On exact superpotentials in confining vacua

Frank FERRARI[†]

Institut de Physique, Université de Neuchâtel
rue A.-L. Bréguet 1, CH-2000 Neuchâtel, Switzerland
`frank.ferrari@unine.ch`

We consider the $\mathcal{N} = 1$ super Yang-Mills theory with gauge group $\mathrm{U}(N)$ or $\mathrm{SU}(N)$ and one adjoint Higgs field with an arbitrary polynomial superpotential. We provide a purely field theoretic derivation of the exact effective superpotential $W(S)$ for the glueball superfield $S = -W^{a\alpha}W^a_\alpha/(32N\pi^2)$ in the confining vacua. We show that the result matches with the Dijkgraaf-Vafa matrix model proposal. The proof brings to light a deep relationship between non-renormalization theorems first discussed by Intriligator, Leigh and Seiberg, and the fact that $W(S)$ is given by a sum over planar diagrams.

September 2002

[†]On leave of absence from Centre National de la Recherche Scientifique, Laboratoire de Physique Théorique de l'École Normale Supérieure, Paris, France.

1   Introduction

In a recent insightful paper, Dijkgraaf and Vafa [1] have proposed a very simple recipe to calculate the exact quantum effective superpotential $W(S)$ for the glueball superfield

$$S = -\frac{\operatorname{tr} W^\alpha W_\alpha}{16N\pi^2} = -\frac{W^{a\alpha}W^a_\alpha}{32N\pi^2} \tag{1}$$

in the confining vacua of a large class of $\mathcal{N} = 1$ supersymmetric Yang-Mills theories. The superpotential $W(S)$ contains highly non-trivial information about the non-perturbative dynamics of the theory. For example, it can be used to derive dynamical chiral symmetry breaking and calculate the tension of the associated domain walls. The work of Dijkgraaf and Vafa is the outcome of a long series of work on a large N string theory duality, first proposed in [2] and further developed in [3] and [4]. The result is particularly useful because field theoretic derivations of exact glueball superpotentials in non-trivial cases have not appeared.

The purpose of the present paper is to remedy this situation, by providing a full field theoretic derivation in the archetypal example of the U(N) or SU(N) theory with one adjoint Higgs supermultiplet Φ and a tree level superpotential of the general form

$$W_{\text{tree}} = \sum_{p \geq 1} \frac{g_p}{p} \operatorname{tr} \Phi^p = \sum_{p \geq 1} g_p u_p \,. \tag{2}$$

The lagrangian of the theory is

$$\mathcal{L} = \frac{N}{4\pi} \operatorname{Im} \operatorname{tr} \tau \left[\int \mathrm{d}^2\theta\, W^\alpha W_\alpha + 2 \int \mathrm{d}^2\theta \mathrm{d}^2\bar\theta\, \Phi^\dagger e^{2V} \Phi \right] + 2N \operatorname{Re} \int \mathrm{d}^2\theta\, W_{\text{tree}}(\Phi) \,, \tag{3}$$

where τ is the complexified 't Hooft coupling constant. The classical glueball superpotential can be read off from (3),

$$W_{\text{cl}}(S) = 2i\pi N\tau S \,. \tag{4}$$

We will focus in Section 2 on the field theory calculation yielding the exact quantum superpotential $W(S)$ in the confining vacua where classically $\langle \phi \rangle = 0$. Non-trivial field theoretic results on this theory can be found in [3, 5]. Our main tools are the Seiberg-Witten solution for the $\mathcal{N} = 2$ theory which is obtained by turning off W_{tree} [6, 7], and the Intriligator-Leigh-Seiberg linearity principle [8]. For the sake of clarity, and since this is an important aspect of our work, we give a self-contained account of this principle. We then analyse in Section 3 the Dijkgraaf-Vafa proposal for our model. We use some matrix model technology to put the solution in a simple form. We can then straightforwardly compare the matrix model and the field theory results,

and we find perfect agreement. In Section 4 we recapitulate our findings and add a few comments on future directions of research.

The original motivation for the present work was actually to use the results in [1] to work out some new interesting physics along the lines of preceding papers by the author [9]. These developments will appear in a separate publication [10].

2 Field theoretic analysis

2.1 The Intriligator-Leigh-Seiberg linearity principle

The basic tools to analyse $\mathcal{N} = 1$ supersymmetric gauge theories were provided by Seiberg long ago [11]. An excellent review with a list of references is [12]. A general procedure consists in using holomorphy, symmetries, and various known asymptotics and consistencies to derive the most general form of the quantum 1PI superpotential for a set of fields X_r as a function of the complex mass scale Λ governing the one-loop running of the gauge coupling constant (we limit the discussion for simplicity to the case where only one mass scale is present). In a wealth of examples, particularly when there is no tree level superpotential W_{tree}, those constraints actually fix the superpotential $W_{\text{q}}^0(X_r, \Lambda)$ uniquely. An interesting problem is then to turn on

$$W_{\text{tree}} = \sum_r g_r \tilde{X}_r \,, \tag{5}$$

where the $\tilde{X}_r$ form a subset of the X_r which can be expressed locally in terms of the elementary fields and can thus be included consistently in the bare lagrangian. In some examples, the use of the general constraints may still determine uniquely $W_{\text{q}}(X_r, \Lambda, g_r)$, but usually this is no longer the case. Intriligator, Leigh and Seiberg (ILS for short) then proposed in [8] that the exact superpotential should be a simple linear function in the bare couplings g_r,

$$W_{\text{q}}(X_r, \Lambda, g_r) = W_{\text{q}}^0(X_r, \Lambda) + \sum_r g_r \tilde{X}_r \,. \tag{6}$$

This states that the tree level superpotential for the $\tilde{X}_r$ is not renormalized, neither perturbatively (as is well-known) nor non-perturbatively. Of course, that does not preclude a possible renormalization of the fields and the couplings taken individually. When such a renormalization does not occur perturbatively, we will also assume that it does not non-perturbatively. The non-trivial renormalizations forbidden by the ILS principle are not to be confused with possible vacuum independent ambiguities in the definition of operators, as discussed for example in [13].

A particularly important physical consequence is that, since there is no renormalization at work, the fields $\tilde{X}_r$ can be integrated out without loosing any information. Mathematically, this comes from the linearity in the couplings g_r. This linearity implies that the integrating out can be reversed by a simple Legendre transform, a procedure called "integrating in" [14]. In practice, one can then work with a superpotential for which all the fields have been integrated out,

$$W_{\text{low}}(\Lambda, g_r) = W_{\text{q}}(\langle X_r \rangle, \Lambda, g_r) \,. \tag{7}$$

If necessary, the fields $\tilde{X}_r$ can then be integrated in by using the relation

$$\frac{\partial W_{\text{low}}}{\partial g_r} = \tilde{X}_r \,. \tag{8}$$

To understand more concretely the significance of the ILS hypothesis, let us consider the U(N) theory with lagrangian (3). In that case, a standard set of fields X_r includes the monopole fields M_m and $\tilde{M}_m$, $1 \leq m \leq N - 1$, and the $u_p = \text{tr}\,\phi^p/p$, $1 \leq p \leq N$. The fields $\tilde{X}_r$ correspond to the u_ps only. When $W_{\text{tree}} \to 0$, the effective superpotential is given by [6]

$$W_{\text{q}}^0(\tilde{M}_m M_m, u_p, \Lambda) = \sqrt{2} \sum_{m=1}^{N-1} \tilde{M}_m M_m A_{D,m}(u_p, \Lambda) \,, \tag{9}$$

where the $A_{D,m}$ are dual $\mathcal{N} = 2$ U(1) vector multiplets scalars, which are known functions of the moduli u_p and scale Λ [6, 7]. Let us now add the tree level superpotential (2). The ILS hypothesis implies that the new exact effective superpotential is

$$W_{\text{q}} = \sqrt{2} \sum_{m=1}^{N-1} \tilde{M}_m M_m A_{D,m}(u_p, \Lambda) + \sum_{p \geq 1} g_p u_p \,. \tag{10}$$

Because we have $\mathcal{N} = 2$ supersymmetry when $W_{\text{tree}} = 0$, and with the normalization for the kinetic term of Φ in (3), there is no renormalization for the individual fields u_p. The u_ps appearing in (10) are thus *the same as the UV operators u_p of the $W_{\text{tree}} = 0$ theory.* These facts can actually be proven from symmetry and regularity arguments for a purely quadratic tree level superpotential [6]. However, in general, the same symmetry and regularity arguments are no longer able to fully determine W_{eff}. For example, if g_2 and g_3 are turned on, one can form the parameter $r = g_3^2 \Lambda^2 / g_2^2$ which is neutral under all the symmetries of the problem. Non-trivial renormalizations involving arbitrary functions of this parameter are possible in principle. The ILS hypothesis implies that those renormalizations do not occur.

An important fact is that the gauge kinetic term can be viewed as an F-term with a superpotential given by (4). Of course, this is also a D-term, and is thus renormalized. By working with renormalized matter fields in the path integral measure (the kinetic terms then include the non-trivial wave function renormalization factors Z), the renormalization of the gauge kinetic term is exhausted at one loop,

$$N\tau = \frac{iNb}{\pi} \ln \frac{\Lambda_0}{\Lambda} \, . \tag{11}$$

The scale Λ_0 is the ultraviolet cutoff, b a number of order one given by the one-loop β function ($b = 1$ for (3)), and Λ a complexified mass scale. This is the famous perturbative non-renormalization theorem for the holomorphic Wilson gauge coupling [15]. Extending this result to the non-perturbative realm by applying the ILS principle, we find that the exact 1PI superpotential including the glueball superfield (1) is of the general form

$$W_{1\mathrm{PI}}(X_r, S, \Lambda, g_r) = W^0(X_r, S) + S \ln \Lambda^{2Nb} + \sum_r g_r \tilde{X}_r \, , \tag{12}$$

where we have indicated the dependence in all the variables explicitly. A most crucial point for our purposes is that the "universal," coupling independent, superpotential W^0 can be obtained, without making any further assumption, from W^0_{q} in (6) by integrating in. This is because the full Λ dependence is always taken into account in (6), even though the S field does not appear (technically, there is no W_I in the notations of [14], equation 2.2). Explicitly, one solves

$$\frac{\partial W^0_{\mathrm{q}}}{\partial \ln \Lambda^{2bN}} = S \tag{13}$$

to express Λ as a function $f(X_r, S)$, and writes

$$W^0(X_r, S) = -S \ln f(X_r, S)^{2Nb} + W^0_{\mathrm{q}}\left(X_r, \Lambda = f(X_r, S)\right) \, . \tag{14}$$

Another subtle issue, that must be kept in mind when dealing with general $\mathcal{N} = 1$ models, is that the fields S or X_r, scale Λ, and couplings g_r, that enter formulas like (12) or the Dijkgraaf-Vafa superpotential $W(S)$ [1], are not physical, RG invariant quantities. In particular, they do not have a finite limit when the UV cut-off Λ_0 is taken to infinity. (Of course the effective superpotentials themselves are physical and RG invariant). The reason is that, to maintain holomorphicity, one must work with the Wilson gauge coupling (11) and renormalized fields and couplings that are related to the physical quantities through non-holomorphic Z factors (a useful discussion of this problem, with a complete list of references, can be found in [16]). For the model

(3) studied in the present paper, as well as for any model that can be viewed as a perturbation of an $\mathcal{N} = 2$ theory, the Z factors are trivial and thus this aspect is irrelevant. In particular, Λ corresponds in that case to a physical dynamically generated mass scale.

2.2 Exact superpotentials

We could now go on and, from the superpotential (10), obtain the universal superpotential $W^0(\tilde{M}_m M_m, u_p, S)$. This would amount to calculating the Legendre transform of the periods $A_{D,m}$. An elegant formula for $\partial A_{D,m}/\partial \ln \Lambda^{2N}$ can be found by using the exact RG equations derived in [17], but it is not clear how to find a useful expression for W^0. Since our purpose is mainly to compute the superpotential $W(S)$ for S alone, we will first integrate out the monopole fields from (10), and then integrate in S. This is of course strictly equivalent to integrating in S first and then integrating out the monopoles. We thus have to solve the equations $\partial W_{\mathrm{q}}/\partial(\tilde{M}_m M_m) = 0$, which yield

$$A_{D,m}(u_p, \Lambda) = 0 \,, \quad 1 \leq m \leq N - 1 \,. \tag{15}$$

As explained in [6, 7], the $A_{D,m}$ are given by integrals of a known differential form over cycles of the genus $N - 1$ Riemann surface defined by the equation

$$y^2 = P(x) = \prod_{k=1}^{N}(x - x_k)^2 - 4\Lambda^{2N} \,, \tag{16}$$

where the x_ks are such that

$$u_p = \frac{1}{p}\sum_{k=1}^{N} x_k^p \,. \tag{17}$$

We have normalized the scale Λ in (16) to obtain a simple match with the matrix model in the next Section. The $A_{D,m}$ are zero when the corresponding cycles vanish, and this yields a factorization constraint on the polynomial $P(x)$. This constraint was solved for the case of SU(N) in [18] with the help of Chebyshev polynomials. There are N solutions, corresponding to the N vacua of our $\mathcal{N} = 1$ theory. We will generally use the solution

$$x_k = 2\Lambda \cos\frac{\pi(k - 1/2)}{N} \quad \Longleftrightarrow \quad pu_p = \begin{cases} 0 & \text{if } p \text{ is odd,} \\ N\Lambda^p \mathrm{C}_p^{p/2} & \text{if } p \text{ is even,} \end{cases} \tag{18}$$

with the understanding that the other vacua are obtained by 2π shifts of the θ angle, $\Lambda^2 \to \Lambda^2 e^{2i\pi k/N}$, $1 \leq k \leq N - 1$. The C_n^p are the binomial coefficients. The case

of U(N) can be boiled down to the case of SU(N) by shifting the variables $x_k \to x_k + u_1/N$. The most general solution to (15) is then straightforwardly obtained,

$$u_p = U_p(z, \Lambda^2) = \frac{N}{p} \sum_{q=0}^{[p/2]} C_p^{2q} C_{2q}^q \Lambda^{2q} z^{p-2q}, \tag{19}$$

where the variable z is defined to be

$$z = u_1/N. \tag{20}$$

The exact superpotential for the field z can then be written down explicitly by replacing the solution (19) in (10),

$$\mathcal{W}(z, \Lambda^2, g_p) = \sum_{p \geq 1} g_p U_p(z, \Lambda^2). \tag{21}$$

Integrating in S then yields the effective superpotential W_{eff} for z and S,

$$W_{\text{eff}}(z, S, \Lambda^2, g_p) = S \ln \Lambda^{2N} + N g_1 z - S \ln \Delta(z, S, g_{p \geq 2})^N + \sum_{p \geq 2} g_p U_p(z, \Lambda^2 = \Delta), \tag{22}$$

where $\Delta(z, S, g_{p \geq 2})$ satisfies the condition

$$\Lambda^2 \partial_{\Lambda^2} \mathcal{W}(z, \Lambda^2 = \Delta, g_p) = NS, \tag{23}$$

together with the requirement that in the classical limit $S \to 0$, $\Delta \to 0$.

The superpotential W_{eff} is very convenient to use. For example, in the case of a cubic tree level superpotential, whose physics is discussed in details in [10], it takes the form

$$W_{\text{eff}}^{\text{cubic}}(z, S, \Lambda^2, g_p) = N\left(g_1 z + \frac{1}{2}g_2 z^2 + \frac{1}{3}g_3 z^3\right) + S \ln \left(\frac{e\Lambda^2(g_2 + 2g_3 z)}{S}\right)^N. \tag{24}$$

The superpotential (24) can be used to describe all the vacua, because only Λ^{2N} enters the formula.

The derivatives of W_{eff} take simple forms,

$$\partial_S W_{\text{eff}}(z, S, \Lambda^2, g_p) = -\ln\left(\Delta(z, S, g_{p \geq 2})/\Lambda^2\right)^N, \tag{25}$$

$$\partial_z W_{\text{eff}}(z, S, \Lambda^2, g_p) = \partial_z \mathcal{W}\left(z, \Lambda^2 = \Delta(z, S, g_{p \geq 2}), g_p\right). \tag{26}$$

We can use the condition

$$\partial_z W_{\text{eff}}(z, S, \Lambda^2, g_p) = 0 \tag{27}$$

to integrate out z and obtain the superpotential for the glueball superfield only. This yields

$$W(S, \Lambda^2, g_p) = -S \ln \left(\Delta/\Lambda^2\right)^N + \sum_{p \geq 1} g_p U_p(z, \Delta), \tag{28}$$

where the polynomials U_p are defined in (19), and Δ and z are expressed in terms of S by using the two conditions (23) and (27). Explicitly, those conditions are

$$S = \sum_{p \geq 2} g_p \sum_{q=1}^{[p/2]} \frac{q}{p} \mathrm{C}_p^{2q} \mathrm{C}_{2q}^q z^{p-2q} \Delta^q, \tag{29}$$

$$0 = \sum_{p \geq 1} g_p \sum_{q=0}^{[(p-1)/2]} \frac{p-2q}{p} \mathrm{C}_p^{2q} \mathrm{C}_{2q}^q z^{p-2q-1} \Delta^q. \tag{30}$$

To compare with the matrix model result discussed in the next Section, it is convenient to use the derivative of W. From (25) we deduce the fundamental field theory formula

$$\partial_S W(S, \Lambda^2, g_p) = -\ln \left(\Delta/\Lambda^2\right)^N. \tag{31}$$

Note that the full dependence in Λ is explicit in (31), since the equations (29) and (30) that determine Δ are independent of Λ. The linearity in $\ln \Lambda^{2N}$ is of course a direct consequence of the ILS principle. Equation (31) shows that to calculate $\langle S \rangle$, one must simply set $\Delta = \Lambda^2$ in (29) and (30) and solve the algebraic equations so obtained.

The formulas simplify when W_{tree} is an even function, because the solution to (30) is then simply $z = 0$, and thus (28) and (29) reduce to

$$W(S, \Lambda^2, g_{2p}) = -S \ln \left(\Delta/\Lambda^2\right)^N + N \sum_{p \geq 1} \frac{1}{2p} g_{2p} \mathrm{C}_{2p}^p \Delta^p, \tag{32}$$

$$S = \frac{1}{2} \sum_{p \geq 1} g_{2p} \mathrm{C}_{2p}^p \Delta^p. \tag{33}$$

The preceding equations also give the solution of the $\mathrm{SU}(N)$ theory for an *arbitrary* tree level superpotential. This is proven by treating the coupling g_1 as a Lagrange multiplier in (28), which automatically sets $z = 0$.

3 Matrix model analysis

3.1 The Dijkgraaf-Vafa proposal

Dijkgraaf and Vafa have conjectured in [1] that the superpotential $W(S)$ can actually be computed by summing the zero momentum *planar* diagrams of the $\mathcal{N} = 1$ theory

under consideration. In our case, their ansätz for the U(N) theory is simply an holomorphic integral over $n \times n$ complex matrices ϕ,

$$\exp\left(n^2 \mathcal{F}/S^2\right) = \int_{\text{planar}} \mathrm{d}^{n^2}(\phi/\Lambda)\, \exp\left[-\frac{n}{S} W_{\text{tree}}(\phi, g_p)\right], \tag{34}$$

from which the superpotential can be deduced,

$$W(S, \Lambda^2, g_p) = -N \partial_S \mathcal{F}(S, g_p). \tag{35}$$

It is convenient to introduce the dummy variable n (which is not to be confused with the number of color N) because the planar diagrams can be extracted by taking the $n \to \infty$ limit. The full N dependence of the superpotential is then given explicitly in (35). Strictly speaking, the integral (34) involves complex matrices and couplings g_p, but the calculation is the same as for hermitian matrices and real couplings. There is no ambiguity in the analytic continuation because we restrict ourselves to planar diagrams. This implies that the standard matrix model techniques, which are reviewed for example in [19], do apply. For the SU(N) gauge theory, the integral (34) must be restricted to traceless matrices, or equivalently one must treat g_1 as a Lagrange multiplier. There is no difference between the U(N) and SU(N) theory when the function W_{tree} is even, because the U(1) part of ϕ in the U(N) theory has then zero vev and its couplings are subleading. In general, however, the U(N) and SU(N) theories are very different [10]. This is perfectly consistent with the field theory results discussed at the end of the preceding Section.

It is convenient to work with dimensionless variables and to write

$$\exp\left(n^2 \mathcal{F}/S^2\right) = \int_{\text{planar}} \mathrm{d}^{n^2}\varphi\, \exp\left[-\frac{n}{\sigma} \operatorname{tr} V(\varphi, \lambda_p)\right], \tag{36}$$

where

$$\sigma = S/\Lambda_{\mathrm{d}}^3, \quad V(\varphi, \lambda_p) = \sum_{p \geq 1} \frac{\lambda_p}{p} \varphi^p, \quad \lambda_p = \frac{g_p \Lambda_{\mathrm{d}}^{3(p-2)/2}}{g_2^{p/2}}. \tag{37}$$

The scale

$$\Lambda_{\mathrm{d}}^3 = g_2 \Lambda^2 \tag{38}$$

is the dynamically generated scale of the low energy pure $\mathcal{N} = 1$ super Yang-Mills theory.

3.2 Matrix model technology

The method to calculate explicitly (36) has been known for a long time [20, 19]. The eigenvalues of ϕ, which are all zero classically in the vacua we consider, are described

in the planar limit by a continuous distribution $\rho(\varphi, \sigma)$ with support on a finite interval $[a, b]$ containing zero (we will no longer indicate explicitly the dependence in the couplings λ_p). The solution is easily expressed in terms of ρ,

$$\frac{\mathcal{F}}{\Lambda_{\mathrm{d}}^6} = -\sigma \int \mathrm{d}\varphi \, \rho(\varphi, \sigma) V(\varphi) + \sigma^2 \int \mathrm{d}\varphi \mathrm{d}\psi \, \rho(\varphi, \sigma) \rho(\psi, \sigma) \ln|\varphi - \psi| \, . \tag{39}$$

For our purposes, the most useful way to present the solution for ρ, a and b is in terms of the following three equations (see [19] for a derivation),

$$\partial_\sigma \left(\sigma \rho(\varphi, \sigma) \right) = \frac{1}{\pi \sqrt{(\varphi - a)(b - \varphi)}} \, , \tag{40}$$

$$\sigma = \int_a^b \frac{\mathrm{d}\varphi}{2\pi} \frac{\varphi V'(\varphi)}{\sqrt{(\varphi - a)(b - \varphi)}} \, , \tag{41}$$

$$0 = \int_a^b \mathrm{d}\varphi \frac{V'(\varphi)}{\sqrt{(\varphi - a)(b - \varphi)}} \, . \tag{42}$$

The formulas (41) and (42) give two algebraic equations that determine a and b as a function of σ and the couplings, and (40) may then be used to deduce ρ. The superpotential (35) is related to the derivative of $\mathcal{F}$. To make this calculation,[1] we will need two identities,

$$\int_a^b \frac{\mathrm{d}\varphi}{\pi} \frac{\ln|\varphi - \psi|}{\sqrt{(\varphi - a)(b - \varphi)}} = \ln \frac{b - a}{4} \quad \text{for any } \psi \in [a, b] \, , \tag{43}$$

$$\partial_\sigma \int_a^b \frac{\mathrm{d}\varphi}{\pi} \frac{V(\varphi)}{\sqrt{(\varphi - a)(b - \varphi)}} = 2\sigma \partial_\sigma \ln(b - a) \, , \tag{44}$$

that we derive in the Appendix. By using (39), (40) and (43), we get

$$\frac{W}{N \Lambda_{\mathrm{d}}^3} = -\frac{\partial_\sigma \mathcal{F}}{\Lambda_{\mathrm{d}}^3} = \int_a^b \frac{\mathrm{d}\varphi}{\pi} \frac{V(\varphi)}{\sqrt{(\varphi - a)(b - \varphi)}} - 2\sigma \ln \frac{b - a}{4} \, . \tag{45}$$

Then, by using (44), we finally obtain our fundamental matrix model equation

$$\partial_S W(S, \Lambda^2, g_p) = -\ln \left(\frac{b - a}{4} \right)^{2N} \, . \tag{46}$$

This equation shows in particular that the condition for a critical point, that yields $\langle S \rangle$, is simply $b - a = 4$.

---

[1]This calculation is undoubtedly known to experts in the field of matrix models, since the result (with a minor misprint) appears for example in [21], but I have been unable to find a derivation in the literature.

3.3 Field theory and matrix model results are equivalent

Comparison between (31) and (46) immediately yields the relationship between field theory and matrix model variables,

$$\frac{(b-a)^2}{16} = \frac{\Delta}{\Lambda^2} \,. \tag{47}$$

It remains to check that the equations (29) and (30) that determine Δ are equivalent to the equations (41) and (42) that determine $b - a$. The simplest way to do that is to work out explicitly the integrals in (41) and (42). This is very elementary. One changes the variable from φ to $\psi = 2\left(\varphi - (a + b)/2\right)/(b-a)$, expands the polynomial V' as a power series in ψ, and uses the identity

$$\int_{-1}^{1} \frac{\mathrm{d}\psi}{2\pi} \frac{\psi^{2p}}{\sqrt{1 - \psi^2}} = \frac{\mathrm{C}_{2p}^{p}}{2^{2p+1}} \,. \tag{48}$$

Formulas (41) and (42) are then exactly mapped onto (29) and (30) respectively, provided one identifies

$$\frac{1}{2}(a + b) = \frac{z}{\Lambda} \,. \tag{49}$$

This completes the proof.

4 Conclusion

The non-trivial part of the field theory calculation made in Section 2 is the Intriligator-Leigh-Seiberg non-renormalization hypothesis [8], while the non-trivial part of the matrix model calculation made in Section 3 is the assumption that only planar diagrams contribute [1]. In the model studied in the present paper, those two hypothesis turn out to be equivalent. The fact that the linearity in the couplings g_p is implemented in the large n matrix model was a priori non-trivial. It is remarkable that the planar matrix model formulas (41) and (42), which are manifestly linear, could be identified with the integrating in relation (23) for S and the integrating out relation (27) for z, with the suitable mapping between field theory and matrix model variables (47) and (49). Linearity would be violated by non-planar contributions. More generally, we conjecture that the ILS linearity principle can be deduced from corresponding linearity properties of the sum over *planar* diagrams.

There are two obvious directions of research that open up. The first is to try to generalize the approach of the present paper. It should not be too difficult, for example, to study the most general vacua of the theory (3), in particular by using

the results of [5]. From the matrix model point of view, this amounts to generalizing equations like (46) to the multi-cut solutions. The second is to try to use the exact superpotentials to work out some new interesting physics. For example, we have shown in [10] that (24) has several unexpected consequences. One can also use the Dijkgraaf-Vafa proposal to full power, for theories that are not simple perturbations of $\mathcal{N} = 2$. The interesting case of a Leigh-Strassler deformation of $\mathcal{N} = 4$ has been treated in [13]. More generally, for pure $\mathcal{N} = 1$ models, the holomorphic variables that enter the DV superpotential are not physical, RG invariant, quantities. Nevertheless, the superpotential itself is physical, and it should contain some very interesting information. We are presently investigating a two-matrix model [22] of this type.

Appendix

We wish first to derive equation (43),

$$I(\psi) = \int_a^b \mathrm{d}\varphi \, \frac{\ln|\varphi - \psi|}{\sqrt{(\varphi - a)(b - \varphi)}} = \pi \ln \frac{b - a}{4} \quad \text{for any } \psi \in [a, b] \,. \tag{50}$$

The derivative of $I(\psi)$ can be expressed as

$$\frac{\mathrm{d}I}{\mathrm{d}\psi} = \frac{1}{2} \left(f(\psi + i\epsilon) + f(\psi - i\epsilon) \right) \tag{51}$$

for

$$f(\psi) = \int_a^b \frac{\mathrm{d}\varphi}{(\psi - \varphi)\sqrt{(\varphi - a)(b - \varphi)}} \,. \tag{52}$$

From

$$\frac{1}{2i\pi} \left(f(\psi + i\epsilon) - f(\psi - i\epsilon) \right) = -\frac{1}{\sqrt{(\psi - a)(b - \psi)}} \tag{53}$$

and

$$f(\psi) \underset{\psi \to \infty}{\sim} \frac{1}{\psi} \int_a^b \frac{\mathrm{d}\varphi}{\sqrt{(\varphi - a)(b - \varphi)}} = \frac{\pi}{\psi} \,, \tag{54}$$

we deduce

$$f(\psi) = \frac{\pi}{\sqrt{(\psi - a)(\psi - b)}} \,. \tag{55}$$

The relation (51) then implies

$$\frac{\mathrm{d}I}{\mathrm{d}\psi} = \begin{cases} 0 & \text{for } \psi \in [a, b] \,, \\ f(\psi) & \text{for } \psi \notin [a, b] \,. \end{cases} \tag{56}$$

It is then straightforward to compute $I(\psi)$ for $\psi > b$ by integrating (56) and using the asymptotics at infinity $I(\psi) = \pi \ln |\psi| + \mathcal{O}(1/\psi)$. The formula (50) follows from continuity.

We now turn to the derivation of equation (44). We have

$$\partial_\sigma \int_a^b \frac{\mathrm{d}\varphi}{\pi} \frac{V(\varphi)}{\sqrt{(\varphi - a)(b - \varphi)}} = \frac{\partial a}{\partial \sigma} \oint_\gamma \frac{\mathrm{d}\varphi}{4i\pi} \frac{V(\varphi)}{(\varphi - a)^{3/2}(\varphi - b)^{1/2}} + (a \to b), \qquad (57)$$

where γ is a contour encircling the cut $[a, b]$ counterclockwise. We then use

$$0 = \oint_\gamma \mathrm{d}\Big[\frac{V(\varphi)}{\sqrt{(\varphi - a)(\varphi - b)}}\Big] \qquad (58)$$

and the condition (42) to deduce

$$\oint_\gamma \frac{\mathrm{d}\varphi\, V(\varphi)}{(\varphi - a)^{3/2}(\varphi - b)^{1/2}} = -\oint_\gamma \frac{\mathrm{d}\varphi\, V(\varphi)}{(\varphi - a)^{1/2}(\varphi - b)^{3/2}}, \qquad (59)$$

and we use

$$0 = \oint_\gamma \mathrm{d}\Big[\frac{\varphi V(\varphi)}{\sqrt{(\varphi - a)(\varphi - b)}}\Big], \qquad (60)$$

together with (41) and (59), to deduce

$$\oint_\gamma \frac{\mathrm{d}\varphi}{2i\pi} \frac{V(\varphi)}{(\varphi - a)^{3/2}(\varphi - b)^{1/2}} = \frac{4\sigma}{a - b}. \qquad (61)$$

Equation (44) then follows immediately from (61), (59) and (57).

Acknowledgements

References

[1] R. Dijkgraaf and C. Vafa, *A Perturbative Window into Non-Perturbative Physics,* HUTP-02/A034, ITFA-2002-34, hep-th/0208048.

[2] R. Gopakumar and C. Vafa, *Adv. Theor. Math. Phys.* **3** (1999) 1415,
C. Vafa, *J. Math. Phys.* **42** (2001) 2798.

[3] F. Cachazo, K. Intriligator and C. Vafa, *Nucl. Phys.* **B 603** (2001) 3.

[4] R. Dijkgraaf and C. Vafa, *Matrix Models, Topological Strings, and Supersymmetric Gauge Theories,* HUTP-02/A028, ITFA-2002-22, hep-th/0206255,
R. Dijkgraaf and C. Vafa, *On Geometry and Matrix Models,* HUTP-02/A030, ITFA-2002-24, hep-th/0207106.

[5] F. Cachazo and C. Vafa, $\mathcal{N} = 1$ *and* $\mathcal{N} = 2$ *Geometry from Fluxes,* HUTP-02/A021, hep-th/0206017.

[6] N. Seiberg and E. Witten, *Nucl. Phys.* **B 426** (1994) 19, erratum **B 430** (1994) 485,
N. Seiberg and E. Witten, *Nucl. Phys.* **B 431** (1994) 484.

[7] P. C. Argyres and A. E. Faraggi, *Phys. Rev. Lett.* **74** (1995) 3931,
A. Klemm, W. Lerche, S. Yankielowicz and S. Theisen, *Phys. Lett.* **B 344** (1995) 169.

[8] K. Intriligator, R.G. Leigh and N. Seiberg, *Phys. Rev.* **D 50** (1994) 1092.

[9] F. Ferrari, *Nucl. Phys.* **B 612** (2001) 151,
F. Ferrari, *Nucl. Phys.* **B 617** (2001) 348,
F. Ferrari, *Four dimensional non-critical strings,* Les Houches summer school 2001, Session LXXVI, *l'Unité de la Physique fondamentale: Gravité, Théorie de Jauge et Cordes,* A. Bilal, F. David, M. Douglas and N. Nekrasov editors, hep-th/0205171.

[10] F. Ferrari, *Quantum parameter space and double scaling limits in* $\mathcal{N} = 1$ *super Yang-Mills theory,* NEIP-02-008, LPTENS-02/49, hep-th/0211069.

[11] N. Seiberg, *Phys. Lett.* **B 318** (1993) 469.

[12] K. Intriligator and N. Seiberg, *Lectures on supersymmetric gauge theories and electric-magnetic duality, Nucl. Phys. Proc. Suppl.* 55 (1997) 1, hep-th/9509066.

[13] N. Dorey, T.J. Hollowood, S.P. Kumar and A. Sinkovics, *Exact Superpotentials form Matrix Models,* SWAT-350, hep-th/0209089,
N. Dorey, T.J. Hollowood, S.P. Kumar and A. Sinkovics, *Massive Vacua of* $\mathcal{N} = 1^*$ *Theory and S-duality from Matrix Models,* SWAT-352, hep-th/0209099.

[14] K. Intriligator, *Phys. Lett.* **B 336** (1994) 409.

[15] M.A. Shifman and A.I. Vainshtein, *Nucl. Phys.* **B 277** (1986) 456,
M.A. Shifman and A.I. Vainshtein, *Nucl. Phys.* **B 359** (1991) 571.

[16] N. Arkani-Hamed and H. Murayama, *Phys. Rev.* **D 57** (1998) 6638,
N. Arkani-Hamed and H. Murayama, *J. High Energy Phys.* **06** (2000) 30,
M.A. Shifman, *Exact results in gauge theories: putting supersymmetry to work,*
the 1999 Sakurai Prize lecture, *Int. J. Mod. Phys.* **A 14** (1999) 5017.

[17] M. Matone, *Phys. Lett.* **B 357** (1995) 342,
T. Eguchi and S.K. Yang, *Mod. Phys. Lett.* **A 11** (1996) 131,
J. Sonnenschein, S. Theisen and S. Yankielowicz, *Phys. Lett.* **B 367** (1996) 145.

[18] M.R. Douglas and S.H. Shenker, *Nucl. Phys.* **B 447** (1995) 271.

[19] P. Di Francesco, P. Ginsparg and J. Zinn-Justin, *Phys. Rep.* **254** (1995) 1.

[20] É. Brézin, C. Itzykson, G. Parisi and J.-B. Zuber, *Comm. Math. Phys.* **59** (1978) 35.

[21] B. Eynard, *Gravitation quantique bidimensionnelle et matrices aléatoires,* PhD thesis, University of Paris VI, 1995.

[22] F. Ferrari and R. Hernández, in preparation.