On the unitarity problem in space/time noncommutative theories

D. Bahns*, S. Doplicher†, K. Fredenhagen‡, G. Piacitelli§

*‡II. Institut für Theoretische Physik, Universität Hamburg, Luruper Chaussee 149, D - 22761 Hamburg, Germany – bahns@mail.desy.de, fredenha@x4u2.desy.de

†Dipartimento di Matematica , Università di Roma "La Sapienza", P.le Aldo Moro 2, 00185 Roma, Italy – dopliche@mat.uniroma1.it

§Dipartimento di Matematica Pura ed Applicata, Università di Padova, Via G. Belzoni 7, 35131 Padova, Italy – piacitel@math.unipd.it

Abstract

It is shown that the violation of unitarity observed in space/time noncommutative field theories is due to an improper definition of quantum field theory on noncommutative spacetime.

Quantum field theory on noncommutative spacetimes is an interesting modification of the standard formalism which takes into account possible deviations from the smoothness of spacetime at small distances. A very convincing reason for such a modification is that uncertainty relations for spacetime coordinates are suggested by an analysis of the principles of quantum theory together with those of general relativity. Such uncertainty relations can be implemented in the framework of noncommutative spaces [1]. A further motivation was established in the context of string theory in the analysis of D-branes in background magnetic fields [2].

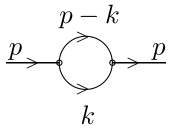
Quantum field theory on the standard noncommutative spacetime is equivalent to a nonlocal theory on a commutative spacetime [1,§ 6]. Owing to the nonlocality especially in time, certain structural properties of ordinary quantum field theories are lost. In particular, the various equivalent formulations of quantum field theory on Minkowski space cease to be equivalent on noncommutative spaces.

Most of the current literature on the subject is based on a set of modified Feynman rules first formulated in [3]. The basic idea in this approach is that the usual

*Supported by the Graduiertenkolleg "Zukünftige Entwicklungen in der Teilchenphysik".
†Research supported by MIUR and GNAMPA - INdAM.

Feynman rules of quantum field theory on Minkowski space may be used and that the transition to a noncommutative spacetime is achieved by a mere replacement of all pointwise products by Moyal star products and subsequent symmetrization. In momentum space this rule entails the appearance of oscillating factors at the vertices, which depend on the order of the in- and out-going momenta.

The formalism unfortunately suffers from a violation of unitarity, as was first remarked in [4] (cf. also [5, 6]). As proposed there, let us consider as an illustrative example the fish graph in a model with ϕ^3 self-interaction in the setting of the modified Feynman rules, where



$$\propto \quad \int d^4k \frac{1 + \cos p\,\sigma k}{((p-k)^2 - m^2 + i\varepsilon)(k^2 - m^2 + i\varepsilon)}$$

with $i\sigma^{\mu\nu} = [x^\mu \overset{*}{,}_\sigma x^\nu]$ defining the Moyal star product $f *_\sigma g(x) = e^{\frac{i}{2}\overleftarrow{\partial}_\mu \sigma^{\mu\nu} \overrightarrow{\partial}_\nu} f(x)g(x)$. The above expression for the fish graph is the Fourier transform of a pointwise product and a star product of two Feynman propagators,

$$\Delta_F(x)\Delta_F(x) + \Delta_F(x) * \Delta_F(x)\,.$$

Here, as well as in the sequel, the $*$ denotes the Moyal star product defined by 2σ, whereas the symbol $*_\sigma$ is reserved for the star product defined by σ.

The first term in the above expresssion, also referred to as the planar contribution, is the same as in the commutative case and thus satisfies unitarity,

$$\begin{aligned} \Delta_F^2 + \overline{\Delta_F^2} &= (\theta\Delta_+ + (1-\theta)\Delta_-)^2 + (\theta\Delta_- + (1-\theta)\Delta_+)^2 \\ &= \theta\Delta_+^2 + (1-\theta)\Delta_-^2 + \theta\Delta_-^2 + (1-\theta)\Delta_+^2 \\ &= \Delta_-^2 + \Delta_+^2\,, \end{aligned} \tag{1}$$

where $\theta\Delta_\pm$ abbreviates the product $\theta(x_0)\Delta_\pm(x)$ of Heaviside function and positive/negative frequency parts of the propagator. On the contrary, the analogous calculation for the second term, the "non-planar contribution", yields (since the Moyal star product is Hermitean)

$$\begin{aligned} \Delta_F * \Delta_F + \overline{\Delta_F * \Delta_F} &= (i\theta\Delta + \Delta_-) * (i\theta\Delta + \Delta_-) + (-i\theta\Delta + \Delta_+) * (-i\theta\Delta + \Delta_+) \\ &= \Delta_+ * \Delta_+ + \Delta_- * \Delta_- + \\ &\quad + i\,\theta\Delta * (i\theta\Delta - i\Delta) + (i\theta\Delta - i\Delta) * i\,\theta\Delta \\ &= \Delta_+ * \Delta_+ + \Delta_- * \Delta_- + \Delta_{ret} * \Delta_{av} + \Delta_{av} * \Delta_{ret}\,, \end{aligned} \tag{2}$$

with $i\Delta$ denoting the commutator function $\Delta_+ - \Delta_-$, and $\Delta_{ret/av}(x) = \theta(\pm\,x_0)\Delta(x)$ the retarded and the advanced propagator, respectively. The violation of unitarity is therefore due to the fact that in general, the (Fourier transform of the) star product of retarded and advanced propagator does not vanish.

These terms disappear, however, when the time is assumed to commute with the space variables, i.e. when there is a timelike vector n with $\sigma^{\mu\nu}n_\nu = 0$. In these cases we have

$$\Delta_{ret/av}(x) = \theta(\pm nx)\Delta(x) = \theta(\pm nx) * \Delta(x)$$

and hence

$$\Delta_{ret} * \Delta_{av} = \Delta * \theta * (1-\theta) * \Delta$$

and

$$\theta * (1-\theta) = \theta(1-\theta) = 0 \ .$$

By continuity, this remains true when n approaches a lightlike vector. This situation has been termed lightlike noncommutativity in [7]. It occurs as a scaling limit for a generic σ.

In [1] a different definition of a scalar field theory on noncommutative spacetime has been proposed, which does lead to a unitary S-matrix. It is based on the introduction of an interaction Hamiltonian in Fock space, defined as

$$H_I(t) = \int_{x_0=t} d^3x \ :\phi *_\sigma \cdots *_\sigma \phi(x):$$

where the integration at $x_0 = t$ is given a precise meaning as a positive map. Since $H_I(t)$ is formally self-adjoint, the corresponding perturbative expansion must be formally unitary.

For the sake of completeness let us also mention that in [1] the value of σ is not fixed, since a fully Lorentz invariant description of the noncommutative spacetime requires that *all* values of σ which are compatible with the spacetime uncertainty relations appear on the same footing, i.e. as points in the set Σ of joint eigenvalues of the coordinates' commutators. It is desirable to rid the Hamiltonian of the dependence on Σ. But since no Lorentz invariant average exists on Σ, the best one can do is the rotation invariant integration over $\Sigma^{(1)}$, the doubled sphere obtained when both the electric and magnetic part of σ have modulus one and hence are parallel, cf. [1].

The question of unitarity, however, is independent of whether the Hamiltonian depends on Σ or not, and in order to clarify the relation between the Hamiltonian approach and the modified Feynman rules, we shall not perform the integration over $\Sigma^{(1)}$ in this note[1]. In [8] we will further investigate the Hamiltonian approach.

[1]Note however, that if the integration over $\Sigma^{(1)}$ is performed, the σ variables in the various factors $H_I(t)$ in the perturbative expansion are treated as *independent* variables, whereas in the approaches outlined in this note they are all identified with one another. Such points will be discussed in detail elsewhere.

From the Dyson series given in [1] we deduce that the corresponding graph theory (for fixed σ) again entails a planar and a nonplanar contribution to the fish graph of ϕ^3 theory, where the planar contribution is again identical to the fish graph of ordinary quantum field theory. The nonplanar contribution, however, differs from the one obtained in the setting of the modified Feynman rules. This is due to the fact that in the Hamiltonian approach the time ordering is performed with respect to the t variables in each factor $H_I(t)$ of the perturbative expansion, such that the resulting Heaviside functions are not involved in the nonlocal products [1, eqn. 6.15]. In fact, the full fishgraph in the Hamiltonian approach is proportional to[2]

$$\int dt_1 dt_2 \int\limits_{x_0=t_1} d^3x \int\limits_{y_0=t_2} d^3y \; \theta(x_0-y_0) \cdot [\, \langle p| :\phi(x)\phi(y): |p\rangle \overset{sym}{\ast_\sigma} \Delta_+(x-y) \overset{sym}{\ast_\sigma} \Delta_+(x-y)\,]$$
$$+ \int dt_1 dt_2 \int\limits_{x_0=t_1} d^3x \int\limits_{y_0=t_2} d^3y \; \theta(y_0-x_0) \cdot [\, \langle p| :\phi(x)\phi(y): |p\rangle \overset{sym}{\ast_\sigma} \Delta_+(y-x) \overset{sym}{\ast_\sigma} \Delta_+(y-x)\,] \,.$$

Here, $\overset{sym}{\ast_\sigma}$ stands for the symmetrized star product with respect to both x and y. Note that the Heaviside function θ is multiplied *pointwise* to the threefold star product and can in general not be combined with the propagators $\Delta_\pm$ to yield the Feynman propagator. For this reason and since

$$\overline{\langle p| \ldots |p\rangle \overset{sym}{\ast_\sigma} \Delta_\pm \overset{sym}{\ast_\sigma} \Delta_\pm} \;=\; \langle p| \ldots |p\rangle \overset{sym}{\ast_\sigma} \Delta_\mp \overset{sym}{\ast_\sigma} \Delta_\mp \,,$$

we can deduce that the unitarity condition at second order is indeed satisfied, as

$$\begin{aligned} & \theta \cdot [\, \langle p| \ldots |p\rangle \overset{sym}{\ast_\sigma} \Delta_+ \overset{sym}{\ast_\sigma} \Delta_+ \,] \;+\; (1-\theta) \cdot [\, \langle p| \ldots |p\rangle \overset{sym}{\ast_\sigma} \Delta_- \overset{sym}{\ast_\sigma} \Delta_- \,] \\ + \;& \theta \cdot [\, \overline{\langle p| \ldots |p\rangle \overset{sym}{\ast_\sigma} \Delta_+ \overset{sym}{\ast_\sigma} \Delta_+} \,] \;+\; (1-\theta) \cdot [\, \overline{\langle p| \ldots |p\rangle \overset{sym}{\ast_\sigma} \Delta_- \overset{sym}{\ast_\sigma} \Delta_-} \,] \\ = \;& \langle p| \ldots |p\rangle \overset{sym}{\ast_\sigma} \Delta_- \overset{sym}{\ast_\sigma} \Delta_- \;+\; \langle p| \ldots |p\rangle \overset{sym}{\ast_\sigma} \Delta_+ \overset{sym}{\ast_\sigma} \Delta_+ \,, \end{aligned}$$

the last expression being the one which also arises in the product of two tree graphs.

The Hamiltonian approach, however, not only breaks Lorentz invariance explicitly, but moreover does not seem to allow for a simple definition of the interacting field. A better way to define the latter, which actually has already been investigated in the context of nonlocal theories in e.g. [9, 10], is to solve the field equation perturbatively [11]. As an illustrative example let us again consider a massive scalar field with ϕ^3 self-interaction.

[2]We will comment on the proper definition of Wick monomials in field theories on a noncommutative spacetime elsewhere.

Let $\phi = \sum g^n \phi_n$ be an expansion of the interacting field as a power series with respect to the coupling constant. Then the field equation is

$$(\Box - m^2)\,\phi_n = -\sum_{k=0}^{n-1} \phi_k *_\sigma \phi_{n-k-1}\,.$$

Hence, ϕ_0 is a free field. If it is identified with the incoming field, then ϕ_1 is given by

$$\phi_1 = \Delta_{ret} \times (\phi_0 *_\sigma \phi_0),$$

$\times$ being the ordinary convolution, and for ϕ_2 we obtain

$$\begin{aligned}\phi_2 &= \Delta_{ret} \times (\phi_0 *_\sigma \phi_1 + \phi_1 *_\sigma \phi_0)\\ &= \Delta_{ret} \times (\phi_0 *_\sigma (\Delta_{ret} \times (\phi_0 *_\sigma \phi_0))) + \Delta_{ret} \times ((\Delta_{ret} \times (\phi_0 *_\sigma \phi_0)) *_\sigma \phi_0)\,.\end{aligned}$$

The once contracted terms in ϕ_2 yield the fish graph, and again we find two different contributions,

$$\int dy\, \Delta_{ret}(y)\,(\Delta_+(y) + \Delta_-(y))\,\phi_0(x-y) \qquad \text{planar part} \tag{3}$$

$$\int dy\, \Delta_{ret}(y)\,(\Delta_+(y) * \phi_0(x-y) + \phi_0(x-y) * \Delta_-(y)) \qquad \text{nonplanar part} \tag{4}$$

Since the Moyal star product is not only strongly closed [12], but also has the special property that one star product under an integral may be replaced by a pointwise product, the nonplanar part is equal to

$$\int dy\, (\Delta_{ret} * \Delta_+(y) + \Delta_- * \Delta_{ret}(y))\,\phi_0(x-y)\,. \tag{5}$$

The theory is unitary as long as the interacting field is Hermitean, which is true by construction as long as ϕ_0 is Hermitean. In particular, Hermiticity is clearly fulfilled for the above expressions (3) and (4), (5). For the planar part this means

$$\begin{aligned}\Delta_{ret}\,(\Delta_+ + \Delta_-) &= (\theta\Delta)\,(\Delta_+ + \Delta_-) \;=\; -\,i\,(\theta\Delta_+^2 - \theta\Delta_-^2)\\ &= -i\,(\theta\Delta_+^2 + (1-\theta)\Delta_-^2) + i\Delta_-^2\\ &= -\,i\,\Delta_F^2 + i\Delta_-^2\\ &= +\,i\,\bar\Delta_F^2 - i\Delta_+^2\,.\end{aligned}$$

Hence, for the planar part the Hermiticity condition at second order is identical to the unitarity condition (1) for the Feynman propagator in ordinary field theory.

The nonplanar part is Hermitean by construction as well. Obviously, $\Delta_{ret} * \Delta_+ + \Delta_- * \Delta_{ret}$ is its own complex conjugate and in particular,

$$
\begin{aligned}
\Delta_{ret} * \Delta_+ + \Delta_- * \Delta_{ret} &= -i\,[\,\theta\Delta_+ * \Delta_+ - \theta\Delta_- * \Delta_+ + \Delta_- * \theta\Delta_+ - \Delta_- * \theta\Delta_-\,] \\
&= +i\,\Delta_- * \Delta_- - i\,[\,\Delta_F * \Delta_+ + \Delta_- * \Delta_F - \Delta_- * \Delta_+\,] \\
&= +i\,\Delta_- * \Delta_- - i\,\Delta_F * \Delta_F - i\,[\,i\,\Delta_{ret} * \Delta_+ - i\,\Delta_{ret} * \Delta_F\,] \\
&= +i\,\Delta_- * \Delta_- - i\,\Delta_F * \Delta_F + i\,\Delta_{ret} * \Delta_{av} \\
&= -i\,\Delta_+ * \Delta_+ + i\,\bar{\Delta}_F * \bar{\Delta}_F - i\,\Delta_{av} * \Delta_{ret}\,.
\end{aligned}
$$

We conclude that the Yang Feldman approach modifies the "scattering amplitude" of ordinary quantum field theory by the additional term $\Delta_{ret} * \Delta_{av}$. It is precisely this term which renders the theory unitary; as we have seen in (2), its absence in the setting of the modified Feynman rules entails a violation of unitarity.

Again, if σ is chosen such that there is a time- or lightlike vector n with $\sigma^{\mu\nu} n_\nu = 0$, we recover the unitarity condition $\Delta_F * \Delta_F + \bar{\Delta}_F * \bar{\Delta}_F \;=\; \Delta_+ * \Delta_+ + \Delta_- * \Delta_-$.

We have thus seen that, as long as a proper perturbative setup is employed, field theories on time/space noncommutative spacetimes may well be unitary in the sense that probabilities are always conserved. The more subtle problem of asymptotic completeness in such theories, which has also been studied before in the context of nonlocal theories, will be pursued elsewhere.

Acknowledgement
We would like to thank Bert Schroer and Wolfhart Zimmermann for valuable hints to publications on nonlocal field theories.

References

[1] S. Doplicher, K. Fredenhagen and J. E. Roberts, "The Quantum structure of spacetime at the Planck scale and quantum fields," Commun. Math. Phys. **172** (1995) 187.

[2] V. Schomerus, "D-branes and deformation quantization," JHEP **9906** (1999) 030 [arXiv:hep-th/9903205].

[3] T. Filk, "Divergencies in a field theory on quantum space," Phys. Lett. B **376** (1996) 53.

[4] J. Gomis and T. Mehen, "Space-time noncommutative field theories and unitarity," Nucl. Phys. B **591** (2000) 265 [arXiv:hep-th/0005129].

[5] L. Alvarez-Gaume, J. L. Barbon and R. Zwicky, "Remarks on time-space noncommutative field theories," JHEP **0105** (2001) 057 [arXiv:hep-th/0103069].

[6] C. S. Chu, J. Lukierski and W. J. Zakrzewski, "Hermitian analyticity, IR/UV mixing and unitarity of noncommutative field theories," [arXiv:hep-th/0201144].

[7] O. Aharony, J. Gomis and T. Mehen, "On theories with light-like noncommutativity," JHEP **0009** (2000) 023 [arXiv:hep-th/0006236].

[8] D. Bahns, S. Doplicher, K. Fredenhagen and G. Piacitelli, in preparation

[9] P. Kristensen, C. Moeller, "On a convergent meson theory: I," Matematisk-fysiske meddedelser 27,7 (1952)

[10] R. Marnelius, "Can The S Matrix Be Defined In Relativistic Quantum Field Theories With Nonlocal Interaction?," Phys. Rev. D **10**, 3411 (1974).

[11] C. N. Yang and D. Feldman, "The S Matrix In The Heisenberg Representation," Phys. Rev. **79** (1950) 972.

[12] A. Connes, M. Flato, D. Sternheimer, "Closed Star Products and Cyclic Cohomology" Lett. Math. Phys. **24**, 1 (1992).