THE BISOGNANO-WICHMANN THEOREM FOR MASSIVE THEORIES

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Abstract. The geometric action of modular groups for wedge regions (Bisognano–Wichmann property) is derived from the principles of local quantum physics for a large class of Poincaré covariant models in $d = 4$. As a consequence, the CPT theorem holds for this class. The models must have a complete interpretation in terms of massive particles. The corresponding charges need not be localizable in compact regions: The most general case is admitted, namely localization in spacelike cones.

Introduction.

In local relativistic quantum theory [23], a model is specified in terms of a net of local observable algebras and a representation of the Poincaré group, under which the net is covariant. The Bisognano-Wichmann theorem [2, 3] intimately connects these two, algebraic and geometric, aspects. Namely, it asserts that under certain conditions *modular covariance* is satisfied: The modular unitary group of the observable algebra associated to a wedge region coincides with the unitary group representing the boosts which preserve the wedge. Since the boosts associated to all wedge regions generate the Poincaré group, modular covariance implies that the representation of the Poincaré group is encoded intrinsically in the net of local algebras. It has further important consequences: It implies the spin-statistics theorem [22, 27] and, as Guido and Longo have shown [22], the CPT theorem. It also implies essential Haag duality, which is an important input to the structural analysis of charge superselection sectors [16, 17].

Counterexamples [10,11,32] demonstrate that modular covariance does not follow from the basic principles of quantum field theory without further input. But its remarkable implications assign a significant role to this property, and it is desirable to find physically transparent conditions under which it holds. Bisognano and Wichmann have shown modular covariance to hold in theories where the field algebras are generated by finite-component Wightman fields [2, 3]. In the framework of algebraic quantum field theory, Borchers has shown that the modular objects associated to wedges have the correct[1] commutation relations with the translation operators as a consequence of the positive energy requirement [4]. Based on his result, Brunetti, Guido and Longo derived modular covariance for conformally covariant theories [10]. In the Poincaré covariant case, sufficient conditions for modular covariance of technical nature have been found by several authors [6, 8, 21, 26, 29] (see [8] for a review of these results).

In the present article, we derive modular covariance in the setting of local quantum physics for a large class of massive models. Specifically, the models must contain massive particles whose scattering states span the whole Hilbert space (asymptotic completeness). Further, within each charge sector the occurring particle masses must be isolated eigenvalues of the mass operator. The corresponding representation of the covering group of the Poincaré group must have no accidental degeneracies; *i.e.* for each mass and charge there is one single

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[1]namely, as expected from modular covariance

particle multiplet under the gauge group (the group of inner symmetries). We admit the most general localization properties for the charges carried by these particles, namely localization in spacelike cones [13].

A byproduct of our analysis is that the CPT theorem holds under these rather general and transparent conditions. It must be mentioned that Epstein has already proved a rudimentary version of the CPT theorem for massive theories in the framework of local quantum physics [20]. But it refers only to the S-matrix (and not to the local fields), and is derived only for compactly localized charges. It must also be mentioned that the spin-statistics theorem, which is a consequence of modular covariance and needs not be assumed for our derivation, has already been proved by Buchholz and Epstein [12] for massive theories with charges localizable in spacelike cones.

The article is organized as follows. In Section 1, the general framework is set up and our assumptions concerning the particle spectrum are made precise, as well as our notion of modular covariance. We state our main result in Theorem 2. The proof will proceed in two steps: In Section 2, single-particle versions of the Bisognano-Wichmann and the CPT theorems are derived (Theorem 5). This is an extension of Buchholz and Epstein's proof [12] of the spin-statistics theorem for topological charges. In Section 3 we prove modular covariance via Haag-Ruelle scattering theory (Proposition 7). As mentioned, this already implies the CPT theorem [22]. Yet for the sake of self consistency, we show in Section 4 that the CPT theorem can be derived directly from our assumptions via scattering theory (Proposition 9).

1. Assumptions and Result.

In the algebraic framework, the fundamental objects of a quantum field theory are the observable algebra and the physically relevant representations of it. The set of equivalence classes of these representations, or charge superselection sectors, has the structure of a semigroup. We will assume that it is generated by a set of "elementary charges" which correspond to massive particles. Then all relevant charges are localizable in spacelike cones [13]. Under these circumstances and if Haag duality holds, it is known [19] that the theory may be equivalently described by an algebra of (unobservable) charged field operators localized in spacelike cones, and a compact gauge group acting on the fields. The observable algebra is then the set of gauge invariant elements of the field algebra, and modular covariance of the former is equivalent to modular covariance of the latter [22, 27]. We take the field algebra framework as the starting point of our analysis. It is noteworthy that then essential Haag duality needs not be assumed for our result, but rather follows from it.

Let us briefly sketch this framework. The Hilbert space $\mathcal{H}$ carries a unitary representation U of the universal covering group $\tilde{P}_+^\uparrow$ of the Poincaré group which has positive energy, *i.e.* the joint spectrum of the generators P_μ of the translations is contained in the closed forward lightcone. There is a unique, up to a factor, invariant vacuum vector Ω. Further, there is a compact group G (the gauge group) of unitary operators on $\mathcal{H}$ which commute with the representation U and leave Ω invariant.

For every spacelike cone[2] $\mathcal{C}$ there is a von Neumann algebra $\mathcal{F}(\mathcal{C})$ of so-called field operators acting in $\mathcal{H}$. The family $\mathcal{C} \to \mathcal{F}(\mathcal{C})$, together with the representation U and the group G, satisfies the following properties.

0) *Inner symmetry:* For all $\mathcal{C}$ and all $V \in G$

$$V\,\mathcal{F}(\mathcal{C})\,V^{-1} = \mathcal{F}(\mathcal{C})\,.$$

i) *Isotony:* $\mathcal{C}_1 \subset \mathcal{C}_2$ implies $\mathcal{F}(\mathcal{C}_1) \subset \mathcal{F}(\mathcal{C}_2)$.

[2] a spacelike cone is a region in Minkowski space of the form $\mathcal{C} = a + \cup_{\lambda>0}\lambda\mathcal{O}$, where $a \in \mathbb{R}^4$ is the apex of $\mathcal{C}$ and $\mathcal{O}$ is a double cone whose closure does not contain the origin.

ii) *Covariance:* For all $\mathcal{C}$ and all $g \in \tilde{P}_+^\uparrow$

$$U(g)\,\mathcal{F}(\mathcal{C})\,U(g)^{-1} = \mathcal{F}(g\,\mathcal{C})\,.$$

iii) *Twisted locality:* There is a Bose-Fermi operator κ in the center of G with $\kappa^2 = 1$, determining the spacelike commutation relations of fields: let $Z \doteq \frac{1+i\kappa}{1+i}$. Then

$$Z\mathcal{F}(\mathcal{C}_1)Z^* \subset \mathcal{F}(\mathcal{C}_2)'$$

if $\mathcal{C}_1$ and $\mathcal{C}_2$ are spacelike separated.

iv) *Reeh-Schlieder property:* For every $\mathcal{C}$, $\mathcal{F}(\mathcal{C})\,\Omega$ is dense in $\mathcal{H}$.

v) *Irreducibility:* $\bigcap_{\mathcal{C}} \mathcal{F}(\mathcal{C})' = \mathbb{C}\cdot 1$.

Note that twisted locality (iii) is equivalent to normal commutation relations [15]: Two field operators which are localized in causally disjoint cones anticommute if both operators are odd under the adjoint action of κ (fermionic) and commute if one of them is even (bosonic) and the other one is even or odd.

Let

$$\mathcal{H} = \bigoplus_{\alpha\in\Sigma} \mathcal{H}_\alpha$$

be the factorial decomposition of G'', with Σ the set of equivalence classes of irreducible unitary representations of G contained in the defining representation [3]. The subspaces $\mathcal{H}_\alpha$ will be referred to as (charge) sectors, and two sectors corresponding to conjugate representations $\alpha, \bar{\alpha}$ of G will be called conjugate sectors. We denote by E_α the projection in $\mathcal{H}$ onto $\mathcal{H}_\alpha$, and by d_α the (finite) dimension of the class α. Note that the Poincaré representation commutes with E_α. Let $\Sigma^{(1)} \subset \Sigma$ be the set of charges carried by the particle types of the theory: $\alpha \in \Sigma^{(1)}$ if, and only if, the restriction of the mass operator $\sqrt{P^2}$ to $\mathcal{H}_\alpha$ has non–zero eigenvalues.

vi) *Massive particle spectrum.* There are no massless particles, *i.e.* no eigenvectors of the mass operator with eigenvalue zero apart from the vacuum vector. For each $\alpha \in \Sigma^{(1)}$, there is exactly one[4] non–zero eigenvalue m_α of $\sqrt{P^2}E_\alpha$, which is in addition isolated. Further, the corresponding subrepresentation of $\tilde{P}_+^\uparrow$ contains only one irreducible representation, with multiplicity equal to d_α. Thus, for each $\alpha \in \Sigma^{(1)}$, there is one multiplet under G of particle types with the same charge α, mass and spin.

vii) *Asymptotic completeness:* The scattering states span the whole Hilbert space (see equation (3.5)).

The property of modular covariance, which we are going to derive from these assumptions, means the following. Due to the Reeh-Schlieder property and locality, for every spacelike cone $\mathcal{C}$ there is a Tomita operator [9] $S_{\rm Tom}(\mathcal{C})$ associated to $\mathcal{F}(\mathcal{C})$ and Ω : It is the closed antilinear involution satisfying

$$S_{\rm Tom}(\mathcal{C})\,B\Omega = B^*\Omega \quad \text{ for all } B \in \mathcal{F}(\mathcal{C})\,.$$

Its polar decomposition is denoted as

$$S_{\rm Tom}(\mathcal{C}) = J_{\mathcal{C}}\,\Delta_{\mathcal{C}}^{\frac{1}{2}}\,.$$

The anti-unitary involution $J_{\mathcal{C}}$ in this decomposition is called the *modular conjugation*, and the positive operator $\Delta_{\mathcal{C}}$ gives rise to the so-called *modular unitary group* $\Delta_{\mathcal{C}}^{it}$ associated to $\mathcal{C}$. Modular covariance means, generally speaking, that the Tomita operators associated to a

[3] In fact, Σ contains all irreducible representations of G, and is in 1–1 correspondence with the superselection sectors of the observable algebra [15].

[4] Our results still hold if no restriction is imposed on the number of (isolated) mass values in each sector.

distinguished class of space-time regions have geometrical significance. This class is the set of wedge regions, *i.e.* Poincaré transforms of the standard wedge region W_1 : [5]

$$W_1 \doteq \{ x \in \mathbb{R}^4 : \, |x^0| < x^1 \} ,$$

and the geometrical significance is as follows. Let $\Lambda_1(t)$ denote the Lorentz boost in x^1-direction, acting on the coordinates x^0, x^1 as

$$\begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} ,$$

and $\lambda_1(t)$ its lift to the covering group $\tilde{P}_+^\uparrow$.

Definition 1. A theory is said to to satisfy *modular covariance* if

$$\Delta_{W_1}^{it} = U(\lambda_1(-2\pi t)) . \tag{1.1}$$

Other notions of modular covariance have been proposed in the literature (see, *e.g.* [14]), but this is the strongest one. In particular, it implies [22] that the modular conjugation J_{W_1} has the geometric significance of representing the reflexion j at the edge of W_1, which inverts the sign of x^0 and x^1 and acts trivial on x^2, x^3. More precisely, Guido and Longo have shown in [22] that equation (1.1) implies that the operator $\Theta \doteq Z^* \, J_{W_1}$ acts geometrically correctly, *i.e.* satisfies

$$\Theta \, \mathcal{F}(W) \, \Theta^{-1} = \mathcal{F}(jW) \tag{1.2}$$

for all wedge regions W, and has the representation properties

$$\Theta \, U(g) \, \Theta^{-1} = U(jgj) \quad \text{for all } g \in \tilde{P}_+^\uparrow , \quad \Theta^2 = 1 . \tag{1.3}$$

Here we have denoted by $g \mapsto jgj$ the unique lift of the adjoint action of j on the Poincaré group to an automorphism of the covering group [31]. Since Θ also sends each sector to its conjugate, equations (1.2) and (1.3) exhibit Θ as a CPT operator[6]. Thus, modular covariance (1.1) implies the CPT theorem. Further, the last two equations imply, by the Tomita-Takesaki theorem, that the theory satisfies *twisted Haag duality for wedges*, *i.e.*

$$Z \mathcal{F}(W') Z^* = \mathcal{F}(W)' . \tag{1.4}$$

Our main result is the following theorem.

Theorem 2. *Let the assumptions 0) , ... , vii) be satisfied. Then modular covariance, the CPT theorem as expressed by equations* (1.2) *and* (1.3)*, and twisted Haag duality for wedges hold.*

It is noteworthy that equation (1.2) holds not only for wedge regions, but also for spacelike cones if one replaces the family $\mathcal{F}$ with the so-called dual family $\mathcal{F}^d$. Namely, twisted Haag duality for wedge regions implies that the dual family $\mathcal{F}^d(\mathcal{C}) \doteq \cap_{W \supset \mathcal{C}} \mathcal{F}(W)$ is still local. On this family, Θ acts geometrically correctly, *i.e.* equation (1.2) holds for all $\mathcal{F}^d(\mathcal{C})$ [22].

Our proof of the theorem will proceed in two steps: In the next section, modular covariance is shown to hold in restriction to the single particle space (Theorem 5). In Section 3 we show that modular covariance extends to the space of scattering states (Proposition 7). By the assumption of asymptotic completeness this space coincides with $\mathcal{H}$, hence modular covariance holds, implying the CPT theorem.

[5] Wedges W will be considered as limiting cases of spacelike cones. $\mathcal{F}(W)$ is the von Neumann algebra generated by all $\mathcal{F}(\mathcal{C})$ with $\mathcal{C} \subset W$.

[6] Here we consider j as the PT transformation. The total space-time inversion arises from j through a π-rotation about the 1-axis, and is thus also a symmetry (if combined with charge conjugation C). In odd-dimensional space-time, j is the proper candidate for a symmetry in combination with C, while the total space-time inversion is not.

2. Modular Covariance on the Single Particle Space.

As a first step, we prove single-particle versions of the Bisognano-Wichmann and the CPT theorems. Let E_I, $I \subset \mathbb{R}$, be the spectral projections of the mass operator $\sqrt{P^2}$. We denote by $E^{(1)}$ the sum of all $E_{\{m_\alpha\}}$, where α runs through the set $\Sigma^{(1)}$ of single particle charges and m_α are the corresponding particle masses. The range of $E^{(1)}$ is called the single particle space.

An essential step towards the Bisognano-Wichmann theorem is the mentioned result of Borchers [4, 5], namely that the modular unitary group and the modular conjugation associated to the wedge W_1 have the correct commutation relations with the translations. In particular, they commute with the mass operator, which implies that $S_{\rm Tom}(W_1)$ commutes with $E^{(1)}$. Let us denote the corresponding restriction by

$$S_{\rm Tom} \doteq S_{\rm Tom}(W_1)\, E^{(1)} .$$

Similarly, the representation $U(\tilde{P}_+^\uparrow)$ leaves $E^{(1)}\mathcal{H}$ invariant, giving rise to the subrepresentation

$$U^{(1)}(g) \doteq U(g)\, E^{(1)} ,$$

and one may ask if modular covariance holds on $E^{(1)}\mathcal{H}$. We show in this section that this is indeed the case, the line of argument being as follows. Let K denote the generator of the unitary group of 1-boosts, $U(\lambda_1(t)) = e^{itK}$. We exhibit an anti-unitary "PT-operator" $U^{(1)}(j)$ representing the reflexion j on $E^{(1)}\mathcal{H}$, and show that $S_{\rm Tom}$ coincides with the "geometric" involution

$$S_{\rm geo} \doteq U^{(1)}(j)\; e^{-\pi K}\, E^{(1)} \tag{2.1}$$

up to a unitary "charge conjugation" operator which commutes with the representation $U^{(1)}$ of $\tilde{P}_+^\uparrow$. By uniqueness of the polar decomposition, this will imply modular covariance on $E^{(1)}\mathcal{H}$.

We begin by exploiting our knowledge about $U^{(1)}(\tilde{P}_+^\uparrow)$. By assumption, for each $\alpha \in \Sigma^{(1)}$ the subrepresentation $U(g)E^{(1)}E_\alpha$ contains only one equivalence class of irreducible representations. As is well–known, the latter is fixed by the mass m_α and a number $s_\alpha \in \frac{1}{2}\mathbb{N}_0$, the spin of the corresponding particle species.

We briefly recall the so-called covariant irreducible representation $U_{m,s}$ for mass $m > 0$ and spin $s \in \frac{1}{2}\mathbb{N}_0$. The universal covering group of the proper orthochronous Lorentz group $L_+^\uparrow$ is identified with $SL(2,\mathbb{C})$ [30]. Explicitly, the boosts $\Lambda_k(\cdot)$ in k-direction and rotations $R_k(\cdot)$ about the k-axis, $k = 1, 2, 3$, lift to

$$\lambda_k(t) = e^{\frac{1}{2}t\,\sigma_k} \quad \text{ and } \quad r_k(\omega) = e^{\frac{i}{2}\omega\,\sigma_k} , \quad k = 1, 2, 3 , \tag{2.2}$$

respectively. The universal covering group $\tilde{P}_+^\uparrow$ of the proper orthochronous Poincaré group $P_+^\uparrow$ is the semidirect product of $SL(2,\mathbb{C})$ with the translation subgroup $\mathbb{R}^4$, elements being denoted by $g = (x, A)$. The representation $U_{m,s}$ of $\tilde{P}_+^\uparrow$ for $m > 0$ and $s \in \frac{1}{2}\mathbb{N}_0$ acts on a Hilbert space $\mathcal{H}_{m,s}$ of functions from the positive mass shell H_m into $\mathbb{C}^{2s+1}$. The latter, viewed as the space of covariant spinors of rank $2s$, is acted upon by an irreducible representation V_s of $SL(2,\mathbb{C})$ satisfying

$$V_s(A^*) = V_s(A)^* \quad \text{ and } \quad V_s(\bar{A}) = \overline{V_s(A)} .$$

Let $\langle\ ,\ \rangle$ denote the scalar product in $\mathbb{C}^{2s+1}$, and $\mathrm{d}\mu(p)$ the Lorentz invariant measure on the mass shell H_m, and let, for $p=(p^0,\boldsymbol{p})\in\mathbb{R}^4$,

$$\tilde{p}\doteq p^0 1-\boldsymbol{p}\cdot\boldsymbol{\sigma} \quad\text{and}\quad \underset{\sim}{p}\doteq p^0 1+\boldsymbol{p}\cdot\boldsymbol{\sigma}\ ,$$

where $\boldsymbol{\sigma}=(\sigma_1,\sigma_2,\sigma_3)$ are the Pauli matrices. Then the scalar product in $\mathcal{H}_{m,s}$ is defined as

$$(\,\psi_1,\psi_2\,)=\int \mathrm{d}\mu(p)\,\langle\,\psi_1(p)\,,\,V_s(\frac{1}{m}\tilde{p})\,\psi_2(p)\,\rangle\,. \tag{2.3}$$

$U_{m,s}$ acts on $\mathcal{H}_{m,s}$ according to

$$\big(U_{m,s}(x,A)\psi\big)(p)=\exp(ix\cdot p)\;V_s(A)\,\psi(\Lambda(A^{-1})\,p)\ , \tag{2.4}$$

where $\Lambda: SL(2,\mathbb{C})\to L_+^\uparrow$ denotes the covering homomorphism. To this representation an anti–unitary operator $U_{m,s}(j)$ can be adjoined satisfying the representation properties

$$U_{m,s}(j)^2=1 \quad\text{and}\quad U_{m,s}(j)\,U_{m,s}(g)\,U_{m,s}(j)=U_{m,s}(jgj) \tag{2.5}$$

for all $g\in\tilde{P}_+^\uparrow$. Namely, it is given by [7]

$$\big(U_{m,s}(j)\psi\big)(p)\doteq V_s\big(\frac{1}{m}\underset{\sim}{p}\;\sigma_3\big)\;\overline{\psi(-j\,p)}\,. \tag{2.6}$$

By our assumption vi), we may identify the subrepresentation $U(g)E^{(1)}E_\alpha$ with the direct sum of d_α copies of $U_{m_\alpha,s_\alpha}(g)$. Then there are mutually orthogonal projections $E^{(1)}_{\alpha,k}\subset E_\alpha$, $k=1,\dots d_\alpha$, in $\mathcal{H}$ onto irreducible subspaces such that

$$E^{(1)}\,E_\alpha=\sum_{k=1}^{d_\alpha}E^{(1)}_{\alpha,k}\,, \tag{2.7}$$

$$U(g)\,E^{(1)}_{\alpha,k}=U_{m_\alpha,s_\alpha}(g)\,E^{(1)}_{\alpha,k}\quad\text{for all } g\in\tilde{P}_+^\uparrow\,.$$

We define a "PT–operator" $U^{(1)}(j)$ on $E^{(1)}\mathcal{H}$ as the anti–linear extension of

$$U^{(1)}(j)\,E^{(1)}_{\alpha,k}\doteq U_{m,s}(j)\,E^{(1)}_{\alpha,k}.$$

Note that this definition of $U^{(1)}(j)$ depends on the choice of the decomposition (2.7).

We define now a closed antilinear operator $S_{\rm geo}$ in terms of the representation $U^{(1)}$, as anticipated, by equation (2.1). Note that the group relation $j\,\lambda_1(t)\,j=\lambda_1(t)$ implies that $S_{\rm geo}$ is, like $S_{\rm Tom}$, an involution: it leaves its domain invariant and satisfies $(S_{\rm geo})^2\subset 1$. The following proposition is a corollary of the article [12] of Buchholz and Epstein.

Proposition 3. *There is a unitary "charge conjugation" operator C on $E^{(1)}\mathcal{H}$ satisfying*

$$C E_\alpha\,E^{(1)}=E_{\bar\alpha}C\,E^{(1)}\quad\textit{and}\quad [C,U(g)]\,E^{(1)}=0\ \textit{ for all } g\in\tilde{P}_+^\uparrow, \tag{2.8}$$

such that

$$C\,S_{\rm geo}=S_{\rm Tom}\,. \tag{2.9}$$

Proof. Let $\alpha\in\Sigma^{(1)}$. Corresponding to the decomposition (2.7) of the particle multiplet α into particle types (α,k) there is, for each k in $\{1,\dots,d_\alpha\}$, a family of linear subspaces $\mathcal{C}\to\mathcal{F}_{\alpha,k}(\mathcal{C})\subset\mathcal{F}(\mathcal{C})$ satisfying

$$E^{(1)}\,\mathcal{F}_{\alpha,k}(\mathcal{C})\,\Omega=E^{(1)}_{\alpha,k}\,\mathcal{F}(\mathcal{C})\,\Omega\,, \tag{2.10}$$

[7] A proof, as well as explicit formulae for the relevant group relations jgj, are given in the Appendix for the convenience of the reader.

see *e.g.* [15, 18]. Note that the closures of the above vector spaces are independent of $\mathcal{C}$ by the Reeh–Schlieder property and span $E^{(1)}\mathcal{H}_\alpha$ if k runs through $\{1,\ldots,d_\alpha\}$. Similarly, the "anti–particle" Hilbert spaces

$$E^{(1)}\,\mathcal{F}_{\alpha,k}(\mathcal{C})^*\,\Omega^- \tag{2.11}$$

are independent of $\mathcal{C}$, mutually orthogonal for different k, and span $E^{(1)}\mathcal{H}_{\bar\alpha}$ if k runs through $\{1,\ldots,d_\alpha\}$ (note that $d_{\bar\alpha}=d_\alpha$). Buchholz and Epstein [12] have shown the particle – anti-particle symmetry in this situation: For each $\alpha\in\Sigma^{(1)}$ and $k=1,\ldots,d_\alpha$ there is a unitary map $C_{\alpha,k}$ from the closure $E^{(1)}_{\alpha,k}\,\mathcal{H}$ of the vector space (2.10) onto the space (2.11) intertwining the respective (irreducible) subrepresentations of $\tilde P_+^\uparrow$. We now recall in detail the relevant result of Buchholz and Epstein. Denote by $\mathcal{F}^\infty_{\alpha,k}(\mathcal{C})$ the set of field operators $B\in\mathcal{F}_{\alpha,k}(\mathcal{C})$ such that the map $g\mapsto U(g)\,B\,U(g)^{-1}$ is smooth in the norm topology. Buchholz and Epstein consider a special class of spacelike cones: Let $\underline{\mathcal{C}}\subset\mathbb{R}^3$ be an open, salient cone in the $x^0=0$ plane of Minkowski space, with apex at the origin. Then its causal completion $\mathcal{C}=\underline{\mathcal{C}}''$ is a spacelike cone. Its dual cone $\mathcal{C}^*$ is defined as the set

$$\mathcal{C}^*\doteq\left\{\,(p^0,\boldsymbol{p})\in\mathbb{R}^4\ :\boldsymbol{p}\cdot\boldsymbol{x}>0\ \text{ for all }\boldsymbol{x}\in\underline{\mathcal{C}}^-\setminus\{0\}\,\right\}.$$

Lemma 4 (Buchholz, Epstein)**.** *Let $B\in\mathcal{F}^\infty_{\alpha,k}(\mathcal{C})$, where $\mathcal{C}$ is a spacelike cone as above and $\alpha\in\Sigma^{(1)}$. Then $p\mapsto\big(E^{(1)}\,B\,\Omega\big)(p)$ is, considered as a function on the mass shell H_{m_α}, the smooth boundary value of an analytic function $k\mapsto\big(E^{(1)}\,B\,\Omega\big)(k)$ on the simply connected subset*

$$\Gamma_{\mathcal{C},\alpha}\doteq\{k\in\mathbb{C}^4\mid k^2=m_\alpha^2\ ,\ \mathrm{Im}k\in-\mathcal{C}^*\ \}$$

of the complex mass shell. Further, its boundary value on $-H_{m_\alpha}$ satisfies

$$\omega_{\alpha,k}\,V_{s_\alpha}(\frac{1}{m_\alpha}\,\underset{\sim}{p}\,\sigma_2)\,\overline{\big(E^{(1)}B\,\Omega\big)(-p)}=\big(C^*_{\alpha,k}E^{(1)}B^*\,\Omega\big)(p)\ , \tag{2.12}$$

where $\omega_{\alpha,k}$ is a complex number of unit modulus which is independent of B and $\underline{\mathcal{C}}$, and $C^*_{\alpha,k}$ is the mentioned intertwiner from $E^{(1)}\,\mathcal{F}_{\alpha,k}(\mathcal{C})^*\,\Omega^-$ onto $E^{(1)}_{\alpha,k}\,\mathcal{H}$.

Note that equation (2.12) coincides literally with equation (5.13) in [12]. We reformulate this result as follows. Denote by $\mathcal{K}_1$ the class of spacelike cones $\mathcal{C}$ contained in W_1 which are of the form $\underline{\mathcal{C}}''$ as in the lemma and contain the positive x^1-axis. Let further

$$D_0\doteq\operatorname*{span}_{\mathcal{C}\in\mathcal{K}_1,\alpha,k}\ E^{(1)}\,\mathcal{F}^\infty_{\alpha,k}(\mathcal{C})\,\Omega \tag{2.13}$$

where α runs through $\Sigma^{(1)}$ and $k=1,\ldots,d_\alpha$. The lemma asserts that on this domain an operator S_0 may be defined by

$$\big(S_0E^{(1)}_{\alpha,k}\psi\big)(p)\doteq V_{s_\alpha}(\frac{1}{m_\alpha}\,\underset{\sim}{p}\,\sigma_2)\,\overline{\big(E^{(1)}_{\alpha,k}\psi\big)(-p)}\,,\quad \psi\in D_0\,. \tag{2.14}$$

Further, the intertwiners $C_{\alpha,k}$, modified by the factors $\omega_{\alpha,k}$ appearing in (2.12), extend by linearity to a unitary "charge conjugation" operator C on $E^{(1)}\mathcal{H}$,

$$C\,E^{(1)}_{\alpha,k}\doteq\omega_{\alpha,k}C_{\alpha,k}\,E^{(1)}_{\alpha,k}\,,$$

which satisfies the equations (2.8) of the proposition. Now equation (2.12) may be rewritten as

$$C\,S_0\subset S_{\mathrm{Tom}}\,. \tag{2.15}$$

This inclusion implies in particular that S_0 is closable, its closure satisfying the same relation. But this closure is an extension of the operator $S_{\rm geo}$, as we show in the Appendix (Lemma 11). Hence we have

$$C\, S_{\rm geo} \subset S_{\rm Tom}\,, \tag{2.16}$$

and it remains to show the opposite inclusion. To this end, we refer to the opposite wedge $W_1' = R_2(\pi)\, W_1$. Let

$$\begin{aligned} S'_{\rm geo} &\doteq U^{(1)}(r_2(\pi))\; S_{\rm geo}\; U^{(1)}(r_2(\pi))^{-1}\,, \\ S'_{\rm Tom} &\doteq U^{(1)}(r_2(\pi))\, S_{\rm Tom}\, U^{(1)}(r_2(\pi))^{-1} \;= S_{\rm Tom}(W_1')\, E^{(1)}\,. \end{aligned}$$

We claim that the following sequence of relations holds true:

$$S_{\rm Tom} \subset \kappa^{-1}\, (S'_{\rm Tom})^* \subset \kappa^{-1}\, C\, (S'_{\rm geo})^* = C\, S_{\rm geo}\,, \tag{2.17}$$

where κ is the Bose-Fermi operator. Twisted locality and modular theory imply that

$$Z S_{\rm Tom}(W_1) Z^* \subset S_{\rm Tom}(W_1')^*.$$

Applying $E^{(1)}$, this yields $Z S_{\rm Tom} Z^* \subset (S'_{\rm Tom})^*$. But $Z S_{\rm Tom} Z^* = Z^2 S_{\rm Tom}$, because κ commutes with the modular operators. Using $Z^2 = \kappa$, this proves the first inclusion. Since C commutes with $U^{(1)}(\tilde P_+^\uparrow)$ and both $S_{\rm geo}$ and $S_{\rm Tom}$ are involutions, the inclusion (2.16) implies that

$$C U^{(1)}(j) = U^{(1)}(j) C^*\,. \tag{2.18}$$

Thus, the adjoint of relation (2.16) reads ${S_{\rm Tom}}^* \subset C {S_{\rm geo}}^*$, which implies the second of the above inclusions. Finally, the group relations (A.2) and $\lambda_1(t)\, r_2(\pi) = r_2(\pi)\, \lambda_1(-t)$ imply that $(S'_{\rm geo})^* = U^{(1)}(r_2(2\pi))\, S_{\rm geo}$. But the spin-statistics theorem [12] asserts that $U(r_2(2\pi)) = \kappa$. (Namely, both operators act on $\mathcal{H}_\alpha$ as multiplication by the statistics sign $\kappa_\alpha = e^{2\pi i s_\alpha}$.) Hence the last equation in (2.17) holds. This completes the proof of (2.17) and hence of the proposition. □

By uniqueness of the polar decomposition, equation (2.9) of the proposition implies the equations

$$\Delta_{W_1}^{\frac12}\; E^{(1)} = e^{-\pi K}\; E^{(1)}\,, \quad J_{W_1}\; E^{(1)} = C\, U^{(1)}(j)\; E^{(1)}\,.$$

Since the unitary C commutes with $U^{(1)}(\tilde P_+^\uparrow)$ and satisfies equation (2.18), we have shown the single particle version of the Bisognano-Wichmann theorem:

Theorem 5. *Let the assumptions 0), …, vi) of Section 1 hold. Then*
i) Modular Covariance holds on the single particle space:

$$\Delta_{W_1}^{it}\; E^{(1)} = U(\lambda_1(-2\pi t))\; E^{(1)}\,. \tag{2.19}$$

ii) $J_{W_1}\, E^{(1)}$ is a "CPT operator" on $E^{(1)}\mathcal{H}$:

$$J_{W_1}\, U(g) J_{W_1}\; E^{(1)} = U(jgj)\; E^{(1)} \quad \text{for all } g \in \tilde P_+^\uparrow\,. \tag{2.20}$$

3. Modular Covariance on the Space of Scattering States.

Having established modular covariance on the single particle space, we now show that it extends to the space of scattering states. The argument is an extension of Landau's analysis [28] on the structure of local internal symmetries to the present case of a symmetry which does not act strictly local in the sense of Landau. The method to be employed is Haag-Ruelle scattering theory [24, 25], whose adaption to the present situation of topological charges has been developed in [13].

This method associates a multi-particle state to n single particle vectors, which are created from the vacuum by quasilocal field operators carrying definite charge. Recall [15] that for every $\alpha \in \Sigma$, there is a family of linear subspaces $\mathcal{C} \to \mathcal{F}_\alpha(\mathcal{C}) \subset \mathcal{F}(\mathcal{C})$ of field operators carrying charge α :

$$\mathcal{F}_\alpha(\mathcal{C})\,\Omega = E_\alpha\,\mathcal{F}(\mathcal{C})\,\Omega\,.$$

Operators in $\mathcal{F}_\alpha(\mathcal{C})$ are bosons or fermions w.r.t. the normal commutation relations according as κ takes the value 1 or -1 on $\mathcal{H}_\alpha$. The mentioned quasilocal creation operators are constructed as follows. For $\alpha \in \Sigma^{(1)}$, let $B \in \mathcal{F}_\alpha(\mathcal{C})$ be such that the spectral support of $B\Omega$ has non–vanishing intersection with the mass hyperboloid H_{m_α}. Further, let $f \in \mathcal{S}(\mathbb{R}^4)$ be a Schwartz function whose Fourier transform $\tilde{f}$ has compact support contained in the open forward light cone V_+ and intersects the energy momentum spectrum of the sector α only in the mass shell H_{m_α}. Recall that the latter is assumed to be isolated from the rest of the energy momentum spectrum in the sector $\mathcal{H}_\alpha$. For $t \in \mathbb{R}$, let f_t be defined by

$$f_t(x) \doteq (2\pi)^{-2} \int \mathrm{d}^4p\, e^{i(p_0-\omega_\alpha(\boldsymbol{p})t}\, e^{-ip\cdot x}\, \tilde{f}(p)\,, \tag{3.1}$$

where $\omega_\alpha(\boldsymbol{p}) \doteq (\boldsymbol{p}^2 + m_\alpha^2)^{\frac{1}{2}}$. For large $|t|$, its support is essentially contained in the region $t\,V_\alpha(f)$, where $V_\alpha(f)$ is the velocity support of f,

$$V_\alpha(f) \doteq \{\, \big(1, \frac{\boldsymbol{p}}{\omega_\alpha(\boldsymbol{p})}\big)\,,\; p = (p^0, \boldsymbol{p}) \in \mathrm{supp}\tilde{f}\,\}\,. \tag{3.2}$$

More precisely [7,24], for any $\varepsilon > 0$ there is a Schwartz function f_t^ε with support in $t\,V_\alpha(f)^\varepsilon$, where V^ε denotes an ε–neighbourhood of V, such that $f_t - f_t^\varepsilon$ converges to zero in the Schwartz topology for $|t| \to \infty$. Let now

$$B(f_t) \doteq \int \mathrm{d}^4x\, f_t(x)\, U(x) B U(x)^{-1}\,.$$

For large $|t|$, this operator is essentially localized in $\mathcal{C} + t\,V_\alpha(f)$. Namely, for any $\varepsilon > 0$, it can be approximated by the operator

$$B(f_t^\varepsilon) \in \mathcal{F}\big(\mathcal{C} + t\,V_\alpha(f)^\varepsilon\big) \tag{3.3}$$

in the sense that $\|B(f_t^\varepsilon) - B(f_t)\|$ is of fast decrease in t. Further, it creates from the vacuum a single particle vector

$$B(f_t)\,\Omega = (2\pi)^2 \tilde{f}(P)\, B\,\Omega \quad \in\, E^{(1)}\mathcal{H}_\alpha\,,$$

which is independent of t, and whose velocity support is contained in that of f. Here we understand the velocity support $V(\psi)$ of a single particle vector to be defined as in equation (3.2), with the spectral support of ψ taking the role of supp$\tilde{f}$. To construct an outgoing scattering state from n single particle vectors, pick n localization regions $\mathcal{C}_i$, $i = 1, \ldots, n$ and compact sets V_i in velocity space, such that for suitable open neighbourhoods $V_i^\varepsilon \subset \mathbb{R}^4$ the regions $\mathcal{C}_i + t\,V_i^\varepsilon$ are mutually spacelike separated for large t. Next, choose $B_i \in \mathcal{F}_{\alpha_i}(\mathcal{C}_i)$, and Schwartz functions f_i as above with $V_{\alpha_i}(f_i) \subset V_i$. Then the standard lemma of scattering theory asserts the following: The limit

$$\lim_{t\to\infty} B_n(f_{n,t}) \cdots B_1(f_{1,t})\,\Omega \doteq \big(\psi_n \times \cdots \times \psi_1\big)^{\mathrm{out}} \tag{3.4}$$

exists and depends only on the single particle vectors $\psi_i \doteq B_i(f_{i,t})\,\Omega$, justifying the above notation. The convergence in (3.4) is of fast decrease in t, and the limit vector depends continuously on the single particle states, as a consequence of the cluster theorem. Further, the normal commutation relations survive in this limit.

Let us write $\mathcal{H}^{(1)} \doteq E^{(1)}\mathcal{H}$, and denote by $\mathcal{H}^{(n)}$, $n \geq 2$, the closed span of outgoing n-particle scattering states and by $\mathcal{H}^{(\mathrm{ex})}$ the span of these spaces:

$$\mathcal{H}^{(\mathrm{ex})} = \mathbb{C}\,\Omega \oplus \bigoplus_{n\in\mathbb{N}} \mathcal{H}^{(n)} . \tag{3.5}$$

Asymptotic completeness means that $\mathcal{H}^{(\mathrm{ex})}$ coincides with $\mathcal{H}$. Our proof that modular covariance extends from $\mathcal{H}^{(1)}$ to $\mathcal{H}^{(\mathrm{ex})}$ relies on the following observation.

Lemma 6. *In each $\mathcal{H}^{(n)}$, $n \geq 2$, there is a total set of scattering states as in equation (3.4), with the localization regions chosen such that $\mathcal{C}_1, \dots, \mathcal{C}_{n-1} \subset W_1'$ and $\mathcal{C}_n = W_1$.*

In particular, for these scattering states the regions $\mathcal{C}_i + tV(\psi_i)^\varepsilon$, $i = 1, \dots, n-1$, are spacelike separated from $W_1 + tV(\psi_n)^\varepsilon$ for large t.

Proof. Consider the set M^n of velocity tupels $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n) \in \mathbb{R}^{3n}$ satisfying the requirements that a) one of the velocities, say $\boldsymbol{v}_{i_0}$, has the strictly largest 1-component:

$$(\boldsymbol{v}_{i_0})_1 > (\boldsymbol{v}_i)_1 \quad \text{ for } i \neq i_0 ,$$

and b) the relative velocities w.r.t. $\boldsymbol{v}_{i_0}$ have different directions:

$$\mathbb{R}^+ (\boldsymbol{v}_i - \boldsymbol{v}_{i_0}) \neq \mathbb{R}^+ (\boldsymbol{v}_j - \boldsymbol{v}_{i_0}) \quad \text{ for } i \neq j .$$

Given such $(\boldsymbol{v}_1, \dots, \boldsymbol{v}_n)$, let $\mathcal{C}_{i_0} = W_1$. For $i \neq i_0$, let $\underline{\mathcal{C}_i}$ be a cone in the $t = 0$ plane of $\mathbb{R}^4$ containing the ray $\mathbb{R}^+ (\boldsymbol{v}_i - \boldsymbol{v}_{i_0})$ and with apex at the origin, and let then $\mathcal{C}_i$ be its causal closure. Then, having chosen sufficiently small opening angles, the regions $\mathcal{C}_i + t\{(1, \boldsymbol{v}_i)\}$, $i = 1, \dots, n$, are mutually spacelike separated for all $t > 0$, and further $\mathcal{C}_i \subset W_1'$ for $i \neq i_0$. Now M^n exhausts the set of all velocity tupels in $\mathbb{R}^{3n}$ except for a set of measure zero. Hence, a scattering state $(\psi_n \times \cdots \times \psi_1)^{\mathrm{out}}$ as in (3.4) can be approximated by a sum of scattering states $(\psi_n^\nu \times \cdots \times \psi_1^\nu)^{\mathrm{out}}$, whose localization regions satisfy that $\mathcal{C}_{i_0}^\nu = W_1$ for some i_0, and $\mathcal{C}_i^\nu \subset W_1'$ for $i \neq i_0$. This is accomplished by a standard argument [1] taking into account the continuous dependence of $(\psi_n \times \cdots \times \psi_1)^{\mathrm{out}}$ on the ψ_i and the Reeh-Schlieder theorem. But due to the normal commutation relations obeyed by the scattering states, $(\psi_n^\nu \times \cdots \times \psi_1^\nu)^{\mathrm{out}}$ coincides with $\pm\, (\psi_{i_0}^\nu \times \cdots \psi_n^\nu \cdots \times \psi_1^\nu)^{\mathrm{out}}$ and hence is of the form required in the lemma. $\square$

Proposition 7. *If the unitary groups $\Delta_{W_1}^{it}$ and $U(\lambda_1(-2\pi t))$ coincide on $\mathcal{H}^{(1)}$, they also coincide on the space $\mathcal{H}^{(\mathrm{ex})}$ of scattering states.*

Proof. Let $U_t \doteq \Delta_{W_1}^{it}\, U(\lambda_1(2\pi t))$. Considering this operator as an internal symmetry, it should act multiplicatively on the scattering states as shown by Landau in [28]. The complication is that U_t does not act strictly local, but only leaves $\mathcal{F}(W_1)$ invariant. We generalize Landau's argument to this case utilizing the last lemma. By induction over the particle number n we show that U_t is the unit operator on each $\mathcal{H}^{(n)}$. Let $(\psi_n \times \cdots \times \psi_1)^{\mathrm{out}}$ be a scattering state with $\psi_i = B_i(f_{i,t})$, where the localization regions $\mathcal{C}_i$ are as in the above lemma. Since $\|B_{n-1}(f_{n-1,t}) \cdots B_1(f_t)\Omega - (\psi_{n-1} \times \cdots \times \psi_1)^{\mathrm{out}}\|$ is of fast decrease in t, while $\|B_n(f_{n,t})\|$ increases at most like $|t|^4$, one concludes as Hepp in [24]:

$$(\psi_n \times \cdots \times \psi_1)^{\mathrm{out}} = \lim_{s\to\infty} B_n(f_{n,s})\, (\psi_{n-1} \times \cdots \times \psi_1)^{\mathrm{out}} . \tag{3.6}$$

Hence

$$U_t\, (\psi_n \times \cdots \times \psi_1)^{\mathrm{out}} = \lim_{s\to\infty} U_t B_n(f_{n,s}) U_t^{-1}\, (\psi_{n-1} \times \cdots \times \psi_1)^{\mathrm{out}} , \tag{3.7}$$

where we have put in the induction hypothesis that U_t acts trivially on $\mathcal{H}^{(n-1)}$. Due to Borchers' result, U_t commutes with the translations, which implies that $U_t B_n(f_{n,s}) U_t^{-1}$ coincides with $(U_t B_n U_t^{-1})(f_{n,s})$. But modular theory and covariance guarantee that $U_t B_n U_t^{-1}$

is, like B_n, in $\mathcal{F}(W_1)$. In addition $(U_t B_n U_t^{-1})(f_{n,s})\,\Omega = U_t\psi_n$, hence by the standard lemma of scattering theory, equation (3.7) may be rewritten as

$$U_t\,(\psi_n \times \cdots \times \psi_1)^{\text{out}} = ((U_t\psi_n) \times \psi_{n-1} \times \cdots \times \psi_1)^{\text{out}} .$$

By assumption of the proposition, U_t acts trivially on ψ_n, and hence on the scattering state. By linearity and continuity, the same holds on $\mathcal{H}^{(n)}$, completing the induction. □

The hypothesis of this proposition has been shown in Theorem 5 to hold under our assumptions 0), . . . , vi). Hence, we have now derived modular covariance from these assumptions and asymptotic completeness. As mentioned, Guido and Longo have shown that modular covariance generally implies covariance of the modular conjugations, and hence the CPT theorem [22, Prop. 2.8, 2.9]. Thus, the proof of Theorem 2 is now completed.

4. The CPT Theorem.

We show here that the CPT theorem can also be derived directly from our assumptions in Section 1, via the single particle result and scattering theory. This should in particular turn out useful for a derivation of the CPT theorem in a theory of massive particles with non–Abelian braid group statistics (plektons) in $d = 2+1$, where the methods of [22] cannot be applied in an obvious way since one has no field algebra.

Recall that incoming scattering states can be constructed as in equation (3.4), where now $t \to -\infty$ and the condition for the limit to exist is that the regions $\mathcal{C}_i - |t|\,V_i^\varepsilon$ be mutually spacelike separated for large $|t|$. The following result holds under the assumptions of Section 1, but without restrictions on the degeneracies of the mass eigenvalues in each sector. Like Proposition 7, it is an extension of Landau's argument [28].

Lemma 8. *J_{W_1} maps outgoing scattering states to incoming ones and vice versa according to*

$$J_{W_1}\left(\psi_n \times \cdots \times \psi_1\right)^{\text{out}} = \left(J_{W_1}\psi_n \times \cdots \times J_{W_1}\psi_1\right)^{\text{in}} . \tag{4.1}$$

Let us put the statement of the lemma into a more concise form. Recall that the spaces of incoming and outgoing scattering states are isomorphic to an appropriately symmetrized Fock space over $\mathcal{H}^{(1)}$ via the operators $W_{\text{in,out}}$ which map $\psi_n \otimes \cdots \otimes \psi_1$ to $(\psi_n \times \cdots \times \psi_1)^{\text{in,out}}$, respectively. Lemma 8 then asserts that

$$J_{W_1}\,W_{\text{out}} = W_{\text{in}}\,\Gamma(J_{W_1}E^{(1)})\,, \tag{4.2}$$

where $\Gamma(U)$ denotes the second quantization of a unitary operator U on $\mathcal{H}^{(1)}$. Note that the same equation holds with W_{out} and W_{in} interchanged.

Proof. We proceed by induction along the same lines as in the last proposition. Let $\psi_i = B_i(f_{i,t})\Omega$ be the single particle states appearing in equation (4.1), with velocity supports contained in compact sets V_i, and with localization regions $\mathcal{C}_i$, such that $\mathcal{C}_i + tV_i^\varepsilon$ are mutually spacelike separated for suitable $\varepsilon > 0$. According to Lemma 6, we may assume that the localization regions satisfy $\mathcal{C}_1, \ldots, \mathcal{C}_{n-1} \subset W_1'$ and $\mathcal{C}_n = W_1$. By the same arguments as in the last proof, we have

$$J_{W_1}\,(\psi_n \times \cdots \times \psi_1)^{\text{out}} = \lim_{t\to\infty} J_{W_1}\,B_n(f_{n,t})\,J_{W_1}\,(J_{W_1}\psi_{n-1} \times \cdots \times J_{W_1}\psi_1)^{\text{in}}\,, \tag{4.3}$$

where we have put in the induction hypothesis that J_{W_1} acts as in equation (4.1) on $\mathcal{H}^{(n-1)}$. Now by Borchers' result [4] we know that the commutation relations $J_{W_1}\,U(x)J_{W_1} = U(jx)$ hold. From these we conclude that the spectral supports of $\psi \in \mathcal{H}$ and $J_{W_1}\psi$ are related by the transformation $-j$, and hence their velocity supports are related by

$$V(J_{W_1}\psi) = -r\,V(\psi)\,, \tag{4.4}$$

where r denotes the inversion of the sign of the x^1-coordinate. By virtue of the Reeh-Schlieder theorem and the continuity of the scattering states, we may assume that for $i = 1, \ldots, n-1$ there are $\hat{B}_i \in \mathcal{F}(r\,\mathcal{C}_i)$ and $\hat{f}_i$ such that $\hat{B}_i(\hat{f}_{i,-t})\Omega = J_{W_1}\psi_i$. Further, $\hat{f}_i$ can be chosen such that $V(\hat{f}_i) \subset V(J_{W_1}\psi_i)^\varepsilon$, which in turn is contained in $-r\,V_i^\varepsilon$ due to equation (4.4). Then $\hat{B}_i(\hat{f}_{i,-t})$ can be approximated by an operator $A_i^\varepsilon(t)$ localized in the region $r\,\{\mathcal{C}_i + tV_i^\varepsilon\}$. These regions are mutually spacelike separated for large positive t, and hence the incoming $n-1$ particle state in equation (4.3) may be written as $\lim_{t\to-\infty} \hat{B}_{n-1}(\hat{f}_{n-1,t}) \cdots \hat{B}_1(\hat{f}_{1,t})\,\Omega$. Similarly, Borchers' commutation relations imply that

$$J_{W_1}\, B_n(f_{n,t})\, J_{W_1} = \big(J_{W_1} B_n J_{W_1}\big)\,(\hat{f}_{n,-t})\ , \qquad \text{where} \quad \hat{f}_n(x) = \overline{f_n(jx)}\,.$$

Now $J_{W_1}\, B_n\, J_{W_1}$ is in $\mathcal{F}(W_1)'$, and $V(\hat{f}_n) = -r\,V(f_n)$, and therefore the discussion around equation (3.3) implies that the above operator may be approximated by an operator $A_n^\varepsilon(t) \in \mathcal{F}\big(W_1 + t\cdot rV_n^\varepsilon\big)'$. Recall that the operators $A_i^\varepsilon(t)$, $i = 1, \ldots, n-1$, are localized in the regions $r\,\{\mathcal{C}_i + tV_i^\varepsilon\}$. For large positive t, these regions are spacelike to $r\,\{W_1 + tV_n^\varepsilon\}$ and are hence contained in $W_1 + t\cdot rV_n^\varepsilon$. Hence the $A_i^\varepsilon(t)$ (anti-) commute with $A_n^\varepsilon(t)$ for large t. Thus the standard arguments of scattering theory [17, 24] apply, yielding that the vector (4.3) may be written as

$$\lim_{t\to-\infty} \big(J_{W_1} B_n J_{W_1}\big)(\hat{f}_{n,t})\ \hat{B}_{n-1}(\hat{f}_{n-1,t}) \cdots \hat{B}_1(\hat{f}_{1,t})\,\Omega\ ,$$

and only depends on the single particle vectors. But these are $\big(J_{W_1} B_n J_{W_1}\big)\,(\hat{f}_{n,t})\,\Omega = J_{W_1}\psi_n$, and $\hat{B}_i(\hat{f}_{i,t})\,\Omega = J_{W_1}\psi$ for $i = 1, \ldots, n-1$. Hence the limit coincides with the right hand side of equation (4.1), completing the induction. □

Proposition 9 (CPT). *Let Θ be the the anti–unitary involution $\Theta \doteq Z^*\, J_{W_1}$.*
i)If the representation property

$$\Theta\, U(g)\, \Theta = U(jgj) \qquad \text{for all } g \in \tilde{P}_+^\uparrow \tag{4.5}$$

holds on $\mathcal{H}^{(1)}$, then it is also satisfied on the space $\mathcal{H}^{(\mathrm{ex})}$ of scattering states.
ii) In this case, and if in addition asymptotic completeness holds, Θ acts geometrically correctly on the family of wedge algebras $\mathcal{F}(W)_W$ in the sense of equation (1.2).

Note that equation (4.5) is equivalent to $J_{W_1}U(g)J_{W_1} = U(jgj)$, since Z commutes with $U(g)$ and satisfies $Z^*J_{W_1} = J_{W_1}Z$. In Proposition 5, we have shown that J_{W_1} satisfies this representation property on $\mathcal{H}^{(1)}$ if the assumptions 0),…,vi) of Section 1 hold. Hence Proposition 9 is a CPT theorem, holding under these assumptions and asymptotic completeness.

Proof. i) Let J_{W_1} have the above representation property on $\mathcal{H}^{(1)}$. As is well known [17], the restriction of $U(\tilde{P}_+^\uparrow)$ to the space of scattering states is equivalent to the second quantization of its restriction to $\mathcal{H}^{(1)}$: $U(g)\, W_{\mathrm{out,in}} = W_{\mathrm{out,in}}\ \Gamma(U(g)E^{(1)})$. By virtue of Lemma 8, see equation (4.2), the assumption thus implies

$$J_{W_1}U(g)J_{W_1}\ W_{\mathrm{out}} = W_{\mathrm{out}}\ \Gamma\big(J_{W_1}U(g)J_{W_1}E^{(1)}\big) = U(jgj)\ W_{\mathrm{out}}\ ,$$

which proves the claim. ii) By twisted locality and modular theory, one has

$$\mathcal{F}(W_1') \subset Z^*\,\mathcal{F}(W_1)'\,Z = \Theta\,\mathcal{F}(W_1)\,\Theta\,. \tag{4.6}$$

Now recall that $U(r_2(\pi))\,\mathcal{F}(W_1)\,U(r_2(\pi))^{-1} = \mathcal{F}(W_1')$ and that $jr_2(\pi)j = r_2(-\pi)$, see equation (A.2). One therefore obtains, by applying $\mathrm{Ad}\big(U(r_2(\pi))\Theta\big)$ to the inclusion (4.6) and using equation (4.5), the opposite inclusion. Hence equality holds in (4.6). Since every wedge region arises from W_1 by a Poincaré transformation, the claimed equation (1.2) follows by covariance of the field algebras and the representation property (4.5) of Θ. □

Appendix A. Single-Particle PT Operator and Geometric Involution.

We provide an explicit formula for the group relations jgj and a proof of the representation property of the "PT operator" $U_{m,s}(j)$ defined in equation (2.6). As before, we denote by $g \mapsto jgj$ the unique lift [31] of the adjoint action of j on the Poincaré group to an automorphism of the covering group. An explicit formula for jgj follows from the observation that j coincides with the proper Lorentz transformation $-R_1(\pi)$: Hence, for all $A \in SL(2,\mathbb{C})$

$$j\,\Lambda(A)\,j = R_1(\pi)\,\Lambda(A)\,R_1(\pi)^{-1} = \Lambda(\sigma_1\,A\,\sigma_1)\,.$$

This shows that the lift jgj is given by

$$j\,(x,A)\,j = (j\,x,\ \sigma_1\,A\,\sigma_1) \quad \text{for all } (x,A) \in \tilde{P}_+^\uparrow\,. \tag{A.1}$$

Using equation (2.2), one has in particular the relations

$$j\,r_2(\omega)\,j = r_2(-\omega)\,, \quad j\,\lambda_1(t)\,j = \lambda_1(t)\,. \tag{A.2}$$

Lemma 10. *The operator $U_{m,s}(j)$ defined in equation* (2.6) *is anti–unitary and satisfies the representation properties* (2.5)*.*

Proof. We prove the second of the equations (2.5) for $g = (0,A)$ with $A \in SL(2,\mathbb{C})$. The other assertions are shown along the same lines. Recall that the covering homomorphism $\Lambda : SL(2,\mathbb{C}) \to L_+^\uparrow$ is characterized by $\underset{\sim}{\Lambda(A)\,p} = A\,\underset{\sim}{p}\,A^*$. We have

$$\begin{aligned}&\big(U_{m,s}(j)\,U_{m,s}(g)\,U_{m,s}(j)\,\psi\big)(p)\\ &\qquad = V_s\big(m^{-2}\,\underset{\sim}{p}\,\sigma_3\,\bar{A}\,\overline{(\underset{\sim}{\Lambda(A^{-1})(-j)p})}\,\sigma_3\big)\,\psi(j\Lambda(A^{-1})jp)\,.\end{aligned} \tag{A.3}$$

Using the identity

$$\underset{\sim}{\Lambda(A^{-1})(-j)p} = \underset{\sim}{\Lambda(A^{-1}i\sigma_1)p} = A^{-1}\sigma_1\,\underset{\sim}{p}\,\sigma_1(A^*)^{-1}\,,$$

which follows from $-j = R_1(\pi)$, and the well-known relation

$$\sigma_2\bar{A}\sigma_2 = (A^*)^{-1} \quad \text{for } A \in SL(2,\mathbb{C})\,,$$

one verifies that the argument of V_s in equation (A.3) equals $\sigma_1\,A\,\sigma_1$. By equation (A.1), this proves the claim. ☐

We now relate the geometric involution $S_{\rm geo} = U^{(1)}(j)e^{-\pi K}E^{(1)}$ with the closable operator S_0 defined in equation (2.14).

Lemma 11. *The closure of S_0 is an extension of $S_{\rm geo}$.*

Proof. Recall that for $f \in \mathcal{S}(\mathbb{R})$ the bounded operator $f(K)$, where K denotes again the generator of the boosts $\lambda_1(\cdot)$, may be written as

$$f(K) = \int \mathrm{d}t\,\tilde{f}(t)\,U(\lambda_1(t))\,.$$

Here $\sqrt{2\pi}\tilde{f}$ is the Fourier transform of f, and the integral is understood in the weak sense. Let now c be a smooth function with compact support, and let $\psi = E^{(1)}B\Omega$, where $B \in \mathcal{F}^\infty_{\alpha,k}(\mathcal{C})$

for some $\mathcal{C} \in \mathcal{K}_1$. Applying the above formula to $c_\pi(K) \doteq e^{-\pi K} c(K)$ one finds, using that $\tilde{c}$ is analytic and $\widetilde{c_\pi}(t) = \tilde{c}(t - i\pi)$,

$$\big(S_{\rm geo}\, c(K)\, \psi\,\big)(p) = \int \mathrm{d}t\, \overline{\tilde{c}(t - i\pi)} \big(U^{(1)}(j)\, U\big(\lambda_1(t)\big)\, \psi\,\big)(p) \tag{A.4}$$

$$= \int \mathrm{d}t\, \overline{\tilde{c}(t - i\pi)}\, V_{s_\alpha}\big(\frac{1}{m_\alpha} \underset{\sim}{p}\, \sigma_3\, \overline{e^{\frac{t}{2}\sigma_1}}\big)\, \overline{\psi\big(\Lambda_1(-t)(-jp)\big)}\,. \tag{A.5}$$

The one-parameter group $\Lambda_1(\cdot)$ extends to an entire analytic function satisfying

$$\Lambda_1(-t - it') = \Lambda_1(-t)\big(j_{t'} - i \sin t'\, \sigma\big)\ ,$$

where $j_{t'}$ acts as multiplication by $\cos t'$ on the coordinates x^0 and x^1 and leaves the other coordinates unchanged, and σ acts as σ_1 on (x^0, x^1) and as the zero projection on (x^2, x^3) [23]. Note that in particular

$$\Lambda_1(-t - i\pi) = \Lambda_1(-t)\, j\,.$$

Further, one easily verifies that for any $q \in H_{m_\alpha}$, the vector $\sigma\, q$ is in the dual cone $\mathcal{C}^*$. Hence for all $t' \in (0, \pi)$ and all $p \in H_{m_\alpha}$, the complex vector $\Lambda_1(-t - it')(-jp)$ is in $\Gamma_{\mathcal{C},\alpha}$, the domain of analyticity of ψ, and approaches $\Lambda_1(-t)(-p)$ as $t' \to \pi$. It follows that the integrand in the expression (A.5) is anti–holomorphic in t in the strip $0 < \mathrm{Im} t < \pi$, and that (A.5) coincides with

$$\int \mathrm{d}t\, \overline{\tilde{c}(t)}\, V_{s_\alpha}\big(\frac{1}{m_\alpha} \underset{\sim}{p}\, \sigma_3 \frac{1}{i} \sigma_1\, \overline{e^{\frac{t}{2}\sigma_1}}\big)\, \overline{\psi(\Lambda_1(-t)(-p))}$$

$$= \int \mathrm{d}t\, \overline{\tilde{c}(t)}\, \big(S_0 U\big(\lambda_1(t)\big)\psi\,\big)(p)\,. \tag{A.6}$$

Here we have used that for all t, $U(\lambda_1(t))\,\psi$ is again in the domain D_0 of S_0 due to the covariance of the field algebra. This is so because for all t, there is some $\mathcal{C}_t \in \mathcal{K}_1$ such that $\Lambda_1(t)\, \mathcal{C} \subset \mathcal{C}_t$. Let now ϕ be in the (dense) domain of S_0^*, and let $\psi \in D_0$. We have shown from (A.4) to (A.6), that

$$\big(\,\phi\,,\, S_{\rm geo}\, c(K)\psi\,\big) = \int \mathrm{d}t\, \overline{\tilde{c}(t)} \big(\,\phi\,,\, S_0\, U\big(\lambda_1(t)\big)\, \psi\,\big) = \big(\,c(K)\psi\,,\, S_0^*\phi\,\big)\,.$$

Let D denote the set of finite linear combinations of vectors of the form $c(K)\,\psi$, where $c \in C_0^\infty(\mathbb{R})$ and $\psi \in D_0$. Then the above equation shows that D is in the domain of S_0^{**}, and that $S_0^{**} = S_{\rm geo}$ on D. But D is a core for $S_{\rm geo}$, hence S_0^{**} is an extension of $S_{\rm geo}$. □

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