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Non-extremal fractional branes[†]

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Abstract

We construct non-extremal fractional D-brane solutions of type-II string theory at the \mathbb{Z}_2 orbifold point of K3. These solutions generalize known extremal fractional-brane solutions and provide further insights into $\mathcal{N}=2$ supersymmetric gauge theories and dual descriptions thereof. In particular, we find that for these solutions the horizon radius cannot exceed the non-extremal enhançon radius. As a consequence, we conclude that a system of non-extremal fractional branes cannot develop into a black brane. This conclusion is in agreement with known dual descriptions of the system.

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1 Introduction

Fractional D-branes [1–3] have proved an interesting and rich subject in string theory, generalizing the ordinary D-branes of type-II string theory. They differ from the latter not only because they carry fractional charges but also in that their world-volume gauge theories are in general non-conformal. In particular, fractional D-branes with eight supercharges can be obtained from type-II string theory on an orbifold limit of K3, the simplest one being T^4/\mathbb{Z}_2 , or on the orbifold limit of an ALE space, which is \mathbb{C}^2/Γ where Γ corresponds to the ADE-classified finite symmetry group of the ALE space [4, 5].

In the present paper we consider the type-II string theories on the T^4/\mathbb{Z}_2 orbifold limit of K3. For these cases, fractional D-branes have half the charge of the usual regular D-branes. Their world-volume theories contain only a vector multiplet, while the regular branes carry in addition a hypermultiplet. The extremal supergravity solutions for this class of fractional branes were found in ref. [6]. Here we study the non-extremal generalizations of these solutions.

The three main reasons for studying this problem are as follows. The first is that it is interesting to determine whether fractional branes in string theory are dynamical objects

in the same sense as ordinary D-branes. Making the branes non-extremal tests whether it is possible to consider thermally excited fractional D-branes.

The second reason is related to the fact that the gauge theories living on the fractional branes that we consider are 3+1-dimensional pure $\mathcal{N}=2$ super-Yang–Mills (SYM) theory or dimensional reductions thereof. These are non-conformal theories and it is of principal interest to extend the Maldacena conjecture [7–9] to such settings. To make the fractional branes non-extremal would in principle enable us to obtain a dual description of finite-temperature pure SYM theories with eight supercharges.

The study of a supergravity/gauge-theory duality for pure SYM with eight supercharges was initiated in ref. [10] by considering D-branes wrapping K3, a setup which is T-dual to the one with fractional D-branes [10–12]. The upshot of the analysis in ref. [10] is that the supergravity solution dual to pure SYM with eight supercharges has a repulson singularity [13–15] near the center. However, this repulson can be excised from the solution by noticing that at a certain distance from the center, the so-called *enhancement radius*, an abelian field in the effective theory becomes non-abelian, the gauge symmetry being enhanced from U(1) to SU(2). This means that the low-energy effective theory contains additional massless fields which have to be taken into account. Moreover, a probe computation shows that D-brane probes become massless as one reaches the enhancement radius. The interpretation of this phenomenon is that there is a spherical distribution of D-branes at the enhancement radius with flat space inside the sphere [10]. Similar results have subsequently also been obtained for other configurations of wrapped branes [16–18] as well as for fractional brane configurations both on non-compact orbifolds like \mathbb{C}^2/Γ [19–24] and on the compact T^4/\mathbb{Z}_2 orbifold limit of K3 [6].

The third main reason for our interest in non-extremal fractional branes is the interplay between the non-extremality features and the enhancement mechanism; since extremal fractional branes on T^4/\mathbb{Z}_2 have an enhancement radius [6], it is of interest to see whether a horizon covering the enhancement can develop and thus provide another kind of excision mechanism for the fractional branes. For systems of D6-branes wrapped on K3, this question has been addressed in refs [10, 25] where it was found that a horizon can indeed be formed. However, in the supergravity/gauge-theory duality the horizon radius corresponds to energies that are beyond reach in the pure SYM theory. This observation has been regarded as evidence for the conjecture that the dual of pure SYM with eight supercharges is a non-gravitational theory [10, 26]. As we are going to show, fractional branes provide further corroboration of this conjecture.

For $\mathcal{N}=1$ SYM in 3+1 dimensions the thermal version of the Klebanov–Strassler setup [27] that builds on the paper [28] has been explored in refs [29–31]. Here it has proved very hard to find an exact non-extremal solution. For the $\mathcal{N}=1^*$ SYM theory progress has been made in refs [32–35], building on the setup of ref. [36] describing zero-

temperature $\mathcal{N} = 1^*$ SYM. In the latter case, decoupling the scale of confinement from the string scale in the NS5-brane world-volume theory has turned out to be problematic, making it difficult to do computations. However, the problems encountered in $\mathcal{N} = 1$ and $\mathcal{N} = 1^*$ SYM seem rather unrelated to the problems of finding a non-extremal dual to $\mathcal{N} = 2$ SYM.

The summary and organization of the paper are as follows. In section 2 we present the supergravity solutions describing non-extremal fractional branes at low energy. More precisely, we consider a bound state of M fractional D_p -branes of type-II string theories on the orbifold T^4/\mathbb{Z}_2 . Our solutions thus generalize the fractional-brane solutions of ref. [6]. Their structure turns out to be somewhat simpler than that of those found in analogous investigations of the $\mathcal{N} = 1$ theory in refs [29–31]. In section 3 we discuss the physical properties of the non-extremal solutions, focusing on the interplay between the non-extremal version of the enhançon and the black-hole horizon r_0 . In particular, we find that the horizon radius cannot exceed the enhançon radius. As a consequence, we conclude that these systems of non-extremal fractional branes cannot develop into black branes. The consequences of the latter conclusion and further interpretation of the results are our primary concerns in section 4. Finally, some computational details are given in an appendix. There we also discuss another solution branch, with well-defined black-hole thermodynamics but the physical interpretation of which is presently not clear.

2 The non-extremal solutions

In this section we present the non-extremal low-energy supergravity solutions for fractional D_p -branes on K3 in the T^4/\mathbb{Z}_2 orbifold limit. For the type-IIA case, the relevant truncated six-dimensional supergravity action was obtained in ref. [6] by compactification of type-IIA supergravity on T^4/\mathbb{Z}_2 , and used to construct extremal solutions for fractional D0- and D2-branes. After deriving the corresponding model on the type-IIB side, we will consider the non-extremal generalizations of these solutions for the full range $p = 0, 1, 2, 3$.

2.1 Fractional branes

Fractional branes [1–3] are certain types of BPS D-branes that one encounters when considering string theory in singular backgrounds. They can be defined in many different (but equivalent) ways. Probably, the most intuitive way to understand their properties is through their description as higher-dimensional D-branes wrapped on a vanishing cycle of the singular manifold [37, 38]. This geometric picture makes manifest the characteristic feature of fractional branes as being stuck at the singularity while free to move in the flat transverse directions only. This viewpoint also shows why they lack the world-volume degrees of freedom associated to fluctuations in the orbifold directions which correspond,

in general, to hypermultiplet excitations. Moreover, as will become clear below, this also automatically accounts for the coupling of fractional branes with the twisted sector of string theory on the orbifold. We refer to ref. [39] for a recent review on the properties of fractional branes on orbifolds for theories with eight supercharges.

Let us now focus on our main case of interest. The K3 manifold has $\dim(H_2(K3)) = 22$ two-cycles, on a space of signature (19,3). At the \mathbb{Z}_2 orbifold point, there are three self-dual and three anti-selfdual cycles from the six two-cycles in $H_2(T^4)$, which are invariant under the \mathbb{Z}_2 involution. In addition, there are 16 anti-selfdual cycles that come from the collapsed spheres at the 16 orbifold singularities. A fractional $D_{\mathbf{p}}$ -brane is then a $D(\mathbf{p}+2)$ -brane wrapped on one of the these cycles, \mathcal{C} , in the \mathbb{Z}_2 orbifold limit of K3. The presence of a background NS-NS two-form flux through the shrinking cycle makes the D-brane tension non-vanishing. The background value of this flux is dictated to be

$$b_0 = \int_{\mathcal{C}} B_{(2)} = \frac{1}{2}(2\pi\sqrt{\alpha'})^2 \quad (2.1)$$

by the requirement of conformal invariance of the string world-sheet in the orbifold background [40]. The effective theory describing the dynamics of such a brane at low energy is a $(\mathbf{p}+1)$ -dimensional pure SYM theory with eight supercharges. From the closed string theory point of view, this translates into the fact that fractional branes couple to both the untwisted and the twisted sector of string theory on the orbifold.

At low energy, in the infinite-volume limit of T^4/\mathbb{Z}_2 , a given fractional $D_{\mathbf{p}}$ -brane simply couples to the metric, the dilaton and the RR $(\mathbf{p}+1)$ -form potential in the untwisted sector, and to a scalar field and a $(\mathbf{p}+1)$ -form potential in the twisted sector. The two latter correspond, respectively, to the zero mode of the NS-NS $B_{(2)}$ field and of the RR $(\mathbf{p}+3)$ -form potential when “dimensionally reduced” on the shrinking two-cycle, and belong to a non-gravitational multiplet of the effective supergravity theory [37, 38]. Denoting by $\omega_{(2)}$ the closed differential two-form Poincaré dual to the vanishing two-cycle \mathcal{C} on which the branes are wrapped, the relations between higher-dimensional forms and twisted fields read

$$B_{(2)} = b \omega_{(2)}, \quad C_{(\mathbf{p}+3)} = \sqrt{2V} A_{(\mathbf{p}+1)} \wedge \omega_{(2)}. \quad (2.2)$$

In the compact case (anticipated here by the introduction of the dimensionful constant \mathbf{V} to be defined shortly) the effective six-dimensional theory is augmented by the zero modes of massless fields on the internal manifold. However, these come only from the untwisted sector since, by construction, the twisted fields have no dynamics on the internal space. For $\omega_{(2)}$ we adopt the following conventions:

$$\int_{\mathcal{C}} \omega_{(2)} = 1, \quad \int_{\mathbb{C}^2/\mathbb{Z}_2} * \omega_{(2)} \wedge \omega_{(2)} = \frac{1}{2}, \quad \omega_{(2)} + * \omega_{(2)} = 0. \quad (2.3)$$

2.2 The actions

Let us start by fixing our conventions and presenting the consistently truncated six-dimensional actions describing the dynamics of the supergravity fields to which the fractional D p -branes couple. As just discussed, the gauge fields that enter the solution for a given p are the two $(p+1)$ -form potentials $C_{(p+1)}$ and $A_{(p+1)}$. From the twisted scalar field arising from the NS-NS two-form potential we separate out a fluctuating part \tilde{b} according to

$$b = \frac{1}{2}\sqrt{V_*} + \sqrt{2V}\tilde{b}, \quad V_* = (2\pi\sqrt{\alpha'})^4, \quad (2.4)$$

where V is the volume of the compact space T^4/\mathbb{Z}_2 and V_* is introduced as a shorthand notation. Furthermore, the Kaluza–Klein reduction of the metric gives rise to four scalar fields, $e^{2\eta_a}$, $a = 6, \dots, 9$. Assuming homogeneity in the compactified directions these are equal and can be replaced by a single scalar field defined as $\eta = \sum_{a=6}^9 \eta_a$. The dimensionally reduced dilaton $\phi = \phi|_{d=10} - \frac{1}{2}\eta$ completes the scalar field content.

For the case $p=0$ the truncated bosonic $d=6$ action governing the dynamics of these fields was derived in ref. [6], together with the electro-magnetically dual two-brane case. From these results the corresponding action for the fractional D3-brane on the type-IIB side can readily be inferred. These three actions take the form¹

$$S_{\text{bulk}}^{(p=0,2,3)} = \frac{V}{2\kappa^2} \int \left\{ d^6x \sqrt{-g} R + *d\phi \wedge d\phi + \frac{1}{4} *d\eta \wedge d\eta + \frac{1}{2} e^{-\eta} *d\tilde{b} \wedge d\tilde{b} \right. \\ \left. + e^{(1-p)\phi} \left[\frac{1}{2} e^\eta *G_{(p+2)} \wedge G_{(p+2)} + \frac{1}{2} *\tilde{F}_{(p+2)} \wedge \tilde{F}_{(p+2)} \right] \right\}, \quad (2.5)$$

where $\kappa = 8\pi^{7/2}g_s\alpha'^2$ is the ten-dimensional gravitational coupling and

$$G_{(p+2)} = dC_{(p+1)}, \quad \tilde{F}_{(p+2)} = dA_{(p+1)} + d\tilde{b} \wedge C_{(p+1)} \quad (2.6)$$

are the gauge-invariant field strengths of the untwisted- and twisted-sector potentials $C_{(p+1)}$ and $A_{(p+1)}$, respectively.

For the remaining case, $p=1$, some subtleties arise because the field strength $\tilde{F}_{(3)}$ is self-dual, a property inherited from its ten-dimensional type-IIB five-form parent. As usual, the self-duality condition has to be imposed on shell. With this proviso in mind we can nevertheless write a truncated $d=6$ gravity action also for the fractional D-string:²

$$S_{\text{bulk}}^{(p=1)} = \frac{V}{2\kappa^2} \int \left\{ d^6x \sqrt{-g} R + *d\phi \wedge d\phi + \frac{1}{4} *d\eta \wedge d\eta + \frac{1}{2} e^{-\eta} *d\tilde{b} \wedge d\tilde{b} \right. \\ \left. + \frac{1}{2} e^\eta *G_{(3)} \wedge G_{(3)} + \frac{1}{4} *\tilde{F}_{(3)} \wedge \tilde{F}_{(3)} - \frac{1}{2} A_{(2)} \wedge G_{(3)} \wedge d\tilde{b} \right\}. \quad (2.7)$$

¹Our conventions are $*\xi_{(6-n)} = \frac{1}{n!} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_n} \epsilon_{\mu_1 \dots \mu_n}^{\mu_{n+1} \dots \mu_6} \xi_{\mu_{n+1} \dots \mu_6}$ and $\epsilon^{012345} = +1$. The RR field strengths in ten dimensions were defined as $\tilde{G}_{(n+1)} = d\tilde{C}_{(n)} + \tilde{H}_{(3)} \wedge \tilde{C}_{(n-3)}$.

²This action results from taking a type-IIB action in ten dimensions compatible with an *anti*-selfdual five-form field strength $\tilde{G}_{(5)}$, and using the conventions (2.3) in the Kaluza–Klein reduction on T^4/\mathbb{Z}_2 .

The $C_{(2)}$ equation of motion is compatible with the self-duality constraint $*\tilde{F}_{(3)} = \tilde{F}_{(3)}$, a fact which relies crucially on the presence of the Chern–Simons term.

In solving the equations of motion obtained from the actions (2.5) and (2.7) one should also specify the boundary conditions at infinity that the various fields should satisfy (i.e. the mass and the charges of the soliton). We are interested in describing a general bound state of non-extremal fractional D_p -branes whose corresponding supergravity solution should match the extremal one at infinity. The boundary conditions at infinity for the latter are encoded in the bosonic world-volume action describing the low-energy dynamics of an extremal fractional brane. For the case at hand this action has been derived in ref. [6] and reads

$$S_{\text{wv}} = -\frac{T_p}{2\kappa} \int d^{p+1}\xi \sqrt{-g} e^{-\eta/2} e^{-(1-p)\phi/2} \left(1 + 2\sqrt{\frac{2V}{V_*}} \tilde{b} \right) \\ [1ex] + \frac{T_p}{2\kappa} \int \left[C_{(p+1)} \left(1 + 2\sqrt{\frac{2V}{V_*}} \tilde{b} \right) + 2\sqrt{\frac{2V}{V_*}} A_{(p+1)} \right], \quad (2.8)$$

where $T_p = \sqrt{\pi}(2\pi\sqrt{\alpha'})^{3-p}$. The world-volume action for a stack of M coincident fractional D_p -branes is obtained simply by multiplying the above action by M (this will be implicitly assumed in what follows).

2.3 Ansätze and solutions

Below we present the solutions of the equations of motion, given in appendices A.1 and A.2, referring to appendix A.3 for an outline of their derivation. We will wherever possible treat the cases $p = 0, 1, 2, 3$ in parallel. As far as the starting-point—i.e. the ansatz for the metric—is concerned, the three-brane (being of codimension two in six dimensions) however needs some special consideration. The solutions, nevertheless, share a common structure for all four cases as will become clear below.

Let us first discuss the lower-dimensional cases, $p = 0, 1, 2$, for which the standard non-extremal p -brane ansatz applies:

$$ds^2 = H^{\frac{p-3}{4}} \left(-f dt^2 + \sum_{i=1}^p (dx^i)^2 \right) + H^{\frac{p+1}{4}} (f^{-1} dr^2 + r^2 d\Omega_{4-p}^2). \quad (2.9)$$

The non-extremality is introduced by the function f which, like H , depends on the transverse radial coordinate only. It is constrained by the equations of motion to satisfy the harmonic equation

$$f'' + \frac{4-p}{r} f' = 0. \quad (2.10)$$

Requiring the non-extremal solution to approach the extremal one at infinity, the solution to eq. (2.10) takes the form

$$f = 1 - \left(\frac{r_0}{r} \right)^{3-p}, \quad (2.11)$$

with the horizon radius r_0 governing the degree of non-extremality.³

The upshot of the analysis of appendix A.3 is that the non-extremal solutions can be expressed entirely in terms of r and two additional harmonic functions, h_1 and h_2 , which in turn depend on three parameters: the horizon radius r_0 and two charges, q_1 and q_2 , depending linearly on the number of branes which act as sources for the solution. More specifically, the scalar fields are given by

$$e^\phi = H^{\frac{1-p}{4}}, \quad e^\eta = \frac{H}{h_1^2}, \quad \tilde{b} = \frac{q_2}{q_1} \left(\frac{h_2}{h_1} - 1 \right), \quad (2.12)$$

while the gauge potentials take the form

$$C_{0\dots p} = -\frac{q_2}{r^{3-p}} \frac{h_3}{H}, \quad A_{0\dots p} = -\frac{q_1}{r^{3-p}} \frac{1}{h_1}, \quad (2.13)$$

the associated field strengths (defined in eq. (2.6) above) being

$$G_{r0\dots p} = \frac{(3-p)q_2}{r^{4-p}} \frac{h_1 h_2}{H^2}, \quad \tilde{F}_{r0\dots p} = \frac{(3-p)q_1}{r^{4-p}} \frac{1}{H}. \quad (2.14)$$

The functions H and h_3 entering the solution read

$$H = \left(1 + \frac{1}{2} \frac{q_2^2}{q_1^2} \right) h_1^2 - \frac{1}{2} \frac{q_2^2}{q_1^2} h_2^2, \quad h_3 = \frac{1}{2} (h_1 + h_2), \quad (2.15)$$

where the two basic harmonic functions are

$$h_i = 1 - \left(\frac{r_i}{r} \right)^{3-p}, \quad i = 1, 2. \quad (2.16)$$

The radial parameters of these functions are given by

$$r_1^{3-p} = \frac{1}{2} r_0^{3-p} + \frac{\sqrt{2q_1^4 + (q_1^2 + q_2^2)r_0^{2(3-p)} - 2q_1^2\Lambda}}{2\sqrt{2q_1^2 + q_2^2}}, \quad (2.17)$$

$$[2ex]r_2^{3-p} = \frac{1}{2} r_0^{3-p} + \frac{\sqrt{2q_1^4 + (q_1^2 + q_2^2)r_0^{2(3-p)} + 2q_1^2\Lambda}}{2\sqrt{q_2^2}}, \quad (2.18)$$

where

$$\Lambda \equiv \sqrt{q_1^4 + (q_1^2 + q_2^2)r_0^{2(3-p)} + \frac{1}{4}r_0^{4(3-p)}}. \quad (2.19)$$

As already noticed, the solution we have found is completely fixed by extracting the relation between the free-supergravity values of the charges q_1 and q_2 and those dictated by the world-volume action (2.8) for the M fractional branes. Using eqs (A.71) and (A.72)

³Although the metric (2.9) develops a horizon at r_0 we shall see in the next section that the supergravity approximation ceases to be valid at a radius r_h strictly larger than r_0 . Keeping this in mind, we shall nevertheless refer to r_0 as the horizon radius.

and equating the corresponding charges with the coupling to the twisted and untwisted potentials $A_{0\dots p}$ and $C_{0\dots p}$ in the WZ action one gets

$$q_1 = \sqrt{\frac{2V}{V_*}} Q_p M, \quad q_2 = Q_p \frac{M}{2}, \quad (2.20)$$

where

$$Q_p = \frac{2}{k_p} \frac{T_p \kappa}{\Omega_{4-p} V} = \frac{1}{k_p \Omega_{4-p}} \frac{V_*}{V} (2\pi\sqrt{\alpha'})^{3-p} g_s, \quad (2.21)$$

with Ω_{4-p} denoting the volume of the unit $(4-p)$ -sphere surrounding the p -branes and $k_p = 3-p$ for $p < 3$ while $k_3 = 1$.⁴

As a consistency check let us also take the extremal limit, $r_0 = 0$. Assuming the charges to be positive, as we will from now on, we obtain

$$h_1^{\text{extr}} = 1, \quad h_2^{\text{extr}} = 1 - \frac{q_1^2}{q_2} \frac{1}{r^{3-p}}, \quad H_{\text{extr}} = 1 + \frac{q_2}{r^{3-p}} - \frac{q_1^2}{2r^{2(3-p)}}, \quad (2.22)$$

so that the solution of ref. [6] is recovered:

$$e^{\eta_{\text{extr}}} = H_{\text{extr}}, \quad \tilde{b}_{\text{extr}} = -\frac{q_1}{r^{3-p}} = A_{0\dots p}^{\text{extr}}, \quad C_{0\dots p}^{\text{extr}} = H_{\text{extr}}^{-1} - 1. \quad (2.23)$$

Note that the structure of the function H that determines the non-extremal metric (together with f) and the dilaton is the same as for the extremal case; while the coefficients of course differ, introducing a non-extremality parameter gives no terms beyond the $r^{-2(3-p)}$ correction, the latter being the usual fractional brane modification to the harmonic function governing the regular-brane solution. For the other fields the non-extremal modifications are somewhat more intricate. Nevertheless, for our class of non-extremal fractional branes, these modifications are entirely due to the non-triviality of the harmonic function h_1 for $r_0 > 0$. In particular, we note that, contrary to the extremal case, the non-extremal twisted fields *do* get corrections with respect to their harmonic asymptotic behavior. The absence of such corrections was taken as input for the ansatz relevant to extremal fractional branes [6], and was argued (for the NS-NS twisted scalar \tilde{h}) to be a manifestation of the fact that $\mathcal{N} = 2$ SUSY only allows for one-loop perturbative corrections. The fact that this property ceases to hold for our non-extremal generalization suggests that the non-extremality parameter r_0 indeed does switch on a temperature in the system.

Turning to the case $p = 3$, the appropriate non-extremal ansatz for the metric reads [41]

$$ds^2 = -f dt^2 + \sum_{i=1}^3 (dx^i)^2 + \left(\frac{\tilde{r}}{r}\right)^2 H (f^{-1} dr^2 + r^2 d\theta^2), \quad (2.24)$$

⁴The solution describing a composite bound state of M fractional and N regular D_p -branes is identical to the one we have discussed so far, the only difference being that the untwisted charge q_2 will now be $q_2 = Q_p(\frac{M}{2} + N)$. By taking $M = 0$ one gets $q_1 = 0$ and $q_2 = Q_p N$, giving back the usual regular brane solution, with no coupling to twisted fields.

where r is an (as yet) undetermined radial parameter. Using this ansatz, the analysis in appendix A.3 can be done in parallel with the lower-dimensional cases. As a consequence, the expressions listed above for the metric, the scalars and the gauge field strengths (upon substituting $(3-p) \mapsto 1$) in terms of the functions h_1 and h_2 are valid also for $p=3$. Although the reason is slightly less obvious, by using the mapping $r^{-(3-p)} \mapsto \log(r_\Lambda/r)$ the same turns out to be the case for the gauge potentials in (2.13), giving

$$C_{0123} = -q_2 \frac{h_3}{H} \log \frac{r_\Lambda}{r}, \quad A_{0123} = -\frac{q_1}{h_1} \log \frac{r_\Lambda}{r}. \quad (2.25)$$

Here r_Λ can be interpreted as a long-distance cut-off in the transverse radial direction of the three-brane world volume, corresponding to a UV cut-off on the dual gauge-theory side. This mapping originates from the harmonic functions which in two dimensions involve a logarithm. The function f , for instance, again satisfies the harmonic equation (2.10) but the solution now reads

$$f = 1 - a_0 \log \frac{r_\Lambda}{r}. \quad (2.26)$$

The dimensionless non-negative parameter a_0 governs the degree of non-extremality. As will become evident below, a_0 can be viewed as the direct formal analogue of the parameter r_0 in the solutions for the lower-dimensional branes. The “horizon radius”⁵ for the three-brane is $r_0 = r_\Lambda e^{-1/a_0}$ and the extremal limit is obtained by letting $a_0 \rightarrow 0$ with r_Λ fixed. Moreover, it is convenient to identify the parameter r in (2.24) with r_Λ since the latter sets the length scale of the transverse geometry.

Hence, the conditions that h_1 and h_2 be harmonic now imply

$$h_1 = 1 - a_1 \log \frac{r_\Lambda}{r}, \quad h_2 = 1 - a_2 \log \frac{r_\Lambda}{r}. \quad (2.27)$$

In an exact analogy with the results (2.17) and (2.18) for $p < 3$ the parameters a_1 and a_2 are given by

$$a_1 = \frac{1}{2}a_0 + \frac{\sqrt{2q_1^4 + (q_1^2 + q_2^2)a_0^2 - 2q_1^2\Lambda}}{2\sqrt{2q_1^2 + q_2^2}}, \quad (2.28)$$

$$[1.5ex]a_2 = \frac{1}{2}a_0 + \frac{\sqrt{2q_1^4 + (q_1^2 + q_2^2)a_0^2 + 2q_1^2\Lambda}}{2\sqrt{q_2^2}}, \quad (2.29)$$

with

$$\Lambda = \sqrt{q_1^4 + (q_1^2 + q_2^2)a_0^2 + \frac{1}{4}a_0^4}. \quad (2.30)$$

The reason for these close formal similarities between $p=3$ and the lower-dimensional cases is explained in section A.3.

⁵We write the term within quotes since the transverse space is two-dimensional and black holes therefore can never develop for $p=3$.

Taking the extremal limit, the gauge potentials (2.25) simplify to

$$C_{0123}^{\text{extr}} = H_{\text{extr}}^{-1} - 1, \quad A_{0123}^{\text{extr}} = -q_1 \log \frac{r_\Delta}{r}, \quad (2.31)$$

where $H_{\text{extr}} = 1 + q_2 \log \frac{r_\Delta}{r} - \frac{1}{2} q_1^2 \left(\log \frac{r_\Delta}{r} \right)^2$. In addition we have $b_{\text{extr}} = -q_1 \log \frac{r_\Delta}{r}$, with the remaining fields formally identical to their lower-dimensional counterparts. Note that in the extremal limit of the three-brane solution we have $\phi = -\frac{1}{2}\eta$. From the definition of ϕ this relation immediately implies that the ten-dimensional dilaton is constant, in agreement with the fractional D3-brane solution of refs [19,20] for the non-compact orbifold spacetime $\mathbb{R}^{1,5} \times \mathbb{C}^2/\mathbb{Z}_2$.

3 Enhançon versus horizon

In this section we first review the enhançon mechanism and describe its manifestation in the case at hand. Then we examine the interplay between the event horizon and the enhançon shell.

3.1 Review of the enhançon

Supergravity solutions of brane configurations which have pure SYM with eight supercharges as their low-energy world-volume theories are in general plagued by naked singularities. This is in particular true for fractional branes and, more generally, for D-branes wrapped on topologically non-trivial cycles. The naked singularities one encounters are of repulson type [13–15] and can be excised by the so-called enhançon mechanism [10].

The logic of this mechanism is as follows. Far away from the source we have a perfectly valid supergravity solution. However, when approaching the source there is a certain radius r_e , dubbed the enhançon radius, at which the effective supergravity description needs to be augmented by additional massless degrees of freedom, the appearance of which leads to an enhancement of the gauge symmetry from $U(1)$ to $SU(2)$.⁶ A brane-probe calculation shows that the tension of the probe vanishes precisely at the enhançon radius. The interpretation of this phenomenon is that the branes are not located at the origin but instead smeared over the spherical shell $r = r_e$. The geometry for $r < r_e$ is consequently completely different, and, in particular, the singularity is excised [10, 25].

For fractional branes, the symmetry enhancement occurs when the NS-NS two-form flux through the vanishing cycle on which the brane is wrapped flows from the value $1/2$ at infinity to 0 at the enhançon radius; when the flux vanishes all parameters of the cycle are zero, which corresponds to a point of enhanced symmetry in the moduli space. At

⁶While on the type-IIA side the symmetry enhancement occurs for an ordinary vector field, the enhanced A_1 gauge symmetry in the type-IIB case is carried by a self-dual two-form potential [42].

this point, the fractional branes become tensionless. In particular, the massless degrees of freedom responsible for the enhanced gauge symmetry correspond to tensionless fractional D-particles for type IIA and to tensionless fractional D-strings for type IIB [43].

For the extremal fractional D-branes on T^4/\mathbb{Z}_2 the enhançon mechanism has been examined in ref. [6]. As we are going to discuss in the next section, the presence of the enhançon has crucial consequences for the decoupling limit and for the nature of the gauge-theory/gravity duality for this kind of systems. It is therefore important to investigate if and how the enhançon is modified in the non-extremal case we are discussing, and more specifically what the relation between the event horizon and the enhançon is.

As discussed in the previous section, fractional $D_{\mathbf{p}}$ -branes are a particular kind of wrapped $D(\mathbf{p}+2)$ -branes arising in orbifold compactifications of string theory. While the geometric volume of the compact two-cycle characterizing the orbifold is identically zero, there is a non-trivial NS-NS two-form background flux on it, displayed in (2.1), which makes the effective stringy volume asymptotically non-vanishing. In fact, as already explained in the previous section, this asymptotic value is modified by the presence of the fractional branes (either extremal or non-extremal). Let us recall the relation between the fields entering the solution discussed in section 2 and the NS-NS two-form flux:

$$b = \int_{\mathcal{C}} B_{(2)} = \frac{1}{2} \sqrt{V_*} + \sqrt{2V} \tilde{b}(r). \quad (3.1)$$

Here we recall that V is the volume of the compact orbifold and $V^* = (2\pi\sqrt{\alpha'})^4$ as defined in (2.4). From the above considerations it follows that the enhançon is located at the radius r_e determined by

$$1 + 2\sqrt{\frac{2V}{V_*}} \tilde{b}(r_e) = 0. \quad (3.2)$$

This equation gives the position of the enhançon in the general case. In the extremal limit it reduces to the result found in ref. [6].

Keeping in mind that it is eq. (3.2) that *defines* the enhançon shell, we can also check that fractional $D_{\mathbf{p}}$ -brane probes indeed become tensionless there. One simply takes the DBI action for the probe evaluated in the background generated by the source branes, chooses static gauge with the transverse coordinates depending on time only, and finally expands the action up to terms quadratic in the velocities. The appropriate DBI action is

$$S_{\text{DBI}} = -\frac{T_p}{2\kappa} \int d^{p+1}\xi \sqrt{-g} e^{-\eta/2} e^{-(1-p)\phi/2} \left(1 + 2\sqrt{\frac{2V}{V_*}} \tilde{b} \right). \quad (3.3)$$

By proceeding as just outlined one easily sees that the probe brane becomes tensionless when eq. (3.2) is satisfied. More precisely, for slow radial motion ($\dot{r}^2 \ll f^2/H$) the kinetic part of its effective lagrangian is

$$T(r, \dot{r}) = \frac{T_p V_p}{4\kappa} \frac{h_1(r)}{f^{3/2}(r)} \left(1 + 2\sqrt{\frac{2V}{V_*}} \tilde{b}(r) \right) \dot{r}^2, \quad (3.4)$$

showing that the effective tension of the probe is μ -dependent and that, as promised, the brane becomes tensionless at the distance r_e given by eq. (3.2).⁷

3.2 Application to non-extremal fractional branes

For our class of non-extremal fractional branes, we can obtain a very simple expression for the enhançon. By substituting the solution discussed in the previous section, in particular eqs (2.12) and (2.16), in eq. (3.2) we find

$$r_e^{3-p} = 2\sqrt{\frac{2V}{V_*}} \frac{q_2}{q_1} \left(r_2^{3-p} - r_1^{3-p} \right) + r_1^{3-p} = r_2^{3-p}, \quad (3.5)$$

where in the last step we have used the relations (2.20). A completely analogous expression holds for a_e , and thus for r_e , in the three-brane case (recall that $r_e = r_\Lambda e^{-1/a_e}$).⁸

We now turn to examining the position of the enhançon shell relative to that of the event horizon. Using eqs (2.18) and (2.19), it is easy to see from eq. (3.5) that the enhançon always lies *outside* the horizon, no matter the value of r_0 . This means that in the region of validity of the supergravity approximation, the bound state never develops into a black brane. This might seem puzzling, since one would think that for large enough mass the system would indeed develop into a black brane, while the above equations show that the enhançon increases with r_0 faster than r_0 itself. However, one has to remember that the energy density of this configuration is not concentrated in the center of the solution, but rather spread out on the enhançon shell. Indeed, the fact that we cannot arrange for the horizon to lie outside the enhançon shell is nicely consistent with the fact that, while the mass is not bounded, the density of mass is. To see this, note first that the density of mass is the total mass M_p divided by the volume V_{tot} that we can fit the system into. The mass M_p goes like r_0^{3-p} while the volume V_{tot} goes like r_e^{5-p} , and hence M_p/V_{tot} actually decreases as we increase r_0 , since r_0 is smaller than r_e .

One can also try to extend the solution to the interior of the enhançon shell. Fractional branes cannot get inside the enhançon since their tension vanishes there and their energy (and charge) is believed to be distributed on the enhançon shell. Therefore, the extremal solution is flat in the interior and the energy density vanishes there [10, 44]⁹. However, by making the solution non-extremal one could imagine creating a neutral black hole on the inside, characterized by an internal horizon radius r'_0 . One could then try to increase r'_0

⁷This argument rests on r_e being larger than both r_0 and a_e , which, as we shall see, is always the case.

⁸As already mentioned, black holes cannot appear in 2+1 dimensions so we exclude the three-brane case from the following discussion.

⁹By studying the Seiberg–Witten curve it has been explicitly shown that the supergravity fields should be constant in the interior [45]. On the other hand, to discuss the system at the enhançon scale one should include the extra massless fields into the analysis.

enough to make it larger than the enhançon, thus allowing the system to develop into a black brane.¹⁰

If we demand that such an interior black hole be in equilibrium with the branes at the enhançon shell, we can in principle find r'_0 in terms of r_0 and M . This raises the interesting question whether the interior horizon can reach the enhançon radius. Unfortunately, we cannot answer this question in a precise way since we do not understand non-extremal fractional branes sufficiently well to determine when we have thermodynamical equilibrium. However, the fact that the density M/V_{tot} , as defined above, decreases for increasing r_0 makes it seem unlikely that by increasing r_0 and thereby decreasing the density of mass M/V_{tot} , one could make the system collapse into a black hole. In the next section we interpret the fact that the non-extremal fractional-brane system on the orbifold under study never collapses into a black hole from the perspective of the SYM theory living on the brane and its supergravity dual.

At this point we would like to alert the reader to an apparent puzzle.¹¹ The discussion so far can readily be extended to the more general bound states where N regular branes are also present. As noticed in the previous section, the only modification of the solution is that we now have $q_2 = Q_2(M/2 + N)$. Of course, probing the geometry with regular branes does not give any information on the enhançon locus since regular branes do not couple to the $B_{(2)}$ -flux and are insensitive to the enhançon. On the other hand, revisiting the fractional-brane probe computation, the enhançon radius is now found to be

$$r_e = r_2 + 2 \frac{N}{M} (r_2 - r_1) . \quad (3.6)$$

Surprisingly, when examining the relative positions of the enhançon and the horizon, the conclusions do not change with respect to the pure fractional-brane case. Indeed, this follows from a simple inspection of eqs (2.17)–(2.18) which gives at hand that $r_2 - r_1$ is always positive for our solution. Thus, r_0 is always smaller than r_e , also in the presence of regular branes and regardless of the relative value of N with respect to M . This contrasts with the result found in refs [10, 25] for branes wrapped on K3. However, the two systems are different. In our case, as already noticed, regular branes do not feel the enhançon and a regular D-brane probe can thus go all the way to the center $r = 0$. Indeed, one finds that $r_e \sim M$ at extremality not only in eq. (3.5) but also in eq. (3.6). The extremal N -regular/ M -fractional brane configuration can therefore be thought of as composed of N regular branes at the origin and M fractional branes smeared on the enhançon shell. In contrast, for the case discussed in refs [10, 25] the unwrapped branes (which correspond to

¹⁰Obviously, we would then have to jump to another branch of the solution. In terms of the four branches of the solution summarized in table 1 in section A.4 we should jump from branch I to branch IV.

¹¹For simplicity of notation we give expressions appropriate for the two-brane case although the discussion applies generally.

regular branes here) do indeed influence the enhançon at extremality, causing it to decrease in size. When the number of unwrapped branes exceeds the number of wrapped ones, the enhançon is small enough that, for a sufficiently large non-extremality parameter r_0 , the system can be turned into a black hole with r_0 larger than r_e . In our case the situation is different. Nevertheless, one would expect a limit $N \gg M$ in which the enhançon at extremality would be quantitatively irrelevant and thus should not sensibly affect the regular-brane thermodynamics. In this respect, it would be very interesting to find the precise relation between r_0 and the external parameters when imposing thermodynamical equilibrium between the internal black hole with horizon radius r_0 and the enhançon shell.

Having extensively discussed the possibility of our system developing a horizon, we should also mention that there is an additional characteristic radius whose relative position with respect to the enhançon is of interest. We are referring here to the radius where the potential for a massive particle probe feeling *only* the string-frame geometry becomes repulsive. This radius (denoted r_d in ref. [25]) is readily obtained as the stationary point of the effective potential for a radially moving probe with world-line action given by the string-frame Nambu–Goto action. The resulting condition is $(g_{tt}^{\text{s.f.}})'(r_d) = 0$, which for our non-extremal case translates to $(fH^{-1/2})'(r_d) = 0$. The solution reads (still for $p=2$)

$$r_d = \frac{(2q_1^2 + q_2^2)(r_0 - r_1)r_1 + q_2^2(r_2 - r_0)r_2}{2q_1^2(r_0 - r_1) + q_2^2(r_2 - r_1)}. \quad (3.7)$$

It is straightforward to verify that this radius is always smaller than or (at extremality) equal to r_e , and that the enhançon hence excises the unphysical region.

Finally, let us recall that D_p -brane probes (fractional or regular) moving in a background generated by the same objects also feel non-trivial potentials out of extremality, where supersymmetry is broken. Although these potentials are not due solely to gravity, we have nevertheless observed that the regions where they would have become repulsive are excised by the enhançon. In fact, it is amusing to notice that for a *regular* D_p -brane probing a non-extremal *fractional*-brane background, the radius at which the potential changes sign actually coincides with the enhançon radius.

4 Discussion and conclusions

The upshot of the previous section is that a system of non-extremal fractional branes on the orbifold T^4/\mathbb{Z}_2 cannot collapse into a black hole. We discuss in the following a possible interpretation of this observation in terms of the pure SYM theory living on the fractional D-branes and the theories dual to them in the sense of the AdS/CFT correspondence [7–9].

It is well known [11] that a transverse T-duality on a fractional brane gives a Hanany–Witten setup [46]. In particular, a fractional D_p -brane on the orbifold T^4/\mathbb{Z}_2 is T-dual

to a $D(p+1)$ -brane stretched between two NS5-branes, the distance between them being proportional to the flux H of the NS-NS two-form of our scenario. Moreover, from the NS5-brane setup one obtains the wrapped brane setting of refs [10,18] by a transverse T-duality. All of these brane setups describe at low energies a pure SYM with eight supercharges. It was argued in refs [10, 26] that the dual of pure SYM with eight supercharges is a non-gravitational theory. This means that the gravitational multiplet decouples for a fractional D-brane solution in type-II string theory on T^4/\mathbb{Z}_2 in the decoupling limit of the pure SYM on the brane. In the T-dual Hanany–Witten setup this is the same as saying that gravity decouples from the NS5-branes so that the dual theory is described by the non-gravitational theory living on the NS5-brane [47–49]. In the fractional-brane setup, the 5+1-dimensional fields living on the NS5-branes correspond to the fields of the twisted sector, which indeed have 5+1-dimensional dynamics. In fact, these are the only fields entering all the relevant gauge-theory quantities in the correspondence while the contribution from the gravitational multiplet always cancels as shown for instance in refs [19, 21–23, 50–52].

Now, in the decoupling limit r/r_e is fixed (this because $r \sim \alpha'$ and $r_e \sim \alpha' g_{\text{YM}}^2 M$ and we have $g_{\text{YM}}^2 M$ fixed in the limit). Hence, if there had been a horizon $r_0 > r_e$, it would have remained after taking the decoupling limit, and we would thus have had a black hole in the dual theory. This would have been in contradiction with the conjecture that the dual is a non-gravitational theory. Hence, our analysis can be seen as a further piece of evidence that the dual theory is indeed non-gravitational. In ref. [10] a similar consideration was made for D-branes wrapping K3. For that case it was found that the solution can collapse into a black hole. However, it was subsequently shown that the energies needed for the solution to collapse to a black hole correspond via the gauge-theory/gravity duality to energies that are beyond reach in the pure SYM theory. We intend to return to the decoupling-limit issue for this kind of pure SYM theories [53].

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A Details on the solutions and their derivation

In this appendix we present the derivation of the non-extremal solutions discussed in the main text. We begin by giving the equations that they solve.

A.1 Equations of motion

The equations of motion for the scalar fields encoded in the actions (2.5) for $p = 0, 2, 3$ and (2.7) for $p = 1$ turn out to be identical in form:

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \phi) = \frac{1-p}{4} e^{(1-p)\phi} \left(e^\eta (G_{(p+2)})^2 + (\tilde{F}_{(p+2)})^2 \right), \quad (\text{A.1})$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} \partial^\mu \eta) = e^{(1-p)\phi} e^\eta (G_{(p+2)})^2 - e^{-\eta} \partial_\mu \tilde{b} \partial^\mu \tilde{b}, \quad (\text{A.2})$$

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} e^{-\eta} \partial^\mu \tilde{b}) = -e^{(1-p)\phi} G_{(p+2)} \cdot \tilde{F}_{(p+2)}. \quad (\text{A.3})$$

Here we have introduced the notation $(G_{(n)})^2 \equiv G_{(n)} \cdot G_{(n)} \equiv \frac{1}{n!} G_{\mu_1 \dots \mu_n} G^{\mu_1 \dots \mu_n}$.

For the gauge fields $A_{(p+1)}$ and $C_{(p+1)}$ the case $p = 1$ differs slightly from the others; while for $p = 0, 2, 3$, the equations of motion all take the common form

$$\partial_{\mu_1} (\sqrt{-g} e^{(1-p)\phi} \tilde{F}^{\mu_1 \dots \mu_{p+2}}) = 0, \quad (\text{A.4})$$

$$\partial_{\mu_1} (\sqrt{-g} e^{(1-p)\phi} e^\eta \tilde{G}^{\mu_1 \dots \mu_{p+2}}) = 0, \quad (\text{A.5})$$

we obtain instead for $p = 1$ (in form notation)

$$d * (\tilde{F}_{(3)} + \tilde{b} * G_{(3)}) = 0, \quad (\text{A.6})$$

$$[1ex] \tilde{F}_{(3)} - * \tilde{F}_{(3)} = 0, \quad (\text{A.7})$$

$$[1ex] d * \left(e^\eta \tilde{G}_{(3)} + \frac{1}{2} \tilde{b}^2 * G_{(3)} \right) = 0. \quad (\text{A.8})$$

Here we included the self-duality condition for $\tilde{F}_{(3)}$, as this constraint on the solution has the status of an equation of motion.¹² For convenience we have introduced the notation

$$\tilde{G}_{(p+2)} = G_{(p+2)} - e^{-\eta} \tilde{b} \tilde{F}_{(p+2)}. \quad (\text{A.9})$$

To the above equations should be added the Einstein equations which we shall give once we have introduced the spherically symmetric ansatz that we will employ. Let us first, however, mention that under such an ansatz the self-dual fractional D1-brane case can alternatively be solved by taking a standard electric field-strength ansatz for $\tilde{F}_{(3)}$ and using the equations of motion obtained for $p = 1$ from the “naive” action (2.5). The dual, magnetic, components of $\tilde{F}_{(3)}$ (as well as the full potential $A_{(2)}$), if required, can

¹²Eqs (A.6) and (A.7) are not independent; the self-duality condition is stronger and implies (A.6).

be obtained by imposing the self-duality condition at the very end. We will adopt this (standard) effective procedure below, allowing us to discuss the case $p = 1$ in parallel with the others.

A.2 The spherical p -brane ansatz

A general ansatz for a metric possessing the symmetries of the non-extremal solution is

$$ds^2 = -B^2 dt^2 + C^2 \sum_{i=1}^p (dx^i)^2 + F^2 dr^2 + G^2 r^2 d\Omega_{4-p}^2, \quad (\text{A.10})$$

where B , C , F and G are functions of the transverse radial coordinate r only. The equations of motion (A.4)–(A.5) for the two gauge fields $A_{(p+1)}$ and $C_{(p+1)}$ can trivially be integrated to give¹³

$$\tilde{F}_{r0\dots p} = e^{-(1-p)\phi} B C^p F \frac{\hat{q}_1}{(Gr)^{4-p}}, \quad (\text{A.11})$$

$$[2ex]\tilde{G}_{r0\dots p} = e^{-(1-p)\phi} e^{-\eta} B C^p F \frac{\hat{q}_2}{(Gr)^{4-p}}, \quad (\text{A.12})$$

where we used the result $\sqrt{-g} = \sqrt{g\Omega_{4-p}} r^{4-p} B C^p F G^{4-p}$. The two “hatted” charges, \hat{q}_1 and \hat{q}_2 , introduced as a shorthand notation for this appendix, each absorbs a p -dependent factor according to

$$\hat{q}_1 = k_p q_1, \quad \hat{q}_2 = k_p q_2, \quad k_p = \begin{cases} 3-p, & p = 0, 1, 2 \\ 1, & p = 3 \end{cases}, \quad (\text{A.13})$$

with q_1 and q_2 being the charges which appear in the solutions and which are natural from the physical perspective. Defining

$$\hat{b} = 1 + \frac{q_1}{q_2} \tilde{b}, \quad (\text{A.14})$$

we have

$$G_{r0\dots p} = e^{-(1-p)\phi} e^{-\eta} B C^p F \frac{\hat{q}_2 \hat{b}}{(Gr)^{4-p}}, \quad (\text{A.15})$$

so that

$$e^{(1-p)\phi} e^{\eta} (G_{(p+2)})^2 = -e^{-(1-p)\phi} e^{-\eta} \hat{b}^2 \frac{\hat{q}_2^2}{(Gr)^{2(4-p)}}, \quad (\text{A.16})$$

$$[2ex]e^{(1-p)\phi} (\tilde{F}_{(p+2)})^2 = -e^{-(1-p)\phi} \frac{\hat{q}_1^2}{(Gr)^{2(4-p)}}. \quad (\text{A.17})$$

¹³In form notation this corresponds to $\tilde{F}_{(p+2)} = (-1)^p \hat{q}_1 e^{-(1-p)\phi} * d\Omega_{4-p}$, so that $*d*[e^{(1-p)\phi} \tilde{F}_{(p+2)}] = -\hat{q}_1 * d^2 \Omega \equiv 0$ and $\int_{S^{4-p}} e^{(1-p)\phi} * \tilde{F}_{(p+2)} = -\hat{q}_1 \Omega_{4-p}$.

Introducing the notation

$$L \equiv BC^p F^{-1} (Gr)^{4-p}, \quad (\text{A.18})$$

$$Y \equiv -e^{(1-p)\phi} \left(e^\eta (G_{(p+2)})^2 + (\tilde{F}_{(p+2)})^2 \right) = \frac{e^{-(1-p)\phi}}{(Gr)^{2(4-p)}} \left(\hat{q}_1^2 + \hat{q}_2^2 e^{-\eta} \hat{b}^2 \right), \quad (\text{A.19})$$

the scalar-field equations of motion (A.1)–(A.3) can be written compactly as

$$\phi'' + \phi' (\log L)' = -\frac{1-p}{4} F^2 Y, \quad (\text{A.20})$$

$$\eta'' + \eta' (\log L)' = -F^2 e^{-(1-p)\phi} e^{-\eta} \hat{b}^2 \frac{\hat{q}_2^2}{(Gr)^{2(4-p)}} - \frac{q_2^2}{q_1^2} e^{-\eta} (\hat{b}')^2, \quad (\text{A.21})$$

$$L^{-1} (L e^{-\eta} \hat{b}')' = F^2 e^{-(1-p)\phi} e^{-\eta} \frac{\hat{q}_1^2}{(Gr)^{2(4-p)}} \hat{b}. \quad (\text{A.22})$$

Finally, the Einstein equations read (no summation over indices)

$$R^t_t = \frac{p-3}{8} Y, \quad (\text{A.23})$$

$$R^i_i = \frac{p-3}{8} Y, \quad (\text{A.24})$$

$$R^r_r = F^{-2} \left((\phi')^2 + \frac{1}{4} (\eta')^2 + \frac{1}{2} \frac{q_2^2}{q_1^2} e^{-\eta} (\hat{b}')^2 \right) + \frac{p-3}{8} Y, \quad (\text{A.25})$$

$$R^\alpha_\alpha = \frac{p+1}{8} Y. \quad (\text{A.26})$$

Here (for $p > 0$) $i = 1, \dots, p$ are the spatial world-volume directions while α runs over the $4-p$ transverse angular coordinates. The Ricci-tensor components for a metric of the form (A.10) are

$$R^t_t = \frac{1}{F^2} \left(-(\log B)'' - (\log B)' (\log L)' \right), \quad (\text{A.27})$$

$$R^i_i = \frac{1}{F^2} \left(-(\log C)'' - (\log C)' (\log L)' \right), \quad (\text{A.28})$$

$$\begin{aligned} R^r_r = \frac{1}{F^2} & \left(-(\log B)'' - ((\log B)')^2 + (\log F)' (\log B)' \right. \\ & + p \left[-(\log C)'' - ((\log C)')^2 + (\log F)' (\log C)' \right] \\ & \left. + (4-p) \left[-(\log Gr)'' - ((\log Gr)')^2 + (\log F)' (\log Gr)' \right] \right), \end{aligned} \quad (\text{A.29})$$

$$R^\alpha_\alpha = \frac{1}{F^2} \left(-(\log Gr)'' - (\log Gr)' (\log L)' + (3-p) \frac{F^2}{(Gr)^2} \right). \quad (\text{A.30})$$

A.3 Solving the equations of motion

In order to find the non-extremal versions of the supergravity solutions for fractional D0- and D2-branes on T^4/\mathbb{Z}_2 of ref. [6] that reduce to Schwarzschild black-brane metrics for vanishing charges, the natural ansatz to employ is

$$ds^2 = H^{\frac{p-3}{4}} \left(-f dt^2 + \sum_{i=1}^p (dx^i)^2 \right) + H^{\frac{p+1}{4}} (f^{-1} dr^2 + r^2 d\Omega_{4-p}^2) . \quad (\text{A.31})$$

This ansatz applies equally well for $p=1$, while for the three-brane, as mentioned in section 2, it is preferable to use the adapted ansatz

$$ds^2 = -f dt^2 + \sum_{i=1}^3 (dx^i)^2 + \left(\frac{r_\Lambda}{r} \right)^2 H (f^{-1} dr^2 + r^2 d\theta^2) . \quad (\text{A.32})$$

From these metrics we can read off expressions for B , C , F and G in terms of the radial functions H and f , and plug them in the equations of motion listed above. Solving the so obtained equations is the objective of the present section.

We start by observing that the harmonic equation (2.10) for f follows from the Einstein equations (A.23) and (A.24) (for $p=0$, eq. (A.24) is not present and other equations need to be used instead). Equipped with this result, we combine the dilaton equation (A.20) with the angular Einstein equation (A.26) to get the equation

$$\chi'' + \chi'(\log L)' = 0 , \quad (\text{A.33})$$

where we have defined $\chi = \log(e^\phi H^{-(1-p)/4})$ and where now $L = f r^{4-p}$. Requiring that the dilaton behave in a non-singular manner at the horizon we find that χ must be constant. Asking furthermore that at infinity we recover flat Minkowski space with vanishing dilaton, we obtain

$$e^\phi = H^{\frac{1-p}{4}} . \quad (\text{A.34})$$

For the case $p=1$, eq. (A.33) simply gives that the dilaton is constant, as appropriate for the string in six dimensions.

It remains to solve the scalar-field equations (A.21)–(A.22) together with the radial Einstein equation (A.25), using also either (A.20) or (A.23). With

$$F^2 Y = \frac{1}{H f r^{2(4-p)}} \left(\hat{q}_1^2 + \hat{q}_2^2 e^{-\eta \hat{b}^2} \right) , \quad (\text{A.35})$$

the two latter amount to the single equation

$$e^{-\eta \hat{b}^2} = \gamma , \quad (\text{A.36})$$

where we have introduced

$$\gamma \equiv -\frac{q_1^2}{q_2^2} - \frac{1}{\hat{q}_2^2} H r^{4-p} \left(f r^{4-p} \frac{H'}{H} \right)' . \quad (\text{A.37})$$

From (A.21) and (A.22) we, on the other hand, obtain the equation

$$\left(L\eta' + \frac{q_2^2}{q_1^2} L e^{-\eta} \hat{b} \hat{b}' \right)' = 0, \quad (\text{A.38})$$

which can trivially be integrated to

$$\left(e^\eta + \frac{1}{2} \frac{q_2^2}{q_1^2} \hat{b}^2 \right)' = q_3 L^{-1} e^\eta, \quad (\text{A.39})$$

q_3 being a constant of integration. Since $L = f r^{4-p}$ we see that the right-hand side becomes singular at the horizon unless $q_3 = 0$. Requiring the scalars η and \hat{b} to be non-singular at the horizon we are hence led to set q_3 to zero. Imposing, furthermore, that these scalars vanish at infinity we arrive at the relation

$$e^\eta = 1 + \frac{1}{2} \frac{q_2^2}{q_1^2} - \frac{1}{2} \frac{q_2^2}{q_1^2} \hat{b}^2, \quad (\text{A.40})$$

which when combined with the dilaton equation written in the form (A.36) finally yields the expressions

$$e^\eta = \left(1 + \frac{1}{2} \frac{q_2^2}{q_1^2} \right) \left[1 + \frac{1}{2} \frac{q_2^2}{q_1^2} \gamma \right]^{-1}, \quad (\text{A.41})$$

$$[2ex]\hat{b} = \sqrt{\gamma} \left(1 + \frac{1}{2} \frac{q_2^2}{q_1^2} \right)^{1/2} \left[1 + \frac{1}{2} \frac{q_2^2}{q_1^2} \gamma \right]^{-1/2}. \quad (\text{A.42})$$

At this point, it remains only to determine the function H by solving the equations (A.21) and (A.25) with the above results as input. To this end, we define the functions h_1 and h_2 by

$$\gamma = \frac{h_2^2}{H}, \quad \frac{1 + \frac{1}{2} \frac{q_2^2}{q_1^2} \gamma}{1 + \frac{1}{2} \frac{q_2^2}{q_1^2}} = \frac{h_1^2}{H}, \quad (\text{A.43})$$

so that

$$H = \left(1 + \frac{1}{2} \frac{q_2^2}{q_1^2} \right) h_1^2 - \frac{1}{2} \frac{q_2^2}{q_1^2} h_2^2, \quad e^\eta = \frac{H}{h_1^2}, \quad \hat{b} = \frac{h_2}{h_1}. \quad (\text{A.44})$$

Using these expressions the gauge field strengths take the form

$$G_{r0\dots p} = \frac{k_p q_2}{r^{4-p}} \frac{h_1 h_2}{H^2}, \quad \tilde{G}_{r0\dots p} = \frac{k_p q_2}{r^{4-p}} \frac{h_1^2}{H^2}, \quad \tilde{F}_{r0\dots p} = \frac{k_p q_1}{r^{4-p}} \frac{1}{H}. \quad (\text{A.45})$$

Having expressed all fields in terms of h_1 and h_2 , we rewrite also the remaining equations of motion which determine these functions. Equation (A.25) thus reads

$$\left(1 + \frac{1}{2} \frac{q_2^2}{q_1^2} \right) h_1 \left(h_1'' + \frac{4-p}{r} h_1' \right) = \frac{1}{2} \frac{q_2^2}{q_1^2} h_2 \left(h_2'' + \frac{4-p}{r} h_2' \right), \quad (\text{A.46})$$

while (A.21) may be written as

$$0 = \frac{\hat{q}_1^2}{r^{2(4-p)}} + H f \left(\log \frac{H}{f} \right)' \left(\log \frac{h_2}{h_1} \right)' \\ [1ex] + H f \left[\frac{1}{h_1} \left(h_1'' + \frac{4-p}{r} h_1' \right) - \frac{1}{h_2} \left(h_2'' + \frac{4-p}{r} h_2' \right) \right]. \quad (\text{A.47})$$

Clearly, the simplest non-trivial way to satisfy eq. (A.46) is by taking both h_1 and h_2 to be harmonic, like f , and we will do so here. Taking the boundary conditions at infinity into account we thus have for $p = 0, 1, 2$

$$f = 1 - \frac{r_0^{3-p}}{r^{3-p}}, \quad h_1 = 1 - \frac{r_1^{3-p}}{r^{3-p}}, \quad h_2 = 1 - \frac{r_2^{3-p}}{r^{3-p}}. \quad (\text{A.48})$$

For $p = 3$ the three harmonic functions governing the non-extremal solution instead take the form

$$f = 1 - a_0 \log \frac{r_\Lambda}{r}, \quad h_1 = 1 - a_1 \log \frac{r_\Lambda}{r}, \quad h_2 = 1 - a_2 \log \frac{r_\Lambda}{r}, \quad (\text{A.49})$$

where r_Λ is a large-radius cut-off. In both cases, we are left with two undetermined parameters— $r_{1,2}$ and $a_{1,2}$ respectively—and in both cases these parameters are fixed by eq. (A.47). However, before analysing this equation, let us derive the solutions for the gauge potentials $C_{(p+1)}$ and $A_{(p+1)}$.

Starting with the untwisted sector, we get an integral for $C_{0\dots p}$ from eq. (A.45) by recalling that $G_{r0\dots p} = C'_{0\dots p}$. For $p = 0, 1, 2$, this integral can readily be evaluated with the result

$$C_{0\dots p} = -\frac{q_2}{r^{3-p}} \frac{h_3}{H}, \quad h_3 \equiv \frac{1}{2}(h_1 + h_2). \quad (\text{A.50})$$

To obtain this result we used the fact that h_1 and h_2 being harmonic implies the identity

$$h_3 H + \frac{r}{3-p} (h_3 H' - h_3' H) = h_1 h_2. \quad (\text{A.51})$$

With $C_{0\dots p}$ in hand, we similarly obtain $A_{0\dots p}$ by integrating the equation

$$A'_{0\dots p} = \tilde{F}_{r0\dots p} - \tilde{b}' C_{0\dots p} = (3-p) \frac{q_1}{r^{4-p}} \frac{1}{H} \left[1 + \frac{r}{3-p} \frac{q_2^2}{q_1^2} h_3 \left(\frac{h_2}{h_1} \right)' \right], \quad (\text{A.52})$$

The solution is found to be

$$A_{0\dots p} = -\frac{q_1}{r^{3-p}} \frac{1}{h_1}. \quad (\text{A.53})$$

Here we used the identity

$$H = h_1^2 + \frac{r}{3-p} \frac{q_2^2}{q_1^2} h_3 (h_1 h_2' - h_1' h_2), \quad (\text{A.54})$$

which, like (A.51), follows directly from the form of H and h_3 in terms of h_1 and h_2 .

For the three-brane the situation is somewhat more subtle since there are no analogues of the first-order differential equations (A.51) and (A.54). Instead, led by the form of the harmonic functions, we apply the mapping $r^{-(3-p)} \mapsto \log(r_\Lambda/r)$ to the potentials in (A.50) and (A.53) above. It is then straightforward to check that the so obtained expressions give the correct gauge potentials:

$$C_{0123} = -\frac{q_2 h_3}{H} \log \frac{r_\Lambda}{r}, \quad A_{0123} = -\frac{q_1}{h_1} \log \frac{r_\Lambda}{r}. \quad (\text{A.55})$$

Let us then, finally, address eq. (A.47), which for harmonic h_1 and h_2 immediately simplifies to

$$\frac{(k_p q_1)^2}{r^{2(4-p)}} + H f \left(\log \frac{H}{f} \right)' \left(\log \frac{h_2}{h_1} \right)' = 0. \quad (\text{A.56})$$

By performing the p -dependent substitutions of variables and parameters

$$\sigma = \begin{cases} r^{-(3-p)} & p = 0, 1, 2 \\ \log \frac{r_\Lambda}{r} & p = 3 \end{cases}, \quad \rho_{0,1,2} = \begin{cases} r_{0,1,2}^{3-p} & p = 0, 1, 2 \\ a_{0,1,2} & p = 3 \end{cases}, \quad (\text{A.57})$$

this equation reduces to the single, p -independent, equation

$$q_1^2 + \hat{H} \hat{f} \left(\log \frac{\hat{H}}{\hat{f}} \right)' \left(\log \frac{\hat{h}_2}{\hat{h}_1} \right)' = 0, \quad (\text{A.58})$$

where

$$\hat{f} = 1 - \rho_0 \sigma, \quad \hat{h}_{1,2} = 1 - \rho_{1,2} \sigma, \quad \hat{H} = \left(1 + \frac{1}{2} \frac{q_2^2}{q_1^2} \right) \hat{h}_1^2 - \frac{1}{2} \frac{q_2^2}{q_1^2} \hat{h}_2^2, \quad (\text{A.59})$$

and the prime now denotes differentiation with respect to σ . Note that only the “unhatted” charges q_1 and q_2 enter in eq. (A.58), showing that these are the charges that appear in the harmonic functions. Inserting the expressions (A.59) we find that the two undetermined parameters ρ_1 and ρ_2 are required by (A.58) to satisfy

$$\rho_1 = u, \quad \rho_2 = \frac{u \rho_0}{2u - \rho_0}, \quad (\text{A.60})$$

where u is a solution of the quartic equation

$$4(2q_1^2 + q_2^2)u^4 - 8(2q_1^2 + q_2^2)\rho_0 u^3 + 2(2q_2^2\rho_0^2 + 5q_1^2\rho_0^2 - 2q_1^4)u^2 + 2(2q_1^4\rho_0 - q_1^2\rho_0^3)u - q_1^4\rho_0^2 = 0. \quad (\text{A.61})$$

This equation has the four solutions

$$u = \frac{1}{2}\rho_0 + \epsilon_1 \frac{\sqrt{2q_1^4 + (q_1^2 + q_2^2)\rho_0^2 - \epsilon_2 2q_1^2\Lambda}}{2\sqrt{2q_1^2 + q_2^2}}, \quad (\text{A.62})$$

where $\epsilon_1 = \pm 1$, $\epsilon_2 = \pm 1$ and

$$\Lambda = \sqrt{q_1^4 + (q_1^2 + q_2^2)\rho_0^2 + \frac{1}{4}\rho_0^4}. \quad (\text{A.63})$$

Consequently, the solutions to the equations of motion have four branches given by

$$\rho_1 = \frac{1}{2}\rho_0 + \epsilon_1 \frac{\sqrt{2q_1^4 + (q_1^2 + q_2^2)\rho_0^2 - \epsilon_2 2q_1^2 \Lambda}}{2\sqrt{2q_1^2 + q_2^2}}, \quad (\text{A.64})$$

$$[2ex]\rho_2 = \frac{1}{2}\rho_0 + \epsilon_1 \frac{\sqrt{2q_1^4 + (q_1^2 + q_2^2)\rho_0^2 + \epsilon_2 2q_1^2 \Lambda}}{2\sqrt{q_2^2}}, \quad (\text{A.65})$$

where, again, $\epsilon_1 = \pm 1$ and $\epsilon_2 = \pm 1$. There are thus four solutions, which may be summarized by the expressions given in section 2.3, with $\rho_{1,2}$ as in (A.64), (A.65). By taking the limit $\rho_0 \rightarrow 0$ in these expressions, one can easily see that the branch which correctly reduces to the extremal fractional-brane solution—i.e. the branch which satisfies the boundary conditions imposed by the action (2.8)—and which thus represents the non-extremal fractional brane solution, is the branch with $\epsilon_1 = +1$ and $\epsilon_2 = +1$. This choice corresponds to (2.17)–(2.18) ($p = 0, 1, 2$) and (2.28)–(2.29) ($p = 3$) in the text.

A.4 Some properties of the four branches

To understand the properties of these four branches it is useful to consider the sign of H at the horizon. (Since, as pointed out in section 2.3, the three-brane case is special, we restrict the discussion to $p < 3$.) Thus, setting $r = r_0$ in eq. (A.56) we find that

$$H(r_0) = (3 - p) \frac{q_1^2}{r_0^{7-2p}} \left(\frac{h'_2}{h_2} - \frac{h'_1}{h_1} \right)^{-1} \Big|_{r=r_0}. \quad (\text{A.66})$$

Using eq. (A.48) we then obtain

$$\text{sign}(H(r_0)) = \text{sign}(r_2 - r_1) \text{sign}(r_0 - r_1) \text{sign}(r_0 - r_2). \quad (\text{A.67})$$

Hence, $H(r_0)$ is positive if either $r_0 < r_1 < r_2$ or $r_1 < r_2 < r_0$ or $r_2 < r_0 < r_1$, while it is negative if either $r_1 < r_0 < r_2$ or $r_0 < r_2 < r_1$ or $r_2 < r_1 < r_0$.

One can now check that the sign of $H(r_0)$ cannot be changed within a particular branch. From eq. (A.67) we see that this precisely means that r_0 , r_1 and r_2 cannot cross within a particular branch. In table 1 we have listed the restrictions on the ranges for r_0 , r_1 and r_2 , along with the sign of $H(r_0)$ for the four branches. The solution discussed in the text corresponds to branch I.

In order to further examine the physics of the four branches obtained above we compute the ADM mass (per unit p -volume)

$$M_p = \frac{\Omega_{4-p}}{16\pi G_6} \left[(4 - p)r_0^{3-p} + (3 - p)\xi \right], \quad 16\pi G_6 = \frac{2\kappa^2}{V} \quad (\text{A.68})$$

branch	ϵ_1	ϵ_2	restrictions	$\text{sign}(H(r_0))$
I	$+1$	$+1$	$r_1 < r_0 < r_2$	$-$
II	-1	$+1$	$r_2 < r_1 < r_0$	$-$
III	$+1$	-1	$r_2 < r_0 < r_1$	$+$
IV	-1	-1	$r_1 < r_2 < r_0$	$+$

Table 1: Properties of the four branches.

where Ω_{4-p} denotes the volume of the unit $(4-p)$ -sphere and

$$\xi = \frac{1}{q_1^2} \left[-(2q_1^2 + q_2^2)r_1^{3-p} + q_2^2 r_2^{3-p} \right] \quad (\text{A.69})$$

is the coefficient of the leading $1/r^{3-p}$ term in the function H in eq. (A.44). Focusing first on the four branches at $r_0 = 0$, it is not difficult to see that branches II and III are unphysical since they both have negative mass, and hence should be discarded. On the other hand, for branches I and IV we find

$$\xi = \begin{cases} q_2, & (\text{I}) \\ \sqrt{2q_1^2 + q_2^2}, & (\text{IV}) \end{cases} \quad (\text{A.70})$$

so that branch I at $r_0 = 0$ has the lowest mass. To obtain (A.70) we have used (2.22) for branch I while for branch IV at $r_0 = 0$ we have $h_1 = 1 + q_1^2/(\sqrt{2q_1^2 + q_2^2}r^{3-p})$, $h_2 = 1$. The charges are

$$\hat{Q}_1 = \frac{-1}{16\pi G_6} \int_{S^{4-p}} e^{(1-p)\phi} * \tilde{F}_{(p+2)} = (3-p) q_1 \frac{\Omega_{4-p}}{16\pi G_6}, \quad (\text{A.71})$$

$$[1ex]\hat{Q}_2 = \frac{-1}{16\pi G_6} \int_{S^{4-p}} e^{(1-p)\phi} e^\eta * \tilde{G}_{(p+2)} = (3-p) q_2 \frac{\Omega_{4-p}}{16\pi G_6}, \quad (\text{A.72})$$

(recall that $16\pi G_6 = 2\kappa^2/V$). Since the untwisted charge is \hat{Q}_2 irrespective of the branch, we conclude that branch I at extremality is BPS ($M_p = \hat{Q}_2$), while branch IV apparently describes a system that is non-BPS even in the limit $r_0 = 0$.

Although it is presently not clear what the precise physical meaning, if any, of the supergravity solution of branch IV is, we note that this solution has well-defined black-brane thermodynamics. Using standard methods of black-hole thermodynamics, we compute the temperature and entropy:¹⁴

$$T = \frac{3-p}{4\pi} \frac{1}{r_0 \sqrt{H(r_0)}}, \quad S = \frac{\Omega_{4-p}}{4G_6} r_0^{4-p} \sqrt{H(r_0)}. \quad (\text{A.73})$$

¹⁴We note that the following expressions are algebraically valid for all branches. However, we have already excluded branches II and III on the grounds of positivity of the mass, while for branch I it is seen from table 1 that $H(r_0) < 0$ and hence the expressions below are not physically meaningful. Alternatively, we have already argued extensively in the text that branch I does not develop into a black brane.

Using the Wess–Zumino term of the world-volume action (2.8) the corresponding chemical potentials, dual to the charges in (A.71) and (A.72), are

$$\mu_1 = -(A_{01\dots p} + \tilde{b} C_{01\dots p}) \Big|_{r=r_0}, \quad \mu_2 = -C_{01\dots p} \Big|_{r=r_0}. \quad (\text{A.74})$$

More explicitly, using (A.44), (A.50) and (A.53) the chemical potentials read

$$\mu_1 = \frac{q_1}{r_0^{3-p}} \frac{1}{h_1(r_0)} + \frac{q_2}{q_1} \left(\frac{h_2(r_0)}{h_1(r_0)} - 1 \right) \frac{q_2}{r_0^{3-p}} \frac{h_3(r_0)}{H(r_0)}, \quad \mu_2 = \frac{q_2}{r_0^{3-p}} \frac{h_3(r_0)}{H(r_0)}, \quad (\text{A.75})$$

in terms of the harmonic functions h_i , $i=1,2,3$ given in eqs (A.48) and (A.50).

As a check, we note that the first law of thermodynamics $dM_p = TdS + \mu_i d\hat{Q}_i$ can be integrated to yield Smarr’s formula

$$(3-p)M_p = (4-p)TS + (3-p)(\mu_1 \hat{Q}_1 + \mu_2 \hat{Q}_2). \quad (\text{A.76})$$

One may verify explicitly that this law holds since

$$\xi = \mu_1 q_1 + \mu_2 q_2, \quad (\text{A.77})$$

where ξ is defined in (A.69). To prove (A.77) one uses the non-trivial identity

$$\frac{1}{r_0^{3-p}} \frac{1}{h_1(r_0)} \left[q_1^2 + q_2^2 \frac{h_2(r_0)h_3(r_0)}{H(r_0)} \right] = \xi. \quad (\text{A.78})$$

which holds for all branches. To verify this statement one uses the form of H in (A.44) along with h_3 in terms of $h_{1,2}$, and subsequently employs the relation (A.60) to eliminate r_2 . The identity then reduces exactly to the quartic equation (A.61) satisfied by r_1 .

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