Nonlinear and Quantum Origin of Doubly Infinite Family of Modified Addition Laws for Fourmomenta *

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April 25, 2020

Abstract

We show that infinite variety of Poincaré bialgebras with nontrivial classical r-matrices generate nonsymmetric nonlinear composition laws for the fourmomenta. We also present the problem of lifting the Poincaré bialgebras to quantum Poincaré groups by using e.g. Drinfeld twist, what permits to provide the nonlinear composition law in any order of dimensionfull deformation parmeter λ (from physical reasons we can put $\lambda = \lambda_p$ where λ_p is the Planck lenght). The second infinite variety of composition laws for fourmomentum is obtained by nonlinear change of basis in Poincaré algebra, which can be performed for any choice of coalgebraic sector, with classical or quantum coproduct. In last Section we propose some modification of Hopf algebra scheme with Casimir-dependent deformation parameter, which can help to resolve the problem of consistent passage to macroscopic classical limit.

1 Introduction

The standard generators of classical relativistic symmetries are described by classical Poincaré algebra[1] ($\eta_{\mu\nu} = (1, -1, -1, -1)$)

$$
\begin{aligned}
[M^{(0)}_{\mu\nu}, M^{(0)}_{\rho\tau}] &= i(\eta_{\mu\tau} M^{(0)}_{\nu\rho} - \eta_{\nu\tau} M^{(0)}_{\mu\rho} \\
&\quad + \eta_{\nu\rho} M^{(0)}_{\mu\tau} - \eta_{\kappa\rho} M^{(0)}_{\nu\tau})\,,
\end{aligned}
$$

*Supported by KBN grant 5PO3B05620

[1]We shall consider further the case D=4, but all arguments in principle can be extended to any D

$$
\begin{aligned}
[M^{(0)}_{\mu\nu}, P^{(0)}_{\rho}] &= i(\eta_{\nu\rho} P^{(0)}_{\kappa} - \eta_{\mu\rho} P^{(0)}_{\nu})\,, \\
[P^{(0)}_{\kappa}, P^{(0)}_{\nu}] &= 0\,. \qquad (1.1)
\end{aligned}
$$

where fourmomenta generators $P^{(0)}_{\nu} = (\overrightarrow{P}^{(0)}, E^{(0)} = cP^{(0)}_{0})$ provide the physical three-momentum as well as energy operators, and Lorentz generators $M^{(0)}_{\mu\nu} = (\overrightarrow{M}, \overrightarrow{N})$describe relativistic angular momentum. The classical Poincaré symmetries are described by a bialgebra with the relations (1.1) satisfied by the primitive coproduct ($I^{(0)}_{A} = (P^{(0)}_{\mu}, M^{(0)}_{\mu\nu})$)

$$
\Delta(I^{(0)}_{A}) = I^{(0)}_{A} \otimes 1 + 1 \otimes I^{(0)}_{A}\,. \qquad (1.2)
$$

The coproduct (1.1) describes classical Abelian addition law of the fourmomenta and the relativistic angular momenta. For the system described by tensor product $|1> \otimes |2>$ of the irreducible representations $|i>$ ($i = 1, 2$) of the algebra (1.1) one gets

$$
\begin{aligned}
p^{(0)}_{\mu;1+2} &= p^{(0)}_{\mu;1} + p^{(0)}_{\mu;2}\,, \qquad (1.3) \\
M^{(0)}_{\mu\nu} &= M^{(0)}_{\mu\nu;1} + M^{(0)}_{\mu\nu;2}\,. \qquad (1.4)
\end{aligned}
$$

where $P^{(0)}_{\mu}|i>= p^{(0)}_{\mu;i}|i>$ and $P_{\mu}(|1> \otimes |2>) = p^{(0)}_{\mu;1+2}(|1> \otimes |2>)$ etc.

If the tensor product $|1> \otimes |2>$ describe a composite system, from (1.3–1.4) follows that both its constituents $|1>$ and $|2>$ are not interacting with each other. It appears however interesting to look for modification of classical relativistic symmetry scheme in such a way that the derived composition law will produce at least in the relation (1.3) some primary geometric interaction term

$$
p_{\mu;1+2} = p_{\mu;1} + p_{\mu;2} + p_{\mu;1+2}(p_{\mu;1}, p_{\mu,2})\,. \qquad (1.5)
$$

If we assume that the origin of this geometric interaction is due to the replacement of classical Minkowski geometry by quantum geometry generated by quantum gravity effects it is plausible to assume that the "corrections" to the formula (1.3) are of order $\frac{1}{M_p}$ (the inverse of Planck mass).

There are two ways of modifying the standard framework of description of fourmomenta in classical relativistic theory, which leads to the modification (1.5):

i) One can assume that the physical generators P_{μ} describing energy and three–momenta are nonlinear functions of the classical generators $P^{(0)}_{\mu}$[2]. The additional requirement of D=3 nonrelativistic classical covariance ($M_i = \frac{1}{2}\varepsilon_{ijk} M_{jk}$)

$$
[M_i, M_j] = i\varepsilon_{ijk} M_k\,, \qquad [M_i, P_j] = i\varepsilon_{ijk} P_k\,, \qquad (1.6)
$$

implies that

$$
P_{\mu} = P_{\mu}(\overrightarrow{P}^{(0)2}, E^{(0)})\,, \qquad M_{\mu\nu} = M^{(0)}_{\mu\nu}\,. \qquad (1.7)
$$

Such nonlinear transformations, considered recently in [1]–[7] shall be discussed in Sect. 2. We would like to stress here that in such an approach the symmetries

[2] We shall restrict our considerations to the case when the Lorentz sector of (1.1) is not modified.

remain classical i.e. they are described by the same classical Poincaré bialgebra (1.1–1.2). The formulae (1.7) one gets if we choose different basis in enveloping classical Poincaré bialgebra.

ii) Second way consists in introducing nonclassical Poincaré bialgebras differing from (1.1) in coalgebra sector, with coproduct determined by classical Poincaré r-matrces. These matrices were almost completely classified by S. Zakrzewski [8]. In Sect. 3 we shall select the infinite variety of them, which can be used for the consistent deformation of the addition law (1.3).

It appears that the knowledge of classical r-matrix only permits to determine the modified addition law in lowest order in $\frac{1}{M_p}$. In order to obtain the addition law for fourmomenta valid in any order in the powers of the deformation parameter (chosen here to be $\frac{1}{M_p}$, with the classical "no deformation" limit given by $M_p \to \infty$) we should extend the quantum bialgebra to Hopf algebra[3].

If we keep the classical Poincaré algebra sector unchanged, the Poincaré r-matrices which are solutions of classical Yang-Baxter equation (CYBE) imply the Hopf-algebraic structure in coalgebraic sector described by the twist function, firstly introduced by Drinfeld (see e.g. [10]–[13]). Unfortunately we do not know explicitly all Hopf-algebraic counterparts of known Poincaré bialgebras listed in [8]. We recall here that the example for which the complete Hopf algebra structure is known in closed form and applied to deformations of relativistic symmetries is the so–called κ-deformed quantum Poincaré algebra (see e.g. [14]–[17]) and its generalizations [18], [19][4].

One can say that the Drinfeld twists which do not commute with the classical coproducts (1.2) for P_μ introduce different nonlinear extensions of the addition law (1.3) for the fourmomenta. It should be stressed however, that to the variety of quantum deformations one can supplement the variety of nonlinear transformations of basic generators. In this paper we wish to point out clearly this two basic origins of possible deformed addition laws. Finally we shall also comment on recent efforts to describe the addition formulae for fourmomenta outside of Hopf-algebraic scheme which can be defined as proposals of representation-dependent Hopf algebra structure. It appears that in such a scheme one can look for the solution of the problem of consistent transition from quantum-deformed Planck scale regime to classical relativistic systems at macroscopic distances.

2 Nonlinear Fourmomentum Basis

Let us consider firstly the classical Poincaré bialgebra described by (1.1)[5]. We shall assume that the transformation (1.7) has its inverse

$$P^{(0)}_\mu = P^{(0)}_\mu(\overrightarrow{P}^2, E)\,. \tag{2.1}$$

Having formulae (1.7) and (2.1) one can calculate the coproduct of P_μ. Taking into consideration that in the framework of bialgebras and Hopf algebras the

[3]Sometimes this extension is called "quantization of bialgebra" (see e.g. [9]).

[4]We would like to point out here that several authors restricted possible deformations of addition law for relativistic fourmomenta only to the ones coming from classical bialgebra (1.1–1.2) and κ-deformed Poincaré algebra. In principle there is however infinite variety of such deformations.

[5]In fact the classical Poincaré bialgebra is also a Hopf algebra, with antipode (coinverse) given by formula $S(I_A) = I_A$.

coproduct is homomorphic, i.e. $\Delta(f(I_A)) = f(\Delta(I_A))$[6], one gets

$$\Delta(P_\mu) = P_\mu(P_\mu^{(0)}(P) \otimes 1 + 1 \otimes P_\mu^{(0)}(P)) \,. \tag{2.2}$$

and we obtain the following addition law for the classical fourmomenta in nonlinear basis (1.7)

$$P_{\mu;1+2} = P_\mu(P_\mu^{(0)}(P_{\mu;1}) + P_\mu^{(0)}(P_{\mu;2})) \,. \tag{2.3}$$

The rule (2.3) has been firstly demonstrated for nonlinear bases of classical Poincaré algebra in [2], and further rediscovered in [3]. The characteristic property of the composition law (2.3) is its symmetry, what discloses the classical nature of symmetries. It should by also stressed that due to the coassociativity of the coproduct

$$(\Delta \otimes 1)\Delta(P_\mu) = \Delta \cdot (\Delta \otimes 1)(P_\mu) \,, \tag{2.4}$$

the addition law (2.3) is coassociative, i.e. if we denote [20]

$$\Delta(P_\mu) = \sum_i f^i(P_\mu) \otimes g^i(P_\mu) \Longleftrightarrow P_{\mu;1+2} = \sum_i f^i(P_{\mu;1}) \cdot g^i(P_{\mu;2}) \,,$$

the coassociativity implies that for any nonlinear basis we get

$$P_{\mu;1+2+3} = P_{\mu;1+(2+3)} = P_{\mu;(1+2)+3} \,, \tag{2.5}$$

where

$$\begin{aligned} P_{\mu;1+(2+3)} &= \sum_i f^i(P_{\mu;1}) \cdot g^i(P_{\mu;2+3}) \,, \\ P_{\mu;(1+2)+3} &= \sum_i f^i(P_{\mu;1+2}) \cdot g^i(P_{\mu;3}) \,. \end{aligned} \tag{2.6}$$

One of the consequences of the nonlinear change (2.1) of fourmomentum basis is the modification of mass Casimir

$$C_2 = P_\mu^{(0)} P^{(0)\mu} \to C_2 = P_\mu^{(0)}(P) P^{(0)\mu}(P) \,. \tag{2.7}$$

In physical applications one usually assumes that

$$P_\mu(P_\mu^{(0)}) = P_\mu^{(0)} + \frac{1}{M_p} f_\mu^{(1)}(P_\mu^{(0)}) + \mathcal{O}(\frac{1}{M_p^2}) \tag{2.8}$$

implying

$$P_\mu^{(0)}(P_\mu) = P_\mu - \frac{1}{M_p} f_\mu^{(1)}(P_\mu) + \mathcal{O}(\frac{1}{M_p^2}) \,, \tag{2.9}$$

where $(f_\mu^{(1)})^{-1} f_\nu^{(1)} = \delta_{\mu\nu} + \mathcal{O}(\frac{1}{M_p^2})$. A known example of the transformations (1.7) and (2.1) relating the standard mass-shell condition with κ-deformed mass-shells [17][7] (we assume the light velocity $c = 1$)

$$C_2 = \left(2M_p \sinh \frac{E}{2M_p}\right)^2 - \overrightarrow{p}^2 \, e^{\pm \frac{E}{M_p}} \,, \tag{2.10}$$

[6] The proof is demonstrated rigorously for holomorphic functions f and with some requirements on the domain of generators I_A.

[7] We put following further applications $\kappa = M_p$

are given in [21].

It should be stressed that the change of nonlinear basis, defining the explicite form of modified mass shell condition (see (2.7)) does not imply the form of the coproduct, which depends on the choice of coalgebraic sector, determined by classical r-matrix. In particular doubly special relativistic theories proposed in [18,19] for any choice of deformed mass shell condition can be endowed by infinite variety of fourmomentum addition laws, parametrized in lowest order of the deformation parameter by Poincaré classical r-matrices [8].

3 Poincaré Bialgebras, Poincaré Hopf Algebras and Addition Law for the Fourmomenta

A second cause which provides the modification of addition law for the fourmomenta is due to different possible choices of coalgebra sector for the Poincaré bialgebras. The class of Poincaré r-matrices which describe the deformation with dimensionfull parameters has the following general form[8]

$$\mathbf{r}_{(i)} = i\, r^{\mu;\rho\tau}_{(i)} P^{(0)}_{\mu} \wedge M^{(0)}_{\rho\tau} \,. \tag{3.1}$$

and satisfying the Yang–Baxter equation (see [8]). Let us observe that in order to obtain the dimensionless quantity we should multiply (3.1) by an inverse of mass parameter χ, which may be put equal to $\frac{1}{M_p}$.

In particular we obtain the following modification of the coproduct (1.1)

$$\Delta^{\kappa}_{(i)}(I_A) = I_A \otimes 1 + 1 \otimes I_A + \frac{1}{\kappa}[I_A \otimes 1 + 1 \otimes I_A, \mathbf{r}_{(i)}] \,. \tag{3.2}$$

where lower index (i) enumerates different choices of classical r-matrices. In particular we obtain

$$\Delta^{\kappa}_{(i)}(P_\mu) = P_\mu \otimes 1 + 1 \otimes P_\mu + \frac{2}{\kappa} r^{\mu;\rho\nu}_{(i)} (P_\nu \otimes P_\rho - P_\rho \otimes P_\nu) \,, \tag{3.3}$$

implying the following addition rule

$$p_{\mu;1+2} = p_{\mu;1} + p_{\mu;2} + \frac{2}{\kappa} r^{\mu;\rho\nu}_{(i)} (p_{\nu;1} p_{\rho;2} - p_{\rho;1} p_{\nu;2}) \,. \tag{3.4}$$

All these modification addition laws are nonsymmetric. Among the variety of classical r -matrices in [8] there is one determining the κ-deformation, given by the formula ($N_i \equiv M_{i0}$) (see e.g. [16])

$$\mathbf{r}_{(\kappa)} = \frac{1}{\kappa} N_i \wedge P_i \,. \tag{3.5}$$

It provides the following fourmomentum composition law

$$E_{1+2} = E_1 + E_2 \,, \quad \vec{p}_{1+2} = \vec{p}_1 + \vec{p}_2 + \frac{1}{\kappa}(E_1 \vec{p}_2 - E_2 \vec{p}_1) \,. \tag{3.6}$$

[8] The classical r-matrix $r = r^{\mu;\nu} P^{(0)}_{\mu} \wedge P^{(0)}_{\nu}$ describes so-called soft deformation [24] with trivial CYBE and will be not considered here.

The formulae (3.4) and (3.6) describe the modified coproduct as well as the addition law for the fourmomenta only in first order of the deformation parameter[9]. The complete deformation formula is provided by promoting bialgebra to quantum algebra – a noncocommutative Hopf algebra (see e.g. [25]). In such a case the quantum nonsymmetric coproduct satisfies the relations

$$\tau(\Delta(I_A)) = \widehat{R}^{-1} \otimes \Delta(I_A) \otimes \widehat{R} \,. \tag{3.7}$$

where τ denotes flip operator ($\tau(A \otimes B) = B \otimes A$) and $\widehat{R}$ is the quantum r-matrix (called also universal R-matrix), which satifies quantum Yang-Baxter equation and in first order in $1/\kappa$ is desribed by a classical r-matrix:

$$\widehat{R} = 1 \otimes + \frac{1}{\kappa} r + \mathcal{O}(\frac{1}{\kappa^2}) \,. \tag{3.8}$$

If the classical r-matrix satisfies classical YB equation, the quantum coproduct and quantum r-matrix can be described by a twist function T_κ in the following way:

$$\Delta_\kappa(I_A) \quad = \quad T_\kappa^{-1} \circ (I_A \otimes 1 + 1 \otimes I_A) \circ T_\kappa \,, \tag{3.9}$$

$$R_\kappa \quad = \quad T^{-\kappa} \otimes (\tau T_\kappa) \,. \tag{3.10}$$

and one obtains coassociative Hopf algebra. In general case when r satisfies modified Yang-Baxter equation the twist (3.9) can be also performed [11], but the coproduct is only quasicoassociative, and the formula (3.9) describes a coproduct of quasi-Hopf quantum group [11], with twist T_κ called Drinfeld twist.

The problem of explicite quantizaton of classical Poincaré r-matrices, listed in [8], in particular the ones belonging to the class (3.1), is not solved. Because in the case of κ-deformed Poincaré algebra the Hopf-algebraic structure is well known, mainly this quantum algebra is used to describe the explicite formulae for deformed relativistic symmetries. In particular it has been recently promoted a version of doubly special relativistic theories [26], [27] with the coalgebra structure determined by κ-deformed Poincaré algebra.

4 Discussion

In this lecture we presented the variety of deformed addition laws of fourmomentum based on Hopf–algebra structure. It should be pointed out that recently due to some physical arguments[10] it was conjectured [6], [28] that the addition law describing the eigenstates of symmetry generators for composite systems should not be described by a coproduct of Hopf-algebraic symmetry scheme. In particular it has been argued in the framework of doubly special relativity theory that the idea of having maximal momentum or energy invariant under generalized Lorentz transformations can not be sustained consistently with Hopf-algebraic coproduct rule.

In principle there are possible two approaches to the problem:

[9] One can say that bialgebras describe infinitesimal form of quantum group.

[10] The oldest one is related with validity of κ-deformed mass-shall condition for large fourmomenta describing macroscopic bodies, firstly pointed out to the authors by I. Białynicki-Birula in 1997.

i) the satisfactory solution should not violate Hopf algebra structure. Quantum groups as Hopf algebras represent the extension of the notion of symmetry to the case of symmetry transformations with noncommutative symmetry parameter and describe the same symmetries in arbitrary multiparticle sector. In such a case we repeat the universal status of Poincaré group valid for any multicomponent system, but we face some problems related with physical interpretation.

ii) It should be however recalled that also without deformation there were problems in description of interacting relativistic multiparticle systems with exact classical Poincaré invariance (see e.g. [29], [30]). In second approach one can introduce therefore weaker nonuniversal space-time symmetries. In such a case the modification of classical relativistic symmetries needs a new algebraic concept which will reconcile e.g. physics at macro-distances with quantum – deformed Planck lenght scale physics, without complete discarding the structure of quantum group [25]. Such a possibility is provided by the representation-dependent form of coalgebra, depending e.g. on the number of particles in the tensor product representation (see [6]) or in general on the labels of considered representation. One can obtain such a framework by postulating the deformation parameter κ depends on (κ-deformed) Casimir operators $\hat{C}_i$

$$\kappa \Longrightarrow \kappa(\hat{C}_i)\,. \tag{4.1}$$

In particular representation $|j_k>$, where

$$\hat{C}_i|j_k >= j_i|j_k > \ . \tag{4.2}$$

the operator-valued deformation parameter $\kappa(\hat{C}_i)$ is replaced by its eigenvalue $\kappa(j_i)$.

The problem which should be considered further is the physical choice the form of the function $\kappa(j_i)$. In particular if the collection of labels j_i describes quantum states which converge to classical state, the function $\kappa(j_i)$ should diverge to infinity, i.e. to the classical relativistic no-deformation limit.

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