Exact form factors in integrable quantum field theories: the sine-Gordon model (II)

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Abstract

A general model independent approach using the 'off-shell Bethe Ansatz' is presented to obtain an integral representation of generalized form factors. The general techniques are applied to the quantum sine-Gordon model alias the massive Thirring model. Exact expressions of all matrix elements are obtained for several local operators. In particular soliton form factors of charge-less operators as for example all higher currents are investigated. It turns out that the various local operators correspond to specific scalar functions called p-functions. The identification of the local operators is performed. In particular the exact results are checked with Feynman graph expansion and full agreement is found. Furthermore all eigenvalues of the infinitely many conserved charges are calculated and the results agree with what is expected from the classical case. Within the frame work of integrable quantum field theories a general model independent 'crossing' formula is derived. Furthermore the 'bound state intertwiners' are introduced and the bound state form factors are investigated. The general results are again applied to the sine-Gordon model. The integrations are performed and in particular for the lowest breathers a simple formula for generalized form factors is obtained.

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1 Introduction

Form factors for integrable model in 1+1 dimensions were first investigated by Vergeles and Gryanik [1] for the sinh-Gordon model and by Weisz [2] for the sine-Gordon model. The 'form factor program' was formulated in [3] where the concept of generalized form factors was introduced. In that article consistency

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equations were formulated which are expected to be satisfied by these objects. Thereafter this approach was developed further and studied in the context of several explicit models by Smirnov [4] who proposed the form factor equations $(i) - (v)$ (see below) as extensions of similar formulae in the original article [3]. The formulae were proven in [5]. In the last decade a large number of articles were published on form factors (see e.g. references in [5]). More recent papers on solitonic matrix elements in the sine-Gordon model are [6, 7] and for the $SU(2)$-Thirring model [8, 9, 10]. Also there is a nice application [11, 12] of form factors in condensed matter physics. The one dimensional Mott insulators can be described in terms of the quantum sine-Gordon model.

In the present article the new approach to the 'form factor program' presented in [5] is developed further. It uses the 'off-shell Bethe Ansatz' to obtain an integral representation of generalized form factors. The approach applies also to general integrable models in 1+1 dimensions where the nested version [13] of the 'off-shell Bethe Ansatz has to be used. Applications of the general case will be published elsewhere [14, 15], here we restrict ourselves essentially to the simple no-nested version. That means the general techniques are applied to the quantum sine-Gordon alias massive Thirring model. The article is a continuation of the previous one [5] where the soliton field (an operator with nonvanishing charge) has been investigated. Here exact expressions of all matrix elements are obtained for several charge-less local operators.

We repeat the investigation of the current and the energy momentum tensor which have been discussed before by Smirnov[4].[1] Our main new results are: 1) We propose the form factors of the local operators $\overline{\psi}\psi(x)$, $\overline{\psi}\gamma^5\psi(x)$ and of the infinitely many conserved currents. 2) In order to identify the operators we perform Feynman graph expansions and compare the results with expansions of the exact expressions. 3) We calculate all eigenvalues of the infinitely many conserved charges. 4) Within the frame work of integrable quantum field theories we derive a general model independent 'crossing' formula (correcting for a sign mistake in a similar formula proposed by Smirnov[4]). 5) We develop the concept of the 'bound state intertwiner' in the context of bound state form factors and prove several relations. 6) Using these techniques we derive a formula for breather form factors.

The 'form factor program' is part of 'bootstrap program' for integrable quantum field theories in 1+1-dimensions. This program *classifies* integrable quantum field theoretic models and in addition it provides their explicit exact solutions in term of all Wightman functions. These results are obtained in three steps:

1. The S-matrix is calculated by means of general properties as unitarity and crossing, the Yang-Baxter equations (which are a consequence of integrability) and the additional assumption of 'maximal analyticity'. This means that the two-particle S-matrix is an analytic function in the physical plane (of the Mandelstam variable $(p_1+p_2)^2$) and possesses there only those poles which are of physical origin.

[1] In this work similar integral representations of form factors were proposed. We have checked that for small coupling the results for the four-particle matrix element of the current and the energy momentum tensor agree with ours. We could not prove that both representations agree in general.

2. Generalized form factors which are matrix elements of local operators

$$^{out}\langle\, p'_m, \ldots, p'_1 \,|\mathcal{O}(x)|\, p_1, \ldots, p_n \,\rangle^{in}$$

are calculated using the S-matrix obtained in 1. More precisely, the equations $(i) - (v)$ given below on page 7 are used as solved. These equations follow from LSZ-assumptions and again the additional assumption of 'maximal analyticity' (see also [5]).

3. The Wightman functions are obtained by inserting a complete set of intermediate states. In particular the two point function for an hermitian operator $\mathcal{O}(x)$ reads

$$\langle\, 0 \,|\mathcal{O}(x)\, \mathcal{O}(0)|\; 0 \,\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \int \cdots \int \frac{dp_1 \ldots dp_n}{(2\pi)^n 2\omega_1 \ldots 2\omega_n} \times \left| \langle\, 0 \,|\mathcal{O}(0)|\, p_1, \ldots, p_n \,\rangle^{in} \right|^2 e^{-ix\sum p_i}. \quad (1)$$

The on-shell program i.e. the exact determination of the scattering matrix using the Yang-Baxter equation was formulated in [16, 17] (for reviews see also [18, 19]). Off-shell considerations were carried out in [1, 2] and in [3, 20], where the concept of a generalized form factor was introduced. The explicit evaluation of all the integrals and the sum in (1) remains an open challenge. A progress towards a solution of this problem has recently been achieved by Korepin et al. [21]. Up to now it has even not been proven that the sum over the intermediate states converges.[2] We expect that our new representations of form factors will help to solve these problems. The 'bootstrap program' does not use classical Lagrangians and any quantization procedure to construct the quantum models. We have contact with the classical models only, when at the end we compare our exact results with Feynman graph expansions which are based on these Lagrangians.

In the previous paper [5] an integral representation for general soliton matrix elements of the fundamental fermi-field of the massive Thirring model has been proposed. In the present paper we generalize this formula and investigate in particular charge-less local operators. The strategy is as follows:

For a state of n particles of kind α_i with momenta $p_i = m \sinh \theta_i$ and a local operator $\mathcal{O}(x)$ the generalized form factor is defined by

$$\langle\, 0 \,|\, \mathcal{O}(x) \,|\, \alpha_1(p_1), \ldots, \alpha_n(p_n) \,\rangle^{in} = e^{-ix(p_1 + \cdots + p_n)} \mathcal{O}_{\underline{\alpha}}(\underline{\theta}) \;, \quad \text{for } \theta_1 > \cdots > \theta_n.$$

where the short notation $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\underline{\theta} = (\theta_1, \ldots, \theta_n)$ has been used. We make the Ansatz

$$\mathcal{O}_{\underline{\alpha}}(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m \, h(\underline{\theta}, \underline{z}) \, p^{\mathcal{O}}(\underline{\theta}, \underline{z}) \, \Psi_{\underline{\alpha}}(\underline{\theta}, \underline{z})$$

with the Bethe state $\Psi_{\underline{\alpha}}(\underline{\theta}, \underline{z})$ defined by eq. (12) and the integration contours $\mathcal{C}_{\underline{\theta}}$ of figure 1. The scalar function $h(\underline{\theta}, \underline{z})$ (see eqs. (14)-(17)) is uniquely determined by the S-matrix and the 'p-function' $p^{\mathcal{O}}(\underline{\theta}, \underline{z})$ depends on the operator

[2] However, it is known [22] that the higher particle contributions are very small compared to the leading ones.

$\mathcal{O}(x)$. By means of the Ansatz we transform the properties $(i)-(v)$ of the co-vector valued function $\mathcal{O}_{\underline{\alpha}}(\underline{\theta})$ to properties $(i')-(v')$ of the scalar function $p^{\mathcal{O}}(\underline{\theta},\underline{z})$ which are easily solved. In particular we obtain the p-functions for the local operators[3] $\mathcal{N}\left[\overline{\psi}\psi\right](x)$, $\mathcal{N}\left[\overline{\psi}\gamma^5\psi\right](x)$, the current $j^\mu(x)=\mathcal{N}\left[\overline{\psi}\gamma^\mu\psi\right](x)$, the energy momentum tensor $T^{\mu\nu}(x)=\frac{i}{2}\mathcal{N}\left[\overline{\psi}\gamma^\mu\overset{\leftrightarrow}{\partial^\nu}\psi\right](x)-g^{\mu\nu}\mathcal{L}^{MT}$ and the infinitely many higher conserved currents $J_L^\mu(x)$

$$\begin{array}{lrcl}
1) & p^{\overline{\psi}\psi}(\underline{\theta},\underline{z}) & = & N_n^{\overline{\psi}\psi}\,q_-(\underline{\theta},\underline{z}) \\
2) & p^{\overline{\psi}\gamma^5\psi}(\underline{\theta},\underline{z}) & = & N_n^{\overline{\psi}\gamma^5\psi}\,q_+(\underline{\theta},\underline{z}) \\
3) & p^{j^\pm}(\underline{\theta},\underline{z}) & = & \pm N_n^j\left(\sum_{i=1}^n e^{\mp\theta_i}\right)^{-1} q_+(\underline{\theta},\underline{z}) \\
4) & p^{T^{\pm\pm}}(\underline{\theta},\underline{z}) & = & N_n^T\sum_{i=1}^n e^{\pm\theta_i}\left(\sum_{i=1}^n e^{\mp\theta_i}\right)^{-1} q_-(\underline{\theta},\underline{z}) \\
 & p^{T^{+-}}(\underline{\theta},\underline{z}) & = & -N_n^T\,q_-(\underline{\theta},\underline{z}) \\
5) & p^{J_L^\pm}(\underline{\theta},\underline{z}) & = & \pm N_n^{J_L}\sum_{i=1}^n e^{\pm\theta_i}\sum_{i=1}^m e^{Lz_i}\,, \quad (L=\pm1,\pm3,\dots)\,.
\end{array} \tag{2}$$

$$\text{where}\quad q_\pm(\underline{\theta},\underline{z})=\sum_{i=1}^n e^{-\theta_i}\sum_{i=1}^m e^{z_i}\pm\sum_{i=1}^n e^{\theta_i}\sum_{i=1}^m e^{-z_i}\,.$$

The identification with the operators is made by comparing the exact results with Feynman graph expansions. Properties as charge, behavior under Lorentz transformations etc. will also become obvious.

The article is organized as follows: In section 2 we recall some formulae of [5] which we need in the following. In section 3 we present a general formula for solitonic form factors. In section 4 we discuss several explicit examples and perform perturbative checks for two- and four-particle form factors. As an example we also investigate the asymptotic behavior when one soliton momentum goes to infinity and compare the result with typical bosonic behavior. Section 5 contains the derivation of a general crossing formula. Again LSZ-assumptions are used. Also the charges of the infinitely many higher local conservation laws on all states are calculated. In sections 6 we formulate the 'bootstrap' principle and introduce the 'bound state intertwiners' (see also [24]). Using these techniques we derive in section 7 from the pure soliton anti-soliton form factors the mixed breather soliton and the pure breather form factors. Section 8 contains conclusions and an outlook. Several proofs and explicit calculations are delegated to appendices.

2 Recall of formulae

In this section we recall some formulae which we shall need in the following sections to present our results. All this material can be found in [5] including the original references. Coleman [25] had shown that the sine-Gordon and the

[3] The symbol $\mathcal{N}$ refers to normal products of local quantum fields. In perturbation theory they are defined by Zimmerman's [23] subtraction method, for example.

massive Thirring model are equivalent on the quantum level. The corresponding classical models are defined by their Lagrangian's

$$\begin{aligned}\mathcal{L}^{SG} &= \tfrac{1}{2}\partial_\mu\varphi\partial^\mu\varphi + \frac{\alpha}{\beta^2}(\cos\beta\varphi - 1)\\ \mathcal{L}^{MT} &= \overline{\psi}(i\gamma\partial - M)\psi - \tfrac{1}{2}g\, j^\mu j_\mu\,, \quad \left(j^\mu = \overline{\psi}\gamma^\mu\psi\right).\end{aligned} \tag{3}$$

2.1 The S-matrix

The sine-Gordon model alias massive Thirring model describes the interaction of several types of particles: solitons, anti-solitons alias fermions and anti-fermions and a finite number of charge-less breathers, which may be considered as bound states of solitons and anti-solitons. In this work we will concentrate on states consisting of solitons and anti-solitons. Integrability of the model implies that the n-particle S-matrix factorizes into two particle S-matrices. In particular scattering conserves the number of particles and even their momenta. The two particle S-matrix contains the following scattering amplitudes: the two-soliton amplitude $a(\theta)$, the forward and backward soliton anti-soliton amplitudes $b(\theta)$ and $c(\theta)$:[4]

$$\begin{gathered} b(\theta) = \frac{\sinh\theta/\nu}{\sinh(i\pi-\theta)/\nu}\,a(\theta)\,, \qquad c(\theta) = \frac{\sinh i\pi/\nu}{\sinh(i\pi-\theta)/\nu}\,a(\theta)\,,\\ a(\theta) = \exp\int_0^\infty \frac{dt}{t}\,\frac{\sinh\frac{1}{2}(1-\nu)t}{\sinh\frac{1}{2}\nu t\,\cosh\frac{1}{2}t}\,\sinh t\frac{\theta}{i\pi}\,. \end{gathered} \tag{4}$$

The parameter θ is the absolute value of the rapidity difference $\theta = |\theta_1 - \theta_2|$ where θ_i are the rapidities of the particles given by the momenta $p_i = M\sinh\theta_i$. The parameter ν is related to the sine-Gordon and the massive Thirring model coupling constant by

$$\nu = \frac{\beta^2}{8\pi - \beta^2} = \frac{\pi}{\pi + 2g}$$

where the second equality is due to Coleman [25].

We list some general properties of the two-particle S-matrix. As usual in this context we use in the notation

$$v^{1\dots n} \in V^{1\dots n} = V_1 \otimes \cdots \otimes V_n$$

for a vector in a tensor product space. The vector components are denoted by $v^{\alpha_1\dots\alpha_n}$. A linear operator connecting two such spaces with matrix elements $A_{\alpha_1\dots\alpha_n}^{\alpha'_1\dots\alpha'_{n'}}$ is denoted by

$$A_{1\dots n}^{1'\dots n'} : \; V^{1\dots n} \to V^{1'\dots n'}$$

where we omit the upper indices if they are obvious. All vector spaces V_i are isomorphic to a space V whose basis vectors label all kinds of particles (here solitons and anti-solitons, i.e. $V \cong \mathbb{C}^2$). An S-matrix such as S_{ij} acts nontrivial only on the factors $V_i \otimes V_j$.

[4]This S-matrix has been obtained first by Zamolodchikov [26] extrapolating a semiclassical result and by means of the 'bootstrap program' using the Yang-Baxter relations in [16].

The physical S-matrix in the formulas above is given for positive values of the rapidity parameter θ. For later convenience we will also consider an auxiliary matrix $\dot{S}(\theta_1, \theta_2)$ regarded as a function depending on the individual rapidities of both particles θ_1, θ_2 or some times also on the difference $\theta_1 - \theta_2$

$$\dot{S}_{12}(\theta_1, \theta_2) = \dot{S}_{12}(\theta_1 - \theta_2) = \begin{cases} (\sigma S)_{12}(|\theta_1 - \theta_2|) & \text{for} \quad \theta_1 > \theta_2 \\ (S\sigma)_{21}^{-1}(|\theta_1 - \theta_2|) & \text{for} \quad \theta_1 < \theta_2 \end{cases}$$

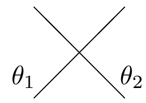
with σ taking into account the statistics of the particles. It is a diagonal matrix σ_{12} with entries -1 if both particles are fermions and $+1$ otherwise (see [27]). The matrix $\dot{S}(\theta_1, \theta_2)$ is an analytic function in terms of both variables θ_1 and θ_2. This follows from unitarity $S^\dagger S = 1$ and the fact that the physical S-matrix is the boundary value of a real analytic function $S(s + i\epsilon)$ as a function of the Mandelstam variable $s = (p_1 + p_2)^2$ such that $S^\dagger(s + i\epsilon) = S(s - i\epsilon)$ or

$$S^\dagger(\theta) = S^{-1}(\theta) = S(-\theta) \tag{5}$$

The auxiliary matrix $\dot{S}_{12}$ acts on the factors $V_1 \otimes V_2$ and in addition exchanges these factors, e.g.

$$\dot{S}_{12}(\theta) \,:\, V_1 \otimes V_2 \to V_2 \otimes V_1 \,.$$

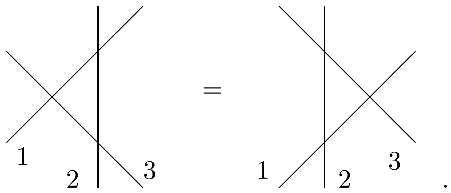
It may be depicted as

$\dot{S}_{12}(\theta_1, \theta_2) \quad =$ 

Here and in the following we associate a rapidity variable $\theta_i \in \mathbb{C}$ to each space V_i which is graphically represented by a line labeled by θ_i or simply by i. In terms of the auxiliary S-matrix the Yang-Baxter equation has the general form

$$\dot{S}_{12}(\theta_{12})\, \dot{S}_{13}(\theta_{13})\, \dot{S}_{23}(\theta_{23}) = \dot{S}_{23}(\theta_{23})\, \dot{S}_{13}(\theta_{13})\, \dot{S}_{12}(\theta_{12}) \tag{6}$$

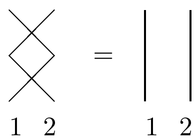
which graphically simply reads



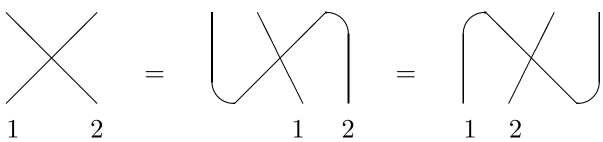
.

Unitarity and crossing may be written and depicted as

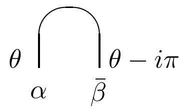
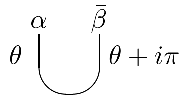
$$\dot{S}_{21}(\theta_{21})\dot{S}_{12}(\theta_{12}) = 1 \;: \tag{7}$$



$$\dot{S}_{12}(\theta_1 - \theta_2) = \mathbf{C}^{2\bar{2}}\, \dot{S}_{\bar{2}1}(\theta_2 + i\pi - \theta_1)\, \mathbf{C}_{\bar{2}2} = \mathbf{C}^{1\bar{1}}\, \dot{S}_{2\bar{1}}(\theta_2 - (\theta_1 - i\pi))\, \mathbf{C}^{\bar{1}1} \tag{8}$$



where $\mathbf{C}^{1\bar{1}}$ and $\mathbf{C}_{1\bar{1}}$ are charge conjugation matrices. For the sine-Gordon model the matrix elements are $\mathbf{C}^{\alpha\bar{\beta}} = \mathbf{C}_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$ where $\bar{\beta}$ denotes the anti-particle of β. We have introduced the graphical rule that a line changing the "time direction" also interchanges particles and anti-particles and changes the rapidity as $\theta \to \theta \pm i\pi$. We depict this as

$\mathbf{C}_{\alpha\bar{\beta}} =$  , $\mathbf{C}^{\alpha\bar{\beta}} =$  .

Similar crossing relations will be used below to formulate the properties of form factors.

Finally we note a property of the two-particle S-matrix

$$\dot{S}_{\alpha\beta}^{\delta\gamma}(0) = -\delta_{\alpha}^{\delta}\delta_{\beta}^{\gamma} \tag{9}$$

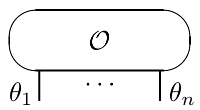
which turns out to be true for all examples. This means that $\dot{S}$ for zero momentum difference is equal to minus the permutation operator.

2.2 Form factors

For a state of n particles of kind α_i with momenta p_i and a local operator $\mathcal{O}(x)$ we define the form factor functions $\mathcal{O}_{\alpha_1,\dots,\alpha_n}(\theta_1,\dots,\theta_n)$ by

$$\langle\, 0 \,|\, \mathcal{O}(x) \,|\, \alpha_1(p_1),\dots,\alpha_n(p_n)\,\rangle^{in} = e^{-ix(p_1+\dots+p_n)}\mathcal{O}_{\underline{\alpha}}(\underline{\theta})\,,\ \text{for } \theta_1 > \dots > \theta_n. \tag{10}$$

For all other arrangements of the rapidities the functions $\mathcal{O}_{\underline{\alpha}}(\underline{\theta})$ are given by analytic continuation. Note that in general this analytic continuation does <u>not</u> provide the physical values of the form factor. These are given for ordered rapidities as indicated above and for other orders of course by the statistics of the particles. The $\mathcal{O}_{\underline{\alpha}}(\underline{\theta})$ are considered as the components of a co-vector valued function $\mathcal{O}_{1\dots n}(\underline{\theta}) \in V_{1\dots n} = \left(V^{1\dots n}\right)^{\dagger}$ which may be depicted as

$\mathcal{O}_{1\dots n}(\underline{\theta}) =$  .

Now we formulate the main properties of form factors in terms of the functions $\mathcal{O}_{1\dots n}(\underline{\theta})$ which follow from general LSZ-assumptions and "maximal analyticity". The later condition means that $\mathcal{O}_{1\dots n}(\underline{\theta})$ is a meromorphic function with respect to all θ's and all poles in the 'physical' strips $0 < \operatorname{Im}\theta_{ij} < \pi$ ($\theta_{ij} = \theta_i - \theta_j$ $i < j$) are of physical origin, as for example bound state poles as discussed in section 6.

Properties: The co-vector valued function $\mathcal{O}_{1\dots n}(\underline{\theta})$ is meromorphic with respect to all variables $\theta_1,\dots,\theta_n$ and

(i) it satisfies the symmetry property under the permutation of both the variables θ_i, θ_j and the spaces i, j at the same time

$$\mathcal{O}_{\dots ij\dots}(\dots,\theta_i,\theta_j,\dots) = \mathcal{O}_{\dots ji\dots}(\dots,\theta_j,\theta_i,\dots)\,\dot{S}_{ij}(\theta_i - \theta_j)$$

for all possible arrangements of the θ's, (*ii*) it satisfies the periodicity property under cyclic permutation of the rapidity variables and spaces

$$\mathcal{O}_{1\dots n}(\theta_1,\theta_2,\dots,\theta_n,) = \mathcal{O}_{2\dots n1}(\theta_2,\dots,\theta_n,\theta_1-2\pi i)\sigma_{\mathcal{O}1}$$

(*iii*) and it has poles determined by one-particle states in each sub-channel. In particular the function $\mathcal{O}_{\underline{\alpha}}(\underline{\theta})$ has a pole at $\theta_{12} = i\pi$ such that

$$\operatorname*{Res}_{\theta_{12}=i\pi} \mathcal{O}_{1\dots n}(\theta_1,\dots,\theta_n) = 2i\,\mathbf{C}_{12}\,\mathcal{O}_{3\dots n}(\theta_3,\dots,\theta_n)\,(\mathbf{1} - S_{2n}\dots S_{23})$$

where $\mathbf{C}_{12}$ is the charge conjugation matrix.

(*iv*) If the model also possesses bound states, the function $\mathcal{O}_{\underline{\alpha}}(\underline{\theta})$ has additional poles. If for instance the particles 1 and 2 form a bound state (12), there is a pole at $\theta_{12} = iu_{12}^{(12)}$ $(0 < u_{12}^{(12)} < \pi)$ such that

$$\operatorname*{Res}_{\theta_{12}=iu_{12}^{(12)}} \mathcal{O}_{12\dots n}(\theta_1,\theta_2,\dots,\theta_n) = \mathcal{O}_{(12)\dots n}(\theta_{(12)},\dots,\theta_n)\,\sqrt{2}\Gamma_{12}^{(12)}$$

where the bound state intertwiner $\Gamma_{12}^{(12)}$ and the relations of the rapidities $\theta_1,\theta_2,\theta_{(12)}$ and the fusion angle $u_{12}^{(12)}$ will be discussed in section 6 below.

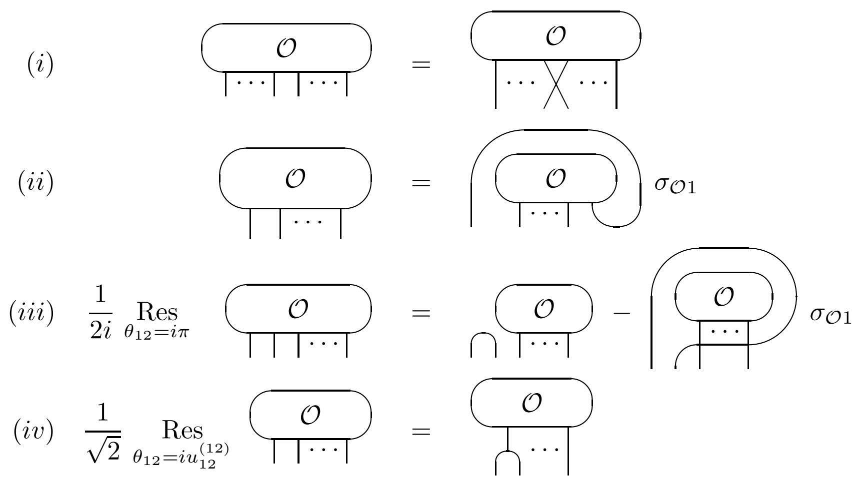
(*v*) Since we are dealing with relativistic quantum field theories Lorentz covariance in the form

$$\mathcal{O}_{1\dots n}(\theta_1+\mu,\dots,\theta_n+\mu) = e^{s\mu}\,\mathcal{O}_{1\dots n}(\theta_1,\dots,\theta_n)$$

holds if the local operator transforms as $\mathcal{O} \to e^{s\mu}\mathcal{O}$ where s is the "spin" of $\mathcal{O}$.

In the formulae (*i*) the statistics of the particles is taken into account by $\dot{S}$ which means that $\dot{S}_{12} = -S_{12}$ if both particles are fermions and $\dot{S}_{12} = S_{12}$ otherwise. In (*ii*) the statistics of the operator $\mathcal{O}$ is taken into account by $\sigma_{\mathcal{O}1} = -1$ if both the operator $\mathcal{O}$ and particle 1 are fermionic and $\sigma_{\mathcal{O}1} = 1$ otherwise.

The properties (*i*) − (*iv*) may be depicted as



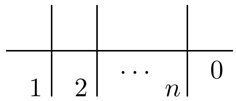
As was shown in [5] the properties $(i)-(iii)$ follow from general LSZ-assumptions and "maximal analyticity". The bound state form factor given by (iv) was discussed in [5] for special cases. In section 6 we investigate the general case and show that the bound state form factor is consistent with the 'bootstrap principle' which means that it also satisfies $(i)-(iii)$ if the constituents do. In section 5 we derive from the same assumptions a general crossing relation which implies (ii) and (iii). Conversely, it has been shown [4, 28, 24] that functions satisfying the properties $(i)-(v)$ and the general crossing relation represent *local* operators i.e. they are form factors of x-dependent operators $\mathcal{O}(x)$ which commute (anti-commute) for space like differences of the arguments.

We will now provide a constructive and systematic way of how to solve the properties $(i)-(v)$ for a co-vector valued function $f_{1\ldots n}(\underline{\theta})$ once the scattering matrix is given. These solutions are candidates of form factors. To capture the vectorial structure of the form factors we will employ the techniques of the algebraic Bethe Ansatz which we briefly explain now.

2.3 The 'off-shell' Bethe Ansatz co-vectors

As usual in the context of algebraic Bethe Ansatz we define the monodromy matrix as

$$T_{1\ldots n,0}(\underline{\theta},\theta_0) = \dot{S}_{10}(\theta_1-\theta_0)\,\dot{S}_{20}(\theta_2-\theta_0)\cdots\dot{S}_{n0}(\theta_n-\theta_0) \tag{11}$$

$=$ 

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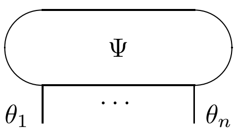
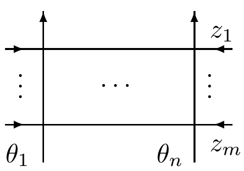
It is a matrix acting in the tensor product of the "quantum space" $V^{1\ldots n} = V_1 \otimes \cdots \otimes V_n$ and the "auxiliary space" V_0 (all $V_i \cong \mathbb{C}^2$ = soliton-anti-soliton space). The sub-matrices A, B, C, D with respect to the auxiliary space are defined by

$$T_{1\ldots n,0}(\underline{\theta}, z) \equiv \begin{pmatrix} A_{1\ldots n}(\underline{\theta}, z) & B_{1\ldots n}(\underline{\theta}, z) \\ C_{1\ldots n}(\underline{\theta}, z) & D_{1\ldots n}(\underline{\theta}, z) \end{pmatrix}.$$

A Bethe Ansatz co-vector in $V_{1\ldots n}$ is given by

$$\Psi_{1\ldots n}(\underline{\theta}, \underline{z}) = \Omega_{1\ldots n} C_{1\ldots n}(\underline{\theta}, z_1)\cdots C_{1\ldots n}(\underline{\theta}, z_m)$$



$=$


(12)

where $\underline{z} = (z_1, \ldots, z_m)$. Usually one has the restriction $2m \leq n$ and the charge of the state is $q = n - 2m$ = number of solitons minus number of anti-solitons. The solitons are depicted by $\uparrow$ or $\leftarrow$ and anti-solitons by $\downarrow$ or $\rightarrow$. The co-vector $\Omega_{1\ldots n}$ is the "pseudo-vacuum" consisting only of solitons (highest weight states)

$$\Omega_{1\ldots n} = \uparrow \otimes \cdots \otimes \uparrow .$$

It satisfies

$$\begin{aligned}
\Omega_{1\dots n}\, B_{1\dots n}(\underline{\theta}, z) &= 0 \\
\Omega_{1\dots n}\, A_{1\dots n}(\underline{\theta}, z) &= \prod_{i=1}^{n} \dot{a}(\theta_i - z)\Omega_{1\dots n} \\
\Omega_{1\dots n}\, D_{1\dots n}(\underline{\theta}, z) &= \prod_{i=1}^{n} \dot{b}(\theta_i - z)\Omega_{1\dots n}\,.
\end{aligned}$$

The eigenvalues of the matrices A and D, i.e. the functions $\dot{a} = -a$ and $\dot{b} = -b$ are given by the amplitudes of the scattering matrix (4). In the following we use the co-vector $\Psi_{1\dots n}(\underline{\theta}, \underline{z})$ in its 'off-shell' version which means that we do not fix the parameters $\underline{z}$ by means of Bethe Ansatz equations but we integrate over the z's [29, 30, 31].

3 The general form factor formula

In this section we present our main result. We derive a general formula in terms of an integral representation which allows to construct form factors i.e. matrix elements of local fields given by eq. (10). More precisely, we construct co-vector valued functions which satisfy the properties $(i) - (v)$ on page 7.

As a candidate of a generalized form factor of a local operator $\mathcal{O}(x)$ we make the following Ansatz for the co-vector valued function

$$\boxed{\mathcal{O}_{1\dots n}(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m\, h(\underline{\theta}, \underline{z})\, p^{\mathcal{O}}(\underline{\theta}, \underline{z})\, \Psi_{1\dots n}(\underline{\theta}, \underline{z})} \tag{13}$$

with the Bethe Ansatz state $\Psi_{1\dots n}(\underline{\theta}, \underline{z})$ defined by eq. (12). For all integration variables z_j $(j = 1, \dots, m)$ the integration contours $\mathcal{C}_{\underline{\theta}}$ consists of several pieces (see figure 1):

a) A line from $-\infty$ to ∞ avoiding all poles such that $\operatorname{Im}\theta_i - \pi - \epsilon < \operatorname{Im} z_j < \operatorname{Im}\theta_i - \pi$.

b) Clock wise oriented circles around the poles (of the $\phi(\theta_i - z_j)$) at $z_j = \theta_i$ $(i = 1, \dots, n)$.

Let the scalar function (c.f. [5])

$$h(\underline{\theta}, \underline{z}) = \prod_{1\le i<j\le n} F(\theta_{ij}) \prod_{i=1}^{n}\prod_{j=1}^{m} \phi(\theta_i - z_j) \prod_{1\le i<j\le m} \tau(z_i - z_j)\,, \tag{14}$$

be given by

$$\tau(z) = \frac{1}{\phi(z)\,\phi(-z)}\;, \qquad \phi(z) = \frac{1}{F(z)\,F(z+i\pi)} \tag{15}$$

and

$$F(\theta) = \sin\frac{1}{2i}\theta\ \exp\int_0^\infty \frac{dt}{t}\frac{\sinh\frac{1}{2}(1-\nu)t}{\sinh\frac{1}{2}\nu t\,\cosh\frac{1}{2}t}\,\frac{1-\cosh t(1-\theta/(i\pi))}{2\sinh t}. \tag{16}$$

The function $F(\theta)$ is the soliton-soliton form factor. It is a solution of Watson's equations

$$F(\theta) = F(-\theta)\,\dot{a}(\theta) = F(2\pi i - \theta) \tag{17}$$

with $\dot{a}(\theta) = -a(\theta)$ where $a(\theta)$ is the soliton-soliton scattering amplitude. It is the uniquely defined 'minimal' solution [3] which has no poles and no zeroes in the 'physical strip' $0 < \operatorname{Im}\theta \le \pi$ and at most a simple zero at $\theta = 0$.

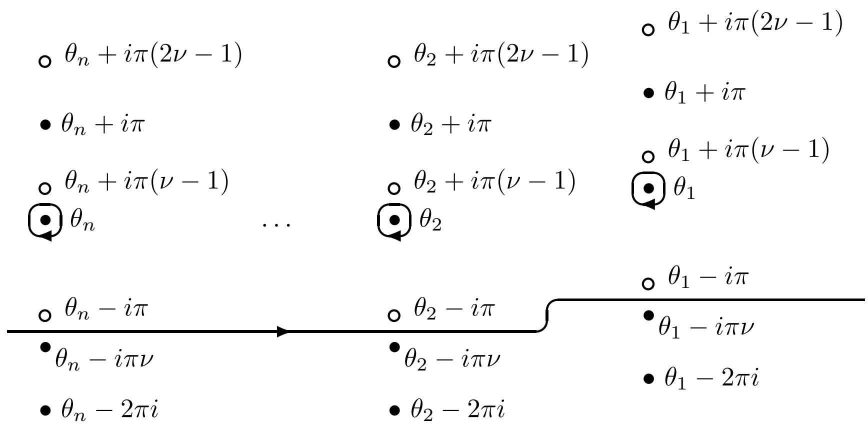


Figure 1: *The integration contour $\mathcal{C}_{\underline{\theta}}$ (for the repulsive case $\nu > 1$). The bullets belong to poles of the integrand resulting from $u(\theta_i - u_j)\,\phi(\theta_i - u_j)$ and the small open circles belong to poles originating from $t(\theta_i - u_j)$ and $r(\theta_i - u_j)$.*

Remarks:

- Using Watson's equations (17) for $F(z)$, crossing (8) and unitarity (7) for the sine-Gordon amplitudes one derives the following identities for the scalar functions $\phi(z)$ and $\tau(z)$ from the definitions (15)

$$\phi(z) = \phi(i\pi - z) = \frac{1}{\dot{b}(z)}\,\phi(z - i\pi) = \frac{\dot{a}(z - 2\pi i)}{\dot{b}(z)}\,\phi(z - 2\pi i)\ , \qquad (18)$$

$$\tau(z) = \tau(-z) = \frac{b(z)}{a(z)}\,\frac{a(2\pi i - z)}{b(2\pi i - z)}\,\tau(z - 2\pi i)$$

  where $b(z)$ is the soliton-anti-soliton scattering amplitude related to $a(z)$ by crossing $b(z) = a(i\pi - z)$.

- The functions $\phi(z)$ and $\tau(z)$ are of the form

$$\phi(z) = const.\ \frac{1}{\sinh z}\exp\int_0^\infty \frac{dt}{t}\,\frac{\sinh\frac{1}{2}(1-\nu)t\,\left(\cosh t(\frac{1}{2} - z/(i\pi)) - 1\right)}{\sinh\frac{1}{2}\nu t\sinh t}$$

$$\tau(z) = const.\ \sinh z\sinh z/\nu$$

- The function $h(\underline{\theta}, \underline{z})$ and the state $\Psi_{1\dots n}(\underline{\theta}, \underline{z})$ are completely determined by the S-matrix.

In contrast to the functions $F(z), \phi(z)$ and $\tau(z)$ the 'p-function' $p^{\mathcal{O}}(\underline{\theta}, \underline{z})$ in the integral representation (13) depends on the local operator $\mathcal{O}(x)$, in particular on the spin, the charge and the statistics. The number of the particles n and the number of integrations m are related by $q = n - 2m$ where q is the charge of the operator $\mathcal{O}(x)$. The p-functions is an entire function in the z_j $(j = 1, \dots, m)$ and in order that the form factor satisfies the properties $(i) - (v)$ it has to satisfy the following

Conditions: The p-function $p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$ (where n is the number of particles and the number of variables θ) satisfies

(i') $p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$ is symmetric with respect to the θ's and the z's.

(ii') $p_n^{\mathcal{O}}(\underline{\theta}, \underline{z}) = \sigma_{\mathcal{O}i} p_n^{\mathcal{O}}(\dots, \theta_i - 2\pi i, \dots, \underline{z})$ and it is a polynomial in $e^{\pm z_j}$ $(j = 1, \dots, m)$.

The statistics factor $\sigma_{\mathcal{O}i}$ is -1 if the operator $\mathcal{O}(x)$ and the particle i are both fermionic and $+1$ otherwise.

$$(iii') \quad \left\{ \begin{array}{l} p_n^{\mathcal{O}}(\theta_1 = \theta_n + i\pi, \underline{\tilde{\theta}}, \theta_n; \underline{\tilde{z}}, z_m = \theta_n) = \dfrac{\varkappa}{m} p_{n-2}^{\mathcal{O}}(\underline{\tilde{\theta}}, \underline{\tilde{z}}) + \tilde{p}^{(1)}(\underline{\theta}) \\ p_n^{\mathcal{O}}(\theta_1 = \theta_n + i\pi, \underline{\tilde{\theta}}, \theta_n; \underline{\tilde{z}}, z_m = \theta_1) = \sigma_{\mathcal{O}1} \dfrac{\varkappa}{m} p_{n-2}^{\mathcal{O}}(\underline{\tilde{\theta}}, \underline{\tilde{z}}) + \tilde{p}^{(2)}(\underline{\theta}) \end{array} \right.$$

where $\underline{\tilde{\theta}} = (\theta_2, \dots, \theta_{n-1})$, $\underline{\tilde{z}} = (z_1, \dots z_{m-1})$ and where $\tilde{p}^{(1,2)}(\underline{\theta})$ are independent of the z's and non-vanishing only for charge-less operators $\mathcal{O}(x)$. The constant $\varkappa$ depends on the coupling and is given by (see formula (B.7) in [5])

$$\varkappa = -\left(F'(0)\right)^2 / \pi \,. \tag{19}$$

(iv') the bound state p-functions are investigated in section 6

(v') $p_n^{\mathcal{O}}(\underline{\theta} + \mu, \underline{z} + \mu) = e^{s\mu} p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$ where s is the 'spin' of the operator $\mathcal{O}(x)$.

As an extension of theorem 4.1 in [5] we prove the following theorem which allows to construct generalized form factors.

Theorem 1 *The co-vector valued function $\mathcal{O}_{1\dots n}(\underline{\theta})$ defined by (13) fulfills the properties (i), (ii) and (iii) on page 7 if the functions $F(\theta), \phi(z)$ and $\tau(z)$ are given by definition (14) – (16) and if the p-function $p_n^{\mathcal{O}}(\underline{\theta}, \underline{z})$ satisfies the conditions $(i') - (iii')$.*

Proof. The properties (i) and (ii) follow as in [5]. The proof of (iii) is also the same as in [5], if the functions $\tilde{p}^{(1)}(\underline{\theta})$ and $\tilde{p}^{(2)}(\underline{\theta})$ in (iii') vanish. For the case of charge-less operators they are in general non-vanishing. Then the proof of (iii) in the form of

$$\operatorname*{Res}_{\theta_{1n} = i\pi} \mathcal{O}_{1\dots n}(\theta_1, \dots, \theta_n) = -2i\, \mathbf{C}_{1n}\, \mathcal{O}_{2\dots n-1}(\theta_2, \dots, \theta_{n-1}) \times \left(\mathbf{1}_{2\dots n-1} - S_{2n} \cdots S_{n-1n}\right)$$

has to be modified as follows: Considering the z_m-integration as in [5] the terms involving $p_{n-2}^{\mathcal{O}}(\underline{\tilde{\theta}}, \underline{\tilde{z}})$ yield the desired result and those proportional to $\tilde{p}^{(1,2)}(\underline{\theta})$ yield terms with a factor $\mathbf{1}_{2\dots n-1} + S_{2n} \cdots S_{n-1n}$ which, however, vanish due to the following lemma.

Lemma 2 *The integral given by (13) and (14) vanishes if the p-function $p(\underline{\theta}, \underline{z})$ is independent of the integration variables z_i and if the number of particles n and the number of C-operators m are related by $n = 2m$ which means that the charge $q = n - 2m$ vanishes.*

This lemma is proven in appendix A.

Remarks:

- The number of C-operators m depends on the charge $q = n - 2m$ of the operator $\mathcal{O}$, e.g. $m = (n-1)/2$ for the soliton field $\psi(x)$ with charge $q = 1$ and $m = n/2$ for charge-less operators like $\overline{\psi}\psi$ or the energy momentum tensor $T^{\mu\nu}$.
- Note that other sine-Gordon form factors can be calculated from the general formula (13) using the bound state formula (iv).
- The general representation of form factors by formula (13) is not specific to the sine-Gordon model. It may be applied to all integrable quantum field theoretic model. So the main task is to solve the corresponding Bethe Ansatz.

4 Examples

In this section we propose the p-functions corresponding to some charge-less local operators. For two-particle form factors the single integration is performed explicitly and compared with known results of [3] and with Feynman graph expansion in lowest order. For 4-particle form factors the double integrals are calculated approximately by expansion in small couplings to lowest order. The results are again checked against Feynman graph expansions. In all cases agreement is obtained.

For the massive Thirring model with the Lagrangians and field equation

$$\mathcal{L}^{MT} = \overline{\psi}(i\gamma\partial - M)\psi - \tfrac{1}{2}g\, j^\mu j_\mu\ , \quad \left(j^\mu = \overline{\psi}\gamma^\mu\psi\right)$$
$$(i\gamma\partial - M)\psi(x) - g\, j^\mu(x)\gamma_\mu\psi(x) = 0$$

we are looking for all matrix elements of the quantum operators corresponding to the following classical fields

1) $\overline{\psi}(x)\psi(x)$
2) $\overline{\psi}(x)\gamma^5\psi(x)$
3) $j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x)$ the topological (electro magnetic) current
4) $T^{\mu\nu}(x) = \frac{i}{2}\overline{\psi}\gamma^\mu \overleftrightarrow{\partial^\nu}\psi - g^{\mu\nu}\mathcal{L}^{MT}$ the energy momentum tensor.

   The light cone components of this tensor are

   $$T^{\pm\pm} = T^{00} \pm 2T^{01} + T^{11} = \overline{\psi}\gamma^\pm \frac{i}{2}\overleftrightarrow{\partial^\pm}\psi$$
   $$T^{+-} = T^{-+} = T^{00} - T^{11} = \overline{\psi}\gamma^+ \frac{i}{2}\overleftrightarrow{\partial^-}\psi - g\, j^\mu j_\mu = M\overline{\psi}\psi$$

   where $\partial^\pm = \partial^0 \pm \partial^1$ and $\gamma^\pm = \gamma^0 \pm \gamma^1$. For the last equality the field equation has been used .

5) The higher conserved currents [32, 33] for $L = 3, 5, \ldots$

   $$J_L^\pm(x) = \begin{cases} i\psi_1^\dagger \left(\partial^+\right)^L \psi_1 + h.c. + O(\psi^4) \\ M\psi_2^\dagger \left(\partial^+\right)^{L-1} \psi_1 + h.c. + O(\psi^4) \end{cases} \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.$$

   A second set of higher conserved currents is obtained by exchanging $\partial^+ \leftrightarrow \partial^-$ and $\psi_1 \leftrightarrow \psi_2$ which we associate to $L = -3, -5, \ldots$.

4.1 Examples of 'p-functions'

In this subsection we propose the p-functions for various local operators. Since the charge of the operators which we consider is zero, the number of integrations m and the number of particles n are related by $m = n/2$ and form factors are non-vanishing only for even number of particles $n = 2, 4, \ldots$. We consider p-functions of the form

$$p_n^{\mathcal{O}}(\underline{\theta}, \underline{z}) = N_n^{\mathcal{O}} \left(p_+^{\mathcal{O}}(P^\mu) \sum_{j=1}^m e^{Lz_j} + p_-^{\mathcal{O}}(P^\mu) \sum_{j=1}^m e^{-Lz_j} \right) \tag{20}$$

where P^μ is the total energy momentum vector of all particles. The integrals in (11) converge for $L < (1/\nu + 1)(n/2 - m + 1) + 1/\nu$. For large values of L the form factors are defined in general as analytic continuations of the integral representation from sufficiently small values of ν to other values. Obviously the p-functions (20) satisfy the conditions $(i') - (iii')$ on page 12. From the property (iii') we obtain the recursion relation for the normalization constants

$$N_n^{\mathcal{O}} = N_{n-2}^{\mathcal{O}} \frac{\varkappa}{m} \quad \Rightarrow \quad N_n^{\mathcal{O}} = N_2^{\mathcal{O}} \frac{1}{m!} \varkappa^{m-1}. \tag{21}$$

The absolute normalizations follow from the two-particle form factors (see eq. (25) below).

We propose that the p-functions of equations (2) on page 4 are associated to the local operators $\mathcal{N}\left[\overline{\psi}\psi\right](x)$, $\mathcal{N}\left[\overline{\psi}\gamma^5\psi\right](x)$, $j^\pm(x)$, $T^{\pm\pm}(x)$, $T^{+-}(x)$ and $J_L^\pm(x)$ where $\pm$ denote the light cone components (e.g. $j^\pm = j^0 \pm j^1$). The vector operator $j^\mu(x) = \mathcal{N}\left[\overline{\psi}\gamma^\mu\psi\right](x)$ is the topological (electro-magnetic) current, $T^{\mu\nu}(x)$ is the energy momentum tensor and the $J_L^\pm(x)$ are the higher conserved currents. The normalization constants are obtained in the next subsection, where we calculate the exact two-particle form factors. The fundamental sine-Gordon bose field $\varphi(x)$ which correspond to the lowest breather is related to the current by Coleman's formula [25]

$$\epsilon^{\mu\nu}\partial_\nu\varphi = -\frac{2\pi}{\beta} j^\mu \quad \text{or} \quad \partial^\pm\varphi = \pm\frac{2\pi}{\beta} j^\pm. \tag{22}$$

This implies the following representation for the p-function

$$p_n^{\varphi}(\underline{\theta}, \underline{z}) = N_n^j \frac{2\pi i}{\beta M} \left(\sum_{i=1}^n e^{\theta} \sum_{i=1}^n e^{-\theta} \right)^{-1} \left(\sum_{i=1}^n e^{-\theta} \sum_{i=1}^m e^{z} + \sum_{i=1}^n e^{\theta} \sum_{i=1}^m e^{-z} \right).$$

Using the integral representation (13) with these p-functions we calculate in the following subsections the exact two-particle form factors and the four-particle form factors in lowest order with respect to the coupling g.

Conservation of higher charges The higher currents satisfy in terms of matrix elements $\mathcal{O}_{1\ldots n}(\underline{\theta}) = \langle\, 0 \,|\, \mathcal{O} \,|\, p_1, \ldots, p_n \,\rangle_{1\ldots n}^{in}$ for all $L \in \mathbb{Z}$ the equation

$$\partial^+ J_L^-(x) + \partial^- J_L^+(x) = 0,$$

such that the higher charges are conserved

$$\frac{d}{dt} Q_L = \frac{d}{dt} \int dx J_L^0(x) = 0.$$

Proof. ¿From the definition we get the correspondence of operators and p-functions

$$\partial^+ J_L^- + \partial^- J_L^+ \leftrightarrow N_n^{J_L} iM \left(\sum_{i=1}^n e^{\theta_i} \sum_{i=1}^n e^{-\theta_i} - \sum_{i=1}^n e^{-\theta_i} \sum_{i=1}^n e^{\theta_i} \right) \sum_{i=1}^m e^{Lz_i} = 0.$$

■

In the following subsection we calculate the charges on 1-particles states and in section 5 on arbitrary n-particles states. It turns out that for even L the charges vanish as in the classical case. The energy momentum tensor $T^{\mu\nu}$ is given by $J_{\pm1}^{\pm}$ and the momentum operator by $P^0 \pm P^1 = Q_{\pm1}$.

4.2 Examples of two particle form factors

Two particle form factors may in general be obtained by diagonalization of the two-particle S-matrix [3]. For several examples we shall show of p-functions that the integral representation in this case reduces to known results for two particle form factors for specific operators. This allows us to confirm the association of p-functions and local operators as proposed above. Also the normalization constants can be calculated. Moreover we check our results in lowest order of perturbation theory.

We consider the form factor for charge-less operators

$$\mathcal{O}_{12}(\theta_1, \theta_2) = \langle\, 0 \,|\, \mathcal{O} \,|\, p_1, p_2 \,\rangle_{12}^{in}.$$

Non-vanishing matrix elements contain one soliton and one anti-soliton. Choosing $n = 2$ and $m = 1$ in the general formula (13) we obtain

$$\mathcal{O}_{12}(\underline{\theta}) = F(\theta_{12}) \int_{\mathcal{C}_{\underline{\theta}}} dz \prod_{i=1}^{2} \phi(\theta_i - z)\, p^{\mathcal{O}}(\underline{\theta}, z)\, \Omega_{12}\, C_{12}(\underline{\theta}, z). \tag{23}$$

The integration can be performed. As a special case we take a p-function of the form (20) for $L = 1$ and prove the following lemma:

Lemma 3 *For the simple p-functions* $e^{\pm z}$ *the integral in (23) can be performed yielding the result*

$$f_{12}^{\pm}(\underline{\theta}) = \frac{2\sinh\frac{1}{2}\theta_{12}}{\nu\varkappa}\, e^{\pm\frac{1}{2}(\theta_1+\theta_2)} \left(\pm \frac{f_+(\theta_{12}) E_{12}^+}{\cosh\frac{1}{2}\theta_{12}} - \frac{f_-(\theta_{12}) E_{12}^-}{\sinh\frac{1}{2}\theta_{12}} \right)$$

where $E^{\pm} = (s \otimes \bar{s} \pm \bar{s} \otimes s)$ *is the symmetric (anti-symmetric) soliton anti-soliton state and the constant* $\varkappa$ *is defined in eq. (19). The functions* $f_{\pm}(\theta)$ *are the positive and negative C-parity two-particle sine-Gordon form factors calculated in [3]*

$$(f_+(\theta)\,,\, f_-(\theta)) = \left(\frac{\tanh\frac{1}{2}(i\pi - \theta)}{\sinh\frac{1}{2\nu}(i\pi - \theta)}\,,\, \frac{1}{\cosh\frac{1}{2\nu}(i\pi - \theta)} \right) F(\theta)$$

with $f_{ss}^{(0)}(\theta)$ *given by eq. (16).*

Proof. We consider the expression

$$
\begin{aligned}
I^{\pm} &= \frac{a(\theta_{12})}{c(\theta_{12})} \int_{C_{\underline{\theta}}} dz\, \tilde{I}(z)\, e^{\pm z}\, \Omega \tilde{C}(\underline{\theta}, z) \\
&= \frac{a(\theta_{12})}{c(\theta_{12})} \int_{C_{\underline{\theta}}} \tilde{I}(z)\, e^{\pm z} \left\{ \frac{b(\theta_1 - z)}{a(\theta_1 - z)} \frac{c(\theta_2 - z)}{a(\theta_2 - z)}\, s \otimes \bar{s} + \frac{c(\theta_1 - z)}{a(\theta_1 - z)}\, \bar{s} \otimes s \right\}
\end{aligned}
$$

with $\tilde{I}(z) = \prod_{i=1,2} \tilde{\phi}(\theta_i - z)$, $\tilde{\phi}(\theta) = \phi(\theta) a(\theta)$ and
$\tilde{C}(\underline{\theta}, z) = C(\underline{\theta}, z) \prod_{i=1,2} 1/a(\theta_i - z)$. Inserting the identities (which follow from Yang-Baxter relations for the soliton S-matrix)

$$
\begin{aligned}
\frac{a(\theta_{12})}{c(\theta_{12})} &= \frac{a(\theta_1 - z)}{c(\theta_1 - z)} \frac{a(\theta_2 - z)}{c(\theta_2 - z)} - \frac{b(\theta_1 - z - 2\pi i)}{c(\theta_1 - z - 2\pi i)} \frac{b(\theta_2 - z)}{c(\theta_2 - z)} \\
&= -\frac{a(\theta_1 - z)}{c(\theta_1 - z)} \frac{a(\theta_2 - z + 2\pi i)}{c(\theta_2 - z + 2\pi i)} + \frac{b(\theta_1 - z)}{c(\theta_1 - z)} \frac{b(\theta_2 - z)}{c(\theta_2 - z)}
\end{aligned}
$$

into the two components of the integral, respectively and using the shift property (18)

$$
\tilde{\phi}(\theta - z - 2\pi i) = \frac{b(\theta - z)}{a(\theta - z)}\, \tilde{\phi}(\theta - z)
$$

we obtain

$$
\begin{aligned}
I^{\pm} &= \left[\int_{C_{\underline{\theta}}} - \int_{C_{\underline{\theta}} + 2\pi i} \right] dz\, \tilde{I}(z)\, e^{\pm z} \left\{ \frac{b(\theta_1 - z)}{c(\theta_1 - z)}\, s \otimes \bar{s} - \frac{a(\theta_2 - z + 2\pi i)}{c(\theta_2 - z + 2\pi i)}\, \bar{s} \otimes s \right\} \\
&= -2\pi i \left[\tilde{I}(z)\, e^{\pm z} \left\{ \frac{b(\theta_1 - z)}{c(\theta_1 - z)}\, s \otimes \bar{s} - \frac{a(\theta_2 - z + 2\pi i)}{c(\theta_2 - z + 2\pi i)}\, \bar{s} \otimes s \right\} \right]_{-\infty}^{\infty} .
\end{aligned}
$$

There are no poles inside the integration contour. However, there are contributions at $\pm\infty$. With the asymptotic formulae for $\operatorname{Re} z \to \pm\infty$

$$
\begin{aligned}
a(\theta - z)/b(\theta - z) &\approx e^{\pm i\pi(1/\nu - 1)} \\
b(\theta - z)/c(\theta - z) &\approx \frac{\mp 1}{2i \sin(\pi/\nu)}\, e^{\mp(\theta - z)/\nu} \\
\tilde{\phi}(\theta - z) &\approx \frac{4}{\sqrt{4\pi\nu\varkappa}} e^{\pm \frac{i}{2}\pi(1/\nu - 1)}\, e^{\pm\frac{1}{2}(1/\nu + 1)(\theta - z - i\pi/2)}.
\end{aligned}
$$

which are derived in appendix B and with $c(\theta)/a(\theta) = i \sin\frac{\pi}{\nu} / \sinh\frac{\pi}{\nu}(i\pi - \theta)$ we obtain the claim. ■

4.2.1 The exact form factors

We use the following conventions for the γ-matrices and the spinors

$$
\begin{aligned}
\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \gamma^1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \gamma^5 = \gamma^0 \gamma^1 \\
u(p) = \sqrt{M} \begin{pmatrix} e^{-\theta/2} \\ e^{\theta/2} \end{pmatrix},\ v(p) = \sqrt{M}\, i \begin{pmatrix} e^{-\theta/2} \\ -e^{\theta/2} \end{pmatrix}.
\end{aligned}
$$

For the examples 1) – 4) above we calculate the two-particle form factors

$$
\mathcal{O}_{s\bar{s}}(\theta_1, \theta_2) = \langle\, 0 \,|\, \mathcal{O}(0) \,|\, p_1, p_2 \,\rangle_{s\bar{s}}^{in}
$$

applying lemma 3 and using the p-functions (2) explicitly[5]

$$\begin{array}{llcl}
1) & \langle\, 0\,|\,\mathcal{N}\left[\overline{\psi}\psi\right](0)\,|\,p_1,p_2\,\rangle_{s\bar{s}}^{in} &=& \bar{v}(\theta_2)u(\theta_1)\,f_+(\theta_{12})/\nu\\
2) & \langle\, 0\,|\,\mathcal{N}\left[\overline{\psi}\gamma^5\psi\right](0)\,|\,p_1,p_2\,\rangle_{s\bar{s}}^{in} &=& \bar{v}(\theta_2)\gamma^5u(\theta_1)\,f_-(\theta_{12})/\nu\\
3) & \langle\, 0\,|\,j^\pm(0)\,|\,p_1,p_2\,\rangle_{s\bar{s}}^{in} &=& \bar{v}(\theta_2)\gamma^\pm u(\theta_1)\,f_-(\theta_{12})\\
4) & \langle\, 0\,|\,T^{\rho\sigma}(0)\,|\,p_1,p_2\,\rangle_{s\bar{s}}^{in} &=& \bar{v}(\theta_2)\gamma^\rho u(\theta_1)\,\frac{1}{2}(p_1^\sigma-p_2^\sigma)\,f_+(\theta_{12})/\nu\,.
\end{array} \tag{24}$$

For the higher conserved currents see the next paragraph. The normalization constants $N_2^{\mathcal{O}}$ in the expressions for the two-particle form factors of (24) have been determined by the following normalization conditions

$$\begin{array}{llcl}
1) & {}_s\langle\, p\,|\,\mathcal{N}\,\left[\overline{\psi}\psi\right](0)\,|\,p\,\rangle_s &=& \bar{u}(\theta)u(\theta)=2M\\
3) & {}_s\langle\, p\,|\,\mathcal{N}\,\left[\overline{\psi}\gamma^\mu\psi\right](0)\,|\,p\,\rangle_s &=& \bar{u}(\theta)\gamma^\mu u(\theta)=2p^\mu\\
4) & {}_s\langle\, p\,|\,T^{\mu\nu}(0)\,|\,p\rangle_s^{in} &=& \bar{u}(\theta)\gamma^\mu u(\theta)p^\nu=2p^\mu p^\nu
\end{array}$$

which are the free field values and which are natural due to the corresponding 'charges' of the operators. The crossing relations and $f_+(i\pi)=\nu$ and $f_-(i\pi)=1$ have been used. Since there is no 'charge' for $\mathcal{N}\left[\overline{\psi}\gamma^5\psi\right](x)$ we take $N_2^{\overline{\psi}\gamma^5\psi}=-N_2^{\overline{\psi}\psi}$ which implies the natural relation ${}_s\langle\, p'\,|\,\mathcal{N}\,\left[\overline{\psi}\gamma^5\psi\right](0)\,|\,p\,\rangle_s\approx\bar{u}(\theta')\gamma^5u(\theta)$ for $\theta'\approx\theta$ for small couplings. This normalization is also consistent with the desired identification (see [25])

$$\mathcal{N}\left[\,\overline{\psi}\left(1\pm\gamma^5\right)\psi\right](x)=-\frac{2\alpha}{M\beta^2}\left(1-\frac{\beta^2}{8\pi}\right):e^{\pm\beta\varphi(x)}:$$

where $:\cdots:$ means normal ordering with respect to the physical vacuum (see also [34, 35]). The constant on the right hand side is obtained from the trace of the energy momentum tensor calculated in [34, 35]. For the cases 3) of the topological (electro-magnetic) current and 4) the energy momentum tensor the given normalization are equivalent to the eigenvalue relations

$$\begin{array}{rcl}
\int dxj^0(x)|\,p\,\rangle_s &=& |\,p\,\rangle_s\\
\int dxT^{\mu0}(x)|\,p\,\rangle_s &=& p^\mu|\,p\,\rangle_s.
\end{array}$$

Finally we obtain from the recursion relation (21) and lemma 3 the normalization constants:

$$N_n^{\overline{\psi}\psi}=-N_n^{\overline{\psi}\gamma^5\psi}=\frac{1}{2\nu}N_n^j=-\frac{1}{M}N_n^T=-iM\frac{1}{4m!}\varkappa^m \tag{25}$$

where the constant $\varkappa$ is defined in eq. (19). Note that these normalizations together with the proposed p-functions (2) imply in particular the quantum version of the classical field operator relation

$$\operatorname{tr}T(x)=T^{+-}(x)=M\mathcal{N}\left[\overline{\psi}\psi\right](x)$$

[5]For the cases 3) and 4) these results agree with those of [2] and [4], respectively, which have been obtained by solving the scalar Watson's equations due to diagonalization of the S-matrix.

for all matrix elements. For the sine-Gordon field Coleman's relation (22) yields [3]

$$\varphi_{s\bar{s}}(\theta_1,\theta_2) = \langle\, 0\,|\,\varphi(0)\,|\,p_1,p_2\,\rangle_{s\bar{s}}^{in} = -\frac{2\pi}{\beta}\frac{f_-(\theta_{12})}{\cosh\frac{1}{2}\theta_{12}}.$$

The higher conserved currents: We will use a generalized version of lemma 3 where $e^{\pm z}$ is replaced by e^{Lz}. The two particle form factors then turn out to be

$$\begin{aligned}\left[J_L^\pm\right]_{12}(\theta_1,\theta_2) &= \langle\, 0\,|\,J_L^\pm\,|\,p_1,p_2\,\rangle_{12}^{in} = \pm N_2^{J_L}\,\sinh\tfrac{1}{2}\theta_{12}\left(\sum_{i=1}^{2}e^{\pm\theta_i}\right)e^{\frac{1}{2}L(\theta_1+\theta_2)}\\ &\times k_L(\theta_{12})\left(\frac{\operatorname{sgn} L}{\cosh\frac{1}{2}\theta_{12}}\,f_+(\theta_{12})E_{12}^+ - \frac{1}{\sinh\frac{1}{2}\theta_{12}}\,f_-(\theta_{12})E_{12}^-\right).\end{aligned} \tag{26}$$

The asymptotic behavior of $\phi(\theta)$ yields the functions (see appendix B)

$$k_L(\theta) = const\sum_{i=0}\sum_{j=0}\delta_{i+j,|L|-1}\,A_iA_j e^{\frac{1}{2}(i-j)\theta}$$

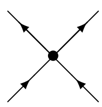
where the constants A_i are also given in appendix B. Only the first term proportional to $f_+(\theta_{12})$ contributes to the higher charges $Q_L = \int dx J_L^0$. With a suitable normalization ($N_2^{J_L}\,k_L(i\pi)\,\nu = M^{L+1}$) and for odd $L = \pm1,\pm3,\ldots.$ one obtains

$$\begin{aligned}{}_s\langle\, p_2\,|\,Q_L\,|\,p_1\,\rangle_s &= 2\pi\delta(p_1^1-p_2^1)\tfrac{1}{2}\left(\left[J_L^+\right]_{s\bar{s}}(\theta_1,\theta_1-i\pi)+\left[J_L^-\right]_{s\bar{s}}(\theta_1,\theta_1-i\pi)\right)\\ &= \langle\, p_2\,|\,p_1\,\rangle\,\left(p_1^+\right)^L.\end{aligned} \tag{27}$$

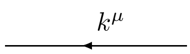
Note that $k_L(i\pi)=0$ for L even.

4.2.2 Feynman graph expansion

From the Lagrangian (3) we get the Feynman rules of figure 2. The two-particle



$$= -ig\gamma^\mu\otimes\gamma_\mu\ ,$$



$$= \frac{i}{\gamma k - M}.$$

Figure 2: *The Feynman rules for the massive Thirring model.*

form factors for charge-less operators of the form $\mathcal{O}(x) = \mathcal{N}\,\overline{\psi}\Gamma\psi$ are given in lowest order perturbation theory by the Feynman graph depicted in figure 3. We consider several examples for Γ:

1) $$\langle\, 0\,|\,\overline{\psi}\psi\,|\,p_1,p_2\,\rangle_{\bar{s}s} = -\,\bar{v}(\theta_1)u(\theta_2) = -2Mi\sinh\tfrac{1}{2}\theta_{12}$$

2) $$\langle\, 0\,|\,\overline{\psi}\gamma^5\psi\,|\,p_1,p_2\,\rangle_{\bar{s}s} = -\,\bar{v}(\theta_1)\gamma^5u(\theta_2) = 2Mi\cosh\tfrac{1}{2}\theta_{12}$$

3) $$\langle\, 0\,|\,\overline{\psi}\gamma^\pm\psi\,|\,p_1,p_2\,\rangle_{\bar{s}s} = -\,\bar{v}(\theta_1)\gamma^\pm u(\theta_2) = \mp 2Mie^{\pm\frac{1}{2}(\theta_1+\theta_2)}$$

4) $$\begin{aligned}\langle\, 0\,|\,\overline{\psi}\gamma^+\tfrac{i}{2}\overset{\leftrightarrow}{\partial}{}^+\psi\,|\,p_1,p_2\,\rangle_{\bar{s}s} &= -\bar{v}(\theta_1)\gamma^+u(\theta_2)\tfrac{1}{2}(p_1-p_2)^+\\ &= 2M^2i\sinh\tfrac{1}{2}\theta_{12}e^{\pm(\theta_1+\theta_2)}\end{aligned}$$

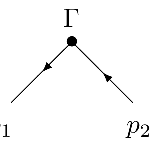


Figure 3: The Feynman graph for two-particle form factors of charge-less operators.

Note that the energy momentum tensor light cone components are $T^{\pm\pm} = \frac{i}{2}\mathcal{N}\left[\overline{\psi}\gamma^{\pm}\overleftrightarrow{\partial}^{\pm}\psi\right]$ and $T^{\pm\mp} = M\mathcal{N}\left[\overline{\psi}\psi\right]$. For all these examples we have agreement with the exact expressions (24) when $g \to 0$ or $\nu \to 1$.

4.3 Examples of 4-particle form factors

4.3.1 Expansion of the exact formula

We investigate the integral (13) (for $n = 4$ and $m = 2$)

$$\mathcal{O}_{\bar{s}\bar{s}ss}(\underline{\theta}) = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \int_{\mathcal{C}_{\underline{\theta}}} dz_2\, h(\underline{\theta},\underline{z})\, p^{\mathcal{O}}(\underline{\theta},\underline{z})\, \Psi_{\bar{s}\bar{s}ss}(\underline{\theta},\underline{z})$$

with the scalar function

$$h(\underline{\theta},\underline{z}) = \prod_{1\le i<j\le 4} F(\theta_{ij}) \prod_{i=1}^{4}\prod_{j=1}^{2}\phi(\theta_i - z_j)\,\tau(z_1 - z_2)\,,$$

and the Bethe Ansatz state component

$$\begin{aligned}\Psi_{\bar{s}\bar{s}ss}(\underline{\theta},\underline{z}) = (\Omega C(\underline{\theta},z_1)C(\underline{\theta},z_2))_{\bar{s}\bar{s}ss} &= \prod_{i=1}^{4}\prod_{j=1}^{2} a(\theta_i - z_j)\\ \times\Big(\tilde{c}(\theta_1 - z_1)\tilde{c}(\theta_2 - z_2) &+ \tilde{b}(\theta_1 - z_1)\tilde{c}(\theta_2 - z_1)\tilde{c}(\theta_1 - z_2)\tilde{b}(\theta_2 - z_2)\Big)\,.\end{aligned}$$

with $\tilde{b} = b/a$, $\tilde{c} = c/a$ for small couplings. We consider first the simple p-functions $\sum_{i=1}^{2} e^{\pm z_i}$ and using the $z_1 \leftrightarrow z_2$ symmetry we calculate the following integral in lowest order with respect to the coupling constant g

$$\begin{aligned} I^{\pm} &= \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \int_{\mathcal{C}_{\underline{\theta}}} dz_2 \left(\prod_{i=1}^{4}\prod_{j=1}^{2}\tilde{\phi}(\theta_i - z_j)\right)\tilde{c}(\theta_1 - z_1)\,\tilde{c}(\theta_2 - z_2)\\ &\quad\times\Big(1 + \tilde{b}(\theta_1 - z_2)\tilde{b}(\theta_2 - z_1)\Big)\,\tau(z_1 - z_2)\left(e^{\pm z_1} + e^{\pm z_2}\right)\\ &= \pm\frac{8ig\sinh\frac{1}{2}\theta_{12}\sinh\frac{1}{2}\theta_{34}}{\prod_{i<j}(\sinh\frac{1}{2}\theta_{ij}\cosh\frac{1}{2}\theta_{ij})}\frac{e^{\pm\frac{1}{2}(\theta_1+\theta_2)}}{e^{\pm\frac{1}{2}(\theta_3+\theta_4)}}\sum_{i=1}^{4} e^{\pm\theta_i} + O(g^2)\,. \end{aligned}\tag{28}$$

The derivation of this result is quite involved. A sketch of the calculation is delegated to appendix C.

Finally we use the p-functions (2) and the normalization constants given by eq. (25) and $F(\theta) = -i \sinh \frac{1}{2}\theta + O(g)$ and obtain the four particle form factors

$$\mathcal{O}_{\bar{s}\bar{s}ss}(\underline{\theta}) = \langle\, 0 \,|\, \mathcal{O} \,|\, p_1, p_2, p_3, p_4 \,\rangle^{in}_{\bar{s}\bar{s}ss}$$

for the various operators in lowest order in the coupling g as

$$\begin{aligned}
\left[\overline{\psi}\psi\right]_{\bar{s}\bar{s}ss}(\underline{\theta}) &= -\frac{1}{2} g M \frac{\sinh \frac{1}{2}\theta_{12} \sinh \frac{1}{2}\theta_{34}}{\prod_{i<j} \cosh \frac{1}{2}\theta_{ij}} \cosh \tfrac{1}{2}\left(\theta_{13} + \theta_{24}\right) \sum_{i=1}^{4} e^{\theta_i} \sum_{i=1}^{4} e^{-\theta_i} \\
\left[\overline{\psi}\gamma^5\psi\right]_{\bar{s}\bar{s}ss}(\underline{\theta}) &= \frac{1}{2} g M \frac{\sinh \frac{1}{2}\theta_{12} \sinh \frac{1}{2}\theta_{34}}{\prod_{i<j} \cosh \frac{1}{2}\theta_{ij}} \sinh \tfrac{1}{2}\left(\theta_{13} + \theta_{24}\right) \sum_{i=1}^{4} e^{\theta_i} \sum_{i=1}^{4} e^{-\theta_i} \\
\left[j^{\pm}\right]_{\bar{s}\bar{s}ss}(\underline{\theta}) &= \mp g M \frac{\sinh \frac{1}{2}\theta_{12} \sinh \frac{1}{2}\theta_{34}}{\prod_{i<j} \cosh \frac{1}{2}\theta_{ij}} \sinh \tfrac{1}{2}\left(\theta_{13} + \theta_{24}\right) \sum_{i=1}^{4} e^{\pm\theta_i} \\
\left[T^{\pm\pm}\right]_{\bar{s}\bar{s}ss}(\underline{\theta}) &= \frac{1}{2} g M^2 \frac{\sinh \frac{1}{2}\theta_{12} \sinh \frac{1}{2}\theta_{34}}{\prod_{i<j} \cosh \frac{1}{2}\theta_{ij}} \cosh \tfrac{1}{2}\left(\theta_{13} + \theta_{24}\right) \left(\sum_{i=1}^{4} e^{\pm\theta_i}\right)^2 \\
\left[T^{+-}\right]_{\bar{s}\bar{s}ss}(\underline{\theta}) &= -\frac{1}{2} g M^2 \frac{\sinh \frac{1}{2}\theta_{12} \sinh \frac{1}{2}\theta_{34}}{\prod_{i<j} \cosh \frac{1}{2}\theta_{ij}} \cosh \tfrac{1}{2}\left(\theta_{13} + \theta_{24}\right) \sum_{i=1}^{4} e^{\theta_i} \sum_{i=1}^{4} e^{-\theta_i}
\end{aligned}$$

which agrees with the Feynman graph result calculated in the next subsection.

4.3.2 Feynman graph expansion

The graphs of figure 4 give the lowest order contributions to the matrix element

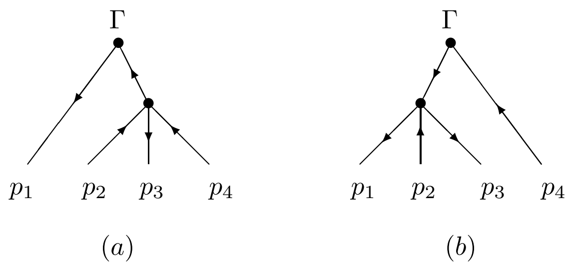


Figure 4: Feynman graphs for four-particle form factors of charge-less operators.

$$\begin{aligned}
\langle\, 0 \,|\, \overline{\psi}\Gamma\psi \,|\, p_1, p_2, p_3, p_4 \,\rangle^{in}_{\bar{s}s\bar{s}s} &= -i\tfrac{1}{2} g \,\langle\, 0 \,|\, \overline{\psi}\Gamma\psi \int d^2x j^{\mu} j_{\mu} |\, p_1, p_2, p_3, p_4 \,\rangle_{\bar{s}s\bar{s}s} \\
&= -ig\left[\left(G_a + G_b\right) - (1 \leftrightarrow 3)\right] - (2 \leftrightarrow 4)
\end{aligned}$$

up to terms of order $O(g^2)$. We obtain

$$\begin{aligned}
G_a &= \bar v_1 \Gamma \frac{i}{\gamma(p_2+p_3+p_4)-M} \mathbf{1} \gamma^\mu u_2\; \bar v_3 \gamma_\mu u_4 \\
&= i\frac{1}{2M} \bar v_1 \Gamma \frac{\gamma(p_2+p_3+p_4)+M}{(p_2+p_3+p_4)^2-M^2} u_3\; \bar u_3 \gamma^\mu u_2\; \bar v_3 \gamma_\mu u_4 + symm \\
G_b &= \bar v_1 \gamma^\mu u_2\; \bar v_3 \gamma_\mu \mathbf{1} \frac{i}{\gamma(-p_1-p_2-p_3)-M} \Gamma u_4 \\
&= -i\frac{1}{2M} \bar v_1 \gamma^\mu u_2\; \bar v_3 \gamma_\mu v_2\; \bar v_2 \frac{\gamma(-p_1-p_2-p_3)+M}{(p_1+p_2+p_3)^2-M^2} \Gamma u_4 + symm
\end{aligned}$$

with $u_i = u(\theta_i)$ etc. The relation $u\bar u - v\bar v = 2M\mathbf{1}$ has been used. The terms *symm* vanish after anti-symmetrization. We obtain for $\Gamma = \mathbf{1}, \gamma^5, \gamma^\pm, \gamma^\rho \frac{i}{2}\overleftrightarrow{\partial^\sigma}$

$$\begin{aligned}
G_a &= Mi \tanh\tfrac{1}{2}\theta_{24} \frac{\sinh\frac{1}{2}\theta_{12}}{\cosh\frac{1}{2}\theta_{34}} - (2 \leftrightarrow 4) \\
G_b &= Mi \tanh\tfrac{1}{2}\theta_{13} \frac{\sinh\frac{1}{2}\theta_{14}}{\cosh\frac{1}{2}\theta_{23}} - (1 \leftrightarrow 3)
\end{aligned}$$

$$\begin{aligned}
G_a^5 &= -Mi \tanh\tfrac{1}{2}\theta_{24} \frac{\cosh\frac{1}{2}\theta_{12}}{\cosh\frac{1}{2}\theta_{34}} - (2 \leftrightarrow 4) \\
G_b^5 &= -Mi \tanh\tfrac{1}{2}\theta_{13} \frac{\cosh\frac{1}{2}\theta_{14}}{\cosh\frac{1}{2}\theta_{23}} - (1 \leftrightarrow 3),
\end{aligned}$$

$$\begin{aligned}
G_a^\pm &= \pm Mi \tanh\tfrac{1}{2}\theta_{24} \frac{e^{\pm\frac{1}{2}(\theta_1+\theta_2)}}{\cosh\frac{1}{2}\theta_{34}} - (2 \leftrightarrow 4) \\
G_b^\pm &= \pm Mi \tanh\tfrac{1}{2}\theta_{13} \frac{e^{\pm\frac{1}{2}(\theta_1+\theta_4)}}{\cosh\frac{1}{2}\theta_{23}} - (1 \leftrightarrow 3),
\end{aligned}$$

$$\begin{aligned}
G_a^{\rho\sigma} &= \rho M^2 i \tanh\tfrac{1}{2}\theta_{24} e^{\rho\frac{1}{2}(\theta_1+\theta_2)} \left(-\sigma \frac{\sinh\frac{1}{2}\theta_{12}}{\cosh\frac{1}{2}\theta_{34}} e^{\sigma\frac{1}{2}(\theta_1+\theta_2)} + e^{\sigma\frac{1}{2}(\theta_3+\theta_4)} \right) \\
&\qquad -(2 \leftrightarrow 4) \\
G_b^{\rho\sigma} &= \rho M^2 i \tanh\tfrac{1}{2}\theta_{13} e^{\rho\frac{1}{2}(\theta_1+\theta_4)} \left(-\sigma \frac{\sinh\frac{1}{2}\theta_{14}}{\cosh\frac{1}{2}\theta_{23}} e^{\sigma\frac{1}{2}(\theta_1+\theta_4)} - e^{\sigma\frac{1}{2}(\theta_2+\theta_3)} \right) \\
&\qquad -(1 \leftrightarrow 3)
\end{aligned}$$

which gives after anti-symmetrization the same expression as the one calculated from the integral representation of the exact form factors (exchanging $2 \leftrightarrow 3$ and the sign due fermi statistics). In particular for the energy momentum tensor we obtain with $T^{++} = \overline{\psi}\gamma^+ \frac{i}{2}\overleftrightarrow{\partial^+}\psi$ in lowest order in the coupling g

$$\begin{aligned}
&\langle\, 0 \,|\, T^{++} \,|\, p_1, p_2, p_3, p_4 \,\rangle^{in}_{\bar s s \bar s s} \\
&\qquad = -\frac{1}{2} g M^2 \frac{\sinh\frac{1}{2}\theta_{13} \sinh\frac{1}{2}\theta_{24} \cosh\frac{1}{2}(\theta_{12}+\theta_{34})}{\prod_{i<j}\cosh\frac{1}{2}\theta_{ij}} \left(\sum_{i=1}^{4} e^{\theta_i} \right)^2 .
\end{aligned}$$

Furthermore we have agreement with the classical relation for the trace of the energy momentum tensor: $T^{+-} = \overline{\psi}\gamma^+ \frac{i}{2}\overleftrightarrow{\partial}^- \psi - g\, j^\mu j_\mu = M\overline{\psi}\psi$ since the Feynman graph calculation implies

$$
\begin{aligned}
\langle\, 0 \,|\, \overline{\psi}\gamma^+ \frac{i}{2}\overleftrightarrow{\partial}^- \psi - M\overline{\psi}\psi \,|\, p_1, p_2, p_3, p_4 \,\rangle^{in}_{\bar{s}s\bar{s}s} &= 16gM^2 \sinh\frac{1}{2}\theta_{13} \sinh\frac{1}{2}\theta_{24} \\
&= g\langle\, 0 \,|\, j^\mu j_\mu \,|\, p_1, p_2, p_3, p_4 \,\rangle^{in}_{\bar{s}s\bar{s}s}
\end{aligned}
$$

in the coupling g as it should be.

4.3.3 Asymptotic behavior

We are interested in the asymptotic behavior of form factors when one or more rapidities tend to infinity. In perturbation theory for pure bosonic models one may use Weinberg's power counting theorem for Feynman graphs [35][6]. For the exponentials of the boson field $\mathcal{O} = \mathcal{N} e^{i\gamma\varphi}$ this yields in particular the asymptotic behavior

$$
\mathcal{O}_n(\theta_1,\theta_2,\dots) = \mathcal{O}_1(\theta_1)\,\mathcal{O}_{n-1}(\theta_2,\dots) + O(e^{-\operatorname{Re}\theta_1}) \tag{29}
$$

as $\operatorname{Re}\theta_1 \to \infty$ in any order of perturbation theory. This behavior is also assumed to hold for the exact form factors and it was used e.g. in [34] to obtain the normalization of exponentials of fields. For fermionic models the asymptotic behavior is more complicated. As an example we investigate a component of the four-particle form factor of the operators $\mathcal{O}^\pm = \overline{\psi}\left(1 \pm \gamma^5\right)\psi$ for the massive Thirring model

$$
\mathcal{O}^\pm_{\bar{s}\bar{s}ss}(\underline{\theta}) = \langle\, 0 \,|\, \mathcal{O}^\pm(0) \,|\underline{p}\,\rangle^{in}_{\bar{s}\bar{s}ss} = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \int_{\mathcal{C}_{\underline{\theta}}} dz_2\, h(\underline{\theta},\underline{z})\, p^\pm(\underline{\theta},\underline{z})\, \Psi_{\bar{s}\bar{s}ss}(\underline{\theta},\underline{z})
$$

with the p-functions of eqs. (2,25)

$$
p^\pm(\underline{\theta},\underline{z}) = \mp N \sum_{i=1}^4 e^{\pm\theta_i} \sum_{i=1}^2 e^{\mp z_i}\,.
$$

After some calculation we finally obtain the asymptotic behavior for $\operatorname{Re}\theta_1 \to \infty$

$$
\mathcal{O}^\pm_{\bar{s}\bar{s}ss}(\underline{\theta}) \approx const^\pm\, e^{\frac{1}{4}(3-1/\nu)\theta_1} \times \int_{\mathcal{C}_{\underline{\theta}'}} dz\, h(\underline{\theta}',z)\, q^\pm(\underline{\theta}',z)\, \Psi_{\bar{s}ss}(\underline{\theta}',z)
$$

with $\underline{\theta}' = (\theta_2,\theta_3,\theta_4)$ and

$$
\begin{aligned}
const^\pm &= const \int_{\mathcal{C}_0} dz\, e^{-\frac{1}{2}(1/\nu\pm1)z} \phi(-z)\, c(-z) \\
q^\pm(\underline{\theta}',z) &= e^{-\frac{1}{2}(1/\nu+1)z} \prod_{i=2}^4 e^{\frac{1}{4}(1/\nu+1)\theta_i} \begin{cases} e^{-z} \\ e^{-\theta_2} + e^{-\theta_3} + e^{-\theta_4} \end{cases} \\
\Psi_{\bar{s}ss}(\underline{\theta}',z) &= c(\theta_2 - z)\, a(\theta_3 - z)\, a(\theta_4 - z)
\end{aligned}
$$

Obviously this is not of the form given by relation (29), in particular the functions $q^\pm(\underline{\theta}',z)$ are not valid p-functions since they do not satisfy the conditions on page 12. This means that they do not correspond to local operators.

[6] This type of arguments has also been used in [3, 36, 37, 38].

5 “Crossing”

5.1 General crossing relations

In order to obtain the general matrix element ${}^{out}\langle\, \phi' \,|\, \mathcal{O}(x) \,|\, \phi \,\rangle^{in}$ of a local operator from that case where the state ϕ' is the vacuum one uses ‘crossing’. This means one shifts in the matrix element particles from the right hand side to the left hand side. It is well known from the theory of Feynman graphs and more general also from LSZ-reduction formulas, that these shifts are related to analytic continuation. We will now derive by means of LSZ-assumptions and ‘maximal analyticity’ a formula[7] which gives a general matrix element of a local operator $\mathcal{O}(x)$ in terms of an analytic continuation of the form factor function $\mathcal{O}_{1\dots n}(\underline{\theta})$. As a generalization of the co-vector valued function $\mathcal{O}_{1\dots n}(\underline{\theta})$ we introduce the short notation $\mathcal{O}_I^J(\underline{\theta}'_J;\underline{\theta}_I)$ given as follows.

Let the array of indices $I = (i_1,\dots,i_{|I|})$ denote the factors of the tensor product of vector spaces $V_I = V_{i_1\dots i_{|I|}} = V_{i_1}\otimes\cdots\otimes V_{i_{|I|}}$ and correspondingly for J. For a local operator $\mathcal{O}(x)$ and for ordered sets of rapidities $\theta_{i_1} > \cdots > \theta_{i_{|I|}}$ and $\theta'_{j_1} < \cdots < \theta'_{j_m}$ we write

$$\mathcal{O}_I^J(\underline{\theta}'_J;\underline{\theta}_I) := {}^{out}\langle j_{|J|}(p'_{j_{|J|}}),\dots,j_1(p'_{j_1}) \,|\, \mathcal{O}(0) \,|\, i_1(p_{i_1}),\dots,i_{|I|}(p_{i_{|I|}})\,\rangle^{in} \tag{30}$$

where $\underline{\theta}_I = (\theta_{i_1},\dots,\theta_{i_{|I|}})$ and $\underline{\theta}'_J = (\theta'_{j_1},\dots,\theta'_{j_{|J|}})$. The function $\mathcal{O}_I^J(\underline{\theta}'_J;\underline{\theta}_I)$ intertwines the spaces $V_I \to V_J$ and may be depicted as in figure 5. Similar to

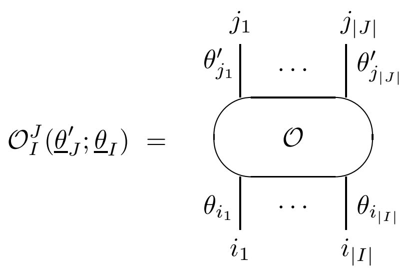


Figure 5: The general matrix element of a local operator.

$\mathcal{O}_{1\dots n}(\underline{\theta})$ this function is given for general order of the rapidities by the symmetry property (i) for both the *in*- and *out*-states which takes the general form:

$$\mathcal{O}_I^J(\underline{\theta}'_J;\underline{\theta}_I) = \mathcal{O}_K^J(\underline{\theta}'_J;\underline{\theta}_K)\dot{S}_I^K(\underline{\theta}_I) = \dot{S}_K^J(\underline{\theta}'_K)\mathcal{O}_I^K(\underline{\theta}'_K;\underline{\theta}_I).$$

For the *connected* contributions this follows again from analytic continuation $\theta_{i_1} > \theta_{i_2} \to \theta_{i_1} < \theta_{i_2}$ which can be proven similarly as for $\mathcal{O}_{1\dots n}(\underline{\theta})$ [5]. For the disconnected contributions this may be considered as a convenient definition. As a generalization of the two-particle S-matrix (including the statistics factor) $\dot{S}_{12}(\theta_1,\theta_2)$ we have introduced the more general object $\dot{S}_I^J(\underline{\theta}_I)$ given by the following definition.

Definition 1 *Let $J = \pi(I)$ be a permutation of I. Then $\dot{S}_I^J(\underline{\theta}_I)$ is the matrix representation of the permutation group $\mathcal{S}_{|I|}$ generated by the simple transposi-*

[7] In [4] a similar formula was proposed which differs from the results of this paper by sign factors. Our proof of the crossing formula follows that of [24].

tions $\sigma_{ij} : i \leftrightarrow j$ for any pair of nearest neighbor indices $i, j \in I$ as[8]

$$\sigma_{ij} \to \dot{S}_I^{\sigma_{ij}(I)}(\underline{\theta}_I) = \dot{S}_{ij}(\theta_{ij})$$

Because of the Yang-Baxter relation and unitarity of the S-matrix the representation $\dot{S}_I^{\pi(I)}(\underline{\theta}_I)$ for all $\pi \in \mathcal{S}_{|I|}$ is well defined. We will also use the notation

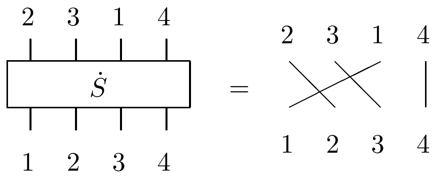
$$\dot{S}_I^{KM}(\underline{\theta}_I) = \dot{S}_I^{\pi(I)}(\underline{\theta}_I)$$

if π is that permutation which reorders the array I such that it coincides with the combined arrays of K and M which means that

$$\pi(I) = KM = (k_1, \ldots, k_{|K|}, m_1, \ldots, m_{|M|})\,.$$

As an example consider the case $I = (1,2,3,4), K = (2,3)$ and $M = (1,4)$

$$\dot{S}_{1234}^{2314}(\theta_1, \theta_2, \theta_3, \theta_4) = \dot{S}_{13}(\theta_{13})\dot{S}_{12}(\theta_{12})$$



Similarly $\dot{S}_{LM}^{J}(\underline{\theta}_L, \underline{\theta}_M)$ is defined as an inverse by the formula $\dot{S}_{LM}^{J}(\underline{\theta}_L, \underline{\theta}_M)\dot{S}_J^{LM}(\underline{\theta}_J) = \delta_J^J$ where δ_J^J is the unit matrix in the vector space V_J.

Analogously to eq. (30) for the general matrix $\mathcal{O}_I^J(\underline{\theta}'_J; \underline{\theta}_I)$ for local operators $\mathcal{O}(x)$ we write for the unit operator

$$\begin{aligned}
\mathbf{1}_M^N(\underline{\theta}'_N; \underline{\theta}_M) &= {}^{out}\langle n_{|N|}(p'_{n_{|N|}}), \ldots, n_1(p'_{n_1}) \mid m_1(p_{m_1}), \ldots, m_{|M|}(p_{m_{|M|}}) \rangle^{in} \\
&= \dot{S}_M^N(\underline{\theta}_M) \prod_{i=1}^{|M|} 4\pi\delta(\theta'_{n_{|N|}-n_i+1} - \theta_{m_i})
\end{aligned}$$

if the rapidities are ordered as $\theta_{m_1} > \cdots > \theta_{m_{|M|}}$ and $\theta'_{n_1} < \cdots < \theta'_{n_{|N|}}$ and if N is the completely reordered array M. This object has obviously no analytic properties, since it is completely disconnected. However, we define it for other orders of the rapidities again by the form factor property (i)

$$\mathbf{1}_M^N(\underline{\theta}'_N; \underline{\theta}_M) = \mathbf{1}_K^N(\underline{\theta}'_N; \underline{\theta}_K)\dot{S}_M^K(\underline{\theta}_M) = \dot{S}_K^N(\underline{\theta}'_K)\mathbf{1}_M^K(\underline{\theta}'_K; \underline{\theta}_M).$$

This implies in particular that

$$\mathbf{1}_M^N(\underline{\theta}'_N; \underline{\theta}_M) = \delta_M^N \prod_{i=1}^{|M|} 4\pi\delta(\theta'_{n_i} - \theta_{m_i})$$

[8] Note that this definition is quite analogous to that of representations of the braid group by means of spectral parameter independent R-matrices.

for $\theta_{m_1} > \cdots > \theta_{m_{|M|}}$ and $\theta'_{n_1} > \cdots > \theta'_{n_{|N|}}$. Here δ_M^N is the unit matrix in the space $V_M = V_N$. For example if $\theta_1 > \theta_2$ and $\theta'_1 > \theta'_2$ one has the two cases

$$\begin{aligned}
\mathbf{1}_{\alpha\beta}^{\alpha'\beta'}(\theta'_1, \theta'_2; \theta_1, \theta_2) &= {}^{in}\langle\, \beta'(\theta'_2), \alpha'(\theta'_1) \,|\, \alpha(\theta_1), \beta(\theta_2)\,\rangle^{in} \\
&= \delta_{\alpha\alpha'}\delta_{\beta\beta'} \prod_{i=1}^{2} 4\pi\delta(\theta'_i - \theta_i) \\
\mathbf{1}_{\alpha\beta}^{\beta'\alpha'}(\theta'_2, \theta'_1; \theta_1, \theta_2) &= {}^{out}\langle\, \alpha'(\theta'_1), \beta'(\theta'_2) \,|\, \alpha(\theta_1), \beta(\theta_2)\,\rangle^{in} \\
&= \dot{S}_{\alpha\beta}^{\beta'\alpha'}(\theta_1, \theta_2) \prod_{i=1}^{2} 4\pi\delta(\theta'_i - \theta_i).
\end{aligned}$$

Theorem 4 *For any local operator $\mathcal{O}(x)$ the 'intertwiner valued' function $\mathcal{O}_I^J(\underline{\theta}'_J; \underline{\theta}_I)$ defined by eq. (30) satisfies the general crossing relations*[9]

$$\begin{aligned}
&\mathcal{O}_I^J(\underline{\theta}'_J; \underline{\theta}_I) \\
&= \sigma_{\mathcal{O}J} \sum_{\substack{L\cup N=J \\ K\cup M=I}} \dot{S}_{NL}^J(\underline{\theta}'_N, \underline{\theta}'_L)\, \mathbf{1}_M^N(\underline{\theta}'_N, \underline{\theta}_M)\, \mathbf{C}^{L\bar{L}} \mathcal{O}_{\bar{L}K}(\underline{\theta}'_{\bar{L}} + i\pi_-, \underline{\theta}_K)\, \dot{S}_I^{MK}(\underline{\theta}_I) \\
&= \sum_{\substack{L\cup N=J \\ K\cup M=I}} \dot{S}_{LN}^J(\underline{\theta}'_L, \underline{\theta}'_N)\, \mathcal{O}_{K\bar{L}}(\underline{\theta}_K, \underline{\theta}'_{\bar{L}} - i\pi_-) \mathbf{C}^{\bar{L}L}\, \mathbf{1}_M^N(\underline{\theta}'_N, \underline{\theta}_M)\, \dot{S}_I^{KM}(\underline{\theta}_I)
\end{aligned} \tag{31}$$

where K, L, M, N and $\underline{\theta}_K, \underline{\theta}_L, \underline{\theta}_M, \underline{\theta}_N$ are defined analogously to I and $\underline{\theta}_I$. However, $\bar{L} = (\bar{l}_{|L|}, \ldots, \bar{l}_1)$ and $\underline{\theta}_{\bar{L}} = (\theta_{\bar{l}_{|L|}}, \ldots, \theta_{\bar{l}_1})$ where the bar denotes the anti-particles and $\mathbf{C}^{\bar{L}L}$ is a multi-particle charge conjugation matrix. The general crossing relations are depicted in figure 6.

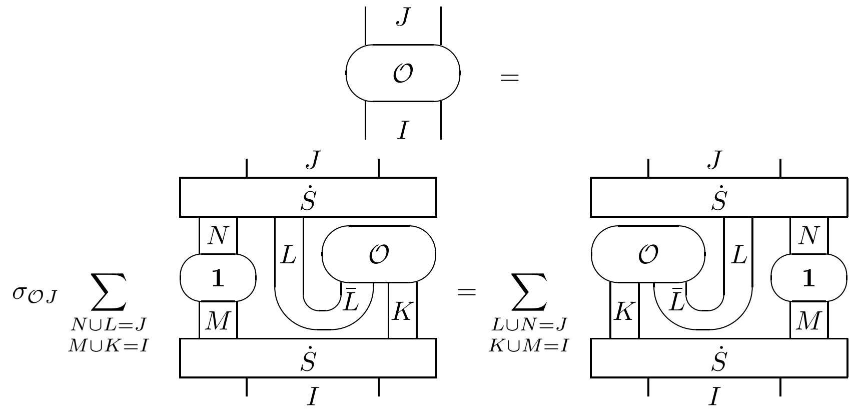


Figure 6: The general crossing relations

[9] These crossing formulae are generalizations of those in [24].

Proof. In [5] the crossing formula was proven for the case of only one out-going particle $|J| = 1$ using LSZ-reduction formulae and the assumption of maximal analyticity. General LSZ-reduction formulae for bosons take the form

$$
\begin{aligned}
{}^{out}\langle\, \phi' \,|\, \mathcal{O} \,|\, p, \phi \,\rangle^{in}_{\alpha} = {}&{}^{out}\langle\, \phi' \,|\, a^{out\,\dagger}_{\alpha}(p)\, \mathcal{O} \,|\, \phi \,\rangle^{in} \\
&+ i \int d^2x \; {}^{out}\langle\, \phi' \,|\, T\left[\mathcal{O} j^{\dagger}_{\alpha}(x)\right] \,|\, \phi \,\rangle^{in}\, e^{-ipx}
\end{aligned}
$$

$$
\begin{aligned}
{}^{out}_{\bar\alpha}\langle\, \phi', p' \,|\, \mathcal{O} \,|\, \phi \,\rangle^{in} = {}&{}^{out}\langle\, \phi' \,|\, \mathcal{O}\, a^{in}_{\alpha}(p') \,|\, \phi \,\rangle^{in} \\
&+ i \int d^2x \; {}^{out}\langle\, \phi' \,|\, T\left[\mathcal{O} j^{\dagger}_{\alpha}(x)\right] \,|\, \phi \,\rangle^{in}\, e^{ip'x}
\end{aligned}
$$

and for fermions the form

$$
\begin{aligned}
{}^{out}\langle \phi' \,|\, \mathcal{O} \,|\, p, \phi \,\rangle^{in}_{\alpha} = {}&\sigma_{\mathcal{O}\alpha} \;{}^{out}\langle \phi' \,|\, a^{out\,\dagger}_{\alpha}(p)\, \mathcal{O} \,|\, \phi \,\rangle^{in} \\
&+ i \int d^2x \; {}^{out}\langle\, \phi' \,|\, T\,[\mathcal{O} \bar{j}_{\alpha}(x)] \,|\, \phi \,\rangle^{in}\, u(\theta)\, e^{-ipx}
\end{aligned}
$$

$$
\begin{aligned}
{}^{out}_{\bar\alpha}\langle \phi', p' \,|\, \mathcal{O} \,|\, \phi \rangle^{in} = {}&\sigma_{\mathcal{O}\bar\alpha} \;{}^{out}\langle\, \phi' \,|\, \mathcal{O}\, a^{in}_{\alpha}(p') \,|\, \phi \,\rangle^{in} \\
&- i\, \sigma_{\mathcal{O}\bar\alpha} \int d^2x \; {}^{out}\langle\, \phi' \,|\, T\,[\mathcal{O} \bar{j}_{\alpha}(x)] \,|\, \phi \,\rangle^{in}\, v(\theta')\, e^{ip'x}\,.
\end{aligned}
$$

Using these relations one can derive similar crossing formulae for the general case that $\phi' \neq \emptyset$. In these more general formulae one particle (from several particles) in the *out*-state is crossed to the *in*-state. Iterating these relations one obtains the general crossing relations. ■

Remarks:

1. Note that the equivalence of both formulations of the crossing relation follows from the properties $(i) - (iii)$.
2. As was shown in [5] the properties (ii) and (iii) on page 12 follow from the crossing formula for $|J| = 1$.

5.2 Eigenvalues of higher charges

In this subsection and as a simple application of the general crossing formula we investigate the higher conservation laws given by the currents $J^{\mu}_{L}(x)$.

Proposition 5 *In terms of the matrix elements* $\mathcal{O}_{1\ldots n}(\underline{\theta}) = \langle\, 0 \,|\, \mathcal{O} \,|\, p_1, \ldots, p_n \,\rangle^{in}_{1\ldots n}$ *the higher charges*

$$
Q_L = \int dx J^0_L(x) = \int dx \tfrac{1}{2}\left(J^+_L(x) + J^-_L(x)\right)
$$

satisfy for odd $L = \pm 1, \pm 3, \ldots$ *the eigenvalue equation*

$$
\left(Q_L - \sum_{i=1}^{n} \left(p^+_i\right)^L \right) |\, p_1, \ldots, p_n \rangle^{in}_{1\ldots n} = 0
$$

for suitable normalizations constants $N^{J_L}_n$. *For even* L *the charges vanish as in the classical case.*

Proof. As a generalization of equation (27) for $n = 1$ we prove for arbitrary $n = |I|$ and $n' = |J|$ that

$$[Q_L]_I^J(\underline{\theta}';\underline{\theta}) = \sum_{i=1}^{n} \left(p_i^+\right)^L \mathbf{1}_I^J(\underline{\theta}';\underline{\theta}). \tag{32}$$

First we show that for $n + n' > 2$ the connected part of the matrix element $[Q_L]_{1\ldots n}^{1'\ldots n'}(\underline{\theta}',\underline{\theta})$ vanishes. This connected part is obtained by the analytic continuation $\left[J_L^0\right]_{n'\ldots 1' 1\ldots n}(\underline{\theta}' + i\pi,\underline{\theta})$. From the correspondence of operator and p-function

$$Q_L = \int dx J_L^0(x) \leftrightarrow 2\pi\delta(P'-P) N_{n'+n}^{J_L} \sum_{\pm} \tfrac{\mp 1}{2} \left(\sum_{i=1}^{n'} e^{\pm\theta_i'} - \sum_{i=1}^{n} e^{\pm\theta_i}\right) \sum_{i=1}^{r} e^{Lz_i}$$

the claim follows since for $n' + n > 2$ there are no poles which may cancel the zero at $P' = P$ where $P^{(\prime)} = \sum p_i^{(\prime)}$. Note that only for $n = n' = 1$ the factor $1/\cosh\frac{1}{2}\theta_{12}$ in (26) cancels the zero. Therefore contributions to (32) come from disconnected parts which contain (analytically continuated) two-particle form factors (c.f. eq. (27))

$$[Q_L]_i^j(\theta_j,\theta_i) = \left(p_i^+\right)^L \mathbf{1}_i^j(\theta_j,\theta_i)\,.$$

It follows that in the general crossing formula only those terms with $K = \{i\}\,, M = I\setminus\{i\}\,, L = \{j\}\,, N = J\setminus\{j\}$ contribute

$$\begin{aligned}
[Q_L]_I^J(\underline{\theta}';\underline{\theta}) &= \sum_{i=1}^{n}\sum_{j=1}^{n} \dot S_{LN}^J(\underline{\theta}_L',\underline{\theta}_N')\,\left(p_i^+\right)^L\,\mathbf{1}_i^j(\theta_j,\theta_i)\,\mathbf{1}_M^N(\underline{\theta}_N',\underline{\theta}_M)\,\dot S_I^{KM}(\underline{\theta}) \\
&= \sum_{i=1}^{n} \left(p_i^+\right)^L \mathbf{1}_I^J(\underline{\theta}';\underline{\theta})
\end{aligned}$$

where the obvious relation $\sum_{j=1}^n \dot S_{LN}^J(\underline{\theta}_L',\underline{\theta}_N')\,\mathbf{1}_i^j(\theta_j,\theta_i)\,\mathbf{1}_M^N(\underline{\theta}_N',\underline{\theta}_M)\,\dot S_I^{KM}(\underline{\theta}) = \mathbf{1}_I^J(\underline{\theta}';\underline{\theta})$ has been used. ■

6 Bound states

Before we define and investigate the properties of bound state form factors we recall some facts on bound state S-matrices and define the "bound state intertwiners". The two-particle S-matrix satisfies real analyticity (5) unitarity (7), crossing (8), the Yang-Baxter relations (6) and the permutation property at vanishing argument (9).

Let the two particles labeled by α and β of mass m_α and m_β, respectively form a bound state labeled by γ of mass m_γ. The mass of the bound state γ is given by

$$m_\gamma = \sqrt{m_\alpha^2 + m_\beta^2 + 2m_\alpha m_\beta \cos u_{\alpha\beta}^\gamma}\quad,\quad (0 < u_{\alpha\beta}^\gamma < \pi).$$

where $u_{\alpha\beta}^\gamma$ is the so called fusion angle and $iu_{\alpha\beta}^\gamma$ is equal to the pure imaginary relative rapidity of the constituents α and β. The two-particle S-matrix may be diagonalized

$$\dot{S}^{\delta\gamma}_{\alpha\beta}(\theta) = \sum_{\epsilon} \varphi^{\delta\gamma}_{\epsilon}(\theta)\, \dot{S}(\alpha,\beta,\epsilon,\theta)\, \varphi^{\epsilon}_{\alpha\beta}(\theta) \tag{33}$$

where the projections onto the eigenspaces (labeled by ϵ) are given by the intertwiners $\varphi^{\delta\gamma}_{\epsilon}(\theta)$ and $\varphi^{\epsilon}_{\alpha\beta}(\theta)$ with

$$\sum_{\epsilon} \varphi^{\delta\gamma}_{\epsilon}(\theta)\varphi^{\epsilon}_{\alpha\beta}(\theta) = \delta_{\alpha\gamma}\delta_{\beta\delta}\,, \quad \sum_{\alpha\beta} \varphi^{\epsilon'}_{\alpha\beta}(\theta)\varphi^{\alpha\beta}_{\epsilon}(-\theta) = \delta_{\epsilon'\epsilon}\,.$$

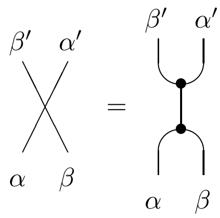
The eigenvalue of the S-matrix $\dot{S}(\alpha,\beta,\gamma,\theta)$ which correspond to a bound state $(\alpha\beta) = \gamma$ has a pole at $\theta = iu^{\gamma}_{\alpha\beta}$, a fact which will be used to define the 'bound state intertwiners'.

6.1 Bound state intertwiners

Following the investigations of [27] (see also [24]) we use in addition to the intertwines $\varphi^{\epsilon}_{\alpha\beta}$ which are defined for all eigenstates ϵ of the S-matrix also similar ones $\Gamma^{\gamma}_{\alpha\beta}(u^{\gamma}_{\alpha\beta})$ which are defined for all fusion angles. They are therefore only defined for an eigenstate γ which correspond to bound states i.e. an to eigenvalue of the two-particle S-matrix $S(\theta)$ which has a pole at $\theta = iu^{\gamma}_{\alpha\beta}$.

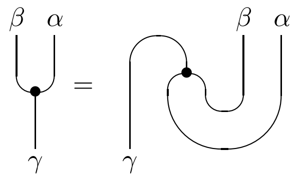
Definition 2 *The matrix elements $\Gamma^{\gamma}_{\alpha\beta}(\theta^{\gamma}_{\alpha\beta})$ of the **bound state intertwiner** are defined by the residue of the S-matrix*

$$i \operatorname*{Res}_{\theta = iu^{\gamma}_{\alpha\beta}} \dot{S}^{\beta'\alpha'}_{\alpha\beta}(\theta) = \Gamma^{\beta'\alpha'}_{\gamma}(u^{\gamma}_{\alpha\beta})\Gamma^{\gamma}_{\alpha\beta}(u^{\gamma}_{\alpha\beta}) : \qquad i\,\mathrm{Res} \tag{34}$$



where the dual intertwiner is defined by the crossing relation

$$\Gamma^{\beta\alpha}_{\gamma}(u^{\gamma}_{\alpha\beta}) = \mathbf{C}_{\gamma\gamma'}\Gamma^{\gamma'}_{\alpha'\beta'}(u^{\gamma}_{\alpha\beta})\mathbf{C}^{\beta'\beta}\mathbf{C}^{\alpha'\alpha} : \tag{35}$$

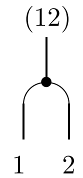


with the charge conjugation matrix $\mathbf{C}$ (e.g. for the sine-Gordon model $\mathbf{C}^{\alpha'\alpha} = \mathbf{C}_{\alpha'\alpha} = \delta_{\alpha'\bar{\alpha}}$).

Remarks:

1. For the bound state intertwiner given by the matrix elements $\Gamma^{\gamma}_{\alpha\beta}(u^{\gamma}_{\alpha\beta})$ and defined by eq. (34) we will also use the notation

$$\Gamma^{(12)}_{12}(u^{(12)}_{12}) =$$



where $u_{12}^{(12)} = u_{\alpha\beta}^{\gamma}$. It intertwines the spaces $V_1 \otimes V_2$ and $V_{(12)}$

$$\Gamma_{12}^{(12)} : V_1 \otimes V_2 \to V_{(12)} \,. \tag{36}$$

2. Note that the bound state intertwiners $\Gamma_{12}^{(12)}(u_{12}^{(12)})$ and the dual ones $\Gamma_{(12)}^{21}(u_{12}^{(12)})$ are defined for $u_{12}^{(12)} > 0$. In addition one may define the 'inverse' ones $\Gamma_{21}^{(12)}(u_{21}^{(12)})$ with $u_{21}^{(12)} = -u_{12}^{(12)} < 0$ such that
$$\begin{aligned}\Gamma_{(12)}^{12}(u_{21}^{(12)})\Gamma_{12}^{(12)}(u_{12}^{(12)}) &= P_{12}(12)\\ \Gamma_{12}^{(12)}(u_{12}^{(12)})\Gamma_{(12)}^{12}(u_{21}^{(12)}) &= \delta_{(12)}\end{aligned} \tag{37}$$
where $P_{12}(12)$ projects onto that subspace of $V_1 \otimes V_2$ defined by (36) on which $\Gamma_{12}^{(12)}(u_{12}^{(12)})$ is nonzero and $\delta_{(12)}$ is the unit matrix in $V_{(12)}$. The dual 'inverse' intertwiner is again defined by a crossing relation analogously to (35).

3. Obviously the bound state intertwiners are defined only up to some phase factors. From eqs. (33,34) follows that the components are given as
$$\Gamma_{\alpha\beta}^{\gamma} = \varepsilon(\alpha,\beta,\gamma) i \left| \operatorname*{Res}_{\theta = iu_{12}^{(12)}} \dot{S}(\alpha,\beta,\gamma,\theta) \right|^{1/2} \varphi_{\alpha\beta}^{\gamma} \,.$$
Here $\dot{S}(\alpha,\beta,\gamma,\theta)$ is the eigenvalue of the S-matrix $\dot{S}_{\alpha\beta}^{\beta'\alpha'}$ which correspond to the bound state γ and the ε's are phase factors. In [27] it was shown that $i \operatorname{Res} \dot{S}_{\gamma}(\theta)$ is real where the sign is related to the parity of the particles involved. The phase factors ε fulfill $\varepsilon(\alpha,\beta,\gamma)\varepsilon(\bar{\alpha},\bar{\beta},\bar{\gamma})$ $= -\operatorname{sgn}\left(i \operatorname{Res} \dot{S}(\alpha,\beta,\gamma,\theta)\right)$.

Example 1 *For the sine-Gordon model we fix the phase factors by $\varepsilon(s,\bar{s},k) = (-1)^k$ and $\varphi_{s\bar{s}}^k = (-1)^k \varphi_{\bar{s}s}^k = 1/\sqrt{2}$ [10]. The two-particle S-matrix has poles at $\theta = iu^{(k)} = i\pi(1-k\nu)$ which correspond to the soliton anti-soliton bound states alias breathers b_k. Using the short notation $\Gamma_{s\bar{s}}^k = \Gamma_{s\bar{s}}^{b_k}(u^{(k)})$ we obtain*
$$\Gamma_{s\bar{s}}^k = (-1)^k \Gamma_{\bar{s}s}^k = (-1)^k i \left| \tfrac{1}{2} \operatorname*{Res}_{\theta = i\pi(1-k\nu)} \dot{S}_{\pm}(\theta) \right|^{1/2} \tag{38}$$
where $+$ and $-$ correspond to even or odd k, respectively. The residues have been calculated in [17]
$$\begin{aligned}\tfrac{1}{2} \operatorname*{Res}_{\theta = i\pi(1-k\nu)} \dot{S}_{\pm}(\theta) = \operatorname*{Res}_{\theta = i\pi(1-k\nu)} \dot{b}(\theta) &= (-1)^k \operatorname*{Res}_{\theta = i\pi(1-k\nu)} \dot{c}(\theta)(-1)^k\\ &= 2i(-1)^k \cot \tfrac{\pi}{2} k\nu \prod_{l=1}^{k-1} \cot^2 \tfrac{\pi}{2} l\nu.\end{aligned} \tag{39}$$
All bound state intertwiners Γ involving solitons are uniquely given by the crossing relations
$$\begin{aligned}\Gamma_s^{ks} = \Gamma_{ks}^s = \Gamma_k^{s\bar{s}} = \Gamma_{\bar{s}k}^{\bar{s}} = \Gamma_{\bar{s}}^{\bar{s}k} = \Gamma_{s\bar{s}}^k\\ \Gamma_{\bar{s}}^{k\bar{s}} = \Gamma_{k\bar{s}}^{\bar{s}} = \Gamma_k^{\bar{s}s} = \Gamma_{sk}^s = \Gamma_{\bar{s}}^{sk} = \Gamma_{\bar{s}s}^k = (-1)^k \Gamma_{s\bar{s}}^k.\end{aligned}$$

[10] This choice is motivated by the fact that with this convention the breather matrix elements $\langle 0| : \varphi^k : (0)|b_k\rangle$ are positive (see [35, 34]).

Proposition 6 on page 32 and the general relation (34) imply up to a sign

$$\operatorname*{Res}_{\theta=i\pi(1-k\nu)}\dot{b}(\theta)=\operatorname*{Res}_{\theta=i\pi\frac{1}{2}(1+k\nu)}S_{ks}(\theta)$$

which may easily checked directly.

Example 2 *For the breather-breather bound states we fix the phase factors* $\varepsilon(k,l,k+l)=1$ *and* $\varphi_{kl}^{k+l}=1$ *(for* $k+l<1/\nu$*) and get*

$$\Gamma_{kl}^{k+l}=i\left|\operatorname*{Res}_{\theta=i\pi\frac{1}{2}(k+l)\nu}S_{kl}(\theta)\right|^{1/2}$$

where for $k\le l$

$$\operatorname*{Res}_{\theta=i\pi\frac{1}{2}(k+l)\nu}S_{kl}(\theta)=2i\tan\tfrac{\pi}{2}(k+l)\nu\frac{\tan\frac{\pi}{2}l\nu}{\tan\frac{\pi}{2}k\nu}\prod_{j=1}^{k-1}\frac{\tan^2\frac{\pi}{2}(k+l-j)\nu}{\tan^2\frac{\pi}{2}j\nu}.$$

All further breather bound state intertwiners Γ *are again uniquely given by the crossing relations using that the breather are self-conjugate*

$$\Gamma_{k+lk}^{l}=\Gamma_{k+l}^{lk}=\Gamma_{lk+l}^{k}=\Gamma_{l}^{kk+l}=\Gamma_{kl}^{k+l}=\Gamma_{lk}^{k+l}.$$

Again proposition 6 and the general relation (34) imply that up to a sign

$$\operatorname*{Res}_{\theta=i\pi\frac{1}{2}(k+l)\nu}S_{kl}(\theta)=\operatorname*{Res}_{\theta=i\pi\frac{1}{2}(1-l\nu)}S_{kk+l}(\theta)\,,$$

which also may easily checked directly.

6.1.1 The bootstrap principle

If there exist bound states in a quantum field theoretic model, the bootstrap principle means that all particles are to be considered on the same footing, in particular:

1. The space of all kinds of particles V is closed under bound state fusion which means that there exist for each fusion angle a bound state intertwiner $\Gamma_{12}^{(12)}(u_{12}^{(12)})$ such that
$$\Gamma_{12}^{(12)}:V_1\otimes V_2\to V_{(12)}\,,\quad\text{with}\quad V_1=V_2=V,\ V_{(12)}\subseteq V.$$
2. If the fusion process $\alpha+\beta\to\gamma$ exists then also the fusions $\beta+\bar\gamma\to\bar\alpha$ and $\bar\gamma+\alpha\to\bar\beta$ must exist. The corresponding fusion angles may be read off figure 7 where the euclidean momenta $(p^0,\operatorname{Im}p^1)$ are depicted and where $\hat u_{\alpha\beta}^{\gamma}=\pi-u_{\alpha\beta}^{\gamma}$ etc. In particular the rapidities of the constituents are given in terms of the rapidity of the bound state by
$$\begin{aligned}\theta_\alpha&=\theta_\gamma+i\hat u_{\beta\bar\gamma}^{\bar\alpha}\\ \theta_\beta&=\theta_\gamma-i\hat u_{\bar\gamma\alpha}^{\bar\beta}\end{aligned}\qquad(40)$$
and similar for the other fusion processes. The fusion angles satisfy the relation $u_{\alpha\beta}^{\gamma}+u_{\beta\bar\gamma}^{\bar\alpha}+u_{\bar\gamma\alpha}^{\bar\beta}=2\pi$ or $\hat u_{\alpha\beta}^{\gamma}+\hat u_{\beta\bar\gamma}^{\bar\alpha}+\hat u_{\bar\gamma\alpha}^{\bar\beta}=\pi$.
3. The various **bound state intertwiners** are related by crossing as depicted in figure 8 on page 32.

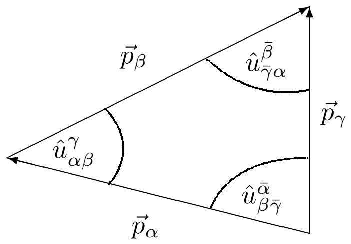
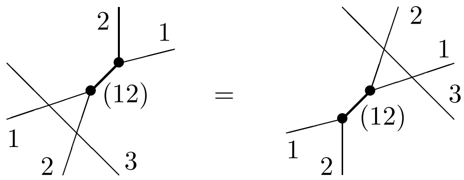


Figure 7: *The euclidian momenta* $(p^0, \operatorname{Im} p^1)$ *of the fusion process* $\alpha + \beta \to \gamma$ *in the center of mass system of* γ.

6.2 Bound state S-matrix

We show that the 'bootstrap principle' provides a consistent scheme. In particular we prove that the definition of the bound state intertwiners by the relation (34) is consistent with crossing symmetry. First we define the bound state S-matrix (see [27]). Using the Yang-Baxter equation and the definition of the bound state intertwiners we have

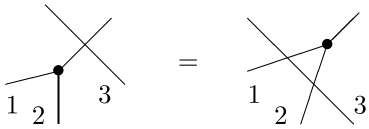
$$\begin{aligned} \operatorname{Res} S_{12}S_{13}S_{23} &= \operatorname{Res} S_{23}S_{13}S_{12} \\ \Gamma^{21}_{(12)}\Gamma^{(12)}_{12} S_{13}S_{23} &= S_{23}S_{13}\Gamma^{21}_{(12)}\Gamma^{(12)}_{12} \end{aligned}$$



Therefore the following definition of the bound state S-matrix is natural.

Definition 3 *[27]The bound state S-matrix which describes the scattering of a bound state* (12) *with another particle* 3 *is given by*

$$\dot{S}_{(12)3}(\theta_{(12)3})\Gamma^{(12)}_{12} = \Gamma^{(12)}_{12}\dot{S}_{13}(\theta_{13})\dot{S}_{23}(\theta_{23})\Big|_{\theta_{12}=iu^{(12)}_{12}} \tag{41}$$



where the rapidity of the bound state $\theta_{(12)}$ *is defined by the relation of the 2-momenta* $p_1 + p_2 = p_{(12)}$ *(see also eqs. (40)).*

It was shown in [27] that the bound state S-matrix defined by (41) fulfills unitarity, crossing and the Yang-Baxter equation. The equation (41) is also called a 'bootstrap equation' [39] (also referred to as a 'pentagon equation'). It relates different two-particle S-matrices and therefore implies strong restrictions on the complete two-particle S-matrix which may be used to calculate the S-matrix for integrable models [39].

We show that the definition of the bound state intertwiners by means of the residue of the two-particle S-matrix (34) is consistent with the 'bootstrap principle' and crossing symmetry.

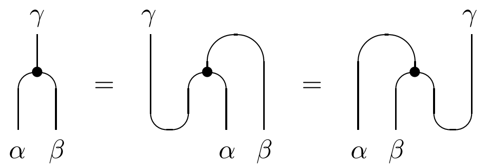


Figure 8: *Crossing relations of the fusion intertwiners*

Proposition 6 *If the bound state intertwiners satisfy the crossing relations*

$$\Gamma^{\gamma}_{\alpha\beta} = \mathbf{C}^{\gamma\bar\gamma'}\Gamma^{\bar\beta'}_{\bar\gamma'\alpha}\mathbf{C}_{\bar\beta'\beta} = \mathbf{C}_{\alpha\bar\alpha'}\Gamma^{\bar\alpha'}_{\beta\bar\gamma'}\mathbf{C}^{\bar\gamma'\gamma} \tag{42}$$

(see figure 8) and if the relation (34) holds for the fusion process $\alpha+\beta\to\gamma$ then it also holds for the fusion processes $\beta+\bar\gamma\to\bar\alpha$ i.e.

$$i \operatorname*{Res}_{\theta_{\beta\bar\gamma}=iu^{\bar\alpha}_{\beta\bar\gamma}} \dot S^{\bar\gamma'\beta'}_{\beta\bar\gamma}(\theta_{\beta\bar\gamma}) = \Gamma^{\bar\gamma'\beta'}_{\bar\alpha}\Gamma^{\bar\alpha}_{\beta\bar\gamma}\,.$$

and correspondingly for $\bar\gamma+\alpha\to\bar\beta$.

The proof of this proposition is delegated to appendix D.

6.3 Bound state form factors

If there are no bound states in the model there exist only the 'annihilation poles' according to property (*iii*) on page 12 which follow from the crossing formula. If there are also bound states there are additional poles [3] and we have also property (*iv*). Such additional poles have been discussed for simple cases in [3, 5]. Here we give the arguments for more general cases. Let us consider a model with a bound state (12) of two particles of 1 and 2 such that the attractive region is connected analytically (by a coupling constant) to a repulsive region, where the bound state decays.

We start in the repulsive region and consider the two-point Wightman function (1) $\langle 0|\mathcal{O}'\mathcal{O}|\,0\rangle = \langle 0|\mathcal{O}'(x)\mathcal{O}(0)|\,0\rangle$. We use the symmetry given by the statistics of the particles to express the *in*-state matrix elements in terms of the form factor functions $\mathcal{O}_{1\dots n}(\underline{\theta})$ for ordered rapidities

$$\begin{aligned}
\langle 0|\mathcal{O}'\mathcal{O}|\,0\rangle &= \sum_{n=0}^{\infty}\frac{1}{n!}\int\cdots\int\frac{dp_1\dots dp_n}{(2\pi)^n 2\omega_1\dots 2\omega_n} \\
&\quad\times\langle 0|\mathcal{O}'|p_1,\dots,p_n\rangle^{in\;in}\langle p_n,\dots,p_1|\mathcal{O}|0\rangle \\
&= \sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{d\theta_1}{4\pi}\int_{-\infty}^{\theta_1}\frac{d\theta_2}{4\pi}\cdots\int_{-\infty}^{\theta_{n-1}}\frac{d\theta_n}{4\pi}\,\mathcal{O}'_{1\dots n}(\underline{\theta})\,\mathcal{O}^{1\dots n}(\underline{\theta})\,e^{-iPx} \\
&= \sum_{n=0}^{\infty}\frac{1}{n!}\int_{-\infty}^{\infty}\frac{d\theta_1}{4\pi}\int_{-\infty}^{\infty}\frac{d\theta_2}{4\pi}\cdots\int_{-\infty}^{\infty}\frac{d\theta_n}{4\pi}\,\mathcal{O}'_{1\dots n}(\underline{\theta})\,\mathcal{O}^{1\dots n}(\underline{\theta})\,e^{-iPx}
\end{aligned}$$

where $P = p_1 + \cdots + p_n$. For the last equality the symmetry (i) of the form factor functions has been used, in particular we have

$$\mathcal{O}'_{12\dots n}\mathcal{O}^{12\dots n} = \mathcal{O}'_{21\dots n}\dot{S}_{12}\mathcal{O}^{12\dots n} = \mathcal{O}'_{12\dots n}\dot{S}_{21}\mathcal{O}^{21\dots n}\,. \tag{43}$$

In the repulsive case the S-matrices $\dot{S}_{12}$ and $\dot{S}_{21}$ have poles at $\theta_{12} = \theta_1 - \theta_2 = iu_{12}^{(12)}$ and $\theta_{21} = \theta_2 - \theta_1 = iu_{12}^{(12)}$, respectively, in the 'unphysical region' $u_{12}^{(12)} < 0$. Therefore also the left hand side has these poles. There are $\binom{n}{2}$ such pairs of poles. By analytic continuation with respect to the coupling constant to the attractive region where $u_{12}^{(12)} > 0$ these poles cross the integration contours and we obtain additional contributions from residues

$$\begin{aligned}\langle 0\,|\,\mathcal{O}'\,\mathcal{O}\,|\,0\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\Bigg\{&\int_{-\infty}^{\infty}\frac{d\theta_1}{4\pi}\int_{-\infty}^{\infty}\frac{d\theta_2}{4\pi}\cdots\int_{-\infty}^{\infty}\frac{d\theta_n}{4\pi}\\ +\binom{n}{2}&\left[-\oint_{iu_{12}^{(12)}}+\oint_{-iu_{12}^{(12)}}\right]\frac{d\theta_{12}}{4\pi}\int_{-\infty}^{\infty}\frac{d\theta_{(12)}}{4\pi}\cdots\int_{-\infty}^{\infty}\frac{d\theta_n}{4\pi}\Bigg\}\\ &\times\mathcal{O}'_{1\dots n}(\underline{\theta})\mathcal{O}^{1\dots n}(\underline{\theta})\,e^{-iPx}.\end{aligned}$$

The substitution $(\theta_1,\theta_2) \to (\theta_{12} = \theta_1 - \theta_2, \theta_{(12)})$ has been applied for the additional residue terms (if the masses of 1 and 2 are equal the bound state rapidity is $\theta_{(12)} = \frac{1}{2}(\theta_1 + \theta_2)$, for the general case see eqs. (40)). Using (43) and the residue formula (34) we obtain

$$\begin{aligned}-\oint_{iu_{12}^{(12)}}\frac{d\theta_{12}}{4\pi}\mathcal{O}'_{12\dots n}(\underline{\theta}) &= -\frac{1}{2}\,\mathcal{O}'_{21\dots n}(\underline{\theta})\Gamma_{(12)}^{21}\Gamma_{12}^{(12)}\Big|_{\theta_{12}=iu_{12}^{(12)}}\\ \oint_{-iu_{12}^{(12)}}\frac{d\theta_{12}}{4\pi}\mathcal{O}^{12\dots n}(\underline{\theta}) &= -\frac{1}{2}\,\Gamma_{(12)}^{21}\Gamma_{12}^{(12)}\mathcal{O}^{21\dots n}(\underline{\theta})\Big|_{\theta_{12}=iu_{12}^{(12)}}\end{aligned}$$

which means that both residues give the same contribution. Therefore the additional term may be written as

$$\begin{aligned}&-\sum_{n=2}^{\infty}\frac{1}{n!}\binom{n}{2}\int_{-\infty}^{\infty}\frac{d\theta_{(12)}}{4\pi}\cdots\int_{-\infty}^{\infty}\frac{d\theta_n}{4\pi}\,\mathcal{O}'_{21\dots n}\Gamma_{(12)}^{21}\Gamma_{12}^{(12)}\mathcal{O}^{12\dots n}\Big|_{\theta_{12}=iu_{12}^{(12)}}\\ &\quad=\int_{-\infty}^{\infty}\frac{d\theta_{(12)}}{4\pi}\sum_{n=2}^{\infty}\frac{1}{(n-2)!}\int_{-\infty}^{\infty}\frac{d\theta_3}{4\pi}\cdots\int_{-\infty}^{\infty}\frac{d\theta_n}{4\pi}\mathcal{O}'_{(12)\dots n}\mathcal{O}^{(12)\dots n}\,e^{-iPx}\end{aligned}$$

where we have introduced the bound state form factor

$$\mathcal{O}_{(12)3\dots n}(\theta_{(12)},\underline{\theta}') = \frac{1}{i\sqrt{2}}\mathcal{O}_{21\dots n}(\underline{\theta})\Gamma_{(12)}^{21}\Big|_{\theta_{12}=iu_{12}^{(12)}} \tag{44}$$

with $\underline{\theta}' = (\theta_3,\dots,\theta_n)$ and the rapidity $\theta_{(12)}$ of the bound state is given by $p_1 + p_2 = p_{(12)}$. This identification is obviously unique up to a sign which may be absorbed into the bound state intertwiner. Using again the property (i) and the residue formula for the S-matrix (34) we obtain the property (iv) of form factors on page 7

$$\begin{aligned}\operatorname*{Res}_{\theta_{12}=iu_{12}^{(12)}}\mathcal{O}_{123\dots n}(\underline{\theta}) &= \operatorname*{Res}_{\theta_{12}=iu_{12}^{(12)}}\mathcal{O}_{213\dots n}(\underline{\theta})\dot{S}_{12}\\ &= -i\mathcal{O}_{213\dots n}(\underline{\theta})\Gamma_{(12)}^{21}\Gamma_{12}^{(12)}\Big|_{\theta_{12}=\theta_{12}^{(12)}}\\ &= \mathcal{O}_{(12)3\dots n}(\theta_{(12)},\underline{\theta}')\,\sqrt{2}\Gamma_{12}^{(12)}\end{aligned}$$

In order to show that the formula (44) for the bound state form factor is consistent with the bootstrap principle we prove the following proposition.

Proposition 7 *The bound state form factor defined by the property* (iv) *or by* (44) *satisfies properties* (i) (ii) *and* (iii)

$$\begin{array}{llcl} (i) & \mathcal{O}_{(12)3\dots n}(\theta_{(12)},\theta_3,\dots) & = & \mathcal{O}_{3(12)\dots n}(\theta_3,\theta_{(12)},\dots)S_{(12)3} \\ (ii) & & = & \mathcal{O}_{3\dots n(12)}(\theta_3,\dots,\theta_{(12)}-2\pi i)\sigma_{\mathcal{O}(12)} \\ (iii) & & \approx & \frac{2i}{\theta_{(12)3}-i\pi}\mathbf{C}_{(12)3}\mathcal{O}_{4\dots n}(\theta_4,\dots) \\ & & & \quad\times(\mathbf{1}-S_{3n}\dots S_{34}) \end{array}$$

Proof. These relations follow directly from the corresponding relations for the form factor before taking the residue, by using in addition the Yang-Baxter equation, the fusion equation (41) and the crossing relations of the bound state intertwiners. For (i) and (ii) the proofs are obvious. For (iii) the proof is quite involved. It is delegated to appendix E.

7 Soliton Breather form factors

In this section we apply the results of the previous section to the sine-Gordon model. We calculate breather form factors starting with the general formula (13) for the soliton form factors.

The b_k-breather-$(n-2)$-soliton form factor is obtained from $\mathcal{O}_{123\dots n}(\underline{\theta})$ by means of the fusion procedure (iv) with the fusion angle given by $u_{12}^{(12)}=u^{(k)}=\pi(1-k\nu)$ the bound state rapidity $\xi=\theta_{(12)}=\frac{1}{2}(\theta_1+\theta_2)$ and $\underline{\theta}'=\theta_3,\dots,\theta_n$

$$\operatorname*{Res}_{\theta_{12}=iu^{(k)}}\mathcal{O}_{123\dots n}(\underline{\theta})=\mathcal{O}_{(12)3\dots n}(\xi,\underline{\theta}')\sqrt{2}\Gamma_{12}^{(12)}(iu^{(k)})$$

where the bound state intertwiner is given by eqs. (38) and (39).

For $\theta_{12}\to iu^{(k)}$ there will be pinchings of the integration contours in formula (13) at the poles $z_i=z^{(l)}=\theta_2-i\pi l\nu=\xi-\frac{1}{2}i\pi(1-k\nu+2l\nu)$ for $l=0,\dots,k$ and $i=1,\dots,m$. Using the pinching rule of contour integrals and the symmetry with respect to the m z-integrations we obtain

$$\begin{aligned}\operatorname*{Res}_{\theta_{12}=iu^{(k)}}\mathcal{O}_{123\dots n}(\underline{\theta}) &= \operatorname*{Res}_{\theta_{12}=iu^{(k)}}(-2\pi i)\,m\sum_{l=0}^{k}\operatorname*{Res}_{z_1=z^{(l)}}\int_{\mathcal{C}_{\underline{\theta}}}dz_2\cdots\int_{\mathcal{C}_{\underline{\theta}}}dz_m \\ &\quad\times h(\underline{\theta},\underline{z})p^{\mathcal{O}}(\underline{\theta},\underline{z})\,\Psi_{1\dots n}(\underline{\theta},\underline{z}).\end{aligned}$$

After a lengthy calculation (see appendix F) we obtain for the case of the lowest breather ($k=1$, $u^{(1)}=\pi(1-\nu)$) the one-breather-$(n-2)$-soliton form factor

$$\begin{aligned}\mathcal{O}_{3\dots n}(\xi,\underline{\theta}') &= \prod_{2<i}F_{sb}(\xi-\theta_i)\prod_{2<i<j}F(\theta_{ij}) \\ &\quad\times\sum_{l=0}^{1}(-1)^l\prod_{2<i}\rho(\xi-\theta_i,l) \\ &\quad\times\int_{\mathcal{C}_{\underline{\theta}}}dz_2\cdots\int_{\mathcal{C}_{\underline{\theta}}}dz_m\prod_{1<j}\chi(\xi-z_j,l) \\ &\quad\times\prod_{2<i}\prod_{1<j}\phi(\theta_i-z_j)\prod_{1<i<j}\tau(z_{ij})\,\tilde{p}^{\mathcal{O}}(\xi,\underline{\theta}',z^{(l)},\underline{z}')\,\Psi_{3\dots n}(\underline{\theta}',\underline{z}')\end{aligned}$$

with $\underline{z}' = (z_2, \dots, z_m)$. The soliton-breather form factor has been introduced as

$$\begin{aligned} F_{sb}(\theta) &= K_{sb}(\theta) \sin \tfrac{1}{2i}\theta \ \exp \int_0^\infty \frac{dt}{t}\, 2 \frac{\cosh \frac{1}{2}\nu t}{\cosh \frac{1}{2} t} \frac{1 - \cosh t(1-\theta/(i\pi))}{2\sinh t} \\ K_{sb}(\theta) &= \frac{-\cos \frac{\pi}{4}(1-\nu)/E(\frac{1}{2}(1-\nu))}{\sinh \frac{1}{2}(\theta - \frac{i\pi}{2}(1+\nu)) \sinh \frac{1}{2}(\theta + \frac{i\pi}{2}(1+\nu))}\,. \end{aligned}$$

The normalization has been chosen such that $F_{sb}(\infty) = 1$. The function $E(\nu)$ was used in [3, 5] and is defined in appendix F. Also we have introduced the short notations

$$\begin{aligned} \rho(\xi, l) &= \frac{\sinh \frac{1}{2}\left(\xi - (-1)^l \frac{i\pi}{2}(1+\nu)\right)}{\sinh \frac{1}{2}\xi} = (-1)^l \frac{\sinh \frac{1}{2}\left(\xi - \frac{i\pi}{2}(1+(-1)^l\nu)\right)}{\sinh \frac{1}{2}\xi} \\ \chi(\xi, l) &= (-1)^l \frac{\sinh \frac{1}{2}(\xi + \frac{i\pi}{2}(1+(-1)^l\nu))}{\sinh \frac{1}{2}(\xi - \frac{i\pi}{2}(1+(-1)^l\nu))} \end{aligned}$$

The following identities have been used

$$\begin{aligned} F(\theta_1 - \theta_i) F(\theta_2 - \theta_i) \tilde{\phi}(\theta_i - z^{(l)}) &= F_{sb}(\xi - \theta_i)\rho(\xi - \theta_i, l) \\ \phi(\theta_1 - z_j)\phi(\theta_2 - z_j) S_{sb}(\xi - z_j)\tau(z^{(l)} - z_j) &= \chi(\xi - z_j, l) \end{aligned}$$

for $\theta_{1/2} = \xi \pm \frac{i\pi}{2}(1-\nu)$, $z^{(l)} = \xi - \frac{i\pi}{2}(1-(-1)^l\nu)$. The new p-function is obtained from the old one by

$$\tilde{p}^{\mathcal{O}}(\xi, \underline{\theta}', z^{(l)}, \underline{z}') = m\, d(\nu)\, p^{\mathcal{O}}(\xi + \tfrac{1}{2}iu^{(1)}, \xi - \tfrac{1}{2}iu^{(1)}, \underline{\theta}', z^{(l)}, \underline{z}')$$

where the constant $d(\nu)$ is given by

$$d(\nu) = \frac{\sqrt{E(\nu)}}{\varkappa\sqrt{\sin \frac{1}{2}\pi\nu}}$$

(see appendix F).

Iterating the above procedure we obtain the r-breather-s-soliton form factor with $2r + s = n$, the breather rapidities $\underline{\xi} = (\xi_1, \dots, \xi_r)$ and the soliton rapidities $\underline{\theta} = (\theta_1, \dots, \theta_s)$

$$\begin{aligned} \mathcal{O}_{1\dots s}(\underline{\xi}, \underline{\theta}) = {} & \prod_{1\le i<j\le r} F_{bb}(\xi_{ij}) \prod_{i=1}^r \prod_{j=1}^s F_{sb}(\xi_i - \theta_j) \prod_{1\le i<j\le s} F(\theta_{ij}) \\ & \times \sum_{l_1=0}^1 \cdots \sum_{l_r=0}^1 (-1)^{l_1+\dots+l_r} \prod_{1\le i<j\le r}\left(1 + (l_i - l_j)\frac{i\sin\pi\nu}{\sinh\xi_{ij}}\right) \\ & \times \prod_{i=1}^r \prod_{j=1}^s \rho(\xi_i - \theta_j, l) \\ & \times \int dz_{r+1} \cdots \int dz_m \prod_{i=1}^r \prod_{j=r+1}^m \chi(\xi_i - z_j, l) \\ & \times \prod_{i=1}^s \prod_{j=r+1}^m \phi(\theta_i - z_j) \prod_{r<i<j\le m} \tau(z_{ij})\, \tilde{p}(\underline{\xi}, \underline{\theta}, \underline{z^{(l)}}, \underline{z})\, \Psi_{1\dots s}(\underline{\theta}, \underline{z}) \end{aligned}$$

again with $z_i^{(l_i)} = \xi_i - \frac{i\pi}{2}(1-(-1)^{l_i}\nu), (i = 1, \dots, r)$. The two-breather form factor has been introduced as

$$
\begin{aligned}
F_{bb}(\theta) &= K_{bb}(\theta)\,\sin\tfrac{1}{2i}\theta\,\exp\int_0^\infty \frac{dt}{t}\,2\frac{\cosh(\frac{1}{2}-\nu)t}{\cosh\frac{1}{2}t}\,\frac{1-\cosh t(1-\theta/(i\pi))}{2\sinh t}\\
K_{bb}(\theta) &= \frac{-\cos\frac{1}{2}\pi\nu/E(\nu)}{\sinh\frac{1}{2}(\theta-i\pi\nu)\sinh\frac{1}{2}(\theta+i\pi\nu)}
\end{aligned}
$$

The normalization has been chosen such that $F_{bb}(\infty) = 1$. The relation

$$
\begin{aligned}
&F_{sb}(\xi_1-\theta_3)F_{sb}(\xi_1-\theta_4)\,\rho(\xi_1-\theta_3,l_1)\rho(\xi_1-\theta_4,l_1)\chi(\xi_1-z_2^{(l_2)},l_1)\\
&\qquad\qquad = F_{bb}(\xi_{12})\left(1+(l_1-l_2)\frac{i\sin\pi\nu}{\sinh\xi_{12}}\right)
\end{aligned}
$$

has been used for $\theta_{3/4} = \xi_2 \pm \frac{i\pi}{2}(1-\nu)$. The new p-function is obtained from the old one by

$$
\tilde p(\underline{\xi},\underline{\theta}'',\underline{z^{(l)}},\underline{z}'') = \binom{m}{r} r!\,d^r(\nu)\,p\left(\xi_1+\tfrac{1}{2}\theta^{(1)},\xi_1-\tfrac{1}{2}\theta^{(1)},\dots,\underline{\theta}'',z_1^{(l_1)},\dots,\underline{z}''\right).
$$

In particular for $n = 2r = 2m$ we get the pure lowest breather form factor

$$
\begin{aligned}
\mathcal{O}(\underline{\xi}) = \prod_{i<j} F_{bb}(\xi_{ij}) &\sum_{l_1=0}^{1}\cdots\sum_{l_r=0}^{1}(-1)^{l_1+\cdots+l_r}\\
&\qquad\times\prod_{1=i<j}^{r}\left(1+(l_i-l_j)\frac{i\sin\pi\nu}{\sinh\xi_{ij}}\right)\tilde p(\underline{\xi},\underline{z^{(l)}})
\end{aligned}
$$

and the pure breather p-function

$$
\tilde p(\underline{\xi},\underline{z^{(l)}}) = r!\,d^r(\nu)\,p\left(\xi_1+\tfrac{1}{2}iu^{(1)},\xi_1-\tfrac{1}{2}iu^{(1)},\dots,z_1^{(l_1)},\dots\right).
$$

8 Conclusion

In a forthcoming paper [35] we investigate extensively the pure breather form factors of the sine-Gordon model. Some results have been published previously [34]. Furthermore other integrable models of quantum field theory will be analyzed. In particular the $SU(N)$-chiral-Gross-Neveu [14] and the $O(N)$-Gross-Neveu [15] model is under investigation. In these models the particles possess anyonic statistics. The form factor program as considered in the present article may easily extended to anyonic fields and particles. Mainly the statistics factors in the form factor equations $(i)-(v)$ have to be replaced by more general phase factors. This has been discussed in [4, 40, 6, 41].

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Appendix

A Proof of the lemma 2

Proof. We start the proof of lemma 2 by establishing some algebraic identities. The formula

$$\prod_{1\le i<j\le m}(x_j-x_i)=\begin{vmatrix} 1 & x_1 & x_1^2 & \dots & x_1^{m-1}\\ 1 & x_2 & x_2^2 & \dots & x_2^{m-1}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_m & x_m^2 & \dots & x_m^{m-1}\end{vmatrix}$$

follows from the fact that both sides are polynomials in x_m of degree $m-1$ with the same $m-1$ zeros and the same asymptotic behavior. Further taking the anti-symmetric sum $[\cdots]_a$ with respect to $x_1,\dots,x_m$ one gets the relations

$$\left[x_m^k\prod_{1\le i<j\le m}(x_j-x_i)\right]_a=\begin{vmatrix} 1 & x_1 & \dots & x_1^{m-2} & x_1^k\\ 1 & x_2 & \dots & x_2^{m-2} & x_2^k\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 1 & x_m & \dots & x_m^{m-2} & x_m^k\end{vmatrix}$$
$$=\delta_{k,m-1}\prod_{1\le i<j\le m}(x_j-x_i)$$

for $k=0,1,\dots,m-1$. Therefore for any set of constants A_k one has

$$\left[\left(\sum_{k=0}^{m-1}A_kx_m^k\right)\prod_{1\le i<j<m}(x_j-x_i)\right]_a=A_{m-1}\prod_{1\le i<j\le m}(x_j-x_i)$$

In particular with $x_i=e^{2z_i}$ and $a_j=e^{2\theta_j}$ we may finally write

$$\prod_{1\le i<j\le m}\sinh z_{ij}\propto$$
$$\left[\left(\prod_{i=1}^{2m}\cosh\tfrac12(\theta_i-z_m)-\prod_{i=1}^{2m}\sinh\tfrac12(\theta_i-z_m)\right)\prod_{i=1}^{m-1}e^{-z_i}\prod_{1\le i<j<m}\sinh z_{ij}\right]_a$$

because the difference on the right hand side is of the form $\sum_{k=0}^{m-1}A_ke^{2kz_m}$. Using $\tau(z)\propto\sinh z\sinh(z/\nu)$ the right hand side of formula (11) for $p^{\mathcal{O}}(\underline{\theta},\underline{z})=$ independent of $\underline{z}$ and $n=2m$ is proportional to

$$\int_{\mathcal{C}_{\underline{\theta}}}dz_1\cdots\int_{\mathcal{C}_{\underline{\theta}}}dz_m\prod_{i=1}^{n}\prod_{j=1}^{m}\phi(\theta_i-z_j)\prod_{1\le i<j\le m}\sinh\frac{1}{\nu}(z_i-z_j)\ \Psi_{1\dots n}(\underline{\theta},\underline{z})$$
$$\times\left[\left(\prod_{i=1}^{n}\cosh\tfrac12(\theta_i-z_m)-\prod_{i=1}^{n}\sinh\tfrac12(\theta_i-z_m)\right)\prod_{i=1}^{m-1}e^{-z_i}\prod_{1\le i<j<m}\sinh z_{ij}\right]_a$$

The z_m-integral may be written as

$$\left(\int_{\mathcal{C}_{\underline{\theta}}} - \int_{\mathcal{C}_{\underline{\theta}}+i\pi\nu}\right) dz_m \prod_{i=1}^{n} \tilde{\phi}(\theta_i - z_m) \prod_{i=1}^{n} \cosh \tfrac{1}{2}(\theta_i - z_m)$$
$$\times \prod_{i=1}^{m-1} \sinh \frac{z_i - z_m}{\nu} \tilde{\Psi}(\underline{\theta}, \underline{z}) = 0$$

where $\tilde{\phi}(\theta) = a(\theta)\phi(\theta)$ and $\tilde{\Psi}(\underline{\theta}, \underline{z}) = \Psi(\underline{\theta}, \underline{z}) / \left(\prod_{i=1}^{n} \prod_{j=1}^{m} a(\theta_i - z_j)\right)$. The shift relations

$$\tilde{\phi}(\theta - i\pi\nu) \cosh \frac{1}{2}(\theta - i\pi\nu) = \tilde{\phi}(\theta)\, i \sinh \frac{1}{2}(\theta)$$

$$\tilde{\Psi}(\underline{\theta}, z_1, \ldots, z_m + i\pi\nu) = -\tilde{\Psi}(\underline{\theta}, z_1, \ldots, z_m)$$

have been used. They can easily be derived from (18), the definition of the Bethe Ansatz state (12) and the S-matrix (4). The contour $\mathcal{C}_{\underline{\theta}} - \mathcal{C}_{\underline{\theta}} + i\pi\nu$ may be closed at $\pm$ infinity since the integrand behaves like $e^{-2|z_m|/\nu}$ for $z_2 \to \pm\infty$ (see appendix B). There are no poles inside the closed contour, therefore the integral vanishes by Cauchy's theorem. ■

B Asymptotic formulae

The asymptotic behavior of the soliton-soliton scattering amplitude is easily obtained from its integral representation

$$a(\theta) = \exp \int_0^\infty \frac{dt}{t} \frac{\sinh \frac{1}{2}(1-\nu)t}{\sinh \frac{1}{2}\nu t \cosh \frac{1}{2}t} \sinh t(\theta/(i\pi)) = e^{\mp i \frac{1}{2}(1/\nu - 1)} + o(1)$$

for $\operatorname{Re}\theta \to \pm\infty$, $(|\operatorname{Im}\theta| < \frac{\pi}{2}(1 + \nu - |1 - \nu|))$. This implies for the other amplitudes

$$b(\theta) = e^{\pm i \frac{1}{2}(1/\nu - 1)} + o(1)\,, \quad c(\theta) = \pm 2i \sin(\pi/\nu)\, e^{\pm i \frac{1}{2}(1/\nu - 1)} e^{\mp\theta/\nu} \left(1 + o(1)\right).$$

The asymptotic behavior of the 'minimal' two-soliton form factor function is given by

$$F(\theta) = \cosh \tfrac{1}{2}\left(i\pi - \theta\right) \exp \int_0^\infty \frac{dt}{t} \frac{\sinh \frac{1}{2}(1-\nu)t}{\sinh \frac{1}{2}\nu t \cosh \frac{1}{2}t} \frac{1 - \cosh t(1 - \theta/(i\pi))}{2 \sinh t}$$
$$= \mathit{const}\, e^{\pm(1/\nu + 1)(\theta - i\pi)}(1 + o(1))$$

for $\operatorname{Re}\theta \to \pm\infty$, $(|\operatorname{Im}\theta - \pi| < \frac{\pi}{2}(3 + \nu - |1 - \nu|))$ with the constant

$$\mathit{const} = \tfrac{1}{2} \exp \tfrac{1}{2} \int_0^\infty \frac{dt}{t} \left(\frac{\sinh \frac{1}{2}(1-\nu)t}{\sinh \frac{1}{2}\nu t \cosh \frac{1}{2}t \sinh t} - \frac{1-\nu}{\nu t} \right)$$

which satisfies

$$\mathit{const}^4 = -\tfrac{1}{4}\nu \left(F'(0)\right)^2 = \tfrac{1}{4}\nu\pi\varkappa$$

(for $\varkappa$ see eq. (19)). Similarly one has for $\operatorname{Re}\theta \to \pm\infty$, $(|\operatorname{Im}\theta - \frac{\pi}{2}| < \frac{\pi}{2}(2+\nu - |1-\nu|))$ the asymptotic expansion

$$\phi(\theta) = \frac{4}{\sqrt{4\pi\nu\varkappa}} e^{\mp\frac{1}{2}(1/\nu+1)(\theta - i\pi/2)} \left(\sum_{n=0}^{\tilde{n}} A_n e^{\mp n\theta} + o\left(e^{\mp\tilde{n}\theta}\right) \right)$$

for any integer $\tilde{n}$ and $\nu < 2/\tilde{n}$. The constants A_n are determined by the expansion

$$\begin{aligned}
&\exp \tfrac{1}{2} \left[\int_{-\infty}^{\infty} \frac{dt}{t} \left(\frac{\sinh \frac{1}{2}(1-\nu)t}{\sinh \frac{1}{2}\nu t \, \sinh t} - \frac{1-\nu}{\nu t} \right) e^{it(\theta - i\pi/2)/\pi} \right] \\
&= \exp \left(\sum_{n=1}^{\tilde{n}} a_n e^{\mp n\theta} + o\left(e^{\mp\tilde{n}\theta}\right) + \sum_{m=1}^{\tilde{m}} b_m e^{\mp 2m\theta/\nu} + o\left(e^{\mp 2\tilde{m}\theta/\nu}\right) \right) \\
&= \sum_{n=0}^{\tilde{n}} A_n e^{\mp n\theta} + o\left(e^{\mp\tilde{n}\theta}\right)
\end{aligned}$$

where $a_n = (-i)^{n-1} \sin \frac{1}{2}\pi(1-\nu)n / \left(\pi n \sin \frac{1}{2}\pi\nu n\right)$. Note that $A_0 = 1$.

C Expansion of the integral representation

In order to compare the exact result for 4-particle form factors with the Feynman graph result the integral representation has to be expanded for small couplings. We calculate the integral $I^{\pm} = \int_{\mathcal{C}_{\underline{\theta}}} dz_1 \int_{\mathcal{C}_{\underline{\theta}}} dz_2 I^{\pm}(z_1, z_2)$ to prove formula (28). The integrands are

$$\begin{aligned}
I^{\pm}(z_1, z_2) = &\left(\prod_{i=1}^{4} \prod_{j=1}^{2} \tilde{\phi}(\theta_i - z_j) \right) \tilde{c}(\theta_1 - z_1)\tilde{c}(\theta_2 - z_2) \\
&\times \left(1 + \tilde{b}(\theta_1 - z_2)\tilde{b}(\theta_2 - z_1) \right) \tau(z_1 - z_2) \left(e^{\pm z_1} + e^{\pm z_2} \right)
\end{aligned}$$

up to order $O(g^2)$. The functions $I^{\pm}(z_1, z_2)$ have poles at $z_i = \theta_1, \theta_2, \theta_3, \theta_4$ and $\theta_i - i\pi, \theta_i + i\pi(\nu - 1)$ for $i = 1, 2$. By means of Cauchy's theorem we have

$$\int_{\mathcal{C}_{\underline{\theta}}} dz_i I(z_1, z_2) = \left[\oint_{\theta_i - i\pi} - \sum_{j=1}^{4} \oint_{\theta_j} + \int_{\mathcal{C}_0} \right] dz_i I(z_1, z_2)$$

where $\mathcal{C}_0$ goes from $-\infty$ to ∞ such that $\operatorname{Im}\theta_i - \pi < \operatorname{Im} z < \operatorname{Im}\theta_i$. Then shifting by the integration contour $\mathcal{C}_0$by $i\pi$ one obtains

$$\int_{\mathcal{C}_0} dz_i I(z_1, z_2) = \left[\sum_{j=1}^{4} \oint_{\theta_j} + \oint_{\theta_i + i\pi(\nu - 1)} + \int_{\mathcal{C}_0 + i\pi} \right] dz_i I(z_1, z_2)$$

which implies finally

$$\begin{aligned}\int_{\mathcal{C}_{\underline{\theta}}} dz_i I(z_1,z_2) = \Bigg[&-\oint_{\theta_i} + \oint_{\theta_i-i\pi} + \tfrac{1}{2}\Bigg[\oint_{\theta_i} + \oint_{\theta_i+i\pi(\nu-1)}\Bigg] \\ &- \tfrac{1}{2}\sum_{j\neq i}\oint_{\theta_j}\Bigg] dz_i I(z_1,z_2) + \tfrac{1}{2}\int_{\mathcal{C}_0} dz_i\left(I(z_i) + I(z_i+i\pi)\right).\end{aligned}$$

The last term $\int_{\mathcal{C}_0} dz_i\left(I(z_i)+I(z_i+i\pi)\right)$ is of order $O(g^2)$. Therefore we may consider up to $O(g^2)$

$$\begin{aligned}I^{\pm} = &\Bigg[-\oint_{\theta_1} + \oint_{\theta_1-i\pi} + \tfrac{1}{2}\Bigg[\oint_{\theta_1} + \oint_{\theta_1+i\pi(\nu-1)}\Bigg] - \tfrac{1}{2}\sum_{j\neq 1}\oint_{\theta_j}\Bigg] dz_1 \\ &\times \Bigg[-\oint_{\theta_2} + \oint_{\theta_2-i\pi} + \tfrac{1}{2}\Bigg[\oint_{\theta_2} + \oint_{\theta_2+i\pi(\nu-1)}\Bigg] - \tfrac{1}{2}\sum_{j\neq 2}\oint_{\theta_j}\Bigg] dz_2 I^{\pm}(z_1,z_2)\,.\end{aligned}$$

The $\left[-\operatorname{Res}_{\theta_1} + \operatorname{Res}_{\theta_1-i\pi}\right]_{z_i}$ terms have $O(1)$ and $O(g)$ contributions the other ones are of order $O(g)$ only. The $O(1)$-terms cancel and after a long but straight forward calculation all $O(g)$-terms give formula (28).

D Proof of proposition 6

In this appendix we prove that the 'bootstrap principle' provides a consistent scheme. In particular we prove proposition 6 which states that the definition of the bound state intertwiner is consistent with crossing symmetry.

Proof. The idea of the proof may read off figure 9: We use the crossing property of the S-matrix (8), the S-matrix bound state formula (41), the residue formula (34) and the crossing relations of the bound state intertwiners (35). We make

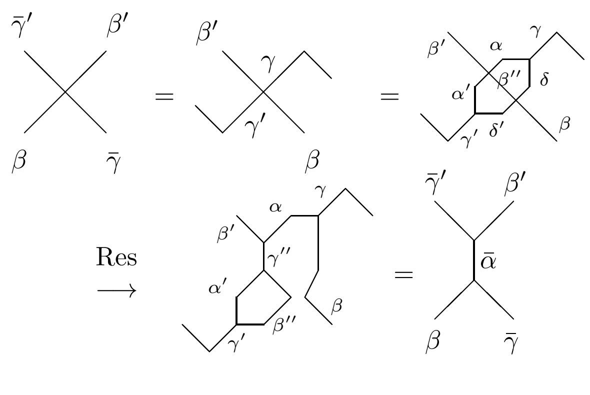


Figure 9: *Proof of proposition 6.*

the convention that rapidities as $\theta_\alpha, \theta_{\alpha'}$ etc. belong to particles with the same mass and other invariant quantum numbers. If δ denotes a particle with the same mass as that of β then we have (see fig. 9)

$$\begin{aligned}
i \operatorname*{Res}_{\theta_{\beta\bar\gamma}=\theta^{\bar\alpha}_{\beta\bar\gamma}} \dot S^{\bar\gamma'\beta'}_{\beta\bar\gamma}(\theta_{\beta\bar\gamma}) &= i \operatorname*{Res}_{\theta_{\beta\bar\gamma}=\theta^{\bar\alpha}_{\beta\bar\gamma}} \mathbf{C}^{\bar\gamma'\gamma'}\mathbf{C}_{\gamma\bar\gamma}\dot S^{\beta'\gamma}_{\gamma'\beta}(i\pi-\theta_{\beta\bar\gamma}) \\
&= i \operatorname*{Res}_{\theta_{\beta\bar\gamma}=\theta^{\bar\alpha}_{\beta\bar\gamma}} \mathbf{C}^{\bar\gamma'\gamma'}\Gamma^{\alpha\delta}_{\gamma'}\Gamma^{\gamma''}_{\alpha\delta}\mathbf{C}_{\gamma\bar\gamma}\dot S^{\beta'\gamma}_{\gamma''\beta}(i\pi-\theta_{\beta\bar\gamma}) \\
&= -i \operatorname*{Res}_{\theta_{\alpha\beta}=\theta^{\gamma}_{\alpha\beta}} \mathbf{C}^{\bar\gamma'\gamma'}\Gamma^{\alpha\delta}_{\gamma'}\dot S^{\beta'\alpha'}_{\alpha\beta''}(\theta_{\alpha\beta})\dot S^{\beta''\delta'}_{\delta\beta}(0)\Gamma^{\gamma}_{\alpha'\delta'}\mathbf{C}_{\gamma\bar\gamma} \\
&= \mathbf{C}^{\bar\gamma'\gamma'}\Gamma^{\alpha\beta''}_{\gamma'}\Gamma^{\beta'\alpha'}_{\gamma''}\Gamma^{\gamma''}_{\alpha\beta''}\Gamma^{\gamma}_{\alpha'\beta}\mathbf{C}_{\gamma\bar\gamma} \\
&= \mathbf{C}^{\bar\gamma'\gamma'}\Gamma^{\beta'\alpha'}_{\gamma'}\Gamma^{\gamma}_{\alpha'\beta}\mathbf{C}_{\gamma\bar\gamma} = \Gamma^{\bar\gamma'\beta'}_{\bar\alpha}\Gamma^{\bar\alpha}_{\beta\bar\gamma}
\end{aligned}$$

where the crossing relation of the S-matrix, the orthogonality relation of the bound state intertwiners (37), the residue formula (34), the permutation property (9) of the S-matrix at $\theta = 0$ and the crossing relations of the intertwiners (42) have been used. Obviously the rapidities satisfy the equivalence

$$\left.\begin{array}{l} \theta_{\beta\bar\gamma} = iu^{\bar\alpha}_{\beta\bar\gamma} \\ \theta_{\alpha\delta} = iu^{\gamma}_{\alpha\delta} \end{array}\right\} \Leftrightarrow \left\{\begin{array}{c} \theta_{\alpha\beta} = iu^{\gamma}_{\alpha\beta} \\ \theta_\beta = \theta_\delta \end{array}\right.$$

(note that $\theta_{\alpha\beta} = \theta_\alpha - \theta_\beta$ etc.). An analogous relation can be shown for the process $\bar\gamma + \alpha \to \bar\beta$. The relation of the three fusion angles may be read off figure 7. ■

E Proof of proposition 7

In this appendix we prove that the bound state form factors given by property (iv) on page 7 satisfy property (iii) which is the recursion relation of n-particle form factors to $(n-2)$-particle form factors. We consider (iii) in the form

$$\operatorname*{Res}_{\theta_{(12)(34)}=i\pi} \mathcal{O}_{(12)(34)5\dots n}(\theta_{(12)},\theta_{(34)},\underline{\theta}') = 2i\mathbf{C}_{(12)(34)}\mathcal{O}_{5\dots n}(\underline{\theta}') \times \left(1 - S_{(34)n}\dots S_{(34)5}\right)$$

with $\underline{\theta}' = \theta_5,\dots,\theta_n$. First we note that the bound state formula (iv) on page 8 for form factors may also be written in a form without taking a residue. This follows if we make the bound state pole at $\theta_{12} = iu$ explicit by using (i) such that

$$\begin{aligned}
\mathcal{O}_{(12)3\dots n}(\theta_{(12)},\underline{\theta}') &= \frac{1}{\sqrt2}\operatorname*{Res}_{\theta_{12}=iu} \mathcal{O}_{123\dots n}(\theta_1,\theta_2,\dots,\theta_n)\Gamma^{12}_{(12)}(-u) \\
&= \frac{1}{\sqrt2}\operatorname*{Res}_{\theta_{12}=iu} \mathcal{O}_{213\dots n}(\underline\theta)\dot S_{12}(\theta_{12})\Gamma^{12}_{(12)}(-u) \\
&= -i\frac{1}{\sqrt2}\mathcal{O}_{213\dots n}(\underline\theta)\Big|_{\theta_{12}=iu}\Gamma^{21}_{(12)}(u). \qquad \text{(E.1)}
\end{aligned}$$

where also the definition (34) and the orthogonality relation (37) of the bound state intertwiners have been used.

The form factor $\mathcal{O}_{1234\ldots n}(\underline{\theta})$ has poles at $\theta_{12}=iu$, $\theta_{34}=iu$, $\theta_{13}=i\pi$, $\theta_{24}=i\pi$. We extract the two first poles by applying formula (E.1) twice and introduce the function

$$G(\theta_{12},\theta_{34},\theta)=\mathcal{O}_{21435\ldots n}(\underline{\theta}')\,\Gamma^{21}_{(12)}\Gamma^{43}_{(34)}\,,\quad \left(\theta=\theta_{(12)(34)}=\theta_{(12)}-\theta_{(34)}\right)$$

where we suppress the co-vector structure and the variables $\theta_4,\ldots,\theta_n$. If for simplicity we assume that all masses are equal then $\theta_{(12)}=\frac{1}{2}(\theta_1+\theta_2)$ and $\theta_{(34)}=\frac{1}{2}(\theta_3+\theta_4)$ are the center-of-mass rapidities. Extracting the poles at $\theta_{13}=i\pi$ and $\theta_{24}=i\pi$, the remainder is analytic and can be expanded as

$$G(\theta_{12},\theta_{34},\theta)=\frac{A+(\theta_{12}-iu)B+(\theta_{34}-iu)C+(\theta-i\pi)D}{(\theta_{13}-i\pi)(\theta_{24}-i\pi)}+\ldots \tag{E.2}$$

with certain constants A,B,C,D. We consider three limiting procedures:

1. Let first $\theta_{13}\to i\pi$, then $\theta_{34}\to iu$ and finally $\theta\to i\pi$:
This means that also $\theta_{12}\to iu$ and $\theta_{24}\to i\pi$. Due to (i) we have $G(\theta_{12},\theta_{34},\theta)=\mathcal{O}_{1324\ldots n}\dot S_{23}\dot S_{21}\dot S_{43}\Gamma^{21}_{(12)}\Gamma^{43}_{(34)}$ and therefore with (iii) for the particle pair 13

$$\begin{aligned}
G_1(\theta)&=\left[\operatorname*{Res}_{\theta_{13}=i\pi}G(\theta_{12},\theta_{34},\theta)\right]_{\theta_{34}=iu}\\
&=2i\left[\mathbf{C}_{13}\mathcal{O}_{24\ldots n}\left(\mathbf{1}-S_{3n}\ldots S_{34}S_{32}\right)\dot S_{23}\dot S_{21}\dot S_{43}\Gamma^{21}_{(12)}\Gamma^{43}_{(34)}\right]_{\theta_{34}=iu}\\
&=2i\mathbf{C}_{13}\mathcal{O}_{24\ldots n}\sigma_{43}\left(-S_{3n}\ldots S_{35}S_{21}\right)\Gamma^{21}_{(12)}\Gamma^{43}_{(34)}
\end{aligned}$$

$$\begin{aligned}
&\approx\frac{(2i)^2}{\theta_{24}-i\pi}\mathbf{C}_{13}\mathbf{C}_{24}\sigma_{43}\mathcal{O}_{5\ldots n}(1-S_{4n}\ldots S_{45})\left(-S_{3n}\ldots S_{35}S_{21}\right)\Gamma^{21}_{(12)}\Gamma^{43}_{(34)}\\
&\approx(2i)^2i\frac{\theta_{12}-iu}{\theta_{24}-i\pi}\mathbf{C}_{13}\mathbf{C}_{24}\sigma_{43}\sigma_{12}\mathcal{O}_{5\ldots n}(1-S_{4n}\ldots S_{45})\left(-S_{3n}\ldots S_{35}\right)\Gamma^{12}_{(12)}\Gamma^{43}_{(34)}\\
&\approx-(2i)^2i\mathbf{C}_{13}\mathbf{C}_{24}\mathcal{O}_{5\ldots n}\left(-S_{3n}\ldots S_{35}\Gamma^{43}_{(34)}+\Gamma^{43}_{(34)}S_{(34)n}\ldots S_{(34)5}\right)\Gamma^{12}_{(12)}
\end{aligned}$$

In deriving the third line unitarity (7) and crossing (8) of the S-matrix has bee used. Also for $(i,j)=(3,4)$ and $(1,2)$ we have used that

$$S_{ji}\Gamma^{ji}_{(ij)}(u)\approx i(\theta_{ij}-iu)\sigma_{ij}\Gamma^{ij}_{(ji)}(-u) \tag{E.3}$$

which follows from the unitarity (7) of the S-matrix, the definition (34) and the orthogonality (37) of the bound state intertwiners. Further we have used again (iii) for the particle pair 24 and the fact that $(\theta_{12}-iu)/(\theta_{24}-i\pi)=-1$ holds for $\theta_{13}=i\pi$ and $\theta_{34}=iu$. The statistics factors $\sigma_{43}\sigma_{12}$ cancel since $1=\bar 2$ and $3=\bar 4$. Also the fusion rule $\dot S_{4i}\dot S_{3i}\Gamma^{43}_{(34)}=\Gamma^{43}_{(34)}\dot S_{(34)i}$ has been applied. On the other hand we obtain from eq. (E.2) using $\theta_{24}-i\pi=2(\theta-i\pi)$ for $\theta_{13}=i\pi$ and $\theta_{12}-iu=-2(\theta-i\pi)$ for $\theta_{13}=i\pi$ and $\theta_{34}=iu$

$$\begin{aligned}
G_1(\theta)&=\frac{A-2(\theta-i\pi)B+(\theta-i\pi)D}{2(\theta-i\pi)}+\ldots\\
&=\frac{A}{2(\theta-i\pi)}-B+\tfrac{1}{2}D+\ldots
\end{aligned}$$

Since $G_1(\theta)$ is non-singular for $\theta \to i\pi$ we conclude that $A = 0$ and

$$
\begin{aligned}
G_1(i\pi) &= -B + \tfrac{1}{2}D \\
&= -(2i)^2 i \mathbf{C}_{13}\mathbf{C}_{24}\mathcal{O}_{5\ldots n}\left(-S_{3n}\ldots S_{35}\Gamma^{43}_{(34)} + \Gamma^{43}_{(34)}S_{(34)n}\ldots S_{(34)5}\right)\Gamma^{12}_{(12)}
\end{aligned}
$$

2. Let now $\theta_{24} \to i\pi$, then $\theta_{34} \to iu$ and finally $\theta \to i\pi$:
Due to (i) we have $G(\theta_{12}, \theta_{34}, \theta) = \mathcal{O}_{2413\ldots n}\dot{S}_{14}\Gamma^{21}_{(12)}\Gamma^{43}_{(34)}$ and therefore similarly as above with (iii) for the particle pair 24

$$
\begin{aligned}
G_2(\theta) &= \left[\operatorname*{Res}_{\theta_{24}=i\pi} G(\theta_{12}, \theta_{34}, \theta)\right]_{\theta_{34}=iu} \\
&= 2i\left[\mathbf{C}_{24}\mathcal{O}_{13\ldots n}\left(\mathbf{1} - S_{4n}\ldots S_{45}S_{43}S_{41}\right)\dot{S}_{14}\Gamma^{21}_{(12)}\Gamma^{43}_{(34)}\right]_{\theta_{34}=iu} \\
&\approx 2i\mathbf{C}_{24}\mathcal{O}_{13\ldots n}\sigma_{21}S_{21}\Gamma^{21}_{(12)}\Gamma^{43}_{(34)} \\
&\approx (2i)^2 i\frac{\theta_{12} - iu}{\theta_{13} - i\pi}\mathbf{C}_{13}\mathbf{C}_{24}\mathcal{O}_{5\ldots n}(1 - S_{3n}\ldots S_{35})\Gamma^{12}_{(12)}\Gamma^{43}_{(34)} \\
&\approx (2i)^2 i\mathbf{C}_{13}\mathbf{C}_{24}\mathcal{O}_{5\ldots n}(1 - S_{3n}\ldots S_{35})\Gamma^{12}_{(12)}\Gamma^{43}_{(34)}
\end{aligned}
$$

again we have used (E.3) and the fact that $(\theta_{12} - iu)/(\theta_{13} - i\pi) = 1$ holds for $\theta_{24} = i\pi$ and $\theta_{34} = iu$. On the other hand we obtain from eq. (E.2) using $\theta_{13} - i\pi = 2(\theta - i\pi)$ for $\theta_{24} = i\pi$ and $\theta_{12} - iu = 2(\theta - i\pi)$ for $\theta_{13} = i\pi$ and $\theta_{34} = iu$

$$
\begin{aligned}
G_2(\theta) &= \frac{A + 2(\theta - i\pi)B + (\theta - i\pi)D}{2(\theta - i\pi)} + \ldots \\
&= \frac{A}{2(\theta - i\pi)} + B + \tfrac{1}{2}D + \ldots
\end{aligned}
$$

such that with $A = 0$

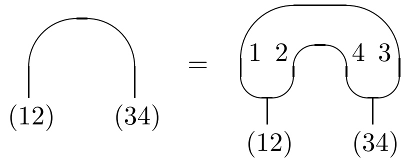
$$
\begin{aligned}
G_2(i\pi) &= B + \tfrac{1}{2}D \\
&= (2i)^2 i\mathbf{C}_{13}\mathbf{C}_{24}\mathcal{O}_{5\ldots n}(1 - S_{3n}\ldots S_{35})\Gamma^{12}_{(12)}\Gamma^{43}_{(34)}
\end{aligned}
$$

3. Let finally $\theta_{12} = \theta_{34} = iu$ and then $\theta \to i\pi$:
Applying formula (E.1) twice the form factor of two bound states and arbitrary other particles is obtained as $\mathcal{O}_{(12)(34)5\ldots n}(\theta_{(12)}, \theta_{(34)}, \underline{\theta}') = -\frac{1}{2}G(iu, iu, \theta)$. Therefore using $\theta_{13} = \theta_{24} = \theta = \theta_{(12)(34)}$ for $\theta_{12} = \theta_{34}$ we obtain

$$
\begin{aligned}
\operatorname*{Res}_{\theta_{(12)(34)}=i\pi} \mathcal{O}_{(12)(34)5\ldots n}(\theta_{(12)}, \theta_{(34)}, \underline{\theta}') &= -\tfrac{1}{2}\operatorname*{Res}_{\theta=i\pi} G(iu, iu, \theta) \\
&= -\tfrac{1}{2}D = -\tfrac{1}{2}(G_1(i\pi) + G_2(i\pi)) \\
&= 2i\mathbf{C}_{13}\mathbf{C}_{24}\Gamma^{12}_{(12)}\Gamma^{43}_{(34)}\mathcal{O}_{5\ldots n}(\underline{\theta}')(1 - S_{(34)n}\ldots S_{(34)5})
\end{aligned}
$$

which proves claim with the charge conjugation matrix for the bound states

$$
\mathbf{C}_{(12)(34)} = \mathbf{C}_{13}\mathbf{C}_{24}\Gamma^{12}_{(12)}\Gamma^{43}_{(34)}
$$



■

F Residue formula

In order to obtain the breather form factors from the soliton form factors we have to calculate the residue

$$\operatorname*{Res}_{\theta_{12}=iu^{(k)}} \mathcal{O}_{123\ldots n}(\underline{\theta}) = \operatorname*{Res}_{\theta_{12}=iu^{(k)}} (-2\pi i)\, m \sum_{l=0}^{k} \operatorname*{Res}_{z_1=z^{(l)}} \int_{\mathcal{C}_{\underline{\theta}}} dz_2 \cdots \int_{\mathcal{C}_{\underline{\theta}}} dz_m \\ \times h(\underline{\theta},\underline{z}) p^{\mathcal{O}}(\underline{\theta},\underline{z})\, \Psi_{1\ldots n}(\underline{\theta},\underline{z})$$

where the pinching rule of contour integrals has been applied.

Lemma 8 *For any analytic function $P(z)$ the following identity holds*

$$\operatorname*{Res}_{\theta_{12}=iu^{(k)}} \operatorname*{Res}_{z_1=z^{(l)}} P(z)\, \phi(\theta_1-z_1)\phi(\theta_2-z_1)\, \Psi_{1\ldots n}(\underline{\theta},\underline{z}) \\ = c(k,l)(-1)^l P(z^{(l)}) \prod_{i=3}^{n} \dot{a}(\theta_i - z^{(l)}) \prod_{j=2}^{m} S_{b_k s}(\xi - z_j) \Gamma_{12}^{b_k} \Psi_{3\ldots n}(\underline{\theta}',\underline{z}')$$

with $\underline{\theta}' = \theta_3,\ldots,\theta_n, \underline{z}' = z_2,\ldots,z_m$ *and*

$$\Psi_{3\ldots n}(\underline{\theta}',\underline{z}') = \Omega_{3\ldots n} C_{3\ldots n}(\underline{\theta}',z_2)\cdots C_{3\ldots n}(\underline{\theta}',z_m).$$

The constant is

$$c(k,l) = R_{k-l} R_l\, i \left| \tfrac{1}{2} \operatorname*{Res}_{\theta=i\pi(1-k\nu)} \dot{S}_{\pm}(\theta) \right|^{-1/2}$$

with

$$R_l = \operatorname*{Res}_{z=i\pi l\nu} \phi(z)\dot{a}(z).$$

Proof. Using the decomposition

$$C_{1\ldots n} = C_{12} A_{3\ldots n} + D_{12} C_{3\ldots n}$$

and the action on the pseudo ground state Ω

$$\begin{aligned} \Omega_{12} C_{12} &= \dot{b}(\theta_1 - z)\dot{c}(\theta_2 - z)\, (s\otimes\bar{s})_{12} + \dot{c}(\theta_1 - z)\dot{a}(\theta_2 - z)\, (\bar{s}\otimes s)_{12} \\ \Omega_{12} D_{12} &= \Omega_{12} \dot{b}(\theta_1 - z)\dot{b}(\theta_2 - z). \end{aligned}$$

we obtain

$$\operatorname*{Res}_{\theta_{12}=iu^{(k)}} \operatorname*{Res}_{z_1=z^{(l)}} P(z)\, \phi(\theta_1-z_1)\phi(\theta_2-z_1)\, \Omega_{1\ldots n} C_{1\ldots n}(\underline{\theta},\underline{z}) \\ = P(z^{(l)}) R_{k-l} R_l (-1)^l \left((s\otimes\bar{s})_{12} + (-1)^k\, (\bar{s}\otimes s)_{12} \right) \Omega_{3\ldots n} A_{3\ldots n}$$

The relations

$$\begin{aligned} \operatorname*{Res}_{\theta_{12}=i\pi(1-k\nu)} \phi(\theta_1 - z^{(l)})\dot{b}(\theta_1 - z^{(l)}) &= \operatorname*{Res}_{\theta=i\pi(1-(k-l)\nu)} \phi(\theta)\dot{b}(\theta) \\ &= -\operatorname*{Res}_{\theta=i\pi(k-l)\nu} \phi(\theta)\dot{a}(\theta) = -R_{k-l} \\ \operatorname*{Res}_{z=z^{(l)}} \phi(\theta_2 - z)\dot{a}(\theta_2 - z) &= \operatorname*{Res}_{z=-i\pi l\nu} \phi(-z)\dot{a}(-z) = -R_l \end{aligned}$$

and

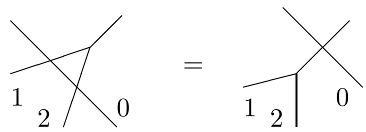
$$\frac{c(i\pi l\nu)}{a(i\pi l\nu)} = (-1)^l, \quad \frac{c(i\pi(1-(k-l)\nu)}{b(i\pi(1-(k-l)\nu)} = (-1)^{k-l}, \quad \frac{b(i\pi l\nu)}{a(i\pi l\nu)} = 0$$

have been used. Further we use that the bound state intertwiner may be written as

$$\Gamma_{12}^{b_k} = (-1)^k i \left| \tfrac{1}{2} \operatorname*{Res}_{\theta=i\pi(1-k\nu)} \dot{S}_{\pm}(\theta) \right|^{1/2} \left(s \otimes \bar{s} + (-1)^k\, \bar{s} \otimes s\right)_{12}.$$

Applying further C-operators we use the fusion relation

$$\Gamma_{12}^{(12)} S_{10}(\theta + \tfrac{1}{2}\theta_{12}^{(12)}) S_{20}(\theta - \tfrac{1}{2}\theta_{12}^{(12)}) \quad = \quad S_{(12)0}(\theta)\Gamma_{12}^{(12)}$$



which implies

$$\Gamma_{12}^{b_k} C_{1\ldots n}(\underline{\theta}, z)|_{\theta_{12}=iu^{(k)}} = S_{b_k s}(\theta_{(12)} - z)\Gamma_{12}^{b_k} C_{3\ldots n}(\underline{\theta}', z)\,.$$

Together with the eigenvalue equation

$$\Omega_{3\ldots n} A_{3\ldots n}(\underline{\theta}', z) = \prod_{i=3}^{n} \dot{a}(\theta_i - z)\Omega_{3\ldots n}$$

the claim follows. ■

Using the shift relation

$$\phi(z + i\pi\nu)\dot{a}(z + i\pi\nu) = -\frac{\cos\frac{1}{2i}z}{\sin\frac{1}{2i}(z + i\pi\nu)}\phi(z)\dot{a}(z)$$

the residues R_l can be calculated

$$R_l = R_0 \frac{(-1)^l}{\sin\frac{1}{2}\pi l\nu} \prod_{j=1}^{l-1} \cot\tfrac{1}{2}j\pi\nu$$

with

$$R_0 = \operatorname*{Res}_{z=0} \phi(z)\dot{a}(z) = \operatorname*{Res}_{z=0} \frac{\dot{a}(z)}{F(z)F(i\pi - z)} = \frac{-1}{F'(0)}.$$

In particular for the lowest breather $k = 1$ we obtain the constants independent of l

$$c(1,l) = R_0 R_1\, i \left| \tfrac{1}{2} \operatorname*{Res}_{\theta=i\pi(1-k\nu)} \dot{S}_{\pm}(\theta) \right|^{-1/2} = \frac{i}{\pi\varkappa\sqrt{\sin\pi\nu}}$$

where $\varkappa$ is defined by eq. (19).Using these results the form factors for arbitrary numbers of lowest breathers and solitons are calculated explicitly in section 7. There we use the constant

$$\begin{aligned} d(\nu) &= \frac{-2\pi i}{\sqrt{2}} F(i\pi(1-\nu))\, c(1,0) = \frac{-2\pi i}{\sqrt{2}} \sqrt{\cos\tfrac{\pi}{2}\nu E(\nu)} \frac{i}{\pi\varkappa\sqrt{\sin\pi\nu}} \\ &= \frac{\sqrt{E(\nu)}}{\varkappa\sqrt{\sin\frac{1}{2}\pi\nu}}. \end{aligned}$$

where the function $E(\nu)$ is defined as

$$E(\nu) = \exp \int_0^\infty \frac{dt}{t} \frac{\sinh \nu t}{2\cosh^2 \frac{1}{2}t} = \exp \frac{1}{\pi} \int_0^{\pi\nu} \frac{tdt}{\sin t}\,.$$

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