

# Fermion Casimir energy for a de Sitter brane in $\text{AdS}_5$

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## Abstract

Based on some recent work of the authors, we focus on the relationship between the Casimir energy of a Majorana spinor field for a Euclidean Einstein universe  $S^4 \times R$  and for a Euclidean de Sitter brane ( $S^4$ ) embedded in  $\text{AdS}_5$ . This is for a conformally coupled massless field. Interestingly, the one brane effective potential is zero and the results are equivalent, as for the scalar case, when evaluated on the conformally related cylinder. However, using the actual metric this equivalence no longer holds because a non-trivial contribution from the path integral measure (known as the cocycle function) is non-zero.

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## 1 Introduction

This short talk is based on a small part of some recent (and continuing) work [1, 2, 3, 4] relating to possible quantum effects in brane world cosmology (BWC) models. The ideas relate to the one brane Randall-Sundrum model [5] and its generalisation to include a curved brane [6, 7]. An interesting BWC scenario has been developed in [8, 9], known as the bulk inflaton model, where it is possible to obtain inflation on a single positive tension brane solely due to the effect of a bulk gravitational scalar field. In this regard, the vacuum energy of the bulk scalar field could have some affect on the cosmological evolution of the brane, (depending on the size and sign of this quantity) and is an interesting subject in its own right.

Employing  $\zeta$ -function methods we evaluate the one-loop effective potential for a de Sitter brane embedded in a bulk 5-dimensional anti-de Sitter spacetime. In fact, for this case, on the conformally related cylinder the effective potential reduces to the Casimir energy on  $S^4$ , see [1]. Thus, first we also evaluate the vacuum (Casimir) energy on the Euclidean Einstein universe  $S^4 \times R$ . We focus on massless conformally coupled Majorana spinor fields and compare with the scalar field results. This is based on results from [4] which continues from previous work [2], for scalar fields. For references relating to flat brane calculations see those cited in [2].

In what follows we mention that there are no zero modes to deal with. This is because for fermion fields the relevant boundary conditions are mixed, meaning half the field components satisfy Neumann and the other half satisfy Dirichlet boundary conditions. This cancels any zero modes (by zero modes, we mean  $n=0$  modes in our mode sum and not null eigenvectors).

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## 2 Casimir energy

Before evaluating the vacuum energy, we must find the eigenvalues to be employed in the  $\zeta$ -function. We start with the Euclidean metric suitable for de Sitter branes [10]

$$ds^2 = dr^2 + \ell^2 \sinh^2(r/\ell) d\Omega_4^2, \quad (1)$$

where  $\ell = (-6/\Lambda_5)^{1/2}$  is the anti-de Sitter radius and  $d\Omega_4^2$  is the metric on the unit 4-sphere. At the classical level, boundary or junction conditions relate the location of the single positive tension brane (located at  $r_0$ ) to the brane tension  $\sigma$  according to [10]

$$\sigma = \frac{3}{4\pi G_5 \ell} \coth(r_0/\ell). \quad (2)$$

The quantum corrections will introduce other sources of stress-energy on the brane which modify these relations. The metric can be written in conformal form

$$ds^2 = a^2(z)(dz^2 + d\Omega_4^2) \quad a(z) = \frac{\ell}{\sinh(z_0 + |z|)}, \quad (3)$$

where the coordinates are chosen to have the positive tension brane at  $z = 0$ . The above form of the metric is conformal to a cylinder  $I \times S^4$ . However, as in the bulk inflaton model, it may also be of interest to consider the compact space  $S^1 \times S^4$  [10]. If two branes are present then the other brane is placed at  $|z| = L$ , with the one brane limit given by  $L \rightarrow \infty$ .

For spin- $1/2$  fermion fields,  $\psi$ , the massless Dirac equation is automatically conformally covariant. We shall concentrate on Majorana spinors, see [4]. Taking the square of the Dirac operator to evaluate the effective action gives

$$W^M = -\frac{1}{4} \log \det \Delta \quad (4)$$

for Majorana fermions, where

$$\Delta = -\nabla^2 + \frac{1}{4} R^{(5)} \quad (5)$$

The squared operator is not conformally invariant, but it is nevertheless still possible to relate the effective actions of fermions with conformally related metrics [11].

On the cylinder and noting that for conformal coupling all delta function contributions cancel,

$$\Delta = \left( -\partial_z^2 - \Delta_f^{(4)} + 3 \right), \quad (6)$$

where on  $S^4$ ,  $R^{(4)} = 12$ . The eigenvalues of the spinor Laplacian,  $\Delta_f^{(4)}$ , on the 4-sphere are well known (e.g., see [12]) and are given by  $(m+2)^2 - 3$ . Half of the field components satisfy Dirichlet boundary conditions and half satisfy Neumann boundary conditions, commonly known as mixed boundary conditions. The eigenvalues of  $\Delta$  are

$$\lambda_{n,m}^M = \left( \frac{\pi n}{L} \right)^2 + (m+2)^2. \quad (7)$$

The degeneracy for the 8 component Majorana spinors is given by [12]

$$d^M(m) = 8 \times \frac{1}{6} (m+1)(m+2)(m+3) \quad (8)$$

where we take  $n \in \mathbb{Z}$ . Then, the  $\zeta$ -function method can be employed to find the contribution to the effective action from the cylinder. The generalised zeta function is given by

$$\zeta^M(s) = \sum_{m,n=0}^{\infty} d(m) \lambda_{m,n}^{-s}. \quad (9)$$

with the one-loop effective action related to  $\zeta(s)$  by (e.g., see [12, 15])

$$W^M = \frac{1}{4} \zeta^{M'}(0) + \frac{1}{4} \zeta(0) \log \mu^2 \quad (10)$$

for Majorana fermions, where  $\mu$  is the renormalisation scale. The effective potential is then obtained by dividing by the total volume.

As in [2] we first begin with the simpler case of the vacuum energy for the Einstein universe  $S^4 \times R$ . The evaluation of such quantities on  $S^3 \times R$  using an exponential cut off in the mode sum was performed by Ford [13, 14], for scalar and spinor fields respectively. This is equivalent to finding the zero point energy of the equation

$$\partial_t^2 \psi - \Delta_f^{(4)} \psi + \frac{1}{4} R^{(4)} \psi = 0. \quad (11)$$

The eigenvalues and degeneracy are given by Eq. (7,8), with  $n=0$ . From our definition of  $W^M$  (Eq. (10)) the vacuum energy for a Majorana field is defined as

$$E_0^M = -\frac{1}{4} \zeta^M(-1/2) = -\frac{1}{3} (\zeta(2s-3, 2) - \zeta(2s-1, 2))_{s \rightarrow -1/2} = 0, \quad (12)$$

where  $\zeta(a, b)$  is the generalised (or Hurwitz)  $\zeta$ -function. This result is expected because in an odd number of dimensions there is no conformal anomaly [15], i.e. as in the case of  $S^2 \times R$ .

Now, we come to evaluate the effective potential for a one brane configuration on the cylinder, where the discrete  $n$  modes become continuous. It is then possible to show that

$$\zeta^M(s) = \frac{2L}{\pi} \int_0^\infty dk \sum_{m=0}^\infty d^M(m) (k^2 + (m+2)^2)^{-s}, \quad (13)$$

where the factor of 2 is because we choose to undo the  $Z_2$  symmetry as in [10]. For large  $s$  we can interchange the order of the sum and the integral and perform the  $k$  integration,

$$\begin{aligned} \zeta^M(s) &= \frac{2L}{\pi} \frac{\sqrt{\pi}}{2} \sum_{m=0}^\infty \frac{\Gamma(s-1/2)}{\Gamma(s)} d^M(m) (m+2)^{1-2s}, \\ &= \frac{L}{\pi} \frac{4}{3} \sqrt{\pi} \frac{\Gamma(s-1/2)}{\Gamma(s)} (\zeta(2s-4, 2) - \zeta(2s-2, 2)), \end{aligned} \quad (14)$$

where in the second step we have used simple algebra to rewrite the equation in terms of generalised (Hurwitz)  $\zeta$ -functions. The analytic continuation to  $s=0$  is contained naturally in the definition of the  $\zeta$ -function, in this case. For  $s=0$  it is clear that  $\zeta^M(0)=0$  because  $\frac{1}{\Gamma(s)} = s + \gamma s^2 + O(s^3)$ , where  $\gamma$  is Euler's constant. Thus,

$$\zeta^{M'}(0) = \frac{L}{\pi} \frac{4}{3} \sqrt{\pi} \Gamma(-1/2) (\zeta(-4, 2) - \zeta(-2, 2)) = 0. \quad (15)$$

Therefore, as for scalar fields [1, 2], the one loop effective potential on the cylinder is zero in the one brane case. Just to see the relationship between the vacuum energy Eq. (12) and the effective potential, let us note that

$$W = -\frac{2}{3} L (\zeta(-4, 2) - \zeta(-2, 2)) = 0. \quad (16)$$

Thus, apart from the extra factor of 2 due to our mode doubling, the effective potential agrees with the vacuum energy density on  $S^4 \times R$ .

### 3 Summary

In this talk we have evaluated, on the conformally related cylinder, the effective potential for a de Sitter brane embedded in a bulk  $AdS_5$  spacetime. We also evaluated the vacuum energy on the Euclidean Einstein universe  $S^4 \times R$  and found the two results to agree. However, there is a slight subtlety concerning our calculation, because when boundaries are present conformal transformations induce a non-trivial Jacobian in the path integral measure. This is taken care of by evaluating an extra term, known as the cocycle function [4]. Working on the cylinder, defined in Eq. (3), the cocycle function is found to vanish, because the half-cylinder volume is infinite, as first argued in [1]. However, using the actual metric,

defined in Eq. (1), the cocycle function does not vanish because the volume of the Euclidean spacetime is finite, see [4].

As the metric in Eq. (1) is the metric of physicality we conclude that the cocycle function must be included. However, there is still a close relationship between the vacuum energy density on  $S^4 \times R$  and the effective potential for a single  $S^4$  brane in  $AdS_5$ .

Aside from the cocycle function, the fact that the vacuum energy on  $S^4 \times R$  with a mass or non-conformal curvature coupling will be non-zero implies that this will also be true of the de Sitter brane case. This considerably simplifies our analysis for the one brane case, which on the conformally related cylinder is equivalent to the evaluation of the Casimir energy on  $S^4 \times R$ . When applying the conformal transformation technique to the most general case, i.e. any given mass or curvature coupling, the potential is non-homogenous on the cylinder. In this case, we must find the vacuum energy for a non-constant field configuration including background distributional sources.

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