

# The Lax pairs for elliptic $C_n$ and $BC_n$ Ruijsenaars-Schneider models and their spectral curves

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## Abstract

We study the elliptic  $C_n$  and  $BC_n$  Ruijsenaars-Schneider models which is elliptic generalization of system given in [1]. The Lax pairs for these models are constructed by Hamiltonian reduction technology. We show that the spectral curves can be parameterized by the involutive integrals of motion for these models. Taking nonrelativistic limit and scaling limit, we verify that they lead to the systems corresponding to Calogero-Moser and Toda types.

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## 1 Introduction

The Ruijsenaars-Schneider(RS) and Calogero-Moser(CM) models as integrable many-body models recently have attracted remarkable attention and have been extensively studied. They describe one-dimensional  $N$ -particle system with pairwise interaction. Their importance lies in various fields ranging from lattice models in statistics physics[2, 3], to the field theory and gauge theory[4, 5]. e.g. to the Seiberg-Witten theory[6] *et al.* In particular, the study of the RS model is of great importance since it is the integrable relativistic generalization of the corresponding CM model[7, 8].

Recently, the Lax pairs for the CM models in various root system have been constructed by Olshanetsky and Perelomov[9] using reduction on symmetric space, further given by Inozemtsev in [10]. Afterwards, D'Hoker and Phong[11] succeeded in constructing the Lax pairs with spectral parameter for each of the finite dimensional Lie algebra, as well as the introduction of untwisted and twisted Calogero-Moser systems. Bordner *et al.*[12, 13, 14] give

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two types universal realization for the Lax pairs associated to all of the Lie algebra: the root type and the minimal type, with and without spectral parameters. Even for all of the Coxeter group, the construction has been obtained in [15]. All of them do not apply the reduction method for under which condition one will confront some obstruction[16] but using pure Lie algebra construction. In [16], Hurtubise and Markman utilize so called “structure group”, which combines semi-simple group and Weyl group, to construct CM systems associate with Hitchin system, which in some degree generalizes the result of Refs. [11, 12, 13, 14, 15]. Furthermore, the quantum version of the generalization have been developed in [17, 18] at least for degenerate potentials of trigonometric after the works of Olshanetsky and Perelomov[19].

So far as for the RS model, only the Lax pair of the  $A_{N-1}$  type RS model was obtained [7, 3, 26, 27, 28, 29] and succeeded in recovering it by applying Hamiltonian reduction procedure on two-dimensional current group[20]. Although the commutative operators for RS model based on various type Lie algebra have been given by Komori and co-workers [21, 22], Diejen[23, 24] and Hasegawa *et al.* [2, 25], the Lax integrability (or Lax pair representation) of the other type RS model is still an open problem[6] except few degenerate case[30, 1].

In the work of Refs. [30] and [1], we have succeeded in constructing the Lax pair for  $C_n$  and  $BC_n$  RS systems only with the degenerate case. The r-matrix structure for them have been derived by Avan *et al.* [31]. In this paper, we study the Lax pair for the most general  $C_n$  and  $BC_n$  RS models—the elliptic  $C_n$  and  $BC_n$  RS models. We shall give the explicit forms of Lax pairs for these systems by Hamiltonian reduction. We calculate the spectral curves for these systems, which are showed to be parameterized by a set of involutive integrals of motion. In particular, taking their non-relativistic limit and scaling limit, we shall recover the systems of corresponding CM and Toda types, respectively. The other various degenerate cases are also be discussed and the connection between the Lax pair with spectral parameter and the one without spectral parameter is remarked.

The paper is organized as follows. The basic materials of the  $A_{N-1}$  RS model are reviewed in Section 2, where we propose a Lax pair associating with the Hamiltonian which has a reflection symmetry with respect to the particles in the origin. This includes construction of Lax pair for  $A_{N-1}$  RS system together with its symmetry analysis etc al. The main results are showed in Sections 3 and 4. In Section 3, we present the Lax pairs for the elliptic  $C_n$  and  $BC_n$  RS models by reducing from that of  $A_{N-1}$  RS model. The explicit forms for the Lax pairs are given in Section 4. Section 5 is devoted to deriving the spectral curves for these systems and their non-relativistic counterpart, the Calogero-Moser model and scaling limit of the Toda model. Section 6 is to show the various degenerate limits: the trigonometric, hyperbolic and rational cases. The last section is a brief summary and discussion.

## 2 The $A_{N-1}$ -type Ruijsenaars-Schneider Model

As a relativistic-invariant generalization of the  $A_{N-1}$ -type nonrelativistic Calogero-Moser model, the  $A_{N-1}$ -type Ruijsenaars-Schneider systems are completely integrable. The system's integrability was first showed by Ruijsenaars[7, 8]. The Lax pair for this model has been constructed in Refs. [7, 3, 26, 27, 28, 29]. Recent progress have showed that the compactification of higher dimension SUSY Yang-Mills theory and Seiberg-Witten theory can be described by this model[6]. Instanton correction of the prepotential associated with the  $sl_2$  RS system has been calculated in Ref. [32].

### 2.1 Model and equations of motion

Let us briefly give the basics of this model. In terms of the canonical variables  $p_i, x_i (i, j = 1, \dots, N)$  enjoying the canonical Poisson bracket

$$\{p_i, p_j\} = \{x_i, x_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij}, \quad (2.1)$$

the Hamiltonian of the  $A_{N-1}$  RS system reads as

$$\mathcal{H}_{A_{N-1}} = \sum_{i=1}^N \left( e^{p_i} \prod_{k \neq i} f(x_i - x_k) + e^{-p_i} \prod_{k \neq i} g(x_i - x_k) \right), \quad (2.2)$$

where

$$\begin{aligned} f(x) &: = \frac{\sigma(x - \gamma)}{\sigma(x)}, \\ g(x) &: = f(x)|_{\gamma \rightarrow -\gamma}, \quad x_{ik} := x_i - x_k, \end{aligned} \quad (2.3)$$

and  $\gamma$  denotes the coupling constant. Here,  $\sigma(x)$  is the Weierstrass  $\sigma$ -function which is an entire, odd and quasiperiodic function with a fixed pair of the primitive quasiperiods  $2\omega_1$  and  $2\omega_2$ . It can be defined as the infinite product

$$\sigma(x) = x \prod_{w \in \Gamma \setminus \{0\}} \left( 1 - \frac{x}{w} \right) \exp \left[ \frac{x}{w} + \frac{1}{2} \left( \frac{x}{w} \right)^2 \right],$$

where  $\Gamma = 2\omega_1\mathbb{Z} + 2\omega_3\mathbb{Z}$  is the corresponding period lattice. Defining a third dependent quasiperiod  $2\omega_2 = -2\omega_1 - 2\omega_3$ , one has

$$\sigma(x + 2\omega_k) = -\sigma(x)e^{2\eta_k(x+\omega_k)}, \quad \zeta(x + 2\omega_k) = \zeta(x) + 2\eta_k, \quad k = 1, 2, 3,$$

where

$$\zeta(x) = \frac{\sigma'(x)}{\sigma(x)}, \quad \wp(x) = -\zeta'(x),$$

and  $\eta_k = \zeta(\omega_k)$  satisfy  $\eta_1 \omega_3 - \eta_3 \omega_1 = \frac{\pi i}{2}$ .

Notice that in Ref. [7] Ruijsenaars used another “gauge” of the momenta such that two are connected by the following canonical transformation:

$$x_i \longrightarrow x_i, \quad p_i \longrightarrow p_i + \frac{1}{2} \ln \prod_{j \neq i}^N \frac{f(x_{ij})}{g(x_{ij})}. \quad (2.4)$$

The Lax matrix for this model has the form (for the general elliptic case)

$$L(\lambda) = \sum_{i,j=1}^N \frac{\Phi(x_i - x_j + \gamma, \lambda)}{\Phi(\gamma, \lambda)} \exp(p_j) b_j E_{ij}, \quad (2.5)$$

where

$$\Phi(x, \lambda) := \frac{\sigma(x + \lambda)}{\sigma(x)\sigma(\lambda)}, \quad b_j := \prod_{k \neq j} f(x_j - x_k), \quad (E_{ij})_{kl} = \delta_{ik} \delta_{jl}. \quad (2.6)$$

and  $\lambda$  is the spectral parameter. It is shown in Ref. [29, 33, 34] that the Lax operator satisfies the quadratic fundamental Poisson bracket

$$\{L_1, L_2\} = L_1 L_2 a_1 - a_2 L_1 L_2 + L_2 s_1 L_1 - L_1 s_2 L_2, \quad (2.7)$$

where  $L_1 = L_{A_{N-1}} \otimes Id$ ,  $L_2 = Id \otimes L_{A_{N-1}}$  and the four matrices read as

$$\begin{aligned} a_1 &= a + w, & s_1 &= s - w, \\ a_2 &= a + s - s^* - w, & s_2 &= s^* + w. \end{aligned} \quad (2.8)$$

The forms of  $a, s, w$  are

$$\begin{aligned} a(\lambda, \mu) &= -\zeta(\lambda - \mu) \sum_{k=1}^N E_{kk} \otimes E_{kk} - \sum_{k \neq j} \Phi(x_j - x_k, \lambda - \mu) E_{jk} \otimes E_{kj}, \\ s(\lambda) &= \zeta(\lambda) \sum_{k=1}^N E_{kk} \otimes E_{kk} + \sum_{k \neq j} \Phi(x_j - x_k, \lambda) E_{jk} \otimes E_{kk}, \\ w &= \sum_{k \neq j} \zeta(x_k - x_j) E_{kk} \otimes E_{jj}. \end{aligned} \quad (2.9)$$

The asterisk symbol means

$$r^* = \Pi r \Pi \quad \text{with} \quad \Pi = \sum_{k,j=1}^N E_{kj} \otimes E_{jk}. \quad (2.10)$$

Noticing that

$$L(\lambda)^{-1}_{ij} = \frac{\sigma(\gamma + \lambda) \sigma(\lambda + (N-1)\gamma)}{\sigma(\lambda) \sigma(\lambda + N\gamma)}$$

$$\times \sum_{i,j=1}^N \frac{\Phi(x_i - x_j - \gamma, \lambda + N\gamma)}{\Phi(-\gamma, \lambda + N\gamma)} \exp(-p_i) b_j' E_{ij}, \quad (2.11)$$

$$b_j' \quad : \quad = \prod_{k \neq j} g(x_j - x_k), \quad (2.12)$$

(the proof of the above identity is sketched in the appendix) one can get the characteristic polynomials of  $L_{A_{N-1}}$  (Ref. [35])

$$\det(L(\lambda) - v \cdot Id) = \sum_{j=0}^N \Phi(\gamma, \lambda)^{-j} (-v)^{N-j} \frac{\mathcal{H}_j^+}{\sigma^j(\gamma)} \times \frac{\sigma(\lambda + j\gamma)}{\sigma(\lambda)}, \quad (2.13)$$

and that of  $L_{A_{N-1}}^{-1}$  by using formula given in Eq. (A.8)

$$\begin{aligned} & \det\left(\frac{\sigma(\lambda)\sigma(\lambda - N\gamma)}{\sigma(\lambda - \gamma)\sigma(\lambda - (N-1)\gamma)} \times L(\lambda - N\gamma)^{-1} - v \cdot Id\right) \\ &= \sum_{j=0}^N \Phi(-\gamma, \lambda)^{-j} (-v)^{N-j} \times \frac{(\mathcal{H}_j^-)}{\sigma^j(-\gamma)} \frac{\sigma(\lambda - j\gamma)}{\sigma(\lambda)}, \end{aligned} \quad (2.14)$$

where  $(\mathcal{H}_0^\pm)_{A_{N-1}} = (\mathcal{H}_N^\pm)_{A_{N-1}} = 1$  and

$$(\mathcal{H}_i^+)_{A_{N-1}} = \sum_{\substack{J \subset \{1, \dots, N\} \\ |J|=i}} \exp\left(\sum_{j \in J} p_j\right) \prod_{\substack{j \in J \\ k \in \{1, \dots, N\} \setminus J}} f(x_j - x_k), \quad (2.15)$$

$$(\mathcal{H}_i^-)_{A_{N-1}} = \sum_{\substack{J \subset \{1, \dots, N\} \\ |J|=i}} \exp\left(\sum_{j \in J} -p_j\right) \prod_{\substack{j \in J \\ k \in \{1, \dots, N\} \setminus J}} g(x_j - x_k). \quad (2.16)$$

Define

$$(\mathcal{H}_i)_{A_{N-1}} = (\mathcal{H}_i^+)_{A_{N-1}} + (\mathcal{H}_i^-)_{A_{N-1}}, \quad (2.17)$$

from the fundamental Poisson bracket Eq. (2.7), we can verify that

$$\{(\mathcal{H}_i)_{A_{N-1}}, (\mathcal{H}_j)_{A_{N-1}}\} = \{(\mathcal{H}_i^\varepsilon)_{A_{N-1}}, (\mathcal{H}_j^{\varepsilon'})_{A_{N-1}}\} = 0, \quad \varepsilon, \varepsilon' = \pm, \quad i, j = 1, \dots, N. \quad (2.18)$$

In particular, the Hamiltonian Eq. (2.2) can be rewritten as

$$\begin{aligned} \mathcal{H}_{A_{N-1}} &\equiv \mathcal{H}_1 = (\mathcal{H}_1^+)_{A_{N-1}} + (\mathcal{H}_1^-)_{A_{N-1}} = \sum_{j=1}^N (e^{p_j} b_j + e^{-p_j} b_j') \\ &= Tr(L(\lambda) + \frac{\sigma(\lambda)\sigma(\lambda + N\gamma)}{\sigma(\gamma + \lambda)\sigma(\lambda + (N-1)\gamma)} L(\lambda)^{-1}) \end{aligned} \quad (2.19)$$

It should be remarked the set of integrals of motion Eq. (2.17) have a reflection symmetry which is the key property for the later reduction to  $C_n$  and  $BC_n$  cases. i.e. if we set

$$p_i \longleftrightarrow -p_i, \quad x_i \longleftrightarrow -x_i, \quad (2.20)$$

then the Hamiltonians flows  $(\mathcal{H}_i)_{A_{N-1}}$  are invariant with respect to this symmetry.

The canonical equations of motion associated with the Hamiltonian flows  $\mathcal{H}_1^+$  in its generic (elliptic) form read

$$\ddot{x}_i = \sum_{j \neq i} \dot{x}_i \dot{x}_j (V(x_{ij}) - V(x_{ji})), \quad i = 1, \dots, N, \quad (2.21)$$

where the potential  $V(x)$  is given by

$$V(x) = \zeta(x) - \zeta(x + \lambda), \quad (2.22)$$

in which  $\zeta(x) = \frac{\sigma'(x)}{\sigma(x)}$ . Here,  $x_i = x_i(t)$ ,  $p_i = p_i(t)$  and the superimposed dot denotes  $t$ -differentiation.

## 2.2 The construction of Lax pair for the $A_{N-1}$ RS model

As for the  $A_{N-1}$  RS model, a generalized Lax pair has been given in Refs. [3, 7] and [26]-[29]. But there is a common character that the time-evolution of the Lax matrix  $L_{A_{N-1}}$  is associated with the Hamiltonian  $(\mathcal{H}_1^+)_{A_{N-1}}$ . We will see in section 3 that the Lax pair cannot reduce from that kind of forms directly. Instead, we give a new Lax pair in which the evolution of  $L_{A_{N-1}}$  is associated with the Hamiltonian  $\mathcal{H}_{A_{N-1}}$ ,

$$\dot{L}_{A_{N-1}} = \{L_{A_{N-1}}, \mathcal{H}_{A_{N-1}}\} = [M_{A_{N-1}}, L_{A_{N-1}}], \quad (2.23)$$

where  $M_{A_{N-1}}$  can be constructed with the help of  $(r, s)$  matrices as follows

$$M_{A_{N-1}} = \text{Tr}_2((s_1 - a_2)(1 \otimes (L(\lambda) - \frac{\sigma(\lambda)\sigma(\lambda + N\gamma)}{\sigma(\gamma + \lambda)\sigma(\lambda + (N-1)\gamma)}L(\lambda)^{-1}))), \quad (2.24)$$

The explicit expression of entries for  $M_{A_{N-1}}$  is

$$M_{ij} = \Phi(x_{ij}, \lambda)e^{p_j}b_j - \Phi(x_{ij}, \lambda + N\gamma)e^{-p_i}b'_j, \quad i \neq j, \quad (2.25)$$

$$M_{ii} = \left(\zeta(\lambda) + \zeta(\gamma)\right)e^{p_i}b_i - \left(\zeta(\lambda + \gamma) - \zeta(\gamma)\right)e^{-p_i}b'_i \quad (2.26)$$

$$+ \sum_{j \neq i} \left( \left( \zeta(x_{ij} + \gamma) - \zeta(x_{ij}) \right) e^{p_j} b_j + \frac{\Phi(x_{ji} + \gamma, \lambda)}{\Phi(\gamma, \lambda)} \Phi(x_{ij}, \lambda + N\gamma) e^{-p_i} b'_j \right). \quad (2.27)$$

### 3 Hamiltonian Reduction of $C_n$ and $BC_n$ RS Models from $A_{N-1}$ -type Ones

Let us first mention some results about the integrability of Hamiltonian (2.2). In Ref. [8] Ruijsenaars demonstrated that the symplectic structure of the  $C_n$ - and  $BC_n$ -type RS systems can be proved integrable by embedding their phase space to a submanifold of the  $A_{2n-1}$  and  $A_{2n}$  type RS ones, respectively, while in Refs. [23, 24] and [22], Diejen and Komori, respectively, gave a series of commuting difference operators which led to their quantum integrability. However, there are not any results about their Lax representations so far except for the special degenerate case [30, 1]. In this section, we concentrate our treatment on the exhibition of the explicit forms for general  $C_n$  and  $BC_n$  RS systems.

For the convenience of analysis of symmetry, let us first give vector representation of  $A_{N-1}$  Lie algebra. Introducing an  $N$  dimensional orthonormal basis of  $\mathbb{R}^N$ ,

$$e_j \cdot e_k = \delta_{j,k}, \quad j, k = 1, \dots, N. \quad (3.1)$$

Then the sets of roots and vector weights of  $A_{N-1}$  type are:

$$\Delta = \{e_j - e_k : j, k = 1, \dots, N\}, \quad (3.2)$$

$$\Lambda = \{e_j : j = 1, \dots, N\}. \quad (3.3)$$

The dynamical variables are canonical coordinates  $\{x_j\}$  and their canonical conjugate momenta  $\{p_j\}$  with the Poisson brackets of Eq. (2.1). In a general sense, we denote them by  $N$  dimensional vectors  $x$  and  $p$ ,

$$x = (x_1, \dots, x_N) \in \mathbb{R}^N, \quad p = (p_1, \dots, p_N) \in \mathbb{R}^N,$$

so that the scalar products of  $x$  and  $p$  with the roots  $\alpha \cdot x$ ,  $p \cdot \beta$ , etc. can be defined. The Hamiltonian of Eq. (2.2) can be rewritten as

$$\mathcal{H}_{A_{N-1}} = \sum_{\mu \in \Lambda} \left( \exp(\mu \cdot p) \prod_{\Delta \ni \beta = \mu - \nu} f(\beta \cdot x) + \exp(-\mu \cdot p) \prod_{\Delta \ni \beta = -\mu + \nu} g(\beta \cdot x) \right), \quad (3.4)$$

Here, the condition  $\Delta \ni \beta = \mu - \nu$  means that the summation is over roots  $\beta$  such that for  $\exists \nu \in \Lambda$

$$\mu - \nu = \beta \in \Delta. \quad (3.5)$$

So does for  $\Delta \ni \beta = -\mu + \nu$ .

#### 3.1 The $C_n$ model

The set of  $C_n$  roots consists of two parts, long roots and short roots:

$$\Delta_{C_n} = \Delta_L \cup \Delta_S, \quad (3.6)$$

in which the roots are conveniently expressed in terms of an orthonormal basis of  $\mathbb{R}^n$ :

$$\begin{aligned}\Delta_L &= \{\pm 2e_j : j = 1, \dots, n\}, \\ \Delta_S &= \{\pm e_j \pm e_k : j, k = 1, \dots, n\}.\end{aligned}\quad (3.7)$$

In the vector representation, vector weights  $\Lambda$  are

$$\Lambda_{C_n} = \{e_j, -e_j : j = 1, \dots, n\}.\quad (3.8)$$

The Hamiltonian of the  $C_n$  model is given by

$$\mathcal{H}_{C_n} = \frac{1}{2} \sum_{\mu \in \Lambda_{C_n}} \left( \exp(\mu \cdot p) \prod_{\Delta_{C_n} \ni \beta = \mu - \nu} f(\beta \cdot x) + \exp(-\mu \cdot p) \prod_{\Delta_{C_n} \ni \beta = -\mu + \nu} g(\beta \cdot x) \right). \quad (3.9)$$

From the above-mentioned data, we notice that either for  $A_{N-1}$  or  $C_n$  Lie algebra, any root  $\alpha \in \Delta$  can be constructed in terms with vector weights as  $\alpha = \mu - \nu$  where  $\mu, \nu \in \Lambda$ . By simple comparison of representation between  $A_{N-1}$  or  $C_n$ , one can find that if replacing  $e_{j+n}$  with  $-e_j$  in the vector weights of  $A_{2n-1}$  algebra, we can obtain the vector weights of  $C_n$  one. This also holds for the corresponding roots. This gives us a hint that it is possible to get the  $C_n$  model by this kind of reduction.

For the  $A_{2n-1}$  model let us set restrictions on the vector weights with

$$e_{j+n} + e_j = 0, \quad \text{for } j = 1, \dots, n, \quad (3.10)$$

which correspond to the following constraints on the phase space of the  $A_{2n-1}$ -type RS model with

$$\begin{aligned}G_i &\equiv (e_{i+n} + e_i) \cdot x = x_i + x_{i+n} = 0, \\ G_{i+n} &\equiv (e_{i+n} + e_i) \cdot p = p_i + p_{i+n} = 0, \quad i = 1, \dots, n,\end{aligned}\quad (3.11)$$

Following Dirac's method[36], we can show

$$\{G_i, \mathcal{H}_{A_{2n-1}}\} \simeq 0, \quad \text{for } \forall i \in \{1, \dots, 2n\}, \quad (3.12)$$

i.e.  $\mathcal{H}_{A_{2n-1}}$  is the first class Hamiltonian corresponding to the constraints in Eq. (3.11). Here the “weak equal” symbol  $\simeq$  represents that, only after calculating the result of left-hand side of the identity, could we use the conditions of constraints. It should be pointed out that the most necessary condition ensuring Eq. (3.12) is the symmetry property of formula (2.20) for the Hamiltonian Eq. (2.2). So for an arbitrary dynamical variable  $A$ , we have

$$\begin{aligned}\dot{A} &= \{A, \mathcal{H}_{A_{2n-1}}\}_D = \{A, \mathcal{H}_{A_{2n-1}}\} - \{A, G_i\} \Delta_{ij}^{-1} \{G_j, \mathcal{H}_{A_{2n-1}}\} \\ &\simeq \{A, \mathcal{H}_{A_{2n-1}}\}, \quad i, j = 1, \dots, 2n,\end{aligned}\quad (3.13)$$



where

$$\Delta_{ij} = \{G_i, G_j\} = 2 \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}, \quad (3.14)$$

and  $\{, \}_D$  denote the Dirac bracket. By straightforward calculation, we have the nonzero Dirac brackets of

$$\begin{aligned} \{x_i, p_j\}_D &= \{x_{i+n}, p_{j+n}\}_D = \frac{1}{2}\delta_{i,j}, \\ \{x_i, p_{j+n}\}_D &= \{x_{i+n}, p_j\}_D = -\frac{1}{2}\delta_{i,j}. \end{aligned} \quad (3.15)$$

Using the above-mentioned data together with the fact that  $\mathcal{H}_{A_{N-1}}$  is the first class Hamiltonian[see Eq. (3.12)], we can directly obtain a Lax representation of the  $C_n$  RS model by imposing constraints  $G_k$  on Eq. (2.23)

$$\begin{aligned} \{L_{A_{2n-1}}, \mathcal{H}_{A_{2n-1}}\}_D &= \{L_{A_{2n-1}}, \mathcal{H}_{A_{2n-1}}\}|_{G_k, k=1, \dots, 2n}, \\ &= [M_{A_{2n-1}}, L_{A_{2n-1}}]|_{G_k, k=1, \dots, 2n} = [M_{C_n}, L_{C_n}], \end{aligned} \quad (3.16)$$

$$\{L_{A_{2n-1}}, \mathcal{H}_{A_{2n-1}}\}_D = \{L_{C_n}, \mathcal{H}_{C_n}\}, \quad (3.17)$$

where

$$\begin{aligned} \mathcal{H}_{C_n} &= \frac{1}{2}\mathcal{H}_{A_{2n-1}}|_{G_k, k=1, \dots, 2n}, \\ L_{C_n} &= L_{A_{2n-1}}|_{G_k, k=1, \dots, 2n}, \\ M_{C_n} &= M_{A_{2n-1}}|_{G_k, k=1, \dots, 2n}, \end{aligned} \quad (3.18)$$

so that

$$\dot{L}_{C_n} = \{L_{C_n}, \mathcal{H}_{C_n}\} = [M_{C_n}, L_{C_n}]. \quad (3.19)$$

Nevertheless, the  $(\mathcal{H}_1^+)_{A_{N-1}}$  is not the first class Hamiltonian, so the Lax pair given by many authors previously cannot reduce to the  $C_n$  case directly in this way.

### 3.2 The $BC_n$ model

The  $BC_n$  root system consists of three parts: long, middle and short roots:

$$\Delta_{BC_n} = \Delta_L \cup \Delta \cup \Delta_S, \quad (3.20)$$

in which the roots are conveniently expressed in terms of an orthonormal basis of  $\mathbb{R}^n$ :

$$\begin{aligned} \Delta_L &= \{\pm 2e_j : j = 1, \dots, n\}, \\ \Delta &= \{\pm e_j \pm e_k : j, k = 1, \dots, n\}, \\ \Delta_S &= \{\pm e_j : j = 1, \dots, n\}. \end{aligned} \quad (3.21)$$

In the vector representation, vector weights  $\Lambda$  can be

$$\Lambda_{BC_n} = \{e_j, -e_j, 0 : j = 1, \dots, n\}. \quad (3.22)$$

The Hamiltonian of  $BC_n$  model is given by

$$\mathcal{H}_{BC_n} = \frac{1}{2} \sum_{\mu \in \Lambda_{BC_n}} \left( \exp(\mu \cdot p) \prod_{\Delta_{BC_n} \ni \beta = \mu - \nu} f(\beta \cdot x) + \exp(-\mu \cdot p) \prod_{\Delta_{BC_n} \ni \beta = -\mu + \nu} g(\beta \cdot x) \right). \quad (3.23)$$

By similar comparison of representations between  $A_{N-1}$  or  $BC_n$ , one can find that if replacing  $e_{j+n}$  with  $-e_j$  and  $e_{2n+1}$  with 0 in the vector weights of the  $A_{2n}$  Lie algebra, we can obtain the vector weights of the  $BC_n$  one. The same holds for the corresponding roots. So by the same procedure as  $C_n$  model, we could get the Lax representation of the  $BC_n$  model.

For the  $A_{2n}$  model, we set restrictions on the vector weights with

$$\begin{aligned} e_{j+n} + e_j &= 0, \quad \text{for } j = 1, \dots, n, \\ e_{2n+1} &= 0, \end{aligned} \quad (3.24)$$

which correspond to the following constraints on the phase space of  $A_{2n}$ -type RS model with

$$\begin{aligned} G'_i &\equiv (e_{i+n} + e_i) \cdot x = x_i + x_{i+n} = 0, \\ G'_{i+n} &\equiv (e_{i+n} + e_i) \cdot p = p_i + p_{i+n} = 0, \quad i = 1, \dots, n, \\ G'_{2n+1} &\equiv e_{2n+1} \cdot x = x_{2n+1} = 0, \\ G'_{2n+2} &\equiv e_{2n+1} \cdot p = p_{2n+1} = 0. \end{aligned} \quad (3.25)$$

Similarly, we can show

$$\{G_i, \mathcal{H}_{A_{2n}}\} \simeq 0, \quad \text{for } \forall i \in \{1, \dots, 2n+1, 2n+2\}. \quad (3.26)$$

i.e.  $\mathcal{H}_{A_{2n}}$  is the first class Hamiltonian corresponding to the above-mentioned constraints Eq. (3.25). So  $L_{BC_n}$  and  $M_{BC_n}$  can be constructed as follows

$$\begin{aligned} L_{BC_n} &= L_{A_{2n}}|_{G'_k, k=1, \dots, 2n+2}, \\ M_{BC_n} &= M_{A_{2n}}|_{G'_k, k=1, \dots, 2n+2}, \end{aligned} \quad (3.27)$$

while  $\mathcal{H}_{BC_n}$  is

$$\mathcal{H}_{BC_n} = \frac{1}{2} \mathcal{H}_{A_{2n}}|_{G_k, k=1, \dots, 2n+2}, \quad (3.28)$$

due to the similar derivation of Eqs. (3.13-3.19).

## 4 Lax Representations of the $C_n$ and $BC_n$ RS Models

### 4.1 The $C_n$ model

The Hamiltonian of the  $C_n$  RS system is Eq. (3.9), so the canonical equations of motion are

$$\dot{x}_i = \{x_i, \mathcal{H}\} = e^{p_i} b_i - e^{-p_i} b'_i, \quad (4.1)$$

$$\begin{aligned} \dot{p}_i = \{p_i, \mathcal{H}\} = & \sum_{j \neq i}^n \left( e^{p_j} b_j (h(x_{ji}) - h(x_j + x_i)) \right. \\ & + e^{-p_j} b'_j (\hat{h}(x_{ji}) - \hat{h}(x_j + x_i)) \Big) \\ & - e^{p_i} b_i \left( 2h(2x_i) + \sum_{j \neq i}^n (h(x_{ij}) + h(x_i + x_j)) \right) \\ & - e^{-p_i} b'_i \left( 2\hat{h}(2x_i) + \sum_{j \neq i}^n (\hat{h}(x_{ij}) + \hat{h}(x_i + x_j)) \right), \end{aligned} \quad (4.2)$$

where

$$\begin{aligned} h(x) &:= \frac{d \ln f(x)}{dx}, & \hat{h}(x) &:= \frac{d \ln g(x)}{dx}, \\ b_i &= f(2x_i) \prod_{k \neq i}^n \left( f(x_i - x_k) f(x_i + x_k) \right), \\ b'_i &= g(2x_i) \prod_{k \neq i}^n \left( g(x_i - x_k) g(x_i + x_k) \right). \end{aligned} \quad (4.3)$$

The Lax matrix for the  $C_n$  RS model can be written in the following form

$$(L_{C_n})_{\mu\nu} = e^{\nu p} b_\nu \frac{\Phi((\mu - \nu) \cdot x + \gamma, \lambda)}{\Phi(\gamma, \lambda)}, \quad (4.4)$$

which is a  $2n \times 2n$  matrix whose indices are labelled by the vector weights, denoted by  $\mu, \nu \in \Lambda_{C_n}$ ,  $M_{C_n}$  can be written as

$$M_{C_n} = D + Y, \quad (4.5)$$

where  $D$  denotes the diagonal part and  $Y$  denotes the off-diagonal part

$$\begin{aligned} Y_{\mu\nu} &= e^{\nu p} b_\nu \Phi(x_{\mu\nu}, \lambda) \\ &\quad + e^{-\mu p} b'_\nu \Phi(x_{\mu\nu}, \lambda + N\gamma), \\ D_{\mu\mu} &= \left( \zeta(\lambda) + \zeta(\gamma) \right) e^{\mu p} b_\mu - \left( \zeta(\lambda + \gamma) - \zeta(\gamma) \right) e^{-\mu p} b'_\mu \end{aligned} \quad (4.6)$$

$$\begin{aligned}
& + \sum_{\nu \neq \mu} \left( \left( \zeta(x_{\mu\nu} + \gamma) - \zeta(x_{\mu\nu}) \right) e^{\nu \cdot p} b_\nu \right. \\
& \left. + \frac{\Phi(x_{\nu\mu} + \gamma, \lambda)}{\Phi(\gamma, \lambda)} \Phi(x_{\mu\nu}, \lambda + N\gamma) e^{-\mu \cdot p} b'_\nu \right)
\end{aligned} \tag{4.7}$$

and

$$\begin{aligned}
b_\mu &= \prod_{\Delta_{C_n} \ni \beta = \mu - \nu} f(\beta \cdot x), \\
b'_\mu &= \prod_{\Delta_{C_n} \ni \beta = \mu - \nu} g(\beta \cdot x), \\
x_{\mu\nu} &: = (\mu - \nu) \cdot x.
\end{aligned} \tag{4.8}$$

The  $L_{C_n}, M_{C_n}$  satisfies the Lax equation

$$\dot{L}_{C_n} = \{L_{C_n}, \mathcal{H}_{C_n}\} = [M_{C_n}, L_{C_n}], \tag{4.9}$$

which equivalent to the equations of motion Eq. (4.1) and Eq. (4.2). The Hamiltonian  $\mathcal{H}_{C_n}$  can be rewritten as the trace of  $L_{C_n}$

$$\mathcal{H}_{C_n} = \text{tr} L_{C_n} = \frac{1}{2} \sum_{\mu \in \Lambda_{C_n}} (e^{\mu \cdot p} b_\mu + e^{-\mu \cdot p} b'_\mu). \tag{4.10}$$

## 4.2 The $BC_n$ model

The Hamiltonian of the  $BC_n$  model is expressed in Eq. (3.23), so the canonical equations of motion are

$$\dot{x}_i = \{x_i, \mathcal{H}\} = e^{p_i} b_i - e^{-p_i} b'_i, \tag{4.11}$$

$$\begin{aligned}
\dot{p}_i &= \{p_i, \mathcal{H}\} = \sum_{j \neq i}^n \left( e^{p_j} b_j (h(x_{ji}) - h(x_j + x_i)) \right. \\
&\quad \left. + e^{-p_j} b'_j (\hat{h}(x_{ji}) - \hat{h}(x_j + x_i)) \right) \\
&\quad - e^{p_i} b_i \left( h(x_i) + 2h(2x_i) + \sum_{j \neq i}^n (h(x_{ij}) + h(x_i + x_j)) \right) \\
&\quad - e^{-p_i} b'_i \left( \hat{h}(x_i) + 2\hat{h}(2x_i) + \sum_{j \neq i}^n (\hat{h}(x_{ij}) + \hat{h}(x_i + x_j)) \right) \\
&\quad - b_0 (h(x_i) + \hat{h}(x_i)),
\end{aligned} \tag{4.12}$$

where

$$\begin{aligned}
b_i &= f(x_i)f(2x_i) \prod_{k \neq i}^n \left( f(x_i - x_k)f(x_i + x_k) \right), \\
b'_i &= g(x_i)g(2x_i) \prod_{k \neq i}^n \left( g(x_i - x_k)g(x_i + x_k) \right), \\
b_0 &= \prod_{i=1}^n f(x_i)g(x_i).
\end{aligned} \tag{4.13}$$

The Lax pair for the  $BC_n$  RS model can be constructed as the form of Eq. (4.4)-(4.8) where one should replace the matrices labels with  $\mu, \nu \in \Lambda_{BC_n}$ , and roots with  $\beta \in \Delta_{BC_n}$ .

The Hamiltonian  $\mathcal{H}_{BC_n}$  can also be rewritten as the trace of  $L_{BC_n}$

$$\mathcal{H}_{BC_n} = \text{tr} L_{BC_n} = \frac{1}{2} \sum_{\mu \in \Lambda_{BC_n}} (e^{\mu \cdot p} b_\mu + e^{-\mu \cdot p} b'_\mu). \tag{4.14}$$

## 5 Spectral Curves of the $C_n$ and $BC_n$ RS Systems

It is recently pointed out in [5, 6, 37, 38] that the  $SU(N)$  RS model is related to five dimensional gauge theories. In the context of Seiberg-Witten theory, the elliptic RS integrable system can be linked with the relevant low energy effective action when a compactification from five dimension to four dimension is imposed with all of the fields belong to the adjoint representation of  $SU(N)$  gauge group[6]. More evidence for this correspondence between SYM and RS model is depicted by calculating instanton correction of prepotential for  $SU(2)$  Seiberg-Witten theory in [32].

As for the spectral curve and its relation with the Seiberg-Witten spectral curve, much progress have been made in the correspondence of “Calogero-Moser integrable theories and gauge theories”. See recent review in [39], [40] and references therein. In this section we will give the spectral curves for  $C_n$  and  $BC_n$  systems, which are shown to be parameterized by the integrals of motion of the corresponding system. We will also see that the elliptic Calogero-Moser, Toda (affine and non-affine) ones are particular limits of these systems.

### 5.1 Spectral curve of the $C_n$ RS system

Given the Lax operator with spectral parameter for the Calogero-Moser system and of the RS system associated with Lie algebras  $\mathcal{G}$ , the spectral curve for the given system is defined as

$$\Gamma : R(v, \lambda) = \det(L(\lambda) - v \cdot Id) \equiv 0. \tag{5.1}$$

It is naturally that the function  $R(v, z)$  is invariant under time evolution,

$$\frac{d}{dt}R(v, \lambda) = \{\mathcal{H}, R(v, \lambda)\} = 0. \quad (5.2)$$

Thus,  $R(v, \lambda)$  must be a function of only the  $n$  independent integrals of motion, which in super-Yang-Mills theory play the role of *moduli*, parameterizing the supersymmetric vacua of the gauge theory. This has been confirmed in the case of the elliptic Calogero-Moser system for general Lie algebra in Refs. [41, 42] and in the case of the elliptic  $SU(N)$  RS system for the perturbative limit and some nonperturbative special point[6].

As for the  $C_n$  RS system, the spectral curve of can be generated by Lax matrix  $L(\lambda)_{C_n}$  as follows

$$\det(L(\lambda)_{C_n} - v \cdot Id) = \sum_{j=0}^{2n} \frac{(\sigma(\lambda))^{(j-1)} \sigma(\lambda + j\gamma)}{(\sigma(\gamma + \lambda))^j} (-v)^{2n-j} (H_j)_{C_n} = 0, \quad (5.3)$$

where  $(\mathcal{H}_0)_{C_n} = (\mathcal{H}_{2n})_{C_n} = 1$  and  $(\mathcal{H}_i)_{C_n} = (\mathcal{H}_{2n-i})_{C_n}$  who Poisson commute

$$\{(\mathcal{H}_i)_{C_n}, (\mathcal{H}_j)_{C_n}\} = 0, \quad i, j = 1, \dots, n. \quad (5.4)$$

This can be deduced by verbose but straightforward calculation to verify that the  $(\mathcal{H}_i)_{A_{2n-1}}$ ,  $i = 1, \dots, 2n$  is the first class Hamiltonian with respect to the constraints Eq. (3.11), using Eq. (2.18), (3.13) and the first formula of Eq. (3.18).

The explicit form of  $(\mathcal{H}_l)_{C_n}$  are

$$(\mathcal{H}_l)_{C_n} = \sum_{\substack{J \subset \{1, \dots, n\}, |J| \leq l \\ \varepsilon_j = \pm 1, j \in J}} \exp(p_{\varepsilon J}) F_{\varepsilon J; J^c} U_{J^c, l-|J|}, \quad l = 1, \dots, n, \quad (5.5)$$

with

$$\begin{aligned} p_{\varepsilon J} &= \sum_{j \in J} \varepsilon_j p_j, \\ F_{\varepsilon J; K} &= \prod_{\substack{j, j' \in J \\ j < j'}} f^2(\varepsilon_j x_j + \varepsilon_{j'} x_{j'}) \prod_{\substack{j \in J \\ k \in K}} f(\varepsilon_j x_j + x_k) f(\varepsilon_j x_j - x_k) \prod_{j \in J} f(2\varepsilon_j x_j), \\ U_{I, p} &= \sum_{\substack{I' \subset I \\ |I'| = [p/2]}} \prod_{\substack{j \in I' \\ k \in I \setminus I'}} f(x_{jk}) f(x_j + x_k) g(x_{jk}) g(x_j + x_k) \begin{cases} 0, & (p \text{ odd}) \\ 1, & (p \text{ even}) \end{cases}. \end{aligned} \quad (5.6)$$

Here,  $[p/2]$  denotes the integer part of  $p/2$ . As an example, for  $C_2$  RS model, the independent Hamiltonian flows  $(\mathcal{H}_1)_{C_2}$  and  $(\mathcal{H}_2)_{C_2}$  generated by the Lax matrix  $L_{C_2}$  are[30]

$$(\mathcal{H}_1)_{C_2} = \mathcal{H}_{C_2} = e^{p_1} f(2x_1) f(x_{12}) f(x_1 + x_2)$$

$$\begin{aligned}
& +e^{-p_1}g(2x_1)g(x_{12})g(x_1+x_2) \\
& +e^{p_2}f(2x_2)f(x_{21})f(x_2+x_1) \\
& +e^{-p_2}g(2x_2)g(x_{21})g(x_2+x_1), \tag{5.7}
\end{aligned}$$

$$\begin{aligned}
(\mathcal{H}_2)_{C_2} = & e^{p_1+p_2}f(2x_1)(f(x_1+x_2))^2f(2x_2) \\
& +e^{-p_1-p_2}g(2x_1)(g(x_1+x_2))^2g(2x_2) \\
& +e^{p_1-p_2}f(2x_1)(f(x_{12}))^2f(-2x_2) \\
& +e^{p_2-p_1}g(2x_1)(g(x_{12}))^2g(-2x_2) \\
& +2f(x_{12})g(x_{12})f(x_1+x_2)g(x_1+x_2). \tag{5.8}
\end{aligned}$$

## 5.2 Spectral curve of the $BC_n$ model

Similar to the calculation of the  $C_n$  case, the spectral curve of the  $BC_n$  RS system can be generated by Lax matrix  $L(\lambda)_{BC_n}$  as follows

$$\det(L(\lambda)_{BC_n} - v \cdot Id) = 0. \tag{5.9}$$

The explicit form of the spectral curve is

$$\det(L(\lambda)_{BC_n} - v \cdot Id) = \sum_{j=0}^{2n+1} \frac{(\sigma(\lambda))^{(j-1)} \sigma(\lambda + j\gamma)}{(\sigma(\gamma + \lambda))^j} (-v)^{2n+1-j} (\mathcal{H}_j)_{BC_n} = 0,$$

where  $(\mathcal{H}_0)_{BC_n} = (\mathcal{H}_{2n})_{BC_n} = 1$  and  $(\mathcal{H}_i)_{BC_n} = (\mathcal{H}_{2n+1-i})_{BC_n}$  Poisson commute

$$\{(\mathcal{H}_i)_{BC_n}, (\mathcal{H}_j)_{BC_n}\} = 0, \quad \forall i, j \in \{1, \dots, n\}. \tag{5.10}$$

This can be deduced similarly to the  $C_n$  case to verify that  $(\mathcal{H}_i)_{A_{2n}}, i = 1, \dots, 2n$  is the first class Hamiltonian with respect to the constraints Eq. (3.25).

The explicit forms of  $(\mathcal{H}_l)_{BC_n}$  are

$$(\mathcal{H}_l)_{BC_n} = \sum_{\substack{J \subset \{1, \dots, n\}, |J| \leq l \\ \varepsilon_j = \pm 1, j \in J}} \exp(p_{\varepsilon J}) F_{\varepsilon J; J^c} U_{J^c, l-|J|}, \quad l = 1, \dots, n, \tag{5.11}$$

with

$$\begin{aligned}
p_{\varepsilon J} &= \sum_{j \in J} \varepsilon_j p_j, \\
F_{\varepsilon J; K} &= \prod_{\substack{j, j' \in J \\ j < j'}} f^2(\varepsilon_j x_j + \varepsilon_{j'} x_{j'}) \prod_{\substack{j \in J \\ k \in K}} f(\varepsilon_j x_j + x_k) f(\varepsilon_j x_j - x_k) \prod_{j \in J} f(2\varepsilon_j x_j) \prod_{j \in J} f(\varepsilon_j x_j), \tag{5.12} \\
U_{I, p} &= \sum_{\substack{I' \subset I \\ |I'| = [p/2]}} \prod_{\substack{j \in I' \\ k \in I \setminus I'}} f(x_{jk}) f(x_j + x_k) g(x_{jk}) g(x_j + x_k) \begin{cases} \prod_{i \in I \setminus I'} f(x_i) g(x_i), & (p \text{ odd}) \\ \prod_{i' \in I'} f(x_{i'}) g(x_{i'}), & (p \text{ even}) \end{cases}.
\end{aligned}$$

### 5.3 Limit to the Calogero-Moser system and Toda system

The Calogero-Moser system can be achieved by taking so-called “the nonrelativistic limit”. The procedure is by rescaling  $p_\mu \mapsto \beta p_\mu$ ,  $\gamma \mapsto \beta \gamma$  and letting  $\beta \mapsto 0$ , followed by making a canonical transformation

$$p_\mu \mapsto p_\mu + \gamma \sum_{\Delta \ni \eta = \mu - \nu} \zeta(\eta \cdot x), \quad (5.13)$$

here  $p_\mu = \mu \cdot p$ , such that

$$L \mapsto Id + \beta L_{CM} + O(\beta^2), \quad (5.14)$$

and

$$\mathcal{H} \mapsto N + 2\beta^2 \mathcal{H}_{CM} + O(\beta^2). \quad (5.15)$$

where  $N = 2n$  for  $C_n$  model and  $N = 2n + 1$  for  $BC_n$  model.

$L_{CM}$  can be expressed as

$$L_{CM} = p \cdot H + X, \quad (5.16)$$

where

$$H_{\mu\nu} = \mu \delta_{\mu\nu}, X_{\mu\nu} = \gamma \Phi(x_{\mu\nu}, \lambda)(1 - \delta_{\mu\nu}). \quad (5.17)$$

The Hamiltonian  $\mathcal{H}_{CM}$  of  $CM$  model can be given by

$$\mathcal{H}_{CM} = \frac{1}{2}p^2 - \frac{\gamma^2}{2} \sum_{\alpha \in \Delta} \wp(\alpha \cdot x) = \frac{1}{4}tr L^2 + const, \quad (5.18)$$

where  $const = -\frac{N(N-1)\gamma^2}{4}\wp(\lambda)$ .

All of the above results are identified with the results of Refs. [9, 11, 13, 14, 15] up to a suitable choice of coupling parameters.

Now the degenerate RS spectral curve reduce to

$$\Gamma : R(v, \lambda) = \det(L(\lambda)_{CM} - v \cdot Id) \equiv 0, \quad (5.19)$$

which exactly identified with the spectral curve analyzed in [39, 41].

Starting from the CM system to the Toda system is more directly due to the progress that the limit to Toda for the general Lie algebra has been studied extensively in [43, 44, 45]. The main idea is making suitable scaling limit with the following parameterization

$$\omega_1 = -i\pi, \quad \omega_3 \in \mathbf{R}_+, \quad \tau \equiv \frac{\omega_3}{\omega_1} = i\omega_3/\pi, \quad (5.20)$$

and shift the dynamical variable  $x$

$$\begin{aligned} x &\rightarrow Q - 2\omega_3 \delta \rho^\vee, & p &\rightarrow P, \\ \lambda &\rightarrow \log Z - \omega_3, & Z &\in \mathbf{R}_+, \end{aligned} \quad (5.21)$$



in which  $h_{\mathcal{G}}$  is the Coxeter number for the corresponding root system  $\mathcal{G}$ ,  $\rho^\vee$  the dual of the Weyl vector defined as  $\rho^\vee = \frac{1}{2} \sum_{\alpha \in \Delta_+} 2\alpha/\alpha^2$  and  $\delta$  satisfies  $\delta \leq 1/h_{\mathcal{G}}$ .

For convenience, we give the basics of these root system as showed in Table 1.

$\mathcal{G}$	all roots	simple roots $\Pi$	$h_{\mathcal{G}}$	dual Weyl vector $\rho^\vee$	vector weights
$A_{n-1}$	$\pm e_i \pm e_j,$ $1 \leq i, j \leq n,$ $i \neq j$	$e_i - e_{i+1},$ $i = 1, \dots, n-1$	$n$	$\sum_{j=1}^n (n-j)e_j$	$e_i,$ $i = 1, \dots, n$
$C_n$	$\pm e_i \pm e_j, \pm 2e_i,$ $1 \leq i, j \leq n,$ $i \neq j$	$e_i - e_{i+1}, 2e_n,$ $i = 1, \dots, n-1$	$2n$	$\sum_{j=1}^n (n + \frac{1}{2} - j)e_j$	$e_i, -e_i,$ $i = 1, \dots, n$
$BC_n$	$\pm e_i \pm e_j, \pm 2e_i, \pm e_i$ $1 \leq i, j \leq n,$ $i \neq j$	$e_i - e_{i+1}, e_n,$ $i = 1, \dots, n-1$	$2n+1$	$\sum_{j=1}^n (n+1-j)e_j$ (define)	$e_i, -e_i, 0,$ $i = 1, \dots, n$

Table 1: Root system of  $A_{n-1}$ ,  $C_n$  and  $BC_n$  types

As for the  $C_n$  model, selecting  $\rho^\vee = \rho_{C_n}^\vee$ ,  $\gamma = im e^{\omega_3 \delta}$ , one have the non-affine  $C_n$  Toda model from the Hamiltonian of the CM model Eq. (5.18)

$$\mathcal{H}_{C_n}^{Toda} = \frac{1}{2}P^2 + m^2 \sum_{j=1}^{n-1} e^{Q_j - Q_{j+1}} + m^2 e^{2Q_n}, \quad (5.22)$$

for  $\delta < 1/h_{C_n}$  and  $C_n^{(1)}$  Toda model

$$\mathcal{H}_{C_n^{(1)}}^{Toda} = \frac{1}{2}P^2 + m^2 e^{-2Q_1} + m^2 \sum_{j=1}^{n-1} e^{Q_j - Q_{j+1}} + m^2 e^{2Q_n}, \quad (5.23)$$

for  $\delta = 1/h_{C_n}$ .

The same holds for the  $BC_n$  model. Selecting  $\rho^\vee = \rho_{BC_n}^\vee$ ,  $\gamma = im e^{\omega_3 \delta}$ , one have the non-affine  $B_n$  Toda model from the Hamiltonian of the CM model Eq. (5.18)

$$\mathcal{H}_{B_n}^{Toda} = \frac{1}{2}P^2 + m^2 \sum_{j=1}^{n-1} e^{Q_j - Q_{j+1}} + m^2 e^{Q_n}, \quad (5.24)$$

for  $\delta < 1/h_{BC_n}$  and  $BC_n$  Toda model

$$\mathcal{H}_{BC_n}^{Toda} = \frac{1}{2}P^2 + m^2 e^{-2Q_1} + m^2 \sum_{j=1}^{n-1} e^{Q_j - Q_{j+1}} + m^2 e^{Q_n}, \quad (5.25)$$

for  $\delta = 1/h_{BC_n}$ .

If we use the following gauge for  $\Phi(x, \lambda)$  [46]

$$\Phi(x, \lambda) \rightarrow \frac{\sigma(x + \lambda)}{\sigma(\lambda)\sigma(x)} \exp(\zeta(\lambda)x), \quad (5.26)$$

which does not destroy the validity [45] for the Lax pair, We have the following limit for  $\gamma\Phi(\alpha \cdot x, \lambda)$

$$\begin{aligned} \gamma\Phi(\alpha \cdot x, \lambda) &\rightarrow -m \exp(\frac{\alpha \cdot Q}{2}) && \text{for } \alpha \in \Pi \quad (\delta \leq 1/h_{\mathcal{G}}), \\ &\rightarrow mZ \exp(-\frac{\alpha \cdot Q}{2}) && \text{for } \alpha = \alpha_h \quad (\delta = 1/h_{\mathcal{G}}), \\ &\rightarrow 0 && \text{otherwise,} \\ \gamma\Phi(-\alpha \cdot x, \lambda) &\rightarrow m \exp(\frac{\alpha \cdot Q}{2}) && \text{for } \alpha \in \Pi \quad (\delta \leq 1/h_{\mathcal{G}}), \\ &\rightarrow -\frac{m}{Z} \exp(-\frac{\alpha \cdot Q}{2}), && \text{for } \alpha = \alpha_h \quad (\delta = 1/h_{\mathcal{G}}), \\ &\rightarrow 0 && \text{otherwise.} \end{aligned} \quad (5.27)$$

So the Lax operator now reads

$$L_{Toda} = P \cdot H - im \sum_{\alpha \in \Pi} \exp(\frac{\alpha \cdot Q}{2}) [E(\alpha) - E(-\alpha)] + im \exp(\frac{\alpha_0 \cdot Q}{2}) [ZE(-\alpha_0) - Z^{-1}E(\alpha_0)], \quad (5.28)$$

where  $E(\alpha)_{\mu\nu} = \delta_{\mu-\nu, \alpha}$ . This Lax operator holds for all the root system of  $A_{n-1}(A_{n-1}^{(1)})$ ,  $C_n(C_n^{(1)})$ ,  $B_n(BC_n)$  and coincide with the standard form given in [9]. It is not difficult to find that the parameter  $Z$  now plays the role of a spectral parameter for the affine Toda model based on  $\mathcal{G}^{(1)}$ . When we refer to the Toda models based on a finite Lie algebra  $\mathcal{G}$ , we should only drop the terms containing the affine root  $\alpha_0$ .

So the degenerate spectral curve for the Toda  $A_{n-1}^{(1)}$ ,  $C_n^{(1)}$  and  $BC_n(A_{2n}^{(2)})$  systems can be defined

$$\Gamma : R(v, \lambda) = \det(L(\lambda)_{Toda} - v \cdot Id) \equiv 0, \quad (5.29)$$

which is identical to the one given in [47, 48].

## 6 Degenerate Cases

Let us now consider the other various special degenerate cases. As is well known, if one or both the periods of Weierstrass sigma function  $\sigma(x)$  become infinite, there will occur three degenerate cases associated with trigonometric, hyperbolic and rational systems. The degenerate limits of the functions  $\Phi(x, \lambda)$ ,  $\sigma(x)$  and  $\zeta(x)$  will give corresponding Lax pairs which include spectral parameter. Moreover, when the spectral parameter value on certain limit, the Lax pairs without spectral parameter will be derived.

## 6.1 Trigonometric limit

The limit can be obtained by sending  $\omega_3$  to  $i\infty$  with  $\omega_1 = \frac{\pi}{2}$ , so that

$$\begin{aligned}\sigma(x) &\rightarrow e^{\frac{1}{6}x^2} \sin x, \\ \zeta(x) &\rightarrow \cot x + \frac{1}{3}x,\end{aligned}\tag{6.1}$$

and the function  $\Phi(x, \lambda) \equiv \frac{\sigma(x+\lambda)}{\sigma(x)\sigma(\lambda)}$  reduce to

$$\Phi(x, \lambda) \rightarrow (\cot \lambda - \cot x)e^{\frac{1}{3}x\lambda}.\tag{6.2}$$

By replacing the corresponding functions  $\Phi(x, \lambda)$ ,  $\sigma(x)$  and  $\zeta(x)$  to the form given above for the Lax pairs, we will get the corresponding spectral parameter dependent Lax pairs. For the simplicity, we notice that the exponential part of the above functions can be removed by applying suitable "gauge" transformation of the Lax matrix on which condition the functions can be valued as follows:

$$\begin{aligned}\sigma(x) &\rightarrow \sin x, \\ \zeta(x) &\rightarrow \cot x, \\ \Phi(x, \lambda) &\rightarrow (\cot \lambda - \cot x).\end{aligned}\tag{6.3}$$

As for the spectral parameter independent Lax pair, furthermore, we can take the limit  $\lambda \rightarrow i\infty$ , so the function

$$\Phi(x, \lambda) \rightarrow \frac{1}{\sin x},\tag{6.4}$$

while the corresponding Lax matrix become to

$$L_{\mu\nu} = e^{\nu p} b_\nu \frac{\sin \gamma}{\sin((\mu - \nu) \cdot x + \gamma)},\tag{6.5}$$

which are exactly the same as the spectral parameter independent Lax matrix given in [1].

## 6.2 Hyperbolic limit

In this case, the periods can be chosen as by sending by sending  $\omega_1$  to  $i\infty$  with  $\omega_3 = \frac{\pi}{2}$ , so following all the procedure in achieving the result of trigonometric case, we can find the hyperbolic Lax pairs by simple replacement of the functions appeared in trigonometric Lax pair as follows:

$$\begin{aligned}\sin x &\rightarrow \sinh x, \\ \cos x &\rightarrow \cosh x, \\ \cot x &\rightarrow \coth x.\end{aligned}\tag{6.6}$$

The same as for the trigonometric case, we can get the Lax pairs with and without spectral parameter.

### 6.3 Rational limit

As far as the form of the Lax pair for the rational-type system is concerned, we can achieve it by making the following substitutions

$$\begin{aligned}\sigma(x) &\rightarrow x, \\ \zeta(x) &\rightarrow \frac{1}{x}, \\ \Phi(x, \lambda) &\rightarrow \frac{1}{x} + \frac{1}{\lambda}.\end{aligned}\tag{6.7}$$

for the spectral parameter dependent Lax pair, while furthermore, taking the limit  $\lambda \rightarrow i\infty$ , we can obtain the spectral parameter independent Lax pair. The explicit form of Lax matrix without spectral parameter is

$$L_{\mu\nu} = e^{\nu \cdot p} b_{\nu} \frac{\gamma}{(\mu - \nu) \cdot x + \gamma}.\tag{6.8}$$

which completely coincide with the spectral parameter independent Lax matrix given in [1].

**Remark:** As for the various degenerate cases for the CM and Toda systems, one can follow the same procedure as for the RS model (please refer to Eq. (6.1)-(6.8)).

## 7 Concluding Remarks

In this paper, we have proposed the Lax pairs for elliptic  $C_n$  and  $BC_n$  RS models. The spectral parameter dependent and independent Lax pairs for the trigonometric, hyperbolic and rational systems can be derived as the degenerate limits of the elliptic potential case. The spectral curves of these systems are given and shown by depicted by the complete sets of involutive constant integrals of motion. They would be related to the integrable system associated with the 5-dimensional gauge theory[5, 6]. In the nonrelativistic limit (scaling limit), the system leads to CM(Toda) systems associated with the root systems of  $C_n$  and  $BC_n$ . There are still many open problems, for example, it seems to be a challenging subject to carry out the Lax pairs with as many independent coupling constants as independent Weyl orbits in the set of roots, as done for the Calogero-Moser systems (see Refs. [9], [12]-[16]). What is also interesting may generalize the results obtained in this paper to the systems associated with all of other Lie Algebras even to those associated with all the finite reflection groups[15]. Moreover, The issue for getting the r-matrix structure for these system is deserved due to the success of calculating for the trigonometric  $BC_n$  RS system by Avan et al. in [31].

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## Appendix

In this appendix we prove the identity Eq. (2.11) and then derive the relation between the Lax operator  $L(\lambda)$  and its inverse of  $L(\lambda)^{-1}$ .

Using the result given in [7] of Eq. (B5), we have the following conclusion:

Let

$$C_{ij} = \frac{\sigma(q_i - r_j + \lambda)}{\sigma(q_i - r_j + \mu)}, \quad i, j = 1, \dots, N, \quad (\text{A.1})$$

then one has

$$\begin{aligned} \det(C) &= \sigma(\lambda - \mu)^{N-1} \sigma(\lambda + (N-1)\mu + \Sigma) \\ &\quad \times \prod_{i < j} \sigma(q_i - q_j) \sigma(r_j - r_i) \prod_{i,j} \frac{1}{\sigma(q_i - r_j + \mu)}, \end{aligned} \quad (\text{A.2})$$

where

$$\Sigma = \sum_{i=1}^N (q_i - r_j). \quad (\text{A.3})$$

So it is straightforward to compute the inverse of matrix  $C$ ,

$$\begin{aligned} (C^{-1})_{ij} &= \text{the cofactor of } C \text{ with respect to } C_{ji} \\ &= \frac{\sigma(\lambda + (N-2)\mu + q_i - q_j)}{\sigma(\lambda - \mu) \sigma(\lambda + (N-1)\mu) \sigma(q_i - q_j - \mu)} \\ &\quad \times \frac{\prod_l \sigma(q_j - q_l + \mu) \prod_l \sigma(q_i - q_l - \mu)}{\prod_{k \neq i} \sigma(q_i - q_k) \prod_{k \neq j} \sigma(q_j - q_k)}. \end{aligned} \quad (\text{A.4})$$

From Eq. (2.5), we have

$$\begin{aligned} L(\lambda) &= \sum_{i,j=1}^N \frac{\Phi(x_{ij} + \gamma, \lambda)}{\Phi(\gamma, \lambda)} \exp(p_j) b_j E_{ij} \\ &= \frac{1}{\Phi(\gamma, \lambda)} \sum_{i,j=1}^N \frac{\sigma(x_{ij} + \gamma + \lambda)}{\sigma(x_{ij} + \gamma) \sigma(\lambda)} \exp(p_j) b_j \\ &= \frac{1}{\Phi(\gamma, \lambda)} \sum_{i,j=1}^N G_{ij} \exp(p_j) b_j, \end{aligned} \quad (\text{A.5})$$

where

$$G_{ij} := \Phi(x_{ij} + \gamma, \lambda) = \frac{\sigma(x_{ij} + \gamma + \lambda)}{\sigma(x_{ij} + \gamma)\sigma(\lambda)},$$

with the help of Eq. (A.4), one has

$$\begin{aligned} (G^{-1})_{ij} &= \frac{\sigma(\lambda + (N-1)\gamma + x_{ij})}{\sigma(\lambda + N\gamma)\sigma(x_{ij} - \gamma)} \\ &\quad \times \frac{\prod_k \sigma(x_{jk} + \gamma) \prod_k \sigma(x_{ik} - \gamma)}{\prod_{k \neq i} \sigma(x_{ik}) \prod_{k \neq j} \sigma(x_{jk})}, \end{aligned} \quad (\text{A.6})$$

So that

$$\begin{aligned} L(\lambda)^{-1}_{ij} &= \Phi(\gamma, \lambda)(G^{-1})_{ij} b_j^{-1} \exp(-p_i) E_{ij} \\ &= \frac{-\sigma(\gamma)^2 \sigma(\lambda + \gamma) \sigma(\lambda + (N-1)\gamma + x_{ij})}{\sigma(\lambda) \sigma(\gamma) \sigma(\lambda + N\gamma) \sigma(x_{ij} - \gamma)} \\ &\quad \times \exp(-p_i) \prod_{k \neq j} \frac{\sigma(x_{jk} + \gamma)}{\sigma(x_{jk})} \\ &= \frac{\sigma(\gamma + \lambda) \sigma(\lambda + (N-1)\gamma)}{\sigma(\lambda) \sigma(\lambda + N\gamma)} \\ &\quad \times \frac{\Phi(x_{ij} - \gamma, \lambda + N\gamma)}{\Phi(-\gamma, \lambda + N\gamma)} \exp(-p_i) b'_j E_{ij}. \end{aligned} \quad (\text{A.7})$$

By comparing the forms of  $L(\lambda)$  and  $L(\lambda)^{-1}_{ij}$ , we find  $L(\lambda)^{-1}_{ij}$  can be expressed with  $L(\lambda)$  as:

$$L(\lambda)^{-1}_{ij} = L(\lambda)_{ij} \big|_{\gamma \rightarrow -\gamma, \lambda \rightarrow \lambda + N\gamma} \times \frac{\sigma(\gamma + \lambda) \sigma(\lambda + (N-1)\gamma)}{\sigma(\lambda) \sigma(\lambda + N\gamma)} \exp(-p_i - p_j). \quad (\text{A.8})$$

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