Homotopic Classification of Yang–Mills Vacua Taking into Account Causality

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Abstract

In this letter we study the existence of θ-vacuum states in Yang–Mills theories defined over asymptotically flat, stationary space-times taking into account not only the topology but the complicated causal structure of these space-times, too. By a result of Chruściel and Wald, apparently causality makes all vacuum states, seen by a distant observer, homotopically equivalent making the introduction of θ-terms unnecessary in (causally) effective Lagrangians.

But a more careful study shows that certain twisted classical vacuum states survive even in this case eventually leading to the conclusion that the concept of "θ-vacua" is meaningful in the case of general Yang–Mills theories. We give a classification of these vacuum states based on Isham's results showing that the Yang–Mills vacuum has the same complexity as in the flat Minkowskian case hence the general CP-problem is not more complicated than the well-known flat one.

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1 Introduction

The famous solution of the long-standing $U(1)$-problem in the Standard Model via instanton effects was presented by 't Hooft about two decades ago [1]. This solution demonstrated that *instantons*, i.e. finite-action solutions of the *Euclidean* Yang–Mills-equations discovered by Belavin et al. [2] should be taken seriously in gauge theories. Another problem arose in these models over *flat* space-times, however: if instantons really exist, they induce a P- hence CP-violating so-called θ-term in the effective Yang–Mills action. But according to accurate experimental results, such a CP-violation does not exists in QCD, for instance. The most accepted solution to this problem is the so-called *Peccei–Quinn mechanism* [3]. A consequence of this mechanism is the existence of a light particle, the so-called *axion*. This particle has not been observed yet, however.

The question naturally arises whether or not such problematic θ-term must be introduced over more general space-times. The aim of our paper is to claim that the answer is yes.

First, let us summarize the vacuum structure of a gauge theory over Minkowski space-time. We are going to use an Yang–Mills theory framework over $(\mathbb{R}^4, g)$ where g is a Lorentzian metric. Let E be a (trivial) complex vector bundle over $\mathbb{R}^4$ belonging to a finite dimensional complex representation of G. Without loss of generality we choose the gauge group G to be a compact

Lie group. Consider a G-connection on this bundle; choosing a particular frame on E, this connection can be identified globally with a $\mathfrak{g}$-valued 1-form A with curvature F_A. We choose the usual Yang–Mills action (by fixing the coupling to be 1):

$$S(A,g) = -\frac{1}{8\pi^2}\int\limits_M \mathrm{tr}\,(F_A \wedge *F_A)\,, \tag{1}$$

in this case $M=\mathbb{R}^4$ and $*$ denotes the Hodge-operator induced by the metric g on $\mathbb{R}^4$. Usually the metric g is fixed and supposed to be the Minkowski metric on $\mathbb{R}^4$. The Euler–Lagrange equations of (1) are the Yang–Mills equations and read as follows:

$$\mathrm{d}_A F_A = 0, \quad \mathrm{d}_A * F_A = 0.$$

The simplest solution is the vacuum. The gauge field A is a vacuum field if $F_A=0$ i.e. its field strength or curvature is equal to zero. By simply connectedness of $\mathbb{R}^4$ such gauge fields can be written in the form $A=f^{-1}\mathrm{d}f$, where $f:\mathbb{R}^4\to G$ is a smooth function.

But by the existence of a global temporal gauge and the stationarity of the flat metric on $\mathbb{R}^4$ it is enough to consider the restriction of f to a space-like submanifold of Minkowski space-time, i.e. $f:\mathbb{R}^3\to G$. Minkowski space-time is asymptotically flat as well, so there is a point i^0 called space-like infinity. This point represents the "infinity of space" hence can be added to $\mathbb{R}^3$ completing it to the three-sphere $\mathbb{R}^3\cup\{i^0\}=S^3$. It is well-known that vacuum fields (possibly after a null-homotopic gauge-transformation around i^0) extend to the whole S^3 consequently classical vacua are classified by maps $f:S^3\to G$. These maps up to homotopy are given by elements of $\pi_3(G)$. For typical compact Lie groups $\pi_3(G)\cong\mathbb{Z}$. This fact can be interpreted as classical vacua are separated from each other by barriers of finite height i.e. it is impossible to develop two vacua of different winding numbers into each other only through vacuum states. Hence homotopy equivalence reflects the dynamical structure of the theory.

On the other hand if f_1, f_2 are vacua of winding numbers n_1, n_2 respectively, there is a gauge transformation $g:S^3\to G$ of winding number n_2-n_1 satisfying $gf_1=f_2$. Consequently we can see that the concept of *dynamical equivalence* of vacua given by the *dynamics* of the theory (i.e. the *homotopy equivalence* of maps $f:S^3\to G$) is different from that of *symmetry-equivalence* of vacua provided by the *symmetry* of the gauge theory (i.e. the *gauge equivalence* of the above maps).

To avoid this discrepancy, we proceed as follows. Suppose we have constructed the Hilbert space $\mathcal{H}_{\mathbb{R}^4}$ of the corresponding quantum gauge theory. If $|n\rangle\in\mathcal{H}_{\mathbb{R}^4}$ denotes the quantum vacuum state belonging to a classical vacuum f of winding number n, the simplest way to construct a state which is invariant (up to phase) under both dynamical (i.e. homotopy) and symmetry- (i.e. gauge) equivalence is to introduce the state

$$|\theta\rangle := \sum_{n=-\infty}^{\infty} e^{in\theta}|n\rangle \;\in\mathcal{H}_{\mathbb{R}^4}, \quad \theta\in U(1). \tag{2}$$

These quantum vacuum states are referred to as "θ-vacua".

From the physical point of view, the introduction of θ-vacua is also necessary because the vacuum states of different winding numbers can be joined semi-classically i.e. by a tunneling process induced by non-trivial instantons of the corresponding *Euclidean* gauge theory. Indeed, as it is well known [4][5], in the case of $G=SU(2)$, the instanton number of an instanton is an element $k \in H^4(S^4, \mathbb{Z}) \simeq \mathbb{Z}$ (here S^4 is the one-point conformal compactification of the Euclidean flat $\mathbb{R}^4$. Note that the notion of "instanton number" comes from a very different compactification process comparing to the derivation of "vacuum winding number"). If two vacua, $|n_1\rangle$, $|n_2\rangle$ ($n_1, n_2 \in \pi_3(SU(2)) \simeq \mathbb{Z}$) are given then there is an instanton of instanton number $n_2 - n_1 \in H^4(S^4, \mathbb{Z}) \simeq \mathbb{Z}$ tunneling between them in temporal gauge [4][5]. In other words the true vacuum states are linear combinations of the vacuum states of unique winding numbers yielding again (2).

But the value of θ cannot be changed in any order of perturbation, i.e. it should be treated as a physical parameter of the theory; this implies that tunnelings induce the effective term

$$\frac{\theta}{8\pi^2} \int\limits_{\mathbb{R}^4} \mathrm{tr}\,(F_A \wedge F_A)$$

in addition to action (1). But it is not difficult to see that such a term violates the parity symmetry P of the theory resulting in the violation of the CP-symmetry.

In summary, we have seen that there are at least three different ways to introduce θ-parameters in Yang–Mills theories *over Minkowskian space-time*:

(i) θ is introduced to fill in the gap between the notions of dynamical (i.e. homotopy) and symmetry- (i.e. gauge) equivalence of Yang–Mills vacua. This approach is pure mathematical in its nature;

(ii) θ must be introduced because by instanton effects vacua of definite winding numbers are superposed in the underlying semi-classical Yang–Mills theory;

(iii) θ must be introduced by "naturality arguments", i.e. nothing prevents us to extend the Yang–Mills action at the full quantum level by a P-violating term $\mathrm{tr}\,(F_A \wedge F_A)$ with coupling constant θ.

There is a correspondence between the above three characterizations of the θ in *flat Minkowskian space-time* but in the case of general space-times, clear and careful distinction must be made until a relation or correspondence between the three notions is established. Clearly, (i) is related to the *topology* of the space-time and the gauge group hence it is relatively easy to check whether or not it remains valid in the general case. Concept (ii) is related to the semi-classical structure of the general Yang–Mills theory especially to the existence of instanton solutions in the Wick-rotated theory and their relationship with vacuum tunneling. The validity of concept (iii) is the most subtle one: we need lot of information on global non-perturbative aspects of the general quantum Yang–Mills theory to check if any θ-term survives quantum corrections. In the present state of affairs, having no adequate general theory of Wick rotation, instantons and their physical interpretation, non-perturbative aspects of general Yang–Mills theories, we can examine only the validity of concept (i) in the general case. Its validity or invalidity may serve as a good indicator for the existence and role of θ-terms in general Yang–Mills theories.

The analysis of the vacuum structure of general Yang–Mills theories from the point of view of (i) was carried out by Isham et al. [6][7][8][9][10]. In these papers Isham et al. argue that in the general case concept (i) for introducing θ-terms still continue to hold due to the complicated topology of the spatial surface S and the gauge group G [6]. The classical vacuum structure of these theories becomes more complicated and we cannot avoid the introduction of various CP-violating terms into the effective Lagrangian [10].

We have to emphasize that the approach of Isham et al. to the problem is pure topological in its nature, however. By a result of Witt [11] every oriented, connected three-manifold S appears as Cauchy-surface of a physically reasonable initial data set. It is well-known that the complicated topology of the space-like submanifold S leads to appearance of singularities in space-time if it arises as the Cauchy development of S. Indeed, an early result of Gannon [12] shows that the Cauchy development of a non-simply connected Cauchy surface is geodesically incomplete i.e. singularities occur. If we accept the Cosmic Censorship Hypothesis, these singularities are hidden behind event horizons resulting in a non-trivial causal structure for these space-times, too. A theorem of Chruściel–Wald [13] shows that distant observers can observe only simply-connected portions of asymptotically flat space-times: all topological properties are hidden behind event horizons, eventually resulting again in a topologically simple *effective* space-time. Hence one may doubt if Isham's conclusions remain valid.

In Section 2 we formulate Yang–Mills theories with an arbitrary compact gauge group over general asymptotically flat, stationary space-times. This model provides a good framework for studying classical Yang–Mills vacua over causally non-trivial space-times. In this setup we simply mimic the above analysis concerning classical Yang–Mills vacua and find that although all vacua are topologically equivalent on the causally connected regime of the space-time, the appearance of a natural boundary condition on the event horizons (also a consequence of the causal structure) finally makes concept (i) still remain valid.

In Section 3 we calculate explicitely the homotopy classes of vacua for the classical groups. A modification appears compared with Isham and other's pure topological considerations in the sense that generally the vacuum structure in our case has the same complexity as in the flat Minkowskian case. This demonstrates the "stability" of the θ-problem.

The idea of studying relationship between micro- or virtual black holes, wormholes and θ-vacua is not new. For example, see Hawking [14] and Preskil et al. [15].

2 Vacua: A General Space-Time Model

The general reference for this chapter is [16]. Let $\widetilde{S}$ be a connected, oriented, closed three-manifold and let $S := \widetilde{S} \setminus \{i^0\}$ where $i^0 \in \widetilde{S}$ is a point. Using the result of Witt [11] we can choose an asymptotically flat initial data set (S, h, k), where h is a smooth Riemannian metric while k is a symmetric $(0,2)$-type tensor field on S both satisfying suitable fall-off conditions in a neighbourhood of i^0 (this point is called *spatial infinity*). We suppose these initial data are given by some matter field represented by a stress-energy tensor $T|_S$ obeying the dominant energy condition. Consider the maximal Cauchy development of (S, h, k) denoted by (M, g). This space-time is *globally hyperbolic* with Cauchy surface S by construction and $M \cong S \times \mathbb{R}$.

Choose a complex vector bundle E over M associated to the gauge group G via a complex representation of G and a G-connection A on it. Consider a Yang–Mills theory with action (1). We will focus on *vacuum solutions on a gravitational background* i.e. pairs (A, g) where A is a smooth flat G-connection on the bundle E while g is a smooth Lorentzian metric on M (which is a solution of the Einstein's equations with a matter field given by a stress-energy tensor T. For technical reasons we suppose T satisfies the strong energy condition). We will refer the collection (E, A, M, g) to as an *Yang–Mills vacuum setup*.

We take two more restrictions. First, we will assume that (M, g) is *asymptotically flat*. At a first look this means that there is a conformal embedding $i : (M, g) \to (\widetilde{M}, \widetilde{g})$ such that the image of the Cauchy surface can be completed to a maximal space-like submanifold $\widetilde{S}$ by adding the space-like infinity $i^0 \in \widetilde{M}$ to it: $i(S) \cup \{i^0\} = \widetilde{S}$. Moreover the infinitely distant points of M, represented by $\partial i(M)$ are divided naturally into three classes: the future and past null infinities $\mathcal{I}^\pm$ and the already mentioned spatial infinity i^0. The *asymptotically flat outer region* of M is defined to be the set $N := J^-(\mathcal{I}^+) \cap i(M)$ which is a manifold (here $J^\pm(A)$ denotes the causal past or future of a set $A \subset \widetilde{M}$). The metric $\widetilde{g}$ is related by a conformal factor Ω to the original metric; although the details are unrelevant in our considerations, we remark that $\widetilde{g}$ is not necessarily smooth in i^0. Note that if $M \setminus N =: B$ is not empty then (M, g) contains a *black hole region* B. We denote by $H \subset M$ its *event horizon*. Clearly, $\partial N = \partial B = H$. For details, see [16].

Secondly, we assume that (M, g) is *stationary*.

In summary, we focus our attention to each stationary, asymptotically flat, globally hyperbolic Yang–Mills vacuum setups (E, A, M, g) with an arbitrary compact gauge group G. We address the problem of describing the topology of Yang–Mills vacua *seen by an observer in the asymptotically flat region* of the space-time (M, g). Clearly, at least classically, only this part of the space-time can be relevant for ordinary macroscopic observers. To achieve this, we refer to a general result of Chruściel and Wald on asymptotically flat outer regions [13].

Theorem 1. (Chruściel–Wald). *Let (M, g) be a globally hyperbolic, asymptotically flat, stationary space-time with a matter field represented by a stress-energy tensor T satisfying the strong energy condition. Then the outer asymptotically flat region N of (M, g) is simply connected, i.e. $\pi_1(N) = 1$.*

Moreover, if M contains a black hole region, then all connected components of the event horizon $H \subset M$ are homeomorphic to $S^2 \times \mathbb{R}$. $\diamondsuit$

By global hyperbolicity, there is a global time function $T : M \to \mathbb{R}$. Let $S_t := T^{-1}(t)$ ($t \in \mathbb{R}$) be a Cauchy surface and $\widetilde{S}_t = i(S_t) \cup \{i^0\}$ its conformal completion. Of course $\widetilde{S}_t \cong \widetilde{S}$ for all $t \in \mathbb{R}$.

In light of the above theorem, $V_t := N \cap S_t$, a Cauchy surface for the outer region, is an oriented simply connected three-manifold. If M contains black hole domains then $\partial V_t \neq \emptyset$ and all boundary components are homeomorphic to a two-sphere S^2 ("the event horizon of a stationary black hole has no handles"). By conformal completion we may consider rather the three-manifold $\widetilde{V}_t := \widetilde{N} \cap \widetilde{S}_t$ which is moreover compact.

Now we are ready to describe the Yang–Mills-vacuum structure over $(N, g|_N)$. The simply connectedness of N implies that a restricted flat Yang–Mills bundle $E|_N$ is trivial whatever G is hence a vacuum Yang–Mills connection $A|_N$ may be regarded as a $\mathfrak{g}$-valued 1-form on $E|_N$. In an appropriate trivialization of the bundle $E|_N$, we may regard $A|_N$ as a $\mathfrak{g}$-valued 1-form over N instead of $E|_N$. Moreover, also by simply connectedness, a smooth Yang–Mills field $A|_N$ is vacuum if and only if there is a smooth function $f : N \to G$ obeying $A|_N = f^{-1}\mathrm{d}f$. Using again the simply connectedness of N, there is a gauge-transformation over N such that all vacuum fields can be transformed into temporal gauge, i.e. $A_0|_N = 0$ where the A_0 component is defined by the time function T. Consequently in temporal gauge, by exploiting the stationarity of the metric g, all flat connections are characterized by smooth functions $f_t : V_t \to G$. It is easily seen that $\pi_2(G) = 0$ implies that always exists a *null-homotopic* gauge transformation in the neighbourhood of the space-like infinity $i^0 \in \widetilde{V}_t$ such that the gauge-transformed function $\tilde{f}_t$ extends as the identity to i^0 hence we are dealing with smooth functions $\tilde{f} : \widetilde{V} \to G$ (we denote $\widetilde{V}_t$ and $\tilde{f}_t$ simply by $\widetilde{V}$ and $\tilde{f}$ if there is no danger of confusion).

A pure Yang–Mills theory being conformally invariant, we may consider our original Einstein–Yang–Mills theory over $(\widetilde{M, \widetilde{g}})$ instead of the original space-time. The restriction of the extended flat Yang–Mills bundle $\widetilde{E}|_{\widetilde{N}}$ is trivial even in this case. Certain physical quantities of the extended theory may suffer from singularities on the boundary $\partial i(M)$ but classical Yang–Mills vacua extend smoothly to the whole $(\widetilde{M, \widetilde{g}})$ as we have seen. In other words the studying of the vacuum sector of the extended Yang–Mills theory is correct.

Summing up, we can see that dynamically (i.e. homotopically) inequivalent vacua of the Yang–Mills theory are classified by the homotopy classes of smooth maps $\tilde{f}:\widetilde{V}\to G$ satisfying $\tilde{f}(i^0)=e\in G$, usually written as

$$\left[(\widetilde{V},i^0),(G,e)\right]. \tag{3}$$

Now suppose that (M,g) contains black hole(s). In this case $\widetilde{V}$ is a simply connected manifold *with boundary*. Such manifolds, considered as CW-complexes, have only cells of dimension less than three. Hence by the Cellular Approximation Theorem, every map $\tilde{f}:\widetilde{V}\to G$ descends to a homotopic map with values only on the cells of G having dimension less than three. Consequently, being $\pi_2(G)=0$, G can be replaced by the simple Postnikov-tower $P_2=K(\pi_1(G),1)$ where $K(\pi_1(G),1)$ is an Eilenberg–Mac Lane space yielding

$$\left[(\widetilde{V},i^0),(G,e)\right]\cong\left[\widetilde{V},K(\pi_1(G),1)\right]\cong H^1(\widetilde{V},\pi_1(G))=0. \tag{4}$$

The result is zero because $\widetilde{V}$ is simply connected. For details, see for instance [17]. Consequently all vacuum states are homotopy equivalent i.e. can be deformed into each other only through vacuum states *over an outer, asymptotically flat portion N of the space-time (M,g)*. Clearly, classically only this part is relevant for a distant observer.

This result can be explained from a different point of view as well. Since the outer part N of M is also globally hyperbolic, the space-like submanifold V forms a Cauchy surface for N. Consequently if we know the initial values of two gauge fields, A and A' say, on $V\subset S$, we can determine their values over the whole *outer* space-time $N\subset M$ by using the field equations. This implies that the values of the fields A and A' "beyond" the event horizon in a moment are irrelevant for an observer outside the black hole. But we just proved that every vacuum fields restricted to V are homotopic. Roughly speaking, homotopical differences between Yang–Mills vacua "can be swept" into a stationary black hole.

Via (4) for arbitrary smooth functions $\tilde{f}_0:\widetilde{V}\to G$ and $\tilde{f}_1:\widetilde{V}\to G$ there is a homotopy

$$\widetilde{F}_T:\widetilde{V}\times[0,1]\to G \tag{5}$$

satisfying $\widetilde{F}_T(x,0)=\tilde{f}_0(x)$ and $\widetilde{F}_T(x,1)=\tilde{f}_1(x)$ for all $x\in\widetilde{V}$. Taking two Cauchy-surfaces $V_0\subset S_0:=T^{-1}(0)$ and $V_1\subset S_1:=T^{-1}(1)$ we can regard the two functions as vacua $\tilde{f}_0:\widetilde{V}_0\to G$ and $\tilde{f}_1:\widetilde{V}_1\to G$. In the homotopy $\widetilde{F}_T$ the subscript T shows that the "time" required for the homotopy is measured by the time function T naturally associated to the globally hyperbolic space-time (M,g). We call this homotopy as T-*homotopy*.

But on physical grounds, such a deformation or homotopy is effective only if it can be carried out *in finite proper time according to a distant observer's clock*. Such homotopies will be referred to as *finite γ-homotopies* or *effective homotopies*.

Let $\gamma:\mathbb{R}^+\to N$ be a smooth, time-like, future directed curve in the outer region $N\subset M$ with $\gamma(0)\in V_0$ representing an observer moving in N. We denote by τ its proper time i.e. the natural affine parameter of the curve γ obeying $g(\dot\gamma(\tau),\dot\gamma(\tau))=-1$. Moreover, let $\beta_1:[0,1]\to V_1$ be a continuous space-like curve in the fixed Cauchy-surface V_1 approaching one connected component of the horizon i.e. $\beta_1(0)\notin H_1=H\cap V_1$ while $\beta_1(1)\in H_1$. Define

$$\tau_{\beta_1(s)}:=\inf_{\tau\in\mathbb{R}}\left\{\gamma(\tau)\in J^+(\beta_1(s))\right\}.$$

We prove the following simple lemma:

Lemma. *Let (M,g) be an asymptotically flat, stationary space-time with outer asymptotically flat region N and black hole region B and event horizon $H = \partial N = \partial B$. Consider the curves $\beta_1 : [0,1] \to V_1$ and $\gamma : \mathbb{R}^+ \to N$ defined above. Then*

$$\lim_{s\to 1} \tau_{\beta_1(s)} = \infty.$$

Proof. The proof is very simple. Clearly, if for a non-space-like curve $\alpha : \mathbb{R} \to M$ the condition $\mathrm{im}\alpha \cap B \neq \emptyset$ holds then $\mathrm{im}\alpha \subset B$ by the definition of the black hole $B \subset M$. Consequently if by assumption $\beta_1(1) \in H_1 \subset B$ then all non-space-like curves α with the property $\alpha(\tau') = \beta_1(1)$ ($\tau' \in \mathbb{R}$) never enter N hence never meet γ showing $\gamma(\tau) \notin J^+(\beta_1(1))$ for all $\tau \in \mathbb{R}^+$ i.e. $\tau_{\beta_1(1)} = \infty$. By continuity we get the result. $\diamond$

Now we are ready to understand the above homotopy from the point of view of a distant observer. Fix an observer γ. Clearly, the T-homotopy (5) can be written as a γ-homotopy as

$$\widetilde{F}_T(x,t) = \tilde{f}_t(x) = \widetilde{F}_\gamma(x,\tau).$$

Take $x := \beta_1(s)$ then we get

$$\widetilde{F}_T(\beta_1(s),1) = \tilde{f}_1(\beta_1(s)) = \widetilde{F}_\gamma(\beta_1(s), \tau_{\beta_1(s)}).$$

But by the Lemma we can see that as we approach the horizon, i.e. $s \to 1$, $\tau_{\beta_1(s)}$ diverges hence typically the T-homotopy cannot be a finite γ-homotopy in other words it cannot be "finished" in finite proper time measured by γ.

From here we can see that given a T-homotopy (5), it gives rise to an effective homotopy if and only if there is a neighbourhood $H \subset U \subset N$ of the horizon in N with the property $\tilde{f}_0|_{U_0} = \tilde{f}_1|_{U_1}$. This implies

$$\tilde{f}_0|_{H_0} = \tilde{f}_1|_{H_1}.$$

This result can be interpreted as a natural boundary condition on each connected component of the horizon for effectively deformable vacua. Since each boundary component is homeomorphic to the two-sphere S^2 and $\pi_2(G) = 0$ we may select a function in each homotopy class obeying $\tilde{f}_0|_{H_0} = e$. We just remark that exactly this is the physical reason for keeping the functions as identity in the space-like infinity i^0 when we discuss homotopy classes of vacua over Minkowskian space-time.

Taking into account that $\partial\widetilde{V}_t = H_t$, *the classes of effectively deformable vacua* are given by the homotopy classes of functions $\tilde{f} : \widetilde{V} \to G$ with the property $\tilde{f}(\partial\widetilde{V}) = \tilde{f}(i^0) = e \in G$. The homotopy is restricted to obey these boundary conditions. This set is denoted by

$$\left[(\widetilde{V}, \partial\widetilde{V}, i^0), (G, e)\right] \tag{6}$$

and replaces (3). To get a more explicit description of this set, we proceed as follows.

3 Homotopic Classification

First taking into account that a function $\tilde{f} : \widetilde{V} \to G$ we are interested in satisfies that it sends each connected component of $\partial\widetilde{V}$ into the unit element $e \in G$, we can replace the three-manifold-with-boundary $\widetilde{V}$ with an oriented, closed, simply connected three-manifold W_k in the following way. Let denote by $k > 0$ the number of connected components of $\partial\widetilde{V}$ (i.e. the number of black holes). As we have seen, all such component is an S^2. Hence we can glue to each such component a three-ball B^3 using the identity function of S^2 to get

$$W_k := \widetilde{V} \cup_{\partial\widetilde{V}} \underbrace{B^3 \cup ... \cup B^3}_{k}.$$

Clearly, $\tilde{f}$ extends as the identity to each ball giving rise to the function $f : W_k \to G$. Consequently, if we fix a point x_n in each ball ($n \leq k$), then we may equivalently consider functions obeying $f(x_1) = ... = f(x_k) = f(i^0) = e$. Modifying the allowed homotopies to obey this constraint, we can replace the homotopy set (6) by

$$[(W_k, x_1, ..., x_k, i^0), (G, e)]$$

(of course if $k = 0$ then no point except i^0 is distinguished in W_0). We prove the following proposition:

Proposition. *Fix a number $k > 0$ and consider the connected, closed, simply connected three-manifold with $k + 1$ distinguished points $(W_k, x_1, ..., x_k, i^0)$ constructed above. Denote by (W_k, i^0) the same space with only one distinguished point. Then there is a natural bijection*

$$[(W_k, x_1, ..., x_k, i^0), (G, e)] \cong [(W_k, i^0), (G, e)] .$$

Proof. Fix a number $k \geq 0$. First it is straightforward that if two functions, f_0 and f_1 are homotopic in $[(W_k, x_1, ..., x_k, i^0), (G, e)]$ then they represent the same homotopy class in $[(W_k, i^0), (G, e)]$ i.e. they are homotopic in the later set as well. This is because the allowed homotopies in $[(W_k, i^0), (G, e)]$ are less restrictive than in $[(W_k, x_1, ..., x_k, i^0), (G, e)]$.

Consider the opposite way. It is not difficult to see that in each class $[f] \in [(W_k, i^0), (G, e)]$ there is a representant which belongs to $[(W_k, x_1, ..., x_k, i^0), (G, e)]$. Indeed, choose an arbitrary representant $f \in [f] \in [(W_k, i^0), (G, e)]$ and consider the pre-image $f^{-1}(e) \subset W_k$. This pre-image contains the point $i^0 \in W_k$ by construction. Taking into account that W_k is path-connected, we can deform $f^{-1}(e)$ to contain the points $x_1, ..., x_k$ as well producing a representant which belongs to $[(W_k, x_1, ..., x_k, i^0), (G, e)]$.

Now suppose that there are two functions f_0 and f_1 which are homotopic in $[(W_k, i^0), (G, e)]$ i.e. there is a continuous function $F : (W_k, i^0) \times [0, 1] \to (G, e)$ with

$$F(x, 0) = f_0(x), \quad F(x, 1) = f_1(x), \quad F(i^0, t) = e.$$

For the sake of simplicity, assume they represent elements in $[(W_k, x_1, ..., x_k, i^0), (G, e)]$, too. Then we have to prove that they are also homotopic in $[(W_k, x_1, ..., x_k, i^0), (G, e)]$ i.e. there is a function $F' : (W_k, x_1, ..., x_k, i^0) \times [0, 1] \to (G, e)$ with the property

$$F'(x, 0) = f_0(x), \quad F'(x, 1) = f_1(x), \quad F'(x_1, t) = ... = F'(x_k, t) = F'(i^0, t) = e.$$

From here we can see that the orbit of an arbitrary distinguished point x_n is a loop $l_n : [0, 1] \to G$ under the homotopy F while the constant loop in the case of F'. Hence if these loops are homotopically trivial in G then we can deform F into the homotopy F' by shrinking the loops l_1,...,l_k.

Now we prove that this is always possible. First, if $\pi_1(G) = 1$ i.e. the compact Lie group is simply connected then certainly each loop l_n is homotopic to the constant loop. Consequently assume $\pi_1(G) \neq 1$. Consider a distinguished point $x_n \in W_k$ and two path $a_n : [0, 1/2] \to W_k$ with $a_n(0) = i^0$ and $a_n(1/2) = x_n$ and $b_n : [1/2, 1] \to W_k$ with $b_n(1/2) = x_n$ and $b_n(1) = i^0$. These give rise to a continuous loop $b_n * a_n : [0, 1] \to W_k$ with $b_n * a_n(0) = b_n * a_n(1) = i^0$. Here $*$ refers to juxtaposition of curves, loops etc. The loop $b_n * a_n$ is homotopic to the trivial loop since W_k is simply connected. Consider the maps $\alpha^0_n := f_0 \circ a_n : [0, 1/2] \to G$ and $\beta^0_n := f_0 \circ b_n : [1/2, 1] \to G$. These are loops in G hence so is their product $\beta^0_n * \alpha^0_n$. Construct the same kind of loops $\alpha^1_n := f_1 \circ a_n$ and $\beta^1_n := f_1 \circ b_n$. The product loop $\beta^1_n * \alpha^1_n$ is homotopic in G to $\beta^0_n * \alpha^0_n$ i.e. $[\beta^0_n * \alpha^0_n] = [\beta^1_n * \alpha^1_n]$ because f_0 is homotopic to f_1. It is clear that

$$\beta^1_n * \alpha^1_n = \beta^0_n * l_n * \alpha^0_n.$$

Consequently we can write for the homotopy classes in question

$$[\beta^1_n * \alpha^1_n] = [\beta^0_n * l_n * \alpha^0_n] = [\beta^0_n * \alpha^0_n * l_n] = [\beta^0_n * \alpha^0_n][l_n] = [\beta^1_n * \alpha^1_n][l_n].$$

In the second step we exploited the fact that a topological group always has commutative fundamental group. This shows that $[l_n] = 1$ that is the loop l_n is contractible in G for all $0 \leq n \leq k$ in other words the homotopy F is deformable into a homotopy F' yielding f_0 and f_1 are homotopic in $[(W_k, x_1, ..., x_k, i^0), (G, e)]$ as well. $\diamond$

The above Proposition enables us to give a more explicit description of the set (6).

Theorem 2. *Let G be a typical classical compact Lie group i.e. let G be $U(n)$ with $n \geq 2$, or $SO(n)$,* Spin(n) *with $n \neq 4$, or $SU(n)$, $Sp(n)$ for all n, or G_2, F_4, E_6, E_7, E_8. Then we have*

$$\left[(\widetilde{V}, \partial\widetilde{V}, i^0), (G, e)\right] \cong \mathbb{Z}.$$

Moreover we have

$$\left[(\widetilde{V}, \partial\widetilde{V}, i^0), (U(1), e)\right] \cong 0,$$

and

$$\left[(\widetilde{V}, \partial\widetilde{V}, i^0), (SO(4), e)\right] \cong \left[(\widetilde{V}, \partial\widetilde{V}, i^0), (\mathrm{Spin}(4), e)\right] \cong \mathbb{Z} \oplus \mathbb{Z}$$

for the remaining cases.

Proof. In light of the above considerations and the Proposition, we have

$$[(\widetilde{V}, \partial\widetilde{V}, i^0), (G, e)] \cong [(W_k, x_1, ..., x_k, i^0), (G, e)] \cong [(W_k, i^0), (G, e)].$$

Hence we can use the results of Isham who classified the set $[(W_k, i^0), (G, e)]$ and it is summarized in [6] in Table 1 on pp. 207. But in our case W_k is a connected, closed, simply connected three manifold hence the above result follows. $\diamond$

Remark. We mention that assuming the validity of the three dimensional Poincaré conjecture i.e. if $W_k \cong S^3$, then Theorem 2 can be derived without using Isham's result since in this case we have simply $[(\widetilde{V}, \partial\widetilde{V}, i^0), (G, e)] \cong \pi_3(G)$.

We can see by Theorem 2 that although the homotopy set (6) of effectively deformable vacua is typically non-trivial, it is remarkable more simple than in the original calculations of Isham et al. based on topological considerations only. The homotopy sets listed in Theorem 2 are exactly the same as for the flat Minkowskian case. Being these vacua of definite winding numbers non gauge invariant, we have to introduce again linear combinations as (2) in this more general situation. Consequently we can see that approach (i) to the θ-parameter, mentioned in the Introduction, still makes sense in the general case.

4 Concluding Remarks

In this letter we have studied the concept of θ-vacua in general Yang–Mills theories. In light of our results, we can see that for outer observers in stationary, asymptotically flat space-times θ-vacua do occur in a Yang–Mills theory. Despite the possible complicated topology of the underlying Cauchy surface however, their structure is similar to the flat Minkowskian case, due to the causal structure of these space-times which is complicated in the general case, too. Hence the introduction of the various new CP-violating terms studied in [10] is unnecessary.

The suppression of the topology of the underlying Cauchy surface is due to Theorem 1 which is a consequence of the so-called Topological Censorship Theorem of Friedman–Schleich–Witt [18]. Consequently, the reduction of the problem of the general CP-violation to the flat Minkowskian case is essentially due to this result.

The natural question arises: are there instanton solutions in the corresponding Wick-rotated theories? What is the physical relevance of these solutions? Do they induce semi-classical tunneling between the vacuum states of different effective winding numbers? If yes, beyond (i) we have another, more physical, reason to introduce θ-vacua because of concept (ii), also mentioned in the Introduction.

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