Some solutions of linearized 5-d gravity with brane

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Abstract

We consider linearized 5-d gravity in the Randall-Sundrum brane world. The class of static solutions for linearized Einstein equations is found. Also we obtaine wave solutions describing radiation from an imaginary point source located at the Planck distance from the brane. We analyze the fields asymptotic behavior and peculiarities of matter sources.

1 Introduction and Summary

For a long time the idea is investigated that the number of spacial dimensions exceeds three. This supposition was used in efforts to unify electromagnetic and gravitational interactions. String theory predicts an existence of more than three space dimensions. One interesting attempt [1, 2] based on more old ideas [3, 4, 5] is the proposition to use the higher-dimensional mechanism for solving the hierarchy problem. It is assumed that the visible flat $(1+3)$-d spacetime (brane) is embedded within a 5-d spacetime. Apart from other approaches an additional dimension is considered to be noncompact. Gravity on the brane resembles the usual 4-dimensional Einstein gravity for long distances due to the special choice of metric, cosmological constant and brane tension. The investigation of this scenario shows that some problems exist [6].

It is important to verify a physical noncontradiction of the RS-metric: $ds^2 = (k|z|+1)^{-2}\eta_{MK}dX^M dX^K$. In this work we consider small gravitational fields and in particular the question: is metric perturbations remain small everywhere in the 5-dimensional spacetime? We assume that the brane is spherically symmetric. By r/k and z denote coordinates on the brane and normal to the brane respectively. Using variables $R = \sqrt{r^2 + (kz+1)^2}$ and $p = r/R$, we separate linearized Einstein equations and get an extensive class of static and wave solutions. Nonstationary solutions depend on time t as $\exp(i(t-R))$ and describe waves radiating outward from an imaginary source located at the Planck distance $1/k$ from the brane. Obtained solutions may be of any degree of decreasing at infinity. A matter on the brane and stringlike formations lined orthogonally to the brane represent field sources.

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2 Linearized equations

The action describing 5-dimensional void spacetime including the 4-dimensional brane with matter is

$$S = \int d^5X \left[\sqrt{|g_{(5)}|} \left(\frac{R}{2} - \Lambda \right) + \sigma\delta(z)\sqrt{|g_{(4)}|}(1 + \mathcal{L}^{\text{matter}}_{(4)}) \right], \tag{1}$$

where $ds^2 = g_{MK}dX^M dX^K$ is a 5-dimensional interval of spacetime with a metric $g_{MK}(M, K = 0, 1, 2, 3, 5)$ and coordinates $X^M = (x^\mu, x^5 = z), (\mu, \kappa = 0, 1, 2, 3)$; $g_{(4)\mu\kappa}$ and x^μ are a metric and coordinates on the brane located at $z = 0$.

5-dimensional Einstein equations are

$$R_{MK} - \frac{1}{2}g_{MK}R = \Lambda g_{MK} + \sigma\delta(z)\left(g^{\mu\kappa}_{(4)} + T^{\text{matter }\mu\kappa}_{(4)}\right)\sqrt{\left|\frac{g_{(4)}}{g_{(5)}}\right|}g_{\mu M}g_{\kappa K}. \tag{2}$$

In the absence of gravitational waves and a matter the solution of (2) is given by the background RS-metric

$$ds^2 = H^{-2}(z)\eta_{MK}dX^M dX^K, \tag{3}$$

where η_{MK} is a Minkowski metric with the signature $(+, -, -, -, -)$ and $H(z) = k|z| + 1$. The cosmological constant Λ and the brane tension σ relate to the constant k as $\Lambda = -6k^2, \sigma = 6k$ [1].

Consider background metric perturbations parameterized by a tensor h_{MK}

$$g_{MK} = H^{-2}(z)(\eta_{MK} + h_{MK}). \tag{4}$$

In the axial gauge [1, 2, 7, 8]$(h^\kappa_\kappa = 0, h_{55} = h_{5\mu} = 0)$ linearized Einstein equations take the form

$$\left(-\Box_{(4)} + \partial_z^2 - \frac{3\,\mathrm{sgn}(z)}{k|z| + 1}\partial_z\right)h_{\mu\kappa} = 12k\delta(z)T^{\text{matter}\,(1)}_{(4)\;\mu\kappa}, \tag{5}$$

where $T^{\text{matter}\,(1)}_{(4)\;\mu\kappa}$ is the 1-st order contribution of matter on the brane to an energy-momentum tensor calculated using unperturbed metric. We suppose that there is no contribution from a matter on the brane in zero-order. This equation comes also from the other gauge choices [9]. Since the operator acting on $h_{\mu\kappa}$ does not depend on indices μ, κ, we shall drop them for simplicity and use further the definitions h and T for $h_{\mu\kappa}$ and $T^{\text{matter}\,(1)}_{(4)\;\mu\kappa}$ respectively.

3 Solution of equations

Consider first the equation (5) in the area outside the brane. For $z > 0$, we have

$$\left(-\Box_{(4)} + \partial_z^2 - \frac{3k}{kz + 1}\partial_z\right)h = 0. \tag{6}$$

Suppose the brane is spherically symmetric; then the dependence of h on x_1, x_2, x_3 reduces to the dependence on $r = k\sqrt{x_1^2 + x_2^2 + x_3^2}$ only. Let $t = kx_0, R = \sqrt{r^2 + (kz + 1)^2}, p = r/R$ be new variables; then equation (6) written for the function $f = rh$ takes the form

$$-\frac{\partial^2 f}{\partial t^2} + \frac{\partial^2 f}{\partial R^2} - \frac{2}{R}\frac{\partial f}{\partial R} + \frac{1}{R^2}\left((1 - p^2)\frac{\partial^2 f}{\partial p^2} + 2p\frac{\partial f}{\partial p}\right) = 0. \tag{7}$$

To find a solution we use the method of dividing variables. Let f takes the form

$$f = e^{i\epsilon t} R^{3/2} (1 - p^2) H(R) P(p), \tag{8}$$

where $\epsilon = 0, \pm 1$. For the function H we obtain the equation containing an arbitrary constant ν

$$R^2 H'' + RH' + \left(R^2 \epsilon^2 - \left(\nu + \frac{1}{2} \right)^2 \right) H = 0. \tag{9}$$

If $\epsilon = \pm 1$, then (9) is a Bessel equation. The function P satisfy the Legendre equation

$$(1 - p^2) P'' - 2pP' + \left(\nu(\nu + 1) - \frac{4}{1 - p^2} \right) P = 0. \tag{10}$$

A general solution $P(p, \nu)$ of (10) is a linear combination of Legendre functions $P_\nu^2(p)$ and $Q_\nu^2(p)$ of the first and second kinds respectively.

The explicit form of $P(p, \nu)$ for $\nu = 0, 1, 2$ is

$$\begin{aligned}
P(p, 0) &= c_{01} \frac{p}{1 - p^2} + c_{02} \frac{1 + p^2}{1 - p^2}, \\
P(p, 1) &= c_{11} \frac{1}{1 - p^2} + c_{12} \frac{p - \frac{p^3}{3}}{1 - p^2}, \\
P(p, 2) &= c_{21} (1 - p^2) + c_{22} \left(\frac{4p}{1 - p^2} + 6p + 3(1 - p^2) \ln \left(\frac{1 + p}{1 - p} \right) \right).
\end{aligned} \tag{11}$$

Here and further, $a_i, c_{ij},$ are arbitrary constants.

4 Static solutions

If we assume $\epsilon = 0$ in (8), then h does not depend on time and takes the form

$$h = \left(a_1 R^{\nu+1} + a_2 R^{-\nu} \right) \frac{1 - p^2}{p} P(p, \nu). \tag{12}$$

A solution have physical significance if h decreases as R increases, so $a_1 = 0$ in (12) for $\nu \geq 0$.

First consider solutions corresponding to the case $\nu = \ 1$,

$$h_1 = a_1 \frac{1}{r}, \tag{13}$$

$$h_2 = a_2 \frac{1}{R} \left(1 - \frac{1}{3} \left(\frac{r}{R} \right)^2 \right). \tag{14}$$

In order to find the energy-momentum tensor T corresponding to h, we replace z by $|z|$ in h and act on h by the operator from the left part of the equation (5).A nonzero result appears in two cases.

In the fist case solution has a singularity of $1/r$ type as $r \to 0$. This is the case of the solution h_1. The result of action on $1/r$ by operator $\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} \right)$ from $\Box_{(4)}$ is $\delta(r)$. It signifies that the source of the field is a string located at $r = 0$ along z direction normally to brane hyperplane. The action (1) does not presuppose an existence of similar sources however corresponding local matter term can be easily included. Since initially it was assumed that perturbations are small, such solutions are correct in the $r > r_0$ area only. The condition that h is small in comparison with η determines the value of r_0. The area $r < r_0$ is occupied by a source. Notice that similar sources are typical for the considered geometry, they arises for example in nonperturbative solutions of the "black cigar" type [10].

The second case corresponds to the action of $\partial^2/\partial z^2$ operator on h. Taking into account that h depends on $|z|$, we have

$$\frac{\partial^2 h}{\partial z^2} = \frac{\partial^2 h}{\partial |z|^2} + \frac{\partial h}{\partial |z|} 2\delta(z). \tag{15}$$

From (15) and (5) we find T

$$T = \frac{1}{6k}\frac{\partial h}{\partial |z|}\ (z = 0). \tag{16}$$

Since h depends on z by means of dependence on R, we obtain

$$T = \frac{1}{6R}\frac{\partial h}{\partial R}\ (R = \sqrt{r^2 + 1}). \tag{17}$$

In the case of solution h_2 determined by (14) we get

$$T_2 = -\frac{a_2}{6(r^2+1)^{\frac{5}{2}}}. \tag{18}$$

As a second example consider the case $\nu = 2$. Solutions take the forms

$$h_3 = a_3 \frac{1}{rR}\left(1 - \frac{r^2}{R^2}\right)^2, \tag{19}$$

$$h_4 = a_4 \left(\frac{10}{R^2} - \frac{6r^2}{R^4} + \frac{3}{rR}\left(1 - \frac{r^2}{R^2}\right)^2 \ln\left(\frac{R+r}{R-r}\right)\right). \tag{20}$$

Substituting h_3 in (5), we receive following expression for the right part

$$2a_3 \left(\frac{\delta(r)}{k|z|+1} + \delta(z)\frac{4r^2 - 1}{r(r^2+1)^{\frac{7}{2}}}\right). \tag{21}$$

This source consists from a matter distributed on the brane and a string. The energy-momentum tensor T_4 corresponding to the solution h_4 describes a matter located on the brane only and looks like

$$T_4 = a_4 \left(\frac{4(r^2-5)}{(r^2+1)^3} + \frac{3(4r^2-1)}{r(r^2+1)^{\frac{7}{2}}} \ln\left(\frac{\sqrt{r^2+1}+r}{\sqrt{r^2+1}-r}\right) - \frac{6}{(r^2+1)^{\frac{5}{2}}}\right). \tag{22}$$

5 Wave solutions

Now we assume $\epsilon = 1$ in (9). Then (8) depends on time and (9) has a form of the Bessel equation. We choose the Hankel function $H(R) = H^{(2)}_{\nu+\frac{1}{2}}(R)$ as a solution of (9).

Then a general solution for h describes outgoing wave. This is clearly seen in the case of integer ν when a simple explicit expression for $H^{(2)}_{\nu+\frac{1}{2}}(R)$ exists

$$H^{(2)}_{\nu+\frac{1}{2}}(R) = const\ R^{\nu+\frac{1}{2}} \left(\frac{d}{RdR}\right)^{\nu} \frac{e^{-iR}}{R}. \tag{23}$$

We present solutions and their asymptotics for some values of ν.

Case $\nu = 0$

$$h = e^{i(t-R)} \left(c_{01} + c_{02}\left(\frac{R}{r} + \frac{r}{R}\right)\right). \tag{24}$$

As $r \to \infty$

$$h \to e^{i(t-r)} \left(c_{01} + 2c_{02} + \frac{c_{02}}{4}\left(\frac{k|z|+1}{r}\right)^4\right), \tag{25}$$

as $r \to 0$

$$h \to e^{i(t-k|z|-1)} c_{02} \frac{k|z|+1}{r}, \tag{26}$$

as $|z| \to \infty$

$$h \to e^{i(t-k|z|)} \left(c_{02}\frac{k|z|}{r} + c_{01} + \frac{3c_{02}}{2}\frac{r}{k|z|} - \frac{5c_{02}}{8}\left(\frac{r}{k|z|}\right)^3\right). \tag{27}$$

In this case a solution exists containing an arbitrary function g.

$$h = g(t-R)\left(c_{01} + c_{02}\left(\frac{R}{r} + \frac{r}{R}\right)\right). \tag{28}$$

Case $\nu = 1$

$$h = e^{i(t-R)} \left(i + \frac{1}{R}\right)\left(c_{11}\frac{R}{r} + c_{12}\left(1 - \frac{1}{3}\left(\frac{r}{R}\right)^2\right)\right). \tag{29}$$

As $r \to \infty$

$$h \to e^{i(t-r)} \left(i + \frac{1}{r}\right)\left(c_{11} + \frac{2}{3}c_{12}\right), \tag{30}$$

as $r \to 0$

$$h \to e^{i(t-k|z|-1)} \left(i + \frac{1}{k|z|+1}\right)\left(c_{11}\frac{k|z|+1}{r} + c_{12}\right), \tag{31}$$

as $|z| \to \infty$

$$h \to e^{i(t-k|z|)} \left(ic_{11}\frac{k|z|}{r} + ic_{12} + c_{11}\frac{1}{r} + \frac{ic_{11}}{2}\frac{r}{k|z|} + c_{12}\frac{1}{k|z|}\right). \tag{32}$$

Case $\nu = 2$

$$h = e^{i(t-R)}\frac{R}{r}\left(-1+3i\frac{1}{R}+3\frac{1}{R^2}\right)\left(c_{21}\left(1-\left(\frac{r}{R}\right)^2\right)^2\right.$$
$$\left.+\ c_{22}\left(4\frac{r}{R}+6\frac{r}{R}\left(1-\left(\frac{r}{R}\right)^2\right)+3\left(1-\left(\frac{r}{R}\right)^2\right)^2\ln\left(\frac{R+r}{R-r}\right)\right)\right). \tag{33}$$

As $r \to \infty$

$$h \to e^{i(t-r)}4c_{22}\left(-1+3i\frac{1}{r}\right), \tag{34}$$

as $r \to 0$

$$h \to e^{i(t-k|z|-1)}c_{21}\frac{k|z|+1}{r}\left(-1+3i\frac{1}{k|z|+1}+3\frac{1}{(k|z|+1)^2}\right), \tag{35}$$

as $|z| \to \infty$

$$h \to e^{i(t-k|z|)}\left(-c_{21}\frac{k|z|}{r}-16c_{22}+3ic_{21}\frac{1}{r}+\frac{1}{k|z|}\left(\frac{3c_{21}}{2}r+48ic_{22}+3c_{21}\frac{1}{r}\right)\right). \tag{36}$$

Some solutions asymptotically diverge but linear combinations of solutions always exist having any desirable degree of convergence at infinity. For example we can linear combine six above presented solutions into two independent expressions having zero asymptotics at infinity.

$$h_5 = const\ e^{i(t-R)}\left(\frac{r^2}{R^2}-\frac{r^3}{R^3}+\frac{3i}{R}\left(1-2\frac{r}{R}-\frac{1}{3}\frac{r^2}{R^2}+\frac{r^3}{R^3}\right)\right.$$
$$\left.+\frac{3}{R^2}\frac{R}{r}\left(1-\frac{r^2}{R^2}\right)^2\right). \tag{37}$$

As $r \to \infty$

$$h_5 \to -const\ e^{i(t-r)}\frac{i}{r}, \tag{38}$$

as $r \to 0$

$$h_5 \to const\ e^{i(t-k|z|-1)}\frac{3}{(k|z|+1)r}, \tag{39}$$

as $|z| \to \infty$

$$h_5 \to const\ e^{i(t-k|z|)}\frac{3}{k|z|}\left(i+\frac{1}{r}\right). \tag{40}$$

And

$$h_6 = const\ e^{i(t-R)}\left(6-6\frac{r^2}{R^2}-\frac{6i}{R}\left(1+\frac{r^2}{R^2}\right)+\frac{6}{R^2}\left(5-3\frac{r^2}{R^2}\right)\right.$$
$$\left.+\frac{3R}{r}\left(-1+\frac{3i}{R}+\frac{3}{R^2}\right)\left(1-\frac{r^2}{R^2}\right)^2\ln\left(\frac{R+r}{R-r}\right)\right). \tag{41}$$

As $r \to \infty$

$$h_6 \to -const\ e^{i(t-r)}\frac{12i}{r}, \tag{42}$$

as $r \to 0$

$$h_6 \to const\ e^{i(t-k|z|-1)} \frac{12}{k|z|+1} \left(i + \frac{4}{k|z|+1} \right), \tag{43}$$

as $|z| \to \infty$

$$h_6 \to const\ e^{i(t-k|z|)} \left(\frac{12i}{k|z|} + \frac{3}{(kz)^2} \left(16 + r^2 \right) \right). \tag{44}$$

Obtained solutions describe waves outgoing from an imaginary point source located outside of the brane. For waves in the regions $z > 0$ and $z < 0$ the source is located at ($r = 0, z = -1/k$) and ($r = 0, z = 1/k$) respectively. Notice that in accordance with [1] $1/k$ is of the Planck length order ($\sim 10^{-33}$ sm), so for greater distances, one may suggest that the source is located on the brane.

Since our solution describes waves propagating with the speed of the light, we can consider them as gravitational waves in the 5-dimensional spacetime. However since a wave propagates with the speed of the light in the $\boldsymbol{R}$ direction, the speed of the wave in the $\boldsymbol{r}$ direction i.e. on the brane must be greater due to the pure geometrical reason. Indeed, an four-dimensional observer residing on the brane does not see any tachyons. A wave on the brane corresponds to variations of a matter distributed on the brane. In order to imitate a gravitational radiation of a point imaginary source in the 5-dimensional spacetime, a variations phase speed must exceed the speed of the light. Notice that contrary to the group speed the wave's phase speed may be of any value. At distances $>> 1/k$ the difference between the speed of the light and the speed of the wave on the brane is $\sim 1/r^2$.

Above we in detail analyzed some solutions which have simple analytical representations. The general solution (8) gives possibilities to study various gravitational fields and sources. It is demonstrated that perturbations exist which remain small everywhere in the 5-dimensional spacetime. It signifies that at this point the RS-metric is not self-contradictory. On the other hand existing of wave solutions suggests that the brane may radiate in the outer space and a violation of the brane energy-momentum conservation may take place. Static solutions may give some information about possible nonperturbative solutions.

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