# Interacting fermions and domain wall defects in $2+1$ dimensions

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Abstract

We consider a Dirac field in $2+1$ dimensions with a domain wall like defect in its mass, minimally coupled to a dynamical Abelian vector field. The mass of the fermionic field is assumed to have just one linear domain wall, which is externally fixed and unaffected by the dynamics. We show that, under some general conditions on the parameters, the localized zero modes predicted by the Callan and Harvey mechanism are stable under the electromagnetic interaction of the fermions.

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1   Introduction

It is a well-known fact that, in an odd dimensional spacetime, a domain wall defect in the mass term of a Dirac field induces a fermionic zero mode localized on the defect [1]. This effect is known to occur even in the presence of an external gauge field, if the corresponding electromagnetic field is contained in the defect hyperplane. Different aspects of this kind of system have been studied both for static [2, 3], and dynamical [4] defects. As far as we know, however, possible effects due to interactions between the fermions have not been considered for this system. In this article, we shall study the stability of this kind of configuration when the electromagnetic interaction between the fermions is turned on. That the localization phenomenon should survive this interaction is not *apriori* evident. For example, for a static configuration, the Coulomb repulsion between the localized charges could be so important as to spread the charge density out over a large region, since the charge density due to the zero mode shall induce an electromagnetic field normal to the defect hypersurface. On the other hand, we note that our study may be thought of as a domain-wall analog of the consideration of the self consistent vacuum currents in the presence of vortices [5].

This paper is organized as follows: in section 2, we introduce the model and derive a self-consistent equation based in some approximations. This equation is solved for two different mass profiles in section 3. Finally, in section 4 we discuss the effects of the non-zero modes and present our conclusions.

2   The model

The Euclidean action S, for the system we shall consider, is given by

$$S = S_F + S_G \tag{1}$$

where

$$S_F = \int d^3x \, \bar{\psi}(x) [\slashed{\partial} + ie \, \slashed{A}(x) + M(x)] \psi(x) \tag{2}$$

is the fermionic action, and

$$S_G = \int d^3x \, \frac{1}{4} F_{\mu\nu} F_{\mu\nu} \tag{3}$$

the Maxwell action, which defines the gauge field dynamics. $x = (x_0, x_1, x_2)$ denote the Euclidean coordinates, and the Hermitian γ matrices are assumed to be in an irreducible 2×2 representation of the Dirac algebra, verifying the

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anticommutation relations $\{\gamma_\mu, \gamma_\nu\} = 2\,\delta_{\mu\nu}$. The complete Green's functions can be derived from the generating functional

$$\mathcal{Z}[j_\mu; \bar\eta, \eta] = \int \mathcal{D}A_\mu \mathcal{D}\bar\psi \mathcal{D}\psi \, \exp\{-S[\bar\psi, \psi; A]$$

$$+ \int d^3x [j_\mu(x) A_\mu(x) + \bar\eta(x)\psi(x) + \bar\psi(x)\eta(x)\} \tag{4}$$

where we included source terms for the gauge and fermionic fields. The fermion mass is regarded as an external classical 'field', dependent on the x_2 coordinate only. We also fix the number of defects to one, by requiring $M(x)$ to cross 0 once, at $x_2 = 0$, say.

By applying the property that the functional integral of a (functional) derivative vanishes to equation (4), we derive the 'quantum equations of motion'

$$0 = \int \mathcal{D}A_\mu \mathcal{D}\bar\psi \mathcal{D}\psi \left[\frac{\delta S}{\delta A_\mu(x)} - j_\mu(x) \right] \exp\{-S[\bar\psi, \psi; A_\mu]$$

$$+ \int d^3x [j_\mu(x) A_\mu(x) + \bar\eta(x)\psi(x) + \bar\psi(x)\eta(x)]\} \tag{5}$$

for A_μ, and

$$0 = \int \mathcal{D}A_\mu \mathcal{D}\bar\psi \mathcal{D}\psi \left[\frac{\delta S}{\delta\bar\psi(x)} - \eta(x) \right] \exp\{-S[\bar\psi, \psi; A_\mu]$$

$$+ \int d^3x [j_\mu(x) A_\mu(x) + \bar\eta(x)\psi(x) + \bar\psi(x)\eta(x)]\} , \tag{6}$$

for ψ (the adjoint equation is trivially obtained). Taking the functional derivative with respect to $\eta(y)$ in (6), and putting all the external sources equal to zero afterwards, we find that equations (5) and (6) reduce to:

$$\partial_\mu F_{\mu\nu}(x) = J_\nu(x) \tag{7}$$

and

$$\langle\, [\slashed\partial + ie\,\slashed A(x) + M(x)]\psi(x)\bar\psi(y) \,\rangle = \delta(x-y) \,, \tag{8}$$

where:

$$J_\nu(x) = ie\langle\bar\psi(x)\gamma_\nu\psi(x)\rangle \tag{9}$$

and

$$F_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu \,, \quad \mathcal{A}_\mu = \langle A_\mu\rangle \,. \tag{10}$$

Equation (7) is an inhomogeneous 'classical' Maxwell equation, with the average gauge field $\mathcal{A}_\mu = \langle A_\mu\rangle$ playing the role of the classical gauge field,

and the average (vacuum) fermionic current J_μ as its source. Equation (8) involves the expectation values $\langle\psi\bar\psi\rangle$ and $\langle A\psi\bar\psi\rangle$. Of course, an exact treatment would require the use of an infinite set of coupled equations involving all the different Green's functions of the system. In order to find a simpler and closed system of equations, we make the following approximation:

$$\langle A_\mu(x)\psi(x)\bar\psi(y)\rangle \;\simeq\; \langle A_\mu(x)\rangle\langle\psi(x)\bar\psi(y)\rangle \;=\; \mathcal{A}_\mu(x)\,S_\mathcal{A}(x,y)\;, \qquad (11)$$

where we introduced $S_\mathcal{A}(x,y)$, which denotes the fermionic propagator in the presence of an 'external field' $\mathcal{A}(x)$, which corresponds to the average gauge field. This amounts to a sort of mean field approximation, where the gauge field is treated classically. To make the approximation involved more explicit, we note that the (exact) three point function appearing in (11) can be written in the equivalent form:

$$\langle A_\mu(x)\psi(x)\bar\psi(y)\rangle \;=\; \int \mathcal{D}A\, A_\mu(x)\,\langle x|(\slashed\partial + ie\,\slashed A + M)^{-1}|y\rangle\, e^{-S_G[A]-\Gamma_F[A]} \quad (12)$$

where

$$\Gamma_F[A] \;=\; -\log\det[\slashed\partial + ie\,\slashed A + M]\;. \qquad (13)$$

The approximation (11) is obtained from (12) by replacing A by its saddle point value. Namely, the approximation amounts to using the (leading) saddle point approximation, where the 'action' which is minimized at the saddle point is the bare Maxwell action plus an effective contribution $\Gamma_F[A]$ coming from the fermionic determinant.

Equation (11) is sufficient to close the system of equations, since then (8) becomes:

$$[\slashed\partial + ie\,\mathcal{A}(x) + M(x)]S_\mathcal{A} \;=\; \delta(x-y)\;. \qquad (14)$$

It is now important to realize that the average current can be expressed as a functional of $\mathcal{A}$, as follows:

$$J_\mu(x) \;=\; ie\,\mathrm{tr}\left[\gamma_\mu\langle\psi(x)\bar\psi(x)\rangle\right] \;=\; -ie\,\mathrm{tr}\left[\gamma_\mu S_\mathcal{A}(x,x)\right]\;. \qquad (15)$$

Equation (7), together with (15), define a closed system of equations, which allows us to find the average gauge field $\mathcal{A}$, and then the current density induced in that background. The equation that determines $\mathcal{A}$ is obtained by replacing J_μ by its expression (15) in (7):

$$\partial_\mu F_{\mu\nu}(x) \;=\; -ie\,\mathrm{tr}\left[\gamma_\mu S_\mathcal{A}(x,x)\right]\;, \qquad (16)$$

which, in general, and depending on the approximation used to evaluate $S_\mathcal{A}$, will be a non-linear integro-differential equation. The non-linearity comes from the fermionic propagator $S_\mathcal{A}$, which is defined as :

$$S_{\alpha\beta}(x,y) = \langle x,\alpha \mid \mathcal{D}^{-1} \mid y,\beta\rangle, \qquad (17)$$

where $\mathcal{D} = (\not{\partial} + ie\not{A} + M)$.

We shall now look for particular solutions of the coupled set of equations, under some restrictions and simplifying approximations. We shall restrict ourselves to *static*, purely electric solutions, with no electric current (hence, no magnetic field). In the Coulomb gauge, the only remaining component for the (average) gauge field is $\mathcal{A}_0$, which is determined by the equation

$$\nabla^2 V \;=\; -ie\, tr[\gamma_0 S_V(x,x)] \;, \tag{18}$$

where $V \;=\; \mathcal{A}_0$.

Our approach to solve the system of equations shall be to first evaluate the fermionic propagator in the external potential V. Then, we shall find the corresponding vacuum charge density as a functional of V, and insert it into the Gauss law (18) to determine V. The resulting V can then be used to fix the precise form of the charge density. We will be able to say that there are localized modes if the system admits solutions where the charge density is confined to a small region around the defect. Of course, we shall have to make some assumptions also on the allowed boundary conditions for the fields. The choice of these conditions is also part of the kind of ansatz used, and also on the amount of generality one wants to introduce into the treatment.

To find the fermion propagator in the presence of the external field V, we shall use the perturbative expansion of $\mathcal{D}^{-1}$ in powers of V, namely, we decompose $\mathcal{D}$ as follows:

$$\mathcal{D} = \mathcal{D}_0 + \mathcal{V}, \tag{19}$$

where

$$\mathcal{D}_0 = \not{\partial} + M(x) \tag{20}$$

and

$$\mathcal{V} \;=\; ie\gamma_0 V(x) \;. \tag{21}$$

Thus, $\mathcal{D}^{-1}$ is naturally expanded as:

$$\mathcal{D}^{-1} = \mathcal{D}_0^{-1} - \mathcal{D}_0^{-1}\mathcal{V}\mathcal{D}_0^{-1} + \mathcal{D}_0^{-1}\mathcal{V}\mathcal{D}_0^{-1}\mathcal{V}\mathcal{D}_0^{-1} - ... \tag{22}$$

We note that the 'free' propagator $\mathcal{D}_0^{-1}$ includes the mass field and its space dependence exactly. This must be so, since the defect changes the spectrum of the Dirac field, an effect that cannot be described perturbatively. To find the inverse of $\mathcal{D}_0$, we use the equivalent expression:

$$\mathcal{D}_0^{-1} = (\mathcal{D}_0^\dagger \mathcal{D}_0)^{-1}\mathcal{D}_0^\dagger. \tag{23}$$

which requires finding the inverse of the Hermitian operator

$$\mathcal{H}_0 \;=\; \mathcal{D}_0^\dagger \mathcal{D}_0\,. \tag{24}$$

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This is a much simpler task than inverting $\mathcal{D}_0$, and it allows one to dimensionally reduce the problem. To see this, we follow the procedure of [2], of which we give a lightning review here. First we write:

$$\mathcal{D}_0 = (a + \widehat{\slashed{\partial}})P_L + (a^\dagger + \widehat{\slashed{\partial}})P_R, \tag{25}$$

where $\widehat{\slashed{\partial}} = \gamma_0 \partial_0 + \gamma_1 \partial_1$. We define the operators $a^\dagger$ and a, that act on functions of the x_2 coordinate as

$$a = \partial_2 + M \qquad a^\dagger = -\partial_2 + M, \tag{26}$$

and the projectors P_L, P_R:

$$P_L = \frac{1 + \gamma_2}{2}, \qquad P_R = \frac{1 - \gamma_2}{2}. \tag{27}$$

These projectors behave like chirality projectors from the point of view of the $1 + 1$ dimensional theory which describes the chiral zero mode. This decomposition makes it possible to disentangle the dynamics corresponding to the x_2 coordinate from the coordinates $\hat{x} = (x_0, x_1)$. The 'dimensional reduction' can be seen to arise at the level of the operator $\mathcal{H}_0$:

$$\mathcal{H}_0 = (h - \widehat{\slashed{\partial}}^2)P_L + (\tilde{h} - \widehat{\slashed{\partial}}^2)P_R, \tag{28}$$

where

$$h = a^\dagger a \qquad \tilde{h} = a\,a^\dagger. \tag{29}$$

To expand the fermionic fields, we define ϕ_n and $\widetilde{\phi}_n$, eigenstates of the operators h and $\tilde{h}$, respectively. We denote by λ_n^2 their (common) eigenvalues:

$$h\phi_n = \lambda_n^2 \phi_n, \qquad \tilde{h}\widetilde{\phi}_n = \lambda_n^2 \widetilde{\phi}_n, \tag{30}$$

$$\langle \phi_n \mid \phi_m \rangle = \delta_{nm}, \qquad \langle \widetilde{\phi}_n \mid \widetilde{\phi}_m \rangle = \delta_{nm}, \tag{31}$$

since the spectra coincide, except for $\lambda_n = 0$, and the eigenvalues are of course positive. The $\lambda_n = 0$ eigenvalue will, by assumption, be present only for h. This will depend of course on the mass profile near the defect, i.e. the zero of the mass. Since the sign of λ_n is arbitrary, we take it positive by convention.

Thus, the fermionic fields can be expanded as:

$$\psi(\hat{x}, x_2) = \sum_n [\phi_n(x_2)\psi_L^{(n)}(\hat{x}) + \widetilde{\phi}_n(x_2)\psi_R^{(n)}(\hat{x})], \tag{32}$$

$$\overline{\psi}(\hat{x}, x_2) = \sum_n [\overline{\psi}_L^{(n)}(\hat{x})\phi_n^\dagger(x_2) + \overline{\psi}_R^{(n)}(\hat{x})\widetilde{\phi}_n^\dagger(x_2)]. \tag{33}$$

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The spinors that carry the dependence on $\hat{x}$ are defined by:

$$\psi_{L,R}^{(n)}(\hat{x}) = P_{L,R}\psi^{(n)}(\hat{x}) \ , \qquad \overline{\psi}_{L,R}^{(n)}(\hat{x}) = \overline{\psi}^{(n)}(\hat{x})P_{R,L} \ , \tag{34}$$

where $\psi_{L,R}^{(n)}$ denotes a general bidimensional fermionic field (one for each value of the index n). In terms of this expansion, the fermionic action becomes:

$$S = S_L^{(0)} + \sum_n S^{(n)} \tag{35}$$

where $S_L^{(0)}$ denotes the action for a chiral left-handed fermion in $1+1$ dimensions, while $S^{(n)}$ is a massive Dirac action, also in $1+1$ dimensions, with a mass equal to λ_n (the sign of the mass is irrelevant in $1+1$ dimensions).

Since

$$\mathcal{H}_0^{-1} = (h - \hat{\overline{\partial}}^2)^{-1}P_L + (\tilde{h} - \hat{\overline{\partial}}^2)^{-1}P_R, \tag{36}$$

the free propagator becomes

$$\mathcal{D}_0^{-1} = (h - \hat{\overline{\partial}}^2)^{-1}P_L(a^\dagger - \hat{\overline{\partial}}) + (\tilde{h} - \hat{\overline{\partial}}^2)^{-1}P_R(a - \hat{\overline{\partial}}) \ . \tag{37}$$

Translation invariance along the x_0 and x_1 coordinates suggests the use of a potential depending only on x_2, $\mathcal{V} = \mathcal{V}(x_2)$. To find the propagator in configuration space, we need to evaluate the following expression:

$$S_{\alpha\beta}(x,y) = (\mathcal{D}_0^{-1})_{\alpha\beta}(x,y) - (\mathcal{D}_0^{-1}\mathcal{V}\mathcal{D}_0^{-1})_{\alpha\beta}(x,y) + ... \tag{38}$$

with $\mathcal{V} = \mathcal{V}(x_2)$. In the perturbative expansion for the propagator, we insert expansions of the identity constructed with intermediate states corresponding to eigenstates of the operator $\mathcal{H}_0$. Using the fact that each eigenvalue λ_n corresponds to the effective mass of a two dimensional mode, and that the lowest mode is massless (the zero mode), it is natural to keep only the zero mode in the intermediate states as a first approximation. Note that the mass λ_n of the non zero modes is separated from the zero mode by a finite gap whose magnitude is controlled by the profile of the mass near the defect (see ref. [2]). With this in mind, we shall first use the leading approximation of keeping just the zero mode, and then make a quantitative evaluation of the error involved in this procedure, by including the correction corresponding to the lowest massive mode. On the other hand, we shall keep the full dependence in the potential, namely, we shall use no truncation for the perturbative series in $\mathcal{V}$. To implement this approximation, we introduce projectors P_0 along the zero mode. They are explicitly given by

$$P_0 = \phi_0\phi_0^\dagger \sum_n \psi_L^{(n)}\overline{\psi}_L^{(n)} \ . \tag{39}$$

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Taking this into account, after some algebra one can show that, in this approximation, the propagator is given by:

$$S_{\alpha\beta}(x,y) \simeq \phi_0^\dagger(x_2)\phi_0(y_2)\langle x_0, x_1, \alpha \mid \frac{\widehat{\slashed{\partial}} + ie\,\gamma_0 V_{0,0} P_L}{(\widehat{\partial} + ie\,\gamma_0 V_{0,0})^2} \mid y_0, y_1, \beta\rangle\,. \tag{40}$$

In this expression there appears the average of V in the zero mode which is denoted by:

$$V_{0,0} \;=\; \langle\phi_0|V|\phi_0\rangle\,. \tag{41}$$

It is worth noting that this result is approximate in the sense that only the zero mode has been included, but all the powers of A_μ have been added, as it is evident from the non-linear dependence of the propagator on A_μ. The charge density is evaluated by multiplying by γ_0, taking the Dirac trace, and finally calculating the coincidence limit $x \to y$. Inserting the result so obtained for the charge density as a functional of the potential into (18) yields:

$$\frac{\partial^2}{\partial x_2^2}V(x_2) \;=\; \phi_0(x_2)\phi_0^\dagger(x_2)\int \frac{d^2k}{2\pi}\frac{-i\,k_0 + ie\,V_{0,0}}{(-i\,k_0 + ie\gamma_0 V_{0,0})^2}\,. \tag{42}$$

The momentum space integral has both linear and logarithmic divergences. Using a symmetric limit kind of regularization, we see that (42) can be expressed as:

$$\frac{\partial^2}{\partial x_2^2}V(x_2) = \phi_0(x_2)\phi_0^\dagger(x_2)\frac{e^2\,V_{0,0}}{2}\,. \tag{43}$$

It is remarkable that, as a consequence of the fact that we are only keeping the zero mode, the expression for the charge density becomes linear in the potential. This happens in spite of the fact that we have kept all the powers of the potential in the fermionic propagator, since the result is a consequence of the fact that massless two dimensional QED is exactly solvable [6], with the exact fermionic determinant being quadratic in the gauge field.

We have obtained an integro-differential equation involving derivatives of V and its average on the lowest energy mode. To solve it self-consistently, we first derive from (43) (by integration) an equation for V, depending also the average of the potential. Then, as a second step, we shall insert this average into (43) in order to obtain the explicit profile of the potential as a function of x_2. At this point, it is clear that the existence of a self-consistent solution depends on the particular form of the zero modes appearing in eq. (43). This differential equation will have a solution only if the charge density is localized in such a way that the integrals involved are well defined. In particular, the zero modes need to be localized around the defect. It was

shown in reference [1] that in $2n+1$ dimensions the zero mode has the form

$$\eta \, e^{-\int_a^{x_2} dy \, M(y)},\tag{44}$$

where η is an spinor independent of x_2.

3   Examples

In what follows we will discuss the possible solutions of eq. (43) for two different kinds of mass profiles.

- Step-like defect.

  Given a mass of the form:

  $$M(x_2) \;=\; \Lambda(2\Theta(x_2)-1)\,.\tag{45}$$

  where Λ is a constant with the dimensions of a mass, and Θ is the Heaviside function, there is only one zero mode [2], which can be explicitly written as:

  $$\phi_0(x_2) = \Lambda^{\frac{1}{2}} \, e^{-\Lambda|x_2|}\,.\tag{46}$$

  In this case, the differential equation becomes

  $$\frac{\partial^2}{\partial x_2^2} V(x_2) \;=\; \frac{1}{2}\,\Lambda e^2 V_{0,0}\, e^{-2\Lambda|x_2|}\,,\tag{47}$$

  and integrating it twice we obtain for the potential V

  $$V(x_2) = a + \frac{1}{8\Lambda}\,e^2 V_{0,0}\, e^{-2\Lambda|x_2|}\,,\tag{48}$$

  where a is a constant, to be related later to the chemical potential. In this expression we have not included a term that corresponds to a constant electric field in the x_2 direction, because it could be eliminated by choosing appropriate boundary conditions (such as vanishing density of charges at infinity).

  In order to find a self-consistent solution for the potential we evaluate the expectation value of V, which is expressed by (48), in the zero mode

  $$V_{0,0} \;=\; a + \frac{1}{8\Lambda}\,e^2 V_{0,0} \int_{-\infty}^{\infty} dx |\phi_0(x)|^2 \, e^{-2\Lambda|x|}\,.\tag{49}$$

  Thus, $V_{0,0}$ is easily seen to be given by

  $$V_{0,0} \;=\; \frac{a}{1 - \frac{e^2}{16\Lambda}}\,.\tag{50}$$

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Therefore, the potential written in terms of the zero modes results

$$V(x_2) = |\phi_0(x)|^2 \frac{2ae^2}{16\Lambda^2 - \Lambda e^2}. \tag{51}$$

Notice that the solution is only stable if the electromagnetic coupling constant and the mass coupling constant satisfy the bound: $e^2 < 16\Lambda$, which means that the strength of the interaction (repulsion) between the electrons cannot be larger than the scale given by the height of the defect. We note that 'stability' refers here to the property of having a confining potential. We see that in this case, i.e., for an step-like mass and keeping only the zero energy mode, there exist a self-consistent solution for the fermionic interaction potential. In other words, even in the case of interacting electrons, the fermions are localized in the x_2 direction and can only move along the defect.

The interpretation of a as a chemical potential proceeds from the fact that the Gauss law (47), combined with (50), means that the charge density of the configuration is

$$\rho(x_2) = a \frac{\Lambda e^2}{2(1 - \frac{e^2}{16\Lambda})} e^{-2\Lambda|x_2|}, \tag{52}$$

and (by integrating over x_2) one sees that the total charge is proportional to the constant a.

- Linear defect.

  Assuming than the mass can be expanded as a power series in x_2, for small enough x_2 we only keep the first order term:

  $$M(x_2) = M'(0)\, x_2, \tag{53}$$

  where we assume $M'(0) \neq 0$ being M' the first derivative of the mass. For this mass profile we can still find the zero mode by defining [2]

  $$h = -\partial_2^2 - M' + M^2 x_2^2. \tag{54}$$

  which is an harmonic oscillator Hamiltonian. The lowest energy mode is:

  $$\phi_0(x_2) = \left(\frac{|M'|}{\pi}\right)^{1/4} e^{-\frac{|M'|}{2}x_2^2}. \tag{55}$$

  Following the same steps as in the previous example we find that the potential can be written in terms of the zero mode as

  $$V(x_2) = a + \int_B^{x_2} dy \int_A^y dz |\phi_0(z)|^2 \left(\frac{e^2 a}{2 - e^2 C}\right), \tag{56}$$

where

$$C = \int_{-\infty}^{\infty} dx_2 |\phi_0(x_2)|^2 \int_B^{x_2} dy \int_A^y dz |\phi_0(z)|^2. \qquad (57)$$

Thus we see that also in this case there exists a self-consistent solution for the Gauss law, for a charge density localized around the defect. However there is a necessary condition for the existence of this localized mode. The wave function of the zero mode has to vanish rapidly outside the region of the space where the mass can be approximated linearly. A quantitative criterion for the validity of this condition can be found in reference [2].

In summary, up to know we have shown the existence of localized solutions if we keep only the lowest energy modes in the expansion of the fermionic propagator. This solution depends on the mass profile, and it is non-perturbative in the electromagnetic interaction between the fermions. We have neglected the (more energetic) massive modes based on the fact that the terms on the action that come from these modes go as $\frac{1}{\lambda_n^2}$, where λ_n is the mass of the mode [2]. Therefore, for a large and steep enough mass, our approximation will be valid. In particular, for a linear defect, the mass of the modes is proportional to the slope of the mass profile. Therefore, by changing this slope we could make λ_n arbitrarily large.

4   Effect of the massive modes

We shall now study the problem of including one massive mode in our calculation in order to check whether there still exist localized solutions or not. It will provide also a quantitative idea about the error involved in considering only the lowest energy modes.

We proceed as follows: in the perturbative expansion for the fermion propagator (22), we consider only the projection of the operators $\mathcal{D}_0^{-1}$ and $\mathcal{V}$ onto the two lowest energy modes. Then each factor $\mathcal{D}_0^{-1}$ contributes with

$$\mathcal{D}_0^{-1} \simeq \widetilde{\phi}_1 \phi_1^\dagger \lambda_1 P_L - \widetilde{\phi}_1 \widetilde{\phi}_1^\dagger \widehat{\slashed{\partial}} P_R + \phi_1 \widetilde{\phi}_1^\dagger \lambda_1 P_R - \phi_1 \phi_1^\dagger \widehat{\slashed{\partial}} P_L - \phi_0 \phi_0^\dagger \widehat{\slashed{\partial}} P_L. \qquad (58)$$

For an even $V(x_2)$, selection rules imply the vanishing of the matrix elements $V_{n,m}$, $V_{n,\tilde{m}}$ and $V_{\tilde{n},\tilde{m}}$.

Replacing (58) into (22), and keeping only the non-vanishing matrix elements of $\mathcal{V}$, the first order correction in $\mathcal{V}$ to the fermion propagator (i.e., correction to (58)) is:

$$\mathcal{D}_0^{-1} \mathcal{V} \mathcal{D}_0^{-1} \simeq \frac{ie\widetilde{\phi}_1 \phi_1^\dagger}{(\lambda_1^2 - \widehat{\partial}^2)^2} [-\widehat{\slashed{\partial}} \lambda_1 \gamma^0 V_{\tilde{1},\tilde{1}}^0 - \lambda_1 \gamma^0 V_{1,1} \widehat{\slashed{\partial}}] P_L$$

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$$+ \frac{ie\widetilde{\phi}_1\phi_0^\dagger}{(\lambda_1^2 - \widehat{\partial}^2)\widehat{\partial}^2}[-\lambda_1\gamma^0 V_{1,0}\widehat{\partial}\!\!\!/]P_L + \frac{ie\widetilde{\phi}_1\widetilde{\phi}_1^\dagger}{(\lambda_1^2 - \widehat{\partial}^2)^2}[\lambda_1^2\gamma^0 V_{1,1} + \widehat{\partial}\!\!\!/\gamma^0 V_{\widetilde{1},\widetilde{1}}\widehat{\partial}\!\!\!/]P_R$$

$$+ \frac{ie\phi_1\widetilde{\phi}_1^\dagger}{(\lambda_1^2 - \widehat{\partial}^2)^2}[-\widehat{\partial}\!\!\!/\lambda_1\gamma^0 V_{1,1} - \lambda_1\gamma^0 V_{\widetilde{1},\widetilde{1}}\widehat{\partial}\!\!\!/]P_R + \frac{ie\phi_0\widetilde{\phi}_1^\dagger}{(\lambda_1^2 - \widehat{\partial}^2)\widehat{\partial}^2}[-\widehat{\partial}\!\!\!/\lambda_1\gamma^0 V_{1,1}]P_R$$

$$+ \frac{ie\phi_1\phi_1^\dagger}{(\lambda_1^2 - \widehat{\partial}^2)^2}[\lambda_1^2\gamma^0 V_{\widetilde{1},\widetilde{1}} + \widehat{\partial}\!\!\!/\gamma^0 V_{1,1}\widehat{\partial}\!\!\!/]P_L + \frac{ie\phi_1\phi_0^\dagger}{(\lambda_1^2 - \widehat{\partial}^2)\widehat{\partial}^2}[\widehat{\partial}\!\!\!/\gamma^0 V_{1,0}\widehat{\partial}\!\!\!/]P_L$$

$$+ \frac{ie\phi_0\phi_1^\dagger}{(\lambda_1^2 - \widehat{\partial}^2)\widehat{\partial}^2}[\widehat{\partial}\!\!\!/\gamma^0 V_{0,1}\widehat{\partial}\!\!\!/]P_L + \frac{ie\phi_0\phi_0^\dagger}{(\widehat{\partial}^2)^2}[\widehat{\partial}\!\!\!/\gamma^0 V_{0,0}\widehat{\partial}\!\!\!/]P_L. \tag{59}$$

Notice that in this case it is not possible to obtain a non-perturbative expression for the fermion propagator due to the fact that we are taking into account massive modes as well as the massless one. In order to write the Gauss law we need to compute

$$tr(\gamma_0 \mathcal{D}^{-1}) \simeq -\frac{ie(\widetilde{\phi}_1\widetilde{\phi}_1^\dagger + \phi_1\phi_1^\dagger)}{(\lambda_1^2 - \widehat{\partial}^2)}\partial_0 + \frac{ie\phi_0\phi_0^\dagger}{(\widehat{\partial}^2)}\partial_0$$

$$+ \frac{ie\widetilde{\phi}_1\widetilde{\phi}_1^\dagger}{(\lambda_1^2 - \widehat{\partial}^2)^2}[\lambda_1^2 V_{1,1} + (2\partial_0^2 - \widehat{\partial}^2)V_{\widetilde{1},\widetilde{1}}] + \frac{ie\phi_1\phi_1^\dagger}{(\lambda_1^2 - \widehat{\partial}^2)^2}[\lambda_1^2 V_{\widetilde{1},\widetilde{1}} + (2\partial_0^2 - \widehat{\partial}^2)V_{1,1}]$$

$$+ \frac{ie(\phi_1\phi_0^\dagger + \phi_0\phi_1^\dagger)}{(\lambda_1^2 - \widehat{\partial}^2)\widehat{\partial}^2}[(2\partial_0^2 - \widehat{\partial}^2)V_{0,1}] + \frac{ie\phi_0\phi_0^\dagger}{(\widehat{\partial}^2)^2}[(2\partial_0^2 - \widehat{\partial}^2)V_{0,0}]. \tag{60}$$

Taking the Fourier transform in the above expression and regularizing the integrals by a symmetric limit, the Gauss law becomes

$$\frac{\partial^2}{\partial x_2^2}V(x_2) = \widetilde{\phi}_1(x_2)\widetilde{\phi}_1^\dagger(x_2)\frac{e^2\, V_{1,1}}{2} + \phi_1(x_2)\phi_1^\dagger(x_2)\frac{e^2\, V_{\widetilde{1},\widetilde{1}}}{2}. \tag{61}$$

Thus we have obtained a differential equation whose solution will depend on the localization properties of the fermionic modes around the defect.

In the case of a mass that can be approximated by a linear function of x_2 near the defect, it is simple to check that [2]

$$\phi_n = \widetilde{\phi}_{n+1}, \tag{62}$$

therefore there is only one zero mode, and the Gauss equation becomes

$$\frac{\partial^2}{\partial x_2^2}V(x_2) = \phi_0(x_2)\phi_0^\dagger(x_2)\frac{e^2\, V_{1,1}}{2} + \phi_1(x_2)\phi_1^\dagger(x_2)\frac{e^2\, V_{0,0}}{2}. \tag{63}$$

Integrating this expression we find

$$V(x_2) = a + \left(\frac{e^2 V_{1,1}}{2}\right) \int_B^{x_2} dy \int_A^y dz |\phi_0(z)|^2 + \left(\frac{e^2 V_{0,0}}{2}\right) \int_B^{x_2} dy \int_A^y dz |\phi_1(z)|^2.$$

(64)

Once again, we look for the self-consistent solutions for the expectation values of the potential. When computed on the two lowest energy modes, they are given by the solution of the equations:

$$V_{i,i} = a + \left(\frac{e^2 V_{1,1}}{2}\right) D_{i0} + \left(\frac{e^2 V_{0,0}}{2}\right) D_{i1},$$

(65)

where $i = 0, 1$ and D_{ij} are:

$$2D_{ij} = \int_{-\infty}^{\infty} dx_2 |\phi_i(x_2)|^2 \int_B^{x_2} dy \int_A^y dz |\phi_j(z)|^2.$$

(66)

Solving (65) we obtain

$$V_{0,0} = a \frac{1 - e^2 D_{10} + e^2 D_{00}}{(1 - e^2 D_{01})(1 - e^2 D_{10}) - e^4 D_{11} D_{00}},$$

(67)

$$V_{1,1} = a \frac{1 - e^2 D_{01} + e^2 D_{11}}{(1 - e^2 D_{01})(1 - e^2 D_{10}) - e^4 D_{11} D_{00}}.$$

(68)

We have found that, in the case of a linear mass, there exist a self-consistent solution of the Gauss equation to first order in the interaction potential, if we include apart from the zero mode, one massive mode. Notice that, for a linear mass around the defect, ϕ_n and $\widetilde{\phi}_n$ are harmonic oscillator eigenstates. Far enough from the defect, the eigenstates decay exponentially (as a Gaussian function), ensuring that the charge density is localized around the defect in such a way that there is a solution for the Gauss equation. Obviously all the caveats regarding the range of validity of approximating the mass by a linear function, that we mention in the previous case, must be taken into account here.

Summarizing, we have considered a Dirac field in $2 + 1$ dimensions with a domain wall like defect in its mass, minimally coupled to a dynamical Abelian vector field. The mass of the fermionic field is assumed to have just one linear domain wall, externally fixed and unaffected by the dynamics. In the absence of electromagnetic interactions among the fermions, it is a well known fact that localized zero modes exist on the defect [1]. We have studied here the effect of the fermionic interactions on these modes showing that, under some general conditions on the parameters, the localized zero modes stable under the electromagnetic interactions of the fermions.

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Acknowledgments

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