

# Non-commutative gauge theory of twisted D-branes

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## Abstract

In this work we propose new non-commutative gauge theories that describe the dynamics of branes localized along twisted conjugacy classes on group manifolds. Our proposal is based on a careful analysis of the exact microscopic solution and it generalizes the matrix models (“fuzzy gauge theories”) that are used to study e.g. the bound state formation of point-like branes in a curved background. We also construct a large number of classical solutions and interpret them in terms of condensation processes on branes localized along twisted conjugacy classes.

# 1 Introduction

Branes on group manifolds have received a lot of attention throughout the last years. Even though the group manifold of  $SU(2) \cong S^3$  is the only one that can appear directly as part of a string background (e.g. in the background of NS5-branes), other group manifolds provide simple toy models for studying the behavior of branes in curved backgrounds. In fact, the symmetry of group manifolds allows to control string perturbation theory beyond the supergravity regime and hence it renders investigations of brane spectra and brane dynamics at finite curvature possible. Moreover, group manifolds are the starting point for coset and orbifold constructions and thereby allow to obtain many less symmetric string backgrounds.

Open strings on group manifolds are modeled with the help of the Wess-Zumino-Witten (WZW) theory on the upper half  $\Sigma = \{\Im z \geq 0\}$  of the complex plane. Boundary conditions giving rise to a consistent open string spectrum were first proposed by Cardy [1] but their geometric interpretation in terms of brane geometry was only uncovered much later in [2]. There it was shown that Cardy's boundary theories describe branes that wrap certain integer conjugacy classes on the group manifold  $G$  (see also [3, 4]).

The Born-Infeld action has been used in [5, 6] to explain the stability of such branes in a weakly curved background (see also [7] for more general cases). Once several branes are placed on a group manifold, they tend to form new bound states. These dynamics are described by non-commutative gauge theories on the quantized (“fuzzy”) conjugacy classes. The latter were derived from the exact boundary conformal field theory in [8]. Even though these actions are applicable only at large level  $k$ , many of the condensation processes they encode are known to possess a deformation to finite  $k$  (see [9, 10]).

Branes localized along conjugacy classes are not the only ones that admit an exact solution. In [11] Birke, Fuchs and Schweigert constructed the open string spectrum for new sets of branes that were shown later to be localized along so-called twisted conjugacy classes [12]. These extra series of branes come associated with non-trivial outer automorphisms  $\omega_G$  of the group  $G$ .

In the present work we aim at understanding the dynamical properties of such twisted D-branes on group manifolds. After a brief introduction to the geometry and conformal field theory of maximally symmetric branes on group manifolds, we will use the microscopic information on open string spectra to construct the space of functions on all these branes in the limit where the level  $k$  is sent to infinity.

We show that the space of functions can be equipped with an associative and non-commutative product. For non-trivial twists  $\omega_G \neq \text{id}$ , the geometry of these branes is not just a simple matrix geometry as in the case of branes on conjugacy classes but is based on a certain algebra of *matrix valued functions* on the group. The associated gauge theories are then easily obtained by copying the computations from [8]. In the final section we analyze classical solutions. Most importantly, we shall establish that for a given twist  $\omega_G$ , all the branes appear as bound states from stacks of one distinguished elementary twisted brane. This is analogous to the situation of branes wrapping ordinary conjugacy classes which can all be built from condensates of point-like branes [13] (see also [14, 15]) by some analogue of Myers dielectric effect [16]. Let us remark that the new condensation processes we find are consistent with the charge conservation laws proposed in [10].

## 2 D-branes on group manifolds

This section is devoted to the description of maximally symmetric branes on group manifolds. Following [2, 12], we will begin with a brief review of their classical geometry. Then, in the second subsection, we shall present some basic results on the boundary conformal field theory of such branes.

### 2.1 The geometry of branes on group manifolds

Strings on the group manifold of a simple and simply connected group  $G$  are described by the WZW-model. Its action is evaluated on fields  $g : \Sigma \rightarrow G$  taking values in  $G$  and it involves one (integer) coupling constant  $k$ , which is known as the “level”. For our purposes it is most convenient to think of  $k$  as controlling the size (in string units) of the background. Large values of  $k$  correspond to a large volume of the group manifold. When dealing with open strings at tree level, the 2-dimensional world sheet  $\Sigma$  is taken to be the upper half plane  $\Sigma = \{z \in \mathbb{C} | \Im z \geq 0\}$ .

Along the boundary of this world sheet we need to impose some boundary condition. Here we shall analyze boundary conditions that preserve the full bulk symmetry of the model, i.e. the affine algebra  $\hat{\mathfrak{g}}_k$ . These boundary conditions are formulated in terms of the chiral currents

$$J(z) = k g^{-1}(z, \bar{z}) \partial g(z, \bar{z}) \quad , \quad \bar{J}(\bar{z}) = -k \bar{\partial} g(z, \bar{z}) g^{-1}(z, \bar{z}) \quad . \quad (1)$$

Note that  $J$  and  $\bar{J}$  take values in the finite dimensional Lie algebra  $\mathfrak{g}$  of the group  $G$ . Along the real line we glue the holomorphic and the anti-holomorphic currents ac-

ording to

$$J(z) = \Lambda \bar{J}(\bar{z}) \quad \text{for all } z = \bar{z} \quad (2)$$

where  $\Lambda$  is an appropriate automorphism of the current algebra  $\hat{\mathfrak{g}}_k$  (see e.g. [17]). The choice of  $\Lambda$  is restricted by the requirement of conformal invariance which means that  $T(z) = \bar{T}(\bar{z})$  all along the boundary. Here  $T, \bar{T}$  are the non-vanishing components of the stress energy tensor. They can be obtained, as usual, through the Sugawara construction.

The allowed automorphisms  $\Lambda$  of the affine Lie algebra  $\hat{\mathfrak{g}}_k$  are easily classified. They are all of the form

$$\Lambda = \Omega \circ \text{Ad}_g \quad \text{for some } g \in G \quad (3)$$

Here,  $\text{Ad}_g$  denotes the adjoint action of the group element  $g$  on the current algebra  $\hat{\mathfrak{g}}_k$ . It is induced in the obvious way from the adjoint action of  $G$  on the finite dimensional Lie algebra  $\mathfrak{g}$ . The automorphism  $\Omega$  does not come from conjugation with some element  $g$ . More precisely, it is an outer automorphism of the current algebra. Such outer automorphisms  $\Omega = \Omega_\omega$  come with symmetries  $\omega$  of the Dynkin diagram of the finite dimensional Lie algebra  $\mathfrak{g}$ . One may show that the choice of  $\omega$  and  $g \in G$  in eq. (3) exhausts all possibilities for the gluing automorphism  $\Lambda$  (see e.g. [18]).

So far, our discussion of the admissible types of gluing automorphisms  $\Lambda$  has been fairly abstract. It is possible, however, to associate some concrete geometry with each choice of  $\Lambda$ . This was initiated in [2] for  $\omega = \text{id}$  and extended to non-trivial symmetries  $\omega \neq \text{id}$  in [12] (see also [3], [4]).

Let us assume first that the element  $g$  in eq. (3) coincides with the group unit  $g = e$ . This means that  $\Lambda = \Omega = \Omega_\omega$  is determined by  $\omega$  alone. The diagram symmetry  $\omega$  induces an (outer) automorphism  $\omega$  of the finite dimensional Lie algebra  $\mathfrak{g}$  through the unique correspondence between vertices of the Dynkin diagram and simple roots. After exponentiation,  $\omega$  furnishes an automorphism  $\omega_G$  of the group  $G$ . One can show that the gluing conditions (2) force the string ends to stay on one of the following  $\omega$ -twisted conjugacy classes

$$C_u^\omega := \{ hu \omega_G(h^{-1}) \mid h \in G \} \quad .$$

The subsets  $C_u^\omega \subset G$  are parametrized by equivalence classes of group elements  $u$  where the equivalence relation between two elements  $u, v \in G$  is given by:  $u \sim_\omega v$  iff  $v \in C_u^\omega$ . Note that this parameter space  $\mathcal{U}^\omega$  of equivalence classes is not a manifold, i.e. it contains singular points.

To describe the topology of  $C_u^\omega$  and the parameter space  $\mathcal{U}^\omega$  (at least locally), we need some more notation. By construction, the action of  $\omega$  on  $\mathfrak{g}$  can be restricted to an action on the Cartan subalgebra  $\mathcal{T}$ . We shall denote the subspace of elements which are invariant under the action of  $\omega$  by  $\mathcal{T}^\omega \subset \mathcal{T}$ . Elements in  $\mathcal{T}^\omega$  generate a torus  $T^\omega \subset G$ . One may show that the generic  $\omega$ -twisted conjugacy class  $\mathcal{C}_u^\omega$  looks like the quotient  $G/T^\omega$ . Hence, the dimension of the generic submanifolds  $C_u^\omega$  is  $\dim G - \dim T^\omega$  and the parameter space has dimension  $\dim \mathcal{T}^\omega$  almost everywhere. In other words, there are  $\dim \mathcal{T}^\omega$  directions transverse to a generic twisted conjugacy class. This implies that the branes associated with the trivial diagram automorphism  $\omega = \text{id}$  have the largest number of transverse directions. It is given by the rank of the Lie algebra.

As we shall see below, not all these submanifolds  $C_u^\omega$  can be wrapped by branes on group manifolds. There exists some integrality requirement that can be understood in various ways, e.g. as quantization condition within a semiclassical analysis [2] of the brane's stability [5, 6, 7]. This implies that there is only a finite set of allowed branes (if  $k$  is finite). The number of branes depends on the volume of the group measured in string units.

Let us become somewhat more explicit for  $G = SU(N)$ . The simplest case is certainly  $N = 2$  because there exists no non-trivial diagram automorphism  $\omega$ . The conjugacy classes  $C_u^{\text{id}}$  are 2-spheres  $S^2 \subset S^3 \cong SU(2)$  for generic points  $u$  and they consist of a single point when  $u = \pm e$  in the center of  $SU(2)$ . More generally, the formulas  $\dim SU(N) = (N-1)(N+1)$  and  $\text{rank } SU(N) = (N-1)$  show that the generic submanifolds  $C_u^{\text{id}}$  have dimension  $\dim C_u^{\text{id}} = (N-1)N$ . In addition, there are singular cases,  $N$  of which are associated with elements  $u$  in the center  $\mathbb{Z}_N \subset SU(N)$ . The corresponding submanifolds  $C_u^{\text{id}}$  are 0-dimensional. Note that all the submanifolds  $C_u^{\text{id}}$  are even dimensional. Similarly, the generic manifolds  $C_u^\omega$  for the non-trivial diagram symmetry  $\omega$  have dimension  $\dim C_u^\omega = (N-1)(N+1/2)$  for odd  $N$  and  $\dim C_u^\omega = N^2 - N/2 - 1$  whenever  $N$  is even. For some exceptional values of  $u$ , the dimension can be lower. For a complete discussion we refer the reader to [19].

So far we restricted ourselves to  $\Lambda = \Omega_\omega$  being a diagram automorphism. As we stated before, the general case is obtained by admitting an additional inner automorphism of the form  $\text{Ad}_g$ . Geometrically, the latter corresponds to rigid translations on the group induced from the left action of  $g$  on the group manifold (see e.g. [20, 21]). The freedom of translating branes on  $G$  does not lead to any new charges or to essentially new physics and we shall not consider it any further, i.e. we shall assume

$g = e$  in what follows.

## 2.2 The conformal field theory description

The branes we considered in the previous subsection may be described through an exactly solvable conformal field theory. In particular, there exists a detailed knowledge about their open string spectra based on the work of Cardy [1] and of Birke, Fuchs and Schweigert [11].

We shall use  $\alpha, \beta, \dots \in \mathcal{B}_k^\omega$  to label different conformal boundary conditions of the boundary conformal field theories associated with the gluing conditions eq. (2) on the currents. The set  $\mathcal{B}_k^\omega$  depends on the choice of the diagram automorphism  $\omega$  and on the level  $k$ . For the trivial diagram automorphism  $\omega = \text{id}$ ,  $\mathcal{B}_k^{\text{id}} = P_k^+$  coincides with the set of primaries of the affine Kac-Moody algebra  $\hat{\mathfrak{g}}_k$ . The latter is well known to form a subset in the space  $P^+$  of dominant integral weights which label equivalence classes of irreducible representations for the finite dimensional Lie algebra  $\mathfrak{g}$ . To keep notations simple, we will not distinguish in notation between elements of  $P^+$  and  $P_k^+$  and denote them both by capital letters  $A, B, C, \dots$ .

The automorphism  $\omega$  generates a map  $\omega_k : P_k^+ \rightarrow P_k^+$ . In fact, given an irreducible representation  $\tau$  of  $\mathfrak{g}$ , we can define another representation by composition  $\tau \circ \omega$ . The class of  $\tau \circ \omega$  is independent of the choice of  $\tau \in [\tau]$  and so we obtain a map  $\omega : P^+ \rightarrow P^+$ . The latter descends to the subset  $P_k^+ \subset P^+$ . A weight  $A \in P_k^+$  is said to be  $(\omega)$ -symmetric if it is invariant under the action of  $\omega$ , i.e. if  $\omega(A) = A$ . According to the results of [1, 11], the labels  $\alpha$  for branes associated with the diagram automorphism  $\omega$  take values in a certain subset  $\mathcal{B}_k^\omega \subset \mathcal{B}^\omega = \mathcal{P}_\omega(P^+)$  of dominant fractional symmetric weights. Here  $\mathcal{P}_\omega = \frac{1}{N}(1 + \dots + \omega^{N-1})$  is the projection of weights onto their  $(\omega)$ -symmetric part with  $N$  denoting the order of  $\omega$ . Finally let us briefly mention that the number of boundary conditions  $\alpha \in \mathcal{B}_k^\omega$  is equal to the number of symmetric weights in  $P_k^+$ . On the other hand,  $\mathcal{B}_k^\omega \neq \mathcal{P}_\omega(P_k^+)$  in contrast to what one might expect naively. We will not need these details here and refer the interested reader to [11] (see also [22]). Our considerations below will mostly take place in the limiting regime of large level  $k$  where we can identify  $\mathcal{B}_\infty^\omega = \mathcal{B}^\omega = \mathcal{P}_\omega(P^+)$ .

Before we continue to outline the conformal field theory results let us briefly summarize some conventions we will be using in the limit  $k \rightarrow \infty$ . In this case, the fractional symmetric weights which label boundary conditions can be described explicitly by

$$\mathcal{B}^\omega = \left\{ \alpha = \sum \lambda_i \omega_i \mid \lambda_i = \lambda_{\omega(i)}, l_i \lambda_i \in \mathbb{N}_0 \right\} . \quad (4)$$

The numbers  $l_i$  denote the length of the orbit of the fundamental weights  $\omega_i$  under the automorphism  $\omega$ . A distinguished element of this set of fractional symmetric weights is  $\rho_\omega = (1/l_1, 1/l_2, \dots) \in \mathcal{B}^\omega$ . It can be considered as a twisted counterpart of the Weyl vector  $\rho = (1, 1, \dots) \in P^+$ . Under the assumption of infinite level  $k$ , the representations of  $\mathfrak{g}$  and  $\hat{\mathfrak{g}}_k$  are both labeled by the same set  $P^+$ . With the identifications of this section in mind, we will always assume that we fix the weights before we let the level run to infinity.

Our main goal now is to explain the open string spectra that come with the maximally symmetric branes on group manifolds. For a pair of boundary labels  $\alpha, \beta \in \mathcal{B}_k^\omega$  associated with the same diagram automorphism  $\omega$ , the partition function is of the form

$$Z_{\alpha\beta}^\omega(q) = \sum_{A \in P_k^+} (n_A^\omega)_\alpha^\beta \chi_A(q) . \quad (5)$$

Here,  $\chi_A(q)$  denote the characters of the current algebra  $\hat{\mathfrak{g}}_k$  and  $\alpha, \beta \in \mathcal{B}_k^\omega$ . Their appearance in the expansion (5) reflects the fact that all the (twisted) conjugacy classes admit an obvious action of the Lie group  $G$  by (twisted) conjugation. Consistency requires the numbers  $(n_A^\omega)_\alpha^\beta$  to be non-negative integers.

There exists a very simple argument due to Behrend et al. [23, 24] which shows that the matrices  $(n_A^\omega)_\alpha^\beta$  give rise to a representation of the fusion algebra of  $\hat{\mathfrak{g}}_k$ . This means that they obey the relations

$$\sum_{\beta \in \mathcal{B}_k^\omega} (n_A^\omega)_\alpha^\beta (n_B^\omega)_\beta^\gamma = \sum_{C \in P_k^+} N_{AB}^C (n_C^\omega)_\alpha^\gamma , \quad (6)$$

where  $N_{AB}^C$  are the fusion rules of the current algebra  $\hat{\mathfrak{g}}_k$ . The argument of [23, 24] starts from a general Ansatz for the boundary state assigned to  $\alpha \in \mathcal{B}_k^\omega$ . Using world sheet duality, one can express the matrices  $n_A^\omega$  in terms of the coefficients of the boundary states and the modular matrix  $S$  for the current algebra  $\hat{\mathfrak{g}}_k$ . The general form of this expression is then sufficient to check the relations (6) (see [23, 24] for details).

The expression for the matrices  $n_A^\omega$  that is given in [11, 25] resembles the familiar Verlinde formula,

$$(n_A^\omega)_\alpha^\beta = \sum_{\substack{\lambda \in P_k^+ \\ \omega(\lambda) = \alpha}} \frac{\bar{S}_{\lambda\beta}^\omega S_{\lambda\alpha}^\omega S_{\lambda A}}{S_{\lambda 0}} \quad \text{for} \quad \alpha, \beta \in \mathcal{B}_k^\omega \quad \text{and} \quad A \in P_k^+ . \quad (7)$$

It contains a unitary matrix  $S^\omega$  whose entries  $S_{\lambda\alpha}^\omega$  are indexed by two  $\omega$ -symmetric labels  $\lambda \in P_k^+, \alpha \in \mathcal{B}_k^\omega$ , i.e. they obey  $\omega(\lambda) = \lambda$  and  $\omega(\alpha) = \alpha$ . When  $\omega = \text{id}$ , the matrix  $S^\omega$  coincides with the usual  $S$ -matrix so that Verlinde's formula [26] implies

$$(n_A^\omega)_\alpha^\beta = N_{A\alpha}^\beta \quad \text{for all } \alpha, \beta, A \in P_k^+ = \mathcal{B}_k^{\text{id}}.$$

This reproduces Cardy's results on the boundary partition functions [1]. For non-trivial automorphism  $\omega$ , the matrix  $S^\omega$  describes modular transformations of twined characters. An explicit formula for  $S^\omega$  can be found in [11]

$$S_{\lambda\alpha}^\omega \sim \sum_{w \in W_\omega} \epsilon_\omega(w) \exp\left(-\frac{2\pi i}{k+g^\vee}(w(\lambda+\rho), \alpha+\rho_\omega)\right). \quad (8)$$

Here,  $W_\omega \subset W$  is the  $\omega$ -invariant part in the Weyl group of  $\mathfrak{g}$ . As  $W_\omega$  can be considered as the Weyl group of another Lie algebra [27] it comes with a natural sign function  $\epsilon_\omega$ .

There exists a generalized state-field correspondence that associates to every highest weight state  $A \in P_k^+$  in the Hilbert space (5) of the  $(\alpha, \beta)$  boundary conformal field theory a boundary primary field  $\psi_A^{(\alpha\beta)}$  living between boundaries  $\alpha, \beta \in \mathcal{B}_k^\omega$ . The general structure of the boundary operator product expansion (OPE) is given by

$$\psi_A^{(\alpha\beta)}(x) \psi_B^{(\beta\gamma)}(y) \sim \sum_C (x-y)^{h_C-h_A-h_B} C_{AB}^{(\alpha\beta\gamma)C} \psi_C^{(\alpha\gamma)}(y) \quad \text{for } x < y \quad (9)$$

where the numbers  $h_A$  denote the conformal weights of the fields which, in the case at hand, are given by

$$h_A = \frac{C_A}{2(k+g^\vee)}. \quad (10)$$

Here,  $C_A$  is the quadratic Casimir of the representation  $A \in P_k^+$  (see eq. (23) below) and  $g^\vee$  denotes the dual Coxeter number of  $\mathfrak{g}$ . For a consistent conformal field theory, the structure constants  $C_{AB}^{(\alpha\beta\gamma)C}$  have to satisfy so-called sewing constraints [28] (see also [29, 30, 31]). One of these constraints expresses the associativity of the OPE,

$$C_{AB}^{(\alpha\beta\gamma)E} C_{CD}^{(\gamma\delta\alpha)E^+} C_{EE^+}^{(\alpha\gamma\alpha)0} = \sum_H C_{BC}^{(\beta\gamma\delta)H} C_{AH}^{(\alpha\beta\delta)D^+} C_{D^+D}^{(\alpha\delta\alpha)0} F_{HE} \begin{bmatrix} B & C \\ A & D \end{bmatrix} \quad (11)$$

where the symbol  $F$  denotes the fusing matrix. For the  $\hat{\mathfrak{g}}_k$ -WZW model, this fusing-matrix is closely related to the  $6j$ -symbol of the corresponding quantum group at  $(k+g^\vee)^{th}$  root of unity. In the limit  $k \rightarrow \infty$ , it thus reduces to the  $6j$ -symbol



of  $\mathfrak{g}$ . Solutions of the sewing constraints for the standard case  $\omega_G = \text{id}$  can be found in [30, 8, 12, 23]. The boundary OPE for non-trivially twisted branes, on the other hand, is not yet known. Our results below implicitly contain these solutions in the limit  $k \rightarrow \infty$ .

The spectrum of ordinary conjugacy classes can be explained in detail. For simplicity, we shall restrict to  $G = SU(2)$ . In this case, generic conjugacy classes are 2-spheres and the space of functions thereon is spanned by spherical harmonics  $Y_m^{j/2}$ ,  $|m| \leq j/2$  and  $j = 0, 2, 4, \dots$ <sup>1</sup> The space of spherical harmonics is precisely reproduced by ground states in the boundary theory  $\alpha$  when we send  $\alpha$  (and hence  $k$ ) to infinity. For finite  $\alpha$ , the angular momentum  $j$  is cut off at a finite value  $j = \min(2\alpha, 2k - 2\alpha) \leq 2\alpha$ . This means that the brane's world-volume is “fuzzy”, since resolving small distances would require large angular momenta. The relation between branes on  $SU(2)$  and the familiar non-commutative fuzzy 2-spheres [32, 33] was fully analyzed in [8] and it provides a very important example of an open string non-commutative geometry that goes beyond the familiar case of branes in flat space [34, 35, 36]. The analysis of [8] goes much beyond the study of partition functions as it employs detailed information on the operator product expansions of open string vertex operators based on [30]. Using the results in [12, 37, 38] it is easy to generalize all these remarks on ordinary conjugacy classes to other groups (see also [32] for more details and explicit formulas on fuzzy conjugacy classes).

Twisted conjugacy classes are more difficult to understand. This is related to the fact that they are never “small”. More precisely, it is not possible to fit a generic twisted conjugacy class into an arbitrarily small neighborhood of the group identity unless the twist  $\omega$  is trivial. This implies that the spectrum of angular momenta in  $Z_{\alpha\alpha}^\omega$  is not cut off before it reaches the obvious large momentum cut-off that is set by the volume of the group, i.e. by the level  $k$ . For large  $\alpha \in \mathcal{B}_k^\omega$  (and large  $k$ ) the ground states in the boundary theory span the space of functions on the generic twisted conjugacy classes  $C_u^\omega$  [12]. The non-commutative geometry associated with twisted conjugacy classes with finite  $\alpha$ , however, was unknown and it is the main subject of the next section.

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<sup>1</sup>To be consistent with our treatment of other groups below, we use a convention in which the representations are labeled by Dynkin labels rather than spins.

$\mathfrak{g}$	order	$\mathfrak{g}^\omega$	$x_e$	$G$	$G^\omega$
$A_2$	2	$A_1$	4	$SU(3)$	$SO(3)$
$A_{2n-1}$	2	$C_n$	1	$SU(2n)$	$Sp(2n, \mathbb{C}) \cap SU(2n)$
$A_{2n}$	2	$B_n$	2	$SU(2n+1)$	$SO(2n+1)$
$D_4$	3	$G_2$	1	$Spin(8) = \widetilde{SO}(8)$	$\widetilde{G}_2$
$D_n$	2	$B_{n-1}$	1	$Spin(2n) = \widetilde{SO}(2n)$	$Spin(2n-1) = \widetilde{SO}(2n-1)$
$E_6$	2	$F_4$	1	$\widetilde{E}_6$	$\widetilde{F}_4$

Table 1: Simple Lie algebras, groups and data related to outer automorphisms.

### 3 Non-commutative geometry and gauge theory

The existing information on the open string spectra of twisted D-branes on group manifolds can be turned into a proposal for the algebra of functions on these branes. The algebras turn out to be non-commutative due to the presence of a non-vanishing B-field. In the second subsection we shall spell out gauge theories that encode the dynamics of twisted D-branes.

#### 3.1 The world-volume algebra

According to the procedure suggested in [36, 8] (see also [39, 40] for similar proposals in case of closed strings), the world-volume geometry of branes can be read off from the correlators of boundary operators in the decoupling regime  $k \rightarrow \infty$ <sup>2</sup>. Note that the conformal dimensions (10) of the boundary fields vanish in this limit so that the operator product expansion (9) becomes independent of the world sheet coordinates. In particular, all conformal families in the boundary theory contribute to the massless sector and thus to the gauge theory which governs the low energy dynamics of the D-branes we are studying. The program we have just sketched has been carried out successfully for the untwisted branes on compact group manifolds. In case of  $A_1$ , this leads to the well known fuzzy spheres [33, 8]. We will now describe the generalization to arbitrary  $\omega$ -twisted D-branes on compact simply-connected simple group manifolds  $G$ . Note that all simple Lie groups of ADE type (except from  $A_1$ ) admit non-trivial outer automorphisms.

In section 2.2 we explained that all possible  $\omega$ -twisted D-branes are labeled by the set  $\mathcal{B}^\omega$  of fractional symmetric weights of  $\mathfrak{g}$ . In this section we propose an alter-

<sup>2</sup>When taking the limit  $k \rightarrow \infty$ , we keep the open string data stable. In particular, the low energy spectrum of open string modes does not change. A limit where the closed string data is kept stable instead has been considered in [12].

native in which the same boundary conditions are labeled by representations  $P_{G^\omega}^+$  of the invariant subgroup  $G^\omega = \{g \in G \mid \omega(g) = g\}$ . Let us stress that we use representations of the group  $G^\omega$  and not of its Lie algebra. The two sets of representations agree only if  $G^\omega$  is simply-connected. This is the case for all Lie algebras but the  $A_{2n}$  series where  $G^\omega = SO(2n+1)$  and  $P_{G^\omega}^+ = \{A \in P_{G^\omega}^+ \mid A_n \text{ even}\}$ . An overview over the relevant groups and Lie algebras can be found in Table 1. It is far from obvious that such a labeling of twisted boundary conditions through representations of the invariant subgroup exists. But as we explain in Appendix A one can indeed construct a structure preserving map  $\Psi : P_{G^\omega}^+ \rightarrow \mathcal{B}^\omega$  between the set of irreducible representations of  $G^\omega$  and the boundary labels for  $\omega$ -twisted branes. For this reason, we will henceforth use both kinds of labels  $a = \Psi^{-1}(\alpha), b = \Psi^{-1}(\beta), \dots$  equivalently when referring to boundary conditions<sup>3</sup>.

Once the existence of  $\Psi$  is established, we can formulate our proposal for the world-volume algebra. To this end, let  $V_a, V_b$  be two representation spaces for irreducible representations  $a, b \in P_{G^\omega}^+ \cong \mathcal{B}^\omega$  of  $G^\omega$ . As we will argue below, the relevant algebraic structure governing strings stretching between two D-branes of type  $a$  and  $b$ , respectively, is given by

$$\mathcal{A}^{(a,b)} \cong \text{Inv}_{G^\omega}(\mathcal{F}^{(a,b)}) \quad \text{where} \quad \mathcal{F}^{(a,b)} := \mathcal{F}(G) \otimes \text{Hom}(V_a, V_b) \quad . \quad (12)$$

Here  $\mathcal{F}(G)$  denotes the algebra of functions on the group  $G$  and  $\text{Hom}(V_a, V_b)$  is the vector space of linear transformations from  $V_a$  to  $V_b$ . The auxiliary space  $\mathcal{F}^{(a,b)}$  can be regarded as a vector space of matrix valued functions on the group  $G$ . It carries an action of the product group  $G \times G^\omega$  defined by

$$A^{(g,h)}(g') = R_b(h)A(g^{-1}g'h)R_a(h)^{-1}, \quad (13)$$

where  $R_a(h) \in \text{GL}(V_a), R_b(h) \in \text{GL}(V_b)$  are the representation matrices of  $h$ . In our construction of  $\mathcal{A}^{(a,b)}$ , we restrict to matrix valued functions  $\text{Inv}_{G^\omega}(\mathcal{F}^{(a,b)})$  which are invariant under the action of  $\{\text{id}\} \times G^\omega \subset G \times G^\omega$ . Let us note that this leaves us with an action of  $G$  on the space  $\mathcal{A}^{(a,b)}$  of  $G^\omega$  invariants. The latter will become important later on.

We can realize the  $G$ -module  $\mathcal{A}^{(a,b)}$  explicitly in terms of  $G^\omega$ -equivariant functions on the group  $G$ ,

$$\mathcal{A}^{(a,b)} \cong \left\{ A \in \mathcal{F}^{(a,b)} \mid A(gh) = R_b(h)^{-1}A(g)R_a(h) \text{ for } h \in G^\omega \right\}. \quad (14)$$

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<sup>3</sup>The map  $\Psi$  specifies the way in which we treat the boundary labels while sending  $k$  to infinity. We refer the reader to Appendix A for details.

When the two involved representations are trivial, i.e.  $a = b = 0$ , elements of  $\mathcal{A}^{(a,b)}$  are simply invariant under right translations with respect to  $G^\omega \subset G$ .

There exists more structure on the spaces  $\mathcal{A}^{(a,b)}$  if  $a = b$ . In fact,  $\mathcal{A}^{(a)} = \mathcal{A}^{(a,a)}$  inherits an associative product from the pointwise multiplication of elements in  $\mathcal{F}^{(a,a)}$ . This turns the subspace  $\mathcal{A}^{(a)}$  of  $G^\omega$ -invariants into an associative matrix algebra.

The constructions we have outlined so far may easily be generalized to arbitrary superpositions of D-branes. To this end we replace the irreducible representations  $V_a, V_b$  in (12) by reducible ones. Let  $V_Q$  be such a reducible representation, i.e.  $V_Q \cong \oplus Q^a V_a$ . It represents a superposition of  $\sum Q^a$  D-branes in which  $Q^a$  branes of type  $a \in P_{G^\omega}^+ \cong \mathcal{B}^\omega$  are placed on top of each other. Strings ending on such a brane configuration  $Q$  give rise to an algebra  $\mathcal{A}^{(Q)} = \mathcal{A}^{(Q,Q)}$  analogous to (12). For a stack of  $N$  identical branes of type  $a \in P_{G^\omega}^+ \cong \mathcal{B}^\omega$ , the constructions specialize and produce the typical Chan-Paton factors,

$$\mathcal{A}^{N(a)} \cong \text{Inv}_{G^\omega}(\mathcal{F}^{(a,a)}) \otimes \text{Mat}(N) \quad . \quad (15)$$

Obviously, the left translation of the group  $G$  turns this into a  $G$ -module with trivial action of  $G$  on  $\text{Mat}(N)$ .

This concludes the formulation of our proposal for the algebra of “functions on twisted D-branes”. There exist two different kinds of evidence which we can use to motivate and support our claim. Let us begin with a simple semiclassical argument. It is not difficult to see that a twisted conjugacy class “close to” the twisted conjugacy class of the group unit can be represented in the form  $C^\omega = G \times_{G^\omega} C'$ . Here,  $C'$  is a conjugacy class of  $G^\omega$  “close to” the group unit and  $G^\omega$  acts on  $G$  by multiplications on the right. In other words,  $C^\omega$  can be considered as a bundle over  $G/G^\omega$  with fiber  $C'$ . In the  $k \rightarrow \infty$  limit, we keep  $C'$  small by rescaling its radius. As discussed in [8],  $C'$  then turns into a co-adjoint orbit. At the same time, the volume of  $G/G^\omega$  grows with  $k$  so that the corresponding Poisson bi-vector scales down. Hence, the  $G/G^\omega$  part of  $C^\omega$  becomes classical. After quantization, we obtain a bundle with non-commutative fibers. While the co-adjoint orbits turn into  $\text{Hom}(V_a, V_a)$ , the base  $G/G^\omega$  stays classical in agreement with our claim for  $a = b$ .

More substantial support comes from the exact CFT results described in the last section. In particular, we shall confront the formula (14) with the CFT-spectrum of boundary fields (5). Before we carry out the details, let us note that the sewing constraints (11) are automatically satisfied by our construction if we manage to show

that the spectra match. In fact, associativity is manifest in our proposal and it is the only content of the sewing constraints when we send the level  $k$  to infinity.

Hence, it remains to discuss the spectrum of open strings. The CFT description provides an expression eq. (5) for the spectrum of strings stretching between D-branes of type  $\alpha, \beta \in \mathcal{B}_k^\omega$  in terms of characters of  $\hat{\mathfrak{g}}_k$  which explicitly shows the  $G$ -module structure of the space of ground states that emerges in the limit  $k \rightarrow \infty$ . We claim that in this limit, the  $G$ -module of ground states is isomorphic to the  $G$ -module  $\mathcal{A}^{(a,b)}$  where  $a = \Psi^{-1}(\alpha)$  and  $b = \Psi^{-1}(\beta)$  are the pre-images of the boundary labels  $\alpha$  and  $\beta$  under the map  $\Psi : P_{G^\omega}^+ \rightarrow \mathcal{B}_k^\omega$  (see above). We will prove this by decomposing  $\mathcal{A}^{(a,b)}$  into irreducibles. To do so, let us note that there is a canonical isomorphism  $\text{Hom}(V, W) \cong V^* \otimes W$ . Furthermore, we may apply the Peter-Weyl theorem to decompose the algebra  $\mathcal{F}(G)$  with respect to the regular action of  $G \times G$  into

$$\mathcal{F}(G) \cong \bigoplus_A U_A^* \otimes U_A$$

where  $A$  runs over all irreducible representations of  $G$  and the two factors of  $G \times G$  act on the two vector spaces  $U_A^*, U_A$ , respectively. To make contact with our definition of  $\mathcal{A}^{(a,b)}$ , we have to restrict the right regular  $G$  action to the subgroup  $G^\omega$ , which leaves us with the  $G \times G^\omega$ -module

$$\mathcal{F}(G) \cong \bigoplus_{A,c} b_A^c U_A^* \otimes V_c \quad .$$

The numbers  $b_A^c \in \mathbb{N}_0$  are the so-called *branching coefficients* which count the multiplicity of the  $G^\omega$ -module  $V_c$  in  $U_A$ . Combining these remarks we arrive at

$$\mathcal{F}^{(a,b)} \cong \bigoplus_{A,c} b_A^c U_A^* \otimes V_c \otimes V_a^* \otimes V_b \quad .$$

It remains now to find the invariants under the  $G^\omega$ -action. Note that  $G^\omega$  acts on the last three tensor factors. The number of invariants in the triple tensor product of irreducible representations is simply given by the fusion rules  $N_{cb}^a$  of  $G^\omega$ . Hence, as a  $G$ -module, we have shown that

$$\mathcal{A}^{(a,b)} \cong \bigoplus_{A,c} b_A^c N_{cb}^a U_A^*$$

This decomposition is now to be compared with the formula (5) for the CFT partition functions. A careful analysis shows that both decompositions agree in the limit  $k \rightarrow \infty$  provided one uses the appropriate identification map  $\Psi$ , i.e.

$$(n_A^\omega)_\alpha^\beta = \sum_c b_A^c N_{ca}^b \quad (16)$$

where the indices are related by  $\alpha = \Psi(a)$  and  $\beta = \Psi(b)$ . The proof is quite technical and not relevant for the further developments in this paper. It can be found in Appendix A. Let us only mention at this place that our proof takes some benefit of results found in [41, 22].

As a simple cross-check we consider the case of trivial automorphism  $\omega = \text{id}$  where we can make contact to well known results (see [8]). First we observe that the construction above simplifies considerably since  $G^\omega \cong G$ . This implies that all the lower case labels can be replaced by capital letters. In particular, the boundary conditions are now labeled by representations of  $G$  itself which is a well established fact. The corresponding  $G$ -module structure is now given by

$$\mathcal{A}^{(A,B)} \cong \bigoplus_C N_{A+B}{}^C U_C \quad .$$

This is in complete agreement with the known CFT results in *Cardy's case* [1].

We close this section with an interesting side-remark. Since we were interested in the analysis of twisted branes, our presentation has focused on the subgroup  $G^\omega$  in  $G$ . The right hand side of our central formula (16), however, gives rise to an infinite-dimensional analogue of a NIM-rep<sup>4</sup> for the fusion algebra<sup>5</sup> of the Lie group  $G$  and any choice of a subgroup  $H \hookrightarrow G$ . This has been discussed extensively in [41] where also the connection of the  $k \rightarrow \infty$  limit of NIM-reps for twisted boundary conditions in WZW models to these NIM-reps has been established. Recently, new explicit expressions for NIM-reps of twisted D-branes have been proposed in [42, 43] for finite level  $k$ . They bear some similarity with our formula (16) but generically do not involve branching coefficients of the invariant subgroup  $G^\omega$ . Where they do, i.e. for the  $A_n$ -series in [42] as well as the  $D_n$ -series and the  $A_{2n}$ -series in [43], they reduce to relation (16) in the limit  $k \rightarrow \infty$ . It seems to us that so far a systematic understanding of the proper choice for the relevant subgroup is only available at infinite level.

### 3.2 The non-commutative gauge theory

Equipped with the exact solution of the boundary WZW model in the limit  $k \rightarrow \infty$ , we are finally prepared to calculate the low-energy effective action for massless open string modes. Compared to the case of D-branes in flat space with background  $B$ -field which leads to a Yang-Mills theory on a non-commutative space [44], there are

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<sup>4</sup>Non-negative Integer valued Matrix Representation.

<sup>5</sup>The latter is obtained from the fusion algebra of the WZW model in the limit  $k \rightarrow \infty$ .

two important changes in the computation. First, the non-commutative Moyal-Weyl product gets replaced by the product of  $G^\omega$ -equivariant matrix valued functions (15) described in the previous subsection. Moreover, there appears a new term  $f_{\mu\nu}{}^\sigma J_\sigma$  in the operator product expansion of the currents. This term leads to an extra contribution of the form  $f_{\mu\nu\sigma} A^\mu A^\nu A^\sigma$  in the scattering amplitude of three massless open string modes. Consequently, the resulting effective action is not only given by Yang-Mills theory on a non-commutative space but also involves a Chern-Simons like term.

For  $N$  branes of type  $a \in P_{G^\omega}^+ \cong \mathcal{B}^\omega$  on top of each other, the fields  $A^\mu(g)$  are elements of  $\mathcal{A}^{N(a)}$ , i.e. they are functions on  $G$  with values in  $\text{End}(V_a) \otimes \text{Mat}(N)$  and equivariance property as formulated in eq. (14). We denote the dimension of  $V_a$  by  $d_a$ . The results of the computation [13] may easily be transferred to the new situation and can be summarized in the following action

$$\mathcal{S}_{N(a)} = \mathcal{S}_{\text{YM}} + \mathcal{S}_{\text{CS}} = \frac{\pi^2}{k^2 d_a N} \left( \frac{1}{4} \int \text{tr} (F_{\mu\nu} F^{\mu\nu}) - \frac{i}{2} \int \text{tr} (f^{\mu\nu\sigma} \text{CS}_{\mu\nu\sigma}) \right) \quad (17)$$

where we defined the “curvature form”  $F_{\mu\nu}$  by the expression

$$F_{\mu\nu}(A) = i L_\mu A_\nu - i L_\nu A_\mu + i [A_\mu, A_\nu] + f_{\mu\nu\sigma} A^\sigma \quad (18)$$

and a non-commutative analogue of the Chern-Simons form by

$$\text{CS}_{\mu\nu\sigma}(A) = L_\mu A_\nu A_\sigma + \frac{1}{3} A_\mu [A_\nu, A_\sigma] - \frac{i}{2} f_{\mu\nu\rho} A^\rho A_\sigma \quad (19)$$

Gauge invariance of (17) under the infinitesimal gauge transformations

$$\delta A_\mu = i L_\mu \Lambda + i [A_\mu, \Lambda] \quad \text{for} \quad \Lambda \in \mathcal{A}^{N(a)} \quad (20)$$

follows by straightforward computation. The operator  $L$  is the usual Lie derivative as defined in (25) below. Note that the “mass term” in the Chern-Simons form (19) guarantees the gauge invariance of  $\mathcal{S}_{\text{CS}}$ . On the other hand, the effective action (17) is the unique combination of  $\mathcal{S}_{\text{YM}}$  and  $\mathcal{S}_{\text{CS}}$  in which mass terms cancel.

In contrast to earlier work, the trace is now normalized by  $\text{tr}(\text{id}) = d_a N$ . Moreover, we use conventions in which only pure Lie algebraic quantities appear. The remaining part of this section is devoted to presenting these conventions which follow [45]. Indices are raised and lowered using the Killing form

$$\kappa(x, y) = \text{tr}(\text{ad}_x \circ \text{ad}_y) \quad \kappa^{\mu\nu} = \frac{1}{I_\theta} \kappa(T^\mu, T^\nu) = \frac{1}{I_\theta} f^{\mu\rho\sigma} f^\nu{}_{\rho\sigma} \quad (21)$$

Here,  $I_B$  denotes the Dynkin index of a representation  $B \in P^+$  and  $T^\mu$  are the generators of  $\mathfrak{g}$  satisfying

$$[T^\mu, T^\nu] = if^{\mu\nu}{}_\sigma T^\sigma . \quad (22)$$

For the adjoint representation  $\theta$  of  $\mathfrak{g}$  entering (21), the Dynkin index may also be expressed by the quadratic Casimir and by the dual Coxeter number according to  $I_\theta = C_\theta = 2g^\vee$ . The quadratic Casimir and the index of an arbitrary representation  $B \in P^+$  with dimension  $d_B$  are given by

$$C_B = (B, B + 2\rho) \quad \text{and} \quad I_B = \frac{d_B}{\dim \mathfrak{g}} C_B . \quad (23)$$

It will be of particular importance for us to evaluate the quadratic Casimir in a given representation  $B \in P^+$  according to

$$\text{tr } R_B(T^\mu)R_B(T^\nu) = I_B \kappa^{\mu\nu} . \quad (24)$$

Finally, let us note that the  $G$ -action on the algebra  $\mathcal{A}^{N(a)}$  allows us to define the derivatives

$$L^\mu A(g) = \frac{1}{i} \frac{d}{dt} \left( A(e^{-itT^\mu} g) \right) \Big|_{t=0} \quad (25)$$

for  $A \in \mathcal{A}^{N(a)}$ . These Lie derivatives appear in the construction of the action above and they satisfy the same Lie algebra relations as in eq. (22).

For a calculation in the next section we will also need derivatives on functions  $K : G \rightarrow \text{Mat}(N)$  which are defined via the group action on the argument from the right,

$$L^{\text{R}\mu} K(g) = \frac{1}{i} \frac{d}{dt} \left( K(g e^{itT^\mu}) \right) \Big|_{t=0} . \quad (26)$$

The connection between the different derivatives (25),(26) is given by the adjoint representation of  $G$ ,

$$L_\mu K(g) = -\text{Ad}(g^{-1})_\mu{}^\nu L_\nu^{\text{R}} K(g) . \quad (27)$$

## 4 Condensates of branes on group manifolds

The study of classical solutions of the non-commutative gauge theories constructed in the previous sections provides insights into the dynamics of twisted branes on group manifolds. In the first subsection we shall exhibit a rather large class of solutions. The symmetry preserving solutions are then interpreted geometrically in the second subsection. In particular, we shall see that all twisted D-branes appear as bound states from stacks of a distinguished elementary twisted brane.



## 4.1 Classical solutions

A simple variation of the action (17) with respect to the gauge fields allows us to derive the following equations of motion,

$$L^\mu F_{\mu\nu} + [A^\mu, F_{\mu\nu}] = 0 \quad , \quad (28)$$

which express that the curvature has to be covariantly constant. Note that the space of gauge fields, and hence the equations of motions, depends on the brane configuration we are looking at.

Solutions to the equations (28) describe condensation processes on a brane configuration  $Q$  which drive the whole system into another configuration  $Q'$ . To identify the latter, we have two different types of information at our disposal. On the one hand, we can compare the tension of D-branes in the final configuration  $Q'$  with the value of the action  $\mathcal{S}_Q(A)$  at the classical solution  $A(g)$ . On the other hand, we can look at fluctuations around the chosen solution and compare their dynamics with the low energy effective theory  $\mathcal{S}_{Q'}$  of the brane configuration  $Q'$ . In formulas, this means that

$$\mathcal{S}_Q(A + \delta A) \stackrel{!}{=} \mathcal{S}_Q(A) + \mathcal{S}_{Q'}(\delta A) \quad \text{with} \quad \mathcal{S}_Q(A) \stackrel{!}{=} \ln \frac{g_{Q'}}{g_Q} \quad . \quad (29)$$

The second requirement expresses the comparison of tensions in terms of the  $g$ -factors of the involved conformal field theories (see e.g. [13] for more details). All equalities must hold to the order in  $(1/k)$  that we used when constructing the effective actions. We say that the brane configuration  $Q$  decays into  $Q'$  if  $Q'$  has lower mass, i.e. whenever  $g_{Q'} < g_Q$ .

In terms of the world-sheet description, each classical solution of the effective action is linked to a conformal boundary perturbation in the CFT of the brane configuration  $Q$ . Adding the corresponding boundary terms to the original theory causes the boundary condition to change so that we end up with the boundary conformal field theory of another brane configuration  $Q'$ . Recall, however, that all these statements only apply to a limiting regime in which the level  $k$  is sent to infinity.

We are especially interested in processes that connect maximally symmetric configurations of D-branes, i.e. those configurations for which the spectrum decomposes into representations of  $\hat{\mathfrak{g}}_k$ . For such configurations we know the low energy effective actions so that we are able to compare them with the dynamics of fluctuations. We shall argue that solutions possessing a  $G$ -symmetric fluctuation spectrum are

associated with gauge fields  $A(g)$  whose curvature is proportional to the identity, or equivalently

$$[F_{\mu\nu}(A), \cdot] = 0 \quad . \quad (30)$$

The  $G$ -module structure of the fluctuation spectrum around the solution can be characterized by the simple rule

$$L_\mu^{Q'} := L_\mu^Q + [A_\mu, \cdot] \quad . \quad (31)$$

Our notation suggests that we want to think of  $L_\mu^{Q'}$  as derivative associated with some new brane configuration  $Q'$ . Note, however, that by construction it acts on the space  $\mathcal{A}^{(Q)}$  of gauge fields on  $Q$ . To check our claims, we must show that the derivatives (31) satisfy the Lie algebra commutation relations and that the action for the fluctuation fields involves these new derivatives rather than  $L^Q$ . With the Ansatz (31) it is easy to show that

$$[L_\mu^{Q'}, L_\nu^{Q'}] = i f_{\mu\nu}{}^\sigma L_\sigma^{Q'} - i [F_{\mu\nu}(A), \cdot] \quad .$$

Furthermore, the expansion of the action around the solution  $A(g)$  reads

$$\mathcal{S}_Q(A + \delta A) = \mathcal{S}_Q(A) + \mathcal{S}_{Q'}(\delta A) + \frac{i}{2} \text{tr}([F_{\mu\nu}(A), \delta A^\mu] \delta A^\nu) \quad .$$

In a slight abuse of notation we have denoted the action for the fluctuation fields  $\delta A_\mu \in \mathcal{A}^{(Q)}$  by  $\mathcal{S}_{Q'}$ . For the moment, this only means that all derivatives  $L^Q$  are replaced by  $L^{Q'}$ . As one reads off from the previous equation, gauge fields satisfying (30) do indeed lead to a  $G$ -symmetric fluctuation spectrum. A more careful investigation shows that the converse is also true, which is to say, any solution which produces a symmetric fluctuation spectrum is necessarily of the form we have described. The crucial observation in proving this statement is that the relation (31) between the derivatives is dictated by comparing third-order terms in (29).

Before we dive into the discussion of these symmetric solutions, let us briefly mention some solutions of (28) which break the  $G$ -symmetry. One particular type of non-symmetric solution can be obtained for each choice of a subalgebra  $\mathfrak{h}$  in  $\mathfrak{g}$ . We label the subalgebra generators by  $i, j, \dots$ , the directions orthogonal to  $\mathfrak{h}$  by  $r, s, \dots$ . Now let  $R$  be a  $d_R$ -dimensional representation of  $\mathfrak{h}$ . Then the equations of motion for a stack of  $d_R$  identical branes possess a constant solution of the form  $A^i = \text{id} \otimes R(T^i)$  and  $A^r = 0$ , where  $\text{id}$  is the identity matrix in the space-time degrees of freedom. The curvature of these solutions is given by  $F_{ij} = 0$ ,  $F_{ir} = 0$  and  $F_{rs} = f_{rs}{}^i R(T_i)$ . Obviously, this solution does not satisfy (30) and hence it

breaks the  $G$ -symmetry. One can identify solutions of this type with some of the symmetry breaking boundary theories that were obtained in [46]. The associated boundary states preserve the affine Lie algebra of  $H$  along with the coset chiral algebra for  $G/H$ .

There is another more trivial class of constant solutions which break the  $G$ -symmetry. In fact, any set of commuting constant matrices  $A_\mu$  is a solution to (28). The corresponding curvature is given by  $F_{\mu\nu} = f_{\mu\nu}^\sigma A_\sigma$ , so that in general  $[F_{\mu\nu}, \cdot] \neq 0$ . The solutions have vanishing action and allow for an easy geometric interpretation [20, 13]: they describe translations of the branes in the group manifold. It is not surprising that these translations break the symmetry, unless the whole stack is translated at once. In the latter case, the fields  $A$  and thus  $F$  are multiples of the identity as it should be the case for a symmetric solution. Let us observe that uniform translations on the group manifold are a rather trivial operation, and we are only interested in analyzing solutions “up to translations”. One can show that it suffices to study solutions with traceless fields  $A(g)$ . From these it is possible to generate any other solution by a uniform translation.

After this digression we come back to the study of symmetric solutions. It follows from the last remark in the previous paragraph that there is no loss in restricting ourselves to symmetric solutions with vanishing curvature  $F = 0$ . Let us now start with a stack of  $N$  branes corresponding to the trivial representation  $0 \in P_{G^\omega}^+$  of  $G^\omega$ . A whole class of solutions of (28) with  $F = 0$  can be constructed out of a function  $K : G \rightarrow \text{GL}(N)$  satisfying

$$\begin{aligned} K(gh) &= K(g)R(h) \quad \text{for all} \quad h \in G^\omega \quad \text{by} \\ A_\mu(g) &:= -(L_\mu K)(g)K(g)^{-1} = K(g)(L_\mu K^{-1})(g) \end{aligned} \tag{32}$$

with  $R$  being an  $N$ -dimensional representation of  $G^\omega$ . The first property guarantees the necessary invariance property  $A(gh) = A(g)$  of the physical fields  $A(g)$ . Moreover, the curvature of these gauge fields vanishes because

$$\begin{aligned} F_{\mu\nu} &= i L_\mu A_\nu - i L_\nu A_\mu + i [A_\mu, A_\nu] + f_{\mu\nu}^\sigma A_\sigma \\ &= -i L_\mu (L_\nu K K^{-1}) + i L_\nu (L_\mu K K^{-1}) \\ &\quad - i (L_\mu K L_\nu K^{-1} - L_\nu K L_\mu K^{-1}) - f_{\mu\nu}^\sigma L_\sigma K K^{-1} \\ &= -i [L_\mu, L_\nu] K K^{-1} - f_{\mu\nu}^\sigma L_\sigma K K^{-1} = 0 . \end{aligned}$$

We should mention at this point that  $F = 0$  is precisely the integrability condition of the linear system  $L^\mu K = -A^\mu K$  so that any solution to  $F = 0$  is of the form (32).

The covariance law  $K(gh) = K(g)R(h)$  for  $h \in G^\omega$  is then necessary to ensure the invariance property  $A(gh) = A(g)$  for  $h \in G^\omega$ . Consistency implies that  $R(h)$  satisfies the representation property.

Our formula (32) for symmetric solutions leaves us with the problem to find a  $\text{GL}(N, \mathbb{C})$  valued function satisfying the covariance condition  $K(gh) = K(g)R(h)$  for  $h \in G^\omega$ . As we shall argue now, such a function does always exist.

To begin with, let us rephrase the existence of  $K$  in the language of vector bundles. For a given representation  $R$ , we consider the vector bundle  $E \rightarrow G/G^\omega$ . Here  $E$  is the associated bundle  $G \times_{G^\omega} V_R^\mathbb{C}$  which consists of pairs  $(g, v)$  along with the identification  $(g, v) \sim (gh, v R(h))$ . Suppose now that we are given the function  $K : G \rightarrow \text{GL}(N)$ . By construction, the  $N$  row-vectors  $k_i(g)$  are linearly independent and satisfy  $k_i(gh) = k_i(g)R(h)$ . In other words, they provide a linearly independent set of global sections of  $E$ , i.e. a global trivialization. The converse is also true. Hence, the existence of  $K$  is equivalent to the triviality of  $E$ .

It is well known that the bundle  $G \times_{G^\omega} V_R$  is trivial if the representation  $R$  of  $G^\omega$  can be obtained by restricting a representation of  $G$ . This condition is too strong for our purposes. Note, however, that we construct our bundle  $E$  with the complexified space  $V_R^\mathbb{C}$  and this makes a huge difference. In fact,  $E$  turns out to be trivial for all representations  $R$ .

The proof proceeds in two steps. First one shows that all the bundles  $E$  are stably equivalent to a trivial bundle. Since stable equivalence is preserved under tensoring representations of  $G^\omega$ , one can concentrate on the fundamental representations and hence reduce the problem to a finite number of checks which have to be performed case-by-case. Passing from stable equivalence to isomorphism is possible if the rank of the bundle is sufficiently large compared to the dimension of the base manifold. Again, this has to be decided by going through all the cases separately. The details of this proof can be found in Appendix B.

After these remarks on the existence of  $K$ , let us briefly address the issue of uniqueness. On the face of it, formula (32) seems to provide a very large set of solutions with many continuous parameters. For a given representation  $R$ , however, all these solutions are gauge equivalent. To see this, let  $K, A$  and  $K', A'$  be two such solutions with a covariance law involving the same  $R$ . Then we find

$$A'_\mu = U A_\mu U^{-1} - (L_\mu U) U^{-1} \quad (33)$$

with  $U = K' K^{-1}$ , i.e.  $A$  and  $A'$  are related by a gauge transformation  $U$  (note that this requires invariance of our theory under global gauge transformations which is

granted as long as we do not consider the automorphism of order 3). In conclusion, our discussion in this subsection provides for each  $R$  a unique solution with vanishing curvature.

The case of  $K \equiv R$  being a representation of  $G$  can be considered as a special example of our general construction as it may be restricted to a (reducible) representation of  $G^\omega$ . The corresponding field  $A(g)$  turns out to be constant and it is given by

$$A^\mu = R(T^\mu)$$

where  $T^\mu$  are generators of the Lie algebra  $\mathfrak{g}$ . It is easy to see that all constant solutions to  $F = 0$  have this form. This class of solutions and the corresponding brane processes have already been discussed in [10]. In the case of untwisted branes, these solutions form a complete set of solutions for a stack of branes of type  $0 \in P_G^+$ . For twisted branes, however, we just presented new non-constant solutions which give rise to new brane processes.

## 4.2 Interpretation of the solutions

In the last section we presented a large number of new stationary points for the action of  $N = d_a$  branes of type  $0 \in P_{G^\omega}^+ \cong \mathcal{B}^\omega$ . Here,  $d_a$  is the dimension of an irreducible representation  $R_a$  of  $G^\omega$ . We shall present evidence that such solutions correspond to processes of the type

$$\text{Stack of } d_a \text{ branes of type } 0 \in P_{G^\omega}^+ \longrightarrow \text{single brane of type } a \in P_{G^\omega}^+ . \quad (34)$$

Any twisted D-brane can be obtained as a condensate of a stack of elementary branes of type  $0 \in P_{G^\omega}^+$ . This implies in particular that any two configurations whose corresponding representations of  $G^\omega$  have the same dimension can be related by a process. A similar observation has been made for untwisted branes [13, 10] and the new processes might provide further insights to the contribution of twisted D-branes to the twisted K-groups which seem to describe the D-brane charges on group manifolds (see [47, 10, 48]).

According to eq. (29), there are essentially two checks that we must perform in order to test the conjecture (34). We will compare the value of the action at the solution to the CFT-prediction below and start with an analysis of the fluctuations around the solution.

The general form of the fluctuation spectrum follows from the discussion in section 4.1 and is summarized in eq. (29). It is implicitly contained in the  $G$ -action

entering the derivatives (31) which we used to compute the action  $\mathcal{S}_{Q'}(\delta A)$  of the fluctuation fields. But as we have stressed earlier, the fields  $\delta A$  are elements of the algebra  $\mathcal{A}^{d_a(0)}$ . Let us be precise and introduce a new symbol  $\tilde{\mathcal{A}}^{d_a(0)}$  for the  $G$ -module  $(\mathcal{A}^{d_a(0)}, L_{Q'})$ . Our aim then is to identify the functional  $\mathcal{S}_{Q'}(\delta A)$  on  $\tilde{\mathcal{A}}^{d_a(0)}$  with the action functional  $\mathcal{S}_{(a)}$  on  $\mathcal{A}^{(a)}$  which governs the dynamics of a brane of type  $(a)$ . This comes down to providing a  $G$ -module isomorphism  $\Phi$  between  $\tilde{\mathcal{A}}^{d_a(0)}$  and  $\mathcal{A}^{(a)}$ . In particular, the derivatives have to match,

$$\Phi(\tilde{L}_\mu^{d_a(0)} \delta A) = L_\mu^{(a)} \Phi(\delta A) .$$

Such an isomorphism can be obtained with the help of the function  $K$  from which we constructed our solution,

$$\Phi(\delta A)(g) = K^{-1}(g) \delta A(g) K(g) \in \mathcal{A}^{(a)} .$$

One can check that the action of the derivatives  $L_\mu^{(a)}$  on  $\mathcal{A}^{(a)}$  precisely coincides with the action of the shifted derivatives on  $\tilde{\mathcal{A}}^{d_a(0)}$ ,

$$\tilde{L}_\mu^{d_a(0)} = L_\mu^{d_a(0)} + [A_\mu, \cdot] \stackrel{\Phi}{\cong} L_\mu^{(a)} ,$$

where the shift is given by our solution  $A_\mu = -L_\mu K K(g)^{-1}$  of the equations of motion. Thus we have proved that the theory of fluctuations around our solution coincides with  $\mathcal{S}_{(a)}$  as we anticipated in eq. (34).

Our second check involves a comparison between the value of the action at the solution and the  $g$ -factors of a CFT description. For technical reasons, we restrict ourselves to automorphisms  $\omega$  of order 2, thereby excluding only the case of triality for  $D_4$ . The existence of  $\omega$  implies strong constraints on the form of the structure constants. To be specific, by diagonalization of  $\omega$ , we may choose a basis in which only the constants  $f_{ijk}, f_{rsi}$  and cyclic permutations thereof do not vanish. Here,  $i, j, k, \dots$  denote indices for elements in the invariant subalgebra  $\mathfrak{g}^\omega$  and  $r, s, t, \dots$  label directions orthogonal to  $\mathfrak{g}^\omega$ . We are now able to compute the action using no more than the properties of the solution we have specified. For a solution with  $F = 0$ , the action reduces to

$$\mathcal{S}_{d_a(0)}(A) = \kappa f^{\mu\nu\sigma} \int \text{tr } A_\mu A_\nu A_\sigma$$

with  $\kappa = i\pi^2/6k^2 d_a$ . We now express the solution (32) in terms of the right derivatives (26),

$$A_\mu = \text{Ad}(g^{-1})_\mu{}^\nu (L_\nu^R K) K^{-1} .$$

Since we have to contract the  $A_\mu$ s with  $f^{\mu\nu\sigma}$ , the adjoint action of  $G$  can be dropped to obtain

$$\mathcal{S}_{d_a(0)}(A) = \kappa f^{\mu\nu\sigma} \int \text{tr } A'_\mu A'_\nu A'_\sigma$$

with  $A'_\mu = K^{-1}(L_\mu^R K)$ . Our previous remarks on the form of the structures constants suggest to split this formula for the action into two terms

$$\mathcal{S}_{d_a(0)}(A) = \kappa f^{ijk} \int \text{tr } A'_i A'_j A'_k + 3\kappa f^{rsi} \int \text{tr } A'_r A'_s A'_i . \quad (35)$$

The first term can be computed easily since  $A'_i(g) = R_a(T_i)$ . As for the second term, one proceeds by expressing the gauge field components  $A'_r(g)$  through  $K(g)$ . Integration by parts and use of the anti-symmetry of the structure constants results in

$$\mathcal{S}_{d_a(0)}(A) = \kappa f^{ijk} \int \text{tr } R_a(T_i) R_a(T_j) R_a(T_k) + \frac{3\kappa}{2} f^{rsi} \int \text{tr } K^{-1} [L_r^R, L_s^R] K R_a(T_i) .$$

Because of the constraints on the structure constants, the commutator of the derivatives lies in the direction of  $\mathfrak{g}^\omega$  so that we obtain a factor  $if_{rs}^j R_a(T_j)$  within the trace. After a bit of algebra using eqs. (21), (23) and (24), the two terms in the expression for the action can be combined into the following result

$$\mathcal{S}_{d_a(0)}(A) = -\frac{\pi^2}{12k^2} \frac{C_a}{x_e} \left( 3C_\theta - 2\frac{C_{\bar{\theta}}}{x_e} \right) . \quad (36)$$

The numbers  $C_a, C_\theta, C_{\bar{\theta}}$  are the quadratic Casimirs of the representation  $a$  of  $\mathfrak{g}^\omega$  and of the adjoint representations of  $\mathfrak{g}, \mathfrak{g}^\omega$ , respectively ( $\theta, \bar{\theta}$  are the highest roots). The constant  $x_e$  denotes the embedding index of the embedding  $\mathfrak{g}^\omega \hookrightarrow \mathfrak{g}$ . It appears due to different normalization of the quadratic forms  $\kappa_{\mathfrak{g}^\omega}$  and  $\kappa_{\mathfrak{g}}|_{\mathfrak{g}^\omega}$ . It turns out that the value of the action (36) is always negative.

As we recalled before, this result for the value of the action must be compared with the difference between two logarithms of the g-factors [49] in the CFT-description. For the branes  $(a)$ , the g-factor is given by [23, eq. (4.2)]

$$g_{(a)} = \frac{S_{0\Psi(a)}^\omega}{\sqrt{S_{00}}} \quad (37)$$

where we explicitly used the identification  $\Psi : P_{G^\omega}^+ \rightarrow \mathcal{B}^\omega$  to emphasize that the second argument of the twisted S matrix is a fractional symmetric weight. This leads to the following expression for the logarithm of ratio of g-factors

$$\ln \frac{g_{(a)}}{d_a g_{(0)}} = \ln \frac{S_{0\Psi(a)}^\omega}{d_a S_{0\Psi(0)}^\omega} = -\frac{\pi^2}{12k^2} \frac{\dim \mathfrak{g}}{\dim \mathfrak{g}^\omega} \frac{C_a}{x_e} C_\theta . \quad (38)$$

To derive this result one shows first that the quotient in the argument of the logarithm is an ordinary character of  $\mathfrak{g}^\omega$ ,

$$\frac{S_{0\Psi(a)}^\omega}{S_{0\Psi(0)}^\omega} = \chi_a\left(-\frac{2\pi i}{k+g^\vee}\frac{1}{x_e}\mathcal{P}(\rho)\right) .$$

Using (a generalized version of) the asymptotic expansion in [50, eq. (13.175)] one then recovers (38) to lowest order in  $1/k$ .

Although it is not obvious, formula (38) agrees with the previous expression (36) for the value of the effective action at the solution which was obtained under the assumption that  $\omega$  is an order 2 automorphism. This can be checked with the help of Table 1 and well known Lie algebra data (see for example [45, p. 44]). To summarize we have shown (except for the case  $G_2 \hookrightarrow D_4$ ) that our solutions describe processes of the type (34).

## 5 Conclusions and Outlook

In this work we have extended the previous analysis of brane dynamics [8, 13, 51] to brane configurations of all maximally symmetric branes on group manifolds including the so-called twisted D-branes. In particular, we exploited the CFT data to construct the algebra of gauge fields on such branes and provided a formula for the effective action. Moreover, we found a large number of solutions and their geometric interpretation. The new condensation processes turn out to be consistent with the charge conservation laws formulated in [10] (see also [48]). In particular they suggest that the charge of an arbitrary  $\omega$ -twisted D-brane of type  $a \in P_{G^\omega}^+ \cong \mathcal{B}^\omega$  is given by the dimension of the group representation  $a$ .

All this analysis is performed in the limiting regime where the level  $k$  is sent to infinity. It would be interesting to investigate how the described condensation processes deform when we go to finite values of the level  $k$ . For constant gauge fields, such deformations have been studied in [9, 10] based on the “absorption of the boundary spin”-principle (see [52, 53]). These investigations lead to very strong constraints on the structure of the charges that are carried by untwisted D-branes. For non-trivial  $\omega$ , however, constant condensates provide only a small number of processes which are difficult to evaluate. The bound state formation (34) we found in the last section of this paper suggests some obvious extensions that might place the analysis of conservation laws for twisted and untwisted D-branes on an equal footing.



Geometrically, the situations with finite  $k$  are associated with a non-vanishing NSNS 3-form H-field. As has been argued in [8, 54, 55] this might lead to new phenomena in the world-volume geometry. For branes on group manifolds they seem to be related to quantum groups or appropriate modifications thereof (see [8, 56, 57]).

Let us also mention that the results we have obtained here can be of direct relevance for the study of branes in coset models that has recently attracted some attention [58, 59, 60, 61, 62, 63]. In fact, it was shown in [59, 63] that many results on the dynamics of branes on group manifolds descend to cosets through some reduction procedure. The main ideas of this reduction are not specific to trivial gluing conditions and generalize immediately to twisted branes in coset theories. Finally, following the ideas of [60, 46], the coset construction is an essential tool to obtain symmetry breaking boundary conditions e.g. on group manifolds. It is therefore tempting to conclude that a combination of all these results can provide a rather exhaustive picture of brane dynamics on group manifolds, even when we go beyond maximally symmetric branes.

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## A Correspondence between boundary labels and representations

The aim of this section is to provide the correspondence between exactly solvable boundary conditions and representations of the invariant subgroup  $G^\omega$ . This is needed to relate the results of the more geometric picture to those of CFT. We will proceed in the first subsection by comparing the explicit expressions for the NIM-reps (7) to recent results for branching coefficients of arbitrary semi-simple Lie algebras [41]. We are thus able to distinguish a certain subalgebra  $\mathfrak{h}^\omega$  and identify boundary labels with representations of  $\mathfrak{h}^\omega$  such that we get a purely Lie algebra theoretic expression for the NIM-reps in the limit  $k \rightarrow \infty$ . In almost all cases  $\mathfrak{h}^\omega$  is given by the invariant subalgebra  $\mathfrak{g}^\omega$  and we also have  $P_{G^\omega}^+ = P_{\mathfrak{g}^\omega}^+$ . The only case where this procedure does not work is  $\mathfrak{g} = A_{2n}$  where we are lead to the orbit Lie algebra  $\mathfrak{h}^\omega = C_n$  and not to the invariant subalgebra  $\mathfrak{g}^\omega = B_n$ . It was argued in [41], however, that in this specific case a second identification is possible which leads to the invariant subalgebra. We reserve a second subsection to review these results and to discuss the case of  $A_{2n}$  in detail. Note that results closely related to those presented in this section have also been found independently in [42, 43] for finite  $k$  using different methods. In all cases for which such a finite  $k$  extension exists, i.e. for the  $A_n$ -series in [42] as well as the  $D_n$ -series and the  $A_{2n}$ -series in [43], our results may also be recovered from the existing literature by taking  $k$  to infinity. Let us emphasize, however, that in the cases of  $A_{2n-1}$ ,  $D_4$  (triality) and  $E_6$  our treatment seems to indicate stronger statements, i.e. larger subgroups, for finite  $k$  than those proposed in [43]. The geometrical interpretation of these results which was described in the main text and the identification of the invariant subgroup as the relevant structure seem to be new. The results of this appendix have already been announced in [41] where they were used to derive new representations for branching coefficients.

### A.1 The generic correspondence

In this first subsection we propose an identification of boundary labels with representations of a distinguished subalgebra  $\mathfrak{h}^\omega$  of  $\mathfrak{g}$  such that formula (16) is satisfied for the embedding  $\mathfrak{h}^\omega \hookrightarrow \mathfrak{g}$  after taking the limit  $k \rightarrow \infty$ . Neither is obvious that this will work a priori nor is clear which subalgebra one should take. Starting from certain assumptions we will first derive a set of consistency relations. Afterwards we will show that for each  $\mathfrak{g}$  there is indeed a unique solution  $\mathfrak{h}^\omega$  to these consistency

$\mathfrak{g}$	order	$\mathfrak{g}^\omega$	orbit Lie algebra $\check{\mathfrak{g}}$	relevant subalgebra $\mathfrak{h}^\omega$
$A_2$	2	$A_1 (x_e = 4)$	$A_1$	$A_1 (x_e = 1)$
$A_{2n-1}$	2	$C_n$	$B_n$	$C_n$
$A_{2n}$	2	$B_n$	$C_n$	$C_n \hookrightarrow A_{2n-1}$
$D_4$	3	$G_2$	$G_2$	$G_2 \hookrightarrow B_3$
$D_n$	2	$B_{n-1}$	$C_{n-1}$	$B_{n-1}$
$E_6$	2	$F_4$	$F_4$	$F_4$

Table 2: More data related to outer automorphisms of simple Lie algebras.

equations and that in most cases it is given by the invariant subalgebra  $\mathfrak{g}^\omega$ .

Let us begin our discussion with a few general remarks about embeddings of Lie algebras. Any embedding map  $\iota : \mathfrak{h}^\omega \hookrightarrow \mathfrak{g}$  induces a “projection” (or rather “restriction”) in weight space  $\mathcal{P} : L_w^{(\mathfrak{g})} \rightarrow L_w^{(\mathfrak{h}^\omega)}$ . A representation is completely defined by its character, i.e. by the weight system of the given highest weight. Under the projection  $\mathcal{P}$ , this weight system is mapped to a sum of weight systems of highest weights of the subalgebra. This process may be summarized in the decomposition formula  $U_A = \oplus_c b_A^c V_c$  of modules (both sides considered as modules of  $\mathfrak{h}^\omega$ ) where we introduced the so called branching coefficients  $b_A^c \in \mathbb{N}_0$ .

It is now important to specify which structure our identification of boundary labels and representations is supposed to preserve. Let  $\langle \mathcal{B}^\omega \rangle$  be the integer linear span of the set of boundary conditions  $\mathcal{B}^\omega$ , i.e. the full lattice generated by elements of  $\mathcal{B}^\omega$ . Both the lattice  $\langle \mathcal{B}^\omega \rangle$  and the weight lattice  $L_w^{(\mathfrak{h}^\omega)}$  permit an action of Weyl groups. In the first case this group is given by the symmetric part  $W_\omega = \{w \in W_{\mathfrak{g}} | \omega \circ w = w \circ \omega\}$  of the Weyl group of  $\mathfrak{g}$  and in the other case it is naturally given by  $W_{\mathfrak{h}^\omega}$ . Furthermore, in both cases we have a projection  $\mathcal{P} : L_w^{(\mathfrak{g})} \rightarrow L_w^{(\mathfrak{h}^\omega)}$  and  $\mathcal{P}_\omega : L_w^{(\mathfrak{g})} \rightarrow \langle \mathcal{B}^\omega \rangle$ , respectively. The latter is given by the projection  $\mathcal{P}_\omega = \frac{1}{N}(1 + \omega + \dots + \omega^{N-1})$  onto the symmetric part of the weights,  $N$  being the order of  $\omega$ . As we will see, we have to find an isomorphism  $\Psi : L_w^{(\mathfrak{h}^\omega)} \rightarrow \langle \mathcal{B}^\omega \rangle$  between the fractional lattice generated by the boundary conditions and the weight lattice of the subalgebra which preserves all of these structures. In particular it should be accompanied with an isomorphism  $\Psi : W_{\mathfrak{h}^\omega} \rightarrow W_\omega$  of the corresponding Weyl groups. To summarize, we have to find a subalgebra  $\mathfrak{h}^\omega$  and a functor-like map  $\Psi$  such that the diagrams (39) commute. It turns out that the answer for both  $\mathfrak{h}^\omega$

and  $\Psi$  is unique.

$$\begin{array}{ccc}
L_w^{(\mathfrak{g})} & \xrightarrow{\mathcal{P}} & L_w^{(\mathfrak{h}^\omega)} \\
\downarrow = & & \downarrow \Psi \\
L_w^{(\mathfrak{g})} & \xrightarrow{\mathcal{P}_\omega} & \langle \mathcal{B}^\omega \rangle
\end{array}
\qquad
\begin{array}{ccc}
L_w^{(\mathfrak{h}^\omega)} & \xrightarrow{w \in W_{\mathfrak{h}^\omega}} & L_w^{(\mathfrak{h}^\omega)} \\
\downarrow \Psi & & \downarrow \Psi \\
\langle \mathcal{B}^\omega \rangle & \xrightarrow{\Psi(w) \in W_\omega} & \langle \mathcal{B}^\omega \rangle
\end{array}
\tag{39}$$

In the following we restrict ourselves to some non-trivial outer automorphism  $\omega \neq \text{id}$  since our statements become trivial otherwise.

Remember that we only consider the limit  $k \rightarrow \infty$ . For  $\alpha = \Psi(a)$ ,  $\beta = \Psi(b)$  we want to proof the equality

$$(n_A^\omega)_\beta^\alpha = \sum_c b_A^c N_{cb}^a$$

where the last expression contains some branching coefficients for  $\mathfrak{h}^\omega \hookrightarrow \mathfrak{g}$  and the tensor product coefficients of the subalgebra. This relation explicitly assumes the existence of the bijection  $\Psi : a \in P_{\mathfrak{h}^\omega}^+ \leftrightarrow \alpha \in \mathcal{B}^\omega$  conjectured above. Using a result of [22] and a useful identity for branching coefficients (which is related to [64], see however [41] for a recent proof using affine Kac-Moody algebra techniques) we may write

$$\begin{aligned}
(n_A^\omega)_\beta^\alpha &= \sum_{B \in \text{wts}(A), w \in W_\omega} \epsilon_\omega(w) \delta_{\alpha, w(\mathcal{P}_\omega B + \beta + \rho_\omega) - \rho_\omega} \\
\sum_c b_A^c N_{cb}^a &= \sum_{B \in \text{wts}(A), w \in W_{\mathfrak{h}^\omega}} \epsilon(w) \delta_{a, w(\mathcal{P}B + b + \rho) - \rho} \quad .
\end{aligned}
\tag{40}$$

The abbreviation  $B \in \text{wts}(A)$  means that  $B$  runs over all weights in the weight system of  $A$ . Both expressions are obviously equal to each other if the bijection  $\Psi$  is structure preserving, i.e.

$$\begin{aligned}
\Psi(\rho) &= \rho_\omega \\
\Psi \circ \mathcal{P} &= \mathcal{P}_\omega \\
\Psi(wa) &= \Psi(w)\Psi(a) \quad .
\end{aligned}$$

The last condition already constrains the possible subalgebras to a large extent. Indeed, the Weyl group  $W_\omega$  can be described as the Weyl group of the so-called orbit Lie algebra of  $\mathfrak{g}$  with respect to the automorphism  $\omega$  (see [27]). In some special cases this orbit Lie algebra coincides with the invariant subalgebra  $\mathfrak{g}^\omega$  while

it does not for the whole  $A_n$  and  $D_n$  series. A survey of these relations can be found in Table 2 which in part has been taken from [11]. Note however that the Weyl groups of  $B_n$  and  $C_n$  are isomorphic to each other (see e.g. [45, p. 74]) which can most easily be seen by treating them as abstract Coxeter groups. Thus by imposing the last constraint we only have to decide which of possibly two subalgebras - the orbit Lie algebra or the invariant subalgebra - and which specific kind of embedding we should take. This choice is uniquely determined by the other two conditions. In the cases of  $A_{2n-1}$  and  $D_n$  the orbit Lie algebra not even is a subalgebra so that this possibility is ruled out immediately.

We will show in the most simple example of the Lie algebra  $A_2$  how this procedure works and then state only results for all the other cases. Let us consider  $\mathfrak{g} = A_2$  with outer automorphism  $\omega(a_1, a_2) = (a_2, a_1)$ . The relevant subalgebra is given by  $\mathfrak{h}^\omega = A_1$  and the projection to fractional symmetric weights - which describe the boundary conditions of the theory - reads  $\mathcal{P}_\omega(a_1, a_2) = \frac{1}{2}(a_1 + a_2, a_1 + a_2)$ . There are two inequivalent embeddings  $A_1 \hookrightarrow A_2$  given by projections  $\mathcal{P}_{x_e}(a_1, a_2) = \sqrt{x_e}(a_1 + a_2)$  with embedding index  $x_e = 1$  and  $x_e = 4$  respectively [50, p. 534]. Imposing the first condition we see that

$$\Psi \circ \mathcal{P}_{x_e}(a_1, a_2) = \sqrt{x_e} \Psi(a_1 + a_2) . \quad (41)$$

This only equals  $\mathcal{P}_\omega(a_1, a_2)$  for

$$\Psi(a) = \frac{1}{2\sqrt{x_e}}(a, a) . \quad (42)$$

The condition  $\Psi(\rho) = \rho_\omega = \frac{1}{2}(1, 1)$  forces us to use the projection with  $x_e = 1$ . One can also check explicitly that the Weyl groups correspond to each other. This is the first example where the relevant subalgebra is not given by the invariant subalgebra (which has embedding index  $x_e = 4$ ) but by the orbit Lie algebra. The same statement holds for the whole  $A_{2n}$  series as we will see. As mentioned already the results of this section are summarized in Table 2.

One can treat the whole ADE series using a case by case study. Let us emphasize that we use the labeling conventions for weights which can be found in [50, p. 540]. The projections have been found using [65, p. 57-61] and the programs LiE [66] and SimpLie [67]. Note that LiE uses a different labeling convention for the weights. For a useful table of branching rules see also [68].

1. The case of  $A_{2n-1}$  is straightforward. The relevant subalgebra is given by the invariant subalgebra  $C_n \hookrightarrow A_{2n-1}$ . This is a maximal embedding and the

identification reads

$$\mathcal{P}_\omega = \frac{1}{2} \left( \begin{array}{c} 1 \quad \quad 1 \\ \diagdown \quad \diagup \\ \quad 1 \quad 2 \quad 1 \\ \diagup \quad \diagdown \\ 1 \quad \quad 1 \end{array} \right) = \frac{1}{2} \left( \begin{array}{c} 1 \quad \quad 1 \\ \diagdown \quad \diagup \\ \quad 1 \quad 2 \\ \diagup \quad \diagdown \\ 1 \quad \quad 1 \end{array} \right) \left( \begin{array}{c} 1 \quad \quad 1 \\ \diagdown \quad \diagup \\ \quad 1 \end{array} \right) = \Psi \circ \mathcal{P} .$$

2. The case  $A_{2n}$  is exceptional. Here the relevant subalgebra is given by the orbit Lie algebra which can be described by the sequence of maximal embeddings  $C_n \hookrightarrow A_{2n-1} \hookrightarrow A_{2n}$  (for  $n = 1$  we have  $A_1 \hookrightarrow A_2$ ). The identification reads

$$\mathcal{P}_\omega = \frac{1}{2} \left( \begin{array}{c} 1 \quad \quad 1 \\ \diagdown \quad \diagup \\ \quad 1 \quad 1 \\ \diagup \quad \diagdown \\ 1 \quad \quad 1 \end{array} \right) = \frac{1}{2} \left( \begin{array}{c} 1 \quad \quad 1 \\ \diagdown \quad \diagup \\ \quad 1 \\ \diagup \quad \diagdown \\ 1 \quad \quad 1 \end{array} \right) \left( \begin{array}{c} 1 \quad \quad 1 \\ \diagdown \quad \diagup \\ \quad 1 \quad 1 \end{array} \right) = \Psi \circ \mathcal{P} .$$

There is a second identification related to the subalgebra  $B_n \hookrightarrow A_{2n}$  which will be discussed in the next subsection and which is the relevant one for the main part of the paper.

3. The order 3 diagram automorphism of  $D_4$  leads to the sequence of maximal embeddings  $G_2 \hookrightarrow B_3 \hookrightarrow D_4$  and to the identification

$$\mathcal{P}_\omega = \frac{1}{3} \left( \begin{array}{cccc} 1 & 0 & 1 & 1 \\ 0 & 3 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{array} \right) = \frac{1}{3} \left( \begin{array}{cc} 0 & 1 \\ 3 & 0 \\ 0 & 1 \\ 0 & 1 \end{array} \right) \left( \begin{array}{cccc} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{array} \right) = \Psi \circ \mathcal{P} .$$

4. For the order 2 automorphism of the  $D$ -series one obtains the maximal embedding  $B_{n-1} \hookrightarrow D_n$  and

$$\mathcal{P}_\omega = \frac{1}{2} \left( \begin{array}{c} 2 \quad \quad \\ \diagdown \quad \diagup \\ \quad 2 \quad 1 \quad 1 \\ \diagup \quad \diagdown \\ \quad 1 \quad 1 \end{array} \right) = \frac{1}{2} \left( \begin{array}{c} 2 \quad \quad \\ \diagdown \quad \diagup \\ \quad 2 \quad 1 \\ \diagup \quad \diagdown \\ \quad 1 \end{array} \right) \left( \begin{array}{c} 1 \quad \quad \\ \diagdown \quad \diagup \\ \quad 1 \quad 1 \end{array} \right) = \Psi \circ \mathcal{P} .$$

5. Also the last case  $E_6$  behaves regular and yields the maximal embedding  $F_4 \hookrightarrow E_6$  with

$$\mathcal{P}_\omega = \frac{1}{2} \left( \begin{array}{cccccc} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{array} \right) = \frac{1}{2} \left( \begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & 0 & 0 \end{array} \right) \left( \begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{array} \right) = \Psi \circ \mathcal{P} .$$

These considerations show that in all cases but  $A_{2n}$  we may identify the boundary labels with representations of the invariant subgroup  $G^\omega$ . Let us emphasize that it is possible to identify the set of Lie algebra representations  $P_{\mathfrak{g}^\omega}^+$  with the set of group representations  $P_{G^\omega}^+$  in these cases as the corresponding groups  $G^\omega$  are all simply-connected. A detailed discussion of the identification one should use in the exceptional case of  $A_{2n}$  is postponed to the next subsection.

## A.2 The special case $A_{2n}$

In the last section it was shown that under certain natural and well-motivated assumptions the relevant subalgebra  $\mathfrak{h}^\omega$  in the case of  $A_{2n}$  which describes the boundary labels is uniquely given by the orbit Lie algebra  $C_n$  and not by the invariant subalgebra  $B_n$ . This contradicts, however, our geometrical intuition as we are expecting the invariant subalgebra (or even better the invariant subgroup) to be the relevant structure. In [41], however, it was argued that one is lead to the invariant subalgebra by taking a different inductive limit, i.e. an identification which involves the level  $k$  explicitly. Indeed, in writing (40) we already took a very special limit implicitly. That taking the limit  $k \rightarrow \infty$  may have nontrivial effects can already be seen from simple current symmetries in fusion rules. In this limit all simple current symmetries get lost and it becomes important on which “branch” of the simple current orbits one sits while taking the limit. There is also another point on which we have not been careful enough in the last subsection. The geometric picture suggests that we should work with representations of the invariant subgroup, not necessarily with those of the invariant subalgebra. These two sets may differ as can easily be seen from the familiar example of  $SO(3)$  which only allows  $SU(2)$  representations of integer spin. Note that the automorphism  $\omega$  in the case of  $SU(3)$  is just given by charge conjugation and that  $SO(3)$  exactly is the invariant subgroup. Similar remarks hold for the whole series  $SO(2N+1) \subset SU(2N+1)$ , i.e. the whole  $A_{2n}$  series. All this should be reflected in the new identification in a certain way.

Let us review the new identification of [41] and see whether it fits our requirements. The construction only works for even values of the level  $k$ . This fact may be reminiscent of the  $D$ -series modular invariants of  $SU(2)$  describing a  $SO(3)$  WZW model. Therefore we will assume  $k$  to be even in what follows. This restriction will not be relevant in the limit  $k \rightarrow \infty$ . The labels for the twisted boundary conditions in the WZW model based on  $A_{2n}$  are given by half-integer symmetric weights  $\alpha$  of  $A_{2n}$ . To be more specific, the Dynkin labels have to satisfy the relations  $2\alpha_i \in \mathbb{N}_0$ ,  $\alpha_i = \alpha_{2n+1-i}$  and  $\sum_{i=0}^n \alpha_i \leq k/4$ . These labels may be interpreted as labels of the invariant subalgebra  $B_n$  of  $A_{2n}$ . The map from weights of  $B_n$  to the boundary labels is given by [41]

$$\Psi(a_1, \dots, a_n) = \frac{1}{4} \left( 2a_{n-1}, \dots, 2a_1, k - 2 \sum_{i=1}^{n-1} a_i - a_n, \dots, 2a_{n-1} \right) . \quad (43)$$

Note that this map involves  $k$  explicitly and is only well-defined for weights whose last Dynkin label  $a_n$  is even. This last condition has two interpretations. From the

group theoretical point of view it restricts to representations of the Lie algebra  $B_n$  which may be integrated to single-valued representations of the group  $SO(2n+1)$ . From the Lie algebra point of view it corresponds to the branching selection rule of the embedding  $B_n \hookrightarrow A_{2n}$ . The relevant projection for this embedding reads  $\mathcal{P}(i_1, \dots, i_{2n}) = (i_1 + i_{2n}, i_2 + i_{2n-1}, \dots, 2(i_n + i_{n+1}))$ .

Actually, there is a geometric reason why we should use a  $k$ -dependent identification map in the case of  $A_{2n}$ . In the limit  $k \rightarrow \infty$ , the twisted conjugacy classes in the vicinity of the group unit have boundary labels close to  $(0, \dots, 0, k/4, k/4, 0, \dots, 0)$ . This can be inferred from an analysis of brane geometry seen by closed strings [12].

Let us summarize these results. Using the new identification map (43) we can identify the labels of twisted branes in  $SU(2n+1)$  sitting close to the group unit with representations of the invariant subgroup  $G^\omega \hookrightarrow G$ . In particular this identification knows about the fact that certain representations of  $\mathfrak{g}^\omega$  may not be lifted to representations of  $G^\omega$  and drops them from the set of boundary labels.

## B Proof of bundle triviality

In this appendix we will prove that all complexified vector bundles  $G \times_{G^\omega} V_R^\mathbb{C}$  over the base manifold  $G/G^\omega$  associated to representations  $V_R$  of  $G^\omega$  are trivial. Here,  $G$  is any simple simply-connected compact Lie group and  $G^\omega$  the subgroup invariant under a diagram automorphism. All possible cases are summarized in table 1 on page 10.

Before we start with the actual proof, let us note that representations  $V_R$  which arise by restricting representations of  $G$  to  $G^\omega$  always lead to trivial bundles. We will use this extensively to proof the triviality of all other bundles.

The proof can be divided into five parts. We will first present these five main statements and then enter the detailed discussion of the single steps.

**Statement 1:** Consider the K-ring  $K_\mathbb{C}(G/G^\omega)$  of complex vector bundles over the base manifold  $G/G^\omega$ . The map  $\mathcal{K} : V_R \rightarrow K(G \times_{G^\omega} V_R^\mathbb{C})$  which sends a representation  $V_R$  of  $G^\omega$  to the K-class of its associated complexified vector bundle, is a *ring homomorphism* from the representation ring  $\text{Rep}(G^\omega)$  to  $K_\mathbb{C}(G/G^\omega)$ .

**Statement 2:** The representation ring of  $G^\omega$  is a polynomial ring on the fundamental representations,  $\text{Rep}(G^\omega) = \mathbb{Z}[V_{\omega_1}, \dots, V_{\omega_r}]$ ,  $r = \text{rank } G^\omega$ . Therefore any K-class of a bundle  $G \times_{G^\omega} V_R^\mathbb{C}$  can be expressed as a polynomial in the K-classes of  $G \times_{G^\omega} V_{\omega_i}^\mathbb{C}$ .



**Statement 3:** All complexified vector bundles associated to fundamental representations of  $G^\omega$  have trivial K-class,  $K(G \times_{G^\omega} V_{\omega_i}^\mathbb{C}) = 0$ , i.e. all these bundles are stably equivalent to trivial bundles.

From the previous remark it follows then that all bundles  $G \times_{G^\omega} V_R^\mathbb{C}$  are stably equivalent to trivial bundles.

**Statement 4:** Two stably equivalent complex vector bundles of rank  $d$  over a base manifold of dimension  $n$  satisfying  $2d \geq n$  are isomorphic.

**Statement 5:** All representations  $V_R$  that are not a restriction of a representation of  $G$  obey the inequality

$$2 \cdot \dim V_R \geq \dim G/G^\omega \quad . \quad (44)$$

We had seen that all bundles are stably equivalent to trivial bundles, from the last two statements we can thus conclude that all bundles are trivial. This ends the main line of argumentation. Note that it was important that we considered complexified vector bundles, otherwise there would appear a much stronger inequality in statement 4 which in many cases could not be fulfilled any more.

Let us now take a closer look at the single statements. The first statement follows from the fact that the bundles associated to the tensor product of two representations  $V_R \otimes V_{R'}$  is the tensor product of the associated bundles,

$$G \times_{G^\omega} (V_R \otimes V_{R'}) \simeq (G \times_{G^\omega} V_R) \otimes_{G/G^\omega} (G \times_{G^\omega} V_{R'}) \quad .$$

Statement 2 is a structure theorem which can be found e.g. in [69, Theorem 23.24].

The third statement is much more technical. We have to check its validity case by case. We know that restrictions of representations of  $G$  give rise to trivial bundles and thus to trivial K-classes. Studying the appearance of fundamental representations in the decomposition of  $G$ -representations, we deduce in an inductive way that all fundamental representations of  $G^\omega$  correspond to bundles of trivial K-classes. Let us discuss the way it works in an example.

$B_3 \hookrightarrow D_4$ : The fundamental representations of  $B_3$  have the dimensions 7, 8 and 21. The first fundamental representation of  $D_4$  is 8-dimensional and decomposes as  $8 \rightarrow 7 + 1$ . The corresponding bundle is trivial, as well as the bundle associated to the trivial representation 1, hence the bundle associated to the 7-dimensional representation is stably equivalent to a trivial bundle. The next fundamental representation of  $D_4$  decomposes as  $28 \rightarrow 21 + 7$  so that we find by an analogous argument

as above that the bundle associated to the fundamental 21 of  $B_3$  is stably equivalent to a trivial bundle. The remaining 8-dimensional fundamental representation of  $B_3$  is the restriction of one of the 8-dimensional representations of  $D_4$  and thus gives rise to a trivial bundle.

In a similar way we will prove statement 3 for all cases at the end of this section. But before we do so, we want to discuss the last two statements. After we have shown that the fundamental and thus all representations give rise to bundles which are stably equivalent to trivial bundles, we want to show that they are actually really trivial. Statement 4 is a theorem that can be found e.g. in [70, Theorem 9.1.5] which tells us that “s-equivalent” and “isomorphic” have the same meaning if the rank of the bundles is high enough. The last statement 5 ensures that all our bundles indeed comply with this requirement, hence they are trivial. To prove this statement we determine for all cases the lowest-dimensional representation not arising as a restriction of a  $G$ -representation. We show then that these satisfy the inequality (44).

As an example take again  $B_3 \hookrightarrow D_4$ . The lowest dimensional non-trivial representation has dimension 7 which is not a restriction of a representation of  $D_4$ . The base manifold has dimension  $\dim D_4 - \dim B_3 = 28 - 21 = 7$ . As  $2 \cdot 7 = 14 \geq 7$ , the inequality holds.

We will now show the validity of this statement for all cases together with statement 3 in a case-by-case study.

- $B_n \hookrightarrow A_{2n}$  [ $SO(2n+1) \subset SU(2n+1)$ ]:  
Decomposition of fundamental representations of  $A_{2n}$  (Dynkin label notation):

$$\begin{array}{ccc} (0 \dots 1 \dots \dots \dots 0) & \longrightarrow & (0 \dots 1 \dots 0) \\ \quad \quad \quad i & & \quad \quad \quad i \quad \quad \quad i < n \\ (0 \dots \dots 1 \dots \dots 0) & \longrightarrow & (0 \dots \dots 2) \\ \quad \quad \quad n & & \end{array}$$

All fundamental weights of  $SO(2n+1)$  are restrictions of fundamental representations of  $SU(2n+1)$ .  $\Rightarrow$  **Statement 3**

The lowest dimensional representation not obtained from a restriction is the representation  $(20 \dots 0)$  of  $B_n$  with dimension  $2n^2 + 3n$ . The base manifold has dimension  $\dim SU(2n+1) - \dim SO(2n+1) = 2n^2 + 3n$ , so inequality (44) holds.  $\Rightarrow$  **Statement 5**

- $C_n \hookrightarrow A_{2n-1}$  [ $Sp(2n) \subset SU(2n)$ ]:

Decomposition of fundamental weights of  $A_{2n-1}$ :

$$\begin{aligned}
(1 \dots \dots \dots 0) &\longrightarrow (1 \dots \dots \dots 0) \\
(01 \dots \dots \dots 0) &\longrightarrow (01 \dots \dots \dots 0) \oplus (0 \dots \dots \dots 0) \\
(001 \dots \dots \dots 0) &\longrightarrow (001 \dots \dots \dots 0) \oplus (1 \dots \dots \dots 0) \\
(0 \dots \dots \underset{i}{1} \dots \dots \dots 0) &\longrightarrow (0 \dots \dots \underset{i}{1} \dots 0) \oplus (0 \dots \dots \underset{i-2}{1} \dots 0) \oplus \dots \\
&\qquad \qquad \qquad (i \leq n) \qquad \qquad \qquad \dots \oplus \begin{cases} (1 \dots \dots \dots 0) & i \text{ odd} \\ (0 \dots \dots \dots 0) & i \text{ even} \end{cases}
\end{aligned}$$

Proceeding inductively, we see that all bundles associated to fundamental representations of  $C_n$  have trivial K-class.  $\Rightarrow$  **Statement 3**

The lowest dimensional representation that cannot be obtained from a restriction is  $(010 \dots 0)$  and has dimension  $2n^2 - n - 1$ . The base manifold has dimension  $\dim SU(2n) - \dim Sp(2n) = 2n^2 - n - 1$ .  $\Rightarrow$  **Statement 5**

- $B_{n-1} \hookrightarrow D_n$  [ $Spin(2n-1) \subset Spin(2n)$ ]:

Decomposition of fundamental representations of  $D_n$ :

$$\begin{aligned}
(10 \dots \dots \dots 0) &\longrightarrow (10 \dots \dots \dots 0) \oplus (0 \dots \dots \dots 0) \\
(01 \dots \dots \dots 0) &\longrightarrow (01 \dots \dots \dots 0) \oplus (10 \dots \dots \dots 0) \\
(0 \dots \dots \underset{i}{1} \dots 0) &\longrightarrow (0 \dots \dots \underset{i}{1} \dots 0) \oplus (0 \dots \dots \underset{i-1}{1} \dots 0) \\
&\qquad \qquad \qquad i \ (i < n-1) \qquad \qquad \qquad i \qquad \qquad \qquad i-1 \\
(0 \dots \dots \dots 10) &\searrow \\
(0 \dots \dots \dots 01) &\nearrow (0 \dots \dots \dots 1)
\end{aligned}$$

$\Rightarrow$  **Statement 3**

The lowest dimensional non-trivial representation of  $B_{n-1}$  is  $(10 \dots 0)$  and has dimension  $2n - 1$ . The dimension of the base manifold is  $\dim Spin(2n) - \dim Spin(2n-1) = 2n - 1$ .  $\Rightarrow$  **Statement 5**

- $F_4 \hookrightarrow E_6$ :

Decomposition of representations of  $E_6$ :

$$\begin{aligned}
(000010) &\longrightarrow (0001) \oplus (0000) \\
(000001) &\longrightarrow (1000) \oplus (0001) \\
(000100) &\longrightarrow (0010) \oplus (1000) \oplus (0001)
\end{aligned}$$

So we see that the three fundamental representations  $(1000)$ ,  $(0010)$ ,  $(0001)$  of  $F_4$  give rise to bundles of vanishing K-class. From the tensor product

$$(0001) \otimes (1000) = (0001) \oplus (0010) \oplus (1001)$$

we can deduce that the same is valid for  $(1001)$ .

Now we look at the decomposition

$$(001000) \longrightarrow (0100) \oplus (1001) \oplus (0010) \oplus (0010) \oplus (1000)$$

and we find a vanishing K-class also for the fourth fundamental representation  $(0100)$ .  $\Rightarrow$  **Statement 3**

The lowest dimensional representation of  $F_4$  is  $(0001)$  and has dimension 26. The dimension of the base manifold is  $\dim E_6 - \dim F_4 = 26$ .  $\Rightarrow$  **Statement 5**

- $G_2 \hookrightarrow D_4$ :

Decompositions of representations of  $D_4$ :

$$\begin{aligned} (1000) &\longrightarrow (01) \oplus (00) \\ (0100) &\longrightarrow (10) \oplus (01) \oplus (01) \end{aligned}$$

$\Rightarrow$  **Statement 3**

The lowest dimensional non-trivial representation of  $G_2$  is  $(01)$  and has dimension 7. The base manifold has dimension  $\dim D_4 - \dim G_2 = 28 - 14 = 14$ .  $\Rightarrow$  **Statement 5**

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