Noncommutative Planar Particles: Higher Order Versus First Order Formalism and Supersymmetrization

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Abstract

We describe the supersymmetrization of two formulations of free noncommutative planar particles – in coordinate space with higher order Lagrangian [1] and in the framework of Faddeev and Jackiw [2,3], with first order action. In nonsupersymmetric case the first formulation after imposing subsidiary condition eliminating internal degrees of freedom provides the second formulation. In supersymmetric case one can also introduce the split into "external" and "internal" degrees of freedom both describing supersymmetric models.

1 Introduction

In [1] the present authors introduced the following nonrelativistic higher order action for D=2 (planar) particle:

$$L_1^{(0)} = \frac{m\dot{x}_1^2}{2} - k\epsilon_{ij}\dot{x}_i\ddot{x}_j \,. \tag{1}$$

*Talk given by J. Lukierski

The canonical quantization of (1) implies the consideration of $x_i, \dot{x}_i$ as independent degrees of freedom (see e.g. [4]) with the following canonically conjugated two momenta [1,4]

$$p_i = \frac{\partial L^{(0)}}{\partial \dot{x}_i} - \frac{d}{dt}\frac{\partial L^{(0)}}{\partial \ddot{x}_i} = m\dot{x}_i - 2k\epsilon_{ij}\ddot{x}_j\,, \tag{2}$$

$$\widehat{p}_i = k\epsilon_{ij}\dot{x}_j\,. \tag{3}$$

The relation (3) introduces a second class constraint, i.e. after the introduction of Dirac brackets the Lagrangian system (1) is described by six degrees of freedom $Y_A = (x_i, p_i, v_i = \dot{x}_i)$. One gets the following set of Dirac brackets [1]:

$$\{Y_A, Y_B\} = \begin{pmatrix} 0 & 1_2 & 0 \\ -1_2 & 0 & 0 \\ 0 & 0 & -\frac{1}{2k}\epsilon \end{pmatrix} \tag{4}$$

In order to get the first order formulation of the action (1) one can use the technique proposed by Faddeev and Jackiw [2,3]. The action (1) is equivalent to the following one[1]

$$L^{(0)} = \frac{mv_i^2}{2} - k\epsilon_{ij}v_i\dot{v}_j + p_i(\dot{x}_i - v_i)\,, \tag{5}$$

with six canonical variables (x_i, v_i, p_i). The canonical quantization of (5) using Dirac brackets leads again to the relations (4).

Next we introduce the variables [5,6]

$$\begin{array}{rcl} Q_i & = & -2k(v_i - p_i)\,, \qquad P_i = p_i \\ X_i & = & x_i + \epsilon_{ij}Q_j\,, \end{array} \tag{6}$$

one gets the following set of canonical Poisson brackets (PB)

$$\begin{array}{rcl} \{X_i, X_j\} & = & -2k\epsilon_{ij}\,, \qquad\qquad \{P_i, P_j\} = 0\,, \\ \{X_i, P_j\} & = & \delta_{ij}\,, \end{array} \tag{7}$$

and

$$\{Q_i, Q_j\} = 2k\,\epsilon_{ij}\,. \tag{8}$$

The action (5) takes the form

$$L^{(0)} = L^{(0)}_{\rm ext} + L^{(0)}_{\rm int}\,, \tag{9}$$

where

$$L^{(0)}_{\rm ext} = P_i\dot{X}_i - k\epsilon_{ij}\,P_i\dot{P}_j - \frac{1}{2}\overrightarrow{P}^2\,, \tag{10}$$

$$L^{(0)}_{\rm int} = -\frac{1}{4k}\,\epsilon_{ij}Q_i\dot{Q}_j + \frac{1}{8k^2}\overrightarrow{Q}^2\,. \tag{11}$$

[1]The equivalence of (5) and (1) can be seen in a clear way if we consider the generating functionals based on both actions (5) and (1) - the last term in (5) shall introduce the functional Dirac delta function replacing v_i by $\dot{x}_l$.

We note that the external and internal degrees of freedom are dynamically independent, and following Duval and Horvathy [7] we can consider the part (10) of the action as the first order action describing noncommutative particles. We observe that our model permits easily the consistent introduction of a scalar potential [5], electromagnetic interactions [5–7] and general Lagrangian framework [8].

We see that the action (10) describes an invariant sector of the model (1) in "external" phase space X_i, P_i. The internal degrees of freedom (11) can be related with nonvanishing anyonic spin [9]. The aim of this note is to supersymmetrize both actions (1) and (10) and discuss the relation between such supersymmetric models. We will show that the split (9) into dynamically independent supersymmetric parts with external and internal degrees of freedom can be performed again.

2 Supersymmetrization of Higher Order Action and its First Order Form

Let us consider for simplicity N=1 supersymmetric quantum mechanics. We introduce the real field $X_i(t,\theta)$ with one Grassmann variable θ

$$x_i(t) \longrightarrow X_i(t,\theta) = x_i(t) + i\theta\psi(t)\,, \tag{12}$$

where $\theta^2 = \theta\psi_j + \psi_j\theta = \psi_i\psi_j + \psi_j\psi_i = 0$. Introducing the supersymmetric covariant derivative

$$D = \frac{\partial}{\partial\theta} - i\theta\frac{\partial}{\partial t} \Rightarrow D^2 = -i\frac{\partial}{\partial t} = -H\,, \tag{13}$$

we get the following supersymmetric extension of (1)

$$\begin{aligned} L^{(0)}_{\text{SUSY}} &= i\int d\theta \Big(\frac{m}{2}\dot{X}_i D X_i - k\epsilon_{ij}\ddot{X}_i D X_j\Big) \\ &= \frac{m}{2}\Big(\dot{x}_i^2 + i\dot{\psi}_i\psi_i\Big) - k\epsilon_{ij}\Big(\dot{x}_i\ddot{x}_j - i\dot{\psi}_i\dot{\psi}_j\Big)\,. \end{aligned} \tag{14}$$

If $k=0$ we obtain the standard case of N=1 nonrelativistic spinning particle, with fermionic second class constraints. If $k \neq 0$ the fermionic momenta become independent from fermionic coordinates ψ_i.

Using the Faddeev-Jackiw method we extend supersymmetrically the action (5) as follows:

$$\begin{aligned} L^{(0)}_{\text{SUSY}} &= \frac{mv_i^2}{2} - k\epsilon_{ij}v_i\dot{v}_j + \frac{im}{2}\psi_i\rho_i \\ &\quad + i\,k\epsilon_{ij}\rho_i\rho_j + p_i(\dot{x}_i - v_i) + \chi_i(\dot{\psi}_i - \rho_i)\,. \end{aligned} \tag{15}$$

The field equation for ρ_i is purely algebraic where ρ_i and χ_i are fermionic. Substituting

$$\chi_i = \frac{im}{2}\psi_i - 2ik\,\epsilon_{ij}\rho_j\,, \tag{16}$$

we find that (we put for simplicity further $m = 1$)

$$
\begin{aligned}
L^{(0)}_{\rm SUSY} &= \frac{v_i^2}{2} - k\epsilon_{ij} v_i \dot{v}_j + \frac{i}{2}\psi_i\dot{\psi}_i \\
&\quad + 2ik\epsilon_{ij}\dot{\psi}_i\rho_j - ik\epsilon_{ij}\rho_i\rho_j + p_i(\dot{x}_i - v_i)\,. \qquad (17)
\end{aligned}
$$

For the fermionic sector of the action (17) one obtains with the use of Dirac brackets the following PB algebra

$$
\{\psi_i, \psi_j\} = 0\,, \qquad \{\psi_i, \rho_j\} = \frac{i}{2k}\,\epsilon_{ij}\,, \qquad \{\rho_i, \rho_j\} = i\frac{1}{4k^2}\,\delta_{ij} \qquad (18)
$$

In order to split (15) into external and internal sector we introduce besides the variables (6) also new fermionic variables

$$
\psi_i \longrightarrow \widetilde{\psi}_i = \psi_i - 2k\epsilon_{ij}\rho_j\,. \qquad (19)
$$

We get

$$
L^{(0)}_{\rm SUSY} = L^{(0)}_{\rm SUSY;ext} + L^{(0)}_{\rm SUSY;int}\,, \qquad (20)
$$

where

$$
L^{(0)}_{\rm SUSY;ext} = P_i\dot{X}_i - k\epsilon_{ij}P_i\dot{P}_j - \frac{1}{2}P_i^2 + \frac{i}{2}\widetilde{\psi}_i\dot{\widetilde{\psi}}_i\,, \qquad (21)
$$

$$
L^{(0)}_{\rm SUSY;int} = -\frac{1}{4k}\epsilon_{ij}Q_i\dot{Q}_j + \frac{1}{8k^2}\overrightarrow{Q}^2 - 2ik^2\rho_k\dot{\rho}_k - ik\epsilon_{ij}\rho_i\rho_k\,, \qquad (22)
$$

The new fermionic coordinates satisfy the following PB algebra

$$
\left\{\widetilde{\psi}_i, \widetilde{\psi}_j\right\} = i\delta_{ij}\,, \qquad \left\{\widetilde{\psi}_i, \rho_j\right\} = 0\,. \qquad (23)
$$

We see therefore that again the supersymmetric action (15) or (20) can be split into dynamically independent external and internal sectors (see (21)–(22)).

3 Supersymmetry in External and Internal Sectors

The actions (21) and (22) describe the supersymmetric extensions respectively of external and internal actions (10) and (11). These actions are invariant under the following set of supersymmetry transformations:

i) in external sector (see (21))

$$
\begin{aligned}
\delta\, X_i &= i\,\epsilon\,\widetilde{\psi}_i\,, \\
\delta\,\widetilde{\psi}_i &= -\,\epsilon\,P_i\,, \\
\delta P_i &= 0 \qquad (24)
\end{aligned}
$$

ii) in internal sector (see (22))

$$\begin{aligned} \delta\, Q_i &= 2i\, k\, \epsilon\, \epsilon_{ij}\, \rho_j\, , \\ \delta \rho_i &= \frac{1}{4k^2}\, \epsilon\, Q_i\, , \end{aligned} \tag{25}$$

where ϵ is a constant Grassmann number.

The supercharge corresponding to (22) is given by the formula

$$Q_{\rm ext} = i\widetilde{\psi}_i P_i \tag{26}$$

and we get consistently the external Hamiltonian (see (21))

$$-\frac{i}{2}\{Q_{\rm ext}, Q_{\rm ext}\} = \frac{1}{2}P_i^2 = H^{(0)}_{\rm SUSY;ext}\, . \tag{27}$$

Similarly from (7) and (22) we obtain the tranformation (25) if

$$Q_{\rm int} = iQ_i\, \rho_i \tag{28}$$

and our internal Hamiltonian (see (22)) is given by

$$-\frac{i}{2}\{Q_{\rm int}, Q_{\rm int}\} = -\frac{1}{8k^2}\overrightarrow{Q}^2 + ik\, \epsilon_{ij}\rho_i\rho_j = H^{(0)}_{\rm SUSY;int}\, . \tag{29}$$

We would like to add here that one could consider only the external part, described by the action (20), as describing supersymmetric planar particles. The quantum mechanical states describing the internal sector can be eliminated by subsidiary conditions.

4 Final Remarks

We would like to mention that

i) We have considered here the N=1 world line supersymmetry. It is quite straightforward to extend the above considerations to N=2 by employing the N=2 D=1 superfields.

ii) We have discussed here, for simplicity, only the free case. The supersymmetrization of the models with gauge intersections considered in [6] is under active consideration.

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