Free Field Equations For Space-Time Algebras With Tensorial Momentum

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Abstract

Free field equations, with various spins, for space-time algebras with second-rank tensor (instead of usual vector) momentum are constructed. Similar algebras are appearing in superstring/M theories. The most attention is payed to the gauge invariance properties, particularly the spin two equations with gauge invariance are constructed for dimensions 2+2 and 2+4 and connection to Einstein equation and diffeomorphism invariance is established.

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1 Introduction

We will consider unitary field theories with space-time symmetry algebra, differing from usual Poincare algebra and consisting from two sets of generators $M_{\mu\nu}$ and $Z_{\mu\nu}$ combined into semidirect product of $so(p,q)$ ($M_{\mu\nu}$) and an Abelian group of second-rank antisymmetric tensors $Z_{\mu\nu}$. Thus the algebra is:

$$\begin{array}{l} [M_{\mu\nu}, M_{\lambda\sigma}] = \eta_{\mu\lambda} M_{\nu\sigma} - \eta_{\nu\lambda} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\lambda} - \eta_{\mu\sigma} M_{\nu\lambda} \\ [M_{\mu\nu}, Z_{\lambda\sigma}] = \eta_{\mu\lambda} Z_{\nu\sigma} - \eta_{\nu\lambda} Z_{\mu\sigma} + \eta_{\nu\sigma} Z_{\mu\lambda} - \eta_{\mu\sigma} Z_{\nu\lambda} \\ [Z_{\mu\nu}, Z_{\lambda\sigma}] = 0 \end{array} \tag{1}$$

where the signature is $\eta_{\mu\nu} = (+,..+,-,-,-...)$ and indices are running over $d = p + q$ time+space values. This algebra differs from Poincare one in two respects: first, role of translation generators are now played by second-rank tensor, not vector, and, second, signature of metric is different. Actually we shall concentrate on a specific values of p, q, (p=2, particularly) but some considerations will be applicable to more general cases. This algebra is inspired by development of superstring/M theory. In the supersymmetry algebras of that theories [1] p=1, q=10,9,... and tensorial charges (branes charges) appear, particularly $Z_{\mu\nu}$, as well as other tensors. The most general is the M-theory supersymmetry algebra at d=11, with the anticommutator of supercharges equal to

$$\begin{aligned} \left\{\bar{Q}, Q\right\} &= \Gamma^i P_i + \Gamma^{ij} Z_{ij} + \Gamma^{ijklm} Z_{ijklm}, \\ i, j, ... &= 0, 1, 2, ..10 \end{aligned} \tag{2}$$

so our algebra is particular case when only $Z_{\mu\nu} \neq 0$, and $p = 1, q = 10$. Energy-momentum P_μ naturally disappeared the other case, with p=2, q=10. This algebra appeared first time in [2] as a possible generalization of space-time symmetry algebra of M theory. Namely, interpreting Majorana spinor Q as a Majorana-Weyl with respect to 2+10 dimensional Lorentz group, combining P_i and Z_{ij} into one 12d tensor $P_{\mu\nu}$, and interpreting Z_{ijklm} as a self-dual 12d sixth-rank antisymmetric tensor, we obtain:

$$\begin{aligned} \left\{\bar{Q}, Q\right\} &= \Gamma^{\mu\nu} P_{\mu\nu} + \Gamma^{\mu\nu\lambda\rho\sigma\delta} Z^+_{\mu\nu\lambda\rho\sigma\delta} \\ \mu\nu, ... &= 0', 0, 1, ...10 \end{aligned} \tag{3}$$

with (1) as a bosonic subalgebra of corresponding superalgebra. Let's stress that it is not possible to add a vector P_μ in the r.h.s., due to the properties of gamma-matrixes. So, the study of algebra (1) can be useful, in different ways of study of superstring/M theory. The dimensions higher than 11 and different supersymmetry algebras appeared many times in the study of M-theory. Papers [4] represent some of that investigations.

In this paper we shall restrict ourselves mainly to purely bosonic case (1), without fermionic part (3).

Our aim is, taking seriously algebra (1), to construct an interacting field theories with such a symmetry algebras. Like for standard Poincare algebra, the first step to interacting field theories may be the construction of relativistic field equations. This is another way of the description of the unitary representations of the algebra, which is adapted for the further construction of the invariant interacting theories, since all elements in that field equations are covariant. As is well-known, one of the main features for such equations is the gauge invariance, which is responsible for removal of superfluous fields components. In previous paper [3] we constructed field equations for scalar, spinor and vector fields. Particularly, it was found the adequate definition of gauge invariance. In this paper we shall consider mainly the free spin two field. In Sect. 2 we recall some facts from the [3] on the field equations and gauge invariance for the algebra (1). In Sect.3 the gauge-invariant gravitino equation in dimension 2+2 is presented. Sections 4, 5 and 6 are devoted to the free gravity equations in 2+q dimensions, and their gauge invariance properties in dimensions 2+4 and 2+6. Conclusion contains discussion of the possible development of present theory.

2 Basic elements of d=2+q theory

The important feature of (1) is the presence of many invariants (Casimir's operators), constructed from $P_{\mu\nu}$:

$$\begin{aligned} TrP^2 &= P_{\mu\nu}P^{\nu\mu}, \\ TrP^4 &= P_{\mu\nu}P^{\nu\lambda}P_{\lambda\rho}P^{\rho\mu}, \\ &\ldots \end{aligned} \tag{4}$$

instead of the single one in algebras with momentum only: $P^2 = P_iP^i$. This means, that now, even in the simplest case of scalar theory,we have to introduce few equations of motion instead of the one Klein-Gordon:

$$\begin{aligned} (TrP^2 - 2m_1^2)\Phi(P_{\mu\nu}) &= 0, \\ (TrP^4 - 2m_2^4)\Phi(P_{\mu\nu}) &= 0, \\ &\ldots \end{aligned} \tag{5}$$

We will call these equations respectively "first level", "second level", etc. Evidently, in these equations Φ is the scalar function, Fourier transform of which depends on a coordinates $x^{\mu\nu}$ conjugate to momenta $P_{\mu\nu}$. So, these equations, as well as others below, are differential equations in the $d(d-1)/2$ dimensional "space-time" with coordinates $x^{\mu\nu}$.

According to the Wigner's little group's method of construction of unitary representations of such a semidirect products, we have to choose an orbit (5) of a Lorentz group in the space $P_{\mu\nu}$, take some arbitrary point on that orbit and then find a stabilizer of that point (=little group). The unitary representations of little group are giving rise to the unitary representations of initial group, through the procedure of induction. The special orbit is one with all $m_i = 0$. For p=2, q=10 this orbit corresponds to the massless multiplet of d=11 supergravity. The possible choice of point on this orbit is:

$$
\begin{aligned}
P_{\mu\nu} &= (P_{0'i}, P_{ij} = 0), && (6)\\
P^2 &= P_{0'i}P^{0'i} = 0, && (7)\\
P^{0'i} &= (1, 1, 0, ..., 0) && (8)
\end{aligned}
$$

The Lie algebra of little group of this orbit can be found as follows. Consider all elements of $so(2,q)$ algebra, which leave the tensor (5) unchanged. Actually (5) itself, with raised second index is an element of $so(2,q)$, so we are seeking its stabilizer in this algebra. It is easy to show, that the following matrices of $so(2,q)$ are exactly all matrices, commuting with (5):

$$
\begin{pmatrix}
0 & a & a & 0 & 0 & ... & 0 \\
-a & 0 & 0 & b & c & ... & d \\
a & 0 & 0 & -b & -c & ... & -d \\
0 & b & b & 0 & e & ... & f \\
0 & c & c & -e & 0 & ... & g \\
... & ... & ... & ... & ... & ... & ... \\
0 & d & d & -f & -g & ... & 0
\end{pmatrix} \qquad (9)
$$

The algebra of matrixes (9) is a direct sum of the $so(1,1)$ algebra of matrices (9) with only non-zero entry a, and an algebra, which is a semidirect sum of $so(d-3)$ (represented by matrixes (9) with $a = b = c = ... = d = 0$) and an Abelian algebra of matrices (9) with non-zero elements $b, c, ..., d$ only. The unitary finite-dimensional representations of this little group algebra are those of $so(d-3)$ subalgebra, with other generators represented by zero, which is possible due to the structure of the algebra.

The role of equations (5) is to bring the general function of the momenta $P_{\mu\nu}$ to a function on the orbit. For non-trivial representations of little group the relativistic fields contain additional components, which can be removed by gauge transformations. The notion of gauge symmetry generalizes in this case, as shown in [3], to the single equation that a sum of variations of these actions is equal to zero.

The first example of fields with gauge invariance is the vector field theory. In momentum representation, introducing the vector field $A_\mu(P_{\lambda\nu})$ and the "field strength"

$$F_{\mu\nu\lambda} = P_{\mu\nu}A_\lambda + P_{\nu\lambda}A_\mu + P_{\lambda\mu}A_\nu \tag{10}$$

we can define the following set of equations of motion:

$$\begin{aligned} P^{\mu\nu}F_{\mu\nu\lambda} &= 0 && (11)\\ (P^3)^{\mu\nu}F_{\mu\nu\lambda} &= 0 && (12)\\ &... && (13)\\ (P^{d-1})^{\mu\nu}F_{\mu\nu\lambda} &= 0 && (14) \end{aligned}$$

An equivalent set of equations is:

$$\begin{aligned} P^{\mu\nu}F_{\mu\nu\lambda} &= 2\left(P^{2\mu}_\lambda - \frac{1}{2}TrP^2\delta^\mu_\lambda\right)A_\mu = 0 && (15)\\ G_{2i}(P)A_\mu &= 0 \quad i = 1,2,3,... && (16) \end{aligned}$$

Here G_i are combinations of invariants (5) coinciding with the coefficients in the expansion of the characteristic polynomial for matrix $P_\mu{}^\nu$:

$$\begin{aligned} f(x) &= \det(P - x) && (17)\\ &= x^d + x^{d-2}G_2 + x^{d-4}G_4 + ... && (18)\\ G_2 &= -\frac{1}{2}TrP^2 && (19) \end{aligned}$$

The form of G_i is independent of dimensionality d. Again this set is equivalent to the ordinary Maxwell system under the condition $P_{ij} = 0$ and therefore reproduces the masslessness condition $P^2 = 0$.

We can define the set of actions corresponding to each set of equations. For (15), (16) these are:

$$\begin{aligned} S_1 &= \int [dX^{\mu\nu}]\left(-\frac{1}{6}F^{\mu\nu\lambda}F_{\mu\nu\lambda}\right), && (20)\\ S_i &= k_{2i}\int [dX^{\mu\nu}]\left(\frac{1}{2}A_\lambda G_{2i}(\partial_{X^{\mu\nu}})A^\lambda\right), && (21)\\ i &= 1,2,.. \end{aligned}$$

Here we introduced coupling constants k_{2i} with appropriate dimensionality. For study of gauge invariance properties of this theory, let's define the following variation of field A_μ (take d=2+10, for definiteness):

$$\delta A_\mu(P_{\lambda\rho}) = P_{\mu\nu}\alpha^\nu \delta(G_2(P))...\delta(G_6(P)) \tag{22}$$

Eqs. of motion (16) are evidently invariant with respect to this transformation, the (15) is also invariant, because delta-functions in (22) put $P_{\mu\nu}$ into the form (5), and (15) produces usual Maxwell equation with usual gauge transformation (22). It is easy to see, that gauge transformation (22) permits one to gauge away $A_{0'}$ component and longitudinal part of A_i (taking into account that A_μ is non-zero only on the shell of delta-functions $\delta(G_2(P))$, ...,$\delta(G_6(P))$):

$$\begin{aligned} \delta A_{0'}(P_{0'i}) &= P_{0'\nu}\alpha^\nu(P_{0'i}) = \beta && (23) \\ \delta A_i(P_{0'i}) &= P_{i\nu}\alpha^\nu(P_{0'i}) = P_{i0'}\gamma && (24) \end{aligned}$$

So, these equations describe the vector representation of SO(9) group, as desired. Note that actions (20), (21) are invariant, also.

Another way of realization of the same idea of gauge invariance is the following. We can define the following set of variations corresponding to the each equation of the set:

$$\begin{aligned} \delta_1 A_\mu(P_{\lambda\rho}) &= P^{10}_{\mu\nu}\alpha^\nu(P_{\lambda\rho}) \\ \delta_2 A_\mu(P_{\lambda\rho}) &= k_2^{-1}P^{8}_{\mu\nu}\alpha^\nu(P_{\lambda\rho}) \\ &. \\ &. \\ \delta_6 A_\mu(P_{\lambda\rho}) &= k_6^{-1}\alpha_\mu(P_{\lambda\rho}) \end{aligned} \tag{25}$$

Then it is easy to see that the following equation is satisfied:

$$\delta_1 S_1 + \delta_2 S_2 + ... + \delta_6 S_6 = 0 \tag{26}$$

due to the well-known Hamilton-Cayley identity for characteristic polynomials:

$$f(P) = 0 \tag{27}$$

where

$$f(x) = \det(P - x)$$

As above, assuming that higher eqns. $\delta S_i/A_\mu$ $(i = 2, ..., 6)$ are satisfied, eqn. (26) gives the usual statement of gauge invariance of Maxwell's action. (The equation similar to (26) is valid also for other sets of actions, with different variations $\delta_i A_\mu$.) Then, under the condition $P_{ij}=0$ the remaining symmetry is:

$$\delta A_{0'}(P_{0'i}) = P^{10}_{0'\nu}\alpha^\nu(P_{0'i}) = \beta \tag{28}$$
$$\delta A_i(P_{0'i}) = P^{10}_{i\nu}\alpha^\nu(P_{0'i}) = P_{i0'}\gamma \tag{29}$$

The first one can be used for gauging away the additional twelfth component $A_{0'}$, the second one gives the usual gauge transformation for the remaining $11d$ Abelian gauge field. The subtlety in (28), (29) is that: if we consider the on-shell condition $P^2=0$ from the beginning, it is not possible to gauge away $A_{0'}$, as is seen from (28).

Let's finish this section with few remarks. First one is that there is another possibility for gauge-invariant actions. The last equation $\det(p)=0$ in (16) can be replaced by equation $Pfaff(p)h=0$, where $Pfaff(p) = \epsilon_{\mu\nu\ldots\alpha\beta}p^{\mu\nu}\ldots p^{\alpha\beta}$. It is possible to find corresponding variations of fields in that set of equations such that Eq.(26) again will be satisfied.

Next, Eqn. (26) in this form is appropriate for generalization to non-quadratic Lagrangians, but we are not aware on any general considerations proving that such equations are direct consequence of a multilagrangian nature of the theory. Moreover, taking into account that the main sense of the equation (26) is that when fields are satisfying the second and higher level equations of motion, then (26) reduces to standard equation of symmetry of first action, one can suggest other, nonlinear relations between variations of actions, such that variation of first one is zero when others are zero. E.g. $\delta_1 S_1 = \sqrt{(\delta_2 S_2)}+\ldots$. In this expression parameters of infinitesimal variations are implied to be removed from variations.

3 Free gravitino field.

Next field with gauge invariance can be the gravitino field. Equations are given in [3]. We shall consider that in dimension $d=(2+2)$:

$$\gamma^{\mu\nu\lambda\rho}\partial_{\nu\lambda}\psi_\rho = 0 \tag{30}$$
$$Pfaff(p)\psi_\mu = 0 \tag{31}$$

Gauge transformations parameter, as in the vector case, has one more vector index in comparison with case of usual Rarita-Shwinger field, i.e. is spin-vector:

$$\delta_1\psi_\rho = \partial_{\rho\nu}\varepsilon_\nu \tag{32}$$
$$\delta_2\psi_\nu \backsim \varepsilon_\nu \tag{33}$$

Check of gauge invariance (26):

$$\delta_1(\gamma^{\mu\nu\lambda\rho}\partial_{\nu\lambda}\psi_\rho) = \gamma^5\widetilde{p}^{\mu\rho}p_{\rho\upsilon}\varepsilon_\nu \backsim Pfaff(p)\varepsilon_\mu = \delta_2(Pfaff(p)\psi_\mu) \tag{34}$$

After reduction, gauge transformations give the gauge invariances of reduced equation ((1+2) gravitino equation)

$$\delta_1\psi_i = -p_i\varepsilon_{0'} \tag{35}$$

which coincide with gravitino gauge transformations. Remaining part of gauge invariance

$$\delta_1\psi_{0'} = p_k\varepsilon_k \tag{36}$$

permits one to gauge away $\psi_{0'}$.

4 Free Gravity

We suggested in [3] the following quadratic first level Lagrangian for the symmetric tensor field $h_{\mu\nu}$:

$$\begin{array}{l} L_1 = -\frac{1}{4}h_{\mu\nu}(\partial_{\lambda\rho}\partial^{\rho\lambda})h^{\mu\nu} - \frac{1}{2}h_{\mu\nu}(\partial^{\mu\lambda}\partial^{\nu\rho})h_{\lambda\rho} \\ +h_{\mu\nu}(\partial^{\mu\lambda}\partial_{\lambda\rho})h^{\rho\nu} - h^\nu_\mu(\partial^{\mu\lambda}\partial_{\lambda\nu})h + \frac{1}{4}h(\partial^{\mu\nu}\partial_{\nu\mu})h \end{array} \tag{37}$$

This expression is unique among those of second order over derivatives and over field $h_{\mu\nu}$, which goes into quadratic part of General Relativity Lagrangian after reduction (6). That contains no additional degrees of freedom. More exactly, reduction means that fields are independent of coordinates x^{ij}, and we shall use notations:

$$\mu = (0', i) \tag{38}$$
$$h_{0'0'} = a, \tag{39}$$
$$h_{0'i} = b_i, \tag{40}$$
$$h^\mu_\mu = h, \tag{41}$$
$$\partial_i = \partial_{0'i} \tag{42}$$
$$h^i_i = u, \tag{43}$$
$$h = a + u \tag{44}$$

After reduction L_2 goes into

$$\widetilde{L}_1 = -\partial_i h_{ij}\partial_j u + (1/2)\partial_i u\partial_i u + \partial_i h_{ij}\partial_k h_{kj} - (1/2)\partial_k h_{ij}\partial_k h_{ij} \tag{45}$$

which is quadratic part of Einstein-Gilbert Lagrangian. Note that components $h_{0'0'} = a, h_{0'i} = b_i$ disappeared.

Corresponding to (45) equation of motion is (in momentum representation):

$$E_{\mu\nu} - \frac{1}{3}E^{\lambda}_{\lambda}\eta_{\mu\nu} = 0 \tag{46}$$

$$E_{\mu\nu} = (-1/2)p_{\alpha\beta}p^{\beta\alpha}h_{\mu\nu} + (php)_{\mu\nu} + (p^2h + hp^2)_{\mu\nu} - (p^2)_{\mu\nu}h^{\lambda}_{\lambda} \tag{47}$$

So, at $d = 3$ equation is trivial, which is in agreement with desired correspondence of this theory with trivial $d = 2$ Einstein gravity. Reduction of Eq.(46) gives the linearized Einstein equation

$$p^2h_{ij} + p_ip_jh^k_k - p_ip_kh^k_j - p_jp_kh^k_i = 0 \tag{48}$$

and nothing else, in agreement with reduction of Lagrangian (37). It is reasonable to assume, that (37) has a gauge invariance, which after reduction removes a and b_i and becomes a usual diffeomorphism invariance of (45) and (48). That is discussed in next Sections.

5 Gauge Invariance of Free Gravity at d=2+2

For this dimensionality we have to add only one, second level, Lagrangian to (37):

$$S_2 = \int L_2 = \frac{1}{2}\int h_{\mu\nu}Pfaff(\partial)h_{\mu\nu} \tag{49}$$

$$\frac{\delta S_2}{\delta h_{\mu\nu}} = K^{\mu\nu} = Pfaff(p)h^{\mu\nu} = 0 \tag{50}$$

where

$$\begin{aligned} Pfaff(p) &= p_{\mu\nu}p_{\lambda\sigma}\varepsilon^{\mu\nu\lambda\sigma} = \widetilde{p}_{\mu\nu}p^{\mu\nu} && (51)\\ \widetilde{p}_{\mu\nu} &= \varepsilon_{\mu\nu\lambda\sigma}p^{\lambda\sigma} && (52)\\ \det(p) &= (1/64)(Pfaff(p))^2 && (53) \end{aligned}$$

This theory has the following two gauge invariances

$$\delta_1 S_1 + \delta_2 S_2 = 0 \tag{54}$$

with

$$\delta_1 h_{\mu\nu} = (p^2\xi + \xi p^2)_{\mu\nu} \tag{55}$$
$$\delta_2 h_{\mu\nu} = \frac{1}{4}(\widetilde{p}\xi p + p\xi\widetilde{p})_{\mu\nu} \tag{56}$$

and

$$\delta_1 h_{\mu\nu} = (p\sigma p)_{\mu\nu} \tag{57}$$
$$\delta_2 h_{\mu\nu} = -\frac{1}{64}\sigma_{\mu\nu} Pfaff(p) \tag{58}$$

where $\xi_{\mu\nu}$ and $\sigma_{\mu\nu}$ are symmetric tensors, parameters of transformation. The reduction of these gauge invariances gives the standard gauge invariance of $\widetilde{L}_1$ and possibility to gauge away superfluous components of $h_{\mu\nu}$:

$$\delta h_{0'0'} = -2p^2\xi_{0'0'} - p^i p^j \sigma_{ij} \tag{59}$$
$$\delta h_{0'k} = -p^2\xi_{0'k} - p_k(p^l\xi_{0'l}) + p_k(p^l\sigma_{0'l}) \tag{60}$$
$$\delta h_{ij} = -p_i p^k \xi_{kj} - p_j p^k \xi_{ki} - p_i p_j \sigma_{0'0'} \tag{61}$$

6 d=2+4 Free Gravity

Lagrangian and equation of motion for field $h_{\mu\nu}$, which reduces to Einstein ones are given above - (37)(46). Next level Lagrangians and equations, in analogy with above, can be assumed as:

$$G_{\mu\nu} = G_4 h_{\mu\nu} = 0 \tag{62}$$
$$K_{\mu\nu} = Pfaff(p) h_{\mu\nu} = 0 \tag{63}$$

The problem is in gauge invariance. It should be of the following form:

$$\delta_1 S_1 + \delta_2 S_2 + \delta_3 S_3 = 0 \tag{64}$$
$$\delta_1 E_{\mu\nu} + \delta_2 G_{\mu\nu} + \delta_3 K_{\mu\nu} = 0 \tag{65}$$

with some variations δ_i, $i = 1, 2, 3$ of field $h_{\mu\nu}$.

We have an analog of (57):

$$\delta_1 h = p^2\xi p^2 - \frac{1}{3}p^2 tr(p^2\xi) \tag{66}$$

$$\delta_2 h = -p\xi p - \frac{1}{3}tr(p^2\xi) \tag{67}$$

$$\delta_3 h = \frac{1}{384}(p\xi q + q\xi p) - \frac{1}{384}Pfaff(p)(\xi) \tag{68}$$

where

$$q_{\mu\nu} = \epsilon_{\mu\nu\lambda\sigma\rho\tau}p^{\lambda\sigma}p^{\rho\tau} \tag{69}$$

After reduction this invariance gives the following invariance of Einstein equation:

$$\delta h_{ij} \backsim p_i p_j \rho \tag{70}$$

where ρ is constructed from p and ξ. This is only a part of diffeomorphism invariance of Einstein equation.

Simultaneously, we argue, that there exist gauge invariance of the first kind (analog of (55)), which is already sufficient for all purposes on removing the superfluous components, and which transforms into the full diffeomorphism invariance of Einstein equation. This can be seen from the following. Let's assume the "diagonalized" form of matrix p:

$$p = \begin{matrix} 0 & a & 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b & 0 & 0 \\ 0 & 0 & -b & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 & -c & 0 \end{matrix}$$

The assumed gauge transformation is

$$\delta h = p^4\xi + \xi p^4 \tag{71}$$

where $\xi_{\mu\nu}$ is symmetric second rank tensor. Variation of equation of motion (46) gives (we present only first column of equation, due to lack of space):

$$\begin{matrix} 2b^2a^4\xi_{11} + 2c^2a^4\xi_{11} + 2a^2b^4\xi_{33} + 2a^2b^4\xi_{44} + 2c^4\xi_{55}a^2 + 2a^2c^4\xi_{66} \\ 2a^4\xi_{12}b^2 + 2a^4\xi_{12}c^2 \\ -a^5b\xi_{24} - ab^5\xi_{24} + c^2a^4\xi_{13} + c^2b^4\xi_{13} \\ a^5b\xi_{23} + ab^5\xi_{23} + c^2a^4\xi_{14} + c^2b^4\xi_{14} \\ -a^5c\xi_{26} - ac^5\xi_{26} + b^2a^4\xi_{15} + b^2c^4\xi_{15} \\ a^5c\xi_{25} + ac^5\xi_{25} + b^2a^4\xi_{16} + b^2c^4\xi_{16} \end{matrix} \tag{72}$$

As is easily seen, this expression is equal to zero when two from three "eigenvalues" a, b or c are equal to zero. This last condition follows from equations (62), (63). So, the variation of Lagrangian (37) is zero when variations of two others are zero. Unfortunately, we didn't find such a relation. The problem can be formulated as follows: 6d relation has to pass into 4d relation when all objects are restricted to 4d, tentative 6d relation has a G_4 which goes into $det_4(p)$ when restricted to surface $det_6(p) = 0$, but in 4d gauge invariance relation (54) enters Pfaffian, not determinant.

The solution we are suggesting for this problem is the following: we change the second level action S_2, writing instead

$$\begin{array}{l} \mathrm{S}_2 = \int (dx)(c(h_{\mu\nu}\partial^4_{\nu\lambda}h_{\lambda\mu} - \frac{1}{2}h_{\mu\lambda}\partial^2\partial^2_{\mu\nu}h_{\nu\lambda}))+ \\ h_{\mu\sigma}\partial_{\mu\nu}\partial^3_{\lambda\sigma}h_{\nu\lambda} - (h_{\mu\sigma}\partial^2\partial_{\mu\nu}\partial_{\lambda\sigma}h_{\nu\lambda}) + (1+c)hG_4h) \end{array} \tag{73}$$

And corresponding equation in momentum representation:

$$\begin{array}{l} c(p^4h + hp^4 - \frac{1}{2}Sp(p^2)(p^2h + hp^2))+ \\ (p^3hp + php^3 - Sp(p^2)php) + (1+c)G_4h = 0 \end{array} \tag{74}$$

where $c = 1 \pm \sqrt{2}$. This equation itself is a consequence of (62), (63), and vice versa: from (73) together with (63) follows that two from three eigenvalues a, b, c are equal to zero. Namely, as shown below in (85), (73) goes into combination of Pfaffian and determinant terms under reduction.

Now we have desired gauge transformation:

$$\delta_1 h_{\mu\nu} = (\partial^4\xi)_{\mu\nu} + (\mu \leftrightarrow \nu) - \frac{2}{3}\partial^2_{\mu\nu}\partial^2_{\alpha\beta}\xi_{\alpha\beta} \tag{75}$$

$$\delta_2 h_{\mu\nu} = (\partial^2\xi)_{\mu\nu} + (\mu \leftrightarrow \nu) + (1-c)\partial_{\mu\sigma}\partial_{\nu\lambda}\xi_{\sigma\lambda} \tag{76}$$

$$\begin{aligned} \delta_3 h_{\mu\nu} &= \frac{1}{48}(1+c)Pfaff(\partial)(q_{\lambda\nu}\partial_{\mu\sigma}\xi_{\sigma\lambda} + (\mu \leftrightarrow \nu)) \\ &+ det(\partial)(2c\xi + 2Sp(\xi))_{\mu\nu} \end{aligned} \tag{77}$$

In momentum representation:

$$\delta_1 h_{\mu\nu} = (p^4\xi + \xi p^4)_{\mu\nu} - \frac{2}{3}p^2_{\mu\nu}Sp(p^2\xi) \tag{78}$$

$$\delta_2 h_{\mu\nu} = (p^2\xi + \xi p^2)_{\mu\nu} + (1-c)(p\xi p)_{\mu\nu} \tag{79}$$

$$\delta_3 h_{\mu\nu} = \frac{1}{48}(1+c)(q\xi p + p\xi q)_{\mu\nu} + \frac{1}{48}Pfaff(p)(2c\xi + 2Sp(\xi))_{\mu\nu} \tag{80}$$

where

$$q_{\mu\nu} = \frac{1}{8}\varepsilon_{\mu\nu\lambda\rho\sigma\tau}p^{\lambda\rho}p^{\sigma\tau} \tag{81}$$

The gauge transformation (66) doesn't disappear in this new set of equations, instead we have invariance with respect to:

$$\delta_1 h_{\mu\nu} = (p^3hp + php^3)_{\mu\nu} + (c-1)(p^2hp^2)_{\mu\nu} - \tfrac{1}{3}(1+c)p^2_{\mu\nu}Sp(p^2h) \quad (82)$$
$$\delta_2 h_{\mu\nu} = c(php)_{\mu\nu} \quad (83)$$
$$\delta_3 h_{\mu\nu} = -\tfrac{1}{48}Pfaff(p)(1+c)(h-Sp(h))_{\mu\nu} \quad (84)$$

These gauge transformations, as in d=2+2, give after reduction the five-dimensional diffeomorphism transformation, as well as possibility to exclude the superfluous components of tensor $h_{\mu\nu}$. The other check is that when reducing to d=2+2 we obtain some combination of invariances (55) (54). Exactly, equation (73) under that reduction goes into combination of Pfaffian and determinant terms (instead of pure Pfaffian in (49)):

$$\begin{aligned} &(p^3hp + php^3 - Sp(p^2)php) + (c-1)det(p)h = \\ &\tfrac{1}{8}Pfaff(p)(phq+qhp) + (c-1)det(p)h = 0 \end{aligned} \quad (85)$$

Evidently, if we choose this equation in 2+2 instead of pure Pfaffian term, only some combination of (54),(55) will survive. The question is whether the interaction will prefer this combination (if any).

7 Conclusion

Generalization of previous results to arbitrary dimension should be possible. Another direction of development is consideration of other orbit of algebra, particularly that corresponding to membrane. Membrane corresponds, from the point of view of algebra (2), to orbit with e.g. $P_{0'0} = m$, $P_{12} = m$ [1]. Invariants are equal to:

$$\begin{aligned} TrP^2 &= -4m^2 \\ TrP^4 &= 4m^4 \\ TrP^6 &= -4m^6 \\ &\dots. \end{aligned} \quad (86)$$

The algebra of little group of this orbit in dimension $d = 2 + (d-2)$ is $so(2) + so(2,1) + so(d-4)$. As always, first factor corresponds to matrix $P_{\mu\nu}$ itself. Matrices from the little group algebra have the form (second index raised):

$$
\begin{array}{cccccccc}
0 & p_{12} & p_{13} & p_{14} & 0 & 0 & 0 & 0 \\
-p_{12} & 0 & -p_{14} & p_{13} & 0 & 0 & 0 & 0 \\
p_{13} & -p_{14} & 0 & p_{34} & 0 & 0 & 0 & 0 \\
p_{14} & p_{13} & -p_{34} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -p_{56} & -p_{57} & -p_{58} \\
0 & 0 & 0 & 0 & p_{65} & 0 & -p_{67} & -p_{68} \\
0 & 0 & 0 & 0 & p_{75} & p_{76} & 0 & -p_{78} \\
0 & 0 & 0 & 0 & p_{85} & p_{86} & p_{87} & 0
\end{array} \tag{87}
$$

The right low block of matrix is algebra $so(d-4)$ (drawn for the case $d=8$), the left upper block is the direct sum of $so(2)$ matrix (with $p_{12} = p_{34}$,other elements are zero) , and $so(2,1)$ (with $p_{12} = -p_{34}$).

The simplest unitary representations of the little group is, as usual, the scalar one, when group is represented trivially, then we can consider non-trivial unitary finite representations of factors $so(2)$ and $so(d-4)$. So, for scalar field φ equations of motion are

$$
\begin{aligned}
(TrP^2 + 4m^2)\varphi &= 0 \\
(TrP^4 - 4m^4)\varphi &= 0 \\
(TrP^6 + 4m^6)\varphi &= 0 \\
&\dots.
\end{aligned} \tag{88}
$$

As in above, the other representations can be considered.

Next important problem is construction of interaction terms. It is reasonable to approach this problem by Nether procedure, starting from linear equations and their gauge invariance transformations, then constructing next order term both in equations and in transformations from the requirement of maintaining the gauge invariance property.

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