

# Brane induced supersymmetry breakdown and restoration

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*Dedicated to Stefan Pokorski on his 60th birthday*

## Abstract

We investigate the phenomenon of brane induced supersymmetry breakdown on orbifolds in the presence of a Scherk–Schwarz mechanism. General consistency conditions are derived for arbitrary dimensions and the results are illustrated in the specific example of a 5–dimensional theory compactified on  $S^1/\mathbb{Z}_2$ . This includes a discussion of the Kaluza–Klein spectrum and the possibility of a brane induced supersymmetry restoration.

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# 1 Introduction

The search for a satisfactory breakdown of supersymmetry is one of the most important challenges in higher dimensional quantum field theories and string theories. Mechanisms at our disposal so far are the Scherk–Schwarz mechanism [1, 2], orbifold twists [3, 4, 5] (or more generally spaces with non trivial holonomy groups [6]) as well as brane induced supersymmetry breaking [7, 8]. As in all these cases extra dimensions are involved these mechanisms show some similarities, but there are also decisive differences. One important difference concerns the question of the possible appearance of chiral fermions as a consequence of supersymmetry breaking. As the compactification of higher dimensional supersymmetric theories usually leads to  $\mathcal{N}$ -extended supersymmetry in  $d=4$ , we need a mechanism that breaks supersymmetry to  $\mathcal{N}=1$  or  $\mathcal{N}=0$  while allowing for a chiral fermion spectrum via the mechanism of orbifold twists.

In specific models we are very often confronted with a situation that a certain combination of the above mentioned mechanisms is at work, a complicated situation that needs a careful analysis to identify the most general properties of such a scheme. Also, it has been suggested that brane induced supersymmetry breakdown is related to the Scherk–Schwarz mechanism [9, 10], a conjecture based on the fact that supersymmetry broken at a given brane could be restored by a similar mechanism on a different brane [7, 11, 12]. Such a mechanism has been analysed in detail in [12] in the framework of the heterotic M-theory of Hořava and Witten [13]. As this is a rather complicated set-up and as some approximations are involved, the explicit calculations were quite difficult and not that easy to present in a simple way. In the present paper we would like to discuss the set-up of the combined action of the three mechanisms in the most general way and illustrate the schemes in the framework of simple 5-dimensional examples. Our general formulae are valid also for  $d > 5$ , but explicit solutions are much harder to obtain. Meanwhile a similar effect has been investigated in [14, 15] that has some overlap with the present work. The situation discussed there is, however, less general than the one considered in the present work (and we also differ in some of the explicit formulae).

As these mechanisms are in some sense similar, we start in section 2 with a careful definition of Scherk–Schwarz mechanism and orbifolding. Section 3 then gives the general conditions for Scherk–Schwarz mechanism on orbifolds and explains the additional restrictions as compared to the one on manifolds. In section 4 we discuss the most general spectrum of the fermion masses on orbifolds with a Scherk–Schwarz mechanism. Here it is crucial to display the dependence of these masses on the higher dimensional coordinates to classify the possibilities for brane induced supersymmetry breakdown. In section 5 we give solutions for specific simple examples of interest, and point out some subtleties in the discussion of brane located mass terms. Section 6 summarizes our main results.

## 2 Compactification, Scherk–Schwarz mechanism and orbifolding

To explain the mutual relations among compactification, orbifolding and the Scherk–Schwarz mechanism it is useful to describe all these three constructions using the same mathematical language. In this section we compare definitions of these three phenomena. We use a one–dimensional example to illustrate all important points.

### 2.1 Compactification

Let us consider a theory defined in  $D$  dimensions. Its action is given by an integral of an appropriate  $D$ –dimensional Lagrangian depending on  $D$ –dimensional fields

$$S_D = \int_{\mathbf{F}} d^D z \mathcal{L}_D(\Phi(z)) . \quad (1)$$

Such a theory is effectively  $d$ –dimensional at low energies if the coordinates  $z$  of the  $D$ –dimensional space  $\mathbf{F}$  can be split into two sets

$$z^M = \{x^\mu, y^m\} , \quad (2)$$

( $M = 1, \dots, D$ ;  $\mu = 1, \dots, d$ ;  $m = d+1, \dots, D$ ) in such a way that coordinates  $y^m$  describe some  $(D-d)$ –dimensional compact space  $\mathbf{C}$ . In the simplest case this means that the full space–time is a product of two factors

$$\mathbf{F} = \mathbf{M} \times \mathbf{C} \quad (3)$$

where  $\mathbf{M}$  is non–compact and  $d$ –dimensional (it should be just the 4–dimensional Minkowski space in realistic models). Integrating over the compact coordinates  $y$  one can obtain an effective  $d$ –dimensional theory valid for energies much smaller than the inverse of the length scale characteristic for the size of  $\mathbf{C}$ .

The simplest way to compare the Scherk–Schwarz mechanism and the ordinary compactification is to consider a case when the compact space  $\mathbf{C}$  can be obtained from a non–compact covering space  $\mathbf{N}$  using some group  $\mathbf{G}$ . Let  $\mathbf{G}$  be a discrete group acting freely on  $\mathbf{N}$ . The action of this group is represented by some operators  $\mathcal{T}_g$  mapping  $\mathbf{N}$  into itself. For all  $g_1, g_2, g_3 \in \mathbf{G}$  they satisfy the condition

$$(g_1 g_2 = g_3) \Rightarrow (\mathcal{T}_{g_1} \mathcal{T}_{g_2} = \mathcal{T}_{g_3}) . \quad (4)$$

The action of  $\mathbf{G}$  is free which means that  $\mathcal{T}_g$  has fixed points in  $\mathbf{N}$  only when  $g$  is the identity element of  $\mathbf{G}$ . We identify points which differ by the action of  $\mathcal{T}_g$  for any  $g \in \mathbf{G}$

$$\mathcal{T}_g(y) \sim y . \quad (5)$$

In other words: we identify two points if they belong to the same orbit in  $\mathbf{N}$ . In this way we obtain a compact space

$$\mathbf{N} \rightarrow \mathbf{C} = \mathbf{N}/G. \quad (6)$$

But identification of points in the space is not enough. We have also to demand that “physics” at two identified points is the same. More precisely: we have to allow only such configurations in the non compact space  $\mathbf{N}$  for which the contribution to the action from a given point is the same as from any other point identified with it (same for each point of a given orbit in the covering space). This should be true at the quantum level in the full theory but in order to simplify the notation we write it as the classical level condition for the Lagrangian at two identified points:

$$\mathcal{L}(\Phi(x, \mathcal{T}_g(y))) = \mathcal{L}(\Phi(x, y)). \quad (7)$$

Only then the action for the non compact space  $\mathbf{N}$  is equivalent to that for the compact space  $\mathbf{C}$  (they differ only by an unimportant normalization constant).

In the ordinary compactification the above requirement is fulfilled by demanding that all the fields have the analogous periodicity property under  $T_g$ :

$$\Phi(x, \mathcal{T}_g(y)) = \Phi(x, y). \quad (8)$$

This condition is of course sufficient to satisfy eq. (7) but in general it is not necessary. It is enough to demand that fields at  $\mathcal{T}_g(y)$  are related to fields at  $y$  by some transformations:

$$\Phi(x, \mathcal{T}_g(y)) = T_g \Phi(x, y) \quad (9)$$

where operations  $T_g$  are elements of the *global*<sup>4</sup> symmetry group of the theory (which again we write at a classical level only):

$$\mathcal{L}(T_g \Phi) = \mathcal{L}(\Phi). \quad (10)$$

## 2.2 Scherk–Schwarz mechanism

Now it is easy to define the Scherk–Schwarz mechanism: it is such compactification for which at least some of twist transformations  $T_g$  are different from identity. The ordinary compactification is the very special case when  $T_g = \mathbb{1}$  for all  $g \in G$ .

Of course the transformations  $T_g$  in the field space can not be arbitrary. They, similarly to the transformations  $\mathcal{T}_g$  in the physical space (4), must respect the group structure of  $G$ :

$$(g_1 g_2 = g_3) \Rightarrow (T_{g_1} T_{g_2} = T_{g_3}). \quad (11)$$

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<sup>4</sup>If  $G$  is an element of a local symmetry group than we have a Hosotani mechanism [16, 17] which is equivalent to gauge symmetry breaking by nontrivial Wilson lines.

In other words, the transformations  $T_g$  must form an appropriate representation of  $\mathbf{G}$ . This is obvious because for every  $\{x, y\}$  and  $g_3 = g_1 g_2$  we get

$$T_{g_1} T_{g_2} \Phi(x, y) = T_{g_1} \Phi(x, \mathcal{T}_{g_2}(y)) = \Phi(x, \mathcal{T}_{g_1} \mathcal{T}_{g_2}(y)) = \Phi(x, \mathcal{T}_{g_3}(y)) = T_{g_3} \Phi(x, y). \quad (12)$$

No additional twists are allowed because  $\mathcal{T}_{g_1} \mathcal{T}_{g_2}(y)$  and  $\mathcal{T}_{g_3}(y)$  denote the same point in the covering space  $\mathbf{N}$  (and not two different points which are identified).

The above definitions are quite simple but nevertheless there is some confusion about the Scherk–Schwarz mechanism in the literature. It seems that the reason is the following: In both kinds of compactification the fields  $\Phi$  are functions on the non-compact space  $\mathbf{N}$ . In the ordinary compactification they are also functions on the compact space  $\mathbf{C}$  because of the condition (9). On the other hand, in the presence of some nontrivial Scherk–Schwarz twists  $T_g$ , at least some of the fields can not be described by (single-valued) functions on  $\mathbf{C}$ . Instead, they can be described by sections of some nontrivial fiber bundle with the compact space  $\mathbf{C}$  as a base space. Of course the structure of that fiber bundle is not arbitrary, it is determined by the twist operators  $T_g$ .

Using the notion of fiber bundles it is possible to define theories on the compact space  $\mathbf{C}$  even without referring to the non-compact space  $\mathbf{N}$  (sometimes  $\mathbf{C}$  can not be obtained as  $\mathbf{N}/\mathbf{G}$ ). We use only  $\mathbf{C}$  and define fields as sections of fiber bundles over  $\mathbf{C}$ . Ordinary compactification corresponds to a case when this fiber bundle is trivial, i.e. just a product of the field space and the base space. Using this formalism one can also check when a nontrivial Scherk–Schwarz mechanism is at all possible. To apply this mechanism we need a nontrivial fiber bundle. Such bundles exist only when the base space ( $\mathbf{C}$  in our case) is a non contractible one.

Let us illustrate our discussion with the simplest possible example, that of the one dimensional circle  $\mathbf{S}^1$ . It can be obtained from the one dimensional real space,  $\mathbf{N} = \mathbb{R}$ , by using the group of addition of integer numbers,  $\mathbf{G} = \mathbb{Z}$ . The  $n$ -th element of  $\mathbb{Z}$  is represented on  $\mathbb{R}$  by the translation by  $2\pi nR$ :

$$\mathcal{T}_n(y) = y + 2\pi nR. \quad (13)$$

Identifying points which differ by the action of any of these translations we obtain a fundamental domain of length  $2\pi R$  which can be described by  $y \in [y_0, y_0 + 2\pi R]$  or  $y \in [y_0, y_0 + 2\pi R)$  for arbitrary  $y_0$ . The interval must be open at one end because  $y = y_0$  and  $y = y_0 + 2\pi R$  describe the same point in the compact space and should not be counted twice.

The group  $\mathbb{Z}$  has infinitely many elements but all of them can be obtained from just one, represented by translation by  $2\pi R$ . Thus we need only one independent twist transformation  $T$ :

$$\Phi(x, y + 2\pi R) = T\Phi(x, y). \quad (14)$$

Other transformations are powers of this one:  $T_n = T^n$ . Of course  $T$  must be a global symmetry of the Lagrangian.

Let us simplify this even further and consider only one real field  $\phi$ . For ordinary compactification  $T = \mathbb{1}$ , and our theory is described by real functions on  $S^1$ :  $\phi(y)$  (we drop dependence on  $x$ ). Non trivial Scherk–Schwarz mechanism is obtained e.g. for  $T = -1$  (of course the Lagrangian must be invariant under  $\phi \rightarrow -\phi$ ) and the theory can be described in terms of sections of a Möbius strip. How can we go back to a description in term of functions? We need a fundamental domain in the covering space  $\mathbb{R}$ . As we discussed after eq. (13) it can be chosen to be  $[y_0, y_0 + 2\pi R[$ . One can use any  $y_0$  but  $y_0 = -\pi R$  is a good choice if one wants to use even and odd functions. So sections of the Möbius strip are represented by single-valued functions on  $[-\pi R, \pi R]$  (we may add the endpoint  $y = \pi R$  and define that the value of a function at  $y = \pi R$  is equal to an appropriate limit) with additional condition

$$\phi(\pi R) = T\phi(-\pi R) = -\phi(-\pi R). \quad (15)$$

In practical calculations it is usually more convenient to work with these functions than with sections of the Möbius strip. But of course one has to remember that they are single-valued functions on the interval  $I = [-\pi R, \pi R]$ , *but in general are NOT single-valued functions on the circle  $S^1$ .*

It is important to remember also that the position of “the point of discontinuity” ( $y_0 = \pm\pi R$  in the above example), has no real meaning – one can not say in a meaningful way at which point the Möbius strip is twisted. The Scherk–Schwarz mechanism is related to global properties of the fields and does not distinguish any particular point(s) in the compact space.

## 2.3 Orbifolding

Let us now discuss the orbifolding. It is a very important construction applied in some higher dimensional theories. It can be used to obtain chiral fermions starting from a model with only non chiral ones. It has to be contrasted with the compactification which does not change the chiral structure of the theory to which it is applied. Nevertheless, using the language introduced in this section, it is possible to define the orbifolding in a very similar way to that used to analyse the Scherk–Schwarz compactification. We start with a space described by a manifold  $\mathbf{P}$  and some discrete group  $H$  which is represented by operations  $Z_h$  transforming  $\mathbf{P}$  into itself. We identify points in  $\mathbf{P}$  which differ by the action of  $Z_h$  for any  $h \in H$  and demand that the fields at such two points differ by some transformation  $Z_h$ :

$$Z_h(y) \sim y, \quad (16)$$

$$[6pt]\Phi(x, Z_h(y)) = Z_h\Phi(x, y), \quad (17)$$

and all these transformations  $Z_h$  must be global symmetries of the theory. The only difference with the compactification is that the group  $H$  does not act freely in  $\mathbf{P}$ . Some

of the transformations  $\mathbb{Z}_h$  have fixed point in  $\mathbf{P}$  and the resulting space is in general not a manifold but an orbifold

$$\mathbf{P} \rightarrow \mathbf{O} = \mathbf{P}/H. \quad (18)$$

Contrary to the Scherk–Schwarz compactification, there are special points in the space obtained by orbifolding. The resulting space (orbifold) is no longer a smooth manifold.

The simplest and very popular example is that of the circle  $S^1$  divided by the two element group  $\mathbb{Z}_2$ . The action of the only nontrivial element of  $\mathbb{Z}_2$  is represented by the reflection

$$\mathcal{Z}(y) = -y. \quad (19)$$

This operation squares to identity so the same must be true for the corresponding operation  $\mathbb{Z}$  in the field space. Thus it is always possible to choose a basis in which all fields have well defined parities

$$\Phi(x, \mathcal{Z}(y)) = \Phi(x, -y) = Z\Phi(x, y) \quad (20)$$

where  $\mathbb{Z}$  is a diagonal matrix with eigenvalues  $\pm 1$ .

This one-dimensional example is somewhat special. The orbifold  $S^1/\mathbb{Z}_2$  is equivalent to a manifold with a boundary (the fixed points have codimension 1 and can be treated as boundaries). In general, orbifolds are not equivalent to manifolds with boundaries.

### 3 Consistency conditions for Scherk–Schwarz mechanism on orbifolds

Let us now discuss the situation when we perform orbifolding and Scherk–Schwarz compactification together in one theory. Many models of this type, especially 5-dimensional ones, have been recently proposed in the literature. The orbifolding is necessary to obtain chiral fermions and also breaks some supersymmetry while the Scherk–Schwarz mechanism can be used to break the remaining supersymmetry. We will see that it is quite simple to analyse both of these mechanisms simultaneously using the formalism of the previous section. In the first subsection we present the consistency conditions which must be fulfilled for a general Scherk–Schwarz compactification on orbifolds. In the second subsection we discuss in more detail the important case of the  $S^1/\mathbb{Z}_2$  orbifold.

#### 3.1 General case

As we discussed in the previous section, the Scherk–Schwarz compactification and the orbifolding, despite important differences between them, can be described using the

same formalism. Also the situation when both constructions appear simultaneously is quite straightforward to analyse. We start with a non compact space  $\mathbb{N}$  and a discrete group  $\mathbb{F}$  acting on it. This group  $\mathbb{F}$  must be only a little bit more complicated than in the previous cases. It contains a non trivial subgroup which acts freely on  $\mathbb{N}$  (and is used to make the resulting space compact like in the Scherk–Schwarz mechanism) but has also non trivial elements which have fixed points when acting on  $\mathbb{N}$  (like in orbifolding). Let us denote those different types of elements of  $\mathbb{F}$  by  $\mathbf{g}$  and  $\mathbf{h}$ , respectively. They are represented by transformations  $\mathcal{T}_{\mathbf{g}}$  and  $\mathcal{Z}_{\mathbf{h}}$ . They in fact form one representation of  $\mathbb{F}$  and we use different letters only to distinguish those transformations which have fixed points.

As usually we identify points which differ by the action of any (combination) of those transformations. We allow also for, in general non trivial, twists in the field space:

$$\mathcal{T}_{\mathbf{g}}(y) \sim y, \quad (21)$$

$$[6pt]\mathcal{Z}_{\mathbf{h}}(y) \sim y, \quad (22)$$

$$[6pt]\Phi(x, \mathcal{T}_{\mathbf{g}}(y)) = T_{\mathbf{g}}\Phi(x, y), \quad (23)$$

$$[6pt]\Phi(x, \mathcal{Z}_{\mathbf{h}}(y)) = Z_{\mathbf{h}}\Phi(x, y). \quad (24)$$

The twist operators  $\mathcal{T}_{\mathbf{g}}$  and  $\mathcal{Z}_{\mathbf{h}}$  must of course form a representation of the group  $\mathbb{F}$  and must be global symmetries of the theory. In full analogy to eq. (11) they have to satisfy the appropriate consistency conditions also for the “mixed” products, e.g.

$$(g_1 h_2 = h_3) \Rightarrow (T_{g_1} Z_{h_2} = Z_{h_3}). \quad (25)$$

Now we can easily compare the Scherk–Schwarz mechanism without and with orbifolding. In both cases we start with the same non compact space  $\mathbb{N}$ . In the first case we use a group  $\mathbb{G}$  which free action on  $\mathbb{N}$  is represented by operators  $\mathcal{T}_{\mathbf{g}}$ . The action of that group in the field space is represented by  $T_{\mathbf{g}}$ . Then we enlarge the group in such a way that some of its elements have fixed points when acting on  $\mathbb{N}$ . For definiteness we may chose it to be a direct product:  $\mathbb{F} = \mathbb{G} \times \mathbb{H}$ . The second subgroup  $\mathbb{H}$  is represented by some non freely acting operators  $\mathcal{Z}_{\mathbf{h}}$ . What is the influence of the orbifolding group  $\mathbb{H}$  on the Scherk–Schwarz twists  $\mathcal{T}_{\mathbf{g}}$ ? Are they more restricted or can they be more general? The answer is obvious: after orbifolding the Scherk–Schwarz twists are more restricted as compared to the same theory without orbifolding. The reason is that there are additional consistency conditions of type shown in eq. (25).

It occurs that those additional consistency conditions can be quite restrictive. To show this we investigate now the interplay between the Scherk–Schwarz mechanism and orbifolding in the important case of the one–dimensional circle.



### 3.2 Scherk–Schwarz mechanism on $S^1/\mathbb{Z}_2$ orbifold

We obtain the orbifold  $S^1/\mathbb{Z}_2$  by dividing the real axis  $\mathbb{R}$  by the group  $\mathbb{Z} \times \mathbb{Z}_2$ . It is enough to consider one element of  $\mathbb{Z}$  and one element of  $\mathbb{Z}_2$ . The equations (21-24) take the following form:

$$\mathcal{T}(y) = y + 2\pi R, \quad (26)$$

$$[6pt]\mathcal{Z}(y) = -y, \quad (27)$$

$$[6pt]\Phi(x, y + 2\pi R) = T\Phi(x, y), \quad (28)$$

$$[6pt]\Phi(x, -y) = Z\Phi(x, y). \quad (29)$$

Now we want to find the additional consistency conditions of the form presented in eq. (25). There is one such condition and it follows from the fact that translation  $\mathcal{T}$  and reflection  $\mathcal{Z}$  for arbitrary  $y$  fulfil the condition:

$$\mathcal{T}\mathcal{Z}\mathcal{T}(y) = \mathcal{Z}(y), \quad (30)$$

from which it follows that

$$\begin{aligned} TZT\Phi(x, y) &= TZ\Phi(x, \mathcal{T}(y)) = T\Phi(x, \mathcal{Z}\mathcal{T}(y)) \\ [6pt] &= \Phi(x, \mathcal{T}\mathcal{Z}\mathcal{T}(y)) = \Phi(x, \mathcal{Z}(y)) = Z\Phi(x, y). \end{aligned} \quad (31)$$

So the operators in the field space must satisfy the relation

$$TZT = Z. \quad (32)$$

We will show below that the above condition puts quite strong restrictions on the possible form of the twist  $T$ .

We know that the operator  $Z$  must square up to identity so its eigenvalues must be equal to  $1$  or  $-1$ . Let us start with a basis in which the first  $n$  eigenvalues of  $Z$  are  $+1$  and the last  $m$  eigenvalues are  $-1$ . In such a basis  $Z$  and  $T$  matrices have the form

$$Z = \begin{pmatrix} \mathbb{1}_n & \\ & -\mathbb{1}_m \end{pmatrix}, \quad T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \quad (33)$$

Multiplying eq. (32) with  $T^\dagger$  and using the fact that  $T$  is unitary we get the following condition

$$ZT = T^\dagger Z \quad (34)$$

Substituting  $Z$  and  $T$  in the form (33) to this equation we find that the diagonal blocks,  $A$  and  $D$ , are hermitian while the off-diagonal ones fulfil the condition  $C = -B^\dagger$ . Thus we can change the basis in two ( $n \times n$  and  $m \times m$ ) subspaces in such a way that  $T$  has the form

$$T = \begin{pmatrix} A & B \\ -B^\dagger & D \end{pmatrix} \quad (35)$$

with diagonal and real  $\mathbf{A}$  and  $\mathbf{D}$ . Now we again use eq. (32); multiplying it with  $\mathbf{Z}$  we find

$$(TZ)^2 = (TZZ)Z = Z^2 = \mathbb{1}, \quad (36)$$

which in terms of the matrices  $\mathbf{A}$ ,  $\mathbf{B}$  and  $\mathbf{D}$  reads

$$A^2 + BB^\dagger = \mathbb{1}_n, \quad (37)$$

$$[6pt] D^2 + B^\dagger B = \mathbb{1}_m, \quad (38)$$

$$[6pt] AB - BD = 0. \quad (39)$$

The last equation can be rewritten using the components of the matrices as

$$B_{ij} (A_{ii} - D_{jj}) = 0 \quad (40)$$

for all  $i = 1, \dots, n$  and  $j = 1, \dots, m$ . This means that the elements of the off-diagonal matrix  $\mathbf{B}$  can be non zero only in subspaces in which the diagonal matrices ( $\mathbf{A}$  and  $\mathbf{D}$ ) have equal eigenvalues. So now we can change the basis in such a way that the matrices  $\mathbf{Z}$  and  $\mathbf{T}$  have the following form:

$$Z = \begin{pmatrix} \mathbb{1}_{n_1} & & & & \\ & -\mathbb{1}_{m_1} & & & \\ & & \mathbb{1}_{n_2} & & \\ & & & -\mathbb{1}_{m_2} & \\ & & & & \dots \end{pmatrix}, \quad (41)$$

$$[6pt] T = \begin{pmatrix} a_1 \mathbb{1}_{n_1} & B_1 & & & \\ -B_1^\dagger & a_1 \mathbb{1}_{m_1} & & & \\ & & a_2 \mathbb{1}_{n_2} & B_2 & \\ & & -B_2^\dagger & a_2 \mathbb{1}_{m_2} & \\ & & & & \dots \end{pmatrix}, \quad (42)$$

where all  $n_i$  are different. Let us concentrate on the first  $(n_1 + m_1) \times (n_1 + m_1)$  block of  $\mathbf{T}$ . We can perform two arbitrary changes of basis, one in the  $n_1$  dimensional subspace and second in the  $m_1$  dimensional one, and the structures of  $\mathbf{Z}$  and  $\mathbf{T}$  matrices remain the same. We can use this freedom to put the  $B_1$  matrix in the form (for  $n_1 < m_1$ ; the other case can be analysed in an analogous way)

$$B_1 = \begin{pmatrix} b_1 & & & & \\ & b_2 & & & \\ & & \ddots & & \\ & & & b_{n_1} & \\ & & & & 0 & 0 & \dots \end{pmatrix}, \quad (43)$$

with real  $b_i$ . Now we can use the conditions (37) and (38). It is easy to see that there are two possibilities: either  $a_1 = \pm 1$ ,  $B_1 = 0$  or  $n_1 = m_1$ ,  $B_1 = b_1 \mathbb{1}_{n_1}$  with the constant  $b_1$  satisfying

$$a_1^2 + b_1^2 = 1. \quad (44)$$

We can perform now the last change of the basis: we permute appropriately the coordinates in subspaces with  $n_i > 1$ . Now the matrices  $Z$  and  $T$  have their final form:

$$Z = \begin{pmatrix} \sigma_3 & & & \\ & \sigma_3 & & \\ & & \ddots & \\ & & & \pm \mathbb{1}_{|n-m|} \end{pmatrix}, \quad (45)$$

$$[6pt]T = \begin{pmatrix} R(2\pi\alpha_1) & & & \\ & R(2\pi\alpha_2) & & \\ & & \ddots & \\ & & & I_{|n-m|} \end{pmatrix}, \quad (46)$$

where  $I_{|n-m|}$  is a diagonal matrix of dimension  $|n-m|$  with diagonal entries equal  $\pm 1$  while  $R(2\pi\alpha_i)$  is a matrix describing rotation by an angle  $2\pi\alpha_i$ :

$$R(2\pi\alpha_i) = \begin{pmatrix} \cos(2\pi\alpha_i) & -\sin(2\pi\alpha_i) \\ \sin(2\pi\alpha_i) & \cos(2\pi\alpha_i) \end{pmatrix}. \quad (47)$$

Observe that now  $T$  is block-diagonal with only 2- and 1-dimensional subspaces. If any of the dimensions  $n_i$  in the form (42) is bigger than 1 then the corresponding  $2n_i \times 2n_i$  subspace decomposes to  $n_i$  block-diagonal entries with the same rotation angle  $2\pi\alpha_i$ .

We see that the possible Scherk–Schwarz mechanism in the case of the one dimensional orbifold  $S^1/\mathbb{Z}_2$  is quite restricted. The only allowed twists are: the rotations in two-dimensional subspaces consisting of one field which is even under the  $\mathbb{Z}_2$  parity and one field which is odd; and the change of sign of some fields which are not rotated. It should be stressed that (contrary to some claims in the literature [14]) orbifolding of the circle does not open any possibilities for generalizing the Scherk–Schwarz mechanism. The situation is just opposite: additional, quite strong constraints must be fulfilled. One can no longer use any arbitrary global symmetry of the theory for the twists, only twists of the form (46) can be consistently used. In particular one can not “generalize” the Scherk–Schwarz mechanism by allowing for extra discontinuities of the fields at the fixed points of the orbifold. Any discontinuities, as well as other local features of the fields, are determined by appropriate equations of motion. As we have already stressed, the Scherk–Schwarz mechanism determines only the global properties of the fields.

## 4 Fermion spectrum on $S^1/\mathbb{Z}_2$ with Scherk–Schwarz mechanism and mass terms

Originally the Scherk–Schwarz mechanism [1, 2] was used to break supersymmetry. The masses of all levels of the Kaluza–Klein tower (especially for gravitini) were shifted by

a constant. It is interesting to check how mass levels are changed by the Scherk–Schwarz mechanism on orbifolds. We will concentrate on the compactification from 5 to 4 dimensions on  $S^1/\mathbb{Z}_2$ .

#### 4.1 Kinetic Lagrangian in 5 dimensions

Many 5-dimensional models using the compactification on  $S^1/\mathbb{Z}_2$  have been discussed recently in the literature. But instead of choosing any specific model of this type we will consider a rather general situation. Thus our analysis can be used to investigate many different, not only 5-dimensional theories, by specifying some parameters.

Let us consider one 5-dimensional fermion field. Usually it is described by a pair of spinors satisfying the symplectic Majorana condition

$$(\lambda^i)^* = C_5 \epsilon_{ij} \lambda^j \quad (48)$$

where  $C_5$  is the 5-dimensional charge conjugation matrix. In the case of the  $S^1/\mathbb{Z}_2$  compactification those spinors have the following  $\mathbb{Z}_2$  parity properties

$$\lambda^1(-y) = + \Gamma^5 \lambda^1(y), \quad (49)$$

$$[6pt] \lambda^2(-y) = - \Gamma^5 \lambda^2(y). \quad (50)$$

But it is not very convenient to use these 5-dimensional symplectic Majorana spinors. We are interested in the compactified, effectively 4-dimensional theory. So let us define two new spinors via the relations

$$\psi^1 = \frac{1 + \Gamma^5}{2} \lambda^1 - \frac{1 - \Gamma^5}{2} \lambda^2, \quad (51)$$

$$[6pt] \psi^2 = \frac{1 - \Gamma^5}{2} \lambda^1 + \frac{1 + \Gamma^5}{2} \lambda^2. \quad (52)$$

It is easy to check that these new spinors fulfil the 4-dimensional Majorana condition

$$(\psi^i)^* = C_4 \psi^i \quad (53)$$

(with  $C_4$  being the 4-dimensional charge conjugation matrix) and have well defined parities under  $\mathbb{Z}_2$ :

$$\psi^1(-y) = + \psi^1(y), \quad (54)$$

$$[6pt] \psi^2(-y) = - \psi^2(y). \quad (55)$$

Then the 5-dimensional kinetic term for our spinor  $\lambda^i$  can be rewritten in terms of  $\psi^i$

$$-\frac{1}{2} \overline{\lambda^i} \Gamma^M \partial_M \lambda_i = -\frac{1}{2} \left[ \overline{\psi^1} \gamma^\mu \partial_\mu \psi^1 + \overline{\psi^2} \gamma^\mu \partial_\mu \psi^2 - \overline{\psi^1} \partial_y \psi^2 + \overline{\psi^2} \partial_y \psi^1 \right] \quad (56)$$

where we dropped eventual couplings to gauge bosons. We add also direct mass terms for the fermions. To be as general as possible we allow for the  $y$  dependence in these mass terms. There can be two kinds of such mass terms: even and odd under the  $\mathbb{Z}_2$  parity:

$$m_{\pm}(-y) = \pm m_{\pm}(y). \quad (57)$$

Taking into account the parity properties of  $\psi^i$  (54), (55) we get the following  $\mathbb{Z}_2$  invariant kinetic Lagrangian

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2} \left[ + \bar{\psi}^1 \gamma^\mu \partial_\mu \psi^1 + \bar{\psi}^2 \gamma^\mu \partial_\mu \psi^2 \right. \\ \left. [6pt] - \bar{\psi}^1 \partial_y \psi^2 + \bar{\psi}^2 \partial_y \psi^1 - m_+ \left( \bar{\psi}^1 \psi^1 + \bar{\psi}^2 \psi^2 \right) - m_- \left( \bar{\psi}^1 \psi^2 + \bar{\psi}^2 \psi^1 \right) \right]. \quad (58)$$

The last four terms in the square bracket will give effective 4-dimensional mass terms after compactification (integration over the 5-th coordinate  $y$ ).

One could think about further generalization of the above Lagrangian by allowing for two independent  $\mathbb{Z}_2$ -even mass terms, one for  $\psi^1$  and another for  $\psi^2$ . This could be an option for two independent spinors  $\psi^i$  but not in models discussed here. The spinors  $\psi^1$  and  $\psi^2$  are related. They are just different components of one 5-dimensional spinor. Before orbifolding, all the interactions for  $\psi^2$  are strictly determined by those for  $\psi^1$  simply by 5-dimensional Lorentz invariance. After orbifolding there is only one quantum number which differentiate between  $\psi^1$  and  $\psi^2$ ; the  $\mathbb{Z}_2$  parity which is even for one field and odd for the other. But this does not influence terms quadratic in any of these fields because such terms are  $\mathbb{Z}_2$ -even anyway. So, in 5-dimensional theories compactified on  $S^1/\mathbb{Z}_2$  the  $\mathbb{Z}_2$ -even mass term  $m_+$  should be the same for both fermions.<sup>5</sup> Similar conclusions can be obtained also for higher dimensional theories. One common  $m_+$  mass term appears e.g. in the case of the 11-dimensional heterotic M-theory [12] which is a practical realization of the situation discussed in this paper.

The rest of this section is devoted to the analysis of the effective 4-dimensional spectrum of fermions coming from this 5-dimensional Lagrangian (58) after compactification on  $S^1/\mathbb{Z}_2$  with possible Scherk-Schwarz twists.

A few remarks about the possible origin of such a Lagrangian and  $y$ -dependent mass terms are in order. The even mass terms, constant or located at the fixed points of the orbifold, can be explicitly present in the model under consideration. Other mass terms can not appear directly because they are not allowed by the symmetries of the theory (e.g. the direct odd mass term is forbidden by  $\mathbb{Z}_2$  parity). But they can appear indirectly when some fields develop non zero vacuum expectation values (VEVs) which break those symmetries. More generally, the Lagrangian (58) should be treated as a part of an effective Lagrangian obtained in a given theory after some

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<sup>5</sup>The authors of refs. [14, 15] also consider Lagrangians which do not agree with this conclusion. In particular in eq. (3.1) in [15] they assume that there is a delta like mass term for  $\psi^1$  but not for  $\psi^2$ . It is unclear how such a situation could be realized in a 5-dimensional model.

operations. Such operations can be e.g.: taking into account non zero VEVs of the background fields, redefinitions of fields, reduction from higher dimensions (if we start with a theory which is more than 5-dimensional), changing from a possible warped metric to an effective flat one etc. Our analysis can be applied to all situations when after all necessary redefinitions we can get the Lagrangian in the form of (58). That Lagrangian is written for a spin 1/2 fermion but it can be also easily generalized to the case of spin 3/2 (we have to add two gamma matrices between  $\psi^i$  and  $\psi^j$  in an appropriate way). So our results are valid also for the very interesting case of gravitini in supersymmetric models.

A very good example of the above mentioned redefinitions is that of the heterotic M-theory. We analysed the massless gravitino in such a model in the presence of brane located gaugino condensates in our previous paper [12]. In this model it is necessary to perform several field redefinitions to take into account six extra dimensions compactified on a Calabi–Yau manifold. Some redefinitions are connected to the fact that the background metric is warped. In the effective 5-dimensional Lagrangian we obtained two types of masses for the gravitino field, analogous to those present in (58). Both are generated by non zero VEVs of some components of the 4-th rank tensor field  $G_{ABCD}$  present in the 11-dimensional supergravity. VEV of  $G_{11abc}$  gives an even mass term in 5-dimensions while VEV of  $G_{a\bar{a}bb}$  gives an odd one ( $a, b$  and  $\bar{a}, \bar{b}$  are, respectively, holomorphic and anti-holomorphic coordinates on the Calabi–Yau manifold). Thus we see that the  $\mathbb{Z}_2$  even and odd, coordinate-dependent mass terms can quite naturally appear in higher dimensional models.

## 4.2 Mass eigenstate equations and Scherk–Schwarz boundary conditions

Before we look for the spectrum of fermions which can be obtained from the Lagrangian (58) we have to specify the properties of the fields under the Scherk–Schwarz twist. In the previous section we have proved that the most general twist can be decomposed into rotations in 2-dimensional subspaces, each consisting of one even and one odd field. The two Majorana fermions  $\psi^1$  and  $\psi^2$  form such a 2-dimensional subspace. In principle it is possible that  $\psi^1$  and  $\psi^2$  belong to two different such subspaces if there are more fields with appropriate quantum numbers (remember that any Scherk–Schwarz twist must be a global symmetry of the theory so it can not mix arbitrary fields). In such a case one should consider  $\psi^1, \psi^2$  to be vectors and  $m_+, m_-$  to be matrices in some type of a flavor space. However we are not going to consider here such a complication especially because it is not important for the most interesting case of the gravitino in supersymmetric models. We concentrate on a 2-dimensional subspace for which the

twists are given by<sup>6</sup>

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} \cos(2\pi\alpha) & -\sin(2\pi\alpha) \\ \sin(2\pi\alpha) & \cos(2\pi\alpha) \end{pmatrix}, \quad (59)$$

in the basis  $\Phi = (\psi^1, \psi^2)^T$ . In this case the twist condition (28) reads

$$\begin{pmatrix} \psi^1(x, \pi R) \\ \psi^2(x, \pi R) \end{pmatrix} = \begin{pmatrix} \cos(2\pi\alpha)\psi^1(x, -\pi R) - \sin(2\pi\alpha)\psi^2(x, -\pi R) \\ \sin(2\pi\alpha)\psi^1(x, -\pi R) + \cos(2\pi\alpha)\psi^2(x, -\pi R) \end{pmatrix}. \quad (60)$$

When the twist parameter  $\alpha$  is equal to zero, we have the standard compactification in which  $\psi^1(y)$  and  $\psi^2(y)$  are (periodic) functions on the circle.

Now we are ready to analyse the spectrum. First we decompose the 5-dimensional fields  $\psi^i$  in the following way:

$$\begin{pmatrix} \psi^1(x, y) \\ \psi^2(x, y) \end{pmatrix} = \sum_n \chi_n(x) \begin{pmatrix} u_n^1(y) \\ u_n^2(y) \end{pmatrix}. \quad (61)$$

We are looking for such a decomposition for which  $\chi_n(x)$  is the  $n$ -th 4-dimensional Majorana fermion with a definite masses  $M_n$ . The vector of functions  $(u_n^1(y), u_n^2(y))^T$  describes the shape of this  $n$ -th mass eigenstate in the 5-th dimension. We need both components because in general mass eigenstates do not have definite parities. Substituting this decomposition into the Lagrangian (58) we find that the functions  $u_n^i(y)$  must satisfy the following differential equations

$$\frac{\partial u_n^1(y)}{\partial y} + [M_n - m_+(y)]u_n^2(y) - m_-(y)u_n^1(y) = 0, \quad (62)$$

$$[6pt] \frac{\partial u_n^2(y)}{\partial y} - [M_n - m_+(y)]u_n^1(y) + m_-(y)u_n^2(y) = 0. \quad (63)$$

They can be rewritten in a more compact form as

$$\frac{\partial \mathbf{u}_n(y)}{\partial y} + [M_n - m_+(y)]i\sigma_2 \mathbf{u}_n(y) - m_-(y)\sigma_3 \mathbf{u}_n(y) = 0 \quad (64)$$

where  $\sigma_i$  are Pauli matrices in a space in which the even and odd components form a vector  $\mathbf{u}_n$ . The equations alone are not enough, we have to specify also the boundary conditions. Parity properties of the fields determine the boundary condition at  $y = 0$ :

$$\begin{pmatrix} u_n^1(0) \\ u_n^2(0) \end{pmatrix} = \begin{pmatrix} c_n \\ 0 \end{pmatrix} \quad (65)$$

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<sup>6</sup> $\chi$  of this form may be an element of the  $SU(2)_R$  automorphism group of the  $\mathcal{N} = 5$  supersymmetry.

where  $c_n$  are constants which should be adjusted in order to have the correct normalization of the 4-dimensional fields. The boundary condition at  $y = \pm\pi R$  can be obtained from eq. (60). Substituting expansion (61) into (60) and using the parity properties of  $u_n^i(y)$  we get:

$$\frac{u_n^2(\pi R)}{u_n^1(\pi R)} = \tan(\pi\alpha). \quad (66)$$

The masses and shapes of the 4-dimensional modes can in principle be found by solving the above differential equations (62), (63) with the boundary conditions given by (65), (66). It is possible to simplify this problem if we are interested only in the masses. To this end we consider only the ratio of the odd component to the even component of the wave function:  $t_n(y) = u_n^2(y)/u_n^1(y)$ . The differential equation for this function decouples from that for the other independent combination of  $u_n^2(y)$  and  $u_n^1(y)$  and reads

$$\frac{\partial t_n(y)}{\partial y} = [M_n - m_+(y)] (1 + t_n^2(y)) - 2m_-(y)t_n(y). \quad (67)$$

The appropriate boundary conditions

$$t_n(0) = 0, \quad (68)$$

$$[6pt]t_n(\pi R) = \tan(\pi\alpha), \quad (69)$$

can be used to obtain the discrete spectrum of masses  $M_n$ . Unfortunately for arbitrary mass terms  $m_{\pm}(y)$  it is not possible to find the solutions either for  $t_n(y)$  or for the separate components  $u_n^i(y)$  in a closed form.

## 5 Fermion spectrum for some types of models

In this section we discuss some situations when exact solutions can be found or when at least some important features of the solutions can be analysed.

### 5.1 Arbitrary $m_+(y)$ with vanishing $m_-$

The situation is very simple when the odd mass term is absent:  $m_-(y) = 0$ . Then the equations for the modes can be easily solved. Using the form (64) we immediately find

$$\mathbf{u}_n(y) = \exp \left\{ -i\sigma_2 \int_0^y ds [M_n - m_+(s)] \right\} \mathbf{u}_n(0). \quad (70)$$

Observe that the above exponent is just equal to the rotation matrix with the rotation angle given by the integral of  $[M_n - m_+]$ . Thus the mass eigenstates are given by

$$\begin{pmatrix} u_n^1(y) \\ u_n^2(y) \end{pmatrix} = \frac{1}{\sqrt{\pi R}} \begin{pmatrix} \cos [M_n y - \int_0^y ds m_+(s)] \\ \sin [M_n y - \int_0^y ds m_+(s)] \end{pmatrix}, \quad (71)$$



where we used the boundary condition (65) at  $y=0$ . It is very easy to solve also the boundary condition (66) at  $y=\pi R$ ; the masses  $M_n$  must satisfy the following equality:

$$\tan\left(\int_0^{\pi R} dy [M_n - m_+(y)]\right) = \tan(\pi\alpha), \quad (72)$$

hence, they are given by a simple formula

$$M_n = \frac{n + \alpha + \alpha_+}{R} \quad (73)$$

where  $\alpha$  is the Scherk–Schwarz twist parameter and  $\alpha_+$  is defined by

$$\alpha_+ = \frac{1}{\pi} \int_0^{\pi R} dy m_+(y). \quad (74)$$

From the above formulae we can see that the Scherk–Schwarz twist parameter  $\alpha$  and the integrated 5-dimensional,  $\mathbb{Z}_2$ -even mass term  $\alpha_+$  have exactly the same influence on the 4-dimensional mass eigenvalues. They both shift the masses of all the standard Kaluza–Klein levels by a constant:  $\alpha/R$  and  $\alpha_+/R$ , respectively.

From eq. (73) it is obvious that there are several possibilities when those two effects produce no net effect leaving the masses of the KK states unchanged. In the case of a gravitino field in a supersymmetric model this corresponds to unbroken supersymmetry. This happens when the mass term and the Scherk–Schwarz parameter satisfy the condition

$$\alpha + \frac{1}{\pi} \int_0^{\pi R} dy m_+(y) = k \in \mathbb{Z} \quad \text{for } m_-(y) = 0. \quad (75)$$

This of course does not mean that the Scherk–Schwarz mechanism is equivalent to arbitrary mass terms satisfying the above equations. As we have already stressed, the Scherk–Schwarz mechanism is related to global properties of the fields and not to their local behavior. The Scherk–Schwarz twist parameter appears directly only in the mass formula. On the other hand the mass term  $m_+(y)$  enters explicitly also the equations determining the shapes of the modes. And those shapes can be important for example if one considers interactions with other fields. In fact the Scherk–Schwarz mechanism is equivalent to the mechanism of adding a mass term only if this mass term is  $\mathbb{Z}_2$ -even and constant, and for sure not when the mass terms are localized at the fixed points of the orbifold (such delta-like localization of the mass terms is quite typical for models considered in the literature).

Let us now discuss the possibility of vanishing fermion (gravitino) mass. The simplest possibility occurs when  $k=0$  in eq. (75). In such a case the effects due to the Scherk–Schwarz twist and the even mass term cancel exactly against each other in the mass formula (73) for each KK level (this was observed in ref. [14] for delta-like mass terms). A different interesting situation occurs when the sum of  $\alpha$  and  $\alpha_+$  is a non

zero integer. The structure of the whole KK tower remains unchanged but the masses of all individual states do change. If we consider a smooth increase of parameters  $\alpha$  and  $\alpha_+$  from zero to their final values: the initially massless mode gets non zero mass while one of the massive modes becomes massless.

This crossing of levels can not occur if  $\alpha, \alpha_+ \ll 1$ . At least one of these parameters must be comparable to 1. The Scherk–Schwarz twist parameter  $\alpha$  can have values only in the range  $[-1/2, 1/2]$ . The reason is that, as we shown in the previous section, the Scherk–Schwarz twist in the case of  $S^1/\mathbb{Z}_2$  is just a rotation by an angle  $2\pi\alpha$  and the boundary condition (66) depends only on  $\tan(\pi\alpha)$ . So,  $\alpha$  and  $(\alpha + n)$  describe in fact exactly the same model. The situation with the mass term  $m_+(y)$  and its (normalized) integral  $\alpha_+$  is different. Two models in which the value of  $\alpha_+$  differs by an integer are really different. But one should be careful. The crossing of levels can occur when the average of  $m_+(y)$  over the 5–th coordinate satisfies

$$\frac{\int_0^{\pi R} dy m_+(y)}{\int_0^{\pi R} dy} = \mathcal{O}\left(\frac{1}{R}\right) \quad (76)$$

and it is necessary to check whether this is still in the range of validity of used approximations and/or assumptions.

## 5.2 Comments on delta like terms on orbifolds

In many models discussed in the literature the even mass terms  $m_+(y)$  have the form of Dirac delta sources located at the branes (fixed points of orbifolds used in those models). Such terms are sometimes taken improperly into account, so let us make some comments to clarify this issue. There is a problem which quite often appears in the literature, namely when one has to multiply  $\delta(y)$  by functions vanishing at  $y=0$ , e.g. functions odd in  $y$ . The naive, and incorrect, way is to assume that all such products are zero because  $\delta(y) \neq 0$  only at the point for which the other functions do vanish. To clarify this we start with reminding the obvious fact that the Dirac delta “function” is not a function but a distribution. So one should treat it consistently as a distribution or as a limit of some appropriate functions. Using any of these approaches one can easily solve the above mentioned problem of multiplying  $\delta(y)$  by odd functions. For arbitrary set of (not necessary different) functions  $g_i(y)$  odd in  $y$  the following relations hold:

$$\delta(y) \prod_{i=1}^{2n+1} g_i(y) = 0, \quad (77)$$

$$[6pt] \delta(y) \prod_{i=1}^{2n} g_i(y) = \frac{1}{2n+1} \delta(y) \lim_{y \rightarrow 0} \prod_{i=1}^{2n} g_i(y). \quad (78)$$

From the last equation it follows in particular that the delta like mass sources couple not only to even but also to odd parity fields because the odd fields can have jumps at  $y = 0$  so also nonzero limits for  $y \rightarrow 0$ . And there is no obvious way to forbid such couplings by some additional symmetries because the odd and even fields are just components of one 5-dimensional field with definite quantum numbers. Because of that we have a common,  $\mathbb{Z}_2$ -even mass term  $m_+$  for both fields,  $\psi^1$  and  $\psi^2$ , in the kinetic Lagrangian (58). Such situation is realized e.g. for the gravitino mass terms in the heterotic M-theory [12].

Another kind of problems can appear when a delta like mass term has large magnitude. From eq. (71) we see that the eigenfunctions in the case of vanishing  $m_-$  correspond to a unit vector rotating in the  $(u^1, u^2)$  space with the  $y$ -dependent phase angle given by

$$\varphi(y) = \left[ M_n y - \int_0^y ds m_+(s) \right]. \quad (79)$$

Let us discuss the behavior of this angle close to  $y = 0$ . We regularize  $\delta(y)$  by some functions  $f_\epsilon(y)$  which integrate to 1 and have support for  $|y| < \epsilon$  with  $\epsilon \rightarrow 0$ . The mass term is approximated by  $m_+(y) = c f_\epsilon(y)$ . For very small  $y$  we can neglect the  $M_n y$  contribution in eq. (79). Then for each function  $f_\epsilon$  the phase at  $y = \epsilon$  is given by:

$$\varphi_\epsilon(\epsilon) = - \int_0^\epsilon dy c f_\epsilon(y) = -\frac{c}{2}. \quad (80)$$

The phase is the same for all  $\epsilon$  so it has this value also in the limit  $\epsilon \rightarrow 0$ . Therefore infinitesimally close to the brane the state vector satisfies the condition

$$u_n^1(\epsilon) = \cos\left(\frac{c}{2}\right) u_n^1(0), \quad (81)$$

$$[6pt] u_n^2(\epsilon) = -\sin\left(\frac{c}{2}\right) u_n^1(0). \quad (82)$$

The values of  $\epsilon$  which differ by multiples of  $4\pi$  give the same values of  $u^1(\epsilon)$  and  $u^2(\epsilon)$ . They give however different behavior of the state vector over the interval  $y \in [0, \epsilon]$  for  $\epsilon \neq 0$ . The state vector corresponding to a bigger value of  $\epsilon$  rotates more times by the full angle  $2\pi$  between points  $y = 0$  and  $y = \epsilon$ . Thus, the equations (81), (82) taken at  $\epsilon = 0$  are alone not enough to describe a given eigenstate. For small values of the magnitude  $\epsilon$  the eigenstate may be described by functions  $u^1(y)$  and  $u^2(y)$  which are just discontinuous at  $y = 0$ . But for large values of  $\epsilon$  a more careful treatment is necessary.

The last remark on delta-like terms on orbifolds concerns their normalization. Performing the integral in eq. (80) we get only  $-c/2$  and not  $-c$  because we integrate only over the “half” of the delta’s support, that for positive  $y$ . This is the so called down-stairs approach to the  $S^1/\mathbb{Z}_2$  orbifold in which we integrate over the interval  $y \in [0, \pi R]$  using the additional prescription that Dirac deltas located at the fixed

points give after integration  $1/2$  instead of 1. In the up-stairs approach one integrates over full circle  $S^1$  but the final result must be divided by 2.

### 5.3 Constant, simultaneously non zero, $m_+$ and $m_-$

Let us now turn to more complicated cases of the Scherk–Schwarz mechanism on orbifolds, when the odd mass term in the Lagrangian (58) is different from zero. Now it is not possible to solve the differential equation for the mass eigenstates (64) by simple integration and exponentiation (as we did to get eq. (70) in the case of  $m_- = 0$ ). The reason is that matrices  $\sigma_2$  and  $\sigma_3$  do not commute and they are multiplied by functions of  $y$ . The formal solution of (64) involves an appropriate ordering operator and not just the ordinary exponential function. Because of that, it is not possible to write the mass eigenstates, given by solutions of (64), in an explicit, closed form. In the case of arbitrary  $m_-(y)$  it is also not possible to solve explicitly eq. (67) which determines the masses eigenvalues (without determining the shapes of the states). But there are some simple cases in which we can solve some of the equations or at least be able to find some properties of the solutions on which we concentrate in the rest of this section.

One of the cases when it is possible to find explicitly the eigenstates is when both mass terms  $m_+$  and  $m_-$  are constant. Then the solution to (64) is given by

$$\mathbf{u}_n(y) = \exp \{ (-i\sigma_2[M_n - m_+] + \sigma_3 m_-) y \} \mathbf{u}_n(0) \quad (83)$$

which can be rewritten as

$$\begin{pmatrix} u_n^1(y) \\ u_n^2(y) \end{pmatrix} \propto \begin{pmatrix} \cos(\mu_n y) + \frac{m_-}{\mu_n} \sin(\mu_n y) \\ \frac{M_n - m_+}{\mu_n} \sin(\mu_n y) \end{pmatrix} \quad (84)$$

where

$$\mu_n = \sqrt{(M_n - m_+)^2 - m_-^2} \quad (85)$$

and we used the boundary condition at  $y = 0$ , eq. (65). The solution (84) is valid even for imaginary  $\mu_n$  (then trigonometrical functions are replaced with appropriate hyperbolic ones). Now we have to implement also the second boundary condition, the one at  $y = \pm\pi R$ . This condition can not be fulfilled for imaginary  $\mu_n$  (with one exception which we discuss later). For real  $\mu_n$  we can find that the masses  $M_n$  are given by the solution of the following equation

$$(M_n - m_+) \tan(\pi\alpha) = \mu_n \tan \left[ \mu_n \pi R - \arctan \left( \frac{m_-}{\mu_n} \right) \right] + m_- \quad (86)$$

in which  $M_n$  appears implicitly on the r.h.s. via the parameters  $\mu_n$  defined in eq. (85). The same condition for mass eigenvalues can be obtained from equation (67)

with boundary conditions (68)–(69). The reality condition for  $\mu_n$  means that the mass eigenstates are given by (84) with mass eigenvalues satisfying (86) when

$$|M_n - m_+| > |m_-|. \quad (87)$$

Now we discuss the additional solution which corresponds to one exception from the reality condition for  $\mu_n$  mentioned above. It exists only when the Scherk–Schwarz twist is trivial:  $\alpha = 0$ . In such a case there is a state with mass  $M = m_+$  and shape given by

$$\begin{pmatrix} u^1(y) \\ u^2(y) \end{pmatrix} \propto \begin{pmatrix} \exp \left\{ \int_0^y ds m_-(s) \right\} \\ 0 \end{pmatrix}. \quad (88)$$

Observe that we have used an integral of  $m_-(y)$ , and not just the product of a constant  $m_-$  with  $y$ , in the above formula. It is not difficult to check that the solution of such form is valid for arbitrary  $y$ -dependent mass term  $m_-(y)$ . It is quite important because it describes the massless mode in the case of vanishing  $m_+$ . It should be stressed that such a state appears only when  $\alpha = 0$ . This means that in the case of a non trivial Scherk–Schwarz twist there can be no massless mode for arbitrary  $m_-(y)$  if  $m_+ \neq 0$ .

Let us now go back to the mass eigenvalue condition (86) and the mass eigenstates (84). They allow to extend our discussion of (non)equivalence of the Scherk–Schwarz mechanism and direct 5-dimensional mass terms. When considering the situation with vanishing  $\mathbb{Z}_2$ -odd mass  $m_-$  (see discussion after eqs. (74) and (75)) we have observed that in such a case the Scherk–Schwarz mechanism is equivalent to adding a constant  $\mathbb{Z}_2$ -even mass  $m_+$ . Moreover, by a suitable choice of the Scherk–Schwarz twist parameter  $\alpha$  it is possible to reproduce the spectrum (but not the shapes of the mass eigenstates) obtained in the case of vanishing  $\alpha$  and given  $y$ -dependent  $m_+$ . The situation changes for  $m_- \neq 0$  (also if it is constant). From eq. (86) it is clear that the masses in the case of the Scherk–Schwarz mechanism are, for arbitrary  $\alpha$ , different from the masses obtained in a theory with direct mass terms. The twist parameter  $\alpha$  and the  $\mathbb{Z}_2$ -even mass  $m_+$  enter the mass formula (86) in quite different ways. The relation between  $m_+$  and mass eigenvalues is very simple. The constant  $m_+$  appears only in the combination  $(M_n - m_+)$  so it just shifts all the mass states by a constant. On the other hand, the relation between  $\alpha$  and the mass spectrum is quite complicated and in general can not be solved explicitly. But one feature of that relation can be read from the mass formula (86). For arbitrary  $m_+$  and  $m_-$  it is possible to choose  $\alpha$  in such a way that one state with any given mass (within some range of values) is present in the spectrum. A state with mass  $M_0$  appears in the spectrum if the Scherk–Schwarz twist parameter is given by

$$\alpha = \frac{1}{\pi} \arctan \left\{ \frac{\mu_0}{M_0 - m_+} \tan \left[ \mu_0 \pi R - \arctan \left( \frac{m_-}{\mu_0} \right) \right] + m_- \right\}, \quad (89)$$

with  $\mu_0 = \sqrt{(M_0 - m_+)^2 - m_-^2}$  if this mass satisfies the condition  $|M_0 - m_+| > |m_-|$ . In particular there is a value of  $\alpha$  for which one state is massless if  $|m_+| > |m_-|$ .

## 5.4 Delta like $m_+$ in the presence of arbitrary $m_-(y)$

So far we discussed three special situations: arbitrary  $m_+$  with vanishing  $m_-$ ; constant  $m_+$  and  $m_-$ ; arbitrary  $m_-$  with vanishing  $m_+$  (only the massless mode). In general it is not possible to find explicit solutions for more complicated cases. However, there is a very interesting class of models which can be analysed using the results obtained so far. Let us consider a situation when the  $\mathbb{Z}_2$  even and odd mass terms are described by some arbitrary functions of  $y$  subject to the constraint  $m_+(y)m_-(y) = 0$  for all  $y$ . This condition is fulfilled for example in all models in which the  $\mathbb{Z}_2$ -even mass terms are located at the branes. The reason is that arbitrary  $m_-(y)$  vanishes at  $y = 0$  and  $y = \pi R$ .

For such models it is possible to check whether a massless state exists in the spectrum. It is very important because this way one can determine under what conditions supersymmetry remains unbroken in the presence of the Scherk–Schwarz twists and the direct mass terms.

First we divide the interval  $[0, \pi R]$  into pieces in such a way that at each piece only one of the mass terms,  $m_+(y)$  or  $m_-(y)$ , is non vanishing. We find a possible solution corresponding to  $M = 0$  for each of these sub-intervals. The solution for  $y_i < y < y_{i+1}$  is given by

$$\begin{aligned} \mathbf{u}_0(y) &= \exp \left\{ i\sigma_2 \int_{y_i}^y ds m_+(s) \right\} \mathbf{u}_0(y_i), & \text{if } m_-(y) = 0 \text{ for } y \in [y_i, y_{i+1}] & (90) \\ [6pt] \mathbf{u}_0(y) &= \exp \left\{ \sigma_3 \int_{y_i}^y ds m_-(s) \right\} \mathbf{u}_0(y_i), & \text{if } m_+(y) = 0 \text{ for } y \in [y_i, y_{i+1}] & (91) \end{aligned}$$

We build the full solution from those partial ones starting from the boundary condition at  $y = 0$  (65) and demanding that it is continuous at points  $y_i$  where the character of the solution changes from (90) to (91) or vice versa. At the end we check whether  $\mathbf{u}_0(\pi R)$  obtained this way, fulfills the boundary condition (66). Such a procedure allows us to check what value of the Scherk–Schwarz twist angle is compatible with a massless fermion for given  $y$ -dependent mass terms  $m_+(y)$  and  $m_-(y)$ .

Let us illustrate this by an example of  $m_+(y)$  terms located at the branes:

$$m_+(y) = 2m_0\delta(y) + 2m_\pi\delta(y - \pi R). \quad (92)$$

All we have to know about  $m_-(y)$  is its integral

$$M_- = \int_0^{\pi R} dy m_-(y). \quad (93)$$

We divide the interval  $[0, \pi R]$  into three regions (two of them infinitesimally small close

to the branes) and use the procedure described above. The result for  $\mathbf{u}_0(\pi R)$  reads

$$\begin{aligned}\mathbf{u}_0(\pi R) &= \exp \{i\sigma_2 m_\pi\} \exp \{\sigma_3 M_-\} \exp \{i\sigma_2 m_0\} \mathbf{u}_0(0) \\ [6pt] &= \begin{pmatrix} \cos(m_\pi) & \sin(m_\pi) \\ [6pt] -\sin(m_\pi) & \cos(m_\pi) \end{pmatrix} \begin{pmatrix} e^{M_-} & 0 \\ 0 & e^{-M_-} \end{pmatrix} \begin{pmatrix} \cos(m_0) & \sin(m_0) \\ [6pt] -\sin(m_0) & \cos(m_0) \end{pmatrix} \begin{pmatrix} c \\ [6pt] [6pt] \end{pmatrix} \\ [6pt] &= c \begin{pmatrix} \cos(m_\pi) \cos(m_0) e^{M_-} - \sin(m_\pi) \sin(m_0) e^{-M_-} \\ [6pt] -\sin(m_\pi) \cos(m_0) e^{M_-} - \cos(m_\pi) \sin(m_0) e^{-M_-} \end{pmatrix}.\end{aligned}$$

This solution agrees with the Scherk–Schwarz twist condition (66) if

$$\frac{\sin(m_\pi) \cos(m_0) e^{M_-} + \cos(m_\pi) \sin(m_0) e^{-M_-}}{\cos(m_\pi) \cos(m_0) e^{M_-} - \sin(m_\pi) \sin(m_0) e^{-M_-}} = -\tan(\pi\alpha). \quad (95)$$

In the case of the gravitino this formula – relating the mass terms located at each of the branes, the bulk ( $\mathbb{Z}_2$ -odd) mass term and the Scherk–Schwarz twist – must be fulfilled if supersymmetry in the effective 4-dimensional theory is to be unbroken.

The above formula can be applied also in the case of ordinary compactification without the Scherk–Schwarz mechanism. Putting  $\alpha$  to zero we obtain the condition

$$\frac{\tan(m_\pi)}{\tan(m_0)} = -e^{-2M_-}. \quad (96)$$

When the  $\mathbb{Z}_2$ -odd mass term integrates to zero,  $M_- = 0$ , we get the usual condition for the unbroken supersymmetry:  $m_\pi = -m_0$  (up to terms which are multiples of  $2\pi$  which we discussed after eq. (75)). In the case of vanishing  $M_-$  the brane located mass terms generated for the gravitino must add up to zero. In the presence of non zero  $M_-$  the situation is more complicated and supersymmetry can be unbroken when the condition (96) is satisfied.

The condition (95) for the existence of a massless mode is a quite complicated relation between the Scherk–Schwarz twist angle and the three mass parameters:  $m_0$ ,  $m_\pi$ ,  $M_-$ . Let us now consider a situation when all the mass parameters are small. Expanding eq. (95) and keeping only the leading terms we get a much simpler relation

$$m_0 + m_\pi = -\tan(\pi\alpha). \quad (97)$$

Observe that the integral of the  $\mathbb{Z}_2$ -odd mass,  $M_-$ , dropped out from the formula. Moreover, in the case of ordinary compactification ( $\alpha = 0$ ) we get a very simple equality  $m_\pi = -m_0$  for arbitrary but small  $M_-$ . These approximate conditions should be used e.g. in cases when one does not know the full theory but only some terms of an expansion in some small parameter.

Let us illustrate this with the heterotic M-theory. The field theoretical limit of this theory is known only up to the first order of the expansion in  $\kappa^{2/3}$  (related to the 11-dimensional coupling constant). We have shown in [12] that effectively there are

two kinds of 5-dimensional mass terms for the gravitino: the  $\mathbb{Z}_2$ -even one generated by  $\langle G_{11abc} \rangle$  and the  $\mathbb{Z}_2$ -odd one coming from  $\langle G_{a\bar{a}bb} \rangle$ . Both of them are linear in the expansion parameter  $\kappa^{2/3}$  so one should use the approximate formula (97) and not the full one (96). Thus the condition for unbroken supersymmetry is just that the brane located mass terms should sum up to zero as was shown in [12].

It is possible to gain a qualitative understanding why in more complicated models the condition for unbroken supersymmetry (96) differs from the simple global cancellation of the sources,  $m_\pi + m_0 = 0$ , even without the Scherk–Schwarz twist. In the presence of the non zero mass term  $M_-$  the massless gravitino is not a constant mode even without any brane terms. In general its amplitude is different at each brane. Thus, the gravitino couples with different strength to sources present at each of the branes. In general, a simple algebraic cancellation of sources is not enough to leave supersymmetry unbroken. From eq. (96) it is clear that the details of the shape of the zero mode – determined by the details of the function  $m_-(y)$  – are not important. Important is only the relation between the gravitino wave function at both branes – determined by the integral of  $m_-(y)$ .

## 6 Conclusions

As we have seen, the consistency conditions for a Scherk–Schwarz mechanism on orbifolds are available in general form (subsection 3.1). The structure of possible Scherk–Schwarz twists which can be applied on orbifolds is more restricted than in the case of manifolds. We found the most general form of such twists which can be consistently used in 5-dimensional theories compactified on  $S^1/\mathbb{Z}_2$ . Only rotations in 2-dimensional subspaces (each consisting of one field with positive and one with negative parity under  $\mathbb{Z}_2$ ) or multiplication by  $-1$  are allowed.

The consistency conditions can be used to obtain differential equations and boundary conditions which determine the spectrum of the effective theory after compactification. Solutions, however, are difficult to obtain in closed form due to the complexity of the problem. We have therefore used a simple  $S^1/\mathbb{Z}_2$  orbifold to illustrate our results. One of the important ingredients in the discussion is the general form of the possible mass terms (and their dependence on the higher dimensional coordinates) that make the connection to brane induced supersymmetry breaking. Of particular importance is the appearance of the mass term  $m_-$  in eq. (58). Its existence depends on the presence of VEVs of higher dimensional background fields, as was discussed already in [12]. The investigation in [14, 15] did not consider such a term. In addition, their ansatz for the brane located mass terms and interactions differs from the one adopted here.

Explicit solutions have been presented for two classes of models: those with constant mass terms  $m_+$  and  $m_-$  and those with arbitrary  $y$ -dependent  $m_+(y)$  but with vanishing  $m_-$ . Closed solutions for general mass terms  $m_+(y)$  and  $m_-(y)$  are difficult



to obtain. Fortunately some of the most interesting cases, as e.g. those with delta-like mass terms located at the branes can be analysed with sufficient accuracy. It is possible to find conditions determining whether massless fermions are present in the spectrum. This allows to check if supersymmetry is broken in a theory with given form of the mass terms  $m_+(y)$  and  $m_-(y)$  in the presence of non trivial Scherk–Schwarz twists.

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