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BPS equations in Six and Eight Dimensions

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We consider the maximal supersymmetric pure Yang-Mills theories on six and eight dimensional space. We determine, in a systematic way, all the possible fractions of supersymmetry preserved by the BPS states and present the corresponding 'self-dual' BPS equations. In six dimensions the intrinsic one has 1/4 supersymmetry, while in eight dimensions, 1/16, 2/16, $\cdots$, 6/16. We apply our results to some explicit BPS configurations of finite or infinite energy on commutative or noncommutative spaces.

1 Introduction

Recently there has been considerable interest in understanding the possible supersymmetric states of D0 branes on D2, D4, D6 and D8 in IIA string theories. While the D0-D2 and D0-D4 systems are relatively well understood with or without the NS-NS B field background, the D0-D6 and D0-D8 cases remain veiled.

A pioneering work on higher than four dimensional gauge theories was done by Corrigan, *et al.* [1]. They investigated the higher dimensional analogues of 'self-duality', seeking linear relations amongst components of the field strength

$$F_{ab} + \frac{1}{2} T_{abcd} F_{cd} = 0 \,, \tag{1}$$

with constant four form tensor T_{abcd}. In four spatial dimensions it is essentially unique, i.e. $\pm\epsilon_{abcd}$. One immediate consequence of the above equation is that the Yang-Mills field equations $D_a F_{ab} = 0$ are automatically followed due to the Jacobi identity. The constant four form tensor was introduced by hand in order to match the indices of the field strength. On the Euclidean space of dimension $D > 4$, it inevitably breaks the $\mathrm{SO}(D)$ invariance and the resulting first order equations may be classified by the unbroken rotational symmetries [1, 2].

There have been much attention to these higher dimensional 'self-dual' equations. Especially notable ones include the ADHM-like construction in $4k$ dimensions [3] and the octonionic instantons in eight dimensions [4, 5]. The solutions constructed in this context have infinite energy. In fact, by using the conservation of the energy momentum tensor, one can show that there exits no static finite energy solution in Yang-Mills theories on commutative space of dimension $D > 4$ [6].

However, the above theorem does not apply to the noncommutative Yang-Mills theories essentially due to the scale symmetry breaking. In the decoupling limit of string theory with a constant NS-NS B field background, the field theories describing the worldvolume of D branes become non-commutative [7]. Several localized solutions with unbroken supersymmetries have been found to have a finite energy or action in the non-commutative gauge theories [8, 9, 10]. Some exact localized solutions are also constructed by directly solving full field equations [10, 11, 12, 13] and as a result they are not always stable. Thus even with explicit solutions the number of the preserved supersymmetries remains sometimes unclear.

In this note, we shall classify all the possible BPS equations in the higher dimensional Yang-Mills theories, which are eventually related to the first order linear equations above. In particular, we consider the super Yang-Mills theories on Euclidean space of even dimension, D, which may be obtained by a dimensional reduction of the $D = 10$ $N = 1$ super Yang-Mills theory. The $D = 2, 4, 6, 8$ theories are realized as the field theoretic description of D2, D4, D6, and D8 branes. For the supersymmetry counting in the intersecting brane picture see [14, 15].

The methods we employ here is fairly straightforward. The unbroken supersymmetries are basically determined by setting the gaugino variation of the supersymmetry to vanish. Namely, for the unbroken supersymmetries, the gaugino transformation should have non-vanishing kernel or the space of zero eigenspinors. The vacuum solution, for example, is invariant under all supersymmetry transformations and so the dimension of the kernel is maximal. We shall classify all the possible kernels and obtain the corresponding set of BPS equations. This is done by analyzing the projection operator to the kernel and in turn the complete characterization of the above constant four form tensor naturally emerges. One of the technical key observations will be to find out the 'canonical form' for the four form tensor which enables us to figure out the corresponding projection operator for each kernel. In six dimensions, as the four form is dual to a two form, the analysis is relatively simple. In eight dimensions the general constant four form tensor has 70 independent components. We decompose them into chiral and anti-chiral sectors of 35 components. We utilize the SO(8) triality among $\mathbf{8}_{\rm v}$, $\mathbf{8}_{+}$, $\mathbf{8}_{-}$ to show that the SO(8) rotation which has 28 independent parameters can reduce any sector down to 7 independent components but not simultaneously. In general we shall prove that the generic projection operators are built up by the elementary building blocks which lead to the BPS equations for the minimally supersymmetric BPS states. The higher supersymmetric BPS equations then follow from imposing multi-sets of the minimal BPS equations. Our analysis holds for both commutative and non-commutative spaces as well as for both Abelian and non-Abelian gauge groups.

In the field theory on four dimensional space the BPS configurations should satisfy self-dual or anti-self-dual equations carrying 1/2 of the original supersymmetry. In the six dimensional case, the minimal supersymmetric one preserves non-chiral 1/4 of the original supersymmetry. This is the only genuinely six dimensional one. The 1/2 supersymmetric configurations also exist but they are four dimensional in its character. The dimension eight shows more variety. The minimal supersymmetry is chiral 1/16. The bit of octonions appears here. From the minimal we build up to chiral 6/16. For the chirally mixed supersymmetries, the story is more complicated and our analysis may not be complete.

The plan of the paper is as follows: In section 2 we study the general property of the projection operators in even dimensions. We briefly note the two and four dimensional cases. In sections 3 and 4 the complete analysis on six and eight dimensional spaces are presented separately. In section 5, by utilizing the results on the projection operators and the BPS equations, we present some identities which spell the Yang-Mills Lagrangian as a positive definite term plus a total derivative term. This shows the "energy bound" of the BPS configurations as the positive definite term vanishes when the BPS equations are satisfied. In the section 6 we apply our results to some known explicit BPS configurations of finite or infinite energy on commutative or non-commutative spaces. We conclude with some remarks in section 7.

There is one caution in the interpretation of the BPS equations; the fact that a certain configuration satisfies the BPS equations of a given fraction implies that the solution preserves *at least* the fraction of supersymmetries.

2 The BPS Condition

The BPS state refers a field configuration which is invariant under some supersymmetry. In super Yang-Mills theories on Euclidean space of even dimension D, a bosonic configuration is BPS if there exits a nonzero constant spinor, ε of dimension $2^{D/2}$ such that the infinitesimal supersymmetric transformation of the gaugino field vanishes

$$\delta\lambda = F_{ab}\gamma_{ab}\varepsilon = 0\,. \tag{2}$$

Such zero eigenspinors of the $2^{D/2} \times 2^{D/2}$ matrix $F_{ab}\gamma_{ab}$ form the kernel space V. We consider global supersymmetry and so we take the kernel V to be independent of the spatial coordinates. The kernel V could be different from the "local" kernel of $F_{ab}(x)\gamma_{ab}$ at each point x. The dimension of the kernel counts the unbroken supersymmetry of a given BPS configuration. For the vacuum $F_{ab} = 0$, the unbroken supersymmetry is maximal, while for the non-BPS configuration V is simply null.

The BPS field strength should satisfy certain consistency conditions or the BPS equations in order to have a given number of unbroken supersymmetries. The key tool we employ here is the projection operator Ω to the kernel space. With an orthonormal basis for the kernel, $V = \{|l\rangle\}$, $1 \le l \le N = \dim V \le 2^{D/2}$, the projection operator is

$$\Omega = \sum_{l=1}^{N} |l\rangle\langle l|\,, \tag{3}$$

satisfying

$$F_{ab}\gamma_{ab}\Omega = 0\,, \qquad \Omega^2 = \Omega\,, \qquad \Omega^\dagger = \Omega\,. \tag{4}$$

Note that Ω is basis independent or unique, as it is essentially the identity on the kernel space.

In the even dimensional Euclidean space the gamma matrices can be chosen to be Hermitian, $\gamma_a^\dagger = \gamma_a$, and the charge conjugation matrix, C satisfies

$$\gamma_a^T = (\gamma_a)^* = C^\dagger\gamma_a C\,, \qquad CC^\dagger = 1\,. \tag{5}$$

It follows from counting the number of symmetric $2^{D/2} \times 2^{D/2}$ matrices [16]

$$C^T = (-1)^{\frac{1}{8}D(D-2)}\,C\,. \tag{6}$$

We let

$$\gamma_{D+1} = i^{D/2}\,\gamma_{12\cdots D}\,, \qquad {\gamma_{D+1}}^\dagger = \gamma_{D+1}\,, \qquad \gamma_{D+1}^2 = 1\,. \tag{7}$$

Then from

$$[\gamma_{ab}, \gamma_{D+1}] = 0\,, \qquad \gamma_{ab}C = C(\gamma_{ab})^*\,, \tag{8}$$

$|l\rangle \in V$ implies

$$\gamma_{D+1}|l\rangle \in V\,, \qquad C|l\rangle^* \in V\,, \tag{9}$$

so that each of $\{\gamma_{D+1}|l\rangle\}$ and $\{C|l\rangle^*\}$ also forms an orthonormal basis for V separately. Consequently

$$[\gamma_{D+1}, \Omega] = 0\,, \qquad C\Omega^* C^\dagger = \Omega\,. \tag{10}$$

As the anti-symmetric products of the gamma matrices form a basis for the $2^{D/2}\times 2^{D/2}$ matrices, one can expand Ω in terms of them. The first equation in (10) indicates only the even products contribute, and with Ω being Hermitian the second equation implies that Ω should be a sum of foursome products of the gamma matrices with real coefficients

$$\Omega=\nu\left(1+\sum_{1\leq n\leq\frac{D}{4}}\frac{1}{(4n)!}T_{a_1a_2\cdots a_{4n}}\gamma_{a_1a_2\cdots a_{4n}}\right). \tag{11}$$

Furthermore with the chiral and anti-chiral projection operators

$$P_\pm=\frac{1}{2}(1\pm\gamma_{D+1})\,, \tag{12}$$

one can decompose the projection operator as

$$\Omega=\Omega_++\Omega_-\,,\qquad \Omega_\pm\equiv\Omega P_\pm=P_\pm\Omega\,, \tag{13}$$

satisfying

$$\Omega_\pm=\Omega_\pm^2=\Omega_\pm^\dagger\,,\qquad \Omega_\pm\Omega_\mp=0\,. \tag{14}$$

When combined with the charge conjugation,

$$C\Omega_\pm^*C^\dagger=\begin{cases}\Omega_\pm & \text{for } D\equiv 0 \text{ mod } 4\\ \Omega_\mp & \text{for } D\equiv 2 \text{ mod } 4\,.\end{cases} \tag{15}$$

Essentially the action $|v\rangle\rightarrow C|v\rangle^*$ preserves the chirality in $D\equiv 0$ mod 4 while it flips in $D\equiv 2$ mod 4. This implies that in $D\equiv 2$ mod 4, the dimension N of the kernel V must be even and the projection operator can be written as

$$\Omega=\sum_{l=1}^{N/2}(\,|l_+\rangle\langle l_+|+|l_-\rangle\langle l_-|\,)\,,\qquad \gamma_{D+1}|l_\pm\rangle=\pm|l_\pm\rangle\,,\qquad |l_-\rangle=C|l_+\rangle^*\,. \tag{16}$$

Since the eigenvalues of Ω are either 0 or 1 and the non-trivial products of the gamma matrices are traceless, we have

$$N=\mathrm{tr}\,\Omega=\nu\times 2^{D/2}\,. \tag{17}$$

The constant ν denotes the fraction of the unbroken supersymmetry so that $0\leq\nu\leq 1$. The $\nu=0$ or 1 cases are trivial, either meaning the non-BPS state or the vacuum, $F_{ab}=0$.

The remaining constraint to ensure Ω be the projection operator is $\Omega^2=\Omega$. In the below we will focus on obtaining its general solutions in each dimension. Once we solve the constraint completely we are able to obtain all the possible BPS equations. With a given projection operator, the formula, $F_{ab}\gamma_{ab}\Omega=0$ can be expanded as a sum of the anti-symmetric products of even number of gamma matrices. The BPS equations stem from requiring each coefficient of these products to vanish. After a brief review on $D=2,4$ cases, we explore the $D=6,8$ separately. Our analysis holds for both commutative and non-commutative spaces as well as for both Abelian and non-Abelian gauge groups.

In the $D=2$ case the projection matrix is trivial. Either $\Omega=0$ or 1. Thus there is no BPS configuration. Of course this is well known. In the $D=4$ case apart from the trivial ones Ω is given by the chiral or anti-chiral projection operator itself

$$\Omega = \tfrac{1}{2}(1 \pm \gamma_{1234})\,, \tag{18}$$

which gives the usual self-dual equations

$$F_{ab} = \pm\tfrac{1}{2}\,\epsilon_{abcd}F_{cd}\,. \tag{19}$$

3 On Six Dimensional Space

In $D=6$ case as the four form is dual to a two form, we can rewrite the projection operator as

$$\Omega = \nu\left(1+\frac{1}{2}T_{ab}\gamma_{ab}\gamma_{12\cdots 6}\right)\,, \tag{20}$$

from which we get

$$\Omega^2 = \nu^2\left[1+\tfrac{1}{2}T_{ab}T_{ab}+(T_{ab}+\tfrac{1}{8}\,\epsilon_{abcdef}T_{cd}T_{ef})\gamma_{ab}\gamma_{12\cdots 6}\right]\,. \tag{21}$$

The condition $\Omega^2=\Omega$ implies

$$\nu^{-1}-1=\tfrac{1}{2}T_{ab}T_{ab}\,,\qquad (2-\nu^{-1})T_{ab}+\tfrac{1}{4}\,\epsilon_{abcdef}T_{cd}T_{ef}=0\,. \tag{22}$$

For a nontrivial $\nu\neq 0$, the supersymmetric condition becomes

$$0=F_{ab}\gamma_{ab}\Omega=\nu(F_{ab}+\frac{1}{4}\epsilon_{abcdef}T_{cd}F_{ef})\gamma_{ab}-\nu(2F_{ac}T_{bc}\gamma_{ab}+F_{ab}T_{ab})\gamma_{12\cdots 6}\,, \tag{23}$$

so that the BPS states satisfy

$$F_{ab}+\frac{1}{4}\epsilon_{abcdef}T_{cd}F_{ef}=0\,, \tag{24}$$

$$F_{ac}T_{bc}-F_{bc}T_{ac}=0\,, \tag{25}$$

$$F_{ab}T_{ab}=0\,. \tag{26}$$

The first equation guarantees the on shell condition, $D_aF_{ab}=0$ due to the Jacobi identity. The others follow from the first equation and the conditions (22) as we shall see below.

To solve Eq.(22) we use the global SO(6) transformations to rotate the real two form T_{ab} to the block diagonal canonical form so that the only non-vanishing components are $T_{12}=-T_{21}$, $T_{34}=-T_{43}$ and $T_{56}=-T_{65}$. In terms of these variables the constraints (22) become

$$\begin{array}{ll}\nu^{-1}-1=T_{12}^2+T_{34}^2+T_{56}^2\,, & (\frac{1}{2}\nu^{-1}-1)T_{12}-T_{34}T_{56}=0\,,\\[2ex] (\frac{1}{2}\nu^{-1}-1)T_{34}-T_{56}T_{12}=0\,, & (\frac{1}{2}\nu^{-1}-1)T_{56}-T_{12}T_{34}=0\,.\end{array} \tag{27}$$

Solving the above equations is straightforward. The nontrivial solutions are $\nu = 1/4,\ 2/4,\ 3/4$. The dimension of the kernel N is then $2, 4, 6$. There is no projection operator of odd dimension in six dimensions as argued after Eq.(15).

For the $\nu = 1/4$ case the two form tensor is

$$T_{12} = \alpha_1\alpha_2\,, \qquad T_{34} = \alpha_1\,, \qquad T_{56} = \alpha_2\,, \tag{28}$$

where α_1, α_2 are two independent signs

$$\alpha_1^2 = \alpha_2^2 = 1\,. \tag{29}$$

There are four possibilities of $\alpha = (\alpha_1, \alpha_2)$ as $(++)$, $(+-)$, $(-+)$, $(--)$, and for each α there exists a corresponding projection operator which is orthogonal to each other

$$\Omega_\alpha \Omega_{\alpha'} = \delta_{\alpha\alpha'}\Omega_\alpha\,. \tag{30}$$

They are also complete as

$$\sum_\alpha \Omega_\alpha = 1_{8\times 8}\,. \tag{31}$$

Explicitly the BPS equations (24) become

$$\begin{array}{cc} \multicolumn{2}{c}{F_{12} + \alpha_2 F_{34} + \alpha_1 F_{56} = 0\,,} \\ F_{13} + \alpha_2 F_{42} = 0\,, & F_{14} + \alpha_2 F_{23} = 0\,, \\ F_{15} + \alpha_1 F_{62} = 0\,, & F_{16} + \alpha_1 F_{25} = 0\,, \\ F_{35} + \alpha_1\alpha_2 F_{64} = 0\,, & F_{36} + \alpha_1\alpha_2 F_{45} = 0\,. \end{array} \tag{32}$$

The remaining equations (25) and (26) are equivalent to the above equations and so do not provide any additional restriction.

To analyze the general BPS states of $\nu = 1/2,\ 3/4$ we first note that $\gamma_{2s-1\,2s}\gamma_{123456}$, $s = 1, 2, 3$ can be written as a linear combination of the $\nu = 1/4$ projection operators

$$\gamma_{2s-1\,2s}\gamma_{123456} = \Omega_{++} + (-1)^s\Omega_{+-} + (-1)^{\frac{1}{2}s(s+1)}\Omega_{-+} + (-1)^{\frac{1}{2}s(s-1)}\Omega_{--}\,. \tag{33}$$

Thus the general $\nu = N/8$ projection operator in the canonical form is also a linear combination, in fact $N/2$ sum of the Ω_α's as in (16).

The $\nu = 1/2$ projection operator is a sum of any two different $1/4$ BPS projection operators, $\Omega_{1/2} = \Omega_\alpha + \Omega_{\alpha'}$. The BPS equations for this $\Omega_{\frac{1}{2}}$ are naturally obtained by imposing two sets of the $\nu = 1/4$ conditions (32) with α and α'. For example with $\alpha = (++)$ and $\alpha' = (-+)$, the non-vanishing two form components are $T_{56} = -T_{65} = 1$ so that the projection operator is $\Omega_{1/2} = \frac{1}{2}(1 - \gamma_{1234})$ and the BPS equations are those of the four dimensional $1/2$ BPS equations on the first four indices while the rest of the field strength vanishes

$$F_{12} + F_{34} = 0\,, \qquad F_{13} + F_{42} = 0\,, \qquad F_{14} + F_{23} = 0\,, \qquad F_{a5} = F_{a6} = 0\,. \tag{34}$$

One can see a different choice of α and α' leads to a self-dual or anti-self-dual BPS equations on a different four dimensional subspace. Essentially different choices are related by the $\mathrm{O}(6)$ rotations.

The $\nu = 3/4$ projection operator can be built from a sum of any three different $1/4$ BPS projection operators. Each of them imposes the $1/4$ BPS equations on the field strength. Nevertheless three sets of BPS equations are too much and the only possible solution is the vacuum, $F_{ab} = 0$.

Attempts to construct ADHM-like solutions of the 1/4 BPS equations (32) would end up with the 1/2 BPS solutions which are essentially four dimensional in its character [17]. This is of course consistent with the string theory prediction that there exists no D0-D6 bound state without the background B field.

4 On Eight Dimensional Space

In $D = 8$, the projection operator can be split to chiral, $+$ and anti-chiral, $-$ parts

$$\Omega_\pm = \nu(2P_\pm - \frac{1}{4!}T_{\pm abcd}\gamma_{abcd})\,, \tag{35}$$

where $T_{\pm abcd}$ is a self-dual or anti-self-dual four form tensor depending on the chirality, $\pm$

$$T_{\pm abcd} = \pm\frac{1}{4!}\epsilon_{abcdefgh}T_{\pm efgh}\,, \tag{36}$$

so that $T_{\pm abcd}\gamma_{abcd}\gamma_9 = \pm T_{\pm abcd}\gamma_{abcd}$. The possible ν values are $\nu = N/16, \;\; N = 1, 2, \cdots 8$.

The square of $\Omega_\pm$ can be simplified by using the self-duality of the four form tensor and noticing, for example $T_{abcd}T_{abce}\gamma_{de} = 0$ due to the symmetries of the term. We get

$$\Omega_\pm^2 = \nu^2\left[2(2 + \frac{1}{4!}T_{\pm abcd}T_{\pm abcd})P_\pm - (\frac{1}{6}T_{\pm abcd} + \frac{1}{8}T_{\pm abef}T_{\pm cdef})\gamma_{abcd}\right]. \tag{37}$$

The condition $\Omega_\pm^2 = \Omega_\pm$ implies

$$\begin{array}{c} \nu^{-1} = 2 + \frac{1}{4!}T_{\pm abcd}T_{\pm abcd}\,, \\ \\ (\nu^{-1} - 4)T_{\pm abcd} = T_{\pm abef}T_{\pm cdef} + T_{\pm acef}T_{\pm dbef} + T_{\pm adef}T_{\pm bcef}\,. \end{array} \tag{38}$$

On the other hand the supersymmetric condition reads

$$0 = F_{ab}\gamma_{ab}\Omega_\pm = 2\nu(F_{ab} + \frac{1}{2}T_{\pm abcd}F_{cd})\gamma_{ab}P_\pm - \frac{1}{3}\nu F_{ae}T_{\pm ebcd}\gamma_{abcd}\,, \tag{39}$$

so that the BPS states satisfy

$$F_{ab} + \frac{1}{2}T_{\pm abcd}F_{cd} = 0\,, \tag{40}$$

$$F_{ae}T_{\pm ebcd} + F_{be}T_{\pm ecad} + F_{ce}T_{\pm eabd} + F_{de}T_{\pm ecba} = 0\,. \tag{41}$$

As shown in Eq.(A.4), the second equation follows from the first one with the properties of the four form tensor (38).

Solving Eq.(38) utilizes the SO(8) triality among $\mathbf{8}_\mathrm{v}$, $\mathbf{8}_+$, $\mathbf{8}_-$ which will enable us to rotate the self-dual or anti-self-dual four form tensor to a 'canonical form'. Eight dimensional Euclidean space admits Majorana spinors so that one can take the gamma matrices to be real and symmetric i.e. $C=1$. Further, the choice, $\gamma_9 = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ forces the gamma matrices to be off block diagonal

$$\gamma_a = \left(\begin{array}{cc} 0 & \rho_a \\ \rho_a^T & 0 \end{array}\right) . \tag{42}$$

The real 8×8 matrix ρ_a satisfies

$$\rho_a \rho_b^T + \rho_b \rho_a^T = 2\delta_{ab} , \tag{43}$$

which in turn implies $\rho_a \in \mathrm{O}(8)$ and $\rho_a^T\rho_b + \rho_b^T\rho_a = 2\delta_{ab}$. With the above choice of gamma matrices the spinors are real, and the traceless chiral matrix $T_\pm \equiv \frac{1}{4!}T_{\pm abcd}\gamma_{abcd}$ is also real and symmetric. Consequently $T_\pm$ and $\Omega_\pm$ are diagonalizable by the SO(8) transformations of the eight dimensional chiral or anti-chiral real spinors, $\mathbf{8}_\pm$.

The SO(8) triality is apparent when we write the $\mathbf{8}_\mathrm{v}$ generators in the basis

$$\gamma_{ab} = \left(\begin{array}{cc} \rho_{[a}\rho_{b]}^T & 0 \\ 0 & \rho_{[a}^T\rho_{b]} \end{array}\right) . \tag{44}$$

Clearly $\rho_{[a}\rho_{b]}^T$ and $\rho_{[a}^T\rho_{b]}$ are the $\mathbf{8}_+$ and $\mathbf{8}_-$ generators respectively. Thus T_+ or T_- can be diagonalized, though not simultaneously, by a SO(8) transformation of the vectors, $\mathbf{8}_\mathrm{v}$.

To proceed further it is convenient to define the following seven quantities

$$\begin{array}{llll} E_{\pm 1} = \gamma_{8127}P_\pm , & E_{\pm 2} = \gamma_{8163}P_\pm , & E_{\pm 3} = \gamma_{8246}P_\pm , & E_{\pm 4} = \gamma_{8347}P_\pm , \\ & & & \\ E_{\pm 5} = \gamma_{8567}P_\pm , & E_{\pm 6} = \gamma_{8253}P_\pm , & E_{\pm 7} = \gamma_{8154}P_\pm . & \end{array} \tag{45}$$

Here we organize the subscript spatial indices of the gamma matrices such that the three indices after the common 8 are identical to those of the totally anti-symmetric octonion structure constants (e.g. [18])

$$\begin{array}{c} e_i e_j = -\delta_{ij} + c_{ijk}\, e_k , \qquad i,j,k = 1,2,\cdots,7 \\ \\ 1 = c_{127} = c_{163} = c_{246} = c_{347} = c_{567} = c_{253} = c_{154} \quad (\text{others zero}) . \end{array} \tag{46}$$

It is easy to see that $E_{\pm i}$ forms a representation for the "square" of the octonions

$$E_{\pm i}E_{\pm j} = \delta_{ij} \pm c_{ijk}^2\, E_{\pm k} , \qquad E_{\pm i} \equiv \pm e_i \otimes e_i . \tag{47}$$

As they commute each other, they form a maximal set of the mutually commuting traceless symmetric and real matrices of a definite chirality, $\gamma_9 E_{\pm i} = \pm E_{\pm i}$. Thus, again using the SO(8) transformations one can choose an orthonormal real basis where $E_{\pm i}$'s are simultaneously diagonal.

All together, $\mathbf{8}_\pm$ can transform $T_\pm$ to a linear combination of $E_{\pm i}$'s. The $\mathrm{SO}(8)$ triality (44) then implies that $\mathbf{8}_\mathrm{v}$ can rotate the self-dual or anti-self-dual four form tensor to a 'canonical form' where the non-vanishing components are only seven as $T_{\pm 1278}$, $T_{\pm 1638}$, $T_{\pm 2468}$, $T_{\pm 3478}$, $T_{\pm 5678}$, $T_{\pm 2538}$, $T_{\pm 1548}$ up to permutations and the duality. This is consistent with the parameter counting as $35 = 28 + 7$, where 35 is the number of independent self-dual or anti-self-dual components $T_{\pm abcd}$, and 28 is the dimension of $\mathrm{so}(8)$.

Apparently for a general $\nu = N/16$ BPS states of a definite chirality, the corresponding projection operator $\Omega_\pm$ is invariant under the $\mathrm{SO}(N) \times \mathrm{SO}(8-N)$ subgroup of $\mathbf{8}_\pm$. The $\mathrm{SO}(8)$ triality then shows that the self-dual or anti-self-dual four form tensor is invariant under $\mathrm{SO}(N) \times \mathrm{SO}(8-N)$ subgroup of $\mathbf{8}_\mathrm{v}$.

(1) $\nu = 1/16$, $\mathrm{SO}(7)$

For the minimal case $\nu = 1/16$, as shown in the appendix, the projection operator is of the general form

$$\Omega_\pm = \frac{1}{8}\Big[P_\pm \pm (\alpha_1\alpha_2 E_{\pm 1} + \alpha_1\alpha_3 E_{\pm 2} + \alpha_3 E_{\pm 3} + \alpha_2 E_{\pm 4} + \alpha_1 E_{\pm 5} + \alpha_1\alpha_2\alpha_3 E_{\pm 6} + \alpha_2\alpha_3 E_{\pm 7})\Big] , \quad (48)$$

where $\alpha_1, \alpha_2, \alpha_3$ are three independent signs

$$1 = \alpha_1^2 = \alpha_2^2 = \alpha_3^2 \, . \quad (49)$$

The corresponding $1/16$ BPS states satisfy

$$\begin{array}{l}
F_{12} + \alpha_1 F_{34} + \alpha_2 F_{56} \pm \alpha_1\alpha_2 F_{78} = 0 \, , \\
F_{13} + \alpha_1 F_{42} + \alpha_3 F_{57} \pm \alpha_1\alpha_3 F_{86} = 0 \, , \\
F_{14} + \alpha_1 F_{23} + \alpha_1\alpha_2\alpha_3 F_{76} \pm \alpha_2\alpha_3 F_{85} = 0 \, , \\
F_{15} + \alpha_2 F_{62} + \alpha_3 F_{73} \pm \alpha_2\alpha_3 F_{48} = 0 \, , \\
F_{16} + \alpha_2 F_{25} + \alpha_1\alpha_2\alpha_3 F_{47} \pm \alpha_1\alpha_3 F_{38} = 0 \, , \\
F_{17} + \alpha_3 F_{35} + \alpha_1\alpha_2\alpha_3 F_{64} \pm \alpha_1\alpha_2 F_{82} = 0 \, , \\
\pm F_{18} + \alpha_1\alpha_2 F_{27} + \alpha_1\alpha_3 F_{63} + \alpha_2\alpha_3 F_{54} = 0 \, .
\end{array} \quad (50)$$

They are seven BPS equations for 28 components of F_{ab}, each of which appears once.

Especially when $\alpha_1 = \alpha_2 = \alpha_3 = 1$,

$$T_{\pm ijk8} = \pm c_{ijk} \qquad \text{and} \qquad F_{i8} \pm \tfrac{1}{2} c_{ijk} F_{jk} = 0 \, . \quad (51)$$

Three independent signs leads to eight possible combinations, covering all chiral or anti-chiral spinor spaces. For each set of $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ there exists a corresponding zero eigenspinor, say $|\alpha_\pm\rangle$ which forms an orthonormal real basis for the chiral or anti-chiral spinor spaces. Accordingly the projection operator (48) can be rewritten as $\Omega_{\pm\alpha} = |\alpha_\pm\rangle\langle\alpha_\pm|$ satisfying the orthogonal completeness

$$\sum_\alpha \Omega_{\pm\alpha} = P_\pm \,, \qquad \Omega_{\pm\alpha}\Omega_{\pm\alpha'} = \delta_{\alpha\alpha'}\Omega_{\pm\alpha} \,. \tag{52}$$

From Eq.(48) $E_{\pm i}$ can be expressed as a linear combination of $\Omega_{\pm\alpha}$'s and hence they are diagonal in the basis. Consequently the general $\nu = N/16$ projection operator in the canonical form is an N sum of the $\Omega_{\pm\alpha}$'s. Furthermore, from the triality, the $\frac{8!}{N!(8-N)!}$ possibilities for the N sum are equivalent to each other up to $\mathrm{SO}(8)$. Higher supersymmetric BPS states then satisfy N copies of the $1/16$ BPS equations of different α choices.

(2) $\nu = 2/16$, $\mathrm{SO}(2)\times\mathrm{SO}(6)$
With the α choices as $(+++),\ (++-)$

$$\begin{array}{ccc}
 & F_{12}+F_{34}+F_{56}\pm F_{78}=0\,, & \\
F_{13}+F_{42}=0\,, & F_{57}\pm F_{86}=0\,, & F_{15}+F_{62}=0\,, \\
F_{14}+F_{23}=0\,, & F_{76}\pm F_{85}=0\,, & F_{16}+F_{25}=0\,, \\
F_{73}\pm F_{48}=0\,, & F_{17}\pm F_{82}=0\,, & F_{35}+F_{64}=0\,, \\
F_{47}\pm F_{38}=0\,, & F_{18}\pm F_{27}=0\,, & F_{63}+F_{54}=0\,.
\end{array} \tag{53}$$

(3) $\nu = 3/16$, $\mathrm{SO}(3)\times\mathrm{SO}(5)$
With the α choices as $(+++),\ (++-),\ (+-+)$

$$\begin{array}{ccc}
F_{12}+F_{34}=0\,, & F_{13}+F_{42}=0\,, & F_{14}+F_{23}=0\,, \\
F_{56}\pm F_{78}=0\,, & F_{75}\pm F_{68}=0\,, & F_{67}\pm F_{58}=0\,, \\
 & F_{15}=F_{26}=F_{37}=\pm F_{48}\,, & \\
 & F_{16}=F_{52}=F_{47}=\pm F_{83}\,, & \\
 & F_{17}=F_{53}=F_{64}=\pm F_{28}\,, & \\
 & \pm F_{18}=F_{72}=F_{36}=F_{54}\,. &
\end{array} \tag{54}$$

Some relevant references include [19, 20].

(4) $\nu = 4/16$, $\mathrm{SO}(4)\times\mathrm{SO}(4)$
With the α choices as $(+++)$, $(++-)$, $(+-+)$, $(+--)$

$$\begin{array}{ccc} F_{12}+F_{34}=0\,, & F_{13}+F_{42}=0\,, & F_{14}+F_{23}=0\,, \\ F_{56}\pm F_{78}=0\,, & F_{75}\pm F_{68}=0\,, & F_{67}\pm F_{58}=0\,, \\ \multicolumn{3}{c}{F_{ab}=0 \quad \text{for} \quad a\in\{1,2,3,4\} \quad b\in\{5,6,7,8\}\,.} \end{array} \tag{55}$$

(5) $\nu = 5/16$, $\mathrm{SO}(5)\times\mathrm{SO}(3)$
With the α choices as $(+++)$, $(++-)$, $(+-+)$, $(+--)$, $(-++)$

$$\begin{array}{c} F_{12}=F_{43}=F_{65}=\pm F_{78}\,, \\ F_{13}=F_{24}=F_{75}=\pm F_{86}\,, \\ F_{14}=F_{32}=F_{76}=\pm F_{58}\,, \\ F_{ab}=0 \quad \text{for} \quad a\in\{1,2,3,4\} \quad b\in\{5,6,7,8\}\,. \end{array} \tag{56}$$

(6) $\nu = 6/16$, $\mathrm{SO}(6)\times\mathrm{SO}(2)$
With the α choices as $(+++)$, $(++-)$, $(+-+)$, $(+--)$, $(-++)$, $(-+-)$

$$F_{12}=F_{43}=F_{65}=\pm F_{78}\,, \tag{57}$$

and other components are zero.

(7) The $7/16$ BPS states do not exist, since the seven sets of $1/16$ BPS equations have only the vacuum solution, $F_{ab}=0$ which does not break any supersymmetry.

We have not analyzed the generic BPS states having both chiralities, $\Omega=\Omega_+ + \Omega_-$, $\Omega_+\neq 0$, $\Omega_-\neq 0$. In general, the global $\mathrm{SO}(8)$ transformations can take only one of Ω_+, Ω_- to the canonical form, but not both simultaneously. Nevertheless, the special case where both projection operators are in the canonical form is manageable from our results. One can check that the case involves a dimensional reduction. For example, for $\nu=(\frac{1}{16})_+ + (\frac{1}{16})_-$ with $\alpha=(+++)$ we get

$$\begin{array}{ll} F_{12}+F_{34}+F_{56}=0\,, & F_{13}+F_{42}+F_{57}=0 \\ F_{14}+F_{23}+F_{76}=0\,, & F_{15}+F_{62}+F_{73}=0\,, \\ F_{16}+F_{25}+F_{47}=0\,, & F_{17}+F_{35}+F_{64}=0\,, \\ F_{27}+F_{63}+F_{54}=0\,, & F_{a8}=0\,. \end{array} \tag{58}$$

This is essentially seven dimensional.

5 Energy Bound

Pure Yang-Mills theories in the Minkowski spacetime of the dimension $D+1$, D for space, admit no local static solution having finite energy when $D \neq 4$, which can be seen easily from a scaling argument [6]. However this is for the commutative case and on the non-commutative space local static configurations can exist. In this section we present some identities for $\frac{1}{4}\mathrm{tr}(F_{ab}^2)$ on both six and eight dimensional space which will show the energy "bound" of the BPS states.

(1) $D=6$

Using the two form tensor solution of the minimal case $\nu = 1/4$, by noticing for example $T_{ab}T_{bc} = -\delta_{ac}$, one can obtain the following identity for the generic configurations

$$\frac{1}{4}\mathrm{tr}(F_{ab}^2) = \frac{1}{8}\mathrm{tr}(F_{ab} + \frac{1}{4}\epsilon_{abcdef}T_{cd}F_{ef} + \kappa T_{ab}T_{cd}F_{cd})^2 - \frac{1}{16}\epsilon_{abcdef}T_{ab}\mathrm{tr}(F_{cd}F_{ef}) \,, \tag{59}$$

where $\kappa = -\frac{1}{2} \pm \frac{1}{\sqrt{6}}$. As usual, with the convention $F_{ab} = \partial_a A_b - \partial_b A_a - i[A_a, A_b]$ the last term is topological as

$$-\frac{1}{16}\epsilon_{abcdef}T_{ab}\mathrm{tr}(F_{cd}F_{ef}) = -\frac{1}{4}\epsilon_{abcdef}T_{ab}\,\partial_c\mathrm{tr}\left(A_d\partial_e A_f - i\frac{2}{3}A_dA_eA_f\right) . \tag{60}$$

From Eq.(22) the vanishing of the first term of the right hand side of (59) actually implies the $1/4$ BPS equation (24) itself and vice versa. As any BPS state satisfies at least one set of the $1/4$ BPS equations, the above equation shows the energy bound of the generic BPS states.

(2) $D=8$

With the four form tensor of the minimal case $\nu = 1/16$, a straightforward manipulation gives using (A.6) and (A.12)

$$\begin{aligned} \frac{1}{4}\mathrm{tr}(F_{ab}^2) &= \frac{1}{16}\mathrm{tr}\left(F_{ab} + \frac{1}{2}T_{\pm abcd}F_{cd}\right)^2 - \frac{1}{8}T_{\pm abcd}\mathrm{tr}(F_{ab}F_{cd}) \\ &= \frac{1}{16}\mathrm{tr}\left(F_{ab} + \frac{1}{2}T_{\pm abcd}F_{cd}\right)^2 - \frac{1}{2}T_{\pm abcd}\,\partial_a\mathrm{tr}\left(A_b\partial_c A_d - i\frac{2}{3}A_bA_cA_d\right) . \end{aligned} \tag{61}$$

As in $D=6$, since any BPS state satisfies at least one set of $1/16$ BPS equations, this equation shows the energy bound of the generic BPS states.

Apart from the above quadratic topological charge, there is another topological charge

$$Q = \frac{1}{(2\pi)^{D/2}(D/2)!}\int \mathrm{tr}(\underbrace{F\wedge F\wedge\cdots\wedge F}_{D/2 \ \text{product}}) \,. \tag{62}$$

which does not need to be related to the above quadratic topological charge directly.

6 Examples in $D = 6$ and $D = 8$

Here we comment on the known solutions of pure Yang-Mills theories on commutative or non-commutative spaces with gauge group $\mathrm{U}(1)$ or non-Abelian. As stated before, there is no static finite energy configurations on commutative space of dimension higher than four.

6.1 Octonionic Instantons

In eight dimensional Euclidean space, with the gauge group $SO(7)^+$ as a subgroup of $SO(8)$, an explicit solution of the octonionic BPS equation (51) has been found [4]. It has infinite energy, but its four-form charge over the corresponding four dimensional hyperplane is finite. Namely the integration of $\mathrm{tr}(F_{ab}F_{cd})\,\mathrm{d}x_a \wedge \mathrm{d}x_b \wedge \mathrm{d}x_c \wedge \mathrm{d}x_d$ over (x_a, x_b, x_c, x_d) hyperplane is finite and in fact proportional to $T_{\pm abcd}$. Our analysis in the previous section tells us that it has $1/16$ supersymmetry.

6.2 Constant Field Strength

One can think of a constant field strength and so they are independent of space. Let us consider the $\mathrm{U}(1)$ theory first. Strictly speaking, in this case there exit non-linearly realized additional supersymmetries in the super Yang-Mills theories where the gaugino transforms as $\delta\lambda = \varepsilon'$ with the gauge fields fixed. As a result any constant field strength preserves all the supersymmetries. Let us then consider the $\mathrm{SU}(2)$ case where all the the constant fields are diagonal i.e. $F_{ab} = f_{ab}\sigma_3$ with σ_3 being the third Pauli matrix. The additional supersymmetries do not play any role here. In this special case one may not have to go through all the previous projection operator analysis. Using the global rotation, $\mathrm{O}(D)$, one can block diagonalize the field strength so that $f_{ab}\gamma_{ab} = \sum_{s=1}^{D/2} 2f_{2s-1\,2s}\gamma_{2s-1\,2s}$. As the $\gamma_{2s-1\,2s}$, $s = 1, 2, \cdots, D/2$ are commuting each other and have eigenvalues ± 1, the constant configuration, $f_{2s-1\,2s} \equiv f_s$ is BPS if and only if

$$\pm f_1 \pm f_2 \pm f_3 \pm \cdots \pm f_{D/2} = 0\,. \tag{63}$$

The number of possible sign combinations out of $2^{D/2}$ choices counts the number of unbroken supersymmetries. The multiplication of all the signs determines the chirality of the corresponding zero eigenspinor. Due to the freedom to flip the over all signs, the number of unbroken supersymmetries is always even. This over all sign change leaves the chirality invariant for even $D/2$, while it flips for odd $D/2$. Surely this is consistent with the results in section 2, and Eq.(63) corresponds to (32) or (53).

More explicitly, fully using the $\mathrm{O}(D)$ rotation we arrange $f_1 \geq f_2 \geq f_3 \geq \cdots \geq f_{D/2} \geq 0$. For $D = 6$ the configuration is BPS if $f_1 - f_2 - f_3 = 0$. As there are two possible over all signs, it corresponds to the $1/4$ BPS state. For $D = 8$ there are several possibilities. For $f_1 > f_2 \geq f_3 \geq f_4 > 0$, it can only satisfy either $f_1 - f_2 - f_3 + f_4 = 0$ or $f_1 - f_2 - f_3 - f_4 = 0$, which has $1/8$ supersymmetry with the positive or negative chirality respectively. For $f_1 > f_2 \geq f_3 > f_4 = 0$, it satisfies both so that $\nu = 1/4$. For $f_1 = f_2 > f_3 = f_4 > 0$, it can satisfy

$f_1 - f_2 \pm (f_3 - f_4) = 0$, which has also 4 supersymmetries and hence $\nu = 1/4$ with positive chirality. For $f_1 = f_2 > f_3 = f_4 = 0$, we have $\nu = 1/2$. Finally for $f_1 = f_2 = f_3 = f_4 > 0$, it can satisfy $f_1 - f_2 + f_3 - f_4 = 0$ and others so that it has six supersymmetries of the positive chirality. This agrees with the previous $3/8$ BPS equations (57).

For the non-Abelian gauge group, we have $F_{ab} = F^{\alpha}_{ab} t^{\alpha}$ and one can not block diagonalize all of F^{α}_{ab}'s simultaneously by a single $\mathrm{SO}(D)$ rotation in general. Nevertheless, we point out that if the gauge group is semi-simple of rank two or higher, a constant field strength configuration in eight dimensions can be a $1/16$ BPS state. For example, if $[t^1, t^2] = 0$, one can construct a non-Abelian constant field strength of which the non-vanishing components are $F^1_{12}, F^1_{34}, F^1_{56}, F^1_{78}$ and $F^2_{13}, F^2_{42}, F^2_{57}, F^2_{86}$ only up to the anti-symmetric property. When F^1_{ab} and F^2_{ab} satisfy the first and second equations of the $1/16$ BPS equations (50) respectively with a unique choice of α, it is certainly a $1/16$ BPS state.

6.3 Non-commutative Exact Solutions

The non-commutative space is specified by the commutation relation

$$[x_a, x_b] = i\theta_{ab}\,. \tag{64}$$

Using the global $\mathrm{O}(D)$ rotation one can take the block diagonal canonical form for θ so that the non-vanishing components are $\theta_{2s-1\,2s} \equiv \theta_s > 0$, $s = 1, 2, \cdots, D/2$ only up to the anti-symmetric property. This choice clearly manifests the $D/2$ pairs of harmonic oscillators, $[a_s, a^{\dagger}_r] = \delta_{sr}$, $a_s = \frac{1}{\sqrt{2\theta_s}}(x_{2s-1} + ix_{2s})$.

On non-commutative space any $\mathrm{U}(n)$ gauge theories are equivalent to another, in particular to a $\mathrm{U}(1)$ theory provided that all the fields are in the adjoint representation [21]. Hence we restrict on the $\mathrm{U}(1)$ gauge group only here without loss of generality. Writing $A_a = Y_a - \theta^{-1}_{ab} x_b$ gives

$$F_{ab} = \theta^{-1}_{ab} - i[Y_a, Y_b]\,. \tag{65}$$

1) Shift Operator Solitons
The shift operator, S in the harmonic oscillator Hilbert space is almost unitary

$$SS^{\dagger} = 1\,, \qquad S^{\dagger}S = 1 - P_0\,. \tag{66}$$

where P_0 is a projection operator of finite dimension to the Hilbert space. The shift operator solution satisfying $D_a F_{ab} = 0$ reads [8, 9, 11, 12]

$$Y_a = \theta^{-1}_{ab}\, S^{\dagger} x_b S \qquad \Longrightarrow \qquad F_{ab} = \theta^{-1}_{ab}\, P_0\,. \tag{67}$$

Similar to the constant solutions, this localized field configuration is supersymmetric if and only if

$$\pm\frac{1}{\theta_1} \pm \frac{1}{\theta_2} \pm \frac{1}{\theta_3} \pm \cdots \pm \frac{1}{\theta_{D/2}} = 0\,, \tag{68}$$

and hence the counting of the supersymmetry proceeds identically as in the constant solutions [9, 22]. The energy of the solution is

$$E_0 = \frac{1}{2e^2}(2\pi)^{D/2}\left(\prod_{s=1}^{D/2}\theta_s\right)\left(\frac{1}{\theta_1^2}+\frac{1}{\theta_2^2}+\cdots+\frac{1}{\theta_{D/2}^2}\right)\mathrm{Tr}P_0\,, \tag{69}$$

and the D-form topological charge is

$$Q_0 = \left(\prod_{s=1}^{D/2}\theta_s\right)\mathrm{Tr}(F_{12}F_{34}\cdots F_{D-1\,D}) = (-1)^{D/2}\,\mathrm{Tr}P_0\,. \tag{70}$$

2) Kraus and Shigemori Solutions

From (65) $D=6$ $\nu=1/4$, or $D=8$ $\nu=1/8$ BPS equations (32,53) are equivalent, with the positive α choice, to

$$\sum_{s=1}^{D/2}[Z_s,\bar{Z}_s] = \sum_{s=1}^{D/2}\frac{1}{\theta_s}\,, \qquad\qquad [Z_s,Z_r]=0\,, \tag{71}$$

where we complexified Y_a so that $Z_s=\frac{1}{\sqrt{2}}(Y_{2s-1}+iY_{2s})$, $s,r=1,2,\cdots,D/2$.

Kraus and Shigemori found a class of finite energy configurations satisfying $D_aF_{ab}=0$ in the special case where all the θ_s's are equal, $\theta_s=\theta$. Their solutions are specified by two non-negative numbers, $L\geq l\geq 0$ and the energy is minimized when $L=l$. In this case their solution reduces to

$$Z_r = \frac{1}{\sqrt{\theta}}\,Sf(N)\,a_rS^\dagger\,, \tag{72}$$

where with total number operator, $N=\sum_s a_s^\dagger a_s$,

$$f(N)=\sqrt{1-\prod_{j=1}^{D/2}\left(\frac{l+j}{N+j}\right)}\,, \tag{73}$$

and S is a shift operator satisfying $1-S^\dagger S=P_l$, a projection operator to the states, $N\leq l$. It is straightforward to show that this is a solution of the above non-commutative BPS equation (71) having $\nu=(\frac{1}{2})^{(D-2)/2}$ supersymmetry. Its energy is

$$E_{K\text{-}S} = \frac{1}{8e^2}D(D-2)(2\pi)^{D/2}\theta^{(D-4)/2}\,\mathrm{Tr}P_l\,, \tag{74}$$

while the D-form topological charge is

$$Q_{K\text{-}S} = -(-1)^{D/2}\,\mathrm{Tr}P_l\,. \tag{75}$$

This is exactly the "opposite" of Q_0 in (70).

3) Comparison

Let us now compare the shift operator solutions with the K-S solutions. First from (69) and (74) we get $E_{K\text{-}S} \geq E_0$. The equality holds only in $D = 4$, and for $D = 6, 8$ the K-S solitons are heavier. In six dimensions both of them can have two supersymmetries or 1/4 of the original one. However the θ_{ab} condition of the K-S solutions, $\theta_1 = \theta_2 = \theta_3$ forbid the shift operator solution from being supersymmetric, which would require $1/\theta_1 = 1/\theta_2 + 1/\theta_3$. In eight dimensions both can be supersymmetric at the same time. The shift operator solution carries $3/8$ of the original supersymmetry while the K-S solution carries $1/8$. As their topological charges are "opposite", they can be thought as $D0 - D8$ and $\overline{D0} - D8$ systems.

7 Conclusion

In this note, we considered the maximally supersymmetric pure Yang-Mills theories on Euclidean space of six and eight dimensions. We determine, in a systematic way, all the possible fractions of the supersymmetry preserved by the BPS states and present the corresponding BPS equations. In six dimensions, the intrinsic one has 1/4 of supersymmetry while the 1/2 BPS states here are essentially four dimensional. In eight dimensions, the minimal fraction is chiral 1/16. Within chiral or anti-chiral sector one could build up to 6/16 without dimensional reduction. We have applied our BPS equations to various known solutions, counted the numbers of supersymmetries and compared their topological charges.

In our eight dimensional analysis we decomposed the constant four form tensor into chiral and anti-chiral sectors as the BPS conditions work separately. Using the SO(8) triality, we were able to bring any chiral sector into the canonical form, but not both simultaneously. Considering both chiralities in the canonical form simultaneously, we have obtained the BPS equations that are essentially seven dimensional in its character. There is no reason not to consider the generically mixed chiralities where only one sector is in the canonical form. Further clarification is necessary in this direction.

As we considered the Euclidean pure Yang-Mills theories in even dimensions, we turned on only the magnetic field strength for the D brane systems and did not included the effect of electric fields. One such example involving the electric fields is the BPS equations describing dyons in three spatial dimensions. In this respect it would be interesting to generalize our results to the case where non-vanishing electric fields are allowed. Others we did not consider includes the super Yang-Mills theory of nine spatial dimensions. From the view point of the dimensional reduction, this corresponds to the eight dimensional theory plus one adjoint Higgs. Our project will be complete in some sense if one is able to classify all the possible BPS equations in the $9+1$ dimensional super Yang-Mills theory. Nevertheless our eight dimensional results are ready for the reduction to the lower dimensions to generate some adjoint Higgs.

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Appendix

There is an identity for an arbitrary self-dual or anti-self-dual four form tensor in $D=8$. From the relation

$$
\begin{aligned}
T_{\pm acde}T_{\pm bcde} &= \left(\frac{1}{4!}\right)^2 \epsilon_{acdefghi}\epsilon_{bcdejklm}T_{\pm fghi}T_{\pm jklm} \\
&= \frac{1}{4}\,\delta_{ab}T_{\pm cdef}T_{\pm cdef} - T_{\pm acde}T_{\pm bcde}\,,
\end{aligned}
\tag{A.1}
$$

we obtain

$$
T_{\pm acde}T_{\pm bcde} = \frac{1}{8}\,\delta_{ab}\,T_{\pm cdef}T_{\pm cdef}\,. \tag{A.2}
$$

With the four form tensor in $D=8$ projection operator we set

$$
Q_{abcd} = F_{ae}T_{\pm ebcd} + F_{be}T_{\pm ecad} + F_{ce}T_{\pm eabd} + F_{de}T_{\pm ecba}\,. \tag{A.3}
$$

From the identity (A.2) and the properties of the four form tensor (38) which come from $\Omega_\pm^2 = \Omega_\pm$ we obtain

$$
\begin{aligned}
0 \;\leq\; Q_{abcd}Q_{abcd} &= 4F_{ab}F_{ac}T_{\pm bdef}T_{\pm cdef} - 12F_{ab}T_{\pm bcde}F_{cf}T_{\pm fade} \\
&= 12(\nu^{-1}-2)F_{ab}F_{ab} - 6\Big[F_{ab}F_{cd}T_{\pm abef}T_{\pm cdef} + (4-\nu^{-1})F_{ab}F_{cd}T_{\pm abcd}\Big] \\
&= 12(F_{ab} + \tfrac{1}{2}T_{\pm abcd}F_{cd})\Big[(\nu^{-1}-2)F^{ab} - T_{\pm abef}F_{ef}\Big]\,.
\end{aligned}
\tag{A.4}
$$

Therefore Eq.(40) makes Q_{abcd} vanish.

Henceforth we solve $\Omega_\pm = \Omega_\pm^2$ for the case $D=8$, $\nu = 1/16$. With the canonical choice of the self-dual or anti-self-dual four form tensor we write

$$
\Omega_\pm = \frac{1}{8}P_\pm\left(1 \pm \sum_{i=1}^{7}\beta_i E_{\pm i}\right), \tag{A.5}
$$

where

$$
\begin{aligned}
&\beta_1 = T_{\pm 3456} = \pm T_{\pm 1278}\,, \quad \beta_2 = T_{\pm 2475} = \pm T_{\pm 1638}\,, \quad \beta_3 = T_{\pm 1357} = \pm T_{\pm 2468}\,, \\
&\beta_4 = T_{\pm 1256} = \pm T_{\pm 3478}\,, \quad \beta_5 = T_{\pm 1234} = \pm T_{\pm 5678}\,, \quad \beta_6 = T_{\pm 1476} = \pm T_{\pm 2538}\,, \\
&\beta_7 = T_{\pm 2376} = \pm T_{\pm 1548}\,.
\end{aligned}
\tag{A.6}
$$

Direct calculation using Eq.(47) gives

$$
\Omega_\pm^2 = \frac{1}{64}P_\pm\left(1 + \sum_{i=1}^{7}\beta_i^2 \pm (2\beta_i + \sum_{j,k} c_{ijk}^2\beta_j\beta_k)E_{\pm i}\right)\,. \tag{A.7}
$$

As the square of the projection operator is itself, we obtain eight equations to solve

$$7 - \sum_{i=1}^{7} \beta_i^2 = 0\,, \tag{A.8}$$

$$6\beta_i - \sum_{j,k} c_{ijk}^2 \beta_j \beta_k \equiv \kappa_i = 0\,. \tag{A.9}$$

Among the latter seven, with seven distinct indices (i, j, k, l, m, r, s), typical four of $\kappa_i \pm \kappa_j = 0$, $\kappa_m \pm \kappa_r = 0$ give

$$\begin{aligned} (3 \pm \beta_k)(\beta_i \mp \beta_j) &= (\beta_l \mp \beta_s)(\beta_m \mp \beta_r)\,, \\ (3 \pm \beta_k)(\beta_m \mp \beta_r) &= (\beta_l \mp \beta_s)(\beta_i \mp \beta_j)\,, \end{aligned} \tag{A.10}$$

which in turn imply

$$\Big[(3 \pm \beta_k)^2 - (\beta_l \mp \beta_s)^2\Big](\beta_i \mp \beta_j) = 0\,. \tag{A.11}$$

If $\beta_i^2 \neq \beta_j^2$ then $9 + \beta_k^2 = \beta_l^2 + \beta_s^2$ and $3\beta_k + \beta_l \beta_s = 0$ so that either $\beta_l^2 = 9$ or $\beta_s^2 = 9$. However, this violates the constraint on the size of β_i^2 (A.8). Thus $\beta_i^2 = \beta_j^2$ and hence, in general, all the coefficients are of the same norm. From (A.8) we get $\beta_i^2 = 1$ for all $i = 1, 2, \cdots, 7$. Now noticing $\beta_i \beta_j \beta_k = 1 \text{ or } -1$, the seven equations (A.9) reduce to

$$1 = \beta_1\beta_2\beta_7 = \beta_1\beta_6\beta_3 = \beta_1\beta_5\beta_4 = \beta_2\beta_5\beta_3 = \beta_2\beta_4\beta_6 = \beta_3\beta_4\beta_7 = \beta_5\beta_6\beta_7\,. \tag{A.12}$$

Essentially there remain three independent signs. Our choice of the solution is $\beta_5 = \alpha_1$, $\beta_4 = \alpha_2$, $\beta_3 = \alpha_3$ and

$$\beta_1 = \alpha_1\alpha_2\,, \quad \beta_2 = \alpha_1\alpha_3\,, \quad \beta_6 = \alpha_1\alpha_2\alpha_3\,, \quad \beta_7 = \alpha_2\alpha_3\,. \tag{A.13}$$

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