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Fermion zero-modes on brane-worlds

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Abstract

We study localization of bulk fermions on a brane with inclusion of Yang-Mills and scalar backgrounds in higher dimensions and give the conditions under which localized chiral fermions can be obtained.

Introduction. Suggestions that extra dimensions may not be compact [1]-[5] or large [6, 7] can provide new insights for a solution of gauge hierarchy problem [7], of cosmological constant problem [2, 4], and give new possibilities for model building. One of the interesting questions, related to these ideas, is localization of different fields on a brane. It has been shown that the graviton [5] and the massless scalar field [8] have normalizable zero modes on branes of different types, that the abelian vector fields are not localized in the Randall-Sundrum (RS) model in five dimensions but can be localized in some higher-dimensional generalizations of it [9]. In contrast, in [8] it was shown that fermions do not have normalizable zero modes in five dimensions, while in [9] the same result was derived for a compactification on a string [10, 11] in six dimensions[1]. It is known, though, that fermion interaction with a scalar domain wall in five dimensions can lead to localization of chiral fermions [1]. Gauge field localization by confinement effects were discussed in [13], bulk fields in a slice of AdS in [14].

In this note we shall prove that under quite general assumptions about the geometry and topology of the internal manifold of the higher-dimensional warp factor compactification there exist massless Dirac fermions. However, these fermionic modes are generically non-normalizable. On the other hand if we include a Yukawa-type coupling to a scalar field of a domain-wall type we can ensure chirality as well as localization of the fermions. To generate chiral fermions by this mechanism the topology of the internal Kaluza-Klein manifold and the gauge field defined on it should be such that the index of the Dirac operator defined on this manifold is non-zero. At the end of this note we shall mention the example of the K_3 surface and S^4 with a background instanton configuration defined on it.

Branes with gauge and gravity backgrounds. We shall consider $D = D_1 + D_2 + 1$ - dimensional manifolds with the geometry

$$\mathrm{d}s^2 = \mathrm{e}^{A(r)}\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{B(r)}g_{mn}(y)\mathrm{d}y^m \mathrm{d}y^n + \mathrm{d}r^2, \tag{1}$$

where $\mu, \nu = 0, 1 \ldots, D_1-1, \quad m, n = 1, \ldots, D_2$. The coordinates y^m cover an internal manifold K with the metric $g_{mn}(y)$. The D-dimensional Dirac equation is

$$\Gamma^A E^M_A(\partial_M - \Omega_M + A_M)\Psi(x, y, r) = 0, \tag{2}$$

where E^M_A is the vielbein, $\Omega_M = \frac{1}{2}\Omega_{M[AB]}\Sigma^{AB}$ is the spin connection, $\Sigma_{AB} = \frac{1}{4}[\Gamma_A, \Gamma_B]$, and A_M is a Yang-Mills field in the algebra of some gauge group G. The RS model is the special case with $D_2 = 0$ and $A_M = 0$.

The non-vanishing components of Ω_M are

$$\Omega_\mu = \frac{1}{4}A'\mathrm{e}^{\frac{A}{2}}\delta^a_\mu\Gamma_r\Gamma_a \ , \tag{3}$$

$$\Omega_m = \frac{1}{4}B'\mathrm{e}^{\frac{B}{2}}e^{\underline{a}}_m\Gamma_r\Gamma_{\underline{a}} + \omega_m \ , \tag{4}$$

[1]Bulk fermions were localized on brane world studied some years ago in [12] by the use of a bulk magnetic field, which falls out of the framework of our ansatz. However, these solutions generally have problems with normalizability of the graviton modes.

where $\Gamma_r, \Gamma_a,\ a = 0, 1, \ldots, D_1 - 1$ and $\Gamma_{\underline{a}},\ \underline{a} = 1, \ldots, D_2$ are the constant Dirac matrices and $\omega_m = \frac{1}{8}\omega_{m[\underline{a},\underline{b}]}[\Gamma_{\underline{a}}, \Gamma_{\underline{b}}]$ is the spin connection derived from the metric $g_{mn}(y) = e^{\underline{a}}_m e^{\underline{b}}_n \delta_{\underline{ab}}$.

Assume $A_\mu = A_r = 0$. The Dirac equation then becomes

$$\left\{ \mathrm{e}^{-\frac{A}{2}}\ \partial\!\!\!/_x + \Gamma^r \left(\partial_r + \frac{D_1}{4}A' + \frac{D_2}{4}B' \right) + \mathrm{e}^{-\frac{B}{2}}\ \Delta\!\!\!\!/_y \right\} \Psi = 0, \tag{5}$$

where $\Delta\!\!\!\!/_y = \Gamma^{\underline{a}} \mathrm{e}^m_{\underline{a}}(\partial_m - \omega_m + A_m)$ is the Dirac operator on the internal manifold K and in the background of the gauge field A_m. With an appropriate choice of K and A_m this operator will have zero modes [15]. Denote these modes by $\psi(y)$. We can then write

$$\Psi(x, y, r) = \psi(y) f(r) \phi(x)\ , \tag{6}$$

where f and ϕ satisfy

$$\partial\!\!\!/_x \phi(x) \quad = \quad 0\ , \tag{7}$$

$$f' + \left(\frac{D_1}{4}A' + \frac{D_2}{4}B' \right) f \quad = \quad 0, \tag{8}$$

or

$$f(r) = \text{const.}\ \mathrm{e}^{-\left(\frac{D_1}{4}A + \frac{D_2}{4}B\right)}. \tag{9}$$

The effective Lagrangian for ϕ then becomes

$$\int \mathrm{d}r\mathrm{d}y \sqrt{-G}\bar{\Psi}\Gamma^A E^M_A(\partial_M - \Omega_M + A_M)\Psi = \bar{\phi}(x)\ \partial\!\!\!/_x \phi(x) \times \int \mathrm{e}^{-\frac{A}{2}} \mathrm{d}r\mathrm{d}y \sqrt{g}\psi^\dagger(y)\psi(y). \tag{10}$$

This should be compared with the expression of D_1-dimensional Newton constant G_{D_1} in terms of the D-dimensional one,

$$G^{-1}_{D_1} = G^{-1}_D V_{D_2} \int d\, r\ \exp\left((\frac{D_1}{2} - 1)A + \frac{B}{2} \right)\ , \tag{11}$$

where V_{D_2} is the volume of the manifold K. Thus, to have the localization of gravity and finite kinetic energy for ϕ, both integrals (10,11) must be simultaneously finite. This is not the case for the exponential warp-factor $A \propto -|r|$ considered in the literature so far. In fact, for such A and B, the function f in (9) diverges as $r \to \infty$.

So, for presently known solutions, the bulk fermions cannot be localized on a brane with the use of gravity and gauge fields only.

Yukawa Coupling and Chiral Fermions. Now let us include a real scalar field χ in our problem. The modification of the Dirac equation will be through some Yukawa term, with the coupling λ, viz.

$$\left\{ \mathrm{e}^{-\frac{A}{2}}\ \partial\!\!\!/_x + \Gamma^r \left(\partial_r + \frac{D_1}{4}A' + \frac{D_2}{4}B' \right) + \lambda\chi + \mathrm{e}^{-\frac{B}{2}}\ \Delta\!\!\!\!/_y \right\} \Psi = 0. \tag{12}$$

The details of the χ-field dynamics will not be important for our discussion. We shall only assume that its equation of motion admits a localized r-dependent solution such that $\chi(r) \to |v|\epsilon(r)$ as $|r| \to \infty$, where $v = \langle\chi\rangle$, and $\epsilon(r)$ is the sign function. With this assumption and imposing the chirality condition $\Gamma^r\Psi = +\Psi$, far away from the core region we need to solve

$$\begin{aligned} \not{\partial}_x\phi &= 0, \\ \left(\partial_r + \frac{D_1}{4}A' + \frac{D_2}{4}B' + \lambda|v|\epsilon(r)\right) f &= 0, \\ \not{\Delta}_y\psi &= 0. \end{aligned} \tag{13}$$

The solution of the above equation can be written as

$$\psi(x,y,r) = \mathrm{e}^{-\left(\frac{D_1}{4}A + \frac{D_2}{4}B\right) - \lambda|v|r\epsilon(r)} \cdot \psi(y)\phi(x), \tag{14}$$

where $\not{\Delta}_y\psi(y) = 0$.

Thus, to have localized $\Psi(x,y,r)$ it is sufficient that

$$-\frac{A}{2} - 2\lambda|v|r\epsilon(r) < 0. \tag{15}$$

This can be achieved for large enough values of $\lambda|v|$. For example, for solutions of Einstein equations with $A = B = cr\epsilon(r)$, where $c < 0$, that can be obtained for a string in 6 dimensions [10] or on $K = K3$ in higher dimensions [16], it is sufficient to have $\lambda|v| > -c/4$.

Now we come to the issue of chirality of the normalizable zero modes. First we note that for an even D_2 normalizable solutions of $\not{\Delta}_y\psi(y) = 0$ have definite chirality. The index theorem gives the difference $n_+ - n_-$, where n_+ and n_- are respectively the number of positive and negative chirality zero modes of the operator $\not{\Delta}_y$. Since we have imposed $\Gamma^r = 1$, the chiralities of $\psi(y)$ and $\phi(x)$ will be identical[2]. Thus the number of chiral families will be equal to $n_+ - n_-$. This mechanism is identical to the one which generates chiral fermions in the standard Kaluza-Klein compactification [17].

On the example of a $K = K_3$ compactification we obtain two chiral families, while for $K = S^4$ with an SU(2) instanton on it there will be $\frac{2}{3}t(t+1)(2t+1)$ chiral localized families in 4 dimensions, where t is the spin of the fermion representations. For $D = 7$ we can take $K = S^2$ with a U(1) magnetic monopole field of charge n on it. The number of chiral families will then be equal to n [18].

Conclusions. We defined conditions under which bulk fermions can be localized on a brane in models with localized gravity in higher dimensional generalizations of the RS model if only gauge and gravitational backgrounds are considered. We show how the domain-wall scalar field structures can insure localization and chirality at the same time. The number of chiral fermions is related to the topology of the manifold K and the gauge field background.

[2] This is due to the fact that for an odd D $\Gamma^r = \Gamma_{D_1} \cdot \Gamma_{D_2}$ where Γ_{D_1} and Γ_{D_2} are the chirality matrices in D_1 and D_2 dimensions.

It remains to be seen if one can find solutions which incorporate all the required features, namely, localized fields of various spins with the correct standard model quantum numbers in a non-singular background geometry. The non-singularity of the localized geometry seems to be rather difficult to achieve, at least without precence of a brane. It has been shown in [4] that for the metrics of the type given in eq.(1) which are regular at $r = 0$ the vacuum Einstein equations for $D_2 > 1$ produce generally singular solutions, although with a finite volume in the y, r subspace. It has been recently argued by Witten [19] that such naked singularities make the physical interpretation of these solutions problematic.

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