Noncommutative Quantum Gravity

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Abstract

The possible role of gravity in a noncommutative geometry is investigated. Due to the Moyal *-product of fields in noncommutative geometry, it is necessary to complexify the metric tensor of gravity. We first consider the possibility of a complex Hermitian, nonsymmetric $g_{\mu\nu}$ and discuss the problems associated with such a theory. We then introduce a complex symmetric (non-Hermitian) metric, with the associated complex connection and curvature, as the basis of a noncommutative spacetime geometry. The spacetime coordinates are in general complex and the group of local gauge transformations is associated with the complex group of Lorentz transformations $CSO(3,1)$. A real action is chosen to obtain a consistent set of field equations. A Weyl quantization of the metric associated with the algebra of noncommuting coordinates is employed.

1 Introduction

The concept of a quantized spacetime was proposed by Snyder [1], and has received much attention over the past few years [2-6]. There has been renewed interest recently in noncommutative field theory, since it makes its appearance in string theory, e.g. noncommutative gauge theories describe the low energy excitations of open strings on D-branes in a background two-form B field [5,7-12]. Noncommutative Minkowski space is defined in terms of spacetime coordinates $x^\mu, \mu = 0, ...3$, which satisfy the following commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}, \tag{1}$$

where $\theta^{\mu\nu}$ is an antisymmetric tensor. In what follows, we can generally extend the results to higher dimensions $\mu = 0, ...d$.

An associative noncommuting algebra, M^4_θ, is constructed with elements given by ordinary continuous functions on M^4 and with a deformed product of functions given by the Moyal bracket or $\star$-product of functions [13]

$$f(x)\star g(x) = \exp\left[\frac{i}{2}\theta^{\mu\nu}\frac{\partial}{\partial\alpha^\mu}\frac{\partial}{\partial\beta^\nu}\right] f(x+\alpha)g(x+\beta)|_{\alpha=\beta=0}$$

$$= f(x)g(x) + \frac{i}{2}\theta^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x) + \mathcal{O}(\theta^2). \tag{2}$$

Conventional field theories are generalized to noncommutative spacetime by replacing the usual product of fields by the Moyal bracket or $\star$-product.

Since the $\star$-product of fields involves an infinite number of derivatives, the resulting field theories are nonlocal. The lack of commutativity of the spacetime coordinates gives rise to a spacetime uncertainty relation

$$\Delta x^\mu \Delta x^\nu \geq \frac{1}{2}|\theta^{\mu\nu}|. \tag{3}$$

It appears that in a perturbative context, the noncommutative theories have a unitary S-matrix for space-space noncommutativity, $\theta^{ij} \neq 0, \theta^{0i} = 0$, while the S-matrix is not unitary for space-time noncommutativity, $\theta^{0i} \neq 0, \theta^{ij} = 0$ [14]. Moreover, there are indications that conventional renormalizable field theories remain renormalizable, when generalized to noncommutative spacetimes [15, 16].

The product of two operators $\hat{f}$ and $\hat{g}$ can be defined and can be shown to lead to the Moyal $\star$-product [17]

$$f(x) \star g(x) = \frac{1}{(2\pi)^4} \int d^4k d^4p \exp[i(k_\mu + p_\mu)x^\mu - \frac{i}{2}k_\mu \theta^{\mu\nu} p_\nu] \tilde{f}(k)\tilde{g}(p)$$

$$= \exp\left[\frac{i}{2}\frac{\partial}{\partial x^\mu}\theta^{\mu\nu}\frac{\partial}{\partial y^\nu}\right] f(x)g(y) \mid_{y\to x}, \tag{4}$$

where $\tilde{f}(k)$ is the Fourier transform

$$\tilde{f}(k) = \frac{1}{(2\pi)^2} \int d^4x \exp(-ik_\sigma x^\sigma) f(x). \tag{5}$$

The noncommutative Yang-Mills action is defined by

$$S_{YM} = -\frac{1}{4}\int d^4x F^a_{\mu\nu} \star F^{a\mu\nu}, \tag{6}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i\epsilon^{abc} A_{b\mu} \star A_{c\nu}. \tag{7}$$

The action is invariant under the gauge transformations

$$A^a_\mu \to [U \star A_\mu \star U^{-1} - \partial_\mu U \star U^{-1}]^a, \tag{8}$$

where

$$U \star U^{-1} = U^{-1} \star U = 1. \tag{9}$$

The definition (7) of $F^a_{\mu\nu}$ requires the gauge fields to be complex and these fields should be invariant under the transformations of a general group of noncommutative gauge transformations $NCU(3,1)$. When we contemplate a noncommutative extension of spacetime, we do not appear to be able to proceed with the quantization of fields as in commutative quantum field theory, in which the classical fields are real for neutral charged fields and complex for charged fields. Moreover, the classical field variables are treated in the quantization procedure as operators in a Hilbert space, and they satisfy commutation or anticommutation relations. Within this standard scenario, spacetime plays a passive role as far as quantization is concerned.

Let us now consider the role of gravity in a noncommutative spacetime. We can employ the Weyl quantization procedure [17, 18] to associate the metric tensor operator $\hat{g}_{\mu\nu}$ with the classical metric $g_{\mu\nu}$. This prescription can be used to associate an element of the noncommutative algebraic structure M_x, which defines a noncommutative space in terms of the coordinate operators $\hat{x}$. Using the Fourier transform of the metric

$$\tilde{g}_{\mu\nu}(k) = \frac{1}{(2\pi)^2}\int d^4x \exp(-ik_\sigma x^\sigma) g_{\mu\nu}(x), \tag{10}$$

we can define the metric operator

$$\hat{g}_{\mu\nu}(\hat{x}) = \frac{1}{(2\pi)^2} \int d^4k \exp(ik_\sigma \hat{x}^\sigma) \tilde{g}_{\mu\nu}(k). \tag{11}$$

Thus, the operators $\hat{g}_{\mu\nu}$ and $\hat{x}$ replace the variables $g_{\mu\nu}$ and x. If the $\hat{x}$ have complex symmetry properties, then the $\hat{g}_{\mu\nu}$ will inherit these properties for a classical metric $g_{\mu\nu}$.

When we turn to the geometry of spacetime in the presence of matter, the situation with regards to noncommutativity of spacetime coordinates is much more problematic. In this case, it seems unlikely that we can retain our conventional notions of a real pseudo-Riemannian space within a real manifold. If we define the metric of spacetime in terms of vierbeins v^a_μ using the Moyal $\star$-product

$$g_{\mu\nu} = v^a_\mu \star v^b_\nu \eta_{ab}, \tag{12}$$

where $a, b = 0, ...3$ denote the flat, fiber bundle tangent space (anholonomic) coordinates, and η_{ab} denotes the Minkowski metric: $\eta_{ab} = \mathrm{diag}(1, -1, -1, -1)$, then the metric of spacetime is forced to be complex.

We shall investigate two possibilities for complex gravity. First we consider the possibility that the fundamental tensor $g_{\mu\nu}$ is complex Hermitian satisfying $g^\dagger_{\mu\nu} = g_{\mu\nu}$, where $\dagger$ denotes the Hermitian conjugate. This was recently proposed by Chamseddine [19, 20], as a possible way to complexify the gravitational metric. The second possibility we shall consider is a complex pseudo-Riemannian geometry, based on a complex symmetric (non-Hermitean) tensor $g_{\mu\nu}$ [21, 22, 23].

2 Nonsymmetric Gravity

The nonsymmetric field extension of Einstein gravity has a long history. It was originally proposed by Einstein, as a unified field theory of gravity and electromagnetism [24, 25]. But it was soon realized that the antisymmetric part $g_{[\mu\nu]}$ in the decomposition

$$g_{\mu\nu} = g_{(\mu\nu)} + g_{[\mu\nu]} \tag{13}$$

could not describe physically the electromagnetic field. It was then suggested that the nonsymmetric field structure describes a generalization of Einstein gravity, known in the literature as the nonsymmetric gravitational theory (NGT) [27-32].

NGT faces some difficulties which have their origin in the lack of a clear-cut gauge symmetry in the antisymmetric sector of the theory. Even in the linear approximation, the antisymmetric field equations are not invariant under the transformation

$$g'_{[\mu\nu]} = g_{[\mu\nu]} + \partial_\nu\lambda_\mu - \partial_\mu\lambda_\nu, \tag{14}$$

where λ_μ is an arbitrary vector field. This lack of overall gauge symmetry in the antisymmetric sector of the theory gives rise to two basic problems [32, 33, 34]. In the weak antisymmetric field approximation, an expansion about a classical general relativity (GR) background, reveals that there are ghost poles, tachyons and higher-order poles associated with the asymptotic boundary conditions [32].

However, this problem can be resolved by a careful choice of the NGT action [30, 31]:

$$S_{NGT} = \int d^4x\sqrt{-g}[g^{\mu\nu}R_{\mu\nu}(W) - 2\lambda - \frac{1}{4}\mu^2 g^{\mu\nu}g_{[\nu\mu]} - \frac{1}{6}g^{\mu\nu}W_\mu W_\nu]. \quad (15)$$

Here, we choose units so that $G = c = 1$, $g = \mathrm{Det}(g_{\mu\nu})$, λ is the cosmological constant, μ is a mass associated with the skew field $g_{[\mu\nu]}$ and $R_{\mu\nu}(W)$ is the contracted curvature tensor:

$$R_{\mu\nu}(W) = \partial_\beta W^\beta_{\mu\nu} - \frac{1}{2}(\partial_\nu W^\beta_{\mu\beta} + \partial_\mu W^\beta_{\nu\beta}) - W^\beta_{\alpha\nu}W^\alpha_{\mu\beta} + W^\beta_{\alpha\beta}W^\alpha_{\mu\nu}, \quad (16)$$

defined in terms of the unconstrained nonsymmetric connection

$$W^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} - \frac{2}{3}\delta^\lambda_\mu W_\nu, \quad (17)$$

where

$$W_\mu = \frac{1}{2}(W^\lambda_{\mu\lambda} - W^\lambda_{\lambda\mu}). \quad (18)$$

Eq.(17) leads to the result

$$\Gamma_\mu = \Gamma^\lambda_{[\mu\lambda]} = 0. \quad (19)$$

A nonsymmetric matter source is added to the action

$$S_M = -8\pi\int d^4x g^{\mu\nu}\sqrt{-g}T_{\mu\nu}, \quad (20)$$

where $T_{\mu\nu}$ is a nonsymmetric source tensor.

This action will lead to a physically consistent Lagrangian and field equations for the antisymmetric field in the linear, weak field approximation, with field equations of the form of a massive Proca-type theory which is free of tachyons, ghost poles and higher-order poles [30, 31].

However, the problems do not end here with this form of NGT. A Hamiltonian constraint analysis performed on NGT by Clayton, showed that when the NGT field equations are expanded about a classical GR background, the resulting theory is unstable [33, 34]. This problem appears to be a generic feature of any fully geometrical NGT-type of theory, including the vierbein derivation of NGT based on the Hermitian fundamental tensor [28, 29, 19]

$$g_{\mu\nu} = e^a_\mu \tilde{e}^b_\nu \eta_{ab}, \quad (21)$$

where e^a_μ is a complex vierbein and $\tilde{e}^a_\mu$ denotes the complex conjugate of e^a_μ. Basically, this result means that we cannot consider GR as a sensible limit of NGT for weak antisymmetric fields.

Another serious problem with the complex version of NGT, in which $g_{[\mu\nu]} = if_{[\mu\nu]}$ and $f_{[\mu\nu]}$ is a real antisymmetric tensor, is that the linear, weak field approximation to the NGT field equations produces generic, negative energy ghost poles. This prompted a proposal that the nonsymmetric vierbeins be described by hyperbolic complex variables [28, 29]. For a sesquilinear, hyperbolic complex $g_{\mu\nu}$, there exists a local $GL(4,R)$ gauge symmetry, which corresponds to $g_{\mu\nu}$ preserving rotations of generalized linear frames in the tangent bundle. This symmetry should not be confused with the linear (global) subgroup $GL(4)$

of the diffeomorphism group of the manifold $\mathcal{M}^4$ under which NGT is also invariant. The hyperbolic complex vierbeins are defined by

$$e^a_\mu = \mathrm{Re}(e^a_\mu) + \omega \mathrm{Im}(e^a_\mu), \tag{22}$$

while the $g_{\mu\nu}$ is given by

$$g_{\mu\nu} = e^a_\mu \tilde{e}^b_\nu \eta_{ab} = e^a_\mu \tilde{e}_{\nu a}, \tag{23}$$

where $\omega^2 = +1$ is the pure imaginary element of the hyperbolic complex Clifford algebra Ω [35]. The $g_{\mu\nu}$ and the connexion $\Gamma^\lambda_{\mu\nu}$ are hyperbolic complex Hermitian in μ and ν, while the spin connection $(\Omega_\sigma)_{ab}$ is hyperbolic complex skew-Hermitian in a and b. The hyperbolic complex unitary group $U(3,1,\Omega)$ is isomorphic to $GL(4,R)$. The spin connection $(\Omega_\sigma)^a_b$ is invariant under the $GL(4)$ transformations provided

$$(\Omega_\sigma)^a_b \to [U_G \Omega_\sigma U_G^{-1} - (\partial_\sigma U_G) U_G^{-1}]^a_b, \tag{24}$$

where U_G is an element of the unitary group $U(3,1,\Omega)$. The curvature tensor $(R_{\mu\nu})^a_b$ is invariant under the $GL(4)$ transformations when

$$(R_{\mu\nu})^a_b \to U^a_{Gc} (R_{\mu\nu})^c_d (U_G^{-1})^d_b. \tag{25}$$

The field equations can be found from the action [28, 29]

$$S_{\mathrm{grav}} = \int d^4 x e R(e), \tag{26}$$

where $e = \sqrt{-g}$ and $R = e^{\mu a} \tilde{e}^{\nu b} (R_{\mu\nu})_{ab}$. Although the particle spectrum is now free of negative energy ghost states in the weak field approximation, the theory still suffers from the existence of dipole ghost states due to the asymptotic boundary conditions for $g_{[\mu\nu]}$, unless we use the action (15), rewritten in the language of vierbeins. However, a Hamiltonian constraint analysis for this theory will still reveal serious instability problems [33, 34].

3 Complex Symmetric Riemannian Geometry

We shall now consider choosing a complex manifold of coordinates $\mathcal{M}^4_C$ and a complex symmetric metric defined by [21, 22, 23]

$$g_{\mu\nu} = s_{\mu\nu} + a_{\mu\nu}, \tag{27}$$

where $a_{\mu\nu} = i b_{\mu\nu}$ and $b_{\mu\nu}$ is a real symmetric tensor. This corresponds to having two copies of a real metric in an eight-dimensional space. The real diffeomorphism symmetry of standard Riemannian geometry is extended to a complex diffeomorphism symmetry under the group of complex coordinates transformations with $z^\mu = x^\mu + i y^\mu$. The metric can be expressed in terms of a complex vierbein $E^a_\mu = \mathrm{Re}(E^a_\mu) + i \mathrm{Im}(E^a_\mu)$ as

$$g_{\mu\nu} = E^a_\mu E^b_\nu \eta_{ab}. \tag{28}$$

The real contravariant tensor $s^{\mu\nu}$ is associated with $s_{\mu\nu}$ by the relation

$$s^{\mu\nu} s_{\mu\sigma} = \delta^\nu_\sigma \tag{29}$$

and also

$$g^{\mu\nu}g_{\mu\sigma} = \delta^{\nu}_{\sigma}. \tag{30}$$

With the complex spacetime is also associated a complex symmetric connection

$$\Gamma^{\lambda}_{\mu\nu} = \Delta^{\lambda}_{\mu\nu} + \Omega^{\lambda}_{\mu\nu}, \tag{31}$$

where $\Omega^{\lambda}_{\mu\nu}$ is purely imaginary.

We shall determine the $\Gamma^{\lambda}_{\mu\nu}$ according to the $g_{\mu\nu}$ by the equations

$$g_{\mu\nu;\lambda} = \partial_\lambda g_{\mu\nu} - g_{\rho\nu}\Gamma^{\rho}_{\mu\lambda} - g_{\mu\rho}\Gamma^{\rho}_{\nu\lambda} = 0. \tag{32}$$

By commuting the two covariant differentiations of an arbitrary complex vector A_μ, we obtain the generalized curvature tensor

$$R^{\lambda}_{\mu\nu\sigma} = -\partial_\sigma\Gamma^{\lambda}_{\mu\nu} + \partial_\nu\Gamma^{\lambda}_{\mu\sigma} + \Gamma^{\lambda}_{\rho\nu}\Gamma^{\rho}_{\mu\sigma} - \Gamma^{\lambda}_{\rho\sigma}\Gamma^{\rho}_{\mu\nu} \tag{33}$$

and a contracted curvature tensor $R_{\mu\nu} = R^{\sigma}_{\mu\nu\sigma}$:

$$R_{\mu\nu} = A_{\mu\nu} + B_{\mu\nu}, \tag{34}$$

where $B_{\mu\nu}$ is a purely imaginary tensor. From the curvature tensor, we can derive the four complex (eight real) Bianchi identities

$$(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R)_{;\nu} = 0. \tag{35}$$

We must choose a real action to guarantee a consistent set of field equations. There is, of course, a degree of arbitrariness in choosing this action due to the complex manifold of coordinate transformations and the complex Riemannian geometry. We shall choose [22]

$$S_{\rm grav} = \frac{1}{2}\int d^4x[\mathbf{g}^{\mu\nu}R_{\mu\nu} + \text{compl.conj.}] = \int d^4x[\mathbf{s}^{\mu\nu}A_{\mu\nu} + \mathbf{a}^{\mu\nu}B_{\mu\nu}], \tag{36}$$

where $\mathbf{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu} = \mathbf{s}^{\mu\nu} + \mathbf{a}^{\mu\nu}$. The variation with respect to $\mathbf{s}^{\mu\nu}$ and $\mathbf{a}^{\mu\nu}$ yields the twenty field equations of empty space

$$A_{\mu\nu} = 0, \quad B_{\mu\nu} = 0, \tag{37}$$

or, equivalently, the ten complex field equations

$$R_{\mu\nu} = 0. \tag{38}$$

We can add a real matter action to (36):

$$S_{\rm matter} = -4\pi\int d^4x[\mathbf{g}^{\mu\nu}T_{\mu\nu} + \text{compl.conj.}], \tag{39}$$

where $T_{\mu\nu} = S_{\mu\nu} + C_{\mu\nu}$ is a complex symmetric source tensor, and $S_{\mu\nu}$ and $C_{\mu\nu}$ are real and pure imaginary tensors, respectively.

We shall assume that the line element determining the physical gravitational field is of the real form

$$ds^2 = s_{\mu\nu}dx^\mu dx^\nu. \tag{40}$$

Let us consider the static spherically symmetric line element

$$ds^2 = \alpha dt^2 - \eta dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{41}$$

where the real α and η are functions of real r only. Our complex fundamental tensor $g_{\mu\nu}$ is determined by

$$\begin{aligned} g_{11}(r) &\equiv -\mu(r) = -[\eta(r) + i\zeta(r)], \\ g_{22}(r) &\equiv s_{22}(r) = -r^2, \\ g_{33}(r) &\equiv s_{33}(r) = -r^2\sin^2\theta, \\ g_{00}(r) &\equiv \gamma(r) = \alpha(r) + i\beta(r). \end{aligned} \tag{42}$$

Solving the $\Gamma^\lambda_{\mu\nu}$ from Eq.(32) and substituting into the field equation (38), we get the solutions

$$\alpha = 1 - \frac{2m}{r}, \quad \beta = \frac{2\epsilon}{r}, \tag{43}$$

where $2m$ and 2ϵ are constants of integration, and we have imposed the boundary conditions

$$\alpha \to 1, \quad \eta \to 1, \quad \beta \to 0, \quad \zeta \to 0, \tag{44}$$

as $r \to \infty$. Solving for $\eta(r)$ and $\zeta(r)$ from the solution $\mu(r) = 1/\gamma(r)$ we get [23]

$$ds^2 = (1 - \frac{2m}{r})dt^2 - \frac{1 - \frac{2m}{r}}{(1 - \frac{2m}{r})^2 + \frac{4\epsilon^2}{r^2}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{45}$$

$$\zeta = -\frac{2\epsilon/r}{(1 - \frac{2m}{r})^2 + \frac{4\epsilon^2}{r^2}}. \tag{46}$$

When $\epsilon \to 0$, we regain the Schwarzschild solution of Einstein gravity.

The weak field approximation obtained from the expansion about Minkowski spacetime

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{47}$$

where $h_{\mu\nu} = p_{\mu\nu} + ik_{\mu\nu}$, leads to an action which is invariant under the local, linear gauge transformation

$$h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\theta_\nu + \partial_\nu\theta_\mu, \tag{48}$$

where θ_μ is an arbitrary complex vector. Thus, two spin 2 massless gravitons describe the complex bimetric gravity field. The physical spin 2 graviton is described by a real mixture of the basic spin two particles associated with $p_{\mu\nu}$ and $k_{\mu\nu}$. Moreover, the physical null cone with $ds^2 = 0$ will be a real mixture of the two null cones associated with $s_{\mu\nu}$ and $a_{\mu\nu}$.

The linearized action $S_{\rm grav}$, obtained from (36) in the weak field approximation, contains negative energy ghost states coming from the purely imaginary part of the metric $k_{\mu\nu}$. To avoid this unphysical aspect of the theory, we can base the geometry on a hyperbolic complex metric

$$g_{\mu\nu} = s_{\mu\nu} + \omega a_{\mu\nu}, \tag{49}$$

where ω is the pure imaginary element of a hyperbolic complex Clifford algebra Ω [35]. We have $\omega^2 = +1$ and for $z = x + \omega y$, we obtain $z\tilde{z} = x^2 - y^2$

where $\tilde{z} = x - \omega y$, so that there exist lines of zeros, $z\tilde{z} = 0$, in the hyperbolic complex space and it follows that Ω forms a ring of numbers and not a field as for the usual system of complex numbers with the pure imaginary element $i = \sqrt{-1}$. The linearized action $S_{\rm grav}$ should now have positive energy and with the invariance under the hyperbolic complex group of transformations $CSO(3,1,\Omega)$ the gravitons should be free of ghost states.

4 Noncommutative Complex Symmetric Gravity

We must now generalize the complex symmetric gravity (CSG) to noncommutative coordinates by replacing the usual products by Moyal-products. We shall use the complex vierbein formalism based on the metric (28) and a complex spin connection $(\omega_\mu)_{ab}$, since this formalism is closer to the standard gauge field formalism of field theory.

In order to accommodate a complex symmetric metric, we shall treat the coordinates as "fermionic" degrees of freedom and use the anticommutation relation

$$\{\hat{x}^\mu, \hat{x}^\nu\} = 2x^\mu x^\nu + i\tau^{\mu\nu}, \tag{50}$$

where $\tau^{\mu\nu} = \tau^{\nu\mu}$ is a symmetric tensor. The Moyal product of two operators is now given by the $\diamond$-product

$$f(x) \diamond g(x) = \exp\left[\frac{i}{2}\tau^{\mu\nu}\frac{\partial}{\partial\alpha^\mu}\frac{\partial}{\partial\beta^\nu}\right] f(x+\alpha)g(x+\beta)|_{\alpha=\beta=0}$$

$$= f(x)g(x) + \frac{i}{2}\tau^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x) + \mathcal{O}(\tau^2). \tag{51}$$

If we form $x^\mu \diamond x^\nu$ and add it to $x^\nu \diamond x^\mu$, then we obtain the anticommutator expression (50).

The complex symmetric metric is defined by

$$g_{\mu\nu} = E^a_\mu \diamond E^b_\nu \eta_{ab}. \tag{52}$$

The spin connection is subject to the gauge transformation

$$(\omega_\sigma)^a_b \rightarrow [U_C \diamond \omega_\sigma \diamond U_C^{-1} - (\partial_\sigma U_C) \diamond U_C^{-1}]^a_b, \tag{53}$$

where U_C is an element of a complex noncommutative group of orthogonal transformations $NCSO(3,1)$.

The curvature tensor is now given by

$$(R_{\mu\nu})^a_b = \partial_\mu(\omega_\nu)^a_b - \partial_\nu(\omega_\mu)^a_b + (\omega_\mu)^a_c \diamond (\omega_\nu)^c_b - (\omega_\nu)^a_c \diamond (\omega_\mu)^c_b, \tag{54}$$

which transforms as

$$(R_{\mu\nu})^a_b \rightarrow U^a_{Cc} \diamond (R_{\mu\nu})^c_d \diamond (U_C^{-1})^d_b. \tag{55}$$

The real action is

$$S_{\rm grav} = \frac{1}{2}\int d^4x [E \diamond E^\mu_a \diamond (R_{\mu\nu})^a_b \diamond E^{\nu b} + \text{compl.conj.}], \tag{56}$$

where $E = \sqrt{-g}$. The action $S_{\rm grav}$ is locally invariant under the transformations of the complex noncommutative, fiber bundle tangent space group $NCSO(3,1)$, i.e. the group of complex noncommutative homogeneous Lorentz transformations. It has been shown by Bonora et al., [36] that attempting to define a noncommutative gauge theory corresponding to a subgroup of $U(n)$ is not trivially accomplished. This is true in the case of a string-brane theory configuration. We must find a $NCSO(3,1)$, which reduces to $CSO(3,1)$ and, ultimately, to $SO(3,1)$ when $\theta \to 0$. Bonora et al., were able to show that it is possible to impose constraints on gauge potentials and the corresponding gauge transformations, so that ordinary commutative orthogonal and symplectic gauge groups were recovered when the deformation parameter θ vanishes. These constraints were defined in terms of a generalized gauge theory charge conjugation operator, and a generalization of connection-based Lie algebras in terms of an antiautomorphism in the corresponding C^*-algebra. The problem of deriving an explicit description of $NCSO(d,1)$ will be investigated in a future article.

To avoid possible problems with negative energy ghost states, we can develop the same formalism based on the hyperbolic complex vierbein $E^a_\mu = \mathrm{Re}(E^a_\mu) + \omega \mathrm{Im}(E^a_\mu)$ and a hyperbolic complex spin connection and curvature. The noncommutative spacetime is defined by

$$\{\hat{x}^\mu, \hat{x}^\nu\} = 2x^\mu x^\nu + \omega\tau^{\mu\nu}, \tag{57}$$

and a hyperbolic complex Moyal $\diamond$-product in terms of the pure imaginary element ω:

$$\begin{aligned} f(x) \diamond g(x) &= \exp\left[\frac{\omega}{2}\tau^{\mu\nu}\frac{\partial}{\partial\alpha^\mu}\frac{\partial}{\partial\beta^\nu}\right] f(x+\alpha)g(x+\beta)|_{\alpha=\beta=0} \\ &= f(x)g(x) + \frac{\omega}{2}\tau^{\mu\nu}\partial_\mu f(x)\partial_\nu g(x) + \mathcal{O}(\tau^2). \end{aligned} \tag{58}$$

A hyperbolic complex vierbein formalism can be developed along the same lines as the complex vierbein formalism and the invariance group of transformations $NCSO(3,1,C)$ is extended to $NCSO(3,1,\Omega)$.

The noncommutative field equations and the action contain an infinite number of derivatives, and so constitute a nonlocal theory at the quantum level. When the coordinates assume their classical commutative structure, then the theory regains standard, classical causal properties. The nonlocal nature of the noncommutative quantum gauge field theory and quantum gravity theory presents potentially serious problems. Such theories can lead to instabilities, which render them unphysical [37], and in the case of space-time noncommutativity, $\theta^{i0} \neq 0$, the perturbative S-matrix may not be unitary [14]. Moreover, it is unlikely that one can apply the conventional, canonical Hamiltonian formalism to quantize such gauge field theories. Of course, we know that the latter formalism already meets serious obstacles in applications to quantum gravity for commutative spacetime. In ref. [11], it was shown that open string theory in a background electric field displays space-time noncommutativity, but it was argued that stringy effects conspire to cancel the acausal effects that are present for the noncommutative field theory.

5 Conclusions

The deformed product of functions given by the Moyal $\star$-product of functions leads to complex gauge fields and a complex Riemannian or non-Riemannian geometry. We first considered the possibility that the fundamental tensor $g_{\mu\nu}$ is Hermitian nonsymmetric, leading to a nonsymmetric gravitational theory (NGT). This theory does not at present form a viable gravitational theory, because of instability difficulties that arise when an expansion is performed about a GR background. The $g_{\mu\nu}$ had to be chosen to be hyperbolic complex to avoid basic negative energy ghost problems. Even though this hyperbolic complex nonsymmetric theory could be shown to be self-consistent as far as asymptotic boundary conditions are concerned and to be free of higher-order ghost poles, the instability problems discovered by Clayton [33, 34] remain. It is possible that these instability problems can be removed by some new formulation of NGT, but so far no one has succeeded in discovering such a formulation.

We then investigated a complex symmetric (non-Hermitian) $g_{\mu\nu}$ on a complex manifold with the local gauge group of complex Lorentz transformations $CSO(3,1)$ (or $CSO(d,1)$ in (d+1)-dimensions). From a real action, we obtained a consistent set of field equations and Bianchi identities in a torsion-free spacetime. By assuming that the physical line element was given by the real form (40), we derived a static spherically symmetric solution with two gravitational "charges" $2m$ and 2ϵ. When $\epsilon = 0$, we regained the standard Schwarzschild solution of GR, so there is a well-defined classical gravity limit of the theory, which agrees with all the know gravitational experimental data. We also considered a complex geometry based on a hyperbolic complex metric and connection. This geometry would avoid potential problems of negative ghost states in the linearized equations of gravity.

By formulating the vierbein and spin connection formalism on a flat tangent space by means of complex vierbeins and a complex spin connection and curvature tensor, we generalized the complex symmetric geometry to anticommuting coordinates by replacing the usual products by Moyal $\diamond$-products and complex gauge transformations, and by Moyal $\diamond$-products for a hyperbolic complex noncommutative geometry. We extended the complex group of gauge transformations $CSO(3,1)$ of the commutative spacetime to a noncommutative group of complex orthogonal gauge transformations $NCSO(3,1,C)$. For the geometry based on a noncommutative Clifford algebra Ω, the invariance group was extended to the hyperbolic complex group $NCSO(3,1,\Omega)$.

The CSG theory we have developed can be the basis of a quantum gravity theory, which contains classical GR as the limit of CSG when the new gravitational charge $\epsilon \to 0$. It would be interesting to consider possible experimental consequences of CSG for strong gravitational fields, for gravitational wave experiments and for cosmology.

A noncommutative quantum gravity theory as well as a noncommutative quantum gauge field theory pose potential difficulties, because of the nonlocal nature of the interactions at both the nonperturbative and perturbative levels. It is not clear whether a physically consistent quantum gauge field theory or quantum gravity theory, based on noncommutative coordinates, can be found at either the perturbative or nonperturbative levels. This is an open question that requires more intensive investigation. It remains to be seen whether our standard physical notions about the nature of spacetime can be profoundly changed at the quantum level, without leading to unacceptable physical consequences.

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