Critical analyses of order parameter and phase transitions at high density in Gross-Neveu model*

Bang-Rong Zhou
Department of Physics, Graduate School of the
Chinese Academy of Sciences, Beijing 100039, China and
CCAST(World Lab.), P.O. Box 8730, Beijing 100080, China

By critical analyses of the order parameter of symmetry breaking, we have researched the phase transitions at high density in $D = 2$ and $D = 3$ Gross-Neveu (GN) model and shown that the gap equation obeyed by the dynamical fermion mass has the same effectivenesss as the effective potentials for such analyses of all the second order and some special first order phase transitions. In the meantime we also further ironed out a theoretical divergence and proven that in $D = 3$ GN model a first order phase transition does occur in the case of zero temperature and finite chemical potential.

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I. INTRODUCTION

The symmetry restoring phase transition at high temperature and high density has been an interesting research topic which is closely relative to evolution of early universe and phase transition of the nuclear matter [1–4]. A good laboratory to research the phase transition is the models of four-fermion interactions including the Gross-Neveu(GN) models in $D = 2$ and $D = 3$ [5] time-space dimensions. On the phase transitions in these models, extensive investigations have been made [6–13], some common conclusions have been reached but there is still particular question, e.g. whether first order phase transition exists or not in $D = 3$ GN model, which needs to be further clarified.

For determining the order of a phase transition, one can analyze critical behaviors of dynamical fermion mass as the order parameter of symmetry breaking at a critical temperature and/or at a critical chemical potential. The phase transition will be second or first order if the dynamical fermion mass goes to zero at the critical point continuously or discontinuously.

The fundamental approach of the critical analyses should be using an effective potential. However, in many cases, the gap equation is a more simple and direct means. So called the gap equation we mean the essential part of the Schwinger-Dyson equation obeyed by the dynamical fermion mass as the order parameter of symmetry breaking. It can be independently derived from the fermion condensates induced by the four-fermion interactions without using an effective potential, though it has been proven that the gap equation can also come from the extreme value condition of an effective potential [14]. If a phase transition is second order and this normally happens when the gap equation has the only non-zero solution, then the critical analysis can be made simply by means of gap equation. This has been carried out in the demonstrations on high temperature second order phase transition for fixed chemical potential in $D = 4$ Nambu-Jona-Lasinio model [15],$D = 3$ [16] and $D = 2$ [17] GN model. However, if a phase transition is first order and this is usually the case when the gap equation has two or more non-zero solutions, then how should one make the critical analysis ? Whether or in what conditions can one do that simply based on the gap equation ?

To answer the above questions, in this paper we will generalize critical analyses to the case including first order phase transitions. As far as the four-fermion models are concerned, it is most possible that the first order phase transition occurs in the case of high density. Hence we will investigate symmetry restoring phase transitions at high density for a given temperature, especially for zero temperature case, in $D = 2$ and $D = 3$ GN models. We will use both gap equations and effective potentials, and specially examine effectiveness and limitations of the gap equation for critical analyses of first order phase transition.

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The Lagrangian of the models will be expressed by

$$\mathcal{L}(x) = \sum_{k=1}^{N} \bar{\psi}^k(x) i\gamma^\mu \partial_\mu \psi_k(x) + \frac{g}{2} \sum_{k=1}^{N} [\bar{\psi}^k(x)\psi_k(x)]^2, \tag{1.1}$$

where $\psi_k(x)$ are the fermion fields with N "color" components and g is the four-fermion coupling constant. The discussions will be made in the fermion bubble diagram approximation which is equivalent to the leading order of the $1/N$ expansion. It is assumed that higher-order corrections in $1/N$ do not change the qualitative feature of the models, since our models involve only some discrete symmetries which will be spontaneously broken or restored [18]. Our basic starting point is the gap equations coming from the fermionic bilinear condensates induced by the four-fermion interactions in Eq.(1.1), besides the corresponding effective potentials. The gap equations have been given in Ref.[16, 17]. The corresponding effective potentials may be obtained from the relations between the gap equations and the effective potentials proven in Ref.[14]. Here we will directly use the known explicit results of them.

The paper is arranged as follows. In Sect. II we discuss $D = 2$ GN model, including analyzing critically the second order high density phase transition at high temperature T by the gap equation and in particular, the first order high density phase transition at $T = 0$ by the gap equation combined with the corresponding effective potential. In Sect. III, similar discussions to ones in Sect. II will be made in $D = 3$ GN model. The first order high density phase transition at $T = 0$ in this model will be proven respectively by critical analyses based on the gap equation and the effective potential. A difference between it and general first order phase transition and this difference's significance for usefulness of the gap equation analysis will be indicated. Finally in Sect. IV we come our conclusions.

II. $D = 2$ GROSS-NEVEU MODEL

In this model[17], there exists a matrix $\gamma_5 = -\sigma^3$ anticommunicating with $\gamma^0 = \sigma^1$ and $\gamma^1 = i\sigma^2$, where $\sigma^i (i = 1, 2, 3)$ are the Pauli matrices. By the Mermin-Wagner-Coleman theorem, a continuous symmetry cannot be spontaneously broken in 2 dimensions [19]. Dynamical generation of the fermion mass will spontaneously break only the discrete chiral symmetry $\chi_D : \psi(t, x) \xrightarrow{\chi_D} \gamma_5\psi(t, x)$ and the special parity $\mathcal{P}_1 : \psi(t, x) \xrightarrow{\mathcal{P}_1} \gamma^1\psi(t, -x)$. Hence possible phase transition at high temperature and high density will relate to restoration of only the χ_D and $\mathcal{P}_1$. However, in 2 dimensions, symmetry restoration could occur only in the mean-field approximation [6, 20] which is equivalent to fermion bubble diagram approximation [17].Because the former can be used only if $N \to \infty$ or N is large and only a finite segment of one-space dimension system is considered [20], here it is assumed that the use of the fermion bubble diagram approximation will be in the above conditions. In 2 dimensions, Lagrangian (1.1) is perturbatively renormalizable. Assume the four-fermion interactions in Eq.(1.1) could lead to the condensates $\sum_{k=1}^{N}\langle\bar{\psi}^k\psi_k\rangle \neq 0$ when $T = \mu = 0$ and the thermal condensates $\sum_{k=1}^{N}\langle\bar{\psi}^k\psi_k\rangle_T \neq 0$ when $T \neq 0$, where μ is the chemical potential of fermions, then we will obtain the renormalized gap equations in respective cases and finally reduce the gap equation at $T \neq 0$ to the following form

$$\ln\frac{m(0)}{m} = F_1(T, \mu, m) = I_1(y, -r) + I_1(y, r) \tag{2.1}$$

with the denotations

$$I_1(y, \mp r) = \int_0^\infty \frac{dx}{\sqrt{x^2 + y^2}} \frac{1}{\exp(\sqrt{x^2 + y^2} \mp r) + 1}, \tag{2.2}$$

where $y = m/T$, $r = \mu/T$, $m(0)$ and $m \equiv m(T, \mu)$ are respectively the dynamical fermion masses at $T = \mu = 0$ and at $T \neq 0$ and/or $\mu \neq 0$. It is easy to verify that $F_1(T, \mu, m)$ is an increasing function of T and μ but a decreasing function of m, and from Eq.(2.1), we can assume that at some critical temperature T_c and critical chemical potential μ_c, the dynamical fermion mass m becomes zero. This will lead to a critical equation. At a high temperature T, by using the high temperature expansion of $F_1(T, \mu, m)$ [15] we may reduce the gap equation (2.1) to

$$\begin{aligned} \ln\tfrac{m(0)}{T\pi} + \gamma &= g(T, \mu)(\tfrac{y}{2\pi})^2 + h(T, \mu) + \mathcal{O}\left[(\tfrac{y}{2\pi})^4\right], \\ g(T, \mu) &= \tfrac{7}{2}\zeta(3) - 93\zeta(5)(\tfrac{r}{2\pi})^2 + \tfrac{1905}{2}\zeta(7)(\tfrac{r}{2\pi})^4, \\ h(T, \mu) &= 7\zeta(3)(\tfrac{r}{2\pi})^2 - 31\zeta(5)(\tfrac{r}{2\pi})^4 + 127\zeta(7)(\tfrac{r}{2\pi})^6, \end{aligned} \tag{2.3}$$

where $\zeta(s)(s=3,5,7)$ are the Riemann zeta functions and γ is the Euler constant. Noting that the $\ln m$ term in Eq.(2.1) has been cancelled, thus we can set $y=0$ in Eq.(2.3) and obtain the critical equation

$$\ln\frac{m(0)}{T_c\pi}+\gamma=h(T_c,\mu_c). \tag{2.4}$$

In the derivation of Eq. (2.4) an implicit fact is that Eq. (2.3) has the only solution m and m can go to zero continuously as T and μ vary. This is just the characteristic of a second order phase transition. In view of this, Eq. (2.4) can be called the critical equation of a second order phase transition. The critical points determined by Eq. (2.4) are all second order ones. For verifying this conclusion, we can discuss from Eqs.(2.3) and (2.4) critical behaviors of the squared order parameter m^2 at μ_c when T is fixed. Replacing T_c by T in Eq.(2.4) and substituting it into Eq. (2.3), we get

$$m^2\simeq(\mu_c^2-\mu^2)f(T,\mu_c)/g(T,\mu),\quad \text{when } \mu\sim\mu_c, \tag{2.5}$$

where

$$f(T,\mu_c)=7\zeta(3)-62\zeta(5)(\frac{\mu_c}{2\pi T})^2+381\zeta(7)(\frac{\mu_c}{2\pi T})^4. \tag{2.6}$$

Eq.(2.5) shows that the squared order parameter m^2 goes to zero continuously when $\mu\to\mu_c$ in the form of $\mu_c-\mu$, hence for a given high temperature T, the symmetry restoration phase transition at μ_c is second order indeed.

Eq.(2.4) comes from high temperature expansion of $F_1(T,\mu,m)$ and is inapplicable to discussion of high density phase transition at low temperature and $T=0$. For discussion of the latter, we have to use the effective potential. In the following we will only focus on the high density phase transition at $T=0$. Let us first go back to Eq.(2.1). Through changing the integral variable by $z=(x^2/y^2+1)^{1/2}$ we can express

$$F_1(T,\mu,m)=\int_1^\infty\frac{dz}{\sqrt{z^2-1}}\left[\frac{1}{e^{y(z-\alpha)}+1}+(-\alpha\to\alpha)\right], \tag{2.7}$$

where $\alpha=\frac{\mu}{m}$. In the limit $T\to0$ or $y\to\infty$, we have

$$F_1(T=0,\mu,m)=\theta(\mu-m)\ln\frac{\mu+\sqrt{\mu^2-m^2}}{m}. \tag{2.8}$$

Substituting Eq.(2.8) into Eq.(2.1) we obtain

$$\ln\frac{m(0)}{m}=\theta(\mu-m)\ln\frac{\mu+\sqrt{\mu^2-m^2}}{m} \tag{2.9}$$

which has the solutions

$$m=m(0),\ \text{when } \mu\le m \tag{2.10}$$

and

$$\begin{aligned} m &= \{m(0)[2\mu-m(0)]\}^{1/2}\equiv m_1,\\ &\quad \text{when } \mu>m \text{ and } \frac{m(0)}{2}\le\mu<m(0). \end{aligned} \tag{2.11}$$

On the other hand, the effective potential $V_{eff}^{(2)}(T,\mu,m)$ has been proven to satisfy the equation [14]

$$\frac{\partial V_{eff}^{(2)}(T,\mu,m)}{\partial m}=\frac{m}{\pi}\left[\ln\frac{m}{m(0)}+F_1(T,\mu,m)\right], \tag{2.12}$$

noting that the gap equation (2.1) multiplied by m is just the extreme conditions given by $\partial V_{eff}^{(2)}(T,\mu,m)/\partial m=0$. Hence, when $T=0$, $V_{eff}^{(2)}(T=0,\mu,m)$ for $m(0)/2\le\mu<m(0)$ will have the three possible extreme points $m=0$, m_1 and $m(0)$. To determine which one of them corresponds to a minimum or a maximum of $V_{eff}^{(2)}(T=0,\mu,m)$, we may calculate and obtain directly from Eq. (2.12)

$$\frac{\partial^2V_{eff}^{(2)}(T=0,\mu,m)}{\partial m^2}=\frac{1}{\pi}\begin{cases}\ln\frac{m}{m(0)}+1 & \text{when}\quad \mu\le m\\ \ln\frac{\mu+\sqrt{\mu^2-m^2}}{m(0)}-\frac{m^2}{(\mu+\sqrt{\mu^2-m^2})\sqrt{\mu^2-m^2}} & \text{when}\quad \mu>m\end{cases}, \tag{2.13}$$

where Eq. (2.8) has been used. Eq. (2.13), combined with Eqs. (2.10) and (2.11), give that

$$\left.\frac{\partial^2 V_{eff}^{(2)}(T=0,\mu,m)}{\partial m^2}\right|_{m=0} = \frac{1}{\pi}\ln\frac{2\mu}{m(0)}\begin{cases} <0 & \text{when} \quad \mu < m(0)/2 \\ >0 & \text{when} \quad \mu > m(0)/2 \end{cases}, \tag{2.14}$$

$$\left.\frac{\partial^2 V_{eff}^{(2)}(T=0,\mu,m)}{\partial m^2}\right|_{m=m_1} = -\frac{1}{\pi}\frac{2\mu - m(0)}{m(0)-\mu} < 0, \text{ when } \frac{m(0)}{2} < \mu < m(0) \tag{2.15}$$

and

$$\left.\frac{\partial^2 V_{eff}^{(2)}(T=0,\mu,m)}{\partial m^2}\right|_{m=m(0)} = \frac{1}{\pi} > 0, \text{ when } \mu < m(0). \tag{2.16}$$

Eqs. (2.14)-(2.16) indicate that
1) When $\mu < \frac{m(0)}{2}$, $V_{eff}^{(2)}(T=0,\mu,m)$ will have the maximum point $m=0$ and the minimum point $m=m(0)$, thus the model has the same spontaneous symmetry breaking as one at $\mu=0$.
2) When $\frac{m(0)}{2} < \mu < m(0)$, $V_{eff}^{(2)}(T=0,\mu,m)$ will have three extreme points: $m=0$ (minimum),$m=m_1$ (maximum) and $m=m(0)$ (minimum), and this is just the characteristic of possible emergence of a first order phase transition.

It is emphasized that in the above analyses we only use the known relation (2.12) between the gap equation and the effective potential and the explicit expression of the gap equation (2.1), and did not use an explicit expression of $V_{eff}^{(2)}(T,\mu,m)$. However, for determining the critical point, we have to find out the explicit expression of $V_{eff}^{(2)}(T,\mu,m)$ from Eq.(2.12). The result is [14]

$$\begin{aligned} V_{eff}^{(2)}(T=0,\mu,m) &= \frac{m^2}{2\pi}\left[\ln\frac{m}{m(0)} - \frac{1}{2}\right] + \frac{\mu^2}{2\pi} + \frac{1}{2\pi}\theta(\mu-m) \\ &\quad \times \left(m^2 \ln\frac{\mu+\sqrt{\mu^2-m^2}}{m} - \mu\sqrt{\mu^2-m^2}\right). \end{aligned} \tag{2.17}$$

Assuming that $\frac{m(0)}{2} < \mu < m(0)$, then for $m=m(0)$ among the three extreme points, we will get

$$V_{eff}^{(2)}[T=0,\mu,m=m(0)] = \frac{1}{4\pi}[2\mu^2 - m^2(0)] < 0,$$

$$\text{when } \mu < \frac{1}{\sqrt{2}}m(0). \tag{2.18}$$

Hence, if $\mu < m(0)/\sqrt{2}$, $m=m(0)$ will be the global minimum point of $V_{eff}^{(2)}(T=0,\mu,m)$, because in this case, the minimum $V_{eff}^{(2)}(T=0,\mu,m=0)=0$ and $V_{eff}^{(2)}(T=0,\mu,m=m_1)$ is a maximum. Obviously, the first order phase transition point should be determined by

$$V_{eff}^{(2)}[T=0,\mu,m=m(0)] = V_{eff}^{(2)}(T=0,\mu,m=0) = 0 \tag{2.19}$$

and this will lead to

$$\mu = \frac{1}{\sqrt{2}}m(0) \equiv \mu_c, \tag{2.20}$$

i.e. when μ crosses over μ_c, the global minimum point of $V_{eff}^{(2)}(T=0,\mu,m)$ will jump from $m(0)$ down to 0 through a barrier tunneling between the minimum points $m=m(0)$ and $m=0$. The critical chemical potential μ_c of the first order phase transition at $T=0$ in $D=2$ GN model in the leading order of $1/N$ expansion given by Eq. (2.20) is consistent with the one obtained by other authors [7–12]. Since the global minimum of $V_{eff}^{(2)}(T=0,\mu,m)$ is also at $m=m(0)$ when $\mu < m(0)/2$, we can write the critical behavior of the order parameter m by

$$m = \begin{cases} m(0) & \text{when} \quad \mu \le \frac{1}{\sqrt{2}}m(0) \\ 0 & \text{when} \quad \mu > \frac{1}{\sqrt{2}}m(0) \end{cases}. \tag{2.21}$$

The above discussions show that for critical analyses of the order parameter in a general first order phase transition, it is not sufficient only using the gap equation, we must use the gap equation combined with corresponding effective potential. This is also applicable to critical analyses of the phase transitions at low temperature $T \neq 0$ in $D = 2$ GN model which are known to be first order. The effective potential at $T \neq 0$ may be derived from Eq.(2.12). By means of it and the gap equation (2.1), we will be able to obtain other results of the phase transition at low temperature in this model including location of the tricritical point etc., following similar methods to ones used in Ref [6, 7, 11, 12].

III. $D = 3$ GROSS-NEVEU MODEL

In this case [16], the dynamical fermion mass will spontaneously break the special parities $\mathcal{P}_1$ and $\mathcal{P}_2$: $\psi(t, x^1, x^2) \xrightarrow{\mathcal{P}_1} \gamma^1\psi(t, -x^1, x^2)$ and $\psi(t, x^1, x^2) \xrightarrow{\mathcal{P}_2} \gamma^2\psi(t, x^1, -x^2)$ and the time reversal $\mathcal{T}$: $\psi(t, x^1, x^2) \xrightarrow{\mathcal{T}} \gamma^2\psi(-t, x^1, x^2)$[21]. After the gap equation at $T = \mu = 0$ is substituted, the gap equation at $T \neq 0$ becomes

$$m(0) = m + F_2(T, \mu, m) \tag{3.1}$$

with

$$F_2(T, \mu, m) = T\left\{\ln\left[1 + e^{-(m-\mu)/T}\right] + (-\mu \to \mu)\right\}, \tag{3.2}$$

and the critical equation of second order phase transition will be

$$m(0) = T_c\left[\ln(1 + e^{\mu_c/T_c}) + \ln(1 + e^{-\mu_c/T_c})\right] \tag{3.3}$$

which determines critical points of second order phase transitions. For example, for a given finite T, when $\mu \sim \mu_c$, $m \approx 0$, so we will have $m/T \ll 1$. Then Eq.(3.1) with (3.2) will lead to

$$\begin{aligned} m(0) &= T\ln\left(1 + e^{\mu/T}\right) + T\ln\left(1 + e^{-\mu/T}\right) \\ &\quad + \frac{m^2}{2T[1 + \cosh(\mu/T)]} + \mathcal{O}\left(\frac{m^3}{T^3}\right). \end{aligned} \tag{3.4}$$

Substituting Eq.(3.3) with T_c replaced by T into Eq.(3.4) we get

$$m^2 = 2T\sinh(\mu/T)(\mu_c - \mu), \text{ for } T \neq 0. \tag{3.5}$$

Eq.(3.5) shows that for a given finite temperature, the symmetry restoration phase transition at μ_c is second order. The same has been shown for the phase transition at T_c for a given μ [16]. These conclusions are valid for $T \neq 0$. For $T = 0$, we must go back the gap equation (3.1). We note that when $\mu \leq m$, $\lim_{T\to 0} F_2(T, \mu, m) = 0$ and by Eq.(3.1) this implies that $m = m(0)$, if $\mu \leq m$, or furthermore, $m = m(0)$, if $\mu \leq m(0)$. On the other hand, if $\mu > m$, then $\lim_{T\to 0} F_2(T, \mu, m) = \mu - m$. Substituting this result into Eq.(3.1) we will obtain that $m(0) = \mu$, but the presupposition leading to this result in fact should be $\mu > m = m(0)$, the both apparently contradict each other. This indicates that when $\mu > m = m(0)$, the gap equation (3.1) can not be satisfied, thus there is no longer symmetry breaking and the order parameter m must be zero. The above results can be summarized as

$$m = \begin{cases} m(0) & \text{when} \quad \mu \leq m(0) \\ 0 & \text{when} \quad \mu > m(0) \end{cases}. \tag{3.6}$$

Furthermore, when $T \to 0$, the derivative $\partial m/\partial\mu$ will be

$$\begin{aligned} &\lim_{T\to 0} \frac{\partial m}{\partial \mu}\bigg|_{m=m(0)} \\ &= -\lim_{T\to 0} \frac{\partial F_2(T, \mu, m)}{\partial \mu} \Big/ \left[1 + \frac{\partial F_2(T, \mu, m)}{\partial m}\right]\bigg|_{m=m(0)} \\ &= -\lim_{T\to 0} e^{-[m(0)-\mu]/T} = \begin{cases} 0, & \text{when } \mu < m(0) \\ -1, & \text{when } \mu = m(0) \\ -\infty, & \text{when } \mu \gtrsim m(0) \end{cases} \end{aligned} \tag{3.7}$$

(noting that when $\mu > m(0)$, the gap equation (3.1) is no longer valid). Eqs.(3.6) and (3.7) show that when $\mu < m(0)$, m does not change as μ varies and keeps its value $m(0)$ at $\mu = 0$; however as soon as μ crosses over $m(0)$, $\partial m/\partial\mu$ will change from 0 into $-\infty$ and m will jump discontinuously from $m(0)$ to zero. This implies that when $T = 0$ the phase transition at the critical chemical potential $\mu_c = m(0)$ is first order, i.e. $(T, \mu) = (0, m(0))$ will be a first order phase transition point in $T - \mu$ phase diagram. On the other hand, we note that $(T, \mu) = (0, m(0))$ is a solution of the critical equation (3.3) corresponding to second order phase transition when we take the limit $T \to 0$ in Eq.(3.3). Hence we may reasonably conclude that $(T, \mu) = (0, m(0))$ will be a tricritical point in the $T - \mu$ phase giagram of $D = 3$ GN model.

This conclusion coincides with the one in Ref. [13] but different from the one in Ref. [12] where the authors claimed that there is no first order phase transition in $D = 3$ GN model. To clarify this problem, we can make further analysis of the phase transition at $T = 0$ in the model by means of the effective potential which may be derived from the gap equation, as was made in $D = 2$ GN model. It has been proven that the extreme value condition of the effective potential $V^{(3)}_{eff}(T, \mu, m)$ is [14]

$$\frac{\partial V^{(3)}_{eff}(T, \mu, m)}{\partial m} = 0 \tag{3.8}$$

with

$$\frac{\partial V^{(3)}_{eff}(T, \mu, m)}{\partial m} = \frac{m}{2\pi}\left[m - m(0) + F_2(T, \mu, m)\right], \tag{3.9}$$

noting that Eq.(3.8) is just the gap equation (3.1) multiplied by m. From Eq. (3.9) we may obtain $V^{(3)}_{eff}(T, \mu, m)$ satisfying the condition $V^{(3)}_{eff}(T, \mu, m = 0) = 0$ expressed by

$$V^{(3)}_{eff}(T = 0, \mu, m) = \frac{1}{2\pi}\left\{\frac{m^2}{2}[\mu\theta(\mu - m) - m(0)] + \theta(m - \mu)\left(\frac{m^3}{3} + \frac{\mu^3}{6}\right)\right\}. \tag{3.10}$$

For fixed μ, from Eq.(3.10) or the $T \to 0$ limit of Eq.(3.9),we find out

$$\frac{\partial V^{(3)}_{eff}(T = 0, \mu, m)}{\partial m} = \frac{m}{2\pi}\{[\mu - m(0)]\theta(\mu - m) + [m - m(0)]\theta(m - \mu)\} \tag{3.11}$$

and

$$\frac{\partial^2 V^{(3)}_{eff}(T = 0, \mu, m)}{\partial m^2} = \frac{1}{2\pi}\{[\mu - m(0)]\theta(\mu - m) + [2m - m(0)]\theta(m - \mu)\}. \tag{3.12}$$

Variation of $V^{(3)}_{eff}(T = 0, \mu, m)$ as μ increases may be discussed by Eqs. (3.10)-(3.12).

1) $\mu < m(0)$. In this case, $V^{(3)}_{eff}[T = 0, \mu < m(0), m]$ will have the maximum point $m = 0 (m < \mu)$ and the minimum point $m = m(0) (m > \mu)$. It is indicated that location of the minimum point keeps at $m = m(0)$ which does not change as μ increases. However, the minimum of the effective potential at $m = m(0)$

$$V^{(3)}_{eff}[T = 0, \mu < m(0), m]|_{m=m(0)} = \frac{1}{12\pi}[\mu^3 - m^3(0)] \tag{3.13}$$

will rise as μ increases. Nevertheless, these results only show the same spontaneous symmetry breaking as one at $T = \mu = 0$.

2) $\mu = m(0)$. In this case, the effective potential will take the following form

$$V^{(3)}_{eff}[T = 0, \mu = m(0), m] = \frac{1}{2\pi}\theta[m - m(0)]$$

$$\times\left[\frac{m^3(0)}{6} - \frac{m^2}{2}m(0) + \frac{m^3}{3}\right] \tag{3.14}$$

which indicates that in the total real axis segment $0 \le m \le m(0)$, $V^{(3)}_{eff}[T = 0, \mu = m(0), m] = 0$, thus the total segment $0 \le m \le m(0)$ including the original $m = m(0)$ when $\mu < m(0)$, will be the minimum points of $V^{(3)}_{eff}[T =$

$0, \mu = m(0), m]$. This situation corresponds to that when $\mu = m(0)$, the $T \to 0$ limit of the gap equation (3.1) has infinite non-zero solutions.
3) $\mu > m(0)$. In this case, $\partial V_{eff}^{(3)}[T = 0, \mu > m(0), m]/\partial m = 0$ will have the only solution $m = 0$ and by Eq. (3.12),

$$\left.\frac{\partial^2 V_{eff}^{(3)}[T = 0, \mu > m(0), m]}{\partial m^2}\right|_{m=0} = \frac{1}{2\pi}[\mu - m(0)] > 0. \tag{3.15}$$

Thus $m = 0$ will be the only minimum point of $V_{eff}^{(3)}[T = 0, \mu > m(0), m]$ and this implies that the broken symmetries will be restored. The critical chemical potential μ_c should be determined by the condition

$$\left.\frac{\partial^2 V_{eff}^{(3)}(T = 0, \mu, m)}{\partial m^2}\right|_{m=0} = 0, \tag{3.16}$$

i.e. at μ_c, $m = 0$ must change from being a maximum point into being a minimum point. The result is that $\mu_c = m(0)$.

The above discussions show that as μ increases and finally crosses over $\mu_c = m(0)$, the global minimum point of the effective potential $V_{eff}^{(3)}(T = 0, \mu, m)$ will jump from $m(0)$ down to 0 thus Eq. (3.6) is reproduced. The phase transition is first order indeed. The analysis based on effective potential leads to the same conclusion as the one only based on the gap equation, however the latter is obviously more simple and direct in this model. We also indicate that the first order phase transition at $T = 0$ in $D = 3$ GN model has a distinct feature, i.e. the transit from broken phase to symmetry phase does not undergo a barrier tunneling, since when μ is increasing, as we have seen above, no barrier emerges from between the minimum points $m = 0$ and $m = m(0)$ of $V_{eff}^{(3)}(T = 0, \mu, m)$. This is apparently different from general first order phase transition, e.g. the one in $D = 2$ GN model discussed in Sect. II. It is assumed that it is just this feature that makes one be able to determine the critical behavior of the order parameter in the first order phase transition in $D = 3$ GN model merely by the gap equation approach.

IV. CONCLUSIONS

In this paper, we have discussed symmetry restoring phase transitions at high density in $D = 2$ and $D = 3$ GN model by means of analyses of critical behaviors of the dynamical fermion mass as order parameter of symmetry breaking. We have used both gap equation and effective potential approach. In these approaches, the gap equations multiplied by the dynamical fermion mass are simply the Schwinger-Dyson equations obeyed by the dynamical fermion mass and the effective potentials will be derived from the gap equations based on the fact that a gap equation multiplied by the dynamical fermion mass can also come from the extreme value condition of corresponding effective potential. This shows that the Schwinger-Dyson equations obeyed by the dynamical fermion mass in fact contain the essential elements of the phase structure of the discussed models. We have found that for second order phase transitions and some specific first order phase transitions e.g. the one at high density and $T = 0$ in $D = 3$ GN model in which the jumping of the order parameter does not correspond to a barrier tunneling in an effective potential, the gap equation analysis alone has the same effect as the effective potential one. However, for general first order phase transitions, e.g. the one at high density and $T = 0$ and low $T \neq 0$ in $D = 2$ GN model, we must combine the gap equation with the corresponding effective potential so as to obtain correct conclusions.

Our analyses also reproduce some of known essential results of the phase transitions in $D = 2$ and $D = 3$ GN model. They include that in $D = 2$ GN model, the phase transitions at high density will be second order when T is high and first order when $T \to 0$; in $D = 3$ GN model, the phase transition at high density will be second order when T is finite and first order when $T = 0$ thus one can conclude that $(T, \mu) = (0, m(0))$ is a tricritical point. The latter further clarifies the theoretical divergence of that if first order phase transition exists in $D = 3$ GN model.

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[21] We no longer include the discrete symmetry Z_2: $\psi_k^1(x) \stackrel{Z_2}{\longleftrightarrow} \psi_k^2(x)$, if $\psi_k(x) = \left(\psi_k^1(x), \psi_k^2(x)\right)$, as was stated in Ref.[16], since the kinetic energy terms in Eq. (1.1) actually have not separate Z_2 symmetry.