

# Relativistic wave equations with fractional derivatives and pseudo-differential operators

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## Abstract

The class of the free relativistic covariant equations generated by the fractional powers of the D'Alembertian operator ( $\square^{1/n}$ ) is studied. Meanwhile the equations corresponding to  $n = 1$  and  $2$  (Klein-Gordon and Dirac equations) are local in their nature, the multicomponent equations for arbitrary  $n > 2$  are non-local. It is shown, how the representation of generalized algebra of Pauli and Dirac matrices looks like and how these matrices are related to the algebra of  $SU(n)$  group. The corresponding representations of the Poincaré group and further symmetry transformations on the obtained equations are discussed. The construction of the related Green functions is suggested.

## 1 Introduction

The relativistic covariant wave equations represent an intersection of ideas of the theory of relativity and quantum mechanics. The first and best known relativistic equations, the Klein-Gordon and particularly Dirac equation, belong to the essentials, which our present understanding of the microworld is based on. In this sense it is quite natural, that the searching for and the study of the further types of such equations represent a field of stable interest. For a review see e.g. [1] and citations therein. In fact, the attention has been paid first of all to the study of equations corresponding to the higher spins ( $s \geq 1$ ) and to the attempts to solve the problems, which have been revealed in the connection with these equations, e.g. the acausality due to external fields introduced by the minimal way.

In this paper we study the class of equations obtained by the 'factorization' of the D'Alembertian operator, i.e. by a generalization of the procedure, by which the Dirac equation is obtained. As the result, from each degree of extraction  $n$  we get a multi-component equation, hereat the case  $n = 2$  corresponds to the Dirac equation. However the equations for  $n > 2$  differ substantially from the cases  $n = 1, 2$  since they contain fractional derivatives (or pseudo-differential operators), so in the effect their nature is non-local.

In the first part (Sec. 2), the generalized algebras of the Pauli and Dirac matrices are considered and their properties are discussed, in particular their relation to the algebra of the  $SU(n)$  group. The second, main part (Sec. 3) deals with the covariant wave equations generated by the roots of the D'Alembertian operator, these roots are defined with the use of the generalized Dirac matrices. In this section we show the explicit form of the equations, their symmetries and the corresponding transformation laws. We also define the scalar product and construct the corresponding Green functions. The last section (Sec. 4) is devoted to the summary and concluding remarks.

Let us remark, the application of the pseudo-differential operators in the relativistic equations is nothing new. The very interesting aspects of the scalar relativistic equations based on the square root of the Klein-Gordon equation are pointed out e.g. in the papers [2]-[4]. Recently, an interesting approach for the scalar relativistic equations based on the pseudo-differential operators of the type  $f(\square)$  has been proposed in the paper [5]. One can mention also the papers [6], [7] in which the square and cubic roots of the Dirac equation were studied in the context of supersymmetry. The cubic roots of the Klein-Gordon equation were discussed in the recent papers [8], [9].

It should be observed, that our considerations concerning the generalized Pauli and Dirac matrices (Sec. 2) have much common with the earlier studies related to the generalized Clifford algebras (see e.g. [10]-[12] and citation therein) and with the paper [13], even if our starting motivation is rather different.

## 2 Generalized algebras of Pauli and Dirac matrices

Anywhere in the next by the term *matrix* we mean the square matrix  $n \times n$ , if not stated otherwise. Considerations of this section are based on the matrix pair introduced as follows.

**Definition 1** For any  $n \geq 2$  we define the matrices

$$S = \begin{pmatrix} 0 & & & & & 1 \\ 1 & & & & & \\ & 1 & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & 1 & 0 \end{pmatrix}, \quad (2.1)$$

$$T = \begin{pmatrix} 1 & & & & & \\ & \alpha & & & & \\ & & \alpha^2 & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & \alpha^{n-1} \end{pmatrix} \quad (2.2)$$

where  $\alpha = \exp(2\pi i/n)$  and in the remaining empty positions are zeros.

**Lemma 2** *Matrices  $X = S, T$  satisfy the following relations*

$$\alpha ST = TS, \quad (2.3)$$

$$X^n = I, \quad (2.4)$$

$$XX^\dagger = X^\dagger X = I, \quad (2.5)$$

$$\det X = (-1)^{n-1}, \quad (2.6)$$

$$\text{Tr } X^k = 0, \quad k = 1, 2, \dots, n-1, \quad (2.7)$$

where  $I$  denotes the unit matrix.

*Proof:*

All the relations easily follow from the *Definition 1*.

**Definition 3** *Let  $\mathcal{A}$  be some algebra on the field of complex numbers,  $(p, m)$  be a pair of natural numbers,  $X_1, X_2, \dots, X_m \in \mathcal{A}$  and  $a_1, a_2, \dots, a_m \in C$ . The  $p$ -th power of the linear combination can be expanded:*

$$\left( \sum_{k=1}^m a_k X_k \right)^p = \sum_{p_j} a_1^{p_1} a_2^{p_2} \dots a_m^{p_m} \{X_1^{p_1}, X_2^{p_2}, \dots, X_m^{p_m}\}; \quad p_1 + \dots + p_m = p,$$

where the symbol  $\{X_1^{p_1}, X_2^{p_2}, \dots, X_m^{p_m}\}$  represents the sum of the all possible products created from elements  $X_k$  in such a way that each product contains element  $X_k$  just  $p_k$  - times. This symbol we shall call combinator.

**Example 4**

$$\{X, Y\} = XY + YX, \quad (2.8)$$

$$\{X, Y^2\} = XY^2 + YXY + Y^2X, \quad (2.9)$$

$$\{X, Y, Z\} = XYZ + XZY + YXZ + YZX + ZXY + ZYX. \quad (2.10)$$

Now, we shall prove some useful identities.

**Lemma 5** *Let us assume  $z$  is a complex variable,  $p, r \geq 0$  and denote*

$$q_p(z) = (1-z)(1-z^2)\dots(1-z^p), \quad q_0(z) = 1, \quad (2.11)$$

$$F_{rp}(z) = \sum_{k_p=0}^r \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} z^{k_1} z^{k_2} \dots z^{k_p}, \quad (2.12)$$

$$G_p(z) = \sum_{k=0}^p \frac{z^k}{q_{p-k}(z^{-1})q_k(z)}, \quad (2.13)$$

$$H_p(z) = \sum_{k=0}^p \frac{1}{q_{p-k}(z^{-1})q_k(z)}. \quad (2.14)$$

Then the following identities hold for  $z \neq 0$ ,  $z^j \neq 1$ ;  $j = 1, 2, \dots, p$ :

$$q_p(z) = (-1)^p z^{\frac{p(p+1)}{2}} q_p(z^{-1}), \quad (2.15)$$

$$G_p(z) = 0, \quad (2.16)$$

$$H_p(z) = 1, \quad (2.17)$$

$$F_{rp}(z) = \sum_{k=0}^p \frac{z^{k \cdot r}}{q_{p-k}(z)q_k(z^{-1})} \quad (2.18)$$

and in particular for  $z^{p+r} = 1$

$$F_{rp}(z) = 0. \quad (2.19)$$

*Proof:*

1) Relation (2.15) follows immediately from definition (2.11):

$$\begin{aligned} q_r(z) &= (1-z)(1-z^2)\dots(1-z^r) = z \cdot z^2 \dots z^r (z^{-1} - 1) \dots (z^{-r} - 1) \\ &= (-1)^r z^{\frac{r(r+1)}{2}} q_r(z^{-1}). \end{aligned}$$

2) Relations (2.16), (2.17):

First, if we invert the order of adding in the relations (2.13), (2.14) making substitution  $j = p - k$ , then

$$G_p(z) = \sum_{k=0}^p \frac{z^k}{q_{p-k}(z^{-1})q_k(z)} = z^p \sum_{j=0}^p \frac{z^{-j}}{q_j(z^{-1})q_{p-j}(z)} = z^p G_p(z^{-1}), \quad (2.20)$$

$$H_p(z) = \sum_{k=0}^p \frac{1}{q_{p-k}(z^{-1})q_k(z)} = \sum_{j=0}^p \frac{1}{q_j(z^{-1})q_{p-j}(z)} = H_p(z^{-1}). \quad (2.21)$$

Now, let us calculate

$$\begin{aligned} H_p(z) - H_{p-1}(z) &= \sum_{k=0}^p \frac{1}{q_{p-k}(z^{-1})q_k(z)} - \sum_{k=0}^{p-1} \frac{1}{q_{p-1-k}(z^{-1})q_k(z)} \\ &= \frac{1}{q_p(z)} + \sum_{k=0}^{p-1} \frac{1}{q_{p-k}(z^{-1})q_k(z)} - \sum_{k=0}^{p-1} \frac{1}{q_{p-k-1}(z^{-1})q_k(z)} \\ &= \frac{1}{q_p(z)} + \sum_{k=0}^{p-1} \frac{1 - (1 - z^{k-p})}{q_{p-k}(z^{-1})q_k(z)} = \sum_{k=0}^p \frac{z^{k-p}}{q_{p-k}(z^{-1})q_k(z)} = G_p(z^{-1}). \end{aligned} \quad (2.22)$$

The last relation combined with Eq. (2.21) implies

$$G_p(z^{-1}) = G_p(z), \quad (2.23)$$

which compared with Eq. (2.20) gives

$$G_p(z^{-1}) = 0; \quad z \neq 0, \quad z^j \neq 1, \quad j = 1, 2, \dots, p. \quad (2.24)$$

So the identity (2.16) is proved. Further, relations (2.24), (2.22) imply

$$H_p(z) - H_{p-1}(z) = 0, \quad (2.25)$$

therefore

$$H_p(z) = H_{p-1}(z) = \dots = H_0(z) = 1 \quad (2.26)$$

and the identity (2.17) is proved as well.

3) The relation (2.18) can be proved by the induction, therefore first let us assume  $p = 1$ , then its l.h.s. reads

$$\sum_{k_1=0}^{k_2} z^{k_1} = \frac{1 - z^{k_2+1}}{1 - z}$$

and r.h.s. gives

$$\frac{1}{q_1(z)} + \frac{z^{k_2}}{q_1(z^{-1})} = \frac{1}{1-z} + \frac{z^{k_2}}{1-z^{-1}} = \frac{1 - z^{k_2+1}}{1 - z},$$

so for  $p = 1$  the relation is valid. Now let us suppose the relation holds for  $p$  and calculate the case  $p + 1$

$$\begin{aligned} & \sum_{k_{p+1}=0}^{k_{p+2}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} z^{k_1} z^{k_2} \dots z^{k_{p+1}} = \sum_{k_{p+1}=0}^{k_{p+2}} z^{k_{p+1}} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} z^{k_1} z^{k_2} \dots z^{k_p} \\ &= \sum_{k_{p+1}=0}^{k_{p+2}} z^{k_{p+1}} \sum_{k=0}^p \frac{z^{k \cdot k_{p+1}}}{q_{p-k}(z) q_k(z^{-1})} = \sum_{k=0}^p \frac{1}{q_{p-k}(z) q_k(z^{-1})} \sum_{k_{p+1}=0}^{k_{p+2}} z^{(k+1) \cdot k_{p+1}} \\ &= \sum_{k=0}^p \frac{1}{q_{p-k}(z) q_k(z^{-1})} \frac{1 - z^{(k+1) \cdot (k_{p+2}+1)}}{1 - z^{k+1}} = \sum_{k=0}^p \frac{z^{-k-1} - z^{(k+1) \cdot k_{p+2}}}{q_{p-k}(z) q_k(z^{-1}) (z^{-k-1} - 1)} \\ &= \sum_{k=0}^p \frac{z^{(k+1) \cdot k_{p+2}} - z^{-k-1}}{q_{p-k}(z) q_{k+1}(z^{-1})} = \sum_{k=1}^{p+1} \frac{z^{k \cdot k_{p+2}} - z^{-k}}{q_{p+1-k}(z) q_k(z^{-1})} = \sum_{k=0}^{p+1} \frac{z^{k \cdot k_{p+2}} - z^{-k}}{q_{p+1-k}(z) q_k(z^{-1})} \\ &= \sum_{k=0}^{p+1} \frac{z^{k \cdot k_{p+2}}}{q_{p+1-k}(z) q_k(z^{-1})} - \sum_{k=0}^{p+1} \frac{z^{-k}}{q_{p+1-k}(z) q_k(z^{-1})}. \end{aligned}$$

The last sum equals  $G_{p+1}(z^{-1})$ , which is zero according to Eq. (2.16), so we have proven relation (2.18) for  $p + 1$ . Therefore the relation is valid for any  $p$ .

4) The relation (2.19) is a special case of Eq. (2.18). The denominators in the sum (2.18) can be with the use of the identity (2.15) expressed

$$q_{p-k}(z) q_k(z^{-1}) = (-1)^p z^s q_{p-k}(z^{-1}) q_k(z), \quad s = \left(\frac{p}{2} - k\right) (p+1)$$

and since  $z^{r \cdot k} = z^{-p \cdot k}$ , the sum can be rewritten

$$\begin{aligned} & \sum_{k=0}^p \frac{z^{k \cdot r}}{q_{p-k}(z) q_k(z^{-1})} = (-1)^p \sum_{k=0}^p \frac{z^{-s} z^{-p \cdot k}}{q_{p-k}(z^{-1}) q_k(z)} \\ &= (-1)^p z^{-\frac{p(p+1)}{2}} \sum_{k=0}^p \frac{z^k}{q_{p-k}(z^{-1}) q_k(z)}. \end{aligned}$$

Obviously, the last sum coincides with  $G_p(z)$ , which is zero according to already proven identity (2.16).

Let us remark, last lemma implies also the known formula

$$x^n - y^n = (x - y)(x - \alpha y)(x - \alpha^2 y) \dots (x - \alpha^{n-1} y), \quad \alpha = \exp(2\pi i/n). \quad (2.27)$$

The product can be expanded

$$x^n - y^n = \sum_{j=0}^n c_j x^{n-j} (-y)^j$$

and one can easily check that

$$c_0 = 1, \quad c_n = \alpha \alpha^2 \alpha^3 \dots \alpha^{n-1} = (-1)^{n-1}.$$

For the remaining  $j$ ,  $0 < j < n$  we get

$$c_j = \sum_{k_j=j-1}^{n-1} \dots \sum_{k_2=1}^{k_3-1} \sum_{k_1=0}^{k_2-1} \alpha^{k_1} \alpha^{k_2} \dots \alpha^{k_j}$$

and after the shift of summing limits we obtain

$$c_j = \alpha \alpha^2 \alpha^3 \dots \alpha^{j-1} \sum_{k_j=0}^{n-j} \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \alpha^{k_1} \alpha^{k_2} \dots \alpha^{k_j}.$$

This multiple sum is a special case of the formula (2.12) and since  $\alpha^n = 1$ , the identity (2.19) is satisfied. Therefore for  $0 < j < n$  we get  $c_j = 0$  and formula (2.27) is proved.

**Definition 6** Let us have a matrix product created from some string of matrices  $X, Y$  in such a way that matrix  $X$  is in total involved  $p$  - times and  $Y$  -  $r$  - times. By the symbol  $P_j^+$  ( $P_j^-$ ) we denote permutation, which shifts the leftmost (rightmost) matrix to right (left) on the position in which the shifted matrix has  $j$  matrices of different kind left (right). (Range of  $j$  is restricted by  $p$  or  $r$  if the shifted matrix is  $Y$  or  $X$ ).

**Example 7**

$$P_3^+ \circ XYXYXYXY = YXYXYXY \quad (2.28)$$

Now, we can prove the following theorem.

**Theorem 8** Let  $p, r > 0$  and  $p + r = n$  (i.e.  $\alpha^{p+r} = 1$ ). Then the matrices  $S, T$  fulfill

$$\{S^p, T^r\} = 0. \quad (2.29)$$

*Proof:*

Obviously, all the terms in the combinator  $\{S^p, T^r\}$  can be generated e.g. from the string

$$\underbrace{SS \dots S}_p \underbrace{TT \dots T}_r = S^p T^r$$

by means of the permutations  $P_j^+$

$$\{S^p, T^r\} = \sum_{k_p=0}^r \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} P_{k_1}^+ \circ P_{k_2}^+ \dots P_{k_p}^+ \circ S^p T^r. \quad (2.30)$$

Now the relation (2.3) implies

$$P_j^+ \circ S^p T^r = \alpha^j S^p T^r$$

and Eq. (2.30) can be modified

$$\{S^p, T^r\} = \left( \sum_{k_p=0}^r \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \alpha^{k_1} \alpha^{k_2} \dots \alpha^{k_p} \right) S^p T^r. \quad (2.31)$$

Apparently the multiple sum in this equation coincides with r.h.s. of Eq. (2.12) and satisfies the condition for Eq. (2.19), thereby the theorem is proved.

Let us remark, that alternative use of permutations  $P_j^-$  instead of  $P_j^+$  would lead to the equation

$$\{S^p, T^r\} = \left( \sum_{k_r=0}^p \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \alpha^{k_1} \alpha^{k_2} \dots \alpha^{k_r} \right) S^p T^r. \quad (2.32)$$

Comparison of Eqs. (2.31), (2.32) with the relation for  $F_{pr}$  defined by Eq. (2.12) implies

$$F_{pr}(\alpha) = F_{rp}(\alpha). \quad (2.33)$$

Obviously this equation is valid irrespective of the assumption  $\alpha^{p+r} = 1$ , i.e. it holds for any  $n$  and  $\alpha = \exp(2\pi i/n)$ . It follows, that Eq. (2.33) is satisfied for any  $\alpha$ .

**Definition 9** By the symbols  $Q_{pr}$  we denote  $n^2$  matrices

$$Q_{pr} = S^p T^r, \quad p, r = 1, 2, \dots, n \quad (2.34)$$

**Lemma 10**

$$Q_{rs} Q_{pq} = \alpha^{s \cdot p} Q_{kl}; \quad k = \text{mod}(r+p-1, n) + 1, \quad l = \text{mod}(s+q-1, n) + 1, \quad (2.35)$$

$$Q_{rs} Q_{pq} = \alpha^{s \cdot p - r \cdot q} Q_{pq} Q_{rs}, \quad (2.36)$$

$$(Q_{rs})^n = (-1)^{(n-1)r \cdot s} I, \quad (2.37)$$

$$Q_{rs}^\dagger Q_{rs} = Q_{rs} Q_{rs}^\dagger = I, \quad (2.38)$$

$$Q_{rs}^\dagger = \alpha^{r \cdot s} Q_{kl}; \quad k = n - r, \quad l = n - s, \quad (2.39)$$

$$\det Q_{rs} = (-1)^{(n-1)(r+s)} \quad (2.40)$$

and for  $r \neq n$  or  $s \neq n$

$$\text{Tr } Q_{rs} = 0. \quad (2.41)$$

*Proof:*

The relations follow from the definition of  $Q_{pr}$  and relations (2.3)-(2.7).

**Theorem 11** The matrices  $Q_{pr}$  are linearly independent and any matrix  $A$  (of the same dimension) can be expressed as their linear combination

$$A = \sum_{k,l=1}^n a_{kl} Q_{kl}, \quad a_{kl} = \frac{1}{n} \text{Tr}(Q_{kl}^\dagger A). \quad (2.42)$$

*Proof:*

Let us assume matrices  $Q_{kl}$  are linearly dependent, i.e. there exists some  $a_{rs} \neq 0$  and simultaneously

$$\sum_{k,l=1}^n a_{kl} Q_{kl} = 0,$$

which with the use of the previous lemma implies

$$\text{Tr} \sum_{k,l=1}^n a_{kl} Q_{rs}^\dagger Q_{kl} = a_{rs} n = 0.$$

This equation contradicts our assumption, therefore the matrices are independent and obviously represent a base in the linear space of matrices  $n \times n$ , which with the use of the previous lemma implies the relations (2.42).

**Theorem 12** For any  $n \geq 2$ , among  $n^2$  matrices (2.34) there exists the triad  $Q_\lambda, Q_\mu, Q_\nu$  for which

$$\{Q_\lambda^p, Q_\mu^r\} = \{Q_\mu^p, Q_\nu^r\} = \{Q_\nu^p, Q_\lambda^r\} = 0; \quad 0 < p, r, \quad p + r = n \quad (2.43)$$

and moreover if  $n \geq 3$ , then also

$$\{Q_\lambda^p, Q_\mu^r, Q_\nu^s\} = 0; \quad 0 < p, r, s, \quad p + r + s = n. \quad (2.44)$$

*Proof:*

We shall show the relations hold e.g. for indices  $\lambda = 1n$ ,  $\mu = 11$ ,  $\nu = n1$ . Let us denote

$$X = Q_{1n} = S, \quad Y = Q_{11}, \quad Z = Q_{n1} = T, \quad (2.45)$$

then the relation (2.36) implies

$$YX = \alpha XY, \quad ZX = \alpha XZ, \quad ZY = \alpha YZ. \quad (2.46)$$

Actually the relation  $\{X^p, Z^r\} = 0$  is already proven in the Theorem 8, obviously the remaining relations (2.43) can be proved exactly in the same way.

The combinator (2.44) can be similarly as in the proof of Theorem 8 expressed

$$\begin{aligned} & \{X^p, Y^r, Z^s\} \\ &= \sum_{j_p=0}^{r+s} \dots \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} P_{j_1}^+ \circ P_{j_2}^+ \dots P_{j_p}^+ \circ X^p \sum_{k_p=0}^s \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} P_{k_1}^+ \circ P_{k_2}^+ \dots P_{k_r}^+ \circ Y^r Z^s, \end{aligned} \quad (2.47)$$

which for matrices obeying relations (2.46) give

$$\begin{aligned} & \{X^p, Y^r, Z^s\} \\ &= \left( \sum_{j_p=0}^{r+s} \dots \sum_{j_2=0}^{j_3} \sum_{j_1=0}^{j_2} \alpha^{j_1} \alpha^{j_2} \dots \alpha^{j_p} \right) \left( \sum_{k_p=0}^s \dots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \alpha^{k_1} \alpha^{k_2} \dots \alpha^{k_r} \right) X^p Y^r Z^s. \end{aligned}$$

Since the first multiple sum (with indices  $j$ ) coincides with Eq. (2.12) and satisfy the condition for Eq. (2.19), r.h.s. is zero and the theorem is proved.



Now let us make few remarks to illuminate content of the last theorem and meaning of the matrices  $Q_\lambda$ . Obviously, the relations (2.43), (2.44) are equivalent to the statement: any three complex numbers  $a, b, c$  satisfy

$$(aQ_\lambda + bQ_\mu + cQ_\nu)^n = (a^n + b^n + c^n)I. \quad (2.48)$$

Further, the theorem speaks about existence of the triad but not about their number. Generally for  $n > 2$  there is more than one triad defined by the theorem, but on the other hand not any three various matrices from the set  $Q_{rs}$  comply with the theorem. Simple example are some  $X, Y, Z$  where e.g.  $XY = YX$ , which happens for  $Y \sim X^p, 2 \leq p < n$ . Obviously in this case at least the relation (2.43) surely is not satisfied. Computer check of the relation (2.47) which has been done with all possible triads from  $Q_{rs}$  for  $2 \leq n \leq 20$  suggests, that a triad  $X, Y, Z$  for which there exist the numbers  $p, r, s \geq 1$  and  $p + r + s \leq n$  so that  $X^p Y^r Z^s \sim I$  also does not comply with the theorem. Further, the result on r.h.s. of Eq. (2.47) generally depends on the factors  $\beta_k$  in relations

$$XY = \beta_3 YX \quad YZ = \beta_1 ZY \quad ZX = \beta_2 XZ \quad (2.49)$$

and computer check suggests the sets in which for some  $\beta_k$  and  $p < n$  there is  $\beta_k^p = 1$  also contradict the theorem. In this way the number of different triads obeying the relations (2.43), (2.44) is rather complicated function of  $n$  - as shown in the table

$n :$	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$\#3 :$	1	1	1	4	1	9	4	9	4	25	4	36	9	16	16	64	9	81	16

Here the statement triad  $X, Y, Z$  is different from  $X', Y', Z'$  means that after any rearrangement of the symbols  $X, Y, Z$  for marking of matrices in the given set, there is always at least one pair  $\beta_k \neq \beta'_k$ .

Naturally, one can ask if there exists also the set of four or generally  $N$  matrices, which satisfy a relation similar to Eq. (2.48)

$$\left( \sum_{\lambda=0}^{N-1} a_\lambda Q_\lambda \right)^n = \sum_{\lambda=0}^{N-1} a_\lambda^n. \quad (2.50)$$

For  $2 \leq n \leq 10$  and  $N = 4$  the computer suggests the negative answer - in the case of matrices generated according to the *Definition 9*. However, one can verify: if  $U_l, l = 1, 2, 3$  is the triad complying with the theorem (or equivalently with the relation (2.48)), then the matrices  $n^2 \times n^2$

$$Q_0 = I \otimes T = \begin{pmatrix} I & & & & \\ & \alpha I & & & \\ & & \alpha^2 I & & \\ & & & \ddots & \\ & & & & \alpha^{n-1} I \end{pmatrix}, \quad (2.51)$$

$$Q_l = U_l \otimes S = \begin{pmatrix} 0 & & & & U_l \\ U_l & & & & \\ & U_l & & & \\ & & \ddots & & \\ & & & U_l & 0 \end{pmatrix} \quad (2.52)$$

satisfy relation (2.50) for  $N = 4$ . Generally, if  $U_\lambda$  are matrices complying with Eq. (2.50) for some  $N \geq 3$ , then the matrices created from them according to the rule (2.51), (2.52) will satisfy Eq. (2.50) for  $N + 1$ . The last statement follows from the following equalities. Let us assume

$$\sum_{k=0}^N p_k = n,$$

then

$$\begin{aligned} \{Q_0^{p_0}, Q_1^{p_1}, \dots, Q_N^{p_N}\} &= \sum_{j_{p_N}=0}^{n-p_N} \dots \sum_{j_1=0}^{j_2} \sum_{j_0=0}^{j_1} P_{j_0}^- \circ P_{j_1}^- \dots P_{j_{p_N}}^- \circ \{Q_0^{p_0}, \dots, Q_{N-1}^{p_{N-1}}\} Q_N^{p_N} \\ &= \sum_{j_{p_N}=0}^{n-p_N} \dots \sum_{j_1=0}^{j_2} \sum_{j_0=0}^{j_1} P_{j_0}^- \circ P_{j_1}^- \dots P_{j_{p_N}}^- \circ \{(U_0 \otimes S)^{p_0}, \dots, (U_{N-1} \otimes S)^{p_{N-1}}\} (I \otimes T)^{p_N} \\ &= \sum_{j_{p_N}=0}^{n-p_N} \dots \sum_{j_1=0}^{j_2} \sum_{j_0=0}^{j_1} \alpha^{j_0} \alpha^{j_1} \dots \alpha^{j_{p_N}} \{(U_0 \otimes S)^{p_0}, \dots, (U_{N-1} \otimes S)^{p_{N-1}}\} (I \otimes T)^{p_N} \\ &= \left( \sum_{j_{p_N}=0}^{n-p_N} \dots \sum_{j_1=0}^{j_2} \sum_{j_0=0}^{j_1} \alpha^{j_0} \alpha^{j_1} \dots \alpha^{j_{p_N}} \right) \{U_1^{p_1}, \dots, U_{N-1}^{p_{N-1}}\} \otimes S^{n-p_N} T^{p_N}, \end{aligned}$$

where the last multiple sum equals zero according to the relations (2.12) and (2.19). Obviously for  $n = 2$  the matrices (2.45) and (2.51), (2.52) created from them correspond, up to some phase factors, to the Pauli matrices  $\sigma_j$  and Dirac matrices  $\gamma_\mu$ .

Obviously, from the set of matrices  $Q_{rs}$  (with exception of  $Q_{nn} = I$ ) one can easily make the  $n^2 - 1$  generators of the fundamental representation of  $SU(n)$  group

$$G_{rs} = a_{rs} Q_{rs} + a_{rs}^* Q_{rs}^+, \quad (2.53)$$

where  $a_{rs}$  are suitable factors. For example the choice

$$a_{kl} = \frac{1}{\sqrt{2}} \alpha^{[kl+n(k+l-1/4)]/2} \quad (2.54)$$

gives commutation relations

$$\begin{aligned} [G_{kl}, G_{rs}] &= i \sin(\pi(kl - rs)/n) \\ &\cdot \left\{ \text{sg}(k+r, l+s, n) (G_{k+r, l+s} - (-1)^{n+k+l+r+s} G_{-k-r, -l-s}) \right. \\ &\quad \left. - \text{sg}(k-r, l-s, n) (G_{k-r, l-s} - (-1)^{n+k+l+r+s} G_{r-k, s-l}) \right\}, \end{aligned} \quad (2.55)$$

where

$$\text{sg}(p, q, n) = (-1)^{p \cdot m_q + q \cdot m_p - n}, \quad m_x = \frac{x - \text{mod}(x-1, n) - 1}{n}.$$

and indices at  $G$  (on r.h.s.) in Eq. (2.55) are understood in the sense of mod - like in the relation (2.35). One can easily check e.g. for  $n = 2$  the matrices (2.53) with the factors  $a_{rs}$  according to Eq. (2.54) are the Pauli matrices - generators of the fundamental representation of the  $SU(2)$  group.

### 3 Wave equations generated by the roots of D'Alambertian operator $\square^{1/n}$

Now, using the generalized Dirac matrices (2.51), (2.52) we shall assemble the corresponding wave equation as follows. These four matrices with normalization

$$(Q_0)^n = -(Q_l)^n = I, \quad l = 1, 2, 3, \quad (3.1)$$

allow to write down the set of algebraic equations

$$(\Gamma(p) - \mu I) \Psi(p) = 0, \quad (3.2)$$

where

$$\Gamma(p) = \sum_{\lambda=0}^3 \pi_\lambda Q_\lambda. \quad (3.3)$$

If the variables  $\mu, \pi_\lambda$  represent the fractional powers of the mass and momentum components

$$\mu^n = m^2, \quad \pi_\lambda^n = p_\lambda^2, \quad (3.4)$$

then

$$\Gamma(p)^n = p_0^2 - p_1^2 - p_2^2 - p_3^2 \equiv p^2 \quad (3.5)$$

and after  $n - 1$  times repeated application of the operator  $\Gamma$  on Eq. (3.2) one gets the set of Klein-Gordon equations in the  $p$ -representation

$$(p^2 - m^2) \Psi(p) = 0. \quad (3.6)$$

The Eqs. (3.2) and (3.6) are the sets of  $n^2$  equations with solution  $\Psi$  having  $n^2$  components. Obviously, the case  $n^2 = 4$  corresponds to the Dirac equation. For  $n > 2$  the Eq. (3.2) is new, more complicated and immediately invoking some questions. In the present paper we shall attempt to answer at least some of them. One can check, that the solution of the set (3.2) reads

$$\Psi(p) = \begin{pmatrix} \mathbf{h} \\ \frac{U(p)}{\alpha\pi_0 - \mu} \mathbf{h} \\ \frac{U^2(p)}{(\alpha\pi_0 - \mu)(\alpha^2\pi_0 - \mu)} \mathbf{h} \\ \vdots \\ \frac{U^{n-1}(p)}{(\alpha\pi_0 - \mu) \dots (\alpha^{n-1}\pi_0 - \mu)} \mathbf{h} \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}, \quad (3.7)$$

where

$$U(p) = \sum_{l=1}^3 \pi_l U_l, \quad (U_l)^n = -I,$$

( $U_l$  is the triad from which the matrices  $Q_l$  are constructed in accordance with Eqs. (2.51), (2.52)) and  $h_1, h_2, \dots, h_n$  are arbitrary functions of  $p$ . At the same time,  $\pi_\lambda$  satisfy the constraint

$$\pi_0^n - \pi_1^n - \pi_2^n - \pi_3^n = \mu^n = m^2. \quad (3.8)$$

First of all, one can bring to notice, that in Eq. (3.2) the fractional powers of the momentum components appear, which means that the equation in the  $x$ -representation will contain the fractional derivatives:

$$\pi_\lambda = (p_\lambda)^{2/n} \rightarrow (i\partial_\lambda)^{2/n}. \quad (3.9)$$

Our primary considerations will concern  $p$ -representation, but afterwards we shall show how the transition to the  $x$ -representation can be realized by means of the Fourier transformation, in accordance with the approach suggested in [14].

Further question concerns relativistic covariance of Eq. (3.2): How to transform simultaneously the operator

$$\Gamma(p) \rightarrow \Gamma(p') = \Lambda \Gamma(p) \Lambda^{-1} \quad (3.10)$$

and the solution

$$\Psi(p) \rightarrow \Psi'(p') = \Lambda \Psi(p) \quad (3.11)$$

to preserve the equal form of the operator  $\Gamma$  for initial variables  $p_\lambda$  and the boosted ones  $p'_\lambda$ ?

### 3.1 Infinitesimal transformations

First let us consider the infinitesimal transformations

$$\Lambda(d\omega) = I + id\omega \cdot L_\omega, \quad (3.12)$$

where  $d\omega$  represents the infinitesimal values of the six parameters of the Lorentz group corresponding to the space rotations

$$p'_i = p_i + \epsilon_{ijk} p_j d\varphi_k, \quad i = 1, 2, 3 \quad (3.13)$$

and the Lorentz transformations

$$p'_i = p_i + p_0 d\psi_i, \quad p'_0 = p_0 + p_i d\psi_i, \quad i = 1, 2, 3, \quad (3.14)$$

where  $\tanh \psi_i = v_i/c \equiv \beta_i$  is the corresponding velocity. Here, and anywhere in the next we use the convention, that in the expressions involving the antisymmetric tensor  $\epsilon_{ijk}$ , the summation over indices appearing twice is done. From the infinitesimal transformations (3.13), (3.14) one can obtain the finite ones. For the three space rotations we get

$$p'_1 = p_1 \cos \varphi_3 + p_2 \sin \varphi_3, \quad p'_2 = p_2 \cos \varphi_3 - p_1 \sin \varphi_3, \quad p'_3 = p_3, \quad (3.15)$$

$$p'_2 = p_2 \cos \varphi_1 + p_3 \sin \varphi_1, \quad p'_3 = p_3 \cos \varphi_1 - p_2 \sin \varphi_1, \quad p'_1 = p_1, \quad (3.16)$$

$$p'_3 = p_3 \cos \varphi_2 + p_1 \sin \varphi_2, \quad p'_1 = p_1 \cos \varphi_2 - p_3 \sin \varphi_2, \quad p'_2 = p_2 \quad (3.17)$$

and for the Lorentz transformations similarly

$$p'_0 = p_0 \cosh \psi_i + p_i \sinh \psi_i, \quad i = 1, 2, 3, \quad (3.18)$$

where

$$\cosh \psi_i = \frac{1}{\sqrt{1 - \beta_i^2}}, \quad \sinh \psi_i = \frac{\beta_i}{\sqrt{1 - \beta_i^2}}. \quad (3.19)$$

The definition of the six parameters implies that the corresponding infinitesimal transformations of the reference frame  $p \rightarrow p'$  changes a function  $f(p)$  :

$$f(p) \rightarrow f(p') = f(p + \delta p) = f(p) + \frac{df}{d\omega} d\omega, \quad (3.20)$$

where  $d/d\omega$  stands for

$$\frac{d}{d\varphi_i} = -\epsilon_{ijk} p_j \frac{\partial}{\partial p_k}, \quad \frac{d}{d\psi_i} = p_0 \frac{\partial}{\partial p_i} + p_i \frac{\partial}{\partial p_0}, \quad i = 1, 2, 3. \quad (3.21)$$

Obviously, the equation

$$p' = p + \frac{dp}{d\omega} d\omega \quad (3.22)$$

combined with Eq. (3.21) is identical to Eqs. (3.13), (3.14). Further, with the use of formulas (3.12) and (3.21) the relations (3.10), (3.11) can be rewritten in the infinitesimal form

$$\Gamma(p') = \Gamma(p) + \frac{d\Gamma(p)}{d\omega} d\omega = (I + id\omega \cdot L_\omega) \Gamma(p) (I - id\omega \cdot L_\omega), \quad (3.23)$$

$$\Psi'(p') = \Psi'(p) + \frac{d\Psi'(p)}{d\omega} d\omega = (I + id\omega \cdot L_\omega) \Psi(p). \quad (3.24)$$

If we define

$$\mathbf{L}_\omega = L_\omega + i \frac{d}{d\omega}, \quad (3.25)$$

then the relations (3.23), (3.24) imply

$$[\mathbf{L}_\omega, \Gamma] = 0, \quad (3.26)$$

$$\Psi'(p) = (I + id\omega \cdot \mathbf{L}_\omega) \Psi(p). \quad (3.27)$$

The six operators  $\mathbf{L}_\omega$  are generators of the corresponding representation of the Lorentz group, so they have to satisfy the commutation relations

$$[\mathbf{L}_{\varphi_j}, \mathbf{L}_{\varphi_k}] = i\epsilon_{jkl} \mathbf{L}_{\varphi_l}, \quad (3.28)$$

$$[\mathbf{L}_{\psi_j}, \mathbf{L}_{\psi_k}] = -i\epsilon_{jkl} \mathbf{L}_{\varphi_l}, \quad (3.29)$$

$$[\mathbf{L}_{\varphi_j}, \mathbf{L}_{\psi_k}] = i\epsilon_{jkl} \mathbf{L}_{\psi_l}, \quad j, k, l = 1, 2, 3. \quad (3.30)$$

How this representation looks like, in other words, what operators  $\mathbf{L}_\omega$  satisfy Eqs.(3.28) - (3.30) and (3.26)? First, one can easily check, that for  $n > 2$  there do not exist matrices  $L_\omega$  with constant elements representing the first term in r.h.s. of equality (3.25) and satisfying the Eq. (3.26). If one assumes, that  $L_\omega$  consist only of constant elements, then the elements of matrix  $\frac{d}{d\omega}\Gamma(p)$  involving the terms like  $p_i^{2/n-1}p_j$  certainly cannot be expressed through the elements of the difference  $L_\omega\Gamma - \Gamma L_\omega$  consisting only of the elements proportional to  $p_k^{2/n}$  - in contradistinction to the case  $n = 2$ , i.e. the case of Dirac equation. In this way the Eq. (3.26) cannot be satisfied for  $n > 2$  and  $L_\omega$  constant. Nevertheless, one can show, that the set of Eqs. (3.26),(3.28) - (3.30) is solvable, provided that we accept the elements of the matrices  $L_\omega$  are not constants, but the functions of  $p_i$ . To prove this, let us first make a few preparing steps.

**Definition 13** Let  $\Gamma_1(p), \Gamma_2(p)$  and  $X$  be the square matrices of the same dimension and

$$\Gamma_1(p)^n = \Gamma_2(p)^n = p^2.$$

Then for any matrix  $X$  we define the form

$$Z(\Gamma_1, X, \Gamma_2) = \frac{1}{np^2} \sum_{j=1}^n \Gamma_1^j X \Gamma_2^{n-j}. \quad (3.31)$$

One can easily check, that the matrix  $Z$  satisfies e.g.

$$\Gamma_1 Z = Z \Gamma_2, \quad (3.32)$$

$$Z(Z(X)) = Z(X) \quad (3.33)$$

and in particular for  $\Gamma_1 = \Gamma_2 \equiv \Gamma$

$$[\Gamma, Z] = 0, \quad (3.34)$$

$$[\Gamma, X] = 0 \Rightarrow X = Z(X). \quad (3.35)$$

**Lemma 14** The Eq. (3.2) can be expressed in the diagonalized (canonical) form

$$(\Gamma_0(p) - \mu) \Psi_0(p) = 0; \quad \Gamma_0(p) \equiv (p^2)^{1/n} Q_0, \quad (3.36)$$

where  $Q_0$  is the matrix (2.51), i.e. there exists the set of transformations  $Y$ , that

$$\Gamma_0(p) = Y(p) \Gamma(p) Y^{-1}(p); \quad Y = Z(\Gamma_0, X, \Gamma) \quad (3.37)$$

and a particular form reads

$$Y = y \cdot Z(\Gamma_0, I, \Gamma), \quad Y^{-1} = y \cdot Z(\Gamma, I, \Gamma_0), \quad (3.38)$$

where

$$y = \sqrt{\frac{n \left[ 1 - (p_0^2/p^2)^{1/n} \right]}{1 - p_0^2/p^2}}.$$

*Proof:*

The Eq. (3.32) implies

$$\Gamma_0 = Z(\Gamma_0, X, \Gamma) \Gamma Z(\Gamma_0, X, \Gamma)^{-1}, \quad (3.39)$$

therefore, if the matrix  $X$  is chosen in such a way that  $\det Z \neq 0$ , then  $Z^{-1}$  exists and the transformation (3.39) diagonalizes the matrix  $\Gamma$ . Let us put  $X = I$  and calculate the following product

$$C = Z(\Gamma_0, I, \Gamma) Z(\Gamma, I, \Gamma_0) = \frac{1}{n^2 p^4} \sum_{i,j=1}^n \Gamma_0^i \Gamma^{n-i+j} \Gamma_0^{n-j}. \quad (3.40)$$

The last sum can be rearranged, instead of the summation index  $j$  we use the new one

$$k = i - j \quad \text{for } i \geq j, \quad k = i - j + n \quad \text{for } i < j; \quad k = 0, \dots, n-1,$$

then the Eq. (3.40) reads

$$C = \frac{1}{n^2 p^4} \sum_{k=0}^{n-1} \left( \sum_{i=k+1}^n \Gamma_0^i \Gamma^{n-k} \Gamma_0^{n+k-i} + \sum_{i=1}^k \Gamma_0^i \Gamma^{2n-k} \Gamma_0^{k-i} \right) \quad (3.41)$$

and if we take into account that  $\Gamma_0^n = \Gamma^n = p^2$ , then this sum can be simplified

$$C = \sum_{k=0}^{n-1} C_k = \frac{1}{n^2 p^2} \sum_{k=0}^{n-1} \sum_{i=1}^n \Gamma_0^i \Gamma^{n-k} \Gamma_0^{k-i}. \quad (3.42)$$

For the term  $k = 0$  we get

$$C_0 = \frac{1}{n^2 p^2} \sum_{i=1}^n \Gamma_0^i \Gamma^n \Gamma_0^{-i} = \frac{1}{n} \quad (3.43)$$

and for  $k > 0$ , using Eqs. (3.3), (2.51), (2.52), (3.36) and *Definition 3* one obtains

$$\begin{aligned} C_k &= \frac{1}{n^2 p^2} \sum_{i=1}^n \Gamma_0^i \Gamma^{n-k} \Gamma_0^{k-i} = \frac{1}{n^2 p^2} \sum_{i=1}^n \Gamma_0^i \left( \sum_{\lambda=0}^3 \pi_\lambda Q_\lambda \right)^{n-k} \Gamma_0^{k-i} \\ &= \frac{1}{n^2 p^2} \sum_{i=1}^n \Gamma_0^i \left( \pi_0 \cdot I \otimes T + \left[ \sum_{\lambda=1}^3 \pi_\lambda U_\lambda \right] \otimes S \right)^{n-k} \Gamma_0^{k-i} \\ &= \frac{1}{n^2 p^2} \sum_{i=1}^n \Gamma_0^i (\pi_0 \cdot I \otimes T + U \otimes S)^{n-k} \Gamma_0^{k-i} \\ &= \frac{1}{n^2 p^2} \sum_{i=1}^n \Gamma_0^i \left( \sum_{p=0}^{n-k} \pi_0^p \cdot U^{n-k-p} \otimes \{T^p, S^{n-k-p}\} \right) \Gamma_0^{k-i} \\ &= \frac{(p^2)^{k/n}}{n^2 p^2} \sum_{p=0}^{n-k} \pi_0^p \cdot U^{n-k-p} \otimes \sum_{i=1}^n T^i \{T^p, S^{n-k-p}\} T^{k-i}. \end{aligned} \quad (3.44)$$

For  $p < n - k \equiv l$  the last sum can be with the use of the relation (2.3) modified

$$\begin{aligned} \sum_{i=1}^n T^i \{T^p, S^{l-p}\} T^{k-i} &= \{T^p, S^{l-p}\} T^k \sum_{i=1}^n \alpha^{i \cdot (l-p)} \\ &= \{T^p, S^{l-p}\} T^k \alpha^{(l-p)} \frac{1 - \alpha^{n \cdot (l-p)}}{1 - \alpha^{(l-p)}} = 0, \end{aligned} \quad (3.45)$$

therefore only the term  $p = n - k$  contributes:

$$C_k = \frac{(p^2)^{k/n}}{n^2 p^2} (p_0^2)^{(n-k)/n} n = \frac{1}{n} \left( \frac{p_0^2}{p^2} \right)^{(n-k)/n}. \quad (3.46)$$

So the sum (3.42) gives in total

$$C = \frac{1}{n} \left[ 1 + \left( \frac{p_0^2}{p^2} \right)^{1/n} + \left( \frac{p_0^2}{p^2} \right)^{2/n} + \dots + \left( \frac{p_0^2}{p^2} \right)^{(n-1)/n} \right] \quad (3.47)$$

$$= \frac{1 - p_0^2/p^2}{n \left[ 1 - (p_0^2/p^2)^{1/n} \right]},$$

therefore Eq. (3.37) is satisfied with  $Y, Y^{-1}$  given by Eq. (3.38) and the proof is completed.

Solution of the Eq. (3.36) reads

$$\Psi_0(p) = \begin{pmatrix} \mathbf{0} \\ \cdot \\ \mathbf{0} \\ \mathbf{g} \\ \mathbf{0} \\ \cdot \\ \cdot \\ \mathbf{0} \end{pmatrix}; \quad \mathbf{g} \equiv \begin{pmatrix} g_1 \\ g_2 \\ \cdot \\ g_n \end{pmatrix}, \quad \mathbf{0} \equiv \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \quad (3.48)$$

i.e. the sequence of non zero components can be only in one block, whose location depends on the choice of the phase of the power  $(p^2)^{1/n}$ . The  $g_j$  are arbitrary functions of  $p$  and simultaneously the constraint  $p^2 = m^2$  is required. Now we shall try to find the generators satisfying the covariance condition for Eq. (3.36)

$$[\mathbf{L}_\omega, \Gamma_0(p)] = 0 \quad (3.49)$$

together with the commutation relations (3.28)-(3.30). Some hint can be obtained from the Dirac equation transformed to the diagonal form in accordance with the relations (3.37), (3.38). We shall use the current representation of the Pauli and Dirac matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (3.50)$$

$$\gamma_0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \quad \gamma_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ -\sigma_j & \mathbf{0} \end{pmatrix}; \quad j = 1, 2, 3, \quad (3.51)$$

where the bold  $\mathbf{0}, \mathbf{1}$  stand for zero and unit matrices  $2 \times 2$ . The Dirac equation

$$(\Gamma(p) - m) \Psi(p) = 0, \quad \Gamma(p) \equiv \sum_{\lambda=0}^3 p_\lambda \gamma_\lambda \quad (3.52)$$

is covariant under the transformations generated by

$$\mathbf{L}_{\varphi_j} = \frac{i}{4} \epsilon_{jkl} \gamma_k \gamma_l + i \frac{d}{d\varphi_j} = L_{\varphi_j} + i \frac{d}{d\varphi_j}; \quad L_{\varphi_j} = \frac{1}{2} \begin{pmatrix} \sigma_j & \mathbf{0} \\ \mathbf{0} & \sigma_j \end{pmatrix}, \quad (3.53)$$

$$\mathbf{L}_{\psi_j} = \frac{i}{2} \gamma_0 \gamma_j + i \frac{d}{d\psi_j} = L_{\psi_j} + i \frac{d}{d\psi_j}; \quad L_{\psi_j} = \frac{i}{2} \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix}, \quad (3.54)$$

where  $j, k, l = 1, 2, 3$ . Obviously, to preserve covariance, one has with the transformation  $\Gamma \rightarrow \Gamma_0 = Y \Gamma Y^{-1}$  perform also

$$\mathbf{L}_\omega \rightarrow \mathbf{M}_\omega = Y(p) \mathbf{L}_\omega Y^{-1}(p). \quad (3.55)$$



For the space rotations  $\mathbf{L}_{\varphi_j}$  commuting with both  $\Gamma_0, \Gamma$  and with the  $Y$  from the relation (3.38) the result is quite straightforward

$$\mathbf{M}_{\varphi_j} = \mathbf{L}_{\varphi_j} = L_{\varphi_j} + i \frac{d}{d\varphi_j}, \quad (3.56)$$

i.e. the generators of the space rotations are not changed by the transformation (3.55). The similar procedure with the Lorentz transformations is slightly more complicated, nevertheless after calculation of the commutator  $[\mathbf{L}_{\psi_j}, \Gamma_0 / \sqrt{1 + p_0 / \sqrt{p^2}}]$  and a few further steps one obtains

$$\mathbf{M}_{\psi_j} = M_{\psi_j}(p) + i \frac{d}{d\psi_j}; \quad M_{\psi_j}(p) = \epsilon_{jkl} \frac{p_k L_{\varphi_l}}{p_0 + \sqrt{p^2}}. \quad (3.57)$$

So the generators (3.56), (3.57) guarantee the covariance of the diagonalized Dirac equation obtained from Eq. (3.52) according to Lemma 14. At the same time it is obvious, that having the set of generators  $L_{\varphi_j}$  (with constant elements) of space rotations, one can according to Eq. (3.57) construct the generators of Lorentz transformations  $M_{\psi_j}(p)$  (or  $\mathbf{M}_{\psi_j}$ ), which satisfy commutation relations (3.28) - (3.30). Obviously this recipe is valid for any representation of infinitesimal space rotations. Let us make a remark, that the algebra given by Eqs. (3.56), (3.57) appears in a slightly modified form already in [15]. Now, we shall show, that if one requires a linear relation between the generators  $M_{\psi_j}$  and  $L_{\varphi_l}$ , like in Eq. (3.57), then this relation can have a more general shape, than that in Eq. (3.57).

**Lemma 15** *Let  $L_{\varphi_j}$  be matrices with constant elements satisfying commutation relations (3.28). Then the operators*

$$\mathbf{M}_{\psi_j} = M_{\psi_j}(p) + i \frac{d}{d\psi_j}; \quad M_{\psi_j}(p) = \frac{\kappa L_{\varphi_j} + \epsilon_{jkl} p_k L_{\varphi_l}}{p_0 + \sqrt{p^2 - \kappa^2}}, \quad (3.58)$$

where  $\kappa$  is any complex constant, satisfy the commutation relations (3.29), (3.30).

*Proof:*

After insertion of the generators (3.58) into the relations (3.29), (3.30) one can check, that the commutation relations are satisfied. In fact, it is sufficient to verify e.g. the commutators  $[\mathbf{L}_{\varphi_1}, \mathbf{L}_{\psi_2}]$ ,  $[\mathbf{L}_{\varphi_1}, \mathbf{L}_{\psi_1}]$  and  $[\mathbf{L}_{\psi_1}, \mathbf{L}_{\psi_3}]$ , the remaining follow from the cyclic symmetry.

Let us note, the formula (3.58) covers also the limit case  $|\kappa| \rightarrow \infty$ , then

$$M_{\psi_j} = i L_{\varphi_j}. \quad (3.59)$$

On the other hand, the relation (3.57) corresponds to  $\kappa = 0$ . The representations of the Lorentz group defined by the generators (3.56), (3.58) and differing only in the parameter  $\kappa$  should be equivalent in the sense

$$\mathbf{M}_{\omega}(\kappa') = X^{-1}(p) \mathbf{M}_{\omega}(\kappa) X(p). \quad (3.60)$$

We shall not make a general proof of this relation, but rather we shall show, that the representations defined in the Lemma 15 and differing only in  $\kappa$ , can be

classified by the same mass  $m^2 = p^2$  and spin  $\mathbf{s}^2 = s(s+1)$ . First, let us note, that the six generators considered in the lemma together with the four generators  $p_\alpha$  of the space-time translations form the set of generators of the Poincaré group. One can easily check, that the corresponding additional commutation relations are satisfied:

$$[p_\alpha, p_\beta] = 0, \quad [\mathbf{M}_{\varphi_j}, p_0] = 0, \quad [p_\alpha, \Gamma_0] = 0, \quad (3.61)$$

$$[\mathbf{M}_{\varphi_j}, p_k] = i\epsilon_{jkl}p_l, \quad [\mathbf{M}_{\psi_j}, p_k] = i\delta_{jk}p_0, \quad [\mathbf{M}_{\psi_j}, p_0] = ip_j. \quad (3.62)$$

Further, the generators  $\mathbf{M}_\omega$  can be rewritten in the covariant notation

$$\mathbf{M}_{jk} = \epsilon_{jkl}\mathbf{M}_{\varphi_l}, \quad \mathbf{M}_{j0} = \mathbf{M}_{\psi_j}, \quad \mathbf{M}_{\alpha\beta} = -\mathbf{M}_{\beta\alpha}. \quad (3.63)$$

Now the Pauli - Lubanski vector can be constructed:

$$V_\alpha = \epsilon_{\alpha\beta\gamma\delta}\mathbf{M}^{\beta\gamma}p^\delta/2, \quad (3.64)$$

which has satisfy

$$V_\alpha V^\alpha = -m^2 s(s+1), \quad (3.65)$$

where  $s$  is the corresponding spin number. One can check, that after inserting the generators (3.63) into relations (3.64), (3.65), the result does not depend on  $\kappa$

$$V_\alpha V^\alpha = -p^2 (M_{\varphi_1}^2 + M_{\varphi_2}^2 + M_{\varphi_3}^2) = -m^2 s(s+1).$$

So the generators of the Lorentz group, which satisfy Eq. (3.49), can have the form

$$\mathbf{R}_\omega = R_\omega + i\frac{d}{d\omega}; \quad R_\omega = \begin{pmatrix} M_\omega & & & & \\ & M_\omega & & & \\ & & M_\omega & & \\ & & & \ddots & \\ & & & & \ddots & \\ & & & & & M_\omega \end{pmatrix}, \quad (3.66)$$

where  $M_\omega$  are the  $n \times n$  matrices defined in accordance with the Lemma 15. There are  $n$  such matrices on the diagonal and apparently these matrices may not be identical.

Finally, it is obvious that Eq. (3.36) is covariant also under any infinitesimal transform

$$\Lambda(d\xi) = I + id\xi \cdot K_\xi, \quad (3.67)$$

where the generators  $K_\xi$  have the similar form as the generators (3.66)

$$K_\xi = \begin{pmatrix} k_\xi & & & & \\ & k_\xi & & & \\ & & k_\xi & & \\ & & & \ddots & \\ & & & & \ddots & \\ & & & & & k_\xi \end{pmatrix} \quad (3.68)$$

and generally their elements may depend on  $p$ . Obviously, one can put the question: If the generators  $L_\varphi, k_\xi$  from Eqs. (3.56), (3.68) with constant elements represent the algebra of some group (containing the rotation group as a

subgroup), then what linear combination  $M_{\psi_j}(p)$  of these generators satisfy the commutation relations (3.29), (3.30) for the generators of Lorentz transformations? In the other words, what are the coefficients in summation

$$M_{\psi_j}(p) = \sum_{k=1}^3 c_{jk}(p) L_{\varphi_k} + \sum_{\xi} c_{j\xi} k_{\xi} \quad (3.69)$$

satisfying the commutation relations for the generators of the Lorentz transformations? In this paper we shall not discuss this more general task, for our present purpose it is sufficient, that we proved existence of the generators of infinitesimal Lorentz transformations, under which the Eq. (3.36) is covariant.

### 3.2 Finite transformations

Now, having the infinitesimal transformations, one can proceed to finite ones, corresponding to the parameters  $\omega$  and  $\xi$ :

$$\Psi'_0(p') = \Lambda(\omega)\Psi_0(p), \quad \Psi'_0(p) = \Lambda(\xi)\Psi_0(p), \quad (3.70)$$

where  $p \rightarrow p'$  is some of the transformations (3.15) - (3.18). The matrices  $\Lambda$  satisfy

$$\Lambda(\omega + d\omega) = \Lambda(\omega)\Lambda(d\omega), \quad \Lambda(\xi + d\xi) = \Lambda(\xi)\Lambda(d\xi), \quad (3.71)$$

which for the parameters  $\varphi$  (space rotations only) and  $\xi$  imply

$$\frac{d\Lambda(\varphi_j)}{d\varphi_j} = i\Lambda(\varphi_j)R_{\varphi_j}, \quad \frac{d\Lambda(\xi)}{d\xi} = i\Lambda(\xi)K_{\xi}. \quad (3.72)$$

Assuming the constant elements of the matrices  $R_{\varphi_j}$  and  $K_{\xi}$ , the solutions of the last equations can be written in the usual exponential form

$$\Lambda(\varphi_j) = \exp(i\varphi_j R_{\varphi_j}), \quad \Lambda(\xi) = \exp(i\xi K_{\xi}). \quad (3.73)$$

The space rotation by an angle  $\varphi$  about the axis having the direction  $\vec{u}$ ,  $|\vec{u}| = 1$  is represented by

$$\Lambda(\varphi, \vec{u}) = \exp \left[ i\varphi \left( \vec{u} \cdot \vec{R}_{\varphi} \right) \right]; \quad \vec{R}_{\varphi} = (R_{\varphi_1}, R_{\varphi_2}, R_{\varphi_3}). \quad (3.74)$$

For the Lorentz transformations we get instead of Eq. (3.72)

$$\frac{d\Lambda(\psi_j)}{d\psi_j} = i f_j(\psi_j) \Lambda(\psi_j) N_j, \quad (3.75)$$

where in accordance with Eqs. (3.18) and (3.58) there stand for

$$f_j(\psi_j) = \frac{1}{p_0 \cosh \psi_j + p_j \sinh \psi_j + \sqrt{p^2 - \kappa^2}}, \quad (3.76)$$

$$N_j = \kappa R_{\varphi_j} + \epsilon_{jkl} p_k R_{\varphi_l}. \quad (3.77)$$

The solution of Eq. (3.75) reads

$$\Lambda(\psi_j) = \exp(iF(\psi_j)N_j); \quad F(\psi_j) = \int_0^{\psi_j} f_j(\eta) d\eta. \quad (3.78)$$

The Lorentz boost in a general direction  $\vec{u}$  with the velocity  $\beta$  is represented by

$$\Lambda(\psi, \vec{u}) = \exp(iF(\psi)N), \quad \tanh \psi = \beta, \quad (3.79)$$

where

$$F(\psi) = \int_0^\psi \frac{d\eta}{p_0 \cosh \eta + \vec{p}\vec{u} \sinh \eta + \sqrt{p^2 - \kappa^2}}, \quad (3.80)$$

$$N = \kappa \vec{u} \vec{R}_\varphi + (\vec{u} \times \vec{p}) \cdot \vec{R}_\varphi. \quad (3.81)$$

The corresponding integrals can be found e.g. in the handbook [16].

Let us note, from the technical point of view, solution of the equation

$$\frac{d\Lambda(t)}{dt} = \Omega(t)\Lambda(t), \quad (3.82)$$

where  $\Lambda, \Omega$  are some square matrices, can be written in the exponential form

$$\Lambda(t) = \exp\left(\int_0^t \Omega(\eta) d\eta\right) \quad (3.83)$$

only if the matrix  $\Omega$  satisfies

$$\left[\Omega(t), \int_0^t \Omega(\eta) d\eta\right] = 0. \quad (3.84)$$

This condition is necessary for differentiation

$$\begin{aligned} \frac{d\Lambda(t)}{dt} &= \frac{d}{dt} \sum_{j=0}^{\infty} \frac{\left(\int_0^t \Omega(\eta) d\eta\right)^j}{j!} = \Omega(t) \sum_{j=0}^{\infty} \frac{\left(\int_0^t \Omega(\eta) d\eta\right)^j}{j!} \\ &= \Omega(t)\Lambda(t) = \Lambda(t)\Omega(t). \end{aligned} \quad (3.85)$$

Obviously the condition (3.85) is satisfied for the generators of all the considered transformations, including the Lorentz ones in Eq. (3.79), since the matrix  $N$  does not depend on  $\psi$ . ( $N$  depends only on the momenta components perpendicular the direction of the Lorentz boost.)

### 3.3 Equivalent transformations

Now, from the symmetry of the Eq. (3.36) one can obtain the corresponding transformations for the Eq. (3.2). The generators (3.66) satisfy the relations (3.49) and (3.28) - (3.30), it follows that the generators

$$\mathbf{R}_\omega(\Gamma) = Y^{-1}(p)\mathbf{R}_\omega(\Gamma_0)Y(p) = R_\omega(\Gamma) + i\frac{d}{d\omega}, \quad (3.86)$$

$$R_\omega(\Gamma) = Y^{-1}(p)R_\omega(\Gamma_0)Y(p) + iY^{-1}(p)\frac{dY(p)}{d\omega},$$

where  $\mathbf{R}_\omega(\Gamma_0), R_\omega(\Gamma_0)$  are generators (3.66) and  $Y(p)$  is the transformation (3.38), will satisfy the same conditions, but with the relation (3.26) instead of the relation (3.49). Similarly the generators  $K_\xi(\Gamma_0)$  in relation (3.68) will be for Eq. (3.2) replaced by

$$K_\xi(\Gamma) = Y^{-1}(p)K_\xi(\Gamma_0)Y(p). \quad (3.87)$$

The finite transformations of the Eq. (3.2) and its solutions can be obtained as follows. First let us consider the transformations  $\Lambda(\Gamma_0, \omega, \vec{u})$  given by Eqs. (3.74) and (3.79). In accordance with Eq. (3.37) we have

$$\Gamma(p) = Y^{-1}(p)\Gamma_0(p)Y(p), \quad \Gamma(p') = Y^{-1}(p')\Gamma_0(p')Y(p') \quad (3.88)$$

and correspondingly for the solutions of Eqs. (3.2), (3.36)

$$\Psi(p) = Y^{-1}(p)\Psi_0(p), \quad \Psi'(p') = Y^{-1}(p')\Psi'_0(p'). \quad (3.89)$$

Since

$$\Psi'_0(p') = \Lambda(\Gamma_0, \omega, \vec{u})\Psi_0(p), \quad (3.90)$$

then Eqs. (3.89) imply

$$\Psi'(p') = \Lambda(\Gamma, \omega, \vec{u})\Psi(p); \quad \Lambda(\Gamma, \omega, \vec{u}) = Y^{-1}(p')\Lambda(\Gamma_0, \omega, \vec{u})Y(p). \quad (3.91)$$

Similarly, the transformations  $\Lambda(\Gamma_0, \xi)$  given by Eq. (3.73) are for Eq. (3.2) replaced by

$$\Lambda(\Gamma, \xi) = Y^{-1}(p)\Lambda(\Gamma_0, \xi)Y(p). \quad (3.92)$$

Let us note, all the symmetries of Eq. (3.2) like the transformation (3.92), which are not connected with a change of the reference frame ( $p$ ), can be in accordance with the relation (3.34) expressed

$$\Lambda(\Gamma, X) = Z(\Gamma, X, \Gamma), \quad (3.93)$$

where  $Z$  is defined by Eq. (3.31) and  $X(p)$  is any matrix for which there exists  $Z(\Gamma, X, \Gamma)^{-1}$ . Further, it is obvious that if we have some set of generators  $\mathbf{R}_\omega(\Gamma)$  satisfying the relations (3.26) and (3.28) - (3.30), then also any set

$$\hat{\mathbf{R}}_\omega(\Gamma) = Z(\Gamma, X, \Gamma)^{-1}\mathbf{R}_\omega(\Gamma)Z(\Gamma, X, \Gamma) \quad (3.94)$$

satisfies these conditions. For the finite transformations one gets correspondingly

$$\hat{\Lambda}(\Gamma, \omega, \vec{u}) = Z(\Gamma(p'), X(p'), \Gamma(p'))^{-1} \Lambda(\Gamma, \omega, \vec{u}) Z(\Gamma(p), X(p), \Gamma(p)). \quad (3.95)$$

In the same way, the sets of equivalent generators and transformations can be obtained for the diagonalized equation (3.36).

Let us remark, according to the Lemma 14 there exists the set of transformations  $\Gamma(p) \leftrightarrow \Gamma_0(p)$  given by the relation (3.37). We used its particular form (3.38), but how will the generators

$$\mathbf{R}_\omega(\Gamma, X_k) = Z(\Gamma_0, X_k, \Gamma)^{-1}\mathbf{R}_\omega(\Gamma_0)Z(\Gamma_0, X_k, \Gamma); \quad k = 1, 2 \quad (3.96)$$

differ for the two different matrices  $X_1$  and  $X_2$ ? The last relation implies

$$\mathbf{R}_\omega(\Gamma, X_1) = Z(\Gamma_0, X_1, \Gamma)^{-1}Z(\Gamma_0, X_2, \Gamma)\mathbf{R}_\omega(\Gamma, X_2)Z(\Gamma_0, X_2, \Gamma)^{-1}Z(\Gamma_0, X_1, \Gamma) \quad (3.97)$$

and according to the relation (3.32)

$$\begin{aligned} \Gamma Z(\Gamma_0, X_1, \Gamma)^{-1}Z(\Gamma_0, X_2, \Gamma) &= Z(\Gamma_0, X_1, \Gamma)^{-1}\Gamma_0 Z(\Gamma_0, X_2, \Gamma) \\ &= Z(\Gamma_0, X_1, \Gamma)^{-1}Z(\Gamma_0, X_2, \Gamma)\Gamma, \end{aligned} \quad (3.98)$$

which means

$$[Z(\Gamma_0, X_1, \Gamma)^{-1}Z(\Gamma_0, X_2, \Gamma), \Gamma] = 0. \quad (3.99)$$

It follows that there must exist a matrix  $X_3$  [e.g. according to implication (3.35) one can put  $X_3 = Z(\Gamma_0, X_1, \Gamma)^{-1}Z(\Gamma_0, X_2, \Gamma)$ ] so that

$$Z(\Gamma, X_3, \Gamma) = Z(\Gamma_0, X_1, \Gamma)^{-1}Z(\Gamma_0, X_2, \Gamma), \quad (3.100)$$

then the relation (3.97) can be rewritten

$$\mathbf{R}_\omega(\Gamma, X_1) = Z(\Gamma, X_3, \Gamma)\mathbf{R}_\omega(\Gamma, X_2)Z(\Gamma, X_3, \Gamma)^{-1}, \quad (3.101)$$

i.e. the generators  $\mathbf{R}_\omega(\Gamma, X_1), \mathbf{R}_\omega(\Gamma, X_2)$  are equivalent in the sense of the relation (3.94).

### 3.4 Scalar product and unitary representations

**Definition 16** *The scalar product of the two functions satisfying Eq. (3.2) or (3.36) is defined:*

$$(\Phi(p), \Psi(q)) = \begin{cases} 0 & \text{for } p \neq q \\ \Phi^\dagger(p)W(p)\Psi(q) & \text{for } p = q \end{cases}, \quad (3.102)$$

where the metric  $W$  is the matrix, which satisfies

$$W^\dagger(p) = W(p), \quad (3.103)$$

$$R_\omega^\dagger(p)W(p) - W(p)R_\omega(p) + i\frac{dW}{d\omega} = 0, \quad (3.104)$$

$$K_\xi^\dagger(p)W(p) - W(p)K_\xi(p) = 0. \quad (3.105)$$

The conditions (3.104), (3.105) in the above definition imply, that the scalar product is invariant under corresponding infinitesimal transformations. For example for the Lorentz group the transformed scalar product reads

$$\begin{aligned} & \Phi'^\dagger(p')W(p')\Psi'(p') \\ &= \Phi^\dagger(p) \left( I - id\omega R_\omega^\dagger(p) \right) \left( W(p) + d\omega \frac{dW}{d\omega} \right) \left( I + id\omega R_\omega(p) \right) \Psi(p) \end{aligned} \quad (3.106)$$

and with the use of the condition (3.104) one gets

$$\Phi'^\dagger(p')W(p')\Psi'(p') = \Phi^\dagger(p)W(p)\Psi(p). \quad (3.107)$$

According to a general definition, the transformations conserving the scalar product are unitary. In this way the Eqs. (3.104), (3.105) represent the condition of unitarity for representation of the corresponding group generated by  $\mathbf{R}_\omega$  and  $K_\xi$ .

How to choose these generators and the matrix  $W(p)$  to solve Eqs. (3.104), (3.105)? Similarly as in the case of solution of Eqs. (3.26) and (3.28) - (3.30) it is convenient to begin with the representation related to the canonical equation (3.36). Apparently the generators of the space rotations can be chosen Hermitian

$$R_{\varphi_j}^\dagger(\Gamma_0) = R_{\varphi_j}(\Gamma_0). \quad (3.108)$$

Then also for the Lorentz transformations one gets

$$R_{\psi_j}^\dagger(\Gamma_0) = R_{\psi_j}(\Gamma_0), \quad (3.109)$$

provided that the constant  $\kappa$  in Eq. (3.58) is real and  $|\kappa| \leq m$ . Also the generators  $K_\xi$  can be chosen in the same way:

$$K_\xi^\dagger(\Gamma_0) = K_\xi(\Gamma_0). \quad (3.110)$$

It follows, that instead of the conditions (3.104), (3.105) one can write

$$[\mathbf{R}_\omega(\Gamma_0), W(\Gamma_0)] = 0, \quad [K_\xi(\Gamma_0), W(\Gamma_0)] = 0. \quad (3.111)$$

The structure of the generators  $\mathbf{R}_\omega(\Gamma_0), K_\xi(\Gamma_0)$  given by Eqs. (3.66), (3.68) suggests, that the metric  $W$  satisfying the condition (3.111) can have a similar structure, but in which the corresponding blocks on the diagonal are occupied by unit matrices multiplied by some constants. Nevertheless, let us note, that the condition (3.111) in general can be satisfied also for some other structures of  $W(\Gamma_0)$ .

From  $W(\Gamma_0)$  we can obtain matrix  $W(\Gamma)$ — the metric for the scalar product of the two solutions of Eq. (3.2). One can check, that after the transformations

$$\mathbf{R}_\omega(\Gamma_0) \rightarrow \mathbf{R}_\omega(\Gamma, X) = Z(\Gamma_0, X, \Gamma)^{-1} \mathbf{R}_\omega(\Gamma_0) Z(\Gamma_0, X, \Gamma), \quad (3.112)$$

$$K_\xi(\Gamma_0) \rightarrow K_\xi(\Gamma, X) = Z(\Gamma_0, X, \Gamma)^{-1} K_\xi(\Gamma_0) Z(\Gamma_0, X, \Gamma) \quad (3.113)$$

and simultaneously

$$W(\Gamma_0) \rightarrow W(\Gamma, X) = Z(\Gamma_0, X, \Gamma)^\dagger W(\Gamma_0) Z(\Gamma_0, X, \Gamma) \quad (3.114)$$

the unitarity in the sense of conditions (3.104), (3.105) is conserved - in spite of the fact that equalities (3.108) - (3.110) may not hold for  $R_\omega(\Gamma, X), K_\xi(\Gamma, X)$ .

### 3.5 Space-time representation and Green functions

If we take the solutions of the wave equation (3.2) or (3.36) in the form of the functions  $\Psi(p)$ , for which there exists the Fourier picture

$$\tilde{\Psi}(x) = \frac{1}{(2\pi)^4} \int \Psi(p) \delta(p^2 - m^2) \exp(-ipx) d^4p, \quad (3.115)$$

then the space of the functions  $\tilde{\Psi}(x)$  constitutes the  $x$ - representation of wave functions. Correspondingly, for all the operators  $D(p)$  given in the  $p$ - representation and discussed in the previous paragraphs, one can formally define their  $x$ - representation:

$$\tilde{D}(z) = \frac{1}{(2\pi)^4} \int D(p) \exp(-ipz) d^4p, \quad (3.116)$$

which means

$$\tilde{D}\tilde{\Psi}(x) = \frac{1}{(2\pi)^4} \int D(p) \Psi(p) \delta(p^2 - m^2) \exp(-ipx) d^4p \quad (3.117)$$

$$= \frac{1}{(2\pi)^4} \int D(p) \exp(-ipx) \tilde{\Psi}(y) \exp(ipy) d^4 y d^4 p = \frac{1}{(2\pi)^4} \int \tilde{D}(x-y) \tilde{\Psi}(y) d^4 y.$$

In this way we get for our operators:

$$\Gamma_0(p) \rightarrow \tilde{\Gamma}_0(z) = Q_0 \frac{1}{(2\pi)^4} \int (p^2)^{1/n} \exp(-ipz) d^4 p; \quad pz \equiv p_0 z_0 - \vec{p} \vec{z}, \quad (3.118)$$

$$\Gamma(p) \rightarrow \tilde{\Gamma}(z) = \sum_{\lambda=0}^3 Q_\lambda \frac{1}{(2\pi)^4} \int p_\lambda^{2/n} \exp(-ipz) d^4 p, \quad (3.119)$$

$$\mathbf{R}_{\varphi_j}(\Gamma_0) \rightarrow \tilde{\mathbf{R}}_{\varphi_j}(\Gamma_0) = R_{\varphi_j}(\Gamma_0) + i \frac{d}{d\tilde{\varphi}_j}; \quad \frac{d}{d\tilde{\varphi}_j} = -\epsilon_{jkl} x_k \frac{\partial}{\partial x_l}, \quad (3.120)$$

$$\begin{aligned} \mathbf{R}_{\psi_j}(\Gamma_0) &\rightarrow \tilde{\mathbf{R}}_{\psi_j}(z) \\ &= \frac{1}{(2\pi)^4} \int \frac{\kappa R_{\varphi_j}(\Gamma_0) + \epsilon_{jkl} p_k R_{\varphi_l}(\Gamma_0)}{p_0 + \sqrt{p^2 - \kappa^2}} \exp(-ipz) d^4 p + i \frac{d}{d\tilde{\psi}_j}; \\ \frac{d}{d\tilde{\psi}_j} &= -x_0 \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_0} \end{aligned} \quad (3.121)$$

and in the same way

$$\begin{aligned} \mathbf{R}_\omega(\Gamma) &\rightarrow \tilde{\mathbf{R}}_\omega(z) \\ &= \frac{1}{(2\pi)^4} \int Z(\Gamma_0, X, \Gamma)^{-1} \mathbf{R}_\omega(\Gamma_0) Z(\Gamma_0, X, \Gamma) \exp(-ipz) d^4 p. \end{aligned} \quad (3.122)$$

Apparently, the similar relations are valid also for the remaining operators  $K_\xi, W, Z, Z^{-1}$  and the finite transformations  $\Lambda$  in the  $x$ -representation. Concerning the translations, the usual correspondence is valid:  $p_\alpha \rightarrow i\partial_\alpha$ .

Further, the solutions of the inhomogeneous version of the Eqs. (3.36), (3.2)

$$(\Gamma_0(p) - \mu) G_0(p) = I, \quad (\Gamma(p) - \mu) G(p) = I \quad (3.123)$$

can be obtained with the use of formula (2.27):

$$G_0(p) = \frac{(\Gamma_0 - \alpha\mu)(\Gamma_0 - \alpha^2\mu) \dots (\Gamma_0 - \alpha^{n-1}\mu)}{p^2 - m^2}, \quad (3.124)$$

$$G(p) = \frac{(\Gamma - \alpha\mu)(\Gamma - \alpha^2\mu) \dots (\Gamma - \alpha^{n-1}\mu)}{p^2 - m^2} \quad (3.125)$$

and Eq. (3.37) implies

$$G(p) = Z(\Gamma_0, X, \Gamma)^{-1} G_0(p) Z(\Gamma_0, X, \Gamma). \quad (3.126)$$

Apparently, the functions

$$\tilde{G}_0(x) = \frac{1}{(2\pi)^4} \int G_0(p) \exp(-ipx) d^4 p, \quad (3.127)$$

$$\tilde{G}(x) = \frac{1}{(2\pi)^4} \int G(p) \exp(-ipx) d^4 p \quad (3.128)$$



formally satisfy Eqs. (3.123) in the  $x$ -representation

$$\int \tilde{\Gamma}_0(x-y)\tilde{G}_0(y)d^4y - \mu\tilde{G}_0(x) = \delta^4(x), \quad (3.129)$$

$$\int \tilde{\Gamma}(x-y)\tilde{G}(y)d^4y - \mu\tilde{G}(x) = \left[ \sum_{\lambda=0}^3 Q_\lambda (i\partial_\lambda)^{2/n} - \mu \right] \tilde{G}(x) = \delta^4(x). \quad (3.130)$$

The last equation contains the fractional derivatives defined in [14]. Obviously, the functions  $\tilde{G}_0, \tilde{G}$  can be identified with the Green functions related to  $x$ -representation of Eqs. (3.36), (3.2).

With the exception of the operators  $\tilde{\mathbf{R}}_{\varphi_j}(\Gamma_0), \tilde{W}(\Gamma_0)$  and  $i\partial_\alpha$  all the remaining operators considered above are pseudo-differential ones, which are in general non-local. The ways, how to deal with such operators, are suggested in [3],[5],[14]. A more general treatise of the pseudo-differential operators can be found e.g. in [17]-[19]. In our case it is significant, that the corresponding integrals will depend on the choice of passing about the singularities and the choice of the cuts of the power functions  $p^{2j/n}$ . This choice should reflect contained physics, however corresponding discussion would exceed the scope of this paper.

## 4 Summary and concluding remarks

In this paper we have first studied the algebra of the matrices  $Q_{pr} = S^p T^r$  generated by the pair of matrices  $S, T$  with the structure given by the *Definition 1*. We have proved, that for a given  $n \geq 2$  one can in the corresponding set  $\{Q_{pr}\}$  always find a triad, for which Eq. (2.48) is satisfied, hereat the Pauli matrices represent its particular case  $n = 2$ . On this base we have got the rule, how to construct the generalized Dirac matrices [Eqs. (2.51), (2.52)]. Further we have shown, that there is a simple relation [Eqs. (2.53), (2.54)] between the set  $\{Q_{pr}\}$  and the algebra of generators of the fundamental representation of the  $SU(n)$  group.

In the further part, using the generalized Dirac matrices we have demonstrated, how one can from the roots of the D'Alembertian operator generate a class of relativistic equations containing the Dirac equation as a particular case. In this context we have shown, how the corresponding representations of the Lorentz group, which guarantee the covariance of these equations, can be found. At the same time we have found additional symmetry transformations on these equations. Further, we have suggested how one can define the scalar product in the space of the corresponding wave functions and make the unitary representation of the whole group of symmetry. Finally, we have suggested, how to construct the corresponding Green functions. In the  $x$ -representation the equations themselves and all the mentioned transformations are in general non-local, being represented by the fractional derivatives and pseudo-differential operators in the four space-time dimensions.

In line with the choice of the representation of the rotation group used for the construction of the unitary representation of the Lorentz group according to which the equations transform, one can ascribe to the related wave functions the corresponding spin - and further quantum numbers connected with the additional symmetries. Nevertheless it is obvious, that before more serious physical speculations, one should answer some more questions requiring a further study.

Perhaps the first could be the problem how to introduce the interaction. The usual direct replacement

$$\partial_\lambda \rightarrow \partial_\lambda + igA_\lambda(x)$$

would lead to the difficulties, first of all with the rigorous definition of the terms like

$$(\partial_\lambda + igA_\lambda(x))^{2/n}.$$

On the end one should answer the more general question: Is it possible on the base of the discussed wave equations to build up a meaningful quantum field theory?

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