Living on the edge: cosmology on the boundary of anti-de Sitter space

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Abstract

We sketch a particularly simple and compelling version of D-brane cosmology. Inspired by the semi-phenomenological Randall–Sundrum models, and their cosmological generalizations, we develop a variant that contains a single (3+1)-dimensional D-brane which is located on the boundary of a single bulk (4+1)-dimensional region. The D-brane boundary is itself to be interpreted as our visible universe, with ordinary matter (planets, stars, galaxies) being trapped on this D-brane by string theory effects. The (4+1)-dimensional bulk is, in its simplest implementation, adS_{4+1}, anti-de Sitter space. We demonstrate that a $k = +1$ closed FLRW universe is the most natural option, though the scale factor could quite easily be so large as to make it operationally indistinguishable from a $k = 0$ spatially flat universe. (With minor loss of elegance, spatially flat and hyperbolic FLRW cosmologies can also be accommodated.) We demonstrate how this model can be made consistent with standard cosmology, and suggest some possible observational tests.

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1 Introduction

In this article we develop what we feel is a particularly simple and compelling cosmological model based on the semi-phenomenological Randall–Sundrum models for low-energy string theory [1, 2]. For some early tentative steps along these lines see the papers of Gogberashvili [3], plus more recent developments in [4] and [5].[1] In developing our cosmology, we wish to minimize the number of baroque features coming from the underlying string theory, and maximize the use of symmetry principles, in order to develop a picture that is as simple and attractive as possible, with good prospects for being observationally testable.

Perhaps the most compelling model along these lines can be built by considering a (4+1)-dimensional manifold with a single (3+1)-dimensional boundary. This boundary is taken to be a D-brane (a membrane on which the fundamental string fields satisfy Dirichlet type boundary conditions), and the D-brane is assigned an intrinsic energy density and pressure arising both from some underlying brane tension *and* from matter [ordinary (3+1)-dimensional matter] that is trapped on the D-brane by stringy effects.

Since this point has the capacity to cause serious confusion, let us try to make it a little more explicit:[2] We are viewing ordinary matter as open-string excitations of the D-brane boundary. But since open strings by definition have their end-points on the D-brane, an open string of energy E is strictly limited in how far it can stretch off the D-brane: Its maximum extension into any higher-dimensional bulk is simply $L_{\text{stringy}} < E/(2\alpha')$ where α' is the fundamental open string tension.[3] In contrast, gravitons are represented by closed string loops which are not trapped on the D-brane — gravitons (and non-perturbative gravity) can very easily penetrate finite distances into the higher-dimension bulk. Thus gravity is in our model fundamentally a (4+1)-dimensional effect and we will be using the (4+1)-dimensional Einstein equations to deduce the analog of the Friedmann equations of motion for the (3+1)-dimensional D-brane boundary.

Now while gravitons can easily penetrate into the bulk, one does not want them to be too effective at doing so. Once one turns away from the large-scale average properties of the cosmological FLRW geometry, to consider the gravitational field generated by astrophysical perturbations (planets, stars, galaxies) one does not want the virtual-graviton cloud surrounding these objects to be completely free to move into the (4+1)-dimensional bulk, since then one would see an inverse-cube law for gravity in lieu of the observed inverse-square law. This is where the Randall–Sundrum mechanism is critical — virtual gravitons generated by matter perturbations are (weakly) trapped near the D-brane, not by stringy effects, but rather by the bulk gravitational field and the tightly constrained location of the sources.[4] We belabor this point because we have seen it generate consid-

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[1]Note that many aspects of this recent work can be viewed as extending domain-wall physics in (3+1) dimensions to brane physics in (4+1) dimensions, and so owes much to early papers on domain-wall physics [6].

[2]We are trying to make this article comprehensible to string theorists, relativists, and astrophysicists. Accordingly *some* comments may be trivial to one of the three communities, but we would rather err on the side of clarity and simplicity than either impenetrable brevity or excessive technical detail.

[3]This whole D-brane picture only makes sense for string excitations of low energy compared to the string scale: $E < \sqrt{\hbar c\, \alpha'}$. So the thickness of the cloud of excitations surrounding the D-brane is at most of order $L_{\text{stringy}} < \sqrt{\hbar c/(2\alpha')}$.

[4]The distance scale on which gravitons are trapped is generically set by the Riemann curvature

erable confusion within the relativity and astrophysics communities: Gravity is *not* used to trap matter on the D-brane and the Randall–Sundrum models have more in common with the field-theory-based trapping mechanisms of Akama [7] and Shaposhnikov [8] than they do with the gravity-based trapping mechanism of [9].

In the interests of simplicity and clarity the (4+1)-dimensional bulk will always be taken to be static and hyper-spherically symmetric, though we shall quickly specialize to Reissner–Nordström–de Sitter space, or even more particularly, to anti-de Sitter space. The boundary will always be taken to be hyper-spherically symmetric in the (4+1)-dimensional sense, with this hyper-spherical symmetry reducing to translation invariance when viewed from the (3+1)-dimensional point of view. In picking this particular starting point we have been guided by many recent publications; including the Randall–Sundrum scenarios [1, 2] (which will used to describe the physics near the brane), the single-brane models of Gogberashvili [3, 4], various previous versions of Randall–Sundrum based cosmology [5], and by a desire to have a framework that is at least plausibly connectable to the complex of ideas going under the name of the adS/CFT correspondence [10, 11].

We start the analysis by a discussion of what it means to apply the Einstein equations to a *manifold with boundary*, interpreting this process in terms of an extension of the Israel–Lanczos–Sen thin shell formalism [12, 13, 14]. This permits us to write down an analog of the usual Friedmann equation of FLRW cosmology, and in the next section we discuss how to make this cosmologically viable. Going beyond the FLRW cosmological fluid approximation we verify that the essential portion of the Randall–Sundrum model (having to do with the weak trapping of perturbatively generated gravitons near the brane) continues to work in the present context. Finally we indicate some possible variants on the present model and describe areas where the present ideas may lead to observational tests.

2 D-brane surgery

2.1 Extrinsic and intrinsic geometries:

We start by considering a rather general static hyper-spherically symmetric geometry in (4+1) dimensions. (This is not the most general such metric, but quite sufficient for our purposes.)[5]

$$ds^2_{4+1} = -F(r)\,dt^2 + \frac{dr^2}{F(r)} + r^2\,d\Omega_3^2. \tag{1}$$

$$d\Omega_3^2 \equiv d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2). \tag{2}$$

To build the class of (3+1)-dimensional geometries we are interested in, we start by simply truncating the (4+1)-dimensional geometry at some time-dependent radius $a(t)$, keeping only the interior portion and discarding the exterior. Kinematically, the surface of this truncated geometry (which we take to be the location of the D-brane) is automatically

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of the higher-dimensional bulk; in the Randall–Sundrum models the relevant parameter is $L_{\text{graviton}} = \sqrt{6/|\Lambda_{4+1}|}$, defined by the cosmological constant in the higher-dimensional bulk.

[5]Note that the technical computations closely parallel those for spherically symmetric (2+1)-dimensional domain walls symmetrically embedded in a spherically symmetric (3+1)-dimensional spacetime. See, for instance, references [15, 16, 17].

a (3+1)-dimensional closed ($k = +1$, positive spatial curvature) FLRW geometry with induced metric

$$ds_{3+1}^2 = -\left[F(a(t)) - \frac{1}{F(a(t))}\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)^2\right] \mathrm{d}t^2 + a(t)^2 \, \mathrm{d}\Omega_3^2. \tag{3}$$

Now consider radial motion of the D-brane; this is radial motion in the embedding (4+1)-dimensional hyperspace. We start the analysis by first parameterizing the motion in terms of proper time along a curve of fixed χ, θ, and ϕ (these are comoving coordinates in the FLRW cosmology). That is: the D-brane sweeps out a world-volume

$$X^\mu(\tau, \chi, \theta, \phi) = (t(\tau), a(\tau), \chi, \theta, \phi). \tag{4}$$

The 5-velocity of the (χ, θ, ϕ) element of the D-brane can then be defined as

$$V^\mu = \left(\frac{\mathrm{d}t}{\mathrm{d}\tau}, \frac{\mathrm{d}a}{\mathrm{d}\tau}, 0, 0, 0\right). \tag{5}$$

Using the normalization condition and the assumed form of the metric, and defining $\dot{a} = \mathrm{d}a/\mathrm{d}\tau$,

$$V^\mu = \left(\frac{\sqrt{F(a) + \dot{a}^2}}{F(a)}, \dot{a}, 0, 0, 0\right); \qquad V_\mu = \left(-\sqrt{F(a) + \dot{a}^2}, \frac{\dot{a}}{F(a)}, 0, 0, 0\right). \tag{6}$$

The unit normal vector to the hypersphere $a(\tau)$ is

$$n^\mu = \left(-\frac{\dot{a}}{F(a)}, -\sqrt{F(a) + \dot{a}^2}, 0, 0, 0\right); \qquad n_\mu = \left(+\dot{a}, -\frac{\sqrt{F(a) + \dot{a}^2}}{F(a)}, 0, 0, 0\right). \tag{7}$$

[We shall take the unit normal to be inward pointing, into the bulk of the (5+1) geometry.] The extrinsic curvature can be written in terms of the normal derivative[6]

$$K_{\mu\nu} = \frac{1}{2}\frac{\partial g_{\mu\nu}}{\partial \eta} = \frac{1}{2} \, n^\sigma \, \frac{\partial g_{\mu\nu}}{\partial x^\sigma}. \tag{8}$$

If we go to an orthonormal basis, the $\hat{\chi}\hat{\chi}$ component is easily evaluated

$$K_{\hat{\chi}\hat{\chi}} = K_{\hat{\theta}\hat{\theta}} = K_{\hat{\phi}\hat{\phi}} = -\frac{1}{2}\sqrt{F(a) + \dot{a}^2} \, \frac{\partial g_{\chi\chi}}{\partial r} \, g^{\chi\chi} = -\frac{\sqrt{F(a) + \dot{a}^2}}{a} \tag{9}$$

The $\tau\tau$ component is a little messier, but generalizing the calculation of [16] (which amounts to calculating the five-acceleration of the brane, this is explained in more detail in [15]) quickly leads to

$$K_{\hat{\tau}\hat{\tau}} = +\frac{1}{2}\frac{1}{\sqrt{F(a) + \dot{a}^2}}\left(\frac{\mathrm{d}F(r)}{\mathrm{d}a} + 2\ddot{a}\right) = +\frac{\mathrm{d}}{\mathrm{d}a}\left(\sqrt{F(a) + \dot{a}^2}\right). \tag{10}$$

In contrast to the extrinsic geometry, the intrinsic geometry of the D-brane is in these coordinates simply

$$ds_{3+1}^2 = -\mathrm{d}\tau^2 + a(\tau)^2 \, \mathrm{d}\Omega_3^2. \tag{11}$$

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[6]Unfortunately sign conventions differ on this point. We follow [15].

2.2 The D-brane as boundary:

A perhaps unusual (and for us very useful) feature of D-brane physics is that the D-brane can be viewed as an actual physical boundary to spacetime, with the "other side" of the D-brane being empty (null and void).[7] In general relativity, as it is normally formulated, the notion of an actual physical boundary to spacetime (that is, an accessible boundary reachable at finite distance) is complete anathema. The reason that spacetime boundaries are so thoroughly deprecated in general relativity is that they are artificial special places in the manifold where some sort of boundary condition has to be placed on the physics. Without such a postulated boundary condition all predictability is lost, and the theory is not physically acceptable. Since without some deeper underlying theory there is no physically justifiable reason for picking any one particular type of boundary condition (Dirichlet, Neumann, Robin, or something more complicated), the attitude in standard general relativity has been to simply exclude boundaries.

The key difference when a D-brane is used as a boundary is that now there is a specific and well-defined boundary condition for the physics: D-branes (remember that "*D is for Dirichlet*") are defined as the loci on which the fundamental open strings end (and satisfy Dirichlet-type boundary conditions). D-branes are therefore capable of providing both a physical boundary for the spacetime *and* a plausible boundary condition for the physics residing in the spacetime.[8]

When it comes to specific calculations, this is however not be the best mental picture to have in mind — after all, how would you try to calculate the Riemann tensor for the edge of spacetime? And what would happen to the Einstein equations at the edge? There is a specific technical trick that clarifies the situation: Take the manifold with D-brane boundary and make a second copy (including a second D-brane boundary), then sew the two manifolds together along their respective D-brane boundaries, creating a single manifold without boundary that contains the doubled D-brane, and exhibits a Z_2 symmetry on reflection around the D-brane. Because this new manifold is a perfectly reasonable no-boundary manifold containing a (thin shell) D-brane, the gravitational field can be analyzed using a slight generalization of the usual Israel–Lanczos–Sen thin-shell formalism of general relativity [12, 13, 14]. We now need to consider (3+1) shells propagating in (4+1) space, but this merely changes a few integer coefficients. The metric is continuous, the connection exhibits a step-function discontinuity, and the Riemann curvature a delta-function at the D-brane. The dynamics of the D-brane can then be investigated in this Z_2-doubled manifold, and once the dynamical equations and their solutions have been investigated the second surplus copy of spacetime can quietly be forgotten (effectively halving the strength of the delta-function contribution to the Riemann tensor). That is, as long as one is working in the Z_2-doubled manifold the discontinuity in the extrinsic

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[7]A brief sketch of these ideas, from the (3+1)-dimensional point of view where one is dealing with holes in spacetime (voids), was presented in [17]. Here we expand on these ideas in a more explicit manner.

[8]A word of warning: D-branes by definition provide boundary conditions directly on the fundamental string states, and so, since all physics in string theory can be viewed in terms of some combination of string states, D-branes will *in principle* provide boundary conditions for all the physics. In practice the route from string state to low-energy effective field theory may be rather indirect, and elucidation of the proper boundary condition may be a little obscure; when in doubt use symmetry as much as possible, and be prepared to keep at least a few adjustable constants as part of the low-energy semi-phenomenological theory.

curvature is twice the extrinsic curvature as seen from either side

$$\kappa_{\alpha\beta}(Z_2) = [K_{\alpha\beta}] = K^+_{\alpha\beta} - K^-_{\alpha\beta} = 2K_{\alpha\beta}. \tag{12}$$

Consequently the Riemann tensor in the Z_2-doubled manifold is

$$
\begin{aligned}
R(Z_2)_{\alpha\beta\gamma\delta} &= -2\delta(\eta) \left[K_{\alpha\gamma}\, n_\beta\, n_\delta + K_{\beta\delta}\, n_\alpha\, n_\gamma - K_{\alpha\delta}\, n_\beta\, n_\gamma - K_{\beta\gamma}\, n_\alpha\, n_\delta \right] \\
&\quad + \Theta(\eta)\, R^+_{\alpha\beta\gamma\delta} + \Theta(-\eta)\, R^-_{\alpha\beta\gamma\delta}.
\end{aligned}
\tag{13}
$$

We now *define* the Riemann tensor of the manifold with boundary by throwing away half of the Z_2-doubled manifold, and in view of the manifest symmetry of the situation, also throwing away half the delta-function contribution.[9]

After doing all this, near the D-brane boundary the Riemann tensor takes the form

$$R_{\alpha\beta\gamma\delta} = -\delta(\eta) \left[K_{\alpha\gamma}\, n_\beta\, n_\delta + K_{\beta\delta}\, n_\alpha\, n_\gamma - K_{\alpha\delta}\, n_\beta\, n_\gamma - K_{\beta\gamma}\, n_\alpha\, n_\delta \right] + R^{\text{bulk}}_{\alpha\beta\gamma\delta}. \tag{14}$$

This is the relevant generalization of equation (14.23) of [15] to a manifold with boundary; note that there is only one side to the boundary and that we explicitly use only the extrinsic curvature of that one side (which is half the extrinsic curvature discontinuity in the Z_2-doubled manifold). This particular formula is valid for any ([n–1]+1)-dimensional boundary to a (n+1)-dimensional bulk. It does assume that the normal n is spacelike, though no symmetry assumptions are made. If we introduce the general projection tensor

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu\, n_\nu, \tag{15}$$

then this projection tensor *is* the induced metric on the boundary and in the particular application we have in mind will be the physical spacetime metric of our universe. Performing the relevant contractions, and still working in an arbitrary number of bulk dimensions

$$R_{\mu\nu} = -\delta(\eta) \left[K_{\mu\nu} + K\, n_\mu\, n_\nu \right] + R^{\text{bulk}}_{\mu\nu}; \tag{16}$$

$$R = -2K\,\delta(\eta) + R^{\text{bulk}}; \tag{17}$$

$$G_{\mu\nu} = -\delta(\eta) \left[K_{\mu\nu} - K\, h_{\mu\nu} \right] + G^{\text{bulk}}_{\mu\nu}. \tag{18}$$

These formulae generalize (14.25)–(14.27) of [15] to a manifold with boundary. With hindsight this makes perfectly good sense since if we now integrate over the complete manifold (bulk plus boundary)

$$
\begin{aligned}
\oint d^{n+1}x\, \sqrt{-g_{n+1}}\, R &= \int_{\text{bulk}} d^{n+1}x\, \sqrt{-g_{n+1}}\, R_{\text{bulk}} \\
&\quad - 2\int_{\text{boundary}} d^{[n-1]+1}x\, \sqrt{-g_{[n-1]+1}}\, K.
\end{aligned}
\tag{19}
$$

Which means that we have automatically recovered the Gibbons–Hawking surface term for the gravitational action, in addition to the Einstein–Hilbert bulk term.

We also take the total stress-energy tensor to be given by a combination of surface and bulk components

$$T_{\mu\nu} = \delta(\eta)\, T^{\text{surface}}_{\mu\nu} + T^{\text{bulk}}_{\mu\nu}, \tag{20}$$

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[9]If for whatever reason one does not wish to work with the Z_2-doubled manifold, there is an alternative construction that leads to the same result that we present in Appendix A.

and normalize our (n+1)-dimensional bulk Newton constant G_{n+1} by

$$G_{\mu\nu} = 8\pi G_{n+1} \, T_{\mu\nu}. \tag{21}$$

Then in particular, picking off the surface contribution to both the Einstein tensor and the stress-energy

$$8\pi G_{n+1} \, T_{\mu\nu}^{\text{surface}} = - \left[K_{\mu\nu} - K \, h_{\mu\nu} \right] \tag{22}$$

Whether or not this surface stress tensor satisfies the energy conditions depends on the signs of the eigenvalues of the extrinsic curvature. By looking at the Z_2-doubled geometry it is a general result [15] that a convex boundary (when viewed from the bulk) violates the null energy condition (NEC), while a concave boundary satisfies it. (This is intimately related to the fact that traversable wormholes violate the null energy condition, see [15, 16, 18, 19, 20, 21].)

2.3   Cosmology

Now particularize to the (4+1)-dimensional version of the thin-shell formalism, and use the FLRW symmetries of the D-brane (some of the integer coefficients and exponents appearing below are dimension dependent):

$$8\pi \, G_{4+1} \, \rho_{3+1} = 3 \, \frac{\sqrt{F(a) + \dot{a}^2}}{a}. \tag{23}$$

(Note that the energy density is positive definite, in agreement with the fact that this boundary is concave when viewed from the bulk.)[10]

$$8\pi \, G_{4+1} \, p_{3+1} = -\frac{1}{a^2} \, \frac{\mathrm{d}}{\mathrm{d}a} \left(a^2 \sqrt{F(a) + \dot{a}^2} \right). \tag{24}$$

These equations can easily be seen to be compatible with the conservation of the stress energy localized on the D-brane[11]

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\rho_{3+1} \, a^3 \right) + p_{3+1} \, \frac{\mathrm{d}}{\mathrm{d}\tau} (a^3) = 0. \tag{25}$$

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[10]Because of this feature the D-brane occurring here is guaranteed to have positive tension, and we do not need to worry (at least not at the cosmological level) about the possibility of energy-condition-violating negative tension D-branes, and the somewhat peculiar features [traversable wormholes, *etc.*] that negative tension D-branes can introduce into the low energy effective theory [17].

[11]There is another potential source of confusion here: Since the (3+1)-dimensional D-brane is sweeping through the (4+1)-dimensional bulk, why is it that the D-brane does not exchange energy with the bulk? One might at first glance expect violations of (3+1)-dimensional stress-energy conservation due to (4+1)-dimensional matter entering or leaving the D-brane. In fact, in general this might happen, and it is potentially an interesting observational signal to look for — but in the present cosmological context this effect is zero: as the D-brane moves through (4+1)-space, it is the "flux" of (4+1)-dimensional matter onto the brane, defined by

$$J_\mu = n_\alpha \, T^{\alpha\beta} \, [g_{\beta\mu} - n_\beta \, n_\mu],$$

that determines whether or not (4+1)-dimensional stress-energy conservation holds [15]. In all of the bulk geometries considered in this article, this flux is identically zero (in fact the stress-energy tensor is diagonal).

So as usual, *two* of these three equations are independent, and the third is redundant.

The conservation equation is identical to that for standard cosmology, while the D-brane version of the Friedmann equation, obtained by rearranging the equation for the surface energy density that was given above, is seen to be

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{F(a)}{a^2} + \left(\frac{8\pi\,G_{4+1}\,\rho_{3+1}}{3}\right)^2. \tag{26}$$

In contrast the *standard* Friedmann equation (for a $k = +1$ closed FLRW universe) is

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{1}{a^2} + \frac{\Lambda}{3} + \frac{8\pi\,G_{3+1}\,\rho}{3}. \tag{27}$$

To get a brane cosmology that is not wildly in conflict with observation, we split the (3+1)-dimensional energy into a constant ρ_0 determined by the brane tension, plus ordinary matter ρ, with $\rho \ll \rho_0$ to suppress the quadratic term in comparison to the linear [5]. Then with $\rho_{3+1} = \rho_0 + \rho$ we have

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{F(a)}{a^2} + \left(\frac{8\pi\,G_{4+1}\,\rho_0}{3}\right)^2 + \left(\frac{16\pi\,G_{4+1}\,\rho_0}{3}\right)\left(\frac{8\pi\,G_{4+1}}{3}\right)\left[\rho + \frac{1}{2}\frac{\rho^2}{\rho_0}\right]. \tag{28}$$

Picking out the term linear in ρ, this permits us to identify

$$G_{3+1} = G_{4+1}\left(\frac{16\pi\,G_{4+1}\,\rho_0}{3}\right); \qquad \text{that is} \qquad G_{4+1} = \sqrt{\frac{3\,G_{3+1}}{16\pi\,\rho_0}}. \tag{29}$$

Therefore

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{F(a)}{a^2} + \left(\frac{8\pi\,G_{3+1}}{3}\right)\left[\frac{1}{2}\rho_0 + \rho + \frac{1}{2}\frac{\rho^2}{\rho_0}\right]. \tag{30}$$

Since we want $\rho_0 \gg \rho$ to suppress the quadratic term, this leaves us with a large (3+1)-dimensional cosmological constant that we will need to eliminate by cancelling it (either fully or partially) with some term in $F(a)$ [5]. This result is in its own way quite remarkable: up to this point no assumptions had been made about the size of the brane tension, or even whether or not the brane tension was zero. Nor had any assumption been made up to this point about the existence or otherwise of any cosmological constant in the (4+1)-dimensional bulk. It is observational cosmology that first forces us to take ρ_0 large [electro-weak scale or higher to avoid major problems with nucleosynthesis], and then forces us to deduce the presence of an almost perfectly countervailing cosmological constant in the bulk [5].

In the next section we shall make use of this still relatively general formalism by specializing $F(r)$ to the Reissner–Nordström–de Sitter form.

2.4   Reissner–Nordström–de Sitter surgery

For the (4+1)-dimensional Reissner–Nordström–de Sitter geometry

$$F(r) = 1 - \frac{2M_{4+1}}{r^2} + \frac{Q_{4+1}^2}{r^4} - \frac{\Lambda_{4+1}\,r^2}{6}. \tag{31}$$

Here M_{4+1} is a (4+1)-dimensional "mass" parameter, corresponding to the mass of the central object in (4+1)-space — it does not have a ready (3+1)-dimensional interpretation and is best carried along as an extra free parameter that from the 4-dimensional point of view can be adjusted to taste. Similarly, Q_{4+1} corresponds to an "electric charge" in the (4+1)-dimensional sense. Our universe, the boundary D-brane, must then be viewed as carrying an equal but opposite charge to allow field lines to terminate. From the (3+1)-dimensional view this may be taken to be a second free parameter. The (4+1)-dimensional cosmological constant combines with the term coming from the D-brane tension to give an effective (3+1)-dimensional cosmological constant

$$\Lambda = \frac{\Lambda_{4+1}}{2} + 4\pi G_{3+1}\, \rho_0. \tag{32}$$

In the original Randall–Sundrum models these two terms were fine-tuned by hand to obtain complete cancellation. In view of the recent observational evidence for a small cosmological constant in our observable universe we need merely assert that this effective cosmological constant is presently relatively small ($\Lambda \lesssim 8\pi G_{3+1}\, \rho_{\text{critical}}$; this is small by particle physics standards, but can be quite significant by cosmological standards).[12]

Since ρ_0 is guaranteed positive,[13] this implies that Λ_{4+1} should be negative, and so if this model is correct we are living on the edge of a bulk anti-de Sitter space. The D-brane dynamical equation now reads

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{1}{a^2} + \frac{2M_{4+1}}{a^4} - \frac{Q_{4+1}^2}{a^6} + \frac{\Lambda}{3} + \left(\frac{8\pi\, G_{3+1}}{3}\right)\left[\rho + \frac{1}{2}\frac{\rho^2}{\rho_0}\right]. \tag{33}$$

It is clear that by tuning these parameters appropriately one can recover standard cosmology to arbitrary accuracy. The M_{4+1} parameter can be used to mimic an arbitrary quantity of what would usually be called "radiation" (relativistic fluid, $\rho = 3p$), while the Q_{4+1} parameter mimics "stiff" matter ($\rho = p$) [22], though with an overall minus sign. An observational astrophysicist or cosmologist could now simply forget about the underlying string theory and D-brane physics, take this expression as the D-brane inspired generalization of the Friedmann equations, and treat M_{4+1}, Q_{4+1}, Λ, and ρ_0 as parameters to be observationally determined.

Since we actually want to do more than just reproduce standard cosmology we should seek some additional constraints on these parameters — and this is where the phenomenon of weak localization of the graviton near the brane comes into play.

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[12] A small effective cosmological constant would indeed imply deviations from the original Randall–Sundrum scenario, but on a distance scale determined by this effective cosmological constant (and observationally this distance scale would be of order Giga-parsecs or larger). So we are not too concerned about this issue in that the implications for particle phenomenology are negligible.

[13] Tricky point: actually it is ρ_{3+1} that is guaranteed to be positive, and this holds because the D-brane universe is taken to be convex as seen from the bulk. Then the same logic that leads to energy condition violations for traversable wormholes now applies in reverse, and the (3+1) null energy condition is generically satisfied in this type of cosmological model. (Violating the strong energy condition, which is relevant for cosmological inflation, is much easier [22].)

3   Weak localization of perturbative gravity

Suppose that the observable universe is large compared to the natural distance scale in the (4+1)-dimensional bulk, that is: $a(\tau) \gg \sqrt{6/|\Lambda_{4+1}|}$ (so that the universe has "grown up"), and both M_{4+1} and Q_{4+1} are sufficiently small to allow us to approximate

$$F(r) \approx \frac{|\Lambda_{4+1}|\, r^2}{6}; \qquad \text{for} \qquad r \approx a. \tag{34}$$

Then near the D-brane we can write

$$ds_{4+1}^2 \approx -\frac{|\Lambda_{4+1}|\, r^2}{6}\, dt^2 + \frac{6}{|\Lambda_{4+1}|\, r^2}\, dr^2 + r^2\, d\Omega_3^2. \tag{35}$$

In terms of the normal distance (proper distance) from the D-brane,

$$\eta \approx \sqrt{\frac{6}{|\Lambda_{4+1}|}}\, \ln(r/a), \tag{36}$$

this implies

$$ds_{4+1}^2 \approx +d\eta^2 + \exp\left(-2\sqrt{\frac{|\Lambda_{4+1}|}{6}}\,\eta\right)\left[-\frac{|\Lambda_{4+1}|\, a^2}{6}\, dt^2 + a^2\, d\Omega_3^2\right]. \tag{37}$$

If we now re-label our time parameter in terms of proper time measured along the D-brane (that is, use the proper time of a cosmologically comoving observer),

$$\tau \approx \sqrt{\frac{|\Lambda_{4+1}|\, a^2}{6}}\, t, \tag{38}$$

and introduce quasi-Cartesian coordinates to the tangent space at any arbitrary point point of the D-brane then

$$ds_{4+1}^2 \approx +d\eta^2 + \exp\left(-2\sqrt{\frac{|\Lambda_{4+1}|}{6}}\,\eta\right)\left[-d\tau^2 + dx^2 + dy^2 + dz^2\right]. \tag{39}$$

Thus in this approximation the near-brane metric is precisely of the Randall–Sundrum form [1, 2] and we know from their analysis that there is a graviton bound state attached to the brane with an exponential falloff controlled by the distance scale parameter[14]

$$L_{\text{graviton}} = \sqrt{\frac{6}{|\Lambda_{4+1}|}}. \tag{40}$$

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[14]Of course this is little more than the statement that if we are interested in laboratory physics in the here and now, then a tangent space approximation to cosmology had better work: Minkowski space is an excellent approximation for physics here on Earth and so the D-brane *must* exhibit at least approximate Lorentz symmetry if it is to be acceptable as a model of empirical reality. Moving off the D-brane and into the bulk, the only essential item is that at large enough distances [from the (4+1) "centre"] we must have $F(r) \propto r^2$. Thus as long as the (4+1) geometry is asymptotically anti-de Sitter space we will recover Randall–Sundrum phenomenology on small scales. (And eventually, on large enough distance scales, the simple Randall–Sundrum phenomenology will break down either because of cosmological expansion, or because of the small effective (3+1)-dimensional cosmological constant, or simply because of the positive spatial curvature [$k = +1$ and a is finite].)

Now the experimental fact that we do not see short distance deviations from the inverse square law of gravity at least down to centimetre scales implies that L_{graviton} is certainly less than one centimetre (and many would argue that it is at most one millimetre). Numerous experiments designed to tighten this limit are currently planned and in progress. Within the approximation that the (3+1)-dimensional effective cosmological constant is negligible we get

$$G_{3+1} = G_{4+1} \; \frac{2}{L_{\text{graviton}}}. \tag{41}$$

The importance of these results for cosmology is that, given the observed almost perfect cancellation of the net cosmological constant,

$$\rho_0 \approx \frac{3}{4\pi \; G_{3+1} \; L_{\text{graviton}}^2} = \frac{3}{4\pi} \frac{L_{\text{Planck}}^2}{L_{\text{graviton}}^2} \; \rho_{\text{Planck}} \tag{42}$$

While this number is certainly large on a usual astrophysics scale, and is rather large even compared with nuclear densities, it could still be much less than the Planck scale and yet be compatible with experiment. Indeed if L_{graviton} is as large as a centimetre then the quadratic terms in the density become important once temperatures reach the electroweak scale (about 100 GeV). The good news is that this implies the model is compatible with standard cosmology at least back to the electroweak scale; the better news is that there are possibilities of seeing deviations from the standard cosmology as we go further back. The larger L_{graviton} is, the better things are with regard to the hierarchy problem of particle physics [1, 2] and the lower the brane tension needs to be. On the other hand, the lower L_{graviton} is the better the brane is at trapping gravitational perturbations and the less risk there is of conflict with gravity-based experiment.

4   Discussion

The Randall–Sundrum scenarios [1, 2], and earlier tentative steps along these lines [3], have engendered a tremendous amount of activity, both in terms of particle physics and in terms of cosmology [4, 5]. In this paper we have sketched what we feel is perhaps the simplest most symmetric cosmology that can be based on these ideas: We have reduced the number of D-branes to exactly one, and have only one bulk (4+1)-dimensional region. The D-brane (which our observable universe lives on) is here viewed as an actual physical boundary to the higher-dimensional spacetime, and we have demonstrated how to write down both curvature tensor and field equations for a manifold with boundary.

We have verified that standard $k = +1$ FLRW cosmology can very easily be reproduced, and that we do not have massive present day violations of observational constraints. If you absolutely insist on a spatially flat $k = 0$ geometry (or even a spatially hyperbolic $k = -1$ geometry) that can also be achieved along the lines of this article, but at some cost in elegance, and for very little real purpose. Remember that for $a(\tau)$ large enough a $k = +1$ spatial slice mimics $k = 0$ to arbitrary accuracy. In Appendix B and Appendix C we sketch how one could nevertheless force spatially flat or spatially hyperbolic FLRW cosmologies into this framework.

As is by now not unexpected [1, 2, 5], likely places to look for observational signatures are in short-distance (centimetre) deviations from the gravitational inverse square law,

and in very early universe cosmology (before densities drop to the electro-weak scale; this is the region where the quadratic density term in the generalized Friedmann equation might come into play).[15]

Because we are viewing the D-brane as an actual boundary, the conjectured connections between the Randall–Sundrum models and Maldacena's adS/CFT conjecture are perhaps more compelling [10, 11] — we no longer have to deal with a Z_2-doubled version of the adS/CFT conjecture, but can work directly on a boundary of the (asymptotic) anti-de Sitter space. As the universe evolves in time the D-brane boundary moves further out into the asymptotic anti-de Sitter region, and this hints at a possible connection between cosmological time, the holographic hypothesis, and renormalization group flow [11].

Acknowledgments

Appendix A: Alternative construction for the Riemann tensor of a manifold with boundary.

Take your original manifold $\mathcal{M}$, with boundary $\partial\mathcal{M}$, and join to the boundary a hyper-tube of topology $\mathcal{H} = (-\infty, 0) \otimes \partial\mathcal{M}$. Let the metric on this hyper-tube be specified in terms of the induced metric on the boundary and the flat 1-dimensional metric:

$$g(\mathcal{H}) = d\eta^2 \oplus g(\partial\mathcal{M}). \tag{43}$$

Then by construction $K_{\alpha\beta}^- = 0$ and $K_{\alpha\beta}^+ = K_{\alpha\beta}$, so that in this geometry

$$\kappa_{\alpha\beta}(\mathcal{M} \cup \mathcal{H}) = [K_{\alpha\beta}] = K_{\alpha\beta}^+ - K_{\alpha\beta}^- = K_{\alpha\beta}, \tag{44}$$

leading to the Riemann tensor [15]

$$R(\mathcal{M} \cup \mathcal{H})_{\alpha\beta\gamma\delta} = -\delta(\eta) \left[K_{\alpha\gamma}\, n_\beta\, n_\delta + K_{\beta\delta}\, n_\alpha\, n_\gamma - K_{\alpha\delta}\, n_\beta\, n_\gamma - K_{\beta\gamma}\, n_\alpha\, n_\delta \right] \\ + \Theta(\eta)\, R(\mathcal{M})_{\alpha\beta\gamma\delta}^{\mathrm{bulk}} + \Theta(-\eta)\, R(\mathcal{H})_{\alpha\beta\gamma\delta}^{\mathrm{bulk}}. \tag{45}$$

Now truncate the geometry by simply throwing away the hyper-tube $\mathcal{H}$. The Riemann tensor in the remaining manifold $\mathcal{M}$ is, as before

$$R(\mathcal{M})_{\alpha\beta\gamma\delta} = -\delta(\eta) \left[K_{\alpha\gamma}\, n_\beta\, n_\delta + K_{\beta\delta}\, n_\alpha\, n_\gamma - K_{\alpha\delta}\, n_\beta\, n_\gamma - K_{\beta\gamma}\, n_\alpha\, n_\delta \right] + R(\mathcal{M})_{\alpha\beta\gamma\delta}^{\mathrm{bulk}}. \tag{46}$$

There is now no symmetry to suggest that one should perform any particular splitting of the delta-function contribution at the boundary, and in fact the observation that the extrinsic curvature is by construction zero on the hyper-tube side of the boundary is an indication that you should assign all the delta-function contribution to $\mathcal{M}$, the resulting manifold with boundary. Either construction (hyper-tube addition or Z_2-doubling) leads to the same result for the Riemann tensor, but some may be happier with one construction over the other.

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[15]In particular, for $\rho \gg \rho_0$ even ordinary radiation ($\rho \propto a^{-4}$) acts as though it has a a^{-8} behaviour, and this is enough to drive an epoch of power-law inflation with $a(t) \propto t^{1/4}$ [5].

Appendix B: Spatially flat FLRW cosmology.

By a little guess-work based on hyper-spherically symmetric Reissner–Nordström–de Sitter space one is led to consider the metric

$$ds^2_{4+1} = -F(r)\,dt^2 + \frac{dr^2}{F(r)} + r^2 \left[dx^2 + dy^2 + dz^2\right]. \tag{47}$$

with (note the *absence* of the leading **1**!)

$$F(r) = -\frac{2M_{4+1}}{r^2} + \frac{Q^2_{4+1}}{r^4} - \frac{\Lambda_{4+1}\,r^2}{6}. \tag{48}$$

This metric still satisfies the (4+1)-dimensional Einstein–Maxwell equations, but with a hyper-planar symmetry instead of a hyper-spherical symmetry. You can now re-do the analysis of this note by placing a spatially flat D-brane boundary at $r = a(t)$ and will obtain very similar results to those of this article. The intrinsic geometry of the D-brane will now be

$$ds^2_{3+1} = -d\tau^2 + a^2(\tau) \left[dx^2 + dy^2 + dz^2\right]. \tag{49}$$

It is not clear to us that the marginal change in the Friedmann equation is worth the loss of hyper-spherical symmetry. The point $r = 0$ is still (for $M_{4+1} \neq 0$ or $Q_{4+1} \neq 0$) a curvature singularity of the (4+1)-dimensional bulk, but whether you really want to call it the "center" of the bulk (as opposed to say a "focal point") is somewhat less than clear.

Appendix C: Spatially hyperbolic FLRW cosmology.

Inspired by the previous guess-work one is led to consider the metric (note the presence of the sinh function)

$$ds^2_{4+1} = -F(r)\,dt^2 + \frac{dr^2}{F(r)} + r^2 \left[d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2)\right]. \tag{50}$$

with (note the presence of the leading *minus* **1**!)

$$F(r) = -1 - \frac{2M_{4+1}}{r^2} + \frac{Q^2_{4+1}}{r^4} - \frac{\Lambda_{4+1}\,r^2}{6}. \tag{51}$$

This metric also satisfies the (4+1)-dimensional Einstein–Maxwell equations, but with a hyperbolic symmetry instead of either hyper-spherical or hyper-planar symmetry. You can now re-do the analysis of this note by placing a spatially hyperbolic D-brane boundary at $r = a(t)$ and will again obtain very similar results to those of this article. The intrinsic geometry of the D-brane will now be

$$ds^2_{3+1} = -d\tau^2 + a^2(\tau) \left[d\chi^2 + \sinh^2\chi\,(d\theta^2 + \sin^2\theta\,d\phi^2)\right]. \tag{52}$$

For all three cases, $k = +1, 0, -1$, the formal dynamical equation for the brane motion [equation (30), valid for arbitrary $F(r)$] is unchanged, while the explicit dynamical equation after Reissner–Nordström–de Sitter surgery (the generalized Friedman equation) becomes

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2} + \frac{2M_{4+1}}{a^4} - \frac{Q^2_{4+1}}{a^6} + \frac{\Lambda}{3} + \left(\frac{8\pi\,G_{3+1}}{3}\right)\left[\rho + \frac{1}{2}\frac{\rho^2}{\rho_0}\right]. \tag{53}$$

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