

# Field spectrum and degrees of freedom in AdS/CFT correspondence and Randall Sundrum model

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## Abstract

Compactified AdS space can not be mapped into just one Poincare coordinate chart. This implies that the bulk field spectrum is discrete despite the infinite range of the coordinates. We discuss here why this discretization of the field spectrum seems to be a necessary ingredient for the holographic mapping. For the Randall Sundrum model we show that this discretization appears even without the second brane.

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# 1 Introduction

The interest of theoretical physicists in studying fields in anti de Sitter (AdS) space is not new[1]. In particular the question of the quantization of fields in this space circumventing the problem of the lack of a Cauchy surface was addressed in[2, 3]. There was, however, a remarkable increase in the attention devoted to this subject since the appearance of two recent models where this geometry plays a special role. The first, the so called AdS/CFT correspondence, was motivated by the Maldacena [4] conjecture on the equivalence (or duality) of the large  $N$  limit of  $SU(N)$  superconformal field theories in  $n$  dimensions and supergravity and string theory in anti de Sitter spacetime in  $n+1$  dimensions. This correspondence was elaborated by Gubser, Klebanov and Polyakov [5] and Witten [6] interpreting the boundary values of bulk fields as sources of boundary theory correlators. The Maldacena conjecture and the subsequent work on AdS/CFT correspondence[7] strongly reinforced the relevance of understanding the subtleties of field theories in AdS spaces.

The second one, the Randall and Sundrum model[8], proposes a solution to the hierarchy problem between the mass scale of the standard model and the Plank scale. Their model is essentially defined in a slice of the five dimensional AdS space bounded by two 3-branes and can be regarded as an alternative to usual compactification, as they discussed in [9]. The standard model fields are confined to one of the branes while the non factorizable form of the metric makes it possible to have propagation of gravity in the extra dimension without spoiling Newton's law up to experimental precision. One important property of this model is that none of the two branes is located at the AdS boundary. This way, it is possible to include gravitational fluctuations on the branes[10], in contrast to the standard AdS/CFT scenario, where the singular form of the metric on the boundary forbids the inclusion of normalizable gravity fluctuations.

The AdS/CFT correspondence and the Randall Sundrum model can be understood as complementary models. In particular, Duff and Liu have shown that they share equivalent corrections to the Newton's gravitational law[11]. A common feature of both AdS/CFT and Randall Sundrum scenarios is that the appropriate description of AdS space involves the use of Poincare coordinate system. This system allows a very useful definition of the bulk/boundary mapping in the AdS/CFT case and also a simple localization for the branes in the Randall Sundrum model. However, the use of such a coordinate system involves

some subtle aspects. We have considered in a recent letter the quantization of fields in AdS space using Poincare coordinates [12] and found the non trivial result that despite the infinite range of the axial coordinate, the corresponding spectrum is discrete. This happens because a consistent quantization in AdS space is only possible if one includes a boundary where a vanishing flux of information from (or to) outside the AdS space can be imposed<sup>1</sup>. In terms of Poincare coordinates, this requires the introduction of an extra point associated with the axial coordinate infinity. This is only possible by using more than one Poincare coordinate chart. One has to define the axial coordinate of the first chart stopping at some (arbitrary) value, leaving the rest of the space to be mapped by other charts. Here we will discuss this problem further detailing a case where one can see that the spectrum of eigenfunctions of some space changes from discrete to continuous if we map it to coordinate that exclude a "point at infinity". We will also discuss the fact that we have different possible choices for the quantum fields corresponding to solutions of different boundary value problems and not only the particular kind of solution that we presented in [12].

Considering the case of the Randall Sundrum model, the four dimensional world as we perceive it corresponds to one of the 3-branes whereas the other acts as a regulating brane. So, the slice of AdS space between the branes is compact in the axial direction and bulk fields in this region will have a discrete spectrum, as discussed by Goldberger and Wise [13]. These authors proposed also that such fields with quartic interactions on the branes can lead to a mechanism of stabilization of the radius  $r_0$  of the second brane [14]. The possibility of defining the Randall Sundrum model without the second brane was discussed in [15]. A very recent discussion about bulk fields in the Randall Sundrum scenario can be found in [16]. In order to gain some understanding on the role of the second brane, we will discuss here what happens with the field spectrum when we take the limit of infinite distance between the two branes taking into account the non triviality of the Poincare coordinates.

The article is organized as follows. In section 2 we will review the basic properties of AdS and discuss the counting of degrees of freedom in this space and its relation with the bulk/boundary correspondence. In section 3 we are going to discuss the mapping of a

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<sup>1</sup>We will comment in section 5 on the case of D-brane approach to black p-branes in which case the spectrum is continuous and its relation with AdS/CFT correspondence.

compact space into an open space plus a point at infinity, that illustrates the case of AdS space when represented in Poincare coordinates. In section 4 we discuss the quantization of fields in AdS bulk taking into account the need of multiple charts. We also discuss how one can choose different boundary conditions to define complete sets of eigenfunctions and then build up quantum fields. In section 5 we comment on the difference between the physical setting of the present model and that of absorption of particles by black p-branes where a continuous spectrum is found. In section 6 we will study the implication of our results on the spectrum of fields living in the bulk of a Randall-Sundrum scenario. In particular we will discuss the case when the second brane goes to infinity. Some final remarks and conclusions are presented in section 7.

## 2 AdS space

The anti-de Sitter spacetime of  $n+1$  dimensions is a space of constant negative curvature that can be represented as the hyperboloid ( $\Lambda = \text{constant}$ )

$$X_0^2 + X_{n+1}^2 - \sum_{i=1}^n X_i^2 = \Lambda^2 \quad (1)$$

embedded in a flat  $n+2$  dimensional space with metric

$$ds^2 = -dX_0^2 - dX_{n+1}^2 + \sum_{i=1}^n dX_i^2. \quad (2)$$

Two coordinate systems are often used for  $AdS_{n+1}$ . First, the so called global coordinates  $(\rho, \tau, \Omega_i)$  can be defined as [7, 17]

$$\begin{aligned} X_0 &= \Lambda \sec \rho \cos \tau \\ X_i &= \Lambda \tan \rho \Omega_i \quad \left( \sum_{i=1}^n \Omega_i^2 = 1 \right) \\ X_{n+1} &= \Lambda \sec \rho \sin \tau, \end{aligned} \quad (3)$$

with ranges  $0 \leq \rho < \pi/2$  and  $0 \leq \tau < 2\pi$ . Quantization of fields in compactified AdS space (including the hypersurface boundary  $\rho = \pi/2$ ) using these coordinates was discussed in [2, 3]. The coordinate  $\tau$ , identified with the time coordinate, has a finite range in the above prescription. This is commonly remedied by considering copies of the

compact AdS space glued together along the  $y$  direction resulting in the covering AdS space in which the time coordinate is non compact. When we refer to AdS space in this article we will actually mean this covering space.

Second, the Poincaré coordinates  $(z, \vec{x}, t)$  can be introduced by

$$\begin{aligned} X_0 &= \frac{1}{2z} (z^2 + \Lambda^2 + \vec{x}^2 - t^2) \\ X_i &= \frac{\Lambda x^i}{z} \\ X_n &= -\frac{1}{2z} (z^2 - \Lambda^2 + \vec{x}^2 - t^2) \\ X_{n+1} &= \frac{\Lambda t}{z}, \end{aligned} \tag{4}$$

where  $\vec{x}$  with  $n-1$  components and  $t$  range from  $-\infty$  to  $+\infty$ , while  $0 \leq z < \infty$ . These coordinates are useful for the AdS/CFT correspondence [5, 6] and in this case the  $AdS_{n+1}$  measure with Lorentzian signature reads<sup>2</sup>

$$ds^2 = \frac{\Lambda^2}{(z)^2} (dz^2 + (d\vec{x})^2 - dt^2). \tag{5}$$

Then, in these coordinates, the  $AdS$  boundary corresponds to the region  $z=0$ , described by usual Minkowski coordinates  $\vec{x}$ ,  $t$  plus a “point” at infinity ( $z \rightarrow \infty$ ). It should be remarked that the metric is not defined at  $z=0$  and actually all the calculations in the AdS/CFT correspondence are actually done first at some small  $z=\delta$  that in some cases, as when defining boundary correlators, is taken to zero after the calculations[18, 19, 20]

The holographic mapping between the AdS bulk and the corresponding boundary [21, 22, 23], that are two manifolds of different dimensionality, is only possible because the metric is such that ”volumes” are proportional to ”areas”. We can understand this by counting the degrees of freedom of the bulk volume and those of the boundary hypersurface. In order to compare these quantities that are actually both infinite we take a discretized version where the space is not continuous but rather a discrete array of volume cells. We take the boundary at  $z=\delta$ . There we take a hypersurface of area  $\Delta A$  corresponding to variations  $\Delta x^1 \dots \Delta x^{n-1}$  in the space coordinates:

$$\Delta A = \left(\frac{\Lambda}{\delta}\right)^{n-1} \Delta x^1 \dots \Delta x^{n-1} \tag{6}$$

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<sup>2</sup>Another form for Poincare coordinates is possible through the mapping  $y = \Lambda \ln(z/\Lambda)$ , which is commonly used in the Randall-Sundrum model, as we are going to discuss in section 6.

and calculate the corresponding volume from  $z = \delta$  to  $\infty$ , finding

$$\Delta V = \Lambda \frac{\Delta A}{n-1}. \quad (7)$$

This is the expected result that the volume is proportional to the area in the bulk / boundary correspondence for a fixed  $\Lambda$  (see Fig. 1).

Figure 1

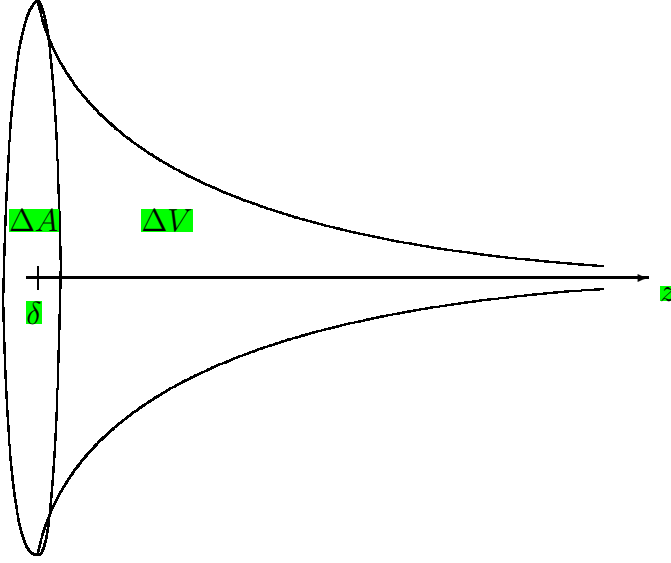


Fig. 1: The Area  $\Delta A$  on the hypersurface at  $z = \delta$  and the corresponding AdS volume  $\Delta V$ .

Now we can count the degrees of freedom by splitting  $\Delta V$  in  $\ell$  pieces of equal volume corresponding to cells whose boundaries are hypersurfaces located at

$$z_j = \frac{\delta}{\sqrt[n-1]{1-j/\ell}}, \quad (8)$$

with  $j = 0, 1, \dots, \ell-1$ . Note that the last cell extends to infinity. These volume cells can be mapped into the area  $\Delta A$  by also dividing it in  $\ell$  parts. We can take  $\ell$  to be arbitrarily large. This way we find a one to one mapping between degrees of freedom of bulk and boundary (see Fig. 2). This analysis shows why it is possible to holographically map the

degrees of freedom of the bulk on the boundary despite the fact that the variable  $z$  has an infinite range. One could then think of the AdS space as corresponding to a "box" in terms of degrees of freedom with respect to  $z$ . By this we mean that effectively the coordinate  $z$  behaves as if it had a finite range. Then the field spectrum associated with this coordinate should be discrete contrarily to ones expectation for a variable of infinite range.

Figure 2

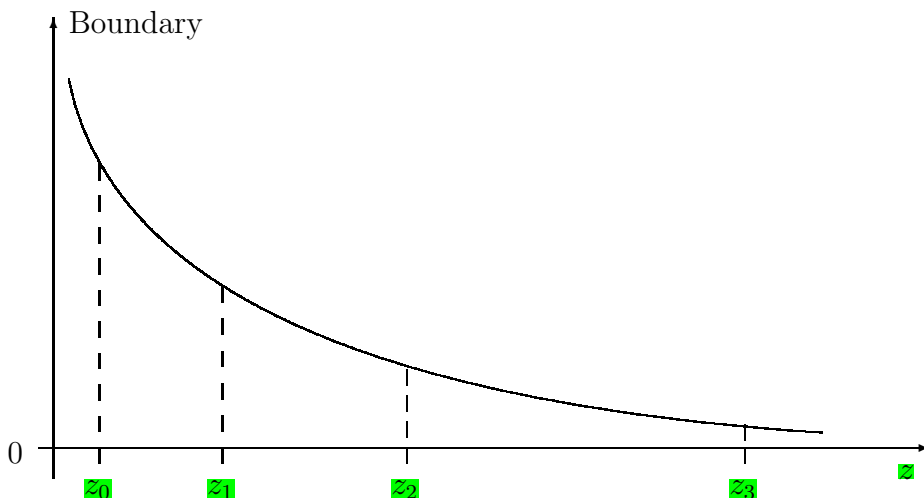


Fig. 2:  $z$  section of equal AdS volume cells.

In the global coordinate context, where the coordinates have finite ranges it was shown that a consistent quantization can only be obtained if one supplements AdS space with a boundary ("the wall of the box") in order to impose vanishing flux of particles and information there[2, 3]. Otherwise massless particles would be able to go to (or come from) spatial infinity in finite times and it would thus be impossible to define a Cauchy surface. This boundary is the hypersurface  $\rho = \pi/2$  that corresponds in Poincare coordinates to the surface  $z = 0$  plus a point defined by the limit  $z \rightarrow \infty$ . Thus a consistent quantization would require the inclusion of this extra point at infinity where the flux should be required to vanish. This is not possible just using one set of Poincaré coordinates because the mapping between these two coordinate systems is not one to one. The non trivial topology of AdS space and problems related to using coordinates like Poincare are also

discussed in [24]. In order to understand more precisely this problem and its consequences for the field spectrum we will consider in the next chapter a simpler illustrative problem.

### 3 Mapping of a compact space into an open space plus a point at infinity

If we start with a description of compactified AdS space (including the hypersurface  $\rho = \pi/2$ ) in terms of global coordinates and then transform to Poincare coordinates we will be considering a mapping that is not one to one. In order to have a clear understanding of this point and to illustrate its consequences for the field spectrum we will first consider a simpler example where a similar situation happens: the case of a one dimensional compact space where a change of coordinates apparently turns the spectrum of eigenfunctions from discrete to continuous. We will see that this happens precisely when we map the compact space into an open set of infinite range plus a point at infinity. At least, two coordinate charts will be necessary in order to find a one to one mapping and thus a correct description of the associated functional space.

Let us consider the compact interval  $S = \{x | 0 \leq x \leq L\}$  along the cartesian coordinate  $x$ . Considering the class of functions  $f$  that are not singular in  $S$  we know that a basis can be formed by solving a boundary value problem for some Sturm Liouville operator on  $S$ , corresponding in general to some linear combination of the functions and their derivatives vanishing at  $x = 0$  and  $x = L$ . Any function  $f$  can be represented in the open set  $S - \{0\} - \{L\}$  as, for example:

$$f(x) = \sum_n a_n \sin\left(\frac{n\pi x}{L}\right) \quad (9)$$

or, alternatively, changing the boundary condition

$$f(x) = \sum_n b_n \cos\left(\frac{n\pi x}{L}\right) \quad (10)$$

or some sinusoidal function corresponding to a mixed boundary condition. The point is that, given the compact interval  $S$ , an arbitrary non-singular function in  $S$  can be expanded, except possibly at the end points, as a discrete series of eigenfunctions. This in some sense defines the "dimension" of this functional space.



Let us however see what happens if we introduce the variable

$$u = \frac{1}{L-x} \quad (11)$$

and map the interval  $\mathbf{S}$  in the interval  $\mathbf{S}' = \{u | 1/L \leq u < \infty\}$ . This map induces a metric in  $\mathbf{S}'$  (considering the original set  $\mathbf{S}$  as of unity measure).

$$ds^2 = dx^2 = \frac{du^2}{u^4} \quad (12)$$

The point  $x=L$  is "mapped" to  $u \rightarrow \infty$ . Now if we naively consider the spectrum of eigenfunctions by just looking at  $\mathbf{S}'$  we would conclude that it is continuous because the interval is not compact. An interesting way of understanding what is happening is just to try reversing the mapping between the two sets. If we want to naively reverse the mapping from  $\mathbf{S}'$  to  $\mathbf{S}$  by just looking at eq. (11) we would not get the point  $x=L$  back. We would then find an open (at one side) set, corresponding to a continuous spectrum of eigenfunctions. This difference in the spectrum would be a consequence of the absence of point  $x=L$  and thus the lack of a condition of non-singularity there and thus an increase in the set of admissible functions. So we would be changing the functional space by naively representing the set by the coordinate  $u$  because the mapping of  $\mathbf{S}$  on  $\mathbf{S}'$  is not one to one. It is interesting to observe also that the point  $x=L$  is mapped into a limit of singular (vanishing) metric in the  $u$  coordinate. Even in the case that such a singular point could be included in the coordinate system, a second chart should be introduced[25].

So it is not possible to map the whole interval  $\mathbf{S}$  into  $\mathbf{S}'$  with just one chart of the coordinate  $u$  but rather we should consider at least two charts. We can, for example define the range of  $u$  to be:  $\frac{1}{L} \leq u \leq \frac{1}{L-R}$  that would map the set  $0 \leq x \leq R$ . Then we can map the rest of  $\mathbf{S}$  with another variable:  $v = 1/x$  with range  $1/L \leq v \leq 1/R$  and induced metric

$$ds^2 = dx^2 = \frac{dv^2}{v^4}. \quad (13)$$

Now the mapping

$$x \mapsto \begin{cases} u = 1/(L-x) & 0 \leq x \leq R \\ v = 1/x & R \leq x \leq L \end{cases} \quad (14)$$

is one to one and in both charts we can find discrete basis of eigenfunctions. We can take  $R$  arbitrarily close to  $L$  but not equal. This mapping does really reproduces the interval  $\mathbf{S}$

and indeed gives the same kind of spectrum of eigenfunctions. In other words they share to the same functional space.

The first tentative mapping was not one to one and we were loosing the possibility of imposing any kind of boundary condition at  $x=L$ , and thus considering a different (larger) functional space. This point plays the role of a point at infinity for the coordinate  $u$  and it is interesting to observe that removing one point we indeed get a larger functional space. This fact will be very important when we consider the AdS case where the functional space has to be such that a (one to one) mapping with the functional space of the boundary should be possible. A representation with just one Poincare chart would, in the same way as in the present example, not be appropriate as it would actually lead to a larger functional space than that of the compactified AdS.

Also important to remark, for understanding the AdS situation, is the fact that we are not saying that we must include some cut off in the interval  $S$  itself. We just have to use at least two coordinate charts. So the distance  $R$  has no significance for the set  $S$  although some value should be chosen in order to be possible to find a one to one representation in terms of the coordinates  $u$  and  $v$ .

For completeness we observe also that if our original interval did not include the point  $x=L$  the mapping with  $S$  would be one to one and there is no need for two charts. Even if we decide to do so, the second chart would be an open set and thus we would conclude that the spectrum would be continuous as expected for an original open set that really would correspond to a larger functional space.

## 4 Quantum fields in the AdS space

Now we turn to the question of quantizing fields in the AdS bulk. This problem was originally investigated in [2, 3] using the global coordinates, eq. (3). The main problem that one finds when quantizing fields in AdS is that this space does not admit a Cauchy surface and consequently suffers from the Cauchy problem [25]. This problem was circumvented by compactifying the AdS space including the hypersurface  $\rho = \pi/2$ . With this prescription it is possible to show that energy and information are conserved and a consistent quantization is possible. The problem we want to discuss here is the quantization

of fields in AdS bulk described by Poincare coordinates, eq. (4). If the mapping between compactified AdS in global and Poincare coordinates were one to one this problem would be trivial. However, as discussed in the previous sections, these coordinates have infinite range and the infinite limit of the axial coordinate ( $z \rightarrow \infty$ ) corresponds to a point in the AdS boundary that is not properly represented by the chart given by eq. (4). So using the discussion of section 3 on the mapping of a compact space (here the compactified AdS space in global coordinates) into an open set plus a point at infinity (here the AdS in Poincare coordinates) we conclude that a consistent quantization in the whole AdS space must involve more than one Poincare chart in order to preserve the dimension of the functional space.

Considering the interval  $\delta < z < \infty$ , we find a way of introducing another chart, with the help of the example of the previous section, by first introducing an auxiliary coordinate  $\alpha$  as

$$z = \frac{1}{\frac{1}{\delta} - \alpha}. \quad (15)$$

with  $0 \leq \alpha \leq 1/\delta$ .

In order to have a proper definition of the point at infinity  $z \rightarrow \infty$ , we can introduce a new chart as

$$z' = \frac{1}{\alpha}. \quad (16)$$

Now the Poincare charts correspond to the system of eqs. (4), (5) plus a second one where  $z$  is rewritten in terms of  $z'$  from

$$\frac{1}{z'} = \frac{1}{\delta} - \frac{1}{z} \quad (17)$$

with ranges  $\delta < z < R$  and  $\delta < z' < R'$ , where  $R' = \delta R / (R - \delta)$  (see Fig. 3).

We can take  $R$  arbitrarily large and map as much of AdS space as we want into just one chart but the fact that  $R$  is finite implies the discretization of the spectrum associated with the coordinate  $z$ . This reduces the dimensionality of the functional space. A similar problem was discussed by Gell-Mann and Zwiebach[26] in the context of dimensional reduction induced by a sigma model.

Note that the region  $0 < z < \delta$  is not covered by the above charts. In order to describe this region with the auxiliary variable  $\alpha$  we can still use eq. (15), but with  $-\infty < \alpha \leq 0$

(see Fig. 4). However in this case a non compact region  $\alpha$  is mapped into a compact  $z$  region, so that the mapping is not one to one. If this is required one has to define another  $z'$  chart.

Figure 3

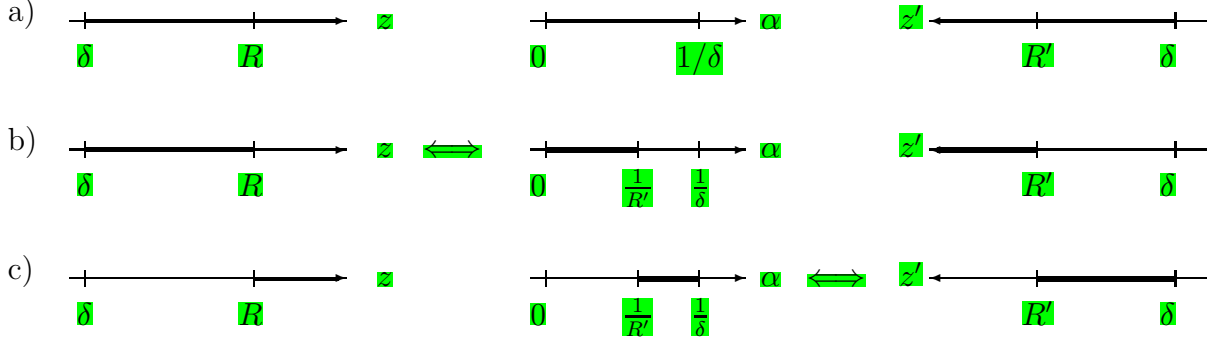


Fig. 3: The corresponding intervals on  $z$ ,  $\alpha$  and  $z'$ . a) The mappings are not one to one, since  $z$  and  $z'$  are non compact while  $\alpha$  is compact. b) The compact region  $\delta < z < R$  has a one to one mapping (indicated by the double arrow) with  $0 \leq \alpha \leq 1/R'$ , corresponding to one Poincare chart. c)  $z'$  in the region  $\delta \leq z' \leq R'$  has a one to one mapping with  $1/R' \leq \alpha \leq 1/\delta$ , corresponding to a second Poincare chart.

Figure 4



Fig. 4: Corresponding intervals for  $0 < z < \delta$ , into  $\alpha$  and  $z'$ .

In [12] we discussed the case of quantum scalar fields in anti de Sitter space in Poincare coordinates but considering a chart where the axial coordinate has the range  $0 \leq z < R$  and choosing as a functional basis for the fields a particular boundary value problem

corresponding to functions vanishing at  $z=0$  and  $z=R$ . We will study here different boundary conditions on an interval  $\delta \leq z \leq R$ . This way we avoid the surface  $z=0$  where the metric is not defined. Also by an appropriate choice of  $\delta$  and boundary conditions we reproduce the Randall Sundrum scenario.

Let us consider a massive scalar field  $\phi$  in the  $AdS_{n+1}$  spacetime described by Poincaré coordinates with action

$$I[\phi] = \frac{1}{2} \int d^{n+1}x \sqrt{g} \left( \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 \right), \quad (18)$$

where we take  $x^0 \equiv z, x^n \equiv t$ ,  $\sqrt{g} = (x^0)^{-n-1}$  and  $\mu = 0, 1, \dots, n$ . The classical equation of motion then reads

$$\left( \nabla_\mu \nabla^\mu - m^2 \right) \phi = \frac{1}{\sqrt{g}} \partial_\mu \left( \sqrt{g} \partial^\mu \phi \right) - m^2 \phi = 0 \quad (19)$$

and one finds solutions[27, 28] for the interval  $\delta \leq z \leq R$  as a linear combination ( $A$  and  $B$  are arbitrary constants)

$$\Phi(z, \vec{x}, t) = e^{-i\omega t + i\vec{k} \cdot \vec{x}} z^{n/2} \left( c J_\nu(uz) + d Y_\nu(uz) \right), \quad (20)$$

where  $J_\nu(uz)$  and  $Y_\nu(uz)$  are, respectively, the Bessel and Neumann functions of order  $\nu = \frac{1}{2} \sqrt{n^2 + 4m^2}$  and  $u \equiv \sqrt{\omega^2 - \vec{k}^2}$  with  $\omega^2 > \vec{k}^2$ .

In the region considered:  $\delta \leq z \leq R$  we can find a functional basis for representing fields by integrating over  $\vec{k}$  the solutions of any boundary value problem in the coordinate  $z$  of the general form

$$A \Phi(z = \delta, \vec{x}, t) + B \frac{\partial \Phi}{\partial z}(z, \vec{x}, t)|_{z=\delta} = F(\vec{x}, t) \quad (21)$$

$$C \Phi(z = R, \vec{x}, t) + D \frac{\partial \Phi}{\partial z}(z, \vec{x}, t)|_{z=R} = G(\vec{x}, t). \quad (22)$$

Here in this work we will choose for simplicity  $F(\vec{x}, t) = G(\vec{x}, t) = 0$ . Any non trivial choice of the constants  $A, B, C, D$  leads to a basis of eigenfunctions for the operator appearing in the equation of motion (19). If we choose, for example,  $B = D = 0$  we find the condition

$$J_\nu(u_p \delta) Y_\nu(u_p R) - J_\nu(u_p R) Y_\nu(u_p \delta) = 0. \quad (23)$$

The roots of this equation define the possible values of  $u_p$ , restricting the spectrum of the field. Choosing these eigenfunctions, we can write the quantum fields as

$$\begin{aligned} \Phi(z, \vec{x}, t) = & \sum_{p=1}^{\infty} \int \frac{d^{n-1}k}{(2\pi)^{n-1}} z^{n/2} N(\vec{k}, u_p) \left[ J_\nu(u_p z) - \left( \frac{J_\nu(u_p \delta)}{Y_\nu(u_p \delta)} \right) Y_\nu(u_p z) \right] \\ & \times \{ a_p(\vec{k}) e^{-i w_p(\vec{k}) t + i \vec{k} \cdot \vec{x}} + a_p^\dagger(\vec{k}) e^{i w_p(\vec{k}) t - i \vec{k} \cdot \vec{x}} \} \end{aligned} \quad (24)$$

where  $w_p(\vec{k}) = \sqrt{u_p^2 + \vec{k}^2}$  and  $N(\vec{k}, u_p)$  is a normalization constant.

Imposing that the operators  $a_p(\vec{k}), a_{p'}^\dagger(\vec{k}')$  satisfy the commutation relations

$$\begin{aligned} [a_p(\vec{k}), a_{p'}^\dagger(\vec{k}')] &= 2 (2\pi)^{n-1} w_p(\vec{k}) \delta_{pp'} \delta^{n-1}(\vec{k} - \vec{k}') \\ [a_p(\vec{k}), a_{p'}(\vec{k}')] &= [a_p^\dagger(\vec{k}), a_{p'}^\dagger(\vec{k}')] = 0 \end{aligned} \quad (25)$$

we find, for example, for the equal time commutator of the field and its time derivative

$$\left[ \Phi(z, \vec{x}, t), \frac{\partial \Phi}{\partial t}(z', \vec{x}', t) \right] = i z^{n-1} \delta(z - z') \delta(\vec{x} - \vec{x}'). \quad (26)$$

Other boundary conditions corresponding to different choices of the constants in eqs. (21),(22) can be chosen and would lead to different basis for representing the field  $\Phi$ . For example, the case  $A = C = 0$  is connected with the Randall-Sundrum model as we are going to discuss in the section 6. Before continuing this discussion let us make some comments on the difference between the situation described above, which implies a discrete spectrum and that of black p-branes, also related to AdS/CFT correspondence, but where particles have a continuous spectrum.

## 5 AdS, Black p-Branes and boundary conditions

### 5.1 AdS and Black p-Branes

The study of the D-brane formulation of black p-branes was a source of motivation for the discovery of the AdS/CFT correspondence. The comparison of the absorption cross section of infalling particles on D3 branes with perturbative calculations on the brane world volume[29] indicated that Greens functions of Yang Mills theory (with extended supersymmetry) could be calculated from supergravity (see [7],[30] for a review and a wide list of references).

The extremal 3-brane metric in  $D = 10$  dimensions[31] may be written as

$$ds^2 = \left(1 + \frac{R^4}{r^4}\right)^{-1/2} (-dt^2 + d\vec{x}^2) + \left(1 + \frac{R^4}{r^4}\right)^{1/2} (dr^2 + r^2 d\Omega_5^2) \quad (27)$$

where  $0 < r < \infty$  with a horizon at  $r = 0$ . In the near-horizon region the metric takes the simpler form

$$ds^2 = \frac{R^2}{z^2} (-dt^2 + d\vec{x}^2 + dz^2) + R^2 d\Omega_5^2 \quad (28)$$

which is an  $AdS_5 \times S_5$  geometry, where  $z = R^2/r$ .

In this study of absorption cross section of particles on D3 branes of [29] a transmitted wave at  $z \rightarrow \infty$  is associated with particles falling into the brane. The flux of particles at infinity is related to the probability of absorption by the horizon. It is important to realise the difference between this physical picture and the one considered here. This way we will understand the difference of the field spectrum in the two cases.

## 5.2 Boundary conditions

Let us see what is precisely the physical picture that we are considering here and what is the physical reason that leads to a discrete field spectrum. We are considering, like the approach of [6], a realization of AdS/CFT correspondence defined in just a purely AdS space (or more precisely, its covering space). That means: there is nothing to absorb particles, or energy, at infinity. The definition of a consistent quantum field theory in AdS space requires adding a boundary at infinity where one imposes a vanishing flux of energy. This is a necessary condition in order to have a well posed Cauchy problem with a unique solution[2, 3]. Thus one actually needs a compactified version of the space, as we discussed in section 2.

In a simple way, we can think that considering just the AdS space, as we are doing here, there should be nothing to absorb particles, or energy at spatial infinity. Thus, considering that massless particles would go to spatial infinity in finite times, one needs to incorporate the idea of [2, 3] of "closing" (compactifying) the AdS space in the Poincare coordinate framework. The spectrum associated with a compactified coordinate clearly has to be discrete, but the way one realizes this in the case of the axial Poincare coordinate  $z$  is

subtle. A compactification of this coordinate is not possible by using just one Poincare coordinate chart because one needs to add an extra point at infinity. As discussed in section 4 we need to stop the coordinate  $z$  at some value  $R$  and map the rest of the space in a second coordinate chart. It is important to stress that we are not imposing any special kind of boundary condition at  $z = R$ . However the fact that we have to cut the coordinate  $z$  at some finite value (or equivalently the fact the manifold is compact in this direction) implies a discretization of the corresponding spectrum.

This difference in the physical content explains why do we find a field spectrum that is discrete in contrast to the case of absorption by branes where it is continuous. The important feature in this compactification that we are considering is that it makes it possible to map degrees of freedom of bulk and boundary theories holographically[12].

## 6 Quantized fields in the Randall Sundrum scenario

Now we will discuss the implications of the results of section 4 about quantum fields in AdS space for the Randall Sundrum model. First, we can change from the variable  $z$  to  $y$  defined by  $z = \Lambda \exp\{y/\Lambda\}$ . The metric (5) then takes the form

$$ds^2 = e^{-2ky} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \quad (29)$$

where  $k = 1/\Lambda$ .

The Randall-Sundrum model[8, 9] corresponds to two 3-branes located at  $y = 0$  and  $y = r_c$ , respectively, and the slice of  $AdS_5$  space between them (or also copies of it) with the background metric (29), plus metric fluctuations. In this scenario the standard model fields live in the main brane ( $y = 0$ ) while gravity propagates in all dimensions including the fifth. Goldberger and Wise [13] studied the quantization of scalar fields in this model (in the AdS slice between the two branes).

In order to show the relation between the discussion on quantum fields in AdS space of section 4 and the proposal of Goldberger and Wise we can choose  $\delta = \Lambda$ , i. e., now the main brane is sitting on the beginning of one Poincare chart and the other brane is located at the end of the same chart ( $R = \Lambda \exp\{r_c/\Lambda\}$ ). The presence of the 3-branes in the Randall-Sundrum model leads to boundary conditions to be imposed on the bulk



fields corresponding in our eqs. (21),(22) to  $A=C=0$ . In this case one obtains solutions of the form of eq. (20), given in section 4, but now the condition (23) that determines the eigenvalues  $u_p$  is replaced by (in terms of the  $z$  variable)

$$\begin{aligned} & \left(2J_\nu(u_p R) + u_p R J'_\nu(u_p R)\right) \left(2Y_\nu(u_p \Lambda) + u_p \Lambda Y'_\nu(u_p \Lambda)\right) \\ & - \left(2Y_\nu(u_p R) + u_p R Y'_\nu(u_p R)\right) \left(2J_\nu(u_p \Lambda) + u_p \Lambda J'_\nu(u_p \Lambda)\right) = 0. \end{aligned} \quad (30)$$

One could then repeat the discussion of section 4 following eq.(23) with the above condition. This would give the solutions found in [13] written in terms of the coordinate  $z$  and our parameters  $R$  and  $\Lambda$ .

Once established this equivalence we can see what does our results on the field spectrum teaches us about the behavior of the model when the second brane goes to infinity. The main brane accommodates the standard model fields and the observable physics lives there. So one can think that we observe a projection of the extra dimension. The existence of the second brane defines a compact AdS slice and this implies a discrete spectrum of bulk fields. As explained in [13] this discretization makes a bulk scalar field looks like a tower of scalars for an observer on the brane in a Kaluza Klein compactification mechanism. However, if we take the coordinate  $z$  (or  $y$ ) as of infinite range, removing the second brane, the spectrum would still be discrete. This happens because, as studied in the previous chapters, in this case we must use more than one coordinate chart.

## 7 Concluding Remarks

We have seen that when we use Poincare coordinates to describe quantum fields in AdS space we must be careful about the fact that one can not map the whole compactified space into just one chart. This explains why the field spectrum in Poincare coordinates is also discrete despite the infinite coordinate ranges. This discretization of the spectrum makes it possible to define a one to one correspondence between the degrees of freedom of the bulk and the boundary. We can understand this if we realize for example that the phase space of the  $AdS_5$  will correspond to a series of tridimensional hypersurfaces that can be mapped into the tridimensional phase space of fields on the boundary. If the spectrum were not discrete, the bulk phase space would correspond to a four dimensional

manifold that would not map into the boundary phase space, violating the holographic principle.

As a remark, we note that we have only discussed the problem of the compactification of the axial coordinate  $z$  because, as we saw here, it is essentially related to the possibility of mapping bulk and boundary degrees of freedom. However, if we want a complete one to one mapping among Poincare coordinates and global coordinates of the compactified AdS space we should also consider the compactification of the  $x^i$  coordinates as well. This happens because given some fixed finite  $z$  and taking the limit  $x^i \rightarrow \infty$  (or  $t \rightarrow \infty$ ) we also reach the boundary. This would require introducing more charts and imply that the field spectrum would be completely discrete.

Regarding the Randall Sundrum model we have seen that even if we remove the second brane we should introduce a second coordinate chart which still implies a discrete field spectrum in the axial direction. Then, for an observer in the main brane the bulk field would still effectively be represented as a tower of fields. This mechanism can be though as a realization of the holographic principle in the Randall Sundrum model.

In conclusion regarding  $z$  as a coordinate associated with the renormalization group scale[10, 16] we can think that the choice of the end of the Poincare chart could correspond to the introduction of an energy scale.

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