

# Analytic continuation of the Hurwitz Zeta Function with physical application.

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**Abstract:** A new formula relating the analytic continuation of the Hurwitz zeta function to the Euler gamma function and a polylogarithmic function is presented. In particular, the values of the first derivative of the real part of the analytic continuation of the Hurwitz zeta function for even negative integers and the imaginary one for odd negative integers are explicitly given. The result can be of interest both on mathematical and physical side, because we are able to apply our new formulas in the context of the Spectral Zeta Function regularization, computing the exact pair production rate per space-time unit of massive Dirac particles interacting with a purely electric background field.

## 1 Introduction

In 1951, Schwinger [1] computed the implicit Effective Lagrangian for a Dirac charged spinor in general electromagnetic background field using proper time approach. In particular, he applied the result to the physically case of a constant and uniform electric background and he was able to evaluate the exact pair-production rate per space-time unit, namely

$$w(E, e, m) = \frac{e^2 E^2}{4\pi^3} \sum_{n=1}^{+\infty} \frac{e^{-n \frac{\pi m^2}{eE}}}{n^2} . \quad (1)$$

This result is well known and can be obtained by other ways (e.g. Itzykson and Zuber [2], using the S-matrix approach, and by Beneventano and Santangelo [3]). In 1990 Blau, Visser and Wipf [4], using the techniques of the Spectral Zeta Function regularization, tried to obtain the same results obtained by Schwinger. However they were able only to obtain non exact results for the four dimensional case, using asymptotic analysis. In 1996, Soldati and Sorbo [5] obtained a new expression, but once again their methods were based on asymptotic analysis and the results were expressed in terms of asymptotic series.

The problem for general electromagnetic external field has been recently discussed by Schubert [6] and by Cho and Pak [7]. Schubert obtained the same action by Schwinger using proper time method and techniques of computation inspired by String Theory. Cho and Pak obtained a renormalized action from the general one (non renormalized) found by Schwinger, using the so-called Sitacaramachandrarao identity. Recently, Beneventano and Santangelo [8] have compared their results obtained using Spectral Zeta Function methods with the general result obtained by Schwinger, showing that they agree.

In this paper, we shall compute the imaginary part of the Effective Lagrangian related to a Dirac field in a constant and uniform external electric field by making use of new formulas concerning the analytic continuation of the first derivative of the Hurwitz zeta function.

The content of the paper is the following. In Section 2, we briefly introduce the Spectral Zeta Function regularization procedure and illustrate the result obtained by Blau Visser and Wipf, which will be our starting point for the application of the new formulas. In Section 3, we derive a new expression for the analytic continuation of the Hurwitz zeta function, viewed as an analytic function of two complex variables. In Section 4, we use that result to recover the rate of pair creation of Dirac particles in constant and uniform electric background.

## 2 Zeta Function regularization

In this Section, we shall review the relation between the mathematical problem that we have solved and the physical problem associated with the Dirac pair creation. For a review of the method, see, for example, [9], [10], [11] and references quoted therein.

Within the context of Quantum Field Theory (QFT) interacting with a classical gauge background field, one is forced to confront with the determinants of differential operators.

In the context of a Klein Gordon field interacting with an external gauge field  $A$ , this determinant is related to some physical quantities formally obtained from the *Euclidean* functional integral:

$$Z[A_E] = \int D\varphi e^{-\frac{1}{2} \int d^4x \varphi A_E \varphi}, \quad (2)$$

where  $A_E$  is the Euclidean Klein Gordon operator. In the definition of this integral we have to analytically continue some global Minkowski temporal coordinate  $x^0$  into imaginary value  $x^0 \rightarrow ix^0$  and consider the analytical continuation of all relevant quantities.

The above Gaussian functional integral could be interpreted as a Wiener measure, but, for our purposes, we may interpret it in terms of a functional determinant, and rewrite the definition

of (2) as:

$$Z[A_E] = \left[ \det \left( \frac{A_E}{\mu} \right) \right]^{-\frac{1}{2}}. \quad (3)$$

where  $\mu$  is a constant with the same physical dimension of the operator  $A_E$ .

It is sometime useful to introduce two physical quantities: the Effective Action and the Effective Lagrangian. The former is defined as the logarithm of  $Z$ . The latter is a function of space-time points and gives, after an integration on the whole space-time, the Effective Action. The physical interpretation of this determinant is found after a re-analytical continuation of the imaginary time into a real time. The result is the vacuum to vacuum transition amplitude.

We can use Zeta Function regularization to give a rigorous meaning to functional determinants. This regularization techniques was introduced by Ray-Singer for elliptic differential operators [12]. Within the Quantum Field Theory, it was used by by Dowker and Chritchley [13] and Hawking [14]. Given a compact Riemannian manifold  $M$  and for elliptic and second order operators acting on  $L_2(M)$ , it can be proved that such definition gives a useful extension of the notion of functional determinant.

Since the square of the Euclidean Dirac operator is a elliptic second order differential operator, making use of the zeta regularization technique, Blau, Visser and Wipf [4] arrived at the following Effective Lagrangian for a Dirac field in an external constant and uniform electric field:

$$\begin{aligned} L_{eff}(E, 0) = & -\frac{e^2 E^2}{2\pi^2} \left\{ \left[ 1 - \ln \left( \frac{-2ieE}{\mu^2} \right) \right] \zeta_H(-1; 1 + i\frac{m^2}{2eE}) + \right. \\ & \left. + \frac{d}{ds} \zeta_H(s; 1 + i\frac{m^2}{2eE})_{s=-1} \right\} + i\frac{m^2 eE}{8\pi^2} \left[ \ln \frac{m^2}{\mu^2} - 1 \right]. \end{aligned} \quad (4)$$

where  $\zeta_H(s; x)$  is the Hurwitz zeta function defined by:

$$\zeta_H(s; x) := \sum_{n=0}^{\infty} \left( \frac{1}{n+x} \right)^s, \quad \text{Res} > 1.$$

They were not able to find an explicit form for the analytic continuation of the derivatives of the Hurwitz zeta function at  $s = -1$ . Thus, a direct comparison between their result and the one obtained by Schwinger was missing.

### 3 Analytic continuation of the Hurwitz zeta function

In this Section, we discuss the analytical properties of the Hurwitz zeta function. Following [15], we obtain a new identity involving the Hurwitz zeta function, the Euler  $\Gamma(s)$  function and a polylogarithmic function.

Recall that the Hurwitz zeta function is defined as follows:

$$\zeta_H(s; x) := \sum_{n=0}^{+\infty} (n+x)^{-s}, \quad (5)$$

where  $\text{Re } s > 1$ ,  $x \neq 0, -1, -2, \dots$ . One can analytically extend it into an analytic function of two complex variables  $s$  and  $x$ .

In order to search for the analytical continuation in the double complex plane, we introduce the function  $F(s; z)$  defined by the following series:

$$F(s; z) := \lim_{k \rightarrow \infty} \sum_{n=-k}^k (-in + z)^{-s}. \quad (6)$$

Notice that the series above is absolutely convergent for  $\text{Re } s > 1$ ,  $z \neq \pm ik, k \in \mathbb{N}$ . Thus, one may write:

$$F(s; z) := \lim_{k \rightarrow \infty} \left[ \sum_{n=0}^k (-in + z)^{-s} + \sum_{n=0}^k (in + z)^{-s} \right] - z^{-s}. \quad (7)$$

Using this definition for the series (6), we proceed in searching for its analytical continuation. With regard to this issue, previous attempts can be found in [16], [17]. Here the author obtained an expression different from our one, requiring, however, as far as physical applications were concerned, an ad hoc finite renormalization.

It is straightforward to prove, the following relation between  $F(s; z)$  and  $\zeta_H(s; z)$ :

$$F(s; z) = i^s \zeta_H(s; iz) + i^{-s} \zeta_H(s; -iz) - z^{-s}. \quad (8)$$

First, let us investigate the function  $F(s; x)$ . The following theorem gives an expression for the analytic continuation of  $F(s; z)$  in terms of polylogarithmic function.

**Theorem:** Following the definition stated above, the analytic continuation of  $F(s, z)$  for each  $s \in \mathbb{C}$  and  $\text{Re } z > 0$  is given by

$$F(s; z) = \frac{(2\pi)^s}{\Gamma(s)} \sum_{n=1}^{+\infty} \frac{e^{-2n\pi z}}{n^{1-s}}. \quad (9)$$

Proof: Define the sequence  $F_k(s; z)$  :

$$F_k(s; x) := \sum_{n=-k}^k (-in + z)^{-s}, \quad (10)$$

which obviously converges to  $F(s; z)$ .

Recall the definition of the Euler  $\Gamma(s)$  function for  $\text{Re } s > 0$ ,

$$\Gamma(s) := \int_0^{\infty} t^{s-1} e^{-t} dt.$$

Consider the product of this function and the sequence  $F_k(s; x)$  :

$$F_k(s; z) \Gamma(s) = \sum_{n=-k}^{+k} (-in + z)^{-s} \int_0^{\infty} t^{s-1} e^{-t} dt = \int_0^{\infty} \sum_{n=-k}^{+k} (-in + z)^{-s} t^{s-1} e^{-t} dt.$$

We may change variables according to  $t = \tau(-in + z)$ , which is allowed for  $z$  not purely imaginary integer since  $|\frac{dt}{d\tau}| \neq 0$ , and we obtain:

$$F_k(s; z) = \frac{1}{\Gamma(s)} \int_0^{+\infty} d\tau e^{-\tau z} \tau^{s-1} \sum_{n=-k}^{+k} e^{in\tau}. \quad (11)$$

The part of the integrand in (11) multiplying the series *vanishes* in  $\tau = 0$  for  $\text{Re } s > 1$ . This implies that the integration may be interpreted as the action of the  $k$ -element of a sequence of distributions  $g_k(\tau)$ ,

$$g_k(\tau) := \sum_{n=-k}^{+k} e^{in\tau} \quad (12)$$

on the continuous function  $e^{-\tau z} \tau^{s-1}$ . It is a well-known result of the theory of distribution that this sequence converges and gives the Poisson summation formula:

$$\lim_{k \rightarrow \infty} g_k(\tau) = \lim_{k \rightarrow \infty} \sum_{n=-k}^{+k} e^{in\tau} = 2\pi \sum_{n=-\infty}^{+\infty} \delta(\tau - 2n\pi).$$

Thus we obtain in the limit  $k \rightarrow \infty$ :

$$\begin{aligned} F(s; z) &= \frac{1}{\Gamma(s)} \int_0^{+\infty} e^{-\tau z} \tau^{s-1} \sum_{n=-\infty}^{+\infty} e^{in\tau} = \\ &= \frac{(2\pi)^s}{\Gamma(s)} \sum_{n=1}^{+\infty} \frac{e^{-2n\pi z}}{n^{1-s}}, \end{aligned}$$

which, as stated in the hypotheses of the theorem, makes sense as analytic continuation for each  $s \in C$  and  $\text{Re } z > 0$ .

Recalling the relation (8) between  $F(s; z)$  and  $\zeta_H(s; z)$ , we have the following corollary:

**Corollary.** The following formula is valid for each  $s \in C$  and  $\text{Re } z > 0$  in the sense of the analytic continuation,

$$(i)^s \zeta_H(s; iz) + i^{-s} \zeta_H(s; -iz) - z^{-s} = \frac{(2\pi)^s}{\Gamma(s)} \sum_{n=1}^{+\infty} \frac{e^{-2n\pi z}}{n^{1-s}}. \quad (13)$$

Proof: Direct use of equations (8) and (9).

Equation (13) can be checked for  $s = -n$ . In this case the right hand side is vanishing and equation (13) gives:

$$i^{-n} \zeta_H(-n, iz) + i^n \zeta_H(-n, -iz) - z^{-s} = 0. \quad (14)$$

Recall that the values of the Hurwitz zeta function are known for negative integers and are related to the Bernoulli's polynomials by the following formula,

$$\zeta_H(-n, z) = -\frac{B_{n+1}(z)}{n+1}. \quad (15)$$

It is easy to show that this equation is identically fulfilled.

**Remark.** We now notice that the series:

$$\sum_{n=1}^{+\infty} \frac{e^{-2n\pi z}}{n^{1-s}} \quad (16)$$

and its derivatives are uniformly convergent in the variable  $s$  for  $\operatorname{Re} z \geq 0$  and  $\operatorname{Re} s \in (-\infty, a)$  with  $a < \infty$ . A straightforward computation leads to the following lemma:

**Lemma 1.** The following formula holds for each  $s \in \mathbb{C}$  and  $\operatorname{Re} z > 0$  in the sense of the analytic continuation,

$$\begin{aligned} (i)^s \frac{d}{ds} \zeta_H(s; iz) + (i)^{-s} \frac{d}{ds} \zeta_H(s; -iz) &= (2\pi)^s \left( \frac{d}{ds} \frac{1}{\Gamma(s)} \right) \sum_{n=1}^{+\infty} \frac{e^{-2n\pi z}}{n^{1-s}} + \frac{(2\pi)^s}{\Gamma(s)} \sum_{n=1}^{+\infty} \frac{e^{-2n\pi z} \ln(2n\pi)}{n^{1-s}} + \\ &+ i \frac{\pi}{2} [-2(i)^s \zeta_H(s; iz) + z^{-s} + \frac{(2\pi)^s}{\Gamma(s)} \sum_{n=1}^{+\infty} \frac{e^{-2n\pi z}}{n^{1-s}}] - (z)^{-s} (\ln|z| + i \arg z). \end{aligned} \quad (17)$$

Proof: One simply applies the derivative on both sides of the formula obtained in the corollary above, then uses the remark above to exchange the series with the derivative with respect to  $s$ , and then analytically continues the result for all the values of  $s$ .

As a result, we shall derive formulas for the real and imaginary values of the first derivative of the Hurwitz zeta function respectively for even and odd negative integers.

Using parity properties of the real and the imaginary part of this polynomials it is easy to prove that,

**Lemma 2.** For  $x \in \mathbb{R}$ ,  $x > 0$ , and for  $n \in \mathbb{N}$ ,  $n < 0$ , the Hurwitz zeta function has the following parity properties:

$$\operatorname{Re} \zeta_H(-n; ix) = \operatorname{Re} \zeta_H(-n; -ix) \quad (18)$$

$$\operatorname{Im} \zeta_H(-n; ix) = -\operatorname{Im} \zeta_H(-n; -ix) \quad (19)$$

We have

**Proposition.** For  $x \in \mathbb{R}$  and for  $m \in \mathbb{N}$ , the following formulas two are valid for any natural number:

$$\operatorname{Im} \frac{d}{ds} \zeta_H(-(2m+1); ix) = \pi \operatorname{Re} \frac{B_{2m+2}(ix)}{4(m+1)} + \frac{(-1)^{m+1} x^{2m+1}}{2} \ln x + \frac{(-1)^{m+1} (2m+1)!}{2(2\pi)^{2m+1}} \sum_{n=1}^{+\infty} \frac{e^{-2n\pi x}}{n^{2(m+1)}} \quad (20)$$

$$\operatorname{Re} \frac{d}{ds} \zeta_H(-2m; ix) = -\pi \operatorname{Im} \frac{B_{2m+1}(ix)}{2(2m+1)} + \frac{(-1)^{m+1} x^{2m}}{2} \ln x + \frac{(-1)^m (2m)!}{2(2\pi)^{2m}} \sum_{n=1}^{+\infty} \frac{e^{-2n\pi x}}{n^{1+2m}} \quad (21)$$

Proof: Simple computation using Lemma 1 and 2.

**Note.** It is possible to obtain other identities involving higher order derivatives of the Hurwitz zeta function starting from (13). As a consequence, it is possible to obtain the values of higher derivatives of either the imaginary or real part of the Hurwitz zeta function for negative integers using recursive formulas.

As far as we know only asymptotic values have been found by E.Elizalde [18, 19] for the analytic continuation of the derivatives of the Hurwitz zeta function. Thus (20) and (21) are new formulas. In the next Section, we use the second one for the application to the study of the Dirac pair creation in pure electrical background field.

## 4 Application to Schwinger pair creation

In this last Section we shall apply the formula (20) to the problem of Dirac pair creation in a purely electrical background field, recovering the Schwinger's result. The starting point is the effective Lagrangian obtained by Blau, Visser and Wipf,

$$L_{eff}(E, 0) = -\frac{e^2 E^2}{2\pi^2} \left\{ \left[ 1 - \ln\left(\frac{-2ieE}{\mu^2}\right) \right] \zeta_H(-1; 1 + i\frac{m^2}{2eE}) + \frac{d}{ds} \zeta_H(s; 1 + i\frac{m^2}{2eE})_{s=-1} \right\} + i\frac{m^2 eE}{8\pi^2} \left[ \ln\frac{m^2}{\mu^2} - 1 \right]. \quad (22)$$

We have to find the imaginary part of the right-hand side of (22). Making use of Eq. (20) to  $s = -1$ , we get:

$$\operatorname{Im} \frac{d}{ds} \zeta_H(-1; ix) = -\frac{1}{4\pi} \sum_{n=1}^{+\infty} \frac{e^{-2n\pi x}}{n^2} - \frac{x}{2} \ln x + \frac{\pi}{24} - \frac{\pi x^2}{4}. \quad (23)$$

If we plug this formula in the effective Lagrangian expression, we obtain the Schwinger's result:

$$w(E, e, m) = \frac{e^2 E^2}{4\pi^3} \sum_{n=1}^{+\infty} \frac{e^{-n \frac{\pi m^2}{eE}}}{n^2}. \quad (24)$$

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