A model for time-dependent cosmological constant and its consistency with the present Friedmann universe

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We use a model where the cosmological term can be related to the chiral gauge anomaly of a possible quantum scenario of the initial evolution of the universe. We show that this term is compatible with the Friedmann behavior of the present universe.

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I. INTRODUCTION

The idea of a varying cosmological constant is not new but has been systematically examined in the last two decades [1]. In particular, with the advent of the inflation mechanism and with the difficulties related to the graceful exit problem [2], various authors were forced to consider Λ as ground state of a scalar theory [3]. However, we may say that the first concrete mechanism for a varying Λ was proposed by Dolgov [4] based on a nominimally coupled scalar field. A sequence of other papers come afterwards, but all of them also based on scalar fields [5].

In a previous work [6] we have presented a different model for a time dependent cosmological "constant" based on gauge fields where its origin was related to a possible quantum scenario of the initial evolution of the universe. This was achieved by showing that the chiral gauge anomaly could be conveniently adapted in order to generate a cosmological *constant*. In this way, the presence of this term today would be a reminiscence of that initial quantum behavior of the universe.

In the present paper, we are going to consider a particular scenario where the geometry is initially Bianchi-like (spatially homogeneous but anisotropic) [7]. We show that the evolution obtained from the Einstein equations in the presence of the cosmological term (as well as the Maxwell one) leads to an asymptotic solution that is compatible with the Friedmann universe [8, 9].

Our paper is organized as follow. In Sec. II we review the general features of the model. The application is done in Sec. III. In Sec. IV we deal with the solution of the particular Einstein equations. We left Sec. V for some concluding remarks and introduce an Appendix to show some details of the calculations.

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II. REVIEW OF THE MODEL

Let us consider an action with the following general form [6]

$$S_\Lambda = \int d^4x\,\sqrt{-g}\;Y(\mathcal{G}) \tag{2.1}$$

where g is the determinant of the metric tensor and Y is some function of an invariant quantity $\mathcal{G}$ which is constructed in terms of a gauge field strength $G^{\rm a}_{\mu\nu}$ and its dual ${}^*G^{\rm a}_{\mu\nu} = \frac{1}{2}\,\eta_{\mu\nu\rho\lambda}\,G^{{\rm a}\rho\lambda}$ as

$$\mathcal{G} = {}^*\,G^{a\mu\nu}G^a_{\mu\nu} \tag{2.2}$$

We are using the following definition for $\eta_{\mu\nu\rho\lambda}$

$$\eta_{\mu\nu\rho\lambda} = \sqrt{-g}\,\epsilon_{\mu\nu\rho\lambda} \tag{2.3}$$

Consequently,

$$\eta^{\mu\nu\rho\lambda} = -\frac{1}{\sqrt{-g}}\,\epsilon^{\mu\nu\rho\lambda} \tag{2.4}$$

where $\epsilon_{\mu\nu\rho\lambda}$ and $\epsilon^{\mu\nu\rho\lambda}$ are the usual Levi-Civita tensor densities ($\epsilon_{0123}=1$).

The variation of S_Λ with respect the metric tensor leads to

$$\begin{aligned}\frac{\delta S_\Lambda}{\delta g^{\mu\nu}} &= \int d^4x\left(\frac{\delta\sqrt{-g}}{\delta g^{\mu\nu}}\,Y + \sqrt{-g}\,\frac{dY}{d\mathcal{G}}\,\frac{\delta\mathcal{G}}{\delta g^{\mu\nu}}\right)\\ &= \int d^4x\Big(-\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\,Y + \sqrt{-g}\,\frac{dY}{d\mathcal{G}}\,\frac{1}{2}\mathcal{G}\,g_{\mu\nu}\Big)\\ &= \frac{1}{2}\int d^4x\,\sqrt{-g}\,\Big(\frac{dY}{d\mathcal{G}}\,\mathcal{G} - Y\Big)\,g_{\mu\nu}\end{aligned} \tag{2.5}$$

We then observe that the action S_Λ contributes to the energy-momentum tensor with a term that is proportional to the metric tensor. This is interpreted in the

Einstein General Relativity theory as a spacetime dependent cosmological *constant*,

$$\Lambda = \frac{dY}{d\mathcal{G}}\,\mathcal{G} - Y \tag{2.6}$$

For a question of simplicity, we shall consider the field strength of the Maxwell electromagnetic theory, where there is a natural realization of this model coming from the chiral gauge anomaly. For instance, from the path integral formalism of quantum matter and gauge fields in a classical curved background one obtains to the following effective action[10, 11]

$$\begin{aligned} S_{\text{eff}} &= \frac{\theta}{4}\int d^4x\,\sqrt{-g}\,\alpha(x)\,\eta^{\mu\nu\rho\lambda}F_{\mu\nu}F_{\rho\lambda} \\ &= \frac{\theta}{4}\int d^4x\,\sqrt{-g}\,\alpha(x)\,\mathcal{G} \end{aligned} \tag{2.7}$$

where θ is a constant and $\alpha(x)$ is the gauge parameter related to the chiral gauge transformation.

The manner in which the effective action above is presented it is not appopriated to be directly related to S_Λ. This is so because since it is linear in $\mathcal{G}$ and cannot generate a cosmological term Λ, as can be verified in Eq. (2.6). However, this problem can be circumvented by conveniently taking the generic function $\alpha(x)$ of the action (2.7) as $\mathcal{G}^p$, where p is, at first, any rational number [1]. In this way, the cosmological action S_Λ turns to be

$$S_\Lambda = \frac{\theta}{4}\int d^4x\,\sqrt{-g}\,\mathcal{G}^{\,p+1} \tag{2.8}$$

We observe that for $p=-1$ we have an actual cosmological constant in the Einstein equation.

The next natural step is to consider this idea in some cosmological model in order to see the way that the cosmological term, coming from Eq. (2.8), modifies the dynamics of the Einstein equation.

III. APPLICATION OF THE MODEL

Let us start from the general action

$$\begin{aligned} S = \int d^4x\,\sqrt{-g}\,\Big[&\frac{1}{2\kappa}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ \frac{1}{4}\theta(F_{\mu\nu}F^{*\mu\nu})^{p+1} \\ &+ i\bar\psi\gamma^\mu(\nabla_\mu - ieA_\mu)\psi\Big] \end{aligned} \tag{3.1}$$

[1] The way of circumventing this problem here is slightly different of the original paper [6]. There, we have redefined the gauge field in order to incorporate a nonlinearity of $\mathcal{G}$.

The equations of motion $\delta S/\delta g^{\mu\nu}=0$, $\delta S/\delta A^\mu = 0$, and $\delta S/\delta\psi = 0$ lead respectively to

$$\begin{aligned} G_{\mu\nu} = -F_{\mu\alpha}F^{\alpha}{}_{\nu} - \frac{1}{4}\,g_{\mu\nu}\,F_{\alpha\beta}F^{\alpha\beta} \\ - \frac{p\theta}{2}g_{\mu\nu}(F_{\alpha\beta}F^{*\alpha\beta})^{p+1} + T^{\psi}_{\mu\nu} \end{aligned} \tag{3.2}$$

$$\begin{aligned} F^{\mu\nu}{}_{;\nu} - (p+1)\theta\big[(F_{\alpha\beta}F^{*\alpha\beta})^p\big]_{,\nu}F^{*\mu\nu} \\ = e\,\bar\psi\gamma^\mu\psi \end{aligned} \tag{3.3}$$

$$\gamma^\mu(\nabla_\mu - ieA_\mu)\,\psi = 0 \tag{3.4}$$

where $G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor and the constant κ of (3.1) was taken 1. In the obtainment of Eq. (3.3) it was used the (Bianchi) identity

$$F^{*\mu\nu}{}_{;\nu} = 0 \tag{3.5}$$

We shall consider that $F^{\mu\nu}=F^{\mu\nu}(t)$. So, taking $\nu=0$ in Eq. (3.3), as well as $\mu=0$, we get

$$\begin{aligned} &\bar\psi\gamma^0\psi = \psi^\dagger\psi = 0 \\ \Rightarrow\ &\psi = 0 \end{aligned} \tag{3.6}$$

From Eq. (3.5) we also have that only F^{*i0} can be obtained. Using the Bianchi-like metric

$$ds^2 = dt^2 - a^2(t)dx^2 - b^2(t)(dy^2+dz^2) \tag{3.7}$$

and just taking that $F^{*10}\neq 0$, we have

$$\begin{aligned} &F^{*01}{}_{;0} = 0 \\ \Rightarrow\ &\sqrt{-g}\,F^{*01}{}_{;0} = 0 \\ \Rightarrow\ &\big(\sqrt{-g}\,F^{*01}\big)_{,0} = 0 \\ \Rightarrow\ &\sqrt{-g}\,F^{*01} = constant \\ \Rightarrow\ &F^{*01} = \frac{B_0}{ab^2} \end{aligned} \tag{3.8}$$

where, in the last step, the *constant* was identified as B_0 and we have used the metric given by (3.7). From this result and using Eq. (2.4) we directly infer that

$$F_{23} = -B_0 \tag{3.9}$$

However, the solution for F^{10} depends on p. In fact, taking $\nu=0$ and $\mu=1$ in Eq. (3.3) and using the results given by (3.6) and (3.8), we obtain

$$ab^2\,F^{10} + (p+1)\theta\,B_0^{p+1}\left(\frac{2aF^{10}}{b^2}\right)^p = E_0 \tag{3.10}$$

where E_0 was chosen to identify the constant that appears in the solution of the corresponding differential equation.

Our goal from now on is to look for if there exists some value of p that renders a consistent solution for the Einstein equations (3.2) having in mind the Friedmann behavior of the present universe. This will be the subject of next section. We conclude the present section by emphasizing the importance of the Maxwell term, $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, into the initial action (3.1). Without it, the only possible solution of (3.3) would just be an actually constant. Indeed, taking $\psi = 0$ and $\nu = 0$ in Eq. (3.3), we get

$$F^{i0}{}_{;0} - (p+1)\theta\big[(F_{\alpha\beta}F^{*\alpha\beta})^p\big]_{,0}\, F^{*i0} = 0 \tag{3.11}$$

where the first term of the expression above comes from the Maxwell term. We then observe that without it we would obtain that $(F_{\alpha\beta}F^{*\alpha\beta})^p$ should be a constant for $p \neq -1$ [for $p = -1$, the cosmological term is also a constant according to(2.8)].

IV. PARTICULAR SOLUTION OF THE EINSTEIN EQUATION

We have seen that to avoid an actual cosmological constant and the trivial case without any cosmological term, p cannot have the values zero and minus one, respectively (and the Maxwell term has to be present in the Lagrangian). In order to have a general view of the physical behavior of the Einstein solution with the value of p, let us initially choose $p = 1$. This corresponds to one of the simplest solution of Eq. (3.10), that is

$$F^{10} = \frac{E_0 b^2}{a(b^4 + 4\theta\, B_0^2)} \tag{4.1}$$

Combining this result with the ones given by (3.6), (3.8), and (3.9), we obtain that the Einstein equation (3.2) leads to

$$\begin{aligned}
2\,\frac{\dot a \dot b}{ab} + \left(\frac{\dot b}{b}\right)^2 &= -\frac{1}{2}\left[\frac{B_0^2}{b^4} + \frac{E_0^2 b^4}{(b^4+2\theta B_0)^2}\right] \\
&\quad + \frac{2\theta E_0^2 B_0^2}{(b^4+2\theta B_0)^2}
\end{aligned} \tag{4.2}$$

$$\begin{aligned}
2\,\frac{\ddot b}{b} + \left(\frac{\dot b}{b}\right)^2 &= -\frac{1}{2}\left[\frac{B_0^2}{b^4} + \frac{E_0^2 b^4}{(b^4+2\theta B_0)^2}\right] \\
&\quad - \frac{2\theta E_0^2 B_0^2}{(b^4+2\theta B_0)^2}
\end{aligned} \tag{4.3}$$

$$\begin{aligned}
\frac{\ddot b}{b} + \frac{\ddot a}{a} + \frac{\dot a \dot b}{ab} &= \frac{1}{2}\left[\frac{B_0^2}{b^4} + \frac{E_0^2 b^4}{(b^4+2\theta B_0)^2}\right] \\
&\quad - \frac{2\theta E_0^2 B_0^2}{(b^4+2\theta B_0)^2}
\end{aligned} \tag{4.4}$$

We observe that the term related to the cosmological *constant* has a behavior of b^{-8} when b goes to infinity (the behavior of the radiation term is with b^{-4}). Consequently, the cosmological term should disappear before the radiation era, what is not consistent with the Friedmann universe, where $a = b$ (notice that it is the Maxwell term which does not permit to have $a = b$ in the last two equations).

From the above results, it is not difficult to conclude that possible values of p that should be compatible with this asymptotic behavior must stay between zero and minus one. However, for these values of p, the solution of Eq. (3.10) is not so direct as in the previous case. For example, taking $p = -1/2$, where Eq. (3.10) becomes a cubic equation with one real root that is given by (see Appendix A)

$$F^{10} = \frac{B_0}{2a}\left(\frac{\theta}{2bB_0}\right)^{\frac{2}{3}} \left\{\left[1-\left(1-\frac{32E_0^3}{27\theta^2 B_0 b^4}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}} + \left[1+\left(1-\frac{32E_0^3}{27\theta^2 B_0 b^4}\right)^{\frac{1}{2}}\right]^{\frac{1}{3}}\right\}^2 \tag{4.5}$$

For $\theta \neq 0$, the terms $32E_0^3/27\theta^2 b^4 B_0$ is $\ll 1$ as $b \to \infty$. Making appropriate expansions in the above relation, we obtain t

F^{10},

With th

$$\begin{aligned}
2\,\frac{\dot a \dot b}{ab} + \left(\frac{\dot b}{b}\right)^2 &= -\frac{1}{2}\left[\frac{B_0^2}{b^4} + \frac{\theta}{4b}\left(\frac{\theta B_0^2}{b}\right)^{\frac{1}{3}}\right] \\
&\quad + \frac{\theta}{4\sqrt{2}\,b}\left(\frac{\theta B_0^2}{b}\right)^{\frac{1}{3}}
\end{aligned} \tag{4.7}$$

$$\begin{aligned}
2\,\frac{\ddot b}{b} + \left(\frac{\dot b}{b}\right)^2 &= -\frac{1}{2}\left[\frac{B_0^2}{b^4} + \frac{\theta}{4b}\left(\frac{\theta B_0^2}{b}\right)^{\frac{1}{3}}\right] \\
&\quad - \frac{\theta}{4\sqrt{2}\,b}\left(\frac{\theta B_0^2}{b}\right)^{\frac{1}{3}}
\end{aligned} \tag{4.8}$$

$$\begin{aligned}
\frac{\ddot b}{b} + \frac{\ddot a}{a} + \frac{\dot a \dot b}{ab} &= \frac{1}{2}\left[\frac{B_0^2}{b^4} + \frac{\theta}{4b}\left(\frac{\theta B_0^2}{b}\right)^{\frac{1}{3}}\right] \\
&\quad - \frac{\theta}{4\sqrt{2}\,b}\left(\frac{\theta B_0^2}{b}\right)^{\frac{1}{3}}
\end{aligned} \tag{4.9}$$

Now, the cosmological term behaves as $b^{-\frac{4}{3}}$, that decay slower than the previous b^{-4} of the radiation term. However, there is also a term with the behavior of $b^{-\frac{4}{3}}$ in the Maxwell counterpart and this last term avoids the obtainment of a Friedmann behavior we are looking for.

From this analysis, we see that the solution of (3.10) leads to a behavior for the cosmological term that is lower than b^{-4} for $-1 < p < 0$. However, there is also a term with this behavior in the Maxwell counterpart. What may happen is that, depending on the value of p, one term may dominate relatively to the other. To see if this actually occurs we need to know the solution of (3.10) for a general p (between zero and minus one). Of course, this is not an easy task. However, we observe that the asymptotic solution for F^{10} given by (4.6) could have been directly inferred from (3.10) by discarding the constant term E_0. It is then easily seen that any asymptotic solution for any p between zero and minus one can be obtained in the same way. The result is

$$F^{10} = \frac{1}{2a}\left[-2(1+p)\theta\left(\frac{B_0}{b^2}\right)^{1+p}\right]^{\frac{1}{1-p}} \tag{4.10}$$

which is in agreement with (4.6) if one takes $p = -1/2$. We should not worry about the minus sign inside the $\frac{1}{1-p}$-root of equation above because F^{10} always appears squared in the calculations of significant quantities. Now, the obtainment of the Einstein equations is just a matter of algebraic work. The result is (we conveniently put the two terms with the same behavior for higher b together)

$$2\,\frac{\dot a\dot b}{ab} + \left(\frac{\dot b}{b}\right)^2 = -\frac{B_0^2}{2b^4} - \left(\frac{p2^{-p}}{1+p}+1\right)\left[2^{\frac{3p-1}{2}}(1+p)\theta\left(\frac{B_0}{b^2}\right)^{1+p}\right]^{\frac{2}{1-p}} \tag{4.11}$$

$$2\,\frac{\ddot b}{b} + \left(\frac{\dot b}{b}\right)^2 = -\frac{B_0^2}{2b^4} - \left(\frac{p2^{-p}}{1+p}+1\right)\left[2^{\frac{3p-1}{2}}(1+p)\theta\left(\frac{B_0}{b^2}\right)^{1+p}\right]^{\frac{2}{1-p}} \tag{4.12}$$

$$\frac{\ddot b}{b} + \frac{\ddot a}{a} + \frac{\dot a\dot b}{ab} = \frac{B_0^2}{2b^4} - \left(\frac{p2^{-p}}{1+p}-1\right)\left[2^{\frac{3p-1}{2}}(1+p)\theta\left(\frac{B_0}{b^2}\right)^{1+p}\right]^{\frac{2}{1-p}} \tag{4.13}$$

(4.14)

where in the terms that appear $(\frac{p2^{-p}}{1+p}+1)$ or $(\frac{p2^{-p}}{1+p}-1)$, the first part comes from the cosmological term and the other one from the Maxwell counterpart. We then observe that as p is closed to minus one the cosmological term is more and more dominant and, consequently, the solution tends more and more to the Friedmann scenario. In the limit case of $p = -1$, the Maxwell term disappears, and just remains an actual cosmological constant as it had already been pointed out in the beginning.

V. CONCLUSION

In this paper we have analyzed further the recent proposal[6] of a cosmological scenario in which the cosmological "constant" is spacetime dependent and whose origin is related to a primordial era, supposed dominated by quantum effects.

In order to see if this model is actually compatible with the observable universe, where the anisotropy rate is considerable low [like Friedmann-Robertson-Walker (FRD)], we leave a parameter free (p) in the theory. We conclude that it can vary from minus one to zero, where these limits mean an actual cosmological constant and no cosmological term, respectively. We have show that as p is closed to minus one as the solution is compatible with the Friedmann scenario.

A next natural step in this research line is to look for the solution of the Einstein equations (4.11) - (4.13). We are presently work in this problem and possible results shall be reported elsewhere [12].

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Appendix A: Solution of (3.10) for p=-1/2

The Eq. (3.10) for $p = -1/2$ reads

$$ab^2\,F^{10} + \frac{\theta b}{2}\sqrt{\frac{B_0}{2aF^{10}}} = E_0 \tag{A.1}$$

This expression can be written in a cubic equation whose general form is

$$u^3 + a_1 u^2 + a_2 u + a_3 = 0 \tag{A.2}$$

where

$$\begin{aligned} u &= \sqrt{\frac{2aF^{10}}{B_0}} \\ a_1 &= 0 \\ a_2 &= -\frac{2E_0}{B_0 b^2} \\ a_3 &= \frac{\theta}{B_0 b} \end{aligned} \tag{A.3}$$

From the discriminant relation

$$D = Q^3 + R^2 \tag{A.4}$$

where

$$Q = \frac{3a_2 - a_1^2}{9} = -\frac{2E_0}{3B_0 b^2} \tag{A.5}$$

$$R = \frac{9a_1 a_2 - 27a_3 - 2a_1^3}{54} = -\frac{\theta}{2B_0 b} \tag{A.6}$$

we obtain

$$D = -\frac{8E_0^3}{27B_0^3 b^6} + \frac{\theta^2}{4B_0^2 b^2} \tag{A.7}$$

We observe that in the region for higher b, D is positive (for $\theta \neq 0$). Consequently, just one root is real for this region and it is given by

$$u = S + T - \frac{a_1}{3} \tag{A.8}$$

where

$$\begin{aligned} S &= \sqrt[3]{R + \sqrt{D}} \\ T &= \sqrt[3]{R - \sqrt{D}} \end{aligned} \tag{A.9}$$

The combination of (A.6) - (A.9) leads to the solution given by (4.5).

[1] For a detailed study concerning the cosmological constant as well as the possibility of its dependence on time see J.M. Overduin and F.I. Cooperstock, Phys. Rev. **D58** (1998) 043506 and references therein.

[2] N. Kaloper, R. Madden, and K.A. Olive, Nucl. Phys. **B452** (1995) 677; Phys. Lett. **B371** (1996) 34; R. Brustein and R. Madden, Phys. Lett. **B410** (1997) 110

[3] See for example G.F.R. Ellis, D.C. Roberts, D. Solomons, and P.K.S. Dunsby, Phys. Rev. **D62** (2000) 084004 and references therein.

[4] See A.D. Dolgov, in *The very early universe*, edited by G.W. Gibbons, S.W. Hawking, and S.T.C. Siklos (Cambridge University Press, Cambridge, England, 1983) and references therein.

[5] We mention, among others, M. Özer and M.O. Taha, Phys. Lett. **B171** (1986) 363; O. Bertolami, Nuovo Cimento **B93** (1986) 36; M.O. Calvão, H.P. Oliveira, D. Pavón, and J.M. Salim, Phys. Rev. **D45** (a992) 3869.

[6] M. Novello, J. Barcelos-Neto and J.M. Salim, Class. Quantum Grav. **18** (2001) 1261.

[7] See, for example, V.A. Belinsky, E.M. Lifshitz, and I.M. Khalatnikov, Adv. Phys. **19** (1970) 525, and references therein.

[8] S. Weinberg, Rev. Mod. Phys. **61** (1989) 1.

[9] For other isotropization mechanisms, see M. Novello and S.L.S. Duque, Physica **A168** (1990) 1073, and references therein.

[10] N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge (1982).

[11] See reference [6] for details.

[12] M. Novello, J. Barcelos-Neto and J.M. Salim, work in progress.