The Geometry of Quantum Mechanics

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Abstract

A recent notion in theoretical physics is that not all quantum theories arise from quantising a classical system. Also, a given quantum model may possess more than just one classical limit. These facts find strong evidence in string duality and M–theory, and it has been suggested that they should also have a counterpart in quantum mechanics. In view of these developments we propose *dequantisation*, a mechanism to render a quantum theory classical. Specifically, we present a geometric procedure to *dequantise* a given quantum mechanics (regardless of its classical origin, if any) to possibly different classical limits, whose quantisation gives back the original quantum theory. The standard classical limit $\hbar \to 0$ arises as a particular case of our approach.

Contents

1 Introduction

1.1 Motivation

Appraoching quantum mechanics from a geometric viewpoint is a very interesting topic. The goal is a geometrisation of quantum mechanics [1], similar in spirit to that of classical mechanics [2, 3]. Beyond this similarity, however, there are numerous deep reasons. One of them is motivated in string duality and M–theory [4, 5]. In plain words, we are confronted with the fact that not all quantum theories arise from quantising a classical system. Also, a given quantum model may possess more than just one classical limit. These two facts are in sharp contrast with our current understanding of quantum mechanics. While it is true that these two phenomena originally arise in the theories of strings and branes [6], some authors [5] have expressed the opinion that they should somehow be reflected at the fundamental level of quantum mechanics as well. Let us describe the general setup.

Quantisation may be understood as a prescription to construct a quantum theory from a given classical theory. As such, it is far from being unique. Beyond canonical quantisation and functional integrals, a number of different, often complementary approaches to quantisation are known, each one of them exploiting different aspects of the underlying classical theory. For example, geometric quantisation [7, 8, 9, 10] relies on the geometric properties of classical mechanics. Systems whose classical phase space $\mathcal{C}$ is a Kähler manifold can be quantised as in refs. [11, 12, 13]. If $\mathcal{C}$ is just a Poisson manifold, then the approach of ref. [14], based on deformation quantisation [15, 16], can always be applied. A path–integral counterpart to these mathematical techniques has been developed in ref. [17].

A common feature to these approaches is the fact that they all take a classical mechanics as a starting point. Thus the classical limit is *a fortiori* unique: it reduces to letting $\hbar \to 0$. If we want to allow for the existence of more than one classical limit, we are led to considering a quantum mechanics that is not based, at least primarily, on the the quantisation of a given classical dynamics. In such an approach one would not take first a classical model and then quantise it. Rather, quantum mechanics itself would be the starting point: a parent quantum theory would give rise, in a certain limit, to a classical theory. If there are several different ways of taking this limit, then there will be several different classical limits.

1.2 Summary

This article puts forward a geometric proposal by which quantum mechanics can be rendered classical, or *dequantised*, in more than one way, thus yielding different classical limits. Under *dequantisation* we understand the following.

Assume that classical phase space $\mathcal{C}$ is $\mathbf{R}^{2n}$. Then, starting from the quantum phase space $\mathcal{Q}$ of standard quantum mechanics [1], the usual classical limit $\hbar \to 0$ is obtained as the quotient of $\mathcal{Q}$ by a certain equivalence relation $\sim$, *i.e.* $\mathcal{Q}/\sim = \mathbf{R}^{2n}$, and we have a trivial fibre bundle $\mathcal{Q} \to \mathbf{R}^{2n}$. We will construct classical phase spaces $\mathcal{Q}/G = \mathcal{C}$, where G is a Lie group acting on $\mathcal{Q}$, and $\mathcal{Q} \to \mathcal{C}$ will be a (not necessarily trivial) G–bundle. The associated vector bundle will have $\mathcal{H}$, the Hilbert space of quantum states, as its typical fibre. In order to qualify as a classical phase space, $\mathcal{C}$ must be a symplectic manifold whose quantisation must give back the original quantum theory on $\mathcal{Q}$. Different choices for G will give rise to different classical limits.

1.3 Outline

This article is organised as follows. Section 2 summarises the standard Hilbert space formulation of quantum mechanics, following the geometric presentation of ref. [1]. We will recall how the standard classical limit $\hbar \to 0$ is taken. In this analysis, a natural mechanism will arise that will allow more than one classical limit to exist. This is presented in section 3. We illustrate our technique with some specific examples in section 4, where one given quantum mechanics is explicitly *dequantised*. The physical implications of our proposal are discussed in section 5. Some technical mathematical aspects of our construction are elucidated in section 6.

2 A geometric approach to quantum mechanics

For later purposes let us briefly summarise the geometric approach to quantum mechanics presented in ref. [1]. Throughout this section our use of the terms *classical* and *quantum* will be the standard one [18].

2.1 The Hilbert space as a Kähler manifold

The starting point is an infinite–dimensional, complex, separable Hilbert space of quantum states, $\mathcal{H}$, that is most conveniently viewed as a real vector space equipped with a complex structure J. Correspondingly, the Hermitian inner product can be decomposed into real and imaginary parts,

$$\langle \phi, \psi \rangle = g(\phi, \psi) + \mathrm{i}\omega(\phi, \psi), \tag{1}$$

with g a positive–definite, real scalar product and ω a symplectic form. The metric g, the symplectic form ω and the complex structure J are related as

$$g(\phi, \psi) = \omega(\phi, J\psi), \tag{2}$$

which means that the triple (J, g, ω) endows the Hilbert space $\mathcal{H}$ with the structure of a Kähler space [2].

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Thus any Hilbert space naturally gives rise to a symplectic manifold: it is the *quantum* phase space $\mathcal{Q}$, or the space of rays in $\mathcal{H}$. Let $\omega_\mathcal{Q}$ denote the restriction of ω to $\mathcal{Q}$. On $\mathcal{Q}$, the inverse of $\omega_\mathcal{Q}$ can be used to define Poisson brackets and Hamiltonian vector fields. This is done as follows.

Any function $f_\mathcal{C} : \mathcal{C} \to \mathbf{R}$ defined on classical phase space $\mathcal{C}$ has associated a self–adjoint quantum observable F on $\mathcal{H}$. The latter gives rise to a quantum function $f_\mathcal{Q} : \mathcal{Q} \to \mathbf{R}$ on quantum phase space $\mathcal{Q}$, defined as the expectation value of the operator F:

$$f_\mathcal{Q}(\psi) = \langle \psi, F\psi \rangle. \tag{3}$$

Now every function $f : \mathcal{Q} \to \mathbf{R}$ defines a Hamiltonian vector field X_f through the equation [3]

$$i_{X_f} \omega_\mathcal{Q} = \mathrm{d}f. \tag{4}$$

In this way the Poisson bracket $\{,\}_\mathcal{Q}$ on $\mathcal{Q}$ is defined by [3]

$$\{f_\mathcal{Q}, g_\mathcal{Q}\}_\mathcal{Q} = \omega_\mathcal{Q}(X_f, X_g). \tag{5}$$

Let us now consider the classical coordinate and momentum functions $q^j_\mathcal{C}$ and $p^k_\mathcal{C}$ satisfying the canonical Poisson brackets on $\mathcal{C}$. Through the above construction one arrives at the quantum coordinate and momentum functions $q^j_\mathcal{Q}$ and $p^k_\mathcal{Q}$ satisfying the canonical Poisson brackets on $\mathcal{Q}$

$$\{q^j_\mathcal{Q}, p^k_\mathcal{Q}\}_\mathcal{Q} = \delta^{jk}, \qquad \{q^j_\mathcal{Q}, q^k_\mathcal{Q}\}_\mathcal{Q} = 0 = \{p^j_\mathcal{Q}, p^k_\mathcal{Q}\}_\mathcal{Q}. \tag{6}$$

It turns out that Hamilton's canonical equations of motion on $\mathcal{Q}$ are equivalent to Schrödinger's wave equation, while the Riemannian metric g accounts for properties such as the measurement process and Heisenberg's uncertainty relations.

We are thus dealing with two phase spaces, that we denote $\mathcal{C}$ (for *classical*) and $\mathcal{Q}$ (for *quantum*). $\mathcal{Q}$ is always infinite–dimensional, as it derives from an infinite–dimensional Hilbert space. On the contrary, $\mathcal{C}$ may well be finite–dimensional. Furthermore, while both $\mathcal{C}$ and $\mathcal{Q}$ are symplectic manifolds, the latter is always Kähler, while the former need not be Kähler.

Two questions arise naturally. First, what is the geometric relation between $\mathcal{C}$ and $\mathcal{Q}$ as manifolds? Second, how are $\mathcal{C}$ and $\mathcal{Q}$ related as *symplectic* manifolds, *i.e.*, how are their respective symplectic forms $\omega_\mathcal{C}$ and $\omega_\mathcal{Q}$ related? When $\mathcal{C} = \mathbf{R}^{2n}$, the answer is provided in ref. [1] and summarised below.

2.2 Quantum phase space as a fibre bundle over classical phase space

For a classical system with n degrees of freedom, let us collectively denote by f_r, $r = 1, \ldots, 2n$, the quantum coordinate and momentum functions $q^j_\mathcal{Q}$ and $p^k_\mathcal{Q}$. We define an equivalence relation on $\mathcal{Q}$ as

$$x_1 \sim x_2 \qquad \text{iff} \qquad f_r(x_1) = f_r(x_2) \quad \forall r. \tag{7}$$

Through this equivalence relation, the quantum phase space $\mathcal{Q}$ becomes a trivial fibre bundle with fibre $\mathcal{H}$ over the classical phase space $\mathbf{R}^{2n}$:

$$\mathcal{Q} \longrightarrow \mathcal{Q}/\sim = \mathbf{R}^{2n}. \tag{8}$$

2.3 Relation between the classical and the quantum symplectic forms

A tangent vector $v \in T_x\mathcal{Q}$ is said vertical at $x \in \mathcal{Q}$ if $v(f_r) = 0 \; \forall r$. Therefore the vertical directions are those in which the quantum coordinate and momentum functions assume constant values. Equivalently, the vertical subspace $\mathcal{V}_x$ at $x \in \mathcal{Q}$ may be defined as

$$\mathcal{V}_x = \{v \in T_x\mathcal{Q} : \omega_\mathcal{Q}\left(X_{f_r}(x), v\right) = 0 \; \forall r\}. \tag{9}$$

Let $\mathcal{V}_x^\perp$ denote the $\omega_\mathcal{Q}$–orthogonal complement of the vertical subspace at $x \in \mathcal{Q}$. Each tangent space splits as the direct sum

$$T_x\mathcal{Q} = \mathcal{V}_x \oplus \mathcal{V}_x^\perp, \tag{10}$$

and the tangent vectors that lie in $\mathcal{V}_x^\perp$ are said horizontal at x. It turns out that the quantum states lying on a horizontal cross section of the bundle (8) are precisely the generalised coherent states of refs. [19, 20].

Now, if u and v are vectors on $\mathcal{C} = \mathbf{R}^{2n}$, denote by u^h and v^h their horizontal lifts to $\mathcal{Q}$. Then the classical symplectic structure $\omega_\mathcal{C}$ is related to its quantum counterpart $\omega_\mathcal{Q}$ through

$$\omega_\mathcal{C}(u, v) = \omega_\mathcal{Q}(u^h, v^h), \tag{11}$$

i.e., $\omega_\mathcal{C}$ is the horizontal part of $\omega_\mathcal{Q}$.

3 Taking a classical limit

The geometric presentation summarised in section 2 makes it clear that the quantum theory contains all the information about the classical theory. In this sense, as explained in section 1, we should think of quantum mechanics as being prior to classical mechanics. Rather than quantising a classical theory, rendering quantum mechanics classical, or *dequantising* it, appears to be the key issue. How can one *dequantise*?

3.1 Symplectic reduction

Our primary concern will be to obtain a classical *symplectic* manifold $(\mathcal{C}, \omega_\mathcal{C})$ from its quantum counterpart $(\mathcal{Q}, \omega_\mathcal{Q})$, in such a way that the quantisation of $(\mathcal{C}, \omega_\mathcal{C})$ will reproduce $(\mathcal{Q}, \omega_\mathcal{Q})$ as a *symplectic* manifold, regardless of the Riemannian metric $g_\mathcal{C}$ on $\mathcal{C}$, if any. In principle, *dequantisation* may be thought of as the symplectic reduction from $(\mathcal{Q}, \omega_\mathcal{Q})$ to a symplectic submanifold $(\mathcal{C}, \omega_\mathcal{C})$; a more general definition will be given in section 3.3. In having $\mathcal{C}$ as a reduced symplectic manifold of $\mathcal{Q}$ we are assured that the quantisation of $\mathcal{C}$ reproduces $\mathcal{Q}$. See refs. [3, 21] for a treatment of symplectic reduction.

We do not require the metric $g_\mathcal{Q}$ on $\mathcal{Q}$ to descend to a metric $g_\mathcal{C}$ on $\mathcal{C}$. Disregarding the metric $g_\mathcal{C}$ is justified, as the metric $g_\mathcal{Q}$ of eqn. (1) can always be obtained from the symplectic form $\omega_\mathcal{Q}$ through the Kähler condition (2).

On the contrary, the symplectic structure is an essential ingredient to keep in the passage from quantum to classical, as classical phase space is always symplectic. In what follows we will consider symplectic structures as in refs. [22, 23] but, more generally, one could relax $\mathcal{C}$ to be a Poisson manifold.

3.2 Reduction via fibre bundles

A useful approach to symplectic reduction is via fibre bundles. When $\mathcal{C} = \mathbf{R}^{2n}$, the classical limit arises in ref. [1] as the base space of a trivial fibre bundle with fibre $\mathcal{H}$ and total space $\mathcal{Q}$. This suggests considering fibre bundles $\mathcal{Q} \to \mathcal{C}$, with fibre $\mathcal{H}$ and total space $\mathcal{Q}$, over some other finite–dimensional base manifold $\mathcal{C}$. If the classical phase space $\mathcal{C}$ so obtained is a symplectic manifold whose quantisation reproduces the initial quantum theory on $\mathcal{Q}$, then associated with that fibre bundle there is one classical limit.

Let us first examine trivial fibre bundles. The equivalence relation of section 2.2 is singled out because it is well suited to obtain the standard coherent states of refs. [19, 20]. We will see in section 4.2 one particular example of a certain group G acting on $\mathcal{Q}$ such that $\mathcal{Q}/G = \mathcal{C}$ coincides with the result of taking the standard classical limit $\hbar \to 0$. The procedure of section 4.2 is in fact quite general in order to replace equivalence relations with group actions.

Nontrivial fibre bundles may also be considered. They provide a realisation of the statement presented in ref. [24], to the effect that one can always choose local coordinates on classical phase space, in terms of which quantisation becomes a local expansion in powers of $\hbar$ around a certain local vacuum. This expansion is local in nature: it does not hold globally on classical phase space when the fibre bundle is nontrivial. In this sense, quantisation is mathematically reminiscent of the local triviality property satisfied by every fibre bundle [25] while, physically, it is reminiscent of the equivalence principle of general relativity [26].

3.3 Definition of *dequantisation*

For our purposes, *dequantisation* will mean the following. Let G a Lie group acting on $\mathcal{Q}$. Modding out by the action of G we will construct principal G–bundles

$$\mathcal{Q} \longrightarrow \mathcal{Q}/G = \mathcal{C} \tag{12}$$

over finite–dimensional symplectic manifolds $\mathcal{C}$. We require the associated vector bundle to have $\mathcal{H}$ as its fibre. Moreover the lift of $\omega_\mathcal{C}$ to $\mathcal{Q}$ must equal $\omega_\mathcal{Q}$.

Eqn. (11) expressed the property that, when $\mathcal{C} = \mathbf{R}^{2n}$, $\omega_\mathcal{C}$ was simply the horizontal part of $\omega_\mathcal{Q}$. Horizontality was closely related to coherence. Here we have no notion of horizontality because *any* $\omega_\mathcal{C}$ will work, provided its lift to $\mathcal{Q}$ equal $\omega_\mathcal{Q}$ (as is the case, *e.g.*, in symplectic reduction). In general, the best we can do is to find local canonical coordinates on $\mathcal{C}$ in terms of which

$$\omega_\mathcal{C} = \mathrm{d}p_k \wedge \mathrm{d}q^k. \tag{13}$$

With respect to these local coordinates, local coherent states $|z_k\rangle$ can be defined simply as eigenvectors of the local annihilation operator $a_k = Q^k + \mathrm{i}P_k$, where Q^k and P_k are the quantum observables corresponding to q^k and p_k. How do Q^k and P_k *dequantise* to q^k and p_k?

3.4 Classical functions from quantum observables

When *dequantising*, instead of having classical functions $f_\mathcal{C} : \mathcal{C} \to \mathbf{R}$ to turn into quantum observables F, we have quantum observables F out of which we would like to obtain classical functions. We can use eqn. (3) in order to define

the quantum function $f_{\mathcal{Q}} : \mathcal{Q} \to \mathbf{R}$ corresponding to the observable F. Now, in the examples that follow, $\mathcal{C}$ is a submanifold of $\mathcal{Q}$. Hence the restriction of $f_{\mathcal{Q}}$ to $\mathcal{C}$ gives rise to a well–defined classical function $f_{\mathcal{C}} : \mathcal{C} \to \mathbf{R}$ whose quantisation reproduces the quantum observable F.

4 Examples of different classical limits

In the following we give some examples of the *dequantisation* of the nonrelativistic quantum mechanics of n degrees of freedom. We will concentrate on some specific nonlinear choices for the manifold $\mathcal{C}$, namely complex projective spaces $\mathbf{CP}^n$ and complex submanifolds thereof. Linear classical phase spaces have been dealt with in sections 2.2, 2.3. Coherent states on spheres have been constructed in ref. [27].

4.1 The standard coherent states

Points in $\mathbf{CP}^n$ may be specified by homogeneous coordinates $[w_0 : \ldots : w_n]$ on $\mathbf{C}^{n+1}$. Alternatively, holomorphic coordinates on $\mathbf{CP}^n$ in the chart with, say, $w_0 \neq 0$, are given by $z_k = w_k/w_0$, with $k = 1, \ldots, n$.

In order to discuss coherent states it is convenient to use homogeneous coordinates. Then we have a Kähler form

$$\omega = \mathrm{i} \sum_{k=0}^{n} \mathrm{d} w^k \wedge \mathrm{d} \bar{w}^k, \tag{14}$$

which we take to define a symplectic structure with invariance group $U(n+1)$. As we are working in homogeneous coordinates we still have to mod out by $U(1)$, so the true invariance group of the Kähler form is $G = U(n+1)/U(1) \simeq SU(n+1)$. Let $G' \subset G$ be a maximal isotropy subgroup [20] of the vacuum state $|0\rangle$. Coherent states $|\zeta\rangle$ are parametrised by points ζ in the coset space G/G' [20]. Set $n = 1$ for simplicity, so $\mathbf{CP}^1 \simeq S^2$. Then $G' = U(1)$, and coherent states $|u\rangle$ are parametrised by points u in the quotient space $S^2 = SU(2)/U(1)$.

We will find it convenient to recall Berezin's quantisation [11] of the Riemann sphere. The Hilbert space is most easily presented in holomorphic coordinates $z, \bar{z}$, which have the advantage of being almost global coordinates on S^2. The Kähler potential

$$K_{S^2}(z, \bar{z}) = \log\left(1 + |z|^2\right) \tag{15}$$

produces an integration measure

$$\mathrm{d}\mu(z, \bar{z}) = \frac{1}{2\pi\mathrm{i}} \frac{\mathrm{d}z \wedge \mathrm{d}\bar{z}}{(1 + |z|^2)^2}. \tag{16}$$

The Hilbert space of states is the space $\mathcal{F}_\hbar(S^2)$ of holomorphic functions on S^2 with finite norm, the scalar product being

$$\langle \psi_1 | \psi_2 \rangle = \left(\frac{1}{\hbar} + 1\right) \int_{S^2} \mathrm{d}\mu(z, \bar{z}) \left(1 + |z|^2\right)^{-1/\hbar} \overline{\psi}_1(z)\, \psi_2(z). \tag{17}$$

It turns out that $\hbar^{-1}$ must be an integer. For ψ to have finite norm, it must be a polynomial of degree less than $\hbar^{-1}$. In fact, setting $\hbar^{-1} = 2j + 2$, $\mathcal{F}_\hbar(S^2)$ is the representation space for the spin–j representation of $SU(2)$, which is the isometry group of S^2. The semiclassical regime corresponds to $j \to \infty$.

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4.2 Trivial fibre bundles: global coherent states

Let $\mathcal{Q}$ be the manifold of rays in $\mathcal{H}$. We define an action of the group of unitary operators $U(\mathcal{H})$ on $\mathcal{Q}$ as follows: first lift $\mathcal{Q}$ to $\mathcal{H}$, then apply a $U(\mathcal{H})$ transformation. In this way we obtain a fibre bundle whose base is $\mathcal{C} = \mathcal{Q}/U(\mathcal{H})$. Now any two points in $\mathcal{Q}$ can always be connected by means of a transformation in $U(\mathcal{H})$, so this $\mathcal{C}$ reduces to a point. This is an instance of the situation mentioned in section 3, that *not* every bundle will give rise to a reasonable classical limit.

A sensible classical limit is the following. $\mathcal{H}$ being infinite–dimensional, we may require the action of $U(\mathcal{H})$ to act as the identity along, say, the first $n+1$ complex dimensions of $\mathcal{H}$, while allowing it to act nontrivially on the rest. In this way the resulting $\mathcal{C} = \mathcal{Q}/U(\mathcal{H})$ is the complex n–dimensional projective space $\mathbf{CP}^n$. It is the base of a principal fibre bundle whose total space is $\mathcal{Q}$ and whose fibre is $U(\mathcal{H})$. This bundle is trivial by construction. Triviality may also be proved recalling that, when the structure group is contractible, the bundle is automatically trivial [25]. Now $U(\mathcal{H})$ is contractible [28] (see also section 6), so all principal $U(\mathcal{H})$–bundles are trivial.

Next consider the trivial vector bundle, with fibre $\mathcal{H}$, that is associated with this trivial principal bundle. Triviality implies that one has the globally defined diffeomorphism $\mathcal{Q} \simeq \mathcal{C} \times \mathcal{H}$. Now coherent states lie on sections of this bundle. Hence the triviality of this bundle ensures that these coherent states are globally defined on $\mathcal{C}$. An equivalent phrasing of this statement is to say that the semiclassical regime is globally defined on $\mathcal{C}$. Upon quantisation, all observers on $\mathcal{C}$ will agree on what is a *semiclassical* vs. what is a *strong quantum* effect. Setting $n = 1$ for simplicity, if one observer on $\mathcal{C}$ measures $j < \infty$, then so will all other observers. If the measure is $j \to \infty$, then so will it be for all other observers, too.

Now $U(\mathcal{H})$ is the invariance group of the Kähler form on $\mathcal{Q}$

$$\omega = i \sum_{k=0}^{\infty} \mathrm{d}w^k \wedge \mathrm{d}\bar{w}^k. \tag{18}$$

The Kähler form on the resulting $\mathbf{CP}^n$ is given in eqn. (14), *i.e.*, it is the one obtained by quotienting (18) with this group action. Incidentally, the metric g on $\mathcal{Q}$ also descends to the quotient $\mathbf{CP}^n$, and we can now apply Berezin's quantisation [11]. In fact we have picked our group action precisely so as to obtain a *dequantisation* of $\mathcal{Q}$ to $\mathbf{CP}^n$ that exactly reproduces the standard clasical limit $\hbar \to 0$ for $\mathbf{CP}^n$. Similarly, the corresponding coherent–state quantisation [20] is the one summarised in section 4.1. This example also illustrates the power of fibrating $\mathcal{Q}$ by means of a group action. Yet another choice for the group action will lead to another different *dequantisation*.

4.3 Nontrivial fibre bundles: nonglobal coherent states

Let us consider the Hopf bundle

$$S^{2n+1}/U(1) \simeq \mathbf{CP}^n, \tag{19}$$

where the sphere S^{2n+1} is the submanifold of $\mathbf{C}^{n+1}$ defined by

$$|z_0|^2 + \ldots + |z_n|^2 = 1, \tag{20}$$

and the $U(1)$ action is

$$(z_0, \ldots, z_n) \mapsto \mathrm{e}^{\mathrm{i}\alpha} (z_0, \ldots, z_n). \tag{21}$$

This fibre bundle is nontrivial [29] (it describes a magnetic monopole of nonzero charge [30]).

Let us consider the group $U(\infty)$

$$U(\infty) = \lim_{n \to \infty} U(n), \tag{22}$$

which is not to be confused with the group $U(\mathcal{H})$ of section 4.2. Elements of $U(\infty)$ are $n \times n$ unitary matrices u in any dimension n. In order to let them act on $\mathcal{H}$, which is infinite dimensional, we may think of u as being tensored with an infinite–dimensional identity matrix, $u \otimes \mathbf{1}$. Therefore $U(\infty)$ is a subgroup of $U(\mathcal{H})$. As was the case with $U(\mathcal{H})$, any two points in $\mathcal{Q}$ are always connected by means of an $U(\infty)$–transformation.

Now we have the fundamental group (see section 6)

$$\pi_1 (U(\infty)) = \mathbf{Z}, \tag{23}$$

so $U(\infty)$ is *not* contractible to a point. $\mathbf{CP}^n$ is also noncontractible. We conclude that principal $U(\infty)$ bundles over $\mathbf{CP}^n$ may be nontrivial [25].

We define an action of $U(\infty)$ on $\mathcal{Q}$ as follows: first lift $\mathcal{Q}$ to the infinite–dimensional sphere S^∞, then embed S^∞ into $\mathcal{H}$ using equation (20) in the limit $n \to \infty$, then apply a $U(\infty)$ transformation. We require that this action be given by eqn. (21) on the first $n+1$ dimensions of $\mathcal{H}$, *i.e.*, only a $U(1)$ subgroup of $U(\infty)$ will act on them. Along the remaining infinite dimensions we let $U(\infty)$ act unconstrained. In this way we obtain a principal $U(\infty)$ fibre bundle whose base $\mathcal{C}$ is $\mathbf{CP}^n$ and whose total space is $\mathcal{Q}$. This $\mathbf{CP}^n$ inherits its symplectic structure (14) by quotienting (18) with the group action, so its standard quantisation reproduces the original quantum theory on $\mathcal{Q}$, up to an important difference. Coherent states (regarded as sections of the associated vector bundle whose fibre is $\mathcal{H}$) are no longer globally defined on $\mathbf{CP}^n$ because this bundle is nontrivial by construction, and therefore it admits no global section.

The physical implications of the local character of these coherent states are easy to interpret. Again set $n = 1$ for simplicity. In the case of the trivial bundle of section 4.2, the cross section of coherent states above *any* observer on the base $\mathbf{CP}^1$ was globally defined. Hence the semiclassical regime was universally defined for all observers on $\mathbf{CP}^1$. On the contrary, the nontriviality of the bundle considered here implies that the semiclassical regime is defined only locally, and it *cannot* be extended globally over $\mathbf{CP}^1$. What to one observer appears to be a semiclassical effect need *not* appear so to a different observer.

For illustrative purposes we have explicitly constructed one particular nontrivial bundle. It should not be difficult to construct other nontrivial bundles such that, *e.g.*, one observer actually perceives as strong quantum ($j < \infty$) the same effect that another observer calls semiclassical ($j \to \infty$).

4.4   Submanifolds of complex projective space

Any smooth, complex algebraic manifold M given by a system of polynomial equations in $\mathbf{CP}^n$ has a natural symplectic structure [2]. Let $\iota : M \to \mathbf{CP}^n$ be

an embedding of the complex manifold M into complex projective space. Then the symplectic form ω on $\mathbf{CP}^n$ can be pulled back to a symplectic form $\iota^*\omega$ on M. The fibre bundles of sections 4.2, 4.3, when pulled back to M, naturally suggest new instances of classical limits. The submanifold M must satisfy the integrality conditions [10].

5 Physical discussion

The deep link existing between classical and quantum mechanics has been known for long. Perhaps its simplest manifestation is that of coherent states. More recent is the notion that not all quantum theories arise from quantising a classical system. Furthermore, a given quantum model may possess more than just one classical limit. These facts find strong evidence in string duality and M–theory.

The geometric formulation of standard quantum mechanics presented in ref. [1] naturally suggests a procedure by which the passage to a classical limit may be performed in more than one different way. We believe this may provide a clue towards solving some of the conceptual problems mentioned in section 1.

We would like to point out that we do not propose a new approach to quantum mechanics, nor do we cast a doubt on its conceptual framework. On the contrary, we stand by its standard textbook interpretation as presented, *e.g.*, in refs. [18, 31]. Using the geometric formulation of standard quantum mechanics given in ref. [1], we have simply observed that what is usually called *the* classical limit in fact corresponds to a very specific choice of a fibre bundle whose total space is the quantum phase space. This, in turn, univocally determines the classical phase space to be the expected one. Historically the opinion has prevailed that *the classical limit is always uniquely and globally defined*. However, as hinted at in ref. [5], we believe this latter statement must be revised in the light of recent developments. In fact, nowhere in the axiomatics of standard quantum mechanics is such a statement to be found; it probably has its origins in the chronological order of developments in theoretical physics. Removing the statement that *the classical limit is always uniquely and globally defined* alters neither the foundations nor the standard interpretation of quantum mechanics.

The standard definition of classical limit is $\hbar \to 0$. However, the notion of duality suggests enlarging this definition in order to cover other cases that, on first sight, do not fall into that category. One possible generalisation of such a definition is the one we have considered here, namely, acting on the quantum phase space $\mathcal{Q}$ by means of a group G, so as to obtain a new phase space $\mathcal{Q}/G = \mathcal{C}$. Calling the latter *classical* is justified if $\mathcal{C}$ is a symplectic manifold whose quantisation gives back the original quantum theory on $\mathcal{Q}$. If that is the case, then $\mathcal{C}$ truly is a classical limit, even if we did not arrive at $\mathcal{C}$ by letting $\hbar \to 0$.

We have emphasised the key role played by the symplectic structure in switching back and forth between $\mathcal{Q}$ and $\mathcal{C}$. On the contrary, the role played by the Riemannian metric $g_{\mathcal{C}}$ has been reduced to that of providing quantum numbers once a certain classical limit has been fixed. It is precisely through lifting the metric dependence that we have succeeded in obtaining different classical limits for a given quantum theory. In this sense, as suggested in ref. [24], implementing duality transformations in quantum mechanics is very reminiscent of topological field theory.

Lifting the metric dependence in favour of diffeomorphism invariance, as in topological theories, is also important for the following reasons. We have made no reference to coupling constants or potentials, with the understanding that the Hamilton–Jacobi method has already placed us, by means of suitable coordinate transformations, in a coordinate system where all interactions vanish. At least under the standard notions [18] of *classical* vs. *quantum*, this is certainly always possible at the classical level [31]. At the quantum level, the approach of ref. [32], which contains standard quantum mechanics as a limiting case, rests precisely on the possibility of transforming between any two states by means of diffeomorphisms. Diffeomorphism invariance is a very powerful tool. It can be used [32] in the passage from *classical* to *quantum*. It can also be applied in the passage from *quantum* to *quantum*, as in ref. [33], where Hamiltonian quantum theories are constructed from functional integrals in the Osterwalder–Schrader framework [34, 35]. The viewpoint advocated here is that it can also be successfully applied in the passage from *quantum* to *classical*.

Then the only truly *quantum* ingredient we have at hand is $\hbar$. In fact one can think of quantisation, especially of deformation quantisation [15, 16], as performing an infinite expansion in powers of $\hbar$ around a classical theory. This full infinite expansion gives the full quantum theory. *Dequantisation* may then be interpreted as the truncation of this infinite expansion to a given finite order. As we have argued, if the quantum fibre bundle $\mathcal{Q} \to \mathcal{C}$ is nontrivial, this expansion in powers of $\hbar$ is local instead of global, so the notion of *classical* vs. *quantum* may not be globally defined for all observers.

6   Mathematical discussion

To conclude we would like to comment on some interesting mathematical points of our construction.

The following theorem holds [25]: a sufficient condition for a fibre bundle to be trivial is that either the stucture group or the base manifold be contractible to a point. Hence the classical limit may be nonglobal only if both the structure group and the base manifold are noncontractible. Concerning the uniqueness of the classical limit, one can in principle fibrate $\mathcal{Q}$ in many different ways, according to the symmetries of the problem.

We need to act with infinite–dimensional groups G on $\mathcal{Q}$ in order to obtain a finite–dimensional quotient $\mathcal{Q}/G$ as a classical phase space. When working with infinite–dimensional groups, the issue of contractibility deserves some care. Indeed one may topologise the group $U(\mathcal{H})$ with different, nonequivalent topologies, so the contractibility of $U(\mathcal{H})$ may depend on what topology one chooses for $U(\mathcal{H})$. Two popular choices are the norm topology and the strong operator topology [36]. It turns out that both of them render $U(\mathcal{H})$ contractible [28, 36].

Concerning $U(\infty)$, the best way to topologise it is the following. Enlarge an $n \times n$ unitary matrix to an $(n+1) \times (n+1)$ unitary matrix by adding one row and one column. The group $U(\infty)$ is defined by performing this enlargement infinitely many times. Now the direct limit topology [37] renders every matrix inclusion $U(n) \subset U(\infty)$ continuous, and it is the *maximal* topology that enjoys this property. Moreover, this topology also respects the fundamental group $\pi_1(U(n)) = \mathbf{Z}$ in the passage $n \to \infty$, as stated in eqn. (23).

One could wonder, why not use a $U(1)$ subgroup of $U(\mathcal{H})$ instead of $U(\infty)$,

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in order to construct a Hopf bundle in section 4.3? In fact one could do so, but at the cost of rendering the whole infinite–dimensional bundle over $\mathcal{C}$ trivial; only the finite–dimensional subbundle corresponding to the Hopf bundle would remain nontrivial. There would be no contradiction, since the triviality of a given bundle does not prevent the existence of nontrivial subbundles. For example, given any vector bundle $E \to \mathcal{C}$ over a (compact and Hausdorff) base manifold $\mathcal{C}$, there always exists a complementary vector bundle $F \to \mathcal{C}$ such that $E \oplus F$ is trivial [38].

However, the situation just described is precisely what we want to avoid. We need the complete, infinite–dimensional bundle over $\mathcal{C}$ to be nontrivial in order for the classical limit not to be globally defined; a finite–dimensional subbundle will not suffice. In retrospective, this argument also justifies our choice of $U(\infty)$ in section 4.3. The topologies considered above on $U(\mathcal{H})$, while rendering every inclusion $U(n) \subset U(\mathcal{H})$ continuous, are not the *maximal* topology enjoying that property. On the contrary, the direct limit topology on $U(\infty)$ is the maximal one with that property. This ensures that the addition of an infinite number of (spectator) dimensions to the n–dimensional Hopf bundle (19) does *not* render the complete infinite–dimensional bundle trivial, as would be the case with $U(\mathcal{H})$.

Quantum–mechanical symmetries are usually implemented by the action of unitary operators on $\mathcal{H}$. The group $U(\mathcal{H})$ thus arises naturally in this setup. However, any principal bundle with structure group $U(\mathcal{H})$ is necessarily trivial. In retrospective, this explains why the classical limit is always considered to be globally defined. In order to bypass this difficulty we have considered the subgroup $U(\infty) \subset U(\mathcal{H})$ and endowed it with a topology of its own (the direct limit topology) that is different from the induced topology it would inherit from $U(\mathcal{H})$. Only so do we have a chance of rendering $U(\infty)$–bundles nontrivial. It is interesting to observe that $U(\infty)$, instead of $U(\mathcal{H})$, is the right group that contains all $U(n)$ groups, in a way that naturally respects their topologies. $U(n)$ groups arise naturally in theories with solitons and instantons. In supersymmetric Yang–Mills theories and superstring theory, solitons and instantons lie at the heart of the notion of duality. This supports the notion that implementing duality transformations in quantum mechanics is in fact possible through mechanisms such as the one proposed here. It would also be very interesting to extend our mechanism to more general quantum–mechanical structures such as rigged Hilbert spaces [39].

Acknowledgments

References

[1] A. Ashtekar and T. Schilling, `gr-qc/9706069`; A. Ashtekar, `qr-qc/9901023`.

[2] V. Arnold, *Mathematical Methods of Classical Mechanics*, Springer, Berlin (1989).

[3] J. Marsden and T. Ratiu, *Introduction to Mechanics and Symmetry*, Springer, Berlin (1999).

[4] J. Schwarz, *Nucl. Phys. Proc. Suppl.* **55B** (1997) 1; `hep-th/0007130`; `hep-ex/0008017`.

[5] C. Vafa, `hep-th/9702201`.

[6] M. Kaku, *Introduction to Superstrings and M–Theory*, Springer, Berlin (1999); *Strings, Conformal Fields and M–Theory*, Springer, Berlin (2000).

[7] B. Kostant, *Lectures in Modern Analysis and Applications III*, ed. C. Taam, Springer, Berlin (1970); *Group Representations in Mathematics and Physics*, ed. V. Bargmann, Springer, Berlin (1970).

[8] J.-M. Souriau, *Group Theoretical Methods in Physics*, eds. A. Janner, T. Janssen and M. Boon, Springer, Berlin (1976); *Structure of Dynamical Systems*, Birkhäuser, Basel (1997).

[9] Sniatycki, *Geometric Quantization and Quantum Mechanics*, Springer, Berlin (1980).

[10] N. Woodhouse, *Geometric Quantization*, Oxford University Press, Oxford (1991).

[11] F. Berezin, *Sov. Math. Izv.* **38** (1974) 1116; *Sov. Math. Izv.* **39** (1975) 363; *Comm. Math. Phys.* **40** (1975) 153; *Comm. Math. Phys.* **63** (1978) 131.

[12] M. Schlichenmaier, `q-alg/9601016`; `q-alg/9611022`; `math.QA/9902066`; `math.QA/9910137`.

[13] N. Reshetikhin and L. Takhtajan, `math.QA/9907171`.

[14] M. Kontsevich, `q-alg/9709040`; *Lett. Math. Phys.* **48** (1999) 35.

[15] For a review see D. Sternheimer, `math.QA/9809056`.

[16] C. Zachos, `hep-th/0110114`.

[17] A. Cattaneo and G. Felder, *Comm. Math. Phys.* **212** (2000) 591.

[18] A. Galindo and P. Pascual, *Quantum Mechanics*, vols. I, II, Springer, Berlin (1990).

[19] J. Klauder and B.–S. Skagerstam, *Coherent States*, World Scientific, Singapore (1985); R. Gilmore, *Ann. Phys. (N.Y.)* **74** (1972) 391.

[20] A. Perelomov, *Generelized Coherent States and their Applications*, Springer, Berlin (1986).

[21] J. Marsden and A. Weinstein, *Rep. Math. Phys.* **5** (1974) 121.

[22] V. Guillemin and S. Sternberg, *Symplectic Techniques in Physics*, Cambridge University Press, Cambridge (1984).

[23] O. Müller, `math-ph/0108016`.

[24] J.M. Isidro, `quant-ph/0105012`.

[25] N. Steenrod, *The Topology of Fibre Bundles*, Princeton University Press, Princeton (1970).

[26] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity*, Wiley, New York (1972).

[27] B. Hall and J. Mitchell, `quant-ph/0109086`.

[28] N. Kuiper, *Topology* **3** (1965) 19.

[29] A. Schwarz, *Topology for Physicists*, Springer, Berlin (1994).

[30] A. Schwarz, *Quantum Field Theory and Topology*, Springer, Berlin (1993).

[31] W. Dittrich and M. Reuter, *Classical and Quantum Dynamics*, Springer, Berlin (1994).

[32] A. Faraggi and M. Matone, *Int. Journ. Mod. Phys.* **A15** (2000) 1869; G. Bertoldi, A. Faraggi and M. Matone, *Class. Quant. Grav.* **17** (2000) 3965.

[33] A. Ashtekar, D. Marolf, J. Mourão and T. Thiemann, *Class. Quant. Grav.* **17** (2000) 4919.

[34] K. Osterwalder and R. Schrader, *Comm. Math. Phys.* **31** (1973) 83; *Comm. Math. Phys.* **42** (1975) 281.

[35] J. Glimm and A. Jaffe, *Quantum Physics*, Springer, Berlin (1980).

[36] J. Dixmier, *Les $C^{\prime\star}$–Algèbres et leurs Représentations*, Gauthier–Villars, Paris (1969).

[37] N. Bourbaki, *Topologie Générale*, Hermann, Paris (1958).

[38] M. Atiyah, *K–Theory*, Benjamin (1967).

[39] R. de la Madrid, A. Bohm and M. Gadella, `quant-ph/0109154`.