

Duality and interacting families in models with the inverse-squared interaction

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Abstract

Weak-strong coupling duality relations are shown to be present in the quantum-mechanical many-body system with the interacting potential proportional to the pair-wise inverse-squared distance in addition to the harmonic potential. Using duality relations we have solved the problem of families interacting by the inverse-squared interaction. Owing to duality, the coupling constants of the families are mutually inverse. The spectrum and eigenfunctions are determined mainly algebraically owing to $O(2,1)$ dynamical symmetry. The constructed Hamiltonian for families and appropriate solutions are of hierarchical nature.

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I. INTRODUCTION

Duality is an important generalization of symmetry for studying relations between seemingly different theories. This symmetry is as old as the Maxwell equation, where it appeared for the first time. In field theory and in theories in higher dimensions there is a web of various dualities between several theories. With more degrees of freedom, duality is enlarged. With the exception of spin systems, there exist the Calogero [1], the Sutherland [2] and the Moser [3] type of rare quantum-mechanical models with duality properties. These are weak-strong coupling dualities, which relate various physical quantities depending on the constants of the interaction λ and $1/\lambda$. These symmetries were found for the Sutherland [4, 5] model and the Calogero model without harmonic interaction [6]. Our purpose here is to demonstrate that the duality of the same type operates in the Calogero model with harmonic interaction. Then there is an efficient use of duality relations to solve the old problem of interacting families of particles, including the inverse-squared interaction acting between particles belonging to different families, as well as between particles belonging to the same family with strength that may be different for different families [1].

The system under consideration is described by the Hamiltonian

$$-\frac{\hbar^2}{2m} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \frac{\hbar^2}{2m} \sum_{i \neq j}^N \frac{\lambda(\lambda-1)}{(x_i - x_j)^2} + \frac{\omega^2}{2} \sum_{i,j=1}^N (x_i - x_j)^2, \quad (1)$$

which has been solved, both classically [7] and quantum-mechanically, and has been intensively studied. This system is also related to one-matrix models [8, 9] and to two-dimensional Yang-Mills theory [10]. In the large-N limit, the system possesses soliton states [11] which are related to edge states in the quantum Hall system [12] and the Chern-Simons theory [13]. The models are also relevant to two-dimensional gravity [14] and to the Seiberg-Witten theory [15]. There is a remarkable connection with the physics of the black hole. The behavior near the horizon of the black hole is described by (1). Further analyzes based on (1) have been used to explore horizon states [16, 17] and shed light on black hole thermodynamics [17]. In solving (1) we have restricted our attention to the case where the coupling $\lambda(\lambda-1)$ is not strongly negative, in order to avoid the 'fall to the center'. The case of the strong coupling region has been analyzed using renormalization group techniques [18] and a new bound state appears. The Calogero solution was found assuming the vanishing of the wave function when coordinates of any two particles coincide. Such a boundary

condition is represented by the Jastrow factor $\prod_{i<j}(x_i - x_j)^\lambda$. A more general boundary condition leads to new bound states [18, 19].

II. $SO(2,1)$ ALGEBRA

We shall determine the eigenvalues and eigenstates of (1) by constructing the representation of a spectrum generating algebra, similarly as it was done in Refs.[6, 20]. Owing to the translational invariance of the model we should introduce completely invariant variables [21]:

$$\xi_i \equiv x_i - X, \quad X = \frac{1}{N} \sum_{i=1}^N x_i, \quad \frac{\partial}{\partial \xi_i} \xi_j = \delta_{ij} - \frac{1}{N}. \quad (2)$$

The wave function of the problem will contain the Jastrow factor . Therefore, it is convenient to perform a similarity transformation of the Hamiltonian into $(\hbar = 1, m = 1)$

$$\begin{aligned} \prod_{i<j}^N (x_i - x_j)^{-\lambda} \left[-\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \sum_{i \neq j}^N \frac{\lambda(\lambda-1)}{(x_i - x_j)^2} \right] \prod_{i<j}^N (x_i - x_j)^\lambda = \\ = -\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \frac{\lambda}{2} \sum_{i \neq j}^N \frac{1}{(x_i - x_j)} \left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j} \right). \end{aligned} \quad (3)$$

Eliminating the center-of-mass degrees of freedom we obtain the generator of time translation

$$T_+ = \frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial \xi_i^2} + \frac{\lambda}{2} \sum_{i \neq j}^N \frac{1}{(\xi_i - \xi_j)} \left(\frac{\partial}{\partial \xi_i} - \frac{\partial}{\partial \xi_j} \right) \quad (4)$$

and the generators of scale and special conformal transformations, respectively, are

$$T_0 = -\frac{1}{2} \left(\sum_{i=1}^N \xi_i \frac{\partial}{\partial \xi_i} + E_0 - \frac{1}{2} \right), \quad T_- = \frac{1}{2} \sum_{i=1}^N \xi_i^2. \quad (5)$$

Using Eqs.(4,5) we can verify that

$$[T_+, T_-] = -2T_0, \quad [T_0, T_\pm] = \pm T_\pm. \quad (6)$$

This is the $so(2,1) \sim su(1,1)$ algebra. In the definition of the operator T_0 the constant E_0 is $E_0 = \frac{\lambda}{2}N(N-1) + \frac{N}{2}$ and $-\frac{1}{2}$ appears after removing the center-of-mass degrees of freedom. The important solution found by Calogero are zero-energy solutions P_m :

$$T_+ P_m(\xi_1, \dots, \xi_N) = 0, \quad T_0 P_m = \mu_m P_m, \quad (7)$$

where $\mu_m = -\frac{1}{2} \left(m + E_0 - \frac{1}{2} \right)$. Calogero has proved that the zero-energy solutions $P_m(\xi_1, \dots, \xi_N)$ are scale and translationally invariant homogeneous polynomials of degree m , written in the center-of-mass variables. Now we shall express the Hamiltonian (1) in terms of the generators (4) and (5). Performing the similarity transformation (3) on the Hamiltonian (1) and eliminating CM degrees of freedom we obtain

$$\prod_{i < j}^N (\xi_i - \xi_j)^{-\lambda} \frac{1}{\omega} \left[-\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial \xi_i^2} + \frac{1}{2} \sum_{i \neq j}^N \frac{\lambda(\lambda-1)}{(\xi_i - \xi_j)^2} + \frac{\omega^2}{2} \sum_{i=1}^N \xi_i^2 \right] \prod_{i < j}^N (\xi_i - \xi_j)^\lambda = -\frac{1}{\omega} T_+ + \omega T_- \equiv 2L_0. \quad (8)$$

The diagonalization of the Hamiltonian (1) can be achieved by diagonalizing L_0 . In addition to L_0 we introduce raising and lowering operators [22]:

$$L_\pm = \frac{1}{2} \left(\frac{1}{\omega} T_+ + \omega T_- \right) \pm T_0, \quad (9)$$

which satisfy commutation relations of the $so(2, 1)$ algebra:

$$[L_0, L_\pm] = \pm L_\pm, \quad [L_+, L_-] = -2L_0. \quad (10)$$

The L operators are 'rotated' T operators:

$$L_0 = -ST_0S^{-1}, \quad L_\pm = (2\omega)^{\pm 1} ST_\mp S^{-1}, \quad (11)$$

where

$$S = e^{-\omega T_-} e^{-\frac{1}{2\omega} T_+}. \quad (12)$$

From these equations we derive that a new set of vacua are 'rotated' L -vacua:

$$|0, \mu_m\rangle = SP_m = e^{-\omega T_-} P_m, \quad (13)$$

such that

$$L_- |0, \mu_m\rangle = 0 \quad (14)$$

and

$$L_0|0, \mu_m\rangle = -\mu_m|0, \mu_m\rangle. \quad (15)$$

The value of the Casimir operator

$$J^2 = -L_+L_- + L_0(L_0 - 1) \quad (16)$$

on those vacuua is

$$J^2|0, \mu_m\rangle = \mu_m(\mu_m + 1)|0, \mu_m\rangle. \quad (17)$$

We shall diagonalize L_0 in terms of T_- variables, assuming that eigenstates are functions $l(T_-)$ acting on the vacuum. From the eigenvalue equation

$$L_0l(T_-)|0, \mu_m\rangle = El(T_-)|0, \mu_m\rangle, \quad (18)$$

we obtain the operator equation

$$T_-l'' + (-2\mu_m - 2\omega T_-)l' + (2\mu_m\omega + \omega E)l = 0, \quad (19)$$

by use of Eq.(9) and the formula from Ref.[6], valid for the function of T_- :

$$[T_+, f(T_-)] = T_-f''(T_-) - 2f'(T_-)T_0. \quad (20)$$

Solutions of Eq.(18) are the well-known Laguerre polynomials:

$$l \sim L_n^{-2\mu_m-1}(2\omega T_-) \quad (21)$$

with the eigenvalues $n - \mu_m$:

$$L_0L_n^{-2\mu_m-1}(2\omega T_-)|0, \mu_m\rangle = (n - \mu_m)L_n^{-2\mu_m-1}(2\omega T_-)|0, \mu_m\rangle. \quad (22)$$

In terms of the raising operators L_+ , the diagonalization of L_0 is achieved by

$$L_0L_+^n|0, \mu_m\rangle = (n - \mu_m)L_+^n|0, \mu_m\rangle. \quad (23)$$

This result is identical to Eq.(18) because acting on a vacuum, the raising operators develop a Laguerre polynomial in $2\omega T_-$ owing to

$$L_+^n |0, \mu_m\rangle = S(2\omega T_-)^n P_m = e^{-\omega T_-} e^{-\frac{1}{2\omega} T_+} (2\omega T_-)^n P_m = L_n^{-2\mu_m-1} (2\omega T_-) |0, \mu_m\rangle. \quad (24)$$

III. DUALITY

weak-strong coupling duality relations for the Sutherland model were first established for the Hamiltonians in Refs. [4, 5] and used to relate the dynamical density correlation function for the coupling constants λ and $\frac{\kappa^2}{\lambda}$. In Ref. [6] duality relations were used to solve the problem of interacting families. From previous investigations in Ref.[11] we know that duality maps particles into holes, so the wave function should contain the prefactor of the form

$$\prod (x - z)^\kappa = \prod_{i,\alpha=1}^{N,M} (x_i - z_\alpha)^\kappa, \quad \begin{matrix} i = 1, \dots, N \\ \alpha = 1, \dots, M \end{matrix}, \quad (25)$$

where z_α denotes M zeros of the wave function describing the positions of M holes. Let us recall the relevant duality relations found in Ref.[6]:

$$\begin{aligned} T_0(x, \lambda) \prod (x - z)^\kappa &= \left\{ -T_0(z, \frac{\kappa^2}{\lambda}) - \frac{1}{2} \left[\kappa M N + \epsilon_0(N, \lambda) + \epsilon_0(M, \frac{\kappa^2}{\lambda}) \right] \right\} \prod (x - z)^\kappa, \\ T_+(x, \lambda) \prod (x - z)^\kappa &= \left[-\frac{\lambda}{\kappa} T_+(z, \frac{\kappa^2}{\lambda}) + \frac{1 + \frac{\lambda}{\kappa}}{2} \sum_{i,\alpha=1}^{N,M} \frac{\kappa(\kappa - 1)}{(x_i - z_\alpha)^2} \right] \prod (x - z)^\kappa, \end{aligned} \quad (26)$$

where $T_{0,\pm}(z, \frac{\kappa^2}{\lambda})$ denotes an operator with the same functional dependence on z_α as that of the operator $T_{0,\pm}(x, \lambda)$ on x_i and with the coupling constant λ replaced by $\frac{\kappa^2}{\lambda}$. Let us remind that solving of the problems of the Calogero type requires just Eq.(7) to be satisfied. From duality relations we can construct generators for both families from $T(x)$'s and $T(z)$'s:

$$\begin{aligned} \mathcal{T}_+ &= T_+(x, \lambda) + \frac{\lambda}{\kappa} T_+(z, \frac{\kappa^2}{\lambda}) - \frac{(\lambda + \kappa)(\kappa - 1)}{2} \sum_{i,\alpha=1}^{N,M} \frac{\kappa(\kappa - 1)}{(x_i - z_\alpha)^2}, \\ \mathcal{T}_0 &= T_0(x, \lambda) + T_0(z, \frac{\kappa^2}{\lambda}), \\ \mathcal{T}_- &= T_-(x, \lambda) + \frac{\kappa}{\lambda} T_-(z, \frac{\kappa^2}{\lambda}). \end{aligned} \quad (27)$$

These generators satisfy the $so(2, 1)$ algebra in spite of the extension by the interaction term:

$$[\mathcal{T}_+, \mathcal{T}_-] = -2\mathcal{T}_0, \quad [\mathcal{T}_0, \mathcal{T}_\pm] = \pm\mathcal{T}_\pm. \quad (28)$$

The duality relations (26) in terms of the generators $\mathcal{T}_{0,\pm}$ turn out to be a sufficient condition for solving models of the Calogero type with two families. The action of \mathcal{T} 's on $\prod(x-z)^\kappa$ is given by

$$\mathcal{T}_+ \prod(x-z)^\kappa = 0, \quad (29)$$

$$\mathcal{T}_0 \prod(x-z)^\kappa = -\frac{(N+M)(\kappa+1)-2}{4} \prod(x-z)^\kappa. \quad (30)$$

The states on which duality relations are displayed are prefactors of the ground-state wave function. To diagonalize the problem of two families with harmonic interaction, we 'rotate' the generators (28) according to Eq.(11) to obtain \mathcal{L} operators:

$$\mathcal{L}_0 = -\mathcal{S}\mathcal{T}_0\mathcal{S}^{-1}, \quad \mathcal{L}_\pm = (2\omega)^{\pm 1}\mathcal{S}\mathcal{T}_\mp\mathcal{S}^{-1}, \quad (31)$$

where

$$\mathcal{S} = e^{-\omega\mathcal{T}_-} e^{-\frac{1}{2\omega}\mathcal{T}_+}. \quad (32)$$

The ground state is given by

$$\mathcal{L}_-\mathcal{S} \prod(x-z) = 0 \quad (33)$$

and the discrete states of \mathcal{L}_0 are given by

$$\mathcal{L}_0\mathcal{L}_+^n|0, \Pi\rangle = (n-\mu)\mathcal{L}_+^n|0, \Pi\rangle, \quad (34)$$

or in terms of the Laguerre polynomials

$$L_n^{-2\mu_\kappa-1}(2\omega\mathcal{T}_-). \quad (35)$$

We interpret the $\mathcal{L}_0(x, z) = -\mathcal{S}\mathcal{T}_0(x, z)\mathcal{S}^{-1}$ as a Hamiltonian (up to similarity transformation) for two interacting families. After performing similarity transformation we obtain

$$\begin{aligned}
H &= 2 \prod_{\alpha < \beta}^M (z_\alpha - z_\beta)^{\frac{\kappa^2}{\lambda}} \prod_{i < j}^N (x_i - x_j)^\lambda \mathcal{L}_0 \prod_{i < j}^N (x_i - x_j)^{-\lambda} \prod_{\alpha < \beta}^M (z_\alpha - z_\beta)^{-\frac{\kappa^2}{\lambda}} = \\
&= \frac{1}{\omega} \left\{ \left[-\frac{1}{2} \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + \frac{1}{2} \sum_{i \neq j}^N \frac{\lambda(\lambda-1)}{(x_i - x_j)^2} + \frac{\omega^2}{2} \sum_{i=1}^N x_i^2 \right] + \right. \\
&+ \frac{\lambda}{\kappa} \left[-\frac{1}{2} \sum_{\alpha=1}^M \frac{\partial^2}{\partial z_\alpha^2} + \frac{\kappa^2}{2\lambda} \left(\frac{\kappa^2}{\lambda} - 1 \right) \sum_{\alpha \neq \beta}^M \frac{1}{(z_\alpha - z_\beta)^2} + \frac{\omega^2 \kappa^2}{2\lambda^2} \sum_{\alpha=1}^M z_\alpha^2 \right] + \\
&\left. + \frac{1}{2} \left(1 + \frac{\lambda}{\kappa} \right) \sum_{i,\alpha}^{N,M} \frac{\kappa(\kappa-1)}{(x_i - z_\alpha)^2} \right\}. \tag{36}
\end{aligned}$$

The wave function of this two-family system is

$$\Psi(x, z, n) \sim \prod_{\alpha < \beta}^M (z_\alpha - z_\beta)^{\frac{\kappa^2}{\lambda}} \prod_{i < j}^N (x_i - x_j)^\lambda \prod_{i,\alpha=1}^{N,M} (x_i - z_\alpha)^\kappa L_n^{-2\mu_\kappa-1}(2\omega\mathcal{T}_-). \tag{37}$$

The Hamiltonian (36) describes two families in interaction. The first family has particles with masses all equal to 1 and the coupling parameter λ . In the second family, particles have masses $\frac{\kappa}{\lambda}$ and the coupling parameter is $\frac{\kappa^2}{\lambda}$. Both physical parameters of the second family are of nonperturbative origin. Now it is straightforward to construct new families. Each new family will appear when the new prefactor in the zero-energy solution is introduced and new extended duality relations for \mathcal{T}_+ and \mathcal{L}_0 are established. A new \mathcal{T}_+ generator will be enlarged with an additional singular interaction. These interactions have the same scaling dimensions as the kinetic term, so the commutation relations of the type $[\mathcal{T}_0, \mathcal{T}_+] = \mathcal{T}_+$ will remain the same even if \mathcal{T}_0 is also enlarged. We construct \mathcal{T}_- by adding the corresponding \mathcal{T}_- 's in order to keep $so(2,1)$ algebra commutation relations unchanged. A new master Hamiltonian is obtained, after performing similarity transformation with an appropriate product of Jastrow factors from \mathcal{L}_0 which is 'S-rotated' \mathcal{T}_0 .

IV. CONCLUSION

Using the $so(2,1)$ algebra and its generators we have first given an algebraic/group theoretical rederivation of known results on the Calogero model with harmonic interaction. It closely follows the exposition of de Alfaro et al. [22]. This algebraic treatment was also used in the analyzis of the magnetic monopole and the vortex [23]. We have then demonstrated

that there exists the duality relations formulated in terms of the generators. This has enabled us to construct the master Hamiltonian for the problem of two interacting families and to construct a unique vacuum. This algebraic approach can be generalized to other variants of the Calogero-Sutherland-Moser type of models.

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