The non-commutative Landau problem

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April 25, 2020

Abstract

The Landau problem is discussed in two similar but still different non-commutative frameworks. The "standard" one, where the coupling to the gauge field is achieved using Poisson brackets, yields all Landau levels. The "exotic" approach, where the coupling to the gauge field is achieved using the symplectic structure, only yields lowest-Landau level states, as advocated by Peierls and used in the description of the ground states of the Fractional Quantum Hall Effect. The same reduced model also describes vortex dynamics in a superfluid ^{4}He film. Remarkably, the spectrum depends crucially on the quantization scheme. The system is symmetric w. r. t. area-preserving diffeomorphisms.

Ann. Phys. (N.Y.) **299**, 128-140, 10 July 2002

1 Introduction

The non-commutative Landau problem, (i.e., a particle in the non-commutative plane, coupled to a constant magnetic field B and an electric potential V) has genereted a considerable amount of recent interest [1, 2, 3, 4, 5, 6, 7]. The starting point of the "standard" approach [4, 5, 6, 7] is to consider the commutation relations and Hamiltonian

$$\begin{aligned} &[x_1, x_2] = i\theta, \qquad [x_i, p_j] = i\delta_{ij}, \qquad [p_1, p_2] = iB, \\ &H = \tfrac{1}{2}\vec{p}^2 + V \end{aligned} \tag{1.1}$$

and observe that the behaviour of the system depends qualitatively on the sign of the parameter

$$m^* = 1 - B\theta. \tag{1.2}$$

When $m^* = 0$, the representation of the Heisenberg algebra of the x_i alone becomes irreducible [6], so that the Casimirs are constants (chosen to be zero in [7]),

$$\pi_i = p_i - \frac{\varepsilon_{ij}x_j}{\theta} = 0. \tag{1.3}$$

Then the problem becomes explicitly solvable; the exact Landau-type energy spectrum is $E_n = \theta^{-1}(n + \frac{1}{2}) + \epsilon_n, \ n = 0, 1, \dots,$ where ϵ_n is the eigenvalue of the potential alone [7] (further discussed below).

Non-commuting coordinates have arised before in the study of the ground states of the Fractional Quantum Hall Effect [8, 9] : seventy years ago, Peierls [10] argued in fact that in a strong magnetic field and a sufficiently weak potential the lowest Landau level retains its identity. Putting the mass and the electric charge again to unity, the energy eigenvalues are $E_n = B/2 + \epsilon_n$, where the ϵ_n are the eigenvalues of the operator $\hat{V}(\hat{p}, \hat{q})$, obtained from the potential alone, but such that $\hat{p}$ and $\hat{q}$ are canonically conjugate, $[\hat{p}, \hat{q}] = -i/B$.

This "Peierls substitution" can be justified taking the $m \to 0$ limit of ordinary quantum mechanics [1, 2]. A slightly different derivation [3] starts with a classical particle associated with the "exotic" two-parameter central extension of the planar Galilei group [11]. The extension parameters combine with the magnetic field into and effective mass, which is, indeed, the parameter m^* in (1.2); when this latter vanishes, the Peierls substitution is once again recovered using Hamiltonian reduction [12].

Despite their deceptive similarity, the standard NC [4, 5, 7] and the Peierls-type "exotic" [3] approaches are, nevertheless, *different* : in the critical case $m^* = 0$, the spectrum of [7] exhibits all Landau levels, while the Peierls spectrum [1, 2, 3] only consists of LLL (Lowest Landau Level) states.

Analyzing the strong similarities and subtle differences of the two approaches using a unified framework, we find that the "standard" coupling is not entirely satisfactory, while the "exotic" approach works perfectly.

A major issue is the spectrum : surprisingly, it depends crucially on the quantization scheme we use, and we have not been able to find the "good" answer.

A further remarkable fact is that the classical counterpart of the Peierls model [1, 3] also describes another, related, but physically different, situation, namely the effective dynamics of vortices in a thin superfluid ^{4}He film [13, 14]. Our remarks provide us with yet another indication for the intimate relationship of supefluidit and the Hall effect. Physically, the "exotic" model describes Laughlin's quasiparticles, which in turn correspond to the vortices of the field theory [8].

2 The spectra

Firstly, we rederive the spectra in a unified framework. Let us first assume that V (but not B) vanishes, $V = 0$, $B \neq 0$.

2.1 The "standard" NC model

Let us start with the standard NC approach [4, 5, 7]. Let $m^* \neq 0$. The classical counterparts of the commutation relations obtained by replacing (1.1) by Poisson brackets are associated with the symplectic form

$$\Omega_{NC} = \frac{1}{m^*}\Big[d\vec{p} \wedge d\vec{x} + \theta dp_1 \wedge dp_2 + B dx_1 \wedge dx_2\Big]. \tag{2.1}$$

Note that the condition $d\Omega_{NC} = 0$ requires the magnetic field B to be constant. Introducing the complex coordinates

$$\begin{aligned} z &= \frac{1}{\sqrt{m^*}}\left(\sqrt{B}(x_1 + ix_2) + \frac{1}{\sqrt{B}}(-ip_1 + p_2)\right), \\ w &= \frac{1}{\sqrt{B}}(-ip_1 - p_2), \end{aligned} \tag{2.2}$$

(2.1) is rewritten as $\Omega_{NC} = (2i)^{-1}(d\bar{z} \wedge dz + d\bar{w} \wedge dw)$, which shows that z and w are canonical coordinates on phase space. Note that these definitions "mostly use" the magnetic field B; the non-commutative paremeter θ only enters z, namely through the pre-factor $(m^*)^{-1/2}$. Then, choosing the antiholomorphic polarization yields the Bargmann-Fock wave functions $f(z,w)e^{-(|z|^2+|w|^2)/4}$ with $f(z,w)$ holomorphic in both variables. Expressed as acting on the holomorphic functions alone, the fundamental operators read

$$\begin{aligned} \widehat{z} &= z\cdot \qquad & \widehat{\bar{z}} &= 2\partial_z \\ \widehat{w} &= w\cdot \qquad & \widehat{\bar{w}} &= 2\partial_w \end{aligned} \tag{2.3}$$

and satisfy

$$[\widehat{\bar{z}}, z] = 2 = [\widehat{\bar{w}}, \widehat{w}], \qquad [\widehat{z}, \widehat{w}] = 0 = [\widehat{\bar{z}}, \widehat{\bar{w}}]. \tag{2.4}$$

The classical kinetic Hamiltonian, $H_0 = \vec{p}^2/2 = \frac{B}{2} w\bar{w}$, is quantized into

$$\widehat{H}_0 = \tfrac{1}{2} B(\widehat{w}\widehat{\bar{w}} + 1) = B\left(w\partial_w + \tfrac{1}{2}\right). \tag{2.5}$$

Now if we let m^* go to zero, both the symplectic form Ω_{NC} and the coordinate z blow up. Regularity can be maintained, though, if we require that

$$z = 0. \tag{2.6}$$

Therefore, f is a function of w alone. (Note that w is essentially $\vec{p}$). Remarkably, $z = 0$ is precisely the condition of the vanishing of the Casimirs, $\pi_i = 0$, in (1.3) [7, 6], since $B = \theta^{-1}$ in the critical case. The eigenfunctions, w^n, have eigenvalues $E_n = B(n + 1/2)$, i. e., the first term in the spectrum in [7].

2.2 The exotic model

Let us now turn briefly to the "exotic" model; for more details the Reader is referred to [3]. The symplectic form is simply

$$\Omega_E = d\vec{p} \wedge d\vec{x} + \theta dp_1 \wedge dp_2 + B dx_1 \wedge dx_2, \tag{2.7}$$

but the associated Poisson brackets acquire an $(m^*)^{-1}$ factor:

$$\{x_1, x_2\} = \frac{\theta}{m^*}, \qquad \{x_i, p_j\} = \frac{\delta_{ij}}{m^*}, \qquad \{p_1, p_2\} = \frac{B}{m^*}. \tag{2.8}$$

We modify therefore (2.2) as

$$\begin{aligned} z &= \sqrt{B}(x_1 + ix_2) + \frac{1}{\sqrt{B}}(-ip_1 + p_2) \\ w &= \sqrt{\frac{m^*}{B}}(-ip_1 - p_2), \end{aligned} \tag{2.9}$$

so that now $(2i)^{-1}(d\bar{z} \wedge dz + d\bar{w} \wedge dw) = \Omega_E$. The new z and w are hence again canonical coordinates, quantized, for $m^* \neq 0$, as in (2.3); the relations (2.4) still hold. The kinetic hamiltonian, $H_0 = \frac{B}{2m^*} w\bar{w}$, is quantized to

$$\widehat{H}_0 = \frac{B}{2m^*}(\widehat{w}\widehat{\bar{w}} + 1) = \frac{B}{m^*}(w\partial_w + \tfrac{1}{2}), \tag{2.10}$$

which differs from (2.5) in the pre-factor $(m^*)^{-1}$.

Unlike Ω_{NC}, the symplectic form (2.7) does not diverge as $m^* \to 0$; it becomes, however, singular, since $\det(\Omega_E) = (m^*)^2$. Equivalently, the associated Poisson brackets (2.8) diverge. These divergences can be avoided by eliminating w,

$$\tfrac{1}{2}\widehat{\bar{w}} f \equiv \partial_w f = 0, \tag{2.11}$$

so that f becomes a function of the "position" coordinate z alone: we recover hence the Laughlin prescription used in the FQHE context [8]. Restricting ourselves to the subspace $\partial_w f = 0$ we find, for $m^* \neq 0$, that any $f(z)$ has the ground state energy,

$$E_0 = \frac{B}{2m^*}. \tag{2.12}$$

All Laughlin wavefunctions belong hence to the LLL. Their energy diverges as $m^* \to 0$, though.

2.3 Quantization of the potential

Restoring the potential, we observe that the total Hamiltonian is, in both cases,

$$H \equiv H_{NC} = H_E = \tfrac{1}{2}\vec{p}^2 + V. \tag{2.13}$$

Our task is to quantize it on the respective Hilbert spaces. Owing to the non-commutativity of the coordinates, quantizing the potential $V = V(\bar{z}, z, \bar{w}, w)$ is, in general, a rather difficult problem. We restrict ourselves therefore to the critical case $B\theta = 1$ and assume that the potential is radial, $V = U(r^2)$.

In the NC model we eliminate the coordinate z. Then $|\vec{x}|^2 = |\vec{p}|^2/B^2 = \bar{w}w/B$, so that our task is to quantize $U(\bar{w}w/B)$ as acting on the analytic functions $f(w)$, subject to $[\widehat{\bar{w}}, \widehat{w}] = 2$.

In the exotic case instead, it is w which is eliminated and we are left with analytic functions $f(z)$, subject to $[\widehat{\bar{z}}, \widehat{z}] = 2$. Now $V = V(\bar{z}, z) = U(\bar{z}z)$, so the task is, in both cases, consistent with the kinetic part in the Hamiltonian. We can treat hence both problems simultaneously, with Z denoting either of the remaining complex variable, w or z, respectively. For simplicity, we choose $B = \theta = 1$.

A remarkable fact is now that the quantum operator $\widehat{V}$ depends crucially on the chosen quantization scheme.

Some people [7, 13, 15] claim that the spectrum is simply $U(\theta(2n+1))$. This statement can be justified, e. g., by excluding the origin and considering the real polarization $r = \text{const}$. in the plane. The flow of any function of r preserves this polarization, so that $\widehat{V}$ is simply multiplication by $U(r^2)$, and the spectrum formula follows at once.

Other quantization schemes yield other results, though. The main difficulty in constructing the quantum operator $\widehat{V}$ is in fact to resolve the ambiguity in ordering the non-commutative fundamental variables. In fact, for an arbitrary classical observable A with associated operator $\widehat{A}$, and an arbitrary real function U, $\widehat{U(A)} \neq U(\widehat{A})$ in general.

Now, according to one proposal [1, 2, 9], $\widehat{V}$ is obtained, in the holomorphic Bargmann-Fock representation, by *anti-normal ordering*, which amounts to "putting all the $\widehat{\bar{Z}}$ to the left and all the $\widehat{Z}$ to the right" [9, 2]. In the radially-symmetric polynomial case

$$V_N = \left(\tfrac{1}{2}\bar{Z}Z\right)^N, \tag{2.14}$$

this rule yields the recurrence relation $\widehat{V}^{an}_N = \widehat{V}^{an}_{N-1}(Z\partial_Z + N)$. Hence, using $\widehat{V}^{an}_0$=1,

$$\widehat{V}^{an}_N = (Z\partial_Z + 1)\dots(Z\partial_Z + N) \tag{2.15}$$

The eigenfunction of $\widehat{V} = Z\partial_Z + 1$ are $f_n = Z^n$ with eigenvalues $n+1$. The spectrum of $\widehat{V}_N^{an}$ is therefore

$$\epsilon_n^{an} = (n+1)\dots(n+N). \tag{2.16}$$

It is easy to see that the prescription requires modification, though: in the simplest case $N = 1$ i. e. for a quadratic potential, $V = \frac{1}{2}Z\bar{Z}$, for example, $\epsilon_n^{an} = n+1$. Observing, however, that this V is actually the *full* Hamiltonian of a 1*–dimensional* oscillator with *phase-space* variable Z, we conclude that the spectrum should be rather $n+1/2$. This discrepancy has been noticed by the authors of [1], who, as they say, have no *a priori* determination for it.

Yet another quantization scheme due to Bergman [16] identifies the quantum operator associated to $V(Z,\bar{Z})$ as

$$\widehat{V}\psi(Z) = \int \exp\left[\tfrac{1}{2}\bar{z}(Z-z)\right](V - \partial_z\partial_{\bar{z}}V)\,\psi(z)dzd\bar{z}. \tag{2.17}$$

This rule has also been derived rigorously, using geometric quantization by Tuynman in Ref. [16].

When V is a polynomial, the formula simplifies as follows: first calculate $\widetilde{V} = V - \partial_Z\partial_{\bar{Z}}V$, and then quantize $\widetilde{V}$ by anti-normal ordering. In the quadratic case above the correction term subtracts $1/2$, yielding the correct formula, namely

$$\widehat{V} = Z\partial_Z + \frac{1}{2}, \tag{2.18}$$

whose spectrum is $\epsilon_n = n + 1/2$. (In this case we agree with [5, 7]).

More generally, in the radially-symmetric polynomial case with $N \geq 2$ we have

$$\widetilde{V}_N = \left(\tfrac{1}{2}\bar{Z}Z\right)^N - \frac{N^2}{2}\left(\tfrac{1}{2}\bar{Z}Z\right)^{N-1},$$

from which one infers that $\widehat{V}_N = \widehat{V}_N^{an} - \frac{N^2}{2}\widehat{V}_N^{an}$. Bergman quantization yields hence

$$\widehat{V}_N = (Z\partial_Z + 1)\dots(Z\partial_Z + N - 1)(Z\partial_Z + N - \frac{N^2}{2}), \tag{2.19}$$

whose spectrum is

$$\epsilon_n^{\text{crit}} = (n+1)\dots(n+N-1)(n+N-\frac{N^2}{2}) \tag{2.20}$$

that disagrees with both previous results, $U(2n+1) = (n+\frac{1}{2})^N$, of [7], as well as (2.16).

The result is unsatisfactory though : for $N \geq 3$, the first few eigenvalues in (2.20) become *negative*. This is absurd, and one would need a rule to exlude those eigenvalues. Unfortunately, we have not been able to derive such a rule from first principles. We can not adopt, therefore, any final answer to the problem of finding the "good" quantization.

Returning to our models, the total Hamiltonian $\vec{p}^2/2+V$ acts, for $m^* \neq 0$, on the "unreduced" Hilbert space as

$$\widehat{H}_{NC} = B(w\partial_w + 1/2) + \widehat{V}(z,\bar{z},w,\bar{w})$$

in the NC case, and as

$$\widehat{H}_E = (B/m^*)(w\partial_w + 1/2) + \widehat{V}(z,\bar{z},w,\bar{w})$$

in the "exotic" case, respectively. Restricting our attention to the subspaces $\hat{z}=0$ and $\partial_w f=0$, respectively, we see that $f_n(w)=w^n$ (resp. $f_n(z)=z^n$) are simultaneous eigenfunctions of both terms, yielding the total eigenvalues $E_n=B(n+1/2)+\epsilon_n$ in the standard NC case, and $E_n=B/2m^*+\epsilon_n$ in the exotic case, where ϵ_n is the eigenvalue of $\hat{V}(z,\bar{z},w,\bar{z})$, restricted to the respective subspaces.

Note that ϵ_n is not yet known, since $\hat{V}(z,\bar{z},w,\bar{z})$ has not yet been calculated. Letting m^* go to zero, however, one or other of the variables z and w become redundant, and V takes the same form $V(Z,\bar{Z})$. The NC spectrum becomes hence $E_n=B(n+1/2)+\epsilon_n^{\rm crit}$, cf. [7], while the exotic spectrum is instead in the LLL, $E_n=(B/2m^*)+\epsilon_n^{\rm crit}$, as suggested by Peierls. In the latter case the first term diverges as $m^*\to 0$, though, and has to be removed by hand. A similar behaviour has been observed by Dunne et al. [1] in the oscillator case $N=1$ who found that their $m\neq 0$ energy eigenvalues diverge.

It is worth noting that the projection to the Hilbert subspace (2.6) can also be obtained by Bargmann-Fock quantization of the w plane, endowed with its canonical symplectic structure and Hamiltonian,

$$\begin{aligned}\Omega^{\rm red}_{NC}&=(2i)^{-1}d\bar{w}\wedge dw,\\ H^{\rm red}_{NC}&=\tfrac{1}{2}Bw\bar{w}+V(w,\bar{w}).\end{aligned}\tag{2.21}$$

In the "exotic" case, the projected theory can be obtained directly from a similar study in the z-plane [1, 3].

3 Classical aspects

Studying the classical mechanics of the two models provides us with further insight.

3.1 The standard case

The motion of a NC particle is governed by Hamilton's equations, $\dot{\xi}=\{\xi,H\}$, associated with the Poisson bracket (1.1), i.e.,

$$\begin{aligned}\dot{x}_i&=p_i-\theta\varepsilon_{ij}E_j,\\ \dot{p}_i&=B\varepsilon_{ij}p_j+E_i,\end{aligned}\tag{3.1}$$

$\vec{E}=-2U'(r^2)\vec{x}$ in the radial case $V=U(r^2)$. According to the first equation, the velocity, $\dot{\vec{x}}$, and the momentum, $\vec{p}$, are not in general the same or even parallel; in the second equation the Lorentz force involves $\vec{p}$ and not $\dot{\vec{x}}$. Eliminating p_i, we get

$$\ddot{x}_i=B\varepsilon_{ij}\dot{x}_j+m^*E_i-\theta\varepsilon_{ij}\dot{E}_j\tag{3.2}$$

with

$$\dot{E}_i=-4U''(r^2)x_k\dot{x}_kx_i-2U'(r^2)\dot{x}_i$$

for $V=U(r^2)$. Let us now put $\varpi_i=p_i-B\varepsilon_{ij}x_j$. The equations of motion, (3.1), imply that

$$\dot{\varpi}_i=(1-B\theta)E_i\equiv m^*E_i,\tag{3.3}$$

so that for $m^*=0$ ϖ_i becomes, for *any* E_i, a *constant of the motion*. The dynamics can, therefore, be consistently restricted to the two-dimensional surface

$$\varpi_i\equiv p_i-B\varepsilon_{ij}x_j=c_i.\tag{3.4}$$

The equations of motion retain their form (3.1), and our constraint (3.4) with $c_i = 0$ reproduces that of Bellucci et al., as our ϖ_i becomes their π_i in (1.3). What we have found is the classical counterpart of the irreducibility of the x_i representation [6].

Curiously, the "new" conserved quantity ϖ is related to the *translational invariance*: in the absence of an $\vec{E}$-field, the NC system in a constant B-field would plainly be invariant w. r. t. $\vec{x} \to \vec{x} + \vec{\gamma}$, with associated conserved linear momenta ϖ_i/m^*. Adding an arbitrary electric field breaks this symmetry in general; the conservation of ϖ is, however, restored when $B = \theta^{-1}$, since, in the critical case, the electric field decouples, cf. (3.3). This restauration of the translational symmetry is quite remarkable, since it holds for *any* electric field. Let us remind the Reader to the crucial rôle played by the restauration of [discrete] translational symmetry corresponding to a torus geometry in deriving the quantization of the Hall conductivity [8].

In what follows we shall also consider $c_i = 0$ for simplicity, although this is not mandatory: another reduced theory could be obtained for each value of the constants c_i. The general one is obtained by a straightforward modifications ; the complex coordinate z in (2.2) should become, e. g., $(Bm^*)^{-1/2}\Big((-i\varpi_1 + \varpi_2) - (-ic_1 + c_2)\Big)$, etc.

In the radial case the particle moves along circles : the electric field is radial, so (3.1) implies that $\dot{r} = 2x_k\dot{x}_k = 2x_kp_k = 0$ upon use of (3.4) with $c_i = 0$.

Eliminating the position $\vec{x}$ using the constraint (3.4) allows us to view the force as a function of p_i alone, $E_i = E_i(-\varepsilon_{jk}p_k/B)$, $E_i = (2/B)\varepsilon_{ij}p_jU'(\vec{p}^2/B^2)$ in the radial case. Therefore, the second equation in (3.1) is actually a first-order equation for p_i ; the first equation of in (3.1) is merely a consequence of the constraint and of the second equation in (3.1). This latter is actually Hamilton's equation associated with the reduced symplectic structure/Poisson bracket and Hamiltonian defined on $\vec{p}$-space,

$$\begin{aligned} &\Omega^{\rm red}_{NC} = \frac{1}{B}dp_1 \wedge dp_2 \qquad \text{i.e.} \qquad \{f,g\}^{\rm red}_{NC} = B\left(\frac{\partial f}{\partial p_1}\frac{\partial g}{\partial p_2} - \frac{\partial g}{\partial p_1}\frac{\partial f}{\partial p_2}\right), \\ &H^{\rm red}_{NC} = \tfrac{1}{2}\vec{p}^2 + V(-\varepsilon_{ij}p_j/B). \end{aligned} \tag{3.5}$$

Note that the reduced Hamiltonian is simply the restriction of the "original" expression to the constraint surface (3.4); note also that (3.5) is consistent with the complex expressions in (2.21). In the radial case the equation of motions is simply

$$\dot{p}_i = \widetilde{B}\varepsilon_{ij}p_j, \qquad \widetilde{B} = B + \frac{1}{B}2U'(\vec{p}^2/B^2). \tag{3.6}$$

Thus, the potential is transmuted into a (generally position-dependent) effective magnetic field $\widetilde{B}$, as observed before in the quadratic case [4]. In fact, the force term in (3.2) is switched off, and the $\dot{E}_i$ merely contributes to the Lorentz force by yielding an effective magnetic field $\widetilde{B}$. It follows that $\vec{p}$ rotates uniformly in the "hodograph" ($\equiv \vec{p}$)-plane with uniform angular velocity $\omega = \widetilde{B}$. By (3.4) the position, $\vec{x}$, performs the same type of motion. In terms of the complex coordinate (2.2), the equations of motion associated with (2.21) are solved by $w(t) = e^{i\widetilde{B}t}\zeta$, with [$\sqrt{B}$ times] ζ the initial $\vec{p}$.

3.2 The exotic case

Turning to the exotic case, we focus our attention to the differences with the NC model. The equations of motion,

$$\begin{aligned} m^*\dot{x}_i &= p_i - \theta\,\varepsilon_{ij}E_j, \\ \dot{p}_i &= B\,\varepsilon_{ij}\dot{x}_j + E_i, \end{aligned} \tag{3.7}$$

[cf. (3.1)] can also be presented as

$$m^*\ddot{x}_i = \Big(B\varepsilon_{ij}\dot{x}_j + E_i\Big) - \theta\Big(\varepsilon_{ij}\dot{E}_j - \dot{B}\dot{x}_i\Big) \tag{3.8}$$

($\dot{B} = \partial_\ell B\dot{x}_\ell + \partial_t B$), which is rather different from (3.2). (The results here do *not* require B to be constant [3]). In the critical case $m^* = 0$ the system becomes singular; Hamiltonian reduction [12] performed in [3] requires the Hall constraint

$$p_i = \varepsilon_{ij}\frac{E_j}{B} \tag{3.9}$$

analogous to but in general different from (3.4). Then the $4D$ phase space reduces to a two-dimensional one with coordinate z, consistent with (2.9). The classical phase space is hence the complex plane with canonical symplectic structure $(2i)^{-1}dz \wedge dz$, and the reduced Hamiltonian is $H_E^{\rm red} = V = V(z,\bar{z})$ alone: this is the classical counterpart of the Peierls substitution. The second-order equations (3.8) reduce to a first-order ones, namely to the *Hall law*,

$$\dot{Q}_i = \varepsilon_{ij}\frac{E_j}{B}. \tag{3.10}$$

These latters can be obtained from the reduced symplectic structure and Hamiltonian

$$\begin{aligned} \Omega_E^{\rm red} &= B dQ_1 \wedge dQ_2, \\ H_E^{\rm red} &= V(Q_1, Q_2) \end{aligned} \tag{3.11}$$

where the $Q_i = x_i - E_i/B^2$ are suitable coordinates [3].

Hence, consistently also with the conservation of the reduced energy $H_E^{\rm red} = V$, the motions follow equipotentials. For a radial potential $V = U(r^2)$ in particular, the trajectories are again circles, with (radius-dependent) uniform angular velocity $\omega = 2U'(r_0^2)/B$ (*different* from that in the NC case). In complex notations, $z(t) = e^{-i\omega t}\zeta$, where $\omega = N(\zeta\bar{\zeta}/2)^{N-1}/B$.

3.3 Comparison of the models

The difference between the models originates in the way the particle is *coupled to the gauge field*. In the standard NC case [7, 4], the rule is to *replace the commutation relation of the momenta*, $\{p_1, p_2\} = 0$, by the last one in (1.1), viz. $\{p_1, p_2\} = B$. Note that this only works for a constant B, otherwise the posited Poisson bracket will not satisfy the Jacobi identity : e.g., $\{x_i, \{p_1, p_2\}\}_{\rm cyclic} = \theta\varepsilon_{ij}\partial_j B$. This is an unpleasant restriction, since the requirement of having a strictly constant B is rather unphysical.

The recipe followed in the exotic case is instead that of Souriau [17], who first unifies both the symplectic structure and the Hamiltonian into a single two-form, viz. $\sigma = \Omega - dH \wedge dt$. Then his rule says that the minimally coupled two-form should be obtained by adding the electromagnetic tensor F to the free two-form σ_0. In this framework, the Jacobi identity holds for any gauge field: it comes from that σ_E is a closed 2-from, $d\sigma_E = 0$, which follows in turn from the homogeneous Maxwell equation $dF = 0$.

The two rules are only equivalent for $\theta = 0$ or for $B = 0$, as it can be readily seen remembering that the Poisson bracket involves the *inverse* of the symplectic matrix Ω,

$$\{f, g\} = P^{\alpha\beta}\partial_\alpha f \partial_\beta g, \qquad P^{\alpha\beta} = (\Omega^{-1})_{\alpha\beta}. \tag{3.12}$$

Explicitely, Souriau's rule yields

$$\sigma_E = d\vec{p} \wedge d\vec{x} + \theta dp_1 \wedge dp_2 + B dx_1 \wedge dx_2 - (\vec{p} \cdot d\vec{p} + dV) \wedge dt, \tag{3.13}$$

whereas the Poisson bracket posited in the NC approach corresponds to the manifestly different two-form

$$\sigma_{NC} = \frac{1}{m^*}\Big[d\vec{p} \wedge d\vec{x} + \theta dp_1 \wedge dp_2 + B dx_1 \wedge dx_2\Big] - (\vec{p} \cdot d\vec{p} + dV) \wedge dt. \tag{3.14}$$

3.4 Variational aspects

A new light is shed on the two models by studying the variational aspects. The classical action can in fact be written as the integral of Cartan's 1-form, $\int L dt = \int \Theta$ [17]; then the associated Euler-Lagrange equations say that the motions curves are tangent to the kernel of the 2-form $\sigma = d\Theta$ [17]. For $\sigma = \Omega - dH \wedge dt$, this means

$$\Omega_{\alpha\beta}\dot{\xi}_\beta = \frac{\partial H}{\partial \xi_\alpha} \tag{3.15}$$

($\xi_\alpha = (p_i, x_j)$). When the matrix $\Omega_{\alpha\beta}$ is regular, (3.15) can be inverted, and we get Hamilton's equations, written in terms of the Poisson bracket (3.12) as $\dot{\xi}_\alpha = \{\xi_\alpha, H\}$. When $\Omega_{\alpha\beta}$ is singular, however, one can only derive a Poisson bracket-formulation after Hamiltonian [12] (alias symplectic [17]) reduction. Conversely, when a Poisson bracket and a Hamiltonian are posited, one can only reconstruct a Lagrangian provided the matrix $P^{\alpha\beta}$ which defines the Poisson bracket is regular.

In the NC case the posited Poisson structure (1.1) only leads to the 2-form (3.14) and hence to a variational formulation *off* the critical case. In fact

$$\Theta_{NC} = \frac{1}{m^*}\left[(p_i - A_i)dx^i + \tfrac{1}{2}\theta\varepsilon_{ij}p_i dp_j\right] - (\tfrac{1}{2}\vec{p}^2 + V)dt \tag{3.16}$$

works (contrary to claims [18]) when $m^* \neq 0$. It blows up, however, when $m^* \to 0$ — although Hamilton's equations behave regularly. This latter can hence only by derived from a variational principle after reduction. The Hamiltonian structure (3.5) corresponds indeed to the first-order Cartan 1-form [Lagrangian]

$$\Theta^{red}_{NC} = \frac{1}{2B}\varepsilon_{ij}p_i dp_j - \Big(\tfrac{1}{2}\vec{p}^2 + V(-\varepsilon_{ij}p_j/B)\Big)dt, \tag{3.17}$$

cf. [18]. The exotic Cartan form is instead

$$\Theta_E = (p_i - A_i)dx^i + \tfrac{1}{2}\theta\varepsilon_{ij}p_i dp_j - (\tfrac{1}{2}\vec{p}^2 + V)dt, \tag{3.18}$$

whose exterior derivative, $d\Theta_E = \sigma_E$, becomes singular at the critical point $m^* = 0$. Then there is no associated Poisson bracket structure, and a Hamiltonian formulation is only possible after "Faddeev-Jackiw" reduction [12, 3]. These latter are consistent with the variational 1-form

$$\Theta^{red}_E = \frac{B}{2}\varepsilon_{ij}Q_i dQ_j - V(Q_1, Q_2)dt, \tag{3.19}$$

used before by Dunne at al. in their $m \to 0$ derivation of the Peierls rule [1].

4 Vortex dynamics in superfluid helium

The interest of the Peierls-type model is underlined by that it also describes the effective dynamics of point-like flux lines in a thin film of superfluid ^{4}He [13, 14]. For the sake of simplicity, we restrict ourselves to two vortices of identical vorticity. The center-of-vorticity coordinates are constants of the motion. Introducing the relative coordinates $x = x_1 - x_2$ and $y = y_1 - y_2$ and the respectively, the vortex equations become [13]

$$(\rho\delta\kappa)\frac{dx}{dt} = \frac{\partial H}{\partial y}, \qquad (\rho\delta\kappa)\frac{dy}{dt} = -\frac{\partial H}{\partial x}, \tag{4.1}$$

where ρ and δ are the density and the thickness of the film, respectively; κ, the vorticity, is quantized in units h/m, where h is Planck's constant and m is the mass of the helium. (In what follows we choose units where $\hbar = 1$). The effective Hamiltonian here has the form

$$H = -(\rho\delta\kappa^2/4\pi)\ln[x^2 + y^2]. \tag{4.2}$$

The system has fractional angular momentum and obeys fractional statistics [13, 14].

A key observation for our purposes [14] is that (4.1) is in fact a Hamiltonian system, $\dot{\xi} = \{\xi, H\}$ ($\xi = (x, y)$), consistent with the Poisson bracket associated with the symplectic structure $\theta^{-1}dx\wedge dy$, $\theta^{-1} = \rho\delta\kappa$ of the plane (x, y), cf. (3.11) with $B = \theta^{-1}$. Vortex dynamics in superfluid helium provides us hence with yet another physical instance of non-commuting coordinates. Let us emphasize that the symplectic plane should be viewed as the classical *phase space* for the vortex motion.

Another peculiarity is the absence of a mass term in the hamiltonian (4.2). The analogy with the motion of *massless* particles in a magnetic field, noticed [13, 14] before, can be further amplified: (4.2) is indeed the "reduced Hamiltonian" $H_E^{\rm red} \propto \ln r^2$ in (3.11).

The identity of the underlying mathematical structures allows us to transfer the analysis in presented above to the superfluid case. For any radial Hamiltonian $H = U(r^2)$ in particular, the equipotentials are circles, so the relative motion of the vortices is a rotation with (separation-dependent) uniform angular velocity $\omega = 2U'(r_0^2)/\rho\delta\kappa$. For the effective vortex Hamiltonian (4.2) in particular, the angular velocity is inversely proportional to the square of the radius, $\omega = -(\kappa/2\pi)r^{-2}$. (This is also consistent with the conservation of vorticity).

As explained above, quantization is conveniently carried out in the Bargmann-Fock framework. The spectrum can be found by quantizing $H = -(\rho\delta\kappa^2/4\pi)\ln r^2$. Leinaas and Myrheim claim that it is simply

$$\epsilon_n = -\frac{\rho\delta\kappa^2}{4\pi}\ln(8\theta\, j_n), \tag{4.3}$$

where the j_n are the eigenvalues of the conserved angular momentum $J = \frac{1}{2}\rho\delta\kappa r^2 = \frac{1}{2}z\bar{z}$. This ignores the ordering problem, and agrees with the first quantization scheme discussed in Sect. 2.3. Although we have not been able to calculate it explicitely, the answer provided the other schemes will definitely be different from (4.3).

Leinaas and Myrheim derive the spectrum of the angular momentum from the representation theory. They observe in fact that $J = A/2$, where the

$$A = (z\bar{z}), \qquad B = \frac{1}{2}(z^2 + \bar{z}^2) \qquad C = \frac{1}{2i}(z^2 - \bar{z}^2) \tag{4.4}$$

span an $\mathrm{sl}(2, \mathbf{R}) \simeq \mathrm{o}(2, 1) \simeq \mathrm{sp}(1)$ algebra, whose representation theory yields the fractional eigenvalues $j_n = (\alpha_0 + n)$ of the angular momentum [14].

The arisal of this algebra is easy to understand: the operators in (4.4) are the generators of the symplectic group sp(1) in the plane. We argue, however, that this only works for the angular momentum, but not for a general Hamiltonian, since $H = U(r^2)$ does not belong to sp(1), except in the trivial case $U(r^2) \propto r^2$, and going from r^2 to $U(r^2)$ is precisely the problem discussed in Sect. 2.3. Its spectrum can not be derived therefore from the representation theory of sp(1) in general.

5 Infinite symmetry

The point is that sp(1) belongs to a much larger — actually infinite-dimensional — symmetry algebra, found before for the edge currents in the Quantum Hall context [19]. The reduced system (3.11) is, namely, symmetric w. r. t. w_∞, the algebra of all area-preserving diffeomorphisms. This is the best explained in terms of Souriau's "*espace des mouvements*" [= space of motions]). Here we only present a brief outline; for details the Reader is referred to Souriau's book [17]. The space of motions we denote here by $\mathcal{M}$ consists of entire motion curves of the system. The classical dynamics is encoded into the symplectic form of $\mathcal{M}$ we denote by σ. This latter incorporates both the phase space symplectic structure and the Hamiltonian : in fact, σ is the projection along the motion curves of the closed two-form $\sigma = d\Theta = \Omega - dH \wedge dt$ [justifying our abuse of notation].

It is then obvious that *any* function $f(\zeta)$ on $\mathcal{M}$ is a constant of the motion, since it only depends on the motion ζ. Integrating the equations of motions, $f(\zeta)$ can be expressed using the phase space variable ξ and time, t, $f(\xi, t) = f(\zeta)$; such a function can look rather complicated, though, owing to the complicated relation between ζ and ξ, t. Its conservation means

$$\partial_t f = \{f, H\}, \tag{5.1}$$

where $\{\cdot, \cdot\}$ is the Poisson bracket associated with the phase-space symplectic form Ω.

To any such function $f(\zeta)$ corresponds,through $-\partial_\mu f = \sigma_{\mu\nu} Z^\nu$, a ("Hamiltonian") vectorfield Z^μ, which generates in turn, at least locally, a 1-parameter group of diffeomorphisms of $\mathcal{M}$ which leaves σ invariant. All such diffeomorphisms form an infinite dimensional group, namely the group of symplectomorphisms of $\mathcal{M}$. Any symplectic transformation is a symmetry of the system : it merely permutes the motions. This group is generated by all expressions of the form $\bar{z}^n z^m$, generalizing the quadratic subalgebra (4.4).

For any radial potential $U(r^2)$ the initial position $\zeta = e^{i\omega t} z(t)$ (where ω is the angular velocity calculated above) is a good coordinate on the space of motions. We find furthermore that

$$\sigma = \frac{d\bar{z} \wedge dz}{2i} - dH \wedge dt = \frac{d\bar{\zeta} \wedge d\zeta}{2i}, \tag{5.2}$$

proving that the space of motions is, for *any* radial potential, the symplectic ζ-plane. But in two dimensions, the Ω-preserving transformations are the same as the *area preserving diffeomorphisms*. The Lie algebra structure is defined by the Poisson bracket associated with (5.2).

The Hamiltonian, the generator of time translations, is a constant of the motion, and is hence defined on $\mathcal{M}$. It belongs therefore to our algebra w_∞. The spectrum of the corresponding quantum operator should result therefore also from the representation theory of the quantum-deformed version of w_∞.

6 Conclusion

In this paper we studied the Landau problem in two non-commutative frameworks. The models are equivalent in the free case [3], but lead to different (albeit similar) conclusions when interactions are introduced. The difference comes from the way the gauge coupling is defined. In the standard NC approach [7, 4, 5], the Poisson bracket — a *contravariant* structure — is modified; in the Peierls-type one [1, 2, 3], the coupling is achieved using Souriau's *covariant* two-form ϖ. The two "minimal coupling" rules are hence the *duals* of each other. The first one yields a complete Landau-type specturm, and the "exotic" one only yields LLL states. The latter applies to the Fractional Quantum Hall Effect. The standard NC framework may be relevant instead for the Integer Effect [8, 21]

Remarkably, the "NC" Poisson bracket (1.1) and the "exotic" two-form (2.7) [or (3.13)] become both singular for the same critical value $m^* = 0$, necessitating reduction from $4D$ to $2D$ phase space. In the NC case the reduced manifold corresponds to fixing the value of the conserved linear momentum ϖ, and is parametrized by the canonical momentum, $\vec{p}$. The reduced dynamics is given by (3.5). In the exotic case we get a "coordinate" picture, with dynamics (3.11). The motion obeys the Hall law. The trajectories in the two theories are similar; the difference arises owing to the extra kinetic term in the (reduced) Hamiltonian $H^{\rm red}_{NC}$.

A singular Poisson structure with related variational problems has been exhibited in hydrodynamics [22], and in the study of quantum Hall fluids [23]. The models discussed in this paper provide further examples.

The fact that the same simple Peierls-type model also describes superfluid vortex dynamics underlines the physical importance of a classical study. Our results provide further evidence for the intimate relation between the fractional Quantum Hall Effect and superfluidity [14]. Let us remind the Reader that w_∞ is also the symmetry of incompressible fluids [20] in general and of Quantum Hall fluids in particular [19, 23].

It is also worth mentionning that this same infinite-dimensional symmetry is the starting point in another, related approach [24].

Acknowledgement. I am indebted to Professors J. Balog, G. Dunne, J. Gamboa, R. Jackiw, J.-M. Leinaas, L. Martina, G. Tuynman, and E. Varoquaux for correspondence, and to M. Niedermayer and F. Schaposnik for discussions. Special thanks are due to C. Duval for his constant advice, help and interest.

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