RELATIVISTIC WAVE EQUATION FOR FIELDS WITH TWO MASS AND SPIN STATES

S.I. Kruglov [1]

International Educational Centre, 2727 Steeles Ave. W, # 104, Toronto, Ontario, Canada M3J 3G9

Abstract

I suggest wave equations for the scalar, pseudoscalar, vector, and pseudovector fields with different masses for spin zero and one states. Tensor, matrix, and quaternion formulations of fields with two mass and spin states are considered. This is the generalization of the Dirac-Kähler equation on the case of different masses of fields with spin one and zero. The equation matrices obtained are simple linear combinations of matrix elements in the 16-dimensional space. Spin projection operators and solutions of equations (for spin one) in the form of matrix-dyads are obtained. The canonical quantization of fields under consideration is studied. The anomalous interaction of the scalar, pseudoscalar, vector, and pseudovector fields with the external electromagnetic field is considered. Three constants which characterize the anomalous magnetic moment and quadrupole electric moment of a particle are introduced.

1 Introduction

The modern high energy physics require the consideration of fields with higher spins. So, supersymmetrical (SUSY) field models introduce superpartners of ordinary particles which are high spin particles. Cosmological models also deal with fields possessing different spins.

The theory of relativistic wave equations describing particles with arbitrary spins has a long history [1], but is not complete yet. The simple high spin fields are fields with spin one. Ordinarily, spin-1 particles are described by Proca [2] or Petiau-Duffin-Kemmer equations [3], [4], [5]. Anomalous interactions of vector particles with electromagnetic fields were considered in [6], [7], [8]. Other schemes to the theory of spin one particles were developed by [9], [10].

Kähler [11] considered an equation for inhomogeneous differential forms which is equivalent to a system of scalar, vector, antisymmetric tensor, pseudovector and pseudoscalar fields [12], [13]. Now, the Kähler equation (called

[1]E-mail: skrouglo@utm.utoronto.ca

Dirac-Kähler equation), in the framework of differential forms is intensively used for description of quarks on the lattice [14]. Note that some authors [15] considered equations for the system of antisymmetric tensor fields which are equivalent to the Kähler equation before the appearance of [11]. It should be mentioned that the Lagrangian of a system of scalar, vector, antisymmetric tensor, pseudovector and pseudoscalar fields (equivalent to Dirac-Kähler fields) is invariant under the internal symmetry group $SO(4,2)$ (or locally isomorphic group $SU(2,2)$) [12], [13].

In this paper, we generalize the Dirac-Kähler equation on the case that scalar, pseudoscalar fields having different masses, m_0, compared with vector, pseudovector fields, m. At the particular case $m_0 = m$, one arrives at the Dirac-Kähler fields.

The paper is organized as follows: In Section 2, we introduce the field equations which are the generalization of the Dirac-Kähler equations. It is prove that fields introduced describe scalar, pseudoscalar fields with mass m_0 and vector, pseudovector fields with mass m. We show that introduced 16-component system of equations can be represented as two 8-component independent sub-systems with self-dual, and antiself-dual antisymmetrical tensors. At parity transformations, these systems are converted to each other. In Section 3, we construct the matrix form of 16-component relativistic wave equation. It is shown that matrices of the equation obey the Dirac commutation relation. This equation also contains two projection operators which are connected with two possible mass states. Section 4 is devoted to the matrix form of 8-component equations. In the case of equal field masses, $m_0 = m$, the 8-component matrices also satisfy the Dirac commutation relation. In section 5, we study the Lorenz covariance of equations under consideration. The one-parameter symmetry group of the Lagrangian and the corresponding conserve current are investigated here. The quaternion form of fields is constructed in section 6. This quaternion form includes different particular cases of field schemes such as Proca, Maxwell, etc. In section 7, spin projection operators and solutions of equations for spin one in the form of matrix-dyads are found. The quantization of fields is carried out in Section 8. In section 9, we consider the possible anomalous electromagnetic field interaction. Section 10 contains the discussion. The appendix is devoted to the computation of products of 16-component matrices.

The system of units $\hbar = c = 1$, $\alpha = e^2/4\pi = 1/137$, $e > 0$ is used.

2 Field equations

Now, I suggest new field equations for the set of bosonic fields which have two spin one states and two spin zero states with differen masses. In particular cases, these equations are transformed to one considered in literature.

Let us introduce and investigate the system of equations for free scalar $\varphi(x)$, pseudoscalar $\widetilde{\varphi}$, vector $\varphi_\mu(x)$, pseudovector $\widetilde{\varphi}_\mu$ and antisymmetric tensor $\varphi_{\mu\nu}$ fields:

$$\partial_\nu\varphi_{\mu\nu}(x)-\partial_\mu\varphi(x)+m^2\varphi_\mu(x)=0, \qquad \partial_\nu\widetilde{\varphi}_{\mu\nu}(x)-\partial_\mu\widetilde{\varphi}(x)+m^2\widetilde{\varphi}_\mu(x)=0, \quad (1)$$

$$a\partial_\mu\varphi_\mu(x)-\varphi(x)=0, \qquad a\partial_\mu\widetilde{\varphi}_\mu(x)-\widetilde{\varphi}(x)=0, \quad (2)$$

$$\varphi_{\mu\nu}(x)=\partial_\mu\varphi_\nu(x)-\partial_\nu\varphi_\mu(x)-\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha\widetilde{\varphi}_\beta(x), \quad (3)$$

where $\partial_\mu=(\partial_m,-i\partial_0)$, $x_\mu=(x_m,ix_0)$; x_0 is the time, and

$$\widetilde{\varphi}_{\mu\nu}(x)=\frac{1}{2}\varepsilon_{\mu\nu\alpha\beta}\varphi_{\alpha\beta}(x) \quad (4)$$

is the dual tensor, $\varepsilon_{\mu\nu\alpha\beta}$ is an antisymmetric tensor Levy-Civita; $\varepsilon_{1234}=-i$; a and m are parameters which are connected with the masses of scalar, pseudoscalar and vector, pseudovector fields. Equations (1)-(4) are the generalization of the tensor form of the Dirac-Kähler equation [11]. At $a=1$, we come to Dirac-Kähler equations [12] (see also [13]); the case $\widetilde{\varphi}(x)=0$ and $\widetilde{\varphi}_\mu(x)=0$ was considered in [16]; at $a=1$, $\widetilde{\varphi}(x)=0$, $\widetilde{\varphi}_\mu(x)=0$, we arrive at the Stueckelberg equations [9] (see also [17]); the case $a=0$, corresponding vector and pseudovector fields, was investigated in [18]; in the case $\varphi(x)=0$, $\widetilde{\varphi}(x)=0$, $\widetilde{\varphi}_\mu(x)=0$ one arrives at the Proca equations [2]; at $a=1$, $\widetilde{\varphi}(x)=0$, $\widetilde{\varphi}_\mu(x)=0$, $m=0$, we obtain generalized Maxwell's equations investigated in [19]. So, field equations (1)-(4) include many important cases of equations for bosonic fields with different number of freedom degrees.

To clear up the physical meaning of constants a, m, one can get from Eqs. (1)-(4) the second order equations for vector and pseudovector fields

$$\begin{aligned}&\partial_\nu^2\varphi_\mu(x)-(1-a)\partial_\mu\partial_\nu\varphi_\nu(x)-m^2\varphi_\mu(x)=0,\\ &\partial_\nu^2\widetilde{\varphi}_\mu(x)-(1-a)\partial_\mu\partial_\nu\widetilde{\varphi}_\mu(x)-m^2\widetilde{\varphi}_\mu(x)=0.\end{aligned} \quad (5)$$

Because two equations in (5) are identical, masses of vector and pseudovector fields are the same. To obtain the masses of vector (and pseudovector) fields, we write the equation for vector field (5) in momentum space (see [16]):

$$\left[\left(p_\lambda^2+m^2\right)\delta_{\mu\nu}-(1-a)p_\mu p_\nu\right]\varphi_\nu(p)=0, \quad (6)$$

where $p^2 = p_\lambda^2 = \mathbf{p}^2 + p_4^2 = \mathbf{p}^2 - p_0^2$. The matrix

$$M = \left(p^2 + m^2\right) I_4 - (1-a)\,(p.p)\,, \tag{7}$$

where the I_4 is the unit 4×4-matrix and the matrix-dyad $(p.p)$ has the matrix elements $(p.p)_{\mu\nu} = p_\mu p_\nu$, obeys the equation

$$\left(M - m^2 - p^2\right)\left[M - m^2 - ap^2\right] = 0. \tag{8}$$

It follows from Eq. (8) that the matrix M possesses the eigenvalues $\lambda_1 = p^2 + m^2$, $\lambda_2 = ap^2 + m^2$. So, Eq. (6) has nontrivial solutions at $\det M = 0$ or $\lambda_1 = \lambda_2 = 0$. This leads to the mass spectrum of the vector field $\varphi_\nu(x)$:

$$p^2 = -m^2, \qquad\qquad p^2 = -\frac{m^2}{a}. \tag{9}$$

We can draw a conclusion from Eq. (9) that one state of the field $\varphi_\nu(x)$ corresponds to the mass m, and the second state corresponds to the squared mass

$$m_0^2 = \frac{m^2}{a}. \tag{10}$$

The requirement that the masses are real ($m^2 > 0$, $m_0^2 > 0$), we get the condition $a > 0$. To identify the the mass states with the spin states, one finds from Eqs. (1)-(4) the second order equation for the scalar and pseudoscalar fields:

$$\begin{aligned} \partial_\nu^2\varphi(x) - \frac{m^2}{a}\varphi(x) = 0,\\ \partial_\nu^2\widetilde{\varphi}(x) - \frac{m^2}{a}\widetilde{\varphi}(x) = 0. \end{aligned} \tag{11}$$

From Eqs. (10), (11) we conclude that scalar $\varphi(x)$ and pseudoscalar $\widetilde{\varphi}(x)$ fields with spin $s = 0$ possess the mass m_0. Therefore, the vector and pseudovector states with spin $s = 1$ have the mass m. In the case of $a = 1$, states with spin $s = 1$ and $s = 0$ possess the same mass m. For arbitrary variable $a > 0$ the system of equations (1)-(4) describes fields which may have the vector and pseudovector states with the mass m and scalar and pseudoscalar states with the mass m_0. The Dirac-Kähler equation [11] is obtained by setting $m = m_0$ ($a = 1$).

Introducing the variables

$$M(x) = \frac{1}{\sqrt{2}}\left(\varphi(x) - i\widetilde{\varphi}(x)\right), \qquad N(x) = \frac{1}{\sqrt{2}}\left(\varphi(x) + i\widetilde{\varphi}(x)\right),$$

$$M_\mu(x) = \frac{1}{\sqrt{2}}\left(\varphi_\mu(x) - i\widetilde{\varphi}_\mu(x)\right), \qquad M_{\mu\nu}(x) = \frac{1}{\sqrt{2}}\left(\varphi_{\mu\nu}(x) - i\widetilde{\varphi}_{\mu\nu}(x)\right),$$

$$N_\mu(x) = \frac{1}{\sqrt{2}}\left(\varphi_\mu(x) + i\widetilde{\varphi}_\mu(x)\right), \qquad N_{\mu\nu}(x) = \frac{1}{\sqrt{2}}\left(\varphi_{\mu\nu}(x) + i\widetilde{\varphi}_{\mu\nu}(x)\right),$$

and adding and subtracting Eqs. (1)-(4), we obtain equations

$$\begin{gathered}\partial_\nu M_{\mu\nu}(x) - \partial_\mu M(x) + m^2 M_\mu(x) = 0, \qquad a\partial_\mu M_\mu(x) = M(x),\\ M_{\mu\nu}(x) = \partial_\mu M_\nu(x) - \partial_\nu M_\mu(x) - i\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha M_\beta(x),\end{gathered} \tag{12}$$

$$\begin{gathered}\partial_\nu N_{\mu\nu}(x) - \partial_\mu N(x) + m^2 N_\mu(x) = 0, \qquad a\partial_\mu N_\mu(x) = N(x),\\ N_{\mu\nu}(x) = \partial_\mu N_\nu(x) - \partial_\nu N_\mu(x) + i\varepsilon_{\mu\nu\alpha\beta}\partial_\alpha N_\beta(x).\end{gathered} \tag{13}$$

The tensor $M_{\mu\nu}(x)$ is self-dual tensor, $M_{\mu\nu}(x) = -i\widetilde{M}_{\mu\nu}(x)$, and the tensor $N_{\mu\nu}(x)$ is the antiself-dual tensor, $N_{\mu\nu}(x) = i\widetilde{N}_{\mu\nu}(x)$.

The self-dual tensor $M_{\mu\nu}$ possesses 3 independent components and realizes the $(1,0)$-representation of the Lorentz group. The antiself-dual tensor $N_{\mu\nu}$ also has 3 independent components and is transformed under the $(0,1)$-representation of the Lorentz group. Eqs. (12) and (13)) are 8-component equations which describes eight independent variables ($M(x)$, $M_\nu(x)$, $M_{ab}(x)$), ($N(x)$, $N_\nu(x)$, $N_{ab}(x)$) are not invariant under the parity transformations separately. At P-transformations, Eqs. (12) are transferred into Eqs. (13) and vice versa. It is possible to have Lagrangian formulation only for the whole system of equations (12), (13) or (1)-(4). P-noninvariant equations (12) (or (13)) have no the Lagrangian formulation. Eqs. (1)-(4) realize the $(0,0)\oplus(1/2,1/2)\oplus(1,0)\oplus(0,1)\oplus(1/2,1/2)\oplus(0,0)$-representation of the Lorentz group, and are equivalent to equations (12), (13)).

3 Matrix form of 16-component equations

Now we obtain the matrix form of equations (1)-(4). To get the relativistic wave equation in the matrix form, we introduce new variables

$$\psi_0(x) = -\varphi(x), \quad \psi_\mu(x) = m\varphi_\mu(x), \quad \psi_{[\mu\nu]}(x) = \varphi_{\mu\nu}(x),$$

$$\widetilde{\psi}_\mu(x) = im\widetilde{\varphi}_\mu(x), \quad \widetilde{\psi}_0(x) = -i\widetilde{\varphi}(x), \quad e_{\mu\nu\alpha\beta} = i\varepsilon_{\mu\nu\alpha\beta} \quad (e_{1234} = 1).$$

Then Eqs. (1)-(4) with the help of Eq. (10) are rewritten as

$$\partial_\mu\widetilde{\psi}_\mu(x) + \frac{m_0^2}{m}\widetilde{\psi}_0(x) = 0,$$

$$\partial_\nu\psi_{[\mu\nu]}(x) + \partial_\mu\psi_0(x) + m\psi_\mu(x) = 0, \tag{14}$$

$$\partial_\nu \psi_\mu(x) - \partial_\mu \psi_\nu(x) - e_{\mu\nu\alpha\beta}\partial_\alpha \widetilde{\psi}_\beta(x) + m\psi_{[\mu\nu]}(x) = 0,$$

$$\partial_\mu \psi_\mu(x) + \frac{m_0^2}{m}\psi_0(x) = 0.$$

With $m_0 = m$, we arrive at the Dirac-Kähler equations. It is convenient to introduce the 16-component wave function

$$\Psi(x) = \{\psi_A(x)\}, \qquad A = 0, \mu, [\mu\nu], \widetilde{\mu}, \widetilde{0}, \tag{15}$$

where $\psi_{\widetilde{\mu}} \equiv \widetilde{\psi}_\mu$, $\psi_{\widetilde{0}} \equiv \widetilde{\psi}_0$. Let us introduce the the elements of the entire algebra $\varepsilon^{A,B}$ [20] with dimension 16×16. The product of these matrices and their matrix elements are given by

$$\varepsilon^{A,B}\varepsilon^{C,D} = \varepsilon^{A,D}\delta_{BC}, \qquad \left(\varepsilon^{A,B}\right)_{CD} = \delta_{AC}\delta_{BD}, \tag{16}$$

where $A, B, C, D = 1, 2, ..., 16$. With the help of the matrices $\varepsilon^{A,B}$ and Eq. (15), equations (14) become

$$\left\{\partial_\nu \left[\varepsilon^{\mu,[\mu\nu]} + \varepsilon^{[\mu\nu],\mu} + \varepsilon^{\nu,0} + \varepsilon^{0,\nu} + \varepsilon^{\widetilde{\nu},\widetilde{0}} + \varepsilon^{\widetilde{0},\widetilde{\nu}} + \right.\right.$$

$$\left. + \frac{1}{2}e_{\mu\nu\rho\omega}\left(\varepsilon^{\widetilde{\mu},[\rho\omega]} + \varepsilon^{[\rho\omega],\widetilde{\mu}}\right)\right]_{AB} + \tag{17}$$

$$\left. + \left[m\left(\varepsilon^{\mu,\mu} + \varepsilon^{\widetilde{\mu},\widetilde{\mu}} + \frac{1}{2}\varepsilon^{[\mu\nu],[\mu\nu]}\right) + \frac{m_0^2}{m}\left(\varepsilon^{0,0} + \varepsilon^{\widetilde{0},\widetilde{0}}\right)\right]_{AB}\right\}\Psi_B(x) = 0.$$

We note that the matrices

$$P_1 = \varepsilon^{\mu,\mu} + \varepsilon^{\widetilde{\mu}\widetilde{\mu}} + \frac{1}{2}\varepsilon^{[\mu\nu],[\mu\nu]},, \qquad P_0 = \varepsilon^{0,0} + \varepsilon^{\widetilde{0},\widetilde{0}} \tag{18}$$

are the projection matrices which obey the equations

$$P_0 P_1 = P_1 P_0 = 0, \qquad P_1 + P_0 = I_{16},$$

where the I_{16} is the unit $16 \times 16-$matrix. Introducing the matrix

$$\Gamma_\nu = \varepsilon^{\mu,[\mu\nu]} + \varepsilon^{[\mu\nu],\mu} + \varepsilon^{\nu,0} + \varepsilon^{o,\nu} + \varepsilon^{\widetilde{\nu},\widetilde{0}} + \varepsilon^{\widetilde{0},\widetilde{\nu}} + \frac{1}{2}e_{\mu\nu\rho\omega}\left(\varepsilon^{\widetilde{\mu},[\rho\omega]} + \varepsilon^{[\rho\omega],\widetilde{\mu}}\right), \tag{19}$$

Eq. (16) becomes

$$\left(\Gamma_\nu\partial_\nu + mP_1 + \frac{m_0^2}{m}P_0\right)\Psi(x) = 0. \tag{20}$$

Eq. (20) is the matrix relativistic wave equation which describes the system of the vector (pseudovector) and scalar (pseudoscalar) fields with the masses m and m_0, respectively. The $16 \times 16-$matrix Γ_ν can be cast as follows:

$$\Gamma_\nu = \beta_\nu^{(+)} + \beta_\nu^{(-)}, \qquad \beta_\nu^{(+)} = \beta_\nu^{(1)} + \beta_\nu^{(\widetilde{0})}, \qquad \beta_\nu^{(-)} = \beta_\nu^{(\widetilde{1})} + \beta_\nu^{(0)},$$

$$\beta_\nu^{(1)} = \varepsilon^{\mu,[\mu\nu]} + \varepsilon^{[\mu\nu],\mu}, \qquad \beta_\nu^{(\widetilde{1})} = \frac{1}{2} e_{\mu\nu\rho\omega} \left(\varepsilon^{\widetilde{\mu},[\rho\omega]} + \varepsilon^{[\rho\omega],\widetilde{\mu}} \right), \tag{21}$$

$$\beta_\nu^{(\widetilde{0})} = \varepsilon^{\widetilde{\nu},\widetilde{0}} + \varepsilon^{\widetilde{0},\widetilde{\nu}}, \qquad \beta_\nu^{(0)} = \varepsilon^{\nu,0} + \varepsilon^{0,\nu}.$$

The 10-dimensional matrices $\beta_\nu^{(1)}$, $\beta_\nu^{(\widetilde{1})}$ and 5-dimensional matrices $\beta_\nu^{(0)}$, $\beta_\nu^{(\widetilde{0})}$ obey the algebra [3], [4], [5]:

$$\beta_\mu \beta_\nu \beta_\alpha + \beta_\alpha \beta_\nu \beta_\mu = \delta_{\mu\nu} \beta_\alpha + \delta_{\alpha\nu} \beta_\mu. \tag{22}$$

The 16-dimensional matrices $\beta_\nu^{(+)}$, $\beta_\nu^{(-)}$ realize the reducible representations of the Petiau-Duffin-Kemmer algebra (22) [21]. The 16×16-matrix Γ_ν satisfies the Dirac algebra:

$$\Gamma_\nu \Gamma_\mu + \Gamma_\mu \Gamma_\nu = 2\delta_{\mu\nu}. \tag{23}$$

In the case when the masses of all states are the same, $m_0 = m$, Eq. (23) is transformed into the matrix form of the the Dirac-Kähler equations [12]:

$$\left(\Gamma_\nu \partial_\nu + m \right) \Psi(x) = 0. \tag{24}$$

4 Matrix form of 8-component equations

To get the matrix form of 8-component equations (12), we introduce four-component wave functions:

$$\xi(x) = -im \begin{pmatrix} M_a(x) \\ M_4(x) \end{pmatrix}, \qquad \chi(x) = \begin{pmatrix} \widetilde{M}_a(x) \\ M(x) \end{pmatrix}, \tag{25}$$

where $\widetilde{M}_a(x) = (1/2)\epsilon_{amn} M_{mn}(x)$. With the help of Eqs. (25), the system of P-noninvariant wave equations (12), may be cast in the form

$$\alpha_\mu \partial_\mu \xi(x) = m\chi(x), \tag{26}$$

$$\overline{\alpha}_\mu \partial_\mu \chi(x) = m\xi(x),$$

where we introduce matrices:

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ ia & 0 & 0 & 0 \end{pmatrix}, \qquad \alpha_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & ia & 0 & 0 \end{pmatrix},$$
$$\alpha_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & ia & 0 \end{pmatrix}, \qquad \alpha_4 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & ia \end{pmatrix}, \tag{27}$$

$$\overline{\alpha}_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \qquad \overline{\alpha}_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix},$$
$$\overline{\alpha}_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, \qquad \overline{\alpha}_4 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \tag{28}$$

It is easy to verify that four-dimensional matrices $\overline{\alpha}_a$ ($a = 1, 2, 3$) obeys the Pauli commutation relations:

$$\{\overline{\alpha}_i, \overline{\alpha}_k\} = 2\delta_{ik}, \qquad [\overline{\alpha}_i, \overline{\alpha}_k] = 2i\epsilon_{ikl}\overline{\alpha}_l, \tag{29}$$

where $\epsilon_{123} = 1$. Note, that matrices $\overline{\alpha}_a$ may be obtained from matrices α_a by setting $a = 1$, and $\overline{\alpha}_4 = -iI_4$. Equations (26) can be also written in the form of one 8-component equation:

$$\beta_\mu \partial_\mu \varphi(x) + m\varphi(x) = 0, \tag{30}$$

where the matrices of the equation β_μ and 8-component wave function $\varphi(x)$ are given by

$$\varphi(x) = \begin{pmatrix} \chi(x) \\ \xi(x) \end{pmatrix}, \qquad \beta_\mu = -\begin{pmatrix} 0 & \alpha_\mu \\ \overline{\alpha}_\mu & 0 \end{pmatrix}. \tag{31}$$

In the case $a = 1$ when the masses scalar and vector states equal each other, $m = m_0$, matrices β_μ (at $a = 1$) obey the Dirac commutation relations (23) (see [13], [22]).

In the same manner, to obtain the matrix form of 8-component equations (13), we introduce four-component wave functions:

$$\xi'(x) = -im \begin{pmatrix} N_a(x) \\ N_4(x) \end{pmatrix}, \qquad \chi'(x) = \begin{pmatrix} \widetilde{N}_a(x) \\ N(x) \end{pmatrix}, \tag{32}$$

where $\widetilde{N}_a(x) = (1/2)\epsilon_{amn}N_{mn}(x)$. Then Eqs. (13) can be represented by

$$\begin{gathered} \alpha'_\mu \partial_\mu \xi'(x) = m\chi'(x), \\ \overline{\alpha}'_\mu \partial_\mu \chi'(x) = m\xi'(x), \end{gathered} \tag{33}$$

where

$$\begin{gathered}
\alpha'_1 = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ ia & 0 & 0 & 0 \end{pmatrix}, \qquad \alpha'_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & i \\ -i & 0 & 0 & 0 \\ 0 & ia & 0 & 0 \end{pmatrix}, \\
\alpha'_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & ia & 0 \end{pmatrix}, \qquad \alpha'_4 = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & ia \end{pmatrix}, \\
\overline{\alpha}'_1 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \overline{\alpha}'_2 = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, \\
\overline{\alpha}'_3 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \end{pmatrix}, \qquad \overline{\alpha}'_4 = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}.
\end{gathered} \tag{34}$$

Two equations (33) are rewritten by one 8-component matrix equation

$$\beta'_\mu \partial_\mu \varphi'(x) + m\varphi'(x) = 0, \tag{35}$$

where matrices β'_μ and columns $\varphi'(x)$ are

$$\varphi'(x) = \begin{pmatrix} \chi'(x) \\ \xi'(x) \end{pmatrix}, \qquad \beta'_\mu = -\begin{pmatrix} 0 & \alpha'_\mu \\ \overline{\alpha}'_\mu & 0 \end{pmatrix}. \tag{36}$$

The matrices β'_μ also at $a = 1$ obey the Dirac algebra (23) (see [13], [22]. One may combine Eqs. (30), (35) in the 16-component relativistic wave equation, as follows

$$\left(\Pi_\mu \partial_\mu + m\right)\Psi(x) = 0, \tag{37}$$

where

$$\Psi(x) = \begin{pmatrix} \varphi(x) \\ \varphi'(x) \end{pmatrix}, \qquad \Pi_\mu = \begin{pmatrix} \beta_\mu & 0 \\ 0 & \beta'_\mu \end{pmatrix}. \tag{38}$$

The 16×16 - matrices Π_μ obey at $a = 1$ the Dirac algebra (23). The whole system of equations (12), (13) is equivalent to Eq. (37) or Eq. (20). Equation (12) (and (13)) as well as Eq. (30) (and (35)) are parity noninvariant separately and at the same time the system of the two equations (12), (13) (or Eq. (37)) are P-invariant.

We note that at the particular case $m = m_0$ ($a = 1$), Eq. (30) and Eq. (35) are invariant under the $GL(2, c)$ symmetry group [22], [13]. In this case Eq. (20) and Eq. (37) are equivalent to the Dirac-Kähler equation and possess the $GL(4, c)$ internal symmetry group, but the Lagrangian is invariant under the $SO(4, 2)$ group [12], [13]. The condition $m \neq m_0$ ($a \neq 1$) eliminates the degeneracy and destroys the symmetry.

5 The Lorentz covariance and symmetry of 16-component wave equation

At the Lorentz transformations of the coordinates:

$$x'_\mu = L_{\mu\nu} x'_\nu, \tag{39}$$

the matrix $L = \{L_{\mu\nu}\}$ obeys the equation

$$L_{\mu\alpha} L_{\nu\alpha} = \delta_{\mu\nu}. \tag{40}$$

The wave function (15) under the Lorentz coordinates transformations (39) is converted into

$$\Psi'(x') = T\Psi(x), \tag{41}$$

where the T is the 16-dimensional reducible tensor representation of the Lorentz group. Taking into account that at the Lorentz transformations (39) the derivatives ∂_μ transforms as $\partial'_\mu = L_{\mu\nu}\partial_\nu$, the first order wave equation (20) becomes

$$\begin{gathered}\left(\Gamma_\nu \partial'_\nu + mP_1 + \frac{m_0^2}{m} P_0\right) \Psi'(x') \\ = \left(\Gamma_\mu L_{\mu\nu} \partial_\mu + mP_1 + \frac{m_0^2}{m} P_0\right) T\Psi(x) = 0.\end{gathered} \tag{42}$$

Eq. (20) is form-invariant if the equalities

$$\Gamma_\mu T L_{\mu\nu} = T\Gamma_\nu, \qquad P_1 T = TP_1, \qquad P_0 T = TP_0 \tag{43}$$

are satisfied. Let us use the infinitesimal Lorentz transformations (39) with the matrix

$$L_{\mu\nu} = \delta_{\mu\nu} + \varepsilon_{\mu\nu}, \qquad \varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \tag{44}$$

and the infinitesimal transformations (41) with the matrix

$$T = 1 + \frac{1}{2}\varepsilon_{\mu\nu}J_{\mu\nu}, \tag{45}$$

where $1 = I_{16}$, $J_{\mu\nu}$ are the generators of the Lorentz group in $16-$dimensional space, $\varepsilon_{\mu\nu}$ are six parameters defining rotations and boosts. Replacing equations (44), (45) into Eqs. (43) and taking into consideration the smallness of parameters $\varepsilon_{\mu\nu}$, we arrive at the equations

$$\Gamma_\mu J_{\alpha\nu} - J_{\alpha\nu}\Gamma_\mu = \delta_{\alpha\mu}\Gamma_\nu - \delta_{\nu\mu}\Gamma_\alpha, \qquad P_1 J_{\mu\nu} = J_{\mu\nu}P_1, \qquad P_0 J_{\mu\nu} = J_{\mu\nu}P_0 \tag{46}$$

It is easy to verify with the help of formulas of Appendix A, that generators

$$J_{\mu\nu} = \frac{1}{4}\left(\Gamma_{[\mu}\Gamma_{\nu]} + \overline{\Gamma}_{[\mu}\overline{\Gamma}_{\nu]}\right), \tag{47}$$

satisfy Eqs. (46). The matrices $\overline{\Gamma}_\nu$ are given by [21]

$$\overline{\Gamma}_\nu = \beta_\nu^{(+)} - \beta_\nu^{(-)}, \tag{48}$$

and also obey the Dirac algebra (23) and commute with the matrices Γ_μ:

$$\Gamma_\mu\overline{\Gamma}_\nu = \overline{\Gamma}_\nu\Gamma_\mu.$$

Generators of the Lorentz representation (47) are also valid in the case of $m = m_0$ [12], [13]. The 16-dimensional matrix of the finite transformations (41) is given by

$$T = \exp\left(\frac{1}{2}\varepsilon_{\mu\nu}J_{\mu\nu}\right). \tag{49}$$

Let us consider a relativistically invariant bilinear form

$$\overline{\Psi}(x)\Psi(x) = \Psi^+(x)\eta\Psi(x), \tag{50}$$

where Ψ^+ is the Hermitian-conjugate wave function and η is the Hermitianizing matrix in 16-dimensional space. As for the case of Dirac-Kähler equation in the matrix form [21], [12], [13], it is given by:

$$\eta = \Gamma_4\overline{\Gamma}_4. \tag{51}$$

The matrix η obeys the necessary (see, for example [20], [23]) equations

$$\eta\Gamma_i = -\Gamma_i\eta \qquad (i = 1, 2, 3), \qquad \eta\Gamma_4 = \Gamma_4\eta.$$

Now we consider the internal symmetry group of Eq. (20). With the help of Eqs. (16), (18), (21), (48) it is easy to show that the matrix $\overline{\Gamma}_5 = \overline{\Gamma}_1\overline{\Gamma}_2\overline{\Gamma}_3\overline{\Gamma}_4$ commutes with the operator of Eq. (20):

$$\overline{\Gamma}_5\left(\Gamma_\mu\partial_\mu + mP_1 + \frac{m_0^2}{m}P_0\right) = \left(\Gamma_\mu\partial_\mu + mP_1 + \frac{m_0^2}{m}P_0\right)\overline{\Gamma}_5. \tag{52}$$

This means that the matrix $\overline{\Gamma}_5$ can be considered as the generator of the symmetry group $GL(1,c)$ of Eq. (20):

$$\Psi'(x) = \exp\left(\omega\overline{\Gamma}_5\right)\Psi(x), \tag{53}$$

where ω is the complex group parameter. The Lagrangian corresponding to the first order relativistic wave equation (20) is given by

$$\mathcal{L} = -\frac{1}{2}\left[\overline{\Psi}(x)\left(\Gamma_\mu\partial_\mu + mP_1 + \frac{m_0^2}{m}P_0\right)\Psi(x)\right]. \tag{54}$$

The Lagrangian (54) is invariant under the transformations (53) at the restriction: $\omega^* = \omega$, i.e ω is a real parameter. This constrain leaves $GL(1,R)$ subgroup of the Lagrangian symmetry. The transformations of the Lorentz group (49) commute with transformations (53) of the internal symmetry group. As a result, the parameter of this group is a scalar.

According to Noether's theorem, the invariance of the Lagrangian (54) under the transformation (53) provides the conservation tensor (see also [12], [13] for the case $m = m_0$):

$$K_\mu = \overline{\Psi}(x)\Gamma_\mu\overline{\Gamma}_5\Psi(x), \qquad \partial_\mu K_\mu = 0. \tag{55}$$

The subgroup $U(1)$ of gauge transformations: $\Psi'(x) = \exp(i\alpha)\Psi(x)$ ($\alpha^* = \alpha$) leads to the conservation of four-current $J_\mu = i\overline{\Psi}(x)\Gamma_\mu\Psi(x)$: $\partial_\mu J_\mu = 0$. In the case of the degeneracy $m = m_0$, the symmetry group of the Lagrangian (54) is the $SO(4,2)$ group [12], [13].

Using the formulas of Appendix A, we obtain the matrix $\overline{\Gamma}_5 = \overline{\Gamma}_1\overline{\Gamma}_2\overline{\Gamma}_3\overline{\Gamma}_4$:

$$\overline{\Gamma}_5 = -\varepsilon^{\widetilde{0},0} - \varepsilon^{0,\widetilde{0}} - \varepsilon^{\widetilde{\mu},\mu} - \varepsilon^{\mu,\widetilde{\mu}} - \frac{1}{2}e_{\mu\nu\rho\omega}\varepsilon^{[\mu\nu],[\rho\omega]}. \tag{56}$$

With the aid of Eqs. (15), (56) the infinitesimal transformation (53): $\delta\Psi(x) = \overline{\Gamma}_5\Psi(x)\delta\omega$ leads to the transformations of the fields (14):

$$\delta\psi_0(x) = -\widetilde{\psi}_0(x)\delta\omega, \qquad \delta\widetilde{\psi}_0(x) = -\psi_0(x)\delta\omega, \qquad \delta\psi_\mu(x) = -\widetilde{\psi}_\mu(x)\delta\omega, \tag{57}$$

$$\delta\widetilde{\psi}_\mu(x) = -\psi_\mu(x)\delta\omega, \qquad \delta\psi_{[\mu\nu]}(x) = -\frac{1}{2}e_{\mu\nu\varrho\sigma}\psi_{[\varrho\sigma]}(x)\delta\omega.$$

The finite transformation (53) can be represented as

$$\Psi'(x) = \left(\cosh\omega + \overline{\Gamma}_5\sinh\omega\right)\Psi(x),$$

which is an analog of dual transformations in electrodynamics but with the imaginary value of the angle ω [13].

6 Quaternion form of field equations

Now we obtain the quaternion form of equations (1)-(3). Quaternions may be considered as a generalization (doubling) of the complex numbers (see, for instance, [24]). Quaternions are useful tools for analyzing the symmetry of fields.

The quaternion algebra consists of four basis elements $e_\mu = (e_k, e_4)$ (see, for example [24]) with the products:

$$e_4^2 = 1, \quad e_1^2 = e_2^2 = e_3^2 = -1, \quad e_1e_2 = -e_2e_1 = e_3, \quad e_2e_3 = -e_3e_2 = e_1,$$
$$e_3e_1 = -e_1e_3 = e_2, \quad e_4e_m = e_me_4 = e_m \quad (m = 1, 2, 3), \tag{58}$$

where e_4 is the unit element. The biquaternion (or complex quaternion) q is defined as

$$q = q_\mu e_\mu = q_m e_m + q_4 e_4, \tag{59}$$

where q_μ are complex numbers. With the help of Eqs. (58), the product of two quaternions, q, q', is given by:

$$qq' = \left(q_4q_4' - q_mq_m'\right)e_4 + \left(q_4'q_m + q_4q_m' + \epsilon_{mnk}q_nq_k'\right)e_m. \tag{60}$$

It should be noted that the combined law for three quaternions: $(q_1q_2)\,q_3 = q_1\,(q_2q_3)$ holds. The quaternion conjugation (hyperconjugation) is defined as

$$\overline{q} = q_4e_4 - q_me_m \equiv q_4 - \mathbf{q}, \tag{61}$$

with the equalities $\overline{q_1 + q_2} = \overline{q}_1 + \overline{q}_2$, $\overline{q_1q_2} = \overline{q}_2\overline{q}_1$.

To get the quaternion form of Eqs. (1), we rewrite them as follows:

$$\partial_0E_m(x) + \partial_m\varphi(x) - \epsilon_{mnk}\partial_nH_k(x) - m^2\varphi_m(x) = 0,$$
$$\partial_mE_m(x) + \partial_0\varphi(x) + m^2\varphi(x) = 0, \tag{62}$$

$$\partial_0 H_m(x) + \partial_m \widetilde{\varphi}(x) + \epsilon_{mnk}\partial_n E_k(x) - m^2 \widetilde{\varphi}_m(x) = 0,$$

$$\partial_m H_m(x) + \partial_0 \widetilde{\varphi}(x) + m^2 \widetilde{\varphi}(x) = 0,$$

where $\varphi_\mu(x) = (\varphi_m(x), i\varphi_0(x))$, $H_m(x) = \frac{1}{2}\epsilon_{mnk}\varphi_{nk}(x)$, $E_m(x) = i\varphi_{m4}(x)$, $\partial_0 = \partial_t$ (t is the time). Introducing the biquaternions

$$\begin{gathered} \nabla = e_\mu \partial_\mu, \qquad F = F_\mu e_\mu, \qquad G = G_\mu e_\mu, \\ F_m = H_m(x) - iE_m(x), \quad F_4 = -\varphi(x) - i\widetilde{\varphi}(x), \quad G_\mu = \varphi_\mu(x) + i\widetilde{\varphi}_\mu(x), \end{gathered} \tag{63}$$

with the help of Eqs. (60), we represent Eqs. (62) as follows

$$\nabla F + m^2 G = 0. \tag{64}$$

So, quaternion equation (64) is equivalent to Eqs. (1). To represent Eqs. (2), (3) in quaternion form, one may cast them into

$$\begin{gathered} \varphi(x) = a\left(\partial_0 \varphi_0(x) + \partial_m \varphi_m(x)\right), \quad \widetilde{\varphi}(x) = a\left(\partial_0 \widetilde{\varphi}_0(x) + \partial_m \widetilde{\varphi}_m(x)\right), \\ H_m(x) = -\partial_m \widetilde{\varphi}_0(x) - \partial_0 \widetilde{\varphi}_m(x) + \epsilon_{mnk}\partial_n \varphi_k(x), \\ E_m(x) = -\partial_m \varphi_0(x) - \partial_0 \varphi_m(x) - \epsilon_{mnk}\partial_n \widetilde{\varphi}_k(x). \end{gathered} \tag{65}$$

It easy to show with the help of Eq. (60) that equations

$$\overline{\nabla} G = \partial_\mu G_\mu e_4 + \left(\partial_4 G_m - \partial_m G_4 - \epsilon_{mnk}\partial_n G_k\right) e_m, \quad \overline{\nabla} = \overline{e}_\mu \partial_\mu,$$

$$\overline{G}\overleftarrow{\nabla} = \partial_\mu G_\mu e_4 - \left(\partial_4 G_m - \partial_m G_4 - \epsilon_{mnk}\partial_n G_k\right) e_m, \quad \overline{e}_\mu = (e_4, -e_m)$$

are valid. The arrow above the ∇ shows the action of the differential operator. Then Eqs. (65) take the form

$$F = \frac{1-a}{2}\overline{G}\overleftarrow{\nabla} - \frac{1+a}{2}\overline{\nabla} G, \tag{66}$$

At the case $a = 1$ ($m = m_0$), one arrives at the quaternion form (Eqs. (64), (66)) of Dirac-Kähler equation [12], [13].

Under the the Lorentz group transformations, the quaternions G, F, ∇ and $\overline{\nabla}$ are converted into (see [12], [13])

$$\begin{gathered} G^L = \overline{L}^* G L, \qquad F^L = \overline{L} F L, \\ \nabla^L = \overline{L}^* \nabla L, \qquad \overline{\nabla}^L = \overline{L}\,\overline{\nabla} L^*, \end{gathered} \tag{67}$$

where quaternions L obey the constrain $L\overline{L} = \overline{L}L = 1$. The field equations (64), (66) are invariant under the Lorentz transformations (67).

From Eqs. (64), (66) one can obtain the quaternion form of some equations for the vector fields (the Proca equations, Maxwell's equations and so on).

7 Spin projection operators and density matrix

To obtain all independent solutions of the matrix equation (20) in the form of matrix-dyads, one needs to construct projection operators [23] extracting "pure" states. With the help of generators of the Lorentz group (47), we find the spin projection operator (see [12], [13]):

$$\sigma_p = -\frac{i}{2\mid \mathbf{p}\mid}\epsilon_{abc}p_a J_{bc} = -\frac{i}{4\mid \mathbf{p}\mid}\epsilon_{abc}p_a\left(\Gamma_b\Gamma_c + \overline{\Gamma}_b\overline{\Gamma}_c\right) \qquad (68)$$

This operator obeys the "minimal" equation as follows:

$$\sigma_p\left(\sigma_p - 1\right)\left(\sigma_p + 1\right) = 0. \qquad (69)$$

According to the general method [23], one obtains the projection operators extracting spin projections ± 1 and 0:

$$\widehat{S}_{(\pm 1)} = \frac{1}{2}\sigma_p\left(\sigma_p \pm 1\right), \qquad \widehat{S}_{(0)} = 1 - \sigma_p^2. \qquad (70)$$

These operators satisfy the ordinary relationships: $\widehat{S}^2_{(\pm 1)} = \widehat{S}_{(\pm 1)}$, $\widehat{S}_{(\pm 1)}\widehat{S}_{(0)} = 0$, $\widehat{S}^2_{(0)} = \widehat{S}_{(0)}$. We consider fields with different spins, one and zero. To separate these states, one has to construct the squared spin operator. Such an operator is given by the squared Pauli-Lubanski vector σ^2:

$$\sigma^2 = \left(\frac{1}{2m}e_{\mu\nu\alpha\beta}p_\nu J_{\alpha\beta}\right)^2 = \frac{1}{m^2}\left(\frac{1}{2}J^2_{\mu\nu}p^2 - J_{\mu\sigma}J_{\nu\sigma}p_\mu p_\nu\right). \qquad (71)$$

One can verify with the aid of Eqs. (47) (see Appendix) that this operator satisfies the matrix equation

$$\sigma^2\left(\sigma^2 - 2\right) = 0. \qquad (72)$$

Eq. (72) shows that eigenvalues of the squared spin operator σ^2 are $s(s+1) = 0$ and $s(s+1) = 2$; that corresponds to spins $s = 0$ and $s = 1$. The projection operators separating these states are given by

$$S^2_{(0)} = 1 - \frac{\sigma^2}{2}, \qquad S^2_{(1)} = \frac{\sigma^2}{2}. \qquad (73)$$

Operators (73) possess the properties

$$S^2_{(0)}S^2_{(1)} = 0, \quad \left(S^2_{(0)}\right)^2 = S^2_{(0)}, \quad \left(S^2_{(1)}\right)^2 = S^2_{(1)}, \quad S^2_{(0)} + S^2_{(1)} = 1.$$

The projection operators $S^2_{(0)}$, $S^2_{(1)}$ acting on the wave function extract pure states with spin 0 and 1, respectively.

The system of field equations under consideration includes two fields with spin one (vector ψ_μ, and pseudovector $\widetilde{\psi}_\mu$, fields), and two fields with spin zero (scalar ψ_0, pseudoscalar $\widetilde{\psi}_0$, fields). So, there is a doubling of the spin states of fields, i.e. degeneracy. It is necessary to introduce additional projection operators to separate these states and to have pure spin states. For this purpose, one may explore the internal symmetry of fields which mixes scalar and pseudoscalar fields as well as vector and pseudovector fields, and is given by Eq. (53). We introduce the projection operator as follows:

$$\Lambda_\lambda = \frac{1}{2}\left(1 + \lambda\overline{\Gamma}_5\right), \tag{74}$$

where $\lambda = \pm 1$. Acting this operator on the wave function (15), one obtains two subsystems of equations (12) and (13) corresponding to additional quantum number $\lambda = \pm 1$. The operator (74) obeys the relation $\Lambda^2_\lambda = \Lambda_\lambda$ required. It is not difficult to check that all introduced projection operators (70), (73) and (74) commute with each other and with the operator of the wave equation (20). So, the product of these operators

$$\Lambda_\lambda \widehat{S}_{(\pm 1),(0)} S^2_{(1),(0)}, \tag{75}$$

extracts the states with definite spin, spin projection and additional quantum number $\lambda = \pm 1$ which is connected with doubling of spin states. It should be noted that these states are not states with the definite parity because the operator Λ_λ mixes states with the opposite parity. The parity operator is given by $P = \eta_P \Gamma_4 \overline{\Gamma}_4$, and it does not commute with the operator Λ_λ. To have the states with definite energy, we consider Eq. (20) in the momentum space:

$$\left(\varepsilon i\widehat{p} + mP_1 + \frac{m_0^2}{m}P_0\right)\Psi(p) = 0, \tag{76}$$

where $\widehat{p} = p_\mu \Gamma_\mu$, $\varepsilon = 1$ corresponds to positive energy, and $\varepsilon = -1$ – to negative energy. It easy to verify, using Eqs. (22), (23), that the projection operator

$$\begin{aligned} M^{(1)}_\varepsilon &= \frac{i\left(p^{(1)} + p^{(\widetilde{1})}\right)\left[i\left(p^{(1)} + p^{(\widetilde{1})}\right) - \varepsilon m\right]}{2m^2} \\ &= \frac{ip^{(1)}\left(ip^{(1)} - \varepsilon m\right)}{2m^2} + \frac{ip^{(\widetilde{1})}\left(ip^{(\widetilde{1})} - \varepsilon m\right)}{2m^2}, \end{aligned} \tag{77}$$

where $p^{(1)} = p_\nu \beta^{(1)}_\nu$, $p^{(\widetilde{1})} = p_\nu \beta^{(\widetilde{1})}_\nu$, obeys the equality $(M^{(1)}_\varepsilon)^2 = M^{(1)}_\varepsilon$, is a solution of Eq. (76), and corresponds to states with spin one (see [12], [13]).

Using Eqs. (21), (56), one obtains the equalities

$$\beta_\nu^{(1)}\overline{\Gamma}_5 = \overline{\Gamma}_5\beta_\nu^{(\widetilde{1})}, \quad \beta_\nu^{(\widetilde{1})}\overline{\Gamma}_5 = \overline{\Gamma}_5\beta_\nu^{(1)}. \tag{78}$$

It is easy to see with the help of Eqs. (78) that operator (77) commutes with the operators (74). Using the properties of the entire algebra (16), and Eqs. (21), (70), (73), one may verify the relations [12], [13]:

$$\begin{gathered}
\widehat{S}_{(\pm 1)}p^{(0)} = \widehat{S}_{(\pm 1)}p^{(\widetilde{0})} = 0, \quad S^2_{(1)}p^{(0)} = S^2_{(1)}p^{(\widetilde{0})} = 0, \quad S^2_{(0)}p^{(1)} = S^2_{(0)}p^{(\widetilde{1})} = 0, \\
p^{(1)}p^{(0)} = p^{(0)}p^{(1)} = p^{(\widetilde{1})}p^{(\widetilde{0})} = p^{(\widetilde{0})}p^{(\widetilde{1})} = 0, \\
p^{(1)}p^{(\widetilde{1})} = p^{(\widetilde{1})}p^{(1)} = p^{(0)}p^{(\widetilde{0})} = p^{(\widetilde{0})}p^{(0)} = 0, \\
\widehat{S}_{(0)}p^{(0)} = p^{(0)}, \quad \widehat{S}_{(0)}p^{(\widetilde{0})} = p^{(\widetilde{0})}, \quad S^2_{(1)}p^{(1)} = p^{(1)}, \quad S^2_{(1)}p^{(\widetilde{1})} = p^{(\widetilde{1})}.
\end{gathered} \tag{79}$$

It is checked with the help of Eqs. (79) that the operators (77) commute with the operators (75). As a result, from Eqs. (75), (77), we obtain projection operators, in the form of matrix-dyads, extracting pure states with spin one, spin projection ± 1, 0, definite energy, and quantum number $\lambda = \pm 1$:

$$\begin{gathered}
\Delta^{(1)} = M^{(1)}_\varepsilon S^2_{(1)}\widehat{S}_{(\pm 1)}\Lambda_\lambda = \Psi^{(1)}\cdot\overline{\Psi}^{(1)}, \\
\Delta^{(1)}_0 = M^{(1)}_\varepsilon S^2_{(1)}\widehat{S}_{(0)}\Lambda_\lambda = \Psi^{(1)}_0\cdot\overline{\Psi}^{(1)}_0,
\end{gathered} \tag{80}$$

The matrix-dyads $\Delta^{(1)}$, $\Delta^{(1)}_0$, correspond to the states with spin one, spin projections ± 1 and 0, correspondingly. The construction of matrix-dyads for the states with spin zero requires finding other solutions of Eq. (76) corresponding to spin zero, and here it is not considered. The dyad representation (80) can be applied to quantum electrodynamic calculations of possesses with the presence of fields under consideration. The method of computing the traces of 16-dimensional matrix products was developed in [12], [13].

8 Canonical quantization of fields

To apply the canonical quantization [25], we consider the fields $\Psi_A(x)$ in the Lagrangian (54) as "coordinates" and the variables $\partial_0\Psi_A(x)$ as "velocities". The momenta are defined as

$$\pi_A(x) = \frac{\partial\mathcal{L}(x)}{\partial\left(\partial_0\Psi_A(x)\right)} = i\left(\overline{\Psi}(x)\Gamma_4\right)_A = i\left(\Psi^+(x)\overline{\Gamma}_4\right)_A. \tag{81}$$

It should be noted that there are no constraints here. With the help of the Poisson brackets between "coordinates" $\Psi_A(x)$ and momenta $\pi_A(x)$, from Eq. (81), one obtains

$$\{\Psi_A(\mathbf{x},t), i\left(\Psi^+(\mathbf{x}',t)\overline{\Gamma}_4\right)_B\} = \delta_{AB}\delta\left(\mathbf{x}-\mathbf{x}'\right). \tag{82}$$

Using the equality $\overline{\Gamma}_4^2 = 1$, and the transition to the quantum theory by the substitution

$$\{\Psi,\pi\} = -i\left[\Psi,\pi\right],$$

where $[\Psi,\pi] = \Psi\pi - \pi\Psi$, we arrive at the quantum commutators

$$\left[\Psi_A(x), \Psi_B^+(x')\right]_{t=t'} = \left(\overline{\Gamma}_4\right)_{AB}\delta(\mathbf{x}-\mathbf{x}'), \tag{83}$$

where A, $B = 1$,2,...,16. The quantization is performed here in the same manner as in the case $m = m_0$ (a=1) [12], [13]. Using Eqs. (15), (19), one finds the commutation relations as follows:

$$\begin{gathered}
[\varphi(x), m\varphi_0^*(x')]_{t=t'} = i\delta(\mathbf{x}-\mathbf{x}'), \qquad [\widetilde{\varphi}(x), m\widetilde{\varphi}_0^*(x')]_{t=t'} = -i\delta(\mathbf{x}-\mathbf{x}'),\\
[m\widetilde{\varphi}_k(x), H_m^*(x')]_{t=t'} = i\delta_{km}\delta(\mathbf{x}-\mathbf{x}'),\\
[m\varphi_k(x), E_n^*(x')]_{t=t'} = i\delta_{kn}\delta(\mathbf{x}-\mathbf{x}'),
\end{gathered} \tag{84}$$

where $H_m = (1/2)\epsilon_{mnk}\varphi_{nk}$, $E_m = i\varphi_{m4}$. Expressing the components $\varphi_0(x)$, $\widetilde{\varphi}_0(x)$, $\varphi_{\mu\nu}$ from Eqs. (1)-(3), replacing them into Eqs. (84), and taking into consideration the commutators of fields, one arrives at the relations

$$[\varphi(x), \partial_0\varphi^*(x')]_{t=t'} = -im\delta(\mathbf{x}-\mathbf{x}'), \tag{85}$$

$$[\widetilde{\varphi}(x), \partial_0\widetilde{\varphi}^*(x')]_{t=t'} = im\delta(\mathbf{x}-\mathbf{x}'), \tag{86}$$

$$[\varphi_k(x), \partial_0\varphi_m^*(x')]_{t=t'} = \frac{i}{m}\delta_{km}\delta(\mathbf{x}-\mathbf{x}'), \tag{87}$$

$$[\widetilde{\varphi}_k(x), \partial_0\widetilde{\varphi}_n^*(x')]_{t=t'} = -\frac{i}{m}\delta_{kn}\delta(\mathbf{x}-\mathbf{x}'). \tag{88}$$

It follows from Eqs. (85)-(88) that vector and pseudoscalar fields have the positive metric and pseudovector and scalar fields lead to negative metric because of the additional sign (-) [13]. The density of the Hamiltonian is defined by the relation $\mathcal{H}(x) = \pi_A(x)\partial_0\Psi_A(x) - \mathcal{L}(x)$. With the help of Eq. (81), (15), (48), we find

$$\mathcal{H}(x) = i\Psi^+(x)\overline{\Gamma}_4\partial_0\Psi(x)$$

$$= m\Big[\varphi_m^*(x)\partial_0 E_m(x) - E_m^*(x)\partial_0\varphi_m(x) + \varphi_0^*(x)\partial_0\varphi(x) - \varphi^*(x)\partial_0\varphi_0(x) \tag{89}$$

$$- \widetilde{\varphi}_0^*(x)\partial_0\widetilde{\varphi}(x) + \widetilde{\varphi}^*(x)\partial_0\widetilde{\varphi}_0(x) - \widetilde{\varphi}_m^*(x)\partial_0 H_m(x) + H_m^*(x)\partial_0\widetilde{\varphi}_m(x)\Big].$$

From the expression for four-current $J_\mu(x) = i\overline{\Psi}(x)\Gamma_\mu\Psi(x)$, one obtains, with the aid of Eqs. (15), (48), the total electric charge

$$Q = \int d^3x\overline{\Psi}(x)\Gamma_4\Psi(x) = \int d^3x\Psi^+(x)\overline{\Gamma}_4\Psi(x)$$

$$= -im\int d^3x\Big[\varphi_m^*(x)E_m(x) - E_m^*(x)\varphi_m(x) + \varphi_0^*(x)\varphi(x) - \varphi^*(x)\varphi_0(x) \tag{90}$$

$$- \widetilde{\varphi}_0^*(x)\widetilde{\varphi}(x) + \widetilde{\varphi}^*(x)\widetilde{\varphi}_0(x) - \widetilde{\varphi}_m^*(x)H_m(x) + H_m^*(x)\widetilde{\varphi}_m(x)\Big].$$

The quantizing procedure here is similar to one in quantum electrodynamics (QED) for the bispinor fields. But the difference is in the statistics: bispinor fields obey the Fermi-Dirac statistics, and 16-component fields $\Psi(x)$ under consideration, are bosonic fields satisfying the Bose-Einstein statistics. As a result, we have here commutators (83) instead of anticommutators in QED.

It follows from Eqs. (89), (90) that neither energy nor charge of the classical fields under consideration have positive values. Therefore in quantized theory we have to introduce indefinite metric [13]. The total space of states has two subspaces with positive (H_p) and negative (H_n) square norms. According to Eqs. (85)-(88) the vector and pseudoscalar states possess a positive square norm, and pseudovector and scalar states have a negative square norms. The total space of states represents the direct sum of the two subspaces H_p and H_n.

9 Electromagnetic interaction of fields

The "minimal" electromagnetic interaction is introduced by the substitution $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$, where A_μ is the vector-potential of the electromagnetic field. The relativistic non-minimal electromagnetic interaction may be taken into account by consideration of the operators (see Appendix):

$$\overline{\Gamma}_{[\mu}\overline{\Gamma}_{\nu]}\mathcal{F}_{\mu\nu}, \quad \Gamma_{[\mu}\Gamma_{\nu]}\mathcal{F}_{\mu\nu}, \quad \overline{\Gamma}_{[\mu}\Gamma_{\nu]}\mathcal{F}_{\mu\nu}, \tag{91}$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength of the electromagnetic field. Introducing phenomenological parameters κ_1, κ_2, κ_3 in the operators (91), and the replacement $\partial_\mu \to D_\mu$, one obtains from Eq. (20) the relativistic wave equation of the first order for the interaction of bosonic fields under consideration with the electromagnetic field:

$$\begin{gathered}\left[\Gamma_\nu D_\nu + mP_1 + \frac{m_0^2}{m}P_0\right. \\ \left. + \frac{1}{4}\left(\kappa_1\overline{\Gamma}_{[\mu}\overline{\Gamma}_{\nu]} + \kappa_2\Gamma_{[\mu}\Gamma_{\nu]} + \kappa_3\overline{\Gamma}_{[\mu}\Gamma_{\nu]}\right)\mathcal{F}_{\mu\nu}\right]\Psi(x) = 0.\end{gathered} \tag{92}$$

To clear up the physical meaning of constants κ_1, κ_2, κ_3 introduced, we rewrite Eq. (92) in tensor form. With the help of Eqs. (97)-(99) (see Appendix), (15), (19), from Eq. (20), one can obtain the system of equations for interacting scalar, vector, pseudoscalar and pseudovector fields with electromagnetic field:

$$\begin{gathered}D_\mu\psi_\mu + \frac{m_0^2}{m}\psi_0 + \frac{1}{2}\left(\kappa_2 - \kappa_1 - \kappa_3\right)\mathcal{F}_{\mu\nu}\psi_{[\mu\nu]} = 0, \\ D_\mu\widetilde{\psi}_\mu + \frac{m_0^2}{m}\widetilde{\psi}_0 + \frac{i}{2}\left(\kappa_2 - \kappa_1 + \kappa_3\right)\widetilde{\mathcal{F}}_{\mu\nu}\psi_{[\mu\nu]} = 0, \\ D_\nu\psi_{[\mu\nu]} + D_\mu\psi_0 + m\psi_\mu + \left(\kappa_2 + \kappa_1\right)\mathcal{F}_{\mu\nu}\psi_\nu + i\left(\kappa_1 - \kappa_2 - \kappa_3\right)\widetilde{\mathcal{F}}_{\mu\nu}\widetilde{\psi}_\nu = 0, \\ iD_\nu\widetilde{\psi}_{[\mu\nu]} + D_\mu\widetilde{\psi}_0 + m\widetilde{\psi}_\mu + \left(\kappa_2 + \kappa_1\right)\mathcal{F}_{\mu\nu}\widetilde{\psi}_\nu + i\left(\kappa_1 - \kappa_2 + \kappa_3\right)\widetilde{\mathcal{F}}_{\mu\nu}\psi_\nu = 0, \\ D_\nu\psi_\mu - D_\mu\psi_\nu + e_{\mu\nu\alpha\beta}D_\beta\widetilde{\psi}_\alpha + m\psi_{[\mu\nu]} + \left(\kappa_2 + \kappa_1\right)\left(\mathcal{F}_{\nu\beta}\psi_{\mu\beta} - \mathcal{F}_{\mu\beta}\psi_{\nu\beta}\right) \\ + \left(\kappa_1 - \kappa_2 - \kappa_3\right)\mathcal{F}_{\mu\nu}\psi_0 + i\left(\kappa_1 - \kappa_2 + \kappa_3\right)\widetilde{\mathcal{F}}_{\mu\nu}\widetilde{\psi}_0 = 0,\end{gathered} \tag{93}$$

where dual tensors are defined by $\widetilde{\mathcal{F}}_{\mu\nu} = (1/2)\varepsilon_{\mu\nu\alpha\beta}\mathcal{F}_{\alpha\beta}$, $\widetilde{\psi}_{[\mu\nu]} = (1/2)\varepsilon_{\mu\nu\alpha\beta}\psi_{[\alpha\beta]}$ ($\varepsilon_{\mu\nu\alpha\beta} = (-i)e_{\mu\nu\alpha\beta}$). Replacing the variables in Eq. (93) with the help of equalities (see Eqs. (1)-(4)) $\psi_0 = -\varphi$, $\psi_\mu = m\varphi_\mu$, $\psi_{[\mu\nu]} = \varphi_{\mu\nu}$, $\widetilde{\psi}_\mu = im\widetilde{\varphi}_\mu$, $\widetilde{\psi}_0 = -i\widetilde{\varphi}$, one can express the scalar and pseudoscalar fields from Eqs. (93) as follows:

$$\begin{gathered}\varphi = \frac{m^2}{m_0^2}D_\mu\varphi_\mu + \frac{m}{2m_0^2}\left(\kappa_2 - \kappa_1 - \kappa_3\right)\mathcal{F}_{\mu\nu}\varphi_{\mu\nu}, \\ \widetilde{\varphi} = \frac{m^2}{m_0^2}D_\mu\widetilde{\varphi}_\mu + \frac{m}{2m_0^2}\left(\kappa_2 - \kappa_1 + \kappa_3\right)\widetilde{\mathcal{F}}_{\mu\nu}\varphi_{\mu\nu}.\end{gathered} \tag{94}$$

Excluding the scalar and pseudoscalar fields in Eqs. (93) with the aid of Eqs. (94) (with the appropriate replacement of fields), one obtains the system of interacting vector, pseudovector and tensor fields:

$$D_\nu\varphi_{\mu\nu} - \frac{m^2}{m_0^2}D_\mu D_\alpha\varphi_\alpha - \frac{m}{2m_0^2}\left(\kappa_2 - \kappa_1 - \kappa_3\right)D_\mu\left(\mathcal{F}_{\alpha\nu}\varphi_{\alpha\nu}\right)$$

$$+ m^2\varphi_\mu + m(\kappa_1+\kappa_2)\mathcal{F}_{\mu\nu}\varphi_\nu - m(\kappa_1-\kappa_2-\kappa_3)\widetilde{\mathcal{F}}_{\mu\nu}\widetilde{\varphi}_\nu = 0,$$

$$D_\nu\widetilde{\varphi}_{\mu\nu} - \frac{m^2}{m_0^2}D_\mu D_\alpha\widetilde{\varphi}_\alpha - \frac{m}{2m_0^2}(\kappa_2-\kappa_1+\kappa_3)D_\mu\left(\widetilde{\mathcal{F}}_{\alpha\nu}\varphi_{\alpha\nu}\right)$$

$$+ m^2\widetilde{\varphi}_\mu + m(\kappa_1+\kappa_2)\mathcal{F}_{\mu\nu}\widetilde{\varphi}_\nu + m(\kappa_1-\kappa_2+\kappa_3)\widetilde{\mathcal{F}}_{\mu\nu}\varphi_\nu = 0, \qquad (95)$$

$$\varphi_{\mu\nu} = D_\mu\varphi_\nu - D_\nu\varphi_\mu - \varepsilon_{\mu\nu\beta\alpha}D_\beta\widetilde{\varphi}_\alpha - \frac{\kappa_1+\kappa_2}{m}\left(\mathcal{F}_{\nu\beta}\varphi_{\mu\beta} - \mathcal{F}_{\mu\beta}\varphi_{\nu\beta}\right)$$

$$+ \frac{m}{m_0^2}\left[(\kappa_1-\kappa_2-\kappa_3)\mathcal{F}_{\mu\nu}D_\alpha\varphi_\alpha - (\kappa_1-\kappa_2+\kappa_3)\widetilde{\mathcal{F}}_{\mu\nu}D_\alpha\widetilde{\varphi}_\alpha\right]$$

$$+ \frac{(\kappa_1-\kappa_2)^2-\kappa_3^2}{2m_0^2}\left(\widetilde{\mathcal{F}}_{\mu\nu}\widetilde{\mathcal{F}}_{\alpha\beta} - \mathcal{F}_{\mu\nu}\mathcal{F}_{\alpha\beta}\right)\varphi_{\alpha\beta}.$$

It follows from Eqs. (95) that constants κ_1, κ_2, κ_3 are connected with the anomalous magnetic moment (AMM) and quadrupole electric moment (KEM) of a particle [8]. The terms in Eqs. (95) containing m_0 are contributed from the scalar and pseudoscalar states. At the limiting case $m_0 \to \infty$ (masses of a scalar and pseudoscalar states are infinite), one neglects this contribution because the corresponding terms approach to zero. At the particular case $m = m_0$, $\kappa_1 = ie/2m$, $\kappa_2 = \kappa_3 = 0$, we arrive at the description of vector particles with gyromagnetic ratio $g = 2$ [26], [13]. So, the usage of the representation of the Lorentz group with high dimension (in our case dimension is 16) for the description of vector fields allows us to introduce more phenomenological constants of electromagnetic interaction.

10 Conclusion

Equations describing fields which may exist in two mass (m, m_0) and spin (one and zero) states have been considered. Such fields possess the additional symmetry due to the doubling of spin states. This symmetry is an analog of dual transformations in electrodynamics. At the equal masses of spin one and zero fields, we recover the internal symmetry group $SO(4,2)$ (or locally isomorphic to the $U(2,2)$ group) investigated in [12], [13]. This symmetry allows us to construct gauge theories with the non-compact groups [13].

The matrices of the relativistic wave equation obtained obey the Dirac commutation relations. However, the matrix equation contains the additional projection operators which are connected with two mass states of scalar and vector fields. At the particular case of equal masses of scalar and vector states, $m = m_0$, we arrive at the Dirac-like 16-dimensional equation which is equivalent to the Dirac-Kähler equation. Such equation is widely used for describing quarks (spin 1/2 fields) on the lattice. However, we consider here bosonic fields which correspond to the Bose-Einstein statistics. The quantization procedure has been carried out similar to QED, but with the commutation relations of fields instead of the anticommutators. Although the quantization of such fields requires the introduction of the indefinite metric, the theory of fields under consideration can be considered as an effective theory.

The density matrices obtained for spin one fields may be applied for calculations of probabilities of different quantum processes in a covariant manner. The method considered in [23] allows us to make evaluations of physical quantities without using the matrices of first-order equations.

The considered quaternion form of equations is very convenient for the study of internal and Lorentz group symmetries. In these particular cases, one may get from Eqs. (64), (66) well-known and important Maxwell, Proca, etc. equations in the quaternion form.

We may speculate that the fields under consideration can be used for theoretical schemes describing different objects including sub-quark matter. This, however, requires to solve the problem of the physical interpretation of quantum fields with indefinite metric.

We have also studied fields describing particles possessing AMM and KEM and, therefore, having the internal structure. Possibly, this scheme may be applied for the description of composite systems in nuclear and particle physics (see [28]).

APPENDIX. Products of 16-Dimensional Matrices

Consider products of Petiau-Duffin-Kemmer matrices. It is implied that all matrices act in 16-dimensional space. Using the properties of the entire algebra matrices (16), we obtain

$$\begin{gathered}
\beta_\mu^{(1)}\beta_\nu^{(1)}=\varepsilon^{[\alpha\mu],[\alpha\nu]}+\delta_{\mu\nu}\varepsilon^{\alpha,\alpha}-\varepsilon^{\nu,\mu}, \qquad \beta_\mu^{(\widetilde{1})}\beta_\nu^{(1)}=e_{\mu\nu\rho\omega}\varepsilon^{\widetilde{\omega},\rho},\\
\beta_\mu^{(0)}\beta_\nu^{(1)}=\varepsilon^{0,[\mu\nu]}, \qquad \beta_\mu^{(\widetilde{0})}\beta_\nu^{(\widetilde{0})}=\delta_{\mu\nu}\varepsilon^{\widetilde{0},\widetilde{0}}+\varepsilon^{\widetilde{\mu},\widetilde{\nu}},\\
\beta_\mu^{(\widetilde{1})}\beta_\nu^{(\widetilde{0})}=\frac{1}{2}e_{\nu\mu\rho\omega}\varepsilon^{[\rho\omega],\widetilde{0}}, \qquad \beta_\mu^{(1)}\beta_\nu^{(\widetilde{1})}=e_{\mu\nu\rho\omega}\varepsilon^{\omega,\widetilde{\rho}}, \qquad (96)\\
\beta_\mu^{(\widetilde{0})}\beta_\nu^{(\widetilde{1})}=\frac{1}{2}e_{\mu\nu\rho\omega}\varepsilon^{\widetilde{0},[\rho\omega]}, \quad \beta_\mu^{(\widetilde{1})}\beta_\nu^{(\widetilde{1})}=\delta_{\mu\nu}\left(\varepsilon^{\widetilde{\alpha},\widetilde{\alpha}}+\frac{1}{2}\varepsilon^{[\rho\omega],[\rho\omega]}\right)-\varepsilon^{\widetilde{\nu},\widetilde{\mu}}+\varepsilon^{[\alpha\nu],[\mu\alpha]},\\
\beta_\mu^{(1)}\beta_\nu^{(0)}=\varepsilon^{[\nu\mu],0}, \qquad \beta_\mu^{(0)}\beta_\nu^{(0)}=\delta_{\mu\nu}\varepsilon^{0,0}+\varepsilon^{\mu,\nu}.
\end{gathered}$$

We write out only nonzero elements of matrices in 16-dimensional space. With the help of Eqs. (16), (21), (48), one finds the antisymmetric product of 16-dimensional Dirac matrices

$$\frac{1}{2}\Gamma_{[\mu}\Gamma_{\nu]} \equiv \frac{1}{2}\left(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu\right) = \varepsilon^{[\alpha\mu],[\alpha\nu]} - \varepsilon^{[\alpha\nu][\alpha\mu]} + \varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu} + \varepsilon^{\widetilde{\mu},\widetilde{\nu}} - \varepsilon^{\widetilde{\nu},\widetilde{\mu}}$$
$$+ e_{\mu\nu\rho\omega}\left(\varepsilon^{\widetilde{\omega},\rho} - \varepsilon^{\rho,\widetilde{\omega}}\right) + \varepsilon^{0,[\mu\nu]} - \varepsilon^{[\mu\nu],0} + \frac{1}{2}e_{\mu\nu\rho\omega}\left(\varepsilon^{\widetilde{0},[\rho\omega]} - \varepsilon^{[\rho\omega],\widetilde{0}}\right). \tag{97}$$

$$\frac{1}{2}\overline{\Gamma}_{[\mu}\overline{\Gamma}_{\nu]} \equiv \frac{1}{2}\left(\overline{\Gamma}_\mu\overline{\Gamma}_\nu - \overline{\Gamma}_\nu\overline{\Gamma}_\mu\right) = \varepsilon^{[\alpha\mu],[\alpha\nu]} - \varepsilon^{[\alpha\nu][\alpha\mu]} + \varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu} + \varepsilon^{\widetilde{\mu},\widetilde{\nu}} - \varepsilon^{\widetilde{\nu},\widetilde{\mu}}$$
$$- e_{\mu\nu\rho\omega}\left(\varepsilon^{\widetilde{\omega},\rho} - \varepsilon^{\rho,\widetilde{\omega}}\right) - \varepsilon^{0,[\mu\nu]} + \varepsilon^{[\mu\nu],0} - \frac{1}{2}e_{\mu\nu\rho\omega}\left(\varepsilon^{\widetilde{0},[\rho\omega]} - \varepsilon^{[\rho\omega],\widetilde{0}}\right), \tag{98}$$

$$\frac{1}{2}\overline{\Gamma}_{[\mu}\Gamma_{\nu]} \equiv \frac{1}{2}\left(\overline{\Gamma}_\mu\Gamma_\nu - \overline{\Gamma}_\nu\Gamma_\mu\right) = e_{\mu\nu\rho\omega}\left(\varepsilon^{\widetilde{\rho},\omega} + \varepsilon^{\omega,\widetilde{\rho}}\right)$$
$$+ \varepsilon^{0,[\nu\mu]} + \varepsilon^{[\nu\mu],0} + \frac{1}{2}e_{\mu\nu\rho\omega}\left(\varepsilon^{\widetilde{0},[\rho\omega]} + \varepsilon^{[\rho\omega],\widetilde{0}}\right). \tag{99}$$

Using Eqs. (47), we obtain the generators of the Lorentz transformations in the 16-dimensional space of wave function (20):

$$J_{\mu\nu} = \frac{1}{4}\left(\Gamma_{[\mu}\Gamma_{\nu]} + \overline{\Gamma}_{[\mu}\overline{\Gamma}_{\nu]}\right)$$
$$= \varepsilon^{[\alpha\mu],[\alpha\nu]} - \varepsilon^{[\alpha\nu][\alpha\mu]} + \varepsilon^{\mu,\nu} - \varepsilon^{\nu,\mu} + \varepsilon^{\widetilde{\mu},\widetilde{\nu}} - \varepsilon^{\widetilde{\nu},\widetilde{\mu}}. \tag{100}$$

References

[1] M. Fierz and W. Pauli, Proc. Roy. Soc. (London) **A173**, (1939), 211; W. Rarita, J. Schwinger, Phys. Rev. **60** (1941), 61; H. J. Bhabha, Rev. Mod. Phys. **17** (1945), 200; V. Bargman, E. Wigner, Proc. Am. Acad. Sci. **34** (1948), 211; H. Joos, Forschr. Physik **10** (1962), 65; S. Weinberg, Phys. Rev. **B133** (1964), 1318; **B134** (1964), 882; D. L. Weaver, C. L. Hammer, R. H. Good jr., Phys. Rev. **135** (1964), B241; W. J. Hurley. Phys. Rev. **D4** (1971), 3605 ; **D10** (1974), 1185; Phys. Rev. Lett. **29** (1972), 1475; M. A. K. Khalil, Progr. Theor. Phys. **58** (1977), 1538.

[2] A. Proca, Compt. Rend. **202** (1936), 1420.

[3] E. Petiau, Tesis, Paris, 1936.

[4] R. J. Duffin, Phys. Rev. **54** (1939), 1114.

[5] H. Kemmer, Proc. Roy. Soc. **173** (1939), 91.

[6] M. Taketani, S. Sakata, Proc. Phys. Math. Soc. Japan **22** (1939), 757; Progr. Theor. Phys. **22** (1955), 84.

[7] H. C. Corben, J. Schwinger, Phys. Rev. **58** (1940), 953.

[8] J. A. Joung, S. A. Bludman, Phys. Rev. **131** (1963), 2326.

[9] E. C. G. Stueckelberg, Helv. Phys. Acta, **11** (1938), 299.

[10] T. D. Lee, C. N. Yang, Phys. Rev. **128** (1962), 885; L. M. Nath, Nucl. Phys. **57** (1964), 611; A. Sankaranarayanan, R. H. Good jr., Nuovo Cim. **36** (1965), 1303; Phys. Rev. **140** (1965), B509; A. Sankaranarayanan, Nuovo Cim. **38** (1965), 889; D. Shay, R. H. Good jr., Phys. Rev. **179** ((1969)), 1410; F. I. Fedorov, Dokl. Akad. Nauk. SSSR **82** ((1952)), 37; Y. Takahachi, R. Palmer, Phys. Rev. **D1** (1970), 2974; R. H. Tucker, C. L. Hammer, Phys. Rev. **D3** (1971), 2448; E. Kyriakopoulos, Phys. Rev. **D1** (1972), 2207.

[11] E. Kähler, Rendiconti di Matematica (3-4) **21** (1962), 425.

[12] S. I. Kruglov, Int. J. Theor. Phys. **41** (2002), 653; arXiv:hep-th/0110251.

[13] S. I. Kruglov, Symmetry and Electromagnetic Interaction of Fields with Multi-Spin (Nova Science Publishers, Huntington, New York, 2001, 216 pp).

[14] P. Becher, Phys. Lett. **B104** (1981), 221; J. M. Rabin, Nucl. Phys. **B201** (1982), 315; P. Becher, H. Joos, Z. Phys. **C15** (1982), 343; T. Banks, Y. Dothan and D. Horn, Phys. Lett. **B117** (1982), 413; H. Aratyn, A. H. Zimerman, Phys. Rev. **D33** (1986), 2999; H. Joos, M. Schaefer, Z. Phys. **C34** (1987), 365; R. Edwards, D. Ritchie, D. Zwanziger, Nucl. Phys. **B296** (1988), 961; A. N. Jourjine, Phys. Rev. **D35** (1987), 757.

[15] D. Ivanenko and L. Landau, Zeitsch. für Phys. **48** (1928), 340; C. Lanczos, Z. Physik **57** (1929), 447, 474, 484; A. W. Conway, Proc. Roy. Soc. **A162** (1937), 145; A. Ericsson, Ark. f. math., astron. och fys., 34/21 (1948), 1; B. Bruno, Ark. f. math., astron. och fys., 34/22 (1948), 1; H. Feschbach and W. Nickols, Ann. of Phys. **4** (1958), 448.

[16] S. I. Kruglov, Int. J. Mod. Phys. **A16** (2001), 4925; arXiv:hep-th/0110083.

[17] S. I. Kruglov, Hadronic J. **24** (2001), 167; arXiv:quant-ph/0110042.

[18] S. I. Kruglov, Int. J. Theor. Phys. **40** (2001), 515; arXiv:hep-ph/9908410.

[19] S.I. Kruglov, Annales Fond. Broglie **26**, no.1 (2001), 125; arXiv:math-ph/0110008.

[20] A. A. Bogush and L. G. Moroz, Introduction to the Theory of Classical Fields (Nauka i Tekhnika, Minsk, 1968) (in Russian).

[21] A. A. Borgardt, Zh. Eksp. Teor. Fiz. **24** (1953), 24, 284; **30** (1956), 334 [Sov. Phys. - JETP **3**, 238 (1956)]; **33** (1957), 791 [Sov. Phys. - JETP **6**, 608 (1958)].

[22] S. I. Kruglov, Eur. Phys. J. **C22** (2001), 89; arXiv:hep-ph/0110100.

[23] F. I. Fedorov, Sov. Phys. - JETP **35**(8) (1959), 339 [Zh. Eksp. Teor. Fiz. **35** (1958), 493]; The Lorentz Group (Nauka, Moscow, 1979) (in Russian).

[24] Gaston Casanova, L'algèbre Vectorielle (Presses Universitaires de France, 1976).

[25] P. A. M. Dirac, Lectures on Quantum Mechanics (Yeshiva University, New York, 1964).

[26] E. Durand, Phys. Rev. **D11** (1975), 3405.

[27] N. N. Bogolyubov and D. V. Shirkov (1980). Introduction to the Theory of Quantized Fields (John Wiley & Sons Ltd.).

[28] M. Kirchbach, Int. J. Mod. Phys. **A15** (2000), 1435.