Spin and exotic Galilean symmetry

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Abstract

A slightly modified and regularized version of the non-relativistic limit of the relativistic anyon model considered by Jackiw and Nair yields particles associated with the twofold central extension of the Galilei group, with independent spin and exotic structure.

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1 Introduction

Strange things can indeed happen in the plane: for example, a particle (called an anyon) can have fractional spin and intermediate statistics [1]. Another one is that the planar Galilei group admits a non-trivial two-parameter central extension [2] leading to an "exotic" model (equivalent to non-commutative mechanics) [3, 4].

Soon after its introduction, Jackiw and Nair [5] rederived our "exotic" model in [3] by taking the non-relativistic limit of their relativistic spinning anyon [6]. This may suggest that spin and exotic structure are related: one can trade one for the other, but one cannot have both. Here, we argue that, at least non-relativistically, spin and "exotic" structure can coexist independently. We show this, first, by reviewing the most general group-theoretical construction associated with the the twofold extended Galilei group, next, by slightly modifying the contraction considered by Jackiw and Nair [5]. Our abstract construction is illustrated by the acceleration-dependent model of [7] and by Moyal field theory [8, 9].

2 Exotic models

The free "exotic" particle model of [3] consists of a five-dimensional "evolution space" $T^*\mathbf{R}^2\times\mathbf{R}$ described by position, $\vec{x}$, momentum, $\vec{p}$, and time, t, (see [10]) which is endowed with the presymplectic two-form

$$\omega = -\,dp_i \wedge dx_i - \frac{\kappa}{2m^2}\,\epsilon_{ij}\,dp_i \wedge dp_j + dh \wedge dt, \qquad \text{with} \qquad h = \frac{\vec{p}^{\,2}}{2m}, \tag{2.1}$$

where $i, j = 1, 2$ and m, κ are given constants interpreted as mass and "exotic" parameter.[1] The equations of motion are then given by the null foliation of ω.

The manifest Galilean invariance of the two-form (2.1) provides us, via the (pre)symplectic version of Nœther's theorem [10], with conserved quantities, most of which take the standard form. The angular momentum, j, and the Galilean boost, $\vec{g}$, contain, however, new terms, namely

$$\begin{aligned} j &= \vec{q} \times \vec{p} + \frac{\kappa}{2m^2}\,\vec{p}^{\,2}, \\ g_i &= m x_i - p_i t + \frac{\kappa}{m}\,\epsilon_{ij} p_j. \end{aligned} \tag{2.2}$$

Those terms change the Poisson brackets of the Galilean boosts, which now satisfy $\{g_i, g_j\} = -\kappa\,\epsilon_{ij}$ rather than commute, as usual.

The system is *bosonic*: the quantum angular momentum operator is, in the momentum representation, $\widehat{\jmath} = -i\epsilon_{jk} p_j \partial_{p_k}$ [2, 3, 11]. Owing to the inherent ambiguity of planar angular momentum, an arbitrary constant, s_0, representing anyonic spin, can be freely added to the angular momentum, though. Let us therefore discuss all related classical models.

All "elementary" systems associated with the doubly-centrally extended Galilei group were determined by Grigore [11] who, following Souriau [10], identifies them with coadjoint orbits of the group, endowed with their canonical symplectic structure. (Here the word "elementary" means that the action of the Galilei group should be transitive; at the quantum level this means that the representation be irreducible.)

These orbits are 4-dimensional, and depend on four parameters denoted by s_0, h_0, m and κ. Their symplectic structure are pulled-back, on "evolution space", precisely as the closed two-form (2.1). This latter only depends on m and κ; the two other parameters only show up in the way the Galilei group acts on the orbit, i.e., in the associated conserved quantities. These latter are found to be the linear momentum, $\vec{p}$, the boost, $\vec{g}$, in (2.2), together with the angular momentum and energy, viz

$$\begin{aligned} j &= \vec{x} \times \vec{p} + \frac{\kappa}{2m^2}\vec{p}^{\,2} + s_0, \\ h &= \frac{\vec{p}^{\,2}}{2m} + h_0, \end{aligned} \tag{2.3}$$

supplemented with the Casimir invariants m and κ. The parameter s_0 is, hence, interpreted as *anyonic spin*, and h_0 is *internal energy*. Let us insist that s_0 and the exotic parameter κ are, *a priori*, independent quantities. Plainly, the model of [3] realizes (2.3) with $s_0 = h_0 = 0$.

Nonvanishing spin and internal energy also arise for the acceleration-dependent system considered by Lukierski et al. [7]. Here, phase space is 6-dimensional with coordinates $\vec{X}, \vec{p}, \vec{Q}$, and

[1] As a rule, in the non-relativistic theory, all spatial indices i, j, k are lower indices. Ordinary convention is, however, adopted for the position of the relativistic indices λ, μ, ν.

endowed with the following symplectic structure and Hamiltonian

$$\begin{aligned} \Omega &= -dp_i \wedge dX_i - \frac{\theta}{2}\epsilon_{ij} dp_i \wedge dp_j - \frac{1}{2\theta}\epsilon_{ij}\, dQ_i \wedge dQ_j, \\ h &= \frac{\vec{p}^2}{2m} - \frac{\vec{Q}^2}{2m\theta^2}, \end{aligned} \tag{2.4}$$

respectively. This system is also invariant with respect to the planar Galilei group; the associated conserved quantities, namely the linear momentum, $\vec{p}$, the boost, $\vec{g}$, the energy h in (2.4), augmented with the angular momentum,

$$j = \vec{X} \times \vec{p} + \frac{\theta}{2}\vec{p}^2 + \frac{1}{2\theta}\vec{Q}^2, \tag{2.5}$$

provide us with a more general moment map (2.3) with internal energy $h_0 = -\vec{Q}^2/(2m\theta^2)$ and anyonic spin $s = \vec{Q}^2/(2\theta)$. Once again, the exotic parameter $\kappa = -m^2\theta$ is non-trivial. Let us mention that the spin can be made totally independent from the exotic parameter by adding a suitable term, whereas the constraint $s = -m\theta h_0$ is also relaxed. See the third reference in [7].

It is worth pointing out that the appearance of these two models is consistent with Souriau's general "décomposition barycentrique" (Theorem (13.15) in [10]). Let us indeed consider the planar Galilei group G. The translations and the boosts span an invariant abelian subgroup $\widetilde{G}$ of G (with Lie algebra $\widetilde{\mathfrak{g}} \subset \mathfrak{g}$), so that the quotient group, $G_0 = G/\widetilde{G}$, is the direct product of rotations and time translations.

Let us consider a classical system represented by a manifold M endowed with a symplectic two-form, Ω, upon which the (planar) Galilei group G acts by symmetries, i.e., in a Hamiltonian fashion. Then the direct product $G \times G_0$ is also a symmetry [10]. Now, if dim $M = 4$, and the symplectic action is transitive, then M can be identified with an affine-coadjoint orbit endowed with its canonical symplectic structure. In fact, M is the dual of the invariant abelian Lie subalgebra $\widetilde{\mathfrak{g}}$, viz $M \cong \widetilde{\mathfrak{g}}^*$. In our case, we get precisely the free exotic particle model (2.1), considered in [3]. If, however, dim$M \geq 6$, then the system decomposes into the direct product of our "free exotic particle" phase space and a symplectic manifold, M_0, which describes the internal motions (this latter is distinguished by the vanishing of the "standard" conserved quantities), namely

$$M \cong \widetilde{\mathfrak{g}}^* \times M_0. \tag{2.6}$$

The extended group $G \times G_0$ respects this decomposition: the rotations and time translations act independently on the internal space, contributing (anyonic) spin and internal energy.

This is exactly what happens for the extended model of [7], whose phase space is decomposed into that of our "free exotic particle" times the symplectic plane. (An additional constraint links the rotations with the time translations, as highlighted by the linked extra terms in (2.4) and (2.5).)

A field theoretical model with anyonic spin and non-trivial exotic structure is provided by the Moyal-Schrödinger field theory [8] given by the non-local Lagrange density

$$L_{NC} = \frac{i}{2}\left(\bar{\psi} \star \partial_t \psi - \partial_t \bar{\psi} \star \psi\right) - \frac{1}{2m}\vec{\nabla}\bar{\psi} \star \vec{\nabla}\psi \tag{2.7}$$

with the Moyal star-product associated with the deformation parameter θ. As shown recently [9], the theory is Galilei invariant, and the boosts

$$\mathcal{G}_i = m \int x_i(\bar{\psi} \star \psi)\, d^2\vec{x} - t\mathcal{P}_i - \frac{1}{2}m\theta\epsilon_{ij}\mathcal{P}_j, \tag{2.8}$$

(where $\vec{\mathcal{P}} = \displaystyle\int \vec{\jmath}\, d^2\vec{x}$, with $\vec{\jmath} = [\bar{\psi}(\vec{\nabla}\psi) - (\vec{\nabla}\bar{\psi})\psi]/(2i)$ is the conserved momentum) satisfy the "exotic" commutation relation $\{\mathcal{G}_1, \mathcal{G}_2\} = -\kappa \int |\psi|^2 d^2\vec{x}$ with $\kappa = -m^2\theta$. Note that the angular momentum

$$\mathcal{J} = \int \left[\vec{x} \times \vec{\jmath} - \frac{\theta}{2}|\vec{\nabla}\psi|^2 + s_0|\psi|^2\right] d^2\vec{x} \tag{2.9}$$

is anyonic, cf. (2.3).

Yet another illustration is provided by the first-order non-local model given by the Lagrangian

$$\Im\left\{\psi^\dagger \star \left(\tfrac{1}{2}(1+\sigma_3)\partial_t - \vec{\sigma}\cdot\vec{\nabla} - im(1-\sigma_3)\right)\psi\right\} \tag{2.10}$$

where $\psi = \begin{pmatrix}\Phi\\ \chi\end{pmatrix}$ is a two-component Pauli spinor. Despite the presence of the Moyal product, the associated Euler-Lagrange equation is the "non-relativistic Dirac equation" of Lévy-Leblond [12],

$$\begin{aligned}(\partial_1 + i\partial_2)\Phi + 2im\chi &= 0\\ \partial_t\Phi - (\partial_1 - i\partial_2)\chi &= 0.\end{aligned} \tag{2.11}$$

These are Galilei invariant when the boosts are implemented as [12]

$$\psi(\vec{x},t) = \begin{pmatrix}1 & 0\\ -\frac{1}{2}(b_1+ib_2) & 0\end{pmatrix} \exp im[\vec{b}\cdot\vec{x} - t\vec{b}^{\,2}/2]\psi(\vec{x}-\vec{b}t, t). \tag{2.12}$$

Using the same technique as in [9], we find that the Moyal star results in a new term in the associated conserved quantity, viz.

$$\mathcal{G}_i = m \int x_i(\bar{\Phi} \star \Phi)\, d^2\vec{x} - t\mathcal{P}_i - \frac{1}{2}m\theta\epsilon_{ij}\mathcal{P}_j, \tag{2.13}$$

where $\vec{\mathcal{P}} = \displaystyle\int [\bar{\Phi}(\vec{\nabla}\Phi) - (\vec{\nabla}\bar{\Phi})\Phi]/(2i)\, d^2\vec{x}$, cf. (2.8). The $\mathcal{G}_i$'s satisfy the "exotic" commutation relation.

3 Non-relativistic limit by group contraction

Let us now turn to the non-relativistic limit of the relativistic anyon model [6] considered by Jackiw and Nair in [5]. Expanding to first order in $1/c$ the restriction to $p_0 \simeq mc^2 + \vec{p}^{\,2}/(2m)$, the relativistic symplectic structure on 6-dimensional phase space

$$\Omega_{\rm JN} = -dp_\mu \wedge dx^\mu + \frac{Sc^2}{2}\epsilon^{\mu\nu\rho}\frac{p_\mu dp_\nu \wedge dp_\rho}{(p^2)^{3/2}} \tag{3.1}$$

(where the Greek indices range from 0 to 2) yields indeed our presymplectic two-form (2.1), when their "spin", S, is identified with our coefficient κ.

The relativistic model admits, furthermore, the symmetry with generators

$$J_\mu = \epsilon_{\mu\nu\rho}x^\nu p^\rho + Sc^2\frac{p_\mu}{\sqrt{p^2}} \tag{3.2}$$

which satisfy the $\mathrm{o}(1,2)$ commutation relations relations, $\{J^\mu, J^\nu\} = \epsilon^{\mu\nu\rho}J_\rho$. Then the quantities $\widetilde{g}_i = \epsilon_{ij}J_j/c$, satisfy, to leading order in $1/c$, the exotic commutation relation above, allowing Jackiw and Nair to identify their $\widetilde{g}_i$ with the Galilean boosts. In particular, S becomes the exotic parameter κ. Their angular momentum,

$$J_0 \simeq \epsilon_{ij}x_i p_j + Sc^2 + \frac{S}{m^2}\,\vec{p}^{\,2}, \tag{3.3}$$

does not admit, however, a well-defined limit as $c \to +\infty$, unless the divergent constant Sc^2 is removed. Only at this price are the exotic quantities, namely the angular momentum and the boosts, (2.2), recovered.

Let us now present a mathematical construction that allows us to absorb the divergent terms into some extra coordinates. Extending the approach of Jackiw and Nair [5], our clue is to work with differential forms and group actions, rather than with the associated conserved quantities.

To get an insight, let us first discuss the model of a massive, spinless, relativistic particle in the plane. Following Souriau [10], it corresponds to a certain coadjoint orbit of the Poincaré group (we still denote by G), symplectomorphic to $T^*\mathbf{R}^2$. The pull-back of its canonical symplectic structure, ω, to G is indeed $d\alpha$, where $\alpha = p_\mu dx^\mu$ is a one-form on G. Using physical coordinates in a Lorentz frame, we obtain

$$\alpha = -\vec{p}\cdot d\vec{x} + mc^2\sqrt{1+\frac{\vec{p}^{\,2}}{m^2c^2}}\,dt, \tag{3.4}$$

which is the "Cartan-form" [10] of the free relativistic Lagrangian, $\mathcal{L}_0 = mc^2\sqrt{1-\vec{v}^{\,2}/c^2}$.

Let us emphasize that the one-form α (as well as the Lagrangian $\mathcal{L}_0$) diverges in the Galilean limit $c \to \infty$, as clearly seen by writing (3.4) as

$$\alpha = -\vec{p}\cdot d\vec{x} + \left[mc^2 + \frac{\vec{p}^{\,2}}{2m} + \mathcal{O}\left(\frac{1}{c^2}\right)\right]dt. \tag{3.5}$$

The two-form $d\alpha$ has, nevertheless, a well-behaved limit, namely the familiar presymplectic two-form of an "ordinary" free, spinless, non-relativistic particle of mass m, namely

$$d\alpha \approx -dp_i \wedge dx_i + dh \wedge dt \qquad \text{with} \qquad h = \frac{\vec{p}^{\,2}}{2m}, \tag{3.6}$$

where the notation "$\approx$" stands for "up to higher-order terms in $1/c^2$".

In order to cure the pathology of (3.5), we consider the *trivial* central extension $\widehat{G} = G \times \mathbf{R}$ of the planar Poincaré group G. The new one-form to consider on $\widehat{G}$ is the left-invariant one-form

$$\widehat{\alpha} = \alpha + \varepsilon d\tau \tag{3.7}$$

where τ is a coordinate on the centre $(\mathbf{R}, +)$ and $\varepsilon \in \mathbf{R}$. We now posit, for the sake of convenience, $\tau = t + u/c^2$ where u is a new real parameter that replaces our old τ. Also, let us introduce a new real constant, h_0, via

$$\varepsilon = -mc^2 + h_0. \tag{3.8}$$

Then the divergence in the one-form $\widehat{\alpha}$ disappears (cf. (3.5)), and the latter simply reads

$$\widehat{\alpha} \approx -\vec{p} \cdot d\vec{x} + \left[\frac{\vec{p}^{\,2}}{2m} + h_0\right] dt - m\, du. \tag{3.9}$$

The one-form $\widehat{\alpha}$ hence converges in the limit $c \to +\infty$. Note that we also get the "internal energy", h_0, specific to Galilean Hamiltonian mechanics. It is worth mentioning that (3.8) is the most general Ansatz for a power series in c^2 that guarantees convergence of $\widehat{\alpha}$ while bringing non-trivial contribution to the non-relativistic limit.

Thus, the divergent term in α is absorbed into an exact one-form that involves the extra coordinate that drops out. The remaining part is regular and yields the correct conserved quantities.

The advantage of this approach is that trivially extended Poincaré group now contracts nicely to the singly extended Galilei group. The infinitesimal action of the Poincaré group G on space-time reads, in fact,

$$\left\{\begin{array}{rcl} \delta x_i &=& \omega \epsilon_{ij} x_j + \beta_i t + \gamma_i, \\ \delta t &=& \vec{\beta} \cdot \vec{x}/c^2 + \epsilon, \end{array}\right. \tag{3.10}$$

with $\omega \in \mathbf{R}$ generating rotations, $\vec{\beta} \in \mathbf{R}^2$ boosts, $\vec{\gamma} \in \mathbf{R}^2$ space translations and $\epsilon \in \mathbf{R}$ time translations.

The infinitesimal action of the centre $(\mathbf{R}, +)$ of the extended group $\widehat{G}$ is readily written as $\delta\tau = \eta/c^2 + \epsilon'$ with $\eta, \epsilon' \in \mathbf{R}$, so that the above definition of u yields, in the limit $c \to +\infty$, $\epsilon' = \epsilon$ and

$$\delta u = -\vec{\beta} \cdot \vec{x} + \eta, \tag{3.11}$$

corresponding precisely to the infinitesimal action of the non-trivial 1-parameter central extension of the Galilei group $\widehat{G}_\infty$ on the "vertical" fiber parametrized by u, while the usual Galilei action on spacetime is plainly deduced from (3.10) in the limit $c \to +\infty$, viz

$$\left\{\begin{array}{rcl} \delta x^i &=& \omega \epsilon_{ij} x_j + \beta_i t + \gamma_i, \\ \delta t &=& \epsilon. \end{array}\right. \tag{3.12}$$

4 Spin and exotic parameter

Let us now extend our improved contraction to spin. Starting with spinning massive particles moving in Minkowski spacetime $\mathbf{R}^{2,1}$, a similar procedure leads to three items analogous to those previously highlighted.

For a particle with spin $s \in \mathbf{R}$ and mass $m > 0$, dwelling in Minkowski spacetime, the one-form to start with on the Poincaré group, G, is given [10] by $\alpha = p_\mu dx^\mu + s\, e_{1\mu}\, de_2^\mu$ where

$p^\mu = me_0^\mu$ and $(e_0, e_1, e_2)_x$ is an orthonormal Lorentz frame at the point $x \in \mathbf{R}^{1,2}$. This one-form retains, in an adapted coordinate system, the rather complicated-looking expression

$$
\begin{aligned}
\alpha &= -\vec{p}\cdot d\vec{x} + mc^2\sqrt{1+\frac{\vec{p}^{\,2}}{m^2c^2}}\,dt \\
&+ s\,\frac{\sqrt{1+\dfrac{\vec{p}^{\,2}}{m^2c^2}}}{1+\dfrac{(\vec{p}\times\vec{u})^2}{m^2c^2}}\Bigg[\left(1+\frac{\vec{p}^{\,2}}{m^2c^2}\right)\vec{u}\times d\vec{u} + \frac{\vec{p}\cdot\vec{u}}{m^2c^2}\,d\vec{u}\times\vec{p} \\
&\quad + \frac{\vec{u}\times\vec{p}\,(\vec{p}\cdot\vec{u})\,\vec{p}\cdot d\vec{p}}{m^4c^4\left(1+\dfrac{\vec{p}^{\,2}}{m^2c^2}\right)} - \frac{\vec{u}\times\vec{p}}{m^2c^2}\,d(\vec{p}\cdot\vec{u})\Bigg]
\end{aligned}
\tag{4.1}
$$

with $\vec{u} \in S^1$, an arbitrary unit vector in the plane. Note that $\vec{u}\times d\vec{u} = d\phi$, where ϕ is the argument of $\vec{u}$ (actually the rotation angle of the SO(2) subgroup of the Lorentz group SO(1,2)). A tedious calculation yields furthermore a presymplectic two-form, $\omega = d\alpha$, similar to (3.1), whose behaviour to order c^{-2} is

$$
\omega \approx -dp_i\wedge dx_i + dh\wedge dt + \frac{s}{m^2c^2}\,dp_1\wedge dp_2 \tag{4.2}
$$

where h is as in (2.1).

Note that if one considers s as being independent of c, then the two-form ω in (4.2) plainly tends, as $c\to+\infty$, to that of an "ordinary" non-relativistic particle, (3.6). Our clue is to posit, instead, the Jackiw-Nair-inspired Ansatz, cf. (3.8),

$$
s = \kappa c^2 + s_0 \tag{4.3}
$$

where κ and s_0 are new constants. Then, in the non-relativistic limit we recover precisely our exotic two-form (2.1). This is just like in [5], up to a minor difference in the interpretation: it is our s, and not their $S = sc^2$ in (3.1) that should be called relativistic spin. The parameter S has indeed physical dimension $[S] = [\hbar/c^2]$, and cannot represent spin, whose dimension, $[\hbar]$, is carried correctly by our s. The constants κ in (4.3) (whose dimension is $[\hbar/c^2]$) is hence interpreted as the exotic parameter; also s_0 will turn out to be Galilean anyonic spin.

Presenting the spin term in equation (4.1) as

$$
\kappa c^2 d\phi + s_0\,d\phi + \frac{\kappa}{m^2}\left[\left(\frac{3}{2}\vec{p}^{\,2} - (\vec{p}\times\vec{u})^2\right)d\phi + (\vec{p}\cdot\vec{u})\;d\vec{u}\times\vec{p} - \vec{u}\times\vec{p}\,d(\vec{p}\cdot\vec{u})\right] + \mathcal{O}\left(\frac{1}{c^2}\right) \tag{4.4}
$$

yields

$$
\alpha \approx -\vec{p}\cdot d\vec{x} + \left[mc^2 + \frac{\vec{p}^{\,2}}{2m}\right]dt + \kappa c^2 d\phi + s_0 d\phi + \frac{\kappa}{2m^2}\,\vec{p}\times d\vec{p} \tag{4.5}
$$

modulo a closed one-form. Thus, while the exterior derivative $d\alpha$ behaves correctly, the one-form (4.5) contains instead two divergent terms, namely mc^2dt and $\kappa c^2 d\phi$. Removing them and setting $s_0 = 0$ would yield the Cartan-form of the Lagrangian used in [3].

This time, regularization is achieved by the (trivial) *double central extension* $\widehat{\widehat{G}} = \widehat{G}\times\mathbf{R}$ of the Poincaré group, endowed with the canonical one-form: $\widehat{\widehat{\alpha}} = \widehat{\alpha} + \chi d\theta$ where $\widehat{\alpha}$ is as in (3.7) and θ parametrizes the new $(\mathbf{R},+)$-factor, the coordinate χ having the dimension of $\hbar$. The divergence associated with the energy having been removed by the first trivial central extension, let us posit[2] $\chi = -\kappa c^2$ and $\theta = \phi - w/c^2$ where $w \in \mathbf{R}$ is the new parameter to consider in place of θ. Then, unlike α in (4.5), the one-form $\widehat{\widehat{\alpha}}$ *is* well behaved in the limit $c \to +\infty$ and retains, (modulo a closed one-form), the expression

$$\widehat{\widehat{\alpha}} \approx -\vec{p}\cdot d\vec{x} + \left[\frac{p^2}{2m} + h_0\right] dt - m\,du + s_0 d\phi + \frac{\kappa}{2m^2}\,\vec{p}\times d\vec{p} + \kappa dw. \tag{4.6}$$

In order to make sense of the group contraction $\widehat{\widehat{G}}_\infty = \lim_{c\to+\infty}\widehat{\widehat{G}}$, we need to compute the infinitesimal action of the Lorentz group on the rotations, parametrized by ϕ, which serve to define our second extension coordinate, w. Using the known form of the matrices spanning the Lie algebra $\mathrm{o}(1,2)$, one writes the infinitesimal Lorentz action on itself to obtain finally

$$\delta\phi \approx \omega + \frac{1}{2c^2}\,\vec{\beta}\times\vec{b}. \tag{4.7}$$

The infinitesimal action of the extra central subgroup $(\mathbf{R},+)$ of $\widehat{\widehat{G}}$ being written as $\delta\theta = \varrho/c^2 + \omega'$ with $\varrho, \omega' \in \mathbf{R}$, we finally get from the definition of w, in the limit $c \to +\infty$, firstly $\omega' = \omega$ and, secondly, the expression

$$\delta w = \varrho + \frac{1}{2}\,\vec{\beta}\times\vec{v} \tag{4.8}$$

where we have put $\vec{v} = \vec{p}/m$ for the velocity.

Formula (4.8) gives the infinitesimal action of the exotic Galilei group $\widehat{\widehat{G}}_\infty$ on the new, exotic, extension fiber with coordinate w. This latter can be viewed as providing an extended evolution space. Notice, though, that the action of $\widehat{\widehat{G}}_\infty$ on this evolution space—whose infinitesimal form is given by (3.12), (3.11), and (4.8)—is *not* the lift of an action on some extended space-time, as it involves the velocity $\vec{v}$ in (4.8).

Let us mention that the conserved quantity associated to a symmetry is obtained, in this framework, by simply contracting the (invariant) one-form $\widehat{\widehat{\alpha}}$ on the group with the infinitesimal generator of the symmetry. Using (4.6), we recover in particular the general conserved quantities in (2.3). No divergences occur.

Our modified contraction with the Ansatz (4.3) yields hence a non-relativistic model with both an exotic structure and anyonic spin.

5 Discussion

The derivation of the exotic model presented by Jackiw and Nair [5] has some subtle points: their "spin", which becomes our exotic parameter, has not the expected physical dimension, and their angular momentum diverges as c tends to infinity. Such divergences are indeed familiar in the theory of group contraction; our mathematical construction, here, allows us to eliminate the

[2]Adding a c-independent constant to χ would just modify the form of the (anyonic) spin s_0 by an overall additive constant.

divergent terms by resorting to a double group extension of the Poincaré group and to recover, for free, the twice centrally extended Galilei group of planar physics.

In a recent paper, Hagen [13] also discusses the relation of spin and exotic extension. He considers the Lévy-Leblond equation (2.11) but with ordinary Lagrangian, i.e. (2.10) with ordinary product, which has no non-trivial second extension [12].

In conclusion, let us stress that fractional spin arises due to the commutativity of the planar rotation group alone, independently of any further symmetry. This is why one can have relativistic as well as non-relativistic anyons. Although the commutative structure of plane rotations plays a rôle for the "exotic" structure also, this latter clearly involves more symmetries, namely the two-dimensional group-cohomology of the Galilei group. On the other hand, this phenomenon definitely does not arise in the relativistic context since the Poincaré group has trivial group-cohomology.

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