On the AdS/CFT Correspondence and Logarithmic Operators

S. Moghimi-Araghi *, S. Rouhani †and M. Saadat ‡

Department of Physics, Sharif University of Technology,
Tehran, P.O.Box: 11365-9161, Iran
Institute for studies in Theoretical physics and Mathematics,
Tehran, P.O.Box: 19395-5531, Iran

April 25, 2020

Abstract

Logarithmic conformal field theory is investigated using the ADS/CFT correspondence and a novel method based on nilpotent weights. Using this device we add ghost fermions and point to a BRST invariance of the theory.

PACS: 11.25.Hf
Keywords: conformal field theory

1 Introduction

Much work has been done in the last few years based on the AdS/CFT correspondence with the aim of understanding conformal field theories [1]. Within this framework, the correlation functions of operators on the boundary of Anti de Sitter space are determined in terms of appropriate bulk propagators. While the form of the two and three point functions within CFT are fixed by conformal invariance, it is interesting to find actions in the bulk which result in the desired boundary green functions. In particular it is interesting to discover which actions give rise to logarithmic conformal field theories (LCFTs) which leads to AdS/LCFT correspondence. Two points should be clarified, first what is meant by ordinary AdS/CFT correspondence and second what is an LCFT, and how does it fit into the correspondence.

The conjecture states that a correspondence between theories defined on AdS_{d+1} and CFT_d can be found. Suppose that a classical theory is defined on the AdS_{d+1} via the action $S[\Phi]$. On the boundary of this space the field is constrained to take certain

*e-mail: samanimi@rose.ipm.ac.ir
†e-mail: rouhani@karun.ipm.ac.ir
‡e-mail: saadat@mehr.sharif.ac.ir

boundary value $\Phi|_{\partial AdS} = \Phi_0$. With this constraint, one can calculate the partition function

$$Z[\Phi_0] = e^{-S_{Cl}[\Phi]}|_{\Phi_{\partial AdS}=\Phi_0}. \tag{1}$$

On the other hand in the CFT$_d$ space, there exist operators like $\hat{O}$ which belong to some conformal tower. Now the correspondence states that the partition function calculated in AdS is the generating function of the theory in CFT with Φ_0 being the source, that is

$$Z[\Phi_0] = \left\langle e^{\int \hat{O}\Phi_0} \right\rangle. \tag{2}$$

How do logarithmic conformal field theories fit into this picture? A decade after the seminal paper by Belavin, Polyakov and Zamolodchikov [2] on the determining role of conformal invariance on the structure of two dimensional quantum field theories, Gurarie [3] pointed to the existence of (LCFT's). Correlation functions in an LCFT may have logarithmic as well as power dependence [4]. Such logarithmic terms were ruled out earlier due to requirements such as unitarity or non existence of null states. The literature on LCFT is already very long, for a survey see some of the recent papers on LCFT for example [5], [6].

The bulk actions defined on AdS$_3$ which give rise to logarithmic operators on the boundary where first discussed in [7, 8] and have consequently been discussed by a number of authors [6, 9]. More recently a connection with world sheet supersymmetry has been discussed in [10].

In an LCFT, degenerate operators exist which form a Jordan cell under conformal transformations. In the simplest case, one has a pair $\hat{A}$ and $\hat{B}$ transforming as

$$\begin{aligned} \hat{A}(\lambda z) &= \lambda^{-\Delta}\hat{A}(z), \\ \hat{B}(\lambda z) &= \lambda^{-\Delta}[\hat{B}(z) - \hat{A}(z)\ln\lambda]. \end{aligned} \tag{3}$$

In references [11, 12] we modified the method developed by Flohr [13] in which nilpotent variables were introduced. We defined a 'superfield' :

$$\hat{O}(z,\eta) = \hat{A}(z) + \hat{\bar{\zeta}}(z)\eta + \bar{\eta}\hat{\zeta}(z) + \bar{\eta}\eta\hat{B}(z), \tag{4}$$

using different components of a logarithmic pair and adding fermionic fields. The word 'superfield' here does not mean that a supersymmetric invariance exists, rather it is a convenient tool.

Now we observe that $\hat{O}(z,\eta)$ has the following transformation law under scaling

$$\hat{O}(\lambda z,\eta) = \lambda^{-(\Delta+\bar{\eta}\eta)}\hat{O}(z,\eta). \tag{5}$$

To find out what this scaling law means, one should expand both sides of equation (5) in terms of η and $\bar{\eta}$. Doing this and comparing the two sides of equation (5), it is found that $\hat{A}(z)$ and $\hat{B}(z)$ transform as equation (3) and ζ and $\bar{\zeta}$ are ordinary fields of dimension Δ. The appearance of such fields has been proposed by Kausch [14], within the $c = -2$ theory. As discussed in [11, 12] using this structure one can derive most of the properties of LCFTs.

This paper is organized as follows: in section 2 the correspondence is explained explicitly and the two point correlation functions of different fields of CFT are derived. In section 3 we state that there should be a BRST invariance in the theory due to existence ghost fields and and find the BRST transformation in both AdS and LCFT spaces and finally show the compatibility of the correlation functions derived in section 2 with the BRST symmetry.

2 AdS/LCFT Correspondence and Correlation Functions

To begin, one should propose an action on AdS. As we will have an operator just like the one in equation (4) in LCFT part of the theory, there should be a corresponding field in AdS, $\Phi(\eta)$, which can be expanded as

$$\Phi(x,\eta) = C(x) + \bar{\eta}\alpha(x) + \bar{\alpha}(x)\eta + \bar{\eta}\eta D(x), \tag{6}$$

where x is $(d+1)-$dimensional with components $x^0, \cdots, x^d$. Of course $d = 2$ is the case we are most interested in, any how the result can be applied to any dimension and so we will consider the general case in our calculations.

Let us then consider the action:

$$S = -\frac{1}{2}\int d^{d+1}x \int d\bar{\eta}d\eta[(\nabla\Phi(x,\eta)).(\nabla\Phi(x,-\eta)) + m^2(\eta)\Phi(x,\eta)\Phi(x,-\eta)]. \tag{7}$$

This action seems to be the simplest non-trivial action for the field $\Phi(\eta)$. To write it explicitly in terms of the four components of the the field, one should expand equation (7) in powers of $\bar{\eta}$ and η using equation (6). Integrating over $\bar{\eta}$ and η one finds

$$S = -\frac{1}{2}\int d^{d+1}x[2(\nabla C).(\nabla D) + 2{m^2}_1 CD + {m^2}_2 C^2 + 2(\nabla\bar{\alpha}).(\nabla\alpha) + 2{m^2}_1\bar{\alpha}\alpha]. \tag{8}$$

To derive expression (8) we have assumed $m^2(\eta) = {m^2}_1 + {m^2}_2\bar{\eta}\eta$. Note that the bosonic part of this action is the same as the one proposed by [7, 8] with ${m^2}_1 = \Delta(\Delta - d)$ and ${m^2}_2 = 2\Delta - d$. In our theory with proper scaling of the fields, one can recover these relations.

The equation of motion for the field Φ is

$$(\nabla^2 - m^2(\eta))\Phi(x,\eta) = 0. \tag{9}$$

The Drichlet Green function for this system satisfies the equation

$$(\nabla^2 - m^2(\eta))G(x,y,\eta) = \delta(x,y) \tag{10}$$

together with the boundary condition

$$G(x,y,\eta)|_{x\in\,\partial_{AdS}} = 0. \tag{11}$$

With this Green function, the Drichlet problem for Φ in AdS can be solved readily. However, near the boundary of AdS, that is $x^d \simeq 0$, the metric diverges so the problem should be studied more carefully. One can first solve the problem for the boundary at $x^d = \varepsilon$ and then let ε tend to zero. With properly redefined scaled fields at the boundary one can avoid the singularities in the theory. So we first take the boundary at $x^d = \varepsilon$ and find the Green function [15]

$$G(x, y, \eta)|_{y^d=\varepsilon} = -a(\eta)\varepsilon^{\Delta+\bar{\eta}\eta-d}\left(\frac{x^d}{(x^d)^2 + |\mathbf{x}-\mathbf{y}|^2}\right)^{\Delta+\bar{\eta}\eta}. \tag{12}$$

where the bold face letters are d-dimensional and live on the boundary. The field in the bulk is related to the boundary fields by

$$\Phi(x, \eta) = 2a(\eta)(\Delta + \bar{\eta}\eta)\varepsilon^{\Delta+\bar{\eta}\eta-d}\int_{y^d=\varepsilon} d^d y\, \Phi(\mathbf{y}, \varepsilon, \eta)\left(\frac{x^d}{(x^d)^2 + |\mathbf{x}-\mathbf{y}|^2}\right)^{\Delta+\bar{\eta}\eta} \tag{13}$$

with $a = \frac{\Gamma(\Delta+\bar{\eta}\eta)}{2\pi^{d/2}\Gamma(\alpha+1)}$ and $\alpha = \Delta + \bar{\eta}\eta - d/2$. To compute Gamma functions in whose argument appears $\bar{\eta}\eta$, one should make a Taylor expansion for the function, that is:

$$\Gamma(a + \bar{\eta}\eta) = \Gamma(a) + \bar{\eta}\eta\Gamma'(a). \tag{14}$$

There is no higher terms in this Taylor expansion because $(\bar{\eta}\eta)^2 = 0$.

Now defining $\Phi_b(\mathbf{x}, \eta) = \lim_{\varepsilon\to 0}(\Delta + \bar{\eta}\eta)\varepsilon^{\Delta+\bar{\eta}\eta-d}\Phi(\mathbf{x}, \varepsilon, \eta)$, we have

$$\Phi(x, \eta) = \int d^d\mathbf{y}\left(\frac{x^d}{(x^d)^2 + |\mathbf{x}-\mathbf{y}|^2}\right)^{\Delta+\bar{\eta}\eta}\Phi_b(\mathbf{y}, \eta). \tag{15}$$

Using the solution derived, one should compute the classical action. First note that the action can be written as (using the equation of motion and integrating by parts)

$$S_{cl.} = \frac{1}{2}\lim_{\varepsilon\to 0}\varepsilon^{1-d}\int d\bar{\eta}d\eta\int d^d\mathbf{y}\left[\Phi(\mathbf{y}, \varepsilon, \eta)\frac{\partial\Phi(\mathbf{y}, \varepsilon, -\eta)}{\partial x^d}\right] \tag{16}$$

putting the solution (13) into equation (14), the classical action becomes

$$S_{cl.}(\Phi_b) = \frac{1}{2}\int d\bar{\eta}d\eta\int d^d\mathbf{x}d^d\mathbf{y}\frac{a(\eta)\Phi_b(\mathbf{x}, \eta)\Phi_b(\mathbf{y}, -\eta)}{|\mathbf{x}-\mathbf{y}|^{2\Delta+2\bar{\eta}\eta}}. \tag{17}$$

The next step is to derive correlation functions of the operator fields on the boundary by using AdS/CFT correspondence, i.e. equation (2). In our language the operator $\hat{O}$ has an η dependence, in addition to its usual coordinate dependence. It lives in the LCFT space and can be expanded as

$$\hat{O}(\mathbf{x}, \eta) = \hat{A}(\mathbf{x}) + \bar{\eta}\hat{\zeta}(\mathbf{x}) + \hat{\bar{\zeta}}(\mathbf{x})\eta + \bar{\eta}\eta\hat{B}(\mathbf{x}), \tag{18}$$

so the AdS/CFT correspondence becomes

$$\left\langle exp\left(\int d\bar{\eta}d\eta\int d^d\mathbf{x}\hat{O}(\mathbf{x}, \eta)\Phi_b(\mathbf{x}, \eta)\right)\right\rangle = e^{S_{cl}(\Phi_b)}. \tag{19}$$

Expanding both sides of equation (19) in powers of Φ_b and integrating over η's, the two-point function of different components of $\hat{O}(\mathbf{x},\eta)$ can be found

$$\begin{aligned}
\langle \hat{A}(\mathbf{x})\hat{A}(\mathbf{y})\rangle &= 0 && (20)\\
\langle \hat{A}(\mathbf{x})\hat{B}(\mathbf{y})\rangle &= \frac{a_1}{(\mathbf{x}-\mathbf{y})^{2\Delta}} && (21)\\
\langle \hat{B}(\mathbf{x})\hat{B}(\mathbf{y})\rangle &= \frac{1}{(\mathbf{x}-\mathbf{y})^{2\Delta}}(a_2-2a_1\log(\mathbf{x}-\mathbf{y})) && (22)\\
\langle \hat{\bar{\zeta}}(\mathbf{x})\hat{\zeta}(\mathbf{y})\rangle &= \frac{-a_1}{(\mathbf{x}-\mathbf{y})^{2\Delta}} && (23)
\end{aligned}$$

with all other correlation functions being zero. These correlation functions can be obtained in another way. Knowing the behaviour of the fields under conformal transformations, the form of two-point functions are determined. The scaling law is given by equation (5). Using this scaling law, most of the correlation functions derived here are fulfilled, however it does not lead to vanishing correlation functions of $\langle \hat{A}(\mathbf{x})\hat{\zeta}(\mathbf{y})\rangle$ and $\langle \hat{B}(\mathbf{x})\hat{\zeta}(\mathbf{y})\rangle$. These correlation functions are found to be:

$$\begin{aligned}
\langle \hat{A}(\mathbf{x})\hat{\zeta}(\mathbf{y})\rangle &= \frac{b_1}{(\mathbf{x}-\mathbf{y})^{2\Delta}} && (24)\\
\langle \hat{B}(\mathbf{x})\hat{\zeta}(\mathbf{y})\rangle &= \frac{1}{(\mathbf{x}-\mathbf{y})^{2\Delta}}\left(b_2-2b_1\log\left(\mathbf{x}-\mathbf{y}\right)\right). && (25)
\end{aligned}$$

Of course, assuming $b_1=b_2=0$ one finds the forms derived above. However, the vanishing value of such correlators comes from some other properties of the theory. What forces these constants to vanish is the fact that the total fermion number is odd [1]. One way of seeing this is to look at the OPE as given in [12]. The OPE of two $\hat{O}$-fields has been proposed to be:

$$\hat{O}(z)\hat{O}(0)\sim z^{\bar{\eta}_1\eta_2+\bar{\eta}_2\eta_1}\frac{\hat{\Phi}_0(\eta_3)}{z^{2(\Delta+\bar{\eta}_3\eta_3)}} \tag{26}$$

where $\eta_3=\eta_1+\eta_2$ and $\hat{\Phi}_0$ is the identity multiplet:

$$\hat{\Phi}_0(\eta)=\hat{\Omega}+\bar{\eta}\hat{\xi}+\hat{\bar{\xi}}\eta+\bar{\eta}\eta\hat{\omega} \tag{27}$$

with the property

$$\langle\hat{\Phi}_0(\eta)\rangle=\bar{\eta}\eta. \tag{28}$$

Note that the ordinary identity operator is $\hat{\Omega}$ which has the unusual property that $\langle\hat{\Omega}\rangle=0$, but its logarithmic partner, $\hat{\omega}$, has nonvanishing norm.

Calculating the expectational value of both sides of equation (26), the correlation functions of different fields inside $\hat{O}$ are found which leads to vanishing correlators $\langle \hat{A}(\mathbf{x})\hat{\zeta}(\mathbf{y})\rangle$ and $\langle \hat{B}(\mathbf{x})\hat{\zeta}(\mathbf{y})\rangle$, just the same result as derived by ADS/LCFT correspondence.

[1] this observation is due to M. Flohr

3 BRST symmetry of the theory

The existence of some ghost fields in the action of the theory considered so far, is reminiscent of BRST symmetry. A few remarks clarifying the word "ghost" may be in order here. The action defined by equation (8) has two types of fields in it. The usual scalars as used by other authors C and D, and fermionic fields α which have been added due to the superfield structure. We have called the fermionic fields "ghosts" because they are scalar fermions and also because they participate in the BRST symmetry as we shall see below.

As the action is quadratic, the partition function can be calculated explicitly:

$$Z = \int \mathbf{D}C(x)\mathbf{D}D(x)\mathbf{D}\bar{\alpha}(x)\mathbf{D}\alpha(x)e^{-S[C,D,\bar{\alpha},\alpha]}. \tag{29}$$

The bosonic and fermionic parts of the action are decoupled and integration over each of them can be performed independently. For bosonic part one has

$$Z_b = \int \prod_p dC_p \int \prod_p dD_p \exp\left\{ -\frac{1}{2}\begin{pmatrix} C_p & D_p \end{pmatrix} G(p) \begin{pmatrix} C_p \\ D_p \end{pmatrix} \right\} \tag{30}$$

where a Fourier transform has been done and

$$G(p) = \begin{pmatrix} p^2 + {m^2}_1 & 0 \\ {m^2}_2 & p^2 + {m^2}_1 \end{pmatrix}. \tag{31}$$

These integrals are simple Gaussian ones. So, apart from some unimportant numbers, this partition function becomes

$$Z_b = \prod_p \left[\det \begin{pmatrix} p^2 + {m^2}_1 & 0 \\ {m^2}_2 & p^2 + {m^2}_1 \end{pmatrix} \right]^{-1/2} = \prod_p (p^2 + {m^2}_1)^{-1}. \tag{32}$$

For the fermionic part the the same steps can be done and the result is

$$Z_f = \prod_p (p^2 + {m^2}_1). \tag{33}$$

Now it is easily seen that the total partition function is merely a number, independent of the parameters of the theory and this is the signature of BRST symmetry. Before proceeding, it is worth mentioning that this symmetry will be induced onto LCFT part of the correspondence and the correlation functions in that space will also be invariant under proper transformations.

In BRST transformation, the fermionic and bosonic fields should transform into each other. In our case this this can be done using η and $\bar{\eta}$. Also one needs an infinitesimal anticommuting parameters. Now let ϵ_1 and ϵ_2 be infinitesimal anticommuting parameters and consider the following infinitesimal transformation of the the field Φ

$$\delta\Phi(\mathbf{x}, \eta) = (\bar{\epsilon}\eta + \bar{\eta}\epsilon)\Phi(\mathbf{x}, \eta). \tag{34}$$

It can be easily seen that this transformation leaves the action invariant, because in the action the only terms which exist are in the form of $\Phi(\eta)\Phi(-\eta)$ and under such a transformation this term will become

$$\delta(\Phi(\eta)\Phi(-\eta)) = \Phi(\eta)(-\bar{\epsilon}\eta - \bar{\eta}\epsilon)\Phi(-\eta) + (\bar{\epsilon}\eta + \bar{\eta}\epsilon)\Phi(\eta)\Phi(-\eta) \tag{35}$$

which is identically zero. We can therefore interpret this transformation as the action of two charges Q and $\bar{Q}$. The explicit action of Q for each component of Φ is :

$$QC = 0 \tag{36}$$
$$Q\alpha = 0 \tag{37}$$
$$Q\bar{\alpha} = C \tag{38}$$
$$QD = -\alpha \tag{39}$$

As expected the bosonic and fermionic fields are transformed into each other, and the square of Q vanishes. The action of $\bar{Q}$ is similar except that it vanishes on $\bar{\alpha}$ and not on α.

To see how this symmetry is induced onto the LCFT part of the theory one should first find the proper transformation. Going back to equation (19) and using the symmetry obtained for the classical action one finds

$$\exp(S_{Cl}[\Phi]) = \left\langle \exp\left(\int \hat{O}(\Phi + \delta\Phi)\right)\right\rangle \tag{40}$$

As the transformation of Φ is $(\bar{\epsilon}\eta + \bar{\eta}\epsilon)\Phi$ the integrand on the right hand side of equation (40) is just $\hat{O}\Phi + (\bar{\epsilon}\eta + \bar{\eta}\epsilon)\hat{O}\Phi$ which can be regarded as $(\hat{O} + \delta\hat{O})\Phi$ with $\delta\hat{O} = (\bar{\epsilon}\eta + \bar{\eta}\epsilon)\hat{O}$. So equation (40) can be rewritten as

$$\left\langle \exp\left(\int \hat{O}\Phi\right)\right\rangle = \left\langle \exp\left(\int (\hat{O} + \delta\hat{O})\Phi\right)\right\rangle. \tag{41}$$

This shows that the correlation functions of the $\hat{O}$ field are invariant under the BRST transformation, that is $\delta\langle\hat{O}_1\hat{O}_2\cdots\hat{O}_n\rangle = 0$ if the BRST transformation is taken to be

$$\delta\hat{O} = (\bar{\epsilon}\eta + \bar{\eta}\epsilon)\hat{O}. \tag{42}$$

Again one can rewrite this transformation in terms of the components of $\hat{O}$ and the result is just the same as equations (36)-(39).

This invariance can be tested using two point correlation functions derived in previous section. These correlation functions are easily found to be invariant under the transformation law (42), as an example

$$Q\langle B\bar{\zeta}\rangle = \langle (QB)\bar{\zeta}\rangle + \langle B(Q\bar{\zeta})\rangle = \langle\bar{\zeta}\zeta\rangle - \langle BA\rangle = 0. \tag{43}$$

Conclusion

Logarithmic conformal field theory has been investigated using a novel method based on nilpotent weights. This method also allows investigation of LCFTs within the AdS/CFT correspondence. Although the emergence of LCFTs through this correspondence had been noted before, but the present method is much easier to implement. Also the transparency of the present method allows inclusion of fermionic fields, thus pointing to a novel symmetry both in the Ads and in LCFT.

Acknowledgements

We wish to thank Michael Flohr for constructive comments.

References

[1] O. Aharony, S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, Phys. Repts. 323 (2000) 183

[2] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Nucl. Phys. **B241** (1984) 333

[3] V.Gurarie, Nucl. Phys. **B410** (1993) 535 [hep-th/9303160]

[4] J. S. Caux, I. I. Kogan and A. M. Tsvelik, Nucl. Phys. **B466** (1996) 444

[5] M. Flohr, Conference on "Non-perturbative quantum effects 2000", Paris, September 2000 [hep-th/0009137]

[6] A. Lewis, Phys. Lett. **B480** (2000) 348; [hep-th/0009096]

[7] A.M.Ghezelbash, M.Khorrami and A.Aghamohammadi, Int. J. Mod. Phys. **A14** (1999) 2581; [hep-th/9807034]

[8] I.I.Kogan, Phys. Lett. **B458** (1999) 66; [hep-th/9903162]

[9] Y.S.Myung and H.W.Lee, JHEP **9910** (1999) 009 [hep-th/9904056]; J.Ellis, N.E.Mavromatos and E.Winstanley, Phys. Lett. **B476** (2000) 165 [hep-th/9909068]; N.E.Mavromatos and E.Winstanley, Int. J. Mod. Phys. **A16** (2001) 251 [hep-th/0006022]; A.Nichols and Sanjay, [hep-th/0007007]

[10] I. I. Kogan and D. Polyakov [hep-th/0003029]

[11] S. Moghimi-Araghi, S. Rouhani and M. Saadat, Nucl.Phys. **B599** (2001) 531-546, [hep-th/0008165]

[12] S. Moghimi-Araghi, S. Rouhani and M. Saadat, [hep-th/0012149] (to appear in Lett. Math. Phys.)

[13] M. Flohr, Nucl.Phys. **B514** (1998) 523-552 [hep-th/9707090]

[14] H. G. Kausch, Nucl.Phys. **B583** (2000) 513-541

[15] W. Mueck and K. S. Wisvanthan, Phys.Rev. **D58** (1998) 41901