Comment on 'Must a Hamiltonian be hermitian'

André van Hameren
Institute of Nuclear Physics, NCSR Demokritos, 15310 Athens, Greece
andrevh@inp.demokritos.gr

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Abstract

A small comment on the paper with the mentioned title by Carl M. Bender, Dorje C. Brody and Hugh F. Jones.

As argued in [1], the eigenfunctions ϕ_n of the Sturm-Liouville eigenvalue problem (8) are fixed up to a constant phase factor. Instead of ϕ_n, one can also choose

$$\psi_n(x) = i^n\phi_n(x) \quad ,$$

to be the eigenfunctions under considerations, which satisfy

$$\mathcal{PT}\psi_n(x) = \psi_n^*(-x) = (-i)^n\phi_n^*(-x) = (-i)^n\phi_n(x) = (-)^n\psi_n(x) \quad ,$$

and

$$\delta(x-y) = \sum_n (-)^n\phi_n(x)\phi_n(y) = \sum_n \psi_n(x)\psi_n(y) \quad .$$

This formula, then, looks more familiar than (6). Only if the functions ψ_n are non-real, one usually has a different formula: $\sum_n \psi_n^*(x)\psi_n(y) = \delta(x-y)$. The operator $\mathcal{C}$ can be represented by

$$\mathcal{C}(x,y) = \sum_n (-)^n\psi_n(x)\psi_n(y) \quad .$$

It acts on the functions ψ_n as $\mathcal{C}\psi_n = (-)^n\psi_n$, so that they are invariant under $\mathcal{CPT}$:

$$\mathcal{CPT}\psi_n = \psi_n \quad ,$$

and are orthonormal under

$$\langle\psi_n|\psi_m\rangle = \int [\mathcal{CPT}\psi_n(x)]\,\psi_m(x)\,dx = \int \psi_n(x)\,\psi_m(x)\,dx \quad ,$$

which looks like a *real* inner product[1]. So we see that $\mathcal{CPT}$-invariance of a Hamiltonian, as introduced in [1], is equivalent with the existence of a set of eigenfunctions that are

[1] *i.e.* $\langle f|g\rangle = \langle g|f\rangle$ instead of $\langle f|g\rangle = \langle g|f\rangle^*$

complete and orthonormal under a *real* inner product. The (complex) inner product in the whole Hilbert space can be defined by

$$\langle f|g\rangle = \sum_n f_n^* g_n \quad \text{with} \quad f_n = \int \psi_n(x)\, f(x)\, dx \ \ , \ \ g_n = \int \psi_n(x)\, g(x)\, dx \ \ .$$

References

[1] Carl M. Bender, Dorje C. Brody and Hugh F. Jones, *Must a Hamiltonian be hermitian*, `hep-th/0303005`.