ϕ^4-Field theory on a Lie group

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Abstract

The ϕ^4 field model is generalized to the case when the field $\phi(x)$ is defined on a Lie group: $S[\phi] = \int_{x\in G} L[\phi(x)]d\mu(x)$, $d\mu(x)$ is the left-invariant measure on a locally compact group G. For the particular case of the affine group $G : x' = ax+b, a \in \mathbb{R}_+, x, b \in \mathbb{R}^n$ the Feynman perturbation expansion for the Green functions is shown to have no ultra-violet divergences for certain choice of $\lambda(a) \sim a^\nu$.

1 Introduction

The ultra-violet (UV) divergences appearing in quantum field theory at small distances (high momentum $\Lambda \to \infty$) are well known to be intimately related to the properties of the theory with respect to the group of scale transformations. For a wide class of theories, known as *multiplicatively renormalizable* theories, the problem can be essentially simplified by the scale transformation of fields (ϕ) and coupling constants (g)

$$\phi_R = Z_\phi^{-1}\phi, \quad g = g_0 Z_g^{-1}.$$

The renormalized Green functions

$$G_n^R(x_1,\ldots,x_n;g,\Lambda) = Z_\phi^{-n} G_n(x_1,\ldots,x_n;g_0,\Lambda)$$

become finite in the limit $\Lambda \to \infty$, with all divergences hidden in infinite renormalization constants $Z_\phi(g,\Lambda), Z_g(g,\Lambda)$.

The independence of physical results on scale transformations

$$\Lambda' = e^l\Lambda, \quad x' = e^{-l}x \qquad (1)$$

is known as renormalization group (RNG) equation.

The modern quantum field theory has become inconceivable without RNG methods. Most of the results obtained phase transitions, quantum electrodynamics, quantum chromodynamics etc. are direct consequences of RNG methods. Therefore, we may have a temptation to base the theory on some kind of covariance with respect to scale transformations (1) from very beginning, not after facing the UV divergences problem.

The best way to study any physical system is to choose a functional basis with the symmetry properties as close to the symmetry of the system as possible. For this reason we choose the spherical functions to study the hydrogen atom, and for the same reason we use plane waves to describe a particle moving in homogeneous space. Of course it is also possible to apply plane waves to SO_3 symmetrical problem, but one can hardly expect any use of it.

It is important what is implied by "the symmetry of the problem". We assume that the system is described by a set of complex-valued functions ϕ^α defined on a manifold $\mathcal{M}$, $\phi^\alpha := \phi^\alpha(x), x \in \mathcal{M}$. A system is said to have a symmetry group G if the action of the group G on the independent variables (*coordinates*) and dependent variables (*fields*)

$$x \to x' = \hat{T}x, \quad \phi^\alpha(x) \to {\phi'}^\alpha(x') = \hat{M}^\alpha_\beta \phi^\beta(x),$$

where $\hat{T}$ and $\hat{M}$ are operators, does not change the action functional (or any other functional which is believed to determine the dynamics of the system).

If the transformation group does not affect the fields themselves ($\hat{M}^\alpha_\beta \equiv \hat{1}$), but only coordinates

$$\phi^\alpha(x) \to {\phi'}^\alpha(x') = \phi^\alpha(\hat{T}^{-1}x'),$$

the field ϕ^α is called a scalar with respect to the transformation group G.

The most important group of transformations used in physics is the Poincare group ${x_\mu}' = \Lambda^\nu_\mu x_\nu + b_\mu$. The wave functions of elementary particles - electrons, photons, quarks etc. - are not Poincare scalars. They have nontrivial transformation properties under Lorenz rotations Λ^ν_μ and are classified according to their spin. However, it is possible to consider certain simplistic models with scalar fields, which do have, or may have physical implications for real systems. One of the most known models is the scalar theory of critical behavior, where magnetization $\phi(x)$ is considered as a function of the coordinate in Euclidean space. The scalar field theory was application point of the Wilson renormalization group awarded by a Nobel prize in 1982. The scalar field theory in Euclidean space is an analytical continuation ($\tau = it$) of a field theory in Minkovski space, and is receiving a lot of attention.

In this paper we restrict ourselves to the theory of complex-valued scalar field. Usually, the scalar field theory is defined on the n-dimensional Euclidean space $\mathbb{R}^n$, which is isomorphic to the group of translations

$$x' = x + b, \quad x, b \in \mathbb{R}^n \tag{2}$$

The representation of the translation group (2) on the space of square-integrable functions is given by $U(b)\phi(x) = \phi(x-b)$. The unitary representation of the translation group is defined on the space of periodic functions

$$U(b)e^{-imx} = e^{imb}e^{-imx}, \quad U(-b) = U^*(b).$$

Thus it is possible to decompose a function $\phi(x)$ with respect to the representations of translation group G

$$\phi(x) = \int_G e^{ixb}\hat{\phi}(b)db. \tag{3}$$

This is Fourier decomposition. Similarly, a function may be decomposed with respect to SO_3 rotations, Poincare group [1] and other groups.

Since the concept of the group is just more general than the concept of the Euclidean or Minkovski space, a question naturally arises: *For what groups it is physically meaningful to construct a decomposition like* (3) *and use may we have of it in field theoretic calculations*?

From physical point of view, the coordinates (x) can not be measured with arbitrary high precession, and it seems more reasonable to speak about the values of the fields ϕ^α measured at a position x with the finite resolution Δx. *The claim of the present paper is that an adequate description of this situation, which inherits the RNG ideas, can be achieved if we use an analog of* (3) *decomposition on the base of the affine group*

$$x' = ax + b, \tag{4}$$

where, as it will be seen later, a can be understood as resolution and b as a coordinate.

The goal of the present paper is to construct a ϕ^4 model where the scalar field $\phi(a, b)$ is defined on the affine group (in the sense that a, b a coordinates on the affine group (4)) and to link the new model with renormalization properties of the known ϕ^4 model in $\mathbb{R}^n$.

The paper is organized as follows. In *section 2* we review the basic formalism of ϕ^4 theory in $\mathbb{R}^n$. In *section 3* we remind the technique of wavelet transform with respect to a locally compact Lie groups. In *section 4* the ϕ^4 theory on the affine group is presented.

2 ϕ^4 Field theory

The scalar field theory with the forth power interaction $\frac{\lambda}{4!}\phi^4(x)$ defined on Euclidean space $x \in \mathbb{R}^n$ is one of the most instructive models any textbook in field theory starts with, see e.g. [2]. Often called a Ginsburg-Landau model for its ferromagnetic counterpart, the model describes a quantum field with the (Euclidean) action

$$S[\phi] = \int d^n x \frac{1}{2}(\partial_\mu \phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4(x). \tag{5}$$

in n-dimensional Euclidean space. Alternatively, the theory of quantum field in Euclidean space is equivalent to the theory of classical fluctuating field with the probability measure $\mathcal{D}P = e^{-S[\phi]}\mathcal{D}\phi$. In this case m^2 is the deviation from critical temperature $m^2 = |T - T_c|$, and λ is the fluctuation interaction strength. To some extent, the ϕ^4 model considered in this way describes a second type phase transition at zero external field in any system with one-component order parameter $\phi = \phi(x)$ and symmetry $\phi \to -\phi$.

The Green functions (correlation functions)

$$G_m(x_1, \ldots x_m) \equiv \langle \phi(x_1) \ldots \phi(x_m) \rangle = \frac{1}{W_E[J]} \frac{\delta^n}{\delta J(x_1) \ldots \delta J(x_m)}\Big|_{J=0} W_E[J] \tag{6}$$

are evaluated as functional derivatives of the generating functional

$$W_E[J] = \mathcal{N} \int \mathcal{D}\phi \exp\left[-S[\phi(x)] + \int d^n x J(x)\phi(x)\right]. \tag{7}$$

(The formal constant $\mathcal{N}$ associated with functional measure $\mathcal{D}\phi$ dropped hereafter.)

The straightforward way to calculate G_m is to factorize the interaction part $V(\phi)$ of generation functional (7) in the form

$$W_E[J] = \exp\left[-V\left(\frac{\delta}{\delta J}\right)\right] W_0[J], \tag{8}$$

where

$$W_0[J] = \int \mathcal{D}\phi \exp\left(J\phi - \frac{1}{2}\phi D\phi\right) = \exp\left(-\frac{1}{2}JD^{-1}J\right), \quad D = -\partial^2 + m^2 \tag{9}$$

is the free part of the generating functional. The perturbation expansion is then evaluated in k-space, where

$$\widehat{D}^{-1}(k) = \frac{1}{k^2 + m^2}.$$

The perturbative calculation of the correlation functions in $n > 2$ dimensions suffers ultra-violet divergence starting from one-loop approximation

$$I_1(n) = -\frac{\lambda}{2} \int \frac{d^n k}{(2\pi)^n} \frac{1}{k^2 + m^2}. \tag{10}$$

The ϕ^4 is renormalizable, i.e. the divergences can be eliminated by renormalization of the fields and parameters

$$\phi = Z_\phi \phi_R, \quad \lambda_0 = \lambda m^{2\epsilon} Z_\lambda, \quad m_0^2 = m^2 Z_m, \tag{11}$$

where all divergences are hidden in infinite renormalization constants Z_ϕ, Z_λ, Z_m.

Technically, the elimination of divergences is related to the evaluation of the loop integral (10) in the spherical domain in k-space limited from above $|k| < \Lambda$, with substitution of fixed coupling constant λ_0 to running coupling constant $\lambda = \lambda(\Lambda)$. In the case of ϕ^4 theory in the dimension $n = 4 - \epsilon$, the renormalization, as it was shown by K.Wilson [4], leads to the exact scaling of the coupling constant

$$\lambda(\Lambda) = \lambda_0 \Lambda^\epsilon \tag{12}$$

at the limit of infinite cutoff momentum $\Lambda \to \infty$. Similar type scaling takes place for other types of 4-th power interactions, say for Fermi interaction $G_0(\bar\psi\psi)^2$.

If we believe, that the power-law dependence of coupling constant on the cutoff momentum, really means the dependence of interaction strength on the scale $a = \Lambda^{-1}$, rather than a pure mathematical trick, we should find a way to incorporate this dependence at the level of the basic model, rather than at technical level. Doing so, after reviewing some necessary facts from group representation theory in next section, we will use the decomposition (often referred to as *wavelet transform*) with respect to the affine group for this purpose.

3 Partition of the unity

From the group theory point of view, the reformulation of the theory from the coordinate representation $\phi(x)$ to the momentum representation $\tilde{\phi}(k)$ by means of Fourier transform (3), is only a particular case of decomposition of a function with respect to representation of a Lie group G. $G : x' = x + b$ for the case of Fourier transform, but other groups may be used as well, depending on the physics of a particular problem.

Let us remind briefly how the decomposition with respect to the given representation of a Lie group is performed [5, 6]. Let $\mathcal{H}$ be a Hilbert space. Let $U(g)$ be a square-integrable representation of a locally-compact Lie group G acting transitively on $\mathcal{H}$, $\forall \phi \in \mathcal{H}, g \in G : U(g)\phi \in \mathcal{H}$. Let there exist such a vector $|\psi\rangle \in \mathcal{H}$, that satisfies *the admissibility condition*:

$$C_\psi = \|\psi\|^{-2} \int_{g \in G} |\langle \psi | U(g) \psi \rangle|^2 d\mu(g) < \infty, \tag{13}$$

where $d\mu(g)$ is the left-invariant Haar measure on G.

Then, any vector $|\phi\rangle$ of a Hilbert space $\mathcal{H}$ can be represented in the form:

$$|\phi\rangle = C_\psi^{-1} \int_G |U(g)\psi\rangle d\mu(g) \langle \psi U^*(g)|\phi\rangle, \tag{14}$$

The equation (14) is also known as *the partition of the unity* with respect to a Lie group G and is often written in the form

$$\hat{1} = C_\psi^{-1} \int_G |U(g)\psi\rangle d\mu(g) \langle \psi U^*(g)|.$$

The basic vector ψ used to construct decomposition (14) is often called *fiducial vector*, or *basic wavelet*.

3.1 Translation group: Fourier transform

The most familiar case of the unity partition is the decomposition with respect to the momentum eigenstates

$$|\phi\rangle = \int |k\rangle dk \langle k|\phi\rangle. \tag{15}$$

In the later case G is the group of translations and the Haar measure is simply dk. There is no need in explicit notation of any fiducial vector ψ there, because the group of translations is Abelian and the representation $U(k)$ is just a mapping between the vectors $k \in \mathbb{R}^n$ and the eigenvectors of the momentum operator. However, for the case of other Lie groups we have to put ψ explicitly. Of course, the final physical results of the theory should be independent on fiducial vector ψ.

3.2 Affine group: Wavelet transform

For the case of affine group (4), $x, b \in \mathbb{R}^n$, with the SO_n rotations dropped for simplicity, the left-invariant Haar measure is $d\mu(a, b) = a^{-n-1} da db$, the representation induced by a basic wavelet ($\psi(x) \in L^2(\mathbb{R}^n)$ for definiteness) is $U(g) = a^{-n/2}\psi((x-b)/a)$. So,

$$\begin{array}{rcl} \phi_a(b) & = & \int a^{-n/2}\bar{\psi}\left(\frac{x-b}{a}\right)\phi(x)d^nx, \\ \phi(x) & = & C_\psi^{-1}\int a^{-n/2}\psi\left(\frac{x-b}{a}\right)\phi_a(b)\frac{dadb}{a^{n+1}}, \end{array} \tag{16}$$

where

$$C_\psi = \int \frac{|\hat{\psi}(k)|^2}{|k|}d^nk. \tag{17}$$

See e.g. [7] for detailed explanation.

4 ϕ^4 model on the affine group

Let us turn to the fourth power interaction model with the (Euclidean) action functional

$$\begin{array}{rcl} S[\phi] & = & \frac{1}{2}\int \phi(x_1)D(x_1,x_2)\phi(x_2)dx_1dx_2 \\ & + & \frac{1}{4!}\int V(x_1,x_2,x_3,x_4)\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)dx_1dx_2dx_3dx_4 \end{array} \tag{18}$$

Using the notation

$$U(g)|\psi\rangle \equiv |g,\psi\rangle, \quad \langle\phi|g,\psi\rangle \equiv \phi(g), \quad \langle g_1,\psi|D|g_2,\psi\rangle \equiv D(g_1,g_2)$$

we can rewrite the generating functional (7) for the field theory with action (18) in the form

$$\begin{array}{rcl} Z_G[J] & = & \int \mathcal{D}\phi(g)\exp\Big(-\frac{1}{2}\int_G \phi(g_1)D(g_1,g_2)\phi(g_2)d\mu(g_1)d\mu(g_2) \\ & - & \frac{\lambda_0}{4!}\int_G \tilde{V}(g_1,g_2,g_3,g_4)\phi(g_1)\phi(g_2)\phi(g_3)\phi(g_4)d\mu(g_1)d\mu(g_2)d\mu(g_3)d\mu(g_4) \\ & + & \int_G J(g)\phi(g)d\mu(g)\Big), \end{array} \tag{19}$$

where $\tilde{V}(g_1,g_2,g_3,g_4)$ is the result of the application of the transform

$$\tilde{\phi}(g) := \int \overline{U(g)\psi(x)}\phi(x)dx$$

to $V(x_1,x_2,x_3,x_4)$ in all arguments x_1,x_2,x_3,x_4.

Let us turn to the particular case of the affine group (4). The restriction imposed by the admissibility condition (13) on the fiducial vector ψ (the basic wavelet) is rather loose: Only the finiteness of the integral C_ψ given by (17) is required. This practically implies only that $\int\psi(x)dx = 0$ and that $\psi(x)$ has compact support. For this reason the wavelet transform (16) can be considered as a microscopic slice of the function $\phi(x)$ taken at a position b and resolution a with "aperture" ψ. Each particular aperture $\psi(x)$ of course has its own view, but the physical observable should be independent on it. In practical applications of WT very often either of the derivatives of the Gaussian $\psi_n(x) = (-1)^n d^n/dx^n e^{-x^2/2}$ is used, but for the purpose of the present paper only the admissibility condition is important but not the shape of $\psi(x)$.

So, for the case of decomposition of scalar free field in $\mathbb{R}^n$ with respect to affine group, the inverse free field propagator matrix element is

$$\begin{aligned}\langle a_1, b_1; \psi | D | a_2, b_2; \psi \rangle &= \int d^n x (a_1 a_2)^{-\frac{n}{2}} \bar{\psi}\left(\frac{x-b_1}{a_1}\right) D\psi\left(\frac{x-b_2}{a_2}\right) \\ &= \int \frac{d^n k}{(2\pi)^n} e^{ik(b_1-b_2)} (a_1 a_2)^{\frac{n}{2}} \overline{\hat{\psi}(a_1 k)} (k^2+m^2) \hat{\psi}(a_2 k) \\ &\equiv \int \frac{d^n k}{(2\pi)^n} e^{ik(b_1-b_2)} D(a_1, a_2, k).\end{aligned}$$

Assuming the homogeneity of the free field in space coordinate, i.e. that matrix elements depend only on the differences $(b_1 - b_2)$ of the positions, but not the positions themselves, we can use (a, k) representation:

$$\begin{aligned} D(a_1, a_2, k) &= a_1^{n/2} \overline{\hat{\psi}(a_1 k)} (k^2+m^2) a_2^{n/2} \hat{\psi}(a_2 k) \\ D^{-1}(a_1, a_2, k) &= a_1^{n/2} \overline{\hat{\psi}(a_1 k)} \left(\frac{1}{k^2+m^2}\right) a_2^{n/2} \hat{\psi}(a_2 k) \\ d\mu(a, k) &= \frac{d^n k}{(2\pi)^n} \frac{da}{a^{n+1}}. \end{aligned} \tag{20}$$

So, we have the same diagram technique as usual, but with extra "wavelet" term $a^{n/2}\hat{\psi}(ak)$ term on each line and the integration over $d\mu(a,k)$ instead of dk.

Concerning the Lorenz covariance of the resulting theory (i.e. invariance under rotations, since Euclidean version of the theory is considered), the introduction of the new scale variable a, practically means that instead one scalar field $\phi(x)$, we have to deal with a collection of fields labeled by the resolution parameter $\{\phi_a(x)\}_a$. For each of them the invariance under rotations and translations holds of course. The things are so simple only if we assume quantization = functional integration in the space of numeric-valued functions only, without paying any special attention to possible commutation relations $[\phi_{a_1}(x), \phi_{a_2}(y)]$. What happens for the case of operator-valued functions is not clear enough [8].

Now, turning back to the coordinate representation (16), where a is the resolution ("window width") and recalling the power law dependence (12) obtained by Wilson expansion, we can define the ϕ^4 model on affine group, with the coupling constant dependent on scale. The simplest case of fourth power interaction of this type is

$$V_{int} = \int \frac{\lambda(a)}{4!} \phi_a^4(b) d\mu(a,b), \quad \lambda(a) \sim a^\nu. \tag{21}$$

The one-loop order contribution to the Green function G_2 in the theory with interacion (21) can be easily evaluated (for isotropic wavelet $\hat{\psi}(\mathbf{k}) = \hat{\psi}(k)$, otherwise the constants will be different) by integration over a scalar variable $z = ak$:

$$\int \frac{a^\nu a^n |\hat{\psi}(ak)|^2}{k^2+m^2} \frac{d^n k}{(2\pi)^n} \frac{da}{a^{n+1}} = C_\psi^{(\nu)} \int \frac{d^n k}{(2\pi)^n} \frac{k^{-\nu}}{k^2+m^2}, \tag{22}$$

where

$$C_\psi^{(\nu)} = \int |\hat{\psi}(\mathbf{k})|^2 k^{\nu-1} dk.$$

Therefore, there are no ultra-violet divergences for $\nu > n - 2$.

In the next orders of perturbation expansion each vertex will contribute $k^{-\nu}$ to the formal divergence degree of each diagram. This is quite natural from dimensional consideration, for a - is a "scale" (window width), and k is "inverse scale". So far, for the ϕ^N theory in n dimensions a diagram with E external lines and V vertexes has a formal divergence degree

$$D = n + E\left(1 - \frac{n}{2}\right) + V\left(\frac{n}{2}(N-2) - N - \nu\right). \quad (23)$$

5 Conclusion

Of course, the power law behavior of the coupling constant $\lambda(a) = a^{\nu}$ is not quite realistic. It seems more natural if λ vanish outside a limited domain of scales. The physical meaning of considering a field theory on the affine group seems more important. Doing so we acquire two parameters: the *coordinate* b and the *resolution* a. The former is present in any field theory, but the later is not. Such model can be considered as a continuous counterpart of lattice theory, but now we consider the grid size, or scale, as a physical parameter of interaction and there is no need to get rid of it at the end of calculations.

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