UTTG-02-00
hep-th/0001073

Dirac Monopole in Non-Commutative Space

Li Jiang

Theory Group, Department of Physics
University of Texas at Austin
Austin, TX 78712, USA
jiang@physics.utexas.edu

Abstract

We consider static $U(1)$ monopole in non-commutative space. Up to the second order in the non-commutativity scale θ, we find no non-trivial corrections to the Dirac solution, the monopole mass remains infinite. We argue the same holds for any arbitrary higher order. Some speculation about the nature of non-commutative spacetime and its relation to the cosmological constant is made.

Non-commutative geometry arises naturally in string theory when the Neveu-Schwarz B field is turned on. In a certain limit, the low-energy effective theory of the worldvolume is described by gauge theory in non-commutative space [1]. Motivated by the attempt to investigate localized structures, some researchers have recently studied topological field configuration in non-commutative geometry, namely the non-Abelian monopoles [2, 3, 4]. In this letter, we consider another elementary example, the Dirac monopole [5].

It would be helpful to make some comparisons first. Non-Abelian monopole can be visualized as D-string stretched between branes [6, 7]. In the $U(2)$ case, when a background B field is turned on along the branes, the string is tilted because the two endpoints carry opposite charges [2]. This leads to a dipole structure in the magnetic field of the monopole. Explicit calculation of this effect to the $O(\theta)$ order has been carried out in [3, 4]. However, $U(1)$ monopole does not admit such a simple geometric picture. First of all, there is no need to introduce a Higgs field ϕ. The singularity is put in by hand: one simply adds a source term to the Bianchi identity, then the base manifold becomes $\mathbb{R}^3 \setminus \{0\}$, which deformation-retracts to S^2. The Wu-Yang method [8] is applied to yield the quantization of magnetic monopole charge ($\pi_1(U(1)) = \mathbb{Z}$). By contrast the topological invariant of a non-Abelian monopole is defined by the asymptotic behavior of the Higgs field. Second, the energy of Dirac monopole diverges, while that of 't Hooft-Polyakov monopole [9, 10] is finite ($m \propto 1/g_{\mathrm{YM}}$). Although it's difficult to make topological argument in non-commutative space, it has been shown that the BPS bound [11, 12] still exists [2, 3, 4]. One naturally asks whether non-commutativity will render Dirac monopole a finite mass. This is interesting since all the attempts to find magnetic monopole have failed. We can even pose the question: why should one treat a "particle" with infinite mass as a physical entity? The case should be compared to that of electron. Although the field energy due to a point electric charge diverges in the same way, it's renormalizable as electron has an experimentally measurable mass about half MeV (one simple way to get rid of the infinity is to replace the point source with a smooth compactly-supported charge, while the same procedure is not applicable to monopole [13]).

If Dirac monopole became finitely heavy in non-commutative space, it would be of great interests to theorists and experimentalists alike. However, our calculation gives a negative answer. Up to the $O(\theta^2)$ order, we show explicitly that there is no correction to the $U(1)$ gauge connection A (except for θ-dependent gauge transformation), therefore the mass remains infinite. We argue the same is true for any higher order. In the following, we will present our calculation by adopting a mixed notation of tensor and differential form. We also discuss another formulation of the $U(1)$ monopole in non-commutative space. Finally, we ponder over the nature of non-commutative spacetime and its relation to the cosmological constant.

The field strength in non-commutative gauge theory is defined as usual with replacement of the ordinary product by the $\star$-product[1]

$$F = dA - \frac{i}{2}[A, A]_\star. \tag{1}$$

Up to the second order $O(\theta^2)$, F in component form is[2]

$$F_{ij} \simeq \partial_i A_j - \partial_j A_i + \theta_{mn}\partial_m A_i \partial_n A_j, \tag{2}$$

where i, j run from 1 to 3. One notices there is no $O(\theta^2)$ term. In fact, all the even powers of θ vanish because of its antisymmetric nature. This can also be seen from the definition of F: all the even powers contain an i, but F is a real number. Then what do we mean by the second order correction? Of course presumably the gauge potential A itself should receive some corrections due to non-commutativity. Expanding A to $A \simeq A^0 + A^1 + A^2$ (the upper indices denote the order in θ), we rewrite (2) as $F_{ij} \simeq F^0_{ij} + F^1_{ij} + F^2_{ij}$, where

$$\begin{aligned} F^0_{ij} &= \partial_i A^0_j - \partial_j A^0_i \\ F^1_{ij} &= \partial_i A^1_j - \partial_j A^1_i + \theta_{mn}\partial_m A^0_i \partial_n A^0_j \\ F^2_{ij} &= \partial_i A^2_j - \partial_j A^2_i + \theta_{mn}\partial_m A^0_i \partial_n A^1_j + \theta_{mn}\partial_m A^1_i \partial_n A^0_j. \end{aligned} \tag{3}$$

For later convenience, we define $f^{1,2} = dA^{1,2}$. The goal is to find the solution to the modified Bianchi identity (in Gauss units)

$$DF = 4\pi g \delta^3(\vec{r}) *1 \tag{4}$$

where g is the monopole charge and $*1$ is the volume form. The covariant derivative of F in non-commutative space is similarly defined

$$DF = dF - i[A, F]_\star. \tag{5}$$

Order by order in θ, (4) is

$$\begin{aligned} dF^0 &= 4\pi g \delta^3(\vec{r}) *1 \\ dF^1 &= -\theta_{mn}\partial_m A^0 \wedge \partial_n F^0 \\ dF^2 &= -\theta_{mn}\partial_m A^1 \wedge \partial_n F^0 - \theta_{mn}\partial_m A^0 \wedge \partial_n F^1. \end{aligned} \tag{6}$$

[1] $f(x) \star g(x) = \exp(\frac{i}{2}\theta_{ij}\partial_i\partial'_j) f(x) g(x')|_{x=x'}$ and $[x_i, x_j] = i\theta_{ij}$.

[2] We only consider static situation and set $A_0 = 0$ since there is no electric source.

The zeroth order equation is just the familiar $\nabla \cdot \vec{B}^0 = 4\pi g \delta^3(\vec{r})$, with $B^0 = *F^0$. Its solution is simply $\vec{B}^0 = g\vec{r}/r^3$, and A^0 has singularities (Dirac string) when expressed in a single coordinate system. One then solves for $A^{1,2}$ by plugging A^0 into the first and the second order equations. In component form, the first order equation reads

$$\begin{aligned}\epsilon_{ijk}\partial_i f^1_{jk} &= -\epsilon_{ijk}\partial_i(\theta_{mn}\partial_m A^0_j \partial_n A^0_k) - \epsilon_{ijk}\theta_{nm}\partial_n A^0_k \partial_m F^0_{ij} \\ &= -\epsilon_{ijk}\theta_{mn}\Big(\partial_m\partial_i A^0_j \partial_n A^0_k + \partial_m A^0_j \partial_n \partial_i A^0_k - \partial_n A^0_k \partial_m(\partial_i A^0_j - \partial_j A^0_i)\Big). \end{aligned} \tag{7}$$

The first and the third terms cancel each other; while shuffling the indices of the fourth term ($i \to k$, $j \to i$, $k \to j$ and $m \leftrightarrow n$) makes it cancel the second one. So the right hand side of (7) vanishes, *i.e.*, $df^1 = 0$. This is a source-free equation with solution $f^1 = 0$ (because of the boundary condition at infinity), therefore we find that A^1 is a pure gauge. We proceed to the second order

$$\begin{aligned}\epsilon_{ijk}\partial_i f^2_{jk} = &-\epsilon_{ijk}\partial_i\Big(\theta_{mn}(\partial_m A^0_j \partial_n A^1_k - \partial_m A^0_k \partial_n A^1_j)\Big) \\ &-\epsilon_{ijk}\theta_{nm}\partial_n A^1_k \partial_m F^0_{ij} - \epsilon_{ijk}\theta_{mn}\partial_m A^0_k \partial_n F^1_{ij}. \end{aligned} \tag{8}$$

It's easy to show that most of the terms in (8) cancel out by either shuffling the indices or directly setting A^1 to be zero. The only non-trivial term comes from the A^0 part in F^1. More precisely we are left with

$$\epsilon_{ijk}\partial_i f^2_{jk} = -\epsilon_{ijk}\theta_{mn}\theta_{pq}\partial_m A^0_k \partial_n(\partial_p A^0_i \partial_q A^0_j). \tag{9}$$

The term on the right hand side is just $-2\epsilon_{ijk}(\theta_{mn}\theta_{pq}\partial_m A^0_k \partial_q A^0_j \partial_n \partial_p A^0_i)$ with the part in brackets symmetric in j and k, so it vanishes identically. Once again, we are led to the usual Bianchi identity $df^2 = 0$, which shows A^2 is trivial up to a gauge transformation. One may argue that $f^{1,2}$ are closed from the previous definition. But we note $f^{1,2} = dA^{1,2}$ are only defined *locally*, just like in the commutative case $F^0 = dA^0$ is not valid *globally* on S^2. *A priori*, it is not obvious the source terms due to A^0 in (7) and (8) should vanish.

It is tempting to generalize the above result to higher order: the gauge potential A does not receive any corrections at all. Since the mere existence of a non-trivial A is due to the monopole at origin, while it's hard to see why a physical quantity defined at a *single* point should be modified because of the spatial non-commutativity (this is also the reason why we do not put $g \simeq g^0 + g^1 + g^2$ in (4)). In consequence, the monopole mass still diverges. Explicitly, $|A^0| \sim 1/r$, $|F^0| \sim 1/r^2$, $|F^1| \sim 1/r^4$ and $|F^2| = 0$, so

$$m = \int |F|^2 \sim \int_0^\infty r^2 dr \left|\frac{1}{r^2} + \frac{1}{r^4}\right|^2 = \infty. \tag{10}$$

While this result is disappointing, there exists an alternative way to address the monopole problem because of the self-interaction of photons in non-commutative space [14]. As already remarked, the singularity at origin is put in by hand and results in the infinite mass. Instead of introducing the singularity beforehand, one can try to seek finite energy solution to the equation of motion. In other words, we are not looking for a point monopole, but an extended field configuration just like in commutative non-Abelian gauge theory. Now one faces the question as whether a Higgs field should be introduced. In the brane picture, a non-trivial ϕ describes the shape of the brane. But it's not clear whether the topological charge depends on ϕ. A detailed analysis is beyond the scope of this short paper and may appear in a subsequent report.

There is the important question about gauge invariance. Due to the $\star$-product structure, the $U(1)$ field strength is not gauge invariant any more

$$\begin{aligned}\delta F &\simeq [i\lambda, F]_\star \\ &\simeq 0 - \theta_{mn}\partial_m\lambda^0\partial_n F^0 - (\theta_{mn}\partial_m\lambda^1\partial_n F^0 + \theta_{mn}\partial_m\lambda^0\partial_n F^1),\end{aligned} \tag{11}$$

where $\lambda = \lambda^0 + \lambda^1 + \cdots$ is the gauge parameter. This makes it difficult to interpret what is the magnetic field generated by the monopole. For instance, consider an electron moving in the monopole background. Even on the classical level, one has to modify the Lorentz force formula $\vec{f} = q\vec{v}\times\vec{B}$ in order to make sure $\delta\vec{f} = 0$ under $\delta B = *\delta F$ (or equivalently it imposes constraints on the possible form of gauge transformation). This situation differs too much from the familiar physical facts, thus raises a basic question one bears in mind: what is the reality of spacetime non-commutativity?

To achieve non-commutativity, one has to give a non-zero expectation value to the Neveu-Schwarz tensor field, hence the associated field energy (we do not consider the so-called *transverse non-commutativity* [15] here). This adds to the total vacuum energy density ρ_{V} and therefore is related to the long-standing cosmological constant problem (for a review, see [16]). The non-commutativity scale is determined by the background field $\theta = 1/B$. If we assume, for a time, the vacuum energy comes solely from B field, we can calculate θ by using the observed value of ρ_{V}. Assuming the energy density of B takes the same form as that of magnetic field[3], we have

$$\rho_{\mathrm{V}} = \frac{1}{8\pi}B^2 = \frac{1}{8\pi\theta^2}, \tag{12}$$

[3]One needs to be a little cautious: although B resembles a magnetic field, it is a two-form *potential*, not a *field strength* in string theory.

which gives

$$\sqrt{\theta} = \left(\frac{1}{8\pi\rho_{\rm V}}\right)^{1/4} \simeq \left(\frac{1}{8\pi \times 10^{-47}\ {\rm GeV}^4}\right)^{1/4} \simeq 5.0 \times 10^{-3}\ {\rm cm}. \tag{13}$$

This numerical value is hardly believable as otherwise it would be verified easily by experiment (note a portion of $\rho_{\rm V}$ only makes the result worse). Of course the above simple calculation does not exclude the possibility that spacetime may be non-commutative on the fundamental level, since the cosmological constant itself has a 120 orders of magnitude contrast between theoretical estimation and experimental observation! Nevertheless, it indicates the nature of non-commutative spacetime is far from clear. A plausible test bed of non-commutativity is superconductivity. Unlike the monopoles, which largely remain to be theorists' fond toys, the magnetic vortices have been observed in type II superconductors and the associated Abelian Higgs model has been well studied (see, for instance, [13]). Non-commutativity could show its signature via some modifications to the conventional flux lattice.

In conclusion, we studied Dirac monopole in non-commutative space and found no non-trivial corrections. A more interesting solution may appear in an alternative formulation of the problem. In the process, we raised more questions than we could answer. While spacetime non-commutativity seems too remote from the real world, we have no doubts about this increasing abstraction in mathematics and physics as remarked in Dirac's original paper [5].

Acknowledgements

We would like to thank Miao Li for bringing to our attention the paper [6]. This work was supported in part by NSF Grant PHY9511632 and the Robert A. Welch Foundation.

References

[1] N. Seiberg and E. Witten, *String Theory and Noncommutative Geometry*, JHEP **09** (1999) 032, `hep-th/9908142`.

[2] A. Hashimoto and K. Hashimoto, *Monopoles and Dyons in Non-Commutative Geometry*, JHEP **11** (1999) 005, `hep-th/9909202`.

[3] K. Hashimoto, H. Hata and S. Moriyama, *Brane Configuration from Monopole Solution in Non-Commutative Super Yang-Mills Theory*, JHEP **12** (1999) 021, `hep-th/9910196`.

[4] D. Bak, *Deformed Nahm Equation and a Noncommutative BPS Monopole*, `hep-th/9910135`.

[5] P. A. M. Dirac, *Quantised Singularities in the Electromagnetic Field*, Proc. R. Soc. Lond. **A133** (1931) 60.

[6] M. R. Douglas and M. Li, *D-Brane Realization of $\mathcal{N} = 2$ Super Yang-Mills Theory in Four Dimensions*, `hep-th/9604041`.

[7] D.-E. Diaconescu, *D-branes, Monopoles and Nahm Equations*, Nucl. Phys. **B503** (1997) 220, `hep-th/9608163`.

[8] T. T. Wu and C. N. Yang, *Concept of Nonintegrable Phase Factors and Global Formulation of Gauge Fields*, Phys. Rev. **D12** (1975) 3845.

[9] G. 't Hooft, *Magnetic Monopoles in Unified Gauge Theories*, Nucl. Phys. **B79** (1974) 276.

[10] A. M. Polyakov, *Particle Spectrum in Quantum Field Theory*, JETP Lett. **20** (1974) 194.

[11] E. B. Bogomol'nyi, *The Stability of Classical Solutions*, Sov. J. Nucl. Phys. **24** (1976) 449.

[12] M. K. Prasad and C. M. Sommerfield, *An Exact Classical Solution for the 't Hooft Monopole and the Julia-Zee Dyon*, Phys. Rev. Lett. **35** (1975) 760.

[13] A. Jaffe and C. Taubes, *Vortices and Monopoles*, Birkhäuser, Boston (1980).

[14] M. Hayakawa, *Perturbative Analysis on Infrared and Ultraviolet Aspects of Noncommutative QED on R^4*, `hep-th/9912167`.

[15] E. Witten, *Bound States of Strings and p-Branes*, Nucl. Phys. **B460** (1996) 335, `hep-th/9510135`.

[16] S. Weinberg, *The Cosmological Constant Problem*, Rev. Mod. Phys. **61** (1989) 1.