Gauge Theoretical Construction
of Non-compact Calabi-Yau Manifolds

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Abstract

We construct the non-compact Calabi-Yau manifolds interpreted as the complex line bundles over the Hermitian symmetric spaces. These manifolds are the various generalizations of the complex line bundle over $\mathbf{C}P^{N-1}$. Imposing an F-term constraint on the line bundle over $\mathbf{C}P^{N-1}$, we obtain the line bundle over the complex quadric surface Q^{N-2}. On the other hand, when we promote the $U(1)$ gauge symmetry in $\mathbf{C}P^{N-1}$ to the non-abelian gauge group $U(M)$, the line bundle over the Grassmann manifold is obtained. We construct the non-compact Calabi-Yau manifolds with isometries of exceptional groups, which we have not discussed in the previous papers. Each of these manifolds contains the resolution parameter which controls the size of the base manifold, and the conical singularity appears when the parameter vanishes.

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1 Introduction

Two-dimensional $\mathcal{N} = 2$ supersymmetric nonlinear sigma models on Calabi-Yau (Ricci-flat Kähler) manifolds are considered as the most important models of the superstring theory [1, 2, 3, 4]. While the supergravity is now regarded as the low-energy effective theory of string/M-theory, the supergravity solutions preserving some of the supersymmetry are considerable issues in the exploration both of perturbative and of non-perturbative properties in string and M-theory. In particular, the AdS/CFT correspondence is one of the most powerful conjectures to obtain these properties. A useful example of this correspondence has been to study D3-branes on the manifold with or without conical singularities. Recently, some resolution procedures of singularities have been extensively discussed (see, for example, [5, 6, 7, 8, 9, 10] and references therein), because such non-singular examples may provide important supergravity dual solutions of four-dimensional $\mathcal{N} = 1$ super-Yang-Mills theory in the infrared regime. It is also important to understand real manifolds with and without singularities in recent study of compactification of M-theory on manifolds with G_2 and $Spin(7)$ holonomies, which presents field theories with less supersymmetry (see, for example, [11, 12, 13, 14, 15]). Returning to the nonlinear sigma models and the string world-sheet on Calabi-Yau manifolds, it is important that we represent these manifolds in the complex coordinates from requirement of manifestation of supersymmetry.

In [16], we presented a simple construction of the $O(N)$ symmetric Ricci-flat Kähler metric which coincides with the Stenzel metric on the cotangent bundle over S^{N-1} [17, 10]. The conical singularity of the conifold is resolved by S^{N-1} with a radius being the deformation parameter. Low dimensional manifolds coincide with the Eguchi-Hanson gravitational instanton [18] or the six-dimensional deformed conifold [19, 20]. A new way of replacing the node of the conifold, which is different from neither the small resolution nor the deformation discussed in [19, 20], was found in [7, 21]. The new Ricci-flat Kähler manifold is identified as the complex line bundle over the complex quadric surface $Q^{N-2} = SO(N)/[SO(N-2) \times U(1)]$ [21]. In [22], we constructed new manifolds whose conical singularities are resolved by other Hermitian symmetric spaces (HSS, see [23, 24]) with classical groups, and found that these manifolds are the complex line bundles over the Grassmann manifold $G_{N,M} = SU(N)/[SU(N-M) \times U(M)]$, $SO(2N)/U(N)$ and $Sp(N)/U(N)$. All of these are gerenalizations of the line bundle over the complex projective space $\mathbf{C}P^{N-1} = SU(N)/[SU(N-1) \times U(1)]$ [25]. In this paper we present the new conifolds with isometries of E_6 and E_7. The conical singularities of these conifolds are resolved by the HSS of exceptional groups, $E_6/[SO(10) \times U(1)]$ and $E_7/[E_6 \times U(1)]$, and new manifolds are identified as the complex line bundles over these HSS. Then we summarize the non-compact Calabi-Yau manifolds interpreted as the complex line bundles over all of the HSS.

This paper is organized as follows. In section 2 we discuss the $\mathbf{C}P^{N-1}$ model and the complex line bundle over $\mathbf{C}P^{N-1}$. In section 3, we discuss the conifold as the complex line bundle over the complex quadric surface Q^{N-2}, which is the simple generalization of the line bundle over $\mathbf{C}P^{N-1}$. In

section 4, we discuss the *new* conifolds with isometries of the exceptional groups, E_6 and E_7. As in the conifold, the conical singularity of E_6 or E_7 conifold is resolved by the HSS of the exceptional group, $E_6/[SO(10) \times U(1)]$ or $E_7/[E_6 \times U(1)]$. In section 5, we review the gauge theoretical construction of the line bundle over the Grassmann manifold $G_{N,M}$ and its generalizations; the line bundles over $SO(2N)/U(N)$ and $Sp(N)/U(N)$. These models are the non-abelian generalizations of the line bundle over $\mathbf{C}P^{N-1}$. Section 6 is devoted to the conclusion. In appendix A, we define the $SO(10)$ γ-matrices and the charge conjugation matrix in the Weyl spinor basis. These representations are useful in performing explicit calculations in section 4. In appendix B, we review the E_6 and E_7 algebras.

2 Complex Line Bundle over $\mathbf{C}P^{N-1}$

In this section we review the gauge theoretical construction of the $\mathbf{C}P^{N-1}$ model [26]. Then we derive the complex line bundle over $\mathbf{C}P^{N-1}$ [25].

$\underline{\mathbf{C}P^{N-1} \text{ Model}}$.

We consider the global symmetry $G = SU(N)$ and the $U(1)_{\text{local}}$ symmetry. We introduce chiral superfields $\vec{\phi}(x, \theta, \bar{\theta})$, belonging to the fundamental representation of $SU(N)$, and a vector superfield $V(x, \theta, \bar{\theta})$ of the $U(1)_{\text{local}}$ gauge symmetry as an auxiliary field. The gauge transformation of $U(1)_{\text{local}}$ is given by

$$\vec{\phi} \rightarrow \vec{\phi}' = \vec{\phi} e^{-i\Lambda}, \quad e^V \rightarrow e^{V'} = e^{i\Lambda} e^V e^{-i\Lambda^\dagger}, \tag{2.1}$$

where $\Lambda(x, \theta, \bar{\theta})$ is a chiral superfield. Note that the local invariance group is enlarged to the complexification $U(1)_{\text{local}}^{\mathbf{C}}$ of the $U(1)_{\text{local}}$ gauge group, because the scalar component of $\Lambda(x, \theta, \bar{\theta})$ is a complex field. The Lagrangian invariant under the global $SU(N)$ and the $U(1)_{\text{local}}$ symmetries is given by

$$\mathcal{L} = \int d^4\theta (\vec{\phi}^\dagger \vec{\phi} e^V - cV). \tag{2.2}$$

The integrand is a Kähler potential, in which c is a real positive constant, and the term cV is called a Fayet-Iliopoulos (FI) D-term. Integrating out the vector superfield V, we obtain the following Kähler potential:

$$\Psi(\vec{\phi}, \vec{\phi}^\dagger) = c \log \left(\vec{\phi}^\dagger \vec{\phi} \right), \tag{2.3}$$

where we have omitted constant terms because they disappear under the integration over θ. The complexified gauge symmetry (2.1) can be fixed by choosing

$$\vec{\phi} = \begin{pmatrix} 1 \\ \varphi^i \end{pmatrix}, \tag{2.4}$$

where $\varphi^i(x, \theta, \bar{\theta})$ are chiral superfields $(i = 1, 2, \cdots, N-1)$. Substituting this into (2.3), we obtain the Kähler potential for the Fubini-Study metric on $\mathbf{C}P^{N-1}$:

$$\Psi(\varphi, \varphi^\dagger) \ = \ c \log \left(1 + |\varphi^i|^2\right), \tag{2.5}$$

where the summation over the index i is implied. Our choice of gauge (2.4) breaks global $SU(N)$ symmetry down to $SU(N-1) \times U(1)$ which preserves the vacuum expectation value $\langle \phi^1 \rangle = 1$. φ^i represent the massless Nambu-Goldstone bosons corresponding to the coordinates of the coset manifold $G/H = SU(N)/[SU(N-1) \times U(1)]$.

Complex Line Bundle over $\mathbf{C}P^{N-1}$.

Now let us construct the non-compact Calabi-Yau manifold. In general, the Ricci-flat condition is a set of partial differential equations difficult to solve. If we impose a global symmetry, however, this condition often reduces to a more tractable ordinary differential equation. In our example, we assume the Kähler potential $\mathcal{K}$ is a function of a single variable

$$X(\vec{\phi}, \vec{\phi}^\dagger) \ \equiv \ \log \vec{\phi}^\dagger \vec{\phi}, \tag{2.6}$$

which is invariant under the global $SU(N)$ symmetry. Here the logarithm in the definition of X is just a convention.

To construct the line bundle over $\mathbf{C}P^{N-1}$, we treat $U(1)_{\text{local}}$ symmetry in $\mathbf{C}P^{N-1}$ as a global symmetry by removing the vector superfield V. Although we do not have $U(1)_{\text{local}}$ symmetry, it is convenient to use the following parametrization to compare with the $\mathbf{C}P^{N-1}$ model,

$$\vec{\phi} \ = \ \sigma \begin{pmatrix} 1 \\ \varphi^i \end{pmatrix}, \tag{2.7}$$

where $\varphi^i(x, \theta, \bar{\theta})$ and $\sigma(x, \theta, \bar{\theta})$ are chiral superfields. The invariant superfield X is decomposed as

$$X \ = \ \log |\sigma|^2 + \Psi, \quad \Psi \ = \ \log \xi, \quad \xi \ \equiv \ 1 + |\varphi^i|^2. \tag{2.8}$$

Ψ is the Kähler potential (2.5) of $\mathbf{C}P^{N-1}$. (Hereafter we set $c = 1$.)

We make a comment on the symmetry breaking. We find that the total space can be regarded as

$$\mathbf{R} \times \frac{SU(N)}{SU(N-1)}, \tag{2.9}$$

at least locally. The part of $SU(N)/SU(N-1)$ is parametrized by the Nambu-Goldstone bosons arising from the spontaneous breaking of the global symmetry $SU(N)$ down to $SU(N-1)$, whereas the factor $\mathbf{R}$ is parametrized by the so-called quasi-Nambu-Goldstone boson [23, 27, 28].

Let us calculate the Kähler metric and the Ricci tensor. From now on we use the same _letters_ for chiral superfields _and_ their complex scalar components. The metric of the complex coordinates

is defined by $g_{\mu\nu^*} = \partial_\mu \partial_{\nu^*} \mathcal{K}$, where we express holomorphic coordinates by $z^\mu = (\sigma, \varphi^i)$ and the differentiation with respect to them by $\partial_\mu = \partial/\partial z^\mu$. The explicit expression of the metric is

$$g_{\mu\nu^*} = \begin{pmatrix} g_{\sigma\sigma^*} & g_{\sigma j^*} \\ g_{i\sigma^*} & g_{ij^*} \end{pmatrix} , \tag{2.10a}$$

with each block being

$$g_{\sigma\sigma^*} = \mathcal{K}'' \frac{\partial X}{\partial \sigma} \frac{\partial X}{\partial \sigma^*} , \quad g_{\sigma j^*} = \mathcal{K}'' \frac{\partial X}{\partial \sigma} \frac{\partial X}{\partial \varphi^{*j}} , \quad g_{ij^*} = \mathcal{K}'' \frac{\partial X}{\partial \varphi^i} \frac{\partial X}{\partial \varphi^{*j}} + \mathcal{K}' \frac{\partial^2 X}{\partial \varphi^i \partial \varphi^{*j}} , \tag{2.10b}$$

where the prime denotes the differentiation with respect to the argument X. Here we have used the following equations: $\frac{\partial^2 X}{\partial\sigma\partial\sigma^*} = \frac{\partial^2 X}{\partial\sigma\partial\varphi^{*j}} = 0$ $(\sigma \neq 0)$. The determinant of this metric is given by

$$\det g_{\mu\nu^*} = \frac{1}{|\sigma|^2} \mathcal{K}'' \cdot \det_{i,j} \left(\mathcal{K}' \frac{\partial^2 X}{\partial \varphi^i \partial \varphi^{*j}} \right) . \tag{2.11}$$

Since the Ricci tensor is defined by $(Ric)_{\mu\nu^*} = -\partial_\mu \partial_{\nu^*} \log \det g_{\kappa\lambda^*}$, the Ricci-flat condition $(Ric)_{\mu\nu^*} = 0$ implies

$$\det g_{\mu\nu^*} = (\text{constant}) \times |F|^2 , \tag{2.12}$$

where F is a holomorphic function. In order to obtain a concrete expression of the determinant (2.11), we need some derivatives of X with respect to φ^i:

$$\frac{\partial X}{\partial \varphi^{*j}} = \partial_{j^*} \Psi = \varphi^j \xi^{-1} , \quad \frac{\partial^2 X}{\partial \varphi^i \partial \varphi^{*j}} = \partial_i \partial_{j^*} \Psi = \delta_{ij} \xi^{-1} - \varphi^{*i} \varphi^j \xi^{-2} . \tag{2.13}$$

Note that the second quantity is just the Fubini-Study metric of $\mathbf{C}P^{N-1}$.

Although it is not difficult to evaluate the determinant (2.11), it is far simpler if we use its symmetry property. The determinant is invariant under the transformation $\varphi \to g\varphi$, where g belongs to a complex isotropy group $SU(N-1)^{\mathbf{C}} = SL(N-1, \mathbf{C})$ that leaves vacuum expectation values invariant. With suitable choice of g, we can assume only φ^1 has a non-vanishing value as a vacuum expectation value. Then the second derivative of X, (2.13), reduces to an $(N-1) \times (N-1)$ diagonal matrix:

$$\frac{\partial^2 X}{\partial \varphi^i \partial \varphi^{*j}} = \text{diag.} \big(\xi^{-2}, \underbrace{\xi^{-1}, \xi^{-1}, \cdots, \xi^{-1}}_{N-2} \big) . \tag{2.14}$$

Substituting this into (2.11), we have

$$\det g_{\mu\nu^*} = |\sigma|^{2N-2} e^{-NX} \mathcal{K}'' (\mathcal{K}')^{N-1} , \tag{2.15}$$

where we have used the relation $\xi = |\sigma|^{-2} e^X$ from (2.8).

The Ricci-flat condition (2.12) reduces to the desired ordinary differential equation

$$e^{-NX} \frac{d}{dX} (\mathcal{K}')^N = a , \tag{2.16}$$

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where a is a constant. The solution of (2.16) for $\mathcal{K}'$ is

$$\mathcal{K}' = \left(\lambda e^{NX} + b\right)^{\frac{1}{N}}, \tag{2.17}$$

where λ is a constant related to a and N, and b is an integration constant interpreted as a *resolution parameter* of the conical singularity. Although it is sufficient for us to obtain the metric from (2.17), we can calculate the Kähler potential itself:

$$\mathcal{K}(X) = \left(\lambda e^{NX} + b\right)^{\frac{1}{N}} + b^{\frac{1}{N}} \cdot I\left(b^{-\frac{1}{N}}\left(\lambda e^{NX} + b\right)^{\frac{1}{N}}; N\right), \tag{2.18}$$

where the function $I(y; n)$ is defined by

$$
\begin{aligned}
I(y; n) \equiv \int^y \frac{dt}{t^n - 1} &= \frac{1}{n}\left[\log\left(y - 1\right) - \frac{1 + (-1)^n}{2}\log\left(y + 1\right)\right] \\
&+ \frac{1}{n}\sum_{r=1}^{\left[\frac{n-1}{2}\right]} \cos\frac{2r\pi}{n} \cdot \log\left(y^2 - 2y\cos\frac{2r\pi}{n} + 1\right) \\
&+ \frac{2}{n}\sum_{r=1}^{\left[\frac{n-1}{2}\right]} \sin\frac{2r\pi}{n} \cdot \arctan\left[\frac{\cos(2r\pi/n) - y}{\sin(2r\pi/n)}\right].
\end{aligned} \tag{2.19}
$$

Let us calculate the components of the metric tensor. The component $g_{\sigma\sigma^*}$ is calculated as

$$g_{\sigma\sigma^*} = \lambda\left(\lambda e^{NX} + b\right)^{\frac{1-N}{N}} e^{N\Psi}|\sigma|^{2N-2}, \tag{2.20}$$

where Ψ is the Kähler potential obtained in (2.5) or (2.8). This metric has a singularity at the $\sigma = 0$ surface: $g_{\sigma\sigma^*} = 0$. However this singularity is just a coordinate singularity of $z^\mu = (\sigma, \varphi^i)$. To find a regular coordinate system we perform the coordinate transformation

$$\rho \equiv \sigma^N/N, \tag{2.21}$$

with φ^i being unchanged. Then the metric in the regular coordinates $z'^\mu = (\rho, \varphi^i)$ is calculated as

$$
\begin{aligned}
g_{\rho\rho^*} &= \lambda\left(\lambda e^{NX} + b\right)^{\frac{1-N}{N}} e^{N\Psi}, \\
g_{\rho j^*} &= \lambda N\left(\lambda e^{NX} + b\right)^{\frac{1-N}{N}} e^{N\Psi}\rho^* \cdot \partial_{j^*}\Psi, \\
g_{ij^*} &= \lambda N^2\left(\lambda e^{NX} + b\right)^{\frac{1-N}{N}} e^{N\Psi}|\rho|^2 \cdot \partial_i\Psi\partial_{j^*}\Psi + \left(\lambda e^{NX} + b\right)^{\frac{1}{N}} \cdot \partial_i\partial_{j^*}\Psi,
\end{aligned} \tag{2.22}
$$

where $\partial_{j^*}\Psi$ and $\partial_i\partial_{j^*}\Psi$ are given in (2.13).

The metric of the submanifold of $\rho = 0$ ($d\rho = 0$) is obtained from (2.22):

$$g_{ij^*}\big|_{\rho=0}(\varphi, \varphi^*) = b^{\frac{1}{N}}\partial_i\partial_{j^*}\Psi. \tag{2.23}$$

Since Ψ is the Kähler potential (2.5), $\partial_i\partial_{j^*}\Psi$ is nothing but the Fubini-Study metric (2.13) of $\mathbf{C}P^{N-1}$. Therefore we find that the total space is the complex line bundle over $\mathbf{C}P^{N-1}$ with fiber ρ. This can

be expected from the fact that there exists a Ricci-flat metric on the complex line bundle over any Kähler-Einstein manifolds [29].

In the limit of $b \to 0$, this base manifold shrinks to zero-size and a singularity appears. The Kähler potential (2.18) reduces to

$$\mathcal{K}\big|_{b=0} \;=\; \lambda^{\frac{1}{N}}|\sigma|^2\big(1+|\varphi^i|^2\big) \;=\; \lambda^{\frac{1}{N}}\vec{\phi}^\dagger\vec{\phi}\,, \tag{2.24}$$

where $\phi^1 = \sigma$ and $\phi^i = \sigma\varphi^{i-1}$ $(i = 2,3,\cdots,N)$. One might consider that this Kähler potential would give a flat metric, but it is not the case; We need a coordinate identification $\rho = \sigma^N/N$ (2.21). The range of $\arg\rho$ has to be $0 \le \arg\rho \le 2\pi$ to avoid a conical singularity at $\rho = 0$ (in the case of $b \ne 0$), that is, the range of $\arg\sigma$ is $0 \le \arg\sigma \le 2\pi/N$ and all points with $\arg\sigma$ and $\arg\sigma + 2\pi k/N$ $(k = 1,2,\cdots,N)$ are identified. Therefore, the manifold of the singular limit is an orbifold $\mathbf{C}^N/\mathbf{Z}_N$. Imposing the parameter b as a non-zero value, we can replace the conical singularity with $\mathbf{C}P^{N-1}$ of radius $b^{\frac{1}{2N}}$. In the case of $N = 2$, it is the Eguchi-Hanson space [18].

3   Conifold

In this section we discuss the gauge theoretical construction of the complex quadric surface $Q^{N-2} = SO(N)/[SO(N-2) \times U(1)]$ [23, 24, 30], followed by the discussion on the Ricci-flat metric on the conifold, which can be regarded as the complex line bundle over Q^{N-2} [21].

Imposing an appropriate constraint on the $\mathbf{C}P^{N-1}$ model, we can obtain the complex quadric surface Q^{N-2} [23, 24, 30]. The Lagrangian is given by

$$\mathcal{L} \;=\; \int d^4\theta\big(\vec{\phi}^\dagger\vec{\phi}e^V - cV\big) + \Big(\int d^2\theta\,\phi_0\vec{\phi}^T J\vec{\phi} + \text{c.c.}\Big)\,, \tag{3.1}$$

where the Kähler potential is the same as the one of the $\mathbf{C}P^{N-1}$ model in (2.2). The integrand of the second term is the superpotential, in which $\phi_0(x,\theta,\bar{\theta})$ is an auxiliary chiral superfield, and J is the $SO(N)$-invariant rank-2 symmetric tensor, given by

$$J \;=\; \begin{pmatrix} 0 & \mathbf{0} & 1 \\ \mathbf{0} & \mathbf{1}_{N-2} & \mathbf{0} \\ 1 & \mathbf{0} & 0 \end{pmatrix}\,, \tag{3.2}$$

where $\mathbf{1}_{N-2}$ is an $(N-2) \times (N-2)$ unit matrix. By the integration over V, we obtain the Kähler potential (2.3) in the same form with $\mathbf{C}P^{N-1}$. The integration over ϕ_0 gives the constraint

$$\vec{\phi}^T J\vec{\phi} \;=\; 0\,. \tag{3.3}$$

This can be solved as

$$
\vec{\phi} \;=\; \begin{pmatrix} 1 \\ \varphi^i \\ -\tfrac{1}{2}(\varphi^i)^2 \end{pmatrix} \,,
\tag{3.4}
$$

where we have chosen the gauge of $\phi^1 = 1$ using the complexified gauge symmetry, and $\varphi^i(x, \theta, \bar{\theta})$ are chiral superfields ($i = 1, 2, \cdots, N - 2$), whose scalar components parametrize the manifold. The Kähler potential written in terms of φ^i is the one of Q^{N-2} in the standard coordinates:

$$
\Psi(\varphi, \varphi^\dagger) \;=\; c \log \left(1 + |\varphi^i|^2 + \frac{1}{4}(\varphi^i)^2(\varphi^{\dagger j})^2 \right) \,.
\tag{3.5}
$$

Let us construct the Ricci-flat metric on the conifold as the complex line bundle over Q^{N-2}, as in the same manner as in the case of $\mathbf{C}P^{N-1}$. We assume that the Kähler potential $\mathcal{K}$ is a function of the invariant superfield $X = \log \vec{\phi}^\dagger \vec{\phi}$:

$$
\mathcal{L} \;=\; \int d^4\theta \, \mathcal{K}(X) + \left(\int d^2\theta \, \phi_0 \vec{\phi}^T J \vec{\phi} + \text{c.c.} \right) \,.
\tag{3.6}
$$

Since the holomorphic constraint from the integration over ϕ_0 is again (3.3), this non-compact manifold is a conifold. The constraint (3.3) can be solved as

$$
\vec{\phi} \;=\; \sigma \begin{pmatrix} 1 \\ \varphi^i \\ -\tfrac{1}{2}(\varphi^i)^2 \end{pmatrix} \,,
\tag{3.7}
$$

where $\sigma(x, \theta, \bar{\theta})$ is a chiral superfield. By comparing this expression with (3.4), we expect that σ is a fiber, and φ^i parametrize the base manifold Q^{N-2}, with the total space being the complex line bundle over Q^{N-2}. With this parametrization, the invariant superfield X is decomposed as

$$
X \;=\; \log |\sigma|^2 + \Psi \,, \quad \Psi \;=\; \log \xi \,,
\tag{3.8a}
$$

$$
\xi \;\equiv\; 1 + |\varphi^i|^2 + \frac{1}{4}(\varphi^i)^2(\varphi^{\dagger j})^2 \,,
\tag{3.8b}
$$

where Ψ is the Kähler potential of Q^{N-2} (3.5). This non-compact manifold can be regarded as $\mathbf{R} \times SO(N)/SO(N-2)$ at least locally.

As in the last section, we use the same letters for the chiral superfields and their scalar components. The metric is defined in (2.10) where the holomorphic coordinates are given by $z^\mu = (\sigma, \varphi^i)$. The determinant $\det g_{\mu\nu^*}$ is written as

$$
\det g_{\mu\nu^*} \;=\; \frac{1}{|\sigma|^2} \mathcal{K}'' \cdot \det_{i,j} \left(\mathcal{K}' \frac{\partial^2 X}{\partial \varphi^i \partial \varphi^{*j}} \right) \,,
\tag{3.9}
$$

with the second derivative of X being

$$
\frac{\partial^2 X}{\partial \varphi^i \partial \varphi^{*j}} \;=\; \partial_i \partial_{j^*} \Psi \;=\; \left(\delta_{ij} + \varphi^i \varphi^{*j} \right) \xi^{-1} - \left(\varphi^{*i} + \varphi^i (\varphi^{*k})^2 \right) \left(\varphi^j + (\varphi^l)^2 \varphi^{*j} \right) \xi^{-2} \,.
\tag{3.10}
$$

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Under the complex isotropy transformation of $SO(N-2)^\mathbf{C}$, we can calculate the determinant in (3.9) with ease:

$$\det g_{\mu\nu^*} = |\sigma|^{2N-6} e^{-(N-2)X} \mathcal{K}''(\mathcal{K}')^{N-2} . \tag{3.11}$$

The Ricci-flat condition (2.12) can be solved for $\mathcal{K}'$:

$$\mathcal{K}' = \left(\lambda e^{(N-2)X} + b\right)^{\frac{1}{N-1}} , \tag{3.12}$$

where λ is a constant and b is an integration constant interpreted as a resolution parameter of the conical singularity. Since it is sufficient for us to construct the metric from (3.12), we do not write down the explicit expression of the Kähler potential itself (see [21]).

We can immediately obtain the Ricci-flat metric from (3.12). The component $g_{\sigma\sigma^*}$ is given by

$$g_{\sigma\sigma^*} = \frac{N-2}{N-1}\lambda\left(\lambda e^{(N-2)X} + b\right)^{-\frac{N-2}{N-1}} e^{(N-2)\Psi} |\sigma|^{2N-6} . \tag{3.13}$$

This component is not regular at the $\sigma = 0$ surface: $g_{\sigma\sigma^*}|_{\sigma=0} = 0$. However this is just a coordinate singularity of $z^\mu = (\sigma, \varphi^i)$. As in the case of the line bundle over $\mathbf{C}P^{N-1}$, we perform the following transformation:

$$\rho \equiv \frac{\sigma^{N-2}}{N-2} , \tag{3.14}$$

with φ^i being unchanged. Under this transformation, the metric in this coordinate system becomes regular in the whole region (including $\rho = 0$):

$$g_{\rho\rho^*} = \lambda\frac{N-2}{N-1}\left(\lambda e^{(N-2)X} + b\right)^{\frac{2-N}{N-1}} e^{(N-2)\Psi} , \tag{3.15a}$$

$$g_{\rho j^*} = \lambda\frac{(N-2)^2}{N-1}\left(\lambda e^{(N-2)X} + b\right)^{\frac{2-N}{N-1}} e^{(N-2)\Psi} \rho^* \cdot \partial_{j^*}\Psi , \tag{3.15b}$$

$$g_{ij^*} = \lambda\frac{(N-2)^3}{N-1}\left(\lambda e^{(N-2)X} + b\right)^{\frac{2-N}{N-1}} e^{(N-2)\Psi} |\rho|^2 \cdot \partial_i\Psi\partial_{j^*}\Psi$$
$$+ \left(\lambda e^{(N-2)X} + b\right)^{\frac{1}{N-1}} \cdot \partial_i\partial_{j^*}\Psi . \tag{3.15c}$$

The metric of the submanifold of $\rho = 0$ $(d\rho = 0)$ is obtained as

$$g_{ij^*}\big|_{\rho=0}(\varphi, \varphi^*) = b^{\frac{1}{N-1}}\partial_i\partial_{j^*}\Psi . \tag{3.16}$$

This is the metric of Q^{N-2} given in (3.10), since Ψ is its Kähler potential (3.5). In the limit of $b \to 0$, this submanifold shrinks to zero-size and the total space becomes a conifold. When $b \neq 0$, the conical singularity is resolved by Q^{N-2} of radius $b^{\frac{1}{2(N-1)}}$. Thus this non-compact Calabi-Yau manifold can be regarded as the complex line bundle over Q^{N-2} with fiber ρ.

Let us make some comments. When $N = 3$, the line bundle over Q^1 coincides with the Eguchi-Hanson gravitational instanton. In the case of $N = 4$, the manifold becomes the line bundle over $Q^2 \simeq S^2 \times S^2$ (the radii of these two S^2 coincide) [7]. The way of removing this conical singularity is different from either the deformation by S^3 [19, 20] or the small resolution by S^2 [19] known in the six-dimensional conifold.

4 Conifolds with Isometries of E_6 and E_7

In the last section we have explained the $O(N)$ symmetric conifold which can be regarded as the complex line bundle over Q^{N-2}. In this section, we construct the *new* conifolds whose isometries are the exceptional groups E_6 and E_7, in the same way as the previous sections.

4.1 Hermitian Symmetric Spaces $E_6/[SO(10) \times U(1)]$ and $E_7/[E_6 \times U(1)]$

In this subsection we give the gauge theoretical construction of the HSS of the exceptional groups, $E_6/[SO(10) \times U(1)]$ and $E_7/[E_6 \times U(1)]$ [23]. We start from the Lagrangian

$$\mathcal{L} \;=\; \int d^4\theta \, (\vec{\phi}^\dagger \vec{\phi} e^V - c\,V) + \left(\int d^2\theta \, W(\vec{\phi}_0, \vec{\phi}) + \text{c.c.} \right), \tag{4.1}$$

where $\vec{\phi}(x, \theta, \bar{\theta})$ are chiral superfields belonging to the fundamental representation of the global symmetry E_6 or E_7, and $V(x, \theta, \bar{\theta})$ is an auxiliary vector superfield of $U(1)_{\text{local}}$ gauge symmetry. Later we define the superpotential W as a function of $\vec{\phi}$ and auxiliary chiral superfields $\vec{\phi}_0(x, \theta, \bar{\theta})$, which also belong to the fundamental representation.

$\underline{E_6/[SO(10) \times U(1)]}.$

We consider the global symmetry $G = E_6$ and a $U(1)_{\text{local}}$ gauge symmetry. The chiral superfields $\vec{\phi}$ belong to the fundamental representation $\mathbf{27}$ of E_6. We decompose E_6 under its maximal subgroup $SO(10) \times U(1)$.

Since the fundamental representation can be decomposed as $\mathbf{27} = (\mathbf{1}, 4) \oplus (\mathbf{16}, 1) \oplus (\mathbf{10}, -2)$ [31], where the second entries are the $U(1)$ charges, the fields $\vec{\phi}$ can be written as

$$\vec{\phi} \;=\; \begin{pmatrix} x \\ y_\alpha \\ z^A \end{pmatrix}. \tag{4.2}$$

Here, x, y_α ($\alpha = 1, \cdots, 16$) and z^A ($A = 1, \cdots, 10$) are an $SO(10)$ scalar, a Weyl spinor and a vector, respectively.

The decomposition of the tensor product, $\mathbf{27} \otimes \mathbf{27} = \overline{\mathbf{27}}_{\text{s}} \oplus \cdots$, implies that there exist the rank-3 symmetric invariant tensor Γ_{ijk} and its complex conjugate Γ^{ijk}, whose components can be read from the invariant [32]

$$I_3 \;\equiv\; \Gamma_{ijk}\phi^i\phi^j\phi^k \;=\; xz^2 + \frac{1}{\sqrt{2}}z^A(yC\sigma_A^\dagger y), \tag{4.3}$$

where we have used the decomposition (4.2). Here the indices i, j, k run from 1 to 27. For the product of the invariant tensors, some identities

$$\Gamma_{ijk}\Gamma^{ijl} \;=\; 10\delta_k^l, \tag{4.4a}$$

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$$\Gamma_{ijk}(\Gamma^{jl\{m}\Gamma^{np\}k}) \;=\; \delta_i{}^{\{l}\Gamma^{mnp\}} \,, \tag{4.4b}$$

hold, where we have used the notation $A^{\{ij\cdots\}} = A^{ij\cdots} + A^{ji\cdots} + \cdots$. (The second one is called the Springer relation.) These identities are used many times in the analysis of the E_7 algebra.

Then we define the superpotential by

$$W(\vec{\phi}_0, \vec{\phi}) \;=\; \Gamma_{ijk}\phi_0{}^i \phi^j \phi^k \,, \tag{4.5}$$

where $\vec{\phi}_0$ represent auxiliary fields whose $U(1)_{\text{local}}$ charge is -2 when the $U(1)_{\text{local}}$ charge of $\vec{\phi}$ is 1. The equations of motion for the auxiliary fields $\phi_0{}^i$, $\partial W/\partial \phi_0{}^i = \Gamma_{ijk}\phi^j\phi^k = 0$, are

$$\partial W/\partial z_0{}^A \;=\; 2z_A x + \frac{1}{\sqrt{2}}(yC\sigma_A^\dagger y) \;=\; 0 \,, \tag{4.6a}$$

$$\partial W/\partial y_{0\alpha} \;=\; \sqrt{2}(C\sigma_A^\dagger y)^\alpha z^A \;=\; 0 \,, \tag{4.6b}$$

$$\partial W/\partial x_0 \;=\; z^2 \;=\; 0 \,, \tag{4.6c}$$

In the second equation, we have used the fact that $(C\sigma_A^\dagger)^{\alpha\beta}$ is symmetric. The first equation can be solved to yield

$$z_A \;=\; -\frac{1}{2\sqrt{2}x}(yC\sigma_A^\dagger y) \,. \tag{4.7}$$

We can show that the last two equations are not independent of the first [23]. Therefore we can write the fundamental field as

$$\vec{\phi} \;=\; \begin{pmatrix} x \\ y_\alpha \\ -\frac{1}{2\sqrt{2}x}(yC\sigma_A^\dagger y) \end{pmatrix} \,. \tag{4.8}$$

Integrating out the auxiliary field V, we obtain the Kähler potential as the same form of (2.3). Using complexified gauge symmetry, we can choose the gauge of $x = 1$:

$$\vec{\phi} \;=\; \begin{pmatrix} 1 \\ \varphi_\alpha \\ -\frac{1}{2\sqrt{2}}(\varphi C\sigma_A^\dagger \varphi) \end{pmatrix} \,, \tag{4.9}$$

where we have rewritten y_α as φ_α. Substituting this into (2.3), we obtain the following expression:

$$\Psi(\varphi, \varphi^\dagger) \;=\; c\log\left(1 + |\varphi_\alpha|^2 + \frac{1}{8}|\varphi C\sigma_A^\dagger \varphi|^2\right) \,. \tag{4.10}$$

This is the Kähler potential of $E_6/[SO(10) \times U(1)]$ in the standard coordinates.

$E_7/[E_6 \times U(1)]$.

The global symmetry in this case is $G = E_7$ and the local symmetry is $U(1)_{\text{local}}$. The chiral superfields

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$\vec{\phi}$ belong to the fundamental representation **56** of E_7. Since the maximal subgroup of E_7 is $E_6 \times U(1)$, we construct E_7 from $E_6 \times U(1)$.

The fundamental representation can be decomposed as $\mathbf{56} = (\mathbf{27}, -\frac{1}{3}) \oplus (\overline{\mathbf{27}}, \frac{1}{3}) \oplus (\mathbf{1}, -1) \oplus (\mathbf{1}, 1)$. Thus we write $\vec{\phi}$ as

$$\vec{\phi} = \begin{pmatrix} x \\ y^i \\ z_i \\ w \end{pmatrix} , \tag{4.11}$$

where y^i and z_i are **27** and $\overline{\mathbf{27}}$ representations of E_6, respectively; x and w are scalars.

There exists the rank-4 symmetric invariant tensor $d_{\alpha\beta\gamma\delta}$, whose components can be read from the invariant [23]

$$\begin{aligned} I_4 &\equiv d_{\alpha\beta\gamma\delta}\phi^\alpha\phi^\beta\phi^\gamma\phi^\delta \\ &= -\frac{1}{2}(xw - y^i z_i)^2 - \frac{1}{3}w\Gamma_{ijk}y^i y^j y^k - \frac{1}{3}x\Gamma^{ijk}z_i z_j z_k + \frac{1}{2}\Gamma^{ijk}\Gamma_{ilm}z_j z_k y^l y^m , \end{aligned} \tag{4.12}$$

where I_4 is invariant due to the Springer relation for the E_6 invariant tensor (4.4b).

By using this invariant tensor, the superpotential invariant under $E_7 \times U(1)_{\text{local}}$ is given by

$$W(\vec{\phi}_0, \vec{\phi}) = d_{\alpha\beta\gamma\delta}\phi_0{}^\alpha\phi^\beta\phi^\gamma\phi^\delta , \tag{4.13}$$

where $\phi_0{}^\alpha$ are auxiliary fields belonging to $(\mathbf{56}, -3)$. Here the second component is the $U(1)_{\text{local}}$ charge assigned to cancel the $U(1)_{\text{local}}$ charge of ϕ^α. The integration over $\phi_0{}^\alpha$ gives the constraints $\partial W/\partial\phi_0{}^\alpha = d_{\alpha\beta\gamma\delta}\phi^\beta\phi^\gamma\phi^\delta = 0$:

$$\partial W/\partial y_0{}^i = w(xz_i - \Gamma_{ijk}y^j y^k) - z_i y^j z_j + \Gamma^{jkl}\Gamma_{jim}z_k z_l y^m = 0 , \tag{4.14a}$$

$$\partial W/\partial w_0 = xy^i z_i - wx^2 - \frac{1}{3}\Gamma_{ijk}y^i y^j y^k = 0 , \tag{4.14b}$$

$$\partial W/\partial z_{0i} = x(wy^i - \Gamma^{ijk}z_j z_k) - y^i y^j z_j + \Gamma^{jik}\Gamma_{jlm}z_k y^l y^m = 0 , \tag{4.14c}$$

$$\partial W/\partial x_0 = wy^i z_i - xw^2 - \frac{1}{3}\Gamma^{ijk}z_i z_j z_k = 0 . \tag{4.14d}$$

First two equations can be solved as [23]

$$z_i = \frac{1}{2x}\Gamma_{ijk}y^j y^k , \quad w = \frac{1}{6x^2}\Gamma_{ijk}y^i y^j y^k . \tag{4.15}$$

It was shown in [23] that the last two equations are not independent. Then, we have

$$\vec{\phi} = \begin{pmatrix} x \\ y^i \\ \frac{1}{2x}\Gamma_{ijk}y^j y^k \\ \frac{1}{6x^2}\Gamma_{ijk}y^i y^j y^k \end{pmatrix} . \tag{4.16}$$

We obtain the Kähler potential (2.3) from the integration over V. The complexified gauge symmetry can be fixed as $x = 1$:

$$\vec{\phi} = \begin{pmatrix} 1 \\ \varphi^i \\ \frac{1}{2}\Gamma_{ijk}\varphi^j\varphi^k \\ \frac{1}{6}\Gamma_{ijk}\varphi^i\varphi^j\varphi^k \end{pmatrix} , \tag{4.17}$$

where we have rewritten y^i as φ^i. Substituting this into (2.3), we obtain

$$\Psi(\varphi, \varphi^\dagger) = c \log\left(1 + |\varphi^i|^2 + \frac{1}{4}|\Gamma_{ijk}\varphi^j\varphi^k|^2 + \frac{1}{36}|\Gamma_{ijk}\varphi^i\varphi^j\varphi^k|^2\right) . \tag{4.18}$$

This is the Kähler potential of $E_7/[E_6 \times U(1)]$ in the standard coordinates.

4.2 Construction of Line Bundles

In this subsection, we construct the conifolds with isometries of E_6 and E_7, which can be regarded as the complex line bundles over $E_6/[SO(10) \times U(1)]$ and $E_7/[E_6 \times U(1)]$, respectively. As in the previous sections, we obtain these manifolds when $U(1)_{\text{local}}$ is not gauged.

First we discuss the case of $E_6/[SO(10) \times U(1)]$. Since the $U(1)_{\text{local}}$ symmetry is not gauged, the chiral superfield satisfying only the F-term constraints (4.6) can be written as (2.7):

$$\vec{\phi} = \sigma \begin{pmatrix} 1 \\ \varphi_\alpha \\ -\frac{1}{2\sqrt{2}}(\varphi C\sigma_A^\dagger\varphi) \end{pmatrix} , \tag{4.19}$$

where $\varphi_\alpha(x, \theta, \bar{\theta})$ are chiral superfields belonging to an $SO(10)$ Weyl spinor ($\alpha = 1, 2, \cdots, 16$), and $\sigma(x, \theta, \bar{\theta})$ is a chiral superfield. We find that σ and φ_α parametrize a fiber and a base manifold, with the total space being a complex line bundle over $E_6/[SO(10) \times U(1)]$. σ_A are $SO(10)$ γ-matrices in the Weyl spinor basis ($A = 1, 2, \cdots, 10$) and C is a charge conjugation matrix (represented in appendix A).

Under the expression (4.19), the invariant superfield $X = \log \vec{\phi}^\dagger\vec{\phi}$ is decomposed as

$$X = \log|\sigma|^2 + \Psi , \quad \Psi = \log\xi , \tag{4.20a}$$

$$\xi \equiv 1 + |\varphi_\alpha|^2 + \frac{1}{8}|\varphi C\sigma_A^\dagger\varphi|^2 , \tag{4.20b}$$

where Ψ is the Kähler potential of $E_6/[SO(10) \times U(1)]$ defined in (4.10). This non-compact manifold can be regarded as $\mathbf{R} \times E_6/SO(10)$ at least locally.

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In the case of $E_7/[E_6 \times U(1)]$, the chiral superfields satisfying only the F-term constraints (4.14) can be written in the same way as (4.19):

$$
\vec{\phi} = \sigma \begin{pmatrix} 1 \\ \varphi^i \\ \frac{1}{2}\Gamma_{ijk}\varphi^j\varphi^k \\ \frac{1}{6}\Gamma_{ijk}\varphi^i\varphi^j\varphi^k \end{pmatrix} .
\tag{4.21}
$$

Here chiral superfields $\varphi^i(x,\theta,\bar{\theta})$, belonging to the **27** representation of E_6, parametrize a base manifold, and $\sigma(x,\theta,\bar{\theta})$ is a chiral superfield parametrizing a fiber, with the total space being a complex line bundle over $E_7/[E_6 \times U(1)]$. Γ_{ijk} is the rank-3 symmetric tensor invariant of E_6. Under the expression (4.21), the invariant superfield $X = \log \vec{\phi}^\dagger \vec{\phi}$ is decomposed as

$$
X = \log |\sigma|^2 + \Psi , \quad \Psi = \log \xi ,
\tag{4.22a}
$$

$$
\xi \equiv 1 + |\varphi^i|^2 + \frac{1}{4}|\Gamma_{ijk}\varphi^j\varphi^k|^2 + \frac{1}{36}|\Gamma_{ijk}\varphi^i\varphi^j\varphi^k|^2 ,
\tag{4.22b}
$$

where Ψ is the Kähler potential of $E_7/[E_6 \times U(1)]$ given by (4.18). This manifold can be also regarded as $\mathbf{R} \times E_7/E_6$, at least locally.

The metric is defined in (2.10) where the holomorphic coordinates (and their conjugates) are $z^\mu = (\sigma, \varphi_\alpha)$ $[z^{*\mu} = (\sigma^*, \varphi_\alpha^*)]$ for $E_6/[SO(10) \times U(1)]$ or $z^\mu = (\sigma, \varphi^i)$ $[z^{*\mu} = (\sigma^*, \varphi_i^*)]$ for $E_7/[E_6 \times U(1)]$. (The notation of φ_i^* is because of the fact that φ^i belong to a complex representation of E_6.) The determinant $\det g_{\mu\nu^*}$ is written as

$$
\det g_{\mu\nu^*} = \begin{cases} \dfrac{1}{|\sigma|^2}\mathcal{K}'' \cdot \det\limits_{\alpha,\beta}\left(\mathcal{K}'\dfrac{\partial^2 X}{\partial\varphi_\alpha\partial\varphi_\beta^*}\right) & \text{for } E_6/[SO(10) \times U(1)] , \\[4mm] \dfrac{1}{|\sigma|^2}\mathcal{K}'' \cdot \det\limits_{i,j}\left(\mathcal{K}'\dfrac{\partial^2 X}{\partial\varphi^i\partial\varphi_j^*}\right) & \text{for } E_7/[E_6 \times U(1)] . \end{cases}
\tag{4.23}
$$

The X differentiated once or twice are calculated as

$$
\frac{\partial X}{\partial\varphi_\alpha} = \partial_\alpha\Psi = \frac{1}{\xi}\left\{\varphi_\alpha^* + \frac{1}{4}(C\sigma_A^\dagger\varphi)^\alpha(\varphi^*\sigma^A C^\dagger\varphi^*)\right\} ,
\tag{4.24a}
$$

$$
\begin{aligned}
\frac{\partial^2 X}{\partial\varphi_\alpha\partial\varphi_\beta^*} = \partial_\alpha\partial_{\beta^*}\Psi = {} & \frac{1}{\xi}\left\{\delta_{\alpha\beta} + \frac{1}{2}(\sigma^A C^\dagger\varphi^*)^\beta(C\sigma_A^\dagger\varphi)^\alpha\right\} \\
& - \frac{1}{\xi^2}\left\{\varphi_\alpha^* + \frac{1}{4}(C\sigma_A^\dagger\varphi)^\alpha(\varphi^*\sigma^A C^\dagger\varphi^*)\right\}\left\{\varphi_\beta + \frac{1}{4}(\sigma^A C^\dagger\varphi^*)^\beta(\varphi C\sigma_A^\dagger\varphi)\right\} ,
\end{aligned}
\tag{4.24b}
$$

for $E_6/[SO(10) \times U(1)]$ or

$$
\frac{\partial X}{\partial\varphi^i} = \partial_i\Psi = \frac{1}{\xi}\left\{\varphi_i^* + \frac{1}{2}(\Gamma_{ijk}\varphi^k)(\Gamma^{jlm}\varphi_l^*\varphi_m^*) + \frac{1}{12}(\Gamma_{ijk}\varphi^j\varphi^k)(\Gamma^{lmn}\varphi_l^*\varphi_m^*\varphi_n^*)\right\} ,
\tag{4.25a}
$$

$$
\frac{\partial^2 X}{\partial\varphi^i\partial\varphi_j^*} = \partial_i\partial_{j^*}\Psi = \frac{1}{\xi}\left\{\delta_{ij} + (\Gamma_{ikl}\varphi^l)(\Gamma^{jkm}\varphi_m^*) + \frac{1}{4}(\Gamma_{ikl}\varphi^k\varphi^l)(\Gamma^{jmn}\varphi_m^*\varphi_n^*)\right\}
$$

$$- \frac{1}{\xi^2}\left\{\varphi_i^* + \frac{1}{2}(\Gamma_{ikl}\varphi^l)(\Gamma^{kmn}\varphi_m^*\varphi_n^*) + \frac{1}{12}(\Gamma_{ikl}\varphi^k\varphi^l)(\Gamma^{mnp}\varphi_m^*\varphi_n^*\varphi_p^*)\right\}$$

$$\times \left\{\varphi^j + \frac{1}{2}(\Gamma_{klm}\varphi^l\varphi^m)(\Gamma^{jkn}\varphi_n^*) + \frac{1}{12}(\Gamma_{klm}\varphi^k\varphi^l\varphi^m)(\Gamma^{jnp}\varphi_n^*\varphi_p^*)\right\}, \quad (4.25b)$$

for $E_7/[E_6 \times U(1)]$.

Using the complex isotropy transformation of $SO(10)^{\mathbf{C}}$ $[E_6{}^{\mathbf{C}}]$, we can put one component of φ_α $[\varphi^i]$ to non-zero with others being zero values, as a vacuum expectation value. This is because the determinant $\det g_{\mu\nu^*}$ is invariant under this transformation. Thus the partial differential equation (2.12) reduces to an ordinary differential equation. Here we put $\varphi_1 \neq 0$ $[\varphi^1 \neq 0]$, and others to zero:

$$\varphi_1 \neq 0, \quad \varphi_2 = \varphi_3 = \cdots = \varphi_{16} = 0 \qquad \text{for } E_6/[SO(10) \times U(1)], \qquad (4.26a)$$

$$\varphi^1 \neq 0, \quad \varphi^2 = \varphi^3 = \cdots = \varphi^{27} = 0 \qquad \text{for } E_7/[E_6 \times U(1)]. \qquad (4.26b)$$

Under the representations (A.1) and (A.3), we calculate second derivatives (4.24b) and (4.25b):

$$\frac{\partial^2 X}{\partial \varphi_\alpha \partial \varphi_\beta^*} = \begin{cases} \xi^{-2} & \alpha = \beta = 1 \\ \xi^{-1} & \alpha = \beta = 2,3,4,5,10,11,12,14,15,16 \\ 1 & \alpha = \beta = 6,7,8,9,13 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } E_6/[SO(10) \times U(1)], \quad (4.27a)$$

$$\frac{\partial^2 X}{\partial \varphi^i \partial \varphi_j^*} = \begin{cases} \xi^{-2} & i = j = 1 \\ \xi^{-1} & i = j = 2,3,\cdots,17 \\ 1 & i = j = 18,19,\cdots,27 \\ 0 & \text{otherwise} \end{cases} \quad \text{for } E_7/[E_6 \times U(1)]. \quad (4.27b)$$

Then the determinant is obtained as

$$\det g_{\mu\nu^*} = \begin{cases} |\sigma|^{22} e^{-12X} \mathcal{K}''(\mathcal{K}')^{16} & \text{for } E_6/[SO(10) \times U(1)], \\ |\sigma|^{34} e^{-18X} \mathcal{K}''(\mathcal{K}')^{27} & \text{for } E_7/[E_6 \times U(1)], \end{cases} \qquad (4.28)$$

where we have used $\xi = |\sigma|^{-2} e^X$ from (4.20) or (4.22). The Ricci-flat condition (2.12) can be solved for $\mathcal{K}'$:

$$\mathcal{K}' = \begin{cases} \left(\lambda e^{12X} + b\right)^{\frac{1}{17}} & \text{for } E_6/[SO(10) \times U(1)], \\ \left(\lambda e^{18X} + b\right)^{\frac{1}{28}} & \text{for } E_7/[E_6 \times U(1)], \end{cases} \qquad (4.29)$$

where λ is a constant and b is an integration constant regarded as a resolution parameter of the conical singularity. The Kähler potential can be also calculated using (2.19), to give

$$\mathcal{K}(X) = \begin{cases} \dfrac{17}{12}\left[\left(\lambda e^{12X} + b\right)^{\frac{1}{17}} + b^{\frac{1}{17}} \cdot I\left(b^{-\frac{1}{17}}\left(\lambda e^{12X} + b\right)^{\frac{1}{17}}; 17\right)\right] & \text{for } E_6/[SO(10) \times U(1)], \\[4mm] \dfrac{14}{9}\left[\left(\lambda e^{18X} + b\right)^{\frac{1}{28}} + b^{\frac{1}{28}} \cdot I\left(b^{-\frac{1}{28}}\left(\lambda e^{18X} + b\right)^{\frac{1}{28}}; 28\right)\right] & \text{for } E_7/[E_6 \times U(1)]. \end{cases}$$

$$(4.30)$$

Now we can immediately obtain the Ricci-flat metric from the solution (4.29) or (4.30). The component $g_{\sigma\sigma^*}$ is calculated as

$$
g_{\sigma\sigma^*} = \begin{cases} \dfrac{12}{17}\lambda\big(\lambda e^{12X} + b\big)^{-\frac{16}{17}} e^{12\Psi}|\sigma|^{22} & \text{for } E_6/[SO(10)\times U(1)] \,, \\[3mm] \dfrac{9}{14}\lambda\big(\lambda e^{18X} + b\big)^{-\frac{27}{28}} e^{18\Psi}|\sigma|^{34} & \text{for } E_7/[E_6\times U(1)] \,. \end{cases}
\tag{4.31}
$$

Each is not regular at the $\sigma = 0$ surface: $g_{\sigma\sigma^*}|_{\sigma=0} = 0$. However it is just a coordinate singularity; If we perform the coordinate transformation

$$
\rho \equiv \begin{cases} \sigma^{12}/12 & \text{for } E_6/[SO(10)\times U(1)] \,, \\ \sigma^{18}/18 & \text{for } E_7/[E_6\times U(1)] \,, \end{cases}
\tag{4.32}
$$

with φ_α or φ^i being unchanged, the components of the metrics in new coordinates become

$$
\begin{aligned}
g_{\rho\rho^*} &= \frac{12}{17}\lambda\big(\lambda e^{12X} + b\big)^{-\frac{16}{17}} e^{12\Psi} \,, \\
g_{\rho\beta^*} &= \frac{144}{17}\lambda\big(\lambda e^{12X} + b\big)^{-\frac{16}{17}} e^{12\Psi}\rho^* \cdot \partial_{\beta^*}\Psi \,, \\
g_{\alpha\beta^*} &= \frac{1728}{17}\lambda\big(\lambda e^{12X} + b\big)^{-\frac{16}{17}} e^{12\Psi}|\rho|^2 \cdot \partial_\alpha\Psi\partial_{\beta^*}\Psi + \big(\lambda e^{12X} + b\big)^{\frac{1}{17}} \cdot \partial_\alpha\partial_{\beta^*}\Psi \,,
\end{aligned}
\tag{4.33}
$$

for $E_6/[SO(10)\times U(1)]$, and

$$
\begin{aligned}
g_{\rho\rho^*} &= \frac{9}{14}\lambda\big(\lambda e^{18X} + b\big)^{-\frac{27}{28}} e^{18\Psi} \,, \\
g_{\rho j^*} &= \frac{81}{7}\lambda\big(\lambda e^{18X} + b\big)^{-\frac{27}{28}} e^{18\Psi}\rho^* \cdot \partial_{j^*}\Psi \,, \\
g_{ij^*} &= \frac{1458}{7}\lambda\big(\lambda e^{18X} + b\big)^{-\frac{27}{28}} e^{18\Psi}|\rho|^2 \cdot \partial_i\Psi\partial_{j^*}\Psi + \big(\lambda e^{18X} + b\big)^{\frac{1}{28}} \cdot \partial_i\partial_{j^*}\Psi \,,
\end{aligned}
\tag{4.34}
$$

for $E_7/[E_6\times U(1)]$. Hence, each metric is regular in the whole region (including $\rho = 0$).

The metric of each submanifold of $\rho = 0$ $(d\rho = 0)$ is

$$
\begin{aligned}
g_{\alpha\beta^*}\big|_{\rho=0}(\varphi,\varphi^*) &= b^{\frac{1}{17}}\partial_\alpha\partial_{\beta^*}\Psi && \text{for } E_6/[SO(10)\times U(1)] \,, \tag{4.35a} \\
g_{ij^*}\big|_{\rho=0}(\varphi,\varphi^*) &= b^{\frac{1}{28}}\partial_i\partial_{j^*}\varphi && \text{for } E_7/[E_6\times U(1)] \,. \tag{4.35b}
\end{aligned}
$$

Since each Ψ is the Kähler potential of $E_6/[SO(10)\times U(1)]$ or $E_7/[E_6\times U(1)]$, $\partial_\alpha\partial_{\beta^*}\Psi$ or $\partial_i\partial_{j^*}\Psi$ is the metric of this manifold, given in (4.24b) and (4.25b). Therefore we find that the total space is the complex line bundle over $E_6/[SO(10)\times U(1)]$ or $E_7/[E_6\times U(1)]$ with a fiber ρ. In the limit of $b \to 0$, each submanifold shrinks to zero-size and the conical singularity appears. Each conical singularity is resolved by $E_6/[SO(10)\times U(1)]$ or $E_7/[E_6\times U(1)]$ of radius $b^{\frac{1}{34}}$ or $b^{\frac{1}{56}}$.

5 Non-compact Calabi-Yau Manifolds from Non-abelian Gauge Theories

In this section we review the construction of non-compact Calabi-Yau manifolds using the non-abelian gauge theories, which provides other generalizations of the complex line bundle over $\mathbf{C}P^{N-1}$ [22]. First we construct the Grassmann manifold $G_{N,M}$ using non-abelian gauge theory [33, 34]. Imposing holomorphic constraints on $G_{2N,N}$, we obtain the rests of HSS, $SO(2N)/U(N)$ and $Sp(N)/U(N)$ [23]. Second we study the non-compact Calabi-Yau manifolds as the complex line bundles over these compact manifolds in detail.

Let $\Phi(x, \theta, \bar{\theta})$ be an $N \times M$ matrix-valued chiral superfield and $V(x, \theta, \bar{\theta}) = V^A T_A$ be a vector superfield taking a value in the Lie algebra of $U(M)$. The global symmetry $SU(N)$ acts on Φ from the left: $\Phi \rightarrow \Phi' = g\Phi$ $[g \in SU(N)]$; on the other hand, the gauge symmetry $U(M)$ acts on Φ from the right:

$$\Phi \rightarrow \Phi' = \Phi e^{-i\Lambda}, \quad e^V \rightarrow e^{V'} = e^{i\Lambda} e^V e^{-i\Lambda^\dagger}, \tag{5.1}$$

where $\Lambda(x, \theta, \bar{\theta})$ is a parameter chiral superfield, taking a value in the Lie algebra of $U(M)$. Note that the local invariance group is enlarged to the complexification of the gauge group, $U(M)^{\mathbf{C}} = GL(N, \mathbf{C})$, because the scalar component of $\Lambda(x, \theta, \bar{\theta})$ is complex. The Lagrangian invariant under the global $SU(N)$ and the local $U(M)$ symmetries is given by

$$\mathcal{L} = \int d^4\theta \left[\text{tr}(\Phi^\dagger \Phi e^V) - c\,\text{tr}V \right], \tag{5.2}$$

where $c\,\text{tr}V$ is the FI D-term.

Integrating out the auxiliary vector superfield V, we obtain

$$\mathcal{K}(\Phi, \Phi^\dagger) = c \log \det(\Phi^\dagger \Phi), \tag{5.3}$$

where we have omitted constants, since they disappear under the integration over θ. Since the gauge symmetry is complexified, we can write the gauge as

$$\Phi = \begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix}, \tag{5.4}$$

where $\varphi(x, \theta, \bar{\theta})$ is an $(N - M) \times M$ matrix-valued chiral superfield. Substituting this into (5.3), we obtain the Kähler potential of $G_{N,M} = SU(N)/[SU(N - M) \times U(M)]$:

$$\mathcal{K}(\varphi, \varphi^\dagger) = c \log \det \left(\mathbf{1}_M + \varphi^\dagger \varphi \right). \tag{5.5}$$

Next, we construct the non-compact Calabi-Yau manifolds by restricting the gauge degrees of freedom from $U(M)$ to $SU(M)$. Let $V(x, \theta, \bar{\theta}) = V^A T_A$ be a vector superfield taking a value in the

Lie algebra of $SU(M)$, whose generators are T_A. The Kähler potential is

$$\mathcal{K}_0(\Phi, \Phi^\dagger, V) \;=\; f(\mathrm{tr}(\Phi^\dagger \Phi e^V)) \,, \tag{5.6}$$

where f is an arbitrary function[4]. The equations of motion for V read

$$\frac{\partial \mathcal{L}}{\partial V} \;=\; f'(\mathrm{tr}(\Phi^\dagger \Phi e^V)) \cdot \mathrm{tr}(\Phi^\dagger \Phi e^V T_A) \;=\; 0 \,, \tag{5.7}$$

where the prime denotes the differentiation with respect to the argument of f. Then, we obtain

$$f'(\mathrm{tr}(\Phi^\dagger \Phi e^V)) \cdot \Phi^\dagger \Phi e^V \;=\; C \mathbf{1}_M \,, \tag{5.8}$$

where $C(x, \theta, \bar\theta)$ is a vector superfield. There is an alternative way to obtain this equation [22]. The trace and the determinant of (5.8) are

$$f'(\mathrm{tr}(\Phi^\dagger \Phi e^V)) \cdot \mathrm{tr}(\Phi^\dagger \Phi e^V) \;=\; MC \,, \tag{5.9a}$$

$$\left[f'(\mathrm{tr}(\Phi^\dagger \Phi e^V)) \right]^M \cdot \det(\Phi^\dagger \Phi) \;=\; C^M \,, \tag{5.9b}$$

respectively, where $\det e^V = 1$ because of the tracelessness of the $SU(M)$ gauge field V. Eliminating C from these equations, we obtain the solution of V as

$$\mathrm{tr}(\Phi^\dagger \Phi e^V) \;=\; M \left[\det(\Phi^\dagger \Phi) \right]^{\frac{1}{M}} \,. \tag{5.10}$$

Substituting this back into (5.6), we obtain the Kähler potential

$$\mathcal{K}_0(\Phi, \Phi^\dagger, V(\Phi, \Phi^\dagger)) \;=\; f\left(M[\det(\Phi^\dagger \Phi)]^{\frac{1}{M}} \right) \;\equiv\; \mathcal{K}(X(\Phi, \Phi^\dagger)) \,, \tag{5.11}$$

where we have defined a vector superfield

$$X(\Phi, \Phi^\dagger) \;\equiv\; \log \det \Phi^\dagger \Phi \,, \tag{5.12}$$

which is invariant under the global $U(N)$ and the local $SU(M)$ symmetries, $\mathcal{K}(X)$ is a real function of X related to f. The result (5.11) can be also obtained from the view point of the moduli space of supersymmetric gauge theories [34, 35]. Since the gauge symmetry is complexified as $SU(M)^{\mathbf{C}} = SL(M, \mathbf{C})$, we can write the matrix-valued chiral superfield Φ as

$$\Phi \;=\; \sigma \begin{pmatrix} \mathbf{1}_M \\ \varphi \end{pmatrix} \,, \tag{5.13}$$

where $\varphi(x, \theta, \bar\theta)$ is an $(N - M) \times M$ matrix-valued chiral superfield whose components are written as φ_{Aa}. Upper case A and lower case a run from 1 to $(N - M)$ and 1 to M, respectively. $\sigma(x, \theta, \bar\theta)$ is a chiral superfield. Then the invariant superfield X defined in (5.12) is decomposed as

$$X \;=\; M \log |\sigma|^2 + \Psi \,, \quad \Psi \;=\; \log \xi \,, \quad \xi \;\equiv\; \det(\mathbf{1}_M + \varphi^\dagger \varphi) \,, \tag{5.14a}$$

---

[4]There exist independent invariants $\mathrm{tr}[(\Phi^\dagger \Phi e^V)^2], \cdots, \mathrm{tr}[(\Phi^\dagger \Phi e^V)^M]$, besides $\mathrm{tr}(\Phi^\dagger \Phi e^V)$. We can show that, even if these are included as the arguments of the arbitrary function of (5.6), we obtain the same result (5.11).

where Ψ is the Kähler potential of $G_{N,M}$ (5.5). We can regard this non-compact manifold locally as $\mathbf{R} \times SU(N)/[SU(N-M) \times SU(M)]$.

Before discussing the Ricci-flat condition on the line bundle over $G_{N,M}$, we consider Kähler coset spaces with other isometries imposing holomorphic constraints on $G_{N,M}$. First we prepare the Lagrangian of $G_{2N,N}$ (5.2), in which $\Phi(x, \theta, \bar{\theta})$ is a $2N \times N$ matrix-valued chiral superfield. We can obtain the HSS of $SO(2N)/U(N)$ or $Sp(N)/U(N)$ by introducing the superpotential

$$W(\Phi_0, \Phi) \;=\; \mathrm{tr}\big(\Phi_0 \Phi^T J' \Phi\big), \quad J' \;=\; \begin{pmatrix} \mathbf{0} & \mathbf{1}_N \\ \epsilon\mathbf{1}_N & \mathbf{0} \end{pmatrix}, \tag{5.15}$$

where $\Phi_0(x, \theta, \bar{\theta})$ is an $N \times N$ matrix-valued auxiliary chiral superfield. Here J' is the rank-2 invariant tensor in which ϵ stands for a sign: $\epsilon = -1$ [$\epsilon = +1$] corresponds to $Sp(N)/U(N)$ [$SO(2N)/U(N)$]. Choosing the same gauge fixing as $G_{N,M}$, we can write $\Phi(x, \theta, \bar{\theta})$ as

$$\Phi \;=\; \sigma \begin{pmatrix} \mathbf{1}_N \\ \varphi \end{pmatrix}. \tag{5.16}$$

Here $\varphi(x, \theta, \bar{\theta})$ is an $N \times N$ matrix-valued chiral superfield whose components are written as φ_{ab}, where $1 \le a < b \le N$ for $SO(2N)/U(N)$ or $1 \le a \le b \le N$ for $Sp(N)/U(N)$. These non-compact manifolds can be locally regarded as $\mathbf{R} \times SO(2N)/SU(N)$ and $\mathbf{R} \times Sp(N)/SU(N)$.

The metrics are defined in (2.10) where holomorphic coordinates are $z^\mu = (\sigma, \varphi_{Aa})$ for $G_{N,M}$ and $z^\mu = (\sigma, \varphi_{ab})$ for $SO(2N)/U(N)$ or $Sp(N)/U(N)$. Thus the determinants $\det g_{\mu\nu^*}$ can be written as

$$\det g_{\mu\nu^*} \;=\; \begin{cases} \dfrac{M^2}{|\sigma|^2}\mathcal{K}'' \cdot \displaystyle\det_{(Aa)(Bb)}\left(\mathcal{K}'\dfrac{\partial^2 X}{\partial\varphi_{Aa}\partial\varphi_{Bb}^*}\right) & \text{for } G_{N,M} \,, \\[4mm] \dfrac{N^2}{|\sigma|^2}\mathcal{K}'' \cdot \displaystyle\det_{(ab)(cd)}\left(\mathcal{K}'\dfrac{\partial^2 X}{\partial\varphi_{ab}\partial\varphi_{cd}^*}\right) & \text{for } SO(2N)/U(N) \text{ or } Sp(N)/U(N) \,. \end{cases} \tag{5.17}$$

The X's differentiated twice can be calculated, to yield

$$\frac{\partial^2 X}{\partial\varphi_{Aa}\partial\varphi_{Bb}^*} \;=\; \partial_{(Aa)}\partial_{(Bb)^*}\Psi \;=\; (\mathbf{1}_M + \varphi^\dagger\varphi)_{ab}^{-1}\Big[\mathbf{1}_{(N-M)} - \varphi(\mathbf{1}_M + \varphi^\dagger\varphi)^{-1}\varphi^\dagger\Big]_{BA}, \tag{5.18}$$

for $G_{N,M}$ and

$$\begin{aligned} \frac{\partial^2 X}{\partial\varphi_{ab}\partial\varphi_{cd}^*} \;&=\; \partial_{(ab)}\partial_{(cd)^*}\Psi \\ &=\; \Big(1 - \tfrac{1}{2}\delta_{ab}\Big)\Big(1 - \tfrac{1}{2}\delta_{cd}\Big)\Big\{(\mathbf{1}_N + \varphi^\dagger\varphi)_{bd}^{-1}\big[\mathbf{1}_N - \varphi(\mathbf{1}_N + \varphi^\dagger\varphi)^{-1}\varphi^\dagger\big]_{ca} \\ &\qquad - \epsilon\,(\mathbf{1}_N + \varphi^\dagger\varphi)_{bc}^{-1}\big[\mathbf{1}_N - \varphi(\mathbf{1}_N + \varphi^\dagger\varphi)^{-1}\varphi^\dagger\big]_{da} + (a \leftrightarrow b, c \leftrightarrow d)\Big\}, \end{aligned} \tag{5.19}$$

for $SO(2N)/U(N)$ ($\epsilon = +1$, $a < b$) or for $Sp(N)/U(N)$ ($\epsilon = -1$, $a \le b$).

Under the complex isotropy transformation of $SL(N - M, \mathbf{C}) \times SL(M, \mathbf{C})$ for $G_{N,M}$ $[SL(N, \mathbf{C})$ for $SO(2N)/U(N)$ or $Sp(N)/U(N)]$, we can calculate the determinant (5.17):

$$
\det g_{\mu\nu^*} = \begin{cases}
M^2 |\sigma|^{2(MN-1)} e^{-NX} \mathcal{K}''(\mathcal{K}')^{M(N-M)} & \text{for } G_{N,M} \,, \\
N^2 2^{\frac{1}{2}N(N-1)} |\sigma|^{2N(N-1)-2} e^{-(N-1)X} \mathcal{K}''(\mathcal{K}')^{\frac{1}{2}N(N-1)} & \text{for } SO(2N)/U(N) \,, \\
N^2 2^{\frac{1}{2}N(N-1)} |\sigma|^{2N(N+1)-2} e^{-(N+1)X} \mathcal{K}''(\mathcal{K}')^{\frac{1}{2}N(N+1)} & \text{for } Sp(N)/U(N) \,,
\end{cases} \tag{5.20}
$$

Therefore the Ricci-flat condition (2.12) can be solved for $\mathcal{K}'$ as

$$
\mathcal{K}' = \begin{cases}
\left(\lambda e^{NX} + b\right)^{\frac{1}{g}} & g \equiv M(N-M)+1 & \text{for } G_{N,M} \,, \\
\left(\lambda e^{(N-1)X} + b\right)^{\frac{1}{f}} & f \equiv \frac{1}{2}N(N-1)+1 & \text{for } SO(2N)/U(N) \,, \\
\left(\lambda e^{(N+1)X} + b\right)^{\frac{1}{h}} & h \equiv \frac{1}{2}N(N+1)+1 & \text{for } Sp(N)/U(N) \,,
\end{cases} \tag{5.21}
$$

where λ is a constant and b is an integration constant. The explicit expression of the Kähler potentials can be found in [22].

The Ricci-flat metric can be calculated by substituting the solution (5.21) into (2.10). The component $g_{\sigma\sigma^*}$ can be calculated as

$$
g_{\sigma\sigma^*} = \begin{cases}
\lambda \dfrac{M^2 N}{g} \left(\lambda e^{NX} + b\right)^{\frac{1}{g}-1} e^{N\Psi} |\sigma|^{2MN-2} & \text{for } G_{N,M} \,, \\
\lambda \dfrac{N^2(N-1)}{f} \left(\lambda e^{(N-1)X} + b\right)^{\frac{1}{f}-1} e^{(N-1)\Psi} |\sigma|^{2N(N-1)-2} & \text{for } SO(2N)/U(N) \,, \\
\lambda \dfrac{N^2(N+1)}{h} \left(\lambda e^{(N+1)X} + b\right)^{\frac{1}{h}-1} e^{(N+1)\Psi} |\sigma|^{2N(N+1)-2} & \text{for } Sp(N)/U(N) \,,
\end{cases} \tag{5.22}
$$

where Ψ is defined in (5.14). Although this component is singular at the $\sigma = 0$ surface: $g_{\sigma\sigma^*}|_{\sigma=0} = 0$, this singularity is just a coordinate singularity of $z^\mu = (\sigma, \varphi)$. By performing the coordinate transformation

$$
\rho \equiv \begin{cases}
\sigma^{MN}/MN & \text{for } G_{N,M} \,, \\
\sigma^{N(N-1)}/N(N-1) & \text{for } SO(2N)/U(N) \,, \\
\sigma^{N(N+1)}/N(N+1) & \text{for } Sp(N)/U(N) \,,
\end{cases} \tag{5.23}
$$

with φ being unchanged, we obtain the regular coordinates. Each metric in the new coordinates $z'^\mu = (\rho, \varphi)$ can be calculated, to give

$$
g_{\rho\rho^*} = \lambda \frac{M^2 N}{g} \left(\lambda e^{NX} + b\right)^{\frac{1}{g}-1} e^{N\Psi} \,, \tag{5.24a}
$$

$$
g_{\rho(Bb)^*} = \lambda \frac{M^2 N^2}{g} \left(\lambda e^{NX} + b\right)^{\frac{1}{g}-1} e^{N\Psi} \rho^* \cdot \partial_{(Bb)^*} \Psi \,, \tag{5.24b}
$$

$$
\begin{aligned}
g_{(Aa)(Bb)^*} = {}& \lambda \frac{M^2 N^3}{g} \left(\lambda e^{NX} + b\right)^{\frac{1}{g}-1} e^{N\Psi} |\rho|^2 \cdot \partial_{(Aa)} \Psi \partial_{(Bb)^*} \Psi \\
& + \left(\lambda e^{NX} + b\right)^{\frac{1}{g}} \cdot \partial_{(Aa)} \partial_{(Bb)^*} \Psi \,,
\end{aligned} \tag{5.24c}
$$

for $G_{N,M}$,

$$g_{\rho\rho^*} = \lambda \frac{N^2(N-1)}{f} \big(\lambda e^{(N-1)X} + b\big)^{\frac{1}{f}-1} e^{(N-1)\Psi} , \tag{5.25a}$$

$$g_{\rho(cd)^*} = \lambda \frac{N^2(N-1)^2}{f} \big(\lambda e^{(N-1)X} + b\big)^{\frac{1}{f}-1} e^{(N-1)\Psi} \rho^* \cdot \partial_{(cd)^*}\Psi , \tag{5.25b}$$

$$g_{(ab)(cd)^*} = \lambda \frac{N^2(N-1)^3}{f} \big(\lambda e^{(N-1)X} + b\big)^{\frac{1}{f}-1} e^{(N-1)\Psi} |\rho|^2 \cdot \partial_{(ab)}\Psi \partial_{(cd)^*}\Psi$$
$$+ \big(\lambda e^{(N-1)X} + b\big)^{\frac{1}{f}} \cdot \partial_{(ab)}\partial_{(cd)^*}\Psi , \tag{5.25c}$$

for $SO(2N)/U(N)$, and

$$g_{\rho\rho^*} = \lambda \frac{N^2(N+1)}{h} \big(\lambda e^{(N+1)X} + b\big)^{\frac{1}{h}-1} e^{(N+1)\Psi} , \tag{5.26a}$$

$$g_{\rho(cd)^*} = \lambda \frac{N^2(N+1)^2}{h} \big(\lambda e^{(N+1)X} + b\big)^{\frac{1}{h}-1} e^{(N+1)\Psi} \rho^* \cdot \partial_{(cd)^*}\Psi , \tag{5.26b}$$

$$g_{(ab)(cd)^*} = \lambda \frac{N^2(N+1)^3}{h} \big(\lambda e^{(N+1)X} + b\big)^{\frac{1}{h}-1} e^{(N+1)\Psi} |\rho|^2 \cdot \partial_{(ab)}\Psi \partial_{(cd)^*}\Psi$$
$$+ \big(\lambda e^{(N+1)X} + b\big)^{\frac{1}{h}} \cdot \partial_{(ab)}\partial_{(cd)^*}\Psi , \tag{5.26c}$$

for $Sp(N)/U(N)$. They are regular in the whole region (including $\rho = 0$) [22].

The metrics of the submanifolds defined by $\rho = 0$ ($d\rho = 0$) are

$$\begin{aligned}
g_{(Aa)(Bb)^*}|_{\rho=0}(\varphi, \varphi^*) &= b^{\frac{1}{g}} \partial_{(Aa)}\partial_{(Bb)^*}\Psi & \text{for } G_{N,M} , \\
g_{(ab)(cd)^*}|_{\rho=0}(\varphi, \varphi^*) &= b^{\frac{1}{f}} \partial_{(ab)}\partial_{(cd)^*}\Psi & \text{for } SO(2N)/U(N) , \\
g_{(ab)(cd)^*}|_{\rho=0}(\varphi, \varphi^*) &= b^{\frac{1}{h}} \partial_{(ab)}\partial_{(cd)^*}\Psi & \text{for } Sp(N)/U(N) .
\end{aligned} \tag{5.27}$$

Since Ψ is the Kähler potential of $G_{N,M}$, $SO(2N)/U(N)$ or $Sp(N)/U(N)$, this is its metric given in (5.18) or (5.19). Therefore we find that the total space is the complex line bundle over $G_{N,M}$, $SO(2N)/U(N)$ or $Sp(N)/U(N)$ as a base manifold with the fiber ρ. In the limit of $b \to 0$, each base manifold shrinks to zero-size and the conical singularity due to the identification (5.23) appears. If $b \neq 0$ the conical singularity is resolved by $G_{N,M}$, $SO(2N)/U(N)$ or $Sp(N)/U(N)$ of a radius $b^{1/2g}$, $b^{1/2f}$ or $b^{1/2h}$, respectively.

6 Conclusion

In this paper we have constructed non-compact Calabi-Yau manifolds interpreted as the complex line bundles over HSS. We have presented the Ricci-flat metrics and their Kähler potentials on these manifolds. In particular, we have presented the new Ricci-flat metrics on the non-compact Calabi-Yau manifolds with the isometries of the exceptional groups E_6 and E_7.

There are several essential points for obtaining these potentials. First, the $U(1)_{\text{local}}$ symmetry, which was gauged for obtaining the compact Kähler manifolds, has been treated as a global symmetry to obtain the non-compact manifolds. Second, we have performed the complex isotropy transformations in order to calculate the determinants of the metrics. Using these transformations, we obtain the ordinary differential equations for solving the Ricci-flat condition. The form of the solutions is written as

$$\mathcal{K}' \;=\; \left(\lambda e^{\mathcal{C}X} + b\right)^{\frac{1}{D}} , \tag{6.1}$$

Here D is the complex dimension of the complex line bundle; $\mathcal{C}$ is a constant related to N (see, Table 1); λ is a constant and b is an integration constant regarded as the resolution parameter of the conical singularity. Third, we have transformed the fiber coordinates from σ to ρ to eliminate the coordinate singularity. The metrics in new coordinates (ρ, φ) are regular. On the basis of these three significant points, we have obtained the complex line bundles over HSS, as summarized in Table 1.

type	$\mathbf{C} \ltimes G/H$	D	$\mathcal{C}$	coordinate transformation
AIII_1	$\mathbf{C} \ltimes \mathbf{C}P^{N-1}$	$1 + (N-1)$	N	$\rho \sim \sigma^N$
AIII_2	$\mathbf{C} \ltimes G_{N,M}$	$1 + M(N-M)$	N	$\rho \sim \sigma^{MN}$
BDI	$\mathbf{C} \ltimes Q^{N-2}$	$1 + (N-2)$	$N-2$	$\rho \sim \sigma^{N-2}$
CI	$\mathbf{C} \ltimes Sp(N)/U(N)$	$1 + \frac{1}{2}N(N+1)$	$N+1$	$\rho \sim \sigma^{N(N+1)}$
DIII	$\mathbf{C} \ltimes SO(2N)/U(N)$	$1 + \frac{1}{2}N(N-1)$	$N-1$	$\rho \sim \sigma^{N(N-1)}$
EIII	$\mathbf{C} \ltimes E_6/[SO(10) \times U(1)]$	$1 + 16$	12	$\rho \sim \sigma^{12}$
EVII	$\mathbf{C} \ltimes E_7/[E_6 \times U(1)]$	$1 + 27$	18	$\rho \sim \sigma^{18}$

Table 1: Line bundles over Hermitian symmetric spaces. We write the classification of coset spaces by Cartan, complex dimensions and the coordinate transformation to avoid the coordinate singularity. The notation $X \ltimes Y$ is used to signify a bundle over Y with fiber X. The numbers D and $\mathcal{C}$ are defined in (6.1).

Before closing the conclusion, let us make a comment. We have sets of the isomorphism between the lower dimensional base manifolds [22]:

$$\text{(i)} \quad \mathbf{C}P^1 \;\simeq\; SO(4)/U(2) \;\simeq\; Sp(1)/U(1) \;\simeq\; Q^1 , \tag{6.2a}$$

$$\text{(ii)} \quad \mathbf{C}P^3 \;\simeq\; SO(6)/U(3) , \tag{6.2b}$$

$$\text{(iii)} \quad Sp(2)/U(2) \;\simeq\; Q^3 , \tag{6.2c}$$

$$\text{(iv)} \quad G_{4,2} \;\simeq\; Q^4 , \tag{6.2d}$$

in addition to the novel duality relation

$$\text{(v)} \quad G_{N,M} \;\simeq\; G_{N,N-M} . \tag{6.2e}$$

We would like to note that each isomorphism of a pair of base manifolds consistently leads the isomorphism of the line bundles over these base manifolds [22].

Acknowledgements

A $SO(10)$ γ-matrices

In this appendix we introduce the $SO(10)$ γ-matrices and the charge conjugation matrices in the Weyl spinor basis. Further discussions are given in [32].

We define the $SO(10)$ γ-matrices σ_A in the Weyl spinor basis ($A = (m, \mu)$, $m = 1, 2, \cdots, 6$ and $\mu = 7, 8, 9, 10$):

$$\sigma_A = (\sigma_m, \sigma_\mu), \quad \sigma_m = \begin{pmatrix} 0 & {}^6\sigma_m \otimes \mathbf{1}_2 \\ -{}^6\sigma_m^\dagger \otimes \mathbf{1}_2 & 0 \end{pmatrix}, \quad \sigma_\mu = \begin{pmatrix} \mathbf{1}_4 \otimes {}^4\sigma_\mu & 0 \\ 0 & \mathbf{1}_4 \otimes {}^4\sigma_\mu^\dagger \end{pmatrix}, \quad \text{(A.1a)}$$

where ${}^6\sigma_m$ are $SO(6)$ γ-matrices in the Weyl spinor basis and ${}^4\sigma_\mu$ are $SO(4)$ γ-matrices in the Weyl spinor basis, say, 2×2 Pauli matrices. They are defined as follows:

$$^6\sigma_m = \left({}^6\sigma_i, {}^6\sigma_{i+3} \right), \quad i = 1, 2, 3, \tag{A.1b}$$

$$({}^6\sigma_i)_{\alpha\beta} = \varepsilon_{i4\alpha\beta} + \delta^i_\alpha \delta^4_\beta - \delta^4_\alpha \delta^i_\beta, \quad ({}^6\sigma_{i+3}) = i\varepsilon_{i4\alpha\beta} - i\delta^i_\alpha \delta^4_\beta + i\delta^4_\alpha \delta^i_\beta, \tag{A.1c}$$

$$^4\sigma_\mu = \left(-i\sigma_1, -i\sigma_2, -i\sigma_3, \mathbf{1}_2 \right), \tag{A.1d}$$

where ε_{ijkl} is the rank-4 totally anti-symmetric tensor ($\varepsilon_{1234} = 1$). Using these γ-matrices, we obtain the various matrices. The spinor rotation matrices σ_{AB} are defined by

$$\sigma_{AB} = \frac{1}{2} \left(\sigma_A \sigma_B^\dagger - \sigma_B \sigma_A^\dagger \right). \tag{A.2}$$

The $SO(10)$ charge conjugation matrix in the Weyl spinor basis is defined by

$$C = \begin{pmatrix} 0 & -\mathbf{1}_4 \otimes i\sigma_2 \\ \mathbf{1}_4 \otimes i\sigma_2 & 0 \end{pmatrix} = C^T = C^{-1} = C^\dagger. \tag{A.3}$$

Because of (A.1) and (A.3), the combination of following matrices are symmetric:

$$(C\sigma_A^\dagger)^{\alpha\beta} = (C\sigma_A^\dagger)^{\beta\alpha}, \quad (\sigma^A C^\dagger)^{\alpha\beta} = (\sigma^A C^\dagger)^{\beta\alpha}. \tag{A.4}$$

B E_6 and E_7 Algebras

In this appendix, we show the short review of the E_6 and E_7 algebras constructed by their maximal subgroups. More discussions are expanded in [36, 32, 23].

B.1 Construction of E_6 algebra

Since the decomposition of the adjoint representation of E_6 under its maximal subgroup $SO(10) \times U(1)$ is $\mathbf{78} = (\mathbf{45}, 0) \oplus (\mathbf{1}, 0) \oplus (\mathbf{16}, 1) \oplus (\overline{\mathbf{16}}, -1)$, we construct the E_6 algebra as $\mathcal{E}_6 = \mathcal{SO}(10) \oplus \mathcal{U}(1) \oplus \mathbf{16} \oplus \overline{\mathbf{16}}$: We prepare the $SO(10)$ generators T_{AB}, the $U(1)$ generator T, spinor generators E_α and $\overline{E}^\alpha = (E_\alpha)^\dagger$, belonging to $\mathbf{16}$ and $\overline{\mathbf{16}}$, respectively. Then their commutation relations can be calculated as follows:

$$
\begin{aligned}
[T_{AB}, T_{CD}] &= -i(\delta_{BC} T_{AD} + \delta_{AD} T_{BC} - \delta_{AC} T_{BD} - \delta_{BD} T_{AC}), \quad [T, T_{AB}] = 0, \\
[T_{AB}, E_\alpha] &= -(\sigma_{AB})_\alpha{}^\beta E_\beta, \quad [T_{AB}, \overline{E}^\alpha] = (\sigma_{AB}^*)^\alpha{}_\beta \overline{E}^\beta, \\
[T, E_\alpha] &= \frac{\sqrt{3}}{2} E_\alpha, \quad [T, \overline{E}_\alpha] = -\frac{\sqrt{3}}{2} \overline{E}^\alpha, \\
[E_\alpha, E_\beta] &= [\overline{E}^\alpha, \overline{E}^\beta] = 0, \quad [E_\alpha, \overline{E}^\beta] = -\frac{1}{2}(\sigma_{AB})_\alpha{}^\beta T_{AB} + \frac{\sqrt{3}}{2} \delta_\alpha{}^\beta T.
\end{aligned}
\tag{B.1}
$$

The $U(1)$ charge of E_α is determined by the difference between $U(1)$ charges of x and y or y and z in (B.2): $\frac{2}{\sqrt{3}} - \frac{1}{2\sqrt{3}} = \frac{1}{2\sqrt{3}} - (-\frac{1}{\sqrt{3}}) = \frac{\sqrt{3}}{2}$. The second coefficient of the last equation has the same value as the $U(1)$ charge of E_α, from the anti-symmetric property of the structure constants. The relative weight of the first and the second terms is determined by the Jacobi identity, $[\overline{E}, [E, E]] + (\text{cyclic}) = 0$, and the non-trivial identity for the spinor generators: $\sum_{A,B}(\sigma_{AB})_\alpha{}^{[\beta}(\sigma_{AB})_\gamma{}^{\delta]} = \frac{3}{2} \delta_\alpha{}^{[\beta} \delta_\gamma{}^{\delta]}$.

The transformation law of $\vec{\phi}$ under the complex extension of E_6 is

$$
\begin{aligned}
\delta \vec{\phi} &= \left(i\theta T + \frac{i}{2} \theta_{AB} T_{AB} + \overline{\epsilon}^\alpha E_\alpha + \epsilon_\alpha \overline{E}^\alpha \right) \vec{\phi} \\
&= \begin{pmatrix} \frac{2i}{\sqrt{3}}\theta & \overline{\epsilon}^\beta & \mathbf{0} \\ \epsilon_\alpha & \frac{i}{2}\theta_{AB}(\sigma_{AB})_\alpha{}^\beta + \frac{i}{2\sqrt{3}}\theta\delta_\alpha^\beta & -\frac{1}{\sqrt{2}}(\overline{\epsilon}\sigma_B C)_\alpha \\ \mathbf{0} & -\frac{1}{\sqrt{2}}(C\sigma_A^\dagger \epsilon)^\beta & \theta_{AB} - \frac{i}{\sqrt{3}}\theta\delta_{AB} \end{pmatrix} \begin{pmatrix} x \\ y_\beta \\ z^B \end{pmatrix},
\end{aligned}
\tag{B.2}
$$

where $\frac{i}{2}\theta_{CD}\rho(T_{CD})^A{}_B = \theta_{AB}$, and $\rho(T_{AB})$ are the vector representation matrices of $SO(10)$. The 16×16 matrices σ_A, σ_{AB} and C are $SO(10)$ γ-matrices, spinor rotation matrices and the charge conjugation matrix, respectively (defined in appendix A). Normalizations are fixed by $\mathrm{tr} T^2 = \mathrm{tr}(T_{AB})^2 = \mathrm{tr} E_\alpha \overline{E}^\alpha = 6$ (no summation). In (B.2) $\overline{\epsilon}_\alpha$ are independent of ϵ^α if we consider the action of $E_6{}^\mathbf{C}$, while $\overline{\epsilon}_\alpha = \epsilon^{*\alpha}$ hold when we consider the real group E_6.

B.2   Construction of E_7 algebra

The decomposition of the adjoint representation of E_7 under the maximal subgroup $E_6 \times U(1)$ is $\mathbf{133} = (\mathbf{78}, 0) \oplus (\mathbf{1}, 0) \oplus (\mathbf{27}, 1) \oplus (\overline{\mathbf{27}}, -1)$, where the second components are the $U(1)$ charges. Hence, we can construct the E_7 algebra by adding generators E^i and $\overline{E}_i (= (E^i)^\dagger)$ $(i = 1, \cdots, 27)$, which belong to the E_6 fundamental and anti-fundamental representations, respectively, to the $E_6 \times U(1)$ algebra, T_A $(A = 1, \cdots, 78)$ and T: $\mathcal{E}_7 = \mathcal{E}_6 \oplus \mathcal{U}(1) \oplus \mathbf{27} \oplus \overline{\mathbf{27}}$. In the same manner as we constructed the E_6 algebra, their commutation relations are obtained as follows:

$$
\begin{aligned}
[T_A, T_B] &= i f_{AB}{}^C T_C \,, & [T, T_A] &= 0 \,, \\
[T_A, E^i] &= \rho(T_A)^i{}_j E^j \,, & [T_A, \overline{E}_i] &= -\rho(T_A)^{T\,j}{}_i \overline{E}_j \,, \\
[T, E^i] &= \sqrt{\frac{2}{3}} E^i \,, & [T, \overline{E}_i] &= -\sqrt{\frac{2}{3}} \overline{E}_i \,, \\
[E^i, E^j] &= [\overline{E}_i, \overline{E}_j] = 0 \,, & [E^i, \overline{E}_j] &= \rho(T_A)^i{}_j T_A + \sqrt{\frac{2}{3}} \delta^i{}_j T \,.
\end{aligned}
\tag{B.3}
$$

Here $\rho(T_A)$ are fundamental representation matrices, and the $f_{AB}{}^C$ are structure constants of E_6. The $U(1)$ charge of E^i is determined from the difference of x and y^i, etc., in (B.4), and the $U(1)$ charge of $\overline{E}_i$ is its conjugate. In the last equation, the coefficient of the second term coincides with the $U(1)$ charge of E^i due to the anti-symmetricity of the structure constants of E_7. The first term is determined by the Jacobi identity $[\overline{E}, [E, E]] + (\text{cyclic}) = 0$ and the non-trivial identity for the E_6 fundamental representation, $\sum_A \rho(T_A)^{[i}{}_j \rho(T_A)^{k]}{}_l = -\frac{2}{3} \delta^{[i}{}_j \delta^{k]}{}_l$.

The action of the E_7 algebra on the fundamental representation is

$$
\begin{aligned}
\delta \vec{\phi} &= \left(i\theta T + i\theta_A T_A + \overline{\epsilon}_i E^i + \epsilon^i \overline{E}_i \right) \vec{\phi} \\
&= \begin{pmatrix}
i\sqrt{\frac{3}{2}}\theta & \overline{\epsilon}_j & \mathbf{0} & 0 \\
\epsilon^i & i\theta_A \rho(T_A)^i{}_j + i\sqrt{\frac{1}{6}}\theta \delta^i{}_j & \Gamma^{ijk}\overline{\epsilon}_k & \mathbf{0} \\
\mathbf{0} & \Gamma_{ijk}\epsilon^k & -i\theta_A \rho(T_A)^{T\,j}{}_i - i\sqrt{\frac{1}{6}}\theta \delta_i{}^j & \overline{\epsilon}_i \\
0 & \mathbf{0} & \epsilon^j & -i\sqrt{\frac{3}{2}}\theta
\end{pmatrix}
\begin{pmatrix}
x \\ y^j \\ z_j \\ w
\end{pmatrix} \,,
\end{aligned}
\tag{B.4}
$$

where $\rho(T_A)$ are the 27×27 representation matrices for the fundamental representation, Γ_{ijk} is the E_6 invariant tensor and Γ^{ijk} is its conjugate. Here normalizations have been determined by $\mathrm{tr} T^2 = \mathrm{tr}(T_A)^2 = \mathrm{tr} E^i \overline{E}_i = 12$ (no summation). In (B.4) $\overline{\epsilon}_i$ are independent of ϵ^i if we consider the action of $E_7{}^{\mathbf{C}}$, while $\overline{\epsilon}_i = \epsilon^i$ hold when we consider the real group E_7.

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