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Quantum group symmetry in sine-Gordon and affine Toda field theories on the half-line

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Abstract

We consider the sine-Gordon and affine Toda field theories on the half-line with classically integrable boundary conditions, and show that in the quantum theory a remnant survives of the bulk quantized affine algebra symmetry generated by non-local charges. The paper also develops a general framework for obtaining solutions of the reflection equation by solving an intertwining property for representations of certain coideal subalgebras of $U_q(\hat{g})$.

1 Introduction

Affine Toda field theories are integrable relativistic quantum field theories in 1+1 dimensions with a rich spectrum of solitons. There is one Toda theory for each affine Kac-Moody algebra $\hat{g}$ and the sine-Gordon model is the simplest example, for $\hat{g} = \widehat{sl(2)} = a_1^{(1)}$. Each of these models possesses a quantum group symmetry, generated by non-local charges. Conservation of these charges determines the S-matrices up to an overall scalar factor, as will be recalled in section 2.1.

The presence of a boundary breaks this quantum group symmetry, and also destroys classical integrability unless the boundary conditions are very carefully chosen. The purpose of this paper is to show that, with such classically integrable boundary conditions, a remnant of the quantum group symmetry nevertheless survives. This residual symmetry is just as powerful for determining the reflection matrices (boundary S-matrices) and

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boundary states as the original symmetry was for determining the particle spectrum and S-matrices in the bulk.

The main results of the paper are the simple expressions (3.25) and (3.26) for the non-local symmetry charges in the sine-Gordon model on the half line and their generalisations (4.11) to $a_n^{(1)}$ affine Toda theories. The sine-Gordon charges were first written down by Mezincescu and Nepomechie [24].

We derive the non-local symmetry charges in two very different ways. The approach in sections 3 and 4 treats the boundary condition as a perturbation of the Neumann boundary condition and employs boundary conformal perturbation theory [18]. These sections thus generalize the approach of [2] which derives the non-local charges on the whole line. The second approach is purely algebraic and is introduced in section 2.3 and applied in section 4.3. While the calculations in boundary perturbation theory presuppose a knowledge of [2], the algebraic techniques described in section 2 are basic and of wider significance.

We confirm the correctness of our symmetry charges by using them to rederive the sine-Gordon soliton reflection matrix [18] in section 3.5 and the reflection matrices for the vector solitons in $a_n^{(1)}$ affine Toda theories [17] in section 4.2. We obtain the reflection matrices by solving linear equations which are a consequence of the symmetry. This is much simpler than the original approaches that relied on solving the reflection equation [4] (boundary Yang-Baxter equation), which is a non-linear equation.

2 Algebraic techniques

2.1 Soliton S-matrices as quantum group intertwiners

This section is a review of the role of the quantum group symmetry in determining the soliton S-matrices and serves as preparation for the following section about reflection matrices.

The affine Toda field theory associated to the affine Lie algebra $\hat{g}$ is known [2, 14] to be symmetric under the quantum affine algebra $U_q(\hat{g})$ with zero center. The solitons in affine Toda field theory are arranged in various multiplets. Let V_θ^μ be the space spanned by the solitons in multiplet μ with rapidity θ. Each such space carries a representation $\pi_\theta^\mu : U_q(\hat{g}) \to \mathrm{End}(V_\theta^\mu)$. The asymptotic two-soliton states span the tensor product spaces $V_\theta^\mu \otimes V_{\theta'}^\nu$ which also carry representations of $U_q(\hat{g})$ built using the coproduct. An incoming two-soliton state in $V_\theta^\mu \otimes V_{\theta'}^\nu$ with $\theta > \theta'$ will evolve during scattering into an outgoing state in $V_{\theta'}^\nu \otimes V_\theta^\mu$ with scattering amplitudes given by the S-matrix

$$S^{\mu\nu}(\theta - \theta') : V_\theta^\mu \otimes V_{\theta'}^\nu \to V_{\theta'}^\nu \otimes V_\theta^\mu . \tag{2.1}$$

The S-matrix has to commute with the symmetry or, in other words, the S-matrix has to be an intertwiner of the tensor product representations,

$$S^{\mu\nu}(\theta-\theta')\circ(\pi^\mu_\theta\otimes\pi^\nu_{\theta'})(\Delta(Q))=(\pi^\nu_{\theta'}\otimes\pi^\mu_\theta)(\Delta(Q))\circ S^{\mu\nu}(\theta-\theta')\quad\text{for all } Q\in U_q(\hat{g}). \tag{2.2}$$

Because the tensor product representations are irreducible for generic rapidities, Schur's lemma tells us that this equation determines the S-matrix uniquely up to an overall scalar factor. This intertwiner can also be obtained from the universal R-matrix $\mathcal{R}$ of $U_q(\hat{g})$ as

$$S^{\mu\nu}(\theta-\theta')\propto\check{P}\circ R^{\mu\nu}(\theta-\theta'),\qquad\qquad R^{\mu\nu}(\theta-\theta')=(\pi^\mu_\theta\otimes\pi^\nu_{\theta'})(R), \tag{2.3}$$

where $\check{P}$ interchanges the tensor factors.

The scattering of three solitons is described by an intertwiner between the tensor product representations $V^\mu_\theta\otimes V^\nu_{\theta'}\otimes V^\lambda_{\theta''}$ and $V^\lambda_{\theta''}\otimes V^\nu_{\theta'}\otimes V^\mu_\theta$ with $\theta>\theta'>\theta''$. There are two ways to build such an intertwiner from the two-soliton S-matrices:

$$\begin{array}{ccccc}
V^\mu_\theta\otimes V^\nu_{\theta'}\otimes V^\lambda_{\theta''} & \xrightarrow{S^{\mu\nu}(\theta-\theta')\otimes\mathrm{id}} & V^\nu_{\theta'}\otimes V^\mu_\theta\otimes V^\lambda_{\theta''} & \xrightarrow{id\otimes S^{\mu\lambda}(\theta-\theta'')} & V^\nu_{\theta'}\otimes V^\lambda_{\theta''}\otimes V^\mu_\theta \\
\Big\downarrow{\scriptstyle \mathrm{id}\otimes S^{\nu\lambda}(\theta'-\theta'')} & & & & {\scriptstyle S^{\nu\lambda}(\theta'-\theta'')\otimes\mathrm{id}}\Big\downarrow \\
V^\mu_\theta\otimes V^\lambda_{\theta''}\otimes V^\nu_{\theta'} & \xrightarrow{S^{\mu\lambda}(\theta-\theta'')\otimes\mathrm{id}} & V^\lambda_{\theta''}\otimes V^\mu_\theta\otimes V^\nu_{\theta'} & \xrightarrow{id\otimes S^{\mu\nu}(\theta-\theta')} & V^\lambda_{\theta''}\otimes V^\nu_{\theta'}\otimes V^\mu_\theta
\end{array} \tag{2.4}$$

Because the tensor product representations are irreducible for generic rapidities, the intertwiner is unique. The above diagram is therefore commutative (up to an overall scalar factor) – that is, the S-matrices satisfy the Yang-Baxter equation.

For the tensor product of two vector representations the intertwining property (2.2) was solved by Jimbo [20] for all non-exceptional quantum affine algebras (both twisted and untwisted). The intertwiners for many other tensor products have been constructed [5] using the tensor product graph method. The complete sets of soliton S-matrices for the algebras $U_q(a_n^{(1)})$ [19], $U_q(c_n^{(1)})$ [15], and $U_q(a_{2n-1}^{(2)})$ [16] have been constructed.

2.2 Soliton reflection matrices as intertwiners

As reviewed in the previous section, quantum group symmetry can be used to obtain the soliton S-matrices. We now want to present a similar technique for obtaining the reflection matrices for solitons hitting a boundary. So far the only way to obtain reflection matrices has been to solve the reflection equation. Because the reflection equation is a non-linear functional matrix equation, solving it is very difficult for anything but the simplest cases. We will instead obtain a linear equation.

We will now restrict the solitons to live on the left half line $x\leq 0$ by imposing suitable integrable boundary conditions at $x=0$. A soliton with positive rapidity θ will eventually hit the boundary and be reflected into another soliton with opposite rapidity $-\theta$. The corresponding quantum process is described by the reflection matrix

$$K^{\mu}(\theta) : V_{\theta}^{\mu} \to V_{-\theta}^{\bar{\mu}}. \tag{2.5}$$

The multiplet $\bar{\mu}$ of the reflected soliton does not necessarily have to be the same as that of the incoming soliton, but it has to have the same mass. It turns out [6] in the case of $a_n^{(1)}$ Toda theory with the usual boundary conditions that solitons in the r-th rank antisymmetric tensor multiplet are converted into solitons in the $(n+1-r)$-th rank antisymmetric tensor multiplet.

There is no $U_q(\hat{g})$ intertwiner between the representations V_{θ}^{μ} and $V_{-\theta}^{\bar{\mu}}$ – that is, there is no $K^{\mu}(\theta)$ satisfying

$$K^{\mu}(\theta)\pi_{\theta}^{\mu}(Q) = \pi_{-\theta}^{\bar{\mu}}(Q)K^{\mu}(\theta) \tag{2.6}$$

for all $Q \in U_q(\hat{g})$. This is not surprising because the boundary should be expected to break the quantum group symmetry down to a subalgebra $\mathcal{B}$ of $U_q(\hat{g})$. The intertwining property (2.6) should only hold for all Q in $\mathcal{B}$. The unbroken symmetry algebra $\mathcal{B}$ should be “large enough” so that the intertwining condition (2.6) determines the reflection matrix uniquely up to an overall scalar factor. The subalgebra $\mathcal{B}$ must also be a left coideal of $U_q(\hat{g})$ in the sense that

$$\Delta(Q) \in U_q(\hat{g}) \otimes \mathcal{B} \quad \text{ for all } Q \in \mathcal{B}. \tag{2.7}$$

This allows it to act on multi-soliton states. Later in this paper we will construct such subalgebras $\mathcal{B}$ describing the residual quantum group symmetries of affine Toda field theories with integrable boundary conditions.

If two solitons are incident on the boundary, they can be reflected in two different orders. Correspondingly there are two ways of constructing intertwiners from $V_{\theta}^{\mu} \otimes V_{\theta'}^{\nu}$ to $V_{-\theta}^{\bar{\mu}} \otimes V_{-\theta'}^{\bar{\nu}}$:

$$\begin{tikzcd}
V_{\theta}^{\mu} \otimes V_{\theta'}^{\nu} \arrow[r, "\mathrm{id}\otimes K^{\nu}(\theta')"] \arrow[d, "S^{\mu\nu}(\theta-\theta')"] & V_{\theta}^{\mu} \otimes V_{-\theta'}^{\bar{\nu}} \arrow[r, "S^{\mu\bar{\nu}}(\theta+\theta')"] & V_{-\theta'}^{\bar{\nu}} \otimes V_{\theta}^{\mu} \arrow[r, "\mathrm{id}\otimes K^{\mu}(\theta)"] & V_{-\theta'}^{\bar{\nu}} \otimes V_{-\theta}^{\bar{\mu}} \arrow[d, "S^{\bar{\nu}\bar{\mu}}(\theta-\theta')"'] \\
V_{\theta'}^{\nu} \otimes V_{\theta}^{\mu} \arrow[r, "\mathrm{id}\otimes K^{\mu}(\theta)"] & V_{\theta'}^{\nu} \otimes V_{-\theta}^{\bar{\mu}} \arrow[r, "S^{\nu\bar{\mu}}(\theta+\theta')"] & V_{-\theta}^{\bar{\mu}} \otimes V_{\theta'}^{\nu} \arrow[r, "\mathrm{id}\otimes K^{\nu}(\theta')"] & V_{-\theta}^{\bar{\mu}} \otimes V_{-\theta'}^{\bar{\nu}}
\end{tikzcd} \tag{2.8}$$

Provided the tensor product representations are irreducible as representations of the subalgebra $\mathcal{B}$ the diagram above is commutative (up to an overall scalar factor) – that is, the reflection matrix automatically satisfies the reflection equation.

Due to the identification (2.3) between the S-matrix and the R-matrix the reflection equation can also be written in the form

$$\check{P}R^{\bar{\nu}\bar{\mu}}(\theta-\theta')\check{P}\,\overset{1}{K}{}^{\mu}(\theta)\,R^{\mu\bar{\nu}}(\theta+\theta')\,\overset{2}{K}{}^{\nu}(\theta') = \overset{2}{K}{}^{\nu}(\theta')\,\check{P}R^{\nu\bar{\mu}}(\theta+\theta')\check{P}\,\overset{1}{K}{}^{\mu}(\theta)\,R^{\mu\nu}(\theta-\theta'), \tag{2.9}$$

where we employ the standard notation $\overset{1}{A} = A \otimes \mathrm{id}$, $\overset{2}{A} = \mathrm{id} \otimes A$. Note that there is one such reflection equation for every pair of soliton multiplets and that generally these reflection equations involve 4 different R-matrices.

In affine Toda theory it is possible for solitons to bind to the boundary, thereby creating multiplets of boundary bound states. The space $V^{[\lambda]}$ spanned by the boundary bound states in multiplet $[\lambda]$ will carry a representation $\pi^{[\lambda]} : \mathcal{B} \to \mathrm{End}(V^{[\lambda]})$ of the symmetry algebra $\mathcal{B}$. The reflection of solitons in multiplet μ with rapidity θ off a boundary bound state in multiplet $[\lambda]$ is described by a reflection matrix $K^{\mu[\lambda]}(\theta) : V^{\mu}_{\theta} \otimes V^{[\lambda]} \to V^{\bar{\mu}}_{-\theta} \otimes V^{[\lambda]}$ which is determined by the intertwining property

$$K^{\mu[\lambda]}(\theta)\,(\pi^{\mu}_{\theta} \otimes \pi^{[\lambda]})(\Delta(Q)) = (\pi^{\bar{\mu}}_{-\theta} \otimes \pi^{[\lambda]})(\Delta(Q))\,K^{\mu[\lambda]}(\theta), \qquad \forall\, Q \in \mathcal{B}. \tag{2.10}$$

2.3 Construction of symmetry algebra

In this paper we will use boundary conformal perturbation theory to construct generators of the coideal subalgebras $\mathcal{B}_{\epsilon} \subset U_q(\hat{g})$ that occur as the symmetry algebras of affine Toda field theories on the half line, where ϵ parameterizes the boundary condition. However we also have an alternative construction which we will describe in this section. The construction has the disadvantage that it requires the a priori knowledge of at least one solution of the reflection equation but it has the advantage that it does not rely on first order perturbation theory. We will use it in section 4.3 to verify the expressions for the symmetry charges derived in section 4.1.

Let us assume that for one particular representation V^{μ}_{θ} we know the reflection matrix $K^{\mu}(\theta) : V^{\mu}_{\theta} \to V^{\bar{\mu}}_{-\theta}$. We define the corresponding $U_q(\hat{g})$-valued L-operators [13] in terms of the universal R-matrix $\mathcal{R}$ of $U_q(\hat{g})$ [21],

$$\begin{aligned} L^{\mu}_{\theta} &= (\pi^{\mu}_{\theta} \otimes \mathrm{id})\,(\mathcal{R}) \in \mathrm{End}(V^{\mu}_{\theta}) \otimes U_q(\hat{g}), \\ \bar{L}^{\bar{\mu}}_{\theta} &= \left(\pi^{\bar{\mu}}_{-\theta} \otimes \mathrm{id}\right)(\mathcal{R}^{\mathrm{op}}) \in \mathrm{End}(V^{\bar{\mu}}_{-\theta}) \otimes U_q(\hat{g}). \end{aligned} \tag{2.11}$$

Here $\mathcal{R}^{\mathrm{op}}$ is the opposite universal R-matrix obtained by interchanging the two tensor factors. Motivated by [31] we construct the matrices

$$B^{\mu}_{\theta} = \bar{L}^{\bar{\mu}}_{\theta}\,(K^{\mu}(\theta) \otimes 1)\,L^{\mu}_{\theta} \in \mathrm{Hom}(V^{\mu}_{\theta}, V^{\bar{\mu}}_{-\theta}) \otimes U_q(\hat{g}). \tag{2.12}$$

It may make things clearer to introduce matrix indices:

$$(B^{\mu}_{\theta})^{\alpha}{}_{\beta} = (\bar{L}^{\bar{\mu}}_{\theta})^{\alpha}{}_{\gamma}(K^{\mu}(\theta))^{\gamma}{}_{\delta}(L^{\mu}_{\theta})^{\delta}{}_{\beta} \in U_q(\hat{g}), \tag{2.13}$$

where we are using the usual summation convention. We will now check that for all θ the $(B^{\mu}_{\theta})^{\alpha}{}_{\beta}$ are elements of a coideal subalgebra $\mathcal{B}$ which commutes with the reflection matrices.

Let us first check that any reflection matrix $K^\nu(\theta') : V^\nu_{\theta'} \to V^{\bar\nu}_{-\theta'}$ which satisfies the appropriate reflection equation commutes with the action of the elements $(B^\mu_\theta)^\alpha{}_\beta$, i.e., that (see eq. (2.6))

$$K^\nu(\theta') \circ \pi^\nu_{\theta'}((B^\mu_\theta)^\alpha{}_\beta) = \pi^{\bar\nu}_{-\theta'}((B^\mu_\theta)^\alpha{}_\beta) \circ K^\nu(\theta'), \tag{2.14}$$

or, in index-free notation,

$$(\mathrm{id} \otimes K^\nu(\theta')) \circ (\mathrm{id} \otimes \pi^\nu_{\theta'})(B^\mu_\theta) = (\mathrm{id} \otimes \pi^{\bar\nu}_{-\theta'})(B^\mu_\theta) \circ (\mathrm{id} \otimes K^\nu(\theta')). \tag{2.15}$$

We observe that

$$\begin{aligned}
(\mathrm{id} \otimes \pi^\nu_{\theta'})(L^\mu_\theta) &= (\pi^\mu_\theta \otimes \pi^\nu_{\theta'})(\mathcal{R}) = R^{\mu\nu}(\theta - \theta'), &(2.16)\\
(\mathrm{id} \otimes \pi^\nu_{\theta'})(\bar L^{\bar\mu}_\theta) &= (\pi^{\bar\mu}_{-\theta} \otimes \pi^\nu_{\theta'})(\mathcal{R}^{\mathrm{op}}) = \check P R^{\nu\bar\mu}(\theta + \theta')\check P, &(2.17)\\
(\mathrm{id} \otimes \pi^{\bar\nu}_{-\theta'})(L^\mu_\theta) &= (\pi^\mu_\theta \otimes \pi^{\bar\nu}_{-\theta'})(\mathcal{R}) = R^{\mu\bar\nu}(\theta + \theta'), &(2.18)\\
(\mathrm{id} \otimes \pi^{\bar\nu}_{-\theta'})(\bar L^{\bar\mu}_\theta) &= (\pi^{\bar\mu}_{-\theta} \otimes \pi^{\bar\nu}_{-\theta'})(\mathcal{R}^{\mathrm{op}}) = \check P R^{\bar\nu\bar\mu}(\theta - \theta')\check P, &(2.19)
\end{aligned}$$

Substituting this into (2.15) and using $\check P R \propto S$ gives

$$\begin{aligned}
&(\mathrm{id} \otimes K^\nu(\theta')) \circ S^{\nu\bar\mu}(\theta + \theta') \circ (\mathrm{id} \otimes K^\mu(\theta)) \circ S^{\mu\nu}(\theta - \theta')\\
&\qquad = S^{\bar\nu\bar\mu}(\theta - \theta') \circ (\mathrm{id} \otimes K^\mu(\theta)) \circ S^{\mu\bar\nu}(\theta + \theta') \circ (\mathrm{id} \otimes K^\nu(\theta')),
\end{aligned} \tag{2.20}$$

which is just the reflection equation (compare with (2.8)). We thus see that every solution of the reflection equation commutes with all the generators $(B^\mu_\theta)^\alpha{}_\beta$, and vice versa: every matrix which satisfies the intertwining equation (2.15) is also a solution of the reflection equation (2.20).

Next we need to check the coideal property. Under the assumption that all $(B^\mu_\theta)^\alpha{}_\beta$ are in $\mathcal{B}$ we need to show that $\Delta\left((B^\mu_\theta)^\alpha{}_\beta\right)$ is in $U_q(\hat g) \otimes \mathcal{B}$. Using that

$$\begin{aligned}
\Delta\left((L^\mu_\theta)^\alpha{}_\beta\right) &= \left((\pi^\mu_\theta)^\alpha{}_\beta \otimes \Delta\right)(\mathcal{R}) &(2.21)\\
&= \left((\pi^\mu_\theta)^\alpha{}_\beta \otimes \mathrm{id} \otimes \mathrm{id}\right)(\mathcal{R}_{13}\mathcal{R}_{12}) &(2.22)\\
&= (L^\mu_\theta)^\gamma{}_\beta \otimes (L^\mu_\theta)^\alpha{}_\gamma, &(2.23)
\end{aligned}$$

and similarly

$$\Delta\left((\bar L^{\bar\mu}_\theta)^\alpha{}_\beta\right) = (\bar L^{\bar\mu}_\theta)^\alpha{}_\gamma \otimes (\bar L^{\bar\mu}_\theta)^\gamma{}_\beta, \tag{2.24}$$

we find that

$$\Delta\left((B^\mu_\theta)^\alpha{}_\beta\right) = (\bar L^{\bar\mu}_\theta)^\alpha{}_\delta (L^\mu_\theta)^\sigma{}_\beta \otimes (B^\mu_\theta)^\delta{}_\sigma, \tag{2.25}$$

which is indeed in $U_q(\hat g) \otimes \mathcal{B}$ as required.

Note that the B-matrices satisfy the quadratic relations

$$\check P R^{\bar\nu\bar\mu}(\theta - \theta')\check P \overset{1}{B^\mu_\theta} R^{\mu\bar\nu}(\theta + \theta') \overset{2}{B^\nu_{\theta'}} = \overset{2}{B^\nu_{\theta'}} \check P R^{\nu\bar\mu}(\theta + \theta')\check P \overset{1}{B^\mu_\theta} R^{\mu\nu}(\theta - \theta'), \tag{2.26}$$

which follow from the FRT relations satisfied by the L-matrices together with the reflection equation (2.9). Algebras with relations of this form are known as reflection equation algebras [31]. Our construction can thus be viewed as an embedding of the reflection equation algebras into the quantized enveloping algebras.

3 The sine-Gordon model

3.1 Review of non-local charges

For the purpose of finding its quantum group symmetry charges one views the sine-Gordon model as a perturbation of a free bosonic conformal field theory by a relevant operator Φ^{pert}[32]. The (Euclidean) action on the whole line is[2]

$$S = \frac{1}{4\pi}\int d^2z\,\partial\phi\bar{\partial}\phi\;+\;\frac{\lambda}{2\pi}\int d^2z\,\Phi^{\text{pert}}(x,t)\,, \tag{3.1}$$

with the perturbing operator

$$\Phi^{\text{pert}}(x,t) = e^{i\hat{\beta}\phi(x,t)} + e^{-i\hat{\beta}\phi(x,t)}\,, \tag{3.2}$$

where $\hat{\beta}$ is the coupling constant[3]. We impose the condition $\phi(-\infty,t)=0$.

The free boson field may be split into holomorphic and antiholomorphic parts, $\phi=\varphi+\bar{\varphi}$, where $\bar{\partial}\varphi = 0 = \partial\bar{\varphi}$, and the two-point functions are

$$\langle\varphi(z)\varphi(w)\rangle_0 = -\ln(z-w), \qquad \langle\bar{\varphi}(\bar{z})\bar{\varphi}(\bar{w})\rangle_0 = -\ln(\bar{z}-\bar{w}), \qquad \langle\varphi(z)\bar{\varphi}(\bar{w})\rangle_0 = 0\,. \tag{3.3}$$

The set of fields in the conformal field theory consists only of those combinations of derivatives and exponentials of the fundamental fields φ and $\bar{\varphi}$ which do not suffer from logarithmic divergences and are local with respect to each other. See [22] for a clear account of the free bosonic theory and its perturbation into the sine-Gordon model.

The $U_q(\hat{sl_2})$ symmetry of the sine-Gordon model is generated by the non-local charges [2]

$$Q_\pm = \frac{1}{4\pi}\int_{-\infty}^{\infty} dx(J_\pm - H_\pm)\,, \qquad \bar{Q}_\pm = \frac{1}{4\pi}\int_{-\infty}^{\infty} dx(\bar{J}_\pm - \bar{H}_\pm)\,, \tag{3.4}$$

where

$$J_\pm =: e^{\pm\frac{2i}{\hat{\beta}}\varphi} :\,, \qquad \bar{J}_\pm =: e^{\mp\frac{2i}{\hat{\beta}}\bar{\varphi}} :\,, \tag{3.5}$$

$$[0.1in]\quad H_\pm = \lambda\tfrac{\hat{\beta}^2}{\hat{\beta}^2-2} : \exp\left(\pm i\left(\tfrac{2}{\hat{\beta}}-\hat{\beta}\right)\varphi \mp i\hat{\beta}\bar{\varphi}\right) : \quad, \tag{3.6}$$

$$[0.1in]\quad \bar{H}_\pm = \lambda\tfrac{\hat{\beta}^2}{\hat{\beta}^2-2} : \exp\left(\mp i\left(\tfrac{2}{\hat{\beta}}-\hat{\beta}\right)\bar{\varphi} \pm i\hat{\beta}\varphi\right) : \tag{3.7}$$

[2]We use the conventions of [2] and denote Euclidean light-cone coordinates as $z=(t+ix)/2$ and $\bar{z}=(t-ix)/2$, where the Euclidean time t is related to Minkowski time t^M by $t=it^M$. The derivatives are then $\partial=\partial_t-i\partial_x, \bar{\partial}=\partial_t+i\partial_x$. We write $d^2z = idz\,d\bar{z} = -dx\,dt/2$.

[3]$\hat{\beta}$ is related to the conventional β by $\hat{\beta}=\beta/\sqrt{4\pi}$.

together with the topological charge

$$T = \frac{\hat{\beta}}{2\pi} \int_{-\infty}^{\infty} dx\, \partial_x \phi\,. \tag{3.8}$$

The time-independence of the charges follows from the current conservation equations

$$\bar{\partial} J_{\pm} = \partial H_{\pm}\,, \quad \partial \bar{J}_{\pm} = \bar{\partial} \bar{H}_{\pm}\,, \tag{3.9}$$

which were obtained in first-order perturbation theory in [2]. In order to derive the $U_q(\hat{sl}_2)$ relations and coproduct, we set $\phi(-\infty) = 0$. The parameter q is then related to the Toda coupling constant $\hat{\beta}$ by

$$q = \exp\left(2i\pi(1-\hat{\beta}^2)/\hat{\beta}^2\right). \tag{3.10}$$

3.2 Neumann boundary condition

We now want to restrict the sine-Gordon model to the half line $x \le 0$ by imposing the Neumann boundary condition $\partial_x \tilde{\phi} = 0$ at $x = 0$. Note that to avoid confusion we will always decorate fields in the theory on the half-line with a tilde. Again we will consider the sine-Gordon model as a perturbation of the free boson.

A simple way to describe the free bosonic field theory on the half-line with Neumann boundary condition is to identify its fields with the subset of parity-invariant fields of the theory on the whole line. Thus for every field $\Phi(x,t)$ on the whole line there exists a field $\tilde{\Phi}(x,t)$ on the half-line defined by

$$\tilde{\Phi}(x,t) = \Phi(x,t) + \bar{\Phi}(-x,t) \qquad \text{for}\ \ x \le 0\,, \tag{3.11}$$

where $\bar{\Phi}(-x,t)$ is the parity-transform of $\Phi(x,t)$. The fundamental field on the half-line, $\tilde{\phi}(x,t) = \phi(x,t) + \phi(-x,t)$, immediately satisfies the Neumann boundary condition. The two-point functions for its chiral components

$$\tilde{\varphi}(x,t) = \varphi(x,t) + \bar{\varphi}(-x,t) \qquad \text{and} \qquad \tilde{\bar{\varphi}}(x,t) = \bar{\varphi}(x,t) + \varphi(-x,t) \tag{3.12}$$

follow immediately from (3.3), and are

$$\langle \tilde{\varphi}(z)\tilde{\varphi}(w)\rangle_0 = -2\ln(z-w)\,, \qquad \langle \tilde{\bar{\varphi}}(z)\tilde{\bar{\varphi}}(w)\rangle_0 = -2\ln(\bar{z}-\bar{w})\,, \tag{3.13}$$

$$\langle \tilde{\varphi}(z)\tilde{\bar{\varphi}}(\bar{w})\rangle_0 = -2\ln(z-\bar{w})\,. \tag{3.14}$$

Note the non-vanishing two-point function between the holomorphic and anti-holomorphic components.

The perturbation $\int d^2z\, \Phi^{\text{pert}}$ is invariant under parity and is thus a valid perturbation of the boson on the half-line too. Note that one does not obtain the sine-Gordon model on the half-line by perturbing with the operator $\exp(i\hat{\beta}\tilde{\phi}(x,t)) + \exp(-i\hat{\beta}\tilde{\phi}(x,t))$, as one might naively have thought[4].

The charges $T, Q_\pm$ and $\bar{Q}_\pm$ described in the previous section transform under parity $\mathcal{P}$ as follows:

$$\mathcal{P} : T \mapsto -T, \qquad Q_\pm \mapsto \bar{Q}_\mp, \qquad \bar{Q}_\pm \mapsto Q_\mp \,. \tag{3.15}$$

The parity-invariant combinations $\tilde{Q}_\pm = Q_\pm + \bar{Q}_\mp$ therefore give conserved charges on the half-line, but T does not. We can express the charges $\tilde{Q}_\pm$ in terms of currents on the half-line,

$$\tilde{Q}_\pm = \frac{1}{4\pi} \int_{-\infty}^{0} dx \left(\tilde{J}_\pm - \tilde{H}_\pm + \tilde{\bar{J}}_\mp - \tilde{\bar{H}}_\mp \right), \tag{3.16}$$

where the half-line currents are defined according to (3.11), for example $\tilde{J}_\pm(x) = J_\pm(x) + \bar{J}_\mp(-x)$. They satisfy the conservation equations

$$\bar{\partial} \tilde{J}_\pm = \partial \tilde{H}_\pm \qquad \text{and} \qquad \partial \tilde{\bar{J}}_\pm = \bar{\partial} \tilde{\bar{H}}_\pm \,. \tag{3.17}$$

3.3 General boundary conditions as boundary perturbations

In [30, 18] the sine-Gordon model was found to be classically integrable with the rather more general boundary condition

$$\partial_x \tilde{\phi} = i\hat{\beta}\lambda_b \left(\epsilon_- e^{i\hat{\beta}\tilde{\phi}(0,t)/2} - \epsilon_+ e^{-i\hat{\beta}\tilde{\phi}(0,t)/2} \right) \qquad \text{at} \;\; x = 0 \,. \tag{3.18}$$

We shall treat this as a boundary perturbation of the sine-Gordon model with Neumann boundary condition:

$$S_\epsilon = S_{\text{Neumann}} + \frac{\lambda_b}{2\pi} \int dt \, \Phi^{\text{pert}}_{\text{boundary}}(t) \,, \tag{3.19}$$

with the boundary perturbing operator

$$\begin{aligned} \Phi^{\text{pert}}_{\text{boundary}}(t) &= \epsilon_- e^{i\hat{\beta}\tilde{\phi}(0,t)/2} + \epsilon_+ e^{-i\hat{\beta}\tilde{\phi}(0,t)/2} \\ &= \epsilon_- e^{i\hat{\beta}\phi(0,t)} + \epsilon_+ e^{-i\hat{\beta}\phi(0,t)} \,. \end{aligned}$$

To check that S_ϵ does indeed produce the boundary condition (3.18) we calculate the correlation functions of $\partial_x \tilde{\phi}(0,t)$ in first-order boundary conformal perturbation theory [18, 29]:

$$\begin{aligned} \langle \partial_x \tilde{\phi}(0,t) \cdots \rangle &= \lim_{x \to 0^-} \langle \partial_x \tilde{\phi}(x,t) e^{-S_{\text{boundary}}} \cdots \rangle_{\text{N}} \\ &= \langle \partial_x \tilde{\phi}(0,t) \cdots \rangle_{\text{N}} - \frac{\lambda_b}{2\pi} \int dt' \lim_{x \to 0^-} \langle \partial_x \tilde{\phi}(x,t) \Phi^{\text{pert}}_{\text{boundary}}(t') \cdots \rangle_{\text{N}} + \mathcal{O}(\lambda_b^2), \end{aligned}$$

[4] See however [1, appendix C] for a different treatment of the boundary perturbation

where $\langle \cdots \rangle_{\rm N}$ denotes the correlation function with Neumann boundary condition. The first term on the right hand side vanishes of course. To evaluate the second we need the operator product expansions

$$\partial\tilde{\varphi}(x,t)\Phi^{\rm pert}_{\rm boundary}(t') = \frac{-\,2i\hat{\beta}}{t-t'+ix}\left(\epsilon_- e^{i\hat{\beta}\tilde{\phi}(0,t)/2} - \epsilon_+ e^{-i\hat{\beta}\tilde{\phi}(0,t)/2}\right) + \text{regular terms},$$
$$\overline{\partial}\tilde{\overline{\varphi}}(x,t)\Phi^{\rm pert}_{\rm boundary}(t') = \frac{-\,2i\hat{\beta}}{t-t'-ix}\left(\epsilon_- e^{i\hat{\beta}\tilde{\phi}(0,t)/2} - \epsilon_+ e^{-i\hat{\beta}\tilde{\phi}(0,t)/2}\right) + \text{regular terms},$$

and thus, using that $\partial_x\phi = \frac{i}{2}(\partial\tilde{\varphi} - \bar{\partial}\tilde{\bar{\varphi}} + \mathcal{O}(\lambda_b)$,

$$\partial_x\phi(x,t)\Phi^{\rm pert}_{\rm boundary}(t') = \hat{\beta}\left(\frac{1}{t-t'+ix} - \frac{1}{t-t'-ix}\right)\left(\epsilon_- e^{i\hat{\beta}\tilde{\phi}(0,t)/2} - \epsilon_+ e^{-i\hat{\beta}\tilde{\phi}(0,t)/2}\right) + \dots. \tag{3.20}$$

We can now use the identity

$$\lim_{x\to 0^-}\left(\frac{1}{(t-t'+ix)^n} - \frac{1}{(t-t'-ix)^n}\right) = \frac{2\pi i}{(n-1)!}\partial_{t'}^{n-1}\delta(t-t'), \tag{3.21}$$

to find

$$\langle \partial_x\tilde{\phi}(0,t)\cdots\rangle = \langle i\hat{\beta}\lambda_b\left(\epsilon_- e^{i\hat{\beta}\tilde{\phi}(0,t)/2} - \epsilon_+ e^{-i\hat{\beta}\tilde{\phi}(0,t)/2}\right)\rangle, \tag{3.22}$$

in agreement with the boundary condition (3.18).

3.4 Quantum group charges for general boundary condition

For the general boundary conditions (3.18) (that is, in the presence of the boundary perturbation (??)) the charges $\tilde{Q}_\pm$ in (3.16) will no longer be conserved. We calculate the time-dependence of the charges:

$$\begin{aligned}
\partial_t\tilde{Q}_\pm &= \frac{1}{4\pi}\int_{-\infty}^{0} dx\,\partial_t\left(\tilde{J}_\pm - \tilde{H}_\pm + \tilde{\bar{J}}_\mp - \tilde{\bar{H}}_\mp\right)\\
&= -\frac{i}{4\pi}\int_{-\infty}^{0} dx\,\partial_x\left(\tilde{J}_\pm + \tilde{H}_\pm - \tilde{\bar{J}}_\mp - \tilde{\bar{H}}_\mp\right)\\
&= -\frac{i}{4\pi}\left(\tilde{J}_\pm(0,t) + \tilde{H}_\pm(0,t) - \tilde{\bar{J}}_\mp(0,t) - \tilde{\bar{H}}_\mp(0,t)\right).
\end{aligned}$$

For the Neumann boundary condition, $J_\pm(0,t) = \bar{J}_\mp(0,t)$ and $H_\pm(0,t) = \bar{H}_\mp(0,t)$ and thus $\partial_t\tilde{Q}_\pm = 0$. But in general we will obtain in first order perturbation theory

$$\langle \partial_t\tilde{Q}_+(0,t)\cdots\rangle = -\frac{\lambda_b}{2\pi}\int dt' \lim_{x\to 0^-}\langle \partial_t\tilde{Q}_+(x,t)\Phi^{\rm pert}_{\rm boundary}(t')\dots\rangle_N + \dots \tag{3.23}$$
$$= -\frac{\lambda_b}{2\pi}\int dt' \lim_{x\to 0^-}\langle \frac{-\,i}{4\pi}\left(\tilde{J}_+(x,t) - \tilde{\bar{J}}_-(x,t)\right)\Phi^{\rm pert}_{\rm boundary}(t')\dots\rangle_N + \dots. \tag{3.24}$$

Because $\tilde{H}_+$ and $\tilde{\bar{H}}_-$ are themselves already of order λ they do not contribute to first order. The necessary operator product expansions are

$$\tilde{J}_+(x,t)\Phi^{\text{pert}}_{\text{boundary}}(t') = \epsilon_+ \frac{1}{(t-t'+ix)^2} : \left(e^{\frac{2i}{\beta}\varphi(x,t)} + e^{\frac{2i}{\beta}\bar{\varphi}(-x,t)}\right) e^{-i\hat{\beta}\phi(0,t)} : + \dots,$$
$$\tilde{\bar{J}}_-(x,t)\Phi^{\text{pert}}_{\text{boundary}}(t') = \epsilon_+ \frac{1}{(t-t'-ix)^2} : \left(e^{\frac{2i}{\beta}\varphi(x,t)} + e^{\frac{2i}{\beta}\bar{\varphi}(-x,t)}\right) e^{-i\hat{\beta}\phi(0,t)} : + \cdots.$$

Using now that at the boundary $\bar{\varphi} = \varphi = \phi/2$ up to order λ terms, we obtain

$$\begin{aligned}
\langle \partial_t \tilde{Q}_+(0,t)\cdots\rangle &= -\frac{\lambda_b\epsilon_+}{2\pi}\frac{1}{2\pi i}\int dt' \lim_{x\to 0^-}\left(\frac{1}{(t-t'+ix)^2} - \frac{1}{(t-t'-ix)^2}\right) \\
&\qquad : e^{\frac{i}{\beta}\phi(x,t)} e^{-i\hat{\beta}\phi(0,t')} : + \cdots \\
&= \frac{\lambda_b\epsilon_+}{2\pi}\frac{\hat{\beta}^2}{\hat{\beta}^2-1}\partial_t e^{i\left(\frac{1}{\beta}-\hat{\beta}\right)\phi(0,t)} + \cdots \\
&= \frac{\lambda_b\epsilon_+}{2\pi}\frac{\hat{\beta}^2}{\hat{\beta}^2-1}\partial_t q^T + \cdots,
\end{aligned}$$

where we used that q has the value given in (3.10). It follows that the charge

$$\hat{Q}_+ = Q_+ + \bar{Q}_- + \hat{\epsilon}_+ q^T \quad \text{with} \quad \hat{\epsilon}_+ = \frac{\lambda_b\epsilon_+}{2\pi}\frac{\hat{\beta}^2}{1-\hat{\beta}^2} \tag{3.25}$$

is conserved to first order in perturbation theory. By similar calculations we obtain a second conserved charge

$$\hat{Q}_- = Q_- + \bar{Q}_+ + \hat{\epsilon}_- q^{-T} \quad \text{with} \quad \hat{\epsilon}_- = \frac{\lambda_b\epsilon_-}{2\pi}\frac{\hat{\beta}^2}{1-\hat{\beta}^2}. \tag{3.26}$$

These charges were first written down in [24]. They generate a coideal subalgebra of $U_q(\hat{g})$ because

$$\Delta(\hat{Q}_\pm) = (Q_\pm + \bar{Q}_\mp)\otimes 1 + q^{\pm T}\otimes \hat{Q}_\pm. \tag{3.27}$$

3.5 Reflection matrices derived from quantum group symmetry

We will now use our knowledge of the conserved charges $\hat{Q}_\pm$ to derive the soliton reflection matrix, up to an overall factor.

The sine-Gordon model only has a single two-dimensional soliton multiplet, spanned by the soliton and anti-soliton states $|A_\pm(\theta)\rangle$. The soliton reflection matrix describes what happens to a soliton during reflection off the boundary: a soliton of type α with rapidity θ is converted into a combination of soliton types β with opposite rapidity $-\theta$ with probability amplitudes $K^\beta{}_\alpha$,

$$K : |A_\alpha(\theta)\rangle \mapsto |A_\beta(-\theta)\rangle K^\beta{}_\alpha(\theta). \tag{3.28}$$

The action of the symmetry charges $\hat{Q}_\pm$ on the soliton states can be obtained from the action of the quantum group charges given in [2]. One finds (after a change of basis with respect to that used in [2])

$$\hat{Q}_\pm : |A_\alpha(\theta)\rangle \mapsto |A_\beta(\theta)\rangle \pi_\theta(\hat{Q}_\pm)^\beta{}_\alpha \tag{3.29}$$

with

$$\pi_\theta(\hat{Q}_\pm)^+{}_+ = \hat{\epsilon}_\pm\, q^{\pm 1}, \qquad \pi_\theta(\hat{Q}_\pm)^-{}_- = \hat{\epsilon}_\pm\, q^{\mp 1}, \tag{3.30}$$

$$\pi_\theta(\hat{Q}_\pm)^+{}_- = c\, e^{\pm\theta/\gamma}, \qquad \pi_\theta(\hat{Q}_\pm)^-{}_+ = c\, e^{\mp\theta/\gamma}, \tag{3.31}$$

where $\gamma = \hat{\beta}^2/(2-\hat{\beta}^2)$ and $c = \sqrt{\lambda\gamma^2(q^2-1)/2\pi i}$. We know that reflection and symmetry transformations have to commute, which leads to the following set of eight homogeneous linear equations for the four entries of the reflection matrix (see (2.6)):

$$K^\gamma{}_\beta(\theta)\ \pi_\theta(\hat{Q}_\pm)^\beta{}_\alpha = \pi_{-\theta}(\hat{Q}_\pm)^\gamma{}_\beta\ K^\beta{}_\alpha, \quad \forall\, \gamma, \alpha \in \{+,-\}. \tag{3.32}$$

This set of equations is very easy to solve and one finds the unique solution (up to an undetermined overall factor $k(\theta)$)

$$K^+{}_-(\theta) = K^-{}_+(\theta) = \left(e^{2\theta/\gamma} - e^{-2\theta/\gamma}\right) k(\theta), \tag{3.33}$$

$$K^\pm{}_\pm = \frac{q-q^{-1}}{c}\left(\hat{\epsilon}_\pm\, e^{\theta/\gamma} + \hat{\epsilon}_\mp\, e^{-\theta/\gamma}\right) k(\theta). \tag{3.34}$$

This agrees with the soliton reflection matrix determined in [18] by solving the reflection equation.

4 Affine Toda theory

To every affine Lie algebra $\hat{g}$ of rank n there is associated an affine Toda field theory [25] with Euclidean action

$$S = \frac{1}{4\pi}\int d^2z\, \partial\phi\bar{\partial}\phi + \frac{\lambda}{2\pi}\int d^2z \sum_{j=0}^{n} \exp\left(-i\hat{\beta}\frac{1}{|\alpha_j|^2}\alpha_j\cdot\phi\right), \tag{4.1}$$

describing an n-component bosonic field ϕ in two dimensions. The exponential interaction potential is expressed in terms of the simple roots α_i, $i = 0,\ldots,n$ of $\hat{g}$ (projected onto the root space of g). The parameter λ sets the mass scale and $\hat{\beta}$ is the coupling constant.

For simplicity we shall restrict our attention to simply-laced algebras, and choose the (slightly unusual) convention that $|\alpha_i|^2 = 1$. With $g = a_1 = su(2)$ this specializes to the sine-Gordon model of Section 3.

4.1 Conserved charges from conformal perturbation theory

The quantum group symmetry algebra $U_q(\hat{g})$ is generated by the topological charges

$$T_j = \frac{\hat{\beta}}{2\pi} \int_{-\infty}^{\infty} dx\, \alpha_j \cdot \partial_x \phi \,. \tag{4.2}$$

together with the non-local conserved charges

$$Q_j = \frac{1}{4\pi c} \int_{-\infty}^{\infty} dx\, (J_j - H_j)\,, \qquad \bar{Q}_j = \frac{1}{4\pi c} \int_{-\infty}^{\infty} dx\, (\bar{J}_j - \bar{H}_j)\,, \qquad j = 0, 1, \ldots, n, \tag{4.3}$$

where

$$J_j =: \exp\left(\tfrac{2i}{\hat{\beta}} \alpha_j \cdot \varphi\right): \,, \qquad \bar{J}_j =: \exp\left(\tfrac{2i}{\hat{\beta}} \alpha_j \cdot \bar{\varphi}\right): \,, \tag{4.4}$$

$$[0.1in] \quad H_j = \lambda \tfrac{\hat{\beta}^2}{\hat{\beta}^2 - 2} : \exp\left(i\left(\tfrac{2}{\hat{\beta}} - \hat{\beta}\right) \alpha_j \cdot \varphi - i\hat{\beta} \alpha_j \cdot \bar{\varphi}\right): \,, \tag{4.5}$$

$$[0.1in] \quad \bar{H}_j = \lambda \tfrac{\hat{\beta}^2}{\hat{\beta}^2 - 2} : \exp\left(i\left(\tfrac{2}{\hat{\beta}} - \hat{\beta}\right) \alpha_j \cdot \bar{\varphi} - i\hat{\beta} \alpha_j \cdot \varphi\right):, \tag{4.6}$$

and we choose the normalization constant $c = \sqrt{\lambda \gamma^2 (q_i^2 - 1)/2\pi i}$ in order to obtain the simple q-commutation relations given later in (4.26). The linear combinations $\tilde{Q}_j = Q_j + \bar{Q}_j$ are parity-invariant and thus yield conserved charges on the half-line with Neumann boundary conditions.

We now add to the action a boundary perturbation,

$$S_\epsilon = S_{\text{Neumann}} + \frac{\lambda_b}{2\pi} \int dt\, \Phi^{\text{pert}}_{\text{boundary}}(t)\,, \tag{4.7}$$

where

$$\Phi^{\text{pert}}_{\text{boundary}}(t) = \sum_{j=0}^{n} \epsilon_j \exp\left(-\frac{i\hat{\beta}}{2} \alpha_j \cdot \tilde{\phi}(0,t)\right)\,, \tag{4.8}$$

which leads to the boundary condition

$$\partial_x \tilde{\phi} = -i\hat{\beta} \lambda_b \sum_{j=0}^{n} \epsilon_j \alpha_j \exp\left(-\frac{i\hat{\beta}}{2} \alpha_j \cdot \tilde{\phi}(0,t)\right) \qquad \text{at} \;\; x = 0. \tag{4.9}$$

By calculations entirely analogous to those of section 3 we find that, due to this perturbation, the $\tilde{Q}_i$ are no longer conserved, but instead satisfy

$$\partial_t \tilde{Q}_i = \frac{\lambda_b \epsilon_i}{2\pi c} \frac{\hat{\beta}^2}{\hat{\beta}^2 - 1} \partial_t q^{T_i}\,, \tag{4.10}$$

so that the new conserved charges are

$$\widehat{Q}_i = Q_i + \bar{Q}_i + \hat{\epsilon}_i q^{T_i}\,, \tag{4.11}$$

where, to first order in perturbation theory,

$$\hat{\epsilon}_i = \frac{\lambda_b \epsilon_i}{2\pi c} \frac{\hat{\beta}^2}{1-\hat{\beta}^2}\,. \tag{4.12}$$

Note that at this stage the boundary parameters $\hat{\epsilon}_i$ can still take arbitrary values. However, we shall see in subsequent sections how the $|\hat{\epsilon}_i|$ are fixed, leaving only a choice of signs.

The symmetry algebra of the boundary affine Toda theory generated by the $\hat{Q}_i$, $i = 0,\ldots,n$, is a coideal subalgebra of $U_q(\hat{g})$ because $\Delta(\hat{Q}_i) = \hat{Q}_i \otimes 1 + q^{T_i} \otimes (\hat{Q}_i - \hat{\epsilon}_i)$.

4.2 Reflection matrices derived from quantum group symmetry

The conserved charges (4.11) derived in the previous section can now be used to derive the soliton reflection matrices, as explained in section 2.2. We will illustrate this here in the example of the vector solitons in $a_n^{(1)}$ Toda theories. The new feature that arises which was not visible in the sine-Gordon model is that solitons are converted into antisolitons upon reflection off the boundary. Thus in particular the vector solitons are reflected into solitons in the conjugate vector representation.

Let V_θ^μ be the space spanned by the vector solitons with rapidity θ. Choosing a suitable basis for V_θ^μ and defining the elementary matrices $e^i{}_j$ to be the matrices with a 1 in the i-th row and the j-th column, the representation matrices of the $U_q(\hat{g})$ generators are

$$\pi_\theta^\mu(Q_i) = x\, e^{i+1}{}_i, \qquad \pi_\theta^\mu(\bar{Q}_i) = x^{-1}\, e^i{}_{i+1}, \qquad \pi_\theta^\mu(T_i) = -e^i{}_i + e^{i+1}{}_{i+1}, \tag{4.13}$$

where $x = e^{\theta/\gamma}$ with $\gamma = \hat{\beta}^2/(2-\hat{\beta}^2)$ and where we identify the indices $n+1=0$, $n+2=1$. The representation matrices for the conjugate representation on $V_\theta^{\bar{\mu}}$ are

$$\pi_\theta^{\bar{\mu}}(Q_i) = x\, e^i{}_{i+1}, \qquad \pi_\theta^{\bar{\mu}}(\bar{Q}_i) = x^{-1}\, e^{i+1}{}_i, \qquad \pi_\theta^{\bar{\mu}}(T_i) = e^i{}_i - e^{i+1}{}_{i+1}, \tag{4.14}$$

The representation matrices of the symmetry generators $\hat{Q}_i$ then immediately follow from (4.11),

$$\pi_\theta^\mu(\hat{Q}_i) = x\, e^{i+1}{}_i + x^{-1}\, e^i{}_{i+1} + \hat{\epsilon}_i\left((q^{-1}-1)\, e^i{}_i + (q-1)\, e^{i+1}{}_{i+1} + \mathbf{1}\right), \tag{4.15}$$

$$\pi_\theta^{\bar{\mu}}(\hat{Q}_i) = x\, e^i{}_{i+1} + x^{-1}\, e^{i+1}{}_i + \hat{\epsilon}_i\left((q-1)\, e^i{}_i + (q^{-1}-1)\, e^{i+1}{}_{i+1} + \mathbf{1}\right), \tag{4.16}$$

where $\mathbf{1}$ denotes the unit matrix. Because the representation matrices are so sparse most of the $(n+1)^3$ components of the intertwining equation (2.6) for the K-matrix are trivial, leaving only the $2n(n+1)$ equations

$$0 = \hat{\epsilon}_i(q^{-1}-q)K^i{}_i + x\, K^i{}_{i+1} - x^{-1}\, K^{i+1}{}_i, \tag{4.17}$$

$$0 = K^{i+1}{}_{i+1} - K^i{}_i, \tag{4.18}$$

$$0 = \hat{\epsilon}_i\, q\, K^i{}_j + x^{-1}\, K^{i+1}{}_j, \quad j \neq i, i+1, \tag{4.19}$$

$$0 = \hat{\epsilon}_i\, q^{-1}\, K^j{}_i + x\, K^j{}_{i+1}, \quad j \neq i, i+1. \tag{4.20}$$

The equations in (4.19) and (4.20) can be used to determine all upper triangular entries in terms of ${K^1}_2$ and all lower triangular entries in terms of ${K^2}_1$. Then the equations (4.17) determine the diagonal entries. Finally the fact that all diagonal entries must be the same according to (4.18) not only determines ${K^2}_1$ in terms of ${K^1}_2$ but also requires that either $\hat{\epsilon}_i = 0$ for all i or that $|\hat{\epsilon}_i| = 1$ for all i. If all $\hat{\epsilon}_i = 0$ then K can be an arbitrary diagonal matrix. If all $|\hat{\epsilon}_i| = 1$ then one obtains the non-diagonal solution

$$\begin{aligned}
{K^i}_i(\theta) &= \left(q^{-1}\,(-q\,x)^{(n+1)/2} - \hat{\epsilon}\,q\,(-q\,x)^{-(n+1)/2}\right)\frac{k(\theta)}{q^{-1}-q}, \\
{K^i}_j(\theta) &= \hat{\epsilon}_i\cdots\hat{\epsilon}_{j-1}\,(-q\,x)^{i-j+(n+1)/2}\,k(\theta), \qquad \text{for } j>i, \\
{K^j}_i(\theta) &= \hat{\epsilon}_i\cdots\hat{\epsilon}_{j-1}\hat{\epsilon}\,(-q\,x)^{j-i-(n+1)/2}\,k(\theta), \qquad \text{for } j>i,
\end{aligned} \tag{4.21}$$

which is unique up to the overall numerical factor $k(\theta)$. We have defined $\hat{\epsilon} = \hat{\epsilon}_0\hat{\epsilon}_1\cdots\hat{\epsilon}_n$. Note that this agrees with the solution found by Gandenberger in [17].

It is interesting to note that the restriction on the possible values of the boundary parameters $\hat{\epsilon}_i$ which we found above agrees with the restrictions which were found in [3] from the requirement of classical integrability.

The reflection matrices for solitons in the multiplets corresponding to the other fundamental representations could be determined either in the same manner as above or by a fusion procedure. Furthermore there will be boundary bound states which may transform non-trivially under the quantum group symmetry and again their reflection matrices can be obtained by using the intertwining property or by boundary fusion. We refer the reader to [9] where similar calculations have been done for the rational reflection matrices which have a Yangian symmetry.

4.3 Construction of conserved charges from reflection matrices

In section 4.1 we derived expressions for the symmetry generators $\hat{Q}_i$ by using first-order boundary conformal perturbation theory. These symmetry generators allowed us to determine the reflection matrices for the vector solitons in $a_n^{(1)}$ Toda theory in section 4.2. This success can be taken as confirmation of the correctness of the expressions (4.11) for the symmetry generators. However one might worry that higher-order perturbation theory might produce additional terms which, while not visible in the vector representation, might be required to guarantee the commutation of the generators with the reflection matrices in higher representations. In order to rule this out, we will now rederive the expressions for the generators $\hat{Q}_i$ using the construction introduced in section 2.3. We will show how to extract the $\hat{Q}_i$ from the B^μ_θ by expanding to first order in $x = \exp(\theta/\gamma)$.

The construction requires the L-matrices that are obtained from the universal R-matrix according to (2.11). Luckily we will not need to work with the rather involved expression for the full universal R-matrix. Instead we will introduce the spectral parameter x and expand to first order in x. For this purpose let us introduce the homomorphism $\Psi_x : U_q(\hat{g}) \to U_q(\hat{g})[x, x^{-1}]$, defined by

$$\Psi_x(Q_i) = x\, Q_i, \qquad \Psi_x(\bar{Q}_i) = x^{-1}\, \bar{Q}_i, \qquad \Psi_x(T_i) = T_i. \tag{4.22}$$

This will be useful later because

$$\pi^\mu_\theta = \pi^\mu_0 \circ \Psi_x \quad \text{for} \quad x = e^{\theta/\gamma}. \tag{4.23}$$

We introduce the spectral parameter dependent universal R-matrix as

$$\mathcal{R}(x) = (\Psi_x \otimes \mathrm{id})(\mathcal{R}). \tag{4.24}$$

It satisfies

$$\mathcal{R}(x)\left((\Psi_x \otimes \mathrm{id}) \circ \Delta\right)(Q) = \left((\Psi_x \otimes \mathrm{id}) \circ \Delta^{\mathrm{op}}\right)(Q)\mathcal{R}(x) \qquad \forall Q \in U_q(\hat{g}). \tag{4.25}$$

We want to use this property to determine the expression for $\mathcal{R}(x)$ to linear order in x. For this purpose we need the relations between the $U_q(\hat{g})$ generators:

$$\begin{aligned} &[T_i, Q_j] = \alpha_i \cdot \alpha_j\, Q_j, \qquad [T_i, \bar{Q}_j] = -\alpha_i \cdot \alpha_j\, \bar{Q}_j \\ &Q_i \bar{Q}_j - q^{-\alpha_i \cdot \alpha_j} \bar{Q}_j Q_i = \delta_{ij}\, \frac{q^{2T_i} - 1}{q_i^2 - 1}, \end{aligned} \tag{4.26}$$

where $q_i = q^{\alpha_i \cdot \alpha_i / 2}$. The generators also satisfy Serre relations which we will not need however. The coproduct Δ is defined by

$$\begin{aligned} \Delta(Q_i) &= Q_i \otimes 1 + q^{T_i} \otimes Q_i, \\ \Delta(\overline{Q}_i) &= \overline{Q}_i \otimes 1 + q^{T_i} \otimes \overline{Q}_i, \\ \Delta(T_i) &= T_i \otimes 1 + 1 \otimes T_i. \end{aligned} \tag{4.27}$$

Using this information we find that

$$\mathcal{R}(x) = \left(1 \otimes 1 + x \sum_{l=0}^{n} (1 - q_l^2) Q_l \otimes \bar{Q}_l\, q^{-T_l}\right) q^{\sum_{j,k=1}^n g_{jk} T_j \otimes T_k} + \mathcal{O}(x^2), \tag{4.28}$$

or, equivalently,

$$\mathcal{R}(x) = q^{\sum_{j,k=1}^n g_{jk} T_j \otimes T_k} \left(1 \otimes 1 + x \sum_{l=0}^{n} (1 - q_l^2)\, q^{-T_l}\, Q_l \otimes \bar{Q}_l\right) + \mathcal{O}(x^2), \tag{4.29}$$

where $g_{jk}\ \alpha_k \cdot \alpha_l = -\delta_{jl}$.

We now specialize to the vector representation of $a_n^{(1)}$ and find the L-operators according to eqns. (2.11),

$$\begin{aligned} L^{\mu}_{\theta} &= (\pi^{\mu}_{\theta} \otimes \mathrm{id})(\mathcal{R}) = (\pi^{\mu}_{0} \otimes \mathrm{id})(\mathcal{R}(x)) && (4.30)\\ &= q^{A}\left(\mathbf{1}\otimes 1 + x\sum_{l=0}^{n}(q^{-1}-q)\, e^{l+1}{}_{l} \otimes \bar{Q}_l\right) + \mathcal{O}(x^2), && (4.31)\\ \bar{L}^{\bar{\mu}}_{\theta} &= (\pi^{\bar{\mu}}_{-\theta} \otimes \mathrm{id})(\mathcal{R}^{\mathrm{op}}) = (\pi^{\bar{\mu}}_{0} \otimes \mathrm{id})(\mathcal{R}(x)^{\mathrm{op}}) && (4.32)\\ &= \left(\mathbf{1}\otimes 1 + x\sum_{l=0}^{n}(q^{-1}-q)\, e^{l+1}{}_{l} \otimes Q_l\right) q^{-A} + \mathcal{O}(x^2), && (4.33) \end{aligned}$$

where $A = \sum g_{jk}(-e^{j}{}_{j} + e^{j+1}{}_{j+1}) \otimes T_k$. We also expand the reflection matrix $K^{\mu}(\theta)$ given in (4.21) to first order in x,

$$K^{\mu}(\theta) = B\left(\mathbf{1} + x\sum_{l=0}^{n}\hat{\epsilon}_l\,(q^{-1}-q)\, e^{l+1}{}_{l}\right) + \mathcal{O}(x^2), \qquad (4.34)$$

where $B = -\hat{\epsilon}\, q\, (-q\, x)^{-(n+1)/2}/(q-q^{-1})$.

Putting it all together according to eq.(2.12) gives

$$B^{\mu}_{\theta} = B + x\sum_{l=0}^{n}(q^{-1}-q)\, e^{l+1}{}_{l} \otimes \left(Q_l + \bar{Q}_l + \hat{\epsilon}_l\, q^{T_l}\right) + \mathcal{O}(x^2). \qquad (4.35)$$

Thus we can read off our generators $\hat{Q}_i$ from the non-vanishing entries of the matrix B^{μ}_{θ} at first order in x. This proves that their action does indeed commute with all reflection matrices because we had shown this for the entries of the matrix B^{μ}_{θ} already in section 2.3. They thus generate a symmetry algebra of $a_n^{(1)}$ affine Toda theory on the half-line.

5 Discussion

In this paper we have derived non-local conserved charges for the sine-Gordon model and $a_n^{(1)}$ affine Toda field theories on the half-line and have shown how to use these to determine the soliton reflection matrices by solving the linear intertwining equations.

The calculations in conformal boundary perturbation theory used to derive the non-local charges in sections 3.4 and 4.1 may be of interest in themselves because of the way in which the perturbation theory for the model on the half-line is embedded into that for the model on the whole line.

The calculations should easily generalize to the Toda theories for arbitrary affine Kac-Moody algebras $\hat{g}$, allowing us to derive the hitherto unknown soliton reflection matrices in these theories. This has recently been carried through for the case of $\hat{g} = d_n^{(1)}$ in [10]. One can then derive also the particle reflection amplitudes, as was done for the $a_n^{(1)}$ Toda particles in [8].

The reconstruction of the symmetry algebra from the reflection matrix described in section 2.3 is of wider applicability. For example we can apply it to the diagonal reflection matrices $K_\theta^\mu : V_\theta^\mu \to V_\theta^\mu$ found in [11]. This gives the symmetry algebra of the $su(n)$ spin chain on the half-line and is the subject of forthcoming work with Phil Isaac. We expect that there will also be new kinds of boundary conditions in affine Toda field theory which preserve this symmetry algebra, generalizing the boundary condition found in [7]. It will be interesting to find these, in particular as some of the corresponding soliton reflection matrices have already been calculated in the continuum limit of the $su(n)$ spin chain [12].

We have demonstrated that one can obtain solutions of the reflection equation by solving intertwining equations for representations of suitable coideal subalgebras of $U_q(\hat{g})$. This is of great practical importance because the linear intertwining equations are much easier to solve than the reflection equation itself.

The interesting mathematical problem therefore now presents itself of classifying all relevant coideal subalgebras of $U_q(\hat{g})$ and their representations. We expect this to lead to a classification of all trigonometric reflection matrices in analogy to the classification of trigonometric R-matrices in terms of representations of quantum affine algebras. The required properties of the coideal subalgebras are that they should be "small enough" so that the intertwiners exist, but also "large enough" so that the tensor product representations are generically irreducible.

In the rational case, generators for the relevant coideal subalgebras of the Yangians $Y(g)$ have been constructed in [9]. In that case one has to consider the involutive automorphisms σ of the Lie algebra g which lead to symmetric pairs (g, g^σ). The coideal subalgebra of $Y(g)$ is then a quantization of the corresponding twisted polynomial algebra. We shall denote these algebras $Y(g, g^\sigma)$ and refer to them as twisted Yangians. Twisted Yangians for $g = su(n)$ have already been described in [27] for $g^\sigma = so(n)$ and $g^\sigma = sp(n)$ and in [26] for $g^\sigma = su(m) \oplus su(n-m) \oplus u(1)$.

One might therefore hope in the trigonometric case to arrive at a theory of twisted quantized affine algebras $U_q(\hat{g}, \hat{g}^\sigma)$. In the classical case twisting by an inner automorphism leads to isomorphic algebras, leaving only the known twisted affine algebras based on Dynkin diagram automorphisms. In the quantum case however, where a particular Cartan subalgebra is singled out by the quantization, new algebras arise.

In the non-affine case the analogous construction of coideal subalgebras of $U_q(g)$ from involutions has been studied in [23]. The motivation in this case is that they lead to quantum symmetric pairs and thus to quantum symmetric spaces and their associated

q-orthogonal polynomials. These algebras include those constructed in [28] by reflection matrix techniques closer to our construction in section 2.3. We will be seeking affine generalizations of these works.

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