

The Kepler problem and non commutativity

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Abstract

We investigate the Kepler problem using a symplectic structure consistent with the commutation rules of the noncommutative quantum mechanics. We show that a noncommutative parameter of the order of 10^{-58}m^2 gives observable corrections to the movement of the solar system. In this way, modifications in the physics of smaller scales implies modifications at large scales, something similar to the UV/IR mixing.

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1 Introduction

Recently there has been a growing interest in physics in the study of non-commutative spaces, i. e., spaces characterized by the commutation rules of the coordinates

$$[\hat{x}_i, \hat{x}_j] = i\hbar\Theta_{ij}, \quad (1)$$

with Θ_{ij} constant, real and antisymmetric. This proposal can be traced to the beginning of Quantum Mechanics with Heisenberg [1] and first appears in the papers of Snyder and Yang [2]. This idea was forgotten many years. However motivated by studies in String Theory the idea of non commutativity reborn and now is studied in several contexts as Field Theory [3], String Theory [4] and Condensed Matter [5].

There are several reasons because it is interesting to study a system in a noncommutative space. For example, mathematically it is possible to construct a new Field Theory changing in the action the standard product by the (Weyl-Moyal) product

$$(f * g)(x) = \exp(\frac{i}{2} \Theta_{ij} \partial_i \partial_j) f(x) g(y)|_{x=y}, \quad (2)$$

with f and g functions infinitely differentiable. The theory build with this new product has interesting properties as the relation between ultraviolet and infrared divergences (mixture UV/IR) [6]. In fact, this property could be required for the formulation of a Field Theory valid at small and large scales [7]. On the other hand, in string theory for some background fields, exists non commutativity in the boundary, this implies to low energies the existence of a noncommutative field theory. Furthermore, from dimensional analysis we can see that $\hbar \Theta_{ij}$ must have units of area and in consequence Θ_{ij} has units of [time/mass]. The units of mass involved in the Θ_{ij} parameter, together with the fact that we have only the fundamental constants c and G to get the units may implies that Θ_{ij} has an effect similar to gravity in these systems. In non-commutative field theory this phenomena appears for some gauge fields [8].

On the other hand, we may construct a non commutative Quantum Mechanics using the commutation rules

$$[\hat{x}_i, \hat{x}_j] = i\hbar \Theta_{ij}, \quad [\hat{x}_i, \hat{p}_j] = i\hbar \delta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0. \quad (3)$$

In this new Quantum Mechanics occurs several interesting phenomena, see for example Ref. [9]. Then one may wonder whether in the classical limit these commutation rules have some interesting physics. So, in analogy with the commutation rules (3) we define a symplectic structure given by

$$\{x_i, x_j\} = \Theta_{ij}, \quad \{x_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0. \quad (4)$$

and we try to analyze the Classical Mechanics associated with these systems [10]. If we consider a Hamiltonian of the form $H = \frac{p_i p_i}{2m} + V(x)$, and using the Poisson brackets (4) we get the Hamilton's equations,

$$\dot{x}_i = \frac{p_i}{m} + \Theta_{ij} \frac{\partial V}{\partial x^j}, \quad (5)$$

$$\dot{p}_i = -\frac{\partial V}{\partial x^i}, \quad (6)$$

i.e, in the configuration space we have

$$m\ddot{x}_i = -\frac{\partial V}{\partial x_i} + m\Theta_{ij}\frac{\partial^2 V}{\partial x_j\partial x_k}\dot{x}_k. \quad (7)$$

Note that there is a correction to the Newton second law that depends of the noncommutative parameter, but also of the variations of the external potential. In other words, the correction term can be seen as a perturbation to the space due to the external potential [10].

In this work we study the equation (7) corresponding to the Kepler problem. We show that there is a perihelion shift of the planets. From the analysis of the case of Mercury we see that the planetary system is highly sensible to the non commutative parameter Θ_{ij} . There, we observe that using a parameter of the order of $10^{-25}s/Kg$ ($\hbar\Theta_{ij} \approx 10^{-58}m^2$) we can explain the observed perihelion shift of Mercury. This shows that in this system there is a connection between the physics at small scales to physics of large scales. Perhaps this phenomena is related to the mixture UV/IR that appears in non commutative Field Theory. Another result of this paper is a lower bound for Θ of the order of $10^{-30}s/Kg$ ($\sqrt{\hbar\Theta} \approx 5 \times 10^3 L_P$), that may implies the possibility of non commutativity in the space before of the Planck scale. Furthermore we show that the corrections to the second and third laws of Kepler, have a similar form to the obtained in the case of the Kerr metric.

2 The Kepler problem

For the Kepler problem the potential is $V(r) = \frac{-k}{r}$, using this expression in (7) we get

$$m\ddot{x}_i = -\frac{x_i}{r}\frac{k}{r^2} + m\epsilon_{ijk}\dot{x}_j\Omega_k + m\epsilon_{ijk}x_j\dot{\Omega}_k \quad (8)$$

where we consider that the non commutative parameter Θ_{ij} , has the form $\Theta_{ij} = \epsilon_{ijk}\Theta_k$, and the angular velocity Ω_i has the expression

$$\Omega_i = \frac{k}{r^3}\Theta_i.$$

In this problem the Hamiltonian is a constant of motion

$$H = \frac{p_i p^i}{2m} + V(r). \quad (9)$$

However, the components of the angular momenta $L^i = m\epsilon^{ijk}x_j\dot{x}_k$ are not conserved. Nevertheless, the generator of rotations about the Θ_i axis L_Θ , see

Ref. [10], given by

$$L_{\Theta} = \Theta_{ij}x_i p_j + \frac{1}{2}\Theta_{ij}p_j \Theta_{ik}p_k. \quad (10)$$

is conserved. In the following we will consider only one independent non-commutative parameter $\Theta = \delta_{i3}\Theta_i$ in this case L_{Θ} can be rewritten in the form

$$L_{\Theta} = \Theta \left[xp_y - yp_x - \Theta mV(r) - \Theta \frac{p_z^2}{2} + \Theta mH \right]. \quad (11)$$

Considering in (11) that the Hamiltonian is a constant of motion, we find that the expression

$$M = xp_y - yp_x - \Theta mV(r) - \Theta \frac{p_z^2}{2} \quad (12)$$

is also conserved. Now, changing variables to spherical coordinates, the equations (8) take the form

$$m(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2 \sin^2(\theta)) = -\frac{dV}{dr} + mr\Omega\dot{\phi}\sin^2(\theta), \quad (13)$$

$$m\frac{d}{dt}(r^2\dot{\theta}) - mr^2\dot{\phi}^2 \sin(\theta)\cos(\theta)\dot{\phi}^2 = mr^2\Omega\dot{\phi}\sin(\theta)\cos(\theta), \quad (14)$$

$$\frac{d}{dt}(mr^2\dot{\phi}\sin^2(\theta)) = -mr\sin(\theta)\frac{d}{dt}(r\Omega\sin(\theta)). \quad (15)$$

The general solution of this system of equations includes the case of no plane orbits. However, the case of a plane orbit is still a valid solution, so we will consider the special case of equatorial orbits, i. e. $\theta = \frac{\pi}{2}$. For this choice the equations (13) through (15) are reduce to

$$m(\ddot{r} - r\dot{\phi}^2) = -\frac{dV}{dr} + mr\Omega\dot{\phi}, \quad (16)$$

$$\frac{d}{dt}(mr^2\dot{\phi}) = -mr\frac{d}{dt}(r\Omega). \quad (17)$$

In terms of these variables the constant of motion M has the expression

$$M = mr^2(\dot{\phi} + \Omega) - \Theta mV. \quad (18)$$

A comparison of Eqs. (17) and (18) shows that we can rewrite (17) as $\dot{M} = 0$. We are thus to conclude that for the Kepler problem the constants of motion have the form

$$M = mr^2\dot{\phi} + \frac{2mk\Theta}{r}, \quad (19)$$

$$E = H = \frac{m}{2}\dot{r}^2 + \frac{M^2}{2mr^2} - \frac{k}{r} - \frac{k\Theta M}{r^3} + \frac{k^2\Theta^2 m}{2r^4}. \quad (20)$$

At this point we notice that the Eq. (19) is very similar to the equation for the angular momentum about the z axis, M_z , for a particle of mass m that follows a plane orbit in the gravitational field of a Kerr black hole, see Ref. [11]. The equation for M_z is

$$M_z = mr^2\dot{\phi} + \frac{2m\mathbf{ma}}{r} \quad (21)$$

where \mathbf{ma} is the geometric angular momentum of the black hole. In this case M_z is conserved and comparing with (19) we see that we can identify in our case the geometric angular momentum with

$$\mathbf{ma} = k\Theta \quad (22)$$

From this comparison we observe that in the same way that in the case of the Kerr metric, for our Kepler problem the second and the third Kepler's laws are not valid. So, due to the corrections induced by the Θ parameter the radius vector drawn from the sun to a planet *not describes equal areas in equal times*.

Returning to the analysis of the radial equation (13), we may rewrite this equation in the form

$$m\ddot{r} - \frac{M^2}{mr^3} + \frac{k}{r^2} + \frac{3kM\Theta}{r^4} - \frac{2\Theta^2k^2m}{r^5} = 0. \quad (23)$$

From this expression we get r in terms of time. However, is more interesting to write r in terms of ϕ , to do that we use the variable

$$u = \frac{1}{r}.$$

Now it turns out that if we use

$$\dot{r} = -\frac{M - 2\Theta km u}{m} \left(\frac{du}{d\phi} \right), \quad (24)$$

the radial equation (23) is reduced to

$$\begin{aligned} & \frac{(M - 2\Theta km u)^2}{m} \frac{d^2u}{d\phi^2} - 2\Theta k(M - 2\Theta km u) \left(\frac{du}{d\phi} \right)^2 \\ & + \frac{M^2}{m} u - 3\Theta kMu^2 + 2\Theta^2k^2mu^3 - k = 0. \end{aligned} \quad (25)$$

Taking the parameters

$$e = \sqrt{1 + \frac{2EM^2}{mk^2}} \quad b = \frac{M^2}{mk},$$

from the standard Kepler's problem, we get to zero order in Θ

$$\frac{d^2 u_0}{d\phi^2} + u_0 - \frac{1}{b} = 0. \quad (26)$$

Solving this equation we obtain

$$u_0 = \frac{(1 + e \cos \phi)}{b}. \quad (27)$$

To the next order in Θ we propose a solution of the form

$$u = u_0 + u_1.$$

Then u_1 must satisfy

$$\frac{d^2 u_1}{d\phi^2} + u_1 = \frac{\Theta m k}{M b^2} \left(2e \cos \phi - \frac{3}{2} e^2 \cos 2\phi + \frac{e^2 + 6}{2} \right), \quad (28)$$

which is solved by

$$u_1 = \frac{\Theta m k}{M b^2} \left(\frac{e^2 + 6}{2} + e \phi \sin \phi + \frac{e^2}{2} \cos 2\phi \right). \quad (29)$$

So, to this order in Θ we have

$$u = \left[\frac{1 + e \cos \phi (1 - \frac{\delta}{b})}{b} \right] + \left(\frac{\delta}{b^2} \right) \left[\frac{e^2 + 6}{2} + \frac{e^2}{2} \cos 2\phi \right] \quad (30)$$

with $\delta = \frac{\Theta m k}{M}$. From the Eq. (30) the perihelion shift per revolution is given by

$$\delta \phi_{NC} = \frac{2\pi \delta}{b} = 2\pi \Theta \left(\frac{m k}{b^3} \right)^{1/2}. \quad (31)$$

On the other hand, using the variable

$$a = \frac{r_{max} + r_{min}}{2} = \left(\frac{b}{1 - e^2} \right), \quad (32)$$

which combined with

$$k = m \mathbf{m} G,$$

where \mathbf{m} is the mass of the sun, tell us that,

$$\delta \phi_{NC} = 2\pi \Theta \left(\frac{m^2 \mathbf{m} G}{a^3 (1 - e^2)^3} \right)^{1/2}. \quad (33)$$

Notice that in Eq. (33) the only constants that are not determined by the system are G and Θ .

In the case of General Relativity, with the Schwarzschild metric the shift to the perihelion is,

$$\delta\phi_{RG} = 6\pi \frac{G\mathbf{m}}{c^2 a(1 - e^2)}. \quad (34)$$

Here, the constants that are not determined by the system are c and G , notice that these are fundamental constants. It therefore appears that Θ has in our problem the role of a fundamental constant.

In particular, in the case of the Mercury planet, using the data,

$$a \approx 6 \times 10^{10} \text{m}, \quad (35)$$

$$e \approx 0.2, \quad (36)$$

$$m \approx 3.3 \times 10^{23} Kg, \quad (37)$$

$$\mathbf{m} \approx 2 \times 10^{30} Kg, \quad (38)$$

$$G \approx 7 \times 10^{-11} \frac{(\text{m})^3}{s^2 Kg}, \quad (39)$$

$$\hbar \approx 6.6 \times 10^{-34} Js. \quad (40)$$

we found that the perihelion shift is of the order,

$$\delta\phi_{NC} \approx 2\pi\Theta(3 \times 10^{17}) \frac{Kg}{s}. \quad (41)$$

Now, considering that the corrections due to the non commutativity must be smaller to agree with the observational results and the number that we get 3×10^{17} is very big. We conclude from (41) that the parameter Θ must be very small. This shows that the planetary system is very sensible to the parameter Θ . So small changes in the non commutativity implies sensible changes to very large scales. In other words, there is a connection between the physics at small scales and the physics at large scales. Perhaps this relationship is an indication of the mixture UV/IR that occurs in noncommutative Field Theory [6].

Armed with the above result and taking into account that the observed perihelion shift for Mercury is [12]

$$\delta\phi_{obs} = 2\pi(7.98734 \pm 0.00037) \times 10^{-8} \frac{\text{rad}}{\text{rev}}, \quad (42)$$

and, if we assume that $\delta\phi_{NC} \approx \delta\phi_{obs}$, we obtain that the Θ parameter is of the order

$$\Theta \approx 3 \times 10^{-25} \frac{s}{Kg}, \quad (43)$$

so that,

$$\hbar\Theta \approx 2 \times 10^{-58} \text{m}^2,$$

or

$$\sqrt{\hbar\Theta} \approx 1 \times 10^{-29} \text{m}. \quad (44)$$

On the other hand, General Relativity predicts for the perihelion shift,

$$\delta\phi_{RG} = 2\pi(7.987344) \times 10^{-8} \frac{\text{rad}}{\text{rev}}. \quad (45)$$

Then, we found a lower bound for Θ requiring that

$$|\delta\phi_{NC}| \leq |\delta\phi_{GR} - \delta\phi_{obs}| \approx 2\pi(1 \times 10^{-12}) \frac{\text{rad}}{\text{rev}}. \quad (46)$$

From this we get,

$$\Theta \leq 3 \times 10^{-30} \frac{s}{Kg},$$

or

$$\hbar\Theta \leq 21 \times 10^{-64} \text{m}^2,$$

$$\sqrt{\hbar\Theta} \leq 5 \times 10^{-32} \text{m} \approx (3 \times 10^3) L_P.$$

In natural units we obtain

$$4 \times 10^{15} \text{GeV} \leq \frac{1}{\sqrt{\hbar\Theta}}.$$

This bound is one order of magnitude larger than the obtained with the Standard Model [9]. However, for a best comparison it will be necessary to study the Kepler problem in a non commutative curved space. But is remarkable that our result is very close to the obtained using High Energy Physics arguments. For example in Ref. [13], using a different symplectic structure they arrive to a lower bound of the order of 10^{-68}m , that is very small compared with the Planck scale.

3 Conclusions

In this work we studied the Kepler problem with a symplectic structure consistent with the commutation rules of non commutative Quantum Mechanics. We get that the corrections to the second and third laws of Kepler, have a similar form to the obtained in the case of the Kerr metric. We show that there is a correction to the perihelion shift of Mercury, and with a $\hbar\Theta$ parameter of the order of 10^{-58}m^2 we obtain an observable deviation. So, in this case there is a connection between the physics at small scales and the physics a large scales. Furthermore we get a lower bound to $\sqrt{\hbar\Theta}$ of the order of 10^{-32}m , that implies that the noncommutative properties of the space appear before of the Planck length scale.

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