KIAS-P01034
hep-th/0108145

On a Matrix Model of Level Structure

Jeong-Hyuck Park *

Korea Institute for Advanced Study
207-43 Cheongryangri-dong, Dongdaemun-gu
Seoul 130-012, Korea

Abstract

We generalize the dimensionally reduced Yang-Mills matrix model by adding $d=1$ Chern-Simons term and terms for a bosonic vector. The coefficient, κ of the Chern-Simons term must be integer, and hence the level structure. We show at the bottom of the Yang-Mills potential, the low energy limit, only the linear motion is allowed for D0 particles. Namely all the particles align themselves on a single straight line subject to κ^2/r^2 repulsive potential from each other. We argue the relevant brane configuration to be D0-branes in a D4 after κ of D8's pass the system.

*Electronic correspondence: jhp@kias.re.kr

1 Introduction

In string theory $N \times N$ matrix models describe the dynamics of N D0-branes. In particular, according to the renowned conjecture by Banks, Fischler, Shenker and Susskind [1], the infinite momentum limit of M-theory is exactly described by the large N limit of U(N) Yang-Mills matrix quantum mechanics. In the limit only degrees of freedom are D0 particles and other higher dimensional extended objects appear as composites of D0-branes. In fact, the dimensionally reduced Yang-Mills action can be regarded as the matrix regularization of the theory of membrane which is the fundamental object in M-theory [2]. On the other hand, considering finite number of D0 particles with various extended objects as background in type IIA string theory may require more generalized form of the matrix action.

An interesting phenomenon for D0 particles near a D8-brane is that, every time the D0 particle crosses the D8-brane, a fundamental string is created connecting those two D-branes [3]. This is in fact U-dual to the Hanany-Witten effect [4]. For κ of D8-branes there appear κ number of fundamental strings and this induces a $d = 1$ Chern-Simons term of level κ in the D0-brane action

$$\kappa \, \mathrm{tr} A_0 \, . \tag{1.1}$$

This Chern-Simons term can be also regarded as a chemical potential for string ends [5].

However adding this term alone in the dimensionally reduced Yang-Mills action causes a problem. In general, at quantum level, the Gauss constraint generates unitary transformations, $U = e^{i\Lambda}$, $\Lambda^\dagger = \Lambda$ on all the arguments in the wavefunction [6]

$$|\Psi'\rangle = e^{i\kappa \mathrm{tr}\Lambda}|\Psi\rangle \, . \tag{1.2}$$

Taking the particular choice, $\Lambda = \mathrm{diag}(2\pi, 0, 0, \cdots, 0)$ gives the identity matrix, $U = 1_{N\times N}$ and the Gauss constraint on wavefunctions successfully works only for integer, κ, leading the level quantization as in noncommutative Chern-Simons theories [7]. This is also consistent with our interpretation of κ as the number of D8-branes. Now the problem is as follows. As all the arguments are matrices meaning they are in the adjoint representation, the overall U(1) transformation will leave the wavefunction invariant, and this is clearly inconsistent with the Gauss constraint for non-zero κ. In fact, any SU(N) singlet wavefunction would be automatically U(N) singlet too [8].

Curing the problem above requires the presence of arguments in other representation. In this paper, we consider a bosonic complex vector, ψ in the fundamental representation as introduced by Polychronakos in the context of quantum Hall physics via a finite matrix Chern-Simons theory [9]. Therefore the action we consider contains the dimensionally reduced Yang-Mills terms, $d=1$ Chern-Simons term of level κ and terms for the vector

$$\mathcal{S} = \int \mathrm{d}t\, \mathrm{tr}\left(\frac{1}{2}\frac{1}{R} D_t X^I D_t X^I - \frac{1}{4}\frac{R}{l_p^6}[X^I, X^J]^2 + \kappa A_0 + i D_t\psi\psi^\dagger - V(\psi\psi^\dagger) \right), \tag{1.3}$$

where $1 \leq I, J \leq D$ are space indices, R is the compactification radius and l_p is the Planck length. As we are mainly interested in the low energy dynamics in this paper, the kinetic term for ψ is taken here as nonrelativistic in the sense that it is linear in time derivative as in the Jackiw-Pi model for the nonrelativistic Chern-Simons vortices [12]. In the last section we will be back to this point.

To realize the relevant brane configuration, one needs a D4-brane, as the light D0-D4 string has both fermionic and bosonic modes while the D0-D8 string is fermionic only [8, 10]. The argument is due to Susskind and Hellerman [11, 5]. Consider D0-branes in a D4-brane and let D8-branes move far away after passing the D0-D4 system. The D0-D8 fundamental strings arising from the Hanany-Witten effect jump one ends from D8's to the D4 forming D0-D4 strings, and this allows bosonic modes for the strings. We may then regard ψ as the effective bosonic degrees. In this picture we put $D=4$.

The covariant derivatives are in our convention

$$D_t X = \tfrac{d}{dt}X + i[A_0, X]\,, \qquad D_t\psi = \tfrac{d}{dt}\psi + iA_0\psi\,, \tag{1.4}$$

and the U(N) gauge symmetry is given by

$$X \to UXU^{-1}, \quad \psi \to U\psi, \quad A_0 \to UA_0U^{-1} + i\tfrac{d}{dt}UU^{-1}. \tag{1.5}$$

The Hamiltonian is with $P^I = \frac{1}{R}D_t X^I$

$$H = \mathrm{tr}\left(\frac{R}{2}P^I P^I + \frac{1}{4}\frac{R}{l_p^6}[X^I, X^J]^2 + V(\psi\psi^\dagger) \right). \tag{1.6}$$

The classical equations of motion are

$$D_t D_t X^I = -\frac{R^2}{l_p^6}[[X^I, X^J], X^J]\,, \quad iD_t\psi = V'(\psi^\dagger\psi)\psi\,, \tag{1.7}$$

and the Gauss constraint is

$$-i[X^I, P^I] + \psi\psi^\dagger = \kappa\, 1_{N\times N}. \tag{1.8}$$

Note that without ψ the Gauss constraint would be problematic at classical level too, since taking trace of it would require $\kappa = 0$. In fact this is the original reason ψ was introduced in [9]. Quantum mechanically it follows that the general wavefunction satisfying the Gauss constraint is of the form (see e.g. [13])

$$|\Psi\rangle = S\left(\mathrm{tr}[g_m(\bar{c})]\right) \prod_{l=1}^{\kappa} \epsilon^{i_1\cdots i_N}(\psi^\dagger f_{l_1}(\bar{c}))_{i_1}\cdots(\psi^\dagger f_{l_N}(\bar{c}))_{i_N}|0\rangle\,, \tag{1.9}$$

where g_m, f_l are arbitrary functions depending on D variables, $\bar{c}^I = \frac{1}{\sqrt{2}}(\frac{1}{R}X^I - iRP^I)$ and $S\,(\mathrm{tr}[g_m(\bar{c})])$ is the U(N) singlet part.

In the next section we solve the equations of motion and the Gauss constraint classically at the bottom of the Yang-Mills potential. Physically this corresponds to the low energy limit, $l_p \to 0$, where D0 particles acquire well defined positions since all the X^I's are simultaneously diagonalizable. We show only the linear motion is allowed for D0 particles. Namely all the particles align themselves on a single straight line subject to $R\kappa^2/r^2$ repulsive potential from each other.

2 Solving the Low Energy Dynamics

Before we solve the dynamics at the bottom of the Yang-Mills potential, we need to specify carefully what limits we are looking at. Converting the Hamiltonian into the length scale, $H = 1/l_E$ we get from Eq.(1.6) $\mathrm{tr}[X,X]^2 \sim l_p^6/(Rl_E)$ or

$$[X,X] \sim \frac{l_p^3}{\sqrt{Rl_E}}\,M\,, \tag{2.1}$$

where M is a dimensionless matrix of order "one". Hence with the unit length, l_{unit} the following limit vanishes the commutator

$$\frac{l_p^3}{\sqrt{Rl_E}} << l_{unit}^2\,. \tag{2.2}$$

Substituting Eq.(2.1) into the equation of motion (1.7) we get

$$D_tD_tX \sim \frac{R^{3/2}}{l_p^3\,l_E^{1/2}}[M,X]\,. \tag{2.3}$$

Thus in the limit

$$\frac{R^{3/2}}{l_p^3\,l_E^{1/2}} << \frac{1}{l_{unit}^2} \tag{2.4}$$

the equation of motion becomes that of free motion. All together, from Eqs.(2.2,2.4) we require

$$\frac{R^3l_{unit}^4}{l_E} << l_p^6 << Rl_El_{unit}^4\,. \tag{2.5}$$

This limit can be realized as taking R constant, l_p small and l_E very large. For example, with a dimensionless small parameter, $\varepsilon << 1$, $l_p = \varepsilon\, l_{unit}$, $l_E = \varepsilon^{-7}l_{unit}$.

At the bottom of the potential, $[X^I,X^J]=0$, all the X^I's are simultaneously diagonalizable and we do so using the gauge symmetry, $X^I=\mathrm{diag}(x^I_1,x^I_2,\cdots,x^I_N)$. Furthermore the remaining $\mathrm{U}(1)^N$ symmetry enables us to set the vector be real and nonnegative, $\psi=\psi^*\geq 0$. Now the (a,b) component of the Gauss constraint is of the form

$$-\tfrac{1}{R}(x_a-x_b)^2A_{0ab}+\psi_a\psi_b=\kappa\,\delta_{ab}\,, \tag{2.6}$$

which determines ψ and the off-diagonal component of A_0

$$\psi_a=\sqrt{\kappa}\,,\qquad\quad A_{0ab}=\frac{\kappa R}{(x_a-x_b)^2}\quad\text{for}\quad a\neq b\,. \tag{2.7}$$

The diagonal element of A_0 is given by the equation of the motion for ψ (1.7)

$$V'(\kappa N)+\sum_b A_{0ab}=0\,,\qquad\quad A_{0aa}=-\sum_{b\neq a}\frac{\kappa R}{(x_a-x_b)^2}-V'(\kappa N)\,. \tag{2.8}$$

Note that ψ freezes becoming non-dynamical and A_0 is real and symmetric.

The remaining main equation of motion at the bottom of the potential is

$$0=D_tD_tX^I=\left(\tfrac{d^2}{dt^2}X^I-[A_0,[A_0,X^I]]\right)+i\left([\tfrac{d}{dt}A_0,X^I]+2[A_0,\tfrac{d}{dt}X^I]\right)\,. \tag{2.9}$$

Here the formula is written as a sum of symmetric part and anti-symmetric part, and we require both to vanish separately.

Using $\frac{d}{dt}A_{0ab}=-2A_{0ab}(x^J_a-x^J_b)(\frac{d}{dt}x^J_a-\frac{d}{dt}x^J_b)/(x_a-x_b)^2$ for $a\neq b$, the anti-symmetric part reads

$$\left(\delta^{IJ}-\frac{(x^I_a-x^I_b)(x^J_a-x^J_b)}{(x_a-x_b)^2}\right)(\tfrac{d}{dt}x^J_a-\tfrac{d}{dt}x^J_b)=0\,, \tag{2.10}$$

which says simply $\vec{x}_a-\vec{x}_b$ is parallel to $\frac{d}{dt}\vec{x}_a-\frac{d}{dt}\vec{x}_b$. On the other hand,

the off-diagonal component of the symmetric part gives the key formula to solve the system. With $a\neq b$

$$(\vec{x}_a-\vec{x}_b)A_{0ab}(A_{0aa}-A_{0bb})=\sum_{c\neq a,b}\left((\vec{x}_a-\vec{x}_c)+(\vec{x}_b-\vec{x}_c)\right)A_{0ac}A_{0bc}\,. \tag{2.11}$$

To see the geometric meaning we consider a $(D-1)$-dimensional hyperplane with a unit vector, $\hat{u}$

$$\hat{u}\cdot\vec{x}=l\geq 0\,. \tag{2.12}$$

Choosing the direction, $\hat{u}$ and the length, l properly one can show that there exists a hyperplane such that it contains at least two points, say $\vec{x}_a,\vec{x}_b$

$$\hat{u}\cdot\vec{x}_a=\hat{u}\cdot\vec{x}_b=l\,, \tag{2.13}$$

and further that for any point, $\vec{x}_c$

$$\hat{u}\cdot\vec{x}_c \leq l \qquad \text{or} \qquad \hat{u}\cdot(\vec{x}_a-\vec{x}_c)=\hat{u}\cdot(\vec{x}_b-\vec{x}_c)\geq 0\,. \tag{2.14}$$

We may regard the hyperplane as a "boundary" for the points. Now the scalar product of Eq.(2.11) with the unit vector shows from $A_{0ac}A_{bc}>0$ that all the points must lie on the plane, $\hat{u}\cdot\vec{x}_c=l$. We can repeat the argument applying to a smaller dimensional hyperplane until we end up with the final statement that all the points must lie on a single straight line! Consequently the anti-symmetric part is automatically satisfied[1].

We set with a D-dimensional unit vector, $\hat{n}$

$$\vec{x}_a=\vec{x}_{CM}+y_a\hat{n}\,, \qquad \sum_a y_a=0\,. \tag{2.15}$$

This reduces all the equations of motion to a single equation

$$\frac{d^2}{dt^2}y_a=\sum_{b\neq a}\frac{2R^2\kappa^2}{(y_a-y_b)^3}\,. \tag{2.16}$$

The center of mass moves with a constant velocity and the unit vector is time independent. The effective Lagrangian for this linear motion is given by $R\kappa^2/r^2$ repulsive potential

$$\mathcal{L}_{eff}=\tfrac{1}{2}\tfrac{N}{R}(\tfrac{d}{dt}\vec{x}_{CM})^2+\sum_a\tfrac{1}{2}\tfrac{1}{R}(\tfrac{d}{dt}y_a)^2-\sum_{a>b}\frac{R\kappa^2}{(y_a-y_b)^2}. \tag{2.17}$$

In particular, for two particle system, $N=2$, we get the general solution in a closed form

$$\vec{x}_1=\vec{x}_{CM}+\hat{n}\sqrt{(vt)^2+(R\kappa/2v)^2}\,, \qquad \vec{x}_2=2\vec{x}_{CM}-\vec{x}_1\,, \tag{2.18}$$

where v is an arbitrary constant corresponding to the eternal velocity.

3 Discussion

We have shown that at the bottom of the Yang-Mills potential only the linear motion is allowed for D0 particles subject to $R\kappa^2/r^2$ repulsive potential from each other. The bottom of the potential is a region where the theory is well described by the classical analysis, since it corresponds to the low energy limit and all the D0 particles acquire well defined positions. Our results tell us that at low energy D0 particles tend to align themselves on a straight line and due to the repulsive potential they move far from each other. This implies they can not form a bound state except $N=1$ case, a single D0 particle.

[1]Similar collinear motion was also observed in a matrix model with a certain ansatz [14].

As a relevant brane configuration we have proposed a D0-D4-D8 system which was originally conceived by Susskind in the context of stringy quantum Hall system. Namely D0-branes in a D4 after κ of D8's pass them. Our results agree with this picture in several aspects.
(i) Classically $\psi_a = \sqrt{\kappa}$ result shows the number of strings attached to each D0 particle is κ, as the number operator for ψ_a is $\psi_a^\dagger \psi_a$. At quantum level we also note the wave-function (1.9) has eigenvalue κN for the number operator's sum $\sum_a \psi_a^\dagger \psi_a$. (ii) The brane configuration where D4 is not parallel to D8's breaks all the supersymmetries leading to a non-BPS configuration. This is also consistent with the presence of the repulsive potential. One subtle case to note is for the system with a single D0 particle, as it can be stable though it is not BPS. (iii) The κ and r dependence in the repulsive potential is consistent with the brane configuration. The electric charge of D0 particles acquired from the κ of string ends is of κ unit, and hence κ^2 coefficient. The electric field is confined in D4, the four-dimensional space, and hence r^{-2} behaviour.

Instead of the nonrelativistic kinetic term for the fundamental strings, we may consider the relativistic form, $D_t\psi D_t\psi^\dagger$. In this case the kinetic energy of the strings contributes to the Hamiltonian contrast to the nonrelativistic case, and after quantization ψ generates two harmonic oscillators corresponding to two different orientations of the strings. This is analogue to the presence of particles and anti-particles in relativistic field theories. A new aspect of the Gauss constraint is that κ is the *difference* between the numbers of those two strings of different orientations ending on each D0 particle. Consequently the total number of strings attached to a D0-brane is not less than κ. This shows at low energy only κ, the minimum number of strings end on a D0 particle and the D0-D4-D8 system can be effectively described by our action having the nonrelativistic kinetic term.

Finally we comment, apart from the stringy interpretation, that imposing the periodic boundary condition our model with $D = 1$ choice reduces to the Sutherland model, a model on a circle with $1/r^2$ potential [15, 16].

Acknowledgements

The author wishes to thank D. Bak, J. Hoppe, S. Hyun, Y. Kiem, H-W Lee, K. Lee, S. Lee, Y. Michishita, J. Park, A. Polychronakos, H. Shin, L. Susskind and P. Yi for stimulating discussions at various stages of writing up this paper. Part of this work was carried out during the fruitful PIMS-APCTP summer workshop at Simon Fraser university.

References

[1] T. Banks, W. Fischler, S. H. Shenker and L. Susskind, Phys. Rev. D **55**, 5112 (1997).

[2] J. Hoppe, Ph.D thesis (Massachusetts Institute of Technology, 1982). For the review see e.g. W. Taylor, Rev. Mod. Phys. **73**, 419 (2001).

[3] U. Danielsson, G. Ferretti and I. R. Klebanov, Phys. Rev. Lett. **79** 1984 (1997).

[4] A. Hanany and E. Witten, Nucl. Phys. B **492** 152 (1997).

[5] S. Hellerman and L. Susskind, "Realizing the quantum Hall system in string theory," hep-th/0107200.

[6] L. Susskind, "The quantum Hall fluid and non-commutative Chern Simons theory," hep-th/0101029.

[7] D. Bak, K. Lee and J-H Park, Phys. Rev. Lett. **87** 030402 (2001); V. P. Nair and A. P. Polychronakos, Phys. Rev. Lett. **87** 030403 (2001).

[8] C. P. Bachas, M. B. Green and A. Schwimmer, JHEP **9801** 006 (1998).

[9] A. P. Polychronakos, JHEP **0104** 011 (2001).

[10] J. Polchinski, "String theory. Vol. 2" *Cambridge, UK: Univ. Pr. (1998).*

[11] L. Susskind, "Noncommutative Geometry and its Application to String Theory and the Quantum Hall System", Lectures given at Korea Institute for Advanced Study, Seoul, Korea, June 2001.

[12] R. Jackiw and S. Y. Pi, Phys. Rev. Lett. **64** 2969 (1990).

[13] S. Hellerman and M. V. Raamsdonk, "Quantum Hall physics equals noncommutative field theory," hep-th/0103179.

[14] A. P. Polychronakos, Phys. Lett. B **408**, 117 (1997).

[15] B. Sutherland, J. Math. Phys. **12** 246 (1971); **12** 251 (1971); Phys. Rev. A **4** 2019 (1971); **5** 1372 (1972).

[16] A. P. Polychronakos, Phys. Lett. B **266**, 29 (1991).