# Quantum Field Theory and Representation Theory: A Sketch

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# 1 Introduction

Ever since the early days of theory there has been a close link between representation theory and quantum mechanics. The Hilbert space of quantum mechanics is a (projective) unitary representation of the symmetries of the classical mechanical system being quantized. The fundamental observables of quantum mechanics correspond to the infinitesimal generators of these symmetries (energy corresponds to time translations, momentum to space translations, angular momentum to rotations, charge to phase changes). The relation between quantum mechanics and representation theory has been formalized as the subject of "geometric quantization" which ideally associates to a classical mechanical phase space (a symplectic manifold M) a complex vector space V in a functorial manner. This functor takes at least some subgroup $G$ of the symplectomorphisms (canonical transformations) of M to unitary transformations of V, making V a unitary $G$-representation.

The theory of geometric quantization has never been very popular among physicists for at least two reasons (in addition to the fact that the mathematical apparatus required is rather extensive and mostly unfamiliar to physicists). The first is that it seems to have very little to say about quantum field theory. The quantum field theory of the standard model of particle physics is built upon the geometrical concepts of gauge fields and the Dirac operator on spinors, concepts which have no obvious relation to those used in geometric quantization.

The second problem is that the most well-developed formalism for doing calculations within the standard model is the path integral formalism. While no rigorous version of this exists, all evidence is that consistent calculations can be performed using this formalism, at least within perturbation theory or outside of perturbation theory with a lattice cut-off. Even in the simple case of quantum mechanics the relationship between the path integral quantization and geometric quantization has been quite unclear making it impossible to see how the ideas of geometric quantization can be useful in the much more complex situation of standard model quantum field theory.

Taking the path integral as fundamental, in its sketchiest form the problem of understanding the standard model quantum field theory comes down to that of making sense of ratios of expressions such as

$$\int [dA](\int [d\Psi]O(A,\Psi)e^{\int_M \Psi \not{D}_A \Psi})e^{\int_M -\frac{1}{g^2}||F_A||^2}$$

Here $\int [dA]$ is supposed to be an integral over the space of connections on a principal bundle over the manifold $M = \mathbf{R}^4$. The variables $\Psi$ are sections of some product of vector bundles, one factor of which is the spinor bundle of M. $\int [d\Psi]$ is supposed to be the linear functional on an infinite dimensional exterior algebra generated by variables $\Psi$ given by taking the coefficient of the top dimensional power of the $\Psi$'s, $\int_M ||F_A||^2$ is the norm-squared of the curvature of the connection $A$ and $\not{D}_A$ is the Dirac operator formed using the covariant derivative determined by $A$. $O(A,\Psi)$ is some functional of the connection and the $\Psi$ variables. One only really expects to make sense of this expression for certain classes of $O(A,\Psi)$. To make such path integrals well-defined one needs to choose a distance scale called a "cut-off" and suppress integration over variables that vary on scales smaller than the cut-off. The parameter $g$ depends on the cut-off and for an asymptotically free theory must be taken to zero in a specified manner as the cut-off goes to zero.

To a mathematician, such path integrals immediately raise a host of questions.

- Why is the space of connections of a principal bundle appearing and why is one trying to integrate over it?
- What is the significance of the Dirac operator acting on sections of the spinor bundle? Does this have anything to do with the same structures that appear in index theory?
- Why is one considering
$$e^{\int_M \Psi \not{D}_A \Psi}$$
and extracting the coefficient of the top dimensional power of the infinite dimensional exterior algebra over the spinor fields?

The present work is motivated by the desire to try and get some answers to these questions by investigating what sort of formal mathematical problem such path integral expressions could represent the solution to. A hint is provided by the deep relationship between quantum theory and representation theory exposed by the theory of geometric quantization. In brief, we are looking for a representation-theoretical interpretation of the kind of quantum field theory that appears in the standard model. The groups involved will be ones whose representation theory is not at all mathematically understood at the present time so unfortunately known mathematics cannot be used to say much about these quantum field theories. On the other hand, the wealth of knowledge that physicists have accumulated about these theories may be useful to mathematicians in trying to understand something about the representation theory of certain important infinite dimensional groups.

The main goal of this paper is to explain and provide a sketch of evidence for the following conjecture:

**Conjecture.** *The quantum field theory of the standard model may be understood purely in terms of the representation theory of the automorphism group of some geometric structure.*

The reader is to be warned that the present version of this document suffers from sloppiness on several levels. If factors of 2, $\pi$ and $i$ seem to be wrong, they probably are. Many of the mathematical statements are made with a blatant disregard for mathematical precision. Sometimes this is done out of ignorance, sometimes out of a desire to simply get to the heart of the matter at hand. The goal has been to strike a balance such that physicists may have a fighting chance of reading this while mathematicians may not find the level of imprecision and simplification too hard to tolerate.

# 2 Quantizing G/T: The Representation Theory of Compact Lie Groups

The standard description in physics textbooks of how to quantize a Hamiltonian classical mechanical system instructs one to first choose "canonical coordinates", (i.e. coordinates $q_i, p_i$ for $i = 1, \ldots, n$) on phase space $\mathbf{R}^{2n}$ satisfying

$$\{q_i, q_j\} = 0, \{p_i, p_j\} = 0, \{q_i, p_j\} = \delta_{ij}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket. Equivalently, one is choosing a symplectic structure

$$\omega = \sum_{i=1}^{n} dq_i \wedge dp_i$$

on $\mathbf{R}^{2n}$.

The corresponding quantum system then is defined in terms of a Hilbert space $\mathcal{H}$ and operators corresponding to the canonical coordinates satisfying

$$[\hat{q}_i, \hat{q}_j] = 0, [\hat{p}_i, \hat{p}_j] = 0, [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}$$

One can construct various explicit Hilbert spaces $\mathcal{H}$ and sets of operators on $\mathcal{H}$ satisfying these conditions, for instance by taking $\mathcal{H}$ to be a set of functions on the coordinates $q_i$, $\hat{q}_i$ to be multiplication by $q_i$, and $\hat{p}_i$ to be the operator $-i\hbar\frac{\partial}{\partial q_i}$.

A more abstract way of looking at this is to note that the $\hat{q}_i, \hat{p}_i$ satisfy the defining relations of the Lie algebra of the Heisenberg group and $\mathcal{H}$ should be a (projective) unitary representation of this group. The Stone-von Neumann theorem tells one that up to equivalence this representation is unique and implies that linear symplectic transformations act (projectively) on $\mathcal{H}$. This projective representation is a true representation of the metaplectic group, a $\mathbf{Z}_2$ extension of the symplectic group.

While this procedure works fine for phase spaces that can be globally identified with $\mathbf{R}^{2n}$, it fails immediately in other simple cases. For instance, the phase space of a fixed spinning particle of spin $\frac{k}{2}$ in $\mathbf{R}^3$ can be taken to be the sphere $S^2$ with area $k$, but there is no way to globally choose canonical coordinates in this case. The corresponding quantum theory should have $\mathcal{H} = \mathbf{C}^{k+1}$ with angular momentum operators $L_i, i = 1, 2, 3$ satisfying the commutation relations of the Lie algebra of the group Spin(3).

While the standard physics quantization procedure fails in this simple case, it is an example of a well-known phenomenon in the representation theory of Lie groups. In this section we will review the representation theory of a compact connected Lie group $G$ from a point of view which makes clear the relationship between the representation and the corresponding phase space. Quite a lot of modern mathematics goes into this story and certainly is not needed for the problem at hand, but has been developed to deal with more general cases such as that of non-compact groups.

## 2.1 The Borel-Weil Theorem

Let $G$ be a compact, connected Lie group, it will carry a left and right invariant Haar measure. A copy $(G_L)$ of $G$ acts from the left on the Hilbert space $L^2(G)$ as the left regular representation. This representation is infinite dimensional and reducible. There is a commuting action on the right by another copy $(G_R)$ of $G$. In the case of $G = U(1)$ the decomposition of the regular representation into irreducible representations is given by Fourier analysis. The irreducible representations are one-dimensional and labelled by an integer. More generally:

**Theorem 1** (Peter-Weyl)**.** *There is a Hilbert space direct sum decomposition*

$$L^2(G) = \sum_{i \in \hat{G}} V_i \otimes V_i^*$$

*where $\hat{G}$ is the set of irreducible representations of $G$.*

Here $V_i$ is an irreducible representation of $G$ on a complex vector space of dimension $d_i$ and $V_i^*$ is the dual representation. $G_R$ acts on the $V_i^*$ factor, $G_L$ acts on the $V_i$ factor. Thus the irreducible $V_i$ occurs in either the right or left regular representation of $G$ on $L^2(G)$ with multiplicity $d_i$.

One can understand this decomposition as a decomposition of functions on $G$ into matrix elements of irreducible representations. To the element

$$(|\alpha>, <\beta|) \in V_i \times V_i^*$$

is associated the function on $G$ given by

$$f(g) = <\beta|\pi_i(g^{-1})|\alpha>$$

where $\pi_i(g)$ is the representation of the group element $g$ on the vector space $V_i$. Under this association the map

$$f(g) \to f(g_L^{-1} g)$$

corresponds to

$$|\alpha> \to \pi_i(g_L)|\alpha>$$

and

$$f(g) \to f(g g_R)$$

corresponds to

$$< \beta| \rightarrow < \overline{\pi}_i(g)\beta|$$

where $\overline{\pi}_i(g)$ is the contragredient representation to $\pi_i(g)$.

The problem still remains of identifying the irreducible representations of $G$ and computing their dimensions $d_i$. This can be done by relating irreducible representations of $G$ to representations of the maximal torus $T$ of $G$, whose representations are given by Fourier analysis. This can be done with the Cartan-Weyl theory of the highest weight. Detailed expositions of the theory can be found in [1, 16].

Different choices of maximal tori $T$ are related by conjugation by an element $g \in G$

$$T \rightarrow gTg^{-1}$$

The subgroup of $G$ that leaves $T$ invariant under this conjugation is called the normalizer $N(T)$ and:

**Definition 1.** *The Weyl group $W(G,T)$ is the group $N(T)/T$ of non-trivial automorphisms of $T$ that come from conjugations in $G$.*

The character $\chi_V$ of a representation is a conjugation invariant function on $G$ which can be computed in a matrix representation $\pi_V$ as a trace

$$\chi_V(g) = tr_{\pi_V}(g)$$

As a function on $T$, $\chi_V$ will always be invariant under the action of $W(G,T)$ on $T$.

In order to find an explicit decomposition of $L^2(G)$ into irreducible representations we will begin by decomposing $L^2(G)$ into pieces that transform under the action of $T$ from the right according to the various weights of $T$. Picking a weight $\lambda$ of $T$, consider the subspace of $L^2(G)$ that satisfies

$$f(gt) = t^{-1} \cdot f(g) = \chi_\lambda^{-1} f(g)$$

An equivalent definition of this space is as a space of sections $\Gamma(L_\lambda)$ of the line bundle

$$\begin{array}{ccc} \mathbf{C} & \longrightarrow & G \times_T \mathbf{C} = L_\lambda \\ & & \downarrow \\ & & G/T \end{array}$$

over the quotient manifold $G/T$ (sometimes known as a flag manifold since in the case $G = U(n)$ it is the space of flags in $\mathbf{C}^n$).

$L_\lambda$ has a curvature 2-form $\omega_\lambda$ which is invariant under the left $G$ action on $G/T$. This is a symplectic structure, so $G/T$ can be thought of as the phase space for a mechanical system. In the simple case $G = SU(2)$, $T = U(1)$, the $\lambda$ are labelled by the integers and

$$L_\lambda = SU(2) \times_{U(1)} \mathbf{C}$$

is the $\lambda$'th power of the tautological line bundle over $\mathbf{C}P_1$.

The left action of $G$ on $L^2(G)$ leaves $\Gamma(L_\lambda)$ invariant and $\Gamma(L_\lambda)$ is an infinite dimensional reducible representation. This representation is sometimes thought of as the "pre-quantization" of the symplectic manifold $G/T$ with symplectic form $\omega_\lambda$. In terms of the decomposition into matrix elements of the Peter-Weyl theorem, $\Gamma(L_\lambda)$ is the subspace of matrix elements that are sums over all irreducible representations of terms of the form

$$<\beta|\pi_i^{-1}(gt)|\alpha>=<\beta|\pi_i^{-1}(t)\pi_i^{-1}(g)|\alpha>=<\beta|\chi_\lambda^{-1}(t)\pi_i^{-1}(g)|\alpha>$$

i.e. the matrix elements such that $<\beta|$ is in the subspace $V_{i,\lambda}^*$ of $V_i^*$ that transforms with character $\chi_\lambda$ under $T$. $V_{i,\lambda}^*$ can equivalently be defined as

$$(V_i^* \otimes \mathbf{C}_\lambda)^T$$

We have shown that $\Gamma(L_\lambda)$ decomposes under the left $G$ action into a direct sum of irreducibles $V_i$ with the multiplicity given by the dimension of $V_{i,\lambda}^*$.

$$\Gamma(L_\lambda)=\sum_{i\in\hat{G}} V_i \otimes (V_i^*\otimes \mathbf{C}_\lambda)^T$$

The true quantization should be an irreducible representation of $G$ and this will require using further structure. To pick out a single irreducible we need to consider the Lie algebra $\mathfrak{g}$ and how it transforms as a real representation under the adjoint action of $T$. The Lie subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ corresponding to $T$ transforms trivially under this action. $T$ acts without invariant subspace on the quotient $\mathfrak{g}/\mathfrak{t}$ so it breaks up into two-dimensional subspaces. Thus $\mathfrak{g}/\mathfrak{t}$ is an even-dimensional real vector space. While $\mathfrak{g}/\mathfrak{t}$ is not a Lie algebra, a choice of complex structure on this space gives a decomposition of the complexification as

$$\mathfrak{g}/\mathfrak{t}\otimes \mathbf{C}=\mathfrak{n}_+\oplus\mathfrak{n}_-$$

Here

$$\mathfrak{n}_+=\sum_{\alpha\in\Phi}\mathfrak{g}^\alpha$$

where $\Phi$ is a set labelling the so-called positive roots and $\mathfrak{n}_-$ is its complex conjugate. The $\mathfrak{g}^\alpha$ are the various one-dimensional complex root spaces on which $T$ acts with weight $\alpha$.

Different choices of the set $\Phi$ corresponding to different invariant complex structures are in one-to-one correspondence with elements of the Weyl group $W(G,T)$.

Globally over $G/T$ one can form

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{t}\otimes\mathbf{C} & \longrightarrow & G\times_T(\mathfrak{g}/\mathfrak{t}\otimes\mathbf{C}) \\ & & \downarrow \\ & & G/T \end{array}$$

which is the complexified tangent bundle of $G/T$. The decomposition into positive and negative roots gives an integrable complex structure since

$$[\mathfrak{n}_+,\mathfrak{n}_+]\subset\mathfrak{n}_+$$

and

$$[\mathfrak{n}_-,\mathfrak{n}_-]\subset\mathfrak{n}_-$$

So for each element of the Weyl group $G/T$ is a complex manifold, but in an inequivalent way. $L_\lambda$ is a holomorphic line bundle and one can show that the holomorphic sections of $L_\lambda$ are precisely those sections of $\Gamma(L_\lambda)$ that are invariant under infinitesimal right translations generated by elements of $\mathfrak{n}_-$, i.e.

$$\Gamma^{hol}(L_\lambda) = \{f \in \Gamma(L_\lambda) : r(X)f = 0 \ \forall X \in \mathfrak{n}_-\}$$

or

$$\Gamma^{hol}(L_\lambda) = \sum_{i \in \hat{G}} V_i \otimes \{v \in (V_i^* \otimes \mathbf{C}_\lambda)^T : \mathfrak{n}_- v = 0\}$$

The condition $\mathfrak{n}_- v = 0$ picks out the lowest weight space in $V_i^*$, so the sum on the right is zero unless the lowest weight space of $V_i^*$ has weight $-\lambda$ or equivalently the highest weight of $V_i$ is $\lambda$. The classification theory of representations by their highest weight implies that for each $\lambda$ in the dominant Weyl chamber there is precisely one irreducible representation with highest weight $\lambda$. For $\lambda$ not in the highest weight chamber there are no such representations. So we have

**Theorem 2** (Borel-Weil)**.** *For $\lambda$ dominant $\Gamma^{hol}(L_\lambda) = H^0(G/T, \mathcal{O}(L_\lambda))$ is the irreducible $G$ representation $V_\lambda$ of highest weight $\lambda$, and for $\lambda$ not dominant this space is zero. This gives all finite dimensional irreducible representations of $G$.*

By demanding that functions on $G$ transform as $\lambda$ under the right $T$ action and using the rest of the right $G$ action to impose invariance under infinitesimal right $\mathfrak{n}_-$ translations, we have constructed a projection operator on functions on $G$ that gives zero unless $\lambda$ is dominant. When $\lambda$ is dominant, this projection picks out a space of functions on $G$ that transforms under the left $G$ action as the irreducible representation with highest weight $\lambda$.

## 2.2 Lie Algebra Cohomology and the Borel-Weil-Bott-Kostant Theorem

The Borel-Weil theorem gives a construction of the irreducible representations of $G$ as $\Gamma(L_\lambda)^{\mathfrak{n}_-}$, the $\mathfrak{n}_-$ invariant part of $\Gamma(L_\lambda)$. The definition of $\mathfrak{n}_-$ depends upon a choice of invariant complex structure. A non-trivial element $w$ of the Weyl group $W(G,T)$ gives a different choice of $\mathfrak{n}_-$ (call this one $\mathfrak{n}_-^w$) and now $\Gamma(L_\lambda)^{\mathfrak{n}_-^w} = 0$ for a dominant weight $\lambda$. Work of Bott [14] and later Kostant [28] shows that the representation now occurs in a higher cohomology group. This is one motivation for the introduction of homological algebra techniques into the study of the representation theory of $G$.

To any finite dimensional representation $V$ of $G$, one can associate a more algebraic object, a module over the enveloping algebra $U(\mathfrak{g})$, which we will also call $V$. If elements of the Lie algebra $\mathfrak{g}$ are thought of as the left-invariant vector fields on $G$, then elements of $U(\mathfrak{g})$ will be the left-invariant partial differential operators on $G$. In homological algebra a fundamental idea is to replace the study of a $U(\mathfrak{g})$ module $V$ with a resolution of $V$, a complex of $U(\mathfrak{g})$ modules where each term has simpler properties, for instance that of being a free $U(\mathfrak{g})$ module.

**Definition 2.** *A resolution of $V$ as a $U(\mathfrak{g})$ module is an exact sequence*

$$0 \leftarrow V \leftarrow M_0 \leftarrow M_1 \leftarrow \cdots \leftarrow M_n \leftarrow 0$$

*of $U(\mathfrak{g})$ modules and $U(\mathfrak{g})$-linear maps. It is a free (resp. projective, injective) resolution if the $M_i$ are free (resp. projective, injective) $U(\mathfrak{g})$ modules.*

Note that deleting $V$ from this complex give a complex whose homology is simply $V$ in degree zero. One is essentially replacing the study of $V$ with the study of a "quasi-isomorphic" complex whose homology is $V$ in degree zero. Now the trivial $U(\mathfrak{g})$ module $V = \mathbf{C}$ becomes interesting, it has a free resolution known as the standard or Koszul resolution:

**Definition 3.** *The Koszul resolution is the exact sequence of $U(\mathfrak{g})$ modules*

$$0 \longleftarrow \mathbf{C} \xleftarrow{\epsilon} Y_0 \xleftarrow{\partial_0} Y_1 \xleftarrow{\partial_1} \cdots \longleftarrow Y_{n-1} \xleftarrow{\partial_{n-1}} Y_n \longleftarrow 0$$

*where*

$$Y_m = U(\mathfrak{g}) \otimes_{\mathbf{C}} \Lambda^m(\mathfrak{g})$$

$$\epsilon(u) = \text{constant term of } u \in U(\mathfrak{g})$$

*and*

$$\partial_{m-1}(u \otimes X_1 \wedge X_2 \wedge \cdots \wedge X_m) = \sum_{i=1}^{m} (-1)^{i+1}(uX_i \otimes X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_m)$$

$$+ \sum_{k<l} (-1)^{k+l}(u \otimes [X_k, X_l] \wedge X_1 \wedge \cdots \wedge \hat{X}_k \wedge \cdots \wedge \hat{X}_l \wedge \cdots \wedge X_m)$$

*(Here $X_i \in \mathfrak{g}$ and $\hat{X}_i$ means delete $X_i$ from the wedge product)*

Applying the functor $\cdot \to Hom_{\mathbf{C}}(\cdot, V)$ to the Koszul complex we get a new exact sequence (the functor is "exact")

$$0 \longrightarrow V \longrightarrow Hom_{\mathbf{C}}(Y_0, V) \longrightarrow \cdots \longrightarrow Hom_{\mathbf{C}}(Y_n, V) \longrightarrow 0$$

which is a resolution of $V$. The $\mathfrak{g}$-invariant part of a $U(\mathfrak{g})$ module can be picked out by the "invariants functor"

$$\cdot \longrightarrow Hom_{U(\mathfrak{g})}(\mathbf{C}, \cdot) = (\cdot)^{\mathfrak{g}}$$

which takes $U(\mathfrak{g})$ modules to vector spaces over $\mathbf{C}$. Applying this functor to the above resolution of $V$ gives the complex

$$0 \longrightarrow V^{\mathfrak{g}} \longrightarrow C^0(\mathfrak{g}, V) \longrightarrow C^1(\mathfrak{g}, V) \longrightarrow \cdots \longrightarrow C^n(\mathfrak{g}, V) \longrightarrow 0$$

where

$$\begin{aligned} C^m(\mathfrak{g}, V) &= Hom_{U(\mathfrak{g})}(\mathbf{C}, Hom_{\mathbf{C}}(Y_m, V)) \\ &= (Hom_{\mathbf{C}}(Y_m, V))^{\mathfrak{g}} \\ &= Hom_{U(\mathfrak{g})}(Y_m, V) \\ &= Hom_{\mathbf{C}}(\Lambda^m(\mathfrak{g}), V) \end{aligned}$$

The invariants functor is now no longer exact and Lie algebra cohomology is defined as its "derived functor" given by the homology of this sequence (dropping the $V^{\mathfrak{g}}$ term). More explicitly

**Definition 4.** *The Lie algebra cohomology groups of* $\mathfrak{g}$ *with coefficients in the* $U(\mathfrak{g})$ *module* $V$ *are the groups*

$$H^m(\mathfrak{g}, V) = \frac{\text{Ker } d_m}{\text{Im } d_{m-1}}$$

*constructed from the complex*

$$0 \longrightarrow V \xrightarrow{d_0} \Lambda^1(\mathfrak{g}) \otimes_{\mathbf{C}} V \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Lambda^n(\mathfrak{g}) \otimes_{\mathbf{C}} V \longrightarrow 0$$

*where the operator* $d_n : C^n(\mathfrak{g}, V) \to C^{n+1}(\mathfrak{g}, V)$ *is given by*

$$d_n\omega(X_1 \wedge \cdots \wedge X_{n+1}) = \sum_{l=1}^{n+1}(-1)^{l+1} X_l(\omega(X_1 \wedge \cdots \wedge \hat{X}_l \wedge \cdots \wedge X_{n+1}))$$

$$+ \sum_{r<s}(-1)^{r+s}\omega([X_r, X_s] \wedge X_1 \wedge \cdots \wedge \hat{X}_r \wedge \cdots \wedge \hat{X}_s \wedge \cdots \wedge X_{n+1})$$

*They satisfy*

$$H^0(\mathfrak{g}) = V^{\mathfrak{g}}$$

For the case $V = \mathbf{C}$ and $G$ a compact Lie group, the $H^m(\mathfrak{g}, \mathbf{C})$ correspond with the de Rham cohomology groups of the topological space $G$, this is not true in general for non-compact groups.

More generally one can consider the functor

$$\cdot \longrightarrow Hom_{U(\mathfrak{g})}(W, \cdot)$$

for some irreducible representation $W$ of $G$. This functor takes $U(\mathfrak{g})$ modules to a vector space of dimension given by the multiplicity of the irreducible $W$ in $V$. The corresponding derived functor is $Ext^*_{U(\mathfrak{g})}(W, V)$ and is equal to $H^*(\mathfrak{g}, Hom_{\mathbf{C}}(W, V))$.

In the previous section we considered $\Gamma(L_\lambda)$ as a $U(\mathfrak{n}_-)$ module using the right $\mathfrak{n}_-$ action on functions on $G$. The $\mathfrak{n}_-$-invariant part of this module was non-zero and an irreducible representation $V_\lambda$ under the left $G$-action. For a non-dominant weight $\lambda$ the $\mathfrak{n}_-$-invariant part was zero, but it turns out we can get something non-zero by replacing the $\Gamma(L_\lambda)$ by its resolution as a $U(\mathfrak{n}_-)$ module. Kostant computed the Lie algebra cohomology in this situation, finding [28]

**Theorem 3** (Kostant)**.** *As a* $T$ *representation*

$$H^*(\mathfrak{n}_-, V_\lambda) = \sum_{w \in W(G,T)} \mathbf{C}_{w(\lambda+\delta)-\delta}$$

*i.e. the cohomology space is a sum of one-dimensional* $T$ *representations, one for each element of the Weyl group, transforming under* $T$ *with weight* $w(\lambda+\delta)-\delta$ *where* $\delta$ *is half the sum of the positive roots. Each element* $w$ *is characterized by an integer* $l(w)$*, its length, and* $C_{w(\lambda+\delta)-\delta}$ *occurs in degree* $l(w)$ *of the cohomology space.*

Replacing $\Gamma(L_\lambda)$ by its resolution as a $U(\mathfrak{n}_-)$ module gives the $\bar{\partial}$ complex $\Omega^{0,*}(G/T, \mathcal{O}(L_\lambda))$ that computes not just $\Gamma^{hol}(L_\lambda) = H^0(G/T, \mathcal{O}(L_\lambda))$ but all the cohomology groups $H^*(G/T, \mathcal{O}(L_\lambda))$ with the result

**Theorem 4** (Borel-Weil-Bott-Kostant)**.**

$$H^*(G/T, \mathcal{O}(L_\lambda)) = H^{0,*}(\Omega^{\cdot,\cdot}(G/T, \mathcal{O}(L_\lambda))) = \sum_{i \in \hat{G}} V_i \otimes (H^*(\mathfrak{n}_-, V_i^*) \otimes C_\lambda)^T$$

*For a given weight $\lambda$, this is non-zero only in degree $l(w)$ where $w(\lambda+\delta)$ is dominant and gives an irreducible representation of $G$ of highest weight $w(\lambda+\delta)-\delta$.*

Knowledge of the T-representation structure of the complex that computes $H^*(\mathfrak{n}_-, V_\lambda)$ allows one to quickly derive the Weyl character formula. The character of $\sum_{i=0}^n (-1)^i C^i(\mathfrak{n}_-, V_\lambda)$ is the product of the character of $V(\lambda)$ and the character of $\sum_{i=0}^n (-1)^i C^i(\mathfrak{n}_-, \mathbf{C})$ so

$$\chi_{V_\lambda} = \frac{\sum_{i=0}^n (-1)^i \chi_{C^i(\mathfrak{n}_-, V_\lambda)}}{\sum_{i=0}^n (-1)^i \chi_{C^i(\mathfrak{n}_-, \mathbf{C})}}$$

By the Euler-Poincaré principle the alternating sum of the characters of terms in a complex is equal to the alternating sum of the cohomology groups, the Euler characteristic, so

$$\chi_{V_\lambda} = \frac{\sum_{i=0}^n (-1)^i \chi_{H^i(\mathfrak{n}_-, V_\lambda)}}{\sum_{i=0}^n (-1)^i \chi_{H^i(\mathfrak{n}_-, \mathbf{C})}} = \frac{\chi(\mathfrak{n}_-, V_\lambda)}{\chi(\mathfrak{n}_-, \mathbf{C})}$$

Where on the right $\chi$ denotes the Euler characteristic, as a $T$-representation. Using the explicit determination of the Lie algebra cohomology groups discussed above, one gets the standard Weyl character formula.

The introduction of homological methods in this section has allowed us to construct irreducible representations in a way that is more independent of the choice of complex structure on $G/T$. Under change of complex structure, the same representation will occur, just in a different cohomological degree. These methods also can be generalized to the case of discrete series representations of non-compact groups, where there may be no complex structure for which the representations occurs in degree zero as holomorphic sections. The general idea of constructing representations on cohomology groups is also important in other mathematical contexts, for instance in number theory where representations of the Galois group $Gal(\bar{\mathbf{Q}}/\mathbf{Q})$ are constructed on $l$-adic cohomology groups of varieties over $\bar{\mathbf{Q}}$.

## 2.3 The Clifford Algebra and Spinors: A Digression

The Lie algebra cohomology construction of the irreducible representations of $G$ requires choosing an invariant complex structure on $\mathfrak{g}/\mathfrak{t}$, although ultimately different complex structures give the same representation. It turns out that by using spinors instead of the exterior powers that appear in the Koszul resolution, one can construct representations by a method that is completely independent of the complex structure and that furthermore explains the somewhat mysterious appearance of the weight $\delta$. For more on this geometric approach to spinors see [17] and [37].

### 2.3.1 Clifford Algebras

Given a real $2n$-dimensional vector space $V$ with an inner product $(\cdot,\cdot)$, the Clifford algebra $C(V)$ is the algebra generated by the elements of $V$ with multiplication satisfying

$$v_1v_2 + v_2v_1 = 2(v_1, v_2)$$

In particular $v^2 = (v,v) = ||v||^2$. $C(V)$ has a $\mathbf{Z}_2$-grading since any element can be constructed as a product of either an even or odd number of generators. For the case of zero inner product $C(V) = \Lambda^*(V)$, the exterior algebra, which has an integer grading. In some sense the Clifford algebra is more basic than the exterior algebra, since given $C(V)$ one can recover $\Lambda^*(V)$ as the associated graded algebra to the natural filtration of $C(V)$ by the minimal number of generating vectors one must multiply to get a given element.

The exterior algebra $\Lambda^*(V)$ is a module over the Clifford algebra, taking Clifford multiplication by a vector $v$ to act on $\Lambda^*(V)$ as

$$\cdot \to v \wedge \cdot + i_v(\cdot)$$

Here $i_v(\cdot)$ is interior multiplication by $v$, the adjoint operation to exterior multiplication. This action can be used to construct a vector space isomorphism $\sigma$

$$x \in C(V) \to \sigma(x) = x1 \in \Lambda^*(V)$$

but this isomorphism doesn't respect the product operations in the two algebras.

$\Lambda^*(V)$ is not an irreducible $C(V)$ module and we now turn to the problem of how to pick an irreducible submodule, this will be the spinor module $S$. The way in which this is done is roughly analogous to the way in which highest weight theory was used to pick out irreducible representations in $L^2(G)$.

### 2.3.2 Complex Structures and Spinor Modules

If one complexifies $V$ and works with $V_{\mathbf{C}} = V \otimes \mathbf{C}$, $C(V_{\mathbf{C}})$ is just the complexification $C(V) \otimes \mathbf{C}$ and is isomorphic to the matrix algebra of $2^n \times 2^n$ complex matrices. $C(V_{\mathbf{C}})$ thus has a single irreducible module and this spinor module $S$ will be a $2^n$-dimensional complex vector space. Knowledge of $C(V_{\mathbf{C}}) = End(S)$ only canonically determines $P(S)$, the projectivization of $S$. To explicitly construct the spinor module $S$ requires some extra structure, one way to do this begins with the choice of an orthogonal complex structure $J$ on the underlying real vector space $V$. This will be a linear map $J : V \to V$ such that $J^2 = -1$ and $J$ preserves the inner product

$$(Jv_1, Jv_2) = (v_1, v_2)$$

If $V$ is given a complex structure in this way, it then has a positive-definite Hermitian inner product

$$(v_1, v_2)_J = (v_1, v_2) + i(v_1, Jv_2)$$

which induces the same norm on $V$ since $(v,v)_J = (v,v)$.

Any such $J$ extends to a complex linear map

$$J : V_{\mathbf{C}} \to V_{\mathbf{C}}$$

on the complexification of $V$ with eigenvalues $+i, -i$ and the corresponding eigenspace decomposition

$$V_{\mathbf{C}} = V_J^+ \oplus V_J^-$$

$V_J^\pm$ will be $n$-complex dimensional vector spaces on which $J$ acts by multiplication by $\pm i$. They are can be mapped into each other by the anti-linear conjugation map

$$v^+ \in V_J^+ \to \overline{v^+} \in V_J^-$$

which acts by conjugation of the complex scalars.

One can explicitly identify $V$ and $V_J^+ \subset V_{\mathbf{C}}$ with the map

$$v \to v^+ = \frac{1}{\sqrt{2}}(1 - iJ)v$$

This map is an isometry if one uses the inner product $(\cdot,\cdot)_J$ on $V$ and the restriction to $V_J^+$ of the inner product $< \cdot,\cdot >$ on $V_{\mathbf{C}}$ that is the sesquilinear extension of $(\cdot,\cdot)$ on $V$ (i.e. $< v_1, v_2 >= (v_1, \overline{v}_2)$).

The space $\mathcal{J}(V)$ of such $J$ is isomorphic to $O(2n)/U(n)$ since each $J$ is in $O(2n)$ and a $U(n)$ subgroup preserves the complex structure. Note that $V_J^+$ and $V_J^-$ are isotropic subspaces (for the bilinear form $(\cdot,\cdot)$) since for $v^+ \in V_J^+$

$$(v^+, v^+) = (Jv^+, Jv^+) = (iv^+, iv^+) = -(v^+, v^+)$$

The set of $J$'s can be identified with the set of maximal dimension isotropic subspaces of $V_{\mathbf{C}}$.

Given such a complex structure, for each $v \in V_{\mathbf{C}}$, one can decompose

$$v = c(v) + a(v)$$

where the "creation operator" $c(v)$ is Clifford multiplication by

$$c(v) = \frac{1}{\sqrt{2}}v^+ = \frac{1}{2}(1 - iJ)v$$

and the "annihilation operator" $a(v)$ is Clifford multiplication by

$$a(v) = \frac{1}{\sqrt{2}}v^- = \frac{1}{2}(1 + iJ)v$$

If one requires that one's representation of the Clifford algebra is such that $v \in V$ acts by self-adjoint operators then one has

$$v^+ = \overline{v^-}, \quad v^- = \overline{v^+}$$

In this case the defining relations of the complexified Clifford algebra

$$v_1 v_2 + v_2 v_1 = 2(v_1, v_2)$$

become the so-called Canonical Anti-commutation Relations (CAR)

$$c(v_1)c(v_2) + c(v_2)c(v_1) = 0$$

$$a(v_1)a(v_2) + a(v_2)a(v_1) = 0$$

$$c(v_1)a(v_2) + a(v_2)c(v_1) = (v_1^+, \overline{v_2^+}) =< v_1^+, v_2^+ >= (v_1, v_2)_J$$

The CAR are well-known to have an irreducible representation on a vector space $S$ constructed by assuming the existence of a "vacuum" vector $\Omega_J \in S$ satisfying

$$a(v)\Omega_J = 0 \quad \forall v \in V$$

and applying products of creation operators to $\Omega_J$. This representation is isomorphic to $\Lambda^*(V^+)$ with $\Omega_J$ corresponding to $1 \in \Lambda^*(V_J^+)$. It is $\mathbf{Z}_2$-graded, with

$$S^+ = \Lambda^{even}(V_J^+), \quad S^- = \Lambda^{odd}(V_J^+)$$

### 2.3.3 Spinor Representations and the Pfaffian Line Bundle

We now have an irreducible module $S$ for $C(V_\mathbf{C})$, and most discussions of the spinor module stop at this point. We would like to investigate how the construction depends on the choice of $J \in \mathcal{J}(V) = O(2n)/U(n)$. One way of thinking of this is to consider the trivial bundle

$$\begin{array}{c} O(2n)/U(n) \times S \\ \downarrow \\ O(2n)/U(n) \end{array}$$

The subbundle of vacuum vectors $\Omega_J$ is a non-trivial complex line bundle we will call (following [37]) $Pf$.

Associating $V_J^+$ to $J$, the space $\mathcal{J}(V)$ can equivalently be described as the space of n-dimensional complex isotropic subspaces of $V_\mathbf{C}$ and is thus also known as the "isotropic Grassmannian". It is an $\frac{n(n-1)}{2}$ complex dimensional algebraic subvariety of the Grassmannian $Gr(n, 2n)$ of complex n-planes in 2n complex dimensions. $\mathcal{J}(V)$ has two connected components, each of which can be identified with $SO(2n)/U(n)$.

$\mathcal{J}(V)$ also has a holomorphic embedding in the space $P(S)$ given by mapping $J$ to the complex line $\Omega_J \subset S$. Elements of $S$ that correspond to $J$'s in this way are called "pure spinors". To any spinor $\psi$ one can associate the isotropic subspace of $v \in V_\mathbf{C}$ such that

$$v \cdot \psi = 0$$

and pure spinors are those for which the dimension of this space is maximal. Pure spinors lie either in $S^+$ or $S^-$, so one component of $\mathcal{J}(V)$ lies in $P(S^+)$, the other in $P(S^-)$.

There is a map from $S^*$, the dual of $S$, to holomorphic sections of $Pf^*$, the dual bundle of $Pf$, given by restricting an element of $S^*$ to the line $\Omega_J$ above the point $J \in \mathcal{J}(V)$. This turns out to be an isomorphism [37] and one can turn this around and define $S$ as the dual space to $\Gamma(Pf^*)$. Besides being a module for the Clifford algebra, this space is a representation of $Spin(2n)$, the spin double cover of $SO(2n)$. This is essentially identical with the Borel-Weil description of the spinor representation of $Spin(2n)$. $\Omega_J$ is a highest weight vector and we are looking at the representation of $Spin(2n)$ on holomorphic sections of the homogeneous line bundle $Pf^*$ over $Spin(2n)/\widetilde{U(n)}$ instead of its pull-back to $Spin(2n)/T$. Here

$$\widetilde{U(n)} = \{(g, e^{i\theta}) \in U(n) \times S^1 : det(g) = e^{i2\theta}\}$$

is the inverse image of $U(n) \subset SO(2n)$ under the projection $Spin(2n) \rightarrow SO(2n)$.

For each $J$, we have an explicit model for $S$ given by $\Lambda^*(V_J^+)$, with a distinguished vector $\Omega_J = 1$. We would like to explicitly see the action of $Spin(2n)$ on this model for $S$. In physical language, this is the "Fock space" and we are looking at "Bogoliubov transformations". The action of the subgroup $\widetilde{U(n)}$ of $Spin(2n)$ is the easiest to understand since it leaves $\Omega_J$ invariant (up to a phase). As a $\widetilde{U(n)}$ representation

$$S = \Lambda^*(V_J^+) \times (\Lambda^n(V_J^+))^{\frac{1}{2}}$$

meaning that $S$ transforms as the product of the standard exterior power representations of $U(n)$ times a scalar factor that transforms as

$$(g, e^{i\theta})z = e^{i\theta}z$$

i.e., the vacuum vector $\Omega_J$ transforms in this way. To see this and to see how elements of $Spin(2n)$ not in $\widetilde{U(n)}$ act we need to explicitly represent $Spin(2n)$ in terms of the Clifford algebra.

The group $Spin(2n)$ can be realized in terms of even, invertible elements $g$ of the Clifford algebra. Orthogonal transformations $T_g \in SO(V)$ are given by the adjoint action on Clifford algebra generators

$$gvg^{-1} = T_g(v)$$

and the adjoint action on the rest of the Clifford algebra gives the rest of the exterior power representations of $SO(V)$. If instead one considers the left action of $g$ on the Clifford algebra one gets a reducible representation. To get an irreducible representation one needs to construct a minimal left ideal and this is what we have done above when we used a chosen complex structure $J$ to produce an irreducible Clifford module $S$.

The Lie algebra generator $L_{ij}$ that generates orthogonal rotations in the $i-j$ plane corresponds to the Clifford algebra element $\frac{1}{2}e_ie_j$ (where $e_i,\ i = 1, \cdots, 2n$ are basis for $V$) and satisfies

$$[\frac{1}{2}e_ie_j, v] = L_{ij}(v)$$

While $Spin(2n)$ and $SO(2n)$ have isomorphic Lie algebras, $Spin(2n)$ is a double cover of $SO(2n)$ since

$$e^{2\pi L_{ij}} = 1$$

but

$$e^{2\pi\frac{1}{2}e_ie_j} = -1$$

A maximal torus $T$ of $Spin(2n)$ is given by the $n$ copies of $U(1)$

$$e^{\theta e_{2k-1}e_{2k}} \text{ for } k = 1, \cdots, n \text{ and } \theta \in [0, 2\pi]$$

corresponding to independent rotations in the $n$ 2-planes with coordinates $2k-1, 2k$. A standard choice of complex structure $J$ is given by a simultaneous $\frac{\pi}{2}$ rotation in these planes. Using this one can show that the vector $\Omega_J$ transforms under $T$ with a weight of $\frac{1}{2}$ for each copy of $U(1)$, thus transforming as

$$(\Lambda^n(V_J^+))^{\frac{1}{2}}$$

under $\widetilde{U(n)}$.

### 2.3.4 The Spinor Vacuum Vector as a Gaussian

There is ([37], Chapter 12.2) an explicit formula for how $\Omega_J$ varies with $J$ as an element of the complex exterior algebra description of the spin module $S$. This requires choosing a fixed $J_0$ and a corresponding decomposition

$$V_{\mathbf{C}} = V^+_{J_0} \oplus V^-_{J_0}$$

This choice fixes a chart on a set of $J$'s containing $J_0$ with coordinate on the chart given by the set of skew-linear maps

$$\omega : V^+_{J_0} \to V^-_{J_0}$$

(skew-linearity of the map implies that its graph is an isotropic subspace, and thus corresponds to a $J$). The space of such maps can be identified with $\Lambda^2(V^+_{J_0})$, the space of antisymmetric two-forms on $V^+_{J_0}$. Under this identification $\Omega_J$ will be proportional to the vector

$$e^{\frac{1}{2}\omega} \in \Lambda^*(V^+_{J_0})$$

The construction we have given here of the spinor representation has a precise analog in the case of the metaplectic representation. In that case there is a similar explicit formula for the highest weight vector as a Gaussian, see [40].

### 2.3.5 Structure of Clifford Algebra Modules

Let $C_k$ be the complexified Clifford algebra $C(\mathbf{R}^k) \otimes \mathbf{C}$ of $\mathbf{R}^k$ and $M_k$ be the Grothendieck group of complex $\mathbf{Z}_2$ graded irreducible $C_k$ modules. Then, using the inclusion

$$i : \mathbf{R}^k \to \mathbf{R}^{k+1}$$

$C_{k+1}$ modules pull-back to $C_k$ modules and one can consider

$$A_k = M_k / i^* M_{k+1}$$

the set of classes of $C_k$ modules, modulo those that come from $C_{k+1}$ modules. $A_*$ is a graded ring, with product induced from the tensor product of Clifford modules. It turns out [6] that

$$A_k = \begin{cases} \mathbf{Z} & k \text{ even} \\ 0 & k \text{ odd} \end{cases}$$

and the generator of $A_{2n}$ is the $n$-th power of the generator of $A_2$.

There is a natural graded version of the trace on $C_{2n}$, the supertrace, which satisfies

$$Str(\alpha) = \begin{cases} Tr_{S^+}(\alpha) - Tr_{S^-}(\alpha) & \alpha \in C_{2n}^{even} \\ 0 & \alpha \in C_{2n}^{odd} \end{cases}$$

This supertrace is (up to a constant depending on conventions) identical with the Berezin integral of the corresponding element of the exterior algebra under the map $\sigma$

$$Str(\alpha) \propto \int \sigma(\alpha)$$

## 2.4 Kostant's Dirac Operator and a Generalization of Borel-Weil-Bott

The Borel-Weil-Bott theorem gave a construction of irreducible representations in terms of the Lie algebra cohomology of an explicit complex built out of the exterior algebra $\Lambda^*(\mathfrak{n}_-)$. This involved an explicit choice of complex structure to define the decomposition

$$\mathfrak{g}/\mathfrak{t} \otimes \mathbf{C} = \mathfrak{n}_+ \oplus \mathfrak{n}_-$$

If we can replace the use of $\Lambda^*(\mathfrak{n}_-)$ by the use of spinors $S_{\mathfrak{g}/\mathfrak{t}}$ associated to the vector space $\mathfrak{g}/\mathfrak{t}$, we will have a construction that is independent of the choice of complex structure on $\mathfrak{g}/\mathfrak{t}$. In addition, the fact that $\Lambda^*(\mathfrak{n}_-)$ and $S_{\mathfrak{g}/\mathfrak{t}}$ differ as $T$ representations by a factor of

$$(\Lambda^n(\mathfrak{n}_-))^{\frac{1}{2}}$$

(here $n$ is the complex dimension of $\mathfrak{n}_-$), explains the mysterious appearance in the Borel-Weil-Bott theorem of the weight of $T$ which is half the sum of the positive roots. This idea goes back to [15].

The use of spinors also allows us to generalize Borel-Weil-Bott from the case of $G/T$ to $G/H$ for an arbitrary $H$ with the same rank as $G$. If $G/H = G_{\mathbf{C}}/P$ ($G_{\mathbf{C}}$ the complexification of $G$, $P$ a parabolic subgroup), then $G/H$ is a complex manifold (actually projective algebraic) and the original Borel-Weil-Bott theorem [14] describes the representations of $G$ that occur in the sheaf cohomology groups of homogeneous vector bundles over $G/H$. Using spinors, one can extend this [25] to the cases where $G/H$ is not even a complex manifold:

**Theorem 5** (Gross-Kostant-Ramond-Sternberg)**.** *In the representation ring $R(H)$*

$$V_\lambda \otimes S^+ - V_\lambda \otimes S^- = \sum_{w \in W(G,H)} sgn(w) U_{w(\lambda+\rho_G)-\rho_H}$$

*Here $U_\mu$ is the representation of $H$ with highest weight $\mu$, $S^\pm$ are the half-spin representations associated to the adjoint action of $H$ on $\mathfrak{g}/\mathfrak{h}$, $W(G,H)$ is the subgroup of $W(G,T)$ that maps the dominant Weyl chamber for $G$ into the dominant Weyl chamber for $H$, and $\rho_G, \rho_H$ are half the sum of the positive roots of $G$ and $H$ respectively.*

This theorem was motivated by consideration of the case $G = F_4$, $H = Spin(9)$, but the simplest non-complex case is that of $G/H$ an even-dimensional sphere ($G = Spin(2n+1)$, $H = Spin(2n)$). In [29] Kostant constructs an algebraic Dirac operator

$$\not{D} : V_\lambda \otimes S^+ \to V_\lambda \otimes S^-$$

such that

$$Ker\not{D} = \sum_{sgn(w)=+} U_{w(\lambda+\rho_G)-\rho_H}, \quad Coker\not{D} = \sum_{sgn(w)=-} U_{w(\lambda+\rho_G)-\rho_H}$$

In the special case $H = T$ [30], the theorem is just the original Lie algebra cohomology version of the Borel-Weil-Bott theorem, but now in terms of a complex involving Kostant's Dirac operator on spinors. As in the Lie algebra cohomology case, one gets a proof of the Weyl character formula, in the form that as an element of $R(T)$,

$$V_\lambda = \frac{\sum_{w\in W(G,T)} sgn(w) U_{w(\lambda+\delta)-\delta}}{S^+ - S^-}$$

Taking $V_\lambda$ the trivial representation shows

$$S^+ - S^- = \sum_{w\in W(G,T)} sgn(w) U_{w(\delta)-\delta}$$

so

$$V_\lambda = \frac{\sum_{w\in W(G,T)} sgn(w) U_{w(\lambda+\delta)-\delta}}{\sum_{w\in W(G,T)} sgn(w) U_{w(\delta)-\delta}} = \frac{\sum_{w\in W(G,T)} sgn(w) U_{w(\lambda+\delta)}}{\sum_{w\in W(G,T)} sgn(w) U_{w(\delta)}}$$

So far we have seen three constructions of the irreducible representations of $G$, of increasing generality. In all cases we are using the right action of $G$ on functions on $G$, separately treating the $T$ subgroup (using Fourier analysis), and the remaining $\mathfrak{g}/\mathfrak{t}$ part of $\mathfrak{g}$. To summarize they are:

- (Borel-Weil): Consider $\mathfrak{n}_-$ invariants (lowest weights), transforming under $T$ with weight $-\lambda$. For those $\lambda$ not in the dominant Weyl chamber we get nothing, but for those in the dominant Weyl chamber this picks out a one dimensional space in $V_\lambda^*$. Out of $L^2(G)$ we get a single irreducible representation
$$V_\lambda = \Gamma^{hol}(L_\lambda)$$
- (Borel-Weil-Bott-Kostant): Consider not just $H^0(\mathfrak{n}_-, V_\lambda^*)$ but all of $H^*(\mathfrak{n}_-, V_\lambda^*)$. Now we get a non-zero space for each $\lambda$, but it will be in higher cohomology for non-dominant $\lambda$. Considering $L^2(G)\otimes\Lambda^*(\mathfrak{n}_-)$, to each $\lambda$ there will be an irreducible representation (of highest weight $w(\lambda+\delta)-\delta$ for some Weyl group element $w$) occurring inside this space as a representative of a cohomology group $H^*(G/T, \mathcal{O}(L_\lambda))$.
- (Kostant): Consider not the Lie algebra cohomology complex $V_\lambda\otimes\Lambda^*(\mathfrak{n}_-)$ but the complex
$$V_\lambda \otimes S^+ \xrightarrow{\not{D}} V_\lambda \otimes S^-$$
Unlike 1. and 2., this does not in any way use the choice of a complex structure on $\mathfrak{g}/\mathfrak{t}$. Weights are organized into "multiplets" of weights of the form $w(\lambda+\delta)-\delta$ for a single dominant weight $\lambda$ and different choices of Weyl group element $w$. Each "multiplet" corresponds to a single irreducible representation of $G$.

In [32], Landweber shows that this third construction implies that irreducible $G$ representations occur as the index of a specific Dirac operator

**Theorem 6** (Landweber)**.** *In the case $H \subset G$ of equal rank there is a geometric Dirac operator*

$$L^2(G\times_H ((S^+)^*\otimes U_\lambda)) \xrightarrow{\not{D}_\lambda} L^2(G\times_H ((S^-)^*\otimes U_\lambda))$$

*such that*

$$Index_G \not{D} = sgn(w)[V_{w(\lambda+\rho_{\mathfrak{h}})-\rho_{\mathfrak{g}}}]$$

*where $w$ is a Weyl group element such that $w(\lambda+\rho_{\mathfrak{h}})-\rho_{\mathfrak{g}}$ is a dominant weight.*

In other words, if $U_\lambda$ is an element of the "multiplet" of representations of $H$ corresponding to a single irreducible representation of $G$, this $G$ representation occurs as the index of the given Dirac operator. This is a special case of a general phenomenon first described by Bott [15]: if

$$L^2(G \times_H M) \xrightarrow{D} L^2(G \times_H N)$$

is an elliptic homogeneous differential operator on $G/H$ then

$$Index_G(D) = [L^2(G \times_H M)] - [L^2(G \times_H N)]$$

is an element of the representation ring $R(G)$ that only depends on the $H$-representations $M$ and $N$ *not* on the specific operator $D$.

Bott's calculations were motivated by the recognition that the case of homogeneous elliptic differential operators is a special case of the general Atiyah-Singer index theorem. In this case the index theorem boils down to purely representation-theoretical calculations and so can be easily checked. We will now turn to the general mathematical context of equivariant K-theory in which the index theorem has its most natural formulation.

# 3 Equivariant K-theory and Representation Theory

In the previous section we have seen an explicit construction of the irreducible representations of compact Lie groups. There is a more abstract way of thinking about this construction which we would now like to consider. This uses the notion of equivariant K-theory, a generalization of topological K-theory.

To a compact space $M$ one can associate a topological invariant, $Vect(M)$, the space of isomorphism classes of vector bundles over $M$. This is an additive semi-group under the operation of taking the direct sum of vector bundles and can be made into a group in the same way that the integers can be constructed out of the natural numbers. One way of doing this is by taking pairs of elements thought of as formal differences $\alpha - \beta$ and the equivalence relation

$$\alpha - \beta \sim \alpha' - \beta' \Leftrightarrow \alpha + \beta' + \gamma = \alpha' + \beta + \gamma$$

for some element $\gamma$. $K(M)$ is the additive group constructed in this way from formal differences of elements of $Vect(M)$. This is often called the "Grothendieck Construction" and $K(M)$ is called the Grothendieck group of $Vect(M)$. The tensor product of vector bundles gives a product on $K(M)$, making it into a ring. Using the suspension of $M$ and Bott periodicity, this definition can be extended to that of a graded ring $K^*(M)$. We will mostly be considering $K^0(M)$ and denoting it $K(M)$.

The Grothendieck construction on isomorphism classes of representations of a compact Lie group $G$ using the direct sum and tensor product of representations gives the representation ring of $G$, $R(G)$. Using characters, $R(G) \otimes \mathbf{C}$

can be studied quite explicitly as a ring of complex-valued functions, either conjugation-invariant on $G$, or Weyl-group invariant on $T$.

Given a space $M$ with an action of a group $G$, one can look at $G$-vector bundles. These are vector bundles $E$ over $M$ such that an element $g$ of $G$ maps the fiber $E_x$ above $x$ to the fiber $E_{gx}$ by a vector space isomorphism. The Grothendieck construction on isomorphism classes of these objects gives the ring $K_G(M)$, the equivariant K-theory of $M$. This specializes to $R(G)$ in the case of $M = pt.$, and to $K(M)$ in the case of trivial $G$-action on $M$.

K-theory is a contravariant functor: to a map $f : M \to N$ pull-back of bundles give a map $f^* : K(N) \to K(M)$. For G-vector bundles and G-maps this also holds in equivariant K-theory.

Given a representation of $H$ one can construct a $G$-vector bundle over $G/H$ by the associated bundle construction and thus a map

$$R(H) \longrightarrow K_G(G/H)$$

This map has an inverse given by associating to a $G$-vector bundle over $G/H$ its fiber over the identity coset which is an $H$ representation. These maps give the induction isomorphism in equivariant K-theory

$$K_G(G/H) = R(H)$$

In our discussion of the Borel-Weil-Bott theorem we were looking at elements of $K_G(G/T)$ corresponding to a weight or representation of $T$. An abstract way of thinking of the Borel-Weil-Bott theorem is that it gives a "wrong-way" map

$$\pi_* : K_G(G/T) = R(T) \to K_G(pt.) = R(G)$$

corresponding to the map $\pi$ that takes $G/T$ to a point. The existence of such a "wrong-way" or "push-forward" map is an indication of the existence of a covariant functor "K-homology", related to K-theory in much the way that homology is related to cohomology. We will see that, from the equivariant K-theory point of view, finding the irreducible representations of a compact Lie group comes down to the problem of understanding Poincaré duality in the $G$-equivariant K-theory of $G/T$.

## 3.1 K-Homology

Unfortunately the definition of K-homology seems to be much more subtle than that of K-theory and all known definitions are difficult to work with, especially in the equivariant context.

Much work on K-theory begins from the point of view of algebra, considering isomorphism classes not of vector bundles, but of finitely generated projective modules over a ring $R$. For any ring $R$, the algebraic K-theory $K(R)$ will be the Grothendieck group of such modules. For the case $R = C(M)$, the continuous functions on a compact space $M$, Swan's theorem gives an equivalence between the algebraic K-theory $K(C(M))$ and the topologically defined $K(M)$.

In algebraic geometry the algebra and geometry are tightly linked. Using the coordinate ring of a variety one has an algebraic K-theory of algebraic vector bundles. Here there is a covariant functor, a "K-homology" in this context given by taking the Grothendieck group of isomorphism classes of coherent algebraic sheaves.

In the very general context of operator algebras one can define the Kasparov K-theory of such an algebra. This is a bivariant functor and so includes a K-homology theory. It can be defined even for non-commutative algebras and is a fundamental idea in non-commutative geometry. For the algebra of continuous functions on a compact space $M$, Kasparov K-theory is identical with topological K-theory and comes with a corresponding covariant K-homology theory. Unfortunately the objects representing classes in this K-homology seem to be difficult to work with and not closely linked to the geometry. Connes [18] has remarked extensively on the importance of this K-homology and the corresponding Poincaré duality for many applications in the non-commutative geometry program.

A more concrete approach to the construction of K-homology, close in spirit to that in the algebraic category, is to define the K-homology of M as $K(\mathbf{R}^n, \mathbf{R}^n - M)$, the Grothendieck group of complexes of vector bundles on $\mathbf{R}^n$, exact off $M$ for some closed embedding of $M$ in $\mathbf{R}^n$. One drawback of this construction is its non-intrinsic nature since it relies on an explicit embedding. In the equivariant case one needs an equivariant embedding in some $G$ representation.

## 3.2 Orientations and Poincaré Duality: Homology and Cohomology

Putting aside for the moment the question the definition of K-homology, we would like to consider abstractly what properties it should have. We would like K-theory ($K^*$) and K-homology ($K_*$) to have some of the same properties as ordinary cohomology ($H^*$) and homology ($H_*$):

- $H^*(M)$ is a contravariant functor for continuous maps, i.e. given

  $$f : M \to N$$

  we have a map

  $$f^* : H^*(N) \to H^*(M)$$

- $H_*(M)$ is a covariant functor for proper maps (maps such that the inverse image of a compact set is compact).

- There is a cup product on $H^*(M)$

  $$(\alpha, \beta) \in H^j(M) \otimes H^k(M) \to \alpha \cup \beta \in H^{j+k}(M)$$

- There is a cap product

  $$(\alpha, \beta) \in H^j(M) \otimes H_k(M) \to \alpha \cap \beta \in H_{k-j}(M)$$

  and it makes $H_*(M)$ an $H^*(M)$ module.

- For $N$ a subspace of $M$, there are relative cohomology groups $H^*(M, N)$.

- (Alexander duality): For $M'$ a smooth manifold of dimension $m'$ and $M$ a closed subspace of dimension $m$ one has

  $$H_i(M) = H^{m'-i}(M', M' - M)$$

In particular, we can choose a closed embedding of $M$ in $M' = S^{n+m}$ for $n$ large enough and then

$$H_i(M) = H^{n+m-i}(S^{n+m}, S^{n+m} - M)$$

- There is a distinguished class $\Omega \in H^*(S^n) = H^*(\mathbf{R}^n, \mathbf{R}^n - 0)$, the generator of $H^n(S^n)$.
- For $M$ a smooth manifold of dimension m, there is a distinguished class in $H_m(M)$, the fundamental class $[M]$.
- (Poincaré duality): For $M$ a smooth manifold of dimension m the map

  $$P.D. : \alpha \in H^*(M) \to \alpha \cap [M] \in H_{m-*}(M)$$

  is an isomorphism. The pairing $< \alpha, \beta >$ given by

  $$(\alpha, \beta) \in H^j(M) \otimes H_j(M) \to < \alpha, \beta >= \pi_*(\alpha \cap \beta) \in H_0(pt.)$$

  where $\pi$ is the map

  $$\pi : M \to pt.$$

  is non-degenerate. Pairing with the fundamental class will be called an integration map and denoted

  $$\int_M \alpha =< \alpha, [M] >$$

- When one has Poincaré duality, given a proper map

  $$f : M \to M'$$

  one can construct a "push-forward" or "umkehrung" map

  $$f^* : H^*(M) \longrightarrow H^*(M')$$

  as follows

  $$H^*(M) \xrightarrow{P.D.} H_{m-*}(M) \xrightarrow{f_*} H_{m-*}(M') \overset{P.D.}{\cong} H^{*-m+m'}(M')$$

  For the case $m > m'$ this is just a generalization of the integration map.
- For the case of a closed embedding

  $$i : M \to M'$$

  the push-forward map $i_*$ can be constructed as follows. First consider the case of a vector bundle

  $$\pi : E \to M$$

  over $M$, with zero-section

  $$i : M \to E$$

  $E$ is said to be oriented if there is a class

  $$i_*(1) \in H^n(E, E - \{zero - section\}) = H^n(E, E - M)$$

whose restriction to each fiber is $\Omega \in H^n(\mathbf{R}^n, \mathbf{R}^n - 0) = H^n(S^n)$. Such a class is called an orientation class or Thom class and if it exists, $E$ is said to be orientable. A manifold is said to be orientable if its tangent bundle is. Now, given a closed embedding

$$i : M \longrightarrow S^{m+n}$$

in a sphere of sufficiently large dimension, $M$ will have a tubular neighborhood which can be identified with the normal bundle $N$. One can use the excision property of relative cohomology and Alexander duality to identify $H^*(N, N-M)$ and $H^*(S^{m+n}, S^{m+n} - M) = H_{m-*}$ and under this identification the fundamental class is just

$$[M] = i_*(1) \in H^n(N, N-M) = H^n(S^{m+n}, S^{m+n} - M)$$

The push-forward $i_*(\alpha)$of an arbitrary $\alpha \in H^*(M)$ can be constructed by using the "Thom isomorphism"

$$\alpha \to \pi^*(\alpha) \cup i_*(1) \in H^*(N, N-M)$$

The Thom class $i_*(1)$ provides us with a sort of "$\delta$-function" localized on $M$. It allows one to relate integration over $M$ to integration over $N$ for $M$ embedded in $N$.

$$\int_M i^*(\alpha) = \int_N i_*(1) \cup \alpha$$

.

Another related application of the Thom class allows one to relate integration over $M$ to integration over $s^{-1}(0)$, the inverse image of zero for some section $s$ of some vector bundle $E$ (transverse to the zero section) as follows

$$\int_{s^{-1}(0)} j^*\alpha = \int_M s^*(i_*(1)) \cup \alpha$$

(Here

$$j : s^{-1}(0) \to M$$

is just the inclusion map).

## 3.3 Orientations and Poincaré Duality: K-theory and K-homology

The properties of cohomology and homology discussed in the last sections are also shared by K-theory and K-homology. We have seen already that $K(M)$ is defined as the Grothendieck group of vector bundles over $M$. Relative K-theory groups $K(M, N)$ can be defined in terms of complexes of vector bundles, exact on $N$, or by pairs of vector bundles, with a bundle map between them that is an isomorphism on the fibers over $N$. $K(M)$ is a contravariant functor: pull-back of vector bundles induces a map $f^!$ on K-theory. The cup product is induced from the tensor product of vector bundles.

$K(M)$ is the degree zero part of a more general $\mathbf{Z}$-graded theory $K^*(M)$ and we will often denote it by $K^0(M)$. A standard cohomology theory has the property

$$H^i(M) = H^{i+1}(\Sigma M), \quad i \geq 1$$

where $\Sigma M$ is the suspension of $M$. K-theory in other degrees is defined by making this suspension property into a definition

$$K^{-i}(M) = \widetilde{K}(\Sigma^i(M))$$

(here on the right we need to use reduced K-theory, K-theory of bundles of virtual dimension zero at a base-point).

This defines K-theory in non-positive degrees, the periodicity properties of K-theory can be used to extend this to an integer grading. We will here only consider complex K-theory, the K-theory of complex vector bundles. There are similar K-theories built using real (KO) and quaternionic (KSp) vector bundles.

The fundamental theorem of the subject is the Bott periodicity theorem which says that K-theory is periodic with period 2:

$$K^{-(i+2)}(M) = K^{-i}(M)$$

and that

$$K^{-i}(pt.) = \begin{cases} \mathbf{Z} & i \text{ even} \\ 0 & i \text{ odd} \end{cases}$$

unlike the case of ordinary cohomology, where a point has non-trivial cohomology only in degree zero.

K-theory and the theory of Clifford algebras are linked together in a very fundamental way [6]. For an expository account of this, see [34]. Recalling the discussion of section 2.3.5, the relation between K-theory and Clifford algebras is given by

**Theorem 7** (Atiyah-Bott-Shapiro)**.** *There is an isomorphism of graded rings*

$$\alpha : A_* \to \sum_{i \geq 0} K^{-i}(pt.)$$

The isomorphism $\alpha$ can be explicitly constructed as follows. First note that

$$M_{2n} = \mathbf{Z} \oplus \mathbf{Z}$$

with generators $[S] = [S^+ \oplus S^-]$, the $\mathbf{Z}_2$ graded spin module and $\widetilde{S}$, the spin module with opposite grading. Similarly

$$i^* M_{2n+1} = \mathbf{Z}$$

and is generated by $[S] + [\widetilde{S}]$. Thus

$$A_{2n} = \mathbf{Z}$$

with generator $[S] - [\widetilde{S}]$. The isomorphism $\alpha$ associates to this generator an element

$$[\mathbf{S}, \widetilde{\mathbf{S}}\ \mu] \in K(\mathbf{R}^{2n}, \mathbf{R}^{2n} - 0) = \widetilde{K}(S^{2n}) = K^{-2n}(pt.)$$

where $\mathbf{S}$ is the product bundle $S \times \mathbf{R}^{2n}$ over $\mathbf{R}^{2n}$, $\widetilde{\mathbf{S}}$ is the product bundle $\widetilde{S} \times \mathbf{R}^{2n}$ and $\mu$ is the tautological bundle map given at a point $v \in \mathbf{R}^{2n}$ by Clifford multiplication by $v$.

This K-theory element $\Omega = [\mathbf{S}, \widetilde{\mathbf{S}}\ \mu]$ plays the role for K-theory that the generator of $H^k(S^k)$ played in ordinary cohomology. In the case $n = 1$ it is sometime known as the "Bott class" since multiplication by it implements Bott periodicity. Note that the K-theory orientation class constructed above has a great deal more structure than that of cohomology. $Spin(2n)$ acts by automorphisms on $C_{2n}$ and on the vector bundle construction of $[\mathbf{S}, \widetilde{\mathbf{S}}\ \mu]$.

There is another purely representation theoretical version of the Atiyah-Bott-Shapiro construction. The representation ring $RSpin(2n+1)$ is a subring of $RSpin(2n)$, the inclusion given by restriction of representations. It turns out that $RSpin(2n)$ is a free $RSpin(2n+1)$ module with two generators which can be taken to be 1 and $S^+$. Now since

$$Spin(2n+1)/Spin(2n) = S^{2n}$$

the associated bundle construction gives a map from $RSpin(2n)$ to vector bundles on $S^{2n}$ and this map defines an isomorphism

$$RSpin(2n)/RSpin(2n+1) \to K(S^{2n})$$

Restricting attention to even-dimensional manifolds $M$ (so we can just use complex K-theory), we can now do the same constructions as in cohomology, defining for a vector bundle $E$ an orientation or Thom class $i_!(1) \in K(E, E-M)$ to be one that restricts to $\Omega$ on the fibers. Choosing a K-theory orientation of the tangent bundle $TM$ of a manifold is equivalent to choosing a $Spin^c$ structure on the manifold. Note that there may be many possible $Spin^c$ structures on a manifold and thus many different K-theory orientations of the manifold. Choosing a closed embedding

$$i : M \longrightarrow S^{2n+2k}$$

the normal bundle $N$ will be orientable if $TM$ and as in the cohomology case the Thom class $i_!(1)$ provides an element

$$K(N, N-M) = K(S^{2n+2k}, S^{2n+2k} - M)$$

Using relative K-theory as a definition of K-homology, this provides a K-homology fundamental class which we will denote $\{M\} \in K_0(M)$.

This definition of the fundamental class in K-homology as a relative K-theory class is useful for topological computations, but there is a very different looking definition of K-homology called analytic K-homology that uses elliptic operators as cycles. The initial suggestion for the existence of this theory can be found in [2], a recent exposition is [27] and many other applications are discussed in [18]. The identity of these two different versions of K-homology is the content of the Atiyah-Singer index theorem. In analytic K-homology, the fundamental class of $M$ is the class of the Dirac operator. The cap product map:

$$\cap : K^0(M) \otimes K_0(M) \to K_0(M)$$

is constructed by twisting the operator by the vector bundle $E$. In particular, cap product by the fundamental class $\{M\} = [\not{D}]$ gives a Poincaré duality map

$$[E] \in K^0(M) \xrightarrow{P.D.} [E] \cap [\not{D}] = [\not{D}_E] \in K_0(M)$$

In analytic K-theory the pushforward map $\pi_*$ associated to the map

$$\pi : M \to pt.$$

takes the class of an elliptic operator $D$ in $K_*(M)$ to its index

$$\pi_*([D]) = index\ D = [ker\ D] - [coker\ D] \in K_0(pt.) = \mathbf{Z}$$

The analog of the cohomology integration map $\int_M$ in K-theory is the map

$$[E] \in K(M) \to \pi_*([E] \cap [\not{D}]) = index\ \not{D}_E \in K_0(pt.) = \mathbf{Z}$$

so the K-theory integral over $M$ of a vector bundle $E$ is just the index of the Dirac operator twisted by $E$.

For $D$ an elliptic pseudodifferential operator, the two versions of K-homology we have considered are related by the notion of the symbol $\sigma(D)$ of the operator. The symbol gives a bundle map between bundles pulled-back to the cotangent bundle $T^*M$, and the map is an isomorphism away from the zero section for the elliptic case. Thus $\sigma(D)$ defines a class in $K(T^*M, T^*M - M)$ and so a class in our topologically defined $K_0(M)$. The symbol of the Dirac operator is the orientation class $i_!(1)$.

The Atiyah-Singer index theorem tells us that one can compute the index of an elliptic operator $D$ by computing the push-forward of $\sigma(D) \in K(T^*M, T^*M - M)$ to $K_0(pt.) = \mathbf{Z}$ They define this by using the fact that $TN$ is a bundle over $TM$ that can be identified with $\pi^*(N) \otimes \mathbf{C}$ and embedding $TN$ in a large sphere $S^{2k}$. Then $i_!(\sigma(D)) \in K(TN, TM)$ will give an element of $K(S^{2k})$ and thus an integer by Bott periodicity. This integer will be the index of $D$.

One would like to be able to compute the index in terms of the more familiar topological invariants of homology and cohomology, we will see later how this can be done using the Chern character and other characteristic classes.

## 3.4 Orientations and Poincaré Duality: Equivariant K-theory

Much of the previous section continues to hold when one generalizes from K-theory to equivariant K-theory. $K^0_G(M)$ is the Grothendieck group of equivariant vector bundles, it again is a ring using tensor product of vector bundles. One major difference is that now $K^0_G(pt.)$ is non-trivial, it is the representation ring $R(G)$. Now $K^0_G(M)$ has additional interesting algebraic structure: it is a module over $K^0_G(pt.) = R(G)$. In order to define a corresponding equivariant homology theory via Alexander duality, one needs to use an equivariant embedding in $V$ for $V$ some representation of $G$ of sufficiently large dimension.

Equivariant K-theory is naturally graded not by $\mathbf{Z}$, but by the representation ring. One gets elements in other degrees than zero by considering $K^0(M \times V)$. The Atiyah-Bott-Shapiro construction can also be generalized. Instead of considering modules over the Clifford algebra of $\mathbf{R}^n$ we need to consider the Clifford algebra of $V$, the ABS construction then gives a complex of equivariant vector bundles over $V$, exact away from zero.

In the equivariant context the Dirac operator continues to provide an equivariant fundamental class and there is an equivariant integration map

$$\pi_! : K_G(M) \to K_G(pt.) = R(G)$$

given by the index map

$$[E] \in K_G(M) \to index\ \not{D}_E \in K_G(pt.) = R(G)$$

since the index of an equivariant operator is a difference of representations, thus an element of $R(G)$.

If we specialize to the case $M = G/T$ we see that our integration map is just the map

$$\pi_! : K_G(G/T) = R(T) \to K_G(pt.) = R(G)$$

that takes a weight belonging to some multiplet to the corresponding representation of $G$. $K_G(G/T) = R(T)$ is a module over $R(G)$, this module structure is the one that comes from taking the restriction of a $G$ representation to $T$ and using multiplication in $R(T)$. $R(T)$ is a free module over $R(G)$ of degree $|W(G,T)|$ and the analog of Poincaré duality in this case is seen in the existence of a map

$$K_G(G/T) \to Hom_{K_G(pt.)}(K_G(G/T), K_G(pt.))$$

or

$$R(T) \to Hom_{R(G)}(R(T), R(G))$$

given by

$$\lambda \to (\mu \to index\ \not{D}_{L_\lambda \otimes L_\mu})$$

A generalization of equivariant K-theory was developed by Atiyah [3] and more recently by Berline and Vergne [12, 13]. This generalization involves "transverse elliptic" operators, i.e. ones whose symbol may not be invertible in directions along $G$-orbits, but is on directions transverse to the $G$-orbits. It allows one to consider a sort of equivariant K-theory integration map which along $G$-orbits is purely a matter of representation theory. The integration map takes values not in the character ring $R(G) \otimes \mathbf{C}$ of conjugation invariant functions on $G$, but in its dual, distributions on $G$ that are conjugation invariant.

# 4 Classifying Spaces, Equivariant Cohomology and Representation Theory

## 4.1 Classifying Spaces and Equivariant Cohomology

We first encountered homological methods in the context of the Lie algebra cohomology $H^*(\mathfrak{n}, V)$. There we studied the $\mathfrak{n}$-invariant part of a $U(\mathfrak{n})$ module $V$ by replacing $V$ with a resolution of $V$ by free $U(\mathfrak{n})$ modules. This resolution is a chain complex whose cohomology is just $V$ in degree zero. Taking the $\mathfrak{n}$-invariant part of the resolution leads to a new chain complex whose degree zero cohomology is the invariant part of $V$. Now the complex may have cohomology in other degrees leading to higher cohomology phenomena.

In this section we will consider another analogous homological construction, this time using topological spaces. Roughly this can abstractly be described as follows [36]. Instead of just a vector space $V$ with a $G$ action, consider a topological space $M$ with a $G$ action. The analog of $V$ here is the chain complex of $M$, it is a differential module over the differential algebra $C_*(G)$ of chains on $G$. To get an analog of a free resolution of $V$, consider a topological space $EG$ which is homotopically trivial (contractible) and has a free $G$ action. The complex of chains on $EG \times M$ will be our analog of a free resolution. Equivariant cohomology will be the derived functor of taking invariants of the $C_*(G)$ action. In a de Rham model of cohomology we are taking forms not just invariant under the $G$ action, but "basic", having no components in the direction of the $G$ action.

With this motivation, the definition of equivariant cohomology is

**Definition 5.** *Given a topological space $M$ with a $G$ action, the $G$-equivariant cohomology of $M$ is*

$$H^*_G(M) = H^*(EG \times_G M)$$

*The classifying space $BG$ of $G$ is the quotient space $EG/G$.*

Here we are using cohomology with real coefficients. Note that $H^*_G(M)$ is not the same as $(H^*(M))^G$, the $G$ invariant part of $H^*(M)$. For $G$ compact taking $G$ invariants is an exact functor so $(H^*(M))^G = H^*(M)$. $H^*_G(M)$ has the property that, for a free $G$ action

$$H^*_G(M) = H^*(G/M)$$

At the other extreme, the equivariant cohomology of a point is now non-trivial: for a compact Lie group

$$H^*_G(pt.) = H^*(EG/G) = H^*(BG) = S(\mathfrak{g}^*)^G$$

the space of conjugation invariant polynomial functions on the Lie algebra $\mathfrak{g}$. Recalling our discussion of the maximal torus $T$ of $G$, these are $W(G,T)$ invariant functions on the Lie algebra of $T$ so

$$H^*_G(pt.) = \mathbf{R}[u_1, u_2, \cdots, u_l]^{W(G,T)}$$

where $u_i$ are coordinates on $\mathfrak{t}$ and $l$ is the rank of $G$. The equivariant cohomology $H^*_G(M)$ is not only a ring, but is a module over the ring $H^*_G(pt.)$.

A simple example to keep in mind is $G = U(1)$ for which in some sense

$$EU(1) = \lim_{n\to\infty} S^{2n+1} = S^\infty$$

and

$$BU(1) = \lim_{n\to\infty} CP^n = CP^\infty$$

the infinite dimensional complex projective space. The cohomology ring of $CP^n$ is

$$H^*(CP^n) = \mathbf{R}[u]/u^{n+1}$$

so this is consistent with

$$H^*_{U(1)}(pt.) = \mathbf{R}[u]$$

Note that the classifying space $BG$ will be infinite dimensional for all compact $G$, but often can be analyzed by taking limits of finite dimensional constructions. We may want to consider cohomology classes with terms in indefinitely high degrees, i.e. formal power series as well as polynomials, writing these as

$$H^{**}_{U(1)}(pt.) = \mathbf{R}[[u]]$$

Just as in equivariant K-theory we have the identity

$$H_G^*(G/T) = H_T^*(pt.)$$

since

$$EG \times_G G/T = EG/T = BT$$

and more generally for $H$ a subgroup of $G$ there is an induction relation

$$H_H^*(M) = H_G^*(G \times_H M)$$

If one wants to work with an explicit de Rham model of equivariant cohomology, one would like to avoid working with differential forms on the infinite dimensional space $EG$. In an analogous fashion to what one does when one computes de Rham cohomology of compact groups, where by an averaging argument one replaces forms on $G$ by left-invariant forms which are generated by left-invariant 1-forms (the Lie algebra $\mathfrak{g}$), one can use a much smaller complex, replacing $\Omega^*(EG)$ by the "equivariant differential forms"

$$\Omega_G^*(M) = \{W(\mathfrak{g}) \otimes \Omega^*(M)\}_{basic} = (S(\mathfrak{g}^*) \otimes \Omega^*(M))^G$$

Here

$$W(\mathfrak{g}) = S(\mathfrak{g}^*) \otimes \Lambda(\mathfrak{g}^*)$$

is the Weil algebra of $G$, a finite dimensional model for forms on $EG$. For more details on equivariant cohomology see [5] and [24] and for a detailed exposition of the formalism of equivariant forms and the equivariant differential, see [26]. There one can also find a translation of this formalism into the language of superalgebras. In this language $\Omega_G^*(M)$ is a superalgebra with action of the Lie superalgebra

$$\tilde{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

where $\mathfrak{g}_0 = \mathfrak{g}$, the $\mathfrak{g}_{-1}$ is a copy of $\mathfrak{g}$ corresponding to the interior products in the $\mathfrak{g}$ directions and $\mathfrak{g}_1$ is one dimensional, corresponding to the differential $d$. Taking basic forms is then equivalent to $\tilde{\mathfrak{g}}$ invariance, roughly corresponding to the motivation of taking $C_*(G)$ invariants given at the beginning of this section.

## 4.2 Classifying Spaces and Equivariant K-Theory

For a general group $G$ the topology of the classifying space of a group $G$ is often closely related to the representation theory of $G$. Given a representation $V$ of $G$, one can form the vector bundle

$$V_G = EG \times_G V$$

over $BG$. In terms of equivariant K-theory classes one has a map

$$V \in K_G(pt.) = R(G) \to [V_G] \in K_G(EG) = K(BG)$$

For compact Lie groups

$$K_G(EG) = K(BG) = \widehat{R}(G)$$

where $\widehat{R}(G)$ is the completion of the representation ring at the ideal of virtual representations of zero dimension [7]. Thinking of $R(G)$ as the ring of conjugation invariant functions on $G$ in this case our map

$$V \in R(G) \to \widehat{R}(G)$$

is just the map that takes a global function to its power series expansion around the identity.

As an explicit example, consider the case $G = U(1)$. Then the representation ring is

$$R(U(1)) = \mathbf{Z}[t, t^{-1}]$$

Writing a representation $t$ in terms of its character $\chi(t)$ as a function on $\mathfrak{g} = \mathbf{R}$

$$\chi(t) = e^u$$

Then $(R(U(1)) \otimes \mathbf{C})$ the subring of the power series ring $\mathbf{C}[[u]]$ generated by $\{e^u, e^{-u}\}$ and the completion at $e^u = 1$ is the power series ring

$$\widehat{R}(U(1)) \otimes \mathbf{C} = \mathbf{C}[[z]]$$

where $z = e^u - 1$.

A lesson of this is that one can study the character ring $R(G)$ by considering the equivariant K-theory of $EG$, although at the cost of only being able to work with power series expansions of characters on the Lie algebra instead of global character functions. Furthermore, when one does this one ends up with something that is equivalent to equivariant cohomology.

In some sense the map

$$\pi_! : K_G(EG) \to K_G(pt.)$$

while only being a homotopy equivalence, not an equivariant homotopy equivalence, still closely relates these two rings. The Baum-Connes conjecture implies that something like this phenomenon occurs even in the context of discrete groups $G$. More specifically they conjecture [9] that

$$K_{G*}(\overline{EG}) = K_*(C_r^*(G))$$

Here $\overline{EG}$ is their classifying space for proper G actions, on the left one is using some version of K-homology, typically one defined in terms of operator algebras, such as Kasparov KK-theory. On the right is the operator algebra K-theory of the reduced $C^*$-algebra of G.

## 4.3 Equivariant Homology and the Fundamental Class

An obvious possible integration map in equivariant cohomology is integration of differential forms which gives a map

$$\int_M : H_G^*(M) \to H_G^*(pt.) = S(\mathfrak{g}^*)^G$$

We can use to this to construct a map similar to the one we saw in equivariant K-theory reflecting Poincaré duality

$$H_G^*(M) \to Hom_{S(\mathfrak{g}^*)^G}(H_G^*(M), S(\mathfrak{g}^*)^G)$$

defined by

$$[\alpha] \to ([\beta] \to \int_M \alpha\beta)$$

This map is an isomorphism in certain cases [24], for instance if $M$ is a compact symplectic manifold and the $G$ action is Hamiltonian (preserves the symplectic form). In other cases it fails to be an isomorphism, most dramatically in the case of a free $G$ action when the map is zero since the integral of an equivariantly closed form vanishes.

This failure is an indication that to get a well-behaved notion of an equivariant homology theory dual to equivariant cohomology in terms of differential forms, one needs to consider not just $S(\mathfrak{g}^*)$, the polynomial functions on $\mathfrak{g}$, but the class of generalized functions on $\mathfrak{g}$, $C^{-\infty}(G)$. We'll begin by considering the case of a point.

### 4.3.1 The Equivariant Homology of a Point

In equivariant K-theory, $K_G(pt.) \otimes \mathbf{C} = R(G) \otimes \mathbf{C}$ is the space of character functions on $G$. There is a natural inner product given by the Haar integral over $G$, and different irreducible representations $V$ and $W$ are orthonormal with respect to this inner product.

$$< \chi_V, \chi_W >= \int_G \chi_V(g)\overline{\chi_W}(g)dg = \begin{cases} 1 & V \simeq W \\ 0 & V \not\simeq W \end{cases}$$

The irreducible representations thus give a distinguished basis for $K_G(pt.)$. For any representation $V$, irreducible or not, its decomposition into irreducibles (coordinates with respect to the distinguished basis) can be found by computing integrals. The integral

$$\int_G \chi_V(g)\overline{\chi_{V_i}(g)}dg$$

gives the multiplicity $mult(V_i, V)$ of the irreducible $V_i$ in $V$. In particular

$$\int_G \chi_V(g)dg$$

gives the multiplicity $mult(1, V)$ of the trivial representation in $V$. For each irreducible $V_i$, the multiplicity $mult(V_i, \cdot)$ is a linear functional on $K_G(pt.)$ giving a distinguished basis for the dual space to $K_G(pt.)$. Another important linear functional is given by evaluating the character at the identity, giving the virtual dimension of an element of $K_G(pt.)$.

In equivariant cohomology $H_G^{**}(pt.)$ consists of power series about zero in the Lie algebra. The (topological) dual vector space is the space

$$C^{-\infty}(\mathfrak{g})_0$$

of distributions supported at 0. Fourier transformation maps $S^*(\mathfrak{g})$ (polynomials on $\mathfrak{g}^*$) to distributions on $\mathfrak{g}$ supported at 0: derivatives of the $\delta$-function. We would like to be able to define analogs of the linear functionals $mult(V_i, \cdot)$, for instance $mult(1, \cdot)$, the multiplicity of the identity representation. This is not so easily defined as in the $R(G)$ case, but one way of doing this is to compute limits of ratios such as the following:

$$\oint_{\mathfrak{g}} f(u) \equiv \lim_{\epsilon \to 0} \frac{\int_{\mathfrak{g}} f(u) e^{-\epsilon ||u||^2}}{\int_{\mathfrak{g}} e^{-\epsilon ||u||^2}}$$

where $|| \cdot ||$ is a conjugation invariant norm on $\mathfrak{g}$.

One could ask what the fundamental class is in the equivariant homology of a point. The fact that $H^*_G(pt) = H^*(BG)$ would indicate that in some sense the answer should be the homology fundamental class of $BG$. The infinite dimensionality of $BG$ makes this very ill defined, a better answer to this question would be the functional $\oint_{\mathfrak{g}}(\cdot)$.

### 4.3.2 The Case of a Free Action

We saw at the beginning of this section that for a free action the obvious integration map $\int_M$ on $H^*_G(M)$ gives zero. To get an integration map with better properties, we proceed much as in the case of the cohomology Thom class. There one could replace integration over $M$ by integration over some $N$ in which $M$ is embedded. The analog of the Thom class in this case is something we will call the Witten-Thom class [44] and to define it we need to work with a generalization of equivariant differential forms. For a definition of the complex $C^{-\infty}(\mathfrak{g}, \Omega^*(M))$ of equivariant differential forms with generalized coefficients and the corresponding generalization $H_G^{-\infty}(M)$, as well as more details of this construction, see [31]. For a related discussion of the fundamental homology class for a free action, see [8].

**Definition 6.** *Given a fibration*

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \big\downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

*with a connection $\omega$ on $P$, define a one-form on $P \times \mathfrak{g}^*$ by*

$$\lambda(p, u) = \omega_p(u)$$

*where $\omega_p$ is the $\mathfrak{g}$-valued connection 1-form at $p \in P$ and $\omega_p(u)$ is its evaluation on the element $u \in \mathfrak{g}^*$. The Witten-Thom class $\gamma \in C^{-\infty}(\mathfrak{g}, \Omega^*(P))$ is defined by*

$$\gamma = \frac{1}{(2\pi)^{dim\ \mathfrak{g}}} \int_{\mathfrak{g}^*} e^{i d_{\mathfrak{g}} \lambda}$$

*where $d_{\mathfrak{g}}$ is the equivariant differential in $C^{-\infty}(\mathfrak{g}, \Omega^*(P))$.*

The Witten-Thom class has the following properties:

- Its class in $H_G^{-\infty}(P)$ is independent of the connection $\omega$ chosen to construct it.
- (Analog of Thom isomorphism): The map

$$H^*(M) \to H_G^{-\infty}(P)$$

on cohomology induced from the map

$$\alpha \to \pi^*(\alpha) \wedge \gamma$$

is an isomorphism of $C^\infty(\mathfrak{g})^G$ modules.

- Evaluation of $\gamma$ on $\phi \in (C^\infty(\mathfrak{g}))^G$ gives

$$\int_{\mathfrak{g}} \gamma(u)\phi(u) = v_\omega \wedge \phi(\Omega)$$

i.e.

$$\gamma(u) = v_\omega \wedge \delta(u - \Omega)$$

Here $\Omega$ is the curvature of $\omega$ and $v_\omega$ is a vertical form on $P$ of degree $dim\ G$ and integral $vol(G)$ over each fiber.

- Integration over $P$ and over $M$ are now related by

$$\int_{\mathfrak{g}} \phi(u) \int_P \gamma \wedge \pi^*\alpha = vol(G) \int_M \alpha \wedge \phi(\Omega)$$

For $\alpha = 1$, this formula gives an expression for the characteristic number

$$\int_M \phi(\Omega)$$

of $P$ corresponding to $\phi$. Taking $\phi$ of the form

$$\phi(u) = e^{-\epsilon||u||^2}$$

and taking the limit as $\epsilon$ goes to zero gives the formula

$$\oint_{\mathfrak{g}} \int_P \gamma \wedge \pi^*\alpha = \int_M \alpha$$

The last of these properties shows that we have an integration map on $H^*_G(P)$ for the case of a free action is given by

$$\alpha \in H^*_G(P) \to \frac{1}{vol(G)} \oint_{\mathfrak{g}} \int_P \gamma \wedge \alpha \in H^*(pt)$$

.

## 4.4 K-theory and (Co)homology: The Chern Character

We have seen that for a point, the relation between equivariant K-theory and equivariant cohomology is just that between a representation $V$ and its character

$$\chi_V = Tr_V(e^u)$$

expressed as a power series about $0 \in \mathfrak{g}$. In general one would like to relate K-theory calculations to more familiar cohomological ones, this is done with a map called the Chern character.

In ordinary K-theory and cohomology, the Chern character is a map

$$ch : K(M) \to H^*(M, \mathbf{C})$$

and is a ring isomorphism. It takes the direct sum of vector bundles to the sum of cohomology classes and the tensor product of vector bundles to the cup product of cohomology classes and takes values in even dimensional cohomology.

A generalization to the odd K-theory groups takes values in odd dimensional cohomology.

Chern-Weil theory gives a de Rham cohomology representative of the Chern character map using the curvature of an arbitrarily chosen connection. If an element $[E]$ of K-theory is the associated bundle to a principal bundle $P$ of the form

$$E = P \times_G V$$

for a representation $V$ of $G$, then the Chern-Weil version of the Chern character is defined by

$$\int_{\mathfrak{g}} \gamma(u)\phi(u) = v_\omega \wedge \phi(\Omega) = v_\omega \wedge ch([E])$$

where $\gamma$ is the Thom-Witten form and

$$\phi(u) = Tr_V(e^u) \in C^\infty(\mathfrak{g})$$

One can define similarly define an equivariant Chern character map

$$ch_G : K_G(M) \to H^*_G(M)$$

a Chern-Weil version of this uses $G$-invariant connections. It is no longer a rational isomorphism since as we have seen even for a point one has to take a completion of $K_G$. Several authors have defined a "de-localized equivariant cohomology" with the goal of having a Chern-character that is an isomorphism. For one version of this, see [20].

Quillen [38] has defined a generalization of the Chern character to the case of relative K-theory, using a generalization of Chern-Weil theory that uses "superconnections" instead of connections. Whereas a connection on a $\mathbf{Z}_2$-graded vector bundle preserves the grading, a superconnection has components that mix the odd and even pieces. He constructs an explicit Chern character map that uses the superconnection

$$d + t\mu$$

on the trivial spin bundles $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ over $\mathbf{R}^{2n}$ and takes the K-theory orientation class

$$[\mathbf{S}, \widetilde{\mathbf{S}}\ \mu] \in K(\mathbf{R}^{2n}, \mathbf{R}^{2n} - 0)$$

to an element

$$Str(e^{(d+t\mu)^2}) \in H^*(\mathbf{R}^{2n}, \mathbf{R}^{2n} - 0)$$

Here $Str$ is the supertrace, $t$ is a parameter, and this differential form becomes more and more peaked at 0 as $t$ goes to infinity.

This construction is generalized in [35] to an equivariant one for the group $G$, where $G$ is some subgroup of $Spin(2n)$. The Atiyah-Bott-Shapiro construction of the orientation class is $G$-equivariant and gives an element in $K_G(\mathbf{R}^{2n}, \mathbf{R}^{2n} - 0)$. Mathai-Quillen use an equivariant superconnection formalism to map this to an equivariant differential form that represents an element of $H_G(\mathbf{R}^{2n}, \mathbf{R}^{2n} - 0)$. Given such an equivariant differential form, for any vector bundle $E$ over $M$ with connection of the form

$$P \times_G V$$

for some $G$-representation $V$, one can use Chern-Weil theory to get an explicit representative for the Chern character of the K-theory orientation class

$$ch(i_!(1)) \in H^*(E, E - M)$$

Mathai and Quillen show that this is not identical with the cohomology orientation class, instead it satisfies

$$ch(i_!(1)) = \pi^*(\hat{A}(E)^{-1})i_*(1)$$

where $\hat{A}(E)$ is the characteristic class of the vector bundle $E$ corresponding to the function

$$\phi(u) = det^{\frac{1}{2}}(\frac{u/2}{e^{u/2} - e^{-u/2}})$$

The Chern character can be used to get an explicit cohomological form of the index theorem. The index of an operator $D$ with symbol $\sigma(D)$ will be given by evaluating

$$ch(i_!(\sigma))$$

on the fundamental class of $TN$, where $i : TM \rightarrow TN$ is the zero-section. Standard manipulations of characteristic classes give the well-know formula

$$index \not{D}_E = \int_M ch(E) \wedge \hat{A}(TM)$$

In the equivariant context, Berline and Vergne [11, 10] found a generalization of this formula expressing the equivariant index (as a power series in $H^*_G(pt.)$) as an integral over equivariant versions of the Chern and $\hat{A}$ classes. They note that in the special case of $M = G/T$ a co-adjoint orbit, their formula reduces to an integral formula for the character due to Kirillov. See [12, 13, 20] for generalizations to the case of transverse elliptic operators and to de-localized equivariant cohomology.

# 5 Witten Localization and Quantization

Given a geometric construction of a representation $V$, a central problem is to understand its decomposition into irreducible representations $V_i$. Since

$$mult(V_i, V) = mult(1, V \otimes V_i^*)$$

once one knows the irreducible representations, one just needs to be able to compute the multiplicity of the identity representation in an arbitrary one. For a compact Lie group, if one knows the character $\chi_V$, this is just the integral over the group. If the character is only known as a power series expansion in the neighborhood of the identity, we have seen that one still may be able to extract the multiplicity from an integral over the Lie algebra.

Witten in [44] considered integrals of this form for integrands such as the equivariant Chern character that occurs in the Berline-Vergne-Kirillov integral formula for an equivariant index. Given a quantizable symplectic manifold $M$ with a Hamiltonian $G$ action, one considers an equivariant line bundle $L$ which gives a class

$$[L] \in K_G(M)$$

As an element of $R(G)$, the quantization here is the K-theory push-forward

$$\pi_!([L]) \in K_G(pt.) = R(G)$$

and one would like to know how this decomposes into irreducibles. This requires evaluating integrals of the form

$$\oint_{\mathfrak{g}} \int_M ch_G([L]) \hat{A}_G(M)$$

The equivariant Chern character is represented by an equivariant differential form that is the exponential of the equivariant curvature form of $L$

$$ch_G([L]) = e^{\omega + i2\pi<\mu,u>}$$

where $\omega$ is the symplectic form (curvature of $L$) and $\mu$ is a the moment map

$$\mu : M \to \mathfrak{g}^*$$

The integral over $\mathfrak{g}$ has exponential terms of the form

$$\int_{\mathfrak{g}} e^{-\epsilon||u||^2 + i2\pi<\mu,u>}(\cdots)$$

and as $\epsilon$ goes to zero the integral will be dominated by contributions from a neighborhood of the subspace of $M$ where $\mu = 0$. The Witten-Thom form relates such integrals to integrals on $\mu^{-1}(0)/G$. Various authors have used this sort of principle to show that "quantization commutes with reduction", i.e. the multiplicity of the trivial representation in $\pi_!([L])$ is the dimension one would get by quantizing the Marsden-Weinstein reduced phase space $\mu^{-1}(0)/G$. For a review of some of these results, see [42].

# 6 Geometrical Structures and Their Automorphism Groups

## 6.1 Principal Bundles and Generalized G-structures

A principal $G$-bundle $P$

$$\begin{array}{ccc} G & \longrightarrow & P \\ & & \downarrow \pi \\ & & M \end{array}$$

can be thought of as a generalization of a group to a family of identical groups parametrized by a base-space $M$. We've already seen two classes of examples: $G$ is a principal $H$-bundle over $G/H$, and $EG$ is a principal $G$-bundle over $BG$. This second example is universal: every $G$ bundle $P$ occurs as the pullback under some map $f : M \to BG$ of $EG$.

To any n-dimensional manifold $M$ one can associate a principal $GL(n, \mathbf{R})$ bundle $F(M)$, the bundle whose fiber above $x \in M$ is the set of linear frames on the tangent space $T_xM$. A sub-bundle of $P \subset F(M)$ with fiber $G \subset GL(n, \mathbf{R})$ will be called a $G$-structure on $M$. The frame bundle $F(M)$ or a $G$-structure $P$ can be characterized by the existence of a tautologically defined equivariant form

$$\theta \in \Omega^1(P) \otimes \mathbf{R}^n$$

such that $\theta_p(v)$ gives the coordinates of the vector $\pi_*(v)$ with respect to the frame $p$.

This classical definition of a $G$-structure is not general enough to deal even with the spinor geometry of $M$ since one needs to consider a structure group $Spin(n)$ or $Spin^c(n)$ at each point and these are not subgroups of $GL(n, \mathbf{R})$. To get a sufficient degree of generality, we'll define

**Definition 7.** *A generalized $G$ structure on a manifold $M$ is a principal $G$-bundle $P$ over $M$ and a representation*

$$\rho : G \to GL(n, \mathbf{R})$$

*together with a $G$-equivariant horizontal $\mathbf{R}^n$-valued one-form $\theta$ on $P$.*

Note that $\rho$ does not have to be a faithful representation, it may have a kernel which can be thought of as an "internal symmetry".

## 6.2 The Automorphism Group of a Bundle

To any principal bundle $P$ one can associate a group, the group $AutP$ of automorphisms of the bundle. This group will be infinite-dimensional except in trivial cases.

**Definition 8.** $Aut(P)$={$f : P \to P$ *such that* $f(pg) = f(p)g)$}

$Aut(P)$ has a subgroup consisting of vertical automorphisms of $P$, those which take a point $p \in P$ to another point in the same fiber. This is the gauge group $\mathcal{G}_P$ and elements of $\mathcal{G}_P$ can be given as maps

$$h : P \to G$$

here $h(p)$ is the group element such that

$$f(p) = ph(p)$$

The condition that $f(p) \in \mathcal{G}_P$ implies that the functions $h$ satisfy a condition $h(pg) = g^{-1}h(p)g$ so

**Definition 9.** $\mathcal{G}_P$={$h : P \to G$ *such that* $h(pg) = g^{-1}h(p)g$}

in other words $\mathcal{G}_P$ consists of the space of sections $\Gamma(AdP)$ where $AdP$ is the bundle

$$AdP = P \times_G G$$

associated to $P$, with fiber $G$ and the action of $G$ on itself the adjoint action.

First consider the case of a $G$ bundle over a point, in other words $P$ is $G$ itself. Describe Aut P in this case, show that it is $G_L \times Z(G)$ May also want to include an explicit discussion of the one-dimensional case.

The gauge group $\mathcal{G}_P$ is a normal subgroup of $Aut(P)$ and the quotient group is the group $Diff(M)$ of diffeomorphisms of the base space $M$. So we have an exact sequence

$$1 \to \mathcal{G}_P \to Aut(P) \to Diff(M) \to 1$$

Taking infinitesimal automorphisms we have the exact sequence

$$0 \to Lie(\mathcal{G}_P) \to Lie(Aut(P)) \xrightarrow{\pi_*} Vect(M) \to 0$$

Here we are looking at the Lie algebra of $G$-invariant vector fields on $P$, it has a Lie subalgebra of vertical $G$-invariant vector fields and a quotient Lie algebra of $G$-invariant vector field modulo vertical ones, which can be identified with $VF(M)$, the vector fields on $M$. The maps in this exact sequence are both Lie algebra homomorphisms and homomorphisms of $C^\infty(M)$ modules.

## 6.3 Connections

The fundamental geometrical structure that lives on a bundle and governs how the fibers are related is called a connection. There are many equivalent ways of defining a connection, perhaps the simplest is

**Definition 10.** *A connection is a splitting $h$ of the sequence*

$$0 \to Lie(\mathcal{G}_P) \to Lie(Aut(P)) \xrightarrow{\pi_*} VF(M) \to 0$$

*i.e. a homomorphism*

$$h : VF(M) \to Lie(Aut(P))$$

*of $C^\infty(M)$ modules such that $\pi_* h = Id$.*

For any vector field $X \in VF(M)$, $hX \in Lie(Aut(P))$ is the horizontal lift of $X$. A connection can also be characterized by the distribution on $P$ of horizontal subspaces $H_p$. $H_p$ is the subspace of $T_pP$ spanned by the horizontal lifts of vector fields.

The map $h$ is not necessarily a Lie algebra homomorphism. When it is the connection is said to be flat. In general the curvature $\Omega$ of a connection is characterized by the map

$$\Omega : VF(M) \times VF(M) \to Lie(\mathcal{G}_P)$$

given by

$$\Omega(X, Y) = [hX, hY] - h[X, Y]$$

A connection can dually be characterized by an equivariant $\mathfrak{g}$ valued one-form

$$\omega \in \Omega^1(P) \otimes \mathfrak{g}$$

which satisfies

$$H_p = ker\ \omega_p$$

The space $\mathcal{A}_P$ is an affine space, the difference of two connections is a one-form on $M$ with values in $\Gamma(Lie(AdP))$. The group $Aut(P)$ acts on $\mathcal{A}_P$ by pull-back of connection one-forms.

# 7 Quantum Field Theories and Representation Theory: Speculative Analogies

We began this article with some fundamental questions concerning the significance of the mathematical structures appearing in the quantum field theory of the Standard Model. Here we would like to provide some tentative answers to these questions by speculating on how these mathematical structures fit into the representation theoretical framework we have developed in previous sections.

## 7.1 Gauge Theory and the Classifying Space of the Gauge Group

Given an arbitrary principal bundle $P$, we have seen that its automorphism group $Aut(P)$ has a normal subgroup $\mathcal{G}_P$ of gauge transformations which acts on $\mathcal{A}_P$, the space of connections on $P$. $\mathcal{G}_P$ has a subgroup $\mathcal{G}_P^0$ of based gauge transformations, those that are the identity on a fixed fiber in $P$. $\mathcal{A}_P$ is contractible and the $\mathcal{G}_P^0$ action is free, so we have

$$E\mathcal{G}_P^0 = \mathcal{A}_P, \quad B\mathcal{G}_P^0 = \mathcal{A}_P/\mathcal{G}_P^0$$

We have seen in the case of a compact Lie group $G$ that the use of the classifying space $BG$ allows us to translate problems about the representation theory of $G$ into problems about the topology of $BG$, which are then studied using K-theory or (co)homology. To any representation $V$ of $G$ is associated an element $[V_G] \in K_G(EG) = K(BG)$ and $K_G^0(EG) = K^0(BG)$ is $\widehat{R(G)}$, the representation ring of $G$, completed at the identity. In the case of the gauge group $\mathcal{G}_P$, our lack of understanding of what $R(\mathcal{G}_P)$ might be is profound, but perhaps quantum field theory is telling us that it can be approached through the study of the topological functors $K^*_{\mathcal{G}_P}(E\mathcal{G}_P)$ and $H^*_{\mathcal{G}_P}(E\mathcal{G}_P)$. The infinite dimensionality of $\mathcal{G}_P$ makes the study of these functors difficult, but if physicist's path integral calculations can be interpreted as formal calculations involving these rings, then there is a large amount of lore about what can be sensibly calculated that will become accessible.

Following this line of thought, the Standard Model quantum field theory path integral would involve a de Rham model

$$\Omega^*_G(\mathcal{A}_P) = \{W(Lie(\mathcal{G}_P) \otimes \Omega^*(\mathcal{A}_P)\}_{\mathrm{b}asic}$$

of the equivariant cohomology of $\mathcal{A}_P$, and elements of the path integral integrand may come from this de Rham model. One obvious objection to this program is that this is what is done in Witten's "topological quantum field theory" (TQFT) formulation of Donaldson invariants [43] (for a review from this point of view, see [19]) and our physical theory should have observables that are not topological invariants. But perhaps the answer is that here one is doing "equivariant topological quantum field theory" and so has observables corresponding to the infinitesimal actions of all symmetries one is considering.

TQFT will correspond to restricting attention to the subspace of $Aut(P)$ invariants, this gives a finite dimensional purely topological problem, corresponding physically to choosing to only study the structure of the vacuum state of the more general equivariant theory.

One also needs to understand why the path integral is expressed as an integral over the classifying space $B\mathcal{G}_P = \mathcal{A}_P/\mathcal{G}_P$. In the Baum-Connes conjecture case described in an earlier section, the conjecture amounts to the idea that the map

$$K_{*G}(EG) \xrightarrow{\pi_*} K_{*G}(pt.)$$

induced from the "collapse" map $\pi : EG \to pt.$ is an isomorphism for $G$ a discrete group. Assuming that something like this is still true for $G = \mathcal{G}_P$, one can perhaps interpret path integral expressions as equivariant homology classes on $\mathcal{A}_P$. Then the integral is a reflection of the existence of some sort of fundamental class providing via cap-product a map

$$K^*_{\mathcal{G}_P}(\mathcal{A}_P) \to K_{*\mathcal{G}_P}(\mathcal{A}_P)$$

Here again the notion of an equivariant fundamental class is the critical one. Our proposal is that the Standard Model path integral involves a $\mathcal{G}_P$-equivariant fundamental class of $\mathcal{A}_P$ and that path integrals are actually calculations in an explicit model of equivariant cohomology related to the abstract equivariant $K$-theory picture by a Chern character map.

Finally one would like to understand the occurrence of the exponential of the Yang-Mills action as a factor in the path integral integrand. This can be understood by a generalization of the arguments discussed in Section 5, which were originally developed by Witten for the sake of applying them formally to the case of connections on a principal bundle $P$ over a Riemann surface $\Sigma$. In that case $\mathcal{A}_P$ is a (infinite dimensional) symplectic manifold with a line bundle $L$ (the determinant bundle for the Dirac operator) whose curvature is the symplectic form. The whole setup is equivariant under the gauge group $\mathcal{G}_P$ and the moment map corresponding to the $\mathcal{G}_P$ action is given by the curvature two-form $F_A$ of the connection $A \in \mathcal{A}_P$.

The localization principle shows that trying to pick out the $Lie\mathcal{G}_P$ by doing an integral

$$\lim_{\epsilon\to 0}\int_{Lie\mathcal{G}_P}\int_{\mathcal{A}_P} e^{-\epsilon||u||^2} ch_{\mathcal{G}_P}(L)$$

leads to integrals of the form

$$\lim_{\epsilon\to 0}\int_{\mathcal{A}_P} e^{-\frac{1}{\epsilon^2}||F_A||^2}(\cdots)$$

a form which is precisely that of the standard Yang-Mills theory. Note that the limit $\epsilon \to$ we would like to take is the same as that of taking the coupling constant to zero in the continuum limit as specified by the asymptotic freedom of the theory. The fact that this limit needs to be taken in a very specific way in order to get a non-trivial continuum limit should be a highly non-trivial idea necessary for the construction of interesting representations of $Aut(P)$ in higher dimensions.

While this specific construction only works for a Riemann surface, for a four dimensional hyperkähler manifold there is an analogous moment map. Here the zeros of the moment map are connections satisfying $F_A^+ = 0$ and a Gaussian factor in this moment map and the Yang-Mills action are related by

$$e^{-\frac{1}{g^2}||F_A||^2} = e^{-\frac{1}{g^2}(||F_A^+||^2+||F_A^-||^2)} = (const.)e^{-\frac{1}{g^2}||F_A^+||^2}$$

since

$$||F_A^+||^2 - ||F_A^-||^2$$

is a constant topological invariant of the bundle $P$.

## 7.2 An Analogy

The fundamental problem faced by any attempt to pursue the ideas of the previous section is our nearly complete ignorance of the representation theory of the group $Aut(P)$ for base spaces of more than one dimension. In previous sections certain techniques for decomposing the space of functions on a compact Lie group $G$ into irreducibles were explained in great detail partly because many of the same techniques may be useful in the $Aut(P)$ case. One can begin by trying to decompose a space of functions on $P$, say $C^\infty(P)$, as an $Aut(P)$ representation. As such it is highly reducible but perhaps the same homological techniques of using Clifford algebras and classifying spaces can be of help.

Some details of the analogy we have in mind are in the following table ($G_L$, $G_R$, $T_L$, $T_R$ are the right and left actions of $G$ or $T$ on $G$).

| Representations of $G$ | Representations of $Aut(P)$ |
|---|---|
| $G,\ C^\infty(G)$ | $P,\ C^\infty(P)$ |
| $T_R$ | $G$ |
| $G/T$ | $P/G = M$ |
| $G_L$ | $AutP$ |
| $T_L$ | $\mathcal{G}_P$ |
| $ET \times_{T_L} G/T$ | $\mathcal{A} \times_{\mathcal{G}_P} P$ |
| $\Gamma(G \times_{T_R} S_{(\mathfrak{g}/\mathfrak{t})_R})$ | $\Gamma(P \times_G S)$ |

The last line of this analogy is the most problematic. In the $G$ case $\mathfrak{g}/\mathfrak{t}_R$ is a finite dimensional space one can use to construct $\mathfrak{n}_+$ and then get a finite dimensional complex using either $\Lambda^*(\mathfrak{n}_+)$ or the spinors $S_{)\mathfrak{g}/\mathfrak{t})_R}$. In the $Aut(P)$ case there is no finite dimensional analog of $\mathfrak{n}_+$. One can still construct an analogous spinor bundle and it will have a space of sections that $Aut(P)$ acts on, but one cannot get non-trivial irreducible representations as subspace of this space of section. Quantum field theory indicates a method for dealing with this problem, that of "Second Quantization", i.e. consider that space of sections as a space that is still to be quantized. The space of sections will be the "one- particle space" and one wants to construct what can variously be thought of as the Fock space or infinite dimensional spinor space associated to this one-particle space.

### 7.3 Hilbert Spaces and Path Integrals for the Dirac Action

A quantum field theory should associate to a manifold with boundary $(M, \partial M)$ a Hilbert space $\mathcal{H}_{\partial M}$ which is associated to the boundary. It should have a vacuum vector $|0>_M \in \mathcal{H}_{\partial M}$ which depends on the bounding manifold $M$, much the way a highest weight vector in a representation of $G$ depends upon the extra data of a complex structure on $G/T$.

We have

$$i : \partial M \to M$$

and a corresponding map

$$i^* : \mathcal{A}_M \to \mathcal{A}_{\partial M}$$

given by restriction of connection to the boundary. Somehow we want to construct a push-forward map

$$i_! : K_{\mathcal{G}}(\mathcal{A}_M) \to K_{\mathcal{G}}(\mathcal{A}_{\partial M})$$

which should give us our $\mathcal{H}_{\partial M}$. We somehow have to deal with the fact that our $\mathcal{H}_{\partial M}$ should be a representation of $\mathcal{G}_{\partial M}$ rather than $\mathcal{G}_M$, these groups are related by the exact sequence

$$1 \to \mathcal{G}_{\partial M,1} \to \mathcal{G}_M \to \mathcal{G}_{\partial M} \to 1$$

where $\mathcal{G}_{\partial M,1}$ is the subgroup of $\mathcal{G}_M$ of gauge transformations that are the identity on $\partial M$. The gauge group on the boundary is naturally a quotient of the gauge group on the entire space.

Recall from section 2.3.3 that the spin representation can be defined as the dual space to $\Gamma(Pf^*)$ where $Pf^*$ is a line bundle over $\mathcal{J}(V)$, the space of maximal isotropic complex subspaces of $V_{\mathbf{C}}$. The line bundle $Pf^*$ can be explicitly constructed on an open subset of $\mathcal{J}(V)$ by associating to an isotropic subspace a skew operator and thus a vector in an exterior algebra according to the Gaussian formula of section 2.3.4.

In [46] Witten shows that, for the case of $\Sigma$ a Riemann surface bounded by $\partial\Sigma = S^1$, the Hilbert space for the theory should be thought of as a space of sections of a bundle $Pf^*$ over the space of maximal isotropic subspaces of the space of spinor field restricted to boundary $\partial\Sigma$. This construction is an infinite dimensional generalization of that of section 2.3.3 and 2.3.4. The formula that associates a vector in an exterior algebra to an isotropic subspace is now just the path integral for the Dirac action

$$\int [d\Psi] e^{\int_\Sigma \Psi \not{D} \Psi}$$

Other vectors in the exterior algebra come from evaluating the path integral with non-trivial operator insertions

$$\int [d\Psi] e^{\int_\Sigma \Psi \not{D} \Psi} \mathcal{O}$$

In higher dimensions one has the same formal structure (at least for the case of $M$ even dimensional, $\partial M$ odd dimensional): one can use the Dirac operator to polarize the space of spinor fields restricted to $\partial M$ and interpret the fermionic path integral as providing a construction of a bundle $Pf^*$ whose sections are the Hilbert space of the theory, including a canonically defined vacuum vector.

# 8 0+1 Dimensions: Representation Theory and Supersymmetric Quantum mechanics

Quantum mechanics is the simplest example of a quantum field theory, one with a zero-dimensional space and one-dimensional time. A path integral formulation of quantum mechanics will involve an integration over paths in some finite dimensional manifold $M$. A beautiful interpretation of supersymmetric quantum mechanical path integrals due to Witten [4] interprets them as an integration map in the $S^1$-equivariant cohomology of the loop space $LM$, where the $S^1$ action is just rotation along the loop. Cohomological orientability of the loop space $LM$ is equivalent to K-theory orientability of the manifold $M$.

In $S^1$-equivariant cohomology there is a localization principle that allows the calculation of integration maps using just data from the neighborhood of the fixed point set of the $S^1$ action. In the $LM$ case the fixed points are just the points of $M$ and the integration map reduces to an integration over $M$, formally giving another derivation of the cohomological formula for the index. This use of equivariant cohomology is not much related to representation theory. The result of the integration map is just a constant in $\mathbf{R}[[u]]$.

In addition, this case is rather different than that of higher dimensions. The space-time symmetry of this theory is dealt with by the $S^1$-equivariant cohomology, but in higher dimensions there is no such group action and the space-time symmetry should be handled by Clifford algebra methods.

If one picks $M = G/T$, for each irreducible representation of $G$ there is a supersymmetric quantum mechanics whose Hilbert space is just this representation. For further details about this and some references to other work, see [49].

# 9 1+1 Dimensions: Loop Group Representations and Two-Dimensional Quantum Field Theories

Two-dimensional quantum field theories involving gauge fields and fermions have been intensively studied over the last 20 years, partly due to the importance of certain such theories as building blocks of conformal field theories. Such conformal field theories can in principle be used to construct perturbative string theories and one might argue that they are the only part of string theory that is reasonably well-defined and well-understood. The Hilbert space of a two-dimensional quantum field theory depends on the fields on the boundary of the two-dimensional base space, a set of circles. Restricting attention to the case of one circle, the Hilbert space will be a representation of a loop group, the group of gauge transformations on the circle. A crucial part of understanding the quantum field theory is understanding how the Hilbert space decomposes into irreducible loop group representations.

There is a large literature on loop group representations, including the book [37]. The class of irreducible loop group representations that is both well understood and of physical significance is that of positive energy representations. These are all projective representations, characterized by a level $k$.

Some quantum field theories that have been studied in terms of loop group representations include:

- Wess-Zumino-Witten (WZW) models. Here the fields are maps from the two dimensional base space to $G$, or equivalently gauge transformations of a trivial bundle. The theory is characterized by a positive integer $k$. The Hilbert space decomposes in a similar way to $L^2(G)$ in the Peter-Weyl theorem. It is a sum of terms of the form

$$\sum_\alpha \overline{V}_\alpha \otimes V_\alpha$$

  where $\alpha$ is a finite set of labels of integrable $LG$ representations of level $k$. One fascinating aspect of these models is "non-abelian bosonization": they are equivalent to fermionic (anti-commuting) quantum field theories.

- $G/H$ "Coset" Models, or equivalently gauged Wess-Zumino-Witten models. Here the fields are gauge transformations of a bundle, as well as connections on the bundle. The Hilbert space of these models is related to that of the standard WZW model, but now one can pick out that part of the representation that is invariant under the subgroup $LH \subset LG$.
- Supersymmetric WZW and gauged WZW models. A supersymmetric two-dimensional quantum field theory has both anti-commuting and commuting fields. The Hilbert space of the theory now may be a complex.

Until recently there was little evidence in the case of these theories for the relevance of the abstract equivariant $K$-theory point of view advocated in the earlier part of this paper. This has changed with the announcement [22] of

**Theorem 8** (Freed-Hopkins-Teleman)**.** *For $G$ compact, simply-connected, simple, there is an equivalence of algebras*

$$V_k(G) \simeq K^{k+h(G)}_{G,dimG}(G)$$

*Here $V_k(G)$ is the Verlinde algebra of equivalence classes of level-$k$ positive energy representations of $LG$ with the fusion product. $K^{k+h(G)}_{G,dimG}(G)$ is the $(k+h(G))$-twisted equivariant $K$-homology of $G$ (in dimension $dimG$) with product induced from the multiplication map $G \times G \to G$. The $G$ action is the conjugation action, and $h(G)$ the dual Coxeter number of $G$.*

While this result is expressed as a statement purely about the finite dimensional compact group $G$, it is related to the loop group and its classifying space as follows. Consider the trivial $G$ bundle over the circle $S^1$. In this case the gauge group $\mathcal{G}$ is the loop group $LG$ and the space of connections $\mathcal{A}$ on the bundle has an $LG$ action. The subgroup $\Omega G$ of based loop group elements acts freely on $\mathcal{A}$. The quotient of $\mathcal{A}/\Omega G$ is just $G$, the group element giving the holonomy of the connection. The remaining $G$ action on the base point acts by conjugation on the holonomy. Assuming that equivariant $K-$theory can be made sense of for loop groups and that it behaves as expected for a free action, we expect

$$K_{\mathcal{G}}(\mathcal{A}) = K_G(\mathcal{A}/\Omega G) = K_G(G)$$

where the $G$ action on $G$ is by conjugation. Our discussion of the relationship of equivariant $K$-theory and representation theory leads us to hope for some sort of relation between the representation ring of $LG$ and this $K_G(G)$. The analog of the representation ring in this context is the Verlinde algebra $V_k(G)$ of level $k$ projective $LG$ representations, and the Freed-Hopkins-Teleman result identifies it with $K^{k+h(G)}_{G,\mathrm{dim}G}(G)$.

The Freed-Hopkins-Teleman theorem fundamentally tells us that in the case of one-dimensional base space, the appropriate representation theory of the gauge group (that of positive-energy representations) can be identified with the equivariant $K$-theory of the classifying space of the gauge group, although this requires using twisted $K$-theory since the representations are projective. We have seen that, for a general base space, quantum field theory path integrals may have an interpretation as calculations in the equivariant $K$-theory of the gauge group. The question of whether there is an extension of the Free-Hopkins-Teleman result to gauge groups in higher dimensions is one of potentially great significance for both mathematics and physics.

Segal has given[41] a map

$$V_k(G) \longrightarrow K^{k+h(G)}_{G,\mathrm{dim}G}(G)$$

that conjecturally is the Freed-Hopkins-Teleman isomorphism. He constructs this map by associating to a loop group representation $E$ the Fredholm complex

$$E \otimes \Lambda_{\frac{1}{2}\infty}(\mathcal{L}\mathfrak{g}^*) \stackrel{d+d^*}{\longrightarrow} E \otimes \Lambda_{\frac{1}{2}\infty}(\mathcal{L}\mathfrak{g}^*)$$

Here $\Lambda_{\frac{1}{2}\infty}L\mathfrak{g}^*$ are the so-called "semi-infinite" left-invariant differential forms on the loop group. He notes that this complex should be thought of as the Hilbert space of a supersymmetric Wess-Zumino-Witten model.

This is formally similar to the Kostant complex

$$V_\lambda \otimes S^+ \stackrel{\not{D}}{\longrightarrow} V_\lambda \otimes S^-$$

and recall that we have argued that

$$S^+ \stackrel{\not{D}}{\longrightarrow} S^-$$

can be thought of as a $K$-homology equivariant fundamental class. It may be possible to recast Segal's construction in the language of Clifford algebras and spinors. Note that the projective factor of the dual Coxeter number contributed by the semi-infinite differential forms is the analog in the loop group case of $\delta$ (half the sum of the positive roots) in the compact group case [21]. The Kostant complex in the loop group case has been studied by Landweber [33].

While Wess-Zumino-Witten models and their supersymmetric extensions have been much studied in the physics literature, much remains to be done to understand fully the relationship between representation theory and these models. The program to understand conformal field theory from representation-theoretical point of view begun in [39] still remains to be developed. In particular it would be most interesting to understand the structure of path integral calculations from this point of view. This might provide inspiration for how to generalize these ideas to the physical case of four dimensional space-time.

# 10 Speculative Remarks About the Standard Model

The motivating conjecture of this paper is that the quantum field theory underlying the Standard Model can be understood in terms of the representation theory of the automorphism group of some geometric structure. Furthermore we have argued that $K$-theory should be the appropriate abstract framework in which to look for these representations. The first question that arises is that of what the fundamental geometric structure should be. $K$-theory is well-known to have a periodicity in dimension of order 8. This is reflected topologically in the Bott periodicity of homotopy groups of Lie groups: for large enough $n$, $\pi_i(Spin(n)) = \pi_{i+8}(Spin(n))$. It is reflected algebraically in the periodicity of the structure of Clifford algebra modules: there is an equivalence between irreducible modules of $C(\mathbf{R}^n)$ and $C(\mathbf{R}^{n+8})$ (with the standard metric). This may indicate that one's fundamental variables can be taken to be geometrical structures on $\mathbf{R}^8$.

There is a long tradition of trying to use the rich structure of the Clifford algebras to classify the particles and symmetries of fundamental physics. Perhaps a $K$-theory point of view will allow this idea to be pursued in a more systematic way. One aspect of the special nature of eight dimensions is the rich geometry of $S^7$, the unit vectors in $\mathbf{R}^8$. The seven sphere has no less than four distinct geometries

- Real:
$$S^7 = Spin(8)/Spin(7)$$
- Octonionic:
$$S^7 = Spin(7)/G_2$$
($G_2$ is the automorphism group of the octonions $\mathbf{O}$).
- Complex:
$$S^7 = Spin(6)/SU(3) = SU(4)/SU(3)$$
- Quaternionic:
$$S^7 = Spin(5)/Sp(1) = Sp(2)/Sp(1) = Sp(2)/SU(2)$$

So by considering automorphisms of the seven-sphere one can naturally get gauge groups $Spin(7)$, $G_2$, $SU(3)$, and $SU(2)$. The last two are sufficient (together with overall phase transformations) for the known Standard Model symmetries. Note that these groups are significantly smaller than the favorite experimentally unobserved internal symmetry groups of GUTs and string theories ($SU(5)$, $SO(10)$, $E_6$, $SO(32)$, $E_8$, etc.).

If spacetime is to be four-dimensional, we are interested in geometrical structures that are bundles over a four-dimensional space. Continuing to use the geometry of the seven-sphere, perhaps one can combine one of the above geometries with the use of the fibration

$$\begin{tikzcd} S^3 \arrow[r] & S^7 \arrow[d] \\ & S^4 \end{tikzcd}$$

so we have another $S^3 = SU(2)$ internal symmetry to consider.

See [48] for an elaboration of some possible ideas about how this geometry is related to the standard model. There it is argued that the standard model should be defined over a Euclidean signature four dimensional space time since even the simplest free quantum field theory path integral is ill-defined in a Minkowski signature. If one chooses a complex structure at each point in space-time, one picks out a $U(2) \subset SO(4)$ (perhaps better thought of as a $U(2) \subset Spin^c(4)$) and in [48] it is argued that one can consistently think of this as an internal symmetry. Now recall our construction of the spin representation for $Spin(2n)$ as $\Lambda^*(\overline{\mathbf{C}^n})$ applied to a "vacuum" vector. Under $U(2)$, the spin representation has the quantum numbers of a standard model generation of leptons

| $\Lambda^*(\overline{\mathbf{C}^2})$ | $SU(2) \times U(1)$ Charges | Particles |
|---|---|---|
| $\Lambda^0(\overline{\mathbf{C}^2}) = \mathbf{1}$ | $(0,0)$ | $\nu_R$ |
| $\Lambda^1(\overline{\mathbf{C}^2}) = \mathbf{C}^2$ | $(\frac{1}{2},-1)$ | $\nu_L, \quad e_L$ |
| $\Lambda^2(\overline{\mathbf{C}^2})$ | $(0,-2)$ | $e_R$ |

A generation of quarks has the same transformation properties except that one has to take the "vacuum" vector to transform under the $U(1)$ with charge $4/3$, which is the charge that makes the overall average $U(1)$ charge of a generation of leptons and quarks to be zero.

The above comments are exceedingly speculative and very far from what one needs to construct a consistent theory. They are just meant to indicate how the most basic geometry of spinors and Clifford algebras in low dimensions is rich enough to encompass the standard model and seems to be naturally reflected in the electro-weak symmetry properties of Standard Model particles.

# 11 On the Current State of Particle Theory

This article has attempted to present some fragmentary ideas relating representation theory and quantum field theory in the hope that they may lead to new ways of thinking about quantum field theory and particle physics and ultimately to progress in going beyond the standard model of particle physics. Some comments about the current state of particle theory and its problems [50, 23] may be in order since these problems are not well known to mathematicians and their severity provides some justification for the highly speculative nature of much of what has been presented here.

For the last eighteen years particle theory has been dominated by a single approach to the unification of the standard model interactions and quantum gravity. This line of thought has hardened into a new orthodoxy that postulates an unknown fundamental supersymmetric theory involving strings and other degrees of freedom with characteristic scale around the Planck length. By some unknown mechanism, the vacuum state of this theory is supposed to be such that low-energy excitations are those of a supersymmetric Grand Unified Theory (GUT) including supergravity. By another unknown mechanism, at even lower energies the vacuum state is supposed to spontaneously break the GUT symmetries down to those of the Standard Model and, again in some unknown way, break the supersymmetry of the theory.

It is a striking fact that there is absolutely no evidence whatsoever for this complex and unattractive conjectural theory. There is not even a serious proposal for what the dynamics of the fundamental "M-theory" is supposed to be or any reason at all to believe that its dynamics would produce a vacuum state with the desired properties. The sole argument generally given to justify this picture of the world is that perturbative string theories have a massless spin two mode and thus *could* provide an explanation of gravity, *if* one ever managed to find an underlying theory for which perturbative string theory is the perturbation expansion. This whole situation is reminiscent of what happened in particle theory during the 1960's, when quantum field theory was largely abandoned in favor of what was a precursor of string theory. The discovery of asymptotic freedom in 1973 brought an end to that version of the string enterprise and it seems likely that history will repeat itself when sooner or later some way will be found to understand the gravitational degrees of freedom within quantum field theory.

While the difficulties one runs into in trying to quantize gravity in the standard way are well-known, there is certainly nothing like a no-go theorem indicating that it is impossible to find a quantum field theory that has a sensible short distance limit and whose effective action for the metric degrees of freedom is dominated by the Einstein action in the low energy limit. Since the advent of string theory, there has been relatively little work on this problem, partly because it is unclear what the use would be of a consistent quantum field theory of gravity that treats the gravitational degrees of freedom in a completely independent way from the standard model degrees of freedom. One motivation for the ideas discussed here is that they may show how to think of the standard model gauge symmetries and the geometry of space-time within one geometrical framework.

Besides string theory, the other part of the standard orthodoxy of the last two decades has been the concept of a supersymmetric quantum field theory. Such theories have the huge virtue with respect to string theory of being relatively well-defined and capable of making some predictions. The problem is that their most characteristic predictions are in violent disagreement with experiment. Not a single experimentally observed particle shows any evidence of the existence of its "superpartner". One can try and explain this away by claiming that an unknown mechanism for breaking the supersymmetry of the vacuum state exists and is precisely such that all superpartners happen to have uncalculable masses too large to have been observed. If one believes this, one is faced with the problem that the vacuum energy should then be of at least the scale of the supersymmetry breaking. Assuming that one's theory is also supposed to be a theory of gravity, there seems to be no way around the prediction that the universe will be a lot smaller than the size of a proton.

Supersymmetry has a complicated relationship with modern mathematics. The general formalism one gets by naively replacing vector spaces by "super vector spaces", groups by "supergroups", etc. produces new structures but does not obviously lead to much new insight into older mathematics. On the other hand, there certainly are crucial parts of the mathematics discussed in this article that fit to a degree into the "super" language. A prime example is seen in the importance of the $\mathbf{Z_2}$ grading of Clifford algebras and its implications for $K$-theory and index theory. In addition, the complexes of equivariant cohomology are naturally $\mathbf{Z_2}$ graded. Whenever one works with a de Rham model of equivariant cohomology one has something like a supersymmetry operator since the Lie derivative is a square

$$\mathcal{L}_X = di_X + i_X d = (d + i_X)^2$$

Perhaps taking into account some of these other mathematical ideas can lead to new insights into which supersymmetric quantum field theories are actually geometrically interesting and help to find new forms of them which actually will have something to do with the real world.

During the past twenty-five years particle physics has been a victim of its own success. The standard model has done an excellent job of explaining all phenomena seen up to the highest energies that can be reached by present-day accelerators. The advent of the LHC at CERN starting in 2007 may change this situation but this cannot be counted on. While historically the attempt to make progress in theoretical physics by pursuing mathematical elegance in the absence of experimental guidance has had few successes (general relativity being a notable exception), we may now not have any choice in the matter.

The exploitation of symmetry principles has lead to much of the progress in theoretical physics made during the past century. Representation theory is the central mathematical tool here and in various forms it has also been crucial to much of twentieth century mathematics. The striking lack of any underlying symmetry principle for string/M-theory is matched by the theory's complete inability to make any predictions about nature. This is probably not a coincidence.

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