth power of a holomorphic, homogeneous polynomial of degree five in the thirteen homogeneous, projective moduli of $V|_{\mathcal{S}}$. This solves the problem of the vanishing structure of Pfaff$(\mathcal{D}_{-})$. In the final section, Section 4, we then argue that the superpotential W must be proportional to this degree twenty polynomial, thereby uniquely determining the superpotential for the vector bundle moduli associated with $V|_{\mathcal{S}}$.

As stated above, for simplicity, we have presented our results in terms of a single, non-trivial example. We have also suppressed much of the relevant mathematics, emphasizing motivation and method over mathematical detail. Our method, however, is, in principle, applicable to any stable, holomorphic vector bundle over any elliptically fibered Calabi-Yau threefold. In [14], we will present a wider range of examples, computing the superpotentials for several different vector bundles and analyzing the structure of their critical points. In addition, we will give a more complete discussion of the mathematical structure underlying our computations. Among other things, an analytical calculation and first principles explanation of the homogeneous polynomials that occur in vector bundle moduli superpotentials of the type under discussion will be presented.

Although at first sight rather complicated to derive, the superpotential for vector bundle moduli potentially has a number of important physical applications. To begin with, it is essential to the study of the stability of the vacuum structure [15, 16] of both weakly coupled heterotic string theory and heterotic M-theory [17, 18, 19]. Furthermore, in both theories it allows, for the first time, a discussion of the dynamics of the gauge bundles. For example, in heterotic M-theory one can determine if a bundle is stable or whether it decays, via a small instanton transition [20], into five-branes. In recent years, there has been considerable research into the cosmology of superstrings and heterotic M-theory [21, 22, 23] . In particular, a completely new approach to early universe cosmology, Ekpyrotic theory [24, 25, 26, 27, 28], has been introduced within the context of brane universe theories. The vector bundle superpotentials discussed in this paper and [14] allow one to study the dynamics of the small instanton phase transitions that occur when a five-brane [24, 25] or an "end-of-the-world" orbifold plane [26, 27, 28] collides with our observable brane, thus producing the Big Bang. These physical applications will be discussed elsewhere.

2 Pfaff$(\mathcal{D}_{-})$ and Superpotential W:

We want to consider $E_8 \times E_8$ heterotic superstring theory on the space

$$M = \mathbb{R}^4 \times X, \tag{2.1}$$

where X is a Calabi-Yau threefold. In general, this vacuum will admit a stable, holomorhic vector bundle V on X with structure group

$$G \subseteq E_8 \times E_8 \tag{2.2}$$

and a specific connection one-form $\mathcal{A}$. It was shown in [29] (see also [30]) that for any open neighborhood of X, the local representative, A, of this connection satisfies the hermitian Yang-Mills equations

$$F_{mn} = F_{\bar{m}\bar{n}} = 0 \tag{2.3}$$

and

$$g^{m\bar{n}} F_{m\bar{n}} = 0, \tag{2.4}$$

where F is the field strength of A.

As discussed in [4], a non-perturbative contribution to the superpotential corresponds to the partition function of a superstring wrapped on a holomorphic curve $C \subset X$. Furthermore, one can show [3] that only a curve of genus zero will contribute. Hence, we will take

$$C = \mathbb{P}^1. \tag{2.5}$$

To further simplify the calculations, we will also assume that C is an isolated curve in X and that the superstring is wrapped only once on C. The spin bundle over C will be denoted by

$$S = S_+ \oplus S_- \tag{2.6}$$

and the restriction of the vector bundle V to C by $V|_C$. Finally, we will assume that the structure group of the holomorphic vector bundle is contained in the subgroup $SO(16) \times SO(16)$ of $E_8 \times E_8$. That is,

$$G \subseteq SO(16) \times SO(16) \subset E_8 \times E_8. \tag{2.7}$$

This condition will be satisfied by any quasi-realistic heterotic superstring vacuum. Briefly, the reason for this restriction is the following. As discussed, for example, in [3, 8] and references therein, when (2.7) is satisfied, the Wess-Zumino-Witten (WZW) term coupling the superstring to the background vector bundle can be written as a theory of thirty-two worldsheet fermions interacting only with the vector bundle through the covariant derivative. In this case, the associated partition function and, hence, the contribution of the WZW term to the superpotential is easily evaluated. When condition (2.7) is not satisfied, this procedure breaks down and the contribution of the WZW term to the superpotential is unknown.

Under these conditions, it can be shown [6] that the non-perturbative superpotential W has the following structure

$$W \propto \mathrm{Pfaff}(\mathcal{D}_-)\exp(\mathrm{i}\int_C B), \tag{2.8}$$

where B is the Neveu-Schwarz two-form field. The Pfaffian of $\mathcal{D}_-$ is defined as

$$\mathrm{Pfaff}(\mathcal{D}_-) = \sqrt{det\mathcal{D}_-} \tag{2.9}$$

where, for the appropriate choice of basis of the Clifford algebra,

$$\mathcal{D}_- = \begin{pmatrix} 0 & D_- \\ i\partial_+ & 0 \end{pmatrix}. \tag{2.10}$$

Here, the operator D_- represents the covariant chiral Dirac operator

$$D_- : \Gamma(C, V|_C \otimes S_-) \rightarrow \Gamma(C, V|_C \otimes S_+), \tag{2.11}$$

whereas $\partial_+ : \Gamma(C, V|_C \otimes S_+) \rightarrow \Gamma(C, V|_C \otimes S_-)$ is independent of the connection $\mathcal{A}$. Pfaff$(\mathcal{D}_-)$ arises as the partition function of the WZW term, as discussed above. Note that we have displayed in (2.8) only those factors in the superpotential relevant to vector bundle moduli. The factors omitted, such as $\exp(\frac{-\mathcal{A}(C)}{2\pi\alpha'})$ where $\mathcal{A}(C)$ is the area of the surface C using the heterotic string Kahler metric on X and α' is the heterotic string parameter, are positive terms dependent on geometric moduli only. Now

$$|det\mathcal{D}_-|^2 = det(\mathcal{D}_-\mathcal{D}_-^\dagger) \propto det\mathcal{D}, \tag{2.12}$$

where the proportionality is a positive constant independent of the connection,

$$\mathcal{D} = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix} \tag{2.13}$$

and

$$D_+ = D_-^\dagger. \tag{2.14}$$

Note that we have absorbed a factor of i into our definition of the Dirac operators D_- and D_+. It follows that

$$det\mathcal{D}_- \propto \sqrt{|det\mathcal{D}|}\mathrm{e}^{\mathrm{i}\phi}, \tag{2.15}$$

where

$$|det\mathcal{D}| = detD_-D_+ \tag{2.16}$$

is a non-negative real number and ϕ is a phase. It is well known that $det\mathcal{D}$ is gauge invariant. However, under both gauge and local Lorentz transformations with infinitesimal parameters ϵ and θ respectively, the phase can be shown to transform as

$$\delta\phi = 2\int_C (-\mathrm{tr}(\epsilon d\mathcal{A}) + \mathrm{tr}(\theta d\omega)), \tag{2.17}$$

where $\mathcal{A}$ and ω are the gauge and spin connections respectively. This corresponds to the worldsheet sigma model anomaly. Fortunately, this anomaly is exactly cancelled by the variation

$$\delta B = \int_C (\mathrm{tr}(\epsilon d\mathcal{A}) - \mathrm{tr}(\theta d\omega)) \tag{2.18}$$

of the B-field [6]. It then follows from (2.8) that the superpotential W is both gauge and locally Lorentz invariant.

We displayed the factor $\exp(\mathrm{i}\int_C B)$ in the superpotential expression (2.8) since it was relevant to the discussion of a gauge invariance. However, as was the case with $\exp(\frac{-\mathcal{A}(C)}{2\pi\alpha'})$, it also does not depend on the vector bundle moduli and, henceforth, we will ignore it. Therefore, to compute the vector bundle moduli contribution to the superpotential one need only consider

$$W \propto \mathrm{Pfaff}(\mathcal{D}_-). \tag{2.19}$$

We now turn to the explicit calculation of $\mathrm{Pfaff}(\mathcal{D}_-)$. To accomplish this, it is necessary first to discuss the conditions under which it vanishes.

3 The Zeros Of Pfaff$(\mathcal{D}_-)$:

Clearly, Pfaff$(\mathcal{D}_-)$ vanishes if and only if $det\mathcal{D}$ does. In turn, it follows from (2.16) that this will be the case if and only if one or both of D_- and D_+ have a non-trivial zero mode. In general, $dim\ kerD_-$ and $dim\ kerD_+$ may not be equal to each other and must be considered separately. However, in this calculation that is not the case, as we now show. Recall that

$$indexD_+ = dim\ kerD_+ - dim\ kerD_-. \tag{3.1}$$

Since $dim_{\mathbb{R}}C = 2$, it follows from the Atiyah-Singer index theorem that

$$indexD_+ = \frac{i}{2\pi}\int_C \mathrm{tr}\mathcal{F}, \tag{3.2}$$

where $\mathcal{F}$ is the curvature two-form associated with connection $\mathcal{A}$ restricted to curve $\mathcal{C}$. Since, in this paper, the structure group of V is contained in the semi-simple group $E_8 \times E_8$, we see that $\mathrm{tr}\mathcal{F}$ vanishes. Therefore,

$$index D_+ = 0 \tag{3.3}$$

and, hence

$$dim\ ker D_+ = dim\ ker D_-. \tag{3.4}$$

It follows that Pfaff$(\mathcal{D}_-)$ will vanish if and only if

$$dim\ ker D_- > 0. \tag{3.5}$$

To proceed, therefore, we must compute the zero structure of $dim\ ker D_-$. This calculation is facilitated using the fact that a holomorphic vector bundle with a hermitian structure admits a unique connection compatible with both the metric and the complex structure (see for example [31]). That is, for a special choice of gauge, one can always set

$$D_- = i\bar{\partial} \tag{3.6}$$

where, for any open neighborhood $\mathcal{U} \subset \mathcal{C}$ with coordinates $z, \bar{z}$,

$$\bar{\partial} = \partial_{\bar{z}}. \tag{3.7}$$

To prove this, note that, for an appropriate choice of complex structure and Dirac γ-matrices one can always set

$$D_- = i(\partial_{\bar{z}} + A_{\bar{z}}) \tag{3.8}$$

in any $\mathcal{U} \subset \mathcal{C}$. Now it follows from (2.3) that the pullback of the gauge connection on X to any open set $\mathcal{U} \subset \mathcal{C}$ is of the form

$$A_z = \partial_z g \cdot g^{-1}, \qquad A_{\bar{z}} = \partial_{\bar{z}} g^{\dagger -1} \cdot g^{\dagger} \tag{3.9}$$

where g is a map from $\mathcal{U}$ to the complexification of structure group G. Now choose a specific open neighborhood $\mathcal{U} \subset \mathcal{C}$ with local coordinates $z, \bar{z}$. Clearly, using a gauge transformation with parameter $g^{\dagger}$ we can set

$$A_{\bar{z}} = 0 \tag{3.10}$$

in this $\mathcal{U}$. Now let $\mathcal{V}$ be any other open subset of $\mathcal{C}$ with local coordinates $z', \bar{z}'$, such that it has a non-empty intersection with $\mathcal{U}$. Denote the connection one-form on $\mathcal{V}$ by A'. Then, the compatibility condition says that

$$A' = f_{\mathcal{U}\mathcal{V}} A f_{\mathcal{U}\mathcal{V}}^{-1} + df_{\mathcal{U}\mathcal{V}} \cdot f_{\mathcal{U}\mathcal{V}}^{-1}, \tag{3.11}$$

where $f_{\mathcal{UV}}$ is the holomorphic transition fuction on $\mathcal{U} \cap \mathcal{V}$. It follows from equation (3.10) and from the fact that $\partial_{\bar{z}} f_{\mathcal{UV}} = 0$ that

$$A'_{\bar{z}'} = 0. \tag{3.12}$$

Continuing this way we see that one can set $A_{\bar{z}}$ to zero globally on C. It then follows from relation (3.8) that $D_- = \bar{\partial}$ as claimed in (3.6). Note, in passing, that with respect to the same complex structure and in the same gauge, the operator D_+ is of the form

$$D_+ = i(\partial_z + A_z) \tag{3.13}$$

where, in general, $A_z \neq 0$ on every open subset of C. Using (3.5) and (3.6), we conclude that Pfaff$(\mathcal{D}_-)$ will vanish if and only if

$$dim\ ker\bar{\partial} > 0. \tag{3.14}$$

However, it follows from equations (2.11) and (3.6) that the zero modes of $\bar{\partial}$ are precisely the holomorphic sections of the vector bundle $V|_C \otimes S_-$. Using the fact that on $C = \mathbb{P}^1$

$$S_- = \mathcal{O}_C(-1), \tag{3.15}$$

and defining

$$V|_C(-1) = V|_C \otimes \mathcal{O}_C(-1), \tag{3.16}$$

we conclude that

$$dim\ ker\bar{\partial} = h^0(C, V|_C(-1)). \tag{3.17}$$

Hence, Pfaff$(\mathcal{D}_-)$ will vanish if and only if

$$h^0(C, V|_C(-1)) > 0. \tag{3.18}$$

Therefore, the problem of determining the zeros of the Pfaffian of $\mathcal{D}_-$ is reduced to deciding whether or not there are any non-trivial global holomorphic sections of the bundle $V|_C(-1)$ over the curve C. An equivalent way of stating the same result is to realize that the condition for the vanishing of Pfaff$(\mathcal{D}_-)$ is directly related to the non-triviality or triviality of the bundle $V|_C$. To see this, note that any holomorphic $SO(16) \times SO(16)$ bundle $V|_C$ over a genus zero curve $C = \mathbb{P}^1$ is of the form

$$V|_C = \bigoplus_{i=1}^{16} \mathcal{O}_{\mathbb{P}^1}(m_i) \oplus \mathcal{O}_{\mathbb{P}^1}(-m_i) \tag{3.19}$$

with non-negative integers m_i. Therefore

$$V|_C(-1) = \bigoplus_{i=1}^{16} \mathcal{O}_{\mathbb{P}^1}(m_i - 1) \oplus \mathcal{O}_{\mathbb{P}^1}(-m_i - 1). \tag{3.20}$$

Using the fact that

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) = m + 1 \tag{3.21}$$

for $m \geq 0$ and

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(m)) = 0 \tag{3.22}$$

for $m < 0$, it follows that

$$h^0(C, V|_C(-1)) = \sum_{i=1}^{16} m_i. \tag{3.23}$$

Therefore $h^0(C, V|_C(-1)) > 0$ if and only if at least one m_i is greater than zero. That is, as first pointed out in [6], $h^0(C, V|_C(-1)) > 0$, and hence Pfaff$(\mathcal{D}_-)$ will vanish, if and only if $V|_C$ is non-trivial. We now turn to the question of how to determine whether or not there are non-trivial sections of $V|_C(-1)$ over C.

The problem of whether or not $h^0(C, V|_C(-1))$ is non-zero can be solved within the context of stable, holomorphic vector bundles over elliptically fibered Calabi-Yau threefolds. In this paper, we will present a single explicit example, prefering to be concrete and to emphasize the method rather than the underlying mathematics. A more detailed discussion, with all the relevant mathematics, will be presented elsewhere [14]. We consider a Calabi-Yau threefold X elliptically fibered over a base

$$B = \mathbb{F}_1, \tag{3.24}$$

where $\mathbb{F}_1$ is a Hirzebruch surface. That is, $\pi : X \to \mathbb{F}_1$. Since X is elliptically fibered, there exists a zero section $\sigma : \mathbb{F}_1 \to X$. We will denote $\sigma(\mathbb{F}_1) \subset X$ simply as σ. The second homology group $H^2(\mathbb{F}_1, \mathbb{R})$ is spanned by two effective classes of curves, denoted by $\mathcal{S}$ and $\mathcal{E}$, with intersection numbers

$$\mathcal{S}^2 = -1, \qquad \mathcal{S} \cdot \mathcal{E} = 1, \qquad \mathcal{E}^2 = 0. \tag{3.25}$$

The first Chern class of $\mathbb{F}_1$ is given by

$$c_1(\mathbb{F}_1) = 2\mathcal{S} + 3\mathcal{E}. \tag{3.26}$$

Over X we construct a stable, holomorphic vector bundle V with structure group

$$G = SU(3). \tag{3.27}$$

This is accomplished [32, 33] by specifying a spectral cover

$$\mathcal{C} = 3\sigma + \pi^*\eta, \tag{3.28}$$

where

$$\eta = (a+1)\mathcal{S} + b\mathcal{E} \tag{3.29}$$

and $a+1$ and b are non-negative integers, as well as a holomorphic line bundle

$$\mathcal{N} = \mathcal{O}_X(3(\lambda + \frac{1}{2})\sigma - (\lambda - \frac{1}{2})\pi^*\eta + (3\lambda + \frac{1}{2})\pi^* c_1(B)), \tag{3.30}$$

where $\lambda \in \mathbb{Z} + \frac{1}{2}$. Note that we use $a+1$, rather than a, as the coefficient of $\mathcal{S}$ in (3.29) to conform with our conventions in [10]. The vector bundle V is then determined via a Fourier-Mukai transformation

$$(\mathcal{C}, \mathcal{N}) \longleftrightarrow V. \tag{3.31}$$

In this paper, we will consider the case

$$a > 5, \qquad b - a = 6, \qquad \lambda = \frac{3}{2}. \tag{3.32}$$

We refer the reader to [10] to show that for such parameters spectral cover $\mathcal{C}$ is both irreducible and positive. In addition, it follows from (3.26), (3.29), (3.30) and (3.32) that

$$\mathcal{N} = \mathcal{O}_X(6\sigma + (9-a)\pi^*(\mathcal{S} + \mathcal{E})). \tag{3.33}$$

Now consider the curve $\mathcal{S} \subset \mathbb{F}_1$. Since $\mathcal{S} \cdot \mathcal{S} = -1$, it is an isolated curve in $\mathbb{F}_1$. Since $\mathcal{S}$ is an exceptional curve

$$\mathcal{S} = \mathbb{P}^1. \tag{3.34}$$

The lift of $\mathcal{S}$ into X, $\pi^*\mathcal{S}$, was determined in [10] to be the rational elliptic surface

$$\pi^*\mathcal{S} = dP_9. \tag{3.35}$$

The curve $\mathcal{S}$ is represented in X by

$$\mathcal{S}_X = \sigma \cdot \pi^*\mathcal{S}. \tag{3.36}$$

By construction, $\mathcal{S}_X$ is isolated in X and $\mathcal{S}_X = \mathbb{P}^1$. We will frequently not distiguish between $\mathcal{S}$ and $\mathcal{S}_X$, referring to both curves as $\mathcal{S}$. Recall that we want to wrap the superstring once over a genus-zero Riemann surface $\mathbb{P}^1$ which is isolated in X. In this example, we will take $\mathcal{S}$ to be this isolated curve.

To proceed, let us restrict the vector bundle data to $\pi^*\mathcal{S}$. The restriction of the spectral cover is given by

$$\mathcal{C}|_{dP_9} = \mathcal{C} \cdot \pi^*\mathcal{S} \tag{3.37}$$

which, using (3.25) and (3.32), becomes

$$\mathcal{C}|_{dP_9} = 3\sigma|_{dP_9} + 5F, \tag{3.38}$$

where F is the class of the elliptic fiber. Note that $\mathcal{C}|_{dP_9}$ is a divisor in dP_9. Similarly

$$\mathcal{N}|_{dP_9} = \mathcal{O}_{dP_9}((6\sigma + (9-a)\pi^*(\mathcal{S}+\mathcal{E})) \cdot \pi^*\mathcal{S}). \tag{3.39}$$

Using (3.25), this is given by

$$\mathcal{N}|_{dP_9} = \mathcal{O}_{dP_9}(6\sigma|_{dP_9}). \tag{3.40}$$

It is useful, as will be clear shortly, to define

$$\mathcal{N}|_{dP_9}(-F) = \mathcal{N}|_{dP_9} \otimes \mathcal{O}_{dP_9}(-F). \tag{3.41}$$

Then

$$\mathcal{N}|_{dP_9}(-F) = \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F). \tag{3.42}$$

Since dP_9 is elliptically fibered, the restriction of V to dP_9, denoted by $V|_{dP_9}$, can be obtained from the Fourier-Mukai transformation

$$(\mathcal{C}|_{dP_9}, \mathcal{N}|_{dP_9}) \longleftrightarrow V|_{dP_9}. \tag{3.43}$$

In a previous paper [10], we showed that the direct image under π of the line bundle on dP_9 associated with $\mathcal{C}|_{dP_9}$, that is

$$\mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 5F) \tag{3.44}$$

is a rank three vector bundle on $\mathcal{S}$. In this case, we find that

$$\pi_*\mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 5F) = \mathcal{O}_{\mathcal{S}}(5) \oplus \mathcal{O}_{\mathcal{S}}(3) \oplus \mathcal{O}_{\mathcal{S}}(2). \tag{3.45}$$

In addition, we proved in [10] that the moduli associated with a small instanton phase transition involving the curve $\mathcal{S}$, the so called transition moduli, are in one-to-one correspondence with the holomorphic sections of this bundle, that is, with elements of

$$H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(5) \oplus \mathcal{O}_{\mathcal{S}}(3) \oplus \mathcal{O}_{\mathcal{S}}(2)). \tag{3.46}$$

It follows that the number of these transition moduli is given by

$$h^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(5) \oplus \mathcal{O}_{\mathcal{S}}(3) \oplus \mathcal{O}_{\mathcal{S}}(2)) = 13, \tag{3.47}$$

where we have used expression (3.21). In this paper, we are interested not in the vector space (3.46) parametrized by the full set of transition moduli but, rather, in its projectivization

$$\mathbb{P}^{12} \simeq \mathbb{P}H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 5F)), \tag{3.48}$$

parametrized by the moduli of the curve $\mathcal{C}|_{dP_9}$. This space, although twelve dimensional, is most easily parameterized in terms of the thirteen homogeneous coordinates a_i. Now note that $\mathcal{C}|_{dP_9}$ is a 3-fold cover of $\mathcal{S}$ with covering map $\pi_{\mathcal{C}|_{dP_9}} : \mathcal{C}|_{dP_9} \to \mathcal{S}$. The image of $\mathcal{N}|_{\mathcal{C}|_{dP_9}}$ under $\pi_{\mathcal{C}|_{dP_9}}$ is also a rank three vector bundle over $\mathcal{S}$. In fact,

$$V|_{\mathcal{S}} = \pi_{\mathcal{C}|_{dP_9}*}\mathcal{N}|_{\mathcal{C}|_{dP_9}}. \tag{3.49}$$

Within this context, we can now consider the question of determining the zeros of Pfaff$(\mathcal{D}_-)$ for the explicit case of a superstring wrapped on $\mathcal{S}$. As discussed in the previous section, we want to study the properties of $h^0(\mathcal{S}, V|_{\mathcal{S}}(-1))$. Now, using (3.49) we have

$$h^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) = h^0(\mathcal{S}, \pi_{\mathcal{C}|_{dP_9}*}\mathcal{N}|_{dP_9} \otimes \mathcal{O}_{\mathcal{S}}(-1)). \tag{3.50}$$

Using a Leray spectral sequence and (3.41), one can show that

$$h^0(\mathcal{S}, \pi_{\mathcal{C}|_{dP_9}*}\mathcal{N}|_{dP_9} \otimes \mathcal{O}_{\mathcal{S}}(-1)) = h^0(\mathcal{C}|_{dP_9}, \mathcal{N}|_{dP_9}(-F)|_{\mathcal{C}|_{dP_9}}) \tag{3.51}$$

where, for our specific example, $\mathcal{N}|_{dP_9}(-F)$ is given by (3.42). In other words, we will find under what circumstances the vector bundle $V|_{\mathcal{S}}(-1)$ has sections by studying under what circumstances the line bundle $\mathcal{N}|_{dP_9}(-F)$ restricted to $\mathcal{C}|_{dP_9}$ does. To accomplish this, consider the short exact sequence

$$0 \to E \otimes \mathcal{O}_{dP_9}(-D) \xrightarrow{f_D} E \xrightarrow{r} E|_D \to 0, \tag{3.52}$$

where E is any holomorphic vector bundle on dP_9 and D is any effective divisor in dP_9. The map f_D from $E \otimes \mathcal{O}_{dP_9}(-D)$ to E is given by multiplication by the section of the line bundle $\mathcal{O}_{dP_9}(D)$ that vanishes precisely on D. The mapping r from E to $E|_D$ is just restriction. If we use the abbreviation

$$E(-D) = E \otimes \mathcal{O}_{dP_9}(-D) \tag{3.53}$$

then, in the usual way, (3.52) implies the long exact sequence of cohomology groups given by

$$\begin{aligned} &0 \to H^0(dP_9, E(-D)) \to H^0(dP_9, E) \to H^0(D, E|_D) \\ &\to H^1(dP_9, E(-D)) \to H^1(dP_9, E)) \to H^1(D, E|_D) \to \dots \quad . \end{aligned} \tag{3.54}$$

In our specific example, we choose

$$E = \mathcal{N}|_{dP_9}(-F) = \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F), \tag{3.55}$$

where we have used (3.42), and

$$D = \mathcal{C}|_{dP_9} = 3\sigma|_{dP_9} + 5F. \tag{3.56}$$

It follows from (3.53) that

$$E(-D) = \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F). \tag{3.57}$$

Let us first first consider the term $H^0(dP_9, E) = H^0(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))$ in the long exact sequence. We compute it in terms of its direct image on $\mathcal{S}$:

$$\pi_* \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F) = \mathcal{O}_{\mathcal{S}}(-1) \oplus \bigoplus_{i=3}^{7} \mathcal{O}_{\mathcal{S}}(-i). \tag{3.58}$$

It follows from (3.22) that this bundle has no global holomorphic sections and, hence, that

$$H^0(dP_9, E) = 0. \tag{3.59}$$

Furthermore, note from (3.42), (3.55) and (3.56) that

$$H^0(D, E|_D) = H^0(\mathcal{C}|_{dP_9}, \mathcal{N}|_{dP_9}(-F)|_{\mathcal{C}|_{dP_9}}). \tag{3.60}$$

Therefore, part of the exact sequence (3.54) is given by

$$\begin{aligned} &0 \to H^0(\mathcal{C}|_{dP_9}, \mathcal{N}|_{dP_9}(-F)|_{\mathcal{C}|_{dP_9}}) \to H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F)) \\ &\to H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F)) \to \cdots \quad . \end{aligned} \tag{3.61}$$

Both H^1 cohomology groups are linear spaces whose structure and dimension we now determine. First consider $H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F))$. Using a Leray spectral sequence and the facts that

$$\pi_* \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F) = \mathcal{O}_{\mathcal{S}}(-6) \oplus \mathcal{O}_{\mathcal{S}}(-8) \oplus \mathcal{O}_{\mathcal{S}}(-9) \tag{3.62}$$

and

$$R^1\pi_*\mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F) = 0, \tag{3.63}$$

it follows that

$$H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F)) = H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-6) \oplus \mathcal{O}_{\mathcal{S}}(-8) \oplus \mathcal{O}_{\mathcal{S}}(-9)). \tag{3.64}$$

Furthemore, using the Serre duality

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(p)) = H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2-p))^*, \tag{3.65}$$

where p is any integer and $*$ signifies the dual linear space, we see that

$$H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-6) \oplus \mathcal{O}_{\mathcal{S}}(-8) \oplus \mathcal{O}_{\mathcal{S}}(-9)) = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(4) \oplus \mathcal{O}_{\mathcal{S}}(6) \oplus \mathcal{O}_{\mathcal{S}}(7))^*. \tag{3.66}$$

If we denote

$$W_1 = H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F)), \tag{3.67}$$

then it follows from (3.64), (3.66) and (3.21) that

$$dim W_1 = 20. \tag{3.68}$$

Now consider $H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))$. Exactly as above, one can show using a Leray spectral sequence that

$$H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F)) = H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-1) \oplus \bigoplus_{i=3}^{7} \mathcal{O}_{\mathcal{S}}(-i)) \tag{3.69}$$

and, by Serre duality, that

$$H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-1) \oplus \bigoplus_{i=3}^{7} \mathcal{O}_{\mathcal{S}}(-i)) = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-1) \oplus \bigoplus_{i=3}^{7} \mathcal{O}_{\mathcal{S}}(-2+i))^*. \tag{3.70}$$

If we denote

$$W_2 = H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F)), \tag{3.71}$$

then it follows from (3.69), (3.70) and (3.21) that

$$dim W_2 = 20 \tag{3.72}$$

as well.

Let us now reconsider the structure of the exact sequence (3.61). This can now be written as

$$0 \to H^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) \to W_1 \overset{f_D}{\to} W_2 \to \cdots \tag{3.73}$$

where we have used (3.50), (3.51),(3.67) and (3.71). We will now display the linear mapping f_D. This is induced from the map f_D in the short exact sequence

$$0 \to E \otimes \mathcal{O}_{dP_9}(-D) \stackrel{f_D}{\to} E \to E|_D \to 0. \tag{3.74}$$

As discussed previously, the map f_D in (3.74) is just multiplication by the unique, up to scaling, element of $H^0(dP_9, \mathcal{O}_{dP_9}(D))$ with the property that it vanishes on D. In the present example, $D = \mathcal{C}|_{dP_9}$.

Exact sequence (3.73) is precisely what we need to solve the problem of whether or not $h^0(\mathcal{S}, V|_{\mathcal{S}}(-1))$ is zero. Since W_1 and W_2 are just linear spaces of the same dimension, and since it follows from (3.73) that the space we are interested in, $H^0(\mathcal{S}, V|_{\mathcal{S}}(-1))$, is the kernel of the map $f_{\mathcal{C}|_{dP_9}}$, we conclude that $h^0(\mathcal{S}, V|_{\mathcal{S}}(-1)) > 0$ if and only if

$$det\ f_{\mathcal{C}|_{dP_9}} = 0. \tag{3.75}$$

Therefore, the solution to this problem, and hence to finding the zeros of Pfaff$(\mathcal{D}_-)$, reduces to computing $det\ f_{\mathcal{C}|_{dP_9}}$, to which we now turn. An arbitrary element of W_1 can be charactarized as follows. Let

$$\tilde{w}_1 = B_{-6} \oplus B_{-8} \oplus B_{-9} \tag{3.76}$$

be an element of $H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-6) \oplus \mathcal{O}_{\mathcal{S}}(-8) \oplus \mathcal{O}_{\mathcal{S}}(-9))$ where B_{-i}, $i = 6, 8, 9$ denotes an arbitrary section in $H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-i))$. We see from the Serre duality relation (3.65) and (3.21) that

$$h^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-i)) = i - 1. \tag{3.77}$$

Now let us lift $\tilde{w}_1$ to w_1 in $H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F))$, using (3.64). We find that

$$w_1 = b_{-6}z + b_{-8}x + b_{-9}y, \tag{3.78}$$

where, from the isomorphism

$$\mathcal{O}_{dP_9}(kF) = \pi^* \mathcal{O}_{\mathcal{S}}(k) \tag{3.79}$$

for any integer k, $b_{-i} = \pi^* B_{-i}$ are elements in $H^1(dP_9, \mathcal{O}_{dP_9}(-iF))$ and we have used the fact that dP_9 has a Weierstrass representation

$$y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3, \tag{3.80}$$

where [10]

$$x \sim \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 2F), \qquad y \sim \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 3F), \qquad z \sim \mathcal{O}_{dP_9}(3\sigma|_{dP_9}) \tag{3.81}$$

and

$$g_2 \sim \mathcal{O}_{dP_9}(4F), \qquad g_3 \sim \mathcal{O}_{dP_9}(6F). \tag{3.82}$$

In the above equations, symbol $\sim$ means "section of".

Expression (3.78) completely characterizes an element $w_1 \in W_1$. In a similar way, any element $w_2 \in W_2 = H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))$ can be written as

$$w_2 = c_{-3}zx + c_{-4}zy + c_{-5}x^2 + c_{-6}xy + c_{-7}y^2 \tag{3.83}$$

where for $j = 3, \cdots 7$, $c_{-j} = \pi^* C_{-j}$ is an element of $H^1(dP_9, \mathcal{O}_{dP_9}(-jF))$ and C_{-j} is a section in the $j-1$-dimensional space $H^1(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(-j))$. Equation (3.83) follows from expressions (3.69), (3.79) and (3.81). Finally, we note from (3.45) and (3.81) that any map $f_{\mathcal{C}|_{dP_9}}$ can be expressed as

$$f_{\mathcal{C}|_{dP_9}} = m_5 z + m_3 x + m_2 y, \tag{3.84}$$

where $m_k = \pi^* M_k$, $k = 2, 3, 5$, is an element in $H^0(dP_9, \mathcal{O}_{dP_9}(kF))$ and M_k is a section in the $k+1$-dimensional space $H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(k))$. Although there are thirteen parameters in m_k, $k = 2, 3, 5$, it must be remembered that they are homogeneous coordinates for the twelve dimensional projective space $\mathbb{P}H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 5F))$.

Putting this all together, we can completely specify the linear mapping $W_1 \overset{f_{\mathcal{C}|_{dP_9}}}{\rightarrow} W_2$. First note that with respect to fixed basis vectors of W_1 and W_2, the linear map $f_{\mathcal{C}|_{dP_9}}$ is a 20×20 matrix. In order to find this matrix explicitly, we have to study its action on these vectors. This action is generated through multiplcation by a section $f_{\mathcal{C}|_{dP_9}}$ of the form (3.84). Suppressing, for the time being, the vector coefficients b_{-i} and c_{-j}, we see from (3.78) that the linear space W_1 is spanned by the basis vector blocks

$$z, \quad x, \quad y \tag{3.85}$$

whereas it follows from (3.83) that the linear space W_2 is spanned by basis vector blocks

$$zx, \quad zy, \quad x^2, \quad xy, \quad y^2. \tag{3.86}$$

The explicit matrix M_{IJ} representing $f_{\mathcal{C}|_{dP_9}}$ is determined by multiplying the basis vectors (3.85) of W_1 by $f_{\mathcal{C}|_{dP_9}}$ in (3.84). Expanding the resulting vectors in W_2 in the basis (3.86) yields the matrix. We find that M_{IJ} is given by

$$\begin{array}{c} \\ xz \\ yz \\ x^2 \\ xy \\ y^2 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} z & x & y \end{array} \\ \begin{pmatrix} m_3 & m_5 & 0 \\ m_2 & 0 & m_5 \\ 0 & m_3 & 0 \\ 0 & m_2 & m_3 \\ 0 & 0 & m_2 \end{pmatrix} \end{array}. \tag{3.87}$$

Of course, M_{IJ} is a 20×20 matrix, so each of the elements of (3.87) represents a $(j-1) \times (i-1)$ matrix for the corresponding $j = 3, 4, 5, 6, 7$ and $i = 6, 8, 9$. For example, let us compute M_{11}. This corresponds to the $xz - z$ component of (3.87) where

$$H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F))|_{b_{-6}} \stackrel{m_3}{\to} H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))|_{c_{-3}}. \tag{3.88}$$

Note, that

$$h^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F))|_{b_{-6}} = 5 \tag{3.89}$$

and

$$h^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))|_{c_{-3}} = 2. \tag{3.90}$$

An explicit matrix for m_3 is most easily obtained if we now use the Leray spectral sequences and Serre duality discussed in (3.64), (3.66) and (3.69), (3.70) to identify

$$H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F))|_{b_{-6}} = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(4))^* \tag{3.91}$$

and

$$H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))|_{c_{-3}} = H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(1))^*. \tag{3.92}$$

If we define the two dimensional linear space

$$\hat{V} = H^0(\mathcal{S}, \mathcal{O}(1)), \tag{3.93}$$

then we see that

$$H^1(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} - 6F))|_{b_{-6}} = Sym^4 \hat{V}^* \tag{3.94}$$

and

$$H^1(dP_9, \mathcal{O}_{dP_9}(6\sigma|_{dP_9} - F))|_{c_{-3}} = \hat{V}^*, \tag{3.95}$$

where by $Sym^k \hat{V}^*$ we denote the k-th symmetrized tensor product of the dual vector space $\hat{V}^*$ of $\hat{V}$. Similarly, it follows from (3.45) and (3.93) that m_3 is an element in

$$H^0(dP_9, \mathcal{O}_{dP_9}(3\sigma|_{dP_9} + 5F))|_{m_3} = Sym^3 \hat{V}. \tag{3.96}$$

Let us now introduce a basis

$$\{u, v\} \in \hat{V} \tag{3.97}$$

and the dual basis

$$\{u^*, v^*\} \in \hat{V}^*, \tag{3.98}$$

where

$$u^* u = v^* v = 1, \qquad u^* v = v^* u = 0. \tag{3.99}$$

Then the space $Sym^k \hat{V}^*$ is spanned by all possible homogeneous polynomials in u^*, v^* of degree k. Specifically,

$$\{u^{*4}, u^{*3}v^*, u^{*2}v^{*2}, u^* v^{*3}, v^{*4}\} \in Sym^4 \hat{V}^* \tag{3.100}$$

is a basis of $Sym^4 \hat{V}^*$ and

$$\{u^3, u^2 v, u v^2, v^3\} \in Sym^3 \hat{V}. \tag{3.101}$$

is a basis of $Sym^3 \hat{V}$. Clearly, any section m_3 can be written in the basis (3.101) as

$$m_3 = \phi_1 u^3 + \phi_2 u^2 v + \phi_3 u v^2 + \phi_4 v^3, \tag{3.102}$$

where $\phi_a, a = 1, \cdots 4$ represent the associated moduli. Now, by using the multiplication rules (3.99), we find that the explicit 2×5 matrix representation of m_3 in this basis, that is, the M_{11} submatrix of M_{IJ}, is given by

$$\begin{array}{cc} & \begin{array}{ccccc} u^{*4} & u^{*3}v^* & u^{*2}v^{*2} & u^* v^{*3} & v^{*4} \end{array} \\ \begin{array}{c} u^* \\ v^* \end{array} & \left(\begin{array}{ccccc} \phi_1 & \phi_2 & \phi_3 & \phi_4 & 0 \\ 0 & \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{array} \right) \end{array}. \tag{3.103}$$

Continuing in this manner, we can fill out the complete 20×20 matrix M_{IJ}. It is not particularly enlightening, so we will not present the matrix M_{IJ} in this paper. What is important is the determinant of M_{IJ}. Let us parametrize the sections m_2 and m_5 as

$$\begin{aligned} m_2 &= \chi_1 u^2 + \chi_2 u v + \chi_3 v^2 \\ m_5 &= \psi_1 u^5 + \psi_2 u^4 v + \psi_3 u^3 v^2 + \psi_4 u^2 v^3 + \psi_5 u v^4 + \psi_6 v^5 \end{aligned} \tag{3.104}$$

where $\chi_b, b = 1, 2, 3$ and $\psi_c, c = 1, \cdots 6$ represent the associated moduli. It is then straightforward to compute the determinant of M_{IJ} using the programs Maple or MATHEMATICA. We find that it has two rather amazing prperties: It factors very simply, and it is independent of some of the variables. Namely

$$det\ f_{\mathcal{C}|_{dP_9}} = det\ M_{IJ} = \mathcal{P}^4, \tag{3.105}$$

where

$$\begin{aligned}
\mathcal{P} = {} & \chi_1^2\chi_3\phi_3^2 - \chi_1^2\chi_2\phi_3\phi_4 - 2\chi_1\chi_3^2\phi_3\phi_1 - \\
& \chi_1\chi_2\chi_3\phi_3\phi_2 + \chi_2^2\chi_3\phi_1\phi_3 + \phi_4^2\chi_1^3 - \\
& 2\phi_2\phi_4\chi_3\chi_1^2 + \chi_1\chi_3^2\phi_2^2 + 3\phi_1\phi_4\chi_1\chi_2\chi_3 + \\
& \phi_2\chi_1\phi_4\chi_2^2 + \phi_1^2\chi_3^3 - \phi_2\chi_2\phi_1\chi_3^2 - \phi_4\phi_1\chi_2^3
\end{aligned} \tag{3.106}$$

is a homogeneous polynomial of degree 5 in the seven transition moduli ϕ_a and χ_b. Note that none of the remaining six moduli ψ_a appear in $\mathcal{P}$. We will see in [14] that the quintic polynomial $\mathcal{P}$ has a simple interpretation, as a resultant. This leads to a geometric description of the hypersurface $D_{\mathcal{P}} \subset \mathbb{P}^{12}$ given by $\mathcal{P} = 0$: There is a smooth eleven-dimensional variety, $\mathcal{F}$, which is a $\mathbb{P}^9$ bundle over $\mathbb{P}^1 \times \mathbb{P}^1$, and a map $i : \mathcal{F} \to \mathbb{P}^{12}$ with the property that i embeds each $\mathbb{P}^9$ fiber as a linear subspace $\mathbb{P}^9$ in $\mathbb{P}^{12}$. Then, we find that

$$D_{\mathcal{P}} = i(\mathcal{F}) \tag{3.107}$$

and, hence, $D_{\mathcal{P}}$ is the union of a two-parameter family of linear subspaces $\mathbb{P}^9$. The singular locus of $D_{\mathcal{P}}$ is the image of the loci of $\mathcal{F}$ on which i is not injective. These singular subspaces of $D_{\mathcal{P}}$ can be analyzed completely. The fact that $\mathcal{P}$ does not depend on the ψ coordinates means that the projective subspace $\mathbb{P}^5 \subset \mathbb{P}^{12}$ given by $\phi = \chi = 0$ is contained in each of the subspaces $\mathbb{P}^9$, and hence $\mathbb{P}^5$ is contained in the singular locus of $D_{\mathcal{P}}$. In fact, $D_{\mathcal{P}}$ is a cone whose vertex is this $\mathbb{P}^5$ and whose base is a hypersurface $\overline{D_{\mathcal{P}}}$ in the complementary (quotient) projective space $\mathbb{P}^6$ with homogeneous coordinates ϕ and χ.

4 The Superpotential and its Critical Points:

In the previous section, we categorized the vanishing locus of $\mathcal{P}$ and, hence, Pfaff$(\mathcal{D}_-)$. However, one can achieve much more than this, actually calculating from the above results the exact expressions for the Pfaffian and the non-perturbative superpotential W. Recall that $\mathcal{P}$ is a section of $\mathcal{O}_{\mathbb{P}^{12}}(D_{\mathcal{P}})$ which vanishes on $D_{\mathcal{P}} \subset \mathbb{P}^{12}$. On the other hand, Pfaff$(\mathcal{D}_-)$ is itself a global holomorphic section of a line bundle over $\mathbb{P}^{12}$. That it is a section, rather than a function, is a reflection of the fact that the Pfaffian is not gauge invariant. Since, from the above results, Pfaff$(\mathcal{D}_-)$ also has $D_{\mathcal{P}}$ as its zero locus, it follows that Pfaff$(\mathcal{D}_-)$ is a section of

$$\mathcal{O}_{\mathbb{P}^{12}}(pD_{\mathcal{P}}), \tag{4.1}$$

where p is a positive integer. Therefore,

$$\text{Pfaff}(\mathcal{D}_-) = c\mathcal{P}^p \tag{4.2}$$

for some constant parameter c. It is possible to define a purely algebraic analogue of $\text{Pfaff}(\mathcal{D}_-)$. A more careful analysis of our argument shows that, quite generally, the algebraic version of $\text{Pfaff}(\mathcal{D}_-)$ equals $det f_{\mathcal{C}|_{dP_9}}$, up to a constant. Furthermore, it was shown in [11, 12, 13] that the algebraic and analytic notions of $\text{Pfaff}(\mathcal{D}_-)$ agree. Therefore,

$$\text{Pfaff}(\mathcal{D}_-) = c\,det f_{\mathcal{C}|_{dP_9}} = c\mathcal{P}^4, \tag{4.3}$$

where $\mathcal{P}$ is given in (3.106). So in our specific case

$$p = 4. \tag{4.4}$$

Thus, up to an overall constant, we have determined $\text{Pfaff}(\mathcal{D}_-)$ as an explicit holomorphic function of the twelve moduli of $\mathbb{P}H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(5) \oplus \mathcal{O}_{\mathcal{S}}(3) \oplus \mathcal{O}_{\mathcal{S}}(2))$. In addition, as will be discussed in [14], polynomial $\mathcal{P}^4$ is closely related to the theta function, Θ, whose natural variables, in this example, can be expressed in terms of the vector bundle moduli.

We can now present the final answer for the vector bundle moduli contribution to the non-perturbative superpotential. Since the superpotential is proportional to the Pfaffian, we conclude that

$$\begin{aligned} W \propto \mathcal{P}^4 = (&\chi_1^2\chi_3\phi_3^2 - \chi_1^2\chi_2\phi_3\phi_4 - 2\chi_1\chi_3^2\phi_3\phi_1 - \\ &\chi_1\chi_2\chi_3\phi_3\phi_2 + \chi_2^2\chi_3\phi_1\phi_3 + \phi_4^2\chi_1^3 - \\ &2\phi_2\phi_4\chi_3\chi_1^2 + \chi_1\chi_3^2\phi_2^2 + 3\phi_1\phi_4\chi_1\chi_2\chi_3 + \\ &\phi_2\chi_1\phi_4\chi_2^2 + \phi_1^2\chi_3^3 - \phi_2\chi_2\phi_1\chi_3^2 - \phi_4\phi_1\chi_2^3)^4 \,, \end{aligned} \tag{4.5}$$

where the thirteen transition moduli ϕ_a, χ_b and ψ_c parameterize the twelve dimensional moduli space $\mathbb{P}H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(5) \oplus \mathcal{O}_{\mathcal{S}}(3) \oplus \mathcal{O}_{\mathcal{S}}(2))$. Summarizing, W in (4.5) is the non-perturbative superpotential induced by wrapping a heterotic superstring once around the isolated curve $\mathcal{S}$ in an elliptically fibered Calabi-Yau threefold with base $B = \mathbb{F}_1$. The holomorphic vector bundle V has structure group $G = SU(3)$ and W is a holomorphic function of the moduli associated with $V|_{\mathcal{S}}$. Note that, in this specific case, the ψ_c moduli do not appear. This is an artifact our our example. Generically, we expect all transition moduli to appear in W. The remaining moduli of V, that is, those not associated with $V|_{\mathcal{S}}$, do not appear in this contribution to the superpotential.

Having found an explicit expression for the non-perturbative superpotential W, it is of interest to find its critical points, that is, those points or submanifolds of the moduli space $\mathbb{P}H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(5) \oplus \mathcal{O}_{\mathcal{S}}(3) \oplus \mathcal{O}_{\mathcal{S}}(2))$ where $N = 1$ supersymmetry is preserved and the cosmological constant vanishes. Recall (see for example [34]) that the Kahler covariant derivative of W is given by

$$D_{a_i}W = \partial_{a_i}W + \frac{1}{M_P^2}(\partial_{a_i}K)W, \tag{4.6}$$

where a_i is any moduli field in W, K is the Kahler potential, M_P is the Planck mass and that $N = 1$ supersymmetry is unbroken if and only if

$$D_{a_i}W = 0. \tag{4.7}$$

Using the expression

$$\mathbf{V} = e^{\frac{K}{M_P^2}}(G^{-1}|DW|^2 - \frac{3}{M_P^2}|W|^2) \tag{4.8}$$

for the potential energy, we see that both DW and $\mathbf{V}$ will vanish if and only if

$$W = dW = 0. \tag{4.9}$$

These equations define the critical points of W. To determine the critical points on a given coordinate patch of $\mathbb{P}H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(5) \oplus \mathcal{O}_{\mathcal{S}}(3) \oplus \mathcal{O}_{\mathcal{S}}(2))$, one performs the differentiations in (4.9) with respect to homogeneous coordinates and then goes to a local patch, setting one of the homogeneous coordinates ϕ_a, χ_b or ψ_c to unity. This proceedure can be justified using the Euler equation for a homogeneous function. Since the superpotential W is given by the fourth power of the polynomial $\mathcal{P}$, the solution to the equations (4.9) is the whole zero locus of $\mathcal{P}$, that is, the eleven-dimensional submanifold

$$D_{\mathcal{P}} \subset \mathbb{P}H^0(\mathcal{S}, \mathcal{O}_{\mathcal{S}}(5) \oplus \mathcal{O}_{\mathcal{S}}(3) \oplus \mathcal{O}_{\mathcal{S}}(2)). \tag{4.10}$$

The singularities of $D_{\mathcal{P}}$ are therefore irrelevant in our specific case. In [14] we will study other examples, where some of the factors of W occur with multiplicity 1, so a detailed geometric understanding of $D_{\mathcal{P}}$ and its singularities becomes important.

5 Conclusion:

In this paper, we have considered non-perturbative superpotentials W generated by wrapping a heterotic superstring once around an isolated holomorphic curve $\mathcal{C}$ of genus-zero in an elliptically fibered Calabi-Yau threefold with holomorphic vector bundle V. We presented a method for calculating the Pfaffian factor in such superpotentials as an explicit function of the vector bundle moduli associated with $V|_C$. For specificity, the vector bundle moduli contribution to W was computed exactly for a Calabi-Yau manifold with base $B = \mathbb{F}_1$ and isolated curve S, and the associated critical points discussed. Our method, however, has wide applicability, as will be shown in [14] where the vector bundle moduli contributions to the superpotentials in a number of different contexts will be exactly computed and analyzed. In addition, we will show in [14] how to compute some of the homogeneous polynomials appearing in these superpotentials analytically and will further extend our results to non-isolated holomorphic curves. Finally, in conjunction with the associated Kahler potential, one can use our superpotential to calculate the potential energy functions of the vector bundle moduli. This potential determines the stability of the vector bundle and has important implications for superstring and M-theory cosmology, as will be discussed elsewhere.

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