Braneworld gravity: Influence of the moduli fields

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Abstract

We consider the case of a generic braneworld geometry in the presence of one or more moduli fields (*e.g.*, the dilaton) that vary throughout the bulk spacetime. Working in an arbitrary conformal frame, using the generalized junction conditions of gr-qc/0008008 and the Gauss–Codazzi equations, we derive the effective "induced" on-brane gravitational equations. As usual in braneworld scenarios, these equations do not form a closed system in that the bulk can exchange both information and stress-energy with the braneworld. We work with an arbitrary number of moduli fields described by an arbitrary sigma model, with arbitrary curvature couplings, arbitrary self interactions, and arbitrary dimension for the bulk. (The braneworld is always codimension one.) Among the novelties we encounter are modifications of the on-brane stress-energy conservation law, anomalous couplings between on-brane gravity and the trace of the on-brane stress-energy tensor, and additional possibilities for modifying the on-brane effective cosmological constant. After obtaining the general stress-energy "conservation" law and the "induced Einstein equations" we particularize the discussion to two particularly attractive cases: for a (n–2)-brane in ([n–1]+1) dimensions we discuss both the effect of (1) generic variable moduli fields in the Einstein frame, and (2) the effect of a varying dilaton in the string frame.

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1 Introduction

The idea that our observable universe might be a submanifold of a higher-dimensional spacetime is an old one, going back at least 18 years [1]. This idea has recently been revived and extensively developed, leading to various versions of the "braneworld" scenario, with two major variants depending on whether the extra dimensions are large but compact [2] or truly non-compact [3, 4, 5]. In particular a key issue is the form of the effective Einstein equations that are induced on the brane by what amounts to a "dimensional reduction" procedure. Particularly important are the papers of Shiromizu, Maeda, and Sasaki [6], Maeda and Wands [7], and Mennim and Battye [8]. See also [9, 10].

In this paper we shall extend this approach and consider the case of a braneworld geometry in the presence of one or more moduli fields (*e.g.*, the dilaton) that vary throughout the bulk spacetime. We generalize the previous calculations by working with an arbitrary number of moduli fields (instead of just the one dilaton field) described by an arbitrary sigma model (instead of limiting attention to kinetic energies that are canonical in at least one conformal frame), with arbitrary curvature couplings (equivalent to working in arbitrary conformal frame), arbitrary self interactions, and arbitrary dimension for the bulk. [While the most popular braneworld scenario involves reduction from (4+1) to (3+1) dimensions as essentially the last step in arriving at a phenomenologically acceptable model, we wish to leave open the possibility of, for instance, performing several braneworld reductions in sequential stages. At each stage of the reduction the braneworld is always codimension one in the corresponding bulk.]

Using the generalized junction conditions of [11], which are the appropriate generalization of the Israel–Lanczos–Sen junction conditions [12] to arbitrary conformal frame, and the Gauss–Codazzi equations we derive effective on-brane gravitational equations. As usual in braneworld scenarios, these equations do not form a closed system in that the bulk can exchange both information and stress-energy with the braneworld. In particular, the on-brane surface stress energy is not conserved in the usual sense: braneworld stress-energy can both flow into and put of the bulk, and in addition braneworld stress-energy can couple to "along the brane" variations in the moduli fields. For the "induced Einstein equations" novelties include anomalous couplings between braneworld gravity and the trace of the braneworld stress-energy tensor, and additional possibilities for modifying the on-brane effective cosmological constant.

After obtaining general "induced Einstein equations" we particularize the discussion to a two particularly attractive cases: for a generic (n–2)-brane in ([n–1]+1) dimensions we discuss (1) generic variable moduli fields in the Einstein frame, and (2) a variable dilaton field in the string frame.

2 Lagrangian and generalized junction conditions

A key technical complication in the current calculation is the use of a general conformal frame. This allows our formalism to be applied equally well in the Einstein frame, Jordan frame, or string frame. There is continuing debate as to which conformal frame is the "most physical", with our own attitude being that it depends on the physical questions you are asking. A nice review, with additional references is given by Faraoni, Gunzig, and Nardone [13]. We will not delve further into this issue, and in this current paper keep the choice of conformal frame arbitrary.

2.1 General conformal frame

We consider an action of the form

$$
\begin{aligned}
\mathcal{S} = {} & \frac{1}{2}\int_{\mathrm{int}(\mathcal{M})} \sqrt{-g}\, \mathrm{d}^n x\, \kappa_n^2\, F(\phi)\, \left[R - 2\Lambda\right] - \int_{\partial\mathcal{M}} \sqrt{-q}\, \mathrm{d}^{n-1}x\, \kappa_n^2\, F(\phi)\, K \\
& + \int_{\mathrm{int}(\mathcal{M})} \sqrt{-g}\, \mathrm{d}^n x \left\{ -\frac{1}{2} H_{ij}(\phi)\, \left[g^{AB}\, \partial_A \phi^i\, \partial_B \phi^j \right] - V(\phi,\psi) + \mathcal{L}_{\mathrm{bulk}}(g_{AB},\phi,\psi) \right\} \\
& + \int_{\mathrm{brane}} \sqrt{-q}\, \mathrm{d}^{n-1}x\, \mathcal{L}_{\mathrm{brane}}(q_{AB},\phi,\psi).
\end{aligned} \tag{2.1}
$$

Here κ_n has dimensions $(length)^{1-n/2} = (energy)^{n/2-1}$; the same as the dimensions of the moduli field ϕ; as usual we choose units so that $\hbar = 1$ and $c = 1$. (Warning: In most of the extant literature κ_n is defined as above, however several key papers, such as [6, 7], invert the definition of κ_n.) We have slightly generalized the action of [11] by explicitly exhibiting the n-dimensional Newton constant (encoded in κ_n), and allowing the bulk Lagrangian to posses additional non-derivative dependence on the moduli fields. A tricky point is that the Gibbons–Hawking boundary term now takes the form [11]

$$-\int_{\partial\mathcal{M}} \sqrt{-q}\; \mathrm{d}^{n-1}x\; \kappa_n^2\; F(\phi)\; K. \tag{2.2}$$

The three arbitrary functions, $F(\phi)$, $H_{ij}(\phi)$, and V, allow us to deal with an extremely wide class of possible moduli fields, one that covers essentially every possibility extant in the literature. To also permit interactions between the brane and the bulk fields, in our analysis we allow the brane Lagrangian to depend arbitrarily on the metric and on the bulk scalar fields, (this is in addition to its dependence on the "matter" fields trapped on or near the brane).

In the bulk region surrounding the brane, we adopt Gaussian normal coordinates with η denoting the spacelike normal to the brane, and $q_{AB} = g_{AB} - n_A\, n_B$ denoting the induced metric. The generalized junction conditions derived in [11] read

$$\pi^+_{AB} - \pi^-_{AB} = \frac{1}{2}\sqrt{-q}\; S_{AB}, \tag{2.3}$$

$$\pi^+_{\phi^i} - \pi^-_{\phi^i} = \sqrt{-q}\; \frac{\partial \mathcal{L}_{\mathrm{brane}}}{\partial \phi^i}, \tag{2.4}$$

$$\pi^+_{\psi} - \pi^-_{\psi} = \sqrt{-q}\; \frac{\partial \mathcal{L}_{\mathrm{brane}}}{\partial \psi}. \tag{2.5}$$

The gravitational "momentum" is defined by [11]

$$\pi_{AB} = \kappa_n^2 \left\{ \frac{1}{2}\sqrt{-q}\; F(\phi)\; \left(K_{AB} - q_{AB}K\right) + \frac{1}{2}\sqrt{-q}\; F'_i(\phi)\; \left(\frac{\partial\phi^i}{\partial\eta}\right) q_{AB} \right\}. \tag{2.6}$$

(Here η, although spacelike, is for technical reasons formally treated as though it were an evolution parameter.) For convenience we have defined

$$F'_i(\phi) \equiv \partial_i F(\phi) \equiv \frac{\partial F(\phi)}{\partial \phi^i}. \tag{2.7}$$

The "momentum" canonically conjugate to ϕ is [11]

$$\pi_{\phi^i} = -\sqrt{-q}\; H_{ij}(\phi)\; \left(\frac{\partial\phi^j}{\partial\eta}\right) - \sqrt{-q}\; \kappa_n^2\; F'_i(\phi)\; K. \tag{2.8}$$

While there are additional junction conditions for the "matter" fields ψ, they are not germane to the present discussion — see [11] for details. Rearranging the previous expressions [(2.3) to (2.8)] and defining the discontinuities

$$\mathcal{K}_{AB} \equiv K^+_{AB} - K^-_{AB}, \qquad\qquad J^i = \frac{\partial\phi^i}{\partial n}^{+} - \frac{\partial\phi^i}{\partial n}^{-}, \tag{2.9}$$

we arrive at

$$F\; \left(\mathcal{K}_{AB} - q_{AB}\; \mathcal{K}\right) + F'_i\; J^i\; q_{AB} = \kappa_n^{-2}\; S_{AB}, \tag{2.10}$$

$$-H_{ij}\; J^j - \kappa_n^2\; F'_i\; \mathcal{K} = \frac{\partial \mathcal{L}_{\mathrm{brane}}}{\partial \phi^i}. \tag{2.11}$$

We have found that it is extremely useful to separate equation (2.10) into a trace-free portion and a trace. That is

$$\mathcal{K}_{AB} - \frac{1}{n-1}\,\mathcal{K}\,q_{ab} = \frac{\kappa_n^{-2}}{F}\,\left(S_{AB} - \frac{1}{n-1}\,S\,q_{ab}\right), \tag{2.12}$$

$$(n-2)\,F\,\mathcal{K} - (n-1)\,F'_i\,J^i = -\kappa_n^{-2}\,S. \tag{2.13}$$

Inverting the general equations (2.11) and (2.13) yields the generalized junction conditions. For the discontinuity in the trace of extrinsic curvature

$$\mathcal{K} = -\frac{\kappa_n^{-2}\,S + (n-1)H^{ij}\,F'_i\,(\mathcal{L}_{\text{brane}})'_j}{(n-2)F + (n-1)H^{kl}\,\kappa_n^2\,F'_k\,F'_l}. \tag{2.14}$$

For the normal discontinuity in the scalar derivative

$$J^i = H^{ij}\left[F'_j\,\frac{S + (n-1)H^{pq}\,\kappa_n^2\,F'_p\,(\mathcal{L}_{\text{brane}})'_q}{(n-2)F + (n-1)H^{kl}\,\kappa_n^2\,F'_k\,F'_l} - (\mathcal{L}_{\text{brane}})'_j\right] \tag{2.15}$$

$$= \frac{H^{ij}\,F'_j}{(n-2)F + (n-1)H^{kl}\,\kappa_n^2\,F'_k\,F'_l}\,S - \left[H^{ij} - \frac{(n-1)\kappa_n^2\,H^{ip}\,F'_p\,H^{jq}\,F'_q}{(n-2)F + (n-1)H^{kl}\,\kappa_n^2\,F'_k\,F'_l}\right]\,(\mathcal{L}_{\text{brane}})'_j. \tag{2.16}$$

Reassembling the trace and trace-free parts of the extrinsic curvature

$$\mathcal{K}_{AB} = \kappa_n^{-2}\,\frac{S_{AB}}{F} - \left\{\frac{[F + H^{ij}\,\kappa_n^2\,F'_i\,F'_j]\kappa_n^{-2}\,S + F\,H^{ij}\,F'_i\,(\mathcal{L}_{\text{brane}})'_j}{(n-2)F + (n-1)H^{ij}\,\kappa_n^2\,F'_i\,F'_j}\right\}\,\frac{q_{AB}}{F}. \tag{2.17}$$

These expressions are rather unwieldy and for computations we have found it useful to introduce dimensionless coefficients $\gamma_{ij}(\phi)$ according to the scheme

$$\mathcal{K} = \frac{\kappa_n^{-2}}{F}\left(\gamma_{11}\,S + \kappa_n\,\gamma_{12}^i\,(\mathcal{L}_{\text{brane}})'_i\right), \tag{2.18}$$

$$J^i = \frac{\kappa_n^{-1}}{F}\left(\gamma_{21}^i\,S + \kappa_n\,\gamma_{22}^{ij}\,(\mathcal{L}_{\text{brane}})'_j\right), \tag{2.19}$$

$$\mathcal{K}_{AB} = \frac{\kappa_n^{-2}}{F}\,\left(S_{AB} + \tilde{\gamma}_{11}\,S\,q_{AB} + \kappa_n\,\tilde{\gamma}_{12}^i\,(\mathcal{L}_{\text{brane}})'_i\,q_{AB}\right). \tag{2.20}$$

These dimensionless coefficients depend only on the value of the moduli fields on the brane itself (and the dimensionality of spacetime). Explicit formulae are given in Appendix A.

Insofar as the junction conditions (and even the bulk Einstein equations) are concerned, there are definite advantages to performing a conformal redefinition of fields and going to the Einstein frame $F_E(\phi) = 1$. However going to the Einstein frame usually carries a cost that causes problems elsewhere in the analysis: For instance the Einstein frame is not the appropriate frame for asking questions about string propagation (the string frame is better adapted to that), while in Brans–Dicke theories (and their relatives) the Einstein frame is inappropriate for discussing Eötvös-type experiments (universality of free fall; the Jordan frame is more appropriate for that). This is why we are keeping the choice of conformal frame arbitrary.

2.2 Einstein frame

To make this a little more explicit, suppose we start with the general Lagrangian above and *define* the Einstein frame metric by

$$[g_E]_{AB} = F(\phi)^{2/(n-2)}\; g_{AB}; \qquad [q_E]_{AB} = F(\phi)^{2/(n-2)}\; q_{AB}. \tag{2.21}$$

Then it is a standard computation to verify that

$$\begin{aligned}
&\frac{1}{2}\int_{\mathrm{int}(\mathcal{M})} \sqrt{-g}\; \mathrm{d}^n x\; F(\phi)\; R(g) - \int_{\partial\mathcal{M}} \sqrt{-q}\; \mathrm{d}^{n-1}x\; F(\phi)\; K(g) \\
&\quad = \frac{1}{2}\int_{\mathrm{int}(\mathcal{M})} \sqrt{-g_E}\; \mathrm{d}^n x\; R(g_E) - \int_{\partial\mathcal{M}} \sqrt{-q_E}\; \mathrm{d}^{n-1}x\; K(g_E) \\
&\qquad -\frac{1}{2}\int_{\mathrm{int}(\mathcal{M})} \sqrt{-g_E}\; \mathrm{d}^n x\; \frac{n-1}{n-2}\; [g_E]^{AB}\; \frac{\partial_A F\; \partial_B F}{F^2}.
\end{aligned} \tag{2.22}$$

Then the action for our general theory expressed in the Einstein frame becomes

$$\begin{aligned}
\mathcal{S}_{\mathrm{Einstein}} = & \frac{1}{2}\int_{\mathrm{int}(\mathcal{M})} \sqrt{-g_E}\; \mathrm{d}^n x\; \kappa_n^2\; \Big[R(g_E) - 2F(\phi)^{-2/(n-2)}\; \Lambda\Big] \\
& - \int_{\partial\mathcal{M}} \sqrt{-q_E}\; \mathrm{d}^{n-1}x\; \kappa_n^2\; K(g_E) \\
& + \int_{\mathrm{int}(\mathcal{M})} \sqrt{-g_E}\; \mathrm{d}^n x \Bigg\{ -\frac{1}{2}[H_E]_{ij}(\phi)\; \big[g^{AB}\; \partial_A\phi^i\; \partial_B\phi^j\big] \\
& \qquad - F(\phi)^{-n/(n-2)}\; V(\phi,\psi) \\
& \qquad + F(\phi)^{-n/(n-2)}\; \mathcal{L}_{\mathrm{bulk}}(F(\phi), [g_E]_{AB}, \phi, \psi) \Bigg\} \\
& + \int_{\mathrm{brane}} \sqrt{-q_E}\; \mathrm{d}^{n-1}x\; F(\phi)^{-(n-1)/(n-2)}\; \mathcal{L}_{\mathrm{brane}}(F(\phi), [q_E]_{AB}, \phi, \psi).
\end{aligned} \tag{2.23}$$

Here the Einstein-frame sigma model metric is

$$[H_E]_{ij}(\phi) \equiv \frac{H_{ij}(\phi)}{F(\phi)} + \kappa_n^2\; \frac{n-1}{n-2}\; \frac{F_i'(\phi)\; F_j'(\phi)}{F^2(\phi)}, \tag{2.24}$$

and we can reduce clutter by defining

$$V_E(\phi,\psi) \equiv F(\phi)^{-n/(n-2)}\; V(\phi,\psi), \tag{2.25}$$

$$\mathcal{L}^E_{\mathrm{bulk}}([g_E]_{AB}, \phi, \psi) \equiv F(\phi)^{-n/(n-2)}\; \mathcal{L}_{\mathrm{bulk}}(F(\phi), [g_E]_{AB}, \phi, \psi), \tag{2.26}$$

and

$$\mathcal{L}^E_{\mathrm{brane}}([q_E]_{AB}, \phi, \psi) \equiv F(\phi)^{-(n-1)/(n-2)}\; \mathcal{L}_{\mathrm{brane}}(F(\phi), [q_E]_{AB}, \phi, \psi). \tag{2.27}$$

The various prices that are paid for making the gravity sector look simple include: (1) what was a simple bulk cosmological constant in the original conformal frame has now become a moduli dependent potential; (2) even if the kinetic energies are canonical in the original conformal frame, there will generally be a non-trivial sigma-model metric in the Einstein frame; (sometimes you can ameliorate this by additionally redefining the moduli fields in a frame-dependent manner); (3) there are now additional (implicitly moduli-dependent) terms in both bulk and brane Lagrangians. If you are willing to live with all this then the gravitational sector at least is considerably simpler and the Einstein-frame junction conditions read

$$[\mathcal{K}_E]_{AB} = \kappa_n^{-2} \left\{ [S_E]_{AB} - \frac{S_E}{n-2} \, [q_E]_{AB} \right\}, \tag{2.28}$$

and

$$[J_E]^i = -[H_E]^{ij} \;\; (\mathcal{L}^E_{\mathrm{brane}})'_j. \tag{2.29}$$

Sometimes (but not always) the gain in the junction conditions is worth the price paid elsewhere in the system. (There is a long and contentious debate in the literature concerning which frame is the most "physical"; see [13] for details and references.)

3 Effective stress-energy tensor (general frame)

For the general systems under consideration (2.1) the bulk Einstein equation reads

$$\begin{aligned} G_{AB} &\equiv R_{AB} - \frac{1}{2} g_{AB} R \\ &= \kappa_n^{-2} \, F^{-1}(\phi) \, H_{ij}(\phi) \left[\partial_A \phi^i \, \partial_B \phi^j - \frac{1}{2} g_{AB} \;\; g^{CD} \, \partial_C \phi^i \, \partial_D \phi^j \right] \\ &\quad + F^{-1}(\phi) \;\left[\nabla_A \nabla_B F(\phi) - g_{AB} \;\; g^{CD} \, \nabla_C \nabla_D F(\phi) \right] \\ &\quad + \kappa_n^{-2} \, F^{-1}(\phi) \, T^{\psi}_{AB} - g_{AB} \, \kappa_n^{-2} \, F^{-1}(\phi) \, V(\phi,\psi) - g_{AB} \, \Lambda. \end{aligned} \tag{3.1}$$

In deriving this we have first varied with respect to the metric, and then systematically rearranged terms until the equation is cast in the form $G_{AB} = \kappa_n^{-2} \, T^{\mathrm{eff}}_{AB}$; see [14] for some examples of this procedure. The double derivatives acting on F can be recast as

$$\nabla_A \nabla_B F(\phi) = F'_i(\phi) \; (\nabla_A \nabla_B \phi^i) + F''_{ij}(\phi) \; (\nabla_A \phi^i) \; (\nabla_B \phi^j). \tag{3.2}$$

This allows us to define the effective bulk stress-energy tensor as

$$\begin{aligned} T^{\mathrm{effective}}_{AB} &= F^{-1}(\phi) \; \left(H_{ij}(\phi) + \kappa_n^2 \, F''_{ij}(\phi) \right) \partial_A \phi^i \, \partial_B \phi^j \\ &\quad - F^{-1}(\phi) \, \frac{1}{2} \; \left(H_{ij}(\phi) + 2\kappa_n^2 \, F''_{ij}(\phi) \right) \, g_{AB} \;\; g^{CD} \, \partial_C \phi^i \, \partial_D \phi^j \\ &\quad + F^{-1}(\phi) \, F'_i(\phi) \, \kappa_n^2 \; \left(\nabla_A \nabla_B \phi^i - g^{CD} \, \nabla_C \nabla_D \phi^i \right) \\ &\quad + F^{-1}(\phi) \, T^{\psi}_{AB} - g_{AB} \, F^{-1}(\phi) \, V(\phi,\psi) - g_{AB} \, \kappa_n^2 \, \Lambda. \end{aligned} \tag{3.3}$$

This is a generalization of the effective stress-energy for ordinary non-minimally coupled scalars, which correspond to $H(\phi) = 1$, and $F(\phi) = 1 - \xi \kappa_n^{-2} \phi^2$. For details see [14]. The bulk scalar field equation reads

$$\begin{aligned} \nabla^A \left(H_{ij}(\phi) \; \nabla_A \phi^j \right) &= V'_i(\phi) - (\mathcal{L}_{\mathrm{bulk}})'_i - \frac{1}{2} \kappa_n^2 \; F'_i(\phi) [R - 2\Lambda] \\ &\quad + \frac{1}{2} (\partial_i H_{jk}) \nabla_A \phi^j \; \nabla^A \phi^k. \end{aligned} \tag{3.4}$$

There is an additional equation of motion for the generic "matter" fields that will not concern us.

4 The Gauss–Codazzi equations

In order to see how effective (n–1)-dimensional gravity is induced on the brane we follow [6] and begin with the equations of Gauss and Codazzi. We use them to relate the extrinsic curvature and (n–1)-dimensional Riemann tensor of a hypersurface to the distribution of bulk matter (this hypersurface being taken to coincide with the location of the brane). The Codazzi equation is related to the conservation of stress-energy on the brane (*non-conservation* in certain circumstances), while the Gauss equation provides an "induced" generalized Einstein equation for the braneworld.

4.1 The Codazzi equation and its implications

Consider the Codazzi equation [6]

$$D^A(K^{\pm}_{AB} - K^{\pm} q_{AB}) = -{}^{(n)}R^{\pm}_{AA'}\, n^A\, q^{A'}{}_B. \tag{4.1}$$

Here D (as opposed to ∇) is the intrinsic covariant derivative with respect to q defined on the brane. We adopt MTW [15] conventions regarding the sign of the extrinsic curvature, which differs from that of [6]. Note that in general there are two Codazzi equations, one for each side of the brane (unless one is adopting the "one-sided" approach of [16, 17]).

First apply the bulk Einstein equation

$$D^A K^{\pm}_{AB} - D_B K^{\pm} = -\kappa_n^{-2}\ {}^{(n)}T^{\pm}_{AA'}\, n^A\, q^{A'}{}_B. \tag{4.2}$$

Now adopt "reduced" Gaussian normal coordinates to simplify this expression: Let indices such as a, b, c run from 0 to $n-2$, so that they denote coordinates on (or parallel to) the brane. In contrast, indices such as A, B, C run from 0 to $n-1$, and denote coordinates in the bulk. The index n will denote the direction normal to the brane. Then

$$g_{AB} = \begin{bmatrix} q_{ab} & 0 \\ 0 & 1 \end{bmatrix}; \qquad q_{AB} = \begin{bmatrix} q_{ab} & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.3}$$

In these coordinates

$$K_{AB} = \begin{bmatrix} K_{ab} & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.4}$$

It is useful to decompose the bulk stress energy tensor as follows:

$$T_{AB} \equiv \begin{bmatrix} T_{ab} & f_a \\ f_b & T_{nn} \end{bmatrix}. \tag{4.5}$$

Here T_{ab} denotes the in-brane part of the bulk stress-energy; $f_a \equiv T_{an}$ denotes the flux onto (or away from) the brane—this implies a shear force applied to the brane by the bulk matter, and T_{nn} denotes the normal compressive force (pressure) on the brane. (These "reduced" on-brane coordinates are often much easier to work with than the full n-dimensional system; however as much of the literature uses the full system we shall aim for maximum comprehensibility by keeping certain key formulas in the "full" system.)

In these "reduced" coordinates

$$D^a\left(K^{\pm}_{ab} - K^{\pm}\, q_{ab}\right) = -\kappa_n^{-2}\, f^{\pm}_b. \tag{4.6}$$

Because the flux term (implying a shear force) this is not quite a conservation equation. Now take differences between the two sides of the brane and define

$$\mathcal{F}_a \equiv f^+_a - f^-_a. \tag{4.7}$$

This represents the discontinuity in the flux of bulk matter as one crosses the brane—it describes how much net bulk matter is accreting onto or evaporating from the brane. The resulting "conservation law" is

$$D^a\left(\mathcal{K}_{ab} - \mathcal{K}\, q_{ab}\right) = -\kappa_n^{-2}\, \mathcal{F}_b. \tag{4.8}$$

Now apply the junction condition to write $\mathcal{K}_{AB}$ in terms of the surface stress-energy tensor and the discontinuity of the normal derivative of the moduli fields. In particular, write [using (2.10)]

$$\begin{aligned}
\mathcal{K}_{ab} - \mathcal{K}\, q_{ab} &= \frac{\kappa_n^{-2}}{F} S_{ab} - \frac{F_i'\, J^i}{F}\, q_{ab} \\
&= \frac{\kappa_n^{-2}}{F} S_{ab} - \frac{F_i'}{F^2} \left(\kappa_n^{-1}\, \gamma_{21}^i\, S + \gamma_{22}^{ij}\, (\mathcal{L}_{\rm brane})_j' \right) q_{ab} \\
&= \frac{\kappa_n^{-2}}{F} \left(S_{ab} + \alpha\, q_{ab}\, S \right) + \frac{\kappa_n^{-1}}{F}\, \beta^i\, (\mathcal{L}_{\rm brane})_i'\, q_{ab}.
\end{aligned} \tag{4.9}$$

This serves as the definition of the dimensionless functions α and β (which depend on the on-brane values of the moduli fields; explicit formulae are given in Appendix A). Then

$$D^a \left(\frac{S_{ab} + \alpha\, q_{ab}\, S + \kappa_n\, \beta^i\, q_{ab}\, (\mathcal{L}_{\rm brane})_i'}{F} \right) = -\mathcal{F}_b. \tag{4.10}$$

Rearranging this

$$\begin{aligned}
D^a S_{ab} &= -F\, \mathcal{F}_b + \frac{D^a F}{F} \left(S_{ab} + \alpha\, q_{ab}\, S + \kappa_n \beta^i\, (\mathcal{L}_{\rm brane})_i' \right) \\
&\quad - D_b \left(\alpha\, S + \kappa_n \beta^i\, (\mathcal{L}_{\rm brane})_i' \right).
\end{aligned} \tag{4.11}$$

Therefore

$$\begin{aligned}
D^a S_{ab} &= -F\, \mathcal{F}_b + \frac{F_i'\, D^a \phi^i}{F} \left(S_{ab} + \alpha\, q_{ab}\, S + \kappa_n \beta^i\, (\mathcal{L}_{\rm brane})_i' \right) \\
&\quad - D_b \phi^j \left(\alpha_j'\, S + \kappa_n (\beta^i)_j'\, (\mathcal{L}_{\rm brane})_i' \right) - \left(\alpha\, D_b S + \kappa_n \beta^i\, D_b (\mathcal{L}_{\rm brane})_i' \right).
\end{aligned} \tag{4.12}$$

In short, stress energy on the brane is *not* conserved—the nonconservation arises both from net bulk fluxes onto the brane, and from variations in the value of the moduli fields as one moves along the brane. If the moduli fields on the brane are "translationally invariant" (as for instance in FLRW brane cosmologies before one switches on perturbations) the second and third term on the RHS vanish (since then $D_a \phi^i = 0$). Even if the moduli fields on the brane are translationally invariant, the last term on the RHS can still pick up contributions from $D_b S$ and $D_b (\mathcal{L}_{\rm brane})_j'$; these are best dealt with on a case by case basis. In most models considered to date one imposes $\mathcal{F}_a = 0$ by symmetry or by fiat (see *e.g.*, [6]) and so recovers a conservation law for the ordinary on-brane surface stress-energy—we now see that in general this is an approximation.

Note that the analysis so far has been very general, we have not had to impose either Z_2 symmetry or "one-sidedness". If one chooses to work directly in the Einstein frame, then

$$D_E^a \left([\mathcal{K}_E]_{ab} - [\mathcal{K}_E]\, [q_E]_{ab} \right) = -\kappa_n^{-2}\, [\mathcal{F}_E]_b. \tag{4.13}$$

The moduli fields have now (apparently) decoupled from the conservation law, but it is more correct to say that they have gone underground by modifying the definition of $\mathcal{F} \to \mathcal{F}_E$. In terms of the surface stress-energy we have the much simpler relation

$$D_E^a [S_E]_{ab} = -[\mathcal{F}_E]_b. \tag{4.14}$$

4.2 The Gauss equation and its implications

In order to see how effective (n–1)-dimensional gravity is 'induced on the braneworld we begin with the Gauss equation and write the (n–1)-dimensional Riemann tensor of the braneworld in the form [6]

$${}^{(n-1)}R_{ABCD} = {}^{(n)}R^{\pm}_{A'B'C'D'}\; q^{A'}{}_A\, q^{B'}{}_B\, q^{C'}{}_C\, q^{D'}{}_D\; + K^{\pm}_{AC}\, K^{\pm}_{BD} - K^{\pm}_{AD}\, K^{\pm}_{BC}. \tag{4.15}$$

This equation applies to each side of the brane independently. After repeated contractions and some rearrangement[1]

$$
\begin{aligned}
{}^{(n-1)}G_{AB} = \frac{n-3}{n-2}\left[G^{\pm}_{CD}\, q_A{}^C\, q_B{}^D + \left(G^{\pm}_{CD}\, n^C n^D - \frac{1}{n-1} G^{\pm C}_{C}\right) q_{AB}\right] \\
+K^{\pm}\, K^{\pm}_{AB} - K^{\pm C}_{A} K^{\pm}_{BC} - \frac{1}{2} q_{AB}\, \left(K^2_{\pm} - K^{\pm}_{CD} K^{CD}_{\pm}\right) - E^{\pm}_{AB}.
\end{aligned}
\tag{4.16}
$$

Here E_{AB} is the "electric" part of the Weyl tensor

$$
E_{AB} = C_{DEHI}\, n^D\, n^H\, q_A{}^E\, q_B{}^I = C_{BAHB}\, n^D\, n^H. \tag{4.17}
$$

Indeed, since the foliation we are dealing with is timelike (spacelike normal) calling this the "electric" part of the Weyl tensor is purely formal.

The above expression (4.16) is purely geometrical, with as yet no physical content. Its meaning can be made clearer by simultaneously substituting the bulk Einstein equations

$$
G_{AB} = \kappa_n^{-2}\, T_{AB}, \tag{4.18}
$$

and by adopting "reduced" Gaussian normal coordinates adapted to the brane. Using the definitions of the previous subsection equation (4.16) becomes

$$
\begin{aligned}
{}^{(n-1)}G_{ab} = \frac{n-3}{n-2}\, \kappa_n^{-2}\, \left[T^{\pm}_{ab} - \frac{1}{n-1}(T^{\pm}_{cd}\, q^{cd})\, q_{ab} + \frac{n-2}{n-1}\, T^{\pm}_{nn}\right] \\
+K^{\pm}\, K^{\pm}_{ab} - K^{\pm c}_{a} K^{\pm}_{bc} - \frac{1}{2} q_{ab}\, \left(K^2_{\pm} - K^{\pm}_{cd} K^{cd}_{\pm}\right) - C^{\pm}_{anbn}.
\end{aligned}
\tag{4.19}
$$

You can take sums and differences of these two equations but the results are not particularly enlightening—it is (finally) at this stage that we find it useful to impose Z_2 symmetry (or adopt the "one-sided" approach of [16, 17] at the cost of introducing a few factors of 2). In the case of Z_2 symmetry

$$
K^{+}_{ab} = K_{ab} = -K^{-}_{ab}; \qquad \mathcal{K}_{ab} = 2K_{ab} = 2K^{+}_{ab} = -2K^{-}_{ab}. \tag{4.20}
$$

For the "one-sided" approach [16, 17]

$$
K^{+}_{ab} = K_{ab}; \qquad K^{-}_{ab} = 0, \text{ “null and void”}; \qquad \mathcal{K}_{ab} = K_{ab} = K^{+}_{ab}. \tag{4.21}
$$

In either case

$$
\begin{aligned}
{}^{(n-1)}G_{ab} = \frac{n-3}{n-2}\, \kappa_n^{-2}\, \left[T_{ab} - \frac{1}{n-1}(T_{cd}\, q^{cd})\, q_{ab} + \frac{n-2}{n-1}\, T_{nn}\right] \\
+K\, K_{ab} - K_a{}^c\, K_{bc} - \frac{1}{2} q_{ab}\, \left(K^2 - K_{cd} K^{cd}\right) - C_{anbn}.
\end{aligned}
\tag{4.22}
$$

In the original situation considered by Shiromizu–Maeda–Sasaki [6] the bulk stress energy was particularly simple, $T_{AB} = -\Lambda\, g_{AB}$ so that $T_{ab} = -\Lambda\, q_{ab}$ and $T_{nn} = -\Lambda$. In that case the terms on the first line above are trivial (corresponding to a bulk-induced contribution to the cosmological constant for the brane), while in the second line the terms involving the extrinsic curvature were uniquely determined by the brane stress-energy. Thus in Shiromizu–Maeda–Sasaki [6] the *only* way the bulk enters into the effective equations of motion is via the Weyl tensor contribution C_{anbn}. In the recent extensions by Maeda and Wands [7] and Mennim and Battye [8] the bulk stress tensor corresponds to a nontrivial dilaton field; and this opens up new avenues for possible bulk influence on the brane.

[1] Comment: Equations (1) through (6) of the Shiromizu–Maeda–Sasaki analysis [6] are dimension independent; the decomposition in equation (7) of the Riemann tensor in terms of the Weyl tensor, Ricci tensor, and Ricci scalar is the first occurrence of dimension-dependent coefficients. Note that in general $G_A{}^A = -[(n-2)/2]R_A{}^A$.

In the more general case of this paper the bulk has in principle the possibility of communicating with the brane in at least five different ways: (1) directly via the Weyl tensor C_{anbn}, (2) via "anisotropies" in the in-brane components of the bulk stress-energy $[T_{ab} - \frac{1}{n-1}(T_{cd}\, q^{cd})\, q_{ab}]$, (3) via the pressure T_{nn}, (4) via the now much more complicated relationship between extrinsic curvature and surface stress-energy (2.17), and (5) via the more complicated "conservation law" of the previous subsection (4.12).

By substituting the bulk Einstein tensor (3.1) in (4.22), deferring for now the task of dealing with the extrinsic curvatures, we obtain

$$\begin{aligned}
{}^{(n-1)}G_{AB} = & \frac{n-3}{n-2}\, F^{-1}\, \left[\kappa_n^{-2}\, H_{ij} + F''_{ij}\right] D_A\phi^i\, D_B\phi^j \\
& -\frac{n-3}{n-2}\, F^{-1}\, q_{AB} \left[\frac{n}{2(n-1)}\, \kappa_n^{-2}\, H_{ij} + F''_{ij}\right] D_C\phi^i\, D^C\phi^j \\
& +\frac{n-3}{n-2}\, F^{-1}\, F'_i \left[D_A D_B\phi^i - q_{AB}\, D_C D^C\phi^i\right] \\
& -\frac{n-3}{n-1}\, (\Lambda + \kappa_n^{-2}\, F^{-1}\, V)\, q_{AB} \\
& +\frac{n-3}{n-1}\, \kappa_n^{-2}\, \frac{1}{2}\, F^{-1}\, H_{ij}\, (n^C\partial_C\phi^i)\, (n^D\partial_D\phi^j)\, q_{AB} \\
& -\frac{n-3}{n-2}\, F^{-1}\, F'_i\, (n^A\partial_A\phi^i)\, (K_{AB} - K\, q_{AB}) \\
& +K\, K_{AB} - K_A{}^C\, K_{BC} - \frac{1}{2}\, q_{AB}\, (K^2 - K_{CD}K^{CD}) \\
& -E_{AB}.
\end{aligned} \tag{4.23}$$

The terms in the first four rows of (4.23) are quantities intrinsically defined on the $(n–1)$-dimensional hypersurface. (D_A is the intrinsic covariant derivative on the brane; acting on braneworld scalars $D_a\phi^i = q_A{}^B\nabla_B\phi^i$; acting on tensors there are additional terms coming from the extrinsic curvature.) The terms in the last four rows depend, a priori, on the extrinsic properties of the hypersurface; we will now use the junction conditions to rewrite them (as much as possible) in terms of intrinsic quantities.

Recall that we are now restricting attention to the case of a Z_2 orbifold identification, (or, modulo a few factors of 2, adopting the "one-sided" approach of [16, 17]). The generalized junction conditions (2.19) and (2.20) will then read

$$(n^A\partial_A\phi^i) = \frac{1}{2}J^i = \frac{1}{2F}\left(\kappa_n^{-1}\, \gamma^i_{21}\, S + \gamma^{ij}_{22}\, (\mathcal{L}_{\rm brane})'_j\right), \tag{4.24}$$

and

$$K_{AB} \quad = \quad \frac{1}{2}\mathcal{K}_{AB} = \frac{\kappa_n^{-2}}{2F}\, \left(S_{AB} + \tilde\gamma_{11}\, S\, q_{AB} + \kappa_n\, \tilde\gamma^i_{12}\, (\mathcal{L}_{\rm brane})'_i\, q_{AB}\right). \tag{4.25}$$

These expressions can now be substituted into (4.23) to give place to $(n–1)$-dimensional Einstein-like equations controlling the gravitational behaviour inside the brane. We find that the most efficient way to proceed is by introducing some generic coefficients $\Gamma_1 \dots \Gamma_7$. Doing this, and adopting "reduced" Gaussian normal coordinates

$$\begin{aligned}
{}^{(n-1)}G_{ab} = & \frac{n-3}{n-2}\, F^{-1}\, \left[\kappa_n^{-2}\, H_{ij} + F''_{ij}\right] D_a\phi^i\, D_b\phi^j \\
& -\frac{n-3}{n-2}\, F^{-1}\, q_{ab} \left[\frac{n}{2(n-1)}\, \kappa_n^{-2}\, H_{ij} + F''_{ij}\right] D_c\phi^i\, D^c\phi^j \\
& +\frac{n-3}{n-2}\, F^{-1}\, F'_i \left[D_a D_b\phi^i - q_{ab}\, D_c D^c\phi^i\right] \\
& -\frac{n-3}{n-1}\, (\Lambda + \kappa_n^{-2}\, F^{-1}\, V)\, q_{ab}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\kappa_n^{-4}}{4F^2}\left\{\Gamma_1\,(S^2)_{ab}+\Gamma_2\,S\,S_{ab}+\Gamma_3\,S^2\,q_{ab}+\Gamma_4\,(S_{cd}\,S^{cd})\,q_{ab}\right\}\\
&+\frac{\kappa_n^{-3}}{4F^2}\left\{\Gamma_5^i\,(\mathcal{L}_{\rm brane})'_i\,S_{ab}+\Gamma_6^i\,(\mathcal{L}_{\rm brane})'_i\,S\,q_{ab}\right\}\\
&+\frac{\kappa_n^{-2}}{4F^2}\left\{\Gamma_7^{ij}\,(\mathcal{L}_{\rm brane})'_i\,(\mathcal{L}_{\rm brane})'_j\,q_{ab}\right\}\\
&-C_{anbn}. \qquad (4.26)
\end{aligned}
$$

This serves as the definition of the dimensionless coefficients $\Gamma_1\dots\Gamma_7$; they are functions of the γ's (and thus implicitly functions of the on-brane values of the moduli fields) and the dimensionality of spacetime. Note that all the terms appearing here are *intrinsic* to the braneworld, *except* for the Weyl tensor C_{anbn}, so that, as explained by Shiromizu–Maeda–Sasaki [6], this system of equations is not closed. The Weyl term depends upon the global behaviour of the bulk spacetime, which itself depends on the bulk scalar field.

Even without explicitly calculating the coefficients $\Gamma_1\dots\Gamma_7$ we see that several key features of the Shiromizu–Maeda–Sasaki analysis carry through—such as the presence of quadratic terms depending on the square of the braneworld stress-energy tensor in these effective Einstein equations. A tedious but straightforward analysis leads to

$$\Gamma_1 = -1; \qquad (4.27)$$

$$\Gamma_2 = +\frac{1}{n-2}; \qquad (4.28)$$

$$\Gamma_3 = -\frac{1}{2(n-2)}+\frac{n-3}{(n-1)^2(n-2)}\,\frac{\mathcal{E}-1}{\mathcal{E}} \qquad (4.29)$$

$$= -\frac{1}{2(n-2)}\left[1-\frac{n-3}{2(n-1)^2}\,\frac{\mathcal{E}-1}{\mathcal{E}}\right]; \qquad (4.30)$$

$$\Gamma_4 = +\frac{1}{2}; \qquad (4.31)$$

$$\Gamma_5^i = 0; \qquad (4.32)$$

$$\Gamma_6^i = -\frac{n-3}{(n-1)(n-2)}\,\frac{\kappa_n\,H^{ij}\,F'_j}{\mathcal{E}} \qquad (4.33)$$

$$= -\frac{n-3}{(n-1)(n-2)}\,\frac{\kappa_n\,[H_E]^{ij}\,F'_j}{F}; \qquad (4.34)$$

$$\Gamma_7^{ij} = +\frac{1}{2}\,\frac{n-3}{n-1}\,[H_E^{-1}]^{ij}; \qquad (4.35)$$

where we have defined

$$\mathcal{E}(\phi) = 1+\frac{n-1}{n-2}\,H^{ij}(\phi)\,\kappa_n^2\,\frac{F'_i(\phi)\,F'_j(\phi)}{F(\phi)}, \qquad (4.36)$$

and $[H_E^{-1}]^{ij}$ is the inverse of the Einstein-frame sigma model metric as defined in (2.24).

To facilitate comparison with the analyses of Shiromizu–Maeda–Sasaki [6], Maeda–Wands [7], and Mennim–Battye [8] it is useful to split the surface stress-energy into a (possibly moduli dependent) "internal cosmological constant" (*not* the "effective cosmological constant"—see below), and "the rest" according to the prescription

$$S_{ab} = -\lambda(\phi)\,q_{ab}+\tau_{ab}(\phi). \qquad (4.37)$$

After this substitution, some rearrangement yields

$${}^{(n-1)}G_{ab} = \frac{n-3}{n-2}\,F^{-1}\,\left[\kappa_n^{-2}\,H_{ij}+F''_{ij}\right]D_a\phi^i\,D_b\phi^j$$

$$
\begin{aligned}
&-\frac{n-3}{n-2}\, F^{-1}\, q_{ab} \left[\frac{n}{2(n-1)}\, \kappa_n^{-2}\, H_{ij} + F''_{ij}\right] D_c\phi^i\, D^c\phi^j \\
&+\frac{n-3}{n-2}\, F^{-1}\, F'_i \left[D_a D_b \phi^i - q_{ab}\, D_c D^c \phi^i\right] \\
&-\frac{n-3}{n-1}\, (\Lambda + \kappa_n^{-2}\, F^{-1}\, V)\, q_{ab} \\
&+\frac{\kappa_n^{-4}}{4F^2} \left\{\Gamma_1\, (\tau^2)_{ab} + \Gamma_2\, \tau\, \tau_{ab} + \Gamma_3\, \tau^2\, q_{ab} + \Gamma_4\, (\tau_{cd}\, \tau^{cd})\, q_{ab}\right\} \\
&+\frac{\kappa_n^{-4}}{4F^2} \left\{-2\, \Gamma_1\, \lambda - (n-1)\, \Gamma_2\, \lambda + \kappa_n\, \Gamma_5^i\, (\mathcal{L}_{\mathrm{brane}})'_i\right\}\, \tau_{ab} \\
&+\frac{\kappa_n^{-4}}{4F^2} \left\{-\Gamma_2\, \lambda - 2\, (n-1)\, \Gamma_3\, \lambda - 2\, \Gamma_4\, \lambda + \kappa_n\, \Gamma_6^i\, (\mathcal{L}_{\mathrm{brane}})'_i\right\}\, \tau\, q_{ab} \\
&+\frac{\kappa_n^{-4}}{4F^2} \Big\{\Gamma_1\, \lambda^2 + (n-1)\, \Gamma_2\, \lambda^2 + (n-1)^2\, \Gamma_3\, \lambda^2 + (n-1)\, \Gamma_4\, \lambda^2 \\
&\qquad -\kappa_n\, \Gamma_5^i (\mathcal{L}_{\mathrm{brane}})'_i\, \lambda - \kappa_n\, \Gamma_6^i (\mathcal{L}_{\mathrm{brane}})'_i\, (n-1)\, \lambda \\
&\qquad +\kappa_n^2\, \Gamma_7^{ij}\, (\mathcal{L}_{\mathrm{brane}})'_i\, (\mathcal{L}_{\mathrm{brane}})'_j \Big\}\, q_{ab} \\
&-C_{anbn}.
\end{aligned}
\tag{4.38}
$$

To interpret this physically, we write it as

$$
\begin{aligned}
{}^{(n-1)}G_{ab} \;=\;& 8\pi\, G_{\mathrm{effective}}\, \tau_{ab} + 8\pi\, G_{\mathrm{anomalous}}\, \tau\, q_{ab} - \Lambda_{\mathrm{effective}}\, q_{ab} \\
&+8\pi\, G_{\mathrm{quadratic}}\, \left\{\Gamma_1\, (\tau^2)_{ab} + \Gamma_2\, \tau\, \tau_{ab} + \Gamma_3\, \tau^2\, q_{ab} + \Gamma_4\, (\tau_{cd}\, \tau^{cd})\, q_{ab}\right\} \\
&+\kappa_n^{-2}\, (T^\phi)_{ab} - C_{anbn}.
\end{aligned}
\tag{4.39}
$$

Here $G_{\mathrm{effective}}$ denotes the effective Newton constant on the brane, in general it is a function of the moduli fields. $G_{\mathrm{anomalous}}$ represents a perhaps unexpected anomalous coupling of the braneworld geometry to the trace of braneworld stress energy: in semi-realistic models this should be made small (and in standard scenarios it often vanishes identically; more on this later). $\Lambda_{\mathrm{effective}}$ is the net effective cosmological constant for matter trapped on the brane—it gets contributions from both the bulk and the brane, according to the formula given below; in realistic models it should be kept as small as possible. $G_{\mathrm{quadratic}}$ governs the quadratic contributions to the effective on-brane Einstein equations; again in realistic models this quantity should be kept small to avoid serious conflict with experiment and observational cosmology. Finally $(T^\phi)_{ab}$ represents the effective stress-energy attributable to along-the-brane variations of the moduli fields. Explicitly

$$
8\pi\, G_{\mathrm{effective}} = \frac{\kappa_n^{-4}}{4F^2} \left\{-2\, \Gamma_1\, \lambda - (n-1)\, \Gamma_2\, \lambda + \kappa_n\, \Gamma_5^i\, (\mathcal{L}_{\mathrm{brane}})'_i\right\}; \tag{4.40}
$$

$$
8\pi\, G_{\mathrm{anomalous}} = \frac{\kappa_n^{-4}}{4F^2} \left\{-\Gamma_2\, \lambda - 2\, (n-1)\, \Gamma_3\, \lambda - 2\, \Gamma_4\, \lambda + \kappa_n\, \Gamma_6^i\, (\mathcal{L}_{\mathrm{brane}})'_i\right\}; \tag{4.41}
$$

$$
8\pi\, G_{\mathrm{quadratic}} = \frac{\kappa_n^{-4}}{4F^2}; \tag{4.42}
$$

$$
\begin{aligned}
\Lambda_{\mathrm{effective}} =\;& \frac{n-3}{n-1}\, (\Lambda + \kappa_n^{-2}\, F^{-1}\, V) \\
&-\frac{\kappa_n^{-4}}{4F^2} \Big\{\Gamma_1\, \lambda^2 + (n-1)\, \Gamma_2\, \lambda^2 + (n-1)^2\, \Gamma_3\, \lambda^2 + (n-1)\, \Gamma_4\, \lambda^2 \\
&\quad -\kappa_n\, \Gamma_5^i\, (\mathcal{L}_{\mathrm{brane}})'_i\, \lambda - \kappa_n\, (n-1)\, \Gamma_6^i\, (\mathcal{L}_{\mathrm{brane}})'_i\, \lambda \\
&\quad +\kappa_n^2\, \Gamma_7^{ij}\, (\mathcal{L}_{\mathrm{brane}})'_i\, (\mathcal{L}_{\mathrm{brane}})'_j \Big\}.
\end{aligned}
\tag{4.43}
$$

We draw some general largely model-independent conclusions: Generically, the induced Einstein equations on the braneworld correspond to a generalization of the notion of a Brans–Dicke theory, with an effective

Newton constant that depends on possibly position-dependent scalar fields (the moduli) which themselves influence both the effective cosmological constant and make direct contributions to the stress-energy. Where the braneworld approach steps well beyond the usual Brans–Dicke theories is in situations where there is significant coupling to the bulk—either through incoming fluxes or a nontrivial bulk Weyl tensor.

Inserting the explicit formulae for $\Gamma_1 \dots \Gamma_7$ we find

$$
\begin{aligned}
8\pi\, G_{\mathrm{effective}} &= +\frac{\kappa_n^{-4}\,\lambda}{4F^2}\,\frac{n-3}{n-2}; && (4.44)\\
8\pi\, G_{\mathrm{anomalous}} &= -\frac{\kappa_n^{-4}}{4F^2}\,\frac{n-3}{(n-1)(n-2)}\left\{\lambda\,\frac{\mathcal{E}-1}{\mathcal{E}} + \frac{\kappa_n^2\, F_i'\, H^{ij}\,(\mathcal{L}_{\mathrm{brane}})_j'}{\mathcal{E}}\right\} && (4.45)\\
&= -\frac{\kappa_n^{-4}}{4F^2}\,\frac{n-3}{(n-1)(n-2)}\left\{\lambda\,\frac{\mathcal{E}-1}{\mathcal{E}} + \frac{\kappa_n^2\, F_i'\, [H_E^{-1}]^{ij}\,(\mathcal{L}_{\mathrm{brane}})_j'}{F}\right\}; && (4.46)\\
8\pi\, G_{\mathrm{quadratic}} &= +\frac{\kappa_n^{-4}}{4F^2}; && (4.47)\\
\Lambda_{\mathrm{effective}} &= \frac{n-3}{n-1}\,(\Lambda + \kappa_n^{-2}\,F^{-1}\,V) + \frac{\kappa_n^{-4}\,\lambda^2}{8F^2}\,\frac{n-3}{n-2}\,\frac{1}{\mathcal{E}}\\
&\quad -\frac{\kappa_n^2}{4F^2}\,\frac{n-3}{n-2}\,\frac{[H_E]^{ij} F_i'\,(\mathcal{L}_{\mathrm{brane}})_j'}{F}\\
&\quad -\frac{\kappa_n^2}{8F^2}\,\frac{n-3}{n-1}[H_E]^{ij}\,(\mathcal{L}_{\mathrm{brane}})_i'\,(\mathcal{L}_{\mathrm{brane}})_j'\big\}. && (4.48)
\end{aligned}
$$

The "quadratic" part of the effective stress tensor is proportional to

$$
-(\tau^2)_{ab} + \frac{\tau\;\tau_{ab}}{n-2} - \frac{\tau^2\; q_{ab}}{2(n-2)}\left[1 - \frac{n-3}{(n-1)^2}\frac{\mathcal{E}-1}{\mathcal{E}}\right] + \frac{1}{2}(\tau^{pq}\;\tau_{pq})\;q_{ab}. \qquad (4.49)
$$

Finally the explicit moduli contribution to the effective stress tensor appearing in equation (4.39) is

$$
\begin{aligned}
(T^\phi)_{ab} &= \frac{n-3}{n-2}\,F^{-1}\,\left[H_{ij} + \kappa_n^2\,F_{ij}''\right] D_a\phi^i\;D_b\phi^j\\
&\quad -\frac{n-3}{n-2}\,F^{-1}\,q_{ab}\left[\frac{n}{2(n-1)}\,H_{ij} + \kappa_n^2\,F_{ij}''\right] D_c\phi^i\;D^c\phi^j\\
&\quad +\frac{n-3}{n-2}\,\kappa_n^2\,F^{-1}\,F_i'\left[D_aD_b\phi^i - q_{ab}\;D_cD^c\phi^i\right]. \qquad (4.50)
\end{aligned}
$$

Some key points to realize are:

—(1) As long as λ is positive (corresponding to a positive brane tension), the effective Newton constant $G_{\mathrm{effective}}$ is also positive, with the sign being independent of total dimensionality. While the use of negative tension branes in supporting roles has gained considerable popularity (see references in [16]), the use of negative tension branes should be viewed with extreme suspicion: Not only is the effective braneworld Newton constant negative for matter trapped on the brane, but negative tension branes also cause disturbing effects in the bulk. It has been known for over a decade that negative tension branes led to traversable wormholes [18], with all their attendant problems [19], a point that we have made more explicit in a braneworld context in [16].

—(2) $G_{\mathrm{anomalous}}$ vanishes whenever $F_i'(\phi) = 0$; in particular, it automatically vanishes in the Einstein frame. This is the reason this contribution has been invisible to date.

—(3) Quadratic terms are unavoidable. They cannot be removed by change of conformal frame or choice of dimensionality. The best that can be done is to set the coefficient of one of the quadratic terms (τ^2) to zero by fine tuning $\mathcal{E} = -(n-3)/(n^2-3n+4)$.

—(4) $\Lambda_{\mathrm{effective}}$ is now much more complicated. The effective cosmological constant can be modified by tuning the moduli fields.

In summary: Many of the qualitative features of the Shiromizu–Maeda–Sasaki analysis [6], plus the extensions by Maeda–Wands [7], and Mennim–Battye [8], continue to hold in this much more general framework. To illustrate what we can learn from the induced braneworld Einstein equations let us now consider some specific examples: (1) we particularize to the Einstein frame (keeping dimensionality, sigma-model, number of moduli fields, and the brane Lagrangian generic); as should be expected, we encounter considerable simplifications, (2) we consider the dilaton field in the string frame, again keeping the discussion as general as is reasonable.

5 Generic moduli fields in the Einstein frame

Suppose we decide to do all calculations in the Einstein frame, but keep the moduli fields arbitrary in all other respects. We have already seen that going to the Einstein frame casts the general action into the considerably simpler form (2.23):

$$\begin{aligned}
\mathcal{S}_{\rm Einstein} = & \frac{1}{2}\int_{{\rm int}(\mathcal{M})} \sqrt{-g_E}\;{\rm d}^n x\;\kappa_n^2\;\left[R(g_E) - 2F(\phi)^{-2/(n-2)}\;\Lambda\right] \\
& - \int_{\partial\mathcal{M}} \sqrt{-q_E}\;{\rm d}^{n-1}x\;\kappa_n^2\;K(g_E) \\
& + \int_{{\rm int}(\mathcal{M})} \sqrt{-g_E}\;{\rm d}^n x \left\{ -\frac{1}{2}[H_E]_{ij}(\phi)\;\left[g^{AB}\;\partial_A\phi^i\;\partial_B\phi^j\right] \right. \\
& \qquad - V_E(\phi,\psi) \\
& \qquad \left. + \mathcal{L}^E_{\rm bulk}([g_E]_{AB},\phi,\psi) \right\} \\
& + \int_{\rm brane} \sqrt{-q_E}\;{\rm d}^{n-1}x\;\mathcal{L}^E_{\rm brane}([q_E]_{AB},\phi,\psi).
\end{aligned} \tag{5.1}$$

Various factors of $F(\phi)$ are now hiding in the definition of V_E, $\mathcal{L}_{\rm bulk}$, and $\mathcal{L}_{\rm brane}$. We explicitly see that what was the cosmological constant in the original frame is now a potential. Similarly if we had a simple brane tension in the original frame, then in this new frame the brane tension will be moduli dependent. (Warning: Because of this you cannot just blindly set $F(\phi) = 1$ everywhere and hope to get meaningful results, going to the Einstein frame is a field redefinition which simplifies the gravitational sector but there is a "conservation of difficulty" phenomenon and the bulk and brane matter Lagrangians are more complicated). The bulk stress-energy tensor simplifies to

$$\begin{aligned}
[T_E]^{\rm effective}_{AB} &= [H_E]_{ij}(\phi)\;\partial_A\phi^i\;\partial_B\phi^j \\
&\quad -\frac{1}{2}\;[H_E]_{ij}(\phi)\;[g_E]_{AB}\;\;[g_E]^{CD}\;\partial_C\phi^i\;\partial_D\phi^j \\
&\quad +[T^\psi_E]_{AB} - [g_E]_{AB}V_E(\phi,\psi) - [g_E]_{AB}\;\kappa_n^2\;\Lambda\;F(\phi)^{-2/(n-2)}.
\end{aligned} \tag{5.2}$$

Note that this stress-energy tensor is defined by variation with respect to g_E and does not necessarily have any simple relationship to the original stress-energy tensor T_{AB}; the same comment applies to the surface stress tensor $[S_E]_{AB}$. Once this redefinition is performed, and provided we agree to phrase questions in terms of $[T_E]_{AB}$ and $[S_E]_{AB}$, the junction conditions also simplify (since there is no longer any mixing between J^i and K). We get

$$\mathcal{K}_E = -\frac{\kappa_n^{-2}\;S_E}{n-2}, \tag{5.3}$$

and

$$J^i \quad = \quad -[H_E]^{ij}\;(\mathcal{L}^E_{\rm brane})'_j. \tag{5.4}$$

As previously noted, the Codazzi equation simplifies to (4.14). Finally the Gauss equation implies that the braneworld geometry satisfies

$$
\begin{aligned}
{}^{(n-1)}[G_E]_{AB} = {} & \frac{n-3}{n-2}\,\kappa_n^{-2}\,[H_E]_{ij}\,D_A\phi^i\,D_B\phi^j \\
& -\frac{n(n-3)}{2(n-1)(n-2)}\,[q_E]_{AB}\,\kappa_n^{-2}\,\,[H_E]_{ij}\,D_C\phi^i\,D^C\phi^j \\
& -\frac{n-3}{n-1}\,(\Lambda\,F(\phi)^{-2/(n-2)}+\kappa_n^{-2}\,V_E)\,[q_E]_{AB} \\
& +\frac{n-3}{n-1}\,\kappa_n^{-2}\,\frac{1}{2}\,[H_E]_{ij}\,(n^C\partial_C\phi^i)\,(n^D\partial_D\phi^j)\,[q_E]_{AB} \\
& +K\,K_{AB}-K_A{}^C\,K_{BC}-\frac{1}{2}\,[q_E]_{AB}\,(K^2-K_{CD}K^{CD}) \\
& -E_{AB}.
\end{aligned}
\tag{5.5}
$$

(With the extrinsic curvature and the Weyl tensor being calculated using the Einstein frame metric.) As in the general case, you can introduce dimensionless coefficients $\Gamma_1\ldots\Gamma_7$, rearrange terms, and introduce an effective Newton constant and effective brane cosmological constant. These are now relatively simple functions of the dimensionality and of the background moduli fields. We collect some technical results in Appendix B, and here merely quote the results for the effective Newton constant and related parameters:

$$8\pi\,G_{\mathrm{effective}} = \frac{\kappa_n^{-4}\,\lambda}{4}\frac{n-3}{n-2}; \tag{5.6}$$

$$8\pi\,G_{\mathrm{anomalous}} = 0: \tag{5.7}$$

$$8\pi\,G_{\mathrm{quadratic}} = \frac{\kappa_n^{-4}}{4}; \tag{5.8}$$

$$
\begin{aligned}
\Lambda_{\mathrm{effective}} = {} & \frac{n-3}{n-1}\,(\Lambda\,F^{-2/(n-2)}+\kappa_n^{-2}\,V_E) \\
& +\frac{\kappa_n^{-4}}{8}\left\{\frac{n-3}{n-2}\lambda^2-\kappa_n^2\,\frac{n-3}{n-1}[H_E]^{ij}\,(\mathcal{L}_{\mathrm{brane}})'_i\,(\mathcal{L}_{\mathrm{brane}})'_j\right\}.
\end{aligned}
\tag{5.9}
$$

In particular note that $G_{\mathrm{anomalous}}=0$. Furthermore the effective cosmological constant picks up contributions from whatever moduli dependence the brane Lagrangian may posses. The coefficients of the "quadratic" pieces of induced Einstein equations are simply

$$-(\tau^2)_{ab}+\frac{\tau\,\tau_{ab}}{n-2}-\frac{\tau^2\,q_{ab}}{2(n-2)}+\frac{1}{2}(\tau^{pq}\,\tau_{pq})\,q_{ab}, \tag{5.10}$$

while the explicit moduli contribution to the effective stress tensor appearing in equation (4.39) now reduces to

$$(T^\phi)_{ab} = \frac{n-3}{n-2}\,[H_E]ij\,D_a\phi^i\,D_b\phi^j-\frac{n(n-3)}{2(n-2)(n-1)}\,q_{ab}\,[H_E]ij\,D_c\phi^i\,D^c\phi^j. \tag{5.11}$$

In comparing with the Maeda–Wands analysis [7] note there are several subtle differences in normalization and sign convention (we have tried to stick to MTW conventions [15]) and that they have specifically chosen the in-brane cosmological constant to be the only moduli-dependent piece of the brane Lagrangian, so that they have

$$\mathcal{L}_{\mathrm{brane}}([g_e],\phi,\psi)=-\lambda(\phi)+\mathcal{L}_{\mathrm{brane}}([g_E],\psi). \tag{5.12}$$

Consequently in their analysis

$$(\mathcal{L}_{\mathrm{brane}})'_i=-\lambda'(\phi)=-\frac{\mathrm{d}\lambda}{\mathrm{d}\phi}. \tag{5.13}$$

Modulo choices of convention, our results (when specialized to $n = 5$, and a single dilaton field with canonical kinetic energy) are in agreement with theirs. See equations (2.18)–(2.22) of [7]

Similarly in comparing with the Mennim–Battye analysis [7] note there are other subtle differences in normalization and sign convention, in particular their $V \to -2\Lambda$ and their $U \to -2\lambda(\phi)$ in our notation. Then consider for instance equation (22) of [8] and compare with our more general formula for the effective cosmological constant as presented above.

6 Dilaton field in the string frame

Consider $F(\phi) = \exp(-2\phi/\kappa_n)$, with $H(\phi) = -4\ \exp(-2\phi/\kappa_n)$. This corresponds to the dilaton field in the string frame [11]. Some key coefficients are collected in Appendix C. The generalized junction conditions become

$$\mathcal{K}_{ab} = \exp(2\phi/\kappa_n)\ \kappa_n^{-2} \left\{ S_{ab} + \frac{\kappa_n}{2}\ (\mathcal{L}_{\mathrm{brane}})'\ q_{ab} \right\}, \tag{6.1}$$

$$J = \exp(2\phi/\kappa_n)\ \kappa_n^{-1} \left\{ \frac{1}{2} S - \frac{n-2}{4}\ \kappa_n\ (\mathcal{L}_{\mathrm{brane}})' b. \right\}. \tag{6.2}$$

Then in particular

$$\mathcal{K}_{ab} - \mathcal{K}\ q_{ab} = \exp(2\phi/\kappa_n) \left\{ S_{ab} - S\ q_{ab} - \frac{n-2}{2} \kappa\ (\mathcal{L}_{\mathrm{brane}})'\ q_{ab} \right\}. \tag{6.3}$$

From the Codazzi equation the "conservation" of braneworld stress-energy reads

$$D^a \left\{ \exp(2\phi/\kappa_n) \left[S_{ab} - S\ q_{ab} - \frac{n-2}{2}\ \kappa_n\ (\mathcal{L}_{\mathrm{brane}})'\ q_{ab} \right] \right\} = -\mathcal{F}_b. \tag{6.4}$$

So if you define the braneworld stress-energy by variation with respect to the string metric, that particular stress-tensor is *not* conserved, both due to explicit interchange of stress energy with the bulk, and (ultimately due to the nontrivial matter-dilaton couplings in the string frame) due to possible variations of the dilaton field along the brane. Even if the dilaton field is constant along the brane (with both $D_a \phi = 0$, and $D_a[(\mathcal{L}_{\mathrm{brane}})'_i] = 0$, appropriate for a "translationally invariant" ground state), one still gets the perhaps unexpected result

$$D^a \left\{ S_{ab} - S\ q_{ab} \right\} = -\mathcal{F}_b. \tag{6.5}$$

It is only if the braneworld stress-energy is additionally traceless, and if there is no net flux onto the brane, that one recovers the naive result

$$D^a \left\{ S_{ab} \right\} = 0. \tag{6.6}$$

If you are trying to do braneworld cosmology in the string frame, you obtain results you might naively expect during the radiation dominated expansion of the universe, but would see what appears to be stress-energy nonconservation during the matter dominated era. This is not an "error" or an "inconsistency" but merely a reflection of the fact that (after taking account of reduction from the bulk to the brane and coupling to the dilation) the string frame braneworld stress-energy tensor does not quite have the properties you might naively expect.

It is now straightforward (given the formalism developed herein) to calculate the braneworld Einstein tensor (4.39). For the effective Newton constant and related quantities

$$8\pi\ G_{\mathrm{effective}} = \frac{\kappa_n^{-4}\ \exp(4\phi/\kappa_n)\ \lambda}{4} \left\{ \frac{n-3}{n-2} \right\}; \tag{6.7}$$

$$8\pi\ G_{\mathrm{anomalous}} = \frac{\kappa_n^{-4}\ \exp(4\phi/\kappa_n)}{4} \left\{ -\lambda \frac{n-3}{n-2} + \frac{n-3}{2(n-1)}\ \kappa_n\ (\mathcal{L}_{\mathrm{brane}})' \right\}; \tag{6.8}$$

$$
\begin{aligned}
8\pi\, G_{\text{quadratic}} &= \frac{\kappa_n^{-4}\ \exp(4\phi/\kappa_n)}{4}; && (6.9)\\
\Lambda_{\text{effective}} &= \frac{n-3}{n-1}\ \left[\Lambda + \kappa_n^{-2} \exp(2\phi/\kappa_n)\ V(\phi)\right] && (6.10)\\
&- (n-3)\frac{\kappa_n^{-4}\ \exp(4\phi/\kappa_n)}{8}\left\{\lambda^2 + \lambda\ \kappa_n\ (\mathcal{L}_{\text{brane}})' + \frac{n-2}{4(n-1)}\kappa_n^2\ \ [(\mathcal{L}_{\text{brane}})']^2\right\}.
\end{aligned}
$$

The "quadratic" part of the effective stress tensor is different from that occurring in the Einstein frame and is now proportional to

$$
-(\tau^2)_{ab} + \frac{\tau\ \tau_{ab}}{n-2} - \frac{\tau^2\ q_{ab}}{(n-1)(n-2)} + \frac{1}{2}(\tau^{pq}\ \tau_{pq})\ q_{ab}. \tag{6.11}
$$

Finally the explicit moduli contribution to the effective stress tensor appearing in equation (4.39) is

$$
(T^\phi)_{ab} = -2\frac{n-3}{n-1}\ q_{ab}\ D_c\phi^i\ D^c\phi^j - 2\frac{n-3}{n-2}\ \kappa_n\ \left[D_a D_b \phi^i - q_{ab}\ D_c D^c \phi^i\right]. \tag{6.12}
$$

Some points to note:
—(1) You still need positive brane tension to make effective gravity in the braneworld attract.
—(2) $G_{\text{anomalous}}$ is generally nonzero though it can be fine-tuned away if you enforce

$$
(\mathcal{L}_{\text{brane}})' = 2\ \frac{n-1}{n-2}\ \lambda. \tag{6.13}
$$

—(3) Predicting even the *sign* of the brane contribution to $\Lambda_{\text{effective}}$ is now a lot trickier.

Let us now consider a more specific example: A "bare" brane (no extra matter apart from the brane tension) with a Lagrangian $\mathcal{L}_{\text{brane}} = -\lambda(\phi)$, and thus with $S_{ab} = -\lambda(\phi)\ q_{ab}$. The Einstein equation for the brane becomes

$$
{}^{(4)}G_{ab} = -\Lambda_{\text{effective}}\ q_{ab}, \tag{6.14}
$$

with

$$
\Lambda_{\text{effective}} = \frac{n-3}{n-1}\Lambda - \frac{(n-3)\kappa_n^{-4}}{8}\ \exp(4\phi/\kappa_n)\ \left\{\lambda(\phi)^2 + \lambda(\phi)\ \kappa_n\ \lambda'(\phi) + \frac{n-2}{4(n-1)}\kappa_n^2[\lambda'(\phi)]^2\right\}. \tag{6.15}
$$

Making an ansatz for the coupling interaction function of the form $\lambda(\phi) = \lambda_0\ \exp(-\alpha\phi/\kappa_n)$, we can see that

$$
\Lambda_{\text{effective}} = \frac{n-3}{n-1}\Lambda - \frac{(n-3)\kappa_n^{-4}}{8}\ \exp(4\phi/\kappa_n)\ \lambda_0^2 \left\{1 - \alpha + \frac{n-2}{4(n-1)}\alpha^2\right\}. \tag{6.16}
$$

Then for α between $2[(n-1)\pm\sqrt{n-1}]/(n-2)$ there is a positive contribution to the effective cosmological constant coming from the brane tension. For other values of α there is a negative contribution to the effective cosmological constant. In order to find a solution in which there is a Poincare invariant brane ($\Lambda_{\text{effective}} = 0$) when $\alpha \in 2[(n-1) \pm \sqrt{n-1}]/(n-2)$ the bulk cosmological constant must be negative (this is the usual situation, corresponding to an anti-de Sitter bulk). That is the case, for example, when $\alpha = 2(n-1)/(n-2)$ (which also corresponds to fine-tuning $G_{\text{anomalous}}$ to zero). On the other hand, when for example $\alpha = 0$, one would need a positive bulk cosmological constant (a de Sitter bulk) if one wishes to accommodate a Poincare invariant brane.

[With hindsight this should not be all that surprising, and in fact the same phenomenon (with slightly different coefficients) also shows up in the Einstein frame. Specifically pick canonical kinetic energies $H_E = 1$ and keep the brane tension ansatz $\lambda(\phi) = \lambda_0\ \exp(-\alpha\phi/\kappa_n)$ used here. Then for α between $\pm\sqrt{(n-1)/(n-2)}$ you need an anti-de Sitter bulk, while for α outside this region you would need a de Sitter bulk to obtain a Poincare invariant brane.]

7 Discussion

In this paper we have developed a general technique for analyzing braneworld geometry in the presence of an arbitrary number of bulk moduli fields. We also permit the use of arbitrary conformal frames, since sometimes one conformal frame may be more useful than others. If one is interested primarily (or solely) in gravitational phenomena, the Einstein frame is often the best choice. For particle physics in Brans–Dicke theories (and their generalizations) the Jordan frame is often the best choice. String theory does not posses a unique Jordan frame, but the string frame is perhaps the best analog in stringy models.

We find that while many of the results known from situations where the dilaton is frozen out by hand, or when one *ab initio* restricts attention to an Einstein frame formulation, continue to hold in this more general context. On the other hand, several things change: (1) in general frames there is a possibility of an anomalous coupling between the trace of stress-energy and braneworld gravity; (2) there is the potential for the exchange of stress-energy and information between the bulk, the on-brane variations of the moduli fields, and the braneworld stress-energy; (3) for moduli-dependent brane tensions, the relationship to the effective cosmological constant is more complex, and in particular one can flip the sign of the brane-tension contribution to the effective Newton constant.

While we have analyzed the behaviour of the gravity sector in some detail, it should be emphasised that there is a lot of flexibility in the class of models we consider. For instance, the use of a generic sigma model for the moduli fields gives you a lot of freedom. Likewise the bulk potential $V(\phi)$ and moduli-dependent brane tension $\lambda(\phi)$ are freely specifiable in this formalism. (More generally, $\mathcal{L}_{\rm brane}(g,\phi,\psi)$ and $\mathcal{L}_{\rm bulk}(g,\phi,\psi)$ are freely specifiable—apart from the fact that they should not contain derivatives of ϕ, because if so they would generate additional terms in the "momenta" used to set up the generalized junction conditions.) We have also had essentially nothing specific to say about the "matter" fields that are assumed to be trapped on or near the brane; from the current perspective they are simply a minor contaminant to be dealt with after the large scale braneworld geometry has been deduced from the interaction between brane tension, bulk moduli fields, and bulk gravity.

Finally, in order to facilitate the analysis of a braneworld cascade, branes within branes within branes, we have kept the dimensionality of the bulk arbitrary.

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Appendix A: Some coefficients (general frame)

As we have seen in the general formula for $\Gamma_1 \dots \Gamma_7$, it is useful to define

$$\mathcal{E}(\phi) \quad = \quad 1 + \frac{n-1}{n-2}\, H^{kl}(\phi)\, \kappa_n^2\, \frac{F'_k(\phi)\, F'_l(\phi)}{F(\phi)}. \tag{A.1}$$

This coefficient shows up repeatedly—ultimately this is due to the fact that $\mathcal{E}$ is intimately related to the determinant of the sigma model metric in the Einstein frame (# denotes the total number of moduli fields)

$$\det[H_E] = \det[H]\; F^{-\#}\; \mathcal{E}. \tag{A.2}$$

As such its vanishing is key to the "exceptional" case of "Cheshire cat branes" (phantom branes) considered in [11]. It also occurs as a sub-piece of the various coefficients enumerated below:

$$\gamma_{11} \quad = \quad -\frac{1}{n-2}\, \mathcal{E}^{-1}; \tag{A.3}$$

$$\gamma^i_{12} = -\frac{n-1}{n-2}\, H^{ij}\;\; \kappa_n F'_j\, \mathcal{E}^{-1}; \tag{A.4}$$

$$\gamma^i_{21} = +\frac{1}{n-2}\, H^{ij}\;\; \kappa_n F'_j\, \mathcal{E}^{-1}; \tag{A.5}$$

$$\gamma^{ij}_{22} = -F\left(H^{ij} - \frac{n-1}{n-2}\frac{\kappa_n^2\,(H^{ik}\;F'_k)\,(H^{jl}\;F'_l)}{F\,\mathcal{E}}\right) = -(H_E^{-1})^{ij}. \tag{A.6}$$

(The explicit factors of κ_n keep all these coefficients dimensionless.) Some derived quantities are

$$\begin{aligned}
\tilde{\gamma}_{11} &= \frac{\gamma_{11}-1}{n-1} && \text{(A.7)}\\
&= -\frac{1}{n-2} + \frac{\mathcal{E}-1}{\mathcal{E}(n-1)(n-2)} && \text{(A.8)}\\
&= -\frac{1}{n-2}\left[1 - \frac{\mathcal{E}-1}{\mathcal{E}(n-1)}\right] && \text{(A.9)}\\
&= -\frac{1}{\mathcal{E}(n-2)} - \frac{\mathcal{E}-1}{\mathcal{E}(n-1)}; && \text{(A.10)}\\
\tilde{\gamma}^i_{12} &= \frac{1}{n-1}\gamma^i_{12} = -\gamma^i_{21} && \text{(A.11)}\\
&= -\frac{1}{n-2}\, H^{ij}\;\; \kappa_n F'_j\, \mathcal{E}^{-1} && \text{(A.12)}\\
&= -\frac{1}{n-2}\, [H_E^{-1}]^{ij}\;\; \kappa_n F'_j\, F^{-1}; && \text{(A.13)}\\
\alpha &= -\frac{\kappa_n\, F'_i}{F}\gamma^i_{21} = -\frac{1}{n-2}\, F^{-1}\, H^{ij}\, \kappa_n^2\, F'_i\, F'_j\, \mathcal{E}^{-1} = -\frac{\mathcal{E}-1}{\mathcal{E}(n-1)}; && \text{(A.14)}\\
\beta^i &= -\frac{\kappa_n\, F'_j}{F}\gamma^{ij}_{22} = +[H_E^{-1}]^{ij}\;\; \kappa_n F'_j\, F^{-1} = +H^{ij}\, \kappa_n\, F'_j\, \mathcal{E}^{-1}. && \text{(A.15)}
\end{aligned}$$

The $\Gamma_1 \ldots \Gamma_7$ coefficients given in the main body of the paper were derived by substitution and rearrangement of (4.25) and (4.24) into (4.23) using these γ coefficients. The calculation in a general frame is tedious, though in the Einstein frame it is relatively simple.

Appendix B: Coefficients in the Einstein frame

Einstein frame calculation:

$$\begin{aligned}
\mathcal{E} &= 1; && \text{(B.1)}\\
\gamma_{11} &= -\frac{1}{n-2}; && \text{(B.2)}\\
\gamma^i_{12} &= 0; && \text{(B.3)}\\
\gamma^i_{21} &= 0; && \text{(B.4)}\\
\gamma^{ij}_{22} &= -(H_E^{-1})^{ij}. && \text{(B.5)}\\
\tilde{\gamma}_{11} &= \frac{\gamma_{11}-1}{n-1} = -\frac{1}{n-2}; && \text{(B.6)}\\
\tilde{\gamma}^i_{12} &= 0; && \text{(B.7)}\\
\alpha &= 0; && \text{(B.8)}\\
\beta^i &= 0. && \text{(B.9)}
\end{aligned}$$

In terms of these γ coefficients the $\Gamma_1 \dots \Gamma_7$ are:

$$\Gamma_1 = -1; \tag{B.10}$$
$$\Gamma_2 = +\frac{1}{n-2}; \tag{B.11}$$
$$\Gamma_3 = -\frac{1}{2(n-2)}; \tag{B.12}$$
$$\Gamma_4 = +\frac{1}{2}; \tag{B.13}$$
$$\Gamma_5^i = 0; \tag{B.14}$$
$$\Gamma_6^i = 0; \tag{B.15}$$
$$\Gamma_7^{ij} = +\frac{n-3}{2(n-1)}\,[H_E]^{ij}. \tag{B.16}$$

When interpreted in terms of the effective Newton constant and related quantities

$$8\pi\, G_{\text{effective}} = \frac{\kappa_n^{-4}}{4}\left\{\frac{n-3}{n-2}\,\lambda\right\}; \tag{B.17}$$
$$8\pi\, G_{\text{anomalous}} = 0; \tag{B.18}$$
$$8\pi\, G_{\text{quadratic}} = \frac{\kappa_n^{-4}}{4}; \tag{B.19}$$
$$\Lambda_{\text{effective}} = \frac{n-3}{n-1}\,(\Lambda\, F^{-2/(n-2)} + \kappa_n^{-2}\, V_E) + \frac{\kappa_n^{-4}}{4}\left\{\frac{n-3}{2(n-2)}\lambda^2 - \kappa_n^2\,\frac{n-3}{2(n-1)}[H_E]^{ij}\,(\mathcal{L}_{\text{brane}})'_i\,(\mathcal{L}_{\text{brane}})'_j\right\}. \tag{B.20}$$

Appendix C: Coefficients in the string frame

String frame calculation: Consider $F(\phi) = \exp(-2\phi/\kappa_n)$, $H(\phi) = -4\ \exp(-2\phi/\kappa_n)$. This corresponds to the dilaton field in the string frame [11]. For simplicity the dilaton is assumed to be the only bulk moduli field. Then

$$\mathcal{E} = -\frac{1}{n-2}; \tag{C.1}$$
$$H_E = \frac{4}{n-2}; \tag{C.2}$$
$$\gamma_{11} = 0; \tag{C.3}$$
$$\gamma_{12} = \frac{n-1}{2}; \tag{C.4}$$
$$\gamma_{21} = \frac{1}{2}; \tag{C.5}$$
$$\gamma_{22} = -\frac{n-2}{4}. \tag{C.6}$$
$$\tilde{\gamma}_{11} = 0; \tag{C.7}$$
$$\tilde{\gamma}_{12}^i = \frac{1}{2}; \tag{C.8}$$
$$\alpha = -1; \tag{C.9}$$
$$\beta^i = -\frac{n-2}{2}. \tag{C.10}$$

In terms of these γ coefficients the $\Gamma_1 \dots \Gamma_7$ are:

$$\Gamma_1 = -1; \tag{C.11}$$
$$\Gamma_2 = +\frac{1}{n-2}; \tag{C.12}$$
$$\Gamma_3 = -\frac{1}{(n-1)(n-2)}; \tag{C.13}$$
$$\Gamma_4 = +\frac{1}{2}; \tag{C.14}$$
$$\Gamma_5^i = 0; \tag{C.15}$$
$$\Gamma_6^i = \frac{n-3}{2(n-1)}; \tag{C.16}$$
$$\Gamma_7^{ij} = +\frac{(n-3)(n-2)}{8}. \tag{C.17}$$

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