Noncommutative version of an arbitrary nondegenerated mechanics.

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Abstract

A procedure to obtain noncommutative version for any nondegenerated dynamical system is proposed and discussed.

It is known that the noncommutative geometry [1, 2] of the position variables in some mechanical models can be obtained [3-8] as the result of direct canonical quantization [9, 10] of underlying dynamical systems with second class constraints. Nontrivial bracket for the position variables appears in this case as the Dirac bracket, after taking into account the constraints presented in the model. In this note we show how to obtain the noncommutative version of any nondegenerated mechanical system. Namely, the following statement will be demonstrated.

Let $S = \int dt L(q^A,\ \dot{q}^A)$ is action of some nondegenerated system, and let $L_1(q^A,\ \dot{q}^A,\ v_A)$ is the corresponding first order Lagrangian (see below). Then the corresponding noncommutative version is $S_N = \int dt \left[L_1(q^A,\ \dot{q}^A,\ v_A) + \dot{v}_A\theta^{AB}v_B\right]$. Namely, the system S_N has the following properties:

1) It has the same number of physical degrees of freedom as the initial system S.

2) Equations of motion of the system are the same as for the initial system S, modulo the term which is proportional to the parameter θ^{AB}.

3) Configuration space variables have the noncommutative brackets: $\{q^A,\ q^B\} = -2\theta^{AB}$.

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We point also that quantization of the system S_N leads to quantum mechanics with ordinary product replaced by the Moyal product, similarly to the case of a particle on noncommutative plane.

Let us present details. Our starting point is some nondegenerated mechanical system with the configuration space variables $q^A(t),\ A = 1, 2, \dots, n$, and the Lagrangian action

$$S = \int dt L(q^A,\ \dot{q}^A). \tag{1}$$

Due to nondegenerate character of the system, there are no constraints in the Hamiltonian formulation. Let p_A are conjugated momentum for q^A, one can write the Hamiltonian action

$$S_H = \int dt \left[p_A\dot{q}^A - H_0(q^A,\ p_A)\right]. \tag{2}$$

Equations of motion which follow from Eq.(1) and (2) are equivalent (they remain equivalent for any degenerated system also [10, 11]). Equivalently, one can describe the initial system (1) by means of the first order Lagrangian action

$$S_1 = \int dt \left[v_A\dot{q}^A - H_0(q^A,\ v_A)\right]. \tag{3}$$

Here $q^A(t),\ v_A(t)$ are the configuration space variables of the formulation [1]. The noncommutative version of the system (1) is described by the following Lagrangian action

$$S_N = \int dt \left[v_A\dot{q}^A - H_0(q^A,\ v_A) + \dot{v}_A\theta^{AB}v_B\right], \tag{4}$$

where θ^{AB} is some constant matrix. It turns out to be the noncommutativity parameter for the variables q^A.

To analyse the physical sector of the Lagrangian system (4), we rewrite it in the Hamiltonian form. All the expressions for determining of the momentum turn out to be the primary constraints of the model $(p_A, \pi^A$ are conjugated momentum for the variables $q^A, v_A)$

$$G_A \equiv p_A - v_A = 0, \qquad T^A \equiv \pi^A - \theta^{AB}v_B. \tag{5}$$

[1] The Lagrangian formulations (1), (3) are equivalent. Actually, denoting the conjugated momentum for the variables $q^A,\ v_A$ as $p_A,\ \pi^A$ one finds, in the Hamiltonian formulation for the action (3), the second class constraints $p_A - v_A = 0,\ \pi^A = 0$. Introducing the corresponding Dirac bracket, one can treat the constraints as the strong equations. Then the Hamiltonian formulation for (3) is the same as for (1), namely Eq.(2).

The Hamiltonian is

$$H = H_0(q^A,\ v_A) + \lambda_1^A G_A + \lambda_{2A} T^A, \tag{6}$$

where λ are the Lagrangian multipliers for the constraints. On the next step of the procedure there are appear only equations for determining of the Lagrangian multipliers

$$\lambda_{2A} = -\frac{\partial H_0}{\partial q^A}, \qquad \lambda_1^A = \frac{\partial H_0}{\partial v_A} - 2\theta^{AB}\frac{\partial H_0}{\partial q^B}. \tag{7}$$

Equations of motion follow from (6), (7)

$$\begin{aligned} \dot{q}^A &= \frac{\partial H_0}{\partial v_A} - 2\theta^{AB}\frac{\partial H_0}{\partial q^B}, \qquad & \dot{p}_A &= -\frac{\partial H_0}{\partial q^A}; \\ \dot{v}_A &= -\frac{\partial H_0}{\partial q^A}, & \dot{\pi}^A &= -\theta^{AB}\frac{\partial H_0}{\partial q^B}. \end{aligned} \tag{8}$$

They are accompanied by the second class constraints (5). Poisson brackets of the constraints are

$$\{G_A, G_B\} = 0, \qquad \{T^A, T^B\} = -2\theta^{AB}, \qquad \{G_A, T^B\} = -\delta_A^B. \tag{9}$$

The constraints can be taken into account by transition to the Dirac bracket. Introducing the Dirac bracket

$$\begin{aligned} \{A, B\}_D = \{A, B\} + 2\{A, G_A\}\theta^{AB}\{G_B, B\} - \\ \{A, G_A\}\{T^A, B\} + \{A, T^A\}\{G_A, B\}, \end{aligned} \tag{10}$$

one finds, in particular, the following brackets for the fundamental variables (all the nonzero brackets are presented)

$$\{q^A, q^B\} = -2\theta^{AB}, \qquad \{q^A, p_B\} = \delta_B^A, \qquad \{p_A, p_B\} = 0; \tag{11}$$

$$\{q^A, v_B\} = \delta_B^A, \quad \{q^A, \pi^B\} = -\theta^{AB}. \tag{12}$$

One has now different possibilities to choose the physical sector: either $(q^A,\ p_A)$, or $(q^A,\ v_A)$, or $(q^A,\ \pi_A)$ (the latter possibility implies that θ is invertible). Let us take the variables $(q^A,\ p_A)$ (the same as for the initial formulation (1)) as the physical one. Since the Dirac bracket has been introduced, the variables v, π can be omitted from consideration. Dynamics of the physical variables is governed now by the equations

$$\dot{q}^A = \frac{\partial H_0}{\partial p_A} - 2\theta^{AB}\frac{\partial H_0}{\partial q^B}, \qquad \dot{p}_A = -\frac{\partial H_0}{\partial q^A}, \tag{13}$$

where $H_0(q,\ p) = H_0(q,\ v)|_{v\to p}$. Modulo the term with θ, they are the same as for the initial system (1). Brackets for the variables $q^A,\ p_A$ are given by Eqs.(11). One can show that other possibilities to choose the physical variables lead to an equivalent description.

To quantize the resulting system, one possibility is to find variables which have the standard brackets. For the case under consideration they are

$$\tilde{q}^A = q^A - \theta^{AB} p_B, \qquad \tilde{p}_A = p_A, \tag{14}$$

and obey $\{\tilde{q},\tilde{q}\} = \{\tilde{p},\tilde{p}\} = 0,\ \{\tilde{q},\tilde{p}\} = 1$. Equations of motion in terms of these variables acquire the standard form

$$\dot{\tilde{q}}^A = \{\tilde{q}^A, \tilde{H}_0\}, \qquad \dot{\tilde{p}}_A = \{\tilde{p}_A, \tilde{H}_0\}, \tag{15}$$

where $\tilde{H}_0 = H_0(\tilde{q} + \theta\tilde{p},\ \tilde{p})$. It leads to quantum mechanics with the Moyal product (see [7] and references therein)

$$H_0(\tilde{q}^A + \theta^{AB}\tilde{p}_B,\ \tilde{p}_B)\Psi(\tilde{q}^C) = H_0(\tilde{q}^A,\ \tilde{p}_B) * \Psi(\tilde{q}^C). \tag{16}$$

In conclusion, let us point that the procedure described above can be applied to some degenerated systems as well. In particular, the noncommutative relativistic particle has been proposed in [8] following a similar line.

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