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Fuzzy Sphere and Hyperbolic Space from Deformation Quantization

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Abstract

We explicitly construct noncommutative $*$ products on circularly symmetric two dimensional space by using the technique of Fedosov's deformation quantization. Especially, on constant curvature spaces i.e., S^2 and H^2, we get $su(2)$ and $su(1,1)$ algebra respectively. These are candidates of $*$ products applicable to noncommutative field theories or noncommutative gauge theories on spaces with nontrivial symplectic structure.

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1 Introduction

Since the relation between string theory and noncommutative geometry was discussed in [1], noncommutative field theories and noncommutative gauge theories have been investigated enthusiastically from various viewpoints.

Many authors use the Moyal product[1] as noncommutative associative $*$ product for explicit calculations. It corresponds to a constant NS-NS B-field background in flat space in the context of string theory. On the other hand, at least formally, more general $*$ products which may correspond to string theory on nonconstant B-field background in curved space are defined by some authors[2]. However, explicit form of $*$ products other than the Moyal product has been scarcely discussed in physical context[3].

In this paper, we use the technique of Fedosov's deformation quantization [3] to get explicit forms of $*$ products on nontrivial backgrounds. For simplicity, we investigate $*$ products on circularly symmetric two dimensional spaces. Specifically, we focus on constant curvature spaces S^2, H^2 and $\mathbb{R}^2$, and explicitly construct $*$ products which are different from the Moyal product. We also discuss some physical applications of our $*$ products.

2 Construction of $*$ product

Here we review the construction of Fedosov's $*$ product very briefly[4], and apply this procedure to circularly symmetric two dimensional spaces.

First, for a given symplectic manifold (M, Ω_0), we define the Weyl algebra bundle W which has $\circ$ product of the Moyal type and its Abelian connection D with some input parameter. For $\mathrm{Ker} D \subset W$ (which is called flat section W_D), we get a one to one correspondence with $C^\infty(M)[[\hbar]]$, where $\hbar$ is the deformation parameter. We denote the map from $C^\infty(M)[[\hbar]]$ to W_D as Q, and its inverse map as σ. Then Fedosov's $*$ product on $C^\infty(M)[[\hbar]]$ is defined by

$$a_0 * b_0 := \sigma(Q(a_0) \circ Q(b_0)), \quad a_0, b_0 \in C^\infty(M)[[\hbar]]. \tag{1}$$

This is a solution of the problem of deformation quantization, i.e.,

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[1] Here we call $* = \exp\left(\frac{i}{2}\frac{\overleftarrow{\partial}}{\partial x^i}\theta^{ij}\frac{\overrightarrow{\partial}}{\partial x^j}\right)$ with constant $\theta^{ij} = -\theta^{ji}$ the Moyal product.

[2] [2],[3], for example.

[3] In [4], nonassociative star product which generalizes [2],[3] is discussed to describe D-brane in curved backgrounds.

[4] See [3],[5] for details.

$*$ is associative and its commutator $[\ ,\ ]_*$ is expanded as

$$[\ ,\ ]_* = i\hbar\{\ ,\ \} + \mathcal{O}(\hbar^2) \tag{2}$$

where $\{\ ,\ \}$ is the Poisson bracket with respect to the symplectic form Ω_0.

Now, we apply this procedure to a two dimensional space M with metric

$$ds^2 = e^{\Phi(r)}(dr^2 + r^2 d\theta^2), \tag{3}$$

where $\Phi(r)$ is some function of r only (i.e. circularly symmetric space) for simplicity. Its volume form is given by

$$\Omega_0 = e^{\Phi(r)} r dr \wedge d\theta, \tag{4}$$

and we identify it with symplectic form. Using Fedosov's procedure with the input[5]

$$\begin{aligned}
&\Omega_0 = \theta^1 \wedge \theta^2 = -\frac{1}{2}\omega_{ij}\theta^i \wedge \theta^j, \\
&\theta^1 = e^{\Phi(r)} dr, \quad \theta^2 = r d\theta, \quad \omega_{ij} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \\
&\Omega_1 = 0, \ \nabla = d, \\
&\mu = \frac{1}{3} e^{-\Phi(r)} r^{-1} (y^1)^2 y^2,
\end{aligned} \tag{5}$$

we get an Abelian connection D as

$$\begin{aligned}
&Da = da - \delta a + \frac{i}{\hbar}(\mathrm{r} \circ a - a \circ \mathrm{r}), \quad a \in W, \\
&\mathrm{r} = e^{-\Phi(r)} r^{-1} y^1 y^2 \theta^1, \\
&\circ := \exp\left(-\frac{i\hbar}{2} \frac{\overleftarrow{\partial}}{\partial y^i} \omega^{ij} \frac{\overrightarrow{\partial}}{\partial y^j} \right), \quad \omega^{ij} := (\omega^{-1})^{ij}.
\end{aligned} \tag{6}$$

For this Abelian connection D, we solve the equation $Da = 0$ and get the map $Q : C^\infty(M)[[\hbar]] \to W_D$ as

$$a = Q(a_0(r,\theta)) = a_0\left(G(r, y^1), \theta + \frac{y^2}{r} \right), \tag{7}$$

where $G(r, y^1)$ is given by

$$\int_r^{G(r,y^1)} e^{\Phi(r')} r' dr' = y^1 r. \tag{8}$$

Then we can define a $*$ product on M by eq.(1).

[5]See [3],[5] for the meaning of $\nabla, \Omega_1, \mu, \delta$. Here we choose these parameters in such a way that the iteration formula (eq.(21) of [5]) which gives an Abelian connection is satisfied trivially, i.e., $\nabla \mathrm{r} + \frac{i}{\hbar}\mathrm{r} \circ \mathrm{r} = 0$. Then we get $\mathrm{r} = \delta\mu + \delta^{-1}(d(\omega_{ij} y^i \theta^j) - \Omega_1)$ for the input (5).

3 S^2 case

In this section we apply the result of §2 to the case $M = S^2$. We consider 2-sphere S^2 with radius R, which is defined as two dimensional surface embedded in $\mathbb{R}^3$:

$$(X^1)^2 + (X^2)^2 + (X^3)^2 = R^2. \tag{9}$$

We parametrize the coordinate $X^i, i = 1, 2, 3$ on S^2 as

$$X^1 = \frac{2R^2 r}{r^2 + R^2}\cos\theta,\ \ X^2 = \frac{2R^2 r}{r^2 + R^2}\sin\theta,\ \ X^3 = R\frac{r^2 - R^2}{r^2 + R^2}, \\ r \geq 0,\ \ 0 \leq \theta \leq 2\pi. \tag{10}$$

Then the metric of S^2, $ds^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2$, is given by

$$ds^2 = \frac{4R^4}{(r^2 + R^2)^2}(dr^2 + r^2 d\theta^2), \tag{11}$$

and the conformal factor e^Φ of eq.(3) is identified as

$$e^{\Phi(r)} = \frac{4R^4}{(r^2 + R^2)^2}. \tag{12}$$

From eqs. (12), (7) and (1), we get the explicit form of our $*$ product on S^2:

$$\begin{aligned} & a_0(r,\theta) * b_0(r,\theta) \\ = & \left(a_0\left(\sqrt{\frac{r^2 + \frac{y^1}{2R^2} r(r^2 + R^2)}{1 - \frac{y^1}{2R^4} r(r^2 + R^2)}}, \theta + \frac{y^2}{r} \right) \exp\left(-\frac{i\hbar}{2}\left(\frac{\overleftarrow{\partial}}{\partial y^1}\frac{\overrightarrow{\partial}}{\partial y^2} - \frac{\overleftarrow{\partial}}{\partial y^2}\frac{\overrightarrow{\partial}}{\partial y^1} \right) \right) \right. \\ & \left. \cdot\, b_0\left(\sqrt{\frac{r^2 + \frac{y^1}{2R^2} r(r^2 + R^2)}{1 - \frac{y^1}{2R^4} r(r^2 + R^2)}}, \theta + \frac{y^2}{r} \right) \right)_{y^1 = y^2 = 0}. \end{aligned} \tag{13}$$

By using this definition, we can calculate $*$ product of the S^2 coordinate X^i (10). In particular, we have

$$[X^i, X^j]_* = i\frac{\hbar}{R}\varepsilon^{ijk}X^k, \tag{14}$$

$$X^1 * X^1 + X^2 * X^2 + X^3 * X^3 = R^2\left(1 - \frac{\hbar^2}{4R^4}\right), \tag{15}$$

where ε^{ijk} is the antisymmetric tensor with $\varepsilon^{123} = +1$. Eq.(14) means that the commutators of X^i's form $su(2)$ algebra which is known as fuzzy sphere algebra, and eq.(15) means that its radius is given by $R\sqrt{1 - \frac{\hbar^2}{4R^4}}$ which is deformed by $\mathcal{O}(\hbar^2)$ from the original radius R of commutative S^2 (9). Namely, we have obtained a fuzzy sphere by deforming S^2 using the $*$ product (13).

4 H^2 case

In this section we apply the result of §2 to the case $M = H^2$. Calculation is quite similar to the S^2 case (§3). We consider two dimensional hyperbolic space H^2 with radius R, which is defined as two dimensional surface embedded in $\mathbb{R}^{1,2}$:

$$-(Y^0)^2 + (Y^1)^2 + (Y^2)^2 = -R^2, \quad Y^0 > 0. \tag{16}$$

We parametrize the coordinates $Y^i, i = 0, 1, 2$ on H^2 as

$$\begin{aligned} &Y^0 = R\frac{R^2 + r^2}{R^2 - r^2},\ Y^1 = \frac{2R^2 r}{R^2 - r^2}\cos\theta,\ Y^2 = \frac{2R^2 r}{R^2 - r^2}\sin\theta, \\ &0 \le r \le R,\ 0 \le \theta \le 2\pi. \end{aligned} \tag{17}$$

Then, the metric of H^2, $ds^2 = -(dY^0)^2 + (dY^1)^2 + (dY^2)^2$, and the conformal factor are given respectively by

$$ds^2 = \frac{4R^4}{(R^2 - r^2)^2}(dr^2 + r^2 d\theta^2), \tag{18}$$

$$e^{\Phi(r)} = \frac{4R^4}{(R^2 - r^2)^2}. \tag{19}$$

From eqs. (19), (7) and (1), we get the explicit form of our $*$ product on H^2:

$$\begin{aligned} &a_0(r,\theta) * b_0(r,\theta) \\ = &\left(a_0 \left(\sqrt{\frac{r^2 + \frac{y^1}{2R^2} r(R^2 - r^2)}{1 + \frac{y^1}{2R^4} r(R^2 - r^2)}}, \theta + \frac{y^2}{r} \right) \exp\left(-\frac{i\hbar}{2} \left(\frac{\overleftarrow{\partial}}{\partial y^1}\frac{\overrightarrow{\partial}}{\partial y^2} - \frac{\overleftarrow{\partial}}{\partial y^2}\frac{\overrightarrow{\partial}}{\partial y^1} \right) \right) \right. \\ &\left. \cdot\, b_0 \left(\sqrt{\frac{r^2 + \frac{y^1}{2R^2} r(R^2 - r^2)}{1 + \frac{y^1}{2R^4} r(R^2 - r^2)}}, \theta + \frac{y^2}{r} \right) \right)\Bigg|_{y^1 = y^2 = 0}. \end{aligned} \tag{20}$$

By using this definition, we obtain the following $*$ products of the H^2 coordinate Y^i (17):

$$[Y^0, Y^1]_* = i\frac{\hbar}{R}Y^2, \quad [Y^2, Y^0]_* = i\frac{\hbar}{R}Y^1, \quad [Y^1, Y^2]_* = -i\frac{\hbar}{R}Y^0, \tag{21}$$

$$-Y^0 * Y^0 + Y^1 * Y^1 + Y^2 * Y^2 = -R^2\left(1 - \frac{\hbar^2}{4R^4}\right). \tag{22}$$

Eq.(21) means that commutators of Y^i's form $su(1,1)$ algebra which corresponds to isometry of H^2, and eq.(22) means that its radius is given by $R\sqrt{1 - \frac{\hbar^2}{4R^4}}$ which is deformed by $\mathcal{O}(\hbar^2)$ from the original radius R of commutative H^2 (16). Namely, we get fuzzy hyperbolic space by deforming H^2 using the $*$ product (20).

5 Large R limit and $\mathbb{R}^2$

Here we consider large radius limit of the results of §3 and §4. The sectional curvature of S^2 (9) (H^2 (16)) is $\frac{1}{R^2}$ ($-\frac{1}{R^2}$), which tends to $+0$ (-0) in the limit $R \to \infty$. Therefore they approach the flat space $\mathbb{R}^2$ in the large R limit in the usual commutative picture. How about it from the noncommutative viewpoint?

For comparison, we construct a $*$ product on $\mathbb{R}^2$ following the method of §2. We adopt as its flat metric

$$ds^2 = 4(dr^2 + r^2 d\theta^2) \tag{23}$$

with its front factor 4 chosen so that (23) coincides with the large R limit of (11) and (18). With $e^\Phi = 4$, we get the explicit form of our $*$ product on $\mathbb{R}^2$:

$$\begin{aligned} & a_0(r,\theta) * b_0(r,\theta) \\ &= \left(a_0 \left(\sqrt{r^2 + \frac{y^1 r}{2}}, \theta + \frac{y^2}{r} \right) \exp\left(-\frac{i\hbar}{2} \left(\frac{\overleftarrow{\partial}}{\partial y^1} \frac{\overrightarrow{\partial}}{\partial y^2} - \frac{\overleftarrow{\partial}}{\partial y^2} \frac{\overrightarrow{\partial}}{\partial y^1} \right) \right) \right. \\ & \left. \cdot\, b_0 \left(\sqrt{r^2 + \frac{y^1 r}{2}}, \theta + \frac{y^2}{r} \right) \right) \Bigg|_{y^1 = y^2 = 0} . \end{aligned} \tag{24}$$

Then, we can calculate the $*$ products of the complex coordinate $z := re^{i\theta},\ \bar{z} := re^{-i\theta}$:

$$\begin{aligned} & z * z = \sqrt{r^4 - \frac{\hbar^2}{16}} e^{2i\theta} = \overline{\bar{z} * \bar{z}}, \quad z * \bar{z} = r^2 - \frac{\hbar}{4}, \quad \bar{z} * z = r^2 + \frac{\hbar}{4}, \\ & [z, \bar{z}]_* = -\frac{\hbar}{2}. \end{aligned} \tag{25}$$

The commutator $[z,\bar{z}]_*$ coincides with that of the usual Moyal product for Cartesian coordinates on $\mathbb{R}^2$, but $*$ product itself is different from the Moyal product. This difference comes from ambiguity of deformation quantization.

We can calculate the commutator $[z,\bar{z}]_*$ also in the S^2 and H^2 cases. For S^2, from eq.(13) we get

$$[z, \bar{z}]_* = \frac{-\frac{\hbar}{2R^4}(r^2 + R^2)^2}{1 - \left(\frac{\hbar}{4R^4}(r^2 + R^2) \right)^2} = -\frac{\hbar}{2R^4}(R^2 + z * \bar{z})(R^2 + \bar{z} * z). \tag{26}$$

And for H^2, from eq.(20) we get

$$[z, \bar{z}]_* = \frac{-\frac{\hbar}{2R^4}(R^2 - r^2)^2}{1 - \left(\frac{\hbar}{4R^4}(R^2 - r^2) \right)^2} = -\frac{\hbar}{2R^4}(R^2 - z * \bar{z})(R^2 - \bar{z} * z). \tag{27}$$

Both eqs.(26) and (27) are reduced to $[z,\bar{z}]_* = -\frac{\hbar}{2}$ (25) as $R \to \infty$. In other words, the $*$ product which we obtained in §2 connects $su(2)$ algebra (or fuzzy S^2) with $su(1,1)$ algebra (or fuzzy H^2) through $R = \infty$.

6 An application

In the previous sections, we explicitly calculated $*$ products by using Fedosov's formulation. They are candidates of $*$ product for defining noncommutative field theory or noncommutative gauge theory on fuzzy S^2, H^2 and $\mathbb{R}^2$.

As an example, we discuss four dimensional noncommutative $U(1)$ gauge theory with one scalar field which is given by the action[6]

$$S = \mathrm{Tr}\left(\frac{1}{4}G^{IJ}G^{KL}F_{IK} * F_{JL} + \frac{1}{2}G^{IJ}D_I\phi * D_J\phi\right). \tag{28}$$

We assume that only two dimensional space is noncommutative (1,2 direction), and use a general formulation of noncommutative gauge theory of [5]:

$$\begin{aligned}
&G^{IJ} = \delta^{IJ}, \ I, J = 1, \cdots, 4, \\
&F_{IJ} = \partial_I A_J - \partial_J A_I - i[A_I, A_J]_* - \frac{J_{IJ}}{\hbar}, \quad J_{12} = -J_{21} = 1, \text{others} = 0, \\
&\partial_I = \frac{i}{\hbar}[-J_{IJ}\tilde{\phi}^J, \ ]_*, \ I = 1, 2, \quad \partial_3 = \frac{\partial}{\partial x^3}, \partial_4 = \frac{\partial}{\partial x^4} \\
&D_I\phi = \partial_I\phi - i[A_I, \phi]_*,
\end{aligned} \tag{29}$$

Here, $\tilde{\phi}^I$ is the "canonical" noncommutative coordinate satisfying

$$\frac{i}{\hbar}[\tilde{\phi}^1, \tilde{\phi}^2]_* = 1. \tag{30}$$

Its explicit form is

$$\tilde{\phi}^1 = \frac{2Rr}{\sqrt{r^2 + R^2}}\cos\theta, \ \tilde{\phi}^2 = \frac{2Rr}{\sqrt{r^2 + R^2}}\sin\theta \tag{31}$$

for fuzzy S^2 (13),

$$\tilde{\phi}^1 = \frac{2Rr}{\sqrt{R^2 - r^2}}\cos\theta, \ \tilde{\phi}^2 = \frac{2Rr}{\sqrt{R^2 - r^2}}\sin\theta \tag{32}$$

for fuzzy H^2 (20), and

$$\tilde{\phi}^1 = 2r\cos\theta, \ \tilde{\phi}^2 = 2r\sin\theta \tag{33}$$

for fuzzy $\mathbb{R}^2$ (24). The action (28) is invariant under noncommutative $U(1)$ gauge transformation:

$$\delta_\lambda A_I = \partial_I\lambda - i[A_I, \lambda]_*, \qquad \delta_\lambda\phi = -i[\phi, \lambda]_*. \tag{34}$$

The equations of motion of (28) are

$$D^I F_{IJ} = -i[\phi, D_J\phi]_*, \ D^I D_I\phi = 0, \tag{35}$$

[6]The symbol Tr is trace for the $*$ product satisfying $\mathrm{Tr} f * g = \mathrm{Tr} g * f$ [3], but we can discuss equations of motion without using the explicit form of the trace.

and we obtain a solution by solving the $U(1)$ noncommutative BPS equation:

$$B_I = D_I\phi, I = 1,2,3, \quad \partial_4 = 0, A_4 = 0, \quad B_I := \frac{1}{2}\varepsilon^{IJK}\left(F_{JK} + \frac{J_{JK}}{\hbar}\right). \tag{36}$$

Under the ansatz

$$\begin{aligned} &A_1 + iA_2 = if_A(l,x^3)(\tilde{\phi}^1 + i\tilde{\phi}^2), \qquad A_3 = 0, \\ &\phi = f(l,x^3), \qquad l := \sqrt{(\tilde{\phi}^1)^2 + (\tilde{\phi}^2)^2 + (x^3)^2}, \end{aligned} \tag{37}$$

eq.(36) can be rewritten as

$$\begin{aligned} &\partial_3 G^{(m)} - 4\partial_L f^{(m)} = \sum_{\substack{2n+k=m,\\ n\geq 1}} \frac{4\partial_L^{2n+1}f^{(k)}}{(2n+1)!} + \sum_{\substack{2n+k+k'\\ =m-1}} \frac{4G^{(k')}\partial_L^{2n+1}f^{(k)}}{(2n+1)!}, \\ &\partial_3 f^{(m)} - \partial_L(LG^{(m)}) = \sum_{\substack{2n+k=m,\\ n\geq 1}} \frac{\partial_L^{2n+1}(LG^{(k)})}{(2n+1)!} \end{aligned} \tag{38}$$

with

$$L := (\tilde{\phi}^1)^2 + (\tilde{\phi}^2)^2, \quad f = \sum_{k=0}^{\infty}\hbar^k f^{(k)}, \quad \left(\frac{1}{\hbar} + f_A\right)^2 = \frac{1}{\hbar^2} + \frac{1}{\hbar}\sum_{k=0}^{\infty}\hbar^k G^{(k)}. \tag{39}$$

We can solve eq.(38) order by order in $\hbar$, and we get

$$\begin{aligned} f &= \frac{g}{l} + \hbar g^2\left(\frac{2x^3}{l^4} - \frac{1}{l^3}\right) + \hbar^2\left(\frac{-8g^3x^3}{l^6} - \frac{g}{4l^5} + \left(\frac{5g}{8} + 10g^3\right)\frac{(x^3)^2}{l^7}\right) + \mathcal{O}(\hbar^3), \\ f_A &= \frac{g}{l(l+x^3)} + \hbar g^2\left(\frac{2}{l^4} - \frac{1}{l^3(l+x^3)} - \frac{1}{2l^2(l+x^3)^2}\right) \\ &\quad + \hbar^2\left(\frac{-8g^3}{l^6} + \frac{4g^3}{l^5(l+x^3)} + \frac{g^3}{l^4(l+x^3)^2} + \frac{g^3}{2l^3(l+x^3)^3} - \left(\frac{5g}{8} + 10g^3\right)\frac{x^3}{l^7}\right) + \mathcal{O}(\hbar^3), \end{aligned} \tag{40}$$

as a solution such that it becomes the $U(1)$ Dirac monopole in the commutative limit (i.e., $\hbar \to 0$). In the fuzzy $\mathbb{R}^2$ case (33), the $\mathcal{O}(\hbar)$ terms coincide with those in [6] which solved the equations of motion with the usual Moyal product.

7 Conclusion and discussion

In this paper we have presented explicit construction of $*$ products on two dimensional constant curvature spaces S^2, H^2 and $\mathbb{R}^2$. We have found that the algebras of the $*$ products represent fuzzy S^2, H^2 and $\mathbb{R}^2$ because the commutators of the $*$ product form $su(2), su(1,1)$

and Heisenberg algebra respectively. The commutators $[z, \bar{z}]_*$ for fuzzy S^2 and H^2 are reduced to that of fuzzy $\mathbb{R}^2$ in the large R limit. In this sense, fuzzy S^2 and H^2 approach to fuzzy $\mathbb{R}^2$ as $R \to \infty$. This is consistent with usual commutative picture.

In §6 we applied explicit form of our $*$ products to $U(1)$ noncommutative BPS equation (36), and obtained its solution to $\mathcal{O}(\hbar^2)$. In eq.(36) the $*$ product appears only in the commutator $[\ ,\ ]_*$. Therefore, eq.(36) is solved unifiedly for fuzzy S^2, H^2 and $\mathbb{R}^2$ by using "canonical" noncommutative coordinate $\tilde{\phi}^I$ (30). In other words, we can get a solution of eq.(36) even if the definition of $*$ is different as long as we use "canonical" noncommutative coordinate $\tilde{\phi}^I$ for the $*$ product.

To study the effects of the difference of $*$ products themselves, we should consider noncommutative equations containing "bare" $*$ products. Its typical example is $\phi * \phi = \phi$ which is essentially the equation for noncommutative soliton [7]. Even for the $\mathbb{R}^2$ case, the $*$ product which we get here is different from the usual Moyal product, and hence $\phi \sim \exp(-r^2)$ is *not* a solution[7] of $\phi * \phi = \phi$. It is a future problem to find an explicit solution of it and to investigate its meaning.

For fuzzy S^2, $*$ product is usually defined by using representation matrix of $su(2)$ and spherical harmonic function, and depends on the size of matrix. On the other hand our $*$ product depends on the deformation parameter $\hbar$, so they are very different in appearance. It is also a future problem to study an explicit relation between them. If the relation becomes clear, our $*$ product may give some suggestions to string theory in the literature [8] for example.

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