

Wilsonian effective action for SU(2) Yang-Mills theory with Cho-Faddeev-Niemi-Shabanov decomposition

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Abstract

The Cho-Faddeev-Niemi-Shabanov decomposition of the SU(2) Yang-Mills field is employed for the calculation of the corresponding Wilsonian effective action to one-loop order with covariant gauge fixing. The generation of a mass scale is observed, and the flow of the marginal couplings is studied. Our results indicate that higher-derivative terms of the color-unit-vector \mathbf{n} field are necessary for the description of topologically stable knotlike solitons which have been conjectured to be the large-distance degrees of freedom.

1 Introduction

The fact that quarks and gluons are not observed as asymptotic states in our world indicates that a description in terms of these fields is not the most appropriate language for discussing low-energy QCD. On the other hand, there seems to be little predictive virtue in describing the low-energy domain only by observable quantities, such as mesons and baryons. A purposive procedure can be the identification of those (not necessarily observable) degrees of freedom of the system that allow for a “simple” description of the observable states. The required “simplicity” can be measured in terms of the simplicity of the action that governs those degrees of freedom. Clearly, a clever guess of such degrees of freedom is halfway to the solution of the theory; the remaining problem is to prove that these degrees of freedom truly arise from the fundamental theory by integrating out the high-energy modes.

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For the pure Yang-Mills (YM) sector of QCD, such a guess has recently been made by Faddeev and Niemi [1] inspired by the work of Cho [2]. For the gauge group $SU(2)$, they decomposed the (implicitly gauge-fixed) gauge potential \mathbf{A}_μ into an “abelian” component C_μ , a unit color vector \mathbf{n} and a complex scalar field φ ; here, C_μ is the local projection of \mathbf{A}_μ onto some direction in color space defined by the space-dependent \mathbf{n} . Faddeev and Niemi conjectured that the important low-energy dynamics of $SU(2)$ YM theory¹ is determined by the \mathbf{n} field; its effective action of nonlinear sigma-model type, the Skyrme-Faddeev model, should then arise from integrating out the further degrees of freedom: C_μ, φ, \dots :

$$\Gamma_{\text{eff}}^{\text{FN}} = \int d^4x \left[m^2 (\partial_\mu \mathbf{n})^2 + \frac{1}{g^2} (\mathbf{n} \cdot \partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})^2 \right]. \quad (1)$$

The additional mass scale m is expected to be generated by the integration process as well; first hints of this mechanism have been observed in a one-loop integration over a reduced set of variables [5, 6]. The associated knotlike solitonic excitations of the Skyrme-Faddeev model are supposed to be identified with glue balls (which are directly observable at least on the lattice).²

The presence of gauge symmetry in YM theory complicates this ambitious conjecture in two ways: first, in order to formulate a quantum theory, the decomposition of \mathbf{A}_μ has to also include the overabundant gauge degrees of freedom; and secondly, the gauge has then to be fixed in a prescribed way, not only to be able to perform functional integration, but also to arrive nevertheless at a unique \mathbf{n} field.³

The first problem was solved by Shabanov [9, 10], who established a one-to-one correspondence between the unfixed gauge field \mathbf{A}_μ and its decomposition, and the quantum theory was formulated; his results are briefly sketched in Sect. 2 and shall serve as the starting point of our investigations. The second problem of gauge fixing implies that a successful realization of the ideas of Faddeev and Niemi will only be meaningful in a certain gauge. In this (a priori unknown) gauge, the important low-energy degrees of freedom might in fact be determined by the \mathbf{n} field and a simple action, whereas in a different gauge, these degrees of freedom may be hidden in a highly complicated structure involving the \mathbf{n} and other fields.

The present paper is dedicated to a calculation of the one-loop Wilsonian effective action for $SU(2)$ Yang-Mills theory in terms of the gauge field decomposition of Shabanov. Our intention is to study the renormalization group flow of the mass scale parameter of Eq. (1), the gauge coupling and further marginal couplings. In view of the second problem mentioned above, our results and their interpretation are strictly tied to the particular

¹Different generalizations of the gauge field decomposition for higher gauge groups can be found in [3], [4] and [9].

²In a very recent paper [7], Faddeev and Niemi generalized their decomposition in order to obtain a manifest duality between the here-considered “magnetic” and additional “electric” variables, involving an abelian scalar multiplet with two complex scalars. This electric sector will not be considered in the present work.

³A different approach was put forward in [8], where the \mathbf{n} field was identified by constructing an unconstrained version of $SU(2)$ Yang-Mills theory in a Hamiltonian context.

gauge we shall choose. We face this problem by fixing the gauge in such a way that Lorentz invariance and global color transformations remain as residual symmetries; these are the symmetries of the Skyrme-Faddeev model and must mandatorily be respected.

The Wilsonian effective action is characterized by the fact that it governs the dynamics of the low-energy modes below a certain cutoff Λ ; it incorporates the interactions that are induced by high-energy fluctuations with momenta between Λ and the ultraviolet (UV) cutoff Λ which have been integrated out. Following the Faddeev-Niemi conjecture, we only retain the \mathbf{n} field as low-energy degree of freedom. Actually, we integrate over the high-energy modes in two different ways: first, we integrate out the $\Lambda < p < \Lambda$ fluctuations of all fields *except for* the \mathbf{n} field, which is left untouched (Sec. 3). Secondly, we integrate all fields *including* the \mathbf{n} field over the same momentum shell (Sec. 4). In this way, we can study the effect of the \mathbf{n} field fluctuations on the flow of the mass scale and the couplings in detail.

The results for both calculations are similar: the mass scale m appearing in Eq. (1) is indeed generated by the renormalization group flow, and the gauge coupling is asymptotically free. As far as the simplicity of the conjectured effective action Eq. (1) is concerned, our results are a bit disappointing: as discussed in Sec. 5, further marginal terms (not displayed in Eq. (1)) are of the same order as the displayed one and therefore have to be included in Eq. (1). Keeping only those terms that involve single derivatives acting on \mathbf{n} results in an action without stable solitons; nevertheless, stability is in fact ensured owing to the presence of higher-derivative terms. The disadvantage is that these terms spoil the desired simplicity of the low-energy effective theory.

Of course, our perturbative results represent only a first glance at the true infrared behavior of the system and are far from providing qualitatively confirmed results, not to mention quantitative predictions. To be precise, the one-loop calculation investigates only the form of the renormalization group trajectories of the couplings in the vicinity of the perturbative Gaussian fixed point. Nevertheless, various extrapolations of the perturbative trajectories can elucidate the question as to whether the Faddeev-Niemi conjecture is realizable or not.

2 Quantum Yang-Mills theory in Cho-Faddeev-Niemi-Shabanov variables

In decomposing the Yang-Mills gauge connection, we follow [2, 9, 10]. Let \mathbf{A}_μ be an SU(2) connection where the color degrees of freedom are represented in vector notation. We parametrize \mathbf{A}_μ as

$$\mathbf{A}_\mu = \mathbf{n} C_\mu + (\partial_\mu \mathbf{n}) \times \mathbf{n} + \mathbf{W}_\mu, \quad (2)$$

where the cross product is defined via the SU(2) structure constants. C_μ is an “abelian” connection, whereas \mathbf{n} denotes a unit vector in color space, $\mathbf{n} \cdot \mathbf{n} = 1$. \mathbf{W}_μ shall be orthogonal to \mathbf{n} in color space, obeying $\mathbf{W}_\mu \cdot \mathbf{n} = 0$, so that $C_\mu = \mathbf{n} \cdot \mathbf{A}_\mu$. For a given \mathbf{n} , C_μ and \mathbf{W}_μ , the connection \mathbf{A}_μ is uniquely determined by Eq. (2). In the opposite direction,

there is still some arbitrariness: for a given \mathbf{A}_μ , \mathbf{n} can generally be chosen at will, but then C_μ and \mathbf{W}_μ are fixed (e.g., $\mathbf{W}_\mu = \mathbf{n} \times D_\mu(\mathbf{A})\mathbf{n}$, where D_μ denotes the covariant derivative).

While the LHS of Eq. (2) describes $3_{\text{color}} \times 4_{\text{Lorentz}} = 12$ off-shell and gauge-unfixed degrees of freedom, the RHS up to now allows for $(C_\mu:)4_{\text{Lorentz}} + (\mathbf{n}:)2_{\text{color}} + (\mathbf{W}_\mu:)3_{\text{color}} \times 4_{\text{Lorentz}} - 4_{\mathbf{n} \cdot \mathbf{W}_\mu=0} = 14$ degrees of freedom. Two degrees of freedom on the RHS remain to be fixed. For example, by fixing \mathbf{n} to point along a certain direction and imposing gauge conditions on \mathbf{W}_μ , we arrive at the class of abelian gauges which are known to induce monopole degrees of freedom in C_μ . In order to avoid these topological defects, we let \mathbf{n} vary in spacetime and impose a general condition on C_μ, \mathbf{n} and \mathbf{W}_μ ,

$$\chi(\mathbf{n}, C_\mu, \mathbf{W}_\mu) = 0, \quad \text{with} \quad \chi \cdot \mathbf{n} = 0, \quad (3)$$

which fixes the redundant two degrees of freedom on the RHS of Eq. (2). Moreover, Eq. (3) also determines how \mathbf{n} , C_μ and \mathbf{W}_μ transform under gauge transformations of \mathbf{A}_μ : by demanding that $\delta\chi(\mathbf{n}, C_\mu(\mathbf{A}), \mathbf{W}_\mu(\mathbf{A})) = 0$ (and $\delta(\chi \cdot \mathbf{n}) = 0$), the transformation $\delta\mathbf{n}$ of \mathbf{n} is uniquely determined, from which δC_μ and $\delta\mathbf{W}_\mu$ are also obtainable.

The thus established one-to-one correspondence between \mathbf{A}_μ and its decomposition (2) allows us to rewrite the generating functional of YM theory in terms of a functional integral over the new fields [9, 10]:

$$Z = \int \mathcal{D}\mathbf{n} \mathcal{D}C \mathcal{D}\mathbf{W} \delta(\chi) \Delta_S \Delta_{\text{FP}} e^{-S_{\text{YM}} - S_{\text{gf}}}. \quad (4)$$

Beyond the usual Faddeev-Popov determinant Δ_{FP} , the YM action S_{YM} and the gauge fixing action S_{gf} , we find one further determinant introduced by Shabanov, Δ_S ; this determinant accompanies the δ functional which enforces the constraint $\chi = 0$, in complete analogy to the Faddeev-Popov procedure:

$$\Delta_S := \det \left(\frac{\delta\chi}{\delta\mathbf{n}} \Big|_{\chi=0} \right). \quad (5)$$

All objects in the integrand of Eq. (4) are understood to be functions of the 14 integration variables \mathbf{n} , C_μ and \mathbf{W}_μ .

By construction, the generating functional (4) is invariant under different choices of χ for the same reason that it is invariant under different choices of the gauge – this is controlled by the Faddeev-Popov procedure.

Nevertheless, the choice of χ crucially belongs to the definition of the decomposition (2) and of the conjectured low-energy degrees of freedom; in other words, even if there is one particular χ that leads to Eq. (1) as the true low-energy effective action after integrating out C_μ and \mathbf{W}_μ , other choices of χ will not lead to the same result, because the low-energy degrees of freedom then are differently distributed over \mathbf{n} , C_μ and \mathbf{W}_μ .

In the present work, χ is chosen in such a way that \mathbf{n} transforms homogeneously under gauge transformations, i.e., \mathbf{n} is orthogonally rotated in color space [2]:

$$\begin{aligned} 0 &= \chi := \partial_\mu \mathbf{W}_\mu + C_\mu \mathbf{n} \times \mathbf{W}_\mu + \mathbf{n}(\mathbf{W}_\mu \cdot \partial_\mu \mathbf{n}), \\ \Rightarrow \quad \delta\mathbf{n} &= \mathbf{n} \times \boldsymbol{\varphi}, \quad \text{under } \delta\mathbf{A}_\mu = D_\mu(\mathbf{A})\boldsymbol{\varphi} = \partial_\mu \boldsymbol{\varphi} + \mathbf{A}_\mu \times \boldsymbol{\varphi}. \end{aligned} \quad (6)$$

Incidentally, the gauge transformation properties of \mathbf{C}_μ and \mathbf{W}_μ also become very simple with the choice (6): \mathbf{W}_μ also transforms homogeneously, and $\delta C_\mu = \mathbf{n} \cdot \partial_\mu \boldsymbol{\varphi}$.

Finally, the choice of the gauge-fixing condition must also be viewed as being part of the definition of the decomposition. Not only does the functional form of Δ_{FP} and S_{gf} depend on this choice, but the discrimination of high- and low-momentum modes is also determined by the gauge fixing. In fact, this gauge dependence of the mode momenta usually is the main obstacle against setting up a Wilsonian renormalization group study. But in the present context, it belongs to the conjecture that the particular gauge that we shall choose singles out those low-momentum modes which finally provide for a simple description of low-energy QCD; in a different gauge, we would encounter different low-momentum modes, but we also would not expect to find the same simple description.

In this work, we choose the covariant gauge condition $\partial_\mu \mathbf{A}_\mu = 0$. This automatically ensures covariance of the resulting effective action and, moreover, allows for the residual symmetry of global gauge transformations, $\boldsymbol{\varphi} = \text{const}$. Together with the choice (6), this residual symmetry coincides with the desired global color symmetry of the Skyrme-Faddeev model (1). This means that the demand for color and Lorentz symmetry of the action (1) is satisfied exactly by a covariant gauge and condition (6).

3 One-loop effective action without \mathbf{n} fluctuations

Our aim is the construction of the one-loop Wilsonian effective action for the \mathbf{n} field by integrating out the \mathbf{C} and \mathbf{W} field over a momentum shell between the UV cutoff Λ and an infrared cutoff $k < \Lambda$. In general, this will induce nonlinear and nonlocal self-interactions of the \mathbf{n} field; since we are looking for an action of the type (1), we represent these interactions in a derivative expansion and neglect higher derivative terms of order $\mathcal{O}(\partial^2 \mathbf{n} \partial^2 \mathbf{n})$ (later, we shall question this approach).

Furthermore, we do not integrate out \mathbf{n} field fluctuations in this section (see Sect. 4) and disregard any induced \mathbf{C} or \mathbf{W} interactions below the infrared cutoff k . From a technical viewpoint, the one-loop approximation of the desired effective action $\hat{\Gamma}_k[\mathbf{n}]$ is obtained by a Gaussian integration of the quadratic \mathbf{C} and \mathbf{W} terms in Eq. (4), neglecting higher-order terms of the action:

$$e^{-\hat{\Gamma}_k[\mathbf{n}]} = e^{-S_{\text{cl}}[\mathbf{n}]} \int_k \mathcal{D}\mathbf{C}\mathcal{D}\mathbf{W} \Delta_{\text{S}}[\mathbf{n}] \Delta_{\text{FP}}[\mathbf{n}] \delta(\boldsymbol{\chi}) \times e^{-\frac{1}{g^2} \int \{ C_\mu \frac{1}{2} M_{\mu\nu}^C C_\nu + \mathbf{W}_\mu \frac{1}{2} M_{\mu\nu}^{\mathbf{W}} \mathbf{W}_\nu + C_\mu \mathbf{Q}_{\mu\nu}^C \cdot \mathbf{W}_\nu + C_\nu K_\nu^C + \mathbf{W}_\mu \cdot \mathbf{K}_\mu^{\mathbf{W}} \}}, \quad (7)$$

where the hat on $\hat{\Gamma}_k[\mathbf{n}]$ indicates that the \mathbf{n} field fluctuations have not been taken into account. Furthermore, any \mathbf{C} or \mathbf{W} dependence of Δ_{S} and Δ_{FP} has been neglected to one-loop order; the various differential operators and currents which all depend on \mathbf{n} (and the gauge parameter α) are defined in Appendix A. The classical action of \mathbf{n} including gauge fixing terms is given by:

$$S_{\text{cl}}[\mathbf{n}] := \int d^4x \left(\frac{1}{4g^2} (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})^2 + \frac{1}{2\alpha g^2} (\partial^2 \mathbf{n} \times \mathbf{n})^2 \right). \quad (8)$$

We treat the δ functional in Eq. (7) in its Fourier representation,

$$\delta(\chi) \rightarrow \int \mathcal{D}\phi e^{-i \int \phi \cdot \partial_\mu \mathbf{W}_\mu + \phi \cdot C_\mu \mathbf{n} \times \mathbf{W}_\mu + (\phi \cdot \mathbf{n})(\partial_\mu \mathbf{n} \cdot \mathbf{W}_\mu)}, \quad (9)$$

where the second term in the exponent, the triple vertex, can actually be neglected, because it leads only to nonlocal terms (cf. later) or terms of higher order in derivatives. Inserting Eq. (9) into Eq. (7), we end up with three functional integrals over \mathbf{C} , \mathbf{W} and ϕ , which can successively be performed, leading to three determinants,

$$e^{-\hat{\Gamma}_k[\mathbf{n}]} \rightarrow e^{-S_{\text{cl}}[\mathbf{n}]} \Delta_S[\mathbf{n}] \Delta_{\text{FP}}[\mathbf{n}] (\det M^C)^{-1/2} (\det \overline{M}^{\mathbf{W}})^{-1/2} (\det -\tilde{Q}_\mu^\phi (\overline{M}^{\mathbf{W}})^{-1}_{\mu\nu} Q_\nu^\phi)^{-1/2}, \quad (10)$$

where we have omitted several nonlocal terms that arise from the completion of the square in the exponent during the Gaussian integration. In Appendix B, we argue that these nonlocal terms are unimportant in the present Wilsonian investigation. Again, details about the various operators in Eq. (10) are given in App. A.

The determinants are functionals of \mathbf{n} only and have to be evaluated over the space of test functions with momenta between k and Λ . The determinants depend also on the gauge parameter α . Only for the Landau gauge $\alpha = 0$ is the gauge-fixing δ functional implemented exactly; in fact, $\alpha = 0$ appears to be a fixed point of the renormalization group flow [11]. But this in turn ensures that the choice of $\alpha = \alpha(k) \equiv \alpha_k$ at the cutoff scale $k \rightarrow \Lambda$ is to some extent arbitrary, since α_k flows to zero anyway as k is lowered. This allows us to conveniently choose $\alpha_{k=\Lambda} = 1$ at the cutoff scale and evaluate the determinants with this parameter choice.

As mentioned above, we evaluate the determinants in a derivative expansion based on the assumption that the low-order derivatives of \mathbf{n} represent the essential degrees of freedom in the low-energy domain. There are various techniques for the calculation at our disposal; it turns out that a direct momentum expansion of the operators is most efficient.⁴ We shall demonstrate this method by means of the third determinant of Eq. (10), the “ \mathbf{C} determinant”; the key observation is that derivatives acting on the space of test functions create momenta of the order of p with $k < p < \Lambda$, whereas derivatives of the \mathbf{n} field are assumed to obey $|\partial \mathbf{n}| \ll k$ in agreement with the Faddeev-Niemi conjecture. This suggests an expansion of the form

$$\begin{aligned} \ln(\det M^C)^{1/2} &= -\frac{1}{2} \text{Tr} \ln(-\partial^2 \mathbb{1}_L + \partial \mathbf{n} \cdot \partial \mathbf{n}) \\ &= -\frac{1}{2} \text{Tr} \left[\ln(-\partial^2 \mathbb{1}_L) + \ln \left(\mathbb{1}_L + \frac{\partial \mathbf{n} \cdot \partial \mathbf{n}}{-\partial^2} \right) \right] \\ &= -\frac{1}{2} \text{Tr} \ln(-\partial^2 \mathbb{1}_L) - \frac{1}{2} \text{Tr} \frac{\partial \mathbf{n} \cdot \partial \mathbf{n}}{-\partial^2} + \frac{1}{4} \text{Tr} \left(\frac{\partial \mathbf{n} \cdot \partial \mathbf{n}}{-\partial^2} \right)^2 + \mathcal{O}((\partial \mathbf{n})^6), \end{aligned} \quad (11)$$

where we suppressed Lorentz (L) indices. Here, we neglected higher-derivative terms of \mathbf{n} , e.g., $\partial^2 \mathbf{n}$, which is in the spirit of the Faddeev-Niemi conjecture; of course, this has to be

⁴As cross-checks, we also employed a proptime representation for the operators which we decomposed with a heat-kernel expansion as well as with a multiple use of the Baker-Campbell-Hausdorff formula.

checked later on. Employing the integral formulas given in App. C, we finally obtain for the Δ determinant

$$\begin{aligned} \ln(\det M^C)^{1/2} \simeq & -\frac{1}{32\pi^2}(\Lambda^2 - k^2) \int_x (\partial_\mu \mathbf{n})^2 \\ & - \frac{1}{32\pi^2} \ln \frac{\Lambda}{k} \int_x (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})^2 + \frac{1}{32\pi^2} \ln \frac{\Lambda}{k} \int_x (\partial_\mu \mathbf{n})^4, \end{aligned} \quad (12)$$

where $\int_x \equiv \int d^4x$. The first term contributes to the desired mass term of Eq. (1), whereas the second and third renormalize the classical action (8).

The remaining four determinants of Eq. (10) have to be evaluated in the same way. The calculation is straightforward though extensive. Here, we shall cite only the final results:

$$\begin{aligned} \ln \Delta_{\text{FP}} &= -\frac{(\Lambda^2 - k^2)}{64\pi^2} \int_x (\partial_\mu \mathbf{n})^2 + \frac{1}{48\pi^2} \ln \frac{\Lambda}{k} \int_x (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})^2 - \frac{1}{32\pi^2} \ln \frac{\Lambda}{k} \int_x (\partial_\mu \mathbf{n})^4, \\ \ln(\det \overline{M}^{\mathbf{W}})^{-1/2} &= -\frac{5(\Lambda^2 - k^2)}{64\pi^2} \int_x (\partial_\mu \mathbf{n})^2 - \frac{5}{24\pi^2} \ln \frac{\Lambda}{k} \int_x (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})^2 + \frac{35}{128\pi^2} \ln \frac{\Lambda}{k} \int_x (\partial_\mu \mathbf{n})^4, \\ \ln(\det -\tilde{Q}^\phi \overline{M}^{\mathbf{W}^{-1}} Q^\phi)^{-1/2} &= \frac{3(\Lambda^2 - k^2)}{128\pi^2} \int_x (\partial_\mu \mathbf{n})^2 + \frac{49}{192\pi^2} \ln \frac{\Lambda}{k} \int_x (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})^2 \\ &\quad - \frac{5}{16\pi^2} \ln \frac{\Lambda}{k} \int_x (\partial_\mu \mathbf{n})^4. \end{aligned} \quad (13)$$

The determinant $\Delta_{\mathbf{S}}$ does not contribute, because it is independent of \mathbf{n} in one-loop approximation. Inserting these results into Eq. (10) leads us to the desired Wilsonian effective action to one-loop order for the \mathbf{n} field in a derivative expansion:

$$\begin{aligned} \hat{\Gamma}_k[\mathbf{n}] &= \frac{13}{8} \frac{\Lambda^2}{16\pi^2} (1 - e^{2t}) \int_x (\partial_\mu \mathbf{n})^2 + \frac{1}{4} \left(\frac{1}{g^2} + \frac{7}{3} \frac{1}{16\pi^2} t \right) \int_x (\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})^2 \\ &\quad - \frac{1}{2} \left(\frac{1}{\alpha g^2} + \frac{5}{4} \frac{1}{16\pi^2} t \right) \int_x (\partial_\mu \mathbf{n})^4, \end{aligned} \quad (14)$$

where $t = \ln k/\Lambda \in [-\infty, 0]$ denotes the “renormalization group time”. We would like to stress once more that $\hat{\Gamma}_k[\mathbf{n}]$ does not contain the result of fluctuations of the \mathbf{n} field itself; in other words, it represents (an approximation to) the “tree-level action” for the complete quantum theory of the \mathbf{n} field.

Indeed, the generation of a “kinetic” term $\sim (\partial_\mu \mathbf{n})^2$ growing under the flow of increasing k as conjectured by Faddeev and Niemi is observed. Moreover, it has the correct sign (+), implying that an “effective field theory” interpretation seems possible. The second term which is proportional to the classical action reveals information about the renormalization of the Yang-Mills coupling:

$$\frac{1}{\hat{g}_{\text{R}}^2} := \frac{1}{g^2} + \frac{7}{3} \frac{1}{16\pi^2} t \quad \Rightarrow \quad \hat{\beta}_{g^2} := \partial_t \hat{g}_{\text{R}}^2 = -\frac{7}{3} \frac{1}{16\pi^2} \hat{g}_{\text{R}}^4. \quad (15)$$

The resulting $\hat{\beta}$ function is a factor of $44/7$ smaller than the β function of full Yang-Mills theory for SU(2). This is an expected result, since we did not integrate over all

degrees of freedom of the gauge field; the \mathbf{n} integration still remains. Nevertheless, the β function implies asymptotic freedom, which indicates that the decomposition of the Yang-Mills field is not a pathologically absurd choice. It is interesting to observe that the \mathbf{C} and \mathbf{W} determinants contribute positively to $\hat{\beta}_{g^2}$, whereas the Faddeev-Popov and the ϕ determinant contribute negatively; the latter, which arises from the \mathbf{W} fixing, even dominates: $-7/3 = [6_C - 4_{\text{FP}} + 40_{\mathbf{W}} - 49_\phi]/3$.

The third term of Eq. (14) contains information about the renormalization of the gauge parameter α under the flow:

$$\frac{1}{\hat{\alpha}_R \hat{g}_R^2} = \frac{1}{\alpha g^2} + \frac{5}{4} \frac{1}{16\pi^2} t, \quad \Rightarrow \quad \partial_t \hat{\alpha}_R = \frac{7}{3} \hat{\alpha}_R \left(1 - \frac{15}{28} \hat{\alpha}_R \right) \frac{\hat{g}_R^2}{16\pi^2}. \quad (16)$$

The RHS of this renormalization group equation is positive for $\alpha < 28/15 \simeq 1.87$; this implies that α runs to zero under the flow as long as $\alpha_\Lambda < 28/15$. Therefore, our starting point $\alpha_\Lambda = 1$ is a consistent choice that ensures a running into the desired Landau gauge $\alpha \rightarrow 0$.

Before we discuss the physical implications of our result Eq. (14), let us study the effective action including the \mathbf{n} field fluctuations. In principle, this action should be obtainable from the present result by inserting $\hat{\Gamma}_k[\mathbf{n}]$ into a functional integral over \mathbf{n} . However, we evaluated $\hat{\Gamma}_k[\mathbf{n}]$ in a derivative expansion, neglecting high-momentum fluctuations of the \mathbf{n} field. But when integrating over \mathbf{n} fluctuations, especially these high-momentum modes are important for the renormalization of the couplings. Hence, their correct running cannot be calculated via such an indirect approach. The direct way is presented in the next section.

4 One-loop effective action including \mathbf{n} fluctuations

In the following, we propose a different way to integrate out the “hard” modes with high momenta \mathbf{p} , $k < p < \Lambda$. This time, we also include the hard fluctuations of the \mathbf{n} field and decompose the complete Yang-Mills field into soft and hard modes,

$$\mathbf{A}_\mu = \mathbf{A}_\mu^{\text{S}} + \mathbf{A}_\mu^{\text{H}}, \quad \mathbf{A}_\mu^{\text{S,H}} = \mathbf{A}_\mu^{\text{S,H}}(C_\mu^{\text{S,H}}, \mathbf{n}^{\text{S,H}}, \mathbf{W}_\mu^{\text{S,H}}). \quad (17)$$

Since the hard modes $\mathbf{A}_\mu^{\text{H}}$ shall be integrated out completely, the explicit use of the decomposition into $\mathbf{C}_\mu^{\text{H}}$, \mathbf{n}^{H} and $\mathbf{W}_\mu^{\text{H}}$ would be a very inconvenient choice of overabundant integration variables; therefore, the decomposition is only adopted for the soft modes $\mathbf{A}_\mu^{\text{S}}$. In the spirit of the Faddeev-Niemi conjecture, we assume that these soft modes are dominated by the \mathbf{n} field:

$$\mathbf{A}_\mu^{\text{S}} = \partial_\mu \mathbf{n}^{\text{S}} \times \mathbf{n}^{\text{S}}. \quad (18)$$

Integrating out the hard modes $\mathbf{A}_\mu^{\text{H}}$ results in two determinants in one-loop approximation,

$$\Gamma_k[\mathbf{A}^{\text{S}}] = \frac{1}{2} \ln \det(\Delta^{\text{YM}}[\mathbf{A}^{\text{S}}])^{-1} - \ln \det \Delta_{\text{FP}}[\mathbf{A}^{\text{S}}], \quad (19)$$

corresponding to the hard gluon and ghost loops; again we dropped the nonlocal terms (cf. App. B). The ghost contribution in the form of the Faddeev-Popov determinant is, of course, identical to the one obtained in the first line of Eq. (13), since the gauge fixing is performed in the same way as before. The gluonic determinant involves the operator

$$(\Delta^{\text{YM}}[\mathbf{A}^{\text{S}}])_{\mu\nu}^{-1} = - \left[D^2 \mathbb{1}_L - 2iF - DD + \frac{1}{\alpha} \partial\partial \right]_{\mu\nu} \Big|_{\mathbf{A}=\mathbf{A}^{\text{S}}}, \quad (20)$$

where D_μ denotes the covariant derivative and $F_{\mu\nu}$ the field strength tensor. The explicit representation of Eq. (20) in terms of the \mathbf{n} field is again given in App. A, Eqs. (A.6) and (A.7). The determinants in Eq. (19) can be calculated in a derivative expansion in the same way as described in the preceding section. Since the computation of the term $\sim (\partial\mathbf{n})^2$ is already very laborious, we do not calculate the marginal terms $\sim (\partial_\mu\mathbf{n} \times \partial_\nu\mathbf{n})^2$ etc. directly, but take over the known one-loop results for the running coupling and the gauge parameter from [11]. The final result for the Wilsonian one-loop effective action for the soft modes of the \mathbf{n} field reads

$$\begin{aligned} \Gamma_k[\mathbf{n}] = & \frac{\Lambda^2}{16\pi^2} (1 - e^{2t}) \int_x (\partial_\mu\mathbf{n})^2 + \frac{1}{4} \left(\frac{1}{g^2} + \frac{44}{3} \frac{1}{16\pi^2} t \right) \int_x (\partial_\mu\mathbf{n} \times \partial_\nu\mathbf{n})^2 \\ & - \frac{1}{2} \left(\frac{1}{\alpha g^2} + \frac{14}{3} \frac{1}{16\pi^2} t \right) \int_x (\partial_\mu\mathbf{n})^4 + \frac{1}{2} \left(\frac{1}{\alpha g^2} + \frac{14}{3} \frac{1}{16\pi^2} t \right) \int_x (\partial^2\mathbf{n} \cdot \partial^2\mathbf{n}), \end{aligned} \quad (21)$$

where we dropped the superscript S. Furthermore, we included for later use a higher-derivative term $\sim \partial^2\mathbf{n} \cdot \partial^2\mathbf{n}$ which is also marginal in the renormalization group sense and accompanied by the $1/(\alpha g^2)$ coefficient in the classical action.

Again, the generation of the “kinetic” term $\sim (\partial\mathbf{n})^2$ with a mass scale is observed; it is smaller by a factor of $8/13$ than in the preceding section. This means that the hard \mathbf{n} field fluctuations that have been taken into account in Eq. (21) reduce the new mass scale slightly; on the other hand, they increase the running of the Yang-Mills coupling by contributing the missing piece to the β function which now obtains the correct SU(2) value, $\beta_{g^2} = \frac{44}{3} \frac{1}{16\pi^2} g_R^4$. The running of the gauge parameter α is also increased, but no qualitative changes compared to Eq. (14) can be observed.

5 Discussion and Conclusions

The main results of our paper are contained in Eqs. (14) and (21), where the Wilsonian one-loop effective actions $\tilde{\Gamma}_k$ and Γ_k for the \mathbf{n} field have been given without and including hard \mathbf{n} field fluctuations, respectively. We were able to demonstrate that a “kinetic” term with a new mass scale for the \mathbf{n} field is indeed generated perturbatively, as was conjectured by Faddeev and Niemi. This term is relevant in the renormalization group sense and perturbatively exhibits a quadratic dependence on the UV cutoff Λ .

Furthermore, we studied the renormalization group flow of the marginal couplings of the \mathbf{n} field self-interactions given by the Yang-Mills coupling and the gauge parameter.

These terms are responsible for the stabilization of possible topological excitations of the \mathbf{n} field, as suggested by the Skyrme-Faddeev model. In total, the difference between $\tilde{\Gamma}_k$ and Γ_k is only of quantitative nature: the inclusion of hard \mathbf{n} field fluctuations increases the running of the marginal couplings and reduces the new mass scale; qualitative features such as stability of possible solitons remain untouched.

In fact, the question of stability turns out to be delicate: truncating our results for $\tilde{\Gamma}_k$ or Γ_k in Eqs. (14) or (21) at the level of the original Faddeev-Niemi proposal Eq. (1) (the first lines of Eqs. (14) and (21), respectively), we find an action that allows for stable knotlike solitons, since the coefficients of both terms are positive (as long as we stay away from the Landau pole, which we consider as unphysical). Taking additionally the $(\partial\mathbf{n})^4$ term of $\tilde{\Gamma}_k$ or Γ_k into account, which is also marginal and does not contain second-order derivatives on \mathbf{n} , stability is lost, since the coupling coefficient is negative in Eqs. (14) and (21); for stable solitons, a strictly positive coefficient would be required for this truncation, as was shown in [12].

Finally dropping the demand for first-order derivatives, we can include one further marginal term $\sim \partial^2\mathbf{n} \cdot \partial^2\mathbf{n}$ as given in Eq. (21) for Γ_k . With the aid of the identity

$$\int_x (\partial^2\mathbf{n} \times \mathbf{n})^2 = \int_x [\partial^2\mathbf{n} \cdot \partial^2\mathbf{n} - (\partial_\mu\mathbf{n})^4], \quad (22)$$

we find that the second line of Eq. (21) represents a strictly positive contribution to the action which again stabilizes possible solitons.⁵

Of course, this game could be continued by including further destabilizing and stabilizing higher-order terms again and again, but such terms are irrelevant in a renormalization group sense; that means their corresponding couplings are accompanied by inverse powers of the UV cutoff Λ and are thereby expected to vanish in the limit of large cutoff.

To summarize, our perturbative renormalization group analysis suggests enlarging the Faddeev-Niemi proposal for the effective low-energy action of Yang-Mills theory by taking all marginal operators of a derivative expansion into account. The original proposal of Eq. (1) was inspired by a desired Hamiltonian interpretation of the action that demands the absence of third- or higher-order time derivatives. But, as demonstrated, the covariant renormalization group does not care about a Hamiltonian interpretation of the final result. In some sense, the desired “simplicity” of the final result is spoiled by the presence of higher-derivative terms; moreover, it remains questionable as to whether the importance of the $\partial^2\mathbf{n} \cdot \partial^2\mathbf{n}$ term is still consistent with the derivative expansion of the action. Unfortunately, this cannot be checked within the present approach.

It should be stressed once again that the perturbative investigation performed here hardly suffices to confirm results about the infrared domain of Yang-Mills theories. On the contrary, it is only a valid approximation in the vicinity of the Gaussian UV fixed point of the theory. Nevertheless, our study might lend some intuition to possible nonperturbative scenarios: for example, let us assume that the Landau gauge $\alpha = 0$ indeed is an infrared

⁵We expect a similar behavior for the action $\tilde{\Gamma}_k$ in Eq. (14), although we have not calculated the coefficient of the $\partial^2\mathbf{n} \cdot \partial^2\mathbf{n}$ term explicitly.

fixed point in covariant gauges. Then the stabilizing term $\sim (\partial^2 \mathbf{n} \times \mathbf{n})^2$ is enhanced in the infrared, provided that the increase of the running coupling g obeys $\alpha g^2 \rightarrow 0$ for $k \rightarrow 0$; this would be realized, e.g., if g approached an infrared fixed point. Such a scenario thus supports the idea of topological knotlike solitons as important infrared degrees of freedom of Yang-Mills theories.

Perhaps the main drawback of our study lies in the fact that the new mass scale is not renormalization-group invariant; for example, we can read off from Eq. (21) that

$$m_k^2 = \frac{1}{16\pi^2} \Lambda^2 (1 - e^{2t}), \quad t \equiv \ln \frac{k}{\Lambda} \leq 0. \quad (23)$$

The new mass scale m_k is necessarily proportional to Λ , because there simply is no other scale in our system. But contrary to the gauge coupling or the gauge parameter, which can be made independent of Λ by adjusting the bare parameters, the Λ dependence of m_k persists, since there is no bare mass parameter to adjust. One may speculate that this problem is solved by “renormalization group improvement” of the kind

$$\Lambda^2 \rightarrow \Lambda^2 e^{-\frac{3 \cdot 16\pi^2}{22g^2(\Lambda)}}, \quad (24)$$

which upon insertion into Eq. (23) leads to a Λ -independent mass scale for $k \rightarrow 0$. Obviously, our perturbative calculation can never produce the RHS of Eq. (24), but a nonperturbative study of the renormalization group flow should result in such a structure (in a different context, such a mechanism has been observed in [13]).

Employing the measured values of the strong coupling constant at various renormalization points, we can determine the order of magnitude of the new mass scale: $m \equiv m_{k \rightarrow 0} = \mathcal{O}(1)$ MeV, e.g., $m \simeq 5.74$ MeV for $\alpha_s(M_Z) = 0.12$ or $m \simeq 0.68$ MeV for $\alpha_s(10 \text{ GeV}) = 0.18$ (the difference between these numbers arises from the fact that the initial values for the coupling are not related by a pure one-loop running). Of physical interest are the masses of the solitonic excitation in this effective theory. Unfortunately, there are no numerical results available for theories with higher-derivative order, so that we have to resort to results for an action identical to the first line of Eq. (21). For this model, the mass of the lowest lying states are approximately given by $M \simeq \mathcal{O}(10^3) \sqrt{q} m$, where q denotes the value of the coefficient in front of the $(\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})^2$ term [12, 14]. For couplings of order 1, we end up with soliton masses of the order of $M \sim \mathcal{O}(1)$ GeV; this is in accordance with lattice results for glue ball masses: e.g., $M_{\text{GB}} \simeq 1.5$ GeV for the lowest lying state in SU(2) [15]. Of course, this rough and speculative estimate should not be viewed as a “serious prediction” of our work.

With all these reservations in mind, the Faddeev-Niemi conjecture about possible low-energy degrees of freedom of Yang-Mills theories provides an interesting working hypothesis which deserves further exploration.

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Appendix

A Differential operators, tensors, currents, etc.

This appendix represents a collection of differential operators and other tensorial quantities which are required in the main text.

The Faddeev-Popov determinant Δ_{FP} in Eq. (7) and (10) for covariant gauges involves the operator (in one-loop approximation)

$$-\partial_\mu D_\mu(\mathbf{A})|_{C=0=\mathbf{W}} = -\partial^2 \mathbb{1}_c + (\partial^2 \mathbf{n} \otimes \mathbf{n} - \mathbf{n} \otimes \partial^2 \mathbf{n}) + (\partial_\mu \mathbf{n} \otimes \mathbf{n} - \mathbf{n} \otimes \partial_\mu \mathbf{n}) \partial_\mu, \quad (\text{A.1})$$

so that $\Delta_{\text{FP}} = \det(-\partial_\mu D_\mu(\mathbf{A})|_{C=0=\mathbf{W}})$.

The objects occurring in the exponent of Eq. (7) are defined as follows:

$$\begin{aligned} M_{\mu\nu}^C &:= -\partial^2 \delta_{\mu\nu} + \partial_\mu \partial_\nu - \frac{1}{\alpha} \partial_\mu \partial_\nu + \frac{1}{\alpha} \partial_\mu \mathbf{n} \cdot \partial_\nu \mathbf{n} \\ M_{\mu\nu}^{\mathbf{W}} &:= -\partial^2 \delta_{\mu\nu} \mathbb{1}_c + \partial_\mu \partial_\nu \mathbb{1}_c - \frac{1}{\alpha} \partial_\mu \partial_\nu \mathbb{1}_c - \partial_\mu \mathbf{n} \otimes \partial_\nu \mathbf{n} + \partial_\nu \mathbf{n} \otimes \partial_\mu \mathbf{n} \\ \mathbf{Q}_{\mu\nu}^C &:= \frac{1}{\alpha} (\partial_\mu \mathbf{n} \partial_\nu + \partial_\nu \mathbf{n} \partial_\mu + \partial_\mu \partial_\nu \mathbf{n}) \\ K_\mu^C &:= \partial_\nu (\mathbf{n} \cdot \partial_\nu \mathbf{n} \times \partial_\mu \mathbf{n}) + \frac{1}{\alpha} \partial_\mu \mathbf{n} \cdot \partial^2 \mathbf{n} \times \mathbf{n} \\ \mathbf{K}_\mu^{\mathbf{W}} &:= \frac{1}{\alpha} \partial_\mu (\mathbf{n} \times \partial^2 \mathbf{n}). \end{aligned} \quad (\text{A.2})$$

The determinants in Eq. (10) employ several composites of these operators. Since we first perform the \mathbf{C} integration, the resulting determinant involves only $M^{\mathbf{C}}$, whereas the \mathbf{W} determinant also receives contributions from the mixing term $\mathbf{Q}^{\mathbf{C}}$,

$$\overline{M}_{\mu\nu}^{\mathbf{W}} = M_{\mu\nu}^{\mathbf{W}} + \tilde{\mathbf{Q}}_{\mu\kappa}^C (M^C)^{-1}_{\kappa\lambda} \mathbf{Q}_{\lambda\nu}^C. \quad (\text{A.3})$$

Here, $\tilde{\mathbf{Q}}$ is defined via partial integration

$$\int (\mathbf{Q}_{\mu\nu}^C \mathbf{W}_\nu) f_\nu \stackrel{\text{i.b.p.}}{=} \int \mathbf{W}_\mu \tilde{\mathbf{Q}}_{\mu\nu}^C f_\nu, \quad (\text{A.4})$$

and f_ν denotes an arbitrary test function.

The last determinant in Eq. (10) arises from the ϕ integration and receives contributions from the relevant parts of the exponent of Eq. (9), which we denote by

$$Q_\mu^\phi := i(-\partial_\mu \mathbb{1}_c + \partial_\mu \mathbf{n} \otimes \mathbf{n}), \quad (\text{A.5})$$

so that $\delta(\chi) \rightarrow \int \mathcal{D}\phi \exp(-\int \mathbf{W}_\mu \cdot Q_\mu^\phi \phi)$ to one-loop order. Employing a notation similar to Eq. (A.4), the differential operator accompanying the term $\sim \phi \phi$ in the exponent finally reads $\tilde{Q}_\mu^\phi (\overline{M}_{\mu\nu}^{\mathbf{W}})^{-1} Q_\nu^\phi$. Integrating the ϕ field along the imaginary axis leads to the last determinant in Eq. (10).

In Sect. 4, we employ the inverse gluon propagator $(\Delta^{\text{YM}}[\mathbf{A}^S])^{-1}$ coupled to all orders to the soft \mathbf{n} field fluctuations. For an explicit representation, we need the covariant derivative,

$$D_\mu[\mathbf{n}] = \partial_\mu \mathbb{1}_c + \mathbf{n} \otimes \partial_\mu \mathbf{n} - \partial_\mu \mathbf{n} \otimes \mathbf{n}, \quad (\text{A.6})$$

where we have inserted the soft gauge potential Eq. (18) into the covariant derivative. The inverse gluon propagator Eq. (20) then reads

$$\begin{aligned} (\Delta^{\text{YM}}[\mathbf{n}])_{\mu\nu}^{-1} &= \mathbb{1}_c \delta_{\mu\nu} - 2(\mathbf{n} \otimes \partial_\lambda \mathbf{n} - \partial_\mu \mathbf{n} \otimes \mathbf{n}) \partial_\lambda \delta_{\mu\nu} + (\mathbf{n} \otimes \partial_\nu \mathbf{n} - \partial_\nu \mathbf{n} \otimes \mathbf{n}) \partial_\mu \\ &\quad + (\mathbf{n} \otimes \partial_\mu \mathbf{n} - \partial_\mu \mathbf{n} \otimes \mathbf{n}) \partial_\nu - (\mathbf{n} \otimes \partial^2 \mathbf{n} - \partial^2 \mathbf{n} \otimes \mathbf{n}) \delta_{\mu\nu} + (\partial_\lambda \mathbf{n})^2 \mathbf{n} \otimes \mathbf{n} \delta_{\mu\nu} \\ &\quad + \partial_\lambda \mathbf{n} \otimes \partial_\lambda \mathbf{n} \delta_{\mu\nu} - (2\partial_\mu \mathbf{n} \otimes \partial_\nu \mathbf{n} - \partial_\nu \mathbf{n} \otimes \partial_\mu \mathbf{n}) \\ &\quad + (\mathbf{n} \otimes \partial_{\mu\nu} \mathbf{n} - \partial_{\mu\nu} \mathbf{n} \otimes \mathbf{n}) - (\partial_\mu \mathbf{n} \cdot \partial_\nu \mathbf{n}) \mathbf{n} \otimes \mathbf{n}. \end{aligned} \quad (\text{A.7})$$

B Nonlocal terms

During the Gaussian integration over the \mathbf{C} , ϕ and \mathbf{W} fields in Sect. 3, several nonlocal terms arise from the completion of the square in the exponent. Here, we shall give reasons why they can be neglected. Let us exemplarily consider the simplest nonlocal contribution arising from the \mathbf{C} integration:

$$K^C (M^C)^{-1} K^C = (\mathbf{n} \cdot \partial_\lambda \mathbf{n} \times \partial_{\lambda\mu} \mathbf{n}) \left(\frac{1}{-\partial^2 + \partial \mathbf{n} \cdot \partial \mathbf{n}} \right)_{\mu\nu} (\mathbf{n} \cdot \partial_\kappa \mathbf{n} \times \partial_{\kappa\nu} \mathbf{n}). \quad (\text{B.8})$$

Within the calculation of the determinants, we expanded the inverse operator assuming that $\partial \mathbf{n} \cdot \partial \mathbf{n} \ll -\partial^2$. This was justified, since the derivative operator acts on the test function space with momenta \mathbf{p} between \mathbf{k} and $\mathbf{\Lambda}$, which are large compared to the conjectured slow variation of the \mathbf{n} field.

In the present case, the situation is different, because the derivative term $-\partial^2$ acts only on the \mathbf{n} field and its derivatives to the right (there is no test function to act on). In other words, the nonlocal terms are only numbers, not operators. The derivatives can thus be approximated by the (inverse) scale of variation of the \mathbf{n} field or its derivatives which is much smaller than \mathbf{k} or $\mathbf{\Lambda}$. This implies that the nonlocal terms do not depend on \mathbf{k} or $\mathbf{\Lambda}$, so that they cannot contribute to the flow of the couplings.

For example, a reasonable lowest-order approximation of the RHS of Eq. (B.8) is given by its local limit,

$$K^C (M^C)^{-1} K^C = (\mathbf{n} \cdot \partial_\lambda \mathbf{n} \times \partial_{\lambda\mu} \mathbf{n}) \left(\frac{1}{\partial \mathbf{n} \cdot \partial \mathbf{n}} \right)_{\mu\nu} (\mathbf{n} \cdot \partial_\kappa \mathbf{n} \times \partial_{\kappa\nu} \mathbf{n}) + \dots, \quad (\text{B.9})$$

where it is obvious that these terms do not contribute to the desired Wilsonian effective action. The same line of argument holds for all nonlocal terms appearing in Sects. 3 and 4.

C Momentum integrals

Several standard integrals appear in the integration over the momentum shell $[k, \Lambda]$ in Sect. 3. One basic formula is given by

$$\int_{[k, \Lambda]} \frac{d^4 p}{(2\pi)^4} \frac{p_\lambda p_\kappa p_\mu p_\nu}{p^8} = \frac{1}{3} \frac{1}{64\pi^2} \ln \frac{\Lambda}{k} (\delta_{\lambda\kappa} \delta_{\mu\nu} + \delta_{\lambda\mu} \delta_{\kappa\nu} + \delta_{\lambda\nu} \delta_{\kappa\mu}). \quad (\text{C.10})$$

From this formula, we can also deduce upon index contraction that

$$\int_{[k, \Lambda]} \frac{d^4 p}{(2\pi)^4} \frac{p_\mu p_\nu}{p^6} = \frac{1}{32\pi^2} \ln \frac{\Lambda}{k}, \quad \int_{[k, \Lambda]} \frac{d^4 p}{(2\pi)^4} \frac{1}{p^4} = \frac{1}{8\pi^2} \ln \frac{\Lambda}{k}. \quad (\text{C.11})$$

The last integral is, of course, standard and can be used to prove Eq. (C.10) in addition to symmetry arguments. The same philosophy applies to the second type of integrals:

$$\int_{[k, \Lambda]} \frac{d^4 p}{(2\pi)^4} \frac{p_\mu p_\nu}{p^4} = \frac{1}{64\pi^2} (\Lambda^2 - k^2) \delta_{\mu\nu}, \quad \int_{[k, \Lambda]} \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2} = \frac{1}{16\pi^2} (\Lambda^2 - k^2). \quad (\text{C.12})$$

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