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Topologizations of Chiral Representations

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Abstract

Recently, two different families of topologies have been proposed for representation spaces of chiral algebras. We prove a theorem that compares the two types of topologies and show that in one of them chiral blocks are continuous functionals.

1 Introduction and Summary

Two-dimensional conformal field theories (CFTs) play a key role in the worldsheet formulation of string theory and in the description of universality classes of critical phenomena. In the attempt to gain a better understanding of their mathematical structure, several axiomatic approaches have been developed. When using an operator calculus, a space of states $\mathcal{V}$ has to be specified that consists of representations of the symmetry algebra. The representation spaces form the basic structure of $\mathcal{V}$, but it is not fully determined by physical requirements what topology should be given to this space, and hence how it should be completed.

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In unitary CFTs, one has, by definition, a positive definite inner product $\langle\ ,\ \rangle$; the conventional approach is then to use the norm $\|.\| = \sqrt{\langle\ ,\ \rangle}$ for completing V so that one obtains a Hilbert space $\mathcal{H}$. Problems usually arise from the fact that domains and ranges for different operators do not coincide. Special care has to be taken when considering operator products. Often this aspect is left aside and one works on the *assumption* that domain issues can be settled.

To define convergence in terms of an inner product is by no means the only possibility, nor is it clear that it provides the best starting point for dealing satisfactorily with domain questions. The theory of distributions shows that it can prove extremely useful to introduce topologies different from that of a (pre-) Hilbert space. It was especially Böhm who argued for the application of such topologies to quantum theory (see [Ma], sec. 1 for references). More recently this idea has reappeared in the context of CFT when Gaberdiel & Goddard [GG] and Huang [Hu] proposed new topologies for representation spaces of chiral symmetry algebras. A central role is played by chiral (or conformal) blocks whose properties lead to the definition of locally convex state spaces that are not Hilbert spaces. In both cases, one deals with a whole family of topologies which are parametrized by suitable subsets of the complex plane. In this article we investigate and compare the two approaches.

Questions of topology naturally occur when it comes to constructing a mathematically rigorous operator formalism. Gaberdiel, Goddard and Huang have achieved that for chiral CFT on the Riemann sphere. One hopes that these results can be further generalized, and that the topological properties of the spaces help in deriving statements that could not be proven so far. Let us mention a few possibilities:

- String theory makes it necessary to deal with CFTs on surfaces of arbitrary genus g. For $g > 1$ and interacting theories, an operator formalism has yet to be developed. In a different approach one defines conformal blocks as linear functionals on tensor products of representation spaces [FB]. The nuclear mapping theorem for nuclear spaces could provide a way to construct vertex operators from these functionals.

- The space of physical superstring states V_{phys} is obtained by taking the BRST cohomology of the combined matter-ghost system. The choice of topology can influence the content of V_{phys}, and should play a role in the construction of picture-changing operators [BZ].

- A cohomological approach to the Verlinde formula has been advocated in the literature [Te, FS]. The correct dimension of chiral blocks is obtained provided a certain sequence of coinvariants is exact. This may be easier to show if one chooses a suitable, possibly nuclear, topology on the state space.

The results of this paper contribute to a better understanding of the topologies given in [GG] and [Hu]. As they were defined in rather different axiomatic settings, we have translated them into a common framework that allows for the construction of both types of topologies. This framework is specified in the next section. Section 3 and 4 explain the definition of the topologies: they are denoted by $\mathcal{T}_{\mathrm{GG}}^{\mathcal{O}}$ and $\mathcal{T}_{\mathrm{Hu}}^{\mathcal{D}}$ and parametrized by open sets $\mathcal{O} \subset \mathbb{C}$ and open disks $\mathcal{D} \subset \mathbb{C}$ centered at 0 respectively. In section 5, we derive some simple properties of $\mathcal{T}_{\mathrm{GG}}^{\mathcal{O}}$ and $\mathcal{T}_{\mathrm{Hu}}^{\mathcal{D}}$: it is shown that $\mathcal{T}_{\mathrm{Hu}}^{\mathcal{D}}$ is nuclear and that $\mathcal{T}_{\mathrm{GG}}^{\mathcal{O}}$ is nuclear if it is Hausdorff. We also take a look at the $\mathcal{O}$- and $\mathcal{D}$-dependence:

$$\mathcal{T}_{\mathrm{GG}}^{\mathcal{O}'} \text{ is coarser than } \mathcal{T}_{\mathrm{GG}}^{\mathcal{O}}, \text{ if } \mathcal{O}' \subset \mathcal{O},$$

whereas $\mathcal{T}_{\mathrm{Hu}}^{\mathcal{D}}$ behaves in the opposite way:

$$\mathcal{T}_{\mathrm{Hu}}^{\mathcal{D}'} \text{ is finer than } \mathcal{T}_{\mathrm{Hu}}^{\mathcal{D}} \text{ for } \mathcal{D}' \subset \mathcal{D}.$$

Section 6 deals with the comparison of the topologies. We prove that

$$\mathcal{T}_{\mathrm{Hu}}^{\mathcal{D}} \text{ is finer than } \mathcal{T}_{\mathrm{GG}}^{\mathcal{O}} \text{ if } \inf_{\zeta \in \mathcal{O}} |\zeta| > r,$$

where r is the radius of the disk $\mathcal{D}$. The techniques employed allow us to show in section 7 that conformal two-point blocks of the vacuum sector are continuous in $\mathcal{T}_{\mathrm{Hu}}^{\mathcal{D}}$ if the radius of $\mathcal{D}$ is less than half the distance of the two points. This result generalizes to an arbitrary number of points. (One can give another proof of continuity which is based on theorem 2.5 in [Hu].) Section 6 and 7 can be read independently of section 5.

We assume that the reader is familiar with the basics of the theory of vertex algebras and the theory of topological vector spaces. The necessary background material can be found in [Ka, FB] and [Na, Tr].

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2 Mathematical Framework

The state space of a CFT is built from representation spaces of a chiral symmetry algebra. The work of Huang and Gaberdiel & Goddard provides us with methods to topologize such spaces. This section fixes the definition of the chiral representations and specifies the additional assumptions needed for their topologization.

2.1 Vertex Algebras and Vertex Algebra Modules

In this article, we use the concept of vertex algebras to formally define the chiral symmetry algebra [FLM, Ka, FB]. The topologies will be defined on certain vertex algebras and modules of them.

Let V be a $\mathbb{Z}_+$-graded vertex algebra consisting of finite-dimensional graded components, that is

$$V = \bigoplus_{h\in\mathbb{Z}_+} V_h$$

and

$$\dim V_h < \infty\,.$$

The vacuum vector is denoted by Ω ($\Omega \in V_0$). The map

$$\begin{aligned} \phi: \quad V &\to \operatorname{End} V\,[[z,z^{-1}]]\,, \\ v &\mapsto \phi(v,z) = \sum_{n\in\mathbb{Z}} (v)_n\, z^{-n-1}\,, \end{aligned}$$

establishes the state-field correspondence. In physical jargon, the endomorphisms $(v)_n$ are called mode operators of the field $\phi(v,z)$ associated to the state v. (The use of the letter ϕ is conventional in quantum field theory; Y is the standard symbol used in the theory of vertex algebras.) In a conformal vertex algebra, the grading corresponds to the assignment of conformal weights. The choice of an integer grading means that we only consider bosonic fields; the restriction to positive values follows from unitarity requirements (see below).

Take W to be a $\mathbb{R}_+$-graded V-module [FB] satisfying

$$W = \bigoplus_{h\in\mathbb{R}_+} W_h$$

and

$$\dim W_h < \infty\,.$$

The fields of the chiral algebra V are represented on W by

$$\begin{aligned} \phi_W: \quad V &\to \operatorname{End} W\,[[z,z^{-1}]]\,, \\ v &\mapsto \phi_W(v,z) = \sum_{n\in\mathbb{Z}} (v)_n\, z^{-n-1}\,. \end{aligned}$$

In the remainder of the text the index W is omitted: it will be clear from the context whether one deals with fields and operators of the vertex algebra or those of its module.

Let $X \subset V$ and $Y \subset W$ be subspaces which generate V and W respectively:

$$V = \operatorname{span}\left\{(x_1)_{n_1}\cdots(x_k)_{n_k}x_{k+1} \mid x_i\in X,\ n_i\in\mathbb{Z}_+,\ k\in\mathbb{N}\right\}, \tag{1}$$

$$W = \operatorname{span}\left\{(v_1)_{n_1}\cdots(v_k)_{n_k}y \mid v_i\in V,\ y\in Y,\ n_i\in\mathbb{Z}_+,\ k\in\mathbb{N}\right\}. \tag{2}$$

We assume that X contains Ω. Line (1) implies that every mode operator $(v)_n$ ($v \in V$, $n \in \mathbb{Z}_+$) is a linear combination of products $(x_1)_{n_1} \cdots (x_k)_{n_k}$. This can be shown by induction and a suitable integration of the operator product expansion (see sec. 1.6 of [FB]). Hence, as a consequence of (1) and (2), W is spanned by vectors of the form

$$(x_1)_{n_1} \cdots (x_k)_{n_k} y \qquad (k \in \mathbb{N})$$

where $x_1, \ldots x_k \in X$, $n_1, \ldots, n_k \in \mathbb{Z}_+$ and $y \in Y$.

2.2 Unitarity, Correlation Functions and Finiteness

Gaberdiel's and Goddard's axioms lead to state spaces of a chiral symmetry algebra that contains at least the Möbius algebra. An additional condition on the amplitudes implies the existence of an inner product and that Möbius transformations are unitary w.r.t. it (see sec. 3.5, [G]). In this paper, we assume that this condition is fulfilled, so effectively one deals with chiral representations that carry a (pseudo-)unitary structure.

We implement these requirements as follows: V and W are equipped with inner products $\langle\ ,\ \rangle$ (antilinear in the first variable), and the Lie algebra $\mathrm{sl}(2, \mathbb{C})$ is unitarily represented on them; the associated operators L_0 and L_{-1} can be identified with the grading and shift operators respectively[1]. It follows that the inner products are compatible with the grading of V and W. Note also that the inner products can be indefinite; for sake of simplicity, we restrict ourselves to unitary CFTs and thus assume that $\langle\ ,\ \rangle$ is positive definite. As a result, the grading of V and W has to be real and positive.

Matrix elements

$$\langle \tilde{w}, \phi(v_1, z_1) \cdots \phi(v_k, z_k) w \rangle \tag{3}$$

of field products are obtained by inserting the formal sum

$$\phi(v_1, z_1) \cdots \phi(v_k, z_k)$$

between states $w, \tilde{w} \in W$ and replacing the formal variables by complex numbers $z_1, \ldots, z_k$. In other words, we consider $k+2$ point blocks on the sphere with in- and out-state taken from the module W and k insertions that are descendants of the vacuum. It follows from the axioms of the vertex algebra module that (3) converges absolutely in the region

$$|z_1| > \ldots > |z_k| > 0\,,$$

and can be analytically extended to a meromorphic function on the domain

$$M^k = \{(z_1, \ldots, z_k) \in (\mathbb{C}^\times)^k \mid z_i \neq z_j \text{ for } i \neq j\}\,.$$

[1]cf. the definition of a Möbius-conformal vertex algebra (see, e.g. [Ka])

M^k is the moduli space of n different ordered points on $\mathbb{C}^\times$.

In the theory of vertex algebras, one frequently considers "matrix elements" of the form

$$w'(\phi(v_1, z_1) \cdots \phi(v_k, z_k)w) , \qquad (4)$$

where w' is an element of the graded dual

$$W' = \bigoplus_{h \in \mathbb{R}_+} (W_h)^* .$$

Since the graded components of W are finite-dimensional, every bra-vector $\langle \tilde{w}|$ can be represented by some dual vector $w' \in W'$, and all theorems for matrix elements (4) apply as well to (3). For later use, we note here that if $W = V$ and $w = \tilde{w} = \Omega$, the amplitude (3) is translation-invariant:

$$\langle \Omega, \phi(v_1, z_1) \cdots \phi(v_k, z_k)\Omega \rangle = \langle \Omega, \phi(v_1, z_1 + z) \cdots \phi(v_k, z_k + z)\Omega \rangle , \quad z \in \mathbb{C} .$$

Given an open set $\mathcal{D} \subset \mathbb{C}$ we define the space of "correlation functions"[2] $\tilde{F}_k^{\mathcal{D}}$ $(k \in \mathbb{N})$ to be the vector space of all functions

$$\langle \tilde{w}, \phi(v_1, .) \cdots \phi(v_k, .)w \rangle ,$$

with $v_1, \ldots v_k \in V$, $w, \tilde{w} \in W$ and arguments $(z_1, \ldots, z_k)$ in the domain

$$M_{\mathcal{D}}^k = \{(z_1, \ldots, z_k) \in \mathcal{D}^k \mid \; z_i \neq z_j \text{ for } i \neq j; \; z_i \neq 0\} .$$

$\tilde{F}_k^{\mathcal{D}}$ is endowed with the topology of compact convergence, i.e. the topology of uniform convergence on compact subsets of $M_{\mathcal{D}}^k$. We denote by $F_k^{\mathcal{D}}$ the completion of $\tilde{F}_k^{\mathcal{D}}$. The topological dual $F_k^{\mathcal{D}*}$ receives the strong topology — the topology of uniform convergence on all weakly bounded subsets of $F_k^{\mathcal{D}}$.

For the construction of Huang's topology it is necessary to impose two additional conditions on the vertex algebra V and the V-module W. Both should be *finitely generated*: the spaces X and Y are assumed to be finite-dimensional; we write $d = \dim X$ and $n = \dim Y$. (Gaberdiel and Goddard only require that X has a countable basis. Various other finiteness conditions have been studied in the literature, see e.g. [NT].)

3 Gaberdiel's and Goddard's Topology

A set of meromorphic and Möbius covariant amplitudes provides the starting point for Gaberdiel's and Goddard's definition of chiral CFT. It allows for a direct construction of vertex operators as continuous maps between topological

[2] Note that these functions are objects of the *chiral* CFT. They are the chiral (or conformal) blocks from which the physical correlators of the full CFT are constructed.

spaces. In sections 4 and 8 of [GG], it is explained how this leads to the more common description in terms of chiral algebras and their representations. We will not discuss this relation and define the topologies directly using the vertex algebra V and the module W. Below we give the construction for the module W; it applies in particular to V, since V is a finitely generated module over itself; in this case, X plays the role of Y.

Let us first sketch the idea: we seek to define seminorms on W; to this end, we fix vectors $\tilde{w} \in W$ and $v_1, \ldots, v_l \in V$, and consider, for each vector w, the correlator

$$\langle \tilde{w}, \phi(v_1, .) \cdots \phi(v_k, .) w \rangle$$

as a function of k arguments in the complex plane. After choosing a suitable domain D for these functions, a seminorm is provided by the supremum norm on compact subsets K of D. In other words, one uses the topology of compact convergence on function spaces in order to topologize the vector space W. To keep the number of seminorms countable, we restrict the choice of out-states and insertions to Y and X respectively. That is, we only consider seminorms of the type

$$\|w\| = \sup_{(\zeta_1, \ldots, \zeta_l) \in K} |\langle y, \phi(x_1, \zeta_1) \cdots \phi(x_l, \zeta_l) w \rangle| \, .$$

The proofs in the following sections require us to put this scheme into more formal language: Let $\mathcal{O}$ be an arbitrary open subset of $\mathbb{C}$ and $l \in \mathbb{N}$. (The spaces $M^l_{\mathcal{O}}$ and $F^{\mathcal{O}}_l$ were defined in sec. 2.2.) There is a linear map

$$g^{\mathcal{O}}_l : \overline{Y} \otimes X^{\otimes l} \otimes W \to F^{\mathcal{O}}_l$$

defined by

$$g^{\mathcal{O}}_l (y \otimes x_1 \otimes \cdots \otimes x_l \otimes w) := \langle y, \phi(x_1, .) \cdots \phi(x_l, .) w \rangle$$

for $y \in \overline{Y}$, $x_1, \ldots, x_l \in X$ and $w \in W$. Here, we use the complex conjugate space $\overline{Y}$ of Y and thereby avoid the antilinearity of the inner product[3]. For fixed $l \in \mathbb{N}$ and $x \in \overline{Y} \otimes X^{\otimes l}$, we obtain a linear map

$$g^{\mathcal{O}}_l (x \otimes .) : W \to F^{\mathcal{O}}_l \, .$$

The family of mappings $g^{\mathcal{O}}_l (x \otimes .)$, $l \in \mathbb{N}$, $x \in \overline{Y} \otimes X^{\otimes l}$, determines an *initial topology* $\mathcal{T}^{\mathcal{O}}_{\mathrm{GG}}$ on W, i.e. the weakest topology with respect to which all $g^{\mathcal{O}}_l (x \otimes .)$ are continuous. It is locally convex, but not necessarily Hausdorff. One can make W separated by dividing out the subspace of vectors that have zero length with regard to all seminorms. In this paper, we will not do so, as we want to compare topologies on W (and not on some quotient space).

[3] $\overline{Y}$ and Y are identical as sets and additive groups, only the scalar multiplications $\bar{\cdot}$ and $\cdot$ differ: they are related by complex conjugation, $a \,\bar{\cdot}\, y = \overline{a} \cdot y \equiv \overline{a}\, y$ for $a \in \mathbb{C}$.

Given bases

$$x_1, \ldots, x_d \tag{5}$$

for X and

$$y_1, \ldots, y_n \tag{6}$$

for Y, multi-indices

$$I = (i_0, i_1, \ldots, i_l) \in \{1, \ldots, n\} \times \{1, \ldots, d\}^l$$

can be used to label a basis

$$x_I = y_{i_0} \otimes x_{i_1} \otimes \cdots \otimes x_{i_l}$$

for $\overline{Y} \otimes X^{\otimes l}$. By linearity, $g_l^{\mathcal{O}}(x \otimes .)$ is continuous for every $x \in \overline{Y} \otimes X^{\otimes l}$ iff it is continuous for every x_I. $\mathcal{T}_{\mathrm{GG}}^{\mathcal{O}}$ is therefore the weakest topology on W for which each $g_l^{\mathcal{O}}(x_I \otimes .)$, $l \in \mathbb{N}$, $I \in \{1, \ldots, n\} \times \{1, \ldots, d\}^l$, is continuous. It is characterized by the family of seminorms

$$\|w\|_{I,K} := \|g_l^{\mathcal{O}}(x_I \otimes .)\|_K = \|\langle y_{i_0}, \phi(x_{i_1}, .) \cdots \phi(x_{i_l}, .) w\rangle\|_K \,, \tag{7}$$

where the multiindex I specifies the basis element x_I and $\|.\|_K$ is the supremum norm on compact subsets $K \subset M_{\mathcal{O}}^l$. This family of seminorms is equivalent to a countable set of seminorms, since we may restrict our choice of K to a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compacta which exhaust $M_{\mathcal{O}}^l$. Note that in the definition we are free to replace the completion $F_l^{\mathcal{O}}$ by $\tilde{F}_l^{\mathcal{O}}$ itself without affecting the topology $\mathcal{T}_{\mathrm{GG}}^{\mathcal{O}}$.

4 Huang's Topology

Huang constructs a topology for finitely generated conformal vertex algebras and for finitely generated modules associated to them [Hu]. His formalism does not rely on the existence of an inner product: the graded dual is employed for defining matrix elements. We use the inner product instead and adapt Huang's scheme accordingly. The differences are pointed out at the end of this section. For the complete proofs, we refer the reader to Huang's paper.

Again, we describe the topology for a finitely generated module W; this includes the specific case $W = V$ where Y is given by X. $\mathcal{T}_{\mathrm{GG}}^{\mathcal{O}}$ was obtained by mapping W into spaces whose topology was already known. Huang takes the reverse approach: he maps a sequence of topological spaces into a vector space containing W, and equips it with the *strict inductive limit topology* (for a definition, see e.g. [Na], chap. 12).

We take $\mathcal{D}$ to be an open disk of arbitrary radius $r > 0$ around 0. Let $\overline{\mathrm{Hom}}(W, \mathbb{C})$ be the space of antilinear functionals on W. Again, the conformal blocks are used as a key input; we specify a map

$$e_k^{\mathcal{D}} : X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{D}*} \to \overline{\mathrm{Hom}}(W, \mathbb{C})$$

by

$$e_k^{\mathcal{D}}(x_1 \otimes \cdots \otimes x_k \otimes y \otimes \mu)(\tilde{w}) := \mu(\langle \tilde{w}, \phi(x_1, .) \cdots \phi(x_k, .)y \rangle) \tag{8}$$

for $x_1, \ldots, x_k \in X$, $y \in Y$, $\mu \in F_k^{\mathcal{D}*}$ and $\tilde{w} \in W$. Here,

$$\langle \tilde{w}, \phi(x_1, .) \cdots \phi(x_k, .)y \rangle$$

is to be understood as a function on the domain $M_{\mathcal{D}}^k$.

Consider the image

$$G_k^{\mathcal{D}} := e_k^{\mathcal{D}}(X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{D}*})$$

and the union over all k

$$G^{\mathcal{D}} := \bigcup_{k \in \mathbb{N}} G_k^{\mathcal{D}} .$$

The construction of the topology proceeds in two steps: first we show that W can be embedded into $G^{\mathcal{D}}$; then a topology is given to $G^{\mathcal{D}}$ and thus also to W.

The space W can be embedded into $G^{\mathcal{D}}$ as follows: for any k-tuple $n_1, \ldots, n_k \in \mathbb{Z}$ one defines functionals $\mu_{n_1,\ldots,n_k} \in F_k^{\mathcal{D}*}$ by

$$\begin{aligned}
&\mu_{n_1,\ldots,n_k}(\langle \tilde{w}, \phi(v_1, .) \cdots \phi(v_k, .)w \rangle) \\
&\quad = \frac{1}{2\pi\mathrm{i}} \oint_{|z_1|=r_1} \cdots \frac{1}{2\pi\mathrm{i}} \oint_{|z_k|=r_k} z_1^{n_1} \cdots z_k^{n_k} \langle v', \phi(v_1, z_1) \cdots \phi(v_k, z_k)w \rangle \, \mathrm{d}z_1 \cdots \mathrm{d}z_k \\
&\quad = \langle \tilde{w}, (v_1)_{n_1} \cdots (v_k)_{n_k} w \rangle ,
\end{aligned}$$

where $r > r_1 > \cdots > r_k > 0$. Note that the inner product provides an isomorphism between W and a subspace of $\overline{\mathrm{Hom}(W, \mathbb{C})}$. As explained in section 1, W is spanned by vectors of the form

$$w = (x_1)_{n_1} \cdots (x_k)_{n_k} y \qquad (k \in \mathbb{N})$$

where $x_1, \ldots x_k \in X$, $n_1, \ldots, n_k \in \mathbb{Z}_+$ and $y \in Y$.

The value of the inner product $\langle\ , w \rangle$ coincides with the value of the functional

$$e_k^{\mathcal{D}}(x_1 \otimes \cdots \otimes x_k \otimes y \otimes \mu_{n_1,\ldots,n_k})$$

on any vector $\tilde{w} \in W$. Indeed,

$$\begin{aligned}
&e_k^{\mathcal{D}}(x_1 \otimes \cdots \otimes x_k \otimes y \otimes \mu_{n_1,\ldots,n_k})(\tilde{w}) \\
&\quad = \mu_{n_1,\ldots,n_k}(\langle \tilde{w}, \phi(x_1, .) \cdots \phi(x_k, .)y \rangle) \\
&\quad = \langle \tilde{w}, (x_1)_{n_1} \cdots (x_k)_{n_k} y \rangle \\
&\quad = \langle \tilde{w}, w \rangle .
\end{aligned}$$

Thus, w can be identified with the vector $e_k^{\mathcal{D}}(x_1 \otimes \cdots \otimes x_k \otimes y \otimes \mu_{n_1,\ldots,n_k})$ in $G_k^{\mathcal{D}} \subset G^{\mathcal{D}}$. This defines our embedding of W into $G^{\mathcal{D}}$.

Next one constructs a canonical embedding of $G_k^{\mathcal{D}}$ into $G_{k+1}^{\mathcal{D}}$: define the linear map

$$\gamma_k : F_{k+1}^{\mathcal{D}} \to F_k^{\mathcal{D}}$$

by

$$\gamma_k(\langle \tilde{w}, \phi(v_1, .) \cdots \phi(v_{k+1}, .)w \rangle) = \langle (v_1)_{-1}^* \tilde{w}, \phi(v_2, .) \cdots \phi(v_{k+1}, .)w \rangle \, .$$

$(v_1)_{-1}^*$ is the adjoint of the -1st mode of $\phi(v_1, z_1)$. For arbitrary $\tilde{w} \in W$, we have

$$\begin{aligned}
& e_k^{\mathcal{D}}(x_1 \otimes \cdots \otimes x_k \otimes y \otimes \mu)(\tilde{w}) \\
& \quad = \mu(\langle \tilde{w}, \phi(x_1, .) \cdots \phi(x_k, .)y \rangle) \\
& \quad = \mu(\gamma_k(\langle \tilde{w}, \phi(\Omega, .)\phi(x_1, .) \cdots \phi(x_k, .)y \rangle)) \\
& \quad = (\gamma_k^*(\mu))(\langle \tilde{w}, \phi(\Omega, .)\phi(x_1, .) \cdots \phi(x_k, .)y \rangle) \\
& \quad = e_{k+1}^{\mathcal{D}}(\Omega \otimes x_1 \otimes \cdots \otimes x_k \otimes y \otimes \gamma_k^*(\mu))(\tilde{w}) \, ,
\end{aligned}$$

where the adjoint

$$\gamma_k^* : F_k^{\mathcal{D}*} \to F_{k+1}^{\mathcal{D}*}$$

has been used. This shows that $G_k^{\mathcal{D}} \subset G_{k+1}^{\mathcal{D}}$, and that the union $G^{\mathcal{D}}$ of all such spaces is a vector space.

How can $G^{\mathcal{D}}$ be made topological? Both X and Y are finite-dimensional and carry a unique Banach space structure. So does the tensor product $X^{\otimes k} \otimes Y$. $F_k^{\mathcal{D}*}$ has the strong topology, and we equip $X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{D}*}$ with the projective tensor product topology. $G_k^{\mathcal{D}}$ is the image of $X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{D}*}$ under the linear and surjective map $e_k^{\mathcal{D}}$, and is given the *final (identification) topology*. It can be shown then that for any $k \in \mathbb{N}$, $G_k^{\mathcal{D}}$ is a topological subspace of $G_{k+1}^{\mathcal{D}}$. We have an increasing sequence of locally convex spaces whose union yields the vector space $G^{\mathcal{D}}$. The topology on $G^{\mathcal{D}}$ is defined as the strict inductive limit determined by this sequence.

As a subspace, W inherits a locally convex and Hausdorff topology from $G^{\mathcal{D}}$; we denote it by $\mathcal{T}_{\mathrm{Hu}}^{\mathcal{D}}$.

The proofs are analogous to those in section 1 and 3 of [Hu] except for the following replacements: $\tilde{G}$ becomes W, i.e. the value $\langle \lambda, w \rangle$ of a functional $\lambda \in \tilde{G}$ on a vector $w \in W$ is replaced by the inner product $\langle \tilde{w}, w \rangle$ between a vector $\tilde{w} \in W$ and w. Instead of the dual space $\tilde{G}^*$ we use the space $\overline{\mathrm{Hom}}(W, \mathbb{C})$ of antilinear functionals on W, equipped with the weak topology. The function spaces in [Hu] correspond to $F_k^{\mathcal{D}}$ with $\mathcal{D}$ the open unit disk in $\mathbb{C}$.

5 Properties of the Topologies

The proofs in section 5 and 6 are again formulated for a general finitely-generated V-module W.

5.1 Nuclearity

We show below that $\mathcal{T}_{\rm Hu}^{\mathcal{D}}$ is nuclear, and that $\mathcal{T}_{\rm GG}^{\mathcal{O}}$ is nuclear if it is Hausdorff. In the proof the following properties of nuclear spaces are used:

1. A linear subspace of a nuclear space is nuclear.
2. The quotient of a nuclear space modulo a closed linear subspace is nuclear.
3. A projective limit of nuclear spaces is nuclear if it is Hausdorff.
4. A countable inductive limit of nuclear spaces is nuclear.
5. The projective tensor product of two nuclear spaces is nuclear.
6. A Fréchet space is nuclear if and only if its strong dual is nuclear. (A topological vector space is called a Fréchet space if it is complete, metrizable, locally convex and Hausdorff.)

For detailed definitions and proofs see, for instance, [Tr], chap. 50. The following theorem provides an alternative characterization for locally convex metrizable spaces:

7. A locally convex space is metrizable iff its topology can be described by a countable family of seminorms.

Given some open subset D of $\mathbb{C}^n$, $n \in \mathbb{N}$, the space $H(D)$ of holomorphic functions on it is nuclear ([Tr], chap. 51). Accordingly, $H(M_{\mathcal{O}}^l)$ and $H(M_{\mathcal{D}}^k)$ are nuclear spaces, and the same holds true for the subspaces $F_l^{\mathcal{O}}$, $F_k^{\mathcal{D}}$ and $\tilde{F}_k^{\mathcal{D}}$. $\mathcal{T}_{\rm GG}^{\mathcal{O}}$ is a projective limit of the spaces $F_l^{\mathcal{O}}$, and hence nuclear if it is Hausdorff (3.).

Consider now Huang's topology: Clearly, $F_k^{\mathcal{D}}$ is an example of a Fréchet space. By 6. the strong dual $F_k^{\mathcal{D}*}$ of $F_k^{\mathcal{D}}$ is nuclear. Note that this conclusion cannot be made for $\tilde{F}_k^{\mathcal{D}*}$, since $\tilde{F}_k^{\mathcal{D}}$ may not be complete and hence not a Fréchet space. It follows from the definition that

$$G_k^{\mathcal{D}} \cong (X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{D}*})/(e_k^{\mathcal{D}})^{-1}(0)\,,$$

where $\cong$ denotes a linear and topological isomorphism. The finite-dimensional space $X^{\otimes k} \otimes Y$ is nuclear and according to 5. the tensor product with $F_k^{\mathcal{D}*}$ is nuclear as well. $e_k^{\mathcal{D}}$ is continuous (Proposition 1.5 in [Hu]) and $(e_k^{\mathcal{D}})^{-1}(0)$ closed, so 2. tells us that $G_k^{\mathcal{D}}$ has the nuclear property. By 4., the latter is preserved under the inductive limit

$$G^{\mathcal{D}} = \bigcup_{k\in\mathbb{N}} G_k^{\mathcal{D}}\,,$$

and W, as a subspace of $G^{\mathcal{D}}$, must again be nuclear.

5.2 Dependence on $\mathcal{O}$ and $\mathcal{D}$

In the case of Gaberdiel's and Goddard's topologies, it is immediate from the definition that

$$\mathcal{T}_{\mathrm{GG}}^{\mathcal{O}'} \text{ is coarser than } \mathcal{T}_{\mathrm{GG}}^{\mathcal{O}} \text{ when } \mathcal{O}' \subset \mathcal{O}.$$

Due to their definition by functionals, Huang's topologies behave in the opposite way:

$$\mathcal{T}_{\mathrm{Hu}}^{\mathcal{C}} \text{ is finer than } \mathcal{T}_{\mathrm{Hu}}^{\mathcal{D}} \text{ for } \mathcal{C} \subset \mathcal{D},$$

This can be seen as follows:
Proof. Suppose that $\mathcal{C} \subset \mathcal{D} \subset \mathbb{C}$ where $\mathcal{C}$ and $\mathcal{D}$ are open disks centered at 0. Consider the map from $F_k^{\mathcal{D}}$ to $F_k^{\mathcal{C}}$ given by restriction to $M_{\mathcal{C}}^k$: it is linear, surjective, and injective, since both pre-image and image are restrictions of a single *meromorphic* function on the domain $M_{\mathbb{C}}^k = M^k$. $F_k^{\mathcal{C}}$ and $F_k^{\mathcal{D}}$ can be identified as vector spaces, but the topology on $F_k^{\mathcal{C}}$ is weaker. Therefore, its dual space $F_k^{\mathcal{C}*}$ is a subspace of $F_k^{\mathcal{D}*}$. Both dual spaces carry the strong topology: the topology of uniform convergence on weakly bounded[4] subsets. Neighbourhood bases at 0 are given by the polar sets

$$B_{\mathcal{D}}^{\circ} = \{\mu \in F_k^{\mathcal{D}*} \mid \sup_{f \in B} |\mu(f)| \leq 1\} \quad \text{for } B \text{ bounded in } F_k^{\mathcal{D}},$$

and

$$B_{\mathcal{C}}^{\circ} = \{\mu \in F_k^{\mathcal{C}*} \mid \sup_{f \in B} |\mu(f)| \leq 1\} \quad \text{for } B \text{ bounded in } F_k^{\mathcal{C}}$$

respectively. The topology induced on $F_k^{\mathcal{C}*}$ by $F_k^{\mathcal{D}*}$ has the base

$$\tilde{B}_{\mathcal{C}}^{\circ} = \{\mu \in F_k^{\mathcal{C}*} \mid \sup_{f \in B} |\mu(f)| \leq 1\} \quad \text{for } B \text{ bounded in } F_k^{\mathcal{D}}.$$

A set B is bounded in $F_k^{\mathcal{D}}$ iff it is bounded w.r.t. each seminorm in $F_k^{\mathcal{D}}$. Hence it is also bounded in $F_k^{\mathcal{C}}$, and $\tilde{B}_{\mathcal{C}}^{\circ} = B_{\mathcal{C}}^{\circ}$. Thus we see that the topology on $F_k^{\mathcal{C}*}$ is finer than that induced by $F_k^{\mathcal{D}*}$.

Furthermore, as topological vector spaces,

$$G_k^{\mathcal{D}} \cong (X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{D}*})/(e_k^{\mathcal{D}})^{-1}(0)\,,$$

and

$$G_k^{\mathcal{C}} \cong (X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{C}*})/(e_k^{\mathcal{C}})^{-1}(0)\,,$$

Since $e_k^{\mathcal{C}}$ is simply the restriction of $e_k^{\mathcal{D}}$ to $X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{C}*}$, $G_k^{\mathcal{C}}$ is a subspace of $G_k^{\mathcal{D}}$. By definition, the topology on $X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{D}*}$ is the projective tensor product of $X^{\otimes k} \otimes Y$ and $F_k^{\mathcal{D}*}$. A neighbourhood base at 0 of the space $X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{D}*}$ is constituted by the sets

$$\mathrm{conv}\,(U \otimes N)$$

[4]Bounded = weakly bounded in locally convex Hausdorff spaces (see [Na], (9.7.6)).

where U and N are neighbourhoods in $X^{\otimes k} \otimes Y$ and $F_k^{\mathcal{D}*}$ respectively. The set $U \otimes N$ consists of all $u \otimes \mu$, $u \in U$, $\mu \in N$, and conv stands for the convex hull. We have

$$\begin{aligned}
&\operatorname{conv}(U \otimes N) \cap (X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{C}*}) \\
&\quad \supset \quad \operatorname{conv}((U \otimes N) \cap (X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{C}*})) \\
&\quad \supset \quad \operatorname{conv}(U \otimes (N \cap F_k^{\mathcal{C}*}))\,,
\end{aligned}$$

and $N \cap F_k^{\mathcal{C}*}$ is a neighbourhood of 0 in $F_k^{\mathcal{C}*}$. This implies that the topology of $X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{C}*}$ is finer than that induced on it by $X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{D}*}$. Therefore, the topology of $G_k^{\mathcal{C}}$ is finer than that induced by $G_k^{\mathcal{D}}$. The same holds true for the inductive limits $G^{\mathcal{C}}$ and $G^{\mathcal{D}}$, and we conclude that $\mathcal{T}_{\text{Hu}}^{\mathcal{C}}$ is finer than $\mathcal{T}_{\text{Hu}}^{\mathcal{D}}$. $\square$

6 Comparison of the Topologies

We would like to show that $\mathcal{T}_{\text{Hu}}^{\mathcal{D}}$ is finer than $\mathcal{T}_{\text{GG}}^{\mathcal{O}}$ for suitable choices of $\mathcal{D}$ and $\mathcal{O}$. For that purpose it suffices to prove that each seminorm of Gaberdiel's & Goddard's topology is continuous in Huang's topology. In the notation of sec. 3, this means that for each $l \in \mathbb{N}$, $I \in \{1, \ldots, n\} \times \{1, \ldots, d\}^l$ and compact subset $K \subset M_{\mathcal{O}}^l$, the seminorm

$$\|.\|_{I,K} := \|g_l^{\mathcal{O}}(x_I \otimes .)\|_K$$

is continuous in $\mathcal{T}_{\text{Hu}}^{\mathcal{D}}$. Let us therefore consider I and K to be fixed. We have to show that for any net $\{w_s\}_{s \in S}$ (S an index set) that converges to 0 in $\mathcal{T}_{\text{Hu}}^{\mathcal{D}}$, the net

$$\begin{aligned}
\|w_s\|_{I,K} &= \|\langle y_{i_0}, \phi(x_{i_1}, .) \cdots \phi(x_{i_l}, .) w_s \rangle\|_K \\
&= \sup_{\zeta \in K} |\langle y_{i_0}, \phi(x_{i_1}, \zeta_1) \cdots \phi(x_{i_l}, \zeta_l) w_s \rangle|
\end{aligned} \tag{9}$$

goes to 0 as well. To simplify notation we drop the index i and write $y, x_1, \ldots, x_l$ from now on.

The proof proceeds in three steps: We specify a neighbourhood base at 0 for $\mathcal{T}_{\text{Hu}}^{\mathcal{D}}$ and express the convergence of $\{w_s\}_{s \in S}$ in terms of it. To apply this convergence property, we need to cast the correlator

$$\langle y_{i_0}, \phi(x_{i_1}, \zeta_1) \cdots \phi(x_{i_l}, \zeta_l) w_s \rangle$$

into a different form. Eq. (19) below provides the desired reordering, and is proved by using the Laurent expansion of correlation functions. This equality is also essential for the proof in sec. 7. The third step consists in choosing a neighbourhood at 0 of $\mathcal{T}_{\text{Hu}}^{\mathcal{D}}$ such that (9) becomes smaller than a given ϵ.

6.1 Convergence in Huang's Topology

Let us recall what spaces were involved in the construction of Huang's topology: $X^{\otimes k} \otimes Y$ is of finite dimension nd^k and has a norm topology. All norms on $X^{\otimes k} \otimes Y$ are equivalent, so we can take it to be the 1-norm w.r.t. some basis (i.e. the sum of the absolute values of the coefficients in this basis). Let $U_\delta(0)$ denote the associated ball of radius $\delta > 0$ around 0. $F_k^{\mathcal{D}*}$ carries the strong topology, and a base for the neighbourhoods of 0 in $F_k^{\mathcal{D}*}$ is given by the polars

$$B^\circ = \{\mu \in F_k^{\mathcal{D}*} \mid \sup_{f \in B} |\mu(f)| \leq 1\}$$

where B is bounded in $F_k^{\mathcal{D}}$. As already mentioned in sec. 5.2, a neighbourhood base at 0 for $X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{D}*}$ is provided by the sets $\operatorname{conv}(U \otimes N)$, where U and N are neighbourhoods in $X^{\otimes k} \otimes Y$ and $F_k^{\mathcal{D}*}$ respectively. Clearly, the sets

$$\operatorname{conv}(U_\delta(0) \otimes B^\circ) \qquad (\delta > 0,\ B \text{ bounded in } F_k^{\mathcal{D}})$$

form an equivalent base. The space $G_k^{\mathcal{D}}$ is the image of $X^{\otimes k} \otimes Y \otimes F_k^{\mathcal{D}*}$ under the map $e_k^{\mathcal{D}}$ and carries the associated *final* or *identification* topology. Therefore, the sets

$$e_k^{\mathcal{D}}(\operatorname{conv}(U_\delta(0) \otimes B^\circ)) \tag{10}$$

provide us with a neighbourhood base at 0 for $G_k^{\mathcal{D}}$. The space

$$G^{\mathcal{D}} = \bigcup_{k \in \mathbb{N}} G_k^{\mathcal{D}}$$

is the strict inductive limit of the spaces $G_k^{\mathcal{D}}$, and induces the topology $\mathcal{T}_{\mathrm{Hu}}^{\mathcal{D}}$ on its (embedded) subspace W. A base at 0 for $G^{\mathcal{D}}$ is constituted by the sets of the form

$$\operatorname{conv}\left(\bigcup_{k \in \mathbb{N}} \mathcal{U}_k\right), \tag{11}$$

where each $\mathcal{U}_k$ is a neighbourhood of 0 in $G_k^{\mathcal{D}}$ (see [Na], p.287, sec. 12.1). Combining (10) and (11), we see that the sets

$$W \cap \operatorname{conv}\left(\bigcup_{k \in \mathbb{N}} e_k^{\mathcal{D}}(\operatorname{conv}(U_{\delta_k}(0) \otimes B_k^\circ))\right)$$

give a base at 0 for Huang's topology[5]. Since $e_k^{\mathcal{D}}$ is linear, the latter simplifies to

$$W \cap \operatorname{conv}\left(\bigcup_{k \in \mathbb{N}} e_k^{\mathcal{D}}(U_{\delta_k}(0) \otimes B_k^\circ)\right). \tag{12}$$

[5] It is understood that $U_{\delta_k}(0)$ and B_k° belong to the spaces $X^{\otimes k} \otimes Y$ and $F_k^{\mathcal{D}*}$ respectively.

Note that in writing so we have identified W with its image under the embedding in $G^{\mathcal{D}} \subset \overline{\mathrm{Hom}}(W,\mathbb{C})$.

Consider now the net $\{w_s\}_{s\in S}$ which converges to 0 in the topology $\mathcal{T}^{\mathcal{D}}_{\mathrm{Hu}}$ on W. Given a sequence of pairs (δ_k, B_k), $k \in \mathbb{N}$, there is an index s_0 such that for each $s \geq s_0$, w_s can be expressed as a finite sum

$$w_s = \sum_i a_i \, e^{\mathcal{D}}_{k_i}(u_i \otimes \mu_i) \tag{13}$$

with

$$u_i \in U_{\delta_{k_i}}(0)\,, \quad \mu_i \in B^{\circ}_{k_i}\,, \qquad k_i \in \mathbb{N}\,,$$

and coefficients obeying

$$\sum_i |a_i| \leq 1\,.$$

To simplify notation we write the right-hand side of (13) without index s.

Laurent Expansion

We want to give an upper estimate for expression (9) when $s \geq s_0$. Let us first consider the case when the sum (13) consists of only one term, i.e.

$$w_s = e^{\mathcal{D}}_k(u \otimes \mu)\,, \qquad u \in U_{\delta_k}(0),\ \mu \in B^{\circ}_k$$

for some $s \geq s_0$ and $k \in \mathbb{N}$. In the following the index k is fixed, so we will omit it from $U_\delta(0)$ and B.

The correlator in (9) can now be written as

$$\langle y_0, \phi(x_1,\zeta_1)\cdots\phi(x_l,\zeta_l) w_s\rangle = \langle y_0, \phi(x_1,\zeta_1)\cdots\phi(x_l,\zeta_l) e^{\mathcal{D}}_k(u\otimes\mu)\rangle\,. \tag{14}$$

In (14) we would like to apply the definition of $e^{\mathcal{D}}_k$ and make the functional μ appear explicitly (see eq. (8)). The operators $\phi(x_1,\zeta_1)\cdots\phi(x_l,\zeta_l)$ prevent us from doing so and should be removed somehow. Given a $\zeta = (\zeta_1,\ldots,\zeta_l) \in M^l_{\mathcal{O}}$ such that

$$|\zeta_1| > \ldots > |\zeta_l| > 0\,,$$

one can expand the correlation function in its natural power series

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^l} \langle y, (x_1)_{m_1}\cdots(x_l)_{m_l} e^{\mathcal{D}}_k(u\otimes\mu)\rangle\, \zeta_1^{-m_1-1}\cdots\zeta_l^{-m_l-1} \\
&\quad= \sum_{m\in\mathbb{Z}^l} \langle (x_l)^*_{m_l}\cdots(x_1)^*_{m_1} y, e^{\mathcal{D}}_k(u\otimes\mu)\rangle\, \zeta_1^{-m_1-1}\cdots\zeta_l^{-m_l-1} \\
&\quad= \sum_{m\in\mathbb{Z}^l} e^{\mathcal{D}}_k(u\otimes\mu)((x_l)^*_{m_l}\cdots(x_1)^*_{m_1} y)\, \zeta_1^{-m_1-1}\cdots\zeta_l^{-m_l-1}
\end{aligned} \tag{15}$$

Next we specify a basis

$$u^j = u^j_1 \otimes \cdots \otimes u^j_k \otimes u^j_{k+1}\,,$$

$$u_1^j, \ldots, u_k^j \in X,\ u_{k+1}^j \in Y, \quad j = 1, \ldots, nd^k\,,$$

for $X^{\otimes k} \otimes Y$, and choose the associated 1-norm to be the norm on $X^{\otimes k} \otimes Y$. Then, each $u \in U_\delta(0) \subset X^{\otimes k} \otimes Y$ is a linear combination

$$u = \sum_{j=1}^{nd^k} b_j\, u^j\,, \qquad |b_j| \leq \delta\,,$$

and after applying the definition of $e_k^{\mathcal{D}}$, the power series (15) becomes

$$\begin{aligned}
&\sum_{j=1}^{nd^k} b_j \sum_{m\in\mathbb{Z}^l} e_k^{\mathcal{D}}(u^j \otimes \mu)((x_l)^*_{m_l} \cdots (x_1)^*_{m_1} y)\, \zeta_1^{-m_1-1} \cdots \zeta_l^{-m_l-1} \\
&\quad = \sum_{j=1}^{nd^k} b_j \sum_{m\in\mathbb{Z}^l} \mu(\langle (x_l)^*_{m_l} \cdots (x_1)^*_{m_1} y, \phi(u_1^j,.) \cdots \phi(u_k^j,.) u_{k+1}^j\rangle) \\
&\quad\quad \times \zeta_1^{-m_1-1} \cdots \zeta_l^{-m_l-1}\,.
\end{aligned} \tag{16}$$

Each term in the sum over j looks like the functional μ applied to

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^l} \langle (x_l)^*_{m_l} \cdots (x_1)^*_{m_1} y, \phi(u_1^j,.) \cdots \phi(u_k^j,.) u_{k+1}^j\rangle\, \zeta_1^{-m_1-1} \cdots \zeta_l^{-m_l-1} \\
&\quad = \sum_{m\in\mathbb{Z}^l} \langle y, (x_1)_{m_1} \cdots (x_l)_{m_l} \phi(u_1^j,.) \cdots \phi(u_k^j,.) u_{k+1}^j\rangle\, \zeta_1^{-m_1-1} \cdots \zeta_l^{-m_l-1}\,.
\end{aligned} \tag{17}$$

Note that (17) is a Hartogs expansion in ζ of[6]

$$f_j = \langle y, \phi(x_1,\zeta_1) \cdots \phi(x_l,\zeta_l) \phi(u_1^j,.) \cdots \phi(u_k^j,.)\, u_{k+1}^j\rangle\,, \tag{18}$$

provided that

$$\sup_{z\in\mathcal{D}} |z| < |\zeta_l|\,.$$

The partial sums of (17) take their values in the dense subspace $\tilde{F}_k^{\mathcal{D}}$ of $F_k^{\mathcal{D}}$. It is a sequence of functions in $\tilde{F}_k^{\mathcal{D}}$, but in general not convergent to a function of $\tilde{F}_k^{\mathcal{D}}$. At this point it becomes important that in the construction of $\mathcal{T}_{\mathrm{Hu}}^{\mathcal{D}}$ we have used the completion $F_k^{\mathcal{D}}$ instead of $\tilde{F}_k^{\mathcal{D}}$. A theorem of complex analysis states that the Hartogs series (17) converges compactly to f_j. As a result, f_j is contained in the completion $F_k^{\mathcal{D}}$, and with μ being an element of $F_k^{\mathcal{D}*}$ the infinite sums in (16) can be written as

$$\begin{aligned}
&\sum_{m\in\mathbb{Z}^l} \mu(\langle y, (x_1)_{m_1} \cdots (x_l)_{m_l} \phi(u_1^j,.) \cdots \phi(u_k^j,.) u_{k+1}^j\rangle)\, \zeta_1^{-m_1-1} \cdots \zeta_l^{-m_l-1} \\
&\quad = \mu(\langle y, \phi(x_1,\zeta_1) \cdots \phi(x_l,\zeta_l) \phi(u_1^j,.) \cdots \phi(u_k^j,.)\, u_{k+1}^j\rangle)\,.
\end{aligned}$$

[6]Be reminded that dots represent variables of the function, whereas ζ_1 to ζ_l are fixed.

Recalling our starting point (eq. (14)) we get

$$\begin{aligned}&\langle y, \phi(x_1,\zeta_1)\cdots\phi(x_l,\zeta_l)\, e_k^{\mathcal{D}}(u\otimes\mu)\rangle\\ &\quad = \sum_{j=1}^{nd^k} b_j\, \mu(\langle y, \phi(x_1,\zeta_1)\cdots\phi(x_l,\zeta_l)\phi(u_1^j,.)\cdots\phi(u_k^j,.)\, u_{k+1}^j\rangle)\,.\end{aligned} \tag{19}$$

Let us recollect what assumptions were needed in order to arrive at the relation (19): The values of $\zeta_1,\ldots,\zeta_l$ are taken from an open subset $\mathcal{O}$ of $\mathbb{C}$ and (18) makes only sense as a function on $M^k_{\mathcal{D}}$ if $\mathcal{O}$ and $\mathcal{D}$ do not overlap. Furthermore, the Hartogs expansion (17) requires that

$$|\zeta_1| > \ldots > |\zeta_l| > \sup_{z\in\mathcal{D}} |z|\,. \tag{20}$$

Eq. (19) continues to hold for arbitrary orderings of $\zeta_1,\ldots,\zeta_l\in\mathcal{O}$ provided

$$\inf_{\zeta\in\mathcal{O}} |\zeta| > \sup_{z\in\mathcal{D}} |z| = r\,.$$

For $i\neq j$, $\zeta_i\neq\zeta_j$, but what about values where $|\zeta_i| = |\zeta_j|$? Let $\boldsymbol{\zeta}_0 = (\zeta_{1,0},\ldots\zeta_{l,0})$ be a point where at least two radii coincide. Clearly, there is a sequence $\boldsymbol{\zeta}_n$ in a region of the type (20) such that

$$\lim_{n\to\infty} \boldsymbol{\zeta}_n = \boldsymbol{\zeta}_0\,.$$

The corresponding sequence of functions

$$\langle y, \phi(x_1,\zeta_{1,n})\cdots\phi(x_l,\zeta_{l,n})\phi(u_1^j,.)\cdots\phi(u_k^j,.)\, u_{k+1}^j\rangle$$

converges compactly to

$$\langle y, \phi(x_1,\zeta_{1,0})\cdots\phi(x_l,\zeta_{l,0})\phi(u_1^j,.)\cdots\phi(u_k^j,.)\, u_{k+1}^j\rangle\,.$$

By continuity of μ it follows that equation (19) is valid for $\boldsymbol{\zeta}_0$ and thus for arbitrary values of $\boldsymbol{\zeta}\in M^l_{\mathcal{O}}$.

Choice of Neighbourhood

Given $\epsilon>0$, we seek neighbourhoods $U_\delta(0)$ and B° such that

$$\sup_{\boldsymbol{\zeta}\in K} |\langle y, \phi(x_1,\zeta_1)\cdots\phi(x_l,\zeta_l)\, e_k^{\mathcal{D}}(u\otimes\mu)\rangle| \leq \epsilon$$

if $u\in U_\delta(0)$ and $\mu\in B^\circ$.

For each point $\boldsymbol{\zeta} = (\zeta_1,\ldots,\zeta_l)\in K$ there is a correlation function

$$f_j = \langle y, \phi(x_1,\zeta_1)\cdots\phi(x_l,\zeta_l)\phi(u_1^j,.)\cdots\phi(u_k^j,.)\, u_{k+1}^j\rangle\,.$$

Let B_j denote the set of these functions. When regarded as a function of $l+k$ variables,

$$\tilde{f}_j = \langle y, \phi(x_1, .) \cdots \phi(x_l, .) \phi(u_1^j, .) \cdots \phi(u_k^j, .)\, u_{k+1}^j \rangle$$

is holomorphic on $M_{\mathcal{O}}^l \times M_{\mathcal{D}}^k$, and for any compact subset K' of $M_{\mathcal{D}}^k$

$$\sup_{(\boldsymbol{\zeta}, \boldsymbol{z}) \in K \times K'} |\tilde{f}_j(\boldsymbol{\zeta}, \boldsymbol{z})| < \infty \,.$$

This means that B_j is bounded in $F_k^{\mathcal{D}}$, and the same holds true for the union over j

$$B = \bigcup_{j=1}^{nd^k} B_j \,.$$

Then, if $\mu \in B^\circ$ and $u \in U_\delta(0)$, $\delta = \epsilon/(nd^k)$, eq. (19) implies that

$$\begin{aligned}
&|\langle y, \phi(x_{i_1}, \zeta_1) \cdots \phi(x_{i_l}, \zeta_l)\, e_k^{\mathcal{D}}(u \otimes \mu) \rangle| \\
&\qquad \leq \sum_{j=1}^{nd^k} |b_j|\, |\mu(\langle y, \phi(x_1, \zeta_1) \cdots \phi(x_l, \zeta_l) \phi(u_1^j, .) \cdots \phi(u_k^j, .)\, u_{k+1}^j \rangle)| \\
&\qquad \leq nd^k \delta \leq \epsilon \qquad \forall \boldsymbol{\zeta} \in K \,,
\end{aligned}$$

as required. In general, w_s is a finite sum of the type (13) for $s \geq s_0$, and one has to consider the expression

$$|\,\langle y, \phi(x_1, \zeta_1) \cdots \phi(x_l, \zeta_l) \sum_i a_i\, e_{k_i}^{\mathcal{D}}(u_i \otimes \mu_i) \rangle\,| \,.$$

Using linearity and

$$\sum_i |a_i| \leq 1$$

we can repeat the same arguments to obtain

$$\sup_{\boldsymbol{\zeta} \in K} |\langle y, \phi(x_1, \zeta_1) \cdots \phi(x_l, \zeta_l) w_s \rangle| \leq \epsilon$$

for $s \geq s_0$. Therefore the seminorms $\|.\|_{I,K}$ are continuous in the topology $\mathcal{T}_{\mathrm{Hu}}^{\mathcal{D}}$. The proof was based on the validity of (19), i.e. we need that

$$\inf_{\zeta \in \mathcal{O}} |\zeta| > \sup_{z \in \mathcal{D}} |z| = r \,,$$

and in that case $\mathcal{T}_{\mathrm{Hu}}^{\mathcal{D}}$ is finer than $\mathcal{T}_{\mathrm{GG}}^{\mathcal{O}}$. $\square$

7 Continuity of Conformal Blocks

Associated to a choice of m points $z_1, \ldots, z_n \in \mathbb{C}$, we define a conformal m-point block as the linear functional

$$\begin{aligned} C_m(z_1, \ldots, z_m): \quad V^{\otimes m} &\to \mathbb{C}\,, \\ v_1 \otimes \cdots \otimes v_m &\mapsto \langle \Omega, \phi(v_1, z_1) \cdots \phi(v_m, z_m)\Omega \rangle\,. \end{aligned} \tag{21}$$

Note that we are now dealing with matrix elements of the vertex algebra V. Physically speaking, these are conformal blocks whose insertions are in the bosonic vacuum sector V; we do not consider conformal blocks of other sectors, since we have not introduced general intertwining operators.

In an arbitrary topology on V, the functionals (21) need not be continuous. We present a proof that two-point blocks are continuous in Huang's topology $\mathcal{T}^{\mathcal{D}}_{\mathrm{Hu}}$, provided $\mathcal{D}$ is small enough. The method can be generalized to arbitrary m-point blocks in principle, though it becomes rather unwieldy for $m > 2$.

For fixed points $z, \tilde{z} \in \mathbb{C}$, the two-point block

$$\begin{aligned} C_2(z, \tilde{z}): \quad V \otimes V &\to \mathbb{C}\,, \\ v \otimes \tilde{v} &\mapsto \langle \Omega, \phi(\tilde{v}, \tilde{z}) \phi(v, z)\Omega \rangle\,. \end{aligned}$$

is continuous on $V \otimes V$ iff it is continuous as a bilinear map from $V \times V$ into the complex numbers. A net $\{(\tilde{v}_s, v_s)\}_{s \in S}$ converges to 0 in $V \times V$ iff both $\{\tilde{v}_s\}_{s \in S}$ and $\{v_s\}_{s \in S}$ converge to 0 in V. Given such nets we want to demonstrate that

$$|\langle \Omega, \phi(\tilde{v}_s, \tilde{z}) \phi(v_s, z)\Omega \rangle| \tag{22}$$

goes to zero.

The proof is similar to the one of sec. 6: We express the convergence of v_s and $\tilde{v}_s$ in terms of neighbourhoods at 0, and manipulate expression (22) such that eq. (19) can be applied. Then we choose suitable neighbourhoods to make the value of (22) smaller than ϵ.

Repeating arguments of sec. 6.1 we see that given a sequence of $\delta_k > 0$ and bounded sets B_k, $\tilde{B}_k$ in $F^{\mathcal{D}}_k$, there is an index s_0 such that for each $s \geq s_0$, v_s and $\tilde{v}_s$ are finite sums of the type

$$v_s = \sum_i a_i\, e^{\mathcal{D}}_{k_i}(u_i \otimes \mu_i)\,, \quad \tilde{v}_s = \sum_i \tilde{a}_i\, e^{\mathcal{D}}_{k_i}(\tilde{u}_i \otimes \tilde{\mu}_i)\,, \tag{23}$$

with

$$u_i, \tilde{u}_i \in U_{\delta_{k_i}}(0), \quad \mu_i \in B^\circ, \tilde{\mu}_i \in \tilde{B}^\circ, \qquad k_i \in \mathbb{N}\,,$$

and coefficients obeying

$$\sum_i |a_i| \leq 1\,, \quad \sum_i |\tilde{a}_i| \leq 1\,.$$

Suppose for the moment that for an $s \geq s_0$ each of the two sums contains only one term, that is

$$v_s = e_k^{\mathcal{D}}(u \otimes \mu)\,, \quad \tilde{v}_s = e_k^{\mathcal{D}}(\tilde{u} \otimes \tilde{\mu})\,,$$

and

$$u, \tilde{u} \in U_{\delta_k}(0)\,, \quad \mu \in B_k^\circ, \tilde{\mu} \in \tilde{B}_k^\circ$$

for some $k \in \mathbb{N}$. Below the index k is omitted from $U_\delta(0)$, B and $\tilde{B}$. Again, we write u and $\tilde{u}$ as linear combinations of orthonormal basis vectors:

$$u = \sum_{j=1}^{nd^k} b_j\, u^j\,, \quad \tilde{u} = \sum_{j'=1}^{nd^k} \tilde{b}_{j'}\, \tilde{u}^{j'},$$

where

$$|b_j| \leq \delta\,, \quad u^j = u_1^j \otimes \cdots \otimes u_k^j \otimes u_{k+1}^j\,,$$

$$u_1^j, \ldots, u_k^j \in X,\ u_{k+1}^j \in Y, \quad j = 1, \ldots, nd^k\,,$$

and

$$|\tilde{b}_{j'}| \leq \delta\,, \quad \tilde{u}^{j'} = \tilde{u}_1^{j'} \otimes \cdots \otimes \tilde{u}_k^{j'} \otimes \tilde{u}_{k+1}^{j'}\,,$$

$$\tilde{u}_1^{j'}, \ldots, \tilde{u}_k^{j'} \in X,\ \tilde{u}_{k+1}^{j'} \in Y, \quad j' = 1, \ldots, nd^k\,.$$

We are now ready to express the two-point correlator $\langle \Omega, \phi(\tilde{v}_s, \tilde{z})\phi(v_s, z)\Omega \rangle$ in terms of the defining maps of Huang's topology. The calculation employs translation invariance (t), locality (l), equation (19) and the state-operator correspondence (s). Within correlation functions $\phi(v, 0)$ stands for the zero limit in the complex variable.

$$\begin{aligned}
&\langle \Omega, \phi(\tilde{v}_s, \tilde{z})\phi(v_s, z)\Omega \rangle \\
&\quad \overset{t}{=} \quad \langle \Omega, \phi(\tilde{v}_s, \tilde{z} - z)\phi(v_s, 0)\Omega \rangle \\
&\quad = \quad \langle \Omega, \phi(\tilde{v}_s, \tilde{z} - z) e_k^{\mathcal{D}}(u \otimes \mu) \rangle \\
&\quad \overset{(19)}{=} \quad \sum_{j=1}^{nd^k} b_j\, \mu(\langle \Omega, \phi(\tilde{v}_s, \tilde{z} - z)\phi(u_1^j, .) \cdots \phi(u_k^j, .)\, u_{k+1}^j \rangle) \\
&\quad \overset{s}{=} \quad \sum_{j=1}^{nd^k} b_j\, \mu(\langle \Omega, \phi(\tilde{v}_s, \tilde{z} - z)\phi(u_1^j, .) \cdots \phi(u_k^j, .)\phi(u_{k+1}^j, 0)\Omega \rangle) \\
&\quad \overset{l,t}{=} \quad \sum_{j=1}^{nd^k} b_j\, \mu(\langle \Omega, \phi(u_1^j, . - \tilde{z} + z) \cdots \phi(u_k^j, . - \tilde{z} + z)\phi(u_{k+1}^j, -\tilde{z} + z)\phi(\tilde{v}_s, 0)\Omega \rangle) \\
&\quad \overset{(19)}{=} \quad \sum_{j=1}^{nd^k} b_j \sum_{j'=1}^{nd^k} \tilde{b}_{j'}\, \mu(\tilde{\mu}(\langle \Omega, \phi(u_1^j, . - \tilde{z} + z) \cdots \phi(u_k^j, . - \tilde{z} + z)\phi(u_{k+1}^j, -\tilde{z} + z)
\end{aligned}$$

$$
\begin{aligned}
& \qquad\qquad \times\ \phi(\tilde u_1^{j'},\tilde{.})\cdots\phi(\tilde u_k^{j'},\tilde{.})\tilde u_{k+1}^{j'}\rangle))\\
\overset{t}{=}\ & \sum_{j=1}^{nd^k} b_j \sum_{j'=1}^{nd^k}\tilde b_{j'}\,\mu(\tilde\mu(\langle\Omega,\phi(u_1^j,z+.)\cdots\phi(u_k^j,z+.)\phi(u_{k+1}^j,z)\\
& \qquad\qquad \times\ \phi(\tilde u_1^{j'},\tilde z+\tilde{.})\cdots\phi(\tilde u_k^{j'},\tilde z+\tilde{.})\phi(\tilde u_{k+1}^{j'},\tilde z)\Omega\rangle))\,. \qquad (24)
\end{aligned}
$$

The notation should be understood as follows: $\tilde\mu$ acts on

$$\langle\Omega,\phi(u_1^j,z+.)\cdots\phi(u_k^j,z+.)\phi(u_{k+1}^j,z)\phi(\tilde u_1^{j'},\tilde z+\tilde{.})\cdots\phi(\tilde u_k^{j'},\tilde z+\tilde{.})\phi(\tilde u_{k+1}^{j'},\tilde z)\Omega\rangle$$

as a function of the variables marked by $\tilde{.}$ while the remaining points are fixed parameters. The expression

$$\tilde\mu(\langle\Omega,\phi(u_1^j,z+.)\cdots\phi(u_k^j,z+.)\phi(u_{k+1}^j,z)\phi(\tilde u_1^{j'},\tilde z+\tilde{.})\cdots\phi(\tilde u_k^{j'},\tilde z+\tilde{.})\phi(\tilde u_{k+1}^{j'},\tilde z)\Omega\rangle)$$

is a function of the variables marked by a dot (without tilde) and serves, in turn, as an argument for the functional μ. Note that for equation (19) to be applicable in the third and sixth equality, it is necessary that

$$|\tilde z - z| > \sup_{\zeta\in\mathcal{D}}|\zeta| \quad \text{and} \quad \inf_{\zeta\in\mathcal{D}}|\zeta-\tilde z+z| > \sup_{\zeta\in\mathcal{D}}|\zeta|\,.$$

This is ensured if the radius r of the disk $\mathcal{D}$ is less than half the distance $|\tilde z - z|$.

Following the same approach as in the previous section we try to make (24) arbitrarily small by a suitable choice of the sets B and $\tilde B$. Take a sequence of compact sets $K_m\subset M_{\mathcal{D}}^k$ such that

$$M_{\mathcal{D}}^k = \bigcup_{m=1}^{\infty} K_m\,.$$

For each m we define B_m to be the set of functions

$$
\begin{aligned}
f_{\boldsymbol\zeta jj'} \ =\ & \langle\Omega,\phi(u_1^j,z+\zeta_1)\cdots\phi(u_k^j,z+\zeta_k)\phi(u_{k+1}^j,z)\\
& \times\ \phi(\tilde u_1^{j'},\tilde z+\tilde{.})\cdots\phi(\tilde u_k^{j'},\tilde z+\tilde{.})\phi(\tilde u_{k+1}^{j'},\tilde z)\Omega\rangle\,,
\end{aligned}
$$

with $\boldsymbol\zeta=(\zeta_1,\ldots,\zeta_l)$ running through K_m and $j,j'=1,\ldots,nd^k$. B_m is bounded in $F_k^{\mathcal{D}}$. For a sequence of bounded sets B_m one can find $\rho_m>0$ such that the union

$$\tilde B = \bigcup_{m=1}^{\infty}\rho_m B_m$$

is again bounded. This is true in any space described by a countable family of seminorms (see [Ko], p.397). If $\tilde\mu$ is taken from $\tilde B^\circ$,

$$\max_{1\le j\le nd^k}\ \max_{1\le j'\le nd^k}\ \sup_{\boldsymbol\zeta\in K_n}|\tilde\mu(f_{\boldsymbol\zeta jj'})| \le \frac{1}{\rho_n}$$

for each $n \in \mathbb{N}$. The set B of functions

$$h_{\tilde{\mu}jj'} : M^k_{\mathcal{D}} \to \mathbb{C},\ \zeta \mapsto \tilde{\mu}(f_{\zeta jj'})\,, \qquad \tilde{\mu} \in \tilde{B}^\circ,\ j, j' = 1, \ldots, nd^k$$

is therefore bounded in $F^{\mathcal{D}}_k$. For $\mu \in B^\circ$, $\tilde{\mu} \in \tilde{B}^\circ$ and $\delta = \epsilon^{1/2}/(nd^k)$, we obtain

$$\begin{aligned} |\langle \Omega, \phi(\tilde{v}_s, \tilde{z})\phi(v_s, z)\Omega\rangle| &\leq \sum_{j=1}^{nd^k} |b_j| \sum_{j'=1}^{nd^k} |\tilde{b}_{j'}|\, |\mu(\tilde{\mu}(f_{\cdot jj'}))| \\ &\leq \sum_{j=1}^{nd^k} \sum_{j'=1}^{nd^k} \delta^2 |\mu(B)| \\ &\leq (nd^k\delta)^2 = \epsilon\,. \end{aligned}$$

The inequality remains valid for v_s and $\tilde{v}_s$ of the form (23). Thus, we arrive at the result that two-point blocks $C_2(z, \tilde{z})$ are continuous in $\mathcal{T}^{\mathcal{D}}_{\mathrm{Hu}}$ if the open disk $\mathcal{D}$ has radius

$$r < \frac{|\tilde{z} - z|}{2}\,.$$

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