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Noncommutative Gauge Theory on Fuzzy Four-Sphere and Matrix Model

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Abstract

We study a noncommutative gauge theory on a fuzzy four-sphere. The idea is to use a matrix model with a fifth-rank Chern-Simons term and to expand matrices around the fuzzy four-sphere which corresponds to a classical solution of this model. We need extra degrees of freedom since algebra of coordinates does not close on the fuzzy four-sphere. In such a construction, a fuzzy two sphere is added at each point on the fuzzy four-sphere as extra degrees of freedom. It is interesting that fields on the fuzzy four-sphere have higher spins due to the extra degrees of freedom. We also consider a theory around the north pole and take a flat space limit. A noncommutative gauge theory on four-dimensional plane, which has Heisenberg type noncommutativity, is considered.

1 Introduction

One of the recent interesting developments in string theory is the appreciation of noncommutative geometry. The first paper which points out a relation between string theory and noncommutative geometry is [1]. In the paper, string field theory was formulated in terms of noncommutative geometry. Some studies of D-branes show further relations between string theory and noncommutative geometry. A system of N coincident D-branes is described by the $U(N)$ Yang Mills theory. In this theory, $U(N)$ adjoint scalars represent the transverse coordinates of this system. Since they are given by $U(N)$ matrices, this fact suggests that the spacetime probed by D-branes may be related to noncommutative geometry. Noncommutative geometry also appears within the framework of toroidal compactification of matrix model[2]. It is discussed in [3] that the world volume theory on D-branes with NS-NS two form background is described by noncommutative gauge theory. These studies suggest that noncommutative geometry may play a fundamental role in string theory.

Matrix models are obtained by the dimensional reduction from Yang-Mills theory and concrete models[4, 5] are proposed to study M theory and string theory. IIB Matrix Model[5] is one of these proposals. The action of this model is given by the dimensional reduction of ten-dimensional $\mathcal{N}$=1 $U(N)$ super Yang-Mills theory to a point;

$$S = -\frac{1}{g^2} Tr \left(\frac{1}{4} [A_\mu, A_\nu] [A^\mu, A^\nu] + \frac{1}{2} \bar{\psi} \Gamma^\mu [A_\mu, \psi] \right), \tag{1}$$

where ψ is a ten dimensional Majorana-Weyl spinor field, and A_μ and ψ are $N \times N$ hermitian matrices. Indices μ, ν run over 1 to 10 and they are contracted by Euclidean metric $\delta_{\mu\nu}$. This model is expected to give the constructive definition of type IIB superstring theory[6].

In the matrix model, eigenvalues of bosonic variables are interpreted as spacetime coordinates, and matter and even spacetime may dynamically emerge out of matrices[6, 7]. Spacetime coordinates are represented by matrices and therefore noncommutative geometry is expected to appear. The idea of the noncommutative geometry is to modify the microscopic structure of the spacetime. This modification is implemented by replacing fields on the spacetime by matrices. A flat noncommutative background appears as a classical solution of the IIB matrix model;

$$[\hat{x}_\mu, \hat{x}_\nu] = -iC_{\mu\nu}\mathbf{1}, \tag{2}$$

where $C_{\mu\nu}$ is a constant second rank tensor. This solution preserves a part of supersymmetry. It was shown[8, 9] that noncommutative Yang-Mills theories in a flat background are obtained by expanding the matrix model around a flat noncommutative background;

$$A_\mu = \hat{x}_\mu + \hat{a}_\mu(\hat{x}). \tag{3}$$

Fields on the background appear as fluctuations around the background. This implies the unification of spacetime and fields.

It is important to study curved backgrounds since the matrix model is expected to be the theory of gravity. Especially how general covariance is embedded in the matrix model is a interesting problem. Some attempts to search general covariance in the matrix model are reported in [10, 11, 12] The IIB matrix model has only flat noncommutative backgrounds as classical solutions. In [13], we have considered a three dimensional supersymmetric matrix model with a third-rank Chern-Simons term. This model has a fuzzy two-sphere as a classical solution. A Fuzzy two-sphere[1] is obtained by introducing the coordinates satisfying the following relations,

$$[\hat{x}_i, \hat{x}_j] = i\alpha\epsilon^{ijk}\hat{x}_k, \quad \hat{x}_i\hat{x}_i = \rho^2, \tag{4}$$

where α is a dimensionful constant. This algebra respects $SO(3)$ symmetry. These matrices are constructed from $SU(2)$ algebra. The second condition is automatically satisfied by the quadratic Casimir. We showed in [13] that expanding the model around the fuzzy two-sphere solution leads to a supersymmetric noncommutative gauge theory on the fuzzy sphere. In [15], a four dimensional bosonic matrix model with a mass term was considered. This model has two classical solutions, a fuzzy two-sphere and a fuzzy two-torus. Using this model, we analyzed noncommutative gauge theories on a fuzzy sphere and a fuzzy torus. A fuzzy torus is obtained by introducing two unitary matrices satisfying the following relation,

$$UV = e^{i\theta}VU. \tag{5}$$

It is natural to use the unitary matrices since the eigenvalues of them are distributed over circles. A fuzzy two-sphere and a fuzzy two-torus are constructed from finite dimensional matrices since the size of matrices represents the number of quanta on the noncommutative manifold. On the other hand, a noncommutative plane cannot be constructed by finite size matrices since the extension of the plane is infinite. It is desirable that a classical solution is described by a finite size matrix since N is considered as a cutoff parameter in the matrix model.

As explained in the previous paragraph, N plays the role of cutoff parameter in the case of the compact manifold. Let us consider fuzzy spheres as examples. When we introduce a cutoff parameter $N-1$ for angular momentum in a two-sphere, the number of independent functions is $\sum_{l=0}^{N-1}({}_3H_l - {}_3H_{l-2}) = \sum_{l=0}^{N-1}(2l+1) = N^2$. Then we can replace the functions with $N \times N$ matrices, and algebras on the sphere become noncommutative. A generalization to a higher dimensional sphere is, however, not straightforward. Let us consider a four-dimensional sphere as an example. When we introduce a cutoff parameter n for angular momentum, the number of independent functions is $\sum_{l=0}^{n}({}_5H_l - {}_5H_{l-2}) = (n+1)(n+2)^2(n+3)/12$. This is not a square of an integer. In this case, we cannot construct a map from functions to matrices. We can restate this difficulty from the algebraic point of view. Algebras of a fuzzy four-sphere are constructed in [17], and the difference from the fuzzy two-sphere case is that the commutators of the coordinates do not close in the fuzzy four-sphere case. This fact makes the analyses of field theories on the fuzzy four-sphere difficult.

Recently there are some developments in this fields. In [19], the authors showed that the matrix description of a fuzzy four-sphere is given by $SO(5)/U(2)$ coset and a fuzzy two-sphere is attached to each point on the fuzzy four-sphere. The stabilizer group of this sphere is not $SO(4)$ but $U(2)$. The authors in [20] considered the quantum Hall effect in a four dimensional sphere. It is well known that noncommutative geometry is naturally realized by the guiding center

[1] There are many papers about fuzzy two-sphere. See, for example, [14].

coordinates of the two-dimensional system of electrons in a constant magnetic field. Their system is composed of particles moving in a four-dimensional sphere under the $SU(2)$ gauge field. The existence of Yang's $SU(2)$ monopole[21] in the system makes the coordinates of particles moving in the four-dimensional space noncommutative. They showed that the configuration space of this system is locally $S^4 \times S^2$. There are further analyses in [22, 23, 24] following these papers.

In this paper, we consider noncommutative gauge theory on a fuzzy four-sphere using a matrix model with a fifth-rank Chern-Simons term following these recent developments. In section two, we explain a matrix description of a fuzzy four-sphere based on [17]. In section three, we consider a noncommutative gauge theory on fuzzy four-sphere using the matrix model. We expand the matrices around a classical solution of fuzzy four-sphere by the same way as (3). It is shown that the Hamiltonian of the quantum Hall system on the four-dimensional sphere appears from this matrix model. In section four, we consider a noncommutative gauge theory on a noncommutative four-dimensional plane by taking a flat limit. We use a technique which is similar to Inönü-Wigner contraction. Section five is devoted to summary. We explain our convention for gamma matrices in Appendix A. In Appendix B, a matrix model with a mass term is considered.

Notations

Indices μ, ν, ... and a, b, ... run over 1 to 5 and 1 to 4 respectively. Indices i,j,k run over 1 to 3 and they are used to parameterize internal two-dimensional spheres.

2 Fuzzy four-sphere construction

A fuzzy four-sphere is considered in [16, 17, 25]. In this paper, we use the construction of [17] and briefly review it here. The fuzzy four-sphere is constructed to satisfy the following two conditions,

$$\epsilon^{\mu\nu\lambda\rho\sigma}\hat{x}_\mu\hat{x}_\nu\hat{x}_\lambda\hat{x}_\rho = C\hat{x}_\sigma, \tag{6}$$

and

$$\hat{x}_\mu\hat{x}_\mu = \rho^2, \tag{7}$$

where ρ is a radius of the sphere. This sphere respects $SO(5)$ invariance. Let us define matrices $\hat{G}_\mu$ as follows,

$$\hat{x}_\mu = \alpha\hat{G}_\mu, \tag{8}$$

where α is a dimensionful constant. These matrices are constructed from the n-fold symmetric tensor product of the five dimensional Gamma matrices [2],

$$\hat{G}_\mu^{(n)} = (\Gamma_\mu \otimes 1 \otimes \cdots \otimes 1 + 1 \otimes \Gamma_\mu \otimes \cdots \otimes 1 + \cdots + 1 \otimes 1 \otimes \cdots \otimes \Gamma_\mu)_{Sym}, \tag{9}$$

where Sym means that we are considering the completely symmetrized tensor product. The dimension of this n-fold symmetrized tensor product space, that is the size of these matrices, is calculated as

$$N =_4 H_n =_{n+3} C_n = \frac{1}{6}(n+1)(n+2)(n+3). \tag{10}$$

[2] Our notation is summarized in Appendix A.

If we replace Γ_μ with Pauli matrices, $\hat{G}_\mu$ becomes the coordinates of a fuzzy two-sphere, that is $(n+1)$-dimensional representation of $SU(2)$. After some calculations, we find that these matrices satisfy the following relations,

$$\hat{G}^{(n)}_\mu \hat{G}^{(n)}_\mu = n(n+4) \equiv c, \tag{11}$$

and

$$\epsilon^{\mu\nu\lambda\rho\sigma}\hat{G}^{(n)}_\mu \hat{G}^{(n)}_\nu \hat{G}^{(n)}_\lambda \hat{G}^{(n)}_\rho = \epsilon^{\mu\nu\lambda\rho\sigma}\hat{G}^{(n)}_{\mu\nu}\hat{G}^{(n)}_{\lambda\rho} = (8n+16)\hat{G}^{(n)}_\sigma, \tag{12}$$

where

$$\hat{G}^{(n)}_{\mu\nu} \equiv \frac{1}{2}[\hat{G}^{(n)}_\mu, \hat{G}^{(n)}_\nu]. \tag{13}$$

(12) is also rewritten as

$$\hat{G}^{(n)}_{\mu\nu} = -\frac{1}{2(n+2)}\epsilon^{\mu\nu\lambda\rho\sigma}\hat{G}^{(n)}_{\lambda\rho}\hat{G}^{(n)}_\sigma = -\frac{1}{2(n+2)}\epsilon^{\mu\nu\lambda\rho\sigma}\hat{G}^{(n)}_\lambda \hat{G}^{(n)}_\rho \hat{G}^{(n)}_\sigma. \tag{14}$$

If we take C as

$$C = (8n+16)\alpha^3, \tag{15}$$

(6) is satisfied. From (7) and (11), ρ and α are related by

$$\rho^2 = \alpha^2 n(n+4) \equiv \alpha^2 c. \tag{16}$$

We also have the following relations

$$\hat{G}^{(n)}_{\mu\nu}\hat{G}^{(n)}_\nu = 4\hat{G}^{(n)}_\mu, \tag{17}$$

$$\hat{G}^{(n)}_{\mu\nu}\hat{G}^{(n)}_{\nu\mu} = 4n(n+4) = 4c, \tag{18}$$

and

$$\hat{G}^{(n)}_{\mu\nu}\hat{G}^{(n)}_{\nu\lambda} = c\delta_{\mu\lambda} + \hat{G}^{(n)}_\mu \hat{G}^{(n)}_\lambda - 2\hat{G}^{(n)}_\lambda \hat{G}^{(n)}_\mu. \tag{19}$$

Commutation relations of these matrices are given by

$$[\hat{G}^{(n)}_\mu, \hat{G}^{(n)}_{\nu\lambda}] \quad = \quad 2\left(\delta_{\mu\nu}\hat{G}^{(n)}_\lambda - \delta_{\mu\lambda}\hat{G}^{(n)}_\nu\right), \tag{20}$$

$$[\hat{G}^{(n)}_{\mu\nu}, \hat{G}^{(n)}_{\lambda\rho}] = 2\left(\delta_{\nu\lambda}\hat{G}^{(n)}_{\mu\rho} + \delta_{\mu\rho}\hat{G}^{(n)}_{\nu\lambda} - \delta_{\mu\lambda}\hat{G}^{(n)}_{\nu\rho} - \delta_{\nu\rho}\hat{G}^{(n)}_{\mu\lambda}\right). \tag{21}$$

These form the $SO(5,1)$ algebra.

Here we comment on the physical meaning of N and n in brane interpretations. The authors of [17] discussed this fuzzy four-sphere as a longitudinal five-brane in the context of BFSS matrix model. They showed that N and n represents the number of D-particles and longitudinal five-branes respectively. We are considering n overlapping longitudinal five-branes.

The area occupied by the unit quantum on *the fuzzy four-sphere* is

$$\hbar = \frac{\frac{8}{3}\pi^2\rho^4 n}{N} = \frac{16\pi^2\rho^4 n}{(n+1)(n+2)(n+3)}, \tag{22}$$

where $8\pi^2\rho^4/3$ is a area of the four-sphere, and we have N quanta in this system. We must draw attention to the factor n. As discussed in the next section, a fuzzy two-sphere is attached to each point on the four-sphere[19], and there are n quanta on the fuzzy two-sphere. Therefore the number of the quanta on the fuzzy four-sphere is N/n. This is a different feature from the fuzzy two-sphere (4) (or the fuzzy plane (2)). There is a degree of freedom at each point and there are N points on the fuzzy two-sphere. On the other hand, there are $n \sim N^{\frac{1}{3}}$ degrees of freedom at each point and there are $n^2 \sim N^{\frac{2}{3}}$ points on the fuzzy four-sphere. From the viewpoint of a noncommutative field theory on the fuzzy four-sphere, n is interpreted as spin degree of freedom. (We explain this point in the next section.) Fields on the fuzzy four-sphere have spin and the rank of it can be up to n. Only N out of N^2 degrees of freedom are assigned at each point and the remaining $N^2 - N$ degrees of freedom are expected to correspond to nonlocal degrees of freedom. If we take large n limit with fixed ρ, a classical sphere is recovered.

Since we obtained the fuzzy four-sphere geometry, the next step is to investigate a field theory on it. The fact that the algebra of $\hat{x}_\mu$ does not, however, close makes investigations difficult. Functions on a usual classical four-sphere can be expanded by the spherical harmonics

$$a(x) = \sum_{l=0}^{\infty} \sum_{m_i} a_{lm_i} Y_{lm_i}(x), \tag{23}$$

where the spherical harmonics is given by

$$Y_{lm_i}(x) = \frac{1}{\rho^l} \sum_a f^{(lm_i)}_{a_1,\ldots,a_l} x^{a_1} \cdots x^{a_l}, \tag{24}$$

where m_i denote relevant quantum numbers. $f_{a_1,\ldots,a_l}$ is a traceless and symmetric tensor. The traceless condition comes form $x_i x_i = \rho^2$. Matrices corresponding to the above functions are

$$\hat{a}(\hat{x}) = \sum_{l=0} \sum_{m_i} a_{lm_i} \hat{Y}_{lm_i}(\hat{x}), \tag{25}$$

where

$$\hat{Y}_{lm_i}(\hat{x}) = \frac{1}{\rho^l} \sum_a f^{(lm_i)}_{a_1,\ldots,a_l} \hat{x}^{a_1} \cdots \hat{x}^{a_l}. \tag{26}$$

Due to the relation (13), algebra among the matrix spherical harmonics does not close. This problem does not appear in the fuzzy two-sphere case. To overcome this difficulty, we have two strategies. The first one is to project out $\hat{G}_{\mu\nu}$[18] and the second one is to include $\hat{G}_{\mu\nu}$ whose counterpart in a usual classical sphere does not exist. In [18] a product which closes without $\hat{G}_{\mu\nu}$ is constructed. This product is, however, non-associative. Since the matrix algebra has associativity, we want to maintain associativity to use matrix models. On the other hand, if we include $\hat{G}_{\mu\nu}$ to maintain associativity, the geometry which is constructed from $\hat{G}_\mu$ and $\hat{G}_{\mu\nu}$ becomes the coset manifold $SO(5)/U(2)$[19]. This coset is not S^4 but locally $S^4 \times S^2$. Throughout this paper, we call a noncommutative space which is given by this coset a fuzzy four-sphere. In the next section, we consider a noncommutative gauge theory on a fuzzy four-sphere using a matrix model .

3 Matrix model and Noncommutative gauge theory on fuzzy four-sphere

To investigate a noncommutative gauge theory on a fuzzy four-sphere, we consider the following matrix model

$$S = -\frac{1}{g^2} Tr\left(\frac{1}{4}[A_\mu, A_\nu][A_\mu, A_\nu] + \frac{\lambda}{5}\epsilon^{\mu\nu\lambda\rho\sigma} A_\mu A_\nu A_\lambda A_\rho A_\sigma\right), \tag{27}$$

where $\mu, \nu, \ldots, \sigma$ run over 1 to 5 and $\epsilon^{\mu\nu\lambda\rho\sigma}$ is the $SO(5)$ invariant tensor. A_μ are $N \times N$ hermitian matrices and λ is a dimensionful constant which depends on N. The indices are contracted by the Euclidean metric $\delta_{\mu\nu}$. Our discussions are restricted only to the bosonic sector. This is a reduced model of Yang-Mills action with a fifth rank Chern-Simons term. The second term is also interpreted as a so called Myers term[26]. In this interpretation, this action represents an effective action of D(-1)-branes in a constant R-R four-form background. Here we do not make use of such an interpretation.

This model has the $SO(5)$ symmetry and the following unitary symmetry;

$$A_\mu \to U A_\mu U^\dagger. \tag{28}$$

It also has the translation symmetry;

$$A_\mu \to A_\mu + c_\mu \mathbf{1}. \tag{29}$$

The equation of motion of this action is as follows,

$$[A_\nu, [A_\mu, A_\nu]] + \lambda \epsilon^{\mu\nu\lambda\rho\sigma} A_\nu A_\lambda A_\rho A_\sigma = 0. \tag{30}$$

There are two classical solutions, firstly diagonal commuting matrices

$$A_\mu = diag(x_\mu^{(N)}, \ldots, x_\mu^{(1)}), \tag{31}$$

and secondly a fuzzy four-sphere

$$A_\mu = \hat{x}_\mu = \alpha \hat{G}_\mu^{(n)}. \tag{32}$$

λ is determined by the condition that the matrix model has a classical solution of the fuzzy four-sphere. We easily find from (20) that λ is determined as the following value,

$$\lambda = \frac{2}{\alpha(n+2)}. \tag{33}$$

We should notice that a system of two four-spheres is *not* a classical soliton.

Although we add the Chern-Simons term to the Yang-Mills action, a reduced model of the Yang-Mills action with a mass term also has a fuzzy four-sphere as a classical solution,

$$S = -\frac{1}{g^2} Tr\left(\frac{1}{4}[A_\mu, A_\nu][A_\mu, A_\nu] + 8\alpha^2 A_\mu A_\mu\right). \tag{34}$$

In this case, both of a two-sphere and a two-torus are also classical solutions. A matrix model with a mass term is investigated in [15]. In this model, a system of multiple fuzzy four-spheres is a classical solution. We give some comments on this model in Appendix B.

We compare classical values of the action for two classical solutions. The value of the action (27) for (31) is

$$S = 0 \tag{35}$$

while

$$S = -\frac{36\alpha^2\rho^2 N}{5g^2} = -\frac{36\rho^4}{5g^2}\frac{N}{n(n+4)} \tag{36}$$

for (32). We conclude that the fuzzy four-sphere solution is more stable than the diagonal commuting matrices at the classical level.

Let us consider noncommutative gauge theory on the fuzzy four-sphere. The idea is to expand the matrices around the classical solution as in [9, 13, 15]. We expand the matrices as follows,

$$A_\mu = \hat{x}_\mu + \alpha\rho\hat{a}_\mu = \alpha\rho\left(\frac{1}{\rho}\hat{G}_\mu + \hat{a}_\mu\right). \tag{37}$$

We define $\hat{w}_{\mu\nu}$ as

$$\hat{w}_{\mu\nu} \equiv i\alpha G_{\mu\nu} = \frac{i\alpha}{2}[\hat{G}_\mu, \hat{G}_\nu] \tag{38}$$

where α has dimension of length. i is added to make $\hat{w}_{\mu\nu}$ hermitian. We have the following noncommutativity on the fuzzy four-sphere

$$[\hat{x}_\mu, \hat{x}_\nu] = -2i\alpha\hat{w}_{\mu\nu}, \tag{39}$$

where $\hat{w}_{\mu\nu}$ satisfies

$$\epsilon^{\mu\nu\lambda\rho\sigma}\hat{w}_{\mu\nu}\hat{w}_{\lambda\rho} = -\alpha(8n+16)\hat{x}_\sigma. \tag{40}$$

We now comment on a classical sphere. It is obtained by a large n limit with the fixed radius of the sphere ρ. In other words, it is the $\alpha \to 0$ limit with the fixed ρ. From (39), the coordinates commute each other in this limit:

$$[\hat{x}_\mu, \hat{x}_\nu] = -2i\alpha\hat{w}_{\mu\nu} \sim O(\alpha\rho) \to 0. \tag{41}$$

The coordinates $\hat{x}_\mu$ and $\hat{w}_{\mu\nu}$ also become commuting matrices:

$$[\hat{x}_\mu, \hat{w}_{\nu\lambda}] = 0, \tag{42}$$

$$[\hat{w}_{\mu\nu}, \hat{w}_{\lambda\rho}] = 0. \tag{43}$$

Fields on a sphere are expanded by the spherical harmonics. A Fuzzy sphere is naturally introduced by giving a cutoff parameter for angular momentum. The spherical harmonics on the higher dimensional fuzzy sphere is considered in [18, 23]. The bases are classified by the $SO(5)$ representations and the matrices are expanded by the irreducible representations of $SO(5)$. The irreducible representation is characterized by the Young diagram. It is labeled by the row length (r_1, r_2) in this case. [3] Only the representations with $r_2 = 0$ correspond to the classical sphere.

[3] The representation of $SO(5)$ is summarized in [18].

Summing up the dimensions of all irreducible representations with the condition $n \geq r_1 \geq r_2$ leads to the square of N. We write the spherical harmonics abstractly as follows,

$$\hat{a}(\hat{x},\hat{w}) = \sum_{r_1=0}^{n} \sum_{r_2,\tilde{m}_i} a_{r_1 r_2 \tilde{m}_i} \hat{Y}_{r_1 r_2 \tilde{m}_i}(\hat{x},\hat{w}), \tag{44}$$

where $\tilde{m}_i$ denote relevant quantum numbers. It is important that we have a cutoff parameter for angular momentum r_1 at n[18]. If we set $w_{\mu\nu} = 0$, $\hat{Y}_{r_1,\tilde{m}_i}$ becomes the usual spherical harmonics (26). We need to assume that the fields depend not only on $\hat{x}$ but also on $\hat{w}$. If we consider a function corresponding to the above matrix,

$$a(x,w) = \sum_{r_1=0}^{n} \sum_{r_2,\tilde{m}_i} a_{r_1 r_2 \tilde{m}_i} Y_{r_1 r_2 \tilde{m}_i}(x,w), \tag{45}$$

a product of fields becomes noncommutative and associative. We note that the noncommutativity is produced by $\hat{w}_{\mu\nu}$. In this construction, $\hat{w}_{\mu\nu}$ form fuzzy two-sphere algebras. This fact means that the noncommutativity on the fuzzy four-sphere is produced by the fuzzy two-sphere, as it will be shown later.

When we consider a field theory corresponding to the matrix model around the noncommutative background, an adjoint action of $\hat{G}_\mu$ is expected to become the following derivative operator;

$$Ad\left(\hat{G}_\mu\right) \to -2i\left(w_{\mu\nu}\frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial w_{\mu\nu}}\right), \tag{46}$$

and adjoint action of $\hat{G}_{\mu\nu}$ becomes

$$Ad\left(\hat{G}_{\mu\nu}\right) \to 2\left(x_\mu \frac{\partial}{\partial x_\nu} - x_\nu \frac{\partial}{\partial x_\mu} - w_{\mu\lambda}\frac{\partial}{\partial w_{\lambda\nu}} + w_{\nu\lambda}\frac{\partial}{\partial w_{\lambda\mu}}\right), \tag{47}$$

where derivative of $w_{\mu\nu}$ is defined as

$$\frac{\partial w_{\lambda\rho}}{\partial w_{\mu\nu}} = \delta_{\mu\lambda}\delta_{\nu\rho} - \delta_{\mu\rho}\delta_{\nu\lambda}. \tag{48}$$

The first two terms in (47) correspond to orbital parts and the last two terms correspond to *isospin* parts.

We next show that a fuzzy two-sphere is attached to each point on the fuzzy four-sphere[19] and it leads to spin of fields. We can always diagonalize a matrix $\hat{G}_\mu$ out of the five matrices. We diagonalize $\hat{x}_5 = \alpha \hat{G}_5$ as in Appendix A. We can construct the $SU(2) \times SU(2)$ algebra from the $SO(4)$ algebra which is a sub-algebra of the $SO(5)$ algebra;

$$[\hat{N}_i, \hat{N}_j] = i\epsilon_{ijk}\hat{N}_k, \tag{49}$$

$$[\hat{M}_i, \hat{M}_j] = i\epsilon_{ijk}\hat{M}_k, \tag{50}$$

$$[\hat{M}_i, \hat{N}_j] = 0, \tag{51}$$

where

$$\begin{aligned}
\hat{N}_1 &= -\frac{i}{4}\left(\hat{G}_{23}-\hat{G}_{14}\right), & \hat{M}_1 = -\frac{i}{4}\left(\hat{G}_{23}+\hat{G}_{14}\right),\\
\hat{N}_2 &= \frac{i}{4}\left(\hat{G}_{13}+\hat{G}_{24}\right), & \hat{M}_2 = \frac{i}{4}\left(\hat{G}_{13}-\hat{G}_{24}\right),\\
\hat{N}_3 &= -\frac{i}{4}\left(\hat{G}_{12}-\hat{G}_{34}\right), & \hat{M}_3 = -\frac{i}{4}\left(\hat{G}_{12}+\hat{G}_{34}\right).
\end{aligned} \tag{52}$$

G_{ab} is written as

$$\begin{aligned}
&\hat{G}_{23} = 2i(\hat{N}_1+\hat{M}_1), & \hat{G}_{14} = -2i(\hat{N}_1-\hat{M}_1),\\
&\hat{G}_{13} = -2i(\hat{N}_2+\hat{M}_2), & \hat{G}_{24} = -2i(\hat{N}_2-\hat{M}_2),\\
&\hat{G}_{12} = 2i(\hat{N}_3+\hat{M}_3), & \hat{G}_{34} = -2i(\hat{N}_3-\hat{M}_3).
\end{aligned} \tag{53}$$

The Casimir of each $SU(2)$ algebra is calculated from (12), (18) and (19) as follows,

$$\begin{aligned}
\hat{N}_i\hat{N}_i &= \frac{1}{16}\left(c+(2n+4)G_5+G_5^2\right)\\
&= \frac{1}{16}\left(n+G_5\right)\left(n+4+G_5\right),
\end{aligned} \tag{54}$$

and

$$\begin{aligned}
\hat{M}_i\hat{M}_i &= \frac{1}{16}\left(c-(2n+4)G_5+G_5^2\right)\\
&= \frac{1}{16}\left(n-G_5\right)\left(n+4-G_5\right).
\end{aligned} \tag{55}$$

Matrices $\hat{M}_i$ and $\hat{N}_i$ are realized by $(n+G_5+2)/2$ and $(n-G_5+2)/2$ dimensional representation [4] of $SU(2)$ respectively. The point $\hat{G}_5 = G_5$ consists of $(n+G_5+2)\cdot(n-G_5+2)/4$ eigenvalues. If we sum up the contributions from $G_5 = n, n-2, \ldots, -n+2, -n$ as

$$\sum_{G_5=-n}^{G_5=n}\left(\frac{n+G_5+2}{2}\right)\cdot\left(\frac{n-G_5+2}{2}\right) = \frac{1}{6}(n+1)(n+2)(n+3), \tag{56}$$

we obtain the size N of the matrix .

At the north pole, the Casimirs of $\hat{N}_i$ and $\hat{M}_i$ are given by

$$\hat{N}_i\hat{N}_i = \frac{n(n+2)}{4} \tag{57}$$

and

$$\hat{M}_i\hat{M}_i = 0 \tag{58}$$

respectively. Then we have a fuzzy two-sphere, which is given by the $(n+1)$-dimensional representation of $SU(2)$, at the north pole. The radius of the two-sphere is given by $\sigma^2 = \alpha^2 n(n+4)/4$ and it is comparable with that of the four-sphere, which is given by $\rho^2 = \alpha^2 n(n+4)$. Since the

[4] $tr\hat{N}_i$ and $tr\hat{M}_i$ represent two-brane charge in string theory interpretation. Since these are given by finite dimensional $SU(2)$ matrices, two-brane charge vanishes on each point on four-sphere.

fuzzy four-sphere has $SO(5)$ symmetry, we can state that a fuzzy two-sphere, which is given by the $(n+1)$-dimensional representation of $SU(2)$, is attached to each point on the fuzzy four-sphere. We can regard this two-sphere as the internal two-dimensional space. Fields have quantum numbers corresponding to the $SU(2)$ angular momentum. We next show that these extra degrees of freedom can be interpreted as spins. This mechanism is similar to the *Kalza-Klein* compactification mechanism.

Generators of Lorentz transformation are given by $\hat{G}_{ab}$. Fields are transformed under the Lorentz transformation as follows

$$\begin{aligned}
& e^{i\hat{G}_{ab}\omega_{ab}}\hat{a}(\hat{x},\hat{w})e^{-i\hat{G}_{ab}\omega_{ab}} \\
= \; & \hat{a}(\hat{x},\hat{w}) + i\omega_{ab}Ad(G)_{ab}\hat{a}(\hat{x},\hat{w}) \\
\to \; & a(x,w) + 2i\omega_{ab}\left(x_a\frac{\partial}{\partial x_b} - x_b\frac{\partial}{\partial x_a} - w_{ac}\frac{\partial}{\partial w_{cb}} + w_{bc}\frac{\partial}{\partial w_{ca}}\right)a(x,w) \\
= \; & a(x,w) + 2i\omega_{ab}\left(x_a\frac{\partial}{\partial x_b} - x_b\frac{\partial}{\partial x_a}\right)a(x,w) - 4i(\theta_i+\omega_i)\epsilon_{ijk}N_j\frac{\partial}{\partial N_k}a(x,w)
\end{aligned} \tag{59}$$

where $\theta_i = (\omega_{23},\omega_{31},\omega_{12})$ and $\omega_i = (\omega_{41},\omega_{42},\omega_{43})$. Transformation from the third line to the fourth line is done at the north pole. It must be noted that such a rewriting is valid on each point on the sphere. The second term in the last equation shows the angular momentum part. The third term shows that the fields have spin angular momentum. The point is that the $SU(2)$ spin takes only the integer values, 0, 1, $\cdots$, $n-1$, n. The rank of the spin is finite since the sizes of $SU(2)$ matrices $\hat{N}_i$ are $n+1$.

When we do Taylor expansion a field with respect to coordinates N_i,

$$\begin{aligned}
a(x,w) &= a(x,0) + N_{i_1}\left.\frac{\partial a(x,N)}{\partial N_{i_1}}\right|_{N=0} + \cdots + \frac{1}{n!}N_{i_1}N_{i_2}\cdots N_{i_n}\left.\frac{\partial^n a(x,N)}{\partial N_{i_1}\partial N_{i_2}\cdots\partial N_{i_n}}\right|_{N=0} \\
&\equiv a(x) + N_{i_1}\tilde{a}_{i_1}(x) + \cdots + \frac{1}{n!}N_{i_1}N_{i_2}\cdots N_{i_n}\tilde{a}_{i_1,i_2,\ldots,i_n}(x)
\end{aligned} \tag{60}$$

the first term is a scalar field and the $m+1$-th term represents a spin m field. This expansion is done at the north pole. Because the noncommutativity is produced by the fuzzy two-sphere, the product between the scalar field has the spin degrees of freedom. If we remove the fuzzy two-sphere from the four-sphere, the product becomes commutative. Such a product is considered in [18] and it is commutative and *non-associative*.

We now consider an action of noncommutative gauge theory on the fuzzy four-sphere. It is obtained form the matrix model action (27) by expanding matrices around the classical solution corresponding to the fuzzy four-sphere as in (37);

$$\begin{aligned}
S = & -\frac{(\alpha\rho)^4}{g^2}Tr\left(\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}_{\mu\nu} - \frac{9\lambda}{40(\alpha\rho)^2}\epsilon^{\mu\nu\lambda\rho\sigma}[A_\mu,A_\nu][A_\lambda,A_\rho]A_\sigma\right. \\
& \left. - \frac{\lambda^2}{16(\alpha\rho)^2}f^{\mu\nu\lambda\rho\sigma\tau}[A_\mu,A_\nu]A_\lambda[A_\rho,A_\sigma]A_\tau\right),
\end{aligned} \tag{61}$$

where

$$f^{\mu\nu\lambda\rho\sigma\tau} = \epsilon^{\alpha\beta\mu\nu\lambda}\epsilon^{\alpha\beta\rho\sigma\tau}$$

$$= \quad 2\delta_{\mu\rho}\left(\delta_{\nu\sigma}\delta_{\lambda\tau}-\delta_{\nu\tau}\delta_{\lambda\sigma}\right)-2\delta_{\mu\sigma}\left(\delta_{\nu\rho}\delta_{\lambda\tau}-\delta_{\nu\tau}\delta_{\lambda\rho}\right)+2\delta_{\mu\tau}\left(\delta_{\nu\rho}\delta_{\lambda\sigma}-\delta_{\lambda\rho}\delta_{\nu\sigma}\right). \quad (62)$$

$\hat{F}_{\mu\nu}$ is a gauge covariant field strength, which is defined by

$$\begin{aligned}
\hat{F}_{\mu\nu} &\equiv \frac{1}{(\alpha\rho)^2}\left([A_\mu, A_\nu]+\lambda\epsilon^{\mu\nu\lambda\rho\sigma}A_\lambda A_\rho A_\sigma\right)\\
&= \frac{1}{(\alpha\rho)^2}\left([A_\mu, A_\nu]+\frac{1}{2}\lambda\epsilon^{\mu\nu\lambda\rho\sigma}[A_\lambda, A_\rho]A_\sigma\right)\\
&= \left[\frac{1}{\rho}\hat{G}_\mu, \hat{a}_\nu\right]-\left[\frac{1}{\rho}\hat{G}_\nu, \hat{a}_\mu\right]+[\hat{a}_\mu, \hat{a}_\nu]\\
&\quad +\alpha\rho\lambda\epsilon^{\mu\nu\lambda\rho\sigma}\left(\frac{1}{\rho^2}\hat{G}_{\lambda\rho}\hat{a}_\sigma+\left[\frac{1}{\rho}\hat{G}_\lambda, \hat{a}_\rho\right]\left(\frac{1}{\rho}\hat{G}_\sigma+\hat{a}_\sigma\right)\right).
\end{aligned} \quad (63)$$

Since A_μ is a covariant quantity (we explain in the next paragraph), the gauge covariance of $\hat{F}_{\mu\nu}$ is manifest. The second and third terms in (61) give gauge invariant interaction terms.

The gauge symmetry in this noncommutative gauge theory comes from the unitary symmetry in the matrix model. For an infinitesimal transformation $U=\exp(i\hat{\lambda}) \sim 1+i\hat{\lambda}$ in (28), a fluctuation around the fixed background transforms as

$$\delta\hat{a}_\mu(\hat{x},\hat{w}) = -\frac{i}{\rho}[\hat{G}_\mu, \hat{\lambda}(\hat{x},\hat{w})]+i[\hat{\lambda}(\hat{x},\hat{w}), \hat{a}_\mu(\hat{x},\hat{w})]. \quad (64)$$

The corresponding transformation in the field theory is

$$\delta a_\mu(x,w) = \frac{2}{\rho}\left(w_{\mu\nu}\frac{\partial}{\partial x_\nu}-x_\nu\frac{\partial}{\partial w_{\mu\nu}}\right)\lambda(x,w)+i[\lambda(x,w), a_\mu(x,w)]_\star. \quad (65)$$

It is known that this gauge transformation contains many degrees of freedom which do not exist in a ordinary commutative gauge theory. When we expand $\hat{\lambda}$ as

$$\hat{\lambda} = \lambda_0+\epsilon^\mu\hat{G}_\mu+\epsilon^{\mu\nu}\hat{G}_{\mu\nu}+O\left(G^2\right), \quad (66)$$

the contribution from the second term gives the rotation (59) and the constant shift of the field in (64). (If we consider the theory around the north pole, $\hat{G}_{ab}$ and $\hat{G}_{5a}$ give rotation and translation respectively.)

Trace in the matrix model action (61) corresponds to the integration on the coset in the corresponding noncommutative field theory. It is noteworthy that this integral is taken over six-dimensional space, the fuzzy four-sphere and the internal two-dimensional space.

We now discuss the Laplacian. It is natural to use $Ad(\hat{G}_{\mu\nu})^2$ as the Laplacian since it is the quadratic Casimir of $SO(5)$ and it is the generator of the rotation. We, however, have another choice $Ad(\hat{G}_\mu)^2$. $\hat{G}_\mu$ and $\hat{G}_{\mu\nu}$ form $SO(5,1)$ algebra. Both of $Ad(\hat{G}_\mu)^2$ and $Ad(\hat{G}_{\mu\nu})^2$ are the invariants of $SO(5)$. From the viewpoint of the matrix model, $Ad(\hat{G}_\mu)^2$ appears naturally as the Laplacian since we are expanding the matrices around the coordinates $\hat{x}_\mu=\alpha\hat{G}_\mu$. After adding a gauge fixing term, the kinetic term becomes

$$S_{kinetic} = \frac{(\alpha\rho)^4}{2g^2}Tr\left(\hat{a}_\nu\left[\frac{\hat{G}_\mu}{\rho},\left[\frac{\hat{G}_\mu}{\rho},\hat{a}_\nu\right]\right]\right)$$

$$
\begin{aligned}
= & -\frac{2(\alpha\rho)^4}{g^2} Tr \left(\hat{a}_\tau \left(\frac{\partial^2}{\partial x_\mu \partial x_\mu} - \frac{x_\mu x_\nu}{\rho^2} \frac{\partial^2}{\partial x_\mu \partial x_\nu} - \frac{4x_\mu}{\rho^2} \frac{\partial}{\partial x_\mu} \right.\right. \\
& \left.\left. -\frac{2w_{\mu\nu}x_\lambda}{\rho^2} \frac{\partial^2}{\partial x_\nu \partial w_{\mu\lambda}} + \frac{x_\nu x_\lambda}{\rho^2} \frac{\partial^2}{\partial w_{\mu\nu} \partial w_{\mu\lambda}} - \frac{w_{\mu\lambda}}{\rho^2} \frac{\partial}{\partial w_{\mu\lambda}} \right) \hat{a}_\tau \right). \qquad (67)
\end{aligned}
$$

The first three terms correspond to the usual Laplacian of a four-sphere. Let us investigate the spectrum of the kinetic term. It is calculated as follows,

$$
\frac{1}{4}[\hat{G}_\mu, [\hat{G}_\mu, \hat{Y}_{r_1 r_2}]] = (r_1(r_1+3) - r_2(r_2+1))\, \hat{Y}_{r_1 r_2}. \qquad (68)
$$

We can also calculate the spectrum of $Ad(\hat{G}_{\mu\nu})^2$ as follows,

$$
-\frac{1}{8}[\hat{G}_{\mu\nu}, [\hat{G}_{\mu\nu}, \hat{Y}_{r_1 r_2}]] = (r_1(r_1+3) + r_2(r_2+1))\, \hat{Y}_{r_1 r_2}. \qquad (69)
$$

The second term is regarded as a mass term from the four-dimensional point of view. $Ad(\hat{G}_\mu)^2$ is related to $Ad(\hat{G}_{\mu\nu})^2$ as

$$
\frac{1}{4}[\hat{G}_\mu, [\hat{G}_\mu, \hat{Y}_{r_1 r_2}]] = -\frac{1}{8}[\hat{G}_{\mu\nu}, [\hat{G}_{\mu\nu}, \hat{Y}_{r_1 r_2}(\hat{G}_\mu, \hat{G}_{\mu\nu})]] - 2r_2(r_2+1)\hat{Y}_{r_1 r_2}. \qquad (70)
$$

This eigenvalue is identical with the eigenvalue of the Hamiltonian which appeared in the context of the quantum Hall system[20]. The Hamiltonian describes the system of a single particle moving on the four-dimensional sphere under the $SU(2)$ instanton background. $Ad(\hat{G}_\mu)$ is interpreted as the covariant derivative under the instanton background.

We have so far discussed the $U(1)$ noncommutative gauge theory on the fuzzy sphere. A generalization to $U(m)$ gauge group is realized by the following replacement:

$$
\hat{x}_\mu \to \hat{x}_\mu \otimes \mathbf{1}_m. \qquad (71)
$$

$\hat{a}$ is also replaced as follows:

$$
\hat{a} \to \sum_{a=1}^{m^2} \hat{a}^a \otimes T^a, \qquad (72)
$$

where $T^a (a = 1, \cdots, m^2)$ denote the generators of $U(m)$.

4 Noncommutative gauge theory on noncommutative four-plane

In this section, we consider a noncommutative gauge theory on a noncommutative four-plane, which arises as a flat limit from the fuzzy four-sphere. This limit corresponds to considering a subspace in this system. Let us consider a theory around the north pole, that is $\hat{x}_5 \sim \rho$ ($\hat{G}_5 \sim \rho/\alpha \sim n$). By virtue of the $SO(5)$ symmetry, this discussion is without loss of generality.

As discussed in (57) and (58), a fuzzy two-sphere, which is given by the $(n+1)$-dimensional representation of $SU(2)$, is attached to the north pole. The commutation relation of $\hat{G}_a$ is rewritten as

$$
[\hat{G}_a, \hat{G}_b] = 2\hat{G}_{ab} = 4i\eta^i_{ab}\hat{N}_i. \qquad (73)
$$

The t' Hooft symbol η^i_{ab} [5] is introduced here as

$$\eta^i_{ab} = \epsilon_{iab4} - \delta_{ia}\delta_{4b} + \delta_{ib}\delta_{4a} \tag{74}$$

where $i = 1, 2, 3$ and $a, b = 1, 2, 3, 4$. It is interesting that the noncommutativity has an internal $SU(2)$ index, and it is used to label the spin indices of the fields. This equation (73) also appeared in [20]. If we use the t' Hooft symbol, the derivative of G_{ab} is rewritten as

$$\frac{\partial}{\partial G_{ab}} = -\frac{i}{2}\eta^i_{ab}\frac{\partial}{\partial N_i}. \tag{75}$$

Commutation relations (20) and (21) are rewritten around the north pole as follows,

$$\begin{aligned}
&[\hat{G}_{ab}, \hat{G}_c] = 2\left(\delta_{bc}\hat{G}_a - \delta_{ac}\hat{G}_b\right), \\
&[\hat{G}_{5a}, \hat{G}_b] = 2\delta_{ab}n,
\end{aligned} \tag{76}$$

$$\begin{aligned}
&[\hat{G}_{ab}, \hat{G}_{cd}] = 2\left(\delta_{bc}\hat{G}_{ad} + \delta_{ad}\hat{G}_{bc} - \delta_{ac}\hat{G}_{bd} - \delta_{bd}\hat{G}_{ac}\right), \\
&[\hat{G}_{5a}, \hat{G}_{5b}] = -2\hat{G}_{ab}, \\
&[\hat{G}_{5a}, \hat{G}_{bc}] = 2\left(\delta_{ab}\hat{G}_{5c} - \delta_{ac}\hat{G}_{5b}\right).
\end{aligned} \tag{77}$$

It is natural to regard $\hat{G}_{5a}$ as the momentum matrices since they are canonical conjugate to $\hat{G}_a$. If we define $\hat{p}_a = \alpha^{-1}i\hat{G}_{5a}$, the commutation relations become

$$\begin{aligned}
&[\hat{x}_a, \hat{x}_b] = 2\alpha^2\hat{G}_{ab}, \\
&[\hat{G}_{ab}, \hat{x}_c] = 2\left(\delta_{bc}\hat{x}_a - \delta_{ac}\hat{x}_b\right), \\
&[\hat{p}_a, \hat{p}_b] = 2\alpha^{-2}\hat{G}_{ab}, \\
&[\hat{p}_a, \hat{G}_{bc}] = 2\left(\delta_{ab}\hat{p}_c - \delta_{ac}\hat{p}_b\right) \\
&[\hat{p}_a, \hat{x}_b] = 2i\delta_{ab}n.
\end{aligned} \tag{78}$$

From (19), we obtain

$$\alpha^2\hat{p}_a\hat{p}_a = \frac{\hat{x}_a\hat{x}_a}{\alpha^2}. \tag{79}$$

Momentum space also form the fuzzy four-sphere. Note that momentum matrices and coordinate matrices are different on the fuzzy four-sphere. In the fuzzy two-sphere case, they are given by the same matrices. They are different in general and we can see such examples in [27, 28, 29].

In order to take a large radius limit, we rescale matrices $\hat{G}_a$, $\hat{G}_{ab}$ and $\hat{G}_{a5}$ as

$$\hat{G}'_a = \frac{1}{\sqrt{n}}\hat{G}_a, \quad \hat{G}'_{ab} = \frac{1}{n}\hat{G}_{ab}, \quad \hat{G}'_{a5} = \frac{1}{\sqrt{n}}\hat{G}_{a5}. \tag{80}$$

It is natural to do the above rescaling since (13) is not changed by this rescaling:

$$[\hat{G}'_a, \hat{G}'_b] = 2\hat{G}'_{ab}, \tag{81}$$

[5] η^i_{ab} satisfies $\eta^i_{ab}\eta^j_{ab} = 4\delta_{ij}$ and $\eta^i_{ab}\eta^i_{ac} = 3\delta_{bc}$

or

$$\epsilon^{abcd}\hat{G}'_a\hat{G}'_b\hat{G}'_c\hat{G}'_d = \epsilon^{abcd}\hat{G}'_{ab}\hat{G}'_{cd} = 8. \tag{82}$$

After the rescaling, the commutation relations become as follows,

$$\begin{aligned} [\hat{G}'_a, \hat{G}'_{bc}] &= \frac{2}{n}\left(\delta_{ac}\hat{G}'_b - \delta_{ab}\hat{G}'_c\right), \\ [\hat{G}'_{5a}, \hat{G}'_b] &= 2\delta_{ab}, \end{aligned} \tag{83}$$

and

$$\begin{aligned} [\hat{G}'_{ab}, \hat{G}'_{cd}] &= \frac{2}{n}\left(\delta_{bc}\hat{G}'_{ad} + \delta_{ad}\hat{G}'_{bc} - \delta_{ac}\hat{G}'_{bd} - \delta_{bd}\hat{G}'_{ac}\right), \\ [\hat{G}'_{5a}, \hat{G}'_{bc}] &= \frac{2}{n}\left(\delta_{ab}\hat{G}'_{5c} - \delta_{ac}\hat{G}'_{5b}\right). \end{aligned} \tag{84}$$

The radius of the four-sphere in the rescaled coordinate is

$$\rho'^2 = \hat{x}'_i\hat{x}'_i = \alpha^2\frac{n(n+4)}{n} = \frac{1}{n}\rho^2 \sim \alpha^2 n. \tag{85}$$

Area of each quantum in the rescaled coordinates is

$$\hbar = \frac{\frac{8}{3}\pi^2\rho'^4 n}{N} = \frac{16\pi^2\rho'^4 n}{(n+1)(n+2)(n+3)} \sim 16\pi^2\alpha^4. \tag{86}$$

After rescaling, α represents a noncommutative scale. To decompactify the four-sphere, we will take α =fixed and $\rho' \to \infty$ (or $n \to \infty$) limit.

From (17), we have

$$\hat{G}_{a5}\hat{G}_5 = 4\hat{G}_a - \hat{G}_{ab}\hat{G}_b. \tag{87}$$

If we use this equation, $\hat{G}_{a5}$ is written in terms of $\hat{G}_a$ and $\hat{G}_{ab}$. Independent matrices are now $\hat{G}_a$ and $\hat{G}_{ab}$.

We consider the adjoint actions of $\hat{G}_a$ and $\hat{G}_{ab}$ around the north pole,

$$\begin{aligned} Ad(\hat{G}_a) &= \frac{2}{i}\left(\sqrt{n}w'_{ab}\frac{\partial}{\partial x'_b} - \frac{1}{\sqrt{n}}x'_b\frac{\partial}{\partial w'_{ab}}\right) \\ &= \frac{2}{i}\rho'\left(\frac{1}{\alpha}w'_{ab}\frac{\partial}{\partial x'_b} - \frac{\alpha}{\rho'^2}x'_b\frac{\partial}{\partial w'_{ab}}\right) \\ &= \frac{2}{i}\rho'\left(\frac{1}{\alpha}w'_{ab}\frac{\partial}{\partial x'_b} + \frac{1}{2\rho'^2}x'_b\eta^i_{ab}\frac{\partial}{\partial N'_i}\right) \end{aligned} \tag{88}$$

and

$$\begin{aligned} Ad\left(\hat{G}_{ab}\right) &= 2\left(x_a\frac{\partial}{\partial x_b} - x_b\frac{\partial}{\partial x_a} - w_{ac}\frac{\partial}{\partial w_{cb}} + w_{bc}\frac{\partial}{\partial w_{ca}}\right) \\ &= 2\left(x'_a\frac{\partial}{\partial x'_b} - x'_b\frac{\partial}{\partial x'_a} - w'_{ac}\frac{\partial}{\partial w'_{cb}} + w'_{bc}\frac{\partial}{\partial w'_{ca}}\right) \\ &= 2\left(x'_a\frac{\partial}{\partial x'_b} - x'_b\frac{\partial}{\partial x'_a} - \eta^i_{ac}\eta^j_{cb}\left(N'_i\frac{\partial}{\partial N'_j} - N'_j\frac{\partial}{\partial N'_i}\right)\right). \end{aligned} \tag{89}$$

The generators of Lorentz transformation are given by $\hat{G}_{ab}$ and the translation generators are given by $\hat{G}_{5a}$. In view of (87), translation generators are given by $\hat{G}_a$ and $\hat{G}_{ab}$;

$$\begin{aligned} Ad\left(\hat{G}_{a5}\right) &= \frac{1}{n}\left(4Ad\left(\hat{G}_a\right) - \hat{G}_{ab}Ad\left(\hat{G}_b\right) - \hat{G}_bAd\left(\hat{G}_{ab}\right)\right) \\ &= 4\frac{\alpha^2}{\rho'^2}Ad\left(\hat{G}_a\right) - \hat{G}'_{ab}Ad\left(\hat{G}_b\right) - \frac{\alpha}{\rho'}\hat{G}'_bAd\left(\hat{G}_{ab}\right) \\ &= -\hat{G}'_{ab}Ad\left(\hat{G}_b\right) + O\left(\frac{1}{\rho'}\right). \end{aligned} \tag{90}$$

If we ignore the $O(1/\rho')$, the translation generator is related to the adjoint action of $\hat{G}_a$.

Here we investigate the eigenvalues of the fuzzy two-sphere in the large n limit. As commented in the previous section, the noncommutativity on the fuzzy four-sphere is produced by the fuzzy two-sphere. One of the three coordinates of the fuzzy two-sphere $\hat{N}_3$ is diagonalized as follows

$$\hat{N}_3 = diag(n/2, n/2-1, \ldots, -n/2+1, -n/2). \tag{91}$$

When we take the large n limit, the two-brane charge no longer vanishes. It is because the contributions from the north pole and the south pole in two-sphere do decouple in this limit. Then $\hat{N}_3$ takes the value $+n/2$ or $-n/2$ in the large n limit. Since we have $N_1'^2 + N_2'^2 \sim 1/2n$, we may regard N'_1 and N'_2 as zero in this limit. After taking the large n limit, the noncommutativity G'_{ab} becomes as follows

$$G'_{12} = i\mathbf{1} \ \text{(or } \ -i\mathbf{1}), \quad G'_{34} = -i\mathbf{1} \ \text{(or } \ i\mathbf{1}), \quad \text{other components are zero.} \tag{92}$$

Then the algebras become Heisenberg type and the symmetry of this plane is broken to $SO(2) \times SO(2)$. The product between fields becomes the so called Moyal product since the noncommutative spherical harmonics has Weyl type ordering.

We next study an action of noncommutative gauge theory on the noncommutative plane, which is obtained from the fuzzy four-sphere. The matrices A_a are given by

$$\begin{aligned} A_a &= \hat{x}_a + \alpha\rho\hat{a}_a \\ &= \alpha\hat{G}_a + \alpha\rho\hat{a}_a \\ &\equiv \alpha\rho' D'_a, \end{aligned} \tag{93}$$

where we have rescaled the field as $\sqrt{n}\hat{a}_\mu = \hat{a}'_\mu$. $Ad(D_a)$ is the covariant derivative on the flat background. Gauge covariant field strength becomes

$$\begin{aligned} \hat{F}_{\mu\nu} &= [D'_\mu, D'_\nu] + (\alpha\rho')\lambda\epsilon^{\mu\nu\lambda\rho\sigma}D'_\lambda D'_\rho D'_\sigma \\ &= \left([D'_\mu, D'_\nu] + \frac{\alpha\rho' 2}{\alpha(n+2)}\epsilon^{\mu\nu\lambda\rho\sigma}D'_\lambda D'_\rho D'_\sigma\right) \\ &= \left([D'_\mu, D'_\nu] + \frac{\alpha^2}{\rho'}\epsilon^{\mu\nu\lambda\rho\sigma}[D'_\lambda, D'_\rho]D'_\sigma\right). \end{aligned} \tag{94}$$

The action around the north pole becomes

$$S = -\frac{(\alpha\rho')^4}{g^2}Tr\left(\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}_{\mu\nu} - \frac{9}{5}\frac{\alpha\rho'}{\alpha(n+2)}\epsilon^{\mu\nu\lambda\rho\sigma}D'_\mu D'_\nu D'_\lambda D'_\sigma D'_\rho\right.$$

$$
\begin{aligned}
& \left. -\frac{(\alpha\rho')^2}{\alpha^2(n+2)^2} f^{\mu\nu\lambda\rho\sigma\tau} D'_\mu D'_\nu D'_\lambda D'_\rho D'_\sigma D'_\tau \right) \\
= & -\frac{(\alpha\rho')^4}{g^2} Tr \left(\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}_{\mu\nu} - \frac{9}{5}\frac{\alpha}{\rho'}\epsilon^{\mu\nu\lambda\rho\sigma}[D'_\mu, D'_\nu][D'_\lambda, D'_\rho]D'_\sigma \right. \\
& \left. - \left(\frac{\alpha}{\rho'}\right)^2 f^{\mu\nu\lambda\rho\sigma\tau} D'_\mu D'_\nu D'_\lambda D'_\rho D'_\sigma D'_\tau \right) \\
= & -\frac{(\alpha\rho')^4}{4g^2} Tr \left(\hat{F}_{ab}\hat{F}_{ab} + 2[D'_a, \hat{\phi}'][D'_a, \hat{\phi}'] + O\left(\frac{1}{\rho'}\right)\right),
\end{aligned}
\tag{95}
$$

where we have rewritten $\hat{a}_5$ as $\hat{\phi}$. The gauge transformation (64) is rewritten as follows,

$$
\delta a'_a(x') = -2G'_{ab}\frac{\partial}{\partial x'_b}\lambda(x') + i[\lambda(x'), a'_a(x')]_\star,
\tag{96}
$$

where G'_{ab} is given by (92).

We now investigate the kinetic term with $O(1/\rho')$. (67) becomes as follows around the north pole,

$$
\begin{aligned}
S_{kinetic} = & \frac{(\alpha\rho')^4}{2g^2} Tr \left(\hat{a}'_b \left[\frac{\hat{G}_a}{\rho'}, \left[\frac{\hat{G}_a}{\rho'}, \hat{a}'_b \right]\right]\right) \\
= & -\frac{2(\alpha\rho')^4}{g^2} Tr \left(\hat{a}'_b \left(\frac{w'_{ab}w'_{ac}}{\alpha^2}\frac{\partial^2}{\partial x'_b \partial x'_c} - \frac{4}{\rho'^2}x'_b\frac{\partial}{\partial x'_b} \right.\right. \\
& \left.\left. -\frac{2}{\rho'^2}w'_{ab}x'_c\frac{\partial^2}{\partial x'_b\partial w'_{ac}} - \frac{1}{\rho'^2}w'_{ab}\frac{\partial}{\partial w'_{ab}} + \frac{\alpha^2}{\rho'^4}x'_bx'_c\frac{\partial^2}{\partial w'_{ab}\partial w'_{ac}} \right)\hat{a}'_b\right) \\
= & -\frac{2(\alpha\rho')^4}{g^2} Tr \left(\hat{a}'_b \left(\frac{\partial^2}{\partial x'_a\partial x'_a} - \frac{x'_ax'_c}{\rho'^2}\frac{\partial^2}{\partial x'_a\partial x'_c} - \frac{4}{\rho'^2}x'_b\frac{\partial}{\partial x'_b} \right.\right. \\
& \left.\left. -\frac{2}{\rho'^2}\eta^i_{ab}\eta^j_{ac}N'_ix'_c\frac{\partial^2}{\partial x'_b\partial N'_j} - \frac{4}{\rho'^2}N'_i\frac{\partial}{\partial N'_i} + \frac{1}{4\rho'^4}x'_bx'_b\frac{\partial^2}{\partial N'_i\partial N'_i}\right)\hat{a}'_b\right),
\end{aligned}
\tag{97}
$$

where we have used the following relation,

$$
G'_{ab}G'_{bc} = \delta_{ac} + \frac{1}{n}\left(-G'_aG'_c - G'_{a5}G'_{5c}\right).
\tag{98}
$$

The first three terms in (97) constitute the usual Laplacian of a four-sphere. Only the first term survives in $\rho' \to \infty$ limit. Extension of the isospin space becomes small and only the four-dimensional space remains. In this case, trace is replaced with the following integral in the noncommutative field theory,

$$
\frac{1}{N}Tr \to \frac{1}{\frac{8\pi}{3}\pi\rho'^4}\int d^4x'.
\tag{99}
$$

From the coefficient in front of the action, we find that the Yang-Mills coupling is given by $g^2_{YM} = 16\pi g^2/\alpha^4 n^3$.

In this section, we have investigated a noncommutative gauge theory on a flat noncommutative background by taking a large radius limit of the fuzzy four-sphere around the north pole. This noncommutative plane has the Heisenberg algebra type noncommutativity and the symmetry of this plane is $SO(2) \times SO(2)$. Although it is desirable to have $SO(4)$ symmetry, it is difficult to construct a noncommutative plane which has higher symmetry. This difficulty may be related to the quantization of Nambu bracket[30]. If a noncommutative plane has $SO(4)$ symmetry, it is expected that the quantization of Nambu bracket is realized on it. Although some trials[31, 32] are implemented, it is difficult to obtain consistent quantization of Nambu bracket.

5 Summary and Discussions

In this paper, we have investigated a noncommutative gauge theory on a fuzzy four-sphere using a five-dimensional matrix model. We considered a matrix model with a fifth-rank Chern-Simons term since this model has a fuzzy four-sphere as a classical solution. By dividing matrices into backgrounds and fields propagating on them, we obtained noncommutative gauge theories on the backgrounds. It is worth noting that we expanded matrices around the coordinates of the fuzzy four-sphere. This facts supports an idea that the eigenvalues of bosonic variables in the matrix model represents spacetime coordinates. A characteristic feature of noncommutative gauge theories or the matrix model is that spacetime and fields are treated on the same footing.

One of the difficulties to consider a noncommutative gauge theory on a fuzzy four-sphere is that algebra of the coordinates does not close. Because of this reason, we need extra degrees of freedom. By adding a fuzzy two-sphere at each point on the fuzzy four-sphere, we can solve this difficulty. These extra degrees of freedom are interpreted as spins. The maximum magnitude of the spin is related to the number of the quanta on a fuzzy two-sphere, which is comparable with number of the quanta on the fuzzy four-sphere.

It is well known that the quantum Hall system is an example of the noncommutative geometry. As is discussed in [20], the quantum Hall system on the four-dimensional sphere is constructed by considering a system of particles under a $SU(2)$ gauge field. From the kinetic term of the noncommutative gauge theory action on the fuzzy four-sphere, we have obtained the same eigenvalue as the Hamiltonian of the quantum Hall system.

The advantage of compact noncommutative manifolds is that one can construct them in terms of finite size matrices while a solution which represents a noncommutative plane cannot be constructed by finite size matrices. (From the viewpoint of the field theories, N plays the role of the cutoff parameter.) We showed that a gauge theory on a noncommutative plane were reproduced from a gauge theory on a fuzzy four-sphere by taking a large n limit. We have considered the theory around the north pole and took a large radius limit. This noncommutative plane has the Heisenberg algebra type noncommutativity and the symmetry is $SO(2) \times SO(2)$. It is difficult to construct a more symmetric noncommutative plane with maintaining the associativity.

Let us comment on the string theory interpretation of the classical solutions. Two classical solutions (31) and (32) may be interpreted as D-instantons and spherical D3-branes [6]. The spherical D3-branes are considered to be deformed by the presence of the R-R background. As is shown in (39), the noncommutativity on the fuzzy four-sphere originates from the presence of the

[6] Time direction is compactified after Wick rotation.

fuzzy two-sphere. The coordinates of the fuzzy two-sphere play the role of the noncommutative parameters. Thus the fuzzy four-sphere is deformed by adding the fuzzy two-sphere on the four-sphere. If we take a commutative, the four-sphere and the fuzzy two-sphere decouple.

We comment on the relation to the IIB matrix model. The second term in the action (27) is interpreted as Myers term from the viewpoint of a D-brane action. On the other hand, we might expect that this five-dimensional matrix model is obtained from IIB matrix model by integrating unnecessary matrices since this model has the same kinds of symmetries as IIB matrix model. It may be alternatively obtained by deforming IIB matrix model. There may be a new model which includes a fuzzy four-sphere as a classical solution and has supersymmetry. Such analyses will be future problems.

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A Notations of Gamma matrices

This appendix is referred to [17]. An explicit form of 4×4 five-dimensional gamma matrices is given by

$$\begin{aligned} \Gamma_\mu &= \begin{pmatrix} 0 & -i\sigma_\mu \\ i\sigma_\mu & 0 \end{pmatrix}, \quad (\mu = 1,2,3) \\ \Gamma_4 &= \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \\ \Gamma_5 &= \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \end{aligned} \tag{A.1}$$

where σ_μ is the Pauli matrices. They satisfy the Clifford algebra:

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\delta_{\mu\nu} \ \ (\mu,\nu = 1,2,3,4,5). \tag{A.2}$$

The matrices $\hat{G}^{(n)}_\mu$ are constructed as in (9). In this notation, $\hat{G}^{(n)}_5$ is diagonalized and the eigenvalues are

$$\hat{G}^{(n)}_5 = diag(n, n-2, \ldots, -n+2, -n), \tag{A.3}$$

where the eigenvalue m has the degeneracy $((n+2)^2 - m^2)/4$.

B Matrix Model with mass term

Let us consider a five-dimensional matrix model with a mass term in this appendix while we considered a five-dimensional matrix model with a Chern-Simons term in the paper. We investigated a four-dimensional matrix model with a mass term in [15]. The action is

$$S = -\frac{1}{g^2} Tr\left(\frac{1}{4}[A_\mu, A_\nu][A_\mu, A_\nu] + 8\alpha^2 A_\mu A_\mu\right), \tag{B.1}$$

where μ,ν run over 1 to 5. The indices are contracted by $\delta_{\mu\nu}$. This model has several classical solutions. The first one is a fuzzy four-sphere;

$$\hat{x}_\mu^{S^4} = \alpha \hat{G}_\mu \quad (\mu = 1, 2, 3, 4, 5), \tag{B.2}$$

the radius of the four-sphere is $\rho_{S^4}^2 = \alpha^2 n(n+4)$. The second one is a fuzzy two-sphere.

$$\begin{aligned} \hat{x}_\mu^{S^2} &= 2\sqrt{2}\alpha \hat{L}_\mu \quad (\mu = 1, 2, 3), \\ &= 0 \quad\quad (\mu = 4, 5), \end{aligned} \tag{B.3}$$

where $\hat{L}_\mu$ is the N-dimensional irreducible representation of $SU(2)$, and the radius of the two-sphere is given by $\rho_{S^2}^2 = 2\alpha^2(N^2-1)$. A system of multiple fuzzy four-spheres is a classical solution while it is *not* in a model with a fifth rank Chern-Simons term. A system of a fuzzy four-sphere and a fuzzy two-sphere is also a classical solution;

$$\begin{aligned} A_\mu &= \begin{pmatrix} \hat{x}_\mu^{S^4} & 0 \\ 0 & \hat{x}_\mu^{S^2} \end{pmatrix}, \quad (\mu = 1, 2, 3) \\ &= \begin{pmatrix} \hat{x}_\mu^{S^4} & 0 \\ 0 & 0 \end{pmatrix}, \quad (\mu = 4, 5). \end{aligned} \tag{B.4}$$

Fuzzy two-torus is also a classical solution,

$$\hat{x}_\mu^{T^2} = \frac{2\sqrt{2}}{\sqrt{1-\cos\left(\frac{2\pi}{N}\right)}} \hat{y}_\mu \quad (\mu = 1, 2, 3, 4), \tag{B.5}$$

where

$$UV = e^{i\frac{2\pi}{N}} VU, \quad U = \hat{y}_1 + i\hat{y}_2, \;\; V = \hat{y}_3 + i\hat{y}_4. \tag{B.6}$$

The values of the action for the fuzzy four-sphere (B.2) and the fuzzy two-sphere (B.3) are

$$S_{S^4} = -\frac{31}{4g^2} n(n+4) N \alpha^4 \tag{B.7}$$

and

$$S_{S^2} = -\frac{8}{g^2} N(N^2-1)\alpha^4 \tag{B.8}$$

respectively. Because the first one is $O(-n^6\alpha^4/g^2)$ and the second one is $O(-n^9\alpha^4/g^2)$, the second one has a lower classical action than the first one.

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