Open string with a background B-field as the first order mechanics and noncommutativity.

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Abstract

To study noncommutativity properties of the open string with constant B-field we construct a mechanical action which reproduces classical dynamics of the string sector under consideration. It allows one to apply the Dirac quantization procedure for constrained systems in a direct and unambiguous way. The mechanical action turns out to be the first order system without taking the strong field limit $B \longrightarrow \infty$. In particular, it is true for zero mode of the string coordinate which means that the noncommutativity is intrinsic property of this mechanical system. We describe the arbitrariness in the relation existent between the mechanical and the string variables and show that noncommutativity of the string variables on the boundary can be removed. It is in correspondence with the result of Seiberg and Witten on relation among noncommutative and ordinary Yang-Mills theories.

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Introduction. Noncommutative Yang-Mills theory arises in a definite limit of string theory [1]. It has been extracted by Seiberg and Witten [2] starting from the open string in the presence of a B-field [3-9], with the action for the corresponding sector being [2]

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \left[\partial_a x^i \partial_a x^i + 2\pi\alpha' \epsilon^{ab} \partial_a x^i \partial_b x^j B_{ij}\right]. \tag{1}$$

An extremum of the action is supplied by

$$(\partial_\tau^2 - \partial_\sigma^2) x^i = 0, \tag{2}$$

$$\{\partial_\sigma x^i + 2\pi\alpha' \partial_\tau x^j B_j{}^i\} \mid_{\sigma=0}^{\sigma=\pi} = 0. \tag{3}$$

Open string propagator with the boundary conditions (3) contains an antisymmetric matrix [3, 6], which gives rise to noncommutativity of the string coordinate on the boundaries [2, 7]. It was suggested [8, 9] that the noncommutative geometry can be reproduced in the elementary framework of constrained systems as a result of the Hamiltonian quantization of the system (2), (3), by analogy with the noncommutativity arising for coordinates of a charged particle in the lowest Landau level [10-12]. To achieve this, there were proposed rather radical modifications [8, 9] of the Dirac procedure for constrained systems. There is some discrepancy among the results obtained in different approaches, which was discussed in [8, 9, 13, 14].

The aim of this work is to quantize the system (2), (3) following the standard methods [15, 16] without any modifications. Canonical analysis of this system presents a problem since the Dirac procedure is initially formulated for the mechanical system (and then can be generalized for a field with vanishing boundary conditions). Application of the Dirac formalism to a field with nontrivial boundary conditions on a compact manifold requires more careful analysis (see [17] for the open string with the Neumann boundary conditions). Consistent treatment of such a system implies necessity to represent the initial dynamics in terms of a mechanical system. Thus we first rewrite Eqs. (2), (3) in the form of equations of motion for mechanical variables $c_n(\tau),\ n \in Z$, and then restore the mechanical action which reproduces this dynamics. After that, the Dirac procedure can be applied to the mechanical system in a direct and unambiguous way. In particular, δ-function regularization is not necessary in this case. The last step is to rewrite the results in terms of the string coordinate $x^i(\tau, \sigma)$, which gives the Hamiltonian formulation associated with the theory (1)-(3).

Some of the results thus obtained are as follows. The corresponding mechanical system turns out to be the first order system without taking the strong field limit $B \longrightarrow \infty$. Thus, as a consequence of the mixed boundary conditions, dynamics of the mechanical variables is governed by equations of the first order in time derivative. In particular, it is true for the zero modes $c^i_0(\tau)$ of the string coordinate $x^i(\tau,\sigma)$. As a consequence, c^i_0 are canonically conjugated to each other: $\{c^i_0, c^j_0\} \ne 0$. It means that the noncommutativity is intrinsic property of this mechanical system. Brackets for the string coordinates turn out to be noncommutative in the bulk and on the boundary. We study freedom in relation between the mechanical and the string variables as well as freedom in choosing brackets which is always present in the Hamiltonian formalism. This allows one to discuss to what extent the noncommutativity of the string coordinates can be avoided. We show that the embedding coordinates of a D-brane can be made commutative, of course one needs in this case to transform simultaneously the Hamiltonian of the system. It is in correspondence with the result of Seiberg and Witten on equivalence of noncommutative and commutative Yang-Mills fields.

Open string with B-field in terms of mechanical variables. To start with, let us continue $x^i(\sigma),\ \sigma \in [0,\pi]$ on the interval $[0,2\pi]$ such that $\tilde{x}^i(\sigma) = x^i(\sigma)$ on $[0,\pi],\ \partial_\sigma \tilde{x} \mid_\pi = \vec{\partial}_\sigma x \mid_\pi, \ldots$, and $\tilde{x}^i(2\pi) = \tilde{x}^i(0),\ \vec{\partial}_\sigma \tilde{x}^i \mid_{2\pi} = \overleftarrow{\partial}_\sigma\ \tilde{x}^i \mid_0, \ldots$. It can be further continued on $\sigma \in (-\infty,\infty)$ as a periodic function $\tilde{x}^i(\sigma + 2\pi n) = \tilde{x}^i(\sigma)$. As a result, any solution of the problem (2), (3) can be presented in the form

$$x^i(\tau,\sigma) = c^i_0(\tau) + \sum_{n=1}^{\infty} \frac{1}{n}\left[c^i_n(\tau)\cos n\sigma + c^i_{-n}(\tau)\sin n\sigma\right], \tag{4}$$

where $c^i_n(\tau),\ n \in Z$, are our mechanical variables. Substitution in Eqs. (2), (3) gives the equations of motion

$$\ddot{c}^i_0 = 0, \qquad \ddot{c}^i_n + n^2 c^i_n = 0, \qquad n \ne 0; \tag{5}$$

$$\dot{c}^i_0 = 0, \qquad \dot{c}^j_n B_j{}^i + \frac{n}{2\pi\alpha'} c^i_{-n} = 0, \qquad n > 0, \tag{6}$$

from which one finds further consequence

$$\dot{c}^i_{-n} - 2\pi\alpha' n c^j_n B_j{}^i = 0. \tag{7}$$

Consequently, the complete dynamics can be rewritten in the equivalent form as

$$\begin{aligned} \dot{c}_0^i &= 0, \\ \dot{c}_n^j B_j{}^i + \frac{n}{2\pi\alpha'} c_{-n}^i &= 0, \\ \dot{c}_{-n}^i - 2\pi\alpha' n c_n^j B_j{}^i &= 0, \qquad n > 0, \end{aligned} \tag{8}$$

which consist of equations of the first order only. Eq. (8) follows from the first order action [1]

$$\begin{aligned} S_f = & \int d\tau \left[\frac{1}{2} \dot{c}_0^i B_{ij} c_0^j + \sum_{n=1}^{\infty} f_n \left(\pi\alpha' \dot{c}_n^i B_{ij} c_n^j - \right. \right. \\ & \left. \left. \frac{1}{4\pi\alpha'} \dot{c}_{-n}^i B_{ij}^{-1} c_{-n}^j + n c_{-n}^i c_n^i \right) \right], \end{aligned} \tag{9}$$

where $f_n \neq 0, \ n > 0$ are real numbers. While they can be removed by shift of the variables c_n, it is convenient to keep them in the action. Starting from any particular S_f, the variables c_n can be taken as the ones which generate the string coordinate (4), the latter will not depend on f_n. At the same time, brackets of c_n and the Hamiltonian H_f will depend on f_n, so there is appear natural arbitrariness in the induced bracket for $x^i(\tau, \sigma)$. Choice of some particular form of the bracket implies that the corresponding Hamiltonian H_f must be associated with the Lagrangian formulation (1)-(3).

Hamiltonian analysis of the machanical action. Direct application of the Dirac algorithm to the action (9), gives us the primary second class constraints

$$\begin{aligned} G_{0i} &\equiv p_{0i} - \frac{1}{2} B_{ij} c_0^j = 0, \\ G_{ni} &\equiv p_{ni} - \pi\alpha' f_n B_{ij} c_n^j = 0, \\ G_{-ni} &\equiv p_{-ni} + \frac{f_n}{4\pi\alpha'} B_{ij}^{-1} c_{-n}^j = 0. \end{aligned} \tag{10}$$

Their Poisson brackets are

$$\begin{aligned} \{G_{0i}, G_{0j}\} &= -B_{ij}, \\ \{G_{ni}, G_{mj}\} &= -2\pi\alpha' f_n B_{ij} \delta_{n-m,0}, \\ \{G_{-ni}, G_{-mj}\} &= \frac{f_n}{2\pi\alpha'} B_{ij}^{-1} \delta_{n-m,0}, \end{aligned} \tag{11}$$

[1] Other possibility is to take as independent the equations (5),(6) with $n > 0$. Then the action can be chosen in the second order form $S = \int d\tau \left[\frac{1}{2} \dot{c}_0 B c_0 + \sum_{n=1}^{\infty} \frac{1}{2} \dot{c}_n \dot{c}_n - \frac{n^2}{2} c_n c_n + (\dot{c}_n B + \frac{n}{2\pi\alpha'} c_{-n}) \lambda_n \right]$. It looks less natural since it involves the Lagrangian multipliers λ_n.

and the corresponding Hamiltonian is

$$
\begin{aligned}
H = \quad & \sum_{n=1}^{\infty} -f_n n c_{-n} c_n + \lambda_0 (p_0 - \frac{1}{2} B c_0) + \\
& \lambda_n (p_n - \pi\alpha' f_n B c_n) + \lambda_{-n} (p_{-n} + \frac{f_n}{4\pi\alpha'} B^{-1} c_{-n}).
\end{aligned} \tag{12}
$$

There are no secondary constraints in the problem. From the consistency conditions that constraints do not evolve in time, $\dot{G} = 0$, one obtains expressions for the Lagrangian multipliers

$$
\lambda_0 = 0, \qquad \lambda_n = -\frac{n}{2\pi\alpha'} c_{-n} B^{-1}, \qquad \lambda_{-n} = 2\pi\alpha' n c_n B. \tag{13}
$$

To take into account the second class constraints (10) one introduces the Dirac bracket

$$
\begin{aligned}
\{K, P\}_D = \quad & \{K, P\} + \{K, G_{0i}\} (B^{-1})^{ij} \{G_{0j}, P\} + \\
& \sum_{n=1}^{\infty} \{K, G_{ni}\} \frac{1}{2\pi\alpha' f_n} (B^{-1})^{ij} \{G_{nj}, P\} - \\
& \{K, G_{-ni}\} \frac{2\pi\alpha'}{f_n} B^{ij} \{G_{-nj}, P\}.
\end{aligned} \tag{14}
$$

After that, the variables p_n^i can be omitted from consideration. The resulting Hamiltonian formulation for (9) consist of the physical variables c_n^i with the Dirac brackets

$$
\begin{aligned}
\{c_0^i, c_0^j\}_D &= -(B^{-1})^{ij}, \\
\{c_n^i, c_m^j\}_D &= -\frac{1}{2\pi\alpha' f_n} (B^{-1})^{ij} \delta_{n-m,0}, \\
\{c_{-n}^i, c_{-m}^j\}_D &= \frac{2\pi\alpha'}{f_n} B^{ij} \delta_{n-m,0}, \qquad n, m > 0,
\end{aligned} \tag{15}
$$

whose dynamics is governed now by the Hamiltonian

$$
H = -\sum_{n=1}^{\infty} f_n n c_{-n}^i c_n^i. \tag{16}
$$

As it should be for the first order system, the Hamiltonian equations of motion which follow from (15), (16) are the equations (8). Note also that the variables c_n^i with n fixed are canonically conjugated to each other.

The equations (8) for the physical variables can be solved now in terms of oscillators

$$\begin{aligned}
c^i_n(\tau) &= \alpha^i_n e^{in\tau} + \alpha^i_{-n} e^{-in\tau}, \\
c^i_{-n}(\tau) &= -2i\pi\alpha'\alpha^i_n B e^{in\tau} + 2i\pi\alpha'\alpha^i_{-n} B e^{-in\tau}, \\
\alpha^{i\,*}_n &= \alpha^i_{-n}.
\end{aligned} \tag{17}$$

From Eq. (15) one finds their brackets

$$\begin{aligned}
\{c^i_0, c^j_0\}_D &= -(B^{-1})^{ij}, \\
\{\alpha^i_n, \alpha^j_m\}_D &= -\frac{1}{4\pi\alpha' f_{|n|}}(B^{-1})^{ij}\delta_{n+m,0}, \qquad n, m \neq 0,
\end{aligned} \tag{18}$$

while the Hamiltonian (16) acquires the form

$$H_f = 4i\pi\alpha' \sum_{n=1}^{\infty} f_n n \alpha^i_n B_{ij} \alpha^j_{-n}. \tag{19}$$

It is worth nothing that this procedure, being applied to the open string with the Neumann boundary conditions, leads to the standard results [18, 19].

Hamiltonian formulation for the open string with a B-field. By using of Eqs. (4), (17) we restore the string coordinate $x^i(\tau, \sigma)$ in terms of oscillators

$$x^i(\tau, \sigma) = c^i_0 + \sum_{n \neq 0} \frac{e^{in\tau}}{|\, n\,|}\left(\alpha^i_n \cos n\sigma - 2i\pi\alpha'\alpha^j_n {B_j}^i \sin n\sigma\right) \tag{20}$$

Thus, with the system (1)-(3) one associates the Hamiltonian formulation which includes the variables $c^i_0,\ \alpha^i_n,\ n \neq 0$ with the brackets (18) and the Hamiltonian (19). As a consequence of Eqs. (18), (20) the induced bracket of the string coordinates turns out to be noncommutative in the bulk and on the boundary

$$\begin{aligned}
\{x^i(\tau,\sigma), x^j(\tau,\sigma'\}_D &= -(B^{-1})^{ij} - \sum_{n=1}^{\infty} \frac{1}{n^2 f_n}\left(\frac{1}{2\pi\alpha'}(B^1)^{ij}\right. \\
&\quad \times \cos n\sigma \cos n\sigma' - 2\pi\alpha' B^{ij} \\
&\quad \left.\times \sin n\sigma \sin n\,\sigma'\right).
\end{aligned} \tag{21}$$

In particular, the coordinates on D-brane obey

$$\begin{aligned}
\{x^i(\tau,0), x^j(\tau,0\}_D = &\quad \{x^i(\tau,\pi), x^j(\tau,\pi\}_D \\
= &\quad -\left(1 + \frac{1}{2\pi\alpha'}\sum_{n=1}^{\infty}\frac{1}{n^2 f_n}\right)(B^{-1})^{ij}.
\end{aligned} \tag{22}$$

It is interesting to note that the standard normalization $f_n \sim \frac{1}{n}$ of the oscillator brackets (18) implies that one needs to use some regularization for Eq. (22). For the choice $f_n = -\frac{\pi}{12\alpha'}$ the D-brane plane becomes commutative, and the Hamiltonian in this case is

$$H = -\frac{i\pi^2}{3}\sum_{n=1}^{\infty}\alpha_n^i B_{ij}\alpha_{-n}^j. \tag{23}$$

Final Discussions. Finally, let us discuss freedom in definition of phase space bracket in the Hamiltonian formalism. Transition from configuration to phase space description is not unique procedure since one needs to define simultaneously the bracket and the Hamiltonian. One possibility is to start from the Poisson bracket, then exact expression for the Hamiltonian is known for a general case [16, 20]. According to the Darboux theorem, one can equally take as the starting point any nondegenerated closed two-form and then try to define a Hamiltonian from the condition that corresponding equations of motion are equivalent to the initial ones. Turning to our case (15), (16) it will be sufficiently to consider the brackets

$$\begin{aligned} \{c_0^i, c_0^j\}_D &= -E^{ij}, \\ \{c_n^i, c_m^j\}_D &= -\frac{1}{2\pi\alpha' f_n}A^{ij}\delta_{n-m,0}, \;\; n, m > 0, \end{aligned} \tag{24}$$

with some antisymmetric nondegenerated constant matrices E and A. From the condition that the dynamics (8) is reproduced in this formulation one finds

$$\begin{aligned} \{c_{-n}^i, c_{-m}^j\}_D &= \frac{2\pi\alpha'}{f_n}(AB^2)^{ij}\delta_{n-m,0}, \\ H &= -\sum_{n=1}^{\infty} f_n n c_{-n}(AB)^{-1}c_n. \end{aligned} \tag{25}$$

One can take Eqs. (24), (25) instead of Eqs. (15), (16) as the Hamiltonian formulation corresponding to the system (9). Repeating the previous analysis, one finds the same expression (19) for the string coordinate in terms of oscillators which obey

$$\begin{aligned} \{c_0^i, c_0^j\}_D &= -E^{ij}, \\ \{\alpha_n^i, \alpha_m^j\}_D &= -\frac{1}{8\pi\alpha' f_{|n|}}\Big(A - sgn(nm)B^{-1}AB\Big)^{ij} \\ &\quad \times \delta_{|n|-|m|,0}, \;\; n, m \neq 0, \end{aligned} \tag{26}$$

The quantities $f_n, \;\; E, \;\; A$ can be chosen in such a way that the noncommutativity parameter on D-brane acquires exactly the same form as it was obtained from the disk propagator [2]. Namely, for the choice

$$
\begin{aligned}
f_n &= \frac{\pi^2}{3(2\pi\alpha')^3}, \\
E &= \frac{(2\pi\alpha')^2}{2} A, \\
A &= \left(1 + (2\pi\alpha')^2 B^2\right)^{-1} B,
\end{aligned} \tag{27}
$$

one obtains the following Hamiltonian formulation for the system (1)-(3)

$$
\begin{aligned}
H &= \frac{i\pi^2}{3(2\pi\alpha')^2} \sum_{n=1}^{\infty} n\alpha_n B^{-1} \left(1 + (2\pi\alpha')^2 B^2\right) \alpha_{-n}, \\
\{c_0^i, c_0^j\}_D &= -\frac{(2\pi\alpha')^2}{2} \left[\left(1 + (2\pi\alpha')^2 B^2\right)^{-1} B\right]^{ij}, \\
\{\alpha_n^i, \alpha_m^j\}_D &= -\frac{3(2\pi\alpha')^2}{2\pi^2} \left[\left(1 + (2\pi\alpha')^2 B^2\right)^{-1} B\right]^{ij} \\
&\quad \times \delta_{n+m,0}, \;\; n, m \neq 0,
\end{aligned} \tag{28}
$$

which gives for the coordinates on D-brane the desired expression

$$
\{x^i, x^j\}_D = -(2\pi\alpha')^2 (1 - 2i\pi\alpha' B)^{-1} B (1 + 2i\pi\alpha' B)^{-1}. \tag{29}
$$

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