

On Noncommutative Merons and Instantons

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Abstract

The Yang-Mills (YM) and self-dual Yang-Mills (SDYM) equations on the noncommutative Euclidean four-dimensional space are considered. We introduce an ansatz for a gauge potential reducing the noncommutative SDYM equations to a difference form of the Nahm equations. By constructing solutions to the difference Nahm equations, we obtain solutions of the noncommutative SDYM equations. They are noncommutative generalizations of the known solutions to the SDYM equations such as the Minkowski solution, the one-instanton solution and others. Using the noncommutative deformation of the Corrigan-Fairlie-'t Hooft-Wilzek ansatz, we reduce the noncommutative YM equations to equations on a scalar field which have meron solutions in the commutative limit and show that they have no such solutions in the noncommutative case. To overcome this difficulty, another ansatz reducing the noncommutative YM equations to a system of difference equations on matrix-valued functions is used. For self-dual configurations this system is reduced to the difference Nahm equations.

1 Introduction

There is a continued interest in noncommutative field theories since many of them appear in a certain zero-slope limit of open strings coupled to a B -field background [1]. Noncommutativity of coordinates gives an opportunity to introduce nonlocality into field theory without losing control over its structure. Before attempting to quantize noncommutative field theories, it is desirable to characterize the moduli space of their classical configurations. The generalizations of classical solutions to the noncommutative case began in 1998 when Nekrasov and Schwarz [2] gave the first examples of noncommutative instantons. Nonperturbative soliton-like solutions also attracted much attention, since they admit a D -brane interpretation in the context of string theory. The consideration of noncommutative solitons was initiated by Gopakumar, Minwalla and Strominger [3]. Since then, a lot of papers on noncommutative instantons and solitons in various field theories has appeared (see e.g. [4] - [10]); it is impossible to mention all of them. For more references see review papers [11] - [14].

In this paper we consider the noncommutative Yang-Mills (ncYM) and noncommutative self-dual Yang-Mills (ncSDYM) equations in four dimensions and discuss the noncommutative generalizations [7, 8] of the Belavin-Polyakov-Schwarz-Tyupkin (BPST) and Corrigan-Fairlie-'t Hooft-Wilzek (CFtHW) ansätze for $U(2)$ instantons. We generalize the noncommutative BPST-like ansatz [7] to the case of $U(n)$ group and reduce the ncSDYM equations to a difference form of Nahm's equations. Our ansatz is a noncommutative generalization of the ansatz from [15] where the BPST ansatz [16] was extended to the $SU(n)$ gauge group and some solutions to the SDYM equations were constructed using the solutions of Nahm's equations [17]. Solving the difference Nahm equations permits us to obtain some explicit solutions of the ncSDYM equations. Among them there are noncommutative generalizations of Minkowski's [18], BPST instanton [16] and abelian 't Hooft [19] solutions. We show that the same ansatz reduces the ncYM equations to a system of difference equations which in the commutative limit have meron solutions.

2 Noncommutative YM and SDYM equations

YM theory on commutative space. We consider the Euclidean space \mathbb{R}^4 with the metric $\delta_{\mu\nu}$, a gauge potential $A = A_\mu(x)dx^\mu$ and the Yang-Mills field $F = dA + A \wedge A$ with the components $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, where $x = (x^\mu) \in \mathbb{R}^4$, $\partial_\mu := \partial/\partial x^\mu$ and $\mu, \nu, \dots = 1, 2, 3, 4$. Both A_μ and $F_{\mu\nu}$ take values in the Lie algebra $u(n)$. The YM equations in \mathbb{R}^4 have the form

$$D_\mu F_{\mu\nu} = 0, \quad (2.1)$$

where $D_\mu := \partial_\mu + \text{ad } A_\mu$. The SDYM equations are [16]

$$F_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F_{\rho\sigma}, \quad (2.2)$$

where $\varepsilon_{\mu\nu\rho\sigma}$ is the completely antisymmetric tensor in \mathbb{R}^4 , with $\varepsilon_{1234} = 1$. By virtue of the Bianchi identity $\varepsilon_{\mu\nu\rho\sigma}D_\nu F_{\rho\sigma} = 0$, each solution of the SDYM equations (2.2) satisfies the YM equations (2.1).

For a set of complex coordinates in \mathbb{R}^4 ,

$$y = x^1 + ix^2, \quad z = x^3 - ix^4, \quad \bar{y} = x^1 - ix^2, \quad \bar{z} = x^3 + ix^4, \quad (2.3)$$

the corresponding components of a gauge potential read

$$A_y = \frac{1}{2}(A_1 - iA_2) , \quad A_z = \frac{1}{2}(A_3 + iA_4) , \quad A_{\bar{y}} = \frac{1}{2}(A_1 + iA_2) , \quad A_{\bar{z}} = \frac{1}{2}(A_3 - iA_4) , \quad (2.4)$$

and the SDYM equations take the form

$$F_{yz} = 0 , \quad F_{\bar{y}\bar{z}} = 0 \quad \text{and} \quad F_{y\bar{y}} + F_{z\bar{z}} = 0 . \quad (2.5)$$

Noncommutative setting. In comparison with the ordinary field theory, in noncommutative field theories (see e.g. [13]) the standard (commutative) product of functions is replaced by the noncommutative star product,

$$(f \star g)(x) = f(x) \exp \left\{ \frac{i}{2} \overleftarrow{\partial}_\mu \theta^{\mu\nu} \overrightarrow{\partial}_\nu \right\} g(x) , \quad (2.6)$$

with a constant antisymmetric tensor $\theta^{\mu\nu}$. In the star-product formulation, the components of the noncommutative field strength have the form

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu \star A_\nu - A_\nu \star A_\mu , \quad (2.7)$$

and the YM and SDYM equations look formally to be unchanged. However, the nonlocality of the star product makes explicit calculations too cumbersome. Therefore, one usually uses the Moyal-Weyl map between ordinary commutative functions f, g from (2.6) and operators \hat{f}, \hat{g} acting in the two-oscillator Fock space.

To be more precise, let us first introduce a four-dimensional noncommutative Euclidean space \mathbb{R}_θ^4 as a space the coordinates x^μ of which satisfy $[x^\mu, x^\nu] = i\theta^{\mu\nu}$. By a change of coordinates we can transform the tensor $\theta^{\mu\nu}$ to the form

$$\theta^{12} = -\theta^{21} = \epsilon\theta^{34} = -\epsilon\theta^{43} = \theta > 0 , \quad (2.8)$$

where $\epsilon = 1$ for the self-dual and $\epsilon = -1$ for the anti-self-dual tensor $\theta^{\mu\nu}$. Then, in complex coordinates (2.3) the choice (2.8) leads to

$$[\hat{y}, \hat{\bar{y}}] = 2\theta , \quad [\hat{z}, \hat{\bar{z}}] = -2\epsilon\theta , \quad (2.9)$$

and other commutators are zero. The coordinate derivatives are now inner derivations of the algebra of functions, i.e.

$$\hat{\partial}_y \hat{f} = -\frac{1}{2\theta} [\hat{y}, \hat{f}] , \quad \hat{\partial}_z \hat{f} = \frac{1}{2\epsilon\theta} [\hat{z}, \hat{f}] , \quad \hat{\partial}_{\bar{y}} \hat{f} = \frac{1}{2\theta} [\hat{\bar{y}}, \hat{f}] , \quad \hat{\partial}_{\bar{z}} \hat{f} = -\frac{1}{2\epsilon\theta} [\hat{\bar{z}}, \hat{f}] , \quad (2.10)$$

where \hat{f} is any function of $\hat{y}, \hat{\bar{y}}, \hat{z}, \hat{\bar{z}}$. The obvious representation space for the Heisenberg algebra (2.9) is the two-oscillator Fock space \mathcal{H} spanned by $\{|n_1, n_2\rangle$ with $n_1, n_2 = 0, 1, 2, \dots\}$. In \mathcal{H} one can introduce an integer ordering of states [20],

$$|k\rangle = |n_1, n_2\rangle = \frac{\hat{y}^{n_1} \hat{\bar{z}}_\epsilon^{n_2} |0, 0\rangle}{\sqrt{n_1! n_2! (2\theta)^{n_1+n_2}}} \quad \text{with} \quad \hat{\bar{z}}_\epsilon := \frac{1-\epsilon}{2} \hat{z} + \frac{1+\epsilon}{2} \hat{\bar{z}} \quad (2.11)$$

and $k = n_1 + \frac{1}{2}(n_1 + n_2)(n_1 + n_2 + 1)$. The coordinates, functions of them and all fields are regarded as operators in \mathcal{H} . The Moyal-Weyl map gives the operator equivalent of star multiplication and integration, i.e.

$$\text{if } f \mapsto \hat{f}, \quad g \mapsto \hat{g} \quad \text{then} \quad f \star g \mapsto \hat{f} \hat{g} \quad \text{and} \quad \int d^4x f = (2\pi\theta)^2 \text{Tr}_{\mathcal{H}} \hat{f}, \quad (2.12)$$

where $\text{Tr}_{\mathcal{H}}$ denotes the trace over the Fock space \mathcal{H} . For explicit formulae of this correspondence see e.g. [11] - [14].

Let us introduce the following operators acting in the Fock space \mathcal{H} :

$$X_\mu := \hat{A}_\mu + i\theta_{\mu\nu} \hat{x}^\nu, \quad (2.13)$$

where \hat{A}_μ are operators in \mathcal{H} corresponding to the components A_μ of a gauge potential A in \mathbb{R}^4 , $\theta_{\mu\sigma}\theta^{\sigma\nu} = \delta_\mu^\nu$. In terms of X_μ the gauge field strength components are

$$\hat{F}_{\mu\nu} = [X_\mu, X_\nu] - i\theta_{\mu\nu}. \quad (2.14)$$

In the operator formulation the noncommutative version of the YM equations (2.1) reads

$$[X_\mu, [X_\mu, X_\nu]] = 0. \quad (2.15)$$

The ncSDYM equations have the form

$$[X_\mu, X_\nu] = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}[X_\rho, X_\sigma] + i(\theta_{\mu\nu} - \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\theta_{\rho\sigma}). \quad (2.16)$$

For $\theta^{\mu\nu}$ from (2.8) the above equations in complex coordinates are equivalent to

$$[X_y, X_z] = 0, \quad [X_{\bar{y}}, X_{\bar{z}}] = 0, \quad [X_y, X_{\bar{y}}] + [X_z, X_{\bar{z}}] + \frac{\epsilon - 1}{2\theta} = 0. \quad (2.17)$$

For convenience we shall omit the hats over the operators.

3 Noncommutative SDYM and difference Nahm equations

Noncommutative generalization of the CFtHW ansatz. In the commutative space \mathbb{R}^4 , the Corrigan-Fairlie-'t Hooft-Wilczek (CFtHW) ansatz for a gauge potential has the form

$$A_\mu = -\bar{\eta}_{\mu\nu}^a \frac{\sigma_a}{2i} \phi^{-1} \partial_\nu \phi, \quad (3.1)$$

where

$$\bar{\eta}_{bc}^a = \epsilon_{bc}^a, \quad \bar{\eta}_{\mu 4}^a = -\delta_\mu^a, \quad \bar{\eta}_{\mu\nu}^a = -\bar{\eta}_{\nu\mu}^a \quad (3.2)$$

is the anti-self-dual 't Hooft tensor [21], $\mu, \nu, \dots = 1, \dots, 4$, $a, b, c = 1, 2, 3$, and σ_a are the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.3)$$

The substitution of (3.1) into the SDYM equations (2.2) reduces them to the equation $\phi^{-1} \square \phi = 0$. Taking the solution of the Laplace equation

$$\phi_n = 1 + \sum_{k=1}^n \frac{\Lambda_k^2}{r_k^2}, \quad (3.4)$$

where $r_k^2 := (x_\mu - a_\mu^k)(x_\mu - a_\mu^k)$ and Λ_k are constants, one obtains a singular multi-instanton 't Hooft solution of the SDYM equations (2.2).

The noncommutative generalization of the CFtHW ansatz (3.1) was introduced¹ in [7] and derived via the twistor approach in [8]. It reads

$$A_\mu = \bar{\eta}_{\mu\nu}^a \frac{\sigma_a}{2i} \left(\phi^{\frac{1}{2}} \partial_\nu \phi^{-\frac{1}{2}} - \phi^{-\frac{1}{2}} \partial_\nu \phi^{\frac{1}{2}} \right) + \frac{1}{2} \left(\phi^{-\frac{1}{2}} \partial_\mu \phi^{\frac{1}{2}} + \phi^{\frac{1}{2}} \partial_\mu \phi^{-\frac{1}{2}} \right) \quad (3.5)$$

and reduces the ncSDYM equations to the equation

$$\phi^{-\frac{1}{2}} (\partial_y \partial_{\bar{y}} \phi + \partial_z \partial_{\bar{z}} \phi) \phi^{-\frac{1}{2}} = 0. \quad (3.6)$$

It is reasonable to assume that a solution ϕ_n of this equation looks as the standard 't Hooft solution (3.4) producing n -instantons, but the direct substitution of such ϕ_n into (3.5) yields a gauge field which is not self-dual on an n -dimensional subspace of the Fock space. This deficiency was analyzed in [7] for the $n=1$ case and in [8] for the case of arbitrary n . In [8] it was proposed to use a suitable Murray-von Neumann transformation after a specific projection of the gauge potential. The proper noncommutative 't Hooft multi-instanton field strength was given explicitly [8], but its gauge potential was not obtained in the explicit form.

Noncommutative generalization of the BPST ansatz. The authors of [7] discussed another possibility of getting over the difficulties connected with the naive extension of the CFtHW ansatz. Namely, they suggested using the BPST ansatz [16] for construction of noncommutative $U(2)$ self-dual field configurations. They have got them explicitly, but the reality of a self-dual gauge potential and the YM field was lost. It seems that there exists a complex gauge transformation of their solution to a real form, since the Lagrangian evaluated on the solution is real and the topological charge Q is equal to one.

Here we want to generalize the BPST-like ansatz of [7] in the way it was done in the commutative case [15]. There, $su(n)$ -valued components of a gauge potential in \mathbb{R}^4 were chosen in the form

$$A_\mu = 2\eta_{\mu\nu}^a x_\nu T_a(\tau) + 2x_\mu T_4(\tau), \quad (3.7)$$

where $\tau := x_\mu x^\mu$, $T_\mu(\tau)$ are $su(n)$ -valued functions of τ and $\eta_{\mu\nu}^a$ are the components of the self-dual 't Hooft tensor [21],

$$\eta_{bc}^a = \epsilon_{bc}^a, \quad \eta_{\mu 4}^a = \delta_\mu^a, \quad \eta_{\mu\nu}^a = -\eta_{\nu\mu}^a. \quad (3.8)$$

A direct substitution of (3.7) into the SDYM equations (2.2) reduces them to Nahm's equations

$$\dot{T}_1 = -[T_2, T_3] - [T_4, T_1], \quad \dot{T}_2 = -[T_3, T_1] - [T_4, T_2], \quad \dot{T}_3 = -[T_1, T_2] - [T_4, T_3], \quad (3.9)$$

¹The nonreal operator form of (3.1) was earlier considered in [2].

where the dot over T_μ denotes the derivative w.r.t. τ , $\dot{T}_\mu := \frac{dT_\mu}{d\tau}$. So, in [15] it was shown that to any solution of Nahm's equations (3.9) there corresponds the explicit solution (3.7) of the SDYM equations (2.2). We shall generalize the ansatz (3.7) to noncommutative $U(n)$ gauge theory in the self-dual space \mathbb{R}_θ^4 .

For the complex coordinates in the self-dual (i.e. $\epsilon = 1$) noncommutative Euclidean space \mathbb{R}_θ^4 we have

$$[y, \bar{y}] = 2\theta, \quad [z, \bar{z}] = -2\theta.$$

Let us consider the noncommutative generalization of the ansatz (3.7)

$$\begin{aligned} A_y &= -2i\bar{z}T_y(\tau) + 2i\bar{y}T_{\bar{z}}(\tau), \quad A_z = -2i\bar{y}T_{\bar{y}}(\tau) - 2i\bar{z}T_z(\tau), \\ A_{\bar{y}} &= 2izT_{\bar{y}}(\tau) - 2iyT_z(\tau), \quad A_{\bar{z}} = 2iyT_y(\tau) + 2izT_{\bar{z}}(\tau), \end{aligned} \quad (3.10)$$

where T_μ are some $u(n)$ -valued functions of the operator $\tau := yy + z\bar{z}$. Notice that for $T_\mu(\tau)$ in \mathbb{R}_θ^4 we have

$$T_\mu y = yT_\mu^-, \quad T_\mu \bar{z} = \bar{z}T_\mu^-, \quad T_\mu \bar{y} = \bar{y}T_\mu^+, \quad T_\mu z = zT_\mu^+, \quad (3.11)$$

where $T_\mu^\pm := T_\mu(\tau \pm 2\theta)$. The ansatz (3.10) gives us a $u(n)$ -valued gauge potential $A = A_\mu^a J_a$ with complex components A_μ^a , since due to (3.11) we have $A^\dagger = (A_\mu^a)^\dagger J_a^\dagger = -(A_\mu^a)^\dagger J_a \neq -A$. Here J_a are the generators of $U(n)$.

Reduction to difference Nahm equations. Let us introduce new variables

$$N_y := 2iT_y, \quad N_{\bar{y}} := 2iT_{\bar{y}}, \quad N_z := 2iT_z - \frac{1}{2\theta}, \quad N_{\bar{z}} := 2iT_{\bar{z}} - \frac{1}{2\theta}, \quad (3.12)$$

in which the operators X_μ from (2.13) for the ansatz (3.10) read

$$X_y = -\bar{z}N_y + \bar{y}N_{\bar{z}}, \quad X_z = -\bar{y}N_{\bar{y}} - \bar{z}N_z, \quad X_{\bar{y}} = zN_{\bar{y}} - yN_z, \quad X_{\bar{z}} = yN_y + zN_{\bar{z}}. \quad (3.13)$$

After some computations we obtain

$$\begin{aligned} [X_y, X_z] &= \bar{z}^2 Q_1 + \bar{y}^2 Q_2 + \bar{y}\bar{z} Q_3, \\ [X_{\bar{y}}, X_{\bar{z}}] &= y^2 Q_1 + z^2 Q_2 - yz Q_3, \\ [X_y, X_{\bar{y}}] + [X_z, X_{\bar{z}}] &= 2y\bar{z} Q_1 - 2z\bar{y} Q_2 + (\bar{y}y - z\bar{z}) Q_3, \end{aligned} \quad (3.14)$$

where

$$Q_1 := N_y^- N_z - N_z^- N_y, \quad Q_2 := N_{\bar{y}}^+ N_{\bar{z}} - N_{\bar{z}}^+ N_{\bar{y}}, \quad Q_3 := N_y^+ N_{\bar{y}} - N_{\bar{y}}^- N_y + N_z^+ N_{\bar{z}} - N_{\bar{z}}^- N_z. \quad (3.15)$$

Recall that $N_\mu^\pm := N_\mu(\tau \pm 2\theta)$. So, a direct substitution of (3.13) into the ncSDYM equations (2.17) in self-dual \mathbb{R}_θ^4 ($\epsilon = 1$) leads to 3 equations

$$Q_1 = 0, \quad Q_2 = 0, \quad Q_3 = 0, \quad (3.16)$$

or, in terms of complex $u(n)$ -valued functions N_μ ,

$$N_y^- N_z - N_z^- N_y = 0, \quad N_{\bar{y}}^+ N_{\bar{z}} - N_{\bar{z}}^+ N_{\bar{y}} = 0, \quad N_y^+ N_{\bar{y}} - N_{\bar{y}}^- N_y + N_z^+ N_{\bar{z}} - N_{\bar{z}}^- N_z = 0. \quad (3.17)$$

Equations (3.17) are similar (but not coincident²) to the discrete Nahm equations introduced in [23] and discussed in the context of hyperbolic monopoles in [24, 25].

In terms of $T_\mu(\tau)$, eqs.(3.17) read

$$\begin{aligned} \frac{1}{2\theta}\{T_{\bar{y}}(\tau+2\theta)-T_{\bar{y}}(\tau)\} &= 2i\{T_{\bar{y}}(\tau+2\theta)T_{\bar{z}}(\tau)-T_{\bar{z}}(\tau+2\theta)T_{\bar{y}}(\tau)\}, \\ \frac{1}{2\theta}\{T_y(\tau+2\theta)-T_y(\tau)\} &= 2i\{T_z(\tau)T_y(\tau+2\theta)-T_y(\tau)T_z(\tau+2\theta)\}, \\ \frac{1}{2\theta}\{T_z(\tau+2\theta)-T_z(\tau)+T_{\bar{z}}(\tau)-T_{\bar{z}}(\tau-2\theta)\} &= \\ = 2i\{T_y(\tau+2\theta)T_{\bar{y}}(\tau)-T_{\bar{y}}(\tau-2\theta)T_y(\tau)+T_z(\tau+2\theta)T_{\bar{z}}(\tau)-T_{\bar{z}}(\tau-2\theta)T_z(\tau)\}. \end{aligned} \quad (3.18)$$

We shall call equations (3.18) *the difference Nahm equations*. Thus, the ansatz (3.10) reduces the noncommutative SDYM equations in the self-dual Euclidean space \mathbb{R}_θ^4 to the difference Nahm equations (3.18). So, to each solution of eqs.(3.18) one may correspond a solution of the ncSDYM equations (2.17).

In the limit $\theta \rightarrow 0$, eqs.(3.18) take the form

$$\dot{T}_{\bar{y}} = 2i[T_{\bar{y}}, T_{\bar{z}}], \quad \dot{T}_y = 2i[T_z, T_y], \quad \dot{T}_z + \dot{T}_{\bar{z}} = 2i\{[T_z, T_{\bar{z}}] + [T_y, T_{\bar{y}}]\}. \quad (3.19)$$

If one introduces

$$T_1 = T_y + T_{\bar{y}}, \quad T_2 = i(T_y - T_{\bar{y}}), \quad T_3 = T_z + T_{\bar{z}}, \quad T_4 = i(T_{\bar{z}} - T_z),$$

then one obtains differential Nahm's equations (3.9). So, in the commutative limit $\theta \rightarrow 0$, the result of our reduction agrees with the one from [15].

Solutions of difference Nahm's equations. Consider $U(2)$ as a gauge group. Its generators have the form $J_a = \sigma_a/2i$, $J_4 = i\mathbf{1}_2/2$. Introduce the matrices

$$\begin{aligned} J_y &:= \frac{1}{2}(J_1 - iJ_2) = \frac{1}{2i} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad J_{\bar{y}} := \frac{1}{2}(J_1 + iJ_2) = \frac{1}{2i} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\ J_z &:= \frac{1}{2}(J_3 + J_4) = -\frac{1}{2i} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad J_{\bar{z}} := \frac{1}{2}(J_3 - J_4) = \frac{1}{2i} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned} \quad (3.20)$$

Calculations give

$$\begin{aligned} J_{\bar{y}}J_{\bar{z}} &= J_yJ_z = J_{\bar{z}}J_y = J_zJ_{\bar{y}} = J_zJ_{\bar{z}} = J_{\bar{z}}J_z = 0, \quad J_{\bar{z}}J_{\bar{y}} = -J_{\bar{y}}J_z = \frac{1}{2i}J_{\bar{y}}, \\ J_zJ_y &= -J_yJ_{\bar{z}} = -\frac{1}{2i}J_y, \quad J_yJ_{\bar{y}} = J_zJ_z = -\frac{1}{2i}J_z, \quad J_{\bar{y}}J_y = J_{\bar{z}}J_{\bar{z}} = \frac{1}{2i}J_{\bar{z}}. \end{aligned} \quad (3.21)$$

²There may exist different nonequivalent discretizations of differential equations. See e.g. discussion in [22] of nonequivalent discretizations of the sine-Gordon equation.

In searching for solutions to the difference Nahm equations (3.18) we consider the following ansatz for $T_\mu(\varphi)$:

$$T_y = \frac{\gamma}{\text{sh}(\varphi)} J_y, \quad T_{\bar{y}} = \frac{\gamma}{\text{sh}(\varphi)} J_{\bar{y}}, \quad T_z = f_z(\varphi) J_z, \quad T_{\bar{z}} = f_{\bar{z}}(\varphi) J_{\bar{z}}, \quad (3.22)$$

where $\varphi := \alpha(r^2 + \beta^2)$, $r^2 \equiv \tau := \bar{y}y + \bar{z}z$ and α, β, γ are some parameters. In this subsection we use the notation r^2 to have more similarity with solutions of the commutative theory in \mathbb{R}^4 . Note that in \mathbb{R}_θ^4 for any function $f(\varphi(r^2))$ we have $f(\varphi(r^2 + 2\theta)) = f(\varphi(r^2) + 2\alpha\theta)$.

Substituting (3.22) into the first two equations of (3.18) and using (3.21), we obtain

$$f_z = \frac{1}{2\theta} (\text{ch}(2\alpha\theta) - 1 + \text{sh}(2\alpha\theta) \text{cth}(\varphi)), \quad f_{\bar{z}} = \frac{1}{2\theta} (1 - \text{ch}(2\alpha\theta) + \text{sh}(2\alpha\theta) \text{cth}(\varphi)). \quad (3.23)$$

To find the value of the parameter γ , we substitute (3.22) with (3.23) into the last equation of (3.18) and obtain

$$\gamma = \frac{\text{sh}(2\alpha\theta)}{2\theta}.$$

So, the solution of the difference Nahm equations (3.18) has the form

$$\begin{aligned} T_y &= \frac{\text{sh}(2\alpha\theta)}{2\theta \text{sh}(\alpha(r^2 + \beta^2))} J_y, \quad T_{\bar{y}} = \frac{\text{sh}(2\alpha\theta)}{2\theta \text{sh}(\alpha(r^2 + \beta^2))} J_{\bar{y}}, \\ T_z &= \frac{1}{2\theta} \{ \text{ch}(2\alpha\theta) - 1 + \text{sh}(2\alpha\theta) \text{cth}(\alpha(r^2 + \beta^2)) \} J_z, \\ T_{\bar{z}} &= \frac{1}{2\theta} \{ 1 - \text{ch}(2\alpha\theta) + \text{sh}(2\alpha\theta) \text{cth}(\alpha(r^2 + \beta^2)) \} J_{\bar{z}}. \end{aligned} \quad (3.24)$$

The trigonometric solution (3.24) has two interesting limits: rational and constant. Namely, for (3.24) in the limit $\alpha \rightarrow 0$, $\beta = \text{const}$, we have

$$T_y = \frac{1}{r^2 + \beta^2} J_y, \quad T_{\bar{y}} = \frac{1}{r^2 + \beta^2} J_{\bar{y}}, \quad T_z = \frac{1}{r^2 + \beta^2} J_z, \quad T_{\bar{z}} = \frac{1}{r^2 + \beta^2} J_{\bar{z}}, \quad (3.25)$$

which solves the difference Nahm equations (3.18). In the limit $\beta \rightarrow \infty$, $\alpha = \text{const}$, from (3.24) we obtain

$$\begin{aligned} T_y &= 0, \quad T_{\bar{y}} = 0, \\ T_z &= \frac{1}{2\theta} \{ \text{ch}(2\alpha\theta) - 1 + \text{sh}(2\alpha\theta) \} J_z, \quad T_{\bar{z}} = \frac{1}{2\theta} \{ 1 - \text{ch}(2\alpha\theta) + \text{sh}(2\alpha\theta) \} J_{\bar{z}} \end{aligned} \quad (3.26)$$

as a simplest constant solution of eqs. (3.18).

In the commutative limit $\theta \rightarrow 0$, from (3.24) we obtain the following solution to the differential Nahm equations (3.19):

$$\begin{aligned} T_y &= \frac{\alpha}{\text{sh}(\alpha(r^2 + \beta^2))} J_y, \quad T_{\bar{y}} = \frac{\alpha}{\text{sh}(\alpha(r^2 + \beta^2))} J_{\bar{y}}, \\ T_z &= \alpha \text{cth}(\alpha(r^2 + \beta^2)) J_z, \quad T_{\bar{z}} = \alpha \text{cth}(\alpha(r^2 + \beta^2)) J_{\bar{z}}. \end{aligned} \quad (3.27)$$

In the limit $\alpha \rightarrow 0$, $\beta = \text{const}$, from (3.27) we obtain the solution to (3.19),

$$T_y = \frac{1}{r^2 + \beta^2} J_y, \quad T_{\bar{y}} = \frac{1}{r^2 + \beta^2} J_{\bar{y}}, \quad T_z = \frac{1}{r^2 + \beta^2} J_z, \quad T_{\bar{z}} = \frac{1}{r^2 + \beta^2} J_{\bar{z}}, \quad (3.28)$$

formally coinciding with the solution (3.25). In the limit $\beta \rightarrow \infty$, $\alpha = \text{const}$, from (3.27) we obtain the simplest constant solution

$$T_y = 0, \quad T_{\bar{y}} = 0, \quad T_z = \alpha J_z, \quad T_{\bar{z}} = \alpha J_{\bar{z}} \quad (3.29)$$

to the differential Nahm equations (3.19).

Above we have described the $u(2)$ -valued solutions of the difference Nahm equations (3.18) and their $\theta \rightarrow 0$ limit. One may try to find nontrivial abelian solutions as well. To have explicit $u(n)$ -valued solutions with $n > 2$, one may take an ansatz (see e.g. [15, 26]) reducing Nahm's equations to the finite Toda lattice equations. In the noncommutative case this will reduce eqs.(3.18) to a discrete variant of the Toda lattice equations which we will not discuss here.

Solutions of ncSDYM equations. Note that the difference Nahm equations (3.18) are integrable and one can easily obtain the Lax pair for them by substituting the ansatz (3.10) into the linear system for the ncSDYM equations written down in [8]. To obtain the explicit form of noncommutative self-dual gauge potentials in \mathbb{R}_θ^4 , one should simply substitute any solution of the difference Nahm equations (3.18) into (3.10). In particular, by substituting the solution (3.24) into (3.10), one obtains the following solution of the ncSDYM equation:

$$\begin{aligned} A_y &= -2i\bar{z} \frac{\text{sh}(2\alpha\theta)}{2\theta \text{sh}(\alpha(r^2 + \beta^2))} J_y + 2i\bar{y} \frac{1}{2\theta} \{1 - \text{ch}(2\alpha\theta) + \text{sh}(2\alpha\theta)\text{cth}(\alpha(r^2 + \beta^2))\} J_{\bar{z}}, \\ A_z &= -2i\bar{y} \frac{\text{sh}(2\alpha\theta)}{2\theta \text{sh}(\alpha(r^2 + \beta^2))} J_{\bar{y}} - 2i\bar{z} \frac{1}{2\theta} \{\text{ch}(2\alpha\theta) - 1 + \text{sh}(2\alpha\theta)\text{cth}(\alpha(r^2 + \beta^2))\} J_z, \\ A_{\bar{y}} &= 2iz \frac{\text{sh}(2\alpha\theta)}{2\theta \text{sh}(\alpha(r^2 + \beta^2))} J_{\bar{y}} - 2iy \frac{1}{2\theta} \{\text{ch}(2\alpha\theta) - 1 + \text{sh}(2\alpha\theta)\text{cth}(\alpha(r^2 + \beta^2))\} J_z, \\ A_{\bar{z}} &= 2iy \frac{\text{sh}(2\alpha\theta)}{2\theta \text{sh}(\alpha(r^2 + \beta^2))} J_y + 2iz \frac{1}{2\theta} \{1 - \text{ch}(2\alpha\theta) + \text{sh}(2\alpha\theta)\text{cth}(\alpha(r^2 + \beta^2))\} J_{\bar{z}}. \end{aligned} \quad (3.30)$$

In the commutative limit $\theta \rightarrow 0$, this solution in the coordinates $(x^\mu) \in \mathbb{R}^4$ has the form

$$\begin{aligned} A_\mu^1 &= 2\alpha\eta_{\mu\nu}^1 \frac{x_\nu}{\text{sh}(\alpha(r^2 + \beta^2))}, \quad A_\mu^2 = 2\alpha\eta_{\mu\nu}^2 \frac{x_\nu}{\text{sh}(\alpha(r^2 + \beta^2))}, \\ A_\mu^3 &= 2\alpha\eta_{\mu\nu}^3 x_\nu \text{cth}(\alpha(r^2 + \beta^2)), \quad A_\mu^4 = -i\partial_\mu(\ln \text{ch}(\alpha r^2 + \alpha\beta^2)). \end{aligned} \quad (3.31)$$

It is the Minkowski solution [18] of the SDYM equations (2.2) with the nontrivial $SU(2)$ components of a gauge potential and the pure gauge extra $U(1)$ piece A_μ^4 . These components A_μ^4 can be transformed to zero by a *complex* gauge transformation $A_\mu^4 \rightarrow A_\mu^4 + \partial_\mu \chi$ with $\chi = i \ln \text{ch}(\alpha r^2 + \alpha\beta^2)$. The existence of such a transformation supports the assumption that all the above-mentioned solutions of the ncSDYM equations can be transformed to a real form by some *complex* gauge transformations.

In the limit $\alpha \rightarrow 0$, $\beta = \text{const}$, the solution (3.24) of the difference Nahm equations takes the form (3.25) and its substitution into (3.10) gives

$$\begin{aligned} A_y &= -2i\bar{z}\frac{1}{r^2 + \beta^2}J_y + 2i\bar{y}\frac{1}{r^2 + \beta^2}J_{\bar{z}}, \quad A_z = -2i\bar{y}\frac{1}{r^2 + \beta^2}J_{\bar{y}} - 2i\bar{z}\frac{1}{r^2 + \beta^2}J_z, \\ A_{\bar{y}} &= 2iz\frac{1}{r^2 + \beta^2}J_{\bar{y}} - 2iy\frac{1}{r^2 + \beta^2}J_z, \quad A_{\bar{z}} = 2iy\frac{1}{r^2 + \beta^2}J_y + 2iz\frac{1}{r^2 + \beta^2}J_{\bar{z}}. \end{aligned} \quad (3.32)$$

It is a solution from [7]. In the commutative limit $\theta \rightarrow 0$, the solution (3.32) in the coordinates $(x^\mu) \in \mathbb{R}^4$ transforms into the BPST one-instanton solution

$$A_\mu^a = 2\eta_{\mu\nu}^a \frac{x_\nu}{r^2 + \beta^2} \quad (3.33)$$

after removing the extra $U(1)$ pure gauge piece $A_\mu^4 = -i\partial_\mu \ln(r^2 + \beta^2)$ by a gauge transformation.

Finally, in the limit $\beta \rightarrow \infty$, $\alpha = \text{const}$, from (3.24) we have obtained the solution (3.26) to the difference Nahm equations. It leads to the following components of the gauge potential:

$$A_y = 2i\alpha\bar{y}J_{\bar{z}}, \quad A_z = -2i\alpha\bar{z}J_z, \quad A_{\bar{y}} = -2i\alpha yJ_z, \quad A_{\bar{z}} = 2i\alpha zJ_{\bar{z}}. \quad (3.34)$$

This solution is a noncommutative $U(1)$ extension

$$A_\mu^1 = 0, \quad A_\mu^2 = 0, \quad A_\mu^3 = 2\alpha\eta_{\mu\nu}^3 x_\nu, \quad A_\mu^4 = -2i\alpha x_\mu \quad (3.35)$$

of the abelian 't Hooft toron solution [19]. The self-dual Yang-Mills field for the components (3.35) of a gauge potential has constant components

$$F_{y\bar{z}} = F_{\bar{y}z} = F_{yz} = F_{\bar{y}\bar{z}} = 0, \quad F_{y\bar{y}} = -2\alpha i J_3, \quad F_{z\bar{z}} = 2\alpha i J_3. \quad (3.36)$$

In the commutative limit, the components A_μ^4 corresponding to the second $U(1)$ group can be gauge transformed to zero.

4 Reductions of ncYM equations

Noncommutative CFtHW ansatz and ncYM equations. In the commutative case, the direct substitution of the CFtHW ansatz (3.1) into the YM equations (2.1) reduces them to the following equation on a scalar field ϕ :

$$\square\phi + \lambda\phi^3 = 0, \quad (4.1)$$

where λ is an arbitrary constant. For $\lambda = 0$, eq.(4.1) reduces to $\square\phi = 0$ and its solutions (3.4) describe 't Hooft n -instantons with $n = 1, 2, \dots$. If $\lambda \neq 0$, the function

$$\phi = \frac{1}{\sqrt{\lambda r^2}} \quad (4.2)$$

satisfies eq.(4.1) and provides a non-self-dual solution of the YM equations called a *meron* [27, 28]. This solution is singular at $r = 0$ and has the topological charge $Q = 1/2$. A two-meron solution can also be obtained by solving (4.1) [27, 28, 29]. These solutions of the YM equations in \mathbb{R}^4 can

be generalized to \mathbb{R}^{4n} [30]. Note that such solutions correspond to unstable $D0$ -branes in the type IIB string theory [31].

The problem of constructing noncommutative generalizations of meron solutions to the YM theory in \mathbb{R}_θ^4 was not considered yet. They should exist at least for small values of the noncommutativity parameter θ due to the standard deformation theory arguments. In attempting to construct them we first check the noncommutative generalization (3.5) of the CFtHW ansatz. After rather lengthy and cumbersome computations we obtain

$$\begin{aligned} \partial_\mu F_{\mu y} + [A_\mu, F_{\mu y}] &= \\ &= -\phi^{-\frac{1}{2}} \begin{pmatrix} \partial_y(\square\phi) - 3(\partial_y\phi)\phi^{-1}\square\phi & 0 \\ 2\partial_{\bar{z}}(\square\phi) - 3(\partial_{\bar{z}}\phi)\phi^{-1}\square\phi - 3(\square\phi)\phi^{-1}\partial_{\bar{z}}\phi & -\partial_y(\square\phi) + 3(\square\phi)\phi^{-1}\partial_y\phi \end{pmatrix} \phi^{-\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} \partial_\mu F_{\mu \bar{y}} + [A_\mu, F_{\mu \bar{y}}] &= \\ &= \phi^{-\frac{1}{2}} \begin{pmatrix} \partial_{\bar{y}}(\square\phi) - 3(\square\phi)\phi^{-1}\partial_{\bar{y}}\phi & 2\partial_z(\square\phi) - 3(\partial_z\phi)\phi^{-1}\square\phi - 3(\square\phi)\phi^{-1}\partial_z\phi \\ 0 & -\partial_{\bar{y}}(\square\phi) + 3(\partial_{\bar{y}}\phi)\phi^{-1}\square\phi \end{pmatrix} \phi^{-\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} \partial_\mu F_{\mu z} + [A_\mu, F_{\mu z}] &= \\ &= -\phi^{-\frac{1}{2}} \begin{pmatrix} \partial_z(\square\phi) - 3(\partial_z\phi)\phi^{-1}\square\phi & 0 \\ 2\partial_{\bar{y}}(\square\phi) - 3(\partial_{\bar{y}}\phi)\phi^{-1}\square\phi - 3(\square\phi)\phi^{-1}\partial_{\bar{y}}\phi & -\partial_z(\square\phi) + 3(\square\phi)\phi^{-1}\partial_z\phi \end{pmatrix} \phi^{-\frac{1}{2}}, \end{aligned}$$

$$\begin{aligned} \partial_\mu F_{\mu \bar{z}} + [A_\mu, F_{\mu \bar{z}}] &= \\ &= \phi^{-\frac{1}{2}} \begin{pmatrix} \partial_{\bar{z}}(\square\phi) - 3(\square\phi)\phi^{-1}\partial_{\bar{z}}\phi & 2\partial_y(\square\phi) - 3(\partial_y\phi)\phi^{-1}\square\phi - 3(\square\phi)\phi^{-1}\partial_y\phi \\ 0 & -\partial_{\bar{z}}(\square\phi) + 3(\partial_{\bar{z}}\phi)\phi^{-1}\square\phi \end{pmatrix} \phi^{-\frac{1}{2}}. \end{aligned}$$

It is easy to see that the ncYM equations are satisfied if the equations

$$\begin{aligned} \partial_y(\square\phi) - 3(\partial_y\phi)\phi^{-1}\square\phi &= 0, & \partial_{\bar{y}}(\square\phi) - 3(\square\phi)\phi^{-1}\partial_{\bar{y}}\phi &= 0, \\ \partial_z(\square\phi) - 3(\partial_z\phi)\phi^{-1}\square\phi &= 0, & \partial_{\bar{z}}(\square\phi) - 3(\square\phi)\phi^{-1}\partial_{\bar{z}}\phi &= 0, \end{aligned} \quad (4.3)$$

and their hermitian conjugate are satisfied. But from (4.3) it follows that

$$\begin{aligned} (\partial_y\phi)\phi^{-1}\square\phi &= (\square\phi)\phi^{-1}\partial_y\phi, & (\partial_z\phi)\phi^{-1}\square\phi &= (\square\phi)\phi^{-1}\partial_z\phi, \\ (\partial_{\bar{y}}\phi)\phi^{-1}\square\phi &= (\square\phi)\phi^{-1}\partial_{\bar{y}}\phi, & (\partial_{\bar{z}}\phi)\phi^{-1}\square\phi &= (\square\phi)\phi^{-1}\partial_{\bar{z}}\phi. \end{aligned} \quad (4.4)$$

For $\square\phi \neq 0$ these formulae are not compatible in the noncommutative case, since $\partial_\mu\phi$ does not commute with ϕ^{-1} and $\square\phi$. In the commutative case, however, eqs.(4.4) become identities and (4.3) with their complex conjugate become

$$\partial_\mu(\square\phi + \lambda\phi^3) = 0. \quad (4.5)$$

Obviously, eqs.(4.5) are satisfied if θ satisfies eq.(4.1).

Noncommutative version of the BPST ansatz and merons. We see that there are some inconsistency in the reduced ncYM equations obtained after the substitution of the ncCFtHW ansatz (3.5) into (2.15). Maybe a noncommutative extension of some other modifications of the CFtHW ansatz (e.g. [32]) will work but this is unclear at the moment. Therefore, we shall try to overcome the above difficulties by using the ncBPST ansatz (3.10) which was successful in the ncSDYM case. Using the ansatz (3.13) and formulae (3.14), (3.15), we obtain

$$\begin{aligned}
[X_\mu, [X_\mu, X_y]] &= -[X_y, [X_y, X_{\bar{y}}] + [X_z, X_{\bar{z}}]] - 2[X_{\bar{z}}, [X_y, X_z]] = \\
&= -\{2(\tau - 2\theta)N_{\bar{z}}^- Q_1^+ - 2(\tau + 4\theta)Q_1^{++}N_{\bar{z}}^+ - (\tau - 2\theta)Q_3N_y^+ + (\tau + 4\theta)N_y^+Q_3^+\} \bar{z} - \\
&\quad - \bar{y} \{2(\tau + 4\theta)N_y^{++}Q_2 - 2(\tau - 2\theta)Q_2^-N_y - (\tau + 4\theta)Q_3^+N_{\bar{z}} + (\tau - 2\theta)N_{\bar{z}}Q_3\} , \\
[X_\mu, [X_\mu, X_{\bar{y}}]] &= [X_{\bar{y}}, [X_y, X_{\bar{y}}] + [X_z, X_{\bar{z}}]] - 2[X_z, [X_{\bar{y}}, X_{\bar{z}}]] = \\
&= \{2(\tau - 2\theta)N_{\bar{y}}^- Q_1^+ - 2(\tau + 4\theta)Q_1^{++}N_{\bar{y}}^+ + (\tau - 2\theta)Q_3N_z^+ - (\tau + 4\theta)N_z^+Q_3^+\} y + \\
&\quad + z \{2(\tau + 4\theta)N_z^{++}Q_2 - 2(\tau - 2\theta)Q_2^-N_z + (\tau + 4\theta)Q_3^+N_{\bar{y}} - (\tau - 2\theta)N_{\bar{y}}Q_3\} , \\
[X_\mu, [X_\mu, X_z]] &= -[X_z, [X_y, X_{\bar{y}}] + [X_z, X_{\bar{z}}]] + 2[X_{\bar{y}}, [X_y, X_z]] = \\
&= \{2(\tau - 2\theta)N_{\bar{y}}^- Q_1^+ - 2(\tau + 4\theta)Q_1^{++}N_{\bar{y}}^+ + (\tau - 2\theta)Q_3N_z^+ - (\tau + 4\theta)N_z^+Q_3^+\} \bar{z} - \\
&\quad - \bar{y} \{2(\tau + 4\theta)N_z^{++}Q_2 - 2(\tau - 2\theta)Q_2^-N_z + (\tau + 4\theta)Q_3^+N_{\bar{y}} - (\tau - 2\theta)N_{\bar{y}}Q_3\} , \\
[X_\mu, [X_\mu, X_{\bar{z}}]] &= [X_{\bar{z}}, [X_y, X_{\bar{y}}] + [X_z, X_{\bar{z}}]] + 2[X_y, [X_{\bar{y}}, X_{\bar{z}}]] = \\
&= \{2(\tau - 2\theta)N_{\bar{z}}^- Q_1^+ - 2(\tau + 4\theta)Q_1^{++}N_{\bar{z}}^+ - (\tau - 2\theta)Q_3N_y^+ + (\tau + 4\theta)N_y^+Q_3^+\} y - \\
&\quad - z \{2(\tau + 4\theta)N_y^{++}Q_2 - 2(\tau - 2\theta)Q_2^-N_y - (\tau + 4\theta)Q_3^+N_{\bar{z}} + (\tau - 2\theta)N_{\bar{z}}Q_3\} .
\end{aligned}$$

So, the ncYM equations (2.15) reduce to the system of difference equations

$$\begin{aligned}
2(\tau + 4\theta)N_z^{++}Q_2 - 2(\tau - 2\theta)Q_2^-N_z + (\tau + 4\theta)Q_3^+N_{\bar{y}} - (\tau - 2\theta)N_{\bar{y}}Q_3 &= 0 , \\
2(\tau - 2\theta)N_{\bar{y}}^- Q_1^+ - 2(\tau + 4\theta)Q_1^{++}N_{\bar{y}}^+ + (\tau - 2\theta)Q_3N_z^+ - (\tau + 4\theta)N_z^+Q_3^+ &= 0 , \\
2(\tau + 4\theta)N_y^{++}Q_2 - 2(\tau - 2\theta)Q_2^-N_y - (\tau + 4\theta)Q_3^+N_{\bar{z}} + (\tau - 2\theta)N_{\bar{z}}Q_3 &= 0 , \\
2(\tau - 2\theta)N_{\bar{z}}^- Q_1^+ - 2(\tau + 4\theta)Q_1^{++}N_{\bar{z}}^+ - (\tau - 2\theta)Q_3N_y^+ + (\tau + 4\theta)N_y^+Q_3^+ &= 0 .
\end{aligned} \tag{4.6}$$

Recall that $\tau \equiv r^2$ and $N_\mu^\pm \equiv N_\mu(\tau \pm 2\theta)$, $N_\mu^{\pm\pm} \equiv N_\mu(\tau \pm 4\theta)$ etc.

It is obvious that for self-dual configurations (satisfying eqs.(3.16)) eqs.(4.6) are also satisfied. Moreover, it is not difficult to show that in the commutative limit $\theta \rightarrow 0$ the one-meron solution is reproduced via the ansatz (3.10) for

$$T_y = \frac{2i}{\sqrt{\lambda r^2}} J_y , \quad T_{\bar{y}} = \frac{2i}{\sqrt{\lambda r^2}} J_{\bar{y}} , \quad T_z = \frac{2i}{\sqrt{\lambda r^2}} J_z , \quad T_{\bar{z}} = \frac{2i}{\sqrt{\lambda r^2}} J_{\bar{z}} . \tag{4.7}$$

To find meron-type solutions of the ncYM equations, one should obtain nontrivial solutions of the system of difference equations (4.6). We postpone this task for a future work.

5 Concluding remarks

In this paper we have discussed the noncommutative CFtHW-like ansatz [7, 8] for a gauge potential and showed that it reduces the ncYM equations to the system (4.3) of equations on a scalar field ϕ . Generically, in the non-self-dual case this system is not compatible. To resolve this problem we generalized the BPST-like ansatz from the paper [7] and derived the system (4.6) of difference equations by substituting this ansatz into the ncYM equations. For the self-dual subcase, the above-mentioned system reduces to the difference Nahm equations. We obtained some new solutions of the ncSDYM equations via solving the difference Nahm equations.

It will also be interesting to consider solutions of the SDYM equations on a noncommutative version of pseudo-Euclidean space with a metric of signature $(2, 2)$, since these equations appear in a zero-slope limit of $N = 2$ strings with a nonzero constant B -field [33]. Moreover, nonlocal dressing symmetries of the self-dual Yang-Mills [34] raise to (dressing) symmetries of open $N = 2$ strings [35] that helps to construct solutions of Berkovits' string field theory [36] via the dressing approach [37]. Therefore, constructing solutions of the (noncommutative) SDYM theory may be helpful for finding solutions in string field theory and studying its nonperturbative properties. Recall that the (noncommutative) SDYM equations are integrable. At the same time the (noncommutative) YM equations are not integrable and there are no general methods of constructing their solutions. Therefore, some guesswork with appropriate ansätze may be useful in producing interesting solutions to the ncYM equations.

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