Conformal Symmetry of Relativistic and Nonrelativistic Systems and AdS/CFT Correspondence

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Abstract

The nonlinear realization of conformal $so(2,d)$ symmetry for relativistic systems and the dynamical conformal $so(2,1)$ symmetry of nonrelativistic systems are investigated in the context of AdS/CFT correspondence. We show that the massless particle in d-dimensional Minkowski space can be treated as the system confined to the border of the AdS_{d+1} of infinite radius, while various nonrelativistic systems may be canonically related to a relativistic (massless, massive, or tachyon) particle on the $AdS_2 \times S^{d-1}$. The list of nonrelativistic systems "unified" by such a correspondence comprises the conformal mechanics model, the planar charge-vortex and 3-dimensional charge-monopole systems, the particle in a planar gravitational field of a point massive source, and the conformal model associated with the charged particle propagating near the horizon of the extreme Reissner-Nordström black hole.

1   Introduction

Being a nontrivial generalization of the Poincaré symmetry, nowadays conformal symmetry is of ever-increasing interest. In the simplest form it appears as a rigid $so(2,d)$ symmetry of massless particles and fields in d-dimensional Minkowski space-time. This symmetry is also enjoyed by many nonrelativistic quantum mechanical [1]-[8] and field theories [9]-[13] in the form of the unexpected $so(2,1)$ symmetry sometimes called dynamical or hidden

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(Schrödinger) symmetry. The recent revival of interest in conformal mechanics is in the context of AdS/CFT correspondence [14, 15, 16] (for review see [17]) and black hole physics [18]-[24].

It is long known that the conformal $so(2,1)$ symmetry of nonrelativistic quantum mechanical systems may be understood as a part of the usual conformal $so(2,d)$ symmetry which survives a tricky nonrelativistic contraction procedure applied to a corresponding relativistic system [25] (see also [8, 12] for the Kaluza-Klein type approach). On the other hand, the AdS/CFT correspondence, or the "holographic principle" relates certain theories on $d+1$ dimensional AdS with conformal field theories in Minkowski space-time of one dimension less. Therefore, the natural question arises as to whether the dynamical conformal symmetry of nonrelativistic systems can be somehow related to the conformal symmetry of certain relativistic systems in the context of AdS/CFT correspondence. The purpose of the present work is to response this question as well as to trace out in detail how the nonlinear realization of the conformal symmetry emerges in the same context of AdS/CFT duality.

More specifically, first, we shall investigate how in the simplest case of the massless scalar particle its conformal symmetry in d-dimensional Minkowski space originates from the Lorentz symmetry of the corresponding ambient Minkowski space having two more dimensions. This will be done with the help of the $d+2$-dimensional model possessing the local (gauge) $so(2,1)$ conformal symmetry. Then we shall reinterpret the model as a massless particle confined to the border of AdS_{d+1} of infinite radius. Secondly, we shall demonstrate how a free nonrelativistic massive particle in one dimension may be canonically related to the massless particle on AdS_2 of finite radius. Generalizing this observation, we shall construct the Lagrangian for the relativistic particle on the $AdS_2 \times S^{d-1}$, which is related to the known nonrelativistic particle systems enjoying the dynamical $so(2,1)$ symmetry. The corresponding list will include the d-dimensional conformal mechanics model [1], the planar charge-vortex and 3-dimensional charge-monopole systems [3, 7], the nonrelativistic particle in a planar gravitational field [26], and the "new" conformal mechanics model corresponding to the charged particle propagating near the horizon of the extreme Reissner-Nordström black hole [18].

The paper is organized as follows. In Section 2 we review the conformal symmetry of the massless particle in d-dimensional Minkowski space-time, while Section 3 is devoted to the discussion of the dynamical conformal $so(2,1)$ symmetry of various mechanical systems. In Section 4 we show how the nonlinearly realized conformal symmetry of the massless particle in d dimensions can be understood in the context of AdS/CFT correspondence. In Section 5 the dynamical conformal symmetry of non-relativistic mechanical systems is analyzed from the perspective of AdS/CFT duality. The obtained results are summarized in Section 6.

2   Conformal Symmetry of Massless Particle

The infinitesimal transformations

$$\delta x^\mu = \omega^\mu{}_\nu x^\nu + \alpha^\mu + \beta x^\mu + 2(x\gamma)x^\mu - x^2 \gamma^\mu \qquad (2.1)$$

generate the conformal symmetry, $ds^2 \to ds'^2 = e^{2\sigma}ds^2$, on the d-dimensional Minkowski space $\mathbb{R}^{1,d-1}$ with metric

$$ds^2 = dx^\mu dx^\nu \eta_{\mu\nu} = -dx_0^2 + \sum_{i=1}^{d-1} dx_i^2.$$

Here the parameters $\omega^\mu{}_\mu$, α^μ, β and γ^μ correspond to the Lorentz rotations, space-time translations, scale (dilatation) and special conformal transformations. Due to a nonlinear (quadratic) in x^μ nature of the two last terms in (2.1), the finite version of the special conformal transformations,

$$x'^\mu = \frac{x^\mu - \gamma^\mu x^2}{1 - 2\gamma x + \gamma^2 x^2}, \tag{2.2}$$

is not defined globally[1], and to be well defined requires a compactification of $\mathbb{R}^{1,d-1}$ by including the points at infinity (for the discussion of the aspects of globality etc., see [27, 16, 17]).

On the classical phase space with canonical Poisson bracket relations $\{x_\mu, p_\nu\} = \eta_{\mu\nu}$, $\{x_\mu, x_\nu\} = \{p_\mu, p_\nu\} = 0$, the transformations (2.1) are generated by

$$M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu, \qquad P_\mu = p_\mu, \qquad D = x^\mu p_\mu, \qquad K_\mu = 2x_\mu(xp) - x^2 p_\mu. \tag{2.3}$$

The generators (2.3) form the conformal algebra

$$\begin{aligned}
\{M_{\mu\nu}, M_{\sigma\lambda}\} &= \eta_{\mu\sigma}M_{\nu\lambda} - \eta_{\nu\sigma}M_{\mu\lambda} + \eta_{\mu\lambda}M_{\sigma\nu} - \eta_{\nu\lambda}M_{\sigma\mu}, \\
\{M_{\mu\nu}, P_\lambda\} &= \eta_{\mu\lambda}P_\nu - \eta_{\nu\lambda}P_\mu, \qquad \{M_{\mu\nu}, K_\lambda\} = \eta_{\mu\lambda}K_\nu - \eta_{\nu\lambda}K_\mu, \\
\{D, P_\mu\} &= P_\mu, \qquad \{D, K_\mu\} = -K_\mu, \\
\{K_\mu, P_\nu\} &= 2(\eta_{\mu\nu}D + M_{\mu\nu}), \\
\{D, M_{\mu\nu}\} &= \{P_\mu, P_\nu\} = \{K_\mu, K_\nu\} = 0.
\end{aligned} \tag{2.4}$$

The algebra (2.4) is isomorfic to the algebra $so(2,d)$, and by defining

$$J_{\mu\nu} = M_{\mu\nu}, \quad J_{\mu d} = \frac{1}{2}(P_\mu + K_\mu), \quad J_{\mu(d+1)} = \frac{1}{2}(P_\mu - K_\mu), \quad J_{d(d+1)} = D, \tag{2.5}$$

can be put in the standard form

$$\{J_{AB}, J_{LN}\} = \eta_{AL}J_{BN} - \eta_{BL}J_{AN} + \eta_{AN}J_{LB} - \eta_{BN}J_{LA} \tag{2.6}$$

with $A, B = 0, 1, \ldots, d, d+1$, and

$$\eta_{AB} = diag(-1, +1, \ldots, +1, -1). \tag{2.7}$$

The phase space constraint $\varphi_m \equiv p^2 + m^2 = 0$ describing the free relativistic particle of mass m in $\mathbb{R}^{1,d-1}$ is invariant under the Poincaré transformations, $\{M_{\mu\nu}, \varphi_m\} = \{P_\mu, \varphi_m\} = 0$. Unlike the P_μ and $M_{\mu\nu}$, the generators of the scale and special conformal transformations commute weakly with φ_m only in the massless case $m = 0$: $\{D, \varphi_0\} = 2\varphi_0 = 0$, $\{K_\mu, \varphi_0\} =$

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[1]Transformation (2.2) is singular at $x^\mu = \gamma^\mu/\gamma^2$ when $\gamma^2 \neq 0$, and at $x\gamma = 1/2$ for $\gamma^2 = 0$.

$4x_\mu \varphi_0 = 0$. This means that D and K_μ are the integrals of motion of the free scalar massless particle whose Lagrangian is

$$L = \frac{\dot{x}^2}{2e}, \tag{2.8}$$

where $\dot{x}_\mu = \frac{d}{d\tau} x_\mu$, and $e = e(\tau)$ is a Lagrange multiplier. Lagrangian (2.8) is invariant under the Poincaré transformations given by Eq. (2.1) with $\beta = \gamma_\mu = 0$ and supplied with the relations $\delta_\omega e = \delta_\alpha e = 0$. The scale and special conformal symmetry transformations act nontrivially on the both x_μ and e. The finite dilatations are

$$x'^\mu = \exp \beta \cdot x^\mu, \qquad e' = \exp 2\beta \cdot e, \tag{2.9}$$

and the special conformal transformations are given by Eq. (2.2) and

$$e' = \frac{e}{(1 - 2x\gamma + \gamma^2 x^2)^2}. \tag{2.10}$$

Lagrangian (2.8) is also invariant under the inversion transformation[2]

$$x^\mu \to \tilde{x}^\mu = \frac{x^\mu}{x^2}, \qquad e \to \tilde{e} = \frac{e}{x^2}. \tag{2.11}$$

The special conformal transformations (2.2), (2.10) can be treated as a combination of (2.11), and of the space-time translation for the finite vector $-\gamma^\mu$ followed by another transformation (2.11).

In correspondence with Eqs. (2.9), (2.10), on the phase space x^μ, p_μ extended by the canonical pair e and p_e, the Noether integrals associated with the scale and special conformal transformations are corrected,

$$D = xp + 2ep_e, \qquad K_\mu = 2x_\mu(xp + 2ep_e) - x^2 p_\mu. \tag{2.12}$$

The additional terms, however, vanish on the physical subspace given by the constraints $p_e = 0$ and $p^2 = 0$, where (2.12) coincide with the D and K_μ from (2.3). Note also that on the physical subspace the generators D and K_μ can be represented in terms of the Poincaré generators,

$$D = \frac{1}{P^0} J_{0\nu} P^\nu, \qquad K_\mu = \frac{1}{(P^0)^2}(2J_{\mu\nu}P_0 - J_{0\nu}P_\mu)J^{0\nu}.$$

The most noticeable feature in the structure of the conformal symmetry is the nonlinear nature of the transformations induced by K_μ. Since the conformal algebra (2.4) is isomorphic to the $so(2, d)$ algebra (2.6), and the latter is the isometry of AdS$_{d+1}$, it is natural to look at the massless particle in the context of the AdS/CFT correspondence. Before doing this, we shall discuss briefly the non-relativistic particle systems possessing the conformal symmetry $so(2, 1)$ in the form of the dynamical symmetry.

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[2]Transformation (2.11) is well defined on the compactified Minkowski space.

3 Dynamical $so(2,1)$ Symmetry

A free non-relativistic particle in $\mathbb{R}^d$,

$$L = \frac{m}{2}\dot{x}_i^2, \qquad \dot{x}_i = \frac{dx_i}{dt}, \tag{3.1}$$

possesses the symmetry

$$\delta x^i = \omega^{ij} x^j + \alpha^i + v^i t + \beta x^i + \gamma x^i t, \qquad \delta t = \epsilon + 2\beta t + \gamma t^2. \tag{3.2}$$

The transformations given by the infinitesimal parameters ω^{ij}, α^i, v^i and ϵ correspond to the Galilei group symmetry, whereas β and γ are associated with the scale and special conformal symmetries. The global form of the last symmetry transformations is

$$t' = \frac{t}{1 - \gamma t}, \qquad x'_i = \frac{x_i}{1 - \gamma t}, $$

that can be compared with its relativistic counterpart (2.2). The integrals of motion corresponding to (3.2) are

$$P_i = p_i, \quad M_{ij} = x_i p_j - x_j p_i, \quad G_i = t p_i - m x_i, \tag{3.3}$$

$$H = \frac{1}{2m}p_i^2, \quad D = \frac{1}{2}x_i p_i - Ht, \quad K = \frac{m}{2}x_i^2 - 2Dt - Ht^2. \tag{3.4}$$

The three transformations touch the time parameter, and their corresponding generators (3.4) form the dynamical symmetry being the conformal symmetry $so(2,1)$,

$$\{D, H\} = H, \quad \{D, K\} = -K, \quad \{H, K\} = -2D. \tag{3.5}$$

With the identification

$$J_{01} = \frac{1}{2}(H - K), \quad J_{02} = \frac{1}{2}(H + K), \quad J_{12} = D,$$

the algebra (3.5) takes the form (2.6) with $A, B = 0, 1, 2$, and $\eta_{AB} = diag(-1, +1, -1)$. In accordance with Eq. (3.4), the $so(2,1)$ quadratic Casimir element is related to the Casimir element of the rotation group $SO(d)$,

$$\mathcal{C} = \frac{1}{2}J_{AB}J^{AB} = -D^2 + HK = \frac{1}{8}M_{ij}^2. \tag{3.6}$$

The $so(2,1)$ is also the dynamical symmetry of the conformal mechanics model of [1],

$$L = \frac{m}{2}\dot{x}_i^2 - \frac{\alpha}{2x_i^2}, \tag{3.7}$$

given on the configuration space $\mathcal{R}^d = \mathbb{R}^d - \{0\}$. In particular case of $d = 1$ (3.7) describes the relative motion in the 2-particle Calogero model [28]. For this system P_i and G_i are not anymore integrals of motion, but the D and K of the form (3.4) with the Hamiltonian

$$H = \frac{1}{2m}p_i^2 + \frac{\alpha}{2x_i^2} \tag{3.8}$$

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are still the integrals of motion which together form the $so(2,1)$ algebra. The $so(2,1)$ quadratic Casimir element is reduced here to

$$\mathcal{C} = \frac{1}{4}\left(\frac{1}{2}M_{ij}^2 + \alpha m\right). \tag{3.9}$$

Another system enjoying the conformal symmetry is the non-relativistic particle in a planar gravitational field of a point massive source [26]. It has the dynamics of the free particle on the cone described by the Lagrangian

$$L = \frac{m}{2}(\dot{r}^2 + \sigma^2 r^2 \dot{\varphi}^2). \tag{3.10}$$

Here we have used the following parametrization of the cone:

$$x_1 = \sigma r \cos\varphi, \quad x_2 = \sigma r \sin\varphi, \quad x_3 = r\sqrt{1-\sigma^2}, \quad 0 < \sigma < 1. \tag{3.11}$$

In terms of the canonically conjugate variables (r, p_r) and (φ, J) the corresponding $so(2,1)$ generators are given by

$$H = \frac{p_r^2}{2m} + \frac{\sigma^{-2}J^2}{2mr^2}, \qquad D = \frac{1}{2}rp_r - Ht, \qquad K = \frac{m}{2}r^2 - 2Dt - Ht^2, \tag{3.12}$$

and the value of the Casimir element, $\mathcal{C} = \frac{1}{4}\sigma^{-2}J^2$, is fixed by the value of the angular momentum integral J.

The list of the non-relativistic systems with the dynamical $so(2,1)$ symmetry includes also the planar charge-vortex [7] and 3-dimensional charge-monopole systems [3, 29]. The corresponding Lagrangian

$$L = \frac{m}{2}\dot{x}_i^2 + qA_i(x)\dot{x}_i \tag{3.13}$$

describes the particle of charge q either on the punctured plane $\mathcal{R}^2$, or in the space $\mathcal{R}^3$ subjected to the magnetic field given by the U(1) gauge potential A_i,

$$\partial_1 A_2 - \partial_2 A_1 = B = \frac{\Phi}{2\pi}\delta^2(x_i), \quad x_i \in \mathcal{R}^2, \tag{3.14}$$

$$\partial_i A_j - \partial_j A_i = \epsilon_{ijk}B_k, \quad B_i = g\frac{x_i}{(x_j^2)^{3/2}}, \quad x_i \in \mathcal{R}^3. \tag{3.15}$$

The Hamiltonian for the both systems can be represented as

$$H = \frac{1}{2m}\mathcal{P}_i^2, \tag{3.16}$$

with $\mathcal{P}_i = p_i - qA_i(x)$. In terms of $\mathcal{P}_i$, the angular momentum integral of the charge-monopole system, $J_i = \frac{1}{2}\epsilon_{ijk}M_{jk} = \epsilon_{ijk}x_jp_k$, takes the form

$$J_i = \frac{1}{2}\epsilon_{ijk}x_j\mathcal{P}_k - qgn_i, \quad n_i = \frac{x_i}{\sqrt{x_l^2}}, \tag{3.17}$$

and the angular momentum integral of the charge-vortex system, $J = \epsilon_{ij}x_ip_j$, is

$$J = \epsilon_{ij}x_i\mathcal{P}_j + \frac{q\Phi}{2\pi}.$$

The Hamiltonian (3.16), and the D and K given by the equations (3.4) with p_i substituted for $\mathcal{P}_i$ are the integrals of motion generating the dynamical conformal symmetry $so(2,1)$. Here, the Casimir element is reduced to $\mathcal{C} = \frac{1}{4}(J - \frac{q\Phi}{2\pi})^2$ for the charge-vortex, and to $\mathcal{C} = \frac{1}{4}(J_i^2 - q^2g^2)$ for the charge-monopole systems.

The conformal symmetry of the non-relativistic systems (3.1), (3.7), (3.10) and (3.13) together with the rotation symmetry makes their dynamics to be very similar. E.g., as follows from Eq. (3.17), the trajectory of the particle in the monopole field is confined to the cone $\vec{J}\vec{n} = -qg$, and due to the equations $\ddot{x}_i = -qgr^{-1}\epsilon_{ijk}x_j\dot{x}_k$, $r = \sqrt{x_l^2}$, its motion over the cone is free [29]. The particle's position being projected to the plane orthogonal to $\vec{J}$ with preservation of the distance from the origin is given by the vector

$$\vec{X} = r\vec{\mathcal{N}}, \quad \vec{\mathcal{N}} = \frac{\vec{n}_\perp}{|\vec{n}_\perp|}, \quad \vec{n}_\perp = \vec{n} - \vec{J}\frac{\vec{n}\vec{J}}{\vec{J}^2},$$

which obeys the same equations of motion as the vector $\vec{x}$ of the particle in the system (3.7) with $\alpha = -q^2g^2/m < 0$,

$$\ddot{\vec{x}} = \frac{\alpha}{m}\frac{\vec{x}}{(\vec{x}^2)^2}.$$

The origin of such a similarity is rooted in the structure of the corresponding Hamiltonians which being written in terms of the canonical radial variables $r = \sqrt{x_i^2}$ and $p_r = r^{-1}p_ix_i$ take a general form

$$H = \frac{p_r^2}{2m} + \frac{\mathcal{J}^2}{2mr^2}, \tag{3.18}$$

where $\mathcal{J}^2$ is equal to $\frac{1}{2}M_{ij}^2$ for the free particle in $\mathbb{R}^d$ (or, in $\mathcal{R}^d$), to $\frac{1}{2}M_{ij}^2 + \alpha m$ for the model (3.7), to $\sigma^{-2}J^2$ for the free particle on the cone[3], to $(J - \frac{q\Phi}{2\pi})^2$ for the planar charge-vortex system, and to $\vec{J}^2 - q^2g^2$ for the charged particle in the monopole field. As it was shown recently [24], by a canonical transformation the Hamiltonian of the "new" conformal mechanics of [18] can be reduced to that of the model (3.7), and so, (3.18) comprises also the model [18] (see Section 5). This general structure will be exploited below under discussion of the AdS/CFT correspondence for non-relativistic systems.

4 Conformal Symmetry of Massless Particle

Let us return to the relativistic case. Having in mind the identification (2.5), we consider a particle in $(d+2)$-dimensional space $\mathbb{R}^{2,d}$ with coordinates $\mathfrak{X}^A$ and metric (2.7), and introduce

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[3]For this case a slightly different definition (3.11) for the radial variable has been used; however, the Hamiltonian takes the same form as in (3.12) if we parametrize the cone in terms of the usual radial variable in the plane, $r^2 = x_1^2 + x_2^2$, and then realize the canonical transformation $r \to \sigma^{-1}r$, $p_r \to \sigma p_r$.

canonical momenta $\mathfrak{P}_A$, $\{\mathfrak{X}_A, \mathfrak{P}_B\} = \eta_{AB}$. In terms of $\mathfrak{X}_A$ and $\mathfrak{P}_A$ the $so(2,d)$ generators are realized quadratically,

$$\mathfrak{J}_{AB} = \mathfrak{X}_A \mathfrak{P}_B - \mathfrak{X}_B \mathfrak{P}_A. \tag{4.1}$$

In order the system would have the same number of degrees of freedom as the massless scalar particle in $\mathbb{R}^{1,d-1}$, we introduce the scalar equations

$$\phi_0 = \mathfrak{P}_A \mathfrak{P}^A = 0, \tag{4.2}$$
$$\phi_1 = \mathfrak{X}^A \mathfrak{X}_A = 0, \qquad \phi_2 = \mathfrak{X}^A \mathfrak{P}_A = 0. \tag{4.3}$$

The relations $\phi_1 = 0$ and $\phi_0 = 0$ could be treated as the constraints only if the hypersurfaces $\mathfrak{X}_0 = \mathfrak{X}_1 = \ldots = \mathfrak{X}_{d+1} = 0$ and $\mathfrak{P}_0 = \mathfrak{P}_1 = \ldots = \mathfrak{P}_{d+1} = 0$ are excluded from the phase space. Only in this case the gauge orbits generated by ϕ_1 and ϕ_0 are regular[4], and then (4.2) and (4.3) form the set of the first class constraints [30, 31]. Therefore, the configuration space of the system (4.2), (4.3) is supposed to be $\mathcal{R}^{2,d} = \mathbb{R}^{2,d} - \{0\}$. Note that up to inessential numerical factors the constraints (4.2), (4.3) are of the form of the non-relativistic generators (3.4) taken at $t = 0$, i.e. they generate the local $so(2,1)$ symmetry.

As in the d-dimensional case, the $so(2,d)$ generators (4.1) can be supplied with the generators $\mathfrak{P}_A$, and

$$\mathfrak{D} = \mathfrak{X}\mathfrak{P}, \qquad \mathfrak{K}_A = 2\mathfrak{X}_A(\mathfrak{X}\mathfrak{P}) - \mathfrak{P}_A \mathfrak{X}^2, \tag{4.4}$$

which together with (4.1) form the $so(3, d+1)$ algebra. However, $\mathfrak{P}_A$ do not commute with the first constraint from (4.3) and so, are not observable. Since the quantities (4.4) are proportional to the constraints (4.3), the global (rigid) symmetries associated with the generators $\mathfrak{D}$ and $\mathfrak{K}_A$ are reduced to the local (gauge) symmetries. On the other hand, the scalar character of the constraints (4.2), (4.3) means that the $so(2,d)$ generators (4.1) are the observable (gauge-invariant) quantities and should be related to the set of the conformal generators (2.3). As we shall see, such a direct relation between the symmetry generators in d and $d+2$ dimensions is due to the special choice of the constraints (4.3), but generally it can be more involved.

In accordance with the structure of the constraints (4.2), (4.3), the system has a sense of the massless particle on the $(d+1)$-dimensional cone $\mathfrak{X}^A \mathfrak{X}_A = 0$ with local scale symmetry generated by the constraint ϕ_2. The Lagrangian can be chosen, e.g., in one of the two alternative forms,

$$L = \frac{\dot{\mathfrak{X}}^2}{2e} + \frac{v}{2}\mathfrak{X}^2, \tag{4.5}$$

or

$$L = \frac{(\dot{\mathfrak{X}} - u\mathfrak{X})^2}{2e} + \frac{v}{2}\mathfrak{X}^2, \tag{4.6}$$

where e, v, u are the scalar Lagrange multipliers. Lagrangian (4.6) produces all the three constraints (4.2), (4.3) as the secondary constraints, whereas for (4.5) the constraint $\phi_2 = 0$ appears as the tertiary one.

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[4]For the massless particle in $\mathbb{R}^{1,d-1}$, the points with $p_0 = p_1 = \ldots = p_{d-1} = 0$, corresponding to the trivial representation of the Poincaré group have also to be excluded from the phase space by the same reasons. This makes a massless particle to be similar to the non-relativistic charge-monopole system [29].

To establish the exact relationship between the $(d+2)$-dimensional massless system (4.2), (4.3) for which the $so(2,d)$ is the Lorentz symmetry, and the d-dimensional massless particle, for which the same plays a role of the conformal symmetry, it is convenient to define

$$\mathfrak{X}^{\pm} = \mathfrak{X}^d \pm \mathfrak{X}^{d+1}, \quad \mathfrak{P}_{\pm} = \frac{1}{2}\left(\mathfrak{P}_d \pm \mathfrak{P}_{d+1}\right) \tag{4.7}$$

having the nontrivial Poisson brackets $\{\mathfrak{X}^+, \mathfrak{P}_+\} = \{\mathfrak{X}^-, \mathfrak{P}_-\} = 1$. Then, the $so(2,d)$ generators (4.1) take the form

$$\mathfrak{J}_{\mu\nu} = \mathfrak{X}_\mu\mathfrak{P}_\nu - \mathfrak{X}_\nu\mathfrak{P}_\mu, \qquad \mathfrak{J}_{d(d+1)} = \mathfrak{X}^+\mathfrak{P}_+ - \mathfrak{X}^-\mathfrak{P}_-,$$
$$\mathfrak{J}_{\mu+} \equiv \mathfrak{J}_{\mu d} + \mathfrak{J}_{\mu(d+1)} = 2\mathfrak{X}_\mu\mathfrak{P}_+ - \mathfrak{P}_\mu\mathfrak{X}^-, \quad \mathfrak{J}_{\mu-} \equiv \mathfrak{J}_{\mu d} - \mathfrak{J}_{\mu(d+1)} = 2\mathfrak{X}_\mu\mathfrak{P}_- - \mathfrak{P}_\mu\mathfrak{X}^+. \tag{4.8}$$

Generically[5] $\mathfrak{X}^- \neq 0$, and having in mind the identification (2.5), the symplectic form of the system, $\Omega = d\mathfrak{P}_A \wedge d\mathfrak{X}^A = d\mathfrak{P}_\mu \wedge d\mathfrak{X}^\mu + d\mathfrak{P}_+ \wedge d\mathfrak{X}^+ + d\mathfrak{P}_- \wedge d\mathfrak{X}^-$, can be represented equivalently as

$$\Omega = d\tilde{\mathfrak{P}}_\mu \wedge d\tilde{\mathfrak{X}}^\mu + d\tilde{\mathfrak{P}}_+ \wedge d\tilde{\mathfrak{X}}^+ + d\tilde{\mathfrak{P}}_- \wedge d\tilde{\mathfrak{X}}^-,$$

where

$$\tilde{\mathfrak{X}}^\mu = -\frac{\mathfrak{X}^\mu}{\mathfrak{X}^-}, \quad \tilde{\mathfrak{X}}^+ = \frac{\mathfrak{X}_A\mathfrak{X}^A}{\mathfrak{X}^-}, \quad \tilde{\mathfrak{X}}^- = \mathfrak{X}^-, \tag{4.9}$$

$$\tilde{\mathfrak{P}}_\mu = \mathfrak{J}_{\mu+}, \quad \tilde{\mathfrak{P}}_+ = \mathfrak{P}_+, \quad \tilde{\mathfrak{P}}_- = \frac{1}{\mathfrak{X}^{-2}}\left(\mathfrak{X}^-\mathfrak{P}_A\mathfrak{X}^A - \mathfrak{P}_+\mathfrak{X}_A\mathfrak{X}^A\right). \tag{4.10}$$

Therefore, the transformation $(\mathfrak{X}^A, \mathfrak{P}_A) \to (\tilde{\mathfrak{X}}^A, \tilde{\mathfrak{P}}_A)$ is canonical, and its inverse is

$$\mathfrak{X}_\mu = -\tilde{\mathfrak{X}}^-\tilde{\mathfrak{X}}_\mu, \quad \mathfrak{X}^- = \tilde{\mathfrak{X}}^-, \quad \mathfrak{X}^+ = \tilde{\mathfrak{X}}^+ - \tilde{\mathfrak{X}}^-\tilde{\mathfrak{X}}_\mu\tilde{\mathfrak{X}}^\mu,$$

$$\mathfrak{P}_\mu = -\frac{\tilde{\mathfrak{P}}_\mu}{\tilde{\mathfrak{X}}^-} - 2\mathfrak{P}_+\tilde{\mathfrak{X}}_\mu, \quad \mathfrak{P}_+ = \tilde{\mathfrak{P}}_+, \quad \mathfrak{P}_- = \tilde{\mathfrak{P}}_- - \tilde{\mathfrak{P}}_+\tilde{\mathfrak{X}}_\mu\tilde{\mathfrak{X}}^\mu - \frac{\tilde{\mathfrak{X}}_\mu\tilde{\mathfrak{P}}^\mu}{\tilde{\mathfrak{X}}^-}.$$

In terms of the new variables the constraints read as

$$\phi_0 = \tilde{\mathfrak{P}}_\mu\tilde{\mathfrak{P}}^\mu + 4\frac{\tilde{\mathfrak{P}}_+}{\tilde{\mathfrak{X}}^{-2}}(\tilde{\mathfrak{X}}^-\phi_2 - \tilde{\mathfrak{P}}_+\phi_1) = 0, \tag{4.11}$$

$$\phi_1 = \tilde{\mathfrak{X}}^+\tilde{\mathfrak{X}}^- = 0, \qquad \phi_2 = \tilde{\mathfrak{X}}^+\tilde{\mathfrak{P}}_+ + \tilde{\mathfrak{X}}^-\tilde{\mathfrak{P}}_- = 0, \tag{4.12}$$

and the Lorentz generators (4.8) take the form

$$\mathfrak{J}_{\mu\nu} = \tilde{\mathfrak{X}}_\mu\tilde{\mathfrak{P}}_\nu - \tilde{\mathfrak{X}}_\nu\tilde{\mathfrak{P}}_\mu, \quad \mathfrak{J}_{\mu+} = \tilde{\mathfrak{P}}_\mu, \quad \mathfrak{J}_{d(d+1)} = \tilde{\mathfrak{X}}_\mu\tilde{\mathfrak{P}}^\mu + 2\frac{\tilde{\mathfrak{P}}_+}{\tilde{\mathfrak{X}}^-}\phi_1 - \phi_2,$$

$$\mathfrak{J}_{\mu-} = 2(\tilde{\mathfrak{X}}_\nu\tilde{\mathfrak{P}}^\nu)\tilde{\mathfrak{X}}_\mu - (\tilde{\mathfrak{X}}_\nu\tilde{\mathfrak{X}}^\nu)\tilde{\mathfrak{P}}_\mu + \frac{1}{\tilde{\mathfrak{X}}^{-2}}(\tilde{\mathfrak{P}}_\mu + 4\tilde{\mathfrak{P}}_+\tilde{\mathfrak{X}}^-\tilde{\mathfrak{X}}_\mu)\phi_1 - 2\tilde{\mathfrak{X}}_\mu\phi_2.$$

The constraints (4.12) single out the surface

$$\tilde{\mathfrak{P}}_\mu\tilde{\mathfrak{P}}^\mu = 0, \tag{4.13}$$

$$\tilde{\mathfrak{X}}^+ = 0, \qquad \tilde{\mathfrak{P}}_- = 0. \tag{4.14}$$

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[5]This always can be achieved by applying the appropriate Lorentz transformation.

Therefore, the variables $\tilde{\mathfrak{P}}_\mu$ and $\tilde{\mathfrak{X}}_\mu$ are the observable (gauge invariant) variables with respect to the constraints (4.3), whereas $\tilde{\mathfrak{X}}^- = \mathfrak{X}^-$ and $\tilde{\mathfrak{P}}_+ = \mathfrak{P}_+$ are the pure gauge variables which can be removed by introducing the constraints

$$\phi_3 \equiv \mathfrak{X}^- + 1 = 0, \quad \phi_4 \equiv \mathfrak{P}_+ = 0 \tag{4.15}$$

as the gauge conditions for (4.3). Reducing the system to the surface given by the set of the second class constraints (4.3), (4.15) being equivalent to the set (4.14) (4.15), we completely exclude from the theory the variables $\mathfrak{X}^\pm$, $\mathfrak{P}_\pm$, and as a result, the canonical variables $\tilde{\mathfrak{X}}^\mu$ and $\tilde{\mathfrak{P}}_\mu$ are reduced to the initial variables $\mathfrak{X}^\mu$ and $\mathfrak{P}_\mu$, the mass shell constraint takes the form $\phi_0 = \mathfrak{P}_\mu \mathfrak{P}^\mu = 0$, and the Lorentz generators (4.1) are reduced to the d-dimensional generators (2.3) of the conformal symmetry.

So, the initial d-dimensional massless system (2.8) can be reinterpreted as the $(d+2)$-dimensional massless system (4.5) or (4.6) living on the cone and possessing local scale symmetry. Under such identification the $so(2,d)$ Lorentz generators of the $(d+2)$-dimensional system correspond to the conformal symmetry generators in accordance with identification (2.5). In this way, the cubic structure of the special conformal symmetry generators K_μ originates from the nonlinearity of the canonical transformation (4.9), (4.10).

The constraints (4.15) and (4.2) forming the set of the first class constraints can be used instead of the set (4.2), (4.3). Having in mind the origin of the relations (4.14), the change of the constraints (4.3) for the constraints (4.15) effectively is a canonical transformation acting nontrivially on the Lorentz generators (4.1). We shall not investigate such an action explicitly, but, instead, look at the rigid symmetries for the case of the choice of the first class constraints (4.15). The variables $\mathfrak{X}_\mu$ and $\mathfrak{P}_\mu$ are the observables and can be identified with x_μ and p_μ of the d-dimensional model. Then, on the surface of the constraints (4.15) in addition to the Poincaré generators $J_{\mu\nu} = \mathfrak{J}_{\mu\nu}$, $P_\mu = \mathfrak{P}_\mu = \mathfrak{J}_{\mu+}$, the scale and the special conformal generators from (2.3) can be identified with the following linear combinations of the $so(3, d+1)$ generators: $D = \mathfrak{P}_- + \mathfrak{D} = P_- - (\mathfrak{K}_d + \mathfrak{K}_{d+1})$, $K_\mu = \mathfrak{K}_\mu + \mathfrak{J}_{\mu-}$, which are the gauge invariant quantities. So, when the system is given by the constraints (4.2), (4.15), the identification of the generators D and K_μ includes the $so(3, d+1)$ generators $\mathfrak{D}$ and $\mathfrak{K}_A$. With such a treatment the nonlinearity of the special conformal symmetry transformations is rooted in the nonquadratic nature of $\mathfrak{K}_A$ given by Eq. (4.4).

The $(d+2)$-dimensional system with the first class constraints (4.2), (4.15) can be described by the Lagrangian

$$L = \frac{\dot{\mathfrak{X}}_A \dot{\mathfrak{X}}^A}{2e} - v(\mathfrak{X}^- + 1). \tag{4.16}$$

Due to the direct identification of the observables of the $(d+2)$-dimensional system (4.16) with the phase space coordinates of the d-dimensional system, the corresponding field formulations for the both systems are also related in a very simple way. In accordance with Eqs. (4.15) and (4.13), the field $\Phi(\mathfrak{X}_A)$ satisfies the equations

$$(\mathfrak{X}^- + 1)\Phi(\mathfrak{X}_A) = 0, \qquad \partial_+ \Phi(\mathfrak{X}_A) = 0, \qquad \partial_B^2 \Phi(\mathfrak{X}_A) = 0,$$

where $\partial_B = \partial/\partial \mathfrak{X}^B$, $\partial_+ = \partial/\partial \mathfrak{X}^+$. Their solution is $\Phi(\mathfrak{X}_A) = \delta(X^- + 1)\varphi(\mathfrak{X}_\nu)$ with the field $\varphi(\mathfrak{X}_\nu)$ obeying the d-dimensional Klein-Gordon equation $\partial_\mu^2 \varphi(\mathfrak{X}_\nu) = 0$, $\partial_\mu = \partial/\partial \mathfrak{X}^\mu$.

The $(d+2)$-dimensional system (4.2), (4.3) can be reinterpreted as a massless particle living on the border of the AdS_{d+1} of infinite radius, whose isometry corresponds to the conformal symmetry of the massless particle in d dimensions. This can be done in the following way. The massless particle on AdS_{d+1} of radius R can be described by the constraints

$$\underline{\phi}_0 = \underline{\mathfrak{P}}_A \underline{\mathfrak{P}}^A = 0, \qquad \underline{\phi}_1 = \underline{\mathfrak{X}}_A \underline{\mathfrak{X}}^A + R^2 = 0, \tag{4.17}$$

where $\underline{\mathfrak{X}}^A$ and $\underline{\mathfrak{P}}_A$ are the canonical variables. The Poisson brackets of the constraints (4.17) are $\{\underline{\phi}_1, \underline{\phi}_0\} = 4\underline{\mathfrak{X}}^A \underline{\mathfrak{P}}_A$. To have the reparametrization-invariant system, the $\underline{\phi}_0$ has to be the first class constraint. This can be achieved by postulating the constraint

$$\underline{\phi}_2 = \underline{\mathfrak{X}}^A \underline{\mathfrak{P}}_A = 0 \tag{4.18}$$

in addition to the constraints (4.17). The constraint (4.18) generates the local scale transformations which due to the relation $\{\underline{\phi}_2, \underline{\phi}_0\} = 2\underline{\phi}_0$ are consistent with the reparametrization invariance generated by the constraint $\underline{\phi}_0$. Since $\{\underline{\phi}_2, \underline{\phi}_1\} = -\underline{\phi}_1 + R^2 \neq 0$, the constraint $\underline{\phi}_1 = 0$ can be understood as a gauge condition for the constraint (4.18). Now, let us realize a canonical transformation $(\underline{\mathfrak{X}}^A, \underline{\mathfrak{P}}_A) \to (\mathfrak{X}^A, \mathfrak{P}_A)$,

$$\mathfrak{X}^A = \frac{\underline{\mathfrak{X}}^A}{R^{1+\varepsilon}}, \quad \mathfrak{P}_A = \underline{\mathfrak{P}}_A R^{1+\varepsilon}, \tag{4.19}$$

with a constant $\varepsilon > 0$, and take a limit $R \to \infty$, $\underline{\mathfrak{X}}^A \to \infty$, $\underline{\mathfrak{P}}_A \to 0$ in such a way that the variables (4.19) would be finite. Then, the constraints (4.17), (4.18) take the form of the first class constraints (4.2), (4.3), and we reproduce the system on the cone. Because of the change of the nature of the constraints from the second to the first class, the described limit procedure has a rather formal character; however, see the next Section for further discussion of the system (4.17), (4.18).

5 AdS/CFT Correspondence for Non-Relativistic Systems

Now, let us show that the dynamical $so(2,1)$ symmetry of the non-relativistic systems discussed in Section 3 can be naturally understood in the context of the CFT/AdS correspondence. We first analyze in detail the simplest system of the free massive particle (3.1), and then generalize the results for other non-relativistic systems.

Let us consider the massive particle (3.1) in $\mathbb{R}^1$. Changing the parametrization of the trajectory, $x^1(t) \to x^1(t(\tau))$, $\dot{t}(\tau) = \frac{dt}{d\tau} > 0$, and introducing the notation $t = x^0$, we pass from (3.1) to the reparametrization-invariant action

$$A = \int L d\tau, \qquad L = \frac{m}{2} \frac{\dot{x}_\mu^2}{\dot{x}^0}. \tag{5.1}$$

Here $\dot{x}_\mu^2 = \dot{x}^\mu \dot{x}^\nu \eta_{\mu\nu}$, $\eta_{\mu\nu} = diag(-1,1)$, and we have added a total derivative term $-\frac{m}{2}\dot{x}^0$. A simple comparison of (5.1) being the reparametrization-invariant representation of the

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non-relativistic massive particle on the one hand with the *massless* particle (2.8) in the 2-dimensional Minkowski space $\mathbb{R}^{1,1}$ on the other hand reveals a similarity of the both systems, and in what follows we shall demonstrate that they are canonically related. Before turning to the Hamiltonian description of the systems, it is instructive to compare them in the Lagrangian picture.

The general solution to the equations of motion for (5.1),

$$\frac{d}{d\tau}\left(\frac{\dot{x}^1}{\dot{x}^0}\right) = 0, \qquad \frac{d}{d\tau}\left(\frac{\dot{x}^1}{\dot{x}^0}\right)^2 = 0, \tag{5.2}$$

can be represented in the form $x^1 = x^1(x^0)$,

$$x^1(x^0) = \mathrm{v}x^0 + a, \tag{5.3}$$

where a and v are the integration constants. This can be compared with the general solution,

$$x^1(x^0) = \epsilon x^0 + b, \qquad e = M^{-1}\dot{x}^0, \tag{5.4}$$

to the equations of motion

$$(\dot{x}^1)^2 - (\dot{x}^0)^2 = 0, \qquad \frac{d}{d\tau}\left(\frac{\dot{x}^1}{e}\right) = 0, \qquad \frac{d}{d\tau}\left(\frac{\dot{x}^0}{e}\right) = 0$$

for the massless particle, where $\epsilon = \pm$, and b and M are the integration constants with M having a dimension of mass. The only formal difference between (5.3) and (5.4) is in the velocity of the particles: for the first system it can be arbitrary, whereas for the second it is the velocity of light. However, at the moment we note that for $\mathrm{v} \neq 0$ the solution (5.3) can be reduced to (5.4) by rescaling x^0.

One can construct the Lagrangian relating the two systems,

$$L = \frac{\dot{x}_\mu^2}{2e} + \frac{\lambda}{l}(e - M^{-1}\dot{x}^0), \tag{5.5}$$

where λ is a Lagrange multiplier and l is a numerical parameter. The equations of motion for the Lagrange multipliers λ and e are algebraic,

$$e - M^{-1}\dot{x}^0, \quad (\dot{x}^1)^2 - (\dot{x}^0)^2 - \frac{\lambda}{l}e^2 = 0, \tag{5.6}$$

the equation for x^1 is of the same form as for the massless particle, and for x^0 it is

$$\frac{d}{d\tau}\left(\dot{x}^0 + \frac{\lambda}{2Ml}\right) = 0.$$

The equations (5.6) can be solved to represent the Lagrange multipliers in terms of other variables, $e = M^{-1}\dot{x}^0$, $\lambda = M^2 l((\dot{x}^1/\dot{x}^0)^2 - 1)$. As a result, the equations for x^1 and x^0 are reduced exactly to the equations (5.2) for the first system, that proves the equivalence of the system (5.5) to the system (5.1) (the substitution of the solution for e and λ into (5.5) with

identification $M = m$ reduces the latter to (5.1)). On the other hand, in the limit $l \to \infty$ the system (2.8) is reproduced from (5.5).

Now, let us compare the two systems within the Hamiltonian picture. In terms of the light-cone variables $x^{\pm} = x^0 \pm x^1$, $p_{\pm} = \frac{1}{2}(p_0 \pm p_1)$, $\{x^+, p_+\} = \{x^-, p_-\} = 1$, the $so(2,2)$ generators (2.3) of the massless particle in $\mathbb{R}^{1,1}$ are represented in the form

$$\mathfrak{D}_+ = \tfrac{1}{2}(D + M^{01}) = x^+ p_+, \quad \mathfrak{H}_+ = \tfrac{1}{2}(P_0 + P_1) = p_+, \quad \mathfrak{K}_+ = \tfrac{1}{2}(K^0 + K^1) = x^{+2} p_+,$$

$$\tag{5.7}$$

$$\mathfrak{D}_- = \tfrac{1}{2}(D - M^{01}) = x^- p_-, \quad \mathfrak{H}_- = \tfrac{1}{2}(P_0 - P_1) = p_-, \quad \mathfrak{K}_- = \tfrac{1}{2}(K^0 - K^1) = x^{-2} p_-.$$

The subsets $\mathfrak{D}_+$, $\mathfrak{H}_+$, $\mathfrak{K}_+$ and $\mathfrak{D}_-$, $\mathfrak{H}_-$, $\mathfrak{K}_-$ mutually commute and every of them generates the algebra of the form (3.5), that reveals explicitly the direct sum structure of the conformal symmetry algebra $so(2,2) = so(2,1) \oplus so(2,1)$. Since the constraint $p^2 = -\frac{1}{4}p_+ p_- = 0$ is equivalent to either $p_+ = 0$ or $p_- = 0$, on the mass shell one of the two sets of the $so(2,1)$ generators turns into zero, and the corresponding rigid $so(2,1)$ symmetry is reduced to the gauge (reparametrization) symmetry, that leaves only one rigid $so(2,1)$ symmetry.

The non-relativistic system (5.1) is characterized by the constraint

$$\varphi = p_0 + \frac{p_1^2}{2m} = 0. \tag{5.8}$$

Here we have shifted the momentum p_0 canonically conjugated to x^0, $p_0 + \frac{m}{2} \to p_0$. This change is a canonical transformation corresponding to omitting the total derivative term $-\frac{m}{2}\dot{x}^0$ from Lagrangian (5.1). The quantities

$$T = p^0, \qquad D = \frac{1}{2}p_1 x^1 + p_0 x^0, \qquad K = \frac{m}{2}x_1^2 - p_0 (x^0)^2 - x^0 p_1 x^1 \tag{5.9}$$

are the integrals of motion being the generators of conformal symmetry $so(2,1)$. The quantization of the system transforms the constraint (5.8) into the Schrödinger equation

$$i\frac{\partial}{\partial x^0}\psi(x^0, x^1) = -\frac{1}{2m}\frac{\partial^2}{\partial x_1^2}\psi(x^0, x^1).$$

In such a way the hidden conformal symmetry of the non-relativistic Schrödinger equation [25] obtains a natural explanation as the symmetry generated by the quantum analogues of the usual integrals of motion (5.9) of the non-relativistic free particle model represented in a reparametrization invariant form.

The equivalent representation of (5.9),

$$T = H - \varphi, \quad D = \frac{1}{2}x^1 p_1 - H x^0 + x^0 \varphi, \quad K = \frac{m}{2}x_1^2 - 2D x^0 - H(x^0)^2 + (x^0)^2 \varphi$$

with $H = \frac{p_1^2}{2m}$, shows that the $so(2,1)$ generators in the reparametrization invariant and usual formulations (see Eqs. (3.4)) coincide up to the terms proportional to the constraint, and, consequently, the transformations generated by them are the same up to the reparametrization.

The two other integrals of motion correspond to the translation and the Galilei boost symmetries,

$$P_1 = p_1, \quad X_1 = x_1 - \frac{x^0}{m} p_1, \tag{5.10}$$

(cf. Eq. (3.3)), from which another set of the $so(2,1)$ generators can be constructed,

$$\bar{T} = \frac{P_1^2}{2m}, \qquad \bar{D} = \frac{1}{2} X_1 P_1, \qquad \bar{K} = \frac{m}{2} X_1^2. \tag{5.11}$$

However, there is only one independent set of the $so(2,1)$ generators since the linear combination of (5.9) and (5.11) vanishes on the physical subspace given by the constraint (5.8),

$$T - \bar{T} = \varphi, \quad D - \bar{D} = x^0 \varphi, \quad K - \bar{K} = (x^0)^2 \varphi.$$

The integrals of motion (5.10) together with the relations

$$X^0 = x^0, \qquad P_0 = p_0 + \frac{p_1^2}{2m}, \tag{5.12}$$

can be treated as the change of the variables $x^\mu, p_\mu \to X^\mu, P_\mu$, which is a canonical transformation. Since the P_0 coincides with the constraint, the canonical pair X^0 and P_0 corresponds to the gauge degree of freedom, while X_1 and P_1 form a pair of the gauge-invariant independent observables.

Supposing $p_1 \neq 0$, one can consider another canonical transformation

$$\tilde{x}_1 = \epsilon m \frac{x_1}{p_1}, \qquad \tilde{p}_1 = \epsilon \frac{p_1^2}{2m}, \qquad \tilde{x}_0 = x_0, \qquad \tilde{p}_0 = p_0, \tag{5.13}$$

where $\epsilon = p_1/|p_1|$. Then the constraint (5.8) takes a form $\varphi = \tilde{p}_0 + \tilde{p}_1 = 0$ for $\tilde{p}_1 > 0$ and $\varphi = \tilde{p}_0 - \tilde{p}_1 = 0$ for $\tilde{p}_1 < 0$. In these variables, the $so(2,1)$ generators (5.9) are

$$T = \tilde{p}^0, \quad D = \tilde{x}^\mu \tilde{p}_\mu, \quad K = 2\tilde{x}_0(\tilde{x}_\mu \tilde{p}^\mu) - \tilde{x}_\mu^2 \tilde{p}_0 + \tilde{x}_\mu^2 \varphi,$$

that can be compared with the generators P^0, D and K_0 from (2.3) for the massless particle. On the other hand, the generators (5.11) being rewritten in the variables (5.13),

$$\bar{T} = \epsilon p_1, \quad \bar{D} = \epsilon(\tilde{x}_0 \tilde{p}_1 - \tilde{x}_1 \tilde{p}_0 + \tilde{x}_1 \varphi), \quad \bar{K} = \epsilon(2\tilde{x}_1 \tilde{x}^\mu \tilde{p}_\mu - \tilde{p}_1 \tilde{x}_\mu^2 + 2\tilde{x}_0 \tilde{x}_1 \varphi),$$

can be compared with the generators P_1, M_{01} and K_1 from (2.3). Note that the $\bar{D}$, being the generator of the scale transformations of X_1 and P_1, corresponds to the Lorentz boost generator M_{01}, while the generator of the special conformal transformations, $\bar{K}$, being quadratic in the generator of the Galilei boosts, is mapped by (5.13) into the special conformal symmetry generator K_1 having a cubic structure in the massless particle's canonical variables.

In accordance with the constraint (5.8), the physical states of the non-relativistic particle with $p_1 \neq 0$ are in the $p_0 < 0$ sector of the phase space. The change $x^0 \to -x^0$ in Lagrangian (5.1) results in the physical subspace with $p_0 > 0$. This corresponds to the two disjoint sectors with $p^0 > 0$ and $p^0 < 0$ in the case of the massless particle. So, generally the physical states of the non-relativistic massive particle obeying the constraint (5.8) can be separated

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into the three sectors: with $p_0 < 0$ or $p_0 > 0$ for which $p_1 \neq 0$, and with $p_0 = p_1 = 0$. In the last case the evolution degenerates, $x_1 = a = const$. For the massless particle, the analogous sector with $p_0 = p_1 = 0$ is excluded from the consideration as corresponding to the trivial representation of the $(1+1)$-dimensional Poincaré group. The two nontrivial sectors with $p_0 \neq 0$ can be characterized by the same rigid $so(2,1)$ symmetry, and there, as we have seen, the two systems, massless and non-relativistic massive, can be related via the canonical transformation (5.13).

The massless particle on AdS_2 of radius R with metric [17]

$$ds^2 = R^2 \frac{dz^2 - dx_0^2}{z^2}$$

is described by the Lagrangian

$$L = R^2 \frac{\dot{z}^2 - \dot{x}_0^2}{2ez^2}, \tag{5.14}$$

where $z > 0$ or $z < 0$. Since Lagrangian (5.14) is reduced to (2.8) by a simple change of the Lagrange multiplier, $ez^2 R^{-2} \to e$, and identification of z with x^1, the massive non-relativistic particle on $\mathcal{R}^1$ can be treated as the system canonically equivalent to the massless particle on AdS_2. Due to exclusion of $x^1 = 0$ from the configuration space, in this case there is neither translation nor Galilei boost invariance in the non-relativistic system, whereas the remaining conformal $so(2,1)$ symmetry corresponds to the isometry of the configuration space AdS_2.

Another form of the Lagrangian for the massless particle on AdS_2 can be chosen similarly to (4.5),

$$L = \frac{\dot{\mathfrak{X}}^2}{2e} + \frac{v}{2}(\mathfrak{X}^2 + R^2), \tag{5.15}$$

where $\mathfrak{X}^2 = \mathfrak{X}^A \mathfrak{X}^B \eta_{AB} = -\mathfrak{X}_0^2 + \mathfrak{X}_1^2 - \mathfrak{X}_2^2$. Lagrangian (5.15) generates the constraints

$$\phi_0 = \mathfrak{P}^2 = 0, \tag{5.16}$$

$$\phi_1 = \mathfrak{X}^2 + R^2 = 0, \qquad \phi_2 = \mathfrak{X}\mathfrak{P} = 0, \tag{5.17}$$

and so, corresponds to the particular $d = 1$ case of the massless system (4.17), (4.18). Due to the scalar nature of the constraints (5.16), (5.17), the $so(2,1)$ generators $\mathfrak{J}_{AB} = \mathfrak{X}_A \mathfrak{P}_B - \mathfrak{X}_B \mathfrak{P}_A$ are the integrals of motion being the isometry generators of AdS_2. The mass shell constraint (5.16) is the first class constraint generating the reparametrization symmetry, whereas the constraints (5.17) form the subsystem of the second class constraints. With taking into account (5.17), the mass shell constraint is represented equivalently in the form $\mathfrak{J}_{AB}\mathfrak{J}^{AB} = 0$ corresponding to Eq. (3.6) for the non-relativistic particle in $\mathcal{R}^1$.

The surface of the second class constraints (5.17) can be parametrized as

$$\mathfrak{X}^0 = R\frac{x^0}{x^1}, \quad \mathfrak{X}^1 - \mathfrak{X}^2 = -\frac{R^2}{x^1}, \quad \mathfrak{X}^1 + \mathfrak{X}^2 = -\frac{x^+ x^-}{x^1}, \tag{5.18}$$

$$\mathfrak{P}_0 = \frac{p_+ x^+ - p_- x^-}{R}, \quad \mathfrak{P}_1 + \mathfrak{P}_2 = p_+ - p_-, \quad \mathfrak{P}_1 - \mathfrak{P}_2 = \frac{p_+ x^{+2} - p_- x^{-2}}{R^2}, \tag{5.19}$$

in terms of the phase space variables $x^\pm = x^0 \pm x^1$, $p_\pm = \frac{1}{2}(p_0 \pm p_1)$, with $x^1 > 0$ or $x^1 < 0$. The parametrization (5.19) reduces the symplectic two-form $d\mathfrak{P}_a \wedge d\mathfrak{X}^a$ to $dp_+ \wedge dx^+ + dp_- \wedge dx^-$,

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i.e. the variables x^+, p_+ and x^-, p_- form the two pairs of canonically conjugate variables on the reduced phase space given by the second class constraints (5.17). On the surface (5.17) the integrals of motion $\mathfrak{J}_{AB}$ are reduced to

$$\mathfrak{J}_{12} = D, \quad \mathfrak{J}_{01} + \mathfrak{J}_{02} = -RT, \quad \mathfrak{J}_{01} - \mathfrak{J}_{02} = -\frac{1}{R}K, \tag{5.20}$$

with $D = p_+ x^+ + p_- x^-$, $T = p_+ + p_-$, $K = p_+ x^{+2} + p_- x^{-2}$. Up to inessential numerical factors the $so(2,1)$ isometry generators (5.20) have the form of the sum of the corresponding generators (5.7). The parametrization (5.18) transforms the Lagrangian (5.15) into (5.14) with identification $z = x^1$, i.e. (5.14) gives the dynamics of the system on the reduced phase space, where the first class constraint (5.16) is represented as

$$\phi_0 = -\frac{1}{2R^2}\mathfrak{J}_{AB}\mathfrak{J}^{AB} = 4x_1^2 p_+ p_- = 0 \tag{5.21}$$

Another alternative form of the Lagrangian for the massless particle on AdS$_2$ is

$$L = -R^2 \frac{\dot{\hat{\mathfrak{X}}}^2}{2e}, \tag{5.22}$$

where we assume that $\mathfrak{X}^2 < 0$, and $\hat{\mathfrak{X}}_A = \frac{\mathfrak{x}_A}{\sqrt{-\mathfrak{X}^2}}$ is a unit vector, $\hat{\mathfrak{X}}^2 = -1$. Lagrangian (5.22) generates the two first class constraints $\phi_0 = 0$ and $\phi_2 = 0$ given by Eqs. (5.16), (5.17). The Lagrangian (5.22) in addition to reparametrization invariance has also local scale invariance. The latter gauge invariance can be fixed by introducing the constraint $\phi_1 = \mathfrak{X}^2 + R^2 = 0$ as a gauge condition to the constraint $\phi_2 = 0$, that means the equivalence of the systems given by Eqs. (5.15) and (5.22). Therefore, Lagrangians (5.14), (5.15), and (5.22) describe the same system of the massless particle on AdS$_2$ being canonically equivalent to the massless particle in $\mathcal{R}^{1,1}$. The latter, as we have seen, can be related to the free non-relativistic massive particle in $\mathcal{R}^1$ via the construction of the reparametrization invariant form of the action and canonical transformation (5.13).

The case of the relativistic massive scalar particle on AdS$_2$ is obtained by adding the "cosmological" term $-\frac{e}{2}\mathcal{M}^2$ to the Lagrangians (5.14), (5.15), or (5.22). This changes the mass shell constraint (5.16) for $\phi_M = \mathfrak{P}^2 + \mathcal{M}^2 = 0$, and its reduced phase space form (5.21) for

$$\phi_M = 4x_1^2 p_+ p_- - \mathcal{M}^2 = 0. \tag{5.23}$$

The constraint (5.23) is equivalent to the relation $\frac{1}{2}\mathfrak{J}_{AB}\mathfrak{J}^{AB} = R^2\mathcal{M}^2$, the comparison of which with (3.9) in the case of 1-dimensional space ($M_{ij} = 0$) reveals the similarity of the relativistic massive particle system on AdS$_2$ with the non-relativistic conformally invariant system (3.7) with the $m\alpha$ identified with $\mathcal{M}^2 R^2$. The relation between the two systems can be established explicitly via identification of z with x_1 and redefinition of the Lagrange multiplier, $ezR^{-2} \to e$, that presents the Lagrangian (5.14) with added "cosmological" term in the form

$$L = \frac{\dot{x}_1^2 - \dot{x}_0^2}{2e} - \frac{e}{2}\frac{\mathcal{M}^2 R^2}{x_1^2}. \tag{5.24}$$

This Lagrangian can be compared with

$$L = m\frac{\dot{x}_\mu^2}{2\dot{x}^0} - \frac{\dot{x}^0}{2}\frac{\alpha}{x_1^2} \tag{5.25}$$

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being the reparametrization invariant version of the Lagrangian (3.7) for the non-relativistic particle in $\mathcal{R}^1$. With the parameter identification $m\alpha = \mathcal{M}^2 R^2$, the canonical equivalence of the non-relativistic system (3.7) in $\mathcal{R}^1$ to the massive relativistic particle on AdS_2 can be established in the same way as it was done for the systems of the free non-relativistic and massless particles (see also [24]).

The generalization of (5.22) for the case of arbitrary dimension is given by the Lagrangian describing a relativistic particle on the $\text{AdS}_2 \times \text{S}^{d-1}$:

$$L = R^2 \frac{\sigma^2 \dot{\hat{\vec{N}}}^2 - \dot{\hat{\mathfrak{X}}}^2}{2e} - \frac{e}{2}\mathcal{M}^2 + \mathfrak{T}_d(\hat{\vec{N}}). \tag{5.26}$$

Here $\hat{N}_i = N_i/\sqrt{\vec{N}^2}$, $\vec{N} \in \mathcal{R}^d$, is a unit vector, $\hat{\vec{N}}^2 = 1$, and $\mathfrak{T}_d(\hat{\vec{N}})$ is a topologically nontrivial term whose $d = 2$ and $d = 3$ form is specified below. The system (5.26) with $\mathcal{M} = 0$, $\mathfrak{T}_d(\hat{\vec{N}}) = 0$, and $\sigma = 1$ may be canonically related to the free non-relativistic particle in $\mathcal{R}^d$, while the choice $0 < \sigma < 1$ corresponds to the d-dimensional generalization of the free non-relativistic particle on the cone (3.10) obtained via the substitution $\sigma^2 r^2 \dot{\varphi}^2 \to \sigma^2 r^2 \dot{\vec{n}}^2$, $\vec{n} \in S^{d-1}$. The case with $\sigma = 1$, $\mathfrak{T}_d(\hat{\vec{N}}) = 0$ and $\mathcal{M} \neq 0$ corresponds to the system (3.7) with $\alpha > 0$, whereas the case $\alpha < 0$ of the model (3.7) is related to (5.26) with $\mathcal{M}^2$ changed for $-\mathcal{M}^2$. Such a modified system (5.26) describes the free tachyon particle on the $\text{AdS}_2 \times \text{S}^{d-1}$. At last, the charge-vortex and the charge-monopole systems correspond to the choice $\sigma = 1$, $\mathcal{M} = 0$ and [7, 29, 31][6]

$$\mathfrak{T}_2(\hat{\vec{N}}) = \frac{q\Phi}{2\pi}\epsilon_{ij}\hat{N}_i\dot{\hat{N}}_j, \qquad \mathfrak{T}_3(\hat{\vec{N}}) = -\frac{qg}{\dot{\hat{\vec{N}}}^2}\epsilon_{ijk}\hat{N}_i\dot{\hat{N}}_j\ddot{\hat{N}}_k.$$

To demonstrate that the Lagrangian (5.26), corresponds, e.g., in the last case to the specified non-relativistic systems, we write the reparametrization invariant version for the Lagrangian (3.13) with the gauge potential given by Eqs. (3.14), and (3.15):

$$L = m \frac{r^2 \dot{\vec{n}}^2 + \dot{r}^2 - \dot{x}_0^2}{2\dot{x}^0} + \mathfrak{T}_d(\vec{n}). \tag{5.27}$$

Here we have presented $x_i \in \mathcal{R}^d$, $d = 2, 3$, as $x_i = rn_i$, $\vec{n}^2 = 1$. Parametrizing the unit vector $\vec{n}$ by the polar, $\vec{n}(\varphi) = (\cos\varphi, \sin\varphi)$, and spherical, $\vec{n}(\varphi, \vartheta) = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$, (local) angle variables, we get the following representation for the interaction term $qA_i\dot{x}_i$: $\mathfrak{T}_2 = \frac{q\Phi}{2\pi}\dot{\varphi}$ and $\mathfrak{T}_3 = qg\cos\vartheta\dot{\varphi}$ for the planar charge-vortex and the $3D$ charge-monopole systems, respectively. Returning to the Lagrangian (5.26), we note that in addition to reparametrization it possesses the two local scale symmetries acting independently on the variables $\mathfrak{X}_A$ and N_i. The scale gauge invariance can be fixed directly in the Lagrangian by putting in it $\mathfrak{X}^2 = -R^2$ and $\vec{N}^2 = 1$. Then, parametrizing $\mathfrak{X}_A$ as in (5.18) and N_i in terms of the polar or spherical coordinates, and identifying x^1 from (5.19) with the radial variable r,

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[6]Up to a numerical coefficient, the term $\mathfrak{T}_d(\hat{\vec{N}})$ has a sense of the highest curvature (torsion) of some smooth curve in $\mathbb{R}^d$ to which the vector $\vec{N}(\tau)$ is tangent with parameter τ treated as the natural parameter [32].

we establish the canonical relation between (5.26) and (5.27) in the same way as it was done for the free non-relativistic and massless particles: by redefinition of the Lagrange multiplier and application of canonical transformation (5.13).

The general case of the $so(2,1) \oplus so(d-1)$ invariant Lagrangian (5.26) corresponds to the superposition of the non-relativistic systems considered in Section 3. E.g., the $d = 2$ case with $0 < \sigma < 1$, $\mathcal{M} \neq 0$ and nontrivial $\mathfrak{T}_2$ describes the charged particle on the cone in the presence of the scalar potential $1/r^2$ and magnetic vortex located at the apex of the cone. The $d = 3$ case of the Lagrangian (5.26) with $\sigma = 1/2$, $\mathcal{M} \neq 0$, and $\mathcal{T}_3 = 0$ describes the dynamics of the charged particle near the horizon of an extreme Reissner-Nordström black hole [18, 24].

6   Discussion and Outlook

In conclusion, let us summarize the obtained results and discuss shortly some problems which deserve further attention.

The massless particle in Minkowski space $\mathbb{R}^{1,d-1}$ can be treated as the massless particle on the $(d+1)$-dimensional conical hypersurface immersed in $\mathcal{R}^{2,d} = \mathbb{R}^{2,d} - \{0\}$. In such a picture, the rigid conformal symmetry $so(2,d)$ of the massless system in $\mathbb{R}^{1,d-1}$ originates from the isometry of the $\mathcal{R}^{2,d}$. The reparametrization, and the local scale and special conformal transformations generated by the three first class $so(2,d)$-scalar constraints (4.2), (4.3) of the extended system constitute its gauge symmetry being the local conformal $so(2,1)$. With this interpretation, the nonlinear nature of the special conformal transformations of the massless particle in (compactified) Minkowski space is rooted in the nonlinearity of the map (4.9), (4.10) establishing the relation between the phase space variables of the d-dimensional system and observables of the system on the cone. The extended system, in turn, can be interpreted as the massless particle confined to the border of the $(d+1)$-dimensional anti de Sitter space of infinite radius.

The rigid conformal $so(2,1)$ symmetry is the dynamical (hidden) symmetry of various non-relativistic systems including the free particle in the (punctured) Euclidean and conical spaces [26], the conformal mechanics model (3.7) [1], the charge-vortex [7] and the charge-monopole [3, 29] systems, and the conformal model corresponding to the dynamics of the charged particle near the horizon of an extreme Reissner-Nordström black hole [18]. The $so(2,1)$ and rotation symmetries of these systems make their dynamics encoded in the universal Hamiltonian structure (3.18) to be very similar.

By representing the Lagrangian of the free non-relativistic particle in $\mathbb{R}^1$ in the reparametrization invariant form, we showed that the system can be canonically related to the massless particle model in $\mathbb{R}^{1,1}$. With this correspondence, in particular, the square of the generator of the Galilei boosts of the non-relativistic particle is mapped into the generator of the conformal boosts of the massless system. The massless particle in $\mathcal{R}^{1,1} = \mathbb{R}^{1,1} - \{0\}$ is related to the massless particle in AdS_2 by a simple canonical transformation. As a result, the free non-relativistic particle in $\mathcal{R}^1 = \mathbb{R}^1 - \{0\}$ is canonically related to the massless particle on AdS_2. On the other hand, we showed that the canonical transformation establishes also the relation between the massive particle on AdS_2 and non-relativistic model (3.7) in one dimension. Generalizing these results, we have demonstrated that all the listed conformally

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invariant non-relativistic d-dimensional systems can be canonically related to the relativistic particle systems on the AdS$_2\times$S^{d-1} described by the Lagrangian of the universal form (5.26).

The quantization of the non-relativistic systems presented in the reparametrization invariant form results in the corresponding Schrödinger equation. In such a way the hidden conformal symmetry of the equation naturally originates from the $so(2,1)$ symmetry of the classical systems. On the other hand, conformal symmetry is the symmetry of the field systems appearing as a result of quantization of the corresponding relativistic systems described classically by the Lagrangian (5.26). Therefore, it would be interesting to investigate the relationship between the non-relativistic and relativistic systems at the level of the corresponding non-relativistic and relativistic field theories. This could be helpful, in particular, for the better understanding of the hidden $so(2,1)$ symmetry of some non-relativistic Chern-Simons field [10, 11, 33] and fluid [12, 13] models. It would also be natural to generalize the present analysis for the conformally invariant models of the particles with spin and for the superconformal systems. In particular, it would be interesting to apply the approach developed here for the analysis of the nonlinear supersymmetry of the fermion-monopole system [34].

Acknowledgements

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