Extended Conformal Symmetry

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Abstract

We show that the grading of fields by conformal weight, when built into the initial group symmetry, provides a discrete, non-central conformal extension of any group containing dilatations. We find a faithful vector representation of the extended conformal group and show that it has a scale-invariant scalar product and satisfies a closed commutator algebra. The commutator algebra contains the infinite Heisenberg and Virasoro algebras. In contrast to the usual treatments of scale invariance, covariant derivatives and gauge transformations automatically incorporate the correct conformal weights when the extended symmetry is gauged.

1 Introduction

Scale invariance provides an successful class of physical (string) models in 2-dim. One element of the success in 2-dim is the infinite character of the conformal Lie algebra in that dimension. However, as we show here, the standard formulation of scale invariance introduced by Weyl [1] is incomplete. By studying scale *covariance* - the grading of fields by conformal weight implicit in Weyl or conformal geometries - we find a noncentral, conformal extension of any Lie group containing dilatations. We find that the infinite Heisenberg and Virasoro algebras are natural adjuncts of scale covariance in any dimension. When the extended groups are gauged the resulting spaces possess a natural commutator product and a scale-invariant inner product.

Initially motivated by the assignment of weights to classical fields, the use of scale covariance is transformed to a necessity by the well-known result that in quantum field theory, scale-invariant theories spontaneously develop scales. Thus, we build a conformally *covariant* theory based on conformal or other scaling symmetries. We start with the observation that it is not only the *units* of length that are arbitrary. In addition, at the outset we can assign an arbitrary conformal *weight* to the coordinates – then the scale invariance of the action fixes the weights of other fields. When this freedom is described as a symmetry we are led to introduce a set of operators that change the assigned weight. *The full collection of such operators forms a noncentral, discrete extension of the conformal group which satisfies any of the standard definitions of conformal transformations* (i.e., preserve angles, preserve ratios of infinitesimal displacements, preserve light cones [2], [3]).

Extended scaling groups and their gauge theories have a range of novel properties. First, they fill some gaps in the previous formulation. In the past, the covariance of fields has been managed by inserting the conformal weight by hand into gauge transformations and the covariant derivative. But in extended conformal gauge theories, the geometric structure automatically recognizes the weights of fields. This provides a check on our development, indicating the correctness of the extension.

What is more interesting than completing Weyl's picture, however, is the potential usefulness of the new structures for building new field theory models. Extended scaling groups are Lie groups with infinitely many connected components. Tangent vectors to the group manifold are both homothetic tensors and elements of an infinite dimensional vector space, which obey a commutator algebra containing the infinite Heisenberg algebra, the Virasoro algebra, homothetic algebras and Kac-Moody algebras as sub-algebras. These vectors also have an invariant scalar product which is positive definite on the conformally self-dual subspace.

We make use of a recent advance in conformal gauge theory in n-dim to display some of these properties of the extended symmetry. In the past, conformal gaugings ([4]-[7]) treated the special conformal transformations as additional symmetries of the final gauged geometry. But *biconformal gauging* requires the special conformal transformations together with the translations to span the base manifold, giving a $2n$-dim space with n-dim Weyl symmetry group [8]. Because the biconformal gauging has a dimensionless volume form, it is possible to write a scale-invariant action linear in the biconformal curvatures. This action leads to two involutions of the $2n$-dim space which show it to be spanned by a Ricci-flat, n-dim Riemannian spacetime together with a flat n-dim Riemannian spacetime [9]. Thus, higher-dim conformal symmetry is consistent with vacuum general relativity in a straightforward way, without requiring a quadratic action or compensating fields. Recently, these results have been extended to include certain matter sources ([10],[11]). Below we illustrate the use of extended conformal symmetry by performing a biconformal gauging of the extended conformal group.

The layout of the paper is as follows. In the next section, we derive the form of extended scaling groups from the usual representation for dimensionful fields. Then, in Sec.(3), we develop a vector representation on which the extended group acts effectively. The vector representation is isomorphic to the space of tangent vectors to the extended group manifold. We show that these vectors satisfy a commutator algebra containing the infinite Heisenberg and Virasoro algebras and define a class of inner products. Finally, in Sec.(4), we indicate a few of the consequences of the structures of Sec.(3) for field theories based on the extended conformal group. Sec.(5) contains a brief summary.

2 Extended Scale Invariance

In this section and the next, we establish our main results. Here we define the extended conformal group, find its center and find their quotient group. The quotient group acts transitively and effectively. In the next section, we find a vector representation of the group and show that the vectors close under commutation and have an invariant, indefinite scalar product.

To build the group, we first note that nothing physical is changed if we assign an arbitrary conformal weight to the spacetime coordinates, adjusting the weights of other fields accordingly. For simplicity, we take the possible weight assignments to lie in the set Z of integers. This much is necessary to account for our ability to step the conformal weight up or down arbitrarily many times by integrating and differentiating fields.

Now we start with the standard definition: *The extended conformal group,* $\mathcal{C}_E$*, is the group of transformations that preserves angles.* We understand this definition to include transformations of conformal weight, so, in addition to the usual conformal transformations, $\mathcal{C}_E$ includes an infinite discrete part which formalizes the conformal grading. The development is as follows.

Suppose $\mathcal{G}$ is a Lie symmetry group including the 1-dim dilatational subgroup, $e^{\lambda D}$, with generator D. Then the generators $\{G_A\} = \{D, G_\alpha\}$ of the Lie algebra, $\mathcal{L}_g$, of $\mathcal{G}$ satisfy commutation relations of the form

$$[D, G_\alpha] = c_{0\alpha}{}^{\beta} G_\beta + c_{0\alpha}{}^{0} D \qquad (1)$$

$$[G_\alpha, G_\beta] = c_{\alpha\beta}{}^{\gamma} G_\gamma + c_{\alpha\beta}{}^{0} D \qquad (2)$$

Examples for $\mathcal{G}$ include the homogeneous and inhomogeneous homothetic (or Weyl) groups and the conformal group. We wish to study tensor field representations $\phi \in \Phi$ of $\mathcal{G}$ which in addition to their transformation properties under $\mathcal{G}$

$$\phi'(x') = e^{\lambda^A G_A} \phi(x) \qquad (3)$$

are assigned geometric units from the set $L = \{(length)^m | m \in Z\}$. This graded field representation therefore takes the product form

$$\phi(x)(length^k) = \phi(x)\ l^k \in \Phi \otimes L \tag{4}$$

We seek an extension, $\mathcal{G}_E$, of the original group $\mathcal{G}$, that acts on the representation $\Phi \otimes L$. The extension should include a subgroup, $\mathcal{J}$, of operators $J_\Sigma : L \to L$. Thus, $\mathcal{J}$ is a subgroup of the automorphism group of L; in addition we ask for $\mathcal{J}$ to be closed under the action of $\mathcal{G}$. Since we expect, for example, Lorentz transformations or translations to commute with changes of assigned conformal weight, this amounts to closure under dilatations, $e^{\lambda D} \mathcal{J} e^{-\lambda D} \subseteq \mathcal{J}$. The largest such group is easily seen (see Appendix A) to be the set $\{J_k | J_k l^m = l^{m+k}\}$ satisfying $e^{\lambda D} J_k e^{-\lambda D} = e^{-\lambda k} J_k$. It follows that

$$\begin{aligned}
[J_k, J_m] &= 0 && (5)\\
[J_m, D] &= m J_m && (6)\\
[J_m, G_\alpha] &= 0 && (7)\\
J_k J_m &= J_{k+m} && (8)
\end{aligned}$$

We form a group containing both D and $\{J_k\}$ by writing

$$h(\alpha, k, \lambda) = e^\alpha J_k e^{\lambda D} \tag{9}$$

Then the group product is

$$\begin{aligned}
h(\alpha, k, \lambda) h(\beta, m, \gamma) &= e^{\alpha+\beta-\lambda m} J_{k+m} e^{(\lambda+\gamma)D} && (10)\\
&= h(\alpha + \beta - \lambda m, k + m, \lambda + \gamma) && (11)
\end{aligned}$$

The identity element is $h(0, 0, 0) = J_0$, and the inverse to $h(\alpha, k, \lambda)$ is $h(-\alpha + \lambda k, -k, -\lambda)$. Notice that the positive real factor e^α is necessary in order to accomodate the factor $e^{-\lambda m}$ that arises in commuting $e^{\lambda D}$ and J_m into standard form after taking a product.

Next, we extend the dilatation factor, $e^{\lambda D}$, to include the rest of the group $\mathcal{G}$. As noted above, the remaining generators G_α may be expected to commute with J_k. Therefore a general element $g \in \mathcal{G}_E$ takes the form

$$g(\alpha, k, \lambda, \lambda^\alpha) = e^\alpha J_k\ e^{\lambda D + \lambda^\alpha G_\alpha} \tag{12}$$

where $\lambda D + \lambda^\alpha G_\alpha$ is a general element of $\mathcal{L}_g$. We easily check that $\mathcal{G}_E$ is a Lie group. In particular, the inverse of g is given by

$$g^{-1} = e^{-\alpha} e^{-\lambda D - \lambda^\alpha G_\alpha} J_{-k} = e^{-\alpha - \lambda k} J_{-k}\ e^{-\lambda D - \lambda^\alpha G_\alpha} \in \mathcal{G}_E \tag{13}$$

The Lie algebra, $\mathcal{L}_{g_E}$, of $\mathcal{G}_E$ is $\mathcal{L}_g$ extended by the identity to include the necessary factor e^α : $\mathcal{L}_{g_E} = \mathcal{L}_g \oplus \mathbf{1}$. The J_k operators which change conformal weights give the group an infinite number of distinct connected components. Clearly, each of these components is a manifold of dimension $(\dim \mathcal{G} + 1)$ which is in $1-1$ correspondence with $\mathcal{G}\otimes R^+$. Thus, as a manifold, $\mathcal{G}_E$ is homeomorphic to the direct product, $\mathcal{G}\otimes R^+\otimes \mathbf{Z}$.

In order to construct a gauge field theory based on $\mathcal{G}_E$, we study the adjoint action of $\mathcal{G}_E$ on the manifold $\mathcal{M} = \mathcal{G}_E$, seeking the maximal effective subgroup. This will be $\mathcal{G}_E$ modulo its center, where the center of $\mathcal{G}_E$ is the set

$$K = \{g | gpg^{-1} = p, \forall p \in \mathcal{G}_E\} = \{e^\alpha\} \tag{14}$$

Notice that the elements J_k do *not* lie in the center because for a general $p = J_m\ e^{\alpha+\lambda D+\lambda^\alpha G_\alpha} \in \mathcal{M}$ we have

$$J_k p J_{-k} = J_k (J_m\ e^{\alpha+\lambda D+\lambda^\alpha G_\alpha}) J_{-k} = e^{\lambda k} p \neq p \tag{15}$$

However, the projective subgroup $\mathcal{G}_E/K$ is isomorphic to the direct product $\mathcal{G} \otimes \mathcal{J}$ because J_k and $g_c = e^{\lambda D+\lambda^\alpha G_\alpha}$ now commute in $\mathcal{G}_E/K$:

$$J_k g_c = J_k e^{\lambda D+\lambda^\alpha G_\alpha} = e^{k\lambda} g_c J_k \cong g_c J_k \tag{16}$$

The quotient $\mathcal{G}_E/K$ is the maximal effective subgroup. The quotient introduces central charges into both the Lie algebra of $\mathcal{G}_E/K$, and into the commutator algebra of the extended representation and tangent space. The central charges for $\mathcal{G}_E/K$ depend only on the original Lie algebra of $\mathcal{G}$. Those of the tangent space are discussed below.

We note also that $\mathcal{G}_E/K$ is transitive, since any two elements $p = J_k g_{c1}$ and $q = J_m g_{c2}$ are connected by the element $r = J_{m-k} g_{c2} g_{c1}^{-1}$:

$$rp = J_{m-k} g_{c2} g_{c1}^{-1} J_k g_{c1} \cong J_m g_{c2} = q \tag{17}$$

We now find a representation for $\mathcal{G}_E$, which turns out to be isomorphic to the tangent space, $T\mathcal{G}_E$ of $\mathcal{G}_E$.

3 The extended representation space

In this section, we continue to study properties of $\mathcal{G}_E$, finding a faithful vector representation for $\mathcal{G}_E$ and developing its properties. Normally, the Lie algebra of a Lie group provides an adequate vector representation for a Lie group. However, when we wish to represent discrete symmetries this representation is not faithful and we must look at infinitesimal transformations in *each* connected component, rather than just a neighborhood of the identity. To illustrate this point, we first consider the trivial example of $O(3)$, including parity. Then we apply the technique to $\mathcal{G}_E$.

For $O(3)$, a general group element may be written in the form $g = P_\alpha e^{\frac{1}{2}\lambda^i M_i}$ where $\alpha \in \{+,-\}$, $P_+ = \mathbf{1}, P_- = -\mathbf{1}$ and the M_i generate rotations. The P_α have product $P_\alpha P_\beta = P_{\alpha\times\beta}$. An element of the Lie algebra is simply $v = v^i M_i$, which is insufficient to represent the action of parity. Instead, we expand about each connected component to find six generators, $P_+ M_i$ and $P_- M_i$, and a representation of the form

$$v = v_+^i (P_+ M_i) + v_-^i (P_- M_i) \tag{18}$$

The enlarged vector space is sufficient to span such common indefinite parity combinations as

$$\vec{y} = \vec{w} + \vec{u} \times \vec{v} = (\vec{u} \times \vec{v})^k (P_+ M_k) + w^k (P_- M_k). \tag{19}$$

Notice that the basis vectors form a commutator algebra under the usual Leibnitz rule for the commutator of a product. For example, the commutator of two vectors in the P_- sector appropriately lie in the P_+ sector since

$$[v, w] = v^i w^j [P_- M_i, P_- M_j] = (\vec{v} \times \vec{w})^k P_+ M_k \tag{20}$$

Similarly, to find a faithful representation of $\mathcal{G}_E$ we consider the expansion of $\mathcal{G}_E$ about each J_k. We find

$$e^\alpha J_k \, e^{\lambda D + \lambda^\alpha G_\alpha} \;\approx\; J_k (1 + \alpha 1 + \lambda D + \lambda^\alpha G_\alpha) \tag{21}$$

so that the representation has the basis

$$A = (J_k \mathbf{1}, J_k D, J_k G_\alpha) \equiv (J_k, L_k, G_\alpha^k) \tag{22}$$

where we have defined

$$L_k \;\equiv\; J_k D \tag{23}$$
$$G_\alpha^k \;\equiv\; J_k G_\alpha \tag{24}$$

It is straightforward to show that vectors of the form

$$v = \sum_{k=-\infty}^{\infty} \left(v^k J_k + v^{0k} L_k + v^{\alpha k} G_\alpha^k \right) \in V \tag{25}$$

give a faithful representation of $\mathcal{G}_E$. In fact, this representation is isomorphic to the tangent space to $\mathcal{G}_E$. The components of v have conformal weights $(k, k, k + w_\alpha)$ where w_α is the weight of G_α.

Since $[J_m, J_n] = 0$ and because of the form of the product $J_k J_m$ given in eq.(8), this space also admits a commutator product between vectors, again defined using $[AB, C] = A[B, C] + [A, C]B$. This algebra is generally *not* a Lie algebra because the Jacobi identities fail. Instead, we have Jacobi relations, given in Appendix B. All of the non-Jacobi terms are proportional to $c_{\alpha\beta}{}^0$ or $c_{0\alpha}{}^0$, so for the case of homogeneous or inhomogeneous homothetic algebras, the usual Jacobi identities do hold. Recalling that the projective quotient also introduces central charges into V, the algebra of V becomes

$$
\begin{aligned}
[G^k_\alpha, G^m_\beta] &= c_{\alpha\beta}{}^\gamma G^{k+m}_\gamma + c_{\alpha\beta}{}^0 L_{k+m} + c_{\alpha\beta}\delta^0_{k+m} && (26)\\
[L_k, G^m_\alpha] &= (c_{0\alpha}{}^\beta - m\delta^\beta_\alpha) G^{k+m}_\beta + c_{0\alpha}{}^0 L_{k+m} && (27)\\
[L_k, L_m] &= (k-m)L_{k+m} + ak(k^2-1)\delta^0_{k+m} && (28)\\
[J_m, L_k] &= mJ_{k+m} + b_k\delta^0_{k+m} && (29)\\
[J_k, J_m] &= ck\delta^0_{k+m} && (30)
\end{aligned}
$$

The algebra has several readily identifiable properties.

1. The subalgebra generated by $\{G^0_\alpha, L_0, J_0 = \mathbf{1}\}$ is the original extended Lie algebra of $\mathcal{G}_E$, while $\{G^0_\alpha, L_0\}$ is the Lie algebra of $\mathcal{G}$.

2. The subalgebra generated by $\{J_k\}$ is the infinite Heisenberg algebra.

3. The subalgebra generated by $\{L_k\}$ is the Virasoro algebra.

4. For any proper Lie subalgebra of $\mathcal{L}_g$, with basis $\{H_\beta\} \subset \{G_\alpha\}$, the set $\{H^k_\beta\}$ is a basis for the associated Kac-Moody algebra. For example, when $\mathcal{G}$ is the conformal group, eq.(26) includes the Poincaré-Kac-Moody algebra. Notice that the commutator algebra does *not* contain the Kac-Moody algebra of $\mathcal{G}$ because of the nontrivial commutator for the L_k. This commutator is nontrivial precisely because the extension is noncentral, i.e., the dilatation generator D measures the weight of J_k the same way it measures the weight of G_A.

We may also define a class of indefinite, weight-m scalar products for V whenever the original group has a nontrivial Killing metric, $K_{AB} = c_{EA}{}^F c_{FB}{}^E$. Using the adjoint representation the Lie algebra of $\mathcal{G}$ we have

$$K_{AB} = tr(G_A G_B) \qquad (31)$$

so that if we define

$$Tr(J_k G_A) = \delta^0_k \; tr(G_A) \qquad (32)$$

we have

$$\begin{aligned} \langle v, w \rangle_m &\equiv Tr(vJ_{-m}w) && (33) \\ &= \sum_k v^{A,k} w^{B,m-k}\ K_{AB} = \langle w, v \rangle_m && (34) \end{aligned}$$

where we have used $tr(G_A) = 0$. The result has weight m because we require the weight of $v = v^m J_m$ to be zero. Of greatest interest is the scale-invariant case, $m = 0$,

$$\langle v, w \rangle \equiv \langle v, w \rangle_0 = Tr(vw) = \sum_k v^{Ak} w^{B,-k} K_{AB} \tag{35}$$

We can find a subspace on which he 0-weight inner product defines a norm. Let the conformal dual of a vector be defined as

$$\bar{v} \equiv \sum v^{kA} J_{-k} G_A \tag{36}$$

and define v to be self-dual if there exists a gauge in which $v = \bar{v}$. Then a gauge-invariant norm is given on the space of self-dual vectors by

$$\parallel v \parallel^2 \equiv \langle v, v \rangle = \sum_k v^{kA} v^{kB} K_{AB} \tag{37}$$

In general, $\parallel v \parallel^2$ shares the signature of K_{AB}. For self-dual vectors of the form $v_v = v^k J_k$ (vertical on the bundle defined below) the norm is positive definite.

As an example of the use of the extended conformal group, we now consider its biconformal gauging.

4 Gauging the extended conformal group

The connected component of the extended conformal group only differs from the conformal group through the presence of the positive real factor, e^{α}, and this part is factored out to produce an effective group action. Therefore, there is no difference between the *local* structure of extended biconformal gauge theory and the biconformal gauging described in ([8]-[11]). For this reason, we give only a brief summary of the biconformal space, then move directly to some properties of graded tensor fields.

We first consider the action of $\mathcal{G}_E/K$ on $\mathcal{M} = \mathcal{G}_E/K$. As just noted, the local structure of $\mathcal{M}$ is described by the original Lie algebra of $\mathcal{G}$. This means that even though $\mathcal{M}$ has multiple connected components, the connection is still a $\mathcal{G}$-valued 1-form on $\mathcal{M}$. Each of the connected components therefore shares the same curvature, and $\mathcal{M}$ is homeomorphic to a direct product, $J \otimes \mathcal{M}_0$. The $\mathcal{G}$-valued connection is sufficient to provide a unique $\mathcal{G}$ mapping along any given curve between any two points on the same connected component, while the discrete operators J_k map uniquely from component to component. Thus, the direct product allows us to define a $\mathcal{G}_E/K$ action on $\mathcal{M}$. Moreover, the tangent bundle to $\mathcal{M}$ includes a copy of the tangent space associated with each connected component, giving an infinite-dimensional irreducible vector representation of $\mathcal{G}_E/K$. The direct product structure of the base manifold, $J \otimes \mathcal{M}_0$, means the manifold itself may be treated as a trivial bundle with projection $\pi : J_k \to \mathbf{1}$. The tangent space $T\mathcal{G}_E$ may be divided into horizontal and vertical vector spaces using this projection.

Our notation follows that of refs [8] and [9], and is based on $O(n, 2)$. The fibre bundle is given by the quotient $\mathcal{C}/\mathcal{W}$, with the connection 1-form

$$\omega = \frac{1}{2}\omega_b^a M_a^b + \omega_0^0 D \tag{38}$$

where the M_b^a generate Lorentz transformations. The biconformal base manifold is spanned by the $2n$ 1-forms (ω_0^a, ω_a^0). Because these basis forms have opposite scaling weights, biconformal geometry has a scale invariant volume form, and allows us to write a scale invariant action linear in the curvature tensors without the use of compensating fields. The resulting field equations, subject to a constraint of minimal torsion, lead to a foliation of the $2n$-dim space by n-dim Riemannian spacetimes satisfying the vacuum Einstein equation. The theory therefore makes close contact with general relativity. The structure equations, curvatures and gravitational field equations are reported in detail in [9].

We now consider tensor fields on such a biconformal geometry. The effect of such matter fields on the results of [9] are under separate investigation [11], so we will limit our focus to a few basic properties of graded matter fields in flat space. For tangent vectors we have

$$v = v^m J_m + v_m^a P_a^m + v_a^m K_m^a \tag{39}$$

where the generators P_a and K^a are for translations and co-translations, respectively. The covariant derivative is given by

$$\begin{aligned}
\mathbf{D}v &= \mathbf{d}v + [\omega, v] &(40)\\
&= (\mathbf{d}v^m + m\omega_0^0 v^m)J_m &(41)\\
&\quad +(\mathbf{d}v_m^a - v_m^c\omega_c^a + (m+1)\omega_0^0 v_m^a)P_a^m &(42)\\
&\quad +(\mathbf{d}v_a^m + \omega_d^c v_c^m + (m-1)\omega_0^0 v_a^m)K_m^a &(43)
\end{aligned}$$

which shows that the scale-covariant derivative, with appropriate weights, emerges correctly. This result confirms our claim at the start, that the extended conformal group completes Weyl's description of scale invariance. We can now differentiate arbitrary weight fields correctly without inserting the weights by hand.

In addition to providing a consistent formalism, the extended group has interesting field theoretic properties. Consider the dynamics of a general weight vector, $v = v^m J_m$. We easily write a massive, scale-invariant action for v, using the invariant inner product:

$$\frac{1}{2}\sum_{n=-\infty}^{\infty}\int (\langle \mathbf{D}v, \mathbf{D}v\rangle + \langle \mathrm{m}^2 v, v\rangle)\mathbf{\Phi} \tag{44}$$

where $\langle \mathbf{D}v, \mathbf{D}v\rangle = \sum K^{AB} D_A v^n D_B v^{-n}$ and $\mathbf{\Phi}$ is the dimensionless biconformal volume element. Then weight of the mass is taken as -1, so that $\langle \mathrm{m}^2 v, v\rangle = \sum \mathrm{m}^2 v^{n+2} v^{-n}$. Neglecting gravitational effects, the field equations are

$$K^{AB} D_A D_B v^n = \mathrm{m}^2 v^{n+2} \tag{45}$$

where indices $A, B, \ldots$ run over the full basis, $\omega^A = (\omega_0^a, \omega_a^0)$ and K^{AB} is the projection of the conformal Killing metric to the base space, $K^{ab} = K_{ab} = 0, K^a{}_b = K_b{}^a = \delta^a_b$. Substituting the form of K^{AB} in the expressions above gives

$$K^{AB} D_A D_B v^n = (D^a D_a + D_a D^a) v^m \tag{46}$$

While eq.(45) appears to be a straightforward classical wave equation, there are two important differences. First, we note that the presence of the mass term couples component fields of different conformal weight. This means that if the theory is not to break conformal invariance, it must be massless. If we keep the mass we necessarily produce mixing between different weight fields. Of course, such mixing is also produced by generic potentials, $U(v)$.

To see the second difference, recall the commutator algebra satisfied by vertical tangent vectors. For simplicity, let the spacetime be flat and let π_m be the canonically conjugate momentum to v^m. Then

$$\pi_m = \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial v^m}{\partial x^0}\right)} = \frac{\partial v^{-m}}{\partial y_0} = \partial^0 v^{-m} \tag{47}$$

Notice that, for scale-invariant actions, canonical conjugacy and conformal conjugacy always coincide. For v^m and π_m, the commutator algebra with central charges gives

$$[v, \pi] = amv^m \partial^0 v^{-m} \tag{48}$$

Therefore, we have a nonvanishing commutation relation between a field and its conjugate momentum. It is shown in [10] that the term on the right is proportional to one component of the Weyl vector. The commutator arises because of the central charge, a. Recall that the central charge, a, is a necessary consequence of the exponential factor in the original group and the demand that the group act effectively.

The commutator in eq.(48) is quadratic in the fields, and therefore of the same order as the source terms for the gravitational sector of the geometry. For this reason, we cannot explore the consequences of eq.(48) further here – the only solution available is the $m = 0$ case of a scalar field (see [11] for a description of this solution, and [10] for the mathematical details). More general classes of solution are under active investigation.

What can be said at this point is that in order to maintain the commutator algebra of the tangent space while using the vectors for field theory, we must use something akin to the techniques of quantum field theory. Indeed, v is already in the form of the usual Hiesenberg operator expansion for a quantum field. Thus, in addition to successfully formalizing certain details of scale invariance, the structures arising from the extended conformal group have unexpected properties which might shed some insight onto our understanding of quantum systems.

Before concluding, we note one further possible form of the scalar field action, in which we take the norm instead of the inner product:

$$\frac{1}{2}\sum_{n=-\infty}^{\infty}\int \|\mathbf{D}v\|^2\mathbf{\Phi} \tag{49}$$

Here we define $\|\mathbf{D}v\|^2 \equiv K^{AB}\langle D_A\bar{v}, D_Bv\rangle$. In this case, we can add a mass term $\mathrm{m}^2\|v\|^2\mathbf{\Phi}$ only for fields v^k of definite weight $k = 1$, but for this one case there is no mixing of conformal weights.

5 Conclusion

We have shown that our freedom to assign an arbitrary conformal weight to spacetime coordinates leads to a noncentral, discrete extension of the conformal group. This discrete extension applies to any Lie group containing dilatations. We examined the properties of the resulting extended scaling groups and their gauge theories.

We showed the presence of central charges in the extended algebra. *A priori*, the extended symmetries necessarily have non-trivial center. The maximal effective and transitive subgroup is therefore projective. When the resulting central charges are introduced, the discrete part of the group leads to the infinite Heisenberg algebra and the dilatational part leads to the Virasoro algebra. These algebras govern tangent vectors when extended groups are gauged, leading to nonvanishing commutators between canonically conjugate fields.

Also, extended scaling groups fill some gaps in the previous formulation of scale invariance. By formalizing the use of conformal weights as part of the initial symmetry, the covariant derivative and gauge properties are automatically appropriate to the weights of fields.

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Appendix A: Scale-covariant weight maps

Here we find the maximal scale-covariant subgroup of the automorphism group of the set of integer-valued conformal weights.

Let Φ be the automorphism group of the integers, $\Phi = \{\varphi \mid \varphi : Z \to Z, bijective\}$. The set $L = \{(length)^m \mid m \in Z\}$ provides a representation Φ_L of Φ by defining $J_\varphi : L \to L$ as

$$J_\varphi l^k = l^{\varphi(k)} \tag{50}$$

We seek the maximal scale-covariant subgroup Φ_D of Φ_L, in the sense that the action of dilations, $e^{\lambda D}$, should be well-defined on Φ_D. That is, Φ_D must be closed under the action of dilatation:

$$e^{\lambda D} J_\varphi e^{-\lambda D} = \sum_{\varphi'} f_{\varphi\varphi'} J_{\varphi'} \in \Phi_D \tag{51}$$

for all J_φ in Φ_D. Consider the action of eq.(51) on L. We compute, for all k,

$$e^{\lambda D} J_\varphi e^{-\lambda D} l^k = \sum_{\varphi'} f_{\varphi\varphi'} J_{\varphi'} l^k \tag{52}$$

$$e^{\lambda\varphi(k) - \lambda k} l^{\varphi(k)} = \sum_{\varphi'} f_{\varphi\varphi'} l^{\varphi'(k)} \tag{53}$$

which is satisfied iff both

$$f_{\varphi\varphi'} = e^{\alpha} \delta_{\varphi\varphi'} \tag{54}$$

$$\lambda\varphi(k) - \lambda k = \alpha(\varphi) \tag{55}$$

with $\alpha(\varphi)$ independent of k. Thus, for all elements of the subgroup, $\varphi = \varphi'$ and $\varphi(k) = k + \alpha(\varphi)/\lambda \in Z$. Therefore, the action of φ is characterized by a single integer $-m = \alpha(\varphi)/\lambda \in Z$ and is given explicityly by $\varphi(k) = k - m$. Labelling J_φ as J_m, we have

$$e^{\lambda D} J_m e^{-\lambda D} = e^{-\lambda m} J_m \tag{56}$$

or infinitesimally, $[J_m, D] = mJ_m$.

We conclude that the set $\Phi_D = \{J_m | J_m l^k = l^{k-m}, m \in Z\}$ with J_m satisfying

$$\begin{aligned} [J_m, J_n] &= 0 && (57) \\ [J_m, D] &= mJ_m && (58) \end{aligned}$$

is the maximal covariant subset. The set is easily seen to form a subgroup since J_0 is the identity and J_k has inverse J_{-k}. Also note that the set $\{D, J_m\}$ is the basis for an infinite Lie algebra.

Appendix B: Nonvanishing Jacobi relations

The non-vanishing Jacobi relations for the tangent commutator algebra involve only the structure constants ${c_{\alpha\beta}}^0$ and ${c_{0\alpha}}^0$. For the homothetic algebras, both of these vanish, while for the conformal group, ${c_{0\alpha}}^0 = 0$. For a general scaling algebra, the Jacobi identities are replaced by the following Jacobi relations:

$$\begin{aligned} [G^k_\alpha, [G^m_\beta, G^n_\gamma]]_{et\ cyc} &= (n{c_{\alpha\beta}}^0 \delta^\rho_\gamma + k{c_{\beta\gamma}}^0 \delta^\rho_\alpha && (59) \\ &\quad + m{c_{\gamma\alpha}}^0 \delta^\rho_\beta)\ G^{k+m+n}_\rho && (60) \\ [G^k_\alpha, [G^m_\beta, L^n]]_{et\ cyc} &= n{c_{\alpha\beta}}^0 L^{k+m+n} && (61) \\ &\quad + (m{c_{0\alpha}}^0 \delta^\rho_\beta - k{c_{0\beta}}^0 \delta^\rho_\alpha) G^{k+m+n}_\rho && (62) \\ [G^k_\alpha, [G^m_\beta, J_n]]_{et\ cyc} &= n{c_{\alpha\beta}}^0 J_{k+m+n} && (63) \\ [G^k_\alpha, [L^m, L^n]]_{et\ cyc} &= (m-n)\ {c_{0\alpha}}^0 L^{k+m+n} && (64) \\ [G^k_\alpha, [L^m, J_n]]_{et\ cyc} &= n{c_{0\alpha}}^0 J_{k+m+n} && (65) \end{aligned}$$

For the homothetic algebras, the usual Jacobi identities hold, so that the extended homothetic algebra is an infinite dimensional Lie algebra. For the conformal group (with a similar simplification for any Lie algebra with a definite weight basis, so that ${c_{0\alpha}}^0 = 0$) the non-vanishing Jacobi relations are

$$\begin{aligned} [P^k_a, [K^m_b, G^n_\gamma]]_{et\ cyc} &= 2(n\eta_{ab} G^{k+m+n}_\gamma && (66) \\ &\quad + k{c_{b\gamma}}^0 P^{k+m+n}_a + m{c_{\gamma a}}^0 K^{k+m+n}_b) && (67) \\ [P^k_a, [K^m_b, L^n]]_{et\ cyc} &= 2n\eta_{ab} L^{k+m+n} && (68) \\ [P^k_a, [K^m_b, J_n]]_{et\ cyc} &= 2n\eta_{ab} J_{k+m+n} && (69) \end{aligned}$$

References

[1] H. Weyl, Sber. preuss. Akad. Wiss. (1918) 465; R. Wietzenbock, Sber. preuss. Akad. Wiss. (1920) 129.

[2] A.O. Barut and R.B. Haugen, Ann. Phys. 71 (1972) 519; “De Sitter and Conformal Groups and Their Applications”, edited by A.O. Barut and W.E. Brittin, (Colorado Associated Press, 1971).

[3] L. P. Hughston and T. R. Hurd, Phys. Rep. 100(5) (1983) 273.

[4] J. Crispim-Romão, A. Ferber and P.G.O. Freund, Nucl. Phys. B126 (1977) 429.

[5] M. Kaku, P.K. Townsend and P. Van Nieuwenhuizen, Phys. Lett. B 69 (1977) 304.

[6] M. Kaku, P.K. Townsend and P. Van Nieuwenhuizen, Phys. Rev. Lett. 39 (1977) 1109; M. Kaku, P.K. Townsend and P. Van Nieuwenhuizen, Phys. Rev. D17 (1978) 3179.

[7] P.K. Townsend and P. Van Nieuwenhuizen, Phys. Rev. D19 (1979) 3166.

[8] J.T. Wheeler, J. Math. Phys. 39 (1998) 299.

[9] André Wehner and James T. Wheeler, Nuc. Phys. B 557 (1999) 380-406.

[10] André Wehner and James T. Wheeler, hep-th/0001061.

[11] André Wehner and James T. Wheeler, hep-th/0001191.