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Boundary One-Point Functions, Scattering, and Background Vacuum Solutions in Toda Theories

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Abstract

The parametric families of integrable boundary affine Toda theories are considered. We calculate boundary one-point functions and propose boundary S-matrices in these theories. We use boundary one-point functions and S-matrix amplitudes to derive boundary ground state energies and exact solutions describing classical vacuum configurations.

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1 Introduction

There is a large class of massive 2D integrable quantum field theories (QFTs) which can be considered as perturbed conformal field theories (CFTs). The short distance behavior of these QFTs is encoded in the CFT data while their long distance properties are determined by $S-$matrix data. A link between these two kinds of data would provide a good viewpoint for a rather complete understanding and description of such theories.

In this paper we study the consistency conditions between CFT and S-matrix data for integrable families of boundary non-simply laced affine Toda theories (ATTs). The similar analysis for simply laced ATTs (where integrable boundary conditions do not contain parametric families) was done in Ref.[9]. These QFTs can be considered as perturbed CFTs (non-affine Toda theories), which possess an extended symmetry generated by $W-$algebra. The integrable boundary conditions preserve this symmetry. This permits us to apply the "reflection amplitude" approach [1] for the calculation of the boundary vacuum expectation values (VEVs) in ATTs. The boundary VEVs (one-point functions), in particular, contain the information about the boundary values of the solutions of classical boundary Toda equations. The information about the long distance behavior of these solutions can be extracted from boundary $S-$matrix. The explicit solutions constructed in this paper provide us the consistency check of CFT and S-matrix data in integrable families of boundary ATTs.

The plan of the paper is as follows: in section 2 we recall some basic facts about Toda theories and one-point functions in ATTs defined on the whole plane. In section 3 we consider integrable boundary ATTs. We give the explicit expressions for boundary one point functions which can be derived by the "reflection amplitude" approach [1]. These functions determine, in particular, the boundary values of the solutions corresponding to the vacuum configurations in ATTs. We use the boundary VEVs to derive a conjecture for quantum boundary ground state energies. In section 4 we construct the boundary scattering theory which is consistent with this conjecture. The semiclassical limits of boundary $S-$matrix amplitudes fix the asymptotics of vacuum solutions in ATTs. These asymptotics determine completely the explicit form of the solutions which we construct in section 5. The solutions can be written in terms of tau-functions associated with multisoliton solutions in ATTs. We check that the boundary values of these solutions agree with the corresponding values given by boundary one-point functions.

2 Affine and non-affine Toda theories

The ATT corresponding to a Lie algebra G of rank r is described by the action[§]

$$\mathcal{A}_b = \int d^2x \left[\frac{1}{8\pi}(\partial_\mu \varphi)^2 + \sum_{i=0}^{r} \mu_i e^{b e_i \cdot \varphi} \right], \tag{1}$$

[§]In this paper we denote as G also the corresponding untwisted affine algebra and use the notation $G^\vee$ for dual affine algebra.

where e_i, $i = 1, ..., r$ are the simple roots of Lie algebra G and $-e_0$ is a maximal root which defines the integers n_i by the relation:

$$\sum_{i=0}^{r} n_i e_i = 0. \tag{2}$$

The fields φ in eq. (1) are normalized so that at $\mu_i = 0$

$$\langle \varphi_a(x)\varphi_b(y)\rangle = -\delta_{ab} \log |x-y|^2 \tag{3}$$

We consider below mainly the non-simply laced ATTs. These theories possess a remarkable property of duality [5]. Under duality transformation $b \to b^\vee = 1/b$ the action (1) transforms to the action of dual ATT characterized by the roots $e_i(G^\vee) = 2e_i(G)/e_i^2(G)$. The parameters μ_i in the dual ATTs are related as follows:

$$\pi\mu_i(G)\gamma(e_i^2 b^2/2) = \left(\pi\mu_i(G^\vee)\gamma(2/(e_i^2 b^2))\right)^{e_i^2 b^2/2} \tag{4}$$

where as usual $\gamma(z) = \Gamma(z)/\Gamma(1-z)$.

The mass ratios in non-simply laced ATT are different from classical ones [5]. For real b the spectrum consists of r particles with masses m_j. These masses of particles are characterized by one mass parameter m and flow from the values $m\lambda_j(G)$ to the values $m\lambda_j(G^\vee)$, where $\lambda_j^2(G)$ and $\lambda_j^2(G^\vee)$ are the eigenvalues of classical mass matrices $M(G)$ and $M(G^\vee)$, i.e.:

$$M_{ab} = \sum_{i=0}^{r} n_i (e_i)^a (e_i)^b. \tag{5}$$

To describe the spectrum it is convenient to introduce the notations:

$$x = \frac{b^2}{1+b^2}; \qquad h = H(G)(1-x) + H(G^\vee)x \tag{6}$$

where H is the Coxeter number. Then the spectrum can be expressed in terms of parameter m as:

$$\begin{aligned} B_r \quad &: \qquad m_r = \sqrt{2}m, \;\; m_j = \sqrt{8}m \sin(\pi j/h); \;\; j = 1, ..., r-1. \\ C_r, BC_r \quad &: \qquad m_j = \sqrt{8}m \sin(\pi j/h); \;\; j = 1, ..., r \end{aligned} \tag{7}$$

The spectrum for the exceptional algebras G can be found in [5].

The exact relation between the parameter m in the above spectra and the parameters μ_i in the action (1) can be obtained by the Bethe ansatz method (see for example Refs. [2],[3]). It was derived in Ref.[4] and has the form:

$$\prod_{i=0}^{r}\left[\frac{-\pi\mu_i}{\gamma(-e_i^2 b^2/2)}\right]^{n_i} = \left[\frac{mk(G)\Gamma\left(\frac{1-x}{h}\right)\Gamma\left(\frac{x}{h}\right)x}{\sqrt{2}\Gamma\left(\frac{1}{h}\right)h}\right]^{2h(1+b^2)}, \tag{8}$$

where integers n_i are defined by the equation (2) and

$$k(B_r) = 2^{-2/h}\Gamma, \quad k(C_r) = 2^{2x/h}, \quad k(BC_r) = 2^{(x-1)/h}. \tag{9}$$

For the exceptional algebras G the numbers $k(G)$ can be found in [4]. The similar relations for the dual ATTs can be easily derived from Eqs.(8,9) and duality relations (4).

The ATTs can be considered as perturbed CFTs. With the parameter $\mu_0 = 0$ the action (1) describes non-affine Toda theories (NATTs), which are conformal field theories. To describe the generator of conformal symmetry we introduce the complex coordinates $z = x_1 + ix_2,\ \bar{z} = x_1 - ix_2$ and vector

$$Q = \rho/b + b\rho^\vee; \qquad \rho = \frac{1}{2}\sum_{\alpha>0}\alpha, \quad \rho^\vee = \sum_{\alpha>0}\alpha/\alpha^2. \tag{10}$$

where the sum in the definition of the Weyl vector ρ runs over all positive roots α of G.

The holomorphic stress energy tensor

$$T(z) = -\frac{1}{2}(\partial_z\varphi)^2 + Q\cdot\partial_z^2\varphi \tag{11}$$

ensures the local conformal invariance of the NATT. Besides conformal invariance NATT possesses an additional symmetry generated by two copies of the chiral $W(G)$-algebras: $W(G)\otimes$ $\overline{W}(G)$ (see Refs. [6],[20] for details). This extended conformal symmetry permits to calculate the VEVs of exponential fields in ATTs, i.e. one point functions

$$\mathbf{G}(a) = \langle 0|\exp(a\cdot\varphi)|0\rangle\,. \tag{12}$$

These VEVs were calculated in [4]. They can be represented in the form:

$$\mathbf{G}(a) = N_G^2(a)\exp\left(\int_0^\infty \frac{dt}{t}[a^2e^{-2t} - \sum_{\alpha>0}\sinh((\alpha^2b^2/2+1)t)F_\alpha(a,t)]\right). \tag{13}$$

In this equation the factor $N_G(a)$ can be expressed in terms of fundamental co-weights $\omega_i^\vee$ of G (i.e. $\omega_i^\vee\cdot e_j = \delta_{ij}$) as:

$$N_G^2(a) = \left[\frac{mk(G)\Gamma\left(\frac{1-x}{h}\right)\Gamma\left(\frac{x}{h}\right)x}{\sqrt{2}\Gamma\left(\frac{1}{h}\right)h}\right]^{2Q\cdot a-a^2}\prod_{i=1}^r\left[\frac{-\pi\mu_i}{\gamma\left(-e_i^2b^2/2\right)}\right]^{-\omega_i^\vee\cdot a} \tag{14}$$

and the function $F_\alpha(a,t)$ is given by

$$F_\alpha(a,t) = \frac{\sinh(ba_\alpha t)\sinh(ba_\alpha - 2bQ_\alpha + h(1+b^2))t)}{\sinh t\,\sinh(\alpha^2b^2t/2)\,\sinh((1+b^2)ht)} \tag{15}$$

(here and below the subscript α denotes the scalar product of the vector with a positive root α, i.e. $a_\alpha = a\cdot\alpha$; $Q_\alpha = Q\cdot\alpha$ and so on).

This expression satisfies many possible perturbative and non-perturbative tests for one-point function in ATT. For example, it can be easily derived from Eqs.(1,8) that the bulk vacuum energy $E(G)$ can be expressed in terms of function $\mathbf{G}(a)$. We have:

$$\mu_i \partial_{\mu_i} E = n_i(1-x)E/h = \mu_i \mathbf{G}(be_i). \tag{16}$$

The values of the function $\mathbf{G}(a)$ at the special points be_i can be calculated explicitly and the result coincides with the expression which was obtained by Bethe ansatz method [4]. In particular, for B, C and BC ATT we obtain that

$$E = \frac{m^2 \sin(\pi/h)}{4\sin(\pi x/h)\sin(\pi(1-x)/h)}. \tag{17}$$

It is convenient to define the field $\widehat{\varphi} = \varphi b/(1+b^2)$. In the limit $b \to 0$ (as well as in the dual limit $b^\vee = 1/b \to 0$) this field can be described by the classical equations. In particular, it can be derived from Eqs.(13,8) that the VEV of this field $\widehat{\varphi}_0 = \langle 0|b\varphi/(1+b^2)|0\rangle$ in that limit can be expressed in terms of fundamental co-weights in the form:

$$\widehat{\varphi}_0 = \sum_{i=1}^{r} \omega_i^\vee \log\left(\frac{m^2}{4\pi b^2 \mu_i}\right) \tag{18}$$

and it coincides with a classical vacuum of ATT (1). After rescaling and shifting of the field φ, the action (1) can be written in terms of the field $\phi = \widehat{\varphi} - \widehat{\varphi}_0$ as the classical action of ATT:

$$\mathcal{A}_b = \frac{1}{4\pi b^2} \int d^2x [\frac{1}{2}(\partial_\mu \phi)^2 + m^2 \sum_{i=0}^{r} n_i e^{e_i \cdot \phi}] + O(1) \tag{19}$$

In a similar way, the opposite (dual) limit $b^\vee = 1/b \to 0$ leads to the classical action of the dual affine Toda theory.

3 Boundary Toda Theories, Boundary One-Point Functions and Classical Vacuum Solutions

In the previous section we considered Toda theories defined on the whole plane R^2. Here we consider these theories defined on the half-plane $H = (x, y; y > 0)$ with integrable boundary conditions. The action of the boundary ATTs can be written in the form:

$$\mathcal{A}_{bound} = \int_H d^2x \left[\frac{1}{8\pi}(\partial_\mu \varphi)^2 + \sum_{i=0}^{r} \mu_i e^{be_i \cdot \varphi}\right] + \int dx \sum_{0}^{r} \nu_i e^{be_i \cdot \varphi/2}. \tag{20}$$

These theories also possess the property of duality. Under the duality transformation $b \to b^\vee = 1/b$ they transform to the QFTs which can be described by the action of the dual ATTs with the parameters $\mu_i(G^\vee)$ given by Eq.(4) and boundary parameters $\nu_i(G^\vee)$ which can be determined in the following way [13]: we define the variables $s_0, \ldots, s_r$ by the relation:

$$\nu_i^2(G) = \mu_i(G)\frac{\cos^2(\pi l_i s_i b/2)}{\sin(\pi l_i^2 b^2/2)}; \qquad l_i = \sqrt{e_i^2(G)} \tag{21}$$

then the dual boundary parameters are determined as follows:

$$\nu_i^2(G^\vee) = \mu_i(G^\vee)\frac{\cos^2(\pi l_i^\vee s_i b^\vee/2)}{\sin(\pi(l_i^\vee b^\vee)^2/2)}; \qquad l_i^\vee = \sqrt{e_i^2(G^\vee)} = 2/l_i. \tag{22}$$

It is easy to derive from these equations that the theory dual to the boundary ATT with the Neumann boundary conditions i.e. all $\nu_i = 0$ or

$$\partial_y \varphi(x,y)|_{y=0} = 0 \tag{23}$$

is specified by the boundary parameters

$$\nu_i = \left(\frac{\mu_i}{2}\cot(\pi l_i^2 b^2/4)\right)^{1/2}. \tag{24}$$

The integrability conditions for classical boundary ATTs were studied in Ref. [12]. In particular it was shown there that contrary to simply laced case, the non-simply laced $B_r, B_r^\vee, C_r, C_r^\vee$ and BC_r ATTs¶ admit parametric families of integrable boundary conditions. These parameters are associated with ν_i corresponding to the non-standard roots ($e_i^2 \neq 2$) of affine Lie algebras. It is convenient to write $\nu_i^{(cl)}$ in the form:

$$\nu_i^{(cl)} = \varepsilon_i \sqrt{\frac{\mu_i}{\pi b^2}} A_i \tag{25}$$

where $\varepsilon_i = \pm 1$. Then integrable boundary conditions can be specified by continuous parameters w_0 and w_r as follows:

$$\begin{aligned}
B_r &: \quad e_r^2 = 1; \quad A_i = 1, i \neq r, \quad A_r = \sqrt{2}\cos(\pi w_r/2). && (26)\\
B_r^\vee &: \quad e_r^2 = 4; \quad A_i = 0, i \neq r, \quad A_r = \cos(\pi w_r)/\sqrt{2}. && (27)\\
C_r &: \quad e_{0,r}^2 = 4; \quad A_i = 0, i \neq 0, r, \quad A_{0,r} = \cos(\pi w_{0,r})/\sqrt{2}. && (28)\\
C_r^\vee &: \quad e_{0,r}^2 = 1; \quad A_i = 1, i \neq 0, r, \quad A_{0,r} = \sqrt{2}\cos(\pi w_{0,r}/2). && (29)
\end{aligned}$$

In the case of BC_r ATT ($e_0^2 = 1, e_r^2 = 4$) there exist two types of integrable boundary conditions:

$$A_0 = \sqrt{2}\cos(\pi w_0/2), \quad A_i = 1, i \neq 0, r, \quad A_r = 1/\sqrt{2} \tag{30}$$

and

$$A_i = 0, i \neq r, \quad A_r = \cos(\pi w_r)/\sqrt{2}. \tag{31}$$

¶ $B_r^\vee$, $C_r^\vee$ and BC_r ATTs are known also as $A_{2r-1}^{(2)}$, $D_{r+1}^{(2)}$ and $A_{2r}^{(2)}$ Toda theories

In this paper we consider the case when in Eq.(25) all signs $\varepsilon_i = 1$. The cases corresponding to different signs of parameters ε_i are more subtle and will be considered in the separate publication.

The quantum version of these integrable boundary conditions can be described in the following way. For $B_r(B_r^\vee)$ ATT the boundary parameters $\nu_i(B_r), i \neq r$ are defined by Eq.(24) ($\nu_i(B_r^\vee) = 0, i \neq r$ for the $B_r^\vee$ case) and for $i = r$:

$$\begin{aligned} \nu_r(B_r) &= \cos(\pi s_r b/2)\sqrt{\mu_r(B_r)/\sin(\pi b^2/2)}; \\ \nu_r(B_r^\vee) &= \cos(\pi s_r b^\vee)\sqrt{\mu_r(B_r^\vee)/\sin(2\pi(b^\vee)^2)}. \end{aligned} \tag{32}$$

For $C_r(C_r^\vee)$ ATT the boundary parameters $\nu_i(C_r) = 0, i \neq 0, r$ ($\nu_i(C_r^\vee)$ for $i \neq 0, r$ are defined by Eq.(24) with the substitution $b \to b^\vee$) and for $i = 0, r$:

$$\begin{aligned} \nu_{0,r}(C_r) &= \cos(\pi s_{0,r} b)\sqrt{\mu_r(C_r)/\sin(2\pi b^2)}; \\ \nu_{0,r}(C_r^\vee) &= \cos(\pi s_{0,r} b^\vee/2)\sqrt{\mu_r(C_r^\vee)/\sin(\pi(b^\vee)^2/2)} \end{aligned} \tag{33}$$

The parameters w_0 and w_r in Eqs.(26-29) coincide with the classical limits of corresponding parameters $s_i b$ or $s_i b^\vee$. It is convenient to define the variables w_0 and w_r by the relation:

$$w_0 = \frac{s_0 b}{1+b^2}, \quad w_r = \frac{s_r b}{1+b^2} \tag{34}$$

These variables are useful for the following quantum calculations. Their dual classical limits coincide with the corresponding limits of parameters $s_i b$ and $s_i b^\vee$. In the classical equations below we consider the parameters w_i as these limits.

We note that quite different classical integrable boundary G and $G^\vee$ ATTs in the quantum case are described by the same theory and are related by a duality transformation ($b \to 1/b$). The same is true for BC_r boundary ATT where quantum boundary conditions (21) for ν_0 and (24) for ν_i with $i \neq 0$ flow from boundary conditions (30) at $b \to 0$ to boundary conditions (31) in the dual limit.

For the analysis of integrable boundary Toda theories it is useful to consider the vacuum expectation values of the boundary exponential fields.

$$\mathcal{G}(a) = \langle \exp(a \cdot \varphi(x,0)/2) \rangle_B . \tag{35}$$

These one-point functions can be calculated by the reflection amplitudes method [1] which uses the extended conformal symmetry of the background boundary CFTs (NATTs). The similar calculations for simply laced ATTs were described in details in Refs.[13],[8],[20]. Here we give the explicit expressions for the one-point functions (Eq.35) corresponding to dual pairs B_r, $B_r^\vee$and C_r, $C_r^\vee$ of integrable boundary ATTs. The boundary one-point functions in BC_r ATT are more involved and will be published elsewhere.

The one-point function for the dual pair of B_r and $B_r^\vee$ ATTs with integrable boundary conditions described above depends on one boundary parameter s_r and can be represented in the form:

$$\mathcal{G}_B(a) = \mathcal{D}_B(a) \exp\left(\int_0^\infty \frac{dt}{t}(\sinh^2(s_r bt) - \sinh^2(b^2t/2)) f(a,t)\right). \tag{36}$$

In this equation $\mathcal{D}_G(a)$ denotes boundary one-point function in ATT corresponding to the Lie algebra G and characterized by dual Neumann boundary conditions (24). This function can be written as follows:

$$\mathcal{D}_G(a) = N_G(a) \exp\left(\int_0^\infty \frac{dt}{t}\left[\frac{a^2}{2}e^{-2t} - \sum_{\alpha>0} \Psi_\alpha(t)\mathcal{F}_\alpha(a,t)\right]\right) \tag{37}$$

where the factor $N_G(a)$ is defined by Eq.(14) and functions Ψ_α and $\mathcal{F}_\alpha$ are:

$$\Psi_\alpha(t) = 2e^t \sinh((1+\alpha^2b^2/2)t)\cosh(\alpha^2b^2t/2); \tag{38}$$

$$\mathcal{F}_\alpha(a,t) = \frac{\sinh(ba_\alpha t)\sinh(ba_\alpha - 2bQ_\alpha + h(1+b^2))t)}{\sinh 2t \sinh(\alpha^2b^2t)\sinh((1+b^2)ht)}. \tag{39}$$

To define the function $f(a,t)$ in Eq.(36), we denote as $\mathbf{B}$ the set of positive short roots of the Lie algebra B_r ($\mathbf{B}: \alpha > 0, \alpha^2 = 1$); then this function has the form:

$$f(a,t) = 2\sum_{\alpha\in\mathbf{B}} \mathcal{F}_\alpha(a,t). \tag{40}$$

To obtain the one point function in the dual integrable boundary $B_r^\vee$ ATT we should make the substitution $b \to 1/b$ in the equations written above and use the duality relations (4) for parameters μ_i.

The one point function for the dual pair of C_r and $C_r^\vee$ ATTs with integrable boundary conditions (33) depends on two boundary parameters s_0, s_r and can be represented in the form:

$$\begin{aligned}\mathcal{G}_C(a) &= \mathcal{D}_C^\vee(a) \exp\left(\int_0^\infty \frac{dt}{t}(\sinh^2(2s_0bt) - \sinh^2 t) f_0(a,t)\right) \\ &\times \exp\left(\int_0^\infty \frac{dt}{t}(\sinh^2(2s_rbt) - \sinh^2 t) f_r(a,t)\right).\end{aligned} \tag{41}$$

In this equation $\mathcal{D}_G^\vee(a)$ denotes the one-point function in ATT corresponding to the Lie algebra G and characterized by Neumann boundary conditions (23). This function is defined by Eq.(37) where we should do the substitution $\Psi_\alpha(t) \to \Psi_\alpha^\vee(t)$ with:

$$\Psi_\alpha^\vee(t) = 2\exp(\alpha^2b^2t/2)\sinh((1+\alpha^2b^2/2)t)\cosh t. \tag{42}$$

To define the functions $f_{0,r}(a,t)$ in Eq.(41) we denote by $\mathbf{C}$ the set of positive long roots of Lie algebra C_r ($\mathbf{C}:\alpha>0, \alpha^2=4$). Then these functions have a form:

$$f_0(a,t) = 2\sum_{\alpha\in\mathbf{C}} \frac{\sinh(ba_\alpha t)\,\sinh((ba_\alpha - 2bQ_\alpha)t)}{\sinh 2t\,\sinh(\alpha^2 b^2 t)\,\sinh((1+b^2)2ht)}, \tag{43}$$

$$f_r(a,t) = 2\sum_{\alpha\in\mathbf{C}} \frac{\sinh(ba_\alpha t)\,\sinh((ba_\alpha - 2bQ_\alpha + 2h(1+b^2))t)}{\sinh 2t\,\sinh(\alpha^2 b^2 t)\,\sinh((1+b^2)2ht)}. \tag{44}$$

To obtain the one point function in the dual integrable boundary $C_r^\vee$ ATT we should do the substitution $b\to 1/b$ and to use the duality relations (4) for parameters μ_i.

To study the classical limit of boundary Toda equations it is convenient to introduce the vector Θ_b which is equal to the difference of the boundary VEV of the field $\widehat{\varphi} = b\varphi/(1+b^2)$ and the VEV of this field in ATT defined on the whole plane (see section 2):

$$\Theta_b(G) = \langle\widehat{\varphi}(x,0)\rangle_B - \langle 0|\widehat{\varphi}|0\rangle = \frac{b}{(1+b^2)}\partial_a(2\mathcal{G}(a) - \mathbf{G}(a))|_{a=0} \tag{45}$$

where $\mathcal{G}$ denotes $\mathcal{G}_B$ or $\mathcal{G}_C$ and $\mathbf{G}$ is defined by Eq.(13). In particular the explicit expression for $\Theta_b(B)$ can be written as follows:

$$\begin{aligned}\frac{\Theta_b(B)}{x(1-x)} \;&=\; -\sum_{\alpha>0}\alpha\int_0^\infty \frac{dt\,\sinh((1-x+\alpha^2 x/2)t)\chi_\alpha(h,t)}{\sinh(\alpha^2 xt/2)\,\cosh((1-x)t)\sinh ht} \\ &+ \sum_{\alpha\in\mathbf{B}} 4\alpha\int_0^\infty \frac{dt(\sinh^2(w_r t) - \sinh^2(xt/2))\chi_\alpha(h,t)}{\sinh(\alpha^2 xt)\sinh(2(1-x)t)\sinh ht}\end{aligned} \tag{46}$$

where $\chi_\alpha(h,t) = \sinh((h - 2x\rho_\alpha - 2(1-x)\rho_\alpha^\vee)t)$; $x = b^2/(1+b^2)$ and variable w_r is defined by Eq.(34). The expression for vector $\Theta_b(C)$ can be easily derived from Eqs.(41-45). It can be written in terms of functions $\chi_\alpha^{(0)} = \chi_\alpha(0,t)$ and $\chi_\alpha^{(r)} = \chi_\alpha(2h,t)$ in the form:

$$\begin{aligned}\frac{\Theta_b(C)}{x(1-x)} \;&=\; -\sum_{\alpha>0}\alpha\int_0^\infty \frac{dt\,\sinh((1-x+\alpha^2 x/2)t)\chi_\alpha(h,t)}{\sinh((1-x)t)\,\cosh(\alpha^2 xt/2)\sinh ht} \\ &+ \sum_{k=0,r}\sum_{\alpha\in\mathbf{C}} 4\alpha\int_0^\infty \frac{dt(\sinh^2(2w_k t) - \sinh^2((1-x)t))\chi_\alpha^{(k)}(t)}{\sinh(\alpha^2 xt)\sinh(2(1-x)t)\sinh 2ht}\end{aligned} \tag{47}$$

The vector Θ_b can be expressed in terms of elementary functions (see section 5) in two dual classical limits $b\to 0$ ($x=0$) and $b\to\infty$ ($x=1$). As it was explained in Ref.[9], in these two limits the values of the vector Θ_b determine the boundary values $\phi(0)$ of the solutions $\phi(y)$ to classical boundary Toda equations for the dual pair of ATTs. These solutions describe the classical vacuum configurations i.e. correspond to the dual classical limits of the correlation function:

$$\Phi_b(y) = \langle\widehat{\varphi}(x,y)\rangle_B - \langle 0|\widehat{\varphi}|0\rangle \tag{48}$$

where the first term in the right hand side of this equations denotes the VEV of the bulk field in boundary ATT and the second is the VEV of field $\widehat{\varphi}$ in the theory defined on the whole plane (see section 2).

After rescaling and shifting (see Eq.(19)) the classical equations which follow from action (20) have a form:

$$\partial_y^2\phi = m^2\sum_{i=0}^{r} n_i e_i \exp(e_i\cdot\phi); \tag{49}$$

with the boundary condition at $y=0$:

$$\partial_y\phi = m\sum_{i=0}^{r}\sqrt{n_i}A_i e_i \exp(e_i\cdot\phi/2). \tag{50}$$

where the coefficients A_i are defined by Eqs.(26-29).

The limiting values of vector Θ_b determine the boundary values $\phi(0)$ of the solution and Eq.(50) defines the derivative $\partial_y\phi(0)$. This gives us the possibility to study Eq.(49) numerically. The numerical analysis of this equation shows that only for these boundary values smooth solutions decreasing at infinity exist.

It is natural to expect that solution $\phi(y)$ to the Eqs.(49,50) can be expressed in terms of tau-functions associated with multi-soliton solutions to classical ATTs equations (see for example [14], [9] and references there). We postpone the explicit construction of these solutions to section 5. Here we consider the classical and quantum boundary ground state energies (BGSEs). The classical BGSE can be expressed in terms of the solution $\phi(y)$ as follows:

$$\begin{aligned}\mathcal{E}^{(cl)} &= \frac{1}{4\pi b^2}[2m\sum_{i=0}^{r}\sqrt{n_i}A_i\exp(e_i\cdot\phi(0)/2)\\ &\quad+\int_0^\infty dy(\frac{1}{2}(\partial_y\phi)^2+m^2\sum_{i=0}^{r}n_i(e^{e_i\cdot\phi}-1))].\end{aligned} \tag{51}$$

For some special values of parameters $w_{0,r}$ classical BGSEs are known. If we put $w_0=w_r=1/2$ in Eqs.(27,28) for the boundary conditions corresponding to $B_r^\vee$ and C_r ATTs we obtain that all coefficients A_i vanish (Neumann boundary conditions). It means that solution $\phi(y)$ also vanishes and:

$$\mathcal{E}^{(cl)}(B_r^\vee) = \mathcal{E}^{(cl)}(C_r) = 0;\quad w_0=w_r=1/2 \tag{52}$$

Putting $w_0=w_r=0$ in boundary conditions (26,29) corresponding to B_r and $C_r^\vee$ ATTs we find that all $A_i=\sqrt{2/e_i^2}$ (classical version of dual to Neumann boundary conditions). The explicit solutions $\phi(y)$ for these values of coefficients A_i were found for all ATTs in [9]. The corresponding BGSEs are:

$$\mathcal{E}^{(cl)}(C_{r-1}^{\vee}) = \mathcal{E}^{(cl)}(B_r) = \frac{rm\cos(\pi/4 - \pi/4r)}{\pi b^2 \cos(\pi/4r)}; \quad w_0 = w_r = 0 \tag{53}$$

To calculate $\mathcal{E}^{(cl)}$ for arbitrary values of boundary parameters we can use the relation:

$$\partial_{w_r}\mathcal{E}^{(cl)}(G) = \frac{2m\sqrt{n_i}(\partial_{w_r}A_r(G))}{4\pi b^2}\exp(e_r \cdot \phi(0)/2) \tag{54}$$

and similar equation with respect the parameter w_0 for $G = C_r, C_r^{\vee}$. The boundary values $\phi(0)$ of the solution are defined by the dual classical limits of vector $\Theta_b(G)$. All boundary values of exponents:

$$E_i = \exp(e_i \cdot \phi(0)/2); \quad i = 0, ..., r \tag{55}$$

are given in section 6. The integration of the Eqs.(54) with "initial conditions" (52,53) will give us the classical BGSEs.

In the quantum case the BGSEs for B_r and $C_r^{\vee}$ ATTs with dual to Neumann boundary conditions (24) which in the quantum case are characterized by the values of variables $w_0 = w_r = x/2$ (and hence also for $B_r^{\vee}$ and C_r ATTs with Neumann boundary conditions: $w_0 = w_r = (1-x)/2$) were conjectured in [9]. Here we accept this conjecture which is:

$$\mathcal{E}^{(q)}(B_r) = \frac{m\cos(\pi/4 - \pi/2h)}{4\sin(\pi x/2h)\,\cos(\pi(1-x)/2h)}; \quad h = 2r - x. \tag{56}$$

To obtain the expression for $\mathcal{E}^{(q)}(B_r^{\vee})$ we should do in Eq.(56) the substitution: $x \to 1-x$. The conjectured form for $\mathcal{E}^{(q)}(C_r^{\vee})$ (with the same substitution for $\mathcal{E}^{(q)}(C_r)$) is also given by Eq.(56) where we should take $h = 2r + 2 - 2x$.

To calculate BGSEs for arbitrary values of variables w_r, w_0 we can find from Eq.(20) that:

$$\partial_{w_i}\mathcal{E}^{(q)}(G) = (b + 1/b)(\partial_{s_i}\nu_i(G))\mathcal{G}_G(be_i); \quad i = 0, r \tag{57}$$

The values of function $\mathcal{G}_G(a)$ at the special points be_i can be calculated explicitly with a result:

$$\partial_{w_r}\mathcal{E}^{(q)}(B_r) = -\frac{\pi m\sin(\pi w_r/h)}{2\sqrt{2}h\sin(\pi x/2h)\sin(\pi(1-x)/h)}; \tag{58}$$

$$\partial_{w_r}\mathcal{E}^{(q)}(C_r^{\vee}) = -\frac{\pi m\sin(\pi w_r/h)\,\cos(\pi w_0/h)}{\sqrt{2}h\sin(\pi x/h)\,\sin(\pi(1-x)/h)}; \tag{59}$$

$$\partial_{w_0}\mathcal{E}^{(q)}(C_r^{\vee}) = -\frac{\pi m\sin(\pi w_0/h)\,\cos(\pi w_r/h)}{\sqrt{2}h\sin(\pi x/h)\,\sin(\pi(1-x)/h)}. \tag{60}$$

The integration of these equations with "initial conditions" (56) gives:

$$\mathcal{E}^{(q)}(B_r) = \frac{m\cos(\frac{\pi}{4} - \frac{\pi}{2h})}{4\sin(\frac{\pi x}{2h})\cos(\frac{\pi(1-x)}{2h})} + \frac{m(\cos(\frac{\pi w_r}{h}) - \cos(\frac{\pi x}{2h}))}{2\sqrt{2}\sin(\frac{\pi x}{2h})\,\sin(\frac{\pi(1-x)}{h})}; \tag{61}$$

$$\mathcal{E}^{(q)}(C_r^\vee) = \frac{m\cos(\frac{\pi}{4}-\frac{\pi}{2h})}{4\sin(\frac{\pi x}{2h})\cos(\frac{\pi(1-x)}{2h})} + \frac{m(\cos(\frac{\pi w_r}{h})\cos(\frac{\pi w_0}{h}) - \cos^2(\frac{\pi x}{2h}))}{\sqrt{2}\sin(\frac{\pi x}{h})\sin(\frac{\pi(1-x)}{h})} \tag{62}$$

The expressions for the BGSEs $\mathcal{E}^{(q)}(B_r^\vee)$ and $\mathcal{E}^{(q)}(C_r)$ can be derived from these equations by the substitution $x \to 1-x$. The classical BGSEs can be derived from Eqs.(61,62) as the main terms ($O(1/x)$ or $O(1/(1-x))$) of the asymptotics of the quantum values.

We note that $C_1^\vee$ and C_1 ATTs coincide with boundary sinh-Gorgon model. In this case BGSE (62) being written in terms of the mass $m_1 = \sqrt{8}m\sin(\pi/h)$ coincides with expression for boundary ground state energy of this model proposed by Al.Zamolodchikov. [16] (see also [17]).

The nonperturbative check of conjectures (61,62) can be made using the boundary Thermodynamic Bethe Ansatz equations [18]. The kernels in these nonlinear integral equations depend on the bulk S-matrices $S_{ij}(\theta)$ and the source (inhomogeneous) terms on the boundary S-matrices $R_j(\theta)$. Both these S-matrices in ATTs are the pure phases. The boundary ground state energy can be expressed through the main terms of the asymptotics of these phases at $\theta \to \infty$ [19]. In particular, boundary ground state energy in ATT can be expressed in terms of the mass m_j of the particle j multiplied by the ratio of Fourier transforms $\Delta_j(\omega)$ and $\Delta_{jj}(\omega)$ of logarithms of boundary S-matrix R_j and bulk amplitude S_{jj} taken at $\omega = i$. To check our conjecture and to construct the explicit solution of Eqs.(49,50) we need the information about the boundary $S-$matrix.

4 Boundary S-matrix

The boundary $S-$matrix (reflection coefficient) for the particle j with mass m_j (7) in integrable boundary ATTs can be defined as:

$$|j,-\theta\rangle_{B,out} = R_j(\theta)|j,\theta\rangle_{B,in} \tag{63}$$

where θ is a rapidity of particle j.

To be consistent with a bulk $S-$ matrix $S_{ij}(\theta)$ the reflection coefficients $R_j(\theta)$ should satisfy the boundary bootstrap equations [11],[10]. These equations can be written in terms of the fusion angles θ_{ij}^k which determine the position of the pole corresponding to the particle k in the amplitude $S_{ij}(\theta)$. We denote as $\overline{\theta}_{ij}^k = \pi - \theta_{ij}^k$, then the consistency equations can be written as:

$$R_k(\theta) = R_i(\theta - i\overline{\theta}_{ik}^j)R_j(\theta + i\overline{\theta}_{jk}^i)S_{ij}(2\theta + i\overline{\theta}_{jk}^i - i\overline{\theta}_{ik}^j). \tag{64}$$

The "crossing unitarity" relation [10] impose additional condition for boundary S-matrix:

$$R_j(\theta)R_j(\theta + i\pi) = S_{jj}(2\theta) \tag{65}$$

The reflection coefficient in ATT is a pure phase $R_j = \exp(i\delta_j(\theta))$.

We give here the conjecture for minimal solutions to these equations which are consistent with boundary ground state energies (61.62). We represent the boundary amplitudes $R_j(\theta)$ in the form of Fourier integrals which is convenient for TBA analysis.

For the particles j with the masses $m_j = \sqrt{8}m\sin(\pi j/h)$ (see Eq.(7)) the reflection coefficients can be written as:

$$R_j(\theta) = \exp\left(i\int_0^\infty \frac{dt}{t}\sin\left(2h\theta t/\pi\right)\left[\Delta_j(G,t)+2\right]\right) \tag{66}$$

where

$$\begin{aligned}\Delta_j(G,t) &= -\frac{4\sinh(1-x)t\,\sinh(h+x)t\,\sin jt\,\cos\left(h/2-j\right)t}{}\sinh t\,\cosh ht/2\,\cosh ht + \\ &+\frac{8\sinh(1-x)t\,\cosh xt\,\sinh jt\,\sinh(j-1)t}{\sinh 2t\,\cosh ht}+\psi_j(G,t).\end{aligned} \tag{67}$$

For the dual pair of B_r and $B_r^\vee$ of integrable boundary ATTs the functions ψ_j ($j=1,...,r-1$) depend on one parameter w_r and have a form:

$$\psi_j(B_r,t) = \frac{8\sinh 2jt\,\cosh xt\,\sinh(w_r+x/2)t\,\sinh(w_r-x/2)t}{\sinh 2t\,\cosh ht}. \tag{68}$$

For the particle r with mass $\sqrt{2}m$ the solution is:

$$\begin{aligned}\Delta_r(B_r,t) &= -\frac{2\sinh(1-x)t\,\sinh 2rt\,\sin(h/2+1)t}{\sinh 2t\,\cosh ht/2\,\cosh ht}\\ &+\frac{4\sinh 2rt\,\sinh(w+x/2)t\,\sinh(w-x/2)t}{\sinh 2t\,\cosh ht}.\end{aligned} \tag{69}$$

The dual pair of $C_r^\vee$ and C_r boundary ATTs is characterized by the functions $\psi_j(C_r^\vee,t)$ ($j=1,...,r$) which depend on two parameters w_0, w_r as follows:

$$\psi_j(C_r^\vee,t) = \frac{4\sinh 2jt(\cosh(2w_1t)\,\cosh(2w_2t)-\cos^2(xt))}{\sinh 2t\,\cosh ht}. \tag{70}$$

All amplitudes R_j in this theory can be obtained using Eq.(64) from the amplitude R_1 corresponding to the lightest particle m_1. It follows from Eqs.(66,67,70) that amplitude $R_1(C_r^\vee)$ can be written in terms of functions $(z)=\frac{\sinh(\theta/2+i\pi z/2h)}{\sinh(\theta/2-i\pi z/2h)}$ as:

$$R_1(\theta) = \frac{(x-1)(h/2+x)(h-x)(h/2+1-x)(h/2)(h/2-1)}{(h-1)(h/2+w_+)(h/2-w_+)(h/2-w_-)(h/2+w_-)} \tag{71}$$

here $w_\pm = w_r \pm w_0$. The amplitude $R_1(B_r)$ can be obtained from this equation if we put $w_0 = x/2$.

We give here also the conjecture for boundary S-matrix in BC_r ATT which flows from boundary conditions (30) to boundary conditions (31). The boundary parameters w_0 and w_r in these equations correspond to dual classical limits of the same parameter which we denote as w. Then amplitudes $R_j(\theta)$ ($j = 1, .., r$) are characterized by functions $\psi_j(BC_r, t)$ which are:

$$\psi_j(BC_r, t) = \frac{4 \sinh 2jt(\cosh t \, \cosh(2wt) - \cos^2(xt))}{\sinh 2t \, \cosh ht} \tag{72}$$

This conjecture is consistent with the following form of boundary ground state energy

$$\mathcal{E}^{(q)}(BC_r) = \frac{m \cos(\frac{\pi}{4} - \frac{\pi}{2h})}{4 \sin(\frac{\pi x}{2h}) \, \cos(\frac{\pi(1-x)}{2h})} + \frac{m(\cos(\frac{\pi w}{h}) \, \cos(\frac{\pi}{2h}) - \cos^2(\frac{\pi x}{2h}))}{\sqrt{2} \sin(\frac{\pi x}{h}) \, \sin(\frac{\pi(1-x)}{h})}. \tag{73}$$

The boundary $S-$matrices described above correspond to the minimal solutions of Eqs.(64,65) which are consistent with GSESs that were derived in the previous section. For $\left|w_{\pm}\right| < 1$ and for $|w_r| < 1/2$ the amplitudes $R_j(\theta)$ have no poles corresponding to the bound states of the particles j with boundary. The only poles with positive residues which appear in these functions are the poles at $\theta = i\pi/2$

$$R_j(\theta) = \frac{iD_j^2(G)}{\theta - i\pi/2} \tag{74}$$

corresponding to the particle boundary coupling with zero binding energy. For example, the lightest particle whose amplitude possesses this pole is the particle m_1. It is easy to derive from Eq.(71) that corresponding residue has a form:

$$D_1^2(C_r^\vee) = \frac{2 \cot\left(\frac{\pi x}{2h}\right) \cot\left(\frac{\pi(1-x)}{2h}\right) \tan^2\left(\frac{\pi w_+}{2h}\right) \tan^2\left(\frac{\pi w_-}{2h}\right)}{\tan\left(\frac{\pi}{2h}\right) \cot\left(\frac{\pi}{4} - \frac{\pi}{2h}\right) \cot\left(\frac{\pi}{4} - \frac{\pi(1-x)}{2h}\right) \tan\left(\frac{\pi}{4} - \frac{\pi x}{2h}\right)}. \tag{75}$$

The residue $D_1^2(B_r)$ can be obtained from this equation if we put $w_0 = x/2$.

It was shown in Ref.[9] that the classical limits of the residues D_j:

$$d_j(G) = \lim_{x \to 0} \sqrt{\pi x(1-x)} D_j(G); \quad d_j(G^\vee) = \lim_{x \to 1} \sqrt{\pi x(1-x)} D_j(G). \tag{76}$$

play the crucial role for the construction of the explicit solutions to Eqs.(4950). Namely, they determine "one particle" contributions to the boundary solution $\phi(y)$ which describes the classical vacuum configuration:

$$\phi(y) = \sum_{j=1}^{r} d_j \xi_j \exp(-m_j y) + ... \tag{77}$$

here ξ_j are the eigenvectors of mass matrix (5) ($m^2 M \xi_j = m_j^2 \xi_j$), satisfying the conditions:

$$\xi_i \cdot \xi_j = \delta_{ij}; \quad \xi_i \cdot \rho^\vee \geq 0. \tag{78}$$

For example, the coefficient d_1 corresponding to the lightest particle m_1 in expansion (77) and defining the main term of the asymptotics at $y \to \infty$ can be extracted from Eq.(75) and has a form:

$$d_1(C_r^\vee) = 2\sqrt{H} \tan\left(\frac{\pi}{4} - \frac{\pi}{2H}\right) \cot\left(\frac{\pi}{2H}\right) \tan\left(\frac{\pi w_+}{2H}\right) \tan\left(\frac{\pi w_-}{2H}\right) \tag{79}$$

$$d_1(C_r) = 2\sqrt{H}\cot\left(\frac{\pi}{2H}\right)\tan\left(\frac{\pi w_+}{2H}\right)\tan\left(\frac{\pi w_-}{2H}\right) \tag{80}$$

where $H(C_r^\vee) = 2r+2$ and $H(C_r) = 2r$. The coefficient $d_1(B_r)$ ($d_1(B_r^\vee)$) can be obtained from Eq.(79) (Eq.(80)) if we put there $w_0 = 0$ ($w_0 = 1/2$).

The coefficients d_j as well as the boundary values $\phi(0)$ fix completely the solution to the Eqs.(49,50). They determine the contribution of the zero modes of the linearized Eq.(49) and make it possible to develop in a standard way the regular expansion of the solution $\phi(y)$ at large distances. If our scattering theory is consistent with conformal perturbation theory this expansion should converge to the boundary value $\phi(0)$. In the next section we use the coefficients d_i to construct the exact solution and to check this consistency condition.

5 Boundary Solutions

It is natural to assume that boundary vacuum solutions can be expressed in terms of tau-functions associated with multisoliton solutions of classical ATTs equations. For all cases that we consider below it is convenient to represent these solutions in the form:

$$\phi(y) = -\frac{1}{2}\sum_{k=0}^{r} n_k e_k \log\tau_k(y); \quad \tau_k(y) \to 0,\ y \to \infty \tag{81}$$

where numbers n_k and roots e_k characterize the corresponding ATT. If we chose the standard basis of roots:

$$e_k \cdot \phi = \phi_k - \phi_{k+1}; \quad k = 1, ..., r-1 \tag{82}$$

and take the roots e_0 and e_r in accordance with extended (affine) Dynkin diagram, we obtain that:

$$\phi_k = \log(\tau_{k-1}/\tau_k); \quad k = 1, ..., r \tag{83}$$

The classical boundary ground state energy can be expressed in terms of boundary values of τ-functions [14] as:

$$\mathcal{E}^{(cl)}(G) = \frac{H}{4\pi b^2}\left(A_k\sqrt{n_k}\exp(e_k\phi(0)/2) + \frac{\tau_k'(0)}{\tau_k(0)}\right); \ k = 0, ..., r. \tag{84}$$

All functions τ_k corresponding to multisoliton solutions of ATT equations are given by finite order polynomials in the variable

$$Z_j(y) = \exp(-m_j y). \tag{85}$$

The coefficients of these polynomials are completely fixed by the equations of motion and by the asymptotics (77). The construction of these polynomials for boundary solutions in simply laced ATTs was described in details in Ref.[9]. In all cases described below it is very similar to construction for boundary solution in D_r ATT. In particular, to simplify the form of tau-functions it is convenient similar to D_r case to introduce the parameters t_j which for $j \leq r$ are related with coefficients d_j as:

$$t_j = \frac{d_j}{2\sqrt{H}\sin(\pi j/H)} \tag{86}$$

This normalization is useful, because the following relations between the roots e_k (that appear in Eq.(81) for $\phi(y)$) and eigenvectors of mass matrix ξ_j (that describe "one particle" contributions (77) to the solution) are valid:

$$-\sum_{k=0}^{r} n_k e_k \cos(\pi j(2k+\kappa(G))/H) = 2\sqrt{H}\sin(\pi j/H)\xi_j \tag{87}$$

where $\kappa(B_r) = \kappa(B_r^\vee) = -1$; $\kappa(C_r^\vee) = 1$ and $\kappa(C_r) = 0$.

Below we give the boundary values of the solutions that follow from classical limits of vector $\Theta_b(G)$, the explicit expressions for coefficients t_j that follow from the boundary $S-$matrices and construct the exact boundary vacuum solutions.

5.1 B$_r$ boundary solution

To specify the boundary values of the solutions it is convenient to introduce the functions:

$$q_l^2(w) = \frac{(\cos(\frac{2\pi w}{H}) + \cos(\frac{2\pi l}{H}))}{(1+\cos(\frac{2\pi l}{H}))} \prod_{i=0}^{l-1} \frac{(\cos(\frac{2\pi w}{H}) + \cos(\frac{2\pi(l-2i)}{H}))}{(\cos(\frac{2\pi w}{H}) + \cos(\frac{2\pi(l-2i-1)}{H}))} \tag{88}$$

and numbers:

$$p_l^2 = \prod_{i=1}^{l} \frac{\cos(\pi(l+1-2i)/H)}{\cos(\pi(l-2i)/H)}. \tag{89}$$

Then boundary values E_i defined by Eq.(55) for B_r solution can be derived from vector $\Theta_0(B)$ (see Eq.(46)) and written as follows:

$$\begin{aligned} E_0 &= E_1 = \frac{\sqrt{8}\cos(\pi w/H)}{H\sin(\pi/H)};\ E_r = \frac{\sqrt{2}\sin(\pi w/H)}{\sin(\pi w/2)\,\sin(\pi/H)}; \\ E_k &= \frac{\cos(\pi k/H)\cos(\pi(r+1-k)/H)p_k^2 p_{r+1-k}^2 q_{k-1}(w_r)}{\cos(\pi(2k-1)/2H)\,\cos\left(\pi(2r-2k+1)/2H\right)q_{k-1}(0)} \end{aligned} \tag{90}$$

where $k = 2, .., r-1$ and $H = 2r$.

The analysis of the boundary $S-$matrix (66-69) gives that for $r > 1$ the amplitudes $R_j(\theta)$ with $j = 1, ...r-1$ have a pole at $\theta = i\pi/2$. The amplitude $R_r(\theta)$ has no such pole and particle m_r does not contribute to the solution. This selection rule follows from $\mathbf{Z}_2$ symmetry of affine Dynkin diagram and integrable boundary conditions (26). The parameters t_j (86) can be derived from explicit form of amplitudes R_j (see section 4) and written as:

$$t_j = \frac{\tan(\frac{\pi}{4} - \frac{\pi j}{2H})}{2\cos^2(\frac{\pi j}{2H})} \prod_{i=0}^{j-1} \frac{\tan^2(\frac{\pi(j-1-2i+w_r)}{2H})}{\tan^2(\frac{\pi(i+1)}{2H})}. \tag{91}$$

Parameters t_j and functions $Z_j = \exp(-ym\sqrt{8}\sin(\pi j/H))$ are defined for $j \le r-1$. To write the solution in the most short form it is convenient, however, to continue these values to $j \le H-1$. To continue parameters t_j we can use Eq.(91). In this way we obtain:

$$t_{H-j} = -t_j; \quad Z_{H-j}(y) = Z_j(y). \tag{92}$$

The exact vacuum solution to Eqs.(49,50) in B_r ATT with boundary conditions (26) can be written in the form (81) where:

$$\tau_k(y) = \sum_{\sigma_1=0}^{1} \dots \sum_{\sigma_{H-1}=0}^{1} \prod_{j=1}^{H-1} \Omega_k^{j\sigma_j} (t_j Z_j)^{\sigma_j} \prod_{m<n}^{H-1} \left(\frac{\sin\left(\frac{\pi(m-n)}{2H}\right)}{\sin\left(\frac{\pi(m+n)}{2H}\right)} \right)^{2\sigma_m\sigma_n} \tag{93}$$

where for B_r solution $\Omega_k = \exp(i\pi(2k-1)/H) = \exp(i\pi(2k-1)/2r)$.

It is easy to derive from Eqs.(92,93) and (86,87) that "one particle" contributions to the solution $\phi(y)$ are given by Eq.(77). It can be also checked that boundary values of the solution $\phi(y)$ defined by Eqs.(81,93) coincide with vector $\Theta_0(B)$ defined by Eq.(90) and the classical boundary ground state energy calculated using Eq.(84) coincides with the main term of the asymptotics ($O(1/x)$) of Eq.(61).

5.2 $B_r^\vee$ boundary solution

The boundary values for this solution are determined by vector $\Theta_\infty(B)$ (see Eq.(46)) and are:

$$E_k^2 = \frac{\cos(2\pi w_r/H) + \cos(\pi(2j-1)/H)}{\cos(\pi/H) + \cos(\pi(2j-1)/H)}; \; E_r = \frac{\sin(\pi w_r/H)}{\sin(\pi w_r)\,\sin(\pi/2H)}. \tag{94}$$

where $k = 0, \dots, r-1$ and $H = 2r-1$.

The residues in the poles of the amplitudes $R_j(\theta)$ ($j = 1, \dots, r-1$) at $\theta = i\pi/2$ determine the following expression for parameters t_j:

$$t_j = \frac{1}{2\cos^2(\frac{\pi j}{2H})} \prod_{i=1}^{j} \frac{\tan(\frac{\pi(2w_r+2i-1)}{4H})\tan(\frac{\pi(2w_r-2i+1)}{4H})}{\tan^2\left(\frac{\pi i}{2H}\right)}. \tag{95}$$

Parameters t_j and functions $Z_j = \exp(-ym\sqrt{8}\sin(\pi j/H))$ are defined for $j \le r-1$. To write the solution in the form (93) we should continue these values to $j \le H-1$. To continue parameters t_j we can use Eq.(95). In this way we obtain: $t_{H-j} = -t_j$; $Z_{H-j}(y) = Z_j(y)$.

The exact vacuum solution to Eqs.(49,50) in $B_r^\vee$ ATT with boundary conditions (27) can be written in the form (81) where functions $\tau_k(y)$ have a form (93) with t_j defined by Eq.(95) and $\Omega_k = \exp(i\pi(2k-1)/H) = \exp(i\pi(2k-1)/(2r-1))$.

It can be checked that boundary values of this solution coincide with vector $\Theta_\infty(B)$ defined by Eq.(94) and the classical boundary ground state energy calculated using Eq.(84) coincides with the main term of the asymptotics ($O(1/(1-x))$) of Eq.(61).

5.3 $\mathrm{C}_r^\vee$ boundary solution

The boundary values of this solution are determined by vector $\Theta_\infty(C)$ (see Eq.(47)) and have a form:

$$\begin{aligned} E_0 &= \frac{\sqrt{2}\sin(\pi w_0/H)\,\cos(\pi w_r/H)}{\sin(\pi w_0/2)\,\sin(\pi/H)};\;\; E_r = \frac{\sqrt{2}\sin(\pi w_r/H)\,\cos(\pi w_0/H)}{\sin(\pi w_r/2)\,\sin(\pi/H)}; \\ E_k &= \frac{\cos(\pi(k+1)/H)\,\cos(\pi(r-k+1)/H)p_{k+1}^2 p_{r+1-k}^2 q_k(w_1)q_{r-k}(w_2)}{\cos(\pi(2k+1)/2H)\,\cos(\pi(2r-2k+1)/2H)q_k(0)q_{r-k}(0)} \end{aligned} \quad (96)$$

where $k=1,...,r-1$ and $H=2r+2$.

The analysis of boundary $S-$ matrix (67,70) gives that for $w_\pm \neq 0$ ($w_\pm = w_r \pm w_0$) all amplitudes $R_j(\theta)$ have a pole at $\theta = i\pi/2$ and all r particles with masses $m_j = \sqrt{8}\sin(\pi j/H)$ contribute to the solution. The parameters t_j can be extracted from the explicit form of these amplitudes and have a form:

$$t_j = \frac{\tan(\frac{\pi}{4} - \frac{\pi j}{2H})}{2\cos^2(\frac{\pi j}{2H})} \prod_{i=0}^{j-1} \frac{\tan(\frac{\pi(j-1-2i+w_+)}{2H})\tan(\frac{\pi(j-1-2i+w_-)}{2H})}{\tan^2(\frac{\pi(i+1)}{2H})}. \quad (97)$$

Parameters t_j and functions $Z_j = \exp(-ym\sqrt{8}\sin(\pi j/H))$ are defined for $j \leq r$. To write the solution in the form (93) it is necessary to continue these values to $j \leq H-1$. To continue parameters t_j we can use Eq.(95). In this way we obtain: $t_{H-j} = -t_j$; $Z_{H-j}(y) = Z_j(y)$.

The vacuum solution in $C_r^\vee$ ATT with boundary conditions (29) can be written in the form (81) (with all $n_k = 2$) where functions $\tau_k(y)$ have a form (93) with t_j defined by Eq.(97) and $\Omega_k = \exp(i\pi(2k+1)/H) = \exp(i\pi(2k+1)/(2r+2))$.

It can be checked that boundary values of this solution coincide with vector $\Theta_\infty(C)$ defined by Eq.(96) and the classical boundary ground state energy calculated using Eq.(84) coincides with the main term of the asymptotics ($O(1/x)$) of Eq.(62).

5.4 C_r boundary solution

The boundary values of this solution are defined by vector $\Theta_0(C)$ and have a form:

$$\begin{aligned} E_0 &= \frac{2\sin(\pi w_0/H)\cos(\pi w_r/H)}{\sin(\pi w_0)\,\sin(\pi/H)};\;\; E_r = \frac{2\sin(\pi w_r/H)\,\cos(\pi w_0/H)}{\sin(\pi w_r)\,\sin(\pi/H)}; \\ E_k^2 &= \frac{(\cos(2\pi w_r/H)+\cos(2\pi k/H))(\cos(2\pi w_0/H)-\cos(2\pi k/H))}{(\cos(\pi/H)+\cos(2\pi k/H))(\cos(\pi/H)-\cos(2\pi k/H))} \end{aligned} \quad (98)$$

where $k=1,...,r-1$ and $H=2r$.

The parameters t_j that follow from the boundary $S-$matrix are:

$$t_j = \frac{1}{2\cos^2(\frac{\pi j}{2H})} \prod_{i=0}^{j-1} \frac{\tan(\frac{\pi(j-1-2i+w_+)}{2H})\tan(\frac{\pi(j-1-2i+w_-)}{2H})}{\tan^2(\frac{\pi(i+1)}{2H})}. \tag{99}$$

Parameters t_j and functions $Z_j = \exp(-ym\sqrt{8}\sin(\pi j/H))$ are defined for $j \leq r$. To write the solution in the form (93) it is necessary to continue these values to $j \leq H-1$. To continue parameters t_j we can use Eq.(99). In this way we obtain: $t_{H-j} = t_j$; $Z_{H-j}(y) = Z_j(y)$.

The vacuum solution in C_r ATT with boundary conditions (28) can be written in the form (81) where functions $\tau_k(y)$ have a form (93) with t_j defined by Eq.(99) and $\Omega_k = \exp(i2\pi k/H) = \exp(i\pi k/r)$.

It can be checked that boundary values of this solution coincide with vector $\Theta_0(C)$ defined by Eq.(98) and the classical boundary ground state energy calculated using Eq.(84) coincides with the main term of the asymptotics ($O(1/(1-x))$) of Eq.(62).

5.5 BC$_r$ boundary solution

In classical case we have two types of integrable boundary conditions (30) and (31). The solution in both cases can be derived in a standard way from the scattering data described in the previous section. However, we can obtain these both solutions by the reduction of $C_{2r}^{\vee}$ and $B_{r+1}^{\vee}$ solutions with respect to the symmetries of corresponding affine Dynkin diagrams (see Ref.[9]) and integrable boundary conditions.

To derive the solution corresponding to boundary conditions (30) we should take the $C_{2r}^{\vee}$ solution and put there $w_0 = w_r$. Then $\mathbf{Z}_2$ symmetry of affine Dynkin diagram and boundary conditions manifest itself in vanishing all coefficients t_j with odd j. The tau-functions in this case possess the symmetry $\tau_j = \tau_{2r-j}$. In the standard basis of roots (82) and $e_0 \cdot \phi = -\phi_1$, $e_r \cdot \phi = 2\phi_r$ the solution can be written in the form (83) where $\tau_0, \ldots, \tau_r$ are the tau-functions for $C_{2r}^{\vee}$ solution. The corresponding classical boundary ground state energy $\mathcal{E}^{(cl)}(BC_r) = \mathcal{E}^{(cl)}(C_{2r}^{\vee})/2$.

To derive the solution corresponding to boundary conditions (31) we can fold the symmetry $\tau_0 = \tau_1$ of $B_{r+1}^{\vee}$ solution. In this way we obtain that BC_r solution can be written in the form (83) where $\tau_k(BC_r) = \tau_{k+1}(B_{r+1}^{\vee})$. The corresponding classical boundary ground state energy coincides with that for $B_{r+1}^{\vee}$ solution.

6 Concluding Remarks

In the previous sections we calculated one-point functions and conjectured boundary scattering theories and boundary ground state energies for the parametric families of integrable boundary Toda theories. The exact solutions corresponding to dual semiclassical limits of correlation function (48) were derived. These solutions were constructed using only boundary scattering data. The boundary values of these solutions are in exact agreement with the same values derived from one-point functions. This gives us the test for the consistency between $S-$matrix and CFT data.

1. To make the similar test for arbitrary values of the coupling constant we can express the long distance expansion for correlation function $\Phi_b(y)$ in terms of the boundary $S-$matrices R_j and form factors of ATTs (see Ref.[21] for details). This expansion should converge to the boundary values which can be extracted from one-point functions, boundary conditions and equations of motion. Namely, $\Phi_b(0) = \Theta_b$ and

$$\partial_y \Phi_b|_{y=0} = 2\pi x \sum_{i=0}^{r} \nu_i e_i \mathcal{G}(be_i); \;\; \partial_y^2 \Phi_b|_{y=0} = 4\pi x \sum_{i=0}^{r} \mu_i e_i \mathcal{G}(2be_i).$$

We suppose to do this test in the subsequent publications.

2. Boundary solutions constructed above correspond to the case when all signs ε_i in Eq.(25) are positive. Together with the results of Ref.[9] they give rather complete description of classical vacuum configurations for this case in integrable boundary ATTs. The situation with different signs ε_i is more sophisticated. For some choice of these signs the solution develops the singularity at the boundary but gives the finite boundary ground state energy [14],[15]. Sometimes the minimum of this energy can not be achieved at the static boundary solution. It looks interesting to analyze carefully how these phenomena can be consistent with classical and quantum integrability. We think that this problem needs the further study.

3. In this paper we did not consider the boundary excited states. These states appear for $|w_\pm| > 1$ in $C_r, C_r^\vee$ and for $|w_\pm| > 1/2$ in $B_r, B_r^\vee$ boundary ATTs. They manifest themselves in the poles of boundary amplitudes $R_j(\theta)$ and can be considered as the bound states of particles with a boundary. The boundary S-matrices for this states and their spectrum can be derived by boundary bootstrap method. In the classical limit these states can be seen as the boundary breather solutions. The quantization of these solutions [22],[23] should be consistent with spectrum of boundary states. This gives an additional test for boundary S-matrix. Another interesting problem related with excited states is the calculation of the expectation values of the boundary fields at these states. In the classical limit these expectation values can be found from the boundary breather solutions. The problem of the quantization of these expectation values as well as other problems mentioned in this section we suppose to discuss in subsequent publications.

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