Isotropic representation of noncommutative $2D$ harmonic oscillator

Anais Smailagic*

INFN, Sezione di Trieste

Euro Spallucci†

Dipartimento di Fisica Teorica, Università di Trieste and INFN, Sezione di Trieste

(Dated: April 25, 2020)

Abstract

We show that $2D$ noncommutative harmonic oscillator has an isotropic representation in terms of commutative coordinates. The noncommutativity in the new mode, induces energy level splitting, and is equivalent to an external magnetic field effect. The equivalence of the spectra of the isotropic and anisotropic representation is traced back to the existence of $SU(2)$ invariance of the noncommutative model.

*Electronic address: `anais@ictp.trieste.it`

†Electronic address: `spallucci@trieste.infn.it`

I. INTRODUCTION

Recent results obtained in the framework of non-perturbative string theory [1],[2], have boosted the interest for a deeper understanding of the role played by noncommutative geometry in different sectors of theoretical physics [3]. Inclusion of noncommutativity in quantum field theory can be achieved in two different ways: via Moyal $*$-product on the space of ordinary functions, or defining the field theory on a coordinate operator space which is intrinsically noncommutative [4], [5]. The equivalence between the two approaches has been nicely described in [6]. While formally well defined, the operator approach is hard to implement in explicit calculations. The analysis of the noncommutative effects is usually performed expanding Moyal $*$-product perturbatively, and take into account additional vertices. In order to get deeper understanding of the way in which noncommutativity affects quantum field theory one tries to understand these effects firstly in exactly solvable models of Noncommutative Quantum Mechanics [7].

The difficulty of performing explicit calculation encountered in operator space formulation of quantum field theory corresponds, in quantum mechanics, to the problem of formulating a Schroedinger equation directly in terms of noncommutative coordinates. The path to follow is to introduce the noncommutativity of coordinates and momenta through the Moyal $*$-product [9]. It turns out that the effect of introducing the $*$-product can be described by suitable shifts of the argument of the wavefunction [10], or of the Hamiltonian [11]. In order to properly treat the noncommutative variables one needs two commuting Heisenberg algebras [8], [12].

In this paper we shall follow an approach where the set of noncommutative coordinates x_i , p_i is expressed as a linear combination of canonical variables of quantum mechanics α_i , β_i. As we shall see, the noncommutativity will manifest itself through redefinitions of various parameters and will produce additional terms in the Hamiltonian of the equivalent commutative description.

As an explicit example we shall study the case of a $2D$ noncommutative harmonic oscillator. The main result of our work is the description of the noncommutative system in terms of *new* set of transformations among noncommutative and canonical variables, that we shall name the "*isotropic representation*". In this mode, the noncommutative $2D$ harmonic oscillator receives a simple and clear physical interpretation. This representation also exhibits the rotational symmetry and leads, in a simple way, to the form of the generator of rotations for the noncommutative representation. Finally, we shall explain the equivalence of the spectra in two different representations in terms of an $SU(2)$ symmetry.

II. CANONICAL COORDINATES

In order to illustrate the general procedure we start with the set of coordinates and momenta satisfying extended commutators as [14]

$$[\, x_k \;, x_j \,] = i\, \Theta_{kj} \tag{1}$$

$$[\, p_k \;, p_j \,] = i\, B_{kj} \tag{2}$$

$$\left[\, x^k \;, p_j \,\right] = i\, \delta^k{}_j \tag{3}$$

with Θ_{kj} and B_{kj} antisymmetric matrices characterizing generalized noncommutativity of the phase space geometry.

We are going to define linear transformations from the noncommutative set of coordinates $(\, x_i \;, p_i \,)$ to a *commutative* set of canonically conjugate coordinates $(\, \alpha_i \;, \beta_i \,)$ which obey

$$[\, \alpha_k \;, \alpha_j \,] = 0 \tag{4}$$

$$[\, \beta_k \;, \beta_i \,] = 0 \tag{5}$$

$$[\, \alpha_k \;, \beta_i \,] = i\, \delta^k{}_j \tag{6}$$

The relation of noncommutative coordinates to conjugate ones is given by

$$x_i = a_{\,ij}\, \alpha_j + b_{\,ij}\, \beta_j \tag{7}$$

$$p_i = c_{\,ij}\, \beta_j + d_{\,ij}\, \alpha_j \tag{8}$$

where, $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ are $N \times N$ transformation matrices. Before going into details of a particular model one needs to determine the conditions that the transformation matrices should satisfy. These conditions are obtained by the requirement of preserving the commutation relations (1),(2),(3) and (4), (5), (6). The resulting conditions are written in matrix form as

$$\mathbf{a}\,\mathbf{b}^T - \mathbf{b}\,\mathbf{a}^T = \mathbf{\Theta} \tag{9}$$

$$\mathbf{c}\,\mathbf{d}^T - \mathbf{d}\,\mathbf{c}^T = -\mathbf{B} \tag{10}$$

$$\mathbf{c}\,\mathbf{a}^T - \mathbf{b}\,\mathbf{d}^T = \mathbf{I} \tag{11}$$

$\mathbf{\Theta}$ and $\mathbf{B}$ are anti-symmetric matrices. Equations (9), (10), (11) determine the structure of the transformation matrices. In order to illustrate in detail how the above procedure works, let us apply it in two dimensional space. As a specific model we choose a noncommutative harmonic oscillator described by the Hamiltonian

$$H \equiv \frac{1}{2}\left[\,(\,p_i\,)^2 + (\,x_i\,)^2\,\right] \tag{12}$$

For simplicity we have chosen the oscillator mass and frequency to be unity.

Inserting (7), (8) in the Hamiltonian (12) one finds its equivalent commutative form as

$$\begin{aligned} H \equiv\; & \frac{1}{2}\,(\,a_{ik}\,a_{im} + d_{ik}\,d_{im}\,)\;\alpha_k\,\alpha_m + \frac{1}{2}\,(\,c_{k\,i}\,c_{im} + b_{k\,i}\,b_{im}\,)\;\beta_k\,\beta_m \\ +\; & \frac{1}{2}\,(\,a_{ik}\,b_{im} + c_{im}\,d_{ik}\,)\;(\,\alpha_k\,\beta_m + \beta_m\,\alpha_k\,) \end{aligned} \tag{13}$$

Notice the appearance of the mixed term in conjugate coordinates and momenta which is induced by noncommutativity of the original system. (9), (10), (11) give six conditions for determining matrices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$. On the other hand, the number of elements of the transformation matrices is sixteen. $2D$ representation for $\mathbf{\Theta}$ and $\mathbf{B}$ is

$$\Theta_{ij} \equiv \theta\,\epsilon_{ij}\;, \qquad B_{ij} \equiv B\,\epsilon_{ij} \tag{14}$$

In order to solve the system one has to match the number of parameters with the number of equations. One way to achieve this, is to assume diagonality of $\mathbf{a}$ and $\mathbf{c}$ as $a_{ij} \equiv a_{(i)}\,\delta_{ij}$, $c_{ij} \equiv c_{(i)}\,\delta_{ij}$. With these assumptions, equation (11) imposes that diagonal elements of matrices $\mathbf{b}$ and $\mathbf{d}$ must be zero. Thus, we are left with 8 unknown parameters and six equations. More equations are needed. Additional equations can be obtained by requiring that the mixed term in (13) be zero. This leads to

$$\mathbf{a}^T\,\mathbf{b}+\mathbf{d}^T\,\mathbf{c}=0 \tag{15}$$

which gives two more equations that we need. The solutions of the complete set of equations can be sought by the following ansatz for **b** and **d**

$$b_{1\,2}=t\,a_{2\,2} \tag{16}$$

$$b_{2\,1}=n\,a_{1\,1} \tag{17}$$

$$d_{1\,2}=p\,c_{2\,2} \tag{18}$$

$$d_{2\,1}=q\,c_{1\,1} \tag{19}$$

where, we have introduced numbers t ,n ,p ,q that can take values ± 1 for the reason that will be clarified later on. Equation (11) imposes condition on these numbers as

$$t\,p=-1 \tag{20}$$

$$n\,q=-1 \tag{21}$$

The complete set of solutions turns out to be

$$c_{1\,1}=\frac{1}{\theta}\left(n\,a_{1\,1}+a_{2\,2}\sqrt{n\,t\,\kappa}\,\right) \tag{22}$$

$$c_{2\,2}=-\frac{1}{\theta}\left(t\,a_{2\,2}+a_{1\,1}\sqrt{n\,t\,\kappa}\right) \tag{23}$$

$$a_{1\,1}^2=\frac{\theta}{2\,n}\left[1+\frac{1}{\sqrt{1-4n\,t\,A^2}}\right] \tag{24}$$

$$a_{2\,2}^2=\frac{\theta}{2t}\left[-1+\frac{1}{\sqrt{1-4n\,t\,A^2}}\right] \tag{25}$$

$$A\equiv-\frac{\sqrt{n\,t\,\kappa}}{n\,t\,(\,1+\kappa+\theta^2)} \tag{26}$$

$$\kappa\equiv 1-B\,\theta \tag{27}$$

Inserting (22), (23), (24), (25) in the Hamiltonian (13) one finds

$$H=\frac{1}{2}\,\Omega_1\,\left[\,(\alpha_1)^2+(\beta_1)^2\,\right]+\frac{1}{2}\,\Omega_2\,\left[\,(\alpha_2)^2+(\beta_2)^2\,\right] \tag{28}$$

$$\Omega_1 \equiv \frac{1}{2\,n}\left[\theta + B + \,n\,t\,\sqrt{4+(\theta-B)^2}\,\right] \tag{29}$$

$$\Omega_2 \equiv -\frac{1}{2\,t}\left[\,\theta + B - n\,t\,\sqrt{4+(\theta-B)^2}\,\right] \tag{30}$$

$$\tag{31}$$

The Hamiltonian (28) is the representation of a noncommutative $2D$ harmonic oscillator in terms of two $1D$ commutative, anisotropic, harmonic oscillators. We shall name this description "anisotropic representation". The above result in terms of parameters $n\ ,t\ ,p\ ,q$ permits to consider the complete range of values of the noncommutative parameter B, assuming $\theta > 0$. In fact, the square root in (22) requires $n\,t\,(\,1 - B\,\theta\,) > 0$, which leads to two different ranges: one where $n = t = 1$, $B < 1/\theta$, and the other where $n = -t = 1$, $B > 1/\theta$. Our result agrees with [8] where the two different regions are described as $\kappa > 0$ and $\kappa < 0$. At this point, we shall prove the existence of anther set of solutions for equations (9), (10), (11) which give particularly nice representation of the $2D$ noncommutative harmonic oscillator in terms of an isotropic oscillator. In this representation the noncommutative effect have a simple and clear physical interpretation. As we have already shown, the complete solution for the transformation matrices can be sought starting with four independent parameters. The anisotropic representation is produced by imposing diagonality of $\mathbf{a}$ and $\mathbf{c}$ together with relations (16), (17), (18), (19). Let us choose a different approach where the matrices $\mathbf{a}$ and $\mathbf{c}$ are kept *diagonal*, but with *single* eigenvalues a and c. The requirement of eigenvalue degeneracy reduces the number of free parameters by half. In order to maintain unaltered the total number of free parameters matrices $\mathbf{b}$ and $\mathbf{d}$, will be chosen *anti-symmetric*:

$$a_{ij} \equiv a\,\delta_{ij}\ , \qquad c_{ij} \equiv c\,\delta_{ij} \tag{32}$$

$$b_{ij} \equiv b\,\epsilon_{ij}\ , \qquad d_{ij} \equiv d\,\epsilon_{ij} \tag{33}$$

Equations (9), (10), (11) reduce to the following conditions

$$a\,b = -\frac{\theta}{2} \tag{34}$$

$$c\,d = \frac{B}{2} \tag{35}$$

$$a\,c - b\,d = 1 \tag{36}$$

The set of equations (34), (35), (36) enables to solve three out of four parameters as

$$b = -\frac{\theta}{2a} \tag{37}$$

$$c = \frac{1}{2a}\left(1 \pm \sqrt{\kappa}\right) \ , \qquad \kappa \equiv 1 - \theta\, B \tag{38}$$

$$d = \frac{a}{\theta}\left(1 \mp \sqrt{\kappa}\right) \tag{39}$$

The above solutions turn equation (11) into

$$(\,\theta + B\,) = 0 \tag{40}$$

Equation (11) cannot be used to determine the remaining parameter a, as it was the case in the anisotropic representation. At most it can impose relation between parameters B and θ. Our intention is to work in full generality and, therefore, we shall assume $\theta + B \neq 0$. Thus, we shall keep the mixed term in the Hamiltonian (13):

$$H = h_1\,(\,\alpha_i\,)^2 + h_2\,(\,\beta_i\,)^2 - \frac{\theta + B}{2}\,\epsilon_{ij}\,\alpha_i\,\beta_j \tag{41}$$

$$h_1 \equiv \frac{a^2}{2}\left[1 + \frac{1}{\theta^2}\left(1 \mp \sqrt{\kappa}\right)^2\right] \tag{42}$$

$$h_2 \equiv \frac{\theta^2}{8a^2}\left[1 + \frac{1}{\theta^2}\left(1 \pm \sqrt{\kappa}\right)^2\right] \tag{43}$$

One can recognize (41) as the Hamiltonian for the commutative, isotropic, $2D$ harmonic oscillator with *additional* term proportional to the two-dimensional angular momentum $L = \epsilon_{ij}\,\alpha_i\,\beta_j$.Thus, we shall name this representation of the noncommutative $2D$ harmonic oscillator "isotropic representation". The term linear in the angular momentum is the reminiscence of the noncommutativity and, thus, it is important for understanding noncommutative effects. Similar term, in Quantum Mechanics, results from the coupling of the angular momentum with an external magnetic field.

Let us find out the spectrum of (41). Schroedinger equation for the stationary states is

$$\left[-h_2\,\frac{\partial^2}{\partial\alpha_i^2} + h_1\,(\,\alpha_i\,)^2 - \frac{1}{2}\,(\,\theta + B\,)\,L\right]\psi\,(\,\alpha_i\,) = E\,\psi\,(\,\alpha_i\,) \tag{44}$$

$$L \equiv -i\,\epsilon_{kj}\,\alpha_k\,\frac{\partial}{\partial\alpha_j} \tag{45}$$

Since Hamiltonian (44) is rotationally symmetric, it is appropriate to work in polar coordinates. In this coordinate system the Schroedinger equation reads:

$$\left[h_2 \left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{L^2}{r^2} \right) - h_2\, r^2 - \frac{1}{2}\,(\theta + B\,)\; L \right] \psi\,(r\; , \phi\,) = E\, \psi\,(r\; , \phi\,) \tag{46}$$

$$L \equiv -i\, \frac{\partial}{\partial \phi} \tag{47}$$

The Schroedinger equation permits separation of variables in the wave-function. We find it convenient to introduce the following redefinitions

$$z \equiv \sqrt{\frac{h_1}{h_2}}\, r^2 \tag{48}$$

$$\widehat{E} \equiv \frac{1}{4\sqrt{h_1\, h_2}} \left[E - \frac{m}{2}\, (\theta + B) \right] - \frac{1}{2} \tag{49}$$

where, we have introduced m as the "magnetic quantum number" of the angular momentum operator L. The wavefunction is of the form

$$\psi(z\; , \phi) = Z(z)\, \exp\left(-\frac{z}{2} + i\, m\, \phi \right) \tag{50}$$

where, the radial wavefunction $Z(z)$ is subject to the equation

$$z\, Z''(z) + (1 - z)\, Z'(z) + \left[\widehat{E} - \frac{m^2}{4z} \right] Z(z) = 0 \tag{51}$$

Equation (51) admits solutions in terms of Generalized Laguerre Polynomials as

$$\psi_{n_r\, m}(z\; , \phi) = N\, z^{|\, m\, |/2}\, L^{|\, m\, |}_{n_r}(z)\, \exp\left(-\frac{z}{2} + i\, m\, \phi \right) \tag{52}$$

$$L^s_n(z) = z^{-s}\, \exp(z) \frac{d_n}{dz^n} \left(z^{n+s}\, \exp(-z) \right) \tag{53}$$

where, N is the proper normalization constant and n_r is the *radial quantum number.* The spectrum of the system is found to be

$$E_{n_r\, m} = 2\sqrt{h_1\, h_2}\, (\, 2n_r + |\, m\, | + 1\,) + \frac{m}{2}\, (\, \theta + B\,) \tag{54}$$

with quantum numbers taking values $n_r = 0\; , 1\; , 2\; , \ldots$, $m = 0\; , \pm 1\; , \pm 2\; , \ldots$.

Using the definitions (42), (43) one finds the frequency in the isotropic case to be

$$\omega \equiv 2\sqrt{h_1\, h_2} = \frac{1}{2}\sqrt{4 + (\theta - B)^2} \tag{55}$$

Equation (55) displayes the independence of the spectrum on the arbitrary parameter a, as advocated earlier. The parameter a simply induces a harmless (global) rescaling of the radial coordinate. Let us introduce a different set of non-negative quantum numbers n_+ , n_- [15] in terms of which the radial and magnetic quantum numbers are written as

$$n_r \equiv n_- + \frac{m - |\, m\,|}{2}\ , \qquad m \equiv n_+ - n_- \tag{56}$$

and the spectrum turns out to be

$$E_{n_+\, n_-} = \omega\, (\, n_+ + n_- + 1\,) + \frac{n_+ - n_-}{2}\, (\, \theta + B\,) \tag{57}$$

The spectrum (57) in the special case $B = 0$ has been studied in [10].
For the sake of transparency, we write down the first two excited states of the noncommutative harmonic oscillator

$$E_{\,0,2} = 3\omega + (\, \theta + B\,) \tag{58}$$

$$E_{\,1,1} = 3\omega \tag{59}$$

$$E_{\,2,0} = 3\omega - (\, \theta + B\,) \tag{60}$$

$$E_{\,0,1} = 2\omega + \frac{1}{2}\, (\, \theta + B\,) \tag{61}$$

$$E_{\,1,0} = 2\omega - \frac{1}{2}\, (\, \theta + B\,) \tag{62}$$

$$E_{\,0,0} = \omega \tag{63}$$

We point out that the spectrum displays the fact that the noncommutativity parameters play the same role as an external magnetic field $\mathcal{H} \equiv \theta + B$. This role of the noncommutative parameters explains the choice made in [9] as corresponding to the absence of the "*magnetic field*" and, thus, re-establishing energy levels degeneracy.

III. DISCUSSION AND CONCLUSIONS

As we have shown, the sets of solutions (37), (38), (39) and (22) , (23), (24), (25) lead to two representations of the noncommutative harmonic oscillator. We would like to show that the spectra of the two modes are identical. Let us re-write (57) as

$$E_{n_+ n_-} = (\,\omega + \theta + B\,)\left(n_+ + \frac{1}{2}\right) + (\,\omega - \theta - B\,)\left(n_- + \frac{1}{2}\right) \tag{64}$$

The energy spectrum (64) matches the one of the Hamiltonian (41), provided one identifies parameters as follows

$$\omega = \frac{1}{2}\,(\,\Omega_1 + \Omega_2\,) \tag{65}$$

$$\theta + B = \frac{1}{2}\,(\,\Omega_1 - \Omega_2\,) \tag{66}$$

The advantage of the representation in terms of isotropic oscillator is that it offers clear identification for the noncommutativity as magnetic field effect. On the other hand, the equivalence to the anisotropic representation displayes that the effect of the magnetic field can be simulated by the frequency difference of the anisotropic oscillators. Similar conclusion, in a different context and in terms of chiral oscillators has been found in [16].
The equivalence of the spectra displays that the two descriptions of the noncommutative harmonic oscillator are equivalent, in spite of the asymmetry with respect to rotations. The generator of rotations, i.e. angular momentum, is defined by the commutators

$$[\,L\ ,\alpha_k\,] = i\,\epsilon_{kj}\,\alpha_j \tag{67}$$

$$[\,L\ ,\beta_k\,] = i\,\epsilon_{kj}\,\beta_j \tag{68}$$

Therefore, our definition of isotropic representation is motivated by the fact that the Hamiltonian (44) commutes with L. Let us express the angular momentum operator (45) in terms of the noncommutative coordinates $(\,x\ ,p\,)$ with the help of (37), (38), (39). One finds that the noncommutative form of L, call it J, is

$$J = \frac{1}{\kappa}\left(\epsilon_{ij}\,x_i\,p_j + \frac{\theta}{2}\,p_i^2 + \frac{B}{2}\,x_i^2\right) \tag{69}$$

The additional terms take into account noncommutativity in θ and B. (69) has the form found in ([8]). J remains the angular momentum in the space of noncommutative coordinates. In fact, it satisfies

$$[\,J\ ,x_k\,] = i\,\epsilon_{kl}\,x_l \tag{70}$$

$$[\,J\ ,p_k\,] = i\,\epsilon_{kl}\,p_l \tag{71}$$

$$\tag{72}$$

In both forms the angular momentum satisfies $[\,H\ ,J\,] = [\,H\ ,L\,] = 0$. Thus, commutative (in terms of α and β) and noncommutative (in terms of x and p) representations are isotropic.

Let us return, now, to the anisotropic representation. We shall name the canonical coordinates of this representation Q_i, P_i for easier distinction from other representations. Rotations are still generated by the angular momentum operator $L = \epsilon_{ij}\,Q_i\,P_j$ satisfying relations (67), (68). On the other hand, one can verify that $[\,H\ ,L\,] \neq 0$ implying the absence of rotational symmetry. The isotropic and anisotropic representations have different symmetry properties with respect to rotations, and the rotational symmetry cannot be responsible for the equivalence of their spectra. We would like to identify the symmetry which leaves the spectra unchanged. For this purpose, let us express the angular momentum operator of the isotropic representation (45) in terms of the coordinates Q_i and P_i. We write down the relation among coordinates of the two representations

$$\alpha_1 = \frac{1}{\sqrt{2}}\left(\frac{h_2}{h_1}\right)^{1/4}(\,Q_1 - P_2\,) \tag{73}$$

$$\alpha_2 = -\frac{1}{\sqrt{2}}\left(\frac{h_2}{h_1}\right)^{1/4}(\,Q_2 - P_1\,) \tag{74}$$

$$\beta_1 = \frac{1}{\sqrt{2}}\left(\frac{h_1}{h_2}\right)^{1/4}(\,Q_2 + P_1\,) \tag{75}$$

$$\beta_2 = -\frac{1}{\sqrt{2}}\left(\frac{h_1}{h_2}\right)^{1/4}(\,Q_1 + P_2\,) \tag{76}$$

which give the form of L in (45), call it now $\widehat{L}$, as

$$\widehat{L} = \frac{1}{2}\left(Q_2^2 + P_2^2 - Q_1^2 - P_1^2\right) \tag{77}$$

One finds

$$\left[\widehat{L} \, , Q_i^2 \right] = 2i \; [\, Q_1 P_1 - Q_2 P_2 \,] \tag{78}$$

$$\left[\widehat{L} \, , P_i^2 \right] = -2i \; [\, Q_1 P_1 - Q_2 P_2 \,] \tag{79}$$

$$\tag{80}$$

which show that (77) is not the generator of rotations. On the other hand, one finds that $\left[\widehat{L} \, , H \right] = 0$. It is possible to verify that

$$\left[\widehat{L} \, , L \right] = -2i \, \overline{L} \equiv 2i \; (\, Q_1 Q_2 + P_1 P_2 \,) \tag{81}$$

and

$$\left[\overline{L} \, , Q_1^2 \right] = 2i \, Q_1 \, P_2 \tag{82}$$

$$\left[\overline{L} \, , Q_2^2 \right] = 2i \, Q_2 \, P_1 \tag{83}$$

$$\left[\overline{L} \, , P_1^2 \right] = -2i \, Q_2 \, P_1 \tag{84}$$

$$\left[\overline{L} \, , P_2^2 \right] = -2i \, Q_1 \, P_2 \tag{85}$$

$$\tag{86}$$

One realizes that the operators L, $\widehat{L}$, $\overline{L}$ form an $SU(2)$ algebra [12]. The equivalence of the spectra in the anisotropic and isotropic representations must, therefore, be result of the invariance of the Hamiltonian with respect to the above $SU(2)$ group. To put in evidence the $SU(2)$ invariance of the Hamiltonian let us calculate the sum of the squares of the three operators L, $\widehat{L}$, $\overline{L}$ in the anisotropic representation. One finds

$$L^2 + \widehat{L}^2 + \overline{L}^2 = \frac{1}{4} \left[Q_i^2 + P_i^2 \right]^2 - 1 \equiv C^2 - 1 \tag{87}$$

The result shows the existence of the operator C which permits the following expression of the Hamiltonian (13)

$$H = \frac{\Omega_1 + \Omega_2}{2} \, C - \frac{\Omega_1 - \Omega_2}{2} \, \widehat{L} \tag{88}$$

Expression (88) indicates that the set of commuting operators, needed to describe the spectrum, is H, C, $\widehat{L}$ which also exhibits invariance of the Hamiltonian under $SU(2)$.

The eigenvalues of these operators can be described in terms of two quantum numbers n_2 and n_1, associated to the operators $Q_2^2 + P_2^2$, $Q_1^2 + P_1^2$ respectively. One can verify that this description leads to the spectrum (64) with $n_1 = n_+$, $n_2 = n_-$. In passing from the anisotropic to the isotropic representation, one rewrites operators C and $\widehat{L}$ in terms of the coordinates of the appropriate representation. In doing so, $\widehat{L}$ becomes the angular momentum operator in the isotropic representation re-establishing rotational symmetry. One can verify that (88) reproduces (41) and (12).

In this paper we have shown the existence of an isotropic representation of the noncommutative harmonic oscillator which goes hand-by-hand with, already known, anisotropic representation. These two representations are different if seen from the point of view of rotational symmetry. The reason for this symmetry breaking can be traced back to the choice of the ansatz for the transformation matrices connecting commutative and noncommutative coordinates. Different eigenvalues of the transformation matrices break rotational symmetry explicitly. Nonetheless, the two representations describe the same physical system, as the equivalence of the spectra show. The symmetry of the spectrum, for any choice of the ansatz, is the $SU(2)$ symmetry described in the discussion above. The isotropic representation has the advantage of giving clear physical meaning to the effect of noncommutativity as being equivalent to an external magnetic field. There may be other representations of the noncommutative system corresponding to different solutions for the transformation matrices, but they should all lead to one of the two forms of the Hamiltonian described in this paper. Finally, we would like to correct generally accepted, but not completely appropriate, use of the term magnetic field when referring ,*only*, to the noncommutative parameter B. This is motivated by the fact that the noncommutative momentum turns into a covariant one in terms of canonical coordinates. However, the role of coordinates and momenta is equivalent in phase space, and thus it is clear that the parameter θ plays the same role as B. This is displayed in (41). Therefore, parameter θ equally deserves the name of “magnetic field” [16].

Acknowledgments

We would like to thanks Prof. A.Jellal and Prof. R.Banerjee for useful discussions, and Prof. M.Crescimanno for providing us a copy of the original paper by V.Fock where the $2D$ harmonic oscillator coupled to a magnetic field was quantized [17].

---

[1] E. Witten Nucl. Phys. B**460** 335 (1996)

[2] N.Seiberg, E. Witten JHEP **9909** 032 (1999)

[3] A. Konechny, A. Schwarz *Introduction to M(atrix) theory and noncommutative geometry, Part II* hep-th/0107251

A. Konechny, A. Schwarz *Introduction to M(atrix) theory and noncommutative geometry* hep-th/0012145

[4] M. R. Douglas, N. A. Nekrasov *Noncommutative Field Theory*; hep-th/0106048

[5] M. Chaichian, A. Demichev, P. Prešnajder Nucl. Phys. B**567** 360 (2000)

[6] L.Alvarez-Gaume, S. R. Wadia Phys. Lett. B **501** 319 (2001)

L. Alvarez-Gaume, J.L.F. Barbon Int. J. Mod. Phys.**A16** 1123 (2001)

[7] R. P. Malik, A. K. Mishra, G. Rajasekaran Int. J. Mod. Phys. **A13**4759 (1998)

V.P. Nair Phys. Lett. B **505** 249 (2001)

B. Morariu, A. P. Polychronakos *Quantum Mechanics on the Noncommutative Torus*; hep-th/0102157

[8] V.P. Nair, A.P. Polychronakos Phys. Lett. B **505** 267 (2001)

[9] A. Hatzinikitas, I. Smyrnakis *The noncommutative harmonic oscillator in more than one dimensions*, hep-th/0103074

[10] J. Gamboa, M. Loewe, F. Mendez, J. C. Rojas *The landau problem and noncommutative Quantum Mechanics*; hep-th/0104224

[11] A. Jellal *Orbital Magnetism of Two-Dimension Noncommutative Confined System* hep-th/0105303

O.F. Dayi, A. Jellal *Landau Diamagnetism in Noncommutative Space and the Nonextensive Thermodynamics of Tsallis;* cond-mat/0103562

[12] S.Bellucci, A. Nersessian, C.Sochichiu *Two phases of the noncommutative quantum mechanics*,

hep-th/0106138

[13] J. Gamboa, M. Loewe, F. Mendez, J. C. Rojas *Noncommutative Quantum Mechanics: The Two-Dimensional Central Field*; hep-th/0106125

[14] The dimensions of various noncommutative constants are $[\,\Theta\, B\,] = \left[\hbar^2\right]$. For simplicity we choose $\hbar = 1$.

[15] A. Messiah *Quantum Mechanics*, Ch.XII, North Hollad Publ. Comp. (1961)

[16] R. Banerjee Phys. Rev.D**60** 085005 (1999)

R. Banerjee *Dissipation and Noncommutativity in Planar Quantum Mechanics* hep-th/0106280

[17] V. Fock "Bemerkung zur Quantelung des HO im Magnetfeld" Zeitschrift Für Physik **84** 446 (1928)