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Holonomies, anomalies and the Fefferman-Graham ambiguity in AdS_3 gravity

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Abstract

Using the Chern-Simon formulation of (2+1) gravity, we derive, for the general asymptotic metrics given by the Fefferman-Graham-Lee theorems, the emergence of the Liouville mode associated to the boundary degrees of freedom of (2+1) dimensional anti de Sitter geometries. Holonomies are described through multi-valued gauge and Liouville fields and are found to algebraically couple the fields defined on the disconnected components of spatial infinity. In the case of flat boundary metrics, explicit expressions are obtained for the fields and holonomies. We also show the link between the variation under diffeomorphisms of the Einstein theory of gravitation and the Weyl anomaly of the conformal theory at infinity.

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1 Introduction

The interest of studying (2+1) dimensional gravity has initially been emphasized in [1] and have recently been revived with the discovery of black holes in spaces with negative cosmological constant [2]. Since then, a large number of studies has been devoted to the elucidation of classical as well as to quantum (2+1) gravity [3]. The major problem resides in the quest for the origin of the black hole entropy. This entropy seems due to a macroscopically large number of physical degrees of freedom, as indicated by the value of the central charge first computed by [4].

In $(2+1)$ dimensions, the field equations of gravity with negative cosmological constant have been proven to be equivalent to those of a Chern-Simons (CS) theory with $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ as gauge group [5]. Assuming the boundary of the space to be a flat cylinder $\mathbb{R}\times S^1$, Coussaert, Henneaux and van Driel

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(CHD) [6] demonstrated the equivalence between this CS theory and a non-chiral Wess-Zumino-Witten (WZW) theory [7, 8], and showed that the AdS_3 boundary conditions as defined in [4] implement the constraints that reduce the WZW model to the Liouville theory [9].

In this paper, we show that, using the less restrictive AdS boundary conditions allowed by the Fefferman-Graham-Lee theorems [10, 11], the CHD analysis can be extended and leads to the Liouville theory formulated on a 2-dimensional curved background (a preliminary version of this analysis can be found in [12]).

In section **2** we recall the Fefferman-Graham (FG) asymptotic expansion of Einstein and AdS_3 metrics. Section **3** contains the main calculation of the paper, consisting of a generalization of the relation between 3-d gravity and Liouville theory to curved boundary metrics. Special attention is paid to the contributions from holonomies, when the topology of the space is cylindrical. An originality of our approach resides in the use of multi-valued gauge group elements, which allows to simply show the correspondence between the Einstein-Hilbert (EH) action and a multi-valued Liouville field action.

In section **4**, we discuss the coupling between the fields living on the different components of spatial infinity, which is induced by the holonomies. We show that this coupling reduces to simple algebraic conditions on-shell. The aim of section **5** is to illustrate the link between Liouville fields, holonomies, AdS_3 metrics and $SL(2,\mathbb{R})$ parallel transport matrices in the framework of flat asymptotic boundaries.

Finally, in section **6**, by taking into account all terms resulting from integrations by parts during the reduction process from the EH action to the Liouville action, performed in section **3**, we make explicitly the connection between the variance of the EH action under diffeomorphisms and Weyl transformations on the boundary of AdS_3 space.

2 Asymptotically anti de Sitter spaces

Graham and Lee [10] proved that, under suitable topological assumptions, Euclidean Einstein spaces with negative cosmological constant Λ are completely defined by the geometry on their boundary. Furthermore, Fefferman and Graham (FG) [11] showed that, whatever the signature, there exists an asymptotic expansion of the metric, which formally solves the Einstein equations with $\Lambda < 0$. The first terms of this expansion may be given by even powers of a radial coordinate r:

$$ds^2 \underset{\bar{r}\to\infty}{\simeq} \ell^2 \frac{dr^2}{r^2} + \frac{r^2}{\ell^2} \overset{(0)}{\mathbf{g}}(x^i) + \overset{(2)}{\mathbf{g}}(x^i) + \cdots \quad . \tag{1}$$

On $(n+1)$-dimensional space-times, the full asymptotic expansion continues with terms of negative even powers of r up to $r^{-2([\frac{n+1}{2}]-2)}$, with in addition a logarithmic term of the order of $r^{-(n-2)}\log r$ when n is even and larger than 2. All these terms are completely defined by the boundary geometry $\overset{(0)}{\mathbf{g}}(x^i)$. It seems thus natural to take them as the definition of asymptotic Einstein metrics. These terms are followed by terms of negative powers starting from $r^{-2([\frac{n+1}{2}]-1)}$; for even n, the trace-free part of the $r^{-(n-2)}$ coefficient is not fully determined by $\overset{(0)}{\mathbf{g}}$. It contains degrees of freedom in Lorentzian spaces [13, 14]. Once this ambiguity is fixed, all subsequent terms become determined.

It is instructive to look at the first iterations of this expansion. We write the metric in terms of forms $\underline{\Theta}^\mu$ as $ds^2 = \underline{\Theta}^0 \otimes \underline{\Theta}^0 + \eta_{ab}\,\underline{\Theta}^a \otimes \underline{\Theta}^b$ with μ, ν [resp. a, b] running from 0 to n [resp. 1 to n] and η_{ab} a flat n-dimensional minkowskian metric *diag.*(1,..., 1, -1). The forms $\underline{\Theta}^\mu$ and the Levi-Civita connection $\underline{\Omega}^{\mu\nu}$ read as[3]:

$$\underline{\Theta}^0 = \ell \frac{dr}{r} \qquad , \qquad \underline{\Theta}^a = \frac{r}{\ell}\underline{\theta}^a + \frac{\ell}{r}\underline{\sigma}^a + o(r^{-2}) \qquad , \tag{2}$$

$$\underline{\Omega}^{a0} = \frac{r}{\ell^2}\underline{\theta}^a - \frac{1}{r}\underline{\tilde{\sigma}}^a + o(r^{-2}) \qquad , \qquad \underline{\Omega}^{ab} = \underline{\omega}^{ab} + o(r^{-1}) \qquad , \tag{3}$$

where the forms $\underline{\theta}^a$, $\underline{\omega}^{ab} \equiv \omega^{ab}{}_c\,\underline{\theta}^c$, $\underline{\sigma}^a \equiv \sigma^a{}_b\,\underline{\theta}^b$ and $\underline{\tilde{\sigma}}^a \equiv \sigma_b{}^a\,\underline{\theta}^b$ are r-independent. These provide the dominant and sub-dominant terms of the metric expansion:

$$\overset{(0)}{\mathbf{g}} = \eta_{ab}\,\underline{\theta}^a \otimes \underline{\theta}^b \quad , \quad \overset{(2)}{\mathbf{g}} \equiv \overset{(2)}{g}{}_{ab}\,\underline{\theta}^a \otimes \underline{\theta}^b = (\sigma_{ab} + \sigma_{ba})\,\underline{\theta}^a \otimes \underline{\theta}^b \qquad . \tag{4}$$

Here and in what follows, the n-dimensional indices and the covariant derivatives are defined with respect to the metric $\overset{(0)}{\mathbf{g}}$. Using these definitions, the $(n+1)$-dimensional Riemann curvature 2-form $\underline{\underline{R}}_{\mu\nu}$ becomes:

$$\begin{aligned}
\underline{\underline{R}}_{a0} &= -\frac{1}{\ell^2}\underline{\Theta}_a \wedge \underline{\Theta}_0 - \frac{1}{r}\left(d\underline{\gamma}_a + \underline{\omega}_{ab} \wedge \underline{\gamma}^b\right) + o(r^{-2}) \qquad , &(5)\\
\underline{\underline{R}}_{ab} &= -\frac{1}{\ell^2}\underline{\Theta}_a \wedge \underline{\Theta}_b + \overset{(0)}{\underline{\underline{R}}}_{ab} + \frac{1}{\ell^2}(\underline{\theta}_a \wedge \underline{\gamma}_b + \underline{\gamma}_a \wedge \underline{\theta}_b) + o(r^{-1}) \qquad , &(6)
\end{aligned}$$

where $\overset{(0)}{\underline{\underline{R}}}_{ab}$ is the n-dimensional curvature 2-form defined by the metric $\overset{(0)}{\mathbf{g}}$, and $\underline{\gamma}_a \equiv \overset{(2)}{g}{}_{ab}\,\underline{\theta}^b$. If we impose the metric of the $(n+1)$-dimensional space to be asymptotically Einsteinian, i.e. $R_{\mu\,\nu} = \frac{2}{n-1}\Lambda\,\eta_{\mu\,\nu}$, these equations, at order r^2, fix:

$$\Lambda = -1/\ell^2 \qquad . \tag{7}$$

Moreover, at order 1 and r^{-1}, they yield:

$$\begin{aligned}
\overset{(0)}{\mathcal{R}}_{ab} + \frac{1}{\ell^2}[(n-2)\overset{(2)}{g}{}_{ab} + \eta_{ab}\overset{(2)}{g}{}^c{}_c] &= 0 \qquad , &(8)\\
\overset{(2)}{g}{}^b{}_{b;a} - \overset{(2)}{g}{}^b{}_{a;b} &= 0 \qquad , &(9)
\end{aligned}$$

where $\overset{(0)}{\mathcal{R}}_{ab}$ are the components of the n-dimensional Ricci tensor.

We define asymptotically AdS spaces by the stronger condition that the Riemann tensor tends to that of AdS, i.e. $\underline{\underline{R}}_{\mu\,\nu} = \Lambda\,\underline{\Theta}_\mu \wedge \underline{\Theta}_\nu$. This implies, in addition to eq. (7):

$$\overset{(0)}{\mathcal{R}}_{ab\,cd} = \frac{1}{\ell^2}\left(\eta_{ad}\overset{(2)}{g}{}_{bc} - \eta_{ac}\overset{(2)}{g}{}_{bd} + \eta_{bd}\overset{(2)}{g}{}_{ac} - \eta_{bc}\overset{(2)}{g}{}_{ad}\right) \qquad , \tag{10}$$

$$\overset{(2)}{g}{}_{ab;c} - \overset{(2)}{g}{}_{ac;b} = 0 \qquad , \tag{11}$$

[3]By definition, $o(r^{-n}) = u_n(r)$ means that $\lim_{r\to\infty} \frac{u_n(r)}{r^n} = 0$

whose traces are the asymptotic Einstein conditions (8, 9). When $n \neq 2$, eq. (8) fully specifies the metric $\overset{(2)}{\mathbf{g}}$ in terms of $\overset{(0)}{\mathbf{g}}$ and eq. (9) becomes the Bianchi identity satisfied by the n-dimensional Einstein tensor. Furthermore eq. (10) is an identity if $n = 3$ and implies that the Weyl tensor of the n-dimensional geometry vanishes if $n > 3$, while eq. (11) implies that the Cotton-York tensor vanishes for $n = 3$ and becomes a consequence of eq. (10) and the Bianchi identities for $n > 3$. Thus, AdS asymptotic spaces are Einstein asymptotic spaces whose $\overset{(0)}{\mathbf{g}}$ metric tensor is conformally flat. The same conclusion holds for $n = 2$ as in three dimensions Einstein spaces with $\Lambda < 0$ are locally AdS and metrics on cylindrical boundaries are conformally flat.

On the other hand, when $n = 2$ only the trace of $\overset{(2)}{\mathbf{g}}$ is fixed by eq. (8):

$$\overset{(2)}{g}{}^{c}_{\ c} = 2\,\sigma^c_c \equiv 2\,\sigma = -\frac{\ell^2}{2}\overset{(0)}{\mathcal{R}} \qquad , \tag{12}$$

and the other components of $\overset{(2)}{\mathbf{g}}$ have only to satisfy the equations:

$$\overset{(2)}{g}{}^{a}_{\ b\,;a} = -\frac{\ell^2}{2}\overset{(0)}{\mathcal{R}}_{,b} \qquad . \tag{13}$$

The subdominant metric components are thus not all determined by the asymptotic metric in three dimensions, but there remains one degree of freedom, which we shall explicit in the next section. Similar indeterminacy, involving $\overset{(n)}{\mathbf{g}}$, arises for all even values of n [13, 14].

To illustrate the meaning of the FG coordinates, let us consider the special case of BTZ black holes [2] whose metric can locally be written as:

$$ds^2 = -(\frac{\rho^2}{\ell^2} - M)dt^2 + \frac{d\rho^2}{\frac{\rho^2}{\ell^2} - M + \frac{J^2}{4\rho^2}} + \rho^2 d\varphi^2 + J d\varphi dt \qquad , \tag{14}$$

where M is the mass and J the angular momentum. To shorten the discussion, we restrict ourselves to the cases $M > |J/\ell| > 0$. In this case, a single coordinate patch (ρ, φ, t), on which the metric is given by (14), covers the three regions denoted I, II and III (see Fig. 1), corresponding to $\rho > \rho_+$, $\rho_+ > \rho > \rho_-$ and $\rho < \rho_-$, where

$$\rho_+ = \frac{\ell}{2}(M + \sqrt{M^2 - \frac{J^2}{\ell^2}}) \qquad \text{and} \qquad \rho_- = \frac{\ell}{2}(M - \sqrt{M^2 - \frac{J^2}{\ell^2}}) \tag{15}$$

are the values of the constant ρ surfaces defining the inner and outer horizons. The complete space-time is obtained by glueing similar overlapping coordinate patches.

The FG radial coordinate r is obtained from ρ by the transformations:

$$r^2 = \frac{1}{2}\left(\rho^2 - \frac{M\ell^2}{2} \pm \ell\,\rho\sqrt{\frac{\rho^2}{\ell^2} - M + \frac{J^2}{4\rho^2}}\right) \qquad , \tag{16}$$

yielding the metric expression:

$$\begin{aligned} ds^2 \quad &= \quad \ell^2\frac{dr^2}{r^2} + \frac{r^2}{\ell^2}(\ell^2 d\varphi^2 - dt^2) + \frac{M}{2}(\ell^2 d\varphi^2 + dt^2) + J dt d\varphi \\ &\quad + \frac{M^2\ell^2 - J^2}{16 r^2}(\ell^2 d\varphi^2 - dt^2) \quad . \end{aligned} \tag{17}$$

This r coordinate is only defined on region I of a coordinate patch (ρ, φ, t), between the exterior horizon $\rho = \rho_+$ and infinity. It can be analytically continued so as to cover two adjacent regions I and I' (see Fig. 1). On the first (I), we have $r^2 > \frac{\ell}{4}\sqrt{M^2 - J^2/\ell^2}$, which is obtained by taking the plus sign in eq. (16); on the other (I') $r^2 < \frac{\ell}{4}\sqrt{M^2 - J^2/\ell^2}$, corresponding to the minus sign. This r coordinate accounts for the global nature of the black holes, where the diffeomorphic regions I and I' are connected by an Einstein-Rosen bridge. Near the space-like infinity of region I $(r \to \infty)$ the metric (17) coincides with the FG asymptotic expansion (1), whereas on region I', where the space-like asymptotic region is given by $r \to 0$, we recover the FG expression of the asymptotic metric after the transformation $r^2 \to (M^2\ell^2 - J^2)\ell^2/(16r^2)$. This transformation leaves the metric (17) invariant, ensuring the consistency of the definitions of mass and angular momentum of the black hole, whatever the asymptotic region is. Of course we may define a single radial coordinate that goes to $\pm\infty$ on the asymptotia of regions I and I' and leads on both ends to the FG expression (1); it is however not related analytically to the r coordinate (16). We shall use such a coordinate (also called r) in section **3**.

Furthermore, if we introduce the radial variable $y = r^{-2}$, corresponding to the original FG radial coordinate [11], we may extend its domain of definition to negative values, which correspond to regions III and III'. Note that, as r has been taken to be intrinsically space-like, it cannot cover regions II, between the horizons, unless we turn it into a time-like coordinate.

3 From Einstein-Hilbert to Liouville action

In this section we perform explicitly the transformations leading from the EH action to the Liouville action, assuming the metric to be asymptotically AdS as defined in the previous section. Our first aim is to show how the asymptotic conditions generalize in case of curved metrics, and lead to a Liouville action on curved background. Our second aim is to keep track of all kinds of boundary contributions that appear during the transformations (which renders this section rather long and detailed), in view of accounting for the effect of holonomies (see section **4**) and showing the precise link between the EH action which is invariant under the combined action of diffeomorphisms and Weyl transformations and the Liouville action which is not (see section **6**).

3.1 First step: Chern-Simon action

The EH (2+1) gravity action with cosmological constant Λ, evaluated on a space-time domain $\mathcal{V}$, consists of the bulk term:

$$S_{EH} = \frac{1}{16\pi G}\int_{\mathcal{V}} (\mathcal{R} - 2\,\Lambda)\,\boldsymbol{\eta} \qquad , \tag{18}$$

where $\boldsymbol{\eta}$ is the volume element 3-form on $\mathcal{V}$.

Let us specify the topological structure of the domain of integration $\mathcal{V}$. Inspired by the BTZ black holes [2], we assume it to consist of the product of a time interval $[t_0, t_1]$ with a spacelike section Σ chosen to be homeomorphic to a disk of radius $\bar{r}$ or to a finite 2-dimensional space-like cylinder of length $2\bar{r}$; afterwards $\bar{r}$ will be pushed to infinity. The disk will provide extremal black holes if we accept a conical singularity at the origin $r = 0$, and global AdS space if not. The cylinder will describe non-extremal black holes, its non trivial topology describing the Einstein-Rosen bridge. In fact, we may not exclude *a priori* more complicated topologies, such as several cylinders attached to an arbitrary compact manifold, but we shall not consider them here.

We use usual cylindrical coordinates (r, φ, t) to parametrize the manifold $\mathcal{V}$, where $|r|$ coincides with the FG radial variable in a neighbourhood of $|r| = \infty$. Accordingly, the **coordinate** domain of integration will be the cube $[t_0, t_1] \times [\bar{r}_L, \bar{r}_R] \times [0, 2\pi]$. In case Σ is a disk, $\bar{r}_L = 0$, $\bar{r}_R = \bar{r}$, and all the considered quantities are periodic in the angular variable φ. In case Σ is a cylinder, we choose $\bar{r}_L = -\bar{r}$ and $\bar{r}_R = \bar{r}$; to account for the holonomies allowed by this non trivial topology, we may not assume *a priori* all quantities to be periodic in φ. Hence, the boundary terms obtained upon application of Stokes theorem will involve when Σ is a disk the geometrical surfaces $t = t_0$, $t = t_1$ and $r = \bar{r}$, with in addition when Σ is a cylinder the extra boundary component $r = -\bar{r}$ and the coordinate surfaces $\varphi = 0$ and $\varphi = 2\pi$.

When $\Lambda < 0$, one can re-express [5] the EH action (18) in terms of the two gauge fields $\mathbf{A} = A_\mu \underline{\Theta}^\mu = J_\mu \underline{A}^\mu$ and $\tilde{\mathbf{A}} = \tilde{A}_\mu \underline{\Theta}^\mu = J_\mu \underline{\tilde{A}}^\mu$, with J_μ generators of the $sl(2, \mathbb{R})$ algebra satisfying $Tr(J_\mu J_\nu) = \frac{1}{2}\eta_{\mu\nu}$ and $Tr(J_\mu J_\nu J_\rho) = \frac{1}{4}\epsilon_{\mu\nu\rho}$, with $\epsilon_{012} = 1$ (see appendix for conventions). These fields are given in terms of the metric by:

$$\underline{A}^\mu = \frac{1}{\ell}\underline{\Theta}^\mu + \frac{1}{2}\epsilon^\mu{}_{\nu\rho}\underline{\Omega}^{\nu\rho} \qquad , \qquad \underline{\tilde{A}}^\mu = -\frac{1}{\ell}\underline{\Theta}^\mu + \frac{1}{2}\epsilon^\mu{}_{\nu\rho}\underline{\Omega}^{\nu\rho} \qquad . \tag{19}$$

They allow to express the action S_{EH} as the difference of two CS actions $S_{SC}[\mathbf{A}]$ and $S_{SC}[\tilde{\mathbf{A}}]$ plus a boundary term that mixes the two gauge fields:

$$S_{EH} = S_{CS} + \mathcal{B}_{CS} \qquad , \qquad S_{CS} = S[\mathbf{A}] - S[\tilde{\mathbf{A}}] \qquad , \tag{20}$$

with

$$S[\mathbf{A}] = \frac{\ell}{16\pi G}\int_{\mathcal{V}} Tr(\mathbf{A}\wedge d\mathbf{A} + \frac{2}{3}\mathbf{A}\wedge\mathbf{A}\wedge\mathbf{A}) \qquad , \tag{21}$$

and

$$\mathcal{B}_{CS} = \frac{\ell}{16\pi G}\int_{\mathcal{V}} Tr\, d(\tilde{\mathbf{A}}\wedge\mathbf{A}) \qquad . \tag{22}$$

The integral $\mathcal{B}_{CS}$ is composed of three terms, noted $\mathcal{B}_{CS}^{(r)}$, $\mathcal{B}_{CS}^{(\varphi)}$ and $\mathcal{B}_{CS}^{(t)}$, coming from integrations over r, φ and t, respectively. As we assume that the metric and thus the fields $\mathbf{A}$ and $\tilde{\mathbf{A}}$ are globally defined on $\mathcal{V}$, i.e. periodic in φ, $\mathcal{B}_{CS}^{(\varphi)}$ vanishes.

We now discuss the large $\bar{r}$ behaviour of the EH and CS actions (18, 20). The insertion of eqs(2, 3) in eq.(19) gives the asymptotic behaviour of the CS fields $\mathbf{A}$ and $\tilde{\mathbf{A}}$. But before we pursue our discussion, the following clarifying remark seems necessary. The FG expression of the metric is, in general, only locally valid. When Σ is a cylinder there are two asymptotic regions and we may always fix the asymptotic behaviour of the fields on one of them according to eqs (2, 19). With such a choice, the frames $\{\Theta^0, \Theta^a\}$ on the other asymptotic region become defined up to a sign that depends on the continuation of the CS fields across the whole manifold. The asymptotic behaviours of $\mathbf{A}$ and $\tilde{\mathbf{A}}$ may indeed have to be exchanged (see section **5** for an explicit example), but this has no consequence on the rest of our analysis.

Let us in a first stage focus on one asymptotic region. To fix the notation, we adopt the convention defined by eqs (2, 19), with the radial coordinate r belonging to a neigbourhood of $+\infty$. We find convenient to write explicitly the large r behaviour of the CS fields using the null frame $\underline{\theta}^{\pm} = \underline{\theta}^1 \pm \underline{\theta}^2$ and its dual vectorial frame $\vec{e}_{\pm} = \frac{1}{2}(\vec{e}_1 \pm \vec{e}_2)$, which are defined on the surfaces $|r| = \bar{r}$. At order r^{-1}, we obtain:

$$A_r \underset{r\to\infty}{\simeq} \frac{1}{2r}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \underset{r\to\infty}{\simeq} -\tilde{A}_r \qquad , \tag{23}$$

$$A_- \underset{r\to\infty}{\simeq} \begin{pmatrix} \frac{\omega_-}{2} & \frac{\sigma}{2r} \\ 0 & -\frac{\omega_-}{2} \end{pmatrix} \equiv K_- \,, \quad \tilde{A}_+ \underset{r\to\infty}{\simeq} \begin{pmatrix} \frac{\omega_+}{2} & 0 \\ -\frac{\sigma}{2r} & -\frac{\omega_+}{2} \end{pmatrix} \equiv \tilde{K}_+ \,, \tag{24}$$

$$A_+ \underset{r\to\infty}{\simeq} \begin{pmatrix} \frac{\omega_+}{2} & \frac{\overset{(2)}{g}_{++}}{r} \\ \frac{r}{\ell^2} + \frac{\sigma_{-+}-\sigma_{+-}}{r} & -\frac{\omega_+}{2} \end{pmatrix} \quad , \quad \tilde{A}_- \underset{r\to\infty}{\simeq} \begin{pmatrix} \frac{\omega_-}{2} & \frac{-r}{\ell^2} + \frac{\sigma_{-+}-\sigma_{+-}}{r} \\ \frac{-\overset{(2)}{g}_{--}}{r} & -\frac{\omega_-}{2} \end{pmatrix} , \tag{25}$$

where we have introduced the null components of the connection 2-form $\underline{\omega}^{12} \equiv \underline{\omega} = \omega_+ \underline{\theta}^+ + \omega_- \underline{\theta}^-$ and for further convenience the notation K_- and $\tilde{K}_+$. Note that the combination $(\sigma_{-+} - \sigma_{+-})/r$ can always be canceled by an infinitesimal (in the limit $r \to \infty$) Lorentz transformation acting on θ^1 and θ^2; this gauge freedom explains that this term never appears in the subsequent calculations. Using these relations, we see that in the large $\bar{r}$ limit, the EH action (18) presents two types of divergences, one in $\bar{r}^2$ and if σ, i.e. $\overset{(0)}{\mathcal{R}}$ (see eq. (12)), does not vanish, another in $\log \bar{r}$. In contrast, the CS action (20) only presents a $\log \bar{r}$ divergence. The quadratic divergence of S_{EH} is found in $\mathcal{B}^{(r)}_{CS}$. Indeed:

$$\mathcal{B}^{(r)}_{CS} = -\frac{\bar{r}^2}{8\,\pi\, G\, \ell^3} \int_{|r|=\bar{r}} \sqrt{\overset{(0)}{\mathbf{g}}}\, d\varphi\, dt \;+\; O(\bar{r}^{\,-2}) \qquad . \tag{26}$$

Note that in the limit $\bar{r} \to \infty$, this term does not contain dynamical degrees of freedom. The logarithmic divergence of S_{EH} appears in both S_{CS} and $\mathcal{B}^{(t)}_{CS}$, which can be written as:

$$\mathcal{B}^{(t)}_{CS} = \underset{\bar{r}\to\infty}{\simeq} \frac{\ell}{32\pi G} \log \bar{r} \int_{|r|=\bar{r}} \overset{(0)}{\mathcal{R}} \sqrt{\overset{(0)}{\mathbf{g}}}\, d\varphi\, dt \qquad . \tag{27}$$

This term will no longer be considered in this section but discussed in section **6**, where the origin of the anomaly will be analyzed.

As a remark *en passant*, let us note that the boundary contribution $\mathcal{B}^{(r)}_{CS}$ (26) can here be expressed in terms of the "extrinsic curvature 2-form" $\underline{\underline{\mathcal{K}}}$, defined as the product of the trace of the extrinsic curvature tensor with the induced surface element 2-form. Indeed, on a surface of constant r, this 2-form is given by:

$$\underline{\underline{\mathcal{K}}} = \frac{1}{2\ell} \epsilon_{\mu\nu\rho} \underline{\Theta}^{\mu} \wedge \underline{\Omega}^{\nu\rho} \qquad , \tag{28}$$

which implies that on this surface:

$$\ell\, Tr(\mathbf{A} \wedge \tilde{\mathbf{A}}) = \underline{\underline{\mathcal{K}}} \qquad . \tag{29}$$

Hence, we may introduce a gravity action S_G suitable for Feynman path integral, which is finite when the boundary metric is flat (and has well-defined functional derivatives) [15]:

$$S_G = \frac{1}{16\pi G}\int_{\mathcal{M}} (\mathcal{R} - 2\Lambda)\boldsymbol{\eta} + \frac{1}{16\pi G}\int_{|r|=\bar{r}} \underline{\underline{\mathcal{K}}} \quad , \tag{30}$$

This gravity action differs from that obtained using the Gibbons-Hawking procedure [16], as the term involving the extrinsic curvature is equal to half of the usual one [17]. This is not in contradiction with [16], as we are working in the Palatini formalism.

Let us now consider the on-shell variation of the actions S_{EH} and S_{CS}. We therefore re-write $S[\mathbf{A}]$ (21) as:

$$S[\mathbf{A}] = \frac{\ell}{16\pi G}\int_{\mathcal{V}} Tr(2\,A_t F_{r\varphi} + A_\varphi \dot{A}_r - A_r \dot{A}_\varphi)\,dr\,d\varphi\,dt + \mathcal{B}^{(r)}[\mathbf{A}] \quad , \tag{31}$$

where the boundary term is:

$$\mathcal{B}^{(r)}[\mathbf{A}] = -\frac{\ell}{16\pi G}\int_{|r|=\bar{r}} Tr(A_t A_\varphi)\,d\varphi\,dt \quad . \tag{32}$$

Using the asymptotic behaviour of the fields $\mathbf{A}$ and $\tilde{\mathbf{A}}$ given by eqs (23, 24, 25), we find that at spatial infinity ($|r| = \bar{r} \to \infty$):

$$\delta A_- = O(r^{-2}) = \delta\tilde{A}_+ \quad , \tag{33}$$

$$Tr(A_-\delta A_+) = O(r^{-2}) = Tr(\tilde{A}_+\delta\tilde{A}_-) \quad , \tag{34}$$

$$Tr(\tilde{A}_-\delta A_+) = O(r^{-2}) = Tr(A_+\delta\tilde{A}_-) \quad . \tag{35}$$

Accordingly, the variations of both actions S_{EH} and S_{CS} vanish:

$$\delta S_{CS} = \int_{|r|=\bar{r}} O(r^{-2})\,d\varphi\,dt \quad , \qquad \delta S_{EH} = \int_{|r|=\bar{r}} O(r^{-2})\,d\varphi\,dt \quad . \tag{36}$$

It is noteworthy that, owing to the boundary conditions (24, 25), the variation of the action S_{CS} vanishes by itself, without needing to add any extra boundary term [18], contrary to what was sometimes stated.

3.2 Second step: Wess-Zumino-Witten action

Let us now return to the action S_{CS} given by eqs (20,31). The time components A_t and $\tilde{A}_t$ play the rôle of Lagrange multipliers and can be eliminated from the bulk action by solving the constraint equations $F_{r\varphi} = 0$ and $\tilde{F}_{r\varphi} = 0$. On the cylinder, their general solutions [8] are given by gauge transforming non trivial flat connections h_i and $\tilde{h}_i$:

$$A_i = G_1^{-1} h_i G_1 + G_1^{-1}\partial_i G_1 \quad , \qquad \tilde{A}_i = G_2^{-1}\tilde{h}_i G_2 + G_2^{-1}\partial_i G_2 \quad , \tag{37}$$

with i labelling the coordinates (r, φ). The flat connection components can be chosen as $h_r = 0$, $\tilde{h}_r = 0$, $h_\varphi = h(t)$ and $\tilde{h}_\varphi = \tilde{h}(t)$, where $h(t)$ and $\tilde{h}(t)$ are $sl(2, \mathbb{R})$ generators that only depend t. The $SL(2, \mathbb{R})/Z_2$ matrices G_1 and G_2 are assumed univoquely defined on the cylinder: $G_1(r, \varphi, t) = G_1(r, \varphi + 2\pi, t)$

and similarly for G_2. Instead of using this represenation of A and $\tilde{A}$, we choose to express them in terms of non-periodic group elements Q_1 and Q_2:

$$A_i = Q_1^{-1}\partial_i Q_1 \qquad , \qquad \tilde{A}_i = Q_2^{-1}\partial_i Q_2 \qquad , \tag{38}$$

where

$$Q_1(r,\varphi,t) = \exp[\varphi\, h(t)]\, G_1(r,\varphi,t) \quad , \quad Q_2(r,\varphi,t) = \exp[\varphi\, \tilde{h}(t)]\, G_2(r,\varphi,t) \quad . \tag{39}$$

These two representations of the gauge fields are obviously equivalent. We choose to use the second one, as it will naturally lead to a single non-periodic Liouville field on the spatial boundary.

Due to the asymptotic behaviour of A_r and $\tilde{A}_r$ (23), Q_1 and Q_2 asymptotically factorize into:

$$Q_1(\bar{r},\varphi,t) \underset{\bar{r}\to\infty}{\simeq} q_1(\varphi,t) S(\bar{r}) \qquad , \qquad Q_2(\bar{r},\varphi,t) \underset{\bar{r}\to\infty}{\simeq} q_2(\varphi,t) S(\bar{r})^{-1} \qquad , \tag{40}$$

with[4]:

$$S(\bar{r}) = \begin{pmatrix} \sqrt{\frac{\bar{r}}{\ell}} & 0 \\ 0 & \sqrt{\frac{\ell}{\bar{r}}} \end{pmatrix} \qquad . \tag{41}$$

On the other hand, the components A_t and $\tilde{A}_t$ in the boundary actions may be eliminated in terms of A_φ, $\tilde{A}_\varphi$, K_- and $\tilde{K}_+$ using the boundary conditions (24). The remaining conditions, given by eq. (25), restrict the matrix elements of q_1 and q_2, but are difficult to implement at this stage, and we choose not to do so for the moment. Dropping these conditions (25) implies that we must add a boundary term to the action S_{SC} ensuring the vanishing of its on-shell variation, independently of (25). This modified action reads as:

$$S'_{CS} = S_{CS} + \mathcal{B}^{(r)}_{CS'} - \mathcal{G}_{CS'} \qquad , \tag{42}$$

The additional dynamical boundary term $\mathcal{B}^{(r)}_{CS'}$ is given by:

$$\mathcal{B}^{(r)}_{CS'} = \frac{\ell}{16\pi G} \int_{|r|=\bar{r}} Tr(A_-\, A_+ + \tilde{A}_-\, \tilde{A}_+)\, \theta\, d\varphi\, dt \qquad , \tag{43}$$

where $\theta = 2\sqrt{-\overset{(0)}{\mathbf{g}}}$. The $\mathcal{G}_{CS'}$ term contains no degrees of freedom in the limit $\bar{r}\to\infty$, is finite and reads as:

$$\mathcal{G}_{CS'} = \frac{\ell}{16\pi G} \int_{|r|=\bar{r}} (\omega_+\omega_- \,+\, \frac{1}{\ell^2}\sigma)\, \theta\, d\varphi\, dt \qquad . \tag{44}$$

This term is equal to $\mathcal{B}^{(r)}_{CS'}$ when all boundary conditions are imposed, and ensures that the value of the action S_{CS} remains unmodified in the limit $\bar{r}\to\infty$. Here and in the following, such boundary terms without degrees on freedom will be qualified as geometrical.

[4]Note that if the asymptotic behaviours of $\mathbf{A}$ and $\tilde{\mathbf{A}}$ are interchanged (as discussed before eq. (23)), $S(\bar{r})$ must be replaced by $S(\bar{r})^{-1}$ in eq. (40).

Inserting in S'_{CS} the expression of the gauge fields in terms of the Q_i matrices (40), we get a sum of four terms:

$$S'_{CS} = S_{WZWC} + \mathcal{B}^{(\varphi)}_{WZWC} + \mathcal{G}_{WZWC} - \mathcal{G}_{CS'} \qquad . \tag{45}$$

The first is the chiral WZW action:

$$\begin{aligned} S_{WZWC} \quad &= \quad -\Gamma[Q_1] + \frac{\ell}{16\pi G}\int_{|r|=\bar{r}} Tr[\frac{1}{e^t_-}\, q'_1\,(q_1^{-1}\partial_- q_1 \,-\, 2\,k_-)]\, dt\, d\varphi \\ &\quad + \Gamma[Q_2] - \frac{\ell}{16\pi G}\int_{|r|=\bar{r}} Tr[\frac{1}{e^t_+}\, q'_2\,(q_2^{-1}\partial_+ q_2 \,-\, 2\,\tilde{k}_+)]\, dt\, d\varphi \qquad , \end{aligned} \tag{46}$$

where the derivatives ∂_+ and ∂_- are taken along the vectors $\vec{e}_+$ and $\vec{e}_-$, $q' = q^{-1}\partial_\varphi q$, $k_- = S(r)\,K_-\,S(r)^{-1}$, $\tilde{k}_+ = S(r)^{-1}\tilde{K}_+\,S(r)$, and the bulk WZW action reads as:

$$\Gamma[Q] \;=\; \frac{\ell}{48\pi G}\int Tr[Q^{-1}dQ\wedge Q^{-1}dQ\wedge Q^{-1}dQ] \qquad . \tag{47}$$

The second term:

$$\mathcal{B}^{(\varphi)}_{WZWC} = \frac{\ell}{16\pi G}\int_{\mathcal{V}} Tr[\partial_\varphi(Q_2^{-1}\partial_r Q_2\, Q_2^{-1}\partial_t Q_2 - Q_1^{-1}\partial_r Q_1\, Q_1^{-1}\partial_t Q_1)]\, dr\, d\varphi\, dt \tag{48}$$

does not vanish in general due to the holonomy encoded in Q_1 and Q_2. The last two terms are geometrical; $\mathcal{G}_{CS'}$ is given by (44) and $\mathcal{G}_{WZWC}$ by:

$$\mathcal{G}_{WZWC} = \frac{\ell}{16\pi G}\int_{|r|=\bar{r}} [\frac{e^t_+}{e^t_-}\,k_-^2 \,+\, \frac{e^t_-}{e^t_+}\,\tilde{k}_+^2]\,\theta\, d\varphi\, dt \qquad . \tag{49}$$

Defining the new variables

$$Q = Q_1^{-1}Q_2 \qquad , \qquad q = q_1^{-1}q_2 \qquad , \tag{50}$$

one of the fields q_1 or q_2 can be eliminated from the action S_{WZWC} using its equation of motion. Indeed, the variables q_1 or q_2 only appear in the chiral WZW action in quadratic expressions of their derivatives with respect to the angular variable φ. Their equations of motion lead to:

$$\begin{aligned} q'_1 \quad &= \quad \theta\,[e^t_-\,(\partial_+ q q^{-1} - q\tilde{k}_+ q^{-1}) + e^t_+ k_- + e^t_+ e^t_- q_1^{-1} n_1(t) q_1] \qquad , & (51)\\ q'_2 \quad &= \quad \theta\,[e^t_+\,(q^{-1}\partial_- q + q^{-1}k_- q) - e^t_-\tilde{k}_+ + e^t_+ e^t_- q_2^{-1} n_2(t) q_2] \qquad , & (52) \end{aligned}$$

where $n_1(t)$ and $n_2(t)$ are $sl(2,\mathbb{R})$ generators depending only on t, appearing upon φ integrations. It is easy to see that these degrees of freedom decouple from q in the action and, using their own equation of motion, may be directly set equal to zero. The resulting action becomes:

$$S_{WZWC} = S_{WZW} + \mathcal{B}^{(\varphi,t)}_{WZW} - \mathcal{G}_{WZWC} \qquad , \tag{53}$$

with the geometrical contribution canceling that occurring in eq. (45). The (non-chiral) WZW action is given by:

$$\begin{aligned} S_{WZW} \quad &= \quad \Gamma[Q] - \frac{\ell}{8\pi G}\int_{|r|=\bar{r}} Tr[\frac{1}{2}q^{-1}\partial_+ q\, q^{-1}\partial_- q + \partial_+ q q^{-1} k_- \\ &\qquad - q^{-1}\partial_- q\tilde{k}_+ - \tilde{k}_+ q^{-1} k_- q]\,\theta\, dt\, d\varphi \qquad . \end{aligned} \tag{54}$$

The boundary term $\mathcal{B}^{(\varphi,t)}_{WZW}$ reads as:

$$\mathcal{B}^{(\varphi,t)}_{WZW} = \frac{\ell}{8\pi G}\int_{\mathcal{V}} Tr[\partial_\varphi(Q_2^{-1}\partial_{[r}Q_2Q^{-1}\partial_{t]}Q)+\partial_t(Q_2^{-1}\partial_{[\varphi}Q_2Q^{-1}\partial_{r]}Q)]dr\, d\varphi\, dt \quad . \tag{55}$$

3.3 Boundary equations of motion as consistency equations

Let us for a moment focus on the equations of motion (51, 52) of q_1 and q_2 as functions of q. Using the remaining boundary conditions (25) and the Gauss decomposition for $SL(2,\mathbb{R})/Z_2$ elements:

$$q = \begin{pmatrix} e^{\phi/2} + xye^{-\phi/2} & xe^{-\phi/2} \\ ye^{-\phi/2} & e^{-\phi/2} \end{pmatrix} \quad , \tag{56}$$

eqs (51, 52) lead to 6 equations. Four of them:

$$x = \frac{\ell}{2}\partial_+\phi \quad , \quad y = \frac{\ell}{2}\partial_-\phi \quad , \tag{57}$$

$$\ell(\partial_- + \omega_-)x + \frac{1}{2}\sigma + e^\phi = 0 \quad , \quad \ell(\partial_+ - \omega_+)y + \frac{1}{2}\sigma + e^\phi = 0 \quad , \tag{58}$$

determine x and y as functions of ϕ and combine to give:

$$\Box\phi + \frac{8}{\ell^2}e^\phi + \frac{4}{\ell^2}\sigma = 0 \quad . \tag{59}$$

This is the Liouville equation on a curved background, the curvature being given by eq. (12). The last two equations are:

$$(\partial_+ + \omega_+)\partial_+\phi - \frac{1}{2}(\partial_+\phi)^2 + \frac{2}{\ell^2}\overset{(2)}{g}_{++} = 0 \quad , \tag{60}$$

$$(\partial_- - \omega_-)\partial_-\phi - \frac{1}{2}(\partial_-\phi)^2 + \frac{2}{\ell^2}\overset{(2)}{g}_{--} = 0 \quad . \tag{61}$$

Using eq. (4) and the expression of the energy-momentum tensor of the Liouville field, these equations and eq.(12) can be summarized as:

$$\overset{(2)}{g}_{ab} = \frac{\ell^2}{2}\,(T_{ab} - \eta_{ab}\overset{(0)}{\mathcal{R}}) \quad . \tag{62}$$

3.4 Third step: Liouville action

We now return to the non-chiral WZW action (54). Using for the matrix Q a Gauss decomposition in terms of Φ, X and Y similar to (56), the bulk term $\Gamma(Q)$ can be written as a sum of three kinds of boundary contributions:

$$\Gamma(Q) = -\frac{\ell}{16\pi G}\int d\wedge[e^{-\Phi}dX\wedge dY] = \mathcal{B}^{(r)}_\Gamma + \mathcal{B}^{(\varphi,t)}_\Gamma \quad . \tag{63}$$

The contribution on surfaces of constant $|r| = \bar{r}$ yields in the limit $\bar{r}\to\infty$:

$$\mathcal{B}^{(r)}_\Gamma = \frac{\ell}{8\pi G}\int_{|r|=\bar{r}} e^{-\phi}(\partial_{[\varphi}y\partial_{t]}x)\, d\varphi\, dt \quad , \tag{64}$$

where we have used eqs (50, 40), which give the asymptotic expressions:

$$e^{-\Phi} \sim \frac{r^2}{\ell^2} e^{-\phi} \qquad , \qquad X \sim \frac{\ell}{r} x \qquad , \qquad Y \sim \frac{\ell}{r} y \qquad . \tag{65}$$

The last contribution is:

$$\mathcal{B}_\Gamma^{(\varphi,t)} = \frac{\ell}{8\pi G} \int_{\mathcal{V}} [\partial_\varphi (e^{-\Phi} \partial_{[t} Y \partial_{r]} X) + \partial_t (e^{-\Phi} \partial_{[r} Y \partial_{\varphi]} X)] dr\, d\varphi\, dt \qquad . \tag{66}$$

Accordingly, the non-chiral WZW action (54) may be written as:

$$S_{WZW} = S_{(x,y,\phi)} + \mathcal{B}_\Gamma^{(\varphi,t)} + \mathcal{G}_{(x,y,\phi)} \qquad , \tag{67}$$

where

$$\begin{aligned} S_{(x,y,\phi)} \quad = \quad & -\frac{\ell}{16\pi G} \int_{|r|=\bar{r}} [\frac{1}{2} \partial_+ \phi \partial_- \phi + \omega_- \partial_+ \phi - \omega_+ \partial_- \phi \\ & + 2 e^{-\phi} (\partial_- x + \omega_- x + \frac{\sigma}{2\ell})(\partial_+ y - \omega_+ y + \frac{\sigma}{2\ell})] \theta\, d\varphi\, dt \,, \end{aligned} \tag{68}$$

and the geometrical term $\mathcal{G}_{(x,y,\phi)}$ is:

$$\mathcal{G}_{(x,y,\phi)} = \frac{\ell}{16\pi G} \int_{|r|=\bar{r}} \omega_+ \omega_- \theta\, d\varphi\, dt \qquad . \tag{69}$$

This term cancels the first term in the geometric contribution appearing upon modifying the CS action, see eqs (42,44).

Finally, the action $S_{(x,y,\phi)}$ (68) can be expressed in terms of ϕ only by eliminating [19, 6] the variables x and y in terms of the constants of motion defined by (58), using the same trick as the one that leads to the Maupertuis action in classical mechanics with conserved energy. From the definition of the connection and eqs (57, 58), we find:

$$\begin{aligned} \theta(\partial_- x + \omega_- x) \quad &= \quad \partial_\varphi (\theta_t^+ x) - \partial_t (\theta_\varphi^+ x) = -\theta (\frac{e^\phi}{\ell} + \frac{\sigma}{2\,\ell}) \qquad , \\ \theta(\partial_+ y - \omega_+ y) \quad &= \quad -\partial_\varphi (\theta_t^- y) + \partial_t (\theta_\varphi^- y) = -\theta (\frac{e^\phi}{\ell} + \frac{\sigma}{2\,\ell}) \qquad . \end{aligned} \tag{70}$$

Using these equations, we add to the action $S_{(x,y,\phi)}$ zero written as:

$$\begin{aligned} 0 \quad = \quad & \frac{\ell}{8\pi G} \int_{|r|=\bar{r}} \Big[e^{-\phi} (\partial_- x + \omega_- x)(\partial_+ y - \omega_+ y + \frac{\sigma}{2\ell}) \\ & \qquad + e^{-\phi} (\partial_+ y - \omega_+ y)(\partial_- x + \omega_- x + \frac{\sigma}{2\ell}) \Big] \theta d\varphi dt \\ & + \frac{\ell}{8\pi G} \int_{|r|=\bar{r}} \left[\partial_\varphi (\theta_t^+ x - \theta_t^- y) - \partial_t (\theta_\varphi^+ x - \theta_\varphi^- y) \right] d\varphi dt \qquad . \end{aligned} \tag{71}$$

Adding this null expression to $S_{(x,y,\phi)}$ allows to eliminate x and y as functions of ϕ using eqs (57, 58), while keeping the correct ϕ field equations and holonomy contributions. Accordingly, we get:

$$S_{(x,y,\phi)} = S_L + B_L + \mathcal{G}_L \qquad , \tag{72}$$

where S_L is the Liouville action on curved background:

$$S_L = -\frac{\ell}{32\pi G}\int_{|r|=\bar{r}}[\frac{1}{2}\overset{(0)}{g}{}^{ab}\partial_a\phi\partial_b\phi - \frac{8}{\ell^2}e^{\phi}+\overset{(0)}{\mathcal{R}}\,\phi]\sqrt{-\overset{(0)}{\mathbf{g}}}\,dt\,d\varphi \qquad . \tag{73}$$

Let us emphasize that the curvature term appearing here comes directly from its definition in terms of the asymptotic metric $\overset{(0)}{\mathbf{g}}$, and not through σ as it is the case in eq. (59). The term B_L is defined on the r,φ and r,t boundaries:

$$B_L = \frac{\ell}{16\pi G}\int_{|r|=\bar{r}}\left[\partial_a(\sqrt{-\overset{(0)}{\mathbf{g}}}\,\overset{(0)}{g}{}^{ab}\partial_b\phi) + \partial_\varphi(\omega_t\phi) - \partial_t(\omega_\varphi\phi)\right]d\varphi dt \qquad , \tag{74}$$

and the geometrical term is given by:

$$\mathcal{G}_L = \frac{\ell}{8\pi G}\int_{|r|=\bar{r}}\frac{\sigma}{\ell^2}\theta d\varphi dt \qquad . \tag{75}$$

4 Holonomies

In the previous section, we have shown that the EH action can be expressed as a sum of terms defined on the r, φ and t boundaries without any remaining bulk terms. The dynamical equation resulting from imposing the stationarity of the action on the spatial boundary $|r| = \bar{r}$, is that of Liouville on a curved background. We now analyze the dynamical content of the φ-boundary terms $\mathcal{B}^{(\varphi)}$ that occur when Σ has the topology of the cylinder.

The first two non vanishing holonomy terms are given by eqs (48, 55) and appear when going from the CS action to the non-chiral WZW action. Together they give, after substitution of Q_1 by Q_2Q^{-1}:

$$\mathcal{B}^{(\varphi)}_{WZWC}+\mathcal{B}^{(\varphi)}_{WZW} = \frac{\ell}{16\pi G}\int_{\mathcal{V}}\partial_\varphi\left[2\,Q_2^{-1}\partial_r Q_2 Q^{-1}\partial_t Q - Q^{-1}\partial_r Q Q^{-1}\partial_t Q\right]\,dr\,d\varphi\,dt\,. \tag{76}$$

Contrary to what happens on the boundary $|r| = \bar{r}$, Q_2 cannot be eliminated in terms of Q by using its equation of motion. When going from the non-chiral WZW action to the $S_{(x,y,\phi)}$ action, another holonomy term appears, $\mathcal{B}^{(\varphi)}_\Gamma$, given by eq. (64). Adding this term to (76), we obtain the φ-boundary action $\mathcal{B}^{(\varphi)}$:

$$\mathcal{B}^{(\varphi)} = \frac{\ell}{16\pi G}\int_{\mathcal{V}}\partial_\varphi\left[2\,Q_2^{-1}\partial_r Q_2 Q^{-1}\partial_t Q - \frac{1}{2}\partial_t\Phi\partial_r\Phi - 2\partial_t X\partial_r Y e^{-\Phi}\right]\,dr\,d\varphi\,dt \qquad , \tag{77}$$

which encodes all bulk terms that survive in addition to the Liouville action (73) on the spatial $|r| = \bar{r}$ boundary.

The equations of motion on the φ-boundary can be obtained from this action by varying Q_2, X, Y and Φ. But it is easier to obtain them by varying Q_1 and Q_2 in the CS action (45) expressed in terms of these fields. We get:

$$\delta\, S'_{CS} = \frac{\ell}{8\pi G}\int_{\mathcal{V}}\partial_\varphi Tr[Q_1^{-1}\partial_r(\partial_t Q_1 Q_1^{-1})\delta Q_1 - Q_2^{-1}\partial_r(\partial_t Q_2 Q_2^{-1})\delta Q_2]dr\,d\varphi\,dt \qquad . \tag{78}$$

In terms of the periodic group elements G_1 and G_2 defined in eqs (39), which factorize in the same way as the non-periodic ones (cf eq. 40):

$$G_1(\bar{r},\varphi,t) \underset{\bar{r}\to\infty}{\simeq} g_1(\varphi,t)S(\bar{r}) \qquad , \qquad G_2(\bar{r},\varphi,t) \underset{\bar{r}\to\infty}{\simeq} g_2(\varphi,t)S(\bar{r})^{-1} \qquad , \tag{79}$$

the variation of the action becomes:

$$\delta\, S'_{CS} = \frac{\ell}{8\pi G}\int_{|r|=\bar{r}} Tr[\partial_t g_1\, g_1^{-1}\,\delta\zeta - \partial_t g_2\, g_2^{-1}\,\delta\tilde{\zeta}\,]|_{\varphi=0}\, dt \qquad , \tag{80}$$

with $\delta\zeta(t) = \exp(-2\pi h)\delta\exp(2\pi h)$ and similarly for $\delta\tilde{\zeta}$. The equations of motion are thus:

$$\partial_t g_1\, g_1^{-1}\,|_{\varphi=0}|_{r=-\bar{r}}^{r=\bar{r}} = 0 \qquad , \qquad \partial_t g_2\, g_2^{-1}\,|_{\varphi=0}|_{r=-\bar{r}}^{r=\bar{r}} = 0 \qquad , \tag{81}$$

whose solutions are simply

$$\begin{aligned} g_1(\varphi=0,\,t)|_{r=-\bar{r}} &= g_1(\varphi=0,\,t)|_{r=+\bar{r}}\,\mathcal{C}_1 \qquad , \\ g_2(\varphi=0,\,t)|_{r=-\bar{r}} &= g_2(\varphi=0,\,t)|_{r=+\bar{r}}\,\mathcal{C}_2 \qquad , \end{aligned} \tag{82}$$

with $\mathcal{C}_1$ and $\mathcal{C}_2$ constant matrices. Hence, the bulk field configuration does not appear in the equations of motion; only the fields defined on the spatial boundary do.

It is interesting at this stage to compare these results with the equivalent ones obtained in [20] on the basis of expression (37) instead of (38) for the gauge potentials. In the latter approach, the CS action reduces to a sum of actions on the spatial boundaries involving globally defined (single-valued) fields which are coupled to the flat connections h and $\tilde{h}$ and hence to each other. In contrast, our analysis based on multi-valued fields appears to be simpler, as the only remaining coupling between the fields in the equations of motion is through eqs (82).

5 Flat boundary metric

To further analyze the significance of the holonomy contributions taken into account in the previous section via φ-boundary terms, we compute the gravitational holonomy encoded in the solutions of Einstein's equations. For this purpose, we restrict our analysis to the flat $\overset{(0)}{\mathbf{g}}$ metric (infinite cylinders have no moduli). For such asymptotic geometry, the expansion described in eq. (2) stops at order r^{-1} and the complete expression of locally AdS metrics reads as [17]:

$$ds^2 = \ell^2\frac{dr^2}{r^2} + \left(\frac{r}{\ell}\,dx^+ + \frac{\ell}{r}\,L_-(x^-)\,dx^-\right)\left(\frac{r}{\ell}\,dx^- + \frac{\ell}{r}\,L_+(x^+)\,dx^+\right) \qquad , \tag{83}$$

This metric describes two asymptotic regions: one in the neighbourhood of $r=\infty$, the other near $r=0$. In order to have a single radial coordinate that leads to the FG metric expression on both asymptotic regions simultaneously we have to introduce a new radial coordinate ρ defined by:

$$\rho = r \qquad \text{for } r > r_2 \qquad , \qquad \rho = -\frac{\ell^2}{r} \qquad \text{for } r < r_1 \qquad , \tag{84}$$

where $r_1 < r_2$ are two arbitrarily chosen positive values of r; between them, ρ is given by any smoothly interpolating increasing function of r. So, near $\rho = \infty$ we may choose as frame:

$$\Theta^0 = \ell\,\frac{d\rho}{\rho}\,, \quad \Theta^+ = \frac{\rho}{\ell}\,dx^+ + \frac{\ell}{\rho}\,L_-(x^-)\,dx^-\,, \quad \Theta^- = \frac{\rho}{\ell}\,dx^- + \frac{\ell}{\rho}\,L_+(x^+)\,dx^+\,. \tag{85}$$

It continues near $\rho = -\infty$ into:

$$\Theta^0 = -\ell\,\frac{d\rho}{\rho}\,, \quad \Theta^+ = -\frac{\rho}{\ell}\,L_-(x^-)\,dx^- - \frac{\ell}{\rho}\,dx^+\,, \quad \Theta^- = -\frac{\rho}{\ell}\,L_+(x^+)\,dx^+ - \frac{\ell}{\rho}\,dx^-\,. \tag{86}$$

To avoid closed causal curves we must assume the functions $L_-(x^-)$ and $L_+(x^+)$ to be non-negative. In this case the geometry (83) presents a singularity on the surface $r^4 = \ell^4\,L_-(x^-)L_+(x^+)$. Such a singularity deserves special attention: in the case of BTZ black holes it corresponds to the troath of the Einstein-Rosen bridge, but it is not clear if it still corresponds to a coordinate singularity in the more general case or becomes a true singularity.

On the boundary $r = \infty$, the Liouville field solution of eq. (59) can be expressed as:

$$\phi = \log\left|\frac{\ell^2\,f'_+\,f'_-}{(f_+ + f_-)^2}\right|\,, \tag{87}$$

where f_+ (resp. f_-) is a function of x^+ (resp. x^-) only and f'_+ (resp. f'_-) its derivative with respect to its argument. To yield the metric (83) these functions must be related to the components of the subdominant term of the asymptotic metric as:

$$L_+ = \frac{\ell^2}{2}\left[\frac{3}{2}\left(\frac{f''_+}{f'_+}\right)^2 - \frac{f'''_+}{f'_+}\right]\,, \qquad L_- = \frac{\ell^2}{2}\left[\frac{3}{2}\left(\frac{f''_-}{f'_-}\right)^2 - \frac{f'''_-}{f'_-}\right]\,. \tag{88}$$

Here we see the important role played by the local character of the gauge fields. Indeed, though $L_+(x^+)$ and $L_-(x^-)$ are periodic in their arguments, the functions $f_\pm$, and thus the Liouville field ϕ, are not necessarily so.

Let us consider the right handed sector. For a given L_+ function, the solution f_+ of eq. (88) can be obtained by posing $f'_+ = 1/w_+^2$, the function w_+ being a solution of the linear second order equation:

$$w''_+ = \frac{L_+(x^+)}{\ell^2}\,w_+\,. \tag{89}$$

Floquet's theory [21, sec. **19.4**] gives us the general functional form of the solutions of this equation. These can be written as linear combinations of particular solutions given by the product of exponential and periodic functions:

$$\mathcal{F}(x^+) = \exp(\mu_+\,x^+)F(x^+) \quad \text{and} \quad \mathcal{K}(x^+) = \exp(-\mu_+\,x^+)K(x^+)\,, \tag{90}$$

where μ_+ is a real or purely imaginary constant. As a consequence, the function f_+ appearing in eq. (88) can, for real values of μ_+, always be chosen as the product of $\exp(2\mu_+\,x^+)$ with a periodic function; but of course more complicated functional forms are also possible and are even required when μ_+ is purely imaginary. The most general expression of f_+ depends on three parameters, and is obtained as the integral of the inverse of the square of the general solution of eq. (89).

The discontinuities of the Liouville field (of the function $f_\pm$) when their argument increases by 2π reflect an important geometrical property of asymptotic AdS_3 spaces. They encode in a simple way the global holonomy properties of the general metric (83). We now attempt to clarify this.

In the $SL(2,\mathbb{R})$ basis, the equation of parallel transport by means of the connections $\mathbf{A}$ is:

$$\partial_\mu \Psi + A_\mu^a J_a \Psi = 0 \qquad , \tag{91}$$

where Ψ has the two components Ψ_1 and Ψ_2. Note that these equations, together with those of the left handed sector, are exactly those defining the Killing spinors of the manifold. The general solutions of eq. (91) for the metric (83) are of the form:

$$\Psi_1 = -\sqrt{\frac{\ell}{r}}\,\ell\,\psi'_+(x^+) \qquad , \qquad \Psi_2 = \sqrt{\frac{r}{\ell}}\psi_+(x^+) \qquad , \tag{92}$$

with the chiral field $\psi_+(x^+)$ satisfying:

$$\psi''_+ = \frac{L_+}{\ell^2}\psi_+ \qquad . \tag{93}$$

This is nothing else but equation (89) defining f_+. Hence, the general solutions of eq. (93) can be written as a linear combination of $1/\sqrt{f'_+}$ and $f_+/\sqrt{f'_+}$. We find:

$$\begin{pmatrix} \Psi_1(r,x^+) \\ \Psi_2(r,x^+) \end{pmatrix} = P(r,x^+;r_0,x_0^+) \begin{pmatrix} \Psi_1(r_0,x_0^+) \\ \Psi_2(r_0,x_0^+) \end{pmatrix} \qquad , \tag{94}$$

where the parallel transport matrix $P(r,x^+;r_0,x_0^+)$ is given by:

$$P = S^{-1}(r)\,p\,S(r_0) \qquad , \tag{95}$$

with (we drop the '+' subscript to lighten the notation):

$$p = \begin{pmatrix} \sqrt{\frac{f'}{f'_0}}\left[1+\frac{f''(f_0-f)}{2\,(f')^2}\right] & \frac{\ell^2}{4}\left[\frac{2(f'^2_0 f''-f'^2 f''_0)+f''_0 f''(f-f_0)}{(f' f'_0)^{3/2}}\right] \\ \frac{1}{\ell^2}\frac{(f_0-f)}{\sqrt{f'_0 f'}} & \sqrt{\frac{f'_0}{f'}}\left[1+\frac{f''_0(f-f_0)}{2\,(f'_0)^2}\right] \end{pmatrix} \qquad , \tag{96}$$

whose eigenvalues are simply $\exp(\pm 2\pi\mu_+)$. Thus, since μ_+ is given by the zeros of an infinite determinant built from all the Fourier coefficients of L_+ [21, sec. **19.4**], the holonomies depend on all the on-shell values of the classical generators of the asymptotic Virasoro algebra. The matrix $\tilde{p}(r,x^-;r_0,x_0^-)$ giving the $\tilde{\Psi}$ solution can be obtained from p^{T-1} by the substitutions $f_+ \to f_-$ and $x^+ \to x^-$.

The matrix p immediately yields the Wilson loop matrix $S^{-1}(r_0)\,w\,S(r_0)$ of parallel transport around close loops, by posing $x^+ = x_0^+ + 2\pi\ell$. They also yield the matrices q_1 and q_2 defining the flat connection. Indeed, inserting eqs (38) into eq. (91), we obtain:

$$q_1(x^+) = q_1(x_0^+)\,p^{-1} \qquad , \qquad q_2(x^-) = q_2(x_0^-)\,\tilde{p}^{-1} \qquad . \tag{97}$$

To make the decomposition (39) explicit, we use as set of solutions of eq. (93) the following linear combination of the quasi-periodic solutions (90):

$$U(x^+) = \ell\left[\mathcal{K}_0'\,\mathcal{F}(x^+) - \mathcal{F}_0'\,\mathcal{K}(x^+)\right] \quad , \tag{98}$$

$$V(x^+) = \mathcal{F}_0\,\mathcal{K}(x^+) - \mathcal{K}_0\,\mathcal{F}(x^+) \quad , \tag{99}$$

where the subscript 0 indicates that the functions must be taken at an arbitrary reference point x_0^+; the functions $\mathcal{F}$ and $\mathcal{K}$ are (partially) normalized so that their wronskian is $1/\ell$. This implies that

$$U(x_0^+) = 1 \quad , \quad U'(x_0^+) = 0 \quad , \quad V(x_0^+) = 0 \quad , \quad V'(x_0^+) = \frac{1}{\ell} \quad . \tag{100}$$

and the wronskian of U and V is also equal to $1/\ell$:

In terms of these solutions, we can write explicitly the non-trivial flat connection $h(t)$ generating the holonomies (eq. 39). When μ_+ is real we have:

$$h(t) = 2\,\mu_+\,q_1(x_0^+)\begin{pmatrix} \ell\mathcal{K}_0' & -\ell\mathcal{F}_0' \\ -\mathcal{K}_0 & \mathcal{F}_0 \end{pmatrix} J_0 \begin{pmatrix} \mathcal{F}_0 & \ell\mathcal{F}_0' \\ \mathcal{K}_0 & \ell\mathcal{K}_0' \end{pmatrix} q_1^{-1}(x_0^+) \quad . \tag{101}$$

Thus $h(t)$ is a just a similarity transform of the matrix $2\,\mu_+\,J_0$. As a consequence, the matrix g_1 (see eq. 79), which depends on both t and φ (and not only on x^+), is globally defined (as it must be), its entries being given by products of exponentials of $\pm\mu_+\,t$ and periodic functions of x^+. When μ_+ is purely imaginary, we pose $\mu_+ = -i\nu_+$, and see that the generators of holonomies $h(t)$ are now similarity transforms of $2\,\nu_+\,J_2$, thus of a timelike generator, whereas the generator found for real μ_+ is spacelike.

Finally note that the expressions (92) and the subsequent developments could also be obtained by limiting ourselves to an asymptotic evaluation of eqs (91). This is due to the special form (83) of the metric, for which the connection form $\underline{\Omega}^{+-}$ vanishes to all orders. As a consequence, the factorization (40) is valid up to $r \mapsto 0$, and eqs (81) are trivially satisfied, i.e. $\mathcal{C}_1 = 1$ and $\mathcal{C}_2 = 1$.

To illustrate the above results, let us consider the special case of BTZ black holes. From its metric written in FG form (17), we immediately obtain the subdominant metric components near $r = \infty$ in the null frame:

$$\overset{(2)}{g}_{++} \equiv L_+ = \frac{1}{4}(M + \frac{J}{\ell}) \quad , \quad \overset{(2)}{g}_{--} \equiv L_- = \frac{1}{4}(M - \frac{J}{\ell}) \quad . \tag{102}$$

For such constant components, eq. (88) is easily integrated in terms of purely exponential solutions:

$$f_+ = \frac{a\,e^{2\mu_+\,x^+} + b}{c\,e^{2\mu_+\,x^+} + d} \quad , \tag{103}$$

where $\mu_+ = \frac{1}{2}\sqrt{M + \frac{J}{\ell}}$ and $a\,d - b\,c = 1$; f_- is obtained by the substitutions $x^+ \to x^-$ and $J \to -J$. As expected, the Liouville field constructed from these solutions is in general not globally defined. Only for $J = 0$ can ϕ be extended periodically (it becomes φ independent) on the interval $[0, 2\pi]$ of the φ-coordinate if we choose $f_+ = \exp(\mu\,x^+)$ and $f_- = \exp(\mu\,x^-)$, $\mu = \sqrt{M}/2$.

Furthermore, we easily obtain (compared with [22] where the same calculation was originally performed in another context) the holonomy matrix w

corresponding to a Wilson loop surrounding the throat of the Einstein-Rosen bridge:

$$w = \begin{pmatrix} \cosh\left(\pi\sqrt{M+\frac{J}{\ell}}\right) & -\frac{\sqrt{M+\frac{J}{\ell}}}{2}\sinh\left(\pi\sqrt{M+\frac{J}{\ell}}\right) \\ -\frac{2}{\sqrt{M+\frac{J}{\ell}}}\sinh\left(\pi\sqrt{M+\frac{J}{\ell}}\right) & \cosh\left(\pi\sqrt{M+\frac{J}{\ell}}\right) \end{pmatrix} \quad , \tag{104}$$

and similarly for $\tilde{w}$ with $J \to -J$. The holonomy is thus always non-trivial, and there exists no globally well defined Killing spinor fields, except in two cases. The first is $J = 0$ and $M = -1$, in which case $w = -2\,J_0$ and the geometry is that of the usual AdS_3 space. The second case is $J = 0$ and $M = 0$, where we have to impose $\Psi_1 = 0 = \tilde{\Psi}_2$ in order to obtain global Killing spinor fields. In the former case, we have thus 4 Killing vector fields, and in the latter 2, in agreement with [23].

6 Anomalies

As a preliminary to the discussion of the link between the diffeomorphism anomaly of the gravitational action and the conformal anomaly of the Liouville action, let us briefly summarize the various steps that led in section **3** from the EH action to the Liouville action, through the CS action. Using eqs (20, 22), and assuming the metric to be globally defined on $\mathcal{V}$, we find that the EH action (18) and CS action (20) are related by:

$$S_{EH} = S_{CS} + \mathcal{B}^{(r)}_{CS} + \mathcal{B}^{(t)}_{CS} \quad , \tag{105}$$

where $\mathcal{B}^{(r)}_{CS}$ is an $\bar{r}^2$-divergent term defined on the spatial boundary given by eq. (26), and $\mathcal{B}^{(t)}_{CS}$ is a $\log \bar{r}$-divergent term corresponding to a total derivative with respect to t and given by eq. (27). The EH action presents an $\bar{r}^2$-divergence, whereas the CS action is at most logarithmically divergent.

Furthermore, using eqs (42,45,53,67,72), we find the relation between the CS action (20) and the Liouville action (73):

$$S_{CS} = S_L + \mathcal{B}^{(\varphi)} + \mathcal{B}^{(t)} + B \quad . \tag{106}$$

$\mathcal{B}^{(\varphi)}$ is defined in eq. (77) and corresponds to the sum of all φ-boundary terms encountered when going from the CS action to the Liouville action. Similarly, $\mathcal{B}^{(t)}$ is the sum of all t-boundary terms and is given by:

$$\begin{aligned} \mathcal{B}^{(t)} &= \mathcal{B}^{(t)}_{WZW} + \mathcal{B}^{(t)}_{\Gamma} \\ &= \frac{\ell}{8\pi G}\int_{\mathcal{V}} \partial_t \left[Tr(Q_2^{-1}\partial_{[\varphi}Q_2 Q^{-1}\partial_{r]}Q) + e^{-\Phi}\partial_{[r}Y\partial_{\varphi]}X\right] dr\, d\varphi\, dt \end{aligned} \tag{107}$$

with $\mathcal{B}^{(t)}_{WZW}$ and $\mathcal{B}^{(t)}_{\Gamma}$ defined by eqs (55, 66). Finally, the term B is sum of all geometrical terms (44, 69, 75) and the boundary term B_L (74). It is equal to:

$$\begin{aligned} B &= \frac{\ell}{16\pi G}\int_{|r|=\bar{r}} \partial_\varphi \left[\sqrt{-\overset{(0)}{\mathbf{g}}}\, \overset{(0)}{g}{}^{\varphi b}\partial_b\phi) + \omega_t\phi\right] \\ &\qquad + \partial_t \left[\sqrt{-\overset{(0)}{\mathbf{g}}}\, \overset{(0)}{g}{}^{tb}\partial_b\phi) - \omega_\varphi\phi + 2\,\omega_\varphi\right] d\varphi dt \quad . \end{aligned} \tag{108}$$

This term is defined on the (r,φ) and (r,t) boundaries.

We now analyze the variation under Weyl transformations and under diffeomorphisms of the EH, CS and Liouville actions. Consider a diffeomorphism generated by the vector field ξ, which moves the boundary $|r|=\bar{r}$ and keeps the metric in the FG form (1). Its radial component is asymptotically of the form $\xi^r=\delta\alpha(\varphi,t)\,r$, where we assume $\delta\alpha$ and all its derivatives vanishing on the space-like boundaries of $\mathcal{V}$; the t and φ components may be assumed to be of order r^{-2}[24, 14]. Under the action of this diffeomorphism, the metric varies as:

$$\delta_D \overset{(0)}{g}_{ab} = 2\overset{(0)}{g}_{ab}\,\delta\alpha \quad , \tag{109}$$

$$\delta_D \overset{(2)}{g}_{ab} = \ell^2\delta\alpha_{,b;a} \quad . \tag{110}$$

The on-shell variation of the EH action (18) under these transformations can be computed as:

$$\delta_D S_{EH} = \frac{\ell}{16\pi G}\int_{|r|=\bar{r}} Tr[\xi^r\{6\,\partial_{[r}(\tilde{A}_\varphi A_{t]}) - 2\,A_r(A_{[\varphi}A_{t]}+\tilde{A}_{[\varphi}\tilde{A}_{t]})\}]\,d\varphi dt \quad . \tag{111}$$

Inserting the boundary conditions (23, 24, 25) in this equation, we obtain by expanding the metric:

$$\delta_D S_{EH} = \frac{\ell}{16\pi G}\int_{|r|=\bar{r}} [\overset{(0)}{\mathcal{R}} - \frac{4\bar{r}^2}{\ell^4}]\sqrt{\overset{(0)}{\mathbf{g}}}\,\delta\alpha\,d\varphi dt \quad , \tag{112}$$

in agreement with [25, 26, 14]. Similarly, the on-shell variation of the CS action (20) is given by:

$$\delta_D S_{CS} = -\frac{\ell}{16\pi G}\int_{|r|=\bar{r}} Tr[2\,\xi^r\,A_r(A_{[\varphi}A_{t]}+\tilde{A}_{[\varphi}\tilde{A}_{t]})]\sqrt{\overset{(0)}{\mathbf{g}}}\,d\varphi dt \quad , \tag{113}$$

and using the boundary conditions (23, 24, 25):

$$\delta_D S_{CS} = \frac{\ell}{32\pi G}\int_{|r|=\bar{r}} \overset{(0)}{\mathcal{R}}\sqrt{\overset{(0)}{\mathbf{g}}}\,\delta\alpha\,d\varphi dt \quad . \tag{114}$$

Using (20), we see that the difference between the variations of the EH and CS actions is transferred in the $\mathcal{B}_{CS}$ term (22). We find indeed:

$$\delta_D\mathcal{B}_{CS} = \frac{\ell}{16\pi G}\int_{|r|=\bar{r}} Tr[\xi^r\{6\,\partial_{[r}(\tilde{A}_\varphi A_{t]})]\,d\varphi dt \tag{115}$$

$$= \frac{\ell}{16\pi G}\int_{|r|=\bar{r}} [\frac{1}{2}\overset{(0)}{\mathcal{R}} - \frac{4\bar{r}^2}{\ell^4}]\sqrt{\overset{(0)}{\mathbf{g}}}\,\delta\alpha\,d\varphi dt \quad , \quad . \tag{116}$$

Thus, the variations of the EH and SC actions under diffeomorphisms are related by:

$$\delta_D S_{EH} = 2\,\delta_D S_{CS} + \frac{\bar{r}^2}{4\pi G\ell^3}\int_{|r|=\bar{r}} \sqrt{\overset{(0)}{\mathbf{g}}}\,\delta\alpha\,d\varphi dt \quad . \tag{117}$$

The variation $\delta_D S_{EH}$ presents a $\bar{r}^2$ divergence whereas $\delta_D S_{SC}$ is finite. More strikingly, the finite contributions of $\delta_D S_{EH}$ and $\delta_D S_{SC}$ differ by a factor of 2.

Since the Liouville action S_L (73) is finite and defined on the spatial boundary $|r| = \bar{r}$, it is invariant under the 3-dimensional diffeomorphisms. Hence, when going from the CS action to the Liouville action, the variation under diffeomorphisms must be totally transferred to the additional terms given in eq. (106), in particular to the $\mathcal{B}$ boundary terms, as the $\mathcal{B}$ term (108) is invariant under 3-dimensional diffeomorphisms. It is easy to check that it is transferred to the $\mathcal{B}^{(t)}$ term (107) that presents a logarithmic divergence in $\bar{r}$. We find indeed that on-shell:

$$\mathcal{B}^{(t)} \underset{\bar{r}\to\infty}{\simeq} -\frac{\ell}{16\pi G}\int_{\mathcal{V}} \frac{1}{r}\partial_t \omega_\varphi \, dr\, d\varphi\, dt \qquad . \tag{118}$$

Using the fact that $\sqrt{\overset{(0)}{\mathbf{g}}}\,\overset{(0)}{\mathcal{R}}$ is equal to $-2\partial_t\omega_\varphi$ up to a φ-derivative, we obtain the expected variation under diffeomorphisms:

$$\delta_D \mathcal{B}^{(t)} = \frac{\ell}{32\pi G}\int_{|r|=\bar{r}} \overset{(0)}{\mathcal{R}}\sqrt{\overset{(0)}{\mathbf{g}}}\,\delta\alpha\, d\varphi dt \qquad . \tag{119}$$

Let us now consider the variation of the actions under a Weyl transformation on the spatial boundary $|r| = \bar{r}$, which compensates the effect of the diffeomorphism (109) on the boundary metric:

$$\delta_W \overset{(0)}{g}_{ab} = -\delta_D \overset{(0)}{g}_{ab} \qquad . \tag{120}$$

The EH action is known to be invariant under the combined action of this Weyl transformation and of the diffeomorphism (109) [13, 14]:

$$\delta_D S_{EH} + \delta_W S_{EH} = 0 \qquad . \tag{121}$$

The variation of the Liouville action S_L (73) under this Weyl transformation is easily computed using (62):

$$\delta_W S_L = -\frac{\ell}{32\pi G}\int_{|r|=\bar{r}} T_{ab}\delta_W \overset{(0)}{g}{}^{ab}\sqrt{\overset{(0)}{\mathbf{g}}}\, d\varphi dt = -\frac{\ell}{16\pi G}\int_{|r|=\bar{r}} \overset{(0)}{\mathcal{R}}\sqrt{\overset{(0)}{\mathbf{g}}}\,\delta\alpha\, d\varphi dt \qquad . \tag{122}$$

Inspection of eqs (105, 106) shows that the only other term which has a non-vanishing contribution to the Weyl variation is $\mathcal{B}^{(r)}_{CS}$. We find:

$$\delta_W \mathcal{B}^{(r)}_{CS} = \frac{\ell}{16\pi G}\int_{|r|=\bar{r}} \frac{4\bar{r}^2}{\ell^4}\sqrt{\overset{(0)}{\mathbf{g}}}\,\delta\alpha\, d\varphi dt \tag{123}$$

We have thus:

$$\delta_W S_{EH} = \delta_W S_L + \delta_W \mathcal{B}^{(r)}_{CS} \qquad . \tag{124}$$

7 Conclusion

In this paper we have explicitly shown that the 3-d EH action with negative cosmological constant is equivalent to a Liouville action on the spatial $\bar{r} \to \infty$ boundaries, plus terms on the t- and φ-boundaries. The equations of motions derived from the φ-boundary terms, in the case of non-trivial topologies such as those of BTZ black holes, relate the two non-connected components constituting spatial infinity. An originality of our approach resides in the fact that we encoded the holonomy in multi-valued functions.

Owing to the fact that we considered arbitrary curved metrics on the spatial r boundaries, we gave an explicit demonstration that the variation under diffeomorphisms of the 3-d EH action is equal to the Weyl anomaly of the asymptotic Liouville theory.

We moreover discussed in detail the asymptotically flat solutions, which generate upon diffeomorphisms all other solutions. We obtained explicitly the link between the subdominant term of the metric on the spatial boundary, which encode the FG ambiguity, and the Liouville field. This reveals that though each regular boundary metric can be expressed in terms of a 3-parameter family of Liouville fields, there are acceptable Liouville fields that lead to singular metrics. The Liouville (and CS) theory thus contains much more solutions than the EH gravity theory. Let us furthermore stress that a Liouville field defines a solution of Einstein equations only if it can be extended to CS connections $\mathbf{A}$ and $\tilde{\mathbf{A}}$ whose difference yield driebein fields that are regular or only present singularities interpretable as coordinate singularities. In the latter case, the topology of the space on which these CS fields are defined will differ from that of AdS.

Note that the developments following eq. (45), which lead from the non-chiral WZW action to the Liouville action (73), are classically valid, but have to be re-examined in the framework of quantum mechanics. Indeed, in quantum mechanics, the changes of variables leading to eq. (54) and the subsequent elimination of the X and Y variables in terms of Φ involve functional determinants that have been completely ignored.

Finally, our analysis does not solve the problem of the degrees of freedom at the origin of the black hole entropy, but leads to formulate the hypothesis that these degrees of freedom could possibly be encoded in a generalized holographic principle involving a multiply connected surface at infinity, each component having its own independent Liouville field only related to the others by consistency relations dictated by the holonomies.

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8 Appendix

In this appendix we recall some 2-d formula that we found useful for the calculations of the main text.

$sl(2,\mathbb{R})$ generators:

$$J_0 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad J_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad , \quad J_2 = \frac{1}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{125}$$

Null frame

$$\begin{aligned} &\underline{\theta}^{\hat{p}} = \theta^{\hat{p}}_i \, dx^i \quad , \quad \vec{e}_{\hat{p}} = \partial_{\hat{p}} = e^i_{\hat{p}} \partial_{x^i} \qquad \hat{p}, \hat{q} \cdots \in \{+,-\} \quad , \quad x^i \in \{\varphi, t\} \\ &\theta \, e^i_{\hat{p}} = \varepsilon_{\hat{p}\,\hat{q}} \varepsilon^{i\,j} \theta^{\hat{p}}_j \qquad \varepsilon_{+\,-} = 1 \quad , \quad \varepsilon^{t\,\varphi} = 1 \end{aligned} \tag{126}$$

Metric in the null frame

$$\begin{aligned}
ds^2 &= \eta_{\hat{p}\,\hat{q}}\underline{\theta}^{\hat{p}}\otimes\underline{\theta}^{\hat{q}} \qquad \hat{p},\,\hat{q}\in\{+,\,-\} \\
\partial_s^2 &= \eta^{\hat{p}\,\hat{q}}\vec{e}_{\hat{p}}\otimes\vec{e}_{\hat{q}} \\
\theta &= \det\left(\theta_i^{\hat{p}}\right) = \theta_t^+\theta_\varphi^- - \theta_\varphi^+\theta_t^- \\
\eta_{\hat{p}\,\hat{q}} &= \begin{pmatrix} 0 & -1/2 \\ -1/2 & 0 \end{pmatrix}
\end{aligned} \tag{127}$$

Connection coefficients

$$\begin{aligned}
&d\underline{\theta}^+ = \underline{\omega}\wedge\underline{\theta}^+ \qquad , \qquad d\underline{\theta}^- = -\underline{\omega}\wedge\underline{\theta}^- \\
&\omega = \omega^-{}_- = -\omega^+{}_+ = \omega_+\theta^+ + \omega_-\theta^- \\
&\varepsilon^{i\,j}\varepsilon_{\hat{p}\,\hat{q}}\partial_{x^i}\theta_j^{\hat{p}} = \omega_{\hat{q}}\,\theta
\end{aligned} \tag{128}$$

Gauss curvature

$$\begin{aligned}
&\mathcal{R} = 4[\partial_-(\omega_+) - \partial_+(\omega_-) + 2\,\omega_+\omega_-] \\
&\theta\,\mathcal{R} = 4\,\partial_{x^i}[\theta(e_-^i\omega_+ - e_+^i\omega_-)]
\end{aligned} \tag{129}$$

Weyl variance

$$\begin{aligned}
&\underline{\theta}^{\hat{p}} \mapsto e^{-\alpha}\,\underline{\theta}^{\hat{p}} \\
&\vec{e}_{\hat{p}} \mapsto e^{\alpha}\,\vec{e}_{\hat{p}} \\
&\theta \mapsto e^{-2\,\alpha}\theta \\
&\underline{\omega} \mapsto \underline{\omega} - \partial_-(\alpha)\,\underline{\theta}^- + \partial_+(\alpha)\,\underline{\theta}^+ \\
&\omega_\pm \mapsto e^{\alpha}\,[\omega_\pm \pm \partial_\pm(\alpha)]
\end{aligned} \tag{130}$$

Dalembertian

$$\Box\phi = 2\,[\partial_+\partial_-\phi - \omega_+\partial_-\phi + \partial_-\partial_+\phi + \omega_-\partial_+\phi] \tag{131}$$

Liouville energy-momentum tensor

$$T_{ab} = \frac{1}{2}\phi_{;a}\phi_{;b} - \phi_{;ab} - \eta_{ab}(\frac{1}{4}\phi_{;c}\phi^{;c} - \phi_{;c}^{;c} - \frac{4}{\ell^2}e^{\phi}) \tag{132}$$

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9 Figure caption

Figure 1: Penrose diagram of BTZ black hole with $M > |J/\ell|$ and domains of definition of the radial FG coordinate.