Twisted $N=8$, $D=2$ super Yang–Mills theory as example of a Hodge–type cohomological theory

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Abstract

It is shown that the dimensional reduction of the $N_T=2$, $D=3$ Blau–Thompson model to $D=2$, i.e., the novel topological twist of $N=8$, $D=2$ super–Yang–Mills theory, provides an example of a Hodge–type cohomological theory. In that theory the generators of the topological shift, co–shift and gauge symmetry, (Q^a, ${}^\star Q^a$, G), together with a discrete duality operation are completely analogous to the de Rham cohomology operators (d, δ, Δ) and the Hodge $\star$–operation.

1. Introduction

Some very enlightening attemps [1] – [8] have been made to incorporate into the gauge–fixing procedure of general gauge theories besides the basic ingredience of BRST cohomology Q also a co–BRST cohomology ${}^\star Q$ which, together with the BRST Laplacian Δ, form the same kind of superalgebra as their counterparts, the de Rham cohomology operators d, $\delta = \pm \star d \star$ and $\Delta = d\delta + \delta d$, in differential geometry [9], namely,

$$\begin{aligned} &Q^2 = 0, \qquad &{}^\star Q^2 = 0, \qquad &{}^\star Q = \pm \star Q \star, \\ &[\Delta, Q] = 0, \qquad &[\Delta, {}^\star Q] = 0, \qquad &\Delta = \{Q, {}^\star Q\} \neq 0. \end{aligned}$$

In Ref. [8] it has been proven that whenever the ghost–extended quantum state space $H_{\rm ext}$ possesses a non–degenerate inner product, which is also non–degenerated upon restriction to the image of Q,Im Q, it allows the following complete and orthogonal Hodge decomposition:

$$H_{\rm ext} = {\rm Ker}\,\Delta + {\rm Im}\,Q + {\rm Im}\,{}^\star Q.$$

In terms of quantum state vectors, this decomposition theorem asserts that an arbitrary state $\psi \in H_{\rm ext}$ can be uniquely decomposed into a sum of a BRST exact, a co–BRST exact and a BRST harmonic state,

$$\psi = \omega + Q\chi + {}^\star Q\phi, \qquad \Delta\omega = 0 \qquad \Leftrightarrow \qquad Q\omega = 0, \quad {}^\star Q\omega = 0.$$

The physical properties of ψ lie entirely within the BRST harmonic state ω which is given by the zero modes of the operator Δ. In proving the consistency of the BRST quantization procedure the essential difficulty consists in showing that the BRST cohomology defines physical (BRST singlet) states $\psi_{\rm phys} \in H_{\rm phys} = {\rm Ker}\,Q / {\rm Im}\,Q$, which are taken to be BRST–closed but not exact. Usually, in order to select from $H_{\rm ext}$ the physical state space $H_{\rm phys}$, one imposes the BRST gauge condition $Q\psi = 0$ [10] – [12]. However, on the class of BRST–closed states $\psi = \omega + Q\chi$,

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which satisfy this condition, besides the harmonic states ω there exists also a set of spurios BRST–exact states, $Q\chi$, which have zero physical norm. On the other hand, by imposing the co–BRST gauge condition, ${}^\star Q\psi = 0$, as well one gets for each BRST cohomology class the uniquely determined harmonic states, $\psi = \omega$.

It has been a long–standing problem to give a prototype example of the BRST gauge–fixing procedure based on harmonic gauges for a field theoretical model. Recently, it has been shown that the $D = 2$ Maxwell theory can be formulated as being a topological field theory of Witten type with underlying symmetries being of Schwarz type [13]. This topological theory provides an example of a Hodge–type cohomological theory where, not only all the de Rham cohomology operators (d, δ, Δ) are expressed in terms of generators of some local symmetries, but even an analogue of the Hodge duality ($\star$) operation exists as discrete symmetry of that theory. On the other hand, pure $D = 2$ Yang–Mills theory does not such a nice structure, because in that theory there is no discrete symmetry inter–relating the BRST and co–BRST charges by some duality operation [14].

In this paper we give another example for a Hodge–type cohomological theory, but this time for a non–abelian gauge theory. More precisely, we consider a Witten type topological theory in $D = 2$ with an underlying $N_T = 4$ equivariant cohomology. This theory is obtained either by a topological twist of $N = 8$, $D = 2$ super–Yang–Mills theory (SYM) [15] or by a dimensional reduction to $D = 2$ of the $N_T = 2$ novel topological twist of $N = 4$, $D = 3$ SYM constructed by Blau and Thompson [16]. The four nilpotent topological supercharges of that theory, denoted by $Q^a = (Q, \bar{Q})$ and ${}^\star Q^a = ({}^\star Q, {}^\star\bar{Q})$, $a = 1, 2$, together with the generator G of the gauge transformations, obey the same kind of superalgebra as their counterparts in differential geometry, the de Rham cohomology operators d, $\delta = \pm \star d \star$ and Δ. More precisely, there exists an direct analogy between the two sets of cohomology operators and the two sets of symmetry operators of the topological theory according to the following assignments:

$$
\begin{aligned}
d \quad &\Leftrightarrow \quad Q & d \quad &\Leftrightarrow \quad \bar{Q} \\
\delta = \star\, d \star \quad &\Leftrightarrow \quad {}^\star\bar{Q} = \star\, Q \star & \delta = -\star\, d \star \quad &\Leftrightarrow \quad {}^\star Q = -\star\, \bar{Q} \star \\
\Delta = \{d, \delta\} \quad &\Leftrightarrow \quad \{Q, {}^\star\bar{Q}\} = G & \Delta = \{d, \delta\} \quad &\Leftrightarrow \quad \{\bar{Q}, {}^\star Q\} = -G.
\end{aligned}
\tag{1}
$$

That is, the exterior and the co–exterior derivatives, d and δ, are related to the nilpotent topological shift and co–shift operators (Q, $\bar{Q}$) and (${}^\star Q$, ${}^\star\bar{Q}$), respectively. Furthermore, it is shown that the analogue of the Laplacian Δ is the gauge generator G. This is due to the fact that for the $N_T = 4$ equivariant cohomology the anti–commutation relations of the topological superalgebra, $\{Q, {}^\star\bar{Q}\} = G$ and $\{\bar{Q}, {}^\star Q\} = -G$, close only modulo the gauge transformation $G = 2\delta_G(\phi)$ generated by a Q^a– and ${}^\star Q^a$–invariant scalar field ϕ. Moreover, it will be shown that the topological supercharges are interrelated by an exact discrete symmetry of that theory which may be interpreted as the corresponding duality ($\star$) operation.

The outline of the paper is as follows. In Section 1 we construct $N_T = 4$, $D = 2$ topological Yang–Mills theory (TYM) by performing, first, a dimensional reduction of $N = 1$, $D = 6$ SYM to $D = 2$ and, afterwards, a topological twist analogous to the one used in constructing the $N_T = 2$ Blau–Thompson model in $D = 3$ [16]. In Section 2 we give a realization of the de Rham cohomology operators in terms of generators of the topological shift and co–shift symmetries. Furthermore, in Section 3, we show that they are related to each other by a discrete symmetry of the theory having the property of a Hodge–type duality operation. In addition, we give the transformation rules for the vector and co–vector supersymmetries.

2. The topological twist of $N = 8$, $D = 2$ super–Yang–Mills theory

In this section we construct the $N_T = 4$, $D = 2$ topological gauge theory with an underlying equivariant cohomology whose four supercharges Q^a and ${}^\star Q^a$, $a = 1, 2$, together with the gauge generator G obey the cohomological algebra (1). In order to get that theory we dimensional reduce and twist the $N=1$, $D=6$ SYM in the Euclidean space–time. Since the details of this procedure are straightforward, we will only focus on the relevant steps.

The Euclidean action of $N=1$, $D=6$ SYM,

$$S^{(N=1)} = \int d^6x \, \mathrm{tr}\Big\{ \tfrac{1}{4} F^{MN} F_{MN} + i\bar{\lambda}\Gamma^M D_M \lambda \Big\}, \tag{2}$$

is build up from an anti–hermitean vector field A_M ($M = 1, \dots 6$) and a complex chiral spinor λ in the adjoint representation of the gauge group. This action is invariant under the following supersymmetry transformations:

$$\delta_Q A_M = i\bar{\lambda}\Gamma_M \epsilon - i\bar{\epsilon}\Gamma_M \lambda, \qquad \delta_Q \lambda = -\tfrac{1}{2}\Gamma^M\Gamma^N F_{MN}\epsilon, \qquad \delta_Q \bar{\lambda} = \tfrac{1}{2}\bar{\epsilon}\Gamma^M\Gamma^N F_{MN}, \tag{3}$$

ϵ being a chiral spinor. For the 8–dimensional Euclidean Dirac matrices, $\{\Gamma_M, \Gamma_N\} = -2\delta_{MN}\mathrm{I}_8$, we choose the representation

$$\begin{aligned}
\Gamma_m &= -i \begin{pmatrix} 0 & \delta_\alpha^{\ \beta} (\sigma_m)_A^{\ B} \\ \delta_\alpha^{\ \beta} (\sigma_m)_A^{\ B} & 0 \end{pmatrix}, \qquad m = 1,2,3, \quad \alpha = 1,2, \quad A = 1,2, \\
\Gamma_{3+m} &= \begin{pmatrix} 0 & -(\sigma_m)_\alpha^{\ \beta} \delta_A^{\ B} \\ (\sigma_m)_\alpha^{\ \beta} \delta_A^{\ B} & 0 \end{pmatrix}, \qquad \Gamma_7 = -i \begin{pmatrix} \delta_\alpha^{\ \beta} \delta_A^{\ B} & 0 \\ 0 & -\delta_\alpha^{\ \beta} \delta_A^{\ B} \end{pmatrix},
\end{aligned}$$

where σ_m are the Pauli matrices. At the first sight this representation seems not to be well adapted to a $4+2$ decomposition of the Euclidean space–time since it does not directly expose the 2–dimensional Dirac matrices. But, let us shortly explain why nevertheless this representation is well suited for our purpose. After performing a dimensional reduction of $N=1$, $D=6$ SYM to $D=3$ the reduced action of $N=4$, $D=3$ SYM has a global symmetry group $SU(2)_E \otimes SU(2)_N \otimes SU(2)_R$ [17], $SU(2)_N$ being the internal Euclidean symmetry group arising from the decomposition $SO(6) \to SU(2)_N \otimes SU(2)_E$. The symmetry group $SU(2)_E$ is the Euclidean group in $D=3$ and $SU(2)_R$ is the symmetry group of $N=1$, $D=6$ SYM. There are two essentially different possibilities to construct topological models with $N_T=2$ topological supercharges, arising from twisting $N=4$, $D=3$ SYM. The standard twist consists in replacing $SU(2)_E \otimes SU(2)_R$ through is diagonal subgroup and gives the $N_T=2$, $D=3$ super–BF model [18]. The second twist, which is intrinsically 3–dimensional, is obtained by taking the diagonal of $SU(2)_E \otimes SU(2)_N$ and leads to the $N_T=2$, $D=3$ novel topological twist introduced by Blau and Thompson [16]. This model, which is the one of interest here, has certain unusual features, e.g., there are no bosonic scalar fields and hence there is no underlying equivariant cohomology.

After dimensional reduction A_M becomes the $D=3$ dimensional complexified gauge field $A_\mu \pm iV_\mu$. The rather striking feature of this model is the strictly nilpotency of the twisted supercharges $Q^a = (Q, \bar{Q})$ even prior to the introduction of the gauge ghosts. Of course, it is not possible to identify Q and $\bar{Q}$ with the exterior and the co–exterior derivative, d and δ, since, by virtue of the relative nilpotency $\{Q, \bar{Q}\} = 0$, the gauge generator G cannot be identified with the Laplacian Δ. But, by dimensional reducing the $N_T=2$ novel topological model further to $D=2$ the third components of $A_\mu \pm iV_\mu$ become scalar fields, $A_3 + iV_3 = \bar{\phi}$ and $A_3 - iV_3 = \phi$, and, therefore, we reach a topological gauge theory with an underlying $N_T=4$ equivariant cohomology. The twisted supercharges $Q^a = (Q, \bar{Q})$ and $^\star Q^a = (^\star Q, ^\star\bar{Q})$ of this theory are nilpotent as before but the anticommutators $\{Q, ^\star\bar{Q}\} = G$ and $\{\bar{Q}, ^\star Q\} = -G$ where the gauge transformations $G = 2\delta_G(\phi)$ are generated by ϕ. Moreover, this theory turns out to be also invariant under a discrete symmetry inter–relating both, ($Q, ^\star\bar{Q}$) and ($\bar{Q}, ^\star Q$), by a duality operation according to (1), so that we have a perfect example of a Hodge–type cohomological theory.

The chiral spinor $\lambda = i\gamma_7\lambda$ can be written as

$$\lambda = \begin{pmatrix} \lambda_{\alpha A} \\ 0 \end{pmatrix}, \qquad \bar{\lambda} = (0, \bar{\lambda}^{\alpha A}), \qquad \alpha = 1,2, \quad A = 1,2,$$

where the unconstrained, complex 4–spinors $\lambda_{\alpha A}$ and $\bar{\lambda}^{\alpha A}$ transform in the fundamental and their conjugate representation of $SU(4)$, respectively. The spinor indices A are raised and lowered as follows, $\lambda^A = \epsilon^{AB}\lambda_B$ and $\lambda_B = \lambda^A\epsilon_{AB}$, with $\epsilon^{AC}\epsilon_{BC} = \delta^A{}_B$ (and analogous for α).

We further define

$$A_M = \Big\{ A_\mu,\ A_3 = \tfrac{1}{2}(\phi + \bar{\phi}),\ A_{\mu+3} = V_\mu,\ A_6 = \tfrac{1}{2}i(\phi - \bar{\phi}) \Big\}, \qquad \mu = 1,2.$$

As a next step, we suppose that no field depends on $x^3, \cdots, x^6$ and decompose the action (2) under the assumption of trivial dimensional reduction. Then, we obtain the reduced action of $N = 8$, $D = 2$ SYM,

$$\begin{aligned} S^{(N=8)} = \int d^{2x}\, \mathrm{tr} \Big\{ & \tfrac{1}{4} F^{\mu\nu}(A - iV) F_{\mu\nu}(A + iV) + \tfrac{1}{2} D^\mu(A) V_\mu D^\nu(A) V_\nu \\ & + \tfrac{1}{2} D^\mu(A)\phi D_\mu(A)\bar{\phi} + \tfrac{1}{2}[V^\mu, \bar{\phi}][V_\mu, \phi] - \tfrac{1}{8}[\phi, \bar{\phi}][\phi, \bar{\phi}] \\ & - \bar{\lambda}^{\alpha C}(\sigma^\mu)_C{}^D D_\mu(A)\lambda_{\alpha D} - i\bar{\lambda}^{\alpha C}(\sigma^\mu)_\alpha{}^\beta [V_\mu, \lambda_{\beta C}] \\ & - \tfrac{1}{2}\bar{\lambda}^{\alpha C}(\sigma_3)_C{}^D[\phi + \bar{\phi}, \lambda_{\alpha D}] + \tfrac{1}{2}\bar{\lambda}^{\alpha C}(\sigma_3)_\alpha{}^\beta[\phi - \bar{\phi}, \lambda_{\beta C}] \Big\}, \end{aligned} \tag{4}$$

where $F_{\mu\nu}(A) = \partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu]$ is the YM field strenght and $D_\mu(A) = \partial_\mu + [A_\mu, \ \cdot\ ]$ the covariant derivative of the gauge field A_μ. Recalling that in the $D = 2$ dimensional Euclidean space–time there are no propagating degrees of freedom associated with A_μ.

In order to get from (4) a topological theory we perform the same twist as in Ref. [1], i.e., we identify the spinor index A with α, and decompose the twisted spinor fields as follows,

$$\begin{aligned} \lambda_{AB} &= \tfrac{1}{\sqrt{2}} \Big\{ \epsilon_{AB}\eta + (\sigma^\mu)_{AB}\psi_\mu + (\sigma_3)_{AB}\zeta \Big\}, \\ \bar{\lambda}^{AB} &= \tfrac{1}{\sqrt{2}} \Big\{ \epsilon^{AB}\bar{\eta} + (\sigma^\mu)_{AB}\bar{\psi}_\mu + (\sigma_3)^{AB}\bar{\zeta} \Big\}; \end{aligned}$$

here, $\eta^a = (\eta, \bar{\eta})$, $\zeta^a = (\zeta, \bar{\zeta})$ and $\psi^a_\mu = (\psi_\mu, \bar{\psi}_\mu)$ form $SU(2)_R$ doublets of Grassmann–odd scalar fields and ghost–for–antighost vector fields, respectively, $SU(2)_R$ being the internal symmetry group of the twisted action. Then, by making use of the following equalities,

$$\begin{aligned} &(\sigma_\mu)_A{}^C(\sigma_\nu)_{CB} = \delta_{\mu\nu}\epsilon_{AB} + i\epsilon_{\mu\nu}(\sigma_3)_{AB}, \qquad \mu = 1,2, \\ &(\sigma_\mu)_A{}^C(\sigma_3)_{CB} = -\, i\epsilon_{\mu\nu}(\sigma^\nu)_{AB}, \qquad (\sigma_3)_A{}^C(\sigma_3)_{CB} = \epsilon_{AB}, \end{aligned}$$

where $\epsilon_{\mu\nu}$, $\epsilon^{\mu\rho}\epsilon_{\nu\rho} = \delta^\mu{}_\nu$, is the anti–symmetric Levi–Civita tensor in $D = 2$, from (4) we arrive at the twisted action of $N_T = 4$, $D = 2$ TYM we are looking for,

$$\begin{aligned} S_T^{(N_T=4)} = \int d^{2x}\, \mathrm{tr} \Big\{ & \tfrac{1}{4} F^{\mu\nu}(A - iV) F_{\mu\nu}(A + iV) + \tfrac{1}{2}(D^\mu(A) V_\mu\, D^\nu(A) V_\nu \\ & - i\epsilon^{\mu\nu}\epsilon_{ab}\zeta^a D_\mu(A + iV)\psi^b_\nu - \epsilon_{ab}\eta^a D^\mu(A - iV)\psi^b_\mu \\ & + \tfrac{1}{2} D^\mu(A - iV)\phi\, D_\mu(A + iV)\bar{\phi} - \tfrac{1}{2} i[\phi, \bar{\phi}] D^\mu(A) V_\mu \\ & - \tfrac{1}{8}[\phi, \bar{\phi}][\phi, \bar{\phi}] - \tfrac{1}{2} i\epsilon^{\mu\nu}\epsilon_{ab}\bar{\phi}\{\psi^a_\mu, \psi^b_\nu\} + \epsilon_{ab}\phi\{\eta^a, \zeta^b\} \Big\}. \end{aligned} \tag{5}$$

The internal indices a are raised and lowered as follows, $\varphi^a = \epsilon^{ab}\varphi_b$ and $\varphi_b = \varphi^a\epsilon_{ab}$, with ϵ_{ab}, $\epsilon^{ac}\epsilon_{bc} = \delta^a{}_b$, being the invariant tensor of $SU(2)_R$.

3. Realization of de Rham cohomology operators and Hodge $\star$–operation

We now want to show that the topological gauge theory constructed above is indeed an example of a Hodge–type cohomological theory. To begin with, it is convenient to cast the action (5) in the form

$$\begin{aligned} S_T = \tfrac{1}{2} \int d^{2x}\, \mathrm{tr} \Big\{ & i\epsilon^{\mu\nu} \bar{B} F_{\mu\nu}(A - iV) - i\epsilon^{\mu\nu} B F_{\mu\nu}(A + iV) - 4B\bar{B} \\ & - 2i\epsilon^{\mu\nu} \epsilon_{ab} \zeta^a D_\mu(A + iV)\psi^b_\nu - 2\epsilon_{ab} \eta^a D^\mu(A - iV)\psi^b_\mu \\ & + D^\mu(A)\phi D_\mu(A)\bar{\phi} + [V^\mu, \phi][V_\mu, \bar{\phi}] - 4Y D^\mu(A) V_\mu - 4Y^2 \\ & - \tfrac{1}{4}[\phi, \bar{\phi}][\phi, \bar{\phi}] - i\epsilon^{\mu\nu} \epsilon_{ab} \bar{\phi}\{\psi^a_\mu, \psi^b_\nu\} + 2\epsilon_{ab}\phi\{\eta^a, \zeta^b\} - 4E^\mu \bar{E}_\mu \Big\}, \end{aligned} \qquad (6)$$

where we have introduced a set of auxiliary fields, namely the Grassmann–even scalar fields B, $\bar{B}$, Y and the Grassmann–even vector fields E_μ, $\bar{E}_\mu$.

Then, the action (6) can be rewritten as a sum of a topological BF–like term and a Q^a–exact term, which bears a close resemblance to the $N_T = 2$ Blau–Thompson model in $D = 3$ [6],

$$S_T = \tfrac{1}{2} \int d^2x\, \mathrm{tr} \Big\{ i\epsilon^{\mu\nu} \bar{B} F_{\mu\nu}(A - iV) \Big\} + \tfrac{1}{2}\epsilon_{ab} Q^a Q^b X, \qquad Q^a = \begin{pmatrix} Q \\ \bar{Q} \end{pmatrix},$$

with the gauge boson

$$X = - \int d^2x\, \mathrm{tr} \Big\{ \tfrac{1}{2}\bar{\phi}\big(\bar{B} + \tfrac{1}{4} i\epsilon^{\mu\nu} F_{\mu\nu}(A + iV)\big) + i\bar{E}^\mu V_\mu + \tfrac{1}{4}\epsilon_{ab}\eta^a\eta^b \Big\},$$

which is invariant under the following strictly nilpotent topological shift symmetry,

$$\begin{aligned} & Q^a A_\mu = \psi^a_\mu, && Q^a E_\mu = 0, \\ & Q^a V_\mu = -i\psi^a_\mu, && Q^a \phi = 0, \\ & Q^a \zeta^b = 2\epsilon^{ab} B, && Q^a \bar{\phi} = 2\zeta^a, \\ & Q^a \eta^b = -2i\epsilon^{ab} Y + \tfrac{1}{2}\epsilon^{ab}[\phi, \bar{\phi}], && Q^a B = 0, \\ & Q^a \psi^b_\mu = 2\epsilon^{ab} E_\mu + i\epsilon^{ab}\epsilon_{\mu\nu} D^\nu(A - iV)\phi, && Q^a \bar{B} = [\eta^a, \phi], \\ & Q^a \bar{E}_\mu = i\epsilon_{\mu\nu} D^\nu(A + iV)\zeta^a - D_\mu(A - iV)\eta^a + i\epsilon_{\mu\nu}[\psi^{\nu a}, \bar{\phi}], && Q^a Y = \tfrac{1}{2} i[\zeta^a, \phi]. \end{aligned} \qquad (7)$$

Alternatively, since now the underlying topological symmetry is $N_T = 4$, the action (6) can be cast also in the following form,

$$S_T = -\tfrac{1}{2} \int d^2x\, \mathrm{tr} \Big\{ i\epsilon^{\mu\nu} B F_{\mu\nu}(A + iV) \Big\} + \tfrac{1}{2}\epsilon_{ab}\, {}^\star Q^a\, {}^\star Q^b \bar{X}, \qquad {}^\star Q^a = \begin{pmatrix} {}^\star Q \\ {}^\star \bar{Q} \end{pmatrix},$$

with

$$\bar{X} = - \int d^2x\, \mathrm{tr} \Big\{ \tfrac{1}{2}\bar{\phi}\big(B - \tfrac{1}{4} i\epsilon^{\mu\nu} F_{\mu\nu}(A - iV)\big) + iE^\mu V_\mu + \tfrac{1}{4}\epsilon_{ab}\zeta^a\zeta^b \Big\},$$

which is invariant under the following strictly nilpotent topological co–shift symmetry,

$$\begin{aligned} & {}^\star Q^a A_\mu = -i\epsilon_{\mu\nu}\psi^{\nu a}, && {}^\star Q^a \bar{E}_\mu = 0, \\ & {}^\star Q^a V_\mu = \epsilon_{\mu\nu}\psi^{\nu a}, && {}^\star Q^a \phi = 0, \\ & {}^\star Q^a \eta^b = 2\epsilon^{ab}\bar{B}, && {}^\star Q^a \bar{\phi} = 2\eta^a, \\ & {}^\star Q^a \zeta^b = -2i\epsilon^{ab} Y - \tfrac{1}{2}\epsilon^{ab}[\phi, \bar{\phi}], && {}^\star Q^a \bar{B} = 0, \\ & {}^\star Q^a \psi^b_\mu = 2i\epsilon^{ab}\epsilon_{\mu\nu}\bar{E}^\nu + \epsilon^{ab} D_\mu(A + iV)\phi, && {}^\star Q^a B = -[\zeta^a, \phi], \\ & {}^\star Q^a E_\mu = -D_\mu(A + iV)\zeta^a + i\epsilon_{\mu\nu} D^\nu(A - iV)\eta^a - [\psi^b_\mu, \bar{\phi}], && {}^\star Q^a Y = -\tfrac{1}{2} i[\eta^a, \phi]. \end{aligned} \qquad (8)$$

This strongly suggest that both topological symmetries are related to each other by a discrete Hodge–type symmetry of the action S_T. To elaborate this suggestion from (6) it can be seen that S_T is indeed form–invariant under the following duality operation,

$$\star\, S_T = S_T, \tag{9}$$

where the ($\star$) operation is defined by

$$\varphi \equiv \begin{bmatrix} \partial_\mu & A_\mu & V_\mu & Y \\ \psi_\mu & \bar{\psi}_\mu & \phi & \bar{\phi} \\ \zeta & \bar{\zeta} & \eta & \bar{\eta} \\ B & \bar{B} & E_\mu & \bar{E}_\mu \end{bmatrix}, \qquad \star\varphi = \begin{bmatrix} -i\epsilon_{\mu\nu}\partial^\nu & -i\epsilon_{\mu\nu}A^\nu & i\epsilon_{\mu\nu}V^\nu & Y \\ \bar{\psi}_\mu & -\psi_\mu & -\phi & \bar{\phi} \\ \bar{\eta} & -\eta & \bar{\zeta} & -\zeta \\ \bar{B} & B & i\epsilon_{\mu\nu}\bar{E}^\nu & -i\epsilon_{\mu\nu}E^\nu \end{bmatrix}. \tag{10}$$

By carring out two successive operations of ($\star$) on the generic expression φ it has the property

$$\star\,(\star\varphi) = \begin{cases} -\varphi & \text{for } \varphi = [\psi^a_\mu, \zeta^a, \eta^a], \\ \varphi & \text{otherwise.} \end{cases}$$

Furthermore, by inspection of (7) and (8) it can be easily verified that the topological supercharges ($Q, {}^\star\bar{Q}$) and ($\bar{Q}, {}^\star Q$) are related to each other by the duality transformation (10) according to

$$\begin{aligned} Q(\varphi) &= {}^\star\bar{Q}\varphi \qquad &\Rightarrow \qquad {}^\star\bar{Q}\varphi &= (\star Q\star)\varphi, \\ \bar{Q}(\star\varphi) &= -{}^\star Q\varphi \qquad &\Rightarrow \qquad {}^\star Q\varphi &= -(\star\bar{Q}\star)\varphi. \end{aligned}$$

Hence, if S_T is invariant under the topological shift symmetry (7), due to the duality invariance (9), it remains invariant under the topological co–shift symmetry (8) as well,

$$\begin{aligned} QS_T &= 0 \qquad &\Rightarrow \qquad {}^\star\bar{Q}S_T &= (\star Q\star)S_T = {}^\star QS_T = 0, \\ \bar{Q}S_T &= 0 \qquad &\Rightarrow \qquad {}^\star QS_T &= -(\star\bar{Q}\star)S_T = -{}^\star\bar{Q}S_T = 0. \end{aligned}$$

Therefore, as anticipated by the scheme (1), the topological supercharges ($Q, \bar{Q}$), (${}^\star Q, {}^\star\bar{Q}$) as well as the ($\star$) operation can be actually identified with the cohomological operators d, δ and the Hodge duality ($\star$) operation, respectively. Moreover, by virtue of the anticommutation relations

$$\{Q, {}^\star\bar{Q}\} \doteq 2\delta_G(\phi), \qquad \{\bar{Q}, {}^\star Q\} \doteq -2\delta_G(\phi) \qquad \Rightarrow \qquad G \doteq 2\delta_G(\phi),$$

where the symbol $\doteq$ means that the relations are satisfied only on–shell, i.e., by taking into account the equations of motions, and where the gauge transformations $\delta_G(\omega)$ are defined by $\delta_G(\omega)A_\mu = -D_\mu(A)\omega$ and $\delta_G(\omega)X = [\omega, X]$, for $X = (V_\mu, \zeta^a, \eta^a, \psi^a_\mu, \phi, \bar{\phi}, B, \bar{B}, E_\mu, \bar{E}_\mu, Y)$, it follows that on–shell the gauge generator G can be identified with the Laplacian Δ. Let us notice, that ϕ remains invariant under both Q^a and ${}^\star Q^a$, whereas the vector fields $A_\mu - iV_\mu$ and $A_\mu + iV_\mu$ are invariant only under either of the both supercharges, namely Q^a in the former and ${}^\star Q^a$ in the latter case.

Finally, let us notice that the action (6) is also invariant under the following vector and co–ve

$$
\begin{aligned}
&Q_\mu^a A_\nu = \delta_{\mu\nu}\eta^a - i\epsilon_{\mu\nu}\zeta^a,\\
&Q_\mu^a V_\nu = -i\delta_{\mu\nu}\eta^a + \epsilon_{\mu\nu}\zeta^a,\\
&Q_\mu^a \bar{\phi} = 0,\\
&Q_\mu^a \phi = 2i\epsilon_{\mu\nu}\psi^{\nu a},\\
&Q_\mu^a \eta^b = 2\epsilon^{ab}\bar{E}_\mu - i\epsilon^{ab}\epsilon_{\mu\nu}D^\nu(A+iV)\bar{\phi},\\
&Q_\mu^a \zeta^b = -2i\epsilon^{ab}\epsilon_{\mu\nu}\bar{E}^\nu - \epsilon^{ab}D_\mu(A-iV)\bar{\phi},\\
&Q_\mu^a \psi_\nu^b = 2i\epsilon^{ab}\epsilon_{\mu\nu}\bar{B} - 2\epsilon^{ab}F_{\mu\nu}(A) - 2i\epsilon^{ab}D_\mu(A)V_\nu - 2i\epsilon^{ab}\delta_{\mu\nu}Y + \tfrac{1}{2}\epsilon^{ab}\delta_{\mu\nu}[\phi,\bar{\phi}],\\
&Q_\mu^a \bar{B} = i\epsilon_{\mu\nu}D^\nu(A+iV)\eta^a,\\
&Q_\mu^a B = 2D_\mu(A)\zeta^a - i\epsilon_{\mu\nu}D^\nu(A-iV)\eta^a + [\psi_\mu^a,\bar{\phi}],\\
&Q_\mu^a \bar{E}_\nu = 0,\\
&Q_\mu^a E_\nu = D_\mu(A+iV)\psi_\nu^a - D_\nu(A+iV)\psi_\mu^a + \delta_{\mu\nu}D_\rho(A-iV)\psi_\rho^a - [\delta_{\mu\nu}\zeta^a + i\epsilon_{\mu\nu}\eta^a,\phi],\\
&Q_\mu^a Y = iD_\mu(A-iV)\eta^a + \tfrac{1}{2}\epsilon_{\mu\nu}[\psi^{\nu a},\bar{\phi}]
\end{aligned}
\tag{11}
$$

and

$$
\begin{aligned}
&{}^\star\bar{Q}_\mu^a A_\nu = i\epsilon_{\mu\nu}\zeta^a - \delta_{\mu\nu}\eta^a,\\
&{}^\star\bar{Q}_\mu^a V_\nu = -\epsilon_{\mu\nu}\zeta^a + i\delta_{\mu\nu}\eta^a,\\
&{}^\star\bar{Q}_\mu^a \bar{\phi} = 0,\\
&{}^\star\bar{Q}_\mu^a \phi = -2i\epsilon_{\mu\nu}\psi^{\nu a},\\
&{}^\star\bar{Q}_\mu^a \zeta^b = -2i\epsilon^{ab}\epsilon_{\mu\nu}E^\nu + \epsilon^{ab}D_\mu(A-iV)\bar{\phi},\\
&{}^\star\bar{Q}_\mu^a \eta^b = 2\epsilon^{ab}E_\mu + i\epsilon^{ab}\epsilon_{\mu\nu}D^\nu(A+iV)\bar{\phi},\\
&{}^\star\bar{Q}_\mu^a \psi_\nu^b = 2i\epsilon^{ab}\epsilon_{\mu\nu}B + 2\epsilon^{ab}F_{\mu\nu}(A) - 2i\epsilon^{ab}\epsilon_{\mu\rho}\epsilon_{\nu\sigma}D^\rho(A)V^\sigma - 2i\epsilon^{ab}\delta_{\mu\nu}Y - \tfrac{1}{2}\epsilon^{ab}\delta_{\mu\nu}[\phi,\bar{\phi}],\\
&{}^\star\bar{Q}_\mu^a B = -D_\mu(A-iV)\zeta^a,\\
&{}^\star\bar{Q}_\mu^a \bar{B} = -2i\epsilon_{\mu\nu}D^\nu(A)\eta^a + D_\mu(A+iV)\eta^a + [\psi_\mu^a,\bar{\phi}],\\
&{}^\star\bar{Q}_\mu^a E_\nu = 0,\\
&{}^\star\bar{Q}_\mu^a \bar{E}_\nu = D_\mu(A-iV)\psi_\nu^a - D_\nu(A-iV)\psi_\mu^a + \delta_{\mu\nu}D_\rho(A+iV)\psi_\rho^a - [\delta_{\mu\nu}\zeta^a + i\epsilon_{\mu\nu}\eta^a,\phi],\\
&{}^\star\bar{Q}_\mu^a Y = \epsilon_{\mu\nu}D^\nu(A+iV)\zeta^a + \tfrac{1}{2}\epsilon_{\mu\nu}[\psi^{\nu a},\bar{\phi}],
\end{aligned}
\tag{12}
$$

respectively. These symmetries, together with the topological shift and co–shift symmetries (7) and (8), obey the topological superalgebra

$$
\begin{aligned}
\{Q^a,Q^b\} &= 0, & \{Q_\mu^a,Q_\nu^b\} &\doteq 2i\epsilon^{ab}\epsilon_{\mu\nu}\delta_G(\bar{\phi}),\\
\{Q^a,{}^\star Q^b\} &\doteq 2\epsilon^{ab}\delta_G(\phi), & \{Q_\mu^a,{}^\star Q_\nu^b\} &\doteq -2i\epsilon^{ab}\epsilon_{\mu\nu}\delta_G(\bar{\phi}),\\
\{{}^\star Q^a,{}^\star Q^b\} &= 0, & \{{}^\star Q_\mu^a,{}^\star Q_\nu^b\} &\doteq 2i\epsilon^{ab}\epsilon_{\mu\nu}\delta_G(\bar{\phi}),\\
\{Q^a,Q_\mu^b\} &\doteq 2\epsilon^{ab}(\partial_\mu + \delta_G(A_\mu - iV_\mu)), & \{{}^\star Q^a,Q_\mu^b\} &\doteq 2i\epsilon^{ab}\epsilon_{\mu\nu}(\partial^\nu + \delta_G(A^\nu + iV^\nu)),\\
\{Q^a,{}^\star Q_\mu^b\} &\doteq -2\epsilon^{ab}(\partial_\mu + \delta_G(A_\mu - iV_\mu)), & \{{}^\star Q^a,{}^\star Q_\mu^b\} &\doteq -2i\epsilon^{ab}\epsilon_{\mu\nu}(\partial^\nu + \delta_G(A^\nu + iV^\nu)).
\end{aligned}
$$

Thereby, the vector supercharges Q_μ^a and ${}^\star Q_\mu^a$ are related to each other by the duality operation (10) according to

$$
Q_\mu^a = \begin{pmatrix} Q_\mu \\ \bar{Q}_\mu \end{pmatrix}, \qquad {}^\star Q_\mu^a = \begin{pmatrix} {}^\star Q_\mu \\ {}^\star\bar{Q}_\mu \end{pmatrix}, \qquad \text{with} \qquad {}^\star\bar{Q}_\mu = \star Q_\mu \star, \qquad {}^\star Q_\mu = -\star\bar{Q}_\mu\star\,.
$$

It is worth mentioning that the form of the action (6) is completely specified by the topological shift symmetries $(Q^a, {}^\star Q^a)$ and the vector supersymmetries $(Q^a_\mu, {}^\star\bar{Q}^a_\mu)$, fixing all the relative numerical coefficients in (6) and, therefore, allowing, in particular, for a single coupling constant.

In conclusion we remark that in a similar spirit one can study another topological twist of $N = 16$, $D = 2$ SYM, i.e., the dimensional reduction of $N = 1$, $D = 10$ SYM to $D = 2$, with an underlying $N_T = 8$ equivariant cohomology, respectively. It describes a topological gauge theory associated to D1–branes on holomorphic curves in K3s [16]. By restricting to the Euclidean space–time this topological model provides other, but more involved example of a Hodge–type cohomological theory. The same model can also be obtained by a dimensional reduction of the $N_T = 4$ equivariant extension of the Blau–Thompson model [19] to $D = 2$.

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