

# A Note on the Noncommutative Wess-Zumino Model \*

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## Abstract

We show that the noncommutative Wess-Zumino (NCWZ) Lagrangian with permutation terms in the interaction parts is renormalizable at one-loop level by only a wave function renormalization. When the non-commutativity vanishes, the logarithmic divergence of the wave function renormalization of the NCWZ theory is the same as that of the commutative one. Next the algebras of noncommutative field theories (NCFT's) are studied. From Neother currents, the field representation for the generators of NCFT's is extracted. Then based on this representation, the commutation relations between the generators are calculated for NCFT's. The symmetry properties of NCFT's inferred from these commutation relations are discussed and compared with those of the commutative ones.

PACS number(s): 11.10.-z, 11.10.Gh, 11.30.-j 12.60.Jv

## 1 Introduction

A deformation, an inverse of contraction (in the sense of Segal-Wigner-Inönü contraction), is one of the methods of generalization of a physical theory [1]. The nondeformed theory can be recovered from the deformed one when taking a limit of deformation parameter to some value, e.g., nonrelativistic, classical physics, the nondeformed theory, is recovered from relativistic physics when taking the velocity of light  $c \rightarrow \infty$ , and from quantum physics when taking the Planck constant  $\hbar \rightarrow 0$ . This naive concept has been applied to field theories on noncommutative (NC) spaces considered as deformations of flat Euclidean or Minkowski spaces. A product of fields on NC spaces can be expressed as a deformed product or star-product [2, 3] of fields on commutative spaces [6, 7, 8]. Nevertheless, a question arises if the commutative field theories can be recovered from their NC counter ones when non-commutativity  $\Theta^{\mu\nu} \rightarrow 0$ . At this moment, there is no conclusive answer to the question, and noncommutative field theories (NCFT's) must be investigated one by one.

An immediate question is the renormalizability of NCFT's. It was shown by Filk [6] that the NC complex scalar field theory has the same kind of divergences as the commutative one, and it was recently conjectured by Minwalla, Van Raamsdonk and Seiberg [8] that if a commutative theory is renormalizable, then the corresponding NC theory is also renormalizable, and if a commutative theory is not renormalizable, the corresponding NC theory is also not renormalizable. In [10], the renormalizability of the NC scalar field theory was studied, and the noncommutative Wess-Zumino (NCWZ) action on superspace was conjectured to be renormalizable. Later, in [11], the deformation aspects of supersymmetric field theories were investigated, and the deformed Wess-Zumino model was again expected to be renormalizable. The superfield formulation

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\*hep-th/0009070, UFIFT-HEP-00-13

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of the NCWZ model is discussed and applied to the renormalization of the theory in [4] and [5]. The detailed calculation of the 1PI diagram at one-loop has not been done yet.

Another interesting issue is that NCFT's have nonlocal interaction terms which explicitly break Lorentz invariance. However, the symmetry must be broken in a particular way by the deformed star products of the fields. Therefore, it is interesting to see how the Lorentz group is deformed in NCFT's.

In this paper, we investigate deformability and renormalizability of the NCWZ theory on Minkowski space. In section 2, we review the wave function renormalization of the NC scalar  $\Phi^4$  theory. In section 3, by adding permutation terms in the interaction part to preserve the supersymmetry transformations, we modify the original Wess-Zumino Lagrangian to be the NCWZ Lagrangian and investigate its renormalizability at one-loop order. In section 4, we extract a representation for the algebras of NCFT's from Noether currents and calculate the commutation relations of the algebras. Then in the last section we compare our results with other works on NCFT's and comment on further research opportunities.

## 2 The NC $\Phi^4$ theory

To serve as an introduction to the renormalization of the NCWZ theory, let us review a result of the NC  $\Phi^4$  theory in the four-dimensional space-time, which is described by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \Phi \star \partial^\mu \Phi - \frac{1}{2} m^2 \Phi \star \Phi - \frac{\lambda}{4!} \Phi \star \Phi \star \Phi \star \Phi. \quad (1)$$

As discussed in [6] [7] and [8], under the integration, the star-product of fields does not affect the quadratic parts of the Lagrangian, whereas it makes the interaction part become nonlocal. Hence, Feynman rules in the momentum space of the NCFT are similar to those of the commutative one, except that the vertices of the NCFT are modified by a phase factor. For the Lagrangian (1), the Feynman rule for the deformed vertex is

$$\begin{aligned} & -\frac{i}{3} \lambda \left( \cos \frac{1}{2} (p_1 \times p_2 + p_1 \times p_3 + p_2 \times p_3) \right. \\ & \quad + \cos \frac{1}{2} (p_1 \times p_2 + p_1 \times p_3 - p_2 \times p_3) \\ & \quad \left. + \cos \frac{1}{2} (p_1 \times p_2 - p_1 \times p_3 - p_2 \times p_3) \right), \end{aligned} \quad (2)$$

where  $p_i$ 's,  $i = 1 \dots 4$ , are the momenta coming out of the vertex, and  $p_i \times p_j \equiv p_{i\mu} \Theta^{\mu\nu} p_{j\nu}$ , where the non-commutativity  $\Theta^{\mu\nu}$  is an anti-symmetric second rank tensor defined by  $[q^\mu, q^\nu] = i\Theta^{\mu\nu}$ . When  $\Theta^{\mu\nu} \rightarrow 0$ , the deformed vertex becomes the non-deformed one. By using the above vertex, one yields a wave function renormalization of the scalar field  $\Phi$  at one-loop order which has only one diagram as follows:

$$\begin{aligned} \Gamma^{(\Phi\Phi)}(p^2) &= i \frac{\lambda}{6} \int \frac{d^4 k}{i(2\pi)^4} \frac{(2 + \cos(p \times k))}{(k^2 + m^2)} \\ &= i \frac{\lambda}{48\pi^2} \int_0^\infty \frac{d\alpha}{\alpha^2} e^{-i\alpha m^2} \left( 1 + \frac{1}{2} e^{-i\frac{\tilde{p}^2}{4\alpha}} \right) e^{\frac{i}{\Lambda^2 \alpha}} \\ &= i \frac{\lambda}{48\pi^2} \left( \Lambda^2 - m^2 \ln\left(\frac{\Lambda^2}{m^2}\right) \right) + i \frac{\lambda}{96\pi^2} \left( \Lambda_{eff}^2 - m^2 \ln\left(\frac{\Lambda_{eff}^2}{m^2}\right) \right) + \dots \end{aligned} \quad (3)$$

The Schwinger parametrization technique to deal with the above integrations can be found in Itzykson and Zuber [13] and Hayakawa [14]. In the second line, the term proportional to  $\exp(-i\tilde{p}^2/4\alpha)$ , where  $\tilde{p}^\mu = \Theta^{\mu\nu} p_\nu$ , is due to the nonplanar contribution, and the factor  $\exp(i/\rho\Lambda^2)$  is introduced to regulate the small  $\rho$  divergence in the planar contribution. Note that the nonplanar contribution is one-half of the planar one. In the third line, we keep only the divergent terms and the effective cutoff,  $\Lambda_{eff}^2 = 1/(1/\Lambda^2 + (-\tilde{p}^2)/4)$ , showing the mixing of ultraviolet (UV) divergence and Infrared (IR) singularity [8]. The above integration can also be done by using the dimensional regularization method, as shown in [16]. Renormalization of the theory at two loops is also discussed in detail in [9].

In the case that  $\Phi$  is a complex scalar field, there are two ways in ordering the fields  $\Phi$  and  $\Phi^*$  in the quartic interaction  $(\Phi^*\Phi)^2$ . So, the general potential of the NC complex scalar field action is

$$g_1\Phi^*\star\Phi\star\Phi^*\star\Phi+g_2\Phi^*\star\Phi^*\star\Phi\star\Phi.$$

The potential is invariant under the global transformation, since the star product has nothing to do with the constant phase transformation. It was shown by Aref'eva, Belov and Koshelev [17] that the theory is not renormalizable for arbitrary values of  $g_1$  and  $g_2$ , and is renormalizable at one-loop level only when  $g_2 = 0$ , or  $g_1 = g_2$ .

### 3 The NCWZ theory

In this section, we focus on renormalization at one loop in the NCWZ theory. We modify the original WZ Lagrangian [12, 18] to be the NCWZ Lagrangian by adding the permutation terms in the interaction part to preserve supersymmetry transformations. Here, we follow the conventions by Sohnius [19]. The NCWZ model is described by the sum of the free off-shell Lagrangian and of the two invariants,

$$\mathcal{L}_{tot} = \mathcal{L}_0 + \mathcal{L}_m + \mathcal{L}_g, \quad (4)$$

where

$$\mathcal{L}_0 = \frac{1}{2} (\partial_\mu A \partial^\mu A + \partial_\mu B \partial^\mu B + i\bar{\Psi} \not{\partial} \Psi + F^2 + G^2), \quad (5)$$

$$\mathcal{L}_m = -m(FA + GB + \frac{1}{2}\bar{\Psi}\Psi), \quad (6)$$

$$\mathcal{L}_g = -\frac{g}{3} (A \star A \star F - B \star B \star F + A \star B \star G + \bar{\Psi} \star (A - \gamma_5 B) \star \Psi + \text{permutation terms}). \quad (7)$$

The off-shell Lagrangians  $\mathcal{L}_0$ ,  $\mathcal{L}_m$  and  $\mathcal{L}_g$  are separately invariant under the supersymmetry transformations:

$$\delta A = \bar{\alpha}\Psi, \quad \delta B = \bar{\alpha}\gamma_5\Psi, \quad \delta F = i\bar{\alpha} \not{\partial} \Psi, \quad \delta G = i\bar{\alpha}\gamma_5 \not{\partial} \Psi, \quad \delta\psi = -(F + \gamma_5 G)\alpha - i \not{\partial}(A + \gamma_5 B)\alpha, \quad (8)$$

where  $\alpha$  and  $\bar{\alpha}$  are the global infinitesimal Majorana spinor parameters.

The Feynman rules in the momentum space can be extracted out directly from the Lagrangians (4). One gets as follows:

#### 1. Propagators

The propagators of the fields and the mixed fields of the NC theory are the same as those of the commutative one.

#### 2. Deformed vertices

- $-\frac{g}{3}(A \star A \star F + \text{permutation terms})$

$$-2ig \cos(\frac{1}{2}p_{A_i} \times p_{A_o}).$$

- $\frac{g}{3}(B \star B \star F + \text{permutation terms})$

$$2ig \cos(\frac{1}{2}p_{B_i} \times p_{B_o}).$$

- $-\frac{g}{3}(A \star B \star G + \text{permutation terms})$

$$-2ig \cos(\frac{1}{2}p_A \times p_B).$$

- $-\frac{g}{3}(\bar{\Psi} \star A \star \Psi + \text{permutation terms})$

$$-2ig \cos(\frac{1}{2}p_i \times p_o).$$

- $\frac{2}{3}(\bar{\Psi} \star \gamma_5 B \star \Psi + \text{permutation terms})$

$$2ig\gamma_5 \cos(\frac{1}{2}p_i \times p_o).$$

where the subscripts  $i$  and  $o$  label incoming and outgoing momenta. The deformed vertices we obtain differ from the nondeformed ones by the factor,  $\cos(\frac{1}{2}p_i \times p_o)$ .

By using the above Feynman rules, one can calculate the one-loop UV divergent contributions to the 1PI 2-point and 3-point functions. The results are summarized as follows:

### 1. Wave function renormalization

- Majorana field  $\Psi$

For the Majorana field, at one loop there are two diagrams. Their sum gives a contribution

$$\begin{aligned} \Gamma^{(\bar{\Psi}\Psi)}(\not{p}) &= -8ig^2 \int \frac{d^4k}{i(2\pi)^4} \cos^2(\frac{1}{2}p \times k) \frac{\not{k}}{(k^2 - m^2)((k+p)^2 - m^2)} \\ &= i \not{p} \frac{g^2}{4\pi^2} \int_0^1 d\alpha (1-\alpha) \int_0^\infty \frac{d\rho}{\rho} e^{-\rho i(m^2 - \alpha(1-\alpha)p^2)} \left(1 + e^{-\frac{1}{\rho} \frac{i\tilde{p}^2}{4}}\right) e^{\frac{1}{\rho} \frac{i}{\Lambda^2}} \\ &= i \not{p} \frac{g^2}{8\pi^2} \left( \ln\left(\frac{\Lambda^2}{m^2}\right) + \ln\left(\frac{\Lambda_{eff}^2}{m^2}\right) \right) + \dots \end{aligned} \quad (9)$$

- Scalar fields  $A, B$

For each field, at one loop there are five diagrams. Their sum gives a contribution

$$\begin{aligned} \Gamma^{(AA)}(p^2) = \Gamma^{(BB)}(p^2) &= -8ig^2 \int \frac{d^4k}{i(2\pi)^4} \cos^2(\frac{1}{2}p \times k) \frac{k \cdot p}{(k^2 - m^2)((k+p)^2 - m^2)} \\ &= ip^2 \frac{g^2}{8\pi^2} \left( \ln\left(\frac{\Lambda^2}{m^2}\right) + \ln\left(\frac{\Lambda_{eff}^2}{m^2}\right) \right) + \dots \end{aligned} \quad (10)$$

- Auxiliary fields  $F, G$

For the  $F$  field, at one loop there are two diagrams. While, for the  $G$  field, at one loop there is only one diagram. However, they give the same contribution

$$\begin{aligned} \Gamma^{(FF)}(p^2) = \Gamma^{(GG)}(p^2) &= 4ig^2 \int \frac{d^4k}{i(2\pi)^4} \cos^2(\frac{1}{2}p \times k) \frac{1}{(k^2 - m^2)((k+p)^2 - m^2)} \\ &= i \frac{g^2}{8\pi^2} \left( \ln\left(\frac{\Lambda^2}{m^2}\right) + \ln\left(\frac{\Lambda_{eff}^2}{m^2}\right) \right) + \dots \end{aligned} \quad (11)$$

- Mixed fields

$$\Gamma^{(FA)}(p^2) = \Gamma^{(GB)}(p^2) = 0. \quad (12)$$

Again, all the integrations can be done directly by using the Schwinger parametrization technique [13, 14]. The divergent terms of the one-loop corrections are the same for all the fields, whereas the finite terms of  $\Gamma^{(FF)}$  and  $\Gamma^{(GG)}$  are different from those of the others. However, all the finite terms are the functions of  $p^2$  and  $\tilde{p}^2$ , and give finite contributions when  $p = 0$ , i.e., there is no IR singularity. Note that in the NCWZ model the planar and nonplanar contributions have the same multiplicative factor, and when  $\Theta^{\mu\nu} \rightarrow 0$ , the right factor of the commutative Wess-Zumino model is retrieved.

### 2. Mass renormalizations

- Since, at one-loop  $\Gamma^{(\bar{\Psi}\Psi)}(\not{p})$  is proportional to only  $\not{p}$ , and both  $\Gamma^{FA}$  and  $\Gamma^{GB}$  are zero, the only mass renormalization is that due to the wave function renormalization.

### 3. Vertex corrections

- $FA^2, FB^2, ABG$

For each vertex, at one loop there are two diagrams, and they are added up to zero. So, there is no correction for each vertex.

- $\bar{\Psi}\Psi A, \bar{\Psi}\gamma_5\Psi B$

Similarly, there is no correction for each of these two vertices, since at one loop there are two diagrams, and they are added up to finite values.

Just as in the  $\Phi^4$  theory, the UV/IR mixing also appears in the NCWZ theory, which is the general consequence of the uncertainty relations between noncommutative coordinates [7]. Renormalization in the NCWZ theory is very similar to the commutative one. Compared with the ordinary Wess-Zumino theory, the counter term for the wave function renormalization reduces one-half, but the cancellations, in particular the absence of mass and vertex corrections, persist due to supergauge invariance.

## 4 The Algebras of NCFT's

In this section the algebras of NCFT's are studied. We'll follow the Noether's procedure to derive the conserved currents, from which the generators are obtained, then the commutation relations between those generators are calculated.

### 4.1 Notations and Identities

To facilitate the calculations involving NC fields star product, we introduce the following notations and list the useful identities.

Define an operator  $\Delta$ , which acts nontrivially on a scalar pair-product  $(f, g)$  as,

$$\begin{aligned}\Delta(f, g) &\equiv \partial_\mu f \tilde{\partial}^\mu g, \\ \Delta^2(f, g) &= \partial_\mu \partial_\nu f \tilde{\partial}^\mu \tilde{\partial}^\nu g, \\ &\vdots = \vdots \\ \Delta^n(f, g) &= \underbrace{\partial_\mu \partial_\nu \cdots \partial_\rho}_n f \underbrace{\tilde{\partial}^\mu \tilde{\partial}^\nu \cdots \tilde{\partial}^\rho}_n g,\end{aligned}\tag{13}$$

where  $\tilde{\partial}^\mu \equiv \frac{i}{2} \Theta^{\mu\nu} \partial_\nu$ <sup>1</sup>.

With our definition, a star product between two scalar fields  $A$  and  $B$  can be written as

$$\begin{aligned}A \star B &= e^\Delta(A, B) \\ &= \left(1 + \Delta + \frac{\Delta^2}{2!} + \frac{\Delta^3}{3!} + \cdots\right)(A, B) \\ &= AB + \partial_\mu \left(E(\Delta)(A, \tilde{\partial}^\mu B)\right),\end{aligned}\tag{14}$$

where the operator  $E(\Delta)$  is

$$E(\Delta) = \frac{e^\Delta - 1}{\Delta} = \sum_{n=0}^{\infty} \frac{\Delta^n}{(n+1)!}.\tag{15}$$

By using the above notations, we obtain some useful identities:

1.  $B \star A = AB - \partial_\mu \left(E(-\Delta)(A, \tilde{\partial}^\mu B)\right).$
2.  $[A, B]_\star \equiv A \star B - B \star A = 2\partial_\mu \left(\frac{\sinh(\Delta)}{\Delta}(A, \tilde{\partial}^\mu B)\right).$
3.  $\{A, B\}_\star \equiv A \star B + B \star A = 2AB + 2\partial_\mu \left(\frac{\cosh(\Delta)-1}{\Delta}(A, \tilde{\partial}^\mu B)\right).$

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<sup>1</sup>We include the factor  $\frac{i}{2}$  here, slightly different from the definition in Section 2.

4.  $(x_\rho A) \star B = x_\rho(A \star B) + A \star \tilde{\partial}_\rho B.$
5.  $B \star (x_\rho A) = x_\rho(B \star A) - \tilde{\partial}_\rho B \star A.$
6.  $[(x_\rho A), B]_\star = x_\rho[A, B]_\star + \{A, \tilde{\partial}_\rho B\}_\star.$
7.  $[B, (x_\rho A)]_\star = x_\rho[B, A]_\star - \{A, \tilde{\partial}_\rho B\}_\star.$
8.  $\{(x_\rho A), B\}_\star = x_\rho\{A, B\}_\star + [A, \tilde{\partial}_\rho B]_\star.$

We assume  $\theta^{0i} = 0$  from now on for causality and unitarity reasons [15]. The immediate consequence is that non-commutativity will not introduce higher order time derivatives of the fields in Lagrangian.

## 4.2 $\Phi^4$ theory

Now let us calculate the Noether currents of the NC  $\Phi^4$  theory following standard technique [22]. Varying the Lagrangian (1), and using the above identities and also the equation of motion, one gets

$$\delta \int d^4x \mathcal{L} = \int d^4x \partial_\mu \left( \frac{1}{2} \{ \partial^\mu \Phi, \delta_0 \Phi \}_\star + \delta x^\mu \mathcal{L} + \frac{\lambda}{12} \frac{\sinh(\Delta)}{\Delta} (\Phi \star \Phi, \tilde{\partial}^\mu [\Phi, \delta_0 \Phi]_\star) \right). \quad (16)$$

Under an infinitesimal translation,  $\delta x^\mu = g^{\mu\nu} \epsilon_\nu$ ,  $\delta_0 \Phi = -\epsilon_\nu \partial^\nu \Phi$ , one yields the energy-momentum tensor,

$$T^{\mu\nu} = \frac{1}{2} \{ \partial^\mu \Phi, \partial^\nu \Phi \}_\star - g^{\mu\nu} \mathcal{L} + \frac{\lambda}{12} \frac{\sinh(\Delta)}{\Delta} (\Phi \star \Phi, \tilde{\partial}^\mu [\Phi, \partial^\nu \Phi]_\star). \quad (17)$$

As explicitly seen, the energy-momentum tensor  $T^{\mu\nu}$  is conserved since its divergence is zero.

Under the infinitesimal Lorentz transformation,  $\delta x^\mu = \epsilon^{\rho\sigma} x_\nu = -\frac{1}{2} \epsilon^{\rho\sigma} (x_\rho g_\sigma^\mu - x_\sigma g_\rho^\mu)$ ,  $\delta_0 \Phi = \frac{1}{2} \epsilon^{\rho\sigma} (x_\rho \partial_\sigma \Phi - x_\sigma \partial_\rho \Phi)$ , where  $\epsilon^{\rho\sigma}$  is an anti-symmetric second rank tensor, one obtains a three-index current

$$\begin{aligned} j_{\rho\sigma}^\mu &= T_\rho^\mu x_\sigma + \frac{1}{2} [\partial_\rho \Phi, \tilde{\partial}_\sigma \partial^\mu \Phi]_\star + \frac{\lambda}{12} (\sinh(\Delta)/\Delta)' (\tilde{\partial}_\sigma (\Phi \star \Phi), \tilde{\partial}^\mu [\Phi, \partial_\rho \Phi]_\star) \\ &\quad - \frac{\lambda}{12} \frac{\sinh(\Delta)}{\Delta} (\Phi \star \Phi, \tilde{g}_\sigma^\mu [\Phi, \partial_\rho \Phi]_\star + \tilde{\partial}^\mu \{ \tilde{\partial}_\sigma \Phi, \partial_\rho \Phi \}_\star) - (\rho \leftrightarrow \sigma), \end{aligned} \quad (18)$$

where  $(\sinh(\Delta)/\Delta)' = (\Delta \cosh(\Delta) - \sinh(\Delta))/\Delta^2$ . The divergence of the three-index current is not equal to zero due to the presence of the terms proportional to the non-commutativity  $\Theta^{\mu\nu}$ . However, note that the Noether currents of the commutative scalar field theory can be obtained by setting  $\Theta^{\mu\nu}$  equal to zero.

In the case of the commutative  $\Phi^4$  theory, one yields the momentum and Hamiltonian generators from the energy-momentum tensor, and the angular momentum and boost generators from the three-index current [22]. These generators form the Poincaré algebra. For the NC  $\Phi^4$  theory, one obtains its generators analogous to those of the commutative one,

$$P^i = \int d^3x (\partial^i \Phi) \dot{\Phi} \equiv \int d^3x \mathcal{P}^i, \quad (19)$$

$$P^0 = \int d^3x \left( \frac{1}{2} (\dot{\Phi}^2 + (\vec{\partial}\Phi)^2 + m^2 \Phi^2) + \frac{\lambda}{4!} \Phi^4 \right) \equiv \int d^3x \mathcal{P}^0, \quad (20)$$

$$M^{0i} = \int d^3x (x^0 \mathcal{P}^i - x^i \mathcal{P}^0), \quad (21)$$

$$M^{ij} = \int d^3x (x^i \mathcal{P}^j - x^j \mathcal{P}^i). \quad (22)$$

The surface terms of  $M^{0i}$  and  $M^{ij}$  are dropped out. These generators generate the translational, rotational and boost transformations on  $\Phi$ .

By using the quantization condition,  $[\Phi(\vec{x}), \dot{\Phi}(\vec{y})] = i\delta^3(\vec{x} - \vec{y})$ , one can easily obtain the following equal-time commutation relations:

$$[P^\mu, P^\nu] = 0, \quad (23)$$

$$[M^{ij}, M^{kl}] = i(\eta^{il} M^{jk} + \eta^{jk} M^{il} - \eta^{ik} M^{jl} - \eta^{jl} M^{ik}), \quad (24)$$

$$[M^{ij}, P^k] = i(\eta^{jk} P^i - \eta^{ik} P^j), \quad (25)$$

$$[M^{0i}, P^j] = i\eta^{ij} P^0. \quad (26)$$

The above commutation relations of the NC  $\Phi^4$  theory are the same as those of the commutative one. In particular, (23) verifies that the NC  $\Phi^4$  Lagrangian has translational invariance and the translation generator  $P^\mu$  is conserved. But the following commutation relations have some additional terms proportional to  $\Theta^{\mu\nu}$ , due to the symmetry-breaking term  $\frac{\lambda}{4!}\Phi^{*4}$ ,

$$[M^{0i}, P^0] = -i\eta^{00}P^i - i\frac{\lambda}{4!} \int d^3x \{ \dot{\Phi}, [\Phi^{*2}, \tilde{\partial}^i \Phi]_* \}_*, \quad (27)$$

$$[M^{ij}, P^0] = -i\frac{\lambda}{3!} \int d^3x x^i (\partial^j \Phi) \Phi^{*3} + (i \leftrightarrow j), \quad (28)$$

$$[M^{0i}, M^{0j}] = -i\eta^{00}M^{ij} + i\frac{\lambda}{4!} \int d^3x \left( x^j \{ \dot{\Phi}, [\Phi^{*2}, \tilde{\partial}^i \Phi]_* \}_* - (i \leftrightarrow j) \right), \quad (29)$$

$$\begin{aligned} [M^{0i}, M^{jk}] &= i(\eta^{ij}M^{0k} - \eta^{ik}M^{0j}) - i\frac{\lambda}{4!} \int d^3x x^i \\ &\quad \times \left( [\partial^k \Phi, \tilde{\partial}^j \Phi^{*3}]_* + \Phi^{*2} \star \partial^k \Phi \star \tilde{\partial}^j \Phi - \tilde{\partial}^j \Phi^{*2} \star \partial^k \Phi \star \Phi \right. \\ &\quad \left. + \Phi \star \partial^k \Phi \star \tilde{\partial}^j \Phi^{*2} - \tilde{\partial}^j \Phi \star \partial^k \Phi \star \Phi^{*2} - (j \leftrightarrow k) \right). \end{aligned} \quad (30)$$

The eqns (27) and (28) explicitly show that the Lorentz generators are not conserved in the theory, and all the deformation terms are directly proportional to  $\Theta^{\mu\nu}$ .

### 4.3 Wess-Zumino model

For the NCWZ model, one start from an on-shell Lagrangian analogous to the commutative one [19],

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial_\mu A \partial^\mu A - m^2 A^2) + \frac{1}{2}(\partial_\mu B \partial^\mu B - m^2 B^2) + \frac{1}{2}(i\bar{\Psi} \not{\partial} \Psi - m\bar{\Psi} \Psi) \\ &\quad - mgA(A^{*2} + B^{*2}) - mgB(A \star B + B \star A) \\ &\quad - g(A\bar{\Psi} \star \Psi - B\bar{\Psi} \star \gamma_5 \Psi) - \frac{1}{2}g^2(A - iB)^{*2}(A + iB)^{*2} \end{aligned} \quad (31)$$

$$\begin{aligned} &= \frac{1}{2}(\partial_\mu \phi \partial^\mu \bar{\phi} - m^2 \phi \bar{\phi}) + \frac{1}{2}(i\psi \sigma^\mu \partial_\mu \bar{\psi} + i\bar{\psi} \sigma^\mu \partial_\mu \psi - m\bar{\psi} \psi - m\psi \bar{\psi}) \\ &\quad - \frac{1}{2}mg(\phi \bar{\phi}^{*2} + \bar{\phi} \phi^{*2}) - g(\phi \bar{\psi} \star \bar{\psi} + \bar{\phi} \psi \star \psi) - \frac{1}{2}g^2 \phi^{*2} \bar{\phi}^{*2}. \end{aligned} \quad (32)$$

where  $\phi \equiv A - iB$ ,  $\bar{\phi} \equiv A + iB$ , and  $\psi, \bar{\psi}$  are the Weyl components of the Majorana field  $\Psi$ , following the notations and conventions by Bailin and Love [23].

Following the similar procedure as done in the  $\phi^4$  theory, the variation of the Lagrangian under the infinitesimal Poincaré and supergauge transformations yields the generators as,

$$P^i = \int d^3x \left( \frac{1}{2} \partial^i \phi \dot{\bar{\phi}} + \frac{1}{2} \dot{\phi} \partial^i \bar{\phi} + i\bar{\psi} \sigma^0 \partial^i \psi \right) \equiv \int d^3x \mathcal{P}^i, \quad (33)$$

$$\begin{aligned} P^0 &= \int d^3x \left( \frac{1}{2} (\dot{\phi} \bar{\phi} + \partial^i \phi \partial^i \bar{\phi} + m^2 \phi \bar{\phi}) + \frac{1}{2} (i\bar{\psi} \sigma^i \partial^i \psi + i\psi \sigma^i \partial^i \bar{\psi} + m\psi \bar{\psi} + m\bar{\psi} \psi) \right. \\ &\quad \left. + \frac{1}{2} mg(\phi \bar{\phi}^{*2} + \bar{\phi} \phi^{*2}) + g(\phi \bar{\psi} \star \bar{\psi} + \bar{\phi} \psi \star \psi) + \frac{1}{2} g^2 \phi^{*2} \bar{\phi}^{*2} \right) \equiv \int d^3x \mathcal{P}^0, \end{aligned} \quad (34)$$

$$M^{0i} = \int d^3x (x^0 \mathcal{P}^i - x^i \mathcal{P}^0), \quad (35)$$

$$M^{ij} = \int d^3x (x^i \mathcal{P}^j - x^j \mathcal{P}^i), \quad (36)$$

$$\chi Q = \chi \int d^3x \left( \dot{\phi} \psi - 2\partial_i \phi \sigma^{0i} \psi + im\phi \sigma^0 \bar{\psi} + ig\phi^{*2} \sigma^0 \bar{\psi} \right), \quad (37)$$

$$\bar{\chi} \bar{Q} = \bar{\chi} \int d^3x \left( \dot{\bar{\phi}} \bar{\psi} - 2\partial_i \bar{\phi} \sigma^{0i} \bar{\psi} + im\bar{\phi} \sigma^0 \psi + ig\bar{\phi}^{*2} \sigma^0 \psi \right) = (\chi Q)^\dagger, \quad (38)$$

where  $\chi$  is an arbitrary Majorana spinor parameter.

In the case of the commutative Wess-Zumino model, the analogs of the above generators are those of the Poincaré algebra and supercharge, which form the  $N = 1$  super-Poincaré algebra. With the representations obtained here in the NCWZ model, one can calculate the commutation relations between those generators,

$$[P^\mu, P^\nu] = 0, \quad (39)$$

$$[M^{ij}, M^{kl}] = i(\eta^{il} M^{jk} + \eta^{jk} M^{il} - \eta^{ik} M^{jl} - \eta^{jl} M^{ik}), \quad (40)$$

$$[M^{ij}, P^k] = i(\eta^{jk} P^i - \eta^{ik} P^j), \quad (41)$$

$$[M^{0i}, P^j] = i\eta^{ij} P^0, \quad (42)$$

The above commutation relations are exactly the same as those obtained in the NC  $\Phi^4$  theory, which suggests the generality of such relations for all NCFT's. In particular, (39) verifies the translational invariance of the theory. Equation (42) is a little surprising. The calculation of it in any way involves the NC interaction terms. Nevertheless it's true for both NCFT's.

Other commutation relations are,

$$[\chi Q, \zeta Q] = [\bar{\chi} \bar{Q}, \bar{\zeta} \bar{Q}] = 0, \quad (43)$$

$$[\chi Q, \bar{\zeta} \bar{Q}] = 2\chi\sigma^\mu\bar{\zeta}P_\mu, \quad (44)$$

$$[P^\mu, \chi Q] = 0, \quad (45)$$

$$[M^{ij}, \chi Q] = -i\chi\sigma^{ij}Q, \quad (46)$$

$$[M^{ij}, \bar{\chi} \bar{Q}] = -i\bar{\chi}\bar{\sigma}^{ij}\bar{Q}. \quad (47)$$

All the above relations are exactly the same as those of the commutative Wess-Zumino model. In particular, one finds the supercharge generators,  $Q$  and  $\bar{Q}$ , and the translation generators  $P^\mu$ 's form a close algebra, and the supercharge generators are conserved.

The rest commutation relations have additional terms proportional to  $\Theta^{\mu\nu}$ , including the similar ones as appears in the NC  $\Phi^4$  theory,

$$\begin{aligned} [M^{0i}, P^0] &= -i\eta^{00}P^i - \int d^3x \left( \frac{i}{2}mg([\phi, \tilde{\partial}^i\bar{\phi}]_\star\dot{\phi} + \dot{\phi}[\bar{\phi}, \tilde{\partial}^i\phi]_\star) + mg([\bar{\phi}, \tilde{\partial}^i\psi]_\star\sigma^0\bar{\psi} + [\phi, \tilde{\partial}^i\bar{\psi}]_\star\bar{\sigma}^0\psi) \right. \\ &\quad - 2ig([\bar{\phi}, \tilde{\partial}^i\psi]_\star\sigma^{0l}\partial_l\psi + [\phi, \tilde{\partial}^i\bar{\psi}]_\star\bar{\sigma}^{0l}\partial_l\bar{\psi}) + \frac{i}{2}g^2(\dot{\phi}[\bar{\phi}^{\star 2}, \tilde{\partial}^i\phi]_\star + [\phi^{\star 2}, \tilde{\partial}^i\bar{\phi}]_\star\dot{\phi}) \\ &\quad \left. + g^2([\phi, \tilde{\partial}^i\psi]_\star\{\bar{\phi}, \psi\}_\star - \{\phi, \bar{\psi}\}_\star[\bar{\phi}, \tilde{\partial}^i\psi]_\star) \right), \end{aligned} \quad (48)$$

$$\begin{aligned} [M^{ij}, P^0] &= \int d^3x \left( \frac{i}{2}mg([\partial^i\phi, \tilde{\partial}^j\phi]_\star\bar{\phi} + [\partial^i\bar{\phi}, \tilde{\partial}^j]_\star\phi) + ig([\partial^i\bar{\psi}, \tilde{\partial}^j\bar{\psi}]_\star\phi + [\partial^i\psi, \tilde{\partial}^j\psi]_\star\bar{\phi}) \right. \\ &\quad \left. + \frac{i}{2}g^2([\partial^i\phi, \tilde{\partial}^j\phi]_\star\bar{\phi}^{\star 2} + [\partial^i\bar{\phi}, \tilde{\partial}^j\bar{\phi}]_\star\phi^{\star 2}) \right) - (i \leftrightarrow j), \end{aligned} \quad (49)$$

$$\begin{aligned} [M^{0i}, M^{0j}] &= -i\eta^{00}M^{ij} + \int d^3x \left( mgx^i(\psi\sigma^0[\phi, \tilde{\partial}^j\bar{\psi}]_\star + [\bar{\phi}, \tilde{\partial}^j\psi]_\star\sigma^0\bar{\psi}) \right. \\ &\quad - \frac{i}{2}mgx^i([\phi, \tilde{\partial}^j\bar{\phi}]_\star\dot{\phi} + \dot{\phi}[\bar{\phi}, \tilde{\partial}^j\phi]_\star) + 2igx^i([\phi, \tilde{\partial}^j\bar{\psi}]_\star\bar{\sigma}^{0l}\partial_l\bar{\psi} + [\bar{\phi}, \tilde{\partial}^j\psi]_\star\sigma^{0l}\partial_l\psi) \\ &\quad \left. - \frac{i}{2}g^2x^i([\phi^{\star 2}, \tilde{\partial}^j\bar{\phi}]_\star\dot{\phi} + \dot{\phi}[\bar{\phi}^{\star 2}, \tilde{\partial}^j\phi]_\star) + g^2\{x^i\phi, \bar{\psi}\}_\star\{\psi, x^j\bar{\phi}\}_\star - (i \leftrightarrow j) \right), \end{aligned} \quad (50)$$

$$\begin{aligned} [M^{0i}, M^{jk}] &= \\ &= i(\eta^{ij}M^{0k} - \eta^{ik}M^{0j}) - \int d^3x \left( \frac{i}{2}mgx^i(\bar{\phi}[\partial^k\phi, \tilde{\partial}^j\phi]_\star + \phi[\partial^k\bar{\phi}, \tilde{\partial}^j\bar{\phi}]_\star) \right. \\ &\quad \left. + igx^i(\phi[\partial^k\bar{\psi}, \tilde{\partial}^j\psi]_\star + \bar{\phi}[\partial^k\psi, \tilde{\partial}^j\bar{\psi}]_\star) + \frac{i}{2}g^2x^i([\partial^k\phi, \tilde{\partial}^j\phi]_\star\bar{\phi}^{\star 2} + [\partial^k\bar{\phi}, \tilde{\partial}^j\bar{\phi}]_\star\phi^{\star 2}) - (j \leftrightarrow k) \right), \end{aligned} \quad (51)$$



and also the transformations of the supercharge generators under the Lorentz boosts,

$$\begin{aligned}
[M^{0i}, \chi Q] &= -i\chi\sigma^{0i}Q + \int d^3x \left( g[\dot{\phi}, \phi]_{\star} \chi\sigma^0 \tilde{\partial}^i \bar{\psi} + g[\phi, \tilde{\partial}^i \partial_l(\phi)] \chi\sigma^l \bar{\psi} \right. \\
&\quad \left. - 2ig\psi \star \psi \chi \psi + img[\phi, \bar{\phi}]_{\star} \chi \tilde{\partial}^i \psi + ig^2[\phi^{\star 2}, \bar{\phi}]_{\star} \chi \tilde{\partial}^i \psi \right), \tag{52}
\end{aligned}$$

$$\begin{aligned}
[M^{0i}, \bar{\chi} \bar{Q}] &= -i\bar{\chi}\bar{\sigma}^{0i}\bar{Q} + \int d^3x \left( g[\dot{\bar{\phi}}, \bar{\phi}]_{\star} \bar{\chi}\bar{\sigma}^0 \tilde{\partial}^i \psi + g[\bar{\phi}, \tilde{\partial}^i \partial_l \bar{\phi}] \bar{\chi}\bar{\sigma}^l \psi \right. \\
&\quad \left. - 2ig\bar{\psi} \star \bar{\psi} \bar{\chi} \bar{\psi} + img[\bar{\phi}, \phi]_{\star} \bar{\chi} \tilde{\partial}^i \bar{\psi} + ig^2[\bar{\phi}^{\star 2}, \phi]_{\star} \bar{\chi} \tilde{\partial}^i \bar{\psi} \right). \tag{53}
\end{aligned}$$

To simplify the expression, we reorder the conjugate fields on the right hand side of the above equations, which induces extra infinite constant terms not explicitly shown here.

In summary, The commutation relations of the Lorentz rotation and boost generators generally have additional terms compared with those of the Poincaré or super-Poincaré algebras. Nevertheless, the results are not surprising, since NCFT's indeed violate the Lorentz invariance. Other commutation relations verify certain symmetries preserved by NCFT's, such as the translational and supergauge invariance. In the limit of  $\Theta^{\mu\nu} \rightarrow 0$ , the Poincaré or Super-Poincaré algebra is retrieved.

## 5 Conclusions

In this paper we first construct a NCWZ Lagrangian, from which the Feynman rules are extracted, then the one-loop UV divergent corrections to the 1PI 2-point functions are explicitly calculated and the renormalization of the theory at one-loop are studied. We found that Girotti and collaborators [21] studied the NCWZ theory by using the Lagrangian similar to ours without using the entirely permutation terms in the interaction parts. However, we arrive at the same conclusion, i.e. the NCWZ model is renormalizable by only a wave function renormalization, as expected by Ferrara and Lledó [11]. But, our calculations explicitly show that the UV/IR mixing still exists in the divergent terms and the renormalization of the wave function of the commutative theory can be recovered by setting  $\Theta^{\mu\nu}$  equal to zero.

Next we turn to the algebras of the NC  $\phi^4$  and Wess-Zumino theory. From Noether currents we extract a representation of the translation, Lorentz and supercharge generators, which is what Dirac called ‘fundamental quantities’ [24] for NCFT's. The commutation relations of those quantities are calculated based on this representation.

The NCFT has non-local interaction terms, which explicitly break the Lorentz invariance, but still preserve the translational and supergauge invariance. It's found that in the NCFT the translation and supercharge generators form the same algebra as in the commutative theory. But, the commutation relations of the Lorentz generators, or between the Lorentz generators and the translation or supercharge generators, generally have extra terms proportional to the non-commutativity  $\Theta^{\mu\nu}$ . In addition to that, there are also other interesting commutation relations, such as  $[M^{0i}, P^j] = i\eta^{ij}P^0$ , still hold true in the NC case.

The role of the representations for the algebras is not clear yet. Since those representations for the fundamental quantities could also construct a theory of a dynamical system [24], questions, like ‘Is the theory so constructed exactly equivalent to the theory with the original Lagrangian?’, ‘Can the extra terms, which appear in the commutation relations of the non-invariant fundamental quantities, actually be expressed by other generators and thus all the generators form a deformed Lorentz algebra?’, have yet to be answered.

## Acknowledgments

We would like to thank Professor P. Ramond and Dr. K. Bering for many stimulating and informative discussions. This work was supported by the Department of Energy under grant DE-FG02-97ER-41029.

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