# Generalization of the $\mathcal{U}_q(gl(N))$ algebra and staggered models

**D. Arnaudon**[1], **A. Sedrakyan**[2], **T. Sedrakyan**[3], **P. Sorba**[4]

*Laboratoire d'Annecy-le-Vieux de Physique Théorique LAPTH*
*CNRS, UMR 5108, associée à l'Université de Savoie*
*BP 110, F-74941 Annecy-le-Vieux Cedex, France*

**Abstract**

We develop a technique of construction of integrable models with a $\mathbb{Z}_2$ grading of both the auxiliary (chain) and quantum (time) spaces. These models have a staggered disposition of the anisotropy parameter. The corresponding Yang–Baxter Equations are written down and their solution for the $gl(N)$ case are found. We analyze in details the $N = 2$ case and find the corresponding quantum group behind this solution. It can be regarded as quantum $\mathcal{U}_{q\mathcal{B}}(gl(2))$ group with a matrix deformation parameter $q\mathcal{B}$ with $(q\mathcal{B})^2 = q^2$. The symmetry behind these models can also be interpreted as the tensor product of the (-1)-Weyl algebra by an extension of $\mathcal{U}_q(gl(N))$ with a Cartan generator related to deformation parameter -1.

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[1]e-mail:*arnaudon@lapp.in2p3.fr*
[2]e-mail:*sedrak@lx2.yerphi.am, Permanent address: Yerevan Physics Institute, Armenia*
[3]e-mail:*tigrans@moon.yerphi.am, Permanent address: Yerevan Physics Institute, Armenia*
[4]e-mail:*sorba@lapp.in2p3.fr*

# 1 Introduction

In some physical problems [1, 2] we have to work with a partition function defined by the action on the Manhattan Lattice $(ML)$. The geometry of $(ML)$ automatically defines a chess like structure for the action. Therefore, when one defines the quantum chain Hamiltonian from this type of action by use of coherent states [3], as it was done in the article [1], we necessarily come to the idea of $\mathbb{Z}_2$ grading of the spaces along the chain and time directions. Precisely we have alternating auxiliary spaces in the chain direction and alternating quantum spaces in the time directions, defined as in article [4]

The aim of the present article is the definition of the corresponding $\mathbb{Z}_2$-structure, the formulation of the conditions (Yang–Baxter Equations $(YBE)$) under which we can construct an integrable model. We have found that the corresponding extended YBE's has a solution for general $gl(N)$ group and that integrable models with staggered parameter anisotropy can be constructed. We study in details the quantum group structure behind this construction for the case of $\mathcal{U}_q(gl(2))$. For spin-1/2 representation of $gl(2)$ group (XXZ chain) this type of model was defined and studied in the article [4] and for spin-1 degrees of freedom (anisotropic $t-J$ model) in [5].

# 2 Basic Definitions

Let us now consider $\mathbb{Z}_2$ graded quantum $V_{j,\rho}(v)$ (with $j = 1, .....N$ as a chain index) and auxiliary $V_{a,\sigma}(u)$ spaces, where $\rho, \sigma = 0, 1$ are the grading indices. Consider $R$-matrices, which act on the direct product of spaces $V_{a,\sigma}(u)$ and $V_{j,\rho}(v)$, $(\sigma, \rho = 0, 1)$, mapping them on the intertwined direct product of $V_{a,\bar{\sigma}}(u)$ and $V_{j,\bar{\rho}(v)}$ with the complementary $\bar{\sigma} = (1-\sigma)$, $\bar{\rho} = (1-\rho)$ indices

$$R_{aj,\sigma\rho}\,(u,v): \quad V_{a,\sigma}(u) \otimes V_{j,\rho}(v) \rightarrow V_{j,\bar{\rho}}(v) \otimes V_{a,\bar{\sigma}}(u). \tag{2.1}$$

Definition. *It is convenient to introduce two transmutation operations $\iota_1$ and $\iota_2$ with the property $\iota_1^2 = \iota_2^2 = id$ for the quantum and auxiliary spaces correspondingly, and to mark the operators $R_{aj,\sigma\rho}$ as follows*

$$\begin{aligned} R_{aj,00} &\equiv R_{aj}, \qquad R_{aj,01} \equiv R_{aj}^{\iota_1}, \\ R_{aj,10} &\equiv R_{aj}^{\iota_2}, \qquad R_{aj,11} \equiv R_{aj}^{\iota_1\iota_2}. \end{aligned} \tag{2.2}$$

The introduction of the $\mathbb{Z}_2$ grading of quantum spaces in time direction means, that we have now two monodromy operators $T_\rho, \rho = 0, 1$, which act on the space $V_\rho(u) = \prod_{j=1}^N V_{j,\rho}(u)$ by mapping it on $V_{\bar{\rho}}(u) = \prod_{j=1}^N V_{j,\bar{\rho}}(u)$

$$T_\rho(v,u) \qquad : V_\rho(u) \rightarrow V_{\bar{\rho}}(u), \qquad\qquad \rho = 0, 1. \tag{2.3}$$

It is clear now, that the monodromy operator of the model, which is defined by translational invariance in two steps in the time direction and determines the partition function, is the product of two monodromy operators

$$T(v,u) = T_0(v,u)T_1(v,u). \tag{2.4}$$

The $\mathbb{Z}_2$ grading of auxiliary spaces along the chain direction means that the $T_0(u,v)$ and $T_1(u,v)$ monodromy matrices are defined according to the following

Definition. *We define the monodromy operators $T_{0,1}(v,u)$ as a staggered product of the $R_{aj}(v,u)$ and $\bar{R}_{aj}^{\iota_2}(v,u)$ matrices:*

$$T_1(v,u) = \prod_{j=1}^N R_{a,2j-1}(v,u)\bar{R}_{a,2j}^{\iota_2}(v,u)$$

$$T_0(v,u)=\prod_{j=1}^{N}\bar{R}^{\iota_1}_{a,2j-1}(v,u)R^{\iota_1\iota_2}_{a,2j}(v,u), \tag{2.5}$$

where the notation $\bar{R}$ denotes a different parametrization of the $R(v,u)$-matrix via spectral parameters $v$ and $u$ and can be considered as an operation over $R$ with property $\bar{\bar{R}}=R$. For the integrable models where the intertwiner matrix $R(v-u)$ simply depends on the difference of the spectral parameters $v$ and $u$ this operation means the shift of its argument $u$ as follows

$$\bar{R}(u)=R(\bar{u}), \qquad \bar{u}=\theta-u, \tag{2.6}$$

where $\theta$ is an additional model parameter. We will consider this case in this paper.

# 3 Staggered Yang–Baxter equations

As it is well known in Bethe Ansatz Technique, the sufficient condition for the commutativity of transfer matrices $\tau(u)=TrT(u)$ with different spectral parameters is the YBE. For our case we have a two sets of equations [4]

$$R_{12}(u,v)\bar{R}^{\iota_1}_{13}(u)R_{23}(v)=R^{\iota_1}_{23}(v)\bar{R}_{13}(u)\tilde{R}_{12}(u,v) \tag{3.1}$$

and

$$\tilde{R}_{12}(u,v)R^{\iota_1\iota_2}_{13}(u)\bar{R}^{\iota_2}_{23}(v)=\bar{R}^{\iota_1\iota_2}_{23}(v)R^{\iota_2}_{13}(u)R_{12}(u,v)\ , \tag{3.2}$$

with $\bar{R}(u)\equiv R(\bar{u})$ and $R^{\iota_2}(u)=R^{\iota_1}(-u)$.

From $R(u)$ above, we follow a procedure which is the inverse of the Baxterisation (debaxterisation) [6]. Let

$$R_{12}(u)=\frac{1}{2i}\left(zR_{12}-z^{-1}R^{-1}_{21}\right) \tag{3.3}$$

with $z=e^{iu}$ and the constant $R_{12}$ and $R^{-1}_{21}$ matrices are spectral parameter independent. Then the Yang–Baxter equations (3.1)–(3.2) for the spectral parameter dependent $R$-matrix $R(u)$ and $R^{\iota_1}(u)$ are equivalent to the following equations for the constant $R$-matrices

$$\begin{aligned}
R_{12}R^{\iota_1}_{13}R_{23} &= R^{\iota_1}_{23}R_{13}R^{\iota_1}_{12} && (3.4)\\
R^{\iota_1}_{12}R_{13}R^{\iota_1}_{23} &= R_{23}R^{\iota_1}_{13}R_{12} && (3.5)\\
R_{12}\left(R^{\iota_1}_{31}\right)^{-1}R_{23}-\left(R_{21}\right)^{-1}R^{\iota_1}_{13}\left(R_{32}\right)^{-1} &= R^{\iota_1}_{23}\left(R_{31}\right)^{-1}R^{\iota_1}_{12}-\left(R^{\iota_1}_{32}\right)^{-1}R_{13}\left(R^{\iota_1}_{21}\right)^{-1} && (3.6)\\
R^{\iota_1}_{12}\left(R_{31}\right)^{-1}R^{\iota_1}_{23}-\left(R^{\iota_1}_{21}\right)^{-1}R_{13}\left(R^{\iota_1}_{32}\right)^{-1} &= R_{23}\left(R^{\iota_1}_{31}\right)^{-1}R_{12}-\left(R_{32}\right)^{-1}R^{\iota_1}_{13}\left(R_{21}\right)^{-1} && (3.7)
\end{aligned}$$

assuming $\tilde{R}=R^{\iota_1}$.

If this modified YBE's have a solution, then one can formulate a new integrable model on the basis of the existing ones. We will hereafter give solutions of these YBE's based on $\mathcal{U}_q(gl(N))$ $R_q$-matrices, for arbitrary $n$.

## 4 $\mathcal{U}_q(gl(2))$ case

As proved in [4] in connection with the staggered XXZ model, a solution of (3.1)–(3.2) is given by

$$R(u) = \begin{pmatrix} \sin(\lambda+u) & 0 & 0 & 0 \\ 0 & \sin(u) & e^{-iu}\sin(\lambda) & 0 \\ 0 & e^{iu}\sin(\lambda) & \sin(u) & 0 \\ 0 & 0 & 0 & \sin(\lambda+u) \end{pmatrix}, \tag{4.1}$$

$$R^{\iota_1}(u) = \begin{pmatrix} \sin(\lambda+u) & 0 & 0 & 0 \\ 0 & -\sin(u) & e^{-iu}\sin(\lambda) & 0 \\ 0 & e^{iu}\sin(\lambda) & -\sin(u) & 0 \\ 0 & 0 & 0 & \sin(\lambda+u) \end{pmatrix}. \tag{4.2}$$

(Notice that we introduced here the off-diagonal factors $e^{iu}$ and $e^{-iu}$ not present in [4] to allow the decomposition (3.3). They are nothing more than a rescaling of the states or a simple gauge transformation.)
A solution of (3.4)–(3.7) is then given by

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{4.3}$$

$$R^{\iota_1} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & q-q^{-1} & -1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{4.4}$$

where (4.3) is the usual $R$-matrix of $\mathcal{U}_q(gl(2))$.

### 4.1 Algebra

The YBE for $R$-matrices (3.1)–(3.2) define the corresponding YBE for $L$ and $L^{\iota}$-operators (the superscript $\iota$ appeared according to definitions by the formula (2.3)), which act on quantum space of the chain. According to formula (3.3) one can introduce $L_a^{\pm}$, $(a = 1, 2)$ operators as

$$L_a(u) = \frac{1}{2i}(zL_a^{+} - z^{-1}L_a^{-}). \tag{4.5}$$

Proposition. *The $R$-matrices (4.3) and (4.4) and the equations (3.4)–(3.7) lead to the following algebra, defined by the generators $L^{\pm}$, $(L^{\pm})^{\iota_1}$*

$$R_{12}L_1^{\pm\iota_1}L_2^{\pm} = L_2^{\pm\iota_1}L_1^{\pm}R_{12}^{\iota_1} \tag{4.6}$$
$$R_{12}L_1^{+\iota_1}L_2^{-} = L_2^{-\iota_1}L_1^{+}R_{12}^{\iota_1} \tag{4.7}$$

$$R_{12}^{\iota_1}L_1^{\pm}L_2^{\pm\iota_1} = L_2^{\pm}L_1^{\pm\iota_1}R_{12} \tag{4.8}$$
$$R_{12}^{\iota_1}L_1^{+}L_2^{-\iota_1} = L_2^{-}L_1^{+\iota_1}R_{12} \tag{4.9}$$

$$R_{12}L_1^{-\iota_1}L_2^{+} - (R_{21})^{-1}L_1^{+\iota_1}L_2^{-} = L_2^{+\iota_1}L_1^{-}R_{12}^{\iota_1} - L_2^{-\iota_1}L_1^{+}(R_{21}^{\iota_1})^{-1} \tag{4.10}$$
$$R_{12}^{\iota_1}L_1^{-}L_2^{+\iota_1} - (R_{21}^{\iota_1})^{-1}L_1^{+}L_2^{-\iota_1} = L_2^{+}L_1^{-\iota_1}R_{12} - L_2^{-}L_1^{+\iota_1}(R_{21})^{-1}. \tag{4.11}$$

Writing the operators $L^{\pm}$ as usual in the form

$$L^+ = \begin{pmatrix} K_{+1} & 0 \\ E & K_{+2} \end{pmatrix}, \qquad L^- = \begin{pmatrix} K_{-1} & F \\ 0 & K_{-2} \end{pmatrix} \tag{4.12}$$

and similarly for $L^{\pm\iota_1}$, we get the relations

$$\begin{array}{cc}
K^{\iota_1}_{+1}K_{-1} = K^{\iota_1}_{-1}K_{+1} & K_{+1}K^{\iota_1}_{-1} = K_{-1}K^{\iota_1}_{+1} \\
K^{\iota_1}_{+2}K_{-2} = K^{\iota_1}_{-2}K_{+2} & K_{+2}K^{\iota_1}_{-2} = K_{-2}K^{\iota_1}_{+2} \\
K^{\iota_1}_{+1}K_{+2} = -K^{\iota_1}_{+2}K_{+1} & K_{+1}K^{\iota_1}_{+2} = -K_{+2}K^{\iota_1}_{+1} \\
K^{\iota_1}_{-1}K_{-2} = -K^{\iota_1}_{-2}K_{-1} & K_{-1}K^{\iota_1}_{-2} = -K_{-2}K^{\iota_1}_{-1} \\
K^{\iota_1}_{+1}K_{-2} = -K^{\iota_1}_{-2}K_{+1} & K_{+1}K^{\iota_1}_{-2} = -K_{-2}K^{\iota_1}_{+1} \\
K^{\iota_1}_{+2}K_{-1} = -K^{\iota_1}_{-1}K_{+2} & K_{+2}K^{\iota_1}_{-1} = -K_{-1}K^{\iota_1}_{+2}
\end{array} \tag{4.13}$$

$$\begin{array}{cc}
K^{\iota_1}_{+1}E = qE^{\iota_1}K_{+1} & K_{+1}E^{\iota_1} = -qEK^{\iota_1}_{+1} \\
K^{\iota_1}_{-1}E = q^{-1}E^{\iota_1}K_{-1} & K_{-1}E^{\iota_1} = -q^{-1}EK^{\iota_1}_{-1} \\
K^{\iota_1}_{+2}E = -q^{-1}E^{\iota_1}K_{+2} & K_{+2}E^{\iota_1} = q^{-1}EK^{\iota_1}_{+2} \\
K^{\iota_1}_{-2}E = -qE^{\iota_1}K_{-2} & K_{-2}E^{\iota_1} = qEK^{\iota_1}_{-2}
\end{array} \tag{4.14}$$

$$\begin{array}{cc}
K^{\iota_1}_{+1}F = -q^{-1}F^{\iota_1}K_{+1} & K_{+1}F^{\iota_1} = q^{-1}FK^{\iota_1}_{+1} \\
K^{\iota_1}_{-1}F = -qF^{\iota_1}K_{-1} & K_{-1}F^{\iota_1} = qFK^{\iota_1}_{-1} \\
K^{\iota_1}_{+2}F = qF^{\iota_1}K_{+2} & K_{+2}F^{\iota_1} = -qFK^{\iota_1}_{+2} \\
K^{\iota_1}_{-2}F = q^{-1}F^{\iota_1}K_{-2} & K_{-2}F^{\iota_1} = -q^{-1}FK^{\iota_1}_{-2}
\end{array} \tag{4.15}$$

$$\begin{array}{l}
E^{\iota_1}F + F^{\iota_1}E = (q-q^{-1})\left(K^{\iota_1}_{-1}K_{+2} - K^{\iota_1}_{+1}K_{-2}\right)\ , \\
EF^{\iota_1} + FE^{\iota_1} = -(q-q^{-1})\left(K_{-1}K^{\iota_1}_{+2} - K_{+1}K^{\iota_1}_{-2}\right).
\end{array} \tag{4.16}$$

These algebraic relations look like those defining $\mathcal{U}_q(gl(2))$, although important sign differences appear, in particular in the exchange relations of the generators $K$. We will refer to it below as algebra $\mathcal{A}$. Now we define the following quadratic operators

$$\begin{array}{rclcrcl}
e &=& K^{\iota_1}_1 E & & f &=& K^{\iota_1}_2 F \\
k_1 &=& K^{\iota_1}_1 K_1 & & k_2 &=& K^{\iota_1}_2 K_2 \\
l_1 &=& K^{\iota_1}_1 K_{-1} & & l_2 &=& K^{\iota_1}_2 K_{-2} \\
m &=& K^{\iota_1}_1 K_2 & & & &
\end{array} \tag{4.17}$$

It is easy to check that they satisfy the relations

$$\begin{array}{rclcrcl}
k_1 e &=& -q^2 e k_1 & & k_1 f &=& -q^{-2} f k_1 \\
k_2 e &=& -q^{-2} e k_2 & & k_2 f &=& -q^2 f k_2 \\
[e,f] &=& (q^2-1)(k_1 l_2 - k_2 l_1) & & & & \\
l_i e &=& -e l_i & & l_i f &=& -f l_i \\
m e &=& -e m & & m f &=& -f m.
\end{array} \tag{4.18}$$

The Cartan subalgebra of this deformed algebra is defined by the generators $k_1, l_1$ and $m$, while $k_2, l_2$ are being fixed by central operators $l_i^2$, $k_1 k_2 = -m^2$, $l_1 l_2$, set to 1. Therefore these relations correspond to the $\mathcal{U}_{q,i}(gl(2))$ algebra, the algebra with two deformation parameters $q$, $-1$. Usual definitions of multiparameter deformations of the enveloping algebra $\mathcal{U}(gl(2))$ were considered in the articles [7].

We can return back to the whole set of generators of the original algebra (4.13)–(4.16), inverting the definitions (4.17)

$$\begin{array}{rclrcl} E &=& K_{+1}k_1^{-1}e & F &=& -K_{+1}m^{-1}f \\ K_{-1} &=& K_{+1}k_1^{-1}l_1 & K_{+2} &=& K_{+1}k_1^{-1}m \\ K_{-2} &=& -K_{+1}m^{-1}l_2 & E^{\iota_1} &=& q^{-1}eK_{+1}^{-1} \\ F^{\iota_1} &=& qk_1m^{-1}fK_{+1}^{-1} & K_{+1}^{\iota_1} &=& k_1K_{+1}^{-1} \\ K_{-1}^{\iota_1} &=& l_1K_{+1}^{-1} & K_{+2}^{\iota_1} &=& -mK_{+1}^{-1} \\ K_{-2}^{\iota_1} &=& k_1m^{-1}l_2K_{+1}^{-1}. & & & \end{array} \tag{4.19}$$

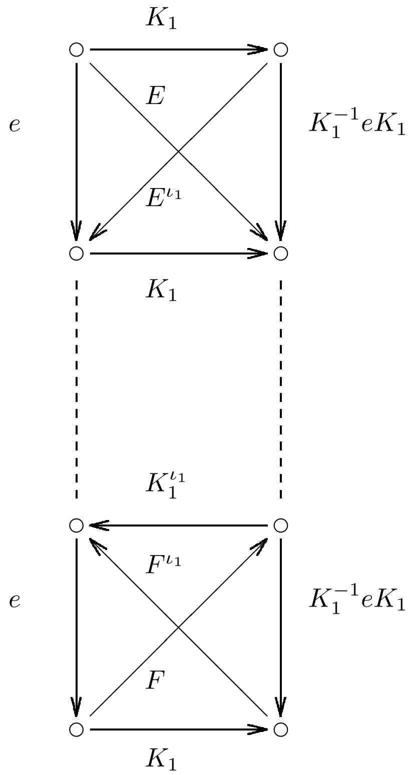
For convenience, we define $E' = qE^{\iota_1}$, $F' = -q^{-1}F^{\iota_1}$, $K'_{\pm 1} = K^{\iota_1}_{\pm 1}$, $K'_{\pm 2} = -K^{\iota_1}_{\pm 2}$ and write all the generators $A$ and $A'$ in the form

$$\begin{array}{rcl} A &=& K_1 a\ , \\ A' &=& k_1 a K_1^{-1}\ . \end{array} \tag{4.20}$$

Let $\mathcal{W}$ be the associative algebra generated by $X$, $Z$, such that $X^2 = Z^2 = 1$ and $XZ + ZX = 0$. ($\mathcal{W}$ is equivalent to the algebra satisfied by the Pauli matrices, which can be written $\sigma_1 = X$, $\sigma_3 = Z$, and hence $\sigma^+ = (1+Z)X/2$, $\sigma^- = (1-Z)X/2$. Then

THEOREM. *The deformed algebra $\mathcal{A}$, defined by the relations (4.13)–(4.16) is isomorphic to the tensor product $\mathcal{W} \otimes \mathcal{U}_{q,i}(gl(2))$*

According to this theorem, the ordinary highest weight irreducible representations of our algebra $\mathcal{A}$ are formed by direct product of $\mathbb{C}^2$ with any irreducible representation of the deformed algebra $\mathcal{U}_{q,i}(gl(2))$. One can draw the picture as below expressing this fact.




We have here two columns of states forming irreps of $\mathcal{U}_{q,i}(gl(2))$ with highest weight states $v_0$ and $v_1$. The operators marked by the capital Latin letters $(K, E, F...)$ maps from one column to the other, while small letter operators $(e, f, k...)$ act inside the columns.

It is known that in the case when $q$ is the root of unity the quantum groups have a so called periodic (and semi-periodic) representations (which are absent in the Lie algebras case). From the above construction of the representations, it follows that when $q^r = 1$ with $r \neq 4s, s = 1, 2...$, the periodic representations here will not be a product of two irreps of $\mathcal{W}$ and $\mathcal{U}_{q,i}(gl(2))$, but will form a joint irrep of double size of one for the $\mathcal{U}_{q,i}(gl(2))$ only.

## 4.2 Alternative description

Let us introduce now the direct sum of spaces $V_0$ and $V_1$ (formula (2.3)) as $V = V_0 \oplus V_1$ and consider the following operators acting there.

Definition. *Let us define*

$$\mathcal{K}_{\pm 1} = \begin{pmatrix} 0 & K_{\pm 1} \\ K_{\pm 1}^{\iota_1} & 0 \end{pmatrix} \qquad \mathcal{K}_{\pm 2} = \begin{pmatrix} 0 & K_{\pm 2} \\ -K_{\pm 2}^{\iota_1} & 0 \end{pmatrix} \tag{4.21}$$

$$\mathcal{E} = \begin{pmatrix} 0 & E \\ -E^{\iota_1} & 0 \end{pmatrix} \qquad \mathcal{F} = \begin{pmatrix} 0 & F \\ F^{\iota_1} & 0 \end{pmatrix} \tag{4.22}$$

$$\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \mathcal{B}^2 = 1 \tag{4.23}$$

After some simple matrix calculations we obtain the following

PROPOSITION. *The deformed algebra defined by the relations (4.13)–(4.16) can be represented as*

$$\begin{array}{rclrcl}
\mathcal{K}_{\pm1}\mathcal{K}_{\pm2} & = & \mathcal{K}_{\pm2}\mathcal{K}_{\pm1} & \mathcal{B}\mathcal{K}_i & = & -\mathcal{K}_i\mathcal{B} \\
\mathcal{B}\mathcal{E} & = & -\mathcal{E}\mathcal{B} & \mathcal{B}\mathcal{F} & = & -\mathcal{F}\mathcal{B} \\
\mathcal{K}_{\pm1}\mathcal{E} & = & q^{\pm1}\mathcal{B}\mathcal{E}\mathcal{K}_{\pm1} & \mathcal{K}_{\pm2}\mathcal{E} & = & q^{\mp1}\mathcal{B}\mathcal{E}\mathcal{K}_{\pm2} \\
\mathcal{K}_{\pm1}\mathcal{F} & = & q^{\mp1}\mathcal{B}\mathcal{F}\mathcal{K}_{\pm1} & \mathcal{K}_{\pm2}\mathcal{F} & = & q^{\pm1}\mathcal{B}\mathcal{F}\mathcal{K}_{\pm2} \\
[\mathcal{E},\mathcal{F}] & = & (q-q^{-1})\mathcal{B}(\mathcal{K}_2\mathcal{K}_{-1}-\mathcal{K}_1\mathcal{K}_{-2}) & & &
\end{array} \tag{4.24}$$

One can recognize in this relations the $\mathcal{U}_q(gl(2))$ algebra, but instead of the ordinary deformation parameter $q$ as a complex number we have here a deformation matrix $q\mathcal{B}$ with the property $(q\mathcal{B})^2 = q^2$. Therefore it is reasonable to mark algebra $\mathcal{A}$ as $sl_{q\mathcal{B}}(2)$.

# 5 Co-algebra structure

Because of the nature of the algebra $\mathcal{A}$, defined by two $R$-matrices and two sets of generators (with and without ${}^{\iota_1}$), we cannot imagine a coproduct

$$\begin{array}{rcl}\Delta & : & \mathcal{A} \longrightarrow \mathcal{A}\otimes\mathcal{A} \\ & & L \longmapsto L\,\dot{\otimes}\,L\end{array} \tag{5.1}$$

as a morphism from $\mathcal{A}$ to $\mathcal{A}\otimes\mathcal{A}$. Actually, the operators obtained as $L\,\dot{\otimes}\,L^{\iota_1}$ satisfy the commutation relations of $\mathcal{U}_q(gl(2))$, and we will use this fact later.
We can however define two (related) notions:

## 5.1 Coproduct $\Delta^{(3)}$

DEFINITION. *Define*

$$\begin{array}{rcl}\Delta^{(3)} & : & \mathcal{A} \longrightarrow \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \\ & & L \longmapsto L\,\dot{\otimes}\,L^{\iota_1}\,\dot{\otimes}\,L\end{array} \tag{5.2}$$

*i.e.*

$$\Delta^{(3)}(K_1) = K_1\otimes K_1^{\iota_1}\otimes K_1 \tag{5.3}$$

$$\Delta^{(3)}(K_2) = K_2\otimes K_2^{\iota_1}\otimes K_2 \tag{5.4}$$

$$\Delta^{(3)}(E) = E\otimes K_1^{\iota_1}\otimes K_1 + K_2\otimes E^{\iota_1}\otimes K_1 + K_2\otimes K_2^{\iota_1}\otimes E \tag{5.5}$$

$$\Delta^{(3)}(F) = K_{-1}\otimes K_{-1}^{\iota_1}\otimes F + K_{-1}\otimes F^{\iota_1}\otimes K_{-2} + F\otimes K_{-2}^{\iota_1}\otimes K_{-2} \tag{5.6}$$

PROPOSITION. $\Delta^{(3)}$ *is a morphism of algebras.*

## 5.2 Left co-action of $\mathcal{U}_q(gl(2))$ on $\mathcal{A}$

Let us define

$$\underline{\Delta} : \mathcal{A} \longrightarrow \mathcal{U}_{q,i}(gl(2))\otimes\mathcal{A} \tag{5.7}$$

by

$$\begin{aligned}
\underline{\Delta}(\mathcal{K}_1) &= k_1 \otimes \mathcal{K}_1 &(5.8)\\
\underline{\Delta}(\mathcal{K}_2) &= k_2 \otimes \mathcal{K}_2 &(5.9)\\
\underline{\Delta}(\mathcal{E}) &= e \otimes \mathcal{K}_1 + k_2 \otimes \mathcal{E} &(5.10)\\
\underline{\Delta}(\mathcal{F}) &= k_1^{\iota_1} \otimes \mathcal{F} + f \otimes \mathcal{K}_2^{\iota_1} &(5.11)
\end{aligned}$$

PROPOSITION. $\underline{\Delta}$ *is a morphism of algebras.*

# 6 Generalization of $gl(N)$

Let us consider now the $gl(N)$ case. The two constant $R$-matrices $R$ and $R^{\iota_1}$ given by

$$R = \sum_{i=1}^{N} q e_{ii} \otimes e_{ii} + \sum_{\substack{i,j=1\\ i\neq j}}^{N} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{\substack{i,j=1\\ i>j}}^{N} e_{ij} \otimes e_{ji} \tag{6.1}$$

$$R^{\iota_1} = \sum_{i=1}^{N} q e_{ii} \otimes e_{ii} + \sum_{\substack{i,j=1\\ i\neq j}}^{N} b_{ij} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{\substack{i,j=1\\ i>j}}^{N} e_{ij} \otimes e_{ji} \tag{6.2}$$

satisfy the four equations (3.4)–(3.7) provided that

$$b_{ij} = b_{ik} b_{kj} \qquad \text{and} \qquad b_{ij}^2 = 1 \tag{6.3}$$

This cocycle condition allows to write $b_{ij} = b_i b_j^{-1}$, with $b_1 = 1$ and $b_i = \pm 1$ for $i > 1$, thus yielding several solutions. Let us now define the matrices

$$\mathcal{B}_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & b_{ij} \end{pmatrix} = \mathcal{B}_i \mathcal{B}_j^{-1}\,, \qquad \mathcal{B}_i = \begin{pmatrix} 1 & 0 \\ 0 & b_i \end{pmatrix}. \tag{6.4}$$

The $RLL$ relations (4.6) together with a gathering of the operators as in (4.12) lead to relations close to that of $\mathcal{U}_q(gl(N))$ with some sign modifications encoded in the $b'_{ij}s$. We denote by $\mathcal{A}$ or $\mathcal{A}_{q,\{b_j\}}$ the algebra generated by the $L$ and $L^{\iota_1}$ operators.
We gather the operators $L^{\pm}$ and $L^{\pm\iota_1}$ in matrices

$$\mathcal{L}^{\pm}_{ij} = \begin{pmatrix} 0 & {L^{\pm}}_{ij} \\ b_i {L^{\pm\iota_1}}_{ij} & 0 \end{pmatrix} \tag{6.5}$$

which acts on a space $V = V_0 \oplus V_1$. The equations $RL_1^{\pm\iota_1} L_2^{\pm} = L_2^{\pm} L_1^{\pm\iota_1} R^{\iota_1}$ and $R^{\iota_1} L_1^{\pm} L_2^{\pm\iota_1} = L_2^{\pm\iota_1} L_1^{\pm} R$ then read as

$$\begin{aligned}
&\mathcal{B}_{im}\mathcal{L}^{\pm}_{im}\mathcal{L}^{\pm}_{in} = q\mathcal{B}_{in}\mathcal{L}^{\pm}_{in}\mathcal{L}^{\pm}_{im} && m<n\\
&\mathcal{B}_{im}\mathcal{L}^{\pm}_{im}\mathcal{L}^{\pm}_{jm} = q\mathcal{B}_{jm}\mathcal{L}^{\pm}_{jm}\mathcal{L}^{\pm}_{im} && i<j\\
&\mathcal{B}_{im}\mathcal{L}^{\pm}_{im}\mathcal{L}^{\pm}_{jn} = \mathcal{B}_{jn}\mathcal{L}^{\pm}_{jn}\mathcal{L}^{\pm}_{im} + (q-q^{-1})\,\mathcal{B}_{jm}\mathcal{L}^{\pm}_{jm}\mathcal{L}^{\pm}_{in} && i<j,\ m<n\\
&\mathcal{B}_{im}\mathcal{L}^{\pm}_{im}\mathcal{L}^{\pm}_{jn} = \mathcal{B}_{jn}\mathcal{L}^{\pm}_{jn}\mathcal{L}^{\pm}_{im} && i<j,\ m>n\\
&\mathcal{B}_{im}\mathcal{L}^{\pm}_{im}\mathcal{L}^{\pm}_{jn} = \mathcal{B}_{jn}\mathcal{L}^{\pm}_{jn}\mathcal{L}^{\pm}_{im} && i>j,\ m<n\\
&\mathcal{B}_{im}\mathcal{L}^{\pm}_{im}\mathcal{L}^{\pm}_{jn} + (q-q^{-1})\,\mathcal{B}_{jm}\mathcal{L}^{\pm}_{jm}\mathcal{L}^{\pm}_{in} = \mathcal{B}_{jn}\mathcal{L}^{\pm}_{jn}\mathcal{L}^{\pm}_{im} && i>j,\ m>n
\end{aligned} \tag{6.6}$$

while the equations $RL_1^{+\iota_1}L_2^- = L_2^- L_1^{+\iota_1} R^{\iota_1}$ and $R^{\iota_1} L_1^+ L_2^{-\iota_1} = L_2^{-\iota_1} L_1^+ R$ give

$$
\begin{array}{ll}
q\mathcal{B}_{im}\mathcal{L}^+_{im}\mathcal{L}^-_{im} = q\mathcal{B}_{im}\mathcal{L}^-_{im}\mathcal{L}^+_{im} & \\
\mathcal{B}_{im}\mathcal{L}^+_{im}\mathcal{L}^-_{in} = q^{-1}\mathcal{B}_{in}\mathcal{L}^-_{in}\mathcal{L}^+_{im} + q^{-1}(q-q^{-1})\,\mathcal{B}_{im}\mathcal{L}^-_{im}\mathcal{L}^+_{in} & m<n \\
\mathcal{B}_{im}\mathcal{L}^+_{im}\mathcal{L}^-_{in} = q^{-1}\mathcal{B}_{in}\mathcal{L}^-_{in}\mathcal{L}^+_{im} & m>n \\
q^{-1}\mathcal{B}_{im}\mathcal{L}^+_{im}\mathcal{L}^-_{jm} = \mathcal{B}_{jm}\mathcal{L}^-_{jm}\mathcal{L}^+_{im} & i<j \\
q^{-1}\mathcal{B}_{im}\mathcal{L}^+_{im}\mathcal{L}^-_{jm} + q^{-1}(q-q^{-1})\,\mathcal{B}_{jm}\mathcal{L}^+_{jm}\mathcal{L}^-_{im} = \mathcal{B}_{jm}\mathcal{L}^-_{jm}\mathcal{L}^+_{im} & i>j \\
\mathcal{B}_{im}\mathcal{L}^+_{im}\mathcal{L}^-_{jn} = \mathcal{B}_{jn}\mathcal{L}^-_{jn}\mathcal{L}^+_{im} + (q-q^{-1})\,\mathcal{B}_{jm}\mathcal{L}^-_{jm}\mathcal{L}^+_{in} & i<j,\ m<n \\
\mathcal{B}_{im}\mathcal{L}^+_{im}\mathcal{L}^-_{jn} = \mathcal{B}_{jn}\mathcal{L}^-_{jn}\mathcal{L}^+_{im} & i<j,\ m>n \\
\mathcal{B}_{im}\mathcal{L}^+_{im}\mathcal{L}^-_{jn} + (q-q^{-1})\,\mathcal{B}_{jm}\mathcal{L}^+_{jm}\mathcal{L}^-_{in} = \mathcal{B}_{jn}\mathcal{L}^-_{jn}\mathcal{L}^+_{im} + (q-q^{-1})\,\mathcal{B}_{jm}\mathcal{L}^-_{jm}\mathcal{L}^+_{in} & i>j,\ m<n \\
\mathcal{B}_{im}\mathcal{L}^+_{im}\mathcal{L}^-_{jn} + (q-q^{-1})\,\mathcal{B}_{jm}\mathcal{L}^+_{jm}\mathcal{L}^-_{in} = \mathcal{B}_{jn}\mathcal{L}^-_{jn}\mathcal{L}^+_{im} & i>j,\ m>n
\end{array}
\tag{6.7}
$$

It is easy to see that if $\mathcal{B}_{ij}=1$, i.e. if all $b_i$ are 1, then this set of algebraic relations simply becomes the set of definition relations of the quantum algebra $\mathcal{U}_q(gl(N))$.
In order to extract the $\mathcal{U}_q(gl(N))$ part of this equations, let us introduce the operator $\mathcal{M}$, defined by the relation

$$\mathcal{M}\mathcal{B}_{ij}\mathcal{L}_{ij} = \mathcal{L}_{ij}\mathcal{B}_{ij}. \tag{6.8}$$

The simple expertise of the equations (6.6) and (6.7) shows that the $\mathcal{B}_{ij}$ matrices always appear there in the first position of the products and with the same indices as the first operator $\mathcal{L}_{ij}$. Therefore, by multiplying all equations by $\mathcal{M}$ from the left and right hand sides and using the relation (6.8) one can absorb the matrices $\mathcal{B}_{ij}$ into operators $\bar{\mathcal{L}}_{ij} = \mathcal{L}_{ij}\mathcal{M}$, for which we get the set of $\mathcal{U}_q(gl(N))$ defining relations.

The solution of the equations (6.8) can be written in the following form. Let us denote as before $l_1 = K_1^{\iota_1}K_{-1} = L_{11}^{+\,\iota_1}L_{11}^-$, $k_1 = K_1^{\iota_1}K_1 = L_{11}^{+\,\iota_1}L_{11}^+$ and also $l_1^{\iota_1} = K_1K_{-1}^{\iota_1}$, $k_1^{\iota_1} = K_1K_1^{\iota_1}$. We will consider in the following that the central operator $l_1^2$ is equal to 1. One can check by direct calculations that the operator

$$\mathcal{M} = \begin{pmatrix} 0 & K_1(k_1l_1)^{-1/2} \\ (k_1l_1)^{-1/2}K_1^{\iota_1} & 0 \end{pmatrix}$$

is fulfilling the equations (6.8) for all pairs $(i,j)$.

The operators $\mathcal{L}_{ij}\mathcal{M}$ have the form $\mathbb{1}\otimes\bar{L}_{ij}$, with $\bar{L}_{ij}$ a generator of standard $\mathcal{U}_q(gl(N))$. Therefore we have proved the following

PROPOSITION. *The algebra $\mathcal{A}_{q,\{b_j\}}$ generated by the relations (6.6) and (6.7) is equivalent to $(\mathbb{1}\otimes\mathcal{U}_q(gl(N)))$ extended by the operator $\mathcal{M}$, satisfying the relations (6.8) with the generators of $(\mathbb{1}\otimes\mathcal{U}_q(gl(N)))$. This supplementary operator can be regarded as an additional Cartan generator, encoding both the deformation parameters $\{b_j\}$ and the doubling of the representation spaces.*

Since the operator $\mathcal{M}$ does not commute with the rest $\mathcal{U}_q(gl(N))$, it seems that the algebra $\mathcal{A}_{q,\{b_j\}}$ can not be represented as a direct product. However one can introduce the operators $I \equiv K_1(k_1l_1)^{-1/2}$ and $I^{\iota_1} \equiv (k_1l_1)^{-1/2}K_1^{\iota_1}l_1^{\iota_1}$ (these operators are always defined in finite dimensional representations) satisfying $I^{\iota_1}I + II^{\iota_1} = 1$ and note that

$$\mathcal{M} = \begin{pmatrix} 0 & I \\ I^{\iota_1} & 0 \end{pmatrix}\begin{pmatrix} l_1^{\iota_1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & l_1 \end{pmatrix}\begin{pmatrix} 0 & I \\ I^{\iota_1} & 0 \end{pmatrix}. \tag{6.9}$$

Let us now consider $\mathcal{U}_{q,\{b_j\}}(gl(N))$, defined as $\mathcal{U}_q(gl(N))$ extended by addition of the generator

$$\bar{l}_1 \equiv \prod_i b_i^{h_{\omega_{i+1}}-h_{\omega_i}} \ . \tag{6.10}$$

In this last expression, $h_{\omega_i}$ is the Cartan generator related with the fundamental weight $\omega_i = \sum (A^{-1})_{ij} h_j$. The commutation properties of $\bar{l}_1$ with the generators of $\mathcal{U}_q(gl(N))$ are naturally deduced from the expression (6.10), taking the usual relations involving the Cartan $h_i$'s (however not supposed to be themselves in $\mathcal{U}_q(gl(N))$). The generator $\bar{l}_1$ will be responsible for the additional deformation parameter $-1$ in $\mathcal{U}_{q,\{b_j\}}(gl(N))$[5], the parameters $b_i$ encoding which directions of the Cartan generators are concerned with this deformation. (Note that the existence of this new generator may change the classification of representations at roots of unity, the size of some periodic representations being for instance enlarged).

PROPOSITION. *The algebra generated by $L_{ij}^{\pm}$ and $L_{ij}^{\pm\iota_1}$ is equivalent to $\mathcal{W}\otimes\mathcal{U}_{q,\{b_j\}}(gl(N))$ with $\mathcal{W}$ defined in section 4.*
The equivalence is indeed provided by the isomorphism

$$\begin{aligned} \phi(\sigma^+ \otimes \mathbb{I}) &= I && (6.11)\\ \phi(\sigma^- \otimes \mathbb{I}) &= I^{\iota_1} && (6.12)\\ \phi(\mathbb{I}\otimes \bar{L}_{ij}) &= L'_{ij} I^{\iota_1} + I^{\iota_1} L_{ij} && (6.13)\\ \phi(\mathbb{I}\otimes \bar{l}_1) &= l_1 + l_1^{\iota_1} && (6.14) \end{aligned}$$

where $\bar{L}_{ij}$ denote the generators of the standard $\mathcal{U}_q(gl(N))$, that we extend with $\bar{l}_1$ defined as above, and $L'_{ij} \equiv b_i L_{ij}^{\iota_1} l_1^{\iota_1}$. We can check that $[\phi(\sigma^{\pm}\otimes\mathbb{I}), \phi(\mathbb{I}\otimes\bar{L}_{ij})] = 0$.

At the end let us make the following remark. It is easy to find out from the algebraic equations (6.6) and (6.7) that multiplying some of them by $\mathcal{B}_{im}$ we can bring all of them to the form, where $\mathcal{B}_{ij}$'s appears only coupled with $q$. Then, as in the $\mathcal{U}_q(gl(2))$ case, one can talk about a quantum algebra with matrix valued deformation parameters $q\mathcal{B}_{ij}$, $((q\mathcal{B}_{ij})^2 = q^2)$.

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