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## THE UNIQUENESS OF THE ABELIAN BORN-INFELD ACTION

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### ABSTRACT

Starting from BPS solutions to Yang-Mills which define a stable holomorphic vector bundle, we investigate its deformations. Assuming slowly varying fieldstrengths, we find in the abelian case a *unique* deformation given by the abelian Born-Infeld action. We obtain the deformed Donaldson-Uhlenbeck-Yau stability condition to all orders in  $\alpha'$ . This result provides strong evidence supporting the claim that the only supersymmetric deformation of the abelian  $d = 10$  supersymmetric Yang-Mills action is the Born-Infeld action.

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<sup>1</sup>Aspirant FWO

# 1 Introduction

An exciting consequence of the discovery of D-branes [1] was their close relation to gauge theories. The worldvolume degrees of freedom of a single  $Dp$ -brane are  $9-p$  scalar fields and a  $U(1)$  gauge field in  $p+1$  dimensions. The former describe the transversal fluctuations of the D-brane while the latter describes an open string longitudinal to the brane. For slowly varying fields, the effective action governing the low-energy dynamics of a D-brane is known through all orders in  $\alpha'$ : it is the ten-dimensional supersymmetric Born-Infeld action, dimensionally reduced to  $p+1$  dimensions [2]. Its supersymmetric extension was obtained in [3]. The knowledge of the full effective action was crucial for numerous applications.

Once several, say  $n$ , D-branes coincide, the gauge group is enhanced from  $U(1)$  to  $U(n)$ , [4]. The non-abelian extension of the Born-Infeld theory is not known yet. The most natural form for it is the symmetrized trace proposal in [5]. However, as shown in [6] and [7], this does not correctly capture all of the D-brane dynamics. Using the mass spectrum as a guideline, partial higher order results were obtained in [8]. In [9],  $\kappa$ -symmetry was shown to be a powerful but technically involved tool to fix the orderings ambiguities.

In [7], it was pointed out that BPS configurations of  $Dp$ -branes at angles, [10], [11], [12], might provide an important tool to probe the structure of the effective action. Upon T-dualizing we end up with  $D2p$ -branes in the presence of constant magnetic background fields. In the large volume limit ( $\alpha' \rightarrow 0$ ) the BPS conditions define a stable holomorphic vector bundle [13]. Moving away from the large volume limit, these conditions receive  $\alpha'$  corrections. As a BPS configuration necessarily solves the equations of motion, we obtain relations between different orders in  $\alpha'$  in the effective action.

In the present paper we start the exploration of the consequences of this idea. As we consider the present paper as a “feasibility study” we will make two simplifying assumptions: we work in the limit of slowly varying fieldstrengths and restrict our attention to the abelian case. The first assumption is translated by the fact that we will ignore terms containing derivatives of the fieldstrength. The second assumption is implemented by taking the magnetic background fields to live in the Cartan subalgebra of  $u(n)$ . As a starting point we take the theory in the  $\alpha' \rightarrow 0$  limit. I.e. we take the Yang-Mills action reduced to the torus of  $U(n)$  in the presence of magnetic background fields which define a stable holomorphic

vector bundle. Subsequently we add arbitrary powers of the fieldstrength to it and demand that the BPS configurations solve the equations of motion. This problem turns out to have a *unique* solution. The resulting action is precisely the abelian Born-Infeld action and the stability condition, also known as the Donaldson-Uhlenbeck-Yau condition [14], acquires  $\alpha'$  corrections which are unique as well.

This result provides a serious incentive to extend the analysis to the much harder non-abelian case [15]. In addition, there is a suspicion that, because of the severely restricted form of the supersymmetry algebra in ten dimensions, the BI action is the only supersymmetric deformation of abelian Yang-Mills<sup>2</sup>. E.g. supersymmetry fixes in the abelian case uniquely the fourth order term in the BI action [16]. As BPS configurations are intimately related to supersymmetry, we believe that our present paper lends strong support to this claim.

Our paper is organized as follows. In the next section we review BPS configurations of  $Dp$ -branes at angles. The subsequent section relates this to supersymmetric Yang-Mills theory. Using an example, we outline our strategy in the third section. Section 4 provides the proof of our assertion. We discuss our results and diverse applications in the final section. Conventions are given in the first appendix while the second gathers some useful results concerning the abelian Born-Infeld action.

## 2 BPS configurations from string theory

Simple BPS configurations of D-branes arise as follows [10], [11], [12]. One starts with two coinciding  $Dp$ -branes. Keeping one of them fixed, one performs a Lorentz transformation on the other one. For all boosts and generic rotations, all supersymmetry gets broken in this way. However there are particular rotations for which some of the supersymmetry is preserved.

Consider two  $Dp$ -branes in the  $(1, 3, \dots, 2p-1)$  directions. Keeping one of them fixed, rotate the other one subsequently over an angle  $\phi_1$  in the  $(1\ 2)$  plane, over an angle  $\phi_2$  in the  $(3\ 4)$  plane, ..., over an angle  $\phi_p$  in the  $(2p-1\ 2p)$  plane. The following table summarizes for various values of  $p$  the BPS conditions on the angles (taken to be non-zero unless stated otherwise) and the number of remaining supersymmetries.

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<sup>2</sup>This was suggested by Savdeep Sethi.

$p$	BPS condition	susy's
2	$\phi_1 + \phi_2 = 2\pi n$	8
3	$\phi_1 + \phi_2 + \phi_3 = 2\pi n$	4
4	$\phi_1 + \phi_2 + \phi_3 + \phi_4 = 2\pi n$	2
	$\phi_1 + \phi_2 = 2\pi n, \phi_3 + \phi_4 = 2\pi m$	4
	$\phi_1 = \phi_2 = \phi_3 = \phi_4$	6

In the table, we took  $n, m \in \mathbb{Z}$ .

In order to make contact with the Born-Infeld theory, we T-dualize the system in the  $2, 4, \dots, 2p$  directions. In this way, we end up with two coinciding D2p-branes with magnetic fields turned on. Indeed, having two D2p-branes extended in the  $1, 2, \dots, 2p$  directions with magnetic flux  $F_{2i-1, 2i}$ ,  $i \in \{1, \dots, p\}$ ,

$$F_{2i-1, 2i} = \begin{pmatrix} g_i + f_i & 0 \\ 0 & g_i - f_i \end{pmatrix}, \quad (2.1)$$

we can choose a gauge such that the potentials have the form,

$$A_{2i-1} = 0, \quad A_{2i} = F_{2i-1, 2i} x^{2i-1}. \quad (2.2)$$

T-dualizing back, we end up with two Dp-branes with transversal coordinates given by

$$X^{2i} = 2\pi\alpha' A_{2i}. \quad (2.3)$$

Using eq. (2.2) in eq. (2.3), we recognize the original configuration with the two Dp-branes at angles with the angles given by

$$\begin{aligned} \phi_i &= \arctan(2\pi\alpha'(g_i + f_i)) - \arctan(2\pi\alpha'(g_i - f_i)) \\ &= \arctan \frac{4\pi\alpha' f_i}{1 + (2\pi\alpha')^2(g_i^2 - f_i^2)}. \end{aligned} \quad (2.4)$$

In the table below we translate the BPS conditions on the angles in BPS conditions on the fieldstrengths, choosing for simplicity the  $U(1)$  part to be zero,  $g_i = 0$ .

$p$	BPS condition	fieldstrengths
2	$\phi_1 + \phi_2 = 2\pi n$	$f_1 + f_2 = 0$
3	$\phi_1 + \phi_2 + \phi_3 = 2\pi n$	$f_1 + f_2 + f_3 = (2\pi\alpha')^2 f_1 f_2 f_3$
4	$\phi_1 + \phi_2 + \phi_3 + \phi_4 = 2\pi n$	$f_1 + f_2 + f_3 + f_4 = (2\pi\alpha')^2 (f_1 f_2 f_3 + f_1 f_3 f_4 + f_1 f_2 f_4 + f_2 f_3 f_4)$
	$\phi_1 + \phi_2 = 2\pi n, \phi_3 + \phi_4 = 2\pi m$	$f_1 + f_2 = f_3 + f_4 = 0$
	$\phi_1 = \phi_2 = \phi_3 = \phi_4$	$f_1 = f_2 = f_3 = f_4$

One notices that for  $p > 2$ , the BPS condition expressed in terms of fieldstrengths corresponding to the angular relation  $\sum_i \phi_i = 2\pi n$ , gets  $\alpha'^2$  corrections. In the next, except when stated otherwise, we will always study BPS conditions of this type. In the remainder of this paper, we will put  $2\pi\alpha' = 1$ .

### 3 BPS configurations in supersymmetric Yang-Mills

The supersymmetric  $U(n)$  Yang-Mills theory in  $d = 10$  is given by<sup>3</sup>

$$\mathcal{S} = \int d^{10}x \operatorname{Tr} \left( -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\psi} \not{D} \psi \right), \quad (3.1)$$

where  $\psi$  is a Majorana-Weyl spinor which transforms in the adjoint representation of  $U(n)$ . The action is invariant under the supersymmetry transformations rules

$$\delta A_\mu = i\bar{\epsilon} \gamma_\mu \psi, \quad (3.2)$$

$$\delta \psi = -\frac{1}{2} F_{\mu\nu} \gamma^{\mu\nu} \epsilon + \eta, \quad (3.3)$$

with  $A_\mu$  the  $U(n)$  gauge potential and  $\epsilon$  and  $\eta$  constant Majorana-Weyl spinors.

The leading term of the effective theory describing  $n$  coinciding D2p-branes ( $p \geq 2$ ) is nothing but eq. (3.1) dimensionally reduced to  $2p + 1$  dimensions. The gauge potentials in the transverse directions appear as  $9 - 2p$  scalar fields in the adjoint representation of  $U(n)$ , which are reinterpreted as the transversal coordinates of the D-branes. As they will not play any significant role in this paper, we drop them from now on.

We now proceed with the analysis of eq. (3.3) in the presence of magnetic background fields and demand that some supersymmetry is preserved. I.e. we investigate whether for certain magnetic background fields there is an  $\epsilon$  such that  $\delta\psi = 0$ . In fact we can use the  $\eta$  transformation in eq. (3.3) to reduce any  $F_{\mu\nu}$  from  $u(n)$  to  $su(n)$ . We start by switching on  $F_{2i-1\,2i} \in su(n)$ ,  $i \in \{1, \dots, p\}$ , which satisfy the BPS condition suggested by D-branes at angles

$$\sum_{i=1}^p F_{2i-1\,2i} = 0. \quad (3.4)$$

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<sup>3</sup>We ignore an overall multiplicative constant.

For further convenience, we switch to complex coordinates (for details, we refer to appendix A), where we have that  $F_{\alpha\bar{\alpha}} = iF_{2\alpha-1\,2\alpha}$ . Eq. (3.4) becomes in complex coordinates,<sup>4</sup>

$$F_{\alpha\bar{\alpha}} \equiv \sum_{\alpha} F_{\alpha\bar{\alpha}} = 0. \quad (3.5)$$

We get that

$$\delta\psi = F_{\alpha\bar{\alpha}}\gamma_{\alpha\bar{\alpha}}\epsilon = 0, \quad (3.6)$$

holds provided that

$$\epsilon = \prod_{\alpha=1}^p (1 + \gamma_{1\bar{1}\alpha\bar{\alpha}})\xi, \quad (3.7)$$

with  $\xi$  an arbitrary Majorana-Weyl spinor. This reduces the number of supersymmetry charges from 16 to  $16/2^{p-1}$ .

It is not hard to check that when *all* magnetic fields are switched on,  $\delta\psi = 0$  still holds provided the magnetic fields do not only satisfy eq. (3.5) but

$$F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0, \quad \alpha, \beta \in \{1, \dots, p\}, \quad (3.8)$$

as well. For  $p = 2$ , eqs. (3.5) and (3.8) are nothing but the well known instanton equations. In general eq. (3.8) defines a holomorphic vector bundle while eq. (3.5), which can be rewritten in a more covariant form,

$$g^{\alpha\bar{\beta}}F_{\alpha\bar{\beta}} = 0, \quad (3.9)$$

is the Donaldson-Uhlenbeck-Yau condition for stability of the vector bundle [13].

Remains to check whether these configurations solve the equations of motion,  $D^\mu F_{\mu\nu} = 0$ . In complex coordinates, this becomes

$$\begin{aligned} 0 &= D_{\bar{\alpha}}F_{\alpha\bar{\beta}} + D_{\alpha}F_{\bar{\alpha}\bar{\beta}} \\ &= D_{\bar{\beta}}F_{\alpha\bar{\alpha}} + 2D_{\alpha}F_{\bar{\alpha}\bar{\beta}}, \end{aligned} \quad (3.10)$$

where we used the Bianchi identities. This is indeed satisfied if eqs. (3.5) and (3.8) hold. Note that magnetic field configurations satisfying eqs. (3.5) and (3.8) always solve the equations of motion and always preserve supersymmetry, even when they are not constant. As a consequence, we will not demand them to be constant anymore.

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<sup>4</sup>Unless stated otherwise, we sum over repeated indices.

## 4 Deformations

A natural question which arises is whether we can deform the Yang-Mills action in such a way that the BPS configurations given in the previous section remain solutions to the equations of motion. Though the discussion in the previous section holds for both the abelian as well as the non-abelian case, we focus in the remainder of this paper on the abelian case. In this way we avoid the additional complication of having to take the different orderings into account. From now on the magnetic fields take values in the Cartan subalgebra of  $u(n)$  and we postpone the study of the non-abelian extension to a future paper [15]. In addition, we will work under the assumption that the fieldstrengths vary slowly. In other words, we add terms polynomial in the fieldstrength to the action and ignore terms containing derivatives of the fieldstrength (acceleration terms). We will further comment on these assumptions in the concluding section. Under these assumptions, we arrive at equations of motion of the form

$$D^\mu F_{\mu\nu} + x D^\mu (F_{\mu\rho} F^{\rho\sigma} F_{\sigma\nu}) + y D^\mu (F_{\mu\nu} F_{\rho\sigma} F^{\sigma\rho}) + \mathcal{O}(F^5) = 0, \quad (4.1)$$

where  $x$  and  $y$  are real constants. As we saw before, the analysis of the leading order term led to the conditions (in complex coordinates)

$$F_{\alpha\beta} = F_{\bar{\alpha}\bar{\beta}} = 0, \quad (4.2)$$

$$g^{\alpha\bar{\beta}} F_{\alpha\bar{\beta}} = F_{\alpha\bar{\alpha}} = 0, \quad (4.3)$$

where in the last line we used the fact that we are working in flat space. Passing to complex coordinates while implementing the holomorphicity conditions eq. (4.2), eq. (4.1) becomes,

$$D_{\bar{\alpha}} F_{\alpha\bar{\beta}} + x D_{\bar{\alpha}} (F_{\alpha\bar{\gamma}} F_{\gamma\bar{\delta}} F_{\delta\bar{\beta}}) + 2y D_{\bar{\alpha}} (F_{\alpha\bar{\beta}} F_{\gamma\bar{\delta}} F_{\delta\bar{\gamma}}) + \mathcal{O}(F^5) = 0. \quad (4.4)$$

Upon using the Bianchi identities and eq. (4.2), this results in

$$D_{\bar{\beta}} (F_{\alpha\bar{\alpha}} + \frac{x}{3} F_{\alpha\bar{\gamma}} F_{\gamma\bar{\delta}} F_{\delta\bar{\alpha}}) + (2y + \frac{x}{2}) F_{\alpha\bar{\beta}} D_{\bar{\alpha}} (F_{\gamma\bar{\delta}} F_{\delta\bar{\gamma}}) + x F_{\gamma\bar{\delta}} F_{\delta\bar{\beta}} D_{\bar{\gamma}} F_{\alpha\bar{\alpha}} + 2y F_{\gamma\bar{\delta}} F_{\delta\bar{\gamma}} D_{\bar{\beta}} F_{\alpha\bar{\alpha}} + \mathcal{O}(F^5) = 0, \quad (4.5)$$

which vanishes if

$$y = -\frac{x}{4}, \quad (4.6)$$

holds and provided that we deform the Donaldson-Uhlenbeck-Yau condition, eq. (4.3), to

$$F_{\alpha\bar{\alpha}} + \frac{x}{3}F_{\alpha\bar{\gamma}}F_{\gamma\bar{\delta}}F_{\delta\bar{\alpha}} + \mathcal{O}(F^5) = 0. \quad (4.7)$$

Rescaling  $F$  and multiplying the equation of motion with a constant, we can put  $x = 1$ . Upon restoring the  $SO(2p)$  invariance, we find that the equations of motion integrate to the action

$$\begin{aligned} \mathcal{S} = \int d^{2p+1}x & \left( \frac{1}{4}F_{\mu_1\mu_2}F^{\mu_2\mu_1} + \frac{1}{8}F_{\mu_1\mu_2}F^{\mu_2\mu_3}F_{\mu_3\mu_4}F^{\mu_4\mu_1} \right. \\ & \left. - \frac{1}{32}(F_{\mu_1\mu_2}F^{\mu_2\mu_1})^2 + \mathcal{O}(F^6) \right), \end{aligned} \quad (4.8)$$

which, modulo an undetermined overall multiplicative constant, we recognize as the Born-Infeld action through order  $F^4$  (see appendix B).

In a similar way, one can push this calculation an order higher by adding the most general integrable terms through fifth order in  $F$  to the equations of motion. Again we require that the (deformed) BPS solutions solve the equations of motion. The scale of the fieldstrengths was already fixed at previous order. In this calculation one needs e.g. that the two last terms in eq. (4.5) get completed to a derivative of eq. (4.7). At the end one finds that the equations of motion get uniquely fixed and they indeed integrate to the Born-Infeld action through sixth order in  $F$ . Furthermore the Donaldson-Uhlenbeck-Yau condition acquires an order  $F^5$  correction,

$$F_{\alpha\bar{\alpha}} + \frac{1}{3}F_{\alpha\bar{\gamma}}F_{\gamma\bar{\delta}}F_{\delta\bar{\alpha}} + \frac{1}{5}F_{\alpha\bar{\gamma}}F_{\gamma\bar{\delta}}F_{\delta\bar{\epsilon}}F_{\epsilon\bar{\zeta}}F_{\zeta\bar{\alpha}} + \mathcal{O}(F^7) = 0. \quad (4.9)$$

These results raise the suspicion that the Born-Infeld action is the *only* deformation of Yang-Mills which allows for BPS solutions of the form eqs. (4.2-4.3). Furthermore one expects that the holomorphicity conditions, eq. (4.2) remain unchanged, while the Donaldson-Uhlenbeck-Yau condition, eq. (4.3) receives  $\alpha'$  corrections. In the next section, we will show that this is indeed the case.

## 5 All order results

In this section we will construct the unique deformation of abelian Yang-Mills which allows for BPS solutions which are in leading order given by eqs. (4.2-4.3).



Consider a general term in the deformed Yang-Mills lagrangian,

$$\lambda_{(p_1, p_2, \dots, p_n)} (\text{tr } F^2)^{p_1} (\text{tr } F^4)^{p_2} \dots (\text{tr } F^{2n})^{p_n}, \quad p_i \in \mathbb{N}, \quad \forall i \in \{1, \dots, n\}, \quad (5.1)$$

where  $\lambda_{(p_1, p_2, \dots, p_n)} \in \mathbb{R}$ . Dropping the overall  $\lambda_{(p_1, \dots, p_n)}$ , this term contributes to the equations of motion by,

$$\sum_{j=1}^n 4j p_j D^\mu \left( (F^{2j-1})_{\mu\nu} (\text{tr } F^2)^{p_1} (\text{tr } F^4)^{p_2} \dots (\text{tr } F^{2j})^{p_{j-1}} \dots (\text{tr } F^{2n})^{p_n} \right). \quad (5.2)$$

Passing to complex coordinates, eq. (A.1), we get

$$\sum_{j=1}^n 4j p_j 2^{p_1 + \dots + p_n - 1} D_{\bar{\alpha}} \left( (F^{2j-1})_{\alpha\bar{\beta}} (F^2)^{p_1} \dots (F^{2j})^{p_{j-1}} \dots (F^{2n})^{p_n} \right), \quad (5.3)$$

where

$$\begin{aligned} (F^m)_{\alpha\bar{\beta}} &\equiv F_{\alpha\bar{\alpha}_2} F_{\alpha_2\bar{\alpha}_3} \dots F_{\alpha_m\bar{\beta}} \\ (F^m) &\equiv F_{\alpha_1\bar{\alpha}_2} F_{\alpha_2\bar{\alpha}_3} \dots F_{\alpha_m\bar{\alpha}_1}. \end{aligned} \quad (5.4)$$

Using the Bianchi identities and eq. (4.2), we find for the action of the derivative operator  $D_{\bar{\alpha}}$  on  $(F^{2j-1})_{\alpha\bar{\beta}}$ :

$$D_{\bar{\alpha}} (F^{2j-1})_{\alpha\bar{\beta}} = \sum_{h=1}^{2j-2} \frac{1}{h} (D_{\bar{\alpha}} F^h) (F^{2j-1-h})_{\alpha\bar{\beta}} + \frac{1}{2j-1} D_{\bar{\beta}} (F^{2j-1}). \quad (5.5)$$

Implementing this result in eq. (5.3) yields,

$$\begin{aligned} &\sum_{j=1}^n 4j p_j 2^{p_1 + \dots + p_n - 1} \left( \left( \sum_{h=1}^{2j-2} \frac{1}{h} (D_{\bar{\alpha}} F^h) (F^{2j-1-h})_{\alpha\bar{\beta}} + \frac{1}{2j-1} D_{\bar{\beta}} (F^{2j-1}) \right) \right. \\ &\quad (F^2)^{p_1} \dots (F^{2j})^{p_{j-1}} \dots (F^{2n})^{p_n} + \\ &\quad \sum_{g=1, g \neq j}^n p_g (D_{\bar{\alpha}} F^{2g}) (F^{2j-1})_{\alpha\bar{\beta}} (F^2)^{p_1} \dots (F^{2j})^{p_{j-1}} \dots (F^{2g})^{p_g-1} \dots (F^{2n})^{p_n} + \\ &\quad \left. (p_j - 1) (D_{\bar{\alpha}} F^{2j}) (F^{2j-1})_{\alpha\bar{\beta}} (F^2)^{p_1} \dots (F^{2g})^{p_{j-2}} \dots (F^{2n})^{p_n} \right). \end{aligned} \quad (5.6)$$

We now study the different types of terms in the equations of motion. We will determine the numerical prefactors such that the (deformed) BPS configurations solve them.

- Terms of the form  $D_{\bar{\beta}} F^{2r-1}$ : there is one of these terms in each order and they add up to,

$$D_{\bar{\beta}} \left( 4 \cdot 1 \lambda_{(1)} F_{\alpha\bar{\alpha}} + \frac{4 \cdot 2}{3} \lambda_{(0,1)} (F^3)_{\alpha\bar{\alpha}} + \frac{4 \cdot 3}{5} \lambda_{(0,0,1)} (F^5)_{\alpha\bar{\alpha}} + \dots \right). \quad (5.7)$$

In leading order it vanishes because of eq. (4.3). It is clear that the all order expression should vanish by itself thereby giving the deformed Donaldson-Uhlenbeck-Yau condition,

$$\frac{1}{1} \lambda_{(1)} F_{\alpha\bar{\alpha}} + \frac{2}{3} \lambda_{(0,1)} (F^3)_{\alpha\bar{\alpha}} + \frac{3}{5} \lambda_{(0,0,1)} (F^5)_{\alpha\bar{\alpha}} + \dots = 0. \quad (5.8)$$

- Terms of the form  $(D_{\bar{\alpha}} F^{2r}) (F^{2l-1})_{\alpha\bar{\beta}}(\text{tail})$ , where

$$(\text{tail}) = (F^2)^{p_1} (F^4)^{p_2} \dots (F^{2n})^{p_n} = \prod_{i=1}^n (F^{2i})^{p_i} : \quad (5.9)$$

as these terms involve traces over even powers of the fieldstrength they can never be cancelled by a condition as eq. (5.8), so they should cancel order by order among themselves. If we look at the first versus the two last terms of eq. (5.6), we see immediately that a term of this form originates from two different terms in the action, namely  $(\text{tr } F^{2l+2r})(\text{tail})$  and  $(\text{tr } F^{2l})(\text{tr } F^{2r})(\text{tail})$ . Suppose first that  $l \neq r$ . Requiring such a term to vanish results, using the first two terms in eq. (5.6), in the following condition,

$$(l+r)(p_{l+r}+1)\lambda_{(\dots, p_l, \dots, p_r, \dots, p_{l+r}+1, \dots)} + 4lr(p_l+1)(p_r+1)\lambda_{(\dots, p_l+1, \dots, p_r+1, \dots, p_{l+r}, \dots)} = 0. \quad (5.10)$$

The Born-Infeld coefficients eq. (B.4) satisfy this condition. Analogously, using the first and third term in eq. (5.6), we get when  $l = r$ ,

$$(p_{2l}+1)\lambda_{(\dots, p_l, \dots, p_{2l}+1, \dots)} + 2l(p_l+2)(p_l+1)\lambda_{(\dots, p_l+2, \dots, p_{2l}, \dots)} = 0, \quad (5.11)$$

again satisfied by the Born-Infeld coefficients eq. (B.4). Note that the two conditions eq. (5.10) and eq. (5.11) are enough to determine all coefficients at a certain order if one is known. We give an example of the chain of relations at order  $F^8$ ,

$$\begin{array}{ccccccc}
(4, 0, 0, 0) & \xrightarrow{(5.11)} & (2, 1, 0, 0) & \xrightarrow{(5.11)} & (0, 2, 0, 0) & \xrightarrow{(5.11)} & (0, 0, 0, 1) \\
& & \uparrow (5.10) & & \nearrow (5.10) & & \\
& & (1, 0, 1, 0) & & & & 
\end{array}$$

So, up until now, we find Born-Infeld modulo a proportionality factor at each order,

$$\lambda_{(p_1, p_2, \dots, p_n)} = \frac{(-1)^{k+1}}{4^k} \frac{1}{p_1! \dots p_n!} \frac{1}{1^{p_1} \dots n^{p_n}} X_{\sum j p_j}, \quad (5.12)$$

where  $X_{\sum_{j=1}^n j p_j} \in \mathbb{R}$  are unknown constants.

- Terms of the form  $(D_{\bar{\beta}} F^{2r-1})$  (tail): they relate different orders in  $F$ . The only way to cancel these terms is by virtue of eq. (5.8). Using eqs. (5.6) and (5.12) we find that such a term appears in the equation of motion as

$$\frac{(-1)^{\sum_l p_l}}{2^{\sum_l p_l} \prod_l (p_l!) \prod_l l^{p_l}} \frac{X_{\sum_l l p_l + r}}{2r-1} (D_{\bar{\beta}} F^{2r-1}) \text{ (tail)}, \quad (5.13)$$

where all summations and products run from  $l = 1$  through  $l = n$ . For a given tail, the sum over  $r$  of such terms has to vanish through the use of eq. (5.8). This determines all unknowns  $X_r$  in terms of two,

$$X_r = X_2 \left( \frac{X_2}{X_1} \right)^{r-2}, \quad r \geq 3. \quad (5.14)$$

We still have the freedom to rescale the fieldstrength in the equations of motion by an arbitrary factor and we also note that the equations of motion are only determined modulo an arbitrary multiplicative factor. In other words, we can only determine the action modulo an overall multiplicative factor. This freedom can be used to put  $X_1 = X_2 = 1$ . Combining this with eq. (5.14), we get

$$X_r = 1, \quad \forall r \geq 1. \quad (5.15)$$

At this point the equations of motion are completely fixed and they are exactly equal to the equations of motion of the abelian Born-Infeld theory,

implying that our action is, modulo an overall multiplicative constant, the Born-Infeld action. This fixes the BPS condition, eq. (5.8), as well,

$$\begin{aligned} 0 &= F_{\alpha\bar{\alpha}} + \frac{1}{3}(F^3)_{\alpha\bar{\alpha}} + \frac{1}{5}(F^5)_{\alpha\bar{\alpha}} + \dots \\ &= \text{tr arctanh } \mathcal{F}, \end{aligned} \quad (5.16)$$

where  $\mathcal{F}$  is a  $p \times p$  matrix with elements  $F_{\alpha\bar{\beta}}$  and the trace is taken over the Lorentz indices.

- Terms of the form  $(D_{\bar{\alpha}} F^{2r-1})(F^{2s})_{\alpha\bar{\beta}}(\text{tail})$ : these are the only terms left and they will cancel because of eq. (5.16). Using eqs. (5.6), (5.12) and (5.15), we get the prefactor of such term,

$$\frac{(-1)^{\sum_l p_l}}{2^{\sum_l p_l} \prod_l (p_l!) \prod_l l^{p_l}} \frac{1}{2r-1} (D_{\bar{\alpha}} F^{2r-1})(F^{2s})_{\alpha\bar{\beta}}(\text{tail}), \quad (5.17)$$

and it is clear that when summing over  $r$  they vanish because of eq. (5.16).

This completes the proof that the abelian Born-Infeld action is the unique deformation of the abelian Yang-Mills action which allows for BPS solutions.

## 6 Discussion and conclusions

Fieldstrength configurations which define a stable, eq. (4.3), holomorphic, eq. (4.2), vector bundle solve the Yang-Mills equations of motion. Such configurations are relevant in the study of BPS solutions for D-branes in the  $\alpha' \rightarrow 0$  limit. In this paper we deformed the *abelian* theory by adding arbitrary powers of the fieldstrength to the Yang-Mills lagrangian. Demanding that a deformation of eqs. (4.2-4.3) still solves the equations of motion, we showed that the deformation is uniquely determined: it is precisely the abelian Born-Infeld theory. The holomorphicity condition eq. (4.2) remains unchanged while the Donaldson-Uhlenbeck-Yau stability condition gets deformed to

$$\text{tr arctanh } \mathcal{F} = 0, \quad (6.1)$$

with  $\mathcal{F}$  a  $p \times p$  matrix with elements  $F_{\alpha\bar{\beta}}$ .

The analysis in section 5 holds not only in flat space but in Kähler geometries as well. Defining  $\mathcal{F}$  to be the  $p \times p$  matrix with elements  $F_{\alpha}^{\beta} \equiv g^{\beta\bar{\gamma}} F_{\alpha\bar{\gamma}}$  with

$g_{\alpha\bar{\beta}}$  the Kähler metric, one finds again eq. (6.1). In this context, it might be worthwhile to mention that it would be interesting to include the transverse scalars in the analysis. This would make it possible to get an all  $\alpha'$  result for the stability condition for branes wrapped around a holomorphic submanifold of a Kähler manifold [17].

Eqs. (4.2) and (6.1) play an important role in the study of BPS configurations for D-branes at finite  $\alpha'$ . As supersymmetry and magnetic field configurations discussed above are closely related, our result provides evidence strengthening the belief that the only supersymmetric deformation of ten-dimensional supersymmetric  $U(1)$  Yang-Mills theory is the supersymmetric Born-Infeld action.

Eq. (6.1) holds in any dimension  $d = 2p$ . It is a  $U(p)$  invariant and therefore it can be rewritten in terms of Casimir invariants. As  $U(p)$  has Casimir invariants of order 1, 2, ...,  $p$ , one can rewrite eq. (6.1) in a less elegant though more familiar form when specifying to particular dimensions. We tabulate the resulting equivalent expressions for the cases relevant to D-brane physics.

$p$	stability condition
2	$F_{1\bar{1}} + F_{2\bar{2}} = 0$
3	$F_{1\bar{1}} + F_{2\bar{2}} + F_{3\bar{3}} + F_{1\bar{1}}F_{2\bar{2}}F_{3\bar{3}} - F_{1\bar{2}}F_{2\bar{1}}F_{3\bar{3}} - F_{1\bar{3}}F_{2\bar{2}}F_{3\bar{1}} + F_{1\bar{2}}F_{2\bar{3}}F_{3\bar{1}} + F_{1\bar{3}}F_{2\bar{1}}F_{3\bar{2}} - F_{1\bar{1}}F_{2\bar{3}}F_{3\bar{2}} = 0$
4	$F_{1\bar{1}} + F_{2\bar{2}} + F_{3\bar{3}} + F_{4\bar{4}} - F_{1\bar{3}}F_{2\bar{2}}F_{3\bar{1}} + F_{1\bar{2}}F_{2\bar{3}}F_{3\bar{1}} + F_{1\bar{4}}F_{4\bar{3}}F_{3\bar{1}} - F_{1\bar{3}}F_{4\bar{4}}F_{3\bar{1}} + F_{1\bar{3}}F_{2\bar{1}}F_{3\bar{2}} - F_{1\bar{1}}F_{2\bar{3}}F_{3\bar{2}} - F_{1\bar{2}}F_{2\bar{1}}F_{3\bar{3}} + F_{1\bar{1}}F_{2\bar{2}}F_{3\bar{3}} - F_{1\bar{4}}F_{2\bar{2}}F_{4\bar{1}} + F_{1\bar{2}}F_{2\bar{4}}F_{4\bar{1}} - F_{1\bar{4}}F_{3\bar{3}}F_{4\bar{1}} + F_{1\bar{3}}F_{3\bar{4}}F_{4\bar{1}} + F_{1\bar{4}}F_{2\bar{1}}F_{4\bar{2}} - F_{1\bar{1}}F_{2\bar{4}}F_{4\bar{2}} - F_{2\bar{4}}F_{3\bar{3}}F_{4\bar{2}} + F_{2\bar{3}}F_{3\bar{4}}F_{4\bar{2}} + F_{2\bar{4}}F_{3\bar{2}}F_{4\bar{3}} - F_{1\bar{1}}F_{3\bar{4}}F_{4\bar{3}} - F_{2\bar{2}}F_{3\bar{4}}F_{4\bar{3}} - F_{1\bar{2}}F_{2\bar{1}}F_{4\bar{4}} + F_{1\bar{1}}F_{2\bar{2}}F_{4\bar{4}} - F_{2\bar{3}}F_{3\bar{2}}F_{4\bar{4}} + F_{1\bar{1}}F_{3\bar{3}}F_{4\bar{4}} + F_{2\bar{2}}F_{3\bar{3}}F_{4\bar{4}} = 0$

We expect that these BPS solutions minimize the energy. Let us briefly investigate this for the case where eq. (4.2) holds<sup>5</sup>. For simplicity, we will only switch on the magnetic fields  $F_{1\bar{1}}, F_{2\bar{2}}, \dots, F_{p\bar{p}}$ . The energy is given by

$$E = \int d^4x |\det(\mathbf{1} - \mathcal{F})| = \int d^4x \prod_{\alpha=1}^p |1 - F_{\alpha\bar{\alpha}}|. \quad (6.2)$$

For  $p = 2$  we get,

$$E = \int d^4x |1 - F_{1\bar{1}} - F_{2\bar{2}} + F_{1\bar{1}}F_{2\bar{2}}| \geq \int d^4x |1 + F_{1\bar{1}}F_{2\bar{2}}| - |F_{1\bar{1}} + F_{2\bar{2}}|. \quad (6.3)$$

---

<sup>5</sup>Throughout this discussion, we assume that the rhs of the inequalities are differences of invariants. This is certainly so for  $p = 4$ , see e.g. [13]. For  $p \geq 6$  this is very probably true as well. We postpone a more detailed examination of the energy of BPS configurations to a future publication.

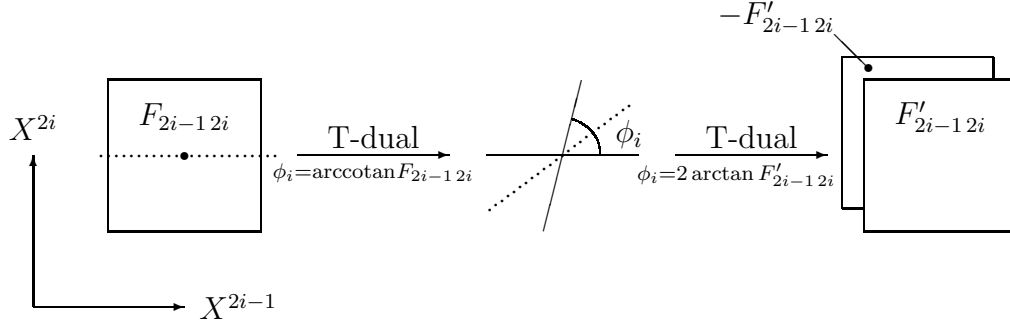


Figure 1: Before T-dualizing, we have a D-brane extending in the  $2i - 1, 2i$  plane with  $u(1)$  magnetic flux  $F_{2i-1, 2i}$  and another D-brane without magnetic flux perpendicular to it. The dotted lines show the directions along which we T-dualize. After T-dualizing twice, we end up with two D-branes which coincide in the  $2i - 1, 2i$  plane with an  $su(2)$  flux  $F'_{2i-1, 2i}\sigma_3$ . This shows that the more exotic BPS conditions minimalizing the energy are T-dual to the ones studied in this paper.

Contrary to the Yang-Mills case, we find two situations in which the relation gets saturated. The first is when  $F_{1\bar{1}} + F_{2\bar{2}} = 0$ , which we recognize as the familiar BPS condition we have been discussing so far. The second configuration is characterized by  $1 + F_{1\bar{1}}F_{2\bar{2}} = 0$ . This corresponds to a D2/D4 system. Though this system is not supersymmetric, it becomes so when we switch on magnetic fields  $F_{1\bar{1}}$  and  $F_{2\bar{2}}$  on the D4-brane which precisely satisfy  $1 + F_{1\bar{1}}F_{2\bar{2}} = 0$ . For  $p = 3$ , we get

$$\begin{aligned}
E &= \int d^6x \left| (1 - F_{1\bar{1}} - F_{2\bar{2}} - F_{3\bar{3}} + F_{1\bar{1}}F_{2\bar{2}} + F_{1\bar{1}}F_{3\bar{3}} + F_{2\bar{2}}F_{3\bar{3}} - F_{1\bar{1}}F_{2\bar{2}}F_{3\bar{3}}) \right| \\
&\geq \int d^6x \left| 1 + F_{1\bar{1}}F_{2\bar{2}} + F_{1\bar{1}}F_{3\bar{3}} + F_{2\bar{2}}F_{3\bar{3}} \right| - \\
&\quad \left| F_{1\bar{1}} + F_{2\bar{2}} + F_{3\bar{3}} + F_{1\bar{1}}F_{2\bar{2}}F_{3\bar{3}} \right|.
\end{aligned} \tag{6.4}$$

Again the result is saturated in two cases. When the last factor vanishes in eq. (6.4) we have the standard BPS condition. When the first factor vanishes, we find a configuration corresponding to either a D0/D6 or a D4/D6 system. By switching on magnetic fields  $F_{1\bar{1}}$ ,  $F_{2\bar{2}}$  and  $F_{3\bar{3}}$  on the D6-brane which satisfy this relation we obtain a BPS configuration. A similar analysis holds for  $p = 4$ . Either one recovers the standard BPS configuration of D8-branes or a D2/D8 system (or equivalently a D6/D8 system) with magnetic fields on the D8-brane such that the result is BPS. Aspects of some of these non-standard BPS configurations

were studied in [18] . Even as these “exotic” BPS configurations have no  $\alpha' \rightarrow 0$  limit, they are in fact T-dual to the BPS configurations studied in section 2 as is demonstrated in figure 1.

Our analysis was performed under the assumption that the fieldstrengths vary slowly, i.e. we ignored terms having derivatives of the fieldstrength. Such terms are expected to be present [19]. It would be very interesting to investigate whether our method can handle such terms as well. However, an additional complication will arise in such an analysis. As explained in [20] and [21], because of field redefinitions, derivative terms are ambiguous. Nonetheless, it is worthwhile to investigate this point as this will further clarify the relation between the commutative and non-commutative pictures [22], [23].

Another point which deserves further attention is the study of the BPS conditions as a function of the string coupling constant  $g_S$ . In this way the method developed in this paper might provide an alternative approach to the study of the effective action as a function of the string coupling constant. In [20], it was shown that through second order in  $g_S$  and in flat space the Born-Infeld action, modulo a renormalization of the tension, still describes the effective dynamics. It would be intriguing if such a claim could be pushed at higher orders (note that in non-trivial geometries this is very probably not true).

Finally, the results in this paper provide sufficient motivation for a detailed investigation of the non-abelian case. As eqs. (4.2) and (4.3) hold both in the abelian and the non-abelian case, we can still use it as a starting point and investigate allowed deformations. Not only do we expect a concrete ordering prescription for the action, but eq. (6.1) should get supplied with an ordering prescription as well. Note that derivative terms might become relevant in this case. We will report on this in [15].

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## A Notations and conventions

Our metric follows the “mostly plus” conventions. Indices denoted by  $\mu, \nu, \dots$  run from 0 to  $2p$ , denoted by  $i, j, \dots$  run from 1 to  $2p$  and denoted by  $\alpha, \beta, \dots$  run from 1 to  $p$ . We use real  $32 \times 32$   $\gamma$ -matrices satisfying  $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$  and  $\gamma_\mu^T = \gamma_0 \gamma_\mu \gamma_0$ . By  $\gamma_{\mu_1 \dots \mu_n}$  we denote the (weighted) completely antisymmetrized product  $\gamma_{[\mu_1} \gamma_{\mu_2} \dots \gamma_{\mu_n]}$  with  $[[\dots]] = [\dots]$ .

Instead of using real spatial coordinates  $x^i$ ,  $i \in \{1, \dots, 2p\}$ , we will often use complex coordinates  $z^\alpha$ ,  $\alpha \in \{1, \dots, p\}$ ,

$$z^\alpha \equiv \frac{1}{\sqrt{2}} (x^{2\alpha-1} + ix^{2\alpha}), \quad \bar{z}^{\bar{\alpha}} \equiv \frac{1}{\sqrt{2}} (x^{2\alpha-1} - ix^{2\alpha}). \quad (\text{A.1})$$

As we work in flat space, the metric is  $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$ ,  $g_{\alpha\bar{\beta}} = \delta_{\alpha\bar{\beta}}$ .

Consider the rotation group  $SO(p)$ . The subgroup preserving the complex structure is  $U(p)$ . If we denote the  $so(2p)$  generators by  $M_{ij} = -M_{ji}$ , the  $u(p)$  generators are given by the subset  $M_{\alpha\bar{\beta}}$ . The  $u(1)$  generator commuting with all the  $u(p)$  generators is given by  $\sum_\alpha M_{\alpha\bar{\alpha}}$ . The remainder of the  $so(2p)$  generators,  $M_{\alpha\beta}$  and  $M_{\bar{\alpha}\bar{\beta}}$  resp., transforms in the  $p(p-1)/2$  and the  $\overline{p(p-1)/2}$  of  $su(p)$  resp.

## B The abelian Born-Infeld action

In this appendix we derive a few properties of the abelian Born-Infeld action needed in section five.

The Born-Infeld lagrangian can be rewritten as<sup>6</sup>

$$\begin{aligned} \mathcal{L}_{BI} &= -\sqrt{\det(\delta^\mu_\nu - F^\mu_\nu)} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{4^k k!} (\text{tr } F^2 + \frac{1}{2} \text{tr } F^4 + \dots + \frac{1}{p} \text{tr } F^{2p} + \dots)^k. \end{aligned} \quad (\text{B.1})$$

A general term in the abelian Born-Infeld Lagrangian

$$\lambda_{(p_1, p_2, \dots, p_n)}^{BI} (\text{tr } F^2)^{p_1} (\text{tr } F^4)^{p_2} \dots (\text{tr } F^{2n})^{p_n}, \quad (\text{B.2})$$

originates from the  $k$ th term in the Taylor expansion, with  $k$  given by

$$k = p_1 + p_2 + \dots + p_n. \quad (\text{B.3})$$

---

<sup>6</sup>The trace denotes a trace over the Lorentz indices.



Hence, the numerical prefactor becomes

$$\lambda_{(p_1, p_2, \dots, p_n)}^{BI} = \frac{(-1)^{k+1}}{4^k} \frac{1}{p_1! \dots p_n!} \frac{1}{1^{p_1} \dots n^{p_n}}. \quad (\text{B.4})$$

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