Quantum Mechanics in Infinite Symplectic Volume

José M. Isidro
Instituto de Física Corpuscular (CSIC–UVEG)
Apartado de Correos 22085, Valencia 46071, Spain
jmisidro@ific.uv.es

April 25, 2020

Abstract

We quantise complex, infinite–dimensional projective space $\mathbf{CP}(\mathcal{H})$. We apply the result to quantise a complex, finite–dimensional, classical phase space $\mathcal{C}$ whose symplectic volume is infinite, by holomorphically embedding it into $\mathbf{CP}(\mathcal{H})$. The embedding is univocally determined by requiring it to be an isometry between the Bergman metric on $\mathcal{C}$ and the Fubiny–Study metric on $\mathbf{CP}(\mathcal{H})$. Then the Hilbert–space bundle over $\mathcal{C}$ is the pullback, by the embedding, of the Hilbert–space bundle over $\mathbf{CP}(\mathcal{H})$.

Keywords: Quantum mechanics, infinite–dimensional projective space, holomorphic vector bundles.

2001 Pacs codes: 03.65.Bz, 03.65.Ca, 03.65.-w.

2000 MSC codes: 81S10, 81P05.

Contents

1 Introduction

1.1 Notations

Throughout this article, $\mathcal{C}$ will denote a complex n–dimensional, connected, classical phase space, endowed with a symplectic form ω and a complex structure $\mathcal{J}$. We will assume that ω and $\mathcal{J}$ are compatible, so holomorphic coordinate charts on $\mathcal{C}$ will also be Darboux charts. Upon quantisation, $\mathcal{H}$ will denote a complex, separable Hilbert space of quantum states. We will assume $\mathcal{C}$ to have an infinite symplectic volume,

$$\int_{\mathcal{C}} \omega^n = \infty, \tag{1}$$

so $\mathcal{H}$ will be infinite–dimensional. Vector bundles over $\mathcal{C}$ with fibre $\mathcal{H}$ will be called *quantum Hilbert–space bundles*, or $\mathcal{QH}$–bundles for short, and denoted $\mathcal{QH}(\mathcal{C})$.

1.2 Summary of results

Classical phase space is the space of all solutions to the classical equations of motion (modulo gauge transformations that may eventually exist). As such it provides a natural starting point for quantisation. Although quantisation is a well–established procedure, recent developments in string and M–theory call for a revision of our understanding of *classical* vs. *quantum* [1]. With this aim in mind we have analysed the quantisation of a complex, compact $\mathcal{C}$ in ref. [2]. Compactness ensures that the symplectic volume of $\mathcal{C}$ is finite and, therefore, the Hilbert space of quantum states is finite–dimensional. Based on the example when $\mathcal{C} = \mathbf{CP}^n$ we have analysed the different equivalence classes of (finite–dimensional) Hilbert–space bundles over $\mathcal{C}$.

It remains to study the case when $\mathcal{C}$ has an infinite symplectic volume, which we do in the present article extending the technique presented in ref. [2]. This corresponds, *e.g.*, to phase spaces such as $\mathcal{C} = T^*\mathcal{Q}$, the cotangent bundle to some configuration space $\mathcal{Q}$. The latter may well be compact, but its cotangent bundle is definitely noncompact. (Noncompactness of $\mathcal{C}$ is a necessary, though not sufficient, condition for its symplectic volume to be infinite). Of course, the quantisation of $T^*\mathcal{Q}$ has been known for long; there is a Hilbert space $\mathcal{H}$ where quantum states are square–integrable functions on $\mathcal{Q}$. We do not intend to alter this picture. Rather, our aim is to generalise it using the language fibre bundles over $\mathcal{C}$ [3]. In this language, the picture described above usually corresponds to the trivial bundle $\mathcal{H} \times T^*\mathcal{Q}$. The notion of quantum–mechanical duality, as elaborated in ref. [2], suggests considering nonflat (and therefore nontrivial) Hilbert–space bundles over $\mathcal{C}$. This is what we do here.

Our approach may be summarised as follows. We first quantise a particular example of a complex, noncompact $\mathcal{C}$, the infinite–dimensional complex projective space $\mathbf{CP}(\mathcal{H})$. Being infinite–dimensional, it cannot be the phase space of a finite number of degrees of freedom. However $\mathbf{CP}(\mathcal{H})$ is easily quantised following ref. [2]. For this we consider a holomorphic atlas on $\mathbf{CP}(\mathcal{H})$. Over each coordinate chart we erect a certain vector–space fibre; fibres are patched together across overlapping charts according to a set of transition functions. In this way a fibre bundle is constructed whose fibre is a Hilbert space of quantum states. The latter fall into two categories. One is the vacuum, the other its excitations. It turns out that all quantum states (except the vacuum) are holomorphic tangent vectors to $\mathbf{CP}(\mathcal{H})$. This identifies the transition functions as jacobian matrices, and the corresponding subbundle as the holomorphic tangent bundle $T(\mathbf{CP}(\mathcal{H}))$. The vacuum $|0\rangle$ appears as the fibrewise generator of a holomorphic line bundle $N(\mathbf{CP}(\mathcal{H}))$. Now $\mathbf{CP}(\mathcal{H})$ is contractible, hence both $T(\mathbf{CP}(\mathcal{H}))$ and $N(\mathbf{CP}(\mathcal{H}))$ are trivial, as well as their sum, which is the complete quantum Hilbert–space bundle $\mathcal{QH}(\mathbf{CP}(\mathcal{H}))$:

$$\mathcal{QH}(\mathbf{CP}(\mathcal{H})) = T(\mathbf{CP}(\mathcal{H})) \oplus N(\mathbf{CP}(\mathcal{H})). \tag{2}$$

Next an n–dimensional, complex phase space $\mathcal{C}$ with infinite symplectic volume is quantised by holomorphically embedding it into $\mathbf{CP}(\mathcal{H})$. The embedding $\iota : \mathcal{C} \rightarrow \mathbf{CP}(\mathcal{H})$ is univocally determined by the condition that it be an isometry between the Bergman metric on $\mathcal{C}$ and the Fubini–Study metric on $\mathbf{CP}(\mathcal{H})$. Then the bundle $\mathcal{QH}(\mathcal{C})$ is the pullback by ι of the bundle $\mathcal{QH}(\mathbf{CP}(\mathcal{H}))$. In particular, $\mathcal{QH}(\mathcal{C})$ is infinite–dimensional as required by the infinite symplectic volume of $\mathcal{C}$.

The differential geometry of $\mathbf{CP}(\mathcal{H})$ was studied long ago in ref. [4], and its deformation quantisation more recently in the nice paper [5]. We have addressed duality in quantum mechanics from different perspectives in previous publications [6]. For background material see, *e.g.*, refs. [7, 8, 9, 10]. Finally we would also like to mention refs. [11, 12, 13].

2 The space of rays in Hilbert space

Realise $\mathcal{H}$ as the space of infinite sequences of complex numbers $Z^1, Z^2, \ldots$ that are square–summable, $\sum_{j=1}^{\infty} |Z^j|^2 < \infty$. The Z^j provide a set of holomorphic coordinates on $\mathcal{H}$. The space of rays $\mathbf{CP}(\mathcal{H})$ is

$$\mathbf{CP}(\mathcal{H}) = \mathcal{H}/(\mathbf{R}^+ \times U(1)). \tag{3}$$

The Z^j provide a set of *projective* coordinates on $\mathbf{CP}(\mathcal{H})$. Now assume that $Z^k \neq 0$, and define $z^j_{(k)} = Z^j/Z^k$ for $j \neq k$. Then $\sum_{j\neq k}^{\infty} |z^j_{(k)}|^2 < \infty$ for every fixed value of k. As $j \neq k$ varies, these $z^j_{(k)}$ cover one copy of $\mathcal{H}$ that we denote by $\mathcal{U}_k$. The open set $\mathcal{U}_k$, endowed with the coordinate functions $z^j_{(k)}$, $j = 1, 2, \ldots \check{k}, \ldots$, where a check over an index indicates omission, provides a holomorphic coordinate chart on $\mathbf{CP}(\mathcal{H})$ for every fixed k. A holomorphic atlas is obtained as the collection of all pairs $(\mathcal{U}_k, z_{(k)})$, for $k = 1, 2, \ldots$ There are nonempty f–fold overlaps $\cap_{m=1}^{f} \mathcal{U}_m$ for all values of $f = 1, 2, \ldots$ Holomorphic tangent vectors and holomorphic 1–forms on $\mathcal{U}_k$ are spanned by $\partial/\partial z^j_{(k)}$ and $\mathrm{d}z^j_{(k)}$. When $f = 2$ above, these vectors and covectors transform according to an (infinite–dimensional) jacobian matrix and its transpose.

$\mathbf{CP}(\mathcal{H})$ is a Kähler manifold. On the coordinate chart $(\mathcal{U}_k, z_{(k)})$, the Kähler potential reads

$$K(z_{(k)}, \bar{z}_{(k)}) = \log\left(1 + \sum_{j\neq k}^{\infty} z^j_{(k)} \bar{z}^j_{(k)}\right), \tag{4}$$

and the corresponding metric $\mathrm{d}s^2_K$ reads on this chart

$$\mathrm{d}s^2_K = \sum_{m,n\neq k}^{\infty} \frac{\partial^2 K(z_{(k)}, \bar{z}_{(k)})}{\partial z^m_{(k)} \partial \bar{z}^n_{(k)}} \mathrm{d}z^m_{(k)} \mathrm{d}\bar{z}^n_{(k)}. \tag{5}$$

Being infinite–dimensional, $\mathbf{CP}(\mathcal{H})$ is noncompact. It is simply connected:

$$\pi_1\left(\mathbf{CP}(\mathcal{H})\right) = 0. \tag{6}$$

It is contractible to a point [14], so its Picard group is trivial:

$$\mathrm{Pic}\left(\mathbf{CP}(\mathcal{H})\right) = 0. \tag{7}$$

It has trivial homology in odd real dimension,

$$H_{2k+1}\left(\mathbf{CP}(\mathcal{H}), \mathbf{Z}\right) = 0, \qquad k = 0, 1, \ldots, \tag{8}$$

while it is nontrivial in even dimension,

$$H_{2k}\left(\mathbf{CP}(\mathcal{H}), \mathbf{Z}\right) = \mathbf{Z}, \qquad k = 0, 1, \ldots \tag{9}$$

3 Quantisation of $\mathbf{CP}(\mathcal{H})$

Since $\mathrm{Pic}\left(\mathbf{CP}(\mathcal{H})\right)$ is trivial, there exists a unique equivalence class of holomorphic lines bundles over $\mathbf{CP}(\mathcal{H})$; using the notations of ref. [2] let us denote it by $N(\mathbf{CP}(\mathcal{H}))$. The latter is a trivial bundle, where the fibre $\mathbf{C}$ is generated by the vacuum state $|0\rangle$. Triviality implies that $|0\rangle$ transforms with the identity when changing coordinates on $\mathbf{CP}(\mathcal{H})$. Let $A_j^{\dagger}(k)$, $A_j(k)$, $j \neq k$, be creation and annihilation operators on the chart $\mathcal{U}_k$, for k fixed. Following ref. [2] we identify $A_j(k) = \partial/\partial z_{(k)}^j$ and $A_j^{\dagger}(k) = \mathrm{d}z_{(k)}^j$. We can now construct the $\mathcal{QH}$–bundle over $\mathbf{CP}(\mathcal{H})$. To this end we will describe the fibre over each coordinate chart $\mathcal{U}_k$, plus the transition functions on the 2–fold overlaps $\mathcal{U}_k \cap \mathcal{U}_m$, for all $k \neq m$.

The Hilbert–space fibre over $\mathcal{U}_k$ is $\mathcal{H}$ itself, the latter being the $\mathbf{C}$–linear span of the infinite set of linearly independent vectors

$$|0\rangle, \qquad A_j^{\dagger}(k)|0\rangle, \qquad j = 1, 2, \ldots, \check{k}, \ldots \tag{10}$$

Reasoning as in ref. [2] one proves that, on the 2–fold overlaps $\mathcal{U}_k \cap \mathcal{U}_m$, the fibre $\mathcal{H}$ can be chosen in either of two equivalent ways. $\mathcal{H}$ is either the $\mathbf{C}$–linear span of the vectors $|0\rangle$, $A_j^{\dagger}(k)|0\rangle$, for $j = 1, 2, \ldots, \check{k}, \ldots$, or the $\mathbf{C}$–linear span of the vectors $|0\rangle$, $A_j^{\dagger}(m)|0\rangle$, for $j = 1, 2, \ldots, \check{m}, \ldots$

Arguments identical to those of ref. [2] prove that all the states of eqn. (10), except the vacuum $|0\rangle$, are (co)tangent vectors to $\mathbf{CP}(\mathcal{H})$ on the chart $\mathcal{U}_k$, and thus transition functions are jacobian matrices. Identifying the tangent and cotangent bundles we can write

$$\mathcal{QH}(\mathbf{CP}(\mathcal{H})) = N(\mathbf{CP}(\mathcal{H})) \oplus T(\mathbf{CP}(\mathcal{H})). \tag{11}$$

Contractibility of $\mathbf{CP}(\mathcal{H})$ implies that the tangent bundle $T(\mathbf{CP}(\mathcal{H}))$ is also trivial, hence the complete $\mathcal{QH}$–bundle is trivial:

$$\mathcal{QH}(\mathbf{CP}(\mathcal{H})) = \mathbf{CP}(\mathcal{H}) \times \mathcal{H}. \tag{12}$$

Of course, triviality of $\mathcal{QH}(\mathbf{CP}(\mathcal{H}))$ could have been established more simply by observing that $\mathbf{CP}(\mathcal{H})$ is contractible. However the previous reasoning was needed in order to elucidate its structure as the direct sum (11), which proves that all quantum states except the vacuum are tangent vectors to $\mathbf{CP}(\mathcal{H})$.

4 Quantisation of $\mathcal{C}$

Sections 4.1, 4.2 present a summary, drawn from ref. [4], on how to embed $\mathcal{C}$ holomorphically within $\mathbf{CP}(\mathcal{H})$. This procedure is applied in section 4.3 in order to quantise $\mathcal{C}$. For background material see, *e.g.*, ref. [15].

4.1 The Bergman metric on $\mathcal{C}$

Denote by $\mathcal{F}$ the set of holomorphic, square–integrable n–forms on $\mathcal{C}$. $\mathcal{F}$ is a separable, complex Hilbert space (finite–dimensional when $\mathcal{C}$ is compact). Let $h_1, h_2, \ldots$ denote a complete orthonormal basis for $\mathcal{F}$, and let z be (local) holomorphic coordinates on $\mathcal{C}$. Then

$$\mathcal{K}(z, \bar{w}) = \sum_{j=1}^{\infty} h_j(z) \wedge \bar{h}_j(\bar{w}) \tag{13}$$

is a holomorphic $2n$–form on $\mathcal{C} \times \bar{\mathcal{C}}$, where $\bar{\mathcal{C}}$ is complex manifold conjugate to $\mathcal{C}$. The form $\mathcal{K}(z, \bar{w})$ is independent of the choice of an orthonormal basis for $\mathcal{F}$; it is called the *kernel form* of $\mathcal{C}$. If $\bar{z}$ is the point of $\bar{\mathcal{C}}$ corresponding to a point $z \in \mathcal{C}$, the set of pairs $(z, \bar{z}) \in \mathcal{C} \times \bar{\mathcal{C}}$ is naturally identified with $\mathcal{M}$. In this way $\mathcal{K}(z, \bar{z})$ can be considered as a $2n$–form on $\mathcal{C}$. One can prove that $\mathcal{K}(z, \bar{z})$ is invariant under the group of holomorphic transformations of $\mathcal{C}$.

Next assume that, given any point $z \in \mathcal{C}$, there exists an $f \in \mathcal{F}$ such that $f(z) \neq 0$. That is, the kernel form $\mathcal{K}(z, \bar{z})$ of $\mathcal{C}$ is everywhere nonzero on $\mathcal{C}$:

$$\mathcal{K}(z, \bar{z}) \neq 0, \qquad \forall z \in \mathcal{C}. \tag{14}$$

Let us write, in local holomorphic coordinates z^j on $\mathcal{C}$, $j = 1, \ldots, n$,

$$\mathcal{K}(z, \bar{z}) = \mathbf{k}(z, \bar{z})\, \mathrm{d}z^1 \wedge \ldots \wedge \mathrm{d}z^n \wedge \mathrm{d}\bar{z}^1 \wedge \ldots \wedge \mathrm{d}\bar{z}^n, \tag{15}$$

for a certain everywhere nonzero function $\mathbf{k}(z, \bar{z})$. Define a hermitean form $\mathrm{d}s^2_B$

$$\mathrm{d}s^2_B = \sum_{j,k=1}^{n} \frac{\partial^2 \log \mathbf{k}}{\partial z^j \bar{z}^k}\, \mathrm{d}z^j \mathrm{d}\bar{z}^k. \tag{16}$$

One can prove that $\mathrm{d}s^2_B$ is independent of the choice of coordinates on $\mathcal{C}$. Moreover, it is positive semidefinite and invariant under the holomorphic transformations of $\mathcal{C}$.

Let us make the additional assumption that $\mathcal{C}$ is such that $\mathrm{d}s^2_B$ is positive definite,

$$\mathrm{d}s^2_B > 0. \tag{17}$$

Then $\mathrm{d}s^2_B$ defines a (Kähler) metric called the *Bergman metric* on $\mathcal{C}$.

4.2 Embedding $\mathcal{C}$ within $\mathbf{CP}(\mathcal{H})$

Let $\mathcal{H}$ be the Hilbert space dual to $\mathcal{F}$. Given $f \in \mathcal{F}$, let its expansion in local coordinates be

$$f = \mathbf{f}\, \mathrm{d}z^1 \wedge \ldots \wedge \mathrm{d}z^n, \tag{18}$$

for a certain function $\mathbf{f}$. Let ι' denote the mapping that sends $z \in \mathcal{C}$ into $\iota'(z) \in \mathcal{H}$ defined by

$$\langle \iota'(z) | f \rangle = \mathbf{f}(z). \tag{19}$$

Then $\iota'(z) \neq 0$ for all $z \in \mathcal{C}$ if and only if property (14) holds. Assuming that the latter is satisfied, and denoting by p' the natural projection from $\mathcal{H} - \{0\}$ onto $\mathbf{CP}(\mathcal{H})$, the composite map $\iota = p' \circ \iota'$

$$\iota : \mathcal{C} \to \mathbf{CP}(\mathcal{H}) \tag{20}$$

is well defined on $\mathcal{C}$, independent of the coordinates, and holomorphic.

One can prove the following results. When property (14) is true, the quadratic differential form $\mathrm{d}s^2_B$ of eqn. (16) is the pullback, by ι, of the canonical Kähler metric $\mathrm{d}s^2_K$ of eqn. (5):

$$\mathrm{d}s^2_B = \iota^*(\mathrm{d}s^2_K). \tag{21}$$

Moreover, the differential of ι is nonsingular at every point of $\mathcal{C}$ if and only if property (17) is satisfied. These two results give us a geometric interpretation of the Bergman metric. Namely, if properties (14) and (17) hold, then ι is an isometric immersion of $\mathcal{C}$ into $\mathbf{CP}(\mathcal{H})$.

The map ι is locally one–to–one in the sense that every point of $\mathcal{C}$ has a neighbourhood that is mapped injectively into $\mathbf{CP}(\mathcal{H})$. However, ι is not necessarily injective in the large. Conditions can be found that ensure injectivity of ι in the large. Assume that, if z, z' are any two distinct points of $\mathcal{C}$, an $f \in \mathcal{F}$ can be found such that

$$f(z) \neq 0, \qquad f(z') = 0. \tag{22}$$

Then ι is injective. Therefore, if $\mathcal{C}$ satisfies assumptions (14), (17) and (22), it can be holomorphically and isometrically embedded into $\mathbf{CP}(\mathcal{H})$.

4.3 Quantisation of $\mathcal{C}$ as a submanifold of $\mathbf{CP}(\mathcal{H})$

Finally we quantise a noncompact $\mathcal{C}$ with infinite symplectic volume. At any point of $\mathcal{C}$ there are only n linearly independent, holomorphic tangent vectors. Applied to $\mathcal{C}$, the construction of ref. [2] provides a finite–dimensional $\mathcal{H}$, contrary to eqn. (1). Therefore we try an alternative route.

We need an infinite–dimensional $\mathcal{QH}$–bundle over $\mathcal{C}$. For this purpose we assume embedding $\mathcal{C}$ holomorphically and injectively within $\mathbf{CP}(\mathcal{H})$ as in eqn. (20). Then the bundle $\mathcal{QH}(\mathbf{CP}(\mathcal{H}))$ of eqn. (12) can be pulled back to $\mathcal{C}$ by the embedding ι. We take this *to define* the bundle $\mathcal{QH}(\mathcal{C})$:

$$\mathcal{QH}(\mathcal{C}) = \iota^* \mathcal{QH}(\mathbf{CP}(\mathcal{H})). \tag{23}$$

Even though $\mathcal{QH}(\mathbf{CP}(\mathcal{H}))$ is trivial, it may contain nonflat (hence nontrivial) subbundles, thus allowing for nontrivial dualities.

A detailed analysis of $\mathcal{QH}(\mathcal{C})$ requires specifying $\mathcal{C}$ explicitly. However some properties can be stated in general. Thus, *e.g.*, the kernel form is the quantum–mechanical propagator [16]. On $\mathbf{C}^n$ it reads

$$\mathcal{K}_{\mathbf{C}^n}(z, \bar z) = N \exp\left(\mathrm{i} \sum_{j=1}^{n} \bar z^j z^j\right) \mathrm{d}z^1 \wedge \ldots \wedge \mathrm{d}z^n \wedge \mathrm{d}\bar z^1 \wedge \ldots \wedge \mathrm{d}\bar z^n, \tag{24}$$

where N is some normalisation. The Bergman metric (16) derived from this kernel is the standard Hermitean metric on $\mathbf{C}^n$. The embedding ι naturally relates physical information (the propagator) and geometric information (the metric on $\mathcal{C}$). In retrospective, this justifies our quantisation of $\mathcal{C}$ by embedding it within $\mathbf{CP}(\mathcal{H})$. An important physical input is the relation between the symplectic volume of $\mathcal{C}$ and the number of linearly independent quantum states. It would be interesting to enquire into a possible geometric origin for this fact.

Acknowledgements

It is a great pleasure to thank J. de Azcárraga, U. Bruzzo and M. Schlichenmaier for encouragement and discussions. This work has been partially supported by research grant BFM2002–03681 from Ministerio de Ciencia y Tecnología and EU FEDER funds.

References

[1] C. Vafa, `hep-th/9702201`.

[2] J.M. Isidro, `hep-th/0304175`.

[3] M. Reuter, *Int. J. Mod. Phys.* **A22** (1998) 3835.

[4] S. Kobayashi, *Trans. Am. Math. Soc.* **92** (1959) 267.

[5] H. García–Compeán, J. Plebański, M. Przanowski and F. Turrubiates, *J. Phys.* **A35** (2002) 4301.

[6] J.M. Isidro, *Int. J. Mod. Phys.* **A16** (2001) 3853; *J. Phys. A: Math. Gen.* **35** (2002) 3305; *Phys. Lett.* **A301** (2002) 210; `quant-ph/0304143`.

[7] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Wiley, New York (1996).

[8] J. de Azcárraga and J. Izquierdo, *Lie Groups, Lie Algebras, Cohomology and some Applications in Physics*, Cambridge University Press, Cambridge (1995).

[9] M. Schlichenmaier, *An Introduction to Riemann Surfaces, Algebraic Curves and Moduli Spaces*, Springer Lecture Notes in Physics **322**, Berlin (1989).

[10] *Geometry and Physics of Branes*, eds. U. Bruzzo, V. Gorini and U. Moschella, Institute of Physics, Bristol (2002).

[11] A. Faraggi and M. Matone, *Int. J. Mod. Phys.* **A15** (2000) 1869;
G. Bertoldi, A. Faraggi and M. Matone, *Class. Quant. Grav.* **17** (2000) 3965.

[12] J. Anandan, *Int. J. Theor. Phys.* **41** (2002) 199.

[13] K. Fujii, *J. Math. Phys.* **41** (2000) 4406; *Rept. Math. Phys.* **48** (2001) 75.

[14] N. Kuiper, *Topology* **3** (1965) 19.

[15] A. Perelomov, *Generalized Coherent States and their Applications*, Springer Texts and Monographs in Physics, Berlin (1986).

[16] W. Dittrich and M. Reuter, *Classical and Quantum Dynamics*, Springer, Berlin (2001).