Quantum Field Theories on Null Surfaces

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Abstract

We study the behaviour of quantum field theories defined on a surface S as it tends to a null surface S_n. In the case of a real, free scalar field theory the above limiting procedure reduces the system to one with a finite number of degrees of freedom. This system is shown to admit a one parameter family of inequivalent quantizations. A duality symmetry present in the model can be used to remove the quantum ambiguity at the self-dual point . In the case of the non-linear σ-model with the Wess-Zumino-Witten term a similar limiting behaviour is obtained. The quantization ambiguity in this case however cannot be removed by any means.

PACS Numbers : 11.10Kk, 11.15Tk, 11.10.Ef

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1. Introduction

Quantum Field Theory (QFT) on null surfaces have been studied in different contexts for a long time. One example of such theories consists of QFT on the light cone [1]. Analysis of these systems has led to the discovery of a rich underlying structure of field theories and gauge theories on null surfaces [2]. Another system of interest in this context involves the dynamics of the degrees of freedom on the horizon of a black hole, the horizon being a null surface. Analysis of the boundary field theories on black hole horizons has recently led to a new understanding of black hole entropy and other related issues [3].

Suppose that a field theory is defined on a surface S which is embedded in a flat Minkowskian manifold $\mathcal{M}$. Let the embedding of S in $\mathcal{M}$ be parametrized by a quantity v whose limiting value v_n corresponds to a null surface S_n. It is natural to ask the question as to how a field theory defined on S evolves as the parameter v varies. In particular, a field theory defined on a null surface S_n can be thought of as a limit of the theory defined on S as $v \to v_n$. It is this limiting case that we propose to investigate in this paper.

The metric h induced on S from $\mathcal{M}$ is a function of the parameter v. When v is away from its limiting value v_n, the induced metric on S is well defined. However, as $v \to v_n$ the induced metric h tends to become degenerate. Correspondingly, any regular metric based action defined on a surface S fails to have a well defined limit as S tends to S_n.

A physical example of where this scenario may occur can be described as follows. Considering a gravitationally collapsing sperical shell S of dust on which some field theory is defined. The collapsing surface S at any stage can be parametrised by a quantity v. If the situation is such that the system eventually tends to a black hole, the surface S finally would tend to a null surface S_n. Field theories on the surface of such a black hole could be studied using the above mentioned limiting procedure.

In this paper we analyze the behaviour of field theories on a surface S as $S \to S_n$. The analysis is based on specific examples where all the issues involved can be seen in an explicit fashion. In Section 2 we study the case of an abelian, free scalar field theory. In the limiting case this model reduces to a system with finite number of degrees of freedom that admits a one parameter family of inequivalent quantizations. This model also exhibits a type of duality symmetry. The quantization ambiguity can be removed if the system is at the self-dual point. Section 3 describes the analysis as applied to the $SU(l)$ Wess-Zumino-Witten (WZW) model. The parameter in front

of the action in this case is constrained from topological considerations. In the limit of $S \to S_n$, the quantum theory in this case is described by a finite degrees of freedom and is characterized by an arbitrary parameter just as in the scalar field theory . We conclude the paper in Section 4 with a summary and outlook.

2. Scalar Field

Let $\mathcal{M}$ be a flat Minkowskian manifold in 2+1 dimensions whose spatial slice has the topology of a cylinder $S^1 \times R$. Let r be the radius of S^1 and let θ be the angle spanning it. Consider the following flat metric in $\mathcal{M}$ given by

$$ds^2 = dt^2 - dz^2 - r^2 d\theta^2, \tag{2.1}$$

Let S given by $z = vt$ be a surface embedded in $\mathcal{M}$ where v is the parameter defining the embedding. The limiting value of this parameter is given by $v = 1$. The pull-back of the above metric in $\mathcal{M}$ to the time-like surface S can be written as

$$ds^2|_{z=vt} = (1 - v^2)dt^2 - r^2 d\theta^2 = h_{ab}dy^a dy^b. \tag{2.2}$$

As $v \to 1$, the surface S tends to the null surface S_n given by $z = t$. From Eqn. (2.2) it is easilly seen that the metric h_{ab} inudced on S is degenerate as $S \to S_n$.

Consider a single real scalar field ϕ which is defined on the surface S. We will assume that the scalar field is valued in a circle. As we shall see later, the degeneracy of the metric in the limit of $v \to 1$ leads to a Hamiltonian that is ill-defined. In order to address this problem we consider a renormalized field $f\phi$ where f is the renormalization parameter. The action for such a real scalar field can be written as

$$S = \int_S \sqrt{h}d^2y \, \mathcal{L} = \frac{f^2}{8\pi} \int_S \sqrt{h}d^2y h^{ab}\partial_a\phi\partial_b\phi. \tag{2.3}$$

As is evident from Eqn. (2.3), f can also be interpreted as the coupling constant of this model.

The field ϕ obeys the equation of motion

$$h^{ab}\partial_a\partial_b\phi = 0. \tag{2.4}$$

In terms of a variable $x = r\gamma\theta$ where $\gamma = \frac{1}{\sqrt{1-v^2}}$, the above action has the form

$$
\begin{aligned}
S &= \int dt \int_0^{x_0} \mathcal{L} \\
&= \frac{f^2}{8\pi} \int dt \int_0^{x_0} dx [(\partial_t \phi)^2 - (\partial_x \phi)^2].
\end{aligned}
$$

(2.5)

where $x_0 = 2\pi r\gamma$ is the period of the variable x. In terms of the variables x and t the action S is that of a free scalar field in 1+1 dimensions with a diagonal metric of signature (1,-1) . The equation of motion following from the action S is

$$
[(\partial_t)^2 - (\partial_x)^2]\phi = 0.
$$

(2.6)

2.1 Canonical Quantization

The mode expansion for the real field ϕ defined on a circle has the form

$$
\phi(t, x) = \phi_0 + \phi_{\mathrm{osc}}(t, x)
$$

(2.7)

where

$$
\phi_0 = Q + \frac{N}{r\gamma}x + \frac{2P}{f^2}t,
$$

(2.8)

and

$$
\begin{aligned}
\phi_{\mathrm{osc}}(t, x) &= \frac{1}{f} \sum_{k>0} [\frac{A_k}{\sqrt{k}}e^{-ikx_+} + \frac{A_k^+}{\sqrt{k}}e^{ikx_+} \\
&\quad + \frac{B_k}{\sqrt{k}}e^{-ikx_-} + \frac{B_k^+}{\sqrt{k}}e^{ikx_-}].
\end{aligned}
$$

(2.9)

In Eqn. (2.9) $x_\pm = t \pm x$ and $B_k = A_{-k}$.

As mentioned before, the field ϕ is assuemd to be valued in a circle for all time t. It therefore satisfies the consistency condition

$$
\phi_0(t, x + 2\pi r\gamma) = \phi_0(t, x) + 2\pi m,
$$

(2.10)

where m is an integer. From the above mode expansion of the field ϕ and Eqn. (2.10) it follows that

$$
kr\gamma = n \qquad (\text{n is an integer}),
$$

(2.11)

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and

$$N = m. \tag{2.12}$$

The canonical momentum conjugate to the field ϕ is defined by

$$\pi(t, x) = \frac{f^2}{4\pi} \partial_t \phi. \tag{2.13}$$

Using Eqns. (7), (8) and (13) we get that

$$\pi(t, x) = \frac{f^2}{4\pi} \partial_t \phi_{\text{osc}} + \frac{P}{2\pi}. \tag{2.14}$$

In the quantum theory, the wave-functional ψ is a function of the field ϕ. Since $\phi(x)$ is identified with $\phi(x) + 2\pi$, the ϕ_0 dependency of the wave-functional ψ satisfies the condition

$$\psi(\phi_0 + 2\pi) = e^{i2\pi\alpha}\psi(\phi_0), \tag{2.15}$$

where α is a real number between 0 and 1. Since we are dealing with bosonic variables alone, it is natural to choose $\alpha = 0$ corresponding to periodic boundary condition. It therefore follows from Eqn. (2.15) that

$$\psi_m(\phi_0) = e^{ip\phi_0}, \quad (\text{ p is an integer }) \tag{2.16}$$

are the eigenfunctions of the operator

$$P = \int dx\pi(x) = -i\frac{\partial}{\partial\phi_0} \tag{2.17}$$

and the corresponding eigenvalues are p. The spectrum of the operator P therefore consists of integers p.

The canonical commutaion relations of the basic field variables are given by

$$[\phi(t, x), \pi(t, y)] = i\delta(x - y) \tag{2.18}$$

and

$$[\phi(t, x), \phi(t, y)] = [\pi(t, x), \pi(t, y)] = 0. \tag{2.19}$$

It follows that

$$[Q, P] = i, \tag{2.20}$$

$$[A_k, A_k'^+] = \delta_{kk'} \tag{2.21}$$

and all other commutation relations are zero .

2.2 Ground State Energy

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The Hamiltonian of the system is given by

$$
\begin{aligned}
H &= \int_0^{x_0} dx [\pi(t,x)\partial_t\phi(t,x) - \mathcal{L}] \\
&= \frac{f^2}{8\pi} \int_0^{x_0} dx [(\partial_t\phi)^2 + (\partial_x\phi)^2] \\
&= H_0 + H_{\text{osc}},
\end{aligned}
\tag{2.22}
$$

where H_0 and H_{osc} are the Hamiltonians for the zero (or winding) and oscillating modes respectively. They are given by

$$
H_0 = \frac{r\gamma}{f^2}P^2 + \frac{f^2}{4r\gamma}m^2
\tag{2.23}
$$

and

$$
H_{\text{osc}} = r\gamma \sum_{k \neq 0} |k| [A_k^+ A_k + \frac{1}{2}]
\tag{2.24}
$$

A given zero mode sector is characterized by the integers p and m. Let the ground state (or the vacuum) in this sector be denoted by $|0>_{pm}$. The vacuum satisfies the condition

$$
H_{\text{osc}}|0\rangle_{pm} = 0
\tag{2.25}
$$

where in the above H_{osc} is assumed to have been normal ordered and the zero-point energy has been subtracted. The ground state energy in a given zero mode sector characterised by the integers p and m satisfies the equation

$$
H|0\rangle_{pm} = (H_0 + H_{\text{osc}})|0\rangle_{pm} = E_G|0\rangle_{pm}
\tag{2.26}
$$

where

$$
E_G = \frac{r\gamma}{f^2}p^2 + \frac{f^2}{4r\gamma}m^2
\tag{2.27}
$$

and p is the eigenvalue of the operator P in the ground state under consideration.

Let us now define the quantity $\tilde{r}$ by

$$
\tilde{r} = \frac{2\gamma r}{f^2}
\tag{2.28}
$$

which is an effective radius for the system. Then the ground state energy can be written as

$$
E_G = \frac{1}{2}[\tilde{r}p^2 + \frac{1}{\tilde{r}}m^2].
\tag{2.29}
$$

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In the limit when $v \to 1$, the induced metric h_{ab} in Eqn. (2.2) tends to blow up. In this limit $\tilde{r}$ and the ground state energy E_G also become undefined. It may thus seem that there is now smooth way of taking the aforementioned limit.

We can however use the following "renormalization group inspired" prescription to make this limit well defined. Let us first note that the quantity v used to define the embedding of S in $\mathcal{M}$ could be thought as a regulator. We are really interested in the situation $S \to S_n$ or $v \to 1$, which can be thought of as the regulator being removed. In order for the ground state energy to have a smooth behaviour in this limit we postulate that the coupling constant f is a function of the regulator v. The functional dependence of f on v is to be determined from the physical condition that as $v \to 1$, the ground state energy should be independent of v. In other words, as $v \to 1$

$$\frac{dE_G}{dv} = 0. \tag{2.30}$$

Using the equation (2.27) we find that

$$\frac{dE_G}{dv} = \frac{1}{2\tilde{r}} \frac{d\tilde{r}}{dv} \left(\tilde{r} p^2 - \frac{1}{\tilde{r}} m^2 \right) \tag{2.31}$$

The second term cannot be zero for all values of v because p and m are fixed numbers . We are hence left with the condition

$$\frac{1}{2\tilde{r}} \frac{d\tilde{r}}{dv} = 0 \tag{2.32}$$

This is possible only if in the limit $v \to 1$,

$$f(v) = F\sqrt{\gamma} \tag{2.33}$$

where F is a constant parameter and can be thought of as the "renormalized" coupling constant.

Using Eqns. (28), (29) and (33), the ground state energy has the form

$$E_G = \frac{r}{F^2} p^2 + \frac{F^2}{4r} m^2. \tag{2.34}$$

In any given zero-mode sector, different choices of the parameter F would lead to different values of the ground state energy and hence to inequivalent quantum field theories. The parameter F is not determined by the above analysis and can presumably be obtained from empirical considerations.

It should be noted that only the zero mode sector has information about the coupling constant f and consequently of the parameter F. The expression

for H_{osc} (cf. Eqn. (2.24)) is independent of f. As $v \to 1$, we see from Eqn. (2.11) that for any given n, $k \to 0$. However, from Eqn. (2.24), only nonzero k contributes to H_{osc}. Hence the contribution to the Hamiltonian coming from the the oscillatory modes become energetically unfavourable as $S \to S_n$. We therefore arrive at the conclusion that as $S \to S_n$, the model under consideration reduces to a quantum mechanical system with a finite number of degrees of freedom given by the zero modes of the original problem. The Hamiltonian of this reduced system is still given by Eqn. (2.23) where f is given by Eqn. (2.33) and the corresponding eigenavlues are given by Eqn. (2.34).

2.3 Duality

A real scalar field system valued in a circle has a well known duality symmetry [4]. A remnant of that can be seen in the reduced system obtained above. For a fixed n, our system is characterized by the numbers p, m and F. Under the transformations

$$
\begin{aligned}
p &\longrightarrow m \\
m &\longrightarrow p \\
\frac{r}{F^2} &\longrightarrow \frac{F^2}{4r}
\end{aligned}
\tag{2.35}
$$

E_G belonging to two different configurations get interchanged. This is analogous to a T-duality.

The duality symmetry by itself imposes no restriction on F. There is however a special configuration, namely the "self-dual" point where the duality symmetry can be used to fix the arbitrariness in F. The "self-dual" point is given by

$$
\frac{r}{F^2} = \frac{F^2}{4r}.
\tag{2.36}
$$

At this special point F therefore satisfies the condition

$$
F^2 = 2r.
\tag{2.37}
$$

Using Eqns. (34) and (37), the ground state energy can them be expressed as

$$
E_G = \frac{1}{2}[p^2 + m^2]
\tag{2.38}
$$

which is independent of F. The limiting procedure described above thus leads to a unique quantization only at the self-dual point.

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3. Non Linear σ - Model

Let $\mathcal{C}$ be a three dimensional manifold whose boundary $\partial\mathcal{C}$ is a two dimensional Minkowskian manifold with a topology of $S^1 \times R$. We identify $\partial\mathcal{C}$ with a surface S on which the induced metric h^{ab} is given by Eqn(2.2) . The action for the non linear σ-model with a Wess-Zumino-Witten term (WZW) [5] for a group G is given by

$$
\begin{aligned}
S &= S_0 + S_{\text{wzw}}, \\
S_0 &= A\int_{\partial\mathcal{C}} d^2 y \sqrt{h} h^{\mu\nu} Tr \partial_\mu g \partial_\nu g^{-1}, \\
S_{\text{wzw}} &= \int_{\mathcal{C}} \Omega = B\int_{\mathcal{C}} Tr(g^{-1}dg)^3,
\end{aligned} \tag{3.1}
$$

where A and B are constants and g takes values in the group G. For simplicity we will assume in this section that $G = SU(l)$ with $l > 1$. The coefficient B of the second term in the action is not arbitrary. From topological considerations $B = \frac{n}{24\pi}$, where n is an integer [5].

Let $T_a, a = 1, ..r$ be the generators of G satisfying the comutation relations

$$
[T_a, T_b] = if_{abc}T_c. \tag{3.2}
$$

These generators are normalised in such a way that

$$
Tr T_a T_b = 2\delta_{ab}. \tag{3.3}
$$

The group G can act on g either from the left or from the right . The left action is given by

$$
g \longrightarrow g' = g + igx_a T_a. \tag{3.4}
$$

where x_a are small parameters . The right action of the group is similarly given by

$$
g \longrightarrow g' = g + ix_a T_a g. \tag{3.5}
$$

For both cases, the variation of S is

$$
\begin{aligned}
\delta S &= \delta S_0 + \delta S_{wzw}, \\
\text{where } \delta S_0 &= 2A\int d^2 x \sqrt{h} h^{\mu\nu} Tr g^{-1}\delta g \partial_\mu(g^{-1}\partial_\nu g) + \text{ Total derivatives} \\
\text{and } \delta S_{wzw} &= -3B\int d^2 x \epsilon^{\mu\nu} Tr g^{-1}\delta g \partial_\mu(g^{-1}\partial_\nu g).
\end{aligned} \tag{3.6}
$$

By setting $\delta S = 0$ we get as equations of motion

$$2A\sqrt{h}h^{\mu\nu}\partial_\mu(g^{-1}\partial_\nu g) - 3B\epsilon^{\mu\nu}\partial_\mu(g^{-1}\partial_\nu g) = 0. \tag{3.7}$$

In terms of the light cone coordinates

$$x_\pm = t \pm x, \tag{3.8}$$

Eqn. (3.7) can be expressed as

$$(2A - 3B)\partial_+(g^{-1}\partial_- g) + (2A + 3B)\partial_-(g^{-1}\partial_+ g) = 0. \tag{3.9}$$

For the choice of $A = \pm\frac{3B}{2}$ the left and right movers decouple from each other and the equations of motion reduces to

$$\partial_\mp(g^{-1}\partial_\pm g) = 0. \tag{3.10}$$

The currents arising from the left action of the group are given by

$$
\begin{aligned}
J_a^\mu &= h^{\mu\nu}TrT_a g^{-1}\partial_\nu g, \\
\text{with}\quad J_a^0 &= \gamma^2 TrT_a g^{-1}\dot{g} \\
\text{and}\quad J_a^1 &= -\frac{1}{r^2}TrT_a g^{-1}g',
\end{aligned}
\tag{3.11}
$$

where $\dot{g} = \partial_t g$ and $g' = \partial_\theta g$. Using Eqn. (3.11) the light-cone components of the currents can be written as

$$
\begin{aligned}
J_a^\pm &= TrT_a g^{-1}\partial_\pm g \\
&= \frac{1}{2\gamma^2}J_a^0 \mp \frac{r}{2\gamma}J_a^1.
\end{aligned}
\tag{3.12}
$$

The currents resulting from the right action of the group can also be found in a similar fashion. They are

$$
\begin{aligned}
\bar{J}_a^\pm &= TrT_a(\partial_\pm g)g^{-1} \\
&= \frac{1}{2\gamma^2}\bar{J}_a^0 \mp \frac{r}{2\gamma}\bar{J}_a^1,
\end{aligned}
\tag{3.13}
$$

where

$$
\begin{aligned}
\bar{J}_a^0 &= \gamma^2 TrT_a \dot{g}g^{-1} \\
\bar{J}_a^1 &= -\frac{1}{r^2}TrT_a g' g^{-1}.
\end{aligned}
\tag{3.14}
$$

3.1 The Canonical Formalism

Let $\xi_i, \ i = 1, ..., \dim G$ be a set of local coordinates parametrizing the elements $g \in G$ [6]. In terms of these local coordinates S_0 can be expressed as

$$
\begin{aligned}
S_0 &= \int d^2x \mathcal{L}_0, \\
\mathcal{L}_0 &= Ar\gamma Tr \frac{\partial g}{\partial \xi_i} \frac{\partial g^{-1}}{\partial \xi_j} [\dot{\xi}_i \dot{\xi}_j - \frac{1}{r^2\gamma^2} \xi'_i \xi'_j].
\end{aligned}
\tag{3.15}
$$

We would also like to express the WZW part of the action in terms of the coordinates ξ. The WZW terms cannot be written globally in terms of a single set of local coordinates. To proceed we assume that the group manifold consists of a number of patches labelled by a parameter u. The restriction of $\Omega = BTr(g^{-1}dg)^3$ to any of these patches can be written as

$$
\Omega^u = d\omega^u,
\tag{3.16}
$$

where ω^u is a two form defined by

$$
\omega^u = \frac{1}{2} \omega^u_{ij} d\xi^i \wedge d\xi^j.
\tag{3.17}
$$

From Eqns. (3.16) and (3.17), Ω^u has the form

$$
\Omega^u = \frac{1}{6} \Omega^u_{ijk} d\xi^i \wedge d\xi^j \wedge d\xi^k,
\tag{3.18}
$$

where Ω^u_{ijk} is given by

$$
\Omega^u_{ijk} = \frac{\partial \omega^u_{jk}}{\partial \xi^i} + \frac{\partial \omega^u_{ki}}{\partial \xi^j} + \frac{\partial \omega^u_{ij}}{\partial \xi^k}.
\tag{3.19}
$$

Ω^u_{ijk} can also be expressed however in terms of the group element g as

$$
\Omega^u_{ijk} = 3BTr[g^{-1}\frac{\partial g}{\partial \xi^i}, g^{-1}\frac{\partial g}{\partial \xi^j}]g^{-1}\frac{\partial g}{\partial \xi^k}.
\tag{3.20}
$$

In terms of the local coordinates the WZW action then takes the form

$$
\begin{aligned}
S_{wzw} &= \sum_u \int_{\partial \mathcal{C}} \mathcal{L}^u_{wzw} d^2x, \\
\text{where} \ \ \mathcal{L}^u_{wzw} &= \frac{1}{2} \omega^u_{ij} \partial_a \xi^i \partial_b \xi^j \epsilon^{ab}.
\end{aligned}
\tag{3.21}
$$

We would next like to compute the canonical momentum P_i conjugate to the coordinate ξ_i. Using Eqns. (3.15) and (3.21) P_i can be written as

$$
\begin{aligned}
P_i &= \frac{\partial \mathcal{L}_0}{\partial \dot\xi_i} + \frac{\partial \mathcal{L}_{wzw}}{\partial \dot\xi_i} \\
&= Ar\gamma Tr[\frac{\partial g}{\partial \xi_i}\frac{\partial g^{-1}}{\partial \xi_j}\dot\xi_j + \frac{\partial g}{\partial \xi_j}\frac{\partial g^{-1}}{\partial \xi_i}\dot\xi_j] - \omega_{ij}^u\partial_\theta\xi^j \\
&= -2Ar\gamma Tr g^{-1}\dot g g^{-1}\frac{\partial g}{\partial x^i} - \omega_{ij}^u\partial_\theta\xi^j.
\end{aligned}
$$

(3.22)

The Hamiltonian density therefore has the form

$$
\begin{aligned}
\mathcal{H} &= P_i\xi_i - \mathcal{L}_0 - \mathcal{L}_{wzw} \\
&= -Ar\gamma Tr[(\dot g g^{-1})^2 + \frac{1}{r^2\gamma^2}(g' g^{-1})^2].
\end{aligned}
$$

(3.23)

As expected, there is no contribution from the WZW term to the Hamiltonian. In terms of the light cone currents defined in Eqn. (3.12) , the Hamiltonian density can be written as

$$
\mathcal{H} = -Ar\gamma[J_a^{+2} + J_a^{-2}].
$$

(3.24)

Next we turn to the commutation relations for this system. The basic commutation relations are given by

$$
[\xi_a(\theta), \xi_b(\theta')] = [P_a(\theta), P_b(\theta')] = 0
$$

(3.25)

and

$$
[\xi_a(\theta), P_b(\theta')] = i\delta_{ab}\delta(\theta - \theta')
$$

(3.26)

From these commutation relations it follows that

$$
[g(\theta), P_b(\theta')] = i\frac{\partial g(\theta)}{\partial \xi_b}\delta(\theta - \theta').
$$

(3.27)

Similarly we have

$$
[g(\theta)^{-1}, P_b(\theta')] = i\frac{\partial g(\theta)^{-1}}{\partial \xi_b}\delta(\theta - \theta').
$$

(3.28)

It is however useful to rewrite these commutation relations in a form that is independent of the local coordinates ξ [6]. To this end, we introduce a new

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set of functions $\xi(x)$ with the condition that $\xi(0) = \xi$. The field $g(\xi)$ can be understood as the value of a field $g(\xi(x))$ at $\xi(0)$. The field $g(\xi(x))$ is defined by

$$g(\xi(x)) = g(\xi(0))\exp(ix_a T_a) \qquad (3.29)$$

Differentiating with respect to x_a and then setting $x = 0$ we get the identity

$$N_b^a \frac{\partial g}{\partial \xi^a} = ig(\xi)T_b \qquad (3.30)$$

where $N_b^a = \frac{\partial \xi^a}{\partial x_b}|_{x=0}$ can be proven to be nondegenerate [6]. Using Eqn. (3.30), we can replace the phase space variables P_a with new variables Π_a defined as

$$\begin{aligned} \Pi_a &= N_a^b P_b \\ &= -2i\frac{Ar}{\gamma}J_a^0 - N_a^b\omega_{bc}^u\partial_\theta\xi_c. \end{aligned} \qquad (3.31)$$

Using Eqns. (3.27), (3.28) ,(3.30) and (3.31) it follows that

$$[g(\theta), \Pi_b(\theta')] = -\delta(\theta - \theta')g(\theta)T_b \qquad (3.32)$$

and

$$[g^{-1}(\theta), \Pi_b(\theta')] = \delta(\theta - \theta')T_b g^{-1}(\theta). \qquad (3.33)$$

The above commutators between g, g^{-1} and Π carry no explicit dependence on the local coordinates ξ. Finally, the commutator of the Π's is given by (see the appendix for the proof)

$$[\Pi_a(\theta), \Pi_b(\theta')] = if_{abc}\Pi_c(\theta)\delta(\theta - \theta') \qquad (3.34)$$

and all other commutation relations are trivial. Eqns (3.32), (3.33) and (3.34) embody the fundamental commutators for this system.

3.2 Current Algebra

We are now ready to calculate the current commutators. First we note that the expression for $J_a^1(\theta)$ contains no time derivative. It therefore follows that

$$[J_a^1(\theta), J_b^1(\theta')] = 0. \tag{3.35}$$

Next, by differentiating Eqn. (3.32) with respect to θ we get

$$[\partial_\theta g(\theta), \Pi_b(\theta')] = -\delta(\theta - \theta')\partial_\theta g(\theta)T_b - \partial_\theta \delta(\theta - \theta')g(\theta)T_b. \tag{3.36}$$

Using the above relation and the definition of the current J_a^1 (cf. Eqn. (3.11)), it can be shown that

$$[\Pi_a(\theta), J_b^1(\theta')] = if_{abc}J_c^1(\theta) + \frac{TrT_aT_b}{r^2}\partial_\theta \delta(\theta - \theta'). \tag{3.37}$$

From Eqn. (3.31) we have $J_a^0 = i\gamma/2Ar[\Pi_a + N_a^b \omega_{bc}^u \partial_\theta \xi_c]$. Using this expression for J_a^0 and Eqn. (3.37) we get the second current commutator as

$$[J_a^0(\theta), J_b^1(\theta')] = -\frac{\gamma}{2Ar}f_{abc}J_c^1(\theta)\delta(\theta - \theta') + \frac{i\gamma}{2Ar^3}TrT_aT_b\partial_\theta \delta(\theta - \theta'). \tag{3.38}$$

Finally, a tedious calculation (the details are shown in the appendix) gives the last current commutator as

$$[J_a^0(\theta), J_b^0(\theta')] = -\frac{\gamma}{2Ar}f_{abc}J_c^0(\theta)\delta(\theta - \theta') - \frac{3B\gamma^2}{4A^2}\delta(\theta - \theta')f_{abc}J_c^1(\theta) \tag{3.39}$$

We would next like to obtain the commutators for the light-cone components of the currents . Using Eqns. (3.35), (3.38) and (3.39) we get the following currents algebra

$$
\begin{aligned}
[J_a^+(\theta), J_b^+(\theta')] &= -\frac{1}{8Ar\gamma^3}f_{abc}\delta(\theta - \theta')J_c^0(\theta) \\
&\quad + \frac{1}{4A\gamma^2}(1 - \frac{3B}{4A})f_{abc}J_c^1(\theta)\delta(\theta - \theta') \\
&\quad + \frac{1}{4iAr^2\gamma^2}TrT_aT_b\partial_\theta \delta(\theta - \theta').
\end{aligned}
\tag{3.40}
$$

Let us now proceed by considering the two cases $A = \frac{3B}{2}$ and $A = -\frac{3B}{2}$ separately . For the first case where $\frac{3B}{2A} = 1$, Eqn. (3.40) reduces to the Kac-Moody algebra

$$[J_a^+(\theta), J_b^+(\theta')] = -\frac{1}{4Ar\gamma}f_{abc}\delta(\theta - \theta')J_c^+ - \frac{i}{2Ar^2\gamma^2}\delta_{ab}\partial_\theta \delta(\theta - \theta'). \tag{3.41}$$

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The currents $\bar{J}_a^- = Tr T_a \partial_- g g^{-1}$ coming from the right action of the group would similarly generate another Kac-Moody algebra. To get the algebra generated by $\bar{J}_a^-$ first note that the action is invariant under the transformations $\theta \longrightarrow -\theta$ and $g \longrightarrow g^{-1}$. Under these transformations $J_a^+ \longrightarrow -\bar{J}_a^-$ and therefore the current commutator in Eqn. (3.41) becomes

$$[\bar{J}_a^-(\theta), \bar{J}_b^-(\theta')] = \frac{1}{4Ar\gamma} f_{abc} \delta(\theta - \theta') \bar{J}_c^- + \frac{i}{2Ar^2\gamma^2} \delta_{ab} \partial_\theta \delta(\theta - \theta'). \quad (3.42)$$

We also have

$$[J_a^+(\theta), \bar{J}_b^-(\theta')] = 0 \quad (3.43)$$

as the two currents come from the two commuting actions of the group on itself .

For the second point $\frac{3B}{2A} = -1$ exactly the same arguments will lead to the two other commuting Kac-Moody algebras given by the currents J_a^- and $\bar{J}_a^+$. The first currents algebra generated by J_a^- has the form (3.41) with the substitution $J_a^+ \longrightarrow -J_a^-$ and $A \longrightarrow -A$. The currents algebra corresponding to J_a^+ is aobtained from (3.42) by a similar substitution $J_a^- \longrightarrow -J_a^+$ and $A \longrightarrow -A$.

3.3 Mode Expansion

Let us consider the case when $\frac{3B}{2A} = 1$ (the treatement of the case $\frac{3B}{2A} = -1$ is exactly similar). We first express the the Hamiltonian density $\mathcal{H}$ in terms of the two commuting set of currents J_a^+ and J_a^- which are relevant to the case under consideration. To this end we note that a given element L in the Lie algebra of G can be written as $L = \frac{T_a}{2} Tr(T_a L)$ Using this we can then check that $J_a^{-2} = (\bar{J}_a^-)^2$. The Hamiltonian density in Eqn. (3.24) can then be expressed as

$$\begin{aligned} \mathcal{H} &= -Ar\gamma[(J_a^+)^2 + (J_a^-)^2] \\ &= -Ar\gamma[(\bar{J}_a^+)^2 + (\bar{J}_a^-)^2] \\ &= -\frac{1}{16Ar\gamma}[(K_a^+)^2 + (\bar{K}_a^-)^2], \end{aligned} \quad (3.44)$$

where K_a^+ and $\bar{K}_a^-$ are defined as

$$\begin{aligned} K_a^+ &= -4Ar\gamma J_a^+, \\ \bar{K}_a^- &= 4Ar\gamma \bar{J}_a^-. \end{aligned} \quad (3.45)$$

They satify the commutation relations

$$[K_a^+(\theta), K_b^+(\theta')] = f_{abc} \delta(\theta - \theta') K_c^+(\theta) - 8iA\delta_{ab} \partial_\theta \delta(\theta - \theta') \quad (3.46)$$

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and

$$[\bar{K}_a^-(\theta), \bar{K}_b^-(\theta')] = f_{abc}\delta(\theta - \theta')\bar{K}_c^-(\theta) + 8iA\delta_{ab}\partial_\theta\delta(\theta - \theta'). \qquad (3.47)$$

Next we proceed with the mode expansion of the currents. Let us first note that in terms of $g \in G$, the current $K_a^+(x)$ has the expression

$$K_a^+(x) = -2Ar\gamma TrT_ag^{-1}\dot{g} - 2ATrT_ag^{-1}g'. \qquad (3.48)$$

A similar expression will hold for the current $\bar{K}^-$. The mode expansion for the two terms in the rhs of Eqn (3.48) are given by

$$\begin{aligned}
TrT_ag^{-1}\dot{g} &= \sum_{k\neq 0} J_a^0(k)e^{i(\omega t - kx)} + J_a^0(0), \\
TrT_ag^{-1}g' &= \sum_{k\neq 0} J_a^1(k)e^{i(\omega t - kx)} + J_a^1(0).
\end{aligned} \qquad (3.49)$$

Next we can check using the periodicity requirement

$$J_a^\mu(t, x + x_0) = J_a^\mu(t, x) \qquad (3.50)$$

that

$$kr\gamma = q \qquad (3.51)$$

where q is an integer . Using the equations of motion (3.10), we see that $\omega = k$ for J_a^+ and $\omega = -k$ for J_a^-.

Now by using Eqn. (3.49) in (3.48), we get

$$\begin{aligned}
K_a^+(x_+) &= \frac{1}{2i\pi}[\sum_{k\neq 0} K_a^+(k)e^{-ikx_+} + \mathcal{P}_a] \\
\text{where } K_a^+(k) &= -4i\pi Ar\gamma[J_a^0(k) + \frac{1}{r\gamma}J_a^1(k)] \\
\text{and } \mathcal{P}_a &= -4i\pi Ar\gamma[J_a^0(0) + \frac{1}{r\gamma}J_a^1(0)].
\end{aligned} \qquad (3.52)$$

Similarly ge wet that,

$$\begin{aligned}
\bar{K}_a^-(x_-) &= \frac{1}{2i\pi}[\sum_{k\neq 0} \bar{K}_a^-(k)e^{ikx_-} + \mathcal{M}_a] \\
\text{where } \bar{K}_a^-(k) &= 4i\pi Ar\gamma[\bar{J}_a^0(k) - \frac{1}{r\gamma}\bar{J}_a^1(k)] \\
\text{and } \mathcal{M}_a &= 4i\pi Ar\gamma[\bar{J}_a^0(0) - \frac{1}{r\gamma}\bar{J}_a^1(0)].
\end{aligned} \qquad (3.53)$$

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In above, $J_a^0(k)$ and $J_a^1(k)$ are the modes corresponding to $TrT_a \dot{g} g^{-1}$ and $TrT_a g' g^{-1}$ respectively .

Using the above currents and the Kac-Moody algebra (3.46) , we get that

$$[\mathcal{P}_a, \mathcal{P}_b] = i f_{abc} \mathcal{P}_c \tag{3.54}$$

and

$$[K_a^+(p), K_b^+(k)] = i f_{abc} K_c^+(p+k) + 16\pi A r \gamma p \delta_{ab} \delta_{p+k,0}. \tag{3.55}$$

In the same way we get from (3.47) ,

$$[\mathcal{M}_a, \mathcal{M}_b] = i f_{abc} \mathcal{M}_c \tag{3.56}$$

and

$$[\bar{K}_a^-(p), \bar{K}_b^-(k)] = i f_{abc} \bar{K}_c^-(p+k) - 16\pi A r \gamma p \delta_{ab} \delta_{p+k,0}. \tag{3.57}$$

From (3.54) and (3.56) we immediately see that $\{\mathcal{P}_a\}$ and $\{\mathcal{M}_a\}$ are two representations of $SU(l)$ generators .

We can now compute the Hamiltonian

$$H = \int_0^{2\pi} d\theta \mathcal{H} \tag{3.58}$$

in terms of the oscillation modes $K_a^+(k)$, $\bar{K}_a^-(k)$ and the zero modes $\mathcal{P}_a$, $\mathcal{M}_a$. The answer turns out to be

$$H = H_{\text{osc}} + H_0 \tag{3.59}$$

where

$$H_{\text{osc}} = -\frac{\pi}{8Ar\gamma} \sum_{k \neq 0} [: K_a^+(k) K_a^+(-k) : + : \bar{K}_a^-(k) \bar{K}_a^-(-k) :] \tag{3.60}$$

and

$$H_0 = -\frac{\pi}{8Ar\gamma} (\mathcal{P}^2 + \mathcal{M}^2). \tag{3.61}$$

The contribution to the Hamiltonian from the oscillatory mode has been normel ordered. $\mathcal{P}^2$ and $\mathcal{M}^2$ are simply the $SU(l)$ Casimirs $\mathcal{P}^2 = \sum_a \mathcal{P}_a^2$ and $\mathcal{M}^2 = \sum_a \mathcal{M}_a^2$ respectively and are given by [8]

$$\begin{aligned} \mathcal{P}^2 &= \frac{N_{adj}}{N_p} p \\ \mathcal{M}^2 &= \frac{N_{adj}}{N_m} m \end{aligned} \tag{3.62}$$

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where N_{adj} is the dimension of the adjoint representation of $SU(l)$. N_p and N_m above are the dimensions of the representaions $\{\mathcal{P}_a\}$ and $\{\mathcal{M}_a\}$ respectively and p (m) is the index of the representations $\{P_a\}$ $(\{M_a\})$. A given zero mode will be characterized by two integers p and m and it will be denoted by $|pm\rangle$. The state $|pm\rangle$ will be annihilated by H_{osc} as the latter is normal ordered. The ground state energy of the system would therefore be given by

$$E_{pm} = -\frac{\pi}{8Ar\gamma}(\mathcal{P}^2 + \mathcal{M}^2).$$

(3.63)

where now $\mathcal{P}^2$ and $\mathcal{M}^2$ are being understood to be equal to the numbers given by the equation (3.62).

We want now to investigate the behaviour of the currents algebras and the Hamiltonian as $v \rightarrow 1$. The currents in the equations (3.52) (3.53) as well as the Hamiltonians (3.60) and (3.61) are functions of the parameter v and tend to become ill defined as $v \rightarrow 1$. As in Section 2.2, we can again use a "renormalization group inspired" technique to get a well defined theory in this limit. The constant B in this case has the allowed values given by $\frac{n}{24\pi}$ where n is an integer. Furthermore A is constrained by the condition $A = \pm\frac{3B}{2}$. We will however assume that A is a function of v, and its dependence on v is to be determined from the condition that in the limit of $v \rightarrow 1$, the ground state energy E_{pm} becomes independent of v, i.e.

$$\frac{dE_{pm}}{dv} = 0.$$

(3.64)

However by using (3.63) it immediately follows that

$$\frac{\pi}{8r\gamma A} = \frac{1}{C}$$

(3.65)

where C is a constant. A in the above equation is constrained as mentioned above. It therefore cannot run continuously with v and changes only in discrete steps always satisfying the constraint. Suppose when $v = 0$, A was given by A_0. The constant C was then given by $C = \frac{8r}{\pi}A_0$. As $v \rightarrow 1$, it follows from Eqn. (3.65) that the limiting value of A actually tends to *zero*. C however is finite in this limit and is given by the same constant value as mentioned above.

In view of the above, as $v \rightarrow 1$, the ground state energy of the system is given by

$$E_{pm} = -\frac{1}{C}(\mathcal{P}^2 + \mathcal{M}^2)$$

(3.66)

corresponding to the Hamiltonian

$$H_0 = -\frac{1}{C}(\mathcal{P}^2 + \mathcal{M}^2).$$

(3.67)

where now $\mathcal{P}_a$ and $\mathcal{M}_a$ are given by :

$$
\begin{aligned}
\mathcal{P}_a &= -\frac{i\pi^2 C}{2} J_a^0(0) \\
\mathcal{M}_a &= \frac{i\pi^2 C}{2} \bar{J}_a^0(0),
\end{aligned}
\tag{3.68}
$$

and they still do satisfy (3.54) and (3.56) respectively . The value of the constant C is related the the value of A when $v = 0$. This value is not determined by the theory and must be obtained from empirical considerations. As in the scalar field case, this system also therefore admits a one parameter family of inequivalent quantizations.

Let us now turn our attention to the oscillatory modes. As $v \to 1$, A satisfies Eqn. (3.65) and the expressions for the oscillatory modes are given by

$$
\begin{aligned}
K_a^+(k) &= -\frac{i\pi^2 C}{2} J_a^0(k) \\
\bar{K}_a^-(k) &= \frac{i\pi^2 C}{2} \bar{J}_a^0(k).
\end{aligned}
\tag{3.69}
$$

The current algebra satisfied by these modes are now given by

$$
[K_a^+(p), K_b^+(k)] = i f_{abc} K_c^+(p+k) + 2\pi^2 C p \delta_{ab} \delta_{p+k,0}
\tag{3.70}
$$

and

$$
[\bar{K}_a^-(p), \bar{K}_b^-(k)] = i f_{abc} \bar{K}_c^-(p+k) - 2\pi^2 C p \delta_{ab} \delta_{p+k,0}.
\tag{3.71}
$$

However, as $v \to 1$, it is clear from Eqn. (3.51) that k must go to zero for any value of the integer q. From Eqn. (3.60), we see that the oscillatory part of the Hamiltonian has contributions only from those modes for which k is not equal to zero. We therefore conclude that as $v \to 1$, the oscillatory modes becomes energetically unfavourable and do not contribute to the Hamiltonian. The entire theory in this limit, just as in the scalar field case, is described by a finite number of degrees of freedom given only by the zero modes.

4. Conclusion

In this paper we have investigated the limitng behaviour of quantum field theories defined on a surface S as the latter tends to a null surface S_n.

In the case of a scalar field theory the above limiting procedure reveals several interesting features. First, as $S \to S_n$, the excitation of the oscillatory degrees of freedom of the system becomes energetically unfavourable.

In this situation, the model reduces to a quantum mechanical system with the winding modes as the only degrees of freedom. Second, in the limit when $S \to S_n$, the renormalized Hamiltonian of the system contains an arbitrary parameter. Hamiltonians with different values of this parameter cannot be related via a unitary transformation. The limiting case of this system therefore admits a one-parameter family of inequivalent quantizations. Finally, this model exhibits a type of T-duality symmetry. This feature can be used to remove the quantization ambiguity only at the self-dual point.

In the case of a non-linear σ-model with a Wess-Zumino-Witten term a similar result is obtained. The parameters of this model are however constrained by topological considerations. However, in the limit when $S \to S_n$, the oscillatory modes of this system also have the same behaviour as in the scalar field case. The renormalized Hamiltonian is described only in terms of a finite number of degrees of freedom given by the zero modes. It is also seen to contain an arbitrary parameter that can be related to one of the constants of the theory when $v = 0$. This observation however is not enough to fix a unique value of this parameter. We can hence say that this system also admits a one-parameter family of inequivalent quantizations.

There seems to be a degree of universality associated with the results obtained above. The suppression of the oscillatory modes in the limit of $v \to 1$ can be traced to Eqn. (2.11) and (3.51) for the scalar field and the non-linear σ-model cases respectively. Such equations would always occur whenever there is periodicity condition on the basic variables of the theory concerned. We therefore conclude that the suppression of the oscillatory modes would be a generic phenomenon in this type of a scenario. This would in turn mean that as $S \to S_n$, the resulting theory on the null surface would generically be described by a finite number of degrees of freedom related to the zero modes of the system.

Both the models considered in this paper admits a one-parameter family of inequivalent quantizations. We have however not found any general argument supporing the universality of this phenomenon.

It would be interesting to perform similar analysis to more realistic models of physical interest, e.g. boundary field theories on a black hole horizon which is a null surface in space-time. The analysis presented in this paper could be adapted to study the dynamics of field theories on such a surface which is currently under investigation.

Acknowledgments

A. Appendix

Here we give some of the identities necessary to derive Eqns. (3.34) and (3.39).

To find the commutation relations among the conjugate momenta Π_a we proceed as follows. Consider the Jacobi identity

$$[[\Pi_a(\theta), \Pi_b(\theta')], g(\theta'')] = -[[\Pi_b(\theta'), g(\theta'')], \Pi_a(\theta)] - [[g(\theta''), \Pi_a(\theta)], \Pi_b(\theta')]. \tag{A.1}$$

Using Eqns. (3.2) and (3.22), the above Jacobi identity gives [6]

$$[\Pi_a(\theta), \Pi_b(\theta')] = i f_{abc} \Pi_c(\theta) \delta(\theta - \theta') + F \tag{A.2}$$

where $[F, g] = 0$. The fact that $[F, g] = 0$ implies that F does not depend on P_i but only on g . Setting $P_i = 0$ in Eqn. (A.2) gives $F = 0$. This proves Eqn. (3.34).

Next we sketch the steps leading to Eqn. (3.39). First we prove the identity

$$f_{abd} N_d^c = -N_a^d \frac{\partial N_b^c}{\partial \xi^d} + N_b^d \frac{\partial N_a^c}{\partial \xi^d} \tag{A.3}$$

which will be used in the proof of the commutator . Using the definition (3.31) of $\Pi_a(\theta)$ we get

$$
\begin{aligned}
[\Pi_a(\theta), \Pi_b(\theta')] &= [N_a^c(\theta) P_c(\theta), N_b^d(\theta') P_d(\theta')] \\
&= N_a^c(\theta)[P_c(\theta), N_b^d(\theta')] P_d(\theta') + N_b^d(\theta')[N_a^c(\theta), P_d(\theta')] P_c(\theta).
\end{aligned}
\tag{A.4}
$$

From Eqn. (A.2) we get,

$$[\Pi_a(\theta), \Pi_b(\theta')] = i f_{abc} \delta(\theta - \theta') \Pi_c(\theta) \tag{A.5}$$

Using Eqns. (A.4), (A.5) and (3.27), Eqn. (A.3) follows easily.

We are now ready to compute $[J_a^0(\theta), J_b^0(\theta')]$. From Eqn. (3.31) we get

$$
\begin{aligned}
[\frac{-2iAR}{\gamma}J_a^0(\theta), \frac{-2iAR}{\gamma}J_b^0(\theta')] \;=\;& [\Pi_a(\theta), \Pi_b(\theta')] \\
+\;& [\Pi_a(\theta), N_b^i(\theta')\omega_{ij}(\theta')\partial_{\theta'}\xi^j] + [N_a^i(\theta)\omega_{ij}(\theta)\partial_\theta\xi^j, \Pi_b(\theta')] \\
+\;& [N_a^i(\theta)\omega_{ij}(\theta)\partial_\theta\xi^j, N_b^i(\theta')\omega_{ij}(\theta')\partial_{\theta'}\xi^j].
\end{aligned} \tag{A.6}
$$

The last commutator is zero as it has no time derivative . The first commutator is given by (A.5) . The third commutator can be obtained from the second by interchanging a with b and θ with θ' then putting an overall minus sign . Let us then compute the second comutator

$$
\begin{aligned}
[\Pi_a(\theta), N_b^i(\theta')\omega_{ij}(\theta')\partial_{\theta'}\xi^j] \;=\;& [\Pi_a(\theta), N_b^i(\theta')]\omega_{ij}(\theta')\partial_{\theta'}\xi^j + N_b^i(\theta')\omega_{ij}(\theta')[\Pi_a(\theta), \partial_{\theta'}\xi^j] \\
+\;& N_b^i(\theta')[\Pi_a(\theta), \omega_{ij}(\theta')]\partial_{\theta'}\xi^j \\
=\;& -i\delta(\theta - \theta')N_a^c\frac{\partial N_b^i}{\partial\xi^c}\omega_{ij}\partial_\theta\xi^j - i\partial_{\theta'}\delta(\theta - \theta')N_b^i(\theta')N_a^j(\theta)\omega_{ij}(\theta') \\
-\;& i\delta(\theta - \theta')N_b^i N_a^c\frac{\partial\omega_{ij}}{\partial\xi^c}\partial_\theta\xi^j
\end{aligned} \tag{A.7}
$$

where we have made use of Eqns. (3.27) and (3.31) . The sum of the second and the third commutator in Eqn. (A.6) is then given by

$$
\begin{aligned}
2 + 3 \;=\;& a + b + c \\
\text{where } \; a \;=\;& -i\delta(\theta - \theta')[N_a^c\frac{\partial N_b^i}{\partial\xi^c} - N_b^c\frac{\partial N_a^i}{\partial\xi^c}]\omega_{ij}\partial_\theta\xi^j \\
b \;=\;& -i\partial_{\theta'}\delta(\theta - \theta')N_b^i(\theta')N_a^j(\theta)\omega_{ij}(\theta') + i\partial_\theta\delta(\theta - \theta')N_a^i(\theta)N_b^j(\theta')\omega_{ij}(\theta) \\
c \;=\;& -i\delta(\theta - \theta')[N_b^i N_a^c - N_a^i N_b^c]\frac{\partial\omega_{ij}}{\partial\xi^c}\partial_\theta\xi^j.
\end{aligned} \tag{A.8}
$$

By using (A.3) we find that

$$
a = i\delta(\theta - \theta')f_{abc}N_c^i\omega_{ij}\partial_\theta\xi^j \tag{A.9}
$$

Careful manipulations with b will give

$$
\begin{aligned}
b \;=\;& -i\partial_{\theta'}\delta(\theta - \theta')N_b^i(\theta')N_a^j(\theta)\omega_{ij}(\theta') + i\partial_\theta\delta(\theta - \theta')N_a^i(\theta)N_b^j(\theta')\omega_{ij}(\theta) \\
=\;& -iN_a^j(\theta)N_b^i(\theta')[\partial_{\theta'}\delta(\theta - \theta')\omega_{ij}(\theta') + \partial_\theta\delta(\theta - \theta')\omega_{ij}(\theta)] \\
=\;& i\delta(\theta - \theta')N_a^j(\theta)N_b^i(\theta')\partial_\theta\omega_{ij} \\
=\;& i\delta(\theta - \theta')N_a^c N_b^i\frac{\partial\omega_{ic}}{\partial\xi^j}\partial_\theta\xi^j.
\end{aligned} \tag{A.10}
$$

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Finally c can be rewitten as

$$c = i\delta(\theta - \theta')N_a^c N_b^i [\frac{\partial \omega_{cj}}{\partial \xi^i} + \frac{\partial \omega_{ji}}{\partial \xi^c}]\partial_\theta \xi^j. \tag{A.11}$$

Putting Eqns. (A.9) , (A.10) and (A.11) together and using Eqns. (3.19) and (3.20), we get

$$
\begin{aligned}
2 + 3 &= i\delta(\theta - \theta')f_{abc}N_c^i \omega_{ij}\partial_\theta \xi^j + i\delta(\theta - \theta')N_a^c N_b^i [\frac{\partial \omega_{ic}}{\partial \xi^j} + \frac{\partial \omega_{cj}}{\partial \xi^i} + \frac{\partial \omega_{ji}}{\partial \xi^c}]\partial_\theta \xi^j \\
&= i\delta(\theta - \theta')f_{abc}N_c^i \omega_{ij}\partial_\theta \xi^j + 3iB\delta(\theta - \theta')N_a^c N_b^i Tr[g^{-1}\frac{\partial g}{\partial \xi^i}, g^{-1}\frac{\partial g}{\partial \xi^c}]g^{-1}g' \\
&= i\delta(\theta - \theta')f_{abc}N_c^i \omega_{ij}\partial_\theta \xi^j + \delta(\theta - \theta')3r^2 B f_{abc}J_a^1. \tag{A.12}
\end{aligned}
$$

Using Eqns. (A.12) and (A.5) in Eqn. (A.6) will lead to the commutation relations

$$[J_a^0(\theta), J_b^0(\theta')] = -\frac{\gamma}{2AR}f_{abc}J_c^0(\theta)\delta(\theta - \theta') - \frac{3B\gamma^2}{4A^2}\delta(\theta - \theta')f_{abc}J_c^1(\theta). \tag{A.13}$$

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