Ginsparg-Wilson Relation, Topological Invariants and Finite Noncommutative Geometry

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We show that the Ginsparg-Wilson (GW) relation can play an important role to define chiral structures in *finite* noncommutative geometries. Employing GW relation, we can prove the index theorem and construct topological invariants even if the system has only finite degrees of freedom. As an example, we consider a gauge theory on a fuzzy two-sphere and give an explicit construction of a noncommutative analog of the GW relation, chirality operator and the index theorem. The topological invariant is shown to coincide with the 1st Chern class in the commutative limit.

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Introduction Quantization of space-time is one of the unsolved issues and lots of attempts have been made. Superstring theory has succeeded in quantizing small fluctuations of the metric, but only little knowledge has been obtained for the space-time itself. Discovery of D-branes has shown us a possibility that space-time coordinates can be noncommutative[1]. Based on this, various matrix models have been proposed as nonperturbative formulations of superstrings[2–4]. Furthermore, noncommutative geometries appear naturally in superstring theories with $B_{\mu\nu}$ background[5] or in the matrix models[6, 7]. These studies have linked the string theory to the old ideas that the concept of the space-time must be modified at the Planck length[8, 9]. In order to understand the origin of our four dimensional space-time[10], we need to analyze superstring numerically, and consider noncommutative geometries with *finite* degrees of freedom.

Another issue is the unification of space-time and the matter. In particular, chiral structures of the fermions may be related to the quantization of space-time as we see in Penrose's twister theory or Connes's noncommutative geometry. If we believe such connection, it is important to construct chiral structures in systems with finite degrees of freedom. This problem is also relevant to a construction of four-dimensional chiral fermions in matrix models. Orbifolding[11] or compactifications in matrix models with a nontrivial index can be candidates to define chiral structures in finite noncommutative geometries. Instantons have been constructed on noncommutative R^4 by using the shift operator[12, 13], whose construction essentially makes use of the infinite dimensionality of the Hilbert space. In finite noncommutative theories, topologically nontrivial configurations have been constructed based on algebraic K-theory and projective modules[8, 14]. It seems difficult, however, to apply these ideas to finite square matrix models such as [3].

In this letter, we propose a new chiral structure in finite noncommutative geometries. We define a chirality and a Dirac operator satisfying the Ginsparg-Wilson (GW) relation, which has been studied in the lattice gauge theory. By using these operators, we can define topological invariants and prove an index theorem for general gauge field configurations, which are constructed from finite square matrices. We first develop a general formalism and then give an explicit example in the case of noncommutative (fuzzy) 2-sphere.

General Formalism In Connes's formulation of noncommutative geometry, a chirality operator and a Dirac operator which anti-commute are introduced. In this letter, we propose to generalize this algebraic relation to the GW relation so that we can define chiral structures in a finite system representing a compact noncommutative manifold.

GW relation has been studied to understand the chiral structure in the lattice gauge theory with a finite number of degrees of freedom. It is expressed as

$$D\Gamma_{n+1} + \Gamma_{n+1}D = aD\Gamma_{n+1}D, \tag{1}$$

where D, Γ_{n+1} and a are an n dimensional lattice Dirac operator, a chirality operator and a lattice spacing respectively. This relation expresses the remnant chiral symmetry on the lattice, which was derived by the block-spin transformation from a continuum theory[15]. This algebraic relation makes it possible to define an extended chiral symmetry on the lattice[16, 17]. The index theorem is also realized on the lattice[16, 18]. Based on the idea of the domain wall fermion, a practical solution to the GW relation was obtained[19]. Chiral gauge theory can be constructed only for an abelian case[20]. Higher dimensional generalization of the domain wall fermion was proposed toward constructing nonabelian chiral gauge theories[21].

Now we apply the algebraic structure of the GW relation to finite noncommutative geometries. In refs.[22] GW relation was pointed out for free fermions on the fuzzy 2-sphere. It was also utilized in unitary matrix models to construct chiral gauge theories on torus[23]. Our proposal here is that the GW relation can be substituted for the simple anticommutation relation, to define chiral structures for *general gauge fields* in finite noncommutative theories.

In order to develop a general formalism, we consider a system that are composed of $N \times N$ matrices. $1/N$ plays a role of the lattice spacing a in the lattice gauge theory. We assume an existence of a chirality operator γ and a Dirac operator in the commutative limit, where N goes to infinity (or a goes to zero). We then introduce two chirality operators Γ and $\hat{\Gamma}$ and a Dirac operator D in our finite system. They are operators acting on $N \times N$ matrices. The Dirac operator D generally depends on background gauge field configurations. Both chirality operators are required to become the chirality operator γ in the commutative limit and satisfy the relations:

$$\Gamma^2 = \hat{\Gamma}^2 = 1, \quad \Gamma^\dagger = \Gamma, \ \hat{\Gamma}^\dagger = \hat{\Gamma}. \tag{2}$$

Γ is assumed to be independent of the gauge field configurations. If the system has chiral anomaly in the commutative limit, we cannot expect that the Dirac operator D anti-commutes with the chirality operator Γ, since, if so, it will contradict with the result in the commutative limit. Hence, in the finite noncommutative geometry, a simple algebraic structure is destroyed and we cannot prove an index theorem only with the simple chirality operator Γ.

The other chirality operator $\hat{\Gamma}$ can be constructed in terms of a hermitian operator H as

$$\hat{\Gamma} \equiv \frac{H}{\sqrt{H^2}}, \ H^\dagger = H. \tag{3}$$

$\hat{\Gamma}$ depends on the gauge field configuration through H.

If we take an appropriate choice of H, we can define a new Dirac operator (GW Dirac operator) D_{GW} by

$$1 - \Gamma\hat{\Gamma} = f(a, \Gamma) D_{GW}. \tag{4}$$

The prefactor f is assumed to be a function of the small parameter a and the chirality operator Γ. These f and H are determined so that the Dirac operator D_{GW} becomes the commutative Dirac operator in the limit $a \to 0$. From (2) and (4), we have the relation

$$\Gamma D_{GW} + D_{GW}\hat{\Gamma} = 0, \tag{5}$$

and then we can prove the following index theorem:

$$2 \ \text{index} D_{GW} \equiv 2(n_+ - n_-) = \mathcal{T}r(\Gamma + \hat{\Gamma}), \tag{6}$$

where $\mathcal{T}r$ is a trace of operators acting on matrices, and $n_\pm$ are numbers of zero eigenstates of D_{GW} with a positive (or negative) chirality (for either Γ or $\hat{\Gamma}$).

It is easy to prove that this index is invariant under a small deformation of any parameter or any configuration (such as gauge field) in the operator H. A necessary check is that this index can take a nontrivial value depending on the background configuration. For some choices of H, this index may become a constant and trivial. For other choices, it gives a nontrivial index as we will see in an example on fuzzy 2-sphere. This is surprising since it seems impossible to define such a nontrivial index out of finite matrices. The resolution lies in the definition of the operator $\hat{\Gamma}$ in (3). It becomes singular when the operator H has a zero-mode. It is a sign function of the operator H, and when an eigenvalue of H crosses zero the index changes by two. Therefore, if there is no symmetry among eigenvalues, the index can take nontrivial values. The configuration space of gauge fields are divided into islands, in each of which the index takes a different value. In the lattice gauge theory, we can exclude a region where H has zero eigenvalues by imposing an admissibility condition for the gauge field. We expect a similar condition for our case.

Another necessary check is to reproduce the well-known index theorem in the commutative limit. Namely, the topological invariant considered here should give an instanton number for nontrivial gauge field configurations. In the case of fuzzy 2-sphere, we will show that the index reproduces the monopole number.

Regarding this commutative limit, it is amusing to point out that we will be able to arrive at different commutative limits with different types of topological invariants though we start from the same size of matrices. Namely, topological invariants in different space-time dimensions can be obtained from the same matrices. Different classical interpretations come from different choices of chirality operators and Dirac operators, which lead to different embeddings of the commutative configurations in the matrices. In refs.[24, 25] they gave simple examples of the embeddings of classical configurations in matrices and obtained classical indices on the lattice. Since an important property of the noncommutative theories is that it contains much more degrees of freedom than ordinary field theories, it is interesting if we can classify the universality classes of more general embeddings in the matrices. We want to discuss the problem in the future[26].

Chiral Transformation We can construct a chiral invariant model by using the Dirac operator D_{GW} as

$$S = \text{t}r(\bar{\Psi} D_{GW} \Psi). \tag{7}$$

Each component of Ψ is an $N \times N$ matrix and tr means a trace over matrices. This action is invariant under the global chiral transformation: $\delta\Psi = i\hat{\Gamma}\Psi, \delta\bar{\Psi} = i\bar{\Psi}\Gamma$. To make it local, we need to specify the ordering of the chiral transformation parameter λ and the fermion field.

The fermion transforms as the fundamental representation under gauge transformations:

$$\Psi \to U\Psi,\ \bar{\Psi} \to \bar{\Psi}U^\dagger. \tag{8}$$

D_{GW} must be constructed to maintain the gauge invariance. Namely we require the transformation, $D_{GW}\Psi \to UD_{GW}\Psi$ under gauge transformations. Then, if we put λ in the left of Ψ, λ should transform covariantly as $\lambda \to U\lambda U^\dagger$ and so does the chiral current. On the contrary, if we put λ in the right, λ and the associated chiral current is invariant under gauge transformations. This ambiguity is specific to the noncommutative field theories and makes the analysis of the Ward-Takahashi(WT) identity complicated. This issue was discussed in refs.[27, 28]. In this letter, we don't go further into this problem and restrict our discussion to the covariant case.

In obtaining the WT identity for the covariant chiral current, the integral measure of the fermion fields is not invariant under the local chiral transformations,

$$\delta\Psi = i\lambda\hat{\Gamma}\Psi,\quad \delta\bar{\Psi} = i\bar{\Psi}\lambda\Gamma, \tag{9}$$

and produces a nontrivial Jacobian. This term gives an integral of the topological charge density (an anomaly term) with the local weight λ in the WT identity:

$$q(\lambda) = \frac{1}{2}\mathcal{T}r(\lambda^L\hat{\Gamma} + \lambda^R\Gamma') = \frac{1}{2}\mathcal{T}r(\lambda^L\hat{\Gamma} + \lambda^L\Gamma). \tag{10}$$

Here the superscript L (R) in λ means that this operator acts from the left (right) on matrices. Γ' is obtained from Γ by exchanging L and R superscript. For a global chiral transformation, we set $\lambda = 1$ and $q(\lambda)$ becomes the index defined in (6).

Examples on Fuzzy 2-Sphere We now consider a simple example on fuzzy 2-sphere. Noncommutative coordinates of the fuzzy 2-sphere is given by $x_i = \alpha L_i$, where L_i's are $2L+1$-dimensional irreducible representation matrices of $SU(2)$ algebra. The radius of the sphere is given by $\rho = \alpha\sqrt{L(L+1)}$. Wave functions on fuzzy 2-sphere can be expanded in terms of noncommutative analogs of the spherical harmonics $\hat{Y}_{lm}$. They are traceless symmetric products of the noncommutative coordinates. There is an upper bound for the angular momentum l for $\hat{Y}_{l,m}$; $l \le 2L$. Any hermitian matrix M can be expanded in terms of these spherical harmonics $\hat{Y}_{l,m}$.

Killing vectors on the fuzzy 2-sphere are expressed by taking a commutator with the $SU(2)$ generator; $\mathcal{L}_i M = [L_i, M] = (L_i^L - L_i^R)M$. An integral over 2-sphere is replaced by taking a trace over matrices $\frac{1}{2L+1}\mathrm{tr} \leftrightarrow \int \frac{d\Omega}{4\pi}$. More detailed correspondence are found in refs.[28, 29].

Now we introduce a Dirac operator D and a chirality operator Γ^R as follows:

$$D = \sigma_i(\mathcal{L}_i + \rho a_i^L) + 1, \tag{11}$$

$$\Gamma^R = \frac{1}{2L+1}(2\sigma_i L_i^R - 1). \tag{12}$$

The free part of D contains no fermion doublers[30]. This Γ^R satisfies the conditions (2). In the commutative limit, it reduces to an ordinary chiral operator on the sphere $\gamma = \sigma_i x_i/\rho$. An anticommutator with the Dirac operator (11) becomes

$$\{\Gamma^R, D\} = \frac{1}{L+1/2}\left(2(\mathcal{L}_i + \rho a_i^L)L_i^R - D\right). \tag{13}$$

Here $\mathcal{L}_i L_i^R$ and $D/(L+1/2)$ vanish in the commutative limit since they are of order $1/L$. Hence the anticommutator becomes proportional to the scalar field:

$$\{\Gamma^R, D\} \to 2a_i x_i = 2\rho\phi. \tag{14}$$

Note that since we embed the 2-sphere in three dimensional flat space, the normal component of the gauge field a_i is interpreted as a scalar field on 2-sphere.

We then introduce the other chirality operator $\hat{\Gamma}$ as in (3). We construct H out of Γ_R and D because these operators commute with the gauge transformations (8). If H contains D less than 2, it becomes

$$H = \Gamma^R + c_1 D + ic_2[D,\Gamma^R] + c_3\left\{\Gamma^R, D\right\} + c_4\Gamma^R D\Gamma^R + c_5. \tag{15}$$

The coefficients c_i are real and of order $1/L$ so that $\hat{\Gamma}$ becomes γ in the commutative limit. Then we can show that, if we define D_{GW} as in (4) with $f = (c_4 - c_1)\Gamma^R + 2ic_2$, it becomes $D_{GW} = (D - \{\Gamma^R, D\}\Gamma^R/2) + \mathcal{O}(1/L)$. It is nothing but the Dirac operator whose coupling to the scalar is projected out, as expected from the GW relation (5).

In the following, we take a simpler H,

$$H = \Gamma^R + aD = \Gamma^L + a\rho\sigma_i a_i^L, \tag{16}$$

where $a = 1/(L+1/2)$ and we defined Γ^L as $\Gamma^L = (2\sigma_i L_i^L + 1)/(2L+1)$. This operator also satisfies $(\Gamma^L)^2 = 1$ and $(\Gamma^L)^\dagger = \Gamma^L$. Since both of Γ^L and a_i^L are operators acting on matrices from the left, this H also acts only from the left. Then the topological charge density (10) becomes

$$q(\lambda) = \frac{1}{2}\mathrm{tr}(1)\mathrm{T}r(\lambda\hat{\Gamma}) + \frac{1}{2}\mathrm{tr}(\lambda)\mathrm{T}r(\Gamma^R). \tag{17}$$

$\mathrm{T}r$ is a trace over matrices and spinors, and Γ^R and $\hat{\Gamma}$ are considered here as mere matrices instead of operators acting on matrices from the right and from the left respectively. If there is no background gauge field, $\hat{\Gamma} = \Gamma^L$ and the topological charge density vanishes. If we assume that the gauge field configuration is weak, we can expand the chirality operator $\hat{\Gamma}$ with respect to the gauge field a_i. Up to the first order in a_i, after taking a trace over σ matrices, we obtain

$$q(\lambda) = -\frac{a^3\rho(2L+1)}{2\alpha}\mathrm{t}r\lambda\left(i\epsilon_{ijk}[L_i, a_j]x_k + 2\rho\phi + \frac{\alpha}{2}[L_i, a_i]\right). \tag{18}$$

In the commutative limit, this becomes

$$q(\lambda) = 2\rho \int \frac{d\Omega}{4\pi} \lambda \epsilon_{ijk} x_i \partial_j a'_k \tag{19}$$

where a'_i is a tangential component of a_i; $a'_i = \epsilon_{ijk} x_j a_k/\rho$. This topological charge density is nothing but the monopole charge density on the 2-sphere. For $\lambda = 1$, this index is quantized to be integers. If we embed classical gauge field configurations with nontrivial monopole charge, our index gives a nontrivial value.

Discussions In this letter we have proposed to use Ginsparg-Wilson relation to formulate the chiral and topological structures in the finite noncommutative geometry, that is, in matrix models. There are still several problems which are listed below.

First the fermion takes a matrix form and it contains N^2 degrees of freedom. Since the number of lattice points is considered to be of order N, the fermion contains much larger degrees of freedom than the ordinary field theories. This is also true for the gauge field configuration. These larger degrees of freedom have been interpreted as an indicator that the noncommutative field theory is related to string theory[31]. Furthermore, it has been expected that space-time is also dynamical in the noncommutative Yang-Mills theories. In the commutative limit, we restrict wave functions to those with momentum smaller than the noncommutativity scale and then the degrees of freedom become of order N. Classical interpretations will be available only for this limiting cases. Different embeddings of classical configurations lead to different classical theories. From this argument, our index is expected to classify even the global topology of space-time. For this purpose, it is important to investigate more examples on higher dimensional noncommutative geometries.

Another problem is the relation to other arguments on the topological properties of the noncommutative field theories[8, 13, 14]. Topologically nontrivial configurations are constructed by projection operators. It will be interesting to investigate relations to them.

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