Causal Poisson Brackets of the $SL(2,\mathbb{R})$ WZNW Model and its Coset Theories

C. Ford[a] [1], G. Jorjadze[b] [2] and G. Weigt[a] [3]

[a] Deutsches Elektronen-Synchrotron DESY Zeuthen,

Platanenallee 6, D-15738 Zeuthen, Germany

[b] Razmadze Mathematical Institute,

M.Aleksidze 1, 380093, Tbilisi, Georgia

Abstract

From the basic chiral and anti-chiral Poisson bracket algebra of the $SL(2,\mathbb{R})$ WZNW model, non-equal time Poisson brackets are derived. Through Hamiltonian reduction we deduce the corresponding brackets for its coset theories.

The analysis of WZNW models is usually reduced to a separate treatment of its chiral or anti-chiral components [1, 2, 3]. But much simpler results can arise when the contributions from the different chiralities are pieced together. This was observed recently for non-equal time Poisson brackets of a gauged WZNW theory, namely the Liouville theory [4]. Since WZNW models and their coset theories turn up in many areas of mathematics and physics, it is worthwhile to add the corresponding results for other WZNW theories.

Here we consider the $SL(2,\mathbb{R})$ WZNW model together with its three cosets, the Liouville theory and both the euclidean and Minkowskian black hole models. The general solution of the WZNW equations of motion gives $g(\tau,\sigma)$ as a product of chiral and anti-chiral fields $g(z)$ and $\bar{g}(\bar{z})$, where $z=\tau+\sigma$, $\bar{z}=\tau-\sigma$ are light cone coordinates. For periodic boundary conditions, the chiral and anti-chiral fields have the monodromies $g(z+2\pi)=g(z)M$ and $\bar{g}(\bar{z}-2\pi)=M^{-1}\bar{g}(\bar{z})$ with $M\in SL(2,\mathbb{R})$.

We use the following basis of the $sl(2,\mathbb{R})$ algebra

$$t_0=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \quad t_1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}, \quad t_2=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}. \tag{1}$$

It satisfies the relations $t_m\, t_n = -\eta_{mn}\, I + \epsilon_{mn}{}^{l}\, t_l$, where I is the unit matrix, $\eta_{mn} = \mathrm{diag}(+,-,-)$ the metric tensor of $3d$ Minkowski space, and $\epsilon_{012}=1$. For the matrices t_n

[1] email: `ford@ifh.de`

[2] email: `jorj@rmi.acnet.ge`

[3] email: `weigt@ifh.de`

one has $\langle t_m\, t_n \rangle = \eta_{mn}$, and for any $a \in sl(2,\mathbb{R})$ $\quad t^n \langle t_n\ a \rangle = a$, where the normalised trace is defined by $\langle \cdot \rangle = -\frac{1}{2}\mathrm{tr}(\cdot)$. We shall also use the nilpotent elements $t_\pm = t_1 \pm t_0$, and the identity

$$\eta_{mn}(t^m)_{ab}(t^n)_{cd} = \delta_{ab}\,\delta_{cd} - 2\delta_{ad}\,\delta_{cb}, \tag{2}$$

which is related to the Casimir.

The monodromy matrix M can be transformed into an abelian subgroup [1] of $SL(2,\mathbb{R})$, and we choose

$$M = \begin{pmatrix} e^{\lambda} & 0 \\ 0 & e^{-\lambda} \end{pmatrix}, \quad \text{with} \quad \lambda \neq 0. \tag{3}$$

For this case the chiral Poisson brackets are [4]

$$\begin{aligned} \{\, g_{ab}(z),\, g_{cd}(y)\,\} = \frac{\gamma^2}{4}\,[\,(\,g(z)\,t_2\,)_{ab}\ (\,g(y)\,t_2\,)_{cd}\ \epsilon(z-y) \\ +\,(\,g(z)\,t_-\,)_{ab}\ (\,g(y)\,t_+\,)_{cd}\,\theta_{-2\lambda}(z-y) \\ +\ (\,g(z)\,t_+\,)_{ab}\ (\,g(y)\,t_-\,)_{cd}\,\theta_{2\lambda}(z-y)]. \end{aligned} \tag{4}$$

Here $\epsilon(z)$ is the stair-step function $\epsilon(z) = 2n+1$ for $2\pi n < z < 2\pi(n+1)$, γ the coupling constant, and

$$\theta_{2\lambda}(z-y) = \frac{e^{\lambda\epsilon(z-y)}}{2\sinh\lambda} \tag{5}$$

is the Green's function for the operator ∂_z acting on functions $A(z)$ with the monodromy $A(z+2\pi) = e^{2\lambda}A(z)$, for $\lambda \neq 0$. In the 'fundamental' interval $z - y \in (-2\pi, 2\pi)$ where $\epsilon(z-y) = sign\,(z-y)$ the Green's function (5) becomes

$$\theta_{2\lambda}(z-y) = \frac{\cosh\lambda}{2\sinh\lambda} + \frac{1}{2}\,\epsilon(z-y), \tag{6}$$

and the chiral Poisson brackets (4) reduce to

$$\begin{aligned} \{\, g_{ab}(z),\, g_{cd}(y)\,\} = -\frac{\gamma^2}{4}\,\epsilon(z-y)\,(\,g(z)\,t^n\,)_{ab}\ (\,g(y)\,t_n\,)_{cd} \\ +\,\frac{\gamma^2}{8}\coth\lambda\,[(\,g(z)\,t_+\,)_{ab}\ (\,g(y)\,t_-\,)_{cd} - (\,g(z)\,t_-\,)_{ab}\ (\,g(y)\,t_+\,)_{cd}]. \end{aligned} \tag{7}$$

The equivalent anti-chiral Poisson brackets are only slightly different

$$\begin{aligned} \{\, \bar{g}_{ab}(\bar{z}),\, \bar{g}_{cd}(\bar{y})\,\} = \ -\frac{\gamma^2}{4}\,\epsilon(\bar{z}-\bar{y})\,(\,t^n\,\bar{g}(\bar{z})\,)_{ab}\ (\,t_n\,\bar{g}(\bar{y})\,)_{cd} \\ -\,\frac{\gamma^2}{8}\coth\lambda\,[(\,t_+\,\bar{g}(\bar{z})\,)_{ab}\ (\,t_-\,\bar{g}(\bar{y})\,)_{cd} - (\,t_-\,\bar{g}(\bar{z})\,)_{ab}\ (\,t_+\,\bar{g}(\bar{y})\,)_{cd}]. \end{aligned} \tag{8}$$

On the line the Poisson brackets (7) and (8) are even simpler; they do not have the $\coth\lambda$ term. For both cases, these Poisson brackets, together with the relation (2), provide the surprisingly simple result

$$\{\, g_{ab}(z,\bar{z}),\, g_{cd}(y,\bar{y})\,\} = \frac{\gamma^2}{4}\,\Theta\,[2g_{ad}(z,\bar{y})\,g_{cb}(y,\bar{z}) - g_{ab}(z,\bar{z})\,g_{cd}(y,\bar{y})], \tag{9}$$

where

$$\Theta = \frac{1}{2}[\epsilon(z-y)+\epsilon(\bar{z}-\bar{y})]. \tag{10}$$

The step character of $\epsilon(z)$ ensures that the bracket is causal, and in particular its equal time form vanishes. Notice that the coefficients in the bracket on the right-hand side of (9) are directly related to those in (2). This indicates what one can expect for a general WZNW theory.

The relation (9) will now be used to calculate causal Poisson brackets for the gauged $SL(2,\mathbb{R})$ theories. The Liouville theory can be obtained by nilpotent gauging, imposing constraints on the chiral and anti-chiral fields separately. For the chiral part the (first class) constraint is

$$J_+(z)+\rho=0, \tag{11}$$

where $\rho>0$ is a fixed parameter, and $J_+=\langle\, t_+\, J\,\rangle$ is the t_+-component of the left Kac-Moody current J. In this case only one component of the WZNW field, $g_{12}(z,\bar{z})$, is gauge invariant, and it is identified with the Liouville exponential [5, 4]

$$g_{12}(z,\bar{z}) = e^{-\gamma\varphi(z,\bar{z})}. \tag{12}$$

Its Poisson bracket algebra can be read off from (9). The generalisation of this result for periodic boundary conditions out of the fundamental domain is given in [4].

The two black hole models [7, 6] are generated by axial and vector gaugings with respect to the time-like element t_0 of the $sl(2,\mathbb{R})$ algebra. For the axial gauge transformations

$$g(z,\bar{z}) \mapsto e^{\epsilon(z,\bar{z})t_0}\, g(z,\bar{z})\, e^{\epsilon(z,\bar{z})t_0}, \tag{13}$$

the gauge invariant components are $v_1(z,\bar{z})=\langle\, t_1\; g(z,\bar{z})\,\rangle$ and $v_2(z,\bar{z})=\langle\, t_2\; g(z,\bar{z})\,\rangle$, and for the vector gauge transformations

$$g(z,\bar{z}) \mapsto e^{\epsilon(z,\bar{z})t_0}\, g(z,\bar{z})\, e^{-\epsilon(z,\bar{z})t_0}, \tag{14}$$

they are given by $v_0(z,\bar z) = \langle\, t_0\, g(z,\bar z)\,\rangle$ and $c(z,\bar z) = \langle\; g(z,\bar z)\,\rangle$.

There is a natural complex structure related to the isomorphism between $SL(2,\mathbb{R})$ and $SU(1,1)$. It is given by the complex coordinates $u = v_1 + iv_2$ and $x = c + iv_0$ which are related by $|x|^2 - |u|^2 = 1$. The $SL(2,\mathbb{R})$ valued field can be written

$$g(z,\bar z) = -\frac{1}{2}\left[x(z,\bar z)\, s + x^*(z,\bar z)\, s^* + u^*(z,\bar z)\, t + u(z,\bar z)\, t^*\right], \tag{15}$$

where t and s are the complex matrices $t = t_1 + it_2$, $s = I + it_0$, which satisfy

$$\begin{array}{llll} s\, s = 2s, & s\, s^* = 0, & s\, t = 0, & s\, t^* = 2t^* \\ t\, s = 2t, & t\, s^* = 0, & t\, t = 0, & t\, t^* = 2s^*. \end{array} \tag{16}$$

These relations provide

$$u(z,\bar z) = \langle\, t\, g(z,\bar z)\,\rangle, \qquad x(z,\bar z) = \langle\, s\, g(z,\bar z)\,\rangle, \tag{17}$$

which are the physical fields for the euclidean and Minkowskian black holes, respectively. Making use of (15)-(17), from (9) we read off the Poisson brackets of the fields $u(z,\bar z)$ and $x(z,\bar z)$

$$\begin{aligned}
\{\, u(z,\bar z),\, u(y,\bar y)\,\} &= \frac{\gamma^2}{2}\,\Theta\,[2u(z,\bar y)\, u(y,\bar z) - u(z,\bar z)\, u(y,\bar y)], \\
\{\, u(z,\bar z),\, u^*(y,\bar y)\,\} &= \frac{\gamma^2}{2}\,\Theta\,[2x(z,\bar y)\, x^*(y,\bar z) - u(z,\bar z)\, u^*(y,\bar y)], \\
\{\, x(z,\bar z),\, x(y,\bar y)\,\} &= \frac{\gamma^2}{2}\,\Theta\,[2x(z,\bar y)\, x(y,\bar z) - x(z,\bar z)\, x(y,\bar y)], \\
\{\, x(z,\bar z),\, x^*(y,\bar y)\,\} &= \frac{\gamma^2}{2}\,\Theta\,[2u(z,\bar y)\, u^*(y,\bar z) - x(z,\bar z)\, x^*(y,\bar y)], \\
\{\, u(z,\bar z),\, x(y,\bar y)\,\} &= \frac{\gamma^2}{2}\,\Theta\,[2x(z,\bar y)\, u(y,\bar z) - u(z,\bar z)\, x(y,\bar y)], \\
\{\, u(z,\bar z),\, x^*(y,\bar y)\,\} &= \frac{\gamma^2}{2}\,\Theta\,[2u(z,\bar y)\, x^*(y,\bar z) - u(z,\bar z)\, x^*(y,\bar y)].
\end{aligned} \tag{18}$$

Unlike in Liouville theory, these are not the Poisson brackets of the black holes, since we still have to take into account the constraints

$$J_0(z) = \frac{1}{\gamma^2}\langle\, t_0\, \partial_z g(z,\bar z)\, g^{-1}(z,\bar z)\,\rangle = 0, \qquad \bar J_0(\bar z) = \frac{1}{\gamma^2}\langle\, t_0\, g^{-1}(z,\bar z)\, \partial_{\bar z} g(z,\bar z)\,\rangle = 0. \tag{19}$$

Since the Kac-Moody algebra includes

$$\{J_0(z_1), J_0(z_2)\} = \frac{1}{2\gamma^2}\delta'(z_1 - z_2), \qquad \{\bar J_0(\bar z_1), \bar J_0(\bar z_2)\} = \frac{1}{2\gamma^2}\delta'(\bar z_1 - \bar z_2), \tag{20}$$

the constraints (19) are second class, therefore the Poisson brackets of the reduced system must be replaced by Dirac brackets [8]

$$\{F,G\}_D = \{F,G\} - \{F,\Phi_\alpha\}A_{\alpha\beta}\{\Phi_\beta,G\}. \tag{21}$$

$A_{\alpha\beta}$ is the inverse of the matrix defined by the Poisson brackets of second class constraints $\{\Phi_\alpha,\Phi_\beta\}$, and according to (20) it is given by

$$A = \gamma^2 \begin{pmatrix} \epsilon(z_1-z_2) & 0 \\ 0 & \epsilon(\bar z_1-\bar z_2) \end{pmatrix}. \tag{22}$$

The Poisson brackets of the currents J_0, $\bar J_0$ with the WZNW field $g(z,\bar z)$ are

$$\{g(z,\bar z),J_0(z_1)\} = \frac{1}{2}\,(t_0\,g(z,\bar z))\,\delta(z-z_1), \quad \{g(z,\bar z),\bar J_0(\bar z_1)\} = \frac{1}{2}\,(g(z,\bar z)\,t_0)\,\delta(\bar z-\bar z_1), \tag{23}$$

and using $t_0t = it = -tt_0$, $t_0s = -is = st_0$ we get

$$\begin{aligned}
\{u(z,\bar z),J_0(z_1)\} &= -\frac{i}{2}\,u(z,\bar z)\,\delta(z-z_1),\\
\{u(z,\bar z),\bar J_0(\bar z_1)\} &= \frac{i}{2}\,u(z,\bar z)\,\delta(\bar z-\bar z_1),\\
\{x(z,\bar z),J_0(z_1)\} &= -\frac{i}{2}\,x(z,\bar z)\,\delta(z-z_1),\\
\{x(z,\bar z),\bar J_0(\bar z_1)\} &= -\frac{i}{2}\,x(z,\bar z)\delta(\bar z-\bar z_1).
\end{aligned} \tag{24}$$

Let us consider the Dirac brackets corresponding to the first four Poisson brackets of (18). Here the correction terms are proportional to the causal factor Θ; the resulting black hole algebra reads

$$\begin{aligned}
\{\,u(z,\bar z),\,u(y,\bar y)\,\}_D &= \gamma^2\,\Theta\,[u(z,\bar y)\,u(y,\bar z) - u(z,\bar z)\,u(y,\bar y)],\\
\{\,u(z,\bar z),\,u^*(y,\bar y)\,\}_D &= \gamma^2\,\Theta\,x(z,\bar y)\,x^*(y,\bar z),\\
\{\,x(z,\bar z),\,x(y,\bar y)\,\}_D &= \gamma^2\,\Theta\,[x(z,\bar y)\,x(y,\bar z) - x(z,\bar z)\,x(y,\bar y)],\\
\{\,x(z,\bar z),\,x^*(y,\bar y)\,\}_D &= \gamma^2\,\Theta\,u(z,\bar y)\,u^*(y,\bar z).
\end{aligned} \tag{25}$$

From these relations the canonical equal-time Poisson brackets can be derived easily.

For the remaining Dirac brackets the corrections are proportional to $\epsilon(z-y)-\epsilon(\bar z-\bar y)$, leading to the non-causal brackets

$$\begin{aligned}
\{\,u(z,\bar z),\,x(y,\bar y)\,\}_D &= \gamma^2\,\Theta\,x(z,\bar y)\,u(y,\bar z) - \frac{\gamma^2}{2}\,\epsilon(z-y)\,u(z,\bar z)\,x(y,\bar y),\\
\{\,u(z,\bar z),\,x^*(y,\bar y)\,\}_D &= \gamma^2\,\Theta\,u(z,\bar y)\,x^*(y,\bar z) - \frac{\gamma^2}{2}\,\epsilon(\bar z-\bar y)\,u(z,\bar z)\,x^*(y,\bar y).
\end{aligned} \tag{26}$$

It is interesting to see that the euclidean, $u(z,\bar{z})$, and Minkowskian, $x(z,\bar{z})$, black hole fields form a coupled algebra.

$u(z,\bar{z})$ and $x(z,\bar{z})$ have the explicit free field realisation

$$\begin{aligned} u(z,\bar{z}) &= e^{\gamma(\phi_1(z)+\bar{\phi}_1(\bar{z}))} e^{i\gamma(\phi_2(z)+\bar{\phi}_2(\bar{z}))} \left(1 - A^*(z)\bar{A}^*(\bar{z})\right), \\ x(z,\bar{z}) &= e^{\gamma(\phi_1(z)+\bar{\phi}_1(\bar{z}))} e^{i\gamma(\phi_2(z)-\bar{\phi}_2(\bar{z}))} \left(1 + A^*(z)\bar{A}(\bar{z})\right), \end{aligned} \tag{27}$$

where the chiral $A(z)$ and anti-chiral $\bar{A}(\bar{z})$ are defined via the differential equations

$$A'(z) = \gamma(\phi_1'(z) - i\phi_2'(z))e^{-2\gamma\phi_1(z)}, \qquad \bar{A}'(\bar{z}) = \gamma(\bar{\phi}_1'(\bar{z}) - i\bar{\phi}_2'(\bar{z}))e^{-2\gamma\phi_1(\bar{z})}. \tag{28}$$

When integrating these equations the boundary conditions must be properly taken into account [6]. $\phi_i(z)$ and $\bar{\phi}_i(\bar{z})$ respectively denote the chiral and anti-chiral part of the free field $\phi_i(z) + \bar{\phi}_i(\bar{z})$ ($i = 1, 2$) and obey standard free field bracket relations. The black hole algebra can also be derived directly from these free field representations and the free field brackets.

We would like to mention that the quantisation of the Liouville theory deforms the Poisson brackets in a unique manner, and preserves the causal structure [4]. These results generalise the exchange algebra to a 'rectangle' relation for physical fields. It is still a challenge to determine the quantum realisation of the black hole algebra. Although each model has its own dynamics, the coupling with its dual partner might have interesting consequences.

Acknowledgements

G.J. is grateful to DESY Zeuthen for hospitality. His research was supported by grants from the DFG, INTAS and RFBR.

References

[1] M.F. Chu, P. Goddard, I Halliday, D. Olive and A Schwimmer, *Phys. Lett.*, **B266** (1991) 71.

[2] K. Gawedzki, *Comm. Math. Phys.* **139** (1991) 201.

[3] J. Balog, L. Feher, L. Palla, *Nucl. Phys.* **B568** (2000) 503.

[4] G. Jorjadze and G. Weigt, Poisson Structure and Moyal Quantisation of the Liouville Theory, hep-th 0105306.

[5] J. Balog, L. Feher, P. Forgacs, L. O'Raifeartaigh, A. Wipf, *Phys.Lett.* **B227** (1989) 214; *Annals Phys.* **203** (1992) 269.

[6] C. Ford, G. Jorjadze, G. Weigt, Integration of the $SL(2,\mathbb{R})/U(1)$ Gauged WZNW Theory by Reduction and Quantum Parafermions, hep-th/0003246, to be published in Theor. Math. Phys.

[7] K. Bardakci, M. Crescimanno, E. Rabinovici, *Nucl. Phys.* **B344** (1990) 344.

[8] P.A.M. Dirac, Lectures on Quantum Mechanics, Yeshiva University Press, New York, 1964.