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Non-commutative Space And Chan-Paton Algebra In Open String Field Algebra

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Abstract

There are several equivalent descriptions for constant B-field background of open string. The background can be interpreted as constant B-field as well as constant gauge field strength or infinitely many D-branes with non-commuting Chan-Paton matrices. In this article, the equivalence of these open string theories is studied in Witten's cubic open string field theory. Through the map between these equivalent descriptions, both algebra of non-commutative coordinates as well as Chan-Paton matrix algebra are identified with subalgebras of open string field algebra.

Contents

1 Introduction

Dirichlet branes (D-branes) [1] have played crucial roles in the study of non-perturbative aspects of string theory. D-branes are often referred to as solitons in string theory. However, to treat them really as solitons, one needs to work in the framework of string field theory. In open string field theory, D-brane exists by definition because open string with Neumann boundary conditions in all space-time directions is equivalent to the open string with its ends on a space-time filling D-brane. Still, it is important to know whether lower dimensional D-branes are contained in open string field theory as solitons [2, 3].

Open string theory in constant B-field background has a description in terms of non-commutative coordinates [4, 5, 6, 7]. Instantons and solitons [8, 9] on such non-commutative spaces have been studied extensively and interesting non-abelian structures similar to Chan-Paton matrix algebra were found. Indeed, these non-abelian structures were regarded as low energy descriptions of Chan-Paton matrix [10], and projection operators [11, 9] or partial isometries [12, 13] appeared as typical elements in the description of these instantons or solitons. On the other hand, Witten's cubic open string field theory itself defines a non-commutative geometry [14]. The 3-string interaction vertex defines a non-commutative associative product in open string field algebra. It is tempting to expect that microscopic description of D-branes can be obtained by applying the techniques in non-commutative solitons to Witten's cubic open string field theory. Some projection operators in open string field algebra have been constructed [15, 16, 17, 18, 19, 20, 21] along this line of thought.

There are several equivalent descriptions for open string in constant background B-field. The background can be regarded as constant B-field, as well as constant gauge field strength or infinitely many lower dimensional D-branes with non-commuting Chan-Paton matrices. In this article, I will call these descriptions as B-picture, F-picture, and D-picture, respectively. The equivalence is most explicitly shown by the method introduced by Ishibashi [22]. These descriptions are mapped to each other by a change of variables.

In this article, I apply the Ishibashi's method to study the equivalence of these three descriptions in Witten's cubic open string field theory. This is a natural direction to pursuit since field theory is an appropriate framework for the description of backgrounds and change of variables.

The organization of this article is as follows. In section 2, I study the equivalence of B-picture and F-picture. Open string field theory in B-picture was also studied in [23, 24, 25], where it was found that the target space coordinates become non-commutative. It was also found that there is a map to the theory with commutative coordinates. I point out this map is essentially the map between B-picture and F-picture. Although important steps have already been made in these preceding works [23, 24, 25], I believe this new viewpoint makes the physical picture of the map clearer. Since F-picture variables are the original variables, i.e. the same to the variables without background B-field, this means that the algebra of non-commutative coordinates is already contained in the original open string field algebra. I also give care to the treatment of subtlety in the map that arises at the boundaries of open string. In section 3, I study the map between B-picture and D-picture. From this map I obtain the expression of Chan-Paton matrices in terms of B-picture variables. Through the map between B-picture and F-picture, it is also possible to express them in terms of F-picture variables. This means that Chan-Paton matrix algebra is contained in the original open string field algebra. Since Chan-Paton matrix is associated with lower dimensional D-branes, the appearance of the Chan-Paton algebra in open string field algebra ensures that lower dimensional D-branes can be described as some field configurations in Witten's cubic open string field theory. Section 4 is devoted to the summary and discussions.

2 Equivalence Of B-Field And Gauge Field Strength In Open String Field Theory

2.1 Review Of Witten's Open String Field Theory In Background B-field

Witten's Cubic Open String Field Theory

I start from recalling the open string theory without B-field background. The worldsheet action is given by

$$S = -\frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \left\{ g_{ij} \eta^{\alpha\beta} \partial_\alpha X^i \partial_\beta X^j \right\}. \tag{2.1}$$

Here, $\eta^{\alpha\beta} \equiv \mathrm{diag}(-1, 1)$ $(\alpha = 0, 1)$, $\sigma^0 = \tau$, $\sigma^1 = \sigma$, and the target space metric g_{ij} $(i, j = 0, \cdots, 25)$ is flat. The equation of motion for $X^i(\tau, \sigma)$ is given by

$$g_{ij} \left(\ddot{X}^j(\tau, \sigma) - X^{j\prime\prime}(\tau, \sigma) \right) = 0, \tag{2.2}$$

where the differentials ∂_τ and ∂_σ are denoted by dot $\dot{}$ and prime $'$, respectively. There are two possible boundary conditions. One is the Neumann boundary condition

$$X^{i\prime}(\tau, \sigma) = 0, \quad (\sigma = 0, \pi) \tag{2.3}$$

and another is the Dirichlet boundary condition

$$\dot{X}^i(\tau, \sigma) = const. \quad (\sigma = 0, \pi) \tag{2.4}$$

Here, I consider the case with Neumann boundary conditions for all directions. Then, $X^i(\tau, \sigma)$ can be expanded as

$$X^i(\tau, \sigma) = \sum_{n=0}^{\infty} x_n^i(\tau) \varphi_n^c(\sigma), \tag{2.5}$$

where $\varphi_n^c(\sigma)$ are ortho-normal complete basis for functions on $0 \leq \sigma \leq \pi$ with Neumann boundary conditions at $\sigma = 0, \pi$:

$$\begin{aligned} \varphi_n^c(\sigma) &= \sqrt{\frac{2}{\pi}} \cos n\sigma, \quad (n \neq 0) \\ \varphi_0^c(\sigma) &= \sqrt{\frac{1}{\pi}} \quad . \end{aligned} \tag{2.6}$$

I will use Hamiltonian formalism on the worldsheet in order to describe string field theory in operator formalism [26, 27, 28, 29]. In Heisenberg picture, I have[1]

$$X^i(\sigma) = \sum_{n=0}^{\infty} x_n^i \varphi_n^c(\sigma). \tag{2.7}$$

[1] I use the same letters for the fields in Lagrangian formalism and Hamiltonian formalism. Readers might not be confused in the context.

The canonical conjugate momentum $P_i(\sigma)$ of $X^i(\sigma)$ is given by

$$P_i(\sigma) = \frac{1}{2\pi\alpha'} g_{ij} \dot{X}^j(\sigma). \tag{2.8}$$

$P_i(\sigma)$ also satisfies Neumann boundary conditions and can be expanded as

$$P_i(\sigma) = \sum_{n=0}^{\infty} p_{in} \varphi_n^c(\sigma). \tag{2.9}$$

The canonical commutation relations between modes turn out to be

$$[x_m^i, x_n^j] = 0, \quad [p_{im}, p_{jn}] = 0, \quad [x_m^i, p_{jn}] = i\delta_j^i \delta_{mn}. \tag{2.10}$$

The Hamiltonian of this system becomes

$$H = \frac{1}{4\pi\alpha'} \sum_{n=0}^{\infty} \left\{ (2\pi\alpha')^2 g^{ij} p_{in} p_{jn} + g_{ij} x_n^i x_n^j n^2 \right\}. \tag{2.11}$$

The equations of motion in Heisenberg picture are given by

$$\dot{x}_n^i = i[H, x_n^i] = 2\pi\alpha' g^{ij} p_{jn}, \tag{2.12}$$

$$\dot{p}_{in} = i[H, p_{in}] = -\frac{1}{2\pi\alpha'} g_{ij} n^2 x_n^j. \tag{2.13}$$

To fix reparametrization invariance, one introduces ghost action S^{gh}:

$$S^{gh} = \int d^2\sigma \left(b_{++} \partial_- c^+ + b_{--} \partial_+ c^- \right). \tag{2.14}$$

The ghost fields have mode expansions

$$c^+(\tau, \sigma) = \sum_{n=-\infty}^{\infty} c_n e^{-in(\tau+\sigma)}, \quad c^-(\tau, \sigma) = \sum_{n=-\infty}^{\infty} c_n e^{-in(\tau-\sigma)}, \tag{2.15}$$

$$b_{++}(\tau, \sigma) = \sum_{n=-\infty}^{\infty} b_n e^{-in(\tau+\sigma)}, \quad b_{--}(\tau, \sigma) = \sum_{n=-\infty}^{\infty} b_n e^{-in(\tau-\sigma)}. \tag{2.16}$$

The kinetic part of string field action is constructed using the BRST charge

$$Q = \sum_{n=-\infty}^{\infty} : c_n \left(L_{-n}^X + \frac{1}{2} L_{-n}^{gh} \right) : \ . \tag{2.17}$$

Here, the matter Virasoro generator L_n^X is given by

$$L_n^X = \frac{1}{2} g_{ij} \sum_{\ell=0}^{\infty} : \alpha_{n-\ell}^i \alpha_\ell^j : , \tag{2.18}$$

where

$$\alpha_n^i = \frac{1}{\sqrt{4\pi\alpha'}} (2\pi\alpha' g^{ij} p_{jn} - inx_n^i). \tag{2.19}$$

and : : denotes the normal ordering. The ghost Virasoro generator L_n^{gh} is given by

$$L_n^{gh} = \sum_{m=-\infty}^{\infty} (n-m) : b_{n+m} c_{-m} : \quad . \tag{2.20}$$

The action of the cubic open string field theory is given as follows [14, 26, 27, 28, 29]

$$\mathbf{S} = \frac{1}{2} \langle R(1,2)| \, |\Psi\rangle_1 \left(Q^{(2)} |\Psi\rangle_2 \right) + \frac{g}{3} \, {}_{123} \langle V_3| \, |\Psi\rangle_1 |\Psi\rangle_2 |\Psi\rangle_3. \tag{2.21}$$

Here, the 3-string interaction vertex is defined by the overlap condition

$$\begin{aligned} \langle V_3| \left(X^{i(r+1)}(\sigma) - X^{i(r)}(\pi - \sigma) \right) &= 0, \quad (0 \le \sigma \le \frac{\pi}{2}) \\ \langle V_3| \left(P_i^{(r+1)}(\sigma) + P_i^{(r)}(\pi - \sigma) \right) &= 0. \quad (0 \le \sigma \le \frac{\pi}{2}) \end{aligned} \tag{2.22}$$

The matter part of the string vertex has a form

$$\langle V_3^X | = {}_{123}\langle 0| \exp \left(-\frac{1}{2} g_{ij} \sum_{\substack{r,s=1,2,3 \\ m,n \ge 0}} a_m^{i(r)} V_{3\,mn}^{rs} a_n^{j(s)} \right), \tag{2.23}$$

where creation operators $a_n^{i\dagger}$ and annihilation operators a_n^i are defined by

$$\begin{aligned} a_n^i &= \frac{1}{\sqrt{4\pi\alpha' n}} (2\pi\alpha' g^{ij} p_{jn} - i n x_n^i), \quad (n \ge 1) \\ a_n^{i\dagger} &= \frac{1}{\sqrt{4\pi\alpha' n}} (2\pi\alpha' g^{ij} p_{jn} + i n x_n^i), \quad (n \ge 1) \\ a_0^i &= \frac{1}{\sqrt{4\pi\alpha'}} (4\pi\alpha' g^{ij} p_{j0} - i x_0^i), \\ a_0^{i\dagger} &= \frac{1}{\sqrt{4\pi\alpha'}} (4\pi\alpha' g^{ij} p_{j0} + i x_0^i). \end{aligned} \tag{2.24}$$

In this article, the explicit form of the coefficients $V_{3\,mn}^{rs}$ is not used, see [26, 27, 28, 29] for details. Also I do not study the ghost part since it is not modified by constant background B-field. The matter part of the identity string field $|I\rangle$, the reflector $\langle R(1,2)|$ and the inverse reflector $|R(1,2)\rangle$ are given by

$$|I\rangle = \exp \left(-\frac{1}{2} g_{ij} \sum_{n,m} a_n^{i\dagger} C_{nm} a_m^{j\dagger} \right) |0\rangle, \tag{2.25}$$

$$\langle R(1,2)| = {}_{12}\langle 0| \exp \left(-g_{ij} \sum_{n,m} a_n^{i(1)} C_{nm} a_m^{j(2)} \right), \tag{2.26}$$

$$|R(1,2)\rangle = \exp \left(-g_{ij} \sum_{n,m} a_n^{i\dagger(1)} C_{nm} a_m^{j\dagger(2)} \right) |0\rangle_{12}, \tag{2.27}$$

where $C_{nm} = (-1)^n \delta_{nm}$. $|I\rangle$ and $\langle R(1,2)|$ satisfy the following overlap conditions

$$\left(X^i(\sigma) - X^i(\pi - \sigma) \right) |I\rangle = 0, \tag{2.28}$$

$$\left(P_i(\sigma) + P_i(\pi - \sigma) \right) |I\rangle = 0, \tag{2.29}$$

$$\langle R(1,2)| \left(X^{i(r+1)}(\sigma) - X^{i(r)}(\pi-\sigma) \right) = 0, \quad (0 \le \sigma \le \frac{\pi}{2}) \tag{2.30}$$

$$\langle R(1,2)| \left(P_i^{(r+1)}(\sigma) + P_i^{(r)}(\pi-\sigma) \right) = 0. \qquad (0 \le \sigma \le \frac{\pi}{2}) \tag{2.31}$$

The star product of string fields is defined by

$$|A \star B\rangle_1 = {}_{234}\langle V_3|A\rangle_3 |B\rangle_4 |R(1,2)\rangle. \tag{2.32}$$

One can define integration $\int$ in string field algebra by

$$\int \Psi = {}_1\langle I|\Psi\rangle_1, \quad {}_1\langle I| = \langle R(1,2)|I\rangle_2. \tag{2.33}$$

The identity string field satisfies[2]

$$|I \star I\rangle = |I\rangle. \tag{2.34}$$

The open string field action (2.21) can be rewritten into the following form

$$\mathbf{S} = \int \frac{1}{2}\Psi \star (Q\Psi) + \frac{g}{3}\Psi \star \Psi \star \Psi. \tag{2.35}$$

Open String Theory In Constant Background B-field

Next, I review the open string theory in constant background B-field B_{ij}. This background is described by the worldsheet action

$$S = -\frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \left\{ g_{ij}\eta^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j - 2\pi\alpha' \epsilon^{\alpha\beta} B_{ij} \partial_\alpha X^i \partial_\beta X^j \right\}. \tag{2.36}$$

Here, $\epsilon^{01} = 1$. To avoid the special features of space-time non-commutativity [30], I only consider the case $B_{0i} \neq 0$. The equations of motion is given by

$$g_{ij} \left(\ddot{X}_B^j(\tau,\sigma) - X_B^{j\,\prime\prime}(\tau,\sigma) \right) = 0, \tag{2.37}$$

where I have introduced the suffix B to explicitely indicate the dependence on the B-field. The boundary conditions are modified from (2.3) by the presence of the B-field:

$$g_{ij} X_B^{j\,\prime}(\tau,\sigma) + b_{ij}\dot{X}_B^j(\tau,\sigma) = 0. \quad (\sigma = 0, \pi) \tag{2.38}$$

Here, $b_{ij} = 2\pi\alpha' B_{ij}$. In Hamiltonian formalism, the boundary conditions (2.38) can be treated as constraints [31] (I use notation similar to [24]). The mode expansions of $X_B^i(\sigma)$ and its canonical variable $\hat{P}_i(\sigma)$ are given as follows:

$$X_B^i(\sigma) = \hat{X}^i(\sigma) + \theta^{ij} Q_j(\sigma), \tag{2.39}$$

[2] I dropped the normalization factor for simplisity.

$$\hat{P}_i(\sigma) = \frac{1}{2\pi\alpha'}\left\{g_{ij}\dot{X}_B^{\ j}(\sigma) + b_{ij}X_B^{j\ \prime}(\sigma)\right\}, \tag{2.40}$$

$$\hat{X}^i(\sigma) = \sum_{n=0}^{\infty}\hat{x}_n^i\varphi_n^c(\sigma), \quad Q_i(\sigma) = \hat{p}_{i0}\sqrt{\frac{1}{\pi}}\left(\sigma - \frac{\pi}{2}\right) + \sum_{n=1}^{\infty}\frac{1}{n}\hat{p}_{in}\varphi_n^s(\sigma), \tag{2.41}$$

$$\hat{P}_i(\sigma) = \sum_{n=0}^{\infty}\hat{p}_{in}\varphi_n^c(\sigma). \tag{2.42}$$

Here, $\varphi_n^s(\sigma)$ are ortho-normal basis for functions on $0 \leq \sigma \leq \pi$ with Dirichlet boundary conditions at $\sigma = 0, \pi$:

$$\varphi_n^s(\sigma) = \sqrt{\frac{2}{\pi}}\sin n\sigma, \tag{2.43}$$

and θ^{ij} is defined by

$$\frac{\theta^{ij}}{2\pi\alpha'} = -\left(\frac{1}{g+b}b\frac{1}{g-b}\right)^{ij}. \tag{2.44}$$

The modes $\hat{x}_m^i$ and $\hat{p}_{jn}$ satisfy the canonical commutation relations

$$[\hat{x}_m^i, \hat{x}_n^j] = [\hat{p}_{im}, \hat{p}_{jn}] = 0, \quad [\hat{x}_m^i, \hat{p}_{jn}] = i\delta_{mn}\delta_j^i. \tag{2.45}$$

These commutation relations lead to the following commutation relations

$$\begin{aligned} &[X_B^i(\sigma), X_B^j(\sigma')] = i\theta^{ij}\left(\delta_{0,\sigma+\sigma'} - \delta_{2\pi,\sigma+\sigma'}\right), \\ &[X_B^i(\sigma), \hat{P}_j(\sigma')] = i\delta_j^i\delta^c(\sigma,\sigma'), \quad [\hat{P}_i(\sigma), \hat{P}_j(\sigma')] = 0, \end{aligned} \tag{2.46}$$

where

$$\begin{aligned} \delta_{0,\sigma} &= 1, \quad (\sigma = 0) \qquad & \delta_{\pi,\sigma} &= 1, \quad (\sigma = \pi) \\ &= 0, \quad (\sigma \neq 0) & &= 0, \quad (\sigma \neq \pi) \end{aligned} \tag{2.47}$$

and

$$\delta^c(\sigma,\sigma') = \sum_{n=0}^{\infty}\varphi_n^c(\sigma)\varphi_n^c(\sigma'). \tag{2.48}$$

Notice the non-commutativity of $X_B^i(\sigma)$ at the boundaries. The Hamiltonian $\hat{H}(B)$ of this system becomes

$$\hat{H}(B) = \frac{1}{4\pi\alpha'}\sum_{n=0}^{\infty}\left\{(2\pi\alpha')^2 G^{ij}\hat{p}_{in}\hat{p}_{jn} + G_{ij}\hat{x}_n^i\hat{x}_n^j n^2\right\}, \tag{2.49}$$

where

$$
\begin{aligned}
G_{ij} &= g_{ij} - (bg^{-1}b)_{ij}, \\
G^{ij} &= \left(\frac{1}{g+b} g \frac{1}{g-b}\right)^{ij}.
\end{aligned} \tag{2.50}
$$

B of $\hat{H}(B)$ indicates the dependence on the background B-field. The equations of motion in Heisenberg picture are given by

$$
\dot{\hat{x}}^i_n = i[\hat{H}(B), \hat{x}^i_n] = 2\pi\alpha' G^{ij} \hat{p}_{jn}, \tag{2.51}
$$

$$
\dot{\hat{p}}_{in} = i[\hat{H}(B), \hat{p}_{in}] = -\frac{1}{2\pi\alpha'} G_{ij} \hat{x}^j_n n^2. \tag{2.52}
$$

2.2 Map Between B-picture and F-picture

In the previous subsection, the effect of the B-field is treated as boundary conditions (2.38). Since the second term in the action (2.36) is a total derivative, it can be integrated to the surface term:

$$
\begin{aligned}
S &= -\frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \left\{ g_{ij}\eta^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j - 2\pi\alpha'\epsilon^{\alpha\beta} B_{ij} \partial_\alpha X^i \partial_\beta X^j \right\} \\
&= -\frac{1}{4\pi\alpha'} \int d\tau \int_0^\pi d\sigma \left\{ g_{ij}\eta^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j \right\} \\
&\quad -\frac{1}{4\pi\alpha'} \int d\tau \; b_{ij} X^i(\pi)\dot{X}^j(\pi) + \frac{1}{4\pi\alpha'} \int d\tau \; b_{ij} X^i(0)\dot{X}^j(0).
\end{aligned} \tag{2.53}
$$

Then, the effect of B_{ij} appears as interactions at the boundaries. It should be regard as open string background, i.e. constant background gauge field strength. I call this description F-picture, and call the description in the previous subsection B-picture. These descriptions should be equivalent since both start from the same action. The equivalence is most explicitly shown by the method introduced by Ishibashi [22], and I will utilize his method. In order to clearly distinguish the boundary interactions from the boundary conditions, one first shifts the positions of the boundary terms to $\sigma = \epsilon$ and $\sigma = \pi - \epsilon$, $(\epsilon > 0)$ and take the limit $\epsilon \to 0$ later. This procedure ensures that the boundary terms contribute to the equation of motion, instead of changing the boundary conditions. Then, the equation of motion is given by

$$
g_{ij}\left(\ddot{X}^j(\tau,\sigma) - X^{j\prime\prime}(\tau,\sigma)\right) + b_{ij}\left(\dot{X}^j(\tau,\pi-\epsilon)\delta^c(\pi-\epsilon,\sigma) - \dot{X}^j(\tau,\epsilon)\delta^c(\epsilon,\sigma)\right) = 0, \tag{2.54}
$$

with Neumann boundary conditions:

$$
g_{ij} X^{j\prime}(\tau,\sigma) = 0. \quad (\sigma = 0, \pi) \tag{2.55}
$$

$X^i(\tau,\sigma)$ can be expanded as

$$
X^i(\tau,\sigma) = \sum_{n=0}^{\infty} x^i_n(\tau)\varphi^c_n(\sigma). \tag{2.56}
$$

In terms of the modes, the equations of motion (2.54) can be rewritten as

$$g_{ij}(\ddot{x}_n^j + n^2 x_n^j) + b_{ij}\left(\dot{X}^j(\pi-\epsilon)\varphi_n^c(\pi-\epsilon) - \dot{X}^j(\epsilon)\varphi_n^c(\epsilon)\right) = 0. \tag{2.57}$$

The canonical momentum $P_i(\sigma)$ of $X^i(\sigma)$ is given by

$$P_i(\sigma) = \frac{1}{2\pi\alpha'}\left(g_{ij}\dot{X}^j(\sigma) + \frac{1}{2}b_{ij}\left(X^j(\pi-\epsilon)\delta^c(\pi-\epsilon,\sigma) - X^j(\epsilon)\delta^c(\epsilon,\sigma)\right)\right). \tag{2.58}$$

$P_i(\sigma)$ also satisfies the Neumann boundary conditions and can be expanded as

$$P_i(\sigma) = \sum_{n=0}^{\infty} p_{in}\varphi_n^c(\sigma). \tag{2.59}$$

The variables in F-picture, x_n^i and p_{in}, are identified with the variables without the background B-field, i.e. (2.7) and (2.8), respectively. What is changed by the B-field is the Hamiltonian. It is given by

$$H_I^\epsilon = \frac{1}{4\pi\alpha'}\sum_{n=0}^{\infty} g_{ij}\left(\dot{x}_n^i\dot{x}_n^j + n^2 x_n^i x_n^j\right), \tag{2.60}$$

where the suffix ϵ indicates the dependence on the cut off and $\dot{x}_n^i$ is solved in terms of p_{in} and x_n^i:

$$\dot{x}_n^i = 2\pi\alpha' g^{ij}p_{jn} - \frac{1}{2}(g^{-1}b)^i{}_j\left(X^j(\pi-\epsilon)\varphi_n^c(\pi-\epsilon) - X^j(\epsilon)\varphi_n^c(\epsilon)\right). \tag{2.61}$$

This is also an equation of motion of x_n^i in Heisenberg picture. The equation of motion for p_{in} is given by

$$\begin{aligned} \dot{p}_{in} &= i[H_I^\epsilon, p_{in}] \\ &= -\frac{1}{2\pi\alpha'}g_{ij}n^2 x_n^j - \frac{1}{2}b_{ij}\left((\dot{X}^j(\pi-\epsilon)\varphi_n^c(\pi-\epsilon) - \dot{X}^j(\epsilon)\varphi_n^c(\epsilon)\right). \end{aligned} \tag{2.62}$$

I will show the equivalence of the two descriptions by constructing a one-to-one map between B-picture variables $(\hat{x}_n^i, \hat{p}_{in})$ and F-picture variables (x_n^i, p_{in}). This map is obtained in the following way [22]. In the interval $0 \le \sigma \le \pi$, one can Fourier expand $(\sigma - \frac{\pi}{2})$ and $\varphi_m^s(\sigma)$ by $\varphi_n^c(\sigma)$, and so as $X_B^i(\sigma)$. Then, one equates $X^i(\sigma) = X_B^i(\sigma)$ in the interval $0 < \sigma < \pi$. By comparing the coefficients of $\varphi_n^c(\sigma)$, one obtains the expression of x_n^i in terms of $\hat{x}_n^i$ and $\hat{p}_{in}$. Here, the limit $\epsilon \to 0$ is taken as

$$\begin{aligned} &\lim_{\epsilon\to 0} X^i(\pi-\epsilon) = X^i(\pi) \neq X_B^i(\pi), \\ &\lim_{\epsilon\to 0} X^i(\epsilon) = X^i(0) \neq X_B^i(0). \end{aligned} \tag{2.63}$$

This prescription can be understood from the corresponding boundary state in closed string theory [22][32, 33]. Then, one can use (2.62) to obtain the expression of p_{in} in terms of $\hat{x}_n^i$

and $\hat{p}_{in}$. The result is as follows:

$$x^i_n = \hat{x}^i_n - \theta^{ij}\sum_{\ell=0}^{\infty}\hat{p}_{j\ell}A_{\ell n}, \tag{2.64}$$

$$p_{in} = \hat{p}_{in} - \frac{1}{2\pi\alpha'}b_{ij}\sum_{\ell=0}^{\infty}\left(\hat{x}^j_\ell - \theta^{jk}\sum_{m=0}^{\infty}\hat{p}_{km}A_{m\ell}\right)A_{\ell n}\frac{\ell^2+n^2}{2}, \tag{2.65}$$

where the matrix A_{mn} is defined as

$$\varphi^c_n(\sigma) = \sum_{m=1}^{\infty}A_{nm}m\varphi^s_m(\sigma), \quad (0 < \sigma < \pi)$$

$$\varphi^s_n(\sigma) = \sum_{m=0}^{\infty}(-n)A_{nm}\varphi^c_m(\sigma). \quad (0 \le \sigma \le \pi) \tag{2.66}$$

Some properties of the matrix A_{mn} is summarized in the appendix. Here, one must be careful since the infinite summations may depend on ordering. In (2.65), the bracket ( ) means that the summation over mode indices inside ( ) should be taken before the summation over mode indices in and outside the bracket. As an illustration, let us consider the matter fields at the "boundary":

$$X^i(\pi) = \sum_{n=0}^{\infty}x^i_n\varphi^c_n(\pi) = \sum_{n=0}^{\infty}\left(\hat{x}^i_n - \theta^{ij}\sum_{\ell=0}^{\infty}\hat{p}_{j\ell}A_{\ell n}\right)\varphi^c_n(\pi) \tag{2.67}$$

$$\neq \ X^i_B(\pi) = \sum_{n=0}^{\infty}\hat{x}^i_n\varphi^c_n(\pi) - \theta^{ij}\sum_{\ell=0}^{\infty}\hat{p}_{j\ell}\left(\sum_{n=0}^{\infty}A_{\ell n}\varphi^c_n(\pi)\right) = \sum_{n=0}^{\infty}\hat{x}^i_n\varphi^c_n(\pi) + \frac{\sqrt{\pi}}{2}\theta^{ij}\hat{p}_{j0}.$$

One must regard $X^i(\pi)$ and $X^i_B(\pi)$ as different operators since while $[X^i(\pi), X^j(\pi)] = 0$, $[X^i_B(\pi), X^j_B(\pi)] = -i\theta^{ij}$. The difference between $X^i(\pi)$ and $X^i_B(\pi)$ may be interpreted as the difference of regularization prescription appeared in [7].

By applying the rule (2.63), the $\epsilon \to 0$ limit of the equations of motion (2.61) and (2.62) become

$$\dot{x}^i_n = 2\pi\alpha' g^{ij}p_{jn} - \frac{1}{2}(g^{-1}b)^i{}_j\left(X^j(\pi)\varphi^c_n(\pi) - X^j(0)\varphi^c_n(0)\right), \tag{2.68}$$

$$\dot{p}_{in} = -\frac{1}{2\pi\alpha'}g_{ij}n^2x^j_n - \frac{1}{2}b_{ij}\left((\dot{X}^j(\pi)\varphi^c_n(\pi) - \dot{X}^j(0)\varphi^c_n(0)\right). \tag{2.69}$$

One can check that the equations of motion in B-picture ((2.51) and (2.52)) lead to the equations of motion in F-picture ((2.68) and (2.69)). Thus the same Hamiltonian $\hat{H}(B)$ gives the equations of motion of both pictures. Hence the two descriptions are equivalent.

The inverse map is given by

$$\hat{x}^i_n = (G^{-1}g)^i{}_jx^j_n + \theta^{ij}\sum_{\ell=0}^{\infty}p_{j\ell}A_{\ell n} + \delta_{n0}\frac{\sqrt{\pi}}{4}(\theta b)^i{}_j\left(X^j(\pi) + X^j(0)\right), \tag{2.70}$$

$$\hat{p}_{in} = p_{in} + \frac{1}{2\pi\alpha'}b_{ij}\sum_{\ell=0}^{\infty}x^j_\ell A_{\ell n}\frac{\ell^2+n^2}{2}. \tag{2.71}$$

The map can be written in the form of Unitary transformation:

$$x_n^i = U_{BF}^{-1} \hat{x}_n^i U_{BF}, \quad p_{in} = U_{BF}^{-1} \hat{p}_{in} U_{BF}. \tag{2.72}$$

Here, U_{BF} is given by

$$U_{BF} = e^N e^M, \tag{2.73}$$

where

$$\begin{aligned} M &= -\frac{i}{2}\theta^{ij}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\hat{p}_{i\ell}\hat{p}_{jm}A_{\ell m} \\ &= -\frac{i}{4}\theta^{ij}\int_0^{\pi}d\sigma\int_0^{\pi}d\sigma'\epsilon(\sigma-\sigma')\hat{P}_i(\sigma)\hat{P}_j(\sigma'), \end{aligned} \tag{2.74}$$

and

$$\begin{aligned} N &= -\frac{i}{4\pi\alpha'}b_{ij}\sum_{\ell=0}^{\infty}\sum_{m=0}^{\infty}\hat{x}_\ell^i A_{\ell m}\frac{\ell^2+m^2}{2}\hat{x}_m^j \\ &= -\frac{i}{4}\int_0^{\pi}d\sigma B_{ij}\left(\hat{X}^i(\sigma)\hat{X}^{j\prime}(\sigma)-\hat{X}^{i\prime}(\sigma)\hat{X}^j(\sigma)\right). \end{aligned} \tag{2.75}$$

Here, $\epsilon(\sigma)$ is the sign function

$$\begin{aligned} \epsilon(\sigma) &= 1, \quad (\sigma > 0) \\ &= 0, \quad (\sigma = 0) \\ &= -1. \quad (\sigma < 0) \end{aligned} \tag{2.76}$$

These operators have appeared in [23, 24, 25].

The 3-string vertex $\langle V_3|$ can be rewritten using the variables $\hat{x}_n^i$ and $\hat{p}_{in}$. From (2.72), I obtain

$$\langle V_3| = \langle \hat{V}_3| e^N e^M, \tag{2.77}$$

where $\langle \hat{V}_3|$ satisfies the following overlap conditions for $\hat{X}^i(\sigma)$ and $\hat{P}_i(\sigma)$:

$$\begin{aligned} &\langle \hat{V}_3|\left(\hat{X}^{i(r+1)}(\sigma)-\hat{X}^{i(r)}(\pi-\sigma)\right)=0, \quad (0\le\sigma\le\frac{\pi}{2}) \\ &\langle \hat{V}_3|\left(\hat{P}_i^{(r+1)}(\sigma)+\hat{P}_i^{(r)}(\pi-\sigma)\right)=0. \quad (0\le\sigma\le\frac{\pi}{2}) \end{aligned} \tag{2.78}$$

Since the difference between the original overlap conditions (2.22) and (2.78) is just a difference of the variables, [3]

$$\langle \hat{V}_3^X| = {}_{123}\langle 0|\exp\left(-\frac{1}{2}g_{ij}\sum_{\substack{r,s=1,2,3\\ m,n\ge 0}}\hat{a}_m^{i(r)}V_{3\,mn}^{rs}\hat{a}_n^{j(s)}\right), \tag{2.79}$$

[3]Since the overlap conditions do not refer to metric, the 3-string vertex is independent from the metric [34, 24].

where the creation and annihilation operators are defined in the same way as in (2.24) :

$$
\begin{aligned}
\hat{a}_n^i &= \frac{1}{\sqrt{4\pi\alpha' n}}(2\pi\alpha' g^{ij}\hat{p}_{jn} - in\hat{x}_n^i), \quad (n \geq 1) \\
\hat{a}_n^{i\dagger} &= \frac{1}{\sqrt{4\pi\alpha' n}}(2\pi\alpha' g^{ij}\hat{p}_{jn} + in\hat{x}_n^i), \quad (n \geq 1) \\
\hat{a}_0^i &= \frac{1}{\sqrt{4\pi\alpha'}}(4\pi\alpha' g^{ij}\hat{p}_{j0} - in\hat{x}_0^i), \\
\hat{a}_0^{i\dagger} &= \frac{1}{\sqrt{4\pi\alpha'}}(4\pi\alpha' g^{ij}\hat{p}_{j0} + in\hat{x}_0^i).
\end{aligned}
\tag{2.80}
$$

Using the defining properties (2.78), one can show $\langle\hat{V}_3|N = 0$. From the overlap conditions,

$$
\begin{aligned}
& \langle\hat{V}_3|\sum_{r=1}^{3}\int_0^\pi d\sigma \hat{X}^{i(r)}(\sigma)\hat{X}^{j(r)\prime}(\sigma) \\
=& \langle\hat{V}_3|\sum_{r=1}^{3} -\left(\int_0^{\frac{\pi}{2}} d\sigma \hat{X}^{i(r)}(\pi-\sigma)\hat{X}^{j(r)\prime}(\pi-\sigma) + \int_{(\frac{\pi}{2})}^{\pi} d\sigma \hat{X}^{i(r)}(\pi-\sigma)\hat{X}^{j(r)\prime}(\pi-\sigma)\right) \\
=& \langle\hat{V}_3|\sum_{r=1}^{3}\left(\int_\pi^{\frac{\pi}{2}} d\sigma \hat{X}^{i(r)}(\sigma)\hat{X}^{j(r)\prime}(\sigma) + \int_{(\frac{\pi}{2})}^{0} d\sigma \hat{X}^{i(r)}(\sigma)\hat{X}^{j(r)\prime}(\sigma)\right) \\
=& -\langle\hat{V}_3|\sum_{r=1}^{3}\int_0^\pi d\sigma \hat{X}^{i(r)}(\sigma)\hat{X}^{j(r)\prime}(\sigma) \\
=& \; 0,
\end{aligned}
\tag{2.81}
$$

where $\int_{(p)}$ means that the integration region starts from the point p with the point p itself being excluded from the integration. The equation (2.81) means $\langle\hat{V}_3|N = 0$. Thus, I obtain

$$
\langle V_3| = \langle\hat{V}_3|e^M. \tag{2.82}
$$

This expression was also found in [23, 24] by requireing the overlap conditions for the B-picture variables:

$$
\begin{aligned}
\langle V_3|\left(X_B^{i(r+1)}(\sigma) - X_B^{i(r)}(\pi-\sigma)\right) &= 0, \quad (0 \leq \sigma \leq \frac{\pi}{2}) \\
\langle V_3|\left(\hat{P}_i^{(r+1)}(\sigma) + \hat{P}_i^{(r)}(\pi-\sigma)\right) &= 0. \quad (0 \leq \sigma \leq \frac{\pi}{2})
\end{aligned}
\tag{2.83}
$$

Note that this 3-string vertex is the same to the original one (2.22), i.e., the one with no background B-field. In [23, 24], it was also shown that the 3-string vertex (2.82) can be rewritten into the following form:

$$
\langle V_3| = \langle\hat{V}_3|\exp\left(-\frac{i}{2}\sum_{r=1}^{3}\theta^{ij}\hat{p}_{i0}^{(r)}\hat{p}_{j0}^{(r+1)}\right). \tag{2.84}
$$

The kinetic term of the open string field theory in the B-field background is given by the B-field dependent BRST charge $\hat{Q}(B)$:

$$
\hat{Q}(B) = \sum_{n=-\infty}^{\infty} : c_n\left(\hat{L}_{-n}^X + \frac{1}{2}L_{-n}^{gh}\right) : , \tag{2.85}
$$

where the matter part of the Virasoro operator is given by

$$\hat{L}_n^X = \frac{1}{2} G_{ij} \sum_{\ell=0}^{\infty} : \hat{\alpha}_{n-\ell}^i \hat{\alpha}_\ell^j : \ . \tag{2.86}$$

Here, : : denotes the normal ordering with respect to the operators

$$\hat{\alpha}_n^i = \frac{1}{\sqrt{4\pi\alpha'}} (2\pi\alpha' G^{ij} \hat{p}_{jn} - in\hat{x}_n^i). \tag{2.87}$$

The open string field action in the B-field background is given by

$$\mathbf{S}(B) = \frac{1}{2} \langle R(1,2)| \, |\Psi\rangle_1 \left(\hat{Q}(B)^{(2)} |\Psi\rangle_2 \right) + \frac{g}{3} \ {}_{123} \langle V_3| \, |\Psi\rangle_1 |\Psi\rangle_2 |\Psi\rangle_3, \tag{2.88}$$

where, as I have explained, the 3-string vertex $\langle V_3|$ is the same to the one without B-field. This is expected from the background independence of the 3-string vertex [35, 36, 37, 38]. The information of the background only appears in the BRST charge $\hat{Q}(B)$.

The algebra of the original coordinates x_0^i is commutative. On the other hand, one may regard $\hat{p}_{i0}$, which commutes with hamiltonian $\hat{H}(B)$, as target space momentum. Then, the canonical conjugate variable $\hat{x}_0^i$ can be regarded as target space "coordinate", and multiplication between coordinates are given by a non-commutative Moyal type product, as one can observe from eq.(2.84). With this "definition" of the coordinates, the hamiltonian $\hat{H}(B)$ determines the subalgebra of open string field algebra which is regarded as algebra of target space coordinates. Notice that under the B-field background, there is no intuitive notion for target space coordinates before one defines them.

To quantize the worldsheet theory, one should also choose Hilbert space. It turns out $|\langle\hat{0}|e^N|\hat{0}\rangle|^2 = 0$ and $|\langle\hat{0}|e^M|\hat{0}\rangle|^2 = 0$. Here, $|\hat{0}\rangle$ is a Fock vacuum for the annihilation/creation operators (2.80). Therefore, $U_{BF} = e^N e^M$ does not act within a single Hilbert space. But one can use linear map (2.64), (2.65) or the inverse map (2.70), (2.71) to describe F-picture operators in terms B-picture operators or vice versa. In open string field algebra, it is not precisely known what set of states should be allowed (see [39] for a recent discussion). It seems that one should allow transformations like U_{BF} to treat different backgrounds in open string field algebra.

I close this section by mentioning that an interpolating picture between B-picture and F-picture might have been obtained if I had partially integrated the second term in (2.36) [7, 40, 32].

3 Equivalence Of B-field And Non-commuting Chan-Paton Matrices In Open String Field Theory

There is an open string worldsheet theory which is equivalent to the open string theory in B-field background discussed in the previous section [22]. This is an open string theory on infinitely many D-branes with non-commuting Chan-Paton matrices. I will call this description D-picture. More precise description is given in the following.

3.1 Map Between B-picture and D-picture

The worldsheet action of what I call D-picture is given by

$$
\begin{aligned}
S \;=\; & -\frac{1}{4\pi\alpha'}\int d\tau \int_0^\pi d\sigma \left\{ g_{ij}\eta^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j - \bar{b}_{ij}\epsilon^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j \right\} \\
& -\frac{1}{2\pi\alpha'}\int d\tau (g_{ij}X^{i\prime}(\pi) - \bar{b}_{ij}\dot{X}^i(\pi))y_\pi^j + \frac{1}{2\pi\alpha'}\int d\tau (g_{ij}X^{i\prime}(0) - \bar{b}_{ij}\dot{X}^i(0))y_0^j \\
& -\frac{1}{4\pi\alpha'}\int d\tau \left(\frac{\theta}{2\pi\alpha'}\right)^{-1}_{ij}(y_\pi^i \dot{y}_\pi^j - y_0^i \dot{y}_0^j), \qquad (3.1)
\end{aligned}
$$

where $\bar{b}_{ij}$ is given by

$$
\bar{b}_{ij} = b_{ij} - \left(\frac{\theta}{2\pi\alpha'}\right)^{-1}_{ij}. \qquad (3.2)
$$

In the above the inverse of θ^{ij} is considered in the appropliate subspace of the target space, i.e. if the rank of θ^{ij} is 24, θ^{-1}_{ij} is the inverse in the 24-dimensional subspace. The action (3.1) can be read off from the corresponding boundary state in closed string theory [22][32, 33]. In the boundary state, the quantum mechanical degrees of freedom y_0^i and y_π^j arise as path integral counterparts of Chan-Paton matrices. Here, they will become infinite dimensional Chan-Paton matrices after the canonical quantization. As in the previous section, I introduce cutoff ϵ to separate interactions from boundary conditions. Then, the equations of motion are given by

$$
\begin{aligned}
g_{ij}(\ddot{X}_D^j(\tau,\sigma) - X_D^{j\prime\prime}(\tau,\sigma)) - g_{ij}(y_\pi^j\delta^{c\prime}(\pi-\epsilon,\sigma) - y_0^j\delta^{c\prime}(\epsilon,\sigma)) & \\
+\bar{b}_{ij}\left(\dot{y}_\pi^j\delta^s(\pi-\epsilon,\sigma) - \dot{y}_0^j\delta^s(\epsilon,\sigma)\right) &= 0, \qquad (3.3) \\
g_{ij}X_D^{j\prime}(\tau,\pi-\epsilon) + \bar{b}_{ij}\dot{X}_D^j(\tau,\pi-\epsilon) + \left(\frac{\theta}{2\pi\alpha'}\right)^{-1}_{ij}\dot{y}_\pi^j &= 0, \qquad (3.4) \\
g_{ij}X_D^{j\prime}(\tau,\epsilon) + \bar{b}_{ij}\dot{X}_D^j(\tau,\epsilon) + \left(\frac{\theta}{2\pi\alpha'}\right)^{-1}_{ij}\dot{y}_0^j &= 0, \qquad (3.5)
\end{aligned}
$$

where the suffix D indicates that $X_D^i(\tau,\sigma)$ satisfies the Dirichlet boundary conditions. In the above, the delta function $\delta^s(\sigma,\sigma')$ for functions with Dirichlet boundary conditions is given by

$$
\delta^s(\sigma,\sigma') = \sum_{n=1}^{\infty}\varphi_n^s(\sigma)\varphi_n^s(\sigma'). \qquad (3.6)
$$

As in the previous section, I move to Hamiltonian formalism. $X_D^i(\sigma)$ can be expanded as

$$
X_D^i(\sigma) = \sum_{n=1}^{\infty}\tilde{x}_n^i\varphi_n^s(\sigma). \qquad (3.7)
$$

The canonical momentum of $X^i_D(\sigma)$ is given by

$$\tilde{P}_i(\sigma) = \frac{1}{2\pi\alpha'}\left(g_{ij}\dot{X}_D{}^j(\sigma) + \bar{b}_{ij}X_D^j{}'(\sigma)\right) + \frac{1}{2\pi\alpha'}\bar{b}_{ij}\left(y^j_\pi\delta^s(\pi-\epsilon,\sigma) - y^j_0\delta^s(\epsilon,\sigma)\right). \quad (3.8)$$

$\tilde{P}_i(\sigma)$ can be expanded as

$$\tilde{P}_i(\sigma) = \sum_{n=0}^{\infty}\tilde{\tilde{p}}_{in}\varphi^s_n(\sigma) + \bar{b}_{ij}\sum_{n=0}^{\infty}n\tilde{x}^j_n\varphi^c_n(\sigma), \quad (3.9)$$

where

$$\tilde{\tilde{p}}_{in} = \frac{1}{2\pi\alpha'}\left(g_{ij}\dot{\tilde{x}}^j_n + \bar{b}_{ij}\left(y^j_\pi\varphi^s_n(\pi-\epsilon) - y^j_0\varphi^s_n(\epsilon)\right)\right). \quad (3.10)$$

This expansion can be obtained in a similar manner that was done in [24] for B-picture to obtain the expansions (2.39)$\sim$(2.42). However, it will turn out to be more convenient to use the variable $\tilde{p}_{in}$ defined by

$$\begin{aligned}\tilde{p}_{in} &= \int_0^\pi d\sigma\frac{1}{2\pi\alpha'}\tilde{P}_i(\sigma)\varphi^s_n(\sigma) \\ &= \tilde{\tilde{p}}_{in} + \frac{1}{2\pi\alpha'}\bar{b}_{ij}\sum_{\ell=0}^{\infty}\tilde{x}^j_\ell\ell A_{\ell n}n,\end{aligned} \quad (3.11)$$

to treat the subtle prescription for summation over mode indices similar to the one appeared in (2.67). An underling reason is that the variable in B-picture corresponding to (3.8) is

$$\frac{1}{2\pi\alpha'}\left(g_{ij}\dot{X}_B{}^j(\sigma) + \bar{b}_{ij}X_B^j{}'(\sigma)\right) = \theta^{-1}_{ij}\hat{X}^j{}'(\sigma), \quad (3.12)$$

and it is expanded by $\varphi^s_n(\sigma)$. Since $\sum_{n=1}^\infty \tilde{p}_{in}\varphi^s_n(\sigma)$ is a part of $\tilde{P}_i(\sigma)$ which can be expanded by $\varphi^s_n(\sigma)$, the matching between them contains no subtlety at boundaries. Notice that the map between $\tilde{\tilde{p}}_{in}$ and $\tilde{p}_{in}$ can be written in the form of unitary transformation

$$\tilde{p}_{in} = e^{\tilde{N}_{\bar{b}}}\tilde{\tilde{p}}_{in}e^{-\tilde{N}_{\bar{b}}}, \quad (3.13)$$

where

$$\begin{aligned}\tilde{N}_{\bar{b}} &= \frac{i}{4\pi\alpha'}\bar{b}_{ij}\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\tilde{x}^i_\ell\ell A_{\ell m}m\tilde{x}^j_m \\ &= -\frac{i}{4\pi\alpha'}\int_0^\pi d\sigma\,\bar{b}_{ij}X^i_D(\sigma)X_D^j{}'(\sigma).\end{aligned} \quad (3.14)$$

The map (3.13) can be regarded as a map from the variables without background B-field to those with constant background B-field $\bar{b}_{ij}$ in open string theory on lower dimensional D-branes. This is another reason that the variable $\tilde{p}_{in}$ is convenient.

The canonical commutation relations between modes are given by

$$[\tilde{x}_n^i, \tilde{p}_{jm}] = i\delta_{nm}, \tag{3.15}$$

$$[y_\pi^i, y_\pi^j] = -i\theta^{ij}, \quad [y_0^i, y_0^j] = i\theta^{ij}, \tag{3.16}$$

others give zero. The Hamiltonian of this system is given by

$$H_D^\epsilon = \frac{1}{4\pi\alpha'} g_{ij} \sum_{n=1}^{\infty} \left(\dot{\tilde{x}}_n^i \dot{\tilde{x}}_n^j + n^2 \tilde{x}_n^i \tilde{x}_n^j \right) - \frac{1}{2\pi\alpha'} g_{ij} \left(y_\pi^j \tilde{x}_n^i n \varphi_n^c(\pi - \epsilon) - y_0^j \tilde{x}_n^i n \varphi_n^c(\epsilon) \right), \tag{3.17}$$

where $\dot{\tilde{x}}_n^i$ is solved in terms of $\tilde{x}_n^i$ and $\tilde{p}_{in}$ by (3.10) and (3.11):

$$\dot{\tilde{x}}_n^i \;=\; 2\pi\alpha' g^{ij} \tilde{p}_{jn} - (g^{-1}\bar{b})^i{}_j \left(\sum_{\ell=0}^{\infty} \tilde{x}_\ell^j \ell A_{\ell n} n - \left(y_\pi^j \varphi_n^s(\pi - \epsilon) - y_0^j \varphi_n^s(\epsilon) \right) \right). \tag{3.18}$$

This is also an equation of motion of $\tilde{x}_n^i$. Other equations of motion become

$$\begin{aligned}
\dot{\tilde{p}}_{in} &= i[H_D^\epsilon, \tilde{p}_{in}] \\
&= -\frac{1}{2\pi\alpha'} \left(g - \bar{b} g^{-1} \bar{b} \right)_{ij} n \left(n \tilde{x}_n^j + y_\pi^j \varphi_n^c(\pi - \epsilon) - y_0^j \varphi_n^c(\epsilon) \right) \\
&\quad + (\bar{b} g^{-1})_i{}^j \sum_{\ell=1}^{\infty} \tilde{p}_{j\ell} \ell A_{\ell n} n,
\end{aligned} \tag{3.19}$$

$$\dot{y}_\pi^i = i[H_D^\epsilon, y_\pi^i] = -\frac{\theta^{ij}}{2\pi\alpha'} \left(g_{jk} X_D^{k\prime}(\pi - \epsilon) + \bar{b}_{jk} \dot{X}_D{}^k(\pi - \epsilon) \right), \tag{3.20}$$

$$\dot{y}_0^i = i[H_D^\epsilon, y_0^i] = -\frac{\theta^{ij}}{2\pi\alpha'} \left(g_{jk} X_D^{k\prime}(\epsilon) + \bar{b}_{jk} \dot{X}_D{}^k(\epsilon) \right). \tag{3.21}$$

Now, I take the $\epsilon \to 0$ limit of the equations of motion (3.19)$\sim$(3.21) with the conditions

$$\begin{aligned}
&\lim_{\epsilon \to 0} X_D^i(\pi - \epsilon) = X_B^i(\pi) \neq X_D^i(\pi), \\
&\lim_{\epsilon \to 0} X_D^i(\epsilon) = X_B^i(0) \neq X_D^i(0).
\end{aligned} \tag{3.22}$$

This prescription can be understood from the corresponding boundary state in closed string field theory [22][32, 33]. Then, I obtain

$$\dot{\tilde{x}}_n^i = 2\pi\alpha' g^{ij} \tilde{p}_{jn} - (g^{-1}\bar{b})^i{}_j \sum_{\ell=0}^{\infty} \tilde{x}_\ell^j \ell A_{\ell n} n, \tag{3.23}$$

$$\begin{aligned}
\dot{\tilde{p}}_{in} &= -\frac{1}{2\pi\alpha'} \left(g - \bar{b} g^{-1} \bar{b} \right)_{ij} n \left(n \tilde{x}_n^j + y_\pi^j \varphi_n^c(\pi) - y_0^j \varphi_n^c(0) \right) \\
&\quad + (\bar{b} g^{-1})_i{}^j \sum_{\ell=1}^{\infty} \tilde{p}_{j\ell} \ell A_{\ell n} n,
\end{aligned} \tag{3.24}$$

$$\dot{y}_\pi^i = -\frac{\theta^{ij}}{2\pi\alpha'} \left(g_{jk} X_B^{k\prime}(\pi) + \bar{b}_{jk} \dot{X}_B{}^k(\pi) \right), \tag{3.25}$$

$$\dot{y}_0^i = -\frac{\theta^{ij}}{2\pi\alpha'} \left(g_{jk} X_B^{k\prime}(0) + \bar{b}_{jk} \dot{X}_B{}^k(0) \right). \tag{3.26}$$

To obtain the map between B-picture and D-picture, one equates $X^i_D(\sigma) = X^i_B(\sigma)$ in the interval $(0 < \sigma < \pi)$. In this interval, one can expand $\varphi^c_n(\sigma)$ by $\varphi^s_m(\sigma)$ and the coefficients of $\varphi^s_m(\sigma)$ in the both side should match and one obtains the expression of $\tilde{x}^i_n$ in terms of B-picture variables. Then, the expression of $\tilde{p}_{in}$, y^i_π and y^i_0 can be obtained from (3.23)$\sim$ (3.26). The result is as follows:

$$\begin{aligned}
\tilde{x}^i_n &= \sum_{\ell=0}^{\infty} \hat{x}^i_\ell A_{\ell n} n + \theta^{ij}\left[\hat{p}_{j0}\frac{A_n}{n} + \hat{p}_{jn}\frac{1}{n}\right], && (3.27)\\
\tilde{p}_{in} &= \theta^{-1}_{ij}\hat{x}^j_n n, && (3.28)\\
y^i_\pi &= \sum_{n=0}^{\infty} \hat{x}^i_n \varphi^c_n(\pi) + \frac{\sqrt{\pi}}{2}\theta^{ij}\hat{p}_{j0} = X^i_B(\pi), && (3.29)\\
y^i_0 &= \sum_{n=0}^{\infty} \hat{x}^i_n \varphi^c_n(0) - \frac{\sqrt{\pi}}{2}\theta^{ij}\hat{p}_{j0} = X^i_B(0). && (3.30)
\end{aligned}$$

Using the commutation relations in B-picture (2.45), one can check that (3.27)$\sim$(3.30) satisfy the commutation relations in D-picture (3.15) and (3.16). One can also check that the equations of motion for the variables in B-picture lead to the equations of motion for D-picture (3.23)$\sim$(3.26). This means that both the equations of motion in B-picture and those in D-picture are given by the same Hamiltonian $\hat{H}(B)$. Thus, these two descriptions are equivalent.

The inverse map is given by

$$\begin{aligned}
\hat{x}^i_n &= \theta^{ij}\frac{1}{n}\tilde{p}_{jn}, \quad (n \neq 0) && (3.31)\\
\hat{x}^i_0 &= \frac{\sqrt{\pi}}{2}\left(y^i_\pi + y^i_0\right) - \frac{\sqrt{\pi}}{2}\theta^{ij}\sum_{\ell=1}^{\infty}\tilde{p}_{j\ell}\frac{\varphi^c_\ell(\pi) + \varphi^c_\ell(0)}{\ell}, && (3.32)\\
\hat{p}_{in} &= -\sum_{\ell=1}^{\infty}\tilde{p}_{i\ell}\ell A_{\ell n} + \theta^{-1}_{ij}\left(n\tilde{x}^j_n + \left(y^j_\pi\varphi^c_n(\pi) - y^j_0\varphi^c_n(0)\right)\right). && (3.33)
\end{aligned}$$

Notice that the swallow of Chan-Paton matrices in (3.33) is similar to the phenomenon appeared in [41]. The similarity becomes clearer if one rewrites the Hamiltonian $\hat{H}(B)$ in terms of D-picture variables.[4]

One can write the map in the form of unitary transformation

$$\begin{aligned}
\tilde{U}_{BD}\hat{x}^i_0\tilde{U}^{-1}_{BD} &= \frac{\sqrt{\pi}}{2}\left(y^i_\pi + y^i_0\right), && (3.34)\\
\tilde{U}_{BD}\hat{p}_{i0}\tilde{U}^{-1}_{BD} &= \frac{1}{\sqrt{\pi}}\theta^{-1}_{ij}\left(y^j_\pi - y^j_0\right), && (3.35)\\
\tilde{U}_{BD}\hat{x}^i_n\tilde{U}^{-1}_{BD} &= \theta^{ij}\frac{1}{n}\tilde{p}_{jn}, \quad (n \neq 0) && (3.36)\\
\tilde{U}_{BD}\hat{p}_{in}\tilde{U}^{-1}_{BD} &= \theta^{-1}_{ij}\tilde{x}^j_n n, \quad (n \neq 0) && (3.37)
\end{aligned}$$

[4] I learned this resemblance from N. Ishibashi.

where

$$\tilde{U}_{BD} = e^{C_1} e^{C_2} e^{C_3} e^{C_4}, \tag{3.38}$$

$$
\begin{aligned}
C_1 &= i\frac{\sqrt{\pi}}{2}\hat{p}_{i0}\sum_{n=1}^{\infty}\hat{x}_n^i\left(\varphi_n^c(\pi)+\varphi_n^c(0)\right), &(3.39)\\
C_2 &= -\frac{i}{\sqrt{\pi}}\theta_{ij}^{-1}\hat{x}_0^i\sum_{n=1}^{\infty}\hat{x}_n^j\left(\varphi_n^c(\pi)-\varphi_n^c(0)\right), &(3.40)\\
C_3 &= -\frac{i}{2}\theta_{ij}^{-1}\sum_{\ell=1}^{\infty}\hat{x}_\ell^i\varphi_\ell^c(\pi)\sum_{m=1}^{\infty}\hat{x}_m^j\varphi_m^c(0), &(3.41)\\
C_4 &= \frac{i}{2}\theta_{ij}^{-1}\sum_{\ell=1}^{\infty}\sum_{m=1}^{\infty}\hat{x}_\ell^i A_{\ell m}\frac{\ell^2+m^2}{2}\hat{x}_m^j. &(3.42)
\end{aligned}
$$

From (3.34)∼(3.37), it is easily understood that one can further transform them by Lorenz rotation, rotation in phase space and scale transformation to obtain U_{BD} which satisfies

$$
\begin{aligned}
U_{BD}\hat{x}_0^i U_{BD}^{-1} &= \frac{\sqrt{\pi}}{2}\left(y_\pi^i + y_0^i\right). &(3.43)\\
U_{BD}\hat{p}_{i0} U_{BD}^{-1} &= \frac{1}{\sqrt{\pi}}\theta_{ij}^{-1}\left(y_\pi^j - y_0^j\right), &(3.44)\\
U_{BD}\hat{x}_n^i U_{BD}^{-1} &= \tilde{x}_n^i, \quad (n\neq 0) &(3.45)\\
U_{BD}\hat{p}_{in} U_{BD}^{-1} &= \tilde{p}_{in}. \quad (n\neq 0) &(3.46)
\end{aligned}
$$

Notice that from (3.27)∼(3.30), the 3-string vertex $\langle V_3| = \langle\hat{V}_3|e^M$ satisfies the following overlap conditions:

$$
\begin{aligned}
&\langle V_3|\left(X_D^{i(r+1)}(\sigma) - X_D^{i(r)}(\pi-\sigma)\right) = 0, \quad (0<\sigma\leq\frac{\pi}{2})\\
&\langle V_3|\left(\tilde{P}_i^{(r+1)}(\sigma) + \tilde{P}_i^{(r)}(\pi-\sigma)\right) = 0, \quad (0<\sigma\leq\frac{\pi}{2})\\
&\langle V_3|\left(y_0^{i(r+1)} - y_\pi^{i(r)}\right) = 0. &(3.47)
\end{aligned}
$$

These are natural overlap conditions for open string field theory on lower dimensional D-branes.

3.2 3-string Vertex In D-picture

In this subsection I study another 3-string vertex which is naturally obtained in D-picture (see [42] for related issues). The 3-string vertex is divided into the overlap vertex for $X_D^i(\sigma)$ part and Chan-Paton matrix part.

I first construct the Chan-Paton matrix multiplication part. In the following, I assume the rank of the matrix θ^{ij} is 24. I set $\theta^{0i} = \theta^{1i} = 0$, and set $\theta^{ij} = 0$ except $\theta^{2i\,2i+1} = -\theta^{2i+1\,2i}$, by an appropriate rotation in the target space. Also I only consider the case where $\theta^{2i\,2i+1} > 0$. Generalization to other cases is straightforward.

First I define creation and annihilation operators in Chan-Paton algebra as follows.

$$z_0^i = \sqrt{\frac{1}{2\theta^{2i\,2i+1}}}\left(y_0^{2i} + iy_0^{2i+1}\right), \quad z_0^{i\dagger} = \sqrt{\frac{1}{2\theta^{2i\,2i+1}}}\left(y_0^{2i} - iy_0^{2i+1}\right),$$
$$z_\pi^i = \sqrt{\frac{1}{2\theta^{2i\,2i+1}}}\left(y_\pi^{2i} - iy_\pi^{2i+1}\right), \quad z_\pi^{i\dagger} = \sqrt{\frac{1}{2\theta^{2i\,2i+1}}}\left(y_\pi^{2i} + iy_\pi^{2i+1}\right). \tag{3.48}$$

Their commutation relations are given by

$$[z_0^i, z_0^{j\dagger}] = \delta^{ij}, \quad [z_\pi^i, z_\pi^{j\dagger}] = \delta^{ij}, \tag{3.49}$$

and otherwise zero. These operators act on the Hilbert space whose basis is labeled by $\mathbf{n} = (n_1, n_2, \cdots, n_{12})$:

$$|\mathbf{n}\rangle_0 = \prod_{i=1}^{12} \frac{1}{\sqrt{n_i!}}(z_0^{i\dagger})^{n_i}|0\rangle_0, \tag{3.50}$$
$$|\mathbf{n}\rangle_\pi = \prod_{i=1}^{12} \frac{1}{\sqrt{n_i!}}(z_\pi^{i\dagger})^{n_i}|0\rangle_\pi, \tag{3.51}$$

where $|0\rangle_0$ and $|0\rangle_\pi$ are defined by the conditions

$$z_0^i|0\rangle_0 = 0, \quad z_\pi^i|0\rangle_\pi = 0. \tag{3.52}$$

Arbitrary open string field in D-picture has Chan-Paton indices

$$|\mathbf{n}\rangle_0|\mathbf{m}\rangle_\pi. \tag{3.53}$$

$|\mathbf{n}\rangle_0|\mathbf{m}\rangle_\pi$ itself is regarded as Chan-Paton matrix. The matrix multiplication is represented by $\langle v_3^{CP}|$ defined as follows:

$$\begin{aligned}\langle v_3^{CP}| &= \prod_{r=1}^{3}\sum_{\mathbf{n}} {}_\pi^{(r)}\langle\mathbf{n}|\ {}_0^{(r+1)}\langle\mathbf{n}| \\ &= \prod_{r=1}^{3} {}_\pi^{(r)}\langle 0|\ {}_0^{(r+1)}\langle 0| \exp\left(\sum_{i=1}^{12} z_\pi^{i(r)} z_0^{i(r+1)}\right).\end{aligned} \tag{3.54}$$

The first line is a definition of the usual matrix multiplication. The form in the second line is a simpler analog of the vertex which appeared in the matrix representation of the 3-string vertex [43]. Precisely speaking, here there appear three kinds of Chan-Paton matrices: those acting on the vector $|\mathbf{n}\rangle_0$, those acting on $|\mathbf{n}\rangle_\pi$, and those whose basis is given by (3.53). But those differences are not essential. The basis of the first type is given by $|\mathbf{n}\rangle_{00}\langle\mathbf{m}|$ and the basis for the second type is given by $|\mathbf{n}\rangle_{\pi\pi}\langle\mathbf{m}|$. One can isomorphically convert them into the third type by multiplying them to the identity matrix in the third type.

$$|1\rangle = \sum_{\mathbf{n},\mathbf{m}} \delta_{\mathbf{nm}} |\mathbf{n}\rangle_0 |\mathbf{m}\rangle_\pi, \tag{3.55}$$

$$|\mathbf{n}\rangle_{00}\langle\mathbf{m}||1\rangle = |\mathbf{n}\rangle_0|\mathbf{m}\rangle_\pi, \quad |\mathbf{n}\rangle_{\pi\pi}\langle\mathbf{m}||1\rangle = |\mathbf{m}\rangle_0|\mathbf{n}\rangle_\pi. \tag{3.56}$$

As an example, let us consider a map

$$z_0^{i\dagger} \mapsto z_0^{i\dagger}|1\rangle = z_\pi^i|1\rangle, \quad z_0^i \mapsto z_0^i|1\rangle = z_\pi^{i\dagger}|1\rangle. \tag{3.57}$$

Then, the algebra (3.49) is isomorphically mapped to

$$[z_0^{i\dagger}|1\rangle, z_0^{j\dagger}|1\rangle]_{CP} = \delta^{ij}|1\rangle, \tag{3.58}$$

where the subscript CP denotes that the multiplication is defined by (3.54).

The 3-string vertex in D-picture can be constructed as

$$\langle V_3^D| = \langle V_3^{X_D}|\langle v_3^{CP}|, \tag{3.59}$$

where $\langle V_3^{X_D}|$ is defined by the overlap conditions

$$\begin{aligned} &\langle V_3^{X_D}| \left(X_D^{i(r+1)}(\sigma) - X_D^{i(r)}(\pi - \sigma) \right) = 0, \quad (0 < \sigma \leq \frac{\pi}{2}) \\ &\langle V_3^{X_D}| \left(\tilde{P}_i^{(r+1)}(\sigma) + \tilde{P}_i^{(r)}(\pi - \sigma) \right) = 0. \qquad (0 < \sigma \leq \frac{\pi}{2}) \end{aligned} \tag{3.60}$$

Since both $\langle V_3|$ and $\langle V_3^D|$ satisfy the same overlap condition, they are identical up to normalization. It would be interesting to check this by a direct calculation using explicit expressions for Neumann coefficients of the 3-string vertex and the map between different pictures.

3.3 Chan-Paton Algebra In Open String Field Algebra

Since the map between F-picture, B-picture and D-picture has been obtained, now I can identify the Chan-Paton matrix subalgebra in the original open string field algebra. Let us extract the Chan-Paton algebra in open string field algebra whose star product is defined by (2.32), without using the expression (3.59). As can be expected from (3.56), to extract Chan-Paton subalgebra from open string field algebra, what I need to do is to just multiply $X_B^i(0)$ to the identity state

$$X_B^i(0) \mapsto X_B^i(0)|I\rangle. \tag{3.61}$$

This is a map from worldsheet variable to open string field. Recall that this kind of map from worldsheet operator algebra to open string field algebra often appears in the cubic open string field theory [44, 43]. I obtain

$$X_B^i(0)|I\rangle \star X_B^j(0)|I\rangle = X_B^i(0)X_B^j(0)|I\rangle, \tag{3.62}$$

$$[X^i_B(0)|I\rangle, X^j_B(0)|I\rangle]_\star = [X^i_B(0), X^j_B(0)]|I\rangle = i\theta^{ij}|I\rangle. \tag{3.63}$$

Here, I have used the relations

$$e^{-M}e^{-N}|I\rangle = |I\rangle, \tag{3.64}$$

$$\left(X^i_B(\sigma) - X^i_B(\pi-\sigma)\right)|I\rangle = 0, \tag{3.65}$$

$$\left(\hat{P}_i(\sigma) + \hat{P}_i(\pi-\sigma)\right)|I\rangle = 0. \tag{3.66}$$

These relations are obtained in the similar way that was done for the 3-string vertex $\langle V_3|$. Thus, the Chan-Paton algebra (3.16) is extracted from the open string field algebra. The map (3.29) and (3.30) confirms that the algebra (3.63) is identified with the Chan-Paton algebra. Once the Chan-Paton subalgebra (3.63) in open string field algebra is obtained, one can apply techniques in non-commutative solitons to obtain exact/approximate solutions. Notice that in the map between the different pictures (F,B,D), no special limit such as the so called Seiberg-Witten limit [7] is assumed.

Let us call the algebra generated by the string coordinate fields at the boundaries as boundary algebra. What I have shown here is that in constant B-field background, the boundary algebra *is* Chan-Paton algebra. It is easy to show that using the defining property of the 3-string vertex and the identity field one can map arbitrary boundary algebra to open string field algebra by the map (3.61). Therefore, I expect that arbitrary boundary algebra can be interpreted as Chan-Paton algebra.

4 Summary And Discussions

In this article, I have studied three equivalent descriptions of open string in constant B-field background in the framework of Witten's cubic open string field theory.

The equivalence of B-picture and F-picture gives the map between the theory with non-commutative coordinates and commutative coordinates. Since F-picture variables are the original variables, i.e. the variables without the B-field background, this means that the algebra of non-commutative coordinates is already contained in the original open string field algebra.

On the other hand, from the equivalence of B-picture and D-picture I obtained the expression of Chan-Paton matrices in terms of B-picture variables. The Chan-Paton matrix in D-picture is shown to be the string coordinate field at the boundary in B-picture. I believe in order to find D-branes in open string field theory, one should not overlook this property, i.e. the Chan-Paton indices are on the boundaries. Using the map, I extracted the Chan-Paton algebra as subalgebra in open string field algebra. Since Chan-Paton algebra is associated with lower dimensional D-branes, this ensures that lower dimensional D-branes can be described as some field configurations in Witten's cubic open string field theory.

There are two directions for the classification of D-branes. The one is to use the highest dimensional branes [45, 46] to classify lower dimensional D-branes as topological defects. The other uses the lowest dimensional branes [47] to construct higher dimensional D-branes. In this article, I started from the open string field theory on space-time filling D-branes, but I could have started from open string theory on infinitely many lower dimensional D-branes. Thus, open string field algebra naturally unifies the two descriptions and appears as the most fundamental framework for the classification of D-branes. In this article, I rewrite Chan-Paton indices, which are fundamental variables in D-picture, to linear combination of variables in another picture. This strategy may be generalized and will be a powerful tool to find D-brane solutions in Witten's cubic open string field theory. However, note that this strategy is applicable to the backgrounds which can be described by the same 3-string vertex. In other words, open string field algebra can be used to classify D-branes which are contained in that paticular open string field algebra [48]. It will be important to investigate to what extent the 3-string vertex can describe different backgrounds.

The equivalence between boundary algebra and Chan-Paton algebra seems to hold more generally. It would be interesting to investigate the relation to the boundary algebra [49] in boundary open string field theory [50] along the line of [51], where a proposal for a concrete definition of the boundary open string field theory was given as a limit of continuous deformation from Witten's cubic open string field theory.

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A Formulas For The Fourier Coefficients

Ortho-normal basis for the functions in $0 \le \sigma \le \pi$ with Neumann boundary conditions at the boundaries $\sigma = 0, \pi$:

$$\begin{aligned} \varphi^c_n(\sigma) &= \sqrt{\frac{2}{\pi}} \cos n\sigma, \quad (n \neq 0) \\ \varphi^c_0(\sigma) &= \sqrt{\frac{1}{\pi}} \quad . \end{aligned} \tag{A.1}$$

Ortho-normal basis for the functions in $0 \le \sigma \le \pi$ with Dirichlet boundary conditions at the boundaries $\sigma = 0, \pi$:

$$\varphi^s_n(\sigma) = \sqrt{\frac{2}{\pi}} \sin n\sigma. \tag{A.2}$$

Definition of the matrix A_{mn}:

$$A_{mn} = \frac{1}{2} \int_0^\pi d\sigma \int_0^\pi d\sigma' \epsilon(\sigma - \sigma') \varphi^c_m(\sigma) \varphi^c_n(\sigma'), \tag{A.3}$$

where $\epsilon(\sigma)$ is the sign function

$$\begin{aligned} \epsilon(\sigma) &= 1, \quad (\sigma > 0) \\ &= 0, \quad (\sigma = 0) \\ &= -1. \quad (\sigma < 0) \end{aligned} \tag{A.4}$$

Explicit value of the components:

$$\begin{aligned} A_{mn} &= \frac{1}{m^2 - n^2} \left(\varphi^c_m(\pi) \varphi^c_n(\pi) - \varphi^c_m(0) \varphi^c_n(0) \right). \quad (m \neq n) \\ &= 0. \quad (m = n) \end{aligned} \tag{A.5}$$

Formulas:

$$\varphi^c_n(\sigma) = \sum_{m=1}^{\infty} A_{nm} m \varphi^s_m(\sigma), \quad (0 < \sigma < \pi) \tag{A.6}$$

$$\varphi^s_n(\sigma) = \sum_{m=0}^{\infty} (-n) A_{nm} \varphi^c_m(\sigma), \quad (0 \le \sigma \le \pi) \tag{A.7}$$

$$\begin{aligned} \sum_{\ell=0}^{\infty} A_{n\ell} A_{m\ell} &= \frac{1}{nm} \delta_{nm}, \qquad (n \neq 0,\, m \neq 0) \\ &= -\frac{\sqrt{\pi}}{2m^2} \left(\varphi^c_m(\pi) + \varphi^c_m(0) \right), \quad (n = 0,\, m \neq 0) \\ &= -\frac{\pi^2}{12}, \qquad (n = m = 0) \end{aligned} \tag{A.8}$$

$$\sum_{\ell=1}^{\infty} A_{n\ell} A_{m\ell} \ell^2 = \delta_{nm}. \tag{A.9}$$

$$\sqrt{\frac{1}{\pi}} \left(\sigma - \frac{\pi}{2}\right) = -\sum_{\ell=1}^{\infty} A_{0\ell} \varphi_\ell^c(\sigma). \quad (0 \leq \sigma \leq \pi) \tag{A.10}$$

$$\sum_{\ell=0}^{\infty} A_{n\ell} \varphi_\ell^c(\pi) = -\delta_{n0} \frac{\sqrt{\pi}}{2}, \quad \sum_{\ell=0}^{\infty} A_{n\ell} \varphi_\ell^c(0) = \delta_{n0} \frac{\sqrt{\pi}}{2}. \tag{A.11}$$

Definition of A_n:

$$\begin{aligned} \frac{1}{n} A_n &= \int_0^\pi d\sigma \sqrt{\frac{1}{\pi}} \left(\sigma - \frac{\pi}{2}\right) \varphi_n^s(\sigma) \\ &= -\frac{1}{\sqrt{2}} \frac{1}{n} \left((-)^n + 1\right) = -\frac{\sqrt{\pi}}{2n} \left(\varphi_n^c(\pi) + \varphi_n^c(0)\right). \end{aligned} \tag{A.12}$$

B Formulas For The Target Space Tensors

$$G_{ij} = g_{ij} - (bg^{-1}b)_{ij}, \quad G^{ij} = \left(\frac{1}{g+b} g \frac{1}{g-b}\right)^{ij}. \tag{B.1}$$

$$\frac{\theta^{ij}}{2\pi\alpha'} = -\left(\frac{1}{g+b} b \frac{1}{g-b}\right)^{ij}. \tag{B.2}$$

$$b_{ij} = \bar{b}_{ij} + \left(\frac{\theta}{2\pi\alpha'}\right)^{-1}_{ij}. \tag{B.3}$$

$$\left(bG^{-1}\right)_i{}^j = -\left(\frac{g\theta}{2\pi\alpha'}\right)_i{}^j, \quad \left(\frac{\bar{b}\theta}{2\pi\alpha'}\right)_i{}^j = -\left(gG^{-1}\right)_i{}^j. \tag{B.4}$$

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