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Derivation of the linearity principle of Intriligator-Leigh-Seiberg

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Abstract: Utilizing the techniques recently developed for $\mathcal{N} = 1$ super Yang-Mills theories by Dijkgraaf, Vafa and collaborators, we derive the linearity principle of Intriligator, Leigh and Seiberg, for the confinement phase of the theories with semi-simple gauge groups and matters in a non-chiral representation which satisfies a further technical assumption.

1 Introduction and Results

Recently Dijkgraaf and Vafa proposed in [1] that the non-perturbative superpotential W_{eff} of $\mathcal{N} = 1$ super Yang-Mills theories is captured by the perturbative calculation for the holomorphic matrix models. This triggered an avalanche of works [2] checking and extending the proposal, and now we have two different, purely field-theoretic derivations for it [3] [4]. Originally the proposal were made for matters in adjoint or bi-fundamental representations, recent works [5] extends this method to matters in the fundamental representation.

In the method of Dijkgraaf and Vafa, one first introduces a bare superpotential $W_{\text{tree}} = \sum g_i O_i$ which leads the system to the confinement phase, and then the effective superpotential for the gaugino condensate S is calculated perturbatively. Therefore, to check the proposal, one has to calculate the effective superpotential by some other methods. Usually this is done by combining two ingredients: one is the exact results such as the Affleck-Dine-Seiberg superpotential or the Seiberg-Witten curves which describes the system without W_{tree}, and the other is the linearity principle of Intriligator, Leigh and Seiberg [6][7] which governs the reaction of the system to the bare superpotential W_{tree}. For example, Ferrari showed in [8] that for the maximally-confined phase of the $\mathcal{N} = 1$ $U(N)$ super Yang-Mills with one adjoint, the effective superpotential calculated by the Dijkgraaf-Vafa method exactly matches that calculated by the linearity principle combined with the Seiberg-Witten solution. This indicates that the linearity principle can be derived generically from the Dijkgraaf-Vafa proposal.

Let us now briefly review the linearity principle. It states that when the effective superpotential W_{eff} is written as a sum of two terms $W_{\text{n.p.}} + W_{\text{tree}}$, the non-perturbatively generated part $W_{\text{n.p.}}$ is independent of g_i and is a function of Λ and $\langle O_i \rangle$ only, where Λ denotes the dynamical scale of the theory and $\langle O_i \rangle$ are the vacuum expectation values of gauge-invariant combinations of matter chiral superfields. Hence W_{eff} is *linear* in g_i's. The authors of [6][7] checked this principle for several simple cases by symmetry and holomorphy, but for matters in more complicated representation, symmetry and holomorphy themselves are not strong enough to ensure the absence of further correction to W_{eff}. In the paper, they proposed that this linearity holds in general, by suitably defining the composite operators. Anomalous global symmetries played a significant role in the analysis. An example is the R symmetry, which transforms matter fermions ψ and gauginos λ as $\psi(x) \to e^{-i\varphi}\psi(x)$ and $\lambda(x) \to e^{i\varphi}\lambda(x)$, respectively.

Now that we have derivations for the Dijkgraaf-Vafa proposal, we can turn the argument around and derive the linearity principle for a general class of theories, using the ideas in [3] and [4]. We show the linearity for theories which satisfy the following three criteria:

First, we consider only the phase where the gauge group is completely Higgsed or completely confined, so that the low energy gaugino condensate for each group is characterized by a single superfield S.

Second, if the R charge of Λ of the theory is non-zero, we further impose the following restriction:

Restriction A There are r basic gauge invariants $F_1,\ldots,F_r$ such that any of the gauge invariants O_i can be written as a polynomial of them, and that no dynamical constraints $P(\langle F_1\rangle, \langle F_2\rangle, \ldots, \langle F_r\rangle) = 0$ are present among F_i's.

Third, we also take the gauge group to be simple for the sake of brevity. The extension to general semi-simple groups should be immediate.

Please note that many theories satisfy these criteria. As an example, let us recall the $Sp(N)$ super Yang-Mills with $2N_f$ fundamentals Q_i. Basic gauge invariants are $T_{ij} = Q_iQ_j$. For $N_f < N + 1$, there are no constraints. For $N_f = N + 1$, there is a dynamical constraint $\mathrm{Pf}\, T_{ij} = \Lambda^{2N_f}$, but in this case the R charge of Λ is zero. For $N_f > N+1$, there are classical constraints analogous to the one above, but they are all lifted dynamically. Therefore they all meet the criteria. The behavior of the $SU(N)$ super Yang-Mills with fundamentals is similar.

The derivation is carried out in the following two steps: In section 2, we show that $W_{\text{n.p.}}$ is equal to $(N_c - N_f)S$ where N_c is the dual Coxeter number of the gauge group and N_f is the index of anomaly of the representation of the matter fields. In section 3, under the restriction **A**, we derive that S is independent of the coupling constants g_i in W_{tree}, when expressed as a function of Λ and the basic gauge invariants $\langle F_i\rangle$. Combining the results obtained in section 2 and 3, the linearity principle follows for the theories considered in this paper. Section 4 contains the conclusion.

We follow the conventions in [3].

2 Determination of $W_{\text{n.p.}}$

Let us consider an $\mathcal{N} = 1$ super Yang-Mills system with a simple gauge group and matter fields in some non-chiral representation so that one can use the Feynman rules which are spelled out in [3].

According to [3], with the bare superpotential W_{tree} introduced, the effective superpotential W_{eff} for the gaugino condensate is calculated by introducing the vector superfield as an external background and integrating the matter superfields out. We only use a few essential features, namely that i) the propagator for each superfield is the inverse of its mass, ii) the vertices come from terms in W_{tree} which is higher than or equal to cubic, and iii) each loop integral brings in two factors of W_α whose lowest component is the gaugino. S is found by extremizing W_{eff}.

Another important feature is that, for pure super Yang-Mills theories with a simple gauge group, the chiral ring is generated by $S = \mathrm{tr}\, W_\alpha W^\alpha/(32\pi^2)$, as noted in [4].

Hence, after integrating out all the matter superfields, any possible contraction of $2n$ of W_α's becomes S^n times some numerical constant. Therefore any n loop diagram is accompanied by the factor S^n.

Let us denote $W_{\rm tree} = \sum m_i Q_i + \sum g_j O_j$, where m_i is the mass of the $i^{\rm th}$ superfield, Q_i are the corresponding quadratic gauge invariants, O_j are gauge invariants which is cubic or higher in matter superfields, and g_j are coupling constants for them. For the $SU(N)$ super Yang-Mills with an adjoint matter Φ, for example, one can introduce $\operatorname{tr}\Phi^n$ and $(\operatorname{tr}\Phi^n)^m$ as gauge invariants. We can also introduce baryonic invariants if one have enough number of fundamental matters. Let us also denote by n_i the index* of the representation of the $i^{\rm th}$ matter superfield, plus that of its conjugate if the representation is not real. For example for the gauge group $SU(N)$, $n_i = N$ for an adjoint and $n_i = 1$ for a pair of the fundamental and the anti-fundamental.

Once one has $W_{\rm eff}$, one can calculate the vacuum expectation values for various operators O_i by just differentiating $W_{\rm eff}$ with respect to g_i because

$$\langle O_i \rangle = \frac{\partial}{\partial g_i} W_{\rm eff}(S(\Lambda, g_j), g_j) = \frac{\partial W_{\rm eff}}{\partial g_i} + \frac{\partial S}{\partial g_i}\frac{\partial W_{\rm eff}}{\partial S} = \frac{\partial W_{\rm eff}}{\partial g_i} \tag{1}$$

where we used the extremization condition $\partial W_{\rm eff}/\partial S = 0$.

Now recall the effective superpotential $W_{\rm eff}$ is a sum of three terms:

$$W_{\rm eff} = W_{\rm VY} + W_{\rm one\ loop} + W_{\rm higher} \tag{2}$$

where $W_{\rm VY}$ is the Veneziano-Yankielowicz term $N_c S(1 - \log(S/\Lambda^3))$, $W_{\rm one\ loop} = \sum n_i S \log m_i/\Lambda$ is the one loop contribution without any vertex insertion, and $W_{\rm higher} = \sum D$ is the sum of diagrams with some vertex insertions (refer [9] for a more detailed explanation). The non-perturbative part of the superpotential $W_{\rm n.p.}$ is defined as

$$W_{\rm n.p.} = W_{\rm eff} - \langle W_{\rm tree} \rangle. \tag{3}$$

Let us note that the number of vertices V, the number of propagators E, and the number of loops L satisfies $V - E + L = 1$ and that V, E and L of each diagram can be counted by

$$\sum g_j \frac{\partial}{\partial g_j}, \qquad \sum m_i^{-1}\frac{\partial}{\partial m_i^{-1}} = -\sum m_i \frac{\partial}{\partial m_i}, \quad \text{and} \quad S\frac{\partial}{\partial S} \tag{4}$$

respectively. Hence, for each diagram D, the value of D satisfies (we denote a diagram and its value by the same letter)

$$\left(1 - S\frac{\partial}{\partial S}\right) D = \left(\sum m_i \frac{\partial}{\partial m_i} + \sum g_j \frac{\partial}{\partial g_j}\right) D, \tag{5}$$

*The index of anomaly $T(r)$ of the representation r is defined by the formula $\operatorname{tr}_r(t^a t^b) = T(r)\delta^{ab}$, where t^a's are the generators of the gauge group normalized so that for the standard $SU(2)$ subgroup they become one half of the Pauli sigma matrices.

but the right hand side is just the contribution of the diagram D to $\langle W_{\rm tree}\rangle$. This means that

$$\begin{aligned}\langle W_{\rm tree}\rangle &= \left(\sum m_i \frac{\partial}{\partial m_i} + \sum g_j \frac{\partial}{\partial g_j}\right)(W_{\text{one loop}} + W_{\rm higher}) \\ &= \left(\sum m_i \frac{\partial W_{\text{one loop}}}{\partial m_i}\right) + \left(1 - S\frac{\partial}{\partial S}\right) W_{\rm higher}. \end{aligned} \tag{6}$$

Combining this with the equation $\partial W_{\rm eff}/\partial S = 0$, one can derive

$$\begin{aligned} W_{\rm n.p.} &= W_{\rm eff} - \langle W_{\rm tree}\rangle \\ &= W_{\rm VY} + \left(1 - \sum m_i \frac{\partial}{\partial m_i}\right) W_{\text{one loop}} + S\frac{\partial}{\partial S} W_{\rm higher} \\ &= \left(1 - S\frac{\partial}{\partial S}\right) W_{\rm VY} + \left(1 - S\frac{\partial}{\partial S} - \sum m_i \frac{\partial}{\partial m_i}\right) W_{\text{one loop}} \\ &= (N_c - \sum n_i) S. \end{aligned} \tag{7}$$

This is the result of this section. As an simple application of this, one can see that there is no non-perturbative superpotential generated for the $SU(N_c)$ super Yang-Mills with $N_f = N_c$ pairs of fundamental flavors.

3 Independence of S from coupling constants

In the following, we assume the theory under consideration satisfies the restriction **A**. We do not need to distinguish quadratic operators and operators which is higher than quadratic, so we collectively denote them as O_i and write $W_{\rm tree} = \sum \lambda_i O_i$. We also take $W_{\rm eff}$ as a function of Λ, S, and coupling constants λ_i. As a convention, we take $O_i = F_i$ for $i = 1, 2, \ldots, r$.

In this section, we show that the gaugino condensate S can be written as a function of Λ and $\langle F_i\rangle$ without explicit dependence on λ_i. In other words, we show that as long as $\langle F_i\rangle$'s are left invariant, S does not change when λ_i's are varied.

To show the claim above, first let us recall that S is determined by extremizing $W_{\rm eff}$, so that

$$-\frac{\partial^2 W_{\rm eff}}{\partial S \partial S}\delta S = \sum_i \delta\lambda_i \frac{\partial^2 W_{\rm eff}}{\partial \lambda_i \partial S}. \tag{8}$$

(Note that $-\partial^2 W_{\rm eff}/\partial S^2$ contains a term N_c/S coming from the Veneziano-Yankielowicz contribution, so generally non-zero.)

Second, the change in vacuum expectation values induced by the change in coupling constants is

$$\delta\langle F_j\rangle = \delta\frac{\partial W_{\text{eff}}}{\partial\lambda_j} = \delta S\frac{\partial^2 W_{\text{eff}}}{\partial S\partial\lambda_j} + \sum_i \delta\lambda_i \frac{\partial^2 W_{\text{eff}}}{\partial\lambda_i\partial\lambda_j} = \sum_i \delta\lambda_i G_{ij} \tag{9}$$

where

$$G_{ij} = \frac{\partial^2 W_{\text{eff}}}{\partial\lambda_i\partial\lambda_j} - \frac{\partial^2 W_{\text{eff}}}{\partial\lambda_i\partial S}\frac{\partial^2 W_{\text{eff}}}{\partial S\partial\lambda_j}\Big/\frac{\partial^2 W_{\text{eff}}}{\partial S\partial S}. \tag{10}$$

Please note that if some dynamical constraints emerge, the rank of G_{ij} is less than r, the number of basic gauge invariants. Therefore, the restriction **A** ensures that the rank of G_{ij} is equal to r.

Let us view $\langle O_i\rangle = \partial W_{\text{eff}}/\partial\lambda_i$ as the expectation values of O_i in the static background of the gaugino condensate S. Then, using the factorization of gauge invariants $\delta\langle O_i\rangle = \sum_k \langle\partial O_i/\partial F_k\rangle\delta\langle F_k\rangle$, one can rewrite δS as

$$-\frac{\partial^2 W_{\text{eff}}}{\partial S\partial S}\delta S = \sum_i \delta\lambda_i\frac{\partial^2 W_{\text{eff}}}{\partial\lambda_i\partial S} = \sum_i \delta\lambda_i\frac{\partial\langle O_i\rangle}{\partial S} = \sum_{k=1}^{r}\left(\sum_i \delta\lambda_i\langle\frac{\partial O_i}{\partial F_k}\rangle\right)\frac{\partial\langle F_k\rangle}{\partial S} \tag{11}$$

and $\delta\langle F_j\rangle$ as

$$\begin{aligned}
\delta\langle F_j\rangle &= \sum_i \delta\lambda_i\left(\frac{\partial\langle O_i\rangle}{\partial\lambda_j} - \frac{\partial\langle O_i\rangle}{\partial S}\frac{\partial^2 W_{\text{eff}}}{\partial S\partial\lambda_j}\Big/\frac{\partial^2 W_{\text{eff}}}{\partial S\partial S}\right) \\
&= \sum_{k=1}^{r}\left(\sum_i \delta\lambda_i\langle\frac{\partial O_i}{\partial F_k}\rangle\right)\left(\frac{\partial\langle F_k\rangle}{\partial\lambda_j} - \frac{\partial\langle F_k\rangle}{\partial S}\frac{\partial^2 W_{\text{eff}}}{\partial S\partial\lambda_j}\Big/\frac{\partial^2 W_{\text{eff}}}{\partial S\partial S}\right) \\
&= \sum_{k=1}^{r}\left(\sum_i \delta\lambda_i\langle\frac{\partial O_i}{\partial F_k}\rangle\right)G_{kj}.
\end{aligned} \tag{12}$$

As already noted, G_{kj} is invertible if the indices k and j are restricted to the range $1, 2, \ldots, r$. Thus, by multiplying (12) by ${G^{-1}}_{jk}$, one obtains

$$\sum_i \delta\lambda_i\langle\frac{\partial O_i}{\partial F_k}\rangle = 0 \tag{13}$$

for $k = 1, 2, \ldots, r$, so that $\delta S = 0$ follows. This is the result of this section.

4 Conclusion

Let us combine the results obtained so far. First, if the R charge of Λ is zero, $N_c - N_f$ is also zero. Hence, the non-perturbatively generated superpotential satisfies $W_{\text{eff}} =$

$(N_c - N_f)S = 0$ by the result of section 2. Second, if the R charge of Λ is non-vanishing, we can use the result of section 3 because we assume the theory satisfies the restriction **A**. Hence $W_{\text{eff}} = (N_c - N_f)S$ does not explicitly depend on the coupling constants, and is a function of Λ and $\langle F_i \rangle$. This is what we wanted to derive.

The extension to a wider class of theories will be interesting and worth studying. Extending the analysis to the chiral matter contents seems to be much harder, because recent developments are mainly focused on theories with matters in a non-chiral representation.

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