Four classes of modified relativistic symmetry transformations

Jerzy Lukierski
Theoretical Physics,
University of Wrocław, pl. Maxa Borna 9,
50-204 Wrocław, Poland

Anatol Nowicki*†
Institute of Physics, University of Zielona Góra,
ul. Podgórna 50, 65-246 Zielona Góra, Poland

Abstract

We discuss the nonlinear transformations of standard Poincaré symmetry in the context of recently introduced Doubly Special Relativity (DSR) theories. We introduce four classes of modified relativistic theories with three of them describing various DSR frameworks. We consider four examples of modified relativistic symmetries, which illustrate each of the considered class.

1 Introduction

The classical relativistic symmetries as described by classical Poincare-Hopf algebra can be deformed due to the following two reasons (see also [1,2]):

i) One can introduce new nonlinear basis in enveloping algebra of classical Poincaré Lie algebra,

ii) One can deform the classical coalgebraic structure, which leads to quantum Poincaré algebras and by considering dual Hopf algebra structure also provides the quantum Poincaré groups.

In this talk we shall consider the modification of relativistic symmetry transformations due to the nonlinear change of basis in the algebraic sector. Recently

*Talk given by A. Nowicki

†Supported by the Polish State Committee for Scientific Research (KBN) grant No 5P03B05620

Amelino-Camelia [3] introduced two categories of modified relativistic symmetries with two parameters c and κ
– light velocity c and fundamental mass κ which can be identified with Planck mass – invariant under the modified Lorentz transformations:

i) Doubly special relativistic theories of first type, denoted by DSR1, with energy E unbounded and momentum $\vec{P}$ bounded by mass-like parameter κ: $|\vec{P}| \leq \kappa\, c$

ii) DSR2 theories, with both energy and moment bounded by κ: $|\vec{P}| \leq \kappa\, c$ and $E \leq \kappa\, c^2$. The first examples of DSR2 theories were provided by Magueijo and Smolin [4,5].

In order to complete the classification we shall introduce further two categories:

iii) DSR3 theories, with momentum unbounded and energy bounded: $E \leq \kappa c^2$.

iv) The theories with momentum as well as energy unbounded. In such a framework two parameters c and κ do not have the meaning of invariant parameters. We shall call these theories Smoothly Modified Special Relativity (SMSR) theories.

2 Nonlinear realizations of relativistic symmetries

We use the following notation in the description of classical D=4 Poincaré algebra:
– Lorentz algebra ($g_{\mu\nu} = (-1, 1, 1, 1)$)

$$[M_{\mu\nu}, M_{\rho\tau}] = i(g_{\mu\rho}M_{\nu\tau} + g_{\nu\tau}M_{\mu\rho} - g_{\mu\tau}M_{\nu\rho} - g_{\nu\rho}M_{\mu\tau}). \tag{1}$$

– covariance relations ($M_i = \frac{1}{2}\epsilon_{ijk}M_{jk}, N_i = M_{i0}$; c is the light velocity)

$$[M_i, \mathcal{P}_j] = i\epsilon_{ijk}\mathcal{P}_k, \quad [M_i, \mathcal{E}] = 0, \quad [N_i, \mathcal{P}_j] \ = \tfrac{i}{c}\delta_{ij}\mathcal{E}, \quad [N_i, \mathcal{E}] = ic\mathcal{P}_i. \tag{2}$$

– commuting four momenta $[P_\mu, P_\nu] = 0$
The momentum variables fulfill standard mass shell condition, described by mass Casimir

$$\mathcal{E}^2 - c^2\mathcal{P}^2 \ = \ \mu^2 c^4. \tag{3}$$

Mass shall (3) is covariant under the relativistic boost transformations

$$\mathcal{E}(\alpha) = \mathcal{E}\cosh(\alpha) - c(\vec{n}\cdot\vec{\mathcal{P}})\sinh(\alpha). \tag{4}$$

$$\vec{\mathcal{P}}(\alpha) \ = \ \vec{\mathcal{P}} + \left((\cosh\alpha - 1)\vec{n}\cdot\vec{\mathcal{P}} - \frac{\mathcal{E}}{c}\sinh\alpha\right)\vec{n}. \tag{5}$$

with the rapidity–velocity relation $\vec{v} = c\vec{n}\tanh\alpha\,, \vec{\alpha} = \alpha\vec{n}$.
We shall consider invertible nonlinear transformations of the momentum space in the form

$$\vec{\mathcal{P}} = \vec{\mathcal{P}}(E,\vec{P}) = \vec{P}\; g\left(\frac{E}{\kappa c^2}, \frac{\vec{P}^{\,2}}{\kappa^2 c^2}\right),\ \mathcal{E} \ = \mathcal{E}(E,\vec{P}) = \ \kappa c^2 f\left(\frac{E}{\kappa c^2}, \frac{\vec{P}^{\,2}}{\kappa^2 c^2}\right). \tag{6}$$

with the dependence on dimensionfull mass-like parameter κ satisfying the conditions

$$\lim_{\kappa\to\infty} g = 1, \qquad \lim_{\kappa\to\infty}(\kappa c^2 f) = E. \tag{7}$$

This form of transformations imply only changes in covariance relations (2) i.e. $[N, P]$ depend on functions f, g but the classical Lorentz algebra (1) is not changed. In transformed variables $E, \vec{P}$ the dispersion relation (3) is given by

$$\kappa^2 c^4 f^2 \left(\frac{E}{\kappa c^2}, \frac{\vec{P}^2}{\kappa^2 c^2}\right) - c^2 \vec{P}^2 g^2 \left(\frac{E}{\kappa c^2}, \frac{\vec{P}^2}{\kappa^2 c^2}\right) = inv. = \mu^2 c^4. \tag{8}$$

and it is invariant under nonlinear transformations of boosts.
For the special choices of functions f and g we obtain three types of DSR and fourth class of SMSR theories, mentioned in the Introduction.

3 3. Doubly special relativity theories

3.1 DSR1 as nonlinear realization of Poincaré algebra - an example

We define nonlinear transformations of the momentum subalgebra as follows

$$\vec{\mathcal{P}} = \vec{P} e^{\frac{E}{\kappa c^2}} \quad , \quad \mathcal{E} = \kappa c^2 \left(\sinh \frac{E}{\kappa c^2} + \frac{\vec{P}^2}{2\kappa^2 c^2} e^{\frac{E}{\kappa c^2}}\right). \tag{9}$$

In this new basis $(E, \vec{P})$ the covariance relations (2) takes the form

$$\begin{aligned}
[M_i, P_j] &= i\epsilon_{ijk} P_k, \qquad [M_i, E] = 0, \\
[N_i, P_j] &= i\kappa c \delta_{ij} \left[\sinh\left(\frac{E}{\kappa c^2}\right) e^{-\frac{P_0}{\kappa c}} + \frac{1}{2\kappa^2 c^2}(\vec{P})^2\right] - \frac{i}{\kappa c} P_i P_j, \\
[N_i, E] &= icP_i.
\end{aligned} \tag{10}$$

DSR1 energy-momentum dispersion relation (κ-deformed mass Casimir) is given by

$$C_2 = \left(2\kappa \sinh \frac{E}{2\kappa c^2}\right)^2 - \frac{1}{c^2} \vec{P}^2 e^{\frac{E}{\kappa c^2}} = M^2. \tag{11}$$

Using the formulae (4-5) we get nonlinearly modified boosts transformations[1]

$$E(\alpha) = E + \kappa c^2 \ln W(\alpha, \vec{n}\vec{P}, E). \tag{12}$$

$$\vec{P}(\alpha) = W^{-1}(...) \left[\vec{P} + \left((\vec{n}\vec{P})(\cosh\alpha - 1) - \kappa c B(m, E) \sinh\alpha\right) \vec{n}\right]. \tag{13}$$

where

$$\begin{aligned}
W(\alpha, \vec{n}\vec{P}, E) &= 1 - \left(\tfrac{1}{\kappa c}(\vec{n} \cdot \vec{P}) \sinh\alpha + B(m, E)(1 - \cosh\alpha)\right), \\
B(m, E) &= 1 - \cosh\left(\tfrac{m}{\kappa}\right) e^{-\frac{E}{\kappa c^2}} = \tfrac{1}{2}\left(1 - e^{-\frac{2E}{\kappa c^2}} + \tfrac{\vec{P}^2}{\kappa^2 c^2}\right).
\end{aligned} \tag{14}$$

[1]The transformation (13) has been firstly described for special choice of boost parameter $\vec{\alpha} = (0, 0, \alpha)$ in [6]; the general formula (13) was obtained firstly in [2] and further discussed in [7].

3.2 DSR2 as nonlinear realization of Poincaré algebra - an example

We assume that the classical Lorentz algebra is given by the formulae (1). We define nonlinear transformations of the momentum subalgebra as follows

$$\vec{\mathcal{P}} = \vec{P}\left(1 - \frac{E}{\kappa c^2}\right)^{-1}, \quad \mathcal{E} = E\left(1 - \frac{E}{\kappa c^2}\right)^{-1}. \tag{15}$$

In this new basis $(E, \vec{P})$ the covariance relations (2) take the form

$$\begin{aligned} [M_i, P_j] \quad &= i\epsilon_{ijk}P_k, \qquad [N_i, P_j] = \frac{i}{c}\left(\delta_{ij}E - \frac{P_iP_j}{\kappa}\right), \\ [M_i, E] \qquad &= 0, \qquad [N_i, E] = 2ic\left(1 - \frac{E}{\kappa c^2}\right)P_i. \end{aligned} \tag{16}$$

DSR2 energy-momentum dispersion relation is given by

$$C_2 = \frac{E^2 - c^2\vec{P}^{\,2}}{\left(1 - \frac{E}{\kappa c^2}\right)^2} \; = \; M^2c^4. \tag{17}$$

Using the formulae (5-6) we get nonlinearly modified boost transformations

$$E(\alpha) = \left(E\cosh\alpha - c(\vec{n}\vec{P})\sinh\alpha\right)\mathcal{W}^{-1}(\alpha, \vec{n}\vec{P}, E). \tag{18}$$

$$\vec{P}(\alpha) = \left(\vec{P} + \vec{n}\left((\cosh\alpha - 1)\vec{n}\cdot\vec{P} - \frac{E}{c}\sinh\alpha\right)\right)\mathcal{W}^{-1}(\alpha, \vec{n}\vec{P}, E). \tag{19}$$

where

$$\mathcal{W}(\alpha, \vec{n}\vec{P}, E) = 1 + \frac{E}{\kappa c^2}(\cosh\alpha - 1) - \frac{(\vec{n}\vec{P})}{\kappa c}\sinh\alpha. \tag{20}$$

3.3 DSR3 as nonlinear realization of Poincaré algebra - an example

We assume that the classical Lorentz algebra is given by the formulae (1). We define nonlinear transformations of the fourmomentum subalgebra as follows

$$\vec{\mathcal{P}} = \vec{P}, \quad \mathcal{E} = E\left(1 + \frac{\vec{P}^2}{\kappa^2c^2}\right)^{1/2}. \tag{21}$$

In this new basis $(E, \vec{P})$ the covariance relations (2) takes the form

$$[M_i, P_j] \qquad = i\epsilon_{ijk}P_k, \qquad [N_i, P_j] = \frac{i}{c}\delta_{ij}E\left(1 + \frac{\vec{P}^2}{\kappa^2c^2}\right)^{1/2},$$

$$[M_i, E] \;= [P_\mu, P_\nu] = 0, \qquad [N_i, E] = icP_i \left(1 + \frac{\vec{P}^2}{\kappa^2 c^2}\right)^{-1/2} \left(1 - \frac{E^2}{\kappa^2 c^4}\right) \tag{22}$$

The energy-momentum dispersion relation is given by

$$C_2 \;=\; E^2 \left(1 + \frac{\vec{P}^2}{\kappa^2 c^2}\right) - c^2 \vec{P}^2 = M^2 c^4. \tag{23}$$

Then using the formulae (4-5) we get the nonlinear boost transformations

$$E(\alpha) = \left(\frac{1 + \vec{P}^2(\alpha)}{1 + \vec{P}^2}\right)^{-\frac{1}{2}} \left[E \cosh\alpha - c(\vec{n}\vec{P}) \left(1 + \frac{\vec{P}^2}{\kappa^2 c^2}\right)^{-1/2} \sinh\alpha\right]. \tag{24}$$

$$\vec{P}(\alpha) \;=\; \vec{P} + \left((\cosh\alpha - 1)\vec{n} \cdot \vec{P} - \frac{E}{c} \left(1 + \frac{\vec{P}^2}{\kappa^2 c^2}\right)^{1/2} \sinh\alpha\right) \vec{n}. \tag{25}$$

4 SMSR as nonlinear realization of Poincaré algebra - an example

We assume that the classical Lorentz algebra is given by the formulae (1). We define nonlinear transformations of the momentum subalgebra as follows

$$\vec{\mathcal{P}} = \vec{P}, \quad \mathcal{E} = 2\kappa c^2 \sinh\left(\frac{E}{2\kappa c^2}\right). \tag{26}$$

In this new basis $(E, \vec{P})$ the covariance relations (2) take the form

$$\begin{aligned} [M_i, P_j] \quad &= i\epsilon_{ijk} P_k, \qquad && [N_i, E] = icP_i \cosh^{-1}\left(\frac{E}{2\kappa c^2}\right), \\ [M_i, E] \;&= [P_\mu, P_\nu] = 0, \qquad && [N_i, P_j] = 2i\kappa c \delta_{ij} \sinh\left(\frac{E}{2\kappa c^2}\right). \end{aligned} \tag{27}$$

dispersion relation is given by

$$C_2 \;=\; \left(2\kappa c^2 \sinh\frac{E}{2\kappa c^2}\right)^2 - c^2 \vec{P}^2 = M^2 c^4. \tag{28}$$

Using the formulae (4-5) we get nonlinear boost transformations

$$E(\alpha) = 2\kappa c^2 \text{arcsinh}\left[\sinh\left(\frac{E}{2\kappa c^2}\right)\cosh\alpha - \frac{(\vec{n}\vec{P})}{2\kappa c}\sinh\alpha\right]. \tag{29}$$

$$\vec{P}(\alpha) = \vec{P} + \left((\cosh\alpha - 1)\vec{n} \cdot \vec{P} - 2\kappa c \sinh\left(\frac{E}{2\kappa c^2}\right)\sinh\alpha\right)\vec{n}. \tag{30}$$

One can see easily that the values of $E(\alpha)$ and $P(\alpha)$ are not bounded.

5 Final Remarks

In this talk we mainly classified different nonlinear bases for classical Poincaré algebra. If we observe that the classical fourmomenta generators are endoved with primitive coproducts ($\mathcal{P}_\mu = (\mathcal{E}/c, \mathcal{P}_i)$)

$$\Delta^{(0)}\mathcal{P}_\mu = \mathcal{P}_\mu \otimes 1 + 1 \otimes \mathcal{P}_\mu \,. \tag{31}$$

then by considering the formulas inverse to (6)

$$E = \kappa\, c^2 F\left(\frac{\mathcal{E}}{\kappa c^2}, \frac{\vec{\mathcal{P}}^2}{\kappa^2 c^2}\right)\,, \qquad \vec{P} = \vec{\mathcal{P}}\, G\left(\frac{\mathcal{E}}{\kappa c^2}, \frac{\vec{\mathcal{P}}^2}{\kappa^2 c^2}\right)\,. \tag{32}$$

one obtains the symmetric nonlinear coproducts for the nonlinear momentum $\vec{P}$ and nonlinear energy E. Such coproducts were considered in [2] as describing the deformed addition law for fourmomenta and further used in [8] in the notation without the notion of coproduct. We get the coproduct formulae:

$$\Delta E = \kappa\, c^2 F\left(\frac{\Delta\mathcal{E}}{\kappa c^2}, \frac{(\Delta\vec{\mathcal{P}})^2}{\kappa^2 c^2}\right)\,, \quad \Delta\vec{P} = \Delta(\vec{\mathcal{P}}) G\left(\frac{\Delta\mathcal{E}}{\kappa c^2}, \frac{(\Delta\vec{\mathcal{P}})^2}{\kappa^2 c^2},\right)\,. \tag{33}$$

where

$$\begin{aligned} \Delta\mathcal{E} &= \mathcal{E}(E, \vec{P}) \otimes 1 + 1 \otimes \mathcal{E}(E, \vec{P}), \\ \Delta\vec{\mathcal{P}} &= \vec{\mathcal{P}}(E, \vec{P}) \otimes 1 + 1 \otimes \vec{\mathcal{P}}(E, \vec{P}). \end{aligned} \tag{34}$$

The quantum nonsymmetric coproduct is obtained if we modify primitive coproduct (31) by introducing Drinfeld twist T [1]

$$\Delta\mathcal{P}_\mu = T^{-1} \circ \Delta^{(0)}(\mathcal{P}_\mu) \circ T = \Delta^{(0)}(\mathcal{P}_\mu) + [r, \mathcal{P}_\mu] + \ldots \,. \tag{35}$$

where r is the classical Poincaré r-matrix [9]. In such a way we obtain the quantum Poincaré algebras with nonlinear fourmomentum basis given by (32) and non-primitive coproduct (33) with inserted formulae (35).

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