

Improved Epstein–Glaser renormalization in coordinate space I. Euclidean framework

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Abstract

In a series of papers, we investigate the reformulation of Epstein–Glaser renormalization in coordinate space, both in analytic and (Hopf) algebraic terms. This first article deals with analytical aspects. Some of the (historically good) reasons for the divorces of the Epstein–Glaser method, both from mainstream quantum field theory and the mathematical literature on distributions, are made plain; and overcome.

1 Introduction

This is the first of a series of papers, the companions [1, 2] often being denoted respectively II and III.

We find it convenient to summarize here the aims of these papers, in reverse order. Ever since Kreimer perceived a Hopf algebra lurking behind the forest formula [3], the question of encoding the systematics of renormalization in such a structure (and the practical advantages therein) has been in the forefront. Connes and Kreimer were able to show, using the \mathcal{O}_\hbar^3 model as an example, that renormalization of quantum field theories in momentum space is encoded in a commutative Hopf algebra of *Feynman graphs* \mathcal{H} , and the Riemann–Hilbert problem with values in the group of loops on the dual of \mathcal{H} [4, 5]. The latter makes sense only in the context of renormalization by dimensional regularization [6, 7], physicists’ method of choice. Now, whereas it is plausible that the Hopf algebra approach to renormalization is consistent with all main renormalization methods, there is much to be learned by a systematic verification of this conjecture. Paper III focuses on combinatorial-geometrical aspects of this approach to perturbative renormalization in QFT within the framework of the Epstein–Glaser (EG) procedure [8].

One can argue that all that experiments have established is (striking) agreement with (renormalized) momentum space integrals [9]. Be that as it may, renormalization on real space is more intuitive, in that momentum space formulations “rather obscure the fact that UV divergences arise from purely short-distance phenomena” [10]. For the questions of whether and how configuration space-based methods exhibit the Hopf algebraic structure, the EG method was a natural candidate. It enjoys privileged rapports with external field theory [11, 12, 13], possesses a stark reputation for rigour, and does not share some limitations of dimensional regularization —allowing for renormalization in curved backgrounds [14], for instance.

In spite of its attractive features, EG renormalization still remains outside the mainstream of QFT. The (rather rigorous) QFT text by Itzykson and Zuber has only the following to say about it: “...the most orthodox procedure of Epstein and Glaser relies directly on the axioms of local field theory in configuration space. It is free of mathematically undefined quantities, but hides the multiplicative structure of renormalization” [15, p. 374]. Raymond Stora, today the chief propagandist of the method, had commented: “In spite of its elegance and accuracy this theory suffers from one defect, namely it does not yield explicit formulae of actual computational value” [16]. Indeed.

Over the years, some of the awkwardness of the original formalism was dispelled in the work by Stora. The “splitting of distributions” was reformulated in [17] as a typical problem of extension (through the boundaries of open sets) in distribution theory. Moreover, in [18] it was made clear that an (easier) Euclidean analog of the EG construction does exist. Beyond being interesting on its own right (for instance for the renormalization group approach to criticality), it allows performing EG renormalization in practice by a (sort of) “Wick rotation” trick —the subject of paper II of this series.

When tackling the compatibility question of EG renormalization and the Connes–Kreimer algebra, two main surviving difficulties are brought to light. The first is that, while the Hopf algebra elucidation of Bogoliubov’s recursive procedure is defined graph-by-graph, in the EG approach it is buried under operator aspects of the time-ordered products and the \mathcal{S} -matrix, not directly relevant for that question. This problem was recently addressed by Pinter [19, 20] and also in [21]; the last paper, however, contains a flaw, examined in III.

The second difficulty, uncovered in the course of the same investigation, has to do with

prior, *analytical* aspects of Epstein and Glaser’s basic method of subtraction. For, it was curious to observe, the extension method by Epstein and Glaser has remained *divorced as well* from the literature on distributions, centered mainly on analytical continuation and “finite part” techniques. One scours in vain for any factual link between EG subtraction and the household names of mathematical distribution theory.

And so the vision of casting all of quantum field theory in the light of distribution analysis [22, 23] has remained unfulfilled.

In the present paper we are concerned with the second of the mentioned difficulties. This means in practice that we deal with primitively UV divergent diagrams. (Nonprimitive diagrams are dealt with in III.) By means of a seemingly minute departure from the letter, if not the spirit, of the EG original prescription we succeed to deliver its *missing link* to the standard literature on extension of distributions. Then we proceed to show the *dominant place* our improved subtraction method occupies with regard to dimensional regularization in configuration space; differential renormalization [24]; “natural” renormalization [25]; and BPHZ renormalization.

The benefits of the improved prescription do not stop there: it goes on to remarkably simplify the task of constructing covariant renormalizations in II, and the Hopf-geometrical constructions in III.

An important sideline of this paper is the use of the theory of Cesàro summability of distributions [26, 27] in dealing with the infrared difficulties; this helps to clarify the logical dependence of the BPHZ procedure on the causal one, already pointed out in [28]. Improved BPHZ methods for massless fields ensue as well.

The main theoretical development is found in Section 2. Afterwards, we proceed by way of alternating discussions and examples. In order to deliver the argument without extraneous complications, we work out diagrams belonging to scalar theories. Most examples are drawn from the massless ϕ^4 model: masslessness is more challenging and instructive, because of the attendant infrared problems, and more interesting for the renormalization group calculations performed in III. Eventually we bring in examples in massive theories as well.

2 Renormalization in configuration space

2.1 The new prescription

All derivatives in this paper, unless explicitly stated otherwise, are in the sense of distributions. We tacitly use the translation invariance of Feynman propagators and amplitudes; in particular, the origin stands for the main diagonal.

Let d denote the dimension of the coordinate space. Typically, d will be $4n$. An unrenormalized Feynman amplitude $f(\Gamma)$, or simply f , associated to a graph Γ , is smooth away from the diagonals. We say that Γ is *primitively* divergent when $f(\Gamma)$ is not locally integrable, but is integrable away from zero. Denote by $\mathcal{F}_{\text{prim}}(\mathbb{R}^d) \hookrightarrow L^1_{\text{loc}}(\mathbb{R}^d \setminus \{0\})$ this class of amplitudes.

By definition, a tempered distribution $\tilde{f} \in \mathcal{S}'(\mathbb{R}^d)$ is an *extension* or *renormalization* of f if

$$\tilde{f}[\phi] := \langle \tilde{f}, \phi \rangle = \int_{\mathbb{R}^d} f(x) \phi(x) d^d x$$

holds whenever ϕ belongs to $\mathcal{S}(\mathbb{R}^d \setminus \{0\})$.

Let

$$f(x) = O(|x|^{-a}) \quad \text{as } x \rightarrow 0, \quad (1)$$

with a an integer, and let $k = a - d > 0$. Then, $f \notin L^1_{\text{loc}}(\mathbb{R}^d)$. But f can be regarded as a well-defined functional on the space $\mathcal{S}_{k+1}(\mathbb{R}^d)$ of Schwartz functions that vanish at the origin at order $k+1$. Thus the simplest way to get an extension of f would appear to be standard Taylor series surgery: to throw away the k -jet of ϕ at the origin, in order to define \tilde{f} by transposition. Denote this jet by $j_0^k \phi$ and the corresponding Taylor remainder by $R_0^k \phi$. We have by that definition

$$\langle \tilde{f}, \phi \rangle = \langle f, R_0^k \phi \rangle. \quad (2)$$

Using Lagrange's integral formula for the remainder:

$$R_0^k \phi(x) = (k+1) \sum_{|\beta|=k+1} \frac{x^\beta}{\beta!} \int_0^1 dt (1-t)^k \partial^\beta \phi(tx),$$

where we have embraced the usual multiindex notation, and exchanging integrations, one appears to obtain an explicit integral formula for \tilde{f} :

$$\tilde{f}(x) = (-1)^{k+1} (k+1) \sum_{|\beta|=k+1} \partial^\beta \left[\frac{x^\beta}{\beta!} \int_0^1 dt \frac{(1-t)^k}{t^{k+d+1}} f\left(\frac{x}{t}\right) \right]. \quad (3)$$

Lest the reader be worried with the precise meaning of (1), we recall that in QFT one usually considers a generalized homogeneity degree, the *scaling degree* [29]. The scaling degree σ_f of a scalar distribution f at the origin of \mathbb{R}^d is defined to be

$$\sigma_f = \inf \{ s : \lim_{\lambda \rightarrow 0} \lambda^s f(\lambda x) = 0 \} \quad \text{for } f \in \mathcal{S}'(\mathbb{R}^d),$$

where the limit is taken in the sense of distributions. Essentially, this means that $f(x) = O(|x|^{-\sigma_f})$ as $x \rightarrow 0$ in the Cesàro average sense [30]. Then σ_f and respectively $\sigma_f - d$ —called the *singular order*—occupy the place of a in (1) and of k .

The trouble with (3) is that the remainder is not a test function, so, unless the infrared behaviour of f is very good, we end up in (2) with an undefined integral. In fact, in the massless theory f is an homogeneous function with an algebraic singularity, the infrared behaviour is pretty bad, and $-d$ is also the critical exponent. A way to avoid the infrared problem is to *weight* the Taylor subtraction. Epstein and Glaser [8] introduced weight functions w with the properties $w(0) = 1$, $w^{(\alpha)}(0) = 0$ for $0 < |\alpha| \leq k$, and projector maps $\phi \mapsto W_w \phi$ on $\mathcal{S}(\mathbb{R}^d)$ given by

$$W_w \phi(x) := \phi(x) - w(x) j_0^k \phi(x). \quad (4)$$

The previous ordinary Taylor surgery case corresponds to $w \equiv 1$, and the identity

$$W_w(w\phi) = w W_1 \phi$$

tells us that W_w indeed is a projector, since $W_w(w x^\gamma) = 0$ for $|\gamma| \leq k$.

Look again at (4). There is a considerable amount of overkilling there. The point is that, in the homogeneous case, a worse singularity at the origin entails a better behaviour at infinity. So we can, and should, weight only the *last* term of the Taylor expansion. This leads to the definition employed in this paper, at variance with Epstein and Glaser's:

$$T_w \phi(x) := \phi(x) - j_0^{k-1}(\phi)(x) - w(x) \sum_{|\alpha|=k} \frac{x^\alpha}{\alpha!} \phi^{(\alpha)}(0). \quad (5)$$

Just $w(0) = 1$ is now required in principle from the weight function.

An amazing amount of mathematical mileage stems from this simple physical observation. To begin with, T_w is also a projector. To obtain an integral formula for it, start from

$$T_w \phi = (1 - w)R_0^{k-1} \phi + wR_0^k \phi,$$

showing that it T_w interpolates between R_0^k , guaranteeing a good UV behaviour, and R_0^{k-1} , well behaved enough in the infrared. By transposition, using (3), we derive

$$\begin{aligned} T_w f(x) = & (-)^k k \sum_{|\alpha|=k} \partial^\alpha \left[\frac{x^\alpha}{\alpha!} \int_0^1 dt \frac{(1-t)^{k-1}}{t^{k+d}} f\left(\frac{x}{t}\right) \left(1 - w\left(\frac{x}{t}\right)\right) \right] \\ & + (-)^{k+1} (k+1) \sum_{|\beta|=k+1} \partial^\beta \left[\frac{x^\beta}{\beta!} \int_0^1 dt \frac{(1-t)^k}{t^{k+d+1}} f\left(\frac{x}{t}\right) w\left(\frac{x}{t}\right) \right]. \end{aligned} \quad (6)$$

This is the central formula of this paper.

2.2 On the auxiliary function

It is important to realize what is (and is not) required of the weight function w , apart from a good behaviour at the origin: in view of the smoothness and good properties of f away from the origin, we have a lot of leeway, and, especially, w does *not* have to be a test function, nor to possess compact support. Basically, what is needed is that w decay at infinity in the weak sense that it sport momenta of sufficiently high order.

We formalize this assertion for greater clarity. First, one says that the distribution f is of order $|x|^l$ (with l not a negative integer) at infinity, in the Cesàro sense, if there exists a natural number N , and a primitive f_N of f of order N , such that f_N is locally integrable for $|x|$ large and the relation

$$f_N(x) = O(|x|^{N+l})$$

as $|x| \uparrow \infty$ holds in the ordinary sense. Now, for any real constant γ , the space \mathcal{K}_γ is formed by those smooth functions ϕ such that $\partial^\alpha \phi(x) = O(|x|^{\gamma-|\alpha|})$ as $|x| \uparrow \infty$, for each $|\alpha|$. A topology for \mathcal{K}_γ is generated by the obvious family of seminorms, and the space \mathcal{K} is defined as the inductive limit of the spaces \mathcal{K}_γ as $\gamma \uparrow \infty$. Consider now the dual space \mathcal{K}' of distributions. The following are equivalent [26, 27]:

- $f \in \mathcal{K}$.
- f satisfies

$$f(x) = o(|x|^{-\infty})$$

in the Cesàro sense as $|x| \uparrow \infty$.

- There exist constants μ_α such that

$$f(\lambda x) \sim \sum_{\alpha \geq 0} \frac{\mu_\alpha \delta^{(\alpha)}(x)}{\lambda^{|\alpha|+1}}$$

in the sense of distributions, as $\lambda \uparrow \infty$.

- All the moments $\langle f(x), x^\alpha \rangle$ exist in the sense of Cesàro summability of integrals (they coincide with the aforementioned constants μ_α).

Any element of \mathcal{K} which is regular and takes the value 1 at zero qualifies as a weight “function”. For instance, one can take for w an exponential function e^{iqx} , with $q \neq 0$. This vanishes at ∞ to all orders, in the Cesàro sense, and so it is a perfectly good infrared problem-buster auxiliary function. The fact that $e^{iqx} \in \mathcal{K}$, for $q \neq 0$, means that, outside the origin in momentum space, the Fourier transform of elements $\phi \in \mathcal{K}$ can be computed by a standard Cesàro evaluation

$$\hat{\phi}(q) = \langle \exp(iqx), \phi(x) \rangle.$$

Of course, for this auxiliary function the original equation (4) no longer applies, since it has no vanishing derivatives at the origin. But (4) can be replaced by the more general

$$W_w \phi(x) := \phi(x) - w(x) \sum_{0 \leq |\alpha| \leq k} \frac{x^\alpha}{\alpha!} \left(\frac{\phi}{w} \right)^{(\alpha)}(0). \quad (7)$$

This was seen, at the heuristic level, by Prange [28]; see the discussion on the BPHZ formalism in subsection 5.3, where the “Cesàro philosophy” comes into its own.

These observations are all the more pertinent because the contrary prejudice is still widespread. For instance, the worthy thesis [20], despite coming on the footsteps of [28], yet unfortunately exhibits it; on its page 30: “... [the exponential] function is not allowed in the \mathcal{W} -operation because it does not have compact support.” Of course it is allowed: then the Fourier transformed subtraction W_w of Epstein and Glaser becomes exactly the standard BPHZ subtraction, around momentum $p = q$. What T_w becomes will be revealed later.

2.3 Properties of the \mathcal{L} -projector

Consider now the functional variation of the renormalized amplitudes with respect to w . One has

$$\left\langle \frac{\delta}{\delta w} T_w f, \psi \right\rangle := \left. \frac{d}{d\lambda} T_{w+\lambda\psi} f \right|_{\lambda=0}$$

for T_w , and similarly for W_w , by definition of functional derivative. It is practical to write now

$$\delta^\alpha := (-)^{|\alpha|} \frac{\delta^{(\alpha)}}{\alpha!},$$

for this combination is going to appear with alarming frequency. From (4) we would obtain

$$\frac{\delta}{\delta w} W_w f[\cdot] = - \sum_{|\alpha| \leq k} f[x^\alpha \cdot] \delta^\alpha, \quad (8)$$

whereas (5) yields

$$\frac{\delta}{\delta w} T_w f[\cdot] = - \sum_{|\alpha|=k} f[x^\alpha \cdot] \delta^\alpha,$$

independently of w in both cases. Malgrange's theorem says that different renormalizations of a primitively divergent graph differ by terms proportional to the delta function and all its derivatives $\delta^{(\alpha)}$, up to $|\alpha| = k$. Thus there is no canonical way to construct the renormalized amplitudes, the inherent ambiguity being represented by the undetermined coefficients of the δ 's, describing how the chosen extension acts on the finite codimension space of test functions not vanishing in a neighborhood of $\mathbf{0}$.

There is, however, a more “natural” way —in which the ambiguity is reduced to terms in the higher-order derivatives of δ , exclusively. This is guaranteed by our choice of T_w .

In practice, one works with appropriate 1-parameter (or few-parameter) families of auxiliary functions, big enough to be flexible, small enough to be manageable. Recall than in QFT, with $c = \hbar = 1$, the physical dimension of length is inverse mass. Let then the variable u have the dimension of mass. We consider the change in $T_w f$ when the variable w changes from $w \equiv w(\mu x)$ to $w((\mu + \delta\mu)x)$, which introduces the Jacobian $\frac{\delta w}{\delta \mu} = \frac{\partial w(\mu x)}{\partial \mu}$, yielding

$$\frac{\partial}{\partial \mu} T_{w(\mu x)} f = - \sum_{|\alpha|=k} \langle f, x^\nu \partial^\nu w(\mu x) x^\alpha \rangle \delta^\alpha. \quad (9)$$

Here we have assumed that f has no previous dependence on u .

Enter now the (rotation-invariant) choice $w_\mu(x) := H(\mu^{-1} - |x|)$, where H is the Heaviside step function: it not only recommends itself for its simplicity, but it turns out to play a central theoretical role. The parameter μ corresponds in our context to 't Hooft's energy scale in dimensional regularization —see subsection 5.1; the limits $\mu \downarrow 0$ and $\mu \uparrow \infty$ correspond to the case $w = 1$ and respectively to the “principal value” of f ; in general they will not exist.

Write $T_\mu f$ for the corresponding renormalizations. With the help of (9), one obtains

$$\mu \frac{\partial}{\partial \mu} T_\mu f = \sum_{|\alpha|=k} f[\delta(\mu^{-1} - |x|) |x| x^\alpha] \delta^\alpha. \quad (10)$$

For f homogeneous (of order $-d - k$ as it happens), the expression is actually *independent* of μ , the coefficients of the δ^α being

$$c^\alpha = \int_{|x|=1} f x^\alpha = \int_{|x|=A} f |x| x^\alpha, \quad (11)$$

with $|\alpha| = k$ and any $A > 0$. Note that similar extra terms, with $|\alpha| < k$, coming out of the formulae (8) would indeed be μ -dependent.

Compute the T_μ in the massless (homogeneous) case, whereupon one can pull f out of the integral sign. We get

$$T_\mu f(x) = (-)^k \left\{ \sum_{|\alpha|=k} \partial^\alpha \left[\frac{x^\alpha f(x)}{\alpha!} (1 - (1 - \mu|x|)^k) \right] + (k+1) \sum_{|\beta|=k+1} \partial^\beta \left[\frac{x^\beta f(x)}{\beta!} \int_1^{\mu|x|} dt \frac{(1-t)^k}{t} \right] \right\}. \quad (12)$$

Formula (12) is simpler than it looks: because of our previous remark on (10), *all* the μ -polynomial terms in the previous expression for $T_\mu f$ must cancel. Let us then denote, for $k > 1$,

$$H_k := \sum_{l=1}^k \frac{(-)^{l+1}}{l} \binom{k}{l} = \binom{k}{1} - \frac{1}{2} \binom{k}{2} + \dots - (-)^k \frac{1}{k}. \quad (13)$$

At least for $\mu|x| \leq 1$, the expression for $T_\mu f$ becomes

$$T_\mu f(x) = (-)^k (k+1) \sum_{|\beta|=k+1} \partial^\beta \left[\frac{x^\beta f(x)}{\beta!} (\log \mu|x| + H_k) \right].$$

By performing the derivative with respect to $\log \mu$ directly on this formula, one obtains in the bargain interesting formulae for distribution theory. Namely, for any f homogeneous of degree $-d-k$:

$$(-)^k (k+1) \sum_{|\beta|=k+1} \partial^\beta \left[\frac{x^\beta f(x)}{\beta!} \right] = \sum_{|\alpha|=k} \left(\int_{|x|=1} f x^\alpha \right) \delta^\alpha(x). \quad (14)$$

Thus the final result is

$$T_\mu f(x) = (-)^k (k+1) \sum_{|\beta|=k+1} \partial^\beta \left[\frac{x^\beta f(x)}{\beta!} \log \mu|x| \right] + H_k \sum_{|\alpha|=k} c^\alpha \delta^\alpha(x), \quad (15)$$

with the c^α given by (11). The resulting expression is actually valid for all μ : away from the origin it reduces to $f(x)$, as it should. It is in the spirit of differential renormalization, since f is renormalized as the distributional derivative of a regular object that coincides with f away from the singularity.

Let us put equation (15) to work at once. When performing multiplicative renormalization in the causal theory [2], the relevant property of a renormalized amplitude turns out to be its *dilatation* or scaling behaviour. This is not surprising in view of the form of our integral equation (6). Now, a further consequence of the choice of operator L_w is that it

modifies the original homogeneity in a minimal way. Had we stuck to W_μ (using (14)), the relatively complicated form

$$W_\mu f(\lambda \cdot) = \lambda^{-k-d} \left(W_\mu f + \log \lambda \sum_{|\alpha|=k} c^\alpha \delta^\alpha + \sum_{|\alpha|<k} a^\alpha (\lambda^{k-|\alpha|} - 1) \delta^\alpha \right),$$

for some a^α , would ensue; whereas for T_μ from (15) one obtains

$$[f]_{R,\mu}(\lambda x) := T_\mu f(\lambda x) = \lambda^{-k-d} \left([f]_{R,\mu}(x) + \log \lambda \sum_{|\alpha|=k} c^\alpha \delta^\alpha(x) \right); \quad (16)$$

or

$$E[f]_{R,\mu}(x) := (-k-d)[f]_{R,\mu}(x) + \sum_{|\alpha|=k} c^\alpha \delta^\alpha(x),$$

where E denotes the Euler operator $\sum_{|\beta|=1} x^\beta \partial^\beta$. From now on, the notations $[f]_{R,\mu}$ and $T_\mu f$ will be interchangeably used; when the dependence on μ need not be emphasized, we can write $[f]_R$ instead.

We invite the reader to prove (16) directly, by reworking the argument used for the case $k=0$ in [31, pp. 307–308].

Let us record here the obvious fact that when we employ two different prescriptions compatible with our renormalization scheme, the difference in the results is given by

$$[f]_{R_1} = [f]_{R_2} + \sum_{|\beta|=k} C^\beta \delta^\beta, \quad (17)$$

for some constants C^β . “Different prescriptions” could mean that we use different weight functions, or different choices of renormalization conditions, or simply different values of the parameter μ . In this respect, we note that the Fourier transforms of our real space renormalized amplitudes (see the end of Section 5) do not obey standard renormalization conditions on momentum space, so in particular contexts they might be in need of modification for this purpose.

2.4 Supplementary remarks

Following [32], with a difference by a factor of 2 in the definition, we conveniently formalize the first order operator from the (larger) space $\mathcal{S}'_1(\mathbb{R}^d)$ of continuous linear functionals on Schwartz functions that vanish at the origin, appearing in the foregoing:

$$S := \sum_{|\beta|=1} \partial^\beta x^\beta.$$

It “regularizes” \mathcal{S}'_1 , mapping it onto \mathcal{S}' . Clearly, on tempered distributions $S = d + E$. Therefore S kills any homogeneous *distribution* of order $-d$, like δ —but not the homogeneous *functionals*! From (15), it is clear that (the analog in \mathcal{S} -space of) S captures the Wodzicki

residue density in the theory of pseudodifferential operators —see for instance the discussion in Chapter 7 of [31]. Analogously, define

$$S_{k+1} := (k+1)! \sum_{|\beta|=k+1} \frac{\partial^\beta x^\beta}{\beta!}.$$

This sends onto \mathbf{S} the space of functionals $\mathbf{S}'_{k+1}(\mathbb{R}^d)$, dual of the space of Schwartz functions which vanish at the origin up to order $k+1$. On well-behaved distributions, it is equivalent to $(S+k) \cdots (S+1)\mathbf{S}$: we think of \mathbf{S}_{k+1} as an ordered power of \mathbf{S}_1 , with the coordinate multiplications remaining to the right of the differential operators. For massless models, our formulae come close to simply iterating \mathbf{S} in this way, as done in [32].

For theoretical purposes, it should be kept in mind that the \mathbf{T} -operator remains in the general framework of the Epstein–Glaser theory: after all, one could always find weight functions $w(\mu)$ such that $W_{w(\mu)}$ is identical to \mathbf{T}_μ . In particular, the singular order of $\mathbf{T}_\mu f$ is the same as that of f [14]. However, we see already that a finer classification needs to be introduced.

Definition 1. A distribution $f = f_1$ is called *associate homogeneous* of order 1 and degree \mathbf{a} when there exists a *homogeneous* distribution f_0 of degree \mathbf{a} such that

$$f_1(\lambda x) = \lambda^a (f_1(x) + g(\lambda) f_0(x)),$$

for some function $g(\lambda)$. It is readily seen that only the logarithm function can foot the bill for g . Then, a distribution f_n is called *associate homogeneous* of order \mathbf{n} and degree \mathbf{a} when there exists an associate homogeneous distribution f_{n-1} of order $\mathbf{n}-1$ and degree \mathbf{a} such that

$$f_n(\lambda x) = \lambda^a (f_n(x) + \log \lambda f_{n-1}(x)).$$

Clearly the renormalization of primitively divergent graphs in massless theories, using \mathbf{T}_μ , gives rise to associate homogeneous distributions of order 1. To pass associate homogeneous distributions of order 1 thru the same machine (12), in order to obtain a renormalized closed expression, is routine. Assuming no prior dependence of \mathbf{f}_1 on μ , one would have to use now

$$\mu \frac{\partial}{\partial \mu} T_\mu f_1 = \sum_{|\alpha|=k} \left(\int f_1 x^\alpha - \log \mu \int f_0 x^\alpha \right) \delta^\alpha$$

instead of (10), to simplify the output of (6). We omit the straightforward details. Of course, if $g = [f]_{R;\mu}$ for f primitive, then $\mathbf{T}_\mu g = g$; but diagrams with a renormalized subdivergence provide less trivial examples. The complete renormalization of massless theories gives rise exclusively to associate homogeneous distributions [2].

As remarked in [33], on the finite dimensional vector space spanned by an homogeneous distribution and its associates up to a given order, the Euler operator takes the Jordan normal form. The \mathbf{S}_{k+1} operators are nilpotent on that space, when $\mathbf{a} = -d - k$. For example,

$$S \left[\frac{1}{x^4} \right]_R = S \left(\frac{1}{x^4} \right),$$

and therefore $S^2\left[\frac{1}{x^4}\right]_R = 0$. In consonance with this, equation (15) remains valid for \mathbb{I} primitively divergent of the associated homogeneous type.

The main theoretical development of the \mathbb{I} -operator closes here. We have found two instances in the QFT literature of the improved causal method advocated by us (5): the “special \mathbb{R} operation” introduced in the outstanding (apparently unpublished) manuscript [34]; and, unwittingly, “natural” renormalization [25] —see subsection 5.2. The scaling properties are discussed in [34]; but no explicit formulae for the renormalized amplitudes are given there. We turn to matters of illustration and comparison.

3 Some examples

We compute the simplest primitive diagrams relevant for the four-point function of ϕ_4^4 theory —for these logarithmically divergent graphs, there is of course no difference between \mathbb{I}_w and \mathbb{W}_w . The following notation will be used in the sequel:

$$\Omega_{d,m} := \int_{|x|=1} x_i^{2m} = \frac{2 \Gamma(m + \frac{1}{2}) \pi^{(d-1)/2}}{\Gamma(m + \frac{1}{2}d)}, \quad (18)$$

where \mathbb{I} labels any component. In particular, $\Omega_{d,0} =: \Omega_d$, the area of the sphere in dimension d . The quotients of the $\Omega_{d,m}$ are rational:

$$\frac{\Omega_{d,m}}{\Omega_d} = \frac{(2m-1)!!}{(2m+d-2)(2m+d-4)\cdots d}.$$

The “propagator”, given by the formula

$$D_F(x) = \frac{|x|^{2-d}}{(d-2)\Omega_d},$$

when $d \neq 2$, and by $D_F(x) = (\log|x|)/\Omega_2$ when $d = 2$, is simply the Green function for the Laplace equation:

$$\Delta D_F(x) = -\delta(x).$$

Consider the “fish” diagram in ϕ_4^4 theory, giving the first correction to the four-point function. The corresponding amplitude is proportional to x^{-4} . On using (15) with $k=0$,

$$\left[\frac{1}{x^4}\right]_R = \partial^\beta \left[\frac{x^\beta \log(\mu|x|)}{x^4}\right] = S_1 \frac{\log(\mu|x|)}{x^4}.$$

Next we look at $\mu \frac{\partial}{\partial \mu} (T_\mu D_F^2)$. By direct computation, on the one hand,

$$\mu \frac{\partial}{\partial \mu} \left[\frac{1}{x^4}\right]_R = \partial^\beta \left[\frac{x^\beta}{x^4}\right] = -\frac{1}{2} \Delta \left(\frac{1}{x^2}\right) = \Omega_4 \delta(x),$$

and on the other, according to (10):

$$\mu \frac{\partial}{\partial \mu} \left[\frac{1}{x^4}\right]_R = c^0 \delta(x), \quad \text{with residue} \quad c^0 = \left\langle \frac{1}{x^4}, \delta(\mu^{-1} - |x|) |x| \right\rangle_{\mathbb{R}^4} = \Omega_4,$$

which serves as a check. In this case, as μ varies from 0 to ∞ , all possible renormalizations of D_F^2 are obtained. One can as well directly check here the equation:

$$T_\mu f(\lambda x) = \lambda^{-4} [T_\mu f(x) + \Omega_4 \log \lambda \delta(x)], \quad (19)$$

for $f(x) = 1/x^4$. Equation (19) contains the single most important information about the fish graph and is *essential* for the treatment of diagrams in which it appears as one of the subdivergences [2].

Next among the primitive diagrams relevant for the vertex correction comes the tetrahedron (also called the “open envelope”) diagram. In spite of appearances, it is a three-loop graph, as one of the circuits depends on the others; it is the lowest-order diagram in ϕ_4^4 theory with the full structure of the four-point function. The unrenormalized amplitude $f \in \mathcal{F}_{\text{prim}}(\mathbb{R}^{12})$ is of the form

$$f(x, y, z) = \frac{1}{x^2 y^2 z^2 (x-y)^2 (y-z)^2 (x-z)^2},$$

which is logarithmically divergent overall. A funny thing about this diagram is that the amplitude for it looks exactly the same in momentum space —see for instance [35]. Denote by \mathbf{x} the collective variable (x, y, z) . Then

$$[f]_R = S_{(x,y,z)} \left[\frac{\log(\mu|s|)}{x^2 y^2 z^2 (x-y)^2 (y-z)^2 (x-z)^2} \right].$$

Again, the most important information from the diagram concerns its dilatation properties. Proceeding as above, we obtain

$$\mu \frac{\partial}{\partial \mu} [f]_R =: I \delta(x),$$

where, for any $A > 0$,

$$I = \int_{s^2=A} \frac{|s| ds}{x^2 y^2 z^2 (x-y)^2 (y-z)^2 (x-z)^2}.$$

This integral is computable with moderate effort. First, one rescales variables: $y = |x|u$ and $z = |x|v$, to obtain

$$I = \int_{\mathbb{S}^3} d\left(\frac{x}{|x|}\right) \int \frac{d^4 u d^4 v}{u^2 v^2 (x/|x| - u)^2 (u - v)^2 (x/|x| - v)^2}.$$

The calculation is then carried out by means of ultraspherical polynomial [36, 37] techniques. We recall that these polynomials are defined from

$$(1 - 2xr + r^2)^{-n} = \sum_{k=0}^{\infty} C_k^n(x) r^k,$$

for $r < 1$. There follows an expansion for powers of the propagator

$$\frac{1}{(x-y)^{2n}} = \frac{1}{|y|^n} \sum_{k=0}^{\infty} C_k^n(xy) \left(\frac{|x|}{|y|} \right)^k,$$

if, for instance, $|x| < |y|$. Using their orthogonality relation

$$\int_{\mathbb{S}^l} d\left(\frac{x}{|x|}\right) C_k^n\left(\frac{xu}{|x||u|}\right) C_l^n\left(\frac{xv}{|x||v|}\right) = \delta_{kl} C_k^n\left(\frac{uv}{|u||v|}\right) \frac{n \Omega_d}{k+n},$$

to perform the angular integrals (in our case $n=1$, $l=3$), we obtain \mathbf{I} as the sum of six radial integrals, corresponding to regions like $|u| < |v| < 1$, and so forth. Each one is equivalent to $2\pi^6 \zeta(3)$. This yields finally the residue $12\pi^6 \zeta(3)$ —the geometrical factor Ω_4 is always present. In consequence, now

$$T_\mu f(\lambda x) = \lambda^{-12} [T_\mu f(x) + 12\pi^6 \zeta(3) \log \lambda \delta(x)].$$

This is the first diagram which has a nontrivial topology, from the knot theory viewpoint, and thus the appearance of a \mathbb{C} -value is expected [35].

Consider now the two-loop “setting sun” diagram that contributes to the two-point function in φ_4^2 theory; it will prove instructive. One has to renormalize $1/x^6$, and the singular order is 2. Off (15) we read that

$$\left[\frac{1}{x^6} \right]_R = \frac{1}{2} S_3 \left[\frac{\log(\mu|x|)}{x^6} \right] + \frac{3\pi^2}{8} \Delta \delta(x). \quad (20)$$

Clearly, our formulae come rather close to simply iterating the operator \mathbf{S} , as done in [32]. The last term obviously does not make a difference for the dilatation properties; but we shall soon strengthen the case for not dropping it. One has

$$\left[\frac{1}{x^6} \right]_R(\lambda x) = \lambda^{-6} \left(\left[\frac{1}{x^6} \right]_R + \frac{\Omega_4}{8} \log \lambda \Delta \delta(x) \right).$$

The reader may check that using W_u instead of T_u would bring to (20) the extra term $\pi^2 \mu^2 \delta(x)$, with an unwelcome μ -power dependence. As we know, this complicates the dilatation properties for the diagram. The terms polynomially dependent on μ are like the “junk DNA” of the Epstein–Glaser formalism, as they carry no useful information on the residues of QFT [2].

More generally, for quadratic divergences (such as also appear in the first (two-vertex) contribution to the two-point function of the φ_3^6 and φ_6^3 theories), one constructs the extension

$$[|x|^{-d-2}]_R = \frac{1}{2} S_3 \left(\frac{\log \mu |x|}{x^{d+2}} \right) + \frac{3\Omega_d}{4d} \Delta \delta(x), \quad (21)$$

and

$$[|x|^{-d-2}]_R(\lambda x) = \lambda^{-d-2} \left([|x|^{-d-2}]_R + \frac{\Omega_d}{2d} \log \lambda \Delta \delta(x) \right).$$

4 The comparison with the mathematical literature

4.1 On the real line

Now we must muster support for the choice of T_m and T_μ . For the basics of distribution theory, we recommend [38]. For concrete computations, a good place to start is the treatment by Hörmander in Section 3.2 of [39], of the extension problem for the distributions

$$f(x) = x_+^{-l}, x_-^{-l}, |x|^{-l}, |x|^{-l} \operatorname{sgn}(x), x^{-l}$$

on the real line. Of course, these are not independent: x_- is just the reflection of x_+ with respect to the origin, then $|x|^{-l} = x_+^{-l} + x_-^{-l}$, and so on; note that x^{-1} is just the ordinary Cauchy principal value of $1/x$.

On our side, for instance, $xf(x) = H(x)$ for $f(x) = x_+^{-1}$; $xf(x) = \operatorname{sgn}(x)$ for $f(x) = |x|^{-1}$; and so on. Then our formulae (15), for l odd, give

$$[x_+^{-l}]_R = \frac{(-)^{l-1}}{(l-1)!} \frac{d^l}{dx^l} (H(x) \log(\mu|x|)) + H_{l-1} \delta^{l-1},$$

or, say,

$$[|x|^{-l}]_R = \frac{(-)^{l-1}}{(l-1)!} \frac{d^l}{dx^l} (\operatorname{sgn}(x) \log(\mu|x|) + 2H_{l-1} \delta^{l-1}),$$

and, for l even, simply

$$[|x|^{-l}]_R = \frac{(-)^{l-1}}{(l-1)!} \frac{d^l}{dx^l} \log|x|. \quad (22)$$

Hörmander invokes the natural method of analytic continuation of x_+^z , with z complex, plus residue subtraction at the simple poles at the negative integers. Our formulae coincide with Hörmander's —see for example his (3.2.5)— provided that (a) we take $\mu = 1$; and (b) H_k defined in (13) equals (as anticipated in the notation) the sum of the first k terms of the harmonic series! This turns out to be the case, although the proof, that the curious reader can find in [40, Ch. 6], is not quite straightforward. Thus we understand that in (15) and similar formulae H_k just means $\sum_{j=1}^k 1/j$.

Encouraged by this indication of being on the right track, we take a closer look at the analytic continuation method. The point is that the function $z \mapsto \int_0^\infty x^z \phi(x) dx$ for $\Re z > -1$ is analytic, its differential being $dz \int_0^\infty x^z \log x \phi(x) dx$. Let us now consider the analytic continuation definition for x_+^z , where for simplicity we first take $-2 < \Re z < -1$. One gleans

$$\langle x_+^z, \phi \rangle = \int_0^\infty x^z R_0^0 \phi(x) dx.$$

We recall the proof of this:

$$\langle x_+^z, \phi \rangle := \left\langle \frac{1}{z+1} \frac{d}{dx} x_+^{z+1}, \phi \right\rangle = -\frac{1}{z+1} \lim_{\epsilon \downarrow 0} \int_\epsilon^\infty x^{z+1} \phi'(x) dx.$$

A simple integration by parts, taking $v = x^{z+1}$ and $u = \phi(x) - \phi(0)$, completes the argument.

Iterating the procedure, one obtains

$$\langle x_+^z, \phi \rangle = \int_0^\infty x^z R_0^{l-1} \phi(x) dx$$

for $-l-1 < \Re z < -l$, with l a positive integer. At $z = -l$, however, this formula fails because of the attendant *infrared* problem. Let us then compute the first two terms of the Laurent development of $\langle x_+^z, \phi \rangle$: in view of

$$\langle x_+^z, \phi \rangle = \int_0^{\mu^{-1}} x^z R_0^{l-1} \phi(x) dx + \int_{\mu^{-1}}^\infty R_0^{l-2} \phi(x) dx + \frac{\phi^{(l-1)}(0) \mu^{-(z+l)}}{(l-1)!(z+l)},$$

the pole part is isolated. Therefore

$$\lim_{z \rightarrow -l} \left[x_+^z - \frac{(-)^{l-1} \delta^{(l-1)}(x)}{(l-1)!(z+l)} \right] = T_\mu(x_+^l) - \delta^{l-1}(x) \log \mu.$$

Hörmander goes on to consider Hadamard's *finite part*: that is, for x_+ , one studies

$$\int_\epsilon^\infty x^z \phi(x) dx,$$

where ϕ is always a test function, for any $z \in \mathbb{C}$, and discards the multiples of powers $\epsilon^{-\theta}$, for nonvanishing θ with $\Re \theta > 0$, and the multiples of $\log \epsilon$. He proves that this finite part coincides with the result of the analytic continuation method.

We do not need to review his proof, as we can show directly the identity of our results with finite part, by the following trick:

$$\begin{aligned} \int_\epsilon^\infty \frac{\phi(x)}{x^l} dx &= \sum_{j=0}^{l-1} \int_\epsilon^{\mu^{-1}} \frac{\phi^{(j)}(0)}{j!} x^{j-l} dx + \sum_{j=0}^{l-2} \int_{\mu^{-1}}^\infty \frac{\phi^{(j)}(0)}{j!} x^{j-l} dx \\ &\quad + \int_\epsilon^{\mu^{-1}} x^{-l} R_0^{l-1} \phi(x) dx + \int_{\mu^{-1}}^\infty x^{-l} R_0^{l-2} \phi(x) dx. \end{aligned}$$

Then, as $\epsilon \downarrow 0$, the two last terms give rise to the $T_\mu(1/x^l)$ renormalization and the surviving finite terms cancel, except for the expected contribution $-\frac{\phi^{(l-1)}(0)}{(l-1)!} \log \mu$, coming from the first sum.

Denote the finite part of x_+^{-l} by $\text{Pf} \frac{H(x)}{x^l}$, where **Pf** stands for *pseudofunction* (or for *partie finite*, according to taste). In summary, we have proved:

Proposition 1. *On the real line, the \mathbf{T} -operator leads to a one-parameter generalization of the finite part and analytic continuation extensions, to wit,*

$$[x_+^{-l}]_R := T_\mu(x_+^{-l}) = \text{Pf} \frac{H(x)}{x^l} + \delta^{l-1}(x) \log \mu.$$

This generalization is in the nature of things. Actually, the finite part and analytic continuation methods are not nearly as uniquely defined as some treatments make them appear. For instance, at the negative integers the definition of the finite part of x_+^z changes if we substitute $A\epsilon$ for ϵ ; and, analogously, one can slip in a dimensionful scale in analytical prolongation formulae. The added flexibility of the choice of μ is convenient.

We parenthetically observe that the nonhomogeneity of T_μ , and then of Pf , is directly related to the presence of logarithmic terms in the asymptotic expansion for the heat kernels of elliptic pseudodifferential operators [26].

Finally, we remark that the Laurent development for $x_+^z|_{z=-l}$ continues:

$$\frac{\phi^{(l-1)}(0)}{\epsilon(l-1)!} + \text{Pf}(x_+^{-l}) + \frac{\epsilon}{2} \text{Pf}(x_+^{-l} \log x_+) + \frac{\epsilon^2}{3!} \text{Pf}(x_+^{-l} \log^2 x_+) + \dots \quad (23)$$

with $\epsilon := z + l$ and the obvious definition for

$$\text{Pf}(x_+^{-l} \log^m x_+) = [x_+^{-l} \log^m x_+]_{R; \mu=1}.$$

4.2 Dimensional reduction

The phrase “dimensional reduction” is used in the sense of ordinary calculus, it does not refer here to the method of renormalization of the same name. The reader may have wondered why we spend so much time on elementary distributions on \mathbb{R} . The reason, as it turns out, is that an understanding of the 1-dimensional case is all that is needed for the renormalization of $|x|^{-d-k}$, for any k and in any dimension d ; thus covering the basic needs of Euclidean field theory. For instance, one can define $[x^{-4}]_R$ on \mathbb{R}^4 from knowledge of x_+^{-1} on \mathbb{R} .

Denote $r := |x|$ and let $f(r)$ be an amplitude on \mathbb{R}^d , depending only on the radial coordinate, in need of renormalization. We are ready now to simplify (15) by a method that generalizes Proposition 1 to any number of dimensions.

Given an arbitrary test function ϕ , consider its projection onto the radial-sum-values function $\phi \mapsto P\phi$ given by

$$P\phi(r) := \int_{|y|=1} \phi(ry).$$

We compute the derivatives of $P\phi$ at the origin: $(P\phi)^{(2m+1)}(0) = 0$ and

$$(P\phi)^{(2m)}(0) = \Omega_{d,m} \Delta^m \phi(0).$$

To prove this, whenever all the β 's, and thus n , are even, use

$$(P\phi)^{(n)}(0) = \sum_{|\beta|=n} \frac{n! \partial^\beta \phi}{\beta!} \Big|_{x=0} \int_{|y|=1} y_1^{\beta_1} \dots y_n^{\beta_n} = \frac{2 \Gamma(\frac{\beta_1+1}{2}) \dots \Gamma(\frac{\beta_n+1}{2})}{\Gamma(\frac{n+d}{2})},$$

in consonance with (18); the integral vanishes otherwise. Note that $P\phi$ can be considered as an even function defined on the whole real line. Then, whenever the integrals make sense,

$$\langle f(r), \phi(x) \rangle_{\mathbb{R}^d} = \langle f(r) r^{d-1}, P\phi(r) \rangle_{\mathbb{R}^+},$$

which in particular means that extension rules for $H(r)f(r)$ on \mathbb{R} give extension rules for $f(r)r^{d-1}$ on \mathbb{R}^d . This we call dimensional reduction.

Before proceeding, let us put the examined real line extensions in perspective, by investigating how satisfactory our results are from a general standpoint, and whether alternative renormalizations with better properties might exist. Note first, from (15):

$$\text{Pf} \frac{H(x)}{x} = \frac{d}{dx}(H(x) \log |x|).$$

For z not a negative integer, the property

$$x x_+^z = x_+^{z+1} \quad (24)$$

obtains; and excluding $z = 0$ as well, we have

$$\frac{d}{dx} x_+^z = z x_+^{z-1}. \quad (25)$$

One can examine how the negative integer power candidates fare in respect of these two criteria: of course, except for x_+^{-l} , which keeps all the good properties, homogeneity is irretrievably lost.

Actually, it is x_+^{-l} that we need. One could *define* a renormalization $[x_+^{-l}]_{\text{diff}}$ of x_+^{-l} simply by

$$[x_+^{-l}]_{\text{diff}} := (-)^{l-1} \frac{1}{(l-1)!} \frac{d^l}{dx^l} (H(x) \log |x|),$$

so $[x_+^{-1}]_{\text{diff}} = \text{Pf} \frac{H(x)}{x}$; and automatically the second (25) of the requirements

$$\frac{d}{dx} [x_+^{-l}]_{\text{diff}} = -l [x_+^{-l-1}]_{\text{diff}}$$

would be fulfilled. This would be “differential renormalization” in a nutshell. It differs from the other extensions studied so far: from our previous results,

$$[x_+^{-l}]_{\text{diff}} = [x_+^{-l}]_R + (-)^l (H_{l-1} + \log \mu) \delta^{l-1}(x).$$

On the other hand, it is seen that

$$\frac{d}{dx} T_\mu(x_+^{-l}) = -l T_\mu(x_+^{-l-1}) + \delta^l(x),$$

so that T_μ does not fulfill that second requirement; but in exchange, it does fulfill the first one (24):

$$x_+ T_\mu(x_+^{-l}) = T_\mu(x_+^{-l+1}).$$

There is no extension of x_+^a for which both requirements simultaneously hold.

It looks as if we are faced with a choice between $[x_+^a]_{\text{diff}}$ and $T_\mu(\cdot)$ —which is essentially $\text{Pf}(\cdot)$ — each one with its attractive feature. But the situation is in truth not symmetrical: in higher dimensional spaces the analogue of the first requirement *can* be generalized to the

renormalization of $|x|^{-k}$; whereas the analog of the second then *cannot* be made to work —have a sneak preview at (26).

Estrada and Kanwal *define* then, for $k \geq 0$ [41, 42],

$$\begin{aligned} \left\langle \text{Pf}\left(\frac{1}{r^{d+k}}\right), \phi(x) \right\rangle_{\mathbb{R}^d} &:= \left\langle \text{Pf}\left(\frac{1}{r^{k+1}}\right), P\phi(r) \right\rangle_{\mathbb{R}^+}; \\ \left\langle \left[\frac{1}{r^{d+k}}\right]_{\text{diff}}, \phi(x) \right\rangle_{\mathbb{R}^d} &:= \left\langle \left[\frac{1}{r^{k+1}}\right]_{\text{diff}}, P\phi(r) \right\rangle_{\mathbb{R}^+}. \end{aligned}$$

In view of (22), the case k odd is very easy, and then all the definitions coincide:

$$\text{Pf}\left(\frac{1}{r^{d+k}}\right) = T_\mu\left(\frac{1}{r^{d+k}}\right) = \left[\frac{1}{r^{d+k}}\right]_{\text{diff}} = r^z|_{z=-d-k},$$

the function r^z having a removable singularity at $-d-k$. However, in most instances in QFT k happens to be even, so we concentrate on this case. We are not in need of new definitions. By going through the motions of changing to radial plus polar coordinates and back, one checks that, assuming a spherically symmetric weight function w , the evaluation $\langle T_w f(r), \phi(x) \rangle$ is equal to

$$\left\langle f(r), \phi(x) - \phi(0) - \frac{\Delta\phi(0)}{2!d}r^2 - \dots - w(r)\frac{\Omega_{d,m}\Delta^m\phi(0)}{(2m)!\Omega_d}r^{2m} \right\rangle;$$

the right hand side being invariant under T_w . This was perhaps clear from the beginning, from symmetry considerations. It means in particular that the different putative definitions of T_μ on \mathbb{R}^d obtained from T_μ on the real line all *coincide* with the original definition, that is:

Proposition 2. *The T_μ operators are mutually consistent under dimensional reduction.*

Moreover,

$$r^{2q} T_\mu(r^{-d-2m}) = T_\mu(r^{-d-2m+2q})$$

follows, by using the easy identity

$$r^2 \Delta^m \delta(x) = 2m(2m+d-2) \Delta^{m-1} \delta(x).$$

Therefore, it is now clear that

$$T_\mu(r^{-d-2m}) = \text{Pf}(r^{-d-2m}) + \frac{\Omega_{d,m}\Delta^m\delta(x)}{\Omega_d(2m)!} \log \mu.$$

It remains to compute the derivatives. A powerful technique, based on “truncated regularization” and calculation of the derivatives across surface jumps, was developed and clearly explained in [41]. It is rather obvious that for $k-d$ odd the “naïve” derivation formulae

(see right below) will apply. Whereas for $k-d=2m$ even, they obtain extra delta function terms; in particular for the powers of the Laplacian

$$\begin{aligned} \Delta^n \left[\frac{1}{r^{d+2m}} \right]_{\text{diff}} &= (d+2m+2n-2) \cdots (d+2m+2)(d+2m)(2m+2) \cdots \\ &\times (2m+2n) \left[\frac{1}{r^{d+2m+2n}} \right]_{\text{diff}} + \frac{\Omega_{d,m}}{(2m)!} \sum_{l=1}^n \frac{\Delta^n \delta(x)}{2m+2l-1}. \end{aligned} \quad (26)$$

The first term is what we termed the “naïve” formula.

Estrada and Kanwal do not explicitly give the powers of Δ for finite part. But from (26) is a simple task to compute

$$\begin{aligned} \Delta^n \left[\frac{1}{r^{d+2m}} \right]_R &= (d+2m+2n-2) \cdots (d+2m+2)(d+2m)(2m+2) \cdots \\ &\times (2m+2n) \left[\frac{1}{r^{d+2m+2n}} \right]_R - \frac{\Omega_{d,m}}{(2m)!} \sum_{l=1}^n \frac{(4(m+l)+d-2) \Delta^n \delta(x)}{2(m+l)(2m+2l+d-2)}. \end{aligned}$$

No one seems to have computed explicitly the distributional derivatives of the $\text{Pf}(x_+^{-l} \log^m x_+)$ and the correspondingly defined $\text{Pf}(r^{-l} \log^m r)$, although they might be quite helpful for Euclidean QFT on configuration space.

We next enterprise to tackle a comparison with methods of renormalization in real space in the physical literature. Of those there are not many: it needs to be said that the flame-keepers of the Epstein–Glaser method [43] actually work in momentum space (using dispersion relation techniques). Euclidean configuration space dimensional regularization, on the other hand, starting from [37], evolved into a powerful calculational tool in the eighties. With the advent of “differential renormalization” [24] in the nineties, regularization-free coordinate space techniques came into their own: they are the natural “market competitors” for the ideas presented here.

We deal first with dimensional regularization.

5 Comparison with the QFT literature

5.1 Dimensional regularization and “minimal subtraction”

Dimensional regularization on real space, for primitively divergent diagrams, can be identified with analytic continuation. To get the basic idea, it is perhaps convenient to perform first a couple of blind calculations. Start from the identity

$$\mu^\epsilon |x|^{-d+\epsilon} = \frac{\mu^\epsilon}{\epsilon} S_x(|x|^{-d+\epsilon}).$$

Then, expanding in ϵ , on use of (15), it follows that

$$\mu^\epsilon |x|^{-d+\epsilon} = \Omega_d \frac{\delta(x)}{\epsilon} + S_x \frac{\log(\mu|x|)}{|x|^d} + O(\epsilon).$$

The first term is a typical infinite (as $1/\epsilon$) counterterm of the dimensionally regularized theory. The order of the delta function derivative, 0 in this case, tells us that we are dealing with a logarithmic divergence. The coefficient Ω_d of the counterterm, or QFT residue, coincides with our *scaling coefficient* of Section 2. The second term is precisely $[1/x^d]_R$, our renormalized expression.

Let us go to quadratic divergences. A brute-force computation establishes for them the differential identity

$$\mu^\epsilon |x|^{-d-2+\epsilon} = \frac{\mu^\epsilon}{2\epsilon(1 - \frac{3}{2}\epsilon + \frac{1}{2}\epsilon^2)} S_3(|x|^{\epsilon-d-2}). \quad (27)$$

On the other hand, from (15):

$$S_3(|x|^{-d-2}) = \frac{\Omega_d}{d} \Delta\delta(x). \quad (28)$$

Performing in (27) the expansion with respect to ϵ , this yields

$$\mu^\epsilon |x|^{-d-2+\epsilon} = \frac{\Omega_d}{2d\epsilon} \Delta\delta(x) + \frac{1}{2} S_3\left(\frac{\log \mu |x|}{x^{d+2}}\right) + \frac{3\Omega_d}{4d} \Delta\delta(x) + O(\epsilon).$$

That is,

$$\mu^\epsilon |x|^{-d-2+\epsilon} = \frac{\Omega_d}{2d} \frac{\Delta\delta(x)}{\epsilon} + [x]^{d-2}_R + O(\epsilon).$$

A pattern has emerged: as before, there is a unique counterterm in $\frac{1}{\epsilon}$; the residue coincides with our scaling coefficient; the order of the delta function derivative reminds us of the order of the divergence we are dealing with; and the “constant” regular term is precisely $[1/x]^{d+2}_R$ constructed in (21) according to our renormalization scheme.

The correspondence between the two schemes, at the present level, is absolute and straightforward. It is then a foregone conclusion that we shall have μ -independent residues, always coincident with the scaling factors, for the simple poles of $1/x^{d+2m}$, and that the first finite term shall coincide with T_μ , provided we identify our scale with 't Hooft's universal one. This is an immediate consequence of the Laurent development (23), transported to \mathbb{R}^d by dimensional reduction. In symbols

$$\mu^\epsilon |x|^{\epsilon-d-2m} = \frac{\Omega_{d,m}}{(2m)!\epsilon} \Delta^m \delta(x) + [x]^{-d-2}_R + O(\epsilon). \quad (29)$$

This substantiates the claim that T_μ effects a kind of minimal subtraction. Let us point out, in the same vein, that already in [26] the analytic continuation of Riemann's zeta function was evaluated as the outcome of a quantum field theory-flavoured renormalization process.

A word of warning is perhaps in order here. Performing the Fourier transform of these identities, we do not quite obtain the usual formulae for dimensional regularization in momentum space. The nonresemblance is superficial, though, and related to choices of “renormalization prescriptions”. The beautiful correspondence is “spoiled” (modified) as well for diagrams with subdivergences, because in dimensional regularization contributions will come to $O(\epsilon^0)$ from the higher terms of the ϵ -expansion, when multiplied by the unavoidable

singular factors; but, again, the difference is not deep: we show in III how one organizes the Laurent expansions with respect to ϵ so as to make the correspondence with the ϵ -subtraction transparent.

Much was made in [44], and rightly so, of the importance of the perturbative residues in the dimensional regularization scheme. Residues for primitive diagrams are the single most informative item in QFT. The coefficients of higher order poles are determined by the residues —consult the discussion in [45]. Now, the appeal of working exclusively with well-defined quantities, as we do, would be much diminished if that information were to disappear in our approach. But we know it is not lost: it is stored in the scaling properties.

5.2 Differential renormalization and “natural renormalization” in QFT

Differential renormalization, in its original form, turns around the following extension of $1/x^4$ (in \mathbb{R}^4):

$$[1/x^4]_{\text{R,FJL}} := -\frac{1}{4} \Delta \frac{\log \mu^2 x^2}{x^2}. \quad (30)$$

At present, two main schools of differential renormalization seem still in vogue: the original and more popular “(constrained) differential renormalization” of the Spanish school —see for instance [46]— and the “Russian school” —inaugurated in [32]. This second method, as already reported, reduces to systematic use of the operators S_{k+1} , i.e., to our formulae (15) without the delta terms. Whereas the first school has its forte in concrete 1-loop calculations for realistic theories, assuming compatibility of differentiation with renormalization, the second initially stressed the development of global renormalization formulae for diagrams with subdivergences, and the compatibility of Bogoliubov’s rules with renormalization.

Hereafter, we refer mainly to the original version. It proceeded from its mentioned starting point to the computation of more complicated diagrams by reductions to two-vertex diagrams. This involves a bewildering series of tricks, witness more of the ingenuity of the inventors than of the soundness of the method. V. gr., the tetrahedron diagram (considered already) is rather inelegantly renormalized by the substitution $1/x^2 \mapsto x^2 [1/x^4]_{\text{R,FJL}}$. They get away with it, in that particular case, because their expression is still not infrared divergent. But in nonprimitive diagrams infrared infinities may arise in relation with the need to integrate the product of propagators over the coordinates of the *internal* vertices in the diagram, and, in general, under the procedures of differential renormalization it is impossible to avoid incurring infrared problems [47].

Even for primitively divergent diagrams, differential renormalization is not free of trouble. In his extremely interesting paper, Schnetz [25] delivers a critique of differential renormalization. In elementary fashion, notice that

$$\frac{x^\mu \log(\mu^2 x^2)}{x^4} = -\frac{1}{2} \partial^\mu \left[\frac{1 + \log(\mu^2 x^2)}{x^2} \right],$$

and so

$$[1/x^4]_R = -\frac{1}{4} \Delta \frac{1 + \log(\mu^2 x^2)}{x^2}.$$

This is to say:

$$[1/x^4]_R - [1/x^4]_{\text{R,FJL}} = \pi^2 \delta(x). \quad (31)$$

We contend that “our” $[1/x^4]_R$ and not $[1/x^4]_{R,FJJ}$ is the right definition. Of course, one is in principle free to add certain delta terms to each individual renormalization and proclaim that to be the “right” definition. However, $1/x^4$ on \mathbb{R}^4 is dimensionally reduced to x_+^{-4} on \mathbb{R}^+ and because, as already pointed out, differential renormalization of this distribution is consistent with $[x_+^{-1}]_R$ for $\mu = 1$, the $[\cdot]_{R,FJJ}$ definition is *inconsistent* with any of the natural alternatives we established in the previous subsection. (It would clearly induce back an extra δ term in the definition of $[x_+^{-1}]_R$ on the real line, fully unwelcome in the context.)

In other words, if we want to make use both of sensible rules of renormalization for the radial integral (namely, including differential renormalization at this level) and of Freedman, Johnson and Latorre’s formulae, we have to relinquish the standard rules of calculus. This Schnetz noticed.

Schnetz proposes instead a “natural renormalization” procedure on \mathbb{R}^4 , boiling down to the rule

$$\Delta^{n+1} \frac{\log(\mu^2 x^2)}{x^2} = - \left[\frac{4^{n+1} n! (n+1)!}{x^{2n+4}} \right]_R + \left(8\pi^2 H_n + \frac{4\pi^2}{n+1} \right) \Delta^n \delta(x), \quad (32)$$

whose first instance is precisely the previous equation (31). This he found by heuristically defining “natural renormalization” as the one that relates renormalization scales at different dimensions without changing the definition of ordinary integrals or generating ϵ -dependence in the renormalization of ϵ -independent integrals; and by elaborate computations to get rid of the angular integrals.

His calculation is any rate correct, and the results can be read off (for $d = 4$, $m = 0$) our (26), taking into account (30) and (31). We have proved that our operator T_w in the context just amounts to “natural renormalization”.

Shortly after the inception of the differential renormalization, it naturally occurred to some people that a definite relation should exist between it and dimensional regularization. However, because of the shortcomings of the former, they landed on formulae both messy and incorrect [48]. The reader is invited to compare them with our (29).

The more refined version of differential renormalization in [32], coincides with our formulae for logarithmic divergences and eludes the main thrust of Schnetz’s critique; however, we have seen that in general it does not yield the Laurent development of the dimensionally regularized theory either. On the other hand, it must be said that the emphasis in [24, 25] in bringing in the Laplacian instead of the less intuitive albeit more fundamental S_k operators has welcome aspects, not only because of the enhanced feeling of understanding, but also in that it makes the transition to momentum space a trivial affair, as soon as the Fourier transform of the (evidently tempered) distribution $x^{-2} \log(\mu^2 x^2)$ is known.

The trinity of basic definitions in differential renormalization is then replaced by the

identities

$$\begin{aligned}\left[\frac{1}{x^4}\right]_R &= -\frac{1}{2}\Delta\frac{\log\mu|x|}{x^2} + \pi^2\delta(x); \\ \left[\frac{1}{x^6}\right]_R &= -\frac{1}{16}\Delta^2\frac{\log\mu|x|}{x^2} + \frac{5}{8}\pi^2\Delta\delta(x); \\ \left[\frac{\log\mu|x|}{x^4}\right]_R &= -\frac{1}{4}\Delta\frac{\log^2\mu|x| + \log\mu'|x|}{x^2} + \frac{\pi^2}{2}\delta(x);\end{aligned}$$

the δ 's being absent in standard differential renormalization. In the next Section 6 we shall see another demonstration of their importance.

The kinship of the EG method with differential renormalization *à la* Smirnov and Zavalov was recognized by Prange [28]; he was stumped for nonlogarithmic divergences, though. See [49] in the same vein.

5.3 The connection with BPHZ renormalization

We still have left some chips to cash. We elaborate next the statement that BPHZ subtraction has no independent status from Epstein–Glaser, and that the validity of that renormalization method is just a corollary of the latter. This involves just a two-line proof.

The Fourier transforms of the causally renormalized amplitudes exist at least in the sense of tempered distributions. They are in fact rather regular. Taking Fourier transforms is tantamount to replacing the test function by an exponential, which, according to the Cesàro theory of [26, 27], can preclude smoothness of the momentum space amplitude only at the origin. The appearance of an (integrable) singularity at $p=0$ is physically expected in a theory of massless particles.

Let us fix our conventions. We define Fourier and inverse Fourier transforms by

$$F[\phi](p) := \hat{\phi}(p) := \int \frac{d^d x}{(2\pi)^{d/2}} e^{-ipx} \phi(x),$$

and

$$F^{-1}[\phi](p) := \check{\phi}(p) := \int \frac{d^d x}{(2\pi)^{d/2}} e^{ipx} \phi(x),$$

respectively. It follows that

$$(x^\mu \phi)^\sim(p) = (-i)^\mu \partial^\mu \check{\phi}(p),$$

where μ denotes a multiindex; so that, in particular,

$$(x^\mu)^\sim(p) = (-i)^\mu (2\pi)^{d/2} \delta^{(\mu)}(p).$$

Also,

$$\partial^\mu \phi(0) = (-i)^\mu (2\pi)^{-d/2} \langle p^\mu, \check{\phi} \rangle. \quad (33)$$

From this and the following consequence of (2):

$$\langle F[\tilde{f}], F^{-1}[\phi] \rangle = \langle F[f], F^{-1}[R_0^k \phi] \rangle,$$

there follows at once

$$F[\tilde{f}](p) = R_0^k F[f](p).$$

This is nothing but the BPHZ subtraction rule in momentum space.

We hasten to add:

- An expression such as $F[f]$ is *not* a priori meaningless: it is a well defined functional on the linear subspace of Schwartz functions ϕ whose first momenta $\int p^\alpha \phi(p) d^d p$ up to order $k+1$ happen to vanish. (This is the counterpart $F\mathcal{S}_{k+1}$, according to (33), of the distributions on real space acting on test functions vanishing up to order $k+1$ at the origin.)
- Moreover, explicit expressions for these functionals on the external variables are given precisely by the unrenormalized momentum space amplitudes!

This circumstance constitutes the (deceptive) advantage of the BPHZ formalism for renormalization. We say deceptive because —as persuasively argued in [34]— the BPHZ method makes no effective use of the recursive properties of renormalization (paper III) and then, when using it, prodigious amounts of energy must go into proving convergence of, and/or computing, the (rather horrendous) resulting integrals, into showing that the Minkowskian counterparts define *bona fide* distributions. . . Much more natural to remain on the nutritious ground of distribution theory on real space, throughout. But this has never been done.

- Also, the $\partial^\mu F[f](0)$ for $|\mu| \leq k$ exist for massive theories.

For zero-mass models, the basic BPHZ scheme runs into trouble; this is due naturally to the failure of $\partial^\mu \tilde{f}(0)$ to exist for $|\mu| = k$, on account of the infrared problem. Now, one can perform subtraction at some external momentum $q \neq 0$, providing a mass scale. This is just the Fourier-mirrored version of standard EG renormalization, with weight function e^{-iqx} ; one only has to remember to use (7) instead of (4).

It is patent, though, that this last subtraction is quite awkward in practice, and will introduce in the Minkowskian context a noncovariance which must be compensated by further subtractions. This prompted Lowenstein and Zimmermann to introduce their “soft mass insertions” [50]. Which amounts to an epicycle too many.

In the light of the approach advocated in this paper, there exist several simpler and more physical strategies.

- One strategy is to recruit our basic formula (5) in momentum space

$$F[f](p) - j_0^{k-1} F[f](p) - \sum_{|\mu|=k} \frac{\partial^\mu F[f w](p) p^\mu}{\mu!}.$$

Still with $w(x) = e^{-iqx}$, this leads at once to

$$F[f](p) - j_0^{k-1} F[f](p) - \sum_{|\mu|=k} \frac{\partial^\mu F[f](q) p^\mu}{\mu!}.$$

Note that the difference between two of these recipes is polynomial in p'' , with $|\mu| = k$ only, as it should. This can be more easily corrected for Lorentz covariance, should the need arise [1].

- A second method is to exploit homogeneity in adapting our recipes for direct use in momentum space, in the spirit of [51] and [52].
- A third one is to perform Fourier analysis on our previous results. One has

$$\int \frac{d^4x}{(2\pi)^2} e^{-ipx} \frac{\log(\mu|x|)}{x^2} = -\frac{1}{p^2} \log\left(\frac{C|p|}{2\mu}\right),$$

where $C := e^\gamma \simeq 1.781072\dots$ with γ the Euler–Mascheroni constant. Then, from (32), for instance for the “fish” diagram in the φ_4^4 model:

$$[\widehat{1/x^4}]_R(p) = \frac{1}{4}[1 - \log(C^2 p^2/4\mu^2)],$$

and more generally:

$$[\widehat{1/x^{2k+4}}]_R(p) = \frac{(-)^{k+1} p^{2k}}{4^{k+1} k! (k+1)!} \left[2 \log \frac{|p|}{2\mu} - \Psi(k+1) - \Psi(k+2) \right],$$

where $\Psi(x) := d/dx(\log \Gamma(x))$ has been invoked, and we recall that

$$\Psi(n) = -\gamma + H_{n-1}.$$

For the setting sun diagram in the φ_4^4 model, in particular:

$$[\widehat{1/x^6}]_R = \frac{p^2}{16} \left(\log \frac{|p|\gamma}{2\mu} - \frac{5}{4} \right). \quad (34)$$

6 Some examples in massive theories

The aim of this short section is to dispel any idea that the usefulness of EG-type renormalization, and in particular of the $\overline{\text{MS}}$ -subtraction, is restricted to massless models. The overall conclusion, though, is that the massless theory keeps a normative character. Our purposes being merely illustrative, we liberally borrow from Schnetz [25], Prange [28] and Haagensen and Latorre [53].

The first example is nothing short of spectacular. Suppose we add to our original Lagrangian for φ_4^4 a mass term $\frac{1}{5}m^2\varphi^2$ and treat it as a perturbation, for the calculation of the new propagator. Then we would have for $D_F(x)$:

$$\frac{1}{x^2} - \int dx' \frac{m^2}{(x-x')^2 x'^2} + \int dx' dx'' \frac{m^4}{(x-x')^2 (x'-x'')^2 x''^2} - \dots$$

This “nonrenormalizable” interaction is tractable with our method. We work in momentum space, so we just have to consider the renormalization of $1/p^{2k}$ for $k > 1$. This is read

directly off (34), by inverting the roles of μ and m , with the proviso that μ gets replaced by $1/\mu$, in order to keep the correct dimensions. Then the result is

$$D_F(x) = \frac{1}{x^2} + \frac{m^2}{2} \sum_{n=0}^{\infty} \frac{m^{2n} x^{2n}}{4^n n!(n+1)!} \left(\log \frac{\mu|x|}{2} - \Psi(n+1) - \Psi(n+2) \right).$$

On naturally identifying the scale $\mu = m$, one obtains on the nose the *exact* expansion of the *exact* result

$$D_F(x) = \frac{m}{|x|} K_1(m|x|).$$

Here K_1 is the modified Bessel function of order 1. Had we kept the original EG subtraction with a $H(\mu - |p|)$ weight, we would earn a surfeit of terms with extra powers of μ , landing in a serious mess.

It is also interesting to see how well or badly fare the other “competitors”. Differential renormalization gives a expression of similar type but with different coefficients:

$$\frac{1}{x^2} + \frac{m^2}{2} \sum_{n=0}^{\infty} \frac{m^{2n} x^{2n}}{4^n n!(n+1)!} \left(\log \frac{\mu|x|}{2} + 2\gamma \right).$$

To obtain the correct result, it is necessary to substitute a different mass scale μ_n for each integral and to adjust ad hoc an *infinity* of such parameters. Dimensional regularization (plus “minimal” subtraction of a pole term for each summand but the first) fares slightly better, as it “only” misses the $\Psi(n+2)$ terms [25]. The distribution-theoretical rationale for the success of the “illegal” expansion performed is explained in [25].

Let us now look at the fish diagram in the massive theory. It is possible to use (3) instead of (6). Make the change of variables:

$$t = \frac{|x|}{s}; \quad dt = -\frac{|x| ds}{s^2}.$$

One gets, for the renormalized amplitude,

$$-S \left[\frac{m^2}{x^4} \int_{|x|}^{\infty} ds s K_1(ms)^2 \right].$$

Now,

$$\int ds s K_1(ms)^2 = \frac{s^2}{2} \left(K_0^2(s) + 2 \frac{K_0(s) K_1(s)}{s} - K_1^2(s) \right)$$

can be easily checked from

$$K_0'(s) = K_1(s); \quad K_1'(s) = -K_0(s) - \frac{K_1(s)}{s}.$$

The final result is then

$$\Delta \left[\frac{m^2}{2} \left(K_0^2(m|x|) - K_1^2(m|x|) + \frac{K_0(m|x|) K_1(m|x|)}{m|x|} \right) \right].$$

Had we used (6), the upper limit of the integral would become $1/\mu$, and the result would be modified by

$$\frac{m^2}{4\mu^2 x^2} (K_1^2(m/\mu) - K_2(m/\mu)K_0(m/\mu)).$$

At the “high energy” limit, as $\mu \uparrow \infty$ and $|x| \downarrow 0$, this interpolates between the previous result and the renormalization in the massless case.

However, this method becomes cumbersome already for renormalizing D_r^3 . It is convenient to modify the strategy, and to use in this context differential renormalization, corrected in such a way that the known renormalized mass zero limit is kept. This idea succeeds because of the good properties of our subtraction with respect to the mass expansion. For instance, away from zero [53],

$$\left(\frac{mK_1(m|x|)}{|x|} \right)^3 = \frac{m^2}{16} (\Delta - 9m^2)(\Delta - m^2) (K_0(m|x|)K_1^2(m|x|) + K_0^3(m|x|)).$$

Note the three-particle “threshold”. To this Haagensen and Latorre add a term of the form

$$\frac{\pi^2}{4} \log \frac{2\mu}{\gamma m} \Delta \delta(x),$$

to which, for reasons sufficiently explained, we should add a term of the form $\frac{5\pi^2}{8} \Delta \delta(x)$. A term proportional to δ (thus a mass correction) is also present. As they indicate, it is better fixed by a renormalization prescription.

7 Conclusion

We have delivered the missing link of the EG subtraction method to the standard literature on extension of distributions. The improved subtraction method sits at the crossroads in regard to dimensional regularization in configuration space; differential renormalization; “natural” renormalization; and BPHZ renormalization. The discussions in the previous sections go a long way to justify the conjecture (made by Connes, and independently by Estrada) that Hadamard’s finite part theory is in principle enough to deal with quantum field theory divergences. To accomplish that feat, however, it must go under the guise of the \mathbb{V} -projector; this gives the necessary flexibility to deal with complicated diagrams with subdivergences [2].

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