SU(N) confining strings

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Abstract

String representations of the $SU(N)$-version of compact QED in three and four dimensions are constructed. Using for concreteness the three-dimensional case, it is demonstrated how the resulting $SU(N)$-generalization of the so-called theory of confining strings can be obtained in various ways. Such an $SU(N)$-theory of confining strings is then compared with the approach to the derivation of the string tension of the Wilson loop defined at the flat contour. This approach is based on the evaluation of the mean magnetic field inside the contour. In the $SU(N)$-version of the four-dimensional compact QED with the θ-term, the critical values of the θ-parameter, at which the problem of crumpling of large world sheets might be solved, are found. These values are only accessible provided the electric coupling constant is larger than a certain N-dependent quantity. This parallels the fact that confinement in the four-dimensional compact QED holds in the strong-coupling regime.

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1   Introduction

One of the most powerful methods, aimed at the description of confinement in QCD, is the method of Abelian projections [1] (see ref. [2] for the latest developments and ref. [3] for reviews). Among various ways to model the grand canonical ensemble of Abelian-projected monopoles, the most natural one is to treat them as the Coulomb plasma, analogous to the one of compact QED [4] (see ref. [5] for a review). The string description of the latter, the so-called theory of confining strings, has been first proposed for the 3D-case in ref. [6] and then generalized to the 4D-case in ref. [7]. A problem remained since the time of these papers is the generalization of the theory of confining strings to the case of an arbitrary number of colors. Some progress in this direction has been achieved in refs. [8, 9]. In particular, in ref. [8], it has been shown that at large-N the high-temperature behavior of the free energy of confining strings matches that of QCD.

A related problem is the investigation of the string representation of the (2+1)D Georgi-Glashow model. This model is different from compact QED due to the presence of the charged W-bosons and due to the Higgs-mediated interaction of monopoles which becomes relevant when the Higgs boson is not infinitely heavy. In this way, the influence of W-bosons to the deconfining phase transition has been analysed in the confining-string language in ref. [10]. The string representation of the BPS-limit [11] (where the mass of the Higgs boson is much smaller than the mass of the W-boson) of the $SU(3)$-version of the (2+1)D Georgi-Glashow model has been found in [12]. In ref. [13], the representation of the $SU(N)$-version of this model in terms of the monopole densities has been derived, and the potential of these densities has been discussed.

The aim of the present paper is to derive and investigate the theory of confining strings which would describe the string content of the $SU(N)$-generalization of 3D compact QED, and to generalize it further to the 4D-case with the θ-term. The paper is organized as follows. Sections 2 and 3 are devoted to the 3D-case. In Section 2, we shall find the string tension of the Wilson loop defined at a flat contour (henceforce called for shortness "flat Wilson loop"). This will be done by the method based on the solution of the system of the saddle-point equations for the dual-photon and the magnetic field. (When the magnetic field is eliminated from these equations, the resulting equation for the dual-photon field is analogous to the one derived for the usual compact QED in ref. [4].) In Section 3, we shall develop the theory of confining strings based on the Kalb-Ramond field. This will be done by two various methods, both of which illustrate how the Kalb-Ramond field unifies the monopole and the free-photonic contributions to the Wilson loop. The string tension derivable in the weak-field limit of this theory (which is nothing else, but the limit of low monopole densities) is parametrically the same as the one obtained in Section 2 for the flat loop. However, the advantage of the formalism based on the Kalb-Ramond field is that it enables one not only to fix the proportionality coefficient at the string tension, but also to derive the coupling constants of the higher-derivative string terms. Section 4 will be devoted to the generalization of the obtained $SU(N)$-theory of confining strings to the 4D-case with the (field-theoretical) θ-term. As it has been found in refs. [7, 14] for the usual compact QED, this term leads to the appearance of the string θ-term. The latter, being proportional to the number of self-intersections of the world sheet, might help in the solution of the problem of crumpling [5] [1]. Provided the electric coupling constant g obeys the inequality $g > 4\sqrt{2\pi N/(N-1)}$ (that parallels the general requirement of the strong coupling in the confining phase of 4D compact QED), this indeed takes place at two certain

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[1]This is simply due to the mutual cancellations of contributions to the string partition function which originate from any two highly crumpled surfaces, whoes numbers of self-intersections differ from each other by one.

values of θ. If $g \gg 4\sqrt{2\pi N/(N-1)}$, one of these values drops out, while the other one takes an especially simple form $\pi(N-1)/(2N)$ (and thus, in particular, it tends to $\pi/2$ at $N \to \infty$). The factor $(N-1)/(2N)$ here is the square of a weight vector of the group $SU(N)$ which describes the charges (in the units of g) of a quark with respect to the diagonal gluons.

In Summary, the main results of the paper will be discussed.

2 String tension of the flat Wilson loop

The string tension is generated by monopoles, and therefore in this Section we shall omit photons. The partition function of the grand canonical ensemble of monopoles in the Euclidean space-time reads

$$\mathscr{Z}_{\text{mon}} = \sum_{\mathcal{N}=0}^{\infty} \frac{\zeta^{\mathcal{N}}}{\mathcal{N}!} \left\langle \exp\left[-\frac{g_m^2}{2} \int d^3x d^3y \, \vec{\rho}^{\,\mathcal{N}}(\vec{x}) D_0(\vec{x}-\vec{y}) \vec{\rho}^{\,\mathcal{N}}(\vec{y}) \right] \right\rangle_{\text{mon}}, \tag{1}$$

where the monopole density is defined as $\vec{\rho}^{\,\mathcal{N}}(\vec{x}) = \sum_{k=1}^{\mathcal{N}} \vec{q}_{i_k} \delta(\vec{x} - \vec{z}_k)$ at $\mathcal{N} \geq 1$ and $\vec{\rho}^{\,\mathcal{N}=0}(\vec{x}) = 0$. In these equations, g_m is the magnetic coupling constant of dimensionality $[\text{mass}]^{-1/2}$, $g g_m = 4\pi$, $\zeta \propto \mathrm{e}^{-\text{const}/g^2}$ is the monopole fugacity of dimensionality $[\text{mass}]^3$, $D_0(\vec{x}) = 1/(4\pi|\vec{x}|)$ is the 3D Coulomb propagator, and it has been taken into account that the monopole charges are distributed along the $(N-1)$-dimensional root vectors $\vec{q}_i$'s of the group $SU(N)$ [15]. Furthermore, the average is defined as follows:

$$\langle \mathcal{O} \rangle_{\text{mon}} = \prod_{n=0}^{\mathcal{N}} \int d^3z_n \sum_{i_n = \pm 1, \ldots, \pm \frac{N(N-1)}{2}} \mathcal{O}. \tag{2}$$

Upon the explicit summation, the partition function (1) can be represented in the sine-Gordon–type form

$$\mathscr{Z}_{\text{mon}} = \int \mathcal{D}\vec{\chi} \exp\left[-\int d^3x \left(\frac{1}{2} \left(\nabla\vec{\chi}\right)^2 - 2\zeta \sum_{i=1}^{N(N-1)/2} \cos\left(g_m \vec{q}_i \vec{\chi}\right) \right) \right], \tag{3}$$

where $\vec{\chi}$ is the dual-photon field. By virtue of the formula [16] $\sum_{i=1}^{N(N-1)/2} q_i^{\alpha} q_i^{\beta} = \frac{N}{2} \delta^{\alpha\beta}$, the following value of the Debye mass of the dual photon has been obtained [13]: $m_D = g_m\sqrt{N\zeta}$.

One can further introduce the monopole field-strength tensor $\vec{F}_{\mu\nu}^{\mathcal{N}}$ violating the Bianchi identity as $\frac{1}{2}\varepsilon_{\mu\nu\lambda}\partial_\mu \vec{F}_{\nu\lambda}^{\mathcal{N}} = g_m \vec{\rho}^{\,\mathcal{N}}$. Since it was argued to omit photons throughout this Section, we obtain for $\vec{F}_{\mu\nu}^{\mathcal{N}}$ the following expression:

$$\vec{F}_{\mu\nu}^{\mathcal{N}}(\vec{x}) = -g_m \varepsilon_{\mu\nu\lambda}\partial_\lambda \int d^3y D_0\left(\vec{x}-\vec{y}\right) \vec{\rho}^{\,\mathcal{N}}\left(\vec{y}\right). \tag{4}$$

Then, by virtue of the Stokes' theorem and the formula

$$\mathrm{tr}\,\exp\left(i\vec{a}\vec{H}\right) = \sum_{\alpha=1}^{N} \exp\left(i\vec{a}\vec{\mu}_\alpha\right),$$

where $\vec{a}$ is an arbitrary $(N-1)$-component vector, $\vec{H}$ is the vector of diagonal generators and $\vec{\mu}_\alpha$'s are the weight vectors of the group $SU(N)$, $\alpha = 1, \ldots, N$, we obtain the following expression for the Wilson loop defined at the $\mathcal{N}$-monopole configuration:

$$W(C)_{\text{mon}}^{\mathcal{N}} = \frac{1}{N} \operatorname{tr} \exp\left(\frac{ig}{2} \vec{H} \int d^3x \vec{F}_{\mu\nu}^{\mathcal{N}} \Sigma_{\mu\nu}\right) = \frac{1}{N} \sum_{\alpha=1}^{N} W_\alpha^{\mathcal{N}}. \tag{5}$$

Here,

$$W_\alpha^{\mathcal{N}} \equiv \exp\left(\frac{ig}{2} \vec{\mu}_\alpha \int d^3x \vec{F}_{\mu\nu}^{\mathcal{N}} \Sigma_{\mu\nu}\right) \tag{6}$$

and

$$\Sigma_{\mu\nu}(\vec{x}) = \int_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi)) \delta(\vec{x} - \vec{x}(\xi))$$

is the vorticity tensor current defined at a certain surface $\Sigma(C)$ bounded by the contour C and parametrized by the vector $\vec{x}(\xi)$ with $\xi = (\xi^1, \xi^2)$ standing for the 2D coordinate. A straightforward calculation with the use of eq. (4) yields

$$W_\alpha^{\mathcal{N}} = \exp\left(i\vec{\mu}_\alpha \int d^3x \vec{\rho}^{\mathcal{N}} \eta\right), \tag{7}$$

where

$$\eta(\vec{x}; C) = \int_{\Sigma(C)} d\sigma_\mu(\vec{x}(\xi)) \partial_\mu \frac{1}{|\vec{x} - \vec{x}(\xi)|}$$

is the solid angle under which the surface $\Sigma(C)$ is seen by an observer located at the point $\vec{x}$, $d\sigma_\mu = \frac{1}{2}\varepsilon_{\mu\nu\lambda} d\sigma_{\nu\lambda}$. (Note that owing to the Gauss' law, $\eta \equiv 4\pi$ for a closed surface, i.e., when C is shrunk to a point.) The ratio of two $W_\alpha^{\mathcal{N}}$'s defined at different surfaces, Σ_1 and Σ_2, bounded by the contour C, reads

$$\frac{W_\alpha^{\mathcal{N}}(\Sigma_1)}{W_\alpha^{\mathcal{N}}(\Sigma_2)} = \exp\left(i\vec{\mu}_\alpha \int_{\Sigma_1 \cup \Sigma_2} d\sigma_\mu(\vec{x}(\xi)) \int d^3x \vec{\rho}^{\mathcal{N}}(\vec{x}) \partial_\mu \frac{1}{|\vec{x} - \vec{x}(\xi)|}\right) =$$

$$= \prod_{k=1}^{\mathcal{N}} \exp\left(-i\vec{\mu}_\alpha \vec{q}_{i_k} \int_{\Sigma_1 \cup \Sigma_2} d\sigma_\mu(\vec{x}(\xi)) \partial_\mu^{\vec{x}(\xi)} \frac{1}{|\vec{z}_k - \vec{x}(\xi)|}\right).$$

Due to the Gauss' law, the last integral in this equation is equal either to -4π or to 0, depending on whether the point $\vec{z}_k$ lies inside or outside the volume bounded by the surface $\Sigma_1 \cup \Sigma_2$. Since the product $\vec{\mu}_\alpha \vec{q}_{i_k}$ is equal either to $\pm\frac{1}{2}$ or to 0, we conclude that $\frac{W_\alpha^{\mathcal{N}}(\Sigma_1)}{W_\alpha^{\mathcal{N}}(\Sigma_2)} = 1$. This fact proves the independence of $W_\alpha^{\mathcal{N}}$ of the choice of the surface Σ in the definition (6). Clearly, this is the consequence of the quantization condition $gg_m = 4\pi$ employed in the derivation of eq. (7).

A representation of the partition function (1), alternative to eq. (3) and more appropriate for the investigation of the Wilson loop, is the one in terms of the dynamical monopole densities [13]. It can be obtained by multiplying eq. (1) by the following unity:

$$\int \mathcal{D}\vec{\rho}\,\delta\left(\vec{\rho} - \vec{\rho}^{\,\mathcal{N}}\right) = \int \mathcal{D}\vec{\rho}\,\mathcal{D}\vec{\chi}\,\exp\left[ig_m \int d^3x\,\vec{\chi}\left(\vec{\rho} - \vec{\rho}^{\,\mathcal{N}}\right)\right],$$

so that the field $\vec{\chi}$ plays the role of the Lagrange multiplier. We obtain for the partition function:

$$\mathcal{Z}_{\text{mon}} = \int \mathcal{D}\vec{\rho}\,\mathcal{D}\vec{\chi}\,\exp\left[-\frac{g_m^2}{2}\int d^3x\,d^3y\,\vec{\rho}(\vec{x})D_0(\vec{x}-\vec{y})\vec{\rho}(\vec{y}) + ig_m\int d^3x\,\vec{\chi}\vec{\rho}\right]\times$$

$$\times\sum_{\mathcal{N}=0}^{\infty}\frac{\zeta^{\mathcal{N}}}{\mathcal{N}!}\left\langle\exp\left(-ig_m\int d^3x\,\vec{\chi}\vec{\rho}^{\,\mathcal{N}}\right)\right\rangle_{\text{mon}}, \tag{8}$$

where the last sum is equal to

$$\exp\left[2\zeta\int d^3x\sum_{i=1}^{N(N-1)/2}\cos\left(g_m\vec{q}_i\vec{\chi}\right)\right]. \tag{9}$$

Accordingly, eq. (7) becomes replaced by

$$W_\alpha^{\mathcal{N}} \to W_\alpha = \frac{1}{\mathcal{Z}_{\text{mon}}}\int \mathcal{D}\vec{\rho}\,\mathcal{D}\vec{\chi}\,\exp\left\{-\frac{g_m^2}{2}\int d^3x\,d^3y\,\vec{\rho}(\vec{x})D_0(\vec{x}-\vec{y})\vec{\rho}(\vec{y})+\right.$$

$$\left.+\int d^3x\left[ig_m\vec{\chi}\vec{\rho} + 2\zeta\sum_{i=1}^{N(N-1)/2}\cos\left(g_m\vec{q}_i\vec{\chi}\right) + i\vec{\mu}_\alpha\vec{\rho}\eta\right]\right\},$$

and the full expression for the monopole contribution to the Wilson loop [instead of eq. (5)] reads $W(C)_{\text{mon}} = \frac{1}{N}\sum_{\alpha=1}^{N}W_\alpha$.

Let us next introduce the magnetic field according to the formulae $\partial_\mu\vec{B}_\mu = \vec{\rho}$, $\varepsilon_{\mu\nu\lambda}\partial_\nu\vec{B}_\lambda = 0$. This yields

$$W_\alpha = \frac{1}{\mathcal{Z}_{\text{mon}}}\int \mathcal{D}\vec{B}_\mu\,\delta\left(\varepsilon_{\mu\nu\lambda}\partial_\nu\vec{B}_\lambda\right)\times$$

$$\times\int \mathcal{D}\vec{\chi}\,\exp\left\{\int d^3x\left[-\frac{g_m^2}{2}\vec{B}_\mu^2 + ig_m\vec{\chi}\partial_\mu\vec{B}_\mu + 2\zeta\sum_{i=1}^{N(N-1)/2}\cos\left(g_m\vec{q}_i\vec{\chi}\right)\right] + 4\pi i\vec{\mu}_\alpha\int_{\Sigma(C)}d\sigma_\mu\vec{B}_\mu\right\}, \tag{10}$$

where, as it was discussed, the surface $\Sigma(C)$ is arbitrary. The magnetic and the dual-photon fields can be integrated out of eq. (10) by solving the respective saddle-point equations. From now on in this Section, we shall consider the contour C located in the (x,y)-plane. This naturally leads to the following Ansätze for the fields to be inserted into the saddle-point equations: $\vec{B}_\mu = \delta_{\mu3}\vec{B}(z)$, $\vec{\chi} = \vec{\chi}(z)$. For the points (x,y) lying inside the contour C, these equations then read:

$$ig_m\vec{\chi}' + g_m^2\vec{B} - 4\pi i\vec{\mu}_\alpha\delta(z) = 0, \tag{11}$$

$$i\vec{B}' - 2\zeta\sum_{i=1}^{N(N-1)/2}\vec{q}_i\sin\left(g_m\vec{q}_i\vec{\chi}\right) = 0, \tag{12}$$

where $' \equiv d/dz$. [Note that differentiating eq. (11) and substituting the result into eq. (12), we obtain the equation

$$\vec{\chi}'' - 2g_m\zeta \sum_{i=1}^{N(N-1)/2} \vec{q}_i \sin\left(g_m\vec{q}_i\vec{\chi}\right) = g\vec{\mu}_\alpha\delta'(z),$$

which is the $SU(N)$-generalization of the respective saddle-point equation obtained in ref. [4].]
These equations can be solved by adapting one extra Ansatz for the saddle-points of the fields, $\vec{B}(z) = \vec{\mu}_\alpha B(z)$, $\vec{\chi}(z) = \vec{\mu}_\alpha\chi(z)$, that due to the formula $\vec{\mu}_\alpha^2 = (N-1)/(2N)$ makes W_α α-independent. Let us next multiply eq. (12) by $\vec{\mu}_\alpha$, taking into account that for any α, $(N-1)$ positive roots yield the scalar product with $\vec{\mu}_\alpha$ equal to $1/2$, while the others are orthogonal to $\vec{\mu}_\alpha$. Equations (11), (12) then go over into

$$2i\psi' + g_m^2 B = 4\pi i\delta(z), \quad B' + 2i\zeta N\sin\psi = 0, \tag{13}$$

where $\psi \equiv g_m\chi/2$. One can straightforwardly check that the solution to this system of equations has the following form:

$$B(z) = i\frac{8m_D}{g_m^2}\frac{\mathrm{e}^{-m_D|z|}}{1+\mathrm{e}^{-2m_D|z|}}, \quad \psi(z) = 4\operatorname{sgn} z \cdot \arctan\left(\mathrm{e}^{-m_D|z|}\right). \tag{14}$$

Taking the value $B(0)$ for the evaluation of the Wilson loop, we obtain for the string tension:

$$\sigma = 4\pi \cdot \frac{N-1}{2N}\frac{4m_D}{g_m^2} = \frac{8\pi}{g_m}\frac{N-1}{\sqrt{N}}\sqrt{\zeta}. \tag{15}$$

This result demonstrates explicitly the nonanalytic behavior of σ with g, but does not yield the correct overall numerical factor. That is because $B(z)$, necessary for this calculation, is defined ambiguously. The ambiguity originates from the exponentially large thickness of the string: $|z| < \mathcal{O}\left(m_D^{-1}\right)$. It is therefore unclear what value of $B(z)$ we should take: either $B(0)$, as it was done, or the one averaged over some range of z. It turns out that in the weak-field (low-density) limit, the problem can be solved for an arbitrarily-shaped surface, by using the representation in terms of the Kalb-Ramond field, which will be described in the next Section.

3 SU(N) confining strings

One of the ways to introduce the Kalb-Ramond field is to do it first for the description of the dual photon, by considering this field as the field-strength tensor corresponding to the field $\vec{B}_\mu$, namely $\vec{B}_\mu = \frac{1}{2g_m}\varepsilon_{\mu\nu\lambda}\vec{h}_{\nu\lambda}$. The Wilson loop (10) then takes the form [2]

$$W_\alpha = \frac{1}{\mathcal{Z}_{\mathrm{mon}}}\int \mathcal{D}\vec{h}_{\mu\nu}\delta\left(\partial_\mu\vec{h}_{\mu\nu}\right)\int \mathcal{D}\vec{\chi}\exp\left\{\int d^3x\left[-\frac{1}{4}\vec{h}_{\mu\nu}^2+\right.\right.$$

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[2]Note the following correspondence between the Coulomb interaction of the monopole densities and the actions of the magnetic- and Kalb-Ramond fields:

$$\frac{1}{4}\int d^3x\vec{h}_{\mu\nu}^2 = \frac{g_m^2}{2}\int d^3x\vec{B}_\mu^2 = \frac{g_m^2}{2}\int d^3x d^3y\vec{\rho}(\vec{x})D_0(\vec{x}-\vec{y})\vec{\rho}(\vec{y}).$$

$$+\frac{i}{2}\vec{\chi}\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda} + 2\zeta\sum_{i=1}^{N(N-1)/2}\cos\left(g_m\vec{q}_i\vec{\chi}\right)\Bigg] + \frac{ig}{2}\vec{\mu}_\alpha\int\limits_{\Sigma(C)}d\sigma_{\mu\nu}\vec{h}_{\mu\nu}\Bigg\}. \tag{16}$$

The field $\vec{\chi}$ can be further integrated out by solving the saddle-point equation of the form (12), where $\vec{B}'$ is replaced by $\frac{1}{2g_m}\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda}$ (and $\vec{\chi}$ depends on all three coordinates). Using the Ansatz $\vec{h}_{\mu\nu} = \vec{\mu}_\alpha h_{\mu\nu}$, we arrive at the following substitution in eq. (16):

$$\int d^3x\left[\frac{i}{2}\vec{\chi}\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda} + 2\zeta\sum_{i=1}^{N(N-1)/2}\cos\left(g_m\vec{q}_i\vec{\chi}\right)\right] \Longrightarrow$$

$$\Longrightarrow -V\left[\vec{h}_{\mu\nu}\right] \equiv 2(N-1)\zeta\int d^3x\left[H_\alpha\mathrm{arcsinh}\,H_\alpha - \sqrt{1+H_\alpha^2}\right]. \tag{17}$$

Here, $H_\alpha \equiv \frac{1}{2g_m\zeta(N-1)}\vec{\mu}_\alpha\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda}$, and $V\left[\vec{h}_{\mu\nu}\right]$ is the multivalued potential of the Kalb-Ramond field (or of monopole densities). Similarly to the case of compact QED [6], it is the summation over branches of this potential which yields the summation over the world sheets $\Sigma(C)$ in eq. (16).

Another way to discuss the connection of the Kalb-Ramond field with the string world sheets is based on the semi-classical analysis of the saddle-point equations analogous to eqs. (13):

$$2i\partial_\mu\psi + \frac{g_m}{2}\varepsilon_{\mu\nu\lambda}h_{\nu\lambda} = 4\pi i\Sigma_\mu, \tag{18}$$

$$\frac{1}{2g_m}\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda} + 2i\zeta N\sin\psi = 0. \tag{19}$$

Here, $\Sigma_\mu \equiv \frac{1}{2}\varepsilon_{\mu\nu\lambda}\Sigma_{\nu\lambda}$, and the field ψ, by means of which the correspondence between the Kalb-Ramond field and the stationary surface is established, is defined by the relation $\vec{\chi} = \frac{2}{g_m}\vec{\mu}_\alpha\psi$. In this approach, based on the (auxiliary) field ψ, the summation over branches of the potential $V\left[\vec{h}_{\mu\nu}\right]$ is equivalent to the following procedure. One should restrict oneself to the domain $|\psi| \leq \pi$ and solve eqs. (18), (19) with the conditions

$$\lim_{|\vec{x}|\to\infty}\psi(\vec{x}) = 0, \tag{20}$$

$$\lim_{\varepsilon\to0}\left\{\psi\left[\vec{x}(\xi) + \varepsilon\vec{n}(\xi)\right] - \psi\left[\vec{x}(\xi) - \varepsilon\vec{n}(\xi)\right]\right\} = 2\pi, \tag{21}$$

where $\vec{n}(\xi)$ is the normal vector to Σ at an arbitrary point $\vec{x}(\xi)$. After that, one should get rid of the so-appearing Σ-dependence of W_α by extremizing the latter with respect to $\vec{x}(\xi)$. Owing to the formula

$$\delta\int d\sigma_{\mu\nu}h_{\mu\nu}\left[\vec{x}(\xi)\right] = \int d\sigma_{\mu\nu}H_{\mu\nu\lambda}\left[\vec{x}(\xi)\right]\delta x_\lambda(\xi),$$

such an extremization is equivalent to the definition of the extremal surface through the equation $H_{\mu\nu\lambda}\left[\vec{x}(\xi)\right] = 0$. By virtue of eqs. (19), (21), $H_{\mu\nu\lambda}$ indeed vanishes on both sides of the stationary surface. The reason for that is the following distinguished property of the extremal surface: ψ is equal to π on one of its sides and to $-\pi$ on the other, whereas for any other surface, $|\psi|$ is smaller than π on one side and larger than π on the other side. Therefore, according to eq. (19), at both sides of the extremal surface, $\varepsilon_{\mu\nu\lambda}\partial_\mu h_{\nu\lambda} = 0$. This is equivalent to the equation $\varepsilon_{\mu\nu\lambda}H_{\mu\nu\lambda} = 0$

and hence (upon the multiplication of both sides by $\varepsilon_{\mu'\nu'\lambda'}$) to $H_{\mu\nu\lambda} = 0$. [In particular, for a flat contour the extremal surface is the flat surface inside this contour. This can explicitly be seen from the solutions (14) of eqs. (13). Namely, at any z such that $|z| \ll m_D^{-1}$, $\psi(z) \simeq \pi \cdot \mathrm{sgn} z$, $B'(z) \simeq 0$, where both equalities hold with the exponential accuracy. The equation $B'(z) \simeq 0$ is equivalent to the condition of vanishing of $H_{\mu\nu\lambda}$ at both sides of the flat surface.]

According to the Hodge decomposition theorem, the general form of $\vec{h}_{\mu\nu}$ is $\vec{h}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu + \varepsilon_{\mu\nu\lambda} \partial_\lambda \vec{\phi}$. The constraint $\partial_\mu \vec{h}_{\mu\nu} = 0$, imposed in eq. (16), is equivalent to setting $\vec{A}_\mu$ equal zero. From now on, we shall promote $\vec{h}_{\mu\nu}$ to include also the fields of free photons, $\vec{A}_\mu$, by abolishing this constraint. Let us further go into the weak-field limit, $|H_\alpha| \ll 1$. Using the Cauchy inequality, we have

$$|H_\alpha| \leq \frac{|\vec{\mu}_\alpha| \, |\vec{\rho}|}{\zeta(N-1)} = \frac{|\vec{\rho}|}{\zeta\sqrt{2N(N-1)}},$$

so that the weak-field limit is equivalent to the following low-density approximation: $|\vec{\rho}| \ll \zeta\sqrt{2N(N-1)}$. To understand in what sense this inequality implies the low-density approximation, note that the mean monopole density stemming from eq. (3) reads:

$$|\vec{\rho}|_{\mathrm{mean}} = \frac{\partial \ln \mathcal{Z}_{\mathrm{mon}}}{\mathcal{V} \partial \ln \zeta} \simeq \zeta N(N-1),$$

where $\mathcal{V}$ is the three-volume occupied by the system. Therefore, at large-N, the low-density approximation implies that $|\vec{\rho}|$ should be of the order N times smaller than its mean value.

In the weak-field limit, we now have the following expression for the total Wilson loop, where the constraint $\partial_\mu \vec{h}_{\mu\nu} = 0$ is removed:

$$W(C,\Sigma)_{\mathrm{weak-field}}^{\mathrm{tot}} = \frac{1}{\mathcal{Z}^{\mathrm{tot}}} \int \mathcal{D}\vec{h}_{\mu\nu} \exp\left\{ -\int d^3x \left[\frac{1}{12m_D^2} \vec{H}_{\mu\nu\lambda}^2 + \frac{1}{4}\vec{h}_{\mu\nu}^2 - \frac{ig}{2}\vec{\mu}_\alpha \vec{h}_{\mu\nu}\Sigma_{\mu\nu} \right] \right\}. \quad (22)$$

Here, $\mathcal{Z}^{\mathrm{tot}}$ is given by the same integral over $\vec{h}_{\mu\nu}$-field, but with $\Sigma_{\mu\nu}$ set to zero, and $\vec{H}_{\mu\nu\lambda} = \partial_\mu \vec{h}_{\nu\lambda} + \partial_\lambda \vec{h}_{\mu\nu} + \partial_\nu \vec{h}_{\lambda\mu}$ is the Kalb-Ramond field-strength tensor. (The apparent Σ-dependence of the r.h.s. of eq. (22) is due to the fact that in course of the $\vec{h}_{\mu\nu}$-expansion of $V\left[\vec{h}_{\mu\nu}\right]$, only one branch of this potential has been taken into account. As it was discussed above, the Σ-dependence disappears upon the summation over all the branches.) Owing to the Hodge decomposition theorem for $\vec{h}_{\mu\nu}$, eq. (22) is obviously factorized as $W(C,\Sigma)_{\mathrm{weak-field}}^{\mathrm{tot}} = W_\alpha W_\alpha^{\mathrm{phot}}$, where the free photonic contribution to the Wilson loop reads

$$W_\alpha^{\mathrm{phot}} = \exp\left[-g^2 \frac{N-1}{4N} \oint_C dx_\mu \oint_C dx'_\mu D_0\left(\vec{x} - \vec{x}'\right) \right]. \quad (23)$$

Doing the integration over $\vec{h}_{\mu\nu}$ in eq. (22), we obtain

$$W(C,\Sigma)_{\mathrm{weak-field}}^{\mathrm{tot}} = \exp\left\{ -g^2 \frac{N-1}{4N} \left[\oint_C dx_\mu \oint_C dx'_\mu D_{m_D}(\vec{x} - \vec{x}') + \right.\right.$$

$$\left.\left. + \frac{m_D^2}{2} \int_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi)) \int_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi')) D_{m_D}\left(\vec{x}(\xi) - \vec{x}(\xi')\right) \right] \right\}, \quad (24)$$

8

where $D_{m_D}(\vec{x}) = \mathrm{e}^{-m_D|\vec{x}|}/(4\pi|\vec{x}|)$ is the 3D Yukawa propagator. Note that the free photonic contribution has completely canceled out of this expression, i.e., it is only the dual photon (of the mass m_D) which mediates the $C \times C$- and $\Sigma \times \Sigma$-interactions.

One can further expand the nonlocal string effective action

$$S_{\mathrm{str}} = (g m_D)^2 \frac{N-1}{8N} \int\limits_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi)) \int\limits_{\Sigma(C)} d\sigma_{\mu\nu}(\vec{x}(\xi')) D_{m_D}(\vec{x}(\xi) - \vec{x}(\xi')) \qquad (25)$$

in powers of the derivatives with respect to the world-sheet coordinates ξ^a's. Note that the actual parameter of this expansion is $1/(m_D R)^2$, where $R \sim \sqrt{\mathrm{Area}(\Sigma)}$ is the size of Σ (see below). The resulting quasi-local action reads (cf. refs. [17, 7]):

$$S_{\mathrm{str}} = \sigma \int d^2\xi \sqrt{\mathbf{g}} + \alpha^{-1} \int d^2\xi \sqrt{\mathbf{g}} \mathbf{g}^{ab}(\partial_a t_{\mu\nu})(\partial_b t_{\mu\nu}) + \kappa \int d^2\xi \sqrt{\mathbf{g}} \mathcal{R} + \mathcal{O}\left(\frac{\sigma}{m_D^4 R^2}\right). \qquad (26)$$

Here, $\partial_a \equiv \partial/\partial\xi^a$, and the following quantities characterize Σ: $\mathbf{g}_{ab}(\xi) = (\partial_a x_\mu(\xi))(\partial_b x_\mu(\xi))$ is the induced-metric tensor, $\mathbf{g} = \det \|\mathbf{g}^{ab}\|$, $t_{\mu\nu}(\xi) = \varepsilon^{ab}(\partial_a x_\mu(\xi))(\partial_b x_\nu(\xi))/\sqrt{\mathbf{g}}$ is the extrinsic-curvature tensor, $\mathcal{R} = \left(\partial^a \partial_a \ln \sqrt{\mathbf{g}}\right)/\sqrt{\mathbf{g}}$ is the (conformal-gauge expression for the) scalar (or intrinsic) curvature [3]. The string coupling constants here read $\sigma = 2\pi^2 \frac{N-1}{\sqrt{N}} \frac{\sqrt{\zeta}}{g_m}$, $\alpha^{-1} = -3\kappa/2 = -\frac{\pi^2(N-1)}{4g_m^3 N^{3/2}\sqrt{\zeta}}$. In the weak-field (or low-density) approximation, we can therefore fix the proportionality factor $2\pi^2$ at σ, that is close to the factor 8π of eq. (15) which, however, could not been fixed within the method of Section 2. Another important fact is that (in the weak-field approximation) the string tension and higher string coupling constants are the same for all large enough surfaces $\Sigma(C)$. [The words "large enough" here mean the validity of the inequality $m_D R \gg 1$, i.e., the string length R should significantly exceed the exponentially large string thickness m_D^{-1}. Indeed, the general n-th term of the derivative (or curvature) expansion of S_{str} is of the order of $\sigma m_D^2 R^4 (m_D R)^{-2n}$.] It is, however, worth noting that some of the terms in the expansion (26) vanish at the surface of the minimal area corresponding to a given contour C (e.g., at the flat surface, in the case considered in Section 2). This concerns, for instance, the rigidity term [4].

Let us finally demonstrate how to introduce the Kalb-Ramond field in the way alternative to its definition via the $\vec{B}_\mu$-field and incorporating automatically the free-photonic contribution to the Wilson loop. It is based on the direct application of eqs. (1) and (7), that yields

$$W_\alpha = \sum_{\mathcal{N}=0}^{\infty} \frac{\zeta^{\mathcal{N}}}{\mathcal{N}!} \left\langle \exp\left[-\frac{g_m^2}{2} \int d^3x d^3y \vec{\rho}^{\mathcal{N}}(\vec{x}) D_0(\vec{x} - \vec{y}) \vec{\rho}^{\mathcal{N}}(\vec{y}) + i\vec{\mu}_\alpha \int d^3x \vec{\rho}^{\mathcal{N}} \eta\right] \right\rangle_{\mathrm{mon}} =$$

$$= \int \mathcal{D}\vec{\varphi} \exp\left\{-\int d^3x \left[\frac{1}{2}\left(\partial_\mu \vec{\varphi} - \frac{\vec{\mu}_\alpha}{g_m} \partial_\mu \eta\right)^2 - 2\zeta \sum_{i=1}^{N(N-1)/2} \cos(g_m \vec{q}_i \vec{\varphi})\right]\right\}, \qquad (27)$$

---

[3] The second term on the r.h.s. of eq. (26) describes the so-called rigidity of the string (see e.g. ref. [5]), while the third term is known to be a full derivative and therefore it does not actually contribute to the string effective action.

[4] Indeed, one can prove the equality $\mathbf{g}^{ab}(\partial_a t_{\mu\nu})(\partial_b t_{\mu\nu}) = (D^a D_a x_\mu)(D^b D_b x_\mu)$, where $D_a D_b x_\mu = \partial_a \partial_b x_\mu - \Gamma_{ab}^c \partial_c x_\mu$, and Γ_{ab}^c is the Christoffel symbol corresponding to the metric $\mathbf{g}^{ab}$. On the other hand, the surface of the minimal area is defined by the equation $D^a D_a x_\mu(\xi) = 0$ together with the respective boundary condition at the contour C.

where $\vec{\varphi} = \vec{\chi} + \frac{\vec{\mu}_\alpha}{g_m}\eta$. One can further use the following formula which shows explicitly how the Kalb-Ramond field unifies the monopole and the free-photonic contributions to the Wilson loop:

$$\exp\left[-\frac{1}{2}\int d^3x \left(\partial_\mu\vec{\varphi} - \frac{\vec{\mu}_\alpha}{g_m}\partial_\mu\eta\right)^2 - g^2\frac{N-1}{4N}\oint_C dx_\mu \oint_C dx'_\mu D_0(\vec{x}-\vec{x}')\right] =$$

$$= \int \mathcal{D}\vec{h}_{\mu\nu}\exp\left[-\int d^3x \left(\frac{1}{4}\vec{h}_{\mu\nu}^2 + \frac{i}{2}\vec{\varphi}\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda} - \frac{ig}{2}\vec{\mu}_\alpha\vec{h}_{\mu\nu}\Sigma_{\mu\nu}\right)\right]. \tag{28}$$

In fact, one can prove that both sides of this formula are equal to [5] $\exp\left[-\frac{1}{2}\int d^3x \left(\partial_\mu\vec{\varphi} + g\vec{\mu}_\alpha\Sigma_\mu\right)^2\right]$. Inserting further eq. (28) into eq. (27), we obtain

$$W_\alpha = \frac{1}{\mathcal{Z}^{\text{tot}}}\int \mathcal{D}\vec{h}_{\mu\nu}\mathcal{D}\vec{\varphi}\exp\left[-\int d^3x \left(\frac{1}{4}\vec{h}_{\mu\nu}^2 + \right.\right.$$

$$\left.\left. +\frac{i}{2}\vec{\varphi}\varepsilon_{\mu\nu\lambda}\partial_\mu\vec{h}_{\nu\lambda} - 2\zeta\sum_{i=1}^{N(N-1)/2}\cos\left(g_m\vec{q}_i\vec{\varphi}\right) - \frac{ig}{2}\vec{\mu}_\alpha\vec{h}_{\mu\nu}\Sigma_{\mu\nu}\right)\right].$$

Denoting $\vec{\chi} \equiv -\vec{\varphi}$, we arrive back to Eq. (16) with the constraint $\partial_\mu\vec{h}_{\mu\nu} = 0$ abolished, as it should be.

4   Generalization to the 4D-case with the θ-term

In this Section, we shall consider the 4D-case and introduce the field-theoretical θ-term. As it has been first found for compact QED in refs. [7, 14], by means of the derivative expansion of the resulting nonlocal string effective action, this term generates the string θ-term. Being proportional to the number of self-intersections of the world sheet, the latter might be important for the solution of the problem of crumpling of large world sheets [5]. In this Section, we shall perform the respective analysis for the general $SU(N)$-case under study. It is worth noting from the beginning that in the lattice 4D compact QED, confinement holds only at strong coupling. On the opposite, the continuum counterpart [7, 14] (of the $SU(N)$-version) of this model, we are going to explore, possesses confinement at arbitrary values of coupling. However, we shall see that the solution to the problem of crumpling due to the θ-term can only be possible in the strong-coupling regime, implied in a certain sense. The continuum sine-Gordon theory of the dual-photon field possesses an UV cutoff which appears in course of the path-integral average over the shapes of monopole loops. However, the θ-parameter is dimensionless, and consequently its values, at which one may expect the disappearance of crumpling, will be cutoff-independent.

The full partition function, including the θ-term and the average over free photons, reads [cf. eq. (1)]

$$\mathcal{Z} = \int \mathcal{D}\vec{A}_\mu e^{-\frac{1}{4}\int d^4x \vec{F}_{\mu\nu}^2}\sum_{\mathcal{N}=0}^{\infty}\frac{\zeta^{\mathcal{N}}}{\mathcal{N}!}\left\langle\exp\left\{-\mathcal{S}\left[\vec{j}_\mu^{\mathcal{N}}, \vec{A}_\mu\right]\right\}\right\rangle_{\text{mon}}, \tag{29}$$

where

---

[5]Obviously, in the noncompact case, when monopoles are disregarded and $\vec{h}_{\mu\nu} = \partial_\mu\vec{A}_\nu - \partial_\nu\vec{A}_\mu$, the r.h.s. of eq. (28) yields the free-photonic contribution to the Wilson loop, eq. (23).

$$S\left[\vec{j}_{\mu}^{\mathcal{N}}, \vec{A}_{\mu}\right] = \frac{1}{2} \int d^4x d^4y \vec{j}_{\mu}^{\mathcal{N}}(x) D_0(x-y) \vec{j}_{\mu}^{\mathcal{N}}(y) + \frac{i\theta g^2}{8\pi^2} \int d^4x \vec{A}_{\mu} \vec{j}_{\mu}^{\mathcal{N}}, \tag{30}$$

and $\vec{F}_{\mu\nu} = \partial_\mu \vec{A}_\nu - \partial_\nu \vec{A}_\mu$. For $\mathcal{N} = 0$, the monopole current $j_\mu^{\mathcal{N}}$ is equal to zero, whereas for $\mathcal{N} \geq 1$, it is defined as $\vec{j}_\mu^{\mathcal{N}} = g_m \sum_{k=1}^{\mathcal{N}} \vec{q}_{i_k} \oint dz_\mu^k(\tau) \delta\left(x - x^k(\tau)\right)$. The couplings g and g_m are now dimensionless and obey the same condition $gg_m = 4\pi$ as in the 3D-case. We have also parametrized the trajectory of the k-th monopole by the vector $x_\mu^k(\tau) = y_\mu^k + z_\mu^k(\tau)$, where $y_\mu^k = \int_0^1 d\tau x_\mu^k(\tau)$ is the position of the trajectory, whereas the vector $z_\mu^k(\tau)$ describes its shape, both of which should be averaged over. The fugacity of a single-monopole loop, ζ, entering eq. (29), has the dimensionality $[\text{mass}]^4$, $\zeta \propto e^{-S_{\text{mon}}}$, where the action of a single k-th loop, obeying the estimate $S_{\text{mon}} \propto \frac{1}{g^2} \int_0^1 d\tau \sqrt{(\dot{z}^k)^2}$, is assumed to be of the same order of magnitude for all loops. Finally, in eq. (29), $D_0(x) = 1/(4\pi^2 x^2)$ is the 4D Coulomb propagator, and the average over monopole loops is defined similarly to eq. (2) as follows:

$$\langle \mathcal{O} \rangle_{\text{mon}} = \prod_{n=0}^{\mathcal{N}} \int d^4 y_n \sum_{i_n = \pm 1, \ldots, \pm \frac{N(N-1)}{2}} \langle \mathcal{O} \rangle_{z_n(\tau)}.$$

The particular form of the path-integral average over the shapes of the loops, $z_n(\tau)$'s, here is immaterial for the final (UV-cutoff dependent) expression for the partition function (see e.g. ref. [18] for a similar situation in the plasma of closed dual strings in the Abelian-Higgs–type models). The only thing which matters is the normalization $\langle 1 \rangle_{z_n(\tau)} = 1$, that will be implied henceforth. The analogue of the partition function (8)-(9) then reads

$$\mathcal{Z}_{\text{mon}}\left[\vec{A}_\mu\right] = \int \mathcal{D}\vec{j}_\mu \exp\left\{-\mathcal{S}\left[\vec{j}_\mu, \vec{A}_\mu\right]\right\} \int \mathcal{D}\vec{\chi}_\mu \exp\left\{\int d^4x \left[2\zeta \sum_{i=1}^{N(N-1)/2} \cos\left(\frac{\vec{q}_i |\vec{\chi}_\mu|}{\Lambda}\right) + i\vec{\chi}_\mu \vec{j}_\mu\right]\right\}, \tag{31}$$

with the action $\mathcal{S}$ given by eq. (30), while the full partition function is

$$\mathcal{Z} = \int \mathcal{D}\vec{A}_\mu e^{-\frac{1}{4}\int d^4x \vec{F}_{\mu\nu}^2} \mathcal{Z}_{\text{mon}}\left[\vec{A}_\mu\right].$$

In eq. (31),

$$|\vec{\chi}_\mu| \equiv \left(\sqrt{\chi_\mu^1 \chi_\mu^1}, \ldots, \sqrt{\chi_\mu^{N-1} \chi_\mu^{N-1}}\right), \tag{32}$$

and $\Lambda \sim \left|y^k\right| / \left(z^k\right)^2 \gg \left|z^k\right|^{-1}$ is the UV cutoff. Clearly, unlike the 3D-case without the θ-term, the $\vec{A}_\mu$-field is now coupled to $\vec{j}_\mu$, making $\mathcal{Z}_{\text{mon}}$ $\vec{A}_\mu$-dependent. This eventually leads to the change of the mass of the dual photon $\vec{\chi}_\mu$, as well as to the appearance of the string θ-term.

After the saddle-point integration over $\vec{\chi}_\mu$, the analogue of eqs. (16)-(17) takes the form

$$W_\alpha = \frac{1}{\mathcal{Z}_{\text{mon}}} \int \mathcal{D}\vec{h}_{\mu\nu} \exp\left\{-S\left[\vec{h}_{\mu\nu}\right] + \frac{ig}{2}\vec{\mu}_\alpha \int_{\Sigma(C)} d\sigma_{\mu\nu} \vec{h}_{\mu\nu}\right\}, \tag{33}$$

where the constraint $\partial_\mu \vec{h}_{\mu\nu} = 0$ was already abolished. Here, the Kalb-Ramond action reads

$$S\left[\vec{h}_{\mu\nu}\right] = \int d^4x \left(\frac{1}{4}\vec{h}_{\mu\nu}^2 - \frac{i\theta g^2}{32\pi^2}\vec{h}_{\mu\nu}\tilde{\vec{h}}_{\mu\nu}\right) + V\left[\vec{h}_{\mu\nu}\right]. \tag{34}$$

The potential V here is given by eq. (17) with the symbol $\int d^3x$ replaced by $\int d^4x$, and

$$H_\alpha = \frac{g\Lambda}{\zeta(N-1)}\vec{\mu}_\alpha\left|\partial_\mu\tilde{\vec{h}}_{\mu\nu}\right|.$$

In these formulae, $\tilde{\mathcal{O}}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\lambda\rho}\mathcal{O}_{\lambda\rho}$, and the absolute value is defined in the same way as in eq. (32), i.e., again with respect to the Lorentz indices only. Note that the form to which the θ-term has been transformed is quite natural, since the respective initial expression of eq. (30) can be rewritten modulo full derivatives as

$$\frac{i\theta g^2}{8\pi^2}\int d^4x \vec{A}_\mu\vec{j}_\mu^{\mathcal{N}} = -\frac{i\theta g^2}{32\pi^2}\int d^4x \left(\vec{F}_{\mu\nu} + \vec{F}_{\mu\nu}^{\mathcal{N}}\right)\left(\tilde{\vec{F}}_{\mu\nu} + \tilde{\vec{F}}_{\mu\nu}^{\mathcal{N}}\right),$$

where [cf. eq. (4)]

$$\vec{F}_{\mu\nu}^{\mathcal{N}}(x) = -\varepsilon_{\mu\nu\lambda\rho}\partial_\lambda\int d^4y D_0(x-y)\vec{j}_\rho^{\mathcal{N}}(y), \quad \partial_\mu\tilde{\vec{F}}_{\mu\nu}^{\mathcal{N}} = \vec{j}_\nu^{\mathcal{N}}.$$

Next, the mass of the Kalb-Ramond field, equal to the Debye mass of the dual photon, which follows from the action (34), reads $m_D = \frac{g\eta}{4\pi}\sqrt{\left(\frac{4\pi}{g^2}\right)^2 + \left(\frac{\theta}{2\pi}\right)^2}$, where $\eta \equiv \sqrt{N\zeta}/\Lambda$. In the extreme strong-coupling limit, $g \to \infty$, this expression demonstrates an important difference of the case $\theta = 0$ from the case $\theta \neq 0$. Namely, since $\eta(g) \propto e^{-\text{const}/g^2} \to 1$, $m_D \to 0$ at $\theta = 0$, whereas $m_D \to \infty$ at $\theta \neq 0$. In another words, in the extreme strong-coupling limit, the correlation length of the vacuum, equal to $1/m_D$, goes large (small) at $\theta = 0$ ($\theta \neq 0$).

In the weak-field limit, $|H_\alpha| \ll 1$, eq. (33) yields [cf. eq. (22)]:

$$W(C,\Sigma)_{\text{weak-field}}^{\text{tot}} = \frac{1}{\mathcal{Z}^{\text{tot}}}\int \mathcal{D}\vec{h}_{\mu\nu}\exp\left\{-\int d^4x\left[\frac{g^2}{12\eta^2}\vec{H}_{\mu\nu\lambda}^2 + \frac{1}{4}\vec{h}_{\mu\nu}^2 - \frac{i\theta g^2}{32\pi^2}\vec{h}_{\mu\nu}\tilde{\vec{h}}_{\mu\nu} - \frac{ig}{2}\vec{\mu}_\alpha\vec{h}_{\mu\nu}\Sigma_{\mu\nu}\right]\right\},$$

where again $\mathcal{Z}^{\text{tot}}$ is the same integral over $\vec{h}_{\mu\nu}$, but with $\Sigma_{\mu\nu}$ set to zero. Integrating over $\vec{h}_{\mu\nu}$, we then obtain:

$$W(C,\Sigma)_{\text{weak-field}}^{\text{tot}} = \exp\left\{-\frac{N-1}{4N}\left[g^2\oint_C dx_\mu\oint_C dx'_\mu D_{m_D}(x-x')+\right.\right.$$

$$\left.\left.+\frac{\eta^2}{2}\int d^4x d^4x' D_{m_D}(x-x')\left(\Sigma_{\mu\nu}(x)\Sigma_{\mu\nu}(x') + \frac{i\theta g^2}{8\pi^2}\Sigma_{\mu\nu}(x)\tilde{\Sigma}_{\mu\nu}(x')\right)\right]\right\}, \tag{35}$$

where $D_{m_D}(x) \equiv m_D K_1(m_D|x|)/(4\pi^2|x|)$ is the 4D Yukawa propagator with K_1 denoting the modified Bessel function. [Note that at $\theta = 0$, eq. (35) takes the form of eq. (24).] Further curvature expansion of the $\Sigma_{\mu\nu} \times \tilde{\Sigma}_{\mu\nu}$ interaction [analogous to the expansion of the $\Sigma_{\mu\nu} \times \Sigma_{\mu\nu}$ interaction in the action (25)] yields the string θ-term equal to $ic\nu$. Here, $\nu \equiv \frac{1}{2\pi}\int d^2\xi\sqrt{g}g^{ab}(\partial_a t_{\mu\nu})(\partial_b \tilde{t}_{\mu\nu})$ is the number of self-intersections of the world sheet, and the coupling constant c reads

$$c = \frac{(N-1)\theta}{8N\left[\left(\frac{4\pi}{g^2}\right)^2 + \left(\frac{\theta}{2\pi}\right)^2\right]}.$$

As it was already discussed, c is Λ-independent, since (similarly to the rigidity coupling constant α) it is dimensionless. We therefore see that at

$$\theta_\pm = \frac{\pi}{2}\left[\frac{N-1}{2N} \pm \sqrt{\left(\frac{N-1}{2N}\right)^2 - \left(\frac{16\pi}{g^2}\right)^2}\right],$$

self-intersections are weighted in the string partition function with the factor $(-1)^\nu$, that might cure the problem of crumpling. It is only possible at $g > 4\sqrt{\frac{2\pi N}{N-1}}$, that parallels the strong-coupling regime which should hold in the lattice version of the model (cf. the discussion in the beginning of this Section). In the extreme strong-coupling limit, understood in the sense $g \gg 4\sqrt{\frac{2\pi N}{N-1}}$, only one critical value, θ_+, survives, $\theta_+ \to \pi\frac{N-1}{2N}$, whereas $\theta_- \to 0$, i.e., θ_- becomes a spurious solution, since $c|_{\theta=0} = 0$.

5   Summary

The present paper has been devoted to the theories of confining strings which describe the string representations of the 3D and 4D $SU(N)$-analogues of compact QED. In the 3D-case, we have first found the string tension of the flat Wilson loop, that has been done by the evaluation of the mean magnetic field inside the contour of the loop, using the saddle-point analysis. The obtained result contains an ambiguity in the overall constant factor, that is due to the large thickness of the confining string which makes the definition of the mean magnetic field ambiguous. We have then developed the $SU(N)$-theory of confining strings which, in its weak-field limit (corresponding to the limit of low monopole densities), enables one to fix the above-mentioned factor. Moreover, this theory enables one also to find the coupling constants of the terms in the expansion of the nonlocal string effective action which are higher in the derivatives than the Nambu-Goto term. The theory of confining strings has been introduced in two various ways. One of these is to consider the Kalb-Ramond field as the field-strength tensor corresponding to the magnetic field produced by monopoles. The free photons can further be included by abolishing a certain constraint imposed on the Kalb-Ramond field. In the other approach, the Kalb-Ramond field appears directly as the field unifying the monopole and free-photonic contributions to the Wilson loop, using the definition of the Wilson loop in the monopole plasma. Note that the above-mentioned results can be readily generalized to the case of 3D Georgi-Glashow model with not-infinitely-heavy Higgs field, along with the lines of ref. [13].

We have further considered the more realistic 4D-case with the field-theoretical θ-term included. The latter generates the string θ-term which, at sufficiently strong coupling, might help in the solution of the problem of crumpling of large world sheets. The respective two values of the θ-parameter at which this happens have been found for the general $SU(N)$-case. In the extreme strong-coupling limit, one of these values becomes spurious, whereas the other one takes an extremely simple form, namely π times the square of a weight vector.

6   Acknowledgments

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