NON-LINEAR SIGMA MODEL ON CONIFOLDS

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Abstract

Explicit solutions to the conifold equations with complex dimension $n = 3, 4$ in terms of *complex coordinates (fields)* are employed to construct the Ricci-flat Kähler metrics on these manifolds. The Kähler 2-forms are found to be closed. The complex realization of these conifold metrics are used in the construction of 2-dimensional non-linear sigma model with the conifolds as target spaces. The action for the sigma model is shown to be bounded from below. By a suitable choice of the 'integration constants', arising in the solution of Ricci flatness requirement, the metric and the equations of motion are found to be *non-singular*. As the target space is Ricci-flat, the perturbative 1-loop counter terms being absent, the model becomes topological. The inherent $U(1)$ fibre over the base of the conifolds is shown to correspond to a gauge connection in the sigma model.

The same procedure is employed to construct the metric for the resolved conifold, in terms of complex coordinates and the action for a non-linear sigma model with resolved conifold as target space, is found to have a minimum value, which is topological. The metric is expressed in terms of the six real coordinates and compared with earlier works. The harmonic function, which is the warp factor in Type II-B string theory, is obtained and the ten-dimensional warped metric has the $AdS_5 \times X_5$ geometry.

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1. Introduction

The original motivation behind the construction of conifolds was to find a non-trivial Calabi-Yau manifold to be used as the transverse space to Minkowski space in superstring theory, instead of the usual flat transverse space. In 1997, Maldacena [1] showed that the large N limit of $\mathcal{N} = 4$ $SU(N)$ gauge theory is related to Type II-B string theory on $AdS_5 \times S^5$. (S^5 preserves maximal number of supersymmetries, namely 32) Subsequently, other backgrounds such as $AdS_5 \times X^5$ for Type II-B theory have been explored with X^5 as Einsteinian, and are found to be related $\mathcal{N} = 1$ superconformal theory in four dimensions [2]. A new example of duality was found by Klebanov and Witten [3] where the X^5 is a smooth Einstein manifold and a coset space such as the one considered in the context of Kaluza-Klein theory [4,5]. The remarkable role played by conifolds acquired recent interest due to the observation of Klebanov and Strassler [6] that certain warped product, with the warp factor depending on the radial coordinate of the conifold, have the interpretation of D3-brane solution of supergravity. This is further confirmed by Herzog, Klebanov and Ouyang [7]. D3-branes on conifold points have been studied [8,9] and related to that of a Type II-B supergravity on a non-compact Calabi-Yau 3-fold which is a deformed conifold, topologically a 6-dimensional cone over $S^2 \times S^3$ base, apex being replaced by S^3. Wrapping of branes on 2-sphere of a conifold leading to fractional D-branes has been considered by Papadopoulos and Tseytlin [10]. A conformal field theory from Calabi-Yau 4-folds is obtained by Gukov, Vafa and Witten [11]. Further, a strong structural similarity between the deformed conifold and the moduli space of CP^1-lumps of unit charge is recently poited out by Speight [12]. Using the Wilson loop construction of Maldacena [1], to an $AdS_5 \times X^5$ background, Caceres and Hernandez [13] study the Higgs phase of large N field theory on the conifold and are able to relate quark-antiquark interaction to the parameter in the harmonic function (warp factor) in the near horizon limit.

While these studies provide a spectacular progress in our understanding of the Calabi-Yau compactification of string theory, it is worth recalling the observation of Strominger [14] that the low energy effective field theories arising from Calabi-Yau compactification are generally inconsistent or ill-defined at the classical level due to the conifold singularities, which arise in any Calabi-Yau compactification. Low energy effective theory, as described by sigma-model whose target space is the moduli space (or equally the conifold, see [12] for their structural similarity), has the equations of motion becoming singular. Therefore, it is

important to first understand non-linear sigma models whose target space is a conifold and this is one of the purposes of this study.

An explicit construction of the metric on Ricci-flat Kähler manifold, was first (to the best of our understanding) made by Candelas and de la Ossa [15] in terms of real coordinates. Non-singular Ricci flat Kähler metrics which generalize the Eguchi-Hanson metric were constructed by Stenzel [16] following the method of Gibbons and Pope [17]. Harmonic forms and Brane resolutions on these manifolds are recently constructed by Cvetic, Gibbons, Lu and Pope [18]. One observation in [18] namely, it is possible to choose the integration constants such that the metric is non-singular and the horizon can be completely eliminated, will be borne out by our explicit construction. This in turn allows us to realize non-linear sigma model on conifolds with the equations of motion not becoming singular at small distances, thereby avoiding the difficulty alluded by Strominger [14]. The construction of the Ricci-flat Kähler metric is based on the strategy: An $O(n)$ invariant quantity is first constructed and the Kähler potential is assumed to be a differentiable function of this invariant. The components of the metric are evaluated in terms of the first and second order differentials of this function with respect to the invariant. The determinant of the resulting metric is equated to a constant times $|\mathcal{F}|^2$, where $\mathcal{F}$ is a holomorphic function. This results in a non-linear differential equation for the function. It is enough to solve this for the first order differential of the function, thereby determining the Ricci-flat metric. This procedure introduces integration constants. The Kähler structure is then verified. This strategy is followed in [19] to construct the metrics on conifold. An alternative procedure is by working out covariantly constant spinor equations [20] as this implies Ricci flatness. In this paper, we follow the differential equation method. In our study, we employ complex coordinates for the conifold, which is not the case in the earlier works [15, 19]. This is partly due to our earlier usage of these coordinates to describe quadrics in CP-space [21] guided by the construction of generalized Gauss maps to describe 2-dimensional surfaces conformally immersed in R^n [22]. The use of complex coordinates facilitates the setting up of the action for non-linear sigma model with the target space as conifold, which is our second motivation. The action for this sigma model will be shown to have a lower bound and the equations of motion do not become singular by a choice of the 'integration constants'.

After the Ricci-flat Kähler metric construction in n=2 and 3 complex dimensional conifolds, and their role in non-linear sigma model, we consider the case of resolved conifold,

using the complex coordinates, as our third motivation. We rewrite the metrics in terms of real coordinates to compare with the earlier results. This gives a better understanding of the conifolds. In particular we clearly see how the choice of integration constants avoid the singularity in conifold in comparison with the resolved conifold. The additional sphere introduced to replace the apex of the conifold in the resolved conifold case, corresponds to the non-zero value of one of the integration constants in the ordinary conifold case.

2. Ricci-Flat Kähler Metric on Conifold

An $O(n)$ symmetric target space with n-complex fields (which can be thought of as coordinates) satisfying the equation,

$$\sum_{i=1}^{n} (\phi^i)^2 = 0, \tag{1}$$

is a conifold of real dimension $(2n-2)$. It is a smooth manifold except for the point $\phi^i = 0$. If ϕ^i solves (1), so also $\psi\phi^i$ for any $\psi \in C$ and so, the surface (1) is made up of complex lines passing through an origin, thereby giving the conical structure. This surface admits the U(1) symmetry group, namely, $\phi^i \to e^{i\theta}\phi^i$. The point $\phi^i = 0$ is a double point singularity. The points ϕ^i and $\psi \phi^i$ are not to be identified. The base of the conifold is given by the intersection of the space of solutions of (1) with a sphere in C^n,

$$\sum_{i=1}^{n} |\phi^i|^2 = r^2. \tag{2}$$

If the right side of (1) is replaced by a^2, then we get a 'deformed conifold' and the metric on this has been obtained in [15] in terms of real coordinates for the case n=4. The general procedure followed is to assume that the metric can be obtained from a scalar differentiable function of r^2, say $K(r^2)$. Then using complex coordinates, $g_{i\bar{j}} = \partial_i \partial_{\bar{j}} K(r^2)$. The determinant of the metric is equated to a constant times square of the modulus of a field (coordinate) so that the Ricci tensor, $R_{i\bar{j}} = -\partial_i \partial_{\bar{j}} log \ det \ g$ vanishes. This procedure yields a second order differential equation for $K(r^2)$ and it will be enough to solve for $K'(r^2) = \frac{dK(r^2)}{dr^2}$ for finding out the metric components.

For the (complex) n-conifold equation (1), we parameterize the n-complex fields ϕ^i sat-

isfying (1), by,

$$\{\phi^1, \phi^2, \cdots \phi^n\} = \psi\Big\{1 - \sum_{i=1}^{n-2}\zeta_i^2, i(1 + \sum_{i=1}^{n-2}\zeta_i^2), 2\zeta_1, 2\zeta_2, \cdots 2\zeta_{n-2}\Big\}, \tag{3}$$

in terms of $(n-1)$ complex fields $\psi, \zeta_1, \zeta_2, \cdots \zeta_{n-2}$. This parameterization has been suggested by Hoffman and Osserman [22] in their study of Gauss map of immersed surfaces in R^n to describe the complex quadric and investigated by us [21] in our study of string theory in which the string world-sheet has been viewed as a 2-dimensional surface in R^n and also in curved space. The base of the conifold is given by the intersection of the space of solutions of (1), that is (3), with a sphere in C^n, given by (2). Using (3), we find,

$$r^2 = 2|\psi|^2\Big\{1 + 2\sum_{i=1}^{n-2}|\zeta_i|^2 + |\sum_{i=1}^{n-2}\zeta_i^2|^2\Big\}. \tag{4}$$

In our parameterization, the limit $r \to 0$ corresponds to $\psi \to 0$. The 'strategy' of finding the Ricci flat metric on the n-conifold yields,

$$g_{\psi\bar\psi} = \frac{a(n-2)}{(n-1)|\psi|^2}(ar^{2n-4} + b)^{\frac{2-n}{n-1}}\ r^{2n-4},$$

$$g_{\psi\bar\zeta_i} = \frac{a(n-2)}{n-1}\ \bar\psi(ar^{2n-4} + b)^{\frac{2-n}{n-1}}\ r^{2n-6}\ \frac{\partial Y}{\partial\bar\zeta_i},$$

$$g_{\zeta_i\bar\zeta_j} = \frac{a(n-2)}{(n-1)|\psi|^4}\ (ar^{2n-4} + b)^{\frac{2-n}{n-1}}\ r^{2n-8}\frac{\partial Y}{\partial\zeta_i}\ \frac{\partial Y}{\partial\bar\zeta_j}$$

$$+ (ar^{2n-4} + b)^{\frac{1}{n-1}}\Big\{\frac{1}{Y}\frac{\partial^2 Y}{\partial\zeta_i\partial\bar\zeta_j} - \frac{1}{Y^2}\frac{\partial Y}{\partial\zeta_i}\ \frac{\partial Y}{\partial\bar\zeta_j}\Big\}, \tag{5}$$

where $Y = r^2/|\psi|^2$, a and b are integration constants. The metric becomes singular when $\psi = 0$, even when $b \neq 0$. This is removed by redefining ψ as $\Psi = \frac{\psi^{n-2}}{n-2}$. Then as long as $b \neq 0$, the metric is smooth. A non-linear sigma model on the target space with the metric (4) can be constructed by taking the $n-1$ complex fields as functions of z and $\bar z$, the 2-dimensional space, by the action,

$$S = \int g_{\alpha\bar\beta}\Big\{\partial_z\alpha\partial_{\bar z}\bar\beta + \partial_{\bar z}\alpha\partial_z\bar\beta\Big\}\Big(\frac{i}{2}\Big)dzd\bar z, \tag{6}$$

where α and β stand for $\psi, \zeta_1, \cdots \zeta_{n-2}$. The Kähler two form $\omega = -2ig_{\alpha\bar\beta}d\alpha \wedge d\bar\beta$ is found to be closed and so the integral of this will be a topological invariant. This is used to show

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that the action (5) is bounded from below and the minimum action is topological. We will now consider explicitly the cases for $n = 3$ and $n = 4$ and show the above results.

2.1: $n = 3$ Conifold

The $O(3)$ symmetric target space with three complex fields satisfying

$$\sum_{i=1}^{3} (\phi^i)^2 = 0, \tag{7}$$

is a 3-conifold. The above equation is solved by the parameterization equations,

$$\phi^1 = \psi(1 - f^2); \quad \phi^2 = i\psi(1 + f^2); \quad \phi^3 = 2\psi f, \tag{8}$$

where ψ and f are complex coordinates. In our description of two dimensional non-linear sigma model, these will be taken as functions of z and $\bar{z}$. The invariant r^2 becomes,

$$r^2 = 2|\psi|^2 (1 + |f|^2)^2. \tag{9}$$

Now using (7) and (8), and following the 'strategy' outlined in the Introduction, the metric on the conifold is obtained in terms of the complex fields ψ, f basis as

$$g = \begin{pmatrix} \dfrac{Ar^2}{2|\psi|^2 \sqrt{Ar^2+b}} & \dfrac{2A\bar{\psi}f(1+|f|^2)}{\sqrt{Ar^2+b}} \\ \dfrac{2A\psi\bar{f}(1+|f|^2)}{\sqrt{Ar^2+b}} & \dfrac{4A|\psi|^2(1+|f|^2)}{\sqrt{Ar^2+b}} + \dfrac{4|\psi|^2 b}{r^2 \sqrt{Ar^2+b}} \end{pmatrix} \tag{10}$$

Here A and b are integration constants ($A \neq 0$, b could be zero). The determinant of the metric is $2A$, thereby making the manifold Ricci flat. The metric is hermitian and satisfies,

$$\frac{\partial g_{\alpha\bar{\beta}}}{\partial \gamma} = \frac{\partial g_{\gamma\bar{\beta}}}{\partial \alpha} \; ; \; \frac{\partial g_{\alpha\bar{\beta}}}{\partial \bar{\gamma}} = \frac{\partial g_{\alpha\bar{\gamma}}}{\partial \bar{\beta}}, \tag{11}$$

where α, β stand for ψ, f. Conditions (11) are necessary and sufficient [23] to show that the Kähler 2-form ω,

$$\omega = -2i \sum_{\alpha,\beta} g_{\alpha\bar{\beta}} \, d\alpha \wedge d\bar{\beta}, \tag{12}$$

is *closed*. This property will be used to obtain a lower bound for the action of the non-linear sigma model with target space described by the above metric.

When the integration constant $b \neq 0$, in the limit $r \rightarrow 0$, i.e., $\psi \rightarrow 0$, the above metric (10) for the conifold, does not become singular. This explicit behaviour is mentioned in the Introduction and is pointed out in [18].

To appreciate the role played by the integration constant b in not making the metric singular at short radial distances, we give the metric in terms of real coordinates. We choose the following representation: $\psi = \frac{r}{\sqrt{2}} e^{2i\phi} \cos^2(\frac{\theta}{2})$ and $f = e^{i\xi} tan(\frac{\theta}{2})$ such that $2|\psi|^2(1 + |f|^2)^2 = r^2$. Then working out the metric, we find that the above metric (10) corresponds to

$$(ds_4)^2 = \frac{A}{2\sqrt{Ar^2 + b}}(dr)^2 + \frac{r^2}{2}\{(d\theta)^2 + sin^2\theta(d\xi)^2\}$$
$$+ \frac{r^2}{4}\Big\{d\tilde{\phi} + cos\theta d\xi\Big\}^2 + \frac{b}{2\sqrt{Ar^2 + b}}\{(d\theta)^2 + sin^2\theta(d\xi)^2\}, \tag{13}$$

where we have introduced $\tilde{\phi} = -2\phi - \xi$. Such a geometrical construction is called 'resolving the singularity' (which will be studied in Section.3). The integration constant b can be interpretted as 'resolution parameter'.

We now consider bosonic non-linear sigma model in two dimensions with the target space as the conifold with metric (10). The action for this model is given by,

$$S = \int \Big\{ g_{\psi\bar{\psi}}(\psi_z\bar{\psi}_{\bar{z}} + \psi_{\bar{z}}\bar{\psi}_z) + g_{\psi\bar{f}}(\psi_z\bar{f}_{\bar{z}} + \psi_{\bar{z}}\bar{f}_z)$$
$$+ g_{f\bar{\psi}}(f_{\bar{z}}\bar{\psi}_z + f_z\bar{\psi}_{\bar{z}}) + g_{f\bar{f}}(f_z\bar{f}_{\bar{z}} + f_{\bar{z}}\bar{f}_z) \Big\} \left(\frac{i}{2}\right) dz \; d\bar{z}, \tag{14}$$

where the metric components $g_{\psi\bar{\psi}}, g_{\psi\bar{f}}, g_{f\bar{\psi}}, g_{f\bar{f}}$ can be read-off from (9), and the subscripts $z, \bar{z}$ denote partial derivatives with respect to z and $\bar{z}$ respectively.

The integral of the Kähler 2-form ω is

$$\int \omega = c \int \Big\{ g_{\psi\bar{\psi}}(\psi_z\bar{\psi}_{\bar{z}} - \psi_{\bar{z}}\bar{\psi}_z) + g_{\psi\bar{f}}(\psi_z\bar{f}_{\bar{z}} - \psi_{\bar{z}}\bar{f}_z)$$
$$+ g_{f\bar{\psi}}(f_z\bar{\psi}_{\bar{z}} - f_{\bar{z}}\bar{\psi}_z) + g_{f\bar{f}}(f_z\bar{f}_{\bar{z}} - f_{\bar{z}}\bar{f}_z) \Big\} \left(\frac{i}{2}\right) dz d\bar{z}, \tag{15}$$

which, in general, is not a topological invariant. However, in view of the fact that the above 2-form ω in (12) is closed, $\int \omega$ is indeed a topological invariant [24]. Then it follows from

(14) and (15) that,

$$S + c^{-1} \int \omega = 2 \int \left\{ g_{\psi\bar\psi} |\psi_z|^2 + g_{\psi\bar f} \psi_z \bar f_{\bar z} \right.$$
$$\left. + g_{f\bar\psi} f_z \bar\psi_{\bar z} + g_{f\bar f} |f_z|^2 \right\} (\frac{i}{2}) dz d\bar z. \tag{16}$$

The integrand on the right hand side of (16) can be simplified using (10), and after some algebra, we find,

$$S + c^{-1} \int \omega = 2 \int \left\{ \frac{A(1 + |f|^2)^2}{\sqrt{Ar^2 + b}} \left| \psi_z + \frac{2\psi \bar f f_z}{1 + |f|^2} \right|^2 \right.$$
$$\left. + \frac{4|\psi|^2}{r^2} \sqrt{Ar^2 + b} |f_z|^2 \right\} \frac{i}{2} dz d\bar z,$$
$$\geq 0. \tag{17}$$

Repeating the steps for $S - c^{-1} \int \omega$, we have,

$$S \geq |c^{-1} \int \omega|, \tag{18}$$

which guarantees minimum for the action S. The equality holds good when the fields ψ, f are either holomorphic or anti-holomorphic, in which case the minimum of the action S corresponds to a topological invariant. In view of the observation that the metric does not become singular as $\psi \to 0$ (at the conical singularity) as long as $b \neq 0$, the classical equations of motion for the action S do not become singular, thereby circumventing the criticism of Strominger [14].

In contrast to the CP^n-models, this invariant need not be an integer. In the limit $\psi \to 0$, the metric is not singular and the above action (14) reduces to,

$$S_{\psi \to 0} = 2\sqrt{b} \int \frac{f_z \bar f_{\bar z} + f_{\bar z} \bar f_z}{(1 + |f|^2)^2}, \tag{19}$$

the standard action for CP^1 non-linear sigma model. The target space of this sigma-model is the base of the conifold, which can be thought of as a quadric of dimension $2\sqrt{b}$. Interestingly, this action coincides with the extrinsic curvature action for the string world-sheet, without the integrability conditions, studied in [21]. The 1-loop partition function of this model [25] corresponds to that of a 2-dimensional Coulomb gas.

Now, returning to the minimum action in (18), which can be written in a compact notation as,

$$S = \int g_{\alpha\bar\beta} \partial_z \phi^\alpha \ \partial_{\bar z} \bar\phi^\beta \ \frac{i}{2} dz d\bar z, \tag{20}$$

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the perturbative quantum 1-loop effects can be studied by fluctuating the fields ψ and f from their classical values. As the metric $g_{\alpha\bar{\beta}}$ involves ψ and f, exercising care for the covariant structure, it has been shown [26] that the 1-loop counter term must be of the form $T_{\alpha\bar{\beta}} = a_1 R_{\alpha\bar{\beta}} + a_2 R g_{\alpha\bar{\beta}}$. As the target space on which the sigma model is defined, is *Ricci flat*, it follows that there will be no perturbative 1-loop corrections. Thus the non-linear sigma model on the conifold (7) is a topological field theory and is scale invariant.

From (17), we see that the term $\psi_z + \frac{2\bar{f}f_z}{(1+|f|^2)}\psi$ can be rewritten as a covariant derivative $D_z\psi \equiv (\partial_z + \mathcal{A}_z)\psi$, with the gauge connection $\mathcal{A}_z = \frac{2\bar{f}f_z}{(1+|f|^2)}$. The appearance of gauge connection within the theory is not surprising, as this corresponds to the $U(1)$ fibre over S^2 as exemplified in (13). So, we are able to see the role of this $U(1)$ fibre over S^2 in the action for the non-linear sigma model. For the case of the fields being holomorphic, this gauge connection is a pure gauge.

2.2: $n = 4$ Conifold

We now take up the conifold described by the quadric in C^4,

$$\sum_{i=1}^{4}(\phi^i)^2 = 0. \tag{21}$$

This is a real-6 dimensional conifold. First, as in Section 2.1, we parameterize the four complex fields (coordinates) satisfying (21) as,

$$\phi^1 = \psi(1 + f_1 f_2) \; ; \; \phi^2 = i\psi(1 - f_1 f_2),$$
$$\phi^3 = \psi(f_1 - f_2) \; ; \; \phi^4 = i\psi(f_1 + f_2.) \tag{22}$$

This parameterization is not exactly as in (3), but is more convenient and used in [21,22]. For later use, we realize, as in [15], that by writing,

$$Z = i\phi^4 I + \vec{\sigma} \cdot \vec{\phi},$$
$$= 2 \begin{pmatrix} -f_2\psi & \psi \\ (f_1 f_2 \psi) & -f_1\psi \end{pmatrix}, \tag{23}$$

we have an identification of the conifold with $SL(2, C)$. In (22), ψ, f_1, f_2 are three independent complex fields characterizing the conifold. The base of the conifold is given by the

intersection of the space of solutions of (21), that is, (22), with a sphere in C^4,

$$\sum_{i=1}^{4} |\phi^i|^2 = 2|\psi|^2(1 + |f_1|^2)(1 + |f_2|^2) = r^2, \quad (say). \tag{24}$$

The apex of the cone is designated by $r = 0$ which corresponds here to $\psi = 0$. The Ricci flat metric is constructed by following the strategy outlined before and it is given by,

$$g_{\alpha\bar{\beta}} = C \begin{pmatrix} \frac{2ar^4}{3|\psi|^2} & \frac{4}{3}a\bar{\psi}f_1(1 + |f_2|^2)r^2 & \frac{4}{3}a\bar{\psi}f_2(1 + |f_1|^2)r^2 \\ \frac{4}{3}a\psi\bar{f}_1(1 + |f_2|^2)r^2 & \frac{2a|f_1|^2r^4/3 + ar^4 + b}{(1 + |f_1|^2)^2} & \frac{4}{3}a|\psi|^2\bar{f}_1 f_2 r^2 \\ \frac{4}{3}a\psi\bar{f}_2(1 + |f_1|^2)r^2 & \frac{4}{3}a|\psi|^2 f_1\bar{f}_2 r^2 & \frac{2a|f_2|^2r^4/3 + ar^4 + b}{(1 + |f_2|^2)^2} \end{pmatrix}, \tag{25}$$

where $C = (ar^4 + b)^{-\frac{2}{3}}$, a and b are integration constants and α, β stand for the fields ψ, f_1, f_2. The determinant can be verified as $\frac{8a}{3}|\psi|^2$, so that the Ricci tensor $R_{\alpha\bar{\beta}} = -\partial_\alpha\partial_{\bar{\beta}}\ell og det g$ is identically zero.

The metric (25) does not become singular as long as the integration constant $b \neq 0$. The apparent coordinate singularity when $\psi \to 0$ in the above metric can be removed by redefining ψ as $\Psi = \frac{1}{2}\psi^2$. We note the appearance of the b term in the (22) and (33) matrix elements of the metric (25). They correspond to 'adding sphere to the apex' of the conifold.

The metric (25), upon using the parameterization, $\psi = \frac{r}{\sqrt{2}}e^{i\xi}cos\frac{\theta_1}{2}cos\frac{\theta_2}{2}$, $f_1 = e^{i\phi_1}tan\frac{\theta_1}{2}$, $f_2 = e^{i\phi_2}tan\frac{\theta_2}{2}$, gives the line element of the 6-dimensional conifold as,

$$(ds_6)^2 = \frac{2}{3}ar^2 (dr)^2 + \frac{1}{4}(ar^4 + b)\Big\{\sum_{i=1}^{2} \Big((d\theta_i)^2 + sin^2\theta_i (d\phi_i)^2\Big)\Big\}$$

$$+ \frac{1}{6}ar^4\Big\{d\tilde{\xi} + cos\theta_1 d\phi_1 + cos\theta_2 d\phi_2\Big\}^2, \tag{26}$$

where we have introduced $\tilde{\xi} = -2\xi - \phi_1 - \phi_2$.

A non-linear sigma model on the target space as the conifold (25) is described by the action,

$$S = \int g_{\alpha\bar{\beta}}\{\partial_z\alpha\partial_{\bar{z}}\bar{\beta} + \partial_{\bar{z}}\alpha\partial_z\bar{\beta}\}\frac{i}{2}dzd\bar{z}, \tag{27}$$

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where α, β stand for ψ, f_1, f_2 and the summation over them is understood. The closed Kähler 2-form ω is used to write the topological invariant,

$$\int \omega = -2i \int g_{\alpha\bar\beta}\{\partial_z\alpha\partial_{\bar z}\bar\beta - \partial_{\bar z}\alpha\partial_z\bar\beta\}\frac{i}{2}dzd\bar z, \tag{28}$$

so that we have,

$$S + c\int\omega = 2\int g_{\alpha\bar\beta}\partial_z\alpha\partial_{\bar z}\bar\beta\frac{i}{2}dzd\bar z. \tag{29}$$

Using the metric components in (25), for $b = 0$, we find,

$$S + c\int\omega = 2\int\Big\{\Big|\psi_z + \frac{\bar f_1 f_{1z}}{(1+|f_1|^2)}\psi + \frac{\bar f_2 f_{2z}}{(1+|f_2|^2)}\psi\Big|^2$$
$$+ (ar^4 + b)\Big\{\frac{|f_{1z}|^2}{(1+|f_1|^2)^2} + \frac{|f_{2z}|^2}{(1+|f_2|^2)^2}\Big\}\Big\}\frac{i}{2} \ dzd\bar z,$$
$$\geq 0. \tag{30}$$

Similar procedure for $S - c\int\omega$ can be combined to get the result that the sigma model action on the conifold satisfies the inequality,

$$S \geq |c\int\omega|. \tag{31}$$

The equality holds good when the fields ψ, f_1, f_2 are either holomorphic or anti-holomorphic, in which case the minimum of the action will be a topological invariant.

The holomorphic classical background fields which are solutions to the equations of motion for the minimum action in (31), can be generically taken as $P(z)/Q(z) = \frac{pz+q}{sz+t}$, as 1-instanton configuration. Then there is a structural similarity between the conifold (21,22,23) and the moduli space of 1-instanton as,

$$Z = \begin{pmatrix} -2f_2\psi & 2\psi \\ (2f_1f_2\psi) & -2f_1\psi \end{pmatrix} \leftrightarrow \begin{pmatrix} p & q \\ s & t \end{pmatrix}. \tag{32}$$

This is recently pointed out by Speight [12].

The limit $\psi \to 0$ of the minimum action in (31) corresponds to Grassmannian non-linear sigma model which was extensively studied in [21] as describing QCD strings.

3. Resolved Conifold

We now consider the case of a resolved conifold. The metric for the resolved conifold, in terms of real coordinates, has been obtained by Pando Zayas and Tseytlin [27]. We will describe it here *in terms of complex coordinates* which will be convenient for our purpose of describing sigma model. Nevertheless, we will give, for comparison with [27], the metric in terms of real coordinates. We follow [15] for the construction of a resolved conifold.

We consider (21, 22, 23) of Section 2.2 for $n = 4$ conifold and realize that (21) is equivalent to $detZ = 0$. Defining $W = \frac{1}{\sqrt{2}}Z$, and dispensing ψ (which will be reintroduced shortly as λ), we have,

$$W = \sqrt{2} \begin{pmatrix} -f_2 & 1 \\ (f_1 f_2) & -f_1 \end{pmatrix} \equiv \begin{pmatrix} X & U \\ V & Y \end{pmatrix}, \tag{33}$$

where $X = -\sqrt{2}f_2, U = \sqrt{2}, V = \sqrt{2}f_1 f_2, Y = -\sqrt{2}f_1$ and $XY - UV = 0$. The resolved conifold is obtained [15] by replacing the preceeding relation by a pair of equations,

$$\begin{pmatrix} X & U \\ V & Y \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = 0, \tag{34}$$

where (λ_1, λ_2) are not both zero. Equations,(34) describe $C_4 \times P_1$, with the node having been replaced by a $P_1 = S^2$. We will work in the region $\lambda_2 \neq 0$ and define $\lambda = \frac{\lambda_1}{\lambda_2}$. A solution to the above equation is given by $U = -X\lambda$, $Y = -V\lambda$. Then, it follows that,

$$W = \sqrt{2}f_2 \begin{pmatrix} -1 & \lambda \\ f_1 & (-f_1\lambda) \end{pmatrix}. \tag{35}$$

Thus f_1, f_2, λ are the three complex coordinates characterizing the resolved conifold *in the patch $\lambda_2 \neq 0$*. The $O(4)$ invariant quantity is given (see Section:2.2 for comparison) by

$$r^2 = Tr(W^\dagger W) = 2|f_2|^2(1 + |f_1|^2)(1 + |\lambda|^2), \tag{36}$$

which is very similar to (24). In here, the scalar differentiable function K is taken to be,

$$K = F(r^2) + 4a^2 \ell og(1 + |\lambda|^2). \tag{37}$$

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That is, in addition to the λ dependence through r^2, we have an additional term that depends on λ only. In the limit $a \to 0$, we obtain the ordinary conifold. The function F is determined by the same strategy used earlier.

Working out the metric as $g_{\alpha\bar{\beta}} = \partial_\alpha \partial_{\bar{\beta}} K$, we find,

$$\begin{aligned}
g_{\lambda\bar{\lambda}} &= 2|f_2|^2(1 + |f_1|^2)\{2F''|\lambda|^2|f_2|^2(1 + |f_1|^2) + F'\} + \frac{4a^2}{(1 + |\lambda|^2)^2}, \\
g_{f_1\bar{\lambda}} &= 2\lambda\bar{f}_1|f_2|^2(r^2F'' + F'), \\
g_{f_2\bar{\lambda}} &= 2\lambda\bar{f}_2(1 + |f_1|^2)(r^2F'' + F'), \\
g_{f_1\bar{f}_1} &= 2|f_2|^2(1 + |\lambda|^2)\{2|f_1|^2|f_2|^2(1 + |\lambda|^2)F'' + F'\}, \\
g_{f_1\bar{f}_2} &= 2\bar{f}_1 f_2(1 + |\lambda|^2)(r^2F'' + F'), \\
g_{f_2\bar{f}_2} &= 2(1 + |\lambda|^2)(1 + |f_1|^2)(r^2F'' + F'),
\end{aligned} \tag{38}$$

where F', F'' represent first and second order differential of F with respect to r^2. The determinant of this metric is found to be,

$$detg = 4|f_2|^2 F'(r^2F'' + F')(r^2F' + 4a^2), \tag{39}$$

agreeing with [27]. In order to realize Ricci flatness, we need to just equate from (39),

$$F'(r^2F'' + F')(r^2F' + 4a^2) = constant. \tag{40}$$

The same equation is obtained in [27] and this ensures that our parameterization is correct. By letting $r^2F' = \gamma$ and after one integration, we convert the above non-linear equation to a cubic algebraic equation,

$$\gamma^3 + 6a^2\gamma^2 - r^4 = 0, \tag{41}$$

where the constant in (39) is taken as $2/3$. The real solution of this is given by,

$$\begin{aligned}
\gamma &= s_1 + s_2 - 2a^2, \\
s_{1,2} &= \left\{\frac{r^4}{2} - 8a^6 \pm \frac{1}{2}\sqrt{r^8 - 32a^6r^4}\right\}^{\frac{1}{3}}.
\end{aligned} \tag{42}$$

Realizing that $s_1 s_2 = 4a^4$, the solution can be written as,

$$\gamma = -2a^2 + 4a^2 s_1^{-1} + s_1. \tag{43}$$

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and so,

$$F' = \frac{\gamma}{r^2},$$

$$F'' = \frac{2}{3\gamma(\gamma + 4a^2)} - \frac{\gamma}{r^4}. \tag{44}$$

Thus, the components of the Ricci flat metric in the complex basis for the resolved conifold are,

$$g_{\lambda\bar{\lambda}} = \frac{1}{(1 + |\lambda|^2)^2}\Big\{\frac{2r^4}{3\gamma(\gamma + 4a^2)}|\lambda|^2 + \gamma + 4a^2\Big\},$$

$$g_{f_1\bar{\lambda}} = \frac{\lambda\bar{f}_1}{(1 + |f_1|^2)(1 + |\lambda|^2)}\frac{2r^4}{3\gamma(\gamma + 4a^2)},$$

$$g_{f_2\bar{\lambda}} = \frac{\lambda\bar{f}_2}{|f_2|^2(1 + |\lambda|^2)}\frac{2r^4}{3\gamma(\gamma + 4a^2)},$$

$$g_{f_1\bar{f}_1} = \frac{1}{(1 + |f_1|^2)^2}\Big\{\frac{2r^4}{3\gamma(\gamma + 4a^2)}|f_1|^2 + \gamma\Big\},$$

$$g_{f_1\bar{f}_2} = \frac{\bar{f}_1 f_2}{(1 + |f_1|^2)|f_2|^2}\frac{2r^4}{3\gamma(\gamma + 4a^2)},$$

$$g_{f_2\bar{f}_2} = \frac{2r^4}{3\gamma(\gamma + 4a^2)|f_2|^2}. \tag{45}$$

Before proceeding to a non-linear sigma model on this resolved conifold, we give a realization of the above metric in terms of real coordinates. Parameterizing the complex coordinates f_1, f_2 and λ, in a manner slightly from the one that led to (26), as,

$$f_1 = e^{-i\phi_1}tan\frac{\theta_1}{2},$$

$$f_2 = \frac{r}{\sqrt{2}}e^{\frac{i}{2}\psi}cos\frac{\theta_1}{2}cos\frac{\theta_2}{2},$$

$$\lambda = e^{-i\phi_2}tan\frac{\theta_2}{2}, \tag{46}$$

such that $2|f_2|^2(1 + |f_1|^2)(1 + |\lambda|^2) = r^2$, we find,

$$(ds_6)^2 = g_{\alpha\bar{\beta}}d\alpha d\bar{\beta},$$

$$= \gamma'(dr)^2 + r^2\gamma'\frac{1}{4}\Big\{d\tilde{\psi} - \sum_{i=1}^{2}cos\theta_i \ d\phi_i\Big\}^2$$

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$$+ \frac{\gamma}{4} \sum_{i=1}^{2} \left\{ (d\theta_i)^2 + sin^2\theta_i \ (d\phi_i)^2 \right\}$$

$$+ a^2 \{ (d\theta_2)^2 + sin^2\theta_2 \ (d\phi_2)^2 \}, \tag{47}$$

the metric on the resolved conifold, agreeing with [27] after the redefinition $\tilde{\psi} = \psi - \phi_1 - \phi_2$. For completeness, we rewrite this metric first by introducing $\rho^2 = \frac{3}{2}\gamma$ and then treat ρ as the radial coordinate, instead of r. This leads to the metric as,

$$(ds_6)^2 = \kappa(\rho)^{-1}(d\rho)^2 + \frac{1}{9}\kappa(\rho)\rho^2 e_{\tilde{\psi}}{}^2 + \frac{\rho^2}{6}(e_{\theta_1}{}^2 + e_{\phi_1}{}^2)$$

$$+ (\frac{\rho^2}{6} + a^2)(e_{\theta_2}{}^2 + e_{\phi_2}{}^2), \tag{48}$$

where $e_{\psi} = \{ d\tilde{\psi} + \sum_{i=1}^{2} cos\theta_i \ d\phi_i \}$, $e_{\theta_i} = d\theta_i$, $e_{\phi_i} = sin\theta_i \ d\phi_i$ and $\kappa(\rho) = \frac{\rho^2 + 9a^2}{\rho^2 + 6a^2}$.

We point out two uses of this metric and its complex realization (45) in the context of D-branes. First, given the Ricci flat metric on the transverse six dimensional space taken as a conifold, the standard brane solution [3,27] is

$$(ds_{10})^2 = H^{-\frac{1}{2}}(y)\eta_{\mu\nu}dx^\mu dx^\nu + H^{\frac{1}{2}}(y)g_{ij}dy^i dy^j, \tag{49}$$

as a warped metric, where y collectively denotes the coordinates of the transverse six dimensional space and the warp factor $H(y)$ is a harmonic in the transverse space, that is,

$$\frac{1}{\sqrt{g}}\partial_i(\sqrt{g}g^{ij}\partial_j H) = 0. \tag{50}$$

Using the metric given in (48) and assuming that H depends on ρ only, one can solve (50) for $H(\rho)$ as,

$$H(\rho) = H_0 + C\left\{ \frac{1}{18a^2\rho^2} - \frac{1}{162a^4}log(1 + \frac{9a^2}{\rho^2}) \right\}. \tag{51}$$

For small values of ρ, it follows that $H(\rho) \to \frac{C}{18a^2\rho^2}$ and $\kappa(\rho) \to \frac{3}{2}$, so that the brane solution becomes,

$$(ds_{10})^2 = \frac{3\sqrt{2}a\rho}{\sqrt{C}}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{2\sqrt{C}}{9\sqrt{2}a}\frac{1}{\rho}(d\rho)^2$$

$$+ \frac{\sqrt{C}}{3\sqrt{2}a}\left\{ \frac{1}{6}\rho(e_\psi^2 + e_{\theta_1}^2 + e_{\theta_2}^2 + e_{\phi_1}^2 + e_{\phi_2}^2) + \frac{a^2}{9}(e_{\theta_2}^2 + e_{\phi_2}^2) \right\}. \tag{52}$$

This solution corresponds to $AdS_5 \times X_5$.

Second, we have seen that the complex coordinates ϕ^i in (21) transform in the four dimensional representation of $SO(4)$ and have unit charge with respect the $U(1)$ symmetry group. Klebanov and Witten [3] have given a holomorphic three form as,

$$\Omega = \frac{d\phi^2 \wedge d\phi^3 \wedge d\phi^4}{\phi^1}, \tag{53}$$

which in our complex parameterization (22) becomes $\Omega = -2\psi \, d\psi \wedge df_1 \wedge df_2$ and has "charge two" under the said $U(1)$. This will be an R-symmetry group.

Now we will consider a 2-dimensional non-linear sigma model with the target space as the above (complex version) resloved conifold. Denoting $\frac{2r^4}{3\gamma(\gamma+4a^2)}$ by $\Gamma(r^2)$, the action becomes,

$$
\begin{aligned}
S = \int \Big\{ &\Big(\frac{1}{(1+|\lambda|^2)^2}(|\lambda|^2\Gamma + \gamma) + \frac{4a^2}{(1+|\lambda|^2)^2} \Big)(\lambda_z\bar{\lambda}_{\bar{z}} + \lambda_{\bar{z}}\bar{\lambda}_z) \\
&+ \frac{\lambda\bar{f}_1\Gamma}{(1+|f_1|^2)(1+|\lambda|^2)}(f_{1z}\bar{\lambda}_{\bar{z}} + f_{1\bar{z}}\bar{\lambda}_z) + h.c \\
&+ \frac{\lambda\bar{f}_2\Gamma}{|f_2|^2(1+|\lambda|^2)}(f_{2z}\bar{\lambda}_{\bar{z}} + f_{2\bar{z}}\bar{\lambda}_z) + h.c \\
&+ \frac{1}{(1+|f_1|^2)^2}(\Gamma|f_1|^2 + \gamma)(f_{1z}\bar{f}_{1\bar{z}} + f_{1\bar{z}}\bar{f}_{1z}) \\
&+ \frac{\bar{f}_1 f_2\Gamma}{|f_2|^2(1+|f_1|^2)}(f_{1z}\bar{f}_{2\bar{z}} + f_{1\bar{z}}\bar{f}_{2z}) + h.c \\
&+ \frac{\Gamma}{|f_2|^2}(f_{2z}\bar{f}_{2\bar{z}} + f_{2\bar{z}}\bar{f}_{2z}) \Big\}\Big(\frac{i}{2}\Big)dzd\bar{z}.
\end{aligned}
\tag{54}
$$

The Kähler 2-form ω can be similarly written and is found to be closed. In order to investigate whether the above action has a minimum, we consider the sum (and difference as well) which turns out to be,

$$
\begin{aligned}
S + c \int \omega = 2 \int \Big\{ &\frac{(4a^2 + \gamma)}{(1+|\lambda|^2)^2}|\lambda_z|^2 + \frac{\gamma}{(1+|f_1|^2)^2}|f_{1z}|^2 \\
&+ \frac{\Gamma|\lambda|^2}{(1+|\lambda|^2)^2}\Big|\lambda_z + \frac{(1+|\lambda|^2)\lambda\bar{f}_1}{|\lambda|^2(1+|f_1|^2)}f_{1z} + \frac{\lambda(1+|\lambda|^2)\bar{f}_2}{|\lambda|^2|f_2|^2}f_{2z}\Big|^2 \Big\}\Big(\frac{i}{2}\Big)dzd\bar{z} \\
&\geq 0.
\end{aligned}
\tag{55}
$$

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Thus, the non-linear sigma model action with the target space as the resolved conifold is bounded below. This action has a smooth behaviour and the equations of motion of the minimum action do not become singular. The structural similarity between the resolved conifold and the moduli space of unit charge instanton (which are the solutions of the equations of motion) is easily seen as in the case of the ordinary conifold.

In contrast to the situation in the non-linear sigma model with the ordinary conifold as the target space, here, the role of the gauge connection needs to be analysed only for the second line in (55). The first term gives the action on S^2 which essentially replaces the apex. Considering the second line, the covariant derivative can be easily noticed with the gauge connection identified with the $U(1)$ fibre on the base of the resolved conifold. As the target space is Ricci flat, perturbative 1-loop corrections are absent and the action becomes topological.

4. Conclusion

We have constructed Ricci-flat Kähler metric for the conifold of complex dimensions $n = 3, 4$ in terms of complex coordinates. This complements the study using real coordinates. The strategy followed consisted in solving the differential equation for the Kähler potential. With the choice of integration constant b not set zero, the metric remains smooth. A realization of the metric in terms of real coordinates is made for both $n = 3, 4$ conifolds. With the intention of constructing a field theory on conifold, we considered two dimensional non-linear sigma model on the conifold, by identifying the complex coordinates as sigma model fields defined on a 2-dimensional space. The closed Kähler 2-form is used to obtain a lower bound for the action for the non-linear sigma model. The minimum action corresponds to the complex fields being either holomorphic or anti-holomorphic in the complex 2-dimensional space. The classical equations of motion are found to be non-singular, by the choice of the integration constant. This suggests a method to overcome the difficulties in deriving a low energy effective action in the case of Calabi-Yau compactification.

The same procedure of using complex coordinates is extended to find the metric (in the complex basis) for the $n = 4$ resolved conifold and its realization in terms of six real coordinates is made. This agrees with [27]. The harmonic function appearing as the warp factor in the solution of Type-IIB string theory is determined and in the $\rho \to 0$ limit, this

solution goes over to $AdS_5 \times X_5$ geometry. A non-linear sigma model on the resolved conifold is constructed using our complex realization of the resolved conifold metric.

Acknowledgements

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