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BCFT and Sliver state

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Abstract

We give a comment on the possible rôle of the sliver state in the generic boundary conformal field theory. We argue that for each Cardy state, there exists at least one projector in the string field theory.

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One of the long standing problem in the string theory is the description of the geometry. Generically we do not need the space-time at all to define the consistent string background. What is necessary instead is the modular invariant representation of the (super) Virasoro algebra with the certain central extension. If we need the geometry, we have to extract it in principle from the algebraic data of the CFT, namely the set of the primary fields, their OPE coefficients, the modular invariant combination of the left and right movers and so on.

In the development of the noncommutative geometry [1, 2], a hint to this problem is given in the context of the noncommutative soliton[3]. In this approach the description of the space-time is replaced by the (in general noncommutative) C^*-algebra $\mathcal{A}$. The noncommutative soliton is defined as the projection operator of $\mathcal{A}$. Immediately after the discovery, it is interpreted as the D-brane in the presence of the background B field [4].

In the noncommutative geometry, a canonical way to extract the geometry from the algebraic data is through the study their K-group [5, 6] which are classified into two types, $K_0(\mathcal{A})$ and $K_1(\mathcal{A})$. The latter is identified as the isomorphic class of the unitary operator in $\mathrm{Mat}_{N\times N}(\mathcal{A})$ for large enough N. On the other hand the former one is represented by the projection operator in $\mathrm{Mat}_{N\times N}(\mathcal{A})$ which fits naturally with the very definition of the noncommutative soliton. In the commutative geometry, the D-brane charge is classified by the topological K-group [7, 8]. The noncommutative soliton gives the representation of K-homology group of the operator algebra and thus describes the D-brane in the noncommutative situation[9, 10].

Since the current examples of the noncommutative geometry is restricted to the rather simple spaces such as the Moyal plane or the noncommutative torus, it is desirable to extend such framework to the full string theory. In this language the operator algebra $\mathcal{A}$ should be replaced by the full (boundary) CFT module. This is, actually, the philosophy advocated by Witten long ago [11] (see also [12]) in his open string field theory.

Recently there is a remarkable progress toward this direction [13, 14, 15, 16, 17, 18]. It is based on the conjecture that we may find the ghost string field which satisfies,

$$Q\Psi_{gh} = \Psi_{gh} \star \Psi_{gh} \tag{1}$$

and if one factories the full string field as $\Psi = \Psi_m \otimes \Psi_{gh}$, the equation of motion for the matter part becomes,

$$\Psi_m \star \Psi_m = \Psi_m\ , \tag{2}$$

namely it is the projector with respect to Witten's $\star$-product. Furthermore there is an explicit representation of the projector [14] which was identified with the sliver state found in [13]. It is conjectured that it describes the $D25$-brane. In [17, 18] the star product is reformulated as the matrix multiplication (see also [11]) and the sliver state is identified as the rank 1 projector in this matrix algebra.

In the boundary conformal field theory description, the D-brane is described by the boundary state (Cardy state). It interesting to see to what extent the information of Cardy state is encoded in the solution to (2).

We start from the general (rational) conformal field theory with the set of the primary fields ϕ_i labeled by the set $i \in \mathcal{I}$ (we use the notation of [19]). The information of the module over the primary field is encoded in Ishibashi state [20]$|i\rangle\rangle$ which satisfies,

$$(L_n - \bar{L}_{-n})|i\rangle\rangle = 0, \qquad \langle\langle i|\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|j\rangle\rangle = \delta_{ij}\chi_i(\tilde{q}) \ , \tag{3}$$

where $\chi_i(\tilde{q})$ is the character of the irreducible representation associated with the primary field ϕ_i and $\tilde{q} = e^{2\pi i/\tau}$. In order to give a well-defined Hilbert space sum in the open string sector, we need to take the linear combination of the Ishibashi state (the Cardy state) [21],

$$|a\rangle = \sum_{j\in\mathcal{E}} \frac{\psi_a^j}{\sqrt{S_{1j}}}|j\rangle\rangle \tag{4}$$

where S_{ij} is the modular transformation matrix of the character (1 represents the identity primary field), ψ_a^j is the coefficients which is determined by the constraints in the following and $a \in \mathcal{V}$ is the set of labels for the boundary state. $\mathcal{E}$ (exponent) is the subset of $\mathcal{I}$ which has the diagonal term in the modular invariant partition function [19]. The inner product between the boundary state is given after the modular transformation,

$$\langle b|\tilde{q}^{\frac{1}{2}(L_0+\bar{L}_0-\frac{c}{12})}|a\rangle = \sum_{i\in\mathcal{I}} \chi_i(q) n_{ia}^b, \tag{5}$$

with $q = e^{2\pi i\tau}$ and

$$n_{ia}^b = \sum_{j\in\mathcal{E}} \psi_a^j (\psi_b^j)^* \frac{S_{ij}}{S_{1j}}. \tag{6}$$

In order that (5) have well-defined interpretation as the trace over open string Hilbert space the coefficients n_{ia}^b must be non-negative integers. The matrices $(n_i)_a^b = n_{ia}^b$ should satisfy the Verlinde fusion algebra [22],

$$n_i n_j = \sum_{k\in\mathcal{I}} N_{ij}^k n_k \ , \qquad N_{ij}^k = \sum_{\ell\in\mathcal{I}} \frac{S_{i\ell}S_{j\ell}S_{k\ell}^*}{S_{1\ell}} \ . \tag{7}$$

As a consequence, the boundary fields ${}^b\Psi_{j,\beta}^a$ need to have four labels, $j \in \mathcal{I}$ which represents the chiral operator, $a, b \in \mathcal{V}$ for the two boundaries and $\beta = 1, \cdots, n^b_{ia}$ to specify the redundancy in this channel. When $j = 1$, namely when the chiral field is the identity operator, there is the natural restriction $a = b$ and $n^b_{1a} = 1$.

For each boundary fields, one may assign the corresponding open string field as,

$$ {}^b\Psi^a_{j,\beta} \to {}^b\hat{\Psi}^a_{j,\beta} \tag{8} $$

which is defined by inserting the field ${}^b\Psi^a_{j,\beta}$ at the origin of the upper half plane and perform the path integral on the half disk [11].

As explicitly discussed in section 3 of [13], one may define the star product between the string field thus defined without the explicit knowledge of the oscillator representation. Indeed the three string vertex operator can be completely specified by the infinite set of the conformal Ward identities,

$$ \langle V_3| \oint_{\mathcal{C}} v(z)T(z)dz = 0, \tag{9} $$

where $\mathcal{C} = \mathcal{C}_1 + \mathcal{C}_2 + \mathcal{C}_3$ is the contour around three punctures and $v(z)$ is the holomorphic vector fields in the interior of the vertex.

It implies that the star product between the string associated with the boundary operator can be derived from the operator product expansion for the boundary operator [19],

$$ {}^b\Psi^c_{i,\alpha_1}(x_1)\,{}^c\Psi^a_{j,\alpha_2}(x_2) = \sum_{p,\beta,t} {}^{(1)}F_{cp} \left[\begin{array}{cc} i & j \\ b & a \end{array} \right]^{\beta\ t}_{\alpha_1\ \alpha_2} \frac{1}{x_{12}^{\Delta_i+\Delta_j-\Delta_p}}\,{}^b\Psi^a_{p,\beta}(x_2) + \cdots, \tag{10} $$

where Δ_i is the dimension of the ith chiral field and ${}^{(1)}F$ is the $3j$ symbol for the boundary field fusion algebra. This algebra should be modified into the star product algebra for the string fields,

$$ {}^b\hat{\Psi}^c_{i,\alpha_1} \star {}^c\hat{\Psi}^a_{j,\alpha_2} = \sum_{p,\beta,t,\{n\}} {}^{(1)}\tilde{F}_{cp} \left[\begin{array}{cc} i & j \\ b & a \end{array} \right]^{\beta\ t}_{\alpha_1\ \alpha_2} (\{n\})\,{}^b\hat{\Psi}^a_{p,\beta}(\{n\}), \tag{11} $$

where the summation over $\{n\}$ stands for the string field associated with the descendants. ${}^{(1)}\tilde{F}$ is the coefficients which should be derived from ${}^{(1)}F$ but their explicit forms will not be necessary at the level of this paper.

To define the projector for the star product is reduced to the problem of finding them in (11), which is rather hopeless unless we know the good handling of the $3j$ symbols. At this point, we would like to indicate that there is an easy but seemingly meaningful solution. Let us consider ${}^a\Psi^b_{1,\alpha}$ which is associated with the identity operator. In this case, we need to put

$a = b$ and the index α can be omitted since $n_{1a}^{a} = 1$. If we write $\mathbf{1}^a \equiv {}^a\Psi_1^a$, eq.(10) is written as,

$$\mathbf{1}^a(x_1)\mathbf{1}^b(x_2) = \delta_{ab}\mathbf{1}^b(x_2)\ . \tag{12}$$

While it is the projection at the level of the operator product, it is quite nontrivial to find the projector at (11) because of the inclusion of the descendants. However, quite remarkably, this is exactly the situation considered in section 6 of [13]. Namely the $n \to \infty$ limit of the conformal mapping,

$$w = \left(\frac{1+iz}{1-iz}\right)^{2/n} \tag{13}$$

gives a finite transformation in terms of the Virasoro operators,

$$U^{sliver} \equiv \exp\left(-\frac{1}{3}L_{-2} + \frac{1}{30}L_{-4} - \frac{11}{1890}L_{-6} + \cdots\right) \tag{14}$$

which defines the infinitely thin wedge (sliver)[2]. We apply this operator to the string field associated with $\mathbf{1}^a$ and denote the corresponding field as Ξ^a. The claim in [13] can be generalized in our situation as

$$\Xi^a \star \Xi^b = \delta_{ab}\,\Xi^b\ . \tag{15}$$

Our argument may be summarized as follows. For each Cardy state (D-brane) labeled by $a \in \mathcal{V}$, there always exists identity operator $\mathbf{1}^a$ in the a-a sector of the open string and one can construct a projector Ξ^a in the string field theory as the sliver state. This result seems to give a possible partial answer to the general expectation that D-branes can be obtained as the solution to the open string field theory. It may also imply that the D-brane charge is classified by the K-group of the boundary conformal field theory.

At this point we would like to make a few comments which seems to be relevant to the future study.

1. We suppose that the solution describes the D-brane of the index $a \in \mathcal{V}$ since it does nothing but project the boundary to a. If there is another solution, it defines the projective module over the D-brane and thus describes the D-branes which may appear after tachyon condensation [23] (see also [24] for the early attempts).

[2]The identity operator $\mathcal{I}^a$ in the string field theory ($n = 1$ case in (13)) gives another solution to this problem satisfying $\mathcal{I}^a \star \mathcal{I}^b = \delta_{ab}\mathcal{I}^a$. At this point, we have no strong reason that the sliver state is better suited than this one except for the arguments in [16, 17]. We would like to come back to this problem in our future study.

2. From the Virasoro algebra view point, it is more natural to consider the projector into the irreducible representation which appeared in the different context in [19],

$$\Pi_i \equiv \sum_{\{n\}} |i\,\{n\}\rangle\langle i\,\{n\}\,| \;, \quad \Pi_i\Pi_j = \delta_{ij}\Pi_j \tag{16}$$

where $|i\,\{n\}\rangle$ is the normalized Virasoro descendant states from the highest weight state $|i\rangle$ and the summation is taken over all the Virasoro module. This projector satisfies

$$L_n\Pi_i = \Pi_i L_n \;, \quad \text{and} \quad \mathrm{Tr}(\Pi_i\, q^{L_0 - c/24}) = \chi_i(q) \tag{17}$$

and it may be regarded as the analogue of the Ishibashi state. It is natural to guess that there is a linear transformation between Π_i and Ξ^a (or possibly $\mathcal{I}^a$) which is similar to (4) but at this moment it is difficult to find the explicit form. The problem comes from the fact that string field we constructed is the element of the Hilbert state and not the operators acting on it. The split string formalism [11, 17, 18, 25] seems to give an important hint to this issue.

3. An interesting question is what is the trace of the projector. To seek the analogy with the noncommutative soliton [4], it seems natural to impose the trace to be one to get a single D-brane. While it is conjectured to be true for the sliver [16, 17], the question is whether this is the necessary constraint. The difference between the usual noncommutative soliton and the string field theory projector is that we need only the zero mode in the former and the full Virasoro module in the latter.

   If the string algebra belongs to the type I von Neumann factor (roughly speaking the matrix algebra) it is quantized and takes the integer value. We think that it is not generally true. In the noncommutative theory on the quantum torus [26], we encountered type II factor and the trace of the projector takes the continuous value. If it belongs to type III, the trace becomes ill-defined. This may be possible since, for example, the trace of the projector (16) is infinite. The subtle point is whether to include the conformal descendants and is related to the definition of the ghost string field [16, 17].

4. More sophisticated use of the projectors to classify the representation of the Virasoro algebra was recently considered in [27]. They applied Ocneanu's paragroup [28, 29] to BCFT. Ocneanu's method is to use

the fusion algebra to define an infinite sequence of von Neumann factors. Accordingly the basic data of the paragroup is the fusion algebra and the $6j$-symbols which satisfy the analogue of Moore-Seiberg relations[30]. It will be quite interesting to find a direct relation between the projectors in the string field theory and generators of Ocneanu's double triangle algebra [27] if it may lead to give a classification of the projector of the fusion algebra (10).

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Note added: After we have written this letter, we found that the similar issue is discussed in [31].

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