Born-Infeld Kinematics and Correction to the Thomas Precession

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Dynamical symmetries of Born-Infeld theory associated with its maximal field strength are encoded in a geometry on the tangent bundle of spacetime manifolds. The resulting extension of general relativity respecting a finite upper bound on accelerations is put to use in the discussion of particle dynamics, first quantization, and the derivation of a correction to the Thomas precession.

1 Introduction

In classical mechanics, the study of phase space geometry yields deep insights which would be hidden in a mere configuration space formulation. In particular, there is no well-defined distinction between coordinates and momenta, as these mix under symplectic transformations. Geometric quantization aims at exploiting the symplectic structure of phase space in order to understand the transition to quantum systems.

In contrast [1], special and general relativity are formulated merely on spacetime. This is of course likewise true for all theories built on this framework, most notably quantum field theory and string theory.

Over the last two decades, there has been some interest in and speculation on a finite upper bound on accelerations. This is mainly due to the fact that although special relativity allows arbitrarily high accelerations, upon quantization, a finite upper bound enters through the back door [2]. This raises the question of whether to use kinematics respecting a distinguished acceleration from start. Indeed, careful quantization of a particle with *dynamically* enforced submaximal acceleration [3] nourishes the hope that a finite upper bound on accelerations might positively influence the convergence behaviour of field theoretic amplitudes. This approach, however, makes an ad-hoc assumption of a maximal acceleration and admittedly lacks a proper kinematical framework.

The aim of this letter is to devise such kinematics, by kinematizing dynamical symmetries of the Born-Infeld action on the velocity phase space of the spacetime manifold, and to derive a correction to the Thomas precession within this framework.

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2 Kinematization

A particle of mass m and electric charge e minimally coupled to Born-Infeld theory [4]

$$\mathcal{L}_{\text{BI}} = \det^{\frac{1}{2}}\left(g_{\mu\nu} + bF_{\mu\nu}\right) \tag{1}$$

can at most experience an acceleration $\mathfrak{a} = eb^{-1}m^{-1}$, as the maximal field strength is given by b^{-1}.

Denoting a point of the tangent bundle TM of the n-dimensional spacetime manifold M by $X^m \equiv (\mathfrak{a}x^\mu, u^\mu)$, where u^μ is the n-velocity of the particle, the Born-Infeld Lagrangian (1) can be written

$$\mathcal{L}_{\text{BI}} = \det^{\frac{1}{4}}\left([X^m, X^n]\right), \tag{2}$$

if one assumes a b^2-suppressed coordinate non-commutativity of spacetime in the presence of an electromagnetic field $F^{\mu\nu}$,

$$[x^\mu, x^\nu] = -ie^{-3}b^2 F^{\mu\nu}, \tag{3}$$
$$[x^\mu, p^\nu] = -ig^{\mu\nu}, \tag{4}$$
$$[p^\mu, p^\nu] = -ieF^{\mu\nu}. \tag{5}$$

Associated with the existence of a distinguished acceleration $\mathfrak{a}$ there must be dynamical symmetries of Born-Infeld theory. The form (2) suggests encoding these in the geometry of the spacetime tangent bundle. We show that this indeed results in a consistent kinematical framework that extends relativity such as to respect a finite upper bound on accelerations. Note that shifting the upper bound $\mathfrak{a}$ to infinity restores coordinate commutativity. Hence spacetime non-commutativity is a signature of a finite upper bound on accelerations.

3 Maximal Acceleration Geometry

Consider the diagonal lift [6] (here given in induced tangent bundle coordinates (x^μ, u^μ))

$$g^D = \begin{pmatrix} g_{ij} + g_{ts}\Gamma^t{}_i\Gamma^s{}_j & \Gamma^t{}_j g_{ti} \\ \Gamma^t{}_i g_{tj} & g_{ij} \end{pmatrix} \tag{6}$$

of the spacetime metric g to the tangent bundle, where $\Gamma^t{}_i \equiv u^a \Gamma_a{}^t{}_i$, and Γ are the Christoffel symbols of g. Requiring positivity of the natural lift [6] $x^* \equiv \left(\mathfrak{a}x^\mu, \frac{dx^\mu}{d\tau}\right)$ of a timelike spacetime curve x,

$$g^D(dx^*, dx^*) > 0 \tag{7}$$

is equivalent to restricting the acceleration of the projection $\pi(x^*) = x$ to values less than $\mathfrak{a}$. This was first observed by Caianiello [5] for flat spacetime.

4 Complex Tangent Bundles?

There were attempts [7, 8] to devise a maximal acceleration modification of special and general relativity by equipping the tangent bundle with the metric g^D and an additional complex structure F, analogous to the phase space structure of non-relativistic systems. However, invoking a strong principle of equivalence, one must require both structures to be covariantly constant,

$$\nabla g^D = 0 \qquad \text{and} \qquad \nabla F = 0, \tag{8}$$

where ∇ is the Levi-Cività connection with respect to g^D. Then according to the Tachibana-Okumura theorem [9], conditions (8) are satisfied if, and only if, the base manifold M is flat. Hence complex tangent bundles can never provide a theory of gravity with finite upper bound on accelerations.

5 Bimetric Tangent Bundles

The Tachibana Okumura no-go theorem can be circumvented by using the horizontal lift (here given in induced tangent bundle coordinates (x^μ, u^μ))

$$g^H = \begin{pmatrix} u^a \partial_a g_{ij} & g_{ij} \\ g_{ji} & 0 \end{pmatrix} \tag{9}$$

of the spacetime metric instead of a complex structure. It can be shown [6] that the horizontal lift ∇^H of the Levi-Cività connection on spacetime is then the unique linear connection on TM satisfying

$$\nabla^H g^D = 0 \qquad \text{and} \qquad \nabla^H g^H = 0 \tag{10}$$

for arbitrary curvatures of the base manifold M. This, however, comes at the cost of introducing torsion to the tangent bundle. A tangent bundle curve $X : \mathbb{R} \longrightarrow TM$ is called an orbit if there exists a frame where $X = \pi(X)^*$. An orbit is called an orbidesic if it is a geodesic with respect to some metric, or an orbiparallel if it is an autoparallel with respect to some connection.

The lifting and projection properties of spacetime geodesics and tangent bundle orbidesics and orbiparallels are well known [6] and summarized in the diagram
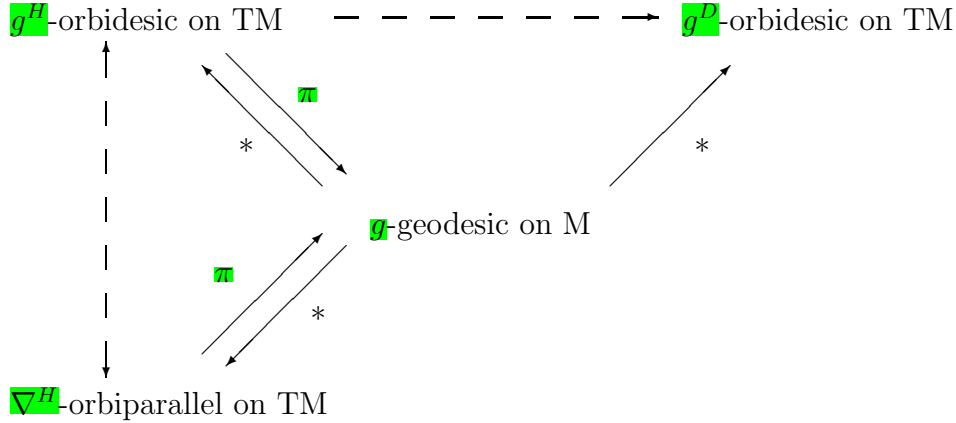
In particular, note that g^H-orbidesics coincide with ∇^H-orbiparallels, despite ∇^H having non-vanishing torsion [6]. Further, one can show that all g^H-orbidesics are g^H-null and g^D-positive.

6 Extended General Relativity

The mathematical structure outlined above allows one to formulate physical postulates for the kinematics of an extension of general relativity respecting a finite upper bound on accelerations.

I. Submaximally accelerated particles are described by orbits X satisfying

$$g^H(dX, dX) = 0, \qquad (11)$$
$$g^D(dX, dX) > 0. \qquad (12)$$

II. In the absence of non-gravitational interaction, orbits of particles are g^H-(null-)geodesics.

III. The physical time experienced by an observer with orbit X is measured by

$$d\omega \equiv \left[g^D(dX, dX)\right]^{\frac{1}{2}}. \qquad (13)$$

Note that g^H-orbidesics on TM coincide with g-geodesics on M, and are automatically g^D-positive. Hence, for unaccelerated motion, the kinematics of standard general relativity are recovered. This explains why for accelerations that are small compared to the upper limit a, we only need one single metric g on spacetime.

For non-g^H-geodesic motion, the metric g^D assumes a non-trivial rôle, as it restricts the deviation of orbits from g^H-orbidesics to the g^D-positive ones.

Condition (11) enforces the orthogonality of the n-velocity and n-acceleration of a particle in any frame. Thus we now recognize that in standard general relativity already, the horizontal lift g^H is a *naturally induced* structure on the associated tangent bundle.

Postulate III predicts a deviation from the relativistic physical time for accelerated particles. From decay experiments [10] we get a lower bound for the maximal acceleration a of about $10^{19}ms^{-2}$. From high-precision measurements of the Thomas precession, we get a better lower bound in section 10.

7 Field Equations

It is conceptually inevitable to formulate the extended theory entirely on the tangent bundle without recurring to spacetime objects. The latter are viewed as derived concepts via the canonical bundle projection. In this spirit we find [11] the lifted Einstein field equations

$$(G^{ambn} - \frac{1}{2}G^{abmn})R^V_{mn} = 8\pi G T^D_{mn}, \qquad (14)$$

where R^V denotes the vertical lift [6] of the spacetime Ricci tensor R, and

$$G^{abcd} \equiv g^{D_{ab}}g^{D_{cd}} + g^{H_{ab}}g^{H_{cd}}. \qquad (15)$$

The lifted equations (14) are equivalent to the Einstein field equations on spacetime.

8 Extended Special Relativity

The case of flat Minkowski spacetime can be studied in a very concise and illuminating way. The diagonal and horizontal lifts of the Minkowski metric η can be written as

$$\eta^D = \eta \otimes 1 \quad \text{and} \quad \eta^H = \eta \otimes I,$$

with

$$1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad I \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{16}$$

This motivates the study of the pseudo-complex numbers

$$\mathbb{P} \equiv \left\{ a + Ib \mid a, b \in \mathbb{R}, I^2 = +1 \right\}.$$

As these build a commutative ring, pseudo-complex Lie algebras can be defined [13]. The pseudo-complexification $V_{\mathbb{P}}$ of a real vectorspace V is a free module of pseudo-complex dimension $\dim_{\mathbb{R}} V$, and we have the isomorphism of $V_{\mathbb{P}} \cong TV$ as real vectorspaces. If V is a representation space for a real Lie algebra L, then $L_{\mathbb{P}}$ acts naturally on $V_{\mathbb{P}}$.

We are particularly interested in the pseudo-complexification of Minkowski spacetime $(\mathbb{R}^n, \eta)$ and the real Lorentz group $SO_{\mathbb{R}}(1, n-1)$. On the algebra level one gets

$$\begin{aligned}
so_{\mathbb{P}}(1, n-1) &= \langle M^{\mu\nu} \rangle_{\mathbb{P}} \\
&= \langle M^{\mu\nu}, IM^{\mu\nu} \rangle_{\mathbb{R}}
\end{aligned} \tag{17}$$

and obtains the connection component of the identity of the pseudo-complex Lorentz group by exponentiation. Changing the basis in (17) to $G^{\mu\nu} \equiv \frac{1}{2}(M^{\mu\nu} + IM^{\mu\nu})$ and $\overline{G}^{\mu\nu} \equiv \frac{1}{2}(M^{\mu\nu} - IM^{\mu\nu})$ we get the decomposition

$$so_{\mathbb{P}}(1, m) = so_{\mathbb{R}}(1, m) \oplus so_{\mathbb{R}}(1, m).$$

Thus, the representation theory in the pseudo-complex case can be easily obtained from the real case.

Clearly, for a pseudo-complex n-vector $U^{\mu} \equiv u^{\mu} + Ia^{\mu}$, the expression

$$\begin{aligned}
\eta(U, U) &\equiv U^{\mu} U^{\nu} \eta_{\mu\nu} \\
&= (u^{\mu} u_{\mu} + a^{\mu} a_{\mu}) + I(2u^{\mu} a_{\mu})
\end{aligned} \tag{18}$$

is $SO_{\mathbb{P}}(1, n-1)$-invariant, separately in the real and pseudo-imaginary parts. This shows the isomorphism of

$$\left(T\mathbb{R}^n, \eta^D, \eta^H \right) \cong (\mathbb{P}^n, \eta) \tag{19}$$

as inner product spaces. Hence, in the flat case we can trade a *bi*metric real vectorspace against a metric module. Essentially, special relativity with pseudo-complex coordinates is extended special relativity.

In particular, according to (17) all pseudo-complex Lorentz transformations can be composed of the standard boost and rotation transformations with pure real and pure pseudo-imaginary arguments. For the interpretation of the pseudo-boosts, it is instructive to consider the orbit X induced by a spacetime curve describing a submaximally accelerated hyperbolic motion. Using the Lorentz-invariance of the theory, we can always arrange for this motion to be in 1-direction in a rest frame at coordinate time zero. As $d\omega$ is an $SO_{\mathbb{P}}(1, 3)$-invariant, we

may study the action of a pseudo-boost on the covariant velocity $U \equiv \frac{dX}{d\omega} \equiv \tilde{u} + I\tilde{a}$. For hyperbolic motion of spacetime curvature $g \equiv \mathfrak{a}\tanh(\alpha)$, the projections π_{01} and π_{10} of U into the $\tilde{u}^0 - \tilde{a}^1$ and $\tilde{u}^1 - \tilde{a}^0$ planes, respectively, are straight lines through the origin of hyperbolic angle $\tanh^{-1}\left(\frac{g}{\mathfrak{a}}\right)$:

$$\tilde{a}^0 = \frac{g}{\mathfrak{a}}\tilde{u}^1, \tag{20}$$

$$\tilde{a}^1 = \frac{g}{\mathfrak{a}}\tilde{u}^0. \tag{21}$$

A pseudo-boost with boost parameter $I\beta$ can be easily seen to hyperbolically rotate both of these lines by an angle β within their respective planes. Hence, the π_{01} and π_{10} projections of the transformed curve coincide with the projections for a U-curve corresponding to a hyperbolic motion with spacetime curvature $\mathfrak{a}\tanh(\alpha + \beta)$. Carefully counting degrees of freedom, one checks that the transformation of the two projections already determines the transformation of the whole U-curve. Hence, the pseudo-boosts are transformations to relatively uniformly accelerated frames, respecting the maximal acceleration $\mathfrak{a}$.

A pseudo-complex Lorentz tranformation of an orbit X clearly induces a transformation of the spacetime projection $\pi(X)$. For not purely real transformation parameters, however, these transformations cannot be understood as maps $M \longrightarrow M$, simply because the components projected out by $\pi : TM \longrightarrow M$ contribute to the transformation.

In other words, spacetime events fail to be well-defined under non-real Lorentz tranformations. Hence, we observe the breakdown of the *classical* spacetime particle concept when changing to an accelerated frame. This anticipates the Unruh effect [12] on a classical level already.

9   Dynamics

Prior approaches [14] to maximal acceleration dynamics enforce the finite upper bound on accelerations *dynamically*, i.e. by modified Lagrangians, but still in the kinematical framework of special or general relativity. The prototypical example is the massive particle studied by Nesterenko et al. [15],

$$L = \sqrt{\ddot{x}^\mu \ddot{x}_\mu - \mathfrak{a}^2}\sqrt{\dot{x}^\mu \dot{x}_\mu}dt^2. \tag{22}$$

This dynamical enforcement inevitably results in Lagrangians containing second order derivatives, being inconvenient for technical and conceptual reasons, especially in the transition to quantum theory.

In contrast, pseudo-complexification of merely relativistic Lagrangians, e.g. for the massive relativistic particle to

$$L_{\mathbb{P}} = \sqrt{\dot{X}^\mu \dot{X}_\mu}dt, \tag{23}$$

where

$$X \equiv \mathfrak{a}x + Iu, \tag{24}$$

results in first order Lagrangians. From relation (24) this might appear to be a mere notational trick. However, this is not the case as u and x are independent degrees of freedom in extended relativity, only linked by the pseudo-complex kinematics, in particular the orthogonality condition 11.

Studying the equations of motion for $L_{\mathbb{P}}$ and making full use of the pseudo-complex Lorentz

symmetry, we indeed get a free particle [11]. A fully $SO_{\mathbb{P}}(1,3)$-invariant coupling term to Born-Infeld theory can be constructed from the diagonal lift of the Kaluza-Klein metric from spacetime to the tangent bundle [11]. It turns out that mere pseudo-complexification of the standard minimal coupling term achieves the same with much less labour.

Pseudo-complexification applies to relativistic spaces, symmetry algebras and Lagrangians alike to give their extended relativistic counterparts. This makes the pseudo-complex formalism so worthwile.

10 Correction to Thomas Precession

Due to the non-commutativity of the boost generators

$$[M^{0i}, M^{0j}] = M^{ij} \tag{25}$$

in special relativity, an observer resting at the center of a uniform circular motion of angular velocity ω and radius R observes the Thomas precession [16]

$$\frac{d\theta_{SR}}{dt} = (\gamma_{SR} - 1)\omega \tag{26}$$

of the spatial coordinate system of an orbiting observer, where $\gamma_{SR} = (1 + R^2\omega^2)^{1/2}$. Exactly along the same lines, the non-commutativity of the pseudo-boost generators

$$[IM^{0i}, IM^{0j}] = M^{ij} \tag{27}$$

leads to an additional precession around the same axis, effecting a total precession rate

$$\frac{d\theta_{ESR}}{dt} = (\gamma_{ESR} - 1)\omega, \tag{28}$$

where $\gamma_{ESR} = (1 + R^2\omega^2 + R^2\omega^4\mathfrak{a}^{-2})^{1/2}$. From experimental data [17], obtained at $R\omega^2 = 10^{18}ms^{-2}$, $\gamma_{SR} = 1.2$, the ratio $\frac{\gamma_{ESR}}{\gamma_{SR}}$ deviates from unity by less than 5×10^{-9}. This yields a lower bound

$$\mathfrak{a} \geq 10^{22}ms^{-2} \tag{29}$$

and hence an upper bound on the Born-Infeld parameter

$$b \leq 10^{-11}\frac{C}{N}. \tag{30}$$

Thus high-precision measurements of the Thomas precession might be able to discriminate between Maxwell and Born-Infeld electrodynamics in the future.

Acknowledgements

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