Spin-Isospin Rotation Dynamics

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The equations for the solitons arbitrarily rotating in the ordinary and isotopic space are obtained. The wave functions of the corresponding dynamic states in the quantum case are found. The generalized matrix of the moments of inertia is degenerate for the O(2)-invariant configurations characteristic for the nucleon and delta-isobar. The equation for such configurations is established. It is shown that the spin-isospin rotation prevents the collapse of the soliton states in the SU(2) sigma-model. The entire consideration is based on the variational approach to the method of collective variables.

1 Introduction

Ordinary and isotopic spins are the fundamental characteristics of nucleons. Therefore the key moment of any realistic theory of nucleons is the search for suitable localized states performing rotations in the ordinary and isotopic spaces. The existing papers describe such states with the help of the adiabatic approximation. Such an approach uses static soliton solutions as a trial ansatz whereas the degeneration parameters of these solutions play the part of collective coordinates. This procedure, also known as the moduli space approximation, found wide application in the field theory. Specifically, in the Skyrme model it was employed in quantizing the hedgehog solution [1], as well as in quantizing the soliton states in higher homotopic classes [2].

At the same time, there exist actual problems whose solution within the framework of the adiabatic approach is impossible. It concerns the exact account of the dynamic deformation of a nucleon, the description of the spectrum of diverse rotational resonances, and the search for the localized states in the SU(2) sigma-model. The formulation of the problems mentioned requires the extended set of collective variables. Their choice, first of all, must reflect the symmetry properties of the initial Lagrangian. This will permit to consider the solitons rotating independently in the ordinary and isotopic spaces. The configuration of the localized states with such an approach must from the very outset be determined with the account for their collective dynamics characterized by the certain values of the ordinary and isotopic spins. These requirements do not assume the obligatory existence of the static soliton solutions. Therefore their realization can revitalize the SU(2) sigma-model as a potential model of nucleons. It is known that in a topological sector due to the Derrick theorem the static soliton solutions degenerate into phantom point states with zero energy. If such states are rotated, they acquire finite size and finite energy. As regards the stability of rotating states, this problem arises when considering any models, and the sigma-model does not differ in this respect from any other. If the quantization scheme with a half-integer spin is used, just a rotation will correspond to the ground state of such a soliton. In order to stabilize the moderate rotations with respect to a disintegration, a mass term should be added to the Lagrangian to provide for the damping asymptotics of the rotating states at the infinity [3]. A consistent study of dynamic soliton states can be performed within the framework of the collective variables formalism.

The exact equations of the collective dynamics are a natural consequence of the Hamiltonian equations of the total system. Let X, P be a set of canonical collective variables reflecting the characteristic properties of the problem considered, and q, p be all other variables tentatively called microscopic. The exact temporal evolution of the total system is described with the help of the Hamiltonian equations

$$\begin{aligned} \frac{\partial H(X,P;q,p)}{\partial p} &= \dot{q}, \quad \frac{\partial H(X,P;q,p)}{\partial q} = -\dot{p}, \\ \frac{\partial H(X,P;q,p)}{\partial P} &= \dot{X}, \quad \frac{\partial H(X,P;q,p)}{\partial Q} = -\dot{P}. \end{aligned} \tag{1}$$

The assumption that the influence of microscopic variables on the collective subsystem is effective only on the time scale considerably exceeding the characteristic time of the own evolution of the collective subsystem forms the ground for using the method of collective variables. In other words, the collective subsystem is actually conservative within a sufficiently broad time interval. In order to separate the principal, governing motions and to exclude completely the minor influence of microscopic variables under these conditions, they should be frozen by putting $\dot{q} = 0,\ \dot{p} = 0$. This requirement transforms the complete set of the Hamiltonian equations (1) into the exact equations of the collective dynamics

$$\frac{\partial H(X,P;q,p)}{\partial p} = 0\,, \quad \frac{\partial H(X,P;q,p)}{\partial q} = 0\,, \tag{2}$$

$$\frac{\partial H_c(X,P)}{\partial P} = \dot{X}, \quad \frac{\partial H_c(X,P)}{\partial Q} = -\dot{P}.$$

The first pair of these equations determines the coherent components of the microscopic variables $\overline{q} \equiv \overline{q}(X,P)$, $\overline{p} \equiv \overline{p}(X,P)$, that minimize the complete Hamiltonian of the system with fixed values of the collective variables X, P. The second pair represents the Hamiltonian equations for the collective subsystem. The collective Hamiltonian $H_c(X,P)$ is defined as the complete Hamiltonian of the system dependent on the coherent components of the microscopic variables:

$$H_c(X,P) \equiv H(X,P;\overline{q}\,(X,P),\overline{p}\,(X,P))\,.$$

Obviously, in such an approach the complete Hamiltonian of the system is presented in the form of the expansion in powers of the fluctuations of microscopic variables $\widetilde{q} = q - \overline{q}\,(X, P)$, $\widetilde{p} = p - \overline{p}\,(X, P)$. The zero term of such an expansion coincides with the collective Hamiltonian, and the account of the fluctuations starts with the quadratic terms. Thus, the approach given provides for the complete description of the total system in the microscopic sense reflecting at the same time the priority of the collective subsystem. The scheme presented clearly demonstrates the variational nature of the method of collective variables noted in the paper [4].

The goal of this paper is to study the dynamic solitons in SU(2) field theories. The general questions of the collective description such as gauging, changing of variables, and variational equations are outlined in Sec. 2 by way of example of the systems described by the Lagrangian quadratic in velocity. Sec. 3 deals with the field SU(2)-theories. The equations for the solitons performing arbitrary rotations in the ordinary and isotopic space are obtained. Sec. 4 discusses the link between the equations of collective dynamics and the exact self-similar solution of the equations of motion. Sec. 5 presents the wave functions of arbitrary dynamic states obtained under spontaneous breakdown of symmetry entangling spin and isotopic variables. Sec. 6 performs the transition from the general equations for the states with arbitrary spin and isospin values to the equations for the configurations with the axial symmetry. Such configurations describe the nucleon and delta-isobar states. The transition noted is not obvious because the matrix of the generalized moments of inertia is degenerate in case of axial symmetry. It is shown in this section that accounting for rotations in the SU(2) sigma-model impedes the collapse of the soliton states characteristic for the static solutions. In Conclusion the main results obtained in this paper are briefly summarized. From the viewpoint of interpretation and physical sense the procedure of the canonical description used in this paper differs from the similar procedure applied within the framework of the adiabatic approximation [1]. However, from the technical viewpoint, the elements of this description are practically identical. These questions are outlined in the Appendix.

2 Equations of collective dynamics

The transition to the collective description is accompanied by the change of variables in the configurational space. The choice of microscopic variables for such a change is not unique. Using this arbitrariness in a proper way can considerably simplify the equations of the variational approach as well as the equations of motion for the fluctuations. In order to demonstrate these and other elements of the variational approach that are common to a broad class of models, let us consider the nonlinear system described by a nondegenerate Lagrangian quadratic in velocity

$$L = \frac{1}{2} \sum g_{ik}(q) q^i_{,t} q^k_{,t} - H(q, 0)\,.$$

Here q^i are the generalized coordinates of the system. The approach outlined below is equally applicable to dynamic systems with a finite number of degrees of freedom as well as to the field theory. As applicable to the field systems, $i \equiv \{i,\ x\}$ is the set of discrete indices and spatial coordinates. It is expedient to present the transition to new variables as follows. We will regard as new microscopic variables the initial coordinates q^i, that simultaneously become the functions of the limited set of collective variables Q^α: $q^i = q^i(Q)$. In order to provide for the nondegenerate nature of the suggested change of variables, we limit the admissible variations of the microscopic variables $\delta q^i(Q = const)$ by the orthogonality conditions

$$\frac{\partial q_0^i}{\partial Q^\alpha}\, g_{ik}(q_0)\, \delta q^k = 0\,. \tag{3}$$

Here $q_0^i \equiv q_0^i(Q)$ is the trial ansatz depending only on collective variables whose form will be defined later. Obviously, the configurational space of the system is considered as a Riemann manifold with the metrics $g_{ik}(q)$ generated by the kinetic term. Let us introduce the projection operator on the subspace formed by the tangent vectors $\partial q_0^i/\partial Q^\alpha$:

$$\mathcal{P}_k^i = g_{ks}(q_0)\frac{\partial q_0^s}{\partial Q^\alpha} g^{\alpha\sigma}(q_0)\frac{\partial q_0^i}{\partial Q^\sigma}\,, \qquad \mathcal{P}^2 = \mathcal{P}\,, \tag{4}$$

where $g^{\alpha\sigma}(q_0)$ is the matrix inverse to the tensor

$$g_{\alpha\sigma}(q_0) = \frac{\partial q_0^i}{\partial Q^\alpha} g_{ik}(q_0)\frac{\partial q_0^k}{\partial Q^\sigma}\,, \tag{5}$$

defining the metrics on the surface of the submanifold $q_0^i(Q)$. In terms of the operator $\mathcal{P}$ the constraint condition (3) can be presented in the form

$$\delta q(1 - \mathcal{P}) = \delta q\,. \tag{6}$$

The variations in this relation can be ridden by a time dependence of microscopic variables $\delta q = \dot{q}\delta t\ (Q = const)$. Therefore the velocities of microscopic variables and their variations will also satisfy to similar constraint conditions

$$\dot{q}(1 - \mathcal{P}) = \dot{q}, \qquad \delta\dot{q}(1 - \mathcal{P}) = \delta\dot{q}\,. \tag{7}$$

Noting that

$$q_{,t} = \dot{q}(Q) + \dot{Q}\frac{\partial q(Q)}{\partial Q}$$

and accounting for Eq. (7) in calculating the variational derivatives, let us find the canonical momenta conjugated to new variables

$$P_\alpha \equiv \frac{\partial L}{\partial \dot{Q}^\alpha} = \frac{\partial q^s}{\partial Q^\alpha} g_{sk}(q)\left(\frac{\partial q^k}{\partial Q^\sigma}\dot{Q}^\sigma + \dot{q}^k\right)\,, \tag{8}$$

$$p_i \equiv \frac{\partial L}{\partial \dot{q}^i} = (1 - \mathcal{P})_i^s g_{sk}(q)\left(\frac{\partial q^k}{\partial Q^\sigma}\dot{Q}^\sigma + \dot{q}^k\right)\,. \tag{9}$$

In these relations the coefficients in front of the velocities $\dot{Q}^{\alpha}$, $\dot{q}^{k}$ define the elements of the block kinetic matrix $G(q)$. Writing the Hamiltonian of the system in new variables

$$H = \dot{q}\frac{\partial L}{\partial \dot{q}} + \dot{Q}\frac{\partial L}{\partial \dot{Q}} - L = \frac{1}{2}(P,\ p)\, G^{-1}(q) \begin{pmatrix} P \\ p \end{pmatrix} + H(q,0)\,, \tag{10}$$

we can turn to the equations for the coherent components $\overline{p}$, $\overline{q}$ (2). Accounting for the constraint $p(1-\mathcal{P}) = p$ (9), the first of them becomes

$$(0, (1-\mathcal{P}))\, G^{-1}(\overline{q}) \begin{pmatrix} P \\ \overline{p} \end{pmatrix} = 0\,. \tag{11}$$

In an extremum point, according to Eq. (2), the arbitrary microscopic coordinates ϱ become the functions of the collective variables $\overline{\varrho}(Q,P)$. Therefore, in an extremum point the collective variables dependence of the initial coordinates $q \equiv q(Q,\varrho)$ can be presented in the form $\overline{q}(Q,P) = q(Q,\overline{\varrho}(Q,P))$. Generalizing the approach offered in the paper [5], let us define the gauge function q_0 and its derivatives in (3) as the limits

$$q_0(Q) = q(Q,\varrho)_{|\varrho\to\overline{\varrho}(Q,P)} = \overline{q}(Q,P), \qquad \frac{\partial q_0(Q)}{\partial Q} = \frac{\partial q(Q,\varrho)}{\partial Q}_{|\varrho\to\overline{\varrho}(Q,P)}\,. \tag{12}$$

Due to this definitions and the constraints (3), (7) the nondiagonal blocks of the kinetic matrix $G(\overline{q})$ (8), (9) vanish in an extremum point. So the matrix itself splits into two independent blocks $g_{\alpha\sigma}(\ \overline{q})$ and $(1-\mathcal{P})^s_i g_{sk} = g_{is}(1-\mathcal{P})^s_k$, relating to the collective and microscopic subsystems, respectively. Eq. (11) comes homogeneous with respect to $\overline{p}$, and we obtain $\overline{p} = 0$. It is the main result of this section. So, in the used gauge the equation determining the coherent components of the coordinates q and the corresponding functional $H_c(q,P)$ will have the forms

$$\frac{\partial H_c(q,P)}{\partial' q^i} = 0, \qquad H_c(q,P) \equiv \frac{1}{2}\, g^{\alpha\sigma}(q) P_\alpha P_\sigma + H(q,0). \tag{13}$$

Here, in contrast to the ordinary derivative $\partial/\partial q$, the derivative over q accounting for the constraint (6) is denoted with a symbol $\partial/\partial' q$. If the collective coordinates are cyclic ones, the dependence of the initial variables from these coordinates is determined obviously. It allows easily to establish the structure of the equation (13). Just such a case is considered further in this paper. In general case, to find the equation for the coherent components $\overline{q}$, let us consider the variation of the functional $H_c(q,P)$ with respect to the variations of microscopic variables δq at Q,P fixed. Since the Hamiltonian $H_c(q,P)$ is the function of the variables q and $q_{,\alpha} \equiv \partial q/\partial Q^{\alpha}$ and accounting that under this conditions $\delta(q_{,\alpha}) = (\delta q)_{,\alpha}$, we obtain

$$\delta H_c(q,P) = \delta q^k \frac{\partial H_c}{\partial q^k} + \frac{\partial}{\partial Q^{\alpha}}(\delta q^k)\frac{\partial H_c}{\partial q^k_{,\alpha}}\,.$$

Carrying out all differentiations over Q at $\overline{\varrho}(Q,P)$ fixed, let us change variables $q \equiv q(Q,\varrho)$ by their coherent components $q(Q,\overline{\varrho}(Q,P))$. Then moving the derivative over Q and noting that in an extremum point

$$\delta q^k \frac{\partial H_c}{\partial q^k_{,\alpha}} = 0\,,$$

we obtain the basic equation of the variational approach

$$(1-\mathcal{P})^k_i \left(\frac{\partial H_c}{\partial q^k} - \frac{\partial}{\partial Q^\alpha} \frac{\partial H_c}{\partial q^k_{,\alpha}} \right) = 0\,. \tag{14}$$

Let us stress that this equation determines the coherent component of the coordinate, and therefore, in view of the choice of the gauge function, one should change the ansatz q_0 in the projection operator $\mathcal{P}$ for the sought function q (12). In the field systems the equation (14) describes the configurations of dynamic solitons. Within limit of $P_\alpha = 0$ it reduces to the equation for the quasistatic configurations

$$(1-\mathcal{P})^k_i \frac{\partial H(q,0)}{\partial q^k} = 0\,.$$

This equation determines static stresses of deformed solitons and in particular allows to establish properties of the intersoliton potentials [6]. The Hamiltonian $H_c(q,P)$ dependent on the components $\overline{q}(Q,P)$ defines the collective Hamiltonian of the system:

$$H_c(Q,P) \equiv \frac{1}{2}\, g^{\alpha\sigma}(\overline{q}(Q,P)) P_\alpha P_\sigma + H(\overline{q}(Q,P),0)\,.$$

Performing the further expansion of the Hamiltonian (10) in powers of the fluctuations of microscopic variables we can go beyond the framework of the purely collective description. The inclusion of the higher-order terms permits to determine the fluctuation spectrum as well as the effect of fluctuations on the collective motion.

Note that the equations obtained are invariant with respect to the general gauge transformation of collective variables $Q \to Q'(Q)$. Specifically, this is seen from the structure of the projection operator $\mathcal{P}$ (4), (5) that is invariant with respect to such transformations. As regards the microscopic variables, their choice is fixed through the gauge condition (3). The invariance properties of the theory with respect to the choice of microscopic parameters are not considered in this paper. We only mention a concrete example of such invariance. It is shown in the paper [5] that the zero mode in the fluctuation spectrum of the one-dimensional problem are removed automatically owing to the inherent properties of the theory and regardless of the gauge form used. So this result solves the "zero-mode problem" appeared in the middle of the seventies in connection with the application of the perturbation theory to the soliton systems.

The gauge (3) used in this paper, in which the gauge function is identified with the extremal of the Hamiltonian (12), reduces to zero the coherent component of the canonical momentum and factorizes the kinetic matrix at the extremum point.

All this simplifies the analysis of the equations for the coherent components as well as equations for fluctuations. Using similar gauges when considering concrete systems permits to formulate a simplified receipt for describing a purely collective motion. That is, using the Lagrange formalism, one can omit the velocities of microscopic variables from the very beginning, what automatically will turn to zero the corresponding canonical momenta. Naturally, this receipt cannot be used if we want to account for the fluctuations of the microscopic variables.

3 Hamiltonian of the spin-isospin rotation

Let us consider the dynamic solitons in the SU(2) field theories. The initial properties of symmetry of these systems and the corresponding integrals of motion play a key role in the collective description of them. It is convenient to present the Lagrangian reflecting these properties of symmetry in the form

$$L(\phi,\dot{\phi}) = 1/2 < g_{ik}(\phi)\dot{\phi}^i\dot{\phi}^k > -H(\phi,0)\,, \qquad < A > \equiv \int d^3x\; A(x)\,. \tag{15}$$

Here the field $\phi^i(x,t)$ belongs to the space of the SU(2) group parameters. This Lagrangian can describe the SU(2) sigma-model as well as the corresponding Skyrme model depending on the concrete form of the kinetic matrix $g_{ik}(\phi)$ and the potential term $H(\phi,0)$. It is only important to stress that the Lagrangian (15) includes the mass term providing for the existence of dynamic soliton states under sufficiently moderate rotation. In the presence of this term the chiral invariance is broken and the actual symmetry properties of the Lagrangian are reduced to its invariance with respect to isotopic rotations $\phi(x) \to T\phi(x)$ and to the invariance with respect to spatial rotations $\phi(x) \to \phi(Tx)$. Here $\boldsymbol{T}$ are the three-dimensional orthogonal matrices. The conserved dynamic functionals of the isotopic and ordinary spin associated with the invariance properties mentioned have the form

$$\begin{aligned} \boldsymbol{I} &= -i < \hat{\boldsymbol{I}}\phi^i g_{ik}\dot{\phi}^k >, \qquad & \hat{I}^i_{ks} &= -i\varepsilon_{iks}\,, \\ \boldsymbol{J} &= -i < \hat{\boldsymbol{l}}\phi^i g_{ik}\dot{\phi}^k >, \qquad & \hat{l}^i_{ks} &= -i\delta_{ks}(\boldsymbol{x}\times\partial/\partial\boldsymbol{x})_i\,, \end{aligned} \tag{16}$$

where $\hat{\boldsymbol{I}}$, $\hat{\boldsymbol{l}}$ are the kinematic operators of the isotopic spin and the angular momentum. The invariance of the Lagrangian with respect to the spatial translations is not important in the problem considered and it will not be dealt with further. In the general case a localized soliton state is a dynamic system with a finite number of degrees of freedom. The essential part of the problem is the determination of the configurational space of this system. Simplifying the consideration, let us deal with the sector of the theory with the unity topological charge and arbitrary values of the angular momentum and the isotopic spin. This sector is elementary with respect to other ones. The localized states in it are not split into inner fragments that may possess their own degrees of freedom including the rotational ones. It may be expected that the lifetime of such states due to the emission of mesons exceeds considerably the times of inner motions associated with the soliton rotation in the ordinary and isotopic spaces. In terms of collective variables such evolution can be described with the substitution

$$\phi(x) = T^{(i)}\varphi(T^{(s)-1}x)\,, \tag{17}$$

that separates the cyclic variables of the system explicitly. In this formula the group parameters $a_\alpha^{(i)}$, $a_\alpha^{(s)}$ of the matrices $T^{(i)}$ and $T^{(s)}$ play the part of the collective coordinates, describing the isotopic and ordinary rotation of the soliton. The field $\varphi(x)$ is a new generalized coordinate associated with the microscopic degrees of freedom. As in this case the collective coordinates are the cyclic ones, it is expedient again to obtain the equation for dynamic solitons not making use of the general formula (14). Developing the canonical procedure on the ground of the substitution (17) and dealing with purely collective motions, we omit the time derivative of the field $\varphi(x)$. According to the prescription of the preceding section, such simplification should be adjust with the gauge conditions. Below, in formulating these conditions, we will identify the gauge function with the coherent component of the field $\varphi(x)$. Thus, differentiating the expression (17) with respect to time yields

$$\dot{\phi}(x) = -iT^{(i)}\left((\boldsymbol{\omega}^{(i)}\hat{\boldsymbol{I}} + \boldsymbol{\omega}^{(s)}\hat{\boldsymbol{l}})\varphi(x)\right)_{|x\to T^{(s)-1}x}\,, \tag{18}$$

where $\boldsymbol{\omega}^{(i)}$, $\boldsymbol{\omega}^{(s)}$ are the left-invariant forms of angular velocities defined by the formulas

$$T^{(a)-1}\dot{T}^{(a)} = -i\hat{\boldsymbol{I}}\boldsymbol{\omega}^{(a)}\,, \qquad a = i,\ s\,. \tag{19}$$

Inserting the formulas (17), (18) into the expression (15) can yield the effective Lagrangian $L_c(a, \boldsymbol{\omega}; \varphi(x))$. This Lagrangian does not contain the velocities $\dot{\varphi}(x)$ and therefore it is degenerate. In the generalized Hamiltonian formalism the field $\varphi(x)$ is the variational parameter minimizing the functional

$$H_c(\varphi) \equiv \sum_{\nu=i,s}\boldsymbol{\omega}^{(\nu)}\frac{\partial L_c}{\partial\boldsymbol{\omega}^{(\nu)}} - L_c \tag{20}$$

with fixed values of other Hamiltonian variables. In order to pass to these variables in the formula (20), it is necessary to employ the linear relations between the velocities $\boldsymbol{\omega}^{(i)}$, $\boldsymbol{\omega}^{(s)}$ and the conserved functionals $\boldsymbol{I}$, $\boldsymbol{J}$. Using the formulas (16), (18), let us present these relations in the matrix form

$$V' = -\Lambda'\Omega\,, \tag{21}$$

where the bi-vectors $\boldsymbol{\Omega}$, $\boldsymbol{V'}$ are defined through the formulas

$$V' = \begin{pmatrix} \boldsymbol{I}^r \\ \boldsymbol{J}^r \end{pmatrix}, \qquad \Omega = \begin{pmatrix} \boldsymbol{\omega}^{(i)} \\ \boldsymbol{\omega}^{(s)} \end{pmatrix}.$$

Here $\boldsymbol{I}^r \equiv -\boldsymbol{I}T^{(i)}$, $\boldsymbol{J}^r \equiv -\boldsymbol{J}T^{(s)}$ are the dynamic functionals having, according to (A.3), the sense of the generators of right shifts in the configurational space of the group parameters $a_\alpha^{(i)}$, $a_\alpha^{(s)}$. The kinetic matrix Λ' in (21) has a block structure. This matrix can be presented in the compact form as a product of a column and a row

$$\Lambda'_{ik}(\varphi) = -\left\langle \begin{pmatrix} \hat{I}^i \\ \hat{l}^i \end{pmatrix} \varphi^s g_{st}(\varphi)(\hat{I}^k, \hat{l}^k)\varphi^t \right\rangle .$$

From the viewpoint of the subsequent study of the axially symmetric field configurations and close to them that are characteristic for the states of the nucleon and delta, it is expedient to use instead of the variables $\boldsymbol{I}^r$, $\boldsymbol{J}^r$ their linear combinations

$$\boldsymbol{R} = (1/2)(\boldsymbol{I}^r - \boldsymbol{J}^r), \qquad \boldsymbol{Q} = \boldsymbol{I}^r + \boldsymbol{J}^r . \tag{22}$$

Introducing into consideration the bi-vector $\widetilde{V} \equiv (\boldsymbol{R}, \boldsymbol{Q})$ and using the formula (21) yield the link between new variables V and the velocities Ω

$$\Omega = -A\Lambda^{-1}V , \tag{23}$$

where the new kinetic matrix Λ and the transformation matrix A have the form

$$\Lambda(\varphi) = \widetilde{A}\Lambda'(\varphi)A = -\begin{pmatrix} (DD) & (DK) \\ (KD) & (KK) \end{pmatrix}, \qquad A = \begin{pmatrix} 1/2 & 1 \\ -1/2 & 1 \end{pmatrix} .$$

The inner blocks of the matrix Λ are defined through the formulas

$$\begin{array}{ll} (D^iD^k) \equiv < \hat{D}^i\varphi^s g_{st}(\varphi)\hat{D}^k\varphi^t >, & (D^iK^k) \equiv < \hat{D}^i\varphi^s g_{st}(\varphi)\hat{K}^k\varphi^t >, \\ (K^iD^k) \equiv < \hat{K}^i\varphi^s g_{st}(\varphi)\hat{D}^k\varphi^t >, & (K^iK^k) \equiv < \hat{K}^i\varphi^s g_{st}(\varphi)\hat{K}^k\varphi^t >, \end{array} \tag{24}$$

where

$$\hat{\boldsymbol{D}} = (1/2)(\hat{\boldsymbol{I}} - \hat{\boldsymbol{l}}), \qquad \hat{\boldsymbol{K}} = \hat{\boldsymbol{I}} + \hat{\boldsymbol{l}}$$

are the kinematic operators acting on the function $\varphi(x)$: $\hat{D}^i\varphi^s(x) \equiv \hat{D}^i_{sb}\varphi^b(x)$. Thus in terms of the variables $\boldsymbol{R}$, $\boldsymbol{Q}$ the functional $H_c(\varphi)$ (20) assumes the form

$$H_c(\varphi) = \frac{1}{2}\widetilde{V}\Lambda^{-1}(\varphi)V + H(\varphi, 0) . \tag{25}$$

In order to obtain the equation for coherent components of the field $\varphi(x)$, we limit the admissible variations of this field by the conditions

$$< \delta_\varphi \phi^q g_{qk}(\phi)\frac{\partial}{\partial a^{(\nu)}}\phi^k >= 0 , \qquad \nu = i,\, s . \tag{26}$$

Here $\delta_\varphi\phi$ are the variations of the total field $\phi(x)$ (17) connected with the variations of the function $\varphi(x)$. In accordance with the prescription of the preceding section, the gauge function in (26), associated with the field $\varphi(x)$, is defined as its coherent component. Under the signs of the trace and the integral the isotopic and spin matrices T in (26) vanish. Besides, if one takes into account the non-degeneracy of the matrices $\theta_{p\alpha}$, defined with the formula

$$T^{-1}\frac{\partial T}{\partial a_\alpha} = -i\hat{I}^p\theta_{p\alpha} ,$$

then the relations (26) can be put in the form

$$< \hat{\boldsymbol{D}}\varphi g\delta\varphi >= 0, \qquad < \hat{\boldsymbol{K}}\varphi g\delta\varphi >= 0\,.$$

If one constructs on the directing vectors $\hat{\mathbf{D}}\varphi(x)$, $\hat{\mathbf{K}}\varphi(x)$ the projection operator

$$\mathcal{P} \equiv -g(\hat{D},\hat{K})\varphi > \Lambda^{-1} < \begin{pmatrix} \hat{D} \\ \hat{K} \end{pmatrix} \varphi, \qquad \mathcal{P}^2 = \mathcal{P}, \tag{27}$$

then the gauge conditions (26) can be rewritten as

$$\delta\varphi(1-\mathcal{P}) = \delta\varphi. \tag{28}$$

With the account of these conditions the coherent components of the field $\varphi(x)$, realizing the extremum of the functional $H_c(\varphi)$, will be determined from the equation

$$(1-\mathcal{P})\frac{\delta H_c(\varphi)}{\delta\varphi} = 0\,. \tag{29}$$

The solutions of this equation depend on the dynamic variables $\boldsymbol{R}$, $\boldsymbol{Q}$ as on the parameters $\varphi_c(x) \equiv \varphi_c(x;\boldsymbol{R},\boldsymbol{Q})$. The functional $H_c(\varphi)$ dependent on these solutions determines the collective Hamiltonian of the spin-isospin rotation

$$H_c(\boldsymbol{R},\boldsymbol{Q}) \equiv H_c(\varphi_c)\,. \tag{30}$$

Let us make explicit the action of the operator $\mathcal{P}$ in equation (29). The functional $H_c(\varphi)$ as a function of the dynamic variables $\boldsymbol{I}^r \equiv -\boldsymbol{I}\,T^{(i)}$, $\boldsymbol{J}^r \equiv -\boldsymbol{J}\,T^{(s)}$ possesses an obvious property

$$H_c\left(\boldsymbol{I}^r,\boldsymbol{J}^r;\varphi(x)\right) = H_c\left(\boldsymbol{I},\boldsymbol{J};T^{(i)}\varphi(T^{(s)-1}x)\right)\,. \tag{31}$$

This relationship is valid for arbitrary $\varphi(x)$, not limited by any additional conditions. Therefore, differentiating it with respect to the parameters $a_\alpha^{(i)}$, $a_\alpha^{(s)}$, one can find the identities relating the conventional variational derivatives $\delta H_c/\delta\varphi(x)$ with the derivatives of the type $\partial H_c(\boldsymbol{I}^r,\boldsymbol{J}^r;\varphi(x))/\partial\boldsymbol{I}^r$. The latter derivatives can be regarded as complete on the solutions of the equation (29), i.e., taking into account the dependence of the solutions $\varphi_c(x)$ on the parameters $\boldsymbol{I}^r$, $\boldsymbol{J}^r$. It is possible, because under such treatment the variational derivative of the collective Hamiltonian with respect to $\varphi_c(x)$ in the left-hand side of (31) should be regarded with the account of the coupling (28), and therefore it falls out due to the equations (29). Thus we obtain from (31) that

$$\int dx\hat{\boldsymbol{I}}\varphi(x)\frac{\delta H_c}{\delta\varphi(x)} = i\dot{\boldsymbol{I}}^r, \qquad \int dx\hat{\boldsymbol{l}}\varphi(x)\frac{\delta H_c}{\delta\varphi(x)} = i\dot{\boldsymbol{J}}^r. \tag{32}$$

Here $\dot{\boldsymbol{I}}^r \equiv \{\boldsymbol{I}^r, H_c\}$, $\dot{\boldsymbol{J}}^r \equiv \{\boldsymbol{J}^r, H_c\}$ are the rates of change of the variables $\boldsymbol{I}^r$, $\boldsymbol{J}^r$ under the action of the collective Hamiltonian $H_c(\varphi_c)$. The Poisson brackets of the functionals $\boldsymbol{I}'$, $\boldsymbol{J}'$ are defined with the expressions (A.2). With the account of the formulas (27), (32) the equation (29) will ultimately assume the form

$$\frac{\delta H_c(\varphi)}{\delta\varphi^n(x)} = -ig_{n\,l}(\varphi)(\hat{D},\hat{K})^i_{ls}\varphi^s(x)\Lambda^{-1}_{ik}\dot{V}^k\,. \tag{33}$$

This equation determines the shape of the rotating soliton. The right-hand part of this equation allows for the self-consistent collective forces reflecting the availability of non-compensated dynamic stresses in the system.

4 Self-similar solutions of evolution equations

We have defined the dynamic soliton states as the solutions of the equation (33). How are these solutions related to the solutions of the exact evolution equations? If the ansatz (17) is substituted into the Lagrange equation of motion

$$\frac{d}{dt}g(\phi(x))\dot{\phi}(x) = \frac{\delta}{\delta\phi(x)}\left(\frac{1}{2} < \dot{\phi}g\dot{\phi} > -H(\phi,0)\right)$$

with the assumption that $\varphi(x)$ in (17) does not depend on time and the conservation laws (16) are taken into account, then we obtain exactly the equation (33) for the field $\varphi(x)$. But the solutions of this equation depend on the functionals $\boldsymbol{R},\boldsymbol{Q}$, which are not the integrals of motion, generally speaking. We will assume that in the systems under consideration a spontaneous breaking of symmetry takes place. Under this breaking the spin and isospin variables entangle and the solutions of the equation (33) remain invariant with respect to arbitrary three-dimensional rotations:

$$\varphi_c(x;\boldsymbol{R},\boldsymbol{Q}) = T^{-1}\varphi_c(Tx;T\boldsymbol{R},T\boldsymbol{Q})\,. \tag{34}$$

In favor of this choice there speaks the fact that in the static limit such solutions are transformed into hedgehog configurations realizing the minimum energy of one-baryon states in the Skyrme model. On the solutions (34) the collective Hamiltonian (30) becomes the function of three invariant functionals

$$H_c = H_c(\boldsymbol{R}^2,\boldsymbol{Q}^2,\boldsymbol{RQ})\,. \tag{35}$$

In agreement with the formulas (A.4), this means that the vector $\boldsymbol{R}$ is precessing uniformly around the vector $\boldsymbol{Q}$ conserved in time according to the equation

$$\dot{\boldsymbol{R}} \equiv \{\boldsymbol{R},H_c\} = \alpha\boldsymbol{Q}\times\boldsymbol{R}, \qquad \alpha \equiv 2\frac{\partial H_c}{\partial\boldsymbol{Q}^2} - \frac{1}{2}\frac{\partial H_c}{\partial\boldsymbol{R}^2}\,.$$

With arbitrary $\boldsymbol{R},\boldsymbol{Q}$ the solutions of the equation (33) do not possess the axial symmetry and depend on time through the parameter $\boldsymbol{R}$. Consequently, the corresponding ansatz (17) is not a solution of the equation of motion, but it describes the true resonant states. In case of axial symmetry (the states with $\boldsymbol{Q}=0$ have physical sense, see below) the vector $\boldsymbol{R}$ is conserved in time, and therefore the ansatz (17) constructed on the solutions of the equation (33) becomes the exact solution of the motion equations. The self-consistent forces vanish in this case, and the equation (33) itself is reduced to the equation for the stationary states of the Hamiltonian $H_c(\boldsymbol{R}^2, 0, 0)$ (35).

5 Quantizing the spin-isospin rotation

In the quantum case the functionals $\boldsymbol{R}^2$, $\boldsymbol{Q}^2$, $\boldsymbol{RQ}$ should be substituted with the corresponding operators. These operators are equivalent to the set of the operators $\boldsymbol{I}^2$, $\boldsymbol{J}^2$, $\boldsymbol{I}^r\boldsymbol{J}^r$. The latter means that in contrast to the spin $\boldsymbol{J}$ and the isotopic spin $\boldsymbol{I}$, the generators of the right shifts $\boldsymbol{J}^r$, $\boldsymbol{I}^r$ are not the integrals of motion of the Hamiltonian (35). Only their sum $\boldsymbol{Q} = \boldsymbol{I}^r + \boldsymbol{J}^r$ is conserved. Consequently, the eigenvalues $Q(Q+1)$, q, $I(I+1)$, n, $J(J+1)$, m of the complete set of the mutually permutable operators $\boldsymbol{Q}^2$, Q_3, $\boldsymbol{I}^2$, I_3, $\boldsymbol{J}^2$, J_3 can be used as the quantum numbers enumerating the eigenstates of the Hamiltonian H_c. Thus, the wave functions of the collective motion corresponding to the noted quantum numbers can be presented in the form

$$|Q, q; I, n; J.m >\equiv \sum_{m^r+n^r=q} < I, n^r; J, m^r|Q, q > \chi^I_{n,n^r}(a^{(i)})\chi^J_{m,m^r}(a^{(s)})\,. \tag{36}$$

Here $< I, n^r; J, m^r|Q, q >$ are the angular coefficients, $\chi^I_{n,n^r}(a^{(i)})$ and $I(I+1), n, n^r$ are the eigenstates and the respective eigenvalues of the isotopic operators $\boldsymbol{I}^2, I_3, I^r_3$. The same sense is also attributed to the wave function of the spin motion $\chi^J_{m,m^r}(a^{(s)})$. The invariant operators in (35) possess the following eigenvalues

$$\begin{aligned}
\boldsymbol{RQ} &= \frac{1}{2}(I(I+1) - J(J+1)),\\
\boldsymbol{Q}^2 &= Q(Q+1), \qquad |I-J| \le Q \le I+J,\\
\boldsymbol{R}^2 &= \frac{1}{2}(I(I+1) + J(J+1) - \frac{1}{2}Q(Q+1)).
\end{aligned}$$

In the sector with the baryon charge $B = 1$ the quantization scheme with half-integer spin and isospin values should be used [7]. In this case the eigenvalues of the operator $\boldsymbol{R}^2$ do not vanish, in contrast to the operators $\boldsymbol{Q}^2$, $\boldsymbol{RQ}$. So the wave function of the nucleon with $I = J = 1/2$, $Q = 0$ will correspond to the ground state. According to Eq.(36) this wave function has the form

$$|n, m >= \frac{1}{\sqrt{2}}\left(\chi_{n\uparrow}(a^{(i)})\chi_{m\downarrow}(a^{(s)}) - \chi_{n\downarrow}(a^{(i)})\chi_{m\uparrow}(a^{(s)})\right)\,. \tag{37}$$

Here $n, m = \pm 1/2$ are the projections of the isotopic and ordinary spins on the quantization axis. The explicit expressions for the components of the vector $\chi^{1/2}_{ks}(a) \equiv$

$\chi_{ks}(a)$ in (37) have the form [1]

$$\begin{aligned}\chi_{\uparrow\uparrow}(a) &= \frac{1}{\pi}(a_1 + ia_2), & \chi_{\uparrow\downarrow}(a) &= -\frac{i}{\pi}(a_0 - ia_3),\\ \chi_{\downarrow\uparrow}(a) &= \frac{i}{\pi}(a_0 + ia_3), & \chi_{\downarrow\downarrow}(a) &= -\frac{1}{\pi}(a_1 - ia_2).\end{aligned} \tag{38}$$

The wave functions of the nucleon (37) differ from the respective wave function of the paper [1]. With the approach considered the spin and isospin rotations are independent, therefore they require a double set of group parameters for their description. The classical states with $Q = 0$ will be described by the axially symmetric configurations and besides the nucleon they will include the delta-isobar states. Depending on the meson mass which limits the possible values of the angular momentum $\boldsymbol{R}^2$, other resonances with $I = J$, $Q = 0$ can also manifest themselves. In order to determine the spectrum of these states it is necessary to treat the equation (33) in the axially symmetrical case with $\boldsymbol{R} \neq 0$, $\boldsymbol{Q} = 0$.

6 O(2)-invariant configurations

In the general case with arbitrary values of the dynamical variables $\boldsymbol{R}$, $\boldsymbol{Q}$ the kinetic matrix Λ, dependent on the solutions of the equation (33), is nondegenerate. If these variables become collinear or one of them vanishes, the system acquires the axially symmetric configuration and the projection of the tangent vector $\hat{\boldsymbol{K}}\varphi_c(x)$ on the axis of symmetry $\boldsymbol{k}$ vanishes: $\boldsymbol{k}\hat{\boldsymbol{K}}\varphi_c(x) = 0$. It means that the matrix of the moments of inertia Λ becomes degenerate on the O(2)-invariant configurations. The bi-vector $(0, \boldsymbol{k})$ is its zero mode. Under these conditions, the perturbation theory and the passage to the limit procedure become ambiguous. They depend on the passage path and can contain nonanalytic terms. In order to find the physically acceptable branches of the solution and to determine the order of the passage to the limit, let us consider the properties of the matrix Λ on the O(2)-invariant configurations in more detail. Obviously, the solutions of the equation (33) are even functions of the vectors $\boldsymbol{R}$, $\boldsymbol{Q}$. That is, in case of axial symmetry these solutions are even functions of the unit vector $\boldsymbol{k}$ directed along the axis of symmetry. This permits to present the elements of the block matrix Λ (24) as follows

$$(D^i D^k) = -(a\nu_{ik} + bk_ik_k), \quad (D^i K^k) = (K^i D^k) = -c\nu_{ik}, \quad (K^i K^k) = -d\nu_{ik},$$

where a, b, c, d are the scalar coefficients, $\nu_{ik} \equiv \delta_{ik} - k_ik_k$. It is possible to determine the roots of the characteristic equation $|\Lambda - \lambda| = 0$:

$$\begin{aligned}\lambda_1 &= 0, \qquad \lambda_2 = b,\\ \lambda_3 &= \lambda_4 = \frac{1}{2}\left(d + a + \sqrt{(d-a)^2 + 4c^2}\right),\\ \lambda_5 &= \lambda_6 = \frac{1}{2}\left(d + a - \sqrt{(d-a)^2 + 4c^2}\right),\end{aligned}$$

and to present the corresponding orthonormalized eigenfunctions of the matrix Λ in the form

$$
\begin{aligned}
\psi_1 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \boldsymbol{k}, & \psi_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \boldsymbol{k}, \\
\psi_3 &= \alpha_3 \begin{pmatrix} c \\ \lambda_3 - a \end{pmatrix} \boldsymbol{p}, & \psi_4 &= \alpha_3 \begin{pmatrix} c \\ \lambda_3 - a \end{pmatrix} \boldsymbol{p}', \\
\psi_5 &= \alpha_5 \begin{pmatrix} c \\ \lambda_5 - a \end{pmatrix} \boldsymbol{p}, & \psi_6 &= \alpha_5 \begin{pmatrix} c \\ \lambda_5 - a \end{pmatrix} \boldsymbol{p}'.
\end{aligned}
$$

Here α_3, α_5 are the normalizing constants, $\boldsymbol{p}$, $\boldsymbol{p}'$ are the unit orthonormalized vectors in the plane orthogonal to the axis of symmetry, viz. $\boldsymbol{pp}' = \boldsymbol{pk} = \boldsymbol{p}'\boldsymbol{k} = 0$. Assuming the axis of symmetry $\boldsymbol{k}$ to be directed along the vector $\boldsymbol{R}$, let us take into account the infinitesimal vector $\boldsymbol{Q}_\perp$ for regularizing the eigenvalue λ_1. Then using the eigenfunctions ψ_r we can establish the asymptotic behavior of the matrix Λ^{-1} in the vicinity of the axially symmetric configurations

$$
\begin{aligned}
\Lambda^{-1} &= \sum_{r=1}^{6} \lambda_r^{-1} \psi_r \tilde{\psi}_r = \\
&= \lambda_1^{-1}(\boldsymbol{Q}_\perp) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \hat{k}\hat{k} + b^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \hat{k}\hat{k} + (\lambda_3^{-1} A_3 + \lambda_5^{-1} A_5)\nu \,.
\end{aligned}
\tag{39}
$$

Here $\lambda_1(\boldsymbol{Q}_\perp)$ is the regularized eigenvalue λ_1, tending to zero with $\boldsymbol{Q}_\perp \to 0$; A_3, A_5 are the degenerate two-dimensional matrices, whose form is inessential in this case. On inserting the expression (39) into the kinetic term $\widetilde{V}\Lambda^{-1}V$, the singular term survives only at $Q_{||} \equiv \boldsymbol{kQ} \neq 0$. This means that the spectrum of classical resonances corresponding to $Q_{||} \neq 0$ possesses a singularity in the asymptotic range of small $\boldsymbol{Q}_\perp^2$. Hence it also follows that for the description of the states corresponding to the nucleon and delta ($\boldsymbol{Q} = 0$) one should first perform the passage to the limit $\boldsymbol{Q}_{||} \to 0$, that will remove the singular term. Only then one should put $\boldsymbol{Q}_\perp = 0$. As a result, the functional $H_c(\varphi)$ (25) will reduce to

$$
H_c(\boldsymbol{R}, \varphi) = \frac{1}{2} \frac{\boldsymbol{R}^2}{b(\varphi)} + H(\varphi, 0) \,,
\tag{40}
$$

where the functional of the moment of inertia $b(\varphi)$ is defined by the formula

$$
b(\varphi) \equiv -k_i k_r (D^i D^r) = - < \boldsymbol{k}\hat{\boldsymbol{I}}\varphi^s g_{st}(\varphi) \boldsymbol{k}\hat{\boldsymbol{I}}\varphi^t > .
$$

The self-consistent forces in the equation (33) vanish with $\boldsymbol{Q} \to 0$ for any order of the passage to the limit. So the equation for the axially symmetric configurations $\boldsymbol{R} \neq 0$, $\boldsymbol{Q} = 0$ will assume the form of the conventional equation for the stationary states of the Hamiltonian (40)

$$
\frac{\delta}{\delta\varphi(x)} \left(\frac{1}{2} \frac{\boldsymbol{R}^2}{b(\varphi)} + H(\varphi, 0) \right) = 0 \,.
\tag{41}
$$

One should seek the solutions of this equation among the class of functions with the topological charge $B = 1$. If one uses the exponential parametrization of SU(2) group: $U(x) = \exp i\boldsymbol{\varphi}(x)\boldsymbol{\tau}$, then the corresponding boundary conditions will have the form

$$|\boldsymbol{\varphi}(\infty)| = 0, \qquad |\boldsymbol{\varphi}(0)| = \pi.$$

That is, these conditions coincide with those for the profile function in case of the static hedgehog configuration.

As the functional $\boldsymbol{R}$ at $\boldsymbol{Q} = 0$ is the integral of motion, then the configuration $\varphi(x, \boldsymbol{R})$ determined by the equation (41) is conserved in time. It means that the corresponding total field (17) is the self-similar solution of the equation of motion. In this solution only the group parameters $a_\alpha^{(i)}$, $a_\alpha^{(s)}$ or, what is the same, the matrices $T^{(i)}$, $T^{(s)}$ depend on time. This dependence can be found directly using the integrals of motion (16). From the definition of the velocity forms (19) it follows that

$$\begin{aligned}
\dot{T}^{(i)} &= -iT^{(i)}\,\hat{\boldsymbol{I}}\boldsymbol{\omega}^{(i)}, \\
\dot{T}^{(s)} &= -iT^{(s)}\,\hat{\boldsymbol{I}}\boldsymbol{\omega}^{(s)}.
\end{aligned}$$

Expressing with the help of the formula (23) the velocities $\boldsymbol{\omega}^{(i)}$, $\boldsymbol{\omega}^{(s)}$ through the functionals $\boldsymbol{R}$, $\boldsymbol{Q}$, let us write these equations in the matrix form

$$\begin{pmatrix} \dot{T}^{(i)} \\ \dot{T}^{(s)} \end{pmatrix} = i \begin{pmatrix} T^{(i)} & 0 \\ 0 & T^{(s)} \end{pmatrix} \hat{\boldsymbol{I}}\, A\, \Lambda^{-1} V.$$

Inserting here the regularized expression for Λ^{-1} (39), and performing first the passage to the limit $\boldsymbol{Q}_{||} \to 0$ and only then putting $\boldsymbol{Q}_{\perp} = 0$ yield the equation for the rotation matrices in case of axial symmetry

$$\begin{pmatrix} \dot{T}^{(i)} \\ \dot{T}^{(s)} \end{pmatrix} = \frac{i}{2b} \begin{pmatrix} T^{(i)} \\ -T^{(s)} \end{pmatrix} \hat{\boldsymbol{I}}\boldsymbol{R}.$$

Hence we find that

$$T^{(i)}(t) = T^{(s)}(-t) = \exp\left(\frac{i}{2\,b}\,\hat{\boldsymbol{I}}\boldsymbol{R}\,t\right).$$

That is, the isotopic and spin subsystems of the classical localized state are rotating uniformly in opposite directions with the angular frequency $\omega = |\boldsymbol{R}|/2b$.

Inserting the solutions of the equation (41) into the functional (40) can determine the classical Hamiltonian of the system on the surface $\boldsymbol{Q} = 0$ for arbitrary $\boldsymbol{R}^2$ values. At the points $I = J = 1/2$ and $I = J = 3/2$ this Hamiltonian determines the exact nucleon and delta masses in the model considered. If the solutions of the equation (41) are many-valued, then in addition to the nucleon and delta this equation can describe other resonances from the series P_{11}, P_{33}. Besides, depending on the meson mass, the spectrum of the solutions of this equation can also contain resonances with $I = J > 3/2$, $Q = 0$. The complete picture of solutions of the equation (41) can be given only by means of its numerical analysis. In the $\boldsymbol{R}^2$ approximation the Hamiltonian (40) reduces to the Hamiltonian of the adiabatic approach [1] for the Skyrme model. The equation (41) makes a strict account of the rotational deformation, and, by doing so, lowers the nucleon and delta masses in comparison with the results of the paper [1]. In particular, one can show that the term of the Hamiltonian (40) of the order $\boldsymbol{R}^4$ is negative. In the sigma-model there are no static soliton solutions and the nucleon mass has a purely quantum origin being due to the spin-isospin rotation of the soliton.

Let us employ the conventional scale analysis and prove that the presence in the formula (40) of the kinetic term

$$H_T(\varphi) \equiv \frac{1}{2}\frac{\boldsymbol{R}^2}{b(\varphi)}$$

impedes the collapse of the rotating soliton state in the SU(2) sigma-model. In this model the potential $H(\varphi, 0)$ has the following form

$$H(\varphi, 0) = H_0(\varphi) + H_m(\varphi)\,,$$

where $H_0(\varphi)$ is the conventional second-order term in derivatives, $H_m(\varphi)$ is the mass term not containing the spatial derivatives. As in the sigma-model the matrix $g(\varphi)$ also does not contain the spatial derivatives, then as a result of the substitution $\varphi(x) \to \varphi(\lambda x)$ the Hamiltonian (40) is reduced to the form

$$H_c(\boldsymbol{R}, \varphi) \to H_{c\lambda} = \lambda^3 H_T + \lambda^{-1} H_0 + \lambda^{-3} H_m\,.$$

It follows from this identity considered for the solutions of the equation (41) that

$$\begin{array}{lcccl} \dfrac{\partial H_{c\lambda}}{\partial \lambda}_{|\lambda=1} & = & 0 & \to & H_T = H_m + \dfrac{1}{3}H_0\,, \\ \dfrac{\partial^2 H_{c\lambda}}{\partial \lambda^2}_{|\lambda=1} & > & 0 & \to & 2H_T + H_m > 0\,. \end{array}$$

That is, the soliton energy $E = H_T + H_0 + H_m$ can be finite only in the presence of the kinetic term $H_T > 0$. Thus, if a soliton can rotate keeping the angular momentum $\boldsymbol{R}$, it will be an object stable with respect to the collapse. In order to estimate the possibility of such a rotation, let us determine the asymptotics of the field $\varphi(x)$ at the spatial infinity. On linearizing the equation (41) with respect to $\varphi(x)$ it assumes the form

$$\left(-\Delta + m_\pi^2 - \left(\frac{\boldsymbol{R}}{b}\right)^2 \nu\right)\varphi(x) = 0\,.$$

Here m_π is the meson mass, $\nu_{ik} \equiv \delta_{ik} - k_i k_k$. This equation is valid for the sigma-model as well as for the Skyrme model, because at small $\varphi(x)$ the contribution of the Skyrme term falls out. Thus the field component in the rotation plane $\nu\varphi(x)$ will have the damping asymptotics only for $\boldsymbol{R}^2 < b^2 m_\pi^2$. The moment of inertia b also depends on $\boldsymbol{R}^2$, and therefore one can determine the exact boundary of admissible values of the moment $\boldsymbol{R}$, at which the soliton remains stable with respect to the disintegration only on the ground of the direct numerical analysis of the equation (41). Thus in the sigma-model, in contrast to the static case, there are conditions for the existence of the dynamic soliton that is stable against the collapse and whose shape is determined from the equation (41).

7 Conclusion

The paper formulates the exact equations for the SU(2)-dynamic solitons performing the rotations in the ordinary as well as isotopic space. Generally, the shape of these solitons is determined with the account of the self-consistent collective forces reflecting the presence of non-compensated dynamic stresses in the system. On assuming the spontaneous breaking of the symmetry, the eigenfunctions of the Hamiltonian of the spin-isospin rotation in the one-baryon sector are established. Within the quantization scheme with a half-integer spin, the axially symmetric field configurations correspond to the ground state. The generalized matrix of the moments of inertia is degenerate on such configurations. With the account of this circumstance, the order of the passage to the limit near singularities is established, and the equation for the O(2)-invariant solitons is found. It is shown that, in contrast to the general case, the O(2)-invariant configurations are equivalent to the exact self-similar solutions of the equation of motion. The temporal dependence of the respective dynamic states is established. The quantum analogues of similar axially symmetric solutions correspond to the states of the nucleon and delta. This paper gives the expression for the state vector of the nucleon reflecting the independent evolution of the soliton in the spin and isospin spaces. The account of rotations is shown to create the condition for the existence of the exact extended solutions of the equations of motion in the SU(2) sigma-model. The treatment performed is based on using the variational approach to the method of collective variables. This paper gives the strict justification of the equations of collective dynamics.

Appendix

The canonical description of the collective motion of the system can be made in a standard way. The elements of the configurational space of the collective subsystem are the parameters $a_\alpha^{(i)}$, $a_\alpha^{(s)}$ of the groups SO(3) associated with the isotopic and spin rotation of the soliton. With the help of the formulas (15), (16), (18) the canonical momenta conjugated to these variables can be related to the functionals $\boldsymbol{I}^r$, $\boldsymbol{J}^r$

$$\pi_\alpha^{(i)} \equiv \frac{\partial L}{\partial \dot{a}_\alpha^{(i)}} = -\boldsymbol{I}^r \frac{\partial \boldsymbol{\omega}^{(i)}}{\partial \dot{a}_\alpha^{(i)}}, \qquad \pi_\alpha^{(s)} \equiv \frac{\partial L}{\partial \dot{a}_\alpha^{(s)}} = -\boldsymbol{J}^r \frac{\partial \boldsymbol{\omega}^{(s)}}{\partial \dot{a}_\alpha^{(s)}}. \tag{A.1}$$

Obviously, the transition to the canonical description is identical for the spin and isospin subsystems. Therefore, the canonical properties of the isotopic subsystem can also be extended to the spin variables. If the elements of the SU(2) group are parametrized by the components of the unit vector on the three-sphere: $A \equiv a_0 + i\boldsymbol{\tau a}$, $a_\alpha^2 \equiv a_0^2 + a_k^2 = 1$, $k = 1\,,2\,,3$, then the orthogonal matrix $\boldsymbol{T}$ and the velocity form $\boldsymbol{\omega}$, can be written as follows

$$\begin{aligned} T_{ik} &= (1/2)trA\tau_i A^{-1}\tau_k = (1 - 2\boldsymbol{a}^2)\delta_{ik} + 2a_i a_k - 2\varepsilon_{iks}a_s a_0\,, \\ \omega_k &= -itr\dot{A}A^{-1}\tau_k = 2(a_0\dot{a}_k - \dot{a}_0 a_k + \varepsilon_{ksr}\dot{a}_s a_r)\,. \end{aligned}$$

In order to determine the classical Poisson brackets of the functionals $\boldsymbol{I}$, $\boldsymbol{I}^r$, it is convenient to take the spatial components of the unit vector a_α: a_k, $k = 1\,,2\,,3$ as the independent collective coordinates. Noting that $\dot{a}_0 = -a_0^{-1}a_k\dot{a}_k$, we find from (A.1) the following expressions for the functionals $\boldsymbol{I}$, $\boldsymbol{I}^r$

$$\begin{aligned} I_k^r &= \frac{1}{2}(\varepsilon_{kqr}a_q^{(i)}\pi_r^{(i)} - a_0^{(i)}\pi_k^{(i)})\,, \\ I_k &= \frac{1}{2}(\varepsilon_{kqr}a_q^{(i)}\pi_r^{(i)} + a_0^{(i)}\pi_k^{(i)})\,. \end{aligned}$$

Hence, using the canonical Poisson bracket $\{a_i, \pi_k\} = \delta_{ik}$, we find the Poisson brackets for the functionals $\boldsymbol{I}$, $\boldsymbol{I}^r$

$$\begin{aligned} \{I_s,\ I_k\} &= \varepsilon_{skn}I_n\ , \qquad \{I_s^r,\ I_k^r\} = \varepsilon_{skn}I_n^r\ , \quad \{I_s,\ I_k^r\} = 0\ , && \text{(A.2)} \\ \{I_k,\ T_{sn}\} &= \varepsilon_{kst}T_{tn}\ , \qquad \{I_k^r,\ T_{sn}\} = \varepsilon_{ikn}T_{si}\,. && \text{(A.3)} \end{aligned}$$

The last two formulas show that the functionals $\boldsymbol{I}$, $\boldsymbol{I}^r$ are the dynamic generators of the left and right shifts in the configurational space of the isotopic variables. Using similar formulas for the respective functionals of the spin subsystem $\boldsymbol{J}$, $\boldsymbol{J}^r$, we find the Poisson brackets for the functionals $\boldsymbol{Q}$, $\boldsymbol{R}$ (22)

$$\{Q_i, Q_j\} = \varepsilon_{ijk}Q_k\ , \qquad \{R_i, R_j\} = \frac{1}{4}\varepsilon_{ijk}Q_k\ , \qquad \{Q_i, R_j\} = \varepsilon_{ijk}R_k\,,$$

as well as the Poisson brackets of these functionals with their invariant combinations $\boldsymbol{R}^2$, $\boldsymbol{Q}^2$, $\boldsymbol{RQ}$

$$\begin{aligned} &\{Q_i, \boldsymbol{RQ}\} = 0\ , \qquad \{Q_i, \boldsymbol{Q}^2\} = 0\ , \qquad \{Q_i, \boldsymbol{R}^2\} = 0\ , \\ &\{R_i, \boldsymbol{RQ}\} = 0\ , \qquad \{R_i, \boldsymbol{Q}^2\} = 2\varepsilon_{iks}Q_k R_s\ , \qquad \{R_i, \boldsymbol{R}^2\} = \frac{1}{2}\varepsilon_{iks}R_k Q_s\ . \end{aligned} \qquad \text{(A.4)}$$

In case of quantum description all components of the unit vector on the three-sphere are conveniently considered as the collective coordinates. Covariant usage of all components of the unit four-vector permits to put the eigenfunctions of the left and right generators in their most symmetric form. Defining the canonical momenta π'_α, $\alpha = 0, 1, 2, 3$ corresponding to these components, we find from (A.1) the following relations

$$
\begin{aligned}
I^r_k &= \frac{1}{2}(\pi'^{(i)}_0 a^{(i)}_k - a^{(i)}_0 \pi'^{(i)}_k + \varepsilon_{kqr} a^{(i)}_q \pi'^{(i)}_r)\,, \\
I_k &= \frac{1}{2}(a^{(i)}_0 \pi'^{(i)}_k - \pi'^{(i)}_0 a^{(i)}_k + \varepsilon_{kqr} a^{(i)}_q \pi'^{(i)}_r)\,.
\end{aligned}
$$

With the account of these solutions the equation (A.1) for π_0 is satisfied identically. Considering the momenta π'_α as the operators on the three-sphere: $\pi'_\alpha = -i\partial/\partial a_\alpha$, one can establish the commutator relations for the operators $\boldsymbol{I},\ \boldsymbol{I^r}$, corresponding to the Poisson brackets (A.2) and find the eigenfunctions of these operators (38) [1].

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